On the exponential bound
in four dimensional simplicial gravity

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Abstract

Simplicial quantum gravity has been proposed as a regularization for four dimensional quantum gravity. The partition function is constructed by performing a weighted sum over all triangulations of the 4-sphere. The model is well-defined only if the number of such triangulations consisting of $N$ simplexes is exponentially bounded. Numerical simulations seem so far to favor such a bound.

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1. Introduction

"Simplicial quantum gravity" has been proposed as a regularization for (Euclidean) quantum gravity in four dimensions [1, 2]. It was suggested as a natural generalization of similar models studied in two dimensions [3, 4] and three dimensions [6, 7, 8, 9, 10, 11, 13, 12]. In all these models the partition functions are given by

$$Z = \sum_T \frac{1}{C_T} e^{-S(T)}$$

(1)

where the summation is over all combinatorially inequivalent triangulations of a manifold of fixed topology. The factor $1/C_T$ is the symmetry factor of the triangulation. It is not important for the following discussion. For pure gravity $S(T)$ is bounded by

$$|S(T)| < \tilde{c}|T|$$

(2)

where $|T|$ denotes the number of simplexes in the triangulation $|T|$. The inequality (2) implies that the partition function is well defined only if the number $\mathcal{N}(|T|)$ of triangulations of a fixed topology (which we in the following will assume spherical) and consisting of $|T|$ simplexes is exponentially bounded:

$$\mathcal{N}(|T|) \leq e^{c|T|}.$$  

(3)

In the case of $2d$ simplicial quantum gravity the exponential bound was proven long ago [14]. In $3d$ a number of plausible arguments was given in favor of an exponential bound [3], but no complete proof exists so far. Most of the arguments of [3] extend to $4d$, but again a proof is missing. In the absence of a proof in $3d$ the question was addressed numerically in [4], and good evidence in favor of an exponential bound was found. The situation in $4d$ seems very similar to the situation in $3d$ and the finite size effects associated with the identification of the smallest possible $c$ in (3) were even considerably smaller than than in $3d$. Recently it has been claimed, however, based on new numerical simulations, that the situation in $4d$ differs profoundly from the situation in $2d$ and $3d$ and that $\mathcal{N}(|T|)$ grows factorially [15]:

$$c\mathcal{N}(|T|) \sim |T|!^{\delta} e^{a|T|}.$$  

(4)

The numerical results they report are in total agreement with earlier simulations in the case where we can check ($\kappa_0 = 0$ in the notation of [15], see later) and the purpose of this short note is to make clear that there are other, in our opinion more plausible, interpretations of the data, which are compatible with the exponential
bound (3). In addition we have used the opportunity to extend the simulations to somewhat larger volume.

2. Results

In 4d the gravity model (1) can be written as

\[ Z = \sum_{T(S^4)} \frac{1}{C_T} e^{-\kappa_4 N_4(T) + k_2 N_2(T)} \]  

where \( N_4(T) \) and \( N_2(T) \) denote the number of 4-simplexes and triangles in the triangulation \( T \). According to Regge calculus \( k_2 \sim 1/G_0 \), the bare gravitational constant. If we have a regular triangulation of \( S^4 \) the number of vertices \( N_0(T) = 2 - N_4(T) + N_2(T)/2 \). This allows us to write

\[ Z \sim \sum_{T(S^4)} \frac{1}{C_T} e^{-\kappa_4 N_4(T) + \kappa_0 N_0(T)}; \quad \kappa_0 = 2k_4, \quad \kappa_4 = k_4 - 2k_2, \]  

in order to make contact with the notation used in [15]. If we assume (3) there will for each value of \( \kappa_0 \) be a critical value, \( \kappa_4^c(\kappa_0) \) of \( \kappa_4 \), such that the partition function is divergent for \( \kappa_4 < \kappa_4^c(\kappa_0) \) and convergent for \( \kappa_4 > \kappa_4^c(\kappa_0) \). The infinite volume limit of the system is obtained by letting \( \kappa_4 \) approach \( \kappa_4^c(\kappa_0) \) from above. Whether this infinite volume limit has an interesting continuum interpretation should then be investigated for various values of \( \kappa_0 \). A second order phase transition in geometry might indicate long range fluctuations and thereby continuum physics. Details of how to determine \( \kappa_4^c(\kappa_0) \) can be found in [16], the only point we want to emphasize here is that one determines a pseudo-critical value \( \kappa_4^c(\kappa_0, N_4) \) by analyzing fluctuations in size around some chosen \( N_4 \). By changing \( N_4 \) this pseudo-critical point will change and in the limit \( N_4 \to \infty \) it should converge to \( \kappa_4^c(\kappa_0) \), again provided (3) is valid. In case (4) should be valid there will be no convergence for \( N_4 \to \infty \).

In order to discuss possible options, let us for simplicity choose \( \kappa_0 = 0 \). We are then directly testing \( N(N_4) \). Let us assume that the number of configurations grows exponentially. There will be subleading corrections. A reasonable trial ansatz for large \( N_4 \) is

\[ \log(N(N_4)) \sim \kappa_4^c N_4 - \tilde{c} N_4^\beta \cdots, \quad \beta < 1. \]

This is known to be true in 2d gravity where \( \beta = 0 \) (and the power is replaced by a logarithm) and (to the extent one can trust numerical simulations) in 3d where \( \beta \approx 2/3 \). This subleading behavior will reflect itself in the determination of \( \kappa_4^c(N_4) \).
which will be given by

\[ \kappa_c^4(N_4) = \kappa_4^c - \frac{c}{N_4^\alpha} - \cdots, \quad \alpha = 1 - \beta. \]  

(8)

If we instead assume that (9) is valid we will get

\[ \kappa_c^4(N_4) = a + \delta \log N_4. \]  

(9)

Let us first remark that in both cases \( \kappa_c^4(N_4) \) is an increasing function of \( N_4 \) and in case \( \alpha \) and \( \delta \) are small it might be difficult to distinguish (8) and (9) when only a limiting range of \( N_4 \) is considered. Secondly one should be aware that both (8) and (9) are asymptotic expressions which might only be valid for large \( N_4 \). Thirdly, (8) and (9) are valid also for \( \kappa_0 \neq 0 \), the only difference being that \( \kappa_c^4, \alpha \) and \( a \) might depend on \( \kappa_0 \), while \( \delta \) is independent of \( \kappa_0 \) since (2) implies that various actions \( S(T) \) can only lead to different exponential corrections. \( (N_4!)^\delta \) is entirely an entropy factor coming from the number of triangulations.

In fig. 1 we show \( \kappa_c^4(N_4, \kappa_0) \) as presented in [13] for three values of \( \kappa_0 \). The curves are just linear interpolations between the data points. We see a clear deviation from the functional form (8) or (9) for \( N_4 < 4000 \) in all three cases. Since this deviation is qualitatively of the same form in the three cases it is natural to assume that there are finite size effects not compatible with (8) or (9) for \( N < 4000 \). In any case it seems not reasonable to determine \( \delta \) from \( N_4 < 4000 \) since the functional form here shows a strong dependence on \( \kappa_0 \) (as also remarked in [13]) and \( \delta \) is independent of \( \kappa_0 \). Let us now concentrate on the case \( \kappa_0 = 0 \) where we can directly compare also with our own old (and new) results. In fig. 2 we show \( \kappa_c^4(N_4) \) for \( \kappa_0 = 0 \) and \( N_4 \geq 4000 \). When data points can be compared directly there is agreement within error bars. The (dashed) straight line is the fit (9) found in [13]. The fully drawn line is a fit to (8) with \( \alpha = 1/4 \) (the reason for this choice will be discussed below). In both cases there are two free parameters in the fit, \( a, \delta \) and \( \kappa_c^4, c \), respectively. For the non-trivial parameters, i.e. \( c \) and \( \delta \) we have \( c \approx 1.23 \) and \( \delta \approx 0.03 \).

3. Conclusions

As is clear from fig. 2 the numerical simulations so far are perfectly consistent with an exponential bound on the entropy of triangulations of \( S^4 \). While it is possible that one can still make (9) compatible with fig. 2 even if we include the new data point corresponding to \( N_4 = 64000 \), we feel that a fit of the form (8) is more natural for \( N > 4000 \) for several reasons. First it is hard to understand why there should be
a factorial growth $(N!)^\delta$ with an extremely small exponent $\delta \sim 0.03$. As mentioned above the origin of such a term is purely combinatorial and if there was a factorial growth one would a priori expect $\delta \sim 1$. Secondly it is difficult for us to understand why the situation for $S^4$ should differ from that of $S^3$. Rather, since there is also an exponential bound in $2d$ and $3d$ it is natural to conjecture that there is an exponential bound on the number of triangulations of $S^d$ for any $d$.

It is interesting to note that data are compatible\footnote{We should note that $\alpha$ is not very well determined from a fit to the data shown in fig. 2. Any $\alpha$ in the range $0.2 < \alpha < 0.3$ will give a reasonable fit.} with the choice $\alpha = 1/4$. This means that the exponent $\beta$ in (7) is $3/4$ and if we tentatively write $N_4 = L^4$ eq. (7) reads:

$$\log \mathcal{N}(L^4) = \kappa_4^c L^4 - cL^3 - \cdots .$$

(10)

In $3d$ we found $\beta \approx 2/3$ and introducing in $3d$ the notation $N_3 = L^3$ one has the analogue of (10):

$$\log \mathcal{N}(L^3) = \kappa_3^c L^3 - cL^2 - \cdots .$$

(11)

This kind of behavior might be related to the fact that the generic triangulation for $\kappa_0 = 0$, both in $3d$ and $4d$ is very singular. There are vertices of very high order (almost comparable with $|T|$) and the $d - 1$ dimensional boundary of the unit ball surrounding such a vertex could maybe act as ”pseudo-boundary” of the closed $d$-dimensional manifold. It would explain the boundary-like terms in (10) and (11). When $\kappa_0$ is increased (decreasing bare gravitational constant) the vertices of very high order disappear. At the same time the value of $\beta$ seems to decrease, maybe approaching zero at the observed phase transition\footnote{The value of $\beta$ (or $\alpha$) is not very well determined from the fits to $\kappa_4^c(N_4)$ by the present data for $\kappa_0$ near or above the phase transition point. In $3d$ the situation is better and it seems as if $\beta \approx 0$ at the transition \cite{12, 13}. The study of baby-universe distributions \cite{17} also supports this picture.}. In addition it would also explain why the corresponding term $cL$ is absent in $2d$. The distribution of high-order vertices in $2d$ can be calculated analytically and the probability of having such vertices falls off exponentially with the order of the vertex.
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Figure caption

**Fig. 1** The data points taken from [15]. The plot is $\log N_4$ versus $\kappa_4^c(N_4, \kappa_0)$ for $\kappa_0 = 0$ (lower curve), $\kappa_0 = 0.5$ (middle curve) and $\kappa_0 = 1.0$ (upper curve).

**Fig. 2** $\log N_4$ versus $\kappa_4^c(N_4, \kappa_0)$ for $\kappa_0 = 0$. Circles indicate the data points from the plot in [15], the size of the circles being approximately the error bars. The dots (with error bars) indicate our own data. We see that there is perfect agreement. The dashed straight line is the fit to (9) as presented in [15]. The fully drawn line is the fit to (8) with $\alpha = 1/4$. 
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