Effect of uncertainty in material properties on wave propagation characteristics of nanorod embedded in elastic medium

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Abstract The effect of uncertainty in material properties on wave propagation characteristics of nanorod embedded in an elastic medium is investigated by developing a nonlocal nanorod model with uncertainties. Considering limited experimental data, uncertain-but-bounded variables are employed to quantify the uncertain material properties in this paper. According to the nonlocal elasticity theory, the governing equations are derived by applying the Hamilton’s principle. An iterative algorithm based interval analysis method is presented to evaluate the lower and upper bounds of the wave dispersion curves. Simultaneously, the presented method is verified by comparing with Monte-Carlo simulation. Furthermore, combined effects of material uncertainties and various parameters such as nonlocal scale, elastic medium and lateral inertia on wave dispersion characteristics of nanorod are studied in detail. Numerical results not only make further understanding of wave propagation characteristics of nanostructures with uncertain material properties, but also provide significant guidance for the reliability and robust design of the next generation of nanodevices.

Keywords Wave propagation · Nanorod · Nonlocal elasticity theory · Uncertain material properties · Interval analysis method

1 Introduction

Nanostructures are made of nano-scale materials. With the merits of the extraordinary mechanical, electrical and thermal properties, the potential application of nanostructures is immense in many branches of science and engineering, such as nanocomposites, nanodevices, nanoelectronics and aerospaces (Wernik and Meguid 2010; Alian et al. 2015a, b). To provide better guidance for the design and application, a deep understanding of the mechanical characteristics of nanostructures is of significance.

Many important physical problems such as optical transition, electrical conductance and stress propagation are directly related to the wave propagation behavior of structures. For nanostructures involving the small-scale effect, molecular dynamic (MD) simulations is an appropriate way to predict wave characteristics of nanostructures, but the substantial difficulty in storing an astronomical amount of data limits its capability to small scale nanostructures. To
overcome this problem, many researchers have paid their attentions to the continuum mechanics theories. The outstanding contribution to this issue dates back to the work of Eringen (Eringen 1972b; Eringen and Edelen 1972). In their studies, nonlocal elasticity theory was first developed based on the assumption that the stress state at a material point is dependent on the strain states of all points in the body. Using nonlocal elasticity theory, Eringen and his coworkers have successfully solved some important problems, such as propagation of plane waves (Eringen 1972a), propagation of Rayleigh waves (Eringen 1973), stress distribution at the tip of a crack (Eringen et al. 1977) and screw and edge dislocations (Eringen 1983). Following his pioneering work, various investigations related to the use of this theory in nanostructures have been reported, including static analysis (Attia and Mahmoud 2016; Wang and Liew 2007), wave propagation (Bahrami and Teimourian 2016; Ghorbanpour Arani et al. 2014; Hu et al. 2008; Liu and Yang 2012), vibration characteristic (Aydogdu and Arda 2016; Brischetto 2014) as well as bending and buckling behaviors (Aranda-Ruiz et al. 2012; Thai 2012) of nanostructures including nanorods (Karličić et al. 2015; Narendar 2012), nanotubes (Bahadini and Hosseini 2016; Natsuki et al. 2008), nanoshfts (Arda and Aydogdu 2014), nanobeams (Eltaher et al. 2016) and nanoplates (Radic’ et al. 2014). In addition, some other size-dependent continuum theories, such as strain gradient elasticity (Fleck and Hutchinson 1997) and modified couple stress elasticity (Yang et al. 2002), have emerged to study the mechanical behaviors of nanostructures (Ebrahimiz and Barati 2017; Ghorbanpour Arani et al. 2016; Khoshshidi et al. 2016; Shaat and Abdelkefi 2016; Togun and Bağdatli 2016).

One-dimensional nanostructures such as carbon nanotubes (CNTs) and nanofibers are widely used as nanofiller for reinforced composites (Alian et al. 2015a, b). They are generally modeled as nanorods embedded in an elastic medium. For instance, Wu and Dzenis (2006) focused on longitudinal and flexural wave propagation in nanofibers by considering surface effects. Using nonlocal elasticity theory, nonlocal scale effects on ultrasonic wave characteristic of single- and coupled-nanorod system were investigated by Narendar and Gopalakrishnan (2010, 2011). Also, Narendar (2011) developed a nonlocal rod model incorporating both small-scale and lateral inertia effects, and wave propagation in uniform nanorods were studied based on nonlocal rod model. More recently, by modeling CNTs as nanorods, Aydogdu (2012b, 2014) investigated longitudinal wave propagation in single- and multi-walled CNTs. Furthermore, the elastic medium effect on mechanical characteristics of nanorods also has been reported (Aydogdu 2012a; Yayli et al. 2015).

Based on deterministic model with well-defined physical properties, numerical prediction on the mechanical behaviors of nanostructures has achieved a great success. However, imperfections and defects in molecular structure such as vacancy (Gass et al. 2008), Stone–Thrower–Wales defect (Lusk and Carr 2009) and pentagon–octagon–pentagon defect (Kotakoski et al. 2011) will introduce unavoidable uncertainty in material properties. Moreover, significant uncertainty in material properties caused by the experimental difficulties in making measurements at nano-scale also should not be neglected. For example, Salvetat et al. (1999) carried out the measurement of the flexural Young’s modulus and shear modulus of nanotube ropes, obtaining values with 50% of error. Based on the test samples of 27 nanotubes, Krishnan et al. (1998) gave the histogram distribution of Young’ modulus with a mean value of 1.3–0.4/0.6TPa. Lee et al. (2008) used atomic force microscope nanoindentation to measure the elastic properties of freestanding monolayer graphene membranes, getting a range of Young’s modulus from 0.8 TPa to 1.25 TPa. A series of other experimental measurements (Enomoto et al. 2006; Ruoff et al. 2003), theoretical predictions (Meo and Rossi 2006; Shokrieh and Rafiee 2010) and MD simulations (Liew et al. 2004; Bao et al. 2004) for the prediction of elastic properties of nanostructures have also presented similar results. Due to inherent uncertainty in material properties, numerical results of the deterministic model cannot perfectly reflect accurate mechanical characteristics of nanostructures. In other words, the reliability of deterministic model cannot be fully guaranteed. As a consequence, a valuable research topic is to develop a more realistic model which takes the uncertainty into account.

A straightforward choice of such a model accounting for these uncertainties could be probabilistic or statistical model (Lv and Qiu 2016). Barzykin and Tachiya (2005) proposed a stochastic model of the carrier dynamics in CNTs. Scarpia and Adhikari (2008)
developed a stochastic reduced order model to predict the natural frequencies of carbon nanotube terahertz oscillators with random parameters. In their study, the statistical data of the flexural modulus is generated using an equivalent atomistic continuum model due to limited available test data from open literatures. More recently, Chang (2013, 2017) applied the stochastic finite element method to investigate the nonlinear dynamic response of fluid-conveying double-walled CNTs, and nonlinear vibration of single-walled carbon nanotubes with random material properties. These investigations promote the analytical model of nanostructures to a new altitude. Also, a major problem in using the probabilistic model is that sufficient information is needed to evaluate the statistical distribution using the probabilistic model is that sufficient information is needed to evaluate the statistical distribution of random parameter. In contrast, a non-probabilistic convex model proposed by Ben-Haim and Elishakoff (1990) only requires the knowledge of bounds of uncertain quantities, which is appropriate to handle the uncertainties with limited available information (Liu et al. 2016; Jiang et al. 2012). The application of non-probabilistic modeling in nanostructures with uncertainties is promising, yet mostly unexplored up to now, which motivates us to do this study.

From the literatures, a great deal of effort has been devoted to capturing the small-scale effects of nanostuctures. Most theoretical researches, however, ignore the significant influence of uncertainty in material properties on mechanical characteristics of nanostructures in reality. Apparently, this may produce a large gap between the analytical model and the real physical system, leading to some difficulties in guiding the research and design of nanostructures. Furthermore, probabilistic modeling requires sufficient statistical data, but collecting sufficient experimental data at nano-scale is a difficult task. In fact, Radebe and Adali (2014) had focused on the analysis of nanoplates with limited data available, and pointed out that non-probabilistic modeling has a good performance in such problem. However, their study is restricted to buckling behaviors. To the authors’ knowledge, there seems no report about the effect of uncertainty in material properties on wave propagation characteristics of nanostructures so far.

In this paper, we develop a nonlocal nanorod model with uncertainties and make the first effort to explore the effect of uncertainties on wave dispersion characteristics of nanorod. This investigation is restricted to nanorods with uncertain-but-bounded material properties. The remainder of this paper is organized as follows. In the next section, based on nonlocal elasticity theory and Hamilton’s principle, the nonlocal nanorod model is developed, which takes nonlocal scale effect, lateral inertia and elastic medium into consideration. Then, Sect. 3 establishes the uncertainty modeling of wave propagation with material uncertainty by virtue of interval-bound convex model. An iterative algorithm based interval analysis method is presented to solve the nonlocal nanorod model for predicting the lower and upper bounds of the wave dispersion curves in Sect. 4. To demonstrate the good performance of the proposed method, in Sect. 5, numerical verification is performed by comparing with Monte-Carlo simulation. Furthermore, we investigate the combined effects of material uncertainties and various parameters such as nonlocal scale, elastic medium and lateral inertia on wave dispersion characteristics of nanorod in detail. Finally, we draw conclusions and give an outlook of the future work in Sect. 6.

2 Nonlocal nanorod model embedded in elastic medium

2.1 Brief review on nonlocal elasticity theory

Nonlocal elasticity theory proposed by Eringen can be formally stated as follows: the stress state at a material point \( \mathbf{x} \) is dependent on the strain states of all points \( \mathbf{x}' \) in the body. This theory is in accordance with the atomic theory of lattice dynamics and the experimental observations on phonon dispersion.

According to Eringen’s nonlocal elasticity theory, the constitutive equation of a homogeneous and isotropic elastic solid with zero body forces is given by

\[
\sigma_{ij}(\mathbf{x}) = C_0 : \int_V \zeta(|\mathbf{x}' - \mathbf{x}|, \tau) \sigma_{ij}^{c}(\mathbf{x}')dV(\mathbf{x}')
\]

where \( \sigma_{ij} \) and \( \sigma_{ij}^{c} \) denote the nonlocal and local stress tensors, respectively. \( C_0 \) denotes the elastic stiffness matrix of classical isotropic elasticity. Symbols ‘:’ denotes the double dot product. \( \tau \) denotes parameter which corresponds to the nonlocalness. \( |\mathbf{x}' - \mathbf{x}| \) is the Euclidean distance. The kernel function \( \zeta(|\mathbf{x}' - \mathbf{x}|, \tau) \) represents nonlocal modulus, which can be expressed as
\[
\sigma(|x|, \tau) = \frac{1}{2\pi \tau^2} K_0 \left( \frac{\sqrt{x \cdot x}}{\tau} \right) \tag{2}
\]

where \( K_0(\cdot) \) is the modified Bessel function. \( l \) is an external characteristic length of the system. Substituting Eq. (2) into Eq. (1) leads to the following constitutive equation

\[
\left( 1 - (e_0 a)^2 \nabla^2 \right) \sigma = C_0 \varepsilon, \quad e_0 a = \tau l \tag{3}
\]

where \( \sigma \) and \( \varepsilon \) are the stress and strain, respectively. \( \nabla^2 \) is the Laplacian operator. \( e_0 \) is called the nonlocal parameter. More specifically, \( a \) is an internal characteristic length, e.g., length of C–C bond in CNT, size of grain, granular distance etc., and \( e_0 \) denotes the small scaling parameter which is considered as a constant appropriate to the material and determined for each material independently. As a matter of fact, an approach to determining the small scaling parameter \( e_0 \) in the nonlocal elasticity theory is not known so far. Eringen (1983) presented \( e_0 = 3.9 \) by matching the plane wave curves with the atomic lattice dynamics. Also, Eringen (1972a) proposed \( e_0 = 0.31 \) for Rayleigh surface wave and lattice dynamics. As for CNTs, the values of \( e_0 \) are scattered in a range from 0.29 to 8.79 (Wang and Hu 2005; Zhang et al. 2006).

Note that Eq. (3) degenerates to the classical (local) constitutive equation when the nonlocal parameter is set to zero. Moreover, the nonlocal elasticity theory is not only applicable to the isotropic material, but also can be extended to the orthotropic and transversely isotropic materials.

2.2 Nonlocal governing partial differential equation of motion

A nanorod (single-walled CNT) embedded in elastic medium is shown in Fig. 1. The nanorod is modeled as a hollow bar which has a circular cross-section with radius \( R \), wall thickness \( h \), and length \( L \). The co-ordinate system of the nanorod is also marked in this figure. The surrounding medium is modeled as a Pasternak foundation with elastic stiffness coefficient \( k_w \) and shear stiffness coefficient \( k_g \). The assumptions adopted in subsequent analysis are given as follows: (1) Displacement in the tangential direction is neglected; (2) Each point along the radial direction of the nanorod has a transverse displacement; (3) The dimension of circular cross-section is much smaller than that of the longitudinal length \( R \ll L \); (4) All material properties are assumed to be time independent.

Based on the assumptions (1) and (2), supposing the axisymmetric deformation axial and transverse displacement fields for nanorod can be expressed as a power series expansion with respect to the radial coordinate \( r \)

\[
u(x, r, t) = \sum_{m=0}^{M} r^{2m} u_{2m}(x, t) \tag{4}
\]

\[
w(x, r, t) = \sum_{n=0}^{N} r^{2n+1} u_{2n+1}(x, t) \tag{5}
\]

where \( r \) is the distance between the element and the center of the cross-section at the location \( x \). \( M \) and \( N \) are the expansion terms of the axial and transverse displacements, respectively. Different models of longitudinal dynamic of nanorod can be obtained by setting different \( M \) and \( N \) in Eqs. (4) and (5), such as Rayleigh-Love model, Rayleigh-Bishop model and Mindlin–Hermann model. The linear elastic strain tensor field of the nanorod could be calculated by

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}, \quad \varepsilon_{rr} = \frac{\partial w}{\partial r}, \quad \varepsilon_{\varphi \varphi} = \frac{w}{r} \\
\varepsilon_{x\varphi} &= \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \varphi} = \frac{1}{r} \frac{\partial u}{\partial \varphi} = 0, \quad \varepsilon_{r\varphi} = \frac{1}{r} \frac{\partial w}{\partial \varphi} = 0.
\end{align*} \tag{6}
\]

Note that axial symmetry leads to \( \varepsilon_{x\varphi} = \varepsilon_{r\varphi} = 0 \).

Combining Eqs. (3) and (6), the components of stress tensor of nanorod can be expressed as

\[
\begin{align*}
\sigma_{xx} &= \lambda (\varepsilon_{xx} + \varepsilon_{rr} + \varepsilon_{\varphi \varphi}) + 2G \varepsilon_{xx} \\
\sigma_{rr} &= \lambda (\varepsilon_{xx} + \varepsilon_{rr} + \varepsilon_{\varphi \varphi}) + 2G \varepsilon_{rr}
\end{align*} \tag{7}
\]
\[
\begin{align*}
(1 - (e_0 a)^2 \nabla^2) \sigma_{\varphi \varphi} &= \lambda (e_{xx} + e_{rr} + e_{\varphi \varphi}) + 2G e_{\varphi \varphi} \\
(1 - (e_0 a)^2 \nabla^2) \sigma_{rr} &= G e_{rr}
\end{align*}
\]  
\tag{9}

\[
\begin{align*}
(1 - (e_0 a)^2 \nabla^2) \sigma_{\varphi r} &= G e_{\varphi r}
\end{align*}
\]  
\tag{10}

where \( \sigma_{ij} \) (\( i, j = x, r, \varphi \)) are the components of stress tensor, \( G \) and \( \lambda \) are Lame’s constants, and they are defined as

\[
G = \frac{E}{2(1 + v)}, \quad \lambda = \frac{E v}{(1 - 2v)(1 + v)},
\]  
\tag{11}

where \( E \) and \( v \) are Young’s modulus and Poisson’s ratio, respectively. According to the Rayleigh–Love theory (Rahmati and Mohammadmehr 2014), the radial stress \( \sigma_{rr} \) will be equal to zero, not only on the free inner and outer wall, but throughout the entire thickness of nanorod. As a result, from Eq. (8), one can get

\[
\lambda \left( \frac{\partial u}{\partial r} + \frac{w}{r} \right) + 2G \frac{\partial w}{\partial r} = 0.
\]  
\tag{12}

By inserting Eqs. (4) and (5) into Eq. (12) and setting \( M = N = 0 \), a relation between axial and transverse displacement fields can be determined with the following form

\[
u(x, r, t) = u_0(x, t) = u(x, t)
\]  
\tag{13}

\[
w(x, r, t) = ru_1(x, t) = -vr \frac{\partial u(x, t)}{\partial x}.
\]  
\tag{14}

Accordingly, by substituting Eqs. (6), (13) and (14) into Eqs. (7)–(10), the constitutive equation relation has a simple form as follows

\[
\begin{align*}
(1 - (e_0 a)^2 \nabla^2) \sigma_{xx} &= E \frac{\partial^2 u}{\partial x^2} \\
(1 - (e_0 a)^2 \nabla^2) \sigma_{xr} &= -G v r \frac{\partial^2 u}{\partial x^2}.
\end{align*}
\]  
\tag{15}

\[
\begin{align*}
(1 - (e_0 a)^2 \nabla^2) \sigma_{xx} &= -G v r \frac{\partial^2 u}{\partial x^2}.
\end{align*}
\]  
\tag{16}

Note that the hoop stress \( \sigma_{\varphi \varphi} \) vanishes naturally.

In our study, both transverse deformation and lateral inertia of the nanorod are considered. The equation of motion for the nanorod embedded in an elastic medium can be given by

\[
\begin{align*}
\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial \sigma_{xx}}{\partial r} + \frac{\sigma_{xr}}{r} + f_x &= \rho \frac{\partial^2 u}{\partial t^2}
\end{align*}
\]  
\tag{17}

\[
\begin{align*}
\frac{\partial^2 \sigma_{rr}}{\partial r^2} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{xx}}{r} + f_x &= \rho \frac{\partial^2 w}{\partial t^2}
\end{align*}
\]  
\tag{18}

where \( \rho \) is the mass density of nanorod, \( f_x \) represents the axial force per unit volume caused by the elastic medium. Considering the condition \( \sigma_{rr} = \sigma_{\varphi \varphi} = 0 \) and eliminating the shear stress \( \sigma_{xr} \) from Eqs. (17) and (18), the second order derivative of axial stress \( \sigma_{xx} \) of the nanorod is achieved as

\[
\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial \sigma_{xx}}{\partial r} + \frac{\sigma_{xx}}{r} = \rho \frac{\partial^2 u}{\partial t^2}.
\]  
\tag{19}

Substituting Eq. (19) into constitutive equation relation (15), the axial stress \( \sigma_{xx} \) expressed in terms of the derivatives of axial displacement \( u \) can be calculated as

\[
\sigma_{xx} = (e_0 a)^2 \rho (1 + 2v) \frac{\partial^3 u}{\partial r^2 \partial x} - (e_0 a)^2 \frac{\partial f_x}{\partial x} + E \frac{\partial u}{\partial x}.
\]  
\tag{20}

In a similar manner, the shear stress \( \sigma_{xr} \) of the nanorod is obtained as

\[
\sigma_{xr} = -\rho v r (e_0 a)^2 \frac{\partial^4 u}{\partial r^2 \partial x^2} - G v r \frac{\partial^2 u}{\partial x^2}.
\]  
\tag{21}

Based on the definition of the strain energy density, one can get

\[
\begin{align*}
e &= \frac{1}{2} (\sigma_{xx} \varepsilon_{xx} + \sigma_{xr} \varepsilon_{xr}) \\
&= \frac{1}{2} \left( (e_0 a)^2 \rho (1 + 2v) \frac{\partial^3 u}{\partial r^2 \partial x} - (e_0 a)^2 \frac{\partial f_x}{\partial x} + E \frac{\partial u}{\partial x} \right)^2 \\
&+ \frac{1}{2} \rho v r (e_0 a)^2 \frac{\partial^4 u}{\partial r^2 \partial x^2} + G \frac{\partial^2 u}{\partial x^2}.
\end{align*}
\]  
\tag{22}

The total potential energy of the nanorod can be calculated by

\[
U = \int_0^L \int_A edA dx
\]  
\tag{23}
where \( A = \int_A \, dA = 2\pi Rh \) denotes the cross-sectional area, and \( J = \int_A \rho^2 dA = \pi Rh(2R^2 + h^2/2) \) denotes the polar moment of inertia of cross-section of the nanorod.

The kinetic energy of the nanorod can be calculated by

\[
T = \frac{1}{2} \rho \int_0^L \int_A \left( \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right) \, dA \, dx
\]

\[
= \frac{1}{2} \rho \int_0^L \int_A \left( \left( \frac{\partial u}{\partial t} \right)^2 + v^2 \left( \frac{\partial^2 u}{\partial t \partial x} \right)^2 \right) \, dA \, dx
\]

\[
= \frac{1}{2} \rho \int_0^L \left( A \left( \frac{\partial u}{\partial t} \right)^2 + v^2 J \left( \frac{\partial^2 u}{\partial t \partial x} \right)^2 \right) \, dx.
\]  

(24)

The work done by external forces can be calculated by

\[
W = \int_0^L \int_A f_x u \, dA \, dx = \int_0^L A f_x u \, dx.
\]  

(25)

In order to obtain the final governing equation, Hamilton’s principle is employed here, which gives the variational form as follows

\[
\delta \int_0^L (W + T - U) \, dt = 0.
\]  

(26)

By performing variational operation, the variation of the potential energy and kinetic energy of nanorod as well as the work done by external forces can be separately derived as

\[
\delta U = \int_0^L \left[ -(e_0 a) \rho A (1 + 2v) \left( \frac{\partial^4 u}{\partial t \partial x^4} \right) + \left( e_0 a \right)^2 A \frac{\partial^2 f_i}{\partial x^2} \right] \, dx + v^2 J \int_0^L \left( \rho e_0 a \left( \frac{\partial^4 u}{\partial t^2 \partial x^2} \right) + G \frac{\partial^4 u}{\partial x^2} \right) \, dx
\]

\[
- E A \left( \frac{\partial^2 u}{\partial x^2} \right) \delta u \, dx + v^2 J \int_0^L \left( \rho (e_0 a)^2 \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + G \frac{\partial^4 u}{\partial x^2} \right) \delta u \, dx
\]  

(27)

\[
\delta T = \int_0^L \left( -\rho A \frac{\partial^2 u}{\partial t^2} + \rho v^2 J \frac{\partial^4 u}{\partial t^2 \partial x^2} \right) \delta u \, dx
\]

\[
\delta W = \int_0^L A f_x \delta u \, dx.
\]  

(28)

(29)

In our study, Pasternak foundation model are used to represent the elastic medium. In this model, only the axial force is produced by elastic medium (Rahmati and Mohammadmehr 2014), and the axial force per unit length of the elastic medium can be given as

\[
\int_A f_x \, dA = A f_x = -k_w u + k_g \frac{\partial^2 u}{\partial x^2}.
\]  

(30)

By substituting Eqs. (27)–(30) into Eq. (26), and setting the coefficient \( \delta u \) to be zero, the nonlocal governing partial differential equation of motion can be derived as

\[
EA \frac{\partial^2 u}{\partial x^2} - v^2 J G \frac{\partial^4 u}{\partial x^4} - (e_0 a)^2 \left( -k_w \frac{\partial^2 u}{\partial x^2} + k_g \frac{\partial^4 u}{\partial x^4} \right)
\]

\[
- k_w u + k_g \frac{\partial^2 u}{\partial x^2}
\]

\[
= \rho A \frac{\partial^2 u}{\partial t^2} - \rho v^2 J \frac{\partial^4 u}{\partial t^2 \partial x^2}
\]

\[
- (e_0 a)^2 \rho A (1 + 2v) \frac{\partial^4 u}{\partial t^2 \partial x^2}
\]

\[
+ (e_0 a)^2 \rho v^2 J \frac{\partial^4 u}{\partial t^2 \partial x^2}.
\]  

(31)

3 Uncertainty modeling of wave propagation with material uncertainty

For a harmonic type of the longitudinal wave propagation, the axial displacement field of nanorod satisfies the following form

\[
\hat{u}(x, t) = \hat{U}(x, \omega) \exp[-j(kx - \omega t)]
\]  

(32)

where \( \hat{U}(x, \omega) \) is the amplitude of the displacement in the frequency domain, \( k \) and \( \omega \) denote the wave number and the circular frequency, respectively, and \( j = \sqrt{-1} \). Substituting Eq. (32) into Eq. (31), leads to the dispersion relation, as follows

\[
E A k^2 + v^2 J G k^4 + (e_0 a)^2 (k_w k^2 + k_g k^4) + k_w + k_g k^2
\]

\[
= \rho A \omega^2 + \rho v^2 J k^2 \omega^2 + \rho (e_0 a)^2 (k_w + 2v) k^2 \omega^2
\]

\[
+ (e_0 a)^2 \rho v^2 J k^4 \omega^2.
\]  

(33)

From Eq. (33), the frequency can be expressed in terms of the wave number, namely

\[
\omega = \sqrt{\frac{E A k^2 + \delta_2 v^2 J G k^4 + \delta_3 (e_0 a)^2 k^2 + 1 (k_w + k_g k^2)}{2 \delta_2 (e_0 a)^2 k^2 v \rho A + (e_0 a)^2 k^2 + 1 (\rho A + \delta_1 \rho v^2 J k^2)}}
\]  

(34)
where $\delta_1$, $\delta_2$ and $\delta_3$ govern the effects of lateral inertia, shear and elastic medium, respectively. They can be set as either 0 or 1.

The cut-off frequency of the nanorod $\omega_{\text{cut-off}}$ can be obtained by setting $k = 0$, which is given as

$$\omega_{\text{cut-off}} = \sqrt{\frac{k_w}{\rho A}}. \quad (35)$$

The escape frequency of the nanorod $\omega_{\text{escape}}$ can be determined by setting $k \rightarrow \infty$, i.e.,

$$\omega_{\text{escape}} = \sqrt{\frac{1}{\delta_1 \rho} \left( \frac{\delta_2 G}{(e_0 a)^2} + \frac{\delta_3 k_g}{\nu^2 J} \right)}. \quad (36)$$

The phase speed $c_p$ can be derived by using the following equation

$$c_p = \frac{\omega}{k}. \quad (37)$$

The cut-off frequency denotes the wave cannot propagate when the wave frequency lower than this value, while the wave whose frequency larger than the value of the escape frequency cannot propagate either.

As known from the literatures (Salvetat et al. 1999; Lee et al. 2008; Kotakoski et al. 2011), uncertainties in material properties are unavoidable due to large amount of parameters associated with their manufacturing and measurement. Radebe and Adali (2014) have made a great contribution to quantifying the material uncertainties in nanostructure by ellipsoidal-bound convex model, which makes the Lagrange multiplier an effective method for performing buckling analysis of uncertain nanoplate in poor information situation. In this section, interval-bound convex model is employed to quantify the uncertain material properties, such as Young’s modulus and mass density. Without any loss of generality, the interval-bound convex model can be described by an $\kappa$-dimensional interval vector filled with all uncertain material properties, as follows

$$b \in b^l = [b_l, \bar{b}] = (b^l_i), b_i \in b_i^l = [b_i^l, \bar{b}_i], i = 1, 2, \ldots, \kappa \quad (38)$$

where $b^l = (b^l_i)$ is an interval vector. $\underline{b} = (\underline{b}_i)$ and $\bar{b} = (\bar{b}_i)$ are the lower and upper bounds of interval vector, respectively.

It is noted from Eq. (34) that the uncertainties in material properties of nanorod will result in significant variations of the frequency. The interval-bound convex model may be thought of as a constraint condition. Obviously, by solving the dispersion relation (33) subjected to the constraint conditions (38), we mean to solve the family of nonlinear equation in which the material parameters $b_i, i = 1, 2, \ldots, \kappa$ are uncertain and ranging inside the certain convex set $b^l_i, i = 1, 2, \ldots, \kappa$.

With the natural interval extension of Eq. (33), the dispersion relation of wave propagation in nanorod with uncertainties can be established as

$$\psi(\omega; b^l) = E^l A k^2 + \nu^2 J G^l k^4$$

$$- 2\nu \rho^l A (e_0 a)^2 k^2 \omega^2$$

$$- \rho^l (e_0 a)^2 k^2 \omega^2 + \omega^2) (A + \nu^2 J k^2) = 0 \quad (39)$$

where $\psi(\cdot)$ is nonlinear function with respect to the frequency $\omega$ and $\psi(\omega; b^l)$ is the natural interval extension of $\psi(\omega; b^l)$, and $b^l = (E^l, \rho^l, G^l)^T$ is interval vector containing all uncertain material properties $b = (E, \rho, G)^T$. Note that Eq. (39) can degenerate to the deterministic dispersion relation of wave propagation when the interval vector $b^l$ reduces to the deterministic one $b$.

By its nature, Eq. (39) is a nonlinear interval equation. The solution of Eq. (39) is a set which is defined as

$$\Gamma = \omega(b^l) = \{ \omega \in \text{Rreal} : \psi(\omega; b) = 0, b \in b^l \} \quad (40)$$

where $\Gamma$ represents the solution set of Eq. (39), $\text{Rreal}$ denotes the real domain.

Due to the strong nonlinearity, a direct solution to Eq. (40) cannot be performed in practice. An equivalence method is to seek the bounds of the frequency, which encloses all possible values of the above mentioned solution set $\Gamma$. The interval of the frequency is defined as

$$\omega = \left[ \begin{array}{c} \min \omega(b), \max \omega(b) \end{array} \right] \quad (41)$$

where $\omega = \min \omega(b)$ and $\bar{\omega} = \max \omega(b)$ are the lower and upper bounds of the frequency, respectively.
4 Interval prediction of wave propagation characteristics

In this section, an iterative algorithm based interval analysis method (IAM) is presented to solve the uncertainty model developed in Sect. 3. Our aim is to employ IAM to evaluate the lower and upper bounds of the wave dispersion curves. To achieve this goal, we first derive two important equations whose solutions are the bounds of the frequency.

4.1 Two important equations for the bounds of the frequency

For convenience, we introduce the following two functions

$$\vartheta(\omega) = \max_{b \in \mathcal{B}} \psi(\omega; b)$$  \hspace{1cm} (42)

$$h(\omega) = \min_{b \in \mathcal{B}} \psi(\omega; b)$$  \hspace{1cm} (43)

where both \( \vartheta(\omega) \) and \( h(\omega) \) are nonlinear functions of the frequency \( \omega \).

Taking the first-order derivative of Eqs. (42) and (43), leads to the following expressions

$$\frac{d\vartheta(\omega)}{d\omega} = \max_{b \in \mathcal{B}} \frac{\partial \psi(\omega; b)}{\partial \omega}$$  \hspace{1cm} (44)

$$\frac{dh(\omega)}{d\omega} = \min_{b \in \mathcal{B}} \frac{\partial \psi(\omega; b)}{\partial \omega}.$$  \hspace{1cm} (45)

Taking the first-order derivative of \( \psi(\omega; b') \) with respect to \( \omega \), one can get

$$\frac{\partial \psi(\omega; b')}{\partial \omega} = -4\nu \rho' A(e_0 a)^2 k^2 \omega - 2\omega \rho' (e_0 a)^2 k^2 + 1) (A + v^2 J).$$  \hspace{1cm} (46)

It is noted from Eq. (43) that the condition \( \frac{\partial \psi(\omega; b')}{\partial \omega} < 0 \) is always satisfied for any \( \omega > 0 \), which can be guaranteed in nanostructures. Combining Eqs. (44) and (45), the inequality, thus, is obtained as follows

$$\frac{dh(\omega)}{d\omega} = \min_{b \in \mathcal{B}} \frac{\partial \psi(\omega; b)}{\partial \omega} < 0.$$  \hspace{1cm} (47)

$$\frac{d\vartheta(\omega)}{d\omega} = \max_{b \in \mathcal{B}} \frac{\partial \psi(\omega; b)}{\partial \omega} < 0.$$  \hspace{1cm} (48)

Equations (47) and (48) indicate that both \( \vartheta(\omega) \) and \( h(\omega) \) are monotonically decreasing function of the frequency \( \omega \).

Suppose that \( \omega = \omega_0 \) is the solution of \( \psi(\omega_0; b) = 0 \). Using inclusion monotonicity, we have

$$\psi(\omega_0; b) \in \psi(\omega_0; b')$$  \hspace{1cm} (49)

where \( \psi(\omega_0; b') \) is an interval extension of \( \psi(\omega_0; b) \). Combining \( \psi(\omega_0; b) = 0 \) and Eqs. (42), (43) and (49), the following inequality is given as

$$h(\omega_0) = \psi(\omega_0; b') \leq 0 \leq \psi(\omega_0; b') = \vartheta(\omega_0).$$  \hspace{1cm} (50)

According to Eqs. (47), (48) and (50), we have reason to believe that the following two equations exist

$$\vartheta(\bar{\omega}) = \lim_{\omega \to \bar{\omega}} \vartheta(\omega) = 0$$  \hspace{1cm} (51)

$$h(\bar{\omega}) = \lim_{\omega \to \bar{\omega}} h(\omega) = 0$$  \hspace{1cm} (52)

where \( \omega \) and \( \bar{\omega} \) are the lower and upper bounds of the frequency.

4.2 Iterative scheme of the bounds of the frequency

Given an initial interval \( \omega_{in} \) enclosing all possible values of the frequency set \( \omega \) and two iterative intervals \( \omega_{in}' \) and \( \omega_{in}'' \) containing the upper and lower bounds of the frequency, respectively. They are defined as

$$\omega_{in}' = [\omega_m, \omega_L], \omega_m \leq \bar{\omega} \leq \omega_L \leq \omega_{in}$$  \hspace{1cm} (53)

$$\omega_{in}'' = [\omega_0, \omega_U], \omega_0 \leq \bar{\omega} \leq \omega_U \leq \omega_{in}$$  \hspace{1cm} (54)

$$\omega_{in}' = [\omega_m, \omega_L], \omega_L \leq \omega \leq \omega_{in}$$  \hspace{1cm} (55)

where \( \omega_m \) and \( \omega_L \) are the lower and upper bounds of \( \omega_{in} \), respectively, \( \omega_0 \) and \( \omega_U \) are the lower and upper bounds of \( \omega_{in}' \), respectively, \( \omega_L \) and \( \omega_L \) are the lower and upper bounds of \( \omega_{in}'' \), respectively.

Taking the first-order Taylor series expansion of Eq. (51) at the frequency \( \omega = \omega_0 \), yields
\[ \vartheta(\tilde{\omega}) = \vartheta(\omega_s) + \frac{d \vartheta(\tilde{\xi})}{d \omega}(\tilde{\omega} - \omega_s) \] (56)

where \( \omega_s \leq \xi \leq \tilde{\omega} \).

Considering Eqs. (44), (53) and (54), and using inclusion monotonicity, gives

\[ \vartheta(\tilde{\omega}) \in \vartheta(\omega_{l_i}^I) \subseteq \vartheta(\omega_s) + \frac{d \vartheta(\tilde{\xi})}{d \omega}(\omega_{l_i}^I - \omega_s) \] (57)

\[ \frac{d \vartheta(\tilde{\xi})}{d \omega} \leq \frac{d \vartheta(\omega_{l_i}^I)}{d \omega} \leq \frac{\widehat{\vartheta}(\omega_{l_i}^I; b^I)}{\partial \omega} \] (58)

where \( \frac{d \vartheta(\omega_{l_i}^I)}{d \omega} \) and \( \frac{\widehat{\vartheta}(\omega_{l_i}^I; b^I)}{\partial \omega} \) are the natural interval extensions of \( \frac{d \vartheta(\omega)}{d \omega} \) and \( \frac{\widehat{\vartheta}(\omega; b)}{\partial \omega} \), respectively. And one has used \( \tilde{\xi} \in \omega_{l_i}^I \) and \( \tilde{\xi} \in \omega_{l_i}^I \) here.

By substituting Eqs. (51) and (58) into Eq. (57), one has

\[ 0 \in \vartheta(\omega_s) + \frac{\widehat{\vartheta}(\omega_{l_i}^I; b^I)}{\partial \omega}(\omega_{l_i}^I - \omega_s). \] (59)

Based on interval mathematics, it is known that the width of right-hand side interval of Eq. (59) relies only on that of \( \omega_{l_i}^I \) due to the widths of both \( \omega_{l_i}^I \) and \( b^I \) are constants. When the width of \( \omega_{l_i}^I \) is close to zero, we have

\[ \text{Wid} \left( \vartheta(\omega_s) + \frac{\widehat{\vartheta}(\omega_{l_i}^I; b^I)}{\partial \omega}(\omega_{l_i}^I - \omega_s) \right) \leq L_c \text{Wid}(\omega_{l_i}^I) \rightarrow 0 \] (60)

where \( \text{Wid}(\cdot) \) denotes the width of interval vector. \( L_c \) is a constant. Obviously, Eq. (60), in principle, provides a feasible method to find the upper bound of frequency by an iterative scheme to reduce the width of \( \omega_{l_i}^I \). The iterative scheme can be established by means of the following equation

\[ \vartheta(\omega_s) + \frac{\widehat{\vartheta}(\omega_{l_i}^I; b^I)}{\partial \omega}(\omega_{l_i}^I - \omega_s) = 0. \] (61)

Solving Eq. (61) leads to the iterative interval \( \omega_{l_i}^I \), which can be further expressed as

\[ \omega_{l_i}^I = [\omega_{l_i}; \bar{\omega}_U] = \left[ \omega_s - \frac{\vartheta(\omega_s)}{d}, \omega_s - \frac{\vartheta(\omega_s)}{d} \right] \] (62)

where

\[ d = \min_{\omega \in \omega_{l_i}^I, b \in b^I} \frac{\partial \vartheta(\omega; b)}{\partial \omega}, \quad \widetilde{d} = \max_{\omega \in \omega_{l_i}^I, b \in b^I} \frac{\partial \vartheta(\omega; b)}{\partial \omega}. \] (63)

Here one has used \( \vartheta(\omega_s) \geq 0 \) and \( \frac{\partial \vartheta(\omega_{l_i}^I; b^I)}{\partial \omega} < 0 \).

By considering \( \omega_{l_i} \leq \tilde{\omega} \leq \bar{\omega}_U \) and \( \vartheta(\omega_s)/d \leq 0 \), one can get the following inequality

\[ \omega_s \leq \omega_s - \frac{\vartheta(\omega_s)}{d} \leq \bar{\omega}_U. \] (64)

Based on inequality (64), the iterative scheme for the upper bound of the frequency can be established by

\[ \omega_{p+1} = \omega_p - \frac{\vartheta(\omega_p)}{d} \] (65)

where \( \omega_p \) is the iterative frequency in the \( p \)-th cycle of the iterative procedure.

Similarly, following the steps in deriving Eq. (65), the iterative scheme for the lower bound of the frequency can be established by

\[ \omega_{q+1} = \omega_q - \frac{h(\omega_q)}{d} \] (66)

where \( \omega_q \) is the iterative frequency in the \( q \)-th cycle of the iterative procedure.

The iterative procedure continues to high-order cycles in the same way until the following convergent conditions are met

\[ |\omega_{p+1} - \omega_p| \leq \varepsilon_1 \text{ and } |\omega_{q+1} - \omega_q| \leq \varepsilon_2 \] (67)

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are the convergence errors for the upper and lower bounds of the iterative frequency, respectively.

Now the iterative scheme for the bounds of the frequency is developed, which is called iterative algorithm based interval analysis method (IAM) in this paper. Based on IAM, the lower and upper bounds of the frequency can be determined if the convergent conditions in Eq. (67) are met. That is to say, interval prediction of wave propagation characteristics of embedded nanorod with uncertain-but-bounded material properties can be performed.
5 Numerical results and discussions

As shown in Fig. 1, a typical nanorod (a single-walled CNT) embedded in elastic medium is considered in order to demonstrate the good performance of the proposed method. Due to uncertainties in material properties, Young’s modulus $E$ and mass density $\rho$ of nanorod are modeled as uncertain-but-bounded parameters, and we have: $E = [0.9785, 1.0815]$ TPa and $\rho = [2185, 2415]$ kg/m$^3$. All the deterministic parameters are listed in Table 1.

5.1 Nonlocal nanorod model verification

The proposed nonlocal nanorod model is verified by comparing with several existing models in this subsection, in which Young’s modulus and mass density are set as the mean value of uncertain parameters, i.e., $E = 1.03$ TPa and $\rho = 2300$ kg/m$^3$. By setting $e_0a$, $\delta_1$, $\delta_2$ and $\delta_3$ in Eq. (34), Table 2 lists three special cases of the proposed model, which are compared with existing models, as shown in Fig. 2.

It is found that the present nonlocal nanorod model can be degenerated to the classical rod model (Shatalov et al. 2011) without wave dispersion, and the nonlocal rod model proposed by Narendar and Gopalakrishnan (2010) where the nonlocal scale effect is introduced. Furthermore, the present model is also compared with Li’s model including lateral inertia and shear effects (Li et al. 2017), which suggests the nonlocal Rayleigh-Bishop model can be obtained from the present model when the elastic medium is neglected.

5.2 Interval analysis method verification

For verification purposes, the wave dispersion curves calculated by the proposed IAM and the classical

\begin{table}[h]
\centering
\caption{Deterministic parameters of nanorod}
\begin{tabular}{cccccc}
\hline
$R$ (nm) & $h$ (nm) & $e_0a$ (nm) & $k_e$ (N) & $k_w$ (N/m$^2$) & $\nu$ \\
\hline
3.5 & 0.35 & 0.5 & $2 \times 10^{-6}$ & $1 \times 10^{11}$ & 0.3 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Three cases of the proposed nonlocal nanorod model}
\begin{tabular}{ccc}
\hline
Cases & Parameters & Description \\
\hline
I & $e_0a = \delta_1 = \delta_2 = \delta_3 = 0$ & Local rod model \\
II & $e_0a = 0.5$, $\delta_1 = \delta_2 = \delta_3 = 0$ & Nonlocal rod model \\
III & $e_0a = 0.5$, $\delta_1 = \delta_2 = 1$, $\delta_3 = 0$ & Nonlocal rod model including lateral inertia and shear \\
\hline
\end{tabular}
\end{table}

Fig. 2 Comparison of wave dispersion curves with different models: a wave frequency; b phase speed
Monte Carlo simulation (MCS) are compared in Fig. 3. The MCS with 90,000 samples is used to generate the reference results, which is taken to represent the “true” bounds. Here, the effects of lateral inertia and elastic medium are not taken into account. For a clear comparison, the lower bound (LB) and upper bound (UB) of the wave frequency at five selected wavenumbers are summarized in Table 3, in which the corresponding relative errors are also given. Some discussions can be summarized as follows:

1. It is clearly observed that the bounds calculated via IAM have excellent agreement with the reference bounds. Specifically, the maximum relative errors are about $8 \times 10^{-10}$ and $10^{-9}\%$ for the lower and upper bounds, respectively, indicating their relative errors are all smaller than $10^{-9}\%$. Another interesting phenomenon is that the lower and upper bounds calculated by IAM are always located in the interval obtained by MCS, which is in accordance with the proposed method. Moreover, the difference between the results obtained by two methods appears after 12–13 effective digits. This is due to the fact that the convergence errors in Eq. (67) are set as $\alpha_1 = \alpha_2 = 10$ Hz, leading to the maximum absolute errors (2.11 Hz for the lower bound and 2.66 Hz for the upper bound) are smaller than 10 Hz. In some sense, the proposed method can evaluate the exact bounds of the wave dispersion curves and could be adopted to study the combined effects of material uncertainties and various parameters on wave dispersion characteristics.

2. Additional useful information exhibited in Table 3 and Fig. 3 is the range of the wave frequency. It is indicated that for very small values of wavenumber ($k < 0.1$ nm$^{-1}$), the range of the wave frequency is very small. With the increase of the wavenumber, it shows a clear deviation between the lower and upper bounds. When the wavenumber increases to 10 nm$^{-1}$, the range of the wave frequency will attain its maximum and remain constant. Note that the phase speed shows an opposite tendency. These numerical results suggest that the uncertainties in material properties will result in significant variations of the wave frequency.

3. The physical meaning can be interpreted from Fig. 3 that the wave in a nanorod starts propagating from zero frequency ($\omega_{\text{cut-off}} = 0$ when $k_w = 0$). With the increase of the wavenumber, the wave frequency increases while the phase speed decreases gradually, which suggests the wave will propagate dispersively. In other words, the wave will change its shape as it propagates. When the wavenumber exceeds a certain value and approaches to infinity, the phase speed is close to zero and the wave frequency remains constant, which can be considered as the escape frequency based on Eq. (36).

5.3 Effect of lateral inertia

In this section, the combined effects of uncertainties and lateral inertia on wave dispersion characteristics
of nanorod will be discussed. The comparison of the wave dispersion curves about lateral inertia is depicted in Fig. 4. From the results, a drastically difference can be found for the wave dispersion curves of nanorod with and without lateral inertia. Then, it comes to some conclusions as follows:

1. In the nonlocal nanorod model with lateral inertia, the wave frequency increases firstly and then decreases as the wavenumber increases, while the lateral inertia is neglected, the wave frequency increases firstly and tends to stable values then. It is clear that the lateral inertia has significant effect on wave dispersion characteristics at higher values of the wavenumber while at lower values of the wavenumber, its effect becomes insignificant. The physical meaning can be interpreted from Fig. 4 that the lateral inertia effect accelerates the wave dispersion at higher values of the wavenumber, and reduces the escape frequency.

2. On the other hand, the range of the wave frequency increases gradually with the growth of the wavenumber, and reaches its maximum value at a particular wavenumber. Then, the range of the wave frequency will decrease instead of being constant, which is very different from the nonlocal model without lateral inertia. Hence, an important conclusion is drawn that the dispersion of the wave frequency in a certain wavenumber range (0.6–3.5 nm\(^{-1}\)) is more sensitive to the uncertainties in material properties than other parts of the wavenumbers.

3. From Fig. 4b, the combined effects of uncertainties and lateral inertia on phase speed of nanorod are easy to understand. The former mainly influences the amplitude of the phase speed and the latter changes the characteristic of the wave shape as the wave propagates.

### 5.4 Effect of elastic medium

The combined effects of uncertainties and elastic medium on wave dispersion characteristics of nanorod are studied in this section. Figures 5 and 6 show the influences induced by the stiffness coefficients \(k_v\) and \(k_g\), respectively. According to the results, the following discussions can be given as:

1. It is observed from Figs. 5a and 6a that the wave frequency without elastic medium is always
smaller than that considering stiffness coefficient $k_w$ for $k < 1 \, \text{nm}^{-1}$ and that considering stiffness coefficient $k_g$ for $k > 1 \, \text{nm}^{-1}$. A similar phenomenon for phase speed is also observed in Figs. 5b and 6b. This implies that the elastic medium forces the nanorod to be stiffer, even though the nonlocal scale effect makes it more flexible. Thus, without the elastic medium, the nonlocal scale effect is found to be more significant.

2. More specifically, the influence of the stiffness coefficient $k_w$ on the wave dispersion characteristics is very different from that of the stiffness coefficient $k_g$. At lower values of the wavenumber, the influence of the former is more dominated than that of the latter, while at higher values of the wavenumber, the opposite tendency can be seen. It can be interpreted that the elastic stiffness $k_w$ changes the cut-off frequency so that the wave can start propagating from a non-zero frequency, while the shear stiffness $k_g$ changes the escape frequency. The influence of elastic medium on the wave dispersion characteristics coincides with the actual physical meaning in Eqs. (35) and (36).

5.5 Effect of nonlocal scale parameter

To investigate the combined effects of material uncertainties and nonlocal scale parameter on wave
dispersion characteristics of nanorod, the nonlocal scale parameter $e_0a$ is initialized in the range of 0–1 nm, where $e_0a = 0$ nm stands for the local nanorod model without considering the nonlocal scale effect. In addition, the effects of lateral inertia and elastic medium are taken into account in this section. Figure 7 plots the wave dispersion curves with different nonlocal scale parameters. Some discussions are summarized as follows:

1. The wave frequency is not sensitive to the nonlocal scale parameters when the wavenumber is small. However, as the wavenumber becomes larger, it is important to capture the nonlocal scale effect. A similar phenomenon is observed for the phase speed.

2. In the local nanorod model ($e_0a = 0$ nm), the wave frequency have a nonlinear variation with the wavenumber, which suggests that the wave is dispersive. Such wave dispersion can be attributed to the effects of the lateral inertia and the elastic medium. When the nonlocal scale effect is taken into account, the wave dispersion becomes serious. As the nonlocal scale parameter increases, the wave frequency will decrease. More specifically, the wave frequency will trend to a certain value for the wavenumber larger than 100 nm$^{-1}$, while it increases firstly and then decreases with increasing the wavenumber. Obviously, the nonlocal scale parameter plays a significant role on the wave dispersion in nanorod.
3. It is found that for a fixed value of $e_0 a$, the dispersion of the wave frequency in a certain wavenumber range (0.6–10 nm$^{-1}$) is more sensitive to the uncertainties than other parts of the wavenumbers. Moreover, the range of the wave frequency at a fixed value of the wavenumber will decrease with an increase in the nonlocal scale parameter. This implies that the effect of the uncertainties on wave dispersion characteristics cannot be neglected for very small values of $e_0 a$.

5.6 Effect of uncertainty level

In order to further investigate the effect of uncertainties on wave propagation characteristics, the uncertainty level $\beta$ is introduced to describe the dispersion of material properties, i.e. $b' = b'I[1 - \beta, 1 + \beta]$ where $b'$ represents the midpoint of interval vector. Figure 8a shows the bounds of the wave frequency for different uncertainty levels $\beta_E$ and $\beta_p$, where the uncertainty level ranges from 0 to 0.2. Let the uncertainty level $\beta_E = \beta_p$. Fig. 8b plots the sectional drawings of (a). The dash line in the figure denotes the wave frequency calculated by using the deterministic model, in which all material properties are deterministic and set as $b = b'$. The wavenumber is set as $k = 1$ nm$^{-1}$. The wave frequency obtained by MCS is also presented to validate the IAM and a good agreement is exhibited.

1. It is clear that the dispersion of the wave frequency increases gradually as the increase of uncertainty level. Both the lower and upper bounds of the wave frequency have an asymmetric relationship about $\beta_E = \beta_p$, which implies that the influence of Young’s modulus on wave frequency is different from that of the mass density. Such difference is in accordance with the actual physical meaning in Eq. (34).

![Figure 8](image_url)

**Figure 8** Bounds of wave frequency for different uncertainty levels: a $\beta_E \neq \beta_p$, b $\beta_E = \beta_p$

**Table 4** The upper bounds of the wave frequency for different uncertainty levels

| $\beta$ | Frequency (Hz) | $\beta$ | Frequency (Hz) | $\beta$ | Frequency (Hz) |
|--------|----------------|--------|----------------|--------|----------------|
| 0      | 15,104,304,315,773.3 | 0.07   | 16,063,507,000,126.6 | 0.14   | 17,111,657,446,058.9 |
| 0.01   | 15,236,305,254,517.5 | 0.08   | 16,207,399,489,720.1 | 0.15   | 17,269,738,268,830.1 |
| 0.02   | 15,370,102,274,481.5 | 0.09   | 16,353,146,337,737.9 | 0.16   | 17,430,097,673,125.0 |
| 0.03   | 15,505,467,551,409.7 | 0.10   | 16,500,801,488,927.3 | 0.17   | 17,592,806,283,478.7 |
| 0.04   | 15,642,444,711,389.1 | 0.11   | 16,650,420,921,831.7 | 0.18   | 17,757,937,646,302.0 |
| 0.05   | 15,781,078,900,667.5 | 0.12   | 16,802,062,753,395.0 | 0.19   | 17,925,568,392,172.9 |
| 0.06   | 15,921,416,859,474.5 | 0.13   | 16,955,787,350,089.5 | 0.20   | 18,095,778,409,152.3 |
2. The deterministic frequency is always located in the frequency range obtained by the proposed nanorod model. Specially, the frequencies obtained by these two models are equal when $\beta = 0$. This suggests that the proposed nanorod model can degenerate to the deterministic one. However, it is difficult for deterministic analysis to predict the true wave dispersion curves of nanorod with uncertain-but-bounded material properties. Under such circumstance, non-probabilistic interval analysis, especially the proposed IAM, shows its superiority to other methods.

3. In addition, Table 4 lists the upper bounds obtained by MCS in Fig. 8b. A good fit to the data in Table 4 can be found with $\tilde{\omega} = 15.1072 + 12.9449 \beta + 9.9046 \beta^2$ THz. It is remarkable that the wave frequency is weakly nonlinear function with respect to the uncertainty level. In this case, the proposed IAM is able to well capture the effect of uncertainty level on wave propagation characteristics of nanorod.

6 Conclusions

This paper is concerned with the effect of uncertainty on wave propagation characteristics of nanorod embedded in an elastic medium. Considering uncertain material properties as uncertain-but-bounded parameters, a nonlocal nanorod model for the uncertain wave propagation analysis is developed on the basis of nonlocal elasticity theory and interval analysis theory. Simultaneously, an IAM is also presented to solve this model. After numerical verification by MCS, the combined effects of material uncertainties and various parameters on wave dispersion characteristics of a typical nanorod are investigated in detail. Main conclusions of the present research are summarized as follows:

1. The nonlocal nanorod model with uncertainties has more excellent capability in mechanical behaviors analysis. The proposed model not only takes the uncertainty in material properties into account, but also degenerates to the deterministic model when the uncertainty level is set to zero. Based on the nonlocal nanorod model with uncertainties, the presented IAM is able to predict the bounds of the wave dispersion curves in the case of limited information available on uncertainties, which cannot be dealt with by the existing deterministic nanorod model.

2. Due to the presence of the nonlocal scale, lateral inertia and elastic medium, the wave is dispersive as it propagates. The effects of them on wave dispersion characteristics of nanorod can be interpreted that both the nonlocal scale and lateral inertia make the nanorod more flexible while the elastic medium forces it to be stiffer.

3. Uncertainties in material properties of nanostructures are inherent. Such uncertainties will result in significant variations of the wave dispersion characteristics. For a fixed uncertainty level, the dispersion of wave propagation behaviors in a certain wavenumber range is more sensitive to uncertainties than other parts of the wavenumbers. In addition, the effects of different material properties on wave propagation behaviors are different. These important conclusions are benefit to guide the reliability and robust design of new nanodevices.

4. Although non-probabilistic modeling, especially the proposed IAM, shows its superiority in wave propagation of nanostructures with uncertainties, the aim of this study is not to replace the probabilistic method by the non-probabilistic method. Our work provides a possible alternative or a supplementary way of the uncertainty analysis in the case of scarce data available. It should be pointed out that the type of uncertainty analysis of nanostructures depends on the type and amount of available data.

Our preliminary investigations suggest that the non-probabilistic model and IAM can also be applied to other nanostructures with uncertainties in various engineering problems. Further work is required to extend the proposed method to solve other mechanical behaviors of nanostructures with uncertain material properties.

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