Internal waves in a compressible two-layer atmospheric model: The Hamiltonian description

V. P. Ruban
Landau Institute for Theoretical Physics, 2 Kosygin Street, 119334 Moscow, Russia

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Slow flows of an ideal compressible fluid (gas) in the gravity field in the presence of two isentropic layers are considered, with a small difference of specific entropy between them. Assuming irrotational flows in each layer [that is \( \vec{v}_{1,2} = \nabla \phi_{1,2} \)], and neglecting acoustic degrees of freedom by means of the conditions \( \text{div}(\vec{\rho}(z)\nabla \phi_{1,2}) \approx 0 \), where \( \vec{\rho}(z) \) is a mean equilibrium density, we derive equations of motion for the interface in terms of the boundary shape \( z = \eta(x,y,t) \) and the difference of the two boundary values of the velocity potentials: \( \psi(x,y,t) = \psi_1 - \psi_2 \). A Hamiltonian structure of the obtained equations is proved, which is determined by the Lagrangian of the form \( L = \int \vec{\rho}(\eta) \eta \psi dxdy - \mathcal{H}(\eta, \psi) \). The idealized system under consideration is the most simple theoretical model for studying internal waves in a sharply stratified atmosphere, where the decrease of equilibrium gas density with the altitude due to compressibility is essentially taken into account. Forplanar flows, a generalization is made to the case when in each layer there is a constant potential vorticity. Investigated in more details is the system with a model density profile \( \bar{\rho}(z) \), and neglecting acoustic degrees of freedom by means of the condition \( \text{div}(\vec{\rho}(z)\nabla \phi_{1,2}) \approx 0 \), for which the Hamiltonian \( \mathcal{H}(\eta, \psi) \) can be expressed explicitly. A long-wave regime is considered, and an approximate weakly nonlinear equation of the form \( u_0 + auu_x - b[-\partial_x^2 + \alpha^2]^{1/2}u_x = 0 \) (known as Smith’s equation) is derived for evolution of a unidirectional wave.

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I. INTRODUCTION

Internal waves constitute an important part in the dynamics of such complex systems as are Atmosphere and Ocean (see, e.g., Refs. [11–17], and references therein). These waves are known to propagate at the background of some inhomogeneity of internal properties of the fluid (gas). In the ocean the main role is played by salt concentration and temperature, while in the atmosphere the most important factors are specific entropy and air moisture. Non-uniformity of shear flows should be mentioned as well. The internal wave dynamics depends essentially on the condition if the stratification is smooth enough in a wide range of altitudes, or the change of internal properties takes place sharply near some surface. The last case, as a rule, is more convenient for a theoretical study, since the spatial dimensionality of the problem is reduced. In many works therefore simplified atmospheric and oceanic models are considered, where the system consists of several layers, with homogeneous fluid within each layer, and then the dynamics of interfaces between the layers is investigated (see, e.g., [11, 17], and references therein). To the best author’s knowledge, in all previous finite-layer models the fluid was assumed to be incompressible, even when the atmosphere was modeled. In the present work, perhaps for the first time, an essentially compressible two-layer atmospheric model is considered. Here it is assumed that there is a sharp boundary \( z = \eta(x,y,t) \) separating two regions of potential flow, with a constant value of specific entropy in each layer. The relative difference of that values is small, and it ensures the slowness of typical flow velocities compared to the local speed of sound. Accordingly, the acoustic degrees of freedom can be effectively “filtered” by the conditions \( \nabla \cdot (\vec{\rho}(z)\vec{v}) = 0 \) in each layer (where \( \vec{\rho}(z) \) is the equilibrium density), instead dealing with the full continuity equation \( \rho_t + \nabla \cdot (\rho \vec{v}) = 0 \). This idea to eliminate relatively fast sound waves was used previously to obtain simplified equations describing convection and internal waves in a continuously stratified compressible fluid [18–20], and also slow isentropic vortex flows in a compressible fluid placed in a static external field [21, 22]. The distinction of the present model is that the potentiality condition in each layer, together with the equation \( \nabla \cdot (\vec{\rho}(z)\nabla \phi) = 0 \) for the velocity potential, allow us to represent equations of motion in terms of the interface shape \( z = \eta(x,y,t) \) itself and the difference of the two boundary values of the velocity potential. Moreover, we succeeded in proving a Hamiltonian structure of the obtained equations, which is a generalization of the canonical structure discovered by V. E. Zakharov in the dynamics of waves at the free surface of an ideal incompressible fluid [23, 24]. In the two-dimensional (2D) case, it is possible to consider in the framework of the two-layer model also shear flows with piecewise constant potential vorticity. The Hamiltonian theory is naturally modified in that case. As applications of the developed theory, we obtained the dispersion relation for internal waves in the two-layer compressible atmosphere, and we derived a nonlinear equation which is intermediate between the Korteweg-de Vries and the Benjamin-Ono equations [26, 27]. This equation determines slow evolution of a unidirectional wave and it takes into account the dispersive correction of a special form, taking place in the model. Previously, a similar equation was derived in a

*Electronic address: ruban@itp.ac.ru
II. APPROXIMATE EQUATIONS AND THEIR HAMILTONIAN STRUCTURE

Let us assume that in the equilibrium state the first layer of gas occupies the region \(0 < z < h\) and has the density \(\bar{\rho}_1(z)\), while the second layer occupies the region \(z > h\) and has the density \(\bar{\rho}_2(z)\) [for simplicity, we have supposed that the lower rigid boundary — “the Earth surface” — is flat, but the more general case of nontrivial topography can be considered in analogous way]. Of course, functions \(\bar{\rho}_1(z)\) and \(\bar{\rho}_2(z)\) cannot be arbitrary, since in fact they are specified by the hydrostatic balance condition together with an equation of state of the gas (see below). For the further derivation of approximate equations describing potential flows in this system which are slow compared with a local speed of sound \(c\), the following condition is very important: 

\[
(\bar{\rho}_1 - \bar{\rho}_2) \ll \bar{\rho}(z) = (\bar{\rho}_1 + \bar{\rho}_2)/2.
\]

The starting-point equations for potential isotropic gas flow in each layer are the non-stationary Bernoulli equation and the continuity equation,

\[
\begin{align*}
\partial_t \varphi + \frac{(\nabla \varphi)^2}{2} & = -w(\rho) - gz + \text{const}, \quad \text{(1)} \\
\partial_t \rho + \nabla \cdot (\rho \nabla \varphi) & = 0, \quad \text{(2)}
\end{align*}
\]

where \(\varphi(\mathbf{r}, t)\) is the potential for the velocity field \(\mathbf{v}\), satisfying the condition of zero normal derivative at the rigid boundary, that is \(\partial_n \varphi(x, y, 0) = 0\); \(\rho(\mathbf{r}, t)\) is the density, \(w(\rho)\) is the specific enthalpy which is defined by the formula

\[
w(\rho) = w_{1,2}(\rho) = \int_0^\rho \frac{dp_{1,2}(\rho)}{\rho}.
\]

Here \(p = p_{1,2}(\rho)\) is the pressure as a function of density in each layer, with \(p_2(\rho) - p_1(\rho) \ll [p_2(\rho) + p_1(\rho)]/2\). In the equilibrium state the velocity potential \(\varphi = 0\), the enthalpy \(w_{1,2}(\bar{\rho}_{1,2}(z))\) is \(\text{const}_{1,2} \approx gz\), and the pressure

\[
\bar{p}_{1,2}(z) = p_0 - g \int_h^z \bar{p}_{1,2}(z)dz.
\]

Let us consider slow flows when \(p_{1,2} = \bar{p}_{1,2}(z) + \tilde{p}_{1,2}\) and \(w_{1,2} \approx \text{const}_{1,2} - gz + \tilde{p}_{1,2}/\rho(z)\), where \(\tilde{p}_{1,2}\) are relatively small corrections to the pressure field due to fluid flow. The equations of slow motion in the main order in \(\nu/c\) take the form

\[
\begin{align*}
\partial_t \varphi_{1,2} + \frac{(\nabla \varphi_{1,2})^2}{2} + \bar{p}_{1,2}(\rho) & = 0, \quad \text{(5)} \\
\nabla \cdot (\bar{p}(z) \nabla \varphi_{1,2}) & = 0. \quad \text{(6)}
\end{align*}
\]

It is the neglect of time derivative \(\partial_t \rho\) in the continuity equation that allows us to exclude from the consideration acoustic degrees of freedom and retain only “soft” modes as the internal waves which are conditioned by the relatively small difference of the two equilibrium density profiles. Compressibility of the medium in this model is manifested in form that a volume of each fluid element at slow motion is effectively “adapted” to the equilibrium density \(\bar{\rho}(z)\), expanding when going up and compressing when going down [since \(\bar{\rho}(z) < 0\)].

Let the shape of disturbed interface be given by equation \(z = \eta(x, t)\), where \(\mathbf{x} = (x, y)\) is the radius-vector in the horizontal plane, and let the boundary values of the velocity potentials be \(\psi_{1,2}(\mathbf{x}, t) = \varphi_{1,2}(\mathbf{x}, \eta(x, y, t), t)\). At the free interface, the normal component \(V_n\) of the velocity field should be continuous, as well as the pressure. It is also clear that a local speed of boundary motion in the normal direction [for definiteness, the normal vector \(\mathbf{n}\) is directed from the first layer to the second one] is equal to \(V_n\). From these considerations, two kinematic conditions and one dynamic condition are derived, which determine evolution of the system:

\[
\begin{align*}
\frac{\partial \varphi_1}{\partial n} & \bigg|_{z=\eta} = \frac{\partial \varphi_2}{\partial n} \bigg|_{z=\eta} = V_n, \quad \text{(7)} \\
\eta_t & = V_n \sqrt{1 + (\nabla \eta)^2}, \quad \text{(8)} \\
\left\{ \bar{p}(\varphi_{1,t} - \varphi_{2,t}) + \frac{\bar{p}}{2} (|\nabla \varphi_1|^2 - |\nabla \varphi_2|^2) \right\} \bigg|_{z=\eta} & + g \int_0^\eta [\bar{\rho}_1(z) - \bar{\rho}_2(z)]dz = 0. \quad \text{(9)}
\end{align*}
\]

It follows form Eq.\(\text{(7)}\) that \(\psi_1\) and \(\psi_2\) are related to each other by a linear integral dependence. Therefore, if we fix the difference \(\psi(x, t) \equiv \psi_1 - \psi_2\), then each potential will be fully determined. Taking into account the equalities

\[
\frac{\partial \psi_{1,2}}{\partial t} = \left[ \frac{\partial \varphi_{1,2}}{\partial t} + (\partial \varphi_{1,2}/\partial z) \eta_t \right] \bigg|_{z=\eta}, \quad \text{(10)}
\]

it is easy to check that the equations of motion for the two main functions \(\eta(x, t)\) and \(\psi(x, t)\) possess the Hamiltonian structure:

\[
\tilde{\rho}(\eta) \eta_t = \frac{\delta H}{\delta \psi}, \quad -\tilde{\rho}(\eta) \psi_t = \frac{\delta H}{\delta \eta}. \quad \text{(11)}
\]
with the corresponding Lagrangian

\[ L = \int \left( \bar{\rho}(\eta) \eta \psi d^{2}x - \mathcal{H}\{\eta, \psi\} \right). \]  

(12)

The Hamiltonian functional \( \mathcal{H}\{\eta, \psi\} \) is given by the following expression:

\[
\mathcal{H} = \int d^{2}x \int_{0}^{\eta(x)} \bar{\rho}(z) \left( \nabla \psi \cdot \nabla \psi \right) dz \\
+ \int d^{2}x \int_{\eta(x)}^{+\infty} \bar{\rho}(z) \left( \nabla \psi \cdot \nabla \psi \right) dz + g \int W(\eta) d^{2}x \\
= \frac{1}{2} \int \bar{\rho}(\eta) \psi V_{n} d^{2}x + g \int W(\eta) d^{2}x, \tag{13}
\]

where

\[ W'(\eta) = \int_{\eta}^{\eta(x)} \left[ \bar{\rho}_{1}(z) - \bar{\rho}_{2}(z) \right] dz, \tag{14}\]

that is the Hamiltonian \( \mathcal{H} \) is the sum of the kinetic energy and an effective potential energy. Let us prove the above statements.

Indeed, the variation \( \delta \psi \) entails some variations \( \delta \varphi_{1,2} \), and consequently — a variation of the kinetic energy. The corresponding variation of the Hamiltonian after integration by parts is determined by a surface integral along the interface \( z = \eta(x) \), and it takes the form

\[
\delta \mathcal{H}_{\delta \psi} = \int S \bar{\rho}(\nabla \varphi_{1} \cdot \mathbf{n}) \delta \psi_{1} dS - \int S \bar{\rho}(\nabla \varphi_{2} \cdot \mathbf{n}) \delta \psi_{2} dS \\
= \int \bar{\rho}(\eta) V_{n} \sqrt{1 + (\nabla \eta)^{2}} \delta \psi d^{2}x. \tag{15}
\]

From here we have \( \delta \mathcal{H}/\delta \psi = \bar{\rho}(\eta) V_{n} \sqrt{1 + (\nabla \eta)^{2}} \) and, making comparison with Eq. (5), we prove the first equation from Eqs. (11). Calculation of variational derivative \( \delta \mathcal{H}/\delta \eta \) is slightly more complicated, because when the integration domain is varied, we have to ensure that after the interface variation the difference \( \psi_{1} - \psi_{2} \) takes at the new boundary the same value \( \psi(x) \) which was before the variation at the old boundary. It is easy to understand that due to the above requirement the values of the potentials at the place of the old boundary are changed after variation \( \delta \eta \) by small quantities \( \delta \psi_{1,2} = -\delta \eta(\partial_{z} \varphi_{1,2}) \big|_{z=\eta} \). Accordingly, variation of the kinetic energy in this case consists of two contributions. The first contribution comes from the change of integration domain:

\[
\delta \mathcal{K}^{(1)}_{\delta \eta} = \int \frac{\bar{\rho}}{2} \left( (\nabla \varphi_{1})^{2} - (\nabla \varphi_{2})^{2} \right) \bigg|_{z=\eta} \delta \eta d^{2}x. \tag{16}
\]

The second contribution is related to the changes of the potentials \( \varphi_{1,2} \) in non-varied domains due to variations of their boundary values by the quantities \( \delta \psi_{1,2}^{\text{old}} \). It is easy to understand that this contribution is equal to

\[
\delta \mathcal{K}^{(2)}_{\delta \eta} = \int \left( \frac{\delta \mathcal{H}}{\delta \psi} \right) \left( \delta \psi_{1}^{\text{old}} - \delta \psi_{2}^{\text{old}} \right) d^{2}x \\
= \int \bar{\rho} V_{n} \sqrt{1 + (\nabla \eta)^{2}} \left( \partial_{z} \varphi_{2} - \partial_{z} \varphi_{1} \right) \bigg|_{z=\eta} \delta \eta d^{2}x. \tag{17}
\]

Taking into account also variation of the effective potential energy, we obtain as the result

\[
\frac{\delta \mathcal{H}}{\delta \eta} = \frac{\bar{\rho}}{2} \left( (\nabla \varphi_{1})^{2} - (\nabla \varphi_{2})^{2} \right) \bigg|_{z=\eta} + g \int_{\eta}^{\eta(x)} \left[ \bar{\rho}_{1}(z) - \bar{\rho}_{2}(z) \right] dz \\
- \bar{\rho} V_{n} \sqrt{1 + (\nabla \eta)^{2}} \left( \partial_{z} \varphi_{1} - \partial_{z} \varphi_{2} \right) \bigg|_{z=\eta}. \tag{18}
\]

Looking at Eqs. (10) and (11), we obtain from here the second equation of the Eqs. (11).

The Hamiltonian nature of the system under consideration in principle allows us to apply to it the standard set of methods [23]. However, a technical difficulty is that the kinetic energy is not expressed directly but through solutions of the partial derivative equation with non-constant coefficients, and in domains with a curved boundary \( z = \eta(x) \). Let us nevertheless suppose that particular solutions of Eq. (10) are known in the form of linear combinations

\[
\Phi_{k}(x, z) = \left[ A_{k}^{(-)}(z) + B_{k}^{(+)}(z) \right] e^{ikx}, \tag{19}
\]

with decaying at \( z \to +\infty \) functions \( \Phi_{k}^{(-)}(z) \), and with growing at \( z \to -\infty \) functions \( \Phi_{k}^{(+)}(z) \). In other words, for every \( k \) a general solution is known for the following equation,

\[
\Phi''(z) + \frac{\bar{\rho}(z)}{\bar{\rho}(z)} \Phi'(z) - k^{2} \Phi(z) = 0. \tag{20}
\]

Then for approximate calculation of the Hamiltonian at small deviations \( \zeta(x, t) = \eta(x, t) - h \), with the condition \( |\nabla \zeta| \ll 1 \), one can write

\[
\varphi_{1}(x, z) = \int \frac{d^{2}k}{(2\pi)^{2}} \left( A_{k}^{(1)}(z) + B_{k}^{(+)}(z) \right) e^{ikx}, \tag{21}
\]

\[
\varphi_{2}(x, z) = \int \frac{d^{2}k}{(2\pi)^{2}} A_{k}^{(2)}(z) e^{ikx}. \tag{22}
\]

After that from the set of boundary conditions 1) \( \partial_{z} \varphi_{1}(x, 0) = 0 \); 2) \( \partial_{z} \varphi_{2}(x, h + \zeta(x)) = \partial_{z} \varphi_{2}(x, h + \zeta(x)) \); and 3) \( \varphi_{1}(x, h + \zeta(x)) = \varphi_{2}(x, h + \zeta(x)) \) = \( \psi(x) \) it is possible to find the unknown function \( A_{k}^{(1)}, B_{k}^{(+)} \), and \( A_{k}^{(2)} \) [and consequently the required quantity \( V_{n}(x) \)] in the form of an expansion in \( \zeta \). Such a method of presenting the Hamiltonian as a series in the small parameter of characteristic wave steepness is generally used in the theory of surface water waves [23–25]. In particular, this method allows us to obtain the dispersion relation for low-amplitude internal waves:

\[
\omega_{k}^{2} = \tilde{g}(h) \frac{D_{1}(h, k) D_{2}(h, k)}{D_{2}(h, k) + D_{1}(h, k)}, \tag{23}
\]

where \( \tilde{g}(h) \) is a renormalized gravity acceleration: \( \tilde{g}(h) = g[\bar{\rho}_{1}(h) - \bar{\rho}_{2}(h)]/\bar{\rho}(h) \), and the short-hand notations have been used:

\[
D_{1}(h, k) = \frac{\Phi_{k}^{(+)}(h) \Phi_{k}^{(-)}(0) - \Phi_{k}^{(-)}(h) \Phi_{k}^{(+)}(0)}{\Phi_{k}^{(+)}(h) \Phi_{k}^{(-)}(0) - \Phi_{k}^{(-)}(h) \Phi_{k}^{(+)}(0)}. \tag{24}
\]
\[ D_2(h, k) = - \frac{\Phi_k^{(-)}(h)}{\Phi_k^{(+)}}(h) \]  

Note that \(D_1(h, k) > 0\) and \(D_2(h, k) > 0\).

As to the system under consideration, here in some cases another way can be suitable how to calculate the Hamiltonian. Since the kinetic energy takes the form

\[ \mathcal{K} = \frac{1}{2} \int (\mathbf{j} \cdot \mathbf{v}) \, dx \, dz, \]

where \(\mathbf{j} = \rho \mathbf{v}\) is the divergence-free field of the current density, we can introduce for \(\mathbf{j}\) a vector potential \(\mathbf{A}\) which satisfies the equation

\[ \text{curl} \left( \frac{1}{\rho} \text{curl} \mathbf{A} \right) = \Omega \equiv \text{curl} \mathbf{v}, \]

with the boundary condition \([\partial_x A_y(x, y, 0) - \partial_y A_x(x, y, 0)] = 0\). After that the kinetic energy can be re-written as follows,

\[ \mathcal{K} = \frac{1}{2} \int \mathbf{A} \cdot \Omega \, dx \, dz \]

\[ = \frac{1}{2} \int G_{ik}(r_1, r_2) \Omega_i(r_1) \Omega_k(r_2) \, dz \, dz, \]

where \(G_{ik}(r_1, r_2)\) is the Green’s function for Eq.\,(22). As far as the (singular) vorticity field \(\Omega\) is totally concentrated at the interface \(z = \eta(x)\), and the vortex lines coincide with levels of the function \(\psi(x)\) at that surface, the half-space integration will reduce to integration along the surface \(z = \eta(x)\) by means of the change

\[ (\Omega^{(x)}(x), \Omega^{(y)}(x), \Omega^{(z)}(x)) \, dz \rightarrow (\psi_y, -\psi_x, \psi_x \eta_x - \psi_y \eta_y) \, dx \, dy. \]

As the simplest example, in this work an exponential profile \(\bar{\rho}(x) = \rho_0 \exp(-2\alpha z)\) of the equilibrium density will be considered, when Eq.\,(22) after substitution \(\mathbf{A} = \rho_0 e^{-2\alpha z} \mathbf{F}\) turns into an equation with constant coefficients. Generally speaking, if taken globally, such a dependence contradicts to adiabatic equations of state for real gases, for those we rather have \(p \approx C_1 \rho^\gamma\), where \(\gamma\) is the adiabatic exponent [for single-atom gases \(\gamma = 5/3\), for gases consisting of two-atom molecules \(\gamma = 7/5\)], and therefore \(\bar{\rho}(z) \approx C_2 (z_0 - z)^{1/(\gamma - 1)}\), where \(z_0\) is the altitude of the upper edge of the atmosphere. Nevertheless, locally on the vertical coordinate near \(z = h\), every realistic dependence \(\bar{\rho}(z)\) is approximated by an exponent, provided not very long waves are considered. We still would like to note that the case \(\bar{\rho}(z) \approx C_2 (z_0 - z)^{1/(\gamma - 1)}\) also admits analytic investigation, though more difficult, since the functions \(\Phi_k^{(\pm)}(z)\) in the particular solutions \(19\) of Eq.\,(3) are expressed in that case through the modified Bessel functions \(I_\nu\) and \(K_\nu\), with the index \(\nu = [(\gamma - 1)^{-1} - 1]/2:\)

\[ \Phi_k^{(-)}(z) = [k(z_0 - z)]^{-\nu} I_\nu(k(z_0 - z)), \]

\[ \Phi_k^{(+))(z) = [k(z_0 - z)]^{-\nu} K_\nu(k(z_0 - z)). \]

### III. The Case of Exponential Profile of Equilibrium Density

Thus, we have to find the Hamiltonian of our system in an explicit form for \(\bar{\rho}(z) = \rho_0 \exp(-2\alpha z)\), and at the beginning we will solve Eq.\,(24). Consider here simpler case \(\alpha h \gg 1\), when the presence of the flat lower boundary at \(z = 0\) is not important, because the corresponding contribution will be shown later to be of the order \(\exp(-2\alpha h)\). More cumbersome 3D solution for the vector potential in the presence of the boundary \(z = 0\) is given in the Appendix. To solve Eq.\,(29), we use the substitution \(\mathbf{A} = \rho_0 e^{-2\alpha z} \mathbf{F}\) and re-write the equation in Fourier representation: \(ik \times [(ik - 2\alpha e_z) \times \mathbf{F}_k] = \Omega_k\).

Applying the well-known formula for the double vector cross-product and choosing the gauge \((\mathbf{k} \cdot \mathbf{F}_k) = 0\), we immediately arrive at a simple equation

\[ [k^2 + 2i\alpha (\mathbf{k} \cdot e_z)] \mathbf{F}_k = \Omega_k. \]

Now we write down the decaying at the infinity solution of the above equation:

\[ F(r) = \frac{\int d^3k}{(2\pi)^3} \frac{\Omega_k e^{ik \cdot r}}{[k^2 + 2\alpha (\mathbf{k} \cdot e_z)]} = \frac{\int \exp[\alpha(z - z_1) - r - r_1]}{4\pi |r - r_1|} \Omega_k \, d^3r_1. \]

Accordingly, the kinetic energy of the 3D system, without taking into account the flat rigid boundary, is given by the following expression:

\[ \mathcal{K} = \frac{\rho_0}{8\pi} \int \frac{e^{-\alpha |r_2 - r_1|}}{|r_2 - r_1|} e^{-\alpha |z_2 - z_1|} \frac{\Omega_{(r_2)} \cdot \Omega_{(r_1)}}{d^3r_1 d^3r_2}. \]

Passing with the help of formula \(28\) from the space integration to the surface integration where singular vorticity field is distributed, we arrive at the expression in terms of \(\eta\) and \(\psi\),

\[ \mathcal{K} = \frac{\rho_0}{8\pi} \int \frac{\exp[-\alpha \sqrt{|x_1 - x_2|^2 + (\eta_1 - \eta_2)^2} - \alpha (\eta_1 + \eta_2)]}{\sqrt{|x_1 - x_2|^2 + (\eta_1 - \eta_2)^2}} \times \{ \nabla \psi_1 \cdot \nabla \psi_2 + [\nabla \psi_1 \times \nabla \eta_1] \cdot [\nabla \psi_2 \times \nabla \eta_2] \} \times d^2x_1 d^2x_2, \]

where \(\nabla \eta\) and \(\nabla \psi\) are 2D gradients. If necessary, a weakly nonlinear regime in the wave dynamics can be easy considered through expansion of the above expression in powers of \(\psi\) and \(\zeta\).

Let us now turn our attention to planar flows. Note that in 2D case \(F_k\) and \(\Omega_k\) are in the essence (pseudo) scalar quantities. The presence of the boundary at \(z = 0\) can be taken into account by a variant of the “image method”, and as the result we have

\[ F(x, z) = \frac{1}{2\pi} \int \left[ K_0(\alpha \sqrt{(x - x_1)^2 + (z - z_1)^2}) - K_0(\alpha \sqrt{(x - x_1)^2 + (z + z_1)^2}) \right] \times e^{\alpha (z - z_1)} \Omega_{(x_1, z_1)} \, dx_1 \, dz_1, \]  

(35)
where $K_0(r)$ is the well-known Macdonald function. We provide below two of many possible integral representations for this function:

$$K_0(\sqrt{a^2 + b^2}) = \int \frac{d^2 k}{2\pi} e^{ik_a x + ik_b y}$$

$$= \int_{-\infty}^{+\infty} \frac{\exp(ika - |b|\sqrt{k^2 + 1})}{2\sqrt{k^2 + 1}} dk. \quad (36)$$

Consequently, the Green’s function in this case takes the form

$$G(x_1, x_2, z_1, z_2) = \frac{\rho_0}{2\pi} \left[ K_0 \left( \alpha \sqrt{(x_1-x_2)^2 + (z_1-z_2)^2} \right) - K_0 \left( \alpha \sqrt{(x_1-x_2)^2 + (z_1+z_2)^2} \right) \right] e^{-\alpha(z_1+z_2)}. \quad (37)$$

The expression for the kinetic energy of the two-layer flow looks as follows:

$$\mathcal{K}_{2D} = \frac{\rho_0}{4\pi} \int \left[ K_0 \left( \alpha \sqrt{(x_1-x_2)^2 + (\eta_1-\eta_2)^2} \right) - K_0 \left( \alpha \sqrt{(x_1-x_2)^2 + (\eta_1+\eta_2)^2} \right) \right] \times e^{-\alpha(\eta_1+\eta_2)} \psi_1^* \psi_2^* dx_1 dx_2, \quad (38)$$

where $\psi' = \partial \psi / \partial x$. Use of formulas (36) allows us to represent the kinetic energy in a slightly different form:

$$\mathcal{K}_{2D} = \frac{\rho_0}{2} \int dx_1 dx_2 \psi_1^* \psi_2^* e^{-\alpha(\eta_1+\eta_2)} e^{ik_1(x_1-x_2)} \times \int \left[ e^{-|\eta_1-\eta_2|\sqrt{k^2 + \alpha^2}} - e^{-|\eta_1+\eta_2|\sqrt{k^2 + \alpha^2}} \right] \frac{dk}{2\pi}. \quad (39)$$

As it will be shown later, such a representation is suitable for consideration of long-wave asymptotics in the nonlinear wave dynamics. Besides that, it also allows us to find easily the dispersion relation for linear waves. Indeed, from Eq. (39) it is obvious that in the quadratic approximation the Hamiltonian is given by the formula

$$\mathcal{H}_{2D}^2 = \frac{\rho_0 e^{-2\alpha h}}{2} \int \left[ \frac{1 - e^{-2\sqrt{k^2 + \alpha^2}}}{2\sqrt{k^2 + \alpha^2}} k^2 \psi_{-k} \psi_k \right] + \hat{g}(h) \zeta_{-k} \zeta_k \frac{dk}{2\pi}. \quad (40)$$

Solving the corresponding linearized equations of motion for the Fourier components $\zeta_k(t)$ and $\psi_k(t)$, we find quite nontrivial expression for the dispersion relation:

$$\omega_k^2 = \hat{g}(h) k^2 \frac{1 - e^{-2\sqrt{k^2 + \alpha^2}}}{2\sqrt{k^2 + \alpha^2}}. \quad (41)$$

Note, the same dispersion law takes place in the 3D case, due to the isotropy of the system in the horizontal plane [it is also confirmed by the formula (39)].

Now we consider the limiting case $\alpha \eta \ll 1$ and typical wave numbers $k$ satisfying the conditions $\alpha \eta \lesssim k \eta \ll 1$. Expanding the exponents in integral (39) in powers of the small arguments, we obtain an approximate kinetic energy functional up to the first order in $\alpha \eta$,

$$\mathcal{K}_\ast \{\eta, \psi\} = \frac{\rho_0}{2} \int \eta(1 - 2\alpha \eta)(\psi')^2 dx - \frac{\rho_0}{2} \int (\psi' \eta)[-\partial_x^2 + \alpha^2]^{1/2}(\psi' \eta) dx. \quad (42)$$

Let us introduce a new unknown function $q = [1 - \exp(-2\alpha \eta)]/(2\alpha)$, which up to the constant factor $\rho_0$ is the canonically conjugate for function $\psi$, and then rewrite the approximate Hamiltonian in terms of $q$ and $\psi$:

$$\mathcal{H}_\ast \{q, \psi\} = \frac{\rho_0}{2} \int q(1 - \alpha q)(\psi')^2 dx - \frac{\rho_0}{2} \int (\psi' q)[-\partial_x^2 + \alpha^2]^{1/2}(\psi' q) dx + \rho_0 \hat{g}(0) \int \left[ \frac{q^2}{2} + \alpha \beta \frac{q^3}{3} \right] dx, \quad (43)$$

where $\beta$ is a dimensionless parameter depending on behavior of the difference $[\bar{p}_1(z) - \bar{p}_2(z)]$ near $z = 0$. Considering propagation of relatively small but finite disturbances $\tilde{q}(x,t) = q(x,t) - \bar{q}$, it is possible by a standard procedure to derive weakly nonlinear equation for $u(x,t) = \psi_x$, which describes a slow evolution of unidirectional wave under the influence of weak dispersion:

$$u_t + c'u_x + \bar{u}u_x - \frac{c\bar{q}}{2} \{[-\partial_x^2 + \alpha^2]^{1/2} - \alpha\} u_x = 0, \quad (44)$$

where the speed of long linear waves is $\bar{c} \approx [\bar{g}(0)\bar{q}]^{1/2}$, and the coefficient $\bar{c} \approx 3/2$. Equation of such kind is called sometimes “Smith’s equation” after the work by Ronald Smith [28] where it arose for the first time in context of continental-shelf oceanic waves. It is interesting to note that the special form of the dispersive term makes the above equation intermediate between the two famous integrable models, namely the Korteweg-de Vries equation and the Benjamin-Ono equation [26, 27]. In this sense the Smith’s equation is similar to the Intermediate Long Wave equation (ILW) (see, e.g., Refs. [11, 12, 23, 30]), but contrary to ILW the Smith’s equation is not integrable, as it was established in Ref. [31].

IV. PLANAR FLOWS WITH PIECEWISE CONSTANT POTENTIAL VORTICITY

Now we would like to make an important generalization of the Hamiltonian theory which is possible for 2D isentropic flows in $(x,z)$ plane, namely we will take into account the fact that potential vorticity $\tilde{\gamma} = -\Omega^{(0)}/\rho$ in the 2D case is governed by the advection equation

$$\tilde{\gamma}_t + \mathbf{v} \cdot \nabla \tilde{\gamma} = 0. \quad (45)$$

This conservation law for the potential vorticity along each fluid particle trajectory allows us at consideration of
planar flows with a piecewise constant function $\tilde{\gamma}(x, z, t)$ to follow only the motion of boundaries where $\tilde{\gamma}$ is discontinuous. In the present paper it is assumed for simplicity that $\tilde{\gamma}$ has a single jump, and this jump coincides with the interface between the layers $z = \eta(x, t)$, but generally this coincidence is not necessary and a separate curve $z = \eta_0(x, z, t)$ can be considered where the jump takes place.

Let (sufficiently small) potential vorticities in the layers be $\gamma_{1,2}$, so that the corresponding stationary shear flows $U_{1,2}(z) \ll c$ satisfy the conditions (we neglect the difference between $\tilde{\rho}_{1,2}$ and $\tilde{\rho}$)

$$-U_{1,2}'(z) = \gamma_{1,2}\tilde{\rho}(z).$$

We shall suppose that in the stationary state the velocity profile has a “break” at $z = h$, that is $U_{1,2}(z) = -\gamma_{1,2}\mu(z)$, where

$$\mu(z) = \int_{h}^{z} \tilde{\rho}(\xi)d\xi.$$  \hspace{1cm} (47)

A 2D velocity field in each layer now takes the form

$$v_{1,2}(x, z, t) = (U_{1,2}(z) + \partial_z\varphi_{1,2}(x, z, t), \partial_z\varphi_{1,2}(x, z, t)),$$

with the potentials $\varphi_{1,2}$ satisfying the same equation \[49\]:

$$\nabla \cdot \tilde{\rho}\nabla \varphi_{1,2} = 0,$$

and it implies the existence of the corresponding stream functions $\psi_{1,2}(x, z, t)$:

$$\tilde{\rho}\partial_x \varphi_{1,2} = \partial_t \psi_{1,2}, \quad \tilde{\rho}\partial_z \varphi_{1,2} = -\partial_x \psi_{1,2}.$$  \hspace{1cm} (49)

Instead of Eq.\[49\], we have to deal now with its generalization:

$$\partial_t \varphi_{1,2} + \gamma_{1,2} \varphi_{1,2} + U_{1,2}(z) \partial_z \varphi_{1,2} + \frac{(\nabla \varphi_{1,2})^2}{2} + \frac{\tilde{\rho}_{1,2}}{\tilde{\rho}(z)} = 0,$$  \hspace{1cm} (50)

which regards the 2D Euler equation in the case of constant potential vorticity under the condition $\nabla \cdot (\tilde{\rho}v) = 0$. Taking into account that the full stream functions of the flows under consideration are

$$\Theta_{1,2}(x, z, t) = \psi_{1,2}(x, z, t) - U_{1,2}(z)/(2\gamma_{1,2}),$$

equation \[50\] can be also represented as follows,

$$\partial_t \varphi_{1,2} + \gamma_{1,2} \Theta_{1,2} + \frac{(\nabla \varphi_{1,2})^2}{2} + \frac{\tilde{\rho}_{1,2}}{\tilde{\rho}(z)} = 0.$$  \hspace{1cm} (52)

Now we note that at the interface $z = \eta(x, t)$ there are the equalities

$$-\partial_z \Theta_1(x, \eta(x, t)) = -\partial_z \Theta_2(x, \eta(x)) = \tilde{\rho}(\eta)\dot{\eta} = \tilde{\rho}(\eta)V_n \sqrt{1 + \eta'^2};$$

where $V_n = (v_1 \cdot n) = (v_2 \cdot n)$.

Demanding the pressure field to be continuous at $z = \eta(x, t)$ and reasoning analogously to the case $\gamma_{1,2} = 0$, we conclude that the evolution equations for the 2D system possess the following structure,

$$\tilde{\rho}(\eta)\dot{\eta} = \delta H/\delta \psi,$$  \hspace{1cm} (54)

$$-\tilde{\rho}(\eta)\psi_t + \gamma \tilde{\rho}(\eta)\partial_z^{-1}\tilde{\rho}(\eta)\dot{\eta} = \delta H/\delta \eta,$$  \hspace{1cm} (55)

where $\gamma = (\gamma_1 - \gamma_2)$, and the Hamiltonian $H$ is equal to the sum of total kinetic energy and the effective potential energy. By a direct calculation it is easy to check that the corresponding Lagrangian for the above equations is

$$\mathcal{L} = \int \psi_t dx + \frac{\gamma}{2} \int \partial_z^{-1} \mu dx - H\{\mu, \psi\},$$

where $\mu = \mu(\eta)$ [see Eq.\[47\]]. For internal waves in an incompressible liquid, an analogous structure was obtained in Refs.\[15, 17\], with the difference that in our case $\mu(\eta)$ is a nonlinear function (see also Ref.\[32\] about waves at the free surface of a 2D incompressible fluid with a constant vorticity).

It is interesting to note that in the quadratic approximation the Lagrangian \[56\] take the form

$$\mathcal{L}^{[2]} = \tilde{\rho}(h) \int \psi \dot{\zeta} dx + \frac{\gamma \overline{\rho^2}(h)}{2} \int \zeta \partial_z^{-1} \zeta dx - \mathcal{H}^{[2]}\{\zeta, \psi\}.$$  \hspace{1cm} (57)

Moreover, it is easy to show that the functional $\mathcal{H}^{[2]}\{\zeta, \psi\}$ does not depend on $\gamma_1$ and $\gamma_2$ [dependence on $\gamma_1$ and $\gamma_2$ appears only in higher orders]:

$$\mathcal{H}^{[2]} = \frac{\overline{\rho}(h)}{2} \int [N(h, k) k^2 \psi - k \psi_k + \overline{g}(h) \zeta - k \overline{\zeta}_k] \frac{dk}{2\pi}. $$

Function $N(h, k)$ is expressed through the Green’s function $G(x_2 - x_1, z_1, z_2)$ by the following formula:

$$\overline{\rho}(h) N(h, k) = \int_{-\infty}^{+\infty} G[x, h] e^{-ikx} dx.$$  \hspace{1cm} (59)

It should be noted that $\omega_{k}^2(k) = \overline{g}(h)k^2 N(h, k)$ is the dispersion law in the case $\gamma_1 = \gamma_2 = 0$ [compare with \[23\]]. For example, with the exponential profile of the equilibrium density the quadratic Hamiltonian is given by expression \[50\]. Solving the corresponding linear equations,

$$\dot{\zeta}_k = N(h, k) k^2 \psi_k,$$  \hspace{1cm} (60)

$$-\psi_k + \gamma \overline{\rho}(h) \frac{\dot{\zeta}_k}{ik} = \overline{g}(h) \zeta_k,$$  \hspace{1cm} (61)

we obtain the dispersion law for linear waves at $\gamma \neq 0$:

$$\omega_k = \frac{1}{2} \gamma \overline{\rho}(h) k N(h, k) + \sqrt{[\gamma \overline{\rho}(h) k N(h, k)]^2/4 + \overline{g}(h) k^2 N(h, k)}.$$  \hspace{1cm} (62)

Since the singular part of the vorticity field (concentrated at the interface) is determined by the relation

$$-\Omega_s = [\psi' - \gamma \mu(\eta)]\delta(z - \eta(x, t)),$$  \hspace{1cm} (63)

where $\delta(z - \eta(x, t))$
is the Dirac’s function, it is convenient to introduce the new unknown variable,

$$p(x, t) = \psi - \gamma \partial_x^{-1} \mu.$$  \hfill (63)

In variables \{\mu, p\} the Lagrangian takes the form (the sign in front of the second term has changed)

$$\mathcal{L} = \int p \mu_t dx - \frac{\gamma}{2} \int \mu \partial_x^{-1} \mu_t dx - \mathcal{H}\{\mu, p\}. \hfill (64)$$

Now, besides the singular part of the vorticity, there is also a distributed part, and the full vorticity field is given by the formula

$$- \Omega(x, z, t) = \frac{\partial}{\partial z} \delta[z - \eta(x, t)] + \gamma_2 \frac{\partial}{\partial z} \delta[z - \eta(x, t) - z], \hfill (65)$$

where $\theta[z - \eta(x, t) - z]$ is the unit jump function (the Heaviside's function). The Hamiltonian of the 2D system is determined with the help of the Green’s function $G'[(x_2 - x_1), z_1, z_2]$ by the following expression:

$$\mathcal{H} = g \int W(x) dx + \frac{1}{2} \int G'[x_2 - x_1, z_1, z_2] \times \Omega(x_1, z_1) \delta[z_2 - \eta(x_1, z_1)], \hfill (66)$$

where Eq. (65) should be substituted, and after the integrations $\eta$ should be expressed through $\mu$. Let us remind that in the case $\delta(z) = \rho_0 \exp(-2\alpha z)$ the Green’s function is given by Eq. (37). Let us also note that in the absence of the density jump a class of flows is possible with $p \equiv 0$. In that case the dynamics of the vorticity waves is determined by the Lagrangian $L_\gamma = -(\gamma/2) \int \mu \partial_x^{-1} \mu dx - \mathcal{H}_\gamma\{\mu\}$, and the dispersion law for such waves is expressed by the formula (62) where $\tilde{g}(h) = 0$ should be put.

V. DISCUSSION

In this work, a compressible two-layer atmospheric model has been suggested, intended for theoretical study of internal waves at the interface between two isentropic layers of a gas with nearly equal values of specific entropy. In the derivation of the approximate equations it was supposed that the flow velocities are small compared with a local speed of sound. It should be noted that this condition puts the lower limit for characteristic wave numbers: $k \gtrsim |\tilde{\rho}(h)|/\tilde{\rho}(h)$, because at longer scales the velocity field penetrates into the upper layer rather far, where in view of constant entropy the temperature is small together with a local speed of sound, and it violates the starting-point assumption of the model. To some extent the above limitation is softened if somewhere above the second layer there is the third layer, with very high temperature, and therefore the boundary between the second layer and the third layer can be effectively treated as a "rigid lid". However one should remember that in the long-wave limit (in the Earth conditions it corresponds to hundreds and thousands kilometers), nonuniform horizontal motions of the whole atmosphere become important. Those flows are approximately described by a "shallow water theory" with adding the Coriolis force, and they lead to variations of a quasi-equilibrium density profile. Besides that, the Coriolis force violates the potentiality of the flow. Thus, the suggested here theory can describe waves with lengths not longer than a few kilometers.

In the present work, only first steps have been made in the study of internal waves in the atmosphere within the compressible two-layer model. Promising directions of further research can be outlined as follows. First, a generalization of the model is evident for more layers and for continuous limit, which will enrich it because an interaction between several interfaces in many cases is able to introduce new interesting effects as instabilities etc. Second, we should mention a wide class of problems about interaction of internal waves and mountains, which also can be studied with the help of this model. Third, nonlinear wave dynamics can be simulated numerically. Fourth, an analogous Hamiltonian formulation is possible for consideration of axisymmetric flows with a piecewise constant generalized potential vorticity. Fifth, it seems likely that analogous finite-layer models are possible not only in the Eulerian hydrodynamics, but in a wider class of conservative hydrodynamic systems as well, for instance, in the hydrodynamics of a relativistic fluid placed in a strong static gravitational field described by a metric 4-tensor. Accordingly, there is a perspective of application of a similar theory to astrophysical problems, where the equilibrium density possesses the spherical symmetry, as a rule.

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Appendix A: Correction to 3D Green’s function due to the flat boundary

To satisfy the boundary condition $[\partial_x F_y(x, y, 0) - \partial_y F_x(x, y, 0)] = 0$, which ensures zero normal component of the velocity field at the rigid flat boundary, we add to the particular solution (32) of the nonhomogeneous equation (31) some specially selected solution of the corresponding homogeneous equation, decaying at $z \rightarrow +\infty$:

$$F^{-}(x, z) = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \mathbf{f}_k \exp[i \mathbf{k} \cdot \mathbf{x} + z(\alpha - \sqrt{k^2 + \alpha^2})]. \hfill (A1)$$

where $\mathbf{f}_k = (f_k^x, f_k^y, 0)$ satisfies the condition of 2D transversal gauge $(\mathbf{k} \cdot \mathbf{f}_k) = 0$. It is not difficult to understand that $\mathbf{f}_k$ should be taken in the following form...
(here and later on $\kappa$ and $\nu$ are tensorial indices in the horizontal plane):

$$f^{(e)}_k = - \left( \delta_{\kappa\nu} - \frac{k_k k_\nu}{k^2} \right) \int \frac{d\xi}{2\pi} \frac{\Omega^{(e)}(k, \xi)}{(k^2 + \xi^2 + 2i\alpha\xi)},$$  \hspace{1cm} (A2)$$

where $\Omega^{(e)}(k, \xi) = \int \Omega^{(e)}(x_1, z_1) e^{-ik\cdot x_1 - i\xi z_1} d^2 x_1 dz_1$ is the Fourier image of the horizontal component of the vorticity field. Now we transform the integral (A2):

$$\int \frac{d\xi}{2\pi} \frac{\Omega^{(e)}(x_1, z_1) e^{-ik\cdot x_1 - i\xi z_1}}{(k^2 + \xi^2 + 2i\alpha\xi)} d^2 x_1 dz_1$$

\hspace{1cm} = \int \frac{\Omega^{(e)}(x_1, z_1) e^{-ik\cdot x_1 - i\xi z_1}}{2\sqrt{k^2 + \alpha^2}} d^2 x_1 dz_1. \hspace{1cm} (A3)$$

For $\xi$-integration we have used the fact that $\Omega(x, z)$ is non-zero only at $z > 0$, and therefore the integration contour can be closed in the lower complex half-plane. Then we substitute the result into Eq. (A1) and see that the conditioned by the flat boundary correction $G^{(-)}_{e\nu}(x_1, x_2, z_1, z_2)$ to the Green's function actually depends on the variables $x = x_2 - x_1$ and $s = z_1 + z_2$, and it is expressed by the following formula:

$$G^{(-)}_{e\nu}(x, s) = \rho_0 e^{-s\alpha} \int \left( \frac{k_k k_\nu}{k^2} - \delta_{\kappa\nu} \right)$$

\hspace{1cm} \times e^{ik\cdot x - s\sqrt{k^2 + \alpha^2}} \left( 2\pi \right)^2
dk$$

\hspace{1cm} = - \rho_0 e^{-s\alpha} \left( \delta_{\kappa\nu} - \delta_{\kappa\nu} \frac{\partial}{\partial r} \right) \frac{1}{4\pi \sqrt{x^2 + s^2}} \exp[-\alpha \sqrt{x^2 + s^2}], \hspace{1cm} (A4)$$

where $\hat{\Delta}^{-1}$ is the inverse 2D Laplace operator. Let us introduce the notation $D(x, s) = \Delta_x^{-1}[\exp(-\alpha \sqrt{x^2 + s^2})/\sqrt{x^2 + s^2}]$. In virtue of the definition, function $D(r, s)$ satisfies the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial D}{\partial r} \right) = \frac{\exp(-\alpha \sqrt{r^2 + s^2})}{\sqrt{r^2 + s^2}}, \hspace{1cm} (A5)$$

from which we obtain by a simple integration

$$r \frac{\partial D}{\partial r} = \left[ \exp(-\alpha s) - \exp(-\alpha \sqrt{r^2 + s^2}) \right] / \alpha. \hspace{1cm} (A6)$$

It should be noted that the second derivatives $\partial_s \partial_r D(x, s)$ can be expressed through the combination $r^{-1} \partial^2 D/\partial r$:

$$\partial_s \partial_r D(x, s) = \frac{x_k x_\nu}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial D}{\partial r} \right) + \delta_{\kappa\nu} \frac{1}{r} \frac{\partial^2 D}{\partial r^2}. \hspace{1cm} (A7)$$

Collecting the obtained expressions and taking into account Eq. (A3), we write the required correction to 3D Green’s function in the final form:

$$G^{(-)}_{e\nu}(x, s) = \rho_0 \left( \delta_{\kappa\nu} - \frac{2 x_k x_\nu}{x^2} \right)$$

\hspace{1cm} \times \left[ \exp(-\alpha s) - \exp(-\alpha \sqrt{x^2 + s^2}) \right]$$

\hspace{1cm} + \rho_0 \left( \frac{x_k x_\nu}{x^2} - \delta_{\kappa\nu} \right) \frac{\exp(-\alpha \sqrt{x^2 + s^2})}{4\pi \sqrt{x^2 + s^2}}, \hspace{1cm} (A8)$$

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