Monodromies at infinity of confluent $A$-hypergeometric functions

Kana ANDO †, Alexander ESTEROV ‡ and Kiyoshi TAKEUCHI §

Abstract

We study the monodromies at infinity of confluent $A$-hypergeometric functions introduced by Adolphson [2]. In particular, we extend the result of [38] for non-confluent $A$-hypergeometric functions to the confluent case. The integral representation by rapid decay homology cycles proved in [9] will play a central role in the proof.

1 Introduction

The theory of $A$-hypergeometric systems introduced by Gelfand-Kapranov-Zelevinsky [11] is a vast generalization of that of classical hypergeometric differential equations. We call their holomorphic solutions $A$-hypergeometric functions. As in the classical case, $A$-hypergeometric functions admit $\Gamma$-series expansions (see [11], [34] etc.) and integral representations ([12]). Moreover they have deep connections with many other fields of mathematics, such as toric varieties, projective duality, period integrals, mirror symmetry, commutative algebra, enumerative algebraic geometry and combinatorics. Also from the viewpoint of the $\mathcal{D}$-module theory (see [20] etc.), $A$-hypergeometric $\mathcal{D}$-modules were very elegantly constructed in [12]. For the recent development of this theory see [34], [35] etc. In [4], [5], [12], [18], [38] etc. the monodromies of their $A$-hypergeometric functions were studied. In particular, in [38] the third author obtained a formula for their monodromies at infinity. The aim of this paper is to generalize it to the confluent $A$-hypergeometric functions introduced by Adolphson [2]. In the confluent case, by the lack of the integral representation, almost nothing was known about the global property of the confluent $A$-hypergeometric functions before [9]. Recently in [9] we established their integral representation by using Hien’s theory of rapid decay homology groups invented

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†Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan. E-mail: ando@graduate.chiba-u.jp

‡National Research University Higher School of Economics
Faculty of Mathematics NRU HSE, 7 Vavilova 117312 Moscow, Russia. E-mail: aesterov@hse.ru.
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§Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan. E-mail: takemicro@nifty.com
in [15] and [16] and obtained a formula for their asymptotic expansion at infinity (see also Hien-Roucairol [17], Saito [33] and Schulze-Walther [36], [37] for related results). We will use this development to prove our main theorem.

In order to introduce our result, first we recall the definition of Adolphson’s confluent $A$-hypergeometric systems in [2]. In this paper, we essentially follow the terminology of [20]. Let $A = \{a(1), a(2), \ldots, a(N)\} \subset \mathbb{Z}^n$ be a finite subset of the lattice $\mathbb{Z}^n$. As in [11] and [12] assume that $A$ generates $\mathbb{Z}^n$. We denote by $\Delta$ the convex hull $\text{conv}(A \cup \{0\})$ of $A \cup \{0\}$ in $\mathbb{R}^n$. By our assumption $\Delta$ is an $n$-dimensional polytope. Let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ be a parameter vector. We identify the set $A$ with the $n \times N$ integer matrix

$$A := (t^i a(1) \ t^i a(2) \ \cdots \ t^i a(N)) = (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq N} \in M(n, N, \mathbb{Z})$$  \hfill (1.1)

whose $j$-th column is $t^i a(j)$. Then Adolphson’s confluent $A$-hypergeometric system on $X = \mathbb{C}^A = \mathbb{C}^N_z$ associated with the parameter vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ is

$$\left\{ \begin{array}{l}
\sum_{j=1}^{N} a_{i,j} z_j \frac{\partial}{\partial z_j} + c_i u(z) = 0 \quad (1 \leq i \leq n), \\
\prod_{\mu_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-\mu_j} u(z) = 0 \quad (\mu \in \text{Ker} A \cap \mathbb{Z}^N). 
\end{array} \right.$$  \hfill (1.2)

This system was introduced first by Gelfand-Kapranov-Zelevinsky [11] when there exists a linear functional $l : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $l(\mathbb{Z}^n) = \mathbb{Z}$ and $A \subseteq l^{-1}(1)$. In such a case, Hotta [19] proved that it is regular holonomic i.e. non-confluent. This is the reason why we call the above system Adolphson’s generalization a confluent $A$-hypergeometric system. Let $D(X)$ be the Weyl algebra over $X$ and consider the differential operators

$$Z_{i,c} := \sum_{j=1}^{N} a_{i,j} z_j \frac{\partial}{\partial z_j} + c_i \quad (1 \leq i \leq n),$$

$$\square_{\mu} := \prod_{\mu_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-\mu_j} \quad (\mu \in \text{Ker} A \cap \mathbb{Z}^N)$$  \hfill (1.4)

in it. Then the above system is naturally identified with the left $D(X)$-module

$$M_{A,c} = D(X)/ \left( \sum_{1 \leq i \leq n} D(X)Z_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathbb{Z}^N} D(X)\square_{\mu} \right).$$  \hfill (1.6)

Let $\mathcal{D}_X$ be the sheaf of differential operators over the “algebraic variety” $X$ and define a coherent $\mathcal{D}_X$-module by

$$\mathcal{M}_{A,c} = \mathcal{D}_X/ \left( \sum_{1 \leq i \leq n} \mathcal{D}_XZ_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathbb{Z}^N} \mathcal{D}_X\square_{\mu} \right).$$  \hfill (1.7)

Then $\mathcal{M}_{A,c}$ is the localization of the left $D(X)$-module $M_{A,c}$ (see [20] Proposition 1.4.4 (ii)] etc.). In [2] Adolphson proved that $\mathcal{M}_{A,c}$ is holonomic. In fact, he proved the following more precise result.
Definition 1.1. ([2] page 274, see also [30] etc.) For \( z \in X = \mathbb{C}^A \) we say that the Laurent polynomial \( h_z(x) = \sum_{j=1}^N z_j x^{a(j)} \) is non-degenerate if for any face \( \Gamma \) of \( \Delta \) not containing the origin \( 0 \in \mathbb{R}^n \) we have

\[
\left\{ x \in T = (\mathbb{C}^*)^n \mid h_z^\Gamma(x) = \frac{\partial h_z^\Gamma}{\partial x_1}(x) = \cdots = \frac{\partial h_z^\Gamma}{\partial x_n}(x) = 0 \right\} = \emptyset,
\]

where we set \( h_z^\Gamma(x) = \sum_{j:a(j) \in \Gamma} z_j x^{a(j)} \).

Let \( \Omega \subset X \) be the Zariski open subset of \( X \) consisting of \( z \in X = \mathbb{C}^A \) such that the Laurent polynomial \( h_z(x) = \sum_{j=1}^N z_j x^{a(j)} \) is non-degenerate. Then Adolphson’s result [2, Lemma 3.3] asserts that the holonomic \( \mathcal{D}_X \)-module \( \mathcal{M}_{A,c} \) is an integrable connection on \( \Omega \). Namely on \( \Omega \subset X \) the characteristic variety of \( \mathcal{M}_{A,c} \) is contained in the zero section of the cotangent bundle \( T^*\Omega \). Now let \( \Gamma \) be the normalized \( \mathcal{D}_X \)-module \( \mathcal{M}_{A,c} \) defined by

\[
\text{Sol}_X(\mathcal{M}_{A,c}) = R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}_{A,c}, \mathcal{O}_{X^\an})
\]

(see [20] etc. for the details). Then the above Adolphson’s result implies that \( \text{Sol}_X(\mathcal{M}_{A,c}) \) is a local system on \( \Omega^\an \). We call the sections of this local system

\[
H^\Gamma\text{Sol}_X(\mathcal{M}_{A,c})_{\Omega^\an} = \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_{A,c}, \mathcal{O}_{X^\an})_{\Omega^\an}
\]

confluent \( \mathcal{A} \)-hypergeometric functions. Moreover Adolphson proved the following fundamental result. Let \( \text{Vol}_\mathbb{Z}(\Delta) \in \mathbb{Z} \) be the normalized \( n \)-dimensional volume of \( \Delta \) i.e. the \( n! \) times of the usual one \( \text{Vol}(\Delta) \in \mathbb{Q} \) with respect to the lattice \( \mathbb{Z}^n \subset \mathbb{R}^n \).

Definition 1.2. (Gelfand-Kapranov-Zelevinsky [12, page 262]) For a face \( \Gamma \) of \( \Delta \) containing the origin \( 0 \in \mathbb{R}^n \) we denote by \( \text{Lin}(\Gamma) \simeq \mathbb{C}^{\dim \Gamma} \subset \mathbb{C}^n \) the \( \mathbb{C} \)-linear span of \( \Gamma \). Then we say that the parameter vector \( c \in \mathbb{C}^n \) is non-resonant if for any face \( \Gamma \) of \( \Delta \) of codimension 1 such that \( 0 \in \Gamma \) we have \( c \notin \{ \mathbb{Z}^n + \text{Lin}(\Gamma) \} \).

Theorem 1.3. (Adolphson [2, Corollary 5.20]) Assume that the parameter vector \( c \in \mathbb{C}^n \) is non-resonant. Then the rank of the local system \( \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_{A,c}, \mathcal{O}_{X^\an})_{\Omega^\an} \) on \( \Omega^\an \) is equal to \( \text{Vol}_\mathbb{Z}(\Delta) \in \mathbb{Z} \).

This is a generalization of the famous result of Gelfand-Kapranov-Zelevinsky in [11] to the confluent case.

Now let us introduce our main result. Fix \( 1 \leq j_0 \leq N \) and let \( \mathbb{L} \simeq \mathbb{C} \) be a complex line in \( X = \mathbb{C}^N \) parallel to the \( j_0 \)-th axis of \( \mathbb{C}^N \) and satisfying the condition

\[
\#(\mathbb{L} \cap (X \setminus \Omega)) < +\infty.
\]

Our result will not depend on the choice of such \( \mathbb{L} \simeq \mathbb{C} \). For \( R > 0 \) such that \( \mathbb{L} \cap (X \setminus \Omega) \subset \{ z \in \mathbb{L} \simeq \mathbb{C} \mid |z| < R \} \) we define a circle \( C \subset \mathbb{L} \simeq \mathbb{C} \) in \( \mathbb{L} \simeq \mathbb{C} \) by \( C = \{ z \in \mathbb{L} \simeq \mathbb{C} \mid |z| = R \} \). We denote the characteristic polynomial of the monodromy of the confluent \( \mathcal{A} \)-hypergeometric functions along \( C \) by \( \lambda_{\infty}(t) \in \mathbb{C}[t] \). Then \( \lambda_{\infty}(t) \) does not depend on the choice of the radius \( R > 0 \). Note also that if the parameter vector
we obtain a morphism neighborhood of \( q \) in it. Then for each point \( Z \) etc. Let \( \Delta_1, \ldots, \Delta_r \prec \Delta \) be the facets of the \( n \)-dimensional polytope \( \Delta \subset \mathbb{R}^n_\varepsilon \) such that \( a(j_0) \in \Delta_i \) and \( 0 \notin \Delta_i \). For \( 1 \leq i \leq r \) let \( \Gamma_{i1}, \Gamma_{i2}, \ldots, \Gamma_{imi} \prec \Delta_i \) be the facets of \( \Delta_i \) such that \( a(j_0) \notin \Gamma_{ij} \). We set
\[
\hat{\Gamma}_{ij} := \text{conv}\{\{0\} \cup \Gamma_{ij}\}, \quad \hat{\Delta}_i := \text{conv}\{\{0\} \cup \Delta_i\}.
\] (1.12)

Let \( \rho_{ij} \in \mathbb{Z} \setminus \{0\} \) be the primitive inner conormal vector of the facet \( \hat{\Gamma}_{ij} \prec \hat{\Delta}_i \) of the polytope \( \hat{\Delta}_i \) and set
\[
h_{ij} := \langle \rho_{ij}, a(j_0) \rangle > 0.
\] (1.13)

We call \( h_{ij} > 0 \) the lattice height of the point \( a(j_0) \) from the facet \( \hat{\Gamma}_{ij} \). By this definition of \( h_{ij} \) we have
\[
\text{Vol}_\varepsilon(\hat{\Delta}_i) = \sum_{j=1}^{m_i} h_{ij} \cdot \text{Vol}_\varepsilon(\hat{\Gamma}_{ij}).
\] (1.14)

Then our main theorem is as follows.

**Theorem 1.4.** Assume that the parameter vector \( c \in \mathbb{C}^n \) is non-resonant. Then we have
\[
\lambda_{j_0}^\infty(t) = \prod_{i=1}^{r} \prod_{j=1}^{m_i} \left\{ t^{h_{ij}} - \exp\left(-2\pi \sqrt{-1} \langle \rho_{ij}, c \rangle \right) \right\}^{\text{Vol}_\varepsilon(\hat{\Gamma}_{ij})}
\times (t - 1)^{\text{Vol}_\varepsilon(\Delta) - \sum_{i=1}^{r} \text{Vol}_\varepsilon(\hat{\Delta}_i)}.
\] (1.15)

We will prove this theorem in Section 3. Our proof is based on an explicit construction of a basis of the corresponding rapid decay homology group. It also enables us to compute the whole monodromy operator (including the nilpotent part) in small dimensions. Note that by the method of [38] which cannot be applied to the confluent case we obtain only the semisimple part of the monodromy.

### 2 A decomposition of a certain rapid decay homology group

In this section, for the preparation of the proof of Theorem 1.4, we show a decomposition of a certain rapid decay homology group associated to a special Laurent polynomial.

First of all, we recall the definition of real oriented blow-ups in Hien [15] and [16] etc. Let \( Z \) be a complex manifold of dimension \( n \) and \( D \subset Z \) a normal crossing divisor in it. Then for each point \( q \in Z \) by taking a local coordinate \((x_1, \ldots, x_n)\) of \( Z \) on a neighborhood of \( q \) such that \( q = (0, \ldots, 0) \) and \( D = \{x_1 \cdots x_k = 0\} \) for some \( 0 \leq k \leq n \) we obtain a morphism
\[
([0, \varepsilon) \times S^1)^k \times B(0; \varepsilon)^{n-k} \rightarrow B(0; \varepsilon)^k \times B(0; \varepsilon)^{n-k}
\] (2.1)

\[
\left\{ (t, e^{\sqrt{-1} \theta}) \right\}_{i=1}^{k} x_{i_{k+1}}, \ldots, x_{n} \rightarrow \left\{ (t, e^{\sqrt{-1} \theta}) \right\}_{i=1}^{k} x_{i_{k+1}}, \ldots, x_{n}
\] (2.2)

to \( B(0; \varepsilon)^{k} \times B(0; \varepsilon)^{n-k} \subset Z \), where we set \( B(0; \varepsilon) = \{x \in \mathbb{C} \mid |x| < \varepsilon\} \) for sufficiently small \( \varepsilon > 0 \). By patching these morphisms together naturally, we can construct a real
Now let us consider the local system $H^0_p(W^o,\mathcal{L})$ for a positive integer $h > 0$. Assume that $g(x)$ is non-degenerate with respect to its Newton polytope $NP(g) \subset \mathbb{R}^{n-1}$ and $\dim NP(g) = n - 1$. Namely we assume that for any face $\Gamma$ of $NP(g)$ the $\Gamma$-part $g^\Gamma$ of $g$ satisfies the condition

$$\left\{ x \in T_0 \subset (\mathbb{C}^e)^{n-1} \mid g^\Gamma(x) = \frac{\partial g^\Gamma}{\partial x_1}(x) = \cdots = \frac{\partial g^\Gamma}{\partial x_{n-1}}(x) = 0 \right\} = \emptyset. \quad (2.3)$$

For a positive integer $h > 0$ we define a meromorphic function $G(x,t)$ on $W = T_0 \times \mathbb{C}_t$ by

$$G(x,t) = \frac{1}{g(x)^{h-1}t^h}. \quad (2.4)$$

Here we consider such a very special function in order to modify our construction of the basis of the rapid decay homology group in [9, Section 7]. Since our results below are technical, for the first reading the reader can skip them and directly go to Section 3. Set $D = \{(x,t) \in W \mid g(x) \cdot t = 0\} \subset W$ and $W^o = W \setminus D$. Let $\pi : \tilde{W} \to W$ be the real oriented blow-up of $W$ along the normal crossing divisor $D \subset W$ and $\iota : W^o \hookrightarrow \tilde{W}$ the inclusion map. We identify $D_0 = \{x \in T_0 \mid g(x) \neq 0\} \subset T_0$ with an open subset of $D$ naturally and set $\tilde{D}_0 = \pi^{-1}(D_0)$, $P = \tilde{D}_0 \cap \{\text{Re}G \geq 0\}$ and $Q = \tilde{D}_0 \setminus P$. Note that the set $Q \subset \tilde{D}_0 \subset \tilde{W}$ consists of the rapid decay directions of the function $\exp(G)$ over $D_0 \subset D$. Now let us consider the local system

$$\mathcal{L} = \mathbb{C}_{W^o} x_1^{\beta_1} \cdots x_{n-1}^{\beta_{n-1}} g(x)^{\beta_n} t^{\beta_n} \quad (2.5)$$

($\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$) on $W^o$. By abuse of notation, for $p \in \mathbb{Z}$ we set

$$H^p_{rd}(W^o) = H_p(W^o \cup Q, Q, \iota_*(\mathcal{L})), \quad (2.6)$$

where $H_p(W^o \cup Q, Q, \iota_*(\mathcal{L}))$ stands for the $p$-th relative twisted homology group of the pair $(W^o \cup Q, Q)$ with coefficients in the rank-one local system $\iota_*(\mathcal{L})$ on $\tilde{W}$ (see [3], [31], [32] etc.). Our aim here is to decompose the vector space $V(p) := H^p_{rd}(W^o)$ into a direct sum with respect to some facets of $NP(g)$. For this purpose, we fix a lattice point $a \in NP(g) \cap \mathbb{Z}^{n-1}$ and impose a condition on the coefficient $z \in \mathbb{C}$ of $x^a$ in $g(x)$ as follows.

First we define a new Laurent polynomial $\tilde{g}(x)$ whose constant term is zero by

$$\tilde{g}(x) = z - x^{-a}g(x). \quad (2.7)$$

Then obviously we have $\{x \in T_0 \mid g(x) = 0\} = \{x \in T_0 \mid \tilde{g}(x) = z\}$. It is well-known that there exists $M \gg 0$ such that the restriction

$$\tilde{g}^{-1}(\mathbb{C} \setminus B(0; M)) \longrightarrow \mathbb{C} \setminus B(0; M) \quad (2.8)$$

of the map $\tilde{g} : T_0 \to \mathbb{C}$ is a locally trivial fibration. Now we require the coefficient $z \in \mathbb{C}$ of $x^a$ in $g(x)$ to satisfy the condition $|z| > M$. This condition on $z \in \mathbb{C}$ in particular implies that the hypersurface $\tilde{g}^{-1}(z) = g^{-1}(0) \subset T_0$ is smooth. We denote by $\Delta_0$ the convex hull of $NP(\tilde{g}) \cup \{0\}$ in $\mathbb{R}^{n-1}$. Then it is easy to see that we have $\Delta_0 = NP(g) - a$. 

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Let $\Sigma_0$ be the dual fan of $\Delta_0$ and $\Sigma$ its smooth subdivision (see [10, 29] etc.). We denote by $X_\Sigma$ the smooth toric variety associated to $\Sigma$. Then $X_\Sigma$ is a compactification of $T_0 = (\mathbb{C}^*)^{n-1}$. Let $D_1, \ldots, D_l \subset X_\Sigma$ be the $T_0$-divisors in $X_\Sigma$ and for $1 \leq i \leq l$ denote by $n_i \geq 0$ the order of the pole of the meromorphic function $\tilde{g}(x)$ on $X_\Sigma$ along $D_i$. Recall that $X_\Sigma \setminus T_0 = \cup_{i=1}^l D_i$ is a normal crossing divisor in $X_\Sigma$. By the non-degeneracy of $g$, the closure $\overline{g^{-1}(0)} \subset X_\Sigma$ of the hypersurface $\tilde{g}^{-1}(0) \subset T_0$ in $X_\Sigma$ intersects any $T_0$-orbit in some $D_i$ such that $n_i > 0$ transversally. More precisely, if

$$q \in (D_{i_1} \cap \cdots \cap D_{i_k} \cap \overline{g^{-1}(0)}) \setminus (\cup_{i \notin \{i_1, \ldots, i_k\}} D_i)$$

(2.9)

for some $1 \leq i_1 < i_2 < \cdots < i_k \leq l$ such that $1 \leq k < \dim X_\Sigma = n - 1$ and $\# \{1 \leq j \leq k \mid n_{i_j} > 0\} \geq 1$, then there exists a local coordinate $y = (y_1, y_2, \ldots, y_{n-1})$ of $X_\Sigma$ on a neighborhood of $q$ such that $q = 0$, $D_{i_j} = \{y_j = 0\}$ $(1 \leq j \leq k)$,

$$\tilde{g}(y) = \frac{y_{n-1}}{y_1^{{n_{i_1}}} \cdots y_k^{{n_{i_k}}}}$$

(2.10)

and $\overline{g^{-1}(0)} = \{y_{n-1} = 0\}$. By this explicit description of $\tilde{g}$ we see that for any $t \in \mathbb{C}$ the closure $\overline{g^{-1}(t)} = \{y_{n-1} = ty_1^{{n_{i_1}}} \cdots y_k^{{n_{i_k}}}\} \subset X_\Sigma$ of the hypersurface $\tilde{g}^{-1}(t) \subset T_0$ in $X_\Sigma$ is smooth in a neighborhood of $q$ and we have

$$D_{i_1} \cap \cdots \cap D_{i_k} \cap \overline{g^{-1}(t)} = D_{i_1} \cap \cdots \cap D_{i_k} \cap \overline{g^{-1}(0)}$$

(2.12)

(see also [10], Section 3.5] etc.). In particular, the hypersurface $\overline{g^{-1}(z)} = \overline{g^{-1}(0)} \subset X_\Sigma$ for the above fixed constant $z \in \mathbb{C}$ with the condition $|z| > M > 0$ has this property and intersects $D_{i_1} \cap \cdots \cap D_{i_k}$ transversally. Moreover, by taking $|z| > M$ large enough, we may assume also that $\overline{g^{-1}(z)} = \overline{g^{-1}(0)}$ intersects any $T_0$-orbit in $X_\Sigma$ transversally.

**Lemma 2.1.** (cf. A’Campo’s lemma) Let $q$ be a point in $X_\Sigma \setminus T_0 = \cup_{i=1}^l D_i$. Assume that for $1 \leq i_1 < i_2 < \cdots < i_k \leq l$ such that $\# \{1 \leq j \leq k \mid n_{i_j} > 0\} \geq 2$ we have

$$q \in (D_{i_1} \cap \cdots \cap D_{i_k}) \setminus (\cup_{i \notin \{i_1, \ldots, i_k\}} D_i \cup \overline{g^{-1}(0)}).$$

(2.13)

Then there exists an open neighborhood $U$ of $q$ in $X_\Sigma$ such that $U \cap \overline{g^{-1}(z)} = U \cap g^{-1}(0) \neq \emptyset$ and

$$H_p\left(\{(U \times \mathbb{C}_t) \cap W^\circ \} \cup Q, Q, \iota_* (\mathcal{L})\right) \simeq 0 \quad (p \in \mathbb{Z})$$

(2.14)

for generic $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$.

**Proof.** By using [9] Lemma 6.1] the proof proceeds completely similarly to that of A’Campo’s lemma for monodromy zeta functions in [1] (see Oka [30], Example (3.7)] etc.). We omit the details. \[\square\]

Let $\Gamma_1, \ldots, \Gamma_m$ be the facets of $\Delta_0$ such that $0 \notin \Gamma_j$. For $1 \leq j \leq m$ let $T_j \simeq (\mathbb{C}^*)^{n-2}$ be the $(n-2)$-dimensional $T_0$-orbit in $X_\Sigma$ associated to $\Gamma_j \smallsetminus \Delta_0$ and $p_j \in \mathbb{Z}^{n-1} \setminus \{0\}$ the primitive outer conormal vector of the facet $\Gamma_j \smallsetminus \Delta_0$. We denote by $d_j > 0$ the value of the linear function $p_j$ on $\Gamma_j$ and call it the lattice height of the origin $0 \in \Delta_0$ from $\Gamma_j$. 

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Then the order of the pole of the meromorphic function $\tilde{g}(x)$ along $T_j$ is equal to $d_j > 0$. Moreover we have

$$\text{Vol}_Z(\Delta_0) = \sum_{j=1}^{m} d_j \cdot \text{Vol}_Z(\Gamma_j).$$

(2.15)

Note that for any $1 \leq j \leq m$ there exists unique $1 \leq i \leq l$ such that $T_j$ is an open subset of $D_i$ and we have $d_j = n_i$. By the above explicit descriptions of $\tilde{g}$ and $\overline{g}^{-1}(z) = g^{-1}(0)$ (see (2.11)), for any $1 \leq j \leq m$ there exists a tubular neighborhood $W_j$ of $T_j^0 = T_j \backslash g^{-1}(0)$ in $X_\Sigma \backslash \overline{g}^{-1}(0)$ such that $W_j \cap g^{-1}(z) = W_j \cap g^{-1}(0) \neq \emptyset$ and $W_j \cap \overline{g}^{-1}(z) = W_j \cap g^{-1}(0)$ is a covering of $T_j^0$ of degree $d_j$. For $p \in \mathbb{Z}$ set

$$V(p)_j := H_p\left(\{(W_j \times \mathbb{C}_t) \cap W^n\} \cup Q, Q, \iota_*(\mathcal{L})\right).$$

(2.16)

Then by Lemma 2.1 and Mayer-Vietoris exact sequences for relative twisted homology groups, for generic $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ we obtain isomorphisms

$$\oplus_{j=1}^{m} V(p)_j \xrightarrow{\sim} V(p) = H^\text{rd}_p(W^n) \quad (p \in \mathbb{Z}).$$

(2.17)

Moreover as in the proof of [9 Proposition 7.5], by using the twisted Morse theory on $T_j \simeq (\mathbb{C}^*)^{n-2}$ (with the help of our figure-8 construction of the rapid decay cycles in the two-dimensional case in [9 Section 6]) for any $1 \leq j \leq m$ we can show that $V(p)_j \simeq 0 \quad (p \neq n)$ and $\dim V(n)_j = h \cdot d_j \cdot \text{Vol}_Z(\Gamma_j)$ for generic $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ and construct a basis of the vector space $V(n)_j$. More precisely, by our Morse theoretical construction of the basis, we obtain a filtration of the $C$-vector space $V(n)_j$ with $\text{Vol}_Z(\Gamma_j)$ subquotients of dimension $h \cdot d_j$.

### 3 The proof of Theorem 1.4

First of all, we briefly recall the results in [9]. In [9] we proved that if the parameter vector $c \in \mathbb{C}^n$ is non-resonant any confluent $A$-hypergeometric function $u(z)$ has an integral representation of the form

$$u(z) = \int_{\gamma^z} \exp(\sum_{j=1}^{N} z_j x^a(j)) x_1^{c_1-1} \cdots x_n^{c_n-1} \, dx_1 \wedge \cdots \wedge dx_n$$

(3.1)

on $\Omega^\text{an} \subset X^\text{an}$, where $\gamma = \{\gamma^z\}$ is a family of real $n$-dimensional topological cycles $\gamma^z$ in the algebraic torus $T = (\mathbb{C}^*)^n$ on which the function $\exp(h_z(x)) = \exp(\sum_{j=1}^{N} z_j x^a(j))$ decays rapidly at infinity. More precisely $\gamma^z$ is an element of Hien’s rapid decay homology group as follows. Assume that $z \in \Omega^\text{an}$. Let $\Sigma_0$ be the dual fan of $\Delta$ in $\mathbb{R}^n$ and $\Sigma$ its smooth subdivision (see [10], [29] etc.). Denote by $Z_\Sigma$ the smooth toric variety associated to the fan $\Sigma$ (see [10], [29] etc.). Then $Z_\Sigma$ is a smooth compactification of $T = (\mathbb{C}^*)^n$ such that $Z_\Sigma \setminus T$ is a normal crossing divisor. Next by using the non-degeneracy of the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_j x^a(j)$, as in [29] Section 3 and [27] Section 3 we construct a complex blow-up $Z := \widehat{Z}_\Sigma$ of $Z_\Sigma$ such that the meromorphic extension of $h_z$ to it has no point of indeterminacy. We say that an irreducible component of the normal crossing divisor $D = Z \setminus T$ is irrelevant if the meromorphic extension of $h_z$ to $Z$ has no
pole along it. Denote by $D' \subset D$ the union of the irrelevant irreducible components in $D$. Let $\pi : \mathcal{Z} \to Z^\text{an}$ be the real oriented blow-up of $Z^\text{an}$ along $D^\text{an}$ (see Section 2 and Hien [15], [16] etc.) and set $\tilde{D} = \pi^{-1}(D^\text{an})$. By the natural open embedding $\iota : T^\text{an} \hookrightarrow \mathcal{Z}$ we consider $T^\text{an}$ as an open subset of $\tilde{Z}$ and set $P_z = \tilde{D} \cap \{x \in T^\text{an} \mid \text{Re} h_z(x) \geq 0\}$ and $Q_z = \tilde{D} \setminus \{P_z \cup \pi^{-1}(D')^\text{an}\}$. Note that $Q_z$ is an open subset of $\tilde{D}$ consisting of the rapid decay directions of the function $\exp(h_z(x))$. Finally let $\mathcal{L}$ be the rank-one local system on $T^\text{an}$ defined by $\mathcal{L} = \mathbb{C}_{T^\text{an}} x_1^{c_1-1} \cdots x_n^{c_n-1}$. Then Hien's rapid decay homology groups that we need for the above integral representation are isomorphic to

$$H^p_{rd}(T)_z := H_p(T^\text{an} \cup Q_z, Q_z; \iota_* (\mathcal{L})) \quad (p \in \mathbb{Z}),$$

where $H_p(T^\text{an} \cup Q_z, Q_z; \iota_* (\mathcal{L}))$ stands for the $p$-th relative twisted homology group of the pair $(T^\text{an} \cup Q_z, Q_z)$ with coefficients in the rank-one local system $\iota_* (\mathcal{L})$ on $\tilde{Z}$ (see [3], [31], [32] etc. and [9] Proposition 3.4)). In the proof of [9, Theorem 4.5] we proved that for any $z \in \Omega^\text{an}$ we have $H^p_{rd}(T)_z \simeq 0 \quad (p \neq n)$ and the dimension of $H^0_{rd}(T)_z$ is Vol$_Z(\Delta)$. Let $\mathcal{H}^\text{rd}_n$ be the local system of rank Vol$_Z(\Delta)$ on $\Omega^\text{an}$ whose stalk at $z \in \Omega^\text{an}$ is isomorphic to $H^0_{rd}(T)_z$ (see also Hien-Roucairol [17]). Namely sections of $\mathcal{H}^\text{rd}_n$ are continuous family of rapid decay $n$-cycles in $T^\text{an}$ for the function $\exp(h_z(x))$. Then one of the main results of [9] is as follows.

**Theorem 3.1.** ([9, Theorem 4.5]) Assume that the parameter vector $c \in \mathbb{C}^n$ is non-resonant. Then we have an isomorphism

$$\mathcal{H}^\text{rd}_n \simeq \text{Hom}_{D_X^\text{an}}((\mathcal{M}_{A,c})^\text{an}, \mathcal{O}_X^\text{an})$$

of local systems on $\Omega^\text{an}$. Moreover this isomorphism is given by the integral

$$\gamma \mapsto \left\{ \Omega^\text{an} \ni z \longmapsto \int_{\gamma^\natural} \exp(\sum_{j=1}^N z_j x^{a(j)}_1 x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n) \right\},$$

where for a continuous family $\gamma$ of rapid decay $n$-cycles in $\Omega^\text{an} \times T^\text{an}$ and $z \in \Omega^\text{an}$ we denote by $\gamma^\natural \subset T^\text{an}$ its restriction to $z \in \Omega^\text{an}$.

By this theorem, the study of the monodromies at infinity of confluent $A$-hypergeometric functions is reduced to that of the rapid homology groups $H_{rd}(T)_z \quad (z \in \Omega^\text{an})$. In [9] Sections 6 and 7 we constructed a basis of $H_{rd}(T)_z$ for generic $c \in \mathbb{C}^n$. From now on, we shall prove Theorem 1.4 by slightly modifying this construction with the help of the results in Section 2.

For the preparation, first let us show that it is enough to prove Theorem 1.4 only for “generic” non-resonant $c \in \mathbb{C}^n$. As in Adolphson [2], fix a point $z^{(0)} \in \Omega$ and consider the $\mathbb{C}[z_1, \ldots, z_N]$-module

$$\mathbb{C}_{z^{(0)}} := \mathbb{C}[z_1, \ldots, z_N]/ \sum_{j=1}^N \mathbb{C}[z_1, \ldots, z_N](z_j - z_j^{(0)}) \simeq \mathbb{C}. \quad (3.5)$$

Then for any non-resonant $c \in \mathbb{C}^n$ the holonomic $D_X$-module $\mathcal{M}_{A,c}$ is an integrable connection of rank Vol$_Z(\Delta)$ on a neighborhood of $z^{(0)} \in \Omega$ and we have an isomorphism

$$\mathbb{C}_{z^{(0)}} \otimes_{\mathbb{C}[z_1, \ldots, z_N]} \mathcal{M}_{A,c} \simeq \mathbb{C}^\text{Vol}_Z(\Delta). \quad (3.6)$$
Let $C$ be a subring of $\hat{C}$ isomorphic to the semigroup algebra $\Gamma[U;\mathcal{M}_{A,c}]$.

There exists a family of bases $s_{1,c}, \ldots, s_{\text{Vol}(\Delta),c} \in \Gamma(U;\mathcal{M}_{A,c})$ (3.7) of the integral connection $\mathcal{M}_{A,c}$ on a neighborhood $U$ of $z^{(0)} \in \Omega$ parametrized by all non-resonant $c \in \mathbb{C}^n$ such that their images $s_{1,c}, \ldots, s_{\text{Vol}(\Delta),c} \in \mathbb{C}[z^{(0)}] \otimes \mathbb{C}[z_1, \ldots, z_N] \mathcal{M}_{A,c} \simeq \mathbb{C}^{\text{Vol}(\Delta)}$ (3.8) generate the $\mathbb{C}$-vector space $\mathbb{C}[z^{(0)}] \otimes \mathbb{C}[z_1, \ldots, z_N] \mathcal{M}_{A,c}$ over $\mathbb{C}$ and the connection matrices of $\mathcal{M}_{A,c}$ with respect to them depend holomorphically on non-resonant $c \in \mathbb{C}^n$.

**Proof.** We recall the constructions in Adolphson [2]. Let $R$ be the $\mathbb{C}$-algebra $\mathbb{C}[x^{a(1)}, \ldots, x^{a(N)}]$ generated by the monomials $x^{a(1)}, \ldots, x^{a(N)}$ over $\mathbb{C}$ and consider the free $\mathbb{C}[z_1, \ldots, z_N]$-module $R[z_1, \ldots, z_N] = \mathbb{C}[z_1, \ldots, z_N] \otimes_{\mathbb{C}} R$. We endow $R[z_1, \ldots, z_N]$ with a structure of a left $D(X)$-module by defining the action of $\frac{\partial}{\partial z_j}$ by

$$\frac{\partial}{\partial z_j} \left( \sum_a c_a(z)x^a \right) = \sum_a \frac{\partial c_a}{\partial z_j}(z)x^a + \sum_a c_a(z)x^{a+a(j)}$$ (3.9)

$(\sum_a c_a(z)x^a \in R[z_1, \ldots, z_N])$. For $1 \leq i \leq n$ and $c \in \mathbb{C}^n$ we define a differential operator $D_{i,c}$ on $R[z_1, \ldots, z_N]$ by

$$D_{i,c} = x_i \frac{\partial}{\partial x_i} + c_i + x_i \frac{\partial h_z}{\partial x_i}(x).$$ (3.10)

Then by [2] Theorem 4.4 there exists an isomorphism

$$\mathcal{M}_{A,c} \simeq \frac{R[z_1, \ldots, z_N]}{\sum_{i=1}^n D_{i,c}R[z_1, \ldots, z_N]}$$ (3.11)

of left $D(X)$-modules. This implies that for the differential operators

$$D_{i,c,z^{(0)}} = x_i \frac{\partial}{\partial x_i} + c_i + x_i \frac{\partial h_z^{(0)}}{\partial x_i}(x)$$ (3.12)

on $R = \mathbb{C}[x^{a(1)}, \ldots, x^{a(N)}]$ we have an isomorphism

$$\mathbb{C}[z^{(0)}] \otimes \mathbb{C}[z_1, \ldots, z_N] \mathcal{M}_{A,c} \simeq \frac{R}{\sum_{i=1}^n D_{i,c,z^{(0)}}R}.$$ (3.13)

Let $C(\Delta) \subset \mathbb{R}^n$ be the cone generated by $\Delta$ and consider the semigroup $C(\Delta) \cap \mathbb{Z}^n$ in it. Let $\hat{R}$ be the $\mathbb{C}$-algebra generated by the monomials $x^a$ (a $\in C(\Delta) \cap \mathbb{Z}^n$). Then $\hat{R}$ is isomorphic to the semigroup algebra $\mathbb{C}[C(\Delta) \cap \mathbb{Z}^n]$ of $C(\Delta) \cap \mathbb{Z}^n$ over $\mathbb{C}$. Note that $R$ is a subring of $\hat{R}$. By [2] Theorem 5.15, if $c \in \mathbb{C}^n$ is non-resonant there exists an isomorphism

$$\frac{R}{\sum_{i=1}^n D_{i,c,z^{(0)}}R} \sim \frac{\hat{R}}{\sum_{i=1}^n D_{i,c,z^{(0)}}\hat{R}}.$$ (3.14)

As explained in the proof of [2] Corollary 5.11, the ring $\hat{R}$ is Cohen-Macaulay for every $A$. So, as explained after [2] Proposition 5.13, the reasoning of the proof of [2] Theorem
with $R$ replaced by $\hat{R}$ is valid for every $A$. In particular, as in [2 (5.7)], one can construct a finite-dimensional subspace $G \subset \hat{R}$ independent of non-resonant $c$, such that

$$\hat{R} = G \oplus (\sum_{i=1}^{n} D_{i,c,z(0)} \hat{R}).$$

(3.15)

Recall that in [2, (5.7)] the basis of $G$ is given by monomials $x^a$ ($a \in C(\Delta) \cap \mathbb{Z}^n$) in $\hat{R}$. In this way, we obtain a basis

$$t_1, \ldots, t_{\text{Vol}_2(\Delta)} \in \mathbb{C}[z(0) \otimes \mathbb{C}[z_1, \ldots, z_N]] M_{A,c}$$

(3.16)

of the $\mathbb{C}$-vector space $\mathbb{C}[z(0) \otimes \mathbb{C}[z_1, \ldots, z_N]] M_{A,c}$. By the proof of [2, (5.7)] it extends to the local one

$$s_{1,c}, \ldots, s_{\text{Vol}_2(\Delta),c} \in \Gamma(U; \mathcal{M}_{A,c})$$

(3.17)

of the integral connection $\mathcal{M}_{A,c}$ on a neighborhood $U$ of $z(0) \in \Omega$. Note also that by the proof of [2, (5.7)] the projection of a given element of $\hat{R}$ to $G$ polynomially depends on $c$. Then by recalling the definitions of the differential operators $\frac{\partial}{\partial z_j}$ on $R$ and $\hat{R}$ we see that the connection matrices of $\mathcal{M}_{A,c}$ with respect to its local basis constructed above depend holomorphically on non-resonant $c \in \mathbb{C}^n$. This completes the proof.

\[\square\]

**Corollary 3.3.** For any $1 \leq j_0 \leq N$ the characteristic polynomial $\lambda^\infty_j(t) \in \mathbb{C}[t]$ depends holomorphically on non-resonant $c \in \mathbb{C}^n$.

**Proof.** Let $\mathcal{M}'_{A,c}$ be the dual of the holonomic $D_X$-module $\mathcal{M}_{A,c}$ (see [20, Section 2.6]). Recall that on $\Omega$ it is nothing but the dual connection of $\mathcal{M}_{A,c}$. Then by [20, Proposition 4.2.1] we have an isomorphism

$$H^0 \text{Sol}_X(\mathcal{M}_{A,c}) \simeq H^{-N}DR_X(\mathcal{M}'_{A,c}).$$

(3.18)

Moreover on $\Omega^{\text{an}}$ the cohomology sheaf $H^{-N}DR_X(\mathcal{M}'_{A,c})$ is isomorphic to the local system consisting of the horizontal sections of the analytic connection $(\mathcal{M}'_{A,c})^{\text{an}}$ associated to $\mathcal{M}'_{A,c}$. Then by Lemma 3.2 and the Cauchy-Kowalevsky theorem (with parameters), for any $z(0) \in \Omega^{\text{an}}$ there exists a basis

$$u_1(z, c), \ldots, u_{\text{Vol}_2(\Delta)}(z, c) \in H^0 \text{Sol}_X(\mathcal{M}_{A,c})_{z(0)}$$

(3.19)

of the $\mathbb{C}$-vector space $H^0 \text{Sol}_X(\mathcal{M}_{A,c})_{z(0)} \simeq \mathbb{C}^{\text{Vol}_2(\Delta)}$ over $\mathbb{C}$ which depends holomorphically on non-resonant $c \in \mathbb{C}^n$. This implies that for any $1 \leq j_0 \leq N$ the monodromy matrix for $\lambda^\infty_j(t)$ has the holomorphic dependence on non-resonant $c \in \mathbb{C}^n$. \[\square\]

By this corollary, it suffices to prove Theorem 1.4 only for “generic” non-resonant $c \in \mathbb{C}^n$.

Now we return to the proof of Theorem 1.4. First let us consider the case $n = 2$. For simplicity, here we treat only the case where $r = 1$ and $a(j_0) \in \text{Int}(\Delta_1)$. The general case can be proved similarly. Let $h > 0$ be the lattice height of the origin $0 \in \Delta \subset \mathbb{R}^2$ from the 1-dimensional facet $\Delta_1 \subset \Delta$ (for the definition see Sections 1 and 2). Then we have the equality $\text{Vol}_2(\Delta_1) = h \cdot \text{Vol}_2(\Delta_1)$. Let $d_1 > 0$ (resp. $d_2 > 0$) be the lattice height of the point $a(j_0) \in \text{Int}(\Delta_1)$ from the vertex facet $\Gamma_{11}$ (resp. $\Gamma_{12}$) of the segment $\Delta_1$. Here
we have Vol$_2$(Δ) = d$_1$ + d$_2$ and
\[ h \cdot d_j = h_{1j} \cdot \text{Vol}_2(\tilde{\Gamma}_{1j}) \quad (j = 1, 2). \] (3.20)

By a suitable automorphism Ψ : (ℝ$^2$, ℤ$^2$) ∼ (ℝ$^2$, ℤ$^2$) of (ℝ$^2$, ℤ$^2$) we may assume that Δ$\subset$ {v = (v$_1$, v$_2$) | v$_2$ = h} ⊂ ℝ$^2$ and Σ$\subset$ {p, h}, Γ$\subset$ {q, h} for some integers p, q ∈ ℤ such that p − q = Vol$_2$(Δ) = d$_1$ + d$_2$. In this situation, let Σ be a smooth subdivision of the dual fan of Δ in ℝ$^2$. Let τ ∈ Σ be the 1-dimensional cone which corresponds to the facet Δ$\subset$ {v$_2$ = h} ⊂ ℝ$^2$ of Δ and σ$_1$, σ$_2$ ∈ Σ the unique 2-dimensional cone containing τ which corresponds to the vertex Γ$\subset$ (resp. Γ$\subset$) of Δ. Note that the primitive vectors on the two edges of σ$_1$ are
\[ b_1 = \begin{pmatrix} -1 \\ k \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \] (3.21)
for some integer k ∈ ℤ. Let Z$_\Sigma$ be the smooth toric variety associated to Σ and C$^2$(σ$_1$) ≃ C$^2$ its affine open subset associated to the 2-dimensional cone σ$_1$ ∈ Σ. On C$^2$(σ$_1$) ≃ C$^2$ the multi-valued function $x_1^{c_1-1}x_2^{c_2-1}$ can be written as
\[ x_1^{c_1-1}x_2^{c_2-1} = y_1^{(b_1.c-e)} \cdot y_2^{(b_2.c-e)}, \] (3.22)
where we set
\[ e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \] (3.23)

Moreover on C$^2$(σ$_1$) ≃ C$^2$ our Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ can be written in the form
\[ h_z(y) = y_1^{kh-p}y_2^{-h} \times \tilde{h}_z(y), \] (3.24)
where $\tilde{h}_z(y)$ is a polynomial i.e. its Newton polytope is contained in the first quadrant ℝ$^2$ of ℝ$^2$. Note also that the T-orbit $T_1 = \mathbb{C}^*$ in Z$_\Sigma$ which corresponds to τ ∈ Σ is \( \{ y = (y_1, y_2) \mid y_1 \in \mathbb{C}^*, y_2 = 0 \} \simeq \mathbb{C}^* \simeq \mathbb{C}^2(\sigma_1). \) Now let g(y$_1$) be the restriction of the polynomial $\tilde{h}_z(y)$ to the T-orbit $T_1 = \mathbb{C}^*$ in Z$_\Sigma$. It is easy to see that this polynomial g(y$_1$) of y$_1$ contains the term $z_j y_1^{k_j}$ and its Newton polytope is the closed interval [0, p − q] = [0, d$_1$ + d$_2$] in ℝ$^1$. Note that g(y$_1$) is naturally identified with the Δ$_1$-part $h_{\Delta_1}$ of $h_z$. By the non-degeneracy of $h_z$ we have \# \{ y$_1$ ∈ T$_1$ | g(y$_1$) = 0 \} = Vol$_2$(Δ) = p − q = d$_1$ + d$_2$. Let Z = Z$_\Sigma$ → Z$_\Sigma$, π : Z → Z, Z$_\Sigma$ ⊂ D, T$^\text{an}$ → Z etc. be as before. We denote the strict transform of T$_1$ ≃ C$^*$ in Z as Z$^\circ$ = W ∩ T$^\text{an}$. Let W be a sufficiently small tubular neighborhood of T$_1$ in Z = Z$_\Sigma$ and set W$^\circ$ = W ∩ T$^\text{an}$. 

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Then in [9] Theorem 6.4 we showed that if $c \in \mathbb{C}^2$ is generic

$$H_2^{rd}(W^o)_z = H_2(W^o \cup Q_z, Q_z; \iota_*(\mathcal{L}))$$

(3.25)
is a linear subspace of $H_2^{rd}(T)_z = H_2(T^{an} \cup Q_z, Q_z; \iota_*(\mathcal{L}))$. Moreover in [9] Proposition 6.2 and the paragraph just after it, we constructed a natural basis of this Vol$_z(\Delta_1)$-dimensional vector space $H_2^{rd}(W^o)_z$. Recall that this basis was constructed by $d_1 + d_2$ figure-8s in $T_1 \simeq \mathbb{C}^*$ associated to the $d_1 + d_2$ points $\{y_1 \in T_1 \mid g(y_1) = 0\}$ in $T_1$. Let $C \subset \mathbb{L} \simeq \mathbb{C}$ be a sufficiently large circle in $\mathbb{L} \simeq \mathbb{C}$ and for a point $z = (z_1, \ldots, z_N) \in C$ set $V = H_2^{rd}(W^o)_z \subset H_2^{rd}(T)_z$. Let $\Phi : V \rightarrow V$ be the automorphism of $V$ induced by the rotation of the point $z_{j_0} \in C \subset \mathbb{L} \simeq \mathbb{C}$ along $C$. Then it suffices to show that the characteristic polynomial of $\Phi$ is equal to

$$\prod_{j=1}^{2} \left\{ t^{h_{1j}} - \exp(-2\pi \sqrt{-1}(\rho_{1j}, c)) \right\}^{\text{Vol}_z(\Gamma_j)}. $$

(3.26)

Now let us set

$$\tilde{g}(y_1) = z_{j_0} - y_1^{-d_1} g(y_1).$$

(3.27)

Then the Newton polytope of the Laurent polynomial $\tilde{g}(y_1)$ is the closed interval $[-d_1, d_2]$ in $\mathbb{R}^1$ and for any $y_1 \in T_1 \simeq \mathbb{C}^*$ we have an equivalence

$$g(y_1) = 0 \iff \tilde{g}(y_1) = z_{j_0}. $$

(3.28)

Since $|z_{j_0}|$ is sufficiently large, the set $\{y_1 \in T_1 \mid g(y_1) = 0\} = \{y_1 \in T_1 \mid \tilde{g}(y_1) = z_{j_0}\}$ splits into two parts, i.e. the one consisting of the $d_1$ points $q_1, \ldots, q_{d_1} \in T_1$ in a neighborhood of the origin of $\mathbb{C}^{y_1}$ and the other consisting of the remaining $d_2$ points $q'_{d_1}, \ldots, q'_{d_2} \in T_1$ at infinity. By this splitting, we can slightly modify the construction of the basis of $V = H_2^{rd}(W^o)_z$ in [9] Section 6 so that we have a direct sum decomposition $V = V_1 \oplus V_2$ of $V$, where $V_1 \simeq \mathbb{C}_y^{h \cdot d_1}$ (resp. $V_2 \simeq \mathbb{C}_y^{h \cdot d_2}$) has a basis consisting of $h \cdot d_1$ (resp. $h \cdot d_2$) rapid decay 2-cycles over the $d_1$ (resp. $d_2$) figure-8s associated to the $d_1$ points $q_1, \ldots, q_{d_1} \in T_1 \simeq \mathbb{C}^*$ (resp. the $d_2$ points $q'_{d_1}, \ldots, q'_{d_2} \in T_2 \simeq \mathbb{C}^*$) as follows. Let us explain the construction of the vector space $V_1$. By homotopy we may assume that for some $0 < \varepsilon \ll 1$ we have

$$q_i = \varepsilon \exp \left( 2\pi \sqrt{-1} \frac{d_1 - i + 1}{d_1} \right) \in T_1 = \mathbb{C}_y^{*} \quad (1 \leq i \leq d_1).$$

(3.29)
Note that by the rotation of the point $z_j \in \mathbb{C} \subset \mathbb{L} \simeq \mathbb{C}$ along $C$ the point $q_i$ ($1 \leq i \leq d_1 - 1$) (resp. $q_{d_1}$) is sent to $q_{i+1}$ (resp. $q_1$). Let $F_i$ ($1 \leq i \leq d_1 - 1$) (resp. $F_{d_1}$) be a figure-8 surrounding the two points $q_i$ and $q_{i+1}$ (resp. $q_{d_1}$ and $q_1$) (see [9, Section 6]).

We take such figure-8s $F_1, F_2, \ldots, F_{d_1} \subset T_1 = \mathbb{C}^*_y$ so that by the rotation of $z_j \in \mathbb{C}$ the $i$-th one $F_i$ ($1 \leq i \leq d_1 - 1$) (resp. the $d_1$-th one $F_{d_1}$) is sent to $F_{i+1}$ (resp. $F_1$). For the unique singular point $\alpha \in F_1$ of the figure-8 $F_1$ choose a normal slice $S_\alpha \simeq \mathbb{C}_{y_1} \subset \mathbb{C}^2(\sigma_1) \simeq \mathbb{C}^2_y$ of $\bar{T}_1 \simeq \mathbb{C}_{y_1} \subset \mathbb{C}^2(\sigma_1) \simeq \mathbb{C}^2_y$ at $\alpha \in \bar{T}_1$. Then $\pi^{-1}(S_\alpha) \subset \bar{Z}$ is isomorphic to the real oriented blow-up $\widehat{\mathbb{C}_{y_2}}$ of the complex plane $\mathbb{C}_{y_2}$ along the origin $\{0\} \subset \mathbb{C}_{y_2}$ and the open subset $Q_z \cap \pi^{-1}(S_\alpha)$ of $\bar{D} \cap \pi^{-1}(S_\alpha) \simeq S^1$ consists of $h$ open intervals $I_1, I_2, \ldots, I_h \subset \bar{D} \cap \pi^{-1}(S_\alpha) \simeq S^1$. We may assume that $I_1, I_2, \ldots, I_h$ are arranged in the clockwise order. Let $\iota_0 : S_\alpha \setminus \{\alpha\} \simeq \mathbb{C}^*_y \hookrightarrow \widehat{\mathbb{C}_{y_2}}$ be the inclusion map. Then by [9, Lemma 3.6(i)] there exists a natural basis of the $h$-dimensional vector space

$$H_1(\mathbb{C}^*_y \cup (I_1 \sqcup \cdots \sqcup I_h), I_1 \sqcup \cdots \sqcup I_h ; (\iota_0)_*(\mathcal{L}|_{S_\alpha \setminus \{\alpha\}})) \tag{3.30}$$

consisting of $h$ (twisted) 1-chains $\gamma_1, \gamma_2, \ldots, \gamma_h$ connecting two consecutive intervals among $I_1, \cdots, I_h$.

Fig. 3: The twisted 1-chains $\gamma_i$
We may assume that \( \gamma_1, \gamma_2, \ldots, \gamma_h \subset \pi^{-1}(S_u) \simeq \mathbb{C}^2 \) are arranged in the clockwise order and the sections of the local system \( \{ \alpha \} \), \( \mathcal{L}|_{S_u \setminus \{ \alpha \}} \) in the “twisted” 1-chains \( \gamma_2, \ldots, \gamma_h \) are defined by the analytic continuations (in the clockwise direction) of that in the first one \( \gamma_1 \). As in the paragraph just after [9, Proposition 6.2], by dragging \( \gamma_1, \gamma_2, \ldots, \gamma_h \) over the figure-8 \( F_1 \) keeping their end points in \( Q \subset \tilde{D} \) we can naturally construct \( h \) rapid decay 2-cycles \( \delta_{11}, \delta_{12}, \ldots, \delta_{1h} \) over \( F_1 \) in \( V = H^d_2(W^\circ)_z = H^d_2(W^\circ \cup Q_z, Q_z \cap \mathcal{L}) \). By the rotation of the point \( z_j \in C \subset L \simeq \mathbb{C} \) along \( C \) and the corresponding analytic continuations of the sections of \( \mathcal{L} \) we obtain \( h \) rapid decay 2-cycles \( \delta_{21}, \delta_{22}, \ldots, \delta_{2h} \) over the figure-8 \( F_2 \) in \( V = H^d_2(W^\circ)_z \). By repeating this construction, for any \( 1 \leq i \leq d_1 \) we obtain
\[
\delta_{i1}, \delta_{i2}, \ldots, \delta_{ih} \in V = H^d_2(W^\circ)_z. \tag{3.31}
\]
As in [9, Section 6] we can show that if \( c \in \mathbb{C}^2 \) is generic these \( h \cdot d_1 \) rapid decay 2-cycles in \( V = H^d_2(W^\circ)_z \subset H^d_2(T)_z \) are linearly independent. We denote by \( V_1 \simeq \mathbb{C}^{h \cdot d_1} \) the linear subspace of \( V \) spanned by them. Similarly by using the remaining \( d_2 \) points \( q_1', \ldots, q_{d_2}' \in T_1 \simeq \mathbb{C}^{y_1} \) at infinity, we obtain a linear subspace \( V_2 \simeq \mathbb{C}^{h \cdot d_2} \) of \( V \) so that we have a direct sum decomposition \( V = V_1 \oplus V_2 \) of \( V \). We can easily see that \( V_1, V_2 \subset V \) are invariant by the automorphism \( \Phi : V \sim V \) of \( V \) induced by the rotation of the point \( z_j \in C \). From now on, we will show that the characteristic polynomial of \( \Phi_1 = \Phi|_{V_1} : V_1 \sim V_1 \) is equal to
\[
\left\{ t^{h_{11}} - \exp(-2\pi \sqrt{-1}(\rho_{11}, c)) \right\}^{\text{Vol}(\Gamma_{11})}. \tag{3.32}
\]
By our construction of the basis \( \delta_{ij} \) (\( 1 \leq i \leq d_1, 1 \leq j \leq h \)) of \( V_1 \subset V = H^d_2(W^\circ)_z \) for any \( 1 \leq i \leq d_1 - 1 \) and \( 1 \leq j \leq h \) we have \( \Phi_1(\delta_{ij}) = \delta_{i+1,j} \). For \( m \in \mathbb{Z} \) we define two integers \( \{m\}_h \) and \( [m]_h \) such that \( 1 \leq [m]_h \leq h \) and \( m = h \cdot \{m\}_h + [m]_h \). Then by (3.24) for any \( 1 \leq j \leq h \) we have \( \Phi_1(\delta_{ij}) = \varepsilon_1 \cdot \varepsilon_2^{(j+1)h} \cdot \delta_{i,j+1} \), where we set \( \varepsilon_1 = \exp(2\pi \sqrt{-1}c_1) \) and \( \varepsilon_2 = \exp(-2\pi \sqrt{-1}c_2) \). Define a square matrix \( K \in M_h(\mathbb{C}) \) of size \( h \) by
\[
K = \varepsilon_1 \begin{pmatrix} 0 & \cdots & 0 & \varepsilon_2 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \varepsilon_2 & \cdots & 0 & 0 \end{pmatrix}^p \in M_h(\mathbb{C}), \tag{3.33}
\]
where \( I_{h-1} \) is the \((h-1) \times (h-1)\) identity matrix. Then by the basis \( \delta_{ij} \) (\( 1 \leq i \leq d_1, 1 \leq j \leq h \)) of \( V_1 \), the automorphism \( \Phi_1 : V_1 \sim V_1 \) of \( V_1 \) is represented by the following square matrix \( L \in M_{h \cdot d_1}(\mathbb{C}) \) of size \( h \cdot d_1 \):
\[
L = \begin{pmatrix} 0 & K \\ I_h & 0 \\ 0 & I_h & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_h & 0 \end{pmatrix} \in M_{h \cdot d_1}(\mathbb{C}), \tag{3.34}
\]
where \( I_h \in M_h(\mathbb{C}) \) stands for the identity matrix of size \( h \). It follows that the characteristic
polynomial of the automorphism $\Phi_1 = \Phi|_{V_1} : V_1 \xrightarrow{\sim} V_1$ of $V_1$ is equal to

$$\prod_{\zeta^h = \exp(-2\pi \sqrt{-1}c_1)} \left\{ t^{d_i} - \exp(2\pi \sqrt{-1}c_1) \cdot \zeta^p \right\}. \quad (3.35)$$

It is easy to see that it can be rewritten as

$$\left\{ t^{h_{11}} - \exp(-2\pi \sqrt{-1}(\rho_{11}, c)) \right\}^{\text{Vol}_Z(\tilde{\Gamma}_{11})}. \quad (3.36)$$

This completes the proof for the case $n = 2$.

Let us consider the general case $n \geq 2$. For simplicity, here we consider only the case where $r = 1$ and $a(j_0) \in \text{Int}(\Delta_1)$. The proof for the general case is similar. Let $\Sigma$, $Z_\Sigma$, $Z = Z_\Sigma \to Z_\Sigma$, $\pi : Z \to Z$, $Q_z \subset \tilde{D}$, $\iota : T^\text{an} \to \tilde{Z}$ etc. be as before. We denote by $T_1 \simeq (\mathbb{C}^*)^{n-1}$ the $T$-orbit in $Z_\Sigma$ associated to the facet $\Delta_1 \prec \Delta$ of $\Delta$. We also denote by $T_1 \simeq (\mathbb{C}^*)^{n-1}$ its strict transform in $Z = Z_\Sigma$. Let $W$ be a sufficiently small tubular neighborhood of $T_1$ in $Z = Z_\Sigma$, and set $W^\circ = W \cap T^\text{an}$. Recall that in \cite[Theorem 7.6]{2} we showed that if $c \in \mathbb{C}^n$ is generic

$$H^r_n(W^\circ)_z = H_n(W^\circ \cup Q_z, Q_z; \tau_S(\mathcal{L})) \quad (3.37)$$

is a linear subspace of $H^r_n(T)_z$. Let $C \subset \mathbb{L} \simeq \mathbb{C}$ be a sufficiently large circle in $\mathbb{L} \simeq \mathbb{C}$ and for a point $z \in C$ set $V = H^r_n(W^\circ)_z \subset H^r_n(T)_z$. Let $\Phi : V \xrightarrow{\sim} V$ be the automorphism of $V$ induced by the rotation of the point $z_{j_0} \in C$. Recall that the dimension of the vector space $V$ is equal to

$$\text{Vol}_Z(\tilde{\Delta}_1) = h \cdot \text{Vol}_Z(\Delta_1) = \sum_{j=1}^{m_1} h_{1j} \cdot \text{Vol}_Z(\tilde{\Gamma}_{1j}), \quad (3.38)$$

where $h > 0$ is the lattice height of the origin $0 \in \Delta \subset \mathbb{R}^n$ from the facet $\Delta_1 \prec \Delta$ (for the definition see Sections 1 and 2). For $1 \leq j \leq m_1$ let $d_j > 0$ be the lattice height of the point $a(j_0) \in \Delta_1$ from the facet $\Gamma_{1j} \prec \Delta_1$. Here we define the lattice heights $d_j$ in the affine span $\text{Aff}(\Delta_1) \simeq \mathbb{R}^{n-1}$ of $\Delta_1$ in $\mathbb{R}^n$ so that we have the equality $h_{1j} \cdot \text{Vol}_Z(\tilde{\Gamma}_{1j}) = h \cdot d_j \cdot \text{Vol}_Z(\Gamma_{1j})$. Then we can slightly modify the construction of the basis of $V$ in \cite[Section 7]{3} with the help of our results in Section 2 to obtain a direct sum decomposition

$$V = \bigoplus_{j=1}^{m_1} V_j \quad (3.39)$$

of $V$ by some vector subspaces $V_j$, where $\dim V_j = h \cdot d_j \cdot \text{Vol}_Z(\Gamma_{1j})$ and $V_j$ has a basis consisting of $h \cdot d_j \cdot \text{Vol}_Z(\Gamma_{1j})$ rapid decay $n$ cycles associated to the facet $\Gamma_{1j}$ of $\Delta_1$. Note that by our Morse theoretical construction of the basis of $V_j$ we have a filtration of the $\mathbb{C}$-vector space $V_j$ with $\text{Vol}_Z(\Gamma_{1j})$ subquotients of dimension $h \cdot d_j$. Moreover as in the case $n = 2$ we can show that the characteristic polynomial of the automorphism of each subquotient induced by $\Phi|_{V_j} : V_j \xrightarrow{\sim} V_j$ is equal to

$$\left\{ t^{h_{1j}} - \exp(-2\pi \sqrt{-1}(\rho_{1j}, c)) \right\}^{l_j}, \quad (3.40)$$

where $l_j > 0$ is the lattice height of the origin $0 \in \tilde{\Gamma}_{1j}$ from the facet $\Gamma_{1j} \prec \tilde{\Gamma}_{1j}$. Then the assertion immediately follows. This completes the proof.
Remark 3.4. For $1 \leq j_1 < j_2 < \cdots < j_k \leq N$ ($k > 0$) assume that no two of the $k$ points $a(j_1), \ldots, a(j_k)$ lie on the same facet of $\Delta$ not containing the origin $0 \in \mathbb{R}^n$. Let $z^{(0)} = (z_1^{(0)}, \ldots, z_N^{(0)}) \in X = \mathbb{C}^N$ be a point such that $z_j^{(0)} \neq 0$ if and only if $j \in \{j_1, \ldots, j_k\}$. Let $L \simeq \mathbb{C}$ be a complex line in $X = \mathbb{C}^N$ parallel to the one generated by the non-zero vector $z^{(0)} \neq 0$ such that
\[
\# \{L \cap (X \setminus \Omega)\} < +\infty
\] (3.41)
and $C \subset L \simeq \mathbb{C}$ a sufficiently large circle in it. Then by the above proof, also for the monodromy of the confluent $A$-hypergeometric functions along such $C$ we obtain a straightforward generalization of Theorem 1.4.

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