Nowhere differentiable intrinsic Lipschitz graphs

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Abstract

We construct intrinsic Lipschitz graphs in Carnot groups with the property that, at every point, there exist infinitely many different blow-up limits, none of which is a homogeneous subgroup. This provides counterexamples to a Rademacher theorem for intrinsic Lipschitz graphs.

The notion of Lipschitz submanifolds in sub-Riemannian geometry was introduced, at least in the setting of Carnot groups, by Franchi, Serapioni and Serra Cassano in a series of seminal papers \[5–7\] through the theory of intrinsic Lipschitz graphs. One of the main open questions concerns the differentiability properties for such graphs: in this paper, we provide examples of intrinsic Lipschitz graphs of codimension 2 (or higher) that are nowhere differentiable, that is, that admit no homogeneous tangent subgroup at any point.

Recall that a Carnot group \(G\) is a connected, simply connected and nilpotent Lie group whose Lie algebra is stratified, that is, it can be decomposed as the direct sum \(\oplus_{j=1}^{s} V_j\) of subspaces such that

\[V_{j+1} = [V_1, V_j]\] for every \(j = 1, \ldots, s - 1\), \([V_1, V_s] = \{0\}\), \(V_s \neq \{0\}\).

We shall identify the group \(G\) with its Lie algebra via the exponential map \(\exp: \oplus_{j=1}^{s} V_j \to G\), which is a diffeomorphism. In this way, for \(\lambda > 0\), one can introduce the homogeneous dilations \(\delta_\lambda: G \to G\) as the group automorphisms defined by \(\delta_\lambda(p) = \lambda^j p\) for every \(p \in V_j\). A subgroup of \(G\) is said to be homogeneous if it is dilation-invariant. Assume that a splitting \(G = \mathbb{W}V\) of \(G\) as the product of homogeneous and complementary (that is, such that \(\mathbb{W} \cap V = \{0\}\)) subgroups is fixed; we say that a function \(\phi: \mathbb{W} \to V\) intrinsic Lipschitz if there is an open nonempty cone \(U\) such that \(V \setminus \{0\} \subset U\) and

\[pU \cap \Gamma_\phi = \emptyset\] for all \(p \in \Gamma_\phi\),

where \(\Gamma_\phi = \{w\phi(w) : w \in \mathbb{W}\}\) is the intrinsic graph of \(\phi\). We say that a set \(\Sigma \subset G\) is a blow-up of \(\Gamma_\phi\) at \(\hat{p} = \hat{w}\phi(\hat{w})\) if there exists a sequence \((\lambda_n)_n\) such that \(\lambda_n \to +\infty\) and the limit

\[\lim_{n \to \infty} \delta_{\lambda_n}(\hat{p}^{-1}\Gamma_\phi) = \Sigma\]

holds with respect to the local Hausdorff convergence. It is worth recalling that, if \(\phi\) is intrinsic Lipschitz, then every blow-up is automatically the intrinsic Lipschitz graph of a map \(\mathbb{W} \to V\). Eventually, we say that \(\phi\) is intrinsically differentiable at \(\hat{w} \in \mathbb{W}\) if the blow-up of \(\Gamma_\phi\) at \(\hat{p} = \hat{w}\phi(\hat{w})\) is unique and it is a homogeneous subgroup of \(G\). See \[8\] for details.

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We say that a group $G$ along with a splitting $\mathbb{W}$ satisfies an intrinsic Rademacher Theorem if all intrinsic Lipschitz maps $\phi: \mathbb{W} \to \mathbb{V}$ are intrinsically differentiable almost everywhere (that is, for almost all points of $\mathbb{W}$ equipped with its Haar measure). It was proved in [6] that this is the case when $\mathbb{V} \simeq \mathbb{R}$ and $G$ is of step two; other partial results for graphs with codimension 1 ($\mathbb{V} \simeq \mathbb{R}$) are contained in [4, 9]. If $\mathbb{V}$ is a normal subgroup, the Rademacher Theorem has been proved for general $G$ by Antonelli and Merlo in [2]. Recently, the third-named author [12] proved that Heisenberg groups (with any splitting) satisfy an intrinsic Rademacher Theorem. The question has been open for a long time if $G$ is the Engel group (which has step 3) and $\mathbb{V} \simeq \mathbb{R}$ (see [1]). In this paper, we prove a result in the negative direction: namely, we provide examples of intrinsic Lipschitz graphs that are nowhere intrinsically differentiable. Let us state our main result:

**Theorem 1.** Let $G$ be a Carnot group with stratification $\bigoplus_{j=1}^{s} V_j$. Let $\mathbb{W}$ be a splitting of $G$ such that $\mathbb{W} \cap V_2 \not\subset [\mathbb{W}, \mathbb{W}]$ and there exists $v_0 \in \mathbb{V} \cap V_1$ such that $v_0 \neq 0$ and $[v_0, \mathbb{W}] = 0$. Then there is an intrinsic Lipschitz function $\phi: \mathbb{W} \to \mathbb{V}$ that is nowhere intrinsically differentiable.

Moreover, $\phi$ can be constructed in such a way that, for every $p \in \Gamma_\phi$, the following properties hold.

(a) There exist infinitely many different blow-ups of $\Gamma_\phi$ at $p$.

(b) No blow-up of $\Gamma_\phi$ at $p$ is a homogeneous subgroup.

The proof of Theorem 1 is postponed in order to first provide some comments.

**Remark 1.** The simplest example of a Carnot group where Theorem 1 applies is $G = \mathbb{H} \times \mathbb{R}$, where $\mathbb{H}$ is the first Heisenberg group. As customary, we consider generators $X, Y, T$ of the Lie algebra of $\mathbb{H}$ such that $[X, Y] = T$, $[X, T] = [Y, T] = 0$ and fix the exponential coordinates $(x, y, t) = \exp(x X + y Y + t T)$. Using coordinates $(x, y, t, r)$ on $\mathbb{H} \times \mathbb{R}$ with $r \in \mathbb{R}$, we can consider the splitting $\mathbb{H} \times \mathbb{R} = \mathbb{W} \mathbb{V}$ given by the vertical subgroup $\mathbb{W} = \{x = r = 0\}$ of $\mathbb{H}$ and the horizontal Abelian subgroup $\mathbb{V} = \{y = t = 0\}$. Then $V_2 \cap \mathbb{W} \not\subset [\mathbb{W}, \mathbb{W}] = \{0\}$ and $v_0 = (0, 0, 0, 1)$ commutes with $\mathbb{W}$. Hence, this splitting of $\mathbb{H} \times \mathbb{R}$ satisfies the conditions of Theorem 1 and it does not satisfy an intrinsic Rademacher Theorem.

It is worth observing that, in this setting, the map $\phi: \mathbb{W} \to \mathbb{V}$ provided in the proof of Theorem 1 takes the form $\phi(y, t) = (0, u(t))$, where $u$ is the $\frac{1}{2}$-Hölder continuous function constructed in the Appendix. In particular, the intrinsic graph $\Gamma_\phi$ is the set $\{(0, y, t, u(t)) : y, t \in \mathbb{R}\}$ and it is contained in the Abelian subgroup $\mathbb{W} \times \mathbb{R}$. One of the properties of $u$ is that the limit

$$\lim_{s \to t} \frac{|u(t) - u(s)|}{|t - s|}$$

does not exist at any $t \in \mathbb{R}$ and this is the ultimate reason for the nondifferentiability of $\phi$.

Similar counterexamples can be constructed in any codimension $k \geq 2$: in fact one can consider $\mathbb{H}^{k-1} \times \mathbb{R} = (\mathbb{R}_{x_{k-1}} \times \mathbb{R}_{y_{k-1}} \times \mathbb{R}_t) \times \mathbb{R}_r$ with splitting $\mathbb{W}$ defined by $\mathbb{W} = \{x = 0, r = 0\}$, $\mathbb{V} = \{y = 0, t = 0\}$. It can be easily checked that the map $\phi(y, t) = (0, u(t))$ defines an intrinsic Lipschitz graph of codimension $k$ for which the properties (a) and (b) in Theorem 1 hold at every point.

**Remark 2.** The measure $\mu = \mathcal{H}^d \res \Gamma_\phi$, where $d$ is the Hausdorff dimension of $\mathbb{W}$ and $\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure, does not have a unique tangent measure at any point. Indeed, first, any tangent measure of $\mu$ is supported on a blow-up of $\Gamma_\phi$. Second, by [7, Theorem 3.9], $\mu$ and all its dilations are uniformly $d$-Ahlfors regular, and thus any tangent measure of $\mu$ is
$d$-Ahlfors regular. We then conclude that if $\mu_1$ and $\mu_2$ are two tangent measures of $\mu$ supported on different blow-ups of $\Gamma_\phi$, then they are two distinct measures. Since blow-ups of $\Gamma_\phi$ are not unique, so are tangent measures. Observe also that no tangent measure can be flat, that is, supported on a homogeneous subgroup. In particular, $\Gamma_\phi$ is purely $C^1_H$-unrectifiable, that is, $\calH^d(\Gamma_\phi \setminus \Sigma) = 0$ for every submanifold $\Sigma$ of class $C^1_H$ (see, for example, [3, § 2.5 and 6.1]).

**Remark 3.** If $\calW$ is a homogeneous subgroup of $\calG$ with codimension 1, then the conditions of Theorem 1 cannot be met because $\bigoplus_{j=2}^3 V_j = [\calW, \calW] + [\calW, V]$. Actually, intrinsic Lipschitz graphs of codimension 1 are boundaries of sets with finite perimeter in $\calG$ (see, for example, [11, Theorem 1.2]), hence at almost every point they possess at least one blow-up which is a homogeneous subgroup of codimension 1, see [1]. Therefore, any possible counterexample to the Rademacher Theorem in codimension 1 cannot be as striking as the one provided by Theorem 1, in the sense that property (b) cannot hold on a set with positive measure.

**Remark 4.** Following the same proof strategy, one can extend Theorem 1 to the case $\calW \cap V_j \not\subseteq [\calW, \calW]$ for some $j > 2$ and $v_0 \in V_j \setminus \{0\}$ with $k < j$ and $[v_0, \calW] = 0$, by taking a $k/j$-Hölder analogue of the function $u$ constructed in the appendix.

**Proof of Theorem 1.** Let $\beta : \calW \to \mathbb{R}$ be a nonzero linear function such that $\calW \cap V_j \subseteq \ker \beta$ whenever $j \neq 2$ and $[\calW, \calW] \subseteq \ker \beta$; such a $\beta$ exists\(^1\) because $\calW \cap V_2 \not\subseteq [\calW, \calW]$. Note that such a function $\beta$ is in fact a group morphism $\calW \to \mathbb{R}$.

Consider a $1/2$-Hölder continuous function $u : \mathbb{R} \to \mathbb{R}$ with the following properties. First, the difference quotients

$$\Delta(s, t) = \frac{u(s) - u(t)}{\sgn(s - t)|s - t|^{1/2}}$$

are bounded, namely,

$$|\Delta(s, t)| \leq 1 \quad \text{for every } s, t \in \mathbb{R}. \quad (1)$$

Second, there exist $c_1 > 0$ and $c_2 > 0$ such that, for every $t_0 \in \mathbb{R}$ and $\delta \in (0, 1]$, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$\sgn(s_1 - t_0) = \sgn(s_2 - t_0)$$
$$c_1 \delta \leq |s_1 - t_0| \leq \delta$$
$$c_1 \delta \leq |s_2 - t_0| \leq \delta$$
$$|\Delta(s_1, t_0) - \Delta(s_2, t_0)| \geq c_2. \quad (2)$$

Such a function exists, as we show in the Appendix.

We can then define $\phi : \calW \to \calV$ as

$$\phi(w) = u(\beta(w))v_0.$$ 

Note that the condition $[v_0, \calW] = 0$ implies

$$vw = uvw \quad \text{for all } w \in \calW, v \in \mathbb{R}v_0. \quad (3)$$

Therefore, by the Baker–Campbell–Hausdorff formula, the intrinsic graph of $\phi$ is the set of points $w\phi(w) = w + u(\beta(w))v_0$ for $w \in \calW$.

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\(^1\)For instance, one can consider $\beta(x) = \langle x, w_0 \rangle$ for some $w_0 \in (\calW \cap V_2) \setminus [\calW, \calW]$ and a scalar product on $\calW$ adapted to the grading $\bigoplus_{j=1}^3 \calW \cap V_j$ of $\calW$. 
Claim 1. The map $\hat{p}$ is intrinsic Lipschitz.

Fix a homogeneous norm $\| \cdot \|$ on $G$. Note that, since $\beta(\delta_\lambda x) = \lambda^2 \beta(x)$ for all $x \in \mathbb{W}$, there is a constant $C$ such that $|\beta(x)| \leq C\|x\|^2$, for all $x \in \mathbb{W}$. We check that $\Gamma_\phi$ has the cone property for the cone (see [7, Definition 10])

$$U = \{ wv : w \in \mathbb{W}, \ v \in V, \|v\| > 2\sqrt{C}\|v_0\|\|w\| \}.$$ 

Given $\tilde{w}, w \in \mathbb{W}$, by (3) we have $(\tilde{w}\phi(\tilde{w}))^{-1}(w\phi(w)) = (\tilde{w}^{-1}w)(\phi(\tilde{w})^{-1}\phi(w))$ and

$$\|\phi(\tilde{w})^{-1}\phi(w)\| = \|u(\beta(w)) - u(\beta(\tilde{w}))\|v_0\| \leq \|\beta(w) - \beta(\tilde{w})\|^{1/2}\|v_0\|$$

Thus, $(\tilde{w}\phi(\tilde{w}))^{-1}\Gamma_\phi \cap U = \emptyset$ for all $\tilde{w} \in \mathbb{W}$, that is, $\Gamma_\phi$ is an intrinsic Lipschitz graph.

Claim 2. For $p \in \Gamma_\phi$, none of the blow-ups of $\Gamma_\phi$ at $p$ is a homogeneous subgroup.

We first observe that, if $V_0 \subset \mathbb{V} \cap V_1$ is the horizontal subgroup generated by $v_0$ and $L : \mathbb{W} \to \mathbb{V}_0$ parameterizes a homogeneous subgroup $\Gamma_L$ of $\mathbb{G}$, then $L|_{\mathbb{W} \cap V_2} = 0$. Indeed, the homogeneity of $\Gamma_L$ implies that for every $w \in \mathbb{W} \cap V_2$ one has $L(2w) = \sqrt{2}L(w)$, because

$$(2w)(\sqrt{2}L(w)) = \delta_{\sqrt{2}}(w)\delta_{\sqrt{2}}(L(w)) = \delta_{\sqrt{2}}(wL(w)) \in \Gamma_L,$$

while the fact that $\Gamma_L$ is a subgroup (plus the fact that $\mathbb{V}_0$ and $\mathbb{W}$ commute) gives $L(2w) = 2L(w)$, because

$$wL(w)L(w) = (wL(w))(wL(w)) \in \Gamma_L.$$ 

This proves that $L = 0$ on $\mathbb{W} \cap V_2$.

We now prove the claim. Assume by contradiction that there exist $p = \hat{w}\phi(\hat{w}) \in \Gamma_\phi$, a map $L : \mathbb{W} \to \mathbb{V}$ such that the intrinsic graph $\Gamma_L$ of $L$ is a homogeneous subgroup and a sequence $(\lambda_n)_n$ with $\lambda_n \to +\infty$, and

$$\lim_{n \to \infty} \delta_{\lambda_n}(p^{-1}\Gamma_\phi) = \Gamma_L.$$ 

Observe that for every $w \in \mathbb{W}$ and every $n$,

$$\delta_{\lambda_n}((\tilde{w}\phi(\tilde{w}))^{-1}(w\phi(w))) = \delta_{\lambda_n}(\tilde{w}^{-1}w\phi(\tilde{w})^{-1}\phi(w))$$

$$\quad = \delta_{\lambda_n}(\tilde{w}^{-1}w)\left(\frac{u(\beta(w)) - u(\beta(\tilde{w}))}{1/\lambda_n}\right).$$ 

If we set $w = \hat{w}\delta_{1/\lambda_n}w'$, then $\beta(w) = \beta(\hat{w}) + \beta(w')/\lambda_n^2$. Therefore, the set $\delta_{\lambda_n}(p^{-1}\Gamma_\phi)$ is the intrinsic graph of the function from $\mathbb{W}$ to $\mathbb{V}$ given by

$$\phi_{\tilde{p},\lambda_n}(w') = \frac{u(\beta(\hat{w}) + \beta(w')/\lambda_n^2) - u(\beta(\hat{w}))}{1/\lambda_n}v_0.$$ 

Since the maps $\phi_{\tilde{p},\lambda_n}$ take values in $\mathbb{V}_0$, $L$ is also $\mathbb{V}_0$-valued and, as we saw above, this implies that $L|_{\mathbb{W} \cap V_2} = 0$.

Write $\hat{t} = \beta(\hat{w})$ and let $w_0 \in \mathbb{W} \cap V_2$ be such that $\beta(w_0) = 1$; then for every $h \in \mathbb{R}$

$$\phi_{\tilde{p},\lambda_n}(hw_0) = (\text{sgn} \ h)|h|^{1/2} \Delta(\hat{t} + h/\lambda_n^2, \hat{t})v_0.$$ 

By (2), there exists a sequence $(h_n)_n$ such that for every $n$

$$|h_n| \in [c_1, 1] \quad \text{and} \quad \|\phi_{\tilde{p},\lambda_n}(h_nw_0)\| \geq \sqrt{c_1} c_2\|v_0\|/2.$$
Up to passing to a subsequence we can also assume that $h_n \to \bar{h}$ with $|\bar{h}| \in [c_1, 1]$; since
\[
\|\phi_{\bar{p}, \lambda_n}(h_n w_0) - \phi_{\bar{p}, \lambda_n}(\bar{h} w_0)\| = \left\| \frac{u(\hat{t} + h_n/\lambda_n^2) - u(\hat{t} + \bar{h}/\lambda_n^2)}{1/\lambda_n} \right\|_{V_0},
\]
\[
\leq |h_n - \bar{h}|^{1/2} \|v_0\|
\]
we obtain
\[
\|L(\bar{h} w_0)\| = \lim_{n} \|\phi_{\bar{p}, \lambda_n}(\bar{h} w_0)\| = \lim_{n} \|\phi_{\bar{p}, \lambda_n}(h_n w_0)\| \geq \sqrt{c_1} c_2 \|v_0\|/2.
\]
This contradicts the fact that $L(\bar{h} w_0) = 0$, and the claim is proved.

Claim 3. For $p \in \Gamma_\phi$, there exist infinitely many different blow-ups of $\Gamma_\phi$ at $p$.

Let $\hat{p} = \hat{w} \phi(\hat{w}) \in \Gamma_\phi$ be fixed and let $\hat{t} = \beta(\hat{w})$: as before, fix also $w_0 \in W \cap V_2$ such that $\beta(w_0) = 1$. By (2), we can find infinitesimal sequences $(s_n^1)_n, (s_n^2)_n$ such that
\[
\sgn(s_n^1) = \sgn(s_n^2) \text{ for every } n,
\]
\[
\Delta(\hat{t} + s_n^1, \hat{t}) \geq \Delta(\hat{t} + s_n^2, \hat{t}) + c_2.
\]
Up to passing to a subsequence, we can assume that there exists $\sigma \in \{1, -1\}$ and $\Delta^1, \Delta^2 \in \mathbb{R}$ such that
\[
\sgn(s_n^1) = \sgn(s_n^2) = \sigma \text{ for every } n,
\]
\[
\Delta(\hat{t} + s_n^1, \hat{t}) \to \Delta^1 \text{ and } \Delta(\hat{t} + s_n^2, \hat{t}) \to \Delta^2 \text{ as } n \to \infty,
\]
\[
\Delta^1 \geq \Delta^2 + c_2.
\]
Due to the continuity of $s \mapsto \Delta(\hat{t} + s, \hat{t})$ for $s \neq 0$, given $\Delta \in (\Delta^2, \Delta^1)$ one can find an infinitesimal sequence $(s_n)_n$ such that, for every $n$, $\sgn(s_n) = \sigma$ and $\Delta(\hat{t} + s_n, \hat{t}) = \Delta$. Now, as in (4) the set $\delta_{[s_n]_{1/2}}(\hat{p}^{-1} \Gamma_\phi)$ is the intrinsic graph of a map $\phi_{\hat{p}, [s_n]_{1/2}} : \mathcal{W} \to \mathcal{V}$ such that
\[
\phi_{\hat{p}, [s_n]_{1/2}}(\sigma w_0) = \sigma \Delta(\hat{t} + s_n, \hat{t}) v_0 = \sigma \Delta v_0.
\]
Since the family $(\phi_{\hat{p}, [s_n]_{1/2}})_n$ is uniformly Hölder continuous, up to extracting a subsequence it converges locally uniformly to a map $\psi : \mathcal{W} \to \mathcal{V}$ such that $\psi(\sigma w_0) = \sigma \Delta v_0$. The arbitrariness of $\Delta \in (\Delta^2, \Delta^1)$ implies that there are infinitely many different blow-ups at $\hat{p}$, and this concludes the proof. \hfill \square

Appendix

We are now going to construct the function $u$ used in the proof of Theorem 1: this function, in a sense, provides a counter-example to a Rademacher property for Lipschitz functions from $(\mathbb{R}, |\cdot|)$ to $(\mathbb{R}, |\cdot|^{1/2})$. We will use a classical procedure producing a self-similar function: although these ideas are well-known (see, for example, [10] and the references therein), we prefer to include a detailed construction because we were not able to find in the literature explicit statements for the precise estimates (2) we need.

We construct a function $u : [0, 1] \to [0, 1]$ whose difference quotients
\[
\Delta(s, t) = \frac{u(s) - u(t)}{\sgn(s - t)|s - t|^{1/2}}
\]
satisfy
\[
|\Delta(s, t)| \leq 1 \quad \text{for every } s, t \in [0, 1]. \tag{A.1}
\]
We will construct $u$ in such a way that there exist $c_1 > 0$ and $c_2 > 0$ with the property that, for every $t \in [0, 1]$ and $\delta \in (0, 1]$, one can find $s_1, s_2 \in [0, 1]$ such that the conditions in (2)
hold. One can then extend $u$ to $\mathbb{R}$ by setting $u(t) = u(-t)$ for $t \in [-1, 0]$ and $u(t + 2n) = u(t)$ for all $n \in \mathbb{Z}$: this extended $u$ does satisfy (1) and (2).

The function $u$ is obtained as the limit of a sequence $(u_n)_{n \in \mathbb{N}}$ where $u_0(t) = t$. The function $u_{n+1}$ is obtained from $u_n$ on setting

$$u_{n+1}(t) = \begin{cases} 
\frac{2}{3}u_n\left(\frac{4}{9}t\right) & \text{if } t \in \left[0, \frac{5}{9}\right], \\
\frac{2}{3} - \frac{1}{3}u_n\left(9(t - \frac{4}{9})\right) & \text{if } t \in \left[\frac{4}{9}, \frac{5}{9}\right], \\
\frac{1}{3} + \frac{2}{3}u_n\left(\frac{2}{3}(t - \frac{5}{9})\right) & \text{if } t \in \left[\frac{5}{9}, 1\right].
\end{cases}$$  

(A.2)

The first few of the functions $u_0, u_1, u_2, \ldots$ are plotted in Figure A.1. Let us note that $u_n(0) = 0$ and $u_n(1) = 1$ for every $n$, hence $u_n(4/9) = 2/3$ and $u_n(5/9) = 1/3$ for every $n \geq 1$.

Note (see Figure A.2) that the graph of $u_{n+1}$ is the union of three affine copies of the graph of $u_n$, via the following maps (acting on $p \in \mathbb{R}^2$):

$$A_0(p) = \begin{pmatrix} 4/9 & 0 \\ 0 & 2/3 \end{pmatrix} p,$$
$$A_{4/9}(p) = \begin{pmatrix} 1/9 & 0 \\ 0 & -1/3 \end{pmatrix} p + \begin{pmatrix} 4/9 \\ 2/3 \end{pmatrix},$$
$$A_{5/9}(p) = \begin{pmatrix} 4/9 & 0 \\ 0 & 2/3 \end{pmatrix} p + \begin{pmatrix} 5/9 \\ 1/3 \end{pmatrix}.$$  

(A.3)

Claim 1. The functions $u_n$ converge uniformly on $[0, 1]$ to a function $u$ for which (A.1) holds.

The fact that $u_n$ uniformly converge to a continuous function $u$ is a consequence of the estimate

$$\|u_{n+1} - u_n\|_{C^0([0,1])} \leq \frac{2}{3}\|u_n - u_{n-1}\|_{C^0([0,1])}.$$
This estimate follows directly from the definition (A.2): for instance, for \( t \in [0, 4/9] \), one has

\[
|u_{n+1}(t) - u_n(t)| = \frac{2}{3}|u_n(9t/4) - u_{n-1}(9t/4)| \leq \frac{2}{3}\|u_n - u_{n-1}\|_{C^0([0,1])}.
\]

Similarly, one can treat the other two cases \( t \in [4/9, 5/9] \) and \( t \in [5/9, 1] \).

The bound (A.1) on the difference quotients of \( u \) follows from the fact that the same is true for all \( u_n \) in the sequence, as we are now going to prove by induction on \( n \). The statement is clearly true for \( n = 0 \). Suppose that \( u_n \) satisfies

\[
|u_n(t) - u_n(s)| \leq |t - s|^{1/2}
\]

for every \( t, s \in [0,1] \), we will prove that also \( |u_{n+1}(t) - u_{n+1}(s)| \leq |t - s|^{1/2} \) for every \( s, t \in [0,1] \). We distinguish several cases depending on which intervals \((0, 4/9), [4/9, 5/9] \) or \([5/9, 1]\) the points \( s \) and \( t \) belong to. We can suppose that \( s < t \).

Case 1: \( s \) and \( t \) are in the same interval. We can use (A.2) and the induction hypothesis to conclude.

Case 2: \( s \in [0, 4/9] \) and \( t \in [4/9, 5/9] \). Since \( 0 \leq u_n \leq 1 \), one sees from the definition of \( u_{n+1} \) that \( \max(u_{n+1}(s), u_{n+1}(t)) \leq 2/3 = u_{n+1}(4/9) \). Thus

\[
|u_{n+1}(t) - u_{n+1}(s)| \leq \max(u_{n+1}(4/9) - u_{n+1}(t), u_{n+1}(4/9) - u_{n+1}(s))
\]

\[
\leq \max((t - 4/9)^{1/2}, (4/9 - s)^{1/2}) \leq (t - s)^{1/2},
\]

where the second inequality follows from Case 1.

Case 3: \( s \in [4/9, 5/9] \) and \( t \in [5/9, 1] \). Due to the symmetry \( u_n(x) = 1 - u_n(1-x) \), this is similar to Case 2.

Case 4: \( s \in [0, 4/9] \) and \( t \in [5/9, 1] \). Then either \( |u_{n+1}(t) - u_{n+1}(s)| \leq 1/3 \), and we are done because \( |t - s| > 1/9 \), or \( |u_{n+1}(t) - u_{n+1}(s)| > 1/3 \), and then necessarily \( u_{n+1}(s) < u_{n+1}(t) \) (otherwise, \( 0 \leq u_{n+1}(s) - u_{n+1}(t) \leq u_{n+1}(4/9) - u_{n+1}(5/9) = 2/3 - 1/3 = 1/3 \) and

\[
|u_{n+1}(t) - u_{n+1}(s)| = u_{n+1}(t) - u_{n+1}(s)
\]

\[
= u_{n+1}(t) - u_{n+1}(5/9) - 1/3 + u_{n+1}(4/9) - u_{n+1}(s)
\]

\[
\leq (t - 5/9)^{1/2} - 1/3 + (4/9 - s)^{1/2},
\]

Thus, we have covered all cases.
where in the last inequality we used Case 1. By squaring the right-hand side of the last inequality, we obtain
\[
(t - 5/9)^{1/2} - 1/3 + (4/9 - s)^{1/2} = (t - s) + 2(t - 5/9)^{1/2}(4/9 - s)^{1/2} - \frac{2}{3}(t - 5/9)^{1/2} - \frac{2}{3}(4/9 - s)^{1/2}
\]
\[
= (t - s) + (t - 5/9)^{1/2}(4/9 - s)^{1/2} - 2/3 + (4/9 - s)^{1/2}(t - 5/9)^{1/2} - 2/3
\]
\[
\leq t - s,
\]
where we used the fact that $4/9 - s \leq 4/9$ and $t - 5/9 \leq 4/9$. This is enough to conclude.

**Claim 2.** There exist $d_1 > 0$ and $d_2 > 0$ such that, for every $t_0 \in [0, 1]$, one can find $s_1, s_2 \in [0, 1]$ such that
\[
\begin{align*}
\text{sgn}(s_1 - t_0) &= \text{sgn}(s_2 - t_0) \\
d_1 &\leq |s_1 - t_0| \leq 1 \\
d_1 &\leq |s_2 - t_0| \leq 1
\end{align*}
\]
(C.4)
In fact, we will prove Claim 2 for $d_1 = 1/18$ and
\[
d_2 = \min \left\{ \frac{1}{3}\left((1 - \frac{4}{81})^{-1/2} - 1\right), \frac{7}{9} - \frac{3}{2}, \frac{1}{\sqrt{3}} \right\}.
\]
We distinguish several cases.

**Case 1:** $t_0 \in [0, 4/9]$. In this case, it suffices to consider $s_1 = 5/9$ and $s_2 = 1$, as we now show. Observe that the distances of $s_1, s_2$ from $t_0$ are both greater than $1/9 > d_1$.

If $u(t_0) \geq 2/9$, by (A.1) and the equality $u(0) = 0$, we have $t_0^{1/2} \geq u(t_0)$, hence $t_0 \geq 4/81$; since $u(t_0) \leq 2/3$, we obtain
\[
\Delta(1, t_0) \geq \frac{1}{\sqrt{1 - \frac{4}{81}}} \quad \text{and} \quad \Delta(5/9, t_0) \leq \frac{3}{\sqrt{\frac{1}{9}}} = \frac{1}{3},
\]
so that $\Delta(1, t_0) - \Delta(5/9, t_0) \geq d_2$.

If $u(t_0) \leq 2/9$, then $(4/9 - t_0)^{1/2} \geq 2/3 - 2/9 = 4/9$, hence $5/9 - t_0 \geq 1/9 + (4/9)^2 = (5/9)^2$ and
\[
\Delta(1, t_0) \geq \frac{7}{9} \quad \text{and} \quad \Delta(5/9, t_0) \leq \frac{3}{5},
\]
and again $\Delta(1, t_0) - \Delta(5/9, t_0) \geq d_2$.

**Case 2:** $t_0 \in [4/9, 1/2]$. In this case, we take $s_1 = 5/9$ and $s_2 = 1$. The distances of $s_1, s_2$ from $t_0$ are both no less than $1/18 = d_1$ and, since $1/3 \leq u(t_0) \leq 2/3$, one gets
\[
\Delta(1, t_0) \geq \frac{1}{\sqrt{\frac{9}{2}}} = \frac{1}{\sqrt{5}} \quad \text{and} \quad \Delta(5/9, t_0) \leq 0.
\]

**Case 3:** $t_0 \in [1/2, 1]$. We proved that, if $t_0 \in [0, 1/2]$, the claim can be proved on choosing $s_1 = 5/9$ and $s_2 = 1$. Therefore, due to the symmetry $u(x) = 1 - u(1 - x)$, when $t_0 \in [1/2, 1]$ it is enough to take $s_1 = 0$ and $s_2 = 4/9$.

**Claim 3.** There exist $c_1 > 0$ and $c_2 > 0$ such that, for every $t \in [0, 1]$ and $\delta \in (0, 1]$, one can find $s_1, s_2 \in [0, 1]$ for which the conditions in (2) hold.
By self-similarity, the graph of $u$ over the interval $[0, 1] \cap [t - \delta, t + \delta]$ contains the image of the graph of $u$ over $[0, 1]$ under an affine map $L : \mathbb{R}^2 \to \mathbb{R}^2$ which is a finite composition $L = A_{j_1} \circ \cdots \circ A_{j_N}$ of maps $(A_{j_k})_{k=1,\ldots,N}$ for $j_k$ in $\{0, 4/9, 5/9\}$. Observe that $L$ is an affine map of the form $L(x, y) = (L_1(x), L_2(y))$ for suitable affine maps $L_1, L_2 : \mathbb{R} \to \mathbb{R}$ and it is not restrictive to assume that the length of the interval $L_1([0, 1])$, which is contained in $[t - \delta, t + \delta]$, is at least $\delta/9$: this implies that there exists $\delta/9 \leq c \leq \delta$ such that $|L_1(x) - L_1(y)| = c|x - y|$ for every $x, y \in [0, 1]$. Let $t_0 \in [0, 1]$ be such that $L(t_0, u(t_0)) = (t, u(t))$. If $s_1, s_2 \in [0, 1]$ are such that (A.4) holds, then we have
\[
\sgn(L_1(s_1) - t) = \sgn(L_1(s_2) - t),
\]
\[
d_{\mathbb{R}}^1 \delta \leq |L_1(s_1) - t| \leq \delta,
\]
\[
d_{\mathbb{R}}^1 \delta \leq |L_1(s_2) - t| \leq \delta.
\]
Since the maps $A_{j_k}$ do not modify the difference quotients, also $L$ has this property, that is,
\[
|\Delta(L_1(s_1), t) - \Delta(L_1(s_2), t)| = |\Delta(s_1, t_0) - \Delta(s_2, t_0)| \geq d_2.
\]
This concludes the proof.

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