Propertime theories and reparametrization-invariant theories

R.R. Lompay*

Abstract

Reparametrization-invariant theories of point relativistic particle interaction with fields of arbitrary tensor dimension are considered. It has been shown that the equations of motion obtained by Kalman [G. Kalman, Phys. Rev. 123, 384 (1961)] are reproduced as the Euler-Lagrange equations for reparametrization-invariant theory in the propertime gauge. The formalism is developed that conserves manifest reparametrization invariance at each stage of calculations. Using the above formalism, the equations of motion are being analyzed and the dynamical variable theories are being constructed. It has been shown that the remained invariance kept after gauge fixation gives an identically-zero invariant Hamiltonian.

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Introduction

The task of obtaining the equations of motion for a point relativistic particle that moves in the external field of specified tensor dimensionality was analyzed by Kalman [1]. On the strength of the covariant Lagrange formalism, the equations of motion of the particle were derived in his approach both by varying the trajectory of the particle and by manifest varying its propertime. This approach agrees well with geometric interpretation in the space-time, but equations of motion obtained by Kalman differ from Euler-Lagrange equations by the terms resulted from the manifest variation of the particle propertime. In this paper, the theories of a point relativistic particle interaction with arbitrary tensor dimensionality field are considered having reparametrization invariance (RI). In such theories, the action functional is not changed at arbitrary functional replacement of the parameter that numerates the points of the particle world line. It appears that the equations of motion of the particle obtained by varying only the particle trajectory with no variation of its time parameter (i.e. relevant Euler-Lagrange equations) coincide with those obtained by Kalman, if one sets in the end of calculations the parameter equal to the propertime of the particle. Thus, the Kalman equations could be reproduced as Euler-Lagrange equations in the reparametrization-invariant theory in the propertime gauge.

This paper consists of two parts. The first part is dedicated to the obtaining and analysis of equations of motion for the relativistic particle that moves in the external field of specified tensor dimensionality. Paragraph 1.1 deals with a brief derivation of the Kalman equations. In paragraph 1.2, the general form of RI-action functional of the particle is considered, Euler-Lagrange equations are derived in the RI-theories and their equivalence with the Kalman equations in the propertime gauge is demonstrated. The formalism that conserves the manifest reparametrization invariance at each stage of calculations is developed. Paragraph 1.3 shows that Noether’s equality corresponding to the RI theory results in the presence in the equations of motion of the particle of the projector onto its proper 3-space. The notions of dynamical (effective) mass of the particle and the 4-force acting on it are introduced.

The second part deals with the construction of the theory of a total dynamical system consisting of the interacting point relativistic particle and of the field of arbitrary tensor dimensionality. The above system has an manifest RI. In paragraph 2.2, the total variation of the action functional for the system is found, the equations of motion of the particle and field are derived. The system 4-momentum vector and the canonical energy-momentum tensor (EMT) are constructed in the paragraph 2.2. The conservative character of canonical EMT is proven. In paragraph 2.3, the 4-angular momentum tensor and the relevant density tensor are constructed. The symmetric EMT is constructed in paragraph 2.4 by generalizing the Belinfante method [2] for the case of the systems consisting of the interacting point particle and fields. It has been shown in the paragraph 2.5 that the remained invariance kept after gauge fixation and related to the freedom of choice of an initial value of the parameter gives the zero Noether’s current, and, hence, does not lead to the integral of motion that is additional to the 10 relativistic ones. All the considerations presented in this part have been carried out by Lorentz-invariant and RI-manner.

*Institute of Electron Physics Nat.Acad.Sci.of Ukraine, Department of Theory of Elementary Interactions, Uzhgorod, Ukraine, email: lompay@zk.arbitr.gov.ua
The following notations are used in this paper. Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots$ run over the values from 0 to 3. The $(-1,1,1,1)$ metric signature is chosen. The velocity of light in vacuum is set equal to unit: $c = 1$.

1 Equations of motion of the particles

1.1 Kalman equations

It seems natural to assume that the action functional for the particle that moves in the external fields has a form:

$$ W[z; \phi] = W^P[z] + W^I[z; \phi] = \int_{\tau_1}^{\tau_2} d\tau L^P(z(\tau), \dot{z}(\tau)) $$

$$ + \sum_{\Sigma_1}^{\Sigma_2} dx \int_{\tau_1}^{\tau_2} d\tau L^I(z(\tau), \dot{z}(\tau); \phi(x), \partial \phi(x)) \delta(x - z(\tau)), $$

where $\tau$ is the propertime of the particle multiplied by $c$. $z \equiv \{z^\mu\} \equiv \{z^0, z^1, z^2, z^3\}$ are pseudo-cartesian coordinates of the particle in the Minkowski space; $\dot{z} \equiv \{\dot{z}^\mu\} \equiv \{dz^\mu(\tau)/d\tau\}$; $\phi$ is a set of external fields; $dx \equiv dx^0 dx^1 dx^2 dx^3$, $\delta(x) \equiv \delta(x^0)\delta(x^1)\delta(x^2)\delta(x^3)$. Let us denote

$$ L \equiv L^P(z, \dot{z}) + \sum_{\Sigma_1}^{\Sigma_2} dx L^I(z, \dot{z}; \phi(x), \partial \phi(x)) \delta(x - z) \equiv L(z, \dot{z}) $$

(1.2)

Let us derive a generalized equation of motion of the point particle using variational principle. Since $\tau$ is an propertime of the particle, then variation of its trajectory also causes relevant variation of the propertime "element". Therefore, assuming boundary points to be non-variable, we find

$$ \delta W = \delta \int_{\tau_1}^{\tau_2} d\tau L = \int_{\tau_1}^{\tau_2} \delta d\tau L + \int_{\tau_1}^{\tau_2} d\tau \delta L, $$

(1.3)

where

$$ \delta L = \frac{\partial L}{\partial z^\mu} \delta z^\mu + \frac{\partial L}{\partial \dot{z}^\mu} \delta \dot{z}^\mu. $$

(1.4)

For $\delta \dot{z}^\mu$

$$ \delta \dot{z}^\mu = \delta \left( \frac{dz^\mu}{d\tau} \right) = \frac{\delta z^\mu}{d\tau} - \frac{dz^\mu}{(d\tau)^2} \delta d\tau. $$

(1.5)

Since $\tau$ is an propertime of the particle, the following relation holds true

$$ d\tau = \sqrt{-dz^\alpha dz_\alpha}, $$

(1.6)

where we find

$$ \delta d\tau = -\frac{dz_\mu}{\sqrt{-dz^\alpha dz_\alpha}} \delta dz^\mu = -\dot{z}_\mu \delta dz^\mu. $$

(1.7)

Substituting (1.7) into (1.5), we obtain

$$ \delta \dot{z}^\mu = \frac{\delta dz^\mu}{d\tau} + \dot{z}_\nu \frac{\delta dz^\nu}{d\tau} = \left( \delta^\mu_\nu + \dot{z}_\nu \dot{z}^\nu \right) \frac{\delta z^\mu}{d\tau}. $$

(1.8)

However, $\delta dz^\mu = d \delta z^\mu$, then

$$ \frac{\delta dz^\mu}{d\tau} = \frac{d}{d\tau} \delta z^\mu. $$

(1.9)

Therefore

$$ \delta d\tau = -\dot{z}_\mu d \delta z^\mu. $$

(1.10)
Thus, according to (1.8), (1.10) and (1.4)
\[ \delta \dot{z}^{\mu} = (\delta^{\mu \nu} + \dot{z}^{\mu} \dot{z}_\nu) \frac{d}{d\tau} \delta z^\nu, \] (1.11)
\[ \delta L = \frac{\partial L}{\partial z^\mu} \delta z^\mu + \frac{\partial L}{\partial \dot{z}^\mu}(\delta^{\mu \nu} + \dot{z}^\mu \dot{z}_\nu) \frac{d}{d\tau} \delta z^\nu. \] (1.12)
Substituting now (1.10) and (1.12) into (1.3) and integrating by parts, we find
\[ \delta W = - \int_{\theta_1}^{\theta_2} L \delta z^\mu + \int_{\theta_1}^{\theta_2} d\tau \left[ \frac{\partial L}{\partial z^\mu}(\delta^{\mu \nu} + \dot{z}^\mu \dot{z}_\nu) \frac{d}{d\tau} \delta z^\nu \right] \]
\[ = \int_{\theta_1}^{\theta_2} d\tau \left[ \frac{\partial L}{\partial \dot{z}^\mu}(\delta^{\mu \nu} + \dot{z}^\mu \dot{z}_\nu) - L \dot{z}_\nu \right] \delta z^\nu \int_{\theta_1}^{\theta_2} \delta W = \int_{\theta_1}^{\theta_2} \delta W \delta z^\nu d\tau, \] (1.13)
where
\[ \frac{\partial W}{\partial \delta z^\nu} = \frac{\partial L}{\partial \delta z^\nu} - \frac{d}{d\tau} \left\{ \frac{\partial L}{\partial \dot{z}^\mu}(\delta^{\mu \nu} + \dot{z}^\mu \dot{z}_\nu) - L \dot{z}_\nu \right\}, \] (1.15)
and, by virtue of the stationary action principle, the following equation of motion result from here
\[ \frac{\partial L}{\partial \dot{z}^\nu} - \frac{d}{d\tau} \left\{ \frac{\partial L}{\partial \dot{z}^\mu}(\delta^{\mu \nu} + \dot{z}^\mu \dot{z}_\nu) - L \dot{z}_\nu \right\} = 0. \] (1.16)
We come just to the Kalman equations [1]. Obviously, they differ from the Euler - Lagrange equations by the terms resulted from the manifest variation of the propertime element.

1.2 Equations of motion of the particle in the reparametrization-invariant theories

Parametrization of the world line of the particle by propertime is too specific. It appears frequently convenient to refuse from the above choice and to numerate the sequence of trajectory points in somehow different manner. In this case, however, the parameter \( \theta \) used for such numeration will not have the clear physical sense as \( \tau \) has, and probably, will not be observable at all. In addition, it may happen that it is convenient to analyze some of the properties of the system at one parametrization choice, while other properties of this system are easier made apparent at the different parameter choice. Therefore it seems expedient to construct initially the theory of point particle interaction with the field in such a way to make it independent on arbitrariness in the particle parametrization, i.e. it should be reparametrizationally invariant.

Let the world line of the particle be parametrized by an arbitrary Lorentz-invariant parameter \( \theta \) with length dimension, i.e. \( z^\mu = z^\mu(\theta) \). Let us denote
\[ z^\mu = \frac{dz^\mu(\theta)}{d\theta}, \quad \nu^\mu(\theta) = \frac{z^{\mu}(\theta)}{\sqrt{-z^{\alpha}(\theta)z'_{\alpha}(\theta)}} \] (1.17)
If the Lagrange function of the particle depends on the \( z(\theta) \) and its derivatives over \( \theta \) not higher than the first order (and we will restrict ourselves to the consideration of just such cases), then the action for the particle that possesses reparametrization invariance must have the following form
\[ W[z] = \int_{\theta_1}^{\theta_2} L(z(\theta), \nu(\theta)) \sqrt{-z^{\alpha}(\theta)z'_{\alpha}(\theta)} d\theta. \] (1.18)
Action (1.18) is indeed reparametrization-invariant, since \( v^\mu \) and \( \sqrt{-z^{\alpha}(\theta)z^{\alpha}(\theta)}d\theta \) do not vary with reparametrization transformations.

Let us find the equation of motion that follows from the action functional (1.18). Let the parameter \( \theta \) be not varying by definition. Then

\[
\delta W = \int_{\theta_1}^{\theta_2} \delta L \sqrt{-z^{\alpha}z^{\alpha}_\theta} d\theta + \int_{\theta_1}^{\theta_2} L \delta \sqrt{-z^{\alpha}z^{\alpha}_\theta} d\theta, \tag{1.19}
\]

\[
\delta L = \frac{\partial L}{\partial z^\mu} \delta z^\mu + \frac{\partial L}{\partial z^\nu} \delta z^\nu = \frac{\partial L}{\partial \nu \mu} \delta v^\nu + \frac{\partial L}{\partial \nu \mu} \delta v^\nu z^\nu. \tag{1.20}
\]

Taking into account that

\[
\frac{\partial \nu ^\mu}{\partial z^\nu} = \frac{1}{\sqrt{-z^{\alpha}z^{\alpha}_\theta}} (\delta ^\mu_\nu + v^\mu v_\nu), \tag{1.21}
\]

and

\[
\delta z^\nu = \delta \left( \frac{dz^\nu}{d\theta} \right) = \frac{d}{d\theta} \delta z^\nu, \tag{1.22}
\]

we find

\[
\delta L = \frac{\partial L}{\partial z^\mu} \delta z^\mu + \frac{\partial L}{\partial \nu \mu} \frac{1}{\sqrt{-z^{\alpha}z^{\alpha}_\theta}} (\delta ^\mu_\nu + v^\mu v_\nu) \frac{d}{d\theta} \delta z^\nu. \tag{1.23}
\]

Furthermore,

\[
\delta \sqrt{-z^{\alpha}z^{\alpha}_\theta} = -v_\nu \frac{d}{d\theta} \delta z^\nu. \tag{1.24}
\]

Substituting (1.23) and (1.24) into (1.19) and integrating by parts, we have

\[
\delta W = \int_{\theta_1}^{\theta_2} \delta L \left[ \sqrt{-z^{\alpha}z^{\alpha}_\theta} \frac{d}{d\theta} \delta z^\mu + \frac{\partial L}{\partial \nu \mu} \delta v^\nu \right] d\theta
\]

\[
= \left[ \frac{\partial L}{\partial \nu \mu} (\delta ^\mu_\nu + v^\mu v_\nu) - L v_\nu \right] \delta z^\nu \bigg|_{\theta_1}^{\theta_2}
\]

\[
+ \int_{\theta_1}^{\theta_2} \left[ \sqrt{-z^{\alpha}z^{\alpha}_\theta} \frac{\partial L}{\partial \nu \mu} - \frac{d}{d\theta} \left( \frac{\partial L}{\partial \nu \mu} (\delta ^\mu_\nu + v^\mu v_\nu) - L v_\nu \right) \right] \delta z^\nu d\theta. \tag{1.25}
\]

Considering, as usually, that at the integration boundary \( \delta z^\nu |_{\theta_1,2} = 0 \), and that the action is extremal at the real trajectories, we find from (1.25) the following equations of motion

\[
\frac{\delta W}{\delta z^\nu} = \sqrt{-z^{\alpha}z^{\alpha}_\theta} \frac{\partial L}{\partial z^\nu} - \frac{d}{d\theta} \left( \frac{\partial L}{\partial \nu \mu} (\delta ^\mu_\nu + v^\mu v_\nu) - L v_\nu \right) = 0, \tag{1.26}
\]

or, after dividing by \( \sqrt{-z^{\alpha}z^{\alpha}_\theta} \),

\[
\frac{\partial L}{\partial z^\nu} - \frac{1}{\sqrt{-z^{\alpha}z^{\alpha}_\theta}} \frac{d}{d\theta} \left( \frac{\partial L}{\partial \nu \mu} (\delta ^\mu_\nu + v^\mu v_\nu) - L v_\nu \right) = 0. \tag{1.27}
\]

It is easy to see that equations (1.27) are the RI-equations.

Let us choose the propertime of the particle as the parameter

\[
\tau = \int_{\tau_0}^{\tau} \sqrt{-z^{\alpha}z^{\alpha}_\theta} d\theta, \tag{1.28}
\]

where \( \tau_0 \) is arbitrary. Then \( z^{\alpha} \) is transformed into \( \dot{z}^{\alpha}, \dot{z}^{\alpha} \dot{z}_\alpha = -1 \), and \( v^\mu \) into \( \dot{z}^\mu \). Hence, in the propertime gauge, equations (1.27) take a form

\[
\frac{\partial L}{\partial z^\nu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \nu \mu} (\delta ^\mu_\nu + \dot{z}^\mu \dot{z}_\nu) - L \dot{z}_\nu \right) = 0, \tag{1.29}
\]
that exactly coincides with the Kalman equations (1.16).

Thus, the Kalman equations derived with the help of manifest variation of the particle propertime element can be reproduced also as the Euler-Lagrange equations in the RI-theory in the propertime gauge.

Let us introduce the following notations to reach manifest RI for all relations:

\[ D_\theta \equiv \sqrt{-z'^\alpha z'^\alpha} d\theta \]  
(1.30)

is the RI-"volume element" (RI-integration measure);

\[ D \equiv \frac{1}{\sqrt{-z'^\alpha z'^\alpha}} \frac{d}{d\theta} \]  
(1.31)

is the RI-derivative. Then

\[ v^\mu = Dz^\mu \]  
(1.32)

and (1.18), (1.25) and (1.26) will be written as

\[ W[z] = \theta_2 \int_{\theta_1} \mathcal{D}\theta L(z(\theta), Dz(\theta)), \]  
(1.33)

\[ \delta W = \left[ \frac{\partial L}{\partial Dz^\nu} (\delta^\mu_\nu + Dz^\mu Dz_\nu) - L Dz_\nu \right] \delta z^\nu \bigg|_{\theta_1}^{\theta_2} \]

\[ + \int_{\theta_1}^{\theta_2} \mathcal{D}\theta \left[ \frac{\partial L}{\partial Dz^\nu} - D \left\{ \frac{\partial L}{\partial Dz^\nu} (\delta^\mu_\nu + Dz^\mu Dz_\nu) - L Dz_\nu \right\} \right] \delta z^\nu, \]  
(1.34)

\[ \frac{\Delta W}{\Delta z^\nu} = \frac{\partial L}{\partial Dz^\nu} - D \left\{ \frac{\partial L}{\partial Dz^\nu} (\delta^\mu_\nu + Dz^\mu Dz_\nu) - L Dz_\nu \right\} = 0, \]  
(1.35)

where

\[ \frac{\Delta W}{\Delta z^\nu} \equiv \frac{1}{\sqrt{-z'^\alpha z'^\alpha}} \frac{\delta W}{\delta z^\nu} \]  
(1.36)

is the RI-variation derivative. In the propertime gauge, \( \mathcal{D}\theta \to d\tau \), \( D \to d/d\tau \), \( Dz^\mu \to \dot{z}^\mu \).

Let us show also the RI-analog of the formula for integration by parts:

\[ \int_{\theta_1}^{\theta_2} \mathcal{D}\theta (Df(\theta)) \ g(\theta) = \left[ f(\theta)g(\theta) \right]_{\theta_1}^{\theta_2} - \int_{\theta_1}^{\theta_2} \mathcal{D}\theta \ f(\theta) \ (Dg(\theta)). \]  
(1.37)

In particular,

\[ \int_{\theta_1}^{\theta_2} \mathcal{D}\theta (Df(\theta)) = f(\theta) \bigg|_{\theta_1}^{\theta_2}. \]  
(1.38)

1.3 Nöether identity. Effective mass and force

The RI can be considered as invariance with respect to the local Abel group of transformations

\[ \begin{cases} 
\theta \to \tilde{\theta} = \theta + \delta \varepsilon (\theta) \\
z^\mu (\theta) \to \tilde{z}^\mu (\tilde{\theta}) = z^\mu (\theta),
\end{cases} \]  
(1.39)

where \( \delta \varepsilon (\theta) \) is an arbitrary infinitesimal function - the transformation parameter. We find from the second formula of (1.39)

\[ \tilde{\delta} z^\mu (\theta) = \tilde{z}^\mu (\tilde{\theta}) - z^\mu (\theta) = 0 \]  
(1.40)
and, since
\[ \delta z^\mu(\theta) = \delta z^\mu(\theta) + z'^\mu(\theta) \delta \varepsilon(\theta) = \delta z^\mu(\theta) + D z^\mu(\theta) \Delta \varepsilon(\theta), \] (1.41)
where
\[ \delta z^\mu(\theta) \equiv z^\mu(\theta) - z'^\mu(\theta), \] (1.42)
\[ \Delta \varepsilon(\theta) \equiv \sqrt{-z'^\alpha z^\alpha} \delta \varepsilon(\theta) \] (1.43)
is the RI-variation of parameter, then at the transformations (1.39)
\[ \delta z^\mu(\theta) = -D z^\mu(\theta) \Delta \varepsilon(\theta). \] (1.44)

On the strength of the second Noether theorem, the following identity takes place
\[ \Delta W \Delta z^\nu = 0, \] (1.45)
that means that the Eulerian \( \Delta W/\Delta z^\nu \) has the following structure
\[ \Delta W \Delta z^\nu = f^\mu_{\nu} (\delta^\mu_{\nu} + Dz^\mu Dz^\nu), \] (1.46)
where \( f^\mu_{\nu} \) is a some RI-4-vector. The quantity
\[ h^\mu_{\nu} \equiv \delta^\mu_{\nu} + Dz^\mu Dz^\nu \] (1.47)
is the projector onto the proper 3-space of the particle, since the above quantity coincides with it in the propertime gauge and is the RI-quantity.

Let us prove formula (1.46) using an manifest form of the variation derivation \( \Delta W/\Delta z^\nu \) (1.35). Using (1.31) and (1.32), it is easy to derive the auxiliary formulae
\[ D^2 z^\mu \equiv DDz^\mu = (\delta^\mu_{\nu} + Dz^\mu Dz^\nu) \frac{z'^\nu}{(\sqrt{-z'^\alpha z^\alpha})^2}, \] (1.48)
\[ Df(z(\theta), Dz(\theta)) = \frac{\partial f(z, Dz)}{\partial z^\mu} Dz^\mu + \frac{\partial f(z, Dz)}{\partial Dz^\mu} D^2 z^\mu, \] (1.49)
where \( f(z(\theta), Dz(\theta)) \) is an arbitrary differentiable function of the above arguments. Formula (1.49) is the RI-analog of the known analysis formula for a differentiation of composite function.

Using (1.35) and (1.49), we find
\[ \frac{\Delta W}{\Delta z^\nu} = \frac{\partial L}{\partial z^\nu} - D \left( \frac{\partial L}{\partial Dz^\alpha} (\delta^\mu_{\nu} + Dz^\mu Dz^\nu) - LDz^\nu \right) = \frac{\partial L}{\partial z^\nu} - D \left( \frac{\partial L}{\partial Dz^\alpha} (\delta^\mu_{\nu} + Dz^\mu Dz^\nu) \right) \]
\[ = \frac{\partial L}{\partial z^\nu} - D \frac{\partial L}{\partial Dz^\nu} (\delta^\mu_{\nu} + Dz^\mu Dz^\nu) + \frac{\partial f(z, Dz)}{\partial Dz^\mu} D^2 z^\nu + LD^2 z^\nu \] (1.50)
Taking into account the formula
\[ D^2 z^\mu Dz^\mu = 0, \] (1.51)
that follows from the identity
\[ Dz^\mu Dz^\mu = -1, \] (1.52)
we obtain finally
\[ \frac{\Delta W}{\Delta z^\nu} = \left[ \frac{\partial L}{\partial z^\mu} - D \frac{\partial L}{\partial Dz^\mu} \right] - \left( \frac{\partial L}{\partial Dz^\alpha} (Dz^\mu - L) \right) \] (1.53)
that coincides with (1.46) at
\[ f^\mu_{\nu} = \left( \frac{\partial L}{\partial z^\mu} - D \frac{\partial L}{\partial Dz^\mu} \right) - \left( \frac{\partial L}{\partial Dz^\alpha} (Dz^\mu - L) \right) D^2 z^\mu. \] (1.54)
Now the equations of motion $\Delta W/\Delta z^\nu = 0$ can be written as
\[
\left( \frac{\partial L}{\partial D z^\alpha} - D \frac{\partial L}{\partial D z^\nu} \right) D^2 z_\nu = \left( \frac{\partial L}{\partial z^\mu} - D \frac{\partial L}{\partial D z^\mu} \right) (\delta^\mu_\nu + D z^\mu D z_\nu).
\] (1.55)

It is natural to call
\[
M \equiv \frac{\partial L}{\partial D z^\alpha} D z^\alpha - L
\] (1.56)
the effective (dynamical) mass of the particle, and
\[
F_\nu \equiv \left( \frac{\partial L}{\partial z^\mu} - D \frac{\partial L}{\partial D z^\mu} \right) (\delta^\mu_\nu + D z^\mu D z_\nu)
\] (1.57)
the effective 4-force that the field acts on the particle with. In the above notations the equations of motion will take a simple form
\[
M D^2 z_\nu = F_\nu.
\] (1.58)

## 2 Total system ”particle+field”

### 2.1 Total action functional variation. Equations of motion of particle and field

Consider now the total system that consists of interacting point relativistic particle and certain set of fields and is described by the action functional of the following form
\[
W[z, \phi; \theta_1, 2, \Sigma_1, 2] = W^P[z; \theta_1, 2] + W^I[z, \phi; \theta_1, 2, \Sigma_1, 2] + W^F[\phi; \Sigma_1, 2]
\]
\[
= \int \mathcal{D}\theta L^P(z(\theta), Dz(\theta)) + \int \mathcal{D}\theta \int dx L^I(z(\theta), Dz(\theta); \phi(x), \partial \phi(x)) \delta(x - z(\theta))
\]
\[
+ \int dx L^F(\phi(x), \partial \phi(x)).
\] (2.1)

If one requires the Lagrange functions $L^{P,I}$ to be form-invariant scalars with respect to the transformations of the Poincaré group, then the condition $\partial L^{P,I}/\partial z^\mu = 0$ must hold true. For the Lagrange function of the particle this means $L^P = L^I(Dz)$. However, on the strength of identity $Dz^\mu Dz_\mu = -1$, it is impossible to construct nontrivial scalar only of $Dz^\mu$. Therefore the only possibility remains: $L^P = C = \text{const}$. Usually $C = -m$ is taken, where $m$ is the (kinematical) mass of the particle. Thus, the action functional for the total relativistic system ”particle+field” must have a form
\[
W = W^P + W^I + W^F
\]
\[
= \int \mathcal{D}\theta m + \int \mathcal{D}\theta \int dx L^I(z(\theta); \phi(x), \partial \phi(x)) \delta(x - z(\theta)) + \int dx L^F(\phi(x), \partial \phi(x)).
\] (2.2)

To derive the equations of motion and to construct dynamical variables we need an expression for the total variation of the action functional (2.2)
\[
\delta W[z, \phi; \theta_1, 2, \Sigma_1, 2] \equiv W[z + \delta z, \phi + \delta \phi; \theta_1, 2, \Sigma_1, 2 + \delta \Sigma_1, 2] - W[z, \phi; \theta_1, 2, \Sigma_1, 2].
\] (2.3)

Consider separately the total variations for each of three terms in (2.2)
\[
\delta W^P[z; \theta_1, 2] = [-m \Delta \theta]^{\theta_2}_{\theta_1} + \delta_1 W^P,
\] (2.4)
where $\Delta \theta \equiv \sqrt{-z^{\alpha}(\theta) z^{\alpha}_*(\theta)} \delta \theta$.
\[
\delta_1 W^P[z; \theta_1, 2] \equiv W^P[z + \delta z; \theta_1, 2] - W^P[z; \theta_1, 2]
\]
\[
= - \int \delta z(\mathcal{D}\theta) m + \int \mathcal{D}\theta m Dz_\nu D(\delta z^\nu) = [m Dz_\nu, \delta z^\nu]^{\theta_2}_{\theta_1} - \theta_1 \mathcal{D}\theta m D^2 z_\nu, \delta z^\nu.
\] (2.5)
When deriving (2.5) we have used formula $\delta_z(\mathcal{D}) = -\mathcal{D} D z_{\nu} D(\delta z^{\nu})$ and performed integration by parts. Substituting (2.5) into (2.4), we find

$$
\delta W^F = [-m \Delta \theta + m D z_{\nu} \delta z^{\nu}]|_{\theta_1}^{\theta_2} - \int_{\Sigma_1}^{\Sigma_2} \mathcal{D} \theta \ m D^2 z_{\nu} \delta z^{\nu}.
$$
(2.6)

For pure field term, as usually,

$$
\delta W^F[\phi; \Sigma_{1,2}] \equiv W^F[\phi + \delta \phi; \Sigma_{1,2} + \delta \Sigma_{1,2}] - W^F[\phi; \Sigma_{1,2}] = \left[ \int_{\Sigma_1}^{\Sigma_2} d\sigma_\mu \mathcal{L}^F \delta x^\mu \right]_{\Sigma_1}^{\Sigma_2} + \delta_\phi W^F,
$$
(2.7)

where

$$
\delta_\phi W^F[\phi; \Sigma_{1,2}] \equiv W^F[\phi + \delta \phi; \Sigma_{1,2}] - W^F[\phi; \Sigma_{1,2}]
= \int_{\Sigma_1}^{\Sigma_2} dx \delta_{\phi} \mathcal{L}^F
= \int_{\Sigma_1}^{\Sigma_2} dx \left[ \frac{\partial \mathcal{L}^F}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}^F}{\partial (\phi_{\mu})} \delta (\phi_{\mu}) \right]
= \int_{\Sigma_1}^{\Sigma_2} dx \left[ \frac{\partial \mathcal{L}^F}{\partial \phi} \delta \phi \right]_{\Sigma_1}^{\Sigma_2} + \int_{\Sigma_1}^{\Sigma_2} dx \left[ \frac{\partial \mathcal{L}^F}{\partial (\phi_{\mu})} \delta (\phi_{\mu}) \right].
$$
(2.8)

Substituting (2.8) into (2.7), we get

$$
\delta W^F = \int_{\Sigma_1}^{\Sigma_2} d\sigma_\mu \mathcal{L}^F \delta x^\mu + \int_{\Sigma_1}^{\Sigma_2} d\sigma_\mu \left[ \frac{\partial \mathcal{L}^F}{\partial \phi} \delta \phi \right]_{\Sigma_1}^{\Sigma_2} + \int_{\Sigma_1}^{\Sigma_2} dx \left[ \frac{\partial \mathcal{L}^F}{\partial (\phi_{\mu})} \delta (\phi_{\mu}) \right].
$$
(2.9)

Furthermore,

$$
\delta W^I[z, \phi; \theta_1, \Sigma_{1,2}] \equiv W^I[z + \delta z, \phi + \delta \phi; \theta_1 + \delta \theta, \Sigma_{1,2} + \delta \Sigma_{1,2}]
= \left[ \int_{\Sigma_1}^{\Sigma_2} dL^I \delta (x - z) \delta \theta \right]_{\theta_1}^{\theta_2} + \int_{\Sigma_1}^{\Sigma_2} d\sigma_\mu \left[ \frac{\partial \mathcal{L}^I}{\partial \phi} \delta \phi \right]_{\Sigma_1}^{\Sigma_2} + \int_{\Sigma_1}^{\Sigma_2} dx \left[ \frac{\partial \mathcal{L}^I}{\partial (\phi_{\mu})} \delta (\phi_{\mu}) \right].
$$
(2.10)

$$
\delta_z W^I[z, \phi; \theta_1, \Sigma_{1,2}] \equiv W^I[z + \delta z, \phi; \theta_1 + \theta_2, \Sigma_{1,2} - \delta \theta, \Sigma_{1,2}]
= \int_{\Sigma_1}^{\Sigma_2} dL^I \delta (x - z) + \int_{\Sigma_1}^{\Sigma_2} d\sigma_\mu \left[ \frac{\partial \mathcal{L}^I}{\partial \phi} \delta \phi \right]_{\Sigma_1}^{\Sigma_2} + \int_{\Sigma_1}^{\Sigma_2} dx \delta_z L^I \delta (x - z).
$$
(2.11)

Using formulae

$$
\delta_z(\mathcal{D}) = -\mathcal{D} D z_{\nu} D(\delta z^{\nu}), \quad \delta_z f(z, Dz) = \frac{\partial f}{\partial z_{\nu}} \delta z^{\nu} + \frac{\partial f}{\partial D z_{\nu}} \delta (D z^{\nu}),
$$
(2.12)

and integrating by parts, we find for the first two terms

$$
\left[ \int_{\Sigma_1}^{\Sigma_2} dL^I \left( \delta \mu_{\nu} + D z_{\nu} D z_{\mu} \right) - \frac{\partial}{\partial z_{\nu}} \frac{\partial f}{\partial z_{\nu}} \right] \delta (x - z) \delta z^{\nu} |_{\theta_1}^{\theta_2}
$$
(2.13)

For the third term

$$
\left[ \int_{\Sigma_1}^{\Sigma_2} dL^I \delta_z (x - z) |_{\theta_1}^{\theta_2} \right] + \int_{\Sigma_1}^{\Sigma_2} d\sigma_\mu \left( \frac{\partial \mathcal{L}^F}{\partial \phi} \delta \phi \right)_{\Sigma_1}^{\Sigma_2} + \int_{\Sigma_1}^{\Sigma_2} dx \frac{\partial \mathcal{L}^F}{\partial \phi} \delta \phi.
$$
(2.14)
where
\[
\frac{dL'}{dx'} = \frac{\partial L'}{\partial \phi(x)} \phi_{\nu}(x) + \frac{\partial L'}{\partial \phi_{\mu}(x)} \phi_{\mu\nu}(x).
\] (2.15)

Compiling together, we obtain
\[
\delta_{\nu}W' = \left[ \int_{\Sigma_1} dx \left\{ \frac{\partial L'}{\partial D^\nu}(\delta^{\nu}_{\mu} + Dz^\mu Dz_{\nu}) - L^1 Dz_{\nu} \right\} \delta(x - z) \delta z' \right]_{\theta_1}^{\theta_2} - \left[ \int d\sigma_{\nu} \left( \int_{\Sigma_1} \frac{\partial L'}{\partial \phi(x)} \partial \phi + \frac{\partial L'}{\partial \phi_{\mu}(x)} \delta \phi_{\mu}(x) \right) \delta(x - z) \right]_{\Sigma_1}^{\Sigma_2} + \int_{\Sigma_1} dL' \delta(x - z) \delta z',
\] (2.16)

Furthermore
\[
\delta_{\phi}W' = W'[z, \phi, \theta_1, \Sigma_1, \Sigma_2] - W'[z, \phi, \theta_1, \Sigma_1, \Sigma_2] = \left[ \int dL' \delta(x - z) \right]_{\Sigma_1}^{\Sigma_2} + \left[ \int d\sigma_{\nu} \left( \int_{\Sigma_1} \frac{\partial L'}{\partial \phi(x)} \partial \phi + \frac{\partial L'}{\partial \phi_{\mu}(x)} \delta \phi_{\mu}(x) \right) \delta(x - z) \right]_{\Sigma_1}^{\Sigma_2}
\] (2.17)

Substituting (2.16), (2.17) into (2.10)
\[
\delta W' = \left[ \int_{\Sigma_1} dx L'(x - z) \theta + \int_{\Sigma_2} dx \left\{ \frac{\partial L'}{\partial D^\nu}(\delta^{\nu}_{\mu} + Dz^\mu Dz_{\nu}) - L^1 Dz_{\nu} \right\} \delta(x - z) \delta z' \right]_{\theta_1}^{\theta_2} - \left[ \int d\sigma_{\nu} \left( \int_{\Sigma_1} \frac{\partial L'}{\partial \phi(x)} \partial \phi + \frac{\partial L'}{\partial \phi_{\mu}(x)} \delta \phi_{\mu}(x) \right) \delta(x - z) \right]_{\Sigma_1}^{\Sigma_2}
\] (2.18)

The term in the second square parenthesis equals to zero, since at any variation of the particle coordinate the equality \(\delta x' = \delta z'\) holds true. Validity of the last statement results from the following simple consideration. Any function (or functional) of the particle coordinate \(f(z)\) can be presented in a form
\[
f(z) = \int_{\Sigma_1} dx f(x) \delta(x - z) \equiv F(z; \Sigma_1, \Sigma_2).
\] (2.19)

Let us take variation of the both parts of (2.19). On the left-hand side we have
\[
\delta_z f(z) = \frac{\partial f(z)}{\partial z^\mu} \delta z^\mu,
\] (2.20)

whereas on the right-hand side
\[
\delta F(z; \Sigma_1, \Sigma_2) = \int_{\Sigma_1} d\sigma_{\mu} f(x) \delta(x - z) \delta x^\mu + \delta_z F,
\] (2.21)
where
\[
\delta_x F(z; \Sigma_{1,2}) \equiv F(z + \delta z; \Sigma_{1,2}) - F(z; \Sigma_{1,2})
\]
\[
= \int dx f(x) \sum_{\Sigma_2} \delta z^\mu = - \int dx f(x) \sum_{\Sigma_1} \delta z^\mu
\]
\[
= \int dx \left[ - \delta_{x \Sigma} \left( f(x)(\delta(x - z)) \right) \delta z^\mu + \sum_{\Sigma_2} \frac{\partial f(x)}{\partial \sigma_{\Sigma}} \delta(x - z) \delta z^\mu \right]
\]
\[
= \int dx \left[ \delta(x - z) \delta z^\mu \right] + \sum_{\Sigma_2} \frac{\partial f(x)}{\partial \sigma_{\Sigma}} \delta z^\mu.
\] (2.22)

Substituting (2.22) into (2.21) and comparing the result with (2.20), we find
\[
\left[ \int dx f(x)(\delta(x - z)(\delta z^\mu - \delta x^\mu) \right]_{\Sigma_2}^{\Sigma_1} = 0,
\] (2.23)

and, on the strength of arbitrariness of the function \( f(z) \) and hypersurfaces \( \Sigma_{1,2} \),
\[
\delta x^\mu = \delta z^\mu.
\] (2.24)

Let us denote
\[
\pi_\mu \equiv \sum_{\Sigma_1} dx \left\{ \frac{\partial L}{\partial \delta_{\Sigma} z^\mu} (\delta_{\Sigma} z^\mu + Dz^\mu Dz_\nu) - L Dz_\nu \right\}
\] (2.25)

\[
\mathcal{L}^I = \int d\theta \mathcal{L}^I \delta(x - z).
\] (2.26)

Then, compiling (2.6), (2.9) and (2.18), we have
\[
\delta \tilde{W} = \left[ \left( - m + \sum_{\Sigma_2} dx L^I \delta(x - z) \right) \Delta \theta + (mDz_\nu + \pi_\nu) \delta z^\nu \right] \bigg|^{\theta_2}_{\theta_1} \]
\[
+ \left[ \int dx \left\{ \mathcal{L}^F \delta x^\mu + \int dx \frac{\partial (\mathcal{L}^I + \mathcal{L}^F)}{\partial \phi} \delta \phi \right\} \sum_{\Sigma_1} \right]_{\theta_2}^{\theta_1}
\]
\[
+ \left[ \int d\theta \left[ - D(mDz_\nu + \pi_\nu) + \int dx \frac{\partial L^I}{\partial \phi} \delta(x - z) \right] \delta z^\nu \right]
\]
\[
+ \left[ \int dx \left\{ \frac{\partial (\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi} - \frac{\partial (\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\Sigma}} \right\} \delta \phi \right].
\] (2.27)

As usually, the boundary terms in (2.27) are expressed via the total variations \( \delta z^\nu, \delta \phi \) of the coordinate of the particle and field function, respectively. If as a result of variation
\[
\theta \to \tilde{\theta} \equiv \theta + \delta \theta, \quad z^\nu(\theta) \to \tilde{z}^\nu(\tilde{\theta}) \equiv z^\nu(\theta) + \delta z^\nu(\theta),
\]
\[
x \to \tilde{x} \equiv x + \delta x, \quad \phi(x) \to \tilde{\phi}(x) \equiv \phi(x) + \delta \phi(x),
\] (2.28)

then
\[
\delta z^\nu(\tilde{\theta}) \equiv \tilde{z}^\nu(\tilde{\theta}) - z^\nu(\theta) = \delta z^\nu(\theta) + Dz^\nu(\theta) \Delta \theta,
\]
\[
\delta \phi(x) \equiv \tilde{\phi}(x) - \phi(x) = \delta \phi(x) + \phi_\nu(x) \delta x^\nu(x),
\] (2.29)

hence
\[
\delta z^\nu(\theta) = \delta z^\nu(\theta) - Dz^\nu(\theta) \Delta \theta, \quad \delta \phi(x) = \delta \phi(x) - \phi_\nu(x) \delta x^\nu(x).
\] (2.30)
Substituting (2.30) into (2.27) and taking into account (2.24), we find finally
\[
\bar{\delta}W = \left. \left[ (mDz_{\nu} + \pi_{\nu}) \tilde{z}^{\nu} \right] \right|_{\theta_1}^{\theta_2} + \left[ \int d\sigma_{\mu} \left\{ \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \phi_{\nu} - \mathcal{L}^F \delta_{\mu} \right\} \delta x^\nu + \int d\sigma_{\mu} \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \tilde{\phi} + \int \frac{\partial}{\partial \theta} \int d\sigma_{\mu} \left\{ \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \phi_{\nu} - \mathcal{L}^F \delta_{\mu} \right\} \delta z^\nu \right|_{\Sigma_1}^{\Sigma_2}.
\]

(2.31)

The following equations of motion for the particle and fields result from (2.31)
\[
\frac{\Delta W}{\Delta z^\nu} = -D(mDz_{\nu} + \pi_{\nu}) + \int d\sigma_{\mu} \frac{dL^{I}}{d\sigma_{\mu}} \delta(x - z) = 0,
\]
\[
\frac{\delta W}{\delta \phi(x)} = \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi} + \frac{\partial}{\partial x^\mu} \left( \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\mu}} \right) = 0.
\]

(2.32)

(2.33)

At the equations of motion
\[
\bar{\delta}W = \left. \left[ (mDz_{\nu} + \pi_{\nu}) \tilde{z}^{\nu} \right] \right|_{\theta_1}^{\theta_2} + \left[ \int d\sigma_{\mu} \left\{ \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \phi_{\nu} - \mathcal{L}^F \delta_{\mu} \right\} \delta x^\nu + \int d\sigma_{\mu} \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \tilde{\phi} \right|_{\Sigma_1}^{\Sigma_2}.
\]

(2.34)

Formula (2.34) makes a basis for constructing the dynamical variables of the system ”particle+field”. Note that variation of the time parameter \(\Delta \theta\) is not involved in (2.34).

### 2.2 4-Momentum vector. Energy-momentum tensor

At the space-time translations
\[
\begin{align*}
\delta \theta &= 0, \\
\delta z^\mu &= \delta x^\mu = \delta z^\mu = \text{const}, \\
\delta \phi &= 0
\end{align*}
\]

(2.35)

and formula (2.34) provides an expression for the 4-momentum vector of the system:
\[
\bar{\delta}W = [P_{\nu}][\Sigma]|_{\Sigma_1}^{\Sigma_2} \delta x^\nu.
\]

(2.36)

Hence,
\[
P_{\nu}[\Sigma] = (mDz_{\nu} + \pi_{\nu}) - \int d\sigma_{\mu} \left\{ \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \phi_{\nu} - \mathcal{L}^F \delta_{\mu} \right\}, \quad (z(\theta) \in \Sigma).
\]

(2.37)

The 4-momentum vector can also be expressed via the energy-momentum tensor (EMT) by the following formula
\[
P_{\nu}[\Sigma] = \int d\sigma_{\mu} T^\mu_{\nu}.
\]

(2.38)

Then
\[
T^\mu_{\nu} = \int_{-\infty}^{+\infty} \partial \theta^\prime Dz^\mu (mDz_{\nu} + \pi_{\nu}) \delta(x - z) - \left\{ \frac{\partial(\mathcal{L}^F + \mathcal{L}^I)}{\partial \phi_{\nu}} \phi_{\nu} - \mathcal{L}^F \delta_{\mu} \right\}
\]

(2.39)
Let us show that EMT (2.39) is conservative.

\[ \partial_\mu T^\mu_\nu = \int_\Sigma \delta W \frac{\partial \delta(x-z)}{\partial z^\mu} \]

\[ - \{ \partial_\mu \left( \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\mu} \right) \phi_\nu + \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\mu} \phi_\mu \phi_\nu - \delta_\nu \mathcal{L}^F \} = - \int_\Sigma \Delta \mathcal{A} \frac{\partial \delta(x-z)}{\partial z^\mu} \]

\[ = - \int_\Sigma \Delta \mathcal{A} \frac{\partial \delta(x-z)}{\partial z^\mu} \]

\[ \frac{\partial \delta(x-z)}{\partial z^\mu} = \delta_\nu \mathcal{L}^F \]

\[ \partial_\mu T^\mu_\nu = 0. \] (2.41)

Note that EMT (2.39) is not symmetrical: \( T_{\mu\nu} \neq T_{\nu\mu} \).

### 2.3 4-Angular momentum tensor. 4-Angular momentum density tensor

At the transformations from the Lorentz group (boosts and rotations)

\[
\begin{aligned}
\delta \theta &= 0 \\
\delta x^\mu &= \frac{1}{2} \delta \varepsilon^{\mu\sigma} (z_\mu \delta \varepsilon^\sigma - z_\sigma \delta \varepsilon^\mu) \\
\delta x^\nu &= \frac{1}{2} \delta \varepsilon^{\mu\sigma} (x_\mu \delta \varepsilon^\sigma - x_\sigma \delta \varepsilon^\mu), \\
\delta \phi &= - \frac{1}{2 \hbar} \delta \varepsilon^{\mu\sigma} (S_{\mu\nu}) \phi,
\end{aligned}
\] (2.42)

In the last formula the truncated notations are used. The total form is \( \tilde{\delta} \phi^A = - \frac{1}{2 \hbar} \delta \varepsilon^{\mu\sigma} (S_{\mu\nu}) A B \phi^B \), where \( A, B = 1, N \); \( N \) is the number of field components \( \phi \). Then formula (2.34) gives the expressions for the 4-angular momentum tensor of the system:

\[ \delta W = \frac{1}{2} \int_\Sigma [J_{\rho\sigma}[\Sigma]] \delta \varepsilon^{\mu\sigma}. \] (2.43)

\[ J_{\rho\sigma}[\Sigma] = \{ z_\rho (mDz_\sigma + \pi_\sigma) - z_\sigma (mDz_\rho + \pi_\rho) \}

\[ - \int d\sigma_\mu \left\{ x_\rho \left( \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\sigma} \phi_\sigma - \mathcal{L}^F \delta^\rho_\sigma \right) - x_\sigma \left( \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\rho} \phi_\rho - \mathcal{L}^F \delta^\mu_\rho \right) \right\} - \frac{1}{\hbar} \int_\Sigma d\sigma_\mu \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\mu} S_{\rho\sigma} \phi. \] (2.44)

The tensor \( J_{\rho\sigma}[\Sigma] \) can also be expressed via the 4-angular momentum density tensor \( J^\mu_{\rho\sigma} \) by the following formula

\[ J^\mu_{\rho\sigma} = \int_\Sigma d\sigma_\mu J^\mu_{\rho\sigma}. \] (2.45)

Then

\[ J^\mu_{\rho\sigma} = \int_\Sigma \Delta \mathcal{A} Dz^\mu \left\{ z_\mu (mDz_\sigma + \pi_\sigma) - z_\sigma (mDz_\rho + \pi_\rho) \right\} \delta(x-z) \]

\[ - \left\{ x_\rho \left( \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\sigma} \phi_\sigma - \mathcal{L}^F \delta^\rho_\sigma \right) - x_\sigma \left( \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\rho} \phi_\rho - \mathcal{L}^F \delta^\mu_\rho \right) \right\} - \frac{1}{\hbar} \int_\Sigma d\sigma_\mu \frac{\partial (\mathcal{F} + \mathcal{L}^I)}{\partial \phi_\mu} S_{\rho\sigma} \phi. \] (2.46)
One can also write down
\[ J^\mu_{\rho\sigma} = J^{P\mu}_{\rho\sigma} + J^{I\mu}_{\rho\sigma} + J^{F\mu}_{\rho\sigma}, \]  
(2.47)

where
\begin{align*}
J^{P\mu}_{\rho\sigma} &= \{x_\rho T^{P\mu}_\sigma - x_\sigma T^{P\mu}_\rho\} \equiv M^{P\mu}_{\rho\sigma}, \quad S^{P\mu}_{\rho\sigma} = 0 \\
J^{I\mu}_{\rho\sigma} &= \{x_\rho T^{I\mu}_\sigma - x_\sigma T^{I\mu}_\rho\} + \left\{ -\frac{i}{\hbar} \frac{\partial L}{\partial \phi}\right\} \rho\sigma \equiv M^{I\mu}_{\rho\sigma} + S^{I\mu}_{\rho\sigma} \\
J^{F\mu}_{\rho\sigma} &= \{x_\rho T^{F\mu}_\sigma - x_\sigma T^{F\mu}_\rho\} + \left\{ -\frac{i}{\hbar} \frac{\partial L}{\partial \phi}\right\} \rho\sigma \equiv M^{F\mu}_{\rho\sigma} + S^{F\mu}_{\rho\sigma}.
\end{align*}
(2.48)

### 2.4 Symmetrical energy-momentum tensor

The task of constructing the symmetrical EMT \( T^{sym}_{\mu\nu} = T^{sym}_{\nu\mu} \) is of certain interest. For a pure field system this task was solved by [2]. Let us generalize the Belinfante method for the case of the system “particle+field”.

It is well known that the density of any preserved quantity is defined with the accuracy to only the term having a form of divergence. Thus, if for certain density \( \rho^\alpha_A \), \( \partial_\mu \rho^\alpha_A = 0 \), where \( A \) is the multi-index that may characterize both space-time and internal properties of the physical system, then for the density \( \tilde{\rho}^\alpha_A = \rho^\alpha_A + \partial_\alpha f^{\alpha\mu}_A \), where \( f^{[\alpha\nu]}_A = f^{\alpha\mu}_A \), also holds true \( \partial_\mu \tilde{\rho}^\alpha_A = 0 \). Both densities lead also to the same integral preserved quantity
\[ \tilde{Q}_A \equiv \int d\sigma^\mu \tilde{\rho}^\alpha_A = \int d\sigma^\mu \rho^\alpha_A + \int d\sigma^\mu \partial_\alpha f^{\alpha\mu}_A = \int d\sigma^\mu \rho^\alpha_A \equiv Q_A, \]  
(2.49)

since, according to the Schwinger’s lemma [3], for the spatially closed systems
\[ \int d\sigma^\mu \partial_\alpha = \int d\sigma_\alpha \partial_\mu, \]  
(2.50)

and the quantity \( f^{\alpha\mu}_A \) is anti-symmetrical over \( \alpha \) and \( \mu \) indices.

Note now that from the momentum conservation law \( \partial_\mu J^{\mu\rho\sigma} = 0 \) the following equality results
\[ T_{[\rho\sigma]} = -\frac{1}{2} \partial_\alpha S^{\alpha}_{\rho\sigma}, \]  
(2.51)

i.e. the antisymmetrical part of EMT is presented in a form of divergence. Substituting (2.39), (2.46) into (2.51), we find
\[ \int_{-\infty}^{+\infty} \partial_\theta D^2z^\mu \frac{\partial L^I}{\partial D^2\phi}\delta(x-z) - \partial(L^F + L^I) \phi = \frac{i}{\hbar} \partial_\alpha \left( \frac{\partial(L^F + L^I)}{\partial \phi_\alpha} S_{\rho\sigma} \phi \right). \]  
(2.52)

It also follows from (2.51) that \( T_{(\rho\sigma)} = T_{\rho\sigma} \), if
\[ J^{\mu\rho\sigma} = x_\rho T^{\mu\sigma} - x_\sigma T^{\mu\rho} + \partial_\alpha F^{\alpha\mu\rho\sigma}, \quad F^{[\alpha\nu]}_{\rho\sigma} = F^{\alpha\nu}_{\rho\sigma} = F^{[\alpha\nu]}_{\rho\sigma} = F^{\alpha\nu}_{\rho\sigma}. \]  
(2.53)

Therefore, one may write down
\[ T^{sym}_{\mu\nu} = T^{sym}_{\mu\nu} - \partial_\alpha f^{\alpha\mu\nu}_\chi, \quad f^{[\alpha\nu]}_\chi = f^{\alpha\nu}_\chi. \]  
(2.54)

Then
\[ J^{sym}_{\mu\rho\sigma} = \{x_\rho T^{sym}_{\mu\sigma} - x_\sigma T^{sym}_{\mu\rho}\} - \{x_\rho f^{sym}_{\mu\sigma} - x_\sigma f^{sym}_{\mu\rho}\} = J^{sym}_{\mu\rho\sigma} - \partial_\alpha (x_\rho f^{sym}_{\mu\sigma} - x_\sigma f^{sym}_{\mu\rho}) \]  
(2.55)

where
\[ J^{sym}_{\mu\rho\sigma} \equiv x_\rho T^{sym}_{\mu\sigma} - x_\sigma T^{sym}_{\mu\rho}. \]  
(2.56)

To satisfy (2.53) we shall require the following equation
\[ f^{\mu\nu}_\rho - f^{\nu\mu}_\rho + S^{\mu\rho\sigma} = 0, \]  
(2.57)
or
\[
f_{\mu\sigma} - f_{\sigma\mu} = -S_{\mu\rho\sigma}.
\]  
(2.58)
to be valid. Hence,
\[
f_{\alpha\mu} = \frac{1}{2} \left[ (f_{\alpha\mu} - f_{\mu\alpha}) - (f_{\mu\alpha} - f_{\alpha\mu}) + (f_{\nu\alpha} - f_{\mu\nu}) \right] = -\frac{1}{2} (S_{\mu\nu\alpha} + S_{\nu\alpha\mu} - S_{\alpha\mu\nu}),
\]  
(2.59)
or, recalling formula (2.46)
\[
f_{\alpha}^{\mu} = \frac{1}{2} i \hbar \left( \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\mu}} S_{\alpha}^{\nu} \phi + \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\nu}} S_{\alpha}^{\mu} \phi - \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\alpha}} S_{\mu\nu} \phi \right).
\]  
(2.60)
Using formulae (2.54), (2.59), (2.60) and (2.52), we construct \( \text{sym} T_{\mu\nu} \), in the manifest form
\[
\text{sym} T_{\mu\nu} = T_{\mu\nu} + \partial_{\alpha} f_{\alpha}^{\mu\nu} = \int^{+\infty}_{-\infty} \mathcal{D}\tilde{\theta}' Dz_{\mu}(m Dz_{\nu} + \pi_{\nu}) \delta(x - z) - \left( \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\nu}} \right) S_{\mu\nu} \phi
\]
\[
+ \frac{1}{2} i \hbar \left( \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\nu}} S_{\alpha}^{\mu} \phi + \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\mu}} S_{\alpha}^{\nu} \phi \right) - \frac{1}{2} i \hbar \partial_{\alpha} \left( \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\alpha}} S_{\mu\nu} \phi \right)
\]
\[
= \int^{+\infty}_{-\infty} \mathcal{D}\tilde{\theta}' M Dz_{\mu} Dz_{\nu} \delta(x - z) + \int^{+\infty}_{-\infty} \mathcal{D}\tilde{\theta}' Dz_{\mu} \frac{\partial}{\partial z_{\nu}} \delta(x - z) - \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\nu}} \phi_{\mu} + \eta_{\mu\nu} \mathcal{L}^{F}
\]
\[
+ \frac{1}{2} i \partial_{\alpha} \left( \frac{\partial (\mathcal{L}^{F} + \mathcal{L}^{I})}{\partial \phi_{\alpha}} S_{\mu\nu} \phi \right) = T_{\mu\nu} - \partial_{\alpha} S_{\alpha}^{\mu\nu}.
\]  
(2.61)
Formula (2.61) makes obvious symmetry of \( \text{sym} T_{\mu\nu} \).

### 2.5 Invariant Hamiltonian

Even if one fixes gauge by any Poincaré-invariant way (e.g. imposing the condition \( z_{\alpha}' z_{\alpha}^{\prime} = -1 \)), then the action functional (2.2) would be invariant with respect to the choice of an initial value of the parameter, i.e. with respect to the transformations
\[
\begin{align*}
\theta &\to \tilde{\theta} = \theta + \delta \varepsilon, \quad \delta \varepsilon = \text{const} \\
\delta z^{\mu} (\theta) &= 0 \\
\delta x^{\mu} &= 0 \\
\delta \phi &= 0.
\end{align*}
\]  
(2.62)
then (2.34) gives an expression for the preserved invariant Hamiltonian of the particle (which is the generator of time parameter translations):
\[
\delta W = [\mathcal{H}][\Sigma]|^{\delta z^{\alpha} \Delta \varepsilon}.
\]  
(2.63)
However, as noted above, variation of the time parameter \( \Delta \theta \) is not included into (2.34). Therefore, substituting (2.62) into (2.34) and comparing the result with (2.63), we obtain
\[
\mathcal{H} = 0.
\]  
(2.64)
Thus, arbitrariness in the choice of an initial value of the time parameter \( \theta \) does not lead to the integral of motion (additional to ten relativistic ones).

Equality to zero of the invariant Hamiltonian also indicates impossibility to construct manifestly Poincaré-invariant and RI-Hamiltonian formalism for the relativistic system "particle+field", similarly to the relativistic particle that moves in the external field. This difficulty is similar to the "zero-Hamiltonian problem" in the generally covariant canonical formulation of the general relativity. It arises in the theory of strings, too.

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References

[1] G. Kalman, Phys. Rev. 123, 384 (1961).
[2] F. G. Belinfante, Physica 6, 887 (1939).
[3] J. Schwinger, Phys. Rev. 74, 1439 (1948).
[4] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[5] M. Kaku, Introduction to Superstrings (Springer-Verlag, New-York, 1992).