We consider gravitational lensing by a generic extended mass distribution. We represent the static external gravitational field of the lens as a potential via an infinite set of symmetric trace free (STF) moments. This leads to a formulation of the diffraction integral where the gravitational phase shift has the most general form. We use this integral to describe the optical properties of extended gravitational lenses. We discuss the possibility of determining the physical characteristics of the lens including its shape, orientation and composition via gravitational lensing. To do that, we consider STF multipole moments for several well-known solids with uniform density. We discuss the caustics formed by the point spread function (PSF) of such lenses, and also the view seen by an imaging telescope placed in the strong interference region of the lens. We show that at each STF order, all the bodies produce similar caustics that are different only by their magnitudes and orientations. Furthermore, there is ambiguity in determining the shape of the lens and its mass distribution if only a limited number of moments are used in the model. This result justifies the development of more comprehensive lens models that contain a greater number of multipole moments. At the same time, inclusion of higher multipole moments leads to somewhat limited improvements as their contributions are suppressed by corresponding powers of the small parameter \((R/b)^l\), where \(R\) characterizes the body’s physical size and \(b\) is the impact parameter, resulting in a weaker signature from those multipole moments in the PSF. Thus, in realistic observations there will always be some ambiguity in the optical properties of a generic lens, unless the properties of the lens can be determined independently, as in the case of the solar gravitational lens (SGL). Our results are novel and offer new insight into gravitational lensing by realistic astrophysical systems.

I. INTRODUCTION

We recently developed a new approach to study generic extended gravitational lenses \([1]\). For that, we considered the propagation of high-frequency electromagnetic (EM) waves in the vicinity of an extended gravitating body. Using the Mie theory \([2, 3]\), we solved the Maxwell equations on the background of a static gravitational field, while working within the first post-Newtonian approximation of the general theory of relativity. The result was the 3-dimensional EM field that describes the diffraction of the EM field on an opaque extended gravitating obscuration.

The new solution describes the EM field deposited on the image plane located in various regions of interest, including those of strong and weak interference and that of geometric optics. We have shown that deviations from spherical symmetry in the lensing object’s gravitational field is evident only in the strong interference region, where it leads to caustics of various orders appearing in the lens’ point spread function (PSF) \([1, 4]\). In the two remaining regions, the optical properties of the lens are consistent with those of a point mass \([5, 6, 10]\). Thus, to capture the most interesting behavior in lensing by a body with arbitrary mass distribution, one needs to consider the strong interference region.

Further generalizing the newly developed wave-optical treatment, we extended the description of gravitational lensing to a generic mass distribution \([11]\). To do that, we modeled the external gravitational field of an extended object in the most general case, taking the potential in the form of an infinite series of symmetric trace-free (STF) multipole moments called “mass multipoles” in the literature (in order to distinguish contributions to the gravitational field by velocity-dependent terms, i.e., “current multipoles”). Such a representation of the gravitational potential in terms of the STF Cartesian tensors is equivalent to that expressed in the form of spherical harmonics. The advantage of using the STF formalism is that it allows us to derive the gravitational phase shift for arbitrary mass distributions, not restricted to, e.g., axial symmetry. This generalizes our previous results \([1, 10, 12, 13]\).

Recognizing the value of this development, there is a need to study possible practical applications of the new wave-optical treatment. There are important questions regarding the fidelity of prospective models of gravitational lenses using STF multipole moments. What is the number of moments needed to achieve the best modeling accuracy? To what extent is it possible to determine the shape and distribution of matter within the lens from examining images produced by it? Ultimately, there is the need to assess the best strategy in the development of a physically justified model. Our paper aims to provide guidance by presenting specific, idealized examples of gravitational lenses and investigating some of their properties in detail.

This paper is organized as follows: In Section \([11]\) we summarize the the solution for the EM field that was obtained on the background of a gravitational field with a generic mass distribution. For that we take the external

Multipole representation of a generic gravitational lens

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gravitational potential of an extended lens expressed via an infinite set of STF mass multipole moments. In Section III we consider lensing by bodies of arbitrary composition, with their gravitational fields represented by STF tensor moments. We derive the gravitational phase shift and discuss the optical properties of several exotic gravitational lenses corresponding to simple geometric shapes. In Section IV we comment on the projection into the lens plane and the consequent loss of information, and the extent to which the lens can still be reconstructed from observing its PSF. In Section V we discuss results and outline the next steps in our investigation. To streamline the main text, we moved some computational details to the Appendices.

II. OPTICAL PROPERTIES OF AN EXTENDED LENS

We consider an extended object, an arbitrary mass distribution, acting as a gravitational lens [11]. The object’s Schwarzschild radius is \( r_g = 2GM/c^2 \), where \( M \) is the object’s mass. We study the propagation of a high-frequency plane EM wave (i.e., neglecting terms \( \alpha (kr)^{-1} \), where \( k = 2\pi/\lambda \) is the wavenumber and \( \lambda \) is the wavelength) in the vicinity of the lens. We assume that the wave is emitted by a point source, which is located at a large distance \( r \) from the lens, such that \( r_g/r \ll 1 \). We consider that this field is observed in an image plane also located at a large distance \( r \) from the lens, such that \( r_g/r \ll 1 \).

A. The EM field on the image plane

Following closely the notation in [11], we represent the trajectory of an incident light ray as

\[
x(t) = x_0 + k(x - t_0) + \mathcal{O}(r_g),
\]

where \( k \) is a unit wave vector in the direction of propagation of the incident light ray and \( x_0 \) represents the origin of the ray. We use \( b = [k \times x_0] \times k \) as the vector impact parameter corresponding to the unperturbed light ray’s trajectory. We use the affine parameter \( \tau = \tau(t) \) to characterize light ray’s path (see details in Appendix B in [5]):

\[
\tau = (k \cdot x) = (k \cdot x_0) + c(t - t_0).
\]

If the \( z \)-axis of the chosen Cartesian coordinate system is oriented along the incident direction of the light ray, we have \( \tau = z \). The value of \( \tau \) starts as negative at the originating point of the light ray (\( \alpha \approx \pi \)), reaches zero at the point of closest approach to the lens (characterized by \( \alpha = \pi/2 \)) and progresses through increasing positive values after departing the vicinity of the lens (\( \alpha \approx 0 \)). With the help of this parameter, we can rewrite (1) as

\[
x(\tau) = b + k\tau + \mathcal{O}(r_g), \quad \text{with} \quad ||x(\tau)|| \equiv r(\tau) = \sqrt{b^2 + \tau^2} + \mathcal{O}(r_g).
\]

In our lens-centric cylindrical coordinate system \((\rho, \phi, z)\) [11], we then have:

\[
\begin{align*}
  k & = (0, 0, 1), \\
  b & = b(\cos \phi, \sin \phi, 0) = b m, \\
  x & = \rho(\cos \phi, \sin \phi, 0).
\end{align*}
\]

Using this parametrization, we solved to the Mie problem (see discussion in [3, 6]) to the required order \((\rho \lesssim r_g \ll r)\) and found that the EM field is given by [1, 10]:

\[
\begin{pmatrix}
  E_\phi \\
  H_\rho
\end{pmatrix} = \begin{pmatrix}
  H_\phi \\
  -E_\phi
\end{pmatrix} = \frac{E_0}{r_0} e^{i\Omega(t)} A(x) \begin{pmatrix}
  \cos \phi \\
  \sin \phi
\end{pmatrix} + \mathcal{O}(r_g^2, \rho^2/z^2),
\]

where \( \Omega(t) = (k(r_0 + r) - \omega t) \). The remaining components of the EM field are small, \((E_z, H_z) = \mathcal{O}(\rho/z)\).

The amplification factor of the EM field, \( A(\rho, \phi) \equiv A(x) \), is given as

\[
A(x) = \frac{k}{ir} \frac{1}{2\pi} \int d^2b \exp \left[ ik \left( \frac{1}{2r} (b - x)^2 + \frac{2}{c^2} \int_{\tau_0}^{\tau} U(b, \tau') d\tau' \right) \right],
\]

where the phase of the integral of [8] is familiar as the Fermat potential that describes gravitational lensing. The first term in the phase of [8] is the geometric delay [14]. The remainder of the expression represents the gravitational delay accumulated by the EM wave as it travels through the gravitational potential \( U \) from the source to the image plane.
B. Computing the eikonal phase for a generic gravitational field

Considering a generic case, it was shown \[13\]–\[20\] that the scalar gravitational potential may equivalently be given in the following form:

\[
U(x) = G \int \frac{\rho(x') d^3x'}{|x - x'|} = GM \sum_{\ell \geq 0} \frac{(2\ell - 1)!!}{\ell!} T_{L \ell} \hat{n}_{L \ell}, \tag{9}
\]

where \(r = |x|\), \(M\) is the mass of the body and \(T_L \equiv T^{<a_1\ldots a_L>}\) are the body’s normalized Newtonian STF mass multipole moments, defined as

\[
M = \int d^3x' \rho(x'), \quad T^{<a_1\ldots a_L>} = \frac{1}{M} \int d^3x' \rho(x') x^{<a_1\ldots a_L>}, \tag{10}
\]

where \(x^{<a_1\ldots a_L>} = x^{<a_1 x_{a2} \ldots x_{aL}>} \equiv \hat{x}^L\), while the angle brackets \(\langle \ldots \rangle\) denote the STF operator \[21\]. Without loss of generality, we set the origin of the coordinate system at the body’s center-of-mass, which allows us to eliminate the dipole moment \(T^a\) from the expansion \(9\).

The first few terms of \(9\) are given as

\[
U(r) = GM \left\{ \frac{1}{r} + \frac{3T^{<ab>}}{2r^3} x^a x^b + \frac{5T^{<abc>}}{2r^5} x^a x^b x^c + \frac{35T^{<abcd>}}{8r^7} x^a x^b x^c x^d + O(r^{-6}) \right\}. \tag{11}
\]

This Cartesian multipole expansion of the Newtonian gravitational potential is equivalent to expansion in terms of spherical harmonics (e.g., see discussion in \[11\]).

Using the light trajectory parametrization \(x = x(b, r)\) from \[3\], one obtains the following expression for the gravitational eikonal phase shift (see derivation details in \[11\]):

\[
\varphi(b) = \frac{2k}{c^2} \int_{r_0}^r U(b, r') dr' = kr_g \ln 4k^2 r r_0 - 2kr_g \left( \ln kb - \sum_{\ell=2}^{\infty} \frac{(2\ell - 2)!!}{\ell! b^\ell} \sqrt{t_\ell^+ + t_\ell^x} \cos[\ell(\phi_\ell - \phi)] \right) + O(r_g^2), \tag{12}
\]

where \(t_\ell^+\) and \(t_\ell^x\) are the transverse-trace free (TT) components of the \(T^{<a_1\ldots a_L>}\) tensor and the angle \(\phi_\ell\) is given by

\[
\cos[\ell\phi_\ell] = \frac{t_\ell^+}{\sqrt{t_\ell^+ + t_\ell^x}}, \quad \sin[\ell\phi_\ell] = \frac{t_\ell^x}{\sqrt{t_\ell^+ + t_\ell^x}}. \tag{13}
\]

In \[11\], we computed several low order terms in \[12\], namely for \(\ell = 2, 3, 4\). We use parameterizations for the vectors \(k\) and \(m\) as given by \[13\]–\[15\]. Thus, the lowest order \(t_2^+\) and \(t_2^x\) are given as

\[
t_2^+ = \frac{1}{2} (T_{11} - T_{22}), \quad t_2^x = T_{12}, \tag{14}
\]

\[
t_3^+ = \frac{1}{2} (T_{11} - 3T_{22}), \quad t_3^x = \frac{1}{4} (3T_{112} - T_{222}), \tag{15}
\]

\[
t_4^+ = \frac{1}{8} (T_{1111} + T_{2222} - 6T_{1122}), \quad t_4^x = \frac{1}{2} (T_{1112} - T_{1222}). \tag{16}
\]

We observe that at each order, the gravitational phase shift is determined by only the two degrees of freedom of the corresponding TT-projected STF multipole moment, \(t_\ell^+\) and \(t_\ell^x\). In other words, at each STF order, \(\ell\), the amplitude, \((t_\ell^+ + t_\ell^x)^{\frac{1}{2}}\), and the rotation angle, \(\ell\phi_\ell\), of the gravitational phase shift \[12\]–\[13\] is set by only two combinations of the TT-projected STF mass multipole moments, \(t_\ell^+\) and \(t_\ell^x\) (see details in \[11\]). These parameters may be known from observations and used to improve the overall lens model.

C. Optical properties of the extended lens

Substituting result \[12\]–\[13\] in \[8\], we get

\[
A(x) = e^{ikr_g \ln 4k^2 r r_0} \frac{k}{ivr^2} \int \int d^2b \exp \left\{ ik \left( \frac{1}{2r} (b - x)^2 - \frac{2r_g}{\ln kb - \sum_{\ell=2}^{\infty} \frac{(2\ell - 2)!!}{\ell! b^\ell} \sqrt{t_\ell^+ + t_\ell^x} \cos[\ell(\phi_\ell - \phi)] \right) \right\}. \tag{17}
\]
In general, this integral must be treated numerically. However, there are two important observations: 1) As the contribution of the \( \ell \)-th multipole moment scales as \( 1/b^\ell \), at some distance from the lens, true to the spirit of the oft-heard joke about all cows appearing spherical to a distant observer, the overall lensing potential approaches that of a monopole. 2) For a weakly aspherical lens, multipole moments are small, making it possible to evaluate (17) using the method of stationary phase with respect to the radial variable, \( b \), as we did in \[ 1, 10 \]. Specifically, we express the integration variables in the double integral (17) using the polar coordinates \((b, \phi_x)\) and evaluate the radial integral from a finite value \( R \) that characterizes the extent of the lens. Essentially, this means that we treat the lens as an opaque object, considering only light with impact parameter \( b > R \). Under these conditions, we found that we can evaluate the radial integral in (17) using the method of stationary phase (see \[ 1, 5, 10 \]), which leads to the following form of the amplification factor

\[ A(x) = \sqrt{2\pi kr_\rho} e^{i\sigma_\rho} e^{ik(r_\rho + r_s \ln 4k^2 r_\rho)} B(x), \quad (18) \]

where \( B(x) \) is the generalized complex amplitude of the EM field in case of an arbitrary, weakly aspherical lens:

\[ B(x) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_x \exp \left[ -ik \left( \frac{2r_\rho}{r} \rho_x \cos(\phi_x - \phi) - 2r_\rho \sum_{\ell=2}^{\infty} \frac{(2\ell - 2)!!}{\ell! (\sqrt{2r_\rho r})^\ell} \sqrt{t_x^2 + t_x^2 \cos[\ell(\phi_x - \phi_\ell)]} \right) \right]. \quad (19) \]

Result (19) yields the following PSF for a generic lens (see details in \[ 1, 11 \]):

\[ \text{PSF}(x) = |B(x)|^2. \quad (20) \]

This PSF can be used for the practical modeling of gravitational lenses, especially for imaging of faint sources \[ 12 \].

Finally, to study imaging with an extended lens, we must consider the EM field as it is seen through an imaging telescope. Their optical properties are guided by (19) and (23), correspondingly. Based on our prior research \[ 1 \], we know that at each order \( \ell \) the PSF will exhibit a unique caustic \[ 4 \] with the cusps yielding bright images to be observed by the telescope \[ 12 \]. This result allows for physically-consistent modeling of realistic gravitational lenses. Using the intensity of light observed in the image plane, \( I(x, x_i) \), given by (23) with \( A(x, x_i) \) from (19) we can study imaging with an extended lens associated with a generic gravitational potential.
III. LENSING WITH BODIES OF VARIOUS SHAPES

To demonstrate how our results can be applied in practice, we consider the gravitational fields and corresponding lensing for classic geometric objects with uniform mass density, such as the ellipsoid, the cuboid, the cylinder and the right circular cone. For that, using (12) the gravitational eikonal phase shift expressed via the STF multipole moments that to the order of $O(r^{-2})$ was obtained in (11) in the following form:

$$\xi_b(b) = -kr_g \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} T^{<a_1 \ldots a_\ell>} \partial_{a_1 \ldots \partial_{a_\ell}} \ln \rho b \equiv kr_g \sum_{\ell=2}^{\infty} \frac{(2\ell - 2)!}{\ell! b^\ell} \sqrt{\ell^2 + t^2} \cos[\ell(\phi - \phi_0)].$$  

(25)

Using the first of these two expressions above, the gravitational eikonal phase shifts for the quadrupole ($\ell = 2$), octupole ($\ell = 3$) and hexadecapole ($\ell = 4$) STF multipole mass moments take the form:

$$\xi_b^2(b) = kr_g \frac{1}{2b^2} T^{<ab>} \left(2m^a_m b + k^a k^b - \delta^{ab}\right),$$  

(26)

$$\xi_b^3(b) = kr_g \frac{1}{6b^4} T^{<abc>} \left(8m^a m^b m^c - 2m^a (\delta^{bc} - k^b k^c) - 2m^b (\delta^{ac} - k^a k^c) - 2m^c (\delta^{ab} - k^a k^b)\right),$$  

(27)

$$\xi_b^4(b) = kr_g \frac{1}{4b^6} T^{<abcd>} \left(8m^a m^b m^c m^d + \frac{1}{3} \left(\delta^{bc} - k^b k^c\right) \left(\delta^{ad} - k^a k^d\right) + \frac{1}{3} \left(\delta^{ac} - k^a k^c\right) \left(\delta^{bd} - k^b k^d\right) + \frac{1}{3} \left(\delta^{ab} - k^a k^b\right) \left(\delta^{cd} - k^c k^d\right)\right) +$$

$$+ \frac{1}{3} \left(\delta^{ab} - k^a k^b\right) \left(\delta^{cd} - k^c k^d\right) - \frac{1}{4} \left(m^a m^b \delta^{cd} - k^b k^c\right) + m^a m^c \delta^{bd} - k^b k^d) + m^a m^d \delta^{bc} - k^b k^c) +$$

$$+ m^b m^c \delta^{ad} - k^a k^d) + m^b m^d \delta^{ac} - k^a k^c\right) + m^c m^d \delta^{ab} - k^a k^b\right),$$  

(28)

where $k$ and $m$ are from (34) and (33), correspondingly (see (11) for details.)

Below, we will use these expressions to study lensing by bodies of various shapes.

A. The lowest STF moments

We first define the STF moments in a particular body-centric coordinate system that is convenient for calculations. For that we use the usual definition (20):

$$Q^L = \int d^3x \rho(x) x^L, \quad \text{where} \quad L \in [1, \ell].$$  

(29)

The coordinate combinations needed to compute the lowest Cartesian STF multipole moments (20) are given as:

$$r^2 n_{ab} = \text{STF}_{ab} \left(x^a x^b\right) = x^a x^b - \frac{1}{3} r^2 \delta^{ab},$$  

(30)

$$r^3 n_{abc} = \text{STF}_{abc} \left(x^a x^b x^c\right) = x^a x^b x^c - \frac{1}{5} r^2 \left(\delta^{ab} x^c + \delta^{bc} x^a + \delta^{ca} x^b\right),$$  

(31)

$$r^4 n_{abcd} = \text{STF}_{abcd} \left(x^a x^b x^c x^d\right) = x^a x^b x^c x^d -$$

$$- \frac{1}{7} \left(x^a x^b \delta^{cd} + x^a x^c \delta^{bd} + x^a x^d \delta^{bc} + x^b x^c \delta^{ad} + x^b x^d \delta^{ac} + x^c x^d \delta^{ab} + r^2 \delta^{abcd}\right).$$  

(32)

To consider lensing by bodies of known shapes, we use their STF multipole moments. Note that, technically, these moments are easier to compute in their center-of-gravity coordinate system. Thus, we will distinguish two sets of moments – those computed in a particular coordinate system that simplifies the calculations, $Q^L$, introduced in (20), and those rotated to the arbitrary frame $T^L$, used in (10). Physically, these two are identical. Rotating these moments to the chosen coordinate system generally involves the three Euler angles.

We start by taking the reference orientation so that the principal axes coincide with the basis vectors $(e_j)_{j=1,2,3}$. The Euler angles are based on the fact that any general rotation $R$ can be written in terms of three angles and so that $R$ is the composition of three rotations:

$$R_1(\phi_3) = R_3(\phi_3) R_1(\beta_3) R_3(\phi_3) = \begin{pmatrix} \cos \phi_3 & \sin \phi_3 & 0 \\ -\sin \phi_3 & \cos \phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta_3 & \sin \beta_3 \\ 0 & -\sin \beta_3 & \cos \beta_3 \end{pmatrix} \begin{pmatrix} \cos \phi_3 & \sin \phi_3 & 0 \\ -\sin \phi_3 & \cos \phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  

(33)
where $\mathbf{R}_3(\psi)$ is a right-handed rotation of angle $\psi$ around the $x^3$ axis, $\mathbf{R}_1(\beta_s)$ is a right-handed rotation of angle $\beta_s$ about the $x^1$ axis, and $\mathbf{R}_3(\phi_s)$ a right-handed rotation of angle $\phi_s$ about the $x^3$ axis. Note that conventionally, the set of $(\phi, \theta, \psi)$ angles are used to denote the Euler angles. We choose to denote these angels as $(\phi_s, \beta_s, \psi)$ for consistency with our prior research, e.g., [1, 4, 10–13]. Expression (33) yields the well-known general formula:

$$
\mathbf{R}(\phi_s, \beta_s, \psi) = \begin{pmatrix}
\cos \psi \cos \phi_s - \cos \beta_s \sin \phi_s 
& \cos \psi \sin \phi_s + \cos \beta_s \cos \phi_s 
& \sin \psi \sin \beta_s \\
-\sin \psi \cos \phi_s - \cos \beta_s \sin \phi_s 
& -\sin \psi \sin \phi_s + \cos \beta_s \cos \phi_s 
& 
- \sin \psi \sin \beta_s \\
\sin \beta_s \sin \phi_s 
& \sin \beta_s \cos \phi_s 
& \cos \beta_s
\end{pmatrix} \equiv R^{ij}. \quad (34)
$$

To rotate the STF multipole moments, $Q^L$, from the body coordinate frame to the chosen coordinate frame and to obtain $T^L$, we must rotate these tensors:

$$
T^{ij} = R^i_p R^j_q Q^{pq}, \quad T^{ijk} = R^i_p R^j_q R^k_s Q^{pqs} \quad T^{ijkl} = R^i_p R^j_q R^k_s R^l_w Q^{pqsw}, \quad (35)
$$

where $T^{ij}$, $T^{ijk}$ and $T^{ijkl}$ are the lowest STF multipole tensors transformed to arbitrary inertial coordinates. (As such rotation from body-fixed to an inertial orientation involves all the components of the tensor $T^{ij}$, using the first expression in (25), technically is more convenient, which explains the choice of (26)–(28)).

B. Solid and hollow spheres

First, we consider a solid sphere of radius $R$ and mass $M$. Using the definition (29) with (30) we see that the STF quadrupole mass moment tensor of a sphere vanishes, namely $T^{ab} = 0$. Similarly, the STF quadrupole moment tensor of a hollow sphere of radius $R$ and mass $M$ yielding $T^{ab} = 0$, as expected. It is easy to verify that all higher STF mass moments of solid and hollow spheres also vanish, $T^L = 0, \ell \geq 1$. Thus, the gravitational potential (11) and consequently, the gravitational lensing behavior of these two types of objects – solid and hollow spheres – are identical to those of a monopole or a point mass [5], in accordance with the Newton’s shell theorem. The optical properties of such monopole gravitational lenses are well-established and were extensively discussed in the literature, e.g., [5–8].

C. Solid ellipsoid

The STF quadrupole moment tensor of a solid ellipsoid with a uniform density distribution, semi-axes $a, b, c$ (Fig. 1) and mass $M$ is computed to be

$$
Q^{ij} = \frac{M}{15} \begin{bmatrix}
2a^2 - b^2 - c^2 & 0 & 0 \\
0 & 2b^2 - a^2 - c^2 & 0 \\
0 & 0 & 2c^2 - a^2 - b^2
\end{bmatrix}. \quad (36)
$$

Note that this expression is given in a specific coordinate frame with primary components of the moment of inertia oriented with respect to a particular body-fixed coordinate axis.

To generalize result (36) and to develop an expression for the gravitational phase shift due to an ellipsoid, we first rotate $Q^{ij}$ from (36) to assume a generic orientation with respect to the incident direction of the EM wave propagation, given by $\mathbf{k}$. After that, we rotate $Q^{ij}$ using (35) and substitute the result in (26), while using the parametrization for $\mathbf{b}$ and $\mathbf{k}$ from (4)–(6). To conduct this transformation, we study rotations of STF tensors and derive expressions to
describe arbitrary orientations of a body with respect to the chosen coordinate system. Specifically, using (26) and (35), we have the following result:

\[
\xi_b^{[2]}(\mathbf{b}) = kr_g \frac{1}{1005} \left\{ \cos[2(\phi_\xi - \phi_s)] \left[ \sin^2 \beta_s \left( \frac{1}{2} (a^2 + b^2) - c^2 \right) + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2} (a^2 - b^2) \right] + \cos 2(\phi_\xi - \phi_s) \cos \beta_s \sin 2\psi (a^2 - b^2) \right\},
\]

where \( \phi_s, \beta_s, \psi \) are the three Euler angles for an arbitrary rotation. Also, note that for the ellipsoid, to avoid conflicting notation with the size of one of the semi-axes, we use \( b \) to denote the impact parameter in the denominator.

Introducing magnitude, \( Q_{c2} \), and phase, \( \phi_{c2} \), factors for a generic ellipsoid,

\[
Q_{c2} = \sqrt{\left[ \sin^2 \beta_s \left( \frac{1}{2} (a^2 + b^2) - c^2 \right) + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2} (a^2 - b^2) \right]^2 + \left[ \cos \beta_s \sin 2\psi (a^2 - b^2) \right]^2},
\]

\[
\cos 2\phi_{c2} = \frac{\sin^2 \beta_s \left( \frac{1}{2} (a^2 + b^2) - c^2 \right) + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2} (a^2 - b^2)}{Q_{c2}},
\]

we rewrite (37) in the familiar form,

\[
\xi_b^{[2]}(\mathbf{b}) = kr_g \frac{Q_{c2}}{1005} \cos[2(\phi_\xi - \phi_s - \phi_{c2})].
\]

Thus, the caustic in the PSF of the gravitating lens of an ellipsoid is a function of two parameters, the magnitude \( Q_{c2} = Q_{c2}(a, b, c, \beta_s, \psi) \) and rotation angle \( \phi_s + \phi_{c2} \). Verifying this behavior numerically, we endeavored to model the gravitational field of a cuboid in an arbitrary orientation. For this, utilizing our existing numerical code to analyze the SGL, we modeled an object weighing as much as the Sun, in the form of a homogeneous cuboid with sides of 30, 120 and 270 km, respectively (the reader may recognize the ratio of the sides of the “monolith” from Clarke’s celebrated science-fiction novel.) As we made the cuboid “tumble” through several orientations, as expected, the resulting PSF retains its regular astroid shape even as it rotates and changes size. Correspondingly, a telescopic view of a distant point source, observed through this gravitational lens, shows a regular Einstein cross in various stages of development and phases of rotation (Fig. 3).

When \( a = b \), \( \phi_{c2} = 0 \) and \( Q_{c2} = (a^2 - c^2) \sin^2 \beta_s \). This can also be confirmed by substituting \( a = b \) into expression (36), which then reduces to

\[
Q_{\text{axisym}}^{ij} = \frac{(a^2 - c^2)}{5} M \left[ \frac{1}{3} 0 0 \right],
\]

\[
 \simeq \frac{2}{5} Ma^2 \left( \frac{a - c}{a} \right) \left[ \frac{1}{3} 0 0 \right].
\]
which is consistent with the STF moment of a spheroid (ellipsoid of revolution) \(^{(11)}\). To demonstrate this, we use \((35)\) to rotate \(Q^{ij}\) from \((11)\) to the needed coordinate frame using \((35)\). Then, we substitute the result into \((26)\) (or, equivalently, in \((37)\)) and derive an expression for the gravitational phase shift introduced by lensing on a spheroid:

\[
\xi_b^{[2]}(b) = -k r_y J_2 \frac{a^2}{2 b^2} \sin^2 \beta_s \cos[2(\phi_x - \phi_s)],
\]

where the normalized dimensionless quadrupole is \(J_2 = -\frac{2}{3} (a - c)/a\), as usual and \(b\) now is the impact parameter.

Note that the octupole moment of the ellipsoid vanishes, \(Q^{<ijk>}_{<ijk>} = 0\), thus the next non-vanishing moment is the hexadecapole. We have all the relevant expressions, but are not bringing them here as they are rather lengthy.

One immediate conclusion from this analysis is that, if one considers only the quadrupole moment, there is ambiguity in specifying the lens’ shape. Inclusion of the next order moment reduces that ambiguity, but not completely eliminates it. Furthermore, one would need to include several moments and operate in the strong lensing regime (in the vicinity of the optical axis, where the impact parameters is smallest) to have good constraints on the lens’ shape and mass distribution. We will further discuss this point below when considering bodies of other shapes.

\[\text{D. Solid cuboid}\]

The STF quadrupole moment of a solid homogeneous rectangular block of width \(w\), depth \(d\), height \(h\) (Fig. 3), and mass \(M\) in a body coordinate frame at its center of mass and oriented along the coordinate axes, has the form:

\[
Q^{ij} = \frac{M}{36} \begin{bmatrix}
2w^2 - h^2 - d^2 & 0 & 0 \\
0 & 2d^2 - w^2 - h^2 & 0 \\
0 & 0 & 2h^2 - w^2 - d^2
\end{bmatrix}.
\]  

(43)

FIG. 3: The generic cuboid characterized by width \((w)\), depth \((d)\) and height \((h)\).

For a generic cuboid, we obtain an expression for the gravitational eikonal phase shift by substituting the components \(Q^{ij}\) from \((35)\) into \((35)\) and then into \((26)\). As a result, we have the following result for the eikonal phase shift:

\[
\xi_b^{[2]}(b) = kr_y \frac{1}{216 a^2} \left\{ \cos[2(\phi_x - \phi_s)] \left[ \sin^2 \beta_s \left( \frac{1}{2} (w^2 + d^2) - h^2 \right) + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2} (w^2 - d^2) \right] \right. + \\
+ \sin[2(\phi_x - \phi_s)] \cos \beta_s \sin 2\psi \left( w^2 - d^2 \right) \right\}.
\]

(44)

Comparing this result to that of an ellipsoid given by \((37)\), we see that, as expected, the eikonal phase shift induced by a generic cuboid behaves similarly. To demonstrate this consistency, we again introduce the magnitude, \(Q_{c2}\), and phase, \(\phi_{c2}\), defined for a generic cuboid as

\[
Q_{c2} = \sqrt{\left\{ \sin^2 \beta_s \left( \frac{1}{2} (w^2 + d^2) - h^2 \right) + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2} (w^2 - d^2) \right\}^2 + \left[ \cos \beta_s \sin 2\psi \left( w^2 - d^2 \right) \right]^2},
\]

(45)

\[
\cos 2\phi_{c2} = \frac{\sin^2 \beta_s \left( \frac{1}{2} (w^2 + d^2) - h^2 \right) + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2} (w^2 - d^2)}{Q_{c2}}, \quad \sin 2\phi_{c2} = \frac{\cos \beta_s \sin 2\psi \left( w^2 - d^2 \right)}{Q_{c2}}.
\]

(46)
we rewrite (44) in the familiar form,

$$\xi_{b}^{[2]}(b) = k r g \frac{Q_{c2}}{2 b^2} \cos[2(\phi_{k} - \phi_{b2})],$$

(47)

in which we recognize the familiar zonal harmonic form of the quadrupole moment, with the axis rotated by \(\phi_{b} + \phi_{c2}\).

Fig. 2 depicts the quadrupole PSF of the “tumbling” cuboid (indistinguishable from the ellipsoid at the quadrupole level) with the side ratios of 1 : 4 : 9 (the reader may recognize the ratio of the sides of the “monolith” from Clarke’s celebrated science-fiction novel.)

Note that the STF mass octupole moment of a cuboid vanishes, \(Q_{ijkl} < 0\). Thus, the next non-vanishing term in the external gravitational potential produced by a cuboid will be that due to the hexadecapole, \(Q_{ijkl}^6\). That expression will also depend on the two parameters – the magnitude \(Q_{c4}\) and the phase, \(\phi_{c4}\). Because of their length, we will not be presenting them here. Instead, we will consider a cube with all the equal sides.

It is interesting that when \(w = d = h\), the STF quadrupole moment referred to the coordinate frame at the cube’s center of mass vanishes, \(Q_{ijkl} < 0\), yielding \(\xi_{b}^{[2]}(b) = 0\). Thus, insofar as the quadrupole terms in the potential are concerned, the cube behaves like a sphere. Furthermore, as we show in Appendix A such a behavior continues all the way to hexadecapole STF mass multipole tensor \(Q_{ijkl}^6\), given by (A4), which is the first nonvanishing STF multipole moment of a cube after the monopole. In fact, the eikonal gravitational phase shift introduced by the hexadecapole of a cube was determined by (A6) to be:

$$Q_{cu4} = \left\{ \cos 4\phi_{s}(\sin^{4}\beta_{s} + 8\cos^{2}\beta_{s}) + 7\sin^{4}\beta_{s} \right\}^{2} + 16\sin 4\phi_{s} \cos \beta_{s} \left(1 + \cos^{2}\beta_{s}\right) \right\},$$

(49)

$$\cos 2\phi_{cu4} = \frac{\cos 4\phi_{s}(\sin^{4}\beta_{s} + 8\cos^{2}\beta_{s}) + 7\sin^{4}\beta_{s}}{Q_{cu4}}, \quad \sin 2\phi_{cu4} = -\frac{4\sin 4\phi_{s} \cos \beta_{s} \left(1 + \cos^{2}\beta_{s}\right)}{Q_{cu4}}.$$

(50)

Expression (A6) may be brought to a form:

$$\xi_{b}^{[4]}(b) = -k r g \frac{1}{1920} \left(\frac{a}{b}\right)^{4} Q_{cu4} \cos[4(\phi_{k} + \psi - \phi_{cu4})],$$

(51)

which is again a rather familiar expression resulting in a hexadecapole caustic.

**E. Solid cylinder**

The STF quadrupole moment of a solid cylinder with uniform matter distribution with radius \(r\), height \(h\) (Fig. 3), and mass \(M\) has the form:

$$Q_{ij} = \frac{M}{4} \left( r^{2} - \frac{1}{3} h^{2} \right) \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}.$$

(52)

Again, we use (19) to rotate \(Q_{ij}\) from (52) to the chosen coordinate frame and substitute the result into (23). As a result, we have the following expression for the eikonal phase shift introduced by gravitational lensing on a uniform massive cylinder:

$$\xi_{b}^{[2]}(b) = k r g \frac{1}{8b^2} \left( r^{2} - \frac{1}{3} h^{2} \right) \sin^{2}\beta_{s} \cos[2(\phi_{k} - \phi_{s})].$$

(53)

It is interesting that when \(h = \sqrt{3}r\) the quadrupole contribution of the cylinder vanishes and it behaves like a sphere (of course, if contributions of the higher multipole moments may be neglected).
FIG. 4: The cylinder, parameterized by its radius \((r)\) and height \((h)\).

In Appendix [B] we computed the STF moments of a cylinder. In particular, we realized that the STF octopole moment moment of a cylinder vanishes, thus, \(\xi_b^{[3]}(b) = 0\). However, the hexadecapole is present \([B8]\), yielding the following contribution to the gravitational phase shift due to massive cylinder \([B10]\), repeated here:

\[
\xi_b^{[4]}(b) = kr_g \frac{(10r^4 - 10r^2 h^2 + h^4)}{320 b^4} \sin^4 \beta_s \cos[4(\phi_\xi - \phi_s)].
\] (54)

Clearly, there are higher non-vanishing STF mass multipoles present, but not only they are small, their contribution, being scaled as \(1/b^\ell\), is much suppressed.

**F. Right circular cone**

The STF quadrupole moment of a right circular cone with radius \(r\), height \(h\) (Fig. 5), and mass \(M\), in the coordinate system at its center of gravity has the form:

\[
Q^{ij} = \frac{3}{20} M \left( r^2 - \frac{1}{4}h^2 \right) \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}.
\] (55)

We use \([35]\) to rotate \(Q^{ij}\) from \([55]\) to needed coordinate frame, and substitute the result into \([20]\), we derive an expression for the eikonal phase shift introduced by gravitational lensing on a regular uniform massive cone:

\[
\xi_b^{[2]}(b) = kr_s \frac{3}{40 b^2} \frac{1}{b^2} \left( r^2 - \frac{1}{4}h^2 \right) \sin^2 \beta_s \cos[2(\phi_\xi - \phi_s)].
\] (56)

This result is similar to that of an axisymmetric ellipsoid \([42]\) or a cylinder \([53]\).
FIG. 6: The quadrupole plus octupole PSF of the right circular cone (left) and the corresponding telescopic view (right), each corresponding (clockwise from top left) to the cases $h/r = 0, 1, 2, 3$. Note that the octupole contribution vanishes for $h/r = 0$, leaving us with the familiar astroid caustic of the quadrupole. Correspondingly, for $h/r = 2$ the quadrupole contribution is zero, and the result is a pure octupole PSF, with the corresponding three-sided caustic. For the chosen parametrization, this PSF is sufficiently compact for a near uniform Einstein ring to form in the telescopic view.

Although the right circular cone is axisymmetric, it has no reflection symmetry with respect to the plane of its rotational symmetry (i.e., no “north-south” symmetry), which makes this shape particularly interesting as it results in the presence of odd harmonics. When we look at the octupole moment of this object in the STF representation, we indeed find that $\xi^{[2]}$ vanishes when $h = 2r$, whereas $\xi^{[3]}$ vanishes only for $h = 0$. Using the ratio $h/r$ to characterize the properties of this gravitational lens, we can compute its PSF and the resulting images, which are shown in Fig. 6.

Similarly, in Appendix C we computed the hexadecapole STF mass moment of a cone, see (C11), which results in the contribution to the gravitational phase shift (58), which we also repeat here:

$$
\xi^{[4]}_b(b) = kr_g \frac{3(160r^4 - 72r^2 h^2 + 13h^4)}{35840 b^4} \sin^4 \beta_s \cos[4(\phi_\xi - \phi_s)].
$$

Higher-order moments, which are not calculated here, also contribute of course, but their contribution vanishes rapidly with increasing impact parameter. Therefore, the approximations presented here, in particular the visualization in Fig. 6, remain valid so long as $b \gtrsim (r, h)$.

G. Trirectangular tetrahedron

The STF quadrupole moment of a trirectangular tetrahedron with width $a$, depth $b$, height $c$ (Fig. 7), and mass $M$, in the coordinate system at its center of gravity has the form (see Appendix D for details):

$$
Q^{ij} = \frac{M}{80} \begin{bmatrix}
2a^2 - b^2 - c^2 & 0 & 0 \\
0 & 2b^2 - a^2 - c^2 & 0 \\
0 & 0 & 2c^2 - a^2 - b^2
\end{bmatrix}.
$$

For a generic tetrahedron, we obtain an expression for the gravitational eikonal phase shift by substituting the components $Q^{ij}$ from (59) into (55) and then into (26). As a result, we have the following eikonal phase shift:

$$
\xi^{[2]}_b(b) = kr_g \frac{3}{160b_0^4} \{ \cos[2(\phi_\xi - \phi_s)] \sin^2 \beta_s \left[ \frac{1}{2}(a^2 + b^2) - c^2 \right] + (1 + \cos^2 \beta_s) \cos 2\psi \frac{1}{2}(a^2 - b^2) + \\
\sin[2(\phi_\xi - \phi_s)] \cos \beta_s \sin 2\psi(a^2 - b^2) \},
$$

where $b_0$ is the base of the tetrahedron.
computed the STF mass multipole moment of the trirectangular tetrahedron, see (D7). The resulting gravitational phase shift vanish. To show that even in this simplified case all the components of the octupole are non-vanishing, we respect to the observer.

of the resulted caustics. These two parameters depend on the dimensions so the tetrahedron and its orientation with the same harmonic structure as for other solids with two parameters controlling the magnitude and rotational angle of the resulted caustics. These two parameters depend on the dimensions so the tetrahedron and its orientation with respect to the observer.

In the case when \( a = b = c \), one can see from (D7) that the STF mass quadrupole and the corresponding gravitational phase shift vanish. To show that even in this simplified case all the components of the octupole are non-vanishing, we computed the STF mass multipole moment of the trirectangular tetrahedron, see (D7). The resulting gravitational phase shift due to the octupole of a trirectangular tetrahedron with \( a = b = c \) takes the form:

\[
\xi_b^{[3]}(b) = \frac{kr_g}{1920} \left\{ a \right\}^3 \cos[3(\phi_t - \phi_s)] \left[ \frac{1}{2} \left( \cos \psi - \sin \psi \right) \left( 2 + \sin 2 \beta_s \sin \psi + \cos \psi \left( \sin 2 \beta_s + \left( 5 + 3 \cos 2 \beta_s \right) \sin \psi \right) \right] + \\
+ \sin[3(\phi_t - \phi_s)] \left[ 2 \sin^2 \beta_s \left( 4 + 3 \cos \beta_s (\cos \psi + \sin \psi) \right) + \left( 5 \sin \beta_s + \sin 3 \beta_s \right) \sin 2 \psi - \frac{1}{2} \left( 15 \cos \beta_s + \cos 3 \beta_s \right) \left( \cos 3 \psi - \sin 3 \psi \right) \right],
\]

where the magnitude, \( Q_{t3} \), and the phase, \( \phi_{t3} \), can be readily read-off from (D2).

Similarly, in Appendix D we computed the hexadecapole STF mass moment of the trirectangular tetrahedron, see (D10). The corresponding gravitational phase shift is given by (D12), which we repeat here for convenience:

\[
\xi_b^{[4]}(b) = \frac{kr_g}{10240} \left\{ a \right\}^4 \left\{ \cos(4(\phi_t - \phi_s)) \right\} \left[ \sin^4 \beta_s + 2 \sin^2 \beta_s \sin 2 \beta_s (\cos \psi + \sin \psi) + \right. \\
+ 2 \left( 3 + \cos^2 \beta_s \right) \sin 2 \beta_s \left( \cos 3 \psi - \sin 3 \psi \right) + \\
+ \left. \left[ 4 \sin^4 \beta_s (\sin \psi - \cos \psi) + \sin \beta_s (1 + 3 \cos^2 \beta_s) (\cos 3 \psi + \sin 3 \psi) - \right. \\
- 2 \sin \beta_s \sin 2 \beta_s \cos 2 \psi + \frac{1}{2} \cos \beta_s (1 + \cos^2 \beta_s) \sin 4 \psi \right],
\]

again yielding the familiar harmonic structure:

\[
\xi_b^{[4]}(b) = \frac{kr_g}{10240} \left\{ a \right\}^4 Q_{t4} \cos[4(\phi_t - \phi_s - \phi_{t4})],
\]

where the magnitude, \( Q_{t4} \), and the phase, \( \phi_{t4} \), can be readily read-off from (D4).

Therefore, the ultimate results for the gravitational phase shifts on each of these multipoles consistently exhibit the familiar harmonic structure and optical properties.
IV. APPLICATION OF RESULTS

In our previous paper [11], we demonstrated how strong gravitational lensing by arbitrary lenses, modeled in the weak field approximation and treating the lens as opaque (i.e., considering only light rays with impact parameters greater than the characteristic size of the lens) can be modeled using a formalism that strongly resembles the language of zonal harmonics, but with an additional rotation parameter introduced at each multipole.

Using the approach established in [11], we now presented specific calculations of some of the lower-order multipoles characterizing gravitational lensing by simple geometric shapes. Calculating higher-order multipoles is technically straightforward but tedious, best done using software tools, either numerically or by way of computer algebra.

The choice of simple geometric shapes is, of course, not intended to imply that there are actual astrophysical lenses out there that are shaped like a cuboid or a right circular cone. Rather, our intent was to offer these cases as representative worked examples, showing how, once the tensor moments of inertia of the lens are known, the rest is straightforward: the corresponding lens can be modeled, its PSF and the resulting caustics can be calculated, and the PSF can be convolved with that of an imaging telescope with ease, in a process that is almost mechanistic.

This simplicity is achieved because the complex three-dimensional structure of the lens is projected onto the thin lens plane. Unfortunately it also implies that we can learn only so much about a particular lens by studying its caustics or the images that it projects, from a single vantage point such as the solar system. Fig. 3 offers a good example. It depicts gravitational lensing by a homogeneous cuboid. The quadrupole astroids, however, that this lens projects would look just the same for any other lens dominated by its quadrupole moment beyond the monopole. Simply looking at a quadrupole caustic thus does not allow us to distinguish between a cuboid, a general ellipsoid, or even a right circular cone if its orientation is such that its octupole contribution is suppressed or if the impact parameter is large enough to suppress it.

Therefore, it is instructive to investigate under what circumstances and to what extent it is possible to distinguish between physically different gravitational lenses by studying the images that they form given a known source, such as a point source. For this purpose, consistently with Eq. (84) in [11], let us introduce a convenient shorthand notation

\[ \tau_\ell R^\ell = (2\ell - 2)!! \sqrt{t_\ell^2 + t_\ell^2/\ell!} = (2\ell - 2)!! R^\ell \sqrt{C_{\ell\ell}^2 + S_{\ell\ell}^2} \]

where \( R \) is the size of the lens and \( C_{\ell\ell} \) and \( S_{\ell\ell} \) are the appropriate spherical harmonics of its mass distribution (see [11] for details). This notation allows us to write Eq. (17) in the following compact form:

\[ A(x) = e^{iky} \ln k^2 \tau_\ell \frac{k}{ir} \frac{1}{2\pi} \int d^2b \exp \left[ ik \left( \frac{1}{2\ell} (b - x)^2 - 2r_g \left( \ln kb - \sum_{\ell=2}^{\infty} \left( \frac{R}{b} \right)^\ell \cos \left( \ell (\phi_\ell - \phi_\ell) \right) \right) \right) \right] \]

which also allows us to express [10] using a pair of parameters \( \tau_\ell \) and \( \phi_\ell \) characterizing the contribution of each multipole. An observation of the PSF (given by (20)) or alternatively, an image of a compact (point) source as seen through the lens of an imaging telescope, given by (22), allows us to estimate the values of \( r_g \) (characterizing the monopole mass), \( \tau_\ell \) and \( \phi_\ell \) (characterizing contribution of each \( \ell \)-th multipole) and also \( \phi_i \) and \( \rho_i \) (characterizing the telescope’s position with respect to the optical axis associated with the light source).

As a general rule, if an image of a known source is recovered at sufficient resolution for it to be reconstructed to multipole order \( n \) \( (2 \leq \ell \leq n) \), then by modeling the lens using a suitable numerical optimization method we can recover not just \( r_g \), but an additional \( 2(n - 1) \) degrees of freedom in the form of the \( \tau_\ell \) and \( \phi_\ell \) parameters.

This is clearly less information than the full three-dimensional multipole representation of the lens, in the form of \( 2\ell + 1 \) degrees of freedom (spherical harmonic coefficients) at each multipole order \( \ell \) or the corresponding \( 2\ell + 1 \) independent terms in a 3-dimensional STF tensor of rank \( \ell \).

The question then naturally arises: By studying the images cast by the lens from a single vantage point, how much information can be recovered about the mass distribution of the lens? This question can have direct astrophysical significance, as studying lensed images can help reveal information about the mass profile of the lensing object (e.g., a foreground galaxy or galaxy cluster).

The PSFs that we worked out in the current paper for a variety of simple geometric shapes can serve as useful examples to study this question. Comparing, e.g., the STF quadrupole moments characterizing the ellipsoid, [38–39], and the cuboid, [43–44], we see that they are indistinguishable. This might lead us to the premature conclusion that due to the degeneracy inherent to the transverse-tracefree (TT) projection (see [11]), some shapes may not be distinguishable at all, even when sufficient information is collected to reconstruct the magnitude and rotation angle associated with higher-order multipoles.

However, this is not the case, and that is readily demonstrated by the hexadecapole moment of the cube [51]. The cube is the special case of the cuboid where all sides are equal. Similarly, the sphere is the special case of the ellipsoid with three equal semi-axes. At the level of the quadrupole, cubes and spheres are indistinguishable, and
they both have vanishing octupole moments. However, the cube has a nonvanishing hexadecapole moment. This is different from the sphere, which has vanishing multipole moments at all orders. Therefore, if information about the hexadecapole moment is available, the cube and the sphere can be distinguished.

When the lens has an observable octupole (i.e., in the absence of a recognizable “north-south” symmetry), the situation is even easier. The presence of an odd-numbered multipole moment immediately allows us to exclude shapes like the ellipsoid or the cuboid that do not have such a contribution.

Ultimately, it is not possible to uniquely determine the three-dimensional mass distribution of a lens from lensing data alone. The problem remains underdetermined due to the loss of information as a result of the projection. To use a playful analogy, in the dark all cats are black (or at least various shades of gray). How can we tell them nonetheless apart? Perhaps we can listen to their meows! That is to say, we can perhaps learn more about the nature of a gravitational lens if we have prior knowledge obtained by some other means, constraining its shape.

As mentioned above, observation of each multipole order allows us to recover an additional two degrees of freedom characterizing the lens. If we have prior knowledge of the shape of the lens, this might be sufficient to reconstruct its mass profile.

Consider, for instance, the case of a large elliptical galaxy. Visually observing the galaxy determines its shape: it’s not a cone, not a cuboid, not a tetrahedron, it’s an ellipsoid. Or rather, it is a rotational ellipsoid, aka. a spheroid, which is axisymmetric. It is thus characterized by two intrinsic parameters (the two independent semi-axes) and two of the three Euler angles that determine its spatial orientation with respect to the observing location. These four degrees of freedom can be reconstructed if a lensing observation is accurately modeled using a combination of a quadrupole and a hexadecapole STF contribution, each characterized by two values. In another possible trade-off in the parameter space, the eccentricity of the spheroid may be determined from visual observation, and the degrees of freedom from gravitational lens observations may be used instead to constrain the values of parameters characterizing the galaxy’s radial mass profile.

More generally, it is not possible to reconstruct the full set of spherical harmonics that characterize an arbitrary lens, as the mathematical problem is underdetermined. Without auxiliary information, the nature of the lens remains ambiguous. We need prior knowledge, some physical insight into the shape and basic characteristics of the lens to reconstruct it reliably. In the case of an actual astrophysical lens, this may come in the form of a physically realistic model of the self-gravitating object that constitutes the lens. A realistic astrophysical object may also be modeled as the convolution of multiple gravitational multipoles, perhaps each matched against a library of templates similar in spirit to the simple solids that we investigated, but of course representing physically realistic mass distributions. Such models are necessarily beyond the scope of our current analysis.

V. DISCUSSION AND CONCLUSIONS

We studied gravitational lenses with generic mass distributions. To characterize the external gravitational field produced by such an object, we used the STF tensor representation of multipole moments discussed in [11]. We benefited from the fact that the STF approach offers a technical advantage by allowing for a closed form solution while integrating the equations of light propagation in the vicinity of a generic lens. As a result, we were able to develop a wave-optical treatment of gravitational lenses of the most generic structure and internal mass distribution.

To demonstrate the utility of our approach and to emphasize the physics of the related phenomena, we considered a select set of unusual gravitational lenses in the form of common geometric shapes. We computed the STF multipole mass moments of the ellipsoid, the cuboid, the cylinder, the right circular cone, and the tetrahedron. For the least symmetric of these objects, the cone and the tetrahedron, we also calculated the octupole moment, and for the cube, cylinder, the right cone we computed the hexadecapole moments, and used them in conjunction with our existing numerical codes to calculate its PSF and a simulated view of lensing as seen through an imaging telescope.

Computations of higher-order multipole moments are tedious but, in principle, straightforward: some examples are presented in the appendices. We anticipate that the algebraic part of these efforts can be readily automated using computer algebra tools. These can then be integrated with numerical codes, if needed, to compute the gravitational phase shift to arbitrary accuracy for any lens that can be characterized by using multipole moments of various orders.

The practical value of this approach arises from the fact that once the STF multipoles are known, computation of the PSF is reduced to a single integral [19] with finite integration limits. Same is true for the PSF convolved with that of an imaging instrument [23]. The resulting integrals are manageable with a modest computational cost.

We have shown that at each STF order there are specific caustics formed in the PSF of an extended lens. However, different bodies produce similar caustics which differ only by their magnitudes and orientations. If only few STF mass moments are used to model the lens, this similarity could lead to an ambiguity in determining the shape of the lensing object and its mass distribution – the challenge that may be addressed by including larger number of the moments. On the other hand, the presence of higher mass moments is of limited utility as their contributions are suppressed by
the power of small parameter \((R/b)\ell\), where \(R\) is the body’s physical size and \(b\), resulting in a weaker signature from that multipole moment in the PSF.

Therefore, in realistic observations there will always be some ambiguity in the optical properties of a generic lens. This challenge may be reduced if auxiliary information on the lens is known from other means. For example, the knowledge of the dominant symmetry determining the shape of the lens is known, one can use that information to develop a more realistic model. For example, a spheroidal galaxy is better modeled using a sphere, an elliptical galaxy is better modeled by using an ellipsoid, and etc. By using that information one can select a more optical set of STF moments to improve the modeling fidelity.

In the case of the solar gravitational lens (SGL), all the relevant information on the structure of the Sun is well-known allowing the development of a very realistic model of the extended SGL. This information will be used to benefit the analysis, especially in the context of a prospective space mission for imaging of an exoplanet [5–9, 22].

The results presented here are new and may be used to study gravitational lensing with a wide range of realistic astrophysical lenses, including many cases that previously could only be modeled by geometric optics and ray tracing.

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Appendix A: STF moment tensors for a cube

Using the definition for the STF moments (29) and expressions (30)–(32), we compute STF moments for a cube with uniform density using coordinate system positioned at its center of mass:

\[
Q^0 = M = \int d^3x \rho(x) = \rho \int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-a}^{a} dz = \rho a^3 \quad \Rightarrow \quad \rho = \frac{M}{a^3}.
\]
Compute the dipole moment is straightforward:

\[ Q^1 = \frac{M}{a^3} \int _{-a}^{a} x dx \int _{-a}^{a} y dy \int _{-a}^{a} z dz = 0. \]  \hfill (A2)

We see that the dipole moment vanishes in the center-of-mass reference frame, \( Q^i = 0 \).

To compute the quadrupole, we use the relevant STF expression for coordinate combination given by (30). As a result, with this, we see that the quadrupole moment also vanishes, \( Q^{ij} = 0 \). Similarly, using the combination (31), we compute the octupole moments of the cube to see that all components of octupole vanish \( Q^{ijkl} = 0 \).

Next, using (32), we compute the non-zero components of the hexadecapole:

\[ Q^{ijkl} = \frac{Ma^4}{600} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \]  \hfill (A3)

As a result, in the case of a cube, the gravitational potential starts with the monopole term, but then it misses dipole, quadrupole and octupole terms, continuing only with the hexadecapole term, taking the form

\[ U(x) = G \left\{ \frac{M}{r} + \frac{35Q_{abcd}x^a y^b z^c x^d}{8r^9} + \mathcal{O}(r^{-6}) \right\}, \]  \hfill (A4)

which implies that the gravitational filed of a cube is driven by the monopole, much like that of a sphere. The first difference comes with the hexadecapole moment that behaves \( \propto 1/r^5 \). This result is interesting and is not discussed in the literature.

To generalize the result, we rotate the STF hexadecapole tensor (A3) to an arbitrary coordinate system:

\[ \bar{T}^{<abcd>} = R^a_{\mu} R^b_{\nu} R^c_{\gamma} R^d_{\delta} Q^{<\mu\nu\gamma\delta>}, \]  \hfill (A5)

where \( R^a_{\mu} \) is the rotation matrix (34). Next, we substitute the result in (28) and obtain the following expression for \( \xi^{4i}_b(a) \) of a cube:

\[ \xi^{4i}_b(a) = -kr_2 \frac{1}{1920} \left( \frac{a}{b} \right)^4 \left\{ \cos[4(\phi z + \psi)] \cos[4\phi_4(b \sin^4 \beta_z + 8 \cos^2 \beta_z)] + 7 \sin^4 \beta_z \right\} - 4 \sin[4(\phi z + \psi)] \sin[4\phi_4(1 + \cos^2 \beta_z)] \}. \]  \hfill (A6)

Appendix B: STF moments of a cylinder

We use the definition for the STF moments (29) and expressions (30)–(32) and compute the moments for a cylinder with uniform density using a coordinate system positioned at its base:

\[ Q^0 \equiv M = \int d^3x \rho(x) = \rho \int_0^h dz \int_0^r dr' \int_0^{2\pi} d\phi'_s = \rho \pi r^2 h \quad \Rightarrow \quad \rho = \frac{M}{\pi r^2 h}. \]  \hfill (B1)

We compute the dipole moment:

\[ Q^1 = \frac{M}{\pi r^2 h} \int_0^h dz \int_0^r dr' \int_0^{2\pi} d\phi'_s r' \cos \phi'_s = 0, \]  \hfill (B2)

\[ Q^2 = \frac{M}{\pi r^2 h} \int_0^h dz \int_0^r dr' \int_0^{2\pi} d\phi'_s r' \sin \phi'_s = 0, \]  \hfill (B3)

\[ Q^3 = \frac{M}{\pi r^2 h} \int_0^h (z - z_0) dz \int_0^r dr' \int_0^{2\pi} d\phi'_s = \frac{1}{2} M \left( h - 2z_0 \right). \]  \hfill (B4)
Therefore, the center of mass of a cylinder is on the $z$-axis at the position of $z_0 = \frac{h}{3}$. With this choice of $z_0$, all the components of the dipole moment vanish, $Q^i = 0$.

With this result, using (30), we compute the STF quadrupole mass moment of a cylinder in the coordinate system positioned at its center of mass:

$$Q^{ij} = \frac{1}{12} M \left( r^2 - \frac{1}{3} h^2 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(B5)

To generalize this expression for the STF quadrupole moment of a cylinder, we rotate $Q^{ij}$ from (B5) to an arbitrary coordinate system by using rule given by (34), as below

$$T^{ij} = R^i_p R^j_q Q^{pq},$$

(B6)

and substitute the result into (20). This allows us to derive expresssion for the gravitational phase shift introduced by gravitational lensing on a quadrupole moment of a uniform massive cylinder:

$$\xi^{[2]}_b = kr_g \frac{1}{8b^2} \left( r^2 - \frac{1}{3} h^2 \right) \sin^2 \beta_s \cos[2(\phi_{\xi} - \phi_\alpha)].$$

(B7)

Note that if $h = \sqrt{3}r$, the quadrupole moment (B5) vanishes along with the corresponding eikonal phase shift (B7).

The STF octupole moment of a cylinder is computed using (31), which reveals the fact that $Q^{ijk} = 0$.

Finally, we compute the STF hexadecapole mass moment of a cylinder using (32) in the coordinate system at its center of mass, which results in:

$$Q^{ijkl} = \frac{M (10r^4 - 10r^2h^2 + h^4)}{2800} \begin{bmatrix} (3 & 0 & 0 & 0) \\ (0 & 1 & 0 & 0) \\ (0 & 0 & -4 & 0) \\ (0 & 1 & 0 & 0) \\ (0 & 0 & 0 & 0) \\ (0 & 1 & 0 & 0) \\ (0 & 0 & 0 & 0) \\ (0 & 0 & 0 & 0) \end{bmatrix}.$$  

(B8)

To generalize this expression, we rotate the STF hexadecapole tensor (B8) to an arbitrary coordinate system:

$$T^{ijkl} = R^i_p R^j_q R^k_s R^l_t Q^{pqrs},$$

(B9)

where $R^i_p$ is the rotation matrix (31). By substituting the result in (28), we obtain he following expression for the gravitational phase shift introduced by the STF hexadecapole mass moment of a right circular cone:

$$\xi^{[4]}_b = kr_g \frac{10r^4 - 10r^2h^2 + h^4}{320 b^4} \sin^4 \beta_s \cos[4(\phi_{\xi} - \phi_\alpha)],$$

(B10)

which, again, due to the axial symmetry of the cone is independent on the angle $\psi$, as expected.

**Appendix C: STF moments of a right circular cone**

Again, we use the definition for the STF moments (29) and expressions (30)–(32) and compute these moments for a cone with uniform density using a coordinate system positioned at its base:

$$Q^0 = M = \int d^3x \rho(x) = \rho \int_0^h dz \int_0^{(r/h)z} r' dr' \int_0^{2\pi} d\phi'_s = \frac{\rho}{3} \pi r^2 h \quad \Rightarrow \quad \rho = \frac{M}{\frac{4}{3} \pi r^2 h}.$$  

(C1)

We compute the dipole moment:

$$Q^1 = \frac{M}{\frac{4}{3} \pi r^2 h} \int_0^h dz \int_0^{(r/h)z} r' dr' \int_0^{2\pi} d\phi'_s r' \cos \phi'_s = 0,$$

(C2)

$$Q^2 = \frac{M}{\frac{4}{3} \pi r^2 h} \int_0^h dz \int_0^{(r/h)z} r' dr' \int_0^{2\pi} d\phi'_s r' \sin \phi'_s = 0,$$

(C3)

$$Q^3 = \frac{M}{\frac{4}{3} \pi r^2 h} \int_0^h (z - z_0) dz \int_0^{(r/h)z} r' dr' \int_0^{2\pi} d\phi'_s = 3M \left( \frac{h}{3} - \frac{1}{4}z_0 \right).$$

(C4)
Therefore, the center of mass of a cone is on the z-axis at the position of \( z_0 = \frac{3}{4}h \). With this choice of \( z_0 \), all the components of the dipole moment vanish, \( Q^i = 0 \).

Next, using (30), we compute the STF quadrupole mass moment of a right circular cone in the coordinate system at its center of mass:

\[
Q^{ij} = \frac{1}{20} M \left( r^2 - \frac{1}{4}h^2 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} .
\]

To generalize the expression of the STF quadrupole moment, we rotate \( Q^{ij} \) from (C5) to an arbitrary coordinate system by using (35), given as

\[
T^{ij} = R^i_p R^j_q Q^{pq},
\]

and substitute the result it into (26). This allows us to derive expression for the gravitational phase shift introduced by gravitational lensing on a quadrupole moment of a regular uniform massive cone:

\[
\xi^{[2]}_b(\mathbf{b}) = k r_g \frac{3}{40} b^2 \left( r^2 - \frac{1}{4}h^2 \right) \sin^2 \beta_s \cos[2(\phi_b - \phi_s)].
\]

Note that if \( h = 2r \), the quadrupole moment (C5) vanishes along with the corresponding eikonal phase shift (C7).

We compute the STF octupole moment of a right circular cone using (31). This yields the STF octupole moment of a cone in the form as below:

\[
Q^{ijk} = \frac{3}{400} M \left( r^2 + \frac{1}{6}h^2 \right) h \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} .
\]

Again, to generalize the results, we rotate the STF octupole tensor (C8) to an arbitrary coordinate system:

\[
T^{ijk} = R^i_p R^j_q R^k_s Q^{pqrs},
\]

where \( R^i_s \) is the rotation matrix (34). After that, we substitute the result in (27) and obtain the following expression for the gravitational phase shift introduced by the octupole of a right circular cone:

\[
\xi^{[3]}_b(\mathbf{b}) = -k r_g \left( r^2 + \frac{1}{6}h^2 \right) h \frac{\sin^3 \beta_s \sin[3(\phi_b - \phi_s)]}{80b^3},
\]

which, due to the axial symmetry is independent on the angle \( \psi \), as expected.

Finally, we compute the STF hexadecapole mass moment of a right circular cone using (32), which yields the following result:

\[
Q^{ijkl} = \frac{3M (160r^4 - 72r^2h^2 + 13h^4)}{313600} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} .
\]

To generalize this expression, we rotate the STF hexadecapole tensor (C11) to an arbitrary coordinate system:

\[
T^{ijkl} = R^i_p R^j_q R^k_s R^l_w Q^{pqrs},
\]
where \( R^b \) is the rotation matrix. By substituting the result in \( \xi^b \), we obtain he following expression for the gravitational phase shift introduced by the STF hexadecapole mass moment of a right circular cone:

\[
\xi^{[4]}_b(b) = kr_g \frac{3(160r^4 - 72r^2h^2 + 13h^4)}{35840b^4} \sin^4 \beta \cos[4(\phi - \phi_s)],
\]

which, again, due to the axial symmetry of the cone is independent on the angle \( \psi \), as expected.

### Appendix D: STF moments of a trirectangular tetrahedron

To compute the STF moments of a trirectangular tetrahedron with uniform density, we use the definition for the STF quadrupole moment in a body coordinate frame at its center of mass and oriented along the coordinate axes:

\[
Q^0 = \int d^3x \rho(x) = \rho \int_0^a dx \int_0^{(1-x)/a} dy \int_0^{c(1-x-y)/b} dz = \rho \frac{M}{6abc} \Rightarrow \rho = \frac{M}{6abc}.
\]

With this, we compute the dipole moment:

\[
Q^1 = \frac{M}{6abc} \int_0^a (x-x_0)dx \int_0^{(1-x)/a} dy \int_0^{c(1-x-y)/b} dz = \frac{1}{4}M(a-4x_0),
\]

\[
Q^2 = \frac{M}{6abc} \int_0^a dx \int_0^{(1-x)/a} dy \int_0^{c(1-x-y)/b} (y-y_0)dy = \frac{1}{4}M(b-4y_0),
\]

\[
Q^3 = \frac{M}{6abc} \int_0^a dx \int_0^{(1-x)/a} dy \int_0^{c(1-x-y)/b} (z-z_0)dz = \frac{1}{4}M(c-4z_0).
\]

Therefore, the center of gravity of a trirectangular tetrahedron is at the point with coordinates \( r_0 = \{\frac{1}{4}a, \frac{1}{4}b, \frac{1}{4}c\} \). With this choice of \( r_0 \), all the components of the dipole moment vanish, \( Q^i = 0 \).

Using coordinate system positioned at the center of mass of a solid trirectangular tetrahedron, we compute its STF quadrupole moment in a body coordinate frame at is center of mass and oriented along the coordinate axes:

\[
Q^{ij} = \frac{M}{80} \begin{bmatrix}
2a^2 - b^2 - c^2 & 0 & 0 \\
0 & 2b^2 - a^2 - c^2 & 0 \\
0 & 0 & 2c^2 - a^2 - b^2
\end{bmatrix}.
\]

For a generic tetrahedron, we obtain an expression for the gravitational eikonal phase shift by substituting the components \( Q^{ij} \) from (D5) into (38) and then into (26). As a result, we have the following eikonal phase shift:

\[
\xi^{[2]}_b(b) = kr_g \frac{3}{160h_0^2} \left\{ \cos[2(\phi - \phi_s)] \left[ \sin^2 \beta \left( \frac{1}{2}(a^2 + b^2) - c^2 \right) + (1 + \cos^2 \beta) \cos 2\psi \frac{1}{2}(a^2 - b^2) \right] + \sin[2(\phi - \phi_s)] \cos \beta \sin 2\psi \left( a^2 - b^2 \right) \right\},
\]

where again we used \( b_0 \) to denote the impact parameter.

We have computed the STF hexadecapole of a tetrahedron to verify that all of its components are non-vanishing, making the results rather lengthy. Because of this, we are not being presenting the hexadecapole moment of the trirectangular tetrahedron here, but confirm that the relevant gravitational phase shift has an expected from with magnitude and phase depending not only on the size of the tetrahedron, but also its orientation with respect to observer.

Instead, we provide the results for the trirectangular tetrahedron \( a = b = c \). Clearly, one can see from (D5) that the quadrupole moment in this case vanishes. Using (31) we compute the STF octupole moment of a trirectangular
tetrahedron, yielding the following result:

\[
Q^{ijk} = \frac{Ma^3}{400} \begin{bmatrix}
2 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 \\
-1 & -1 & 2 & -1 \\
-1 & -1 & -1 & 2
\end{bmatrix}.
\]  

(D7)

Again, to generalize the results, we rotate the STF octupole tensor \([CS]\) to an arbitrary coordinate system:

\[
\mathcal{T}^{ijk} = R^i_a R^j_b R^k_c Q^{a b c},
\]  

(D8)

where \(R^a_b\) is the rotation matrix \([CS]\). After that, we substitute the result in \([27]\) and obtain the following expression for the gravitational phase shift introduced by the octupole of a trirectangular tetrahedron with \(a = b = c\):

\[
\xi_{b}^{[3]}(b) = \frac{k r g}{1920} \left(\frac{a}{b}\right)^3 \left\{ \cos[3(\phi_x - \phi_s)] \left[ 4 \left( \cos \psi - \sin \psi \right) \left( 2 + \sin 2 \beta_s \sin \psi + \cos \psi \left( \sin 2 \beta_s + \left( 5 + 3 \cos 2 \beta_s \right) \sin \psi \right) \right] + \right. \\
+ \sin[3(\phi_x - \phi_s)] \left[ 2 \sin^2 \beta_s \left( 4 + 3 \cos \beta_s \cos \psi + \sin \psi \right) + \left( 5 \sin \beta_s + \sin 3 \beta_s \right) \sin 2 \psi - \right. \\
\left. - \frac{1}{2} \left( 15 \cos \beta_s + \cos 3 \beta_s \right) \left( 3 \cos \psi - \sin 3 \psi \right) \right] \right\},
\]  

(D9)

where \(b\) is the impact parameter.

Finally, we compute the STF hexadecapole mass moment of a right circular cone using \([28]\), which yields the following result:

\[
Q^{ijkl} = \frac{Ma^4}{2240} \begin{bmatrix}
\frac{1}{10} & -\frac{3}{10} & -\frac{1}{10} & -\frac{3}{10} \\
-\frac{3}{10} & \frac{1}{10} & -\frac{3}{10} & \frac{1}{10} \\
-\frac{1}{10} & -\frac{3}{10} & \frac{1}{10} & -\frac{3}{10} \\
-\frac{3}{10} & \frac{1}{10} & -\frac{3}{10} & \frac{1}{10}
\end{bmatrix}.
\]  

(D10)

To generalize this expression, we rotate the STF hexadecapole tensor \([D10]\) to an arbitrary coordinate system:

\[
\mathcal{T}^{ijkl} = R^i_p R^j_q R^k_s R^l_w Q^{p q s w},
\]  

(D11)

where \(R^a_b\) is the rotation matrix \([CS]\). By substituting the result in \([28]\), we obtain the following expression for the gravitational phase shift introduced by the STF hexadecapole mass moment of a right circular cone:

\[
\xi_{b}^{[4]}(b) = \frac{k r g}{10240} \left(\frac{a}{b}\right)^4 \left\{ \cos[4(\phi_x - \phi_s)] \left[ \sin^4 \beta_s + 2 \sin^2 \beta_s \sin 2 \beta_s \left( \cos \psi + \sin \psi \right) + \right. \\
+ 2 \left( 3 + \cos^2 \beta_s \right) \sin 2 \beta_s \left( \cos 3 \psi - \sin 3 \psi \right) + \right. \\
+ 8 \left( 1 + \cos^2 \beta_s \right) \sin^2 \beta_s \sin 2 \psi + \frac{1}{2} \left( \sin^4 \beta_s + 8 \cos^2 \beta_s \cos 4 \psi \right) + \right. \\
+ 4 \sin[4(\phi_x - \phi_s)] \left[ \sin^3 \beta_s \sin \psi + \cos \psi \right) + \sin \beta_s \left( 1 + 3 \cos^2 \beta_s \right) \left( \cos 3 \psi + \sin 3 \psi \right) - \right. \\
\left. - 2 \sin \beta_s \sin 2 \beta_s \left( \cos 2 \psi + \frac{1}{2} \cos \beta_s \left( 1 + \cos^2 \beta_s \right) \sin 4 \psi \right) \right\},
\]  

(D12)

where \(b\), again, is the impact parameter.