VECTOR BUNDLES ON CURVES COMING FROM VARIATION OF HODGE STRUCTURES

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Abstract. Fujita’s second theorem for Kähler fibre spaces over a curve asserts that the direct image $V$ of the relative dualizing sheaf splits as the direct sum $V = A \oplus Q$, where $A$ is ample and $Q$ is unitary flat. We focus on our negative answer (9) to a question by Fujita: is $V$ semiample?

We give here an infinite series of counterexamples using hypergeometric integrals and we give a simple argument to show that the monodromy representation is infinite. Our counterexamples are surfaces of general type with positive index, explicitly given as abelian coverings with group $(\mathbb{Z}/n)^2$ of a Del Pezzo surface of degree 5 (branched on a union of lines forming a bianticanonical divisor), and endowed with a semistable fibration with only 3 singular fibres.

The simplest such surfaces are the three ball quotients considered in [3], fibred over a curve of genus 2, and with fibres of genus 4.

These examples are a larger class than the ones corresponding to Shimura curves in the moduli space of Abelian varieties.

Contents

Introduction 2
1. Fujita’s theorems and questions on Vector Bundles on curves arising from Variation of Hodge Structures 3
1.1. Semistable reduction 7
2. Fujita’s second theorem 7
2.1. Sketch of proof of Fujita’s theorem 8
3. Cyclic coverings of the projective line branched on 4 points 9
4. Abelian coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ yielding surfaces which are counterexamples to Fujita’s question 13
5. General observations and relation with Shimura curves 21
6. Appendix 22
Acknowledgments 23
References 23

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In this paper we first begin recalling previous results ([21], [22], [9], [10]) concerning Fujita’s first and second theorem for Kähler fibre spaces over a curve, asserting that the direct image $V$ of the relative dualizing sheaf splits as the direct sum $V = A ⊕ Q$, where $A$ is ample and $Q$ is unitary flat. Then we focus on our negative answer ([9], [10]) to a question posed by Fujita 30 years ago: $V$ does not need to be semiample.

We show here that the two examples of ([9]) fits into an infinite series of counterexamples, again based on the use of hypergeometric integrals à la Deligne-Mostow, for each positive number $n$ and each way to write $n$ as a sum of four positive integers, and yielding a family of cyclic coverings of the line parametrized by $\mathbb{P}^1$.

Following Beukers and Heckman we can show that the monodromy group of $Q$ is infinite without resorting to the classification by H.A. Schwarz.

Under some mild restrictions on $n$ and the four integers (for $n$ the restriction boils down to the fact that $n$ should be coprime to 6), we give a very simple explicit description of fibre surfaces $f: S \to B$ which are obtained from the above family via a cyclic $\mathbb{Z}/n$-base change $B \to \mathbb{P}^1$, and which have the following remarkable properties:

1. The Albanese map $\alpha: S \to \text{Alb}(S)$ has as image a curve of genus $b \geq 2$, and coincides with the fibration $f: S \to B$;
2. all the fibres of $\alpha$ are smooth, except three singular fibres which are constituted of two smooth curves of genus $b$ meeting transversally in one point
3. the surfaces $S$ have all positive index, indeed $K^2_S > 2.5e(S)$
4. the direct image $V = f_*\omega$ of the relative dualizing sheaf splits as the direct sum $V = A ⊕ Q$, where $A$ is ample and $Q$ is unitary flat, and $Q$ corresponds to an infinite monodromy representation of $\pi_1(B)$: hence $V$ is not semiample (since, by the results of [9], a unitary flat bundle is semi-ample if and only if the monodromy representation is finite).

In the previous two examples ([9], [10]) we had $n = 7$, but we used another method to produce a base change yielding a semistable fibration; as a consequence the degree of the base change that we needed was much larger than 7 (42 in the easier case), and the semistable fibrations were not described with full details. The description we give here was motivated by a question by Fujino, who asked whether we could give a completely explicit example of a semistable fibration satisfying property (4).

To underline the simplicity of the present geometric construction, let us observe that the simplest surfaces in our series correspond to writing $5 = 2 + 1 + 1 + 1$, and are therefore surfaces $S$ fibred over a curve of...
The following is our main result.

**Theorem 1.** There exists an infinite series of surfaces with ample canonical bundle, whose Albanese map is a fibration \( f : S \to B \) onto a curve \( B \) of genus \( b = \frac{1}{2}(n - 1) \), and with fibres of genus \( g = 2b = n - 1 \), where \( n \) is any integer relatively prime with 6.

These Albanese fibrations yield negative answers to Fujita’s question about the semiampleness of \( V := f_*\omega_{S|B} \), since here \( V := f_*\omega_{S|B} \) splits as a direct sum \( V = A \oplus Q \), where \( A \) is an ample vector bundle, and \( Q \) is a unitary flat bundle with infinite monodromy group.

The fibration \( f \) is semistable: indeed all the fibres are smooth, with the exception of three fibres which are the union of two smooth curves of genus \( b \) which meet transversally in one point.

For \( n = 5 \) we get three surfaces which are rigid, and are quotient of the unit ball in \( \mathbb{C}^2 \) by a torsion free cocompact lattice \( \Gamma \). The rank of \( A \), respectively \( Q \), is in this case equal to 2.

Finally we end surveying quite briefly relations with existing literature concerning Shimura curves in the moduli space of Abelian varieties: this is work of several people, but especially the work of Moonen ([39]) is related to our easiest examples.

1. **Fujita’s theorems and questions on Vector Bundles on curves arising from Variation of Hodge Structures**

An important progress in classification theory was stimulated by a theorem of Fujita, who showed ([21]) that if \( X \) is a compact Kähler manifold and \( f : X \to B \) is a fibration onto a projective curve \( B \) (i.e., \( f \) has connected fibres), then the direct image sheaf

\[
V := f_*\omega_{X|B} = f_*\left(\mathcal{O}_X(K_X - f^*K_B)\right)
\]

is a nef vector bundle on \( B \), where ‘nef’ means that each quotient bundle \( Q \) of \( V \) has degree \( \deg(Q) \geq 0 \); sometimes the word ‘nef’ is replaced by the word ‘numerically semipositive’.

In the note [22] Fujita announced the following quite stronger result:

**Theorem 2.** (Fujita, [22])

Let \( f : X \to B \) be a fibration of a compact Kähler manifold \( X \) over a projective curve \( B \), and consider the direct image sheaf

\[
V := f_*\omega_{X|B} = f_*\left(\mathcal{O}_X(K_X - f^*K_B)\right).
\]

Then \( V \) splits as a direct sum \( V = A \oplus Q \), where \( A \) is an ample vector bundle and \( Q \) is a unitary flat bundle.

Fujita sketched the proof, but referred to a forthcoming article concerning the positivity of the so-called local exponents (this article was never written, see [2]).
Soon afterwards, using Griffiths’ results on Variation of Hodge Structures, since the fibre of $V := f_∗ω_X|_B$ over a point $b ∈ B$ such that $X_b := f^{-1}(b)$ is smooth is the vector space $V_b = H^0(X_b, Ω_X^{−1})$, Kawamata ([31] [32]) improved on Fujita’s result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

\[ \text{Kod}(X) \geq \text{Kod}(B) + \text{Kod}(F), \]

(here $F$ is a general fibre). Kawamata did this by showing the semipositivity also for the direct image of higher powers of the relative dualizing sheaf

\[ W_m := f_∗(ω^{⊗m}_{X/B}) = f_∗(O_X(m(K_X − f^∗K_B))). \]

Kawamata also extended his result to the case where the dimension of the base variety $B$ is $>1$ in [31], giving later a simpler proof of semipositivity in [33]. There has been a lot of literature on the subject ever since, see the references we cited (see [17] for the ampleness of $W_m$ when $m ≥ 2$ and when the fibration is not birationally isotrivial, see also [20] and [19]). Kawamata introduced a simple lemma, concerning the degree of line bundles on a curve whose metric grows at most logarithmically around a finite number of singular points, which played a crucial role for the proof.

The missing details concerning the proof of the second theorem of Fujita, using Kawamata’s lemma and some crucial estimates given by Zucker ([17]) for the growth of the norm of sections of the $L^2$-extension of Hodge bundles, were provided in [9], where also a negative answer was given to the following question posed by Fujita in 1982 (Problem 5, page 600 of [29], Proceedings of the 1982 Taniguchi Conference).

To understand this question it is not only important to have in mind Fujita’s second theorem, but it is also very convenient to recall the following classical definition used by Fujita in [21], [22].

Let $V$ be a holomorphic vector bundle over a projective curve $B$.

**Definition 3.** Let $p : P := \text{Proj}(V) = \mathbb{P}(V^∨) → B$ be the associated projective bundle, and let $H$ be a hyperplane divisor (s.t. $p_∗(O_P(H)) = V$).

Then $V$ is said to be:

- (NP) numerically semi-positive if and only if every quotient bundle $Q$ of $V$ has degree $\text{deg}(Q) ≥ 0$,
- (NEF) nef if and only if $H$ is nef on $P$,
- (A) ample if and only if $H$ is ample on $P$
- (SA) semi-ample if and only if $H$ is semi-ample on $P$ (there is a positive multiple $mH$ such that the linear system $|mH|$ is base point free).

**Remark 4.** Recall that (A) $⇒$ (SA) $⇒$ (NEF) $⇔$ (NP), the last follows from the following result due to Hartshorne.
**Proposition 5.** A vector bundle $V$ on a curve is nef if and only if it is numerically semi-positive, i.e., if and only if every quotient bundle $Q$ of $V$ has degree $\deg(Q) \geq 0$, and $V$ is ample if and only if every quotient bundle $Q$ of $V$ has degree $\deg(Q) > 0$.

Moreover, we have also:

**Definition 6.** A flat holomorphic vector bundle on a complex manifold $M$ is a holomorphic vector bundle $H := \mathcal{O}_M \otimes \mathbb{C}^H$, where $H$ is a local system of complex vector spaces associated to a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$.

$H := (\tilde{M} \times \mathbb{C}^r)/\pi_1(M)$, $\tilde{M}$ being the universal cover of $M$ (so that $M = \tilde{M}/\pi_1(M)$).

We say that $H$ is unitary flat if it is associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$.

**Question 7. (Fujita)** Is the direct image $V := f_*\omega_X|_B$ semi-ample?

In [9] we established a technical result which clarifies how Fujita’s question is very closely related to Fujita’s II theorem

**Theorem 8.** Let $H$ be a unitary flat vector bundle on a projective manifold $M$, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $H$ is nef and moreover $H$ is semi-ample if and only if $\text{Im}(\rho)$ is finite.

Hence in our particular case, where $V = A \oplus Q$ with $A$ ample and $Q$ unitary flat, the semi-ampleness of $V$ simply means that the flat bundle has finite monodromy (this is another way of wording the fact that the representation of the fundamental group $\rho : \pi_1(B) \to U(r, \mathbb{C})$ associated to the flat unitary rank-$2$ bundle $Q$ has finite image).

The main new result in our joint work [9] was to provide a negative answer to Fujita’s question in general:

**Theorem 9.** There exist surfaces $X$ of general type endowed with a fibration $f : X \to B$ onto a curve $B$ of genus $\geq 3$, and with fibres of genus $6$, such that $V := f_*\omega_X|_B$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where $A$ is an ample rank-$2$ vector bundle, and the flat unitary rank-$2$ summands $Q_1, Q_2$ have infinite monodromy group (i.e., the image of $\rho_j$ is infinite). In particular, $V$ is not semi-ample.

Recall however that in special cases one can conclude that $V$ is semi-ample.

**Corollary 10.** Let $f : X \to B$ be a fibration of a compact Kähler manifold $X$ over a projective curve $B$. Then $V := f_*\omega_X|_B$ is a direct sum $V = A \bigoplus (\bigoplus_{i=1}^h Q_i)$, with $A$ ample and each $Q_i$ unitary flat without any nontrivial degree zero quotient. Moreover,
(I) if $Q_i$ has rank equal to 1, then it is a torsion bundle ($\exists m$ such that $Q_i^\otimes m$ is trivial) (Deligne)

(II) if the curve $B$ has genus 1, then $\text{rank}(Q_i) = 1$, $\forall i$.

(III) In particular, if $B$ has genus at most 1, then $V$ is semi-ample.

Proof. The idea of the proof is as follows:

(I) was proven by Deligne (and by Simpson using the theorem of Gelfond-Schneider), while

(II) Follows since $\pi_1(B)$ is abelian, if $B$ has genus 1: hence every representation splits as a direct sum of 1-dimensional ones.

In our construction for theorem 9, we started from hypergeometric integrals associated to a cyclic group of order 7, and we derived the non finiteness of the monodromy as a consequence of the classification due to Schwarz (13).

However, in order to provide a semistable fibration, we first resolved the singularities of the resulting surface fibres over $\mathbb{P}^1$, then applied blow ups in order to achieve that the reduced divisors associated to the fibres would be normal crossing divisors, and then applied the general method in order to construct a semistable base change.

The final result was that these examples had a base of much larger genus, and the description given was not fully detailed. The novelty of this paper, answering a question by Osamu Fujino, is to provide an explicit semistable fibration without having to take a base change where the genus of the base curve $B$ becomes too large. This will be discussed in the fourth section, where we shall also give a simpler proof.

An interesting observation, concerning the crucial difference of the roles played by unitary flat bundles versus flat bundles in our context, is given by the following result. While a unitary flat bundle is nef, the same does not hold for a flat bundle. This is no surprise, as communicated to the first author by Janos Kollár, in view of the following old theorem of André Weil (45), reproven by Atiyah in [1].

Theorem 11. (Weil-Atiyah) A vector bundle $V$ over a projective curve is (isomorphic to) a flat holomorphic bundle if and only if, in its unique decomposition as a direct sum $V = \oplus_i V_i$ of indecomposable bundles, each of the summands $V_i$ has degree zero.

In our situation we proved (again in [9]):

Theorem 12. Let $f : X \to B$ be a Kodaira fibration, i.e., $X$ is a surface and all the fibres of $f$ are smooth curves not all isomorphic to each other. Then the direct image sheaf $V := f_*\omega_X|_B$ has strictly positive degree hence $\mathcal{H} := R^1f_*(\mathcal{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not nef (i.e., not numerically semipositive).
1.1. **Semistable reduction.** Assume now that $f : X \to B$ is a fibration of a compact Kähler manifold $X$ over a projective curve $B$, and consider the invertible sheaf $\omega := \omega_{X|B} = \mathcal{O}_X(K_X - f^*K_B)$.

By Hironaka’s theorem there is a sequence of blow ups with smooth centres $\pi : \hat{X} \to X$ such that $\hat{f} := f \circ \pi : \hat{X} \to B$ has the property that all singular fibres $F$ are such that $F = \sum_i m_i F_i$, and $F_{\text{red}} = \sum_i F_i$ is a normal crossing divisor.

Since $\pi_* \mathcal{O}_\hat{X}(K_{\hat{X}}) = \mathcal{O}_X(K_X)$ we obtain

$$\hat{f}_* \omega_{\hat{X}|B} = \hat{f}_* \mathcal{O}_\hat{X}(K_{\hat{X}} - \hat{f}^*K_B) = f_* \mathcal{O}_X(K_X - f^*K_B) = f_* \omega_{X|B}.$$ 

Therefore one can assume wlog that all the fibres of $f$ have reduction which is a normal crossing divisor, and the well known semistable reduction theorem, whose statement is here reproduced, shows that one can reduce to the case where the fibration is semistable, i.e., all fibres are reduced and yield normal crossing divisors.

**Theorem 13. (Semistable reduction theorem, [34])** There exists a cyclic Galois covering of $B$, $B' \to B = B'/G$, such that the normalization $X''$ of the fibre product $B' \times_B X$ admits a resolution $X' \to X''$ such that the resulting fibration $f' : X' \to B'$ has the property that all the fibres are reduced and normal crossing divisors.

The following proposition was used in [9] while reducing the proof of Fujita’s second theorem to the semistable case.

**Proposition 14.** The sheaf $V' := f'_* \omega_{X'|B'}$ is a subsheaf of the sheaf $u^*(V)$, where $V := f_* \omega_{X|B}$, and the cokernel $u^*(V)/V'$ is concentrated on the set of points corresponding to singular fibres of $f$.

The proposition shows indeed that, when the fibration is not semistable, then certain unitary flat summands on $B'$ may yield ample summands on $B$; and the precise calculation given in its proof helps to decide exactly when this happens.

2. **Fujita’s second theorem**

The tools used for the proof of Fujita’s second theorem involve differential geometric notions of positivity, which we now recall.
Definition 15. Let \((E, h)\) be a Hermitian vector bundle on a complex manifold \(M\). Take the canonical Chern connection associated to the Hermitian metric \(h\), and denote by \(\Theta(E, h)\) the associated Hermitian curvature, which gives a Hermitian form on the complex vector bundle \(T_M \otimes E\).

Then one says that \(E\) is Nakano positive (resp.: semi-positive) if there exists a Hermitian metric \(h\) such that the Hermitian form associated to \(\Theta(E, h)\) is strictly positive definite (resp.: semi-positive definite).

Remark 16. Umemura proved ([44]) that a vector bundle \(V\) over a curve \(B\) is positive (i.e., Griffiths positive, or equivalently Nakano positive) if and only if \(V\) is ample.

One of the principal positivity property can be summarized through the well known slogan: ‘curvature decreases in subbundles’. Except that one has to formulate the statement properly as follows: curvature decreases in Hermitian subbundles. Indeed the example of Kodaira fibrations produces subbundles of a flat bundle (they have zero curvature) which are positively curved.

We pass now to sketching the ideas used in the proof of Fujita’s second theorem.

Theorem 17. (Fujita, [22])

Let \(f : X \to B\) be a fibration of a compact Kähler manifold \(X\) over a projective curve \(B\), and consider the direct image sheaf

\[
V := f_*\omega_{X/B} = f_* (\mathcal{O}_X(K_X - f^*K_B)).
\]

Then \(V\) splits as a direct sum \(V = A \oplus Q\), where \(A\) is an ample vector bundle and \(Q\) is a unitary flat bundle.

2.1. Sketch of proof of Fujita’s theorem.

I) Thanks to the auxiliary results shown in the previous section, using the semistable reduction theorem (yielding a base change \(B' \to B\) such that all fibres of the pull-back \(X' \to B'\) are reduced with normal crossings) and in particular proposition 14 giving a comparison of the pull-back of \(V\) with the analogously defined \(V'\), it suffices to prove the theorem in the semistable case, i.e., where each fibre is reduced and a normal crossing divisor (see proposition 2.9 of [9] for details).

II) Idea of the proof in the case of no singular fibres.

\(V\) is a holomorphic subbundle of the holomorphic vector bundle \(\mathcal{H}\) associated to the local system \(\mathbb{H}_\mathbb{Z} := R^m f_* (\mathbb{Z}_X)\), \(m := \dim(X) - 1\) (i.e., \(\mathcal{H} = \mathbb{H}_\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_B\)).

The bundle \(\mathcal{H}\) is flat, hence the curvature \(\Theta_{\mathcal{H}}\) associated to the flat connection satisfies \(\Theta_{\mathcal{H}} \equiv 0\).

We view \(V\) as a holomorphic subbundle of \(\mathcal{H}\), while

\[
V^\vee \cong R^m f_* \mathcal{O}_X, \quad m = \dim(X) - 1
\]
is a holomorphic quotient bundle of $H$.

The curvature formula for subbundles gives (σ is the II fundamental form)

$$\Theta_V = \Theta_H|_V + \bar{\sigma} \cdot \sigma = \bar{\sigma} \cdot \sigma,$$

and Griffiths ([24], see also [26] and [48]) proves that the curvature of $V^\vee$ is semi-negative, since its local expression is of the form $ih'(z)d\bar{z} \wedge dz$, where $h'(z)$ is a semi-positive definite Hermitian matrix.

In particular we have that the curvature $\Theta_V$ of $V$ is semipositive and, moreover, that the curvature vanishes identically if and only if the second fundamental form $\sigma$ vanishes identically, i.e., if and only if $V$ is a flat subbundle.

However, by semi-positivity, we get that the curvature vanishes identically if and only its integral, the degree of $V$, equals zero. Hence $V$ is a flat bundle if and only if it has degree 0.

The same result then holds true, by a similar reasoning, for each holomorphic quotient bundle $Q$.

III) The more difficult part of the proof uses some crucial estimates given by Zucker (using Schmid’s asymptotics for Hodge structures) for the growth of the norm of sections of the $L^2$-extension of Hodge bundles, and the following lemma by Kawamata ([32], see also proposition 3.4, page 11 of [41]).

**Lemma 18.** Let $L$ be a holomorphic line bundle over a projective curve $B$, and assume that $L$ admits a singular metric $h$ which is regular outside of a finite set $S$ and has at most logarithmic growth at the points $p \in S$.

Then the first Chern form $c_1(L,h) := \Theta_h$ is integrable on $B$, and its integral equals $\deg(L)$.

The above lemma shows that in the semistable case singularities are ininfluent, and the argument runs as in the case of no singular fibres.

IV) The existence of such a metric follows from the results of Schmid in [42] and Zucker in [47], leading to the following lemma.

**Lemma 19.** For each point $s \in B$ there exists a basis of $V$ given by elements $\sigma_j$ such that their norm in the flat metric outside the punctures grows at most logarithmically.

In particular, for each quotient bundle $Q$ of $V$ its determinant admits a metric with growth at most logarithmic at the punctures $s \in S$, and the degree of $Q$ is given by the integral of the first Chern form of the singular metric.

### 3. Cyclic coverings of the projective line branched on 4 points.

In this section we explain how we obtain explicit examples of fibrations where $V = f_\ast \omega$ has a flat summand. Let $\zeta_n := e^{2\pi i/n}$. Consider a
cyclic covering of the projective line with group $\mathbb{Z}/n$, branched on four points. Hence a curve $C = C_x$ described by an equation

$$z^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - xy_0)^{m_3}, \quad x \in \mathbb{C} \setminus \{0, 1\},$$

where, of course, gcd($m_0, \ldots, m_3, n) = 1$.

The above equation describes a singular curve inside the line bundle over $\mathbb{P}^1$ whose sheaf of sections is the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$, and we denote by $C$ the normalization of this curve. Then $C$ admits a Galois cover $\phi : C \to \mathbb{P}^1$ with cyclic Galois group equal to the group of $n$-th roots of unity in $\mathbb{C}$,

$$G = \{ \zeta \in \mathbb{C}^* | \zeta^n = 1 \},$$

acting by scalar multiplication on $z_1$. The choice of a generator in $G$ yields an isomorphism $G \cong \mathbb{Z}/n$, for instance we have $G = \langle \epsilon \rangle$, where $\epsilon$ acts as $z_1 \mapsto \zeta_n z_1$. The cover $\phi$ is branched at $\mathcal{S} = \{ s_1 = 0, s_2 = 1, s_3 = x, s_0 = \infty \}$ (where, in projective coordinates $[y_0, y_1]$ one has $0 = [1, 0], 1 = [1, 1], x = [1, x], \infty = [0, 1]$).

We shall make the restrictive assumption that

$$0 < m_j \leq n - 3, \quad \text{and} \quad m_0 + m_1 + m_2 + m_3 = n.$$

**Remark 21.** We want to point out that the above is indeed a restriction, even if we allow a change of the generator of $G$ taking the residue class of a number $h$ coprime to $n$. This change of generator has the effect of replacing $m_j$ with the rest modulo $n$ of $hm_j$, which we denote by $[hm_j]$.

Now, take the example where $n = 8$ and $m_1 = m_2 = 4, m_3 = 3, m_4 = 5$. Then, however we change the generator of $G$, the residue class $[h4]$ shall always be equal to 4. Hence the sum $[hm_0] + [hm_1] + [hm_2] + [hm_3] > n = 8$ always, and indeed the sum is then always equal to $2n = 16$ (observe in fact that $\Sigma := ([hm_0] + [hm_1] + [hm_2] + [hm_3]) \in \{n, 2n, 3n\}$, and, changing $m_j$ to $n - m_j$, $\Sigma \mapsto 4n - \Sigma$).

Now, for $j \in \mathbb{Z}/n\mathbb{Z}$, let $\chi_j : G \to \mathbb{C}^*, \zeta \mapsto \zeta^j$ and let $L_j$ denote the rank-one local system on $\mathbb{P}^1 \setminus \mathcal{S}$ whose monodromy matrix $\alpha_s$ at $s_i$ is given by $\zeta_{m_j}^i$.

Let $H^1(C, \mathbb{C})_j$ be the subspace of $H^1(C, \mathbb{C})$ on which $G$ acts as $\chi_j$. Then one has an isomorphism

$$H^1(C, \mathbb{C})_j = H^1(\mathbb{P}^1 \setminus \mathcal{S}, L_j)$$

and moreover

$$H^{1,0}(\mathbb{P}^1 \setminus \mathcal{S}, L_j) = H^{1,0}(C)_j,$$

where $H^{1,0}(C)_j$ is again the part of $H^{1,0}(C)$ on which $G$ acts by the character $\chi_j$ (cf. [14], Section (2.23)). For $j \neq 0$, one has

$$\dim(H^1(C, \mathbb{C})_j) = \dim(H^1(\mathbb{P}^1 \setminus \mathcal{S}, L_j)) = 2.$$
For \( j \neq 0 \) let \( \mu_{i,j} = \frac{[m_i,j]}{n} \). By [14], Equation 2.20.1,
\[
\dim H^{1,0}(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}_j) = -1 + \sum_{i=0}^{3} \mu_{i,j} \quad \text{for} \quad j \in \mathbb{Z}/n\mathbb{Z}, \ j \neq 0.
\]
Hence, under the above assumption \([20]\) we have
\[
H^1(C, \mathbb{C})_{-1} = H^{1,0}(C, \mathbb{C})_{-1} \simeq H^{1,0}(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}_{-1}).
\]
By [14], Prop. 2.20, the Hermitian form \( H_j \) on \( H^1(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}_j) \) is positive definite on \( H^1(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}_j) \) and negative definite on \( H^0(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}_j) \). Hence the positivity and the negativity index of \( H_j \) are given by
\[
\rho_j = (-1 + \sum_{i=0}^{3} \mu_{i,j}, 3 - \sum_{i=0}^{3} \mu_{i,j}).
\]
Varying \( x \in \mathbb{C} \setminus \{0, 1\}, \) one obtains a family of curves \( \pi : C \to \mathbb{C} \setminus \{0, 1\} \) with fibre \( \pi^{-1}(x) = C_x, \) equipped with a compatible action of \( G. \) For each \( j \in \mathbb{Z}/n\mathbb{Z} \) one also obtains a local system \( \mathbb{L}_j' \) on
\[
M := \{(x,y) \in \mathbb{C}^2 \mid x, y \neq 0, 1, x \neq y\}
\]
which extends \( \mathbb{L}_j \) to \( M. \) Let
\[
f : M \to \mathbb{C} \setminus \{0, 1\}, \ (x,y) \mapsto x.
\]
The higher direct image \( \mathbb{H} = R^1\pi_*\mathbb{C} \) decomposes then with respect to the \( G \)-action into \( \chi_j \)-equivariant parts
\[
\mathbb{H} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathbb{H}_j \quad \text{where} \quad \mathbb{H}_j = R^1f_*\mathbb{L}_j'.
\]
If \( j \neq 0, \) then the monodromy representation of \( \mathbb{H}_j \) by [24] respects a Hermitian form \( H_j \) of index
\[
(-1 + \sum_{i=0}^{3} \mu_{i,j}, 3 - \sum_{i=0}^{3} \mu_{i,j}).
\]
We shall prove in the appendix that this monodromy representation is irreducible if the \( m_i \)'s are coprime to \( n \) (as we shall assume in the sequel). This result and the following lemma shall be used to show the existence of a flat unitary summand with infinite monodromy: but we shall also give a self-contained and more elementary proof of the main theorem which only uses the second Fujita theorem.

The index of \( H_j \) is related to finiteness properties of the monodromy of \( \mathbb{H}_j \) by the following result which is a straightforward generalization of [H] Thm. 4.8 (cf. [27]). We remark that the lemma applies in many other contexts, e.g., for more general rigid local systems or motivic local systems whose Hodge numbers can be calculated (cf. [30], [15], [16]).
Lemma 25. Choose an embedding of \( \bar{\mathbb{Q}} \) into \( \mathbb{C} \). Let \( K \) be either a finite abelian extension of \( \mathbb{Q} \) or a totally real Galois extension of \( \mathbb{Q} \). Denote by \( \text{Gal} := \text{Gal}(K/\mathbb{Q}) \) and let \( \mathcal{O}_K \) denote the ring of integers of \( K \). Let \( \Gamma \) be a finitely generated group and let \( \rho : \Gamma \rightarrow \text{GL}_n(\mathcal{O}_K) \) be an absolutely irreducible representation whose image shall be denoted by \( H := \text{Im}(\rho) \). Suppose that \( \rho \) respects a Hermitian form, i.e., there exists a Hermitian matrix \( M = (m_{i,j}) \in K^{n \times n} \) with

\[
\bar{A}^T M A = M \quad \forall A \in H.
\]

For \( \sigma \in \text{Gal} \), let \( M^\sigma = (m^\sigma_{i,j}) \). Then \( H \) is finite if and only if \( M^\sigma \) is a definite Hermitian form for all \( \sigma \in \text{Gal} \).

Proof. If \( H \) is finite then \( H \) leaves the positive definite unitary form

\[
\bar{v}^T M w := \sum_{h \in H} \bar{v}^T \bar{h}^T h w \quad v, w \in K^n
\]

invariant. By our assumptions on \( K \), any \( \sigma \in \text{Gal} \) commutes with complex conjugation, hence \( H^\sigma \) leaves the form defined by the matrix \( M^\sigma \) invariant. Moreover, \( M^\sigma \) is determined up to a constant, since \( H^\sigma \) is again irreducible. Since \( H^\sigma \) is also finite, the matrix \( M^\sigma \) must be definite.

Let now the form \( M^\sigma \) be definite for any \( \sigma \in \text{Gal} \). By the additive isomorphism \( \mathcal{O}_K \cong \mathbb{Z}^d \) (\( d = |\text{Gal}| \)), the representation \( \rho \) gives rise to a representation

\[
\tilde{\rho} : \Gamma \rightarrow \text{GL}_{nd}(\mathbb{Z})
\]

such that the trace of \( \tilde{\rho}(g) \) coincides with the the relative trace of \( \rho(g) : \)

\[
\text{Trace}(\tilde{\rho}(g)) = \text{Trace}_{K/\mathbb{Q}}(\text{Trace}(\rho(g))) = \sum_{\sigma \in \text{Gal}} \text{Trace}(\rho(g))^\sigma.
\]

Extending the scalars from \( \mathbb{Z} \) to \( \mathbb{C} \) we obtain from \( \tilde{\rho} \) a representation

\[
\tilde{\rho} \otimes \mathbb{C} : \Gamma \rightarrow \text{GL}_{nd}(\mathbb{C}).
\]

Since any semisimple representation with values in \( \mathbb{C} \) is determined up to isomorphy by its trace by the theorem of Brauer-Nesbitt, there exists a matrix \( g \in \text{GL}_n(\mathbb{C}) \) such that

\[
(\tilde{\rho} \otimes \mathbb{C})^g = \prod_{\sigma \in \text{Gal}} \rho^\sigma \otimes \mathbb{C},
\]

where \( \rho^\sigma \otimes \mathbb{C} \) denotes the extension of scalars of \( \rho^\sigma \) from \( \mathcal{O}_K \) to \( \mathbb{C} \). By the definiteness of the forms \( M^\sigma (\sigma \in \text{Gal}) \), the latter representation takes its values in the product \( \prod_{\sigma \in \text{Gal}} U(M^\sigma) \) of the compact unitary groups \( U(M^\sigma) \) associated to the Hermitian forms \( M^\sigma (\sigma \in \text{Gal}) \). We conclude that the image of \( \tilde{\rho} \) is contained in the compact and discrete group

\[
\left( \prod_{\sigma \in \text{Gal}} U(M^\sigma)^{-1} \right) \cap \text{GL}_{nd}(\mathbb{Z})
\]
and is hence finite. Therefore the image of $\rho$ is finite. 

The next result is a straightforward consequence of the above lemma (of course, it may also be derived by Schwarz’ list of hypergeometric differential equations with finite monodromy [43]). For simplicity, we restrict ourselves to the cases considered in the next section.

**Corollary 26.** Assume that

$$n \in \mathbb{N}, n \geq 5, \text{such that } GCD(n, 6) = 1, \ m_0, m_1, m_2, m_3 \in \mathbb{N},$$

with $1 \leq m_j \leq n - 1, \ m_0 + m_1 + m_2 + m_3 = n,$

$$m_k, (m_i + m_3) \in (\mathbb{Z}/n\mathbb{Z})^*, \forall i = 0, 1, 2, k = 0, 1, 2, 3.$$ 

Then there is $j \in (\mathbb{Z}/n\mathbb{Z})^*,$ such that the monodromy of the local system $\hat{H}_j$ is infinite.

**Proof.** Since $n - 1$ and $m_0 + m_3$ are invertible in $\mathbb{Z}/n\mathbb{Z},$ there is a $j$ such that $j(m_0 + m_3) \equiv -1 \ (\text{mod } n).$

Define now $m'_j := [m_j], \text{ so that } m'_0 + m'_3 = n - 1.$

We have the obvious inequalities $2 \leq m'_1 + m'_2 \leq 2n - 2.$ Hence

$$n + 1 \leq m'_0 + \cdots + m'_3 \leq 3n - 3$$

and therefore

$$m'_0 + \cdots + m'_3 = 2n.$$ 

Hence the underlying unitary form is indefinite by Formula (24), hence the monodromy is infinite by Lemma 25 and Prop. 39. 

4. **Abelian coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ yielding surfaces which are counterexamples to Fujita’s question**

In this section we shall provide an infinite series of examples of surfaces fibred over a curve, whose fibres are curves with a symmetry of $G := \mathbb{Z}/n \ (\text{and with quotient } \mathbb{P}^1).$

To avoid too many technicalities, we make the following simplifying assumptions, part of which were already mentioned in corollary 26:

$$n \in \mathbb{N}, n \geq 5, \text{such that } GCD(n, 6) = 1, \ m_0, m_1, m_2, m_3, n_0, n_1, n_2 \in \mathbb{N},$$

with $1 \leq n_i, m_j \leq n - 1, \ m_0 + m_1 + m_2 + m_3 = n, n_0 + n_1 + n_2 = n$

$$m_j, n_i, (m_i + m_3) \in (\mathbb{Z}/n\mathbb{Z})^*, \forall i = 0, 1, 2, j = 0, 1, 2, 3.$$ 

**Lemma 27.** Such integers $n_i, m_j$ satisfying the above properties exist if and only if $n$ and 6 are coprime.

**Proof.** If $n$ is even, then if $m_j$ is a unit in $\mathbb{Z}/n,$ then $m_j$ is odd, but then $m_i + m_3$ is even, and cannot be a unit.

If instead $3|n,$ then without loss of generality

$$m_0 \equiv m_3 \ (\text{mod } 3), m_1 \equiv m_2 \equiv -m_3 \ (\text{mod } 3),$$

but then $m_1 + m_3$ is not a unit in $\mathbb{Z}/n.$
Finally, if $GCD(n, 6) = 1$, then we can simply choose
\[ m_0 = m_1 = m_2 = 1, \ m_3 = n - 3, n_0 = n_1 = 1, n_2 = n - 2. \]

\[ \square \]

**Definition 28.** We shall refer to the choice
\[ m_0 = m_1 = m_2 = 1, \ m_3 = n - 3, n_0 = n_1 = 1, n_2 = n - 2 \]
as the **standard case**.

We consider again the equation
\[ z^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - xy_0)^{m_3}, \ x \in \mathbb{C} \setminus \{0, 1\} \]
but we homogenize it to obtain the equation
\[ z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (x_0 y_1 - x_1 y_0)^{m_3} x_0^{n-m_3}. \]

The above equation describes a singular surface $\Sigma'$ which is a cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $G := \mathbb{Z}/n$; $\Sigma'$ is contained inside the line bundle $\mathcal{L}_1$ over $\mathbb{P}^1 \times \mathbb{P}^1$ whose sheaf of holomorphic sections $\mathcal{L}_1$ equals $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.

The first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induces a morphism $p : \Sigma' \to \mathbb{P}^1$ and we consider the curve $B$, normalization of the covering of $\mathbb{P}^1$ given by
\[ w_1^n = x_0^{n_0} x_1^{n_1} (x_1 - x_0)^{n_2}. \]

We consider the normalization $\Sigma$ of the fibre product $\Sigma' \times_{\mathbb{P}^1} B$.

$\Sigma$ is an abelian covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $(\mathbb{Z}/n)^2$, and the local monodromies are as follows:

\[ \{ y = \infty \} = \{ y_0 = 0 \} \mapsto (m_0, 0), \ \{ y = 0 \} = \{ y_1 = 0 \} \mapsto (m_1, 0), \]
\[ \{ y = 1 \} = \{ y_1 = y_0 \} \mapsto (m_2, 0), \]
\[ \{ x = \infty \} \mapsto (n - m_3, n_0), \ \{ x = 0 \} \mapsto (0, n_1), \ \{ x = 1 \} \mapsto (0, n_2), \]
\[ \Delta := \{(x_0 y_1 - x_1 y_0) = 0\} \mapsto (m_3, 0). \]

Since the branch divisor is not a normal crossing divisor, we blow up the three points $P_0 := \{ x_0 = y_0 = 0 \}$, $P_1 := \{ x_1 = y_1 = 0 \}$, $P_2 = \{ x_0 - x_1 = y_0 - y_1 = 0 \}$.

We obtain in this way a del Pezzo surface which we denote by $Z$, we denote by $E_i$ the exceptional $(-1)$-curve inverse image of the point $P_i$, and we notice that the pull back of the branch divisor is now a normal crossing divisor.

The local monodromies around the three exceptional divisors are now
\[ E_0 \mapsto (m_0, n_0), \ E_1 \mapsto (m_1 + m_3, n_1), \ E_2 \mapsto (m_2 + m_3, n_2). \]

We finally define $S$ to be the normalization of the pull back $\Sigma \times_{\mathbb{P}^1 \times \mathbb{P}^1} Z$. 
Proposition 29. The surface $S$ is smooth, and each irreducible component of the branch locus $B$ has local monodromy of order $n$.

Proof. Given two irreducible components $B_i, B_j$, they are smooth and they intersect transversally in exactly one point, or are disjoint. Hence it is sufficient to show

- that the inertia subgroups (image of the local monodromy) are cyclic of order $n$
- if $B_i, B_j$ intersect, the corresponding inertia subgroups generate $(\mathbb{Z}/n)^2$.

By our assumptions all the local monodromies are elements of order $n$, hence the first assertion.

The second assertion follows from the following fact: $(\mathbb{Z}/n)^2$ is generated by pairs of the form

$(a, 0), (0, b), a, b \in (\mathbb{Z}/n)^*$,

$(a, 0), (c, b), a, b \in (\mathbb{Z}/n)^*$,

$(0, a), (b, c), a, b \in (\mathbb{Z}/n)^*$,

or of the form

$(n - m_3, n_0), (m_0, n_0)$,

since their span is the span of $(n - m_3, 0), (m_0, n_0)$, and $n_0, m_3 + m_0$ are units in $(\mathbb{Z}/n)$.

□

Proposition 30. Let $f : S \rightarrow B$ be the morphism induced by the projection $\Sigma \rightarrow B$. Then

1. the genus of the fibres $F$ equals $g = n - 1$
2. the genus of the base curve $B$ equals $b = \frac{n - 1}{2}$
3. all the fibres are smooth, except the fibres over $x = 0, x = 1, x = \infty$, which consists of two smooth curves of genus $b$ intersecting transversally in exactly one point.
4. $f$ is the Albanese map of $S$, i.e., $b = q := h^1(O_S) = h^1(O_S(K_S))$.

Proof. (1) and (2) follow from Hurwitz’ formulae:

$$2(g - 1) = -2n + 4(n - 1), \quad 2(b - 1) = -2n + 3(n - 1),$$

since the general fibre is a $(\mathbb{Z}/n)$-cyclic cover of $\mathbb{P}^1$ totally ramified in 4 points, while $B$ is a $(\mathbb{Z}/n)$-cyclic cover of $\mathbb{P}^1$ totally ramified in 3 points.

(3): the fibres over $x = 0, x = 1, x = \infty$ are the inverse images of two smooth curves meeting transversally in exactly one point $P$ and which are part of the branch locus $B$. The covering is totally branched on $P$, hence these special fibres consist of two smooth curves meeting transversally in exactly one point $P'$. Both are $(\mathbb{Z}/n)$-cyclic covers of $\mathbb{P}^1$ totally ramified in three points, hence their genus equals $b$. 
The other fibres are the inverse image of a $\mathbb{P}^1$ intersecting the branch locus transversally in 4 points, hence they are all smooth of genus $g = n - 1$.

(4) There are several ways to prove that $b = q$, some more explicit, the following one is in the spirit of this paper.

We want to calculate

$$q := h^1(\mathcal{O}_S) = h^1(\mathcal{O}_S(K_S)) = h^1(\omega_{S|B}(K_B))$$

and we denote $\omega_{S|B}$ for short by $\omega$.

We use the Leray spectral sequence for $f$, saying that

$$q = h^0(B, R^1 f_*(\omega))(K_B)) + h^1(f_*(\omega))(K_B))$$

The first term, by relative duality, equals $b = h^0(B, \mathcal{O}_B(K_B))$, while the second vanishes, as $f_*(\omega) = V = A \oplus Q$ by Fujita’s second theorem. Then $h^1(V(K_B)) = h^0(V^\vee)$ by Serre duality, and $h^0(A^\vee) = 0$ since $A$ is ample, while $h^0(Q^\vee) = 0$, else the monodromy of some summand of the unitary flat bundle $Q$ would have trivial monodromy, contradicting the irreducibility of the monodromy representation.

$\square$

**Proposition 31.** The smooth surface $S$ is minimal of general type with $K_S$ ample, and with invariants

$$e(S) = c_2(S) = 3 + 2(n - 2)(n - 3) = 2n^2 - 10n + 15;$$

$$K_S^2 = 5(n - 2)^2.$$  

They have positive index $\sigma(S) = \frac{1}{5} (K_S^2 - 2e(S)) > 0$ and indeed their slope $\frac{K_S^2}{e(S)} \geq 2,5$.

We have that the universal cover of $S$ is the unit ball in $\mathbb{C}^2$ if and only if $n = 5$, which corresponds to the case of three distinct surfaces $S', S'', S'''$.

**Proof.** The calculation for the topological Euler-Poincaré characteristic $e(S)$ follows from the Zeuthen Segre formula asserting that $e(S)$ equals the sum of the product $e(B)e(F) = 4(b - 1)(g - 1)$ with the number $\mu$ of singular fibres counted with multiplicity: here therefore $\mu = 3$ and we get $e(S) = 3 + 2(n - 2)(n - 3) = 2n^2 - 10n + 15$.

To calculate $K_S^2$, we observe that $K_S$ is numerically equivalent to the pull back of $K_Z + \frac{2}{n}B$.

Since $B \equiv 4L_1 + 4L_2 - 2\Sigma_iE_i$, where $L_1, L_2$ are the total transforms of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, and $K_Z = -2L_1 - 2L_2 + \Sigma_iE_i$, we obtain that $B \equiv -2K_Z$, hence $K_S$ is numerically equivalent to the pull back of $K_Z + \frac{2}{n}B = -\frac{2}{n}K_Z$.

Since $-K_Z$ is ample, and $K_Z^2 = 5$, we easily obtain that $S$ has has ample canonical divisor $K_S$, and

$$K_S^2 = 5(n - 2)^2.$$
The surface $S$ is minimal since $K_S$ is ample.

We now calculate the slope as
\[
\frac{5(n-2)^2}{3 + 2(n-2)(n-3)} = \frac{5}{2} \frac{n-2}{n-3 + \frac{3}{2(n-2)}} > \frac{5}{2}.
\]

The same formula shows that the slope is a strictly decreasing function of $n$, tending to $\frac{5}{2}$ as $n \to \infty$, and beginning with slope $= 3$ for $n = 5$. But, by the theorem of Yau, slope equal to 3 is equivalent to having the ball as universal cover.

Consider now the case $n = 5$: the 4-tuple of residue classes modulo 5 is equivalent, modulo simultaneous multiplication by a unit, to $1 + 1 + 2 = 5$, and this is the only representation via integer rests which add up to 5. Also the $n_i$ are uniquely determined as $1 + 2 + 2 = 5$.

There are two different cases: $m_3 \neq m_i$, or (up to renumbering) $m_3 = m_0 = m_1$; in this second case there are two subcases, according to $n_0 = n_1$ or $n_0 \neq n_1$.

\[\square\]

**Remark 32.** The above three surfaces which occur for $n = 5$ have already been constructed in [3].

Being ball quotients, they are rigid.

In joint work of the first author together with Ingrid Bauer it was recently shown that the above surfaces $S$ for $n \geq 5$ are rigid.

Another interesting question is whether the surfaces $S$ are always $K(\pi, 1)$'s, i.e., whether their universal covering is always contractible.

Recall now the following algebraic formula for the Euler number, the so-called Zeuthen-Segre formula (see the lecture notes [6]).

**Definition 33.** Let $f : S \to B$ be a fibration of a smooth algebraic surface $S$ onto a curve of genus $b$, and consider a fibre $F_t = \sum n_i C_i$, where the $C_i$ are irreducible curves.

Then the divisorial singular locus of the fibre is defined as the divisorial part of the critical scheme, $D_t := \sum (n_i - 1) C_i$, and the Segre number of the fibre is defined as
\[
\mu_t := \deg F + D_t K_S - D_t^2,
\]
where the sheaf $F$ is concentrated in the singular points of the reduction of the fibre, and is the quotient of $O_S$ by the ideal sheaf generated by the components of the vector $dt/s$, where $s = 0$ is the equation of $D_t$, and where $\tau$ is the pull-back of a local parameter at the point $t \in B$.

More concretely,
\[
\tau = \Pi_j f^n_j, s = \tau/(\Pi_j f_j),
\]
and the logarithmic derivative yields
\[
d\tau = s[\sum n_j (df_j \Pi_{h \neq j} f_h)].
\]
The following is the refined Zeuthen-Segre formula

**Theorem 34.** Let \( f : S \to B \) be a fibration of a smooth algebraic
surface \( S \) onto a curve of genus \( b \), and with fibres of genus \( g \).
Then
\[
c_2(S) = 4(g - 1)(b - 1) + \mu,
\]
where \( \mu = \sum_{t \in B} \mu_t \), and \( \mu_t \geq 0 \) is defined as above. Moreover, \( \mu_t \) is
strictly positive, except if the fibre is smooth or a multiple of a smooth
curve of genus \( g = 1 \).

**Proposition 35.** Let \( S \) be one of the surfaces considered in this section.
Then any surface \( X \) which is homeomorphic to \( S \) has Albanese map
which is a fibration onto a curve \( B \) of the same genus \( b = 1/2(n - 1) \)
as the Albanese image of \( S \). If moreover \( X \) is diffeomorphic to \( S \), the
Albanese fibres have the same genus \( g = 2b = n - 1 \) and, if the number
of singular points on the fibres is finite, there are only three singularity
on the fibres, counted with multiplicity. In particular, there are at most
three singular fibres.

**Proof.** The first two statements follow directly from \([7]\), theorem A.
For the last statement we invoke the above refined Zeuthen-Segre
formula
\[
e(X) = 4(b - 1)(g - 1) + \mu.
\]

Since \( b, g \) are the same for \( S \) and \( X \), it follows that \( \mu = 3 \), which
shows the third assertion.

The refined version of the Zeuthen-Segre formula implies in particular
that, if \( D \) is the divisorial part of the critical locus, then \( D = D_K X - D^2 \), where \( D_K X - D^2 = 2p(D) - 2 - 2D^2 \) is a positive even
number.

Each non reduced fibre \( F_t = \Sigma i n_i C_i \) gives a contribution \( D_t := \Sigma (n_i - 1)C_i \) to \( D \), and Zariski’s lemma says that, if \( D_t \neq 0 \), then
\( D^2_t < 0 \) unless \( F_t \) is a multiple fibre.

If we had a multiple fibre \( F_t = mC \), then we would have \( D_t K_X - D^2_t = D_t K_X = (m - 1)/m F K_X = (2g - 2)(m - 1)/m \geq (g - 1) \geq 4 \), which
is a contradiction. Hence there are no multiple fibres.

Assume that \( F_t \) is a non reduced fibre, so that \( D_t \neq 0 \): then \( D_t K_X - D^2_t = 2 \), since it is a strictly positive even integer which is not greater
than 3.

So, if there are infinitely many singular points on the fibres, then
there is exactly one non reduced fibre and at most one more singular
point; in particular, there are at most two singular fibres.

\[\square\]

We can summarize our main result in the following theorem, for
which we give two proofs, one self-contained and based on Fujita’s sec-
ond theorem, the other based on the theory of hypergeometric integrals.
Theorem 36. There exists an infinite series of surfaces with ample canonical bundle, whose Albanese map is a fibration $f : S \to B$ onto a curve $B$ of genus $b = 1/2(n - 1)$, with fibres of genus $g = 2b = n - 1$; here $n \geq 5$ can be any integer relatively prime with 6 and $f$ is as in Prop. 30.

These Albanese fibrations yield negative answers to Fujita’s question about the semiampleness of $V := f_*\omega_S|B$, since here $V := f_*\omega_S|B$ splits as a direct sum $V = A \oplus Q$, where $A$ is an ample vector bundle, and $Q$ is a unitary flat bundle with infinite monodromy group.

The fibration $f$ is semistable: indeed all the fibres are smooth, with the exception of three fibres which are the union of two smooth curves of genus $b$ which meet transversally in one point.

For $n = 5$ we get three surfaces which are rigid, and are quotient of the unit ball in $\mathbb{C}^2$ by a torsion free cocompact lattice $\Gamma$.

The rank of $A$, respectively $Q$ is in this case equal to 2.

Proof. By Propositions 29, 30, 31 the only assertion which needs to be shown is that, if we considering the splitting of $V := f_*\omega_S|B$ as a direct sum $V = A \oplus Q$, where $A$ is an ample vector bundle, and $Q$ is a unitary flat bundle, then $Q$ has infinite monodromy group.

We observe that the group $G = \mathbb{Z}/n$ acts on the fibration, thus we have a splitting according to the characters of $G$, $j \in \mathbb{Z}/n$, $V = \bigoplus_{j \in \mathbb{Z}/n} V_j$.

The fact that all the fibres are smooth and that the only singular fibres are two smooth curves intersecting transversally in one point shows that the vanishing cycles are homologically trivial. Hence the local monodromies in cohomology are trivial, thus we have a flat vector bundle $H := R^1 f_* (\mathcal{O})$, which is a holomorphic flat bundle having $V$ as a holomorphic subbundle.

Similarly we have a splitting

$$H = \bigoplus_{j \in \mathbb{Z}/n} \mathcal{H}_j,$$

where the flat bundles $\mathcal{H}_j$ have all rank 2 for $j \neq 0$, as observed in section three.

Moreover, we have the following direct sum of complex vector bundles $\mathcal{H}_j = V_j \oplus V_{-j}$, and there are a priori several possible cases:

- $\mathcal{H}_j = V_j$ and $V_{-j} = 0$, hence $V_j$ is a flat holomorphic bundle; in this case the bundle $\mathcal{H}_j$ carries a flat Hermitian form which is positive definite;

- $\mathcal{H}_j = V_{-j}$ and $V_j = 0$, hence $V_{-j}$ is a flat holomorphic bundle; in this case the bundle $\mathcal{H}_j$ carries a flat Hermitian form which is negative definite;

- $\mathcal{H}_j = V_j \oplus V_{-j}$, both summands have rank 1, and here the bundle $\mathcal{H}_j$ carries a flat Hermitian form which is indefinite. This case...
could a priori bifurcate in the cases $V_j$ is flat, or $V_j$ is ample (i.e., it has strictly positive degree).

**First Proof:**

Step 1: $V$ is not flat.

In fact, otherwise (see for instance theorem 4 of [9])

$$0 = 12 \deg(V) = K_S^2 - 8(g - 1)(b - 1);$$

hence $K_S^2 = 8(g - 1)(b - 1) = 2e(S) - 6$, contradicting Proposition 31.

Step 2: hence $V = \bigoplus_j V_j$ admits an ample rank 1 summand $V_j = A_j$.

Step 3: It suffices to prove the theorem in the case where $n$ is prime.

In fact, if $k$ divides $n$, $n = hk$, we have an analogous fibration $f_k : S_k \to B_k$ for the surface $S_k$ obtained by taking the associated $(\mathbb{Z}/k)^2$ covering. Pulling back the fibration to $B$, under $\psi : B \to B_k$, we obtain a surface $S' = S/G'$, where $G' = \mathbb{Z}/h$; and the fibration $f$ factors through $f' : S' \to B$. Hence $V' = \psi^*(V_k)$ is a direct summand of $V$ and we are done, since $V'$ has a unitary flat summand $Q'$ with infinite monodromy.

Step 4: There is an eigenbundle $H_j$ with infinite monodromy.

This follows from Step 2 and the following lemma.

**Lemma 37.** If $V_j = A_j$ is an ample rank 1 summand, then $H_j$ is irreducible and with infinite monodromy.

**Proof. (of the Lemma).** We first show that the rank two flat vector bundle is irreducible. Otherwise there would be an exact sequence of flat vector bundles

$$0 \to \mathcal{H}' \to \mathcal{H}_j \to \mathcal{H}'' \to 0$$

where both $\mathcal{H}'$, $\mathcal{H}''$ have rank 1.

Since $\mathcal{H}_j = V_j \oplus \mathcal{V}_{-j}$, we get a nontrivial homomorphism $V_j \to \mathcal{H}_j$ which realizes $V_j$ as a holomorphic subbundle. Composing with the above surjection $\mathcal{H}_j \to \mathcal{H}''$ we get a holomorphic homomorphism $V_j \to \mathcal{H}''$, which must be zero since the target has degree zero, while $V_j = A_j$ has positive degree. We deduce a nontrivial holomorphic homomorphism $V_j \to \mathcal{H}'$, which must be zero by the same argument, and we have found a contradiction to the fact that $V_j \to \mathcal{H}_j$ is injective.

Step 4. Observe preliminarily that our surfaces, the Albanese map $f$ and all the bundles $V, Q$, are defined over $\mathbb{Z}$.

Now, by construction we have a flat rank 2 summand $V_{-1} = \mathcal{H}_{-1}$ (since $m_0 + m_1 + m_2 + m_3 = n$). Hence, when $n$ is prime, $\mathcal{H}_{-1}$ and $\mathcal{H}_j$ are Galois conjugate. The condition that the monodromy is infinite is obviously invariant under Galois conjugation (since a finite group of matrices transforms to a finite group under a field automorphism).
Hence also $V_{-1} = \mathcal{H}_{-1}$ has infinite monodromy, and it is a direct summand of $Q$ with infinite monodromy.

**Second Proof:**

By corollary [20] there is $j \in (\mathbb{Z}/n)^*$ such that $\mathbb{H}_j$ carries a monodromy invariant indefinite Hermitian form $H_j$, and is irreducible with infinite monodromy.

Therefore also $\mathcal{H}_j$ has infinite monodromy. Since $j$ is a unit, it follows that $\mathcal{H}_{-1}$ and $\mathcal{H}_j$ are Galois conjugate. Hence $V_{-1} = \mathcal{H}_{-1}$ has also infinite monodromy, and the same holds for $Q$, of which $V_{-1}$ is a direct summand. 

**Remark 38.** In the standard case $V$ has a lot of flat summands.

In fact, $V_j = 0$ for $j \leq \frac{n}{3}$ (since $3j \leq n$ implies $j + j + [j(n-3)] < 2n \Rightarrow j + j + [j(n-3)] = n$); hence $V_{-j}$ is flat for $j \leq \frac{n}{3}$.

On the other hand, for $n=11$ we can take $m_0 = 1, m_1 = 2, m_2 = 3, m_3 = 5$ and then only $V_{10}$ is a flat summand, of course with infinite monodromy.

5. **General observations and relation with Shimura curves**

Consider our surfaces $S \to B$ as yielding a curve inside the compactified moduli space of curves of genus $g = n - 1$. The image of $B$ inside $\overline{\mathcal{M}}_g$ intersects the boundary only in points belonging to the divisor $\Delta_{g/2,g/2}$.

Moreover, under the Torelli map $\mathcal{M}_g \to \mathfrak{A}_g$, the image does not go to the boundary, since the singular fibres have compact Jacobian.

For $n = 5$ we obtain a rigid curve inside $\overline{\mathcal{M}}_g$, a phenomenon which is not new: compare the examples provided by double Kodaira fibrations ([8]).

Now, $B$ parametrizes all the curves with an action of $\mathbb{Z}/n$ whose quotient is $\mathbb{P}^1$, and with branch locus $S$ consisting of 4 points: because all deformations preserving the symmetry come from $H^1(C, \Theta_C)^G$ which is isomorphic to $H^1(\mathbb{P}^1, \Theta(-S))$, the space of logarithmic deformations of the pair consisting of $\mathbb{P}^1$ and the 4 points on it.

$B$ parametrizes, via the Torelli map, also principally polarized Abelian surfaces with such a symmetry.

The question is whether the symmetry-preserving deformations of these Abelian varieties are just the ones parametrized by $B$.

The main point is that (see [11] and [18], especially for more details concerning the relation with Shimura curves) the dual of $H^1(C, \Theta_C)^G$ equals $H^0(2K_C)^G$, while the tangent space to the symmetry preserving deformations of the Abelian varieties is given by

$$\text{Sym}^2(H^0(K_C))^G = \text{Sym}^2(\bigoplus_j V_j)^G = \bigoplus_{j \leq n/2} (V_j \otimes V_{-j}).$$
Observe that $V_0 = 0$, while, for a character $j$, writing as usual $\mu_{i,j} = \frac{1}{n}[m_{i,j}]$, the condition $\sum_i \mu_{i,j} = 2$ is equivalent to $\dim (V_j \otimes V_{-j}) = 1$, else one has $\dim (V_j \otimes V_{-j}) = 0$.

In other words, the number of parameters for the symmetry-preserving deformations of these Abelian varieties is just the number of rank 2 ample bundles in the direct image sheaf $f_*(\omega)$.

If there is only one such ample summand, then this means that we have a Shimura curve in $\mathbb{A}_g$. This situation leads to a finite number of cases, which were classified by Moonen in $[39]$ (see $[18]$ for groups more general than cyclic groups).

Interest in these Shimura curves is due to a conjecture by Oort that there should not be such curves as soon as $g$ is bigger than 7, see $[38]$ and references therein for results in this direction.

6. Appendix

**Proposition 39.** Let $m_0, m_1, m_2, m_3, n \in \mathbb{Z}$ with $0 < m_k \leq n - 3$ ($0 \leq k \leq 3$) and $m_0 + m_1 + m_2 + m_3 = n$. For $j \in 1, \ldots, n - 1$, let $\hat{H}_j$ be the local system as in Section 3. Assume additionally that each of the numbers $m_0, \ldots, m_3$ is coprime to $n$ (resp., assume that that $j$ is coprime to $n$ with no further assumption on $m_0, \ldots, m_3$). Then the local systems $\hat{H}_j$ are irreducible for $j = 1, \ldots, n - 1$ (resp., for $j$ prime to $n$).

**Proof.** By construction, the local sections of $\hat{H}_{-j}$ are variations in $x$ of periods on the desingularizations of the curves

$$z_i^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - xy_0)^{m_3}, \ x \in \mathbb{C} \setminus \{0, 1\}$$

of the form (given on the the affine part belonging to $y_1 = 1$)

$$\int_{\gamma} \frac{y_0^s (1 - y_0)^t (1 - xy_0)^u}{z_1^j} dy_0,$$

where $s, t, u$ are integers, cf. $[40]$, Section 2. It is convenient to introduce integers $A, B, C$ and rational numbers $a, b, c$ by the following conditions:

$A = (1-b)n = m_0, B = (b+1-c)n = m_2, C = an = m_3, n-A-B-C = m_1$.

Therefore

$$a = \frac{m_3}{n}, \ b = 1 - \frac{m_0}{n}, \ c = 2 - \frac{m_0}{n} - \frac{m_2}{n}.$$

If $\gamma$ denotes integration from 0 to 1 then the above integral can be expressed as a hypergeometric function as follows:

$$\int_0^1 \frac{y_0^s (1 - y_0)^t (1 - xy_0)^u}{z_1^j} dy_0 = D_2 F_1(ja-u, jb-j+1+s, jc-2j+2+t+s; x)$$

where $D$ is a constant in $\mathbb{C}$, cf. $[40]$, Formula (7). Hence, in order to show that $\hat{H}_j$ is irreducible, it suffices to show that the hypergeometric
differential equation belonging to $\binom{2}{F_1}(ja - u, jb - j + 1 + s, jc - 2j + 2 + t + s; x)$ is irreducible. This is the case if and only if the values $ja - u$, $jb - j + 1 + s$ and the differences $(jc - 2j + 2 + t + s) - (ja - u)$, $(jc - 2j + 2 + t + s) - (jb - j + 1 + s)$ are not contained in $\mathbb{Z}$, cf. [5], Cor. 3.10. Obviously, the latter condition holds if and only if the values $ja = \frac{jm_3}{n}$, $jb = -\frac{jm_0}{n} + j$ and 

$$ja - jc = \frac{jm_3}{n} + \frac{jm_0}{n} + \frac{jm_2}{n} - 2j = -\frac{jm_1}{n} - j$$

as well as

$$jb - jc = j - \frac{jm_0}{n} + \frac{jm_0}{n} + \frac{jm_2}{n} - 2j = \frac{jm_2}{n} - j$$

are not contained in $\mathbb{Z}$. This holds by our assumptions. □

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