Towards unified theory of 2d gravity

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ABSTRACT

We introduce a new 1-matrix model with arbitrary potential and the matrix-valued background field. Its partition function is a $\tau$-function of KP-hierarchy, subjected to a kind of $L_{-1}$-constraint. Moreover, partition function behaves smoothly in the limit of infinitely large matrices. If the potential is equal to $X^{K+1}$, this partition function becomes a $\tau$-function of $K$-reduced KP-hierarchy, obeying a set of $W_K$-algebra constraints identical to those conjectured in [1] for double-scaling continuum limit of $(K-1)$-matrix model. In the case of $K = 2$ the statement reduces to the early established [2] relation between Kontsevich model and the ordinary 2d quantum gravity. Kontsevich model with generic potential may be considered as interpolation between all the models of 2d quantum gravity with $c < 1$ preserving the property of integrability and the analogue of string equation.
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1 Motivation and results

1.1 Introduction

Matrix models play an increasingly important role in the theory of quantum gravity, being a reasonably simple realization of Regge calculus, adequate at least in essentially to topological theories. Their conventional application to string theory is based on identification of the critical points of peculiar “double-scaling” limit \[3, 4, 5, 6\] of Hermitean multimatrix models \[7\] with the \(c < 1\) minimal conformal field theories \[8\] coupled to 2d gravity. Such models are often supposed \[3\] to exhaust the entire set of consistent bosonic 2-dimensional string models, while bosonic strings beyond 2 space-time dimensions are presumably unstable \[10\]. Technical methods, based on the application of matrix models, appeared rather successful not only in the framework of perturbation theory, but allow one to estimate the entire sums of perturbation series, thus rising the study of string models to a qualitatively new level.

On the other hand, matrix models appeared to possess unexpectedly deep internal mathematical structure \[3\]. In fact, if matrix models are used to describe interpolation between different critical points \(i.e.\) particular \(c < 1\) string models, the interpolating flows are mutually commuting, \(i.e.\) the entire pattern of flows constitutes an integrable hierarchy (some reduction of Kadomtsev-Petviashvili (KP) system). This integrable structure indicates that the somewhat artificial description in terms of matrix integrals should possess a more invariant algebro-geometrical description, which could be used for formulation of the yet unknown dynamical principle of string theory, unifying all available string models into distinguished and (hopefully) physically important entity.

The main obstacles on the way from matrix models to such general principle are:

(i) the sophisticated intermediate step: the double-scaling limit,

(ii) the somewhat obscure origin of integrability: it is not \textit{a priori} obvious, why matrix integrals with generic “potentials” are \(\tau\)-functions of (reduced) KP-hierarchies,

(iii) the non-universal description of different \(c < 1\) string models: some of them are nicely unified as different critical points of particular multimatrix model, but there is no nice interpolation between critical points of models with the different number of
matrices. The naive way out is to use \( \infty \)-matrix model, but it is no longer described in terms of finite-dimensional matrix integrals, and, more important, reductions to the case of particular multimatrix model is too much singular.

In this paper we propose a kind of resolution of all these three problems. Namely, we argue that there is a new one-matrix model \([11]\) (which we call Generalized Kontsevich Model (GKM)), which

(i) for a specially adjusted potential describes properly the double scaling limit of any multimatrix model,

(ii) has the partition function, which is a KP \( \tau \)-function, properly reducible at the points, associated with multimatrix models, and satisfies an additional equation, which reduces to conventional \( \mathcal{L}_{-1} \)-constraints for multimatrix models,

(iii) allows continuous deformation of potential, changing it from the form, associated with a given multimatrix model to those corresponding to the others.

This makes the study of GKM a natural step in the development of string theory.

In the remaining part of this section we describe GKM and its relation to multimatrix models more explicitly. The proofs are presented in sections 2-4. For a more condensed presentation of our results see \([1]\).

1.2 Generalized Kontsevich model

This paper is devoted to the study of 1-matrix model in external matrix field \( \Lambda \), essentially defined by the integral over \( N \times N \) matrix \( X \):

\[
\mathcal{F}_N^{(V)}[\Lambda] \equiv \int dX \ e^{-Tr V(X) + Tr \Lambda X} \tag{1.1}
\]

with arbitrary “potential” \( V(X) \) and \( dX = \prod_{i,j=1}^{N} dX_{ij} \). Such integrals arise in consideration of various problems \([12, 13, 2, 14]\), but this time our purpose is to use (1.1) in order to define a universal matrix model, which could describe interpolation between all the multimatrix models and their critical points, as explained in sect.1.1 above. Namely, we present strong arguments that partition function of such universal model may be identified with \( Z_\infty^{(V)}[M] \), where
\[ Z_N^{\{V\}}[M] \equiv \frac{\int e^{-U(M,Y)}dY}{\int e^{-U_2(M,Y)}dY} \quad (1.2) \]

with
\[ U(M,Y) = Tr[\mathcal{V}(M + Y) - \mathcal{V}(M) - \mathcal{V}'(M)Y] \quad (1.3) \]

and
\[ U_2(M,Y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} U(M,\epsilon Y), \quad (1.4) \]

(i.e. \(U_2(M,Y)\) is the \(Y^2\)-contribution to \(U(M,Y)\)).

After the shift of integration variable
\[ X = M + Y \quad (1.5) \]

the numerator in (1.2) may be rewritten in terms of (1.1):
\[ \int e^{-U(M,Y)}dY = e^{Tr[\mathcal{V}(M) - MY]} \mathcal{F}_N^{\{V\}}[\mathcal{V}'(M)]. \quad (1.6) \]

The denominator in (1.2) is a simple Gaussian integral — a peculiar matrix generalization of \([\mathcal{V}''(\mu)]^{-1/2}\).

We shall call \(Z_N^{\{V\}}[M]\), defined by eq.(1.2), the partition function of \textit{generalized Kontsevich model} (GKM). The reason for this is that for the special choice of potential\(1\):
\[ \mathcal{V}(X) = \mathcal{V}_2(X) = X^3/3, \quad (1.7) \]

eq.(1.2) becomes the partition function of original Kontsevich model\(2\):
\[ Z_N^{\{2\}}[M] = \frac{\int dY e^{-1/3 TrY^3 - TrMY^2}}{\int dY e^{-TrMY^2}} = \]
\[ = e^{-(2/3)TrM^3} \int dX e^{-1/3 TrX^3 + TrM^2X} \frac{1}{\int dX e^{-TrMX^2}}. \quad \text{(1.8)} \]

Eq.(1.8) has a natural generalization, which is also a particular case of (1.2): if \(1\) We do not specify integration contour in matrix integrals (1.1), (1.2), since all what we are going to discuss does not depend on it. To make contact with the literature, note that for potential \(\mathcal{V}_K = \frac{X^{K+1}}{K+1}\) integrals should be over Hermitian or anti-Hermitian matrices if \(K + 1\) is even or odd respectively.
\[ \mathcal{V}(X) = \mathcal{V}_K(X) = \frac{X^{K+1}}{K+1} \]  

(1.9)

Eq. (1.2) becomes

\[ Z_N^{(K)}[M] = \frac{e^{-\frac{1}{K+1}TrM^{K+1}}}{\int dX e^{-\frac{1}{K+1}Tr[\sum_{a+b=K+1}M^aXM^bX]}}. \]  

(1.10)

We shall argue that \( Z_N^{(K)}[M] \), defined by (1.10), is identical to the square root of partition function of \((K-1)\)-matrix model in the double-scaling limit with

\[ T_n = \frac{1}{n}Tr M^{-n}, \quad n \neq 0 \pmod{K}, \]  

(1.11)

playing the role of generalized Kazakov’s time-variables [3, 16]:

\[ Z_N^{(K)}[M] = \sqrt{\Gamma_{\infty}^{(K-1)}(T_n)}. \]  

(1.12)

where [7]

\[ \Gamma_{\infty}^{(K)} = \lim_{\text{double scaling}} \prod_{i=1}^{K} \int dM_i e^{\sum_t t_nTrM_i^n + TrM_iM_{i+1}} \]  

(1.13)

(the term \( M_KM_{K+1} \) in the exponent should be omitted; the change from discrete time-variables \( \{t_n\} \) to continuum \( \{T_n\} \) is implicit in performing the double-scaling limit, see [16, 17]).

Since potential \( \mathcal{V}(X) \) can be continuously deformed from one \( K \) in (1.9) to another, the GKM (1.2) as a corollary of (1.12) may be considered as continuous interpolation between all the models of 2d gravity with \( c < 1 \), described by particular multicritical points of particular multimatrix models (let us remind, that for \( K \) fixed the \( p \)-th multicritical point is defined by the condition [1] that

\[ \text{all } T_n = 0, \text{ except for } T_1 \text{ and } T_{K+p}, \]  

(1.14)

and according to (1.11) this is a constraint on the form of the matrix \( M \).

This is the sense in which the GKM (1.2) may be considered as the universal partition function of bosonic string models. We shall not discuss the implications of this fact here, instead we shall concentrate on the arguments in favour of this conclusion, which
are entirely based on the identity (1.12) (and also motivated by the nice features of interpolating function $Z^{(\mathbb{V})}[M]$ itself — it is usually a KP $\tau$-function, subjected to $\mathcal{L}_{-1}$-constraint, see below). Our strategy is to try to understand as much as possible about $Z^{(K)}[M]$, which stands at the $l.h.s.$ of (1.12) and compare these results with what is conjectured in \cite{1} about $\sqrt{\Gamma_{ds}^{(K-1)}}$ at the $r.h.s.$

1.3 Properties of $Z^{(2)}$

In order to motivate our considerations let us remind first, what is known about $Z^{(K)}[M]$ in the particular case of $K = 2$.

Expression (1.2) has been derived in \cite{15} as a representation of the generating functional of intersection numbers of the stable cohomology classes on the universal module space, \textit{i.e.} it is defined to be a partition function of Witten’s 2$d$ topological gravity \cite{18}. In \cite{2} (see also \cite{13, 19} for alternative derivations) it was shown that as $N \to \infty$ $Z^{(2)}_{\infty}$ considered as a function of time variables (1.11), satisfies the set of Virasoro constraints

$$\mathcal{L}_{n}^{(K=2)} Z^{(2)}_{\infty} = 0, \quad n \geq -1,$$

$$\mathcal{L}_{n}^{(2)} = \frac{1}{2} \sum_{k \text{ odd}} kT_{k} \partial/\partial T_{k+2n} + \frac{1}{4} \sum_{a+b=2n, a,b \text{ odd and }>0} \partial^{2}/\partial T_{a} \partial T_{b} +$$

$$+ \frac{1}{4} \sum_{a+b=2n, a,b \text{ odd and }>0} aT_{a} bT_{b} + \frac{1}{16} \delta_{n,0} - \partial/\partial T_{3+2n}. \quad (1.16)$$

(Notations are slightly changed as compared to \cite{2}. This derivation is partly reproduced in sect.4.2 below.). Sometimes it may be convenient to consider the appearance of the last item at the $r.h.s.$ of (1.16) as result of additional shift of time-variables,

$$T_{n} \to \hat{T}_{n}^{(K)} \equiv T_{n} - \frac{K}{n} \delta_{n,K+1}, \quad (1.17)$$

however, these $\hat{T}$-times are defined in a $K$-dependent way and do not seem to have too much sense. Constraints (1.15) are exactly the equations \cite{1, 20, 16}, imposed on $\sqrt{\Gamma_{ds}^{(1)}}$. Moreover, it is very plausible that they possess a unique solution, so that (1.15) in fact implies that
\[ Z^{(2)}_\infty = \sqrt{\Gamma^{(1)}_{ds}}. \]  

(1.18)

Alternative solution to the constraints (1.15) is given \[1, 20\] by a Galilean-invariant KdV \(\tau\)-function \[21\], and, relying upon the same belief that the solution is unique, one concludes that \(Z^{(2)}_\infty\) is a \(\tau\)-function of KdV-hierarchy, subjected to additional \(\mathcal{L}^{(2)}_{-1}\)-constraint:

\[ Z^{(2)}_\infty = \tau_{KdV} = \tau^{(2)}, \]  

(1.19)

\[ \mathcal{L}^{(2)}_{-1}\tau^{(2)} = 0. \]  

(1.20)

Notation \(\tau^{(2)}\) reflects the fact that any \(\tau_{KdV}\) may be considered as a KP \(\tau\)-function, evaluated at specific points of Grassmannian (see sect.2.5 below for details), corresponding to the 2-reduction of KP hierarchy. Generically, the \(\tau\)-function of \(K\)-reduced hierarchy, \(\tau^{(K)}(T_n)\), is the KP \(\tau\)-function at peculiar points of Grassmannian and possesses the following property:

\[ \tau^{(K)}(T_1...T_{K-1}, T_K, T_{K+1}...T_{2K-1}, T_{2K}, T_{2K+1}, ...) = e^{(\sum_n a_{nK}T_{nK})} \tau^{(K)}(T_1...T_{K-1}, 0, T_{K+1}...T_{2K-1}, 0, T_{2K+1}, ...). \]  

(1.21)

This means that the entire dependence of generic \(\tau^{(K)}\) of all variables \(T_{nK}\) (i.e. \(T_n\) with \(n = 0 \text{ mod } K\)) is exhausted by the simple exponent

\[ \exp(\sum_n a_{nK}T_{nK}) \]  

(1.22)

with time-independent parameters \(a_{nK}\). Moreover, this factor is actually absent in the case of GKM partition function: \(a_{nK}[Z^{(K)}] = 0\).

The significance of the just reported results concerning \(K = 2\) is two-fold. First, they establish the relation (1.7) between two different models of 2d gravity: Witten’s topological gravity \[19\], represented by \(Z^{(2)}_\infty\) \[13\], and ordinary quantum 2d gravity \[4, 5, 6\], represented by \(\Gamma^{(1)}_{ds}\). This was the main emphasize of ref.\[3\]. The second implication, and it is the one of interest for us in this paper, is the possibility to describe the sophisticated double-scaling limit of matrix model in terms of a completely different matrix model (1.2).
It is important that $Z_{N}^{(K)}$ is in fact a “smooth” function of $N$ as $N \to \infty$, and all the properties of $Z_{\infty}^{(K)}$ are just the same as at finite $N$’s, — in variance with the double-scaling limit for ordinary matrix models, the limit $N \to \infty$ for GKM is non-singular. Also, to avoid misunderstanding, we should emphasize that what we mean under double-scaling limit here is not the limit, describing a particular fixed point, but the entire pattern of all the flows between various fixed points of a given $(K-1)$-matrix model, so that partition function $\Gamma_{ds}^{(K)}$ is indeed a function of all time-variables (1.3).

1.4 The case of arbitrary $K$

In order to generalize the identity (1.15) to the case of arbitrary $K$, i.e. to prove (1.12), one can try to prove either an analogue of (1.15) or that of (1.19), (1.20) and compare the result with the what is expected for $\sqrt{\Gamma_{ds}^{(K-1)}}$. This will be our strategy below. Namely, in sect.2 a simple and promising formalism is developed, based on interpretation of (1.11) as Miwa transformation, and we use it to prove that any $Z_{\infty}^{(V)}[M]$, defined by (1.2) and (1.11) (i.e. with arbitrary $V(X)$) is in fact a KP $\tau$-function — in particular, satisfies bilinear Hirota difference equations. Moreover, in the Grassmannian parameterization of KP $\tau$-functions the point of Grassmannian is specified by the choice of potential $V(X)$, so that multimatrix models with different $K$ are (due to (1.12)) associated with different points of Grassmannian. (In terms of the universal module space these points are in fact associated with certain infinite-genus hyperelliptic surface for $K = 2$, and with certain infinite genus abelian coverings of degrees $K$ in the general case. In this sense interpolation with the changing $V(X)$ is some flow at infinity of the universal module space.)

What is specific for a given $K$ is that whenever $V(X)$ is a homogeneous polynomial of degree $K + 1$, i.e. if it is of the form (1.9), the KP $\tau$-function becomes independent of all $T_{Kn}$, i.e. acquires the property (1.21) and may be considered as $\tau$-function $\tau^{(K)}$ of $K$-reduced hierarchy. This will be also explained in more details in sect.2 below.

Though any $Z_{\infty}^{(V)}[M]$ defined by (1.2) is a KP $\tau$-function, the inverse statement is not correct. $Z_{\infty}^{(V)}[M]$ is subjected to the infinite set of constraints, essentially implied by the Ward identities for the integral (1.1) [12, 4]:

10
\[
\left\{ \text{Tr} \epsilon(\Lambda) \left[ \mathcal{V}' \left( \frac{\partial}{\partial \Lambda_{tr}} \right) - \Lambda \right] \right\} \mathcal{F}_N^{(V)} = 0 \tag{1.23}
\]

with any \( \epsilon(\Lambda) \). These identities result from invariance of (1.1) under any shift of integration variable \( X \rightarrow X + \epsilon(\Lambda) \). Eq.(1.23) seems to provide a complete set of differential equations, unambiguously defining \( \mathcal{F}_N^{(V)} \) (up to a \( \Lambda \) and \( N \)-independent constant). This statement is the actual basis for all of the arguments, which make use of the uniqueness of solutions to any constraints.

As shown in [2] in the particular case of \( K = 2 \) the entire set of such identities with all possible \( \epsilon(\Lambda) \) is equivalent to (1.15) and thus implies (1.19) and (1.20). In fact in sect.2 we follow a somewhat different logic and give an alternative direct derivation of relations

\[ Z^{(V)}_\infty = \tau , \quad Z^{(K)}_\infty = \tau^{(K)} \tag{1.24} \]

for any \( \mathcal{V}(X) \) and \( K \), so it remains only to prove the analogue of (1.20). Such direct proof is presented in sect.3, moreover it is valid not only for potentials of the form (1.9), but for any \( \mathcal{V}(X) : \)

\[
\mathcal{L}_{-1}^{(V)} Z^{(V)} = \left\{ \sum_{n \geq 1} \text{Tr} \left[ \frac{1}{\mathcal{V}'(M)M^{n+1}} \right] \frac{\partial}{\partial T_n} + \frac{1}{2} \sum_{i,j} \frac{1}{\mathcal{V}'(\mu_i)\mathcal{V}'(\mu_j)} \frac{\mathcal{V}''(\mu_i) - \mathcal{V}''(\mu_j)}{\mu_i - \mu_j} \frac{\partial}{\partial T_1} \right\} Z^{(V)} = 0 , \tag{1.25}
\]

where \( \mu_i \) are eigenvalues of the matrix \( M \).

For potentials (1.9) \( \mathcal{L}_{-1}^{(V)} \) acquires conventional form of \( \mathcal{L}_{-1}^{(K)} \) (see eq.(1.28) below) and may be easily supplemented by other generators of Virasoro algebra.

Despite of eqs.(1.24) and (1.25) provide a complete description of partition function of GKM, the study of direct consequences of Ward identities (1.23) also seems interesting. In particular, it should lead to a direct derivation of the analogue of Virasoro constraints (1.15). It also deserves noting that the meaning of constraints on \( Z^{(V)}_\infty[M] \) is that \( \mathcal{V}(X) \) parametrizes only some restricted subset of the universal module space. Only this subset is associated with matrix models and interpolations between them. This is also clear from the fact that \( \mathcal{V}(X) = \sum \mathcal{V}_n X^n \) is naturally parameterized by an infinite vector \( \{ \mathcal{V}_n \} \) rather than by a matrix \( A_{mn} \), what would be natural for the entire Grassmannian.
Remarkably enough the “dimension” of this subset is just the same as that of the space of time-variables (the space of potentials is parameterized by a vector \( \{ V_n \} \), while that of time-variables — by \( \{ T_n \} \) or by eigenvalue vector \( \{ \mu_n \} \) of the matrix \( M \)). This clearly suggests that a formalism should exist for so restricted set of \( \tau \)-functions, which would treat potentials and times in a more symmetrical fashion (see also discussion in sect.3.4).

Implications of Ward identities (1.23) for \( Z^{(K)}_\infty[ M] \) with \( K > 2 \) are discussed to some extent in sect.4. This is a straightforward calculation, though much more tedious than in the simplest case of \( K = 2 \). We argue that the proper generalization of (1.15) for the case of \( K = 3 \) is given by the following set of constraints:

\[
W^{(3)}_{3n} Z^{(3)}_\infty = 0, \ n \geq -2;
\]

\[
\left\{ \sum_{k \geq 1} (3k - 1) \hat{T}_{3k-1} W^{(2)}_{3k+3n} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+2}} W^{(2)}_{3b-3} \right\} Z^{(3)}_\infty = 0, \ a, b \geq 0, \ n \geq -2;
\]

\[
\left\{ \sum_{k \geq 1+\delta_{n+3,0}} (3k - 2) \hat{T}_{3k-2} W^{(2)}_{3k+3n} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+1}} W^{(2)}_{3b-3} \right\} Z^{(3)}_\infty = 0, \ a, b \geq 0, \ n \geq -3.
\]

(1.26)

Here \( W^{(j)}_{Kn} \) stands for the \( Kn \)-th harmonics of the \( j \)-th generator of Zamolodchikov’s \( \mathcal{W}_K \)-algebra, and the proper notation would be \( W^{(j)}_{Kn} \) or \( W^{(j)}_{K} \). The way to express these quantities through free fields is described in [1]; for example,

\[
W^{(3)}_{3n} = 3 \sum_{k,l \geq 1} k \hat{T}_k l \hat{T}_l \frac{\partial}{\partial T_{k+l+3n}} + 3 \sum_{k \geq 1} k \hat{T}_k \sum_{a+b=3n} \frac{\partial^2}{\partial T_a \partial T_b} + \sum_{a+b+c=3n} \frac{\partial^3}{\partial T_a \partial T_b \partial T_c} + \sum_{a+b+c=-3n} a \hat{T}_a b \hat{T}_b c \hat{T}_c, \ a, b, c > 0, \ a, b, c, k, l \neq 0 \ mod \ 3
\]

(1.27)

and

\[
W^{(2)}_{Kn} = \mathcal{L}^{(K)}_n = \frac{1}{K} \sum_k k \hat{T}_k \partial/\partial T_{k+Kn} + \frac{1}{2K} \sum_{a+b=Kn, a,b>0} \partial^2/\partial T_a \partial T_b + \sum_{a+b=Kn, a,b>0} \partial^3/\partial T_a \partial T_b \partial T_c
\]

\[\text{Let us also remark that such symmetry exposes itself in the fact that the degree of potential } K \text{ and the order of the multicriticality } p+K \text{ (fixed by the choice of times) give the theory of minimal matter with } c = 1 - \frac{6(p-K)^2}{pK} \text{ coupled to } 2d \text{ gravity, and, thus, } p \text{ and } K \text{ should enter the theory on equal footing.}\]
\[ + \frac{1}{2K} \sum_{a+b=-K_n, a,b>0} a\hat{T}_a b\hat{T}_b + \frac{(K-1)(K+1)}{24K} \delta_{n,0}. \]  

(Note that in order to make formulas a bit more compact, we expressed all these operators through \( \hat{T}_n \) rather than \( T_n \)-variables. These are defined in (1.17), \( \partial/\partial \hat{T} = \partial/\partial T \).) The analogue of (1.26) for \( K > 3 \) is a more or less obvious generalization of (1.26), involving all the generators \( \mathcal{W}_{K_n}^{(K)} \), \( \mathcal{W}_{K_n}^{(K-1)} \), ..., \( \mathcal{W}_{K_n}^{(2)} \) of \( \mathcal{W}_K \)-algebra with the “coefficients” made from \( \hat{T} \)'s and \( (\partial/\partial T) \)'s. The form of these operators is a bit reminiscent of the \( \hat{\mathcal{W}} \)-operators, introduced in \([14, 22]\) as ingredients of Ward identities in discrete multimatrix models.

Eqs.(1.26) (and their analogues for \( K > 3 \)) are obviously resolved by any solution of the simpler set of equations,

\[ \mathcal{W}_{K_n}^{(k)} Z^{(K)} = 0, \ k = 2, 3, ..., K; \ n \geq 1 - k \]  

(eq.(1.20) is the particular case: \( k = 2, n = -1 \)). On the other hand (1.26) are equivalent to (1.23) for the particular form of potential (1.9), and thus are supposed to possess a single solution. Therefore from this point of view it also looks plausible that (1.26) (and its counterpart for any \( K \)) is simply equivalent to (1.29). Unfortunately we do not provide an explicit proof of this equivalence (though we could rely, of course, upon the indirect proof from ref.[1], deducing (1.29) from eqs.(1.19),(1.20), which are proved in sect.2,3 without any reference to (1.23)).

If one believes in this transition from (1.26) to (1.29), we can finally return to multimatrix models. Namely, in \([14]\) it was suggested that \( \sqrt{\Gamma_{ds}^{(K-1)}} \)

(i) is a \( \tau \)-function of \( K \)-reduced hierarchy,

(ii) satisfies the constraint

\[ \mathcal{L}_{-1}^{(K)} \sqrt{\Gamma_{ds}^{(K-1)}} = 0, \ n \geq -1, \]  

and, as a corollary of (i) and (ii),

(iii) satisfies the entire set of eqs.(1.29):

\[ \mathcal{W}_{K_n}^{(k)} \sqrt{\Gamma_{ds}^{(K-1)}} = 0, \ k = 2, 3, ..., K; \ n \geq 1 - k. \]  

Now it remains to use the above-mentioned assertion that this system has a unique solution in order to arrive to our identification
and thus to the central idea of this paper: *universality* of GKM (1.2).

1.5 Comments

Before we proceed to actual derivation of all these statements, let us comment briefly on the very status of the constraints (1.29) imposed on $\sqrt{\Gamma_{ds}^{(K-1)}}$. In order to honestly derive such constraints on the lines of \[16\] it is first necessary to find out their counterparts (loop equations) in the *discrete* multimatrix models (these are expressed in terms of the so-called $\tilde{\mathcal{W}}$ operators in \[2, 22\]), then carefully perform the appropriate reductions (including identification of all the $K - 1$ *a priori* different potentials), and take the double-scaling limit. The actual form of the $\tilde{\mathcal{W}}$-constraints is very similar to (1.26) and therefore it may happen that the final constraints would arise just in the form of (1.26) rather than (1.29). This could make the relation (1.26) even more important.

After this general description of our reasoning and results it remains to prove that

(i) $Z_{\infty}^{(V)}$ as defined by (1.2) (the limit $N \to \infty$ is smooth) is indeed a KP $\tau$-function;
(ii) for any $V(X)$ it is subjected to additional constraint (1.25);
(iii) if $V(X) = const \cdot X^{K+1}$, then $Z_{\infty}^{(K)}$ in fact becomes a $\tau$-function $\tau^{(K)}$ of the $K$-reduced KP-hierarchy;
(iv) in this case also the entire set of constraints like (1.15) for $K = 2$ and (1.26) for $K = 3$ are imposed on $Z_{\infty}^{(K)}$.

The proof of the points (i),(iii) is given in sect.2, (ii) is discussed in sect.3 and (iv) — in sect.4. While (i),(ii),(iii) are proved in full generality — for any potential $V(X)$, — the universal form of the constraints (iv) is not completely understood. Moreover, the proof of (iv) for the case of $V(X) = \frac{X^{K+1}}{K+1}$, as presented in sect.4, is not complete even in the simplest case of $K = 3$. Also the proof of (iv) is not independent of (iii), since eq.(1.21) is used. The deep understanding of these $\mathcal{W}_{\infty}$-constraints and their universal formulation remains the first obvious problem for GKM to be resolved in the future. The second obvious problem is to find out the algebro-geometrical origin of the model (1.2) itself.
2 Partition function of the GKM as KP $\tau$-function

2.1 Short summary

The purpose of this section is to prove that:

(A) The partition function $Z^{(K)}_N[M]$, if considered as a function of time-variables

$$T_n = \frac{1}{n} tr M^{-n}, \quad (2.1)$$

is a KP $\tau$-function for any value of $N$ and any potential $\mathcal{V}[X]$.

(B) As soon as $\mathcal{V}[X]$ is homogeneous polynomial of degree $K + 1$, $Z^{(K)}_N[M]$ is in fact a $\tau$-function of $K$-reduced KP hierarchy. Moreover, actually, $\frac{\partial Z^{(K)}}{\partial T_{nK}} = 0$.

In order to prove these statements, first, in sect. 2.2 we rewrite the r.h.s. of eq.(1.2) in terms of determinant formula (2.21)

$$Z^{(K)}_N[M] = \det \frac{\phi_i(\mu_j)}{\Delta(\mu)} \quad i,j = 1, ..., N. \quad (2.2)$$

We show in sect. 2.3 that any KP $\tau$-function in the Miwa parameterization does have the same determinant form. Finally, as a check of self-consistency, we prove in sect. 2.4 that any determinant formula (2.2) with any set of functions $\{\phi_i(\mu)\}$ satisfies Hirota difference equation.

A brief description of Grassmannian (or Universal module space) language is given in sect. 2.5. The point of Grassmannian corresponding to $\tau$-function (2.2) is defined by the infinite-vector function $\{\phi_i(\mu)\}$. Remarkably, the results of ref. [23] which concern identification of the points in Grassmannian, associated with multi-matrix models, are immediately reproduced from the explicit expressions for the basis $\{\phi_i(\mu)\}$. This exhausts our discussion of (A).

The main thing which distinguishes matrix models from the point of view of solutions to the KP-hierarchy is that the set of functions $\{\phi_i(\mu)\}$ in (2.2) is not arbitrary. Moreover, this whole infinite set of functions is expressed in terms of a single potential $\mathcal{V}[X]$ (i.e. instead of arbitrary matrix $A_{ij}$ in $\phi_i(\mu) = \sum A_{ij} \mu^j$ we have here only a vector $\mathcal{V}_i$ or $\mathcal{V}[\mu] = \sum \mathcal{V}_i \mu^i$). This is the origin of $L_{-1}$ and other $W$- constraints (which in the context of KP-hierarchy may be considered as implications of $L_{-1}$). All these constraints are
in fact contained in the Ward identity (1.23). While the detailed discussion of these constraints is postponed to the sect.3,4, in subsect.2.6 we discuss another implication of (1.23) arising in the case of polynomial $\mathcal{V}[X]$. Namely, whenever $\mathcal{V}[X]$ is a homogeneous polynomial of degree $K + 1$, $\phi_{i+K}(\mu)$ in (2.2) can be substituted by $\mu^k\phi_i(\mu)$ and in such points of Grassmannian KP $\tau$-function is known to possess the property (1.21) and thus may be considered as a $\tau$-function $\tau^{(K)}$ of $K$-reduced KP hierarchy. In sect.2.7 we prove that the factor (1.22) is actually absent in GKM. This is all to be said below about (B).

### 2.2 From GKM to determinant formula

**Numerator of (1.2).** We begin from evaluation of the integral (1.1):

$$F^{[\mathcal{V}]}_N[\Lambda] \equiv \int dX \ e^{-Tr[\mathcal{V}(X) - Tr\Lambda X]}.$$  \hspace{1cm} (2.3)

If eigenvalues of $X$ and $\Lambda$ are denoted by $\{x_i\}$ and $\{\lambda_i\}$ respectively, this integral can be rewritten as

$$\frac{V_N}{\Delta(\Lambda)} \left[ \prod_{i=1}^{N} \int dx_i e^{-\mathcal{V}(x_i) + \lambda_i x_i} \right] \Delta(X).$$  \hspace{1cm} (2.4)

$V_N$ stands for the volume of unitary group $U(N)$ and is unessential in what follows; $\Delta(X)$ and $\Delta(\Lambda)$ are Van-der-Monde determinants, e.g. $\Delta(X) = \prod_{i>j}(x_i - x_j)$. The transformation, leading from (2.3) to (2.4), is a trick, familiar from the study of multi-matrix models in the formalism of orthogonal polynomials [24, 7, 25], and we do not dwell upon this point. For what follows it is very important that the term $Tr\Lambda X$ in (2.3) is linear in $X$ (be it instead, say, $Tr\Lambda X^p$, the polynomial factor $\Delta(X)$ at the r.h.s. of eq.(2.4) would be substituted by $\Delta^2(X)/\Delta(X^p)$ and our reasoning below would be unapplicable).

The r.h.s. of (2.4) is proportional to the determinant of $N \times N$ matrix

$$\Delta^{-1}(\Lambda)\det_{(ij)} F_i(\lambda_j)$$  \hspace{1cm} (2.5)

with
\[ F_{i+1}(\lambda) \equiv \int dx \ x^i e^{-\nu(x)+\lambda x} = \left( \frac{\partial}{\partial \lambda} \right)^i F_1(\lambda). \] (2.6)

Note that

\[ F_1(\lambda) = F_{N=1}^{[\nu]}[\lambda]. \] (2.7)

If we recall that in GKM

\[ \Lambda = \nu'(M) \] (2.8)

and denote the eigenvalues of \( M \) through \( \{\mu_i\} \), eq.(2.6) acquires the form:

\[ F_{N}^{[\nu]}[\nu'(M)] = \frac{\det \Phi_i(\mu_j)}{\prod_{i>j}(\nu'(\mu_i) - \nu'(\mu_j))}, \] (2.9)

where

\[ \Phi_i(\mu) = F_i(\nu'(\mu)). \] (2.10)

**Denominator.** Proceed now to the denominator of (1.2):

\[ I_{N}^{[\nu]}[M] \equiv \int dX \ e^{-U_2(M,X)}. \] (2.11)

Making use of \( U(N) \)-invariance of Haar measure \( dX \) one can easily diagonalize \( M \) in (2.11). Of course, this does not imply any integration over angular variables and provide no factors like \( \Delta(X) \). Then for evaluation of (2.11) it remains to use the obvious rule of Gaussian integration,

\[ \int dX \ \ e^{-\sum_{i,j} U_{ij} X_{ij} X_{ji}} \sim \prod_{i,j} U_{ij}^{-1/2} \] (2.12)

(an unessential constant factor is omitted), and substitute the explicit expression for \( U_{ij}(M) \). If potential is represented as a formal series,

\[ \nu(X) = \sum_{n=1}^{\infty} \nu_n X^n, \] (2.13)

(and thus is supposed to be analytic in \( X \) at \( X = 0 \), eq.(1.4) implies that
\[ U_2(M, X) = \sum_{n=0}^{\infty} (n + 1) V_{n+1} \left\{ \sum_{a+b=n-1} \text{Tr} M^a X M^b X \right\}, \]

and

\[ U_{ij} = \sum_{n=0}^{\infty} (n + 1) V_{n+1} \left\{ \sum_{a+b=n-1} \mu_i^a \mu_j^b \right\} = \sum_{n=0}^{\infty} (n + 1) V_{n+1} \frac{\mu_i^n - \mu_j^n}{\mu_i - \mu_j} = \frac{V'(\mu_i) - V'(\mu_j)}{\mu_i - \mu_j}. \]

Coming back to (1.2), we conclude that

\[ Z_{\{V\}}[N][M] = e^{\text{Tr}[V(M)-M V'(M)]} \mathcal{F}_N[\{V'(M)\}] \Delta(M), \]

\[ \sim \left[ \det \Phi_i(\mu_j) \right] \prod_{i>j} \frac{U_{ij}}{(V'(\mu_i) - V'(\mu_j))} \prod_{i=1}^{N} s(\mu_i) = \frac{[\det \Phi_i(\mu_j)]}{\Delta(M)} \prod_{i=1}^{N} s(\mu_i). \quad (2.14) \]

Here \( s(\mu_i) \equiv [U_{ii}]^{1/2} e^{V(\mu_i)-\mu_i V'(\mu_i)}, \) i.e.

\[ s(\mu) = [V''(\mu)]^{1/2} e^{V(\mu)-\mu V'(\mu)} \quad (2.15) \]

The product of \( s \)-factors at the r.h.s. of (2.16) can be absorbed into \( \Phi \)-functions:

\[ Z_{\{V\}}[N][M] = \frac{[\det \Phi_i(\mu_j)]}{\Delta(M)} \quad (2.16) \]

where

\[ \Phi_i(\mu) = s(\mu) \Phi_i(\mu). \quad (2.17) \]

**Kac-Schwarz operator.** From eqs.(2.6),(2.10) and (2.15) one can deduce that \( \Phi_i(\mu) \) can be derived from the basic function \( \Phi_1(\mu) \) by the relation

\[ \Phi_i(\mu) = [V''(\mu)]^{1/2} \int (y + \mu)^{i-1} e^{-U(\mu,y)} dy = A_{\{V\}i}(\mu) \Phi_1(\mu), \quad (2.18) \]

where \( U(\mu,y) \) is defined by eq.(1.3) and \( A_{\{V\}}(\mu) \) is the first-order differential operator

\[ A_{\{V\}}(\mu) = s \frac{\partial}{\partial \lambda} s^{-1} = \frac{e^{V(\mu)-\mu V'(\mu)}}{[V''(\mu)]^{1/2}} \frac{\partial}{\partial \mu} \frac{e^{-V(\mu)+\mu V'(\mu)}}{[V''(\mu)]^{1/2}} = \frac{1}{V''(\mu)} \frac{\partial}{\partial \mu} + \mu - \frac{V''(\mu)}{2[V''(\mu)]^2}. \quad (2.19) \]
In the particular case (1.9)

$$A_{(K)}(\mu) = \frac{1}{k\mu^{k-1}} \frac{\partial}{\partial \mu} + \mu - \frac{k-1}{2k\mu^k}$$

coincides (up to the scale transformation of $\mu$ and $A_{(K)}(\mu)$) with the operator which determines the finite dimensional subspace of the Grassmannian in ref. [23] (we shall return to this point in sect.2.5). We emphasize that the property

$$\Phi_{i+1}(\mu) = A_{(V)}(\mu)\Phi_i(\mu) \quad (F_{i+1}(\lambda) = \frac{\partial}{\partial \lambda}F_i(\lambda))$$

(2.20)
is exactly the thing which distinguishes partition functions of GKM from expression (2.2) for generic $\tau$-function,

$$\tau_{\{\phi_i\}}^N[M] = \frac{[\text{det } \phi_i(\mu)]}{\Delta(M)}$$

(2.21)

with arbitrary sets of functions $\phi_i(\mu)$. In the next section we demonstrate that the quantity (2.21) is in fact a KP $\tau$-function in Miwa coordinates. Implications of additional constraint (2.20) will be discussed in sect.2.6 and sect.3.4.

$N$-dependence. Before we proceed to discussion of Miwa coordinates, two more things about the formula (2.21) deserve mentioning. First, the entire set $\{\phi_i(\mu)\}$ is naturally $N$-independent and infinite. It is reasonable to require also that $\phi_i$’s are linear independent and

$$\phi_i(\mu)/\phi_1(\mu) = c_i \mu^{i-1}(1 + o(\frac{1}{\mu})),$$

(2.22)

with $\mu$-independent $c_i$ (this is true for functions (2.7)). Then the set $\{\phi_i(\mu)\}$ is identified as projective coordinates of a point of Grassmannian, see sect.2.5. The r.h.s. of eq.(2.21) naturally represents $\tau_{\{\phi_i\}}^\infty[M]$ for an infinitely large matrix $M$. In order to return to the case of finite $N$ it is enough to require that all eigenvalues of $M$, except $\mu_1, \ldots, \mu_N$, tend to infinity. In this sense the function $\tau_{\{\phi_i\}}^N[M]$ in (2.21) is independent of $N$; the entire dependence on $N$ comes from the argument $M$: $N$ is the quantity of finite eigenvalues of $M$. As a simple check of consistency, let us additionally carry $\mu_N$ to infinity in (2.21), then, according to (2.22)
\[ \det_N \phi_i(\mu_j) \sim c_N \phi_1(\mu_N)(\mu_N)^{N-1} \cdot \det_{N-1} \phi_i(\mu_j) \cdot (1 + o(1/\mu_N)) \]

and

\[ \Delta_N(M) \sim (\mu_N)^{N-1} \Delta_{N-1}(M)(1 + o(1/\mu_N)), \]

so that

\[ \tau_{N,N}^{\{\phi_i\}} \sim \tau_{N-1}^{\{\phi_i\}} \cdot [c_N \phi_1(\mu_N)](1 + o(1/\mu_N)). \] (2.23)

This is the exact statement about the \( N \)-dependence of \( \tau_N \). Actually in GKM \( c_N \phi_1(\mu) = 1 \) at \( \mu \to \infty \). In what follows we often omit the subscript \( N \).

**Multiplicities.** The second remark concerns the form of the r.h.s. of (2.21), when some eigenvalues of \( M \) coincide (see also eq.(2.30) below). If eigenvalue \( \mu_i \) appears with the multiplicity \( p_i \), eq.(2.21) looks like

\[ \tau[(\mu_i, p_i)] \sim \frac{\det[\phi_i(\mu_j)A_{\{\nu\}}\phi_i(\mu_j) ... A_{\{\nu\}}^{p_i-1} \phi_i(\mu_j)]}{\prod_{i>j} (\mu_i - \mu_j)^{p_ip_j}}. \] (2.24)

Notation in this formula is not very transparent: it is implied that the matrix in the numerator has rows of the form

\[ \phi_i(\mu_1), A_{\{\nu\}} \phi_i(\mu_1), ... , A_{\{\nu\}}^{p_i-1} \phi_i(\mu_1), \]

\[ \phi_i(\mu_2), A_{\{\nu\}} \phi_i(\mu_2), ... , A_{\{\nu\}}^{p_i-1} \phi_i(\mu_2), \phi_i(\mu_3), A_{\{\nu\}} \phi_i(\mu_3), ... , \]

where operator \( A_{\{\nu\}} \) is defined by eq.(2.19). Expression (2.24) will be used in sect.2.4 in the proof of Hirota identity. Note also that if some \( \mu_i \to \infty \), this may be treated as vanishing of the corresponding multiplicity, \( p_i = 0 \). Such \( \mu_i \) obviously do not contribute to (2.24). Thus, the value of \( N \) (the finite size of the matrix \( M \) in GKM) may be interpreted as the number of \( \mu \)'s which appear with non-vanishing multiplicities.

### 2.3 KP \( \tau \)-function in Miwa parameterization

Generic KP \( \tau \)-function is a correlator of a special form:

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\[ \tau^G \{ T_n \} = \langle 0 | : e^{\sum T_n J_n} : G | 0 \rangle \]  
(2.25)

with

\[ J(z) = \bar{\psi}(z) \psi(z); \quad G = : \exp G_{mn} \bar{\psi}_m \psi_n : \]  
(2.26)

in the theory of free 2-dimensional fermionic fields \( \psi(z), \bar{\psi}(z) \) with the action \( \int \bar{\psi} \partial \psi \).

Vacuum states are defined by conditions

\[ \psi_n | 0 \rangle = 0 \quad n < 0, \quad \bar{\psi}_n | 0 \rangle = 0 \quad n \geq 0 \]  
(2.27)

where \( \psi(z) = \sum_z \psi_n z^n \, dz^{1/2}, \bar{\psi}(z) = \sum_z \bar{\psi}_n z^{-n-1} \, dz^{1/2} \).

The crucial restriction on the form of the correlator, implied by (2.26) is that the operator \( e^{\sum T_n J_n} : G \) is Gaussian exponential (exponent is quadratic in fields), so that insertion of this operator may be considered as modification of \( \bar{\psi} - \psi \) propagator, and Wick’s theorem is usually applicable. Namely, the correlators

\[ C_G(\{ \mu_i \}, \{ \lambda_j \}) \equiv \langle 0 | \prod_i \bar{\psi}(\mu_i) \psi(\lambda_i) G | 0 \rangle \]  
(2.28)

for any relevant \( G \) are expressed through the pair correlators of the same form:

\[ C_G(\{ \mu_i \}, \{ \lambda_j \}) = \det_{(ij)} C_G(\{ \mu_i, \lambda_j \}). \]  
(2.29)

Operator \( G \) in (2.28) can be safely substituted by the entire \( e^{\sum T_n J_n} : G \), but we do not need this for our purposes below. Instead with the help of Miwa transformation we shall express \( e^{\sum T_n J_n} : G \) in (2.25) through insertions of fermionic operators. Then (2.25) acquires the form of (2.28) and after that the application (2.29) provides a representation for \( \tau \)-function in determinant form, to be compared with eq.(2.21) for partition function of GKM.

There are slightly different forms of Miwa transformation. Usually one writes

\[ T_n = \frac{1}{n} \sum_i p_i \frac{1}{\mu_i^h} \]  
(2.30)
and treats the r.h.s. as an integral $\int \frac{p(\mu)}{\mu^n}$ over Riemann sphere with coordinate $\mu$ and singular function $p(\mu) = \sum_i p_i \delta(\mu - \mu_i)$. We use instead the transformation (2.1):

$$T_n = \frac{1}{n} Tr \ M^{-n} = \frac{1}{n} \sum_i \frac{1}{\mu_i^n},$$

interpreting $\mu_i$'s as eigenvalues of the matrix $M$, all coming with unit multiplicities $p_i$ (while $p_i > 1$ in (2.30) may be interpreted as result of coincidence of $p_i$ eigenvalues). The form (2.30) of Miwa transformation is preferable from the point of view of invertibility and also is convenient for the proof of Hirota identities (see sect.2.4 below).

The simplest way to understand what happens to the operator $e \sum T_n J_n$ after the substitution of (2.31), is to use the free-boson representation of the current $J(z) = \partial \varphi(z)$. Then $\sum T_n J_n = \sum \left\{ \sum \frac{1}{n \cdot \mu_i^n} \varphi_n \right\} = \sum \varphi(\mu_i)$, and

$$e \sum \varphi(\mu_i) := \frac{1}{\prod_{i < j}(\mu_i - \mu_j)} \prod_i : e^{\varphi(\mu_i)} : .$$

In fermionic representation it is better to start from

$$T_n = \frac{1}{n} \sum_i \left( \frac{1}{\mu_i^n} - \frac{1}{\tilde{\mu}_i^n} \right)$$

instead of (2.31). Then

$$e \sum T_n J_n := \frac{\prod_{i,j}^{N} (\tilde{\mu}_i - \mu_j)}{\prod_{i,j} (\mu_i - \mu_j) \prod_{i,j} (\tilde{\mu}_i - \tilde{\mu}_j)} \prod_i \tilde{\psi}(\tilde{\mu}_i) \psi(\mu_i) .$$

In order to come back to (2.31) and (2.32) it is necessary to shift all $\tilde{\mu}_i$'s to infinity. This may be expressed by saying that the left vacuum in (2.25) is substituted by

$$\langle N \rangle \sim \langle 0 | \tilde{\psi}(\infty) \tilde{\psi}'(\infty) \cdots \tilde{\psi}^{(N-1)}(\infty) \rangle .$$

The $\tau$-function (2.25) can be represented in various forms:

$$\tau_N^G[M] = \langle 0 | e \sum T_n J_n : G | 0 \rangle = \Delta(M)^{-1} \langle N | \prod \varphi(\mu_i) : G | 0 \rangle = \lim_{\tilde{\mu}_i \to \infty} \frac{\prod_{i,j} (\tilde{\mu}_i - \mu_j)}{\prod_{i,j} (\mu_i - \mu_j) \prod_{i,j} (\tilde{\mu}_i - \tilde{\mu}_j)} \langle 0 \prod \tilde{\psi}(\tilde{\mu}_i) \psi(\mu_i) G | 0 \rangle = \lim_{\tilde{\mu}_i \to \infty} \frac{C_G(\{\tilde{\mu}_i\}, \{\mu_i\})}{C_I(\{\tilde{\mu}_i\}, \{\mu_i\})} .$$

(2.35)
and applying the Wick’s theorem (2.29) we obtain:

$$\tau_{N}^{G}[M] = \lim_{\mu_{j} \to \infty} \det_{(ij)} \frac{C_{G}(\tilde{\mu}_{i}, \mu_{j})}{C_{I}(\tilde{\mu}_{i}, \mu_{j})} = \frac{\det \phi_{i}(\mu_{j})}{\Delta(M)}$$  \hspace{1cm} (2.36)

with functions

$$\phi_{i}(\mu) \sim \langle 0 | \tilde{\psi}(t^{-1}) (\infty) \psi(\mu) G | 0 \rangle.$$  \hspace{1cm} (2.37)

Thus, we proved that KP \(\tau\)-function in Miwa coordinates (2.2) has the determinant form (2.1).

Below we shall need a more detailed expression of the right hand side in the eq.(2.35), when all points \(\tilde{\mu}_{i}\) tend to the same value \(1/t\) with fixed \(t \neq 0\). From (2.35) one can obtain:

$$\tau_{N}^{G}[t|M] \equiv \frac{\Pi_{i}(1-t\mu_{i})^{N}}{\Pi_{i>j}(\mu_{i}-\mu_{j})} \det \left\{ \frac{1}{(i-1)!} \partial_{t}^{i-1} \langle 0 | \tilde{\psi}(t^{-1}) \psi(\mu_{j}) G | 0 \rangle \right\} =$$

$$= \frac{\Pi_{i}(1-t\mu_{i})^{N}}{\Pi_{i>j}(\mu_{i}-\mu_{j})} \det \left\{ \frac{1}{(i-1)!} \partial_{t}^{i-1} \langle 0 | : e^{\Sigma_{n=1}^{k} (\mu_{j}^{-n-} t^n) J_n} : G | 0 \rangle \right\}$$  \hspace{1cm} (2.38)

In the limit of \(t \to 0\) we obtain the formula (2.37) with

$$\phi_{i}(\mu) = \lim_{t \to 0} \phi_{i}(\mu, t) = \frac{1}{(i-1)!} \lim_{t \to 0} \partial_{t}^{i-1} \langle 0 | : e^{\Sigma_{n=1}^{k} (\mu_{j}^{-n-} t^n) J_n} : G | 0 \rangle.$$  \hspace{1cm} (2.39)

Eqs.(2.38) and (2.39) will be used in sect.3.2 in one of our derivations of the \(\mathcal{L}_{-1}\)-constraint.

One more remark is in order now. From the form of the operator expansion for \(\tilde{\psi}(\infty) \psi(\mu)\) it is clear that the function \(\phi_{i}(\mu)\), as given by (2.39), behaves as \(\mu^{i-1}(1+o(1/\mu))\) when \(\mu \to \infty\), in accordance with (2.37), and all the functions \(\phi_{i}(\mu)\) are independent. This last statement should be more or less clear from the fact, that the set \(\{\phi_{i}(\mu)\}\) contains all the information about fermionic propagator \(C_{G}(\tilde{\mu}, \mu)\), which is exactly the same as the information in operator \(G\), which should be of the form (2.26), and defines the actual fermionic action. To make this mutual independence of \(\phi\)'s even more transparent we prove in the next section 2.4 that Hirota identities are satisfied by the quantity (2.36) with any set of functions \(\{\phi_{i}(\mu)\}\). The possibility to choose \(\{\phi_{i}(\mu)\}\) in any way is important for
us to argue without going into any more details, that partition function of GKM, which was expressed in determinant form (2.36) with specific choice of \( \phi_i = \Phi_i \), is indeed a KP \( \tau \)-function.

### 2.4 Hirota equations for \( \tau \)-function in Miwa coordinates

The usual form of bilinear Hirota equations, which are the defining equations of KP \( \tau \)-functions, is:

\[
\sum_{i=0}^{\infty} \mathcal{P}_i (\cdot) \mathcal{P}_{i+1} (\hat{D}_T) e^{[\sum_i u_i D_{T_i}] \cdot \tau} = 0
\]

where \( D_T \) are Hirota symbols, \( \hat{D} \equiv (D_{T_1}, \frac{1}{2} D_{T_2}, \ldots) \) and \( \mathcal{P}_i \) are Schur polynomials.

Note that these equations are in fact more than KP equations themselves, which describe evolution of the functions \( u_i(T_n) \) (\( u(T_n) \equiv u_2(T_n) \equiv \frac{\partial^2 \log \tau}{\partial T^2_1} \) and all other \( u_i(T_n) \) can be determined from the relations \( \frac{\partial}{\partial T_1} \frac{\partial}{\partial T_n} \log \tau = (L^n)_{-1} \) with \( L \equiv \partial + \sum_i u_{i+1} \partial^{-i} \) and \( \partial \equiv \frac{\partial}{\partial T_1} \) — see (3.35) and (3.36)) rather than \( \tau \)-itself. Indeed, any transformation

\[
\tau(T_n) \rightarrow \tau(T_n)e^{H(T_2, T_3, \ldots) + T_1 \tilde{H}}
\]

(2.40)

with \( \partial H / \partial T_1 = 0 \) and \( \tilde{H} = \text{const} \) does not change \( u_i(T_n) \), i.e. \( H \) and \( \tilde{H} \) can not be fixed by the entire set of KP equations. They are, however, fixed by Hirota equations (up to linear function \( H + T_1 \tilde{H} = \sum b_n T_n \), \( b_n = \text{const} \)). This remark is important for the subject of matrix models when ever interpolation between critical points is considered.

For example, one can use the representation

\[
\tau(T_n) = \exp \left\{ \int_{T_1}^{T_n} (T_1 - x) u(x, T_2 \ldots) dx + H(T_2, T_3 \ldots) + T_1 \tilde{H} \right\}
\]

(2.41)

and at any given multicritical point the functions \( H, \tilde{H} \) are unessential. However, they are very important for interpolations, and, in particular, for the relevance of Virasoro and \( W \)-constraints.

Our first task in this section is to rewrite the Hirota equations in Miwa coordinates (2.1). Since this is a widely known transformation (see, for example [26]), we just cite here the answer: Hirota equations in Miwa coordinates state that
\begin{align*}
(\mu_a - \mu_b)\tau(p_a, p_b, p_c + 1)\tau(p_a + 1, p_b + 1, p_c) + \\
+ (\mu_b - \mu_c)\tau(p_a + 1, p_b, p_c)\tau(p_a, p_b + 1, p_c + 1) + \\
+ (\mu_c - \mu_a)\tau(p_a, p_b + 1, p_c)\tau(p_a + 1, p_b, p_c + 1) &= 0, \\
\end{align*}
(2.42)

where \(\tau\)-function is expressed through the Miwa variables (2.30) according to eq.(2.24) and three arbitrary eigenvalues from the set \(\{\mu_i\}\) with corresponding multiplicities are chosen.

The second task is to prove that \(\tau[M] = \Delta(M)^{-1}\det \phi_i(\mu_j)\) with any \(\{\phi_i(\mu)\}\) satisfies (2.42).

All we need to derive the desired equations is the well known Jacobi identity for the determinants. For any \((N + 2) \times (N + 2)\) determinant \(J\) we denote by \(J(i_1 \ldots i_s, j_1 \ldots j_s)\) the determinant obtained from \(J\) with \(s\) rows \(i_1, \ldots, i_s\) and \(s\) columns \(j_1, \ldots, j_s\) omitted.

Then the Jacobi identity reads

\[
J(i,j)J = J(i,j) - J(i,j)J(i,j).
\]

Let us consider \(\tau_N(\{\mu\}) = \tau_N(\mu_1, \ldots, \mu_N)\) and divide the given \(a \text{ priori}\) different set of eigenvalues \(\mu_1, \ldots, \mu_N\) into the \(L\) “clusters” of sizes \(p_1, \ldots, p_L\):

\[
\mu_1, \ldots, \mu_{p_1}; \mu_{p_1+1}, \ldots, \mu_{p_1+p_2}; \ldots; \mu_{p_1+\ldots+p_{L-1}+1}, \ldots, \mu_{p_1+\ldots+p_L},
\]

where \(\sum_j p_j = N\). Then one should introduce two additional eigenvalues \(\mu_{N+1}, \mu_{N+2}\) and apply the Jacobi identity for \(i = N+1, j = N+2\) to the function \(\tau_{N+2}(\{\mu\}, \mu_{N+1}, \mu_{N+2})\).

Then simple calculation gives the following system of equations:

\[
\tau_{N+2}(\{\mu\}, \mu_{N+1}, \mu_{N+2})\tau_N(\{\mu\}) = \\
= \frac{1}{\mu_{N+1} - \mu_{N+2}} \left\{\tau_{N+1}(\{\mu\}, \mu_{N+2})\hat{\tau}_{N+1}(\{\mu\}, \mu_{N+1}) - \right. \\
\left. -\tau_{N+1}(\{\mu\}, \mu_{N+1})\hat{\tau}_{N+1}(\{\mu\}, \mu_{N+2})\right\}
\]
(2.43)

where \(\hat{\tau}_{N+1}\) denotes some new \(\tau\)-function which is obtained from the given \(\tau_{N+1}\) by a change of the last row: \(\phi_{N+1} \rightarrow \phi_{N+2}\). Now let all the eigenvalues in each cluster tend
to the values $\mu_1, \ldots, \mu_L$ respectively and $\mu_{N+1} \to \mu_a$, $\mu_{N+2} \to \mu_b$ where $\mu_a$ and $\mu_b$ belong to different clusters. Then eq.(2.43) acquires the form (in the notations as in eq.(2.42)):

$$
(\mu_a - \mu_b)\tau(p_a + 1, p_b + 1, p_c)\tau(p_a, p_b, p_c) = \\
= \tau(p_a, p_b + 1, p_c)\hat{\tau}(p_a + 1, p_b, p_c) - \tau(p_a + 1, p_b, p_c)\hat{\tau}(p_a, p_b + 1, p_c),
$$

(2.44)

where $p_c$ describes some arbitrary third cluster different from the previous ones. One can multiply this equation by the factor $\frac{\tau(p_a, p_b, p_c + 1)}{\tau(p_a, p_b, p_c)}$ and write down the couple of two another equations obtained by cyclic permutations among indices $(a, b, c)$. The sum of these three equations coincides with eq.(2.42).

Another interesting expression can be derived from the eq.(2.44) as follows. Let us make the shift $p_a \to p_a - 1$, $N \to N - 1$ and put $p_b = 0$ (this means that we have the cluster with the single element $\mu_b$ in the corresponding $\tau$-functions in eq.(2.44)). Then in the limit $\mu_b \to \mu_a$ we have $\tau(p_a, p_b + 1) \to \tau(p_a + 1)$ (notations of other clusters will be omitted since they don’t changed) and equation now takes the form

$$
\tau_{N+1}(p_a + 1)\tau_{N-1}(p_a - 1) = \\
= \tau_N(p_a) \lim_{\mu_b \to \mu_a} \frac{\partial}{\partial \mu_b} \hat{\tau}_N(p_a, \mu_b) - \hat{\tau}_N(p_a) \lim_{\mu_b \to \mu_a} \frac{\partial}{\partial \mu_b} \tau_N(p_a, \mu_b).
$$

Simple calculation gives further that

$$
\lim_{\mu_b \to \mu_a} \frac{\partial}{\partial \mu_b} \tau_N(p_a, \mu_b) = \frac{1}{p_a} \frac{\partial}{\partial \mu_a} \tau_N(p_a)
$$

and now we obtain

$$
p_a\tau_{N+1}(p_a + 1)\tau_{N-1}(p_a - 1) = \tau_N^2(p_a) \frac{\partial}{\partial \mu_a} \hat{\tau}(p_a). \quad (2.45)
$$

This equation holds for every function in the form (2.21). More concrete expression can be obtained for the GKM partition function $\mathcal{F}_N^{(V)}$ defined by eq.(2.9) in the terms of $\lambda$’s variables (see eq.(2.8)). For this quantity the following relation holds:

$$
\hat{\mathcal{F}}_N^{(V)} = \sum_{c=1}^{L} \frac{\partial}{\partial \lambda_c} \mathcal{F}_N^{(V)}.
$$
Therefore we have

\[ p_a \mathcal{F}_{N+1}^{\{v\}}(p_a + 1) \mathcal{F}_{N-1}^{\{v\}}(p_a - 1) = \]

\[ = \mathcal{F}_{N}^{\{v\}}(p_a) \sum_{b=1}^{L} \frac{\partial^2}{\partial \lambda_a \partial \lambda_b} \mathcal{F}_{N}^{\{v\}}(p_a) - \frac{\partial}{\partial \lambda_a} \mathcal{F}_{N}^{\{v\}}(p_a) \sum_{b=1}^{L} \frac{\partial}{\partial \lambda_b} \mathcal{F}_{N}^{\{v\}}(p_a) . \]

We should remark that in the case of \( p_1 = N, \ p_a = 0, \ a \geq 1 (\lambda_i \equiv \lambda) \) this equation reduces to the Toda-chain one. Indeed, now \( \mathcal{F}_{N}^{\{v\}}(p_a) \) has very simple determinant form

\[ \mathcal{F}_{N}^{\{v\}}(\lambda) = \frac{1}{(N - 1)!} \det \partial^{i+j} F_1(\lambda) . \]

It is just determinant of matrix with specific Toda-chain symmetry of constant entries along anti-diagonal [27]. From the other point of view, matrix integral (1.1) reduces in the present case (of proportional to unit matrix \( \Lambda \)) to standard discrete model partition function which is well-known to correspond to Toda-chain hierarchy [28, 17]. If we return to the original description in the terms of \( \mu \)'s (see eqs.(2.16), (2.24)), then our final equation is

\[ p_a \frac{\tau_{N+1}(p_a + 1)\tau_{N-1}(p_a - 1)}{\mathcal{V}(\mu_a)} = \tau^2(p_a) \sum_{b=1}^{L} \frac{\partial^2}{\partial \lambda_a \partial \lambda_b} \log \frac{\tau_N(p_a)}{N} , \quad (2.46) \]

\[ \mathcal{N} = \left( \prod_{m>n} (\mu_m - \mu_n)^{p_m p_n} \right)^{-1} \prod_{n=1}^{L} \left( [\mathcal{V}''(\mu_n)]^{1/2} e^{\mathcal{V}(\mu_n) - \mu_n \mathcal{V}'(\mu_n)} \right)^{p_n} . \]

We shall return to eq.(2.46) in sect.3.3 in the context of the string equation.

### 2.5 Grassmannian description of KP \( \tau \)-function

In this section we present the necessary material about infinite dimensional Grassmannians and their connection to integrable hierarchies. The details and proofs can be found in [29, 30, 31].

**Definitions.** Let us consider the infinite dimensional Hilbert space \( H \) realized as the space of Laurent series \( \phi(\mu) \) on the unit circle \( S^1(\mu \in S^1) \) on the complex plane. This space is naturally represented as the direct sum \( H = H_+ \oplus H_- \), where \( H_+ (H_-) \) is spanned
by the vectors \( \{ \mu^n \}, n \geq 0 \) \( \{ \mu^{-n} \}, n > 0 \), \( n \in \mathbb{Z} \). These vectors form the orthogonal basis in \( H \): \( \langle \mu^n, \mu^m \rangle = \delta_{nm} \).

The infinite dimensional Grassmannian \( Gr \) is the set of subspaces \( W \in H \) (Grassmannian point) satisfy the two conditions:

1) the orthogonal projector \( pr_+ : W \rightarrow H_+ \) is Fredholm operator, \( i.e. \) the kernel and co-kernel are finite dimensional spaces;

2) The orthogonal projector \( pr_- : W \rightarrow H_- \) is compact operator.

This definition corresponds to Sato’s Grassmannian rather than Segal-Wilson one, where \( \phi(\mu) \) should be convergent in some neighborhood of \( \infty \).

The index of \( pr_+ \) is called the relative dimension of \( W \):

\[
\text{ind } pr_+ = \dim(\ker pr_+) - \dim(\text{coker } pr_+).
\]

Grassmannian is decomposed into the direct sum of the connected components:

\[
Gr = \bigoplus_n Gr_n, \tag{2.47}
\]

where \( Gr_n \) consists of \( W \)'s with the relative dimension \( n \). For our purposes it is sufficient to deal with \( Gr_0 \).

It is convenient to represent linear operators acting in \( H \) as block matrices with respect to the decomposition \( H_+ \oplus H_- \):

\[
\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \alpha \in \text{GL}(\infty), \tag{2.48}
\]

where the operators \( a, b, c, d \) act as follows: \( a : H_+ \rightarrow H_+, \) \( b : H_- \rightarrow H_+, \) and so on. Clearly, \( b \) and \( c \) should be compact operators.

For applications to integrable hierarchies the following commutative subgroup \( \Gamma \subset \text{GL}(\infty) \) is of importance. It consists of the mappings from \( S^1 \) to non-zero complex numbers and acts on \( H \) by multiplications of the Laurent series. There are two natural commutative subgroups \( \Gamma_+ \) and \( \Gamma_- \) in \( \Gamma \): the elements of \( \Gamma_+(\Gamma_-) \) can be analytically continued inside (outside) \( S^1 \). Their elements \( \gamma_+(\mu) \in \Gamma_+ \) and \( \gamma_- (\mu) \in \Gamma_- \) can be parameterized through the infinite sets \( \{ T_k \} \) and \( \{ \tilde{T}_k \} \) \( (k \geq 1) \) of independent variables (“times”):
\[
\gamma_+(\mu) = \exp\left(\sum_{k \geq 1} T_k \mu^k\right), \quad \gamma_-(\mu) = \exp\left(\sum_{k \geq 1} \tilde{T}_k \mu^{-k}\right).
\]

Let \(\{\phi_n\} \ (n \geq 0)\) be a basis in \(W\). It is called \textit{admissible} if the operator transforming \(\mu^n\) to \(pr_+(\phi_n)\) in \(H_+\) has a well-defined determinant.

We can write

\[
\phi_n = \sum_{k=1}^\infty (\phi_+ \phi_k \mu^{k-1}) + \sum_{k=1}^\infty (\phi_- \phi_k \mu^{-k}),
\]

where \(\phi_+\) has a determinant, \(\phi_-\) is compact. In particular, for \(W \in Gr_0\) it is always possible to choose \(\phi_+^{(c)}_{mn} = \delta_{mn}\). In this case we call this basis \(\phi^{(c)}\) \textit{canonical} for \(W\). Generally, the convenient choice for admissible \(\phi_+\) (to be used below) is a lower-triangular matrix with unit diagonal elements: \((\phi_+)_{nk} = 0\) if \(n < k\). In this case

\[
\phi_k(\mu) = \mu^{k-1}(1 + O(\mu^{-1})), \quad k \geq 1
\]

(compare to (2.22)).

The \(\tau\)-function of KP hierarchy \(\tau_W\) is defined as determinant of the orthogonal projection \(\gamma_+ W \to H_+\):

\[
\tau_W(\{T_k\}) = \det(pr_+ : \gamma_+(\{T_k\}) W \to H_+),
\]

or, more explicitly,

\[
\tau_W(\{T_k\}) = \det(1 + a^{-1} b B),
\]

where \(\gamma_+\) is represented in the block form (2.48), \(B = \phi_- (\phi_+)^{-1}\) (see (2.50)).

Given a Riemann surface of finite genus one can associate with it a point in \(Gr\) by constructing the admissible basis according to the well-known rules [30]. In this case the formal Laurent series for the basis vectors are in fact convergent. Our situation is different: our basis vectors are asymptotic (formal) series which do not need to converge. This should be interpreted as adequate generalization to infinite genus Riemann surfaces. In this sense we often identify \(Gr\) with the universal module space (of line bundles over Riemann surfaces with punctures).
Instead of $W$ one can work with the equivalent space $\gamma_-(\{\tilde{T}_k\})W$, the $\tau$-function being changed in the trivial way:

$$
\tau_{\gamma_-(\{T_k\})} = \exp\left(\sum_{k=1}^{\infty} kT_k\tilde{T}_k\right)\tau_W(\{T_k\}).
$$

Let us note that in Miwa variables (2.30) the “evolution factor” $\gamma_+(\mu)$ (2.49) looks as a product of pole factors:

$$
\gamma_+(\mu) = \prod_j (1 - \frac{\mu}{\mu_j})^{-p_j}.
$$

The various reductions of the KP hierarchy can be described in these terms as follows. Let $f(\mu)$ be a Laurent series in $\mu$. Those $W$’s which satisfy the condition

$$
f(\mu)W \subset W
$$

give rise to solutions of the “$f$-reduction” of KP. In particular, the choice $f(\mu) = \mu^K$ corresponds to the well known $K$-reductions (KdV, Boussinesq, ...) which are in fact associated with $A_{K-1}$ Lie algebra [32]. In this case the $\tau$-function does not depend on $T_n$ with $n = 0 \mod K$ (modulo trivial exponential factors linear in times like that in (2.54)).

An admissible (but non-canonical) basis can be chosen in such a way that

$$
\phi_{K+n}(\mu) = \mu^K \phi_n(\mu), \ n \geq 1.
$$

Vise versa, if we have a basis of the form (2.56), the corresponding point of $Gr$ leads to a solution of the $K$-reduced KP hierarchy. In the next section 2.6 we show that the GKM (1.2) with arbitrary polynomial potential $V(x)$ corresponds to the $V'(\mu)$-reduction of the KP hierarchy.

Note that $K$-reduction is equivalent to existence of some admissible basis with the property (2.56). However, other admissible bases (including canonical) in this case satisfy a weaker condition:

$$
\phi_{K+n}(\mu) = \mu^K \cdot \sum_{i=1}^{n} a_i \phi_i
$$

with some constant $a_i$’s.
Fermionic formalism. Grassmannian approach is in fact equivalent to the fermionic language used in the section 2.3. Let us briefly describe the relation between them.

In the fermionic interpretation $H$ becomes the one-particle Fock space spanned by $\tilde{\psi}_n$. The fermionic operator $G$ of the form (2.26) makes linear transformations in $H$ according to

$$\tilde{\psi}_n \rightarrow G\tilde{\psi}_n G^{-1} \in H.$$ 

The positive (negative) modes of the current $J(\mu)$ (2.26) are the generators of $\Gamma_+ (\Gamma_-)$. The matrix $G_{mn}$ in (2.26) can be expressed through the canonical basis $\phi_k^{(c)}(\mu)$ of the corresponding $W$ as follows:

$$\delta_{mn} + G_{mn} = \frac{1}{(2\pi i)^2} \oint \frac{d\mu d\nu}{\mu \nu} \mu^m \nu^n G_W(\mu, \nu), \tag{2.57}$$

$$G_W(\mu, \nu) = \sum_{k=1}^{\infty} \mu^{-k} \phi_k^{(c)}(\nu). \tag{2.58}$$

A more invariant expression is

$$G_W(\mu, \nu) = \frac{1}{\mu - \nu} \tau_W(\{\frac{\nu^{-k} - \mu^{-k}}{k}\}).$$

Eqs.(2.57) and (2.58) are very important for any interpretation in terms of (infinite-genus) Riemann surfaces, in particular, for identification of GKM with appropriate Liouville-like models of 2d gravity. Explicit formulae for $G_{\nu \nu}$ can be easily derived, at least, in the case of $\nu = \nu_K$.

$\tau$-functions of reduced hierarchies. Let us note that in the fermionic language it is trivial to understand that the $K$-reduction condition leads to the $\tau$-function $\tau^{(K)}$ almost independent of times $T_{nK}$, i.e. the property (1.21). Indeed, (2.56)–(2.58) imply that the condition of such reduction can be transformed into the matrix $\{G_{mn}\}$:

$$[G, E^K] = 0,$$

where $E$ is the shift matrix, $(E_{mn}) = \delta_{m+1,n}$. On the same “infinite-matrix language” the elements of $\Gamma_\pm$ can be parameterized as $\gamma_+ = \exp(\sum T_k E^k)$, $\gamma_- = \exp(\sum \tilde{T}_k E^{-k})$, i.e. the
currents $J_k$ from eq. (2.26) act as $E^k$. Now it is evident that exponent of currents $J_k$ with $k = 0 \mod K$ can be pulled to the right vacuum and canceled, eliminating the dependence of the corresponding times (up to the exponential factor like that in eq. (2.54) — this is due to the freedom to insert the exponential $\gamma_- = \exp(\sum \tilde{T}_k E^{-k})$ between $G$ and $\gamma_+$).

Now let us discuss the freedom in the definition of the $\tau$-function from the viewpoint of Hirota equation. As it was pointed out in (2.54) there is a freedom to multiply the $\tau$-function by a linear exponential of times. Another possible modification is dealing with other components $Gr_n$ of Grassmannian. It corresponds to the shift of the first basis vector or, equivalently, multiplying it by $\mu^n$. All these can be trivially understood in terms of Hirota equation in Miwa parameterization (2.42). Indeed, it is invariant with respect to multiplying by the product $\prod_i f(\mu_i)$, where $f(\mu)$ is arbitrary Laurent series. If the first term of this series $\sim \mu^0$, one can expand $f(\mu_i) \to \exp\{\sum_k \mu_i^{-k} \tilde{T}_k\}$ and reproduce correct factor in eq. (2.54). If the first term behaves as $\mu^n$, one should deal with $Gr_n$. Thus, in Miwa variables we readily describe all full freedom in the definition of the $\tau$-function.

**GKM and specific points in Grassmannian.** To conclude this subsection let us note that the problem of correspondence between matrix models and the points of Grassmannian has been already addressed in ref. [23]. That paper examined the implications of $\mathcal{L}_{-1}$-constraint in the double-scaling limit of $K$-matrix model and concluded that the corresponding point of Grassmannian is defined by the set of $K$ linear independent vectors $\{A^i \phi_1\}$, where operator $A$ for $V = \frac{X^{K+1}}{K+1}$ essentially coincides with ours (see (2.19), (2.20)). Moreover, from our study of GKM we obtain straightforwardly the explicit form of all basis vectors for any potentials: $\phi_i = \Phi_i$ as given in eq. (2.18). Note also that in [23] a shift of time variables analogous to ours in (1.17) was introduced.

As to ref. [23], we would explain the appearance of the Airy functions (in the case of $K = 2$ model) from constraint (1.20) in the following way. If we put all $\tilde{T}_k = 0$ except for $\tilde{T}_1$ this constraint can be satisfied only if the $\tau$-function is zero. This means that the relevant point of Grassmannian is singular. Instead, we can put $\tilde{T}_3 = c \neq 0$ keeping all other $\tilde{T}_k$ with $k \geq 5$ equal to zero. In this case we have

$$\frac{3}{2} c \cdot \frac{\partial \log \tau}{\partial T_1} + \frac{1}{4} T_1^2 = 0$$
Choosing $c = -2/3$, so that $T_3 = \dot{T}_3 + 2/3 = 0$, and taking the derivative with respect to $T_1$ and using the relation

$$u(T_1) = \frac{\partial^2 \log \tau}{\partial T_1^2},$$

where $u(T_1)$ is the KdV potential, we obtain

$$2u(T_1) = T_1.$$

According to the ideology of the inverse scattering method in the case of 2-reduction we should consider the Schrödinger equation for this potential:

$$L^2 \Psi = \frac{\partial^2 \Psi}{\partial T_1^2} + 2u \Psi = \mu^2 \Psi$$

i.e.

$$\frac{\partial^2 \Psi}{\partial T_1^2} + T_1 \Psi = \mu^2 \Psi.$$ (2.59)

Solving this equation, we obtain Airy function

$$\Psi(\mu | T_1) \big|_{T_k \geq 3 = 0} = \Psi_0(\mu)Ai(-T_1 + \mu^2) \equiv \Psi_0(\mu) \int e^{-x^3/3 + (\mu^2 - T_1)x} dx.$$ The wave function $\Psi$ is nothing but the Baker-Akhiezer function. $\Psi_0$ is adjusted from normalization requirement $\Psi(\mu)e^{-\sum T_n\mu^n} \to 1$ as $\mu \to \infty$. In general $\Psi$ can be expressed through $\tau$-function as \[33\]

$$\Psi(\mu | T_n) = e^{\sum T_n \mu^n} \frac{\tau(T_n - \mu^{-n}/n)}{\tau(T_n)}.$$ (2.60)

(see sect.3.3 below for more details). In eq.(2.59) we have $\Psi$-function of the form (2.60) with $T_k = 0, k \geq 3$. The admissible basis for the corresponding point of $Gr_0$ can be constructed by means of the $\Psi$-function as follows \[30\]:

$$\phi_k(\mu) \sim \left. \frac{\partial^{k-1} \Psi}{\partial T_1^{k-1}} \right|_{\text{all } T_k = 0}.$$ The basis vectors, constructed according to this expression, obviously coincide with those which were obtained in sect.2.2 (eq.(2.6)) for the particular case of $\mathcal{V}(X) = \frac{1}{3}X^3$. 33
For generic $K$ the role of (2.59) is played by a $K$-th order differential equations ($L^K\Psi = \mu^2\Psi$) and the result is the Airy function of level $K$ as defined in (2.6).

## 2.6 GKM and reductions of KP-hierarchy

We prove here that whenever potential $\mathcal{V}(X)$ in GKM is a homogeneous polynomial of degree $K + 1$, i.e. $\mathcal{V}(X) = \text{const} \cdot X^{K+1}$, the partition function $Z^{\{\mathcal{V}\}}[M]$, which is a KP $\tau$-function, in fact can be treated as a $\tau$-function $\tau^{\{K\}}$ of $K$-reduced KP-hierarchy.

For this purpose let us return to the section 2.2 and note that because of (2.7) $F_1(\lambda)$ satisfies the Ward identity (1.23):

$$[\mathcal{V}'(\partial/\partial \lambda) - \lambda]F_1(\lambda) = 0. \quad (2.61)$$

If the potential $\mathcal{V}(X)$ is a polynomial of finite degree $K + 1$, then (2.6) may be used to express $F_{K+1}$ in the form of linear combination of $F_i$'s with $i \leq K$. Namely, if $\mathcal{V}(X) = -\sum_{i=1}^{K+1} V_i X^i$, $V_{K+1} = \frac{1}{K+1}$, eq.(2.6) implies, that

$$F_{K+1} = -\sum_{i=1}^{K} i V_i F_i + \lambda F_1. \quad (2.62)$$

Clearly, all the terms on the r.h.s. of (2.62) except for the last one $\lambda F_1$ will drop out of the determinant (2.9), i.e. in this case $F_{K+1}$ may be defined as $\lambda F_1$ rather than by eq.(2.62). Obviously in the same way any $F_{K+m}$ may be substituted by $\lambda F_m$, while any $F_{nK+m} - by \lambda^n F_m$. In other words, $F_{N}[A]$ is given by eq.(2.9) with the first $K$ functions $F_1 \ldots F_K$ defined by (2.6), while other elements of the basis are given by the recurrent relation:

$$F_{nK+m} \sim \lambda^n F_m. \quad (2.63)$$

Note that this is true when $\mathcal{V}(X)$ is any potential of degree $K + 1$, not obligatory homogeneous. However, we should recall, that $\lambda = \mathcal{V}'(\mu)$ and $\mu$ rather than $\lambda$ is the proper parameter to deal with in Grassmannian picture. Therefore eq.(2.63) implies the appearance of KP-reduction, associated with the function $\mathcal{V}'(\mu)$. If $\mathcal{V}$ is further restricted to be $\mathcal{V}(X) = \frac{X^{K+1}}{K+1}$, $\lambda = \mu^K$, and (2.63) acquires the form:
\[ F_{nK+m} \sim \lambda^n F_m = \mu^{nK} F_m, \] (2.64)

which is characteristic of conventional \( K \)-reduction.

As we already explained, \( K \)-reduction implies the relation (1.21) with some factor of the form (1.22). However, even this factor is absent in the case of GKM (1.10). For particular case of \( V(X) = \text{const} \cdot X^3 \) \((K = 2)\) this was exhaustively proved in [13], using the properties of symplectic structure on (the model of) the universal module space. Since interpretation of entire GKM in such terms is not yet available, the analog of such proof for generic \( K \) is still lacking.

### 2.7 On \( T_{nK} \)-independence of \( Z^{(K)} \)

**Reformulation of the problem.** First of all, let us choose our variables \( \{\mu_i\} \) in such a way that the only non-vanishing times \( T_n = \frac{1}{n} \sum_i \mu_i^{-n} \) are those with \( n = 0 \mod K \). To do this it is enough to have only \( K \) finite parameters \( \mu_1, \ldots, \mu_K \), which are essentially \( K \)-th order roots of unity:

\[ \mu_j = \mu \varepsilon^j, \varepsilon = e^{2\pi i/K}, j = 1 \ldots K. \] (2.65)

For such choice of variables the formula (1.21),

\[ \tau^{(K)}(T_1 \ldots T_{K-1}, T_K, T_{K+1} \ldots T_{2K-1}, T_{2K}, T_{2K+1} \ldots) = e^{\sum_n a_n K T_n K \tau^{(K)}}(T_1 \ldots T_{K-1}, 0, T_{K+1} \ldots T_{2K-1}, 0, T_{2K+1} \ldots), \] (2.66)

turns into:

\[ \tau^{(K)}(0 \ldots 0, T_K, 0 \ldots 0, T_{2K}, 0 \ldots) = e^{\sum_n a_n K T_n K \tau^{(K)}}(0) = e^{\sum_n a_n K \mu^{-nK} \tau^{(K)}}(0). \] (2.67)

This relation is valid for any \( \tau \)-function \( \tau^{(K)} \) of \( K \)-reduced KP hierarchy. Our purpose is to prove that for the particular example of such \( \tau^{(K)} \), namely, for partition function \( Z^{(K)} \) of GKM, associated with potential of the form \( V(X) = X^{K+1}/(K+1) \), all the

\[ a_n K[Z^{(K)}] = \frac{\partial}{\partial T_{nK}} \log Z^{(K)} = 0, \] (2.68)
or, equivalently, the function in the r.h.s. of (2.67) is independent of $\mu$. In other words, we need to prove that

$$\xi^{(K)}(\mu) \equiv Z^{(K)}[\mu_j] \big|_{(2.65)} = 1. \quad (2.69)$$

The l.h.s. of this relation can be evaluated with the help of eq.(2.16),

$$Z^{(K)} = \frac{\det_{ij} \Phi_i(\mu_j)}{\Delta(M)}.$$

We emphasize that in order to prove $T_{nK}$- independence of $Z^{(K)}$ it is enough to examine the determinant of $K \times K$ matrix in (2.16) with entries

$$\Phi_i(\mu) = \left( s(\lambda) \frac{\partial}{\partial \lambda} s(\lambda)^{-1} \right)^{i-1} \Phi_1(\mu) \equiv A^{i-1}_{(K)} \Phi_1 = \mu^{i-1} e^{(i)}(\mu), \quad (2.70)$$

$$e^{(i)}(\mu) = \sum_{\alpha \geq 0} e^{(i)}_{\alpha} \mu^{-\alpha(K+1)}, \quad e^{(i)}_0 = 1 \quad (2.71)$$

(of course, all the quantities $\Phi, s, A$ (see (2.15), (2.18)-(2.20)) and $e^{(i)}$ depend on the form of potential $V$ and thus on $K$).

**Straightforward calculation ($K = 2$).** We begin our proof that $\xi^{(K)}(\mu) = 1$ from direct calculation of this quantity. If we substitute the expansion (2.71) with $\mu$ ’s defined in (2.65) into (2.16) the result reads:

$$\xi^{(K)}(\mu) = \sum_{\{\alpha_i\}} \left( \prod_{i=1}^{K} e^{(i)}_{\alpha_i} \right) \cdot \frac{1}{\mu^{\sum_{i=1}^{K} \alpha_i}} \cdot \frac{\det_{ij} \varepsilon^{(-\alpha_i+i-1)j}}{\det_{ij} \varepsilon^{(i-1)j}}. \quad (2.72)$$

The remaining determinant at the r.h.s. is actually vanishing unless $\sum_i \alpha_i = 0 \text{ mod } K$, so that (2.68) is in fact equivalent to the set of identities

$$\sum_{\{\alpha_i\} = nK} \left( \prod_{i=1}^{K} e^{(i)}_{\alpha_i} \right) \cdot \frac{\det_{ij} \varepsilon^{(-\alpha_i+i-1)j}}{\det_{ij} \varepsilon^{(i-1)j}} = \delta_{n,0} \text{ for all } n \geq 0 \quad (2.73)$$

for coefficients $e^{(i)}_{\alpha}$ of series expansions of modified $K$-level Airy functions

$$\Phi_1^{(K)}(\mu) = \mu^{\frac{K-1}{2}} e^{-\frac{K\mu^{K+1}}{K+1}} \int e^{-\frac{K^{K+1}}{K+1} + \mu^{K+1}} x^{i-1} dx. \quad (2.74)$$

For example, for $K = 2$ identities (2.73) look like
\[
\sum_{\alpha + \beta = 2n} (-)^n e^{(1)}_\alpha e^{(2)}_\beta = \delta_{n,0}.
\] (2.75)

Since for \( K = 2 \)

\[
\Phi_1(\mu) = \sqrt{\frac{\mu}{\pi}} e^{-\frac{2}{3} \mu^3} \int dx e^{-\frac{x^3}{2} + x \mu^2} = \frac{1}{\sqrt{\pi}} \int dz \exp \left\{-z^2 - \frac{z^3}{3 \mu^{3/2}} \right\} = \\
= \sum_k \frac{1}{9^k (2k)!} \frac{\Gamma(3k + 1/2)}{\Gamma(1/2)} \frac{1}{\mu^{3k}},
\]

\[
\Phi_2(\mu) = \sqrt{\frac{\mu}{\pi}} e^{-\frac{2}{3} \mu^3} \int dx \cdot x e^{-\frac{x^3}{2} + x \mu^2} = \frac{1}{\sqrt{\pi}} \int dz (z + \mu) \exp \left\{-z^2 - \frac{z^3}{3 \mu^{3/2}} \right\} = \\
= -\sum_k \frac{1}{9^k (2k)!} \frac{6k + 1}{6k - 1} \frac{\Gamma(3k + 1/2)}{\Gamma(1/2)} \frac{1}{\mu^{3k}},
\]

i.e.

\[
e^{(1)}_\alpha = \frac{1}{9^\alpha (2\alpha)!} \frac{\Gamma(3\alpha + 1/2)}{\Gamma(1/2)}, \quad e^{(2)}_\alpha = -\frac{6\alpha + 1}{6\alpha - 1} e^{(1)}_\alpha.
\]

A more explicit form of (2.75) is

\[
\sum_{\alpha + \beta = 2n} \frac{36 \alpha \beta - 1}{(2\alpha)! (2\beta)!} \frac{\Gamma(3\alpha - 1/2) \Gamma(3\beta - 1/2)}{\Gamma(1/2)^2} = -4[\Gamma(1/2)]^2 \delta_{n,0},
\] (2.76)

which is indeed a valid \( \Gamma \)-function identity.

This calculation, though absolutely straightforward, is hard to repeat for generic \( K \). Still it is useful for illustrative purposes and we included it into this section.

The proof. A much easier approach is to prove that

\[
\mu \frac{\partial}{\partial \mu} \xi^{(K)}(\mu) = 0.
\] (2.77)

Since it is obvious from (2.72) that at least \( \xi^{(K)} \rightarrow 1 \) as \( \mu \rightarrow \infty \), this would provide a complete proof of (2.69). In order to prove (2.78) it is enough to act with \( \mu \partial/\partial \mu \) upon

\[
\xi^{(K)}(\mu) \sim \det_{(ij)\bar{e}^{(i)}_{\alpha \bar{e}^{(j)}}}
\] (2.78)

and then make use of the fact, that \( \mu \frac{\partial}{\partial \mu} e^{(i)}(\mu \bar{e}^{(j)}) \) can be decomposed into linear combination of \( e^{(1)}(\mu \bar{e}^{(j)}) \ldots e^{(K)}(\mu \bar{e}^{(j)}) \). This decomposition, which is, of course, the implication of \( K \)-reduction, can be worked out from the relation (2.20),
\[ \mu^i e^{(i+1)}(\mu) = A(1) \mu^{-1} e^{(i)}(\mu). \] (2.79)

Since

\[ A(1) = \frac{1}{K\mu^{-1}} \frac{\partial}{\partial \mu} + \mu - \frac{K - 1}{2K\mu}, \] (2.80)

(see also [23]), we have:

\[ (\mu \frac{\partial}{\partial \mu})^1 e^{(i)}(\mu) = (\frac{K + 1 - 2i}{2} - K\varepsilon^j \mu^{-1}) e^{(i)}(\mu) + K\varepsilon^j \mu e^{(i+1)}(\mu). \] (2.81)

This should be supplemented by the condition

\[ e^{(K+1)} = e^{(1)} \] (2.82)

(equation \( \lambda e^{(1)} = \mu^K e^{(1)} = A(1) e^{(1)} = s(\lambda) \left( \frac{\partial}{\partial \lambda} \right)^K s(\lambda)^{-1} e^{(1)} \) is nothing but the equation for the level-\( K \) Airy function \( F_1^{(K)} = s^{-1} e^{(1)} = \int e^{-\frac{K+1}{K+1} \mu^K x} \).

Let us begin our study of (2.78) from the familiar example of \( K = 2 \). First of all, one can check up (2.82) with the help of explicit expressions (2.76) for \( e^{(i)}(\alpha) \). Then

\[ (\mu \frac{\partial}{\partial \mu}) e^{(i)}(\mu) = (\frac{1}{2} - 2\mu^3) e^{(1)}(\mu) + 2\mu^3 e^{(2)}(\mu) e^{(1)}(\mu) - e^{(2)}(\mu) e^{(1)}(\mu) = \]

\[ = [(\frac{1}{2} - 2\mu^3) e^{(1)}(\mu) + 2\mu^3 e^{(2)}(\mu)] e^{(1)}(\mu) + e^{(2)}(\mu) e^{(1)}(\mu) - e^{(2)}(\mu) e^{(1)}(\mu) = \]

\[ = [(\frac{1}{2} - 2\mu^3) e^{(1)}(\mu) + 2\mu^3 e^{(2)}(\mu)] e^{(1)}(\mu) + e^{(2)}(\mu) e^{(1)}(\mu) - e^{(2)}(\mu) e^{(1)}(\mu) = \]

and the r.h.s. is identically zero.

If we return to the case of arbitrary \( K \), note that after the substitution of (2.79) into the l.h.s. of (2.78) we obtain a sum of \( K \) determinants like (2.79) with \( e^{(i)}(\mu) \) in the row \( i \) substituted by (2.82):

\[ (\mu \frac{\partial}{\partial \mu}) e^{(i)}(\mu) = (\frac{K + 1 - 2i}{2} - K\varepsilon^j \mu^{-1}) e^{(i)}(\mu) + K\mu \varepsilon^{j+1} e^{(i+1)}(\mu). \]

Next, note that the last item in the r.h.s. coincides with the entry \( e^{(i+1)}(\mu) \) of the next line of the same determinant, and thus can be eliminated. Moreover, the first
item at the r.h.s. implies just that the $i'$-th line is multiplied by a factor of $\frac{K + 1 - 2i'}{2}$ and the effect of this cancels in the sum over $i'$: $\sum_{i'=1}^{K} \frac{K + 1 - 2i'}{2} = 0$. Therefore, we conclude that

$$\mu \frac{\partial}{\partial \mu} \xi^{(K)}(\mu) \sim \mu \frac{\partial}{\partial \mu} \det (ij) \varepsilon^{ij}\varepsilon^{(i)}(\mu \varepsilon^{j}) \equiv \mu \frac{\partial}{\partial \mu} \mathcal{D} = -K \mu^{K+1} \sum_{i'=1}^{K} \mathcal{D}_{i'},$$

(2.83)

where $\mathcal{D}_{i'}$ is just the same determinant $\mathcal{D}$, only in the $i'$-th line $d_{i'j} \equiv \varepsilon^{i'j} \varepsilon^{(i')}(\mu \varepsilon^{j})$ is substituted by $\varepsilon^{j}d_{i'j} = \varepsilon^{(i'+1)j} \varepsilon^{(i')}(\mu \varepsilon^{j})$. The sum of such $\mathcal{D}_{i'}$ (for any original $\mathcal{D}$) is identical zero as a consequence of the identity $\sum_{j=1}^{K} \varepsilon^{j} = 0$.

**Examples:**

$K = 2$:

$$\mathcal{D} = \det \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = d_{11}d_{22} - d_{12}d_{21};$$

$$\mathcal{D}_1 = \det \begin{pmatrix} -d_{11} & (-)^2d_{12} \\ d_{21} & d_{22} \end{pmatrix} = -d_{11}d_{22} - d_{12}d_{21},$$

$$\mathcal{D}_2 = \det \begin{pmatrix} d_{11} & d_{12} \\ -d_{21} & (-)^2d_{22} \end{pmatrix} = d_{11}d_{22} + d_{12}d_{21},$$

and

$$\mathcal{D}_1 + \mathcal{D}_2 = 0.$$

$K = 3$: $\varepsilon = e^{2\pi i/3}$,

$$\mathcal{D} = \det \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} = d_{11}d_{22}d_{33} + d_{12}d_{23}d_{31} + d_{13}d_{32}d_{21} - d_{12}d_{21}d_{33} - d_{13}d_{31}d_{22} - d_{23}d_{32}d_{11},$$

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\[
D_1 = \begin{vmatrix}
\varepsilon d_{11} & \varepsilon^2 d_{12} & \varepsilon^3 d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{vmatrix} = \\
= \varepsilon d_{11}d_{22}d_{33} + \varepsilon^2 d_{12}d_{23}d_{31} + \varepsilon^3 d_{13}d_{32}d_{21} - \varepsilon^2 d_{12}d_{21}d_{33} - \varepsilon^3 d_{13}d_{31}d_{22} - \varepsilon d_{23}d_{32}d_{11},
\]

\[
D_2 = \begin{vmatrix}
d_{11} & d_{12} & d_{13} \\
\varepsilon d_{21} & \varepsilon^2 d_{22} & \varepsilon^3 d_{23} \\
d_{31} & d_{32} & d_{33}
\end{vmatrix} = \\
= \varepsilon^2 d_{11}d_{22}d_{33} + \varepsilon^3 d_{12}d_{23}d_{31} + \varepsilon d_{13}d_{32}d_{21} - \varepsilon d_{12}d_{21}d_{33} - \varepsilon^2 d_{13}d_{31}d_{22} - \varepsilon^3 d_{23}d_{32}d_{11},
\]

\[
D_3 = \begin{vmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
\varepsilon d_{31} & \varepsilon^2 d_{32} & \varepsilon^3 d_{33}
\end{vmatrix} = \\
= \varepsilon^3 d_{11}d_{22}d_{33} + \varepsilon d_{12}d_{23}d_{31} + \varepsilon^2 d_{13}d_{32}d_{21} - \varepsilon^3 d_{13}d_{31}d_{22} - \varepsilon d_{23}d_{32}d_{11},
\]

and

\[
D_1 + D_2 + D_3 = 0.
\]

These two examples are enough to illustrate the general phenomenon. The thing is that determinant \( D \) is an algebraic sum of products \( d_{i_1j_1} \ldots d_{i_Kj_K} \). Let us pick up one of these items. It appears in \( D_{i'} \) with the coefficient \( \varepsilon^{j'} \), where \( j' \) is defined as the second subscript of the element \( d_{i'j'} \) in this product. The map \( i' \to j' \) depends on the particular item we choose, but when we sum over all \( i' \) this is the same as to sum over all \( j' \), since every subscript appears in the product once and only once. Therefore in the sum \( \sum_{i'} D_{i'} \) every item appears with the universal coefficient \( \sum_{j'} \varepsilon^{j'} = 0 \), i.e. \( \sum_{i'} D_{i'} = 0 \).

This completes our proof of (2.78) and thus of (2.68).
Remark. To avoid possible confusion, let us note that it is important that Airy-functions are analytic functions of $\mu$, but not of $\lambda = \mu^K$ (e.g., $e^{(i)}(\mu)$ are expanded in series with integer powers of $\mu^{-(K+1)} = \lambda^{-1-1/K}$). This is what makes the proof slightly non-trivial. Otherwise, if one suggested that the entries of the matrix in (2.79) are analytic functions of $\lambda$, the $\lambda$-derivative of this determinant would be simply a combination of determinants with coincident rows and thus vanishing. However, non-analiticity in $\lambda$ requires one to be more accurate: $\lambda^{1/K}$ takes different values $\mu^j$ when appearing in different places, and a detailed calculation, as given above, is necessary.

3 Universal $L_{-1}$-constraint and string equation

3.1 Motivations

Since it was proved in sect.2 that $Z^{(K)}[M]$ is a $\tau$-function $\tau^{(K)}$ of the $K$-reduced KP hierarchy, in order to define it completely (i.e. to fix the point of Grassmannian) it is enough to deduce somehow a single additional constraint

$$L^{(K)}_{-1} Z^{(K)} = 0, \quad (3.1)$$

with

$$L^{(K)}_{-1} = W^{(2)}_{-1} = \frac{1}{K} \sum_{n \geq 1 \mod K} (n + K)T_{n+K} \partial/\partial T_n - \partial/\partial T_1 + \frac{1}{2K} \sum_{a+b=K \atop a,b \geq 1} aT_a bT_b. \quad (3.2)$$

According to arguments of ref.[1] such constraint (3.1), when imposed on $Z^{(K)} = \tau^{(K)}$, implies the entire set of $W_K$-algebra constraints in the form of (1.29). Another reason for the study of $L_{-1}$- constraint is that it is much simpler than any other ones: $L_{-1}$ is the only operator of interest which does not contain double- and higher- order $T$-derivatives (to be exact, there is one more such generator: $L_0$). Among other things, this means that it is easier to formulate the universal (i.e. for any potential $V(X)$) $L_{-1}$-constraint, than any other corollary of (universal!!) Ward-identity (1.23).

This section contains formulation and derivation of $L^{(V)}_{-1}$-constraint for GKM with arbitrary potential $V(X)$, making use of explicit formulas, derived in sect.2. Namely, we
are going to exploit the fact that the functions $\tilde{\Phi}_i(\mu)$ in (2.10) are not independent, but are rather related by the action of $\partial/\partial \lambda$- operator: see (2.20). The proof itself is described in sect.3.2 and sect.3.3 contains additional remarks about string equation. We emphasize, that not only the $L_{-1}$- constraint is valid for any $\mathcal{V}(X)$ (with the only restriction that $\mathcal{V}(x)$ grows faster than $x$ as $x \to \infty$), it is just the same for any size $N$ of the matrices. Just like the property of integrability (i.e. that $Z^{(V)} = \tau^{(V)}$), this constraint is not sensitive to $N$, and this makes the entire construction behaving smoothly in continuum limit, as $N \to \infty$.

3.2 Direct evaluation of $L_{-1}Z$

It is well known [21, 28], that $L_{-1}$-constraint is closely related to the action of operator

$$Tr \frac{\partial}{\partial \Lambda_{tr}} = Tr \frac{1}{\mathcal{V}''(M)} \frac{\partial}{\partial M}.$$  

Therefore it is natural to examine, how this operator acts on

$$Z^{(V)}[M] = \frac{\det \tilde{\Phi}_i(\mu_j)}{\Delta(M)} \prod_i s(\mu_i),$$  

$$s(\mu) = (\mathcal{V}''(\mu))^{1/2} e^{\mathcal{V}(\mu) - \mu \mathcal{V}'(\mu)},$$  

$$\tilde{\Phi}_i(\mu) = F_i(\lambda) = (\partial/\partial \lambda)^{i-1} F_1(\lambda), \lambda = \mathcal{V}'(\mu).$$

First of all, if $Z^{(V)}$ is considered as a function of $T$-variables,

$$\frac{1}{Z^{(V)}} Tr \frac{\partial}{\partial \Lambda_{tr}} Z^{(V)} = - \sum_{n \geq 1} Tr \frac{1}{\mathcal{V}''(M) M^{n+1}} \frac{\partial \log Z^{(V)}}{\partial T_n}.$$  

In the particular case of $\mathcal{V}(X) = \mathcal{V}_K(X) = \frac{X^{K+1}}{K+1}$, eq.(3.6) turns into

$$\frac{1}{Z^{(K)}} Tr \frac{\partial}{\partial \Lambda_{tr}} Z^{(K)} = - \frac{1}{K Z^{(K)}} \sum_{n \geq 1 \mod K} Tr \frac{1}{M^{n+K}} \frac{\partial Z^{(K)}}{\partial T_n} =$$  

$$= - \frac{1}{Z^{(K)}} \left\{ L^{(K)}_{-1} - \frac{1}{2} \sum_{a+b=K \atop a,b \geq 1} aT_a bT_b + \frac{\partial}{\partial T_1} \right\} Z^{(K)}.$$
We made use of the definition (3.2) of $\mathcal{L}^{(K)}_{-1}$-operator and the fact that $Z^{(K)}$ is independent of all $T_{nK}$ (since it is a $\tau^{(K)}$-function). Time-variables are defined by Miwa transformation $T_n = \frac{1}{n} Tr M^{-n}$. On the other hand, if we apply (3.3) to explicit formula (3.4), we obtain:

$$
\frac{1}{Z^{(v)}} Tr \frac{1}{\partial \Lambda_{tr}} Z^{(v)}
$$

$$
= -Tr M + \frac{1}{2} \sum_{i,j} \frac{\mathcal{V}''(\mu_i) - \mathcal{V}''(\mu_j)}{\mu_i - \mu_j} + Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j),
$$

(3.8)

or, in the case of $V(X) = X^{K+1}/(K+1)$,

$$
\frac{1}{Z^{(V)}} Tr \frac{1}{\partial \Lambda_{tr}} Z^{(K)} = -Tr M + \frac{1}{2} \sum_{a+b=K \atop a,b>1} aT_a bT_b + Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j).
$$

(3.9)

Combination of (3.7) and (3.9) implies, that

$$
\frac{1}{Z^{(K)}} \mathcal{L}^{(K)}_{-1} Z^{(K)} = -\frac{\partial}{\partial T_1} \log Z^{(K)} + Tr M - Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j).
$$

(3.10)

The $r.h.s.$ here is practically independent of $K$, and this may be used, together with eqs.(3.6) and (3.8) in order to suggest the formula for the universal operator $\mathcal{L}^{(v)}_{-1}$: the idea is to preserve the relation (3.10):

$$
\frac{1}{Z^{(v)}} \mathcal{L}^{(v)}_{-1} Z^{(v)} = -\frac{\partial}{\partial T_1} \log Z^{(v)} + Tr M - Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j).
$$

(3.11)

Here

$$
\mathcal{L}^{(v)}_{-1} = \sum_{n \geq 1} Tr \left[ \frac{1}{\mathcal{V}''(M) M^{n+1}} \frac{\partial}{\partial T_n} + \frac{1}{2} \sum_{i,j} \frac{\mathcal{V}''(\mu_i) - \mathcal{V}''(\mu_j)}{\mu_i - \mu_j} - \frac{\partial}{\partial T_1} \right]
$$

(3.12)

and this turns into (3.2) when $V(X) = X^{K+1}/(K+1)$ (note that the items with $i = j$ are included into the sum at the $r.h.s.$ in (3.12)).

So, in order to prove the $\mathcal{L}^{(v)}_{-1}$-constraint, one should prove that the $r.h.s.$ of (3.9) vanishes,
\[
\frac{\partial}{\partial T_1} \log Z_N^{(V)} = Tr M - Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j), \tag{3.13}
\]
for \(Z_N^{(V)} = \tau_N^{(V)}\), defined by (3.4). The l.h.s. may be represented as residue of the ratio

\[
res_\mu \frac{\tau_N^{(V)}(T_n + \mu^{-n}/n)}{\tau_N^{(V)}(T_n)} = \frac{\partial}{\partial T_1} \log \tau_N^{(V)}(T_n). \tag{3.14}
\]

However, if expressed through Miwa coordinates, the \(\tau\)-function in the numerator is given by the same formula (3.4) with one extra parameter \(\mu\), i.e. is in fact equal to \(\tau_{N+1}^{(V)}\).

This idea is almost enough to deduce (3.13). Since (3.13) is valid for any value of \(N\), it is reasonable to begin with an illustrative example of \(N = 1\). Then \(\lambda = \mathcal{V}'(\mu)\)

\[
\tau_1^{(V)}(T_n) = \tau_1^{(V)}[\mu_1] = e^{\mathcal{V}(\mu_1) - \mu_1 \mathcal{V}'(\mu_1)}[\mathcal{V}''(\mu_1)]^{1/2} F(\lambda_1),
\]

\[
\tau_1^{(V)}(T_n + \mu^{-n}/n) = \tau_2^{(V)}[\mu_1, \mu] =
\]

\[
= e^{\mathcal{V}(\mu_1) - \mu_1 \mathcal{V}'(\mu_1)} e^{\mathcal{V}(\mu) - \mu \mathcal{V}'(\mu)} \frac{[\mathcal{V}''(\mu_1) \mathcal{V}''(\mu)]^{1/2}}{\mu - \mu_1} \left[ F(\lambda_1) \partial F(\lambda)/\partial \lambda - F(\lambda) \partial F(\lambda_1)/\partial \lambda_1 \right] =
\]

\[
= \frac{e^{\mathcal{V}(\mu) - \mu \mathcal{V}'(\mu)} [\mathcal{V}''(\mu)]^{1/2} F(\lambda)}{\mu - \mu_1} \tau_1^{(V)}[\mu_1] \cdot [-\partial \log F(\lambda_1)/\partial \lambda_1 + \partial \log F(\lambda)/\partial \lambda]. \tag{3.15}
\]

The function

\[
F(\lambda) = \int dx \ e^{-\mathcal{V}(x)+\lambda x} \sim e^{\mathcal{V}(\mu) - \mu \mathcal{V}'(\mu)} [\mathcal{V}''(\mu)]^{-1/2} \left\{ 1 + O\left(\frac{\mathcal{V}'''(\mu)}{\mathcal{V}''(\mu)}\right) \right\}. \tag{3.16}
\]

If \(\mathcal{V}(\mu)\) grows with \(\mu^n\) as \(\mu \to \infty\), then \(\mathcal{V}'''(\mu)/(\mathcal{V}'(\mu))^2 \sim \mu^{-n}\), and for our purposes it is enough to have \(n > 1\), so that in the braces at the r.h.s. stands \(1 + o(1/\mu)\) as \(\mu \to \infty\). Then numerator at the r.h.s. of (3.15) is \(\sim 1 + o(1/\mu)\), while the second item in square brackets behaves as \(\partial \log F(\lambda)/\partial \lambda \sim \mu (1 + o(1/\mu))\). Combining all this with eq.(3.14), we obtain:

\[
\frac{\partial}{\partial T_1} \log \tau_1^{(V)} = res_\mu \left\{ \frac{1 + o(1/\mu)}{\mu - \mu_1} \left[ -\partial \log F(\lambda_1)/\partial \lambda_1 + \mu (1 + o(1/\mu)) \right] \right\} =
\]

\[
= \mu_1 - \partial \log F(\lambda_1)/\partial \lambda_1. \tag{3.17}
\]

Thus (3.13) is proved for the particular case of \(N = 1\).
The proof is literally the same for any $N$, only instead of a relatively simple expression in square brackets at the r.h.s. of (3.15) one has the ratio:

$$\det \begin{vmatrix} F(\lambda_1) & \partial F(\lambda_1)/\partial \lambda_1 & \ldots & \partial^{N-1} F(\lambda_1)/\partial \lambda_1^{N-1} & \partial^{N} F(\lambda_1)/\partial \lambda_1^{N} \\ F(\lambda_2) & \partial F(\lambda_2)/\partial \lambda_2 & \ldots & \partial^{N-1} F(\lambda_2)/\partial \lambda_2^{N-1} & \partial^{N} F(\lambda_2)/\partial \lambda_2^{N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F(\lambda_N) & \partial F(\lambda_N)/\partial \lambda_N & \ldots & \partial^{N-1} F(\lambda_N)/\partial \lambda_N^{N-1} & \partial^{N} F(\lambda_N)/\partial \lambda_N^{N} \\ F(\lambda) & \partial F(\lambda)/\partial \lambda & \ldots & \partial^{N-1} F(\lambda)/\partial \lambda^{N-1} & \partial^{N} F(\lambda)/\partial \lambda^{N} \end{vmatrix}$$

over

$$\det \begin{vmatrix} F(\lambda_1) & \partial F(\lambda_1)/\partial \lambda_1 & \ldots & \partial^{N-1} F(\lambda_1)/\partial \lambda_1^{N-1} \\ F(\lambda_2) & \partial F(\lambda_2)/\partial \lambda_2 & \ldots & \partial^{N-1} F(\lambda_2)/\partial \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ F(\lambda_N) & \partial F(\lambda_N)/\partial \lambda_N & \ldots & \partial^{N-1} F(\lambda_N)/\partial \lambda_N^{N-1} \end{vmatrix},$$

which is in fact equal to

$$F(\lambda)\mu^{N} \left\{ [1 + o(1/\mu)] - \frac{1}{\mu} [\text{tr} \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda)] \cdot [1 + O(1/\mu)] \right\}.$$ (3.20)

We used here the estimates $\frac{\partial F/\partial \lambda}{F} = \mu(1 + o(1/\mu))$, $\frac{\partial^2 F/\partial \lambda^2}{F} = \left( \frac{\partial F/\partial \lambda}{F} \right)^2 + \frac{\partial}{\partial \lambda} \mu(1 + o(1/\mu)) = \mu^2(1 + o(1/\mu))$, ..., $\frac{\partial^N F/\partial \lambda^N}{F} = \mu^N(1 + o(1/\mu))$, which allow us to pick up only the contributions with $\partial^N F(\lambda)/\partial \lambda^N$ and $\partial^{N-1} F(\lambda)/\partial \lambda^{N-1}$ to (3.18) — all other are lower order in $\mu$ as $\mu \to \infty$. The $N$-th derivative in (3.18) is multiplied by determinant of $N \times N$ matrix, which is exactly (3.19), while the $(N-1)$-th derivative — by

$$\det \begin{vmatrix} F(\lambda_1) & \partial F(\lambda_1)/\partial \lambda_1 & \ldots & \partial^{N-2} F(\lambda_1)/\partial \lambda_1^{N-2} & \partial^{N} F(\lambda_1)/\partial \lambda_1^{N} \\ F(\lambda_2) & \partial F(\lambda_2)/\partial \lambda_2 & \ldots & \partial^{N-2} F(\lambda_2)/\partial \lambda_2^{N-2} & \partial^{N} F(\lambda_2)/\partial \lambda_2^{N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F(\lambda_N) & \partial F(\lambda_N)/\partial \lambda_N & \ldots & \partial^{N-2} F(\lambda_N)/\partial \lambda_N^{N-2} & \partial^{N} F(\lambda_N)/\partial \lambda_N^{N} \end{vmatrix}$$

This is, however, nothing but $\text{tr} \frac{\partial}{\partial \Lambda_{tr}}$ of the logarithm of (3.19), and for this to be true it is absolutely essential, that $F_i \sim (\partial/\partial \lambda)^{i-1} F_i$. 45
With the estimate (3.20) we obtain from (3.14):

\[ \frac{\partial}{\partial T_1} \log \tau_N^{(y)} = \]

\[ = \text{res}_\mu \left\{ \frac{1 + o(1/\mu)}{\prod_{j=1}^N (\mu - \mu_j)} \mu^N \left\{ \left[ 1 + o(1/\mu) \right] - \right. \right. \]

\[ - \frac{1}{\mu} \left[ Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j) \right] \cdot \left[ 1 + O(1/\mu) \right] \left. \right\} = \]

\[ = \sum_{j=1}^N \mu_j - Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j). \quad (3.22) \]

This completes the proof of eq.(3.13) and thus of the universal \( L_{-1}^{(y)} \)-constraint:

\[ L_{-1}^{(y)} \tau^{(y)} = 0. \quad (3.23) \]

We proved the crucial eq.(3.13) by the direct calculation. Now we would like to describe an alternative proof which originates from representation of \( \tau \)-function through the current correlators described in sect.2.3. This approach may be useful for evaluation of the higher \( T \)-derivatives (see sect.3.4 below). Namely, from eqs.(2.35) and (2.38) it is obvious that \( \tau_N^G[t|M] \equiv \tau(t, \{T_n\}) \) depends on \( t \) and \( T_n \) as follows: \( \tau(t, \{T_n\}) = \tau(T_1-Nt, T_2-\frac{1}{2}Nt^2, ...) \) i.e. in eq.(2.38)

\[ \tau_N^G[t|M] = \exp(-N \sum \frac{t^n}{n} \frac{\partial}{\partial T_n}) \tau(T) = \sum_{n} t^n \mathcal{P}_n(\tilde{\partial}) \tau(T) \quad (3.24) \]

where \( \mathcal{P}_n(\tilde{\partial}) \) are the Schur polynomials and symbol \( \tilde{\partial} \) represent the vector with components \( -N\partial/\partial T_1, -\frac{1}{2}N\partial/\partial T_2, ... \). Therefore from eqs.(2.38), (2.39) and (3.24) one can deduce

\[ -N \frac{\partial}{\partial T_1} \tau(T) = \partial_t \tau_N^G[t|M] \big|_{t=0} = -N \sum_{i} \mu_i \cdot \tau(T) + \frac{1}{\Delta(\mu)} \lim_{t \to 0} \partial_t \det \phi_i(\mu_j, t), \]

where \( \phi_i(\mu_j, t) \) are defined by eq.(2.39). From this expression it follows that

\[ \frac{\partial}{\partial T_1} \tau(T) = \sum_{i} \mu_i \cdot \tau(T) - \dot{\tau}(T) \quad (3.25) \]
where \( \hat{\tau}(T) \) denotes some new \( \tau \)-function which is obtained from the given \( \tau(T) \) by the shift of the last row in the determinant: \( \phi_N(\mu_j) \rightarrow \phi_{N+1}(\mu_j) \). Eq.(3.25) is nothing but eq.(3.13)\(^3\).

### 3.3 Universal string equation

The string equation, as implied by (3.23), is by definition:

\[
\frac{\partial}{\partial T_1} \frac{\mathcal{L}_1^{\{\nu\}}}{\tau^{\{\nu\}}} = 0. \tag{3.26}
\]

If we substitute explicit expression (3.12) of the \( \mathcal{L}_1 \)-operator the string equation acquires the form:

\[
\sum_{n \geq -1} T \tau[\frac{1}{\mathcal{V}(M)M^{n+1}}] \frac{\partial^2 \log \tau}{\partial T_1 \partial T_n} = u. \tag{3.27}
\]

Here we introduced a conventional notation \( u \equiv \partial^2 \log \tau / \partial T_1^2 \) and also defined new \( T_0 \)- and \( T_{-1} \)-derivatives, so that \( \partial \log \tau / \partial T_0 \equiv 0 \), \( \partial \log \tau / \partial T_{-1} \equiv T_1 \). The derivation of (3.27) involves taking \( T_1 \)-derivatives of the objects

\[ T^{\{\nu\}}_n = T \tau[\frac{1}{\mathcal{V}(M)M^{n+1}}]. \tag{3.28} \]

In order to get some impression about \( \partial T_n / \partial T_1 \), let us imagine, that \( \mathcal{V}(M) \) is some polynomial of degree \( Q + 1 \), so that \( [\mathcal{V}(M)]^{-1} \sim M^{1-Q}(1 + v_1 Q^{-1} + v_2 Q^{-2} + ...) \). Then \( T_n \sim (Q+n)T_{Q+n} + v_1 (Q+n+1)T_{Q+n+1} + v_2 (Q+n+2)T_{Q+n+2} + ... \). Therefore whenever \( Q + n > 1 \) (i.e. \( Q > 0 \), as it is already necessary for the derivation of (3.23))

\[
\partial T_n / \partial T_1 = 0, \text{ for } n \geq 1. \tag{3.29}
\]

In order to derive (3.27) one also needs \( T_1 \)-derivative of the second item in (3.12),

\[
\frac{1}{2} \sum_{i,j} \frac{1}{\mathcal{V}(\mu_i)\mathcal{V}(\mu_j)} \frac{\mathcal{V}'(\mu_i) - \mathcal{V}'(\mu_j)}{\mu_i - \mu_j}. \tag{3.30}
\]

\(^3\)Let us emphasize that explicit form of r.h.s. of (3.25) does not depend on the choice of basis. Unlike this, the manifest form of derivatives over \( T_m \), with \( m > 1 \), should deal with concrete choice of basis. It is noteworthy to remark that, for \( m \leq K \), this manifest form coincides for the canonical basis and our basis (2.18).
Under the same assumptions about \( V(\mu) \) the second ratio in this sum is a polynomial in \( \mu \)'s of degree \( Q - 2 \). The only contribution to (3.30), which contains pure factors \( \mu_i^{-1} \) or \( \mu_j^{-1} \), is:

\[
\frac{1}{2} \sum_{i,j} \frac{1}{\mu_i} \frac{V''(\mu_i)}{\mu_i} + \frac{V''(\mu_j)}{\mu_j} = T_1 T_{-1},
\]

others are expanded as bilinear series in \( T_m T_n \) with \( m, n > 1 \). Therefore, \( T_1 \)-derivative of (3.30) (if \( Q > 2 \)) is just \( T_1 \) and is described by the \( n = 0 \) term in (3.27).

So, we derived the universal string equation (3.27) in the form:

\[
\sum_{n \geq -1} T_n^{(V)} \frac{\partial^2 \log \tau^{(V)}}{\partial T_1 \partial T_n} = u. \tag{3.31}
\]

If potential \( V(X) = \text{const} \cdot X^{K+1} \), the variables \( T_n = \frac{n+K}{K} T_{n+K} \), and we return to familiar relation [34]:

\[
\frac{1}{K} \sum_{n \geq -1} \sum_{n \neq 0 \text{ mod } K} (n + K) T_{n+K} \frac{\partial^2 \log \tau^{(K)}}{\partial T_1 \partial T_n} = u. \tag{3.32}
\]

Note that even if we restore the factor (1.22), describing the maximal possible \( T_{nK} \)-dependence of \( \tau^{(K)} \) it drops out of the string equation (3.32) after taking the \( \partial/\partial T_1 \)-derivative.

Our next task is reformulation of (3.31) in terms of pseudo-differential operators and in the form of bilinear relation. If

\[
L = \partial + \sum_{i=1}^{\infty} u_{i+1} \partial^{-i} \quad (\partial \equiv \frac{\partial}{\partial T_1} \equiv \frac{\partial}{\partial x}), \tag{3.33}
\]

then [33]

\[
\frac{\partial^2 \log \tau}{\partial T_1 \partial T_n} = (L^n)_{-1}. \tag{3.34}
\]

For \( n = -1, 0, 1 \) we have \((L^{-1})_{-1} = 1\), \((L^0)_{-1} = 0\), \((L)_{-1} = u_2 \equiv u\) in accordance with (3.27). \( K \)-reduction is associated with the condition

\[
(L^K)_{-1} = 0 \tag{3.35}
\]

(this guarantees, that items with \( n = 0 \ mod \ K \) do not appear in (3.32)). Rewritten in these terms, eq.(3.31) turns into:
\[
\frac{1}{\mathcal{V}''(M)} \sum_{n \geq -1} M^{-n-1}(L^n)_{-1} = u .
\]

Recall now, that in the framework of the dressing formalism \[33\] \( L = K\partial K^{-1}, \) \( L^n = K\partial^n K^{-1} \). The eigenfunctions of \( L \) can be defined as \( (Ke^{xz}) \), where brackets denote that operator \( K \) acts on \( e^{xz} \) (i.e. \( (Ke^{xz}) \) is a function of \( z \), not an operator). Baker-Akhiezer function in these terms is given by \[33\] \( \Psi(z|T_n) = e^{\sum T_n z^n} (Ke^{xz}) \), while its conjugate \( \Psi^*(z|T_n) = e^{-\sum T_n z^n} [(K^{-1})^* e^{-xz}] \). These definitions are useful for us, since \[33\]

\[
(L^n)_{-1} = \text{res}_z z^n \Psi^*(z)\Psi(z). \tag{3.37}
\]

We assume that contour integral over \( z \), implicit in the definition of \( \text{res}_z \), is around zero.

If (3.37) is substituted into (3.36), we obtain:

\[
u = -\text{res}_z Tr \frac{M \Psi^*(z)\Psi(z)}{\mathcal{V}''(M)} z(zI - M). \tag{3.38}
\]

Since \( \Psi(z|T_n) = e^{\sum T_n z^n} \frac{\tau(T_n - z^{-n}/n)}{\tau(T_n)} \) and \( \Psi^*(z|T_n) = e^{-\sum T_n z^n} \frac{\tau(T_n + z^{-n}/n)}{\tau(T_n)} \), the product \( \Psi^*(z)\Psi(z) \sim 1 + \mathcal{O}(1/z) \) (as \( z \to \infty \)) and the contour integral over \( z \), when pulled to infinity, picks up the contributions from eigenvalues of \( M \). Finally we obtain our universal string equation in the form of a bilinear relation for Baker-Akhiezer functions:

\[
\sum_i \frac{\Psi^*(\mu_i)\Psi(\mu_i)}{\mathcal{V}''(\mu_i)} + u = 0 . \tag{3.39}
\]

Comparison of eqs.(2.46) and (3.39) enables us to derive a useful representation for \( u \)-function:

\[
u = -\sum_{a,b} \frac{\partial^2}{\partial \lambda_a \partial \lambda_b} \log \frac{\tau}{N} . \tag{3.40}
\]

Note that eqs.(2.45) and (2.46) which follow from the general formula (2.44), provide the connection between residues of the Baker-Akhiezer function and time derivatives of the \( \tau \)-function without reference to the pseudo-differential operator language. Indeed, it can be easily understood if one substitutes eq.(3.25) into eq.(2.45).
3.4 Discussion

In this section 3 we derived the universal $L_{-1}^{(V)}$-constraint (1.25) on partition function $Z^{(V)}$. Its simplest implication, the universal string equation, was considered in sect.3.3. String equation is nothing but $T_1$-derivative of this $L_{-1}$-constraint. Remarkably enough, it appears to be much simpler than the constraint itself (compare (3.12) and (3.40)).

However, string equation is not the only implication of (1.23). Another one should be the entire tower (1.29) of Virasoro and $W$-constraints imposed on $Z^{(V)}$. In this subsection we discuss three different ways to derive these remaining relations for any $V(X)$ in the GKM. Being technically different, they emphasize different (though related) properties of partition function, and therefore each of the three deserves careful investigation, which is, however, beyond the scope of this paper.

**KP-hierarchy approach.** This one is the most popular in the literature (see [1, 20]). The idea is that $L_{-1}$-constraint, when imposed on KP $\tau$-function, automatically implies all other constraints. In other words, $L_{-1}$-constraint is enough to fix the point in Grassmannian with which the KP $\tau$-function is associated and, thus, it fixes this $\tau$-function completely. It would be interesting to find out explicit dependence of the point in Grassmannian on the potential $V(X)$ in GKM. Even more interesting would be any alternative description of the entire subset $U_\infty$ in Grassmannian, associated with GKM with arbitrary $V(X)$, and any reasonable parametrization of $U_\infty$, which would be an alternative parametrization of the space of potentials. Since $V(X)$ can be changed smoothly, $U_\infty$ should be a kind of a manifold, lying at infinity of the universal module space (if the latter is embedded into Sato’s Grassmannian), and surely possess some amusing properties. Unfortunately, the already existing papers on the subject (of course, they concern conventional (multi)matrix models rather than GKM) either emphasize implications of $K$-reduction, like [23], or rely upon the formalism of pseudo-differential operators, like [1, 20], and thus need be translated into Grassmannian language.

A separate question in the framework of this approach is why should the constraints be always associated with $W_\infty$-algebra (as they actually are). Technically this is more or less obvious whenever pseudo-differential operators are used. In KP-Grassmannian language, the crucial ingredient should be the $W_\infty$-covariance of KP-hierarchy, discovered
in [35, 36, 37]. The constraints of interest involve operators from some Borel subalgebra \( \mathcal{W}_\infty^+ \) of this \( \mathcal{W}_\infty \), and the choice of subalgebra depends on the choice of potential.

Reductions of GKM to (double-scaling limit of) multimatrix models, i.e. the what happens if \( \mathcal{V}(X) \sim X^{K+1} \), in Grassmannian language should be attributed to intersections of \( U_\infty \) and submanifolds \( Gr^{(K)} \) in Sato’s Grassmannian, associated with \( K \)-reductions. Somehow at these points a closed Zamolodchikov’s \( \mathcal{W}_K \)-subalgebra emerges from entire \( \mathcal{W}_\infty \). It is certainly interesting to see, how this happens and how the Lie algebra structure is broken. It is also unclear, whether such phenomenon takes place only at \( U_\infty \cap Gr^{(K)} \) or everywhere in \( Gr^{(K)} \).

One more important question is what is the relation between the just discussed \( \mathcal{W}_\infty \)-algebra of [35, 36, 37] and another \( \mathcal{W}_\infty \), which is presumably relevant in \( c = 1 \) models, and is naively generated by operators, which involve finite-differences instead of derivatives in the free-field representation. We use this chance to note, that the problem of how \( c = 1 \) models are included into framework of GKM is still open, and the invariant description of the \( U_\infty \) subset in Grassmannian would be also helpful for its resolution.

**Straightforward derivation of \( \mathcal{L} \)- and \( \mathcal{W} \)-constraints.** This approach is just the straightforward generalization of the what was done in sect.3.2 in the derivation of \( \mathcal{L}^{(V)}_{-1} \)-constraint from the knowledge of explicit expression (2.2) for \( Z^{(V)} \). We sketch here several steps of the derivation for the Virasoro case. The idea is to apply the operator \( Tr \Lambda^{m+1} \frac{\partial}{\partial \Lambda^{m+1}} \) to (2.2). On one hand this is equal to

\[
\sum_{n \geq 1} Tr \left( \frac{(\mathcal{V}'(M))^{m+1}}{\mathcal{V}''(M)M^{n+1}} \right) \frac{\partial}{\partial T_n} Z^{(V)}, \quad (3.41)
\]

on the other hand, it can be explicitly evaluated. Expression (3.41) is very close to \( \mathcal{L}^{(V)}_m Z^{(V)} \). The main difference is that the sum in (3.41) goes from \( n = 1 \) and for small enough \( n \) \( Tr \left( \frac{(\mathcal{V}'(M))^{m+1}}{\mathcal{V}''(M)M^{n+1}} \right) \) contains powers of positive powers of \( M \), which can not be expressed through \( T \)-variables. For \( \mathcal{V} \sim X^{K+1} \) this happens for \( n \leq Km \). In order to get rid of these positive powers of \( M \) one should use the analogues of eq. (3.14), saying that

\[
\frac{\partial}{\partial T_k} \log Z^{(V)} \sim Tr M^k + \ldots. \quad (3.42)
\]
This is the origin of conventional $\partial^2/dT_a dT_b$-terms in operators $L^{(K)}$.

As to generalization of the shift (1.17), it results in additional contribution of the form

$$-\sum_{n \geq 1} \oint \frac{[V'(\mu)]^{m+1}}{\mu^n} \frac{\partial}{\partial T_n}$$

in the definition of $L_m^{(V)}$ - operators. All extra terms cancel, giving rise to the proper universal constraints of the form $L_m^{(V)} Z^{(V)} = 0, m \geq -1$.

$\mathcal{W}$-constraints can be (at least in principle) derived in the same manner, though actual calculations are increasingly sophisticated.

**Implications of Ward identities.** This third approach exploits the fundamental Ward-identity (1.23). Technically its main difference from the previous approach is that the operators $\text{Tr} V'(\frac{\partial}{\partial \Lambda_{tr}})$, non-linear in $\partial/\partial \Lambda_{tr}$, are involved. Conceptually this approach is different, since it does not exploit explicitly any integrable features of partition function. We shall discuss this method in more details in the next section 4.

4 From Ward identities to $W$-constraints

4.1 General discussion

This section is devoted to the derivation (unfortunately, incomplete) of the entire set of Virasoro and $\mathcal{W}$- constraints in GKM. The role of such constraints in the study of any matrix model is two-fold. First of all, they can be considered as complete set of differential equations, which specify partition function of the model as a function of time-variables. Then this set of equations implies, among other things, that the solution is KP $\tau$-function (and sometimes a reduced $\tau^{(K)}$-function). The first application, discussed in details in sect.3 above, is reasonable if one already knows about the integrable structure. Then partition function is identified with *some* (reduced) KP $\tau$-function (*i.e.* evaluated at *some* point of Grassmannian), and the role of Virasoro constraint is to specify this $\tau$-function (*i.e.* fix the point in Grassmannian). In this second type of circumstances it is enough to have a single constraint, namely $L_{-1} Z = 0, -$ all other constraints, if imposed on KP $\tau$-function or $\tau^{(K)}$, are deducible corollaries of this one [1, 20]. With identification of $Z^{(V)}$ with a $\tau$-function in sect.2 and the proof of universal $L_{-1}^{(V)}$- constraint in sect.3
we exhausted this line of reasoning. The subject of this section is to concentrate instead on another approach and ignore almost all what we already studied about integrable structure of GKM. Then the constraints can be considered as a complete set of differential equations, which specify the partition function of the model as a function of time-variables. Among other things, the set of equations implies that the solution is KP $\tau$-function (and sometimes a reduced $\tau^{(K)}$-function). To be more concrete, we shall investigate direct corollaries of Ward identity (1.23),

$$\{Tr \epsilon(\Lambda)[V'(\partial/\partial\Lambda_{tr}) - \Lambda]\}F_N[\Lambda] = 0. \quad (4.1)$$

We shall, however, discuss only specific potentials, $V(X) = \text{const} \cdot X^{K+1}$, and use explicitly the fact that partition function is independent of all $T_{nK}$.

The main problem with the implications of Ward identity (4.1) is that they acquire the form of conventional Virasoro or $\mathcal{W}$-constraints only in the limit of $N = \infty$. The reliable results from our point of view, should, however, be $N$-independent. But as soon as $N$ is finite, the complete set of independent Ward identities (4.1) is also finite. Remarkably enough they can still be expressed through generators of $\mathcal{W}$-algebras, but the constraints arise as a finite number of vanishing conditions for infinite linear combinations of $\mathcal{W}$-operators, acting on partition function. As $N \to \infty$, the number of such conditions tends to infinity, implying that every item in linear combinations vanishes by itself. These items have exactly the form of conventional Virasoro constraints in the particular case of $K = 2$, while for $K = 3$ they rather look like eqs.(1.26), and generalization of (1.26) for $K > 3$ is more or less obvious. In any case the honest statement is that any solution to the $\mathcal{W}$-constraints in the conventional form (1.29) does satisfy the equations, which follow from (4.1), for any $N$ (to make $N$ finite one should take all but the first $N$ eigenvalues of matrix $M$ to infinity, see sect.2.2). In this sense transition from finite to infinite $N$ is smooth. However, inverse can be true, i.e. (4.1) can imply (1.29) only as $N = \infty$. Moreover, since (1.26), which is actually implied by (4.1) as $N = \infty$, is not quite identical to (1.29), one needs also to rely upon the (very plausible) assumption that solutions to (1.26) and (1.29) are unique (up to inessential constant factors) and thus coincide.

Such approach has been already applied in [2] to the study of Kontsevich model itself, i.e. for the case of $K = 2 : \mathcal{V}_2(X) = X^3/3$. Our purpose now is to extend this consider-
ation to other potentials $\mathcal{V}_K(X) = X^{K+1}/(K+1)$. However, while the structure of the answers is very clear (this is obvious from (1.26)), actual calculations are very tedious. Therefore we restrict our presentation below only to the first non-trivial case of $K = 3$. It is considered in sect.4.3. Before, in sect.4.2 we reproduce from $[2]$ the derivation of Virasoro constraints in the case of $K = 2$.

4.2 Virasoro constraints in Kontsevich model ($K = 2$)

The problem. Our purpose in this section is to prove the identity

$$\frac{1}{\mathcal{F}} tr(\epsilon_p \frac{\partial^2}{\partial \Lambda^2} - \epsilon_p \Lambda)\mathcal{F} = \frac{1}{Z} \sum_{n \geq -1} \mathcal{L}_n Z \ tr(\epsilon_p \Lambda^{-n-2})$$

(4.2)

for

$$\mathcal{F}^{(2)} \{\Lambda\} \equiv \int DX \ \exp(-trX^3/3 + tr\Lambda X) = C[\sqrt{\Lambda}] \exp\left(\frac{2}{3} tr\Lambda^{3/2}\right) Z_{1g}^{(2)} (T_m)$$

(4.3)

with

$$C[\sqrt{\Lambda}] = \det(\sqrt{\Lambda} \otimes I + I \otimes \sqrt{\Lambda})^{-\frac{1}{2}}$$

(4.4)

and

$$\mathcal{L}_n^{(2)} = \frac{1}{2} \sum_{k \geq n+1, \ k \ odd} kT_k \frac{\partial}{\partial T_{k+2n}} + \frac{1}{4} \sum_{a+b=2n, a,b \ odd} \frac{\partial^2}{\partial T_a \partial T_b} +$$

$$+ \delta_{n+1,0} \frac{T_2}{4} + \delta_{n,0} \frac{1}{16} - \frac{\partial}{\partial T_{2n+3}}.$$  

(4.5)

Below in this section we omit label $\{2\}$ and do not indicate explicitly the fact that all sums run over only odd times.

While (4.2) is valid for any size of the matrix $\Lambda$, in the limit of infinitely large $\Lambda$ ($N \to \infty$) we can insist that all the quantities

\footnote{To avoid any comparison note that in original version of $[2]$ there were wrong signs in the exponential in (4.3) and in the shift of times (1.17) (normalization of times are also different ).}
(4.6)

\[ \text{tr}(\epsilon_p \Lambda^{-n-2}) \]

(e.g. \( \text{tr}\Lambda^{p-n-2} \) for \( \epsilon_p = \Lambda^p \)) become algebraically independent, so that eq. (4.2) and (4.1) imply that

\[ \mathcal{L}_n Z\{T\} = 0, \quad n \geq -1. \quad (4.7) \]

**Method.** Note that \( \mathcal{F}\{\Lambda\} \) in (4.3), which we have to differentiate in order to prove (4.2), depends only upon eigenvalues \( \{\lambda_k\} \) of the matrix \( \Lambda \). Therefore, it is natural to consider eq.(4.2) at the diagonal point \( \Lambda_{ij} = 0, \; i \neq j \). The only “non-diagonal” piece of (4.2) which survives at this point is proportional to

\[ \frac{\partial^2 \lambda_k}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \bigg|_{\Lambda_{mn}=0, \; m \neq n} = \frac{\delta_{ki} - \delta_{kj}}{\lambda_i - \lambda_j} \quad \text{for} \; \; i \neq j. \quad (4.8) \]

Eq.(4.8) is nothing but a familiar formula for the second order correction to Hamiltonian eigenvalues in ordinary quantum-mechanical perturbation theory. It can be easily derived from the variation of determinant formula:

\[ \delta \log(\det \Lambda) = \text{tr} \frac{1}{\Lambda} \delta \Lambda - \frac{1}{2} \text{tr} \left( \frac{1}{\Lambda^2} \delta \Lambda \delta \Lambda \right) + \ldots . \quad (4.9) \]

For diagonal \( \Lambda_{ij} = \lambda_i \delta_{ij} \), but, generically, non-diagonal \( \delta \Lambda_{ij} \), this equation gives

\[ \sum_k \frac{\delta \lambda_k}{\lambda_k} = -\frac{1}{2} \sum_{i \neq j} \frac{\delta \lambda_i \delta \lambda_j}{\lambda_i \lambda_j} = \frac{1}{2} \sum_{i \neq j} \left( \frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right) \frac{\delta \lambda_i \delta \lambda_{ji}}{\lambda_i - \lambda_j} + \ldots , \]

which proves (4.8).

**Proof.** Now we shall turn directly to the proof of (4.2). Since \( \epsilon_p \) is assumed to be a function of \( \Lambda \), it can be, in fact, treated as a function of eigenvalues \( \lambda_i \). After that, (4.2) can be rewritten in the following way:

\[ \frac{e^{-\frac{2}{3}\text{tr}\Lambda^{3/2}}}{C(\sqrt{\Lambda})Z\{T\}} \left[ \text{tr} \; \epsilon_p \left\{ \frac{\partial^2}{\partial \Lambda^2} - \Lambda \right\} \right] C(\sqrt{\Lambda})e^{\frac{2}{3}\text{tr}\Lambda^{3/2}} Z\{T\} = \]
\[
\begin{align*}
&= \frac{1}{Z} \sum_{a,b \geq 0} \left( \frac{\partial^2 Z}{\partial T_a \partial T_b} \sum_i \epsilon_p(\lambda_i) \frac{\partial T_a}{\partial \lambda_i} \cdot \frac{\partial T_b}{\partial \lambda_i} \right) + \\
&+ \frac{1}{Z} \sum_{n \geq 0} \frac{\partial Z}{\partial T_n} \left[ \sum_{i,j} \epsilon_p(\lambda_i) \frac{\partial^2 T_n}{\partial \Lambda_{ij} \partial \Lambda_{ji}} + 2 \sum_i \epsilon_p(\lambda_i) \frac{\partial T_n}{\partial \lambda_i} \frac{\partial \log C}{\partial \lambda_i} + \\
&+ 2 \sum_i \epsilon_p(\lambda_i) \frac{\partial T_n}{\partial \lambda_i} \left( \frac{2}{3} \right) \frac{\partial \text{tr} \Lambda^{3/2}}{\partial \lambda_i} \right] + \\
&+ \left[ \sum_i \epsilon_p(\lambda_i) \left( \frac{\partial}{\partial \lambda_i} \left( \frac{2}{3} \right) \text{tr} \Lambda^{3/2} \right)^2 - \sum_i \lambda_i \epsilon_p(\lambda_i) + \\
&+ \sum_{i,j} \epsilon_p(\lambda_i) \left( \frac{\partial^2}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \left( \frac{2}{3} \right) \text{tr} \Lambda^{3/2} \right) + \\
&+ 2 \sum_i \epsilon_p(\lambda_i) \left( \frac{2}{3} \right) \frac{\partial \text{tr} \Lambda^{3/2}}{\partial \lambda_i} \frac{\partial \log C}{\partial \lambda_i} + \\
&+ \frac{1}{C} \sum_{i,j} \epsilon_p(\lambda_i) \frac{\partial^2 C}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \right]
\end{align*}
\]

with \( \text{tr} \Lambda^{3/2} = \sum_k \lambda_k^{3/2} \) and \( C = \prod_{i,j} (\sqrt{\lambda_i} + \sqrt{\lambda_j})^{-1/2} \).

First, since
\[
\frac{\partial T_n}{\partial \lambda_i} = -\lambda_i^{-n-\frac{3}{2}},
\]
it is easy to notice that the term with the second derivatives (4.11) can be immediately written in the desired form:
\[
(4.11) = \frac{1}{4} \sum_{n \geq -1} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-n-2} \sum_{a+b=2n} \frac{\partial^2 Z}{\partial T_a \partial T_b} = \\
= \frac{1}{4} \sum_{n \geq -1} \text{tr} \{\epsilon_p(\lambda_i) \Lambda^{-n-2}\} \sum_{a+b=2n} \frac{\partial^2 Z}{\partial T_a \partial T_b}.
\]
\[
\frac{\partial \log C}{\partial \lambda_i} = -\frac{1}{2\sqrt{\lambda_i}} \sum_j \frac{1}{\sqrt{\lambda_i} + \sqrt{\lambda_j}} \tag{4.20}
\]

\[
\frac{\partial}{\partial \lambda_i} tr \Lambda^{3/2} = \frac{3}{2} \sqrt{\lambda_i} \tag{4.21}
\]

and

\[
\frac{\partial^2 T_n}{\partial \Lambda_{ij} \partial \Lambda_{ji}} = \sum_k \frac{\partial^2 \lambda_k}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \frac{\partial T_n}{\partial \lambda_k} + \frac{\partial^2 T_n}{\partial \lambda_i^2} \delta_{ij} \tag{4.22}
\]

Then for (4.12) we have

\[
\frac{1}{2Z} \sum_{n \geq 0} \frac{\partial Z}{\partial T_n} \left[ \sum_{i,j} \epsilon_p(\lambda_i) \left\{ \frac{\lambda_i^{n-3/2} \lambda_j^{n-3/2}}{\sqrt{\lambda_i} + \sqrt{\lambda_j}} + \sum_{a+b=2(n+1)} (\sqrt{\lambda_i})^a (\sqrt{\lambda_j})^b \right. \right.
\]

\[
+ \left. \lambda_i^{n-2} \frac{1}{\sqrt{\lambda_i} + \sqrt{\lambda_j}} \right\} - 2 \sum_i \epsilon_p(\lambda_i) \lambda_i^{n-1} \right] =
\]

\[
= \frac{1}{2Z} \sum_{n \geq 0} \frac{\partial Z}{\partial T_n} \left[ \sum_{i,j} \sum_{a=0}^{n-1} \epsilon_p(\lambda_i) \lambda_i^{n-2+a} \lambda_j^{-a-1/2} - 2 \sum_i \epsilon_p(\lambda_i) \lambda_i^{n-1} \right] =
\]

\[
= \frac{1}{2Z} \sum_{n \geq -1} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-n-2} \sum_{k \geq \delta_{n+1,0}} \left( \sum_j \lambda_j^{-k-1/2} - 2\delta_{k,3} \right) \frac{\partial Z}{\partial T_{2n+k}} =
\]

\[
= \frac{1}{Z} \sum_{n \geq -1} tr(\epsilon_p \Lambda^{-n-2}) \left\{ \frac{1}{2} \sum_{k \geq \delta_{n+1,0}} kT_k \frac{\partial Z}{\partial T_{2n+k}} - \frac{\partial Z}{\partial T_{2n+3}} \right\}. \tag{4.23}
\]

The remaining part contains terms proportional to \(Z\) itself which need a bit more care.

First, it is easy to notice that two items in (4.13) just cancel each other, so the (4.13) gives no contribution to the final result. For (4.14) we have

\[
\sum_i \epsilon_p(\lambda_i) \frac{\partial^2 tr \Lambda^{3/2}}{\partial \lambda_i^2} + \sum_i \epsilon_p(\lambda_i) \frac{\partial tr \Lambda^{3/2}}{\partial \lambda_i} \frac{\partial^2 \lambda_k}{\partial \lambda_i \partial \Lambda_{ji}}. \tag{4.24}
\]

Using (4.21) and (4.8) it can be transformed into

\[
\frac{1}{2} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-1/2} + \sum_{i \neq j} \epsilon_p(\lambda_i) \frac{\sqrt{\lambda_i} - \sqrt{\lambda_j}}{\lambda_i - \lambda_j} =
\]

\[
= \sum_{i,j} \epsilon_p(\lambda_i) \frac{1}{\sqrt{\lambda_i} + \sqrt{\lambda_j}} \tag{4.25}
\]

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and this cancels (4.15) (transformed with the help of (4.20), (4.21)). Thus, the only contribution is (4.16), which gives

\[
\frac{1}{C} \frac{\partial^2 C}{\partial \lambda_{ij} \partial \lambda_{ji}} = \sum_k \frac{\partial^2 \lambda_k}{\partial \lambda_{ij} \partial \lambda_{ji}} \frac{\partial \log C}{\partial \lambda_k} + \frac{1}{C} \frac{\partial^2 C}{\partial \lambda_i^2} \delta_{ij} = \\
= \sum_k \frac{1}{2(\lambda_i - \lambda_j)} \left[ \frac{1}{\sqrt{\lambda_j(\sqrt{\lambda_j} + \sqrt{\lambda_k})}} - \frac{1}{\sqrt{\lambda_i(\sqrt{\lambda_i} + \sqrt{\lambda_k})}} \right] (1 - \delta_{ij}) + \\
+ \delta_{ij} \left[ \frac{\partial^2 \log C}{\partial \lambda_i^2} + \left( \frac{\partial \log C}{\partial \lambda_i} \right)^2 \right].
\]

Now, using (4.8) and (4.20) we obtain for (4.16):

\[
\left\{ \frac{1}{16} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-2} + \\
+ \frac{1}{4} \sum_{i,j} \epsilon_p(\lambda_i) \lambda_i^{-1} \left( \frac{1}{\sqrt{\lambda_i(\sqrt{\lambda_i} + \sqrt{\lambda_j})}} + \frac{1}{(\sqrt{\lambda_i} + \sqrt{\lambda_j})^2} \right) + \\
+ \frac{1}{4} \sum_{i,j,k} \epsilon_p(\lambda_i) \lambda_i^{-1} \left( \frac{1}{\sqrt{\lambda_i(\sqrt{\lambda_i} + \sqrt{\lambda_j})}} + \frac{1}{\sqrt{\lambda_i(\sqrt{\lambda_i} + \sqrt{\lambda_k})}} \right) \right\} = \\
= \frac{5}{16} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-2} + \sum_{i \neq j} \left( \frac{1}{4} \epsilon_p(\lambda_i) \lambda_i^{-1} \lambda_j^{-1} + \frac{1}{2} \epsilon_p(\lambda_i) \lambda_i^{-3/2} \lambda_j^{-3/2} \right) + \\
+ \sum_{i \neq j \neq k} \epsilon_p(\lambda_i) \left( \frac{1}{4 \lambda_i(\sqrt{\lambda_i} + \sqrt{\lambda_j})(\sqrt{\lambda_i} + \sqrt{\lambda_k})} + \\
+ \frac{1}{2(\lambda_i - \lambda_j)} \left[ \frac{1}{\sqrt{\lambda_j(\sqrt{\lambda_j} + \sqrt{\lambda_k})}} - \frac{1}{\sqrt{\lambda_i(\sqrt{\lambda_i} + \sqrt{\lambda_k})}} \right] \right) = \\
= \frac{5}{16} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-2} + \sum_{i \neq j} \left( \frac{1}{4} \epsilon_p(\lambda_i) \lambda_i^{-1} \lambda_j^{-1} + \frac{1}{2} \epsilon_p(\lambda_i) \lambda_i^{-3/2} \lambda_j^{-3/2} \right) + \\
+ \frac{1}{4} \sum_{i \neq j \neq k} \epsilon(\lambda) \lambda_i^{-1} \lambda_j^{-1/2} \lambda_k^{-1/2}.
\]

In the last transformation we used the fact that

\[
\sum_{j \neq k} \frac{1}{(\lambda_i - \lambda_j)} \left[ \frac{1}{\sqrt{\lambda_j(\sqrt{\lambda_j} + \sqrt{\lambda_k})}} - \frac{1}{\sqrt{\lambda_j(\sqrt{\lambda_i} + \sqrt{\lambda_k})}} \right] = \\
= \frac{1}{4} \sum_{j \neq k} \left\{ \frac{1}{(\lambda_i - \lambda_j)} \left[ \frac{1}{\sqrt{\lambda_j(\sqrt{\lambda_j} + \sqrt{\lambda_k})}} - \frac{1}{\sqrt{\lambda_j(\sqrt{\lambda_i} + \sqrt{\lambda_k})}} \right] \right\} + (j \leftrightarrow k).
\]
Finally, (4.27) can be rewritten as

\[
\frac{1}{16} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-2} + \frac{1}{4} \sum_i \epsilon_p(\lambda_i) \lambda_i^{-1} \sum_{j,k} \lambda_j^{-1/2} \lambda_k^{-1/2} = \\
= \sum_n \left( \sum_i \epsilon_p(\lambda_i) \lambda_i^{-n-2} \right) \left\{ \frac{1}{16} \delta_{n,-1} \left[ 2 \sum_j \lambda_j^{-1/2} \right]^2 + \frac{1}{16} \delta_{n,0} \right\}. \quad (4.29)
\]

Now taking together (4.19), (4.23) and (4.29) we obtain our main result:

\[
\frac{e^{-\frac{2}{3} \text{tr} \Lambda^{3/2}}}{C(\sqrt{\Lambda}) Z \{ T \}} \left[ \text{tr} \epsilon_p \left\{ \frac{\partial^2}{\partial \Lambda^2} - \Lambda \right\} \right] C(\sqrt{\Lambda}) e^{\frac{2}{3} \text{tr} \Lambda^{3/2}} Z \{ T \} = \\
= \frac{1}{Z} \sum_{n \geq -1} \text{tr}(\epsilon_p \Lambda^{-n-2}) \left\{ \frac{1}{2} \sum_{k \geq \delta_{n+1,0}} k T_k \frac{\partial}{\partial T_{2n+k}} + \frac{1}{4} \sum_{a+b=2n} \sum_{a \geq 0, b \geq 0} \frac{\partial^2}{\partial T_a \partial T_b} + \right. \\
+ \frac{1}{16} \delta_{n,0} + \left. \frac{1}{4} \delta_{n+1,0} T_1^2 - \frac{\partial}{\partial T_{2n+3}} \right\} Z(T) = 0. \quad (4.30)
\]

This completes the derivation of (4.2) and, thus, of Virasoro constraints for the case \( K = 2 \).

### 4.3 Example of \( K = 3 \)

In the case of generic \( K \) the analogue of the derivation, described in the previous section, should involve the following steps.

- Represent \( F[\Lambda] \) as

\[
F^{(K)}[\Lambda] = g_K[\Lambda] Z^{(K)}(T_n), \quad (4.31)
\]

with

\[
g_K[\Lambda] = \frac{\Delta(M)}{\Delta(\Lambda)} \prod_i \left[ \psi''(\mu_i)^{-1/2} e^{(\psi(\mu_i) - \psi(\mu_i))} \right] = \frac{\Delta(\Lambda^{1/K})}{\Delta(\Lambda)} \prod_i \left[ \lambda_i^{-\frac{K-1}{K}} e^{\alpha^{-1}_K \psi^{K-1/2}(\mu_i)} \right]. \quad (4.32)
\]

Parameter \( \alpha \) is introduced here for the sake of convenience, in fact, \( \alpha = 1 \).

- Substitute this \( F^{(K)}[\Lambda] \) into (4.1), which in the particular case of \( \psi_K(X) = \frac{X^{K+1}}{(K+1)} \), looks like
\begin{align}
\{ \text{Tr } \epsilon(\Lambda)[(\frac{\partial}{\partial \Lambda_{\text{tr}}})^{K} - \alpha^{K} \Lambda] \} g_{K} [\Lambda] Z^{(K)}(T_{n}) = 0. \tag{4.33} \end{align}

When \( \partial/\partial \Lambda_{\text{tr}} \) acts on \( Z(T_{n}) \), the following rule is applied:

\begin{align}
\frac{\partial Z}{\partial \Lambda_{\text{tr}}} = - \frac{1}{K} \sum_{n \geq 1} (n + K) T_{n+K} \frac{\partial Z}{\partial T_{n}}, \tag{4.34} \end{align}

and the sum at the r.h.s. goes over all positive \( n \neq 0 \) mod \( K \). As to higher-order derivatives, \( \frac{\partial^{i} Z}{\partial \Lambda_{\text{tr}}^{i}} \), they are defined with the help of relations like (4.8).

— When all the \( \Lambda \)-derivatives in (4.1) act on exponent in \( g[\Lambda] = g[\mathcal{V}'(M)] \), we get the term, which is equal to \( \mathcal{V}'(\partial \text{Tr}(M \mathcal{V}'(M) - \mathcal{V}(M)) / \partial \Lambda_{\text{tr}}) = \mathcal{V}'(M) = \Lambda \) and cancels \( \Lambda \)-term in (4.1). The next contribution, when all but one of the \( \Lambda \)-derivatives act on the exponent, vanishes. Actually this reflects the fact that there are no \( \mathcal{W}^{(1)} \)-generators among the final \( \mathcal{W} \)-constraints.

— Perform a shift of variables

\begin{align}
T_{n} \to \hat{T}_{n} = T_{n} - \frac{\alpha}{n} K \delta_{n,K+1} \tag{4.35} \end{align}

(this procedure doesn’t change \( \partial/\partial T_{n} \to \partial/\partial \hat{T}_{n} \)).

— After all these substitutions the l.h.s. of eq.(4.33) acquires the form of an infinite series where every item is a product of \( \text{Tr}[\tilde{\epsilon}(M)M^{-p}] \) and a linear combination of generators of \( \mathcal{W}_{K} \)-algebra, acting on \( Z^{(K)}(T_{n}) \). In the case of \( K = 3 \) this equation looks like

\begin{align}
\frac{1}{27} \text{Tr} \left[ \tilde{\epsilon}(M) M^{-3} \left\{ \sum_{n \geq -2} M^{-3n} \mathcal{W}_{3n}^{(3)} + \right. \right.
\left. + 9 \sum_{n \geq -2} M^{-3n-1/3} \left\{ \sum (3k - 2)\hat{T}_{3k-2} \mathcal{W}_{3n+3k}^{(2)} + \sum_{a+b=3n, a \geq 0, n \geq -3} \frac{\partial}{\partial T_{3a+1}} \mathcal{W}_{3b-3}^{(2)} \right\} \right. \right.
\left. + 9 \sum_{n \geq -2} M^{-3n-2/3} \left\{ \sum (3k - 2)\hat{T}_{3k-2} \mathcal{W}_{3n+3k}^{(2)} + \sum_{a+b=3n, a \geq 0, n \geq -3} \frac{\partial}{\partial T_{3a+1}} \mathcal{W}_{3b-3}^{(2)} \right\} \right\} \right] Z^{(3)} = 0, \tag{4.36} \end{align}

(see (1.27), (1.28) for explicit expressions for \( \mathcal{W} \)-operators). This relation is valid for any value of \( N \), and in this sense is an example of identity, which behaves smoothly in the
limit $N \to \infty$.

— If $N = \infty$ all the quantities $\text{Tr} \tilde{\epsilon}(M)M^{-p}$ with given $p$ but varying $\tilde{\epsilon}(M)$ become independent, and (4.36) may be said to imply (1.26).

In what follows we restrict ourselves to an illustrative calculation: we prove that (4.36) holds for $N = 1$ (i.e. when there are numbers instead of matrices). Even this calculation is long enough. Additional technical details, necessary to deal with matrices rather than ordinary numbers, are exhaustively discussed in the previous section 4.2.

Example. When (4.32) with $K = 3$, $N = 1$ is substituted into (4.33) and the rule (4.34) is used, we obtain ($Z \equiv Z^{(3)}_1$):

\[
0 = \tilde{\epsilon}(\mu) \left\{ \sum_{n \geq 1} \frac{\alpha^2}{\mu^{n+1}} \frac{\partial Z}{\partial T_n} - \frac{\alpha}{\mu^5} \left[ \frac{1}{3} \sum_{m,n} \frac{1}{\mu^{m+n}} \frac{\partial^2 Z}{\partial T_m \partial T_n} + \sum_n \frac{n + 4}{3} \frac{1}{\mu^n} \frac{\partial Z}{\partial T_n} + \frac{7}{9} Z \right] + \right. \\
+ \frac{1}{\mu^3} \left[ \sum_{l,m,n} \frac{1}{27 \mu^{l+m+n}} \frac{\partial^3 Z}{\partial T_l \partial T_m \partial T_n} + \sum_{m,n} \frac{n + m + 8}{18} \frac{1}{\mu^{m+n}} \frac{\partial^2 Z}{\partial T_m \partial T_n} + \right. \\
\left. \left. \left. + \sum_n \frac{n^2 + 12n + 39}{27} \frac{1}{\mu^n} \frac{\partial Z}{\partial T_n} + \frac{28}{27} Z \right] \right\} \\
(4.37)
\]

Our purpose now is to compare this expression with the $l.h.s.$ of (4.36). In order to understand the structure of (4.36) let us note, that the first item (with $W^{(3)}$) in (4.36) contains the contribution $\frac{1}{\mu^{l+m+n}} \frac{\partial^3 Z}{\partial T_l \partial T_m \partial T_n}$, just the same as in (4.37), but only under restriction $l + m + n = 0 \mod 3$. However, no such restriction is imposed in (4.37), and in order to restore the equivalence between (4.36) and (4.37), one needs to add the terms with $\partial(W^{(2)} Z)/\partial T$ to the $l.h.s.$ of (4.36), which add the missing contributions $\frac{1}{\mu^{l+m+n}} \frac{\partial^3 Z}{\partial T_l \partial T_m \partial T_n}$ with $l + m + n = 1, 2 \mod 3$.

If we analyze the terms without $\alpha$ in (4.37), it is important that $\alpha$ appears also in the shift (4.35):

\[
n\hat{T}_n^{(3)} = nT_n - 3\alpha \delta_{n,4},
\]

so that at the same time we substitute all $\hat{T}$‘s by $T$‘s in (4.36), (1.26)–(1.28). Then the term $\frac{(K - 1)(3K - 1)(5K - 1)...((2K - 1)K - 1)}{(2K)^K} Z = \frac{28}{27} Z$ in (4.37) should be compared to the contributions without $Z$-derivatives to (4.36). These come from the negative
harmonics of $W^{(3)}$ and $W^{(2)}$ operators. Namely, so modified $W^{(3)}_{-6}$ contains $(2T_2)^3 + 3(T_1)^2(4T_4) = \frac{4}{\mu^6}$, $W^{(3)}_{-3}$ contains $(T_1)^3 = \frac{1}{\mu^3}$, and there is no constant term in $W^{(3)}_0$. As to Virasoro operators, there is $1 + \mathcal{O}(\alpha)$ in $W^{(3)}_{-3}$ and $\mathcal{O}(\alpha)$ in $W^{(3)}_0$. Also the contribution of interest to $\partial W^{(2)}_{-3}/\partial T_1$ is $\frac{1}{6} \cdot 2(T_1)(2T_2) = \frac{1}{3\mu^2}$, and to $\partial W^{(2)}_{-3}/\partial T_2$ is $\frac{1}{6} \cdot 4(T_1) = \frac{2}{3\mu}$. Also $(2T_2)W^{(2)}_{-3}$ contributes $\frac{1}{6} \cdot 2(T_1)(2T_2)^2 = \frac{1}{3\mu^5}$, $(5T_5)W^{(2)}_0 - (5T_5) \cdot \frac{1}{9} = \frac{1}{9\mu^5}$, $(2T_2)W^{(2)}_0 - (2T_2) \cdot \frac{1}{9} = \frac{1}{9\mu^2}$, $(T_1)W^{(2)}_{-3} - \frac{1}{6} \cdot 2(T_1)(2T_2)^2 = \frac{1}{3\mu^4}$, $(4T_4)W^{(2)}_0 - (4T_4) \cdot \frac{1}{9} = \frac{1}{9\mu^4}$, and $(T_1)W^{(2)}_0 - (T_1) \cdot \frac{1}{9} = \frac{1}{9\mu}$. Putting this all together we obtain the coefficient

$$\frac{1}{27} \left\{ [4 + 1] + 9[\frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9}] + 9[\frac{2}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9}] \right\} + \mathcal{O}(\alpha) = \frac{28}{27} + \mathcal{O}(\alpha)$$

in front of $Z$ in (4.36), in accordance with (4.37). In order to reproduce the term $\frac{(K - 1)(K + 1)(2K + 1)}{24K} \alpha^{K - 2} Z = \frac{7}{9} \alpha Z$ in (4.37) it is necessary to restore the shift (4.38) of $T_1$-variable in (4.36). There are also contributions with $T_4^2$ in $W^{(3)}$ and in $(3k - 2)T_{3k-2}W^{(2)}_{3k+3h}$, which are responsible to occurrence of the $\alpha^2$-term in (4.37).

This illustrative comparison of (4.36) and (4.37) is enough to recognize the proper structure of (4.36). After it is found out it is easy to ensure that all the remaining contributions to (4.36) and (4.37) also coincide. Of course, in order to check the trace-structure of (4.36) our simple example of $N = 1$ is not enough — the analogue of detailed consideration of sect.4.2 is required. Such detailed derivation, as well as consideration of the case of $K > 3$, is beyond the scope of this paper.

5 Conclusion

To conclude, we presented enough evidence that the Generalized Kontsevich Model (1.2) interpolates between all the multimatrix models, while preserving the property of integrability and the $\mathcal{L}_{-1}$-constraint. This makes GKM a very appealing candidate for the role of a theory, which could unify all “stable” (i.e. with $c \leq 1$) bosonic string models. However, the study of GKM was at most originated here. Let us list a set of problems, which seem interesting for the future development.
1) Though integrable structure and $L_{-1}$-constraint in GKM have been discussed more or less exhaustively, the situation with generic $W$-constraints and their implications remains less satisfactory. Of course, they may be derived from $L_{-1}$-constraint, as suggested in refs.[1], but still it is desirable:

— to complete the derivation of $W$-constraints from Ward identity (1.23) for $N = \infty$, as suggested in sect.3, for $K \geq 3$;

— to prove the equivalence of these $W$-constraints (which look like (1.26)) to conventional constraints (1.29);

— to describe the subvariety $U_\infty$ in Grassmannian (at infinity of the Universal module space), specified by the universal $L_{-1}^{(\psi)}$-constraint;

— to take double-scaling continuum limit of the properly reduced $\tilde{W}$-constraints in discrete multimatrix models (which look like $\tilde{W}_{q-p}^{(p+1)}[M_1] = \tilde{W}_{p-q}^{(q+1)}[M_2]$ for the 2-matrix case $[\tilde{W}]$) in order to derive the constraints (1.31) for $\sqrt{\Gamma_{d-1}^{(K-1)}}$ (as it was done in ref.[16] for the 1-matrix case);

— also the problems discussed in sect.3.4 should be added to this list.

2) We tried to argue, that particular $(K-1)$-matrix model arises from GKM for a particular choice of potential: $V(X) \equiv V_K(X) = \text{const} \cdot X^{K+1}$. Particular $(CFT + 2d$ gravity)$-model is specified by additional adjustment of $M$-matrix in such a way, that all $T_n = 0$, $n \neq 1$, $p + K$ (for $c = 1 - \frac{6(p - K)^2}{pK}$). These particular points in the space of parameters $\{V(X), M\}$ should be interpreted as critical points of GKM. The relevant questions are:

— what is the proper criterium, distinguishing critical points in terms of GKM itself;

— are there any other critical points?

One can even hope, that the answer to the last question is positive and some other critical points may be identified with non-bosonic string models. Let us remind that the stability criterium like $c \leq 1$ (implying the absence of tachionic instabilities, leading to desintegration of Riemann surfaces through creation of holes), which in bosonic case implies the restriction $d \leq 2$ for the space time dimension, in the case of $N = 2$ superstrings turns into a much more promising condition: $d \leq 4$.

3) A separate class of problems is related to identification of (double scaling limit) $c = 1$ models in the theory of GKM. This could lead to a more natural formulation of
GKM, in particular, help to restore the symmetry between arguments $\mathcal{V}(X)$ and $M$ of $Z^{(\nu)}[M]$. Keeping in mind the topological implications of $K = 2$ model it is also natural to ask,

— what corresponds to Penner model [38, 39].

4) The last group of problems concerns the meaning and generalizations of GKM. First of all, both formulations in terms of topological (like [15, 40, 38, 41, 39]) and conformal (in the spirit of [42, 43]) is required in order to understand explicitly the connection to Liouville theory. Second, the hope that just GKM (1.2) is enough to include $N = 2$ superstrings may be too optimistic, and then one needs to look for more universal models. Third, in any case one should search for a more invariant (algebro-geometric) interpretation of (1.2): GKM is already much simpler than original multi-matrix models (it is a 1-matrix model with a smooth continuum limit), but still it is not simple enough to be appealing as a fundamental theory. The main aesthetic problem is the asymmetry between $M$ and $\mathcal{V}(X)$ and, perhaps, the very relevance of matrix integrals. Fourth, GKM does describe interpolation between sufficiently many string models, but it is not a dynamical interpolation: one may still need to look for an “of-shell” version of GKM, involving a sort of integration over $\mathcal{V}(X)$. This problem can be after all related to the question about critical points. In particular, it may be interesting to find out something similar to the “off-shell” actions of refs.[17, 44, 45], which reproduce Virasoro constraints in the ordinary 1-matrix models as dynamical equations of motion, for the case of GKM. Possible links between GKM and double-loop algebras (in the spirit of [46]) also deserve investigation.

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