Current algebras on $S^3$ of complex Lie algebras  
( Revised version )

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Abstract

This is a full revised version of the previous same titled article. The 2-cocycle of the central extension of the current algebra on $S^3$ is taken the place of a new quaternion valued one, that is defined by the boundary Dirac operator. And we added a discussion on the symmetric invariant bilinear forms associated to the central extension $\hat{g}$.

A current algebra is a Lie algebra of smooth mappings of a given manifold into a Lie algebra, or its central extension. Affine Kac-Moody algebra is an example where the manifold is $S^1 \subset \mathbb{C} \setminus \{0\}$. We shall extend this theory to a Lie algebra of smooth mappings of $S^3 \subset \mathbb{C}^2 \setminus \{0\}$ into a complex simple Lie algebra $g$. Let $L$ be the $\mathbb{C}$--algebra generated by the Laurent polynomial type harmonic spinors over $\mathbb{C}^2 \setminus \{0\}$. Here we mean “harmonic” as the zero-mode of the Dirac operator on $\mathbb{C}^2$. The real part $K$ of $L$ becomes a commutative real subalgebra of $L$. For a simple Lie algebra $g$, the $g$-current algebra is defined as the ( real ) Lie algebra $Lg$ that is generated by $L \otimes_{\mathbb{R}} g$. For the Cartan subalgebra $h$ of $g$, $Kh = K \otimes_{\mathbb{R}} h$ becomes a Cartan subalgebra of $Lg$. We have the weight space decomposition of $Lg$ with respect to the Cartan subalgebra $Kh$. We introduce a quaternion valued 2-cocycle on the $g$-current algebra $Lg$. Then we have the central extension $Lg \oplus Hc$ associated to the 2-cocycle. Adjoining a derivation coming from the radial vector field $\frac{\partial}{\partial n}$ on $S^3$ we obtain the second central extension $\hat{g} = Lg \oplus Hc \oplus Rd$. We shall investigate the root space decomposition and the Chevalley generators of $\hat{g}$.
1 Introduction

An affine Kac-Moody algebra of untwisted type can be realized in terms of a central extension of the loop algebra of a simple Lie algebra, \([K]\). Let

\[ L = \mathbb{C}[t, t^{-1}] \]

be the algebra of Laurent polynomials \(\sum_i a_i t^i\). The residue function \(\text{Res} : L \to \mathbb{C}\) is given by \(\text{Res}(\sum_i a_i t^i) = a_{-1}\). Given a simple Lie algebra \(\mathfrak{g}\), then \(L\mathfrak{g} = L \otimes_{\mathbb{C}} \mathfrak{g}\) may be made into a Lie algebra in a unique way satisfying

\[ [P \otimes x, Q \otimes y] = PQ [x, y] , \quad \text{for } P, Q \in L , x, y \in \mathfrak{g} . \]

\(L\mathfrak{g}\) is called the loop algebra of \(\mathfrak{g}\). Let \(\langle \cdot | \cdot \rangle\) be the invariant bilinear form on \(\mathfrak{g}\). We define a bilinear form

\[ \langle \cdot | \cdot \rangle : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}[t, t^{-1}] \]

by \(\langle P \otimes x | Q \otimes y \rangle = PQ \langle x | y \rangle\). We define a 2-cocycle \(\kappa\) on the Lie algebra \(L\mathfrak{g}\) by the formula

\[ \kappa(P \otimes x, Q \otimes y) = \text{Res}\left( \frac{dP}{dt} Q \right) \langle x | y \rangle . \]

Let \(L\mathfrak{g} \oplus \mathbb{C}c\) be the extension of \(L\mathfrak{g}\) by the 1-dimensional center \(\mathbb{C}c\) associated to the 2-cocycle \(\kappa\) whose Lie multiplication is given by

\[ [a + \lambda c , b + \mu c] = [a , b] + \kappa(a, b)c . \]

The Euler derivation \(t \frac{d}{dt}\) acts on \(L\mathfrak{g} \oplus \mathbb{C}c\) as an outer derivation; \(\Delta(a + \lambda c) = t \frac{d}{dt} a\) for \(a \in L\mathfrak{g}, \lambda \in \mathbb{C}\). Then adjoining the derivation \(\Delta\) to \(L\mathfrak{g} \oplus \mathbb{C}c\) we have the Lie algebra

\[ \hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d \]

by defining the Lie product as

\[ [a + \gamma c + \lambda d , b + \delta c + \mu d] = [a , b] + \kappa(a, b)c + \lambda \Delta(b) + \mu \Delta(a) . \]
We follow this procedure to have a central extension of the current algebra on $S^3$. While the Laurent polynomial type harmonic spinors over $C^2 \setminus \{0\}$ do not constitute an algebra we shall consider the $C-$algebra generated by the Laurent polynomial type harmonic spinors which we call the \textit{algebra of current over $S^3$} and denote by $\mathcal{L}$. It plays the same role as the algebra $L = C[t, t^{-1}]$ of Laurent polynomials over $S^1$ does. The \textit{current algebra of $g$} is the real Lie algebra $L_g$ that is generated by $\mathcal{L} \otimes_C g$. We shall introduce a 2-cocycle on $\mathcal{L}$ that takes values in the quaternions $\mathbb{H}$. Then it is extended to a 2-cocycle on the current algebra $L_g$. For this purpose we prepare in section 2 a rather long introduction to our previous results on analysis of harmonic spinors on $C^2$, \cite{F,GM,Ko1,Ko2,Ko3} and \cite{K-I}, that is, we develop some parallel results as in classical analysis; the separation of variable method for Dirichlet problem, the expansion by eigenfunctions of Laplacian, Cauchy integral formula for holomorphic functions and Laurent expansion of meromorphic functions etc.. For example, the Dirac operator on spinors corresponds to the Cauchy-Riemann operator on complex functions. Let $\Delta = \mathbb{H}^2$ be the 4-dimensional spinor space, that is, an irreducible representation of the complex Clifford algebra $\text{Clif}^c_4 = \text{End}(\Delta)$. The algebraic basis of $\text{Clif}^c_4$ is given by the Dirac matrices: 
\[
\gamma_k = \begin{pmatrix} 0 & -i \sigma_k \\ i \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,
\]
and
\[
\gamma_4 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.
\]
Where $\sigma_k$ are Pauli matrices. Let $S = \mathbb{R}^4 \times \Delta$ be the spinor bundle. The Dirac operator is defined by the following formula:
\[
D = -\frac{\partial}{\partial x_1} \gamma_4 - \frac{\partial}{\partial x_2} \gamma_3 - \frac{\partial}{\partial x_3} \gamma_2 - \frac{\partial}{\partial x_4} \gamma_1 : C^\infty(\mathbb{R}^4, S) \longrightarrow C^\infty(\mathbb{R}^4, S).
\]
Let $S^{\pm} = \mathbb{R}^4 \times \Delta^{\pm}$ be the (even and odd) half spinor bundle corresponding to the decomposition $\Delta = \Delta^+ \oplus \Delta^-$. \cite{F} has the polar decomposition: $D = \gamma_+ \left( \frac{\partial}{\partial n} - \tilde{\varphi} \right)$ with the tangential (nonchiral) component $\tilde{\varphi}$ and the radial vector $\frac{\partial}{\partial n}$. The tangential Dirac operator $\varphi$ on $S^3$ is a self adjoint elliptic differential operator. The eigenvalues of $\varphi$ are $\{ \frac{m}{2}, -\frac{m+3}{2} ; m = 0, 1, \cdots \}$ with multiplicity $(m + 1)(m + 2)$. We have an explicitly written polynomial formula of eigen-spins $\{ \phi^{+(m,l,k)}, \phi^{-(m,l,k)} \}_{0 \leq l \leq m, 0 \leq k \leq m+1}$ corresponding to the eigenvalues $\frac{m}{2}$ and $-\frac{m+3}{2}$ respectively that give rise to a complete orthonormal system in $L^2(S^3, S^+)$, \cite{Ko1,Ko2}. A spinor $\phi$ on a domain $G \subset C^2$ is called a \textit{harmonic spinor on $G$} if $D\phi = 0$. Each $\phi^{+(m,l,k)}$ is extended to a harmonic spinor on $C^2$, while each $\phi^{-(m,l,k)}$ is extended to a harmonic spinor on $C^2 \setminus \{0\}$ that is regular at infinity. Every harmonic spinor $\varphi$ on $C^2 \setminus \{0\}$ has
an expansion by the basis $\phi^{\pm(m,l,k)}$:

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z).$$

We call the above series a *Laurent polynomial type spinor* if finitely many coefficients $C_{-(m,l,k)}$ are non-zero, and the coefficients

$$\begin{pmatrix} C_{-(0,0,1)} \\ C_{-(0,0,0)} \end{pmatrix}$$

is called the *quaternion residue* of $\varphi$ and is denoted by $qRes[\varphi]$. The space of Laurent polynomial type spinors is denoted by $C[\phi^{\pm}]$. Let $H$ be the algebra of quaternion numbers. We look an even spinor also as a $H$-valued smooth function: $C^\infty(S^3,S^+) = C^\infty(S^3,H)$, so that the space of spinors $C^\infty(S^3,S^+)$ is endowed with a multiplication rule:

$$\phi_1 \cdot \phi_2 = \begin{pmatrix} u_1u_2 - \bar{v}_1v_2 \\ v_1u_2 + \bar{u}_1v_2 \end{pmatrix}, \quad \text{for } \phi_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad i = 1,2. \quad (1.1)$$

$C[\phi^{\pm}]$ does not constitute an algebra, as we see by the example $\phi^{+(1,0,0)} \cdot \phi^{-(0,0,0)}$ which is not a harmonic spinor. The *algebra of current $L$ on $S^3$* is a $C-$subalgebra of $C^\infty(S^3,S^+)$ (or rather $C^\infty(C^2 \setminus \{0\},S^+)$) that is generated by $C[\phi^{\pm}]$. In section 3 we introduce a 2-cocycle on $L$. For spinors $\varphi, \psi \in L$ we put

$$A(\varphi, \psi) = qRes[\varphi \cdot \psi - \bar{\varphi} \cdot \psi]. \quad (1.2)$$

Then $A$ gives a $H-$valued skew-symmetric bilinear form on $L$ and satisfies the cocycle condition:

$$A(\phi_1\phi_2, \phi_3) + A(\phi_2\phi_3, \phi_1) + A(\phi_3\phi_1, \phi_2) = 0. \quad (1.3)$$

Hitherto we have prepared the space of spinors $C^\infty(S^3,S^+)$ and the algebra of current $L$ that will play the role of coefficients of our current algebra discussed below. These are complex algebras. On the other hand $C^\infty(S^3,S^+) \simeq C^\infty(S^3,H)$ has a $H$-module structure, while our basic interest is on the real Lie algebra generated by $L \otimes_{C} g$. In such a way it is frequent that we deal with the fields $H$, $C$ and $R$ in one formula. So to prove a steady point of view for our subjects we shall introduce the concept of quaternion Lie algebras, $[Kq]$. First we note that a quaternion module $V = H \otimes_{C} V_o = V_o + JV_o$, $V_o$
being a \( \mathbb{C} \)-module, has two involutions \( \sigma \) and \( \tau \):

\[
\sigma(u + Jv) = u - Jv, \quad \tau(u + Jv) = u + Jv, \quad u, v \in V_0.
\]

A \textit{quaternion Lie algebra} \( q \) is defined as a real submodule of a quaternion module \( V \) that is endowed with a real Lie algebra structure compatible with the involutions \( \sigma \) and \( \tau \):

\[
\sigma q \subset q, \quad \sigma[x, y] = [\sigma x, \sigma y], \quad \tau[x, y] = [\tau x, \tau y] \quad \text{for} \quad x, y \in q.
\]

For a complex Lie algebra \( g \) the \textit{quaternionification} of \( g \) is a quaternion Lie algebra \( g^q \) that is generated (as a real Lie algebra) by \( H \otimes \mathbb{C} g \). For example, \( \mathfrak{so}^*(2n) = H \otimes \mathbb{C} \mathfrak{so}(n, \mathbb{C}) \) is the quaternionification of \( \mathfrak{so}(n, \mathbb{C}) \). \( \mathfrak{sl}(n, \mathbb{H}) \) is the quaternionification of \( \mathfrak{sl}(n, \mathbb{C}) \) though \( H \otimes \mathbb{C} \mathfrak{sl}(n, \mathbb{C}) \) is not a Lie algebra. The algebra of current \( \mathcal{L} \) is a quaternion Lie algebra. In fact \( \mathcal{L} \) is a real submodule of \( C^\infty(S^3, \mathbb{H}) \) that is invariant under the involutions \( \sigma \) and \( \tau \). The real part \( K = \{ \phi \in \mathcal{L}; \sigma \phi = \phi, \tau \phi = \phi \} \) plays an important role. \( K \) is a commutative normal subalgebra of \( \mathcal{L} \), and satisfies the condition \([K, \mathcal{L}] = 0\).

Let \( g \) be a simple Lie algebra that we suppose to be a subalgebra of \( \mathfrak{gl}(n, \mathbb{C}) \). Let \( \mathcal{L}g \) be the quaternion Lie algebra generated by \( \mathcal{L} \otimes \mathbb{C} g \) with the Lie bracket defined by

\[
[\phi_1 \otimes X_1, \phi_2 \otimes X_2] = (\phi_1 \cdot \phi_2) \otimes (X_1 X_2) - (\phi_2 \cdot \phi_1) \otimes (X_2 X_1)
\]

for \( \phi_1, \phi_2 \in \mathcal{L}, X_1, X_2 \in \mathfrak{g} \). Here the right hand side is the bracket of the tensor product of the associative algebra \( \mathcal{L} \) and the matrix algebra \( \mathfrak{g} \). \( \mathcal{L}g \) is called the \( \mathfrak{g} \)-\textit{current algebra}. Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{g} \). Let \( K \mathfrak{h} = K \otimes \mathbb{R} \mathfrak{h} \). We find that \( K \mathfrak{h} \) is a Cartan subalgebra of \( \mathcal{L} \mathfrak{g} \). It extends the adjoint representation \( \text{ad}_\mathfrak{h} : \mathfrak{h} \rightarrow \text{End}_\mathbb{C}(\mathfrak{g}) \) to the adjoint representation \( \text{ad}_{K \mathfrak{h}} : K \mathfrak{h} \rightarrow \text{End}_\mathbb{C}(\mathcal{L} \mathfrak{g}) \). The associated weight space decomposition of \( \mathcal{L} \mathfrak{g} \) with respect to \( K \mathfrak{h} \) will be given. We find that the space of non-zero weights of \( \mathcal{L} \mathfrak{g} \) corresponds bijectively to the root space of \( \mathfrak{g} \). Let \( \mathfrak{g}_\lambda \) be the root space of root \( \lambda \) and let \( \Phi^\pm \) be the set of positive (respectively negative) roots of \( \mathfrak{g} \). Then we have the triangular
decomposition of the $g$-current algebra:

$$L_g = L_h + L_e + L_f, \quad \text{(direct sum)},$$

with

$$L_e = \sum_{\lambda \in \Phi^+} L \otimes_R g_{\lambda}, \quad L_f = \sum_{\lambda \in \Phi^-} L \otimes_R g_{\lambda}.$$ 

$L_h$ has the weight 0: $[K_h, L_h] = 0$.

We discuss in section 5 our central subject to give the central extension of $g$-current algebra. We extend the $H$-valued 2-cocycle $A$ on $L$ to a 2-cocycle on $L_g$ by the formula

$$A(\phi \otimes X, \psi \otimes Y) = (X|Y)A(\phi, \psi), \quad \phi, \psi \in L, X, Y \in g,$$

where $(X|Y) = \text{Trace}(XY)$ is the Killing form of $g$. Then we have the associated central extension:

$$L_g(c) = L_g \oplus Hc,$$

which is a quaternion Lie algebra. The radial vector field $\frac{\partial}{\partial n}$ on $C^2 \setminus 0$ acts on $L_g(c)$ as an outer derivation. Then, adjoining the derivation $\frac{\partial}{\partial n}$, we have the second central extension:

$$\hat{g} = L_g(c) \oplus Cd.$$

(Actually we adopt the prolonged radial derivation; $2|z|^3 \frac{\partial}{\partial m}$.) We shall investigate the root space decomposition of $\hat{g}$. For a root $\alpha \in \Phi$, let $g_\alpha = \{ x \in g; [h, x] = \alpha(h)x, \forall h \in h \}$ denote the root space of $\alpha$. Put

$$\hat{h} = h \oplus Hc \oplus Cd,$$

$\hat{h}$ is a commutative subalgebra of $\hat{g}$ and $\hat{g}$ is decomposed into a direct sum of the simultaneous eigenspaces of $ad(\hat{h})$, $\hat{h} \in \hat{h}$, and $\Phi \subset h^*$ is regarded as a subset of $\hat{h}^*$.

We introduce $\Lambda \in \hat{h}^*$; as the dual elements of $c$, and $\delta \in \hat{h}^*$ as the dual element of $d$. Then $\alpha_1, \cdots, \alpha_l, \delta, \Lambda$ give a basis of $\hat{h}^*$. The set of simple root are

$$\hat{\Phi} = \left\{ \frac{m}{2} \delta + \alpha; \quad \alpha \in \Phi, m \in \mathbb{Z} \right\} \bigcup \left\{ \frac{m}{2} \delta; \quad m \in \mathbb{Z} \right\}.$$

$\hat{g}$ has the weight space decomposition:

$$\hat{g} = \bigoplus_{m \in \mathbb{Z}} \hat{g}_{\frac{m}{2} \delta} \oplus \bigoplus_{\alpha \in \Phi, m \in \mathbb{Z}} \hat{g}_{\frac{m}{2} \delta + \alpha}.$$
Each weight space is given as follows.

\[ \widehat{g}_{\frac{m}{2}\alpha} = \mathcal{L}[m] \otimes \mathfrak{g}_\alpha, \quad \text{for } \alpha \neq 0 \text{ and } m \in \mathbb{Z}, \]

\[ \widehat{g}_0 = (\mathcal{L}[0]\mathfrak{h}) \oplus \mathbb{H} \oplus \mathbb{C} \supset \widehat{\mathfrak{h}}, \]

\[ \widehat{g}_{\frac{m}{2}} = \mathcal{L}[m] \otimes \mathfrak{h}, \quad \text{for } 0 \neq m \in \mathbb{Z}. \]

Where \( \mathcal{L}[m] \) is the subspace of \( \mathcal{L} \) constituting of those elements \( \phi \in \mathcal{L} \) that are of homogeneous degree \( m \). \( \mathcal{L}[0]\mathfrak{h} \) is the Lie subalgebra generated by \( \mathcal{L}[0] \otimes \mathfrak{h} \).

2 Spinor analysis on \( S^3 \subset \mathbb{C}^2 \).

Here we prepare a fairly long preliminary of spinor analysis on \( \mathbb{R}^4 \) because I think various subjects belonging to quaternion analysis or detailed properties of harmonic spinors of the Dirac operator on \( \mathbb{R}^4 \) are not so familiar to the readers. We refer to [F, Ko1] for the exposition on Dirac operators on \( \mathbb{R}^4 \) and to [D-S-Sc, G-M, Ko2] for the function theory of harmonic spinors. Subsections 2.1, 2.2, 2.3 are to remember the theory of harmonic spinors.

2.1 Spinors and the Dirac operator on \( \mathbb{R}^4 \).

2.1.1

Let \( \mathbf{K} \) be the field \( \mathbb{R} \) or \( \mathbb{C} \). Let \( V \) be a \( \mathbf{K} \)-vector space equipped with a quadratic form \( q \) over the field \( \mathbf{K} \). The Clifford algebra \( C_{\mathbf{K}}(V, q) \) is a \( \mathbf{K} \)-algebra which contains \( V \) as a sub-vector space and is generated by the elements of \( V \) subject to the relations

\[ v_1v_2 + v_2v_1 = 2q(v_1, v_2), \]

for \( v_1, v_2 \in V \). In the sequel we denote \( \text{Clif}_n = C_{\mathbb{R}}(\mathbb{R}^n, -x_1^2 - \cdots - x_n^2) \) and \( \text{Clif}^c_n = C_{\mathbb{C}}(\mathbb{C}^n, z_1^2 + \cdots + z_n^2) \). It holds \( \text{Clif}^c_n = \text{Clif}^c_n \otimes_{\mathbb{R}} \mathbb{C} \). We have an important isomorphism:

\[ \text{Clif}^c_{n+2} = \text{Clif}^c_n \otimes_{\mathbb{C}} \mathbb{C}(2). \quad (2.1) \]

Here \( \mathbf{K}(m) \) denotes the algebra of \( m \times m \)-matrices with entries in the field \( \mathbf{K} \). Let \( \mathbf{H} \) be the algebra of quaternion numbers that are formed on \( \mathbb{R} \) by the symbols \( i, j, k \). The left multiplication of \( \mathbf{H} \) yields an endomorphism of \( \mathbf{H} \); \( \mathbf{H} \simeq \text{End}_H \mathbf{H} \simeq \mathbb{C}(2) \). Then the
corresponding matrices to $i, j, k \in H$ are given by $i\sigma_3, i\sigma_2, i\sigma_1$. Where $\sigma_k$, $k = 1, 2, 3$, are Pauli matrices:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The relations $\sigma_i^2 = -1$, $i = 1, 2, 3$, and $\sigma_1\sigma_3 + \sigma_3\sigma_1 = 0$ shows that $\{\sigma_1, \sigma_3\}$ generate $\text{Cliff}_2$, so that $\text{Cliff}_2^2 = H$. Let $\Delta = C^2 \otimes_R C^2$ be the vector space of complex 4-spinors that gives the spinor representation of Clifford algebra $\text{Cliff}_4^2$:

$$
\text{Cliff}_4^2 = \text{End}_C(\Delta) = C(4).
$$

Then $\text{Cliff}_4^2$ is generated by the following Dirac matrices:

$$
\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad \gamma_4 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.
$$

The set

$$
\{\gamma_p, \gamma_p\gamma_q, \gamma_p\gamma_q\gamma_r, \gamma_p\gamma_q\gamma_r\gamma_s; \quad 1 \leq p, q, r, s \leq 4 \}
$$

(2.2)

gives a 16-dimensional basis of the representation $\text{Cliff}_4^2 \simeq \text{End}_C(\Delta)$ with the following relations:

$$
\gamma_p\gamma_q + \gamma_q\gamma_p = 2\delta_{pq}.
$$

The representation $\Delta$ decomposes into irreducible representations $\Delta^\pm = C^2$ of Spin(4).

Let $S = C^2 \times \Delta$ be the trivial spinor bundle on $C^2$. The corresponding bundle $S^+ = C^2 \times \Delta^+$ (respectively $S^- = C^2 \times \Delta^-$) is called the even (respectively odd) half spinor bundle and the sections are called even (respectively odd) spinors. On the other hand, since $\text{Cliff}_4^2 = H(2) \otimes_R C$ and $\Delta = H^2 = H \oplus H$, we may look at an even spinor on $M \subset R^4$ as a $H$ valued smooth function: $C^\infty(M, H) = C^\infty(M, S^+)$. We feel free to use the alternative notation to write a spinor:

$$
C^\infty(M, H) \ni u + jv = p + qi + rj + sk \longleftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} \in C^\infty(M, S^+), \quad u = p + qi, \quad v = -r + si.
$$

We have also the following alternative notation:

$$
C^\infty(M, H) \ni p + qi + rj + sk \longleftrightarrow pI + q i\sigma_3 + r i\sigma_2 + s i\sigma_1 \in C^\infty(M, S^+).
$$

(2.3)
The conjugate quaternion $\overline{x}$ of $x = p + qi + rj + sk$ is defined by $\overline{x} = p - qi - rj - sk$ and conjugation is an anti-involution; $(xy) = \overline{y}\overline{x}$.

The multiplication of two even spinors is defined by

$$\phi_1 \cdot \phi_2 = \begin{pmatrix} u_1 u_2 - \overline{v}_1 v_2 \\ v_1 u_2 + \overline{u}_1 v_2 \end{pmatrix}$$

(2.4)

for $\phi_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$, $i = 1, 2$. It corresponds to the quaternion multiplication:

$$(u_1 + jv_1)(u_2 + jv_2) = (u_1 u_2 - \overline{v}_1 v_2) + j(v_1 u_2 + \overline{u}_1 v_2).$$

2.1.2

The Dirac operator is defined by

$$\mathcal{D} = c \circ d : C^\infty(M, S) \rightarrow C^\infty(M, S).$$

where $d : S \rightarrow T^\ast \mathbb{C}^2 \otimes S \simeq TC^2 \otimes S$ is the covariant derivative which is the exterior differential in this case, and $c : TC^2 \otimes S \rightarrow S$ is the bundle homomorphism coming from the Clifford multiplication. With respect to the Dirac matrices $\{\gamma_j\}_{j=1,2,3,4}$, (2.2), the Dirac operator has the expression:

$$\mathcal{D} = -\frac{\partial}{\partial x_1} \gamma_4 - \frac{\partial}{\partial x_2} \gamma_3 - \frac{\partial}{\partial x_3} \gamma_2 - \frac{\partial}{\partial x_4} \gamma_1.$$

By means of the decomposition $S = S^+ \oplus S^-$ the Dirac operator has the chiral decomposition:

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} : C^\infty(\mathbb{C}^2, S^+ \oplus S^-) \rightarrow C^\infty(\mathbb{C}^2, S^+ \oplus S^-).$$

If we adopt the notation

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_4},$$

then...
$D$ and $D^\dagger$ have the following coordinate expressions;

\[
D = \begin{pmatrix}
\frac{\partial}{\partial z_1} & \frac{\partial}{\partial \bar{z}_2} \\
\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1}
\end{pmatrix}, \quad D^\dagger = \begin{pmatrix}
\frac{\partial}{\partial \bar{z}_1} & \frac{\partial}{\partial \bar{z}_2} \\
-\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1}
\end{pmatrix}.
\]

2.1.3

The right action of $SU(2)$ on $\mathbb{C}^2$ is written by

\[
R_g z = \begin{pmatrix}
a z_1 - b \bar{z}_2 \\
az_2 + b \bar{z}_1
\end{pmatrix}, \quad g = \begin{pmatrix}
a & -\bar{b} \\
b & a
\end{pmatrix} \in SU(2), \quad z = \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} \in \mathbb{C}^2.
\]

Then the infinitesimal action of $su(2)$ on $\mathbb{C}^2$ is

\[
((dR_e)X)F = \frac{d}{dt} \bigg|_{t=0} R_{e^tX}F, \quad X \in su(2).
\]

It yields the following basis of vector fields $(\theta_3, \theta_1, \theta_2)$ on $\{|z| = 1\} \simeq S^3$:

\[
\begin{align*}
\theta_1 &= \frac{1}{2\sqrt{-1}}dR(\sigma_2), \quad \theta_2 = \frac{1}{2\sqrt{-1}}dR(\sigma_1), \quad \theta_3 = -\frac{1}{2\sqrt{-1}}dR(\sigma_3).
\end{align*}
\]

(2.5)

We prefer often the following basis $(e_+, e_-, \theta)$ given by

\[
2\theta_3 = -\sqrt{-1}\theta, \quad 2\theta_1 = e_+ + e_-, \quad 2\theta_2 = \sqrt{-1}(e_+ - e_-).
\]

(2.6)

The local coordinate expression of these vector fields becomes:

\[
e_+ &= -z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2}, \quad e_- = -\bar{z}_2 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_2},
\]

(2.7)

\[
\theta &= \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right),
\]

(2.8)

and the following commutation relations hold;

\[
[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta.
\]
The dual basis are given by the differential 1-forms:

\[
\theta_3^* = \frac{1}{2\sqrt{-1}|z|^2}(z_1 dz_1 + z_2 dz_2 - z_1 dz_1 - z_2 dz_2),
\]
\[
\theta_1^* = \frac{1}{2|z|^2}(e_+^* + e_-^*), \quad \theta_2^* = \frac{1}{2\sqrt{-1}|z|^2}(e_+^* - e_-^*),
\]

where

\[
e_+^* = (-\overline{z}_2 d\overline{z}_1 + \overline{z}_1 d\overline{z}_2), \quad e_-^* = (-z_2 dz_1 + z_1 dz_2),
\]

here we wrote the formulae in the form extended to \( \mathbb{C}^2 \setminus \{0\} \).

\( \theta_k^* \), \( k = 1, 2, 3 \), are real 1-forms: \( \overline{\theta}_k^* = \theta_k^* \). It holds that \( \theta_j^*(\theta_k) = \delta_{jk} \) for \( j, k = 1, 2, 3 \).

The integrability condition becomes

\[
\frac{-1}{2} d\theta_3^* = \theta_1^* \wedge \theta_2^*, \quad \frac{-1}{2} d\theta_1^* = \theta_2^* \wedge \theta_3^*, \quad \frac{-1}{2} d\theta_2^* = \theta_3^* \wedge \theta_1^*. \tag{2.9}
\]

And \( \theta_0^* \wedge \theta_1^* \wedge \theta_2^* = d\sigma_{S^3} \) is the volume form on \( S^3 \).

**Lemma 2.1.**

\[
\int_{S^3} \theta_k f \, d\sigma = 0, \quad k = 1, 2, 3, \tag{2.10}
\]

for any function \( f \) on \( S^3 \).

This is proved as follows. We consider the 2-form \( \beta = f \theta_1^* \wedge \theta_2^* \). By virtue of the integrable condition (2.9) we have

\[
d\beta = (\theta_3 f) \theta_3^* \wedge \theta_1^* \wedge \theta_2^* = \theta_3 f \, d\sigma.
\]

Hence

\[
0 = \int_{S^3} d\beta = \int_{S^3} \theta_3 f \, d\sigma.
\]

Similarly for the integral of \( \theta_k f \), \( k = 1, 2 \), of the base vector fields \( \theta_k \); \( k = 1, 2 \), on \( S^3 \).

On the other hand the lemma is an immediate consequence of the \( SO(4) \)-invariance of the volume form. This is a remark due to Professor T. Iwai of Kyoto university.

**2.1.4**

We introduce the normal vector \( n \) to \( S^3 \) and its \((1,0)\)-components \( \nu \) as follows:

\[
n = \nu + \overrightarrow{\nu}, \quad \nu = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}. \tag{2.11}
\]
Then \( \theta = \nu - \nu \) and we have the following commutation relations:

\[
n e_{\pm} = e_{\pm} n, \quad n \theta = \theta n. \tag{2.12}
\]

The normal derivation is defined by

\[
\frac{\partial}{\partial n} = \frac{1}{2|z|} n = \frac{1}{2|z|} (\nu + \bar{\nu}), \tag{2.13}
\]

We shall denote by \( \gamma \) the Clifford multiplication of the normal vector \( \frac{\partial}{\partial n} \). The multiplication \( \gamma \) changes the chirality: \( \gamma = \gamma_+ \oplus \gamma_- : S^+ \oplus S^- \to S^- \oplus S^+ \), and \( \gamma^2 = 1 \). The matrix forms of \( \gamma_\pm \) are

\[
\gamma_+ = \frac{1}{|z|} \begin{pmatrix} \bar{z}_1 & -z_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad \gamma_- = \frac{1}{|z|} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \tag{2.14}
\]

**Proposition 2.2.** [Ko1] The Dirac operators \( D \) and \( D^\dagger \) have the following polar decompositions:

\[
D = \gamma_+ \left( \frac{\partial}{\partial n} - \vartheta \right) : S^+ \to S^-, \tag{2.15}
\]

\[
D^\dagger = \left( \frac{\partial}{\partial n} + \vartheta + \frac{3}{2|z|} \right) \gamma_- : S^- \to S^+, \tag{2.16}
\]

where the non-chiral Dirac operator \( \vartheta \) is given by

\[
\vartheta = - \sum_{i=1}^3 \left( \frac{1}{|z|} \theta_i \right) \cdot \nabla \frac{1}{|z|} \theta_i \right] = \frac{1}{|z|} \begin{pmatrix} -i \theta_3 & \theta_1 - i\theta_2 \\ \theta_1 - i\theta_2 & i \theta_3 \end{pmatrix}. \tag{2.16}
\]

\( \vartheta \) restricted is called the tangential Dirac operator:

\( \vartheta|S^3 : C^\infty(S^3, S^+) \to C^\infty(S^3, S^+) \)

The tangential Dirac operator on \( S^3 \) is a self adjoint elliptic differential operator. The tangential Dirac operator \( \vartheta \) is written in the formula:

\[
\vartheta = \sqrt{-1} \frac{\vartheta}{|z|} ( - \theta_3 \sigma_3 + \theta_1 \sigma_2 - \theta_2 \sigma_1 ),
\]
and by the quaternion notation it becomes:

$$\mathbf{\phi} = \frac{1}{|z|}(-\theta_3 i + \theta_1 j - \theta_2 k).$$  \hspace{1cm} (2.17)

The following commutation relation holds:

$$n \mathbf{\phi} = \mathbf{\phi} n - \mathbf{\phi}.$$  \hspace{1cm} (2.18)

Lemma 2.1 yields the following Proposition 2.3.

$$\int_{S^3} \mathbf{\phi} \phi d\sigma = 0,$$  \hspace{1cm} (2.19)

for any $\phi \in C^\infty(S^3, S^+)$ $\simeq C^\infty(S^3, H)$.

### 2.2 Harmonic spinors

#### 2.2.1 Harmonic polynomials on $S^3 \subset \mathbb{C}^2$

In the following we denote a function $f(z, \bar{z})$ of variables $z, \bar{z}$ simply by $f(z)$.

**Definition 2.4.** For $m = 0, 1, 2, \cdots$, and $l, k = 0, 1, \cdots, m$, we define the monomials:

$$v^k_{(l,m-l)} = (e_-)^k z_1^{l-1} z_2^{m-l}.$$  \hspace{1cm} (2.20)

$$w^k_{(l,m-l)} = (-1)^k \frac{l!}{(m-k)!} v^{m-k}_{(k,m-k)}.$$  \hspace{1cm} (2.21)

Actually these are polynomials of $z_1, z_2, \bar{z}_1, \bar{z}_2$ but they play the role of monomials in our study. The monomials $v^k_{(l,m-l)}$ in (2.20) come naturally from the right action of $SU(2)$ on $\mathbb{C}^2$, so as the monomials $w^k_{(l,m-l)}$ from the left action of $SU(2)$ on $\mathbb{C}^2 \setminus \{0\}$. $v^k_{(l,m-l)}$ are harmonic polynomials on $\mathbb{C}^2$; $\Delta v^k_{(l,m-l)} = 0$, where $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$. It holds that

$$v^k_{(l,m-l)} = (-1)^{m-l-k} \frac{k!}{(m-k)!} v^{m-k}_{(l,m-l)}.$$  \hspace{1cm} (2.22)

In [Ko0, Ko1] we saw that

$$\frac{1}{\sqrt{2\pi}} \sqrt{\frac{(m+1)(m-k)}{l(l-m)!}} v^k_{(l,m-l)}; \ m = 0, 1, \cdots, 0 \leq k, l \leq m$$
forms a $L^2(S^3)$-complete orthonormal system of the space of harmonic polynomials. The similar assertions hold for

$$\frac{1}{\sqrt{2\pi}} \sqrt{\frac{(m+1)(m-k)}{l(l-m)!}} w^k_{(l,m-l)}; \ m = 0, 1, \cdots, 0 \leq k, l \leq m.$$  

Let $H$ be the space of harmonic polynomials on $S^3$. For each pair $(m, l), 0 \leq l \leq m$, let $H_{(m,l)}$ be the linear subspace generated by the vectors $\{v^k_{(l,m-l)}; 0 \leq k \leq m+1\}$. $H$ is
the direct sum of $H_{(m,l)}$, $0 \leq m$, $0 \leq l \leq m$, and each $H_{(m,l)}$ gives a $(m+1)$-dimensional right representation of $su(2)$ with the highest weight $\frac{m}{2}$. Similarly the subspace $H^\dagger_{(m,l)} = \{w_{(l,m-l)}^k; 0 \leq k \leq m+1\}$ gives a $(m+1)$-dimensional left representation of $su(2)$ with the highest weight $\frac{m}{2}$.

In Lemma 4.1 of [Kol] we proved the following product formula for the harmonic polynomials $v_{(a,b)}^k$:

**Proposition 2.5.**

\[
v_{(a_1,b_1)}^{k_1} v_{(a_2,b_2)}^{k_2} = \sum_{j=0}^{a_1+a_2+b_1+b_2} C_j |z|^j v_{(a_1+a_2-j,b_1+b_2-j)}^{k_1+k_2-j}, \quad (2.23)
\]

for some rational numbers $C_j = C_j(a_1,a_2,b_1,b_2,k_1,k_2)$.

**Proposition 2.6.**

The space of harmonic polynomials $H$ on $S^3$ is given a graded $\mathbb{C}$-algebra structure.

In fact, let $k = k_1 + k_2$, $a = a_1 + a_2$ and $b = b_1 + b_2$. Restricted to $S^3$, the harmonic polynomial $v_{(a,b)}^k$ is equal to a constant multiple of $v_{(a_1,b_1)}^{k_1} \cdot v_{(a_2,b_2)}^{k_2}$ modulo a linear combination of polynomials $v_{(a-j,b-j)}^{k-j}$, $1 \leq j \leq \min(k,a,b)$. So the set of harmonic polynomials becomes a graded $\mathbb{C}$-algebra.

### 2.2.2 Harmonic spinors on $S^3 \subset \mathbb{C}^2$

Now we introduce a basis of the space of even harmonic spinors.

**Definition 2.7.** For $m = 0, 1, 2, \cdots; l = 0, 1, \cdots, m$ and $k = 0, 1, \cdots, m + 1$, we put

\[
\phi^{+(m,l,k)}(z) = \sqrt{\frac{(m+1-k)!}{k!!(m-l)!}} \binom{k!^{k-1}_{(l,m-l)}}{-v_{(l,m-l)}^k},
\]

\[
\phi^{-(m,l,k)}(z) = \sqrt{\frac{(m+1-k)!}{k!!(m-l)!}} \left( \frac{1}{|z|^2} \right)^{m+2} \binom{w_{(m+1-l,l)}^k}{w_{(m-l,l+1)}^k}. \quad (2.24)
\]

**Examples**
\[
\phi^{(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi^{(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \phi^{(1,0,0)} = \begin{pmatrix} 0 \\ -\sqrt{2} z_2 \end{pmatrix}
\]
\[
\phi^{(1,0,1)} = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}, \quad \phi^{(1,1,1)} = \begin{pmatrix} 0 \\ \overline{z_2} \end{pmatrix}, \quad \phi^{(2,0,0)} = \begin{pmatrix} 0 \\ -\sqrt{3} z_2^2 \end{pmatrix}
\]
\[
\phi^{- (0,0,0)} = \frac{1}{|z|^4} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}, \quad \phi^{- (0,0,1)} = \frac{1}{|z|^4} \begin{pmatrix} -z_1 \\ z_2 \end{pmatrix},
\]
\[
\phi^{- (1,0,0)} = \frac{\sqrt{2}}{|z|^6} \begin{pmatrix} z_2^2 \\ 2z_2 z_1 \end{pmatrix}, \quad \phi^{- (1,0,1)} = -\frac{2}{|z|^6} \begin{pmatrix} z_1 z_2 \\ |z_1|^2 - |z_2|^2 \end{pmatrix}.
\]

**Proposition 2.8.** [Ko1]

1. \( \phi^{+(m,l,k)} \) is a harmonic spinor on \( \mathbb{C}^2 \) and \( \phi^{-(m,l,k)} \) is a harmonic spinor on \( \mathbb{C}^2 \{0\} \) that is regular at infinity.

2. On \( S^3 = \{|z| = 1\} \) we have:
\[
\partial \phi^{+(m,l,k)} = \frac{m}{2} \phi^{+(m,l,k)}, \quad \partial \phi^{-(m,l,k)} = -\frac{m+3}{2} \phi^{-(m,l,k)}.
\]

3. The eigenvalues of \( \partial \) are
\[
\frac{m}{2}, \quad -\frac{m+3}{2}; \quad m = 0, 1, \cdots,
\]
and the multiplicity of each eigenvalue is equal to \((m+1)(m+2)\).

4. The set of eigenspinors
\[
\left\{ \frac{1}{\sqrt{2\pi}} \phi^{+(m,l,k)}, \frac{1}{\sqrt{2\pi}} \phi^{-(m,l,k)} ; \quad m = 0, 1, \cdots, 0 \leq l \leq m, 0 \leq k \leq m + 1 \right\}
\]
forms a complete orthonormal system of \( L^2(S^3, S^+) \).

**Remark 2.9.** On \( \mathbb{C}^2 \{0\} \) we have
\[
\partial \phi^{+(m,l,k)} = \frac{m}{2|z|} \phi^{+(m,l,k)}, \quad \partial \phi^{-(m,l,k)} = -\frac{m+3}{2|z|} \phi^{-(m,l,k)}.
\]
The Bergman kernel on the space of harmonic spinors is associated to the basis spinors \( \{ \phi^{+(m,l,k)}, \phi^{-(m,l,k)} \} \). It is given by the following formula:

\[
B(z, \zeta) = \frac{1}{2\pi^2} \sum_{m,l,k} \phi^{+(m,l,k)}(\zeta) \otimes \phi^{+(m,l,k)}(z) + \phi^{-(m,l,k)}(\zeta) \otimes \phi^{-(m,l,k)}(z),
\]

for \( z \in G, \zeta \in \partial G \).

The Cauchy kernel (fundamental solution) of the half Dirac operator \( D : C^\infty(C^2, S^+) \to C^\infty(C^2, S^-) \) is given by

\[
K^\dagger(z, \zeta) = \frac{1}{|\zeta - z|^3} \gamma_-(\zeta - z) : C^\infty(C^2, S^-) \to C^\infty(C^2, S^+), \quad |z| < |\zeta|.
\]

In \([Ko2]\) we showed that the Bergman kernel coincides with the Cauchy kernel:

\[
K^\dagger(z, \zeta) = B(z, \zeta) \quad \text{for } z \in G, \zeta \in \partial G
\]

for any bounded domain \( G \subset C^2 \).

We have the following integral representation of spinors:

**Theorem 2.10.** \([Ko1]\) Let \( G \) be a domain of \( C^2 \) and let \( \varphi \in C^\infty(\overline{G}, S^+) \). We have

\[
\varphi(z) = -\frac{1}{2\pi^2} \int_G K^\dagger(z, \zeta)D\varphi(\zeta)dv(\zeta) + \frac{1}{2\pi^2} \int_{\partial G} K^\dagger(z, \zeta)(\gamma_+\varphi)(\zeta)d\sigma(\zeta), \quad z \in G.
\]

### 3 Laurent polynomial type harmonic spinors on \( C^2 \setminus \{0\} \) and the associated 2-cocycle

#### 3.1 Algebra of Laurent polynomial type harmonic spinors on \( C^2 \setminus \{0\} \)

**3.1.1 Quaternion trace and Quaternion residue**

We have the Laurent expansions of harmonic spinors, that is, a harmonic spinor \( \varphi \) on \( C^2 \setminus \{0\} \) has an expansion by the basic spinors \( \{ \phi^{\pm(m,l,k)} \}_{m,l,k} \);

\[
\varphi(z) = \sum_{m,l,k} C^{+(m,l,k)} \phi^{+(m,l,k)}(z) + \sum_{m,l,k} C^{-(m,l,k)} \phi^{-(m,l,k)}(z), \quad (3.1)
\]
which is uniformly convergent on any compact subset of $C^2 \setminus \{0\}$, \cite{Ko2}. The coefficients $C_{\pm(m,l,k)}$ are given by the following formula:

$$C_{\pm(m,l,k)} = \frac{1}{2\pi^2} \int_{S^3} \langle \varphi, \phi_{\pm(m,l,k)} \rangle \, d\sigma, \quad \text{(3.2)}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $S^+$. 

**Definition 3.1.** We call the series \eqref{SPPT} a spinor of Laurent polynomial type on $C^2 \setminus \{0\}$ if only finitely many coefficients $C_{-(m,l,k)}$ are non-zero. The space of spinors of Laurent polynomial type is denoted by $C[\phi^\pm]$.

The *quaternion trace* of $\varphi$ is defined by

$$qTr \varphi = \begin{pmatrix} C_{+(0,0,1)} \\ -C_{+(0,0,0)} \end{pmatrix}. \quad \text{(3.3)}$$

By the quaternion notation it becomes

$$qTr \varphi = C_{+(0,0,1)} + C_{+(0,0,0)} j.$$

We have

$$qTr \varphi = \frac{1}{2\pi^2} \int_{|\zeta|=1} \varphi(\zeta) \sigma(d\zeta). \quad \text{(3.4)}$$

**Proposition 3.2.**

$$qTr(\phi_1 \cdot \phi_2) = qTr(\phi_2 \cdot \phi_1). \quad \text{(3.5)}$$

In fact, for

$$\phi_i(z) = \sum_{m,l,k} C_{+(m,l,k)}^{(i)} \phi_{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}^{(i)} \phi_{-(m,l,k)}(z), \quad i = 1, 2,$$

we have

$$qTr(\phi_1 \cdot \phi_2) = \sum_m C_{+(m+3,\cdot)}^{(1)} C_{-(m,\cdot)}^{(2)} + C_{+(m,\cdot)}^{(2)} C_{-(m+3,\cdot)}^{(1)} = qTr(\phi_2 \cdot \phi_1).$$

**Definition 3.3.** Let $\varphi$ be a Laurent polynomial type spinor \eqref{SPPT}. We call the vector

$$\begin{pmatrix} C_{-(0,0,1)} \\ -C_{-(0,0,0)} \end{pmatrix}$$

the *quaternion residue of $\varphi$ at $z = 0* and we denote it by $qRes \varphi$. 

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By the quaternion notation it becomes

\[ qRes \varphi = C_{-001} + C_{-000} j . \]

Since \( \phi^{-m,l,k}(z) \sim O(|z|^{-m-3}) \), \( qRes \varphi \) is the coefficient of \( O(|z|^{-3}) \) in the expansion of \( \varphi \). We have the following

**Proposition 3.4.**

\[ qRes \varphi = \frac{1}{2\pi^2} \int_{|\zeta|=1} \gamma_+ (\zeta) \varphi(\zeta) \sigma(d\zeta). \quad (3.6) \]

**Proof.** From (3.1) and (3.2) we have

\[ \varphi(z) = \sum_{m,l,k} C_{+(m,l,k)} \phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)} \phi^{-(m,l,k)}(z), \quad (3.7) \]

with

\[ C_{-(m,l,k)} = \frac{1}{2\pi^2} \int_{|z|=1} \langle \varphi(z), \phi^{-(m,l,k)}(z) \rangle \sigma(dz). \]

Then

\[ C_{-(000)} = \frac{1}{2\pi^2} \int_{|z|=1} \langle (\gamma_+ \varphi)(z), (\gamma_+ \phi^{-(000)})(z) \rangle \sigma(dz) = \frac{1}{2\pi^2} \int_{|z|=1} (\gamma_+ \varphi)_2(z) \sigma(dz), \]

since \( \gamma_+ \phi^{-(000)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), where \( (\phi)_2 \) is the 2nd component of \( \phi \). Similarly we have

\[ -C_{-(001)} = \frac{1}{2\pi^2} \int_{|z|=1} (\gamma_+ \varphi)_1(z) \sigma(dz). \]

\[ \square \]

**Proposition 3.5.**

\[ qRes (\phi_1 \cdot \phi_2) = qRes (\phi_2 \cdot \phi_1). \quad (3.8) \]

In fact, for

\[ \phi_i(z) = \sum_{m,l,k} C^{(i)}_{+(m,l,k)} \phi^{+(m,l,k)}(z) + \sum_{m,l,k} C^{(i)}_{-(m,l,k)} \phi^{-(m,l,k)}(z), \quad i = 1, 2, \]

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the coefficient of the $O(|z|^{-3})$-terms in $\phi_1 \cdot \phi_2$ is
\[ q\text{Res}(\phi_1 \cdot \phi_2) = \sum_m C^{(1)}_{+(m,)} C^{(2)}_{-(m,)} + C^{(2)}_{+(m,)} C^{(1)}_{-(m,)} = q\text{Res}(\phi_2 \cdot \phi_1). \]

We remark the following equivalence of the actions of the boundary Dirac operator and the normal derivation. For a Laurent polynomial type spinor
\[ \varphi(z) = \sum_{m,l,k} C_{+(m,l,k)} \phi^+(m,l,k)(z) + \sum_{m,l,k} C_{-(m,l,k)} \phi^-(m,l,k)(z), \]
we have
\[ \frac{\partial}{\partial n} \varphi(z) = \frac{1}{2|z|} \sum_{m,l,k} mC_{+(m,l,k)} \phi^{+(m,l,k)}(z) - \frac{1}{2|z|} \sum_{m,l,k} (m+3)C_{-(m,l,k)} \phi^{-(m,l,k)}(z), \]
and
\[ \partial \varphi(z) = \frac{1}{2|z|} \sum_{m,l,k} mC_{+(m,l,k)} \phi^{+(m,l,k)}(z) - \frac{1}{2|z|} \sum_{m,l,k} (m+3)C_{-(m,l,k)} \phi^{-(m,l,k)}(z). \]

Hence we have the following

**Proposition 3.6.** For a Laurent polynomial type spinor $\varphi$,
\[ \partial \varphi(z) = \frac{\partial}{\partial n} \varphi(z). \] (3.9)

Since there is no term of order $O(1/|z|^3)$ in the above expansions of $\partial_n \varphi$ and $\partial \varphi$ we have the following

**Lemma 3.7.** For a Laurent polynomial type spinor $\varphi$,
\[ q\text{Res}(\partial \varphi) = 0, \quad q\text{Res} \left( \frac{\partial}{\partial n} \varphi \right) = 0. \] (3.10)

### 3.1.2 Algebra of currents on $S^3$

The space $\mathbb{C}[\phi^\pm]$ of spinors of Laurent polynomial type over $\mathbb{C}^2 \setminus \{0\}$ is not an algebra.

**Definition 3.8.** The subalgebra of $C^\infty(\mathbb{C}^2 \setminus \{0\}, S^+)$ that is generated by $\mathbb{C}[\phi^\pm]$ is called the **algebra of currents** on $\mathbb{C}^2 \setminus \{0\}$ and is denoted by $\mathcal{L}$.

Every spinor $\varphi \in \mathcal{L}$ is written as a $\mathbb{C}$-linear combination of basic spinors $\phi_1,\phi_2,\ldots,\phi_r$ for $\phi_i = \phi^{\pm(m_i,l_i,k_i)}$, $0 \leq m_i, 0 \leq l_i \leq m_i, 0 \leq k_i \leq m_i + 2$, $1 \leq i \leq r$. 


We note that a spinor of $\mathcal{L}$ is not necessarily a harmonic spinor on $\mathbb{C}^2 \setminus \{0\}$. For example,

$$\phi^+(1,0,0) \cdot \phi^-(0,0,0) = \sqrt{2} \begin{pmatrix} 0 \\ -z_2 \end{pmatrix} \cdot \frac{1}{|z|^4} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = \frac{\sqrt{2}}{|z|^4} \begin{pmatrix} z_1 z_2 \\ -z_2 \end{pmatrix}$$

is not harmonic over $\mathbb{C}^2 \setminus \{0\}$.

**Lemma 3.9.** The product $\phi^\pm(m_1,l_1,k_1) \cdot \phi^\pm(m_2,l_2,k_2)$ of two spinors $\phi^\pm(m_1,l_1,k_1)$ and $\phi^\pm(m_2,l_2,k_2)$ is written by a linear combination of spinors $|z|^p \phi^\pm(m,l,k)$, $0 \leq m, l, k, p \leq m_1 + m_2 + 2$.

**Proof.** The multiplication of spinors is defined in (2.4). By virtue of Proposition 2.5 and the formula (2.21) and (2.22) we see that the product $\phi^\pm(m_1,l_1,k_1) \cdot \phi^\pm(m_2,l_2,k_2)$ is decomposed into a linear sum of the following spinors;

$$|z|^p \begin{pmatrix} v^k_{l,-m-l} \\ 0 \end{pmatrix}, \quad |z|^p \begin{pmatrix} 0 \\ v^k_{l,m-l} \end{pmatrix},$$

where $k$, $l$, $m$ and $p$ are numbers that are smaller than or equal to $m_1 + m_2 + 2$. On the other hand a spinor of the form $|z|^p \begin{pmatrix} v^k_{l,m-l} \\ 0 \end{pmatrix}$ or $|z|^p \begin{pmatrix} 0 \\ v^k_{l,m-l} \end{pmatrix}$ is written as a linear combinations of $|z|^p \phi^+(m,l,k)$ and $|z|^p \phi^-(m,l,k)$. For example,

$$\begin{pmatrix} v^k_{l,m-l} \\ 0 \end{pmatrix} = A \phi^+(m,l,k+1) + B \phi^-(m-1,k,l),$$

with $A = \sqrt{\frac{1}{(k+1)!(l-m-l)!} \frac{1}{m+1}}$ and $B = (-1)^l \sqrt{\frac{m-l}{m+1}}$. So any product $\phi^\pm(m_1,l_1,k_1) \cdot \phi^\pm(m_2,l_2,k_2)$ is written as a linear combination of $\{ |z|^p \phi^\pm(m_1+1,m_2-n,\cdots) ; 1 \leq n, p \leq m_1 + m_2 + 2 \}$. \hfill $\square$

We have seen that the $\mathbb{C}$-algebra $\mathcal{L}$ has the basis

$$\{ |z|^p \phi^\pm(m,l,k) ; 0 \leq p, 0 \leq m, 0 \leq l \leq m, 0 \leq k \leq m + 1 \}. \quad (3.11)$$

Moreover, we see from Proposition 2.5 that any $v^k_{l,m-l}$ is written by a linear combination of $|z|^{2r} \phi^\pm(m_1,l_1,k_1) \cdot \phi^\pm(m_2,l_2,k_2)$ for $1 \leq r \leq m$, $0 \leq m_1 + m_2 \leq m - 1$, $0 \leq l_1 + l_2 \leq l$ and $0 \leq k_1 + k_2 \leq k$, so that any $\phi^\pm(m,l,k)$ is written by a linear combination of the products $|z|^{2r} \phi^\pm(m_1,l_1,k_1) \cdot \phi^\pm(m_2,l_2,k_2)$ for $0 \leq r \leq m - 1$, $0 \leq m_1 + m_2 \leq m - 1$, $0 \leq l_1 + l_2 \leq l$ and $0 \leq k_1 + k_2 \leq k$, so that $\mathcal{L}$ is a graded algebra generated by the four spinors
\[ I = \phi^{+(0,0,1)}, \quad J = -\phi^{+(0,0,0)}, \quad \kappa = \phi^{-(0,0,1)}, \quad \lambda = \phi^{-(0,0,0)}. \] Thus we have proved the following:

**Theorem 3.10.** The algebra of current \( \mathcal{L} \) becomes a graded algebra with the generators given by the spinors:

\[
I = \phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = -\phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
\kappa = \phi^{-(0,0,1)} = \frac{1}{|z|^4} \begin{pmatrix} -z_1 \\ z_2 \end{pmatrix}, \quad \lambda = \phi^{-(0,0,0)} = \frac{1}{|z|^4} \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}.
\]

**Example.**

As we saw in the above, \( \phi^{+(1,0,0)} \cdot \phi^{-(0,0,0)} \) is not a harmonic spinor. But

\[
|z|^{4 \phi^{+(1,0,0)} \cdot \phi^{-(0,0,0)}} = \frac{1}{\sqrt{2}} \phi^{+(2,0,2)} + \frac{\sqrt{2}}{3} \phi^{+(2,0,0)} \in C[\phi^{\pm}] \subset \mathcal{L},
\]

and their restrictions to the boundary \( S^3 = \{|z| = 1\} \) are equal.

We put

\[
\mathcal{K} = \{ \phi \in \mathcal{L}; \, \phi = \begin{pmatrix} f \\ 0 \end{pmatrix}, \text{ for } f \in C^\infty(C^2 \setminus \{0\}, R) \}, \tag{3.12}
\]

\[
\mathcal{J} = \{ \phi \in \mathcal{L}; \, \phi = \begin{pmatrix} \sqrt{-1} f \\ g + \sqrt{-1} h \end{pmatrix}, \text{ for } f, g, h \in C^\infty(C^2 \setminus \{0\}, R) \}. \tag{3.13}
\]

As we noticed at the beginning of Theorem 3.10 a spinors of the form \( \begin{pmatrix} v_{l,m-l}^k \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 \\ v_{l,m-l}^{k+1} \end{pmatrix} \) is written by a linear combinations of \( \phi^{+(m,l,k+1)} \) and \( \phi^{-(m-1,k,l)} \). This fact and the relation \( (2.22) \) yield that \( \mathcal{K} \) and \( \mathcal{J} \) are \( R \)-linear suspaces of \( \mathcal{L} \) and \( \mathcal{L} \) is decomposed into the direct sum; \( \mathcal{L} = \mathcal{K} \oplus \mathcal{J} \). Evidently \( \mathcal{K} \) is a commutative subalgebra of \( \mathcal{L} \). By the induced Lie algebra structure on \( \mathcal{L} \) we see that \( \mathcal{J} \) is an ideal of \( \mathcal{L} \) and

\[
[\mathcal{K}, \mathcal{L}] = 0, \quad [\mathcal{L}, \mathcal{L}] = \mathcal{J}. \tag{3.14}
\]

The quarternion trace \( (3.3) \) on \( \mathcal{L} \) is non-degenerate and the restriction to \( \mathcal{K} \) is also non-degenerate.
3.2 Radial derivative on $\mathcal{L}$

The vector fields $\theta_i$, $i = 1, 2, 3$, and the boundary Dirac operator $\bar{\phi}$, (2.15), are tangent to $S^3 = \{|z| = 1\}$:

$$\theta_i |z| = 0, \quad i = 1, 2, 3, \quad \bar{\phi}(|z| \varphi) = |z| \bar{\phi} \varphi.$$

The normal vector field on $\mathbb{R}^4 \setminus \{0\}$ and the normal derivation are defined in (2.11) and (2.13):

$$n = \nu + \tau, \quad \frac{\partial}{\partial n} = \frac{1}{2|z|} n.$$

The action of the normal vector field $n$ on a spinor $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in C^\infty(\mathbb{R}^4 \setminus \{0\}, S^+) \) is defined by

$$n \varphi = \begin{pmatrix} n u \\ n v \end{pmatrix}.$$

Proposition 3.11.

1. $n(\phi_1 \cdot \phi_2) = (n \phi_1) \cdot \phi_2 + \phi_1 \cdot (n \phi_2).$ \hspace{1cm} (3.15)

So $n$ gives a derivation of the algebra $\mathcal{L}$. In particular $\frac{\partial}{\partial n}$ is a derivation of $\mathcal{L}$.

2. $n \phi^{+(m,l,k)} = m \phi^{+(m,l,k)}$, \hspace{1cm} $n \phi^{-(m,l,k)} = -(m + 3) \phi^{-(m,l,k)}.$ \hspace{1cm} (3.16)

3. If $\varphi$ is a spinor of Laurent polynomial type on $\mathbb{C}^2 \setminus \{0\}$:

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)} \phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)} \phi^{-(m,l,k)}(z),$$

$n \varphi$ is also a spinor of Laurent polynomial type:

$$n \varphi = \sum_{m,l,k} m C_{+(m,l,k)} \phi^{+(m,l,k)}(z) - \sum_{m,l,k} (m + 3) C_{-(m,l,k)} \phi^{-(m,l,k)}(z),$$ \hspace{1cm} (3.17)

Proof. The formula (3.16) follows from the definition (2.24). \qed
3.3 Homogeneous decomposition of $\mathcal{L}$

Let $\mathcal{L}[m]$ be the subspace of $\mathcal{L}$ consisting of those spinors that are of homogeneous degree $m$:

$$\varphi(z) = |z|^m \varphi \left( \frac{z}{|z|} \right).$$

$\mathcal{L}[m]$ is spanned by the sum of the spinors $\varphi = \phi_1 \cdots \phi_n$ such that each $\phi_i$ is equal to $\phi_i = \phi^{+(m_i,l_i,k_i)}$ or $\phi_i = \phi^{-(m_i,l_i,k_i)}$, where $m_i \geq 0$, $0 \leq l_i \leq m_i + 1$, $0 \leq k_i \leq m_i + 2$, and such that

$$m = \sum_{i: \phi_i = \phi^{+(m_i,l_i,k_i)}} m_i - \sum_{i: \phi_i = \phi^{-(m_i,l_i,k_i)}} (m_i + 3).$$

It holds that $n \varphi = m \varphi$, so the eigenvalues of $n$ on $\mathcal{L}$ are $\{m; m \in \mathbb{Z}\}$ and $\mathcal{L}[m]$ is the space of eigenspinors for the eigenvalue $m$.

**Example**

$$\phi = \phi^{+(2,0,0)} \cdot \phi^{-(0,0,0)} \in \mathcal{L}[-1], \quad \text{and} \quad n\phi = -\phi.$$ 

Here we note that $-1$ is not an eigenvalue of the tangential Dirac operator $\partial$.

On the other hand $\varphi \in \mathcal{L}$ is also expressed as a linear sum of the spinors of the form $|z|^p \phi^{\pm(m,l,k)}$. For this component we have

$$n |z|^p \phi^{+(m,l,k)} = (p + m)|z|^p \phi^{+(m,l,k)},$$

$$n |z|^p \phi^{-(m,l,k)} = (p - m - 3)|z|^p \phi^{-(m,l,k)}$$

so that $|z|^p \phi^{+(m,l,k)}$ is of homogeneous order $(p + m)$ and $|z|^p \phi^{-(m,l,k)}$ is of homogeneous order $(p - m - 3)$. The normal vector field $n$ acting on $\mathcal{L}$ preserves the homogeneous degree. Thus we have the eigenspace decomposition of $\mathcal{L}$ by the normal derivative $n$:

$$\mathcal{L} = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}[m]$$

Let

$$H_{[m]} = \{ \varphi \in \mathcal{L}[m]; \quad D \varphi = 0 \}.$$ 

**Proposition 3.12.**

1. $H_{-1} = H_{-2} = 0$. 

2. \[ \mathcal{L}[m] = \sum_{p \in \mathbb{Z}} |z|^p H_{m-p} \] (3.21)

So any \( m \)-homogeneous spinor of \( \mathcal{L} \) is equal to a finite sum \( \sum_k |z|^k \phi_{(m-k)} \) with \( \phi_{(m-k)} \in H_{(m-k)} \).

**Proof** Since \( \phi(z) = |z|^m \phi(\frac{z}{|z|}) \) for \( \phi \in \mathcal{L}[m] \), we have

\[
D\phi(z) = |z|^{m-1} \gamma_+(z) \left( \frac{m}{2} \phi - \partial \phi \right) \left( \frac{z}{|z|} \right).
\]

For \( \phi \in H_m \) we have \( \partial \phi = \frac{m}{2} \phi \) on \( S^3 = \{|z|=1\} \). Since the eigenvalues of the boundary Dirac operator \( \partial \) are \( \frac{m}{2} \) for \( m \geq 0 \) and \( m \leq -3 \), we have

\[ H_{-2} = H_{-1} = 0. \]

We shall prove that

\[ H_k \cap |z|\mathcal{L}[k-1] = 0, \quad \forall k \in \mathbb{Z}, \] (3.22)

then by counting the dimensions we have

\[ \mathcal{L}[k] = H_k \oplus |z|\mathcal{L}[k-1]. \] (3.23)

It implies

\[
|z|^{-k} \mathcal{L}[m+k] = |z|^{-k} H_{m+k} \oplus |z|^{-k+1} H_{m+k-1} \oplus \ldots \oplus |z|^m H_0 \oplus |z|^{m+3} H_{-3} \oplus \ldots \oplus |z|^{k-1} H_{m-k+1} \oplus |z|^k \mathcal{L}[m-k].
\]

for any \( k \). For the proof of (3.22) we suppose that \( |z|\phi \in H_k \cap |z|\mathcal{L}[k-1] \) with \( \phi \in \mathcal{L}[k-1] \). Let \( q \) be the biggest number such that \( |z|\phi = |z|^q \psi \) for a \( \psi \in \mathcal{L}[k-q] \). Then

\[
D(|z|^q \psi) = \frac{q}{2} |z|^{q-1} \gamma_+ \psi + |z|^q D\psi = 0.
\]

We have \( |z|\phi = |z|^{q+1} \alpha \) with \( \alpha = -\frac{2}{q} \gamma_- D\psi \in \mathcal{L}[k-q-1] \). This is a contradiction. \( \square \)

**Corollary 3.13.** Let \( m = 0, 1, \ldots \).

1. \( H_m \) has the base \( \{ \phi^{+(m,l,k)}; l = 0, 1, \ldots, m, \ k = 0, \ldots m + 2 \} \).

2. \( H_{-(m+3)} \) has the base \( \{ \phi^{-(m,l,k)}; l = 0, 1, \ldots, m, \ k = 0, \ldots m + 2 \} \).
Proposition 3.14. For each $m \in \mathbb{Z}$ there is a linear isomorphism

$$\mathcal{L}[m] \simeq \mathbb{C}[\phi^\pm].$$

(3.24)

In fact, we have the following isomorphism

$$|z|^p H_{m-p} \simeq \mathbb{C}[\phi^{+(m-p,l,k)}; l = 0, 1, \ldots, m - p, k = 0, \ldots m - p + 1]$$

$$|z|^p H_{-(m-p+3)} \simeq \mathbb{C}[\phi^{-(m-p,l,k)}; l = 0, 1, \ldots, m - p, k = 0, \ldots m - p + 1]$$

that is given by letting $|z| = 1$ and the inverse is defined by

$$\phi \mapsto |z|^m \phi(z).$$

(3.25)

From (3.21) we have our assertion. \hfill \Box

Example 1

$\phi^{+(1,0,1)}\phi^{+(1,1,1)} \in \mathcal{L}[2]$ is decomposed to

$$\phi^{+(1,0,1)}(z)\phi^{+(1,1,1)}(z) = \frac{1}{\sqrt{2}}\phi^{+(2,1,1)}(z) - \frac{1}{3}\phi^{+(2,2,2)}(z) - \frac{1}{3}\sqrt{2}|z|^6\phi^{-(1,1,2)}(z)$$

$$\in H_2 \oplus |z|^6 H_4.$$  

(3.26)

The restriction to $\{|z| = 1\}$ becomes

$$\mathcal{L}[2] \ni \phi^{+(1,0,1)}\phi^{+(1,1,1)} \mapsto \varphi = \frac{1}{\sqrt{2}}\phi^{+(2,1,1)}(z) - \frac{1}{3}\phi^{+(2,2,2)}(z) - \frac{1}{3}\sqrt{2}|z|^6\phi^{-(1,1,2)}(z) \in \mathbb{C}[\phi^\pm]$$

and the inverse is given by

$$\mathbb{C}[\phi^\pm] \ni \varphi \mapsto |z|^2 \varphi(z) \in \mathcal{L}[2].$$

Example 2
\( \phi^{+(1,0,1)} \phi^{-(0,0,0)} \in \mathcal{L}[-2] \) is decomposed to
\[
\phi^{+(1,0,1)}(z) \phi^{-(0,0,0)}(z) = \frac{1}{|z|^4} \left( \frac{2}{3} \phi^{+(2,0,1)}(z) + \frac{\sqrt{2}}{3} \phi^{+(2,1,2)}(z) + \frac{1}{|z|^2} \frac{1}{2} \phi^{+(0,0,1)}(z) \right) \\
+ |z|^2 \left( \frac{1}{6} \phi^{-(1,1,1)}(z) + \frac{1}{3\sqrt{2}} \phi^{-(1,0,0)}(z) \right) \\
\in \frac{1}{|z|^4} H_2 \oplus \frac{1}{|z|^2} H_0 \oplus |z|^2 H_{-4}.
\] (3.27)

The restriction to \( \{|z| = 1\} \) becomes
\[
\mathcal{L}[-2] \ni \phi^{+(1,0,1)} \phi^{-(0,0,0)} \rightarrow \varphi = \frac{2}{3} \phi^{+(2,0,1)} + \frac{\sqrt{2}}{3} \phi^{+(2,1,2)} + \frac{1}{2} \phi^{+(0,0,1)} + \frac{1}{6} \phi^{-(1,1,1)} + \frac{1}{3\sqrt{2}} \phi^{-(1,0,0)} \in \mathbb{C}[\phi^+] \\
\text{and the inverse is given by}
\begin{align*}
\mathbb{C}[\phi^+] \ni \varphi & \rightarrow |z|^{-2} \varphi \left( \frac{z}{|z|} \right) \in \mathcal{L}[-2].
\end{align*}
\]

By virtue of the above isomorphisms \( (3.24) \) and \( (3.25) \) we extend the definitions of a quaternion trace and a quaternion residue that are defined for the Laurent polynomial type spinors to those over the algebra of current:
\[
\mathcal{L} = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}[m], \quad \mathcal{L}[m] = \sum_{p \in \mathbb{Z}} |z|^p H_{m-p}.
\]

**Definition 3.15.** Let \( \varphi = \sum_m \varphi_{[m]} \) with \( \varphi_{[m]} \in \mathcal{L}[m] \) be the decomposition of \( \varphi \) into the homogeneous components \( \varphi_{[m]} \). We denote \( \varphi_{[m]} = \sum_{p \in \mathbb{Z}} |z|^p \phi_{[m-p]} \) with \( \phi_{[m-p]} \in H_{m-p} \).

The quaternion trace \( qTr \varphi \) is defined by the term \( \varphi_{[0]} |_{|z|=1} = \sum_{p \in \mathbb{Z}} \phi_{[-p]} \in \mathbb{C}[\phi^+] \):
\[
qTr \varphi = \frac{1}{2\pi^2} \int_{S^3} \varphi_{[0]}(\zeta) \sigma(d\zeta).
\] (3.28)

And the quaternion residue \( qRes \varphi \) is given by the term \( \varphi_{[-3]} |_{|z|=1} = \sum_{p \in \mathbb{Z}} \phi_{[-3-p]} \in \mathbb{C}[\phi^+] \):
\[
qRes \varphi = \frac{1}{2\pi^2} \int_{S^3} \gamma_+ (\zeta) \varphi_{[-3]}(\zeta) \sigma(d\zeta).
\] (3.29)
Example 1. Let \( \varphi = \phi^{+(1,1,2)} \cdot \phi^{+(1,1,2)} \cdot \phi^{-(1,0,0)} \) with \( \phi^{+(1,1,2)} = \left( -\sqrt{2} \mathbf{z}_2 \right) \) and
\[
\phi^{-(1,0,0)} = \frac{\sqrt{2}}{|z|} \left( \frac{z_2^2}{\mathbf{z}_{2 \mathbf{2} - 1}} \right). \]
Then \( \varphi(z) = \frac{\sqrt{2}}{|z|} \left( \frac{2|z_2|^4 - |z_2|^2 \mathbf{z}_2 \mathbf{z}_1}{2z_2^4 + \mathbf{z}_2^2} \right) \in \mathcal{L} \) and
\[
q\text{Res} \varphi = 2\sqrt{2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]

Example 2. Let \( \psi = |z|^3 \phi^{+(1,1,2)} \cdot \phi^{+(1,1,2)} \cdot \phi^{-(1,0,0)} \).
\[
\psi = |z|^3 \varphi(z) = \frac{\sqrt{2}}{|z|^2} \left( \frac{2|z_2|^4 - |z_2|^2 \mathbf{z}_2 \mathbf{z}_1}{2z_2^4 + \mathbf{z}_2^2} \right) \in \mathcal{L}
\]
it holds that
\[
q\text{Tr} \psi = 2\sqrt{2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]

Proposition 3.16. Lemma 3.10 is valid for the algebra of current \( \mathcal{L} \):
\[
q\text{Res} \phi \varphi = 0, \quad q\text{Res} |z|^{-k} \mathbf{n} \varphi = 0, \quad \forall k \in \mathbb{N}. \tag{3.30}
\]
In particular \( q\text{Res} \frac{\partial}{\partial n} \varphi = 0 \). But \( q\text{Res} \mathbf{n} \varphi = -3q\text{Res} \varphi \).

Lemma 3.17. 1.
\[
q\text{Res} |z|^{-k} \varphi = 0, \quad k = 1, 2, \quad \forall \varphi \in \mathcal{L}. \tag{3.31}
\]
2.
\[
q\text{Res} |z|^{-3} \varphi = q\text{Tr} \varphi. \tag{3.32}
\]

Proof. First let \( \varphi \) be a Laurent polynomial type spinor; \( \varphi = \sum C_{-(m,\cdot)} \phi^{-(m,\cdot)} + \sum C_{+(m,\cdot)} \phi^{+(m,\cdot)} \). The quaternion trace of \( |z|^3 \varphi \) is given by the coefficients of its homogeneous component of order 0, hence is equal to \( q\text{Tr} |z|^3 \varphi = \left( \begin{array}{c} C_{-(0,0,1)} \\ C_{-(0,0,0)} \end{array} \right) = q\text{Res} \varphi \).

Any spinor of \( \mathcal{L} \) is written by a sum of Laurent polynomial type spinors each summand of which may be multiplied by some radial length \( |z|^p \) so that we the proof follows from Definition 3.15. \( \square \)
3.4 H-valued 2-cocycle on $\mathcal{L}$.

3.4.1 2-cocycle on $\mathcal{L}$

We shall introduce a $\mathbf{H}$-valued 2-cocycle on the $\mathbb{R}$-algebra $\mathcal{L}$. Remember the decomposition

$$\mathcal{L} = K \oplus J,$$

$$K = \mathbb{R} I, \quad J = \{ \phi = q i + r j + s k \in \mathcal{L}; \quad q, r, s \in C^\infty(\mathbb{C}^2 \setminus 0, \mathbb{R}) \}.$$

We call $\phi \in \mathcal{L}$ a basic form if it takes one of the following forms:

$$\phi = p I, q i, r j, s k \quad \text{with} \quad p, q, r, s \in C^\infty(\mathbb{C}^2 \setminus 0, \mathbb{R}).$$

The product of two basic forms is also a basic form. So every spinor in $\mathcal{L}$ is written as a sum of basic forms. For basic forms $\phi, \psi \in \mathcal{L}$, we put

$$\epsilon(\phi, \psi) = \begin{cases} +1, & \text{if } \phi \text{ or } \psi \text{ or } \phi \cdot \psi \in K \\ -1, & \text{if } \phi, \psi \text{ and } \phi \cdot \psi \in J. \end{cases} \quad (3.33)$$

That is,

$$\epsilon(p I, t I) = \epsilon(p I, q i) = \epsilon(p I, r j) = \epsilon(p I, s k) = +1,$$

$$\epsilon(q i, u i) = \epsilon(r j, v j) = \epsilon(s k, w k) = +1,$$

$$\epsilon(q i, v j) = \epsilon(q j, w k) = \epsilon(r j, w k) = -1.$$

Lemma 3.18. For basic forms $\phi$ and $\psi$ we have

$$\partial \phi \cdot \psi + \epsilon(\phi, \psi) \partial \psi \cdot \phi = \partial(\phi \cdot \psi) = \epsilon(\phi, \psi) \partial(\psi \cdot \phi). \quad (3.34)$$

Proof. The formulae are proved for each couple of basic forms $\phi, \psi$: For example, for $\phi = p I$ and $\psi = u i$, we have

$$\partial \phi \cdot \psi = -(\theta_3 p) u I + (\theta_4 p) u k + (\theta_2 p) u j,$$

$$\partial \psi \cdot \phi = -(\theta_3 u) p I + (\theta_1 u) p k + (\theta_2 u) p j,$$

$$\partial \phi \cdot \psi + \partial \psi \cdot \phi = -(\theta_3 u) p I + (\theta_1 u) p k + (\theta_2 u) p j = \partial(p u i) = \partial(\phi \cdot \psi).$$
For \( \phi = rj \) and \( \psi = wk \), we have

\[
\begin{align*}
\partial \phi \cdot \psi &= - (\theta_3 r) w I + (\theta_1 r) w k + (\theta_2 r) w j \\
\partial \psi \cdot \phi &= + (\theta_3 w) r I - (\theta_1 w) r k - (\theta_2 w) r j \\
\partial \phi \cdot \psi - \partial \psi \cdot \phi &= - (\theta_3 r w) I + (\theta_1 r w) k + (\theta_2 r w) j \\
&= \partial (r w i) = \partial (\phi \cdot \psi).
\end{align*}
\]

Others follow by similar calculations. \( \square \)

**Definition 3.19.** For \( \phi, \psi \in \mathcal{L} \), we put

\[
A(\phi, \psi) = q \text{Res} \left( \partial \phi \cdot \psi - \partial \psi \cdot \phi \right). \tag{3.35}
\]

From Proposition 3.4 we have

\[
A(\phi, \psi) = \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma_+ (\zeta) \left( \partial \phi \cdot \psi - \partial \psi \cdot \phi \right) \sigma (d\zeta). \tag{3.36}
\]

\( A(\phi, \psi) \) is an antisymmetric bilinear form of \( \phi, \psi \in \mathcal{L} \). From Proposition 3.16 and Lemma 3.18 we see that \( A(\phi, \psi) = 0 \) for basic forms \( \phi, \psi \) with \( \epsilon(\phi, \psi) = -1 \).

**Proposition 3.20.** For \( \phi_1, \phi_2, \phi_3 \in \mathcal{L} \), we have

\[
\begin{align*}
A(\phi_1, \phi_2) + A(\phi_2, \phi_1) &= 0, \tag{3.37} \\
A(\phi_1 \cdot \phi_2, \phi_3) + A(\phi_2 \cdot \phi_3, \phi_1) + A(\phi_3 \cdot \phi_1, \phi_2) &= 0. \tag{3.38}
\end{align*}
\]

**Proof.** Evidently \( A \) is an antisymmetric bilinear form. We put \( a(\phi, \psi) = (\partial \phi \cdot \psi - \partial \psi \cdot \phi) \).

The 2-cocycle property is proved by calculations exploiting the formula \( a(\phi_1 \phi_2, \phi_3) \) for basic forms \( \phi_1, \phi_2, \phi_3 \). For these basic forms we have

\[
\begin{align*}
a(\phi_1 \cdot \phi_2, \phi_3) &= (\partial (\phi_1 \cdot \phi_2)) \cdot \phi_3 - (\partial \phi_3) \cdot (\phi_1 \cdot \phi_2) \\
&= (\partial \phi_1) \cdot (\phi_2 \cdot \phi_3) + \epsilon(\phi_1, \phi_2)(\partial \phi_2) \cdot (\phi_1 \cdot \phi_3) - (\partial \phi_3) \cdot (\phi_1 \cdot \phi_2). \\
a(\phi_2 \cdot \phi_3, \phi_1) &= (\partial (\phi_2 \cdot \phi_3)) \cdot \phi_1 - (\partial \phi_1) \cdot (\phi_2 \cdot \phi_3) \\
&= (\partial \phi_2) \cdot \phi_3 \cdot \phi_1 + \epsilon(\phi_2, \phi_3)(\partial \phi_3) \cdot (\phi_2 \cdot \phi_1) - (\partial \phi_1) \cdot (\phi_2 \cdot \phi_3). \\
a(\phi_3 \cdot \phi_1, \phi_2) &= (\partial \phi_3) \cdot \phi_1 \cdot \phi_2 + \epsilon(\phi_3, \phi_1)(\partial \phi_1) \cdot (\phi_3 \cdot \phi_2) - (\partial \phi_2) \cdot (\phi_3 \cdot \phi_1).
\end{align*}
\]
Then
\[
\begin{align*}
  a(\phi_1 \cdot \phi_2, \phi_3) + a(\phi_2 \cdot \phi_3, \phi_1) + a(\phi_3 \cdot \phi_1, \phi_2) \\
  = \epsilon(\phi_1, \phi_2)(\phi_2 \cdot (\phi_1 \cdot \phi_3)) + \epsilon(\phi_2, \phi_3)(\phi_3 \cdot (\phi_2 \cdot \phi_1)) + \epsilon(\phi_3, \phi_1)(\phi_1 \cdot (\phi_3 \cdot \phi_2)) \\
  = k \partial(\phi_1 \cdot \phi_2 \cdot \phi_3),
\end{align*}
\]
for a constant \(k\). Since
\[
A(\phi, \psi) = q \text{Res.} a(\phi, \psi), \quad \text{for } \phi, \psi \in \mathcal{L},
\]
we have from Proposition 3.16
\[
A(\phi_1 \cdot \phi_2, \phi_3) + A(\phi_2 \cdot \phi_3, \phi_1) + A(\phi_3 \cdot \phi_1, \phi_2) \\
= q \text{Res.} (a(\phi_1 \cdot \phi_2, \phi_3) + a(\phi_2 \cdot \phi_3, \phi_1) + a(\phi_3 \cdot \phi_1, \phi_2)) d\sigma \quad (3.39)
\]
for basic forms \(\phi_1, \phi_2\) and \(\phi_3\). Then by the additivity of \(A(\phi, \psi)\) for the variables \(\phi\) and \(\psi\) we conclude that \(A\) satisfies the condition (3.38).

The bilinear form \(A(\phi, \psi)\) of Definition 3.19 on \(\mathcal{L}\) gives a \(\mathbb{H}\)-valued 2-cocycle.

By the definition of quaternion residue we have the following:
\[
\begin{align*}
  & A(\phi^{+(m_1,l_1,k_1)}, \phi^{+(m_2,l_2,k_2)}) = 0 \quad (3.40) \\
  & A(\phi^{-(m_1,l_1,k_1)}, \phi^{-(m_2,l_2,k_2)}) = 0. \quad (3.41)
\end{align*}
\]

Proposition 3.21.
\[
A(\frac{\partial}{\partial n} \phi, \psi) + A(\phi, \frac{\partial}{\partial n} \psi) = 0. \quad (3.42)
\]

Proof. Let \(a(\phi, \psi) = \phi \psi - \phi \partial \psi \cdot \phi\). It holds from (2.18) that \(\frac{\partial}{\partial n} \phi = \phi \frac{\partial}{\partial n} - \phi\)
on $\mathcal{L}$. So we have

$$a\left(\frac{\partial}{\partial n}\phi, \psi\right) + a\left(\phi, \frac{\partial}{\partial n}\psi\right) = \hat{a}(\frac{\partial}{\partial n}\phi) \cdot \psi - \hat{a}(\phi) \cdot \frac{\partial}{\partial n}\psi + \hat{a}(\frac{\partial}{\partial n}\psi) \cdot \phi$$

$$= \frac{\partial}{\partial n}(\phi) \cdot \psi - \frac{\partial}{\partial n}(\psi) \cdot \phi + \frac{\partial}{\partial n}(\phi) \cdot \frac{\partial}{\partial n}\psi - \frac{\partial}{\partial n}(\psi) \cdot \frac{\partial}{\partial n}\phi$$

$$+ \frac{1}{2|z|}(\hat{a}(\phi \cdot \psi) - \hat{a}(\phi \cdot \phi))$$

$$= \frac{\partial}{\partial n}a(\phi, \psi) + \frac{1}{2|z|}a(\phi, \psi).$$

By virtue of Proposition 3.16, we have

$$A\left(\frac{\partial}{\partial n}\phi, \psi\right) + A\left(\phi, \frac{\partial}{\partial n}\psi\right) = q\text{Res} \left[ \frac{\partial}{\partial n}a(\phi, \psi) + \frac{1}{2|z|}a(\phi, \psi) \right] = 0.$$

\[\square\]

4 $g$-current algebras on $S^3$

4.1 Quarternification of a Lie algebra

Hitherto we have prepared the spaces $C^\infty(S^3, S^+)$ and $\mathcal{L}$ that will play the role of coefficients of current algebras which we shall discuss in the next section. These are complex vector spaces. As we have seen they are given a $\mathbb{C}$–algebra structure. $C^\infty(S^3, S^+) \simeq C^\infty(S^3, \mathbb{H})$ has also a $\mathbb{H}$-module structure, while our basic interest is on $\mathcal{L}$ that is endowed with the real Lie algebra structure. In such a way it is frequent that we deal with the fields $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ in one formula. So to prove a steady point of view for our subjects we introduce here the concept of quarternion Lie algebras, $[Kq].$

First we note that a quarternion module $V = \mathbb{H} \otimes_{\mathbb{C}} V_o = V_o + JV_o$, $V_o$ being a $\mathbb{C}$-module, has two involutions $\sigma$ and $\tau$:

$$\sigma(u + Jv) = u - Jv, \quad \tau(u + Jv) = \overline{u} + J\overline{v}, \quad u, v \in V_o.$$

$$\sigma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix} \quad \text{and} \quad \tau \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}.$$
We have $\mathcal{L} = \mathcal{K} \oplus \mathcal{J}$ with

\[ \mathcal{K} = \{ \varphi \in \mathcal{L}; \sigma \varphi = \tau \varphi = \varphi \}; \]
\[ \mathcal{J} = \{ \varphi \in \mathcal{L}; \sigma \varphi = -\varphi \text{ or } \tau \varphi = -\varphi \}. \]

**Definition 4.1.** A *quaternion Lie algebra* $\mathfrak{q}$ is a $\mathbb{C}$-submodule of a $H$-$\mathcal{L}$ module $V$ that is endowed with a real Lie algebra structure compatible with the involutions $\sigma$ and $\tau$:

\[ \sigma \mathfrak{q} \subset \mathfrak{q}, \]
\[ \sigma [x, y] = [\sigma x, \sigma y], \quad \tau [x, y] = [\tau x, \tau y] \quad \text{for } x, y \in \mathfrak{q}. \]

For a complex Lie algebra $\mathfrak{g}$ the *quaternionification* of $\mathfrak{g}$ is a quaternion Lie algebra $\mathfrak{g}_q$ that is generated (as a real Lie algebra) by $H \otimes_{\mathbb{C}} \mathfrak{g}$.

For example, $\mathfrak{so}^*(2n) = H \otimes_{\mathbb{C}} \mathfrak{so}(n, \mathbb{C})$ is the quaternionification of $\mathfrak{so}(n, \mathbb{C})$. And $\mathfrak{sl}(n, H)$ is the quaternionification of $\mathfrak{sl}(n, \mathbb{C})$ while $H \otimes_{\mathbb{C}} \mathfrak{sl}(n, \mathbb{C})$ is not a Lie algebra.

**Theorem 4.2.** The algebra of current $\mathcal{L}$ is a quaternion Lie algebra by its associated Lie bracket.

In fact $\mathcal{L}$ is a real submodule of $S^3H = C^\infty(S^3, H)$ that is invariant under the involutions $\sigma$ and $\tau$. $\mathcal{L}$ is an associative complex algebra with the multiplication and has an induced Lie algebra structure over $\mathbb{R}$. The involutions $\sigma$ and $\tau$ are given by

\[ \sigma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix} \quad \text{and} \quad \tau \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}. \]

and satisfy $\sigma([x, y]) = [\sigma x, \sigma y]$ and $\tau([x, y]) = [\tau x, \tau y]$.

We have $\mathcal{L} = \mathcal{K} \oplus \mathcal{J}$ with

\[ \mathcal{K} = \{ \varphi \in \mathcal{L}; \sigma \varphi = \tau \varphi = \varphi \}; \]
\[ \mathcal{J} = \{ \varphi \in \mathcal{L}; \sigma \varphi = -\varphi \text{ or } \tau \varphi = -\varphi \}. \]
4.2 \( g \)-Current algebras and its subalgebras

We remember that we called the algebra \( L \) the algebra of current on \( S^3 \). By virtue of Theorem 3.10 \( L \) is a \( C \)-algebra generated by the spinors

\[
I = \phi^+(0,0,1), \quad J = -\phi^+(0,0,0), \quad \kappa = \phi^-(0,0,1), \quad \lambda = \phi^-(0,0,0).
\]

In the sequel we denote \( S^3E \) for a \( C \)-module \( E \) the space of \( E \)-valued smooth functions on \( S^3 \):

\[
S^3E = C^\infty(S^3, E).
\]

The space of positive spinors \( C^\infty(S^3, S^+ \)) is identified with \( S^3H \).

\[
S^3\text{gl}(n, H) = S^3H \otimes \text{gl}(n, C)
\]

becomes a quarternion Lie algebra with the Lie bracket defined by

\[
[\phi_1 \otimes X_1, \phi_2 \otimes X_2] = (\phi_1 \cdot \phi_2) \otimes X_1X_2 - (\phi_2 \cdot \phi_1) \otimes X_2X_1,
\]

for \( \phi_1, \phi_2 \in S^3H \) and \( X_1, X_2 \in \text{gl}(n, C) \). Here the right hand side is in the tensor product of the \( C \)-algebra \( S^3H \) and the matrix algebra \( \text{gl}(n, C) \).

Let \((g, [\ , \ ] )\) be a complex Lie algebra that we suppose to be a subalgebra of \( \text{gl}(n, C) \) for some \( n \). Then \( L \otimes C g \) becomes a \( C \)-submodule of \( S^3\text{gl}(n, H) = S^3H \otimes C \text{gl}(n, C) \). The involutions \( \sigma \) and \( \tau \) on \( L \) are extended to \( L \otimes C g \) by

\[
\sigma(\phi \otimes X) = \sigma(\phi) \otimes X \quad \text{and} \quad \tau(\phi \otimes X) = \tau(\phi) \otimes X
\]

respectively for \( \phi \in L \) and \( X \in g \). Thus \( L \otimes C g \) endowed with the bracket (4.1) generates a quarternion Lie algebra.

**Definition 4.3.** The quarternification of \((L \otimes C g, [\ , \ ] )\), that is, the quarternion Lie algebra generated by \((L \otimes C g, [\ , \ ] )\) is called \( g \)-current algebra and is denoted by \( Lg \).

As the following examples show \( L \otimes C g \) is not necessarily a Lie algebra so that the Lie algebra \( Lg \) is defined as the quarternification generated by \( L \otimes C g \) in the \( H \)-module \( S^3\text{gl}(n, H) \).

**Examples:** The following elements are in \( Lg \oplus (L \otimes C g) \).

1. \[
\sqrt{-1}(X_1X_2 + X_2X_1) \in Lg, \quad \text{for } \forall X_1, X_2 \in g.
\]
In fact we have
\[ \mathcal{L}g \ni [J \otimes X_1, \sqrt{-1} I \otimes X_2] = \sqrt{-1} I \otimes (X_1 X_2 + X_2 X_1). \]

Here the right hand-side is calculated in \( S^3 gl(n, H) \) which gives the left-hand side element of \( \mathcal{L}g \).

2.
\[ \sqrt{-1} J \otimes (X_1 X_2 + X_2 X_1) \in \mathcal{L}g, \quad \text{for } \forall X_1, X_2 \in g. \]

In fact
\[ \mathcal{L}g \ni [J \otimes X_1, \sqrt{-1} I \otimes X_2] = \sqrt{-1} J \otimes (X_1 X_2 + X_2 X_1). \]

Remember that the quarternification of a complex Lie algebra \( g \) is the quarternion Lie algebra \( g^q \) generated by \( H \otimes \mathbb{C} g = g + Jg \). The latter is not a Lie algebra in general. Since \( I = \phi^{+\langle 0,0,1 \rangle} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( J = -\phi^{+\langle 0,0,0 \rangle} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) are in \( \mathcal{L} \), \( g^q \) becomes a subspace of \( \mathcal{L}g \). We have the following relations:

\[ S^3 g^q \supset S^3 g + J (S^3 g) \supset \mathcal{L}g \supset g^q, \]

where \( S^3 g + J (S^3 g) \) is not necessarily a Lie algebra and \( S^3 g^q \) is the Lie algebra with bracket \([4,4,1]\).

The following examples 1 ~ 3 show the case where both \( S^3 H \otimes \mathbb{C} g \) and \( H \otimes \mathbb{C} g \) become Lie algebras (over \( \mathbb{C} \)).

**Examples**

1. \( gl(n, H) = H \otimes \mathbb{C} gl(n, \mathbb{C}) \subset \mathcal{L}gl(n, \mathbb{C}) \subset S^3 H \otimes \mathbb{C} gl(n, \mathbb{C}) = S^3 gl(n, H) \)

2. \( so^*(2n) = H \otimes \mathbb{C} so(n, \mathbb{C}) \subset \mathcal{L}so(n, \mathbb{C}) \subset S^3 H \otimes \mathbb{C} so(n, \mathbb{C}) = S^3 so^*(2n) \)

3. \( sp(2n) = H \otimes \mathbb{C} u(n) \subset \mathcal{L}u(n) \subset S^3 H \otimes \mathbb{C} u(n) = S^3 sp(2n). \)

4. \( sl(n, H) \) is the quarternification of \( sl(n, \mathbb{C}) \) which is not in \( \mathcal{L} \otimes \mathbb{C} sl(n, \mathbb{C}) \).

In fact, let \( \{ h_i = E_{ii} - E_{i+1,i+1}; \ 1 \leq i \leq n - 1, \ E_{ij}, i \neq j \} \) be the basis of \( g = sl(n, \mathbb{C}) \). Then \([\sqrt{-1} J h_1, J h_2] = -2\sqrt{-1} E_{22} \in g^q \subset \mathcal{L}g \) but not in \( \mathcal{L} \otimes \mathbb{C} g \).
4.3 Root space decomposition of $g$-current algebras

4.3.1

Let $g$ be a simple Lie algebra with Cartan matrix $A = (c_{ij})$. Let $h$ be a Cartan subalgebra, $\Phi$ the corresponding root system. Let $\Pi = \{\alpha_i; i = 1, \cdots, l = \dim h\} \subset h^*$ be the set of simple roots and $\{h_i = \alpha_i^\vee; i = 1, \cdots, l\} \subset h$ be the set of simple coroots. The Cartan matrix $A = (c_{ij})_{i,j=1,\cdots,r}$ is given by $c_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. $\alpha(h)$ is real if $h \in h$ is real. The Killing form $(x|y) = tr(ad x ad y)$ gives the symmetric invariant bilinear form on $g$. We have an isomorphism $h \rightarrow h^*$ from $h$ to $h^*$ given by $\langle h^*, x \rangle = (h|x)$. Let $g_\alpha = \{\xi \in g; \text{ad}(h)\xi = \alpha(h)\xi, \forall h \in h\}$ be the root space of $\alpha \in \Phi$. Then $\dim_C g_\alpha = 1$. Let $\Phi_+ \subset \Phi$ be the set of positive (respectively negative) roots of $g$ and put

$$ e = \sum_{\alpha \in \Phi_+} g_\alpha, \quad f = \sum_{\alpha \in \Phi_-} g_\alpha. $$

Fix a standard set of generators $h_i \in h$, $e_i \in g_{\alpha_i}$, $f_i \in g_{-\alpha_i}$. $g$ is generated by the set \{ $e_i, f_i, h_i; i = 1, \cdots, l$ \}, and these generators satisfy the relations:

$$ [h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = c_{ji} e_j, \quad [h_i, f_j] = -c_{ji} f_j. \quad (4.2) $$

This is a presentation of $g$ by generators and relations which depend only on the root system $\Phi$. The triangular decomposition of the simple Lie algebra $g$ becomes $g = f + h + e$, direct sum, with the space of positive root vectors $e$ and the space of negative root vectors $f$.

$g$ is considered as a quarternion Lie subalgebra of the $g$-current algebra $Lg$:

$$ i : g \ni X \longrightarrow \phi^{+(0,0,1)} \otimes X \in Lg, \quad (4.3) $$

$$ [\phi^{+(0,0,1)} \otimes X, \phi^{+(0,0,1)} \otimes Y]_{Lg} = [X, Y]_g. $$

We adopt the following abbreviated notations: For $\phi_i \in L$, $x_i \in g$, $i = 1, \cdots, t$, we put

$$ x_{12\cdots t} = [x_1, [x_2, [\cdots, x_t]], \cdots], $$

$$ \phi_{12\cdots t} \ast x_{12\cdots t} = [\phi_1 \otimes x_1, [\phi_2 \otimes x_2, [\cdots, \phi_t \otimes x_t]], \cdots]. \quad (4.4) $$

Every element of $Lg$ is expressed by a linear combination of $\phi_{12\cdots t} \ast x_{12\cdots t}$’s. We have a
projection from $\mathcal{L}_g$ to $g$ that extends the correspondence:

$$\pi : \mathcal{L}_g \ni \phi_{12\ldots t} \ast \xi_{12\ldots t} \longrightarrow \xi_{12\ldots t} \in g.$$  \hspace{1cm} (4.5)

It is obtained by letting all $\phi_i$'s in (4.4) equal to $\phi^{(0,0,1)}$.

### 4.3.2 The adjoint representation $ad_{K_h} : K_h \rightarrow \text{End}(\mathcal{L}_g)$

We shall investigate the triangular decomposition of $g$-current algebra $\mathcal{L}_g$.

**Definition 4.4.** Let $\mathcal{L}_h$, $\mathcal{L}_e$ and $\mathcal{L}_f$ respectively be the Lie subalgebras of the $g$-current algebra $\mathcal{L}_g$ that are generated by $\mathcal{L} \otimes_R h$, $\mathcal{L} \otimes_R e$ and $\mathcal{L} \otimes_R f$ respectively. Let $K_h$ and $J_h$ be the Lie subalgebra of $\mathcal{L}_g$ generated by $K \otimes_R h$ and $J \otimes_R h$ respectively.

$\mathcal{L}_e$ consists of linear combinations of elements of the form $\phi_{12\ldots t} \ast e_{12\ldots t}$ for $\phi_j \in \mathcal{L}$ and $e_j \in \mathcal{e}$, $j = 1, 2, \ldots, t$. Similarly $\mathcal{L}_f$ is generated by $\phi_{12\ldots t} \ast f_{12\ldots t}$ with $\phi_j \in \mathcal{L}$ and $f_j \in \mathcal{f}$, $j = 1, 2, \ldots, t$. Later we shall see that $\mathcal{L}_e = \mathcal{L} \otimes_R e$ and $\mathcal{L}_f = \mathcal{L} \otimes_R f$, viewed as real Lie algebras.

**Lemma 4.5.** It holds that

$$[\phi \otimes x, \psi \otimes y] = (\phi \psi) \otimes [x, y]$$ \hspace{1cm} (4.6)

for any $\phi \in K$, $\psi \in \mathcal{L}$, and $x, y \in g$.

This is an immediate consequence of (3.14) and (4.1).

**Lemma 4.6.**

1. $h \subset K_h$ and $K_h = K \otimes_R h$. \hspace{1cm} (4.7)

2. $K_h$ is a commutative subalgebra of $\mathcal{L}_g$, and $N(K_h) = K_h$. That is, $K_h$ is a Cartan subalgebra of $\mathcal{L}_g$, where $N(K_h) = \{ \xi \in \mathcal{L}_g; [\kappa, \xi] \in K_h, \forall \kappa \in K_h \}$ is the normalizer of $K_h$.

3. $[K_h, \mathcal{L}_h] = 0$, $[K_h, \mathcal{L}_e] = \mathcal{L}_e$, $[K_h, \mathcal{L}_f] = \mathcal{L}_f$.

**Proof.** Let $\phi_i \in K$ and $h_i \in h$, $i = 1, 2$. We have $[\phi_1 \otimes h_1, \phi_2 \otimes h_2] = (\phi_1 \phi_2) [h_1, h_2] = 0$. So $K_h = K \otimes_R h$, and $K_h$ is a commutative subalgebra of $\mathcal{L}_g$. Now the first assertion
follows from the definitions; \( \phi^{+(0,0,1)} \otimes h \subset K\hbar \). We shall prove \( N(K\hbar) = K\hbar \). Let \( \psi \otimes x \in (L \otimes g) \cap N(K\hbar) \). By hypothesis \( \phi \otimes h, \psi \otimes x = (\phi \psi) \otimes [h, x] \) is in \( K\hbar = K \otimes \hbar \) for any \( \phi \in K \) and \( h \in \hbar \). Then \( \phi \psi \in K \) for all \( \phi \in K \), so \( \psi \in N(K) \), which implies \( \psi \in K \). Hence \( \psi \otimes x \in K\hbar \). \( N(K\hbar) \) being generated by \( (L \otimes g) \cap N(K\hbar) \), it follows \( N(K\hbar) = K\hbar \).

We proceed to the proof of the 3rd assertion. Let \( \phi \otimes h \in K \otimes \hbar \) and \( \psi \otimes h' \in L \otimes \hbar \) with \( \phi \in K, \psi \in L \) and \( h, h' \in \hbar \). We have \( [\phi \otimes h, \psi \otimes h'] = (\phi \psi) \otimes [h, h'] = 0 \). Jacobi identity yields \( [\phi \otimes h, [\psi_1 \otimes h_1, \psi_2 \otimes h_2]] = 0 \) for \( \psi_i \in L, h_i \in \hbar, i = 1, 2, \) and \( [\phi \otimes h, \psi_{12-1} \otimes h_{12-1}] = 0 \). Hence \( [K\hbar, L\hbar] = 0 \). Let \( \psi \otimes e_j \in L \otimes e \). We have

\[
[\phi \otimes h_i, \psi \otimes e_j] = (\phi \psi) \otimes [h_i, e_j] = (\phi \psi) \otimes c_j e_j \in L \otimes e.
\]

So we have \( [\phi \otimes h_i, L \otimes e] \subset L \otimes e \), hence \( [K\hbar, L \otimes e] \subset L \otimes e \). Similarly \( [K\hbar, L \otimes f] \subset L \otimes f \). Conversely any element \( \psi_{1-1} \otimes e_j \) satisfies the relation \((4.8)\) for all \( \phi \otimes h \in K\hbar \) with non-zero \( (c_j + \cdots c_j) \) hence \( [K\hbar, L \otimes e] = L \otimes e \). Similarly \( [K\hbar, L \otimes f] = L \otimes f \).

Let \( i : \hbar \hookrightarrow K\hbar \) be the embedding \((4.3)\). Let \( \pi : Lg \rightarrow g \) be the projection \((4.5)\) and \( \pi_o : L \rightarrow C \) be the projection to the homogeneous degree 0 terms i.e. the trace of Laurent polynomial type spinors, \((3.3)\).

\( K\hbar \) being a Cartan subalgebra of \( Lg \), we shall investigate the adjoint representation \( ad_K \in End_R(Lg) \) and the associated weight space decomposition. The adjoint representation \( ad_K \) is written as follows:

\[
ad_{\phi \otimes h}(\psi \otimes x) = (\phi \psi) \otimes ad_h x,
\]

\[
ad_{\phi \otimes h}(\psi_{1-m} \otimes x_{1-m}) = \sum_{i=1}^{m} [\psi_1 \otimes x_1, [\psi_2 \otimes x_2, \cdots [(\phi \psi_1) \otimes ad_h x_i, [\psi_{i+1} \otimes x_{i+1}, \cdots, \psi_m \otimes x_m] \cdots],
\]

for \( \phi \otimes h \in K\hbar \) and \( \psi \otimes x, \psi_{1-m} \otimes x_{1-m} \in Lg \).

To a \( \lambda \in Hom_R(K\hbar, L) \) there corresponds \( \alpha = \pi_o \circ \lambda \circ i \in Hom(\hbar, C) = \hbar^* \). Conversely given \( \alpha \in \hbar^* \), we have \( \lambda \in Hom_R(K\hbar, L) \) that associates to \( \kappa = \phi \otimes h \in K\hbar \) with an element \( \alpha(h) \phi \in L \). Then \( \lambda(ih) = \alpha(h) \phi^{+(0,0,1)} \), that is, \( \alpha = \pi_o \circ \lambda \circ i \). We put

\[
Hom_R(K\hbar, L) = \{ \lambda : K\hbar \rightarrow L, \lambda(\kappa) = \pi_o \circ \lambda \circ i(h) \phi, \forall \kappa = \phi \otimes h \}.
\]

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\( \lambda(\kappa) \) yields a one dimensional representation of the Lie algebra \( L \mathfrak{g} \), that is given by the multiplication of \( \lambda(\kappa) = \alpha(h) \phi \in \mathcal{L} : \)

\[
L \mathfrak{g} \ni \xi = \psi \otimes x \longrightarrow \lambda(\kappa) \xi = \alpha(h) (\phi \psi) \otimes x \in L \mathfrak{g}, \quad \kappa = \phi \otimes h.
\]

For each \( \lambda \in \text{Hom}_{\mathbb{R}}(K \mathfrak{h}, \mathcal{L}) \), we define

\[
(L \mathfrak{g})_\lambda = \{ \xi \in L \mathfrak{g} : \text{ad}_\kappa \xi = \lambda(\kappa) \xi, \quad \forall \kappa \in K \mathfrak{h} \}.
\] (4.11)

\( \lambda \in \text{Hom}_{\mathbb{R}}(K \mathfrak{h}, \mathcal{L}) \) is called a weight of \( \text{ad}_{K \mathfrak{h}} \) whenever \( (L \mathfrak{g})_\lambda \neq 0 \). \( (L \mathfrak{g})_\lambda \) is called the weight space of weight \( \lambda \). The set of the non-zero weights is denoted by

\[
\Phi_{\mathcal{L}} = \{ \lambda \in \text{Hom}_{\mathbb{R}}(K \mathfrak{h}, \mathcal{L}) : \lambda \neq 0 \}.
\]

The relevance of the root space decomposition \( \mathfrak{g} = \sum_{\alpha \in \Phi} (\mathfrak{g})_\alpha \) is shown by the following proposition.

**Proposition 4.7.** \( L \mathfrak{g} \) is the direct sum of the weight spaces:

\[
L \mathfrak{g} = (L \mathfrak{g})_0 \oplus_{\lambda \in \Phi_{\mathcal{L}}} (L \mathfrak{g})_\lambda.
\] (4.12)

We have

\[\text{ad}_\kappa [\xi_1, \xi_2] = [\text{ad}_\kappa \xi_1, \xi_2] + [\xi_1, \text{ad}_\kappa \xi_2], \]

for all \( \kappa \in K \mathfrak{h}, \xi_i \in L \mathfrak{g}, i = 1, 2 \). This follows inductively from the definition of \( \text{ad}_{K \mathfrak{h}} \), [4.9].

It holds that if \( \xi, \eta \in L \mathfrak{g} \) are weight vectors of weights \( \lambda, \mu \) then \([\xi, \eta]\) is a weight vector of weight \( \lambda + \mu \):

\[
[ (L \mathfrak{g})_\lambda, (L \mathfrak{g})_\mu ] \subset (L \mathfrak{g})_{\lambda+\mu}.
\] (4.14)

**Proposition 4.8.** The adjoint representation \( \text{ad}_h \) of \( \mathfrak{g} \) extends to the adjoint representation \( \text{ad}_{K \mathfrak{h}} \) of \( L \mathfrak{g} \).

**Proof.** \( \phi^{+(0,0,1)} \in \mathcal{K} \) and the abbreviation \( \phi^{+(0,0,1)} \otimes \mathfrak{h} \simeq \mathfrak{h} \) imply the embedding \( i : \mathfrak{h} \longrightarrow K \mathfrak{h} \). The adjoint representation \( \text{ad}_{K \mathfrak{h}} \) restricts to the adjoint representation of \( \mathfrak{h} \) on \( \mathfrak{g} \) if we take \( \phi = \psi = \phi^{+(0,0,1)} \) in [4.9]. Then we have

\[\text{ad}_h \circ \pi = \pi \circ \text{ad}_{ih}, \quad \forall h \in \mathfrak{h}.\] (4.15)
Conversely we see from (4.9) that the action of the representation \(ad_{K_h}\) on \(Lg\) comes from \(ad_h \in \text{End}(g)\). If \(ad_h y = 0\) for a \(h \in h\) and a \(y \in g\) then \(ad_{\phi \otimes h} \psi \otimes y = 0\) for all \(\phi \in K\) and \(\psi \in L\). In fact, since \([K, L] = 0\) we have \([\phi \otimes h, \psi \otimes y] = (\phi \cdot \psi) \otimes [h, y] = 0\).

**Theorem 4.9.**

1. The root spaces of the adjoint representation \(ad_{K_h}\) on \(Lg\) and that of \(ad_h\) on \(g\) correspond bijectively: \(\Phi_L \simeq \Phi\).

2. For \(\lambda \in \Phi_L\), hence \(\alpha = \pi_o \circ \lambda \circ i \in h^*\),
   \[
   (Lg)_\lambda = L \otimes g_\alpha. \tag{4.16}
   \]
   And
   \[
   (Lg)_0 = Lh = Kh \oplus Jh. \tag{4.17}
   \]

3. \(Lg\) is the direct sum of the weight spaces:
   \[
   Lg = Lh \oplus \oplus_{\alpha \in \Phi} (L \otimes g_\alpha). \tag{4.18}
   \]

**Proof.** Let \(\lambda \in \Phi_L\). There exists a weight vector \(\xi \in Lg\) with the weight \(\lambda\): \([\phi \otimes h, \xi] = \lambda(\phi \otimes h)\xi\) for any \(\phi \otimes h \in Kh\). We define \(\tilde{\lambda} \in \text{Hom}(h, R)\) by the formula \(\tilde{\lambda}(h) = \lambda(\phi^{+(0,0,1)} \otimes h)\). Then \(\tilde{\lambda}\) becomes a root of the representation \(ad_h\) on \(g\): \([h, x] = [\phi^{+(0,0,1)} \otimes h, \phi^{+(0,0,1)} \otimes x] = \tilde{\lambda}(h)x\). Conversely let \(\xi = \psi_{i1...im} x_{i1...im} \in Lg\). We suppose that each \(x_i \in g\) is a weight vector with root \(\beta_i \in \Phi, i = 1, \ldots, m\). General elements of \(Lg\) are linear combinations of such vectors. It follows from (4.9) that
   \[
   ad_{\phi \otimes h} \xi = (\Sigma_{i=1}^m \beta_i(h)\phi) \xi, \quad \forall \phi \otimes h \in Kh.
   \]

Hence \(\Sigma_{i=1}^m \beta_i(h)\phi \in \Phi_L\), and \(\xi\) is a weight vector of \(ad_{\phi \otimes h}\). The relation extends linearly to \(Lg\). Thus we have proved the first assertion. From (4.9) we have \(L \otimes g_\alpha \subset (Lg)_\alpha\) for any \(\alpha \in \Phi\). Lemma 4.6 shows that \(Lh \subset (Lg)_0\). Then (4.8) yields that \(\phi_{i1...i_l} \otimes e_{i1...i_l}\) and \(\phi_{i1...i_l} \otimes f_{i1...i_l}\) are weight vectors. Thus all Lie products of generators \(\{\phi \otimes e_i, \phi \otimes f_i, \phi \otimes h_i; \phi \in L, i = 1, \ldots, l\}\) are weight vectors. Since every element of \(Lg\) is a linear combination of products of these weight vectors we deduce from (4.12) and the fact \(\Phi \simeq \Phi_L\) that
   \[
   Lg = (Lg)_0 \oplus \oplus_{\alpha \in \Phi} (Lg)_\alpha. \tag{4.19}
   \]

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Now the simple roots $\alpha_1, \cdots, \alpha_l \in \Phi$ are linearly independent, so the only monomials which have weight $\alpha_j$ are the weight vectors of $L \otimes g_{\alpha_j}$. We conclude

$$\left(L\mathfrak{g}\right)_{\alpha_j} = L \otimes_C g_{\alpha_j}.$$  \hspace{1cm} (4.20)

Hence $(L\mathfrak{g})_{\alpha} = L \otimes_C g_{\alpha}$ for all $\alpha \in \Phi$. Therefore (4.19) becomes

$$L\mathfrak{g} = (L\mathfrak{g})_0 \oplus \oplus_{\alpha \in \Phi} (L \otimes_C g_{\alpha}).$$  \hspace{1cm} (4.21)

Now we shall prove $(L\mathfrak{g})_0 = L\mathfrak{h}$. We regard $L\mathfrak{g}$ as a $K\mathfrak{h}$-module. Hence $L\mathfrak{h}$ is a $K\mathfrak{h}$-submodule. $L\mathfrak{h}$ is contained in $(L\mathfrak{g})_0$ by Lemma 4.6. If $L\mathfrak{h} \neq (L\mathfrak{g})_0$ the $K\mathfrak{h}$-module $(L\mathfrak{g})_0/ L\mathfrak{h}$ will have a 1-dimensional submodule $M/ L\mathfrak{h}$ on which $K\mathfrak{h}$ acts with weight 0. That is, $[K\mathfrak{h}, M/ L\mathfrak{h}] = 0$. Then $[K\mathfrak{h}, M] \subset L\mathfrak{h}$ and $M$ is a $K\mathfrak{h}$-submodule of $L\mathfrak{h}$. That is a contradiction.

We know that any weight $\lambda \in \Phi$ is of the form $\sum_{i=1}^l k_i \alpha_i$, $k_i \in \mathbb{Z}$. Moreover a non-zero weight $\lambda$ has the form $\lambda = \sum_{i=1}^l k_i \alpha_i$, $k_i \in \mathbb{Z}$, with all $k_i \geq 0$ or all $k_i \leq 0$. Therefore

$$L\mathfrak{e} = \sum_{\lambda \in \Phi^+} L \otimes_R g_{\lambda}, \hspace{1cm} (4.22)$$

$$L\mathfrak{f} = \sum_{\lambda \in \Phi^-} L \otimes_R g_{\lambda}, \hspace{1cm} (4.23)$$

From the above discussion we have the following

**Theorem 4.10.** The $\mathfrak{g}$-current algebra $L\mathfrak{g}$ has the following triangular decomposition

$$L\mathfrak{g} = L\mathfrak{e} \oplus L\mathfrak{h} \oplus L\mathfrak{f}. \hspace{1cm} (4.24)$$

It follows from (4.24) that

**Corollary 4.11.**

$$L\mathfrak{g} \oplus (L \otimes g) = \mathcal{J}\mathfrak{h}. \hspace{1cm} (4.24)$$
4.3.3 Symmetric invariant bilinear form on $Lg$

The symmetric invariant bilinear form on $Lg$ is defined by

$$(\phi_1 \mid \phi_2) = q\text{Res}(\phi_1 \phi_2).$$  \hspace{1cm} (4.25)

from Definition 3.3. The invariant bilinear form on $Lg$ is given by the following formula:

$$(\phi_1 \otimes x_1 \mid \phi_2 \otimes x_2) = (\phi_1 \mid \phi_2)(x_1 \mid x_2), \text{ for } \phi_i \in L, x_i \in g, i = 1, 2,$$  \hspace{1cm} (4.26)

where $(x_1 \mid x_2)$ is the Killing form of $g$. In fact, we have

$$([\phi_1 \otimes x_1, \phi_2 \otimes x_2] \mid \phi_3 \otimes x_3) = (\phi_1 \phi_2 \otimes x_1 x_2 \mid \phi_3 \otimes x_3) - (\phi_2 \phi_1 \otimes x_2 x_1 \mid \phi_3 \otimes x_3)$$

$$= (\phi_1 \phi_2 \mid \phi_3)(x_1 x_2 \mid x_3) - (\phi_2 \phi_1 \mid \phi_3)(x_2 x_1 \mid x_3)$$

$$= (\phi_1 \mid \phi_2 \phi_3)(x_1 \mid x_2 x_3) - (\phi_1 \mid \phi_3 \phi_2)(x_1 \mid x_3 x_2)$$

$$= (\phi_1 \otimes x_1 \mid \phi_2 \phi_3 \otimes x_2 x_3 - \phi_3 \phi_2 \otimes x_3 x_2),$$

where the calculation relies on the fact that the Lie algebra $g$ is a subalgebra of $gl(n, \mathbb{C})$, so in particular $(x_1 x_2 \mid x_3) = (x_1 \mid x_2 x_3)$.

The bilinear form $(\xi \mid \eta), \xi, \eta \in Lg$, is non-degenerate. As an immediate consequences we have the following

**Proposition 4.12.**

1. $(Lg)_\lambda$ and $(Lg)_\mu$ are orthogonal with respect to the bilinear form (4.26) unless $\mu + \lambda = 0$.

2. $$(Kg \mid Jg) = 0.$$ \hspace{1cm} (4.27)

**Proof.** Suppose $\lambda + \mu \neq 0$ and let $\xi \in (Lg)_\lambda, \eta \in (Lg)_\mu$. Choose $\kappa \in K\mathfrak{h}$ with $(\lambda + \mu)(\kappa) \neq 0$. Then

$$(\xi, \kappa \mid \eta) = (\xi \mid [\kappa, \eta])$$

implies $-\lambda(\kappa)(\xi \mid \eta) = \mu(\kappa)(\xi \mid \eta)$ that is $(\lambda + \mu)(\kappa)(\xi \mid \eta) = 0$. Hence $(\xi \mid \eta) = 0$. The second assertion is trivial from the definition; $q\text{tr}(\phi \cdot \psi) = 0$ for $\phi \in K$ and $\psi \in J$. \hfill $\Box$

Suppose $\lambda + \mu \neq 0$ For any $\alpha \in \Phi$ there is a unique element $h_\alpha \in \mathfrak{h}$ such that

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\[\alpha(h) = (h_\alpha|h) \text{ for all } h \in \mathfrak{h}.\] Similar assertion holds for \(K\mathfrak{h}\). In fact, let \(\lambda \in \Phi_L\) be the weight of \(ad_{\mathfrak{K}\mathfrak{h}}\) corresponding to \(\alpha = \pi_0 \circ \lambda \circ i\). Then for any \(\kappa = \phi \otimes h \in K\mathfrak{h}\) it holds \(\lambda(\kappa) = \phi \otimes \alpha(h) = (h_\alpha|h)\phi\). Hence, \(\kappa_\lambda = \phi^{+ (0,0,1)} \otimes h_\alpha \in K\mathfrak{h}\) is the unique element that represents \(\lambda \in \Phi_L\):

\[\lambda(\kappa) = (\kappa_\lambda \mid \kappa) \quad \text{for } \forall \kappa \in K\mathfrak{h}. \quad (4.28)\]

We know that \(h_\alpha = [x, y] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}\) for some \(x \in \mathfrak{g}_\alpha\) and \(y \in \mathfrak{g}_{-\alpha}\). But \(\kappa_\lambda\) can not have an analogous formula. Though it holds that

\[[(\mathcal{L}\mathfrak{g})_\lambda, (\mathcal{L}\mathfrak{g})_{-\lambda}] = (\mathcal{L}\mathfrak{g})_0 = \mathcal{L}\mathfrak{h} = K\mathfrak{h} \oplus J\mathfrak{h},\]

from (4.17).

**Proposition 4.13.**

1. Let \(\lambda \in \Phi_L\) be a weight of \(ad_{\mathfrak{K}\mathfrak{h}}\) and \(\alpha = \pi_0 \circ \lambda \circ i \in \Phi\) the corresponding root of \(\mathfrak{g}\).

   Then the vector \(\kappa_\lambda = \phi^{+ (0,0,1)} \otimes h_\alpha \in K\mathfrak{h}\) gives the \(K\mathfrak{h}\)-component of \([ (\mathcal{L}\mathfrak{g})_\lambda, (\mathcal{L}\mathfrak{g})_{-\lambda} ] \).

2. We have the relation:

\[ [\xi, \eta] = (\xi \mid \eta) \kappa_\lambda, \quad (4.29)\]

for \(\xi \in (\mathcal{L}\mathfrak{g})_\lambda\) and \(\eta \in (\mathcal{L}\mathfrak{g})_{-\lambda}\).

**Proof.** In fact, let \(\lambda \in \Phi_L\) and let \(\alpha \in \Phi\) be the corresponding element: \(\alpha = \pi_0 \circ \lambda \circ i\), and let \(R\mathfrak{e}_\alpha\) be the 1-dimensional \(\mathfrak{h}\)-submodule contained in \(\mathfrak{g}_\alpha\). We have \([h, e_\alpha] = \alpha(h)e_\alpha\) for all \(h \in \mathfrak{h}\). Similarly for \(\epsilon_\lambda = \phi^{+ (0,0,1)} \otimes e_\alpha \in \mathcal{L} \otimes e_\alpha = (\mathcal{L}\mathfrak{g})_\lambda\) and \(\kappa = \phi \otimes h \in K\mathfrak{h}\), we have \([\kappa, \epsilon_\lambda] = \alpha(h)\phi \otimes e_\alpha = \lambda(\kappa)\epsilon_\lambda\). Let \(y \in \mathfrak{g}_{-\alpha}\) be such that \((e_\alpha|y) \neq 0\). Such a \(y \in \mathfrak{g}_{-\alpha}\) certainly exists. Then \([e_\alpha, y] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}\). For any \(\psi \in \mathcal{L}\), we have \([\epsilon_\lambda, \psi \otimes y] = \psi \otimes [e_\alpha, y] \in \mathcal{L}\mathfrak{h} = K\mathfrak{h} \oplus J\mathfrak{h}\). We shall verify that the \(K\mathfrak{h}\) part of \([\epsilon_\lambda, \psi \otimes y]\) is given by \((\epsilon_\lambda|\psi \otimes y)\kappa_\lambda\). In fact let \(\xi = [\epsilon_\lambda, \psi \otimes y] = (\epsilon_\lambda|\psi \otimes y)\kappa_\lambda\). Then

\[ (\kappa|\xi) = (\kappa|[\epsilon_\lambda, \psi \otimes y]) = (\epsilon_\lambda|\psi \otimes y)(\kappa|\kappa_\lambda) = ([\kappa, \epsilon_\lambda]|\psi \otimes y) - \lambda(\kappa)(\epsilon_\lambda|\psi \otimes y) = 0 \]

for any \(\kappa = \phi \otimes h \in K\mathfrak{h}\). Thus \(\xi \in J\mathfrak{h}\). Hence \((\epsilon_\lambda|\psi \otimes y)\kappa_\lambda\) is the projection of \([\epsilon_\lambda, \psi \otimes y]\) to \(K\mathfrak{h}\). The first assertion is proved. Now for the proof of the second assertion we consider

\[ [\xi, \eta] = (\xi \mid \eta) \kappa_\lambda. \]

For all \(\kappa \in K\mathfrak{h}\) we have

\[ ([\xi, \eta] - (\xi \mid \eta) \kappa_\lambda | \kappa) = ([\xi, \eta] | \kappa) - (\xi \mid \eta)(\kappa_\lambda | \kappa) \]

\[ = (\xi \mid [\eta, \kappa]) - \lambda(\kappa)(\xi \mid \eta) = 0. \]
Since the form is non-degenerate on $\mathfrak{K}\mathfrak{h}$ we deduce $[\xi, \eta] - (\xi|\eta)\kappa = 0$. \hfill \Box

5 Central extension of the $\mathfrak{g}$-current algebra

5.1 Central extension of the $\mathfrak{g}$-current algebra $L_{\mathfrak{g}}$

Let $(V, [\cdot, \cdot]_V)$ be a quaternion Lie algebra. A central extension of $(V, [\cdot, \cdot]_V)$ is a quaternion Lie algebra $(W, [\cdot, \cdot]_W)$ such that $W = V \oplus Z$ (direct sum) and $Z$ is contained in the center of $W$;

$$Z \subset \{ w \in W : [w, x]_W = 0, \forall x \in W \},$$

and such that $[\cdot, \cdot]_W$ restricts to $[\cdot, \cdot]_V$.

Let $\mathfrak{g}$ be a simple Lie algebra which we suppose to be a subalgebra of a matrix algebra, and let $L_{\mathfrak{g}}$ be the $\mathfrak{g}$-current algebra. There exists a non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ (Killing form), which is given by $(x|y) = \text{Trace}(xy)$. The invariance means; $([x, y]|z) = (x|[y, z])$ for all $x, y, z \in \mathfrak{g}$.

In Proposition 3.19 we introduced a 2-cocycle $A$ on the space of current $\mathcal{L}$ that takes values in $\mathbb{H}$. We extend them to the 2-cocycle on the $\mathfrak{g}$-current algebra $L_{\mathfrak{g}}$ by

$$A(\phi \otimes x, \psi \otimes y) = (x|y) A(\phi, \psi)$$

for $\phi, \psi \in \mathcal{L}$ and $x, y \in \mathfrak{g}$. Then we have a $\mathbb{H}$-valued bilinear form on $L_{\mathfrak{g}}$ that satisfy cocycle conditions:

$$A(u, v) = -A(v, u)$$

$$A([u, v], w) + A([v, w], u) + A([w, u], v) = 0 \quad \text{for } u, v, w \in L_{\mathfrak{g}}.$$

In fact it is enough to check these conditions for $u = \phi \otimes x, v = \psi \otimes y, w = \pi \otimes z$, with $\phi, \psi, \pi \in \mathcal{L}$ and $x, y, z \in \mathfrak{g}$. The first follows from (3.37) and the symmetry of $(\cdot|\cdot)$. The second property follows from (3.38) and the symmetry and invariance of $(\cdot|\cdot)$. Indeed
we have
\[ A([u,v],w) = A((\phi \psi) \otimes x y, \pi \otimes z) - A((\psi \phi) \otimes y x, \pi \otimes z) = (x y | z) A(\phi \psi, \pi) - (y x | z) A(\psi \phi, \pi). \]
\[ A([v,w],u) = A((\psi \pi) \otimes y z, \phi \otimes x) - A((\pi \psi) \otimes z y, \phi \otimes x) = (y z | x) A(\psi \pi, \phi) - (z y | x) A(\pi \psi, \phi). \]
\[ A([w,u],v) = A((\pi \phi) \otimes z x, \psi \otimes y) - A((\phi \pi) \otimes x z, \psi \otimes y) = (z x | y) A(\pi \phi, \psi) - (x z | y) A(\phi \pi, \psi). \]

\((\cdot | \cdot)\) being symmetric invariant bilinear form we have \((x y | z) = (y z | x) = (z x | y)\) etc., then
\[ A([u,v],w) + A([v,w],u) + A([w,u],v) =
(y z | x) \{ A(\phi \psi, \pi) + A(\psi \pi, \phi) + A(\phi \pi, \psi) \} - (y x | z) \{ A(\psi \phi, \pi) + A(\pi \psi, \phi) + A(\phi \pi, \psi) \}. \]

By (3.38) the last formula vanishes.

Associated to the 2-cocycle \(A\), we have the central extension of \(Lg\).

**Theorem 5.1.** Let \(c\) be a indefinite number. Put
\[ Lg(c) = Lg \oplus Hc. \tag{5.2} \]

We endow \((L \otimes g) \oplus Hc\) with the following bracket:
\[ [\phi \otimes x, \psi \otimes y]^c = [\phi \otimes x, \psi \otimes y] + A(\phi \otimes x, \psi \otimes y)c, \]
\[ [c, \phi \otimes x]^c = 0, \tag{5.3} \]
for \(\phi \otimes x, \psi \otimes y \in L \otimes g\). The bracket is extended to \(Lg(c)\), and \(Lg(c)\) becomes a quaternion Lie algebra with the conjugation automorphism \(\sigma\) extended by \(\sigma c = c\).

We shall further complete the central extension of the current algebra \(Lg(c)\) by adjoining the normal derivation coming from the normal vector field. First we extend \(n\) to an outer derivation of the Lie algebra \(Lg\) by
\[ n(\phi \otimes x) = (n\phi) \otimes x, \quad \phi \in L, x \in g. \tag{5.4} \]

Then we extend \(n\) further to \(Lg(c)\) by killing the \(c\); \(nc = 0\). In fact we have the following
Lemma 5.2. Let \( \frac{\partial}{\partial n} = \frac{1}{2z^1} n \) be the normal derivative. We have

\[
\bigl[ \frac{\partial}{\partial n} (\phi_1 \otimes x_1, \phi_2 \otimes x_2) \bigl] + \bigl[ \phi_1 \otimes x_1, \frac{\partial}{\partial n} (\phi_2 \otimes x_2) \bigl] = \frac{\partial}{\partial n} \bigl( \bigl[ \phi_1 \otimes x_1, \phi_2 \otimes x_2 \bigl] \bigr).
\]

(5.5)

Proof. From Propositions \(3.11\) and \(5.21\) we have

\[
\bigl[ \frac{\partial}{\partial n} (\phi_1 \otimes x_1, \phi_2 \otimes x_2) \bigl] + \bigl[ \phi_1 \otimes x_1, \frac{\partial}{\partial n} (\phi_2 \otimes x_2) \bigl] = \bigl[ \phi_1 \cdot \phi_2 \bigr] \otimes x_1 \cdot x_2 - \bigl[ \phi_2 \cdot \phi_1 \bigr] \otimes x_2 \cdot x_2 - \bigl[ \phi_1 \cdot \phi_2 \bigr] \otimes x_2 \cdot x_1
\]

\[
+ \bigl( x_1 | x_2 \bigr) \left( A \left( \frac{\partial}{\partial n} \phi_1, \phi_2 \right) + A \left( \phi_1, \frac{\partial}{\partial n} \phi_2 \right) \right)
\]

\[
= \frac{\partial}{\partial n} \bigl( \phi_1 \cdot \phi_2 \bigr) \otimes x_1 \cdot x_2 - \frac{\partial}{\partial n} \bigl( \phi_2 \cdot \phi_1 \bigr) \otimes x_2 \cdot x_1 = \frac{\partial}{\partial n} \bigl( \bigl[ \phi_1 \otimes x_1, \phi_2 \otimes x_2 \bigl] \bigr).
\]

\[
\square
\]

We have shown that \( \frac{\partial}{\partial n} \) acts on the Lie algebra \( L \mathfrak{g}(c) \). We remark that Lemma 5.2 is valid for the normal vector field \( |z|^{-k} n, \ k \geq 1 \), but not for \( |z|^k n, \ k \geq 0 \). \( |z|^{-k} n, \ k \geq 1 \), acts on the Lie algebra \( L \mathfrak{g}(c) \):

\[
\bigl[ |z|^{-k} n (\phi_1 \otimes x_1, \phi_2 \otimes x_2) \bigr] + \bigl[ \phi_1 \otimes x_1, |z|^{-k} n (\phi_2 \otimes x_2) \bigr] = |z|^{-k} n \bigl( \bigl[ \phi_1 \otimes x_1, \phi_2 \otimes x_2 \bigr] \bigr).
\]

Theorem 5.3. Let \( d \) be an indefinite element. We consider the \( \mathbb{R} \)-vector space:

\[
\hat{\mathfrak{g}} = L \mathfrak{g} \oplus (H \mathfrak{c}) \oplus (C \mathfrak{d}).
\]

(5.6)

We endow \( \hat{\mathfrak{g}} \) with the following extended bracket:

\[
\bigl[ \phi \otimes x, \psi \otimes y \bigr] = \bigl[ \phi \otimes x, \psi \otimes y \bigr]_{\hat{\mathfrak{g}}}
\]

(5.7)

\[
\bigl[ c, \phi \otimes x \bigr] = 0
\]

(5.8)

\[
\bigl[ d, \phi \otimes x \bigr] = \frac{\partial}{\partial n} \phi \otimes x
\]

(5.9)
for \( x, y \in \mathfrak{g} \) and \( \phi, \psi \in \mathcal{L} \). The involution \( \sigma \) is extended to \( \hat{\mathfrak{g}} \) by

\[
\sigma(\phi \otimes x) = \sigma\phi \otimes x, \quad \sigma c = 0, \quad \sigma n = n.
\]

Then we get a quaternion Lie algebra \((\hat{\mathfrak{g}}, [\cdot, \cdot])\).

**Proof.** We write simply \([\cdot, \cdot]\) instead of \([\cdot, \cdot]_{\hat{\mathfrak{g}}}\). It is enough to prove the following Jacobi identity:

\[
[[d, \phi_1 \otimes x_1], \phi_2 \otimes x_2] + [[\phi_1 \otimes x_1, \phi_2 \otimes x_2], d] + [[\phi_2 \otimes x_2, d], \phi_1 \otimes x_1] = 0.
\]

From the defining equation (5.9) and Lemma 5.2 the sum of the 1st and the 3rd terms is equal to

\[
[[d, \phi_1 \otimes x_1], \phi_2 \otimes x_2] + [\phi_1 \otimes x_1, [d, \phi_2 \otimes x_2]] = \frac{\partial}{\partial n} ([\phi_1 \otimes x_1, \phi_2 \otimes x_2]),
\]

which is equal to \(-[[\phi_1 \otimes x_1, \phi_2 \otimes x_2], d]\). \( \Box \)

**Proposition 5.4.** The centralizer of \( d \in \hat{\mathfrak{g}} \) is given by

\[
(\mathcal{L}[0] \mathfrak{g}) \oplus \mathfrak{H} \mathfrak{c} \oplus \mathbb{C} d.
\]

Here \( \mathcal{L}[0] \) is the subspace in \( \mathcal{L} \) generated by \( \phi_1 \cdots \phi_n \) with \( \phi_i \) being \( \phi_i = \phi^{\pm (m_i, l_i, k_i)} \) such that

\[
\sum_{i; \phi_i = \phi^{+(m_i, l_i, k_i)}} m_i - \sum_{i; \phi_i = \phi^{- (m_i, l_i, k_i)}} (m_i + 3) = 0,
\]

and \( \mathcal{L}[0] \mathfrak{g} \) is the subalgebra of \( \hat{\mathfrak{g}} \) generated by \( \mathcal{L}[0] \otimes \mathfrak{g} \). The proposition follows from the definition (5.9).

**Definition 5.5.** We call the quaternion Lie algebra \( \hat{\mathfrak{g}} \) the affine current algebra over \( \mathfrak{g} \):

\[
\hat{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathfrak{H} \mathfrak{c} \oplus \mathbb{C} d.
\]

**5.2 Root space decomposition of the current algebra \( \hat{\mathfrak{g}} \)**

Let \( \hat{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathfrak{H} \mathfrak{c} \oplus \mathbb{C} d \) be the affine current algebra over \( \mathfrak{g} \), Definition 5.5. We introduce the subalgebra

\[
\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{H} \mathfrak{c} \oplus \mathbb{C} d,
\]

(5.11)
where we applied the identification \( h \ni h \xrightarrow{\cong} \phi^{+,(0,0,1)} \otimes h \in Lg \). \( \hat{h} \) is a commutative subalgebra of \( \hat{g} \). The adjoint action of \( \hat{h} \) over \( \hat{g} \) is written as follows. From the discussion in previous sections, in particular by virtue of Theorem 4.10, Corollary 4.11, (4.16) and (5.6), we see that any element \( \xi \in \hat{g} \) is written in the form:

\[
\xi = x + pc + qd, \quad x \in Lg, \quad p \in H, \quad q \in C
\]

\[
x = y + \sum_{\alpha \in \Phi} \varphi_{\alpha} \otimes x_{\alpha}, \quad \varphi_{\alpha} \in L, \quad x_{\alpha} \in g_{\alpha},
\]

\[
y = \kappa + z \in Lh, \quad \kappa \in Kh, \quad z \in Jh
\]

On the other hand any element of \( \hat{h} \) is written in the form

\[
\hat{h} = \phi^{+,(0,0,1)} \otimes h + sc + td, \quad h \in h, \quad s \in H, \quad t \in C.
\]

From Lemma 4.6 follows \([\phi \otimes h, y] = 0\) for any \( \phi \in K, \) \( h \in h \) and \( y \in Lh, \) in particular \([\phi^{+,(0,0,1)} \otimes h, y] = 0\). So we see that the adjoint action of \( \hat{h} = h + sc + td \in \hat{h} \) on \( \xi = y + \sum_{\alpha} \varphi_{\alpha} \otimes x_{\alpha} + pc + qd \in \hat{g} \) becomes

\[
ad(\hat{h})(\xi) = \sum_{\alpha} \alpha(h)\varphi_{\alpha} \otimes x_{\alpha} + t \sum_{\alpha} \left( \frac{\partial}{\partial n} \varphi_{\alpha} \right) \otimes x_{\alpha} + t [d, y].
\]

\[
(5.13)
\]

Let \( \hat{h}^* \) be the dual space of \( \hat{h} \):

\[
\hat{h}^* = Hom_{C}(\hat{h}, C).
\]

An element \( \alpha \) of the dual space \( h^* \) of \( h \) is regarded as a element of \( \hat{h}^* \) by putting

\[
\langle \alpha, c \rangle = \langle \alpha, d \rangle = 0.
\]

\[
(5.14)
\]

So \( \Phi \subset h^* \) is seen to be a subset of \( \hat{h}^* \). We define \( \delta, \Lambda \in \hat{h}^* \), by

\[
\langle \delta, \alpha_i^\vee \rangle = \langle \Lambda, \alpha_i^\vee \rangle = 0, \quad 1 \leq i \leq l,
\]

\[
\langle \delta, c \rangle = 0, \quad \langle \delta, d \rangle = 1,
\]

\[
(5.14)
\]

\[
\langle \Lambda, c \rangle = 1, \quad \langle \Lambda, d \rangle = 0.
\]

Then \( \alpha_1, \ldots, \alpha_l, \delta, \Lambda \), give the basis of \( \hat{h}^* \).
We shall investigate the decomposition of \( \hat{\mathfrak{g}} \) into a direct sum of the simultaneous eigenspaces of \( \text{ad}(\hat{h}) \), \( h \in \hat{\mathfrak{h}} \). For a 1-dimensional representation \( \lambda \in \hat{\mathfrak{h}}^* \) we put
\[
\hat{\mathfrak{g}}_\lambda = \{ \xi \in \hat{\mathfrak{g}}; \ [\hat{h}, \xi]_{\hat{\mathfrak{g}}} = \langle \lambda, \hat{h} \rangle \xi \text{ for } \forall \hat{h} \in \hat{\mathfrak{h}} \}.
\] (5.15)

\( \lambda \) is called a root of the representation \( (\hat{\mathfrak{g}}, \text{ad}(\hat{\mathfrak{h}})) \) if \( \lambda \neq 0 \) and \( \hat{\mathfrak{g}}_\lambda \neq 0 \). \( \hat{\mathfrak{g}}_\lambda \) is called the root space of \( \lambda \). Let \( \hat{\Phi} \) be the set of roots:
\[
\hat{\Phi} = \left\{ \lambda = \alpha + m\Lambda + k_0\delta \in \hat{\mathfrak{h}}^*; \alpha = \sum_{i=1}^{l} k_i \alpha_i \in \Phi, k_i, m \in \mathbb{Z}, 0 \leq i \leq l \right\}.
\]
The set \( \hat{\Pi} = \{ \alpha_1, \cdots, \alpha_l, \Lambda, \delta \} \) forms a fundamental basis of \( \hat{\Phi} \). Thus we have the root space decomposition of \( \hat{\mathfrak{g}} \) with respect to \( \hat{\mathfrak{h}} \):
\[
\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \oplus (\oplus_{\lambda \in \hat{\Phi}} \hat{\mathfrak{g}}_\lambda).
\] (5.16)

There are following types of roots:

(i) \( \lambda = \alpha + k\delta, 0 \neq \alpha \in \Phi \),  
(ii) \( \lambda = k\delta, \ k \neq 0 \),  
(iii) \( \lambda = 0\delta \) and (iv) \( \lambda = 0 \).

**Proposition 5.6.** We have the following relations:

1. \[
[\hat{\mathfrak{g}}^\delta, \hat{\mathfrak{g}}^\delta] \subset \hat{\mathfrak{g}}^m, \ 	ext{for } m, n \in \mathbb{Z}.
\] (5.17)

2. \[
[\hat{\mathfrak{g}}^\delta, \hat{\mathfrak{g}}^\delta] \subset \hat{\mathfrak{g}}^m, \text{for } m, n \in \mathbb{Z}.
\] (5.18)

The Proposition is proved by a standard argument using the properties of Lie bracket.

We now describe each root space \( \hat{\mathfrak{g}}_\lambda \). We may assume that the weight vector \( \xi \in \hat{\mathfrak{g}} \) of each weight \( \lambda \) takes the form \( \xi = y + \sum_{\alpha \in \Phi} \varphi_{\alpha} \otimes x_\alpha \) because others do not contribute to give weight, see (5.13). Let \( x \in \mathfrak{g}_\alpha \) for \( \alpha \in \Phi, \alpha \neq 0 \), and let \( \varphi \in \mathcal{L}[m] \) for \( m \in \mathbb{Z} \), that
is, \( \varphi \) is \( m \)-homogeneous. (3.19). From (5.13) we have

\[
[\phi \otimes h, \varphi \otimes x]_{\hat{g}} = (\phi \varphi) \otimes [h, x] = (\alpha, h) \varphi \otimes x,
\]

\[
[d, \varphi \otimes x]_{\hat{g}} = (\frac{m}{2} \varphi) \otimes x,
\]

for any \( \phi \otimes h \in K \mathfrak{h} \). That is,

\[
[\hat{h}, \varphi \otimes x]_{\hat{g}} = \left( \frac{m}{2} \delta + \alpha, \hat{h} \right) (\varphi \otimes x),
\]

for every \( \hat{h} \in \hat{\mathfrak{h}} \). It implies the relation;

\[
\mathcal{L}[m] \otimes \mathfrak{g}_\alpha \subset \hat{\mathfrak{g}}_{\frac{m}{2} \delta + \alpha}.
\]

Now let \( y \in \mathcal{L} \mathfrak{h} \). It is written by a linear combination of terms of the form \( y' = \phi_{i_1 i_2 \cdots i_t} \otimes h_{i_1 i_2 \cdots i_t} \in \hat{\mathfrak{h}} \) and \( \phi_j \in \mathcal{L}[m_j], \ j = i_1, \cdots, i_t, \) so that

\[
\frac{\partial}{\partial n} y' = \left( \frac{1}{2} \sum_{k=1}^t m_k \right) \phi_{i_1 i_2 \cdots i_t} \otimes h_{i_1 i_2 \cdots i_t},
\]

and we find that \( y' \in \hat{\mathfrak{g}}_{\frac{m}{2} \delta} \) with \( m = \sum_{k=1}^t m_k \in \mathbb{Z} \). Hence

\[
\mathcal{L} \mathfrak{h} \subset \hat{\mathfrak{g}}_{0 \delta} \oplus \oplus_{m \neq 0} \hat{\mathfrak{g}}_{\frac{m}{2} \delta},
\]

with \( \hat{\mathfrak{g}}_{0 \delta} = \mathcal{L}[0] \otimes \mathfrak{h} \), and \( \hat{\mathfrak{g}}_{\frac{m}{2} \delta} = \mathcal{L}[m] \otimes \mathfrak{h} \).

**Theorem 5.7.** 1.

\[
\hat{\Pi} = \left\{ \frac{m}{2} \delta + \alpha; \ \alpha \in \Pi, \ m \in \mathbb{Z} \right\}
\]

\[
\bigcup \left\{ \frac{m}{2} \delta; \ m \in \mathbb{Z} \right\}.
\]

(5.19)

is a base of \( \hat{\Phi} \).

2. For \( \alpha \in \Phi, \ \alpha \neq 0 \) and \( m \in \mathbb{Z} \), we have

\[
\hat{\mathfrak{g}}_{\frac{m}{2} \delta + \alpha} = \mathcal{L}[m] \otimes \mathfrak{g}_\alpha.
\]

(5.20)
\[ \hat{\mathfrak{g}}_{0\delta} = \mathcal{L}[0] \otimes_{\mathfrak{h}} \hat{\mathfrak{h}}, \quad (5.21) \]

\[ \hat{\mathfrak{g}}_{m\delta} = \mathcal{L}[m] \otimes_{\mathfrak{h}} \hat{\mathfrak{h}}, \quad \text{for } 0 \neq m \in \mathbb{Z}. \quad (5.22) \]

4. \( \hat{\mathfrak{g}} \) has the following decomposition:

\[ \hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{0\delta} \oplus (\oplus_{0 \neq m \in \mathbb{Z}} \hat{\mathfrak{g}}_{m\delta}) \oplus (\oplus_{\alpha \in \Phi, m \in \mathbb{Z}} \hat{\mathfrak{g}}_{m\delta + \alpha}). \quad (5.23) \]

5. \[ \hat{\mathfrak{g}}_{0\delta} \oplus (\oplus_{0 \neq m \in \mathbb{Z}} \hat{\mathfrak{g}}_{m\delta}) = \mathcal{L}\mathfrak{h} = \mathcal{K}\mathfrak{h} \oplus \mathcal{J}\mathfrak{h}. \quad (5.24) \]

**Proof.** First we prove the second assertion. We have already proved \( \mathcal{L}[m] \otimes \mathfrak{g}_\alpha \subset \hat{\mathfrak{g}}_{m\delta + \alpha} \).

Conversely, for \( m \in \mathbb{Z} \) and \( \xi \in \hat{\mathfrak{g}}_{m\delta + \alpha} \), we shall show that \( \xi \) has the form \( \phi \otimes x \) with \( \phi \in \mathcal{L}[m] \) and \( x \in \mathfrak{g}_\alpha \). Let \( \xi = \psi \otimes x + \sum p_k a_k + q n \).

Then

\[ [\hat{\mathfrak{h}}, \xi]_\hat{\mathfrak{g}} = [\phi^{+(0,0,1)} \otimes h + \sum s_k a_k + tn, \psi \otimes x + \sum p_k a_k + q n]_\hat{\mathfrak{g}} = \psi \otimes [h, x] \]

\[ + t\left( \sum_{n \in \mathbb{Z}} \frac{n}{2} \psi_n \otimes x \right) \]

for any \( \hat{h} = \phi^{+(0,0,1)} \otimes h + \sum s_k a_k + tn \in \hat{\mathfrak{h}} \), where \( \psi = \sum_n \psi_n \) is the homogeneous decomposition of \( \psi \). From the assumption we have

\[ [\hat{\mathfrak{h}}, \xi]_\hat{\mathfrak{g}} = \left\langle \frac{m}{2} \delta + \alpha, \xi \right\rangle \]

\[ = \langle \alpha, h \rangle \psi \otimes x + \left( \frac{m}{2} t + \langle \alpha, h \rangle \right)(\sum p_k a_k + q n) \]

\[ + \frac{m}{2} t \left( \sum_k \psi_k \right) \otimes x. \]

Comparing the above two equations we have \( p_k = q = 0 \), and \( \psi_k = 0 \) for all \( k \) except for \( k = m \). Therefore \( \psi \in \mathcal{L}[m] \). We also have \([\hat{\mathfrak{h}}, \xi]_\hat{\mathfrak{g}} = \psi \otimes [h, x] = \langle \alpha, h \rangle \psi \otimes x \) for any \( \hat{h} = \phi^{+(0,0,1)} \otimes h + \sum s_k a_k + td \in \hat{\mathfrak{h}} \). Hence \( x \) has weight \( \alpha \) and \( \xi = \psi_m \otimes x \in \hat{\mathfrak{g}}_{m\delta + \alpha} \). We have proved

\[ \hat{\mathfrak{g}}_{m\delta + \alpha} = \mathcal{L}[m] \otimes_{\mathfrak{g}_\alpha} \hat{\mathfrak{g}}. \]

Now we shall show

\[ \mathcal{L}\mathfrak{h} \supset \hat{\mathfrak{g}}_{0\delta} \oplus \oplus_{0 \neq m} \hat{\mathfrak{g}}_{m\delta}. \]
where \( \hat{g}_{0\delta} = \mathcal{L}[0] \otimes_C \mathfrak{h} \), and \( \hat{g}_{m\delta} = \mathcal{L}[m] \otimes_R \mathfrak{h} \). The converse implication has been proved before, so both sides coincide. Let \( \xi = \in \hat{g}_{0\delta} \oplus_{m \neq 0} \hat{g}_{m\delta} \) which we may assume to be the form \( \xi = y + \sum p_k a_k + qn \). It satisfies

\[
[\hat{h}, \xi]_{\hat{g}} = \left( \frac{m}{2}\delta, \hat{h} \right) \xi, \quad \forall \hat{h} \in \hat{h},
\]

for \( m = 0 \) or \( m \neq 0 \). From (5.13) we find \( \xi = y \in \mathcal{L}[m]\mathfrak{h} \). The above discussion yields the first and the fourth assertions.

**Corollary 5.8.**

\[
\oplus_{\Phi \ni \alpha \neq 0} \hat{g}_{m\delta + \alpha} = \mathcal{L}[m] \otimes_C g.
\]

### 5.3 Standard invariant bilinear form on \( \hat{g} \)

Let \( (\cdot | \cdot) \) be the standard invariant form on \( \hat{g} \) and let \( \theta \) be the highest root of the root system \( \Phi \). Normalize the form \( (\cdot | \cdot) \) on \( g \) by the conditionn \( (\theta | \theta) = 2 \) and extend it to the whole \( \hat{g} \) by

\[
(\phi_1 \otimes x_1 | \phi_2 \otimes x_2) = (\phi_1 | \phi_2) (x_1 | x_2), \quad x_i \in g, \phi_i \in \mathcal{L}, \quad i = 1, 2
\]

\[
(\mathcal{H}c + \mathcal{C}d | \mathcal{L}g) = 0, \quad (c|c) = (d|d) = 0, \quad (c|d) = 1,
\]

where we have defined

\[
(\phi | \psi) = q \text{Res}(\phi \cdot \psi).
\]

It extends the invariant bilinear form (4.26) of \( \mathcal{L}g \). Here we shall check only the following invariance property. Since the rests are easy to prove.

**Lemma 5.9.**

\[
( [d, \phi \otimes x] | \psi \otimes y) = (d | [\phi \otimes x, \psi \otimes y])
\]

for \( \phi, \psi \in \mathcal{L} \).

**Proof.** The left hand side of the above equation is

\[
( [d, \phi \otimes x] | \psi \otimes y) = (\frac{\partial}{\partial n} \phi \otimes x | \psi \otimes y) = (\frac{\partial}{\partial n} \phi | \psi) (x | y)
\]

\[
= q \text{Res}(\frac{\partial}{\partial n} \phi \cdot \psi) (x | y) = \frac{1}{2} \left( q \text{Res}(\frac{\partial}{\partial n} \phi \cdot \psi) - q \text{Res}(\frac{\partial}{\partial n} \psi \cdot \phi) \right) (x | y).
\]
Here we used the fact: \( q \text{Res} \frac{\partial}{\partial n} (\psi \cdot \phi) = 0 \). The right hand side of (5.25) is equal to

\[
(d | [\phi \otimes x, \psi \otimes y]) + (x | y) A(\phi, \psi)c = A(\phi, \psi)(x | y).
\]

Proposition 3.10 implies

\[
q \text{Res} \frac{\partial}{\partial n} \phi \cdot \psi = q \text{Res} (\partial \phi \cdot \psi) \quad \text{and} \quad q \text{Res} \frac{\partial}{\partial n} \psi \cdot \phi = q \text{Res} (\partial \psi | \phi).
\]

Hence

\[
q \text{Res} \frac{\partial}{\partial n} \phi \cdot \psi - q \text{Res} \frac{\partial}{\partial n} \psi \cdot \phi = q \text{Res} (\partial \phi \cdot \psi) - q \text{Res} (\partial \psi | \phi) = A(\phi, \psi).
\]

5.4 Chevalley generators of \( \hat{g} \)

We consider the root spaces \( g_\theta \) and \( g_{-\theta} \) where \( \theta \) is the highest root of \( g \). We have \( \dim g_\theta = \dim g_{-\theta} = 1 \), and the bilinear form \( (\cdot | \cdot) \) restricted on \( g_\theta \times g_{-\theta} \), that is, the form restricted on \( (Lg)_\theta \times (Lg)_{-\theta} \), is non-degenerate. Let \( \omega^0 \) be the Chevalley involution of \( g \).

By the natural embedding of \( g \) in \( \hat{g} \) we have the vectors

\[
\begin{align*}
h_i & = \phi^{(0,0,1)} \otimes h_i \in \hat{h}, \\
e_i & = \phi^{(0,0,1)} \otimes e_i \in \hat{g}_{\theta^0 + \alpha_i}, \quad f_i = \phi^{(0,0,1)} \otimes f_i \in \hat{g}_{\theta^0 - \alpha_i}, \quad i = 1, \ldots, l.
\end{align*}
\]

Then

\[
\begin{align*}
[e_i, f_j]_\hat{g} & = \delta_{ij} h_i, \\
h_i, e_j]_\hat{g} & = a_{ij} e_j, \quad [h_i, f_j]_\hat{g} = -a_{ij} f_j, \quad 1 \leq i, j \leq l.
\end{align*}
\]

We have obtained a part of generators of \( \hat{g} \) that come naturally from \( g \). We want to augment these generators to the Chevalley generators of \( \hat{g} \). We take the following generators
of the algebra $\mathcal{L}$:

$$I = \phi^{(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = \phi^{(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad (5.27)$$

$$\kappa = \phi^{(1,0,1)} = \begin{pmatrix} z_2 \\ -\overline{z_1} \end{pmatrix}, \quad \lambda = \phi^{-(0,0,0)} = \frac{1}{|z|^4} \begin{pmatrix} z_2 \\ \overline{z_1} \end{pmatrix} |_{|z|=1}, \quad (5.28)$$

We put

$$\kappa_* = -\frac{1}{\sqrt{2}} \phi^{+(1,1,2)} - \frac{1}{2} \phi^{+(1,0,1)} + \frac{1}{2} \phi^{-(0,0,0)}$$

$$= \begin{pmatrix} \overline{z_2} \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} z_2 \\ -\overline{z_1} \end{pmatrix} + \frac{1}{2 |z|^4} \begin{pmatrix} z_2 \\ \overline{z_1} \end{pmatrix} |_{|z|=1} = \begin{pmatrix} \overline{z_2} \\ \overline{z_1} \end{pmatrix}.$$  

$$\lambda_* = \frac{1}{\sqrt{2}} \phi^{+(1,0,2)} + \frac{1}{2} \phi^{+(1,1,1)} + \frac{1}{2} \phi^{-(0,0,1)}$$

$$= \begin{pmatrix} \overline{z_1} \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} z_1 \\ \overline{z_2} \end{pmatrix} + \frac{1}{2 |z|^4} \begin{pmatrix} -z_1 \\ \overline{z_2} \end{pmatrix} |_{|z|=1} = \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}.$$  

It holds that

$$\kappa \in \mathcal{L}[1], \quad \lambda \in \mathcal{L}[-3], \quad \kappa_*, \lambda_* \in \mathcal{L}[1] \oplus \mathcal{L}[-3]$$

Lemma 5.10.

1.

$$\kappa \cdot \kappa_* = \kappa_* \cdot \kappa = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (5.29)$$

$$\lambda_* \cdot \lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (5.30)$$

2.

$$A(\kappa, \kappa_*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A(\lambda, \lambda_*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (5.31)$$
**Proof.** We have the following equations

\[
\begin{align*}
\varphi \kappa &= \frac{1}{2} \kappa, \quad \varphi \lambda = -\frac{3}{2} \lambda, \\
\varphi \kappa_* &= \left( -\frac{1}{2} \overline{z}_2 - \overline{z}_1 \right) = \frac{1}{2} \kappa_* - \lambda, \quad \varphi \lambda_* = \frac{1}{2} \left( \frac{z_1}{\overline{z}_2} \right) + \left( \frac{z_1}{\overline{z}_2} \right) = \frac{1}{2} \lambda_* - \phi^{-(0,0,1)}. 
\end{align*}
\]

By virtue of these equations we have

\[
\varphi \kappa \cdot \kappa_* - \varphi \kappa_* \cdot \kappa = \lambda \cdot \kappa = \left( \frac{z_2^2 + |z_1|^2}{\overline{z}_1 (z_2 - \overline{z}_2)} \right). 
\]

The quaternion residue of the last term is \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), so \( A(\kappa, \kappa_*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

Similarly we have

\[
\varphi \lambda \cdot \lambda_* - \varphi \lambda_* \cdot \lambda = \left( \frac{-z_1 (\overline{z}_2 + z_2)}{-z_2^2 + |z_2|^2 - \frac{1}{2}} \right). 
\]

The quaternion residue of the last term is \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), so \( A(\lambda, \lambda_*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Let \( \theta \) be the highest root of \( \mathfrak{g} \) and suppose that \( e_\theta \in \mathfrak{g}_{-\theta} \) and \( f_\theta \in \mathfrak{g}_\theta \) satisfy the relations \([e_\theta, f_\theta] = h_\theta\) and \((e_\theta | f_\theta) = 1\). We introduce the following vectors of \( \hat{\mathfrak{g}} \);

\[
\begin{align*}
f_J &= J \otimes f_\theta \in \hat{\mathfrak{g}}_{0\delta+\theta}, \quad e_J = (-J) \otimes e_\theta \in \hat{\mathfrak{g}}_{0\delta-\theta}, \\
f_\kappa &= \kappa \otimes f_\theta \in \hat{\mathfrak{g}}_{\frac{1}{2}\delta+\theta}, \quad e_\kappa = \kappa_* \otimes e_\theta \in \hat{\mathfrak{g}}_{-\frac{3}{2}\delta-\theta} \oplus \hat{\mathfrak{g}}_{\frac{5}{2}\delta-\theta}, \\
f_\lambda &= \lambda \otimes f_\theta \in \hat{\mathfrak{g}}_{-\frac{1}{2}\delta+\theta}, \quad e_\lambda = \lambda_* \otimes e_\theta \in \hat{\mathfrak{g}}_{-\frac{7}{2}\delta-\theta} \oplus \hat{\mathfrak{g}}_{\frac{9}{2}\delta-\theta}. 
\end{align*}
\]

Then we have the generators of \( \mathcal{L} \mathfrak{g} \oplus \mathbf{H} \mathfrak{c} \) that are given by the following triples:

\[
\begin{align*}
\left( \hat{e}_i, \hat{f}_i, h_i \right) &\quad i = 1, 2, \ldots, l, \\
\left( \hat{e}_\lambda, \hat{f}_\lambda, h_\theta \right), \quad \left( \hat{e}_\kappa, \hat{f}_\kappa, h_\theta \right), \quad \left( \hat{e}_J, \hat{f}_J, h_\theta \right).
\end{align*}
\]

These triples satisfy the following relations.
Proposition 5.11.

1. \[ [e_{\pi}, f_{i}]_{\hat{g}} = [f_{\pi}, e_{i}]_{\hat{g}} = 0, \quad \text{for } 1 \leq i \leq l, \quad \text{and } \pi = J, \kappa, \lambda. \] (5.36)

2. \[ [e_{J}, f_{J}]_{\hat{g}} = \hat{h}_{\theta}, \] (5.37)

3. \[ [e_{\lambda}, f_{\lambda}]_{\hat{g}} = \sqrt{-1}\hat{h}_{\theta} - c, \quad [e_{\kappa}, f_{\kappa}]_{\hat{g}} = \sqrt{-1}\hat{h}_{\theta} - c. \] (5.38)

Adding the element \( n \) to these generators of \( \mathcal{L}_{\hat{g}} \oplus \mathbf{H}_{c} \) we have obtained the Chevalley generators of \( \hat{g} \).

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