THE KTH TRAVELING SALESMAN PROBLEM IS PSEUDOPOLYNOMIAL WHEN TSP IS POLYNOMIAL

BRAHIM CHAOURAR

ABSTRACT. Given an undirected graph $G = (V, E)$ with a weight function $c \in \mathbb{R}^E$, and a positive integer $K$, the Kth Traveling Salesman Problem (KthTSP) is to find $K$ Hamilton cycles $H_1, H_2, ..., H_K$ such that, for any Hamilton cycle $H \not\in \{H_1, H_2, ..., H_K\}$, we have $c(H) \geq c(H_i), i = 1, 2, ..., K$. This problem is NP-hard even for $K$ fixed. We prove that KthTSP is pseudopolynomial when TSP is polynomial.

2010 Mathematics Subject Classification: Primary 90C27, Secondary 90C57.
Key words and phrases: K best solutions, Traveling Salesman Problem, Kth best Traveling Salesman Problem, pseudopolynomial.

1. INTRODUCTION

Sets and their characteristic vectors will not be distinguished. We refer to Bondy and Murty [1] and Schrijver [17] about, respectively, graph theory and polyhedra terminology and facts. Given an undirected graph $G = (V, E)$ with a weight function $c \in \mathbb{R}^E$, and a positive integer $K$, the Kth Traveling Salesman Problem (KthTSP) is to find $K$ distinct Hamilton cycles $H_1, H_2, ..., H_K$ such that, for any Hamilton cycle $H \not\in \{H_1, H_2, ..., H_K\}$, we have $c(H) \geq c(H_i), i = 1, 2, ..., K$. Since KthTSP is the famous TSP for $K = 1$, then KthTSP is NP-hard even for $K$ fixed. KthTSP is motivated by searching near optimal solutions with some special properties: when in addition of the TSP constraints, "there are some other which might be difficult to consider explicitly in a mathematical model, or if considered, would increase largely the size of the model. By finding the best, second best, ..., Kth best solution, we are able to sequentially verify these solutions with respect to the additional constraints and stop when a solution that satisfies all of them is found" [19]. Another motivation is that if, for any reason, the route of the best solution is unavailable, then alternate solutions (routes) are desirable [16]. Finding K best solutions of an optimization problem in general has
been studied by few authors \[11,14,15,18\] and almost the same situation happened for particular problems \[2,3,4,5,6,9,10,12,13,15\]. The remainder of the paper is organized as follows: in section 2, we give an algorithm for finding K best solutions for a general model containing KthTSP, then, in section 3, we apply this algorithm to KthTSP and deduce that it is polynomial on \(K\) and \(|E|\) when TSP is polynomial. And we conclude in section 4.

2. An Algorithm for Finding K Best Solutions of a Large Class of Combinatorial Optimization Problems

Let \(P \subseteq \mathbb{R}^m\) be a polyhedra, \(f(m)\) be the number of its facets, \(N(x)\) be the set of all neighbors of an extreme point \(x \in P\), \(x_K\) the Kth best solution in \(P\), regarding to a given weight function and a given positive integer \(K\).

Based on the following property, an algorithm has been used for particular problems \[5,6\].

**Proposition 2.1.** For any positive integer \(j\) such that \(2 \leq j \leq K\),

\[
x_j \in \bigcup_{i=1}^{j-1} N(x_i) \setminus \{x_1, x_2, \ldots, x_{j-1}\}.
\]

Since selecting \(K\) best numbers from a list of \(n\) numbers requires a running time complexity of \(O(n + K\log K)\) \[7\], solving an \(n \times n\) system of linear equations is \(O(n^3)\) \[8\], and if \(C(m)\) is the running time complexity for finding the best solution on \(P\), then we have the following two consequences.

**Corollary 2.2.** The running time complexity for finding \(K\) best solutions of \(P\) regarding to a given weight function is \(O(C(m) + KNm^3 + K\log K)\) where \(N\) is the maximum cardinality of all \(N(x_i), i = 1, \ldots, K - 1\).

Since \(N\) can be bounded by \(mf(m) - m^2\) then:

**Corollary 2.3.** The running time complexity for finding \(K\) best solutions of \(P\) regarding to a given weight function is \(O(C(m) + Km^4f(m) + K\log K)\).

**Corollary 2.4.** If \(C(m)\) and \(f(m)\) are polynomial on \(m\) then finding \(K\) best solutions of \(P\) is pseudopolynomial, i.e., polynomial on \(m\) and \(K\).

We will propose now a new algorithm which generalizes one used in \[11\] for the Kth Best Base of a Matroid (KBBM).
Let us give a general model of combinatorial objects containing Hamilton cycles.

Let $E$ be a finite set and $\mathcal{X} = \subseteq \{0, 1\}^E$. We say that $\mathcal{X}$ is an $\alpha$-bases system, where $\alpha$ is a positive integer, if the following conditions hold:

1. $\alpha = \text{Min}\{x \setminus y \text{ such that } (x, y) \in \mathcal{X}^2, x \neq y\}$;
2. there exists a positive integer $r$ such that $x(E) = r$, for any $x \in \mathcal{X}$;
3. for any $(x, x') \in \mathcal{X}^2$, there exist $t \in N$, $F_i' \subseteq x' \setminus x$, and $F_i \subseteq x \setminus x', i = 1, 2, ..., t$ such that $\alpha \leq |F_i| = |F_i'| \leq \alpha + 1$, $x' = x \setminus \bigcup_{i=1}^t F_i \cup \bigcup_{i=1}^t F_i'$ and $x \setminus \bigcup_{i \in I \subseteq \{1, ..., n\}} F_i \cup \bigcup_{i \in I \subseteq \{1, ..., n\}} F_i' \in \mathcal{X}$.

Such pair $(F_i, F_i')$ verifying the condition (3) is called an $x$-exchangeable pair.

Note that bases of a matroid form a $1$-bases system and we will prove that Hamilton cycles of a complete graph form a $2$-bases system.

We have then the following property for $K$ best solutions of $\alpha$-bases system.

**Theorem 2.5.** Given a weight function $c \in R^E$ and a $j$th $c$-best solution (of $\mathcal{X}$) $x$. If $(F_0, F_0')$ is an $x$-exchangeable pair such that $c(F_0) - c(F_0') = \text{Maximum}\{c(F) - c(F') \text{ such that } (F, F') \text{ is an } x \text{-exchangeable pair and } c(F) - c(F') \leq 0\}$, then $x_0 = (x \setminus F_0) \cup F_0'$ is a $(j+1)$th $c$-best solution of $\mathcal{X}$.

**Proof.** By induction on $j \geq 1$.

By using the condition (3) of the definition of $\alpha$-bases systems, any $x' \in \mathcal{X} \setminus \{x\}$ can be expressed as $x' = x \setminus \bigcup_{i=1}^t F_i \cup \bigcup_{i=1}^t F_i'$ for some $x$-exchangeable pairs $(F_i, F_i') \subseteq (x \setminus x') \times (x' \setminus x)$, $i = 1, ..., t$. Since $x$ is a $c$-best solution then $\mathcal{F}_x = \{(F, F') \text{-exchangeable pairs such that } c(F) - c(F') > 0\} = \emptyset$. Thus $c(x_0) = c(x) - (c(F_0) - c(F_0')) \leq c(x) - \sum_{i=1}^t (c(F_i) - c(F_i')) = c(x')$. So $x_0$ is the $2$nd $c$-best solution.

Suppose now that $j \geq 2$ and let $x_i$ be the $i$th $c$-best solution for $i = 1, 2, ..., j$.

For any subset $X \subseteq \mathcal{F}_x$ we can get a $x_i = x \setminus \bigcup_{(F, F') \in X} F \cup \bigcup_{(F, F') \in X} F'$ and $c(x_i) = c(x) - \sum_{(F, F') \in X} (c(F) - c(F')) \leq c(x)$ ($X = \emptyset$ gives $x = x_j$ itself and $X = \mathcal{F}_x$ gives the $c$-best solution). It follows that $x_{j+1} = x_0 = x \setminus F_0 \cup F_0'$ because of a similar argument as for $j = 1$. \qed

This proof gives an algorithm for finding $K$ best solutions in $\alpha$-bases systems. The algorithm consists of finding the best solution first (O(C(m))) and then the 2nd best by adding a subset to the (best) solution (O(|E| - r)), finding the matched subsets of our (best) solution forming an $x$-exchangeable pair (O(θ)) and choosing the best subset of
this solution forming an exchangeable pair \(O(r))\). By repeating this procedure \(K\) times, the running time complexity of this algorithm is \(O(C(m) + Kr(|E| - r)\theta)\) where \(\theta\) is the running time complexity of the oracle used to find exchangeable pairs.

3. **KthTSP is pseudopolynomial when TSP is polynomial.**

First we need to prove that Hamilton cycles of a complete graph verify the properties (1)-(3) of \(\alpha\)-bases systems.

**Theorem 3.1.** Hamilton cycles of a complete graph form a 2-bases system.

**Proof.** For Hamilton cycles, \(E\) is the set of edges of a given complete graph \(K_n\).

- **Property (1):** It is clear that \(\alpha = 2\).
- **Property (2):** It is clear that \(r = n\).
- **Property (3):** Let \(H\) and \(H'\) two distinct Hamilton cycles and \(d(H, H') = |H\setminus H'|\). We will prove this property by induction on \(d(H, H')\).

  If \(d(H, H') = 2\) (respectively 3) then let \(F = H\setminus H'\) and \(F' = H'\setminus H\).

  It is not difficult to see that \(H' = (H\setminus F) \cup F', |F| = |F'| = 2 = \alpha\) (respectively \(3 = \alpha + 1\)) and \((F, F')\) is an \(H\)-exchangeable pair.

  If \(d(H, H') = 2p\) (respectively \(2p + 1\), with \(p \geq 2\), then there exists a circuit \(C = \{e, e', f, f'\}\) (of cardinality 4) such that \(\{e, f\} \subseteq H\setminus H', e' \in H'\setminus H, f' \notin H\) and \(H'' = H \Delta C = H\setminus \{e, f\} \cup \{e', f'\}\) is a Hamilton cycle. It is clear that \(d(H'', H') \leq d(H, H') - 1\). By induction, \(H'\) can be expressed in means of \(H''\) and \(H''\)-exchangeable pairs. If \(f' \in H'\) then we are done. Else, one of the removed \(H''\)-exchangeable pairs should contain \(f'\) and by substituting \(H''\), we will get an \(H\)-exchangeable pair with components of cardinality 3.

Since finding exchangeable pairs corresponds to choose 2 nonadjacent edges (respectively 3 edges) from a Hamilton cycle and to find 2 nonadjacent edges (respectively 3 edges) such that exchanging between them gives a new Hamilton cycle then \(O(\theta) = O(1)\). It follows that the running time complexity of our algorithm for KthTSP is \(O(C(m) + Knm)\).

Then we can state our main result.

**Corollary 3.2.** KthTSP is pseudopolynomial when TSP is polynomial.

**Proof.** If TSP is polynomial for (special instances of) complete graphs then \(C(m)\) is polynomial and we are done.

If TSP is polynomial for special classes of graphs, then we can put an infinity weight to removed edges from the corresponding complete graph and we get the same result. □
THE KTH TRAVELING SALESMAN PROBLEM IS PSEUDOPOLYNOMIAL WHEN TSP IS POLYNOMIAL

Note that, with a natural modification, our algorithm works for arbitrary weights and for Max KthTSP.

4. Conclusion

We have generalized an algorithm described in [11] for a generalization of bases of a matroid. By applying this algorithm to Hamilton cycles, we have proved that KthTSP is pseudopolynomial when TSP is polynomial. Future investigations can be applying this algorithm for appropriate combinatorial objects.

References

[1] J. A. Bondy and U. S. R. Murty (2008), Graph Theory with Applications, Elsevier, New York.
[2] P. M. Camerini, L. Fratta, and F. Maffioli (1975), Efficient Methods for Ranking Trees, Proceedings 3rd International Symposium on Network Theory, Split, Yugoslavia: 419.
[3] P. M. Camerini, L. Fratta, and F. Maffioli (1980a), Ranking Arborescences in O(K m log n) time, European Journal of Operations Research 4: 235.
[4] P. M. Camerini, L. Fratta, and F. Maffioli (1980b), The K Best Arborescences of a Network, Proceedings 3rd International Symposium on Network Theory, Split, Yugoslavia: 419.
[5] B. Chaourar (2008), On the Kth Best Base of a Matroid, Operations Research Letters 36 (2): 239-242.
[6] B. Chaourar (2010), An O(K n log[K n]) algorithm for the Kth Best Spanning Tree in Series Parallel Graphs, Arabian Journal for Science and Engineering, Arabian Journal for Science and Engineering 35 (1D): 29-35.
[7] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein (2009), Introduction to Algorithms, 3rd Edition, MIT Press, Cambridge, USA.
[8] R. W. Farebrother (1988), Linear Least Squares Computations, M. Dekker, New York, USA.
[9] H. N. Gabow (1977), Two Algorithms for Generating Weighted Spanning Trees in Order, SIAM Journal of Computing 6: 139.
[10] H. W. Hamacher, J. C. Picardi, and M. Queyranne (1985), On Finding the K Best Cuts in a Network, Operations Research Letters 2: 303.
[11] H. W. Hamacher and M. Queyranne (1985), K Best Solutions to Combinatorial Optimization Problems, Annals of Operations Research 4 (6): 123-143.
[12] N. Katoh, T. Ibaraki, and H. Mine (1981), An Algorithm for Finding K Minimum Spanning Trees, SIAM Journal of Computing 10 (2): 247.
[13] N. Megiddo, A. Tamir, and R. Chandrasekaran (1981), An O(nlog^2n) Algorithm for the Kth Longest Path in a Tree with Applications to Location Problems, SIAM Journal of Computing 10 (2): 328.
[14] K. G. Murty (1968a), Solving the Fixed Charge Problem by Ranking the Extreme Points, Operations Research 16 (2): 268-279.
[15] K. G. Murty (1968b), An Algorithm for Ranking all the Assignments in Increasing Order of Cost, Operations Research 16 (3): 682-687.
[16] M. Pollack (1961), *Solutions of the kth Best Route Through a Network - A Review*, Journal of Mathematical Analysis and Applications 3: 547-559.

[17] A. Schrijver (1986), *Theory of Linear and Integer Programming*, John Wiley and Sons, Chichester.

[18] Wolsey (1973), *Generalized Dynamic Programming Methods in Integer Programming*, Mathematical Programming 4: 222-232.

[19] H. H. Yanasse, N. Y. Soma, and N. Maculan (2000), *An Algorithm for Determining the K-Best Solutions of the One-Dimensional Knapsack Problem*, Pesquisa Operacional 20 (1): 117-134.

DEPARTMENT OF MATHEMATICS AND STATISTICS, AL IMAM UNIVERSITY (IMSIU), P.O. BOX 90950, RIYADH 11623, SAUDI ARABIA, CORRESPONDENCE ADDRESS: P.O. BOX 287574, RIYADH 11323, SAUDI ARABIA

E-mail address: bchaourar@hotmail.com