The Horava–Lifshitz modifications of the Casimir effect at finite temperature revisited

Hongbo Cheng

1 Department of Physics, East China University of Science and Technology, Shanghai 200237, China
2 The Shanghai Key Laboratory of Astrophysics, Shanghai 200234, China

Received: 1 May 2022 / Accepted: 26 September 2022 / Published online: 15 November 2022
© The Author(s) 2022

Abstract We investigate the Casimir force for parallel plates at finite temperature in the Horava–Lifshitz (HL) theory. We find that the HL exponent cannot be chosen as an integer, or the Casimir energy will be a constant, and further, the Casimir force between two parallel plates will vanish. The higher temperature causes the attractive Casimir force to weaken, which is consistent with the original results confirmed theoretically and experimentally. We can select the HL factor appropriately to obtain a thermally revised Casimir force similar to the standard results for the parallel plates.

1 Introduction

Under the background fields or within the quantization volume, the Casimir effect as a direct consequence of quantum field theory subject to a change in the spectrum of vacuum oscillations was put forward more than 70 years ago [1]. The Casimir effect shows a shift of the vacuum energy because of the nontrivial boundary conditions or the topology of the space [1–3]. Subsequently, the repulsive Casimir force for a conducting spherical shell was found by Boyer [4]. The nature of the Casimir force depends on the abovementioned conditions. The attractive Casimir force between two parallel plates was varied experimentally by Sparnaay [5]. Further, the precision in measuring the Casimir effect has been greatly improved experimentally, and more precise measurements have been performed [5–11]. The characteristics of the Casimir effect, including the sign of the Casimir energy and the nature of the Casimir force, have been discussed in many contexts, because the Casimir effect is involved in various factors including the boundaries, the structure of spacetime, the topologically nontrivial backgrounds, some types of quantum field theory, and temperature [8,12–15]. Much effort has been devoted to the problems and related topics. The Casimir force can be changed by the geometry of the boundaries [16–19]. The presence of extra dimensions with their size and shape also affects the Casimir effect [20–34]. The quantum field theory at finite temperature shares many effects [35,36]. The thermal influence certainly modifies the Casimir effect and cannot be neglected [37–49].

As a kind of modified field theory, the Horava–Lifshitz (HL) theory proposes a power-counting renormalizable theory of gravity. The HL model enables scaling of coordinates like \( t \to b^z t \) and \( x^i \to b x^i \) to lead to spacetime anisotropy, where \( b \) is a length constant and \( z \) is the critical exponent [50–60]. According to the energy scale, we can adjust the value of \( z \) to restore the renormalizability of the theory [50–60]. The factor \( z \) tends to be unity at low energies and is greater than unity within the micrometer-sized region [50–60]. The HL approach has been used for example to describe black holes and gravitational waves to show its influence [61–66]. The kinetic terms with higher-order spatial derivatives appear in the Lagrangian due to rescaling of the coordinates in the context of HL theory [50–60,67,68]. These new kinetic terms restore the renormalization of the modified model while breaking the Lorentz symmetry [9,48–51,64]. The Lorentz violation may happen at high energies or in an extremely low-energy region [53–60].

In cosmology, the finite action from the HL theory leads to the beginning of the flat and homogeneous universe, while the different versions of the theory can result in different cosmological models [67,68]. The HL gravity only involves regular black holes [67,68] and is considered a candidate for quantum gravity. Horava’s theory relaxes Lorentz invariance in the ultraviolet (UV) limit and is power-counting renormalizable. In the infrared (IR) limit, this kind of quantum gravity reduces to general relativity [50–60].

It is necessary to study the Casimir effect for parallel plates within the framework of HL theory. The Casimir effect has
been widely analyzed in many areas of physics, so the effect can be thought of as a window to probe many kinds of models \cite{8,10,12–15}. As mentioned above, the HL model provides the kinetic terms with higher-order spatial derivatives in the Lagrangian. This extension will violate the Lorentz symmetry to change the dispersion relation of original quantum field theory. The Lorentz-violating extensions certainly lead to a noticeable revision of the Casimir effect. Much effort has been devoted to the Casimir effect in HL-like theories. Experimental investigation on micrometer-size physics has inspired theoretical research on the Casimir effect in the Lorentz-violating generalizations of the standard quantum field theory whose Lorentz symmetry is fully conserved \cite{69–74}. The Casimir effect has been considered in the context of Lorentz-breaking scenarios \cite{75–79}. The Casimir effect for two parallel plates was studied in anisotropic spacetime dominated by the HL theory, and the revised results of the effect deviate greatly from the measurements \cite{76}. The authors of Ref. \cite{77} discussed a massless scalar real field satisfying the Dirichlet, Neumann, and mixed-boundary conditions in the HL-like theory to reflect the dependence of the Casimir effects for parallel plates or two-dimensional rectangular boxes on the HL parameter relating to the breaking of Lorentz symmetry and the types of boundary conditions \cite{77}. The Casimir energy of a scalar field trapped in a box in spacetimes with three spatial Lifshitz dimensions was computed, and the energy is singular or regular according to the value of the critical exponent \cite{80}. The Casimir energy was also evaluated in a box in the background with a Lifshitz extra dimension, and the influence of the Lifshitz parameter was observed \cite{80}. The Casimir energy for parallel plates in a Lifshitz-like field theory was considered, and the relations among the Casimir energy, Lifshitz fixed-point parameter, mass of the scalar field, dimensionality, and temperature were obtained \cite{81}.

It is important to investigate the influence of the HL theory on the Casimir effect with nonzero temperature, because the quantum field theory at finite temperature shares many effects. Thermal corrections to the Casimir energy under the HL field theory were found around the Kehagias–Sfetsos black hole \cite{75,82}. The Casimir free energy for parallel plates at a Lifshitz fixed point (LP) was derived in flat spacetime when the temperature was not equal to zero in order to show the thermal corrections to the Casimir effect, and the results are significant \cite{81}. It is important to scrutinize the Casimir effect at finite temperature within the frame of the HL scenario. We should further extend the research in Ref. \cite{81} to elaborate the Casimir energy and Casimir force in terms of temperature in view of the modified gravity, and we should investigate how the temperature affects their nature. In order to show the thermal corrections to the Casimir effect for parallel plates at a Dirichlet boundary condition, we derive the frequency of a massless real scalar field with thermal corrections by means of the finite-temperature HL scheme. We regularize the frequency to obtain the Casimir energy density with the help of the hypergeometric functions and zeta function. The Casimir force involving the HL factor for parallel plates can also be obtained from the Casimir energy per unit volume at finite temperature. According to the expressions of the Casimir energy and Casimir force, the influence of HL theory and temperature can be examined, and the results will be discussed.

2 The Casimir effect for parallel plates at finite temperature in the Horava–Lifshitz theory

We make use of the imaginary time formalism in the finite-temperature field theory to describe the massless real scalar field in thermal equilibrium \cite{35}. A partition function for the scalar field system introduces \cite{35,36,50–60}

\[
Z = N \int D\phi \exp \left[ \int_0^\beta d\tau \int d^3 x \mathcal{L}(\phi, \partial_E\phi) \right]
\]

where \( \mathcal{L}(\phi, \partial_E\phi) \) is the Lagrangian density corrected by HL theory. \( N \) is a constant. The subindex “period” lets the scalar field \( \phi(\tau, x) \) obey \cite{35,36}

\[
\phi(0, x) = \phi(\beta, x)
\]

where \( \tau = it \) and \( \beta = \frac{1}{\tau} \), the inverse of the temperature. The spatial coordinates are denoted as \( x \). Here, the scalar field \( \phi(\tau, x) \) confined to the interior of a parallel-plate device must satisfy the Dirichlet boundary conditions at the plates. According to the solutions to the Klein–Gordon equations and the boundary conditions, the generalized zeta function for a massless real scalar field can be written as \cite{35,36,50–60,76}

\[
\zeta(s; -\partial_E) = Tr((-\partial_E)^{-s})
= \int \frac{d^2 k}{(2\pi)^2} \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty}
\times \left[ l^2(\varepsilon-1) \left( k_1^2 + k_2^2 + \left( \frac{n\pi}{a} \right)^2 \right)^{\varepsilon} + \left( \frac{2l\pi}{\beta} \right)^2 \right]^{-s}
\]

where

\[
\partial_E = \frac{\partial^2}{\partial\tau^2} - (-1)^z \xi_0^{2(\varepsilon-1)} (\nabla^2)^z
\]

Here, \( k = \sqrt{k_1^2 + k_2^2} \) denotes the transverse components of the momentum. As mentioned above, \( \varepsilon \) is the critical exponent \cite{50–60}; \( a \) is the distance of the plates. We derive the
generalized zeta function (3) to obtain

\[
\zeta(s; -\beta) = \frac{1}{4\pi} \frac{1}{zs - 1} \left( \frac{\pi}{a} \right)^{-2sz+2} \zeta(2sz - 2) + \frac{1}{2\pi} \frac{1}{zs - 1} l_0^{-2z(z-1)} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left( \frac{n_1\pi}{a} \right)^{-2sz+2} \times F \left( s, s - \frac{1}{z}, s - \frac{1}{z} + 1; -\beta_1 \right) \tag{5}
\]

where

\[
\beta_1 = l_0^{-2z(z-1)} \left( \frac{n_1\pi}{a} \right)^{-2z} \left( \frac{2n_2\pi}{\beta} \right)^2,
\tag{6}
\]

and \( F(a, b; c; x) \) is the hypergeometric function [83]. We obtain the total energy density to the system under thermal influence [35, 36]:

\[
\varepsilon = -\frac{\partial}{\partial \beta} \left( \frac{\partial \zeta(s; -\beta)}{\partial s} \right)_{s=0}
= \sum_{n=1}^{\infty} \frac{2 \times (-1)^n}{n\pi + 1} \left( \frac{\pi}{a} \right)^{2nz+2} \beta^{2n-1} \times \zeta(-2(nz + 1)) \zeta(2n) \left( \frac{n}{a} \right) \beta^{2n-1}
+ \frac{1}{\sin \frac{\pi}{z}} \left( \frac{1}{z} \right) l_{-2z-2(n-\frac{1}{2})} \left( 2\pi \right)^{2(\frac{1}{z} - 1)} \zeta(-2z) \times \zeta \left( 2 \left( \frac{1}{z} - 1 \right) \right) \left( \frac{n}{a} \right)^{2z} \beta^{1 - \frac{2}{z}}
- \frac{1}{2\sin \frac{\pi}{z}} l_{2z-2(n-\frac{1}{2})} \left( 2\pi \right)^{2(\frac{1}{z} - 1)} \zeta \left( 2 \left( \frac{1}{z} - 1 \right) \right) \beta^{2 - \frac{2}{z}}
+ \frac{1}{\sin \frac{\pi}{z}} \sum_{n=1}^{\infty} (-1)^n l_{-2z-2(n-\frac{1}{2})} \left( 2\pi \right)^{2(\frac{1}{z} - 1)} \zeta \left( 2 \left( \frac{1}{z} - 1 \right) \right) \beta^{1 - \frac{2}{z}}
\times (2\pi)^{2(\frac{1}{z} - 1)} \left( n + 1 - \frac{1}{z} \right)
\times \zeta(-2(n + 1)z) \zeta \left( 2 \left( n + 1 - \frac{1}{z} \right) \right)
\times \left( \frac{\pi}{a} \right)^{2z+2nz} \beta^{2n+1 - \frac{2}{z}}
+ \frac{1}{\sin \frac{\pi}{z}} \sum_{n=1}^{\infty} (-1)^n l_{-2z-2(n-\frac{1}{2})} \left( 2\pi \right)^{2(\frac{1}{z} - 1)} \zeta \left( 2 \left( \frac{1}{z} - 1 \right) \right) \beta^{2 - \frac{2}{z}}
\times (2\pi)^{2(\frac{1}{z} - 1)} \left( n - 1 \right) \zeta(-2nz)
\times \zeta \left( 2 \left( n - 1 \right) \right) \left( \frac{\pi}{a} \right)^{2nz} \beta^{2(n - \frac{1}{2}) - 1}
\tag{7}
\]

with the help of the properties of the hypergeometric function and Euler’s reflection formula of the gamma function [83]. We regularize the expression by means of a zeta function technique \( \zeta(1 - s) = 2(2\pi)^{-s} \Gamma(s) \zeta(s) \cos \frac{n\pi}{2} \) to obtain the Casimir energy per unit volume for parallel plates at finite temperature as follows.

\[
\varepsilon_C = \sum_{n=1}^{\infty} \frac{(-1)^n}{zn + 1} 2^{-2z(z+1)n} - 2\pi - 2z \zeta(2z(z-1)n) - 2\pi - 2z \zeta(2n-2) \beta^{2n-1 - \frac{2}{z}}
\times \Gamma(2zn + 3) \zeta(2zn + 3) \zeta(2n) \sin zn\pi
\times \zeta \left( 2 \left( 1 - 1 \right) \right) \cos \frac{n\pi}{2} \sin zn\pi
\times \zeta \left( 2 \left( n + 1 \right) \right) \zeta \left( 2 \left( n + 1 \right) \right) \sin zn\pi
\end{equation}

\[
+ \frac{1}{\sin \frac{\pi}{z}} \sum_{n=1}^{\infty} (-1)^n \left( n + 1 - \frac{1}{z} \right) 2^{-2z(z+1)n} \beta^{2n+1 - \frac{2}{z}}
\times \Gamma(2zn + 3) \zeta(2zn + 3) \zeta(2n) \sin zn\pi
\times \zeta \left( 2 \left( n + 1 \right) \right) \zeta \left( 2 \left( n + 1 \right) \right) \sin zn\pi
\times \zeta \left( 2 \left( n + 1 \right) \right) \zeta \left( 2 \left( n + 1 \right) \right) \sin zn\pi
\times \zeta \left( 2 \left( n + 1 \right) \right) \zeta \left( 2 \left( n + 1 \right) \right) \sin zn\pi
\times \zeta \left( 2 \left( n + 1 \right) \right) \zeta \left( 2 \left( n + 1 \right) \right) \sin zn\pi
\tag{8}
\]

The Casimir energy consists of five parts if the HL variable is not chosen to be an integer. Four of the parts contain \( \sin z\pi \) or \( \sin zn\pi \) terms, where \( n \) is a positive integer. If the critical exponent \( z \) is equal to the integer, the Casimir energy will reduce to

\[
\varepsilon_C \bigg|_{z=\text{integer}} = \frac{l_0^{2(\frac{1}{z} - 1)}}{2\pi z} \Gamma \left( 1 + \frac{2}{z} \right) \zeta \left( 1 + \frac{2}{z} \right) > 0
\tag{9}
\]

with the positive sign of the Casimir energy, regardless of whether the thermal influence persists. Within the frame of the HL model, the Casimir effect for parallel plates cannot exist due to the positive Casimir energy, which is not consistent with the original experiment [3, 5–7]. According to Eq. (9), it is obvious that the term left is independent of the plate gap \( a \). Further, the Casimir force between the parallel plates will disappear. We can adjust the parameter \( z \) to simulate the Casimir energy while keeping the critical exponent as a non-integer, and the Casimir energy can be similar to the measurements [3, 5–7].

The Casimir force between the plates is obtained by the derivative of the Casimir energy (8) with respect to the plate distance. The Casimir force per unit area with HL corrections on the plates obeying the Dirichlet boundary conditions can
be written as

\[ f_C = -\frac{\partial \varepsilon_C}{\partial a} = \sum_{n=1}^{\infty} (-1)^n 2^{-2(z+1)n} \text{Li}_{2n-1}(2z) a^{-2n-3} \beta^{2n-1} \]

\[ \times \Gamma(2zn + 3) \zeta(2zn + 3) \cos zn \pi \]

\[ - \frac{\sin z}{\pi} \sum_{n=1}^{\infty} (-1)^n (zn + 1) \sin z \pi \]

\[ - \frac{1}{\sin z} \sum_{n=1}^{\infty} (-1)^n (zn + 1) \sin z \pi \]

\[ \times \Gamma(2zn + 3) \zeta(2zn + 3) \cos zn \pi \]

3 Discussion

We discuss the Casimir force for parallel plates under thermal influence within the framework of the HL field theory to extend the work on the Casimir free energy in Ref. [81]. We derive the Casimir energy of the parallel-plate system which is described at finite temperature under the HL problem, and we obtain the corresponding Casimir force. Within the HL theory, the Casimir energy for the parallel-plate system is positive, and the Casimir force between the two parallel plates vanishes unless the HL factor \( z \) is a non-integer. The magnitude of the Casimir force between two parallel plates with an adequate HL parameter decreases as the thermal influence increases, while the Casimir force remains attractive. When the parallel plate distance is extremely large, the asymptotic value of the Casimir force approaches zero regardless of the temperature, while the terms involving the HL exponent appear in the expression. We argue that the HL exponent from a kind of field theory cannot be chosen as an integer exactly according to the results above. It is possible to select the HL exponent so that the Casimir effect of various models is comparable to the experimental results, and we could explore the special field theory in another direction in the future.

Acknowledgements This work is partly supported by the Shanghai Key Laboratory of Astrophysics.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical study and no experimental data.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indi-
