Time-dependent relaxed magnetohydrodynamics – inclusion of cross helicity constraint using phase-space action

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A new formulation of time-dependent Relaxed Magnetohydrodynamics (RxMHD) is derived variationally from Hamilton’s Action Principle using microscopic conservation of mass, and macroscopic conservation of total magnetic helicity, cross helicity and entropy, as the only constraints on variations of density, pressure, fluid velocity, and magnetic vector potential over a relaxation domain. A novel phase-space version of the MHD Lagrangian is derived, which gives Euler–Lagrange equations consistent with previous work on exact ideal and relaxed MHD equilibria with flow, but generalizes the relaxation concept from statics to dynamics. The application of the new dynamical formalism is illustrated for short-wavelength linear waves, and the interface connection conditions for Multiregion Relaxed MHD (MRxMHD) are derived. The issue of whether \( E + u \times B = 0 \) should be a constraint is discussed.

I. INTRODUCTION

In this paper we are principally concerned with developing a dynamical theory, RxMHD, of plasma relaxation (Rx) within a single domain \( \Omega \) that is closed, of genus at least 1, and whose boundary \( \partial \Omega \) is smooth, gapless, and perfectly conducting. This is part of a larger project, the development of Multiregion Relaxed MHD (MRxMHD) \([1]\), in which \( \Omega \) is but a subregion of a larger plasma region, partitioned into multiple relaxation domains physically separated by interfaces. Thus in general \( \partial \Omega(t) \) is the union of the inward-facing sides of the interfaces \( \Omega \) shares with its neighbors.

We consider these interfaces to be impervious to magnetic flux, implying the tangentiality condition

\[
\mathbf{n} \cdot \mathbf{B} = 0 \quad \text{on } \partial \Omega, \tag{1}
\]

where \( \mathbf{B} \equiv \nabla \times \mathbf{A} \) is the magnetic field and \( \mathbf{n} \) is a unit normal at each point on \( \partial \Omega \). Also, to conserve magnetic fluxes trapped within \( \Omega \), loop integrals of the vector potential \( \mathbf{A} \) within the interfaces must be conserved \([1]\).

We also take the interfaces to be perfectly flexible and impervious to mass and heat transport. However they transmit pressure forces between the subregions so we shall also analyse the interaction between two neighboring regions, \( \Omega \) and \( \Omega' \).

The plasma is modeled as a magnetohydrodynamic (MHD) fluid. Thus we start by considering the well-known ideal magnetohydrodynamic (IMHD) equations over \( \Omega \), which are encapsulated in the Lagrangian, \([2, 3]\),

\[
L_{\Omega}[v, \rho, p, \mathbf{A}] \equiv \int_{\Omega} \frac{\rho u^2}{2} \, dv - W_{\Omega}, \tag{2}
\]

with potential energy

\[
W_{\Omega}[p, \mathbf{A}] \equiv \int_{\Omega} \left( \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) \, dv, \tag{3}
\]

where \( dv \) is the volume element \( d^3x \) and \([v, \rho, p, \mathbf{A}]\) signals that \( L_{\Omega} \) is a functional of the Eulerian fields \( v(x, t), \rho(x, t), p(x, t) \) and \( \mathbf{A}(x, t) \) — the fluid velocity, mass density, pressure and magnetic vector potential, respectively (the constant \( \mu_0 \) being the vacuum permeability constant used in SI units).

We shall later verify that the IMHD equation of motion can be derived from this Lagrangian by defining the action integral

\[
S \equiv \int L_{\Omega} \, dt, \tag{4}
\]

and deriving an Euler–Lagrange equation from Hamilton’s Principle (of stationary action), \( \delta S = 0 \).

In this paper we endeavor to ground the presentation conceptually in standard Lagrangian and Hamiltonian mechanics \([\text{e.g.,}]\), in which one starts with a configuration space of generalized coordinates \( q_i \), some of which, say the \( q_{i_{\text{holo}}} \), may be subject to holonomic constraints, meaning their variations \( \delta q_{i_{\text{holo}}} \) are not free but can be expressed in terms of the remaining variations \( \delta q_{i_{\text{free}}} \). In our
case we can picture the fluid as an infinite set of fluid elements, each labeled by its initial position \( x_0 \) and evolving under the Lagrangian time-evolution map, \( x = r^t(x_0) \), taking fluid elements from their initial to their current positions. (This map is also called a flow in dynamical systems theory. It is the solution of the dynamical system \( \partial_t r^t = v(r^t, t), r^0(x_0) = x_0 \forall x_0 \in \Omega, \) where \( t_0 \) is an arbitrary initial time.)

We review, in Sec. [III A], the microscopic holonomic constraints of IMHD and, in Sec. [III B], its macroscopic global invariants. In Eq. [III C] we explain their relation to the RxMHD concept.

In Sec. [III] we present a representation for a Configuration-Space Lagrangian (CSL) for fluids similar to that used in [8], which allows for arbitrary arrays of both holonomically constrained and free fields and thus forms an appropriately general starting point for developing our relaxation formalism. This allows the integrations by parts to give Euler–Lagrange equations from Hamilton’s Principle to be reused, rather than having to adapt them to different scenarios.

As an example of the use of this formalism, in Sec. [IV] we derive the equation of motion, in momentum-conservation form, from the IMHD Lagrangian. However, we also show that adding to this Lagrangian what should be a redundant constraint term can give physically incorrect Euler–Lagrange equations. This unsatisfactory property of the CSL is the main motivation for developing the Phase-Space Lagrangian (PSL) approach as an alternative.

In Sec. [V] we derive and motivate the Phase-Space Lagrangian (PSL) approach that is the formal basis of our variational relaxation theory. We derive the corresponding Hamiltonian in terms of a canonical momentum density, and then make a noncanonical change of momentum variable to replace it with an Eulerian velocity field \( u \), forming a phase-space Lagrangian, Eq. [46], in terms of \( u \) and the Lagrangian velocity field \( v \). This is applied to the IMHD Hamiltonian with a cross-helicity constraint term. Although this constraint breaks the identification of \( u \) with \( v \), it is shown that \( u \) is the physical flow velocity, while \( v \) is relegated to the role of a relabeling field.

In Sec. [V] we “dynamize” equilibrium relaxation theory by taking as Hamiltonian the relaxed-MHD-equilibrium energy functional [5] to form a PSL. In Subsec. [V H] this is used in the phase-space version of Hamilton’s Principle to give dynamical Euler–Lagrange equations, which are analyzed in Subsec. [V C].

As a preliminary investigation of the physical implications of the our newly derived dynamics, in Sec. [VI] we derive the local dispersion relations for linear waves in the WKB approximation for both IMHD and RxMHD and find them very different. In the RxMHD case, at least some waves break the ideal Ohm’s Law.

In Sec. [VII] we discuss whether and how to make the ideal Ohm’s Law a constraint. Conclusions are given in Sec. [VIII].

In Appendix [A] we review the derivation of relaxed plasma equilibria with field-aligned flow by finding stationary points of an energy functional that includes the helicity and cross helicity constraints, and in Appendix [B] we show that the PSL approach allows a natural extension to axisymmetric equilibria with cross-field flow. [5] [6]. The Grad–Shafranov-Bernoulli equations for such equilibria are given in Sec. [12] and typical ordering of the flow parameters in tokamaks is briefly reviewed in Appendix [??].

Finally, in Appendix [C] the coupling across the interfaces between two neighboring relaxation regions is derived variationally from the phase-space action principle and shown to be the standard pressure-jump condition found previously. [1] Thus generalization to MRxMHD is straightforward.

## II. IDEAL MHD CONSTRAINTS AND INVARIANTS AND THE RELAXATION CONCEPT

### A. Microscopic (holonomic) IMHD constraints

In IMHD the fields \( \rho, p, \) and \( \mathbf{B} \) are not free variables but are constrained holonomically to evolve under the same Lagrangian map as the fluid elements. Thus their variations can be expressed in terms of \( \xi(x, t) \), the variation in the position of a fluid element that has reached the point \( x \) at time \( t \) (see [1] for details),

\[
\begin{align*}
\delta v &= \delta \xi + v \cdot \nabla \xi - \xi \cdot \nabla v, \\
\delta \rho &= -\nabla \cdot (\rho \xi), \\
\delta p &= -\gamma p \nabla \cdot \xi - \xi \cdot \nabla p, \\
\delta \mathbf{B} &= \nabla \times (\xi \times \mathbf{B}) \\
&= -\mathbf{B} \cdot \xi + \mathbf{B} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{B}.
\end{align*}
\]

Equation (5) is an expression of the microscopic (fluid-element-wise) conservation of mass and can be derived by integrating the continuity equation

\[
\partial_t \rho + \nabla \cdot (\rho v) = 0 \quad \Leftrightarrow \quad \frac{dp}{dt} = -\rho \nabla \cdot v
\]

along varied Lagrangian trajectories \( r^t(x_0) \). (Throughout this paper \( d/dt = \partial_t + v \cdot \nabla \), the advective derivative.) Similarly Eq. (7) expresses microscopic entropy conservation by integrating the ideal adiabatic pressure equation

\[
\partial_t p + \nabla \cdot (p v) + (\gamma - 1) p v \cdot \nabla v = 0 \quad \Leftrightarrow \quad \frac{dp}{dt} = -\gamma p \nabla \cdot v
\]

and Eq. (8) expresses the “freezing in” of magnetic flux into microscopic loops, advected by the flow field \( v \). It can be obtained by integrating the curl of the “ideal Ohm’s Law,” \( \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \),

\[
\partial_t (\mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B})) = 0 \quad \Leftrightarrow \quad \frac{d\mathbf{B}}{dt} = -\mathbf{B} \cdot \left( \nabla \cdot (\mathbf{v} \times \nabla v) \right) \]


where $I$ is the unit dyadic.

As a consistency check, one can show that Eqs. (9–12) are preserved under the variations Eqs. (6–8), which thus represent a Lie symmetry of IMHD.

The IMHD equation of motion is

$$
\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}.
$$

(12)

It will be verified in Sec. III C that the equation of motion can be derived from Hamilton’s Principle using the holonomic variations above.

B. Macroscopic (global) IMHD invariants

There is an infinity of microscopic IMHD invariants applying within infinitesimal fluid elements and tubes, but the only macroscopic (i.e. global within a domain $\Omega$) IMHD invariants we shall use as nonholonomic constraints are

- the magnetic helicity $2\mu_0 K_\Omega$, where, [8], we define the invariant $K_\Omega$ as

$$
K_\Omega[A] \equiv \frac{1}{2\mu_0} \int_\Omega A \cdot B
dV.
$$

(13)

- the cross helicity $\mu_0 K_\Omega^{EM}$, where

$$
K_\Omega^{EM}[u, A] \equiv \frac{1}{\mu_0} \int_\Omega u \cdot B
dV.
$$

(14)

(this global invariant derives from a relabelling symmetry in the Lagrangian representation of the fields, [9][15]),

- the total entropy [11]

$$
S_\Omega[p, p] \equiv \int_\Omega \frac{\rho}{\gamma - 1} \ln \left( \frac{\rho p}{\gamma T} \right) dV.
$$

(15)

C. The relaxation concept

By relaxation we mean, qualitatively, an assumed tendency of the plasma subsystem in $\Omega$, in the absence of external forcing, to approach a steady state. As high-temperature plasmas have very low particle collision rates, they cannot be assumed to be in thermodynamic equilibrium. Instead the physical mechanism for plasma relaxation is usually taken to be some kind of small-scale turbulence [16] e.g., though in strongly three-dimensional systems deterministic chaos has also been invoked [17].

Even ignoring the anisotropy created by the strong confining magnetic field, we could perhaps discern the existence of four relaxation timescales, an electromagnetic (or Alfvén) timescale $\tau_{Rx}$, a thermal equilibration timescale $\tau_{Rx}^T$, and an electrostatic potential equilibration timescale. Except perhaps for the latter two effects (which are relevant to the discussion in Sec. VII), we are not concerned with timescales in this paper, but assume simply there is an upper bound, $\tau_{Rx}$, beyond which our relaxation theory becomes applicable.

A serious discussion of the complex physics of relaxation mechanisms is beyond the scope of this paper. Instead we define what we mean by relaxation formalistically, as a generalization of the postulate of Taylor [14] that a relaxed steady state can be found by minimizing (“relaxing”) an energy functional, subject to the constraint that only the most macroscopically robust, global invariant of IMHD, the magnetic helicity, survives for $t \gtrsim \tau_{Rx}$.

While seemingly over simplistic, Taylor’s approach was found to be remarkably effective for describing experimental results from a very turbulent toroidal magnetic confinement experiment, Zeta.

In the MRxMHD equilibrium approach, [1], Taylor relaxation has the great attraction that it reduces the problem of computing $B$ in $\Omega$ to that of solving a well-studied elliptic PDE (the linear-force-free, or Beltrami equation). This solves the long-standing mathematical problem, [21], of the existence of IMHD equilibria in non-axisymmetric toroidal plasmas by regularizing away the singularities that arise if the magnetic field lines are constrained to lie on smoothly nested invariant tori (magnetic surfaces). Instead, because the Beltrami equation is elliptic, solving it requires no assumptions as to the detailed behavior of magnetic field lines, so magnetic islands and chaos cause no problems.

In this paper explore the question: Can the MRxMHD approach be extended to slowly time-dependent problems and equilibria with flow?

We follow Taylor in assuming that most of the microscopic invariants of IMHD are broken even in less turbulent systems, allowing heat transport and magnetic and vorticity reconnection events that allow the system to evolve to a self-organized, relaxed steady state. However, we increase the number of IMHD global invariants used as constraints in energy minimization so as to widen the class of energy minima (see FIG. 1 for a Venn diagram).

![FIG. 1. Constraint sets, spaces of allowed variations, and resulting minima](attachment:venn_diagram.png)

FIG. 1. Constraint sets, spaces of allowed variations, and equilibrium states: Illustrating how reducing the number of constraints, broadening the space of allowed variations, narrows the class of equilibria, and vice versa. (Reprinted with permission from Entropy [22].)

Only a negligible strength of the ideal-invariant-
breaking mechanism should be needed to maintain such a steady state, once formed, in a near-collisionless plasma. Thus we take as an “axiom” that the steady state Euler-Lagrange equations from a variational relaxation principle [22] should be consistent with the original ideal equations. That is, for a mathematical formulation of relaxation to be physically acceptable it should satisfy the Principle of Consistency with Ideal Equilibria (Consistency Principle for short): Relaxed equilibria should be a subset of the stationary solutions of the IMHD equations. (A problem with this principle is discussed in Sec. [II])

To achieve relaxation we lift the microscopic holonomic constraints from and replacing them by the three IMHD constraints in Sec. [II] These include the fluid-magnetic cross helicity Eq. (12), which couples an unconstrained plasma flow and vector potential , thus improving on an earlier attempt at deriving RxMHD. (though at the expense of complicating the PDE for the magnetic field).

Note that we have not lifted the holonomic constraint on as, to be able to talk about fluid elements at all, we need the mass density not only to be defined (perhaps in a weak, coarse-grained sense) but to obey a mass continuity equation pointwise, which we enforce variationally using Eq. (6), (so we do not need to include total mass as a global constraint).

A similar approach, for finding equilibria with flow, was used by Finn and Antonsen [5] using an entropy maximization relaxation principle. However, they also show this leads to the same equations as energy minimization. Thus we take, as in IMHD, the entropy in to be conserved and follow Taylor in defining relaxed states as energy minima.

However, to avoid invoking a dissipation mechanism to minimize energy, we stay within the framework of conservative classical mechanics by developing a formalism that abstracts the problem of defining equilibria to that of finding stationary points of a Hamiltonian. Stability can also be examined by taking the second variation of the Hamiltonian, but in this paper we deal only with first variations.

III. GENERAL CONFIGURATION-SPACE LAGRANGIAN (CSL)

A. General representation of holonomic constraints

As in [2], consider a configuration-space Lagrangian of the general form

\[ L[r,v,\eta] = \int_{\Omega} L(\lambda, \eta, \partial_t \eta, \nabla \eta) \, dV , \]

(16)

where \( \eta \) is a set of freely variable, unconstrained physical fields (scalars and 3-vectors) arranged into a column vector, and \( \lambda \) is a similar set, but holonomically constrained to vary with the Lagrangian map function \( r^d \) discussed in Sec. [II].

In the IMHD Lagrangian, for example, \( \mathbf{B} \) is subject to the holonomic “frozen-in flux” constraint Eq. (8), so it appears in \( \lambda \); whereas in Taylor’s relaxation principle the only local constraint on \( \mathbf{B} \) is \( \nabla \cdot \mathbf{B} = 0 \), which is enforced by the representation \( \mathbf{B} = \nabla \times \mathbf{A} \). The vector potential \( \mathbf{A} \), being a free variable, appears in \( \eta \). Explicit examples of \( \lambda, \eta, \mathbf{L}, \mathbf{V}, \), and \( \mathbf{A} \) are given at the end of this section and in other sections of the paper.

In the assumed absence of an external potential (e.g. gravity) in the system, the \( x \) and \( t \) dependences of \( \mathbf{L} \) arise only from those of the physical fields \( \mathbf{v}, \lambda \), and \( \eta \), and their derivatives.

Using a somewhat more explicit version of the formalism used in [3], we can represent some or all of the holonomic Eulerian variations in Eqs. (6–8) in the general form

\[ \delta \lambda^T \cdot (\mathbf{v} \cdot \nabla \xi - \mathbf{A} \nabla \cdot \xi) - \xi \cdot \nabla \lambda^T , \]  

(17)

where the diagonal, dimensionless constraint structure matrices \( \mathbf{V} \) and \( \mathbf{A} \) have as elements real-numbers, zero 3-vectors, and symmetric dyadic-tensor elements occurring only on their diagonals. (In \( \mathbf{V} \) all nondyadics are zero—for its role is to project out the 3-vector component of \( \lambda^T \).

The dot product \( \cdot \) denotes the usual 3-vector inner product, and also a matrix product where appropriate. (If there is no \( \cdot \) between 3-vectors then they form a dyadic). The transpose operation \( ^T \) acts on both matrices and dyadics, e.g. \( (ab)^T = ba \). Also, the real number \( \nabla \cdot \xi \) distributes multiplicatively over the elements of the matrix \( \mathbf{A} \) in the standard way, and dotting with the dyadic \( \nabla \xi \) likewise distributes over the elements of \( \mathbf{V} \), with the convention that a product of a zero element and a dyadic remains a null element of unchanged type.

B. General CSL Euler–Lagrange equations

As in Sec. [II A] we work in the current coordinates \( x \) and use the Eulerian variations \( \delta \) introduced by Newcomb [2] [8], which are such that \( \delta x \equiv 0 \). Thus, for variations of compact support localized in time and space so that boundary terms can be omitted, the variation in the action integral Eq. (4) is

\[
\delta S = \int \left[ \delta \mathbf{v} \cdot \frac{\partial L}{\partial \mathbf{v}} + \delta \lambda^T \cdot \frac{\partial L}{\partial \lambda} + \delta \eta^T \cdot \frac{\partial L}{\partial \eta} \right] \, dV \, dt,
\]

(18)
with $\delta L/\delta \mathbf{r}$ defined\,[23] by

$$
\frac{\delta L}{\delta \mathbf{r}} = \nabla \cdot \left( \mathbf{1} \lambda^T \mathbf{A} \frac{\partial L}{\partial \lambda} - \mathbf{V} \cdot \lambda^T \frac{\partial L}{\partial \lambda} \right) - (\nabla \lambda^T) \cdot \frac{\partial L}{\partial \lambda},
$$

where we have used the assumed symmetry of its dyadic blocks to commute $\mathbf{V}$ with $\lambda^T$.

Hamilton’s Principle applied to variations in fluid element positions, $\delta S = 0 \forall \xi$, gives the Euler–Lagrange equation

$$
\partial_t \left( \frac{\partial L}{\partial \mathbf{v}} \right) + \nabla \cdot \left( \frac{\partial L}{\partial \mathbf{v}} \right) + \mathbf{V} \cdot \frac{\partial L}{\partial \mathbf{v}} = \frac{\delta S}{\delta \mathbf{r}},
$$

(20)

Using Eq. (19) the equation of motion Eq. (23) can be put in partial conservation form, cf. [3] Eq. (24),

$$
\partial_t \left( \frac{\partial L}{\partial \mathbf{v}} \right) + \nabla \cdot \left( \frac{\partial L}{\partial \mathbf{v}} + \mathbf{V} \cdot \lambda^T \frac{\partial L}{\partial \lambda} + \mathbf{1} \left( \mathbf{L} - \lambda^T \mathbf{A} \frac{\partial L}{\partial \lambda} \right) \right) = \nabla \mathbf{L} - (\nabla \lambda^T) \frac{\partial L}{\partial \lambda} - (\nabla \lambda^T) \frac{\partial L}{\partial \lambda},
$$

(21)

where $\nabla \cdot (\mathbf{L}) \equiv \nabla \mathbf{L}$ has been added to both sides of Eq. (21) so that, in the absence of an external potential, the RHS is the remaining part of $\nabla \mathbf{L}$ obtained by applying the chain rule to all the arguments of $\mathbf{L}$ except $\mathbf{v}$ and $\lambda$. That is [see Eq. (16)]. RHS is $(\nabla \mathbf{L})^T \cdot \partial \mathbf{L}/\partial \mathbf{v} + (\nabla \lambda^T) \cdot \partial \mathbf{L}/\partial \lambda + \mathbf{1} \cdot \partial \mathbf{L}/\partial \lambda$, where the transpose in the last term turns the column vector $\nabla \lambda$ into a row vector containing the transposes of any dyadics in $\nabla \lambda$. (This ensures that “$\mathbf{v}$ contracts with an $\lambda$, $\nabla$ contracts wth a $\nabla$.”)

Except in Eq. (19), the notation $\delta L/\delta f$ represents the standard functional derivative of $\mathbf{L}$ with respect to an arbitrary field $f$, e.g. [24]. When $\mathbf{L}$ depends only on $\mathbf{v}$ and not on $\partial \mathbf{f}$ or $\nabla \mathbf{f}$, then $\delta L/\delta f = \partial \mathbf{L}/\partial f$, which identity has been used extensively to simplify Eq. (18). However $\mathbf{L}$ does not depend so simply on $\lambda$ — instead we have, on integration by parts with respect to $\mathbf{x}$ and $\mathbf{x}$,

$$
\frac{\delta S}{\delta \mathbf{x}} = \frac{\partial L}{\partial \mathbf{x}} - \partial_t \frac{\partial L}{\partial t_x} - \nabla \cdot \frac{\partial L}{\partial \mathbf{x}},
$$

(22)

where $\mathbf{x}$ denotes $\partial \mathbf{x}$ and $\partial \mathbf{L}/\partial \mathbf{x}$ and $\partial \mathbf{L}/\partial \nabla \mathbf{x}$ are column vectors of derivatives of $\mathbf{L}$ with respect to the elements of $\mathbf{n}$ and the gradients of these elements, respectively.

Using Eq. (22), the free-field Euler–Lagrange equations follow from Hamilton’s Principle, $\delta S/\delta \mathbf{x} = 0$,

$$
\partial_t \frac{\partial L}{\partial t_x} + \nabla \cdot \frac{\partial L}{\partial \mathbf{x}} = \frac{\partial L}{\partial \mathbf{x}}.
$$

(23)

Dotting both sides of Eq. (23) with $\nabla \mathbf{v}^T$ from the left and subtracting the results from both sides of Eq. (21), the full momentum conservation result expected from Noether’s Theorem is found to be, cf. [3] Eq. (27),

$$
\partial_t \mathbf{G} + \nabla \cdot \mathbf{T} = 0,
$$

(24)

where

$$
\mathbf{G} = \frac{\partial L}{\partial \mathbf{v}} - (\nabla \mathbf{v}^T) \cdot \frac{\partial L}{\partial \mathbf{x}},
$$

(25)

and

$$
\mathbf{T} = \mathbf{v} \cdot \frac{\partial L}{\partial \mathbf{v}} + \mathbf{V} \cdot \lambda^T \frac{\partial L}{\partial \lambda} + \mathbf{1} \left( \mathbf{L} - \lambda^T \mathbf{A} \frac{\partial L}{\partial \lambda} \right) - \text{Tr} \left( \frac{\partial L}{\partial \nabla \mathbf{v} \cdot \nabla \mathbf{v}^T} \right),
$$

(26)

the trace operator $\text{Tr}$ contracting over the indices of $\mathbf{v}$, but not of $\nabla$. The right-hand side of Eq. (24) vanishes because of the cancellation between the RHS of Eq. (21) and terms arising from the subtraction process. An energy conservation equation can also be derived, as in [3].

C. Example: Ideal MHD CSL with cross helicity constraint

As an explicit example, consider an MHD Lagrangian

$$
L_{\mathcal{O}}[\mathbf{v}, \rho, \mathbf{p}, \mathbf{B}] = \int_{\Omega} dV \frac{\rho v^2}{2} - W_{\Omega} + \nu K_{\Omega}^X[\mathbf{v}, \mathbf{B}],
$$

(27)

with $W_{\Omega}$ given by Eq. (3) and $\rho, \mathbf{p}, \mathbf{B}$ constrained within each fluid element to conserve mass and entropy, and to “freeze-in” magnetic flux. These constraints are expressed in the time evolution equations (9–11) and the holonomic variations given in Eqs. (6–8).

We have also added a global constraint term, $\nu K_{\Omega}^X[\mathbf{v}, \mathbf{B}]$, where $\nu$ is a Lagrange multiplier to enforce constancy of $K_{\Omega}^X$, the cross helicity Eq. (14). As the cross helicity is an IMHD invariant, one might expect this constraint to be redundant but we shall find that it actually leads to an incorrect equation of motion when $\nu \neq 0$. For the purposes of this paper this is a fatal flaw in the CSL approach.

In the compact representation, Eq. (17), the constrained quantities are combined into a matrix column vector $\lambda$, made up of two scalars, $\rho$ and $\mathbf{p}$, and a 3-vector, $\mathbf{B}$,

$$
\lambda^T = [\rho, \mathbf{p}, \mathbf{B}].
$$

(28)

Then the Lagrangian density is

$$
\mathcal{L} = \frac{\rho v^2}{2} - \frac{p}{\gamma - 1} - \frac{\mathbf{B} \cdot \mathbf{B}}{2 \mu_0} + \nu \mathbf{v} \cdot \mathbf{B} \mu_0
$$

$$
= \frac{\lambda_1 v^2}{2} - \frac{\lambda_2}{\gamma - 1} - \frac{\lambda_3 \lambda_3}{2 \mu_0} + \nu \mathbf{v} \cdot \mathbf{B} \mu_0,
$$

(29)

which has no free fields $\mathbf{x}$.

By comparing Eqs. (6–8) and Eq. (17) we see that the structure matrices are

$$
\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

(30)
with \( \mathbf{0} \) denoting the zero 3-vector.

Thus Eq. (25) gives

\[
\mathbf{G} = \lambda_1 \mathbf{v} + \nu \frac{\lambda_3}{\mu_0} \equiv \rho \mathbf{v} + \nu \frac{\mathbf{v} \cdot \mathbf{B}}{\mu_0},
\]

(31)

and Eq. (26) gives

\[
\mathbf{T} = \lambda_1 \mathbf{v} \mathbf{v} + \nu \frac{\lambda_3}{\mu_0} + \left[ 0, 0, \lambda_3 \right] \left[ \frac{\nu^2}{2} \frac{1}{\gamma - 1}, \nu \mathbf{v} - \frac{\lambda_3}{\mu_0} \right]^T
\]

\[
+ \left( \frac{\lambda_1 \nu^2}{2} - \frac{\lambda_2}{\gamma - 1} \frac{\lambda_3 - \lambda_3}{2 \mu_0} + \nu \frac{\lambda_3}{\mu_0} \right)
\]

\[
\equiv \rho \mathbf{v} \mathbf{v} + \left( \frac{\nu + \mathbf{B} \cdot \mathbf{B}}{2 \mu_0} \right) - \frac{\mathbf{B} \mathbf{B}}{\mu_0}
\]

\[
+ \frac{\nu}{\mu_0} \left( \mathbf{v} \mathbf{B} + \mathbf{B} \mathbf{v} \right).
\]

(32)

When the Lagrange multiplier \( \nu = 0 \), \( \mathbf{G} \) and \( \mathbf{T} \) are the standard MHD momentum density and total stress tensor, respectively, providing thus a verification of both the general formalism and of the specific CSL, Eq. (29).

However, when \( \nu \neq 0 \), \( \mathbf{G} \) and \( \mathbf{T} \) have no obvious physical interpretation. If the terms proportional to \( \nu \) cancel out in the momentum conservation equation Eq. (24), the constraint would at least have no physical effect. However, writing the equation of motion for \( \mathbf{B} \) in Eq. (11) as \( \partial \mathbf{B} + \nabla \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) = \mathbf{0} \) we see that cancellation cannot occur because the contribution of the cross-helicity term to the stress tensor is symmetric rather than antisymmetric.

A problem with the globally constrained CSL approach was also found previously, [23], when applied to the Euler flow Lagrangian with fluid helicity as a constraint. This was not found to give a physically correct Bernoulli equation. These examples lead to the conclusion that applying global constraints to a CSL cannot be relied upon to give physically meaningful model, motivating our development of the PSL as an alternative in the following.

IV. PHASE-SPACE ACTION PRINCIPLE FOR GENERAL MHD-LIKE FLUIDS

In developing relaxed MHD (RxMHD) we follow [11] in maintaining the microscopic (holonomic) IMHD constraint Eq. (6) on variations in mass density \( \rho \), intrinsic to the concept of fluid element, so that total mass is automatically conserved under variation. Also as in [11] we vary \( p \) freely, and \( \mathbf{A} \) freely within \( \Omega \) but holonomically constrained on \( \partial \Omega \). Then the above global invariants are enforced by using Lagrange multipliers, the main departure from [11] being the inclusion of the cross helicity as a constraint to couple magnetic field and fluid in the relaxation process.

In this section we develop a general variational principle that uses two velocity fields in representing the motion of the plasma fluid: \( \mathbf{u}(x, t) \), defined purely in the Eulerian picture in a given Lab frame, and \( \mathbf{v}(x, t|x_0, t_0) \), the vector field of the dynamical system \( \mathbf{x} = \mathbf{v} \) that provides a Lagrangian labeling of the fluid elements.

As the global invariants of ideal MHD form such an essential part of our relaxation theory we first review them before deriving the phase-space Lagrangian approach and testing it on IMHD in the presence of an imposed (redundant) cross-helicity constraint.

A. Canonical Hamiltonian Formulation

Building on the general Lagrangian formulation set out in Sec. IIIA we define the canonical momentum densities

\[
\pi \equiv \frac{\partial L}{\partial \mathbf{v}} ,
\]

(33)

\[
\mathbf{\pi}_\eta \equiv \frac{\partial L}{\partial \mathbf{v}} ,
\]

(34)

where \( L \) is a CSL density as in Sec. IIIA. We now suppose these equations to be solved to give \( \mathbf{v} \) and \( \mathbf{\eta} \), as functions of \( \pi \) and \( \mathbf{\pi}_\eta \), with corresponding Hamiltonian defined by the Legendre transformation

\[
H[r, \pi, \mathbf{\eta}, \mathbf{\pi}_\eta, t] = \int_\Omega \mathcal{H} \, dv
\]

where \( \mathcal{H}(r, \pi, \mathbf{\eta}, \mathbf{\pi}_\eta, t) \equiv \mathbf{\pi} \cdot \mathbf{v} + \mathbf{\pi}_\eta \mathbf{\pi}_\eta \mathbf{\eta} - L \).

The general variation of \( H \) is

\[
\delta H = \int dv \left[ \delta \mathbf{\pi} \cdot \mathbf{v} + \left( \mathbf{\pi} - \frac{\partial L}{\partial \mathbf{v}} \right) \cdot \delta \mathbf{v} - \mathbf{\xi} \cdot \frac{\delta L}{\delta \mathbf{v}} + \mathbf{\pi}_\eta \frac{\partial L}{\partial \mathbf{\pi}_\eta} \delta \mathbf{\eta}_t \right.
\]

\[
+ \left( \mathbf{\pi}_\eta - \frac{\partial L}{\partial \mathbf{\pi}_\eta} \right)^T \delta \mathbf{\eta}_t - \delta \mathbf{\eta}_t \left( \frac{\partial L}{\partial \mathbf{\pi}_\eta} - \nabla \cdot \frac{\partial L}{\partial \nabla \mathbf{\eta}} \right) \right]
\]

\[
= \int dv \left[ \delta \mathbf{\pi} \cdot \mathbf{v} - \mathbf{\xi} \cdot \frac{\delta L}{\delta \mathbf{v}} + \mathbf{\pi}_\eta \frac{\partial L}{\partial \mathbf{\pi}_\eta} \right.
\]

\[
+ \delta \mathbf{\pi}_\eta \delta \mathbf{\eta}_t - \delta \mathbf{\eta}_t \left( \frac{\partial L}{\partial \mathbf{\pi}_\eta} - \nabla \cdot \frac{\partial L}{\partial \nabla \mathbf{\eta}} \right) \right) \forall \delta \mathbf{\pi}, \delta \mathbf{\eta}.
\]

(36)

where we used Eq. (33) and Eq. (34). Thus

\[
\mathbf{v} = \frac{\delta H}{\delta \mathbf{\pi}}, \quad \frac{\delta H}{\delta \mathbf{\pi}} = \frac{\delta L}{\delta \mathbf{\pi}},
\]

(37)

\[
\mathbf{\eta}_t = \frac{\delta H}{\delta \mathbf{\pi}_\eta}, \quad \frac{\delta H}{\delta \mathbf{\pi}_\eta} = \frac{\partial L}{\partial \mathbf{\pi}_\eta} + \nabla \cdot \frac{\partial L}{\partial \nabla \mathbf{\eta}} ,
\]

(38)

the first equations of Eq. (37) and Eq. (38) being the obvious generalizations of the canonical Hamilton equation of motion \( \dot{\mathbf{q}}_i = \partial H/\partial p_i \). The generalizations of \( \dot{\mathbf{p}}_i = -\partial H/\partial q_i \), though less standard, are provided by eliminating \( L \) from Eq. (20) and Eq. (23) using Eq. (37).
and Eq. (38),
\[ \partial_t \pi + \nabla \cdot \left( \frac{\delta H}{\delta \pi} \right) + \left( \nabla \frac{\delta H}{\delta \pi} \right) \cdot \pi = -\frac{\delta H}{\delta \pi}, \]
(39)
\[ \partial_t \pi_i = -\frac{\delta H}{\delta \pi_i}. \]
(40)

We shall not elaborate on these canonical equations further, as we do not use them in this paper. Instead we build on the concept of the phase-space Lagrangian (PSL), \( L_{ph}(r, v, \pi, \partial_t \pi, \pi_i, \pi_\eta, \partial_t \pi_\eta) \),
\[ L_{ph} = \int_\Omega \left( \pi \cdot v + \pi_\eta \cdot \eta_e \right) dV - H, \]
(41)
with the corresponding phase-space action,
\[ S_{ph} = \int L_{ph} dt. \]
(42)

It is a standard result in classical mechanics that a “modified Hamilton’s Principle” (see e.g. [4], p. 362), or phase-space action principle
\[ \delta S_{ph} = 0 \]
(43)
for all phase-space variations \( \delta q_i \) and \( \delta p_i \) (\( \xi, \delta \eta_i, \delta \pi \) and \( \delta \pi_\eta \) in our case), yields the canonical Hamiltonian equations of motion and thus provides a valid alternative to the original configuration-space-based Hamilton’s Principle for deriving physical equations of motion.

Note that, as the \( p_i \) (\( \pi \) and \( \pi_\eta \) in our case) are now regarded as freely variable, the dimensionality of the space of allowed variations is doubled in the phase-space action principle, making it much more flexible as the \( p_i \) are now untied from their Lagrangian roots in \( \partial L/\partial q_i \).

That is, by using the PSL action principle we are no longer restricted to canonical Hamiltonian mechanics as the variational principle remains valid under noncanonical changes in phase-space coordinates. By appealing directly to the phase-space action principle, extra formal complications such as noncanonical Poisson brackets [24] can be avoided.

Note particularly that our PSL is of the same general form as the CSL of Sec. III except with the set of free variables \( \eta \) augmented by including \( \pi \) (or its replacement under a change of phase-space variables). Thus, once we have a Hamiltonian, we can reuse the general Euler–Lagrange results of Sec. III B simply by replacing \( L \) with \( L_{ph} \). For such reasons we make the phase-space action principle the basis of the theory developed in this paper.

Historical note: The phase-space action principle has long been used (implicitly) in the generating-function theory of canonical transformations [4], p. 380, though the current terminology and emphasis on its utility in noncanonical transformations is more recent (see e.g. 26 [27]). A more mathematical terminology for \( L_{ph} dt \) is the fundamental [28], or Poincaré-Cartan [29] p. 44, 1-form.

B. PSL for standard form Lagrangians

Although we do not need the canonical equation of motion, we do need to make explicit the canonical Hamiltonian \( H \) in order to form the phase-space Lagrangian Eq. (41) as the starting point. This is greatly simplified by restricting, in this paper, to CSLs of the standard kinetic-minus-potential energy form,
\[ L_{std} = \frac{\rho v^2}{2} - V(\lambda, \eta, \nabla \eta), \]
(44)
where we have assumed \( V \) contains neither \( \eta \) nor \( v \). The former assumption implies \( \partial L/\partial \eta_i = 0 \) and the latter implies \( \partial L/\partial v = \rho v \). Thus, from Eq. (33), we can eliminate \( v \) in terms of \( \pi \) trivially, \( v = \pi/\rho \). Also, from Eq. (34), \( \pi_\eta = 0 \). Thus, from Eq. (35), the canonical Hamiltonian density is \( H_{std} = \pi^2/2\rho + V \).

However, in this paper we do not work with the canonical momentum \( \pi \) but instead exploit the freedom afforded by the PSL to work with a velocity-like phase space variable \( u \) obtained by the noncanonical change of variable \( \pi = \rho u \). Then the Hamiltonian becomes
\[ H_{nc} = \int_\Omega \left( \frac{\rho u^2}{2} + V \right) dV, \]
(45)
and the PSL in noncanonical form becomes, from Eq. (41),
\[ L_{nc} = \int_\Omega \rho u \cdot v dV - H_{nc}. \]
(46)

This equation forms the basis of the development in the remainder of this paper. As \( \pi \) was freely variable using the PSL action principle, so \( u \) is freely variable in the noncanonical phase space. It has the dimensions of a velocity, and, as we shall show, it can indeed be interpreted as an Eulerian flow velocity, freed from the labeling constraint of the Lagrangian flow velocity \( v \).

Note: Typically the only field gradient in \( V \) is that of \( \mathbf{A} \) in \( \mathbf{B} = \nabla \times \mathbf{A} \). As the curl makes it clumsy to work with the general Euler–Lagrange equation Eq. (23) it is useful to give here the functional derivative \( \partial L_{nc}/\partial \mathbf{A} \) when \( V \) is an explicit function of \( \mathbf{A} \) and \( \mathbf{B} \). Interchanging dot and cross in the scalar triple product \( \left( \partial L_{nc}/\partial \mathbf{B} \right) \cdot \nabla \times \delta \mathbf{A} \) and integrating by parts we find
\[ \frac{\delta L_{nc}}{\delta \mathbf{A}} = \frac{\partial L_{nc}}{\partial \mathbf{A}} + \nabla \times \frac{\partial L_{nc}}{\partial \mathbf{B}}. \]
(47)
The corresponding Euler–Lagrange equation is found by setting \( \delta L_{nc}/\delta \mathbf{A} \) to zero.

C. Example: Ideal MHD PSL with cross helicity constraint

In this section we test the PSL approach against the same problem for which the CSL gave unphysical results in Sec. III C. Thus we take as Hamiltonian
Eq. (45) with the potential energy density term \( \mathcal{V} = p/(\gamma - 1) + B \cdot B/2 \mu_0 - \nu u \cdot B/\mu_0 \), the cross-helicity constraint term \( \nu K_{\lambda \gamma} \) being subtracted from the Hamiltonian rather than adding \( \nu K_{\lambda \gamma} [v, B] \) to the CSL Lagrangian. (Constraining the Hamiltonian is more relevant to the constrained energy minimization idea behind the present paper than constraining the CSL action.) The Lagrangian velocity \( v \) remains holonomically constrained as in Sec. IIIC, as do \( \rho, p \), and \( B \), so \( \lambda \) and the structure matrices in Eq. (30) remain unchanged.

Then the PSL density is [cf. Eq. (25)]

\[
\mathcal{L}_{nc} = \lambda_1 u \cdot v - \frac{\lambda_1 u^2}{2} - \frac{\lambda_2}{\gamma - 1} \frac{\lambda_3 \cdot \lambda_3}{2 \mu_0} + \nu \frac{u \cdot \lambda_3}{\mu_0},
\]

which now has \( u \) as a free field, so \( \eta = [u] \).

As a first step toward answering this question, we consider the fluid equation of motion in momentum conservation form—Eq. (29) gives

\[
G = \lambda_1 u \equiv \rho u,
\]

and Eq. (26) gives

\[
T = \lambda_1 v u + [0, 0, \lambda_3] \begin{bmatrix} \frac{\nu^2}{2} & 1 & \nu u - \lambda_3 \end{bmatrix}^T
+ I \left( \lambda_1 u \cdot v - \frac{\lambda_2 u^2}{2} \frac{\lambda_3}{\gamma - 1} - \frac{\lambda_3 \cdot \lambda_3}{2 \mu_0} + \nu \frac{u \cdot \lambda_3}{\mu_0} \right)
- \lambda_1 u \cdot v + \frac{\lambda_3 u^2}{2} + \nu \frac{u \cdot \lambda_3}{\mu_0} - \lambda_3 (\nu u - \lambda_3)
= \rho vu + \frac{\nu B}{\mu_0} u + I \left( \rho + \frac{B \cdot B}{2 \mu_0} - \frac{BB}{\mu_0} \right)
= \rho uu + I \left( \rho + \frac{B \cdot B}{2 \mu_0} \right) - \frac{BB}{\mu_0}.
\]

However, there are no \( \eta \) or \( \nabla \eta \) terms so the Euler–Lagrange equation \( \partial \mathcal{L}_{nc} / \partial u = 0 \), giving (after dividing by \( \lambda_1 \))

\[
u \frac{u \cdot \lambda_3}{\mu_0} = v + u_{Rx},
\]

where the magnetic-field-aligned velocity

\[
u \frac{B}{\mu_0} = \frac{u_{Rx}}{\rho}
\]

is the fully relaxed flow velocity, found in Appendix A to result from the cross helicity constraint when extremizing the Hamiltonian \( H_{\lambda}^{Rx} \), Eq. (A1). Figure 2 gives a visualization of Eq. (49), showing the flow \( u \) (red) as the vector sum of the flow \( v \) (blue) and the field-aligned background flow \( u_{Rx} \) (green).

When \( \nu = 0 \), \( u_{Rx} = 0 \) also, and we may then identify \( u \) and \( v \). However, adding the cross-helicity constraint makes the velocity-like noncanonical momentum field \( u \) and the Lagrangian-map-constrained velocity \( v \) different. Which is the “true” physical fluid velocity?

**Demonstration that \( u \) is the IMHD flow velocity:**

- Unlike the CSL approach in Sec. IIIC, the PSL method gives the physically correct momentum density and stress tensor, even with the redundant cross-helicity constraint, provided we identify \( u \), not \( v \), as the physical flow velocity.

- From Eq. (49), \( \nabla \cdot (\rho v) = \nabla \cdot (\rho u) \), so, from Eq. (9), \( u \) obeys the required mass continuity equation \( \partial_t \rho + \nabla \cdot (\rho u) = 0 \).

- Provided \( p \) is barotropic (i.e. \( p/\rho^\gamma = \text{const throughout } \Omega \)), then the required adiabatic pressure equation of motion, Eq. (10), must be satisfied even when \( v \) is replaced by \( u \).

- From Eq. (49), \( v \times B = u \times B \), so, from Eq. (11), the required “frozen-in flux” equation, \( \partial_t B = \nabla \times (u \times B) \), is satisfied. \( \Box \)
We can now interpret the Lagrangian velocity \( \mathbf{v} \) as an auxiliary dynamical vector field whose inverse flow to \( t = t_0 \) gives a labeling of all the fluid elements being advected by \( \mathbf{u} \): Adding the cross-helicity constraint generates a relabeling transformation. (See Fig. 3. The concept of moving labels was used in Ref. [3] in the context of waves on a background flow.)

V. ELEVATION OF RELAXATION TO A DYNAMICAL THEORY

A. PSL for RxMHD

In Appendix A we verify that constrained noncanonical Hamiltonian \( H_{\text{Rx}}^{(1)} \), Eq. (A1), is an energy functional whose stationary points under variation give a subset of the solutions of isothermal, ideal magnetohydrostatics (IMHS), i.e. they meet the ideal Consistency Principle. Thus they are acceptable relaxed solutions.

However, there are two motivations to generalize this equilibrium result, one reason being that it is more appropriate to extend relaxation theory to a dynamical one.

As \( H_{\text{Rx}}^{(1)} \) gives a satisfactory relaxed magnetostatics it is a natural starting point for a dynamical theory. To do this we replace the Hamiltonian in Eq. (46) with \( H_{\text{Rx}}^{(1)}[\mathbf{u}] \) to form the relaxed PSL

\[
L_{\Omega}^{\text{Rx}}[\mathbf{u}, \mathbf{v}] = \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} \, dV - H_{\Omega}^{\text{Rx}}(\mathbf{u}), \tag{53}
\]

where \( H_{\Omega}^{\text{Rx}} \) is as given in Eq. (A1). Then the PSL density is [cf. Eq. (29)]

\[
\mathcal{L}_{\Omega}^{\text{Rx}} = \rho \mathbf{u} \cdot \mathbf{v} - \frac{\rho u^2}{2} - \frac{p}{\gamma - 1} - \frac{\mathbf{B} \cdot \mathbf{B}}{2\mu_0} + \frac{\mu \mathbf{A} \cdot \mathbf{B}}{2\mu_0} + \frac{\nu_1 \mathbf{u} \cdot \mathbf{B}}{\mu_0}, \tag{54}
\]

In the notation of Sec. IIIA relaxation is implemented by moving all but \( \rho \) to the set of free variables, i.e. the holonomic variables array is just \( \lambda = [\rho] \), and the free variables array is \( \eta^T = [p, \mathbf{u}, \mathbf{A}] \). The structure matrices become trivial, \( \mathbf{V} = [0] \), \( \Lambda = [1] \).

B. RxMHD Euler–Lagrange equations

The \( \mathbf{u} \) component of Eq. (23) gives, as in Sec. IV C

\[
\rho \mathbf{u} - \rho \mathbf{v} = \nu_1 \frac{\mathbf{B}}{\mu_0} \equiv \rho \mathbf{u}^{(\text{Rx})}_{\Omega}, \tag{55}
\]

where \( \mathbf{u}^{(\text{Rx})}_{\Omega} \) is defined in Eq. (50), with \( \nu \) set to \( \nu_1 \) (also see Figure E). Note that \( \nabla \cdot (\rho \mathbf{u}^{(\text{Rx})}_{\Omega}) = 0 \).

The \( \mathbf{B} \) component gives, using Eq. (47), the modified Beltrami equation,

\[
\nabla \times \mathbf{B} = \mu_1 \mathbf{B} + \nu_1 \mathbf{v} \times \mathbf{u}, \tag{56}
\]

and the final, \( p \), component of Eq. (23) gives the isothermal equation of state,

\[
p = \tau_0 \rho. \tag{57}
\]

From Eq. (19),

\[
\frac{\delta L_{\Omega}^{\text{Rx}}}{\delta \mathbf{r}} = \nabla \cdot \left[ \rho \left( \frac{\partial L_{\Omega}^{\text{Rx}}}{\partial \rho} \right) - \frac{\partial L_{\Omega}^{\text{Rx}}}{\partial \rho} \nabla \rho \right] = \rho \nabla \left( \mathbf{u} \cdot \mathbf{v} - \frac{u^2}{2} - \tau_1 \ln \frac{\rho}{\rho_1} \right), \tag{58}
\]

where \( \rho_1 \) is an arbitrary spatial constant. The momentum equation is, from Eq. (20),

\[
\partial_t (\rho \mathbf{u}) = \nabla \cdot (\rho \mathbf{v} \mathbf{u}) + (\nabla \mathbf{v}) \cdot \mathbf{u} = \frac{\delta L_{\Omega}^{\text{Rx}}}{\delta \mathbf{r}} = \rho \nabla \left( \mathbf{u} \cdot \mathbf{v} - \frac{u^2}{2} - \tau_1 \ln \frac{\rho}{\rho_1} \right), \tag{59}
\]

where \( h_\Omega \) is as defined in Eq. (A11), \( u^2/2 + \tau_1 \ln \rho/\rho_1 \).

Taking the divergence of both sides of Eq. (55) we have \( \nabla \cdot (\rho \mathbf{v} \mathbf{u}) = \nabla \cdot (\rho \mathbf{u} \mathbf{v}) \), so \( \mathbf{u} \) obeys the same continuity equation as \( \mathbf{v} \), Eq. (9). That is,

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{60}
\]

We can condense Eq. (56) by writing it in terms of the vorticity, \( \omega = \nabla \times \mathbf{u} \), giving \( \nabla \times \mathbf{B} = \mu_0 \mathbf{B} + \nu_1 \omega \).

Further physical insight is gained by writing Eq. (56) in terms of electric current \( \mathbf{j} \equiv \nabla \times \mathbf{B}/\mu_0 \),

\[
\mathbf{j} = \frac{\mu_0}{\mu_0} \frac{\mathbf{B}}{\mu_0} + \frac{\nu_1}{\mu_0} \omega, \tag{61}
\]

the first term on the RHS of Eq. (61) being the usual parallel electric current of the linear-force-free (Beltrami) magnetic field model while the second term is a vorticity-driven current [20].

The equation of motion Eq. (59) can be written, using continuity, Eq. (9), and dividing by \( \rho \),

\[
\partial_t \mathbf{u} + \mathbf{v} \times \mathbf{u} = -\nabla h_\Omega. \tag{62}
\]

Taking the curl of both sides gives \( \partial_t \omega + \nabla \times (\omega \times \mathbf{v}) = 0 \). Thus, in steady flow there must exist a potential, \( h_\Omega^{(\omega)} \) say, such that \( \omega \times \mathbf{v} = \nabla h_\Omega^{(\omega)} \), which implies \( \nabla (h_\Omega + h_\Omega^{(\omega)}) = 0 \). That is, any steady RxMHD state has a generalized Bernoulli equation \( h_\Omega + h_\Omega^{(\omega)} = \text{const.} \). Note that \( \mathbf{v} \cdot \nabla h_\Omega^{(\omega)} = \omega \cdot \nabla h_\Omega^{(\omega)} = 0 \), so generically the streamlines of \( \mathbf{v} \) and the vorticity lines of \( \mathbf{u} \) must either lie within invariant tori of these two flows or occupy chaotic regions in which \( h_\Omega^{(\omega)} = \text{const.} \).

Eliminating \( \mathbf{v} \) using Eq. (61), this equation can also be written as

\[
\rho (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \mathbf{j} \times \mathbf{B}. \tag{63}
\]
Thus Eq. (65) is simply the standard IMHD equation of motion, Eq. (12), implying that our RxMHD equation of motion is consistent with Newton’s second law.

Note that Eq. (65), and hence Eq. (59), can also be written in the standard conservation form [see Eq. (51), Eq. (52), and Eq. (24)]

\[ \partial_t (\rho \mathbf{u}) + \nabla \cdot \left( \rho \mathbf{u} \mathbf{u} + \left( \rho + \frac{B^2}{2\mu_0} \right) \mathbf{I} - \frac{\mathbf{B} \mathbf{B}}{\mu_0} \right) = 0. \]  

(64)

C. On the RxMHD equations of motion

In the limit \( \nu_\Omega \to 0 \), Eq. (55) shows that \( \mathbf{v} = \mathbf{u} \) and the Euler–Lagrange equations become exactly the same as those in the original dynamical MRxMHD paper [1] — they describe uncoupled Beltrami magnetic fields and Euler flows. Thus, at least in this case, the phase-space Lagrangian method is equivalent to the configuration-space Lagrangian method used in [1]. The physical implication is that, in this limit, both methods describe relaxation of magnetic field, but not fluid (unless we set \( \mathbf{v} = 0 \), in which case Eq. (55) gives the same \( \mathbf{u} \) as the relaxed equilibrium flow given in Appendix A).

To understand the mathematical nature of RxMHD when \( \nu_\Omega \neq 0 \) within a given domain \( \Omega \), we distinguish between the dynamical variables \( \mathbf{u} \) and \( \rho \), whose time evolution is to be found by solving equations of motion, and passive variables, whose time evolution depends only on the time dependence of \( \partial_\Omega \), like the Lagrange multipliers \( \tau_\Omega, \mu_\Omega \), and \( \nu_\Omega \), or fields whose time dependence is, in addition, driven implicitly by that of the dynamical variables. The main example of the latter class is the magnetic field, a functional of \( \mathbf{u} \) found by solving the inhomogeneous modified Beltrami equation, Eq. (66), to give

\[ \mathbf{B} = \mathbf{B}_\Psi + \nu_\Omega (\text{curl} - \mu_0 \mathbf{l})^{-1} \cdot \nabla \times \mathbf{u}, \]  

(65)

where \( \mathbf{B}_\Psi \) is the unique solution of the homogeneous Beltrami equation, Eq. (56), with \( \nu_\Omega = 0 \), given prescribed magnetic fluxes \( \Psi \), a set of constants of the motion whose number depends on the topological genus of \( \Omega \) [29, 31]. (In the above we assumed \( \mu_\Omega \) is not an eigenvalue of the homogeneous Beltrami equation.)

Likewise, the pressure \( p \) is known in terms of \( \rho \) through Eq. (57), so Eq. (60) and Eq. (63) constitute an infinite dimensional dynamical system of the form \( \partial_t (\rho \mathbf{u}) = \mathbf{f}(\rho, \mathbf{u}) \).

The field \( \mathbf{v} \) allows freedom for the initial conditions for \( \mathbf{u} \) to be specified arbitrarily, through Eq. (55), rather than to be constrained to the fully relaxed, field-aligned flow \( \nu_\Omega \mathbf{B}/\mu_0 \rho \), but it should not be regarded as giving cross-field flow only. For example, Eq. (51) shows \( \mathbf{v} \) with both cross-field and field-aligned flow.

We could in principle display the dynamics in terms of \( \mathbf{v} \), instead of \( \mathbf{u} \), but it is considerably more complicated and difficult to interpret. However, a hybrid approach, where one first specifies \( \mathbf{v} \) and then seeks compatible solutions for \( \rho, \mathbf{u} \), and \( \mathbf{B} \), can restrict attention to interesting classes of solutions.

For instance, if we take \( \mathbf{v} \) to be purely field-aligned in such as way as to counteract the fully relaxed flow, i.e. by setting \( \mathbf{v} = -\nu_\Omega \mathbf{B}/\mu_0 \rho \), then Eq. (55) shows \( \mathbf{u} = 0 \) and we recover the Taylor-relaxed state as a special case.

More interesting are

**Solutions with a continuous symmetry:**

Suppose the boundary of \( \Omega \) possesses a continuous geometric symmetry and seek solutions, equilibrium or possibly dynamical, that maintain this symmetry in time.

For specificity, consider the important case of axisymmetric systems, in which scalar quantities are independent of the toroidal angle \( \phi \), so that, for example, \( \mathbf{e}_0 \cdot \nabla \rho = 0 \), where \( \mathbf{e}_0(\phi) \) is the unit vector \( R \nabla \phi, R \) being the distance from the symmetry, \( Z \) axis.

Similarly, \( \mathbf{e}_0 \cdot \nabla h_\Omega = 0 \), so dotting both sides of Eq. (62) with \( \mathbf{e}_0 \) gives

\[ \partial_t (\mathbf{e}_0 \cdot \mathbf{\omega}) + \mathbf{e}_0 \cdot \mathbf{\omega} \times \mathbf{v} = 0. \]  

(66)

If we now choose

\[ \mathbf{v} = |\mathbf{v}| \mathbf{e}_0 \]  

(67)

then Eq. (66) is satisfied for any \( \mathbf{u} \) solution such that the toroidal component of vorticity, \( \mathbf{e}_0 \cdot \mathbf{\omega}(\mathbf{x}) \), is constant in time throughout \( \Omega \). An example of such a solution is examined in detail in Appendix B.

VI. LINEARIZED DYNAMICS IN THE WKB APPROXIMATION

A. Linearization

As a first step toward understanding the dynamical implications of the RxMHD equations, we linearize around a steady flow \( (\partial_t \to 0) \) solution of Eqs. (55), (60), (61), and (62), in a domain \( \Omega \) with either fixed boundaries or with only low-amplitude, short-wavelength perturbations. Thus, insert in these equations the ansatz \( \mathbf{u} = \mathbf{u}^{(0)} + \alpha \mathbf{u}^{(1)} + O(\alpha^2), \mathbf{v} = \mathbf{v}^{(0)} + \alpha \mathbf{v}^{(1)} + O(\alpha^2) \), and similarly for \( \rho^{(0)} \), where \( \alpha \) is the amplitude expansion parameter (for an example of an equilibrium with nonzero \( \mathbf{u}^{(0)} \) and \( \mathbf{v}^{(0)} \) see Appendix B). For fixed boundaries, or for short-wavelength perturbations, the entropy, helicity and cross-helicity integrals are conserved at \( O(\alpha) \), with therefore no perturbation in the Lagrange multipliers. Thus here we take \( \tau_\Omega, \mu_\Omega, \) and \( \nu_\Omega \) as time-independent constants. Also, from here on we take the superscript \((0)\) to be implicit, e.g. \( \rho \) means \( \rho^{(0)} \) etc.

For short wavelength, high frequency velocity perturbations we use the eikonal ansatz

\[ \mathbf{u}^{(1)} = \tilde{\mathbf{u}}(\mathbf{x}, t) \exp \left( i \varphi(\mathbf{x}, t)/\varepsilon \right), \]  

(68)
with similar notations for linear perturbations of other quantities, $\varepsilon$ being the WKB (local plane-wave) expansion parameter. The instantaneous local values of frequency and wave vector are then defined as $\omega(x,t) \equiv -\partial_t \varphi$ and $\mathbf{k} \equiv \nabla \varphi$. Taking $\varphi$ and equilibrium quantities to vary on $O(1)$ spatial and temporal scales, $\omega$, $\mathbf{k}$, $\partial_t \mathbf{u}$, $\nabla \mathbf{u}$, $\mu_\Omega$, $\nu_\Omega$ etc. are $O(1)$, but $\partial_t \mathbf{u}^{(1)}$, $\nabla \mathbf{u}^{(1)}$ etc. are large, $O(\varepsilon^{-1})$.

B. Short-wavelength IMHD perturbations

Dynamical relaxation theory is physically applicable on slow, quasi-equilibrium timescales $t \gtrsim \tau_{\text{Rx}}$. In the opposite limit of fast dynamics, on times $t \ll \tau_{\text{Rx}}$, of perturbations on relaxed equilibria (which are a subset of ideal equilibria) or slowly evolving states, it is more physically consistent to use IMHD than RxMHD.

The three IMHD plane-wave branches are derived in many text books. We shall follow [32] pp. 172–173], where the local eigenvalue equation for IMHD waves is given in the form

$$\mathbf{D} \cdot \mathbf{\bar{u}} = 0,$$

(69)

where, in the isothermal ($\gamma = 1$) case,

$$\mathbf{D} \equiv \rho \omega^2 \mathbf{I} - p \mathbf{k} \mathbf{k} - \mu_0^{-1} (\mathbf{k} \mathbf{B} - \mathbf{B}_k) \cdot (\mathbf{B}_k - \mathbf{k} \cdot \mathbf{B} \mathbf{I}).$$

(70)

with $\omega'$ denoting the Doppler-shifted frequency in a local frame moving with the fluid, i.e. $\omega' \equiv \omega - \mathbf{k} \cdot \mathbf{u}$. The local dispersion relation may be found by representing $\mathbf{D}$ as a matrix using the co and contravariant bases $\{e_i\}$, $\{e_i^\prime\}$,

$$e_1 \equiv \mathbf{k} = k \mathbf{e}^1 + \mathbf{k} \cdot \mathbf{B} \mathbf{e}^2$$

$$e_2 \equiv \mathbf{B} = k \mathbf{B} \mathbf{e}^1 + B^2 \mathbf{e}^2$$

$$e_3 \equiv |\mathbf{k} \cdot \mathbf{B}|^2 \mathbf{e}^3$$

and setting $\det \mathbf{D} = 0$. The resulting dispersion relation has the three roots

Alfvén waves: $\omega'^2 = k^2 \mathbf{c}_A^2$, (72a)

slow MS waves: $\omega'^2 = -k^2 \left(\mathbf{c}_s^2 + \mathbf{c}_A^2\right) \left(1 - \sqrt{1 - \alpha^2}\right)$

$$\approx k^2 \mathbf{c}_s^2$$

(72b)

fast MS waves: $\omega'^2 = \mathbf{c}_s^2 + \mathbf{c}_A^2 \left(1 + \sqrt{1 - \alpha^2}\right)$

$$\approx k^2 \mathbf{c}_A^2$$

(72c)

where “MS” stands for “magnetosonic,”

$$\alpha^2 \equiv \frac{k^2}{k^2} \frac{\mathbf{c}_s^2 + \mathbf{c}_A^2}{(\mathbf{c}_s^2 + \mathbf{c}_A^2)^2},$$

(73)

$k \equiv \mathbf{k} \cdot \mathbf{B} / B$, $c_s \equiv (p / \rho)^{1/2} = \sqrt{\gamma \rho}$ is the isothermal sound speed, and $c_A \equiv (B^2 / \mu_0 \rho)^{1/2}$ defines the local Alfvén speed. The notation $\approx$ refers to the low-$\beta$ approximation $c_s / c_A \ll 1$. Similar simplifications occur for “flute-like” perturbations, i.e. when $k \parallel / k \ll 1$.

C. Short-wavelength RxMHD perturbations

The linearizations of Eqs. [55], [60], [61], and [62] are

$$\rho \mathbf{v}^{(1)} + \rho^{(1)} \mathbf{v} = \rho \mathbf{u}^{(1)} + \rho^{(1)} \mathbf{u} - \nu_\Omega \frac{\mathbf{B}^{(1)}}{\mu_0}$$

(74)

$$\partial_t \rho^{(1)} + \nabla \cdot (\rho \mathbf{u}^{(1)} + \rho^{(1)} \mathbf{u}) = 0$$

(75)

$$\nabla \times \mathbf{B}^{(1)} = \mu_\Omega \mathbf{B}^{(1)} + \nu_\Omega \omega^{(1)}$$

(76)

$$\partial_t \mathbf{u}^{(1)} + \omega \times \mathbf{v}^{(1)} + \mathbf{w} \times \mathbf{v} = -\nabla h_{\Omega}^{(1)},$$

(77)

where $h_{\Omega}^{(1)} = \mathbf{u} \cdot \mathbf{u}^{(1)} + \nu_\Omega \rho^{(1)} / \rho$.

Using Eq. [68] in Eqs. [74], [75], [76], and [77] gives, to leading order in $\varepsilon$ with the orderings $\mu_\Omega$ and $\nu_\Omega = O(\varepsilon^0)$,

$$\mathbf{v} = \mathbf{u} + (\mathbf{u} - \mathbf{v}) \frac{\rho_0}{\rho} - \nu_\Omega \frac{\mathbf{B}}{\mu_0 \rho}$$

(79)

$$\frac{\rho}{\rho_0} = \frac{k \cdot \mathbf{u}}{\omega'}$$

(80)

$$\mathbf{k} \times \mathbf{B} = \nu_\Omega k \mathbf{x} \cdot \mathbf{u}, \quad \mathbf{k} \cdot \mathbf{B} = 0$$

(81)

$$\omega \mathbf{u} = (k \mathbf{x} \cdot \mathbf{u}) \mathbf{v} + \mathbf{k} \left(\mathbf{u} \cdot \mathbf{u} + \rho \tau_\Omega \frac{\mathbf{B}}{\rho_0}ight)$$

(82)

$$\mathbf{x} \cdot \mathbf{k} \times \mathbf{B} = \nu_\Omega \mathbf{k} \times \mathbf{k} \mathbf{u}, \quad \mathbf{k} \cdot \mathbf{B} = 0$$

where $\mu_\Omega \mathbf{B}$ and $\omega \times \mathbf{v}$ have been dropped as higher order in $\varepsilon$ than other terms in Eq. [81] and Eq. [82], respectively and $\omega'$ is as in Eq. [70].

Gathering all terms in Eq. [82] on the LHS and multiplying by $\rho \omega'$ gives the eigenvalue equation

$$\mathbf{D}_{\text{Rx}} \cdot \mathbf{\bar{u}} = 0,$$

(83)

where, using Eq. [71],

$$\mathbf{D}_{\text{Rx}} \equiv \rho \omega^2 \mathbf{I} - p \mathbf{k} \mathbf{k} - \frac{\nu_\Omega \omega'}{\mu_0} (\mathbf{k} \mathbf{B} - \mathbf{k} \cdot \mathbf{B} \mathbf{I})$$

$$= \left(\rho \omega^2 - k^2 p - \frac{\nu_\Omega \omega'}{\mu_0} \mathbf{k} \cdot \mathbf{B}\right) e_1 e^1$$

$$- \left(p k \cdot \mathbf{B} + \frac{\nu_\Omega \omega'}{\mu_0} \mathbf{B}^2\right) e_2 e^2$$

$$+ \left(\rho \omega^2 + \frac{\nu_\Omega \omega'}{\mu_0} k \cdot \mathbf{B}\right) \left(e_2 e^2 + e_3 e^3\right).$$

(84)

There being only one off-diagonal component, the determinant is the product of the diagonals,

$$\left(\rho \omega^2 - \frac{\nu_\Omega k \cdot \mathbf{B}}{\mu_0} \omega' - k^2 p\right) \left(\rho \omega' + \frac{\nu_\Omega k \cdot \mathbf{B}}{\mu_0}\right)^2 \omega'^2.$$
Setting this determinant to zero gives the dispersion relations
\[ \omega_1' = 0, \quad \omega_2' = -\frac{\nu_1 k \cdot B}{\mu_0 \rho}, \quad (i.e. \omega_2 = k \cdot v), \quad and \]
\[ \omega_{3 \pm}' = \frac{1}{2} \left( \frac{\nu_1 k \cdot B}{\mu_0 \rho} \pm \left[ \left( \frac{\nu_1 k \cdot B}{\mu_0 \rho} \right)^2 + 4k^2 c_s^2 \right]^{1/2} \right) \]  
\[ \tag{85} \]
giving the group velocities
\[ \frac{\partial \omega_1}{\partial k} = u, \quad \frac{\partial \omega_2}{\partial k} = v, \quad and \]
\[ \frac{\partial \omega_{3 \pm}}{\partial k} = \frac{1}{2} \frac{\nu_1 k \cdot B}{\mu_0 \rho} \left\{ 1 \pm \frac{\nu_1 k \cdot B}{\mu_0 \rho} \left[ \left( \frac{\nu_1 k \cdot B}{\mu_0 \rho} \right)^2 + 4k^2 c_s^2 \right]^{-1/2} \right\} \]
\[ + 2k c_s^2 \left[ \left( \frac{\nu_1 k \cdot B}{\mu_0 \rho} \right)^2 + 4k^2 c_s^2 \right]^{-1/2} \]  
\[ \tag{86} \]

In the limit \( \nu_1 = 0 \) this formulation of relaxed Euler flow gives entropy waves advected by the flow and simple sound waves, uncoupled to \( B \), as in \[1\] and \[33\] (as expected, since the cross-helicity constraint has been dropped). Even when \( \nu_1 \neq 0 \) we have completely eliminated all the IMHD waves, replacing them with four branches, two being waves advected with \( u \) and \( B \) and two being hybrids of simple sound waves and entropy waves.

We now demonstrate that the ideal Ohm’s Law is not necessarily respected by these linear waves. We proceed by finding a case where the solvability condition, \( \nabla \times (u \times B) = 0 \), for the electrostatic potential [see Eq. (87) in the next section], is not satisfied. For linear waves, the solvability condition becomes \( k \times (u \times B + \tilde{u} \times B) = 0 \).

From Eq. (81), \( \tilde{B} = \nu_1 (1-k \cdot k/\kappa^2) \cdot \tilde{u} \). Consider for example the branch of waves satisfying \( \omega_1' = 0 \), which, from Eq. (83) and Eq. (84), satisfy \( k \cdot \tilde{u}, i.e. have transverse polarization in velocity (and, as with all waves, in magnetic field). Then \( k \times (u \times B + \tilde{u} \times B) = k \cdot (B - \nu_1 u) \cdot \tilde{u} \). This cannot vanish for all \( k \) unless \( B = \nu_1 u \), which is not in general true.

This indicates either that we need to invoke turbulent e.m.f.’s during dynamical RxMHD evolution, or to implement an ideal-Ohm constraint as discussed in Sec. VII.

VII. IDEAL OHM’S LAW AND CROSS-FIELD FLOW

The IMHD-equilibrium Consistency Principle requires that Eq. (11) (with \( \partial_t B = 0 \)) should be satisfied in a relaxed equilibrium. Thus we take the unqualified term relaxed MHD equilibrium to imply that the “ideal Ohm’s Law,” \( \mathbf{E} + u \times B = 0 \), is satisfied with, in general, a nonzero \( \mathbf{E} \).

Given a particular frame (the Lab frame) in which an MHD equilibrium appears as a steady state, so that \( \nabla \times \mathbf{E} = -\partial_t \mathbf{B} = 0 \), one can always choose a gauge in which \( \mathbf{E} = -\nabla \Phi, \Phi \) being a single-valued electrostatic potential.

[One might object that, taking \( \Omega \) to be a simple torus for simplicity, \( \Phi \) should also include a secular “loop voltage” term, \( -E_{\text{ext}} \phi/2\pi, \) induced by an external time-dependent poloidal magnetic flux linking \( \Omega \) (\( \phi \) being the geometric toroidal angle and \( E_{\text{ext}} \) a constant throughout \( \Omega \)). Noting that \( u \times B \cdot d\ell = 0 \) for any line element \( d\ell \) aligned with \( B \), we see that the line integral around any closed field line within \( \Omega \) vanishes, \( -\oint u \times B \cdot d\ell = 0 \), while \( \oint E \cdot d\ell = N E_{\text{ext}}, N \) being the number of toroidal turns before the field line closes on itself. Equating the two shows that \( E_{\text{ext}} = 0 \). As (possibly long) closed field lines almost always exist, this shows that \( \Phi \) is indeed generically single valued. Physically, this is a consequence of the assumption of no gaps in the perfectly conducting interfaces, so that the linking fluxes are always conserved no matter what the genus of \( \Omega \).]

Thus, in the Lab frame, the equilibrium ideal Ohm’s law is electrostatic,
\[ \nabla \Phi = u \times B, \]  
\[ \tag{87} \]
implying \( u = u_\parallel + u_\perp B/B, \) i.e. \( u \) is the vector sum of the \( \mathbf{E} \times \mathbf{B}, \) or cross-field flow \( u_{\perp} = -\nabla \Phi \times \mathbf{B}/B^2 \) and a parallel flow, \( u_\parallel, \) not determined by \( \Phi \). Equation (87) also implies \( \mathbf{B} \cdot \nabla \Phi = 0 \) and \( u \cdot \nabla \Phi = 0, \) i.e. that \( \Phi \) is constant on both magnetic field and flow lines. As a consequence, if \( \Phi \) has smoothly nested level surfaces, then both \( u \) and \( B \) lie in the local tangent plane at each point on each isopotential surface.

Finn and Antonsen [5] after Eq. (29)] conclude from this constancy of \( \Phi \) along a field line that “if the turbulent relaxation has ergodic field lines throughout the plasma volume,” then \( \nabla \Phi = 0 \), which implies from Eq. (87) that \( u \times B = 0 \). We shall call such field-aligned steady flows fully relaxed equilibria. In Appendix A we find that stationary points of the MHD energy, subject only to microscopic mass and macroscopic entropy, magnetic helicity and cross helicity constraints, are indeed fully relaxed equilibria.

However the Consistency Principle applies only to the final, non-turbulent, state of relaxation, where it seems highly unlikely that field lines could ever fill the whole of \( \Omega \) ergodically, though in fully three dimensional plasmas with islands this may be a good model for chaotic separation subregions. In most cases the class of ergodically relaxed equilibria seems unnecessarily restrictive. Indeed, it does not include many equilibria of physical interest, in particular, tokamaks with strong toroidal flow.

Thus Finn and Antonsen [5 Sec. III] go on to construct an axisymmetric equilibrium with cross-field flow by adding the additional constraint of conservation of angular momentum in the relaxed energy principle. As their equilibrium satisfies Eq. (87) it satisfies our IMHD Consistency Principle so it definitely qualifies as a relaxed MHD equilibrium. In Appendix B we show that the rotating equilibrium of [5] can be found within our
RxMHD formalism without the need to invoke angular momentum conservation as a constraint.

In contrast to the axisymmetric equilibrium, we showed for the time-dependent waves in Sec. [VTC] that the solvability condition $\nabla \times (u \times B) = 0$ for the potential $\Phi$ in the ideal electrostatic Ohm’s law Eq. (87) is not in general satisfied dynamically. Nevertheless, as the applicability of our relaxed dynamics is limited to very long timescales, it may be reasonable to assume that the electrostatic approximation $E = -\nabla \Phi$ still holds during dynamical evolution. Then we could build in the ideal electrostatic approximation $E$ as a holonomic constraint by re-dynamical evolution. Then we could build in the ideal Ohm’s law Eq. (87) as initial conditions for dynamical perturbations break consistency with the electrostatic approximation $E$. To implement the relaxation prescription in Sec. [II.C] the global invariants listed in Sec. [II.B] are enforced using Lagrange multipliers to give the constrained energy functional

$$H_{\Omega}^{Rx}[\rho, u, p, A] \equiv H_{\Omega}[\rho, u, p, A] - \tau_\Omega S_\Omega - \mu_\Omega K_\Omega - \nu_\Omega K^X_\Omega [u, A],$$

where $\tau_\Omega$, $\mu_\Omega$, and $\nu_\Omega$ are Lagrange multipliers to enforce conservation, respectively, of entropy, magnetic helicity, and cross helicity in $\Omega$.

The energy variation is now, assuming the support of each variation is localized within $\Omega$,

$$\delta H_{\Omega}^{Rx} = \int_\Omega \left( \delta u \cdot \frac{\delta H_{\Omega}}{\delta u} + \delta A \cdot \frac{\delta H_{\Omega}}{\delta A} + \delta p \frac{\delta H_{\Omega}}{\delta p} + \rho \xi \cdot \nabla \frac{\delta H_{\Omega}}{\delta \rho} \right) dV$$

(A2)

The functional derivatives are

$$\frac{\delta H_{\Omega}^{Rx}}{\delta u} = \rho u - \nu_\Omega \frac{B}{\mu_0},$$

(A3)

$$\frac{\delta H_{\Omega}^{Rx}}{\delta A} = \frac{1}{\mu_0} (\nabla \times B - \mu_\Omega B - \nu_\Omega \nabla \times u),$$

(A4)

$$\frac{\delta H_{\Omega}^{Rx}}{\delta p} = \frac{1}{\gamma - 1} \left( 1 - \frac{\rho}{p} \right),$$

(A5)

$$\frac{\delta H_{\Omega}^{Rx}}{\delta \rho} = \frac{u^2}{2} - \frac{\tau_\Omega}{\gamma - 1} \left[ \ln \left( \frac{\kappa p}{\rho^\gamma} \right) - \gamma \right].$$

(A6)

For $\delta H_{\Omega}^{Rx}$ to be zero for independent variations $\delta u$, $\delta A$, $\delta p$, and $\xi$ the four Euler–Lagrange equations

$$\rho u = \nu_\Omega \frac{B}{\mu_0},$$

(A7)

$$\nabla \times B = \mu_\Omega B + \nu_\Omega \nabla \times u,$$

(A8)

$$p = \tau_\Omega \rho,$$

(A9)

and $\nabla h_{\Omega} = 0$ (A10)

must be satisfied. In the above we have denoted $\delta H_{\Omega}^{Rx}/\delta \rho$ by $h_{\Omega}$, the Bernoulli head, defined by

$$h_{\Omega} \equiv \frac{u^2}{2} - \frac{\tau_\Omega}{\gamma - 1} \left[ \ln \left( \frac{\kappa p}{\rho^\gamma} \right) - \gamma \right] + \text{const}$$

(A11)

where the second line absorbs the arbitrary constant in the definition into $\rho_\Omega$. Equation (A11) shows $h_{\Omega}$ is constant throughout $\Omega$ in relaxed steady flow. Choosing this

VIII. CONCLUSION

We have shown that a Phase Space Lagrangian approach allows the derivation of a dynamical formalism, Relaxed Magnetohydrodynamics (RxMHD), for slow motions of relaxed plasmas. This provides a theoretical basis for dynamical extensions of present Multiregion RxMHD (MRxMHD) equilibrium computations with flow [21]. An axisymmetric steady-flow solution is found that meets the consistency test of also being an ideal-MHD equilibrium with flow, but application of the theory to nonaxisymmetric systems is left to future work. A demonstration that dynamical perturbations break consistency with the ideal Ohm’s Law is briefly discussed, leaving further analysis to future work.

APPENDICES

Appendix A: Energy principle for fully relaxed equilibria with flow

In this section we extend Taylor’s energy minimization principle by including flow and thermal kinetic energies and keeping Taylor’s magnetic helicity constraint, adding an entropy constraint, and adding the cross helicity constraint in order to construct a relaxed state with finite pressure and a steady flow, in a similar way to Finn and Antonsen [5]. We take as energy the Hamiltonian $H_{uc}$, Eq. (45).
constant to be zero, gives us an expression for the physical observable \( \rho \), found from Eq. (A11) to be given by

\[
\rho = \rho_\Omega \exp \left( -\frac{u^2}{2\tau_\Omega} \right). \tag{A12}
\]

In the limit \( u = 0 \), \( \rho \) and hence \( p \) are constant within \( \Omega \), as was assumed in previous MRxMHD work, e.g. in developing the Stepped Pressure Equilibrium Code SPEC [17]. Note that Eq. (A10) can also be written, using Eq. (A7) and Eq. (A8) and the identity \( \nabla (u^2/2) = 2u \nabla u - (\nabla u) \times u \),

\[
\rho u \cdot \nabla u = -\nabla p + (\nabla \times B - \mu_0 B) \times \left( \nu_\Omega \frac{B}{\mu_0} \right) \tag{A13}
\]

where \( j = \nabla \times B / \mu_0 \) by Ampère’s Law (pre Maxwell).

1. Equilibrium Consistency checks

Below we show Eqs. (A7–A10) are IMHD-equilibrium compatible, i.e. consistent with Eqs. (9–12), \( \mathbf{v} \) there being replaced by \( \mathbf{u} \) and terms in \( \partial_t \) being set to zero:

1. Observing that Eq. (A8) is compatible with \( \nabla \cdot \mathbf{B} = 0 \), take the divergence of both sides of Eq. (A7), to find \( \nabla \cdot (\rho \mathbf{u}) = 0 \), thus showing \( \mathbf{u} \) satisfies the continuity equation Eq. (9), at least in the steady flow case.

2. From Eq. (A7) we have \( \mathbf{u} \times \mathbf{B} = 0 \), so the static ideal Ohm’s law, Eq. (A7) is trivially satisfied under the ergodic relaxation condition \( \nabla \Phi = 0 \).

3. The static limit of the ideal equation of motion, Eq. (12), is just Eq. (A13).

4. The final consistency test must be more nuanced, as we have altered the ideal thermodynamics by relaxing the temperature (proportional to \( \tau \)) throughout \( \Omega \). This models the rapid transport of heat in the highly chaotic magnetic field line flow that is required to justify relaxation theory physically, but it has the consequence that \( p/\rho \gamma \) is not constant microscopically (though \( p V_\Omega^\gamma \) is still constant in time if \( \rho \) is spatially constant). This switch to a local isothermal equation of state can be modeled by taking \( \gamma \to 1 \) so the static version of Eq. (10) is of the same form as Eq. (6), i.e. \( \nabla \cdot (\rho \mathbf{u}) = \tau_\Omega \nabla \cdot (\rho \mathbf{u}) = 0 \), which is satisfied because \( \nabla \cdot (\rho \mathbf{u}) = 0 \) was verified in the first of our consistency tests.

(We use the notation \( \Box \) to indicate that an equation has passed a specified validation test.)

Interestingly, Eqs. (A7–A10) would also be compatible with steady, relaxed Euler flow [25], if we could suppose \( \mathbf{B} \) is a harmonic “vacuum” field, so that \( \nabla \times \mathbf{B} = 0 \). Then \( \mathbf{u} \) would obey the nonlinear Beltrami equation \( \nabla \times \mathbf{u} = -(\mu_0 \mu_\Omega / \nu_\Omega^2) \rho \mathbf{u} \) if we could also satisfy \( \nabla \times (\rho \mathbf{u}) = 0 \).

Appendix B: Axisymmetric equilibria with both field-aligned and cross-field flow

The first test of our new phase-space action formulation is whether it can generalize the rather restricted class of flows in the relaxed equilibria derived from an energy principle in Sec. A, in which \( \mathbf{u} \) had to be parallel to \( \mathbf{B} \). As seen from Eq. (55), the new field \( \mathbf{v} \) does indeed provide the possibility of flows with a component perpendicular to \( \mathbf{B} \), even in steady flows, in which it is also appropriate to check for consistency with IMHD. We term such equilibria “semi-relaxed” as they do not obey the ergodic relaxation condition \( \mathbf{u} \times \mathbf{B} = 0 \) discussed in Sec. VII.

Previous authors [6, 35] have inserted toroidal equilibrium flow “by hand” by constraining the \( \mathbf{Z} \) component of the angular momentum in their energy stationarization (or, equivalently, entropy stationarization [5]). However, this is physically consistent only if the system is rotationally symmetric about the \( \mathbf{Z} \) axis. In our more general approach, conserved physical quantities should arise naturally from Noether’s theorem if there is a continuous symmetry, rather than by constraints.

1. Rigid rotation \( \mathbf{v} \)

Nevertheless the previous work on axisymmetric equilibria does provide physically consistent solutions that can be used to validate our equations, so we now test whether we can reproduce the results of Finn and An- tonsen (FA) [5] on relaxed axisymmetric equilibria.

In the following discussion we use the usual cylindrical coordinates \( R, \phi, \mathbf{Z} \), with \( R \) the distance from the vertical \( \mathbf{Z} \) axis and \( \phi \) the toroidal angle, \( \mathbf{e}_{R,\phi,Z} \) being the corresponding right-handed orthonormal set of basis vectors.

Specifically, we seek to show that choosing \( \mathbf{v} \) to be a rigid rotation around the \( \mathbf{Z} \)-axis with angular frequency \( \omega_\Omega \),

\[
\mathbf{v} = R \omega_\Omega \mathbf{e}_\phi(\phi) = R^2 \omega_\Omega \nabla \phi, \tag{B1}
\]

throughout an axisymmetric toroidal domain \( \Omega \), will lead to a time-independent solution of the RxMHD equations [5, 23], that is consistent with FA’s Eqs. (26), (27), and (29), [3]. Transcribed into our notation the FA equations
are
\[ u = \frac{\nu_0}{\mu_0 \rho} \mathbf{B} + R \varpi \Omega e_\phi(\phi) \]
\[ = u^{\text{Rx}} + R \varpi \Omega e_\phi(\phi) \quad \text{(B2)} \]
\[ \nabla \times \left[ (1 - M_A^{\text{Rx}}) \mathbf{B} \right] = \mu_0 \mathbf{B} + 2\nu_0 \varpi \Omega e_\tau \]
\[ u^{\text{Rx}} = \frac{R^2 \varpi^2}{2} + \tau_\Omega \ln \frac{\rho}{\rho_0} = 0 \quad \text{(B4)} \]

where \( u^{\text{Rx}} = \nu_0 \Omega / \mu_0 \rho \) is the fully relaxed flow speed defined in Eq. (50), and
\[ M_A = \frac{u^{\text{Rx}}}{c_A} \quad \text{(B5)} \]
is the \textit{parallel Alfvén Mach number}, \( c_A \equiv (B^2 / \mu_0 \rho)^{1/2} \)
being the local Alfvén speed as in Sec. VIB.

With the choice Eq. (B5), we note first that Eq. (B2) and Eq. (B5) are identical. □

Also, multiplying both sides of Eq. (B2) by \( \rho \) and taking divergences gives
\[ \nabla \cdot (\rho \mathbf{u}) = \varpi \Omega R^2 \rho \nabla \cdot \nabla \phi + \varpi \Omega \nabla \phi \cdot \nabla (R^2 \rho) = 0 \quad \text{(B6)} \]

thus verifying consistency with the continuity equation, Eq. (60). □ [In deriving the above identities we have used \( \nabla \phi = e_\phi / R, \nabla \phi = 0 \), and \( e_\phi \nabla (R^2 \rho) = 0 \), as \( \rho = \rho(R, Z) \) by axisymmetry.]

The vorticity of the toroidal flow is a constant vector in the \( Z \) direction, \( \nabla \times \mathbf{v} = 2 \varpi \Omega e_\tau \). We can also calculate the total vorticity, \( \omega = \nabla \times \mathbf{u} \), by taking the curl of both sides of Eq. (B2).
\[ \omega = \nabla \times \left( \frac{\nu_0}{\mu_0 \rho} \mathbf{B} \right) + 2 \varpi \Omega e_\tau \]
\[ = \left( \frac{\nu_0}{\mu_0 \rho} \right) \nabla \times \mathbf{B} + \nabla \left( \frac{\nu_0}{\mu_0 \rho} \right) \times \mathbf{B} + 2 \varpi \Omega e_\tau . \quad \text{(B7)} \]

With this identification the first line of Eq. (B3) can now, with a little rearranging and multiplying both sides by \( \mu_0 \), be recognized as Eq. (50). \( \nabla \times \mathbf{B} = \mu_0 \mathbf{B} + \nu_0 \omega \). □

It is convenient at this point to introduce a poloidal-toroidal decomposition, i.e. we use two basis vectors spanning the poloidal, \( R, Z \) half-plane at each toroidal angle \( \phi \), and a third basis vector in the orthogonal toroidal direction \( e_\phi(\phi) \). The most general representation of \( \mathbf{B} \) (and similarly other divergence-free fields like \( \nabla \times \mathbf{B}, \rho \mathbf{u}, \) and \( \omega \)) is then
\[ \mathbf{B} = \nabla \phi \times \nabla \psi + F \nabla \psi , \quad \text{(B8)} \]

the first term on the RHS being the \textit{poloidal magnetic field}, \( \mathbf{B}_{\text{pol}} \), and the second is the \textit{toroidal magnetic field}, \( \mathbf{B}_{\text{tor}} \).

Unlike in the Grad–Shafranov representation for flawless MHD equilibria (see e.g. [32] pp. 177–178), the toroidal field strength function \( F \) is not simply a function of \( \psi \)—at this point it is an unspecified function of \( R \) and \( Z \).

Taking the curl of both sides gives, in the expected toroidal-poloidal form,
\[ \nabla \times \mathbf{B} = - \nabla \phi \times \nabla F + \nabla \times (\nabla \phi \times \nabla \psi) \]
\[ = - \nabla \phi \times \nabla F + (\nabla^2 \psi \nabla \phi)
\[ + (\nabla \nabla \phi) \cdot \nabla \psi - (\nabla \phi) \cdot \nabla \nabla \psi \]
\[ = - \nabla \phi \times \nabla F + \Delta^* \psi \nabla \phi \quad \text{(B9)} \]

where, using the identity \( \nabla \psi \cdot \nabla \phi - \nabla \phi \cdot \nabla \psi = -(2/R) \nabla R \nabla \psi \nabla \phi \) [32] pp. 177–178), \( \Delta^* \psi \equiv R^2 \nabla^2 (\nabla \psi / R^2) \).

Multiplying both sides of Eq. (B2) by \( \rho \) and using Eq. (B8) we find
\[ \rho \mathbf{u} = \nabla \phi \times \nabla \left( \frac{\nu_0}{\mu_0} \psi \right) + \left( \frac{\nu_0}{\mu_0} F + \varpi \Omega R^2 \rho \right) \nabla \phi , \quad \text{(B10)} \]

which again is in the expected poloidal-toroidal representation. (Note that the poloidal flow is driven solely by cross helicity—setting \( \nu_0 = 0 \) gives a purely toroidal, rigid-rotational flow.)

Substituting Eq. (B8) and Eq. (B9) in Eq. (56) yields a toroidal-poloidal form for the vorticity,
\[ \omega = \frac{1}{\nu_0} \nabla \times \mathbf{B} - \frac{\mu_0}{\nu_0} \mathbf{B} \]
\[ = - \nabla \phi \times \nabla \left( \frac{\nu_0}{\mu_0} \psi \right) + \left( \frac{\Delta^* \psi - \mu_0 F}{\nu_0} \right) \nabla \phi , \quad \text{(B11)} \]

whereas taking the curl of both sides of Eq. (B2) gives an alternative expression for the vorticity,
\[ \omega = \frac{\nu_0}{\mu_0 \rho} \left( \nabla \times \mathbf{B} - \frac{\nabla \rho \times \mathbf{B}}{\rho} \right) + \nabla \times \mathbf{v} \]
\[ = - \nabla \phi \times \nabla \left( \frac{\nu_0}{\mu_0} \frac{F + \varpi \Omega \rho^2}{\rho} \right)
\[ + \nu_0 \frac{F + \varpi \Omega R^2}{\mu_0 \rho} \nabla \phi \quad \text{(B12)} \]

Substituting Eq. (B1) and Eq. (B12) in the force-balance equation Eq. (62) (with \( \partial_t \mathbf{u} = 0 \)) we get
\[ \nabla \left[ \frac{\partial \mathbf{u}}{\partial_t} - \varpi \Omega \left( \frac{\nu_0}{\mu_0} \frac{F + \varpi \Omega R^2}{\rho} \right) \right] = 0 \quad \text{(B13)} \]

thus generalizing the Bernoulli relation Eq. (A10) to include rigid rotation. Choosing the arbitrary constant \( \rho_\Omega \) appropriately, Eq. (B13) implies
\[ \frac{u^2}{2} + \tau_\Omega \ln \frac{\rho}{\rho_\Omega} - \frac{\nu_0 \varpi \Omega F}{\mu_0} - \varpi \Omega R^2 = 0 \quad \text{(B14)} \]

Then, expanding \( u^2 / 2 \) using Eq. (B12), Eq. (B14) is readily seen to agree with the generalized Bernoulli equation, Eq. (B1)}. □
Another form for the generalized Bernoulli equation may be had by decomposing Eq. (B2) into poloidal and toroidal components, \( u_\omega \equiv \nu_\Omega e_\omega \times \nabla \psi / R \mu_\Omega \rho \) and \( u_R \equiv \nu_\Omega F / R \mu_\Omega \rho + R \omega_\Omega \), respectively. Then \( u^2 = u_\omega^2 + u_R^2 \) and \( \nu_\Omega \omega_\Omega F / \mu_\Omega \rho + \omega_\Omega^2 R^2 = \omega_\Omega R \mu_\Omega \). Thus Eq. (B14) can also be written

\[
\tau_\Omega \ln \frac{\rho}{\rho_\Omega} + \frac{u_\omega^2 + u_R^2}{2} - \omega_\Omega R \mu_\Omega \psi = 0, \tag{B15}
\]

as in the IMHD result of McClements and Hole \cite{35}, Eq. (20)], in the case of isothermal magnetic surfaces (identifying their \( \mathcal{T}/m_i \) with our \( \tau_\Omega \)). This reference discusses the conditions under which the Grad–Shafranov equation, to be derived in the next subsection, changes from elliptic to hyperbolic.

Finally, crossing both sides of Eq. (B2) with \( B \) and using Eq. (B8) we find

\[
\mathbf{u} \times \mathbf{B} = \omega_\Omega R^2 \nabla \phi \times \mathbf{B} = -\omega_\Omega \nabla \psi = \nabla \phi,
\]

where \( \phi = -\omega_\Omega \psi \). Comparing with Eq. (B7) we see that, although we did not impose the ideal Ohm’s law as a constraint, it is nevertheless satisfied in this case. \( \square \)

2. Grad–Shafranov Equation

Comparing the two expressions for \( \omega \) given in Eq. (B11) and Eq. (B12), and choosing the arbitrary baseline for \( \psi \) appropriately, we find two relations

\[
(1 - M_{Rx}^2) F = \nu_\Omega \omega_\Omega R^2 - \mu_\Omega \psi, \tag{B17}
\]

\[
(1 - M_{Rx}^2) \Delta^* \psi = \mu_\Omega F - \frac{\nu_\Omega^2}{\mu_\Omega} \nabla \rho \cdot \nabla \psi, \tag{B18}
\]

the first giving \( F \) in terms of \( \psi \) and \( \rho \) and the second being what we shall call a generalized Grad–Shafranov–Beltrami (GSB) equation for \( \psi \). (A similar system was analyzed in \cite{57}, in the limit \( \nu_\Omega \to 0, \omega_\Omega \to 0 \) and rippled slab geometry.)

To solve Eq. (B18) we need to express \( \rho \) in terms of \( \psi \) and \( R \), which can be done by using the generalized Bernoulli equation Eq. (B5), to give an implicit equation for \( \rho \),

\[
\rho = \rho_\Omega \exp \left[ -\frac{1}{\tau_\Omega} \left( \frac{\nu_\Omega^2}{\mu_\Omega} \frac{B^2}{2 \omega_\Omega^2} - \frac{\omega_\Omega^2 R^2}{2} \right) \right], \tag{B19}
\]

where \( B^2 = (|\nabla \psi|^2 + F^2)/R^2 \).

There is an obvious singularity in Eq. (B18) when the fully-relaxed-flow Alfvén Mach number \( M_{Rx}^2 = 1 \), but plasma flows in toroidal confinement experiments are typically much less than the Alfvén speed, defined using the total magnetic field in the numerator, so it is unlikely that this singularity would be encountered in practice.

However, while not immediately obvious, the dependence of \( \rho \) on \( |\nabla \psi| \), through \( B^2 \) in Eq. (B19), can cause Eq. (B18) to become hyperbolic at much lower plasma flow speeds than \( c_A \), as first found by Lovelace et al. \cite{35} and analyzed in the context of modern low-aspect-ratio tokamaks by McClements and Hole \cite{36}. This is because the \( \nabla \rho \) in Eq. (B18) contributes a factor, \( \nabla \nabla \psi \), having second-order derivatives of \( \psi \) that must be included along with the second-order derivatives in \( \Delta \psi \) to evaluate whether Eq. (B18) is elliptic or hyperbolic. (This is based on the sign of the discriminant \( D = A_{\Omega Z}^2 - 4 A_{\Omega R} A_{\Omega Z} \), with \( A_{\Omega R}, A_{\Omega Z}, A_{\Omega Z} \) the coefficients of \( \partial^2 \psi / \partial^2 R, \partial^2 \psi / \partial R \partial \Omega, \partial^2 \psi / \partial \Omega ^2 \), respectively: If \( D = 0 \) the equation is parabolic, while \( D < 0 \) implies ellipticity and \( D > 0 \) hyperbolicity.)

It seems unlikely that the MHD equilibria studied in this Appendix would be minima of the Hamiltonian \( H_{\Omega}^{Rx} \) in ranges where such transitions occur. If so, while valid MHD equilibria, could not properly be called relaxed. However, determining this is outside of the scope of this paper as we examine only first variations.

Appendix C: Interface Euler–Lagrange equation

To calculate the boundary contribution to the variation of the phase-space action \( \delta S_{\Omega}^{Rx} \equiv \int dL_{\Omega}^{Rx} \), we need to take into account surface terms from integrations by parts omitted in Sec. VB because the support of \( \xi \) was taken not to include \( \partial \Omega \). These integrations by parts do not give any surface terms involving \( \delta \mathbf{u} \), but we need to include the surface term from the variation of the boundary itself, \( \int_{\partial \Omega} d{\bf S} \cdot \xi L_{\Omega}^{Rx} \), where \( L_{\Omega}^{Rx} \) is given in Eq. (54).

Using Eqs. (53), (6), and (A6), in Eq. (53), we now calculate the residual, boundary action variation

\[
\delta S_{\Omega}^{Rx} = \int_{\Omega} \int_{\Omega} dV \left[ \partial_t (\rho \mathbf{u} \cdot \xi) + \nabla \cdot (\rho \mathbf{v} \mathbf{u} - \rho \mathbf{v} \mathbf{u} \cdot \mathbf{u}) \right]
\]

\[
+ \int_{\Omega} \int_{\Omega} \int_{\Omega} dV \nabla \cdot \left( \frac{\rho \Omega}{\gamma - 1} \left[ \ln \left( \frac{\kappa \rho}{\rho^2} \right) - \gamma \right] \right)
\]

\[
- \frac{1}{\mu_\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} dV \nabla \cdot \left[ \delta \mathbf{A} \times \left( \mathbf{B} - \frac{\mu_\Omega}{2} \mathbf{A} - \nu_\Omega \mathbf{u} \right) \right]
\]

\[
+ \int_{\Omega} \int_{\Omega} d{\bf S} \cdot L_{\Omega}^{Rx}.
\] (C1)

To commute \( \int_{\Omega} dV \) and \( \partial_t \) in the first term of the top line (which arose from \( \rho \mathbf{u} \cdot \mathbf{v} \)) we write

\[
\int_{\Omega} \int_{\Omega} dV \partial_t (\rho \mathbf{u} \cdot \xi) = \int_{\Omega} \int_{\Omega} dV \Pi_{\Omega}(x,t) \partial_t (\rho \mathbf{u} \cdot \xi)
\]

\[
= \int_{\Omega} \int_{\Omega} dV \{ \partial_t [\Pi_{\Omega}(x,t) \rho \mathbf{u} \cdot \xi] - \rho \mathbf{u} \cdot \mathbf{u} \partial_t \Pi_{\Omega}(x,t) \}
\]

\[
= - \int_{\Omega} \int_{\Omega} dV \rho \mathbf{u} \cdot \mathbf{u} \partial_t \Pi_{\Omega}(x,t),
\]

where the spatial integration range on the right is now arbitrarily large and the \( \Pi_{\Omega} \) is a unit top-hat, (a.k.a. rectangle or boxcar) function with support on \( \Omega \). Restricting
to variations with support on \( t \) not including the initial and final times, in the last line we have dropped end-point terms from the complete time derivative term \( \partial_\xi \).

Assuming, as an example, \( \Omega \) to be an annular toroid we introduce a right-handed, but generally non-orthonormal, curvilinear coordinate system \( \{ \theta, \zeta, s \} \) where \( \theta \) and \( \zeta \) are poloidal and toroidal angles, respectively, and \( s_\Omega(x,t) \) is a radial coordinate whose level surfaces are tori, with the surface \( s_\Omega = 0 \) the inner torus of \( \partial \Omega \) and \( s = 1 \) the outer one. Then \( \Pi_\Omega(x,t) = \Theta(s_\Omega)\Theta(1 - s_\Omega) \), \( \Theta(\cdot) \) being the Heaviside step function. Thus we write

\[
\int dt \int dV \partial_t (\rho \mathbf{u} \cdot \delta_\xi) = -\int dt \int d\theta d\zeta ds_\Omega J_\Omega \rho \mathbf{u} \cdot \delta_\xi \left[ \delta_\xi(s_\Omega) - \delta_\xi(s_\Omega - 1) \right] ,
\]

where \( J_\Omega = \sqrt{\rho} \) is the Jacobian of the transformation \( \mathbf{x} \rightarrow \{ \theta, \zeta, s_\Omega \} \). The coefficient of the \( \delta \) functions, \( \partial_\xi s_\Omega \), can be evaluated by observing that, as the elements of \( \partial \Omega \) are advected with velocity \( \mathbf{v} \), so are the level surfaces of \( s_\Omega \) defining \( \partial \Omega \). That is,

\[
(\partial_\xi + \mathbf{v} \cdot \nabla)s_\Omega = 0 , \quad (C2)
\]

so \( \partial_\xi s_\Omega = -\mathbf{v} \cdot \nabla s_\Omega \equiv \mathbf{v} \cdot \mathbf{e}^{s_\Omega} \), where \( \mathbf{e}^{s_\Omega} \) is one of the three basis vectors \( \{ \mathbf{e}_\theta, \mathbf{e}_\zeta, \mathbf{e}_s \} \equiv \{ \nabla \theta, \nabla \zeta, \nabla s_\Omega \} \). Thus the final result for the first term of Eq. (C1) is

\[
\int dt \int dV \partial_t (\rho \mathbf{u} \cdot \delta_\xi) = -\int dt \int d\theta d\zeta ds_\Omega J_\Omega \mathbf{e}^{s_\Omega} \cdot \mathbf{v} \rho \mathbf{u} \cdot \delta_\xi \left[ \delta_\xi(s_\Omega) - \delta_\xi(s_\Omega - 1) \right] = -\int dt \int \Omega dS \cdot \mathbf{v} \rho \mathbf{u} \cdot \delta_\xi , \quad (C3)
\]

where we used the differential geometry identity

\[
dS \equiv n dS \equiv \text{sgn}(n \cdot \nabla s_\Omega) \mathbf{e}_s \times \mathbf{e}_\zeta \partial_\zeta \mathbf{e}_s = \text{sgn}(n \cdot \nabla s_\Omega) \mathbf{J} \mathbf{e}^{s_\Omega} \partial_\zeta \mathbf{e}_s , \quad (C4)
\]

\( n \) being the outward normal on \( \partial \Omega \) so the sign function \( \text{sgn} \) gives \( - \) at \( s_\Omega = 0 \) and \( + \) at \( s_\Omega = 1 \).

Using Gauss’ theorem to cast the volume integral over \( \nabla \cdot (\rho \mathbf{v} \mathbf{u} \cdot \delta_\xi) \) as a surface integral, and comparing with the result in Eq. (C3), we see that the first two terms in \( \delta S_\Omega^{Rx} \), Eq. (C1) (which arose from \( \rho \mathbf{u} \cdot \delta_\xi \) cancel). Thus, inserting \( \delta S_\Omega^{Rx} \) explicitly, we now have

\[
\delta S_\Omega^{Rx} = \int dt \int \Omega dS \cdot \left\{ -\rho \mathbf{v} \cdot \mathbf{u} + \rho \mathbf{v} \cdot \mathbf{e}^{s_\Omega} \cdot \mathbf{v} \rho \mathbf{u} \cdot \delta_\xi \right\} \left[ \delta_\xi(s_\Omega) - \delta_\xi(s_\Omega - 1) \right] = -\int dt \int \Omega dS \cdot \mathbf{v} \rho \mathbf{u} \cdot \delta_\xi , \quad (C5)
\]

where we used the boundary condition \( n \cdot \mathbf{B} = 0 \), Eq. (1), to eliminate some terms.

\[
\delta S_\Omega^{Rx} = \int dt \int \partial_\xi dS \cdot \left\{ \mathbf{v} \cdot (\mathbf{v} \mathbf{u} - \mathbf{B}) + \nabla \delta \chi \cdot (\mathbf{B} - \frac{\mu_0}{2} \mathbf{A} - \nu_\Omega \mathbf{u}) \right\} + \frac{1}{\mu_0} \left[ \delta_\chi \mathbf{v} \cdot (\mathbf{B} - \frac{\mu_0}{2} \mathbf{A} - \nu_\Omega \mathbf{u}) \right] , \quad (C6)
\]

\[
\int dt \int t dS \cdot \mathbf{f} = -\int dt \int t dS \cdot \mathbf{g} \cdot \nabla \mathbf{f} , \quad (C6)
\]

the Euler–Lagrange equation Eq. (56) and the tangential boundary condition Eq. (1). (While we have held the Lagrange multipliers fixed in the calculation, in principle they also vary during boundary variations to maintain the constancy of their respective constraint functionals. However, the exact constraints do not contribute to the final result Eq. (C5) as the Lagrange multipliers have dropped out.)

In Multiregion RxMHD (MRxMHD) each point on \( \partial \Omega \) is also on the boundary of a neighboring region, \( \Omega' \) say, with unit normal \( \mathbf{n}' = -\mathbf{n} \). Thus the total action variation from \( \delta_\xi \) localized around such a point is \( \delta S_\Omega^{Rx} + \delta S_\Omega^{Rxx} \)

\[
\int dt \int \Omega dS \cdot \left\{ \frac{p}{2} + \frac{B^2}{2\mu_0} \right\} = 0 . \quad (C7)
\]

Interestingly, the Galilean-invariant pressure-balance equation Eq. (C7), which couples neighboring relaxation regions, contains no time derivatives. Yet it is this equation that imparts the inertia of the plasma fluid to interface dynamics, through the effect of the internal RxMHD dynamics within \( \Omega \) determining changes in \( p \) and \( B^2 \) at the boundary.
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