Adiabatic evolution of Hayward black hole

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Abstract

In this letter we use the Carathéodory’s approach to thermodynamics, to construct the thermodynamic manifold of the Hayward black hole. The Pfaffian form representing the infinitesimal heat exchange reversibly is considered to be $\delta Q_{\text{rev}} \equiv dr_s - F_Hdl$, previously obtained by Molina & Villanueva \textsuperscript{[1]}, where $r_s$ is the Schwarzschild radius, $l$ is the Hayward’s parameter responsible for the possible regularization of the Schwarzschild black hole, and $F_H$ is the intensive variable called the Hayward’s force. By solving the associated Cauchy problem, the adiabatic paths are confined to the non-extremal manifold, and therefore, the status of the second and third laws are preserved. Consequently, the extremal sub-manifold corresponds to the adiabatically disconnected boundary of the manifold. In addition, the merger of two extremal Hayward black holes is analyzed.

Keywords: Black hole thermodynamics, Hayward black hole, Adiabatic processes

1. Introduction

Ever since their advent, the reconciliation of the laws of thermodynamics with black hole mechanics \textsuperscript{[2]}, the entropy assigned by Bekenstein to the black holes \textsuperscript{[3, 4, 5, 6]} and the possibility of black hole evaporation through the Hawking radiation \textsuperscript{[7]}, have been of great interest among physicists. And although it has not been possible to detect such phenomena from direct observations, nevertheless, strong effort have been being made to mimic similar processes in black hole analogs, such as the experimental Unruh radiation \textsuperscript{[8]} in simulated systems, both theoretically and experimentally \textsuperscript{[9, 10, 11, 12, 13, 14, 15, 16, 17]}. On the other hand, while the famous Bekenstein-Hawking (B-H) entropy formula has been applied widely for the regular black holes, nevertheless, its direct application to the extremal black holes (EBHs) is not that simple. In fact, the special conjecture of zero entropy for EBHs \textsuperscript{[18, 19]}, leads to overlooking the direct relationship between the entropy and the event horizon’s area, as demanded by the B-H formula.

Along with the strong interest of the community in the study of the thermodynamics of regular black holes, in this paper, we focus on a particular, non-singular minimal black hole model proposed by Hayward in Ref. \textsuperscript{[20]}, which constructs a static spherically symmetric and asymptotically flat spacetime. Recently, in Ref. \textsuperscript{[21]} this solution has been generalized to certain scalar-tensor theories, and new regular black holes have been reported in Refs. \textsuperscript{[22, 23, 24]}, in the context of quasi-topological electromagnetic theories. In fact, the non-singular nature of this black hole has made it an interesting topic studying its thermodynamics. Accordingly, and in Ref. \textsuperscript{[1]}, the laws of black hole thermodynamics have been investigated for the Hayward black hole (HBH) through studying the relations between its dynamical parameters $\{r_s, l\}$ that define the state of the system. In this paper we construct the correct foliation of the thermodynamics manifold by using the Carathéodory’s approach, for which, the appropriate Pfaffian form $\delta Q_{\text{rev}}$, representing the infinitesimal heat exchanged reversibly, is taken into account. We should, however, state that the present study aims at establishing a new method of analyzing black hole thermodynamics and still is not constrained by the experimental data. Although the method does not require a priori knowledge of any of the so-called laws of thermodynamics, we will use the already known results for physical quantities, such as entropy, temperature, etc. Therefore, the adiabatic surfaces are obtained by solving the Cauchy problem associated to the Pfaffian equation $\delta Q_{\text{rev}} = 0$. 

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2. The Hayward black hole spacetime

The Hayward black hole spacetime, is given by the regular, non-singular, static spherically symmetric metric
\[ ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{1} \]
in which, the lapse function \( f(r) \) is given by \[ f(r) = 1 - \frac{r_s r_i^2}{r^3 + r_s r_i^2}, \tag{2} \]
where \( r_s = 2GM/c^2 \) is the radius of the Schwarzschild black hole (SBH) of mass \( M \), and \( l \) is the Hayward’s parameter (\( 0 \leq l < \infty \)), so that for \( l = 0 \), the SBH is regenerated. The spacetime admits an event horizon, which is obtained representing the thermodynamic limit of the \( l \)th, by solving the cubic equation \( \delta f(r) = 0 \), and is given by \[ r_+ = r_s \left(1 + 2 \cos \alpha \right)^{-1} \equiv r_s R_+, \tag{3} \]
where \( \alpha(r_s, l) = 1/3 \arccos\left(1 - 2\ell_s^2/l_s^2\right), \ell_s = 2r_s/\sqrt{27} \), and \( 0 \leq l < l_s \). In the case that \( l = l_s \), the roots reduce to the two degenerate positive values, and the EHBB is obtained representing the thermodynamic limit of the black hole. Hence, \( l_s \) is the extremal limit of the Hayward’s parameter.

3. The \( [r_s, l] \) thermodynamics in Carathéodory’s approach

The most usual way to describe the thermo-geometric processes in black hole spacetimes, goes through the second law of black hole thermodynamics, which is postulated as \([2, 3, 4, 5, 6, 7]\): The area of the black hole event horizon cannot decrease; it increases during most of the physical processes of the black hole, which, by means of the well-known B-H area-entropy formula
\[ S = \frac{k_B}{4} \frac{4\pi \ell_p^2}{\ell_p^3}, \tag{4} \]
relates the entropy with the event horizon, where \( k_B \) and \( \ell_p \) are, respectively, the Boltzmann constant and the Planck length. Since the event horizon depends on the pair \([r_s, l]\), it is useful to define the metric entropy function
\[ S(r_s, l) = \frac{\ell_p^2}{4\pi k_B} = r_s^2 R_+^2(r_s, l). \tag{5} \]
On the other hand, the thermodynamics of the system can be approached, geometrically, in the context of the Carathéodory’s approach, which postulates the integrability of the Pfaffian form
\[ \delta Q_{\text{rev}} = T dS, \tag{6} \]
representing the infinitesimal heat exchanged reversibly \([25, 26, 27, 28, 29]\), and this way, as highlighted in Ref. \([30]\), it is connected with the Gibbs’s thermodynamics. Here \( T \) is the integrating factor representing the absolute temperature, defined by
\[ \frac{1}{T} = \left(\frac{\partial S}{\partial \alpha}\right)_l > 0, \tag{7} \]
which, by applying Eq. (5), yields \([1]\)
\[ T = \frac{T_s}{R_+(\alpha)[R_s(\alpha) + g_s(\alpha)]}, \tag{8} \]
where \( T_s \equiv (2r_s)^{-1}(4S_{s})^{-1/2} \) is the SBH temperature, and \( g_s(\alpha) = \frac{1}{2} \sin \alpha (\csc 3\alpha - \cot 3\alpha) \).

Therefore, considering \([r_s, l]\) as independent thermodynamic coordinates, the homogeneity of the system is reflected by the integrable Pfaffian form
\[ \delta Q_{\text{rev}} = dr_s - F_H dl. \tag{9} \]
Note that, \( dr_s \) and \(-F_H dl\) represent, respectively, the internal energy and work, and the intensive variable (homogeneous of degree zero)
\[ F_H = -\frac{g_s(\alpha)}{R_s(\alpha) + g_s(\alpha)} \frac{r_s}{T}. \tag{10} \]
was introduced as the generalized Hayward’s force in Ref. \([1]\). Here, an equilibrium geometrical state is compared with the equilibrium states of standard thermodynamics, by taking the infinitesimal variation of \( F_H \) in Eq. (9), along the stationary HBH solution. Then, the open non-extremal manifold \( l < l_s \) corresponds to the thermodynamic domain, and is encompassed by the extremal sub-manifold (thermodynamic limit \( T = 0 \)), formed by \( l = l_s \). The foliation of this thermodynamic manifold can be generated by the integrability property \( \delta Q_{\text{rev}} \wedge d(\delta Q_{\text{rev}}) = 0 \) (which is trivial in a two-variable case), specifically, on the submanifolds of codimension one, which are solutions of the Pfaffian equation \( \delta Q_{\text{rev}} = 0 \) (see the next sections).

Before continuing, let us turn our attention to Fig. 1, which shows the behaviors of the metric entropy and temperature. The physically accepted segment, lies within the domain \( 0 \leq T \leq T_s \) (the blue curve). Note that, if \( r_s \) is fixed, then \( \Delta S > 0 \) implies \( \Delta l < 0 \). Therefore, by varying \( l \) while keeping \( r_s \) fixed (the red
Then, the area for the extremal states become

\[ \delta \mathcal{A} = 4\pi (r_+^2)^2 = \frac{16}{9} \pi r_+^2 \]

which implies that

\[ d\mathcal{A} = 24\pi l dl \]

Thus, the isoareal condition \( d\mathcal{A} = 0 \) is satisfied only if \( l = \text{const.} \), but these states still satisfy the Pfaffian equation \( \delta Q_{\text{rev}} = dr_+ - F_H^r dl = 0 \). Consequently, the adiabatic transformations are not isoareal transformations on the extremal submanifold. We will return to this point later, but for now, the EHBH is regarded as an extremal submanifold that resides in the (adiabatic) integral manifold of \( \delta Q_{\text{rev}} \) in Eq. (9).

For non-extremal states the situation is different, since it is possible to obtain solutions for the isoareal equation \( d\mathcal{A} = 0 \), and therefore, one can generate a foliation of the parameter space of the HBH, whose leaves are the surfaces \( \mathcal{A} = \text{const.} \). In fact, the Carathéodory's approach allows for foliating the thermodynamic manifold by means of the solutions to the Pfaffian equation \( \delta Q_{\text{rev}} = 0 \), that provide a smooth and continuous 1-form field residing in the non-extremal sub-manifold. Accordingly, the integral manifolds of \( \delta Q_{\text{rev}} \) are surfaces with constant \( S \), which together with the paths that solve the Pfaffian equation, construct an isentropic surface (i.e. adiabatic and reversible) [27]. To elaborate on this point, let us apply the changes of variables \( x = r_+^2 \) and \( y \equiv (F_H^r)^2l^2 \), so that Eq. (9) can be recast as

\[ \delta Q_{\text{rev}} = \frac{dy}{2 \sqrt{y}} - \frac{F_H(x,y)}{F_H^r} \frac{dy}{2 \sqrt{y}} \]

which holds as long as \( y \leq x \). Accordingly, the states which are connected adiabatically with the initial black hole state \( (x_0, y_0) \), are solutions to the Cauchy problem

\[ \frac{dy}{dx} = \frac{\sqrt{x}}{x} F_H^r, \]

\[ y(x_0) = y_0. \]

where \( x_0 > y_0 \). Applying Eq. (10), one can rewrite Eq. (15a) as

\[ \frac{dy}{dx} = \frac{y}{x} \left( 1 + \frac{R_r(x,y)}{G_r(x,y)} \right), \]

\[ = \frac{1}{8} \left( 1 + 2 \cos \left( \frac{1}{3} \arccos \left( 1 - \frac{2y}{x} \right) \right) \right)^3. \]

The above problem allows for two solutions, say \( y_{1,2} \), which are given by

\[ y_1(x) = x - y_2(x), \]

in which

\[ y_2(x) = \frac{27\rho^2}{16} \left( 1 - \frac{\rho}{2 \sqrt{x}} \right), \]
For \( \rho \neq \{0, 2^{\sqrt{2n}}\} \) and considering an arbitrary, non-extremal, initial state, the functions given by Eqs. (17) and (18), yield the following equations for \( \rho \):

\[
\begin{align*}
\rho^3 - 2 \sqrt{x_0 \rho^3} + \frac{32}{27} \sqrt{x_0}(x_0 - y_0) &= 0, \\
\rho^3 - 2 \sqrt{x_0 \rho^3} + \frac{32}{27} \sqrt{x_0} y_0 &= 0,
\end{align*}
\]

whose solutions can be written simply as

\[
\rho_k(x_0, y_0) = \frac{2 \sqrt{x_0}}{3} \left[ 1 + 2 \cos \left( \omega + \frac{2k\pi}{3} \right) \right],
\]

where \( k = 0, 1, 2, \) and

\[
\omega = \omega(x_0, y_0) = \frac{1}{3} \arccos \left| 1 - \frac{2y_0}{x_0} \right|.
\]

The above means that, given an initial equilibrium configuration for the HBH, there are six possible curves adiabatically connected with it, say \( y_j(x) \equiv y_j(x; \rho_k) \), with \( j = 1, 2 \) and \( k = 0, 1, 2 \). Let us designate by \( \epsilon_{jk} \) and \( \sigma_{jk} \), the value of the \( x \)-coordinate at the intersection of each curve \( y_j \) with the EHBH (where \( y = x \)), and the SBH (where \( y = 0 \)), respectively (green and yellow dots in Fig. 2). Then, it is no hard to show that \( \epsilon_{1k} = \sigma_{2k} \) and \( \epsilon_{2k} = \sigma_{1k} \). In addition, since \( \rho_0 > \rho_2 > 0 \), one gets \( \epsilon_{10} = \sigma_{20} > \epsilon_{12} = \sigma_{22} > 0 \), and consequently, \( \epsilon_{20} = \sigma_{10} > \epsilon_{22} = \sigma_{12} > 0 \). However, because \( \rho_1 < 0 \), the function \( y_2(x) \) is strictly positive and therefore \( \epsilon_{21} = \sigma_{11} > 0 \) and \( \epsilon_{11} = \sigma_{21} \to \infty \).

The above result is crucial to exclude the leaf \( \mathcal{T} = 0 \) from the adiabatic manifold. In fact, if we delete all the solutions \( y_j \), except \( y_{11} \), then the generated adiabatic surface does not intersect the extremal surface (see Fig. 3). Thus, assuming that the relation \( S \propto \mathcal{A} \) is not followed by the extremal black holes [18], we characterize the thermodynamics of the HBH by

\[
S(r, I) = \begin{cases} 
\mathcal{A}/4, & \text{non-extremal states,} \\
0, & \text{extremal states.}
\end{cases}
\]

To ensure the correct foliation of the thermodynamic manifold, and based on the processes described in the last part of Sect. 2 (cf. Fig. 1), the following conditions must hold:

1. The slope of the \( x-y \) curves is positive,

\[
\frac{dy}{dx} > 0.
\]

2. As the variables decrease, the system evolves towards the SBH, while when they grow, the EHBH is approached.

![Figure 2: Adiabatic solutions for the Cauchy problem given by Eq. (16). Top panel: \( y_1(x) \) given by Eq. (17); Bottom panel: \( y_2(x) \) given by Eq. (18). In both plots we have used as the initial state \( I = (x_0, y_0) = (0.2, 0.1) \), so that \( \rho_0 = 0.815, \rho_1 = -0.218 \) and \( \rho_2 = 0.298 \). Green dots represent the intersection of each function with the EHBH, \( y(x) = x \), whereas the yellow represent the intersection with the SBH, \( y(x) = 0 \).](attachment:figure2.png)

3. In the neighborhood of any arbitrary state, \( i \), of a physical system there are neighboring states \( \mathcal{I} \), which are inaccessible from \( i \) along adiabatic paths (Carathéodory’s principle [31, 32]).

Therefore, the complete solution to the Cauchy problem that satisfies the physical requirements, can be written as

\[
y(x) = x - \frac{27\rho_1^2}{16} \left( 1 + \frac{|\rho_1|}{2\sqrt{x}} \right),
\]

whose asymptotic behavior is

\[
y(x) \approx x - \frac{27\rho_1^2}{16}.
\]

that ensures the condition \( x > y \). Recently, an study on the adiabatic properties of the Bañados–Teitelboim–Zanelli (BTZ) black hole was carried in Ref. [33], showing that the correct thermodynamic foliation is possible by building a piecewise argument, with a physically accepted segment, which is the solution (25).
In the case that the initial states are constituted by the final entropy is greater than the initial one, protecting the final state is a non-extremal state, and thus, the final black hole is as well, extremal and therefore, build a heat engine whose efficiency is equal to one. We, however, were able to find a solution that avoids such unwanted behavior; it generates a manifold that does not include the extremal sub-manifold \( T = 0 \). This can be better understood in the context of the Carathéodory’s principle (cf. condition 3), by assuming such states as being inaccessible.

On the extremal sub-manifold, the condition \( \delta Q_{\text{rev}} = 0 \) is still valid, which gives rise to two kinds of transformations that satisfy the initial extremal condition \( y(x_0) = x_0 \). One of the solutions, adiabatically connects the extremal states with the non-extremal ones, that have the same areas (isoareal transformations). The second law. This obviously involves a detailed study of such a situation that will not be addressed here. Another way that could alleviate the understanding of the process, consists of abandoning the hypothesis of zero entropy to give rise to a law of the form \( S = f(\mathcal{A}) \), where \( f \) is a function of the area, as proposed, for example, in Ref. \[34\] in the context of a thin shell for the case of a rotating uncharged BTZ black hole.

6. Discussion and final remarks

The correct construction of a thermodynamic manifold allows us establishing its known laws with complete property and thus, connecting this realization with the so-called thermo-mechanics of black holes. This makes it possible to identify the isoareal and adiabatic transformations, since for non-extremal black holes, the law \( S \propto \mathcal{A} \) is valid.

In this letter we have constructed the correct foliation of the thermodynamic manifold for the HBH, by applying the Carathéodory axiomatic principle. Accordingly, we solved the Cauchy problem associated with the Pfaffian equation \( \delta Q_{\text{rev}} = 0 \) for the HBH, where \( \delta Q_{\text{rev}} \) represents the infinitesimal heat exchanged reversibly. It is important to note that, even the procedure of applying the B-H formula, does not demand any priori knowledge of the laws of thermodynamics, in order to construct the submanifold, although of course, they are mathematically connected.
other solution connects the extremal states with different areas. This solution, however, can be considered as an adiabatic transformation, by virtue of the null entropy law. This accounts for the disconnection between the leaves $\mathcal{T} = 0$ and $\mathcal{T} > 0$, which is expressed, as well, in the metric entropy (23). In fact, as stated above, such disconnection can be provoked by the exclusion of the solutions $y_{10}$ (except for $y_{11}$), that allows for eliminating any connections between the above varieties, and respects the second and the third laws.

Nevertheless, the scattering process of two static EHBH leads to questioning this issue, even more profoundly, because of the impossibility of the increase in entropy in the final state. As in the other cases, the adding of a new variable seems to be the solution, although the price will be the loss of the homogeneity of the system, which gives rise to quasi-homogeneous potentials that do not allow for fixing the degree of homogeneity of the Pfaffian form, although, a thermodynamic construction is allowed [26].

Finally, all the standard thermodynamic parameters can be connected with each other, in order to obtain different quantities which are of interest. For example, using the Stefan-Boltzmann law together with the corresponding Hayward’s quantities, one can calculate the evaporation time of the HBH, from the differential equation

$$\frac{dr_s}{dt} = -br_s^2 R_s^1 (r_s, l) T^4 (r_s, l),$$

where $b = 1.6C/(120r)$. Expansion of the right hand side of Eq. (30) about the Schwarzschild solution (corresponding to $l = 0$) and doing the integration, yields

$$\Delta \text{evap} \approx \frac{16r_s^3}{3b} \left( 1 + \frac{l^2}{r_s^2} \right),$$

which implies that the lifetime of the HBH is longer than that of the SBH.

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