Massive Neutrinos: Model Building

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Abstract. We discuss models of flavour symmetries that have been proposed in order to explain the neutrino data; to do so, a combined consideration of fermion mass and mixing hierarchies, leptogenesis, CP-violation and lepton-flavour violation is required. Quantum corrections to neutrino masses and mixings have to be adequately accounted for. Although there is a wide range of solutions within different frameworks, it is possible to identify some common characteristics which, in view of the expected neutrino data, may be used to constraint or even exclude certain classes of models.

1. Introduction
In recent years, several experiments have provided convincing evidence for solar and atmospheric neutrino oscillations, giving important input on the neutrino mass differences and mixing angles. By now, we know that the atmospheric neutrino mixing angle, $\theta_{23}$, is large, with a central value around 0.5 and a range $0.31 \leq \sin^2 \theta_{23} \leq 0.72$. The solar angle $\theta_{12}$ is also large, but not maximal, with a central value $\sin^2 \theta_{12} \sim 0.3$. Upper limits from the CHOOZ experiment indicate that the third mixing angle, $\theta_{13}$, is small, with an upper limit of $\sin^2 \theta_{13} \leq 0.1$. Regarding mass differences, we know that $\Delta m^2_{\text{atm}} \sim 2.5 \times 10^{-3} \text{eV}^2$ and $\Delta m^2_{\text{sol}} \sim 8 \times 10^{-5} \text{eV}^2$.

It turns out that the data can be accommodated in minimal schemes with:

- three light neutrinos with hierarchical masses, of the order of the required mass differences for the atmospheric and solar deficits.
- inverted hierarchy solutions, with $|m_1|, |m_2| \gg |m_3|$, where $m_{1,2}^2 \sim \Delta m^2_{\text{atm}}$ and $m_{12}^2 = \Delta m^2_{\text{sol}}$.

The minimal schemes with only three neutrino masses, allow only two independent mass differences and thus the LSND result indicating $\bar{\nu}_\mu - \bar{\nu}_e$ and $\nu_\mu - \nu_e$ oscillations, cannot be simultaneously explained unless a sterile light neutrino state is introduced. In models with light sterile neutrinos, one has to take into account the constraints from cosmological Big Bang Nucleosynthesis: a sterile neutrino that mixes with an active one, thus being in equilibrium at the time of nucleosynthesis, can change the abundance of primordially produced elements, such as $^4\text{He}$ and deuterium. The larger the mixing and the mass differences between the sterile and active neutrinos, the bigger the deviations from the observed light element abundances. This implies that models where the sterile component contributes to solar rather than atmospheric neutrino oscillations, are accommodated easier within the standard nucleosynthesis scenarios. Here, in order to retain a simple connection of the neutrino masses with the known charge lepton and quark hierarchies, we focus on models without sterile neutrinos.
Naturally light neutrinos can be obtained via the see-saw mechanism \[1\]; this relates the light neutrino mass matrix, \(m_{\text{eff}}\), with the Dirac neutrino matrix, \(m_D\), and the heavy right-handed Majorana mass matrix, \(M_R\), via the formula

\[
m_{\text{eff}} = m_D \cdot M_R^{-1} \cdot m_D^T
\]

Finally, the leptonic mixing matrix is given by \(V_{\text{MNS}} = V_\nu^T V_\ell\), where \(V_\nu\) diagonalizes the charged-lepton mass matrix, while \(V_\ell\) diagonalizes the light neutrino mass matrix, \(m_{\text{eff}}\) \[2\].

The various phenomenological textures can be classified by their predictions for the neutrino mixing and mass hierarchies, and a very large number of models have been proposed in the literature \[3\]. In particular, we would like to know the answer to the following questions:

- Is the atmospheric neutrino mixing maximal, or close-to-maximal?
- How large is the solar neutrino mixing?
- What is the magnitude of \(\theta_{13}\)?
- Are the neutrinos degenerate or hierarchical?
- How is their pattern of masses related to those of other fermions?
- Are the neutrino masses Dirac or Majorana \(^1\)?
- In a given model and basis, does the mixing dominantly arise from \(V_\nu\) or \(V_\ell\)?
- Is there significant CP violation in the neutrino and charged-lepton sectors?

Even before passing in detail to phenomenological textures, we can gain some very rough insight by looking at the simple formulas for mass eigenvalues and mixing for a 2 \(\times\) 2 matrix. From

\[
m_{\text{eff}} = \begin{pmatrix} m_{22} & m_{23} \\ m_{23} & m_{33} \end{pmatrix}, \quad \sin^2 2\theta = \frac{4m_{23}^2}{(m_{33} - m_{22})^2 + 4m_{23}^2}
\]

we see that:

(i) For large 23 mixing in \(m_{\text{eff}}\), large hierarchies require \(\theta\)-determinant solutions.
(ii) On the other hand, if the large mixing comes from the charged-lepton sector, we can have large hierarchies without \(\theta\)-determinant solutions.

In any investigation, one has to take into account the effect of quantum corrections. For instance, due to the renormalisation of the neutrino mass operator, it turns out that a given mixing at the GUT scale may be amplified or destroyed at low energies \[4, 5\]. The eigenvalues of the neutrino mass operator are also modified by quantum effects, resulting in severe constraints on neutrino mass textures.

Moreover, the existence of neutrino masses and oscillations may have additional consequences. The decays of heavy neutrinos may have given rise to leptogenesis \[6\], which can provide additional constraints on CP-violating phases and on the magnitudes of certain neutrino Yukawa couplings. Finally, mixing in the charged-lepton sector is enhanced by radiative corrections in supersymmetric models that generate mixing also for the sleptons, which contribute to rare decays and flavour conversions \[7, 8\]. In what follows, we outline some combined considerations of the above.

\(^1\) Some light on this may be shed by experiments on neutrinoless double beta decay \((d + d \rightarrow u + u + e + e)\), which may only occur for Majorana neutrinos. On the other hand, only Dirac masses give rise to diagonal neutrino magnetic moment contributions.
2. Mixing angles and CP-violation

The fact that neutrinos have masses and mix with each other like quarks also implies that, in principle, we can expect non-negligible CP violation in the neutrino sector. This may manifest itself in several different ways in low-energy neutrino physics, including the Maki-Nakagawa-Sakata (MNS) oscillation phase $\delta$ and two possible Majorana phases. Additional phases arise in extensions of the light-neutrino sector to include either heavy singlet neutrinos and/or charged leptons.

The MNS mixing is given by a unitary matrix, $U$, which can be parametrized as:

$$U = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) V \text{diag}(e^{-i\phi_1/2}, e^{-i\phi_2/2}, 1)$$

(2)

where

$$V = \begin{pmatrix}
c_{12}c_{13} & s_{12} & s_{13}e^{-i\delta} \\
-s_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}s_{13}c_{12}e^{i\delta} \\
s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}s_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}$$

(3)

where $c_{ij}$ and $s_{ij}$ stand for $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively. In this formalism, the mixing angles and the phases are given by

$$\theta_{13} = \arcsin(|U_{13}|), \quad \theta_{12} = \arctan\left(\frac{|U_{12}|}{|U_{11}|}\right), \quad \theta_{23} = \arctan\left(\frac{|U_{23}|}{|U_{33}|}\right)$$

$$\delta_\mu = \arg(U_{23}), \quad \delta_\tau = \arg(U_{33}), \quad \delta = -\arg\left(\frac{U^*_{\mu i}U_{\mu j}U^*_{\tau i}}{c_{12}c_{13}c_{23}s_{13} + c_{12}c_{23}s_{13}}\right)$$

(4)

A good measure of the amount of CP violation in the oscillations of light neutrinos is the Jarlskog invariant

$$J_{CP} = \frac{1}{2}|\text{Im}(U^*_{11}U_{12}U^*_{22})| = \frac{1}{2}|\text{Im}(U^*_{11}U_{13}U^*_{31}U^*_{33})|$$

$$= \frac{1}{2}|\text{Im}(U^*_{22}U_{23}U^*_{32}U^*_{33})| = \frac{1}{2}|c_{12}c_{13}c_{23}\sin\delta s_{12}s_{13}s_{23}|$$

(5)

It is clear from this formula that different models predict different amounts of CP violation. For instance, the magnitude of $\theta_{13}$ on which we have so far we only have an upper bound, is closely connected with the expected magnitude of CP violation (which will be zero for models with texture zeroes in the (1,3) entries).

3. Introduction to flavour symmetries

The fact that the fermion mass matrices exhibit a hierarchical structure suggests that they are generated by an underlying family symmetry. Under such a symmetry different generations of fermions have different charges. Requiring that only invariant operators are allowed, will determine the magnitude of masses.

Let us start with the simplest possibility, which is this of an Abelian flavour-symmetry. We denote the charges of the Standard Model fields under the symmetry, as appears in Table 1.

The Higgs charges are chosen so that the terms $f_3f_5^2H$ (where $f$ denotes a fermion and $H$ denotes $H_1$ or $H_2$) have zero charge. Then, only the (3,3) element of the associated mass matrix will be non-zero. The remaining entries are generated when the $U(1)$ symmetry is spontaneously broken, via standard model singlet fields, $\theta$, $\bar{\theta}$, with non-trivial opposite $U(1)$ charges, and equal
Table 1. $U(1)$ charges of the fields, where $i$ stands for a generation index.

| $Q$, $u^c_i$, $d^c_i$, $L_i$, $e^c_i$, $\nu^c_i$, $H_2$, $H_1$ |
|----------|--------|--------|----------|--------|--------|--------|--------|
| $\alpha_i$, $\beta_i$, $\gamma_i$, $c_i$, $c_i$, $d_i$, $-\alpha_3 - \beta_3$, $-\alpha_3 - \gamma_3$ |

The suppression factor for each entry depends on the family charge: the higher the net $U(1)$ charge of a term $f_i H$, the higher the power $n$ in the term $f_i H \left( \frac{\theta}{M} \right)^n$ that has zero charge. For example, if only the 2–3 and 3–2 elements of the matrix are allowed by the symmetry at order $\epsilon \equiv \theta/M$, one has the following hierarchy of masses:

$$M \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}$$

(6)

where $M$ is an intermediate mass scale, determined by the mechanism that generates the non-renormalisable terms. The symmetry breaking arises via an extension of the “see-saw” mechanism, mixing light to heavy states (known as the Froggatt–Nielsen mechanism [9]).

We can now see that the choice $\alpha_i = \beta_i = \gamma_i = (-4, 1, 0)$ and zero Higgs $U(1)$ charges, leads to a solution which reproduces the known quark hierarchies [10]:

$$M_{up} \propto \begin{pmatrix} \epsilon^8 & \epsilon^3 & \epsilon^4 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon & 1 & 1 \end{pmatrix}, M_{down} \propto \begin{pmatrix} \epsilon^8 & \epsilon^3 & \epsilon^4 \\ \epsilon^3 & \epsilon^2 & \bar{\epsilon} \\ \epsilon & \bar{\epsilon} & 1 \end{pmatrix}, \text{ where } \bar{\epsilon} = \sqrt{\epsilon} \approx 0.22$$

Similarly, one may derive predictions for charged lepton and neutrino textures.

It is clear that one can generalise this simple example of an Abelian, left-right symmetric model to GUT and non-Abelian symmetries. In general, in the case of an Abelian flavour symmetry, demanding that the masses and mixings are fully predicted by the flavour and GUT structure leads in the simplest and apparently more natural models to large mass hierarchies and unacceptably small solar mixing, which is correlated with the hierarchies of charged lepton masses. In this case, large solar mixing, as required by the experimental data, has therefore to arise either:

(i) from the see-saw conditions (this can happen for example in models with right-handed singlet neutrino dominance and zero-determinant solutions), or
(ii) by imposing more $U(1)$ symmetries and/or introducing more fields (however, this strategy introduces additional model dependence and loses predictivity). On the other hand, large atmospheric mixing is naturally predicted in a wide class of Abelian models, whereas non-Abelian flavour symmetries typically predict naturally large angles both for solar and atmospheric neutrinos.

The situation changes when passing to non-Abelian flavour symmetries. Let us for instance look at the simple case where the lepton fields are $SO(3)$ triplets. Then, degenerate lepton textures are to be expected. Subsequently we break $SO(3)$ so that large charged lepton but small neutrino mass splitting is generated. This means that in non-Abelian models we expect that:

(i) solutions with almost-degenerate neutrinos can be naturally generated.

(ii) textures with (almost)-bimaximal mixing are mostly predicted.

Successful non-Abelian models with bimaximal mixing and large neutrino hierarchies have been proposed. For instance the $SU(3)$ flavour model of [12], gives rise to matrices of the form

$$M_{\nu_D} \propto \begin{pmatrix} 0 & \lambda \epsilon^3 & \lambda \epsilon^3 \\ -\lambda \epsilon^3 & \epsilon^2 & \epsilon^2 \\ -\lambda \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}, M_{\nu_R} \propto \begin{pmatrix} \epsilon^3 & 0 & \epsilon^3 \\ 0 & \epsilon^3 & \epsilon^3 \\ \epsilon^2 & \epsilon^2 & 1 \end{pmatrix}$$

(7)
with approximate bimaximal mixing and neutrino hierarchies obeying \( \frac{m_2}{m_3} \sim c_\nu < \epsilon \).

In subsequent paragraphs, we focus on Abelian flavour symmetries, since they are simple and arise naturally in a wide class of models. Moreover, understanding Abelian symmetries is a first step towards the understanding of non-Abelian symmetries. Indeed, by combining two or more Abelian symmetries and introducing more than one field whose vev determines the expansion parameter of the mass matrices, one can simulate to some extent the picture obtained from a non-Abelian structure.

4. Quantum corrections to neutrino masses and mixings

In the presence of neutrino masses, the running of the various couplings from the unification scale down to low energies is modified. For the neutrino sector, the Dirac neutrino Yukawa coupling, \( \lambda_N \), runs until the scale \( M_N \). Subsequently it decouples and the quantity that runs is the effective neutrino operator \( m_{\text{eff}} \).

In order to understand the renormalization effects due to a non-zero \( \lambda_N \) between \( M_{\text{GUT}} \) and \( M_N \), we can look at the simple case of the small-\( \tan \beta \) regime of a supersymmetric theory, where only the top and the Dirac neutrino Yukawa couplings contribute in a relevant way. In a diagonal basis, the renormalisation group equations of the Yukawa couplings take the following form [13]:

\[
16\pi^2 \frac{d}{dt} \lambda_t = \left( 6\lambda_t^2 + \lambda_N^2 - G_U \right) \lambda_t \\
16\pi^2 \frac{d}{dt} \lambda_N = \left( 4\lambda_N^2 + 3\lambda_t^2 - G_N \right) \lambda_N \\
16\pi^2 \frac{d}{dt} \lambda_b = \left( \lambda_b^2 - G_D \right) \lambda_b \\
16\pi^2 \frac{d}{dt} \lambda_\tau = \left( \lambda_\tau^2 - G_E \right) \lambda_\tau
\]

where \( \lambda_\alpha : \alpha = t, b, \tau, N \), represent the third-generation Dirac Yukawa couplings for the up and down quarks, charged lepton and neutrinos, respectively, and the \( G_\alpha \equiv \sum_{i=1}^3 c_i^\alpha g_i(t) \) are functions that depend on the gauge couplings, with the coefficients \( c_i^\alpha \) given in [13].

Below the right-handed Majorana mass scale, where \( m_{\text{eff}} \) is formed, \( \lambda_N \) decouples from the renormalisation group equations. However, the effective neutrino mass operator will be a running quantity [4, 5]. For a generic \( \tan \beta \), one finds that

\[
16\pi^2 \frac{d}{dt} m_{\text{eff}}^{ij} = \frac{1}{8\pi^2} \left( -c_i g_i^2 + 3\lambda_t^2 + \frac{1}{2}(\lambda_t^2 + \lambda_\tau^2) \right)
\]

where \( i, j \) are lepton flavour indices, already indicating that large Yukawa terms, which lower the effective couplings, have a larger effect on \( m_{\text{eff}}^{33} \) than on the other entries. Finally, the neutrino mixing angle relevant to the atmospheric neutrino deficit, \( \theta_{23} \), is also a running quantity, given by the following formula:

\[
16\pi^2 \frac{d}{dt} \sin^2 2\theta_{23} = 2\sin^2 2\theta_{23} \left( 1 - 2\sin^2\theta_{23} \right) \left( \lambda_\tau^2 - \lambda_\mu^2 \right) \frac{m_{\text{eff}}^{33} + m_{\text{eff}}^{22}}{m_{\text{eff}}^{33} - m_{\text{eff}}^{22}}
\]

where the initial conditions for the running from \( M_N \) down to low energies are determined by the running of couplings between \( M_{\text{GUT}} \) and \( M_N \).

Already this equation indicates that the mixing angle may significantly change from the GUT scale to low energies (i) if \( \lambda_\tau \) is large, and (ii) if the diagonal entries of \( m_{\text{eff}} \) are close in magnitude. Because of the running of the \( \tau \) Yukawa coupling being larger than those for the other flavours of charged leptons, \( m_{\text{eff}}^{33} \) decreases more rapidly than \( m_{\text{eff}}^{22} \). Then, if we start...
with $m_{\text{eff}}^{22} < m_{\text{eff}}^{33}$, at a later time we may obtain $m_{\text{eff}}^{22} = m_{\text{eff}}^{33}$ and the mixing angle becomes maximal. Subsequently, the running of $\lambda_\tau$ results in $m_{\text{eff},0}^{33} < m_{\text{eff},0}^{22}$, and thus in a decrease of the mixing.

Moreover, we also observe that the neutrino masses will in fact vary non-trivially with the energy. It is convenient for the subsequent discussion to define the integrals

$$I_g = \exp\left[\frac{1}{8\pi^2} \int_{t_0}^{t} (-c_3 g_i^2 dt)\right]$$
$$I_t = \exp\left[\frac{3}{8\pi^2} \int_{t_0}^{t} \lambda_i^2 dt\right]$$
$$I_i = \exp\left[\frac{1}{8\pi^2} \int_{t_0}^{t} \lambda_i^2 dt\right], \quad i = e, \mu, \tau$$

As we have already mentioned, the initial conditions for the entries of the effective mass operator are defined at $M_N$, the scale where the neutrino Dirac coupling $\lambda_N$ decouples from the renormalisation-group equations. Then, [5],

$$m_{\text{eff}} \propto \begin{pmatrix}
m_{0}^{11} I_e & m_{0}^{12} \sqrt{T_\mu} \sqrt{T_e} & m_{0}^{13} \sqrt{T_\mu} \sqrt{T_\tau} \\
 m_{0}^{21} \sqrt{T_\mu} \sqrt{T_e} & m_{0}^{22} I_\mu & m_{0}^{23} \sqrt{T_\mu} \sqrt{T_\tau} \\
 m_{0}^{31} \sqrt{T_e} \sqrt{T_\tau} & m_{0}^{32} \sqrt{T_\mu} \sqrt{T_\tau} & m_{0}^{33} I_\tau
\end{pmatrix}$$

where the initial conditions are denoted by $m_{ij}^0$. As we have already mentioned, these conditions are defined at $M_N$, the scale where the neutrino Dirac coupling $\lambda_N$ decouples from the renormalization-group equations. From (13), we see that the relative structure of $m_{\text{eff}}$ is only modified by the charged-lepton Yukawa couplings. On the contrary, the top and gauge couplings give only an overall scaling factor. We then see that while these renormalization effects are not significant for schemes with hierarchical neutrino masses, in models with degenerate neutrinos at the GUT scale, they can have a dramatic effect. In particular, they can spoil the required neutrino degeneracy, even for small $\tan \beta$ [5].

5. GUTs with Abelian Flavour Symmetries

An interesting question that arises is whether realistic fermion mass structures are consistent with the constraints on a family symmetry in GUT-embedded solutions and, if yes, which GUT schemes would be favoured. Along these lines, a huge number of proposals have appeared in the literature [3].

Here, in order to have a minimal model dependence and increase predictivity, we will assume that the mass textures are entirely determined by the U(1) and the GUT-multiplet structure without additional help from Higgs or heavy GUT fields. This means that the GUT structure is only used in order to constrain the $U(1)$ charges. Moreover, since we have an Abelian flavour symmetry, we will assume a large charged-lepton mixing, and no cancellations in $m_{\text{eff}}$.

5.1. SU(5)

The field structure of SU(5) implies that

$$Q_{(q,u^c,e^c)}_i = Q_i^{10}$$
$$Q_{(l,d^c)}_i = Q_i^5$$
$$Q_{(\nu_R)}_i = Q_i^{\nu_R}$$
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have to count on the right-handed neutrino sector and the see-saw conditions for solutions

that are compatible with the solar neutrino data. Several sets of textures have been proposed,

among others in [14], where several solutions were found through fits to the fermion data and

considerations from anomaly cancellation. For instance, a possible set of textures is:

\[
Y^u = \begin{pmatrix}
\bar{e}^6 & \bar{e}^5 & \bar{e}^3 & \bar{e}^2 & 1 \\
\bar{e}^5 & \bar{e}^4 & \bar{e}^2 & 1 \\
\bar{e}^3 & \bar{e}^2 & 1 \\
\bar{e}^2 & 1 \\
\end{pmatrix},
Y^d = \begin{pmatrix}
\bar{e}^4 & \bar{e}^3 & \bar{e}^3 & \bar{e}^2 & 1 \\
\bar{e}^3 & \bar{e}^2 & 1 \\
\bar{e}^2 & 1 \\
\bar{e}^2 & 1 \\
\end{pmatrix},
Y^e = \begin{pmatrix}
\bar{e}^4 & \bar{e}^3 & \bar{e} \\
\bar{e}^3 & \bar{e}^2 & 1 \\
\bar{e}^2 & 1 \\
\end{pmatrix}
\]

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considerations from anomaly cancellation. For instance, a possible set of textures is:

\[
Y^u = \begin{pmatrix}
a_1^u \bar{e}^{16} & a_2^u \bar{e}^6 & a_3^u \bar{e}^8 \\
a_2^u \bar{e}^6 & a_2^u \bar{e}^4 & a_3^u \bar{e}^2 \\
a_3^u \bar{e}^8 & a_3^u \bar{e}^2 & a_3^u \\
\end{pmatrix},
Y^d = \begin{pmatrix}
a_1^d \bar{e}^{31/2} & a_2^d \bar{e}^{11/2} \\
a_2^d \bar{e}^{11/2} & a_2^d \bar{e}^{9/2} \\
a_3^d \bar{e}^{15/2} & a_3^d \bar{e}^{3/2} \\
\end{pmatrix}
\]

\[
Y^e = \begin{pmatrix}
a_1^e \bar{e}^{16/3} & a_2^e \bar{e}^{16/3} \\
a_2^e \bar{e}^{16/3} & a_3^e \bar{e}^{8/3} \\
a_3^e \bar{e}^{8/3} \\
\end{pmatrix},
Y^\nu = \begin{pmatrix}
a_1^\nu \bar{e}^{11/2} & a_2^\nu & a_3^\nu \\
a_2^\nu & a_2^\nu & a_3^\nu \\
a_3^\nu & a_3^\nu & a_3^\nu \\
\end{pmatrix}
\]

\[
M_{RR} \propto \begin{pmatrix}
\bar{e}^{[2n_1+1]+1/2+n_1+\sigma} & \bar{e}^{[1/2+n_1+\sigma]} \\
\bar{e}^{[1/2+n_1+\sigma]} & \bar{e}^{[1/2+n_1+\sigma]} \\
\bar{e}^{[1/2+n_1+\sigma]} & \bar{e}^{[1/2+n_1+\sigma]} \\
\end{pmatrix}
\]

Among others, it is interesting to investigate the correlations between the neutrino parameters

and the predictions for CP-violation, in different textures. This is presented in Figure 1 [15].

Similar correlations can be obtained for other groups. However, this goes beyond the scope

of the present article, and thus, in what follows we will only make some generic comments on

other models of grand unification.

5.2. Neutrino masses in Left-Right symmetric models

In Left-Right symmetric models, the $U(1)$ family charges are strongly constrained since the

symmetry requires identical $U(1)$ charges of the left- and right-handed fields. In the previous

section, we discussed the predictions for quark masses. For charged leptons, the choice of charges

\[
b_i = c_i = d_i = \left(\begin{array}{c}
\frac{7}{2} \\
\frac{1}{2} \\
0 \\
\end{array}\right)
\]

\[
b_i = c_i = d_i = \left(\begin{array}{c}
\frac{5}{2} \\
\frac{1}{2} \\
0 \\
\end{array}\right)
\]

leads to two possible charged-lepton matrices:

\[
M_\ell \propto \begin{pmatrix}
\bar{e}^7 & \bar{e}^5 & \bar{e}^{7/2} \\
\bar{e}^5 & \bar{e} & \bar{e}^{1/2} \\
\bar{e}^{7/2} & \bar{e}^{1/2} & 1 \\
\end{pmatrix},
M_\ell \propto \begin{pmatrix}
\bar{e}^5 & \bar{e}^3 & \bar{e}^{5/2} \\
\bar{e}^3 & \bar{e} & \bar{e}^{1/2} \\
\bar{e}^{5/2} & \bar{e}^{1/2} & 1 \\
\end{pmatrix}
\]
Both these matrices lead to natural lepton hierarchies for $\bar{\epsilon} \approx 0.22$ and imply large but non-maximal lepton mixing.

What about neutrino masses? The neutrino Dirac mass is specified to be of the same type as for the charged leptons, but with a different expansion parameter. Actually, since neutrinos (charged leptons) and up-type (down-type) quarks couple to the same Higgs, they should have
the same expansion parameter $\epsilon(\bar{\epsilon})$. Then,

\[
m_D \propto \begin{pmatrix}
\epsilon^7 & \epsilon^3 & \epsilon^{7/2} \\
\epsilon^3 & \epsilon & \epsilon^{1/2} \\
\epsilon^{7/2} & \epsilon^{1/2} & 1
\end{pmatrix}, \quad m_D \propto \begin{pmatrix}
\epsilon^5 & \epsilon^3 & \epsilon^{5/2} \\
\epsilon^3 & \epsilon & \epsilon^{1/2} \\
\epsilon^{5/2} & \epsilon^{1/2} & 1
\end{pmatrix},
\]

for the two choices of charges in (14) respectively.

Of course the mass structure of neutrinos is more complicated, due to the heavy Majorana masses of the right-handed components. These arise from a term of the form $\nu_R^T \nu_R \Sigma$, where $\Sigma$ is a $SU(3) \times SU(2) \times U(1)$ invariant Higgs scalar field with $I_W = 0$. The possible choices for the $\Sigma$ charge will give a discrete spectrum of possible forms for the Majorana mass, $M_R$. For example, if $\Sigma$ has the same charge with the Higgs doublets, the form of the heavy Majorana mass matrix will be similar to that of the charged leptons.

5.3. $SO(10)$

In an $SO(10)$ GUT, all quarks and leptons are accommodated in the 16 representation of the group. This implies that the quark and lepton charges for the left- and right-handed fields of a given family are the same. Moreover, since both Higgs fields of the Minimal Supersymmetric Standard Model fit in a single 10-plet of $SO(10)$, in the simplest scheme one would predict left-right-symmetric mass matrices with similar structure for all fermions.

However, this implies the prediction $V_{\mu\tau} \approx V_{cb}$, which may only be reconciled with observations either with the help of coefficients (which is unnatural) or by considering the effects of the additional Higgs multiplets that are required for breaking $SO(10)$ down to $SU(3) \times SU(2) \times U(1)$. This way, one can generate several operators with rank $\geq 4$ in the mass matrices, however the predictivity of the $U(1)$ symmetry is reduced since the choice of an operator at a given entry is only phenomenological.

5.4. Flipped-$SU(5)$

In the case of the flipped-$SU(5)$, the fields $Q_i, d_i$ and $\nu_i^c$ belong to a 10 of $SU(5)$, while $u_i^c$ and $L_i$ belong to a 5. Finally, the $e_i^c$ fields belong to singlet representations of $SU(5)$. This assignment implies symmetric down-quark mass matrices. The structure of the up-quark mass matrix will depend on the charges of the right-handed quarks. However, as these are the same with the charges of the left-handed leptons, the mass matrix will be constrained by the need to generate large mixing for atmospheric neutrinos. In this model, it turns out that the contribution from the up-quark sector to $V_{cb}$ is negligible [11] and therefore $V_{cb} \simeq \sqrt{m_s/m_b}$. This is too large and requires a significant coefficient adjustment. However, it has been shown that the string-embedded flipped $SU(5)$ model, due to its additional (although highly constrained) structure, works in a nice way [16].

5.5. $SU(3)_c \times SU(3)_L \times SU(3)_R$

In $SU(3)_c \times SU(3)_L \times SU(3)_R$, the left- and right-handed quarks belong to a $(3, 3, 1)$ and $(\bar{3}, 1, 3)$ respectively and thus their $U(1)$ charges are not related. On the other hand the left and right-handed leptons belong to the same $(1, 3, \bar{3})$ representation and hence must have the same $U(1)$ charge. Thus, the lepton mass matrices have to be symmetric, and similar to those of a left-right symmetric model. Since the quark mass matrices are asymmetric (with different expansion parameters but similar structure for up- and down-quarks), it is straightforward to choose $U(1)$ charges, such that all quark hierarchies are fulfilled. This choice, does not impose any constraints on the lepton charges.

Overall, from this discussion, we conclude that different types of theoretical models “prefer” different solutions of the solar and atmospheric neutrino deficits and predict different correlations
of the neutrino parameters with those of the other fermions. There is a large number of proposals in the literature, and hopefully, the upcoming neutrino data can help us to constrain or even exclude many of the existing models.

6. Lepton Flavour Violation and Massive Neutrinos
Mixing in the neutrino sector also generates mixing in the sleptons via loop corrections, contributing to rare decays and lepton-flavour conversions [7, 8]. One can evaluate these effects for instance in the context of the CMSSM, where the soft supersymmetry-breaking masses of the charged and neutral sleptons are assumed to be universal at the GUT scale, with a common value $m_0$. In the leading-logarithmic approximation, the non-universal renormalization of the soft supersymmetry-breaking scalar masses is by an amount

$$\left( \delta m^2_L \right)_{ij} \approx -\frac{1}{8\pi^2} \left( 3m_0^2 + A_0 \right) (Y_{\nu} Y_{\nu})_{ij} \log \frac{M_{\text{GUT}}}{M_{N_i}}$$  \hspace{1cm} (16)

The pattern of charged-lepton-flavour violation induced by renormalization depends on the details of the neutrino Yukawa coupling matrix $(Y_{\nu})_{ij}$.

The branching ratio of the charged lepton decays $l_j \to l_i + \gamma$ is given by:

$$BR(l_j \to l_i \gamma) = \frac{48\pi^3\alpha}{G_F} \left( (A^L_M)^2 + (A^R_M)^2 \right)$$ \hspace{1cm} (17)

where

$$A^L_M = A^L_M(n) + A^L_M(c), \quad A^R_M = A^R_M(n) + A^R_M(c)$$ \hspace{1cm} (18)

and

$$A_M(n) : \quad \frac{1}{6(1-x)^4} \left( 1 - 6x + 3x^2 + 2x^3 - 6x^2 \log x \right) \frac{M}{m_{l_j}},$$

$$A_M(c) : \quad \frac{1}{6(1-x)^4} \left( 2 - 9x - 18x^2 - 18x^3 + 6x^3 \log x \right) \frac{M}{m_{l_j}},$$

$$A_E(n) : \quad \frac{1}{(1-x)^4} \left( -6 - 27x^2 + 18x^3 + 6x^3 \log x \right),$$

$$A_E(c) : \quad \frac{1}{(1-x)^4} \left( 16 - 45x + 36x^2 - 7x^3 + 6(2 - 3x) \log x \right)$$ \hspace{1cm} (19)

Here $m_{l_j}$ is the mass of the lepton $l_j$, $M$ is the chargino (neutralino) mass and $m$ the sneutrino (charged slepton) mass.

Another lepton-number-violating observable is $\mu \to e$ conversion on a nucleus, with a rate

$$R(\mu^+ Ti \to e^+ Ti) \approx \frac{\alpha E_{e} p_{e} Z F_{c}^2}{3 \pi m_{\mu}^2 C \Gamma(A, Z)} BR(\mu \to e\gamma)$$

$$\approx 5.6 \times 10^{-3} BR(\mu \to e\gamma)$$ \hspace{1cm} (20)

This process is very interesting, despite the relative suppression by about two orders of magnitude, because of the accuracy possible in future measurements of this process.

A similar suppression is expected for the decay $\mu \to 3e$,

$$\frac{\Gamma(\mu^+ \to e^+ e^+ e^-)}{\Gamma(\mu^+ \to e^+ \gamma)} \approx 6 \times 10^{-3}$$ \hspace{1cm} (21)

However, the present experimental bound on this branching ratio is relatively weak, and the prospects for significant improvement are more distant. Processes such as $\tau \to e\gamma$ and $\mu\gamma$ are also potentially interesting.
Figure 2. Comparison of flavour-changing and -conserving $\chi_2$ decay modes as functions of $m_0$ for (a) $\tan \beta = 10, \mu > 0, m_{1/2} = 600$ GeV and (b) $\tan \beta = 40, \mu > 0, m_{1/2} = 600$ GeV. A non-universality factor $x = 0.9$ and a mixing angle $\phi = \pi/6$ are assumed.

In principle, one can also expect to observe lepton-flavour violation through slepton mixing at the LHC. In this respect, the most promising mode is the flavour-violating decay of $\tilde{\chi}_2^0$:

$$\tilde{\chi}_2^0 \rightarrow \tilde{\ell}_i^+ \ell_j^- \rightarrow \chi_1^0 \ell_i^+ \ell_j^-$$

Here, the $\tilde{\chi}_2^0$ is produced through $\tilde{q}$ and $\tilde{g}$ decays. For instance, it turns out that $\sim 60\%$ of the first and second-generation left-handed squarks decay in Wino-like neutralino and chargino, while, in a wide region of the parameter space, the right-handed slepton masses are expected to be smaller than $m_{\tilde{\chi}_2^0}$. The dominant background arises from $\tilde{\chi}_2^0 \rightarrow \chi_1^0 Z(h) \rightarrow \tilde{\ell}_i^+ \ell_j^-$. In the case of supersymmetric models with $\mu \gg M_{1,2}$, most of the parameter space excluded by rare muon and tau decays. However, if $\mu$ is relatively close to the gaugino masses, the LHC may cover remarkably the range where rare decays are suppressed (due to cancellations between the chargino and neutralino loop diagrams). An example is demonstrated by Figure 2a and 2b, where a comparison of the flavour-changing and -conserving $\chi_2$ decay modes as functions of $m_0$ for different supersymmetric parameters is presented [17].

7. Relevance of leptogenesis

The textures of neutrino Yukawa couplings are also constrained by the requirement of successful leptogenesis, which requires obtaining the correct magnitude of lepton flavour violation from the decays of heavy, right-handed Majorana neutrinos via a difference between the branching ratios for heavy neutrino decays into leptons and antileptons:

$$BR(N_L^c \rightarrow \ell + \bar{\ell}) \neq BR(N_L^c \rightarrow \Phi + \bar{\ell})$$

Since lepton- and baryon-number-violating interactions are in thermal equilibrium up to the time of the electroweak phase transition, a non-zero lepton number gives rise to a non-zero baryon number, by sharing the lepton asymmetry $\Delta L \neq 0$ with a baryon asymmetry $\Delta B \neq 0$.

In order to avoid washout of the initial decay asymmetry, the model must satisfy an out-of-equilibrium condition, namely that the heavy neutrino decay rate is smaller than the Hubble
parameter $H$ at $T \approx M_N$. The tree-level width of the heavy neutrino $N_1$ is: $\Gamma = \frac{(\lambda^\dagger \lambda)_{11}}{8\pi} M_{N_1}$, which should be compared with $H \approx 1.7 \ g_s^{1/2} \frac{T^2}{M_p}$, $(g_s^{MSSM} \approx 228.75, \ g_s^{SM} = 106.75)$, leading to the requirement

$$\frac{(\lambda^\dagger \lambda)_{11}}{14\pi g_s^{1/2}} M_p < M_{N_1}$$

This condition may be implemented more accurately by looking in detail at the Boltzmann equations, but this formula is sufficient for our purposes.

The $CP$-violating decay asymmetry $\epsilon$ arises from the interference between tree-level and one-loop amplitudes, and is

$$\epsilon_j = \frac{1}{8\pi (\lambda^\dagger \lambda)_{11}} \sum_j \Im \left[ (\lambda^\dagger \lambda)_{1j}^2 \right] f \left( \frac{m_{N_j}^2}{m_{N_1}^2} \right)$$

where the kinematic function

$$f(y) = \sqrt{y} \left[ 1 - (1+y) \ln \left( \frac{1+y}{y} \right) \right]$$

To this should be added self-energy corrections $\tilde{\delta} \propto M_{N_1} \frac{M_{N_2} - M_{N_1}}{M_{N_2}}$ which exhibit a resonant enhancement of the lepton asymmetry for models with degenerate Majorana masses.

It is clear that once a specific model is chosen, it is possible to examine whether it is compatible with the requirement for successful leptogenesis and to review for which sets of parameters this can occur.

8. Summary
We discussed different models of neutrino mass textures, in the light of the recent neutrino data. There have been various proposals in the literature, on how the viable phenomenological textures may arise in models with flavour and GUT symmetries. Here, we focus in identifying common characteristics of the various theories, in a way that the new data can constrain or even exclude whole classes of models.

Quantum corrections may drastically modify the neutrino masses and mixings from high to low scales. Finally, low energy lepton-flavour-violating processes, considerations of $CP$-violation and leptogenesis, may in certain frameworks provide additional insight to neutrino mass textures.

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