Area Estimates and Rigidity of Non-compact \( H \)-Surfaces in 3-Manifolds

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Abstract
For appropriately values of \( H \), we obtain an area estimate for a complete non-compact \( H \)-surface of finite topology and finite area, embedded in a three-manifold of negative curvature. Moreover, in the case of equality and under additional assumptions, we prove that a neighbourhood of the mean convex side of the surface must be isometric to a hyperbolic Fuchsian manifold. Also, we show by a counterexample that although that area estimate holds for minimal surfaces, one does not have rigidity for equality in this case.

1 Introduction
Let \( \Sigma \) be a closed \( H \)-surface (a surface of constant mean curvature \( H \)) embedded in a Riemannian 3-manifold \((M, g)\). Imposing a curvature condition on \((M, g)\) and perhaps some extra condition on \( \Sigma \) is possible to obtain area estimates and rigidity results for \( \Sigma \) and \((M, g)\). For example, assuming a lower bound on the scalar curvature and that the surface is minimizing or does have index 1 with respect to the functional \( \text{Area} - H \cdot \text{Volume} \), some rigidity results were obtained in \([7, 26, 5, 6, 22, 16]\). There are also related results for free boundary minimal surfaces in 3-manifolds with boundary, see \([3, 19]\), and for MOTS in a spacetime, see \([12, 20]\). See also \([4, 21]\) for results in higher dimensions.

The case of interest on this work is when one have bounds on the sectional curvature \( K_M \) of \((M, g)\) but no assumption on the index of the \( H \)-surface \( \Sigma \). We will state two important results on this direction.

Theorem (Mazet-Rosenberg, [17]). Let \((M^3, g)\) be a complete oriented Riemannian 3-manifold whose sectional curvatures satisfy \( 0 \leq K_M \leq 1 \). Let \( \Sigma \subset M \) be an embedded minimal 2-sphere, then

\[ |\Sigma| \geq 4\pi. \]
Moreover the equality holds if, and only if, \((M, g)\) is isometric to the round sphere \(S^3_1\) or to a quotient of \(S^2_1 \times \mathbb{R}\) with the round product metric.

To state the second result consider the following construction.

**Fuchsian 3-Manifolds:** Let \(\Sigma\) be a oriented surface which is either closed or non-compact with finite topology, endowed with a complete metric \(h_\Sigma\) of constant curvature \(-1\) and finite area, so we have that \(\chi(\Sigma) < 0\). The Riemannian manifold \((\Sigma, h_\Sigma)\) is a quotient of \(\mathbb{H}^2\) by a cocompact or cofinite Fuchsian group \(\Gamma\) of isometries of \(\mathbb{H}^2\).

The group \(\Gamma\) can be extended to a group of isometries of \(\mathbb{H}^3\), for the details see \cite{11}. The quotient \(\mathbb{H}^3/\Gamma\) is homeomorphic to \(\mathbb{R} \times \Sigma\), and its metric is given by

\[
dt^2 + \cosh^2(t)h_\Sigma.
\]

The surfaces \(\{t\} \times \Sigma\) are totally umbilical with constant mean curvature \(\text{tgh}(t)\), with respect to the unit normal \(-\partial_t\). Fix \(H \in (0, 1)\), and define

\[
t_H = \text{arctgh} \; H. \tag{1}
\]

Applying a diffeomorphism we can rewrite the metric in \(\mathbb{H}^3/\Gamma\) as

\[
g_H = dt^2 + \cosh^2(t_H - t)h_\Sigma. \tag{2}
\]

Now, \(\{0\} \times \Sigma\) has constant mean curvature equal to \(H\).

**Theorem** (Espinar-Rosenberg, \cite{11}). Let \((M^3, g)\) be a complete oriented Riemannian 3-manifold whose sectional curvatures satisfy \(K_M \leq -1\). Let \(\Sigma \subset M\) be an embedded closed oriented \(H\)-surface, \(0 < H < 1\). Then

\[
|\Sigma| \leq \frac{2\pi|\chi(\Sigma)|}{1 - H^2} \tag{3}
\]

Moreover if the equality holds and \(\Sigma\) separates the ambient space, then there are a neighbourhood \(\mathcal{U}\) of \(\Sigma\) in \((M, g)\), and an isometry

\[
\Phi : ([0,t_H] \times \Sigma, g_H) \to (\mathcal{U}, g),
\]
such that \(\Phi(0, \Sigma) = \Sigma\), where \(g_H\) is the metric \([2]\). Moreover, \(\Phi\) can be extended to \([t_H] \times \Sigma\), in such a way that \(\Sigma_m := \Phi(t_H, \Sigma)\) is a totally geodesic surface, and \(\Sigma\) is a covering space of \(\Sigma_m\).

It is important to remark that the area estimate \([3]\) also holds if \(\Sigma\) is a minimal surface, but the proof of the rigidity on the case of equality does
not work if $H = 0$. On the last section we will show by a counter-example that the rigidity is not valid on this case.

In [11], Espinar and Rosenberg also obtained versions of the previous two theorems on the case of compact \textit{capilarity $H$-surfaces} on Riemannian 3-manifolds with boundary.

Observe that all the theorems stated concern compact surfaces. We are interested on the case where one have a complete non compact $H$-surface of finite area inside a 3-manifold. As far as we know there is no area-rigidity theorem in this situation. We can now state our main results.

**Proposition 1.** Let $(M, g)$ be a complete oriented 3-manifold with sectional curvature $K_M \leq -1$. Let $\Sigma \subset M$ be a complete non-compact oriented $H$-surface, $0 \leq H < 1$, with finite topology and finite area. Then:

1. $\Sigma$ has finite total curvature and
   \[ \int_{\Sigma} K_\Sigma \, d\mu = 2\pi \chi(\Sigma); \]
   \[ (4) \]

2. The area of $\Sigma$ is estimated as
   \[ |\Sigma| \leq \frac{2\pi |\chi(\Sigma)|}{1 - H^2}. \]
   \[ (5) \]

Moreover, the equality on (5) is valid if, and only if, $\Sigma$ is totally umbilic, $K_M \equiv -1$ along $\Sigma$, and $K_\Sigma \equiv H^2 - 1$.

We highlight that some versions of proposition (1) were already proved in [9] and [18].

Suppose $H \in (0, 1)$. In the case of equality in equation (5), and under additional assumptions on $\Sigma$ and $(M, g)$, a neighbourhood of the mean convex side of $\Sigma$ must be locally isometric to a Fuchsian manifold. As we shall see in section 3, if $\Sigma$ is a minimal surface one does not have rigidity in $(M, g)$. Precisely, we have:

**Theorem 1.** Let $(M^3, g)$ be a complete oriented Riemannian 3-manifold whose sectional curvatures satisfy $-a^2 \leq K_M \leq -1$. Let $\Sigma \subset M$ be an embedded complete non-compact oriented $H$-surface, $0 < H < 1$, with finite topology. Suppose that
   \[ |\Sigma| = \frac{2\pi |\chi(\Sigma)|}{1 - H^2}. \]
Then there is a neighborhood \( \mathcal{U} \) of \( \Sigma \) in \((M, g)\), and a local isometry 
\[
\Phi : ([0, t_H) \times \Sigma, g_H) \to (\mathcal{U}, g),
\]
such that \( \Phi(0, \Sigma) = \Sigma \), where \( g_H \) is the metric \((2)\).

An important remark about the previous theorem is that the map \( \Phi \) is not necessarily a diffeomorphism. When \( \Sigma \) is closed and embedded in a 3-manifold, it has an embedded tubular neighborhood consisting of normal geodesics of same length, but in the non-compact case this is not always true even if one assumes that the immersion is proper. Consider for example the following situation:

**Example 1:** Let us review a construction done in [18]. Let \( \mathcal{M} \) be a complete hyperbolic 3-manifold of finite volume that contains a properly embedded totally geodesic thrice-punctured 2-sphere \( S \), the existence of such a manifold is proved in [1].

Let \( N \) be a unit normal vector to \( S \). Then, for any \( t > 0 \), the equidistant \( S(t) = \{\exp_x(tN(x)) ; x \in S\} \) is the image of a proper immersion of \( S \) into \( \mathcal{M} \) of constant mean curvature. Moreover, there is some \( \varepsilon_\mathcal{M} > 0 \) such that, for \( t < \varepsilon_\mathcal{M} \), \( S(t) \) is also properly embedded.

However, \( S(t) \) intersects \( S \) for all \( t \), hence the map \( \Psi : [0, \varepsilon_\mathcal{M}) \times S \to \mathcal{M}, \)
\[
\Psi(t, x) = \exp_x((\varepsilon_\mathcal{M} - t)N(x))
\]
is a local diffeomorphism, but is not injective.

It is worth mention that a more general construction allows the totally geodesic surface to have other topology, [2].

In view of this problem we will add more assumptions on \( \Sigma \) which assure that the map \( \Phi \) on theorem [1] is in fact a diffeomorphism, and so is an isometry. For state this result let us fix some notation. Consider a surface \( \Sigma \) properly embedded in a complete Riemannian 3-manifold \((M, g)\) and denote by \( N \) a unit normal along \( \Sigma \). Following chapter 8 of [13], define the *minimal focal distance* of \( \Sigma \) in \( M \) as 
\[
m_f(\Sigma, N) := \inf_{x \in \Sigma} \sup \{ t > 0 ; \text{dist}(\exp_x(tN(x)), \Sigma) = t \}.
\]
If \( m_f(\Sigma, N) > 0 \), then every geodesic of the form \( \gamma_x(t) = \exp_x(tN(x)) \), \( x \in \Sigma \), is minimizing on the interval \([0, m_f(\Sigma, N)]\), therefore \( \Sigma \) has an embedded tubular neighbourhood in \( M \) consisting of normal geodesics of same length.
Observe that the definition of \( m_f(\Sigma, N) \) depends on the choice of a unit normal. In the case of theorem \( \text{[1]} \) we have \( H_\Sigma > 0 \), hence the mean curvature vector \( \vec{H}_\Sigma \) has a well defined direction. So in this case we will choose \( N \) pointing into the direction of \( \vec{H}_\Sigma \) and denote \( m_f(\Sigma) = m_f(\Sigma, N) \). Following this we have:

**Corollary 1.** Let \((M^3, g)\) and \( \Sigma \) satisfy the hypothesis of theorem \( \text{[1]} \). Suppose that \( \Sigma \) separates and \( m_f(\Sigma) > 0 \). Then the map \( \Phi \) of theorem \( \text{[1]} \) is an diffeomorphism, and hence an isometry. Moreover, \( \Phi \) can be extended to \( \{t_H\} \times \Sigma \), in such a way that \( \Sigma_m := \Phi(t_H, \Sigma) \) is a totally geodesic surface, and \( \Sigma \) is a covering space of \( \Sigma_m \).

**Remark:** In general an hyperbolic metric on a surface of finite topology is not unique. So in theorem \( \text{[1]} \) and corollary \( \text{[1]} \) we do not know what is the conformal structure of \( \Sigma \). However, it is well known that a thrice-punctured sphere has a unique hyperbolic metric up to diffeomorphism, so in this case we have only one possibility for the metric \( h_\Sigma \), and so for the metric \( g_H \).

Let us make some comments on the proof of the stated results. We follow the ideas of \( \text{[17]} \) and \( \text{[11]} \), however some difficulties arise due to the fact \( \Sigma \) is not compact. First, the analog of proposition \( \text{[1]} \) in the case \( \Sigma \) is closed is proved using the Gauss equation and the Gauss-Bonnet formula, so we need to prove first the item \( \text{[1]} \) (which is interesting by itself) to obtain the area estimate \( \text{[5]} \). On the proof of theorem \( \text{[1]} \) we work with the equidistants of \( \Sigma \). In the case \( \Sigma \) is closed, the geometry of these surfaces is uniformly controlled in a tubular neighbourhood of fixed radius, but in the non-compact case this is not necessarily true. In view of this, we added the hypothesis that the sectional curvature of \((M, g)\) is also bounded below, so we can use classical comparison theorems to control the geometry of the equidistants of \( \Sigma \). We also needed to obtain a version of the first variation of area for a family of non-compact surfaces with finite area, see equation \( \text{[17]} \).

**Acknowledgements:** This work started while I worked at Instituto Nacional de Matemática Pura e Aplicada (IMPA), under the fund Programa de Capacitação Institucional PCI/MCTI-CNPq. I would like to thank José Espinar, Pedro Gaspar and Harold Rosenberg for discussions and comments on a early version of the manuscript. I also thank Albvao Ramos, for discussions regarding example \( \text{[11]} \).
2 Proof of the main results

Proof of Proposition 1. Denote by $N_{\Sigma}$ a unit vector field normal to $\Sigma$ and by $A_{\Sigma}$ the second fundamental form of $\Sigma$ with respect to $N_{\Sigma}$.

By the Gauss equation and the Arithmetic-Geometric Inequality, we obtain
\[ K_{\Sigma} = K_{M} + \det(A_{\Sigma}) \leq -1 + H^2 < 0. \]  
(6)

Since $\Sigma$ has negative curvature, the curvature integrand $\int_{\Sigma} K_{\Sigma} \ d\sigma$ exists in $[-\infty, 0)$. Also, $\Sigma$ has finite area, then by theorem A in [24] (see also theorem 12 in [15]), $\Sigma$ has finite total curvature and it holds the Gauss-Bonnet Formula:
\[ \int_{\Sigma} K_{\Sigma} \ d\mu = 2\pi \chi(\Sigma). \]  
(7)

Integrating (6) and using (7) we have
\[ (1 - H^2)|\Sigma| \leq - \int_{\Sigma} K_{\Sigma} \ d\mu = 2\pi |\chi(\Sigma)|. \]

Moreover, equality holds if and only if, equality holds in (6), that is, $K_{\Sigma} = H^2 - 1$, $K_{M} = -1$ along $\Sigma$, and $\det(A_{\Sigma}) \equiv H^2$ (which implies that $\Sigma$ is totally umbilic).

Proof of theorem 1. Let $N_{\Sigma}$ be the unit normal along $\Sigma$ pointing into the direction of the mean curvature vector $\vec{H}_{\Sigma}$. Let $\Phi : [0, +\infty) \times \Sigma \to M$ defined by $\Phi(t, x) = \exp_x(tN_{\Sigma}(x))$. Since $K_{M} \leq -1$ and $\Sigma$ is totally umbilic with $H_{\Sigma} < 1$, it follows from proposition 2.3 in [10] that $\Sigma$ has no focal points, so the normal exponential map of $\Sigma$ is a local diffeomorphism. Hence $\Phi$ is a local diffeomorphism.

Using $\Phi$, we pull back the Riemannian metric of $M$ to $[0, +\infty) \times \Sigma$. By the Gauss lemma, the vector field $\partial_t$ is everywhere normal to the surface $\Sigma_t := \{t\} \times \Sigma$, so this new metric can be written as
\[ \Phi^* g = dt^2 + \sigma_t, \]
where $\{\sigma_t\}$ is a smooth family of metrics on $\Sigma$.

This construction makes of $\Phi$ a local isometry, so $\Sigma_0$ is a totally umbilic $H$-surface on the metric $\Phi^* g$ with area $\frac{2\pi |\chi(\Sigma)|}{1 - H^2}$. Moreover $\Sigma_t$ is an equidistant to $\Sigma_0$. 


Fix the notation $t_H = \arctgh H$. We will prove that for any $t \in [0, t_H)$, we have
\[
\sigma_t = \cosh^2(t_H - t)h_\Sigma,
\]
where $h_\Sigma$ is a hyperbolic metric of curvature $-1$, such that,
\[
g|_\Sigma = \cosh^2(t_H)h_\Sigma.
\]

Denote by $A_t$ and $H_t$ respectively the second fundamental form and the mean curvature function of $\Sigma_t$ with respect to the unit normal $\partial_t$. We also define $\lambda_t(x) \geq 0$, such that $H_t + \lambda_t$ and $H_t - \lambda_t$ are the principal curvatures of $\Sigma_t$ at $(x, t)$. We notice that $\lambda_t(x) = 0$ if, and only if, $\Sigma_t$ is umbilical at the point $(x, t)$.

Consider the following pairs

**Model 1:** $(\mathbb{R} \times \Sigma, dt^2 + \cosh^2(t_H - t)\ h_\Sigma)$, $\tilde{\Sigma}_1 = \{0\} \times \Sigma$.

**Model 2:** $(\mathbb{R} \times \Sigma, dt^2 + a^2 \cosh^2(t_a - at)\ h_\Sigma)$, $\tilde{\Sigma}_2 = \{0\} \times \Sigma$, where $t_a = \arctgh (a^{-1}H)$.

The model 1 is just the Fuchsian model of curvature $-1$ we construct in the introduction and the equidistants of $\tilde{\Sigma}_1$ have second fundamental form equal to
\[
tgh (t_H - t) \cdot h_\Sigma.
\]
Observe that for $t = 0$ we have the second fundamental form of $\tilde{\Sigma}_1$ which is equal to $H \cdot h_\Sigma$.

Furthermore, the model 2 is a variation of this where the Riemannian manifold has constant sectional curvature $-a^2$ and the equidistants of $\tilde{\Sigma}_2$ have second fundamental form equal to
\[
a \cdot \tgh (t_a - at) \cdot h_\Sigma.
\]
Observe that for $t = 0$ we have the second fundamental form of $\tilde{\Sigma}_2$ which is equal to $H \cdot h_\Sigma$.

Since $\Sigma_0$ is totally umbilic and
\[
- a^2 \leq K_M \leq -1,
\]
using the comparison theorem 3.1 in [10] (we compare $(M, g)$ with Model 1 and Model 2) we obtain
\[
tgh (t_H - t) \cdot h_\Sigma \leq A_t \leq a \cdot \tgh (t_a - at) \cdot h_\Sigma,
\]
(9)
where the inequalities are in the sense of quadratic forms. So the functions $H_t$, $\lambda_t$ are uniformly bounded independently of $t$. Moreover, theorem 3.2 of [14] implies that

$$|\Sigma_t| \leq \cosh(t_a - at)|\Sigma|,$$

so, if $t \in [0, \epsilon]$, the areas $|\Sigma_t|$ are uniformly bounded independently of $t$.

Finally, we have the Gauss equation

$$K_{\Sigma_t} = K_M(t) + (H_t + \lambda_t)(H_t - \lambda_t),$$  

which together with the inequalities (8) and (9) implies that $K_{\Sigma_t}$ is bounded. In particular the negative part $K_{\Sigma_t}$ is bounded, so using that $\Sigma_t$ has finite area we conclude that $\int_{\Sigma_t} K_{\Sigma_t} \, d\mu_t$ is finite. Then, by theorem 1 in [25] (see also [15]), it follows that $\Sigma_t$ has finite total curvature. Using again theorem A in [24], we also obtain

$$\int_{\Sigma_t} K_{\Sigma_t} \, d\mu_t = 2\pi \chi(\Sigma).$$

Observe that the equation (11) implies

$$K_{\Sigma_t} \leq -1 + H_t^2 - \lambda_t^2.$$  

Since $\Sigma_t$ has finite area, and the functions $H_t^2$ and $\lambda_t^2$ are bounded and non-negative, its respective integrals over $\Sigma_t$ exist and are finite. Integrating inequality (13) and using equation (12), we have:

$$2\pi \chi(\Sigma) = \int_{\Sigma_t} K_{\Sigma_t} \, d\mu_t \leq -|\Sigma_t| + \int_{\Sigma_t} H_t^2 \, d\mu_t - \int_{\Sigma_t} \lambda_t^2 \, d\mu_t.$$  

Thus we obtain the inequality

$$\int_{\Sigma_t} \lambda_t^2 \, d\mu_t \leq -|\Sigma_t| - 2\pi \chi(\Sigma) + \int_{\Sigma_t} H_t^2 \, d\mu_t.$$  

In the following, we denote by $F(t)$ the right hand side of inequality (15).

**Claim:** $F$ vanishes on $[0, t_H]$.

**Proof of Claim 1.** First note that since $\Sigma_0$ is a $H$-surface with area $\frac{2\pi |\chi(\Sigma)|}{1 - H^2}$, we have $F(0) = 0$.

Now, observe that $\{\Sigma_t; t \in \mathbb{R}\}$ is a normal variation of $\Sigma_0$ with variational vector field $X = \partial_t$. Let $\{B_n; n \in \mathbb{N}\}$ be a exhaustion of $\Sigma$ by compact
domains with smooth boundary, and denote by $B_{n,t}$ the image of $B_n$ by the flow of $\partial_t$ which is inside $\Sigma_t$. By the first variation formula

$$\frac{d}{dt}|B_{n,t}| = -2 \int_{B_{n,t}} H_t \, d\mu_t,$$

observe that there is no boundary term since the variational vector field $X$ is everywhere normal to $\Sigma_t$.

By inequality (9)

$$0 \leq H_t \leq a, \quad \forall t \in [0, t_H],$$

which together with the fact $\Sigma_t$ has finite area implies that $\int_{\Sigma_t} H_t \, d\mu_t$ exist and is finite.

Furthermore, using equations (9) and (10), we see there are constants $c, \tilde{c} > 0$ independent of $t$ such that

$$\left| \int_{\Sigma_t} H_t \, d\mu_t - \int_{B_{n,t}} H_t \, d\mu_t \right| \leq c |\Sigma_t \setminus B_{n,t}| \leq \tilde{c} |\Sigma \setminus B_n|.$$

The last term on the above formula converges to 0 when $n$ goes to $+\infty$, hence $\{ \frac{d}{dt}|B_{n,t}| \}_{n \in \mathbb{N}}$ converges uniformly to $-2 \int_{\Sigma_t} H_t \, d\mu_t$. Moreover, $\{|B_{n,t}|\}_{n \in \mathbb{N}}$ converges to $|\Sigma_t|$, for any $t$. By the criterion of differentiability for limits of sequences of functions, we conclude that $|\Sigma_t|$ is differentiable for $t \in [0, t_H]$ and we obtain the

**First Variation of Area:**

$$\frac{d}{dt}|\Sigma_t| = -2 \int_{\Sigma_t} H_t \, d\mu_t. \quad (17)$$

Now, define $F_n(t) = -|B_{n,t}| - 2\pi \chi(\Sigma) + \int_{B_{n,t}} H_t^2 \, d\mu_t$. By the first variation of mean curvature

$$\frac{\partial H_t}{\partial t} = \frac{1}{2} (\text{Ric}_M(\partial_t, \partial_t) + |A_t|^2) = \frac{1}{2} \text{Ric}_M(\partial_t, \partial_t) + H_t^2 + \lambda_t^2. \quad (18)$$

Thus,

$$\frac{dF_n}{dt} = \int_{B_{n,t}} 2H_t \, d\mu_t + \int_{B_{n,t}} (2H_t \frac{\partial H_t}{\partial t} - 2H_t^3) \, d\mu_t = \int_{B_{n,t}} H_t (2 + \text{Ric}_M(\partial_t, \partial_t)) \, d\mu_t + 2 \int_{B_{n,t}} H_t \lambda_t^2 \, d\mu_t,$$
for all \( t \in [0, t_H] \).

We obtain from inequalities (8) and (16) that

\[
2a(1 - a^2) \leq H_t(2 + \text{Ric}_M(\partial_t, \partial_t)) \leq 0, \quad \forall t \in [0, t_H],
\]

and

\[
0 \leq H_t \lambda_t^2 \leq C, \quad \forall t \in [0, t_H],
\]

for some constant \( C > 0 \). Therefore, the integrals \( \int_{\Sigma_t} H_t(2 + \text{Ric}_M(\partial_t, \partial_t)) \, d\mu_t \) and \( \int_{\Sigma_t} H_t \lambda_t^2 \, d\mu_t \) exist and are finite. Arguing as we did to prove the First Variation of Area we conclude that the function \( F \) is differentiable for \( t \in [0, t_H] \) and

\[
\frac{dF}{dt} = \int_{\Sigma_t} H_t(2 + \text{Ric}_M(\partial_t, \partial_t)) \, d\mu_t + 2 \int_{\Sigma_t} H_t \lambda_t^2 \, d\mu_t \leq 2C \int_{\Sigma_t} \lambda_t^2 \, d\mu_t,
\]

where in the last part we used inequalities (19) and (20).

Since

\[
\int_{\Sigma_t} \lambda_t^2 \, d\mu_t \leq F(t),
\]

we obtain \( \frac{dF}{dt} \leq 2CF \) in \([0, t_H] \).

By Gronwall’s Lemma, we get \( F(t) \leq F(0)e^{2Ct} \) on \([0, t_H] \). But \( F(0) = 0 \), so \( F(t) \leq 0 \) on \([0, t_H] \). However, by inequality (9) we have \( F(t) \geq 0, \forall t \).

Then it follows that \( F \equiv 0 \).

Since \( F \equiv 0 \) all the inequalities in formula (21) are in fact equalities, so we obtain:

(a) \( \Sigma_t \) is umbilic;

(b) \( \text{Ric}_M(\partial_t, \partial_t) + 2 \equiv 0 \), which implies that the sectional curvature of any 2-plane orthogonal to \( \Sigma_t \) is equals to \(-1\);

(c) \( \frac{\partial H_t}{\partial t} = H_t^2 - 1 \).

On the one hand, by items (a) and (c) we have

\[
H_t(x) = \tgh(t_H - t) \quad \text{for all } x \in \Sigma,
\]

that is, \( \Sigma_t \) has constant mean curvature for all \( t \in [0, t_H] \).
On the other hand, by the First Variation of the Area \([17]\), we have

\[
\frac{d}{dt} |\Sigma_t| = 2 \tgh (t_H - t) |\Sigma_t| \quad \text{for all } t \in [0, t_H),
\]

which implies that

\[
|\Sigma_t| = 2\pi |\chi(\Sigma)| \cosh^2 (t_H - t) \quad \text{for all } t \in [0, t_H]. \tag{23}
\]

It follows from inequality \((14)\) and from equations \((22)\) and \((23)\) that

\[
2\pi \chi(\Sigma) = \int_{\Sigma_t} K_{\Sigma_t} \leq (H^2_t - 1)|\Sigma_t| = -2\pi |\chi(\Sigma)|,
\]

hence the sectional curvature of the 2-plane tangent to \(\Sigma_t\) is equal to \(-1\) for all \(t \in [0, t_H]\). In view of all this we conclude that

\[
K_{\Sigma_t} = \tgh^2(t_H - t) - 1 = -\cosh^{-2} (t_H - t), \quad \forall t \in [0, t_H].
\]

Therefore the metric \(dt^2 + \sigma_t\) is hyperbolic. Since, \(\Sigma\) is totally umbilical with constant mean curvature \(H\), we conclude that

\[
\sigma_t = \cosh^2(t_H - t) h_{\Sigma}, \quad \forall t \in [0, t_H].
\]

\[\square\]

**Proof of corollary** \([1]\). If \(\Phi\) is injective on \([0, t_H] \times \Sigma\) there is nothing to do, so suppose this is not true and define

\[
\bar{\epsilon} = \sup \{\epsilon \in [0, t_H); \Phi \text{ is injective on } [0, \epsilon) \times \Sigma\}.
\]

Since \(m_f(\Sigma) > 0\), we have \(\bar{\epsilon} > 0\). Suppose that \(\Phi\) is not injective on \([0, t_H] \times \Sigma\). Then there are distinct points \(p, q \in \Sigma\) such that 1 or 2 holds:

1. \(\Phi(\bar{\epsilon}, p) = \Phi(0, p)\) or \(\Phi(0, q)\),
2. \(\Phi(\bar{\epsilon}, p) = \Phi(\bar{\epsilon}, q)\).

In case 1, we will construct a closed curve \(C\) meeting tranversely a equidistant \(\Sigma_{t_0}\) in exactly one point, where \(0 < t_0 < \bar{\epsilon}\), but this contradicts the fact that \(\Sigma\) separates \(M\). To construct \(C\) when \(\Phi(\bar{\epsilon}, p) = \Phi(0, p)\), take \(C(t) = \Phi(t, p), 0 \leq t \leq \bar{\epsilon}\). When \(\Phi(\bar{\epsilon}, p) = \Phi(0, q)\), let \(\Gamma\) be a curve joining \(q\) to \(p\) on \(\Sigma\), and define \(C = \{\Gamma\} \cup \{\Phi(t, p); 0 \leq t \leq \bar{\epsilon}\}. Thus case 1 cannot occur.

Consider case 2, \(\Phi(\bar{\epsilon}, p) = \Phi(\bar{\epsilon}, q) =: x_0\). Given \(W \subset \Sigma_0\) define \(W_t = \{t\} \times W\). Since \(\Phi\) is a local diffeomorphism, there exists neighbourhoods \(U\)
and $V$ of $p$ and $q$ (respectively) in $\Sigma_0$, such that $[0, \bar{\epsilon}) \times U$ and $[0, \bar{\epsilon}) \times V$ are disjoint, $\Phi$ is injective on $[0, \bar{\epsilon}) \times U$ and on $[0, \bar{\epsilon}) \times V$, and $x_0 \in \Phi(U_\epsilon) \cap \Phi(V_\epsilon)$.

There are two possibilities, depending on the direction of the mean curvature vectors, $\vec{H}_{\Phi(\bar{\epsilon},p)}$ and $\vec{H}_{\Phi(\bar{\epsilon},q)}$. Suppose first that $\vec{H}_{\Phi(\bar{\epsilon},p)} = \vec{H}_{\Phi(\bar{\epsilon},q)}$. Then $\Phi(U_\epsilon)$ is on one side of $\Phi(V_\epsilon)$ at $x_0$ (where they are tangent) and they have the same mean curvature vector at this point, so by the maximum principle $\Phi(U_\epsilon) = \Phi(V_\epsilon)$. Consequently, $\Sigma$ is a non trivial covering space of the embedded surface $\Phi(\Sigma_\epsilon)$. However, for $\bar{\epsilon}$ near $\epsilon$, $\Phi(\Sigma_\epsilon)$ is a graph over $\Phi(\Sigma_\epsilon)$ in the trivial normal bundle of $\Phi(\Sigma_\epsilon)$ in $M$. This is impossible when the covering space is non trivial.

Now suppose that $\vec{H}_{\Phi(\epsilon,p)} = -\vec{H}_{\Phi(\epsilon,q)}$. At $x_0$, $\Phi(U_\epsilon)$ is strictly mean convex towards $\vec{H}_{\Phi(\epsilon,p)}$ and $\Phi(\{\epsilon\} \times V)$ is strictly mean convex towards $-\vec{H}_{\Phi(\epsilon,p)}$. Then $\Phi([0, \bar{\epsilon}) \times U)$ is on the mean concave side of $\Phi(\{\epsilon\} \times U)$ near $x_0$, and $\Phi([0, \bar{\epsilon}) \times V)$ is on the mean concave side of $\Phi(V_\epsilon)$ near $x_0$. So, any small neighbourhood of $\Phi(\epsilon, p)$ in $M$ contains an open set on the mean concave side of $\Phi(U_\epsilon)$ which is contained in the mean concave side of $\Phi(V_\epsilon)$. Hence $\Phi([0, \bar{\epsilon}) \times U)$ intersects $\Phi([0, \bar{\epsilon}) \times V)$ which contradicts the fact that $\Phi$ injective on $[0, \epsilon) \times \Sigma$.

Therefore $\Phi$ is injective on $[0, t_H) \times \Sigma$. Thus $\Phi|_{[0, t_H) \times \Sigma}$ is a diffeomorphism. Moreover, we know that $\Phi(t_H, \cdot)$ is a immersion, and by the proof of theorem [1], the surface $\Sigma_{t_H}$ is totally geodesic with respect to the metric $\Phi^*g$. Hence $\Phi(\Sigma_{t_H})$ is a totally geodesic surface. Arguing as in the case 2 discussed before we conclude that $\Sigma$ is a covering space of $\Phi(\Sigma_{t_H})$. \hfill \Box

3 Counterexample in the case $H = 0$

Let $S$ be a oriented surface which is either closed or non-compact with finite topology, endowed with a complete metric $h_S$ of constant curvature $-1$ and finite area. Consider the manifold $M = [0, +\infty) \times S$ endowed with the warped product metric

$$g = dt^2 + (e^t - t)^2 h_S.$$ 

Denote $f(t) = e^t - t$. Consider the surface $\Sigma = \{0\} \times S$. Its induced metric is $g|_\Sigma = h_S$ and its second fundamental form is $A_\Sigma = -f'(0) \frac{\partial f}{\partial t}^* = 0$. So $\Sigma$ is totally geodesic,

$$K_\Sigma \equiv -1 \text{ and } |\Sigma| = 2\pi|\chi(S)|.$$ 

Denote $\xi = \partial_t$. Given a vector field $V$ on $M$, define the projection on $S$ as $V^S := V - g(V, \xi)\xi$. Following the convention in [23] to sign of the curvature,
we have that the curvature tensor $R_M$ of $M$ can be expressed in terms of the curvature tensor $R_S$ of $S$ as (see [23], prop.12.42)

\[ R_M(X,Y)Z = R_S(X^S,Y^S)Z^S - \left( \frac{f'}{f} \right)^2 [g(X,Z)Y - g(Y,Z)X] + (\log f)' g(Z,\xi) [g(Y,\xi)X - g(X,\xi)Y] - (\log f)' [g(X,Z)g(Y,\xi) - g(Y,Z)g(X,\xi)]\xi, \]

where $X, Y, Z$ are smooth vector fields on $M$. Given a plane $\Pi \subset T_{(t,x)}M$, and an orthonormal basis \{X, Y\} of $\Pi$, it follows that the sectional curvature of $\Pi$ is given by

\[ K_M(\Pi) = \frac{K_S - (f')^2}{f^2} + \left[ \frac{-K_S + (f')^2}{f^2} - \frac{f''}{f} \right] [g(X,\xi)^2 + g(Y,\xi)^2] \]

\[ = \frac{-1 - (e^t - 1)^2}{(e^t - t)^2} + \left[ \frac{1 + (e^t - 1)^2}{(e^t - t)^2} - \frac{e^t}{e^t - t} \right] [g(X,\xi)^2 + g(Y,\xi)^2] \]

\[ = \frac{-1 - (e^t - 1)^2}{(e^t - t)^2} + \left[ \frac{e^t(t - 2) + 2}{(e^t - t)^2} \right] [g(X,\xi)^2 + g(Y,\xi)^2]. \]

Since $\xi$ is unitary we have $0 \leq g(X,\xi)^2 + g(Y,\xi)^2 \leq 1$. Moreover, there is $0 < t_0 < 1$ such that the function $u$ defined by $u(t) = \frac{e^t(t - 2) + 2}{(e^t - t)^2}$ is non-positive, for all $t \in [0, t_0]$, and is positive for all $t > t_0$. So, if $t \in [0, t_0]$, the sectional curvature will be maximal when $g(X,\xi)^2 + g(Y,\xi)^2 = 0$, and in this case

\[ K_M(\Pi) = \frac{-1 - (e^t - 1)^2}{(e^t - t)^2}. \]

Define $w(t) = \frac{-1 - (e^t - 1)^2}{(e^t - t)^2}$. We have

\[ w'(t) = \frac{2(e^t - t)(e^t - 1)}{(e^t - t)^4} [e^t(t - 2) + 2], \]

so the sign of $w'(t)$ is determined by the sign of $e^t(t - 2) + 2$, hence $w'(t) \leq 0$ for $t \in [0, t_0]$, and thus

\[ K_M(\Pi) = w(t) \leq w(0) = -1. \]

On the other hand, if $t > t_0$, the sectional curvature will be maximal when $g(X,\xi)^2 + g(Y,\xi)^2 = 1$, and in this case

\[ K_M(\Pi) = \frac{-e^t}{e^t - t} \leq -1, \]
where the inequality holds since \( e^t \geq e^t - t, \forall t \geq 0 \).

Now we want to prove that the sectional curvatures of \((M, g)\) are also bounded below. Observe that, if \( t \in [0, t_0]\), the curvature will be minimal when \( g(X, \xi)^2 + g(Y, \xi)^2 = 1 \), i.e,

\[
K_M(\Pi) = -\frac{e^t}{e^t - t},
\]

since the left hand side defines a continuous function, we have that this function is bounded in \([0, t_0]\). If \( t > t_0 \), the curvature will be minimal when \( g(X, \xi)^2 + g(Y, \xi)^2 = 0 \), i.e,

\[
K_M(\Pi) = -\frac{1 - (e^t - 1)^2}{(e^t - t)^2}.
\]

It follows from equation (24) that \( w \) is increasing on \((t_0, +\infty)\), and then \( w(t) \geq w(t_0), \forall t > t_0 \).

Therefore \((M, g)\) satisfies \(-a^2 \leq K_M \leq -1\), for some \( a > 0 \), but the metric \( g \) is not hyperbolic. To obtain a complete manifold without boundary, just take the double of \( M \), the metric \( g \) in the two copies of \( M \) will glue appropriately because \( \Sigma \) is totally geodesic. This construction provides an counter-example to theorem (1) and corollary (1) in the case \( H = 0 \).

On the other hand, consider the Riemannian metric

\[
\tilde{g} = dt^2 + l(t)^2 h_S,
\]
on \( \mathbb{R} \times S \), where

\[
l(t) = \begin{cases} 
eq t & \text{if } t \geq 0, \\ \cosh t & \text{if } t < 0. \end{cases}
\]

Since the function \( l \) is of class \( C^2 \) (but is not \( C^3 \)), we obtain a \( C^2 \) metric which agrees with the model up to reach the totally geodesic slice, where then the metric agrees with our counter-example.

Concerning this last construction, observe that the proofs of the main results only use \( C^2 \) properties of the metric. It is interesting to know if an example like this can exist with an \( C^\infty \) metric.

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