HARD SCATTERING IN QCD WITH POLARIZED BEAMS

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ABSTRACT
I show that factorization for hard processes in QCD is also valid when the detected particles are polarized, and that the proof of the theorem determines the operator form for the parton densities. Particular attention is given to the case of transversely polarized incoming hadrons.

1. INTRODUCTION

Perturbative QCD is by now a staple ingredient of most phenomenology of high energy collisions. Mostly the applications have been to the case that the incoming particles are unpolarized. However, in recent years much more attention has been devoted to the polarized case. Unfortunately, there has been confusion as to the exact status of the factorization theorem in the polarized case. This is the theorem that underlies almost all perturbative calculations. The confusion has particularly extended to the question of the correct definition of the parton distribution functions [1, 2, 3].

In actuality, when one examines the proofs of the factorization theorem [4, 5], one finds that almost no reference is made to the polarization of the measured particles. So the purpose of this paper is to explain that the factorization theorem does indeed apply, and that it determines, essentially uniquely, the definition of the parton distributions. (The ambiguity in the definition is directly tied to the well-known freedom to choose the renormalization

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and factorization scales, and is entirely independent of the problems of polarization.) There has been particular confusion about the case about the case that the incoming beams have transverse polarization. For example of the supposed problems, see the discussion in the book by Ioffe et al, [3]. Better and more recent discussions of some of the issues can be found in [6, 7, 8, 9, 10, 11].

The processes with which we are concerned are those with a hard scattering: deeply inelastic lepton scattering, the Drell-Yan process, jet production in hadron collisions, etc. For the sake of definiteness, I will treat in this paper two specific cases: the structure functions for deeply inelastic lepton scattering, and the Drell-Yan process.

All the pieces of the discussion can be found in the literature. What has been lacking is a unified presentation. In this paper, I shall be concerned solely with the ‘twist-2’ behavior of hard scattering cross sections. This means that at each order of perturbation theory I consider the terms that scale with energy like their dimension (modified by logarithms). ‘Higher twist’ terms—power suppressed terms—will be ignored; they are very interesting, but their study is much harder than that of the twist-2 terms.

The terminology of twist has its origins in the operator product expansion for deeply inelastic lepton scattering, and is somewhat inappropriate here. But the usage has stuck.

2. STATEMENT OF FACTORIZATION THEOREM

In this section, I will formulate the factorization theorems for deeply inelastic scattering and for the Drell-Yan cross section. These are typical of the most general case of factorization.

2.1 Structure Functions for Deeply Inelastic Scattering

The structure functions for deeply inelastic scattering are defined in terms of the structure tensor $W_{\mu\nu}(p, q)$ by:

$$W_{\mu\nu} = (\omega_{\mu\nu} + q_\mu q_\nu / q^2) W_1(x, Q^2) + (p_\mu - q_\mu p \cdot q / q^2) (p_\nu - q_\nu p \cdot q / q^2) \frac{W_2(x, Q^2)}{M^2}$$

$$+ \frac{i}{M} q_\mu q_\nu \epsilon^{\rho\sigma} \left[ s^\sigma \left( G_1 + \frac{p \cdot q}{M^2} G_2 \right) - s \cdot q p^\sigma \frac{1}{M^2} G_2 \right].$$

(1)

Here, $M$, $p^\mu$ and $s^\mu$ are the mass, momentum and spin vector of the target, and $q^\mu$ is the momentum of the exchanged virtual photon. (The case of a more general exchanged boson,
like a W or a Z gives more structure functions, a situation that is an inessential complication for the present purpose.) As usual, we define $Q^2 = -q^2$, $\nu = p \cdot q$, and $x = Q^2 / 2p \cdot q$. We will be interested in the Bjorken limit: $Q \to \infty$ with $x$ fixed. Our normalization of the spin vector is such that a pure state has $s^2 = -1$.

As for the spin-dependent structure functions, we will consider in this paper only the case that the target has spin $\frac{1}{2}$. Then target’s spin state (no matter whether mixed or pure) is completely determined by its spin vector $s^\mu$, and there are exactly two polarized structure functions, $G_1$ and $G_2$. (For the case of a target of general spin, see [12].) The normalization of $s^\mu$ is that it satisfies $0 \geq s^2 \geq -1$ and $s \cdot p = 0$, and that a pure spin state has $s^2 = -1$.

Later, when we do power counting to determine the sizes of the leading contributions to the structure functions, it will be convenient to work in the center-of-mass frame. Then in the Bjorken limit, we find that components of $p^\mu$ and of $q^\mu$ are of order $Q$. But also the components of $s^\mu$ become large. So, following Ralston and Soper [6], we decompose $s^\mu$ in terms of a helicity $\lambda$ and a transversity $s_\perp$:

$$s^\mu = \lambda \left( \frac{p^\mu M}{M} - \frac{q^\mu M}{p \cdot q} \right) \frac{1}{\sqrt{1 + 2xM^2 / p \cdot q}} + s_\perp^\mu, \quad (2)$$

where the helicity $\lambda$ is defined by

$$\lambda = \frac{q \cdot s M}{q \cdot p} \frac{1}{\sqrt{1 + 2xM^2 / p \cdot q}}, \quad (3)$$

and $s_\perp^\mu$ is orthogonal to both $p^\mu$ and $q^\mu$. Then $-s^\mu s_\mu = \lambda^2 + |s_\perp|^2$, while both $|\lambda|$ and $|s_\perp|$ are less than unity. For a pure state, $\lambda^2 + |s_\perp|^2 = 1$.

It is common to call $s_\perp$ the transverse spin. However, moving particles are not in an eigenstate of transverse spin, as Jaffe and Ji [9] explained, but may be in an eigenstate of the transverse components of the Pauli-Lubanski vector; these transverse components are called transversity. Note that our spin vector $s^\mu$ is proportional to the Pauli-Lubanski vector.) Polarized particles in a high energy accelerator are typically transversely polarized and carry a definite value of the spin vector. Thus they are in a state of definite transversity. Note that the concept of ‘transverse’ in this context is not Lorentz invariant, when referred to a single
particle. It is only defined when one brings another vector into the situation to represent (say) the center-of-mass of the scattering.

At various stages, we will need to take the limit of zero mass, and the decomposition (2) exhibits potential singularities at \( M = 0 \). In this limit, we may approximate eq. (2) by

\[
s^\mu = \frac{\lambda P^\mu}{M} + s_\perp^\mu + \text{power law correction},
\]

The factorization properties are most easily expressed in terms of scaling structure functions which are defined by rewriting eq. (1) as

\[
W_{\mu\nu} = (-g_{\mu\nu} + q_\mu q_\nu/q^2)F_1(x, Q^2) + \frac{(p_\mu - q_\mu p \cdot q/q^2)(p_\nu - q_\nu p \cdot q/q^2)}{p \cdot q} F_2(x, Q^2)
\]

+ \frac{iM}{p \cdot q} \epsilon_{\mu\nu\rho\sigma} q^\rho s^\sigma g_1 + \frac{iM}{(p \cdot q)^2} \epsilon_{\mu\nu\rho\sigma} q^\rho (p \cdot q s^\sigma - s \cdot q p^\sigma) g_2.
\]

Here, \( F_1 \equiv W_1, F_2 \equiv p \cdot q W_2/M^2, g_1 \equiv p \cdot q G_1/M^2 \) and \( g_2 \equiv p \cdot q^2 G_2/M^4 \).

In the Bjorken limit, it is convenient to use the decomposition eq. (2) of the spin, so that the spin-dependent part of \( W_{\mu\nu} \) is:

\[
W_{\mu\nu}^{\text{pol}} = \lambda \frac{i\epsilon_{\mu\nu\rho\sigma} q^\rho p^\sigma}{p \cdot q} \frac{1}{\sqrt{1 + 2xM^2/p \cdot q}} \left( g_1 - \frac{2xM^2}{p \cdot q} g_2 \right)
\]

+ \frac{i\epsilon_{\mu\nu\rho\sigma} q^\rho s_\perp^\sigma M}{p \cdot q} g_2
\]

= \lambda \frac{i\epsilon_{\mu\nu\rho\sigma} q^\rho p^\sigma}{p \cdot q} g_1 + \lambda \frac{i\epsilon_{\mu\nu\rho\sigma} q^\rho s_\perp^\sigma M}{p \cdot q} g_2 + \text{non-leading powers}.
\]

We will see that in the Bjorken limit, \( Q \to \infty \) with \( x \) fixed, each of \( F_1, F_2, g_1, \) and \( g_2 \) scales like \( Q^0 \) times logarithms. Each of these scaling structure functions is dimensionless and, except for \( g_2 \), its definition in terms of the tensor does not involve the mass of the target. The exception for \( g_2 \) is a choice that directly reflects the fact that its leading contribution is associated with operators of twist-3 rather than twist-2: redefining it to remove the factor of \( M \) would reduce its power law by one power of \( Q \).

2.2 Factorization for Unpolarized Deeply Inelastic Scattering

The factorization theorem [4] for deeply inelastic lepton scattering applies in the Bjorken
limit, and for the unpolarized structure functions it gives:

\[
F_1(x, Q^2) = \sum_a \int_x^1 \frac{d\xi}{\xi} f_{a/A}(\xi, \mu) H_{1a} \left( \frac{\xi}{x} \frac{Q}{\mu}, \alpha_s(\mu) \right) + \text{remainder},
\]

\[
\frac{1}{x} F_2(x, Q^2) = \sum_a \int_x^1 \frac{d\xi}{\xi} f_{a/A}(\xi, \mu) \xi H_{2a} \left( \frac{x}{\xi} \frac{Q}{\mu}, \alpha_s(\mu) \right) + \text{remainder}.
\]  

This theorem asserts that in the Bjorken limit, the target may be regarded as a beam of partons, and that the scattering really takes place on these partons. The remainder terms are a power of \( Q \) smaller than the leading terms.

The quantities \( f_{a/A}(\xi) \) are parton distribution functions (or parton densities). Their operator definition, which will be given below, can be interpreted in light front quantization as the number density of partons as a function of the light-cone fraction of the momentum of the parent hadron.

The functions \( H_{1a} \) and \( H_{2a} \) should be regarded as the short-distance part of the structure functions for a parton target of flavor \( a \) (gluon, quark or antiquark). In lowest order in \( \alpha_s \), each \( H_{ia} \) is a delta function at \( \xi = x \) times the charge squared \( e_a^2 \) of the parton:

\[
H_{1a}(x/\xi) = \frac{1}{2} e_a^2 \delta(x/\xi - 1) + O(\alpha_s),
\]

\[
H_{2a}(x/\xi) = e_a^2 \delta(x/\xi - 1) + O(\alpha_s).
\]  

The first term in these perturbation expansions gives the parton model approximation to QCD, with \( F_1 = \frac{1}{2} \sum_a e_a^2 f_{a/p}(x) \) and \( F_2 = \sum_a e_a^2 x f_{a/p}(x) \). In higher order, the \( H_{ia} \) are the structure functions at the parton level, but with subtractions made according to a standard prescription, to remove the non-ultraviolet contributions.

The appearance of the Dirac delta function in eq. (8) (and of more complication generalized functions in higher orders) implies that the asymptotic behavior given by eq. (7) is to be interpreted in the sense of distribution theory– cf. [13].

The factors of \( 1/x \) and of \( \xi/x \) in the equation for \( F_2 \) arise from the dependence on the target momentum of the definition of the structure function \( F_2 \).

2.3 Structure Functions v. Parton Distributions

I make a clear distinction between the concepts of a ‘structure function’ and a ‘parton distribution function’. A structure function is a term in a decomposition of the deep inelastic
cross section, as in eq. (5); it is experimentally measurable. A parton distribution function (or parton density) is a number density of quarks (or gluons) in a fractional momentum variable. Mathematically it is a hadron expectation value of a certain operator; as such, it is a theoretical construct.

However, the parton model gives the structure functions in terms of certain simple linear combinations of quark and antiquark densities. [For our purposes, the parton model is the (useful) approximation in which one neglects the perturbative corrections to the hard scattering coefficients in the factorization formulae.] It has therefore become common in the literature to identify the concepts of structure function and parton distribution.

This identification has particularly disastrous consequences for the discussion of polarized scattering with transversely polarized hadrons: The transverse spin contribution to deep inelastic scattering at the level of approximation of the leading term in eq. (7) is exactly zero (to all orders of perturbation theory), as we will review below. Attempts to identify the structure function $g_2$ with some kind of transverse spin distribution result in inconsistencies.

2.4 Factorization for Drell-Yan

The factorization theorem for the Drell-Yan process is typical of factorization theorems for more general hard scattering processes, and it is formulated as follows.

The process is the inclusive production of a lepton pair of high invariant mass via an electroweak particle. The classical case is with a high-mass virtual photon: $H + H \rightarrow \gamma^* + \text{anything}$, with $\gamma^* \rightarrow e^+e^-$ or $\gamma^* \rightarrow \mu^+\mu^-$. The cases of $W$ and $Z$ production can be treated in an essentially identical fashion.

We let $s$ be the square of the total center-of-mass energy and $q^\mu$ be the momentum of the $\gamma^*$. The kinematic region to which the theorem applies is where $\sqrt{s}$ and $Q$ get large in a fixed ratio. ($Q$ is $\sqrt{q^2}$.) The transverse momentum $q_\perp$ of the $\gamma^*$ is either of order $Q$ or is integrated over.

In the case that $q_\perp$ is integrated over, the factorization theorem for the unpolarized
Drell-Yan cross section reads:

\[
\frac{d\sigma}{dQ^2 dy d\Omega} = \sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \, f_{a/A}(\xi_A, \mu) \, H_{ab}\left(\frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}, \theta, \phi, Q, \frac{\mu}{Q}, \alpha_s(\mu)\right) \, f_{b/B}(\xi_B, \mu) + \text{remainder},
\]

(9)

where \( y \) is the rapidity of the virtual photon and \( d\Omega \) is the element of solid angle for the lepton pair: the polar angles for this decay are \( \theta \) and \( \phi \) relative to some chosen axes. The sums over \( a \) and \( b \) are over parton species, and we write

\[
x_A = e^y \sqrt{\frac{Q^2}{s}}, \quad x_B = e^{-y} \sqrt{\frac{Q^2}{s}}.
\]

(10)

The function \( H_{ab} \) is the ultraviolet-dominated hard scattering cross section, computable in perturbation theory. It plays the role of a parton level cross section and is often written as

\[
H_{ab} = \frac{d\hat{\sigma}}{dQ^2 dy d\Omega},
\]

(11)

where the hat over the \( \sigma \) indicates a hard scattering cross section at the parton level. The parton distribution functions, \( f \), are the same as in deeply inelastic scattering. Fig. 1 illustrates the factorization theorem.

Fig. 1. Factorization theorem for Drell-Yan cross section.
2.5 Twist

In both of the above factorization theorems, the dependence of the hard scattering coefficients ($H_{1a}$ etc) on the large momentum $Q$ is of the form $Q^p$ times logarithms of $Q$, where $p$ is the dimension of the hard scattering coefficient. This is true for each separate order of perturbation theory in $\alpha_s$. For the coefficient $H_{ab}$ for Drell-Yan we have $p = -4$, and for $H_1$ and $H_2$ for deeply inelastic scattering we have $p = 0$. The quantities multiplying the hard scattering coefficients are dimensionless parton densities. Immediate consequences are the standard scaling laws that $F_1$ and $F_2$ behave like $Q^0$ and that the Drell-Yan cross section $d\sigma/dQ^2dy$ behaves like $Q^{-4}$ in the scaling limit, apart from the usual logarithmic scaling violations.

Terms of this kind, we will label ‘twist-2’ as a generalization of the usage in the operator-product expansion for deep inelastic scattering. (Twist is the dimension minus the spin of the operators.) The remainder terms in the factorization theorems are therefore called higher twist.

When we consider scattering with transversely polarized hadrons, there are some processes for which the twist-2 term is exactly zero. A notorious example is the single transverse spin asymmetry of high $p_\perp$ particle production in hadron-hadron scattering. Another case is the structure function $g_2$ for deeply inelastic scattering. For these processes the leading twist is twist-3, and the corresponding asymmetries are proportional to some hadronic mass scale divided by $Q$ at large $Q$. The choice of the dimensional factor multiplying $g_2$ in eq. (5) to be $M/p \cdot q^2$ rather than $1/p \cdot q^{3/2}$ is an expression of this fact.

2.6 Longitudinal Polarization

The factorization theorems stated above are for unpolarized incoming hadrons, and they involve an incoherent sum over parton types. In the case that the incoming hadrons are polarized, the theorems need generalization.

For the case of longitudinal polarization, the factorization statements can be readily formulated simply by simply extending the sum over parton types $a$ (and $b$) to include a sum over parton helicities. The unpolarized parton densities will be a sum over the helicity
densities. (The asserted theorem is still in need of the proof which will be summarized later.)

For Drell-Yan, the hard-scattering coefficient in eq. (9) should be treated as a helicity-dependent cross section at the parton level.

For deeply inelastic scattering, the formulae for $F_1$ and $F_2$ will remain unchanged, since these structure functions and the corresponding parton level structure functions are, by definition, spin independent. But there will be a factorization formula for the helicity dependent structure function $g_1$, and this will involve the helicity asymmetry of the parton densities. Extra polarized structure functions beyond $g_1$ will be needed for the case of a target of spin greater than $\frac{1}{2}$. [12]

2.7 General Polarization, Including Transverse

Now, a characteristic of the quantum mechanical theory of spin is that interference and coherent phenomena occur even in circumstances where the physics is otherwise classical. Such is the case for the factorization of hard processes when the detected hadrons have a general polarization. The problem is one of interference between scattering of partons of different quantum numbers.

For the flavor quantum numbers of partons there is no such interference. For example, we have no contribution to $F_1$ from an interference between scattering on an up quark and a down quark:

$$\langle u + \gamma^* | \text{final state} \rangle \langle \text{final state} | d + \gamma^* \rangle.$$ (12)

The reason is that an examination of the flavor of the final state of the hard scattering is sufficient to determine which kind of parton initiated the hard scattering. Alternatively, one can examine the term in an operator definition of a parton distribution that would be appropriate to an interference term:

$$\langle p | \bar{u} \cdots d | p \rangle.$$ (13)

Quark number conservation forces this to be zero. The dots indicate factors that are irrelevant to the flavor structure.

But for spin, there are no such constraints: The final states that can be produced from the scattering of left-handed quarks can be the same as the final states from right-handed
quarks, and therefore the amplitudes can interfere. To take account of this, we must equip the partons entering the hard scattering with a spin density matrix. Therefore the full specification of a parton distribution is given by the number density of partons together with the parton’s density matrix. The number density times the density matrix has linear dependence on the spin density matrix of the initial hadron. Longitudinal polarization gives the special case that the density matrices of both the parton and the hadron are diagonal in a helicity basis. (Note that the use of a density matrix allows the state of a deeply inelastic scattering parton to be either pure or mixed: In an inclusive cross section, where we sum over unobserved parts of the final state, a parton that enters the hard scattering can be in a mixed state even when its parent hadron is in a pure state.)

The most general form for the factorization theorem for Drell-Yan is

\[
\frac{d\sigma}{dQ^2 dy d\Omega} = \sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \rho_{a(a'\bar{\imath})/A} f_{a/A}(\xi_A, \mu) H_{a(a'\bar{\imath})b(b'\bar{\im\imath})/B} f_{b/B}(\xi_B, \mu) \rho_{b(b'\bar{\im\imath})/B} + \text{remainder.} \tag{14}
\]

Here \(\rho_{i(aa')/H}\) is the density matrix of partons of flavor \(i\) in hadron \(H\), with \(\alpha\) and \(\alpha'\) being the helicity indices of the matrix. The density matrix is of course a function of the same variables \(\xi\) and \(\mu\) as the number density \(f_{i/H}(\xi, \mu)\). The factorization (14) differs from the unpolarized case by the presence of spin density matrices \(\rho\) for the partons and by the dependence of the hard scattering coefficient on the spin indices, \(\alpha, \alpha', \beta, \text{ and } \beta'\). The density matrix of a parton is necessarily a linear function of the spin vector of its parent hadron.

Calculations of amplitudes in perturbation theory are often made in a helicity basis [14]. In that case, it is convenient to work directly with density matrices in a helicity basis. The indices \(\alpha, \alpha'\) etc will take on the values + and − (or \(L\) and \(R\) for left- and right-handed polarization).

Another method is to work with cut Feynman graphs, for the cross section. In that case, one uses a polarization sum for initial state quarks written in terms of the quark’s spin
(Our conventions here are those of Bjorken and Drell [15].) The density matrix of the quark is then
\[
\frac{1}{2} \hat{\rho} \left( 1 - \lambda \gamma_5 + \gamma_5 \hat{\rho}_\perp \right),
\]
(15)

where we have chosen the z-axis to be along the 3-momentum of \( p^\mu \). For the case of a spin-\( \frac{1}{2} \) hadron, the helicity \( \lambda_a \) of a quark is proportional to the helicity \( \lambda_A \) of its parent hadron, and similarly for the transversity:
\[
\lambda_a = \Delta_{L,a/A} \lambda_A,
\]
\[
\sigma_{\perp a} = \Delta_{T,a/A}s_{\perp A}.
\]
(17)
(18)

Here \( \Delta_L \) and \( \Delta_T \) are the longitudinal and transverse spin asymmetries of a quark in a fully polarized hadron. They are functions of the variables \( \xi \) and \( \mu \), of course, and these asymmetries give the quark densities defined by Jaffe and Ji [9] by weighting by the unpolarized densities:
\[
f_{La/A} = \Delta_{La/A} f_{a/A},
\]
\[
f_{Ta/A} = \Delta_{Ta/A} f_{a/A},
\]
(19)
(20)

Jaffe and Ji use the notation \( g_{La/A} \) instead of \( f_{La/A} \), but this identification invites confusion with the \( g_1 \) structure function, to which it is directly related in the parton model. I have also changed Jaffe and Ji’s notation \( h \) for the transversity distribution to \( f_T \) to correspond with \( f_L \).

We will give the operator definitions of the parton densities below, exactly as stated by Jaffe and Ji, and we will justify that these are the correct definitions to use in the factorization theorems. These densities satisfy linear evolution equations (Gribov-Lipatov-Altarelli-Parisi) of a similar structure to the ones in the unpolarized case.

Exactly corresponding considerations apply to gluons. They also have two physical polarization states. (As we will review below, it is the states of a massless parton that are
relevant for the factorization theorem.) However, the gluon has spin one, so that the offdiagonal terms in its density matrix correspond to linear rather than transverse polarization. Moreover, an operator measuring these offdiagonal elements has helicity two, unlike the operator for quarks, which has helicity one, as can be seen from the operator definitions below. Thus it is a consequence of angular momentum conservation, as explained by Artru and Mekhfi [7], that there are no linearly polarized gluon partons in a spin-\( \frac{1}{2} \) hadron. Furthermore, the evolution equation for transversely polarized quarks has no mixing with gluons, and vice versa.

This result depends on the azimuthal symmetry of the operators measuring the parton densities. One way of evading the result is to use a process sensitive to the intrinsic transverse momentum of the partons [16].

2.8 Consequences of Chiral Symmetry

In QCD there are chiral symmetries that if unbroken would actually prohibit some of the interference terms that would otherwise occur with transverse polarization. At the level of the hard scattering, as in (12), one is working in perturbation theory with quark masses neglected. Thus the chiral symmetries are exact for the hard scattering coefficients at the twist 2 level.

An important manifestation of this is conservation of quark helicity in massless perturbation theory. As we will see below, a particular consequence of this is that the twist 2 contribution to the structure function \( g_2 \) is exactly zero. That is, the transverse spin asymmetry in ordinary deeply inelastic scattering is of the order of a hadron mass divided by \( Q \) for large \( Q \).

But one can easily conceive of hard scattering with two partons in the initial state, and then there is no reason for the interference terms to vanish [6].

At first glance, the same reasoning might appear to apply to the parton densities, as defined by (13). But there we are dealing with nonperturbative quantities. Hence the offdiagonal terms in the density matrix are explicitly allowed for two reasons. First the chiral symmetry is definitely broken in the nonperturbative part of QCD. Secondly, in the helicity
basis, which is natural to use, the density matrix for the hadron is itself offdiagonal, so they can be no constraint prohibiting offdiagonal terms in the quark density matrix.

3. SUMMARY OF PROOF

The proof of a factorization theorem is made by considering a cross section as the sum over all cut Feynman graphs for the process in question. Ideally, one would like to extend the proof to handle nonperturbative contributions. But that extension has yet not been made.

The formulation of the theorem is in fact general enough to allow such an extension, and the physical picture it gives is completely reasonable. In particular, the definitions of the parts of the factorization formulae that are to be used nonperturbatively, viz, the parton densities, are valid beyond perturbation theory: The parton densities have a gauge invariant definition in terms of hadron matrix elements of certain operators.

Furthermore, in the whole of our discussion, there is no restriction on which kinds of particle compose the initial state, except that they should be gauge invariant and physical. In perturbative calculations we will typically use on-shell, physically polarized gluon and quark states, but in principal we could also use hadron states with a bound-state wave function. For the general theory, it will make no difference. The hard scattering functions are the only quantities for which a purely perturbative calculation makes sense, and for them the initial states are always on shell, massless partons.

The steps in the proof [4] are:

1. Power counting. Apply the method of Libby and Sterman [17] to determine those regions of integration momenta of the cut graphs that give leading contributions.

2. Cancellation of superleading regions. There are contributions in which all the partons coupling to the hard scattering are gluons with an unphysical scalar polarization. Unfortunately, these give a power law larger than the final result. Ward identity methods are used to show that these contributions exactly cancel among themselves.

3. All remaining contributions have a form like that of the parton model, with two generalizations. First, the hard scattering is not restricted to be in the Born approximation.
Secondly, there may be soft gluons connecting the lines associated with initial hadrons among themselves and with initial and final state lines in the hard scattering, and there may be extra collinear gluons with scalar polarization connecting to the hard scattering.

4. Cancellation of final-state interactions. All interactions that are too late to affect the inclusive cross section must cancel. For example, hadronization of final-state jets does not affect the totally inclusive structure functions. Hence the partons initiating these jets effectively have virtuality of order $Q^2$.

5. Taylor expansion. The hard scattering is expanded in powers of the small components of its external momenta, and in the mass parameters of its internal lines. The subgraphs of collinear lines (‘jet subgraphs’) are expanded in powers of the relatively small components of the soft lines that connect them to other jet subgraphs.

6. Cancellation of soft gluons. A Ward identity argument is used to factorize the soft gluons, after which a unitarity cancellation applies.

7. Factorization of collinear scalar gluons. This again goes by a Ward identity argument.

8. At this point we have jet factors and hard scattering factors. One now applies combinatoric arguments in the same way as in Wilson’s expansion to get the factorization theorem [18, 19, 20].

9. The hard scattering can now be identified as a cross section for scattering of on-shell partons with subtractions to remove the non-ultraviolet contributions.

10. Operator definition of parton densities. The jet factors can now be shown to be exactly hadronic expectation values of certain operators. The precise form of these operators is completely determined by the Taylor expansion at step 5. The operators are bilocal operators that have simple interpretations in light front quantization. The terms that were obtained in step 7 from the collinear scalar gluons turn these into gauge invariant operators.

3.1 Use of Physical Gauge

It is possible to use a physical gauge in trying to prove factorization. Several of the unpleasant steps involving gluons with scalar polarization can then be omitted, and the result appears
to be a much simpler proof.

However, there are unphysical singularities that prevent the unitarity arguments in step 4 from being applied in as strong a manner as is needed. Furthermore, the Ward identity arguments to cancel soft gluons rely on contour deformations that are obstructed by these same singularities. Thus it seems best to work in an ordinary covariant gauge and to accept the added complications [4, 5].

4. POWER COUNTING

Libby and Sterman [17] showed that to classify the important regions of momentum space in a high energy limit it is useful to measure momenta and masses in units of the large momentum scale $Q$. Thus one writes a generic momentum and mass in the form

$$k^\mu = Q\tilde{k}^\mu, \quad m = Q\tilde{m}.$$  \hspace{1cm} (21)

By simple dimensional analysis, the large $Q$ limit is equivalent to the limit of zero mass and on-shell external momenta; for example a cross section might be written

$$\sigma(Q^2; m, k) = Q^{-2}\sigma(1; \tilde{m}, \tilde{k}),$$  \hspace{1cm} (22)

with $\tilde{m} \to 0$ and $\tilde{k}^2 \to 0$ when $Q \to \infty$. Thus the complications of a high-energy limit can be investigated by examining the singularities in zero-mass limit. The method of Coleman and Norton [21] shows in a physically appealing fashion how to determine the configuration of loop momenta that give these singularities. Tkachov and collaborators [13, 19] have shown how systematic exploitation of this idea can considerably streamline proofs of the operator product expansion and of other results on the asymptotics of Euclidean Green functions.

In our case, the significant configurations involve internal lines of three kinds: (a) Lines that carry momenta collinear to momenta of external particles. (b) Lines that carry soft momenta. (c) Lines that carry large ultraviolet momenta. (If only large momentum lines are important, then the cross section under discussion is infra-red safe, and the problem may be treated by classical renormalization-group methods.)
Then the importance of each configuration is determined by expanding in powers of a suitable small variable \( \lambda \) about the singular point in the massless limit.

Let us now examine these arguments as applied to deeply inelastic lepton scattering. We will present them in a sufficiently general manner, that the extension to other processes will be simple.

4.1 Deeply Inelastic Scattering at Twist-2 Level

To explain the power counting, it will be convenient to use light-front coordinates in which

\[
p^+ = Q/x\sqrt{2}, \quad p^- = M^2/Q\sqrt{2}, \quad p_\perp = 0, \tag{23}
\]

and

\[
-q^+ = q^- = Q/\sqrt{2}, \quad q_\perp = 0. \tag{24}
\]

(Our metric is such that \( V^2 = 2V^+V^- - V_\perp^2 \) for any vector \( V^\mu \).)

Let us also assume that we sum over cuts of graphs for the structure tensor before applying the Libby-Sterman argument to deeply inelastic scattering. The sum over cuts means that no final-state interactions need enter our argument.

At the leading power of \( Q \), contributions to the structure tensor come from regions symbolized in fig. 2. In the upper part of this diagram, which we will call the hard subgraph \( H \), all the internal lines have large momenta, that is the scaled momenta have virtuality of order unity. In the lower part, which we will call the collinear or jet subgraph, \( J \), all the lines have momenta collinear with \( p^\mu \). That is, the corresponding scaled momenta are close to a light-like vector with only a nonzero + component. All but two of the lines joining the subgraphs are gluons with scalar polarization.

It is a relatively simple generalization [17] of the arguments below that shows that graphs with extra quarks and/or transversely polarized gluons joining the two subgraphs are suppressed by a power \( Q \) for each extra line.

4.2 Simple Quark Connection

First we consider the case that there is just a single quark line connecting the subgraphs \( H \) and \( J \) on each side of the final-state cut. As we now show, it is fairly easy to massage the
Fig. 2. Regions for twist-2 contributions to deep inelastic scattering.

contribution of fig. 2 into a parton-model-like form that implies the scaling properties of the structure functions that were stated earlier.

We must first find the leading part of the trace over Dirac matrices. For this purpose, we decompose the top part — the hard subgraph $H$ — and the bottom part — the jet subgraph $J$ — according to the Dirac structure on the fermion line connecting the two subgraphs:

$$J = J_S + J_V + J_T + J_P,$$

$$H = H_S + H_V + H_T + H_P.$$

(25)

In the last line we have suppressed the indices $\mu \nu$ of the structure tensor. When we perform the trace over the Dirac matrices and the integral over the explicit loop momentum $k^\mu$, we find that:

$$W = \int \frac{d^4k}{(2\pi)^4} \, \text{tr}(JH)$$

$$= \int \frac{d^4k}{(2\pi)^4} \, 4 \left( J_S H_S + J_V \cdot H_V - 2 J_T \cdot H_T - J_P \cdot H_P + J_P S H_P S \right).$$

(26)

We suppose that we are looking only at the region of momentum appropriate to fig. 2, so that in particular $k^+ = O(Q)$, and $|k^-|, |k_\perp| \ll Q$. The jet part $J$ depends on the momenta $p^\mu$ and $k^\mu$, and on the hadron’s spin state defined by $\lambda$ and $s_\perp$. In the rest frame of the hard scattering, all the vectors involved in $H$ have components of order $Q$, so that we may regard all components of the decomposition of $H$ as being of order $Q^D$, where $D$ is the mass dimension of $H$. 

17
As we increase $Q$ while keeping the longitudinal momentum fraction, virtuality and transverse components of $k^\mu$ fixed, we may consider $J$ as being obtained by a boost from the rest frame of $p^\mu$. Then the terms in the decomposition of $J$ scale with $Q$ as follows:

$$
\begin{align*}
J^+_V, J^+_A, J^+_T &\propto Q^1, \\
J^i_V, J^i_A, J^+_T, J^ij_T, J_S, J_{PS} &\propto Q^0, \\
J^-_V, J^-_A, J^-T_\mu &\propto Q^{-1},
\end{align*}
$$

(27)

the proof following from the effect of boost transformations. The indices $i$ and $j$ refer to purely transverse components.

It follows that the leading terms in the trace in eq. (26) are given by

$$
W = \int d^4k \frac{1}{(2\pi)^4} 4 \left( J^V_H V^V_H - 2 J^T_i H^{T-i} - J^{PV} H^{PV} \right) 
\left(1 + O(\text{mass}/Q \times \text{logarithms})\right).
$$

(28)

4.3 Relation to Quark Distribution

In fig. 2 we aim to associate the subgraph $H$ with a contribution to the hard scattering coefficient and the subgraph $J$ with contribution to a parton distribution. Since all components of momentum inside $H$ are of order $Q$, we may, within $H$, neglect the transverse momentum and virtuality of $k^\mu$ and write

$$
W = \int d\xi \, \text{tr} \, H(q, \xi) \left[ \frac{d^4 k^-}{(2\pi)^4} J(k, p, s) \right] + \text{nonleading power},
$$

(29)

where $\xi \equiv k^+ / p^+$, while $H$ has been approximated by something with an incoming onshell quark that has zero transverse momentum. We also set the quark masses in $H$ to zero.

It remains to discuss the polarization structure.

In the frame we have chosen, it is manifest that the leading power for $\text{tr} J H$ comes from the terms in eq. (28). To relate this to a standard spin projection for Dirac particles, recall the conventional projection for a spinor wave function:

$$(\slashed{p} + m) \left(1 + \gamma_5 \slashed{s}\right).$$

(30)
This is singular in the zero mass limit. After application of the decomposition (2) in terms of helicity and transverse spin, an expression is obtained that has a well-behaved zero-mass limit:

\[ \gamma (1 + \gamma_5 \gamma_\perp - \lambda \gamma_5). \]  

(31)

After some reorganization of the \( \gamma \) matrices implicit in eq. (28), we get a contribution of the form

\[ W_{\mu \nu} = \int_x \frac{d\xi}{\xi} \frac{1}{2} \text{tr} \mathcal{H}_{\mu \nu} \hat{k}[1 + \gamma_5(\lambda_q + \gamma_\perp)]f(\xi) + \text{twist higher than 2}. \]  

(32)

where \( \mathcal{H} \) means the massless on-shell limit of \( H \), and \( \hat{k}^\mu \equiv (\xi p^+, 0, 0_\perp) \). We have restored the \( \mu \nu \) indices that correspond to the external photon.

The quantity \( f(\xi) \) in this equation represents the contribution of the lower part of fig. 2 to the quark density. We need to sum over all possible graphs and all possible regions, and to apply the same combinatoric arguments as for the operator product expansion. Then we should expect to get the following definition of the quark density:

\[ f(\xi) = \int \frac{dk^- d^2k_\perp}{(2\pi)^4} \text{tr} J(k, p, s) \gamma^+ \]  

(33)

while the quark helicity, \( \lambda_q \), and quark transversity, \( s_q \perp \), are defined by

\[ \lambda_q f(\xi) = \int \frac{dk^- d^2k_\perp}{(2\pi)^4} \text{tr} J(k, p, s) \gamma_5 \gamma^+ \]  

(34)

and

\[ s_q \perp f(\xi) = \int \frac{dk^- d^2k_\perp}{(2\pi)^4} \text{tr} J(k, p, s) \gamma^+ \gamma_5 \gamma_\perp. \]  

(35)

The reasoning given above for these to be appropriate definitions can be found in [6], where references to earlier work on unpolarized parton densities in light front quantization can be found.

Equation (32) is clearly of the form of the desired factorization theorem. Moreover, the trace with \( \mathcal{H} \) in eq. (26) is such that the result has the normalization of the structure tensor for deep inelastic scattering off a quark target with momentum \( \xi p \), helicity \( \lambda_q \) and transverse spin \( s_q \perp \). Furthermore \( f_i(\xi) \) must be interpreted as the number density of partons. (The
factor $1/\xi$ in eq. (32) is needed to interpret $f(\xi)$ as a number density because of the relativistic normalization of the states.) There will be technicalities to generalize these results to real QCD, but the power counting arguments will remain unchanged.

4.4 Transverse Polarization

A twist-2 contribution to the structure functions $(M/\sqrt{p \cdot q})g_2$ and $g_1$ is one that is of order $Q^0$ (modulo the usual logarithms), since they are dimensionless: all the factors in the tensors multiplying them in eq. (7) are dimensionless ratios of momenta that are of order $Q$. The power counting argument just presented shows that our basic expectation is that $(M/\sqrt{p \cdot q})g_2$ scales like $Q^0$.

(In doing the power counting, we consider in the first instance, the structure tensor $W_{\mu\nu}$, which is dimensionless. Then we derive results for the structure functions by considering the possible tensors in eq. (5), but treating the tensors in combinations that scale as $Q^0$ in the Bjorken limit. The coefficients of these tensors then also scale as $Q^0$ (times logarithms). The unpolarized structure functions $F_1$ and $F_2$ are such coefficients. Since the spin vector for a longitudinally polarized proton can be taken as $\lambda p^\mu/M$, the coefficient of $g_1$ is $i\lambda\epsilon_{\mu\nu\rho\sigma} q^\rho s^\sigma/p \cdot q$, so that $g_1$ scales as $Q^0$. But the part of the spin vector that goes with $g_2$ is the transverse part $s_{\perp}^\mu$, which is invariant when one goes to the Bjorken limit; thus it is the combination $(M/\sqrt{p \cdot q})g_2$ that must be considered in our scaling argument.)

Now, when one actually performs the calculation of Feynman graphs to the leading power of the hard subgraphs $H$, one gets zero for the part corresponding to $g_2$. The most basic way of seeing this is to observe that the leading power of $Q$ is given by inserting massless propagators everywhere in the hard part. Then $H$ must contain an odd number of Dirac matrices, and this gives zero in the trace with $\gamma_5, p_{\perp}$ in eq. (32). (Similar reasoning shows that the other two terms in eq. (32) are generally nonzero.) This argument works to all orders of perturbation theory, and demonstrates that $(M/\sqrt{p \cdot q})g_2$ is suppressed by at least one power of $Q$. Since the first nonleading power term is presumably nonzero, it is in fact $g_2$, with the conventional definition, that scales.

A fancier way of saying the same thing is to observe that in the zero mass limit, both
QCD and the electromagnetic vertices are chirally invariant. Chiral invariance prohibits helicity flip for the quarks, and $g_2$ corresponds to an offdiagonal term in the density matrix, so that it necessarily involves helicity flip.

If one calculates Feynman graphs for $H$, using an on-shell projection, but leaving the quark masses nonzero, then the result is of course proportional to the quark mass. However, to treat this as the dominant contribution to $g_2$ is an entirely incorrect application both of parton model ideas and of QCD. In the first place, as the above argument makes clear, there are other terms in the projection over Dirac matrices besides the ones that give the twist-2 terms. These terms cannot be interpreted as the product of the scattering of on-shell quarks times a quark number density. Rather they correspond to coupling to the matrix elements of the twist-3 operators that were listed by Jaffe and Ji [9]. Furthermore, there are regions other than those of fig. 2 that contribute at the twist 3 level; the corresponding operators involve, for example, the correlation of a gluon with quarks [10, 11].

When one attempts to force a connection in the standard fashion between a transverse spin dependence of the quark densities and the transverse structure functions, contradictions arise [3, 2]. These have given an undeserved idea in the folklore that transverse spin cannot be treated within the parton model and its QCD realization.

It should also be clear that if one represents the size of the twist 3 contributions as being of order $M/Q$ relative to a typical twist 2 term, then $M$ should be some kind of hadronic mass scale, hundreds of MeV, at the least. The effect of putting a current quark mass into the calculation of the hard subgraph is a rather small effect, and hardly can be expected to be the dominant nonleading contribution.

4.5 Most General Case

To the extent that the gauge properties of QCD are irrelevant, the argument given above can be readily turned into a full proof. (One has a minor generalization that it is necessary also to consider the possibility of gluon lines joining the jet and hard subgraphs.)

Moreover, the argument can be further generalized, for example to processes like Drell-Yan with two hadrons in the initial state. If both partons are both transversely polarized,
then the helicity conservation argument no longer prohibits a transverse spin dependence of the cross section for such a process, rather the contrary.

Our argument shows clearly that there is no problem in defining the concept of a transversely polarized quark. Such an argument was (to my knowledge) first constructed, in the context of the Drell-Yan process, by Ralston and Soper [6]. In that process, transversely polarized quarks do indeed give contributions to the cross section, at the level of twist-2 terms. Ralston and Soper were working at a time before the full proof of the factorization theorem had become worked out.

However, as far as proofs of factorization in a gauge theory are concerned, there are three essential complications. First, if there is a quark connection, then it is possible to have extra gluons connecting $H$ and $J$. Second, it is possible to have Faddeev-Popov ghost connections, for which there is no physical parton distribution. Third, if there are only gluons connecting the two subgraphs, then the leading power is $Q^2$ times the canonical power. We will treat these complications in the next section. The important point is to see that these issues are the same as in unpolarized scattering, so that we need only quote previous results.

5. SUPER-LEADING TERMS

Consider now the case of a gluon connecting the two subgraphs in fig. 2. We may represent this by

$$J^\alpha...(-g_{\alpha\beta})H^\beta...,$$

where the dots (...) represent the indices for the other lines and for the virtual photons attached to $H$. By exactly the same argument as in the previous subsection, the leading power in (36) comes from the term

$$-\frac{J^+}{p^+} p^+ H^-.$$

We have multiplied and divided by $p^+$ to exhibit a jet factor that is boost invariant. It is easy to check if all the lines joining the two subgraphs are gluons, then we get a contribution to the structure functions that is a factor of $Q^2$ larger than the twist-2 contribution that we got in the quark case. We call this contribution ‘super-leading’.
Moreover, if we start with a contribution with just a pair of quark lines, and if we add an extra gluon line joining the top and bottom, then the term (37) in the gluon polarization gives a contribution that has the same power law as before the gluon was added. This is specific to the case of a vector field. If we add an extra fermion or an extra scalar line (in the case that we have a model with elementary spin zero fields), then the numerator factors are insufficient to compensate the extra large denominator in $H$, and we lose a power of $Q$. The factor (37) results in a factor $Q$ greater than ‘normal’.

To handle these contributions, we make the following decomposition of the numerator $-g_{\alpha\beta}$ of the gluon propagator in (36):

$$-g_{\alpha\beta} = \frac{n_{\alpha}}{n \cdot k} + \left(-g_{\alpha\beta} + \frac{n_{\alpha}k_{\beta}}{n \cdot k}\right)$$

$$= \frac{n_{\alpha}k_{\beta}}{n \cdot k} + h_{\alpha\beta}.$$  \hspace{1cm} (38)

Here, $k^{\alpha}$ is the momentum of the gluon, supposed collinear to $p^{\mu}$, and $n^{\alpha}$ is a vector with just a $-\delta^{\alpha}_{\alpha}$ component: $n^{\alpha} = \delta^{\alpha}_{\alpha}$. (This vector gives a covariant definition of the fractional momentum carried by $k$: $\xi = n \cdot k / n \cdot p$.) The leading term (37) is entirely contained in the first term in eq. (38), while the contribution given by $h_{\alpha\beta}$ is exactly one power of $Q$ smaller.

When we expand each of the gluons between $H$ and $J$ by using eq. (38), we will say that the gluons with the $n_{\alpha}k_{\beta}/n \cdot k$ term have scalar polarization, while those with the $h_{\alpha\beta}$ term have transverse polarization.

The largest superleading term is obtained when all the connections between $H$ and $J$ are gluons with the $n_{\alpha}k_{\beta}/n \cdot k$ term. Since every gluon attached to $H$ is given a factor $k^{\beta}$, there is a cancellation [22] by a Ward identity. This cancellation is exact in an abelian theory. But since we must exclude graphs that are one-particle-reducible in each group of collinear external lines, we obtain extra terms in a nonabelian theory. For example, there are commutators between the different scalar gluons. The same argument must be applied recursively to the commutators. The details of such an argument have never been worked out in detail, even in the unpolarized case, to the best of my knowledge. In any event, the argument has nothing to do with the polarization of any of the external particles involved.

When there is only one transverse gluon, with the other connecting lines being scalar
gluons, the same argument applies. The transverse gluon gives some nonzero terms in the Ward identity, involving the transverse gluon. But there is no connection on other side of the final state, which means the process cannot happen. In a nonabelian theory, there should also be terms in the Ward identity that bring in ghosts and that cancel the contributions when Faddeev-Popov ghosts connect the hard and jet subgraphs. Again, the polarization state of the initial hadron is entirely irrelevant.

We are therefore left with contributions that involve either two transverse gluon lines or two quark lines, one on each side of the final-state cut, together with arbitrarily many scalar gluons. The identical argument given above is used to extract the leading term for the quark lines, then a Ward identity is used to move the scalar gluons to the quarks, and thereby build up a gauge-invariant quark operator. The case of two transverse gluons is handled similarly: Only values of the index $\beta$ in $h_{\alpha\beta}$ that are in the transverse plane give a leading contribution, and the combination of one of these factors on each side of the cut is exactly what corresponds to the spin density matrix for a gluon.

6. EXTRACTION OF SOFT GLUONS

Deeply inelastic scattering is special: there are no soft gluons to worry about after the sum over final-state cuts.

7. FACTORIZATION

We now need combinatoric arguments to go from the decomposition given above, region-by-region, to the factorization eq. (7) with an operator definition of the parton densities. These arguments are of exactly the same form as for Wilson's original operator product expansion in the Euclidean case [18, 19, 20]. Once one has established the leading regions to be those symbolized by fig. 2, the fact of having a different kinematic definition of the regions is irrelevant.

Thus we may regard fig. 2, without the extra scalar gluons, as being the factorization. The spin structure is entirely contained in the Dirac structure we elucidated above. That enables us to read off the correct definitions of the parton densities.
8. DEFINITION OF PARTON DISTRIBUTION FUNCTIONS

Elementary manipulations convert the formula (33) for the quark density into the expectation value of a certain nonlocal operator [23]:

\[ f_i(\xi) = \int \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} \langle p | \bar{\psi}_i(0, y^- , 0) \frac{\gamma^+}{2} P e^{-ig \int_0^{y^-} dy'^- A_i^+(0, y'^-, 0) t_\alpha \psi_i(0) | p} \rangle. \] (39)

The path ordered exponential is needed to make the operator gauge invariant, and the manipulations with the Ward identities prove that it is needed. When we work in the light-cone gauge \( A^+ = 0 \), this exponential vanishes.

The quark helicity and transverse spin are functions of \( \xi \). They are defined by replacing the \( \gamma^+ / 2 \) in eq. (39) by \( \gamma^5 \gamma^+ / 2 \) and \( \gamma^+ \gamma_5 \gamma_\perp / 2 \), respectively. (See eqs. (34) and (35).) By conservation of angular momentum and parity, the quark helicity and transverse spin are proportional [7] to the corresponding quantities for the target, so that we can write

\[ \lambda_i f_i(\xi) = \lambda f_{Li}(\xi), \]
\[ s^\mu_{i\perp} f_i(\xi) = s^\mu_{\perp} f_{Ti}(\xi), \] (40)

where the spin variables with and without the subscript \( i \) are for the parton \( i \) and the hadron target, respectively. (Angular momentum conservation here refers to the invariance of both the theory and of definitions like eq. (39) under rotations about the z-axis.

The operator definitions that translate eqs. (34) and (35) are:

\[ \lambda f_{Li}(\xi) = \int \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} \langle p | \bar{\psi}_{Li}(0, y^- , 0) \frac{\gamma^+ \gamma_5}{2} P e^{-ig \int_0^{y^-} dy'^- A_i^+(0, y'^-, 0) t_\alpha \psi_i(0) | p} \rangle. \] (41)

and

\[ s^\mu_{\perp} f_{Ti}(\xi) = \int \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} \langle p | \bar{\psi}_{Ti}(0, y^- , 0) \frac{\gamma^+ \gamma^\mu_\perp \gamma_5}{2} P e^{-ig \int_0^{y^-} dy'^- A_i^+(0, y'^-, 0) t_\alpha \psi_i(0) | p} \rangle. \] (42)

The asymmetries \( \Delta_L = f_{Li}(\xi) / f_i(\xi) \) and \( \Delta_T = f_{Ti}(\xi) / f_i(\xi) \) are, in general, functions of the fractional momentum variable \( \xi \) (and of the scale \( \mu \) at which the densities are defined).

Feynman rules are readily written down, as in fig. 3. We have diagrams in which there is an incoming particle for the state \( | p \rangle \) on the left, and an outgoing particle for the state \( \langle p | \) on the right. There is a cut for the final state. The operator is represented by the double
line crossing the final state. Any number of gluons may attach to the double line. Integrals over all loop momenta are performed, and in addition there is an integral over the \( k^- \) and \( k_\perp \) coming out of the vertex; \( k^+ \) is set equal to \( \xi p^+ \). Finally the product of Dirac matrices for the explicit quark line is traced with \( \gamma^+/2 \) for the unpolarized density, with \( \gamma_5 \gamma^+ /2 \) for the helicity part, and with \( \gamma^+ \gamma_5 \gamma_\perp /2 \) for the transverse polarization part. These rules are equivalent to those written down by Artru and Mekhfi [7] in a helicity basis.

Fig. 3. Feynman rules for quark densities.

The definitions given above have ultra-violet divergences when \( k_\perp \to \infty \). These are renormalized in the same way as the ultra-violet divergences in the ordinary twist-2 local operators. Modulo the anomalies associated with the \( \gamma_5 \), which have nothing specific to do
with the difficulties in defining transverse polarization, these renormalizations are straightforward. The renormalization procedure introduces explicit dependence on a renormalization scale $\mu$. The renormalization group equations for the parton densities are the ordinary evolution equations, the (Gribov-Lipatov)-Altarelli-Parisi equations.

When integer moments are taken of the above quark densities, and combined with the appropriate sign (plus or minus) times the antiquark densities, matrix elements of twist two local operators are obtained. For the unpolarized distributions and and for the helicity distributions, these operators are familiar from the treatment of deeply inelastic scattering by the operator product expansion.

9. COVARIANCE OF PARTON DISTRIBUTIONS

The definitions we have given of the quark distributions depend on the choice of a frame, and might therefore appear not to be Lorentz invariant. So we must now show that this is not so. The choice of coordinates can be specified by a vector

$$ n^\mu \equiv \delta_-^\mu, \tag{43} $$

which is lightlike and future pointing. Then the momentum-space integrals in eq. (29) and its relatives have a covariant formulation:

$$ dk^- d^2k_\perp = d^4k \delta(k \cdot n - \xi p \cdot n), \tag{44} $$

while the coordinate space integrals in eq. (39) etc are

$$ dy^- e^{-i\xi p^+ y^-} = d\lambda e^{-i\xi \lambda p \cdot n}. \tag{45} $$

The coordinate of the antiquark field in eq. (39) is $\lambda n^\mu$, and the matrix $\gamma^+$ is $\gamma \cdot n$, so that all the definitions are invariant under boosts along the $z$-axis, that is, they are invariant under scaling of $n^\mu$ to $C n^\mu$. Finally, the path-ordered exponential in eq. (39) has the boost invariant form

$$ Pe^{-ig \int_0^{\lambda n} d\lambda' n \cdot A_\alpha(\lambda' n)t_\alpha} \tag{46} $$
All the definitions of the parton densities now covariant, and they all explicitly depend on
the fractional momentum variable $\xi$. They also depend on the momentum and spin vector
of the hadron $p$, $s$, and on $n$. The operator expectation values in the definitions depend
linearly on the hadron’s density matrix and so their dependence on $s$ is a constant plus a
linear term.

Let us define the hadron’s helicity by $\lambda = M s \cdot n/p \cdot n$ and its transverse polarization
by $s^\mu_{\perp} = s^\mu - \lambda (p^\mu/M - n^\mu M/p \cdot n) = s^\mu - p^\mu s \cdot n/p \cdot n - n^\mu s \cdot nM/p \cdot n^2$. The transverse polarization satisfies $n \cdot s_{\perp} = 0$.

The distributions for unpolarized quarks and for the helicity dependence, eqs. (39) and
(41) can now be seen to be Lorentz invariant. Since $n \cdot n = 0$ and the distributions are
independent of the scale of $n^\mu$, the only kinematic variable on which they can depend (aside
from $\xi = k \cdot n/p \cdot n$) is the helicity $\lambda$, and that at most linearly. Parity invariance of QCD
then shows that the unpolarized distribution is independent of the hadron polarization, while
the helicity asymmetry is linear, just as we asserted on the left of the defining equations.
(Note that parity invariance is essential to this result: if QCD did not conserve parity, then,
for example, the number of left handed quarks in a unpolarized proton need not equal the
number of right handed quarks.)

As defined in eq. (42), the transverse distribution is given as a Lorentz vector that represents
the polarization vector $s^\mu_q$ of the quark. Rotation invariance forces it to be proportional
to the hadron’s spin vector, and so the right hand side of eq. (42) must be a linear combination
of $s^\mu$, $n^\mu s \cdot n/p \cdot n^2$ and $p^\mu \lambda$, with coefficients that are independent of $n$ and of the
spin of the hadron. We have written the definition in terms of the transverse components of
a gamma matrix $\gamma^\mu_{\perp}$, but we could have used the complete $\gamma^\mu$ to preserve manifest Lorentz
covariance. The definition satisfies $n \cdot s_q = 0$, since $(\gamma \cdot n)^2 = 0$. Thus we get a linear
combination of $s^\mu_{\perp}$ and $n^\mu s \cdot n/p \cdot n^2$. The coefficient of $s^\mu_{\perp}$ we call the transversity part of
the quark density, $f_T(\xi)$.

The term proportional to $n^\mu s \cdot n/p \cdot n^2$ corresponds to one of the twist three distribution
listed by Jaffe and Ji [9], and arises only if one replaces the $\gamma^\mu_{\perp}$ in eq. (42) or eq. (35) by $\gamma^-$. Most importantly, this term gives no contribution in a twist-2 hard scattering calculation,
because one immediately puts the quark spin into a trace calculation involving the quantity

$$k(1 - \lambda_q \gamma_5 + \gamma_5 s_q),$$

(47)

where the spin vector is supposed to satisfy \(s_q \cdot k = 0\), with \(k^\mu\) now being a light-like vector in the + direction.

We have now seen that the definitions we have made of the quark densities, both the unpolarized one and the longitudinal and the transversity spin distributions, are explicitly Lorentz invariant: they are scalar quantities independent of the choice of the vector \(n^\mu\). Essentially the same considerations apply to the gluon distributions to be defined in the next section. The only non-invariance comes into the definition of the hadron helicity \(\lambda\) and the hadron transverse spin vector \(s_{\perp}^\mu\).

Let us now see why this last noninvariance creates no problem. The physics is that we define the parton densities to be used in a conventional twist-2 hard scattering calculation. The presence of other particles in the process gives us the definition of \(n^\mu\). Consider for example a collision of two particles of momenta \(p_1\) and \(p_2\)—fig. 4. Suppose particle 1, which is moving to the right, has left-handed helicity. By boosting so that the reference frame moves faster than the particle, we reverse its velocity and its helicity, but clearly we have also changed the view of the collision: particle 2 now overtakes particle 1.

Fig. 4. Collision of two particles.

From the point-of-view of the rest frame of particle 1, it is being probed by a an almost light-like particle moving in a certain direction. We can create the exactly light-like vector
\(n^\mu\) as a linear combination of \(p_2^\mu\) with a small admixture of \(p_1^\mu\):
\[
n^\mu \propto p_2^\mu - cp_1^\mu,
\]
with \(c \approx m_2^2/s\). The only ambiguity is that it might be convenient to choose one of the other momenta in the hard scattering instead of \(p_2\) in this formula. For example, in deeply inelastic scattering, one often chooses to define transverse coordinates with respect to \(p\) and \(q\), which are the momentum vectors relevant for the hadronic part of the process, whereas the simpler definition for an experiment is to define transverse with respect to \(p\) and the momentum of the incoming lepton \(l\). (This is indeed what one means by transverse polarization in an experiment.) These two possibilities give different definitions of the vector \(n\) in the parton densities. In the rest frame of the incoming hadron, they differ by a rotation through an angle of order \(M/\sqrt{s}\), so that at its largest the difference corresponds to a twist-3 effect.

10. OPERATOR DEFINITION OF POLARIZED GLUON DISTRIBUTIONS

Exactly analogous definitions may be made for the density of gluons and the longitudinal and linear polarization of the gluon. (A pure state that is a linear combination of equal amounts of left and right helicity is called transversely polarized for a spin-\(\frac{1}{2}\) particle, but linearly polarized for a spin-1 particle).

The gauge invariant definitions are
\[
f_g(\xi) = -2 \sum_{j=1}^{2} \int \frac{dy^-}{2\pi \xi p^+} e^{-i\xi p^+ y^-} \langle p|G^{+j}(0, y^-, 0_\perp) \mathcal{P} G^{+j}(0)|p\rangle,
\]
\[
f_{\text{hel}}(\xi) = \sum_{j,j'=1}^{2} P_{\text{hel}}^{j,j'} \int \frac{dy^-}{2\pi \xi p^+} e^{-i\xi p^+ y^-} \langle p|G^{+j}(0, y^-, 0_\perp) \mathcal{P} G^{+j'}(0)|p\rangle,
\]
\[
f_{\text{lin}}(\xi) = \sum_{j,j'=1}^{2} P_{\text{lin}}^{j,j'} \int \frac{dy^-}{2\pi \xi p^+} e^{-i\xi p^+ y^-} \langle p|G^{+j}(0, y^-, 0_\perp) \mathcal{P} G^{+j'}(0)|p\rangle,
\]
where \(G_{\mu\nu}\) is the gluon field strength tensor and \(\mathcal{P}\) denotes the path-ordered exponential of the gluon field along the light-cone that makes the operators gauge-invariant, in exact analogy to eq. (39):
\[
\mathcal{P} = \exp \left[ \int_0^{y^-} dy'^- A_\alpha^+(0, y'^-, 0_\perp) T_\alpha \right].
\]
Here $T_\alpha$ are the generating matrices for the adjoint representation of color SU(3). The $j$ index runs over the two transverse dimensions, and the spin projection operators are defined by

$$P^\text{hel}_{11} = P^\text{hel}_{11} = 0,$$

$$P^\text{hel}_{12} = -P^\text{hel}_{21} = i,$$

$$P^\text{lin}_{n,jj'} = n_j n_j' - \delta_{jj'}/2. \quad (50)$$

By angular momentum conservation, the linear polarization of a gluon is zero in a spin-$\frac{1}{2}$ hadron [7]. (The reason is that the linear polarization is measured by an operator that flips helicity by two units. Since no helicity is absorbed by the space-time part of the definition of the parton densities (the integrals are azimuthally symmetric), the helicity flip in the operator must correspond to a helicity flip term in the density matrix for the hadron.

Just as for the quarks, one can take integer moments of the gluon densities and get matrix elements of local operators.

11. FACTORIZATION FOR DRELL-YAN

There are some complications in the proof of factorization for Drell-Yan as compared with the proof for deeply inelastic scattering. These same complications appear in all other processes with two hadrons in the initial state. The complications are explained in [4, 5], and the proof that they do not wreck the factorization are entirely independent of the polarization issue.

The typical leading region corresponds to fig. 1, which looks like an obvious generalization of the case of deeply inelastic scattering, fig. 2. There are now two jet subgraphs, corresponding to the two initial-state hadrons. There is also the possibility of extra collinear gluons joining the jet subgraphs to the hard subgraph, and these are treated in exactly the same way as in deeply inelastic scattering. However, and much more malignantly, there are leading regions in which soft gluons are exchanged between the two jet subgraphs. (There are also soft gluons exchanged with jets going into the final state from the hard scattering, but these cancel after a sum over the unobserved part of the final state.) The soft gluons can be emitted off internal lines and in the initial state. Note that a soft gluon is defined to be
one that carries momentum much less than $Q$, as measured in the center of mass frame. This is a broader definition than the one that concerns the very well-known infrared divergences in QED, and consequently the soft gluons are not restricted to being emitted from external, onshell colored particles.

Such interactions were known before QCD: they were called Pomeron exchange, and physically they generate the observed final states. The final states corresponding to fig. 1 taken in its naivest interpretation has two jets of hadrons corresponding to the remnants of the two incoming hadrons, together with a large gap in rapidity between them that contains no particles. This gap is completely filled in by the Pomeron.

The leading power for the soft interactions is given by a generalization of the method that generates (37) from (36). After that, a proof of cancellation of these soft interactions involves a tricky combination of Ward identities, analyticity and unitarity [5]. None of this part of the proof depends on the polarization. Note that a key part of the proof rests on analyticity arguments that were first made in nonperturbative Pomeron physics [24]: these involve analyticity (i.e., causality) but not polarization, and are valid to the whole leading power.

Once one has got the soft interactions canceled, and the collinear scalar gluons factored out of the hard part, one is left with the task of determining the leading power part of the traces over Dirac matrices joining the jet subgraphs and the hard subgraph. (There is the same task to perform for gluons.) This just involves two copies of the argument given above for deeply inelastic scattering. Ralston and Soper [6] gave this argument before the full apparatus of factorization was formulated, and we now see their argument must be true in full QCD.

The only difference from deeply inelastic scattering is that the suppression of transverse polarization no longer occurs. If we have both initial hadrons transversely polarized, then there is a twist-2 asymmetry in the hard scattering cross section that is explicitly nonzero at the Born graph level—an asymmetry that is well known in $e^+e^-$ physics.

One error does occur in the Ralston-Soper paper. They attempt to list all the rather numerous structure functions that are permitted in the decomposition of the dilepton angular
distribution, and they miss some (These are not structure functions in the common misusage of the term.) The error was corrected by Donoghue and Gottlieb [25], and does not at all effect the general principles. In any case, only one of the polarized structure functions is actually nonzero for the Born graph.

12. FACTORIZATION FOR OTHER PROCESSES

In this paper, I have restricted attention to inclusive processes with a single large scale. The methods apply to many processes. Although not explicitly treated here, the issues in handling the single particle distribution in a jet are isomorphic to those in the distribution of partons in a hadron. So the methods apply equally to high $p_\perp$ single-particle production in hadron-hadron collisions, to inclusive particle production in $e^+e^-$ annihilation, or to inclusive particle production in deeply inelastic lepton scattering, for example. It would be useful to make a complete characterization of the processes for which a factorization theorem of the standard type holds. All the considerations of the present paper apply to any of these processes.

One interesting possibility is that of measuring the correlation between two particles in a jet. This should be correlated with the spin of the parton that initiates the jet, and may provide a useful handle to probe the transverse spin distribution [26].

There are many situations in which there is a second scale associated with the hard scattering. The simplest is the Drell-Yan process when $q_\perp \ll Q$. A full factorization theorem has been stated [27] for this process, and has been proved [28] for the analogous processes of two-particle production and the energy-energy correlation in $e^+e^-$ annihilation in the back-to-back region. It would be interesting, and not too hard, to extend these theorems to the polarized case. The polarization-specific issues are orthogonal to and decoupled from the complications associated with the low $q_\perp$ region.

Another case is where the hard scattering has a scale $Q$ much less than the center-of-mass energy $\sqrt{s}$. This is called the semi-hard region or the small $x$ region. A leading logarithm statement of a kind of factorization has been stated by Lipatov and coworkers [29]. The proof leaves much to be desired, and does not go beyond the leading logarithm level. Much
work remains here.

Another area is that of exclusive processes. The state of the the factorization theorems and their proofs is not nearly so good as for inclusive processes. There are considerable complications [30] because of regions other than the simplest short-distance scattering: for example in hadron-hadron elastic scattering at large angle, there is competition between the short-distance scattering and the Sudakov-suppressed Landshoff process. Disentangling the polarization dependence would be interesting as a piece of theory, but may not be a high priority because of the minute cross sections at the high values of $Q$ where perturbative methods unambiguously apply.

Perhaps the most interesting recent developments in the theory of polarized hard scattering have been the realization [10, 11] that a generalization of the factorization theorem appears to be provable for the first nonleading twist. Now the first nonleading twist that is relevant in polarized scattering, with transversely polarized beams, is twist 3: that is, the first power corrections are a single power $1/Q$ down from the twist-2 terms. In many cases of single spin asymmetries, the twist-2 term vanishes. An interesting phenomenology should result. Qiu and Sterman [10] have explained the validity of factorization in this case; Jaffe and Ji [9] have listed the operators that are needed to define the single-body parton distributions. Interesting physics also lies in two parton correlations that are an essential parton of the twist-3 results. There are experiments on the single transverse spin asymmetries of single particle production at relatively large $p_{\perp}$ that are greatly in need of theoretical interpretation.

13. CONCLUSIONS

The factorization theorems for hard scattering are as true when the incoming hadrons are polarized as when they are unpolarized. This is also true for processes in which one measures the polarization of hadrons in the final state (from fragmentation of a jet).

The parton densities have unambiguous definitions, which are just matrix elements of gauge invariant generalizations of the quark and gluon number operators that are natural in light-front (or infinite-momentum) quantization. In the case of polarized beams, the
operators are just those that are directly related to a spin density matrix for the partons.

This has particular consequences: The quark transversity distribution that is relevant for twist-2 processes with transversely polarized hadrons is perfectly well defined, contrary to what one might conclude from a superficial reading of the literature [2, 3]. The helicity asymmetry of the gluon density is well defined. Its first moment is in general a nonlocal operator, unless one uses the light cone gauge $A^+ = 0$.

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