Quantum Jet Theory, Observer Dependence, and Multi-dimensional Virasoro algebra

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Abstract

We review some key features of Quantum Jet Theory: observer dependence, multi-dimensional Virasoro algebra, and the prediction that spacetime has four dimensions.

1 Introduction

Every experiment is an interaction between a system and an observer, and the result depends on the physical properties of both. In particular, a real observation depends on the observer’s mass $M$ and charge $e$. QFT predictions are independent of these quantities, which means that some tacit assumptions have been made: the observer’s charge is small (so the observer does not disturb the system) and his mass is large (so he follows a well-defined classical trajectory in spacetime; in particular, the observer’s position and velocity at equal times commute). This assumption is unproblematic for all interactions except gravity, where charge and mass are the same; heavy mass equals inert mass.

The problem with quantum gravity is thus that an obese observer will collapse into a black hole of his own, whereas a skinny observer can not know where he is (he can only know where he was). From this perspective, assuming that their observations are identical becomes absurd. This is the simple physical reason why QFT must be incompatible with gravity. Alas, it also suggests a route to remedy the problem: make the observer into a physical entity with quantum dynamics. The predictions of such a theory...
will depend on $M$ and $e$, and it must reduce to QFT in the limit $G = 0$, $M \to \infty$, $e \to 0$, and to general relativity in the limit $\hbar = 0$, $M \to 0$.

To describe observer-dependent physics we need observer-dependent mathematics. Fortunately, such mathematics is available in the form of Taylor expansions, or jets\footnote{Locally, a $p$-jet can be uniquely coordinatized by a Taylor expansion truncated at order $p$.}. Namely, a Taylor series does not only depend on the function being expanded, but also on the choice of expansion point, i.e. the observer’s position. This motivates the name Quantum Jet Theory (QJT).

Despite the simplicity of the argument above, this is not historically how QJT was discovered. Instead, the story started over twenty years ago when I made two observations:

- The symmetry of general relativity is the algebra of spacetime diffeomorphisms.
- CFT tells us that all interesting (local, unitary) quantum representations of the diffeomorphism algebra on the circle are anomalous, i.e. representations of the Virasoro algebra.

Putting these facts next to each other, it became obvious that the correct symmetry of quantum gravity must be some multi-dimensional generalization of the Virasoro algebra. This algebra was subsequently discovered\footnote{Locally, a $p$-jet can be uniquely coordinatized by a Taylor expansion truncated at order $p$.} \cite{3, 7}, and its off-shell representations were understood in \cite{4}. Unlike the classical representations, which act on tensor fields, the quantum representations do not act on fields, but on spacetime histories of tensor-valued $p$-jets. This is the link between the multi-dimensional Virasoro algebra, QJT, and observer dependence.

An extension of the diffeomorphism algebra is a gauge anomaly, which according to standard wisdom is inconsistent. However, this is not necessarily true. A gauge anomaly turns a classical gauge symmetry into a quantum global symmetry, which acts on the Hilbert space rather than reducing it. The quantum theory may or may not be inconsistent, depending on whether the action of the anomalous gauge symmetry is unitary or not. A toy example of a consistent theory with an anomalous gauge symmetry is the subcritical free string, which according to the no-ghost theorem can be quantized with a ghost-free spectrum, despite its conformal anomaly\footnote{Locally, a $p$-jet can be uniquely coordinatized by a Taylor expansion truncated at order $p$.} \cite{2}.

In this review we follow the historical path to QJT. We start with the multi-dimensional Virasoro algebra in section\footnote{Locally, a $p$-jet can be uniquely coordinatized by a Taylor expansion truncated at order $p$.} \ref{sec:2} and construct its lowest-energy representations in section\footnote{Locally, a $p$-jet can be uniquely coordinatized by a Taylor expansion truncated at order $p$.} \ref{sec:3}. In section\footnote{Locally, a $p$-jet can be uniquely coordinatized by a Taylor expansion truncated at order $p$.} \ref{sec:4} we treat QJT as a regularization, and show how the $p$-jet phase space becomes infinite-dimensional.
because the equations of motion are undefined on the “skin”. Finally we indicate in section 5 how the divergent parts of anomalies can be cancelled. For a realistic choice of field content this uniquely singles out four spacetime dimensions.

2 Multi-dimensional Virasoro algebra

The Virasoro algebra, 

\[ [L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}, \]  

(1)

where \( \delta_m \) is the Kronecker delta, is the unique central extension of the algebra \( \text{vect}(1) \) of vector fields (or infinitesimal diffeomorphisms) in one dimension. We want to find analogous extensions of \( \text{vect}(d) \), the algebra of vector fields in \( d \) dimensions. Taken at face value, the prospects to succeed appear bleak, due to two no-go theorems:

- \( \text{vect}(d) \) does not possess any central extension when \( d > 1 \).
- In QFT, there are no diff anomalies in four dimensions [1].

However, no theorem is stronger than its axioms. If we relax some assumptions above, the no-go theorems can be evaded and a multi-dimensional Virasoro algebra can be constructed. The crucial assumptions are encoded in the keywords “central” and “in QFT”.

- \( \text{vect}(d) \) does not possess any central extension when \( d > 1 \), but it does possess non-central extensions which nevertheless reduce to the usual, central, Virasoro extension when \( d = 1 \). In general, we construct extensions by the module of closed \( (d-1) \)-forms. When \( d = 1 \), a closed zero-form is a constant function, and the extension is central. When \( d > 1 \), a closed \( (d-1) \)-form transforms in a nontrivial way under diffeomorphisms, but nontrivial Lie algebra extensions still exist.

- The construction of diff anomalies in four dimensions is not possible within the framework of OFT, but it can be done in QJT.

To make the connection to the ordinary Virasoro algebra very explicit, it is instructive to write down the brackets in a Fourier basis. It is clear that
the Virasoro algebra (1) can be written in the form
\[
\begin{align*}
[L_m, L_n] &= (n - m)L_{m+n} + cm^2nS_{m+n}, \\
[L_m, S_n] &= (n + m)S_{m+n}, \\
[S_m, S_n] &= 0,
\end{align*}
\]
\(mS_m = 0\). (2)

It is easy to see that this form of the Virasoro algebra is equivalent to (1), apart from a linear cocycle that has been absorbed into a redefinition of \(L_0\). The new formulation (2) immediately generalizes to \(d\) dimensions. The generators \(L_\mu(m) = -i \exp(\im \nu^\nu x^\nu) \partial_\mu\) and \(S^\mu(m)\) satisfy the relations
\[
\begin{align*}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n) \\
&\quad + (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu)m_\rho S^\rho(m + n), \\
[L_\mu(m), S_\nu(n)] &= n_\mu S_\nu(m + n) + \delta_\mu^n m_\rho S^\rho(m + n), \\
[S^\mu(m), S^\nu(n)] &= 0, \\
m_\mu S^\mu(m) &= 0.
\end{align*}
\]

This is an extension of \(\text{vect}(d)\) by the abelian ideal with basis \(S^\mu(m)\). Geometrically, we can think of \(L_\mu(m)\) as a vector field and \(S^\mu(m) = \epsilon^{\mu\nu_2...\nu_d} S_{\nu_2...\nu_d}(m)\) as a dual one-form (and \(S_{\nu_2...\nu_d}(m)\) as an \((d - 1)\)-form); the last condition expresses closedness.

The cocycle proportional to \(c_1\) was discovered by Rao and Moody [7], and the one proportional to \(c_2\) by myself [3]. There is also a similar multi-dimensional generalization of affine Kac-Moody algebras. The multi-dimensional Virasoro and affine algebras are often referred to as “Toroidal Lie algebras” in the mathematics literature.

3 Lowest-energy representations and QJT

In the previous section we constructed Virasoro-like extensions of \(\text{vect}(d)\), thus evading the first no-go theorem. This leaves two major problems: how to build representations, and how to avoid the fact that there are no diff anomalies in 4D in QFT. The solution to both problems is the same: the representations act on trajectories in jet space, to which the no-go theorem does not apply.

Let us compare with the ordinary Virasoro algebra, whose representation theory may be viewed as QFT. Building Fock representations consists of three steps:
1. Start from a classical representation, which acts on primary fields, i.e. scalar densities.

2. Add canonical momenta.

3. Normal order.

Naïvely, one could expect to build a QFT representation of vect$(d)$ in the same manner, except that we could choose any tensor density as the starting point. However, this strategy fails, for several reasons:

1. To define the Fock vacuum we must single out a privileged time or energy direction, which is uncomfortable when studying spacetime diffeomorphisms.

2. Normal ordering of bilinears always leads to a central extension, but the Virasoro extension is noncentral when $d > 1$.

3. Irremovable infinitities arise even after normal ordering, making the approach useless.

These problems have proved unsurmountable, and lead to the conclusion that it is impossible to construct nontrivial lowest-energy representations of vect$(d)$ within the framework of QFT.

The resolution of this paradox appeared in the seminal work by Rao and Moody [7]. In the physics-flavored language of [4], their construction can be described as follows:

1. Start from a classical realization acting on trajectories in $p$-jet space, instead of a representation acting on fields.

2. Add canonical momenta for the jets.

3. Normal order.

The problem with infinities is avoided because a $p$-jet only has finitely many degrees of freedom, and a trajectory in $p$-jet space thus consists of finitely many functions of a single variable. This is precisely the situation where normal ordering works without producing infinities.

Locally, a $p$-jet is essentially the same thing as a Taylor expansion; a $p$-jet has a unique representative which is a polynomial of order at most $p$, which is the Taylor series truncated at order $p$. A $p$-truncated Taylor series around the point $q = (q^\mu)$ takes the form

$$
\phi(x) = \sum_{|m| \leq p} \frac{1}{m!} \phi_m(x - q)^m.
$$

(4)
This formula is written in a form which may appear one-dimensional, but with standard multi-index notation it makes sense also for \(d > 1\). E.g., 
\[ m = (m_0, m_1, ..., m_{d-1}) \]

is a multi-index with length \(|m| = \sum_{\mu=0}^{d-1} m_{\mu}\). A basis for the space of \(p\)-jets consists of all Taylor coefficients \(\phi_{m}\) of order \(|m| \leq p\), together with the expansion point \(q\). Unlike \(q\), the point \(x\) is a c-number which labels the field components \(\phi(x)\).

To illustrate how the Virasoro-like extensions of \(\text{vect}(d)\) arise, it suffices to consider \(-1\)-jets, whose basis consists of the expansion point only. \(q^\mu\) and its conjugate momentum \(p^\mu\) satisfy the canonical commutation relations

\[ [q^\mu, p^\nu] = i\delta^\mu_\nu. \] (5)

It follows immediately from the definition that the diffeomorphism generators on the torus admit the realization

\[ L_\mu(m) = e^{im\cdot q}p_\mu. \] (6)

This equation defines an embedding of \(\text{vect}(d)\) into the universal enveloping algebra of (5), and hence a representation of \(\text{vect}(d)\) on the corresponding Fock space.

Since the Heisenberg algebra (5) is finite-dimensional, no extension can arise from normal ordering. However, an extension does arise with a slight modification of the construction. We consider one-dimensional trajectories in the space of \(-1\)-jets, instead of just the \(-1\)-jets themselves. For technical simplicity, we consider closed trajectories, i.e. circles, even though it may be physically dubious to introduce closed time-like loops. The space of trajectories has the basis \(q^\mu(t), p^\mu(t), t \in S^1\), and satisfy the canonical commutation relations

\[ [q^\mu(t), p^\nu(t')] = i\delta^\mu_\nu\delta(t - t'). \] (7)

The embedding of \(\text{vect}(d)\) into this algebra is completely analogous to (6):

\[ L_\mu(m) = \int dt \ e^{im\cdot q(t)}p_\mu(t). \] (8)

The quantization step consists of constructing lowest-energy representations of the Heisenberg algebra (7). Since \(q^\mu(t)\) and \(p_\mu(t)\) depend on a variable on the circle, they can be divided into positive and negative frequency modes, which we denote by \(q^\mu_>(t)\), \(q^\mu_<\) and \(p^\mu_>(t)\), \(p^\mu_<\), respectively; where the zero modes are assigned is not important. The Fock vacuum is defined by

\[ q^\mu_<(t)|0\rangle = p^\mu_<|0\rangle = 0. \] (9)

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The diffeomorphism generators \( \mathcal{S} \) must be normal ordered to act in a well-defined manner on the Fock vacuum. We define

\[
L_\mu(m) = \int dt \, :e^{im \cdot q(t)}p_\mu(t): = \int dt \left( e^{im \cdot q(t)}p^\mu_\mu(t) + p^\mu_\mu(t)e^{im \cdot q(t)} \right).
\]

By direct calculation, we find that the normal-ordered generators \( \mathcal{G} \) satisfy the Virasoro-like extension \( \mathcal{S} \) of \( \text{vect}(d) \), where

\[
S_\mu(m) = \frac{1}{2\pi} \int dt \, \dot{q}^\mu(t)e^{im \cdot q(t)},
\]

and the parameters are \( c_1 = 2d, c_2 = 0 \).

Two observations are in order. First, the condition \( m_\mu S_\mu(m) = 0 \) is equivalent to demanding that integrals over total derivatives vanish, i.e.

\[
\int dt \frac{d}{dt} F(q(t)) \equiv 0, \quad \text{for every function } F.
\]

This condition is automatically satisfied because the integral runs over a circle. Second, in one dimension there is only one circle; we can therefore use the circle coordinate \( q \) as the independent variable rather than \( t \), and the integral becomes

\[
S(m) = \frac{1}{2\pi} \int dq \, e^{im \cdot q} = \delta_m.
\]

Hence \( \mathcal{G} \) reduces to the central Kronecker-delta extension of the ordinary Virasoro algebra when \( d = 1 \).

More general representations, based on \( p \)-jets of tensor fields instead of \( -1 \)-jets, were constructed in \cite{4}. The parameters \( c_1 \) and \( c_2 \), as well as other “abelian charges”, depend on the truncation order \( p \) as well as a \( gl(d) \) representation, which labels the type of tensor field that we start from. In contrast, the operator \( S_\mu(m) \) is always realized in the same way \( \mathcal{S} \).

This clearly shows why this kind of anomaly can never arise within the framework of QFT. The extension is a functional of the expansion points \( q^\mu(t) \), which are never introduced in QFT. Hence it is impossible to even write down the Virasoro-like anomalies within a QFT framework. But it is also clear what the remedy is: work with the jets instead of the fields, because a jet, or Taylor series, automatically carries information about the expansion point.

7
4 QJT as a regularization

This section closely follows the treatment in [6]. To extract the physical content, it is useful to decompose spacetime into space and time; boldface quantities \((x, q, m)\) are used to denote spatial components. For definiteness, we consider a free scalar field with mass \(\omega\). To the field \(\phi(x, t)\) correspond jet data \(\phi_m(t), q(t)\), related through its Taylor series:

\[
\phi(x, t) = \sum_m \frac{1}{m!} \phi_m(t)(x - q(t))^m. \tag{14}
\]

Here \(m = (m_1, ..., m_{d-1})\) is a spatial multi-index of length \(|m| = \sum_{j=1}^{d-1} m_j\), and all \(m_j \geq 0\).

Jets come equipped with a natural regularization: pass from \(\infty\)-jets to \(p\)-jets, i.e. truncate all Taylor series at order \(p\). This means that the sum in \((14)\) only runs over \(m\) of length \(|m| \leq p\). The correct equations of motion in \(p\)-jet space take the form \(\mathcal{E}_m(t) = 0\), where

\[
\mathcal{E}_m = \begin{cases} 
\phi_m + \omega^2 \phi_m, & |m| \leq p - 2 \\
\text{undefined}, & |m| = p - 1, p 
\end{cases} \tag{15}
\]

Here \(m + n_j = (m_1, ..., m_j + n, ..., m_{d-1})\) and

\[
\phi_m(t), \forall t \in \mathbb{R}, |m| = p - 1, p. \tag{17}
\]

The key observation is that \(\mathcal{E}_m\) is not defined for the modes with \(|m| = p - 1, p\), i.e. the “skin” of the \(p\)-jet. The expression used for \(|m| \leq p - 2\) (the “body”) would involve the components \(\phi_{m+2\mu}\) with \(|m| > p\), which do not belong to \(p\)-jet space (they do belong to \((p+2)\)-jet space). The full \(p\)-jet phase space, i.e. the space of \(p\)-jet histories which solve the equations of motion \((15)\), is hence spanned by

The \(p\)-jet phase space is infinite-dimensional because the equations of motion are unable to determine some histories in terms of data living at \(t = 0\). This is the origin of the new gauge and diff anomalies. The “body” consists of only finitely many degrees of freedom, and can hence not contribute to
anomalies, but the “skin” is infinite-dimensional and normal ordering leads to anomalies in theories with a gauge symmetry. In fact, we can even give the free scalar field a gauge symmetry under reparametrizations of the observer’s trajectory, by not identifying \( x^0 = q^0(t) \) with the parameter \( t \). The algebra of reparametrizations then becomes a Virasoro algebra. The contribution from a single bosonic function of \( t \) to the central charge is \( c = 2 \). The number of different multi-indices with \(|m| \leq r\) in \( d - 1\) space dimensions is \( \binom{d+r-1}{d-1} \). The total central charge for the “skin” is thus

\[
c_{\text{Tot}} = 2 \binom{d + p - 1}{d - 1} - 2 \binom{d + p - 3}{d - 1}.
\]  

(18)

5 Finite anomalies

The passage to \( p \)-jets is a regularization, and in the end we want to eliminate it by taking the limit \( p \to \infty \). However, the expression for \( c_{\text{Tot}} \) in (18) diverges in this limit, provided that \( d \geq 3 \). Whereas finite gauge and diff anomalies may be consistent, an infinite anomaly is certainly a sign of inconsistency. Fortunately, in [5] I discovered a way to avoid this problem.

Consider a theory with fermions with central charge \( c_F \), with bosons with central charge \(-c_B\), and with gauge fields with central charge \(-c_G\). The total central charge is given by

\[
c_{\text{Tot}} = (c_F - c_B) \binom{d + p - 1}{d - 1} - c_F \binom{d + p - 2}{d - 1}
\]

\[
+ (c_B + c_G) \binom{d + p - 3}{d - 1} - c_G \binom{d + p - 4}{d - 1}.
\]

(19)

If we choose the field content such that

\[
c_F = 3c,
\]

\[
c_B = 2c,
\]

\[
c_G = c,
\]

(20)

for some \( c > 0 \), the total central charge (19) reduces to

\[
c_{\text{Tot}} = c \binom{d + p - 4}{d - 4}.
\]

(21)

In particular, \( c_{\text{Tot}} = c \) is independent of \( p \) if \( d = 4 \), and the infinite parts of the reparametrization anomaly have been cancelled.

Hence, if we choose a theory with a natural field content (fermions with first-order equations of motion, bosons with second-order equations of motion, and with gauge symmetries which are not reducible), we can cancel
the infinite part (but not the finite part) of the reparametrization anomaly iff spacetime has four dimensions. The same is true for the infinite parts of gauge and diff anomalies [5]. This strongly suggests that spacetime has four dimensions.

Unfortunately, making a more detailed identification of the field content with experimentally existing matter leads to problems which remain open.

6 Conclusion

We have reviewed key features of Quantum Jet Theory: observer dependence and the multi-dimensional Virasoro algebra. The appearance of new diff anomalies shows that QJT is substantially different from QFT, which is positive given that QFT is incompatible with gravity. Virasoro-like diff anomalies may well be consistent, and are in fact a necessary ingredient in any local, non-holographic, quantum theory of gravity.

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