The Plimpton 322 Tablet and the Babylonian Method of Generating Pythagorean Triples

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Abstract

Ever since it was published by Neugebauer and Sachs in 1945, the Old Babylonian tablet known as Plimpton 322 has been the subject of numerous studies leading to different and often conflicting interpretations of it. Overall, the tablet is more or less viewed as a list of fifteen Pythagorean triplets, but scholars are divided on how and why the list was devised. In this paper, we present a survey of previous attempts to interpret Plimpton 322, and then offer some new insights that could help in sharpening the endless debate about this ancient tablet.

1 Introduction

Plimpton 322 is the catalog name of an Old Babylonian (OB) clay tablet held at Columbia University. The tablet is named after New York publisher George A. Plimpton who purchased it from archaeology dealer Edgar J. Banks in the early nineteen twenties. In the mid thirties, the tablet, along with the rest of Mr. Plimpton collection, was donated to Columbia University. According to Banks, the tablet was found at Tell Senkereh, an archaeological site in southern Iraq corresponding to the ancient Mesopotamian city of Larsa [Robson, 2002].

The preserved portion of the tablet (shown in Figure 1) is approximately 13 cm wide, 9 cm high and 2 cm deep. As can be seen from the picture, a chunk of the tablet is missing from the middle of the right-hand side. Also, the tablet had (before it was baked for preservation) remnants of modern glue on its damaged left-hand side suggesting that it might be a part of a larger tablet, the discovery of the remainder of which, if it ever existed, might settle many of the questions we try to answer in this paper. The exact date of the tablet is not known, but it is generally agreed that it belongs to the second half of the Old Babylonian period, roughly between 1800 and 1600 BCE. More recently, based on the style of cuneiform script used in the
tablet and comparing it with other dated tablets from Larsa, Eleanor Robson has narrowed the date of Plimpton 322 to a period ranging from 1822 to 1784 BCE \cite{Robson2001}.

The preserved part of Plimpton 322 is a table consisting of sixteen rows and four columns. The first row is just a heading and the fourth (rightmost) column of each row below the heading is simply the number of that row. The remaining entries are pure numbers written in sexagesimal (base 60) notation. However, it should be noted that due to the broken left edge of the tablet, it is not fully clear whether or not 1 should be the leading digit of each number in the first column. In Table \ref{table1} we list the numbers on the obverse of the tablet, with numbers in brackets being extrapolated. At all times, it should be kept in mind that the original tablet would still be of acceptable size if one or two columns were added to its left edge.

The numbers on the Plimpton tablet are written in cuneiform script using the sexagesimal number system. Strictly speaking, the Babylonian number system is not a pure sexagesimal system in the modern sense of the word. First, the digits from 1 to 59 are expressed using only two symbols: A narrow wedge representing 1 and a wide wedge representing 10. The numbers from 1 to 9 are expressed by grouping the corresponding number of narrow wedges, and the multiples of ten up to fifty are expressed by grouping the corresponding number of wide wedges. Every other digit is expressed as a group of wide wedges followed by a group of narrow wedges. Second, despite the occasional indication of zero by an empty space, the
Table 1: The numbers on the obverse of Plimpton 322.

| I       | II     | III    | IV    |
|---------|--------|--------|-------|
| [1 59 00] 15 | 1 59  | 2 49 1 | 1     |
| [1 56 56] 58 14 56 15 | 56 07 | 3 12 1 | 2     |
| [1 55 07] 41 15 33 45 | 1 16 41 | 1 50 49 | 3     |
| [1] 5[3] 10 29 32 52 16 | 3 31 49 | 5 09 01 | 4     |
| [1] 48 54 01 40 | 1 05  | 1 37 5 | 5     |
| [1] 47 06 41 40 | 5 19  | 8 01   | [6]   |
| [1] 43 11 56 28 26 40 | 38 11 | 59 01  | 7     |
| [1] 41 33 59 03 45 | 13 19 | 20 49  | 8     |
| [1] 38 33 36 36 | 9 01  | 12 49  | 9     |
| [1] 35 10 02 28 27 24 26 40 | 1 22 41 | 2 16 01 | 10    |
| [1] 33 45 | 45  | 1 15   | 11    |
| [1] 29 21 54 02 15 | 27 59 | 48 49  | 12    |
| [1] 27 [00] 03 45 | 7 12 01 | 4 49   | 13    |
| [1] 25 48 51 35 06 40 | 29 31 | 53 49  | 14    |
| [1] 23 13 46 40 | 56  | 53     | [15]  |

The Babylonian number system does not explicitly specify the power of sixty multiplying the leading digit of a given number. However, we will see that this uncertainty could be an important benefit, especially since the base power can often be deduced from context. To avoid these ambiguities as we transliterate cuneiform numbers into modern symbols, a semicolon will be used to distinguish the whole part from the fractional part of the number, while an empty space will be used as a separator between the digits of the number (other authors use commas or colons to separate the digits). We will also insert a zero wherever necessary.

The Babylonian use of this strange system of counting is still debated, but one sure thing about the number sixty is that it is the smallest number with twelve divisors: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. With such a large number of divisors, many commonly used fractions have simple sexagesimal representations. In fact, it is true that the reciprocal of any number that divides a power of sixty will have a finite sexagesimal expansion. These are the so-called regular numbers. In modern notation, a regular number must be of the form $2^\alpha 3^\beta 5^\gamma$, where $\alpha$, $\beta$ and $\gamma$ are integers, not necessarily positive. The advantage of 60 over 30 = $2 \times 3 \times 5$ is that 60, unlike 30, is divisible by the highly composite number 12.

Let $m$ and $n$ be two natural numbers and denote the reciprocal of $n$ by $\overline{n}$. It follows that
\( \pi \) has a finite representation in base 60 if and only if \( n \) is regular. This is helpful because the Babylonians found \( m/n \) by computing \( m \) times \( \pi \). To facilitate their multiplications, they extensively used tables of reciprocals. In particular, the standard table of reciprocals lists the regular numbers up to 81 along with the sexagesimal expansion of their reciprocals, as shown in Table 2. Fractions like \( \frac{1}{7} \) are omitted from the table because they do not have finite expansions,

| \( n \) | \( \pi \) | \( n \) | \( \pi \) | \( n \) | \( \pi \) |
|-------|-----|-------|-----|-------|-----|
| 2     | 30  | 3     | 345 | 45    | 1 20 |
| 3     | 20  | 18    | 320 | 48    | 1 15 |
| 4     | 15  | 20    | 3   | 50    | 1 12 |
| 5     | 12  | 24    | 230 | 54    | 1 06 40 |
| 6     | 10  | 25    | 224 | 1     | 1     |
| 8     | 7 30| 27    | 213 20 | 1 04  | 56 15 |
| 9     | 6 40| 30    | 2    | 1 12  | 50    |
| 10    | 6   | 32    | 1 52 30 | 1 15  | 48    |
| 12    | 5   | 36    | 1 40 | 1 20  | 45    |
| 15    | 4   | 40    | 1 30 | 1 21  | 44 26 40 |

Table 2: The standard Babylonian table of reciprocals.

but sometimes approximations were used for such small non-regular numbers. For example,

\[
\frac{1}{7} = 13 \times \frac{1}{91} \approx 13 \times \frac{1}{90} = 13 \times (0; 00 40) = 0; 08 40.
\]

The Babylonians went even further than this. On another OB tablet published by A. Sachs, we find a lower and an upper bound for \( \frac{1}{7} \) \[Sachs, 1952\]. It states what we now write as

\[
0; 08 34 16 59 < \frac{1}{7} < 0; 08 34 18.
\]

Amazingly, the correct value of \( \frac{1}{7} \) is \( 0; 08 34 17 \), where the part to the right of the semicolon is repeated indefinitely.

### 2 Interpretation of the Tablet

Originally, Plimpton 322 was classified as a record of commercial transactions. However, after Neugebauer and Sachs gave a seemingly irrefutable interpretation of it as something related to Pythagorean triplets, the tablet gained so much attention that it has probably become the most celebrated Babylonian mathematical artifact \[Neugebauer and Sachs, 1945\]. For some, like Zeeman, the tablet is hailed as an ancient document on number theory, while for others, like
Robson, it is just a school record of a student working on selected exercises related to squares and reciprocals [Zeeman, 1992; Robson, 2001]. But no matter which view one takes, there is no doubt that the tablet is one of the greatest achievements of OB mathematics, especially since we know that it was written at least one thousand year before Pythagoras was even born. That the Old Babylonians knew of Pythagoras theorem (better called rule of right triangle) is evident in the many examples of its use in various problems of the same period [Heyrup, 1999]. Having said this, it should be clear that the tablet is no way a proof of Pythagoras theorem. In fact, the idea of a formal proof is nowhere to be found in extant Babylonian mathematics [Friberg, 1981].

According to Neugebauer and Sachs, the heading of the fourth column is ‘its name’, which simply indicates the line number, from 1 to 15 [Neugebauer and Sachs, 1945]. The headings of columns two and three read something like ‘square of the width (or short side)’ and ‘square of the diagonal’, respectively. An equally consistent interpretation can be obtained if the word ‘square’ is replaced by ‘square root’. These headings make sense only when coupled with the fact that the Babylonian thought of the sides of a right triangle as the length and width of a rectangle whose diagonal is the hypotenuse of the given triangle. Also, the Babylonians used the word square to mean the side of a square as well as the square itself [Robson, 2001]. Let the width, length and diagonal of the rectangle be denoted by \( w \), \( l \) and \( d \). Then the relation between the right triangle and the rectangle is shown in Figure 2(a), while in Figure 2(b) three squares are drawn, one for each side of the triangle. Indeed, Figure 2(b) should look familiar to anyone acquainted with Euclid’s proof of Pythagoras theorem. Such Babylonian influence on Greek mathematics is in accordance with tales that Pythagoras spent more than twenty years of his life acquiring knowledge from the wise men of Egypt and Mesopotamia [Bell, 1991, p. 85].

So, using Neugebauer’s interpretation, the second column represents the short side of a right triangle or the width \( w \) of the corresponding rectangle and the third column represents the hypotenuse of the right triangle or the diagonal \( d \) of the rectangle. The longer side of the triangle or length \( l \) of the rectangle does not appear in the table (maybe it was written on the missing part of the tablet). In such an interpretation, the first column is simply \( d^2/l^2 \) or \((d/l)^2\). To see how this can be applied to the table, let us look at an example. In line 5, the square of the number in Column III minus the square of the number in Column II is a perfect square. That is,

\[
(137)^2 - (105)^2 = 12624 = (112)^2.
\]

Moreover, the number in the first column is nothing but the square of the ratio \((137):(112)\). In decimal notation, we have

\[
97^2 - 65^2 = 5184 = 72^2,
\]
with the number in Column I being $97^2/72^2$.

In general, if we think of the two middle entries in a given row as the width $w$ and diagonal $d$ of a rectangle (or the short side and hypotenuse of a right triangle), then the entry in the first column of that row is $d^2/l^2$, where $l$ is the length of the rectangle. Except for a few errors, which we will say more about in Section 6, the entries in the first three columns of each line of Table 1 are exactly $d^2/l^2$, $d$, and $w$. It is important to note that an equally consistent interpretation can be obtained if we remove the leading 1 in Column I and think of the numbers in that column as $w^2/l^2$. The equivalence of the two interpretations follows from the fact that if $l^2 + w^2 = d^2$, then

$$1 + \frac{w^2}{l^2} = \frac{d^2}{l^2}. \tag{1}$$

This brings us to the heading of the first column. The heading consists of two lines, both of which are damaged at the beginning. Quoting Neugebauer and Sachs:

The translation causes serious difficulties. The most plausible rendering seems to be:

'The takiltum of the diagonal which has been subtracted such that the width. . . . .

[Neugebauer and Sachs, 1945, p. 40]

Building on the work of Hoyrup, Robson took the work of Neugebauer and Sachs a step further [Hoyrup, 1990; Robson, 2001]. By carefully examining the damaged heading of the first column, she was able to render the sensible translation:

The holding-square of the diagonal from which 1 is torn out, so that the short side comes up.

1 After personally inspecting Plimpton 322, Bruins concluded that the apparent unit at the beginning of each line is due to the horizontal line between rows [Bruins, 1957]. However, Friberg has given a more convincing argument in support of the leading one stance [Friberg, 1981].
In addition to being linguistically and contextually sound, the above interpretation, thought of as equation (1) in words, makes perfect mathematical sense. This is in line with the ‘cut-and-paste’ geometry introduced by Høyrup and adopted by Robson. Also, the interpretation clearly speaks in favor of the restoration of the leading 1 at the beginning of each line of the preserved tablet.

The above interpretation of the tablet is the most widely accepted one by scholars because it relates the numbers on the tablet in a meaningful way which is totally drawn from extant OB mathematics. Other, wilder, interpretations of the tablet have also been proposed but they do not carry much weight. One such interpretation suggests that the tablet represents some sort of a trigonometric table [Joyce, 1995; Maor, 2002, pp. 30-34]. This is based on the fact that in each line of the Plimpton table, the entry in the first column is the square of the cosecant of the angle between the long side and the hypotenuse of a right triangle, where the angle decreases from about $45^\circ$ to $30^\circ$ by roughly one degree per line as we move down the table. However, this hypothesis is dismissed by most historians of Babylonian mathematics on many grounds, the least of which is the lack of any traces of trigonometric functions in extant Babylonian mathematics. Moreover, the translation of the heading of the first column offered by Robson totally refutes such an interpretation [Robson, 2001].

3 Previous Methods for Reconstructing the Table

Having determined what the tablet means, it remains to answer the more difficult question of how it was constructed. In this respect, there are two major theories on how the numbers on Plimpton 322 were generated. The first method was proposed by Neugebauer and Sachs in their highly acclaimed book *Mathematical Cuneiform Texts* in which the tablet was originally published [Neugebauer and Sachs, 1945, pp. 38-41]. The method had many proponents including [Gillings, 1953; Price, 1964; Buck, 1980] and others. The second method was first introduced by E. M. Bruins in 1949, but it did not become main stream until it reappeared in the works of Schmidt and Friberg, and more recently in the work of Robson [Schmidt, 1980; Friberg, 1981; Robson, 2001]. The decision of which method was employed in the construction of the tablet is made more difficult by the fact that the two methods are mathematically equivalent to each other. As a general rule, one should pick the method which is more consistent with extant mathematics of the OB period. But before this could be done, we shall give some background information and a summary of each method.

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2 Price mentioned a similar interpretation proposed by Goetze [Price, 1964].
It was known to the ancient Greeks that if \( m > 1 \) is an odd integer, then the numbers

\[
\begin{align*}
m, & \quad \frac{1}{2}(m^2 - 1) \quad \text{and} \quad \frac{1}{2}(m^2 + 1)
\end{align*}
\]

satisfy Pythagoras theorem [Shanks, 2001, p. 121]. That is,

\[
\begin{align*}
m^2 + \left[ \frac{(m^2 - 1)}{2} \right]^2 &= \left[ \frac{(m^2 + 1)}{2} \right]^2.
\end{align*}
\]

The restriction that \( m \) is an odd integer is needed to guarantee that the triplet consists of integers, but the theorem holds for every real number \( \alpha \). In particular, if we replace \( m \) by \( \alpha > 0 \) and divide by \( \alpha^2 \), we obtain the normalized equation

\[
1 + \left( \frac{\alpha - 1/\alpha}{2} \right)^2 = \left( \frac{\alpha + 1/\alpha}{2} \right)^2. \tag{2}
\]

Observe that as \( \alpha \) varies between 1 and \( 1 + \sqrt{2} \), \( (\alpha - 1/\alpha)/2 \) varies between 0 and 1; and when \( \alpha \) increases beyond \( 1 + \sqrt{2} \), \( (\alpha - 1/\alpha)/2 \) increases beyond 1. In the former case, the longer side of the triangle must be 1; while in the latter the reverse is true. The case \( \alpha < 1 \) will be ignored since it leads to negative values of \( \alpha - 1/\alpha \).

Although (2) holds for every positive real \( \alpha \), we are mainly interested in the case when \( \alpha \) is a regular number. But if \( \alpha \) is regular, then it must be of the form \( p/q \), where \( p \) and \( q \) are regular integers. This yields

\[
1 + \left( \frac{p^2 - q^2}{2pq} \right)^2 = \left( \frac{p^2 + q^2}{2pq} \right)^2. \tag{3}
\]

Now multiplying by \( (2pq)^2 \), we see that (3) is the same as saying that the integers

\[
2pq, \quad p^2 - q^2 \quad \text{and} \quad p^2 + q^2 \tag{4}
\]

form an integral Pythagorean triplet, a fact also known to Euclid. More generally, if \( p \) and \( q \) are integers of opposite parity with no common divisor and \( p > q \), then all primitive Pythagorean triplets are generated by [1] [Shanks, 2001, p. 141]. By a primitive triplet we mean a triplet consisting of numbers that are prime to each other.

There is direct evidence that the Babylonians were aware of and even used an identity the like of (2) or (3) [Bruins, 1957]. Also, according to Neugebauer and Sachs, the numbers in the first three columns of the tablet are merely \( d^2/l^2 \), \( w \) and \( d \), where \( l = 2pq \), \( w = p^2 - q^2 \) and \( d = p^2 + q^2 \) are as in (1). In Table 3, we list the numbers on the original tablet (corrected when necessary) along with the missing side \( l \) as well as the generating parameters \( p \) and \( q \). What makes this interpretation particularly attractive is that except for \( p = 205 \) in row four of the table, all values of \( p \) and \( q \) are in the standard table of reciprocals (Table 2). But it happens
that the regular number 2 05 is the first (restored) value in another table of reciprocals found in the OB tablet CBS 29.13.21 [Neugebauer and Sachs, 1945, p. 14]. Moreover, the value of \( \frac{p}{q} \), like that of \( \frac{d^2}{l^2} \), decreases as we move down the table. In fact, if we allow \( p \) and \( q \) to be regular numbers less than or equal to 2 05 and sort the resulting table by \( \frac{p}{q} \) (or by \( \frac{d^2}{l^2} \)), then only one extra triplet that goes between lines 11 and 12 is produced, see Table 4. This led D. E. Joyce to argue that the extra triplet may have been inadvertently left out or that it was dismissed because the magnitudes of the sides in the resulting triplet are too large [Joyce, 1995]. On the other hand, if we consider all regular integers \( p \) and \( q \) such that \( p \leq 205 \) and \( q \leq 100 \), then the first fifteen triplets, sorted by descending value of \( \frac{p}{q} \), are exactly those found in the Plimpton tablet [Price, 1964].

The second widely accepted method for devising the tablet was proposed by E. M. Bruins in 1949, and is often called the ‘reciprocal’ method as opposed to the ‘generating pair’ method of Neugebauer and Sachs. The method is based on the fact that if \( r = \frac{p}{q} \) is regular, then the numbers \( x = \frac{2}{r} - r \) and \( y = \frac{2}{r} + r \) satisfy, on top of being regular, the equation \( 1 + x^2 = y^2 \). It

\[
\begin{array}{cccccccc}
 p & q & l & d^2/l^2 & w & d & n \\
 12 & 5 & 2 00 & 1 59 00 15 & 1 59 & 2 49 & 1 \\
 1 04 & 27 & 57 36 & 1 56 56 58 14 50 06 15 & 56 07 & 1 20 25 & 2 \\
 1 15 & 32 & 1 20 00 & 1 55 07 41 15 33 45 & 1 16 41 & 1 50 49 & 3 \\
 2 05 & 54 & 3 45 00 & 1 53 10 29 32 52 16 & 3 31 49 & 5 09 01 & 4 \\
 9 & 4 & 1 12 & 1 48 54 01 40 & 1 05 & 1 37 & 5 \\
 20 & 9 & 6 00 & 1 47 06 41 40 & 5 19 & 8 01 & 6 \\
 54 & 25 & 45 00 & 1 43 11 56 28 26 40 & 38 11 & 59 01 & 7 \\
 32 & 15 & 16 00 & 1 41 33 45 14 03 45 & 13 19 & 20 49 & 8 \\
 25 & 12 & 10 00 & 1 38 33 36 36 & 8 01 & 12 49 & 9 \\
 1 21 & 40 & 1 48 00 & 1 35 10 02 28 27 24 26 40 & 1 22 41 & 2 16 01 & 10 \\
 2 & 1 & 4 & 1 33 45 & 3 & 5 & 11 \\
 48 & 25 & 40 00 & 1 29 21 54 02 15 & 27 59 & 48 49 & 12 \\
 15 & 8 & 4 00 & 1 27 00 03 45 & 2 41 & 4 49 & 13 \\
 50 & 27 & 45 00 & 1 25 48 51 35 06 40 & 29 31 & 53 49 & 14 \\
 9 & 5 & 1 30 & 1 23 13 46 40 & 56 & 1 46 & 15 \\
\end{array}
\]

Table 3: Plimpton 322 preceded by the generators \( p \) and \( q \) and the long side \( l \).

\[
\begin{array}{cccccccc}
 p & q & l & d^2/l^2 & w & d & n \\
 2 05 & 1 04 & 4 26 40 & 1 31 09 09 25 42 02 15 & 3 12 09 & 5 28 41 & 11a \\
\end{array}
\]

Table 4: The missing line 11a should be inserted between lines 11 and 12.
turned out that for each line in the tablet one can find a regular number \( r \) (called the *generating ratio*) such that when \( x \) and \( y \) are divided by their regular common factors, we end up with the numbers in the second and third columns of the Plimpton tablet. To see how this could be done, let us take a closer look at line five of Table 3. Since \( p = 9 \) and \( q = 4 \), we have \( r = 2;15 \) and \( \overline{r} = 0;26 \ 40 \). This yields \( x = 0;54 \ 10 \) and \( y = 1;20 \ 50 \). Since the rightmost (sexagesimal) digit of \( x \) and that of \( y \) are divisible by 10, we should multiply both \( x \) and \( y \) by 6 (or equivalently divide \( x \) and \( y \) by 10). The resulting numbers are \( 5;25 \) and \( 8;05 \), both of which should be multiplied by 12 (or divided by 5) leading to \( 1 \ 05 \) and \( 1 \ 37 \). In tabular form, we have

\[
\begin{array}{cccc}
6 & \times & 0;54 \ 10 & 1;20 \ 50 & \times & 6 \\
12 & \times & 5;25 & 8;05 & \times & 12 \\
1 \ 05 & & 1 \ 37 & \\
\end{array}
\]

Since the terminating digits of \( 1 \ 05 \) and \( 1 \ 37 \) have nothing in common, the process stops here. Now if we think of \( 1 \ 05 \) and \( 1 \ 37 \) as the width \( w \) and diagonal \( d \) of a rectangle, then the length \( l \) must be \( 1 \ 12 \), which is the product of \( 12 \) and \( 6 \). Equivalently, \( w = p^2 - q^2 \), \( d = p^2 + q^2 \), and \( l = 2pq \). In Table 5 we list the values of \( r \), \( \overline{r} \), \( x \), \( y \) and \( l \). Observe that even though \( p \) and \( q \)

| \( p \) | \( q \) | \( r \) | \( \overline{r} \) | \( x \) | \( y \) | \( l \) | \( n \) |
|---|---|---|---|---|---|---|---|
| 12 | 5 | 2;24 | 0;25 | 0;59 \ 30 | 1;24 \ 30 | 2 \ 00 | 1 |
| 104 | 27 | 2;22 \ 13 \ 20 | 0;25 \ 18 \ 45 | 0;58 \ 27 \ 17 \ 30 | 1;23 \ 46 \ 02 \ 30 | 57 \ 36 | 2 |
| 115 | 32 | 2;20 \ 37 \ 30 | 0;25 \ 36 | 0;57 \ 30 \ 45 | 1;23 \ 06 \ 45 | 1 \ 20 \ 00 | 3 |
| 205 | 54 | 2;18 \ 53 \ 20 | 0;25 \ 55 \ 12 | 0;56 \ 29 \ 04 | 1;22 \ 24 \ 16 | 3 \ 45 \ 00 | 4 |
| 9 | 4 | 2;15 | 0;26 \ 40 | 0;54 \ 10 | 1;20 \ 50 | 1 \ 12 | 5 |
| 20 | 9 | 2;13 \ 20 | 0;27 | 0;53 \ 10 | 1;20 \ 10 | 6 \ 00 | 6 |
| 54 | 25 | 2;09 \ 36 | 0;27 \ 46 \ 40 | 0;50 \ 54 \ 40 | 1;18 \ 41 \ 20 | 45 \ 00 | 7 |
| 32 | 15 | 2;08 | 0;28 \ 07 \ 30 | 0;49 \ 56 \ 15 | 1;18 \ 03 \ 45 | 16 \ 00 | 8 |
| 25 | 12 | 2;05 | 0;28 \ 48 | 0;48 \ 06 | 1;16 \ 54 | 10 \ 00 | 9 |
| 121 | 40 | 2;01 \ 30 | 0;29 \ 37 \ 46 \ 40 | 0;45 \ 56 \ 06 \ 40 | 1;15 \ 33 \ 53 \ 20 | 1 \ 48 \ 00 | 10 |
| 2 | 1 | 2 | 0;30 | 0;45 | 1;15 | 4 \ 00 | 11 |
| 48 | 25 | 1;55 \ 12 | 0;31 \ 15 | 0;41 \ 58 \ 30 | 1;13 \ 13 \ 30 | 40 \ 00 | 12 |
| 15 | 8 | 1;52 \ 30 | 0;32 | 0;40 \ 15 | 1;12 \ 15 | 4 \ 00 | 13 |
| 50 | 27 | 1;51 \ 06 \ 40 | 0;32 \ 24 | 0;39 \ 21 \ 20 | 1;11 \ 45 \ 20 | 45 \ 00 | 14 |
| 9 | 5 | 1;48 | 0;33 \ 20 | 0;37 \ 20 | 1;10 \ 40 | 45 \ 00 | 15 |

Table 5: Reciprocal method: \( r = \frac{p \overline{q}}{q} \), \( x = \overline{2} (r - \overline{r}) \), \( y = \overline{2} (r + \overline{r}) \) and \( l = 2pq \).

change erratically from one line to the next, the ratio \( p/q \) steadily decreases as we move down
the table. Since it is a common Babylonian practice to list numbers in descending or ascending order (the ratios \(d^2/l^2\) in Column I of the tablet decrease from top to bottom), we have the first piece of evidence showing that the \(r\)-method is more in accordance with OB mathematics.

Although the reciprocal method (or \(r\)-method) may seem awkward from a modern point of view, there is ample evidence that the techniques it employs have been used in OB mathematics [Robson, 2001; Friberg, 1981; Bruins, 1967]. One advantage of the method is that the concept of relatively prime numbers becomes unnecessary: Start with any regular number \(r\), the process terminates with a primitive triplet. Moreover, the method is favored because it provides a simple way to compute the ratio \(d^2/l^2\) in the first preserved column. Since \(d/l\) is just \(y\), all the scribe had to do was calculate \(\bar{2}(r + \bar{r})\) and then square the result. The method also makes use of the flexibility inherent in the (ambiguous) Babylonian number system, where multiplication and division can be interchanged at will. This is due to the dismissal of leading and trailing zeros as well as the lack of a symbol that separates the fractional from the whole part of a number. In short, the advantage of Bruins’ method over other methods can be summarized as follows:

So what does make Bruins’ reciprocal theory more convincing than the standard \(p,q\) generating function—or, indeed, the trigonometric table? I have already showed that its starting points (reciprocal pairs, cut-and-paste algebra) and arithmetical tools (adding, subtracting, halving, finding square sides) are all central concerns of Old Babylonian mathematics: it is sensitive to the ancient thought-processes and conventions in a way that no other has even tried to be. For example, in this theory the values in Column I are a necessary step towards calculating those in Column III and may also be used for Column II. And the Column I values themselves are derived from an ordered list of numbers [Robson, 2001].

4 Possible Ways to Complete the Table

Suppose that one wants to list all Pythagorean triplets \((w, l, d)\) such that \(w < l < d < 20000\), \(l < 15000\) is a regular, \(\gcd(l, d) = 1\) and \(d^2/l^2 < 2\). Then the first 15 triplets listed in descending order of \(d^2/l^2\) are exactly those found in Plimpton 322. The only exception is that the triplet \((45, 60, 75)\) in the tablet should be replaced by the equivalent triplet \((3, 4, 5)\), since the greatest common divisor of \(l\) and \(d\) in the former triplet is different from one.

Although the above triplets agree with those in Plimpton 322, no one would be imprudent enough to think that the Babylonians constructed the tablet by following such a predefined set of modern rules, not to mention the enormous number of calculations involved. From the outset, it should be made absolutely clear that it is not enough to provide a set rules that produce the
numbers in the tablet unless those rules can be found, at least implicitly, in the collective body of OB mathematics. So in order to determine how the tablet may have been devised, we must only work with the sort of mathematics that OB scribes had at their disposal.

The Old Babylonians used 1;25 as a rough estimation of root two, and there is strong evidence that they also used the much closer approximation of 1;24 51 10. The evidence for the closer approximation comes from two OB tablets known as YBC 7243 and YBC 7289 [Neugebauer and Sachs, 1945, pp. 42-43]. The first of the two tablets contains a list of coefficients, where on the tenth line appears the number 1 24 51 10 followed by the words ‘Diagonal, square root’. The second tablet is a round tablet circumscribing a (diagonal) square with the cuneiform symbol for 30 written above the middle of the upper-left side. In addition, the sexagesimal numbers 1 24 51 10 and 42 25 35 are inscribed along the horizontal diagonal and across the lower half of the vertical diagonal, see Figure 3. Since the Old Babylonians did not explicitly write trailing zeros and since the result of multiplying 30 by 1 24 51 10 is 42 25 35 00, it is immediately clear that 42 25 35 should be interpreted as the product of 30 and 1 24 51 10. A more meaningful relation between the three numbers can be deduced if we think of some or all of them as fractions and not integers. We can always do this because the Babylonian number system does not distinguish between fractions and whole numbers. Taking 1 24 51 10 as 1;24 51 10 and comparing it with \( \sqrt{2} = 1;24 51 10 \overline{07}\ldots \), there is little doubt that the number at hand is a Babylonian approximation of root two. This makes perfect sense since the length of the diagonal of a square is equal to radical two times the length of its side. Furthermore, if we think of 30 as 0;30 = \( \overline{2} \), then the length of the diagonal will be equal to the reciprocal of root two. In other words, if the length of the side is \( \overline{2} \), then the length of the diagonal is a half times root two, which is the same as the reciprocal of root two. From this we see that the side of the square was so cleverly chosen so that the other two numbers inscribed on the tablet

3In decimal notation, we have 1;24 51 10 = 1.41421296, while \( \sqrt{2} = 1.414213\ldots \).
are nothing but highly accurate approximations of root two and its reciprocal. Knowing the central role reciprocals played in Babylonian mathematics, it is hard to believe that this could have happened by accident [Melville, 2006]. As to how the Babylonians may have found such an extremely good approximation of root two, various ways have been proposed by different authors [Neugebauer and Sachs, 1945, p. 43; Fowler and Robson, 1998].

The YBC 7289 tablet is another attestation of the Old Babylonian understanding of the theorem of Pythagoras. The tablet covers a special case of the theorem where the rectangle is replaced by a square, meaning that the width \( w \) is the same as the length \( l \). But if \( w = l \), then the equation \( w^2 + l^2 = d^2 \) can be rewritten as \( d^2/l^2 = 2 \). The other special case of the theorem is the extreme case with \( d = l \), where the rectangle collapses into a line. It is between these two special cases that the numbers in the first column of the Plimpton tablet should be viewed. Let \( \alpha_0 \) be the generating ratio for the first special case and, for the lack of a better term, \( \alpha_\infty \) be the generating ratio for the other case. It follows from (2) that \( \alpha_0 = 1 + \sqrt{2}, \alpha_\infty = 1 \), and that every positive ratio \( \alpha \) such that \( \alpha - 1/\alpha > 0 \) must satisfy the inequality

\[
\alpha_0 > \alpha > \alpha_\infty, \tag{5}
\]

provided that the length of the rectangle is taken as 1. Moreover, if \( \alpha \) is replaced by the regular number \( r = p/q \), then we can use (3) to show that for \( r \) to satisfy (5) we must have

\[
p^2 - q^2 < 2pq. \tag{6}
\]

A regular ratio \( r \) satisfying (6) will henceforth be called admissible.

If we look at the fifteen pairs \((p, q)\) that generate the numbers inscribed on the Plimpton tablet, we find that their ratios \( p/q \) are all admissible. Moreover, the largest value of \( p \) is 205 and that of \( q \) is 54. Assuming for now that no ratio is allowed to have larger values of \( p \) and \( q \), we get a total of 38 admissible ratios, the first fifteen of which are exactly those of Table 5. The same set of ratios is produced if \( q \) is allowed to be as large as sixty. In Table 6, we list the 38 ratios along with the Pythagorean triplets they generate if the \( r \)-method is used. A similar table was first devised by Price using the \( pq \)-method [Price, 1964]. The lines ending with an asterisk (lines 11, 15, 18 and 36) are those for which the \( pq \)-method deviates from the \( r \)-method. These lines correspond to ratios where \( p \) and \( q \) are both odd, and consequently \( p^2 - q^2 \) and \( p^2 + q^2 \) are both even. It follows that \( p^2 - q^2, p^2 + q^2 \) and \( 2pq \) have a common factor of 2 and so the Pythagorean triplet is not primitive. In such cases, the values produced by the \( pq \)-method will be twice the values produced by the \( r \)-method.

---

4Price made many calculation errors in the complete table. In fact, the value of \( \left(\frac{p^2+q^2}{2pq}\right)^2 \) in his table \((d^2/l^2 \text{ in our table})\) is incorrect in lines 16, 17, 23, 24, 25, 28, 29, 30 and 34.
| \( r \) | \( l \) | \( d^2/l^2 \) | \( w \) | \( d \) | \( n \) |
|---|---|---|---|---|---|
| 2:24 | 2 00 | 1;59 00 15 | 1 59 | 2 49 | 1 |
| 2:22 13 20 | 57 36 | 1;56 56 58 14 50 06 15 | 56 07 | 1 20 25 | 2 |
| 2:20 37 30 | 1 20 00 | 1;55 07 41 15 33 45 | 1 16 41 | 1 50 49 | 3 |
| 2:18 53 20 | 3 45 00 | 1;53 10 29 32 52 16 | 3 31 49 | 5 09 01 | 4 |
| 2:15 | 1 12 | 1;48 54 01 40 | 1 05 | 1 37 | 5 |
| 2:13 20 | 6 00 | 1;47 06 41 40 | 5 19 | 8 01 | 6 |
| 2:09 36 | 45 00 | 1;43 11 56 28 26 40 | 38 11 | 59 01 | 7 |
| 2:08 | 16 00 | 1;41 33 45 14 03 45 | 13 19 | 20 49 | 8 |
| 2:05 | 10 00 | 1;38 33 36 36 | 8 01 | 12 49 | 9 |
| 2:01 30 | 1 48 00 | 1;35 10 02 28 27 24 26 40 | 1 22 41 | 2 16 01 | 10 |
| 2 | 1 00 | 1;33 45 | 45 | 1 15 | 11* |
| 1:55 12 | 40 00 | 1;29 21 54 02 15 | 27 59 | 48 49 | 12 |
| 1:52 30 | 4 00 | 1;27 00 03 45 | 2 41 | 4 49 | 13 |
| 1:51 06 40 | 45 00 | 1;25 48 51 35 06 40 | 29 31 | 53 49 | 14 |
| 1:48 | 45 | 1;23 13 46 40 | 28 | 53 | 15* |
| 1:46 40 | 4 48 | 1:22 09 12 36 15 | 2 55 | 5 37 | 16 |
| 1:41 15 | 14 24 | 1;17 58 56 24 01 40 | 7 53 | 16 25 | 17 |
| 1:40 | 15 | 1;17 04 | 8 | 17 | 18* |
| 1:37 12 | 2 15 00 | 1;15 04 53 43 54 04 26 40 | 1 07 41 | 2 31 01 | 19 |
| 1:36 | 1 20 | 1;14 15 33 45 | 39 | 1 29 | 20 |
| 1:33 45 | 13 20 | 1;12 45 54 20 15 | 6 09 | 14 41 | 21 |
| 1:30 | 12 | 1;10 25 | 5 | 13 | 22 |
| 1:28 53 20 | 36 00 | 1;09 45 22 16 06 40 | 14 31 | 38 49 | 23 |
| 1:26 24 | 30 00 | 1;08 20 16 04 | 11 11 | 32 01 | 24 |
| 1:25 20 | 1 36 00 | 1;07 45 23 26 38 26 15 | 34 31 | 1 42 01 | 25 |
| 1:24 22 30 | 48 00 | 1;07 14 53 46 33 45 | 16 41 | 50 49 | 26 |
| 1:23 20 | 15 00 | 1;06 42 40 16 | 5 01 | 15 49 | 27 |
| 1:21 | 18 00 | 1;05 34 04 37 46 40 | 5 29 | 18 49 | 28 |
| 1:20 | 24 | 1;05 06 15 | 7 | 25 | 29 |
| 1:16 48 | 26 40 | 1;03 43 52 35 03 45 | 6 39 | 27 29 | 30 |
| 1:15 | 40 | 1;03 02 15 | 9 | 41 | 31 |
| 1:12 | 1 00 | 1:02 01 | 11 | 1 01 | 32 |
| 1:11 06 40 | 28 48 | 1;01 44 55 12 40 25 | 4 55 | 29 13 | 33 |
| 1:07 30 | 2 24 | 1:00 50 10 25 | 17 | 2 25 | 34 |
| 1:06 40 | 3 00 | 1:00 40 06 40 | 19 | 3 01 | 35 |
| 1:04 48 | 11 15 | 1:00 21 21 53 46 40 | 52 | 11 17 | 36* |
| 1:04 | 8 00 | 1:00 15 00 56 15 | 31 | 8 01 | 37 |
| 1:02 30 | 20 00 | 1:00 06 00 09 | 49 | 20 01 | 38 |

Table 6: The continuation of Plimpton 322. The first two columns should be inscribed on the broken part of the tablet; while lines 16 to 38 should be inscribed on the reverse of the tablet.
To better understand the difference between the two methods, let us take a closer look at lines 11 and 15. For line 11, Price took $p = 100$ and $q = 30$ so that $pq = 2$, yielding the triplet $(100 00, 45 00, 115 00)$. This seems somewhat contrived since taking $p = 2$ and $q = 1$ produces the equivalent triplet $(4, 3, 5)$. On the other hand, Bruins method yields $x = 0; 45$ and $y = 1; 15$. Since these two numbers along with $l = 1$ make the well known triplet $(100, 45, 115)$ the scribe did not bother do the simplification to obtain the reduced triplet $(4, 3, 5)$. Although it is hard to decide which method was used based only on this case, we still think that the $r$-method better explains why the non-reduced triplet is the one that eventually appeared on the tablet. We take this view in light of the fact that the non-reduced triplet $(1, 45, 115)$ can be found in another OB text from Tell Dhiha’i [Baqir, 1974]. The text poses and solves the following problem: Find the sides of the rectangle whose diagonal is 45 and whose area is 115. The relevance of the problem to our case lies not only in the fact that the calculated sides and the diagonal form the triplet $(1, 45, 115)$, but more importantly in the clear resemblance between the solution algorithm and the $r$-method [Friberg, 1981]. As for line 15, the $pq$-method yields the triplet $(130, 56, 146)$, while the $r$-method yields the triplet $(45, 28, 53)$. Since the part of the triplet inscribed on the tablet is $w = 56$ and $d = 53$, it is not clear which entry should be considered as the wrong one. If the $pq$-method is used then the erroneous value would be that of $d$; while if the $r$-method is used then the incorrect value would be that of $w$ (more on this in Section 6).

Had the scribe continued to line 18 or 36, we would have been able to tell which method was used with a higher degree of certainty. Unfortunately, lines 18 and 36 are not inscribed, forcing us to ponder over the criteria by which the number of lines in the tablet was determined.

If there are 38 admissible ratios with $p \leq 205$ and $q \leq 100$, then one wonders why only 15 of those are found on Plimpton 322. For proponents of the trigonometric table it is because these ratios roughly correspond to angles between 45° and 30°; while for proponents of the incomplete table the missing ratios should have also been inscribed, perhaps on the reverse of the tablet. The latter view is supported by the fact that the lines separating the columns on the obverse of the tablet are continued on the reverse. Also, if the tablet is to list all ratios leading to angles between 45 and 30 degrees, then it should contain an extra line since the sixteenth ratio 16/9 yields the triplet $(175, 288, 337)$, with an angle slightly greater than 31°. All of this suggests that the size of the tablet may be related to the purpose behind it, and possibly to the method employed in its construction. In the remainder of this section, we will closely examine the three main previously proposed procedures for the selection of $p$ and $q$, weigh the pros and cons of each procedure and, based on the conclusion reached, select the most likely procedure to account for the size of the tablet in a way which is consistent with the words and numbers inscribed on it.
**Procedure 1.** In this procedure, first introduced by Price, the regular numbers $p$ and $q$ are chosen so that $1 < q < 60$ and $1 < p/q < \alpha_0$ [Price, 1964]. Since 54 is the largest regular integer less than 60 and since 128 is the largest regular integer less than 54 times $\alpha_0$, the conditions on $p$ and $q$ can be restated as

$$1 < p \leq 128 \quad \text{and} \quad 1 < q \leq 54.$$  \hfill (7)

This leads to a set of 38 admissible ratios (see Column I of Table 6), the first 15 of which, when written in descending order, match exactly with those needed to produce the Plimpton tablet. From this set, only the fourth ratio $125/54$ cannot be written as $p\overline{q}$, where $p$ and $q$ are in the standard table of reciprocals. The simplicity of the procedure plus the fact that it succeeds in producing the correct ratios make it doubly appealing, though not without some shortcomings. One such shortcoming is that for $p$ and $q$ satisfying (7), we get a total of 234 distinct ratios and there is no telling of how the 38 admissible ratios were selected and sorted unless all 234 ratios were written as sexagesimal numbers.\footnote{Since 54 is the largest regular numbers less than 60, taking $q = 60$ generates only one extra ratio, namely $1/30$, but this does not change the set of admissible ratios.} More importantly, it is not explained why $q$ has to be less than 60, especially since, according to Price, entries in the first column of the preserved tablet are calculated by squaring the result of the division of $p^2 + q^2$ by 2$pq$. In other words, if both $\overline{p}$ and $\overline{q}$ are needed in the calculation of the table, then there is no good reason for $p$ and $q$ to have different upper bounds. This problem persists even if the $r$-method is used, since for every $r = p\overline{q}$ we must also compute $\overline{r} = q\overline{p}$.

**Procedure 2.** In this procedure the ratios are chosen so that both $p$ and $q$ are regular integers less than or equal to 125. This leads to 47 admissible ratios, of which the line corresponding to the twelfth ratio $125/64 = 1;57\ 11\ 15$ is missing from the tablet, see Table 4. The argument proposed by Joyce that this ratio may have been dismissed because it leads to large values of $w$ and $d$ does not carry much weight since the ensuing triplet $(11529, 16000, 19721)$ is comparable to the triplet $(12709, 13500, 18541)$ produced by the fourth ratio $125/54$ [Joyce, 1995]. This, however, does not completely rule out the possibility that the ratio was unintentionally overlooked. But even if that is the case, we still have to address the unanswered question: Why was 125 chosen as the upper bound for $p$ and $q$? A possible answer to this crucial question is offered by the following quote from Neugebauer: ‘…The only apparent exception is $p = 2;05$ but this number is again well known as the canonical example for the computation of reciprocals beyond the standard table.’ [Neugebauer, 1969, p. 39] As in Procedure 1, there is no mentioning of how the 47 admissible ratios were sorted out from a total of 303 possible ratios.
Procedure 3. This procedure was first proposed by Bruins and later adopted and improved by Robson [Bruins, 1957; Robson, 2001]. According to Robson, the ratios were chosen so that neither $r = \frac{p}{q}$ nor $\tau = \frac{q}{p}$ has more than four sexagesimal places, with the total number of places in the pair not exceeding seven. The number of ratios satisfying these conditions is 18, of which only 15 found their way to the Plimpton tablet. The decimal values of the three omitted ratios are 288/125, 135/64 and 125/64, and the lines corresponding to them are shown in Table 7 (the letter 'a' is appended to the line number $n$ to indicate that the line at hand should be inserted between line $n$ and the line that follows). The main problem with this procedure is that it is not easily justifiable why the maximum number of sexagesimal places in the pair $r$ and $\tau$ has to be seven. We believe that this is a consequence of rather than the criteria for choosing $r$ and $\tau$. Our view is supported by the fact that apart from the fourth ratio 125/54, every other ratio can be expressed as $p\tau$, where $p$ and $q$ belong to the standard table of reciprocals. But even if we accept that the pair $r$ and $\tau$ should not have more than seven sexagesimal digits, there are still some issues with this procedure that must be addressed.

First, Robson assumes that because they yield what she calls nice (small) Pythagorean triplets, the first and fifteenth ratios were preselected by the scribe as upper and lower bounds for the remaining ratios. But if that is the case, then certainly the ratio $5/3 = 1; 40$, whose reciprocal is $0;36$, makes a better lower bound (see line 18 of Table 6). This is more so since it generates the nicer triplet (15, 8, 17). Second, she talks about the ancient Babylonians being totally oblivious to the notion of a complete table, which is hard to believe in light of the fact that every line corresponding to a standard ratio lying between the largest (top) and smallest (bottom) ratios is included in the tablet. Third, she excludes line 4a on the ground that its short side and diagonal are half a place too long, meaning that they contain tens in the leftmost sexagesimal place. But if the length of the sides is important then so should be the size of the ratio $d^2/l^2$, where a quick glance at Column I of the tablet shows that the value of $d^2/l^2$ in line 10 is two full places longer than its value in line 4a. In addition, she argues that since the long side is two sexagesimal places in every line of the table, line 11a was left out because its

| $r$     | $l$     | $d^2/l^2$ | $w$     | $d$     | $n$   |
|---------|---------|-----------|---------|---------|------|
| 2;18 14 24 | 20 00 00 | 1;52 27 06 59 24 09 | 18 41 59 | 27 22 49 | 4a   |
| 2;06 33 45 | 4 48 00  | 1;40 06 47 17 32 36 15 | 3 55 29 | 6 12 01 | 8a   |
| 1;57 11 15 | 4 26 40  | 1;31 09 09 25 42 02 15 | 3 12 09 | 5 28 41 | 11a  |

Table 7: The three lines corresponding to the ratios 288/125, 135/64 and 125/64.
longer side 4 26 40 has three sexagesimal places. While this is true if we exclude the terminating zero in the longer side, the scribe must be fully aware of this zero, especially since the longer (uninscribed) side in lines 2, 5 and 15 does not end with zero. Finally, she dismisses line 8a because its generating ratio is not easily derivable from the standard table of reciprocals using attested OB techniques. The same argument is given as another reason for dismissing line 4a. We think, however, that the first two procedures provide a better explanation of why the two lines are missing from the tablet.

When all three procedures are considered, Procedure 1 is more likely to have been used by the author of the Plimpton tablet because it produces the complete set of ratios based on a minimal number of tenable assumptions. As for the other two procedures, the weakness of the second stems from its inability to account for the missing ratio 125/64; while the third is plagued by the many contentious rules proposed by Robson, who has otherwise done an excellent job in explaining the Plimpton tablet and in putting it in its proper context.

5 A New Method for Reconstructing the Table

In this section, we will show how the generating ratios $p/q$ can be obtained by a new selection procedure, which is not only easy to implement but is also consistent with extant OB mathematics. In addition, we will demonstrate how upper bounds for $p$ and $q$ like those given by Price may be reached depending solely on the approximation of root two being used.

We begin by noticing that the ratios $p/q$ and $d^2/l^2$ in columns one and three of Table 6 decrease as we move down the table. The same is true for the ratio $w/l$, obtained by dividing an entry in Column IV by the respective entry in Column II. In Figure 4, the three dotted graphs (from top to bottom) represent $p/q$, $d^2/l^2$ and $w/l$: The large dots correspond to lines 1 to 15, found on the obverse of Plimpton 322; the midsize dots correspond to lines 16 to 31; and the small dots correspond to lines 32 to 38. The reason we grouped lines 32 to 38 together is that the ratio $l/w$ in these lines is greater than 5, while for the first 31 lines we have

$$1 < l/w < 5.$$  \(8\)

More precisely, $l/w$ does not exceed 4 until line 30, and even for line 31 it is still less than half way between 4 and 5, 40/9 to be exact. So it seems arguable that the table should end at line 31 since this leaves 16 entries for the reverse of the tablet, as opposed to the 15 lines (plus headings) inscribed on the obverse of the tablet. The argument is strengthened by the fact that

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\(^6\)By allowing both $r$ and $\pi$ to have up to four sexagesimal digits, Robson obtains three more pairs which are then dismissed using similar reasoning.
it is extremely difficult to find right triangles with \( l/w > 5 \) in extant OB mathematics. On the other hand, there are many OB text problems having right triangles (rectangles) whose lengths to widths are around 4 to 1 \cite{Robson1997}. In Figure 5 the rectangles corresponding to lines 1, 15 and 31 are drawn. Observe that in addition to having their lengths to widths satisfy (8), these triangles look like what a teacher (whether ancient or modern) trying to illustrate Pythagoras theorem would draw in front of a group of students.

If an inequality similar to (8) is to hold for all entries in the tablet, then one would have to disagree with Price that the reverse of the tablet should contain the 23 uninscribed lines. This view is supported by the fact that even if we assume that the reverse of the tablet does not contain the two lines occupied by the heading on the obverse, a maximum of 17 or 18 lines could fit on the reverse. In the case of 18 lines, we get a total of 33 triangles (rectangles) with \( 1 < l/w < 6 \). In contrast, the last (38-th) rectangle, shown in Figure 6 has a ratio \( l/w \) greater than 24. Therefore, if the Plimpton tablet is to be a list of practical Pythagorean triplets then it should not contain such a triangle. However if the tablet is thought of as an ancient piece
of number theory then there is no reason why the process should not be continued until all
triplets have been found. We believe that the size of the tablet and the nature of Babylonian
mathematics speak in favor the former point of view. This does not mean that the scribe was
unaware of how the process can be carried out to its fullest. As a rule of thumb, the scribe
had to perform a balancing act between how close is the first generating ratio to \( \alpha_0 \) (regular
numbers can get arbitrarily close to \( \alpha_0 \)) and how large is the corresponding Pythagorean triplet,
keeping in mind that the ratio \( l/w \) of the resulting triangle should not be permitted to increase
boundlessly.

For instance, suppose that the scribe considered all regular integers \( p \) and \( q \) that are less
than or equal 60\(^2\). Then the only ratio \( p/q \) that falls between the first ratio \( 12/5 = 2;24 \) and
\( \alpha_0 \) is \( 3125/1296 = 2;24 \ 40 \ 33 \ 20 \), which shows that 2;24 is an exceptionally good choice for the
first ratio. Consequently, whether the scribe was searching for a regular number greater than
2;24 or just considered regular numbers from a standard (or even extended) table of reciprocals,
the first ratio would be the same provided that the length of the generated line should not
exceed the width of the tablet. In particular, taking 3125/1296 as the first ratio, the first triplet
becomes (8086009, 8100000, 11445241), with the ensuing line shown in Table 8. It is doubtful
that the scribe had performed the tedious calculations necessary to produce such a line; but
even if he did, the sheer size of the numbers involved gave him a compelling reason to reject it.
All of this forces us to be overly cautious as we try to rediscover how the generating ratios may

\[
\begin{array}{cccccccccc}
 p & q & l & d^2/l^2 & w & d & n \\
---- & ---- & ---- & ---- & ---- & ---- & ---- \\
52 & 05 & 21 & 36 & 37 & 30 & 00 & 00 & 1;59 & 47 & 34 & 27 & 58 & 38 & 07 & 21 & 36 & 37 & 26 & 06 & 49 & 52 & 59 & 14 & 01 & 1 \\
\end{array}
\]

Table 8: The line corresponding to \( p\overline{q} = 2;24 \ 40 \ 33 \ 20 \).

have been chosen by a ancient scribe whose mathematical tools and interests are in many ways
alien to ours. It should also be emphasized that when we speak of ‘the scribe’ we mean by that
the group of professionals that has developed these tools, possibly over a period longer than the
lifespan of any individual member of the group.

\[^7 \text{Even if we allow } p \text{ and } q \text{ to be as large as } 60^3, \text{ only one more ratio lying between } 2;24 \text{ and } \alpha_0 \text{ can be found, namely } 19683/8192 = 2;24 \ 09 \ 45 \ 21 \ 05 \ 37 \ 30. \]
**Procedure 4.** In this procedure, the generating ratios are those of the form \( \frac{p}{q} \), where both \( p \) and \( q \) are taken from the standard table of reciprocals (regular numbers up to 81). This leads to 40 admissible ratios, of which 81/64, 75/64 and 81/80 have \( q > 60 \). The three lines produced by these ratios are listed in Table 9. As for the remaining 37 lines, they are exactly those in Table 8 with line 4 deleted. Assuming that line 4 was not added to the tablet by mistake,

| \( r \)  | \( l \) | \( \frac{d^2}{l^2} \) | \( w \) | \( d \) | \( n \) |
|--------|--------|----------------|--------|--------|--------|
| 1;15 56 15 | 2 52 48 | 1;03 23 29 29 33 54 01 40 | 41 05 | 2 57 37 | 30 |
| 1;10 18 45 | 2 40 00 | 1;01 31 19 18 53 26 15 | 25 29 | 2 42 01 | 34 |
| 1;00 45 | 3 36 00 | 1;00 00 33 20 04 37 46 40 | 2 41 | 3 36 01 | 40 |

Table 9: The three extra lines corresponding to the ratios 81/64, 75/64 and 81/80.

it is plausible that it was inserted between line 3 and line 5 to reduce the conspicuously large difference between the third and fifth ratio. The motivation behind this is that among the 40 admissible ratios, the largest difference between successive ratios is 0;05 37 30 and the largest difference between successive values of \( \frac{d^2}{l^2} \) is 0;06 13 39 35 33 45, both of which occurring between the third and fourth lines. In Figure 7 we draw the difference between successive values of \( \frac{d^2}{l^2} \) for the first fifteen ratios, where it is obvious that the greatest difference is the one between line 3 and line 4. In fact, if we restrict ourselves to the first 15 ratios, then 0;05 37 30 is the only difference in \( r \) exceeding \( 12 = 0; 05 \), while 0;06 13 39 35 33 45 is the only difference in \( d^2/l^2 \) exceeding \( 10 = 0; 06 \). Given the importance of the numbers 10 and 12 in the sexagesimal number system, these facts can hardly be ignored. Moreover, the first 15 ratios are exactly those leading to \( 1 > w/d > 0;30 \), that is they cover every rectangle whose width is greater than half its diagonal. Since it is highly likely that the tablet is a copy of an older original, one can see how the original might have been produced by precisely these ratios, but line 4 was added to the tablet either inadvertently by an inexperienced scribe or overtime by someone who has noticed the unusually large gap between the third and fourth ratios.

We still have to answer how the admissible ratios were sorted out. We have seen that the standard table of reciprocals is comprised of the 30 regular numbers less than or equal to 81. It follows that there is a total of 900 regular numbers of the form \( r = \frac{p}{q} \), where \( p \) and \( q \) are taken from the standard table of reciprocals. Of these 900 ratios, 237 are distinct, and so it is

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8The only other difference in \( r \) exceeding \( 12 \) is 0;05 25, taking place between the fifteenth and sixteenth ratios; while no other difference in \( d^2/l^2 \) exceeds \( 10 \).

9If lines 16 to 31 were inscribed on the reverse of the tablet, then they would be exactly those lines satisfying the inequality \( 0;30 > w/d > 0;12 \).
not an easy task to order them. The simplest way would be to write down the sexagesimal representation of each fraction and then sort them in ascending or descending order. But the number of possible ratios can be greatly reduced if we are only interested in ratios between some given bounds. For example, there are only 49 values of \( r \) satisfying the condition \( 1 < r < 3 \). An upper bound of 3 is reasonable because 3 is the smallest integer greater than \( \alpha_0 \). In this case, finding the admissible ratios amounts to choosing \( p \) between \( q \) and \( 3q \), a condition that can be easily checked.

![Figure 7: A chart of the difference between consecutive values of \( d^2/l^2 \).](image)

We believe that the above procedure provides a simple and direct way of producing and sorting the generating ratios. Moreover, the sorting part of the procedure can be modified so that it can be applied to other procedures. In particular, Procedure 1 could become much more plausible provided that we can satisfactorily explain why 60 (apart from being the base number) was taken as an upper bound for \( q \). Indeed, it can be shown that upper bounds for \( p \) and \( q \) similar to those in (7) can be reached in a number of ways, depending only on the approximation of \( \sqrt{2} \) used. To begin with, suppose that the scribe used 1;30 as an estimate of \( \sqrt{2} \). Then his approximation for \( \alpha_0 = 1 + \sqrt{2} \) would be 2;30. But in the Babylonian number system 2;30 is written as 2 30, which in addition to our 5/2 could be read as 150 or even 150/60. The last form points us in the direction of how the scribe may have obtained 2 30 and 1 00 as upper bounds for \( p \) and \( q \), leading to the same 38 ratios obtained using (7). Now the advantage of 2 30 as an upper bound is that it is the smallest regular integer \( p \) such that \( p \) times 50 is greater than \( \alpha_0 \). It follows that for \( q \leq 60 \), no new admissible ratios will be produced for values of \( p \) larger than 2 30. Moreover, to determine whether the ratio \( \frac{p}{q} \) is less than 2;30, all that the scribe had to do is check if \( 2p < 5q \). Alternatively, had the scribe used 1;25 as an approximation of \( \sqrt{2} \), his

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10If we allow \( p \) and \( q \) to be 1, then we get a total of 961 fractions, of which 257 are distinct.
estimate of $\alpha_0$ would be $2;25 = 29/12$. Again, he could think of this as $2\ 25$ divided by $1\ 00$, which in turn could be taken as upper bounds for $p$ and $q$. As in the previous case, the same admissible ratios are generated, but checking for admissibility in this case is not as simple. This may not be a disadvantage since other tests for admissibility, such as (10), may have been used.

In addition to the two approximations of $\sqrt{2}$ used above, the scribe may have used a third approximation, albeit indirectly. Earlier in this section, we have seen that even for $p$ and $q$ as large as $60^2$, the only regular number $p/q$ lying between $2;24$ and $\alpha_0$ is $2;24\ 40\ 33\ 20$. Since every ratio used in the tablet has a maximum of four sexagesimal digits and since no such ratio exists between $2;24$ and $\alpha_0$, the scribe may have been prompted from the outset to consider $2;24$ as the largest admissible ratio. In fact, even if the scribe was oblivious to all of this, the same conclusion could be reached by taking the first two sexagesimal digits of $\alpha_0$, provided that $\sqrt{2}$ is approximated by $1;24\ 51\ 10$. But if we only consider ratios $p/q$ less than or equal to $2;24$ and insist that $q \leq 60$, then $p$ should not exceed 144. As in the previous two cases, we get the 38 ratios listed in Table 6.

### 6 Explaining the Errors

The Plimpton tablet contains a number of errors that when carefully analyzed may give us a clearer understanding of how the numbers on the tablet were generated. The apparent errors can be divided into two categories: Typographical errors and computational errors. The typographical errors, shown in Table 10, can be easily explained. Looking at the first error, it is obvious that the scribe carelessly wrote the symbol for 6 a bit too close to that of 50, and thus the correct number can be obtained by simply inserting a little space between the two symbols.

To undo the second error, the sexagesimal digit 59 should be written as 45 followed by 14, and not as the sum of the two digits. The third error amounts to 8 being miscopied as 9, where it

| Error | Line | Column | Inscribed number | Correct number |
|-------|------|--------|-----------------|----------------|
| 1     | 2    | I      | 58 14 56 15     | 58 14 50 06 15 |
| 2     | 8    | I      | 41 33 59 03 45  | 41 33 45 14 03 45 |
| 3     | 9    | II     | 9 01            | 8 01           |
| 4     | 13   | I      | 27 03 45        | 27 00 03 45    |

Table 10: The typographical errors in Plimpton 322.

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11 Neugebauer considered only two errors; Friberg added two more; and Robson added another one.
12 Friberg offers a reasonable explanation of how the error may have occurred if the $r$-method was used. Friberg, 1981.
is quite easy to make such a mistake due to the similarity between the two symbols (9 has just one more wedge than 8). As for the fourth error, it can be easily discarded if the space between 27 and 3 is transliterated as a zero. At any rate, this is hardly an error since we know that OB scribes did not consistently use a blank space to represent zero.

Having dealt with the typographical errors, the computational errors, listed in Table 11, require a deeper understanding of OB mathematics. To begin with, we have seen that there is a definite error in the last line of the tablet, but scholars are divided on whether the erroneous number should be in the second or third column. More precisely, either the entry in Column III should be 1 46 rather than 53, or the entry in Column II should be 28 instead of 56. This suggests that at some point in the calculation, a multiplication by 2 or $\sqrt{2}$ should have been applied to both entries, but the operation was only performed on one. But if the $pq$-method is employed, then no doubling or halving is needed in calculating $p^2 - q^2$ (Column II) or $p^2 + q^2$ (Column III). This led proponents of the method to propose that in order to obtain a primitive pair, the scribe intended to halve both numbers but forgot to do it for $p^2 - q^2$. On the other hand, if the $r$-method was used, then the process of eliminating the common factors of $x = 0;37\ 20$ and $y = 1;10\ 40$ should look something like this:

\[
\begin{align*}
3 & \times 0;37\ 20 & 1;10\ 40 & \times 3 \\
30 & \times 1;52 & 3;32 & \times 30 \\
30 & \times 56 & 1;46 & \times 30 \\
28 & 00 & 53 & 00
\end{align*}
\]

Since the elimination process is usually carried out on a supplementary tablet, it is easily seen how 56 and 28 could be confused for each other as the bottom numbers are transcribed from the supplementary tablet to the Plimpton tablet.

The above $r$-method explanation of the error is more consistent than the one given by Robson, where the common factors of $x$ and $y$ are eliminated using the multipliers 3, 5 and 3. The problem with Robson’s explanation is that it is not clear why the pair (1;52, 3;32) was multiplied by 5 rather than 30, especially since in explaining the first computational error she multiplied the pair (1;56 54 35, 2;47 32 05) by 12 and not by two.\footnote{Robson’s justification is that the numbers in the first pair terminate in 2, while those in the second pair terminate in 6} To be consistent, she should

| Error | Line | Column | Inscribed number | Correct number |
|-------|------|--------|-----------------|---------------|
| 1     | 2    | III    | 3 12 01         | 1 20 25       |
| 2     | 13   | II     | 7 12 01         | 2 41          |
| 3     | 15   | II     | 56              | 28            |

Table 11: The computational errors in Plimpton 322.
either multiply the latter pair by 2, or preferably multiply the former pair by 15 so that, at each
step, the product of the greatest common factor and the multiplier is sixty. But if the second
multiplier is taken to be 15, then the process terminates after only two steps, putting our choice
of multipliers in question. This, however, can be answered in one of two ways: Either the scribe
failed to notice that 4 is the greatest common divisor of 52 and 32, or he was aware of this but,
since both numbers end in 2, he went for the simpler operation of halving, which is equivalent
to multiplying by 30. In the former case, the error probably occurred at the second step when,
rather than the single correct multiplier of 15, the two multipliers 30 and 15 were respectively
applied to the current numbers 1;52 and 3;32, perhaps because noticing that 52 is a multiple of
4 is not as obvious as noticing that 32 is a multiple of 4 [Friberg, 1981]. But in either case, the
ensuing explanation is more concise than the one advanced by Robson.

Turning to the second computational error, we find that the inscribed number 7 12 01 is
simply the square of the correct value 2 41. This, coupled with the assumption that the entry in
Column III of line 15 should be twice the inscribed number, led Gillings to conclude: ‘Thus to
calculate the numbers of the tablet, the doubling of numbers, and the recording of squares, from
their abundant table texts, must have been part of the procedure.’ [Gillings, 1955] The doubling
part of Gillings’ conclusion is disputed by the fact that our above explanation of the error in line
15 did not require the doubling of numbers to be a necessary part of the procedure. In addition,
Gillings does not specify the stage of the procedure at which the doubling and squaring should
take place. If anything, these facts should encourage us to search for new ways to explain how
the square of the supposed number made it to the tablet. Although the exact way may never
be known, we can still make an educated guess. The first thing that comes to mind is that the
error may be due to a routine check that the scribe performed to make sure that the numbers
in Columns II and III along with the uninscribed side form a Pythagorean triplet. To do so, the
scribe first computes \( w^2 \) and \( d^2 \) (after \( w \) and \( d \) were found by eliminating the common factors
of \( x \) and \( y \)) either directly or by consulting a table of squares. Then he computes the difference
between the two squares, and takes the square root of the answer to get \( l \). At this stage, he
is ready to transfer the results from the rough to the clean tablet, but in the process of doing
so he copied \( w^2 \) instead of \( w \). This does not undermine our explanation of the error in line 15
since transcribing the values of \( w \) and \( d \) on the clean tablet should not be done until after the
prescribed check has been performed. An objection to this argument would be that when the
\( r \)-method is used, then at exactly the same step we reach \( w \) and \( d \), we also get \( l \), which is equal
to the product of the multipliers used. A possible answer to this is that the scribe cleared the
common factors of \( y \) only, and then found \( w \) as the square root of the difference between \( d^2 \) and

in 5 [Robson, 2001, pp. 192-193].
$l^2$. But even if the scribe knew how to find $l$ at the same time he found $w$ and $d$, he may still choose to find it using the rule of right triangle, as we shall see in the next section.

We are left with the first computational error, which is the most difficult one to explain since there is no obvious relation between the inscribed number 3 12 01 and the correct number 1 20 25. According to Neugebauer:

It seems to me that this error should be explicable as a direct consequence of the formation of the numbers of the text. This should be the final test for any hypothesis advanced to explain the underlying theory.

The above quote is taking from a note on page 50 of the 1951 edition of Neugebauer’s now classic book *The Exact Sciences in Antiquity*. In later editions of the same book, Neugebauer changed the note so that it reflects R. J. Gillings attempt to resolve the error [Neugebauer, 1969]. According to Gillings, the error is due to the accumulation of two mistakes made by the scribe [Gillings, 1953; Gillings, 1966]. First, in computing

$$d = p^2 + q^2 = (p + q)^2 - 2pq,$$

the scribe accidently calculated $(p + q)^2 + 2pq$. For $p = 1 04$ and $q = 27$, the calculated value would be

$$d = 2 18 01 + 57 36 = 3 15 37,$$

but the scribe made the second mistake where he took $p = 1 00$ instead of 1 04, obtaining $2pq = 54 00$. Now adding 2 18 01 to 54 00, the inscribed value is reached. A modified version of Gillings’ conjecture was proposed by Price, but Price considered the problem as unresolved. Later Gillings refuted the equivalence between his method and that of Price [Gillings, 1966].

The problem with Gillings’ conjecture is twofold. First, the idea of computing $p^2 + q^2$ as $(p+q)^2 - 2pq$ is doubtful since in this case $p = 1 04$ and $q = 27$ are powers of 2 and 3 respectively, and so the scribe can find $d$ by simply finding the sum of the easily computable squares of $p$ and $q$. But even for arbitrary $p$ and $q$, the argument that the right hand side of (9) should be used to calculate $d$ begs the question since it runs from finding the squares of $p$ and $q$ to finding the square of the larger number $p + q$. Moreover, for the Babylonians, squaring was one of the basic mathematical operation, which is reflected not only in the number but also in the scope of the tables of squares they left behind [Friberg, 2007, 45–52]. In fact, it is known that the Babylonians used squares to find the product of two numbers by applying the formula

$$pq = \frac{1}{4} [(p + q)^2 - (p - q)^2] \quad \text{or} \quad pq = \frac{1}{2} [(p + q)^2 - p^2 - q^2],$$

see [O’Connor and Robertson, 2000]. This clearly supports our argument since in both formulas the squares of $p$ and $q$ are used to calculate the product $pq$ and not the reverse, as proposed
by Gillings. Second, the $pq$-method, and hence Gillings’ explanation of the error, involves the concept of relatively prime integers, which is not only unattested from the historical and archaeological record but also runs contrary to the practical nature of OB mathematics. So, in order to accept Gillings’ conjecture, we have to accept that a questionable method was used to generate the tablet; that an unlikely rule was used to compute the sum of two squares; and that at some point in the calculation subtraction was substituted for addition, and then 104 was inexplicably taken as 100. In light of these facts, we think that Gillings’ explanation of the first computational error is improbable at best.

The other often mentioned method for explaining the error in the second line of the Plimpton tablet is based on the $r$-method introduced by E. M. Bruins in 1949. The advantage of this method is that it employs attested OB techniques, which made it the preferred method for authors like Friberg, Schmidt, Robson and others. Starting with the initial pair $x = 0;58\,27\,17\,30$ and $y = 1;23\,46\,02\,30$, Robson applied a slightly different version of the method to obtain the numbers inscribed in columns two and three as follows:

\[
\begin{array}{ccc}
2 \times & 0;58\,27\,17\,30 & 1;23\,46\,02\,30 \times 2 \\
12 \times & 1;56\,54\,35 & 2;47\,32\,05 \times 12 \\
12 \times & 23;22\,55 & 33;30\,25 \times 12 \\
12 \times & 4\,40;35 & 6\,42;05 \times 12 \\
12 \times & 56\,07 & 1\,20\,25 \times 12 \\
12 \times & 11\,13\,24 & 16\,05\,00 \times 12 \\
2\,14\,40\,48 & 3\,13\,00\,00
\end{array}
\]

Instead of stopping when the numbers 57 07 and 1 20 25 were reached, the scribe, unaware that all common factors have been cleared out, carried out the process two extra steps obtaining $w = 2\,14\,40\,48$ and $d = 3\,13\,00\,00$. Realizing that he went too far, he looked back for the correct pair, but in doing so he made two new mistakes: First, he took the value of $d$ from the last step in place of the correct value obtained two steps earlier, and then he sloppily wrote 3 12 01 instead of 3 13 [Robson, 2001]. There is no quarrel in accepting the assumption that 3 12 01 was written for 3 13 since one will probably find similar mistakes somewhere in this paper. Even Gillings, who staunchly opposed Bruins’ explanation of the error, admitted that

\[\text{\underline{**Exception\underline{**}}}
\]

\[\text{\underline{**Despite the availability of modern computers, both typographical and computational errors are still being made by prominent authors in peer reviewed journals. We have seen that in his version of the complete table Price has made many calculation errors, some of which are similar to those in Plimpton 322 (in Column IV of line 15, 8 should be 9), while others are even more difficult to explain. A more serious mistake was committed by Friberg as he tried to explain the appearance of the number } M = 2\,02\,02\,02\,05\,05\,04 \text{ in the Sippar text Ist.S 428. Friberg erroneously observed, that the square root of } M \text{ is the same as the integral part of the square root of } 2\,02\,02\,02\,02\,02, \text{ while the}\]
the number 3 12 01 may well be 3 13 Gillings, 1958. But the problem with Bruins’ method is that it is hard to see why the scribe would copy the correct value for \(w\) as opposed to the (wrong) value for \(d\) that can only be reached if two unnecessary steps have been performed. One explanation would be if the multiplications were not carried out simultaneously. In such case the error must have occurred as the scribe copied the numbers obtained at the end of the simplification process, not taking into account that the simplification of \(y\) required six steps while that of \(x\) required only four steps.

The above procedure for explaining the first computational error can be slightly modified so that it becomes much more plausible. Assuming that the number 3 12 1 is a miscopy of 3 13, the values of \(w\) and \(d\) given by the scribe can be reached in the following way:

\[
\begin{align*}
30 & \times 58 27 17 30 & 1 23 46 02 30 & \times 30 \\
\frac{5}{5} & \times 1 56 54 35 & 2 47 32 05 & \times \frac{5}{5} \\
\frac{5}{5} & \times 23 22 55 & 33 30 25 & \times \frac{5}{5} \\
5 & \times 4 40 35 & 6 42 05 & \times 5 \\
\frac{5}{5} & \times 23 22 55 & 1 20 25 & \times \frac{5}{5} \\
5 & \times 4 40 35 & 16 05 & \times 5 \\
& & 56 07 & 3 13
\end{align*}
\]

In the first step the scribe correctly multiplied both \(x\) and \(y\) by 30, but in one of the following three steps (say step four) he must have multiplied the left number by 5 and the right number by \(\frac{5}{5}\), instead of multiplying both by 5. The simplicity of the error and the fact that this is the only case where the scribe needs four separate multiplications in order to get rid of all common factors between \(x\) and \(y\) make the modified procedure much more attractive.

Another way to account for the error would be if the wrong value of \(y\) was used at the beginning of the procedure. For example, if \(y = 1 23 46 02 30\) is replaced by \(3 20 01 02 30\), then the inscribed number 3 12 01 would appear opposite to 56 07. On the other hand, if \(y\) is replaced by \(3 21 02 30\), then the process terminates with 56 07 and 3 13. In both cases, the initial number used is quite similar to the correct value of \(y\). Since \(y = \sqrt{r^2 + r}\), the error two supposedly equal numbers are 1 25 34 08 and 1 25 34 07 Friberg, 1981, 290. Even Robson has made a number of errors in her major work on the Plimpton tablet Robson, 2001. Of these errors, two are extremely pertinent to us. The first error occurs in the bottom row of Table 6, where the numbers 1;48, 0;33 20 and 0;33 20 are written under the headings \(x\), \(1/x\) and \((x - 1/x)/2\). The first two entries are correct, but the third entry should have been 0;37 20. The similarity between this and some of the errors on the Plimpton tablet is striking. But most ironic is the error made by Robson while trying to explain the first computational error. When multiplying 11 13 24 by 12, she wrote 2 14 36 48 instead of 2 14 40 48. It seems that humans are still prone to the same mistakes they were prone to 4000 years ago.
would have probably occurred as \( r/2 \) was added to \( 7/2 \). In particular, for \( r = 2;22\ 13\ 20 \) the calculation of \( y \) should look something like this:

\[
\begin{array}{c}
1;11 \quad 06 \quad 40 \\
0;12 \quad 39 \quad 22\ 30 \\
1;23 \quad 46 \quad 02\ 30
\end{array}
\]

Now observe that if the (sexagesimal) digit 46 of the correct \( y \) is ignored, then the resemblance between 1 23 02 30 and 3 20 01 02 30 becomes more pronounced when the two numbers are read out loud. It is even more so between 1 23 02 30 and 3 21 02 30. Moreover, 3 21 02 30 is equal to 12/5 times the correct \( y \), where 12/5 is the value of \( r \) in the previous line. Applying Robson’s procedure with \( y = 3\ 21\ 02\ 30 \) yields:

\[
\begin{array}{c}
2 \times \ 0;58\ 27\ 17\ 30 & 3;21\ 02\ 30 \times \ 2 \\
12 \times \ 1;56\ 54\ 35 & 6;42\ 05 \times \ 12 \\
12 \times \ 23;22\ 55 & 1\ 20;25 \times \ 12 \\
12 \times \ 4\ 40;35 & 16\ 05 \times \ 12 \\
56\ 07 & 3\ 13\ 00
\end{array}
\]

From the above discussion we see that except when we take \( y = 3\ 20\ 01\ 02\ 30 \), the corresponding number on the tablet should be 3 13 (or 3 13 followed by one or two zeros) rather than the apparent 3 12 01. As to how this might have happened, we offer three different explanations. First, the space between 12 and 01 should be ignored as a scribal error, and consequently 3 12 01 should be read as 3 13. Second, being aware of the elusive zero(s) at the end of 3 13, the scribe inserted a space (the OB symbol for zero) in the wrong place and wrote 3 12 01 or what might be 3 12 00 01. Third, as the scribe was copying the results on the tablet, he noticed that 56 07 (the width) is larger than 3 13 (the diagonal). Realizing that this cannot be true, he glanced over his calculations and hastily misread 3 13 as 3 12 01.

Having looked at the computational errors one by one, we now look at the tablet as a whole to see if there is a possible relation between the different errors. In Table 12, we list the lines for which the rightmost digit of at least one of the (correct) values of \( w \) and \( d \) (Columns II and III of the tablet) is divisible by a regular number. Observe that except for line 5, one or both of \( w \) and \( d \) deviate from the expected answer. For lines 2 and 15, the errors would be easily explained if the calculation of \( w \) from \( x \) was not done in parallel with the calculation of \( d \) from \( y \). But even if

\[15\] If the calculations are carried out on a separate tablet, as is usually the case, then it is not difficult to see how \( y = \frac{2}{7}(r + 7) \) could have been multiplied by 12/5, the value of \( r \) in the previous calculation of \( y \).
the calculations were done side by side, it is still not difficult to see how the numbers inscribed on the clean tablet could be out of step with each other, especially since for these lines the number of steps needed to clear out the regular numbers on the $x$ side is different from the number of steps needed on the $y$ side. As for line 11, the numbers inscribed in Columns II and III are those of $x$ and $y$, respectively 45 and 1 15. Since in this case the value of $l$ is the base number 1 00, the scribe was apparently satisfied with the non-reduced triplet (45, 1 00, 1 15), which Melville called the favorite version of the primitive triplet (3, 4, 5) \[Melville, 2004\]. Moreover, since we are fairly sure that the scribe knew of the equivalence between the two triplets, it not unreasonable to think that he kept the non-reduced triplet because, unlike the primitive triplet, its entries are comparable in magnitude to those of other triplets.

### Table 12: The four lines for which $w$ or $d$ is divisible by a regular number.

| $x$  | $y$  | $l$  | $w$  | $d$  | $n$  |
|------|------|------|------|------|------|
| 58   | 27   | 17   | 30   | 57   | 36   |
| 54   | 10   | 1 20 | 50   | 1 12 | 1 05 |
| 45   |      | 1 15 |      | 1 00 | 45   |
| 37   | 20   |      | 1 10 | 40   | 28   |

7 Purpose of the Tablet

The presence of modern glue on the edge of the Plimpton tablet does not completely rule out the possibility that the tablet was damaged while it was still in preparation. In such case, the idea that the reverse was meant for the remaining entries becomes much more plausible. Alternatively, the tablet may have been abandoned because someone had noticed the errors before the reverse had been inscribed. This goes hand in hand with the view that the tablet is just a school exercise about finding right triangles with integral sides, which is normally giving by an accomplished scribe to a group of students who aspire to join the respected profession of their teacher. In fact, the types of computational errors committed added to the similarity between two of the errors suggest in a way that the tablet may have been written by someone who has not fully mastered the techniques involved.

If the tablet is taken to be an exercise from a scribal school, then it is not that hard to see the purpose behind such an exercise. First, the generation of the tablet involves many of the mathematical techniques used by OB scribes: Cut-and-paste geometry, division as multiplication by a reciprocal, squaring and taking the square root of a number, and so on. Second, it is easy for the presiding scribe to check the work of his students by comparing their results with a
master copy that has the correct answers. This will be more so if the tablet contains a complete list of triplets, meaning all triplets corresponding to standard ratios between the first and last ratio of the list, as is the case with the Plimpton tablet. Third, the numbers on the tablet may be related to the solution of another ancient problem about right triangles and upright walls. True, there is no direct evidence that the Babylonians used the rule of right triangle in erecting walls, but there are extant Babylonian (and even Egyptian) problem texts in which the rule is used to measure the length of a cane leaning against a wall [Melville, 2004].

One such Babylonian cane-against-the-wall problem is found in the BM 34 568 tablet from the Seleucid period, roughly 300–100 BCE. The problem as stated by Friberg reads like this:

A cane is leaning against a wall. 3 cubits it has come down, 9 cubits it has gone out. How much is the cane, how much the wall? I do not know their numbers [Friberg, 1981].

Following the statement of the problem, the length of the cane is then found by calculating

\[ l = \frac{d^2 + b^2}{2d}, \]

where for \( d = 3 \) and \( b = 9 \) we get \( l = 15 \), see Figure 8. Using modern notation, the solution is obtained by writing the equation \( h^2 + b^2 = l^2 \) as

\[ (l - d)^2 + b^2 = l^2 \quad \text{or} \quad d^2 + b^2 = 2dl, \]

and then solving for \( l \). Finally, instead of calculating the height of the wall using the formula \( h = l - d = 15 - 3 = 12 \), the scribe used the rule of right triangle to first compute \( h^2 = l^2 - b^2 = 224 \), and then took the square root of 224 to get \( h = 12 \). As mentioned in the previous section, this gives more credibility to our explanation of the second computational error, since the erroneous number inscribed on the tablet is simply the square of the correct number.

![Figure 8: The cane against the wall problem of BM 34 568.](image)

Although the above cane-against-the-wall problem comes from period separated by nearly fifteen centuries from the time of Plimpton 322, it is still similar in style as well as content to
tablets from the OB period. In fact, a similar problem is found on the OB tablet BM 85196, where the triplet (18, 24, 30) provides the correct answer [Melville, 2004]. Moreover, Carlos Gonçalves has recently shown that the solution to the first problem of BM 34 568, which is about finding the diagonal of a rectangle but is not solved using the rule of right triangle, can be reduced to finding a pair of reciprocals either algebraically or preferably using cut-and-paste geometry [Gonçalves, 2008]. While the usefulness of the cane-against-the-wall problem supports the view that the Plimpton tablet can be thought of as a scribal school exercise with some practical applications, the argument given by Gonçalves clearly favors Bruins’ method of generating the numbers on the tablet.

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\[16\] Observe that in both problems, the resulting triplet is a multiple of the primitive triplet (3, 4, 5).
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