A note about extension of functions
belonging to Sobolev - Grand Lebesgue Spaces.

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Abstract

We deduce an extension theorem for the so-called Sobolev - Grand Lebesgue Spaces defined on the suitable subsets of the whole finite-dimensional Euclidean space, and estimate the norms of correspondent extension operator, which may be chosen as linear.

Key words and phrases. Lebesgue - Riesz, Yudovich, ordinary Sobolev and Sobolev - Grand Lebesgue norm and spaces, Euclidean space, Lipschitz domain, ordinary and linear extension, linear bounded operator, norm, measurable functions, semi-space, generating function, estimation.

1 Introduction. Previous results.

Let \( B(G) \), where \( G \) is non-trivial subset of an Euclidean space \( \mathbb{R}^d : G \subset \mathbb{R}^d \) be a family of Banach spaces defined on the class of functions defined in turn on the support \( G \); the norm in this space will be denoted \( \| f \| B(G) \).

It will be presumed henceforth that all the considered domains \( G \) are closures of the non-empty open sets and are Lipschitzian. The case when \( G = \mathbb{R}^d \) is trivial for us and may be excluded.

Put also for definiteness \( B = B(\mathbb{R}^d) \).
For instance, the classical Lebesgue - Riesz $L_p(G)$ spaces equipped with the ordinary norm

$$\|f\|_p(G) = \|f\|_{L_p(G)} := \left( \int_G |f(x)|^p \, dx \right)^{1/p}, \quad p \in [1, \infty), \ f : G \to R; \ x \in G,$$

as well as the famous Sobolev spaces $W^m_p(G)$, $m = 1, 2, \ldots$:

$$\|f\|_{W^m_p(G)} \overset{def}{=} \max_{\alpha, |\alpha| \leq m} \|D^\alpha f\|_{L_p(G)},$$

$$\|f\|_{W^m_p} \overset{def}{=} \|f\|_{W^m_p(R^d)}; \text{ see e.g. } [1], [24], [25], [26], [30], [31].$$

Here as ordinary

$$\alpha = \tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d); \quad \alpha_j = 0, 1, 2, \ldots;$$

$$|\alpha| := \sum_{j=1}^d \alpha_j; \quad D^\alpha f := \frac{D^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}$$

and all the derivatives are understood in the weak (Sobolev) sense. Of course, $D^0 f = f$.

Let the function $f : G \to R$ belongs to some space $B(G)$. By definition, the function $\tilde{f}, R^d \to R$ is named as an extension of the function $f$ (from the set $G$), iff

$$\forall x \in G \Rightarrow \tilde{f}(x) = f(x)$$

and wherein $\|\tilde{f}\|_B < \infty$.

If there exists a linear bounded operator $L : \tilde{f} = Lf = L_G f, \ f \in B(G)$, where again $\tilde{f}$ is an extension for $f$, such that

$$K = K(G) = K(B,G) \overset{def}{=} \sup_{0 \neq f \in B(G)} \left\{ \frac{\|Lf\|_B}{\|f\|_{B(G)}} \right\} < \infty,$$

then the extension $\tilde{f} = Lf = L_G f$ is named linear.

It is known, see e.g. [3], [4], [11], [31] that under imposed conditions on the Lipschitz domain $G$ and for the Sobolev’s spaces $B(G) = W^m_p(G)$, $m \geq 1$, $p \geq 1$ (the case $m = 0$ is trivial) there exist a linear extension operator. See also the works [18], [19], [25], [26], [29] etc.

We will ground in offered report that this linear operator there exists also for the so-called Sobolev - Grand Lebesgue Spaces.


2 Main result.

**Sobolev - Grand Lebesgue Spaces.**

Let \((a, b) = \text{const}, \, 1 \leq a < b \leq \infty\). Let also \(\psi = \psi(p), \, p \in (a, b)\) be certain numerical valued measurable strictly positive: \(\inf_{p \in (a, b)} \psi(p) > 0\) function, not necessary to be bounded. Denotation:

\[(a, b) := \text{supp}(\psi); \, \Psi[a, b] := \{ \psi : \text{supp}(\psi) = (a, b) \}, \]

\[\Psi \overset{\text{def}}{=} \cup_{1 \leq a < b \leq \infty} \Psi[a, b].\]

**Definition 2.1.** see e.g. [28]. The Sobolev - Grand Lebesgue Space \(S[m, G, \psi]\) based on the set \(G, \, G \subset \mathbb{R}^d\) is defined as a set of all (measurable) functions having a finite norm

\[||f||S[m, G, \psi] \overset{\text{def}}{=} \sup_{p \in (a, b)} \left\{ \frac{||f||W^m_p(G)}{\psi(p)} \right\}. \tag{2}\]

The particular case \(m = 0\), i.e. when

\[||f||G\psi = ||f||G\psi(a, b) \overset{\text{def}}{=} ||f||S[0, G, \psi] = \sup_{p \in (a, b)} \left\{ \frac{||f||L_p(G)}{\psi(p)} \right\} \tag{3}\]

and under some additional restrictions on the generating function \(\psi = \psi(p)\) correspondent to the so - called Yudovich spaces, see [32], [33]. These spaces was applied at first in the theory of Partial Differential Equations (PDE), see [7], [8].

A general case of these spaces when \(m = 0\) are named as the classical Grand Lebesgue Spaces (GLS) \(G\psi, \, \psi \in \Psi\). These spaces are investigated in many works, see e.g. [9], [10], [12], [13], [14], [15], [16], [17], [20], [21], [22], [23], [27]. The general case of Sobolev - Grand Lebesgue Spaces appears at first perhaps in the article [28], where was investigated the modulus of continuity of the functions belonging to these spaces.

**Theorem 2.1.** Assume that all the formulated before restrictions are satisfied, indeed: that the Lipschitz domain \(G\) is closure of the non - empty open subset of whole Euclidean space \(\mathbb{R}^d\). We propose that for arbitrary Sobolev - Grand Lebesgue Space \(S[m, G, \psi]\) there exists a linear bounded extension operator \(L = L_G\).

**Proof.** One can suppose \(d \geq 2\) and that \(G = \bar{x} = \{x_j\} = \{x_1, x_2, \ldots, x_{d-1}, x_d\}\), where \(\bar{x} = (\bar{x}, x_d); \, \bar{x} = \{ x_1, x_2, \ldots, x_{d-1} \}; \, \) so that \(\bar{x} \in G \Leftrightarrow x_d \geq 0\); on the other words, upper semi - space; see e.g. [3], [4], [11]. Let us define the following extension operator \(L f(x) := f(x), \, x \in G, \, f(\cdot) \in S[m, G, \psi];\)
\[ L f(x) := \sum_{k=1}^{d+1} c_k f(\bar{x}, -kx_d), \quad x_d < 0. \]

The coefficients \( \{c_k\} \) may be uniquely determined from the following system of linear equations

\[
\sum_{k=1}^{m+1} (-k)^l c_k = 1; \quad l = 0, 1, \ldots, d.
\]

Suppose that \( f \in S[m, G, \psi] \); one can assume without loss of generality \( ||f||_{S[m, G, \psi]} = 1 \). Then for all the values \( p \in (a, b) \)

\[ ||f||_{W^m_p} \leq \psi(p), \quad p \in (a, b), \]

therefore

\[ \forall p \in (a, b), \quad \forall \alpha : |\alpha| \leq m \Rightarrow ||D^\alpha f||_p(G) \leq \psi(p). \]

Introduce the functions

\[ g_k(\bar{x}, y) := f(\bar{x}, -ky), \quad y \leq 0; \]

then

\[ D^\alpha g_k = (-k)^{\alpha_d} f(\bar{x}, -ky), \]

\[ ||D^\alpha g_k||_p(G) = k^{\alpha_d - 1/p} ||D^\alpha f||_p(G) \leq k^{\alpha_d} ||D^\alpha f||_p(G), \]

therefore

\[ \forall \alpha : |\alpha| \leq m \Rightarrow \sum_{k=1}^{m+1} |c_k| k^m \cdot ||D^\alpha f||_p(G) \leq \]

\[ \sum_{k=1}^{m+1} |c_k| k^m \cdot \psi(p), \quad p \in (a, b). \]

Following, by virtue of triangle inequality for Lebesgue - Riesz spaces

\[ ||L[f]||_{W^m_p} (R^d \setminus G) \leq C(d, m) \psi(p), \quad C(d, m) < \infty, \]

\[ ||L[f]||_{S[m, R^d, \psi]} \leq 1 + C(d, m) < \infty, \]

Q.E.D.
3 Concluding remarks.

It is interest in our opinion to compute the exact value of extension constant for Sobolev - Grand Lebesgue Spaces, as well as to generalize the extension theorem on the anisotropic spaces.

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