Affine Factorable Surfaces in Pseudo-Galilean Space

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Abstract. An affine factorable surface of the second kind in the three dimensional pseudo-Galilean space \( G^1_3 \) is studied depending on the invariant theory and theory of differential equation. The first and second fundamental forms, Gaussian curvature and mean curvature of the meant surface are obtained according to the basic principles of differential geometry. Also, some special cases are presented by changing the partial differential equation into the ordinary differential equation to simplify the solving process. The classification theorems of the considered surface with zero and non zero Gaussian and mean curvatures are given. Some examples of such a study are provided.

Keywords: Affine factorable surface; mean curvature; Gaussian curvature; minimal surface.

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1 Introduction

In classical differential geometry, the problem of obtaining Gaussian and mean curvatures of a surface in the Euclidean space and other spaces is one of the most important problems, so we are interested here to study such a problem for a surface known as affine factorable surface in the three dimensional pseudo Galilean space \( G^1_3 \).

The geometry of Galilean Relativity acts like a “bridge” from Euclidean geometry to special Relativity. The Galilean space which can be defined in three-dimensional projective space \( P_3(R) \) is the space of Galilean Relativity [1]. The geometries of Galilean and pseudo-Galilean spaces have similarities, but, of course, are different. In the Galilean and pseudo Galilean spaces, some special surfaces such as surfaces of revolution, ruled surfaces, translation surfaces and tubular surfaces have been studied in [2–10]. For further study of surfaces in the pseudo Galilean space, we refer the reader to Šipuš and Divjak’s paper [9]. Recall that the graph surfaces are also known as Monge surfaces [11]. In this work, we are interested here in studying a special type of Monge surface, namely factorable surface of second kind that is graph of the function \( y(x, z) = f(x)g(z) \). Such surfaces with \( K, H = const. \) in various ambient spaces have been classified (cf [12–16]). In this work, we are interested here in studying a special type of Monge surface, namely factorable surface of second kind that is graph of the function \( y(x, z) = f(x)g(z) \). Such surfaces with \( K, H = const. \) in various ambient spaces have been classified (cf [12–16]). Our purpose is to analyze the factorable surfaces in the pseudo-Galilean space \( G^1_3 \) that is one of real Cayley-Klein spaces (for details, see [17–19]).
There exist three different kinds of factorable surfaces, explicitly, a Monge surface in $G_3^1$ is said to be factorable (so-called homothetical) if it is given in one of the following forms $\Phi_1: z(x,y) = f(x)g(y)$, the first kind $\Phi_2: y(x,z) = f(x)g(z)$, the second kind and $\Phi_3: x(y,z) = f(y)g(z)$ is the third kind where $f, \ g$ are smooth functions [14]. These surfaces have different geometric structures in different spaces such as metric, curvatures, etc.

2 Basic concepts

The pseudo-Galilean space $G_3^1$ is one of the Cayley-Klein spaces with absolute figure that consists of the ordered triple $\{\omega, f, I\}$, where $\omega$ is the absolute plane given by $x_o = 0$, in the three dimensional real projective space $P_3(\mathbb{R})$, $f$ the absolute line in $\omega$ given by $x_o = x_1 = 0$ and $I$ the fixed hyperbolic involution of points of $f$ and represented by $(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : x_2)$, which is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the absolute conic. The metric connections in $G_3^1$ are introduced with respect to the absolute figure. In terms of the affine coordinates given by $(x_o : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ is defined by (see [9][20])

$$d(p,q) = \begin{cases} 
|q_1 - p_1|, & \text{if } p_1 \neq q_1, \\
\sqrt{(q_2 - p_2)^2 - (q_3 - p_3)^2}, & \text{if } p_1 = q_1.
\end{cases}$$

The pseudo-Galilean scalar product of the vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ is given by

$$\langle X, Y \rangle_{G_3^1} = \begin{cases} 
x_1y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\
x_2y_2 - x_3y_3, & \text{if } x_1 = 0 \text{ and } y_1 = 0.
\end{cases}$$

In this sense, the pseudo-Galilean norm of a vector $X$ is $\|X\| = \sqrt{|X \cdot X|}$. A vector $X = (x_1, x_2, x_3)$ is called isotropic (non-isotropic) if $x_1 = 0$ ($x_1 \neq 0$). All unit non-isotropic vectors are of the form $(1, x_2, x_3)$. The isotropic vector $X = (0, x_2, x_3)$ is called spacelike, timelike and lightlike if $x_2^2 - x_3^2 > 0$, $x_2^2 - x_3^2 < 0$ and $x_2 = \pm x_3$, respectively. The pseudo-Galilean cross product of $X$ and $Y$ on $G_3^1$ is given as follows

$$X \wedge_{G_3^1} Y = \begin{vmatrix} 0 & -e_2 & e_3 \\
e_2 & x_1 & x_2 \\
e_3 & y_1 & y_2 
\end{vmatrix},$$

where $e_2$ and $e_3$ are canonical basis.

Let $M$ be a connected, oriented 2-dimensional manifold and $\phi : M \to G_3^1$ be a surface in $G_3^1$ with parameters $(u,v)$. The surface parametrization $\phi$ is expressed by

$$\phi(u,v) = (x(u,v), y(u,v), z(u,v)).$$

On the other hand, we denote by $E, F, G$ and $L, M, N$ the coefficients of the first and second fundamental forms of $\phi$, respectively. The Gaussian $K$ and mean $H$ curvatures are

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN + GL - 2FM}{2|EG - F^2|},$$

(2.1)
where

\[ E = \phi'_u \phi'_u, \quad F = \phi'_u \phi'_v, \quad G = \phi'_v \phi'_v, \]
\[ L = \phi''_{uu} n, \quad M = \phi''_{uv} n, \quad N = \phi''_{vv} n, \]

and

\[ n = \frac{\phi'_u \wedge \phi'_v}{|\phi'_u \wedge \phi'_v|}. \]

3  Factorable surfaces in pseudo-Galilean space \( G^1_3 \)

In what follows, we consider the factorable surface of second kind in \( G^1_3 \) which can be locally written as

\[ \phi(x, z) = (x, f(x)g(z), z). \] (3.1)

For this study it is important to consider the following definition:

**Definition 3.1** An affine factorable surface in pseudo-Galilean space \( G^1_3 \) is defined as a parameter surface \( \phi(u, v) \) which can be written as

\[ \phi(u, v) = (x(u, v), y(u, v), z(u, v)) = (u, f(u)g(v + au), v) = (x, f(x)g(z + ax), z), \] (3.2)

for non zero constant \( a \) and functions \( f(x) \) and \( g(z + ax) \) [27].

Now, from (3.2) by a direct calculation, the first fundamental form with its coefficients of \( \phi \) can be given by

\[ I = Edx^2 + 2F dx dy + G dy^2, \]
\[ E = 1, \quad F = 0, \quad G = (fg')^2 - 1, \]

where

\[ g' = \frac{dg(z + ax)}{dz + ax}. \]

Also, the second fundamental form of \( \phi \) is

\[ II = L dx^2 + 2M dx dy + N dy^2, \]

note that

\[ L = \frac{(f''g + 2af'g' + a^2fg'')}{D}, \]
\[ M = \frac{(f'g' + afg'')}{D}, \quad N = \frac{fg''}{D} \]
where
\[ D(x, z) = \sqrt{1 - (fg')^2}. \]

In addition, the Gaussian and mean curvature of \( \phi \) can be obtained

\[
K = \frac{f'^2g^2 - f''fg'g}{(1 - (fg')^2)^2},
\]

\[
H = \frac{\Omega(x, z)}{2(1 - (fg')^2)^2},
\]

where
\[
\Omega(x, z) = (1 - a^2)fg'' - f''g - 2af'g' + f^2f'g'^2 + 2af'f^2g'^3 + a^2f^3g^2g''.
\]

A surface in \( G_3^1 \) is said to be isotropic minimal (resp. flat) if \( H \) (resp. \( K \)) vanishes identically. Further, it is said to have constant isotropic mean (resp. Gaussian) curvature if \( H \) (resp. \( K \)) is a constant function on whole surface.

### 4 Affine factorable surfaces with zero curvatures

In this section, if the Gaussian and mean curvatures of (3.2) are vanished, then we get the following main result:

**Theorem 4.1** Let \( \phi : I \subset \mathbb{R} \to G_3^1 \) be an affine factorable surface of second kind in the form

\[
\phi(x, z) = (x, f(x)g(z + ax), z),
\]

if its Gaussian curvature is zero, then the surface is one of the following surfaces:

1. \( y(x, z) = f_o g(z + ax) \);
2. \( y(x, z) = g_o f(x) \);
3. \( y(x, z) = ce^{c_5x + c_4z} \);
4. \( y(x, z) = [(1 - k)(c_6x + c_7)]^{1/k} \left( \left( \frac{k-1}{k} \right)(c_8(z + ax) + c_9) \right)^{k/(k-1)} \).

**Proof.** If the Gaussian curvature of \( \phi \) is zero, then from (3.3), we have

\[ f'^2g^2 - f''fg'g = 0. \] (4.1)

To solve this equation we have the following cases to be discussed:

**Case1.** if \( f' = 0 \), then \( f'' = 0, f = f_o = \text{const.} \), then \( y(x, z) = f_o g(z + ax) \).

**Case2.** if \( g' = 0 \), then \( g'' = 0, g = g_o = \text{const.} \), then \( y(x, z) = g_o f(x) \).

**Case3.** if \( f' \neq 0 \) and \( g' \neq 0 \), let
\[
\begin{cases}
  u = x, \\
  v = z + ax,
\end{cases}
\]
where \( \partial(u, v)/\partial(x, z) \neq 0 \). Then (4.1) can be written as

\[
f_u^2 g_v^2 - f f_{uv} g_{vv} = 0,
\]
or

\[
\left( \frac{df}{du} \right)^2 \left( \frac{dg}{dv} \right)^2 = f \frac{df_u}{df} \frac{dg_v}{dg} \frac{d^2 g}{du \ dv}.
\] (4.2)

From (4.2), we have

\[
\frac{df}{du} \frac{dg}{dv} = f \frac{df_u}{df} \frac{dg_v}{dg}.
\]

Since \( \frac{df}{du} \frac{dg}{dv} \neq 0 \) and \( g \frac{dg_v}{dg} \neq 0 \), so

\[
\left( \frac{f \frac{df_u}{df}}{f_u} \right) = \left( \frac{g_v}{g \frac{dg_v}{dg}} \right),
\] (4.3)

let’s write the last equation as follows

\[
\left( \frac{f \frac{df_u}{df}}{f_u} \right) = \left( \frac{g_v}{g \frac{dg_v}{dg}} \right) = k, \quad k = \text{const}.
\] (4.4)

(a) If \( k = 1 \), then from (4.4), we have

\[
\frac{df_u}{f_u} = \frac{df}{f}, \quad \frac{dg_v}{g_v} = \frac{dg}{g}.
\] (4.5)

Solving this equation takes the form

\[
f = c_1 e^{c_2 u}, \quad g = c_3 e^{c_4 v},
\]

where \( c_1, c_2, c_3, c_4 \) are constants. And then

\[
y(x, z) = f(x)g(z + ax) = c_1 e^{c_2 x} c_3 e^{c_4 (z + ax)} = c_5 e^{c_6 x + c_4 z},
\]

where \( c_5 = c_1 c_3 \) and \( c_6 = c_2 + ac_4 \) are constants.

(b) When \( k \neq 1 \), then from (4.4), we have

\[
f \frac{df_u}{df} = kf_u, \quad kg \frac{dg_v}{dg} = g_v,
\]

which has the solution

\[
f(x) = [(1 - k)(c_7 x + c_8)]^{\frac{1}{1-k}},
\]
\[
g(z + ax) = \left[ \left( \frac{k - 1}{k} \right) (c_9(z + ax) + c_{10}) \right]^{\frac{k}{k-1}}.
\]

Therefore we find that

\[
y(x, z) = [(1 - k)(c_7 x + c_8)]^{\frac{1}{1-k}} \left[ \left( \frac{k - 1}{k} \right) (c_9(z + ax) + c_{10}) \right]^{\frac{k}{k-1}},
\]

where \( c_7, c_8, c_9 \) and \( c_{10} \) are constants. □
**Theorem 4.2** For a given affine factorable surface of second kind in a three dimensional pseudo-galilean space in the form

\[ \phi(x, z) = (x, f(x)g(z + ax), z). \]

Let its mean curvature equal zero, then this surface will be one of the following:

1. \[ y(x, z) = f_o(b_1(z + ax) + b_2), \text{ or } y(x, z) = f_o \left( \sqrt{\frac{a^2 - 1}{a^2 f_o^2}}(z + ax) + b_3 \right); \]
2. \[ y(x, z) = g_o(b_4x + b_5); \]
3. \[ y(x, z) = b_8(b_6x + b_7), \text{ or } y(x, z) = (b_6x + b_7)(b_9(z + ax) + b_{10}); \]
4. \[ y(x, z) = (b_{12}x + b_{13})(b_{11}(z + ax) + b_{12}), \text{ or } y(x, z) = \frac{1}{b_{11}}(b_{11}(z + ax) + b_{12}). \]

**Proof.** If \( H = 0 \), then from (3.4), we have

\[ (1 - a^2)fg'' - f''g - 2af'g' + f^2f''g^2g + 2af'f^2g'^3 + a^2f^3g'^2g'' = 0. \]  \hspace{1cm} (4.6)

This equation can be solved by introducing the following:

1. If \( f' = f'' = 0 \), then \( f = f_o = \text{const.} \), and (4.6) becomes

\[ (1 - a^2)f'g'' + a^2f^3g'^2g'' = 0. \]

It can be written in a simple form

\[ g'' = 0 \text{ or } g' = \sqrt{\frac{a^2 - 1}{a^2 f_o^2}}; \]

which gives the solution

\[ g = b_1(z + ax) + b_2 \text{ or } g = \sqrt{\frac{a^2 - 1}{a^2 f_o^2}}(z + ax) + b_3, \]

it is so

\[ y(x, z) = f_o(b_1(z + ax) + b_2), \]

or

\[ y(x, z) = f_o \left( \sqrt{\frac{a^2 - 1}{a^2 f_o^2}}(z + ax) + b_3 \right), \]

where \( b_1, b_2 \) and \( b_3 \) are constants.

2. When \( g' = g'' = 0 \), then \( g = g_o = \text{const.} \), and (4.6) becomes

\[ f''g = 0, \]

which has the solution

\[ f = b_4x + b_5. \]

Using what we got from solutions we can write

\[ y(x, z) = g_o(b_4x + b_5), \]
where \( b_4, b_5 \) are constants.

(3) When \( f'' = 0 \), this leads to \( f' = b_6 \) which gives \( f = b_6 x + b_7 \). From (4.6), we have

\[
(1 - a^2)fg'' - 2af'g' + 2af'f^2g^3 + a^2 f^2 g^2 g'' = 0,
\]

which can be written as

\[
(1 - a^2)fg'' - 2af_u g_v + 2af_u f^2 g^3_v + a^2 f^2 g^2 g'' v = 0.
\]

Differentiating this equation three times with respect to \( u \), we obtain

\[
g_v v g_v \v = 0,
\]

which gives

\[
g_v = 0 \rightarrow g = b_8,
\]

or

\[
g_v v = 0 \rightarrow g = b_9(z + ax) + b_{10},
\]

in light of this, we get

\[
y(x, z) = b_8(b_6 x + b_7),
\]

or

\[
y(x, z) = (b_0 x + b_7)(b_9(z + ax) + b_{10}),
\]

where \( b_6, b_7, b_8, b_9 \) and \( b_{10} \) are constants.

(4) If \( g'' = 0 \), this means that \( g' = b_{11} \rightarrow g = b_{11}(z + ax) + b_{12} \) and then from (4.6), we obtain

\[
f''g + 2af'g' - f^2 f''g^2 g - 2af' f^2 g^3 = 0,
\]

which can be written as

\[
f_u u g + 2af_u g_v - f^2 f_u g_v^2 g - 2af_u f^2 g^3_v = 0.
\]

If we differentiate this equation with respect to \( v \), we get

\[
b_{11} f_{uu} - b_{11}^3 f^2 f_{uu} = 0,
\]

\[
f_{uu} = 0 \rightarrow f = b_{12} x + b_{13},
\]

or

\[
f = \frac{1}{b_{11}}.
\]

So, we have

\[
y(x, z) = (b_{12} x + b_{13})(b_{11}(z + ax) + b_{12}),
\]

or

\[
y(x, z) = \frac{1}{b_{11}}(b_{11}(z + ax) + b_{12}),
\]

taking into consideration \( b_{11}, b_{12} \) and \( b_{13} \) are constants. This completes the proof. \( \blacksquare \)
5 Affine factorable surfaces with non zero curvatures

In this section, we describe the affine factorable surfaces of second kind in $G^1_3$ when $K = \text{const.} \neq 0$ and $H = \text{const.} \neq 0$. So, we start as follows:

**Theorem 5.1** Let $\phi : I \subset R \to G^1_3$ be an affine factorable surface of second kind in $G^1_3$. Let its Gaussian curvature is non-zero constant, then the surface takes the form:

$$y(x,z) = (g_o(z + ax) + \lambda_2) \left( \pm \frac{1}{g_o} \tanh \left[ \sqrt{K_o} x \mp g_o \lambda_1 \right] \right), \quad \lambda_1, \lambda_2 \in R.$$  

**Proof.** Let $K_o$ be a non-zero constant Gaussian curvature. Hence, we get

$$K_o = \frac{f'^2 g'^2 - f'' f' g'' g}{(1 - (fg')^2)^2}, \quad (5.1)$$

from this equation, $K_o$ vanishes identically when $f$ or $g$ is a constant function. Then $f$ and $g$ must be non-constant functions. We distinguish two cases for eq(5.1):

**Case1.** $f' = f_o, f_o \in R - \{0\}$, then from eq(5.1), we can get polynomial equation on ($g'$):

$$K_o - (2K_o f'^2 + f_o^2)g'^2 + K_o f^4 g'^4 = 0,$$

which yields a contradiction.

**Case2.** $g' = g_o, g_o \in R - \{0\}$. Then eq(5.1) leads to

$$f' = \pm \sqrt{K_o - 2K_o g_o^2 f'^2 + K_o g_o^4 f^4} \over g_o,$$

after solving this equation, we obtain

$$f(x) = \pm 1 \frac{1}{g_o} \tanh \left[ g_o \sqrt{K_o} x \mp g_o \lambda_1 \right], \quad \lambda_1 \in R.$$

**Case3.** $f'' \neq 0, g'' \neq 0$. Then eq(5.1) can be arranged as follows:

$$K_o = \frac{f'^2 g'^2 - f'' f' g'' g}{(1 - (fg')^2)^2},$$

let $u = x, v = z + ax$ and $\partial(u,v)/\partial(x,y) \neq 0$, we can obtain

$$K_o = \frac{f_o^2 g_v^2 - f_{uu} f_{vv} g_v g}{(1 - (fg_o)^2)^2}, \quad (5.2)$$

The partial derivative of (5.2) with respect to $u$ and $v$ leads to a polynomial equation

$$\frac{f'}{f^2 f''} + \frac{3 f' f^2}{f'' g'^4} = 0, \quad (5.3)$$

which means that all coefficients must vanish, the contradiction $f' = 0$ is obtained. Thus the proof is completed. ■
Theorem 5.2  For a given affine factorable surface of second kind in $G^1_3$ which has a non-zero constant mean curvature $H_o$. Then the following occurs:

$$y(x, z) = f_o \left( \frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_o^2}}{3f_oH_o} (z + ax) + \lambda_4 \right),$$

$$y(x, z) = \left( -\frac{2H_o}{g_o} x^2 + cx + c \right) g_o.$$ 

Proof. From (3.4), we get

$$H_o = \frac{(1 - a^2)fg'' - f''g - 2af'g' + f^2f''g^2 + 2af'f^2g^3 + a^2f^3g''}{2(1 - (fg')^2)^{3/2}},$$

for solving this equation, the following two cases can be discussed:

Case a. $f = f_o, g'' = \lambda_3 = const.$, we get

$$2H_o \left( 1 - (fg')^2 \right)^{3/2} = (1 - a^2)fg'' + a^2f^3g''g,$$

let $u = x, v = z + ax$ and $\partial(u, v)/\partial(x, y) \neq 0$, we can obtain

$$2H_o \left( 1 - (fg')^2 \right)^{3/2} = (1 - a^2)fg_{vv} + a^2f^3g_v^2g_{vv}, \quad (5.4)$$

which has the partial derivative with respect to $v$ as

$$g_v = \frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_o^2}}{3f_oH_o}.$$ 

Solving this equation gives

$$g = \pm \frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_o^2}}{3f_oH_o} (z + ax) + \lambda_4, \quad \lambda_4 \in R$$

then we have

$$y(x, z) = f_o \left( \frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_o^2}}{3f_oH_o} (z + ax) + \lambda_4 \right).$$

Case b. $g = g_o$, we get

$$2H_o = -f''g,$$

so, we obtain

$$f = -\frac{H_o}{g_o} x^2 + \lambda_5 x + \lambda_6.$$ 

Where $\lambda_5, \lambda_6 \in R$. Then the proof is finised. ■

Here, through the study, which presented on affine factorable surface of second kind in pseudo-Galilean space $G^1_3$, we conclude with the following important theory which relates between its mean and Gaussian curvatures.
Theorem 5.3 Let $\phi : I \subset R \rightarrow G^1_3$ be an affine factorable surface in three dimensional pseudo-Galilean space. The relation between its Gaussian curvature $K$ and its mean curvature $H$ is given by the formula

$$H = A(x, z)K, \quad (5.5)$$

where $A(x, z) = \frac{D^{3}(a^{2}fg''+2af'g'+f''a)-fg''D}{f'''fg''g-\sqrt{1-(fg')^{2}}}$. When $D = 0$, then $\phi$ is isotropic minimal affine factorable surfaces of second kind.

6 Some examples

We illustrate several examples relating to the affine factorable surfaces of second kind with zero and non zero Gaussian ($K$) and mean ($H$) curvatures in the three dimensional pseudo-Galilean space $G^1_3$.

Example 6.1 Let us consider the affine factorable surfaces of second kind in $G^1_3$ given by

(1) $\phi : y(x, z) = 8e^{6x+z}, (x, z) \in [-1, 1] \times [0, 2\pi], \text{ (isotropic flat } (K = 0)),$

(2) $\phi : y(x, z) = \sqrt{\frac{3}{4}(2x + z)} + 9, (x, z) \in [0, 15] \times [-1, 30], \text{ (isotropic minimal } (H = 0)),$

(3) $\phi : y(x, z) = (10x + z) \tanh[x], (x, z) \in [-1, 1], \text{ (}K = \text{ const.} \neq 0),$

(4) $\phi : y(x, z) = -x^2 + 2x + 1, (x, z) \in [-1, 1], \text{ (}H = \text{ const.} \neq 0).$

These surfaces can be drawn respectively as in Figs.1-4.

Figure 1: An isotropic flat affine factorable surface of second kind.
Figure 2: An isotropic minimal affine factorable surface of second kind.

Figure 3: Affine factorable surface of second kind with $K = \text{const.} \neq 0$.

7 Conclusion

In surface theory in the field of differential geometry, especially factorable surfaces, there are three kinds of these surfaces known as first, second and third kind. In this paper, we are interested in studying factorable surface of second kind which has affine form in the three dimensional pseudo Galilean space $G^1_3$. The classification of this surface with zero and non zero Gaussian and mean curvatures is discussed. Also, an important relation between the curvatures of this surface is obtained. Finally, some examples are introduced and plotted.
Figure 4: Affine factorable surface of second kind with $H = \text{const.} \neq 0$.

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