Analysis.

Some explicit formulas for a sequence of secondary measures.

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Abstract.

We study here a sequence of secondary measures, so called because the set of secondary polynomials on a given term become orthogonal for the next measure. The main result is a formula making explicit the density of any term of the sequence, under some hypotheses. We give some applications and also derive an interpretation of the Fourier coefficients as multiple integrals.

1. Introduction and notations.

We consider a probability density function $x \mapsto \rho(x)$ on an interval $I$ bounded with $a$ and $b$. The Stieltjes transform of the measure of density $\rho$ is defined on $C-I$ by the formula:

$$z \mapsto S_\rho(z) = \int_a^b \frac{\rho(t)dt}{z-t}. \ [1].$$

We note $n \mapsto P_n$ an Hilbert base of normalized polynomials for the classic inner product:

$$(f, g) \mapsto <f, g>_\rho = \int_a^b f(t)g(t)\rho(t)dt \text{ on the associated Hilbert’s space } L^2(I, \rho).$$

We call $Q_n(X) = \int_a^b \frac{P_n(t) - P_n(X)}{t-X}\rho(t)dt$ the secondary polynomial associated with $P_n$.

Let us recall the well known result below: [2]

If a positive measure on $I$ associated to a density function $\mu$ having Stieltjes’s transformation given by the formula:

$$S_\mu(z) = z - c_i - \frac{1}{S_p(z)},$$

then secondary polynomials $Q_n$ form an orthogonal family for the inner product induced by $\mu$.

We habusively call $\mu$ the secondary measure associated with $\rho$. The moment of order 0 of this new measure is given by $d_0 = c_2 - (c_1)^2$.

If we normalize $\mu$, we introduce $\rho_1 = \frac{\mu}{d_0}$ and so we can continue the process with $\rho_1$. It thus appears a sequence of probability density functions: $n \mapsto \rho_n$ starting with $\rho_0 = \rho$ and such that for every $n$ integer, $\rho_{n+1}$ is the ‘normalized secondary measure’ of $\rho_n$. 


We adopt now the following notations:

\[ c_i^n = \int_a^b x^i \rho_n(x) \, dx ; \quad c_z^n = \int_a^b x^z \rho_n(x) \, dx \quad \text{and} \quad d_0^n = c_2^n - (c_1^n)^2 . \]

The Stieltjes transform of \( \rho_n \) will be simply represented by \( z \mapsto S_n(z) \).

2. A classical scheme of continuous fractions.

According with previous notations, we have for every \( n \):

\[ S_{n+1}(z) = \frac{1}{d_0^n [z - c_1^n - \frac{1}{S_n(z)}]} . \]

This can also be written as: \( S_n(z) = \frac{1}{z - c_1^n - d_0^n S_{n+1}} \). Thus we obtain a diagram of continuous fractions:

\[ S_p(z) = S_0(z) = \frac{1}{z - c_1^0} \frac{d_0^0}{z - c_1^1} \frac{d_0^1}{z - c_1^2} \frac{d_0^2}{z - \cdots} \frac{d_0^{n-1}}{z - c_1^n - d_0^n S_{n+1}(z)} . \]

According to classical results of the theory \([3,4]\) we have the formula (2.1):

\[ S_p(z) = \frac{u_n(-d_0^n S_{n+1}(z)) + u_{n+1}}{v_n(-d_0^n S_{n+1}(z)) + v_{n+1}} , \]

with the relations (2.2)

\[ \begin{cases} 
  u_n = (z - c_i^n)u_n - d_0^{n-1}u_{n-1} \\
  v_n = (z - c_i^n)v_n - d_0^{n-1}v_{n-1}
\end{cases} \]

starting with:

\[ \begin{cases} 
  u_0 = 0 \\
  u_1 = 1 \\
  v_0 = 1 \\
  v_1 = z - c_1^0
\end{cases} \]

- The determinant \( \Delta_{n+1} = \begin{vmatrix} u_{n+1} & v_{n+1} \\
 u_n & v_n \end{vmatrix} \) simplifies to: \( \Delta_{n+1} = d_0^n d_0^1 \cdots d_0^{n-1} \). (2.3).

We deduce immediately that:

\[ S_p(z) - \frac{u_{n+1}}{v_{n+1}} = \frac{(d_0^0 d_0^1 \cdots d_0^n)S_{n+1}(z)}{v_{n+1} (v_{n+1} - d_0^n v_n S_{n+1}(z))} . \]

Let us study this difference in the neighborhood of infinity.

\( S_{n+1}(z) = S_{\rho_{n+1}}(z) \) is equivalent to \( \frac{1}{z} \). ( \( \rho_{n+1} \) is a density of probability).

\( v_{n+1} \) is obviously a polynomial in \( z \) with leading term equal to \( z^{n+1} \).

Thus, in the neighborhood of infinity:

\[ S_p(z) - \frac{u_{n+1}}{v_{n+1}} \text{ is equivalent to } \frac{d_0^0 d_0^1 \cdots d_0^n}{z^{2n+3}} \]
Noting that the degree of $u_{n+1}$ is $n$ and that of $v_{n+1}$ is $n+1$, we can conclude that the fraction

$$F_n(z) = \frac{u_{n+1}}{v_{n+1}}$$

is a Pade approximant for $S_n(z)$ of type $[n/n+1]$. [4]

According the classic theory, this fraction is simply equal to the quotient $\frac{Q_{n-1}(z)}{P_{n+1}(z)}$. So we have the proportionality : $v_n = \lambda_n P_n(z)$ et $u_n = \lambda_n Q_n(z)$.

Thanks to the recurrence relation, the leading coefficient of $v_n$ is equal to 1, for every integer $n$, so we have $\lambda_n = \frac{1}{a_n}$, with $a_n$ leading coefficient of $P_n$.

From the formula above (2.1) : $S_\rho(z) = \frac{u_n(-d_0^n S_{n+1}(z)) + u_{n+1}}{v_n(-d_0^n S_{n+1}(z)) + v_{n+1}}$ and we can now deduce

$$S_n(z) = \frac{v_n S_0(x) - u_n}{d_0^{n-1} (v_{n-1} S_0(x) - u_{n-1})} = \frac{1}{d_0^{n-1}} \left( \frac{a_{n+1}}{a_n} \right) \frac{Q_n(z) - P_n(z) S_p(z)}{Q_{n-1}(z) - P_{n-1}(z) S_p(z)}$$  (2.4)

3. Calculation of coefficients $d_0^n$.

Another result of the theory of Stieltjes transform states that in the neighborhood of infinity :

$$S_\rho(z) - \frac{Q_n(z)}{P_n(z)}$$

is equivalent to $\frac{\gamma_n}{z^{2n+1}}$, with $\gamma_n = \frac{\gamma_0 d_0^2}{a_n^2} \times \left\| \frac{P_n}{P_0} \right\|^2$, expression in which $a_n$ is the leading coefficient of $P_n$. ( $\gamma_n = \frac{1}{a_n} \int_a^b t^n P_n(t) \rho(t) dt$). [6]

Comparing with the previously obtained equivalence, we deduce :

$$d_0^0 d_0^1 \ldots d_0^{n-1} = \frac{\gamma_n d_0^2}{a_n^2} \times \left\| \frac{P_n}{P_0} \right\|^2$$

and so : $d_0^n = \frac{a_n^2}{a_{n+1}^2} \left\| \frac{P_n}{P_0} \right\|^2 = \frac{a_n^2}{a_{n+1}^2}$  (3.1)

The formula (2.4) simplifies then to : (3.2) $S_n(z) = \frac{a_n}{a_{n-1}} \frac{Q_n(z) - P_n(z) S_p(z)}{Q_{n-1}(z) - P_{n-1}(z) S_p(z)}$

Recall now that in the classical recurrence relation to three terms, as written

$$(3.3) x P_n(x) = t_n P_{n+1}(x) + s_n P_n(x) + t_{n-1} P_{n-1}(x)$$

we have $t_{n-1} = \frac{a_{n-1}}{a_n} \left\| \frac{P_n}{P_{n-1}} \right\|^2 = \frac{a_{n-1}}{a_n}$, and so we obtain (3.4) $S_n(z) = \frac{t_{n-1}}{Q_{n-1}(z) - P_{n-1}(z) S_p(z)} \frac{Q_n(z) - P_n(z) S_p(z)}{Q_{n-1}(z) - P_{n-1}(z) S_p(z)}$
4. Explicitation of the density $\rho_n$. 

We recall the inversion formula of Stieltjes Perron which allows to reconstruct the density from its Stieltjes transform under some hypotheses: \( \rho(x) = \frac{1}{2i\pi} \lim_{\epsilon \to 0^+} (S_{\rho}(x - i\epsilon) - S_{\rho}(x + i\epsilon)) \).

A simple case of applications appears when the density is a continuous function over a compact interval.

We suppose here the initial density $\rho$ satisfying the hypotheses of inversion and we suppose also the existence of $\varphi(x) = \lim_{\epsilon \to 0^+} S_{\rho}(x - i\epsilon) + S_{\rho}(x + i\epsilon)$ for $x$ all over the interval $I$.

This function is called the reducer of the measure of density $\rho$ and allows to explicit the secondary measure associated with $\rho$ as: $\mu(x) = \frac{\rho(x)}{\varphi^2(x) + \pi^2 \rho^2(x)}$. [5]

Under these hypotheses, the density $\rho_n$ can be made explicit thanks Stieltjes Perron and the formula (3.4) above: $S_n(z) = \frac{1}{t_{n-1}} \frac{Q_n(z) - P_n(z)S_{\rho}(z)}{Q_{n-1}(z) - P_{n-1}(z)S_{\rho}(z)}$.

An elementary calculation then leads to:

$$
\rho_n(x) = \frac{1}{t_{n-1}} \times \frac{\rho(x)(P_{n-1}(x)Q_n(x) - P_n(x)Q_{n-1}(x))}{(P_{n-1}(x)\frac{\varphi(x)}{2} - Q_{n-1}(x))^2 + \pi^2 \rho^2(x)P_{n-1}^2(x)} \tag{4.1}
$$

Now recall the formula (2.2): $\Delta_{n+1} = \begin{vmatrix} u_{n+1}(x) & v_{n+1}(x) \\ u_n(x) & v_n(x) \end{vmatrix} = a_0^nd_0^1\cdots d_0^{n-1}$.

From $v_n = \lambda_n P_n(z) ; u_n = \lambda_n Q_n(z) ; \lambda_n = \frac{1}{a_n} ; a_n^2 = \frac{a_{n-1}^2}{a_n^2}$ we easily deduce:

$$Q_{n+1}(x)P_n(x) - P_{n+1}(x)Q_n(x) = \frac{a_{n+1}}{a_n} = \frac{1}{t_n}.$$ 

So, the formula (4.1) simplifies to:

$$\rho_n(x) = \frac{1}{(t_{n-1})^2} \times \frac{\rho(x)}{(P_{n-1}(x)\frac{\varphi(x)}{2} - Q_{n-1}(x))^2 + \pi^2 \rho^2(x)P_{n-1}^2(x)} \tag{4.2}$$
Since all these functions are probability densities, we deduce the value of the following integral as:

\[
\int_a^b \frac{\rho(x)}{(P_{n-1}(x) \frac{\varphi(x)}{2} - Q_{n-1}(x))^2 + \pi^2 \rho^2(x) P_{n-1}^2(x)} \, dx = (t_{n-1})^2
\]

Some examples. [5,6,7]

- For the uniform Lebesgue measure over [0,1]:

\[
\int_0^1 \frac{dx}{[P_n(x) \ln \left( \frac{x}{1-x} \right) - Q_n(x)]^2 + \pi^2 P_n^2(x)} = \frac{(n+1)^2}{4(2n+1)(2n+3)}
\]

\(P_n(x)\) is the shifted (and normalized) Legendre polynomial of order \(n\).

- For \(\rho(x) = e^{-x}\) over [0,\(\infty\]):

\[
\int_0^{\infty} \frac{e^{-x} \, dx}{[P_n(x) e^{-x} \text{Ei}(x) - Q_n(x)]^2 + \pi^2 e^{-2x} P_n^2(x)} = (n+1)^2
\]

\(P_n(x)\) is the Laguerre polynomial of order \(n\) and \(\text{Ei}\) the exponential integral.

- For the Gaussian measure \(\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\) over the real axis.

\[
\int_{-\infty}^{\infty} \frac{\rho(x) \, dx}{\sqrt{2(e^{-\frac{x^2}{2} \text{erfi}(\frac{x}{\sqrt{2}}))}}} = n + 1
\]

\(P_n(x)\) is the Hermite polynomial of order \(n\) and \(\text{erfi}\) the imaginary error function.

- For the Tchebychev measure of the second kind (\(\rho(x) = \frac{2}{\pi} \sqrt{1-x^2}\) over [-1,1])

In this case the sequence of normalized secondary measures is constant. So we find the classical relation connecting the Tchebychev polynomials:

\[
P_n^2(x) - P_{n-1}(x) P_{n+1}(x) = 1
\]
5. An interpretation of the Fourier coefficients.

We recall the following result: [5]

The operator \( f(x) \mapsto g(x) = \int_a^b \frac{f(t) - f(x)}{t - x} p(t) \, dt \) creating secondary polynomials extends to a continuous linear map \( T_p \) linking the space \( L^2(I, \rho) \) to the Hilbert’s space \( L^2(I, \mu) \).

Its restriction to the hyperplane \( H_\rho \) of the function orthogonal with \( P_0 = 1 \) constitutes an isometric function for both norms respectively. So, for any couple \((f, g)\) of elements of \( L^2(I, \rho) \), we have:

\[
(5.1) \quad \langle f / g \rangle_\rho - \langle f / 1 \rangle_\rho \times \langle g / 1 \rangle_\rho = \langle T_p (f) / T_p (g) \rangle_\mu
\]

Because of the normalization performed in the sequence of secondary measures, we lose the isometric character of the associated transforms.

If we note \( H_n \) the space of elements of \( L^2(I, \rho_n) \) orthogonal with \( P_0 = 1 \),

\[
\sqrt{d_0^n} \, T_{p_n} \text{ is now an isometric map linking } (H_n, \rho_n) \text{ to } (L^2(I, \rho_{n+1})
\]

So, for any couple \((f, g)\) of \( L^2(I, \rho_n) \), we can write :

\[
(5.2) \quad \langle f / g \rangle_{p_n} - \langle f / 1 \rangle_{p_n} \times \langle g / 1 \rangle_{p_n} = d_0^n \langle T_{p_n} (f) / T_{p_n} (g) \rangle_{p_{n+1}}
\]

We will now generalize this formula by composing the transforms.

We introduce \( F_n = T_{p_n} \circ T_{p_{n+1}} \circ ... \circ T_{p_1} \circ T_{p_0} \). (\( p_0 = \rho \) and so \( F_0 = T_{\rho} \))

An elementary recurrence led to :

\[
(5.3) \quad F_n(f)(t_n) = \int_a^b f(t) \, dt - \int_a^b \left( \sum_{k=0}^{n+1} \prod_{j \neq k} (t_k - t_j) \right) \rho_0(t_0) ... \rho_n(t_n) \, dt_0 ... dt_n.
\]

(multiple integral of order \( n+1 \))

Let be a couple \((f, g)\) of elements of \( L^2(I, \rho) \) with \( g \perp P_0 \).

We deduce then from (5.2) : \( \langle f / g \rangle_{p_0} = \langle T_{p_0} (f) / T_{p_0} (g) \rangle_{\mu} = d_0^0 \langle F_0 (f) / F_0 (g) \rangle_{p_1} \)

Assume further \( g \) orthogonal with \( P_1 \), still in \( L^2(I, \rho) \). We deduce that \( F_0 (g) \) is orthogonal with \( P_0 = 1 \) in \( L^2(I, \rho_1) \), and so we get : \( \langle f / g \rangle_{p_1} = d_0^1 d_1^0 \langle F_1 (f) / F_1 (g) \rangle_{p_2} \)

More generally, if \( g \) is orthogonal at every term of \((P_0, P_1, ... , P_n)\) in \( L^2(I, \rho) \).

We have the relation \((5.4) : \quad \langle f / g \rangle_{p_0} = d_0^0 d_1^0 ... d_n^0 \langle F_n (f) / F_n (g) \rangle_{p_{n+1}} \)
This formula above will allow us to make explicit the Fourier coefficients of a given function as multiple integral.

Because $P_n \perp P_0, P_{1}, \ldots, P_{n-1}$ we deduce directly from (5.4):

\[(5.5) \quad C_n(f) = \frac{1}{a_n} \int_{a}^{b} \int_{a}^{b} \ldots \int_{a}^{b} \left( \sum_{k=0}^{n} \prod_{j\neq k} \frac{f(t_k)}{(t_k - t_j)} \right) \rho_0(t_0) \ldots \rho_n(t_n) dt_0 \ldots dt_n\]

Recall now a precedent result (3.1):

\[d_0^0 \ldots d_0^0 = \frac{2}{a_n^2} \times \frac{\|P_0\|^2}{\|P_0\|^2} = \frac{1}{a_n^2}\]

We also easily result $F_{n-1}(P_n) = a_n$, leading coefficient of $P_n$, because every function $\rho_n$ is a density of probability.

So the formula (5.5) simplifies to:

\[(5.6) \quad C_n(f) = \frac{1}{a_n} \int_{a}^{b} \int_{a}^{b} \ldots \int_{a}^{b} \left( \sum_{k=0}^{n} \prod_{j\neq k} \frac{f(t_k)}{(t_k - t_j)} \right) \rho_0(t_0) \ldots \rho_n(t_n) dt_0 \ldots dt_n\]

We conclude this paragraph by a particular application of this formula.

We consider here the function: $x \mapsto f(x) = \frac{1}{x + a}$, with $a \notin I$

$f$ is an eigenvector for every operator $T_p$. Specifically, an elementary calculation gives:

\[T_p(f(x)) = \gamma_a f(x) \quad \text{with} \quad \gamma_a = -\int_{I} \frac{\rho(t)dt}{t + a} = S_p(-a)\]

By elementary composition, we get: $F_{n-1}(f) = \gamma_0 \times \gamma_1 \times \ldots \times \gamma_{n-1} \times f$. So, from (5.6):

\[C_n(f) = \frac{1}{a_n} \times \gamma_0 \times \gamma_1 \times \ldots \times \gamma_{n-1} \times \int_{I} \frac{\rho_n(t)dt}{t + a} = -\frac{1}{a_n} \times \gamma_0 \times \gamma_1 \times \ldots \times \gamma_{n-1} \times \gamma_n\]

This can be written as:

\[C_n(f) = - \frac{1}{a_n} \times \prod_{k=0}^{n} S_{p_k}(-a)\]

And more explicitly:

\[\int_{I} \frac{P_n(x)\rho(x)dx}{x + a} = -\frac{1}{a_n} \times \prod_{k=0}^{n} S_{p_k}(-a)\]

By simple quotient: $S_{p_n}(-a) = \frac{a_n}{a_{n-1}} \times \frac{C_n(f)}{C_{n-1}(f)}$

So we have for $z \notin I$:

\[\int_{I} \frac{\rho_n(t)}{t - z} dt = \frac{a_n}{a_{n-1}} \times \int_{I} \frac{P_n(t)\rho(t)dt}{t - z} \int_{I} \frac{P_{n-1}(t)\rho(t)dt}{t - z} (5.7)\]
Apply this formula to the tchebychev measure of the second kind over the interval \([0, 1]\).

We have \(p(x) = \frac{8\sqrt{x(1-x)}}{\pi}\), \(S_{p_n}(x) = -8a - 4 + 8\sqrt{a^2 + a}\), and the leading coefficient of \(P_n\) is \(a_n = 4^n\). As in this case the sequence of normalized secondary measures is constant, we deduce from

\[
(5.7) : -8a - 4 + 8\sqrt{a^2 + a} = 4 \times \int_0^1 \frac{P_n(t)p(t)dt}{t+a}\int_0^1 \frac{P_{n-1}(t)p(t)dt}{t+a}
\]

\(C_n(f) = q^nC_0(f)\), with \(q = 2\sqrt{a^2 + a - 2a - 1}\) and \(C_0(f) = \int_0^1 \frac{P_0(x)dx}{x+a} = -4q\)

So we have in \(L^2([0,1],p)\) \(\frac{1}{x+a} = -4\sum_{n=0}^{\infty} q^n P_n(x)\).

Note then \(q = 2\sqrt{a^2 + a - 2a - 1} \Rightarrow a = \frac{(q+1)^2}{4q}\).

So we get :

\[
\frac{t}{(t+1)^2 - 4tx} = \sum_{n=0}^{\infty} t^n P_n(x)
\]

One recognizes here the classical generating function for the Tchebychev polynomials.

**6. Complements and other formulas.**

- First moment of \(p_n\).

From the formulas (2.2) we get :

\(v_{n+1}(x) = (x - c_n)v_n(x) - d_n^{-1}v_{n-1}(x)\).

Thanks to the proportionality \(v_n = \lambda_n P_n\), this can be written :

\(xP_n(x) = \frac{\lambda_{n+1}}{\lambda_n} P_{n+1}(x) + c_n^n P_n(x) + \frac{d_n^{-1}\lambda_{n-1}}{\lambda_n} P_{n-1}(x)\)

If we recall the classical three-terms relation (3.3) :

\(xP_n(x) = t_n P_{n+1}(x) + s_n P_n(x) + t_{n-1} P_{n-1}(x)\), we deduce immediatly the equalities :

\(c^n_n = s_n\); \(t_n = \frac{\lambda_{n+1}}{\lambda_n}\) and \(t_{n-1} = \frac{d_n^{-1}\lambda_{n-1}}{\lambda_n}\)

Note that the last two give back : \(d_n^n = (t_n)^2\).

Using (4.2), the relation \(c^n_n = s_n\) is reflected in integral form by

\[
\int_0^1 \frac{x p(x)}{(P_{n-1}(x)\frac{\varphi(x)}{2} - Q_{n-1}(x))^2 + \pi^2 p^2 (x) P_{n-1}^2(x)}dx = s_n(t_{n-1})^2
\]

(6.1)
• Reducer of $\rho_n$.

Using formula (3.4) we can make explicit the reducer of $\rho_n$, defined by:

$$\varphi_n(x) = \lim_{\varepsilon \to 0^+} S_n(x - i\varepsilon) + S_n(x + i\varepsilon).$$

After some elementary calculations and thanks to the inversion formula of Stieltjes Perron, we get:

$$\varphi_n(x) = \frac{2}{(t_{n-1})^2} \times \frac{(P_n(x) \varphi(x) - Q_n(x)) \times (P_{n-1}(x) \varphi(x) - Q_{n-1}(x)) + \pi^2 \rho^2(x) P_n(x) P_{n-1}(x)}{(P_{n-1}(x) \varphi(x) - Q_{n-1}(x))^2 + \pi^2 \rho^2(x) P_{n-1}(x)^2} \quad (6.2)$$

• Orthogonal polynomials associated with $\rho_n$.

From the coupling formula $S_\mu(z) = z - c_1 - \frac{1}{S_\rho(z)}$ we can easily express the set of primary $(U_n)$ and secondary $(V_n)$ orthogonal polynomials of $\mu$ through:

$$\begin{align*}
U_n(x) &= Q_{n+1}(x) \\
V_n(x) &= (x - c_1) Q_{n+1}(x) - P_{n+1}(x)
\end{align*} \quad (6.3)$$

From the isometric character of $T_\rho$, we have for $n \geq 1 \|Q_n\|_\mu = \|P_n\|_\rho = 1$.

When we normalize $\mu$ to $\rho_1 = \frac{\mu}{d_0}$, we can introduce $P_n^1(x) = t_0 U_n(x)$ to keep the norm equal to 1. (Because $T_{\rho_1} = \frac{1}{d_0} T_\mu$ and $d_0 = (t_0)^2$)

The same way we change $V_n$ to $Q_n^1(x) = \frac{V_n(x)}{t_0}$. So we get $Q_n^1 = T_{\rho_1}(P_n^1)$ because $T_{\rho_1} = \frac{1}{d_0} T_\mu$ and $T_\mu (U_n) = V_n$.

The relations (6.3) translate in to:

$$\begin{align*}
P_n^1(x) &= t_0 Q_{n+1}(x) \\
Q_n^1(x) &= \frac{1}{t_0} [(x - c_1) Q_{n+1}(x) - P_{n+1}(x)]
\end{align*} \quad (6.4)$$

More generally if we note $n \mapsto (P_n^k(x), Q_n^k(x))$ the set of polynomials associated with $\rho_k$, we can write in matrix form:

$$\begin{pmatrix} P_{n+1}^k(x) \\
Q_{n+1}^k(x)
\end{pmatrix} = \begin{pmatrix} t_k & (t_k)^2 \\
-1 & x - c_1
\end{pmatrix} \begin{pmatrix} P_{n+1}^k(x) \\
Q_{n+1}^k(x)
\end{pmatrix} \quad (6.5)$$
Now introduce $M_k(x) = \frac{1}{t_k} \begin{pmatrix} 0 & t_k^2 \\ -1 & x-s_k \end{pmatrix}$ and $\Pi_k(x) = M_k(x)M_{k-1}(x)\ldots M_0(x)$

We easily deduce from (6.5) : 
$$ \left( \begin{array}{c} P_{n+k}^0(x) \\ Q_{n+k}^0(x) \end{array} \right) = \Pi_k(x) \left( \begin{array}{c} P_{n+k}^0(x) \\ Q_{n+k}^0(x) \end{array} \right) \quad (6.6) $$

If we note $\Pi_k(x) = \begin{pmatrix} A_{k+1}(x) & B_{k+1}(x) \\ C_{k+1}(x) & D_{k+1}(x) \end{pmatrix}$, we obtain the relations :

$$ A_{k+2}(x) = t_{k+1} C_{k+1}(x) $$

$$ B_{k+2}(x) = t_{k+1} D_{k+1}(x) $$

$$ C_{k+2}(x) = -\frac{1}{t_{k+1}} A_{k+1}(x) + \frac{(x-s_{k+1})}{t_{k+1}} C_{k+1}(x) $$

$$ D_{k+2}(x) = -\frac{1}{t_{k+1}} B_{k+1}(x) + \frac{(x-s_{k+1})}{t_{k+1}} D_{k+1}(x) $$

It appears clear that $C_n$ and $D_n$ satisfy the classical three-terms relation for the initial measure : $xP_n(x) = t_{n+1}P_{n+1}(x) + s_n P_n(x) + t_{n-1} P_{n-1}(x)$

$\forall k \geq 1 \quad xC_k(x) = t_k C_{k+1}(x) + s_k C_k(x) + t_{k-1} C_{k-1}(x)$

$\forall k \geq 1 \quad xD_k(x) = t_k D_{k+1}(x) + s_k D_k(x) + t_{k-1} D_{k-1}(x)$

According to the classical theory, we can write :

$$ C_k(x) = aP_k(x) + bQ_k(x) $$

$$ D_k(x) = cP_k(x) + dQ_k(x) $$

The components $a,b,c,d$ are determined using the initial conditions.

$$ P_0^0(x) = 1; \quad P_0^0(x) = \frac{x-s_0}{t_0} $$

$$ Q_0^0(x) = 0; \quad Q_0^0(x) = \frac{1}{t_0} $$

By (6.4) we have $P_2^0(x) = \frac{(x-s_0)(x-s_1)-(t_0)^2}{t_0 t_1} \quad Q_2^0(x) = \frac{x-s_1}{t_0 t_1}$.

A direct calculation with matrices gives us :

$$ C_1(x) = -Q_0^0(x); \quad C_2(x) = -Q_0^0(x); \quad D_1(x) = P_0^0(x); \quad D_2(x) = P_2^0(x) $$

So we get : $a = 0; b = -1; c = 1; d = 0$ and consequently :

$\forall k \geq 1 \quad C_k(x) = -Q_k^0(x) = -Q_k(x) \quad \text{and} \quad D_k(x) = P_k^0(x) = P_k(x)$

$$ \Pi_k(x) = \begin{pmatrix} -t_k Q_k(x) & t_k P_k(x) \\ -Q_{k+1}(x) & P_{k+1}(x) \end{pmatrix} : \begin{pmatrix} P_{n+k}^0(x) = t_k [P_k(x)Q_{n+k+1}(x) - Q_k(x)P_{n+k+1}(x)] \\ Q_{n+k}^0(x) = P_{k+1}(x)Q_{n+k+1}(x) - P_{n+k+1}(x)Q_{k+1}(x) \end{pmatrix} \quad (6.7) $$
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