Derivation of instantaneous unit hydrographs using linear reservoir models
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ABSTRACT

In this research, a new conceptual model for producing instantaneous unit hydrographs (IUHs) is introduced by a linear combination of the Nash model, which assumes that the discharge from a reservoir is a linear function of its storage, and a model called inter-connected linear reservoir model (ICLRM), which assumes that the discharge from a reservoir is a linear function of the difference of its storage and its adjacent downstream reservoir. By employing these assumptions, a system of first-order linear differential equations with three degrees of freedom (storage coefficient, number of reservoirs, and weighting coefficient) is obtained as the governing equation for the proposed model. This model may be considered as the general form of the two models and is therefore capable of simulating IUHs laying between these two models. To show the capabilities of the model, linear and curvilinear soil conservation service (SCS) hydrographs are simulated using dimensionless hydrographs obtained by this model. Moreover, several real hydrographs were simulated by the proposed model and compared with hydrographs obtained by Nash, ICLRM, and SCS models. The results show that the model yields more accurate results compared to other studied models and may be considered as a new model for simulating IUHs.

Key words | instantaneous unit hydrograph, linear reservoir model, Nash model, system of linear first-order differential equations

HIGHLIGHTS

- Nash model and inter-connected linear reservoir model (ICLRM), being two extremes of linear reservoir models, are unified.
- The general model is a linear combination of the Nash and ICLRM models.
- The weighting coefficient gives the proposed model an extra degree of freedom.
- This model yields more accurate results for the simulation of SCS hydrographs.

INTRODUCTION

The number of floods causing damage and financial losses throughout the world is not low. For this reason, studies and investigations of flood prediction models for water emergency measures and management strategies are of great importance (Singh 2013). Among these models, hydrographs have been extensively used for surface runoff estimation.

The unit hydrograph, defined by Sherman (1952) is one of the most widely used methods for generating a direct runoff hydrograph. Because of the lack of real flood records, different methods for developing synthetic unit hydrographs (SUHs) were then proposed. Traditional or empirical models are one of the methods that are used for developing
SUHs. Snyder (1938), Bernard (1935), Taylor & Schwarz (1952), and Soil Conservation Service (Mockus 1957) have presented such models which are established based on empirical equations reflecting watershed characteristics (Bhunya et al. 2011). SUHs are still in use despite not being accurate (Singh et al. 2011). Because of the similarity between probability distribution functions (PDFs) and unit hydrographs, and the fact that the area under these two is equal to unity, different PDFs have also been used for generating SUHs. The potential of the Gamma distribution function to derive SUH was explored by Gray (1961), Croley (1980), Aron & White (1982), Haktanir & Sezen (1990), and Roy & Thomas (2016). Nadarajah (2007), Bhunya et al. (2007, 2008); Rai et al. (2010), and Ghorbani et al. (2013, 2017) used beta, Weibull, log-normal, and Gamma distribution functions to derive SUH. The Nakagami-m function was used by Rai et al. (2010), Sarkar et al. (2010), and Sarkar & Rai (2011) for the derivation of the unit hydrograph. Singh (2000) proposed a simple method for transmuting popular SUHs, such as those of Snyder (1938), the Soil Conservation Service (Mockus 1957), and Gray (1961) into the Gamma distribution. Another modeling technique for deriving SUH is conceptual models. In 1957, Nash suggested a conceptual model based on linear reservoirs with equal storage coefficient \( K \) for the derivation of the instantaneous unit hydrograph (IUH). Later, Dooge (1959) proposed a mathematical model for deriving the IUH based on the conceptual reservoir model. Bhunya et al. (2005) and Singh et al. (2007) represented a hybrid and extended hybrid model based on the linear reservoir model. Nourani et al. (2007) applied the model to simulate the Nash model in the laboratory for rainfall–runoff modeling. Ahmad et al. (2010) presented a new method to determine a unique pair of hydrologic parameters of the Nash model. Khaleghi et al. (2018) introduced a new conceptual model, based on the Nash Model, called the inter-connected linear reservoir model (ICLRM) which is a connection of series of linear reservoirs all at the same level.

Since the Nash model and other similar models such as ICLRMs will produce a unique dimensionless IUH when plotted for a specific number of reservoirs, a general form of Nash and ICLRMs models is shown to produce a wider range of IUHs compared to the other two studied models. The validity of this new model is also investigated by the simulation of SCS linear and curvilinear, and several real hydrographs.

**METHODOLOGY AND FORMULATION**

**Model description**

In this research, three different models for producing the IUH are studied and compared. The first one is the well-known Nash model (Nash 1957), which assumes that a watershed could be simulated using a series of linear reservoirs connected to each other in a cascade form (Figure 1(a)).

Using this assumption, the discharge from a reservoir can be written as:

\[
Q_i = \mathcal{L}(S_i) \rightarrow Q_i = \frac{1}{K_i}S_i
\]

where \( \mathcal{L} \) represents a linear function, \( Q \) is the discharge from the system, \( S \) is the storage of the system, and \( K \) is the storage coefficient.

**Figure 1** | Schematic representation of a watershed using (a) Nash model and (b) ICLRMs.
The second model is called the ICLRM (Khaleghi et al. 2018) and assumes that a watershed can be modeled by a series of reservoirs connected to each other via a linear frictional element, but unlike the Nash model, all reservoirs have the same elevation (Figure 1(b)). This model can be formulated as:

$$Q_i = \mathcal{L} (S_i - S_{i+1}) \rightarrow Q_i = \left( \frac{S_i}{K_i} - \frac{S_{i+1}}{K_{i+1}} \right)$$

(2)

The third model, the modified linear model, being the focus of this study, is a linear combination of the first two models. This model can be mathematically expressed as:

$$Q_i = \omega_{Nash} \left( \frac{S_i}{K_i} \right) + (1 - \omega) \left( \frac{S_i}{K_i} - \frac{S_{i+1}}{K_{i+1}} \right)$$

(3a)

$$Q_i = \omega \left( \frac{S_i}{K_i} \right) + (1 - \omega) \left( \frac{S_i}{K_i} - \frac{S_{i+1}}{K_{i+1}} \right)$$

(3b)

where $\omega$ is a weighting coefficient.

In matrix form, the derivation of the Nash model in a matrix form is discussed. To obtain the governing equation for the Nash model, the continuity equation for the $i$th reservoir is written as:

$$Q_{i-1} - Q_i = \frac{dS_i}{dt}$$

(4)

By differentiating Equation (1) and substituting it into Equation (4), the above equation is written as:

$$Q_{i-1} - Q_i = K_i \frac{dQ_i}{dt}$$

(5)

For the first reservoir, the above equation can be in the form of:

$$-Q_1 = K_1 \frac{dQ_1}{dt}$$

(6)

All of the above equations can be written in a matrix form as:

$$\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_i \\
\vdots \\
Q_{n-1} \\
Q_n
\end{bmatrix} \frac{d}{dt} =
\begin{bmatrix}
-\frac{1}{K_1} & 0 & 0 & \cdots & 0 \\
\frac{1}{K_2} & -\frac{1}{K_2} & 0 & \cdots & 0 \\
0 & \frac{1}{K_2} & -\frac{1}{K_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \frac{1}{K_{n-1}} & \cdots & \frac{1}{K_{n-1}} & -\frac{1}{K_{n-1}} \\
0 & 0 & \cdots & \frac{1}{K_n} & -\frac{1}{K_n}
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_i \\
\vdots \\
Q_{n-1} \\
Q_n
\end{bmatrix}$$

(7)

Models formulation

**Nash model**

The derivation of the Nash model is covered in most textbooks; however, since the governing equations for the ICLRM and the modified linear model are presented in a

**Inter-connected linear reservoir model**

To develop the governing equation for the ICLRM, the continuity equation for the $i$th and $i+1$th reservoirs is written as:

$$Q_{i-1} - Q_i = \frac{dS_i}{dt}$$

(8)
\[
Q_i - Q_{i+1} = \frac{dS_{i+1}}{dt} \tag{9}
\]

Dividing Equations (8) and (9) by \(K_i\) and \(K_{i+1}\) and subtracting them, one gets:

\[
\frac{Q_{i-1}}{K_i} - \left(\frac{1}{K_i} + \frac{1}{K_{i+1}}\right)Q_i + \frac{Q_{i+1}}{K_{i+1}} = \frac{1}{K_i} \frac{dS_i}{dt} - \frac{1}{K_{i+1}} \frac{dS_{i+1}}{dt} \tag{10}
\]

By differentiating Equation (2) and substituting it in the right-hand side of Equation (10), the above equation simplifies to:

\[
\frac{Q_{i-1}}{K_i} - \left(\frac{1}{K_i} + \frac{1}{K_{i+1}}\right)Q_i + \frac{Q_{i+1}}{K_{i+1}} = \frac{dQ_i}{dt} \tag{11}
\]

which is the governing equation for interior reservoirs. Since the first and last reservoirs are only connected to one reservoir, their governing equation does not follow Equation (11). The equation for the first reservoir is obtained by writing the continuity equation for the first and the second reservoirs as:

\[
-Q_1 = \frac{dS_1}{dt} \tag{12}
\]

\[
Q_1 - Q_2 = \frac{dS_2}{dt} \tag{13}
\]

Dividing Equations (12) and (13) by \(K_1\) and \(K_2\) and using the differentiated form of Equation (2) corresponding to the first reservoir results, one gets:

\[
-\left(\frac{1}{K_1} + \frac{1}{K_2}\right)Q_1 + \frac{Q_2}{K_2} = \frac{dQ_1}{dt} \tag{14}
\]

Since the discharge from the last reservoir is as freefall, the discharge–storage relation for it is of the form:

\[
Q_n = \frac{1}{K_n} S_n \tag{15}
\]

By writing the continuity equation for the last reservoir, and differentiating Equation (15), the formulation for the last reservoir is obtained as:

\[
\frac{Q_{n-1}}{K_n} - \frac{Q_n}{K_n} = \frac{dQ_n}{dt} \tag{16}
\]

By writing Equation (11) for interior reservoirs and Equations (14) and (16) for the first and last reservoirs, the general formulation of the model would be as follows:

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_i \\
\vdots \\
Q_{n-1} \\
Q_n
\end{bmatrix}
\begin{bmatrix}
\frac{1}{K_1} + \frac{1}{K_2} & -\frac{1}{K_2} & 0 & \cdots & 0 \\
\frac{1}{K_2} & -\left(\frac{1}{K_2} + \frac{1}{K_3}\right) & \frac{1}{K_3} & 0 & \cdots \\
0 & \cdots & 0 & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{K_n} & \cdots & \cdots & \cdots & -\left(\frac{1}{K_n} + \frac{1}{K_{n-1}}\right) & \frac{1}{K_{n-1}} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_i \\
\vdots \\
Q_{n-1} \\
Q_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_i \\
\vdots \\
Q_{n-1} \\
Q_n
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{K_n} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\]
**Modified linear model**

To derive the governing equation for the modified linear model, Equation (3b) is simplified to:

$$Q_i = \frac{S_i}{K_i} - (1 - \omega) \frac{S_{i+1}}{K_{i+1}}$$

(18)

By writing the continuity equation for all reservoirs and following the same procedures as in the derivation of the ICLRM equation, one yields:

$$\frac{Q_{i-1}}{K_i} - \left( \frac{1}{K_i} + \frac{(1 - \omega)}{K_{i+1}} \right) Q_i + \frac{(1 - \omega)}{K_{i+1}} Q_{i+1} = \frac{dQ_i}{dt}$$

(19)

and the matrix form of the modified linear model would be as follows:

$$\frac{d}{dt} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-1} \\ Q_n \end{bmatrix} = - \begin{bmatrix} \frac{1}{K_1} + \frac{(1 - \omega)}{K_2} & \frac{(1 - \omega)}{K_2} & 0 \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \frac{1}{K_{n-1}} - \frac{(1 - \omega)}{K_{n+1}} & \frac{(1 - \omega)}{K_{n+1}} \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \frac{1}{K_{n-1}} & \frac{(1 - \omega)}{K_{n}} & \frac{(1 - \omega)}{K_{n}} \\ 0 & 0 & 0 & \frac{1}{K_{n}} & \frac{1}{K_{n}} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-1} \\ Q_n \end{bmatrix}$$

(20)

The obtained matrix is a tri-diagonal matrix where the arrays of the main diagonal are obtained by the summation of inversed storage coefficients of the two reservoirs at the left and right sides of the considered pipe with a negative sign and weighting coefficients of 1 and 1 - \omega, respectively; the diagonal below the main diagonal is the inversed storage coefficients of the left-side reservoirs, and the diagonal above the main diagonal is the inversed storage coefficients of the right reservoirs with a weighting coefficient of 1 - \omega. For \omega = 1 and \omega = 0, the above equation would be identical to the Nash model and the ICLRM, respectively.

**Solution and comparison of the models**

As seen from Equations (7), (17), and (20), the governing equations of all these models are a first-order linear differential equation. The general equation of these models may be written as follows:

$$\frac{d}{dt}[Q]_{nx1} = [T]_{nxn} [Q]_{nx1}$$

(21)

where \(T\) is the transmissivity matrix presented in Equations (7), (17), and (20). The general solution of these equations is as follows:

$$[Q]_{nx1} = c_1 [v_1]_{nx1} e^{\lambda_1 t} + c_2 [v_2]_{nx1} e^{\lambda_2 t} + \cdots + c_n [v_n]_{nx1} e^{\lambda_n t}$$

(22a)

where \(\lambda_i\) is the \(i\)th eigenvalue of the transmissivity matrices, and \([v_i]_{nx1}\) is their corresponding eigenvectors. \(c_s\) are constants that are determined using initial conditions as follows:

$$[Q]_{nx1} = \sum_{i=1}^{n} c_i [v_i]_{nx1} e^{\lambda_i t}$$

(22b)

and

$$[Q]_{nx1} = c_1 [v_1]_{nx1} + c_2 [v_2]_{nx1} + \cdots + c_n [v_n]_{nx1}$$

(23a)

$$[c]_{nx1} = [V]_{nxn}^{-1} [Q]_{nx1}$$

(23b)
where \( \{Q_0\}_{n \times 1} \) is the initial condition vector and is calculated by substituting initial elevations of each reservoir into Equation (18), and \( [V]_{n \times n} \) is a matrix whose columns are eigenvectors of the matrix \( T \).

\[
[V]_{n \times n} = [(v_1)_{n \times 1} (v_2)_{n \times 1} \ldots (v_n)_{n \times 1}] 
\] (24)

In this research, the first reservoir is filled instantaneously with a unit volume of water, and thus \( \{Q_0\}_{n \times 1} \) is a zero vector except for the first array which is equal to \( 1/K \).

It should be noted that the solution of the Nash model using the matrix approach is the same as the traditional approach as one expects. These solutions for a value of \( K = 1 \) are plotted in Figure 2. It is seen that as \( \omega \) approaches 1, the graph representing the modified linear model approaches the Nash model, and as \( \omega \) approaches zero, the graph approaches ICLRM, enabling the modified linear model to span the area within these two hydrographs by the variation of \( \omega \) in the range of 0–1. This coverage is indeed the superiority of the modified linear model over the Nash and ICLRM models.

**Linear reservoir assumption**

The definition of the linear reservoir is having an output which is linearly proportional to the storage of the reservoir as stated in Equation (1). Due to this linearity, the governing equation of the model is found to be a linear differential equation with constant variables making it possible to find an analytical solution. More importantly, the linearity of this conceptual model allows for straightforward calculations of watershed response to any precipitation patterns using the superposition principle.

Although linear reservoir models have been extensively used to simulate IUHs, only a few numbers of researchers have mentioned how to construct this model in such a way as to have a linear relationship between storage and discharge. In a prismatic reservoir with an orifice as its outlet, the discharge is in proportion with the square root of storage, thus in order to have a linear relationship, the assumptions of prismatic reservoirs or orifice discharge must be changed. Purcell (2006) used nonprismatic reservoirs to maintain a linear relationship between the storage and discharge of reservoirs. Nourani et al. (2007) used a
frictional element in their proposed liquid analog model to maintain this relation. Their frictional element, consisting of a porous medium, dictated Darcian flow through the outlets of each reservoir, resulting in a linear relation between storage and discharge.

In this research, however, a linear relationship between storage and discharge is provided by assuming laminar flow through each pipe. It must be noted that linear reservoir models only simulate the IUH of a watershed, and therefore neither the hydraulics of these models nor the shape of reservoirs is representative of the watershed.

To have a laminar regime through connecting pipes, it is assumed that flow through each pipe follows the Hagen–Poiseuille equation. Hence, the velocity may be found as follows:

\[ v_i = \frac{\gamma D_i^2}{3\mu L_i} (h_i) \quad \text{Nash model} \]  
\[ v_i = \frac{\gamma D_i^2}{3\mu L_i} (h_i - h_{i+1}) \quad \text{ICLRM} \]

In the above equation, \( \gamma \) is the specific weight of the fluid, \( \mu \) is the dynamic viscosity of the fluid, \( D \) is the diameter of the pipe, \( L \) is the length of the pipe, and \( h \) is the water elevation at reservoirs.

Assuming that reservoirs are all cylindrical, storage may be replaced for \( h \) in Equations (25a) and (25b).

![Figure 4](https://placehold.it/500x300)

**Figure 4** | Effects of increasing the number of reservoirs on (a) the Nash and ICLRM models, and (b) the modified linear model, and their comparison with SCS hydrographs.
as follows:

\[
v_i = \frac{\gamma D_i^2}{32 \mu L_i} \left( \frac{S_i}{A_i} \right) \rightarrow Q_i = \frac{\pi \gamma D_i^4}{128 \mu L_i} \left( \frac{S_i}{A_i} \right) \quad \text{Nash model (26a)}
\]

\[
v_i = \frac{\gamma D_i^2}{32 \mu L_i} \left( \frac{S_i}{A_i} - \frac{S_{i+1}}{A_{i+1}} \right) \rightarrow Q_i = \frac{\pi \gamma D_i^4}{128 \mu L_i} \left( \frac{S_i - S_{i+1}}{A_i - A_{i+1}} \right) \quad \text{ICLRM (26b)}
\]

where \( Q_i \) is the flow rate in a pipe connecting reservoirs \( i \) and \( i + 1 \), \( S \) is the storage of reservoirs, and \( A \) is the cross-sectional area of reservoirs. The flow rate in each pipe is found to be:

\[
Q_i = \frac{S_i}{K_i} \quad \text{Nash model}
\]

\[
\text{(27a)}
\]

**Figure 5**

(a) Two different DIUHs with a peaking factor of 350 as well as SCS 350 hydrograph produced by the modified linear model. 
(b) contour plot of peak rate factor with respect to \( \omega \) and \( n \).
\[ Q_i = \left( \frac{S_i}{K_i} - \frac{S_{i+1}}{K_{i+1}} \right) \text{ ICLRM} \]  

(27b)

where \( K \) is the storage coefficient of reservoirs and is equal to \( K_i = 128\mu L_i A_i/\pi \gamma D_i^4 \).

Equations (27a) and (27b) are a justification of how one may assume a linear relationship between the head difference of two reservoirs and the flow in a pipe connecting them (Equations (1) and (2)).

**RESULTS AND DISCUSSION**

To investigate the applicability of the mentioned models, it is assumed that (1) all of the linear elements in these models are identical, in other words, \( K \) is uniform in these models; and (2) the number of reservoirs, \( n \), must be an integer.

It should be noted that the dimensionless forms of Equations (22a) and (22b) are independent of \( K \) and are unique when plotted for a specified number of reservoirs, while the dimensionless IUH (DIUH) obtained by the

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**Figure 6** | Simulation of linear SCS dimensionless hydrographs by the modified linear model. (continued.)
Figure 6 | Continued.
The proposed model is not unique and depends on \( \omega \) (Figure 3). This degree of freedom, \( \omega \), gives the advantage of obtaining an infinite number of hydrographs ranging from those obtained by the Nash model (the DIUH with the lowest values of \( Q/Q_p, \omega = 1 \)) to those obtained by the ICLRM (the DIUH with the highest values of \( Q/Q_p, \omega = 0 \)).

For comparison, DIUHs of the Nash model, the ICLRM, and the modified linear model, as well as the widely used SCS hydrographs, with peaking factors ranging from 150 to 600, are plotted in Figure 4. It is seen from Figure 4(a) that by increasing the number of reservoirs, \( n \), DIUHs obtained by the ICLRM lay between 150 and 300 SCS hydrographs, while those obtained by the Nash model lay within 200–600 SCS hydrographs. Therefore, it is expected that the modified linear model simulates SCS hydrographs with peak factors ranging from 150 to 600 as \( \omega \) increases from 0 to 1 (Figure 4(b)).

An important aspect of Figure 4(a) is that for a specified value of \( n \), the DIUH obtained by the ICLRM is always on the right side of that obtained by the Nash model. As a result, the hydrograph on the most right side of Figure 4(a) belongs to the ICLRM with \( n = 2 \), and the one on the leftmost side of Figure 4(a) belongs to the Nash model with \( n \) approaching infinity. It is also observed, from Figure 4, that as the number of reservoirs, \( n \), increases, the peaking factor for the DIUHs increases; \( t_{peak}/t_{base} \) decreases.

In general, by increasing one of the parameters \( n \) or \( \omega \), while the other one is kept fixed, \( t_{peak}/t_{base} \) will decrease. Therefore,

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**Figure 7** | Simulation of curvilinear SCS dimensionless hydrographs by the modified linear model. (continued.)
Figure 7 | Continued.
different DIUHs with similar peaking factors may be obtained by different pairs of \(n, \omega\). This is shown in Figure 5(a) where two different DIUHs have resulted in a peak factor of 350. However, as \(n\) increases, the resulting DIUH would be a better simulation for steeper catchments. The contour plot of a peak factor vs. \(n\) and \(\omega\) is shown in Figure 5(b).

To investigate the applicability of the introduced model, at first, the IUHs obtained by the three studied models are compared with the widely used linear and curvilinear SCS dimensionless hydrographs reported in the NRCS National Engineering Handbook (2007). These hydrographs cover peaking factors ranging from 150 to 600. For this comparison, the number of reservoirs varying from 2 to 10 is studied. Using the least square method, the optimum value of \(\omega\) is obtained. These hydrographs are plotted in Figures 6 and 7, respectively.

The root-mean-square error (RMSE) and the Nash–Sutcliffe efficiency for the simulation of these hydrographs are also reported, respectively, in Tables 1 and 2.

**Table 1 | RMSE and Nash-Sutcliffe efficiencies for simulating linear SCS hydrographs with the three investigated models**

| SCS peaking factor | Nash | ICLRM | Modified linear | Nash | ICLRM | Modified linear | Nash | ICLRM | Modified linear |
|--------------------|------|-------|-----------------|------|-------|-----------------|------|-------|-----------------|
| 150                | 2    | 2     | 2               | 0    | 0.2483 | 0.1267          | 0.1267 | 40.8  | 84.6           | 84.6 |
| 200                | 2    | 3     | 3               | 0    | 0.0460 | 0.0291          | 0.0291 | 89.2  | 95.7           | 95.7 |
| 250                | 3    | 3     | 3               | 0.39 | 0.0797 | 0.1032          | 0.0671 | 92.5  | 87.5           | 94.7 |
| 300                | 2    | 8     | 3               | 0.67 | 0.0910 | 0.1342          | 0.0652 | 91.6  | 81.8           | 96.0 |
| 350                | 3    | 10    | 4               | 0.67 | 0.0643 | 0.1858          | 0.0562 | 95.8  | 65.3           | 96.8 |
| 400                | 4    | 10    | 4               | 0.83 | 0.0585 | 0.2020          | 0.0440 | 96.8  | 61.6           | 98.2 |
| 450                | 4    | 10    | 4               | 0.98 | 0.0382 | 0.2275          | 0.0380 | 98.0  | 30.1           | 98.0 |
| 500                | 5    | 10    | 5               | 0.98 | 0.0397 | 0.2512          | 0.0393 | 98.2  | 25.8           | 98.2 |
| 550                | 6    | 10    | 6               | 0.97 | 0.0444 | 0.2696          | 0.0442 | 97.5  | 6.3            | 97.3 |
| 600                | 6    | 10    | 6               | 1    | 0.0508 | 0.3044          | 0.0508 | 95.7  | –54.7          | 95.7 |

**Table 2 | RMSE and Nash-Sutcliffe efficiencies for simulating curvilinear SCS hydrographs with the three investigated models**

| SCS peaking factor | Nash | ICLRM | Modified linear | Nash | ICLRM | Modified linear | Nash | ICLRM | Modified linear |
|--------------------|------|-------|-----------------|------|-------|-----------------|------|-------|-----------------|
| 150                | 2    | 2     | 2               | 0    | 0.1718 | 0.0418          | 0.0418 | 73.0  | 98.4           | 98.4 |
| 200                | 2    | 5     | 2               | 0.65 | 0.0469 | 0.0576          | 0.0145 | 97.1  | 95.6           | 99.7 |
| 250                | 3    | 6     | 3               | 0.52 | 0.0941 | 0.0987          | 0.0316 | 90.1  | 89.1           | 98.9 |
| 300                | 3    | 6     | 3               | 0.78 | 0.0471 | 0.1493          | 0.0164 | 97.9  | 78.8           | 99.7 |
| 350                | 3    | 7     | 3               | 1    | 0.0006 | 0.1992          | 0.0006 | 100.0 | 62.8           | 100.0 |
| 400                | 4    | 9     | 4               | 0.90 | 0.0271 | 0.2476          | 0.0111 | 99.3  | 44.1           | 99.9 |
| 450                | 5    | 9     | 5               | 0.87 | 0.0382 | 0.2892          | 0.0139 | 98.7  | 24.4           | 99.8 |
| 500                | 5    | 10    | 5               | 1    | 0.0030 | 0.3295          | 0.0030 | 100.0 | 5.8            | 100.0 |
| 550                | 6    | 10    | 6               | 0.97 | 0.0117 | 0.3786          | 0.0047 | 99.9  | –6.5           | 100.0 |
| 600                | 7    | 10    | 7               | 0.96 | 0.0118 | 0.3652          | 0.0045 | 99.9  | –34.2          | 100.0 |
From Tables 1 and 2, it is seen that using the modified linear model results in a more accurate estimation of the SCS and studied real hydrographs compared to the ICLRM and the Nash model which is due to the fact that the modified linear model has an extra degree of freedom, \( \omega \).

In addition, several storm events of different watersheds are also simulated using hydrographs obtained by SCS, Nash, ICLRM, and modified linear models. The studied catchments are Amameh watershed, the southern area of central Alborz, Iran (Nourani et al. 2009), Rondout Creek watershed, New York, USA (Kilduff et al. 2014), Indian Creek and Flint Creek watersheds in Lake County, Illinois, USA (Melching & Marquardt 1997), and Martur river catchment, India (Chander & Shanker 1984). A comparison of the simulation of the DIUH of these watersheds by the SCS, Nash, ICLRM, and modified linear models, as well as RMSE and Nash–Sutcliffe efficiencies, is shown and reported in Figures 8–11 and Table 3.

As can be seen in Figures 8–11, the modified linear model gives a better fit to the real hydrographs compared to other studied models. According to Table 3, the RMSE has a smaller value for the modified linear model in comparison to the SCS, Nash, and ICLRM models. Moreover, the Nash–Sutcliffe efficiency of the modified linear model is more appropriate. Therefore, it is observed that the modified linear model yields more accurate results compared to the SCS, Nash, and ICLRM models. The \( \omega \) parameter makes the hydrographs simulated by the modified linear model more consistent with the actual watershed’s data.
CONCLUSIONS

In this research, a linear combination of the Nash model, a series of linear reservoirs in a cascade form, and the ICLRM, a series of linear reservoirs all at the same elevation, were studied. The governing system of the linear differential equation for the modified linear model was derived, and it was shown that for a specified number of reservoirs, the DIUHs obtained by this model, contrary to the Nash and

![Figure 10](image)

Figure 10 | Simulation of IUHs of (a) Indian Creek and (b) Flint Creek watersheds in Lake County by the studied models.

![Figure 11](image)

Figure 11 | Simulation of IUH of Martur river catchment by the studied models.

| Table 3 | RMSE and Nash-Sutcliffe efficiencies for simulating real hydrographs |
|---------|---------------------------------------------------------------|
|         | Nash | ICLRM | SCS   | Modified linear |
| Figure 8(a) Parameters | $n = 4$ | $n = 10$ | 400 | $n = 4$, $w = 0.76$ |
| RMSE     | 0.0916 | 0.2247 | 0.0888 | 0.0711 |
| Nash-Sutcliffe efficiency | 89.94 | 39.43 | 90.54 | 93.94 |
| Figure 8(b) Parameters | $n = 2$ | $n = 10$ | 250 | $n = 3$, $w = 0.63$ |
| RMSE     | 0.0429 | 0.1877 | 0.0316 | 0.0109 |
| Nash-Sutcliffe efficiency | 97.09 | 44.34 | 98.42 | 99.81 |
| Figure 9(a) Parameters | $n = 3$ | $n = 10$ | 250 | $n = 4$, $w = 0.64$ |
| RMSE     | 0.0875 | 0.1452 | 0.0976 | 0.0644 |
| Nash-Sutcliffe efficiency | 91.56 | 71.11 | 86.94 | 94.33 |
| Figure 9(b) Parameters | $n = 3$ | $n = 10$ | 350 | $n = 4$, $w = 0.64$ |
| RMSE     | 0.0327 | 0.1660 | 0.0330 | 0.0225 |
| Nash-Sutcliffe efficiency | 98.60 | 63.82 | 98.57 | 99.34 |
| Figure 10(a) Parameters | $n = 3$ | $n = 10$ | 250 | $n = 3$, $w = 0.71$ |
| RMSE     | 0.0934 | 0.1338 | 0.1095 | 0.0684 |
| Nash-Sutcliffe efficiency | 88.78 | 76.96 | 84.59 | 93.98 |
| Figure 10(b) Parameters | $n = 4$ | $n = 10$ | 400 | $n = 4$, $w = 0.87$ |
| RMSE     | 0.1180 | 0.2625 | 0.1318 | 0.1165 |
| Nash-Sutcliffe efficiency | 82.63 | 14.05 | 78.33 | 83.07 |
| Figure 11 Parameters | $n = 2$ | $n = 10$ | 300 | $n = 3$, $w = 0.61$ |
| RMSE     | 0.0523 | 0.1431 | 0.0568 | 0.0521 |
| Nash-Sutcliffe efficiency | 97.63 | 82.23 | 97.21 | 97.65 |
ICLRM models, are not unique and depend on the weighting coefficient, $\omega$. By the variation of the weighting coefficient, the proposed model is capable of simulating DIUHs laying between the Nash and ICLRM models, which makes this model superior to the Nash and ICLRM models.

Both linear and curvilinear SCS hydrographs with different peaking factors, ranging from 150 to 600, were simulated using the modified linear model. It was shown that this model yields smaller values of RMSE and higher Nash–Sutcliffe efficiencies compared to the Nash model. Furthermore, several real DIUHs were simulated using the proposed model. It was observed that the proposed model yielded more accurate results: smaller RMSE and higher Nash–Sutcliffe efficiencies.

Therefore, due to the simplicity of the proposed model and its capability of simulating a wider range of IUHs over the Nash model, the modified linear model may be considered as an appropriate model for the simulation of IUHs.

DATA AVAILABILITY STATEMENT

All relevant data are included in the paper or its Supplementary Information.

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First received 3 November 2020; accepted in revised form 18 January 2021. Available online 5 February 2021