FINITUDE AND CONSTRUCTION OF LATTICE SURFACES IN STRATA

SLADE SANDERSON

ABSTRACT. The Veech group of a translation surface is the group of Jacobians of orientation-preserving affine automorphisms of the surface. We show that in any given stratum, there are at most finitely many unit-area translation surfaces with a given lattice Veech group. Furthermore, given any stratum and lattice \( \Gamma \leq \text{SL}_2 \mathbb{R} \), we present an algorithm which constructs all unit-area translation surfaces in the stratum with Veech group \( \Gamma \).

Our methods can be applied to obtain obstructions of lattice groups being realized as Veech groups in certain strata; in particular, we show that the square torus is the only translation surface in any minimal stratum whose Veech group is all of \( \text{SL}_2 \mathbb{Z} \).

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1. Introduction

The study of translation surfaces may be approached from numerous different angles—complex analysis, differential geometry, algebraic topology and number theory, for instance, each offer unique insights into these objects. These different perspectives lead to a wealth of questions regarding translation surfaces and their dynamics. One such avenue of inquiry is the well-known action of \( \text{SL}_2 \mathbb{R} \) on the stratum \( \mathcal{H}_1(d_1, \ldots, d_\kappa) \) of unit-area translation surfaces with singularities of orders \( d_1, \ldots, d_\kappa \). Orbits and stabilizers (i.e. Veech groups) of translation surfaces under this action have proven to be of great interest. Analogues of Ratner’s theorems concerning unipotent flows on homogeneous spaces (\([\text{Rat}]\)) are proven in the celebrated work of Eskin, Mirzakhani and Mohammadi; in particular, \( \text{SL}_2 \mathbb{R} \)-orbit closures are affine invariant submanifolds of \( \mathcal{H}_1(d_1, \ldots, d_\kappa) \) (\([\text{EMM}]\)). Despite having been well-studied since as early as the 1980’s, some immediate
questions regarding Veech groups have been non-trivial to address. General methods of computing Veech groups were unknown until recently ([Bo], [BrJ], [Ed], [ESS], [Mu], [Ve3], and the special cases of [Fr] and [Sch]), and while some universal properties of Veech groups are known—for instance, they are necessarily discrete—a complete classification of which subgroups of SL$_2\mathbb{R}$ are realized as Veech groups remains an open problem ([HMSZ]).

Of particular interest within the space of all translation surfaces are so-called lattice surfaces, or Veech surfaces, which are those whose Veech groups are lattices (have finite covolume) in SL$_2\mathbb{R}$. Lattice surfaces admit especially nice dynamics ([Ve]), but the list of known (families of) lattice surfaces is relatively short ([DPU]). Lattice surfaces are also precisely those surfaces whose SL$_2\mathbb{R}$-orbits are closed with respect to the analytic topology on the stratum ([Ve2], [SmW]), and the projection of a closed orbit to the moduli space of Riemann surfaces is an algebraic curve—called a Teichmüller curve—isometrically immersed with respect to the Teichmüller metric (see, say, [Wr]). A number of results have been proven regarding the finiteness of algebraically primitive Teichmüller curves in restricted settings (see [Wr], [DPU] and the references therein). Our main result has a related, but different flavor:

**Theorem 1.1.** There are at most finitely many unit-area lattice surfaces with a given Veech group in any given stratum.

The proof of Theorem 1.1 also provides an algorithm to construct these surfaces.

**Theorem 1.2.** There exists an algorithm which, given a stratum $\mathcal{H}_1(d_1, \ldots, d_\kappa)$ and lattice $\Gamma \leq$ SL$_2\mathbb{R}$, returns all translation surfaces $(X, \omega) \in \mathcal{H}_1(d_1, \ldots, d_\kappa)$ with Veech group $\text{SL}(X, \omega) = \Gamma$.

Our methods essentially reverse the algorithm for computing Veech groups given in [ESS]. There the authors associate to each stratum a canonical (infinite, disconnected in general) translation surface $\mathcal{O}$, within which any closed, connected translation surface $(X, \omega)$ of the stratum may be naturally represented. Data regarding the translation surface and its saddle connections are recorded via pairs of points in $\mathcal{O}$ termed orientation-paired marked segments, and the authors show that $(X, \omega)$ may be recovered from a certain finite subset of these marked segments—the marked Voronoi staples (§3.4 of [ESS]). Furthermore, a classification of the Veech group of $(X, \omega)$ in terms of its orientation-paired marked segments and affine automorphisms of $\mathcal{O}$ is given in Proposition 17 of [ESS].

Rather than beginning with a lattice surface $(X, \omega)$ and using its corresponding orientation-paired marked segments to compute elements of the Veech group $\Gamma = \text{SL}(X, \omega)$, we begin with a lattice $\Gamma \leq$ SL$_2\mathbb{R}$, simulate (normalized) subsets of orientation-paired marked segments in $\mathcal{O}$ from $\Gamma$, and construct candidate translation surfaces with Veech group $\Gamma$ from finite unions of appropriately scaled versions of these ‘simulations.’ Theorem 1.1 follows from the facts that (i) a lattice $\Gamma$ determines only finitely many such simulations (Lemma 4.8), (ii) the orientation-paired marked segments of any unit-area translation surface with Veech group $\Gamma$ are realized as a finite union of scaled simulations (Theorem 4.9), and (iii) for any finite collection of simulations there are only finitely many scalars for which the union of scaled simulations could possibly coincide with the orientation-paired marked segments of a unit-area translation surface. For this latter point, we introduce the technical but important notions of permissible triples and permissible scalars of particular subsets of $\mathcal{O}$ and prove finiteness results regarding them (Proposition 5.5 and Lemma 5.7). In §4.4 these notions and results are applied to orientation-paired marked segments and marked Voronoi staples, and Theorem 1.1 is proven in §4.5.

In §5 we present the algorithm (Algorithm 5.1) asserted in Theorem 1.2. We also collect the results of a running example and use Algorithm 5.1 to construct a lattice surface with Veech group $\langle (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \rangle$ in $\mathcal{H}_1(2)$.

Section 6 gives an example of how these ideas may be used to obtain obstructions for lattices being realized as Veech groups in certain strata. In particular, we prove:

**Theorem 1.3.** The square torus is the only translation surface in $\bigcup_{g > 0} \mathcal{H}(2g-2)$ with Veech group $\text{SL}(X, \omega) = \text{SL}_2\mathbb{Z}$.

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1In [SmW2], the authors provide an implicit algorithm which enumerates all affine equivalence classes of lattice surfaces in terms of a parameter measuring the infimum of areas of triangles within the surface. Our algorithm is similar in that it returns lattice surfaces, but it differs in that the input is both a stratum and the desired lattice Veech group of the returned surfaces.
2. BACKGROUND

2.1. Translation surfaces. We begin with basic definitions and notation; see, say, surveys of \[\text{Wr}\] and \[\text{Zo}\] for more details.

2.1.1. Translation surfaces: various perspectives. A translation surface \((X, \omega)\) is a Riemann surface \(X\) together with a non-zero, holomorphic one-form \(\omega\). Equivalently, a translation surface is defined as a real surface \(X\) such that off of a finite set \(\Sigma\), the surface \(X \setminus \Sigma\) is equipped with a translation atlas, and the resulting flat structure extends to all of \(X\) to give conical singularities at points of \(\Sigma\) whose angles are positive integral multiples of \(2\pi\). Any translation surface may be \emph{polygonally presented} as a collection of polygons in the plane together with an identification of edges in pairs, such that identified edges are parallel, equal length and of opposite orientation. A translation surface is \emph{closed} if it is compact and has no boundary.

In these three perspectives—which we move fluidly between—the zeros of \(\omega\), the set \(\Sigma\) and the vertices of a polygonally presented surface correspond to the same set of singular points of the surface. We also allow for \emph{removable singularities (or marked points)} in \(\Sigma\), i.e. points of cone angle \(2\pi\).

2.1.2. Metric, Lebesgue measure, saddle connections and holonomy vectors. A translation surface \((X, \omega)\) comes equipped with a metric defined as follows: given \(p, q \in X\), the distance from \(p\) to \(q\) is given by

\[
d(p, q) := \inf_{\gamma} \left| \int_{\gamma} \omega \right|,
\]

where the infimum is over all piecewise-smooth curves \(\gamma \subset X\) from \(p\) to \(q\). The infimum is realized by a straight line segment (with respect to the flat structure) on \((X, \omega)\) or by a union of such segments meeting at singularities. Lebesgue measure in the plane pulls back via coordinate maps of the translation atlas of \((X, \omega)\) to give a measure on \(X \setminus \Sigma\); we extend this measure to all of \(X\) by declaring \(\Sigma\) to be a null set.

A \emph{saddle connection} is a straight line segment on \((X, \omega)\) emanating from a singularity. A \emph{saddle connection} is a separatrix (of positive length) which also ends at a singularity and has no singularities in its interior. The \emph{holonomy vector} of a saddle connection \(s\) is defined as \(\text{hol}(s) := \int_s \omega\). The collection of all holonomy vectors of \((X, \omega)\) forms a subset of the plane with no limit points and whose directions are dense in \(S^1\) (Proposition 3.1, \[V0\]).

2.1.3. Voronoi decomposition. Every translation surface \((X, \omega)\) has a (unique) \emph{Voronoi decomposition} subordinate to its singular set \(\Sigma\), which is described as follows (see also \[Ed\], \[MS\]). For any \(x \in X\), let \(d(x, \Sigma)\) denote the minimum of the set \(\{d(x, \sigma)\}_{\sigma \in \Sigma}\). To each \(\sigma \in \Sigma\) there is associated an open, connected 2-cell, \(C_\sigma\), comprised of the points \(x \in X\) for which \(d(x, \Sigma)\) is realized by a unique length-minimizing path ending at \(\sigma\). The boundary of \(C_\sigma\) is the union of 1-cells of points \(x\) for which \(d(x, \Sigma)\) is realized by precisely two length-minimizing paths, at least one of which ends at \(\sigma\), and of 0-cells of points \(x\) for which \(d(x, \Sigma)\) is realized by three or more length-minimizing paths, at least one of which ends at \(\sigma\).

2.2. Veech groups.

2.2.1. Affine diffeomorphisms and Veech groups. An \emph{affine diffeomorphism} \(f : (X_1, \omega_1) \to (X_2, \omega_2)\) between (not necessarily closed nor connected) translation surfaces is a diffeomorphism from \(X_1\) to \(X_2\) sending the singular set \(\Sigma_1\) of \(X_1\) into that of \(X_2\), and which—on the complement of \(\Sigma_1\)—is locally an affine map of the plane of constant linear part\(^2\) (i.e. a map of the form \(v \mapsto Av + b\) for some global \(A \in \text{GL}_2 \mathbb{R}\) and local \(b \in \mathbb{R}^2\)). An \emph{affine automorphism} is an affine diffeomorphism from a translation surface \((X, \omega)\) to itself. The set of all affine automorphisms of \((X, \omega)\), denoted \(\text{Aff}(X, \omega)\), forms a group under composition, as does the subset \(\text{Aff}^+(X, \omega) \subset \text{Aff}(X, \omega)\) of orientation-preserving elements. The map \(\text{der} : \text{Aff}(X, \omega) \to \text{GL}_2 \mathbb{R}\) sending an orientation-preserving affine automorphism to its linear part in local coordinates gives a group.

\(^2\)The translation structures guarantee that the linear part of a locally affine map is constant on components of \((X_1, \omega_1)\); this latter statement in the definition is thus intended for disconnected translation surfaces.
homomorphism. The image of Aff$^+(X,\omega)$ under this map, denoted SL$(X,\omega)$, is called the Veech group of $(X,\omega)$, and its kernel is denoted Trans$(X,\omega)$. Veech shows in [VG] that for closed, connected $(X,\omega)$, the Veech group SL$(X,\omega)$ is a discrete subgroup of SL$_2\mathbb{R}$, and thus its image in PSL$_2\mathbb{R}$ is a Fuchsian group.

2.2.2. Strata, a GL$_2\mathbb{R}$-action and lattices. We call $(X_1,\omega_1)$ and $(X_2,\omega_2)$ translation equivalent if there is an affine diffeomorphism between them with trivial linear part. For non-negative integers $d_1 \leq \cdots \leq d_\kappa$, let $\mathcal{H}(d_1, \ldots, d_\kappa)$ denote the stratum of all closed, connected translation surfaces—up to translation equivalence—with singularities of cone angles $2\pi(d_1+1) \leq \cdots \leq 2\pi(d_\kappa+1)$ and no other singularities (recall that we allow for ‘removable singularities,’ so some $d_i$ may be zero). Abusing notation, we denote an element of a stratum by any of its translation-equivalent representatives. By the Riemann-Roch Theorem, every translation surface $(X,\omega) \in \mathcal{H}(d_1, \ldots, d_\kappa)$ has the same genus $g > 0$, and the sum of the $d_i$ equals $2g-2$.

There is a natural action of GL$_2\mathbb{R}$ on each stratum, given by post-composing the coordinate charts of $(X,\omega) \in \mathcal{H}(d_1, \ldots, d_\kappa)$ with the usual action of a matrix in the plane via a linear transformation (one can verify that this action is well-defined with respect to translation-equivalence). Any translation surface may be normalized by the action of a diagonal matrix so as to have unit-area; we denote the collection of unit-area translation surfaces in $\mathcal{H}(d_1, \ldots, d_\kappa)$ by $\mathcal{H}_1(d_1, \ldots, d_\kappa)$. There is a corresponding action of SL$_2\mathbb{R}$ on $\mathcal{H}_1(d_1, \ldots, d_\kappa)$, and the stabilizer of $(X,\omega)$ under this SL$_2\mathbb{R}$-action is isomorphic to the Veech group SL$(X,\omega)$ as defined in [2.1]. We call $(X,\omega)$ a lattice surface if SL$(X,\omega)$ is a lattice group, i.e. SL$(X,\omega)$ has finite covolume in SL$_2\mathbb{R}$.

3. A canonical translation surface for each stratum

3.1. Canonical surface. Here we recall the canonical (infinite, and disconnected in general) translation surface associated to each stratum $\mathcal{H}(d_1, \ldots, d_\kappa)$ introduced in [Ed] and [ESS] and explore some immediate results. For each $1 \leq i \leq \kappa$, let $\mathcal{O}_i := (\mathbb{C}, z^{d_i}/dz)$, and set

$$\mathcal{O} = \mathcal{O}(d_1, \ldots, d_\kappa) := \bigcup_{i=1}^\kappa \mathcal{O}_i.$$  

Each component $\mathcal{O}_i \subset \mathcal{O}$ is an infinite translation surface with a sole singularity of cone angle $2\pi(d_i+1)$ at the origin, which we denote by $0 \in \mathcal{O}_i$ (the context will be clear as to which component an origin 0 belongs). Intuitively, we may think of $\mathcal{O}_i$ as $d_i+1$ copies of the plane, denoted $c_0^i, \ldots, c_{d_i}^i$, each of which is slit along the non-negative real axis and glued so that the bottom edge, $b_j^i$, of the slit of $c_j^i$ is identified with the top edge, $t_{j+1}^i$, of the slit of $c_{j+1}^i$ for each $j \in \mathbb{Z}_{d_i+1}$; see Figure 1. After making these identifications, we (arbitrarily) consider $t_{j+1}^i \sim b_j^i$ as part of $c_j^i+1$ and not of $c_j^i$. In particular, $c_{j}^i \cap c_{k}^i = \{0\}$ for distinct $j, k \in \mathbb{Z}_{d_i+1}$. Define $\text{proj}_i : \mathcal{O}_i \to \mathbb{C}$ by $\text{proj}_i(p) = \int_0^\gamma z^{d_i}dz$, where $\gamma$ is any piecewise-smooth curve from 0 to $p$ (note that the integral is well-defined, independent of path), and extend to a map $\text{proj} : \mathcal{O} \to \mathbb{C}$ by setting $\text{proj}|_{\mathcal{O}_i} = \text{proj}_i$ for each $i \in \{1, \ldots, \kappa\}$. Denote by $\mathcal{P}$ the image of $p$ under $\text{proj}(\cdot)$, and by $\mathcal{P}_x, \mathcal{P}_y$ the real numbers for which $\mathcal{P} = \mathcal{P}_x + \mathcal{P}_y$. Note that on sufficiently small neighborhoods in $\mathcal{O}$ (and away from singularities), the map $\text{proj}(\cdot)$ restricts to give a coordinate chart for the translation atlas associated to $\mathcal{O}$. Moreover, each $\mathcal{O}_i \setminus \{0\}$ is naturally equipped with (generalized) polar coordinates: every regular point $p \in c_j^i \subset \mathcal{O}_i$ is determined by a magnitude $|p| := d(0,p) = |\mathcal{P}|$ and an argument $\text{arg}(p) := \text{arg}(\mathcal{P}) + 2\pi j$. 

![Figure 1](image_url) 

**Figure 1.** The surface $C_1$, where the bottom of the slit, $b_j^i$, is identified with the top of the slit, $t_{j+1}^i$, for each $j \in \mathbb{Z}_{d_i+1}$.
For each $i \in \{1, \ldots, \kappa\}$, define an affine automorphism $\rho_i : \mathcal{O}_i \to \mathcal{O}_i$ which acts as a counterclockwise rotation by an angle of $2\pi$ about the origin; note that $\rho_i$ belongs to $\text{Trans}(\mathcal{O}_i)$ since, in local coordinates given by $\text{proj}(\cdot)$, $\rho_i$ has trivial linear part. We extend $\rho_i$ to $\mathcal{O}$ by acting as the identity on all other components and observe that $\rho_i \in \text{Trans}(\mathcal{O})$. For each integer $d \geq 0$, let $n(d) = \#\{1 \leq i \leq \kappa \mid d_i = d\}$. Having fixed $d$, let $i_1, \ldots, i_{n(d)}$ be the distinct indices of $d_1, \ldots, d_\kappa$ which equal $d$, and let $S_{n(d)}$ denote the symmetric group on the set $\{i_1, \ldots, i_{n(d)}\}$. Extend $S_{n(d)}$ to act as the identity on all other indices, and for each $\alpha \in S_{n(d)}$, let $f_\alpha : \mathcal{O} \to \mathcal{O}$ permute the indices of $\mathcal{O}_1, \ldots, \mathcal{O}_\kappa$ by $\alpha$, respecting polar coordinates. We again have that $f_\alpha \in \text{Trans}(\mathcal{O})$ for each $\alpha \in S_{n(d)}$ with $n(d) \geq 1$.

**Lemma 3.1.** The group $\text{Trans}(\mathcal{O})$ is in bijective correspondence with

\[
\left( \prod_{d \geq 0, n(d) \geq 1} S_{n(d)} \right) \times \left( \prod_{i=1}^{\kappa} C_{d_i+1} \right),
\]

where $C_{d_i+1}$ is the cyclic group of order $d_i + 1$.

**Sketch.** Lemma 2 of [ESS] shows that $\text{Trans}(\mathcal{O})$ is generated by an action of

\[
\prod_{d \geq 0, n(d) \geq 1} S_{n(d)}
\]

together with an action of $\prod_{i=1}^{\kappa} C_{d_i+1}$ on $\mathcal{O}$, as naturally defined from the definitions of $f_\alpha$ and $\rho_i$ above. That is, any element $\tau \in \text{Trans}(\mathcal{O})$ may be written as a finite composition of various $f_\alpha$’s and $\rho_i$’s. We may rearrange the order of this composition (possibly altering the indices of some $\rho_i$’s) to collect all $f_\alpha$’s with $\alpha$ belonging to the same $S_{n(d)}$ and all $\rho_i$’s corresponding to the same $\mathcal{O}_i$. That $\tau$ may be uniquely written in such a way is verified by considering the indices of images of $2\pi$-sectors $c_j$.

**Lemma 3.2.** Let $f \in \text{Aff}(\mathcal{O})$ with $\text{der}(f) = A$. Then

\[
\text{proj} \circ f = A \cdot \text{proj},
\]

where the notation on the right denotes the usual action of a matrix $A$ on the plane as a linear transformation.

**Proof.** Since $\text{proj}(\cdot)$ restricts on sufficiently small neighborhoods to give a (bijective) coordinate map for the translation atlas of $\mathcal{O}$, we have—by definition of $f$—that for each $v \in \mathbb{R}^2$ in the image of such a neighborhood,

\[
\text{proj} \circ f \circ \text{proj}^{-1}(v) = Av + b
\]

for some $b \in \mathbb{R}^2$. This composition agrees on the images of the intersections of such neighborhoods; in particular, $b$ depends only on the component $\mathcal{O}_i$ to which the neighborhood belongs. Choose some neighborhood as above which contains $0 \in \mathcal{O}_i$ on its boundary. The above composition (and each of the maps comprising it) extends continuously to the closure of its domain. Since $f$ sends singularities to singularities, setting $v = 0$ gives $b = 0$. As $\mathcal{O}_i$ was arbitrary, the result follows from the previous observations.

In Lemma 3 of [ESS], it is shown that $\text{der}(\text{Aff}(\mathcal{O})) = \text{GL}_2^+ \mathbb{R}$. To each $A \in \text{GL}_2^+ \mathbb{R}$ we associate a canonical $f_A \in \text{Aff}^+(\mathcal{O})$ with $\text{der}(f_A) = A$ as follows. Fix some $g_A \in \text{Aff}^+(\mathcal{O})$ with $\text{der}(g_A) = A$, and choose $\tau \in \text{Trans}(\mathcal{O})$ so that the composition $f_A := \tau \circ g_A$ satisfies for each $1 \leq i \leq \kappa$ both (i) $f_A(\mathcal{O}_i) = \mathcal{O}_i$ and (ii) for each point $p \in \mathcal{O}_i$, the angle measured counterclockwise from $p$ to $f_A(p)$ is non-negative and less than $2\pi$. Existence of $\tau$ follows from Lemma 3.1. Notice that $\text{der}(f_A) = A$ since $\text{der}(\cdot)$ is a group homomorphism, $\tau \in \text{Trans}(\mathcal{O}) = \ker(\text{der})$ and $\text{der}(g_A) = A$. Furthermore, conditions (i) and (ii) guarantee that $f_A$ is unique, regardless of the initial choice of $g_A$ (and subsequent choice of $\tau$).

We also note that any $g_A \in \text{Aff}^+(\mathcal{O})$ with $\text{der}(g_A) = A \in \text{GL}_2^+ \mathbb{R}$ may be written uniquely as $g_A = \tau' \circ f_A$ for some $\tau' \in \text{Trans}(\mathcal{O})$ (namely $\tau' = \tau^{-1}$ for $\tau$ as above), with $f_A$ the canonical affine automorphism associated to $A$.

**Definition 3.1.** With notation as above, let

\[
\text{Aff}^+_c(\mathcal{O}) := \{ f_A \in \text{Aff}^+(\mathcal{O}) \mid A \in \text{GL}_2^+ \mathbb{R} \}.
\]
be the collection of canonical affine automorphisms of \( \mathcal{O} \). For any \( r \in \mathbb{R}_+ \) and \( p \in \mathcal{O} \), let
\[
 rp := f_{D(r)}(p),
\]
where \( f_{D(r)} \in \text{Aff}^+(\mathcal{O}) \) with \( D(r) = (r, 0, r) \).

In particular, if \( p \in \mathcal{O} \) with polar coordinates \((|p|, \theta)\), then \( rp \in \mathcal{O} \) with polar coordinates \((r|p|, \theta)\). Note then that \( \overline{rp} = r\overline{p} \).

While \( \text{Trans}(\mathcal{O}) \) is generally non-abelian, we have the following commutativity result:

**Lemma 3.3.** The subgroup \( \text{Trans}(\mathcal{O}) \leq \text{Aff}(\mathcal{O}) \) belongs to the centralizer of \( \text{Aff}^+(\mathcal{O}) \), i.e.
\[
 \tau \circ f_A = f_A \circ \tau
\]
for each \( \tau \in \text{Trans}(\mathcal{O}) \) and \( f_A \in \text{Aff}^+(\mathcal{O}) \).

**Proof.** Let \( \tau \in \text{Trans}(\mathcal{O}) \), \( f_A \in \text{Aff}^+(\mathcal{O}) \) and \( p \in \mathcal{O} \). We must show
\[
 \tau \circ f_A(p) = f_A \circ \tau(p).
\]
Since \( \text{der}(\tau \circ f_A) = \text{der}(f_A \circ \tau) = A \), both sides of the previous line are sent under \( \text{proj}() \) to the same point \( z \in \mathbb{C} \) (Lemma 3.3). Suppose \( p \in c_k^l \) and \( \text{proj}(\mathcal{O}_l) = \mathcal{O}_l \) with \( \tau(c_k^l) = c_{k}^{l} \). Let \( r_k^l \) denote the set of points in \( \mathcal{O}_l \) with argument \( 2\pi k \) (i.e. \( r_k^l \) is the ray along the positive real axis in \( c_k^l \)). Note that \( \text{proj}() \) sends each of \( \tau \circ f_A(c_k^l) \) and \( f_A \circ \tau(c_k^l) \) bijectively onto \( \mathbb{C} \), and these images are completely determined by the images \( \tau \circ f_A(r_k^l) \) and \( f_A \circ \tau(r_k^l) \) in \( \mathcal{O}_l \); the former are the sets of points within angle \( 2\pi \) counterclockwise of the respective images of the ray \( r_k^l \). Since there is only one point in each \( 2\pi \) sector which projects under \( \text{proj}() \) to \( z \in \mathbb{C} \), it suffices to show that \( \tau \circ f_A(r_k^l) = f_A \circ \tau(r_k^l) \). By definition of \( f_A \) and \( r_k^l \), the image \( f_A(r_k^l) \) belongs to \( c_k^l \), and thus \( \tau \circ f_A(r_k^l) \) belongs to \( c_{k}^{l} \). But also \( \tau(r_k^l) \) belongs to \( c_k^l \) by assumption, and again by definition of \( f_A \), the image \( f_A \circ \tau(r_k^l) \) belongs to \( c_{k}^{l} \) as well. Since the images \( \tau \circ f_A(r_k^l) \) and \( f_A \circ \tau(r_k^l) \) are rays emanating from \( 0 \) in \( c_k^l \) and pointing in the same direction, we have \( \tau \circ f_A(r_k^l) = f_A \circ \tau(r_k^l) \) and the result follows.

3.2. **Permissible triples.** Here we introduce terminology and present results regarding particular subsets of the canonical surface \( \mathcal{O} = \mathcal{O}(d_1, \ldots, d_n) \). While the material of this subsection becomes technical, the underlying notions are rooted in elementary Euclidean geometry. As we shall see in 4.1.3 and 4.1.5, the results proven here will be crucial for our main finiteness result regarding lattice surfaces in strata. We begin with definitions and notation.

**Definition 3.2.** The **half-space** determined by \( p \in \mathcal{O}_i \) is
\[
 H(p) := \{ q \in \mathcal{O}_i \mid d(0, q) \leq d(p, q) \}.
\]

Note that \( H(p) \) is convex in the sense that the length-minimizing path between any two points in \( H(p) \) is also contained in \( H(p) \). Unless otherwise noted, a (closed or open) \( \theta \)-sector of \( \mathcal{O}_i \) is a (closed or open) sector of infinite radius, centered at \( 0 \in \mathcal{O}_i \), and of angle \( \theta \). For the following definition, see Figure 2.

**Definition 3.3.** Let \( p, q \in \mathcal{O}_i \) be two points with distinct arguments in the same open \( \pi \)-sector of \( \mathcal{O}_i \). The **triangle** determined by \( p \) and \( q \), denoted \( \triangle(p, q) \), is the union of straight line segments from \( 0 \in \mathcal{O}_i \) to \( p \), from \( p \) to \( q \), and from \( q \) to \( 0 \). Let
\[
 c(p, q) := \partial H(p) \cap \partial H(q)
\]
denote the **circumcenter** determined by \( p \) and \( q \),
\[
 C(p, q) := \{ z \in \mathcal{O}_i \mid d(z, c(p, q)) = |c(p, q)| \}
\]
denote the **circumcircle** determined by \( p \) and \( q \), and
\[
 B(p, q) := \{ z \in \mathcal{O}_i \mid d(z, c(p, q)) < |c(p, q)| \}
\]
denote the **ball** determined by \( p \) and \( q \).
Lemma 4 of [ESS] implies that for any two points \( p, q \in \mathcal{O} \) belonging to the same open \( \pi \)-sector, the distance \( d(p, q) \) between \( p \) and \( q \) in \( \mathcal{O} \) equals the distance between their respective images \( \overline{p} \) and \( \overline{q} \) in the plane.

Hence on any such sector, \( \text{proj}(\cdot) \) is an isometry, so we find that \( \text{proj}(c(p, q)) \) is the center of the Euclidean circle \( \text{proj}(C(p, q)) \) of radius \( |c(p, q)| \), and \( \text{proj}(B(p, q)) \) is the open ball whose boundary is \( \text{proj}(C(p, q)) \). Moreover, \( \text{proj}(C(p, q)) \) is the circumcircle of the Euclidean triangle \( \text{proj}(\Delta(p, q)) \) with vertices \( 0, \overline{p}, \overline{q} \). In particular, \( 0, \overline{p}, \overline{q} \in \text{proj}(C(p, q)) \) implies \( 0, p, q \in C(p, q) = \partial B(p, q) \).

**Remark.** We use the same notation and terminology of circumcenters, circumcircles, and balls determined by two points \( p, q \in \mathbb{R}^2 \), naturally identifying \( \mathbb{R}^2 \) with \((\mathbb{C}, dz)\).

**Definition 3.4.** Let
\[
\mathbb{P}(\mathcal{O}) := \{ \{p, p^\circ\} \subset \mathcal{O} \mid \overline{p} = -\overline{p} \neq 0 \},
\]
and for any subset \( P \subset \mathbb{P}(\mathcal{O}) \), let
\[
P_F := \bigcup_{(p, p^\circ) \in P} \{p, p^\circ\}
\]
(the subscript \( F \) suggests that we ‘forget’ the pairing inherent to \( P \)). For \( r \in \mathbb{R}_+ \) and \( P \subset \mathbb{P}(\mathcal{O}) \), let \( rP := \{\{rp, rp^\circ\} \mid \{p, p^\circ\} \in P\} \) and \( rP_F := \{rp \mid p \in P_F\} \). Call \( P \) limit-point free if \( P_F \) has no limit points in \( \mathcal{O} \), and call \( P \) distinctive if for any \( \{p, p^\circ\} \in P \), the set
\[
\{\{q, q^\circ\} \in P \mid p \in \{q, q^\circ\}\}
\]
equals the singleton \( \{\{p, p^\circ\}\} \).

Notice that for any \( p \in \mathcal{O} \) with \( \overline{p} \neq 0 \), there is precisely one \( p^\circ \) satisfying \( \overline{p^\circ} = -\overline{p} \) for each \( 2\pi \)-sector \( c_j^i \), \( 1 \leq i \leq \kappa \), \( j \in \mathbb{Z}_{d_i+1} \), and hence
\[
|\{p^\circ \mid \{p, p^\circ\} \in \mathbb{P}(\mathcal{O})\}| = \sum_{i=1}^\kappa (d_i + 1) = 2g - 2 + \kappa.
\]
Thus for a distinctive set \( P \) and any \( p \in P_F \), the point \( p^\circ \in P_F \) is uniquely determined from this set of \( 2g - 2 + \kappa \) elements.

**Definition 3.5.** Let \( \{p, p^\circ\}, \{q, q^\circ\} \in \mathbb{P}(\mathcal{O}) \) with \( p \) and \( q \) belonging to the same open \( \pi \)-sector of the same component of \( \mathcal{O} \) and satisfying \( \arg(p) \neq \arg(q) \). Denote by \( [p, q] \) the unique element of \( \mathcal{O} \) for which
\[
(\text{i}) \quad p^\circ \text{ and } [p, q] \text{ belong to the same open } \pi \text{-sector of the same component of } \mathcal{O}, \text{ and}
(\text{ii}) \quad |p, q| = \overline{q} - \overline{p}.
\]
See Figure 3

**Remark.** To avoid cumbersome notation, the point \( p^\circ \) is not explicitly shown in \( [p, q] \); however, the reader should note the implicit dependence of \( [p, q] \) on \( p^\circ \). As our primary interest is in distinctive subsets of \( \mathbb{P}(\mathcal{O}) \), the point \( p^\circ \) shall be made clear by context.
Proof. By Definitions 3.5 and 3.6, we have

\[ p, q \] are determined by \( p, q \) and \( p^o \).

Notice from Definition 3.5 that \([p, q]\) is defined if and only if \([q,p]\) is defined.

**Proposition 3.4.** For any \(\{p, p^o\}, \{q, q^o\} \in \mathbb{P}(\mathcal{O})\) for which \([p, q] \in \mathcal{O}\) is defined, we have \(\{[p, q], [q, p]\} \in \mathbb{P}(\mathcal{O})\). Moreover, \([p^o, [p, q]]\) and \([p, q, [p, p^o]]\) are defined and equal \(q\) and \(q^o\), respectively.

**Proof.** Definition 3.5 gives that \([p, q] = \overline{p} - \overline{q} = -[q, p]\).

Since \(p\) and \(q\) belong to the same \(\pi\)-sector with \(\text{arg}(p) \neq \text{arg}(q)\), we have \(\overline{q} - \overline{p} \neq 0\), so \(\{[p, q], [q, p]\} \in \mathbb{P}(\mathcal{O})\).

Condition (i) of Definition 3.5 guarantees that \([p^o, [p, q]]\) and \([p, q]\) belong to the same open \(\pi\)-sector of the same component. Since \(p\) and \(q\) belong to the same open \(\pi\)-sector different arguments, condition (ii) guarantees that \(\text{arg}(p^o) \neq \text{arg}([p, q])\), so \([p^o, [p, q]]\) and \([p, q, [p, p^o]]\) are defined. Next we show \([p^o, [p, q]] = q\).

Condition (i) of Definition 3.5 (applied to \([p^o, [p, q]]\)) gives that \(p\) and \([p^o, [p, q]]\) belong to the same open \(\pi\)-sector of the same component of \(\mathcal{O}\); by assumption, the same is true of \([p, q]\) and \(q\), so we find that \(q\) and \([p^o, [p, q]]\) belong to the same open 2\(\pi\)-sector. As \(\text{proj}(\cdot)\) is injective on open 2\(\pi\)-sectors, it suffices to show that these latter two points are sent under this map to the same point in the plane. We compute

\[ [p^o, [p, q]] = [p, q] - \overline{p} = \overline{q} - \overline{p} = \overline{q} + \overline{p} = \overline{q} \]

as desired. The proof that \([p, q], [p^o]\) equals \(q^o\) is similar.

The following definition and subsequent results will prove to be essential in §4.4 and §4.5 below.

**Definition 3.6.** Let \(P, Q\) and \(U\) be distinctive, limit-point free subsets of \(\mathbb{P}(\mathcal{O})\). We call \((p, q, u) \in P \times Q \times U\) a permissible triple with permissible scalars \((r, s, t) \in \mathbb{R}_+^3\) if

(i) \([rp, sq]\) and \([sq, rp]\) are defined and equal \(tu\) and \(tu^o\), respectively, and

(ii) \((B(rp, sq) \cup B(tu, rp^o) \cup B(sq^o, tu^o)) \cap (rP \cup sQ \cup tU) = \emptyset\).

See Figure 4.

One immediate, but important, observation regarding permissible triples and permissible scalars is the following:

**Proposition 3.5.** Let \((p, q, u)\) be a permissible triple with permissible scalars \((r, s, t) \in \mathbb{R}_+^3\). The scalars \(s\) and \(t\) are uniquely determined by \(p, q, u \in \mathcal{O}\) and \(r \in \mathbb{R}_+\).

**Proof.** By Definitions 3.5 and 5.6 we have

\[ t\overline{u} = [rp, sq] = sq - rp. \]

Rearranging, we find that \(s\) and \(t\) are solutions to

\[ \begin{pmatrix} \overline{x} & -\overline{x} \\ \overline{y} & -\overline{y} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} r \overline{x} \\ r \overline{y} \end{pmatrix}. \]

The fact that \(rp\) and \(sq\) belong to the same open \(\pi\)-sector with different arguments implies that \(\overline{p}\) and \(\overline{q}\) are \(\mathbb{R}\)-linearly independent. This, together with Equation 1 implies that also \(\overline{u}\) and \(\overline{v}\) are \(\mathbb{R}\)-linearly independent. Hence the matrix on the left side of the previous equation is invertible, so \(s\) and \(t\) are uniquely determined.
that each of these sets have the same cardinality and hence are both infinite. As the set of possible directions
\{p, q, u\} with corresponding permissible scalars \{r, s, t\} is important Definition 3.6 for instance, the first two entries \(p\) and \(q\) must necessarily belong to the same open \(\pi\)-sector of the same component of \(\mathcal{O}\) for condition (i) to hold. Nevertheless, this ordering does admit some flexibility.

**Proposition 3.6.** With notation as in Definition 3.6, the following statements are equivalent:

(a) \((p, q, u)\) is a permissible triple with permissible scalars \((r, s, t)\),

(b) \((q^o, u^o, p)\) is a permissible triple with permissible scalars \((s, t, r)\), and

c) \((u, p^o, q^o)\) is a permissible triple with permissible scalars \((t, r, s)\).

**Proof.** We need only show (a) implies (b): the same argument will give (b) implies (c) and (c) implies (a).
Let \((p, q, u)\) be a permissible triple with permissible scalars \((r, s, t)\). We claim that \((q^o, u^o, p)\) is a permissible triple with permissible scalars \((s, t, r)\).

(i) We must show that \([sq^o, tu^o]\) and \([tu^o, sq^o]\) are defined and equal \(rp\) and \(rp^o\), respectively. We have by assumption that \([sq, rp]\) is defined and equals \(tu^o\). By Proposition 3.5, both \([sq^o, [sq, rp]] = [sq^o, tu^o]\) and \([[sq, rp], sq^o] = [tu^o, sq^o]\) are defined. By the same Proposition, these equal \(rp\) and \(rp^o\), respectively.

(ii) We have

\[(B(sq^o, tu^o) \cup B(rp, sq) \cup B(tu, rp^o)) \cap (sQ_F \cup tU_F \cup rP_F) = \emptyset\]

by assumption.

\[\square\]

One might suspect from Definition 3.6 that it is difficult for some \((p, q, u) \in P_F \times Q_F \times U_F\) to be a permissible triple. Indeed, this suspicion is confirmed by the following:

**Lemma 3.7.** For any distinctive, limit-point free \(P, Q, U \subset \mathbb{P}(\mathcal{O})\), the set of permissible triples \((p, q, u) \in P_F \times Q_F \times U_F\) is finite.

**Proof.** Suppose on the contrary that \(\{(p_k, q_k, u_k)\}_{k \in \mathbb{N}} \subset P_F \times Q_F \times U_F\) is an infinite set of permissible triples with corresponding permissible scalars \(\{(r_k, s_k, t_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}_+^3\). At least one of the sets \(\{p_k\}_{k \in \mathbb{N}}, \{q_k\}_{k \in \mathbb{N}}\) or \(\{u_k\}_{k \in \mathbb{N}}\) is infinite; by Proposition 3.6, we may assume that either \(\{p_k\}_{k \in \mathbb{N}}\) or \(\{p_k^o\}_{k \in \mathbb{N}}\) is infinite. Since \(P\) is distinctive, these sets have the same cardinality and hence are both infinite. As the set of possible directions of points in each \(\mathcal{O}\) is compact and there are only finitely many components \(\mathcal{O}\) of \(\mathcal{O}\), we may also assume that each \(p_k\) (resp. \(p_k^o\)) belongs to some small sector of a fixed component of \(\mathcal{O}\) and that \(\lim_k \arg(p_k) = \theta_P\) (resp. \(\lim_k \arg(p_k^o) = \theta_P^o\)) exists. Rotating each of \(P, Q\) and \(U\) by \(-\theta_P\), assume for simplicity that \(\theta_P = 0\) (and hence \(\theta_P^o = \pi\)).

From Definition 3.6, we see for any \(a \in \mathbb{R}_+\) that \([ap, aq] = a[p, q]\) whenever \([p, q]\) is defined. It follows that if \((r, s, t)\) are permissible scalars for \((p, q, u)\), then so are \((ar, as, at)\) for any \(a \in \mathbb{R}_+\). Rescaling each \((r_k, s_k, t_k)\) as necessary, we further assume that \(|r_k| = 1\) for all \(k\). Under these assumptions, we find
that \( \{r_kp_k\} \subseteq S^1 \) with \( r_kp_k \to (1,0) \). Also note that since \( P \) is limit-point free, \( |p_k| \to \infty \) and hence \( r_k = 1/|p_k| \to 0 \).

Condition (ii) of Definition 3.6 (together with the fact that \( \text{proj}(\cdot) \) is an isometry on \( \pi \)-sectors) guarantees that in the plane, we have for each \( k \in \mathbb{N} \) both

\[
(2) \quad B(r_kp_k, s_kq_k) \cap \{r_kp_k\}_{\ell \in \mathbb{N}} = \emptyset
\]

and

\[
B(t_k\overline{u_k}, r_kp_k) \cap \{r_kp_k\}_{\ell \in \mathbb{N}} = \emptyset.
\]

This latter intersection may be rewritten

\[
(3) \quad B(t_k\overline{u_k}, -r_kp_k) \cap \{-r_kp_k\}_{\ell \in \mathbb{N}} = \emptyset.
\]

For each \( k \), let \( c_k := c(r_kp_k, s_kq_k) \) and \( B_k := B(r_kp_k, s_kq_k) \) be the circumcenter and ball determined by \( r_kp_k \) and \( s_kq_k \). The circumcenter \( c_k \) is the intersection of the perpendicular bisectors of the straight line segments from the origin to \( r_kp_k \) and from the origin to \( s_kq_k \). Since \( \text{arg}(p_k) \to 0 \), for large \( k \) the former perpendicular bisector does not intersect the negative real axis, so for all large \( k \) we have \( \text{arg}(c_k) \in [0, \pi) \cup (\pi, 2\pi) \). Passing to a subsequence, assume without loss of generality that \( \text{arg}(c_k) \in [0, \pi) \) for all \( k \).

We consider two cases (see Figure 5):

(i) Suppose there is a subsequence for which \( (p_k)_y \geq 0 \) for all \( k \). Since \( \text{arg}(p_k) \to 0 \), we may pass to a subsequence to assume \( \arg(p_{k+1}) \leq \arg(p_k) \) for all \( k \). For large enough \( k \), the \( x \)-coordinate of \( r_kp_k \in S^1 \) is greater than \( 1/2 \). Since both 0 and \( r_kp_k \) belong to \( C_k := \partial B_k \), we find that the vertical line \( x = 1/2 \) intersects \( C_k \) at two distinct points; let \( b_k^+ \) denote the point of intersection on the upper-semicircle of \( C_k \). Note that \( b_y := \inf_k \{b_k^+_y\} > 0 \); otherwise \( \arg(c_k) \in (\pi, 2\pi) \) contrary to our assumption. Set \( b := (1/2, b_y) \) and note that for large enough \( k \), we have \( \arg(p_k) < \arg(b) \). Let

\[
S_k = \{p \in \mathbb{R}^2 \mid |p| < 1/2, \ \text{arg}(p) \in (\arg(p_k), \text{arg}(b))\}.
\]

Note that \( S_k \) is contained in the triangle with vertices \( 0 \), \( r_kp_k \) and \( b_k^+ \), which is contained in \( B_k \); hence \( S_k \subset B_k \) for each \( k \). Choose some \( N \in \mathbb{N} \) large enough that \( \arg(p_{k+N}) \in [0, \text{arg}(b)) \). For each \( k > N \), we have \( \arg(p_k) \leq \arg(p_{k+N}) \) by assumption, so \( \arg(p_k) \in [\arg(p_{k+N}), \text{arg}(b)) \). Taking \( k \) large enough, we also have \( |r_kp_k| = |p_{k+N}|/|p_k| < 1/2 \). Hence \( r_kp_k \in S_k \subset B_k = B(r_kp_k, s_kq_k) \), contradicting Equation 2.

(ii) Suppose there is no subsequence for which \( (p_k)_y \geq 0 \) for all \( k \). Then there is some subsequence for which \( (p_k)_y < 0 \)—and hence \( -p_k \) is greater than for all \( k \). Notice that \( c(t_k\overline{u_k}, -r_kp_k) = c_k - r_kp_k \), so the argument of this circumcenter belongs to \( (0, \pi) \). An analogous proof (reflecting about the \( x \)-axis) to that of case (i) implies that for some fixed \( N \) and large enough \( k \), \( -r_kp_k \in B(t_k\overline{u_k}, -r_kp_k) \), which contradicts Equation 3.

We conclude that the set of permissible triples must be finite. \( \square \)
3.3. Marked segments and Voronoi staples. In this subsection we fix a stratum \( \mathcal{H}(d_1, \ldots, d_n) \), its corresponding canonical surface \( \mathcal{O} = \mathcal{O}(d_1, \ldots, d_n) \) and a translation surface \((X, \omega) \in \mathcal{H}(d_1, \ldots, d_n)\). Label the singularities of \((X, \omega)\) as \(\sigma_1, \ldots, \sigma_k\), so that \(\sigma_i\) has cone angle \(2\pi(d_i+1)\). In a sufficiently small neighborhood of \(\sigma_i\) define (generalized) polar coordinates in a fashion analogous to that on \(\mathcal{O}_i\) (see \[3.1\] and the proof of Lemma 5 of \[ESS\]).

Let \(s\) be a separatrix emanating from \(\sigma_i\) of length \(|s| > 0\) and corresponding angle \(\theta \in [0, 2\pi(d_i+1)]\), and denote by \(\tilde{s}\) the point of \(\mathcal{O}_i\) with polar coordinates \((|s|, \theta)\). If \(|s| = 0\), set \(\tilde{s} := 0 \in \mathcal{O}_i\). If \(s\) is in fact a saddle connection, we call \(\tilde{s}\) the marked segment determined by \(s\); note that in this case, hol\((s) = \text{proj}(\tilde{s}) \in \mathbb{C}\). Denote by \(s'\) the identical, but oppositely-oriented saddle connection to \(s\). Let \(\mathcal{M}(X, \omega)\) be the set of all pairs \(\{s, s'\}\) of oppositely-oriented saddle connections of \((X, \omega)\) and

\[
\mathcal{M}_F(X, \omega) := \bigcup \{s, s'\} \in \mathcal{M}(X, \omega)
\]

the set of all oriented saddle connections on \((X, \omega)\). Let \(\tilde{\mathcal{M}}(X, \omega)\) denote the set of all orientation-paired marked segments \(\{\tilde{s}, \tilde{s}'\}\) determined by oppositely oriented saddle connections \(\{s, s'\} \in \mathcal{M}(X, \omega)\).

Proposition 3.8. The set \(\tilde{\mathcal{M}}(X, \omega)\) is a limit-point free subset of \(\mathbb{P}(\mathcal{O})\). Moreover, if \(\tilde{s}\) and \(\tilde{t}\) are marked segments with identical arguments belonging to the same component of \(\mathcal{O}\), then in fact \(\{\tilde{s}, \tilde{s}'\} = \{\tilde{t}, \tilde{t}'\}\). In particular, \(\tilde{\mathcal{M}}(X, \omega)\) is distinctive.

**Proof.** Suppose \(\{\tilde{s}, \tilde{s}'\} \in \tilde{\mathcal{M}}(X, \omega)\). Since the saddle connections \(s\) and \(s'\) are identical but oppositely-oriented, it is clear that \(\overline{s} = -\overline{s}'\), so \(\{\tilde{s}, \tilde{s}'\} \in \mathbb{P}(\mathcal{O})\) and \(\tilde{\mathcal{M}}(X, \omega) \subset \mathbb{P}(\mathcal{O})\). The image of \((\tilde{\mathcal{M}}(X, \omega))_F\) under \(\text{proj}(\cdot)\) equals the image of \(\mathcal{M}_F(X, \omega)\) under hol\((\cdot)\). Since the set of holonomy vectors has no limit points \((\mathbb{P}(\mathcal{O}))_F\) and \(\text{proj}(\cdot)\) is a homeomorphism on sufficiently small neighborhoods of regular points, we have that \((\tilde{\mathcal{M}}(X, \omega))_F\) has no limit points and hence \(\tilde{\mathcal{M}}(X, \omega)\) is limit-point free.

Now suppose \(\tilde{s}\) and \(\tilde{t}\) are marked segments with identical arguments in the same component \(\mathcal{O}_i \subset \mathcal{O}\). Then the underlying saddle connections \(s\) and \(t\) both emanate in the same direction from \(\sigma_i \in \Sigma\). If their lengths differ, say \(|\tilde{s}| < |\tilde{t}|\), then \(|s| < |t|\). This implies that \(t\) has a singularity in its interior, which is impossible. If \(|\tilde{s}| = |\tilde{t}|\), then in fact \(s = t\), and so \(\{\tilde{s}, \tilde{s}'\} = \{\tilde{t}, \tilde{t}'\}\). It follows that \(\tilde{\mathcal{M}}(X, \omega)\) is distinctive. \(\square\)

Let \(\tilde{\mathcal{M}}_F(X, \omega) := (\tilde{\mathcal{M}}(X, \omega))_F\) denote the set of all marked segments determined by saddle connections on \((X, \omega)\).

The **star domain** for \(\sigma_i \in \Sigma\) is

\[
\text{star}_i(X, \omega) := \{\hat{s} \mid s\text{ is a separatrix on } (X, \omega)\text{ emanating from } \sigma_i\} \subset \mathcal{O}_i.
\]

Note that \(\text{star}_i(X, \omega)\) consists of the union of closed rays emanating from \(0 \in \mathcal{O}_i\), which stop only when meeting a marked segment (and thus almost every such ray is infinite). The **star domain** for \((X, \omega)\) is

\[
\text{star}(X, \omega) := \bigcup_{i=1}^{k} \text{star}_i(X, \omega) \subset \mathcal{O}.
\]

Define a map \(\eta : \text{star}(X, \omega) \rightarrow (X, \omega)\), where for each point \(p \in \text{star}_i(X, \omega)\), if \(s\) is the separatrix for which \(p = \hat{s}\), then \(\eta(p)\) is the endpoint of \(s\) on \((X, \omega)\). In other words, if \(p \in \text{star}(X, \omega)\) has polar coordinates \((|p|, \theta)\), then \(\eta(p)\) is the endpoint of the separatrix of length \(|p|\) emanating from \(\sigma_i \in \Sigma\) with angle \(\theta\).

For each \(x\) in the Voronoi 2-cell \(C_i := C_{\sigma_i}\), let \(s_x\) be the unique length-minimizing separatrix from \(\sigma_i\) to \(x\). Note that \(\eta\) is injective—and thus invertible—on the set

\[
\{s_x \mid x \in C_i\} \subset \text{star}_i(X, \omega)
\]

if \(\eta(s_x) = \eta(s_y)\) for \(x, y \in C_i\), then \(x = y\) and \(s_x = s_y\) by the aforementioned uniqueness of these separatrices. Hence \(\hat{s}_x = \hat{s}_y\). Let \(\iota_i : C_i \rightarrow \{s_x \mid x \in C_i\}\) denote the corresponding inverse, namely \(x \mapsto \hat{s}_x\) for each \(x \in C_i\), and define \(\iota : \bigcup_{i=1}^{k} C_i \rightarrow \mathcal{O}\) by setting \(\iota|_{C_i} = \iota_i\) for each \(i\); see Figure 6.

The translation surface \((X, \omega)\) is isometric to the quotient space of \(\bigcup_{i=1}^{k} C_i\) under the equivalence relation defined by identifying shared edges of Voronoi 2-cells. Proposition 7 of \[ESS\] shows that in a similar fashion, \((X, \omega)\) may be recovered from the closure of the image under \(\iota\) of its Voronoi 2-cells by identifying appropriate edges of the various \(\iota(C_i)\). We provide a brief overview of the method by which these edge identifications
are made. Recall that the half-space \( H(p) \) determined by \( p \in \Omega_i \) is convex, and thus so is any intersection of such half-spaces.

**Definition 3.7.** For \( S \) a subset of \( \mathcal{O} \) with no limit points, the *convex body* of \( \mathcal{O} \) subordinate to \( S \) is defined by

\[
\Omega_i(S) := \bigcap_{p \in S \cap \Omega_i} H(p).
\]

The set of essential points of \( \Omega_i(S) \) is the (unique) minimal subset \( E_i(S) \subset S \) for which

\[
\Omega_i(S) = \bigcup_{p \in E_i(S)} H(p).
\]

When \( S = \hat{\mathcal{M}}(X, \omega) \) is the set of all marked segments of \((X, \omega)\), we use the suppressed notation \( \Omega_i(\hat{\mathcal{M}}(X, \omega)) \) and \( E_i(\hat{\mathcal{M}}(X, \omega)) \).

Call distinct points \( p, q \in S \) adjacent within \( S \) if \( p \) and \( q \) belong to the same component \( \mathcal{O}_i \), and either of the two open sectors centered at \( 0 \) between \( p \) and \( q \) contains no points of \( S \).

Proposition 14 of [ESS] shows that the convex body \( \Omega_i \) is precisely the set \( \overline{i(C_i)} \). Furthermore, Proposition 14 and Definition 15 of [ESS] show that there is a subset \( \hat{S}(X, \omega) \) of \( \hat{\mathcal{M}}(X, \omega) \) for which the union of essential points \( E := \bigcup_{i=1}^n E_i \) equals \( \hat{S}(X, \omega) = (\hat{S}(X, \omega))_{\mathcal{F}}, \) i.e. the essential points come equipped with a natural pairing. Elements \( \{s, s'\} \) of \( \hat{S}(X, \omega) \) are called marked Voronoi staples and their underlying pairs of saddle connections \( \{s, s'\} \in \hat{\mathcal{M}}(X, \omega) \) are called Voronoi staples. It follows from the definition of \( E_i \) that each edge on the boundary of \( \Omega_i \) belongs to the boundary of a half-space \( H(\hat{s}) \) for some \( \hat{s} \in E_i \subset \hat{S}(X, \omega) \); conversely, the boundary of the half-space determined by each \( \hat{s} \in \hat{S}(X, \omega) \) contains an edge on the boundary of some \( \Omega_i \). Propositions 7 and 14 of [ESS] show that the edges of the convex bodies of \( \bigcup_{i=1}^n \Omega_i \) corresponding to \( \hat{s} \) and its orientation-paired \( \hat{s}' \) are equal length, and that \((X, \omega)\) is isometric to the quotient space of \( \bigcup_{i=1}^n \Omega_i \) under the equivalence relation given by identifying these edges via translation.

Propositions 7 and 14 of [ESS] show that the edges of the convex bodies of \( \bigcup_{i=1}^n \Omega_i \) corresponding to \( \hat{s} \) and its orientation-paired \( \hat{s}' \) are equal length, and that \((X, \omega)\) is isometric to the quotient space of \( \bigcup_{i=1}^n \Omega_i \) under the equivalence relation given by identifying these edges via translation.

**Remark.** Under this identification of edges, the vertices of the various \( \Omega_i \) are regarded as regular points on the resulting translation surface (they correspond to the Voronoi 0-cells of \((X, \omega)\)). The origins \( 0 \in \mathcal{O}_i \subset \mathcal{O} \) become singularities of the resulting translation surface; we require this also if \( \mathcal{O}_i = (\mathbb{C}, dz) \), in which case the singularity is a marked point.

**Example 3.9.** In Figure 6, the marked Voronoi staples are \( \{a, \hat{a}'\}, \{\hat{b}', b\} \) and \( \{\hat{c}', c\} \). The hexagonal translation surface on the left is recovered by identifying the edges of the convex bodies corresponding to these respective pairs.
Figure 7. The set of eighth roots of unity contains $\Theta_{\Gamma_0}$ with $\Gamma_0 = \langle S, T^2 \rangle$ and $S$ and $T$ defined as in Example 4.1.

We conclude this subsection with a brief observation:

**Proposition 3.10.** Any two adjacent elements $\hat{s}$ and $\hat{t}$ of $\hat{S}_F(X, \omega)$ belong to the same open $\pi$-sector.

**Proof.** If not, then $\partial H(\hat{s}) \cap \partial H(\hat{t}) = \emptyset$, and the corresponding convex body has infinite area. This would imply that the closed translation surface $(X, \omega)$ has infinite area, which is a contradiction. $\square$

4. **Finiteness of lattice surfaces in strata**

4.1. **Fanning groups and surfaces.** While our primary interest is in lattices, many of the methods and results apply to an *a priori* larger collection of translation surfaces (Corollary 4.3).

**Definition 4.1.** Let $\Gamma$ be a discrete subgroup of $\text{SL}_2 \mathbb{R}$ and $\Delta$ the open unit disk. Set

$$\Theta_\Gamma := S^1 \setminus \bigcup_{A \in \Gamma} A \cdot \Delta,$$

where $A \cdot \Delta$ denotes the usual action of a matrix on $\mathbb{R}^2$ as a linear transformation. We call $\Gamma$ a *fanning group* if $\Theta_\Gamma$ is finite and call a translation surface $(X, \omega)$ a *fanning surface* if its Veech group $\text{SL}(X, \omega)$ is fanning.

**Example 4.1.** Let $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and set $\Gamma_0 := \langle S, T^2 \rangle$. Figure 7 shows that $\Theta_{\Gamma_0}$ is contained in the set of eighth roots of unity and thus $\Gamma_0$ a fanning group. One verifies that in fact $\Theta_{\Gamma_0}$ equals this set.

Recall that a parabolic matrix in $\text{SL}_2 \mathbb{R}$ has a unique eigenvalue which belongs to $\{ \pm 1 \}$. We call $v \in S^1$ a *parabolic direction* for a subgroup $\Gamma$ of $\text{SL}_2 \mathbb{R}$ if $v$ is an eigenvector for some parabolic element of $\Gamma$.

**Lemma 4.2.** If $\Gamma$ is a subgroup of $\text{SL}_2 \mathbb{R}$ whose parabolic directions are dense in $S^1$, then $\Gamma$ is a fanning group.

**Proof.** Let $P \in \Gamma$ be parabolic, and let $\pm v \in S^1$ be the corresponding parabolic directions for $P$, i.e. $\{ \pm P \cdot v \} = \{ \pm v \}$. Note that the image $P \cdot \Delta$ is a compact set whose boundary is an ellipse passing through $\pm v$, centered at the origin, and the same is true for $P^{-1} \cdot \Delta$. It follows that

$$S^1 \cap \left( P \cdot \Delta \cup P^{-1} \cdot \Delta \right)$$
consists of two closed arcs in \( S^1 \), each of which contains one of \( \pm v \) in its (non-empty) interior; see Figure 7 with, say, \( v = (1, 0) \) and \( P = T^2 \). Recall that the set of such parabolic directions \( \pm v \) for \( \Gamma \) is dense in \( S^1 \) by assumption. Since \( S^1 \) is compact and the interior of each
\[
S^1 \cap (P \cdot \Delta \cup P^{-1} \cdot \Delta)
\]
is non-empty, there is a finite collection of parabolics \( P_1, \ldots, P_n \in \Gamma \) for which
\[
S^1 = S^1 \cap \bigcup_{i=1}^n (P_i \cdot \Delta \cup P_i^{-1} \cdot \Delta).
\]
Then
\[
S^1 \cap \bigcup_{i=1}^n (P_i \cdot \Delta \cup P_i^{-1} \cdot \Delta)
\]
is all of \( S^1 \) except for finitely many points, and it follows that
\[
\Theta_\Gamma = S^1 \setminus \bigcup_{A \in \Gamma} A \cdot \Delta \subset S^1 \setminus \bigcup_{i=1}^n (P_i \cdot \Delta \cup P_i^{-1} \cdot \Delta)
\]
is finite. Thus \( \Gamma \) is fanning. \( \Box \)

**Remark.** The Veech group of a lattice surface necessarily contains parabolic elements (in particular, there is a parabolic direction in the direction of any saddle connection). In the remainder of the paper, when considering a lattice group \( \Gamma \leq \text{SL}_2 \mathbb{R} \), we thus assume that \( \Gamma \) contains parabolic elements.

**Corollary 4.3.** Every lattice group is a finitely generated fanning group. In particular, every lattice surface is a fanning surface with a finitely generated Veech group.

**Proof.** Let \( \Gamma \leq \text{SL}_2 \mathbb{R} \) be a lattice and \( \overline{\Gamma} = \Gamma / \{ \pm \text{Id} \} \) its image in \( \text{PSL}_2 \mathbb{R} \). By Lemma 4.2 to show that \( \Gamma \) is a fanning group, it suffices to show that the parabolic directions of \( \Gamma \) are dense in \( S^1 \). Note that \( v \in S^1 \) is a parabolic direction of \( \Gamma \) if and only if the reciprocal of the slope of \( v \) is a parabolic fixed point for the action of \( \overline{\Gamma} \) on the upper-half plane via Möbius transformations. Thus it suffices to show that the set of parabolic fixed points of \( \overline{\Gamma} \) is dense in \( \mathbb{R} \cup \{ \infty \} \). Now \( \overline{\Gamma} \) is a Fuchsian group of the first kind, i.e. its limit set \( \Lambda(\overline{\Gamma}) \) is all of \( \mathbb{R} \cup \{ \infty \} \) (Theorem 4.5.2 of [K3]), and \( \Lambda(\overline{\Gamma}) \) is the closure of the \( \overline{\Gamma} \)-orbit of any point in \( \mathbb{R} \cup \{ \infty \} \) (see, say, [De]). If \( x \in \mathbb{R} \cup \{ \infty \} \) is a fixed point of a parabolic \( P \in \overline{\Gamma} \), then for any \( A \in \overline{\Gamma} \), the image \( Ax \) is a fixed point of the parabolic \( APA^{-1} \). Hence the \( \overline{\Gamma} \)-orbit of \( x \) consists of parabolic fixed points, and the closure of this orbit is all of \( \mathbb{R} \cup \{ \infty \} \). Thus \( \Gamma \) is a fanning group.

That \( \Gamma \) is finitely generated follows from the fact that \( \overline{\Gamma} \) is geometrically finite, and the sides of a Dirichlet region for \( \overline{\Gamma} \) determine generators for the group \( \overline{\Gamma} \) (Theorems 3.5.4 and 4.1.1 of [K3]). The group \( \Gamma \) is generated by representatives of the generators of \( \Gamma / \{ \pm \text{Id} \} \) together with, possibly, \( -\text{Id} \). \( \Box \)

**4.2. The group \( \text{Aff}_+^+(O, \omega) \) and its action on marked segments.** Let \( (X, \omega) \in \mathcal{H}(d_1, \ldots, d_n) \) and \( O = O(d_1, \ldots, d_n) \). A central result of [Ed] (Theorem 18) and [ESS] (Proposition 17) is that a matrix \( A \in \text{SL}_2 \mathbb{R} \) belongs to \( \text{SL}(X, \omega) \) if and only if there is some \( f \in \text{Aff}_+^+(O) \) with \( \text{der}(f) = A \) satisfying \( f(\hat{M}(X, \omega)) \subset \hat{M}(X, \omega) \). Theorem 17 of [Ed] shows that same statement holds with the latter subset inclusion replaced by the equality \( f(\hat{M}(X, \omega)) = \hat{M}(X, \omega) \). Note, in particular, that these statements require the affine automorphism \( f \) to respect the orientation-pairing of marked segments—it is not enough that the set of marked segments \( \hat{M}_F(X, \omega) \) is invariant under \( f \) to conclude that \( \text{der}(f) \in \text{SL}(X, \omega) \) (see Example 21 of [ESS]). We introduce the following notation for the collection of such affine automorphisms of \( O \):

**Definition 4.2.** Let
\[
\text{Aff}_+^+(O, \omega) := \{ f \in \text{Aff}_+^+(O) \mid f(\hat{M}(X, \omega)) = \hat{M}(X, \omega) \}.
\]

Note that \( \text{Aff}_+^+(O, \omega) \) is a subgroup of \( \text{Aff}_+^+(O) \). From the comments above we see that the image of \( \text{Aff}_+^+(O, \omega) \) under \( \text{der}(\cdot) \) is precisely the Veech group \( \text{SL}(X, \omega) \) (though \( \text{der}(\cdot) \) restricted to \( \text{Aff}_+^+(O, \omega) \) need not be injective, so \( \text{Aff}_+^+(O, \omega) \) and \( \text{SL}(X, \omega) \) are in general non-isomorphic). Let \( G \) be a set of (possibly infinitely many) generators of \( \text{SL}(X, \omega) \). For each \( A \in G \), choose one \( g_A \in \text{Aff}_+^+(O, \omega) \) with \( \text{der}(g_A) = A \), and let \( \mathcal{G} \) be the subgroup of \( \text{Aff}_+^+(O, \omega) \) generated by the set of these \( g_A \).
Lemma 4.4. With notation as above, there is some subset \( \{\tau_1, \ldots, \tau_n\} \subset \text{Trans}(\mathcal{O}) \) for which the coset representatives of \( \mathcal{G} \) in \( \text{Aff}_C^+(X, \omega) \) are \( \tau_1 \mathcal{G}, \ldots, \tau_n \mathcal{G} \).

**Proof.** Let \( f \mathcal{G} \) be a coset of \( \mathcal{G} \) in \( \text{Aff}_C^+(X, \omega) \). We aim to show that there is some \( \tau \in \text{Trans}(\mathcal{O}) \) for which \( f \mathcal{G} = \tau \mathcal{G} \), or, equivalently, \( \tau^{-1} \circ f \in \mathcal{G} \); that the index of \( \mathcal{G} \) in \( \text{Aff}_C^+(X, \omega) \) is finite will then follow from finiteness of \( \text{Trans}(\mathcal{O}) \) (Lemma 4.3). Write \( f = \tau_f \circ f_A \), where \( \tau_f \in \text{Trans}(\mathcal{O}) \) and \( f_A \in \text{Aff}_C^+(\mathcal{O}) \) with \( \text{der}(f) = A \in \text{SL}(X, \omega) \). Now \( A = A_1 \cdots A_m \) for some \( A_i \in \mathcal{G} \); let \( g := g_{A_1} \circ \cdots \circ g_{A_m} \), where each \( g_{A_i} \) is one of the chosen generators of \( \mathcal{G} \). Note that \( \text{der}(g) = A \), so we may write \( g = \tau_g \circ f_A \) for some \( \tau_g \in \text{Trans}(\mathcal{O}) \). Set \( \tau := \tau_f \circ (\tau_g)^{-1} \in \text{Trans}(\mathcal{O}) \). Then
\[
\tau^{-1} \circ f = (\tau_f \circ (\tau_g)^{-1}) \circ (\tau_f \circ f_A) = \tau_g \circ f_A = g \in \mathcal{G},
\]
as desired. \( \square \)

The group \( \text{Aff}_C^+(X, \omega) \) acts on \( \hat{\mathcal{M}}(X, \omega) \) via
\[
\text{Aff}_C^+(X, \omega) \times \hat{\mathcal{M}}(X, \omega) \to \hat{\mathcal{M}}(X, \omega)
\]
\[(f, \{(s, s')\}) \mapsto \{f(s), f(s')\}.\]

Note in particular that \( f(s') = f(s')' \). For any \( \{s, s'\} \in \hat{\mathcal{M}}(X, \omega) \), let \( \{s, s'\} \) denote the orbit of \( \{s, s'\} \) under this action.

**Lemma 4.5.** If \((X, \omega)\) is a fanning surface, then the orbit space
\[
\hat{\mathcal{M}}(X, \omega)/\text{Aff}_C^+(X, \omega) = \left\{ \left\{ (s, s') \right\} \mid \{s, s'\} \in \hat{\mathcal{M}}(X, \omega) \right\}
\]
is finite.

**Proof.** Let \( \{(s, s')\} \in \hat{\mathcal{M}}(X, \omega)/\text{Aff}_C^+(X, \omega) \). The image under \( \text{proj}(\cdot) \) of \( \{(s, s')\} \) consists of pairs \( \{\pm v\} \) of holonomy vectors in \( \mathbb{C} \), among which there is some pair of minimal length. The preimage under \( \text{proj}(\cdot) \) of this pair is a finite set in \( \mathcal{O} \) containing a pair of orientation-paired marked segments \( \{s, s'\} \) of minimal length in \( \{(s, s')\} \). Rescaling \((X, \omega)\) if necessary, we may assume \(|s| = |s'| = 1\). We claim that
\[
\{\bar{s}, \bar{s}'\} \subset \Theta_{\text{SL}(X, \omega)}.
\]
Note for each \( A \in \text{SL}(X, \omega) \) that \( \bar{s} \notin S^1 \cdot A \cdot \Delta \); otherwise \( A^{-1} \cdot \bar{s} \in \Delta \), but the comments following Definition 4.2 together with Lemma 4.3 imply \( A^{-1} \cdot \bar{s} = \text{proj}(\circ f(\bar{s}) \text{ for some } f \in \text{Aff}_C^+(X, \omega) \text{ with } \text{der}(f) = A^{-1}. \)

Since \( 1 > |\text{proj} \circ f(\bar{s})| = |f(\bar{s})| \), this contradicts the assumption that \( \{s, s'\} \) is a pair of minimal length in \( \{(s, s')\} \). The same is true for \( \bar{s}' \), so we find that
\[
\{\bar{s}, \bar{s}'\} \subset \bigcap_{A \in \text{SL}(X, \omega)} (S^1 \setminus A \cdot \Delta) = S^1 \setminus \bigcup_{A \in \text{SL}(X, \omega)} A \cdot \Delta = \Theta_{\text{SL}(X, \omega)}
\]
as claimed.

For each \( v \in S^1 \), let \( r_v \) denote the infinite open ray emanating from \( 0 \in \mathbb{C} \) in the direction of \( v \). Note that the preimage \( \text{proj}^{-1}(r_v) \subset \mathcal{O} \) consists of \( \sum_{i=1}^n (d_i + 1) = 2g - 2 + \kappa \) infinite open rays emanating from the origins of the various components of \( \mathcal{O} \), and each of these rays contains at most one marked segment of \((X, \omega)\) (Proposition 4.3). From the claim above, the minimal-length representatives of each \( \{(s, s')\} \in \hat{\mathcal{M}}(X, \omega)/\text{Aff}_C^+(X, \omega) \) belong to one of the \( 2g - 2 + \kappa \) open rays of \( \text{proj}^{-1}(r_v) \) for some \( v \in \Theta_{\text{SL}(X, \omega)} \).

Hence
\[
\left| \hat{\mathcal{M}}(X, \omega)/\text{Aff}_C^+(X, \omega) \right| \leq (2g - 2 + \kappa) \cdot |\Theta_{\text{SL}(X, \omega)}|,
\]
where \(|\cdot|\) denotes cardinality. Since \((X, \omega)\) is a fanning surface, the right-hand side is finite. \( \square \)

The proof of Lemma 4.5 also gives the following:

**Corollary 4.6.** Let \((X, \omega)\) be a fanning surface and \( \{(s, s')\} \in \hat{\mathcal{M}}(X, \omega)/\text{Aff}_C^+(X, \omega) \). If \( \{s, s'\} \) are orientation-paired marked segments of minimal length in \( \{(s, s')\} \), then \( \bar{s}/|\bar{s}| \) and \( \bar{s}'/|\bar{s}'| \) belong to the finite set \( \Theta_{\text{SL}(X, \omega)} \).
4.3. Simulating (normalized) $\text{Aff}_+^+(X, \omega)$-orbits. The set of orientation-paired marked segments $\tilde{\mathcal{M}}(X, \omega)$—and the set of $\text{Aff}_+^+(X, \omega)$-orbits into which it partitions—depends intrinsically on $(X, \omega)$ and its geometry. The results of the previous subsection suggest that in the case of a fanning surface, much of this information is encoded in the Veech group $\text{SL}(X, \omega)$ (and the orders $d_1, \ldots, d_\kappa$ of singularities of $(X, \omega)$). This subsection further explores these ideas by introducing simulations of $\text{Aff}_+^+(X, \omega)$-orbits which are constructed via generators of a fanning group. We again fix a stratum $\mathcal{H}(d_1, \ldots, d_\kappa)$.

**Definition 4.3.** Let $\Gamma \leq \text{SL}_2\mathbb{R}$ be a fanning group generated by $G = \{A_1, A_2, \ldots\} \subset \Gamma$, let $C := \{c^j_i \mid 1 \leq i \leq \kappa, j \in \mathbb{Z}_{d_i+1}\}$ be the collection of $2\pi$-sectors of components of $O = O(d_1, \ldots, d_\kappa)$, and set
\[
\mathcal{G}_G := \Theta_\Gamma \times C^2 \times \text{Trans}(O)^{|G|} \times \mathcal{P} \left( \text{Trans}(O) \right),
\]
where $\mathcal{P}(\cdot)$ denotes the power set, not including $\emptyset$. For each $s = (v, (c^j_i, c^k_i), (\tau^m_n)_{n=1}^{|G|}, (\tau^m_n')_{m=1}^M) \in \mathcal{G}_G$, let $\{p, p^o\} \in \mathcal{P}(O)$, where $p^o = -p$ and $p^o \in c^j_i$, let $G_s$ be the subgroup of $\text{Aff}_+^+(X, \omega)$ generated by the set $\{\tau_n \circ f_{A_n}\}_{n=1}^{|G|}$ with $f_{A_n} \in \text{Aff}_+^+(O)$, and set
\[
\text{Aff}_+^+(s) := \bigcup_{m=1}^M \tau^m_n G_s.
\]
Define the simulation determined by $s$, denoted $\text{sim}(s)$, to be the union of images of $\{p, p^o\}$ under all elements of $\text{Aff}_+^+(s)$. For any $r \in \mathbb{R}_+$, the $r$-scaled simulation determined by $s$ is $r \text{sim}(s)$; set $\text{sim}_r(s) := (\text{sim}(s))_r$ (recall Definition 4.4). The set of all simulations determined by $G$ is denoted
\[
\text{Sim}_G := \{\text{sim}(s) \mid s \in \mathcal{G}_G\}.
\]

Recall $S$, $T$ and $G_0$ from Example 4.1.

**Example 4.7.** Let $G_0 := \{A_1, A_2\}$ with $A_1 = S$ and $A_2 = T^2$, let $O = O(2)$, and set
\[
s_0 := ((1, 0), (c_0^1, c_1^2), (\rho_0^2, \text{Id}), \{\text{Id}\}) \in \mathcal{G}_{G_0}.
\]
Then $G_{s_0} = \langle \rho_0^2 \circ f_S, f_{T^2} \rangle$ and $\text{Aff}_+^+(s_0) = G_{s_0}$. The map $\rho_0^2 \circ f_S$ acts as a counterclockwise rotation of $O$ by an angle of $\pi/2 + 4\pi$, and $f_{T^2}$ is a bijective horizontal shear on each $c_1^2$. A subset of the simulation $\text{sim}(s_0)$ is shown in cyan in Figure 8. The points $\{p, p^o\}$ from Definition 4.3 are those satisfying $p = (1, 0)$ and $p^o = (-1, 0)$ with $p \in c_0^1$ and $p^o \in c_1^2$. In red and yellow are subsets of $\text{sim}(s_1)$ and $\text{sim}(s_2)$, respectively, for
\[
s_1 := ((1, 0), (c_0^1, c_0^1), (\rho_1^2, \text{Id}), \{\text{Id}\})
\]
and
\[
s_2 := ((\sqrt{2}/2, \sqrt{2}/2), (c_0^1, c_1^2), (\rho_1^2, \text{Id}), \{\text{Id}\}).
\]

Observe that Definition 4.3 and Corollary 4.3 immediately imply:
Lemma 4.8. If $\Gamma$ is a fanning group generated by a finite set $G \subset \Gamma$, then $\text{Sims}_G$ is finite. In particular, if $\Gamma$ is a lattice, then there exists a set of generators $G$ of $\Gamma$ for which the set of simulations $\text{Sims}_G$ is finite.

Definition 4.4. Let $r \in \mathbb{R}_+$ be the minimal length of orientation-paired marked segments in $[\{\hat{s}, \hat{s}'\}] \in \hat{\mathcal{M}}(X, \omega)/\text{Aff}_G^\pm(X, \omega)$. For any marked segment $\hat{s}$ belonging to a pair in the orbit $[\{\hat{s}, \hat{s}'\}]$, the normalized marked segment corresponding to $\hat{s}$ is $n(\hat{s}) := (1/r)\hat{s}$. Set $n(\hat{s}') := n(\hat{s}')$. Define the normalized $\text{Aff}_G^\pm(X, \omega)$-orbit corresponding to $[\{\hat{s}, \hat{s}'\}]$ to be
\[
n([\{\hat{s}, \hat{s}'\}]) := \{n(\hat{s}), n(\hat{s}') \mid [\hat{s}, \hat{s}'] \in [\{\hat{s}, \hat{s}'\}]\}.
\]

Thus the normalized $\text{Aff}_G^\pm(X, \omega)$-orbit corresponding to $[\{\hat{s}, \hat{s}'\}]$ takes all points of the orbit $[\{\hat{s}, \hat{s}'\}]$ and rescales them (by a constant scalar $1/r$) so that the minimal length of a pair in the orbit is one.

Theorem 4.9. If $(X, \omega)$ is a fanning surface, then $\hat{\mathcal{M}}(X, \omega)$ is a finite union of scaled simulations. In particular, if $\Gamma \leq \text{SL}_2\mathbb{R}$ is a fanning group generated by $G \subset \Gamma$, then
\[
\left\{n([\{\hat{s}, \hat{s}'\}]) \mid [\{\hat{s}, \hat{s}'\}] \in \hat{\mathcal{M}}(X, \omega)/\text{Aff}_G^\pm(X, \omega) \text{ for some } (X, \omega) \text{ with } \text{SL}(X, \omega) = \Gamma\right\} \subset \text{Sims}_G.
\]

Proof. We will prove the second statement; the first follows from this together with Lemma 1.5. Let $(X, \omega)$ be a translation surface with $\text{SL}(X, \omega) = \Gamma$, and let $G = \{A_1, A_2, \ldots \}$ be a (finite or infinite) set of generators for $\Gamma$. For each $1 \leq n \leq |G|$, choose some $g_{An} \in \text{Aff}_G^+(X, \omega)$ with $\text{der}(g_{An}) = A_n$. Write $g_{An} = \tau_n \circ f_{An}$ for some $\tau_n \in \text{Trans}(O)$, where $f_{An} \in \text{Aff}_G^+(O)$, and set $\mathcal{G} := \{g_{An}\}_{n=1}^{|G|} = \{\tau_n \circ f_{An}\}_{n=1}^{|G|}$. By Lemma 4.4 there is some subset $\{\tau'_m\}_{m=1}^M \subset \text{Trans}(O)$ for which
\[
\text{Aff}_G^+(X, \omega) = \bigcup_{m=1}^M \tau'_m \mathcal{G}.
\]

Now let $[\{\hat{s}, \hat{s}'\}] \in \hat{\mathcal{M}}(X, \omega)/\text{Aff}_G^+(X, \omega)$ with $\{\hat{s}, \hat{s}'\}$ a pair of minimal length in $[\{\hat{s}, \hat{s}'\}]$. By Corollary 4.6 $v := \hat{s}/|\hat{s}| = n(\hat{s})$ belongs to the finite set $\Theta_G$. Set $s = (v, (c^j_i, c^k_i), (\tau_n)_{n=1}^{|G|}, \{\tau'_m\}_{m=1}^M) \in \mathcal{G}_G$, where $c^j_i, c^k_i \in \mathcal{C}$ are the $2\pi$-sectors containing $\hat{s}$ and $\hat{s}'$, respectively. Then $\mathcal{G}_s = \mathcal{G}$, $\text{Aff}_G^+(s) = \text{Aff}_G^+(X, \omega)$, and $\text{sim}(s)$ is the $\text{Aff}_G^+(X, \omega)$-orbit of $\{n(\hat{s}), n(\hat{s}')\} \subset \mathcal{O}$; this is precisely the set $n([\{\hat{s}, \hat{s}'\}])$. $\square$

Theorem 4.9 implies that for any fanning group $\Gamma$, the orientation-paired marked segments $\hat{\mathcal{M}}(X, \omega)$ of any $(X, \omega)$ with Veech group $\Gamma$ are given by a finite collection of simulations, each of which is appropriately scaled. Recall that the marked Voronoi staples $\hat{S}(X, \omega)$—from which $(X, \omega)$ may be reconstructed—are determined by $\hat{\mathcal{M}}(X, \omega)$ (4.3). In 4.1.3, we show that for any finite collection of simulations, there are at most finitely many scalars for which the union of scaled simulations agrees with the marked segments of a unit-area translation surface. Together with Lemma 4.8 this will prove the main result of this paper—that there are only finitely many unit-area translation surfaces in any given stratum with a given finitely generated fanning (and hence lattice) Veech group.

4.4. Marked Voronoi staples determine permissible triples. Let $(X, \omega) \in \mathcal{H}(d_1, \ldots, d_n)$ and $\mathcal{O} = \mathcal{O}(d_1, \ldots, d_n)$. In this subsection we apply the results of 4.3 to orientation-paired marked segments and their $\text{Aff}_G^+(X, \omega)$-orbits. In particular, we shall see that adjacent elements of $\hat{S}_F(X, \omega)$ naturally determine permissible triples (Proposition 4.11). To this end, we begin with the following:

Lemma 4.10. For any two adjacent $\hat{s}, \hat{t} \in \hat{S}_F(X, \omega)$, we have
\[
B(\hat{s}, \hat{t}) \cap \hat{\mathcal{M}}_F(X, \omega) = \emptyset.
\]

Moreover, $[\hat{s}, \hat{t}] \in \hat{\mathcal{M}}_F(X, \omega)$ with $[\hat{s}, \hat{s}'] = [\hat{t}, \hat{s}]$.

Proof. Let $i$ be the index for which $\hat{s}, \hat{t} \in \mathcal{O}_i$, and recall from Definition 4.3 that the circumcenter $c := c(\hat{s}, \hat{t})$ determined by $\hat{s}, \hat{t} \in \mathcal{O}_i$ is the intersection of $\partial H(\hat{s})$ and $\partial H(\hat{t})$. Since $\hat{s}, \hat{t} \in \hat{S}_F(X, \omega)$ are essential points of $\Omega_i$, there are two (adjacent) edges of $\Omega_i$ contained in $\partial H(\hat{s})$ and $\partial H(\hat{t})$, respectively. The intersection of
in fact belong to the former set of permissible triples when the Veech group of $(X, \omega)$ is generated by $G$ (Definition 3.6). We introduce notation for the set of all permissible triples arising from any three simulations of $(X, \omega)$.

Recall that permissible triples are defined in terms of three distinct, limit-point free subsets of $\mathcal{P}(O)$ (Definition 3.6). We introduce notation for the set of all permissible triples arising from any three simulations of $(X, \omega)$ in $\text{Sims}_G$ and notation for specific triples determined by adjacent $\hat{s}, \hat{t} \in \mathcal{S}_F(X, \omega)$. These latter triples will in fact belong to the former set of permissible triples when the Veech group of $(X, \omega)$ is generated by $G$ (Proposition 4.11).

**Figure 9.** Illustration of the proof of Lemma 4.10 showing that $\hat{u} = [\hat{s}, \hat{t}]$ is a marked segment for adjacent $\hat{s}, \hat{t} \in \mathcal{S}_F(X, \omega)$.

The edges is precisely the circumcenter $c$, so $c \in \Omega_i$. Suppose for the sake of contradiction that there is some marked segment $\hat{u} \in B(\hat{s}, \hat{t})$. Then

$$d(\hat{u}, c) < |c| = d(0, c),$$

and hence $c \notin H(\hat{u})$. This contradicts the fact that $c \in \Omega_i$, so we have

$$B(\hat{s}, \hat{t}) \cap \hat{M}_F(X, \omega) = \emptyset.$$

This result—together with the convexity of $B(\hat{s}, \hat{t})$ and the fact that $0 \in \partial B(\hat{s}, \hat{t})$—gives that $B(\hat{s}, \hat{t}) \subset \text{star}_i(X, \omega)$, so $\eta$ is defined on all of $B(\hat{s}, \hat{t})$. Let $\gamma$ be the straight line segment in $O_i$ from $\hat{s}$ to $\hat{t}$, and let $\sigma_j, \sigma_k \in \Sigma$ be the singularities at which the oriented saddle connections $s$ and $t$ end, respectively. Then, recycling notation, $u := \eta(\gamma)$ is a saddle connection from $\sigma_j$ to $\sigma_k$, and $\hat{u} \in \hat{M}_F(X, \omega)$. We claim that $\hat{u} = [\hat{s}, \hat{t}]$.

By Proposition 3.10, $\hat{s}$ and $\hat{t}$ belong to the same open $\pi$-sector of $O_i$. We also have by Proposition 3.8 that $\arg(\hat{s}) \neq \arg(\hat{t})$, so $[\hat{s}, \hat{t}]$ is in fact defined. It is clear from the definition of $\hat{u}$ that $\hat{s}'$ and $\hat{u}$ belong to the same open $\pi$-sector of the same component $O_j$. Furthermore, $\overline{u} = \overline{t} - \overline{s}$; see Figure 9. By Definition 3.6, we have $\hat{u} = [\hat{s}, \hat{t}]$. Reversing the orientation of $\gamma$ in the argument above gives that $\hat{u}' = [\hat{t}, \hat{s}]$. □

Recall that permissible triples are defined in terms of three distinct, limit-point free subsets of $\mathcal{P}(O)$ (Definition 3.6). We introduce notation for the set of all permissible triples arising from any three simulations in $\text{Sims}_G$ and notation for specific triples determined by adjacent $\hat{s}, \hat{t} \in \mathcal{S}_F(X, \omega)$. These latter triples will in fact belong to the former set of permissible triples when the Veech group of $(X, \omega)$ is generated by $G$ (Proposition 4.11).

**Definition 4.5.** For $\Gamma \leq \text{SL}_2 \mathbb{R}$ a fanning group generated by $G \subset \Gamma$, let

$$\mathfrak{P}_G := \{(p, q, u) \in \text{sim}_F(r) \times \text{sim}_F(s) \times \text{sim}_F(t) \mid r, s, t \in \mathcal{G}_G \text{ and } (p, q, u) \text{ is a permissible triple}\}$$

be the set of all permissible triples corresponding to simulations in $\text{Sims}_G$.

For $(X, \omega) \in \mathcal{H}(d_1, \ldots, d_n)$, let

$$\hat{P}(X, \omega) := \{(n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}])) \mid \hat{s}, \hat{t} \text{ are adjacent elements of } \mathcal{S}_F(X, \omega)\},$$

and set

$$\hat{P}_F(X, \omega) := \bigcup_{(n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}])) \in \hat{P}(X, \omega)} \{n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}])\}.$$ 

**Proposition 4.11.** For any fanning group $\Gamma \leq \text{SL}_2 \mathbb{R}$ generated by $G \subset \Gamma$ and any $(X, \omega) \in \mathcal{H}(d_1, \ldots, d_n)$ with Veech group $\text{SL}(X, \omega) = \Gamma$,

$$\hat{P}(X, \omega) \subset \mathfrak{P}_G.$$
Proof. Let \((n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}])) \in \widehat{P}(X, \omega)\) for some \((X, \omega)\) with Veech group \(\Gamma\). By Theorem 4.9 there are \(s, \hat{s}, \hat{t}, t \in \mathcal{S}_G\) for which

\[
 n([[\hat{s}, \hat{t}'])) = \text{sim}(s), \quad n([[\hat{t}', \hat{t}'])) = \text{sim}(\hat{s}), \quad \text{and} \quad n([[\hat{s}, \hat{t}],[\hat{s}, \hat{t}''])) = \text{sim}(\hat{s}, t). \]

Hence

\[
 (n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}])) \in \text{sim}_F(s) \times \text{sim}_F(\hat{s}) \times \text{sim}_F(\hat{s}, t), \]

and we must show that \((n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}]))\) is a permissible triple. The corresponding permissible scalars we shall use are the minimal lengths \((r, r_1, r_{[\hat{s}, \hat{t}]} \in \mathbb{R}_+^3)\) of pairs in the respective \(\text{Aff}_G^+(X, \omega)\)-orbits \([[s, s']], [[\hat{t}, \hat{t}']]\) and \([[\hat{s}, \hat{t}],[\hat{s}, \hat{t}']\}). In particular, \(r \sim n(\hat{s}) = \hat{s}, \quad r_1 \sim n(\hat{t}) = \hat{t} \) and \(r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}]) = [\hat{s}, \hat{t}]. \) Moreover, \(r_{[\hat{s}, \hat{t}]} \sim n(\hat{s}) = \{[s, s'], r_{[\hat{s}, \hat{t}]} \sim n(\hat{s}) = \{[\hat{t}, \hat{t}'], r_{[\hat{s}, \hat{t}]} \sim n(\hat{s}) = \{[\hat{s}, \hat{t}],[\hat{s}, \hat{t}']\}).\)

(i) From the observations above, \([r \sim n(\hat{s}), r_1 \sim n(\hat{t})] = [\hat{s}, \hat{t}]\) and \([r_{[\hat{s}, \hat{t}]} \sim n(\hat{s}), r_{[\hat{s}, \hat{t}]} \sim n(\hat{s})] = [\hat{t}, \hat{s}]. \) Since \(\hat{s}\) and \(\hat{t}\) are adjacent elements of \(\widehat{S}_F(X, \omega), \) Lemma 4.10 guarantees these are defined. Moreover,

\[
 [r \sim n(\hat{s}), r_1 \sim n(\hat{t})] = [\hat{s}, \hat{t}] = r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}])
\]

and

\[
 [r_1 \sim n(\hat{t}), r_{[\hat{s}, \hat{t}]} \sim n(\hat{s})] = [\hat{t}, \hat{s}] = [\hat{t}, \hat{s}] = (r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}]))' = r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}])'.
\]

(ii) We must show that

\[
 B(r \sim n(\hat{s}), r_1 \sim n(\hat{t})) \cup B(r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}]), r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}])') \cup B(r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}])', r_{[\hat{s}, \hat{t}]} \sim n([\hat{s}, \hat{t}])')
\]

does not intersect

\[
 r_{[\hat{s}, \hat{t}]} \sim \text{F}(s) \cup r_{[\hat{s}, \hat{t}]} \sim \text{F}(\hat{s}) \cup r_{[\hat{s}, \hat{t}]} \sim \text{F}(\hat{s}, t),
\]

or, equivalently,

\[
 (B(\hat{s}, \hat{t}) \cup B([\hat{s}, \hat{t}], s') \cup B([\hat{t}', [\hat{s}, \hat{t}']]) \cap \{[[\hat{s}, \hat{t}]], [[\hat{t}', \hat{t}']]] \}_{F} \cup \{[[\hat{s}, \hat{t}],[\hat{s}, \hat{t}']]] \}_{F}) = \emptyset.
\]

Since each of the \(\text{Aff}_G^+(X, \omega)\)-orbits are contained in \(\text{M}(X, \omega)\), it suffices to show

\[
 (B(\hat{s}, \hat{t}) \cup B([\hat{s}, \hat{t}], s') \cup B([\hat{t}', [\hat{s}, \hat{t}']]) \cap \text{M}_F(X, \omega) = \emptyset.
\]

Lemma 4.10 gives that

\[
 B(\hat{s}, \hat{t}) \cap \text{M}_F(X, \omega) = \emptyset.
\]

Now suppose for the sake of contradiction that

\[
 B([\hat{s}, \hat{t}], s') \cap \text{M}_F(X, \omega) \neq \emptyset,
\]

and let \(j\) be the index for which \(s' \in O_j. \) Since \(\text{M}_F(X, \omega)\) has no limit points, there are at most finitely many marked segments belonging to this intersection, and by Proposition 3.8 none of these marked segments have the same argument as \(s'. \) Choose some \(\hat{u}\) in the intersection with argument nearest that of \(s'. \) Then the straight-line segment from \(s'\) to \(\hat{u}\) belongs to stem:\(j(X, \omega), \) and a similar argument to that in the proof of Lemma 4.10 gives that \([s', \hat{u}] \in \text{M}(X, \omega). \) By definition of \([s', \hat{u}], \) this marked segment belongs to the same open \(\pi\)-sector as \(\hat{s}\) and hence to \(B(\hat{s}, \hat{t}), \) contradicting Lemma 4.10 (see Figure 10). A similar argument gives

\[
 B([\hat{t}', [\hat{s}, \hat{t}']]) \cap \text{M}_F(X, \omega) = \emptyset.
\]

Thus \((n(\hat{s}), n(\hat{t}), n([\hat{s}, \hat{t}]))\) is a permissible triple, and \(\widehat{P}(X, \omega) \subset \mathfrak{P}_G. \)

\(\square\)
4.5. Main finiteness result. Throughout this subsection, let $G$ be a set of generators of a fanning group $\Gamma$. As noted at the end of §4.4 our goal is to show that for any finite collection of simulations determined by $G$, there are at most finitely many scalars for which the union of scaled simulations coincides with the marked segments of a unit-area translation surface. Propositions 3.5, 3.6 and 4.11 suggest that for adjacent elements $\hat{s}, \hat{t} \in \widehat{S}_F(X, \omega)$ of a translation surface $(X, \omega)$ with Veech group $\Gamma$, the desired scalars for the simulations corresponding to entries of the same permissible triple. The ‘transitive adjacency’ alluded to above refers to all—loosely speaking—‘transitively adjacent’ to one another (or their orientation-paired inverses), then an arbitrary choice of one scalar uniquely determines all other scalars.

We make these vague notions precise by introducing a simulation graph whose vertices are the simulations corresponding to permissible triples of some subset $P$ of $\Psi_G$, and whose edges are defined between simulations corresponding to entries of the same permissible triple. The ‘transitive adjacency’ alluded to above refers to the connectedness of such a graph.

**Definition 4.6.** For $P \subset \Psi_G$, define the simulation graph determined by $P$ to be the graph $\mathcal{G}_P = (\mathfrak{V}_P, \mathfrak{E}_P)$ with vertex set $\mathfrak{V}_P$ consisting of the simulations $\text{sim}(t), \text{sim}(\hat{s}), \text{sim}(t) \in \mathfrak{V}_G$ for which there is some $(p, q, u) \in P$ with $(p, q, u) \in \text{sim}_F(t) \times \text{sim}_F(\hat{s}) \times \text{sim}_F(t)$, and edge set $\mathfrak{E}_P$, where $\{\text{sim}(t), \text{sim}(\hat{s})\} \in \mathfrak{E}_P$ if there is some $(p, q, u) \in P$ with $(p, q, u) \in \text{sim}_F(t) \times \text{sim}_F(u) \times \text{sim}_F(v)$ and $\{t, \hat{s}\} \subset \{t, u, v\}$.

We call $(r_s)_{\text{sim}(s) \in \mathfrak{V}_P} \in \mathbb{R}^{\left|\mathfrak{V}_P\right|}$ consistent scalars for the graph $\mathcal{G}_P$ if $(r_r, r_s, r_t)$ are permissible scalars for each $(p, q, u) \in P$ with $(p, q, u) \in \text{sim}_F(t) \times \text{sim}_F(\hat{s}) \times \text{sim}_F(t)$. We call the graph $\mathcal{G}_P$ consistent if it admits consistent scalars.

**Example 4.12.** Consider again Example 4.7 and Figure 8; let $p \in \text{sim}(\hat{s}_0)$ be the point with argument $3\pi/2$, $q \in \text{sim}(\hat{s}_1)$ the point with argument $\pi$ and $u \in \text{sim}(\hat{s}_2)$ the point with argument $3\pi/4$. Figure 11 shows that $(p, q, u) \in \Psi_{G_0}$ is a permissible triple with permissible scalars $(1, 1, \sqrt{2})$. Let $P := \{(p, q, u)\}$. Then $\mathcal{G}_P$ is the graph with vertex set $\mathfrak{V}_P = \{\text{sim}(\hat{s}_0), \text{sim}(\hat{s}_1), \text{sim}(\hat{s}_2)\}$ and edge set

$$\mathfrak{E}_P = \{\{\text{sim}(\hat{s}_0), \text{sim}(\hat{s}_1)\}, \{\text{sim}(\hat{s}_1), \text{sim}(\hat{s}_2)\}, \{\text{sim}(\hat{s}_2), \text{sim}(\hat{s}_0)\}\},$$

and $(1, 1, \sqrt{2})$ are consistent scalars for $\mathcal{G}_P$.

**Proposition 4.13.** Let $(X, \omega)$ be a fanning surface with Veech group $SL(X, \omega) = \Gamma$, and let $P := \hat{P}(X, \omega)$. Then the simulation graph $\mathcal{G}_P$ is consistent, and if $(r_s)_{\text{sim}(s) \in \mathfrak{V}_P}$ are consistent scalars for $\mathcal{G}_P$, then any other consistent scalars are of the form $(a r_s)_{\text{sim}(s) \in \mathfrak{V}_P}$ with $a \in \mathbb{R}_+$.

**Proof.** Theorem 4.2 implies that the vertex set $\mathfrak{V}_P$ is finite, say $\mathfrak{V}_P = \{\text{sim}(\hat{s}_i)\}_{i=1}^n$. The same theorem also gives that

$$\bigcup_{i=1}^n r_i \text{sim}(\hat{s}_i) \subset \widehat{\mathcal{M}}(X, \omega)$$

for some $(r_i)_{i=1}^n \subset \mathbb{R}_+$. In fact, each $r_i$ is the minimal length of a pair in the $\text{Aff}_+(X, \omega)$-orbit for which $r_i \text{sim}(\hat{s}_i) = \{[\hat{s}, \hat{s}']\}$. The proof of Proposition 4.11 shows that the scalars $(r_i)_{i=1}^n \subset \mathbb{R}_+$ are consistent, so the graph $\mathcal{G}_P$ is consistent.

Now suppose that $(r'_i)_{i=1}^n \subset \mathbb{R}_+$ are also consistent scalars for $\mathcal{G}_P$, and recall the definition of the edge set $\mathfrak{E}_P$. Induction with Propositions 3.5 and 3.6 implies that any of the scalars $r'_i$ corresponding to a simulation.
of a connected component of $\mathfrak{G}_P$ uniquely determines the remaining scalars $r'_j$ of the simulations of the component. In particular, if $r'_j = ar_j$ for some $a \in \mathbb{R}^+$, then $r'_j = ar_j$ for each of these scalars. Hence it suffices to show that $\mathfrak{G}_P$ is connected.

Consider the subgraph $(V, E)$ of $(\mathfrak{M}_P, \mathfrak{E}_P)$ with vertex set $V$ consisting of all simulations in $\mathfrak{M}_P$ which contain a pair of normalized marked Voronoi staples and edge set $E$ consisting of edges $\{\text{sim}(s_i), \text{sim}(s_j)\}$, where $\text{sim}(s_i)$ and $\text{sim}(s_j)$ contain $\{n(\hat{s}), n(\hat{s'})\}$ and $\{n(\hat{t}), n(\hat{t'}), \}$ respectively, for adjacent $\hat{s}, \hat{t} \in \hat{S}(X, \omega)$. Recall that $(X, \omega)$ is isometric to the quotient space of $[\bigcup_{\kappa=1}^n \Omega]$ under the equivalence relation given by identifying the two edges of convex bodies which correspond to a pair of marked Voronoi staples (§3 of [ESS]). If $(V, E)$ were disconnected, then after this identification of edges of convex bodies the resulting surface—and hence $(X, \omega)$—would be disconnected. This is a contradiction, so the subgraph $(V, E)$ is connected. By definition of $P = \hat{P}(X, \omega)$ and $\mathfrak{M}_P$, any vertex $\text{sim}(s_i) \in \mathfrak{M}_P \setminus V$ must contain $\{n(\hat{s}), n(\hat{s'}), n(\hat{t}), n(\hat{t'})\}$ for some adjacent $\hat{s}, \hat{t} \in \hat{S}(X, \omega)$. By definition of $\mathfrak{E}_P$, there are edges of $\mathfrak{F}_P$ from $\text{sim}(s_i)$ to the simulations containing $\{n(\hat{s}), n(\hat{s'})\}$ and $\{n(\hat{t}), n(\hat{t'})\}$, both of which belong to $V \subset \mathfrak{M}_P$. Thus $\mathfrak{G}_P$ is connected, and the result follows.

Theorem 1.1 is now a corollary of the following:

**Lemma 4.14.** There are at most finitely many unit-area translation surfaces with a given finitely generated fanning Veech group in any given stratum.

**Proof.** Let $G$ be a finite set of generators for a fanning group $\Gamma$, and fix $H_1(d_1, \ldots, d_n)$. We must show that the set of all $(X, \omega) \in H_1(d_1, \ldots, d_n)$ for which $\text{SL}(X, \omega) = \Gamma$ is finite. By Proposition 4.11 each of these translation surfaces belongs to the set

$$\{(X, \omega) \in H_1(d_1, \ldots, d_n) \mid \hat{P}(X, \omega) \subset \mathfrak{P}_G\},$$

so it suffices to show that this latter set is finite. By Lemmas 3.7 and 4.8 the set $\mathfrak{P}_G$ of all permissible triples is finite, so it suffices to show for any $P \subset \mathfrak{P}_G$ that the set

$$\{(X, \omega) \in H_1(d_1, \ldots, d_n) \mid \hat{P}(X, \omega) = P\}$$

is finite. Let $\mathfrak{M}_P = \{\text{sim}(s_i)\}_{i=1}^n$ be the vertex set of the simulation graph $\mathfrak{G}_P$. As in the proof of Proposition 4.13 for each $(X, \omega)$ with $\hat{P}(X, \omega) = P$, there are consistent scalars $(r_i)_{i=1}^n \in \mathbb{R}_{\geq 0}$ for which

$$\bigcup_{i=1}^n r_i \text{sim}(s_i) \subset \hat{M}(X, \omega).$$

By definition of $\mathfrak{G}_P$ and its vertex set $\mathfrak{M}_P$, we have that the set of marked Voronoi staples, $\hat{S}(X, \omega)$—from which $(X, \omega)$ may be recovered—is contained in this union of scaled simulations. Again from Proposition...
4.13 if \((Y, \eta) \in \mathcal{H}_1(d_1, \ldots, d_n)\) also satisfies \(\hat{P}(Y, \eta) = P\), then its corresponding consistent scalars are of the form \((ar_i)_{i=1}^n\) for some \(a \in \mathbb{R}_+^n\). From \(\S 3.3\) we have

\[
\bigcup_{j=1}^\kappa \Omega_j(\hat{S}(X, \omega)) = \bigcup_{j=1}^\kappa \Omega_j \left( \bigcup_{i=1}^n r_i \text{sim}(s_i) \right)
\]

and

\[
\bigcup_{j=1}^\kappa \Omega_j(\hat{S}(Y, \eta)) = \bigcup_{j=1}^\kappa \Omega_j \left( \bigcup_{i=1}^n ar_i \text{sim}(s_i) \right) = a \left( \bigcup_{j=1}^\kappa \Omega_j \left( \bigcup_{i=1}^n r_i \text{sim}(s_i) \right) \right).
\]

Since \((X, \omega)\) and \((Y, \eta)\) are isometric to the quotient spaces of these respective sets under an equivalence relation identifying edges, and since both translation surfaces have area one, we conclude that \(a = 1\) and thus \((X, \omega) = (Y, \eta)\).

\[\square\]

**Proof of Theorem 1.1.** This follows immediately from Corollary 4.3 and Lemma 4.14. \[\square\]

5. CONSTRUCTING LATTICE SURFACES WITH GIVEN VEECH GROUPS: AN ALGORITHM

The results of the previous sections naturally suggest an algorithm to construct all unit-area translation surfaces with a given lattice Veech group \(\Gamma\) in any given stratum \(\mathcal{H}_1(d_1, \ldots, d_n)\). We highlight the main steps of this construction here, proving Theorem 1.2.

**Algorithm 5.1.** *Input:* Non-negative integers \(d_1 \leq \cdots \leq d_n\) whose sum is even and a finite set of generators \(G\) of a lattice group \(\Gamma \leq \text{SL}_2 \mathbb{R}\).

*Output:* All \((X, \omega) \in \mathcal{H}_1(d_1, \ldots, d_n)\) with Veech group \(\text{SL}(X, \omega) = \Gamma\).

1. **Compute a finite set** \(\Theta \subset S^1\) containing \(\Theta_{\Gamma}\), and compute \(\Theta_G\) (replacing \(\Theta_{\Gamma}\) with \(\Theta\) in Definition 4.3).

2. **For each** \(n \in \mathbb{N}\):
   (a) **Compute increasing, finite subsets** \(\text{sim}^n(a)\) of each of the simulations \(\text{sim}(a) \in \text{Sim}_G\), and set \(\text{sim}_G^n(a) := (\text{sim}(a))_{\Gamma}\).
   (b) **Let** \(P_G^n\) be the elements \((p, q, u)\) of

   \[
   \bigcup_{r, s, t \in \Theta_G} \text{sim}_G^n(r) \times \text{sim}_G^n(s) \times \text{sim}_G^n(t)
   \]

   satisfying condition (i) of Definition 3.3 for some \((r, s, t) \in \mathbb{R}_{\times}^3\); call the elements \((p, q, u)\) possible permissible triples and the scalars \((r, s, t)\) possible permissible scalars. (Note that for \(n\) large enough, the set of permissible triples \(P_G^n\) belongs to \(\mathcal{P}_G\).)

3. **For each** \(P \subset P_G^n\):
   (i) **Construct the graph** \(\Theta_G^n = (\mathcal{P}_G^n, \mathcal{E}_G^n)\) analogous to the simulation graph in Definition 4.6 (replacing \(\mathcal{P}_G\) with \(\mathcal{P}_G^n\), each \(\text{sim}(a)\) with \(\text{sim}^n(a)\), and ‘permissible scalars’ with ‘possible permissible scalars’ in the definition). If \(\Theta_G^n\) is consistent with consistent scalars \((r_s)_{\text{sim}^n(a) \in \mathcal{P}_G^n}\), then construct the convex bodies

   \[
   \Theta_G^n \left( \bigcup_{\text{sim}^n(a) \in \mathcal{P}_G^n} r_s \text{sim}^n(a) \right)
   \]

   and determine the corresponding essential points

   \[
   \mathcal{E}_G^n \left( \bigcup_{\text{sim}^n(a) \in \mathcal{P}_G^n} r_s \text{sim}^n(a) \right)
   \]

   for each \(i \in \{1, \ldots, \kappa\}\).
   (ii) **If** the essential points come in pairs \(\{p, p^\circ\}\) determined by the simulations \(\text{sim}^n(a) \in \mathcal{P}_G^n\), and the edges of the convex bodies determined by \(p\) and \(p^\circ\) are equal length, then construct the translation surface \((X, \omega)\) by identifying these edges.
   (iii) **Rescale** \((X, \omega)\) if necessary so that it has area one, and verify that \(\text{SL}(X, \omega) = \Gamma\).
While Algorithm 5.1 does produce the finite set of unit-area lattice surfaces asserted by Theorem 1.1 in finite time, it does not give a halting criterion to determine when the set of returned translation surfaces is exhaustive. An explicit test to determine when the reduction has stopped—say when the set of returned translation surfaces is finite—does not give a halting criterion to determine when the set of returned translation surfaces is finite. However, Algorithm 5.1—if run indefinitely—is easily adapted to As the set of all strata is countable, Algorithm 5.1—if allowed to run indefinitely—is easily adapted to include the set of all strata. Here we give an example showing how the ideas of the previous sections may be used—in certain cases—to give obstructions for lattices being realized as Veech groups in strata. In particular, we show that the square torus is the only translation surface with Veech group $\text{SL}_2\mathbb{Z}$ in the collection of all minimal strata $\mathcal{H}(2g-2)$, $g > 0$. Throughout this section, fix $g > 0$ and $(X, \omega) \in \mathcal{H}(2g-2)$. Let $\sigma := \sigma_1$ be the sole singularity of $(X, \omega)$, $\Omega := \Omega_1$ the convex body subordinate to $\hat{\mathcal{M}}(X, \omega)$ and $O = O(2g-2)$ the canonical surface associated to $\mathcal{H}(2g-2)$. We begin with two lemmas.

**Example 5.2.** Examples 4.7 and 4.12 illustrate (parts of) Steps 1, 2.a, 2.b, and 2.c.i of the algorithm. Figure 12 illustrates the construction of convex bodies and corresponding essential points in Step 2.c.i as well as the resulting translation surface $(X, \omega)$ constructed in Step 2.c.ii. Notice that the essential points come only from $\text{sim}(s_0)$ and $\text{sim}(s_1)$; however, $\text{sim}(s_2)$ was necessary to determine the permissible scalars $(1, 1, \sqrt{2})$ (Example 4.13). Rescaling to unit-area, one verifies (using one of the algorithms mentioned in 4.10) that indeed $\text{SL}(X, \omega)$ is $\langle S, T^2 \rangle$.

A number of improvements to Algorithm 5.1 can be made for computational efficiency. We mention three in particular:

- It is optimal to choose a minimal set of generators $G$ for $\Gamma$, as the size of $\mathcal{S}_G$—and hence of $\text{Sims}_G$—depends on $|G|$.
- Non-distinctive simulations $\text{sim}(s) \in \text{Sim}_G$ cannot possibly be a normalized $\text{Aff}_G^+(X, \omega)$-orbit for any translation surface $(X, \omega)$ (Proposition 3.8). Hence if $\text{sim}^n(s)$ is not distinctive for some $n \in \mathbb{N}$, then $s$ may be removed from the set $\mathcal{S}_G$.
- In Step 2.b, one need only consider $(p, q, u) \in \text{sim}_p^r(\tau) \times \text{sim}_q^r(s) \times \text{sim}_u^r(t)$ where $\tau$, $s$ and $t$ are defined using the same elements of $\text{Trans}(\mathcal{O})$, as these elements determine $\text{Aff}_G^+(\tau)$, $\text{Aff}_G^+(s)$ and $\text{Aff}_G^+(t)$, each of which equals $\text{Aff}_G^+(X, \omega)$ (see the proof of Theorem 4.9). A similar restriction applies to the subsets $P \subset \mathcal{P}_G$ considered in Step 2.c.

As the set of all strata is countable, Algorithm 5.1—if allowed to run indefinitely—is easily adapted to produce all unit-area translation surfaces with a given lattice Veech group in any stratum. Similarly, as the set of generators for any fanning group is at most countable, the algorithm—if run indefinitely—may be adapted to return all unit-area translation surfaces with a given fanning Veech group in any stratum.

**6. The Modular Group in Minimal Strata**

Here we give an example showing how the ideas of the previous sections may be used—in certain cases—to give obstructions for lattices being realized as Veech groups in strata. In particular, we show that the square torus is the only translation surface with Veech group $\text{SL}_2\mathbb{Z}$ in the collection of all minimal strata $\mathcal{H}(2g-2)$, $g > 0$. Throughout this section, fix $g > 0$ and $(X, \omega) \in \mathcal{H}(2g-2)$. Let $\sigma := \sigma_1$ be the sole singularity of $(X, \omega)$, $\Omega := \Omega_1$ the convex body subordinate to $\hat{\mathcal{M}}(X, \omega)$ and $O = O(2g-2)$ the canonical surface associated to $\mathcal{H}(2g-2)$. We begin with two lemmas.
Lemma 6.1. Let \( \hat{S}(X, \omega) = \{ \{ \hat{s}_i, \hat{s}'_i \} \}_{i=1}^{n} \) be the marked Voronoi staples of \((X, \omega)\). If the counterclockwise angle measured from \(\hat{s}_i\) to \(\hat{s}'_i\) equals that measured from \(\hat{s}'_i\) to \(\hat{s}_i\) for each \(i\), then \(n = 2g\) for \(n\) even and \(n = 2g + 1\) for \(n\) odd.

Proof. Since \(\hat{S}_F(X, \omega)\) is the set of all essential points of \(\Omega\), the convex body \(\Omega\) has \(2n\) edges. Decompose \(\Omega\) into \(2n\) closed triangular regions with disjoint interiors, where each triangle has one vertex at the origin \(0 \in \mathcal{O}\), and remaining two vertices at the endpoints of an edge of \(\Omega\). The sum of the interior angles of all triangles in this decomposition is \(2n\pi\). The origin \(0 \in \mathcal{O}\) is a singularity of cone angle \(2(2g-1)\pi\), so the sum of the interior angles of \(\Omega\) is

\[
2n\pi - 2(2g-1)\pi = 2(n - 2g + 1)\pi.
\]

Recall that \((X, \omega)\) is isometric to the quotient space of \(\Omega\) under the equivalence relation given by identifying edges determined by the marked Voronoi staples \(\hat{S}(X, \omega)\), and the vertices of \(\Omega\) under this identification correspond to Voronoi 0-cells of \((X, \omega)\) \((\text{3.3})\). In particular, the sum of interior angles of \(\Omega\) which are identified as a single point must equal \(2\pi\). Our assumptions on the counterclockwise angles between \(\hat{s}_i\) and \(\hat{s}'_i\) and between \(\hat{s}'_i\) and \(\hat{s}_i\) imply that when \(n\) is even, the vertices of \(\Omega\) become a single point under the identification, while when \(n\) is odd, the vertices of \(\Omega\) become two distinct points under the identification (the argument here is analogous to that which shows the translation surface constructed by identifying opposite edges of a regular \(2n\)-gon has one singularity when \(n\) is even and two singularities when \(n\) is odd). In particular, for \(n\) even,

\[
2(n - 2g + 1)\pi = 2\pi
\]

implies \(n = 2g\), while for \(n\) odd,

\[
2(n - 2g + 1)\pi = 4\pi
\]

implies \(n = 2g + 1\).

Lemma 6.2. Suppose that the minimal length of a saddle connection on \((X, \omega)\) is one. If the set

\[
P := \{ p \in \mathcal{O} \mid |p| = 1, \ \text{arg}(\overline{p}) \in \{0, \pi/2, \pi, 3\pi/2\} \}
\]

belongs to the set of marked segments \(\hat{M}_F(X, \omega)\), then in fact \(P = \hat{S}_F(X, \omega)\).

Proof. It suffices to show that \(P\) is the set of essential points of the convex body \(\Omega\), i.e. that (i) \(\Omega = \cap_{p \in P} H(p)\) and (ii) for any \(q \in P\), \(\Omega \subseteq \cap_{p \in P \\setminus \{q\}} H(p)\). Condition (ii) is immediate from the fact that \(\Omega\) is compact while \(\cap_{p \in P \\setminus \{q\}} H(p)\) is non-compact for any \(q \in P\) Now

\[
\Omega = \bigcap_{\hat{s} \in \hat{M}_F(X, \omega)} H(\hat{s}) = \left( \bigcap_{p \in P} H(p) \right) \cap \left( \bigcap_{\hat{s} \in \hat{M}_F(X, \omega) \setminus P} H(\hat{s}) \right),
\]

so it suffices to show for each marked segment \(\hat{s} \in \hat{M}_F(X, \omega) \setminus P\), that \(\cap_{p \in P} H(p) \subset H(\hat{s})\).

We claim that for each \(p \in P\), the ball centered at \(p\) of radius one contains no marked segments other than \(p\), i.e.

\[
B_{1}(p) \cap \hat{M}_F(X, \omega) \setminus \{p\} = \emptyset.
\]

Suppose on the contrary that the intersection is non-empty. As in the proof of Proposition 4.11 we can choose some \(\bar{u}\) in the intersection with argument nearest that of \(p\) and find that \([p, \bar{u}]\) is a marked segment of length strictly less than one. Its corresponding saddle connection also has length strictly less than one, contrary to our assumptions, so the claim holds.

Next, we claim that the ball centered at the origin of radius \(\sqrt{2}\) is contained in the union of balls of radius one centered at the various \(p \in P\),

\[
B_{\sqrt{2}}(0) \subset \bigcup_{p \in P} B_{1}(p).
\]

This is clear from elementary Euclidean geometry, restricting to each \(2\pi\)-sector \(c_j^1\); see Figure 13. In particular, the previous two claims imply that

\[
B_{\sqrt{2}}(0) \cap \left( \hat{M}_F(X, \omega) \setminus P \right) = \emptyset.
\]
Figure 13. The ball $B_{\sqrt{2}}(0)$ is contained in the union of balls $B_1(p)$, $p \in P$. Note that this figure shows only a portion of the ball $B_{\sqrt{2}}(0)$, as this ball is centered at the singularity $0 \in O$. Similarly, only the upper half of $B_1(p_0)$ is shown.

Notice that the maximum distance from the origin to any point in $\bigcap_{p \in P} H(p)$ is $\sqrt{2}/2$ (this distance being realized at the vertices of this intersection). Let $\hat{s} \in \hat{M}(X, \omega) \setminus P$ and let $q$ belong to the complement of $H(\hat{s})$. Certainly $|q| > |\hat{s}|/2$. Equation 4 implies $|\hat{s}| \geq \sqrt{2}$, so $|q| > \sqrt{2}/2$. Hence $q \notin \bigcap_{p \in P} H(p)$, so the complement of $H(\hat{s})$ does not intersect $\bigcap_{p \in P} H(p)$, as desired.

With these two lemmas, we now prove the main result of this section.

Proof of Theorem 1.3. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, fix $\tau_S, \tau_T \in \text{Trans}(O)$, and set $g_S = \tau_S \circ f_S$ and $g_T = \tau_T \circ f_T$ where $f_S, f_T \in \text{Aff}(O)$. We first show that all elements of $P := \{p \in O \mid |p| = 1, \arg(p) \in \{0, \pi/2, \pi, 3\pi/2\}\}$ belong to the same $\langle g_S, g_T \rangle$-orbit. Note that $\text{Trans}(O) = \langle \rho \rangle$; let $n \in \mathbb{Z}$ be such that $\tau_T = \rho^n$. Set $P' := \{p \in O \mid |p| = \sqrt{2}, \arg(p) \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}\}$.

Beginning with $p_0 \in P \cap c_0$ and $\arg(p_0) = 0$ and sweeping counterclockwise, label the elements of $P \cup P'$ as $p_0, p_1, \ldots, p_{16g-9}$; see Figure 14. For each $k \in \mathbb{Z}_{16g-8}$, we have

$$f_S(p_k) = p_{k+2},$$

and for $k \not\equiv 1 \pmod{4}$,

$$f_T(p_k) = \begin{cases} p_k & \text{if } k \equiv 0 \pmod{4} \\ p_{k-1} & \text{if } k \equiv 2, 3 \pmod{4}. \end{cases}$$

Moreover, for each $k \in \mathbb{Z}_{16g-8}$,

$$\tau_T(p_k) = p_{k+8n}.$$

Figure 14. Labelling of the points $p_0, \ldots, p_{16g-9}$ of $P \cup P'$ for $j \in \mathbb{Z}_{2g-1}$. 25
Set \( f := g_T^{-1} \circ g_S^{-1} \circ g_T^{-1} \circ g_S \) and \( g := g_S^{-1} \circ g_T^{-1} \circ g_S \circ g_T^{-1} \). Using the relations above—-together with Lemma \( \S \) and the fact that \( \text{Trans}(O) = \langle p_l \rangle \) is abelian—we find for each \( k \in \mathbb{Z}_{16g-8} \) that

\[
\begin{align*}
    f(p_{4k}) &= g_T^{-1} \circ g_S^{-1} \circ g_T^{-1} \circ g_S(p_{4k}) \\
   &= (f_T^{-1} \circ \tau_T^{-1}) \circ (f_S^{-1} \circ \tau_S^{-1}) \circ (f_T^{-1} \circ \tau_T^{-1}) \circ (\tau_S \circ f_S)(p_{4k}) \\
   &= \tau_T^{-2} \circ f_T^{-1} \circ f_S^{-1} \circ f_T^{-1} \circ f_S(p_{4k+2}) \\
   &= \tau_T^{-2} \circ f_T^{-1} \circ f_T^{-1}(p_{4k+3}) \\
   &= \tau_T^{-2}(p_{4k+1}) \\
end{align*}
\]

and

\[
\begin{align*}
g(p_{4k+2}) &= g_S^{-1} \circ g_T^{-1} \circ g_S \circ g_T^{-1}(p_{4k+2}) \\
   &= (f_S^{-1} \circ \tau_S^{-1}) \circ (f_T^{-1} \circ \tau_T^{-1}) \circ (\tau_S \circ f_S)(f_T^{-1} \circ \tau_T^{-1})(p_{4k+2}) \\
   &= \tau_T^{-2} \circ f_T^{-1} \circ f_T^{-1} \circ f_S \circ f_T^{-1}(p_{4k+2}) \\
   &= \tau_T^{-2} \circ f_T^{-1} \circ f_T^{-1} \circ f_S(p_{4k+3}) \\
   &= \tau_T^{-2} \circ f_T^{-1} \circ f_T^{-1}(p_{4k+5}) \\
   &= \tau_T^{-2} \circ f_T^{-1}(p_{4k+6}) \\
   &= \tau_T^{-2}(p_{4k+4}) \\
   &= p_{4k+16n+4} \\
\end{align*}
\]

Then

\[
\begin{align*}
g_T^8 \circ g \circ f \circ g \circ f(p_0) &= g_T^8 \circ g \circ f(p_{-16n+2}) \\
   &= g_T^8 \circ g \circ f(p_{-32n+4}) \\
   &= g_T^8 \circ g(p_{-48n+6}) \\
   &= g_T^8(p_{-64n+8}) \\
   &= \tau_T^2 \circ f_T^6(p_{-64n+8}) \\
   &= \tau_T^8(p_{-64n+8}) \\
   &= ps. \\
\end{align*}
\]

Iterating this latter composition shows that each \( p_{6k} \) belongs to the \( \langle g_S, g_T \rangle \)-orbit. Injectivity of \( g_S \) implies that the union of the images under \( g_S, g_T^2 \) and \( g_T^3 \) of these \( p_{6k} \) is all of \( P \), as claimed.

Now suppose \( (X, \omega) \in \mathcal{H}(2g-2) \) with \( \text{SL}(X, \omega) = \text{SL}_2 \mathbb{Z} = \langle S, T \rangle \) for some \( g > 0 \). Normalize \( (X, \omega) \) if necessary so that its shortest saddle connection has unit length. One computes

\[
S^1 \setminus (T \cdot \Delta \cup T^{-1} \cdot \Delta \cup ST \cdot \Delta \cup ST^{-1} \cdot \Delta) = \{(\pm 1, 0), (0, \pm 1)\}.
\]

This set contains \( \Theta_{\text{SL}_2 \mathbb{Z}} = \Theta_{\text{SL}(X, \omega)} \), so Corollary \( \S \) implies that the shortest saddle connection of \( (X, \omega) \) is horizontal or vertical. Since the orthogonal matrix \( S \) belongs to \( \text{SL}(X, \omega) \) and any affine automorphism of \( (X, \omega) \) sends saddle connections to saddle connections, \( (X, \omega) \) has both horizontal and vertical saddle connections of minimal length one.

Let \( f \in \text{Aff}^+(O) \) and \( \{p, p^r\} \in \hat{P}(O) \). Lemma \( \S \) together with the fact that \( f \) is a self-homeomorphism of \( O = \mathcal{O}(2g-2) \) imply that the counterclockwise angle measured from \( p \) to \( p^r \) equals that measured from \( f(p) \) to \( f(p^r) \). In particular, this is true of any \( f \in \text{Aff}^+(X, \omega) \) and horizontal, unit-length \( \{s, \hat{s}\} \in \hat{M}(X, \omega) \). Since \( S, T \in \text{SL}(X, \omega) \), there exist \( g_S, g_T \in \text{Aff}^+(X, \omega) \) with \( \text{der}(g_S) = S, \text{der}(g_T) = T \). The beginning of this proof shows that the set \( P \) belongs to the \( \langle g_S, g_T \rangle \)-orbit of \( \hat{s} \). Since this orbit belongs to \( \hat{M} \),
Lemma 6.2 implies that $P = \hat{S}_F(X, \omega)$. Moreover, $\hat{s}'$ belongs to this orbit, so the observation above implies that the counterclockwise angle measured from $\hat{s}$ to $\hat{s}'$ equals that measured from $\hat{s}'$ to $\hat{s}$, and the same is true of any pair of marked Voronoi staples in $\hat{S}(X, \omega)$. Since

$$|\hat{S}(X, \omega)| = |\hat{S}_F(X, \omega)|/2 = |P|/2 = 2(2g - 1)$$

is even, Lemma 6.1 implies that $2(2g - 1) = 2g$, which is only true for $g = 1$. When $g = 1$, the convex body $\Omega$ is the unit-square centered at $0 \in \mathcal{O} = (\mathbb{C}, dz)$, and the reconstruction of $(X, \omega)$ from $\Omega$ gives the square torus. □

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