On three-dimensional trace anomaly from holographic local RG

Ken Kikuchi*, Hiroto Hosoda* and Akihiro Suzuki*

*E-mail: kikuchi@eken.phys.nagoya-u.ac.jp, hosoda@eken.phys.nagoya-u.ac.jp, aakihiro@eken.phys.nagoya-u.ac.jp

Received September 26, 2016; Accepted November 4, 2016; Published January 15, 2017

Odd-dimensional quantum field theories (QFTs) can have nonzero trace anomalies if external fields are introduced and some ingredients needed to make Lorentz scalars with appropriate mass dimensions (or weights) are supplied. We have studied a three-dimensional QFT and explicitly computed the trace of the stress tensor using the holographic local renormalization group (RG). We have checked some properties of vector beta functions and the Wess–Zumino consistency condition; however, we have found that the anomalies vanish on fixed points. We clarify what is responsible for the vanishing trace anomalies.

Subject Index B31, B32, B34

1. Introduction

Without a doubt, symmetry plays a central role in physics. For example, spacetime symmetries impose various conservation laws, which govern classical physics almost completely. Poincaré symmetry is one such symmetry and is one of the fundamental assumptions of quantum field theories (QFTs). However, in quantum theories, it happens that some symmetries are violated due to quantum corrections. They are called anomalies. Thus anomalies play important roles in quantum theories. For instance, the chiral anomaly gave human beings an insight into how many colors Nature has.

In this work, we would like to study an anomaly called the trace anomaly. Its general classification was given in Ref. [1]. The anomaly has also been playing a significant role in QFTs. In fact, it is known that in two and four spacetime dimensions, coefficients of some terms in the trace anomaly can be interpreted as the “number of degrees of freedom” in theories [2–4]. See Ref. [5] for the case of three dimensions.

Being important, the trace anomaly has been calculated in many ways. In local RG, we lift a scale parameter into a spacetime dependent function and identify the Weyl transformations of the metric as local scale transformations. In this line, to make the theory consistent, coupling “constants” are forced to have spacetime dependence and are promoted to coupling “functions”. This is why the method is called the “local” renormalization group (LRG). The Weyl variation equation describes the RG flow and we can obtain the trace anomalies from this equation. When we use holography, we can get this LRG equation by the formalism made first in Ref. [6], generalized in Ref. [7] to general even spacetime dimensions, and in Ref. [8] to gauge theories. The explicit relation between LRG and the holographic Hamilton–Jacobi method was first elucidated in Ref. [9].
can derive equations of motion by the Hamilton–Jacobi formalism. In the AdS/CFT correspondence, this equation, the so-called flow equation, can be considered as the LRG equation of boundary field theory. Strictly speaking, we should say that this method is using holography. In this way, we can compute the trace anomalies in $d$-dimensional QFTs.

It is believed that the trace anomalies trivially vanish in odd spacetime dimensions, and the use of the method was limited to even spacetime dimensions. However, as in even spacetime dimensions, by introducing external fields, and furthermore by breaking parity so as to supply an ingredient (i.e., the Levi-Civita tensor) to make Lorentz scalars from odd numbers of Lorentz indices, there is no reason for the trace anomaly to vanish. Nakayama pointed out [10] the possibility and wrote down consistency conditions that the three-dimensional trace anomaly should obey. In this work, limiting our analysis to QFTs with bulk duals, we explicitly computed the trace anomaly and, in contrast to our optimistic expectation, ended up finding that the anomaly vanishes on fixed points. An explicit computation was also done in Ref. [11].

The organization of the paper is as follows: First, we put forward our calculations following the well-known formalism: the Hamilton–Jacobi formalism. In this formalism the so-called flow equation has great importance. We get the trace of the stress tensor through this equation, and at the same time the scalar and vector beta functions. Next, we check some properties of the beta functions and the Wess–Zumino (WZ) consistency conditions of the anomaly coefficients following Ref. [10]. Finally, we conclude our analysis and clarify the reason why the trace anomalies vanish in our situation.

2. Formalism

We start with a simple extension of the bulk action in Ref. [8] by adding the $\theta$-term to break the parity:

$$S = \int_{M_4} d^4x \sqrt{\gamma} \left\{ V(\hat{\phi}) - \hat{R}(4) + \frac{1}{2} L_{I}^{J}(\hat{\phi}) \hat{\gamma}^I \hat{\nabla}^I \hat{\phi} \hat{\gamma}^J \hat{\nabla}^J \hat{\phi} + \frac{1}{4} B(\hat{\phi}) \hat{F}_{\hat{\mu}\hat{\nu}} \hat{F}^{\hat{\mu}\hat{\nu}} + \frac{1}{4} \Theta_{(4)}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{F}_{\hat{\mu}\hat{\nu}} \hat{F}^{\hat{\rho}\hat{\sigma}} \right\} - 2 \int_{\Sigma_3} d^3x \sqrt{\hat{h}} \hat{K},$$

where fields with a hat denote off-shell fields, which are not necessarily solutions of equations of motion. Our notation is collected in Appendix A. Since we want to identify the scalar fields with coupling functions in the (LRG), we restrict gauge symmetries $G$ to groups with real representations such as SO$(N)$. The scalar fields belong to some real representation $r$; therefore we do not distinguish upper and lower indices. The second term on the right-hand side is the Gibbons–Hawking term, which is needed to treat the action in the Hamiltonian formalism and is defined on a three-dimensional hypersurface $\Sigma_3 := \{ X \in M_4 | \tau = \text{const.} \}$. We also define an induced metric $\hat{h}_{\mu\nu}$ and determinants $\hat{\gamma} := -\det(\hat{\gamma}_{\hat{\mu}\hat{\nu}})$ and $\hat{h} := -\det(\hat{h}_{\mu\nu})$. Following the traditional method, i.e., by using the ADM decomposition

$$ds^2 = \hat{\gamma}_{\hat{\mu}\hat{\nu}} dX^\hat{\mu} dX^\hat{\nu} = \hat{N}^2(x, \tau) d\tau^2 + \hat{h}_{\mu\nu}(x, \tau) [dx^\mu + \hat{\lambda}^\mu(x, \tau) d\tau] [dx^\nu + \hat{\lambda}^\nu(x, \tau) d\tau],$$

1 We would like to express our appreciation to Adam Schwimmer for bringing the paper to our attention and elucidating a mechanism for producing trace anomalies.
and defining canonical momenta

\[
\hat{\pi}^{\mu \nu} := \frac{\partial L_4}{\partial (\partial_\tau \tilde{h}_{\mu \nu})} = \tilde{K}^{\mu \nu} - \hat{h}^{\mu \nu} \hat{K},
\]

\[
\hat{\pi}^I := \frac{\partial L_4}{\partial (\partial_\tau \phi^I)} = \frac{1}{N} L_{JI} (\phi) \left( \hat{\nabla}_\tau \phi^J - \hat{\lambda}^\mu \hat{\nabla}_\mu \phi^J \right),
\]

\[
\hat{\pi}^{a \mu} := \frac{\partial L_4}{\partial (\partial_\tau A^{a \mu})} = \frac{1}{N^3} B (\phi) \left[ \hat{N}^2 \hat{h}^{\mu \nu} \hat{F}^a_{\tau \nu} - \hat{\lambda}^\nu \left( \hat{N}^2 \hat{h}^{\rho \mu} + \hat{\lambda}^\rho \hat{\lambda}^\mu \right) \hat{F}^a_{\nu \rho} \right] - \hat{N} \Theta \epsilon_{(4)}^{\mu \nu \rho \tau} \hat{F}^a_{\nu \rho \tau} - \Theta \epsilon_{(3)}^{\mu \nu \rho} \hat{F}^a_{\rho \nu},
\]

where \( S = \int d^3 x d\tau \sqrt{\hat{h}} L_4 + (GH) \), one arrives at the first-order action:

\[
S [\tilde{h}_{\mu \nu}, \phi^I, \hat{A}^{a \mu}, \hat{N}, \hat{\lambda}^\mu; (x, \tau)] = \int d^3 x d\tau \sqrt{\hat{h}} \left\{ \hat{\pi}^{\mu \nu} \partial_\tau \tilde{h}_{\mu \nu} + \hat{\pi}^I \partial_\tau \phi^I + \hat{\pi}^{a \mu} \partial_\tau A^{a \mu} \\
+ \hat{N} \left[ \frac{1}{2} \hat{\pi}^2 - \hat{\pi}^{\mu \nu} - \frac{1}{2} L_{JI} (\phi) \hat{\pi}^I \hat{\pi}^J - \frac{1}{2 B (\phi)} \hat{h}^{\mu \nu} \hat{\pi}^a \hat{\pi}^a \\
- \frac{\Theta}{B (\phi)} \epsilon_{(3)}^{\mu \nu \rho} \hat{\pi}^a \hat{F}^a_{\nu \rho} + V (\phi) - \hat{R}_{(3)} \right] \right. \\
+ \frac{1}{2} L_{JI} (\phi) \hat{h}^{\mu \nu} \hat{\nabla}_\mu \phi^I \hat{\nabla}_\nu \phi^J \\
+ \left. \left( \frac{1}{4} B (\phi) + \frac{\Theta^2}{B (\phi)} \right) \hat{F}^a_{\mu \nu} \hat{F}^{a \mu \nu} \right\} \\
+ \hat{\lambda}^\mu \left[ 2 \hat{\nabla}^\nu \hat{\pi}_{\mu \nu} - \hat{\pi}^I \hat{\nabla}_\mu \phi^I - \hat{\pi}^{a \mu} \hat{\pi}^a \right] \\
+ \hat{A}^{a \mu}_\tau \left[ \hat{\nabla}_\mu \hat{\pi}^{a \mu} - (i T^a \phi) \hat{\pi}^I \right] \\
+ (GH \text{ term}).
\]

As one notices at once, the action does not contain \( \tau \) derivatives of \( \hat{N}, \hat{\lambda}^\mu \), and \( \hat{A}^{a \mu}_\tau \); thus these fields are auxiliary fields, and their equations of motion yield the first-class constraints

\[
\hat{H} := \frac{1}{\sqrt{\hat{h}}} \frac{\delta S}{\delta \hat{N}} \\
= \frac{1}{2} \hat{\pi}^2 - \hat{\pi}^{\mu \nu} - \frac{1}{2} L_{JI} (\phi) \hat{\pi}^I \hat{\pi}^J - \frac{1}{2 B (\phi)} \hat{h}^{\mu \nu} \hat{\pi}^a \hat{\pi}^a - \Theta \epsilon_{(3)}^{\mu \nu \rho \tau} \hat{F}^a_{\nu \rho \tau} \approx 0,
\]

\[
\hat{P}_\mu := \frac{1}{\sqrt{\hat{h}}} \frac{\delta S}{\delta \hat{\lambda}_\mu} \\
= 2 \hat{\nabla}^\nu \hat{\pi}_{\mu \nu} - \hat{\pi}^I \hat{\nabla}_\mu \phi^I - \hat{\pi}^{a \mu} \hat{\pi}^a \approx 0,
\]

\[
\hat{G}^a := \frac{1}{\sqrt{\hat{h}}} \frac{\delta S}{\delta \hat{A}^{a \mu}_\tau} = \hat{\nabla}_\mu \hat{\pi}^{a \mu} - (i T^a \phi) \hat{\pi}^I \approx 0.
\]
Equations (7) and (8) are Hamiltonian and momentum constraints, respectively, which ensure “time” translation invariance and three-dimensional diffeomorphism invariance, respectively. Equation (9) is nothing but Gauss’s law and it guarantees the gauge invariance of the system.

Solving the equations of motion with Dirichlet boundary conditions at \( \tau = \tau_0 \),

\[
\bar{h}_{\mu\nu}(x, \tau = \tau_0) = h(x), \quad \bar{\phi}^I(x, \tau = \tau_0) = \phi^I(x), \quad \bar{A}_\mu^a(x, \tau = \tau_0) = A_\mu^a(x, \tau = \tau_0),
\]

where fields with a bar indicate on-shell fields, one attains an on-shell action:

\[
S[h_{\mu\nu}(x), \phi^I(x), A_\mu^a(x); \tau_0] := S[\bar{h} = \bar{h}, \bar{\phi} = \bar{\phi}, \bar{A} = \bar{A}; (x, \tau_0)] = \int d^3 x \int_{\tau_0}^\infty d\tau \sqrt{\bar{h}} \left\{ \bar{\pi}^{\mu\nu} \partial_\tau \bar{h}_{\mu\nu} + \bar{\pi}^I \partial_\tau \bar{\phi}^I + \bar{\pi}^{a\mu} \partial_\tau \bar{A}_\mu^a \right\}. \tag{10}
\]

Its variation

\[
\delta S[h(x), \phi(x), A(x); \tau_0] = - \int d^3 x \sqrt{h} \left\{ \bar{\pi}^{\mu\nu}(x, \tau_0) \delta h_{\mu\nu}(x) + \bar{\pi}^I(x, \tau_0) \delta \phi^I(x) + \bar{\pi}^{a\mu}(x, \tau_0) \delta A_\mu^a(x) \right\} \tag{11}
\]

yields Hamilton–Jacobi (HJ) equations:

\[
\bar{\pi}^{\mu\nu}(x, \tau_0) = - \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h_{\mu\nu}(x)}, \quad \bar{\pi}^I(x, \tau_0) = - \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \phi^I(x)}, \quad \bar{\pi}^{a\mu}(x, \tau_0) = - \frac{1}{\sqrt{h}} \frac{\delta S}{\delta A_\mu^a(x)}, \quad \frac{\partial S}{\partial \tau_0} = 0. \tag{12}
\]

Substituting the HJ equations into the Hamiltonian constraint (7), we arrive at the flow equation

\[
\{S, S\}(x) = \mathcal{L}_3(x), \tag{13}
\]

where

\[
\{S, S\} := \left( \frac{1}{\sqrt{h}} \right)^2 \left[ - \frac{1}{2} \left( h_{\mu\nu} \frac{\delta S}{\delta h_{\mu\nu}} \right)^2 + \left( \frac{\delta S}{\delta h_{\mu\nu}} \right)^2 + \frac{1}{2} L^{IJ}(\phi) \frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} \right. \\
+ \frac{1}{2} B(\phi) \frac{h_{\mu\nu}}{\delta A_\mu^a \delta A_\nu^a} \frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} \left( 3 \right) - \frac{\Theta}{B(\phi)} \sqrt{h} e^{\mu\nu}_{\phi^I} \frac{\delta S}{\delta A_\mu^a} F_{\mu\nu}^a \right] \tag{14}
\]

and

\[
\mathcal{L}_3 := V(\phi) - R(3) + \frac{1}{2} L^{IJ}(\phi) \frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} + \left( \frac{1}{4} B(\phi) + \frac{\Theta^2}{B(\phi)} \right) F_{\mu\nu}^a F^{a\mu\nu}. \tag{15}
\]

The other constraints,\(^2\) i.e., the momentum constraint and Gauss’s law, can be used to show that three-dimensional diffeomorphism and gauge invariance are realized. In fact, Gauss’s law

\(^2\) Some consequences of these constraints are collected in Appendix C.
constraint (9) and the Hamilton–Jacobi equations give

\[ 0 = \int d^d x \sqrt{h} \alpha^a (\nabla_\mu \pi^{a \mu} - (iT^a \phi)^I \pi^I) \]

\[ = \int d^d x \left\{ \nabla_\mu \alpha^a \frac{\delta S}{\delta A^{a \mu}_A} + \alpha^a (iT^a \phi)^I \frac{\delta S}{\delta \phi^I} \right\} \]

\[ = \int d^d x \left( \delta^\text{gauge}_a A^a_\mu \frac{\delta S}{\delta A^{a \mu}_A} + \delta^\text{gauge}_a \phi^I \frac{\delta S}{\delta \phi^I} \right) = \delta^\text{gauge}_a S. \tag{16} \]

Here,

\[ \delta^\text{gauge}_a A^a_\mu := \nabla_\mu \alpha^a \equiv \nabla_\mu \alpha^a + f^{a b}_c A^b_\mu \alpha^c, \quad \delta^\text{gauge}_a \phi^I := \alpha^a (iT^a \phi)^I, \tag{18} \]

denote an infinitesimal gauge transformation. Further, the momentum constraint (8) and the Hamilton–Jacobi equations lead to

\[ 0 = \int d^d x \sqrt{h} \epsilon^\mu (2 \nabla^\nu \pi^{\mu \nu} - \pi^I \nabla_\mu \phi^I - F^a_{\mu \nu} \pi^{a \nu}) \]

\[ = \int d^d x \left\{ (\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu) \frac{\delta S}{\delta h_{\mu \nu}} + \epsilon^\mu \nabla_\mu \phi^I \frac{\delta S}{\delta \phi^I} + \epsilon^\mu F^a_{\mu \nu} \frac{\delta S}{\delta A^a_{\nu}} \right\} \]

\[ = \delta_\epsilon S - \int d^d x \sqrt{h} \epsilon^\mu A^a_{\mu} \{ \nabla_\nu \pi^{a \nu} - (iT^a \phi)^I \pi^I \}. \tag{19} \]

Here,

\[ \delta_\epsilon \phi^I := \mathcal{L}_\epsilon \phi^I \equiv \epsilon^\mu \partial_\mu \phi^I, \quad \delta_\epsilon A^a_{\mu} := \mathcal{L}_\epsilon A^a_{\mu} \equiv \epsilon^\nu \partial_\nu A^a_{\mu} + \partial_\mu \epsilon^\nu A^a_{\nu}, \]

\[ \delta_\epsilon h_{\mu \nu} := \mathcal{L}_\epsilon h_{\mu \nu} \equiv \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \tag{20} \]

are Lie derivatives with respect to three-dimensional diffeomorphism. Noting that the second term in Eq. (19) vanishes, because of Eq. (9), implies invariance of the on-shell action under three-dimensional diffeomorphism.

Separate the action into local and nonlocal parts:

\[ \frac{1}{2 \kappa^2_4} S[h, \phi, A] \equiv \frac{1}{2 \kappa^2_4} S_{\text{loc}}[h, \phi, A] - \Gamma[h, \phi, A]. \tag{21} \]

Furthermore, so as to study the flow equation systematically, we employ the derivative expansion by assigning an additive number called weight as in Table 1.

| Elements                        | Weight \( w \) |
|--------------------------------|---------------|
| \( h_{\mu \nu}(x) \), \( \phi^I(x) \), \( \Gamma[h, \phi, A] \) | 0             |
| \( \partial_\mu, A^a_{\mu}(x) \) | 1             |
| \( R, R_{\mu \nu}, R_{\mu \nu \rho \sigma}, \partial^2, \frac{\delta}{\delta \phi^I(x)} \) | 2             |
| \( \frac{\delta}{\delta h_{\mu \nu}(x)} \) | 3             |
Then,
\[ S_{\text{loc}}[h, \phi, A] = \int d^3 x \sqrt{h} L_{\text{loc}} = \int d^3 x \sqrt{h} \sum_{w=0,2,3,...} [L_{\text{loc}}]_w. \]

We parametrize the local part as below:
\[ [L_{\text{loc}}]_0 = W(\phi), \tag{22} \]
\[ [L_{\text{loc}}]_2 = -\Phi(\phi) R_{(3)} + \frac{1}{2} M^{IJ}(\phi) \nabla_\mu \phi^I \nabla_\nu \phi^J, \tag{23} \]
\[ [L_{\text{loc}}]_3 = \epsilon^{\mu\nu\rho} D^{JK}(\phi) \nabla_\mu \phi^I \nabla_\nu \phi^J \nabla_\rho \phi^K + \epsilon_{(3)}^{\mu\nu\rho} E^I(\phi)(F_{\mu\nu})^{IJ} \nabla_\rho \phi^J \tag{24} \]
\[ + \epsilon_{(3)}^{\mu\nu\rho} \frac{k_{CS}}{4\pi} \text{tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right). \]

In order to respect flavor symmetry, e.g., \( E^I \) should belong to some representation \( r \) to which \( \phi^I \) also belongs. Thus it must have the form
\[ E^I(\phi(x)) \equiv \phi^I(x) E(\phi(x)) \tag{25} \]
with \( E(\phi) \) a flavor singlet. Similarly, if the gauge group \( G \) has an antisymmetric three-tensor, the form
\[ D^{JK}(\phi(x)) \equiv \epsilon^{JK} D(\phi(x)) \tag{26} \]
with \( D(\phi) \) in a flavor singlet makes the first term of Eq. (24) \( G \) invariant, and the term is allowed only when the group \( G \) has such a three-tensor.

We also define
\[ S_{\text{loc},w-3} := \int d^3 x \sqrt{h} [L_{\text{loc}}]_w. \tag{27} \]

Using the parametrization, flow equation (13) is decomposed as follows:
\[ w = 0 : \quad V(\phi) = -\frac{3}{8} W^2(\phi) + \frac{1}{2} L^{IJ}(\phi) \partial^I W(\phi) \partial^J W(\phi), \tag{28} \]
\[ w = 2 : \quad -1 = \frac{1}{4} W(\phi) - L^{IJ}(\phi) \partial^I W(\phi) \partial^J \Phi(\phi), \tag{29} \]
\[ \frac{1}{2} L^{IJ}(\phi) = -\frac{1}{8} W(\phi) M^{IJ}(\phi) - L^{KL}(\phi) \partial^K W(\phi) \Gamma^{L,IJ}(\phi) \]
\[ - W(\phi) \partial^I \partial^J \Phi(\phi) - \frac{1}{2 B(\phi)} M^{IK}(\phi) M^{JL}(\phi) (T^a \phi)^K (T^a \phi)^L, \tag{30} \]
\[ 0 = W(\phi) \partial^K \Phi(\phi) + L^{IJ}(\phi) \partial^I W(\phi) M^{JK}(\phi), \tag{31} \]
\[ w = 3 : \quad 0 = \left( \frac{1}{\sqrt{h}} \right)^2 \left\{ 2 \kappa^2 \left( h_{\rho\sigma} \frac{\delta S_{\text{loc},0-3}}{\delta h_{\rho\sigma}} \right) h_{\mu\nu} \frac{\delta}{\delta h_{\mu\nu}} \left( \Gamma - \frac{1}{2} \kappa^2 S_{\text{loc},3-3} \right) \right. \]
\[ - 4 \kappa^2 \frac{\delta S_{\text{loc},0-3}}{\delta h_{\mu\nu}} \frac{\delta}{\delta h_{\mu\nu}} \left( \Gamma - \frac{1}{2} \kappa^2 S_{\text{loc},3-3} \right) \tag{32} \]
- $2\kappa_4^2 L^{Ij}(\phi) \frac{\delta S_{\text{loc;0-3}}}{\delta \phi^I} \frac{\delta}{\delta \phi^J} \left( \Gamma - \frac{1}{2\kappa_4^2} S_{\text{loc;3-3}} \right)
- \frac{2\kappa_4^2}{B(\phi)} h_{\mu\nu} \frac{\delta S_{\text{loc;2-3}}}{\delta A_{\mu}^a} \frac{\delta}{\delta A_{\nu}^b} \left( \Gamma - \frac{1}{2\kappa_4^2} S_{\text{loc;3-3}} \right)
- \frac{\Theta}{B(\phi)} \sqrt{h} \epsilon_{\mu\nu\rho} \frac{\delta S_{\text{loc;2-3}}}{\delta A_{\mu}^a} F_{\nu\rho}^a$, \hspace{1cm} (32)

\[ w = 4 : \{S, S\}_{w=4} = \left( \frac{1}{4} B(\phi) + \frac{\Theta^2}{B(\phi)} \right) F_{\mu\nu} F^{\mu\nu}. \hspace{1cm} (33) \]

By defining vevs in the presence of external fields $(h, \phi, A)$ as

\[ \langle T_{\mu\nu}(x) \rangle := \frac{2}{\sqrt{h}} \frac{\delta \Gamma}{\delta h_{\mu\nu}(x)}, \hspace{1cm} \langle O^I(x) \rangle := \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta \phi^I(x)}, \hspace{1cm} \langle J_{\mu}^{a}(x) \rangle := \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta A_{\mu}^{a}(x)}, \hspace{1cm} (34) \]

Eq. (32) can be solved for the trace of the stress tensor:

\[ \langle T^\mu_{\mu}(x) \rangle = \frac{2}{2\kappa_4^2} h_{\mu\nu} \frac{1}{\sqrt{h}} \frac{\delta S_{\text{loc;3-3}}}{\delta h_{\mu\nu}} + \frac{4}{W} L^{Ij}(\phi) \frac{1}{\sqrt{h}} \frac{\delta S_{\text{loc;0-3}}}{\delta \phi^I} \frac{1}{\sqrt{h}} \langle O^J(x) \rangle + \frac{4}{BW} h_{\mu\nu} \frac{1}{\sqrt{h}} \frac{\delta S_{\text{loc;2-3}}}{\delta A_{\mu}^{a}} \langle J_{av}^{a}(x) \rangle + \frac{1}{2\kappa_4^2} \frac{4\Theta}{BW} \epsilon_{\mu\nu\rho} \frac{1}{\sqrt{h}} \frac{\delta S_{\text{loc;2-3}}}{\delta A_{\mu}^{a}} F_{\nu\rho}^{a}, \hspace{1cm} (35) \]

where $\langle O' \rangle$ is defined as a vev of an operator $O$ with counterterms taken into account, e.g.,

\[ \langle O' \rangle := \frac{1}{\sqrt{h}} \frac{\delta}{\delta \phi^I} \left( \Gamma - \frac{1}{2\kappa_4^2} S_{\text{loc;3-3}} \right). \]

This expression allows us to identify the coefficients of the vevs as beta functions:

\[ \beta^I(\phi) := - \frac{4}{W(\phi)} L^{Ij}(\phi) \frac{1}{\sqrt{h}} \frac{\delta S_{\text{loc;0-3}}}{\delta \phi^I} \frac{1}{\sqrt{h}} = - \frac{4}{W} L^{Ij} \phi^I W, \hspace{1cm} (36) \]

\[ \beta_a^a(\phi, A) \equiv \rho_{I}^{a}(\phi) \nabla_{\mu} \phi^I := - \frac{4}{B(\phi) W(\phi)} h_{\mu\nu} \frac{1}{\sqrt{h}} \frac{\delta S_{\text{loc;2-3}}}{\delta A_{\nu}^{a}} = \frac{4}{BW} \nabla_{\mu} (i T^{a} \phi)^J \nabla_{\nu} \phi^J. \hspace{1cm} (37) \]

Furthermore, the first term on right-hand side of Eq. (35) is the only origin of the term so called the "virial current". However, since the three-dimensional theory is topologically $\delta S_{\text{loc;3-3}}/\delta h_{\mu\nu} = 0$, the term trivially vanishes, i.e., there is no virial current in our theory.

3. Analysis

According to Ref. [12], the vector $\beta$ functions must satisfy some properties such as (i) the gradient property, (ii) compensated gauge invariance, (iii) orthogonality, (iv) the Higgs-like relation, and (v) a nonrenormalization condition. Although these properties are confirmed to be satisfied in even dimensions [8], one can see that they are also satisfied in three spacetime dimensions: (i) the gradient property $\beta^a \propto \delta S_{\text{loc}}/\delta A^a$ is manifest in expression (37), (ii) compensated gauge invariance follows trivially since the virial current $v$ vanishes, (iii) orthogonality can be seen via an explicit computation thanks to the gauge invariance of $\Phi(\phi)$ Eq. (C.2) (and Eq. (31)):

\[ \rho_{al} \beta^l = \frac{16}{BW} (i T^{a} \phi)^K \partial^K \Phi = 0, \hspace{1cm} (38) \]
(iv) to show the Higgs-like relations, define the local RG operator
\[
\Delta_\sigma := \int d^3 x \sigma(x) \left\{ 2h_{\mu\nu}(x) \frac{\delta}{\delta h_{\mu\nu}(x)} + \beta_I [\phi(x)] \frac{\delta}{\delta \phi_I(x)} + \rho^{al_1}[\phi(x)] \nabla_\mu \phi^l(x) \frac{\delta}{\delta A^a_{\mu}(x)} \right\},
\]
and by comparing coefficients of n-point functions, one obtains anomalous dimensions
\[
\gamma^U = -\partial^j \beta^I + \rho^{al_1} (iT_a^I)^I, \quad \gamma^a_b = \rho^{cl} \delta_b c_i (iT^a_\phi)^I,
\]
and finally, (v) the equivalence between the vanishing vector beta function and conservation of the current operator can be proved by case analysis using the operator identity (C.3) as in Ref. [8].

The most general form of the trace anomaly is given by
\[
\Delta_\sigma \Gamma[h, \phi, A]_{\text{anomaly}} = \int d^3 \sqrt{h} \epsilon^{\mu\nu\rho}_{(3)} \sigma(x) \left\{ C_{LJK} \nabla_\mu \phi^l \nabla_\nu \phi^j \nabla_\rho \phi^K + C^a_I F^a_{\mu\nu} \nabla_\rho \phi^l \right\}.
\]
Comparing Eq. (41) and Eq. (D.1), the anomaly coefficients are identified:
\[
C_{LJK} = \frac{1}{2 \kappa^4} \left\{ -2E \rho^{al_1} (iT^a_\phi)^I - 2 \rho^{al_1} \partial^I E(\phi (iT^a_\phi)^I) + \epsilon^{LJK} \beta^I \partial^L D \right.
- 3D \rho^{al_1} \epsilon^{JKL}(iT^a_\phi)^I - 3 \partial^I D \epsilon^{JKL} \beta^I \left\}.
\]
\[
C^a_I = \frac{1}{2 \kappa^4} \left\{ - \phi \rho^{al_1} - \frac{k_{CS}}{4\pi} C(r) \rho^{al_1} + 2E (iT^a_\phi)^I + \beta^K \partial^K E(iT^a_\phi)^I \right.
- E \rho^{bl_1} (\phi (T^a_\mu, T^b_\nu) \phi) + \partial^I E(\phi (iT^a_\phi)^I) - 3D \epsilon^{LJK} (iT^a_\phi)^I \beta^K \left\}.
\]
Using these expressions, one can show that they satisfy the WZ consistency conditions, i.e.,
\[
3 \beta^I C_{LJK} + \rho^{aI} C^a_K - \rho^{aI} C^a_I = 0,
\]
\[
\beta^I C^a_I = 0,
\]
by exploiting the orthogonality (38) and antisymmetry of the generators $T^a$.

4. Conclusion

In this paper we have discussed the trace anomaly in three dimensions. We expected we could get nonzero trace anomalies even on conformal fixed points if we broke the parity symmetry, but we eventually showed it is not the case. Now that we have finished the explicit computation, we can easily see why the anomaly vanishes in our situation. We know that the trace of the stress tensor has a weight $w = 3$ and the nonlocal action $\Gamma$ has $w = 0$. Then, since coefficients of vevs of operators are identified with beta functions, the bracket (14) tells us that $\beta^I \propto \delta S_{\text{loc:}0-3}/\delta \phi$ and $\beta_\mu \propto \delta S_{\text{loc:}2-3}/\delta A$. With this knowledge, let us have a closer look at the bracket. The metric and scalar field part is a good place to start. Since functional derivatives with these fields cancel $w = -3$ from the volume element of the local action, only pairs of local Lagrangians whose weights sum to 3 can contribute to $\langle T^\mu_\mu \rangle$. Because of the absence of a local Lagrangian with $w = 1$, there is no pair with weights $3 = 1 + 2$, and all we have is a pair $3 = 0 + 3$. The fact $\delta S_{\text{loc:}3-3}/\delta h_{\mu\nu} = 0$ kills a

\[3\] We have employed a slightly different notation from Ref. [10].
\[4\] Note that the weight analysis tells us that nonlocal action no longer contributes to $\langle T^\mu_\mu \rangle$. 

8/12
potential contribution from the metric part of the pair, and the scalar field part gives the scalar beta function. Thus all contributions to \( \langle T_{\mu}^{\mu} \rangle \) from the metric and scalar field parts of the bracket are proportional to (scalar) beta functions. Next, let us scrutinize the gauge field part. Since functional derivatives with \( A_{\mu}^{a} \) have weights \( w = 2 \), they do not completely cancel \( w = -3 \) from the volume element, and only pairs of local Lagrangians whose weights sum to 5 can contribute to \( \langle T_{\mu}^{\mu} \rangle \). Then just a pair \( 5 = 2 + 3 \) can survive, however, the contribution is again proportional to the (vector) beta function. Therefore, all contributions are proportional to beta functions. The absence of local Lagrangians with \( w = 1 \) is essential. From the above argument we have learned that one needs a term with \( w = 1 \) that respects Lorentz and flavor symmetries in the local action in order to achieve nonzero trace anomalies on fixed points. Following the same analysis, one can also see that with just the simple extension of the bulk action, one cannot get nonzero trace anomalies in general odd dimensions either, because of the absence of local Lagrangians with odd weights.

Acknowledgements

We are grateful to Tadakatsu Sakai for valuable discussions and useful comments on our draft.

Funding

Open Access funding: SCOAP3.

Appendix A. Notation

Let us denote by \([\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}]\) the sign of a permutation \((\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma})\), where we define \([012] \equiv +1\). Then the Levi-Civita tensor is defined by

\[
\epsilon_{(4)}^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} = \frac{1}{\sqrt{|\hat{\gamma}|}} \epsilon^{\hat{\sigma}}_{\hat{\mu} \hat{\nu} \hat{\rho}}.
\]

In this convention, we arrive at the three-dimensional expression

\[
\hat{N} \epsilon_{(4)}^{\mu \nu \rho \tau} = \epsilon_{(3)}^{\mu \nu \rho}.
\] (A.1)

Matrix representations are given by

\[
(A_{\mu})^{IJ} := -A_{\mu}^{a} (iT^{a})^{IJ}, \quad (F_{\mu \nu})^{IJ} := -F_{\mu \nu}^{a} (iT^{a})^{IJ},
\] (A.2)

and covariant derivatives are defined by

\[
\hat{\nabla}_{\hat{\mu}} \hat{\phi}^{I} := \hat{\nabla}_{\hat{\mu}} \hat{\phi}^{I} - A_{\mu}^{a} (iT^{a}) \hat{\phi}^{I},
\] (A.3)

\[
\nabla_{\mu} \phi^{I} := \nabla_{\mu} \phi^{I} - A_{\mu}^{a} (iT^{a}) \phi^{I},
\] (A.4)

\[
\nabla_{\mu} \alpha^{a} := \nabla_{\mu} \alpha^{a} + f^{a}_{bc} A_{\mu}^{b} \alpha^{c}.
\] (A.5)

The generators \( T^{a} \) are normalized to yield the quadratic Casimir

\[
\text{tr}(T^{a} T^{b}) \equiv \delta^{ab} C(r)
\] (A.6)

for some representation \( r \).

---

\( ^{5} \) Lorentz indices of the bulk \( M_{4} \) are denoted by \( \hat{\mu}, \hat{\nu}, \ldots \) and those of the hypersurface by \( \mu, \nu, \ldots \).
Finally, we define the Levi-Civita connection in the theory space as
\[ \Gamma^{IJK} := \frac{1}{2} (\phi^J M^{IK} + \phi^K M^{IJ} - \phi^I M^{JK}). \]  
\( \text{(A.7)} \)

**Appendix B. Some useful formulae**

\[
\frac{\delta S_{\text{loc,0-3}}}{\delta h_{\mu\nu}} = \frac{1}{2} \sqrt{h} h^{\mu\nu} W(\phi),
\]

\[
\frac{\delta S_{\text{loc,0-3}}}{\delta \phi^I} = \sqrt{h} \phi^I W(\phi),
\]

\[
\frac{\delta S_{\text{loc,0-3}}}{\delta A^a_{\mu}} = 0,
\]

\[
\frac{\delta S_{\text{loc,2-3}}}{\delta h_{\mu\nu}} = \sqrt{h} \left\{ 2 R^{\mu\nu} - \frac{1}{2} h^{\mu\nu} R(3) \right\} - \nabla^\mu \nabla^\nu \Phi(\phi) + h^{\mu\nu} \nabla^2 \Phi(\phi) + \frac{1}{2} M^{IJ} (\phi) \left\{ \frac{1}{2} h^{\mu\nu} \nabla^\rho \phi^J \nabla^\rho \phi^J - h^{\mu\nu} \nabla^\rho \phi^J \nabla^\rho \phi^J \right\},
\]

\[
\frac{\delta S_{\text{loc,2-3}}}{\delta \phi^I} = \sqrt{h} \left\{ -\phi^I (\phi) R(3) - \Gamma^{IJK}(\phi) \nabla^\mu \phi^J \nabla^\rho \phi^K - M^{IJ}(\phi) \nabla^2 \phi^J \right\},
\]

\[
\frac{\delta S_{\text{loc,2-3}}}{\delta A^a_{\mu}} = -\sqrt{h} M^{IJ}(\phi) \nabla^\mu \phi^J (i T^a \phi)^J,
\]

\[
\frac{\delta S_{\text{loc,3-3}}}{\delta h_{\mu\nu}} = 0,
\]

\[
\frac{\delta S_{\text{loc,3-3}}}{\delta \phi^I} = \sqrt{h} \epsilon^{\mu\nu\rho\phi}_{(3)} \left\{ \phi^I D^{KL}(\phi) \nabla_\mu \phi^J \nabla_\nu \phi^K \nabla_\rho \phi^L - 3 \nabla_\mu D^{JJK}(\phi) \nabla_\nu \phi^J \nabla_\rho \phi^K - 3 D^{JJK}(\phi) (F_{\mu\nu} \phi)^J \nabla_\rho \phi^K + \phi^I E^J(\phi) (F_{\mu\nu} \phi)^J - \nabla_\rho E^J(\phi) (F_{\mu\nu} \phi)^J \right\},
\]

\[
\frac{\delta S_{\text{loc,3-3}}}{\delta A^a_{\mu}} = \sqrt{h} \epsilon^{\mu\nu\rho}_{(3)} \left\{ -3 D^{JJK}(\phi) (i T^a \phi)^J \nabla_\mu \phi^J \nabla_\nu \phi^K - 2 \nabla_\nu \left[ E^J(\phi) (i T^a \phi)^J \right] - E^J(\phi) (F_{\nu\mu} i T^a \phi)^J \right\} - \frac{k_{\text{CS}}}{4\pi} C(r) F^a_{\nu\mu}.
\]

**Appendix C. Consequences of first-class constraints**

\[
0 = (iT^a \phi)^J \phi^I W(\phi),
\]

\[
0 = (iT^a \phi)^J \phi^I \Phi(\phi),
\]

\[
0 = \nabla_\mu J^a_{\mu} - (iT^a \phi)^J O^J,
\]

\[
0 = \phi^J M^{JK}(\phi) (iT^a \phi)^J + (iT^a \phi)^J M^{IJ} + (iT^a \phi)^J M^{JK},
\]

\[
\nabla_\mu W(\phi) = \nabla_\mu \phi^J \phi^I W(\phi),
\]

\[
0 = \nabla_\nu T_{\mu\nu} - \nabla_\mu \phi^J O^J - F^a_{\mu\nu} j^a_{\nu},
\]

\[
0 = \nabla_\mu \Phi(\phi) - \nabla_\mu \phi^J \phi^I \Phi(\phi).
\]
Appendix D. Explicit form of \( \langle T^\mu_\mu \rangle \)

Substituting some formulae in Appendix B, one obtains an explicit form of the trace of the stress tensor:

\[
\langle T^\mu_\mu \rangle = -\beta^I \langle O^I \rangle - \beta_\mu^a \langle J^a_I \rangle - \frac{\Theta}{2\kappa_4^2} \epsilon^{\mu\nu\rho} \beta^\rho \nu_I a
\]

\[+ \frac{1}{2\kappa_4^2} \beta^I \sqrt{\hat{g}} \delta S_{\text{loc};3-3}^{\mu} + \frac{1}{2\kappa_4^2} \beta_\mu^a \sqrt{\hat{g}} \delta A^{\mu}_{\text{loc};3-3} \]

\[= -\beta^I \langle O^I \rangle - \beta_\mu^a \langle J^a_I \rangle \]

\[+ \frac{1}{2\kappa_4^2} \epsilon^{\mu\nu\rho} \nabla_\mu \phi^I \nabla_\nu \phi^K \left\{ -2E \rho^a (i T^a)^J - 2\rho^a \partial^I E (\phi i T^a)^K + \epsilon^{\mu\nu\rho} \beta_\rho \partial^I D \right. \]

\[\left. - 3D \rho^a \epsilon^{JKL} (i T^a)^L - 3 \partial^I \epsilon^{JKL} \beta^I \right\} \]

\[+ \frac{1}{2\kappa_4^2} \epsilon^{\mu\nu\rho} F^a_{\mu\nu} \nabla_\rho \phi^K \left\{ -\Theta \rho^a - \frac{k_C S}{4\pi} C(r) \rho^a + 2E (i T^a)^I \beta^I + \beta^K \partial^K E (i T^a)^I \right. \]

\[\left. - E \rho^b (\phi (T^a, T^b) + \partial^I E (\phi i T^a) \beta^I - 3D e^{JKL} (i T^a)^J \beta^K \right\}. \quad (D.1) \]

Appendix E. Adding extra terms to the bulk action

If one adds a term

\[\frac{1}{4} \int_{M_4} d^4x \sqrt{\hat{g}} \epsilon^{\mu\nu\rho\sigma} H^{LJKL} \hat{\nabla}^\rho \hat{\nabla}^\sigma \hat{\nabla}^{\mu\nu} \hat{\nabla}^{\beta\gamma} \hat{\nabla}^{\hat{\phi}_L} \hat{\nabla}^{\hat{\phi}_J} \hat{\nabla}^{\hat{\phi}_K} \hat{\nabla}^{\hat{\phi}_M} \quad (E.1)\]

to the bulk action, this term disturbs canonical momentum conjugate to the scalar field as

\[\hat{\pi}^M := \frac{\partial L_4}{\partial (\partial_\mu \phi^M)} = \hat{\pi}^M + \epsilon^{\mu\nu\rho} H^{LJKM} \hat{\nabla}^\mu \hat{\nabla}^\nu \hat{\nabla}^\rho \hat{\nabla}^{\hat{\phi}_M}, \quad (E.2)\]

where “\( \hat{\pi}^M \)” is the same as Eq. (4). Following the same calculation as in the formalism, we can write the first-order action and arrive at first-class constraints. The new term does not change the momentum constraint and the Gauss’s law constraint, however, it changes the Hamiltonian constraint

\[\hat{H}' := \frac{1}{\sqrt{\hat{g}}} \delta S' \]

\[= - \frac{1}{2} L^{IJ} \hat{\pi}^I \hat{\pi}^J + \epsilon^{\mu\nu\rho} L^{IJ} \hat{\nabla}^\mu \hat{\nabla}^\nu \hat{\nabla}^\rho \hat{\nabla}^{\hat{\phi}_M} \]

\[\quad - \frac{1}{2} \epsilon^{\mu\nu\rho} \epsilon^{\alpha\beta\gamma} L^{IJ} H^{JKLM} \hat{\nabla}^\mu \hat{\nabla}^\nu \hat{\nabla}^\rho \hat{\nabla}^{\hat{\phi}_M} + \cdots, \quad (E.3)\]

and these changes are accompanied by modifications of the flow equation. The third term does not contribute to the trace anomaly because the term has \( w = 6 \). The second term can give a nonzero contribution to \( \langle T^\mu_\mu \rangle \), however, since the covariant derivatives already have \( w = 3 \), the term can enter the trace of the stress tensor only when \( \hat{\pi}^I \) gives a term with \( w = 0 \) through the Hamilton–Jacobi equation. It is possible only if the functional derivative \( \delta / \delta \phi^I \) acts on \( S_{\text{loc};0-3} \), and this is nothing but \( \beta^I \). Thus the additional term \( (E.1) \) does not give a nontrivial contribution to the trace anomaly either.
References

[1] S. Deser and A. Schwimmer, Phys. Lett. B 309, 279 (1993) [arXiv:hep-th/9302047] [Search INSPIRE].
[2] A. B. Zamolodchikov, JETP Lett. 43, 730 (1986) [Pis’ma Zh. Eksp. Teor. Fiz. 43, 565 (1986)].
[3] Z. Komargodski and A. Schwimmer, J. High Energy Phys. 1112, 099 (2011) [arXiv:1107.3987 [hep-th]] [Search INSPIRE].
[4] Z. Komargodski, J. High Energy Phys. 1207, 069 (2012) [arXiv:1112.4538 [hep-th]] [Search INSPIRE].
[5] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safidi, J. High Energy Phys. 1106, 102 (2011) [arXiv:1103.1181 [hep-th]] [Search INSPIRE].
[6] J. de Boer, E. P. Verlinde, and H. L. Verlinde, J. High Energy Phys. 0008, 003 (2000) [arXiv:hep-th/9912012] [Search INSPIRE].
[7] M. Fukuma, S. Matsuura, and T. Sakai, Prog. Theor. Phys. 104, 1089 (2000) [arXiv:hep-th/0007062] [Search INSPIRE].
[8] K. Kikuchi and T. Sakai, Prog. Theor. Exp. Phys. 2016, 033B02 (2016) [arXiv:1511.00403 [hep-th]] [Search INSPIRE].
[9] S. Rajagopal, A. Stergiou, and Y. Zhu, J. High Energy Phys. 1511, 216 (2015) [arXiv:1508.01210 [hep-th]] [Search INSPIRE].
[10] Y. Nakayama, Nucl. Phys. B 879, 37 (2014) [arXiv:1307.8048 [hep-th]] [Search INSPIRE].
[11] A. Bzowski, P. McFadden, and K. Skenderis, J. High Energy Phys. 1603, 066 (2016) [arXiv:1510.08442 [hep-th]] [Search INSPIRE].
[12] Y. Nakayama, Int. J. Mod. Phys. A 28, 1350166 (2013) [arXiv:1310.0574 [hep-th]] [Search INSPIRE].