We introduce a novel notion of invariance feedback entropy to quantify the state information that is required by any controller that enforces a given subset of the state space to be invariant. We establish a number of elementary properties, e.g. we provide conditions that ensure that the invariance feedback entropy is finite and show for the deterministic case that we recover the well-known notion of entropy for deterministic control systems. We prove the data rate theorem, which shows that the invariance entropy is a tight lower bound of the data rate of any coder-controller that achieves invariance in the closed loop. We analyze uncertain linear control systems and derive a universal lower bound of the invariance feedback entropy. The lower bound depends on the absolute value of the determinant of the system matrix and a ratio involving the volume of the invariant set and the set of uncertainties. Furthermore, we derive a lower bound of the data rate of any static, memoryless coder-controller. Both lower bounds are intimately related and for certain cases it is possible to bound the performance loss due to the restriction to static coder-controllers by 1 bit/time unit. We provide various examples throughout the paper to illustrate and discuss different definitions and results.

1. Introduction

In this work we study the classical feedback control loop, in which a controller that is feedback connected with a given system is used to enforce a prespecified control task in the closed loop. Unlike in the classical setting, we do not assume that the sensor (or coder) is able to transmit an infinite amount of information to the controller, but is restricted to use a digital noiseless channel with a bounded data rate to communicate with the controller. The closed loop of such a feedback is illustrated in Fig. 1. In this context, we are interested in characterizing the minimal data rate of the digital channel between coder and controller that enables the controller to achieve the given control task. Or equivalently, we are interested in quantifying the information required by the controller to achieve a given control goal.

Data rate constrained feedback is a maturate research topic and has been extensively studied for linear control systems and asymptotic stabilizability, see e.g. [1] and references therein. Remarkably, for this class of synthesis problems, the critical data rate has been characterized in terms of the unstable eigenvalues of the system matrix independent of the particular disturbance model [2–4].

We are interested in minimal data rates necessary for a coder-controller scheme to render a given nonempty subset of the state space invariant. Invariance specifications are one of the most fundamental system requirements and are ubiquitous in the analysis and control of dynamical systems [5, 6]. In [7], Nair et. al extended the well-known notion of topological entropy of dynamical systems [8–10] to discrete-time deterministic...
control systems and showed that the topological feedback entropy characterizes the data rate necessary to achieve invariance. Later Colonius and Kawan [11] introduced a notion of invariance entropy for continuous-time deterministic control systems. While the definition in [7] clearly resembles the definition of entropy for dynamical systems in [8] based on open covers, the invariance entropy introduced in [11] is close to the notion of entropy in [9, 10] based on spanning sets. Both notions coincide for discrete-time control systems provided that a strong invariance condition holds [12, 13].

In this paper, we continue this line of research and introduce a notion of invariance feedback entropy for uncertain control systems to characterize the necessary state information required by any controller to enforce the invariance condition in the closed loop. After we introduce the notation used in this paper in Section 2, we motivate the need of the novel notion of invariance feedback entropy in Section 3. We define invariance feedback entropy and establish various elementary properties in Section 4. We show that the entropy is nonincreasing across two systems that are related via a feedback refinement relation [14]. This result generalizes the fact that the invariance entropy of deterministic control systems cannot increase under semiconjugation [11, Thm 3.5], [13, Prp. 2.13]. We provide conditions that ensure that the invariance feedback entropy is finite and show that we recover the notion of invariance feedback entropy known for deterministic control systems, in the deterministic case. We establish the data rate theorem in Section 5. It shows that the invariance entropy provides a tight lower bound on the data rate of any coder-controller that enforces the invariance specification in the closed loop. To this end, we introduce a history-dependent notion of data rate. We discuss possible alternative data rate definitions and motivate our particular choice by two examples. We continue with the analysis of uncertain linear control systems in Section 6. We derive a lower bound on the invariance feedback entropy. The lower bound depends on the absolute value of the determinant of the system matrix and a ratio involving the volume of the invariant set and the set of uncertainties. The lower bound is invariant under state space transformations and recovers the well-known minimal data rate [1] in the absence of uncertainties. Additionally, we derive a lower bound of the data rate of any static, memoryless coder-controller. Both lower bounds are intimately related and for certain cases it is possible to bound the performance loss due to the restriction to static coder-controllers by \( \log_2(1 + 1/2^{h_{\text{inv}}}) \), where \( h_{\text{inv}} \) is the invariance feedback entropy of the uncertain linear systems, i.e., the best possible (dynamically) achievable data rate. We show that the lower bounds are tight for certain classes of systems.

A preliminary version of the results presented in Sections 3-5 appeared in [15]. The results on uncertain linear systems (Section 6) are currently under review in [16]. This paper provides a detailed and extended elaboration of the results proposed in [15, 16], including the new results presented in Theorem 1 and Theorem 5.
2. Notation

We denote by \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{R} \) the set of natural, integer and real numbers, respectively. We annotate those symbols with subscripts to restrict the sets in the obvious way, e.g. \( \mathbb{R}_{>0} \) denotes the positive real numbers. We denote the closed, open and half-open intervals in \( \mathbb{R} \) with endpoints \( a \) and \( b \) by \([a, b]\), \( ]a, b[\), \( [a, b[\), and \( ]a, b]\), respectively. The corresponding intervals in \( \mathbb{Z} \) are denoted by \([a; b]\), \( ]a; b[\), \([a; b[\), and \( ]a; b]\), i.e., \([a; b] = [a, b] \cap \mathbb{Z} \) and \([0; 0] = \emptyset \).

For a set \( A \), we use \( \#A \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) to denote the number of elements of \( A \), i.e., if \( A \) is finite we have \( \#A \in \mathbb{Z}_{\geq 0} \) and \( \#A = \infty \) otherwise. Given two sets \( A \) and \( B \), we say that \( A \) is smaller (larger) than \( B \) if \( \#A \leq \#B \) (\( \#A \geq \#B \)) holds. A set \( (U_n)_{n \in A} \) of subsets of \( A \), is said to cover \( B \), where \( B \subseteq A \), if \( B \) is a subset of the union of the sets \( (U_n)_{n \in A} \). A cover of \( B \), is a set of subsets of \( B \) that covers \( B \).

Given two sets \( A, B \subseteq \mathbb{R}^n \), we define the set addition by \( A + B := \{ x \in \mathbb{R}^n \mid \exists a \in A, \exists b \in B \; x = a + b \} \). For \( A = \{ a \} \), we slightly abuse notation and use \( a + B = \{ a \} + B \).

The symbols \( \text{cl} \) and \( \text{int} \) denote the closure, respectively, the interior of \( A \). We call a set \( A \subseteq \mathbb{R}^n \) measurable if it is Lebesgue measurable and use \( \mu(A) \) to denote its measure [17].

We follow [18] and use \( f : A \rightrightarrows B \) to denote a set-valued map from \( A \) into \( B \), whereas \( f : A \rightarrow B \) denotes an ordinary map. If \( f \) is set-valued, then \( f \) is strict if for every \( a \in A \) we have \( f(a) \neq \emptyset \). The inverse mapping \( f^{-1} : B \rightrightarrows A \) is defined by \( f^{-1}(b) = \{ a \in A \mid b \in f(a) \} \). The restriction of \( f \) to a subset \( M \subseteq A \) is denoted by \( f|_M \). By convention we set \( f|_{\emptyset} := \emptyset \). The composition of \( f : A \rightrightarrows B \) and \( g : C \rightrightarrows A \), \((f \circ g)(x) = f(g(x)) \) is denoted by \( f \circ g \). We use \( B^A \) to denote the set of all functions \( f : A \rightarrow B \).

The concatenation of two functions \( x : [0; a[ \rightarrow X \) and \( y : [0; b[ \rightarrow X \) with \( a \in \mathbb{N} \) and \( b \in \mathbb{N} \cup \{ \infty \} \) is denoted by \( xy \) which we define by \( xy(t) := x(t) \) for \( t \in [0; a] \) and \( xy(t) := y(t - a) \) for \( t \in [a, a + b[ \).

We use \( \inf \emptyset = \infty \), \( \log_2 \infty = \infty \) and \( 0 \cdot \infty = 0 \).

3. Motivation

We study data rate constrained feedback for discrete-time uncertain control systems described by difference inclusions of the form

\[
\xi(t+1) \in F(\xi(t), \nu(t))
\]

(1)

where \( \xi(t) \in X \) is the state signal and \( \nu(t) \in U \) is the input signal. The sets \( X \) and \( U \) are referred to as state alphabet and input alphabet, respectively. The map \( F : X \times U \rightrightarrows X \) is called the transition function.

We are interested in coder-controllers that force the system \([11]\) to evolve inside a nonempty set \( Q \) of the state alphabet \( X \), i.e., every state signal \( \xi \) of the closed loop illustrated in Fig. [1] with \( \xi(0) \in Q \) satisfies \( \xi(t) \in Q \) for all \( t \in \mathbb{Z}_{\geq 0} \). Specifically, we are interested in the average data rate of such coder-controllers.

Notably, our system description is rather general and, depending on the structure of alphabets \( X \) and \( U \), we can represent a variety of commonly used system models. If we assume \( X \) and \( U \) to be discrete, we can use \([11]\) to represent discrete event systems [19] and digital/embedded systems [20]. Let us consider the following simple example.

[1]If \([11]\) represents a discrete event system, the data rate unit is given in bits/event.
Example 1. Consider a system with state alphabet and input alphabet given by $X := \{0, 1, 2\}$ and $U := \{a, b\}$, respectively. The transition function is illustrated by

![Transition diagram](attachment:image.png)

The set of interest is defined to $Q := \{0, 2\}$. The transitions and states that lead, respectively, are outside $Q$ are indicated by dashed lines. When the system is in state 0 the only valid input is given by $a$. Similarly, if the system is in state 2 the only valid input is given by $b$. If the input $a$ is applied at 0 at time $t$, the system can either be in 0 or 2 at time $t + 1$. Note that the valid control inputs for the states 0 and 2 differ and the controller is required to have exact state information at every point in time. Due to the nondeterministic transition function, it is not possible to determine the current state of the system based on the knowledge of the past states, the past control inputs and the transition function. Therefore, the controller can obtain the state information only through measurement, which implies that at least one bit needs to be transmitted at every time step. □

Current theories \[7, 11, 13, 21\] are unable to explain the minimal data rate of one bit per time step observed in Example 1.

If we allow $X$ and $U$ to be (subsets of) Euclidean spaces, we are able to recover one of the most fundamental system models in control theory, i.e., the class of nonlinear control systems with bounded uncertainties \[6, 22\]. If the system description is given in continuous-time, we can use (1) to represent the sampled-data system \[23\] with sampling time $\tau \in \mathbb{R}_{>0}$ as illustrated in Fig. 2. The disturbance signal $\omega$ is assumed to be bounded $\omega(s) \in W \subseteq \mathbb{R}^p$ for all times $s \in \mathbb{R}_{\geq 0}$. The transition function $F(x, u)$ is defined as the set of states that are reachable by the continuous-time system at time $\tau$ from initial state $x$ under constant input signal $\nu_c(s) = u$ and a bounded disturbance signal $\omega$. If the continuous-time dynamics is linear, the sampled-data system results in a discrete-time system of the form

$$\xi(t + 1) \in A\xi(t) + B\nu(t) + W$$

(2)

where $A$ and $B$ are matrices of appropriate dimension and $W$ is a nonempty set representing the uncertainties.

Example 2. Consider an instance of (2) with $X := \mathbb{R}$, $U := [-1, 1]$ and

$$F(x, u) := \frac{1}{2}x + u + [-3, 3]$$

with the set of constraints given by $Q := [-4, 4]$. □
For Example 2, we establish in Section 6, that the smallest possible data rate of a coder-controller that enforces $Q$ to be invariant is one bit per time step. This is in stark contrast to what is known for data rate constrained feedback control of linear systems with bounded disturbances in the context of asymptotic stabilization (or norm boundedness) [7, Thm. 1], or for data rates of coder-controllers for controlled invariance for deterministic linear systems [11, Thm. 5.1]. Both results suggest that the data rate should be zero, since the eigenvalue of the system matrix in Example 2 is given by $1/2$.

4. Invariance Feedback Entropy

We introduce the notion of invariance feedback entropy and establish some elementary properties.

4.1. The entropy. Formally, we define a system as triple

$$\Sigma := (X, U, F)$$

(3)

where $X$ and $U$ are nonempty sets and $F : X \times U \Rightarrow X$ is assumed to be strict. A trajectory of (3) on $[0; \tau]$ with $\tau \in \mathbb{N} \cup \{\infty\}$ is a pair of sequences $(\xi, \nu)$, consisting of a state signal $\xi : [0; \tau + 1] \rightarrow X$ and an input signal $\nu : [0; \tau] \rightarrow U$, that satisfies (1) for all $t \in [0; \tau]$. We denote the set of all trajectories on $[0; \infty]$ by $B(\Sigma)$.

Throughout the paper, we call a system $(X, U, F)$ finite if $X$ and $U$ are finite. We call $(X, U, F)$ topological if $X$ is a topological space.

We follow [7] and [11, Sec. 6] and define the invariance feedback entropy with the help of covers of $Q$.

Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. A cover $A$ of $Q$ and a function $G : A \rightarrow U$ is called an invariant cover $(A, G)$ of $\Sigma$ and $Q$ if $A$ is finite and for all $A \in A$ we have $F(A, G(A)) \subseteq Q$.

Consider an invariant cover $(A, G)$ of $\Sigma$ and $Q$, fix $\tau \in \mathbb{N}$ and let $S \subseteq A^{[0;\tau]}$ be a set of sequences in $A$. For $\alpha \in S$ and $t \in [0; \tau - 1]$ we define

$$P(\alpha|_{[0;\tau-1]}) := \{A \in A \mid \exists \hat{\alpha} \in S \hat{\alpha}|_{[0;\tau]} = \alpha|_{[0;\tau]} \wedge A = \hat{\alpha}(t + 1)\}.$$

The set $P(\alpha|_{[0;\tau]})$ contains the cover elements $A$ so that the sequence $\alpha|_{[0;\tau]}$ can be extended to a sequence in $S$. For $t = \tau - 1$ we have $\alpha|_{[0;\tau-1]} = \alpha$ and we define for notational convenience the set

$$P(\alpha) := \{A \in A \mid \exists \hat{\alpha} \in S \hat{\alpha} = \alpha(0)\}$$

which is actually independent of $\alpha \in S$ and corresponds to the "initial" cover elements $A$ in $S$, i.e., there exists $\alpha \in S$ with $A = \alpha(0)$. A set $S \subseteq A^{[0;\tau]}$ is called $(\tau, Q)$-spanning in $(A, G)$ if the set $P(\alpha)$ with $\alpha \in S$ covers $Q$ and we have

$$\forall \alpha \in S \forall t \in [0;\tau - 1] \quad F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A' \in P(\alpha|_{[0;\tau]})} A'.$$

(4)

We associate with every $(\tau, Q)$-spanning set $S$ the expansion number $N(S)$, which we define by

$$N(S) := \max_{\alpha \in S} \prod_{t=0}^{\tau-1} #P(\alpha|_{[0;\tau]}).$$
A tight lower bound on the expansion number of any \((\tau, Q)\)-spanning set \(S\) in \((A, G)\) is given by
\[
  r_{\text{inv}}(\tau, Q) := \min \left\{ N(S) \mid S \text{ is } (\tau, Q)\text{-spanning in } (A, G) \right\}.
\]
We define the entropy of an invariant cover \((A, G)\) by
\[
  h(A, G) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q).
\]
As shown in Lemma 1 (stated below), the limit of the sequence in (5) exists so that the entropy of an invariant cover \((A, G)\) is well-defined.

The invariance feedback entropy of \(\Sigma\) and \(Q\) follows by
\[
  h_{\text{inv}} := \inf_{(A, G)} h(A, G)
\]
where we take the infimum over all \((A, G)\) invariant covers of \(\Sigma\) and \(Q\). Let us revisit the examples from the previous section to illustrate the various definitions.

**Example 1** (Continued). First, we determine an invariant cover \((A, G)\) of the system in Example 1 and \(Q\). Since the system is finite, we can set \(A := \{\{x\} \mid x \in Q\}\). Recall that \(Q = \{0, 2\}\) and a suitable function \(G\) is given by \(G(\{0\}) := a\) and \(G(\{2\}) := b\). Suppose that \(S \subseteq A^{[0; r]}\) is \((\tau, Q)\)-spanning with \(\tau \in \mathbb{N}\). Let us check condition (4) for \(t \in [0; \tau - 1]\) and \(\alpha \in S\). If \(\alpha(t) = \{0\}\), we have \(P(\alpha) = \{\{0\}, \{2\}\}\) since \(F(\{0\}, G(\{0\})) = F(0, a) = \{0, 2\}\). If \(\alpha(t) = \{2\}\) the same reasoning leads to \(P(\alpha) = \{\{0\}, \{2\}\}\). Also for \(\alpha \in S\) we have \(P(\alpha) = \{\{0\}, \{2\}\}\) since \(P(\alpha)\) is required to be a cover of \(Q\). It follows that \(S = A^{[0; r]}\) and the expansion number \(N(S) = r_{\text{inv}}(A, G) = 2^r\) so that the entropy of the \((A, G)\) follows to \(h(A, G) = 1\). Since \((A, G)\) is the only invariant cover we obtain \(h_{\text{inv}} = 1\).

**Example 2** (Continued). Let us recall the linear system in Example 2. An invariant cover \((A, G)\) is given by \(A := \{a_0, a_1\}\) with \(a_0 := [-4, 0]\), \(a_1 := [0, 4]\) and \(G(a_0) := 1\), \(G(a_1) := -1\). We use a similar reasoning as in Example 1 to see that for every \(\tau \in \mathbb{N}\) the only \((\tau, Q)\)-spanning set is \(S := A^{[0; r]}\). Since \(#A = 2\), we obtain for the entropy of the invariant cover \(h(A, G) = 1\).

We continue with the subadditivity of \(\log_2 r_{\text{inv}}(\cdot, Q)\).

**Lemma 1.** Consider the system \(\Sigma = (X, U, F)\) and a nonempty set \(Q \subseteq X\). Let \((A, G)\) be an invariant cover of \(\Sigma\) and \(Q\), then the function \(\tau \mapsto \log_2 r_{\text{inv}}(\tau, Q), \mathbb{N} \to \mathbb{R}_{\geq 0}\) is subadditive, i.e., for all \(\tau_1, \tau_2 \in \mathbb{N}\) the inequality
\[
  \log_2 r_{\text{inv}}(\tau_1 + \tau_2, Q) \leq \log_2 r_{\text{inv}}(\tau_1, Q) + \log_2 r_{\text{inv}}(\tau_2, Q)
\]
holds and we have
\[
  \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q) = \inf_{\tau \in \mathbb{N}} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q).
\]

The following lemma might be of independent interest. We use it in the proof of Theorem 1.

**Lemma 2.** Consider an invariant cover \((A, G)\) of \(\Sigma\) and some nonempty set \(Q \subseteq X\). Let \(S\) be a \((\tau, Q)\)-spanning set, then we have \(#S \leq N(S)\).

The proofs of both lemmas are given in the appendix.
4.2. Entropy across related systems. One of the most important properties of entropy of classical dynamical systems is its invariance under any change of coordinates [8, Thm. 1]. In [12] this property has been shown for deterministic control systems in the context of semiconjugation [12, Thm. 3.5]. In the following, we present a result in the context of feedback refinement relations [13], which contains the result on semiconjugation as a special case.

Definition 1. Let $\Sigma_1$ and $\Sigma_2$ be two systems of the form

$$\Sigma_i = (X_i, U_i, F_i) \text{ with } i \in \{1, 2\}. \quad (8)$$

A strict relation $R \subseteq X_1 \times X_2$ is a feedback refinement relation from $\Sigma_1$ to $\Sigma_2$ if there exists a map $r : U_2 \to U_1$ so that the following inclusion holds for all $(x_1, x_2) \in Q$ and $u \in U_2$

$$R(F_1(x_1, r(u))) \subseteq F_2(x_2, u). \quad (9)$$

Theorem 1. Consider two systems $\Sigma_i$, $i \in \{1, 2\}$ of the form (8). Let $Q_1$ and $Q_2$ be two nonempty subsets of $X_1$ and $X_2$, respectively. Suppose that $R$ is a feedback refinement relation from $\Sigma_1$ to $\Sigma_2$ and $Q_1 = R^{-1}(Q_2)$. Then

$$h_{1, \text{inv}} \leq h_{2, \text{inv}} \quad (10)$$

holds, where $h_{i, \text{inv}}$ is the invariance feedback entropy of $\Sigma_i$ and $Q_i$.

Proof. If $h_{2, \text{inv}} = \infty$, the inequality holds and subsequently we consider the case $h_{2, \text{inv}} < \infty$. Then we pick an invariant cover $(A_2, G_2)$ of $\Sigma_2$ and $Q_2$ so that $h(A_2, G_2) < \infty$. We define the set $A_1 := \{A_1 \subseteq Q_1 \mid \exists A_2 \in A_2 \ R^{-1}(A_2) = A_1\}$ and the map $G_1 : A_1 \to U_1$ by $G_1(R^{-1}(A_2)) := r(G_2(A_2))$, where $r : U_2 \to U_1$ is the map associated with the feedback refinement relation in Def. 1. Let us show that $(A_1, G_1)$ is an invariant cover of $\Sigma_1$ and $Q_1$. Clearly $A_1$ is finite since $A_2$ is finite. Moreover, $Q_1 = R^{-1}(Q_2) = R^{-1}(\cup A_2 \in A_2 A_2) = \cup A_2 \in A_2 R^{-1}(A_2) = \cup A_1 \in A_1 A_1$ shows that $A_1$ is a cover of $Q_1$. Let $A_1 \in A_1$. Since there exists $A_2 \in A_2$ so that $A_1 = R^{-1}(A_2)$, we use (9) to derive $R(F_1(x_1, G_1(A_1))) \subseteq F_2(x_2, G_2(A_2))$ since $G_1(A_1) = r(G_2(A_2))$, which shows that $F_1(x_1, G_1(A_1)) \subseteq R^{-1}(F_2(x_2, G_2(A_2))) \subseteq R^{-1}(Q_2) = Q_1$ and we see that $(A_1, G_1)$ is an invariant cover of $\Sigma_1$ and $Q_1$.

Let $S_2$ be a $(\tau, Q)$-spanning set in $(A_2, G_2)$ with $N(S_2) = r_{2, \text{inv}}(\tau, Q)$. We define the set $S_1 \subseteq A_{1[0;\tau]}$ by $\alpha_1 \in S_1$ iff there exists $\alpha_2 \in S_2$ so that $\alpha_1(t) = R^{-1}(\alpha_2(t))$ holds for all $t \in [0; \tau]$. Then we have $P(\alpha_1|_{[0;\tau]}) = P(\alpha_2|_{[0;\tau]})$ for all $t \in [0; \tau]$ and $N(S_1) = N(S_2)$ holds. Let us show that $S_1$ is $(\tau, Q)$-spanning in $(A_1, G_1)$. To this end, let $\alpha_1 \in S_1$. Then there exists $\alpha_2 \in S_2$ so that $\alpha_1(t) = R^{-1}(\alpha_2(t))$ holds for all $t \in [0; \tau]$. We fix $t \in [0; \tau]$ and set $u = G_2(\alpha_2(t))$ and by definition of $G_1$ we have $G_1(\alpha_1(t)) = r(G_2(\alpha_2(t)))$. From (9) it follows $F_1(\alpha_1(t), r(u)) \subseteq R^{-1}(F_2(\alpha_2(t), u))$ and we derive

$$F_1(\alpha_1(t), u) \subseteq R^{-1}(F_2(\alpha_2(t), u))$$

$$= (\cup A'_{1[0;\tau]} P(\alpha_2|_{[0;\tau]}))^{-1}(A'_{2[0;\tau]})$$

$$= \cup A_{1[0;\tau]} P(\alpha_1|_{[0;\tau]}) A'_{1[0;\tau]}$$

which shows that $S_1$ is $(\tau, Q)$-spanning in $(A_1, G_1)$. It follows that $r_{1, \text{inv}}(\tau, Q) \leq r_{2, \text{inv}}(\tau, Q)$. Since this inequality holds for every $\tau \in \mathbb{N}$, we get $h(A_1, G_1) \leq h(A_2, G_2)$ and the assertion follows. \qed
4.3. Conditions for finiteness. We analyze two particular instances of systems – finite systems and topological systems – and provide conditions ensuring that the invariance entropy is finite. The results are based on the following lemma.

Lemma 3. Consider a system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. There exists an invariant cover $(A, G)$ of $\Sigma$ and $Q$ iff $h_{inv} < \infty$.

Proof. It follows immediately from (5) that $h_{inv} < \infty$ implies the existence of an invariant cover of $\Sigma$ and $Q$. For the reverse direction, we assume that $(A, G)$ is an invariant cover of $\Sigma$ and $Q$. We fix $\tau \in \mathbb{N}$ and define $S := \{\alpha \in A^{[0, \tau]} | \alpha(0) \in A \land \forall t \in [0, \tau - 1] \alpha(t + 1) \cap F(\alpha(t), G(\alpha(t))) \neq \emptyset\}$. It is easy to verify that $S$ is $(\tau, Q)$-spanning and $N(S) \leq (\#A)\tau$. An upper bound on $h_{inv}$ follows by $\log_2 \#A$. □

If $\Sigma$ is finite, it is rather straightforward to show that the controlled invariance of $Q$ w.r.t. $\Sigma$ is necessary and sufficient for $h_{inv}$ to be finite. Let us recall the notion of controlled invariance [5].

We call $Q \subseteq X$ controlled invariant with respect to a system $\Sigma = (X, U, F)$, if for all $x \in Q$ there exists $u \in U$ so that $F(x, u) \subseteq Q$.

Theorem 2. Consider a finite system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Then $h_{inv} < \infty$ if and only if $Q$ is controlled invariant.

Proof. Let $h_{inv}$ be finite. Then there exists an invariant cover $(A, G)$ so that $h(A, G) < \infty$. Hence, for every $x \in Q$ we can pick an $A \in A$ with $x \in A$, so that $F(x, G(A)) \subseteq F(A, G(A)) \subseteq Q$. Hence $Q$ is controlled invariant w.r.t. $\Sigma$.

Assume $Q$ is controlled invariant w.r.t. $\Sigma$. For $x \in Q$, let $u_x \in U$ be such that $F(x, u_x) \subseteq Q$. It is easy to check that $(A, G)$ with $A := \{x \mid x \in Q\}$ and $G(\{x\}) := u_x$ is an invariant cover of $\Sigma$ and $Q$, so that the assertion follows from Lemma [3]. □

In general controlled invariance of $Q$ is not sufficient to guarantee finiteness of the invariance feedback entropy as shown in the next example.

Example 3. Consider $\Sigma = (\mathbb{R}, [-1, 1], F)$ with the dynamics given by $F(x, u) := x + u + [-1, 1]$. Let $Q := [-1, 1]$, then for every $x \in Q$ we can pick $u = -x$ so that $F(x, u) = [-1, 1] \subseteq Q$, which shows that $Q$ is controlled invariant. Now suppose that $h_{inv}$ is finite. Then according to Lemma [3] there exists an invariant cover $(A, G)$ of $\Sigma$ and $Q$. Since $A$ is required to be finite, there exists $A \in A$ with an infinite number of elements and therefore we can pick two different states in $A$, i.e., $x, x' \in A$ with $x \neq x'$. However, there does not exist a single $u \in U$ so that $F(x, u) \subseteq Q$ and $F(x', u) \subseteq Q$. Hence, $(A, G)$ cannot be an invariant cover, which implies $h_{inv} = \infty$. □

In the subsequent theorem we present some conditions for topological systems, which imply the finiteness of the invariance entropy. With this conditions, we follow closely the assumptions based on continuity and strong invariance used in [1], [2] to ensure finiteness of the invariance entropy for deterministic systems. We use the following notion of continuity of set-valued maps [24].

Let $A$ and $B$ be topological spaces and $f : A \Rightarrow B$. We say that $f$ is upper semicontinuous, if for every $a \in A$ and every open set $V \subseteq B$ containing $f(a)$ there exists an open set $U \subseteq A$ with $a \in U$ so that $f(U) \subseteq V$.

Theorem 3. Consider a topological system $\Sigma = (X, U, F)$ and a nonempty compact subset $Q$ of $X$. If $F(\cdot, u)$ is upper semicontinuous for every $u \in U$ and $Q$ is strongly
controlled invariant, i.e., for all \( x \in Q \) there exists \( u \in U \) so that \( F(x,u) \subseteq \text{int} \ Q \), then \( h_{\text{inv}} < \infty \).

**Proof.** For each \( x \in Q \), we pick an input \( u_x \in U \) so that \( F(x,u_x) \subseteq \text{int} \ Q \). Since \( F(\cdot, u_x) \) is upper semicontinuous and \( \text{int} \ Q \) is open, there exists an open subset \( A_x \) of \( X \), so that \( x \in A_x \) and \( F(A_x, u_x) \subseteq \text{int} \ Q \). Hence, the set \( \{ A_x \mid x \in Q \} \) of open subsets of \( X \) covers \( Q \). Since \( Q \) is a compact subset of \( X \), there exists a finite set \( \{ A_{x_1}, \ldots, A_{x_m} \} \) so that \( Q \subseteq \bigcup_{i \in [1,m]} A_{x_i} \) [25, Ch. 2.6]. Let \( A := \{ A_{x_1} \cap \ldots \cap A_{x_m} \cap Q \} \) and define for every \( i \in [1;m] \) the function \( G(A_{x_i}) := u_{x_i} \). Then \((A,G)\) is an invariant cover of \( \Sigma \) and \( Q \), and the assertion follows from Lemma 3. \( \square \)

**Example 3 (Continued).** Let \( \varepsilon > 0 \), consider \( \Sigma \) from Example 3 with the modified input set \( U_\varepsilon := [-1 - \varepsilon, 1 + \varepsilon] \). Let \( Q_\varepsilon := [-1 - \varepsilon, 1 + \varepsilon] \) then we see that \( Q_\varepsilon \) is strongly controlled invariant. We construct an invariant cover for \( \Sigma \) and \( Q_\varepsilon \) as follows. We define \( n \) as the smallest integer larger than \( \frac{1}{2\varepsilon} \) and introduce \( \{ x_{-n}, \ldots, x_0, \ldots, x_n \} \) with \( x_i := 2i\varepsilon \) and set \( A_i := (x_i + [-\varepsilon, \varepsilon]) \cap Q_\varepsilon \). For each \( i \in [-n;n] \) we define \( G(A_i) := -x_i \) so that \( F(A_i, G(A_i)) = Q_\varepsilon \). By definition of \( n \) we have \( x_{-n} \leq -1 \) and \( 1 \leq x_n \) and we see that \((A,G)\) with \( A := \{ A_i \mid i \in [-n;n] \} \) is an invariant cover of \( \Sigma \) and \( Q_\varepsilon \). Hence, it follows from Lemma 3 that \( h_{\text{inv}} \) is finite. \( \square \)

**4.4. Deterministic systems.** For deterministic systems we recover the notion of invariance feedback entropy in \([7,12]\).

Let us consider the map \( f : X \times U \to X \) representing a deterministic system

\[ \xi(t+1) = f(\xi(t), \nu(t)). \] (11)

We can interpret (11) as special instance of (3), where \( F \) is given by \( F(x,u) := \{ f(x,u) \} \) for all \( x \in X \) and \( u \in U \) and the notions of a trajectory of (3) extend to (11) in the obvious way. Given an input \( u \in U \), we introduce \( f_u : X \to X \) by \( f_u(x) := f(x,u) \) and extend this notation to sequences \( \nu \in U^{[0,t]} \), \( t \in \mathbb{N} \) by

\[ f_u(x) := f_{\nu(0)}(x). \]

We follow [12] to define the entropy of (11). Consider a nonempty set \( Q \subseteq X \) and fix \( \tau \in \mathbb{N} \). A set \( \mathcal{S} \subseteq U^{[0,\tau]} \) is called \((\tau, Q)\)-spanning for \( f \) and \( Q \), if for every \( x \in Q \) there exists \( \nu \in \mathcal{S} \) so that the associated trajectory \((\xi,\nu)\) on \([0,\tau]\) of (11) with \( \xi(0) = x \) satisfies \( \xi([0;\tau]) \subseteq Q \). We use \( r_{\text{det}}(\tau, Q) \) to denote the number of elements of the smallest \((\tau, Q)\)-spanning set

\[ r_{\text{det}}(\tau, Q) := \inf \{ \# \mathcal{S} \mid \mathcal{S} \text{ is } (\tau, Q)\text{-spanning} \}. \] (12)

The (deterministic) invariance entropy of \((X, U, f)\) and \( Q \) is defined by

\[ h_{\text{det}} := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{det}}(\tau, Q). \] (13)

Again the function \( \tau \mapsto \frac{1}{\tau} \log_2 r_{\text{det}}(\tau, Q) \) is subadditive [12, Prop. 2.2] which ensures that the limit in (13) exists.

We have the following theorem.

**Theorem 4.** Consider the system \( \Sigma = (X, U, F) \) and a nonempty set \( Q \subseteq X \). Suppose \( F \) satisfy \( F(x,u) = \{ f(x,u) \} \) for all \( x \in X \), \( u \in U \) for some \( f : X \times U \to X \). Then the invariance feedback entropy of \( \Sigma \) and \( Q \) equals the deterministic invariance entropy of \((X, U, f)\) and \( Q \), i.e.,

\[ h_{\text{inv}} = h_{\text{det}}. \] (14)
Proof. We begin with the inequality $h_{\det} \geq h_{\text{inv}}$. If $h_{\det} = \infty$ the inequality trivially holds and subsequently we assume that $h_{\det}$ is finite. We fix $\varepsilon > 0$ and pick $\tau \in \mathbb{N}$ so that $\frac{1}{\tau} \log_2 r_{\text{det}}(\tau, Q) \leq h_{\det} + \varepsilon$. We chose a $(\tau, Q)$-spanning set $S_{\det}$ for $f$ and $Q$ with $\#S_{\det} = r_{\text{det}}(\tau, Q)$. For every $\nu \in S_{\det}$ we define the sets

$$A_0(\nu) := Q \cap \bigcap_{t=0}^{\tau-1} f^{-1}_{[\nu, \alpha]}(Q)$$

and for $t \in [0; \tau - 1]$ the sets $A_{t+1}(\nu) := f(A_t(\nu), \nu(t))$. The minimality of $S_{\det}$ implies that $A_0(\nu) \neq \emptyset$ and $A_0(\nu) \neq A_0(\nu')$ for all $\nu, \nu' \in S_{\det}$. Let $A$ be the set of all sets $A_t(\nu)$ for $t \in [0; \tau]$ and $\nu \in S_{\det}$ and set $G(A(\nu)) := \nu(t)$. By definition of $A_t(\nu)$, it is easy to see that $f(A_t(\nu), G(A(\nu))) \subseteq Q$ for all $t \in [0; \tau]$ and $\nu \in S_{\det}$. Moreover, since $S_{\det}$ is $(\tau, Q)$-spanning, for every $x \in Q$ there is $\nu \in S_{\det}$ so that for all $t \in [0; \tau]$ we have $f_{[\nu, \alpha]}(x) \in Q$ which implies $x \in A_0(\nu)$ and we see that $\{A_0(\nu) \mid \nu \in S_{\det}\}$ covers $Q$. It follows that $(A, G)$ is an invariant cover of $(X, U, F)$ and $Q$. For every $\nu \in S_{\det}$ we define the function $\alpha_\nu : [0; \tau] \to A$ by $\alpha_\nu(t) := A_t(\nu)$ and introduce $S_{\text{inv}} := \{\alpha_\nu \mid \nu \in S_{\det}\}$. Since $\alpha_\nu(0) = A_0(\nu)$, we see that $P(\alpha_\nu) = \{A_0(\nu) \mid \nu \in S_{\det}\}$, which shows that $P(\alpha_\nu)$ covers $Q$. Also for every $t \in [0; \tau - 1]$ and $\alpha_\nu \in S_{\text{inv}}$ we have $f(\alpha_\nu(t), G(\alpha_\nu(t))) = f(\alpha_\nu(t), \nu(t)) = \alpha_\nu(t + 1)$ so that $S_{\text{inv}}$ satisfies (3). Therefore, $S_{\text{inv}}$ is $(\tau, Q)$-spanning in $(A, G)$. Moreover, as $\nu \neq \nu'$ implies $\alpha_\nu(0) \neq \alpha_{\nu'}(0)$, we have $\#P(\alpha_\nu) = 1$ for all $\nu \in S_{\text{inv}}$ and $t \in [0; \tau - 1]$. Also $\#P(\alpha_\nu) = \#S_{\det}$, so that $r_{\text{inv}}(\tau, Q) \leq N(S_{\text{inv}}) = \#S_{\det} = r_{\text{det}}(\tau, Q)$ follows. Due to Lemma 1 we have $\log_2 r_{\text{inv}}(n\tau, Q) \leq n \log_2 r_{\text{inv}}(\tau, Q)$ and we see that $\frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q)$ (and therefore $\frac{1}{\tau} r_{\text{inv}}(\tau, Q)$) provides an upper bound for $h(A, Q)$ so that we obtain $h_{\text{inv}} \leq h(A, Q) \leq h_{\det} + \varepsilon$. Since this holds for any $\varepsilon > 0$ we obtain the desired inequality. We conclude with the inequality $h_{\det} \leq h_{\text{inv}}$. If $h_{\text{inv}} = \infty$ the inequality trivially holds and subsequently we assume $h_{\text{inv}} < \infty$. We fix $\varepsilon > 0$ and pick an invariant cover $(A, G)$ of $\Sigma$ and $Q$ so that $h(A, G) \leq h_{\text{inv}} + \varepsilon$. We fix $\tau \in \mathbb{N}$ and pick a $(\tau, Q)$-spanning set $S_{\text{inv}}$ in $(A, G)$ so that $N(S_{\text{inv}}) = r_{\text{inv}}(\tau, Q)$. We define for every $\alpha \in S_{\text{inv}}$ the input sequence $\nu_\alpha : [0; \tau] \to U$ by $\nu_\alpha(t) := G(\alpha(t))$ and introduce the set $S_{\det} := \{\nu_\alpha \mid \alpha \in S_{\text{inv}}\}$. For $x \in Q$ we iteratively construct $\alpha \in A^{|0; \tau]}$ and $\nu \in U^{|0; \tau]}$ as follows: for $t = 0$ we pick $\alpha_0 \in S_{\text{inv}}$ so that $x = \alpha_0(0)$ and set $\nu(0) := G(\alpha_0(0))$. For $t \in [0; \tau - 1]$ we pick $\alpha_{t+1} \in S_{\text{inv}}$ so that $\alpha_{t+1}[0; t] = \alpha_t$ and $f_{[\nu, \alpha]}(x) \in \alpha_{t+1}(t + 1)$ and set $\nu(t + 1) := G(\alpha_{t+1}(t + 1))$. Since $(A, G)$ is an invariant cover of $(X, U, F)$ and $Q$, it is easy to show that $f_{[\nu, \alpha]}(x) \in Q$ holds for all $t \in [0; \tau]$, which implies that $S_{\det}$ is $(\tau, Q)$-spanning for $f$ and $Q$ and we obtain $r_{\text{det}}(\tau, Q) \leq \#S_{\det} \leq \#S_{\text{inv}} \leq N(S_{\text{inv}}) = r_{\text{inv}}(\tau, Q)$, where the inequality $\#S_{\text{inv}} \leq N(S_{\text{inv}})$ follows from Lemma 2. Since this holds for any $\tau \in \mathbb{N}$, we obtain the inequality $\varepsilon + h_{\text{inv}} \geq h(A, G) \geq h_{\det}$ for arbitrary $\varepsilon > 0$ which shows $h_{\text{inv}} \geq h_{\det}$.

4.5. Invariant covers with closed elements. We conclude this section with a result on the topological structure of the cover elements for topological systems with lower semicontinuous transition functions and closed sets $Q$. The result is used in Theorem 7 but might be of interest on its own.

Let $A$ and $B$ be topological spaces and $f : A \Rightarrow B$. We say that $f$ is lower semicontinuous if $f^{-1}(V)$ is open whenever $V \subseteq B$ is open.
**Theorem 5.** Consider a topological system \( \Sigma = (X, U, F) \) and a nonempty set \( Q \subseteq X \). Suppose that \( F(\cdot, u) \) is lower semicontinuous for every \( u \in U \) and \( Q \) is closed. Let \( (A, G) \) be an invariant cover of \( (X, U, F) \) and \( Q \). Consider

\[
C := \{ \text{cl} A \subseteq X \mid A \in A \} \quad \text{and} \quad H(\text{cl} A) := G(A).
\]  
(15)

Then \( (C, H) \) is an invariant cover of \( (X, U, F) \) and \( Q \) and

\[
h(C, H) \leq h(A, G).
\]  
(16)

In the proof of the theorem, we use the following lemma.

**Lemma 4.** Let \( X \) be a topological space and \( f : X \to X \). If \( f \) is lower semicontinuous then \( f(\text{cl} \Omega) \subseteq \text{cl} f(\Omega) \) holds for every nonempty subset \( \Omega \subseteq X \).

**Proof.** For the sake of contradiction, suppose there exists \( x \in \text{cl} \Omega, \ y \in f(x) \) and \( y \notin \text{cl} f(\Omega) \). Then there exists an open set \( V \) so that \( y \in V \) and \( V \cap f(\Omega) = \emptyset \). As \( f \) is lower semicontinuous it follows that \( U := f^{-1}(V) \) is open and from \( V \cap f(\Omega) = \emptyset \) follows that \( U \) is disjoint from \( \Omega \). Hence, we reach a contradiction since \( x \in U \cap \text{cl} \Omega \). Q.E.D.

**Proof of Theorem 5.** Let us first show that \( (C, H) \) is an invariant cover of \( (X, U, F) \) and \( Q \). Let \( C \subseteq \Omega \), then there exists \( A \in A \) so that \( C = \text{cl} A \). As \( Q \) is closed and \( A \subseteq Q \) we have \( \text{cl} A \subseteq Q \) so that \( C \subseteq Q \). Moreover, \( Q \subseteq \{ \text{cl} A \mid A \in A \} = \{ C \mid C \in C \} \) and we see that \( C \) is a finite cover of \( Q \). Again let \( C \in C \) and \( A \in A \) with \( C = \text{cl} A \). Let \( u = H(C) = G(A) \), then it follows from the lower semicontinuity of \( F \) that \( F(\text{cl} A,u) \subseteq \text{cl} F(A,u) \) (see Lemma 4). Hence, \( F(C,H(C)) = F(\text{cl} A,u) \subseteq \text{cl} F(A,u) \subseteq Q \). We conclude that \( (C,H) \) is an invariant cover of \( (X, U, F) \) and \( Q \).

Let \( S_a \) be a \((\tau, Q)\)-spanning set in \((A,G)\) and let \( S_c \subseteq C^{[0;\tau]} \) be given by \( \alpha \in S_c \) iff there exists \( \alpha' \in S_a \) so that \( \alpha(t) = \text{cl} \alpha'(t) \) for all \( t \in [0;\tau] \). Let us first point out that \( P(\text{cl} \alpha_{[0;\tau]}) = P(\alpha_{[0;\tau]}) \) holds for all \( t \in [0;\tau] \) where \( P(\text{cl} \alpha_{[0;\tau]}) \) and \( P(\alpha_{[0;\tau]}) \) is defined with respect to \( S_c \) and \( S_a \), respectively. Hence \( N(S_c) = N(S_a) \). We continue to show that \( S_c \) is \((\tau, Q)\)-spanning in \((C,H)\). It is straightforward to see that \( \{ \alpha(0) \mid \alpha \in S_c \} \) is a cover of \( Q \) since \( \{ \alpha(0) \mid \alpha \in S_a \} \) is a cover of \( Q \). For \( \alpha \in S_c \) let \( \alpha(t) = \text{cl} \alpha(t) \) for all \( t \in [0;\tau] \). We fix \( t \in [0;\tau] \) and set \( C = \alpha(t), A = \alpha(t), u = H(\alpha(t)) = G(\alpha(t)) \) and derive

\[
F(C,u) \subseteq \text{cl} F(A,u) \subseteq \text{cl} \left( \bigcup_{A' \in P(\alpha_{[0;\tau]})} A' \right) \\
= \text{cl} \left( \bigcup_{A' \in P(\alpha_{[0;\tau]})} \text{cl} A' \right)
\]

which shows that \( S_c \) is \((\tau, Q)\)-spanning in \((C,H)\). Given the definition of the entropy of an invariant cover the assertion follows.

**5. Data-Rate-Limited Feedback**

We present the data rate theorem associated with the invariance feedback entropy of uncertain control systems. It shows that the invariance feedback entropy is a tight lower bound of the data rate of any coder-controller scheme that renders the set of interest invariant.

We introduce a history-dependent definition of data rates of coder-controllers with which we extend previously used time-invariant \[1\] and time-varying \[2, 11\] notions. We interpret the history-dependent definition of data rate as a nonstochastic variant of the notion of data rate used e.g. in \[26, \text{Def. 4.1}\] for noisy linear systems, defined as the average of the expected length of the transmitted symbols in the closed loop. We
motivate the particular notion of data rate by two examples; one which illustrates that
the time-varying definition \[7\] results in too large data rates and one which shows that
the notion of data rate based on the framework of nonstochastic information theory,
used in \[27, 28\] for estimation \[28\] and control \[27\] of linear systems, leads to too small
data rates.

5.1. The coder-controller. We assume that a coder for the system \((3)\) is located at
the sensor side (see Fig. 1), which at every time step, encodes the current state of the
system using the finite coding alphabet \(S\). It transmits a symbol \(s_t \in S\) via the discrete
noiseless channel to the controller. The transmitted symbol \(s_t \in S\) might depend on all
past states and is determined by the coder function

\[
\gamma : \bigcup_{t \in \mathbb{Z}_{\geq 0}} X^{[0;t]} \rightarrow S.
\]

At time \(t \in \mathbb{Z}_{\geq 0}\), the controller received \(t + 1\) symbols \(s_0 \ldots s_t\), which are used to
determine the control input given by the controller function

\[
\delta : \bigcup_{t \in \mathbb{Z}_{\geq 0}} S^{[0;t]} \rightarrow U.
\]

A coder-controller for \((3)\) is a triple \(H := (S, \gamma, \delta)\), where \(S\) is a coding alphabet and \(\gamma\) and \(\delta\) is a compatible coder function and controller function, respectively.

Given a coder-controller \((S, \gamma, \delta)\) for \((3)\) and \(\xi \in X^{[0;t]}\) with \(t \in \mathbb{Z}_{\geq 0}\), let us use the mapping

\[
\Gamma_t : X^{[0;t]} \rightarrow S^{[0;t]}
\]

to denote the sequence \(\zeta = \Gamma_t(\xi)\) of coder symbols generated by \(\xi\), i.e., \(\zeta(t') = \gamma(\xi|[0;t'])\)
holds for all \(t' \in [0;t]\). Subsequently, for \(\zeta \in S^{[0;t]}\) with \(t \in \mathbb{N}\), we use

\[
Z(\zeta) := \{s \in S \mid \exists (\xi, \nu) \in B(2) \quad \zeta s = \Gamma(\xi|[0;t]) \land \nu|[0;t] = \delta \circ \zeta\}
\]\n
(17)

to denote the possible successor coder symbols \(s\) of the symbol sequence \(\zeta\) in the closed
loop illustrated in Fig. 1. For notational convenience, let us use the convention \(Z(\emptyset) := S\), so that \(Z(\zeta|[0;t0]) = S\) for any sequence \(\zeta\) in \(S\). For \(t \in \mathbb{N} \cup \{\infty\}\), we introduce the set

\[
Z_t := \{\zeta \in S^{[0;t]} \mid \zeta(0) \in \gamma(X) \land \forall t \in [0;t] \quad \zeta(t) \in Z(\zeta|[0;t])\}
\]

and define the transmission data rate of a coder-controller \(H\) by

\[
R(H) := \lim_{t \rightarrow \infty} \sup_{\zeta \in Z_t} \frac{1}{t} \sum_{t=0}^{t-1} \log_2 \#Z(\zeta|[0;t])
\]\n
(18)
as the asymptotic average numbers of symbols in \(Z(\zeta)\) considering the worst-case of
possible symbol sequences \(\zeta \in Z_t\).

A coder-controller \(H = (S, \gamma, \delta)\) for \((3)\) is called \(Q\)-admissible where \(Q\) is a nonempty
subset of \(X\), if for every trajectory \((\xi, \nu)\) on \([0; \infty[\) of \((3)\) that satisfies

\[
\xi(0) \in Q \quad \text{and} \quad \forall t \in \mathbb{Z}_{\geq 0} \quad \nu(t) = \delta(\Gamma_t(\xi|[0;t]))
\]\n
(19)

we have \(\xi(Z_{\geq 0}) \subseteq Q\). Let us use \(B_Q(H)\) to denote the set of all trajectories \((\xi, \nu)\) on
\([0; \infty[\) of \((3)\) that satisfy (19).
5.1.1. Time-varying data rate definition. We follow \([7]\) and introduce a time-varying notion of data rate for a coder-controller \(H = (S, \gamma, \delta)\) for \((3)\). Let \((S_t)_{t \geq 0}\) be the sequence in \(S\) that for each \(t \in \mathbb{Z}_{\geq 0}\) contains the smallest number of symbols so that \(\gamma(\xi) \in S_t\) holds for all \(\xi \in X^{[0:t]}\). Then the time-varying data rate of \(H\) follows by

\[
R_{tv}(H) := \lim_{\tau \to \infty} \inf_{\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#S_t.
\]

In the following we use an example to show that there exists a \(Q\)-admissible coder-controller \(H\), which satisfies \(R(H) < R_{tv}(\bar{H})\) for any \(Q\)-admissible coder-controller \(\bar{H}\). Note that this inequality is purely a nondeterministic phenomenon: if the control system is deterministic, it follows from the deterministic and the nondeterministic data rate theorem (\([7,\) Thm. 1] and Theorem 6 below) and the equivalence \(h_{det} = h_{inv}\) (Theorem \([4]\)) that the different notions of data rates coincide in the sense that \(\inf_H R(H) = \inf_{H_{tv}} R_{tv}(H)\) (at least if the strong invariance condition in \([7,\) Thm. 1] holds).

**Example 4.** Consider an instance of \((3)\) with \(U := \{a, b\}\), \(X := \{0, 1, 2, 3\}\) and \(F\) is illustrated by

![Diagram](image)

Let \(Q := \{0, 1, 2\}\). The transitions that lead outside \(Q\) and the states that are outside \(Q\) are marked by dashed lines. Consider the coder-controller \(H = (S, \gamma, \delta)\) with \(S := X\) and \(\gamma\) and \(\delta\) are given for \(\xi \in X^{[0:t]}\), \(t \in \mathbb{Z}_{\geq 0}\), by \(\gamma(t) := \xi(t)\) and \(\delta(\xi) := a\) if \(\xi(t) \in \{0, 1, 3\}\) and \(\delta(\xi) := b\) if \(\xi(t) = 2\). We compute the number of possible successor symbols \(Z(\xi)\) for \(\xi \in X^{[0:t]}\), \(t \in \mathbb{Z}_{\geq 0}\), by \(#Z(\xi) = 1\) if \(\xi(t) \in \{0, 2, 3\}\) and \(#Z(\xi) = 2\) if \(\xi(t) = 1\). It is easy to verify that \(H\) is \(Q\)-admissible. Since the state \(\xi(t) = 1\) occurs only every other time step for any element \((\xi, \nu)\) of the closed loop, we compute the data rate to \(R(H) = \frac{1}{2}\). Consider a time-varying \(Q\)-admissible coder-controller \(\bar{H} = (S, \bar{\gamma}, \bar{\delta})\).

Initially, the states \(\{0, 1\}\) and \(\{2\}\) need to be distinguishable at the controller side in order to confine the system to \(Q\) so that \(#S_0 \geq 2\) follows. At time \(t = 1\), the system is possibly again in any of the states \(\{0, 1, 2\}\) (depending on the initial condition) and we have \(#S_1 \geq 2\). By continuing this argument we see that \(#S_t \geq 2\) for all \(t \in \mathbb{Z}_{\geq 0}\) and \(R_{tv}(\bar{H}) \geq 1\) follows. \(\Box\)

5.1.2. Zero-error capacity of uncertain channels. Alternatively to the definition of the data rate of a coder-controller in \((18)\) we could follow \([27, 28]\) and define the data rate of a coder-controller as the zero-error capacity \(C_0\) of an ideal stationary memoryless uncertain channel (SMUC) in the nonstochastic information theory framework presented in \([28,\) Def. 4.1] \(\). The input alphabet of the SMUC equals the output alphabet and is given by \(S\). The channel is ideal and does not introduce any error in the transmission. Hence, the transition function is the identity, i.e., \(T(s) = s\) holds for all \(s \in S\). The input function space \(Z_\infty \subseteq S^{[0:\infty]}\) is the set of all possible symbol sequences that are generated by the closed loop, which represents the total amount of information that needs to be transmitted by the channel. For the ideal SMUC, the zero-error capacity \([28,\) Eq. (25)] for a coder-controller \(H\) results in

\[
C_0(H) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 \#Z_\tau.
\]
We use the following example to demonstrate that the zero-error capacity is too low, i.e., $C_0(H) = 0$ while $R(H) \geq 1$.

**Example 5.** Consider an instance of [3] with $U := \{a, b\}$, $X := \{0, 1, 2, 3\}$ and $F$ is illustrated by

![Diagram](image)

The transitions and states that lead, respectively, are outside the set of interest $Q := \{0, 1, 2\}$ are dashed. Consider the $Q$-admissible coder-controller $H = (S, \gamma, \delta)$ with $S := X$ and $\gamma$ and $\delta$ are given for $\xi \in X^{[0;\tau]}$, $t \in Z_{\geq 0}$ by $\gamma(\xi) := \xi(t)$ and $\delta(\xi) := \begin{cases} a & \text{if } \xi(t) \in \{0, 3\} \\ b & \text{if } \xi(t) = 1 \\ c & \text{if } \xi(t) = 2. \end{cases}$

We pick the trajectory $(\xi, \nu) \in B_Q(H)$ given for $t \in Z_{\geq 0}$ by $\xi(2t) = 0$ and $\xi(2t + 1) = 1$. We obtain $Z(\xi|_{[0;\ell]}) = \{1, 2\}$ if $\xi(t) = 0$ and $Z(\xi|_{[0;\ell]}) = \{0, 2\}$ if $\xi(t) = 1$. Since $\#F(x, u) \leq 2$ for all $x \in X$ and $u \in U$, it is straightforward to see that $\sum_{t=0}^{\tau-1} \log_2 \#Z(\xi|_{[0;\ell]}) = \max_{\xi \in Z_{\tau}} \sum_{t=0}^{\tau-1} \log_2 \#Z(\xi|_{[0;\ell]})$ holds for all $\tau \in N$. Hence, we obtain $R(H) = 1$.

We are going to derive $C_0(H)$. Consider the set $Z_{\tau} \subseteq X^{[0;\tau]}$ and the hypothesis for $\tau \in N$: there exists at most one $\xi \in Z_{\tau}$ with $\xi(\tau - 1) = 1$ and there exists at most one $\xi \in Z_{\tau}$ with $\xi(\tau - 1) = 0$. For $\tau = 1$ we have $Z_1 = X$ and the hypothesis holds. Suppose the hypothesis holds for $\tau \in N$ and let $\xi \in Z_{\tau}$. We have $Z(\xi) = \{0, 2\}$ if $\xi(t) = 1$, $Z(\xi) = \{1, 2\}$ if $\xi(t) = 0$, $Z(\xi) = \{2\}$ if $\xi(t) = 2$ and $Z(\xi) = \{3\}$ if $\xi(t) = 3$, so that the hypothesis holds for $\tau + 1$, which shows that the hypothesis holds for every $\tau \in N$. Therefore, we obtain a bound of the number of elements in $Z_{\tau}$ by $4 + 2(\tau - 1)$ and the zero-error capacity of $H$ follows by $C_0(H) = 0$. 

Example 5 shows that even though, the asymptotic average of the total amount of information that needs to be transmitted (= symbol sequences generated by the closed loop) via the channel is zero, the necessary (and sufficient) data rate to confine the system $\Sigma$ within $Q$ is one. The discrepancy results from the causality constraints that are imposed on the coder-controller structure by the invariance condition, i.e., at each instant in time the controller needs to be able to produce a control input so that all successor states are inside $Q$ see e.g. [20]. Contrary to this observation, the zero-error capacity is an adequate measure for data rate constraints for deterministic linear systems (without disturbances) [27, 28].

### 5.1.3. Periodic coder-controllers

In the proof of the data rate theorem, we work with periodic coder-controllers. Given $\tau \in N$ and a coder-controller $H = (S, \gamma, \delta)$, we say that $H$ is $\tau$-periodic if for all $t \in Z_{\geq 0}$, $\zeta \in S^{[0;\ell]}$ and $\xi \in X^{[0;\ell]}$ we have $\gamma(\xi) = \gamma(\xi|_{[t/\tau;\ell /\tau + t]}), \delta(\zeta) = \delta(\zeta|_{[t/\tau;\ell /\tau + t]}).$
Lemma 5. The transmission data rate of a \( \tau \)-periodic coder-controller \( H = (S, \gamma, \delta) \) for (3) is given by

\[
R(H) = \max_{\zeta \in \mathbb{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0,t]}).
\]

Proof. Let \( L \) denote the right-hand-side of (21). Consider \( T \in \mathbb{N} \), \( \zeta \in \mathbb{Z}_T \) and set \( a := \lceil T/\tau \rceil \) and \( \bar{\tau} := T - a\tau \). We define \( \zeta_i := \zeta|_{[tr,(i+1)r]} \) for \( i \in [0; a[ \) and \( \zeta_a := \zeta|_{[ar:T]} \). Since \( \gamma \) is \( \tau \)-periodic, we see that each \( \zeta_i \) with \( i \in [0; a[ \) is an element of \( \mathbb{Z}_\tau \), and we obtain for \( N_i := \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta_i|_{[0,t]}) \) the bound \( N_i \leq L\tau \) for all \( i \in [0; a[ \). We define \( N_a := \sum_{t=0}^{\bar{\tau}-1} \log_2 \#Z(\zeta_a|_{[0,t]}) \) which is bounded by \( N_a \leq \tau \log_2 \#S \). Note that \( a\tau + \bar{\tau} = T \), so that for \( C := \tau \log_2 \#S \) we have

\[
\frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0,t]}) = \frac{1}{\tau} \left( \sum_{i=0}^{a-1} N_i + N_a \right) \leq \frac{1}{\tau} (aL\tau + L\bar{\tau} + C) = L + \frac{\bar{\tau}}{\tau}.
\]

Since \( C \) is independent of \( T \), the assertion follows. \( \square \)

Lemma 6. For every coder-controller \( H = (S, \delta, \gamma) \) for (3) and \( \varepsilon > 0 \), there exists a \( \tau \)-periodic coder-controller \( \hat{H} = (S, \hat{\delta}, \hat{\gamma}) \) that satisfies

\[
R(\hat{H}) \leq R(H) + \varepsilon.
\]

Proof. For \( \varepsilon > 0 \), we pick \( \tau \in \mathbb{N} \) so that \( \log_2 \#Z_0/\tau \leq \varepsilon/2 \) and

\[
\max_{\zeta \in \mathbb{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0,t]}) \leq R(H) + \varepsilon/2.
\]

We define \( \hat{\gamma} \) and \( \hat{\delta} \) for all \( \xi \in X^{[0:t]}, \zeta \in S^{[0:t]} \) with \( t \in \mathbb{Z}_{\geq 0} \) by

\[
\hat{\gamma}(\xi) := \gamma(\xi|_{[t/r:t/r]}), \quad \hat{\delta}(\zeta) := \delta(\zeta|_{[t/r:t/r]}).
\]

Let \( \hat{Z} \) be defined in (17) w.r.t. \( \hat{\gamma} \). Then we have for all \( \zeta \in S^{[0:t]} \) with \( t \in [0; \tau - 1[ \) the equality \( Z(\zeta) = \hat{Z}(\zeta) \) and for every \( \zeta \in S^{[0:\tau]} \) we have \( \hat{Z}(\zeta) = Z_0 \) which follows from the fact that \( \hat{\gamma} \) is \( \tau \)-periodic. The transmission data rate of \( \hat{H} \) follows by (21) which is bounded by

\[
\max_{\zeta \in \mathbb{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#\hat{Z}(\zeta|_{[0,t]}) + \log_2 \#Z_0 \leq R(H) + \varepsilon.
\]

\( \square \)

5.2. The data rate theorem.

Theorem 6. Consider the system \( \Sigma = (X, U, F) \) and a nonempty set \( Q \subseteq X \). The invariance feedback entropy of \( \Sigma \) and \( Q \) satisfies

\[
h_{inv} = \inf_{H \in \mathcal{H}} R(H)
\]

where \( \mathcal{H} \) is the set of all \( Q \)-admissible coder-controllers for \( \Sigma \).

We use the following two technical lemmas to show the theorem.

Lemma 7. Let \( H = (S, \gamma, \delta) \) be a \( Q \)-admissible \( \tau \)-periodic coder-controller for \( \Sigma = (X, U, F) \). Then there exists an invariant cover \( (\mathcal{A}, G) \) of \( \Sigma \) and \( Q \) and a \( (\tau, Q) \)-spanning set \( S \) in \( (\mathcal{A}, G) \) so that

\[
\frac{1}{\tau} \log_2 N(S) \leq R(H).
\]
Proof. For every \( t \in [0; \tau] \) and every \( \zeta \in Z_{t+1} \) we define \( A(\zeta) := \{ x \in Q \mid \exists (\xi, \nu) \in B_Q(H) \ (\zeta = \Gamma_t(\xi) \land \xi(t) = x) \}, \) \( G(A(\zeta)) := \delta(\zeta) \) and \( \mathcal{A} := \{ A(\zeta) \mid \zeta \in Z_{t+1} \land t \in [0; \tau] \}. \) We show that \((\mathcal{A}, G)\) is an invariant cover of \( \Sigma \) and \( Q.\) Clearly, \( \mathcal{A} \) is finite and every element of \( \mathcal{A} \) is a subset of \( Q.\) Since \( H \) is \( Q\)-admissible, for every \( x \in Q \) there exists \( (\xi, \nu) \in B_Q(H) \) so that \( \xi(0) = x.\) Hence, \( \{ A(s) \mid s \in Z_t \} \) covers \( Q \) and we see that \( \mathcal{A} \) covers \( Q.\) Let \( A \in \mathcal{A} \) and suppose that there exists \( x \in A \) so that \( F(x, G(A)) \not\subseteq Q.\) Since \( A \in A, \) there exists \( t \in [0; \tau], \zeta \in Z_{t+1} \) and \((\xi, \nu) \in B_Q(H) \) so that \( A(\zeta) = \mathcal{A}(\zeta), \zeta = \Gamma_t(\xi) \) and \( x = \xi(t).\) Note that \( \nu \) satisfies \([19] \) so that \( \nu(t) = G(A(\zeta)) \) holds. We fix \( x' \in F(x, G(A)) \) \( \backslash Q \) and pick a trajectory \((\xi', \nu')\) of \( \Sigma \) on \([0; \infty[\) such that \( \xi'(0) = x' \) and \( \nu'(t') = \delta(\Gamma_t((\xi|_{[0:t]}(\xi'))|_{t+t'+1})) \) holds for all \( t' \in Z_{t+1}.\) We define \((\xi, \nu)\) by \( \xi := [\xi|_{[0:t]}(\xi') \land \nu := [\nu|_{[0:t]}(\nu') \), which by construction is a trajectory of \( \Sigma \) on \([0; \infty[\) which satisfies \([19] \) but \( \xi(0; \infty[) \not\subseteq Q.\) This contradicts the \( Q\)-admissibility of \( H \) and we can deduce that \( F(A, G(A)) \subseteq Q \) for all \( A \in \mathcal{A}, \) which shows that \((\mathcal{A}, G)\) is an invariant cover of \( \Sigma \) and \( Q.\)

We are going to construct a \((\tau, Q)\)-spanning set \( S \subseteq A^{0; \tau} \) with the help of \( Z_\tau.\) For each \( \zeta \in Z_\tau \) we define a sequence \( \alpha_\zeta : [0; \tau[ \to A \) by \( \alpha_\zeta(t) := A(\zeta|_{[0:t]}(\zeta)) \) for all \( t \in [0; \tau[ \) and use \( S \) to denote the set of all such sequences \( \{ \alpha_\zeta | \zeta \in Z_\tau \}. \) Note that \( P(\alpha_\zeta) = \{ A(s) \mid s \in Z_t \} \) holds for all \( \alpha_\zeta \in S, \) and we see that \( P(\alpha_\zeta) \) covers \( Q.\) Let us show \([11] \). Let \( \alpha_\zeta \in S, \) \( t \in [0; \tau[ \) so that \( \alpha_\zeta(t) = A(\zeta) \). We define \( \zeta := \zeta|_{[0:t]}(\zeta) \) and fix \( x_0 \in A(\zeta) \) and \( x_1 \in F(x_0, G(A(\zeta))).\) Since \( x_0 \in A(\zeta) \) there exists \((\xi, \nu) \in B_Q(H) \) so that \( \zeta = \Gamma_t(\xi) \) with \( \xi(t) = x_0 \) and we use \([19] \) to see that \( G(A(\zeta)) = \delta(\zeta) \) holds. Therefore, \((\xi, \nu)|_{[0:t]}(\xi') \) can be extended to a trajectory in \((\xi, \nu) \in B_Q(H) \) with \( \xi(t+1) = x_1.\) Let \( s = \gamma(\xi|_{[0:t+1]}(\xi')) \), then we have \( s \in Z(\zeta) \) and \( \zeta_{t+1} = \zeta_{t+1}(\xi|_{[0:t+1]}(\xi')).\) We conclude that \( x_1 \in A(\zeta_{t+1}) \). We repeat this process for \( x_1 \in F(A(\zeta_{t+1}), G(A(\zeta_{t+1}))), \) \( i \in [0; k] \) until \( t + k = \tau - 1 \) at which point we arrive at \( \zeta_{t+k} \in Z_\tau \) and we see that the associated sequence \( \alpha_{\zeta_{t+k}} \) is an element of \( S \) that satisfies \( x_1 \in \alpha_{\zeta_{t+k}}(t+1) \) and \( \alpha_{\zeta_{t+k}}(0; \tau[) = \alpha_{\zeta}(0; \tau[).\) Since such a sequence can be constructed for every \( x_1 \in F(x_0, G(A(\zeta))) \) and \( x_0 \in A(\zeta), \) we see that \([11] \) holds and it follows that \( S \) is \((\tau, Q)\)-spanning in \((\mathcal{A}, G).\)

We claim that \( \#P(\alpha_{\zeta}|_{[0:t]}(\zeta)) \leq \#Z(\zeta|_{[0:t]}(\zeta)) \) for every \( \alpha_{\zeta} \in S \) and \( t \in [0; \tau[.\) Let \( A \in P(\alpha_{\zeta}|_{[0:t]}(\zeta)) \), then there exists \( \alpha_{\zeta} \in S \) such that \( A = \alpha_{\zeta}(t+1) \) and \( \zeta|_{[0:t]}(\zeta) = \zeta|_{[0:t]}(\zeta).\) Hence \( \zeta(t+1) \in Z(\zeta|_{[0:t]}(\zeta)). \) Moreover, for \( A, \tilde{A} \in P(\alpha_{\zeta}|_{[0:t]}(\zeta)) \) with \( A \neq \tilde{A} \) there exists \( \alpha_{\zeta}, \alpha_{\tilde{A}} \in S \) such that \( A = \alpha_{\zeta}(t+1) \) and \( \tilde{A} = \alpha_{\tilde{A}}(t+1) \), which shows that \( \zeta(t+1) \neq \zeta(t+1) \) and \( \zeta(t+1) \in Z(\zeta|_{[0:t]}(\zeta)) \) and we obtain \( \#P(\alpha_{\zeta}|_{[0:t]}(\zeta)) \leq \#Z(\zeta|_{[0:t]}(\zeta)) \) for all \( t \in [0; \tau[ \) and \( \zeta \in Z_t. \) For \( t = \tau - 1 \) we have \( P(\zeta) = \{ A(s) \mid s \in Z_t \}. \) For \( Z(\zeta) \) we have \( Z(\zeta) = \gamma(X), \) since \( H \) is \( \tau\)-periodic and we obtain \( \#P(\zeta) \leq \#Z(\zeta) \) for every \( \zeta \in Z. \) Hence, \( N(S) \leq \max_{\zeta \in Z} \prod_{t=0}^{\tau} \#Z(\zeta|_{[0:t]}(\zeta)) \) follows and we obtain \( \frac{1}{\tau} \log_2 N(S) \leq R(H). \)

In the proof of the following lemma, we use an enumeration of a finite set \( A, \) which is a function \( e : [1; \#A] \to A \) such that \( e(1; \#A) = A. \)

Lemma 8. Consider an invariant cover \((\mathcal{A}, G)\) of \( \Sigma = (X, U, F) \) and some nonempty set \( Q \subseteq X. \) Let \( S \) be a \((\tau, Q)\)-spanning set in \((\mathcal{A}, G).\) Then there exists a \( Q\)-admissible \( \tau\)-periodic coder-controller \( H = (S, \gamma, \delta) \) for \( \Sigma \) so that

\[
\frac{1}{\tau} \log_2 N(S) \geq R(H).
\]
Proof. We define \( S_t := \{ \alpha \in A^{[0; \tau]} \mid \exists z \in S \, \hat{\alpha}|_{[0; \tau]} = \alpha \} \) for \( t \in [0; \tau] \) and observe that \( S_{t-1} = S \) and for every \( \alpha \in S \) we have \( P(\alpha) = S_0 \). For \( \alpha \in S_t \), with \( t \in [0; \tau - 1] \) let \( e(\alpha) \) be an enumeration of \( P(\alpha) \). We slightly abuse the notation, and use \( e(\emptyset) \) to denote an enumeration of \( P(\emptyset) \). We fix \( \gamma \), which we define iteratively. For \( t = 0 \) and \( x \in X \) we set \( \gamma(x) := e(\emptyset)(A) \) if there exists \( A \in S_0 \) with \( x \in A \). If there are several \( A \in S_0 \) that contain \( x \) we simply pick one. If there does not exist any \( A \in S_0 \) with \( x \in A \) we set \( \gamma(x) := 1 \). For \( t \in [0; \tau] \) and \( \xi \in X^{[0; \tau]} \) we define \( \gamma(\xi) := e(\alpha_{[0; \tau]})(\alpha(t)) \) for \( \alpha \in S_t \) that satisfies \( \xi(t) \in \alpha(t) \). If there exists \( \alpha \in S_t \) that satisfies \( \xi(t) \in \alpha(t) \), we set \( \gamma(\xi) := 1 \). We define \( \delta(\xi) := G(\alpha(t)) \), otherwise we set \( \delta(\xi) := u \) for some \( u \in U \). Let us show that the coder-controller is \( Q \)-admissible. We fix \( (\xi, \nu) \in \mathcal{B}_{Q}(H) \) and proceed by induction with the hypothesis parameterized by \( t \in [0; \tau) \) : there exists \( \alpha \in S_t \) so that \( \xi(t) \in \alpha(t) \), \( \gamma(\xi_{[0; t]}) = e(\alpha(\xi_{[0; t]}))(\alpha(t')) \) and \( \nu(t') = G(\alpha(t')) \) for all \( t' \in [0; t] \). For \( t = 0 \), we know that \( S_0 \) covers \( Q \) so that for \( \xi(0) \in Q \) there exists \( A \in S_0 \) with \( x \in A \) and it follows from the definition of \( \gamma \) and \( \delta \) that \( \gamma(\xi(0)) = e(\emptyset)(A) \) for some \( A \in S_0 \) with \( \xi(0) \in A \) and \( \nu(0) = \delta(\gamma(A)) = G(A) \). Now suppose that the induction hypothesis holds for \( t \in [0; \tau - 1] \). Since \( \xi(t) \in \alpha(t) \) and \( \nu(t) = G(\alpha(t)) \) for some \( \alpha \in S_t \), we use \( \emptyset \) to see that there exists \( \alpha \in S_t \) so that \( \xi(0) \in A \) and \( \xi(t + 1) \in \alpha(t + 1) \), so that \( \alpha \) satisfies i) and ii) in the definition of \( \gamma \) and \( \delta \). We have \( \gamma(\xi(0)) = e(\emptyset)(\hat{\alpha}(t + 1)) \) for some \( \hat{\alpha} \in S_{t+1} \) with \( \xi(t + 1) \in \hat{\alpha}(t + 1) \) and \( \hat{\alpha}_{[0; \tau]} = \alpha \). Since \( \hat{\alpha} \) is uniquely determined by the symbol sequence \( \zeta \in S_{[0; \tau]} \) given by \( \zeta(t') = e(\hat{\alpha}|_{[0; t']})(\hat{\alpha}(t')) \) for all \( t' \in [0; t + 1] \), we have \( \nu(t + 1) = \delta(\zeta) = G(\hat{\alpha}(t + 1)) \), which completes the induction. Note that the induction hypothesis implies that \( F(\xi(t), \nu(t)) \subseteq Q \) for all \( t \in [0; \tau] \), since \( \xi(t) \in \alpha(t) \) and \( \nu(t) = G(\alpha(t)) \). We obtain \( \xi([0; \tau]) \subseteq Q \) from the \( \tau \)-periodicity of \( H \) and the \( Q \)-admissibility follows. We derive a bound for \( R(H) \). Since \( H \) is \( \tau \)-periodic, we have for any \( \zeta \in Z \), the equality \( Z(\zeta) = e(\emptyset)(S_0) \) and we see that \( \#Z(\zeta) = \#e(\emptyset)(S_0) = \#P(\alpha) \) for any \( \alpha \in S \). We fix \( \zeta \in Z \) and pick \( \alpha \in S \) so that \( \alpha(t) = e^{-1}(\alpha(t'))(\zeta(t)) \) holds for all \( t \in [0; \tau] \). By definition, the set \( Z(\zeta|_{[0; \tau]}) \) is the co-domain of an enumeration of \( P(\alpha|_{[0; \tau]} \), which shows \( \#Z(\zeta|_{[0; \tau]} = \#P(\alpha|_{[0; \tau]} \). Therefore, we have \( \max_{\zeta \in Z} \prod_{t=0}^{\tau - 1} \#Z(\zeta|_{[0; t]} \leq \max_{\alpha \in S} \prod_{t=0}^{\tau - 1} \#P(\alpha|_{[0; t]} \) and the assertion follows by \( (21) \). \( \square \)

We continue with the proof of Theorem \( (4) \). Let us first prove the inequality \( h_{\text{inv}} \leq \inf_{H \in \mathcal{H}} R(H) \). If the right-hand-side of \( (22) \) equals infinity the inequality trivially holds and subsequently we assume the right-hand-side of \( (22) \) is finite. We fix \( \varepsilon > 0 \) and pick a coder-controller \( H = (S, \gamma, \delta) \) so that \( R(H) \leq \inf_{H \in \mathcal{H}} R(H) + \varepsilon \). According to Lemma \( (6) \) there exists a \( \tau \)-periodic coder-controller \( H = (S, \gamma, \delta) \) so that \( R(H) \leq R(H) + \varepsilon \). It is straightforward to see that for every \( (\xi, \nu) \in \mathcal{B}_{Q}(H) \) and \( \xi_i := \xi|_{[i\tau; (i+1)\tau]} \), \( i \in \mathbb{Z}_{\geq 0} \), there exists \( (\xi, \nu) \in \mathcal{B}_{Q}(H) \), so that \( \xi_i = \xi_i|_{[0; \tau]} \), which shows that \( H \) is \( Q \)-admissible. From Lemma \( (7) \) it
follows that there exists an \((A, G)\) of \(\Sigma\) and \(Q\) and a \((\tau, Q)\)-spanning set in \((A, G)\) so that \(\frac{1}{\tau} \log_2 N(S) \leq R(H)\). We use Lemma 1 to see that \(r_{\text{inv}}(n\tau, Q) \leq nr_{\text{inv}}(\tau, Q)\) so that \(h(A, G) = \lim_{n \to \infty} \frac{1}{n\tau} \log_2 r_{\text{inv}}(n\tau, Q) \leq \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q) \leq \frac{1}{\tau} \log_2 N(S)\). By the choice of \(H\) we obtain \(2\varepsilon + \inf_{H \in \mathcal{H}} R(H) \geq R(H) \geq h_{\text{inv}}\). Since this holds for arbitrary \(\varepsilon > 0\) we arrive at the desired inequality.

We continue with the inequality \(h_{\text{inv}} \geq \inf_{H \in \mathcal{H}} R(H)\). If \(h_{\text{inv}} = \infty\) the inequality trivially holds and subsequently we consider \(h_{\text{inv}} < \infty\). We fix \(\varepsilon > 0\) and pick an invariant cover \((A, G)\) of \(\Sigma\) and \(Q\) so that \(h(A, G) < h_{\text{inv}} + \varepsilon\). We pick \(\tau \in \mathbb{N}\) so that \(\frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q) < h(A, G) + \varepsilon\). Let \(S\) be \((\tau, Q)\)-spanning set that satisfies \(r_{\text{inv}}(\tau, Q) = N(S)\). It follows from Lemma 8 that there exists a \(Q\)-admissible coder-controller \(H\) so that \(\frac{1}{\tau} \log_2 N(S) \geq R(H)\) holds, and hence, we obtain \(2\varepsilon + h_{\text{inv}} \geq R(H)\). This inequality holds for any \(\varepsilon > 0\), which implies that \(h_{\text{inv}} \geq \inf_{H \in \mathcal{H}} R(H)\). \(\square\)

6. Uncertain Linear Control Systems

We derive a lower bound of the invariance feedback entropy of uncertain linear control systems (2) and compact sets \(Q\). In this setting, we also derive a lower bound of the data rate of any static or memoryless coder-controller. We show that for certain systems the lower bounds are tight.

6.1. Universal lower bound.

**Theorem 7.** Consider the matrices \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) and two nonempty sets \(W, Q \subseteq \mathbb{R}^n\) with \(W \subseteq Q\) and suppose that \(W\) is measurable and \(Q\) is compact. Let (3) be given by \(X = \mathbb{R}^n\), \(U \subseteq \mathbb{R}^m\) with \(U \neq \emptyset\) and \(F\) according to

\[
\forall x \in X \forall u \in U \quad F(x, u) = Ax + Bu + W. \tag{23}
\]

Then, the invariance feedback entropy of (3) and \(Q\) satisfies

\[
\log_2 \left( |\det A| \frac{\mu(Q)}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^n} \right) \leq h_{\text{inv}}. \tag{24}
\]

**Proof.** Let us first point out that every compact set has finite Lebesgue measure and we have \(0 \leq \mu(W)^{1/n} \leq \mu(Q)^{1/n}\) as \(W \subseteq Q\). Therefore, \(1 \leq \mu(Q)^{1/n}/(\mu(Q)^{1/n} - \mu(W)^{1/n}) \leq \infty\) and the left-hand-side of (24) is well-defined. If \(|\det A| = 0\) the left-hand-side is \(-\infty\) and (24) holds. In the remainder we consider the case \(|\det A| > 0\). If \(h_{\text{inv}} = \infty\) the inequality (24) holds independent of the left-hand-side and subsequently we assume that \(h_{\text{inv}} < \infty\). We pick \(\varepsilon \in \mathbb{R}_{>0}\) and an invariant cover \((\mathcal{C}, H)\) of (3) and \(Q\), so that \(h(\mathcal{C}, H) \leq h_{\text{inv}} + \varepsilon\). Given Theorem 5 we can assume that the cover elements of \(\mathcal{C}\) are closed, which yields by the compactness of \(Q\) that the cover elements are compact and therefore Lebesgue measurable.

We fix \(\tau \in \mathbb{N}\) and pick a \((\tau, Q)\)-spanning set \(S\) so that \(r_{\text{inv}}(\tau, Q) = N(S)\), which exists, since for fixed \(\tau\), the number of \((\tau, Q)\)-spanning set is finite.

We are going to show that there exists \(\alpha \in S\) that satisfies

\[
\left( |\det A| \frac{\mu(Q)}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^n} \right)^\tau \leq \prod_{t=0}^{\tau-1} \#P(\alpha|_{[0,t]}). \tag{25}
\]

We construct \(\alpha \in S\) iteratively over \(t \in [0; \tau]\). For \(t = 0\) we introduce \(S_0 := \{\alpha(0) \mid \alpha \in S\}\) and define

\[
m_0 := \max \{\mu(\alpha(0))^{1/n} \mid \alpha \in S\}.
\]
We invoke the induction hypothesis and use the inequality $N N N$ to pick $Ω_0$ such that

$$t + 1 = \max \{ \mu(Ω) | Ω \in P(α|_{[0,t+1]} ) \}$$

Then we set $m_{t+1} := \max \{ \mu(Ω) | Ω \in P(α|_{[0,t+1]} ) \}$ and $Ω_{t+1} \in P(α|_{[0,t+1]} )$ so that $m_{t+1} = \mu(Ω_{t+1})$. For $t = τ - 1$ we obtain a sequence $α := Ω_0 \cdots Ω_{t-1}$ that is an element of $S$. Hence, it follows from (24) that $α$ satisfies for all $t \in [0; τ]$ the inclusion

$$Aα(t) + BH(α(t)) + W \subseteq \bigcup_{Ω \in P(α|_{[0,t]} )} Ω.$$

(26)

For $t \in [0; τ - 1]$, we use the Brunn-Minkowsky inequality for compact, measurable sets [20]

$$\mu(Aα(t))^{1/n} + \mu(W)^{1/n} \leq \mu(Aα(t)) + BH(α(t)) + W)^{1/n}$$

and the identity [17]

$$\mu(Aα(t))^{1/n} = | \det A |^{1/n} \mu(α(t))^{1/n}$$

together with $μ(α(t))^{1/n} = m_t$ and (26), to derive

$$| \det A |^{1/n} m_t + \mu(W)^{1/n} \leq m_{t+1}(#P(α|_{[0,t+1]} ))^{1/n}$$

(27)

for all $t \in [0; τ - 1]$. Also, for ever $t \in [0; τ$] we have

$$| \det A |^{1/n} m_t + \mu(W)^{1/n} \leq \mu(Q)^{1/n}$$

(28)

since $Aα(t) + BH(α(t)) + W \subseteq Q$ which follows from the fact that $α(t) \in C$ and $(C, H)$ is an invariant cover. To ease the notation, let us introduce $N_0 := (#P(α))^{1/n}$ and $N_t := (#P(α|_{[0,t]}))^{1/n}$ for $t \in [1; τ]$. We use induction over $τ' \in [0; τ]$ to show

$$( | \det A |^{1/n} \mu(Q)^{1/n} / \mu(W)^{1/n} )^{τ' + 1} \leq \prod_{t=0}^{τ'} N_t$$

(29)

Let us show (29) for $τ' = 1$. Since $P(α)$ is a cover of $Q$ and $#P(α)^{1/n} = N_0$ we obtain

$$\mu(Q)^{1/n} \leq m_0 N_0.$$  

(30)

From (28) we obtain $m_0 \leq (\mu(Q)^{1/n} - \mu(W)^{1/n}) / | \det A |^{1/n}$ and (29) follows for $τ' = 1$. If $τ = 1$ we have shown (29) and subsequently we consider $τ > 1$. We fix $τ'' \in [1; τ - 1]$ and assume that (29) holds for all $τ' \in [0; τ'']$. We use (27) to derive

$$m_0 \leq \left( \frac{m_{τ''}}{| \det A |^{τ''/n} \left( \prod_{t=1}^{τ''} N_t \right) - \sum_{t=1}^{τ''} \frac{\mu(W)^{1/n}}{| \det A |^{1/n} \prod_{t=1}^{t-1} N_t}} \right)^{τ'' + 1}$$

(31)

with the convention that $\prod_{t=a}^{b} x_t = 1$ for $b < a$. Using (30) and rearranging the terms in (31) we obtain

$$\mu(Q)^{1/n} + \sum_{t=1}^{τ''} \frac{\mu(W)^{1/n}}{| \det A |^{1/n} \prod_{t=1}^{t-1} N_t} \prod_{t=0}^{τ''} N_t \leq \frac{m_{τ''}}{| \det A |^{τ''/n} \prod_{t=0}^{τ''} N_t}. $$

(32)

We invoke the induction hypothesis and use the inequality

$$\prod_{t=0}^{t-1} N_t \geq (| \det A | μ(Q)^{1/n} / (\mu(Q)^{1/n} - μ(W)^{1/n}))^t$$

for $t \in [1; τ - 1]$.
to derive
\[
\mu(Q)^{1/n} + \sum_{t=1}^{\tau''} \frac{\mu(W)^{1/n} \mu(Q)^{1/n}}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^t} \leq \frac{m_{\tau''}}{|\det A|^{\tau''/n}} \prod_{t=0}^{\tau''} N_t
\]  
(33)

From Lemma 9 (given in the Appendix) it follows that the left-hand-side of (33) evaluates to
\[
\mu(Q)^{1/n} + \sum_{t=1}^{\tau''} \frac{\mu(W)^{1/n} \mu(Q)^{1/n}}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^t} = \frac{\mu(Q)^{(\tau''+1)/n}}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^{\tau''}}.
\]  
(34)

We combine \(m_{\tau''} \leq (\mu(Q)^{1/n} - \mu(W)^{1/n})/|\det A|^{1/n}\) (that follows from (28)) with (33) and (34) to see
\[
\frac{\mu(Q)^{(\tau''+1)/n}}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^{\tau''}} \leq \frac{\mu(Q)^{1/n} - \mu(W)^{1/n}}{|\det A|^{(\tau''+1)/n}} \prod_{t=0}^{\tau''} N_t
\]  
(35)

which shows that (29) holds for \(\tau' = \tau'' + 1\). Hence, (29) holds for all \(\tau' \in [0; \tau]\). In particular, for \(\tau' = \tau - 1\) and we conclude that (25) holds.

Inequality (25) together with the definition of \(N(S)\) yields
\[
\left(\frac{|\det A|}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^n}\right)^{\tau} \leq N(S) = r_{inv}(\tau, Q)
\]
where the equality follows by our choice of \(S\). From (5) we get
\[
\log_2 \left(\frac{|\det A|}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^n}\right) \leq h(C, H) \leq h_{inv} + \varepsilon
\]  
(36)

which implies (24) since (36) holds for every \(\varepsilon > 0\). \(\square\)

**Remark 1.** Note that the lower bound, i.e., the left-hand-side of inequality (24), is invariant under coordinate transformation. Let \(z = Tx\) for some invertible matrix \(T \in \mathbb{R}^{n \times n}\) so that the transition function \(F\) of the system in the new coordinates is
\[
\bar{F}(z, u) = TAT^{-1}z + TBu + TW
\]  
(37)
and \(\bar{Q} = TQ\). Then we obtain
\[
|\det(TAT^{-1})| \frac{\mu(TQ)}{(\mu(TQ)^{1/n} - \mu(TW)^{1/n})^n} = |\det A| \frac{|\det T| \mu(Q)}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^n}
\]
\[
|\det A| \frac{|\det T| ((\mu(Q))^{1/n} - \mu(W)^{1/n})^n} = |\det A| \frac{\mu(Q)}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^n}.
\]

For the case that all eigenvalues of \(A\) are larger than one, and \(W\) is a singleton set, we recover the well-known result [13, Th. 3.1] for deterministic linear control systems, i.e., the invariance entropy equals \(\log_2 |\det A|\). This matches also other results known from stabilization with rate limited feedback [4].
6.2. Static coder-controllers. We restrict our attention to static coder-controllers and derive a lower bound of the data rate of such coder-controllers.

Let \((C, H)\) be an invariant cover of \(\Omega\) and a nonempty set \(Q \subseteq X\). We define the data rate of \((C, H)\) by
\[
R(C, H) := \log_2 |C|.
\]
(38)
The definition is motivated by the fact that any invariant cover \((C, H)\) immediately provides a static or memoryless coder-controller scheme: given \(x \in Q\) at the coder side, it is sufficient that the coder transmits one of the cover elements \(C \in C\) that contains the current state \(x \in C\), to ensure that the controller is able to confine the successor states of \(x\) to \(Q\), i.e.,
\[
Ax + BH(C) + W \subseteq Q.
\]
(39)
The number of different cover elements that need to be transmitted via the digital, noiseless channel at any time \(t > 0\) is bounded by \(|C|\). Neither the coder nor the controller requires any past information for a correct functioning. Hence, we speak of \((C, H)\) as static or memoryless coder-controller for \((X, U, F)\).

The next result provides a lower bound on the data rate of any static coder-controller.

Theorem 8. Consider the matrices \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) and two nonempty sets \(W, Q \subseteq \mathbb{R}^n\) with \(W \subseteq Q\) and suppose that \(W\) is measurable and \(Q\) is compact. Let \(\Omega\) be given by \(X = \mathbb{R}^n\), \(U \subseteq \mathbb{R}^m\) with \(U \neq \emptyset\) and \(F\) according to \((23)\). Then, we have
\[
\log_2 \left| |\det A| \frac{\mu(Q)}{(\mu(Q)^{1/n} - \mu(W)^{1/n})^{n}} \right| \leq \inf_{(C, H)} R(C, H)
\]
(40)
where we take the infimum over all invariant covers \((C, H)\) of \((\Omega)\) and \(Q\).

Proof. Every compact set has finite Lebesgue measure and from \(W \subseteq Q\) it follows \(0 \leq \mu(W)^{1/n} \leq \mu(Q)^{1/n}\). Hence, \(1 \leq \mu(Q)^{1/n}/(\mu(Q)^{1/n} - \mu(W)^{1/n}) \leq \infty\) and the left-hand-side of \((11)\) is well-defined. If \(|\det A| = 0\) the left-hand-side of \((11)\) evaluates to \(-\infty\) so that \((11)\) holds. Let us consider \(|\det A| > 0\). If the right-hand-side of \((11)\) evaluates to \(\infty\) nothing needs to be shown and we consider \(\inf_{(C, H)} R(C, H) < \infty\). Since \(\inf_{(C, H)} R(C, H)\) is finite, there exists an invariant cover \((D, G)\) of \((X, U, F)\) and \(Q\). Let \((C, H)\) be the invariant cover with closed cover elements as constructed from \((D, G)\) in \((15)\). According to Theorem 3 \((C, H)\) is an invariant cover of \((X, U, F)\) and \(Q\) and we have \(R(C, H) = R(D, G)\).

As \((C, H)\) is an invariant cover of \((X, U, F)\) and \(Q\), we have for every \(\Omega \in C\) the inclusion
\[
A\Omega + BH(\Omega) + W \subseteq Q.
\]
(41)
We use the Brunn-Minkowsky inequality for compact, measurable sets \(\mu(A\Omega)^{1/n} + \mu(W)^{1/n} \leq \mu(A\Omega + BH(\Omega) + W)^{1/n}\) together with the identity \(17\)
\[
\mu(A\Omega)^{1/n} = |\det A|^{1/n} \mu(\Omega)^{1/n}
\]
to derive \(|\det A|^{1/n} \mu(Q)^{1/n} + \mu(W)^{1/n} \leq \mu(Q)^{1/n}\) which yields the bound
\[
\mu(\Omega)^{1/n} \leq \frac{\mu(Q)^{1/n} - \mu(W)^{1/n}}{|\det A|^{1/n}}.
\]
(42)
As \(|C|\) is an upper bound on the number of cover elements needed to cover \(F(\Omega, H(\Omega))\) we have
\[
\mu(Q)^{1/n} \leq R(C, H)^{1/n} \max\{\mu(\Omega)^{1/n} \mid \Omega \in C\}.
\]
(43)
We use (42) (which holds for every $\Omega \in \mathcal{C}$) in (43) and rearrange the result to obtain

$$|\det A|^{\frac{1}{n}} \frac{\mu(Q)^{\frac{2}{n}}}{\mu(Q)^{\frac{2}{n}} - \mu(W)^{\frac{2}{n}}} \leq R(\mathcal{C}, H)^{\frac{1}{n}}.$$ 

Since this inequality holds for every invariant cover $(\mathcal{C}, H)$, we obtain

$$\log_2 \left| \frac{1}{|\det A|^{\frac{1}{n}} \left( \frac{\mu(Q)^{\frac{2}{n}}}{\mu(Q)^{\frac{2}{n}} - \mu(W)^{\frac{2}{n}}} \right)^n} \right| \leq \inf_{(\mathcal{C}, H)} R(\mathcal{C}, H).$$

It is easy to bound the difference between the universal lower bound in (24) and the lower bound of data rates for static coder-controllers in (40) so that we arrive at the following corollary, which allows us to quantify the performance loss due to the restriction to static coder-controllers.

**Corollary 1.** In the context and under the assumptions of Theorem 8, let $a \in \mathbb{R}_{\geq 0}$ be given by

$$a := |\det A| \frac{\mu(Q)}{(\mu(Q)^{\frac{2}{n}} - \mu(W)^{\frac{2}{n}})^{\frac{1}{n}}}.$$ 

Suppose that $a < \infty$ and there exists an invariant cover $(\mathcal{C}, H)$ of (3) and $Q$ with $R(\mathcal{C}, H) = \log_2[a]$. Then, the data rate $R$ of $(\mathcal{C}, H)$ satisfies

$$R \leq h_{\text{inv}} + 1.$$ 

**Proof.** Let $b \in [0, 1]$ be so that $a + b = [a]$. We use $a \leq 2h_{\text{inv}}$ and $0 \leq h_{\text{inv}}$ to derive

$$R = \log_2(a + b) \leq \log_2(2h_{\text{inv}} + b) \leq h_{\text{inv}} + \log_2(1 + 2^{-h_{\text{inv}}}) \leq h_{\text{inv}} + 1. \quad \square$$

### 6.3. Tightness of the lower bounds.

We show for a particular class of scalar linear difference inclusions of the form

$$\xi(t + 1) \in a \xi(t) + \nu(t) + [w_1, w_2]$$ 

with $a \in \mathbb{R}_{\neq 0}$, $w_1, w_2 \in \mathbb{R}$ and $w_1 \leq w_2$ that the lower bounds established in the previous subsections are tight.

Subsequently, we assume that $Q$ is given as an interval containing $[w_1, w_2]$ 

$$Q := [q_1, q_2], \quad q_1, q_2 \in \mathbb{R}, q_1 < w_1, w_2 < q_2.$$ 

We are going to construct a static coder-controller $(\mathcal{C}, H)$ and show that its data rate equals the lower bound in Theorem 8. To this end, we introduce

$$\Delta q := q_2 - q_1, \quad \Delta w := w_2 - w_1, \quad q_c := (q_2 - q_1)/2 \quad \text{and} \quad w_c := (w_2 - w_1)/2$$ 

and consider

$$m := \left| a \right| \frac{\Delta q}{\Delta q - \Delta w} \quad \text{and} \quad d := \frac{\Delta q}{m}. \quad (46a)$$

Given $q_c$ and $d$, we introduce the intervals $\Lambda_i \subseteq \mathbb{R}, i \in \mathbb{Z}$

$$\Lambda_i := \begin{cases} q_c + [id, (i + 1)d] & \text{if } m \text{ is even} \\ q_c + [(i - \frac{1}{2})d, (i + \frac{1}{2})d] & \text{if } m \text{ is odd} \end{cases} \quad (46c)$$

which we use to define

$$\mathcal{C} := \{\Lambda_i \cap Q \mid \Lambda_i \cap (\text{int}Q) \neq \emptyset\}. \quad (46d)$$
The control function follows for every \( C_i \in \mathcal{C} \) by
\[
H(C_i) := q_c - aq_c - w_c - \begin{cases} 
ad(i + \frac{1}{2}) & \text{if } m \text{ is even} \\
adi & \text{if } m \text{ is odd.}
\end{cases}
\tag{46e}
\]

For this construction of \((\mathcal{C}, H)\), we have the following result.

**Theorem 9.** Consider the scalars \( a \in \mathbb{R}_{\geq 0}, w_1, q_1, w_2, q_2 \in \mathbb{R} \) with \( q_1 < w_1 \leq w_2 < q_2 \). Let \((\mathcal{C})\) be given by \( X = U = \mathbb{R} \) and \( F \) by \( F(x, u) = ax + u + [w_1, w_2] \). Then, \((\mathcal{C}, H)\) defined in \((47)\) is an invariant cover of \((\mathcal{C})\) and \([q_1, q_2]\) and we have
\[
\log_2 \left[ |a| \frac{\Delta q}{\Delta q - \Delta w} \right] = R(\mathcal{C}, H).
\tag{47}
\]

**Proof.** We show the theorem for odd \( m \). The case for even \( m \), follows along the same arguments. It is rather straightforward to show that \( C \) is a cover of \( Q \) and subsequently we show that \( \#\mathcal{C} = m \). Note that \( i > m/2 - 1/2 \) implies that the left limit of \( \Lambda_i \) satisfies \( q_c + (i - \frac{1}{2})d \geq q_c + m/2d = q_2 \), which shows that \( i > m/2 - 1/2 \) implies \( \Lambda_i \cap (\text{int}Q) = \emptyset \). Similarly, \( i < -m/2 + 1/2 \) implies \( \Lambda_i \cap (\text{int}Q) = \emptyset \), and we see that \( \Lambda_i \cap (\text{int}Q) \neq \emptyset \) implies \( -m/2 + 1/2 \leq i \leq m/2 - 1/2 \) so that \( \#\mathcal{C} \leq m \) holds.

We continue to show that \( F(C_i, H(C_i)) \subseteq [q_1, q_2] \) holds for every \( C_i \in \mathcal{C} \). Given \((46c)\) we obtain for \( F(C_i, H(C_i)) \) the interval
\[
(a((q_c + d[i - \frac{1}{2}, i + \frac{1}{2}]) \cap Q) + q_c - aq_c - w_c - ad) + [w_1, w_2]
\]
which is a subset of \( I := q_c + |a|d[\lfloor -1, 1 \rfloor] + \Delta w[-1, 1] \). Let us show that \( I \subseteq Q \). Since \( I \) is centered at \( q_c \), it is sufficient to show \( |a|d/2 + \Delta w/2 \leq \Delta q/2 \). Note that \( m \geq |a|\Delta q/(\Delta q - \Delta w) \) so that \( d \leq (\Delta q - \Delta w)/|a| \) follows and we obtain the desired inequality \( |a|d/2 + \Delta w/2 \leq \Delta q/2 \) which shows \( F(C_i, H(C_i)) \subseteq [q_1, q_2] \). Hence \((\mathcal{C}, H)\) is an invariant cover with \( R(\mathcal{C}, H) \leq \log_2 m \), which together with the inequality in Theorem 8 shows the assertion.

**Example 2** (Continued). Let us recall the linear system in Example 2 with \( a = 1/2 \), \( W = [-3, 3] \) and \( Q = [-4, 4] \). For this case, \( m = 2 \) and \( d = 4 \). The cover elements of \( \mathcal{C} \) are given according to \((46c)\) by
\[
C_{-1} = [-4, 0] \text{ and } C_0 = [0, 4].
\]
The inputs follow according to \((46c)\) by
\[
H(C_{-1}) = 1 \text{ and } H(C_0) = -1.
\]
The data rate of \((\mathcal{C}, H)\) is given by \( \log_2 2 = 1 \) bits per time unit.

We can use Corollary 1 to conclude that the performance loss due to the restriction to static coder-controllers in Example 2 is no larger than 1 bit/time unit. However, for this example, and in general for scalar systems of the form \((45)\) for which \( |a|\Delta q/(\Delta q - \Delta w) \) is in \( \mathbb{N} \), we see that the data rate of the proposed static coder-controller matches the best possible data rate \( h_{\text{inv}} \) since in this case \( R(\mathcal{C}, H) \) equals the lower bound in Theorem 7.

The construction of static coder-controllers whose data rate achieves the lower bound in Theorem 5 in a more general setting is currently under investigation.
Appendix A.

Proof of Lemma 9. We fix $\tau_1, \tau_2 \in \mathbb{N}$ and choose two minimal $(\tau_i, Q)$-spanning sets $S_i, i \in \{1, 2\}$ in $(A, G)$ so that $r_{\text{inv}}(\tau_i, Q) = N(S_i)$. Let $S$ be the set of sequences $\alpha : [0; \tau_1 + \tau_2] \to A$ given by $\alpha(t) := \alpha_1(t)$ for $t \in [0; \tau_1[$ and $\alpha(t) := \alpha_2(t - \tau_1)$ for $t \in [\tau_1; \tau_1 + \tau_2[$, where $\alpha_i \in S_i$ for $i \in \{1, 2\}$. We claim that $S$ is $(\tau_1 + \tau_2, Q)$-spanning in $(A, G)$. It is easy to see that $\{A \in A \mid \exists \alpha \in S : A = \alpha(0)\}$ covers $Q$, since $\{A \in A \mid \exists \alpha \in S_i, A = \alpha(0)\}$ covers $Q$. Let $t \in [0; \tau_1 + \tau_2[$ and $\alpha \in S$. If $t \in [0; \tau_1 - 1[$, we immediately see that $F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{\alpha' \in P(\alpha|_{[0; t]}), \alpha' \in S} A'$ since $\alpha_1 := \alpha|_{[0; \tau_1]} \in S_1$ and $S_i$ satisfies (41). Similarly, if $t \in [\tau_1; \tau_1 + \tau_2 - 1[$, we have $F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{\alpha' \in P(\alpha|_{[t; \tau_1]}), \alpha' \in S} A'$ since $\alpha_2 := \alpha|_{[\tau_1; \tau_1 + \tau_2]} \in S_2$ and $S_2$ satisfies (41). For $t = \tau_1 - 1$, we know that $P(\alpha|_{[0; \tau_1]} = \alpha_1 \alpha_2(0) = A)$ which covers $Q$ and the inclusion $F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{\alpha' \in P(\alpha|_{[0; t]}), \alpha' \in S} A'$ follows. Hence, $S$ satisfies (41) and we see that $S$ is $(\tau, Q)$-spanning. Subsequently, for $i \in \{1, 2\}$ and $\alpha \in S_i$, $t \in [0; \tau_1 - 1[$, let us use $P_i(\alpha|_{[0; t]} := \{A \in A \mid \exists \alpha \in S_i, \alpha_i|_{[0; t]} = \alpha|_{[0; t]} \wedge A = \alpha(t + 1)\}$. Then we have $P(\alpha|_{[0; \tau_1]} = P_1(\alpha|_{[0; \tau_1]}$ with $\alpha_1 := \alpha|_{[0; \tau_1]}$ if $t \in [0; \tau_1 - 1[$ and $P(\alpha|_{[0; \tau_1]} = P_2(\alpha|_{[\tau_1 - \tau_1; \tau_1]})$ with $\alpha_2 := \alpha|_{[\tau_1; \tau_1 + \tau_2]}$ if $t \in [\tau_1; \tau_1 + \tau_2 - 1[$, while for $t = \tau_1 - 1$ we have $P(\alpha|_{[0; \tau_1]} = P_2(\alpha_2)$ with $\alpha_2 := \alpha|_{[\tau_1; \tau_1 + \tau_2]}$ and $P(\alpha) := P_1(\alpha_1)$ with $\alpha_1 := \alpha|_{[0; \tau_1]}$. Therefore, $N(S)$ is bounded by $N(S_1) \cdot N(S_2)$ and we have $r_{\text{inv}}(\tau_1 + \tau_2, Q) \leq r_{\text{inv}}(\tau_1, Q) \cdot r_{\text{inv}}(\tau_2, Q)$. Hence, $\tau \mapsto \log_2 r_{\text{inv}}(\tau, Q)$, $\mathbb{N} \to \mathbb{R}_{>0}$, is a subadditive sequence of real numbers and (7) follows by (12, Lem. 2.1).

Proof of Lemma 10. For every $t \in [0; \tau[$, we define the set $S_t := \{\alpha \in A^{[0; t]} \mid \exists \alpha' \in S, \alpha'|_{[0; t]} = \alpha\}$. By definition of $P$, we have for all $\alpha \in S$ the equality $P(\alpha) = S_0$, which shows the assertion for $\tau = 1$ since in this case we have $S_0 = S$. Subsequently, we assume $\tau > 1$. For $t \in [0; \tau[$ and $a_0 \ldots a_t \in S_t$, we use $Y(a_0 \ldots a_t) := \{\alpha \in S \mid a_0 \ldots a_t = \alpha|_{[0; t]}\}$ to denote the sequences in $S$ whose initial part is restricted to $a_0 \ldots a_t$. For $t \in [0; \tau - 1[$ and $a_0 \ldots a_t \in S_t$, we have the inequality

$$\#Y(a_0 \ldots a_t) \leq \#P(a_0 \ldots a_t) \max_{a_{t+1} \in P(a_0 \ldots a_t)} \#Y(a_0 \ldots a_{t+1}).$$

For every $a_0 \ldots a_{\tau-2} \in S_{\tau-2}$ we have $\#Y(a_0 \ldots a_{\tau-2}) = \#P(a_0 \ldots a_{\tau-2})$ and we obtain a bound for $\#Y(a_0)$ by

$$\#P(a_0) \max_{a_1 \in P(a_0)} \#P(a_0 a_1) \cdots \max_{a_{\tau-2} \in P(a_0 \ldots a_2)} \#P(a_0 \ldots a_{\tau-2}),$$

so that $\#Y(a_0) \leq \max_{\alpha \in S} \prod_{t=0}^{\tau-2} \#P(\alpha|_{[0; t]})$ holds for any $a_0 \in S_0$. We use $\bigcup_{a_0 \in S_0} Y(a_0) = S$ and the fact that for every $\alpha \in S$ we have $P(\alpha) = S_0$ to arrive at desired inequality $\#S \leq \max_{\alpha \in S} \prod_{t=0}^{\tau-1} \#P(\alpha|_{[0; t]}).$ \hfill \IEEEQED

Lemma 9. For $a, b \in \mathbb{R}$ and $T \in \mathbb{N}$, it holds

$$a + \sum_{t=1}^{T} \frac{ba^t}{(a - b)^t} = \frac{a^{T+1}}{(a - b)^{T+1}}. \quad (48)$$

Proof. We show the identity by induction over $T$. For $T = 1$, equation (48) is easy to verify and subsequently, we assume that the equality holds for $T - 1$ with $T \in \mathbb{N}_{\geq 2}$.\hfill \IEEEQED
Now we obtain
\[
a + \sum_{t=1}^{T} \frac{ba^t}{(a - b)^t} = \frac{ba^T}{(a - b)^T} + a + \sum_{t=1}^{T-1} \frac{ba^t}{(a - b)^t}
\]
\[
= \frac{ba^T}{(a - b)^T} + \frac{a^T}{(a - b)^{T-1}}
\]
\[
= \frac{ba^T + a^T(a - b)}{(a - b)^T} = \frac{a^{T+1}}{(a - b)^T}
\]
which completes the proof. \(\square\)

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