Improving the Feasibility of Moment-Based Safety Analysis for Stochastic Dynamics

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Abstract—Given a dynamical system modeled via stochastic differential equations (SDEs), we evaluate the safety of the system through its exit-time moments. Using appropriate semidefinite positive matrix constraints, an SDP moment-based approach can be used to compute moments of the exit time. However, the approach is impeded when analyzing higher dimensional physical systems as the dynamics are limited to polynomials. Computational scalability is also poor as the dimensionality of the state grows, largely due to the combinatorial growth of the optimization program. We propose methods to make feasible the safety analysis of higher dimensional physical systems. The restriction to polynomial dynamics is lifted by using state augmentation, which allows one to generate the optimization for a broader class of nonlinear stochastic systems. We then reformulate the constraints to mitigate the computational limitations associated with an increase in state dimensionality. We employ our methods on two example processes to characterize their safety via exit times and show the ability to handle multidimensional systems that were previously unsupported by the existing SDP method of moments.

Index Terms—Autonomous systems, Markov processes, stochastic systems, uncertain systems.

I. INTRODUCTION

Safety verification is an important step to ensure that systems perform in ways that designers intend while mitigating the risks associated with unplanned behavior. In the deterministic scenario, reachability can be applied to the verification problem and produce a Boolean result that indicates whether a system will enter an unsafe state. In the stochastic setting, we may seek an analog to this Boolean safety statement through the distribution of the time of exit from a safe region. For example, consider a vehicle equipped with a controller using uncertain measurements. The exit-time distribution yields insight into when it will fail. Analysis of the variability of the time-to-failure allows designers to reason about systems where the safe operation window may be long. A system may always eventually fail; however, if the time-to-failure is sufficiently large, then the design may still be acceptable.

Our article applies a moment-based approach [1] to obtain exit-time moments for systems modeled via stochastic differential equations (SDEs). We formulate the computation of the moments as an infinite-dimensional convex optimization over the space of measures using a linear evolution equation on the moments of the measures. Following [2], we use semidefinite programming (SDP) to compute bounds on the expected duration of safe operation. This moment-based method has previously seen successful application in financial instrument pricing [3] and optimal control [4].

Applied to SDEs, the existing method is restricted to systems with polynomial drift and diffusion dynamics as it relies on processes whose infinitesimal generators map monomials into polynomials. In particular, the inability to support sinusoidal dynamics prevents application to a large number of physical systems. Indeed, sinusoids can be found in the dynamics of virtually all robotic systems. Examples include robotic arms, tracked robots, wheeled robots, and quadcopters. The existing SDP method of moments also suffers from poor computational scalability as the dimensionality of the system and complexity of the safe set grow. The combinatorial growth of the number of moments results in matrix constraints that quickly drive computational costs upward.

We propose changes to the SDP method of moments to analyze the safety characteristics of physical systems via their distribution of exit times. The following is contributed.

1) We discuss the limitations of the existing SDP moment method when applied to high-dimensional systems and the restrictions that exclude a large class of applications.
2) We introduce a state augmentation method to extend support beyond polynomial dynamics. We then propose a reformulation of the SDP, which replaces large sequences of positive semidefinite (PSD) matrix constraints with smaller sets of scalar equality constraints.
3) We provide examples of our approach in handling a broader class of real-world dynamics and demonstrate the ability to compute accurate bounds on the duration of safe behavior by considering higher degree moments.

II. RELATED WORK

In the stochastic setting, several studies have been proposed in which an initial set satisfying certain (static) safety conditions is obtained. Reachability has been studied through stochastic viability and target problems [5], [6]. A connection between stochastic optimal control (the exit-time problem) and the reach-avoid problem for controlled diffusion processes is presented in [7]. The method computes a set of states where there exists an admissible control scheme driving the system to a desired set prior to hitting an avoid set. Notably, the set of initial states are given by the super level sets of the viscosity solution of a suitable Hamilton–Jacobi–Bellman equation.

Occupation measure approaches for computing the region of attraction (ROA) in deterministic systems have been studied in [8] and [9]. Korda et al. [8] analyze the ROA for deterministic nonlinear (polynomial) systems and show a linear programming approach for approximating the region. A linear evolution equation over measures is used to yield an optimization over occupation measures. Using a similar analysis, the notion of p-safety for stochastic systems is
developed in [10] and is most closely related to our methodology. In p-safety, one analyzes the set of initial states for which the system is safe with probability at least $p$. The evolution equation of the occupation measure is applied to stochastic polynomial dynamics. Similar to the approach we use, p-safety links the initial, final, and occupation measures with a linear evolution and formulates an infinite-dimensional optimization that is solved by the generalized moment method [10].

Numerical approaches for the exit time of Markov processes have also been studied [2], [11]. These methods characterize the exit-time problem via an infinite-dimensional convex program subject to certain martingale constraints [12]. The evolution of functionals is formulated using the moments of the occupation and exit location measures, reducing the analysis to moment sequences. An SDP approach to the moment problem is proposed by Lasserre et al. [1] and showed increased computational efficiency and accuracy. The SDP approach has been utilized in a number of applications, ranging from chemical/biomolecular dynamics to finance and economics [3], [13], [14], [15]. An analysis of stochastic hybrid systems is given in [16] where the state space is augmented with discrete states. In order to derive the appropriate linear constraints, these works consider polynomial dynamics (or require casting to polynomials), which are often naturally afforded due to the underlying systems under test.

Scalability issues are well known in the literature applying the SDP method. The moment approach is used to find bounds of the survival time of chemical systems using PSD constraints in [17]. The examples presented use low-order moments and the authors make explicit the limitations on scalability due to the combinatorial growth of moments with the state dimension. Numerical stability is also questioned as higher moments quickly begin to differ by multiple orders of magnitude. Likewise, researchers applying the SDP method to the analysis of economic-emission dispatch noted the limitations to small-medium-sized problems due to computations involving a large number of high-dimensional matrices [18].

We consider the SDP-based moment method for evaluating the safety of stochastic dynamical systems and use state augmentation to extend beyond polynomial dynamics. By reformulating the constraints of the SDP, we reduce the number of large-dimensional matrices present in the optimization and allow for applications with a greater number of moments that yield bounds with higher accuracy.

III. SYSTEM MODEL

A. Notation

For two values $a, b \in \mathbb{R}$, we define $a \wedge b := \min\{a, b\}$. Given a set $A$, we denote its complement by $A^c$ and its boundary by $\partial A$. The Borel $\sigma$-algebra on a topological space $A$ is denoted by $\mathcal{B}(A)$ and for $B \in \mathcal{B}(A)$, the indicator function is denoted by $\mathbb{1}_B$ and defined as $\mathbb{1}_B(x) = 1$ if $x \in B$ and 0 otherwise. The support of a measure $\mu$ on a measurable space $(A, \mathcal{B}(A))$ is denoted by $\text{supp}(\mu)$. For a process $X = (X_t)_{t \geq 0}$ described via SDE in $\mathbb{R}^n$, we denote $\dot{X}$ as the state augmented version of $X$ given by an SDE in $\mathbb{R}^{n+s}$, $s > 0$. The set of integers $\{1, 2, \ldots, N\}$ is denoted by $[N]$. We represent the $n$-dimensional multi-index $\alpha$ as a tuple such that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and the set of such $n$-dimensional multi-indexes is denoted by $\mathbb{N}^n$. Lastly, the monomial with degree corresponding to the multi-index $\alpha$ is given by $(x_1, x_2, \ldots, x_n)_{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

B. Stochastic Differential Equations

We consider the $\mathbb{R}^n$-valued SDE:

$$
\frac{dX_t}{dt} = h(X_t, t)dt + \sigma(X_t, t)dB_t
$$

where $X_t \in \mathbb{R}^n$, $X_0 = x_0$ is known, $0 \leq t \leq T$, and $T > 0$. Let $B_t$ be a standard $d$-dimensional Brownian motion and $d$ $B_t$ represent its differential form. Let the functions $h: \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times d}$ represent the drift and diffusion terms of the SDE, respectively. Furthermore, let the functions be measurable and satisfy the space variable growth condition

$$
|h(x, t)| + |\sigma(x, t)| \leq C(1 + |x|) \quad x \in \mathbb{R}^n, t \in [0, T]
$$

for some constant $C$, as well as the space variable Lipschitz condition

$$
|h(x, t) - h(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq D|x - y| \quad x, y \in \mathbb{R}^n, t \in [0, T].
$$

Under these circumstances, the SDE (1) has a unique time-continuous solution starting at time $t$ and state $x_0$ [19, Th. 5.2.1]. The stochastic process $X = (X_t)_{t \geq 0}$, given by the SDE (1), with initial condition $X_0 = x_0$ with probability one, is a Markov process with continuous sample paths [19, Th. 7.1.2].

We consider a state space $\mathbb{E} \subseteq \mathbb{R}^n$ that is partitioned into two sets: $S$ and $S^c$. Here, $S$ is an open and bounded safe set and $S^c = E - S$ is its complement (unsafe set). $\tau$ is a stopping time defined with respect to $S^c$ and is the minimum of the first time that the process $X$ reaches the unsafe set

$$
\tau = \inf\{t \mid X_t \in S^c\}.
$$

Going forward, we will be concerned with a finite exit time $\tau \wedge T$. Intuitively speaking, if the exit time $\tau \wedge T$ is strictly less than $T$, then the system has become unsafe within the time horizon we are concerned with. While if $\tau \wedge T = T$, the system has stayed safe almost surely for the entire finite duration we are examining.

IV. COMPUTATION OF EXIT-TIME MOMENTS

A. Linear Evolution Equation

Let $(X_t)_{t \geq 0}$ be a time-homogeneous diffusion in $\mathbb{R}^n$ such that its dynamics are given by the following SDE:

$$
\frac{dX_t}{dt} = h(X_t)dt + \sigma(X_t)dB_t.
$$

The infinitesimal generator with domain $D(A)$ of a time-homogeneous Itô diffusion in $\mathbb{R}^n$ for twice differentiable continuous $f$ is [19, Th. 7.3.3]

$$
Af(x) = \sum_i h_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^t)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}.
$$

To generate linear constraints, the dynamics of (3) are first lifted to the space of measures to define a set of linear evolution equations. The process $(X_t)$ satisfies the martingale problem, we have

$$
\mathbb{E}^{x_0}[f(X_{T\wedge T})] - \mathbb{E}^{x_0}[f(X_0)] = \mathbb{E}^{x_0}\int_0^{T\wedge T} Af(X_s)ds = 0.
$$

Here, the notation $\mathbb{E}^{x_0}$ serves to emphasize that $X_0 = x_0$. Let $\mu_0$ be the expected occupation measure up to the exit time $\tau \wedge T$ of the process $(X_t)$, and $\mu_1$ be its exit location distribution

$$
\mu_0(B) = \mathbb{E} \int_0^{\tau \wedge T} \mathbb{1}_B(X_t)dt \quad \mu_1(B) = \mathbb{P}(X_{\tau \wedge T} \in B).
$$

The measures $\mu_0$ and $\mu_1$ are supported on the safe set $S$ and safe set boundary $\partial S$, respectively (supp($\mu_0$) = $S$, supp($\mu_1$) = $\partial S$). Equation (5) is now rewritten as

$$
\int_S \mathbb{P}(f(x)\mu_1(dx) - f(x_0) - \int_S Af(x)\mu_0(dx) = 0
$$

for every test function $f \in D(A)$ and $X_0 = x_0 \in S$. Equation (6) represents a linear evolution equation linking the occupation and exit
measures of the process \((X_t,\alpha)\), also referred to as the basic adjoint equation \([11]\).

The moments of the measures \(\mu_0\) and \(\mu_1\) are given by

\[
m_i = \int_S x^i \mu_0(dx) \quad \text{and} \quad b_i = \int_S x^i \mu_1(dx)
\]

where each \(i \in \mathbb{N}^n\) is an \(n\)-dimensional multi-index and \(x^i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\). Notice that the first moment of the exit time is \(m_0\). Under processes where monomial test functions \(f\) produce a polynomial infinitesimal generator \(A f\), the conditions imposed by the basic adjoint equation are further relaxed from the space of all functions \(f \in D(A)\) to all monomials \(f\), and expressed through the sequence of moments of \(\mu_0\) and \(\mu_1\): \([m_1]_{i \in \mathbb{N}^n}\) and \([b_i]_{i \in \mathbb{N}^n}\). The relaxed condition

\[
\sum_j [c_j(i) \cdot m_j] + x_0^i - b_i = 0
\]

is imposed for every \(i \in \mathbb{N}^n\) and monomial \(f(x) = x^i\), and \(c_i, s\) are scalar constants. The condition (7) gives a set of linear constraints involving the moments of the exit time and exit distribution.

### B. SDP Moment Constraints

The martingale constraints (7) alone are not able to guarantee that the sequences \([m_1]_{i \in \mathbb{N}^n}\) and \([b_i]_{i \in \mathbb{N}^n}\) are moment sequences with respect to the appropriate occupation and exit measures. In order to enforce that the sequences are moment sequences, additional conditions must be imposed.

Let \([m_\alpha]\) be a sequence where \(\alpha \in \mathbb{N}^n\) is a multi-index. The sequence is sorted according to the graded lexicographic order where \(\alpha\) represents a monomial \(x^\alpha\). Given a moment sequence, the moment matrix \(M_k(m)\) is defined as follows:

\[
M_k(m)(i, j) = m_{\gamma + \xi} \quad \text{if} \quad i, j \neq 1
\]

where \(\gamma\) and \(\xi\) are the multi-indices of the entries at \(m_\alpha[i]\) and \(m_\alpha[j]\), respectively. Else

\[
M_k(m)(1, s) = M_k(m)(s, 1) = m_\alpha[s].
\]

Next the localizing matrix \(M_k(qm)\) is defined with respect to a polynomial \(q\). Let \(\beta(i, j)\) be the multi-index of the \(i, j\)th entry of the moment matrix \(M_k(m)\) and let \([q]\) be the vector of coefficients of the polynomial \(q\) in graded lexicographic order. The entries of the localizing matrix is then given by

\[
M_k(qm)(i, j) = \sum \alpha q_\alpha \cdot m_{\beta(i, j) + \alpha}.
\]

### C. Optimization Program

The upper and lower bounds of the expected exit time \(E[\tau]\) of the system (3) is computed through the following semidefinite program [2].

**Optimization I (Original Constraints)**

Maximize (resp. Minimize):

\[
m_0
\]

Subject to:

\[
\begin{align*}
x_0^i &+ \sum_{i \in \mathbb{N}^n} c_i(k) \cdot m_i - b_k = 0 \\
M_k(m) &\succeq 0 \\
M_k(b) &\succeq 0 \\
M_k(q_0 m) &\succeq 0 \\
M_k(q_1 b) &\succeq 0 \\
\forall k \leq K
\end{align*}
\]

where \(M_k(\cdot)\) and \(M_k(q_0, q_1)\) are the moment and localizing matrices corresponding to the moment sequence of \(\mu_0\) and \(\mu_1\). The polynomials \(q_0\) and \(q_1\) with which the localizing matrices are defined with respect to are derived from the semialgebraic sets \(E_1 := \{ x \in \mathbb{R}^d \mid q_0(x) \geq 0 \}\) and \(E_2 := \{ x \in \mathbb{R}^d \mid q_1(x) \geq 0 \}\), such that the measures \(\mu_0, \mu_1\) are supported on \(E_1\) and \(E_2\), respectively. The optimization is restricted to a finite number of moments \(K\) to make computation tractable.

### V. State Space Augmentation

#### A. Time-Dependent Itô Diffusion

To compute higher order moments, the time dimension must be included within the state. The new state \(\tilde{X}_t = [X_t, t]^T\) is given as \(X_t = [X_t, t]^T\), with dynamics

\[
\begin{align*}
d\tilde{X}_t &= [h(X_t, t)]^T dt + [\sigma(X_t, t)^T] dB_t \\
\tilde{X} = (\tilde{X}_t)_{t \geq 0}\quad \text{is now an Itô diffusion in} \ \mathbb{R}^{n+1} \ \text{with initial condition} \ \tilde{x}_0 = (x_0, 0). \ \text{The safe set of the SDE (8)} \ \tilde{S} = S \times [0, T], \ \text{which guarantees a finite exit time.}
\end{align*}
\]

As in Section IV-A, the martingale constraints in terms of the moment sequences \([m_1]_{i \in \mathbb{N}^{n+1}}\) and \([b_i]_{i \in \mathbb{N}^{n+1}}\) are given by (7) for every monomial \(f(x, s) \in D(A)\), such that \(f(x, s) = (x, s)^k, k \in \mathbb{N}^{n+1}\).

The formulation of the moment and localizing matrix constraints remains the same as that of Section IV-B, while the maximization (minimization) variable when computing the higher order moments \(E[\tau^n]\) is now \(m_0, m_{0,n-1}\).

#### B. Augmentation With Redundant States

The restriction to polynomial drift and diffusion dynamics exclude a large class of real-world systems that exhibit other nonlinear behaviors in their dynamics model. In particular, physical systems operating in multidimensional space often incorporate sinusoidal dynamics to specify force components or rotation matrices.

In order to support these dynamics for \(X\), the assumption of the infinitesimal generator mapping monomial test functions \(f\) to polynomials must be broken. As a result, one is unable to relax the basic adjoint equation and generate constraints based on the moment sequences of \(\mu_0\) and \(\mu_1\).

**Definition 1:** Given a stochastic process \(X\) in \(\mathbb{R}^n\), we say that \(X\) is closed under infinitesimal generation if \(A f(x)\) is a polynomial with respect to the state variables for all monomial functions \(f(x) = x^i, i \in \mathbb{N}^n\).

**Proposition 1:** The time-dependent Itô diffusion \(\tilde{X}\) in \(\mathbb{R}^{n+1}\) described via the SDE (8) is closed under infinitesimal generation if for each drift term \(\tilde{h}_i(\tilde{x}), 0 \leq i < n+1\), and diffusion term \(\tilde{\sigma}_{q, k}(\tilde{x}), 0 \leq j, k < n+1\), there exists \(n+1\) dimensional multi-index sets \(P, Q\), such that \(\tilde{h}_i(\tilde{x}) = \sum_{p \in P} c_p \tilde{x}^p\) and \((\tilde{\sigma}^T)_{q, k}(\tilde{x}) = \sum_{q \in Q} c_q \tilde{x}^q\), where \(c_p, c_q \in \mathbb{R}\).

In the cases where the dynamics of the SDE violate the requirements in Proposition 1, we propose an augmentation technique where the state space is extended with redundant variables. The augmentation is chosen such that the extended state space now includes the nonpolynomial (w.r.t the state) terms of the drift and diffusion components, as well as possibly their derivatives.

Let \((c_j(X_t))\}_{j \in [0, J-1]}\) be the set of coefficients (along with possibly their derivatives) of the generator \(Af(x, s)\) that violate the generator polynomial mapping assumption

\[
f \mapsto Af(x, s) = \sum_{i \in \mathbb{N}^{n+1}} c_i(k) \cdot (x, s)^i
\]
where the $c_i$s are some scalar constants. Note that the $c_i$s are partially determined by the coefficients of the partial derivatives found in the infinitesimal generator and correspond to the drift and diffusion terms of the SDE. We consider the augmented state space $\hat{X}_t \in \mathbb{R}^{J+n+1}$

$$\hat{X}_t = [X_t, t, c_0(X_t), \ldots, c_J(X_t)]^T$$

with corresponding dynamics

$$d\hat{X}_t = \hat{h}(\hat{X}_t)dt + \hat{\sigma}(\hat{X}_t)dB_t$$

where

$$\hat{h}(\hat{X}_t) = \begin{bmatrix} h_1(X_t) \\ h_2(X_t) \\ \vdots \\ h_n(X_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_{1,1}(X_t) & \cdots & \sigma_{1,d}(X_t) \\ \sigma_{2,1}(X_t) & \cdots & \sigma_{2,d}(X_t) \\ \vdots & \cdots & \vdots \\ \sigma_{n,1}(X_t) & \cdots & \sigma_{n,d}(X_t) \end{bmatrix} dB_t$$

In the case of sinusoidal dynamics, the functions $\sin$ and $\cos$ form a length 4 cycle under the derivative operator.

We can employ this characteristic to formulate a general augmentation methodology for all multidimensional SDE dynamics where the drift and diffusion terms are polynomials with respect to sinusoidal dynamics and the state variables.

**Theorem 1:** Let $X$ be a process with state $x \in \mathbb{R}^n$ and sinusoidal drift and diffusion such that the dynamics are

$$dx_t = \begin{bmatrix} h_1(x_t) \\ h_2(x_t) \\ \vdots \\ h_n(x_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_{1,1}(x_t) & \cdots & \sigma_{1,d}(x_t) \\ \sigma_{2,1}(x_t) & \cdots & \sigma_{2,d}(x_t) \\ \vdots & \cdots & \vdots \\ \sigma_{n,1}(x_t) & \cdots & \sigma_{n,d}(x_t) \end{bmatrix} dB_t$$

where

$$h_i(x_t) = \sum_{p} \alpha_{i,p}(\sin(\phi_{p} x^{\gamma_{p}}), \cos(\psi_{p} x^{\gamma_{p}}), x^{\gamma_{p}})^{\beta_{p}}$$

$$\sigma_{ij}(x_t) = \sum_{q} \alpha_{q,j}(\sin(\phi_{q} x^{\gamma_{q}}), \cos(\psi_{q} x^{\gamma_{q}}), x^{\gamma_{q}})^{\beta_{q}}$$

where each $\alpha_{ij} \in \mathbb{R}$ is a scalar coefficient, $\beta_{ij} \in \mathbb{N}^n$ is a multi-index, $x^{\gamma_{ij}}$ is a monomial with respect to the state $x$ given by a multi-index $\gamma_{ij} \in \mathbb{N}^n$, and $\phi_{ij} \in \Phi$ and $\psi_{ij} \in \Psi$ are a finite set of frequencies. Let $\hat{x}$ denote the sinusoidal augmented state such that

$$\hat{x} = [x, \sin(\phi_1 x^{\gamma_1}), \sin(\phi_2 x^{\gamma_2}), \ldots, \sin(\phi_m x^{\gamma_m}), \sin(\psi_1 x^{\gamma_1}), \sin(\psi_2 x^{\gamma_2}), \ldots, \sin(\psi_m x^{\gamma_m}), \cos(\phi_1 x^{\gamma_1}), \cos(\phi_2 x^{\gamma_2}), \ldots, \cos(\phi_m x^{\gamma_m}), \cos(\psi_1 x^{\gamma_1}), \cos(\psi_2 x^{\gamma_2}), \ldots, \cos(\psi_m x^{\gamma_m})]$$

where $\phi_1, \ldots, \phi_m \in \Phi$ and $\psi_1, \ldots, \psi_m \in \Psi$. Then, the augmented state $\hat{x}$ has dimension $2(|\Phi| + |\Psi|) + n$ and the augmented system $\hat{X}$ satisfies Definition 1—in other words, the augmented state includes sine and cosine terms for all unique frequencies found in the dynamics of $X$.

**Proof:** See the Appendix.

With the appropriate state augmentation, we may now obtain the martingale constraints (7) in terms of the moment sequences for previously unsupported nonlinear dynamics. In performing the augmentation involving sinusoidal terms, if the original system dynamics have $|\Phi| + |\Psi|$ unique modes, then the dimension of the state increases by $2(|\Phi| + |\Psi|)$. This poses a challenge for the computation of the optimization program in Section IV-C. As the dynamics increase in complexity with additional sinusoidal modes, the augmentation requires more states resulting in the PSD matrix constraints growing intractable.

**VI. REDUCED CONSTRAINT OPTIMIZATION**

**A. Reformulated Localizing Matrix Constraints**

Following the definition of the moment sequence, it is clear that the number of moments and the size of the moment/localizing matrices scale combinatorially with the dimension of the state space. This poses a challenge to the computational feasibility when high-dimensional (possibly state augmented) systems are considered. In addition, the number of localizing matrix constraints grows linearly with the number of polynomials $q_i$ used to specify the safe set and its boundary (while the size of each matrix grows combinatorially). Given a complex safe set with numerous polynomials, this quickly results in an intractable number of extremely large PSD matrix constraints. Therefore, we propose replacing the sequence of localizing matrix constraints for the exit measure with a set of scalar equality constraints over the moment sequence. We add additional assumptions to the polynomials that form the semialgebraic safe set which, in practice, are easily satisfied.

**Definition 2:** For a polynomial $q_i$, a critical point $x$ is a point where the derivative of $q_i$ vanishes. A critical value of $q_i$ is an element of the codomain in the image of some critical point.

**Proposition 2:** Suppose the semialgebraic safe set is given by $S = \{x \mid q_i(x) \geq 0, i \in [N] \}$ and $0$ is not a critical value of $q_i$, $\forall i \in [N]$, then the boundary is characterized by $\partial S = \{x \mid \text{there exists } i \text{ such that } q_i(x) = 0\}$.

**Proof:** Consider the case when $\forall i \ q_i(x) \neq 0$. If $q_i(x) > 0$, then there must exist $\epsilon > 0$ such that $q_i(x + \epsilon) > 0$ in $S$. As a result, $x$ cannot be on the boundary of $S$. Likewise, if $q_i(x) < 0$, then there must exist $\epsilon > 0$ such that $q_i(x + \epsilon) < 0 \notin S$, $x \notin \partial S$. Therefore, by the contrapositive, if $x \in \partial S$, then there must exist an $i$ such that $q_i(x) = 0$.

Now consider the case when there exists an $i$ such that $q_i(x) = 0$. We are given that $0$ is not a critical value so $\nabla_s q_i \neq 0$; therefore, there exists $\epsilon > 0$ such that either

$$q_i(x) + \epsilon \cdot \nabla s q_i > 0 \quad \text{and} \quad q_i(x) - \epsilon \cdot \nabla s q_i < 0$$

As a result, $x$ must be on the boundary of $S$.

Given a safe set $S = \{x \mid q_i(x) \geq 0, i \in [N] \}$, let the polynomial $q'$ be the product of the $q_i$s. Using Proposition 2, the boundary can be characterized as

$$\partial S = \{x \mid q'(x) \geq 0, -q'(x) \geq 0\}.$$
BM EXIT-TIME MOMENTS (K = 8)

| Moment | Analytical Value | Original Constraints | Reduced Constraints |
|--------|------------------|----------------------|--------------------|
|        |                  | Lower Bound | Upper Bound | Lower Bound | Upper Bound |
| 1      | 0.25000          | 0.24999 | 0.25000 | 0.25000 | 0.25000 |
| 2      | 0.10417          | 0.10410 | 0.10421 | 0.10416 | 0.10418 |
| 3      | 0.06354          | 0.06339 | 0.06389 | 0.06348 | 0.06434 |
| 4      | 0.05153          | 0.04487 | 0.06690 | 0.05131 | 0.05258 |
| 5      | 0.05221          | 0.03460 | 0.30626 | 0.05133 | 0.06491 |
| 6      | 0.06348          | 0.02910 | –       | 0.05861 | 0.20670 |

Upper and lower bounds of the first six moments of a time-space Brownian motion. Bounds are computed using original and reduced constraints (Optimization I and II). A moment sequence with maximum degree K = 8 is used. Dashes (−) indicate SCS did not converge. The SDP with reduced constraints demonstrates tighter bounds, particularly for higher order moments.

**Optimization II (Reduced Constraints)**

Maximize (resp. Minimize):

\[ \max \quad n \cdot m_{0,n-1} \]

Subject to:

\[ \dot{x}_k + \sum_{i \in [n+1]} c_i(k) \cdot m_i - b_k = 0 \]

\[ M_K(m) \geq 0 \]

\[ M_K(b) \geq 0 \]

\[ M_K(q,m) \geq 0 \]

\[ \sum_{\alpha} q_{\alpha} \cdot b_{i(j)+\alpha} = 0 \quad \forall k \leq K \]

where the \( q_i \) s are given by the polynomials of the semialgebraic safe set and \( q' \) by the product of the \( q_i \) s. Due to the symmetry of the remaining moment and localizing matrices, their respective sequences of PSD constraints may be replaced with a single constraint involving the highest moment degree \( K \), further reducing the memory requirements during computation.

**B. Computational Consequences (Splitting Conic Solver)**

For intuition on the computational impact of the reformulated optimization, let us consider a widely used open-source SDP solver such as the splitting conic solver (SCS) [20]. The algorithm consists of three main steps with the primary computational burdens falling upon: 1) Projection onto a subspace by solving a linear system with a coefficient matrix \( I + Q \), and 2) projection onto a cone requiring an eigendecomposition. Suppose we are solving the moment method problem consisting of \( N \) moments given by \( x = [m_0, m_1, m_2, \ldots] \), up to a maximum moment degree of \( K \). The semialgebraic safe set \( S \) consists of \( N_q \) polynomials and the boundary \( \partial S \) consists of \( 2N_q \) polynomials. Let the moment and localizing matrices \( M_K(m) \) and \( M_K(q,m) \) have dimension \( d_K \).

Using Optimization I, the PSD constraints are formed using a block diagonal matrix consisting of \( M_K(m), M_K(b), \{ M_K(q,m) \mid i \in [N_q] \}, \) and \( \{ M_K(q,b) \mid i \in [2N_q] \} \), resulting in a PSD matrix constraint of size \((2 + 3N_q)d_K \) by \((2 + 3N_q)d_K \). On the other hand, by using Optimization II with reduced constraints, the localizing matrices of the boundary are removed, resulting in a PSD matrix constraint of size \((2 + N_q)d_K \) by \((2 + N_q)d_K \). This causes a reduction \( \delta \) in the size of the SCS coefficient matrix \((I + Q)\) that is proportional to the square of the number of polynomials and dimension of moment/localizing matrix: \( \delta = O(N_q^2 d_K^2) \).

Consider a scenario where one is computing the exit time of a multidimensional system with a number of sinusoidal modes. As state augmentation is used to incorporate each of the modes into the state space, the sequence of moments increases combinatorially in length with respect to the new state dimension. This in turn directly increases the dimensions of the moment and localizing matrices \( (d_K) \). The growth in computational costs is further exacerbated when complex safe sets consisting of numerous polynomials are used (growth in \( N_q \)). As a result, even minor applications of state augmentation can have a dramatic impact on the difference in computational feasibility when comparing between the original and reduced constraint formulations.

**VII. EXAMPLES**

**A. Time-Space Brownian Motion**

We first demonstrate the scalability of the reduced constraints SDP to higher order exit-time moments and longer moment sequences through a 2-D time-space Brownian motion example. Let \( Y_t = y_0 + W_t \) where \( W_t \) is a 1-D Brownian motion. The time-space process is given by \( X = \{ (t,Y_t) \} \). The generator is given by

\[ Af(t, y) = \frac{\partial f(t, y)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(y)}{\partial y^2}(t, y). \]

We consider the exit of the process from a safe set given by \( S = \{ (t, y) \mid T \geq t \geq 0, 1 \geq y \geq 0 \} \). The safe set contains the space interval \([0,1]\) and a finite time interval up to time \( T \). The initial condition is given by \( y_0 = 0.5 \). Tables I and II show the computed upper and lower bounds of the first six moments of the exit time.

Using Optimization I (original constraints), we see that SCS returns bounds with minimal spread for lower order moments of the exit time (Table I). Beyond the third moment, the spread begins to increase due to numerical instabilities and an insufficient moment sequence length. With a maximum moment degree \( K = 8 \), the SDP with original constraints is unable to produce an upper bound for the sixth moment.
S where x indicates the highest order moment considered in the optimization. is the vertical position. sin(x) = 8 and shows the tradeoff between accuracy and computational feasibility. Under this scenario, the original formulation fails to converge for all six moments when using SCS. In comparison, the reduced constraint SDP continues to provide bounds for all moments. As expected, we are able to compute values for all six moments and see tighter bounds. On the other hand, when using Optimization II with reduced constraints, we are able to compute values for all six moments and see tighter bounds.

A larger moment sequence is required to compute accurate bounds for higher order moments. Table II uses a max moment degree of K = 14 and shows the tradeoff between accuracy and computational feasibility. Under this scenario, the original formulation fails to converge for all six moments when using SCS. In comparison, the reduced constraint SDP continues to provide bounds for all moments. As expected, we are able to obtain tighter bounds versus those computed with K = 8.

### B. Spring–Mass–Damper With Variable Damping Rate

We consider a spring–mass–damper system with the following parameters: spring constant $k_s = 5.0$, object mass $m_s = 1.0$, and static damper constant $k_c = 1.0$.

To demonstrate the redundant state augmentation technique, we use a variable damper force subject to noise and proportional to both the static damper constant and a sinusoidal term with respect to the position of the mass.

The state space is $X = [x, v, t] \top$ where $x$ is the vertical position of the mass, $v$ is its velocity, and $t$ is the time. The system dynamics are

$$dX = \begin{bmatrix} \frac{k_s}{m_s} x - g \frac{v}{m_s} \sin(x) \\ 0 \\ \frac{k_c}{m_s} \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} dB_t$$

To generate the linear martingale constraints, we use the following state augmentation:

$$\dot{X}_t = \begin{bmatrix} x, v, t, \sin(x), \cos(x) \end{bmatrix} \top$$

$$d\dot{X}_t = \begin{bmatrix} v \\ -\frac{k_s}{m_s} x - g \frac{v}{m_s} \sin(x) \\ 0 \\ v \cos(x) \\ v \sin(x) \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} dB_t$$

We consider two safe sets with differing exit times: $S_1$ where $x \in [-2, 0]$, and $S_2$ where $x \in [-2.5, 0]$. The initial values for vertical position and velocity are $\frac{8}{9}$ and 0, respectively. As before, SCS is used for calculations.

Table III shows the bounds on the expected exit time of the system calculated using both original and reduced constraints. The max degree $K$ indicates the highest order moment considered in the optimization. All other values in the tables have units of seconds. The observations from the table provide evidence that the reduced constraints of Optimization II benefit the application of the moment method on higher dimensional state-augmented systems. With the original constraints, the solver can only handle the state augmented system up to $K = 4$, resulting in a very loose bound on the exit time, which inhibits its use for evaluating whether the system’s true expected exit time satisfies some desired safety requirements.

Comparing the original and reduced constraint SDP formulations, the latter provides significantly more useful information regarding the safety behavior of the state augmented system. The columns on the right-hand side of Table III show tighter bounds for both safe sets $S_1$ and $S_2$. As was seen previously when using the original constraints, the larger expected exit time of $S_2$ poses SCS with a more difficult problem to solve; however, unlike before, we continue to see convergence of the solver for both upper and lower bounds when using the reduced constraints.

### VIII. Conclusion

In this article, we considered a safety analysis of stochastic systems through a moment-based method. Using appropriate SDP moment conditions, a convex optimization problem is formed to compute bounds on a process’ exit-time moments. Noting the strong assumptions on the dynamics required for the moment method, we proposed a state space augmentation technique to support a broader class of systems. The use of state augmentation allows one to go beyond polynomial dynamics and characterize the safety behavior of a wide range of physical systems. We discussed the tradeoff in computational feasibility that comes with state space augmentation and propose a reformulation of the optimization constraints. The method grants an easily automated
proceedure for simplifying the large PSD matrix constraints associated with complex dynamics and safe sets—greatly improving scalability into higher dimensions. Lastly, we presented numerical examples using our methods and presented scenarios where a consideration of exit-time moments granted useful insight into the safety of the system.

APPENDIX

We present the proof of Theorem 1.

Proof: We denote each additional state in the augmented state space as \( f(\xi,x^m) \). Recall that \( x^m(1) \) is a monomial with respect to the state. For an arbitrary \( h, 0 \leq h < m \), the dynamics of the state \( f(\xi,x^m) \) is given by

\[
df(\xi,x^m) = f(\xi,x^m) : \xi + \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} (dx_1, dx_2),
\]

where

\[
dx^m = \sum_{i=1}^{n} \left( x_{i,m} \cdot \prod_{j \neq i} (\gamma_j x_j^m) \right) dx^m_{i,j}.
\]

We note that, by construction, \( dx^m \) has drift \((h,\gamma)\) and diffusion \((\sigma,\phi)\) that is polynomial w.r.t. the sinusoidal terms \( \sin(\phi_j x_j^m, \phi) \), \( \cos(\phi_j x_j^m, \phi) \), \( \sin(\psi_j x_j^m, \phi) \), and \( \cos(\psi_j x_j^m, \phi) \). Furthermore, the augmented state space \( \dot{x} \) includes all sinusoidal terms \( \sin(\phi_j x_j^m, \phi) \), \( \cos(\phi_j x_j^m, \phi) \), \( \sin(\psi_j x_j^m, \phi) \), and \( \cos(\psi_j x_j^m, \phi) \), up to monomials of degree \( \gamma_m \), where \( \gamma_m \) is greater than the order of the highest monomial \( x_j^m \) in the original dynamics of \( X \). Therefore \( dx^m \) is polynomial w.r.t. the augmented state space \( \dot{x} \). We denote the drift and diffusion of \( dx^m \) as \( p^{(1)} \) and \( p^{(2)} \), respectively,

\[
dx^m = \sum_{i=1}^{n} \left( x_{i,m} \cdot \gamma_i x_i^m \right) \left( \prod_{j \neq i} p^{(1)}(j) + \prod_{j \neq i} p^{(2)}(j) \right) dt + \sum_{i=1}^{n} \left( \gamma_i x_i^m \right) \left( \prod_{j \neq i} p^{(1)}(j) + \prod_{j \neq i} p^{(2)}(j) \right) \left( \sum_{i=1}^{m} p^{(2)}(i) \right) \left( \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \right).
\]

We define the following drift submatrix quantities:

\[
h = \begin{bmatrix} h_1, h_2, \ldots, h_n \end{bmatrix}^T
\]

\[
h_{\sin} = \begin{bmatrix} h_{\sin}(\phi_1 x_1^m), \ldots, h_{\sin}(\phi_m x_m^m) \end{bmatrix}^T
\]

\[
h_{\cos} = \begin{bmatrix} h_{\cos}(\phi_1 x_1^m), \ldots, h_{\cos}(\phi_m x_m^m) \end{bmatrix}^T.
\]

Next, we define the following diffusion submatrix quantities:

\[
\sigma = \begin{bmatrix} \sigma_{1,1}, \ldots, \sigma_{1,d} \\
\vdots & \vdots \\
\sigma_{n,1}, \ldots, \sigma_{n,d} \end{bmatrix}
\]

\[
\sigma_{\sin} = \begin{bmatrix} \sigma_{\sin}(\phi_1 x_1^m),1, \ldots, \sigma_{\sin}(\phi_1 x_1^m),d \\
\vdots & \vdots \\
\sigma_{\sin}(\phi_m x_m^m),1, \ldots, \sigma_{\sin}(\phi_m x_m^m),d \\
\sigma_{\cos} = \begin{bmatrix} \sigma_{\cos}(\phi_1 x_1^m),1, \ldots, \sigma_{\cos}(\phi_1 x_1^m),d \\
\vdots & \vdots \\
\sigma_{\cos}(\phi_m x_m^m),1, \ldots, \sigma_{\cos}(\phi_m x_m^m),d \\
\end{bmatrix}
\]

where

\[
h_{\sin}(\xi, x^m) = \sin(\xi, x^m) \cdot \xi,
\]

\[
h_{\cos}(\xi, x^m) = \cos(\xi, x^m) \cdot \xi.
\]

The dynamics of the augmented SDE is now given by

\[
d\dot{X}_i = \left[ h_{\sin} \right] dt + \left[ \sigma_{\sin} \right] dB_t.
\]
We note that all terms in (11)–(14) are polynomial w.r.t. the augmented state space $\hat{x}$. As polynomials are closed under addition and multiplication, the resulting drift and diffusion terms of the augmented system is also polynomial w.r.t. $\hat{x}$.

We apply (4) to obtain the generator $Af(\hat{x})$ of the augmented system. The monomials of $\hat{x}$ are closed under differentiation w.r.t. $x \in \hat{x}$, which results in $\frac{\partial f}{\partial x_1}$ and $\frac{\partial^2 f}{\partial x_1 \partial x_j}$ being monomials of $\hat{x}$ for all $x_1, x_j \in \hat{x}$, and test functions $f = (\hat{x})^\beta, \beta \in \mathbb{N}$. As a result, we see that (4) applied to the augmented system $\hat{X}$ for monomial test functions yields a generator consisting of the sum of products between polynomials and monomials $(h(\cdot), \sigma(\cdot), \beta_\infty(\cdot), \gamma_\infty(\cdot))$ w.r.t. the augmented state space. Again, following the closure properties of polynomials, the resulting generator is polynomial w.r.t. the augmented state space.

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