Allen-Cahn Approximation of Mean Curvature Flow in Riemannian manifolds II, Brakke’s flows
Adriano Pisante* and Fabio Punzo†

Abstract
We prove convergence of solutions to the parabolic Allen-Cahn equation to Brakke’s motion by mean curvature in Riemannian manifolds, generalizing previous results from [15] in Euclidean space. We show that a sequence of measures, associated to energy density of solutions of the parabolic Allen-Cahn equation, converges in the limit to a family of rectifiable Radon measures, which evolves by mean curvature flow in the sense of Brakke. A key role is played by a local almost monotonicity formula (a weak counterpart of Huisken’s monotonicity formula) proved in [22], to get various density bounds for the limiting measures.

Keywords: Allen-Cahn equation, Riemannian manifold, Huisken’s monotonicity formula, mean curvature flow, Brakke’s inequality, varifolds.

1 Introduction
In [22] we started to investigate the Allen-Cahn equation
\begin{equation}
\partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) \text{ in } M \times (0, \infty),
\end{equation}
completed with the initial condition
\begin{equation}
u^\varepsilon = u^\varepsilon_0 \text{ in } M \times \{0\},
\end{equation}
were \( \varepsilon > 0 \) is a small parameter and \( M \) is an \( N \)-dimensional Riemannian manifold with Ricci curvature bounded from below.

We suppose that the nonlinearity is the negative gradient of a double well potential \( F \) with two minima of equal depth. More precisely, we assume for

* Dipartimento di Matematica "G. Castelnuovo", Università di Roma “La Sapienza”, P.le A. Moro 5, I-00185 Roma, Italia (pisante@mat.uniroma1.it).
† Dipartimento di Matematica "G. Castelnuovo", Università di Roma “La Sapienza”, P.le A. Moro 5, I-00185 Roma, Italia (punzo@mat.uniroma1.it).
In addition, of diffuse interfaces, the regions say

\[ \mu(0) = f(\pm 1) = 0, f < 0 \text{ in } (0, 1), f > 0 \text{ in } (1, \infty), \]

\[ f'(0) < 0, f'(\pm 1) > 0; \]

\[ F > 0 \text{ in } \mathbb{R} \setminus \{\pm 1\}, F(\pm 1) = 0; \]

\[ \min_{[a, \infty)} F'' > 0, \text{ for some } \alpha \in (0, 1). \]

For any \( \varepsilon > 0 \) and for any \( (x, t) \in M \times [0, \infty) \), set

\[ E^\varepsilon(x, t) := \frac{1}{2} \nabla u^\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u^\varepsilon) \quad (x \in M, t \geq 0), \]

and define the energy density

\[ d\mu^\varepsilon := \left\{ \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) \right\} d\mathcal{V}(x). \]

Clearly,

\[ d\mu^\varepsilon_t(x) = \varepsilon E^\varepsilon(x, t) d\mathcal{V}(x) \quad (x \in M, t \geq 0). \]

Let \( E_0 \subset M \) be an open bounded subset with smooth boundary \( \partial E_0 = \Sigma_0 \). Hence there exist \( C_0 > 0, R_0 > 0 \) such that

\[ \mathcal{H}^{N-1}(\Sigma_0 \cap B_R(x)) \leq C_0 \omega_{N-1} R^{N-1} \]

for all \( 0 < R < R_0 \). In the end, the hypersurface \( \Sigma_0 \) will be the initial datum for an evolution by mean curvature, in a suitable weak sense, obtained as limit of diffuse interfaces, the regions say \{ \( |u^\varepsilon| \leq \alpha \} \), where \( u^\varepsilon \) solve \((1.1)-(1.2)\).

Concerning the initial conditions \( u_0^\varepsilon \) in \((1.2)\) (and the corresponding \( \mu_0^\varepsilon \equiv \mu^\varepsilon(\cdot, 0) \) given by \((1.3)\)) we always assume the following:

\[ \begin{cases} 
(i) & \mu_0^\varepsilon \to \alpha \mathcal{H}^{N-1}[\Sigma_0] \text{ as } \varepsilon \to 0 \text{ as Radon measures, for some } \alpha \geq 0; \\
(ii) & u_0^\varepsilon \to 2\chi_{E_0} - 1 \text{ as } \varepsilon \to \infty \ast \text{ weakly in } BV_{loc}(M); \\
(iii) & \text{there exists } C_0 > 0 \text{ such that } \frac{\omega_{N-1}}{C_0} \mu_0^\varepsilon(B_R(x)) \leq \omega_{N-1} R^{N-1} \leq C_0 \quad \text{for all } x \in M, 0 < R < R_0, 0 < \varepsilon < 1; \\
(iv) & \text{there exists } k_0 > 0 \text{ such that } \|u_0^\varepsilon\|_\infty \leq H_0; \\
(v) & \text{there exists } C > 0 \text{ such that for any } 0 < \varepsilon < 1 \|\nabla u_0^\varepsilon\|_\infty \leq \frac{C}{\varepsilon}. 
\end{cases} \]

\((H_1)\)

Under hypotheses \((H_0),(H_1)\) in [22] it is shown that problem \((1.1)-(1.2)\) admits a unique bounded solution. Moreover, \( u^\varepsilon \in C^\infty(M \times (0, \infty)) \cap C(M \times [0, \infty)) \);

\[ |u^\varepsilon| \leq k_0 \quad \text{for all } x \in M, t > 0. \]

In addition,

\[ \sup_{\varepsilon > 0} \sup_{t \in (0, \infty)} \int_M E^\varepsilon d\mathcal{V} \leq C_2 \]

where \( C_2 := \sup_{\varepsilon > 0} \mu_0^\varepsilon(M) \); \( t \mapsto \int_M E^\varepsilon(x, t) d\mathcal{V}(x) \) is nonincreasing for \( t > 0 \).
Moreover, recalling the definition of discrepancy Radon measure
\[ d\xi^\varepsilon_t := \left( \frac{\varepsilon}{2} \nabla u^\varepsilon_t^2 - \frac{1}{\varepsilon} F(u^\varepsilon_t) \right) d\nu, \]
it is proved that
\[ \limsup_{\varepsilon \to 0^+} \sup_{(x,t) \in Q} \xi^\varepsilon_t (x) \leq 0, \tag{1.9} \]
for each compact subset \( Q \subset M \times (0, \infty) \). Using (1.9) is then shown that
\[ \frac{d}{dt} \int_M \phi(x,t) d\mu^\varepsilon_t \leq \frac{C_3}{\sqrt{s-t}} \int_M \phi(x,t) d\mu^\varepsilon_t + C_4 + \frac{C_5}{\sqrt{s-t}} \tag{1.10} \]
for all \( 0 \leq t < s \), for some positive constants \( C_3, C_4, C_5 \) independent of \( \varepsilon \).

Here, for any fixed reference point \((y,s) \in M \times (0, \infty)\), \( \phi(x,t) \equiv \phi(x,t; y,s) \) is a suitable kernel, depending explicitly on the Riemannian distance \( d(x,y) \) for \( x,y \in M \) as follows
\[ \phi(x,t) = \tilde{\zeta}(d^2(x))(s-t) - \frac{N-1}{4} e^{-\frac{d^2(x)}{s-t}}. \tag{1.11} \]

In contrast with the case of \( \mathbb{R}^N \), it has a suitably small compact support in space due to the cut-off function \( \tilde{\zeta} \). In addition, we were able to deduce the previous inequality in full generality, without assuming well prepared initial data as in [15].

Clearly, inequality (1.10) does not imply monotonicity for the function \( t \to \int_M \phi(x,t) d\mu^\varepsilon_t(x) \). Nevertheless, it still allows us to control the behavior of \( d\mu^\varepsilon_t \) at small scales. For this reasons, we refer to (1.10) as a local almost monotonicity formula, in analogy with the monotonicity formula valid in the Euclidean space.

As a consequence of (1.10), we were able to obtain the inequality
\[ \mathcal{G}(t) \leq e^{\frac{C_3}{(\sqrt{s-t_0} - \sqrt{s-t})}} [\mathcal{G}(t_0) + C_4(t-t_0) + C_5((\sqrt{s-t_0} - \sqrt{s-t})], \tag{1.12} \]
for all \( 0 \leq t_0 < t < s \), where
\[ \mathcal{G}(t) := \int_M \phi(x,t) d\mu^\varepsilon_t (0 \leq t < s). \]

Actually, (1.9), (1.10) and (1.12) remain true (see [22] for details), if instead of (H1), we only assume that (1.6) is satisfied, and that for each compact subset \( K \subset M, T > 0 \), there holds:
\[ \sup_{\varepsilon > 0} \sup_{t \in (0,T)} \varepsilon \int_K E^\varepsilon d\nu \leq \overline{C} \tag{1.13} \]
for some constant \( \overline{C} > 0 \) depending on the compact subset \( K \) and on \( T \), but independent of \( \varepsilon \). In addition, for each compact subset \( K \subset M \) we have that
\[ (G_1^\varepsilon) \qquad \int_M \phi(x,t;y,s) d\mu^\varepsilon_t(x) \leq \overline{C} \quad \text{for all} \ y \in M, 0 \leq t < s, \]
for some \( \overline{C} = C_K > 0 \), and
\[ (G_2^\varepsilon) \quad \mu^\varepsilon_t(B_R(x)) \leq \omega_{N-1} D_0 R^{N-1} \quad \text{for all} \ x \in K, 0 < R < \tilde{R}, t \geq 0, \]
for some $0 < \tilde{R} < R_0$ and $D_0 = D_0(\mathcal{C}, \tilde{R}) > 0$ independent of $\varepsilon$.

In the present paper, extending results from [15] in the Euclidean space, we describe the asymptotic behaviour of the family of measures $\{\mu^\varepsilon_t\}$ as $\varepsilon \to 0$ on Riemannian manifolds. First of all, adapting the semidecreasing trick from [15]-[16], we prove that we can extract a subsequence $\mu^\varepsilon_{n_t}$ that, for every $t > 0$, converges as Radon measure on $\mathcal{M}$ to a limit Radon measure $\mu_t$ for all $t \geq 0$ as $n \to \infty$.

Using (1.10), a version of Brakke’s Clearing-Out Lemma [8, Lemma 6.3] is established, similar to [15, Lemma 6.1]. Note that our proof is direct and self-contained. In particular, it does not rely on the so-called Empty Spot Lemma [15, Lemma 6.4] and on the related results on propagation of fronts (see [9]), which at present are not available on Riemannian manifolds. However, we also point out that the same strategy as [15] could be adapted to the present situation (see Remark 4.5). Furthermore, we do not use the gradient bound coming from the assumption of well prepared data. In particular, it is shown that if $(y, s) \notin \bigcup_{t \geq 0} \text{supp} \mu^\varepsilon_t \times \{t\}$, then there exists a neighborhood $U \subset \mathcal{M} \times [0, \infty)$ of $(y, s)$ such that $\{u^\varepsilon_n\}$ converges uniformly in $U$ to either 1 or $-1$ as $n \to \infty$. Such a result, well known in the Euclidean case, shows once more absence of evolving interface where there is no energy concentrating in the limiting measures $\mu_t$. In addition, an estimate for the size of the bad set follows, showing that $\mathcal{H}^{N-1}(\text{supp} \mu_t)$ is locally finite for a.e. $t > 0$.

With the local almost monotonicity formula (1.10) at disposal, we adapt to the present situation the strategy of [15] and show that the discrepancy measure $d\xi^\varepsilon_t$ converges to 0 as $\varepsilon \to 0^+$. Indeed, in view of (1.9) it is enough to consider the negative part of the limiting discrepancy $d\xi_t$. In addition, at $|d\xi|-$a.e. point in space-time a suitable (forward) density of $\mu_t$ defined through (1.11) is shown to be both zero (as a consequence of (1.10)) and strictly positive (because of the Clearing-Out lemma), so the discrepancy has to vanish identically.

Thus, we obtain all preliminary results necessary to pass to limit as $\varepsilon \to 0^+$, in the sense of varifolds, in the Brakke’s type equality (7.2) satisfied by $\mu^\varepsilon_t$ (see Section 6). Hence $N-1$-rectifiability for the limiting measures $\mu_t$ for a.e. $t > 0$ and Brakke’s inequality, namely

$$\int M \{ -\phi H^2_t + \langle \nabla \phi, T_x \mu^\varepsilon_t(\tilde{H}_t) \rangle \} d\mu_t,$$

for all $\phi \in C^2_c(M; \mathbb{R}^+)$ and for every $t > 0$, follows at once (all the terms in the formula being actually well defined and finite for a.e. $t > 0$). Here $\overline{\nabla}_t$ is the upper derivative of $
abla \phi(x)d\mu_t(x)$, where $\tilde{H}_t$ is the mean curvature vector associated to the varifold corresponding to $\mu_t$ and $T_x \mu^\varepsilon_t$ is the orthogonal projection onto the normal space to the measure (see Section 2 for precise definitions). Note that in this paper we do not address the issue of integrality for the limiting rectifiable measures $\mu_t$ (or, equivalently, for the corresponding varifolds). This should follow from a careful adaptation of the subtle result in [25] (see also [27]) valid in the Euclidean space. In particular, once integrality is established one would have $\tilde{H}_t \perp T_x \mu_t$ a.e. (see [8]) and in particular no projection operator in (7.3). In addition, we do not investigate
partial regularity property of the solution we construct. In this respect, when trying to discuss this issue, especially in connection with the so-called 'unit density hypothesis' for the limiting varifolds, it would be natural to generalize the recent partial regularity results from [19], [26] valid in the Euclidean case.

As a final remark we observe that, among several possible ways to obtain global weak solutions of the mean curvature flow on manifolds, such as the level-set approach via viscosity solutions (see e.g. [11], [17], [4] or [10]), the method of barriers (see e.g [5], [6]) or the geometric measure theory approach via varifolds, currents or BV functions (see, e.g. [8], [16] or [20]), we decided to adapt the Allen-Cahn approximation from [17]. In our opinion this approach seems more promising in order to flow unbounded initial hypersurfaces with only locally finite area. Such problem arises naturally for example when trying to evolve complete noncompact surfaces in the hyperbolic space. Indeed, unbounded minimal hypersurfaces with prescribed boundary at infinity in \( \mathbb{H}^N \) exist in abundance and can be constructed e.g. by the stationary phase-field approximation analogous to (1.1) (see e.g. [21]). As it was the main motivation for the present research, we plan in a future paper to study convergence to such equilibria under mean curvature flow in \( \mathbb{H}^N \) for unbounded hypersurfaces with fixed boundary at infinity and the connections of such evolution with the renormalized area studied e.g. in [2].

2 Mathematical background: varifolds, rectifiable Radon measures, rectifiable varifolds, first variation, mean curvature

In this Section we recall some preliminaries from Geometric Measure Theory (for more details see, e.g., [3], [23], [16]).

To begin with, recall that by Nash Embedding Theorem, we can assume that \( M \) is isometrically embedded in \( \mathbb{R}^L \) for some \( L \geq N \).

Let \( G(L,k) \) be the Grassman manifold of unoriented \( k \)-planes in \( \mathbb{R}^L \) \((k \leq N)\); let

\[
G_k (M) := \{(x,S) \in M \times G(L,k) : S \subset T_x M \}.
\]

A general \( k \)-varifold is a Radon measure on \( G_k(M) \). We denote by \( \mathbb{V}_k(M) \) the set of all general \( k \)-varifolds, and we give it the topology corresponding to the weak-* convergence of Radon measures. We write:

\[
V(\psi) = \int_{G_k(M)} \psi(x,S)dV(x,S),
\]

where \( \psi \in C^0_c(G_k(M)) \).

Denote by \( \mathcal{M}^+(M) \) the set of all positive Radon measures on \( M \). Given \( \mu \in \mathcal{M}^+(M) \) for any \( x \in M, \lambda > 0, 1 \leq k \leq N \) we can define the scaled Radon measure \( \mu_{x,\lambda} \in \mathcal{M}(\mathbb{R}^L) \) by

\[
\mu_{x,\lambda}(A) = \frac{\mu[M \cap (\lambda A + x)]}{\lambda^k} \quad (A \subseteq \mathbb{R}^L).
\]
Let $\mathcal{P}$ a $k$--plane in $T_x M$ and $\alpha > 0$. We say that $\mathcal{P} \equiv T_x \mu$ is the $k$--dimensional approximate tangent plane of $\mu$ at $x$, if

$$
\lim_{\lambda \to 0^+} \mu_{x, \lambda} = \alpha \mathcal{H}^k[\mathcal{P}],
$$

where $\mathcal{H}^k$ is the $k$--dimensional Hausdorff measure in $\mathbb{R}^k$. The $k$--dimensional approximate tangent space of a set $X \subseteq M$ at $x \in M$ is defined by

$$
T_x X := T_x(\mathcal{H}^k[X]),
$$

if it exists. We say that $X$ is countably $k$--rectifiable, if $X \subseteq C_0 \cup (\cup_{i \geq 1} C_i)$, where $\mathcal{H}^k(C_0) = 0$ and each $C_i \subseteq M$ is an embedded $C^1 k$--submanifold. We say that $X \subseteq M$ is locally $k$--rectifiable if in addition $X$ has locally finite $\mathcal{H}^k$--measure. If $X$ is locally $k$--rectifiable and $\mathcal{H}^k$--measurable, then $T_x X$ exists $\mathcal{H}^k[X]$--a.e.

Let $X$ be an $\mathcal{H}^k$--measurable subset of $M$, and let $\theta : M \to [0, \infty)$ be locally $\mathcal{H}^k$--integrable, with $X = \{\theta > 0\} \mathcal{H}^k$--a.e. The Radon measure $\mu$ on $M$ is a $k$--rectifiable Radon measure, if either of the following equivalent conditions holds:

(a) $\mu$ has $k$--dimensional tangent planes of positive multiplicity $\mu$--a.e.;

(b) $\mu = \mathcal{H}^k[\theta]$ for some $X$ which is $\mathcal{H}^k$--measurable and countably $k$--rectifiable, and $\theta$ locally $\mathcal{H}^k$--integrable.

We denote by $\mathcal{M}_k(M)$ the set of $k$--rectifiable Radon measures on $M$. We call $\mu$ an integer $k$--rectifiable Radon measure, if either of the following equivalent conditions hold:

(c) $\mu$ has $k$--dimensional tangent planes of positive integer multiplicity $\mu$--a.e.;

(d) $\mu = \mathcal{H}^k[\theta]$ for $X \mathcal{H}^k$--measurable and locally $\mathcal{H}^k$--rectifiable, and $\theta$ locally $\mathcal{H}^k$--integrable with values in $\mathcal{N}$.

We write $\mathcal{I}_k \mathcal{M}_k(M)$ for the set of all such $\mu$.

Associated to a varifold $V$ there is a Radon measure $\mu_V$ on $M$, defined by

$$
\mu_V := \pi_2(V),
$$

where $\pi : G_k(M) \to M$ is the natural projection map. On the other hand, if $\mu$ is a $k$--rectifiable Radon measure, then $x \mapsto T_x \mu$ is a $\mu$--measurable section of $G_k(M)$ defined $\mu$--a.e. Therefore, we can define the varifold $V = V_\mu$ by

$$
V_\mu(\psi) = \int_{G_k(M)} \psi(x, S)dV_\mu(x, S) = \int_M \psi(x, T_x \mu)d\mu(x)
$$

for $\psi \in C^0_c(G_k(M))$. Observe that $\mu = \|V\|$, where $\|V\|$ is the Radon measure defined as follows:

$$
\|V\|(A) := V\left(G_k(M) \cap \pi^{-1}(A)\right) \text{ whenever } A \subset M.
$$

We denote by $\mathcal{RV}_k(M)$ the set of $k$--rectifiable varifolds, i.e. the varifolds associated to $k$--rectifiable Radon measures on $M$; whereas, by $\mathcal{IV}_k(M)$ the set of integer $k$--rectifiable varifolds, i.e. the varifolds associated to integer $k$--rectifiable Radon measures on $M$. 

6
When $S$ is a $k$–plane in $G(T_xM,k)$, we also use $S$ to denote the orthogonal projection from $T_xM$ onto $S$. Furthermore, we write $A : B$ for the inner product of endomorphisms $A$ and $B$ on $T_xM$.

Let $U \subseteq M$ be an open subset; we set $C^1_c(U,TU) := \{ Y \in \Gamma(TM) : \text{supp} Y \subset U \}$. When $U = M$, we write $C^1_c(TM)$ instead of $C^1_c(M,TM)$.

Let $Y \in C^1_c(TM)$. Recall the first variation formula for a varifold

$$\delta V(Y) = \int_{G_k(M)} S : DYdV(x,S);$$

here

$$(S : DY)(x) = \sum_{i=1}^k \langle De_i Y(x),e_i \rangle,$$

$\{e_1,\ldots,e_k\}$ being any orthonormal basis of $S$ and $De_i Y$ the corresponding covariant derivatives. For $U \subseteq M$ open, define the total first variation by

$$|\delta V|(U) = \sup \left\{ \delta V(Y) : Y \in C^1_c(U), |Y| \leq 1 \right\}.$$ 

If $\delta V$ is a Radon measure and $|\delta V| < \|V\|$, then

$$\delta V(Y) = \int_{G_k(M)} S : DYdV(x,S) = -\int_M \langle Y(x), \vec{H}(x)\rangle d\|V\|(x),$$

where $\vec{H} : M \to TM$ is the Radon-Nikodym derivative of $\delta V$ with respect to $\|V\|$. By definition $\vec{H}$ is called the mean curvature vector field. When, in addition, $V = V_\mu$ is also a $k$–rectifiable varifold, then

$$\delta V_\mu(Y) = \int_M T_x\mu : DYd\mu = -\int_M \langle \vec{H}, Y \rangle d\mu.$$

### 3 Passing measures to limits

In this section we address compactness of the family of Radon measures $\{\mu^\varepsilon_t\}_{\varepsilon > 0}$.

At first, recall the following lemma (see Lemma 6.6 in [16]).

**Lemma 3.1** Let $\varphi \in C^2_c(M;[0,\infty))$. Then

$$\frac{\|\nabla \varphi\|^2}{\varphi} \leq 2 \max_{\{\varphi > 0\}} |\text{Hess } \varphi| \text{ in } \{\varphi > 0\}.$$ 

The next key-fact is known as the semidecreasing property for the family of measures $\{\mu^\varepsilon_t\}_{\varepsilon > 0}$.

**Lemma 3.2** Let assumption $(H_0)$ be satisfied. Let $u^\varepsilon$ be the solution to problem (1.1)–(1.2). Suppose that (1.6) and (1.13) hold true. Let $\varphi \in C^2(M;\mathbb{R}^+) \text{ with supp } \varphi \text{ compact}$. Then, for any $T > 0$, the function

$$t \mapsto \mu^\varepsilon_t(\varphi) - C(\varphi)t \ (t \in (0,T))$$

is nonincreasing, for some constant $C = C(\varphi,T) > 0$. 


Proof. By (1.1),
\[
\frac{d}{dt} \int_M \varepsilon \varphi(x) \mathcal{E}(x,t) d\mathcal{V}(x) = - \int_M \varepsilon \varphi \left( -\Delta u^\varepsilon + \frac{1}{\varepsilon^2} f(u^\varepsilon) \right)^2 d\mathcal{V}(x)
\]
\[+ \int_M \varepsilon \langle \nabla \varphi, \nabla u^\varepsilon \rangle \left(-\Delta u^\varepsilon + \frac{1}{\varepsilon^2} f(u^\varepsilon) \right) d\mathcal{V}(x). \tag{3.1}\]

Hence
\[
\frac{d}{dt} \int_M \varepsilon \varphi(x) \mathcal{E}(x,t) d\mathcal{V}(x)
\]
\[\leq - \int_M \varepsilon \varphi \left( -\Delta u^\varepsilon + \frac{1}{\varepsilon^2} f(u^\varepsilon) - \frac{\langle \nabla \varphi, \nabla u^\varepsilon \rangle}{2\varphi} \right)^2 d\mathcal{V}(x)
\]
\[+ \int_M \varepsilon |\nabla u^\varepsilon|^2 |\nabla \varphi|^2 d\mathcal{V}(x) \leq C_1(\varepsilon) \mu_t^\varepsilon \{ \varphi > 0 \} \leq C
\]
for all \( t \in (0,T) \), for some positive constant \( C = C(\varphi,T) \) in view of Lemma 3.1 and the uniform bound (1.13). So, the conclusion follows. \( \square \)

Now, we define the kernel \( \phi \) used in (1.10). In fact, let \( K \subset M \) be a compact subset, \( y \in K, s > 0 \). Let \( \hat{\zeta} \in C^2([0,\infty)) \) such that
\[
|\hat{\zeta}| \leq 1, \quad |\hat{\zeta}'| \leq 1, \quad |\hat{\zeta}''| \leq 1 \quad \text{in} \quad [0,\infty), \tag{3.2}\]
\[
\hat{\zeta} = \begin{cases} 1 & \text{in} \quad [0, R_0^2/4) \\ 0 & \text{in} \quad [R_0^2, \infty), \end{cases} \tag{3.3}\]
where \( R_0 := \inf_{y \in K} \text{inj}(y) \). Define
\[
\hat{\eta}(\rho,t) := [(s-t)]^{-\frac{N+1}{2}} e^{-\frac{\rho^2}{4(s-t)}} \quad (\rho \geq 0, 0 \leq t < s).
\]

For any \( x \in M \), let
\[
\eta(x,t) := \hat{\eta}(d^2(x),t) \quad (x \in M, 0 \leq t < s). \tag{3.4}
\]
\[
\zeta(x) := \hat{\zeta}(d^2(x)) \quad (x \in M).
\]

Finally, define
\[
\phi(x,t) \equiv \phi(x,t;y,s) := \eta(x,t)\zeta(x) \quad (x \in M, 0 \leq t < s). \tag{3.5}
\]

In view of (1.13) and the above monotonicity property in Lemma 3.2 we can repeat the argument in [8] (see also [12]), to show the next compactness result.

**Proposition 3.3** Let assumption \((H_0)\) be satisfied. Let \( u^\varepsilon \) be the solution to problem (1.1)–(1.2). Suppose that (1.6) and (1.13) hold true. Then there are a Radon measure \( \mu_t \) on \( M \) and a sequence \( \{ \varepsilon_n \} \subset (0,\infty) \), \( \varepsilon_n \to 0 \) as \( n \to \infty \) such that, for every \( t > 0 \),
\[
\mu_t^\varepsilon_n \to \mu_t \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad n \to \infty \tag{3.6}
\]
as Radon measure on $M$. Furthermore, for each compact subset $K \subset M$ we have

\[
(G_1) \quad \int_M \phi_{y,s}(x,t)d\mu_t(x) \leq C \quad \text{for all } y \in K, 0 \leq t < s,
\]

and

\[
(G_2) \quad \mu_t(B_R(x)) \leq \omega_{N-1} D_0 R^{N-1} \quad \text{for all } x \in K, 0 < R < \tilde{R}, t \geq 0.
\]

**Proof.** In view of $(G_2^2)$ and the above monotonicity property we can repeat the argument in [8] (see also [15]), to show (3.6). Furthermore, under hypothesis $(H_0)$, as a consequence of $(G_1^1)$ and $(G_2^2)$ from the introduction, we get $(G_1)$ and $(G_2)$.

\[\Box\]

### 4 Clearing-Out Lemma

In this section we will prove the *Clearing-Out Lemma*. This result roughly says that energy concentration occurs only near the interface region, e.g. $\{ |u^\varepsilon| \leq \alpha \}$.

In particular, we show that as $\varepsilon \to 0^+$ the solution $u^\varepsilon$ converges to either 1 or $-1$ in a neighborhood of any point which does not belong to $\bigcup_{\varepsilon \geq 0} \text{supp } \mu_\varepsilon \times \{ \varepsilon \}$, where $\mu_t$ is the limit Radon measure obtained in Section 3.

For each $y \in M, s > 0$ we shall write

\[
\phi(x, t; y, s) \equiv \phi_{y,s}(x,t), \quad x \in M, 0 \leq t < s.
\]

Observe that because of (3.5) we clearly have $\phi_{y,s}(x,t) = \phi_{x,s}(y,t)$.

**Lemma 4.1** Let assumption $(H_0)$ be satisfied. Let $u^\varepsilon$ be the solution to problem (1.1)-(1.2). Suppose that (1.6) and (1.13) hold true. Then

(i) there exists $\kappa_1 > 0, \kappa_2 > 0$ depending on $N, R_0$ and $F$ such that, if for some $s > t \geq 0$,

\[
s - t < \kappa_1,
\]

and

\[
\int_M \phi_{y,s}(x,t)d\mu_t(x) < \kappa_2,
\]

then there exists a neighborhood $V \subset M \times [0, \infty)$ of $(y, s)$ such that

\[
|u^\varepsilon| \geq \alpha \quad \text{in } V
\]

for all $\varepsilon > 0$ small enough. As a consequence,

\[
(y, s) \notin \bigcup_{\varepsilon \geq 0} \text{supp } \mu_\varepsilon \times \{ \varepsilon \}.
\]

(ii) If $(y, s) \notin \bigcup_{\varepsilon \geq 0} \text{supp } \mu_\varepsilon \times \{ \varepsilon \}$, then there exists a neighborhood $U \subset M \times [0, \infty)$ of $(y, s)$ such that $\{u^\varepsilon_n\}$ converges uniformly in $U$ to either 1 or $-1$ as $n \to \infty$.

9
To prove the Clearing-Out Lemma we need the following technical lemma, which parallels Lemma 3.4 in [15] and will be used to prove Lemma 4.1. The proof is standard and is omitted for brevity.

Set
\[
\phi_r^y(x) = \phi_{y,s}(x,t) \quad \text{whenever} \quad r^2 = 2(s-t),
\]
so
\[
\phi_r^y(x) := \zeta(x) \frac{1}{r^{N-1}} e^{- \frac{(d(x,y))^2}{2r^2}}, \quad x \in M.
\]

**Lemma 4.2** Let \( \mu \) be a measure satisfying for each \( K \subset M \)
\[
\mu(B_R(y)) \leq \omega_{N-1} \tilde{D}_0 R^{N-1} \quad \text{for all} \quad y \in K, 0 < R < R_0
\]
for some \( \tilde{D}_0 = \tilde{D}_0(K) > 0 \). Then, for some positive constant \( D = D(R_0, K) \), we have:

(i) \( \int_M \phi_r^y d\mu \leq D; \)
(ii) for any \( r > 0, 0 < R < R_0, y \in M \)
\[
\int_{M \setminus B_R(y)} \phi_r^y d\mu \leq 2^{N-1} D e^{- \frac{2r^2}{4r^2}};
\]
(iii) for any \( \delta > 0 \) and \( \bar{r} > 0 \) there exists \( \gamma_2 = \gamma_2(\delta, \bar{r}) > 0 \) such that for any \( r \in (0, \bar{r}), d(y, y_1) \leq \gamma_2 r \) we have
\[
\int_M \phi_{y_1}^r d\mu \leq (1 + \delta) \int_M \phi_r^y d\mu + D\delta;
\]
(iv) for any \( R > 0, 0 < r < R_0 \) with \( r \leq R \) we have:
\[
\int_M \phi_r^y d\mu \leq \left( \frac{R}{r} \right)^{N-1} \int_M \phi_r^R d\mu;
\]
(v) for any \( \delta > 0 \) there exists \( \gamma_3 = \gamma_3(\delta) > 0 \) such that for any \( r \in (0, \bar{r}), d(y, y_1) \leq \gamma_3 r \) and for any \( y \in M \) we have:
\[
\int_M \phi_r^y d\mu \leq (1 + \alpha(\delta)) \int_M \phi_r^y d\mu + D\delta;
\]
(vi) for any \( \delta > 0 \) there exists \( \alpha = \alpha(\delta) > 0 \) such that for all \( r > 0, y \in M \),
\[
\int_M \phi_r^{\alpha(\delta)y} d\mu \leq \frac{\mu(B_r)}{\omega_N [\alpha(\delta)]^{N-1} r^{N-1}} + \delta D.
\]

Now we can prove the Clearing-Out Lemma.

**Proof of Lemma 4.1.** 1. In view of Lemma 4.2 (iii) – (v) we can find a neighborhood \( U \subset M \times [0, \infty) \) of \( (y, s) \) such that for all \((y', s') \in U\)
\[
0 < \inf_U (s' - t) \leq s' - t < \kappa_1, \quad (4.5)
\]
\[
\int_M \phi_{y', s'}(x,t) d\mu(x) < \kappa_2 \quad \text{for all} \quad (y', s') \in U.
\]
We may assume $U \subset M \times (t, \infty)$. By Proposition 3.3 and $(G_1^s)$,

$$\int_M \phi_{y',s'}(x,t) d\mu^\varepsilon_s(x) \leq 3\kappa_2 \quad (4.6)$$

for all $n > n_0$ (for some $n_0 \in \mathbb{N}$) and for all $(y',s') \in U$. By [22] Proposition 3.2], for any compact subset $K \subset M$ and for any $\tau \in (0,T)$ there exists a constant $\tilde{k} > 0$ independent of $\varepsilon$ such that

$$\|\nabla u^\varepsilon(\cdot,t)\|_{L^\infty(K)} \leq \frac{\tilde{k}}{\varepsilon} \quad \text{for all } t \in (\tau,T). \quad (4.7)$$

We claim that, for some $C > 0$, if $d(x,y') \leq \rho$, then

$$[F(u^\varepsilon(y',s'))]^2 \leq C \int_M \phi_{y',s'+\frac{\varepsilon^2}{2}}(x,s') d\mu^\varepsilon_s(x), \quad (4.8)$$

where

$$\rho \equiv \rho(y',s') := \frac{\varepsilon}{2Lk} F(u^\varepsilon(y',s')), \quad (4.9)$$

and $L$ is the Lipschitz constant of $F$ in $[-k_0, k_0]$.

In fact, whenever $d(x,y') \leq \rho$, we have

$$F(u^\varepsilon(y',s')) = F(u^\varepsilon(x,s')) + F(u^\varepsilon(y',s')) - F(u^\varepsilon(x,s'))$$

$$\leq F(u^\varepsilon(x,s')) + L |u^\varepsilon(y',s') - u^\varepsilon(x,s')| \leq F(u^\varepsilon(x,s')) + L\tilde{k}d(x,y')$$

$$\leq F(u^\varepsilon(x,s')) + \frac{L\tilde{k}}{\varepsilon} \rho.$$ 

Hence

$$F(u^\varepsilon(y',s')) \leq 2F(u^\varepsilon(x,s')).$$

This easily implies that, for some $C > 0$,

$$F(u^\varepsilon(y',s')) \leq \frac{C}{\rho^{N-1}} \int_{B_\rho(y')} F(u^\varepsilon(x,s')) d\nu(x) = \frac{C}{\rho^{N-1}} \int_{B_\rho(y')} \frac{F(u^\varepsilon(x,s'))}{\rho} d\nu(x).$$

So,

$$[F(u^\varepsilon(y',s'))]^2 \leq \frac{C}{\rho^{N-1}} \int_{B_\rho(y')} \frac{F(u^\varepsilon(x,s'))}{\varepsilon} d\nu(x) \leq C \int_M \phi_{y',s'+\frac{\varepsilon^2}{2}}(x,s') d\mu^\varepsilon_s(x).$$

By (1.12),

$$\int_M \phi_{y',s'+\frac{\varepsilon^2}{2}}(x,s') d\mu^\varepsilon_s(x)$$

$$\leq e^{C_\delta(\sqrt{s'} + \frac{\rho^2}{2} - \frac{\rho}{\sqrt{2}})} \left[ \int_M \phi_{y',s'+\frac{\varepsilon^2}{2}}(x,t) d\mu^\varepsilon_t(x) + C_4 \left( s' - t \right) + C_5 \left( \sqrt{s'} + \frac{\rho^2}{2} - t - \frac{\rho}{\sqrt{2}} \right) \right]. \quad (4.9)$$
Now if \( r^2 = 2(s' - t), R^2 = 2 \left( s' + \frac{\rho^2}{2} - t \right) \), in view of (4.3) and \( \rho = O(\varepsilon) \), then 
\[
\frac{R}{r} \geq 1, \quad \frac{R}{r} \to 1 \text{ as } \varepsilon \to 0 \text{ uniformly in } U. 
\]
So, from Lemma 4.2 (v) with \( \mu = \mu^\varepsilon \), for any \( \delta > 0 \), for \( \varepsilon > 0 \) small enough, we have
\[
\int_M \phi_{y',s'} \frac{r^2}{2} (x,t) d\mu^\varepsilon_t \leq (1 + \delta) \int_M \phi_{y',s'} (x,t) d\mu^\varepsilon_t + D\delta. \tag{4.10}
\]
In view of (4.8), (4.6) and (4.10), we get
\[
F(\mu^\varepsilon(y',s')) \leq e^{C_3 \sqrt{\kappa_1}} [(1 + \delta)\kappa_2 + D\delta + C_4 \kappa_1 + C_5 \sqrt{\kappa_1}]. \tag{4.11}
\]
If we select \( \kappa_1 > 0, \kappa_2 > 0 \) and \( \delta > 0 \) small enough, from (4.11) and \( (H_0) - (iv) \) it follows that
\[
|u^\varepsilon| \geq \alpha \tag{4.12}
\]
for all \((y',s') \in U\) and for any \( \varepsilon > 0 \) sufficiently small.

4. By the same arguments as in [22, Lemma 3.5], there exists a neighborhood of \((y, s), V := B_R(y) \times I \subset U\), such that
\[
1 - \varepsilon \leq |u^\varepsilon| \leq 1 + \varepsilon \quad \text{in } V. \tag{4.13}
\]
Let \( s' \in I \). From (1.9) we have that for any \( \delta' > 0 \) there exists \( \varepsilon_{\delta'} > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_{\delta'}) \) there holds:
\[
\mu^\varepsilon_{s'}(B_{R/2}(y)) = \int_{B_{R/2}} \left[ \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) \right] d\mathcal{V}(x) \leq \int_{B_R} \left[ \frac{2}{\varepsilon} F(u^\varepsilon) + \delta' \right] d\mathcal{V}(x) \leq \int_{B_R} \left( \frac{2}{\varepsilon} F(1 - \varepsilon) + \frac{2}{\varepsilon} F(1 + \varepsilon) + \delta' \right) d\mathcal{V}(x)
\]
\[
\leq \int_{B_{R/2}} \left[ C \varepsilon \left( \sup_{|s| \leq \kappa_0} |F''(s)| \varepsilon \right)^2 + \delta' \right] d\mathcal{V}(x) \leq \int_{B_R} \left( \tilde{C} \varepsilon + \delta' \right) d\mathcal{V}(x) = \mathcal{V}(B_{R/2}) (\tilde{C} \varepsilon + \delta') ;
\]
here use of (4.13) has been made. Letting \( \varepsilon \to 0 \), and then \( \delta' \to 0^+ \), we get
\[
\mu^\varepsilon_{s'}(B_R(2)(y)) = 0 \quad \text{for } s' \text{ near } s, \]
and
\[
u^\varepsilon_n \to \pm 1 \quad \text{uniformly in a neighborhood of } (y, s);
\]
thus, (i) has been shown.

To prove (ii), let \((y, s) \notin \bigcup_{t \geq 0} \text{supp } \mu_t \times \{t\} \); hence
\[
\int_M \phi_{y,s'} d\mu^\varepsilon_{s'} \to 0 \quad \text{as } s' \to s^-,
\]
thus,
\[
u(y', s') = \lim_{n \to \infty} \nu^\varepsilon_n(y', s') = \pm 1 \quad \text{near } (y, s).
\]
\[\square\]

As a simple consequence we have the following result.
Corollary 4.3 We have
\[ \text{supp } \mu = \bigcup_{t' \geq 0} \text{supp } \mu_{t'} \times \{t'\}, \]
where
\[ d\mu := d\mu_{t'} dt'. \]

Proof. It is obvious that \( \text{supp } \mu \subseteq \bigcup_{t' \geq 0} \text{supp } \mu_{t'} \times \{t'\}. \) Now, let \((y, s) \notin \text{supp } \mu.\) Then we can find an open subset \( U \subseteq M \times [0, \infty) \) such that \((y, s) \in U, U \cap \text{supp } \mu = \emptyset.\) Hence \( \int_M \phi_{y,s}(x,t)d\mu_t \to 0 \) as \( t \to s^- \). By Lemma 4.1, \((y, s) \notin \bigcup_{t' \geq 0} \text{supp } \mu_{t'} \times \{t'\};\) this completes the proof. \( \square \)

Another consequence of the Clearing-Out Lemma is that one can control the size of the set where the energy is concentrating.

Corollary 4.4 Let \( U \subseteq M \) be an open subset. Then there exists \( C_5 > 0 \) and \( C_6 > 0 \) such that if \((\text{supp } \mu)_{t} := \text{supp } \mu \cap (M \times \{t\}),\)
\begin{enumerate}
\item \( \mathcal{H}^{N-1}( (\text{supp } \mu)_{t} \cap K ) \leq C_5 \liminf_{s \to t^-} \mu_s(U) \) for every \( t > 0, \)
\item \( \mathcal{H}^{N-1}( (\text{supp } \mu)_{t} \cap B_R ) \leq C_6 R^{N-1} \) for every \( 0 < R < R_0 \) and \( t \geq 0. \)
\end{enumerate}

Proof. Clearly, (i) follows, if we show that
\[ \mathcal{H}^{N-1}( (\text{supp } \mu)_{t} \cap K ) \leq C \liminf_{s \to t^-} \mu_s(U) \]
for every \( t > 0, \) for every compact subset \( K \subseteq U. \)

Let \((y, t) \in (\text{supp } \mu)_{t} \cap K,\) take any \( \delta > 0 \) and \( \alpha = \alpha(\delta) > 0 \) given by Lemma 4.2(vi). For each \( 0 < r < \sqrt{\kappa}, \) by Lemma 4.1
\[ \kappa_2 \leq \int_M \phi_y^{\alpha r} d\mu_{t-\frac{\alpha r^2}{2}}. \tag{4.14} \]
which combined with Lemma 4.2(vi) yields
\[ \kappa_2 \leq \frac{\mu_{t-\frac{\alpha r^2}{2}}(B_r)}{\omega_N r^{N-1} [\alpha(\delta)]^{N-1}} + D\delta. \]
The previous inequality with \( \delta = \frac{\kappa_2}{2D} \) gives
\[ \kappa_2 \leq \frac{2 \mu_{t-\frac{\alpha r^2}{2}}(B_r)}{\omega_N r^{N-1} [\alpha(\delta)]^{N-1}}. \tag{4.15} \]
Let \( r > 0 \) such that \( \text{dist}(K, \partial U) > r \) and consider the covering of \( \text{supp } \mu_{t} \cap K \)
\[ B = \{ B_r(x) : x \in (\text{supp } \mu)_{t} \}. \]
By the Besicovitch Covering Theorem, which can be applied for compact subset of Riemannian manifolds, (see, e.g., Theorem 1.1.4 and Example 1.15 (c) in [14]), there are finitely many countable subcollections \( B_1, \ldots, B_{\ell} \) of \( B \) such that each \( B_i \) is made of disjoint balls, and
\[ (\text{supp } \mu)_{t} \cap K \subseteq \bigcup_{i=1}^{\ell} \bigcup_{y_j \in B_i} B_r(y_j). \]
We have, for some $\tilde{C} > 0$, an estimate for the pre-Hausdorff measures

$$H^{N-1}_r((\text{supp } \mu)_t \cap K) \leq \tilde{C} \sum_{i=1}^\ell \sum_{B_r(y_j) \in B_i} \omega_N r^{N-1} \left\{ x : \text{dist}(x, K) \leq r \right\} \leq \tilde{C} \sum_{i=1}^\ell \frac{2}{\alpha^{N-1} \kappa_2} \mu_{t-\alpha^2r^2/2}(B_r(y_j)) \leq \frac{2\tilde{C}\ell}{\alpha^{N-1} \kappa_2} \mu_{t-\alpha^2r^2/2}(U).$$

Sending $r \to 0^+$, we obtain

$$H^{N-1}_r((\text{supp } \mu)_t \cap K) \leq C \liminf_{s \to t^-} \mu_s(U),$$

so, (i) has been proven. Furthermore, (ii) follows by (i) and ($G_2$). \hfill \Box

**Remark 4.5** Notice that the Clearing-out Lemma could also be proved analogously to [15]. In that case, we need the so-called Empty Spot Lemma, which could be deduced in the present situation, too. Indeed, it is mainly based on a result given in [9] in Euclidean space, concerning propagation of interfaces, that could be easily shown also in a general Riemannian manifold $M$. To do this the key role is played by an important property of the the signed distance $\tilde{d}(x, t)$ from $\partial \Sigma_t$, $\Sigma_t$ being a family of sets evolving by mean curvature flow, starting from a sphere $\partial B_R(x_0)$, with $R > 0$ small enough. Indeed, as in [4, Section 7], in a tubular neighborhood of $\Sigma_t$, one has:

$$|\partial_t \tilde{d}(x, t) - \Delta \tilde{d}(x, t)| \leq \tilde{C}|\tilde{d}(x, t)|.$$  \hspace{1cm} (4.16)

This is all what is needed to conclude.

## 5 Density lower bound

The result of this section, roughly speaking, shows that a suitable $(N-1)$--density of $\mu$ (forward density in the terminology of [15]) is bounded below on the support of the measure. More precisely, we show that an explicit lower bound holds $H^{N-1}_r$--a.e. on each time-slice of $\text{supp } \mu$.

In the sequel, we take $\kappa_2$ as in Lemma 4.1. Define

$$Z^- := \left\{ (x, t) \in \text{supp } \mu : \limsup_{s \to t^-} \int_{M} \phi_{y,s} d\mu_s(y) < \kappa_2 \right\},$$

$$Z^-_t := Z^- \cap (M \times \{t\}).$$

**Lemma 5.1** For any $\tilde{\sigma} > 0$, $H^{N-2+\tilde{\sigma}}(Z^-_t) = 0$ for a.e. $t \geq 0$. 

14
\textbf{Proof.} 1. It is direct to see that, for each $\tau > 0$,

$$Z^- = \bigcup_{0 < \tau < \bar{\tau}, \kappa_2 < \kappa_3} Z^{\kappa_3, \tau},$$

where

$$Z^{\kappa_3, \tau} := \left\{ (x, t) \in \text{supp } \mu : \int_M \phi_{y,s} d\mu_s(y) \leq \kappa_3 \text{ for all } s \in (t, t + \tau) \right\}.$$

Hence, the thesis will follow, if we prove that $\mathcal{H}_t^{N-2+\bar{\delta}}(Z_{\kappa_3, \tau}) = 0$ for each $\kappa_3 < \kappa_2, 0 < \tau < \kappa_1$.

\textit{Claim.} Let $\delta > 0, s \in [t, t + \tau], \gamma_2 = \gamma_2(\delta, \sqrt{2\tau})$ be the constant given by Lemma \ref{lem:4.2}(iii), $r := \sqrt{2(s - t)}, t' := s + s^2 = t + r^2$. If

$$t' - t \leq 2\tau \quad \text{and} \quad d(x, x') \leq \gamma_2 r,$$

then

either $(x, t) \notin Z^{\kappa_3, \tau}$ or $(x', t') \notin Z^{\kappa_3, \tau}$.

Indeed, more precisely, we are going to show that if $(x, t) \in Z^{\kappa_3, \tau}$, then

$$\mathcal{P}^{(x, t)}_{2\tau} \cap Z^{\kappa_3, \tau} = \{(x, t)\},$$

(5.1)

where

$$\mathcal{P}^{(x, t)}_{2\tau} := \left\{ (x, t) \in M \times [0, \infty) : \frac{d^2(x', x)}{\gamma_2} \leq t' - t \leq 2\tau \right\}.$$

In fact, let $(x, t) \in Z^{\kappa_3, \tau}$. By Lemma \ref{lem:4.2}(iii), for any $x' \in B_{\gamma_2 r}(x)$, we have:

$$\int_M \phi_{x', s + r^2/2}(y, s) d\mu_s(y) = \int_M \phi_{x', y} d\mu_s(y) \leq (1 + \delta) \int_M \phi_{x', y} d\mu_s(y) + D\delta = (1 + \delta) \int_M \phi_{x, y} d\mu_s(y) + D\delta \leq (1 + \delta)\kappa_3 + D\delta < \kappa_2,$$

for $\delta > 0$ sufficiently small. By Lemma \ref{lem:4.1} $(x', t') \notin \bigcup_{\xi \geq 0} \text{supp } \mu_\xi$; thus, $(x', t') \notin Z^{\kappa_3, \tau}$. This proves the claim.

2. For every $x_0 \in M, t_0 > 0$, define

$$Z' := Z^{\kappa_3, \tau} \cap (B_1(x_0) \times [t_0 - \tau, t_0 + \tau]).$$

Since $Z^{\kappa_3, \tau}$ is a countable union of such $Z'$, the thesis follows, if we show that $\mathcal{H}_t^{N-2+\bar{\delta}}(Z'_t) = 0$ for a.e. $t \geq 0$, where $Z'_t := Z' \cap (M \times \{t\})$.

Observe that the set $Z' \cap \{x \times \mathbb{R}\}$ contains at most one point, for $\mathcal{P}^{(x, t)}_{2\tau}$ is higher than $Z'$ when $(x, t) \in Z'$, in view of (5.1).

Let $\pi$ be the (nearest point) projection from $M \times [0, \infty)$ onto $M \times \{0\}$; so $\pi(Z') \subset B_1(x)$. Let $\delta_1 > 0$. There exist sequences $\{x_i\}_{i \in \mathbb{N}} \subset \pi(Z')$ and $\{r_i\}_{i \in \mathbb{N}} \subset (0, \delta_1)$ with

$$\sum_{i=1}^{\infty} \omega_N r_i^N \leq 2V(B_1(x_0)),$$

(5.2)
such that
\[ \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \supseteq \pi(Z'). \]

In view of Step 1,
\[ Z' \subseteq B_{r_i}(x_i) \times [t_i - r_i^2/\gamma^2, t_i + r_i^2/\gamma^2], \]
where \((x_i, t_i) := \pi^{-1}(x_i)\). Thus, for some \( \tilde{C} > 0 \), we have:
\[ \int_{t_0 - \tau}^{t_0 + \tau} H_{\delta_i}^{N-2+\tilde{\sigma}}(Z'_i) dt \leq \sum_{i \in \mathbb{N}: t \in [t_i - r_i^2/\gamma^2, t_i + r_i^2/\gamma^2]} \tilde{C} \gamma^2 r_i^{N+\tilde{\sigma}} \]
\[ \leq 2 \tilde{C} D \delta^2 \mathcal{V}(B_1(x_0)). \]

Sending \( \delta_1 \to 0^+ \), by the monotone convergence theorem
\[ \int_{t_0 - \tau}^{t_0 + \tau} H_{\delta_i}^{N-2+\tilde{\sigma}}(Z'_i) dt = 0. \]
This implies the result. \( \Box \)

6 Vanishing of the limit discrepancy

The purpose of this section is to show that the discrepancy Radon measure vanishes as \( \varepsilon \to 0^+ \), up to subsequences. More precisely, define
\[ d\xi^\varepsilon = d\xi^\varepsilon dt, \quad d\mu^\varepsilon := d\mu^\varepsilon dt. \]
Since \( |\xi^\varepsilon| \leq \mu^\varepsilon \), by Proposition 3.3 we can assume that there exist a Radon measure \( \xi \) on \( M \times [0, \infty) \), and a subsequence of \( \{\varepsilon_n\} \), which will be still denoted by \( \{\varepsilon_n\} \), such that
\[ \xi^\varepsilon_n \to \xi, \quad \mu^\varepsilon_n \to \mu := d\mu dt \quad \text{as} \quad n \to \infty \]
as Radon measure on \( M \times [0, \infty) \). By (1.9), \( \xi \leq 0 \). Indeed, we are going to show the following result.

Proposition 6.1 There holds \( \xi = 0 \).

In order to prove Proposition 6.1 we use the following lemma (see [22]).

Lemma 6.2 Let assumption \((H_0)\) be satisfied. Let \( u^\varepsilon \) be the solution to problem (1.1) - (1.2). Suppose that (1.6) and (1.13) hold true. Let \( K \subset M \) be a compact subset, \( y \in K, s > 0 \). Let \( \phi := \eta \zeta \) with \( \zeta \) and \( \eta \) as in Section 3. Then for every \( \varepsilon > 0 \)
\[ \frac{d}{dt} \int_M \phi(x, t)d\mu^\varepsilon (x) \leq \frac{1}{2(s - t)} \int_M \phi^\varepsilon \{ |\nabla u^\varepsilon|^2 - E^\varepsilon \} d\mathcal{V}(x) \]
\[ + \frac{C_3}{(s - t)^{1/2}} \int_M \phi d\mu^\varepsilon (x) + C_4 \quad \text{for all} \quad 0 < t < s. \]

for some positive constants \( C_3 \) and \( C_4 \) only depending on \( K \).
Proof of Proposition 6.1. Step 1. Given \( x_0 \in M \) and \( K = \overline{B_{R_0}(x_0)} \), in view of \((G_1)\), integrating \( (6.1) \) we get

\[
- \int_0^{s-\bar{\sigma}} \int_M \frac{1}{2(s-t)} \phi_{y,s}(x,t)d\xi(x)dt \leq \int_M \phi_{y,s}(x,0)d\mu_0(x) + C_3C\sqrt{s} + C_4s
\]

for every \((y, s) \in M \times (0, \infty), 0 < \bar{\sigma} < s\). Letting \( \varepsilon \rightarrow 0 \), using Lemma 4.2(i) and \((G_1)\), since \( \xi \leq 0 \), we obtain a uniform bound, for \( s \) in a compact set, namely

\[
\int_0^{s-\bar{\sigma}} \int_M \frac{1}{2(s-t)} \phi_{y,s}(x,t)d\xi((x,t)) \leq \int_M \phi_{y,s}(x,0)d\mu_0(x)
+ C_3C\sqrt{s} + C_4s \leq C + C_3C\sqrt{s} + C_4s =: \bar{C}.
\]

Now, take \( T > 0, 0 < R < R_0 \). Integrating over \( B_R(x_0) \times (0, T + 1) \) we have:

\[
\int_1^{T+1} \int_{B_R(x_0)} \int_0^{s-\bar{\sigma}} \int_M \frac{1}{2(s-t)} \phi_{y,s}(x,t)d\xi((x,t))d\mu_s(y)ds \leq \bar{C}\omega_{N-1}D_0(T + 1)R^{N-1}.
\]

Hence, by Tonelli theorem,

\[
\int_0^{T+1-\bar{\sigma}} \int_M \int_{t+\bar{\sigma}}^{T+1} \frac{1}{2(s-t)} \int_{B_R(x_0)} \phi_{y,s}(x,t)d\mu_s(y)dsd\xi((x,t)) \leq \bar{C}\omega_{N-1}D_0(T + 1)R^{N-1}.
\]

Sending \( \bar{\sigma} \rightarrow 0 \), by the monotone convergence theorem we have

\[
\int_0^{T+1} \int_M \int_t^{T+1} \frac{1}{2(s-t)} \int_{B_R(x_0)} \phi_{y,s}(x,t)d\mu_s(y)dsd\xi((x,t)) \leq \bar{C}\omega_{N-1}D_0(T + 1)R^{N-1}.
\]

So,

\[
\int_t^{T+1} \frac{1}{2(s-t)} \int_{B_R(x_0)} \phi_{y,s}(x,t)d\mu_s(y)ds \leq C(x, t) < \infty
\]

for \( |\xi| \)-a.e. \((x, t) \in M \times (0, T)\).

Step 2. For any \( x \in B_{R/2}(x_0), s > t > 0 \) we have:

\[
\int_M \phi_{y,s}(x,t)d\mu_s(y) = \int_{B_R(x_0)} \phi_{y,s}(x,t)d\mu_s(y) + \int_{M\setminus B_R(x_0)} \phi_{y}^{2(s-t)}(x)d\mu_s(y)
\leq \int_{B_R(x_0)} \phi_{y,s}(x,t)d\mu_s(y) + \int_{M\setminus B_{R/2}(x_0)} \phi_{y}^{2(s-t)}(x)d\mu_s(y)
\leq \int_{B_R(x)} \phi_{y,s}(x,t)d\mu_s(y) + 2^{N-1}e^{-\frac{1}{8}(R/2)^2/2}D;
\]
here the fact that $B_{R/2}(x) \subset B_R(x_0)$ for any $x \in B_{R/2}(x_0)$, and Lemma 4.2 (ii) have been used. Thus for $|\xi|$– a.e. $(x, t) \in B_{R/2}(x_0) \times [0, T]$,

$$\int_t^{t+1} \frac{1}{2(s-t)} \int_M \phi_{y,s}(x,t)d\mu_s(y)ds \leq C(x,t) + \int_t^{t+1} \frac{1}{2(s-t)}2^{N-1}e^{-\frac{\sqrt{e^{\beta_n} - e^\beta}}{\pi}(s-t)}D < \infty.$$  \hspace{1cm} (6.2)

Since $T > 0$ and $x_0 \in M$ were arbitrary, (6.2) holds for $|\xi|$–a.e. $(x, t) \in M \times [0, \infty)$.

**Step 3.** Now, take $(x, t) \in M \times [0, T]$ such that (6.2) holds true. We shall prove that

$$\lim_{s \to t^+} \int_M \phi_{y,s}(x,t)d\mu_s(y) = 0.$$  

Let $\beta(s) := \log(s-t)$, and

$$h(s) := \int_M \phi_{y,s}(x,t)d\mu_s(y).$$

Hence, (6.2) implies

$$\int_{-\infty}^{0} h(t+e^\beta)d\beta < \infty.$$  \hspace{1cm} (6.3)

We are going to show that (6.3) yields $h(t+e^\beta) \to 0$ as $\beta \to -\infty$.

**Step 4.** Let $\gamma \in (0, 1]$ to be specified later. By (6.3), there exists a sequence $\{\beta_n\} \subset (-\infty, 0)$ such that

$$\lim_{n \to \infty} \beta_n = -\infty, \quad 0 < \beta_n - \beta_{n+1} \leq \gamma, \quad h(t+e^{\beta_n}) \leq \gamma.$$  \hspace{1cm} (6.4)

Let $\beta \in (-\infty, \beta_1]$ and assume $\beta \in [\beta_n, \beta_{n-1})$ for some $n \in \mathbb{N}$. Since $\beta_n \leq \beta$, from (4.12) we get

$$h(t+e^\beta) = \int_M \phi_{y,t+e^\beta}(x,t)d\mu_{t+e^\beta}(y)$$

$$= \int_M \phi_{x,t+e^\beta}(y,t+e^\beta)d\mu_{t+e^\beta}(y)$$

$$\leq e^{C_2}\left(\sqrt{e^{\beta_n} - e^\beta} - \sqrt{e^{\beta_n} - e^\beta}\right) \left[ \int_M \phi_{x,t+e^\beta}(y,t+e^\beta)d\mu_{t+e^\beta}(y) \right]$$

$$+ C_4(e^\beta - e^{\beta_n}) + C_5\left(\sqrt{e^{2\beta} - e^{\beta_n}} - \sqrt{e^{2\beta} - e^\beta}\right)$$

$$= e^{C_2}\left(\sqrt{e^{2\beta} - e^{\beta_n}} - \sqrt{e^{2\beta} - e^\beta}\right) \left[ \int_M \phi_{x,t+e^\beta}(y,t+e^\beta)d\mu_{t+e^\beta}(y) \right]$$

$$+ C_4(e^\beta - e^{\beta_n}) + C_5\left(\sqrt{e^{2\beta} - e^{\beta_n}} - \sqrt{e^{2\beta} - e^\beta}\right),$$  \hspace{1cm} (6.5)

where $R := \sqrt{2(e^{\beta_n} - e^\beta)}$. Furthermore, in view of (6.4), we have

$$\gamma \geq h(t+e^{\beta_n}) = \int_M \phi_{y,t+e^{\beta_n}}(x,t)d\mu_{t+e^{\beta_n}}(y) = \int_M \phi_{y}(y)d\mu_{t+e^{\beta_n}}(y),$$  \hspace{1cm} (6.6)
where \( r := \sqrt{2e^{\beta_n}} \). Note that
\[
1 \leq \frac{R}{r} = \sqrt{2e^{\beta_n} - 1} \leq 1 + \tilde{C}\gamma. \tag{6.7}
\]

**Step 5.** Let \( \delta > 0 \) and set \( \gamma = \min\{\delta, \gamma_3(\delta)/\tilde{C}\} \), where \( \gamma_3 \) is given by Lemma 4.2(iv). By (6.5)-(6.6) and Lemma 4.2(v),
\[
\begin{align*}
  h(t + e^\beta) &\leq e^{C_\beta} \left( \sqrt{e^{2\beta} - e^{\beta_n}} - \sqrt{e^{2\beta} - e^{\beta}} \right) \left[ \int_M \phi_x R d\mu_{t+e^\beta} (y) 
    + C_4(e^\beta - e^{\beta_n}) + C_5 \left( \sqrt{e^{2\beta} - e^{\beta_n}} - \sqrt{e^{2\beta} - e^{\beta}} \right) \right] 
    + C_4(e^\beta - e^{\beta_n}) + C_5 \left( \sqrt{e^{2\beta} - e^{\beta_n}} - \sqrt{e^{2\beta} - e^{\beta}} \right) \tag{6.8}
\end{align*}
\]
for all \( \beta \in [\beta_n, \beta_n - 1) \). So, letting \( \delta \to 0^+ \) (and thus \( \gamma \to 0^+ \)), as \( \beta \to -\infty \) (hence \( \beta_n \to -\infty \)) we obtain
\[
\lim_{s \to t^+} h(s) = 0 \quad \text{for } |\xi| - \text{a.e. } (x, t) \in M \times (0, T). \tag{6.8}
\]

**Step 6.** By Lemma 5.1, for any \( \tilde{\sigma} > 0 \),
\[
\limsup_{s \to t^+} h(s) \geq \kappa_2 > 0 \quad d(\mathcal{H}^{N-2+\tilde{\sigma}} | \text{supp } \mu_t ) dt - \text{a.e. } (x, t) \in M \times (0, \infty). \tag{6.9}
\]
On the other hand, in view of \((G_2)\), for all \( \bar{x} \in M, 0 < R < R_0 \), in \( B_R(\bar{x}) \times [0, T] \) for any \( \tilde{\sigma} \in (0, 1) \) there holds:
\[
 d|\xi| \leq d\mu = d\mu_t dt << d(\mathcal{H}^{N-2+\tilde{\sigma}} | \text{supp } \mu_t ) dt. \tag{6.10}
\]
By (6.8)-(6.10),
\[
0 < \kappa_2 \leq \limsup_{s \to t^+} h(s) = 0 \quad |\xi| - \text{a.e. } (x, t) \in M \times (0, T).
\]
This implies \( \xi|_{B_R(\bar{x}) \times (0, T)} = 0 \). Since \( \bar{x} > 0 \) and \( T > 0 \) were arbitrary, the conclusion follows. \( \square \)
7 Brakke’s inequality for the limit measure

In this section we establish the main result of the present paper. In fact, we prove that the limit measure $\mu_t$ evolves by mean curvature flow, in the sense of Brakke. To state this result precisely, we need some notations.

Recall that the upper derivative of a function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ at $x_0 \in \mathbb{R}$ is given by

$$D_{x_0} \psi := \lim \sup_{x \rightarrow x_0} \frac{\psi(x) - \psi(x_0)}{x - x_0}.$$ 

Let $\phi \in C^2_c(M; \mathbb{R}^+)$; for any $\mu \in \mathcal{M}(M)$ define:

$$B(\mu, \phi) \equiv -\infty$$

whenever either of the following holds:

(i) $\mu\{\phi > 0\} \notin \mathcal{M}_k(M)$;
(ii) $|\delta \mu\{\phi > 0\}$ is not absolutely continuous with respect to $\mu\{\phi > 0\}$;
(iii) $\int_M \phi H^2 d\mu = \infty$.

Otherwise,

$$B(\mu, \phi) := \int_M \left\{ -\phi H^2 + \langle \nabla \phi, T x \mu^\perp(\vec{H}) \rangle \right\} d\mu.$$

Define

$$B^\varepsilon(u^\varepsilon, \phi) := -\varepsilon \int_M \left\{ \phi \left( -\Delta u^\varepsilon + \frac{f(u^\varepsilon)}{\varepsilon^2} \right)^2 + \langle \nabla \phi, \nabla u^\varepsilon \rangle \left( -\Delta u^\varepsilon + \frac{f(u^\varepsilon)}{\varepsilon^2} \right) \right\} dV.$$

and observe that, in view of (3.1), for any $t > 0$ we have

$$\frac{d}{dt} \mu_t(\phi) = B^\varepsilon(u^\varepsilon, \phi).$$

Our purpose is pass to the limit as $\varepsilon \rightarrow 0^+$ in (7.2). Indeed, to do it appropriately we shall use suitable varifolds (see Subsection 2) associated to $u^\varepsilon$, and results proved in Sections 3, 5, 6. Our main result is as follows.

**Theorem 7.1** Let assumptions (H_0) be satisfied. Let $u^\varepsilon$ be the solution to problem (1.1)-(1.2). Suppose that (1.6) and (1.13) hold true. Then the family of Radon measures $\{\mu_t\}_{t \geq 0}$ from Proposition 3.3 are $(N-1)$-rectifiable for a.e. $t > 0$ and satisfies the Brakke’s inequality:

$$\nabla_t \mu_t(\phi) \leq B(\mu, \phi)$$

for every $\phi \in C^2_c(M; \mathbb{R}^+)$ and for every $t > 0$, where

$$\mu_t(\phi) \equiv \int_M \phi(x)d\mu_t(x).$$

Before going into the proof of the main result, let us introduce varifolds that will be used in the sequel.
Let $t_0 \geq 0; \{t_n\}_{n \in \mathbb{N}} \subset [0, \infty), t_n \to t_0$. Let $u^\varepsilon$ be a family of equibounded solutions to problem (1.1)-(1.2). Let $\{\varepsilon_n\} \subset (0, 1), \varepsilon_n \to 0$; consider the sequence of functions

$$\{u^n\} = \{u^{\varepsilon_n}(\cdot, t_n)\}.$$  

Define

$$d\mu^n := \left(\varepsilon_n^2 |\nabla u^n|^2 + \frac{F(u^n)}{\varepsilon_n^2}\right) d\mathcal{V};$$

$$d\xi^n := \left(\varepsilon_n^2 |\nabla u^n|^2 - \frac{F(u^n)}{\varepsilon_n^2}\right) d\mathcal{V}.$$  

By standard results in unique continuation for parabolic equations, for each $t > 0$ and $n \in \mathbb{N}$,

$$\mathcal{V}(\{\nabla u^{\varepsilon_n}(\cdot, t) = 0\}) = 0.$$  

(7.4)

So, for all $n \in \mathbb{N}$, we can define the $(N - 1)$-varifold $V^n$ by

$$\int_M \psi(x, S)dV^n(x, S) = \int_M \psi(x, \nabla u^n(x)^\perp)d\mu^n(x)$$

for any $\psi \in C_c^0(G_{N-1}(M); \mathbb{R})$. Note that $\|V\| = \mu^n$. Define

$$\mathfrak{B}^n(u^n, \phi) := -\varepsilon_n \int_M \left\{ \phi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n} \right)^2 
\right.

+ \langle \nabla \phi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n} \right) \right\} d\mathcal{V}.$$  

(7.5)

The next proposition will have a key role in the proof of Theorem 7.1

**Proposition 7.2** Let $\phi \in C^2_c(M; \mathbb{R}^+)$. Assume that

(i) there exists $\mu$ such that $\mu^n \to \mu$ as $n \to \infty$, as Radon measure on $M$;

(ii) $|\xi^n|\{\phi > 0\} \to 0$ as $n \to \infty$;

(iii) there exists a constant $\tilde{C} > 0$ such that $\mathfrak{B}^n(u^n, \phi) \geq -\tilde{C}$ for all $n \in \mathbb{N}$;

(iv) $\mathcal{H}^{N-1}(\text{supp } \mu \cap \{\phi > 0\}) < \infty$.

Then

(a) $\mu\{\phi > 0\}$ is $(N - 1)$-rectifiable;

(b) there exists $V \in \mathcal{R}\mathcal{V}_{N-1}(M)$ such that $V^n\{\phi > 0\} \to V$ as $n \to \infty$, and $\|V\| = \mu\{\phi > 0\}$;

(c) for all $Y \in C^1_c(\{\phi > 0\}; TM)$,

$$\delta V(Y) = -\lim_{n \to \infty} \int_M \varepsilon_n \langle Y, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\mathcal{V};$$

(d) $\mathfrak{B}(\mu, \phi) \geq \limsup_{n \to \infty} \mathfrak{B}^n(u^n, \phi)$.
7.1 Proof of Proposition 7.2

The proof of Proposition 7.2, to which this Subsection is devoted, needs some preliminary results. To begin with, the next density lemma will be used (see Lemma 7.4 in [16]).

**Lemma 7.3** Let $\mu \in \mathcal{M}_k(M)$. For any vector field $Z \in L^2_\mu(TM)$ and any $\delta > 0$, there exists a vector field $Y \in C^1_c(TM)$ such that

$$\|Z - Y\|_{L^2_\mu(TM)} \leq \delta.$$ 

Furthermore, we make use of the following lemma.

**Lemma 7.4** Let $\phi \in C^2_c(M; \mathbb{R}^+)$, $\mu$ be a Radon measure, $C_1(\phi) := \sup_M |\text{Hess} \phi|$. Let $u^\epsilon$ be solution to equation (1.1). Then

$$\int_M \langle \nabla \phi, T_{\epsilon\mu}^{\frac{1}{2}}(H) \rangle d\mu \leq \frac{1}{2} \int_M \phi H^2 d\mu + C_1(\phi) \mu(\{\phi > 0\}),$$  

(7.6)

$$\int_M \phi H^2 d\mu \leq -2\mathcal{B}(\phi, \mu) + 2C_1(\phi) \mu(\{\phi > 0\}),$$  

(7.7)

when these are defined; similarly

$$\int_M \epsilon \langle \nabla \phi, \nabla u \rangle \left(-\Delta u + \frac{1}{\epsilon^2} f(u^\epsilon)\right) d\mathcal{V}(x) \leq \frac{1}{2} \int_M \epsilon \phi \left(-\Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon)\right)^2 d\mathcal{V}(x) + 2C_1(\phi) \int_{\{\phi > 0\}} \frac{\epsilon}{2} |\nabla u^\epsilon|^2 d\mathcal{V}(x);$$  

(7.8)

$$\int_M \epsilon \phi \left(-\Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon)\right)^2 d\mathcal{V}(x) \leq -2\mathcal{B}(u^\epsilon, \phi) + 4C_1(\phi) \int_{\{\phi > 0\}} \frac{\epsilon}{2} |\nabla u^\epsilon|^2 d\mathcal{V}(x).$$  

(7.9)

**Proof.** Inequality (7.6) follows from the Cauchy-Schwartz inequality, since $C_1(\phi) \geq \sup_M |\text{Hess} \phi|$, in view of Lemma 3.1. Moreover, from (7.6) we deduce (7.7). Inequalities (7.8)-(7.9) can be shown similarly. \qed

Note that in view of (7.2), for all $n \in \mathbb{N}$ we can define the unit tangent field

$$\nu^n := \frac{\nabla u^n}{|\nabla u^n|} \ \mathcal{V} \text{ a.e. in } M,$$

and the dual unit cotangent field $\tilde{\nu}^n$ (i.e. $\tilde{\nu}^n(\nu^n) = 1$ a.e. in $M$).

We have the following auxiliary identity.

**Lemma 7.5** Let $\phi \in C^2_c(M; \mathbb{R}^+)$, $U \subset \subset \{\phi > 0\}$, $Y \equiv (Y^1, \ldots, Y^N) \in C^1_c(U; TM)$. There holds:

$$-\epsilon \Delta u^n(\nabla u^n, Y) = \frac{\epsilon}{2} \text{div} (Y|\nabla u^n|^2) - \frac{\epsilon}{2} |\nabla u^n|^2 \text{D} Y : I$$  

$$-\epsilon \text{div}(\nabla u^n(\nabla u^n, Y)) + \epsilon \text{D} Y : \nabla u^n \otimes du^n. $$  

(7.10)
Proof. Write $u$ instead of $u^n$ for brevity. Take any $p \in M$ and fix an orthonormal frame $\{E_i\}_{i=1}^N$ around $p$. We have $\nabla u = \sum_{i=1}^N E_iuE_i$ around $p$; furthermore, if $Z = \sum_{i=1}^N Z^iE_i$, then at $p$ there holds $\text{div} Z = \sum_{i=1}^N E_iZ^i$ and

$$\Delta uE_iu = \left( \sum_{j=1}^N E_jE_ju \right)E_iu = \sum_{j=1}^N \left[ E_j(E_juE_iu) - E_juE_j(E_iu) \right]$$

$$= \sum_{j=1}^N E_j(E_juE_iu) - \frac{1}{2} E_i \sum_{j=1}^N (E_ju)^2;$$

here equality $[E_i, E_j](p) = 0$ has been used. Thus,

$$-\varepsilon \Delta u(\nabla u, Y) = -\varepsilon \Delta u \sum_{i=1}^N E_iuY^i$$

$$= -\varepsilon \sum_{i=1}^N Y^i \sum_{j=1}^N E_j(E_juE_iu) + \frac{\varepsilon}{2} \sum_{i=1}^N Y^iE_i|\nabla u|^2$$

$$= -\varepsilon \sum_{j,i=1}^N Y^iE_j(E_juE_iu) + \frac{\varepsilon}{2} \sum_{i=1}^N \left[ E_i(Y^i|\nabla u|^2) - E_iY^i|\nabla u|^2 \right]$$

$$= \varepsilon \sum_{j,i=1}^N \left[ -E_j(Y^iE_juE_iu) + E_jY^iE_juE_iu \right]$$

$$+ \frac{\varepsilon}{2} \text{div}(Y|\nabla u|^2) - \frac{\varepsilon}{2} |\nabla u|^2 \text{div} Y$$

$$= -\varepsilon \sum_{j=1}^N E_j(E_ju(\nabla u, Y)) + \varepsilon \sum_{i=1}^N (\nabla Y^i, \nabla u)E_iu$$

$$+ \frac{\varepsilon}{2} \text{div}(Y|\nabla u|^2) - \frac{\varepsilon}{2} |\nabla u|^2 \text{div} Y.$$

It is easily seen that

$$DY : I = \text{div} Y,$$

$$DY : \nabla u \otimes du = \sum_{j,i=1}^N E_iY^jE_juE_iu = \sum_{i=1}^N (\nabla Y^i, \nabla u)E_iu. \quad (7.13)$$

From (7.11)-(7.13) we get (7.10).\[\square\]

The following representation formula for $\delta V^n$ holds.

**Lemma 7.6** Let assumptions of Lemma 7.5 be satisfied. There holds:

$$\delta V^n(Y) = \int_M \nu^n \otimes \widetilde{\nu}^n : DYd\kappa^n$$

$$- \int_M \varepsilon_n(Y, \nabla u^n) \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n} \right) d\nu(x). \quad (7.14)$$
Proof. Defining the stress tensor by
\[ T := \left\{ \frac{\varepsilon_n}{2} |\nabla u^n|^2 + \frac{1}{\varepsilon_n} F(u^n) \right\} I - \varepsilon_n \nabla u^n \otimes du^n, \]
we have
\[ T = \frac{\varepsilon_n}{2} |\nabla u^n|^2 (I - 2\nu^n \otimes \tilde{\nu}^n) + \frac{F(u^n)}{\varepsilon_n} I. \quad (7.15) \]
Integrating by parts, by (7.10) and (7.15),
\[ \int_M \varepsilon_n \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n} \right) (\nabla u^n, Y) d\nu = -\int_M T : DYd\nu \]
\[ = -\int_M \left( \frac{\varepsilon_n}{2} |\nabla u^n|^2 + \frac{F(u^n)}{\varepsilon_n} \right) (I - \nu^n \otimes \tilde{\nu}^n) : DYd\nu \]
\[ + \int_M \left( -\frac{F(u^n)}{\varepsilon_n} + \frac{\varepsilon_n}{2} |\nabla u^n|^2 \right) \nu^n \otimes \tilde{\nu}^n : DYd\nu \]
\[ = -\int_M (I - \nu^n \otimes \tilde{\nu}^n) : DYd\mu^n + \int_M \nu^n \otimes \tilde{\nu}^n : DYd\xi^n. \quad (7.16) \]
Since
\[ \delta V^n(Y) = \int_M DY : Sd\nu(x, S) = \int_M DY : (I - \nu^n \otimes \nu^n)d\mu^n, \]
from (7.16) we obtain (7.14). \(\square\)

Now we are ready to prove Proposition 7.2.

Proof of Proposition 7.2. Keep the same notation as above. By compactness theorem for Radon measures with locally equibounded masses, there exist a subsequence of \( \{V_n^k\} \subset \{V^n\} \) and \( \tilde{V} \in \mathcal{V}_{N-1}(M) \) such that \( V^n \to \tilde{V} \) as \( n \to \infty \), as varifolds. Let us write \( \{V_n^k\} = \{V^n\} \).

Claim 1. We have that \( \tilde{V} | \{\phi > 0\} \in \mathcal{R}\mathcal{V}_{N-1}(\{\phi > 0\}) \); moreover, \( ||\tilde{V}|| = \mu \) and \( \mu \) is a \((N-1)\)-rectifiable Radon measure on \( M \).

In fact, for \( U \subset \subset \{\phi > 0\} \) and \( Y \in C^1_c(U; TM) \), sending \( n \to \infty \) in (7.11), in view of hypothesis (ii), we get
\[ \delta \tilde{V}(Y) = \int_M DY : Sd\tilde{V}(x, S), \]
\[ = -\lim_{n \to \infty} \int_M \varepsilon_n(Y, \nabla u^n) \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\nu. \quad (7.17) \]
Furthermore, by hypothesis (iii) and (7.10) we obtain
\[ |\delta \tilde{V}(Y)| \leq ||Y||_\infty \limsup_{n \to \infty} \int_{U} \varepsilon_n |\nabla u^n| \left| -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right| d\nu \]
\[ \leq ||Y||_\infty \limsup_{n \to \infty} \int_{U} \left\{ \frac{\varepsilon_n}{2} \frac{|\nabla u^n|^2}{\phi} + \varepsilon_n \phi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 \right\} d\nu \]
\[ \leq ||Y||_\infty \limsup_{n \to \infty} \left[ 2\tilde{C}(\phi, U)\mu^n(\{\phi > 0\}) + 2\tilde{C} + 4C_1(\phi)\mu^n(\{\phi > 0\}) \right] \]
\[ |\delta \tilde{V}(Y)| \leq C(\phi, U, \mu, \tilde{C})\|Y\|_{\infty}. \] (7.18)

By (7.18), \(|\delta \tilde{V}|(\{\phi > 0\})\) is a Radon measure on \(\{\phi > 0\}\). By Corollary 2) of the rectifiability theorem in [3, pag. 450] and hypothesis (iv), Claim 1 follows. So, (a) has been verified.

In view of rectifiability, the varifold \(V \equiv \tilde{V}\) is uniquely determined by \(\mu\), independently of the subsequence; thus, the all sequence \(\{\tilde{V}^n|\{\phi > 0\}\}\) converges to \(V\) as varifolds. So, (b) follows. From (7.17) we get (c).

It remains to prove (d). To this aim take \(\psi \in C^2_\mathbb{R}(\{\phi > 0\}; \mathbb{R}^+)\) with \(\sqrt{\psi} \in C^1(\{\phi > 0\}; \mathbb{R}^+)\). Since \(\mu\) is rectifiable, in view of Lemma 7.3 we have:

\[ \left( \int_{\{\phi > 0\}} \psi H^2 d\mu \right)^{1/2} = \sup \left\{ (\sqrt{\psi} Y, \tilde{H}) : Y \in C_\infty(\{\phi > 0\}; TM), \|Y\|_{L^2(\mu)} \leq 1 \right\}. \]

By (7.14), with \(Y\) replaced by \(\sqrt{\psi} Y\), and assumption (ii), we get:

\[
\int_{\text{supp } \phi} \sqrt{\psi}(Y, \tilde{H}) d\mu = -\delta V(\sqrt{\psi} Y) = -\lim_{n \to \infty} \delta V^n(\sqrt{\psi} Y) \\
= \lim_{n \to \infty} \int_{\text{supp } \phi} \varepsilon_n \sqrt{\psi}(Y, \nabla u^n) \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\nu \\
- \lim_{n \to \infty} \int_{\text{supp } \phi} \nu^n \otimes \tilde{v}^n : D(\sqrt{\psi} Y) d\xi^n \\
\leq \limsup_{n \to \infty} \left( \int_{\text{supp } \phi} \varepsilon_n |\nabla u^n|^2 |Y|^2 d\nu \right)^{1/2} \left[ \int_{\text{supp } \phi} \varepsilon_n \psi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\nu \right]^{1/2} \\
\leq \limsup_{n \to \infty} \left( \int_{\text{supp } \phi} |Y|^2 d\mu^n \right)^{1/2} \limsup_{n \to \infty} \left[ \int_{\text{supp } \phi} \varepsilon_n \psi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\nu \right]^{1/2} \\
= \|Y\|_{L^2(\mu)} \limsup_{n \to \infty} \left[ \int_{\text{supp } \phi} \varepsilon_n \psi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\nu \right]^{1/2}.
\]

Fixing \(\psi = \psi_k\) and taking the sup on \(Y\), this implies

\[ \int_{\text{supp } \phi} \psi_k H^2 d\mu \leq \limsup_{n \to \infty} \int_{\text{supp } \phi} \varepsilon_n \psi_k \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\nu \] (7.19)

for all \(k \in \mathcal{N}\), where \(\{\psi_k\}_{k \in \mathcal{N}} \subset C^2(\{\phi > 0\})\) with \(\sqrt{\psi_k} \in C^1(\{\phi > 0\})\), \(\psi_k \leq \psi_{k+1}\) for every \(k \in \mathcal{N}\), \(\psi_k \to \phi\) in \(L^1(\{\phi > 0\})\). Letting \(k \to \infty\) in (7.19), in view of the monotone convergence theorem we have:

\[ \int_{\text{supp } \phi} \phi H^2 d\mu \leq \limsup_{n \to \infty} \int_{\text{supp } \phi} \varepsilon_n \phi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\nu \leq C(\phi, \mu, \tilde{C}); \] (7.20)
in the last inequality (7.9) and hypothesis (iii) have been used.

Claim 2. The following equality holds:

\[
\lim_{n \to \infty} \int_{\text{supp } \psi} \epsilon_n \langle \nabla \psi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\epsilon_n^2} \right) dV = \int_{\text{supp } \psi} \langle \nabla \psi, S^\perp (\nabla u^n) \rangle d\mu ,
\]

(7.21)

for any \( \psi \in C^2 \{ \{ \phi > 0 \}; TM \} \); here \( S = S(x) = T_x \mu \).

To prove Claim 2, note that since \( \mu \) is rectifiable, by Lemma 7.3 for each \( \delta > 0 \) we can select \( Y \in C^1 \{ \{ \phi > 0 \}; TM \} \) such that

\[
\int_{\text{supp } \psi} \left| Y(x) - S^\perp (\nabla \psi(x)) \right|^2 d\mu \leq \delta^2 .
\]

(7.22)

From (7.14) we obtain:

\[
\int_{\text{supp } \psi} \langle \nabla \psi, S^\perp (\nabla u^n) \rangle d\mu = \int_{\text{supp } \psi} \langle S^\perp (\nabla \psi), (\nabla u^n) \rangle d\mu = \sum_{i=1}^{6} A_i ,
\]

(7.23)

where

\[
A_1 := \int_{\text{supp } \psi} \left( S^\perp (\nabla \psi) - Y \right) \nabla u^n d\mu ,
\]

\[
A_2 := -\delta V(Y) + \delta V^n(Y)
\]

\[
A_3 := -\delta V^n(Y) - \int_{\text{supp } \psi} \epsilon_n \langle \nabla u^n, Y \rangle \left( -\Delta u^n + \frac{f(u^n)}{\epsilon_n^2} \right) dV
\]

\[
A_4 := \int_{\text{supp } \psi} \epsilon_n \langle Y - (\nabla \psi, \nu^n) \nu^n, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\epsilon_n^2} \right) dV
\]

\[
A_5 := \int_{\text{supp } \psi} \epsilon_n \langle (\nabla \psi, \nu^n) \nu^n - \nabla \psi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\epsilon_n^2} \right) dV
\]

\[
A_6 := \int_{\text{supp } \psi} \epsilon_n \langle \nabla \psi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\epsilon_n^2} \right) dV .
\]

Let us estimate \(|A_i|\) for \( i = 1, \ldots, 6 \).

From (7.20) and (7.22) we get

\[
|A_1| \leq \delta \left( \int_{\text{supp } \psi} H^2 d\mu \right)^{1/2} \leq C(\psi, \phi) \delta .
\]

(7.24)

In view of (b), we get

\[
\lim_{n \to \infty} A_2 = 0 .
\]

(7.25)

By (7.17),

\[
\lim_{n \to \infty} A_3 = \lim_{n \to \infty} \int_{\text{supp } \psi} \nu^n \otimes \nabla u^n : D Y d\xi^n = 0 .
\]

(7.26)
Moreover, by (7.23), hypothesis (iii) and Hölder inequality,

\[
|A_4| \leq \left( \varepsilon_n \int_{\text{supp } \psi} |\nabla u^n|^2 \left| Y - (\nabla \psi, \nu^n) \right|^2 d\mathcal{V} \right)^{1/2} \\
\cdot \left[ \sup_{\text{supp } \psi} \frac{1}{\phi} \int_{\text{supp } \psi} \varepsilon_n \phi \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\mathcal{V} \right]^{1/2}
\]

\[
\leq \left( 2 \int_{\text{supp } \psi} \left| Y(x) - S^1((\nabla \psi)(x)) \right|^2 dV^n(x, S) \right)^{1/2} [C(\text{supp } \psi, \phi)]^{1/2}
\cdot \left[ -2B^n(u^n, \phi) + 4C_1(\phi) \int_{\text{supp } \psi} \frac{\varepsilon_n}{2} |\nabla u^n|^2 d\mathcal{V} \right]^{1/2}
\]

\[
\leq \left( 2 \int_M \left| Y - S^1(\nabla \psi) \right|^2 dV^n \right)^{1/2} [C(\text{supp } \psi, \phi)]^{1/2} [C(\phi, \mu^n_{\phi} \{\phi > 0\}, \tilde{C})]^{1/2}.
\]

Thus,

\[
\limsup_{n \to \infty} |A_4| \leq C(\phi, \psi) \left( 2 \int_{\text{supp } \psi} |Y - S^1(\nabla \psi)|^2 dV^n \right)^{1/2}
\leq C(\phi, \psi, \mu, \tilde{C}) \delta.
\]

(7.27)

Note that by definition of \( \nu^n \), \( \mathcal{I}_3 \equiv 0 \), hence from (7.23)-(7.27) we deduce

\[
\int_{\text{supp } \psi} \varepsilon_n \langle \nabla \psi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\mathcal{V} - \int_{\text{supp } \psi} \langle S^1(\nabla \psi), \vec{H} \rangle d\mu_1 \leq C(\psi, \phi) \delta.
\]

Letting \( \delta \to 0^+ \), we obtain (7.21).

**Claim 3:** The limit (7.21) remains true with \( \psi \) replaced by \( \phi \).

To see this, for any \( \delta > 0 \) we take \( \psi \in C^2(\{\phi > 0\}; \mathbb{R}^+) \) such that \( \psi < \phi \), \( \|\phi - \psi\|_{C^2} < \delta \). We write:

\[
\limsup_{n \to \infty} \int_{\text{supp } \phi} \varepsilon_n \langle \nabla \phi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\mathcal{V} - \int_{\text{supp } \phi} \langle S^1(\nabla \phi), \vec{H} \rangle d\mu \leq \limsup_{n \to \infty} |\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3|,
\]

(7.28)

where

\[
\mathcal{I}_1 := - \int_{\text{supp } \psi} \langle S^1(\nabla \psi), \vec{H} \rangle d\mu + \lim_{n \to \infty} \int_{\text{supp } \phi} \varepsilon_n \langle \nabla \phi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\mathcal{V},
\]

\[
\mathcal{I}_2 := \lim_{n \to \infty} \int_{\text{supp } \phi} \varepsilon_n \langle \nabla \phi - \nabla \psi, \nabla u^n \rangle \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right) d\mathcal{V},
\]

\[
\mathcal{I}_3 := \int_{\text{supp } \phi} \langle S^1(\nabla \phi - \nabla \psi), \vec{H} \rangle d\mu.
\]

27
By (7.21),
\[ \limsup_{n \to \infty} |I_1| = 0. \tag{7.29} \]
By (7.20), Hölder inequality and Lemma 3.1,
\[ \limsup_{n \to \infty} |I_2| \leq \limsup_{n \to \infty} \left( \int_{\{\phi > 0\}} |\nabla \phi - \nabla \psi|^2 d\nu \right)^{1/2} \times \left[ \int_{\{\phi > 0\}} \varepsilon_n (\phi - \psi) \left( -\Delta u^n + \frac{f(u^n)}{\varepsilon_n^2} \right)^2 d\nu \right]^{1/2} \leq 2\mu(\{\phi > 0\}) \sup_{\{\phi > 0\}} |\text{Hess}(\phi - \psi)|^{1/2} C(\phi, \mu, \tilde{C}) \leq C(\phi, \mu, \tilde{C}) \delta. \tag{7.30} \]
Furthermore, it is easily checked that from (7.20) one also has
\[ |I_3| \leq C(\phi) \delta. \tag{7.31} \]
From (7.28)-(7.31), letting \( \delta \to 0^+ \), and combining (7.19) and (7.20) with (7.1) and (7.5), we get Claim 3 and (d). This completes the proof. \( \square \)

### 7.2 Proof of Theorem 7.1

Finally we can prove the main result of the paper.

**Proof of Theorem 7.1.**

For any \( \phi \in C^2_c(M, \mathbb{R}^+) \), from the semidecreasing property in Lemma 3.2, we have that
\[ \overline{D}_{t_0} \mu_t(\phi) > -\infty \tag{7.32} \]
for a.e. \( t_0 > 0 \), the upper derivative being clearly an ordinary derivative. Now, fix any \( t_0 > 0 \) such that (7.32) is satisfied, otherwise there is nothing to be proven. Set
\[ -\infty < \bar{D} \equiv \overline{D}_{t_0} \mu_t(\phi). \tag{7.33} \]
Thus, there exists sequences \( \{\delta_k\} \subset (0, 1), \delta_k \to 0 \) and \( \{t_k\} \subset (0, \infty), t_k \to t_0 \)

as \( k \to \infty \) such that
\[ \bar{D} - \delta_k \leq \frac{\mu_{t_k}(\phi) - \mu_{t_0}(\phi)}{t_k - t_0}; \]
we may assume that \( t_k > t_0 \) for all \( k \in \mathbb{N} \).

Since \( \mu_t^{\varepsilon_n} \to \mu_t \), we can find a sequence \( \{r_k\} \subset (0, \infty), r_k \to \infty \) such that for all \( k \in \mathbb{N} \)
\[ \bar{D} - 2\delta_k \leq \frac{\mu_{t_k}^{\varepsilon_{r_k}}(\phi) - \mu_{t_0}^{\varepsilon_{r_k}}(\phi)}{t_k - t_0} = \frac{1}{t_k - t_0} \int_{t_0}^{t_k} \frac{d}{dt} \mu_t^{\varepsilon_{r_k}}(\phi) dt. \tag{7.34} \]
Note that, for \( d|\xi^{\varepsilon_n}| \to 0 \) in \( M \times [0, \infty) \), we can increase \( r_k \) so that
\[ \int_{t_0}^{t_k} \int_{\{\phi > 0\}} d|\xi^{\varepsilon_{r_k}}| \leq \delta_k^2 (t_k - t_0). \tag{7.35} \]
By the proof of Lemma 3.2 there exists \( C_1 = C_1(\phi) > 0 \) such that
\[ \frac{d}{dt} \mu_t^{\varepsilon_n}(\phi) \leq C_1(\phi) \quad \text{for all} \quad n \in \mathbb{N}, 0 < t < t_0 + 1. \]
For any $k \in \mathbb{N}$, define 

$$Z_k := \left\{ t \in [t_0, t_k] : \frac{d}{dt} \mu^{\varepsilon_{rk}}(\phi) \geq \hat{D} - 3\delta_k \right\}.$$ 

We have 

$$\hat{D} - 2\delta_k \leq \frac{1}{t_k - t_0} \int_{[t_0, t_k] \setminus Z_k} (\hat{D} - 3\delta_k) dt + \frac{1}{t_k - t_0} \int_{Z_k} C_1(\phi) dt.$$ 

So, 

$$\text{meas}(Z_k) \geq \frac{\delta_k(t_k - t_0)}{C_1(\phi) - \hat{D} + 3\delta_k} \geq \frac{\delta_k(t_k - t_0)}{2(C_1(\phi) - \hat{D})},$$

for $k \in \mathbb{N}$ big enough. By (7.35),

$$\text{meas}(Z_k) \inf_{t \in Z_k} \xi_{s_k}^{\varepsilon_{rk}} \left( \{ \phi > 0 \} \right) \leq \delta_n^2(t_k - t_0).$$

Hence, due to (7.2) and (7.13), we can construct a sequence $\{s_k\} \subset Z_k$ such that

$$\hat{D} - 3\delta_k \leq \frac{d}{dt} \xi_{s_k}^{\varepsilon_{rk}}(\phi)|_{t = s_k} = B^{\varepsilon_{rk}}(u^{\varepsilon_{rk}}(\cdot, s_k), \phi),$$

and

$$\left| \xi_{s_k}^{\varepsilon_{rk}} \right| \left( \{ \phi > 0 \} \right) \leq 2(C_1 - \hat{D})\delta_n.$$ 

By hypothesis (1.13) with $K = \text{supp} \phi$ and standard compactness results, there exists a subsequence of $\{\mu^{\varepsilon_{rk}}\}$, which converges to $\tilde{\mu}$, for some Radon measure on $M$. By Lemma 3.2 it is possible to show that (see [16], Section 7.6)

$$\tilde{\mu}(\{ \phi > 0 \}) = \mu_{t_0}(\{ \phi > 0 \}),$$

hence $\mathcal{B}(\tilde{\mu}, \phi) = \mathcal{B}(\mu_{t_0}, \phi)$.

Corollary 4.4 combined with (7.36)-(7.38) implies that hypotheses (i)-(iv) of Proposition 7.2 are satisfied with $\{u^k\}$ replaced by $\{u_k\}$, where $u^k \equiv u^{\varepsilon_{rk}}(\cdot, s_k)$.

By (7.33), (7.36)-(7.38), due to Proposition 7.2 (a), we see that $\mu_{t_0}$ is locally $(N - 1)$-rectifiable (varying $\phi = \phi_i \in C^2(M, \mathbb{R}^+)$ in a countable set of functions such that $\cup \{ \phi_i > 0 \} = M$). Moreover, in view of Proposition 7.2 (d), we obtain

$$\mathfrak{d}_{t_0} \mu_t(\phi) \leq \mathcal{B}(\mu_{t_0}, \phi).$$

This completes the proof. $\square$

References

[1] M. Alfaro, D. Hilhorst and H. Matano, The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system, J. Diff. Eq. 245 (2008), 505-565.
[2] S. Alexakis, R. Mazzeo, *Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds*, Commun. Math. Phys. **297** (2010), 621–651.

[3] W. K. Allard, *On the first variation of a varifold*, Ann. Math. **95** (1972) 417-491.

[4] D. Azagra, M. Jimenez-Sevilla, F. Macia, *Generalized motion of level sets by functions of their curvatures on Riemannian manifolds*, Calc. Var. **33** (2008), 133–1671.

[5] G. Bellettini, M. Novaga, *Comparison results between minimal barriers and viscosity solutions for geometric evolutions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 97–131.

[6] G. Bellettini, M. Paolini, *Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Acc. Naz. Sci. XL Mem. Mat., **XIX** (1995), 43–67.

[7] F. Bethuel, G. Orlandi, D. Smets, *Convergence of the parabolic Ginzburg–Landau equation to motion by mean curvature*, Annals of Math. **163** (2006), 37-163.

[8] K. A. Brakke, "The Motion of a Surface by its Mean Curvature", Princeton University Press, Princeton, NJ, 1978.

[9] X. Chen, *Generation and propagation of the interface for reaction-diffusion equations*, J. Diff. Eq. **96** (1992), 116–141.

[10] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equation*, J. Differential Geom. **33** (1991), 749–786.

[11] L.C. Evans, J. Spruck, *Motion of level sets by mean curvature, I*, J. Diff. Geom. **33** (1991), 635–681.

[12] S. Gallot, D. Hulin, J. Lafontaine, "Riemannian Geometry", Universitext (Springer, 1993).

[13] A. Grigoryan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[14] J. Heinonen, "Lectures on Analysis on Metric Spaces", Springer (2001).

[15] T. Ilmanen, *Convergence of the Allen-Cahn equation to the Brakke’s motion by mean curvature*, J. Diff. Geom. **31** (1993), 417–461.

[16] T. Ilmanen, "Elliptic Regularization and Partial Regularity for Motion by Mean Curvature", Mem. Amer. Math. Soc. **520** (1994).

[17] Ilmanen, *Generalized motion of sets by mean curvature on a manifold*, Indiana Univ. Math. J. **41** (1992), 671–705.

[18] J. Jost, "Riemannian Geometry and Geometric Analysis", Universitext (Springer-Verlag, 2005).

[19] K. Kasai, Y. Tonegawa, *A general regularity theory for weak mean curvature flow*, [http://arxiv.org/pdf/1111.0824.pdf](http://arxiv.org/pdf/1111.0824.pdf)

[20] S. Luckhaus, T. Sturzenhecker, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations **3** (1995), 253–271.
[21] A. Pisante, M. Ponsiglione, Phase Transitions and Minimal Hypersurfaces in Hyperbolic Space, Comm. Part. Diff. Eq. 36 (2011), 819-849.

[22] A. Pisante, F. Punzo, Allen-Cahn Approximation of Mean Curvature Flow in Riemannian manifolds I, uniform estimates, preprint (2013).

[23] L. Simon, ”Lectures on Geometric Measure Theory”, Proc. Centre Math. Anal., Austr. Nat. Univ. 3 (1983).

[24] H.M. Soner, Ginzburg-Landau Equation and Motion by Mean Curvature, I: Convergence, J. Geom. Anal. 7 (1997), 437–475.

[25] Y. Tonegawa, Integrality of varifolds in the singular limit of reaction-diffusion equations, Hiroshima Math. J. 33 (2003), 323–341.

[26] Y. Tonegawa, A second derivative Hölder estimate for weak mean curvature flow, http://arxiv.org/pdf/1204.4571.pdf

[27] K. Takasao, Y. Tonegawa, Existence and regularity of mean curvature flow with transport term in higher dimensions, http://arxiv.org/abs/1307.6629