INTERPOLATIVE GENERALISED MEIR-KEELER CONTRACTION

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Abstract:
Introduction/purpose: The aim of this paper is to introduce the notion of an interpolative generalised Meir-Keeler contractive condition for a pair of self maps in a fuzzy metric space, which enlarges, unifies and generalizes the Meir-Keeler contraction which is for only one self map. Using this, we establish a unique common fixed point theorem for two self maps through weak compatibility. The article includes an example, which shows the validity of our results.

Methods: Functional analysis methods with a Meir-Keeler contraction.

Results: A unique fixed point for self maps in a fuzzy metric space is obtained.

Conclusions: A fixed point of the self maps is obtained.

Key words: Fuzzy metric space, common fixed points, weak compatibility, Interpolative generalised Meir-Keeler contraction.

Introduction

In 1965 L. Zadeh (Zadeh, 1965) introduced the theory of fuzzy sets. Later on, in 1978, the concept of a fuzzy metric space was introduced by Kramosil and Michalek in (Kramosil & Michalek, 1975), which was modified by George and Veeramani (George & Veeramani, 1994) in order...
to obtain a Hausdorff topology for this class of fuzzy metric spaces. Then in year 1988, Grabiec (Grabiec, 1988) gave a fuzzy version of the Banach (Banach, 1922) contraction principle in the setting of a fuzzy metric space. Over the past years, various authors have tried to generalize the fixed point theorem by modifying and varying the contractive condition, see, e.g., (Gregori & Sapena, 2002), (Jain & Jain, 2021), (Mihet, 2008), (Saha et al, 2016), (Tirado, 2012) and (Wardowski, 2013) in the sense of George and Veeramani. In 2019, Zheng and Wang (Zheng & Wang, 2019) introduced a Meir-Keeler contraction in the setting of a fuzzy metric (Schweizer & Sklar, 1983) space and proved some fixed point results for a self map.

Inspired with the interpolative theory, Karapinar and Agrawal (Karapinar & Agarwal, 2019) introduced the notion of an interpolative Rus-Reich-Ćirić type contraction via the simulation function in a metric space. Motivated by this paper, we introduce an interpolative generalised Meir-Keeler contraction (Gregori & Minana, 2014) for two self maps (Rhoades, 2001) in the setting of a fuzzy metric space, which enlarges, unifies and generalizes the existing Meir-Keeler contraction in a fuzzy metric (Mihet, 2010) space through weak compatibility (Banach, 1922).

The structure of the paper is as follows:

After the preliminaries, we introduce a interpolative generalised Meir-Keeler contraction in the setting of a fuzzy metric space. Then we study the Meir-Keeler contractive mapping due to Zheng and Wang (Zheng & Wang, 2019). In section 4, the existence of a unique common fixed point of an interpolative generalised Meir-Keeler contractive mapping has been established through weak compatibility followed by an example.

Preliminaries

**DEFINITION 1.** (George & Veeramani, 1994) A mapping $\ast : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (t-norm for short) if $\ast$ is continuous and satisfies the following conditions:

(i) $\ast$ is commutative and associative, i.e. $a \ast b = b \ast a$ and $a \ast (b \ast c) = (a \ast b) \ast c,$ for all $a, b, c \in [0, 1]$;

(ii) $1 \ast a = a$, for all $a \in [0, 1]$;

(iii) $a \ast c \leq b \ast d$, for $a \leq b, c \leq d$ for $a, b, c, d \in [0, 1]$. 


The well-known examples of the t-norm are the minimum t-norm \( \min \{a, b\} \) written as \( m \) and the product t-norm \( a \times b \).

**Definition 2.** (George & Veeramani, 1994) A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \( X \) is a (nonempty) set, \( \ast \) is a continuous t-norm and \( M \) is a fuzzy set on \( X \times X \times (0, +\infty) \) satisfying the following conditions, for all \( x, y, z \in X \) and \( t, s > 0 \):

1. **(GV1)** \( M(x, y, t) > 0 \);
2. **(GV2)** \( M(x, y, t) = 1 \) if and only if \( x = y \);
3. **(GV3)** \( M(x, y, t) = M(y, x, t) \);
4. **(GV4)** \( M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s) \);
5. **(GV5)** \( M(x, y, t) : (0, +\infty) \rightarrow (0, 1] \) is continuous.

**Lemma 1.** (Grabiec, 1988) Let \((X, M, \ast)\) be a fuzzy metric space. Then \( M(x, y, t) \) is non-decreasing for all \( x, y \in X \).

**Theorem 1.** (George & Veeramani, 1994) Let \((X, M, \ast)\) be a fuzzy metric space. A sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) converges to \( x \in X \) if and only if \( \lim_{n \to \infty} M(x_n, x, t) = 1 \).

**Definition 4.** (George & Veeramani, 1994) \((X, M, \ast)\) (or simply \( X \)) is called \( M \)-complete if every \( M \)-Cauchy sequence in \( X \) is convergent.

**Lemma 2.** (Saha et al., 2016) If \( \ast \) is a continuous t-norm \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences such that \( \alpha_n \to \alpha, \beta_n \to \beta \) and \( \gamma_n \to \gamma \) as \( n \to +\infty \) then

\[
\lim_{k \to +\infty} (\alpha_k \ast \beta_k \ast \gamma_k) = \alpha \ast \lim_{k \to +\infty} \beta_k \ast \gamma,
\]
and
\[ \lim_{k \to +\infty} (\alpha_k \ast \beta_k \ast \gamma_k) = \alpha \ast \lim_{k \to +\infty} \beta_k \ast \gamma. \]

**Lemma 3.** (Saha et al., 2016) Let \( \{ f(k, \cdot) : (0, +\infty) \to (0, 1); k = 0, 1, 2, \ldots \} \) be a sequence of functions such that \( f(k, \cdot) \) is continuous and monotone increasing for each \( k \geq 0 \). Then \( \lim_{k \to +\infty} f(k, t) \) is a left continuous function in \( t \) and \( \lim_{k \to +\infty} f(k, t) \) is a right continuous function in \( t \).

Denote \( \Delta = \{ \delta : (0, 1] \to (0, 1] \} \) where \( \delta \) is right continuous.

**Definition 5.** (Zheng & Wang, 2019) Let \((X, M, \ast)\) be a fuzzy metric space. A mapping \( f : X \to X \) is said to be a fuzzy Meir-Keeler contractive mapping with respect to \( \delta \in \Delta \) if the following condition holds:

\[ \text{for all } \epsilon \in (0, 1), \epsilon - \delta(\epsilon) < M(x, y, t) \leq \epsilon \text{ implies } M(fx, fy, t) > \epsilon, \quad (1) \]

for all \( x, y \in X, t > 0 \).

**Definition 6.** (Jain et al., 2009) Two self maps \( f \) and \( g \) in a fuzzy metric space \((X, M, \ast)\) are said to be weakly compatible if they commute at their coincidence points i.e. for \( x \in X, fx = gx = y \) implies \( gy = fy \).

**Interpolative generalised Meir-Keeler contraction**

**Definition 7.** Let \((X, M, \ast)\) be a fuzzy metric space. A pair \((f, g)\) of self maps in \( X \) is said to be an interpolative generalised Meir-Keeler contractive if there exists \( \alpha, \beta \in [0, 1) \) with \( \alpha + \beta < 1 \) and for all \( x, y \in X, t > 0 \)

\[ \text{for all } \epsilon \in (0, 1), \epsilon - \delta(\epsilon) < M(x, y, t) \leq \epsilon \text{ implies } M(fx, fy, t) > \epsilon, \quad (2) \]

where
\[ M(x, y) = (M(fx, gx, t))^\alpha (M(fy, gy, t))^\beta (M(gx, gy, t))^{1-\alpha-\beta}. \]

**Remark 1.** From equation (2) for all \( x \neq y \in X, t > 0 \) the pair \((f, g)\) is a strict contraction i.e.
\[ M(fx, fy, t) > M(x, y). \]

Thus for \( x \neq y \).
\[ M(fx, fy, t) > (M(fx, gx, t))^\alpha (M(fy, gy, t))^\beta (M(gx, gy, t))^{1-\alpha-\beta} \quad (3) \]
REMARK 2. Taking \( g = I \), the identity map in equation (2) we obtain

for all \( \epsilon \in (0, 1), \epsilon - \delta(\epsilon) < M(x, y) \leq \epsilon \) implies \( M(fx, fy, t) > \epsilon \), \( \epsilon = 1 \)

where

\[
M(x, y) = (M(fx, x, t))^\alpha (M(fy, y, t))^\beta (M(x, y, t))^{(1-\alpha-\beta)}.
\]

which is an interpolative generalised Meir-Keeler contraction, for a self map \( f \).

REMARK 3. Taking \( \alpha = 0, \beta = 0 \) in equation (4) then \( M(x, y) = M(x, y) \)

and we have

for all \( \epsilon \in (0, 1), \epsilon - \delta(\epsilon) < M(x, y, t) \leq \epsilon \) implies \( M(fx, fy, t) > \epsilon \), \( \epsilon = 1 \)

which is precisely the Meir-Keeler contraction, for a self map given by Zheng and Wang (Zheng & Wang, 2019).

LEMMA 4. (Zheng & Wang, 2019) If \( \delta \in \Delta \), then for \( t \in (0, 1) \), there exists \( k = k(t) \in N \) such that

\[
t + \frac{\delta(t)}{k} < 1 \quad \text{and} \quad \delta\left(t + \frac{\delta(t)}{k}\right) - \frac{\delta(t)}{k} > 0.
\]

Before we prove the main result, we prove the following lemma:

LEMMA 5. Let \((f, g)\) be a pair of an interpolative Meir-Keeler contractive mapping with respect to \( \delta \in \Delta \) and \( f(X) \subseteq g(X) \). Construct a sequence \( \{y_n\} \), by \( fx_n = gx_{n+1} = y_n \), for \( n = 0, 1, 2, \ldots \). Then \( \lim_{n \to +\infty} M(y_n, y_{n+1}, t) = 1 \).

Proof. Suppose if possible on the contrary that \( \lim_{n \to +\infty} M(y_n, y_{n+1}, t) = p(\leq 1) \). For \( \alpha, \beta \in [0, 1) \) we have

\[
\lim_{n \to +\infty} M(x_n, x_{n+1}) = \lim_{n \to +\infty} \left\{ \frac{(M(fx_n, gx_n, t))^\alpha (M(fx_{n+1}, gx_{n+1}, t))^\beta}{(M(gx_n, gx_{n+1}, t))^{(1-\alpha-\beta)}} \right\}
\]

\[
= \lim_{n \to +\infty} \left\{ \frac{(M(y_n, y_{n-1}, t))^\alpha (M(y_{n+1}, y_n, t))^\beta}{(M(y_{n-1}, y_{n+1}, t))^{(1-\alpha-\beta)}} \right\}
\]

\[
= \lim_{n \to +\infty} \left( p^\alpha p^\beta p^{(1-\alpha-\beta)} \right) = p.
\]

By using lemma (4), for \( p < 1 \) and \( \delta \in \Delta \) we can find \( k = k(p) \in N \) such that
\[ p + \frac{\delta(p)}{k} < 1 \text{ and } \delta \left( p + \frac{\delta(p)}{k} \right) - \frac{\delta(p)}{k} > 0. \]

Since \( \lim_{n \to +\infty} M(x_n, x_{n+1}) = p \), we can find \( n_0 \) such that when \( n > n_0 \),
\[ M(x_n, x_{n+1}) > p + \frac{\delta(p)}{k} - \delta \left( p + \frac{\delta(p)}{k} \right). \] \( (6) \)

Also, there exists \( n_1 \) such that whenever \( n > n_1 \),
\[ M(x_n, x_{n+1}) < p + \frac{\delta(p)}{k}. \] \( (7) \)

Let \( n > \max\{n_0, n_1\} \). Then both equations (6), and (7) hold for such \( n \).

Taking \( \epsilon = p + \frac{\delta(p)}{k} \), in equation (2) we get
\[ M(f x_n, f x_{n+1}, t) > p + \frac{\delta(p)}{k}, \]

i.e. \( M(y_n, y_{n+1}, t) > p + \frac{\delta(p)}{k} \), which contradicts the fact that
\( \lim_{n \to +\infty} M(y_n, y_{n+1}, t) = p \). Therefore, \( \lim_{n \to +\infty} M(y_n, y_{n+1}, t) = 1. \) \( \square \)

Main results

Our first new result is the next one:

**Theorem 2.** Let \( f \) and \( g \) be self maps in a fuzzy metric space \( (X, M, \ast) \) satisfying the following conditions:

\( (4.11) \) \( f(X) \subseteq g(X); \)

\( (4.12) \) The pair \( (f, g) \) is an interpolative generalised Meir-Keeler contraction;

\( (4.13) \) \( f(X) \) is complete;

\( (4.14) \) The pair \( (f, g) \) is weakly compatible.

Then \( f \) and \( g \) have a unique common fixed point in \( X \) if and only if there exists \( x_0 \in X \) such that \( \bigwedge_{t>0} M(x_0, f(x_0), t) > 0. \)

**Proof.** Suppose the pair \( (f, g) \) has a unique common fixed point \( u \) then \( u = fu = gu. \) Therefore, \( M(u, fu, t) = 1, \forall t > 0. \) Hence
\(\Lambda_{t>0} M(u, fu, t) > 0.\)

Conversely, suppose that there exists \(x_0 \in X\) such that \(\Lambda_{t>0} M(x_0, f(x_0), t) > 0.\) Construct a sequence \(\{y_n\}\), by defining \(f x_n = g x_{n+1} = y_n,\) for \(n = 0, 1, 2, \ldots\) First we show that if the two maps \(f\) and \(g\) have a common fixed point then it is unique. Let \(u\) and \(v\) be two common fixed points of \(f\) and \(g\). Then \(u = fu = gu\) and \(v = fv = gv.\) We show that \(u = v.\)

Suppose, on the contrary that \(u \neq v,\) then \(fu \neq fv.\) Now
\[
M(u, v) = (M(fu, gu, t))^\alpha (M(fv, gv, t))^\beta (M(gu, gv, t))^{1-\alpha-\beta},
\]
\[
= (M(u, u, t))^\alpha (M(v, v, t))^\beta (M(u, v, t))^{1-\alpha-\beta},
\]
\[
= (M(u, v))^\alpha (1-\alpha-\beta),
\]

Now
\[
M(u, v, t) = M(fu, fv, t),
\]
\[
> M(u, v),\quad \text{using (3)}
\]
\[
= (M(u, v))^\alpha (1-\alpha-\beta)
\]
i. e. \(M(u, v, t) > (M(u, v))^\alpha (1-\alpha-\beta),\)

implies
\[
(M(u, v))^\alpha > 1,
\]
which is not true as the left hand quantity is less than 1. So \(u = v.\) Thus, if the pair \((f, g)\) has a common fixed point then it is unique.

**Step 1** To see the existence of a common fixed point of the self maps \(f\) and \(g,\) we consider the following cases.

**CASE I** Suppose any two terms of the sequence \(\{y_n\}\) are equal i. e. for some \(n \in N, y_n = y_{n+1} = y_{n+2} = \ldots\) we have \(f x_n = g x_{n+1} = g x_{n+2} = y_{n+1}\) we have \(f x_{n+1} = g x_{n+1}.\) Let \(f x_{n+1} = g x_{n+1} = z.\) So \(x_{n+1}\) is a point of coincidence of the pair \((f, g).\) As the pair \((f, g)\) is weakly compatible we have \(f z = g z.\) Now we show that \(f z = z.\) Suppose, if possible on the contrary, that \(f z \neq z\) so \(f z \neq f x_{n+1}.\) By using equation (3) we have
\[
M(z, f z, t) = M(f x_{n+1}, f z, t),
\]
\[
> (M(f x_{n+1}, g x_{n+1}, t))^\alpha (M(f z, g z, t))^\beta (M(g x_{n+1}, g z, t))^{1-\alpha-\beta},
\]
\((M(z, z, t))^{\alpha} (M(fz, fz, t))^{\beta} (M(z, fz, t))^{1-\alpha-\beta},\)

\((M(z, fz, t))^{1-\alpha-\beta}.\)

i.e.

\(M(z, fz, t) > (M(z, fz, t))^{(1-\alpha-\beta)}, \) for all \(t > 0\) implies \((M(z, fz, t))^{(\alpha+\beta)} > 1,\) which is not possible. Hence \(fz = z.\) Therefore, \(z\) is a common fixed point of the pair \((f, g)\) in this case.

So we can assume the consecutive terms of the sequence \(\{y_n\}\) are distinct.

Again, to see the existence of a common fixed point in other cases, we first show that all the terms of the sequence \(\{y_n\}\) are distinct.

**CASE II** Suppose \(y_n = y_m,\) for some \(m > (n + 1),\) then as all the consecutive terms of the sequence \(\{y_n\}\) are distinct, we claim that \(y_{n+1} = y_{m+1}.\)

Suppose if possible on the contrary \(y_{n+1} \neq y_{m+1}\) then \(y_n \neq y_{n+1} \neq y_{m+1} \neq y_m\) implies \(y_n \neq y_m\) which contradicts our assumption. So we have \(y_{n+1} = y_{m+1}.\) Also and

\[M(y_{n+1}, y_{n+2}, t) = M(fx_{n+1}, fx_{n+2}, t) \]

\[> \left\{ \begin{array}{l}
(M(fx_{n+1}, gx_{n+1}, t))^{\alpha} (M(fx_{n+2}, gx_{n+2}, t))^{\beta} \\
(M(gx_{n+1}, gx_{n+2}, t))^{(1-\alpha-\beta)}
\end{array} \right\} \]

\[= \left\{ \begin{array}{l}
(M(y_{n+1}, y_{n+2}, t))^{\alpha} (M(y_{n+2}, y_{n+1}, t))^{\beta} \\
(M(y_{n+1}, y_{n+2}, t))^{(1-\alpha-\beta)}
\end{array} \right\} \]

\[= (M(y_{n+1}, y_{n+2}, t))^{1-\beta}(M(y_{n+2}, y_{n+1}, t))^{\beta} \]

i.e.

\([M(y_{n+1}, y_{n+2}, t) > (M(y_{n+1}, y_{n+2}, t))^{1-\beta}(M(y_{n+2}, y_{n+1}, t))^{\beta}.\]

Thus

\[M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t). \quad (8)\]

So

\[M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t) < M(y_{n+2}, y_{n+3}, t) < ... < M(y_m, y_{m+1}, t).\]

i.e. \(M(y_n, y_{n+1}, t) < M(y_n, y_{n+1}, t),\) which is not possible. So this case does not arise.
Thus, we conclude that for distinct \( n, m \in \mathbb{N}, y_n \neq y_m \). Therefore, the elements of the sequence \( \{y_n\} \) are distinct. From equation (8) we have

\[
M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t),
\]

for all \( t > 0 \). Thus, \( \{M(y_n, y_{n+1}, t)\} \), for each \( t > 0 \), is a strictly increasing sequence, which is bounded above by 1. Therefore, by lemma 5, for \( t > 0 \),

\[
\lim_{n \to +\infty} M(y_n, y_{n+1}, t) = 1, \tag{9}
\]

Now we prove that the sequence \( \{y_n\} \) is \( M \)-Cauchy. Suppose if possible on the contrary that it is not true; then there exist \( \eta \in (0, 1), t_0 > 0 \) and the sequences \( \{p(n)\}, \{q(n)\} \) (\( p(n) \) being the smallest ones of the index)

\[
n < p(n) < q(n), M(p(n), q(n), t_0) \leq 1 - \eta, M(p(n) - 1, q(n), t_0) > 1 - \eta. \tag{10}
\]

**STEP 2:** In this step, we show that \( \lim_{n \to +\infty} M(p(n) - 1, q(n) - 1, t_0) = 1 - \eta \). Now for all \( n \geq 1, 0 < \lambda < t_0/2 \), we obtain,

\[
1 - \eta \geq M(p(n), q(n), t_0), \quad \text{using (10)}
\]

\[
\geq M(p(n), p(n) - 1, \lambda) \ast M(p(n) - 1, q(n), t_0, \lambda - 2) \ast M(q(n) - 1, q(n), \lambda). \tag{11}
\]

Let

\[
h_1(t) = \lim_{n \to +\infty} M(p(n) - 1, q(n) - 1, t), \quad t > 0.
\]

Taking the limit supremum on both sides of equation (11), and using the properties of \( M \) and \( \ast \), and by lemma 3, we obtain

\[
1 - \eta \geq 1 \ast \lim_{n \to +\infty} M(p(n) - 1, q(n) - 1, t_0 - 2\lambda) \ast 1 \quad \text{using (10)}
\]

\[
= h_1(t_0 - 2\lambda).
\]

Since \( M \) is bounded with the range in \( (0, 1] \), continuous and non-decreasing in the third variable \( t \), it follows from lemma 3, that \( h_1 \) is continuous from the left. Therefore, for \( \lambda \to 0 \), we obtain

\[
h_1(t_0) = \lim_{n \to +\infty} M(p(n) - 1, q(n) - 1, t_0) \leq 1 - \eta. \tag{12}
\]

Let

\[
h_2(t) = \lim_{n \to +\infty} M(p(n) - 1, q(n) - 1, t), \quad t > 0.
\]

Again, for all \( n \geq 1, \lambda > 0 \)

\[
M(p(n) - 1, q(n) - 1, t_0 + \lambda) \geq M(p(n) - 1, q(n), t_0) \ast M(q(n), q(n) - 1, \lambda)
\]

\[
\geq M(p(n) - 1, q(n) - 1, t_0) \ast M(q(n), q(n) - 1, \lambda) \ast 1.
\]
Taking the limit infimum as \( n \to +\infty \) in the above inequality, we obtain

\[
h_2(\lambda + t_0) = \lim_{n \to +\infty} M(y_{p(n)}-1, y_{q(n)}-1, \lambda + t_0),
\]

\[
\geq (1 - \eta) \lim_{n \to +\infty} M(y_{p(n)}, y_{q(n)}-1, \lambda),
\]

\[
= (1 - \eta) \lim_{n \to +\infty} M(y_{p(n)}-1, y_{q(n)}-1, \lambda),
\]

\[
= (1 - \eta) \lim_{n \to +\infty} M(y_{p(n)}-1, y_{q(n)}-1, \lambda) * \lim_{n \to +\infty} M(y_{q(n)}-1, y_{q(n)}-1, \lambda) = (1 - \eta).
\]

\[
\lim_{n \to +\infty} M(y_{p(n)}-1, y_{q(n)}-1, t_0) \geq (1 - \eta).
\]

Combining the inequalities (12) and (13), we get

\[
\lim_{n \to +\infty} M(y_{p(n)}-1, y_{q(n)}-1, t_0) = (1 - \eta).
\]

**STEP 3** In this step, we show that \( \lim_{n \to +\infty} M(y_{p(n)}, y_{q(n)}, t_0) = 1 - \eta \).

From equation (10) we have

\[
\lim_{n \to +\infty} M(y_{p(n)}, y_{q(n)}, t_0) \leq 1 - \eta.
\]

Also for all \( n \geq 1 \) and \( \lambda > 0 \) we have

\[
M(y_{p(n)}, y_{q(n)}, t_0 + 2\lambda) \geq M(y_{p(n)}, y_{p(n)}-1, \lambda) * M(y_{p(n)}-1, y_{q(n)}-1, t_0) * M(y_{q(n)}-1, y_{q(n)}-1, \lambda)
\]

Taking the limit infimum as \( n \to +\infty \) in the above inequality, using (9), (14) and the properties of \( M \) and \( \ast \) and by lemma 2, we obtain

\[
\lim_{n \to +\infty} M(y_{p(n)}, y_{q(n)}, t_0 + 2\lambda) \geq 1 * \lim_{n \to +\infty} M(y_{p(n)}-1, y_{q(n)}-1, t_0) * 1 = 1 - \eta,
\]

using (14)

(16)

Since \( M \) is bounded with the range in \((0, 1]\), continuous and non-decreasing in the third variable \( t \), it follows from lemma 3 that \( \lim_{n \to +\infty} M(y_{p(n)}, y_{q(n)}, t_0) \) is a continuous function of \( t \) from the right.
Therefore, for $\lambda \to 0$, we obtain
\[
\lim_{n \to +\infty} M(y_p(n), y_q(n), t_0) \geq (1 - \eta). \tag{17}
\]
Combining inequalities (15) and (17), we get
\[
\lim_{n \to +\infty} M(y_p(n), y_q(n), t_0) = (1 - \eta). \tag{18}
\]

**STEP 4** In this step, we show that the sequence $\{y_n\}$ is an M-Cauchy sequence.

Using equations (9) and (14) at $t = t_0$, we have
\[
M(x_p(n), x_q(n)) = \left\{ \begin{array}{ll}
(M(fx_p(n), gx_p(n), t_0)^\alpha (M(fx_q(n), gx_q(n), t_0))\beta \\
(M(gx_p(n), gx_q(n), t_0))^{(1-\alpha - \beta)}
\end{array} \right. \}
\]
\[
= \left\{ \begin{array}{ll}
(M(y_p(n), y_q(n)-1, t_0)^\alpha (M(y_q(n), y_q(n)-1, t_0))\beta \\
(M(y_p(n)-1, y_q(n)-1, t_0))^{(1-\alpha - \beta)}
\end{array} \right. \}
\]
\[
> M(fx_p(n), fx_q(n), t_0)
\]
\[
= M(y_p(n), y_q(n), t_0).
\]  
\[
M(y_p(n), y_q(n), t_0) > M(x_p(n), x_q(n)) \tag{21}
\]
For $n \to +\infty$ and using equations (20) and (21), we have
\[
(1 - \eta) \geq (1 - \eta)^{(1-\alpha - \beta)},
\]
implies that $(1 - \eta)^{(\alpha + \beta)} \geq 1$, which is not possible as $(1 - \eta) < 1$. So, $\{y_n\}$ is an M-Cauchy sequence in $g(X)$ which is M-complete. Therefore, there
exists $z \in g(X)$ such that
\[ \{y_n\} \to z. \tag{22} \]
i. e.
\[ \{fx_n\} \to z \text{ and } \{gx_{n+1}\} \to z. \tag{23} \]

As $z \in g(X)$ there exists $v \in X$ such that
\[ z = gv. \tag{24} \]

**STEP 5** Now we show that $gv = fv$. Suppose, on the contrary, that $fv \neq gv (= w)$. Then exists a positive integer $n_0$ such that $gv \neq gx_n$, for all $n \geq n_0$.

\[
M(x_n, v) = (M(fx_n, gx_n, t))^\alpha(M(fv, gv, t))^\beta(M(gx_n, gv, t)^{(1-\alpha-\beta)} = (M(y_n, y_{n-1}, t))^\alpha(M(fv, z, t))^\beta(M(gx_n, z, t)^{(1-\alpha-\beta)}.
\]

Now
\[
M(fx_n, fv, t) > M(x_n, v) = (M(y_n, y_{n-1}, t))^\alpha(M(fv, z, t))^\beta(M(gx_n, z, t)^{(1-\alpha-\beta)}.
\]

For $n \to +\infty$ and using equations (9), (23) and (24) we get
\[ M(z, fv, t) \geq [M(fv, z, t)]^\beta \text{ i. e. } M(fv, z, t)]^{1-\beta} > 1, \text{ which is not possible if } fv \neq z. \text{ Hence } fv = u \text{ and we have } \]
\[ fv = gv = z. \tag{25} \]

As the pair of self maps $(f, g)$ is weakly compatible, we have
\[ fz = gz. \tag{26} \]

**STEP 6** Now we show that $fz = z$. Suppose, on the contrary that $fz \neq z$. Then $gz \neq z$.

\[
M(z, v) = (M(fz, gz, t))^\alpha(M(fv, gv, t))^\beta(M(gz, gv, t)^{(1-\alpha-\beta)} = (M(fz, z, t)^{(1-\alpha-\beta)} \text{ using (25, 26)}
\]

and
\[
M(fz, z, t) = M(fz, fv, t)
\]
\[ M(z, v), \quad \text{using (25)} \]
\[ = [M(fz, z, t)]^{(1-a-b)}. \]

i.e. \( M(fz, z, t)^{(a+b)} > 1 \) which is not possible as the left hand side is less than 1. Thus, \( fu = gu = u. \)

Taking \( g = I \) in Theorem 2, then the sequence \( \{x_n\} = \{x_0, fx_0, \cdots, f^n x_0, \cdots\} \) becomes a Picard sequence for the self map \( f \) and we have

**Corollary 1.** Let \( f \) be an interpolative fuzzy Meir-Keeler contractive map on a \( M \)-complete fuzzy metric space \( (X, M, \ast) \). Then the map \( f \) has a unique fixed point in \( X \).

**Remark 4.** If we take \( \alpha = 0 \) and \( \beta = 0 \) in the above corollary, we obtain Theorem 3.1 of Zheng and Wang (Zheng & Wang, 2019).

**Example 1.** (of Theorem 4.1) Let \( X = [0, 1] \). Define a self map \( f : X \to X \) by \( f(x) = \frac{x}{2} \), and \( g(x) = x \), the identity map on \( X \). Taking \( M(x, y) = \frac{1}{1 + d(x, y)} \), then \( (X, M, \ast) \) is a complete stationary fuzzy metric space with the product t-norm. Define \( \delta \) as follows:

\[
\delta(t) = \begin{cases} 
\frac{1}{12} & \text{if } 0 < t < \frac{3}{4}, \\
\frac{n-1}{n(n+2)} & \text{if } \frac{n-1}{n} \leq t \leq \frac{n}{n+1}, \text{for } n \geq 4.
\end{cases}
\]

Then \( \delta \in \Delta \).

Taking \( \alpha = 0 = \beta \), observe that for all values of \( x, y \in X, f(x), f(y) \in [0, \frac{1}{3}] \). We show that the quadruple \( (X, M, \delta, f) \) is an interpolative Meir-Keeler contractive. For this we prove the following condition:

\[ \text{For all } \epsilon \in (\frac{3}{4}, 1), \epsilon - \delta(\epsilon) < M(x, y) \leq \epsilon \Rightarrow M(fx, fy) > \epsilon. \]

If \( \epsilon \in (\frac{3}{4}, 1) \Rightarrow \frac{n-1}{n} \leq \epsilon \leq \frac{n}{n+1}, \text{for } n \geq 4, \text{so } \delta(t) = \frac{1}{n(n+2)}. \)

Therefore, the inequality \( \epsilon - \delta(\epsilon) < M(x, y) \leq \epsilon \) gives

\[
\left(\frac{n-1}{n}\right) - \frac{1}{n(n+2)} < \epsilon - \delta(\epsilon) < \frac{1}{1 + d(x, y)} \leq \epsilon < \frac{n}{n+1}. \text{ Therefore } \frac{1}{n} \leq d(x, y) < \frac{2}{n}
\]
which implies that \( x, y \in [0, 1] \). Hence

\[
M(fx, fy) = \frac{1}{1 + d(fx, fy)} = \frac{1}{1 + \frac{d(x, y)}{3}} > \frac{1}{1 + \frac{1}{n}} = \frac{n}{n + 1} > \epsilon.
\]

Thus, the quadruple \((X, M, \delta, f)\) is an interpolative Meir-Keeler contractive and \( x = 0 \) is the unique fixed point of the map \( f \).

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ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введени/цель: Цель данной статьи заключается в введении понятия интерполяционного обобщенного условия сжатия Меира-Келлера для отображений в нечетком метрическом пространстве, которое расширяет, объединяет и обобщает многообразие Меира-Келлера, предназначенное только для одного отображения. При применении устанавливается единая теорема о совместной неподвижной точке для двух отображений через слабую совместимость. В статье приведен пример, доказывающий достоверность результатов исследования.

Методы: Методы функционального анализа с сокращением Меира-Келлера.

Результаты: Получена уникальная неподвижная точка для отображений в нечетком метрическом пространстве.

Выводы: Получена неподвижная точка собственных отображений.

Ключевые слова: нечеткое метрическое пространство, общие фиксированные точки, слабая совместимость, интерполяционное обобщенное сокращение Меира-Келлера.

ИНТЕРПОЛАТИВНА УОШТЕНА МЕИР-КЕЛЕРОВА КОНТРАКЦИЈА

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Сажетак:

Увод/циљ: Циљ овог рада је да се уведе појам интерполативног генерализованог Меир-Келеровог контрактивног угла за пресликавања у фузиметричком простору. Он увећава, обједињује и генерализује Меир-Келерову контракцију и служи за само једно пресликавање. Користећи га, успостављамо јединствену заједничку теорему фиксне тачке за два пресликавања кроз слабу компатибилност. Рад садржи пример који показује валидност наших резултата.

Методе: Методе функционалне анализе са Меир-Келеровом контракцијом.

Резултати: Јединствена фиксна тачка за пресликавања у фузипростору је добијена.

Закључак: Фиксна тачка пресликавања самог у себе је добијена.

Кључне речи: фузиметрички простор, заједничке фиксне тачке, слаба компатибилност, интерполативна генерализована Меир-Келерова контракција.

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