An improved upper bound for the grid Ramsey problem

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Abstract
For a positive integer \( r \), let \( G(r) \) be the smallest \( N \) such that, whenever the edges of the Cartesian product \( K_N \times K_N \) are \( r \)-colored, then there is a rectangle in which both pairs of opposite edges receive the same color. In this paper, we improve the upper bounds on \( G(r) \) by proving
\[
G(r) \leq \frac{(1 - \frac{1}{128}r^{-2}(1 - o(1)))^r}{2},
\]
for \( r \) large enough. Unlike the previous improvements, which were based on bounds for the size of set systems with restricted intersection sizes, our proof is a form of quasirandomness argument.

KEYWORDS
graph Cartesian product, grid Ramsey problem, Ramsey theory, quasirandomness

1 | INTRODUCTION

This paper is concerned with the following question, known as the grid Ramsey problem. For a positive integer \( r \), let \( G(r) \) be the smallest \( N \) such that, whenever an \( r \)-edge-coloring \( \chi \) of the Cartesian product \( K_N \times K_N \) is given, there is a rectangle in which both pairs of opposite edges receive the same color, that is, we may find \((i, j, i', j')\) such that \( \chi((i, j) (i', j')) = \chi((i, j') (i', j)) \) and \( \chi((i, j) (i, j')) = \chi((i', j) (i', j')) \). The grid Ramsey problem is to determine \( G(r) \). (Note that the notation \( \chi((i, j) (i', j')) \) may seem unusual at first, but the vertices of the graph \( K_N \times K_N \) are ordered pairs and if we follow the standard convention of writing \( uv \) for the edge between vertices \( u \) and \( v \) in a graph, then we end up with an expression like \( \chi((i, j) (i', j)) \)). In addition to being interesting in its own right, this function is related to other topics in Ramsey theory. In fact, the trivial bound of \( G(r) \leq r^{c(r^2)} + 1 \) is the simplest form of a crucial ingredient in Shelah’s celebrated proof of primitive recursive bounds in Hales-Jewett theorem [7]. Motivated by this, Graham, Rothschild, and Spencer [5] raised the question of whether \( G(r) \) can be bounded by a polynomial in \( r \). In [3], Conlon, Fox, Lee, and Sudakov answered their question negatively.
by constructing an $r$-edge-coloring of $K_N \times K_N$ without alternating rectangles, where $N \geq 2^{\Omega((\log r)^{3/2} / \sqrt[3]{\log \log r})}$. Moreover, they constructed (Theorem 1.3 (iii) of [3]) an $r$-edge-coloring of $K_M \times K_N$ without alternating rectangles, where $M$ was quadratic in $r$, and $N$ was close to $\Omega((\log r)^{3/2})$, which to some extent indicates the difficulty of improving the upper bounds on $G(r)$.

When it comes to upper bounds, Shelah's original bound is $G(r) \leq r^{(r+1)/2} + 1$. That argument actually uses very little of the structure present in the problem, and we include it here since it is very short and it is our starting point in the proof. Namely, we simply pick our $r + 1$ favorite rows $j_1, \ldots, j_{r+1}$ and for each column $i$, restrict $\chi$ to the complete graph on the vertices $(i, j_1), \ldots, (i, j_{r+1})$. Since there are $r^{(r+1)/2}$-edge-colorings of $K_{r+1}$, by the pigeonhole principle, there are two columns $i$ and $i'$ whose colorings of the fixed $K_{r+1}$ agree. Finally, consider $r + 1$ horizontal edges $(i, j_l) (i', j_l)$ for $l \in [r+1]$. There are two that have the same color and determine the desired rectangle.

The only improvements so far are due to Gyárfás [6] who showed that

$$G(r) \leq r^{(r+1)/2} - r^{(r-1)/2} + 1 = (1 - r^{-2r-1}(1 - o(1)))r^{(r+1)/2}$$

and a very recent one due to Corsten [4],

$$G(r) \leq r^{(r+1)/2} - \left(\frac{1}{4} - o(1)\right)r^{(r+1)/2} = \left(1 - \frac{1}{4}r^{-r}(1 - o(1))\right)r^{(r+1)/2}.$$ 

Our main result is

**Theorem 1.** For large enough $r$, we have

$$G(r) \leq \left(1 - \frac{1}{128}r^{-2}(1 - o(1))\right)r^{(r+1)/2}.$$ 

Unlike the arguments of Gyárfás and Corsten, who used theorems of Fisher and Ray-Chaudhuri–Wilson that bound the size of a set system with restrictions on intersections, our proof is based on quasirandomness. Informally speaking, a graph is quasirandom if it contains roughly the same number of cycles of length 4 as a random graph of the same density. The reason why such graphs are called quasirandom lies in the fact that this requirement forces a graph to behave like a random one, for example, to have roughly the same number of triangles as a random graph of the same density would. This notion was introduced by Thomason [8], and by Chung et al. [1]. We note that quasirandomness has been playing a role in Ramsey theory as well. For example, in [2] Conlon combines an earlier argument of Thomason [9] with quasirandomness to prove the best known bounds on the diagonal Ramsey numbers.

## 2 | PROOF OF THEOREM 1

### 2.1 | Overview of the argument

Suppose that $N$ is very slightly smaller than $r^{(r+1)/2}$ and $\chi$ is a given $r$-edge-coloring of $K_N \times K_N$ without alternating rectangles. Then Shelah’s argument above tells us that every two rows, viewed as $r$-edge-colorings of $K_N$ must not coincide on a copy of $K_{r+1}$, that is, for any choice of
row indices \( a_1, a_2 \in \mathbb{N} \) and vertices of \( K_{r + 1}b_1, \ldots, b_{r + 1} \in \mathbb{N} \), we must have \( \chi((a_1, b_i) (a_1, b_j)) \neq \chi((a_2, b_i) (a_2, b_j)) \) for some \( i \) and \( j \). However, since \( N \) is very close to \( r^{r+1} \) and we do not have a copy of \( K_{r + 1} \) which receives the same edge-coloring in two rows, we may deduce that for any fixed graph \( H \) on \( r + 1 \) vertices in \( [N] \) and \( r \)-edge-coloring \( c: E(H) \to [r] \), we must get roughly \( r^{-e(H)} \) rows in which \( H \) receives exactly the coloring \( c \).

Next, define the intersection graph of two rows to be the graph on the vertex set \( [N] \) whose edges are precisely those that receive the same color in both rows. From arguments above, we deduce that these intersection graphs are quasirandom. In contrast, looking at the vertical edges (edges between vertices in the same column), a similar argument allows us to deduce that the intersection graphs of pairs of rows we considered before are additionally \( r \)-partite with all vertex classes of size close to \( N/r \). We reach a contradiction by showing that \( r \)-partite graphs with vertex classes of almost equal size cannot be very quasirandom. We are now ready to proceed with the proof.

2.2 | Proof

Let \( \chi \) be an arbitrary coloring of the grid with \( r \) colors. For each \( j \in [N] \), we denote by \( H_j \) the complete graph on the vertex set \( [N] \) along with an \( r \)-edge-coloring of the \( j \)th row by \( \chi \), that is,

\[
H_j = \{ (xy, \kappa) \in [N]^{(2)} \times [r] : \chi((x, j)(y, j)) = \kappa \},
\]

where the notation \( X^{(2)} \) stands for \( \{ Y \subseteq X : |Y| = 2 \} \). We misuse the notation slightly, and we define \( H_{j_1} \cap H_{j_2} \) as

\[
H_{j_1} \cap H_{j_2} = \{ xy \in [N]^{(2)} : (\exists \kappa \in [r])(xy, \kappa) \in H_{j_1} \text{ and } (xy, \kappa) \in H_{j_2} \}.
\]

Thus, the notation \( H_{j_1} \cap H_{j_2} \) in our sense is exactly the projection of the usual \( H_{j_1} \cap H_{j_2} \) under the mapping \( (xy, \kappa) \mapsto xy \). Thus, as noted in [3], the condition that \( \chi \) has no alternating rectangles is exactly the same as all graphs \( H_{j_1} \cap H_{j_2} \) on the vertex set \( [N] \) having chromatic number at most \( r \). The following lemma shows that such graphs cannot be too quasirandom. For a graph \( G \), we write \( \text{hom}(C_4, G) \) for the number of homomorphisms from \( C_4 \) to \( G \), that is, the number of \( (x, y, z, w) \in V(G)^4 \) such that \( xy, yz, zw, wx \in E(G) \). This is very similar to counting copies of \( C_4 \) inside \( G \), except that we allow repetitions of vertices, and the order matters, which is helpful when performing calculations.

Lemma 2. Let \( G \) be a graph of order \( N \). Suppose that \( G \) is \( k \)-partite with vertex classes of size at most \( (1 + \varepsilon)N/k \) for some \( \varepsilon \in [0, 1] \). Let \( \delta = 2e(G)/N^2 \) be the density of \( G \). Then we have

\[
\text{hom}(C_4, G) \geq \left( 1 - 4\varepsilon \right) \left( 1 + \frac{1}{(k - 1)^3} \right) \delta^4N^4.
\]

Proof. Let \( V \) be the set of vertices of the graph and let \( V_1, \ldots, V_k \) be the vertex classes. Let \( \delta_{ij} \) be the density of the bipartite graph on vertex classes \( V_i \) and \( V_j \), that is, \( \delta_{ij} = \frac{e(V_i, V_j)}{|V_i||V_j|} \). For vertices \( x \) and \( y \) we write \( d_{x,y} \) for the codegree of \( x \) and \( y \), which is the number of common neighbors. (If \( x = y \), then \( d_{x,y} = d_x \)). We have
hom(C₄, G) = \sum_{x,y \in V} d_{x,y}^2 = \sum_{i \in [k]} \sum_{x,y \in V_i} d_{x,y}^2 + \sum_{i,j \in [k], i \neq j} \sum_{x \in V_i, y \in V_j} d_{x,y}^2 \geq \sum_{i \in [k]} \frac{1}{|V_i|^2} \left( \sum_{x,y \in V_i} d_{x,y} \right)^2 \\
+ \sum_{i \neq j} \frac{1}{|V_i||V_j|} \left( \sum_{x \in V_i, y \in V_j} d_{x,y} \right)^2 = \sum_{i \in [k]} \left( \sum_{x \in V_i} \frac{d_{x,y}}{|V_i|} \right)^2 \\
+ \sum_{i,j \in [k], i \neq j} \left( \sum_{x \in V_i, y \in V_j} \frac{d_{x,y}}{|V_i||V_j|} \right)^2 \geq \frac{1}{k} \left( \sum_{i \in [k]} \sum_{x \in V_i} \frac{d_{x,y}}{|V_i|} \right)^2 \\
+ \frac{1}{k(k-1)} \left( \sum_{i,j \in [k], i \neq j} \sum_{x \in V_i, y \in V_j} \frac{d_{x,y}}{|V_i||V_j|} \right)^2. 

(1)

Let

$$T = \sum_{i,j \in [k]} \sum_{x \in V_i, y \in V_j} \frac{d_{x,y}}{|V_i||V_j|}$$

and

$$S = \sum_{i \in [k]} \sum_{x,y \in V_i} \frac{d_{x,y}}{|V_i||V_i|}.$$

Write \(d_z^i\) to be the number of edges between a vertex \(z\) and class \(V_i\). We thus have

$$S = \sum_{i \in [k]} \sum_{x,y \in V_i} \frac{d_{x,y}}{|V_i||V_i|} = \sum_{i \in [k]} \sum_{z \in V_i} \left( \frac{d_z^i}{|V_i|} \right)^2 = \sum_{i \in [k]} \sum_{z \in V_i} \sum_{i' \in [k], i \neq i'} \frac{d_z^i d_z^{i'}}{|V_i||V_i'|} = \frac{1}{k-1} \sum_{i \in [k]} \sum_{z \in V_i} \sum_{i' \in [k], i \neq i'} \frac{d_z^i}{|V_i||V_i'|} = \frac{1}{k-1} T.$$

Using the elementary fact that, for a fixed \(T\), the quadratic function \(x \mapsto x^2 - \frac{2}{k}Tx + \frac{1}{k} T^2\) is increasing on \([\frac{1}{k}T, \infty)\), the inequality (1) becomes (since \(T \geq 0\) and \(S \geq \frac{1}{k-1} T)\)

$$\text{hom}(C_4, G) \geq \frac{1}{k} S^2 + \frac{1}{k(k-1)} (T - S)^2 = \frac{1}{k-1} S^2 - \frac{2}{k(k-1)} TS + \frac{1}{k(k-1)} T^2$$

$$\geq \frac{1}{k-1} \left( S^2 - \frac{2}{k} TS + \frac{1}{k} T^2 \right) \geq \frac{1}{k-1} \left( \left( \frac{T}{k-1} \right)^2 - \frac{2}{k} T \left( \frac{T}{k-1} \right) + \frac{1}{k} \right)$$

$$= \frac{1}{k^2} \left( 1 + \frac{1}{(k-1)^2} \right) T^2.$$
But

\[
T = \sum_{i,j \in [k]} \sum_{x \in V_i, y \in V_j} \frac{d_{x,y}}{\sqrt{|V_i| |V_j|}} = \sum_{z \in V} \left( \sum_{i \in [k]} \frac{d_{z}^i d_{z}^r}{\sqrt{|V_i| |V_r|}} \right)
\]

\[
= \sum_{j \in [k]} \left( \sum_{z \in V_j} \frac{d_{z}^j}{\sqrt{|V_j|}} \right)^2 \geq \sum_{j \in [k]} \frac{1}{|V_j|} \left( \sum_{z \in V_j} \frac{d_{z}^j}{\sqrt{|V_j|}} \right)^2 \geq \frac{1}{k} \left( \sum_{j \in [k]} \sum_{z \in V_j} d_{z}^j \right)^2 \geq \frac{k^2}{k (1 + \epsilon)^2 N^2} \left( \sum_{j \in [k]} \sum_{z \in V_j} \sum_{i \in [k]} d_{z}^i \right)^2 = \frac{k\delta^2 N^2}{(1 + \epsilon)^2}.
\]

Thus, (using the elementary fact that \((1 + \epsilon)^{-4} \geq (1 - \epsilon)^{4} \geq 1 - 4\epsilon \text{ for } \epsilon \in [0, 1])\) the number of homomorphisms of \(C_4\) to \(G\) is

\[
\text{hom}(C_4, G) \geq (1 - 4\epsilon) \left( 1 + \frac{\delta^2 N^2}{(k - 1)^2} \right)
\]

as required. \(\square\)

For the next proposition, we need to generalize our notation slightly. For a set \(A\), we write \(K_A\) for the complete graph with the vertex set \(A\). For example, the usual notation \(K_N\) could be written as \(K_{[N]}\) instead.

**Proposition 3.** Let \(A, B \subset [N] \text{ with } |B| \geq r^5\). Suppose that \(\chi : E(K_A \times K_B) \to [r]\) is an \(r\)-edge-coloring without alternating rectangles. Then, there are \(a_1, a_2, a_3, a_4 \in A\) and an edge-colored \(C_4\) on vertices \(a_1, a_2, a_3, a_4((a_1a_2, \kappa_1), (a_2a_3, \kappa_2), (a_3a_4, \kappa_3), (a_4a_1, \kappa_4))\) such that for at least \((1 + \frac{1}{4} r^{-3}) \frac{1}{r^2} |B| \text{ of } b \in B \text{ the } b\text{th row contains this pattern (ie, } \chi((a_i, b) (a_{i+1}, b)) = \kappa_b \text{ where } i = 1, 2, 3, 4, \text{ and } a_5 = a_1), \text{ or there are } b_1, b_2 \in B \text{ and a colored edge } (b_1, b_2, \kappa) \text{ such that for at least } (1 + \frac{1}{8} r^{-3} + O(r^{-4})) \frac{1}{r} |A| \text{ of } a \in A \text{ the } a\text{th column contains this pattern.}

**Remark.** We could in principle make this proposition more explicit by writing \(20r^{-4}\) instead of \(O(r^{-4})\), when \(r \geq 100\), for example (the choices of constants 20 and 100 are somewhat arbitrary), but we opt not to do so, to make the exposition clearer.

**Proof.** Suppose the contrary. First of all, for each \((a_1a_2, \kappa_1), (a_2a_3, \kappa_2), (a_3a_4, \kappa_3), (a_4a_1, \kappa_4))\), we get at most \((1 + \frac{1}{4r^2}) \frac{1}{r^2} |B| \text{ of } b \in B \text{ that contain the given pattern in } b\text{th row. Hence, (writing } a_5 = a_1)\)
\[
\sum_{b_1, b_2 \in B} \sum_{a_1, a_2, a_4 \in A} \sum_{x_1, x_2, x_4 \in [r]} \| b \in B : (\forall i \in [4]) \chi ((a_i, b)(a_{i+1}, b)) = x_i \|^2 
\leq r^4 |A|^4 \left( 1 + \frac{1}{4r^3} \right)^2 \frac{1}{r^8} |B|^2 = \left( 1 + \frac{1}{4r^3} \right)^2 \frac{1}{r^5} |A|^4 |B|^2.
\]

For \( b_1, b_2 \in B \), let \( \delta_{b_1, b_2} \) stand for the density of \( H_{b_1} \cap H_{b_2} \), that is, \( \delta_{b_1, b_2} = \frac{2\chi(H_{b_1} \cap H_{b_2})}{|A|^2} \).

Notice that
\[
\sum_{b_1, b_2 \in B} \delta_{b_1, b_2} |A|^4 = 16|A|^{-4} \sum_{b_1, b_2 \in B} \chi(H_{b_1} \cap H_{b_2})^4 
\geq 16|A|^{-4} |B|^{-6} \left( \sum_{b_1, b_2 \in B} \chi(H_{b_1} \cap H_{b_2}) \right)^4
\geq 16|A|^{-4} |B|^{-6} (1 - r |B|^{-1})^4 \left( \sum_{b_1, b_2 \in B} \chi(H_{b_1} \cap H_{b_2}) \right)^4,
\]

where the last inequality follows from the easy bounds \( \sum_{b \in B} \chi(H_b) \leq |B| \binom{|A|}{2} \) and \( \sum_{b_1, b_2} \chi(H_{b_1} \cap H_{b_2}) \geq \frac{1}{r} |B|^2 \binom{|A|}{2} \). Note that since \( r |B|^{-1} \leq 1 \), we also have \( (1 - r |B|^{-1})^4 \geq 1 - 4r |B|^{-1} \). We proceed further
\[
\sum_{b_1, b_2 \in B, b_1 \neq b_2} \delta_{b_1, b_2} |A|^4 \geq (1 - 4r |B|^{-1}) |A|^{-4} |B|^{-6}
\times \left( \sum_{a_1, a_2 \in A} \sum_{x \in [r]} \left( \sum_{b \in B} \chi((a_1, b)(a_2, b)) = x \right) \right)^2
\geq (1 - 4r |B|^{-1}) r^{-4} |A|^{-12} |B|^{-6}
\times \left( \sum_{a_1, a_2 \in A} \sum_{x \in [r]} \sum_{b \in B} \chi((a_1, b)(a_2, b)) = x \right)^8
= (1 - 4r |B|^{-1}) r^{-4} |A|^4 |B|^2.
\]

In particular,
\[
\sum_{b_1, b_2 \in B, b_1 \neq b_2} \delta_{b_1, b_2} |A|^4 \geq (1 - 4r |B|^{-1} - r^{-4}) r^{-4} |A|^4 |B|^2,
\]

where the last inequality follows from the easy bounds \( \sum_{b \in B} \chi(H_b) \leq |B| \binom{|A|}{2} \) and \( \sum_{b_1, b_2} \chi(H_{b_1} \cap H_{b_2}) \geq \frac{1}{r} |B|^2 \binom{|A|}{2} \). Note that since \( r |B|^{-1} \leq 1 \), we also have \( (1 - r |B|^{-1})^4 \geq 1 - 4r |B|^{-1} \). We proceed further
\[
\sum_{b_1, b_2 \in B} \delta_{b_1, b_2} |A|^4 \geq (1 - 4r |B|^{-1}) |A|^{-4} |B|^{-6}
\times \left( \sum_{a_1, a_2 \in A} \sum_{x \in [r]} \left( \sum_{b \in B} \chi((a_1, b)(a_2, b)) = x \right) \right)^2
\geq (1 - 4r |B|^{-1}) r^{-4} |A|^{-12} |B|^{-6}
\times \left( \sum_{a_1, a_2 \in A} \sum_{x \in [r]} \sum_{b \in B} \chi((a_1, b)(a_2, b)) = x \right)^8
= (1 - 4r |B|^{-1}) r^{-4} |A|^4 |B|^2.
\]

In particular,
\[
\sum_{b_1, b_2 \in B, b_1 \neq b_2} \delta_{b_1, b_2} |A|^4 \geq (1 - 4r |B|^{-1} - r^{-4}) r^{-4} |A|^4 |B|^2,
\]

where the last inequality follows from the easy bounds \( \sum_{b \in B} \chi(H_b) \leq |B| \binom{|A|}{2} \) and \( \sum_{b_1, b_2} \chi(H_{b_1} \cap H_{b_2}) \geq \frac{1}{r} |B|^2 \binom{|A|}{2} \). Note that since \( r |B|^{-1} \leq 1 \), we also have \( (1 - r |B|^{-1})^4 \geq 1 - 4r |B|^{-1} \). We proceed further
\[
\sum_{b_1, b_2 \in B} \delta_{b_1, b_2} |A|^4 \geq (1 - 4r |B|^{-1}) |A|^{-4} |B|^{-6}
\times \left( \sum_{a_1, a_2 \in A} \sum_{x \in [r]} \left( \sum_{b \in B} \chi((a_1, b)(a_2, b)) = x \right) \right)^2
\geq (1 - 4r |B|^{-1}) r^{-4} |A|^{-12} |B|^{-6}
\times \left( \sum_{a_1, a_2 \in A} \sum_{x \in [r]} \sum_{b \in B} \chi((a_1, b)(a_2, b)) = x \right)^8
= (1 - 4r |B|^{-1}) r^{-4} |A|^4 |B|^2.
\]
so we get
\[\sum_{b_1,b_2 \in B \atop b_1 \neq b_2} \delta_{b_1,b_2} \geq \frac{1}{r^2}\]

\[\sum_{b_1,b_2 \in B \atop b_1 \neq b_2} \delta_{b_1,b_2} \geq \frac{1}{r^2}\]

Since \(|B| \geq r^5\) we deduce
\[\sum_{b_1,b_2 \in B \atop b_1 \neq b_2} \delta_{b_1,b_2} \geq \frac{1}{r^2}\]

We may thus find distinct \(b_1, b_2 \in B\) such that \(\delta_{b_1, b_2} \geq r^{-2}\) and \(\hom(C_4, H_{b_1} \cap H_{b_2}) \leq (1 + \frac{1}{2} r^{-3} + O(r^{-4})) \delta_{b_1, b_2} |A|^4\). Lemma 2 applies to give a vertex class of \(H_{b_1} \cap H_{b_2}\) of size at least \((1 + \frac{1}{8} r^{-3} + O(r^{-4}))r^{-1}N\). But, this corresponds to \((b_1 b_2, \kappa)\) appearing in that many columns for a suitable \(\kappa \in [r]\), which is a contradiction to the initial assumptions.

We are now ready to prove the main result.

**Proof of Theorem 1.** Suppose that \(\chi: E(K_N \times K_N) \to [r]\) is an \(r\)-edge-coloring without an alternating rectangle. Our goal is to show that \(N \leq (1 - \frac{1}{128} r^{-2}(1 - o(1)))r^{(r+1)}\). We iteratively apply Proposition 3. For \(i \in [0, \lfloor \frac{1}{8} r \rfloor]\), unless we have already deduced the desired bound on \(N\), we show that there are a set of columns \(A_i \subset \{i\}\), a set of rows \(B_i \subset \{i\}\), a set of \(4i\) colored horizontal edges \(E_i^{\text{hor}} \subset \{i\}^2 \times [r]\), a set of \(i\) colored vertical edges \(E_i^{\text{ver}} \subset \{i\}^2 \times [r]\) and an integer \(J_i \in [0, i]\) with the following properties:

- vertices of \(E_i^{\text{hor}}\) are disjoint from \(A_i\), vertices of \(E_i^{\text{ver}}\) are disjoint from \(B_i\),
- for each \((a_1 a_2, \kappa) \in E_i^{\text{hor}}\) and each \(b \in B_i\), we have \(\chi((a_1, b) (a_2, b)) = \kappa\), for each \((b_1 b_2, \kappa) \in E_i^{\text{ver}}\) and each \(a \in A_i\), we have \(\chi((a, b_1) (a, b_2)) = \kappa\),
- \(|E_i^{\text{hor}}| = 4(i - J_i)|\) and \(|E_i^{\text{ver}}| = J_i\) and finally
- \(|A_i| \geq (1 + \frac{1}{8} r^{-3} + O(r^{-4}))4^{-i} r^{-3} N - 4(i - J_i), |B_i| \geq (1 + \frac{1}{4r})i - J_i, \frac{1}{r^{4(i - J_i)}} N - 2J_i|.

We prove this by induction on \(i\), with the base case \(i = 0\) being trivial, where we may take \(A_i = \{i\}, B_i = \{i\}, E_i^{\text{hor}} = \emptyset, E_i^{\text{ver}} = \emptyset, J_i = 0\). Suppose now that the claim holds for some \(i \in [0, \lfloor \frac{1}{8} r \rfloor]\) and let \(A_i, B_i, E_i^{\text{hor}}, E_i^{\text{ver}}, J_i\) the relevant sets and integer for \(i\). Apply Proposition 3 to \(\chi|_{K_N \times K_N}\). We now discuss the two possible outcomes.

Suppose that there is an edge-colored \(C_4((a_1 a_2, \kappa_1), (a_2 a_3, \kappa_2), (a_3 a_4, \kappa_3), (a_4 a_1, \kappa_4))\), with \(a_1, a_2, a_3, a_4 \in A_i\) such that for at least \((1 + \frac{1}{4r})i - J_i\) \(|B_i|\) of \(b \in B_i\) the \(b\)th row contains \(\kappa_i\). Then, we define
\[A_{i+1} = A_i \setminus \{a_1, a_2, a_3, a_4\},\]
\[E_{i+1}^{\text{hor}} = E_i^{\text{hor}} \cup \{(a_1 a_2, \kappa_1), (a_2 a_3, \kappa_2), (a_3 a_4, \kappa_3), (a_4 a_1, \kappa_4)\},\]
\[B_{i+1} = \{b \in B_i : (\forall i \in [4]) \chi((a_i, b) (a_{i+1}, b)) = \kappa_i\}, \text{ (where } a_5 = a_1)\]
\[E_{i+1}^{\text{ver}} = E_i^{\text{ver}}, \quad J_{i+1} = J_i.\]
Otherwise, there is a colored edge \((b_1b_2, \kappa)\) such that for at least
\[(1 + \frac{1}{8}r^{-3} + O(r^{-4}))^\frac{1}{r} |A_i|\]
of \(a \in A_i\) ath column contain this pattern. In this case, we define
\[A_{i+1} = \{a \in A_i : \chi((a, b_1)(a, b_2)) = \kappa\},\]
\[E_{i+1}^{\text{hor}} = E_i^{\text{hor}}, \quad B_{i+1} = B_i \setminus \{b_1, b_2\},\]
\[E_{i+1}^{\text{ver}} = E_i^{\text{ver}} \cup \{(b_1b_2, \kappa)\}, \quad J_{i+1} = J_i + 1.\]

It is easy to check that in both cases the given sets and integer satisfy the desired properties. Notice that this procedure can be completed for \(i \leq \lfloor \frac{1}{8}r \rfloor\), as long as the condition \(|B_i| \geq r^5\) of Proposition 3 is satisfied. In contrast, if that fails, then we certainly have the desired bound on \(N\) for large enough \(r\).

To finish the proof, we look at the cases depending on \(J_i\). Let \(s = \lfloor \frac{1}{8}r \rfloor\).

**Case 1.** \(J_i \geq s/2\). Let \(V \subset [N]\) be any set of size \(r + 1\) that contains all vertices of edges in \(E_s^{\text{ver}}\). List all edges in \(V^{(2)} \setminus E_s^{\text{ver}}\) as \(e_1, \ldots, e_m\), where \(m = (\frac{r+1}{2}) - J_i\). By the pigeonhole principle, we may find colors \(\kappa_1, \ldots, \kappa_m \in [r]\) such that for at least \(r^{-m}|A_s|\) of \(a \in A_s\) we have color \(\kappa_i\) on edge \(e_i\) in the column \(a\). However, if we get at least two such \(a_1, a_2 \in A_s\), then \(\chi\) in these two columns coincides on \(K_V\), which gives an alternating rectangle since \(|V| = r + 1\), as in Shelah’s proof, resulting in a contradiction. Therefore,
\[
1 \geq r^{-m}|A_s| \geq r^{-\left(\frac{r+1}{2}\right)+J_i}\left(1 + \frac{1}{8}r^{-3} + O(r^{-4})\right)^{\frac{J_i}{r} N - 4r} \geq \left(1 + \frac{1}{8}r^{-3} + O(r^{-4})\right)^{\frac{r}{16} - 1 - \left(\frac{r+1}{2}\right) N - 4r^{-(\frac{r+1}{2})+r/8+2}},
\]
completing the proof in this case.

**Case 2.** \(J_i < s/2\). This is very similar to the previous case, except that we now reverse the roles of rows and columns. Let \(V \subset [N]\) be any set of size \(r + 1\) that contains all vertices of edges in \(E_s^{\text{hor}}\). List all edges in \(V^{(2)} \setminus E_s^{\text{hor}}\) as \(e_1, \ldots, e_m\), where \(m = (\frac{r+1}{2}) - 4(s - J_i)\). By the pigeonhole principle, we may find colors \(\kappa_1, \ldots, \kappa_m \in [r]\) such that for at least \(r^{-m}|B_s|\) of \(b \in B_s\) we have color \(\kappa_i\) on edge \(e_i\) in the row \(b\). However, if we get at least two such \(b_1, b_2 \in B_s\), then \(\chi\) in these two rows coincides on \(K_V\), which once again gives an alternating rectangle, resulting in a contradiction. Therefore,
\[
1 \geq r^{-m}|B_s| \geq r^{-\left(\frac{r+1}{2}\right)+4(s-J_i)}\left(1 + \frac{1}{4r^3}\right)^{s-J_i} \frac{1}{r^4(s-J_i)} N - 2r \geq \left(1 + \frac{1}{4r^3}\right)^{\frac{r}{16} - 1 - \left(\frac{r+1}{2}\right) N - 2r^{4r+1-(\frac{r+1}{2})}},
\]
completing the proof. \(\square\)
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