Hydrodynamics for the ABC model with slow/fast boundary

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Abstract

In this article, we consider the ABC model in contact with slow/fast reservoirs. In this model, there is at most one particle per site, which can be of type $\alpha \in \{A,B,C\}$ and particles exchange positions in the discrete set of points $\{1,\cdots,N-1\}$ with a weakly asymmetric rate that depends on the type of particles involved in the exchange mechanism. At the boundary points $x = 1, N-1$ particles can be injected or removed with a rate that depends on the type of particles involved. We prove that the hydrodynamic limit, in the diffusive time scale, is given by a system of non-linear coupled equation with several boundary conditions, that depend on the strength of the reservoir's action.

1 Introduction

The ABC model, introduced by Evans et al. [9, 10], is a system consisting of three species of particles, labeled $A, B,$ and $C$. In its original and most studied version, the particles are located on a one-dimensional discrete torus with $N$ points (one particle per site). The system evolves under nearest neighbor transpositions: $AB \to BA, BC \to CB, CA \to AC$ with rate $q \leq 1$ and $BA \to AB, CB \to BC, AC \to CA$ with rate $1/q$. In particular, for this dynamics defined on the torus, the total number of particles of each species, $N_A, N_B$ and $N_C$, is conserved and $N_A + N_B + N_C = N$. The invariant measure is explicitly computed only when $N_A = N_B = N_C$, in which case the dynamics is reversible with respect to the Gibbs measure of a certain Hamiltonian having long range pair interactions. When $q < 1$, the bias in the dynamics drives, say, a $B$ particle to move to the right when it is inside a region of $A$ particles and to the left when it is inside a region of $C$ particles. Therefore starting from an arbitrary configuration the system will reach, after a relatively short time, a metastable state of the type $\ldots AABBBBC C C C AAAABB \ldots$ with blocks of particles (pure regions) located in the cyclic order $ABC$. As discussed in [9, 10], in the thermodynamic limit $N \to \infty$, with $N_\alpha/N \to r_\alpha$ for $\alpha \in \{A,B,C\}$, and $q < 1$ constant, the system segregates into three pure $A, B$ and $C$ clusters. In [16], in a proper time scale, the asymptotic dynamics among these stable configurations, which differ only by rotation, was studied in the strongly asymmetric regime, where $q \to 0$ when $N \to \infty$.

While there is always strong separation when $q < 1$ is held fixed, particles exchange symmetrically when $q = 1$, in which case all configurations are equally probable. So, a natural scaling to investigate, introduced by Clincy et al. [8], is the weakly asymmetric regime where $q = e^{-\beta/(2N)}, N \to \infty$ and $\beta > 0$ is a fixed control parameter which plays the role of the inverse temperature. In this regime, they observed phase transitions in the invariant measure $\mu_\beta^N$, at a critical value $\beta_c$. For equal densities ($r_A = r_B = r_C = 1/3$) it is known that $\beta_c = 2\pi\sqrt{3}$. This
phase transition, also analysed in [5] and [1], can be stated in terms of the free energy functional \( \mathcal{F}_\beta \) associated to \( \hat{\mu}_N^\beta \). The typical configurations of the system as \( N \to \infty \) are described by the density profiles that minimize \( \mathcal{F}_\beta(\rho^A, \rho^B, \rho^C) \), where for \( \alpha \in \{A, B, C\} \), \( \rho^\alpha \) denotes the density of particles of type \( \alpha \). For \( \beta < \beta_c \) (disordered high temperature phase) \( \mathcal{F}_\beta \) has an unique minimizer; for \( \beta > \beta_c \) (segregated low temperature phase) \( \mathcal{F}_\beta \) has a continuum of minimizers, parameterized by translation. As discussed in [1] this means that in the limit \( N \to \infty \), \( N_A/N \to r_A \), \( \alpha \in \{A, B, C\} \), for \( \beta < \beta_c \) the invariant measure \( \hat{\mu}_N^\beta \) converges in the sense of Definition 2.6 to the deterministic flat profile \( (\rho^A, \rho^B, \rho^C) = (r_A, r_B, r_C) \), while for \( \beta > \beta_c \) the limiting profile in the sense of convergence in distribution, for \( \hat{\mu}_N^\beta \), is random and space-dependent.

In [1] the model was studied on a one-dimensional lattice with \( N \) sites with closed (zero flux) boundary. The dynamics is the same as above, except that a particle at \( x = 1 \) (respectively \( x = N \)) can only jump to the right (respectively left). In contrast with the dynamics defined on the torus, for the system with this boundary mechanism, independently from the values of \( N_A \), \( N_B \) and \( N_C \), the dynamics always satisfies detailed balance with respect to a Gibbs measure \( \hat{\mu}_N^\beta(\cdot) = Z_{N, \beta}^{-1} e^{-\beta H_N(\cdot)} \), where \( H_N(\cdot) \) is an explicit Hamiltonian function of the configuration. It coincides with the invaraint measure on the torus when \( N_A = N_B = N_C \).

As discussed in [8], [5] and [4], for the dynamics on the torus, in the weakly asymmetric regime \( q = e^{-\beta/2N} \), under the diffusive time scale \( tN^2 \) the density profiles of the three species \( (\rho^A, \rho^B, \rho^C) \) evolve according to the system of hydrodynamic equations

\[
\begin{align*}
\partial_t \rho^A &= \Delta \rho^A + \beta \nabla [\rho^A(\rho^B - \rho^C)], \\
\partial_t \rho^B &= \Delta \rho^B + \beta \nabla [\rho^B(\rho^C - \rho^A)], \\
\partial_t \rho^C &= \Delta \rho^C + \beta \nabla [\rho^C(\rho^A - \rho^B)],
\end{align*}
\]

where \( \Delta \) and \( \nabla \) denote, respectively, the Laplacian and gradient operators on the continuous torus. One can make the equivalent choice of interpreting particles of species \( C \) as holes, denoted by \( \emptyset \) and looking only at the evolution of the density profiles for \( A \) and \( B \). As a consequence of the exclusion rule, \( \rho^A(x) + \rho^B(x) + \rho^\emptyset(x) = 1 \) for any \( x \in [0, 1] \), the hydrodynamic equations read as

\[
\begin{align*}
\partial_t \rho^A &= \Delta \rho^A + \beta \nabla [\rho^A(\rho^A + 2\rho^B - 1)] \\
\partial_t \rho^B &= \Delta \rho^B - \beta \nabla [\rho^B(2\rho^A + \rho_B - 1)].
\end{align*}
\]

The hydrodynamic equations for the weakly asymmetric model on the torus have been formally derived in [17].

In the present work, we study a version of the weakly asymmetric ABC model on the interval \( \{1, \ldots, N-1\} \) with a boundary dynamics that allow the exchange of particles with reservoirs on both sides of the bulk, with a strength that can be regulated by some parameters that increase or decrease the reservoirs’ action. The precise definition of the model is given in the next section. For this model we prove the hydrodynamic limit for the density of each type of particle, whose hydrodynamic equation is given by (1.1) and with Dirichlet or Robin boundary conditions, that depend on the rates of exchange of particles with the reservoirs. In this non-conservative model, unlike the case considered in [1], the invariant measure is not reversible and it is not explicitly known.

Our result is an extension of what was considered for the case of one-species in [2] and [7]. The strategy of proof we employ is based on the entropy method introduced in [14]. The idea of the method results of combining the tightness of the sequence of empirical measures with the unique characterization of the limit points, which gives convergence of the whole sequence. This limit point is supported on a trajectory of measures, absolutely continuous with respect to the Lebesgue measure and whose density is the unique weak solution of the corresponding
hydrodynamic equation. Our model has weakly asymmetric rates and a boundary dynamics. These two features bring at the macroscopic level a system of non-linear equations with several boundary conditions. To prove the hydrodynamic limit we need to obtain several replacement lemmas. One is necessary at the bulk in order to overcome the fact that the dynamic is weakly asymmetric and one needs to make a proper linearisation at the microscopic level. Other replacements are needed at the boundary so that one recognizes the boundary conditions of Dirichlet and Robin type. This last replacement, which consists of replacing occupations variables close to the boundary by the respective density reservoirs, is crucial, and we obtain it by making a comparison with the action of the adjoint operator at those occupation variables.

There are many intriguing questions that we leave for future work. The first one has to do with the derivation of the hydrostatic limit, i.e. the hydrodynamic limit starting from the stationary measure of the system. The second one is related to the derivation of the fluctuations around the hydrodynamical profile. Since the equations are coupled, determining the correct fluctuation fields and the limiting equations is a very interesting problem. Our method could also be employed to other variations of the dynamics, as e.g. considering $M$-types of particles and allowing a finite number of particles per site.

Here follows an outline of this work. In Section 2 we introduce the model and we state our main result. In Section 3 we give an heuristic argument which allows recognizing the limit equation and the respective boundary conditions, depending on the boundary strength. Section 4 is devoted to the proof of tightness of the sequence of empirical measures associated to each type of particle. In Section 5 we rigorously characterize the limit point and Section 6 is devoted to the proof of the replacement lemmas that are needed along the proofs. In Appendix A we prove the uniqueness of weak solutions of the equations we derive.

2 The ABC model with slow/fast boundary

Let $\theta \geq \delta$ and $\theta \geq 1$, $\beta, \dot{\beta} > 0$ and $r_{AB}, r_{B\bar{O}}, r_{\bar{O}A}, r_{BA}, r_{\bar{O}B}, r_{A\bar{O}} > 0$. Let $N \in \mathbb{N}$. We consider the one-dimensional two-species weakly asymmetric simple exclusion process with state space $\Omega_N = \{A, B, \emptyset\}^{\Lambda_N}$, where $\Lambda_N = \{1, \ldots, N - 1\}$. A configuration in $\Omega_N$ is denoted by $\eta$, where $\eta(x) = a$ if site $x$ is occupied by a particle of type $a$. We make the convention that $a+1, a+2, \ldots$ denote the particle types that are successors to $a$ in the cyclic order $AB\emptyset$. For $a \in \{A, B, \emptyset\}$, and $x \in \Lambda_N$ we denote by $\xi^a_x : \Omega_N \to \{0, 1\}$ the function that indicates that there is a particle of type $a$ at site $x$, i.e., $\xi^a_x(\eta) = 1$ if $\eta(x) = a$ and $\xi^a_x(\eta) = 0$ otherwise. As discussed in the previous section, this model is known in literature as the ABC model, in which case, holes $\emptyset$ are described as particles of type $C$. The dynamics is the following: a couple of particles $(\eta(x), \eta(x+1)) = (a, a+1)$ swaps positions with rate $1 - \beta_{2N}$, while the configuration $(\eta(x), \eta(x+1)) = (a+1, a)$ is inverted with rate $1 + \beta_{2N}$. This implies that particles prefer to arrange themselves in cyclic alphabetical order. At the boundary, particles are injected or removed with the same behaviour, but the swapping depends on different rates $r_{AB}$ according to the relative positions of the particle at $x = 1, N - 1$ and at the corresponding reservoirs.

The infinitesimal generator $\mathbb{L}_N$ of the Markov process acts on functions $f : \Omega_N \to \mathbb{R}$ as $\mathbb{L}_N = \mathbb{L}_N^B + \mathbb{L}_N^L + \mathbb{L}_N^R$, where the central term corresponds to the dynamics of the bulk, while the terms on the left and on the right are the generators corresponding to the left and the right boundary, respectively. The bulk generator is

$$\mathbb{L}_N^B f(\eta) = \sum_{x=1}^{N-2} c_{x,x+1}(\eta)[f(\eta^{x,x+1}) - f(\eta)]$$

where $\eta^{x,x+1}$ is the configuration obtained from $\eta$ by exchanging particles at sites $x$ and $x+1$. 

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where

\[ \eta^{x,x+1}(y) = \begin{cases} 
\eta(y), & \text{if } y \notin \{x, x + 1\}, \\
\eta(x + 1), & \text{if } y = x, \\
\eta(x), & \text{if } y = x + 1,
\end{cases} \]

and

\[ c_{x,x+1}(\eta) = \begin{cases} 
1 + \frac{\beta}{2N^\theta} & \text{if } (\eta(x), \eta(x + 1)) \in \{(B,A), (\varnothing, B), (A, \varnothing)\}, \\
1 - \frac{\beta}{2N^\theta} & \text{if } (\eta(x), \eta(x + 1)) \in \{(A, B), (B, \varnothing), (\varnothing, A)\}.
\end{cases} \]

The boundary generators are given by

\[
\begin{aligned}
\mathcal{L}^L_N f(\eta) &= c^+_{-1}(\eta)[f(\eta^{1,+}) - f(\eta)] + c^-_{N-1}(\eta)[f(\eta^{N-1,-}) - f(\eta)] \\
\mathcal{L}^R_N f(\eta) &= c^+_{N-1}(\eta)[f(\eta^{N-1,+}) - f(\eta)] + c^-_{N-1}(\eta)[f(\eta^{N-1,-}) - f(\eta)],
\end{aligned}
\tag{2.1}
\]

where

\[
\begin{aligned}
c^+_{-1}(\eta) &= \left(\frac{1}{N^\theta} + \frac{\tilde{\beta}}{2N^\theta}\right) \left(\xi^A_{1}(\eta)r_{AB} + \xi^B_{1}(\eta)r_{BA} + \xi^\varnothing_{1}(\eta)r_{\varnothing A}\right), \\
c^-_{N-1}(\eta) &= \left(\frac{1}{N^\theta} - \frac{\tilde{\beta}}{2N^\theta}\right) \left(\xi^A_{N-1}(\eta)r_{AB} + \xi^B_{N-1}(\eta)r_{BA} + \xi^\varnothing_{N-1}(\eta)r_{\varnothing A}\right), \\
c^+_{N-1}(\eta) &= \left(\frac{1}{N^\theta} + \frac{\tilde{\beta}}{2N^\theta}\right) \left(\xi^A_{N-1}(\eta)r_{AB} + \xi^B_{N-1}(\eta)r_{BA} + \xi^\varnothing_{N-1}(\eta)r_{\varnothing A}\right), \\
c^-_{-1}(\eta) &= \left(\frac{1}{N^\theta} - \frac{\tilde{\beta}}{2N^\theta}\right) \left(\xi^A_{1}(\eta)r_{AB} + \xi^B_{1}(\eta)r_{BA} + \xi^\varnothing_{1}(\eta)r_{\varnothing A}\right),
\end{aligned}
\]

and \(\eta^{x,\pm}\) is obtained from \(\eta\) when at site \(x\) a particle \(\alpha\) is replaced by \(\alpha \pm 1\), i.e,

\[
\eta^{x,\pm}(y) = \begin{cases} 
\eta(y), & \text{if } y \neq x, \\
\eta(x), & \text{if } y = x.
\end{cases}
\]

Observe that when \(\theta > \delta\) the rates given above are well defined for \(N\) sufficiently large, but in the case \(\theta = \delta\) we need to impose the restriction

\[
\tilde{\beta} < 2. \tag{2.2}
\]

A simple computation shows that

\[
\begin{aligned}
\xi^A_{x}(\eta^{x,+}) &= \xi^\varnothing_{x}(\eta), & \xi^B_{x}(\eta^{x,+}) &= \xi^A_{x}(\eta), & \xi^\varnothing_{x}(\eta^{x,+}) &= \xi^B_{x}(\eta), \\
\xi^A_{x}(\eta^{x,-}) &= \xi^\varnothing_{x}(\eta), & \xi^B_{x}(\eta^{x,-}) &= \xi^A_{x}(\eta), & \xi^\varnothing_{x}(\eta^{x,-}) &= \xi^B_{x}(\eta). 
\end{aligned} \tag{2.3}
\]

To understand these relations, let us look at the top and leftmost one in (2.3). The following chain of equivalences holds:

\[
\xi^A_{x}(\eta^{x,+}) = 1 \iff \eta^{x,+}(x) = A \iff \eta(x) = \varnothing \iff \xi^\varnothing_{x}(\eta) = 1.
\]

Let us now describe the dynamics at the boundary. For example, let us look at the leftmost term on the first line of (2.1). If there is a particle of species \(A\) at \(x = 1\), then this particle will be replaced by a particle of species \(B\) with rate \(r_{AB}(\frac{1}{N^\theta} + \frac{\tilde{\beta}}{2N^\theta})\) or by a hole/it will disappear with rate \(r_{\varnothing A}(\frac{1}{N^\theta} - \frac{\tilde{\beta}}{2N^\theta})\). For \(x = 1\) or \(x = N - 1\), we can interpret \(c^+_{x}(\eta)\) as the rate for upgrading a particle at site \(x\) and \(c^-_{x}(\eta)\) as the rate for downgrading.
If we impose the following relationships on the rates

\[
\begin{align*}
  r_{AB} &= r_{\varnothing B} = r_B, & \tilde{r}_{AB} &= \tilde{r}_{\varnothing B} = \tilde{r}_B \\
  r_{B\varnothing} &= r_{A\varnothing} = r_{\varnothing}, & \tilde{r}_{B\varnothing} &= \tilde{r}_{A\varnothing} = \tilde{r}_{\varnothing} \\
  r_{\varnothing A} &= r_{R\varnothing} = r_A, & \tilde{r}_{\varnothing A} &= \tilde{r}_{R\varnothing} = \tilde{r}_A
\end{align*}
\]  

(2.4)

with

\[
\begin{align*}
  r_A + r_B + r_{\varnothing} &= 1 & \tilde{r}_A + \tilde{r}_B + \tilde{r}_{\varnothing} &= 1,
\end{align*}
\]  

(2.5)

we can interpret \( r_A, r_B, r_{\varnothing} \) as the concentration of particles of types \( A, B \) and \( \varnothing \) at the left reservoir, and \( \tilde{r}_A, \tilde{r}_B, \tilde{r}_{\varnothing} \) as the concentrations at the right reservoir. With this choice of rates the boundary generators become

\[
\begin{align*}
  \mathcal{L}_N^L f(\eta) &= \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\theta} \right)(\xi^A_N(\eta)r_B + \xi^B_N(\eta)r_{\varnothing} + \xi^\varnothing_N(\eta)r_A)[f(\eta^{1,+}) - f(\eta)] \\
  &\quad + \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\theta} \right)(\xi^A_N(\eta)r_B + \xi^B_N(\eta)r_{\varnothing} + \xi^\varnothing_N(\eta)r_A)[f(\eta^{1,-}) - f(\eta)],
\end{align*}
\]  

(2.6)

\[
\begin{align*}
  \mathcal{L}_N^R f(\eta) &= \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\theta} \right)(\xi^A_N(\eta)\tilde{r}_B + \xi^B_N(\eta)\tilde{r}_{\varnothing} + \xi^\varnothing_N(\eta)\tilde{r}_A)[f(\eta^{N-1,+}) - f(\eta)] \\
  &\quad + \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\theta} \right)(\xi^A_N(\eta)\tilde{r}_B + \xi^B_N(\eta)\tilde{r}_{\varnothing} + \xi^\varnothing_N(\eta)\tilde{r}_A)[f(\eta^{N-1,-}) - f(\eta)].
\end{align*}
\]

The dynamics can be summarized in the figure below.

![Figure 1: Dynamics of the ABC model with reservoirs at \( x = 0, N \). Particles of species \( A \) in yellow, \( B \) in orange and \( C/\)holes in light cyan.](image)

### 2.1 Hydrodynamic equations

Our goal in this article is to describe the space-time evolution of the density of each species of particles as solutions of a system of partial differential equations (PDEs).

The solutions that we obtain are in the weak sense and to define them properly, we first need to define the set of functions that we test the solutions with. To that end, for \( m, n \in \mathbb{N}_0 \), let \( C^{m,n}([0, T] \times [0, 1]) \) be the set of continuous functions defined on \([0, T] \times [0, 1]\) that are \( m \) times differentiable on the first variable and \( n \) times differentiable on the second variable, and with continuous derivatives. For a function \( \phi \in C^{m,n}([0, T] \times [0, 1]) \), we denote by \( \partial_t \phi \) its derivative with respect to the time variable \( t \) and by \( \nabla \phi \) and \( \Delta \phi \) its first and second derivatives with respect to the space variable. We use \( \phi_t(u) \) as a notation to \( \phi(t, u) \). We also denote \( C^{m,n}_c([0, T] \times [0, 1]) \) as the set of functions \( G \in C^{m,n}([0, T] \times [0, 1]) \) such that for each time \( s,
with compact support. We denote by \( \langle \cdot, \cdot \rangle \) the inner product on \( L^2([0,1]) \) and by \( \langle \cdot, \cdot \rangle_\mu \) the same inner product with respect to a measure \( \mu \).

**Definition 2.1.** Let \( \mathcal{H}^1(0,1) \) be the set of all locally integrable functions \( g : (0,1) \rightarrow \mathbb{R} \) such that there exists a function \( \partial_a g \in L^2(0,1) \) satisfying \( \langle \partial_a, \phi \rangle = -\langle \partial_a g, \phi \rangle \), for all \( \phi \in C_c^\infty((0,1)) \). For \( g \in \mathcal{H}^1(0,1) \) we define the norm by

\[
\|g\|^2_{\mathcal{H}^1(0,1)} = \|g\|^2_{L^2(0,1)} + \|\partial_a g\|^2_{L^2(0,1)}.
\]

The elements in \( \mathcal{H}^1(0,1) \) coincide a.e. with absolutely continuous functions whose derivative (which exist a.e.) belong to \( L^2(0,1) \).

**Definition 2.2.** Let \( L^2([0,T] \times \mathcal{H}^1(0,1)) \) be the set of measurable functions \( f : [0,T] \rightarrow \mathcal{H}^1(0,1) \) such that

\[
\int_0^T \|f_t\|^2_{\mathcal{H}^1(0,1)} dt < \infty.
\]

From here on, we fix an initial measurable profile \( g = (g^A, g^B, g^\emptyset) : [0,1] \rightarrow [0,1]^3 \) with \( g^A + g^B + g^\emptyset = 1 \). This will correspond to the initial condition in all the PDEs that we derive.

**Definition 2.3.** We say that \( \rho = (\rho^A, \rho^B, \rho^\emptyset) : [0,T] \times [0,1] \rightarrow [0,1]^3 \) is a weak solution of the system of parabolic equations with Dirichlet boundary conditions

\[
\begin{cases}
\partial_t \rho^A = \Delta \rho^A + \beta \nabla [\rho^A(\rho^{A+1}-\rho^{A+2})], \\
\rho^A(0) = r_A, \quad \rho^A(1) = \bar{r}_A, \quad t \in (0,T), \quad \alpha \in \{A,B,\emptyset\}, \quad (2.7) \\
\rho^A_0(u) = \rho^A(u), \quad u \in [0,1],
\end{cases}
\]

if for each \( \alpha \in \{A,B,\emptyset\} \) it holds:

**D1.** \( \rho^A \in L^2([0,T] \times \mathcal{H}^1(0,1)) \),

**D2.** \( \rho^A(0) = r_A \) and \( \rho^A(1) = \bar{r}_A \) for a.e. \( t \in (0,T) \),

**D3.** for all \( t \in [0,T] \) and any \( \phi \in C^2_c([0,1]) \), \( F_{\text{Dir}}(t, \phi, \rho, g) = 0 \), where

\[
F_{\text{Dir}}^\alpha(t, \phi, \rho, g) := \langle \rho^\alpha_t, \phi \rangle - \langle \rho^\alpha, \phi \rangle - \int_0^t \langle \rho^\alpha_s, \Delta \phi \rangle - \beta \langle \rho^{A+1}_s - \rho^{A+2}_s, \nabla \phi \rangle ds. \quad (2.8)
\]

**Definition 2.4.** Let \( \kappa_1, \kappa_2 \in \mathbb{R} \). We say that \( \rho = (\rho^A, \rho^B, \rho^\emptyset) : [0,T] \times [0,1] \rightarrow [0,1]^3 \) is a weak solution of the system of parabolic equations with Robin boundary conditions

\[
\begin{cases}
\partial_t \rho^A = \Delta \rho^A + \beta \nabla [\rho^A(\rho^{A+1}-\rho^{A+2})] \\
\nabla \rho^A(0) = -\beta \rho^A(0)(\rho^{A+1}(0) - \rho^{A+2}(0)) \\
-\kappa_1(2r_{a+2} - 1)\rho^A_t(0) - 2r_a \rho^A_t(0) + r_A - \kappa_2(r_A - \rho^A(0)), \quad t \in (0,T), \\
\rho^A_t(1) = -\beta \rho^A(1)(\rho^{A+1}(1) - \rho^{A+2}(1)) \\
-\kappa_1(2r_{a+2} - 1)\rho^A_t(1) - 2r_a \rho^A_t(1) + \bar{r}_A + \kappa_2(\bar{r}_A - \rho^A(1)), \quad t \in (0,T), \\
\rho^A_0(u) = \rho^A(u), \quad u \in [0,1],
\end{cases}
\]

if for each \( \alpha \in \{A,B,\emptyset\} \) it holds:

\[
(2.9)
\]
R1. \( \rho^a \in L^2([0, T] \times \mathcal{H}^1(0, 1)) \),

R2. for all \( t \in [0, T] \) and any \( \phi \in C^{1,2}([0, T] \times [0, 1]) \), \( F^a_{\text{Rob}}(t, \phi, \rho, g) = 0 \), where

\[
F^a_{\text{Rob}}(t, \phi, \rho, g) : = \langle \rho^a_t, \phi_t \rangle - \langle g^a, \phi_0 \rangle - \int_0^t \langle (\rho^a_s, (\Delta + \partial_s^a)) \phi_s \rangle \, ds + \beta \int_0^t \langle \rho^a_s (\rho^a_{s+1} - \rho^a_{s+2}), \nabla \phi_s \rangle \, ds - \int_0^t \nabla \phi_s(0) \rho^a_s(0) - \nabla \phi_s(1) \rho^a_s(1) \, ds - \kappa_1 \int_0^t \phi_s(0) \left[ (2r_{a+2} - 1) \rho^a_s(0) - 2r_a \rho^a_s(1) + r_a \right] \, ds - \kappa_2 \int_0^t \left[ \phi_s(0) (r_a - \rho^a_s(0)) + \phi_s(1) (r_a - \rho^a_s(1)) \right] \, ds.
\]

(2.10)

**Remark 2.5.** We observe that in the Dirichlet regime we use test functions which are only space dependent, while in the Robin regime we use test functions also time dependent. The reason for this is that in the Dirichlet case, this space of test functions is enough to prove uniqueness of the weak solution, while in the Robin case we need a bigger space of test functions.

In Section A.2 we prove the uniqueness of weak solutions of the equation with Dirichlet boundary conditions, as given in Definition 2.3. In Appendix A.3 we also give an heuristic argument for the proof of the uniqueness of weak solutions in the Robin case, by assuming that the solution itself can be taken as a test function, which would need later a regularization both in time and space. Nevertheless, we believe that since we did not impose any boundary condition on the test function, our notion of solution should be unique.

### 2.2 Hydrodynamic limit

Here we state the main theorem concerning the hydrodynamic limit of the empirical measure. Assuming the relations (2.4) and (2.5), denote by \( \{\eta_t : t \geq 0\} \) the continuous-time Markov process on \( \Omega_N \) with generator \( N^2 \mathcal{L}_N \), and by \( \mathbb{P}_{\mu_N} \) the probability measure that this process induces on the Skorohod space \( \mathcal{D}([0, T], \Omega_N) \), when it starts from a distribution \( \mu_N \) on \( \Omega_N \). Denote by \( \mathbb{E}_{\mu_N} \) the expectation with respect to \( \mathbb{P}_{\mu_N} \). Now we fix our initial set of probability measures. Fix an initial measurable profile \( g = (g^A, g^B, g^0) : [0, 1] \to [0, 1]^3 \) with \( g^A + g^B + g^0 = 1 \).

**Definition 2.6.** We say that a sequence of probability measures \( (\mu_N)_{N \geq 1} \) on \( \Omega_N \) is associated with the profile \( g \) if for every continuous function \( \phi \), for any \( \varepsilon > 0 \) and \( a \in \{A, B, 0\} \), it holds

\[
\lim_{N \to \infty} \mu_N \left( \eta \in \Omega_N : \left| \frac{1}{N} \sum_{x \in \Lambda_N} \phi \left( \frac{x}{N} \right) \xi_x^a(\eta) - \int_0^1 \phi(u) g^a(u) \, du \right| > \varepsilon \right) = 0.
\]

(2.11)

**Theorem 2.7.** Let \( g = (g^A, g^B, g^0) : [0, 1] \to [0, 1]^3 \) be a measurable function and let \( (\mu_N)_{N \geq 1} \) be a sequence of probability measures associated with \( g \). For any \( t \in [0, T] \), for any \( \phi \in C^0([0, 1]) \),
any $\alpha \in \{A, B, \emptyset\}$, and any $\varepsilon > 0$ it holds

$$
\lim_{N \to \infty} \mathbb{P}_{\mu_N}\left( \eta : \frac{1}{N} \sum_{x \in \Lambda_N} \phi \left( \frac{x}{N} \right) \xi^\alpha_x(\eta_t) - \int_0^1 \phi(u) \rho^\alpha_t(u) du > \varepsilon \right) = 0, \quad (2.12)
$$

where $\rho = (\rho^A, \rho^B, \rho^\emptyset) : [0, T] \times [0, 1] \to [0, 1]^3$ is the unique weak solution of:

a) $(2.7)$, if $\delta < 1 \leq \theta$;

b) $(2.9)$ with:

b1) $\kappa_1 = \kappa_2 = 0$, if $\theta \geq \delta > 1$.

b2) $\kappa_1 = \bar{\beta}/2$ and $\kappa_2 = 1$, if $\theta = 1 = \delta$ and $\bar{\beta} < 4/3$.

b3) $\kappa_1 = 0$ and $\kappa_2 = 1$, if $\theta > 1$ and $\delta = 1$.

### Remark 2.8.

We note that in item b2) in the statement of the theorem we imposed $\bar{\beta} < 4/3$. Nevertheless, the model is well defined if $\bar{\beta} < 2$, see $(2.2)$. This more restrictive condition comes from the fact that our proof of the uniqueness of weak solutions only works when $\bar{\beta} < 4/3$. If the uniqueness of weak solutions can be proved for $\bar{\beta} < 2$, then our result also holds for those values of $\bar{\beta}$.

### 3 Heuristics for the hydrodynamic equation

In this section we give an heuristic argument which is the basis to derive the weak formulation of the respective hydrodynamic equations. Let $\mathcal{M}$ be the space of positive measures on $[0, 1]$ with total mass bounded by 1 and endowed with the weak topology. For each $\alpha \in \{A, B, \emptyset\}$, denote by $\pi^{N,\alpha} : \Omega_N \to \mathcal{M}$ the function that associates to each configuration $\eta$ the measure obtained by assigning mass $1/N$ to each particle of type $\alpha$:

$$
\pi^{N,\alpha}(\eta, du) = \frac{1}{N} \sum_{x \in \Lambda_N} \xi^\alpha_x(\eta) \delta_x(du).
$$

Thus, the empirical process $\pi^{N,\alpha}_t := \pi^{N,\alpha}(\eta_t)$ is a Markov process on the space $\mathcal{M}$. 

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For a given function $\phi : [0, T] \times [0, 1] \to \mathbb{R}$ we denote by $(\pi^N_t, \phi_t)$ the integral of $\phi_t(\cdot)$ with respect to the measure $\pi^N_t$ and observe that by Dynkin’s formula,

$$\mathcal{M}^N_t(\phi) = \langle \pi^N_t, \phi_t \rangle - \langle \pi^N_0, \phi_0 \rangle - \int_0^t \langle N^2 \mathcal{L}_N + \partial_x \rangle (\pi^N_s, \phi_s) \, ds$$

(3.1)

is a martingale with respect to the natural filtration. A simple, but long, computation shows that last identity can be written as

$$\mathcal{M}^N_t(\phi) = \langle \pi^N_t, \phi_t \rangle - \langle \pi^N_0, \phi_0 \rangle - \int_0^t \langle \pi^N_s, \partial_x \rangle ds - \int_0^t \langle \pi^N_s, \Delta_N \phi_s \rangle ds$$

$$+ \frac{\beta}{N} \int_0^t \sum_{s=1}^{N-2} \nabla^+ \phi_s(\frac{x}{N}) g^a_{x,x+1}(\eta_s) ds$$

$$- \frac{N^2}{N^{1+\theta}} \int_0^t \left[ \phi_s(\frac{1}{N}) f^a_1(\eta_s) + \phi_s(\frac{N-1}{N}) f^a_{N-1}(\eta_s) \right] ds$$

$$- \int_0^t \left[ \nabla^+ \phi_s(0) \xi^a_1(\eta_s) - \nabla^+ \phi_s(\frac{N-1}{N}) \xi^a_{N-1}(\eta_s) \right] ds$$

$$- \frac{\beta N^2}{N^{1+\theta}} \int_0^t \left[ \phi_s(\frac{1}{N}) h^a_1(\eta_s) + \phi_s(\frac{N-1}{N}) h^a_{N-1}(\eta_s) \right] ds,$$

(3.2)

where

$$g^a_{x,x+1}(\eta) = \frac{1}{2} \left[ \xi^a_x(\eta) (\xi^a_{x+1}(\eta) - \xi^a_x(\eta)) + \xi^a_x(\eta) (\xi^a_{x+1}(\eta) - \xi^a_{x+2}(\eta)) \right],$$

$$f^a_1(\eta) = (-r_{a+1} - r_{a+2}) \xi^a_1(\eta) + r_{a+1} \xi^a_{a+1}(\eta) + r_{a+2} \xi^a_{a+2}(\eta),$$

$$f^a_{N-1}(\eta) = (-\bar{r}_{a+1} - \bar{r}_{a+2}) \xi^a_{N-1}(\eta) + \bar{r}_{a+1} \xi^a_{a+1}(\eta) + \bar{r}_{a+2} \xi^a_{a+2}(\eta),$$

$$h^a_1(\eta) = \frac{1}{2} \left[ (r_{a+2} - r_{a+1}) \xi^a_1(\eta) - r_a \xi^a_{a+1}(\eta) + r_a \xi^a_{a+2}(\eta) \right],$$

$$h^a_{N-1}(\eta) = \frac{1}{2} \left[ (\bar{r}_{a+2} - \bar{r}_{a+1}) \xi^a_{N-1}(\eta) + \bar{r}_a \xi^a_{a+1}(\eta) - \bar{r}_a \xi^a_{a+2}(\eta) \right].$$

(3.3)

Above we used the usual notions of discrete Laplacian and discrete derivative as:

$$\nabla^+_N \phi(\frac{x}{N}) = N[\phi(\frac{x+1}{N}) - \phi(\frac{x}{N})]$$

and

$$\Delta_N \phi(\frac{x}{N}) = N^2[\phi(\frac{x+1}{N}) - 2\phi(\frac{x}{N}) + \phi(\frac{x-1}{N})].$$

Using the relations in (2.5) and the exclusion rule $\xi^a_1 + \xi^a_{-1} = 1$, we have

$$f^a_1(\eta) = r_a - \xi^a_1(\eta),$$

$$f^a_{N-1}(\eta) = \bar{r}_a - \xi^a_{N-1}(\eta),$$

$$h^a_1(\eta) = \frac{1}{2} \left[ 2(r_{a+2} - r_{a+1}) \xi^a_1(\eta) - 2r_a \xi^a_{a+1}(\eta) + r_a \right],$$

$$h^a_{N-1}(\eta) = \frac{1}{2} \left[ 2(\bar{r}_{a+2} - \bar{r}_{a+1}) \xi^a_{N-1}(\eta) + 2\bar{r}_a \xi^a_{a+1}(\eta) - \bar{r}_a \right].$$

(3.4)

At this point we need to analyse each term in (3.2). In the next subsection, we explain how to obtain heuristically the weak formulation presented in Definition 2.3. The rigorous proof will be given in Section 5.
3.1 The Dirichlet case

Consider \( \phi \in C^2_c([0, 1]) \). Since \( \phi \) has a compact support, the third, fourth and fifth lines of (3.2) vanish, for \( N \) sufficiently big. This means that we can rewrite (3.2) as

\[
\mathcal{M}^{N, \alpha}_t(\phi) = \langle \pi^{N, \alpha}_t, \phi \rangle - \langle \pi^{N, \alpha}_0, \phi \rangle - \int_0^t ds \langle \pi^{N, \alpha}_s, \Delta_N \phi \rangle + \frac{\beta}{2N} \int_0^t ds \sum_{x=1}^{N-2} \nabla^+_N \phi(\frac{x}{N}) \left[ \xi^{a+1}_x(\eta_s)(\xi^{a+2}_x(\eta_s) - \xi^{a+2}_x(\eta_s)) + \xi^{a}(\eta_s)(\xi^{a+1}_x(\eta_s) - \xi^{a+2}_x(\eta_s)) \right]
\]

(3.5)

plus terms that vanish as \( N \to +\infty \). For \( \ell \in \mathbb{N} \) and \( x \in \mathbb{Z} \) we introduce the averages on boxes of size \( \ell \), one to the right of \( x \), another one to the left of \( x \):

\[
\bar{\xi}_x^{a, \ell}(\eta) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} \xi_y^a(\eta), \quad \bar{\xi}_x^{a, \ell}(\eta) = \frac{1}{\ell} \sum_{y=x-\ell}^{x-1} \xi_y^a(\eta).
\]

(3.6)

First we observe that by paying a price of order \( O(\varepsilon) \), we can restrict the sum in the last line of (3.5) to \( x \in \Lambda^\varepsilon_N \) where

\[
\Lambda^\varepsilon_N := \{ \varepsilon N + 1, \cdots, N - 1 - \varepsilon N \}.
\]

(3.7)

Above and in what follows, \( \varepsilon N \) represents \( \varepsilon N \). Using repeatedly Lemma 6.7 we can rewrite the last line of (3.5) as

\[
\frac{\beta}{2N} \int_0^t ds \sum_{x \in \Lambda^\varepsilon_N} \nabla^+_N \phi(\frac{x}{N}) \left[ \bar{\xi}^{a+1}_x(\eta_s)(\bar{\xi}^{a+2}_x(\eta_s) - \bar{\xi}^{a+2}_x(\eta_s)) + \bar{\xi}^{a}(\eta_s)(\bar{\xi}^{a+1}_x(\eta_s) - \bar{\xi}^{a+2}_x(\eta_s)) \right]
\]

(3.8)

Before we go on we explain how we do it. First, we split the integral on the second line of (3.5), with the sum restricted to \( \Lambda^\varepsilon_N \) in several terms, i.e. we write it as

\[
\frac{\beta}{2N} \int_0^t ds \sum_{x \in \Lambda^\varepsilon_N} \nabla^+_N \phi(\frac{x}{N}) \xi_x^{a+1}(\eta_s) \xi_x^{a+2}(\eta_s) - \frac{\beta}{2N} \int_0^t ds \sum_{x \in \Lambda^\varepsilon_N} \nabla^+_N \phi(\frac{x}{N}) \xi_x^{a+1}(\eta_s) \xi_x^{a+2}(\eta_s)
\]

\[
+ \frac{\beta}{2N} \int_0^t ds \sum_{x \in \Lambda^\varepsilon_N} \nabla^+_N \phi(\frac{x}{N}) \xi_x^a(\eta_s) \xi_{x+1}^{a+1}(\eta_s) - \frac{\beta}{2N} \int_0^t ds \sum_{x \in \Lambda^\varepsilon_N} \nabla^+_N \phi(\frac{x}{N}) \xi_x^a(\eta_s) \xi_{x+1}^{a+2}(\eta_s).
\]

(3.9)

Now in the first integral in last display, we apply Lemma 6.7 with \( G_s(\frac{x}{N}) = \frac{\beta}{2N} \nabla^+_N \phi_s(\frac{x}{N}) \), \( \tau_s \psi(\eta) = \xi^{a+1}_s(\eta) \) and we make the replacement of \( \xi^{a+1}_x \) by \( \xi^{a+1}_x \). In all the remaining terms we do a similar replacement. Noting that for \( u \in [0, 1] \) if

\[
\bar{\tau}_c^e(\frac{x}{N})(u) = \frac{1}{e} 1_{\left[\frac{x}{N}, \frac{x}{N} + e\right]}(u) \quad \text{and} \quad \bar{\tau}_c^e(\frac{x}{N})(u) = \frac{1}{e} 1_{\left[\frac{x}{N} - e, \frac{x}{N}\right]}(u),
\]

(3.10)

then, for any \( s \in [0, T] \)

\[
\langle \pi^{N, \alpha}_s, \bar{\tau}_c^e(\frac{x}{N}) \rangle = \frac{1}{eN} \sum_{z=x+1}^{x+eN} \xi^a_z(\eta_s), \quad \langle \pi^{N, \alpha}_s, \bar{\tau}_c^e(\frac{x}{N}) \rangle = \frac{1}{eN} \sum_{z=x-eN}^{x-1} \xi^a_z(\eta_s),
\]

(3.11)
and we can rewrite (3.9) as

\[
\frac{\beta}{2N} \int_0^t ds \sum_{x \in \Lambda'_N} \nabla^+_N \phi(\xi^x_N) \bigg[ \xi^x_{x+1}(\eta_s) \big( \langle \pi^N_{s,a+1}, \xi^x_{e}(\eta_s) \rangle - \langle \pi^N_{s,a+2}, \xi^x_{e}(\eta_s) \rangle \big) \\
+ \xi^x_{x}(\eta_s) \big( \langle \pi^N_{s,a+1}, \xi^x_{e}(\eta_s) \rangle - \langle \pi^N_{s,a+2}, \xi^x_{e}(\eta_s) \rangle \big) \bigg].
\]

By paying a price of order \(O(\epsilon)\) in order to convert the sum in \(\Lambda'_N\) to the whole sum, we can rewrite the last term as

\[
\frac{\beta}{2} \int_0^t \left( \pi^N_{s,a}, \nabla^+_N \phi(\cdot) \left( \langle \pi^N_{s,a+1}, \xi^x_{e}(\cdot) \rangle - \langle \pi^N_{s,a+2}, \xi^x_{e}(\cdot) \rangle \right) \\
+ \left( \pi^N_{s,a}, \nabla^+_N \phi(\cdot) \left( \langle \pi^N_{s,a+1}, \xi^x_{e}(\cdot) \rangle - \langle \pi^N_{s,a+2}, \xi^x_{e}(\cdot) \rangle \right) \right) ds.
\]

Putting last identity back to (3.5) we have that

\[
\mathcal{M}^N_t(\phi) = \langle \pi^N_t, \phi \rangle - \langle \pi^N_0, \phi \rangle \int_0^t ds \langle \pi^N_s, \Delta_N \phi \rangle \\
+ \frac{\beta}{2} \int_0^t \left( \pi^N_{s,a}, \nabla^+_N \phi(\cdot) \left( \langle \pi^N_{s,a+1}, \xi^x_{e}(\cdot) \rangle - \langle \pi^N_{s,a+2}, \xi^x_{e}(\cdot) \rangle \right) \\
+ \left( \pi^N_{s,a}, \nabla^+_N \phi(\cdot) \left( \langle \pi^N_{s,a+1}, \xi^x_{e}(\cdot) \rangle - \langle \pi^N_{s,a+2}, \xi^x_{e}(\cdot) \rangle \right) \right) ds,
\]

plus terms that vanish as \(N \to +\infty\) and \(\epsilon \to 0\). From the computations of Section 4 we will see that the martingale vanishes in \(L^2(\mathbb{P}_{\mu_N})\), as \(N \to +\infty\). Moreover, from the results of Section 4, we can also assume the weak convergence of \(\pi^N_t(\eta, du)\) to \(\pi^t_t(du) = \rho_t^a(u) du\), for each \(\alpha \in \{A, B, \emptyset\}\). On the other hand, since \(\rho_t^a(u) \in [0, 1]\) for all \(t \in [0, T]\) and \(u \in [0, 1]\), from Lebesgue’s differentiation theorem it holds

\[
\lim_{\epsilon \to 0} \left| \rho^a_t(u) - \frac{1}{t} \int_u^{u+\epsilon} \rho^a_t(\nu) d\nu \right| = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \left| \rho^a_t(u) - \frac{1}{t} \int_{u-\epsilon}^u \rho^a_t(\nu) d\nu \right| = 0
\]

for almost every \(u \in [0, 1]\). As a consequence, (3.1) converges, as \(N \to +\infty\), to

\[
0 = \langle \rho^a_t, \phi_t \rangle - \langle \rho^a_0, \phi_0 \rangle - \int_0^t ds \langle \rho^a_s, (\mathcal{E}_s + \Delta) \phi_s \rangle + \beta \int_0^t ds \langle \rho^a_s (\rho^a_{s+1} - \rho^a_{s+2}), \nabla \phi_s \rangle.
\]

We note that above we gave an heuristic argument to obtain the integral formulation as given in (2.8). We note that the proof of condition D2, is explained in Section 6.6.

In the next subsection, we explain a similar argument as given above, to obtain the weak formulation of (2.9).

### 3.2 The Robin cases

Now we do not assume any condition on the test function, i.e. we consider \(\phi \in C^{1,2}([0, T] \times [0, 1])\). In this case we need to analyse carefully each line in (3.2). The analysis of the first and second lines can be done exactly as in the Dirichlet case of the previous subsection. Now let us analyse the remaining terms. The fourth line does not depend on the value of the parameters \(\theta\) nor \(\delta\). From Lemma 6.6 we can rewrite that term as

\[
- \int_0^t ds \left[ \nabla^+_N \phi_s(0) \xi_{eN}(\eta_s) - \nabla^+_N \phi_s(\eta_s) \xi_{eN}(\eta_s) \right],
\]

(3.16)
plus terms that vanish as \( N \to +\infty \) and \( \epsilon \to 0 \). As in the previous case, we could now try to use (3.14) but as we have seen above, it is only true for almost every \( u \in [0,1] \) and here we need this to be true for \( u = 0 \) and \( u = 1 \). The way to conclude that result, is to use Lemma 6.12, which is similar to Lemma 6.2 in [6]. From those results we conclude that the last display can be written as

\[
- \int_0^t \left[ \nabla \phi_s(0) \rho_s^a(0) - \nabla \phi_s(1) \rho_s^a(1) \right] ds
\]

(3.17)

plus terms that vanish as \( N \to +\infty \). Now we analyse the third line of (3.2). If \( \theta > 1 \), it is simple to conclude that the whole term vanishes as \( N \to +\infty \). Let us now see the case \( \theta = 1 \). Then it rewrites as

\[
- \int_0^t \left[ \phi_s(0)(r_a - \xi_1^a(\eta_1)) + \phi_s(1)(\bar{r}_a - \xi_N^a(\eta_1)) \right] ds.
\]

Applying Lemma 6.6 we can rewrite it as

\[
- \int_0^t ds \left[ \phi_s(0)(r_a - \xi_{\frac{1}{N}} r_1^a(\eta_1)) + \phi_s(1)(\bar{r}_a - \xi_{\frac{1}{N}} N^a(\eta_1)) \right]
\]

plus terms that vanish as \( N \to +\infty \) and \( \epsilon \to 0 \). As before, using the same arguments as the ones we used in the analysis of the previous terms, we can rewrite last display as

\[
- \int_0^t \left[ \phi_s(0)(r_a - \rho_s^a(0)) + \phi_s(1)(\bar{r}_a - \rho_s^a(1)) \right] ds.
\]

Finally we treat the last term in (3.2). For \( \theta > 1 \) it is simple to see that term vanishes as \( N \to +\infty \). For \( \theta = 1 \) and repeating the same arguments as before, that term can be rewritten as

\[
- \frac{\tilde{\beta}}{2} \int_0^t \phi_s(0) \left[ (2r_{a+2} - 1)\rho_s^a(0) - 2r_a \rho_s^{a+1}(0) + r_a \right] ds
\]

\[
- \frac{\tilde{\beta}}{2} \int_0^t \phi_s(1) \left[ (1 - 2\bar{r}_{a+2})\rho_s^a(1) + 2\bar{r}_a \rho_s^{a+1}(1) - \bar{r}_a \right] ds.
\]

Putting together last results we deduce that the limit of (3.2) gives for \( \theta \geq \delta > 1 \)

\[
0 = \langle \rho_s^a, \phi_s \rangle - \langle \rho_s^0, \phi_s \rangle - \int_0^t ds \langle \rho_s^a, (\partial_s + \Delta)\phi_s \rangle + \beta \int_0^t ds \langle \rho_s^a(\rho_s^{a+1} - \rho_s^a), \nabla \phi_s \rangle
\]

\[
- \int_0^t \left[ \nabla \phi_s(0) \rho_s^a(0) - \nabla \phi_s(1) \rho_s^a(1) \right] ds,
\]

(3.18)

which coincides with (2.10) for the choice \( \kappa_1 = \kappa_2 = 0 \).

For \( \theta > \delta = 1 \), the limit of (3.2) gives

\[
0 = \langle \rho_s^a, \phi_s \rangle - \langle \rho_s^0, \phi_s \rangle - \int_0^t ds \langle \rho_s^a, (\partial_s + \Delta)\phi_s \rangle + \beta \int_0^t ds \langle \rho_s^a(\rho_s^{a+1} - \rho_s^a), \nabla \phi_s \rangle
\]

\[
- \int_0^t \left[ \nabla \phi_s(0) \rho_s^a(0) - \nabla \phi_s(1) \rho_s^a(1) \right] ds
\]

\[
- \frac{\tilde{\beta}}{2} \int_0^t \phi_s(0) \left[ (2r_{a+2} - 1)\rho_s^a(0) - 2r_a \rho_s^{a+1}(0) + r_a \right] ds
\]

\[
- \frac{\tilde{\beta}}{2} \int_0^t \phi_s(1) \left[ (1 - 2\bar{r}_{a+2})\rho_s^a(1) + 2\bar{r}_a \rho_s^{a+1}(1) - \bar{r}_a \right] ds
\]

(3.19)
which coincides with (2.10) for the choice \( \kappa_1 = 0 \) and \( \kappa_2 = 1 \).

Finally, for \( \theta = 1 = \delta \), the limit of (3.2) gives

\[
0 = \langle \rho^a_T, \phi \rangle - \langle \rho^a_0, \phi \rangle - \int_0^t ds \langle \rho^a_s, (\partial_s + \Delta) \phi_s \rangle + \beta \int_0^t ds \langle \rho^a_s (\rho^{a+1}_s - \rho^{a+2}_s), \nabla \phi_s \rangle - \int_0^t [\nabla \phi_s (0) \rho^a_s (0) - \nabla \phi_s (1) \rho^a_s ] ds
\]

\[
- \frac{\tilde{\beta}}{2} \int_0^t [\phi_s (0) [(2r_{a+2} - 1) \rho^a_s (0) - 2r_a \rho^a_s (1) + r_a] ds
\]

which coincides with (2.10) for the choice \( \kappa_1 = \tilde{\beta}/2 \) and \( \kappa_2 = 1 \).

4 Tightness

For \( \alpha \in \{A, B, \emptyset\} \) denote by \( \mathbb{Q}^{N,\alpha} = \mathbb{P}_{\mu_N} (\pi^{N,\alpha})^{-1} \) the probability measure on \( \mathcal{G}([0, T], \mathcal{M}) \) induced by the Markov process \( \{\pi_t^{N,\alpha} : t \geq 0\} \) and by the initial measure \( \mu_N \). In this section we will prove that the sequence \( \{\mathbb{Q}^{N,\alpha}\}_{N \geq 1} \) is tight and this will ensure that every subsequence of \( \{\mathbb{Q}^{N,\alpha}\}_{N \geq 1} \) has a further subsequence which is weakly convergent.

**Proposition 4.1.** For \( \alpha \in \{A, B, \emptyset\} \), the sequence of measures \( \{\mathbb{Q}^{N,\alpha}\}_{N \geq 1} \) is tight.

**Proof.** By Proposition 1.7 of Chapter 4 of [15] it is enough to show, for every \( \phi \in C^1([0, 1]) \), that the sequence of measures associated with the real-valued process \( \{\langle \pi_t^{N,\alpha}, \phi \rangle\}_{N \geq 1} \) is tight, and for that it is enough to show that, for every \( \phi \in C([0, 1]) \) and \( \varepsilon > 0 \),

\[
\lim_{\gamma \to 0} \limsup_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{[t-\varepsilon] \leq \gamma} |\langle \pi_t^{N,\alpha}, \phi \rangle - \langle \pi_t^{N,\alpha}, \phi \rangle| > \varepsilon \right] = 0. \quad (4.1)
\]

We begin by showing that (4.1) holds for every \( \phi \in C^2([0, 1]) \), and we note that the extension to \( \phi \in C([0, 1]) \) is a simple argument that can be seen, for example, in Section 4 of [2].

Considering the Dynkin's martingale (3.1), in order to conclude (4.1) it is enough to prove that

\[
\lim_{\gamma \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \sup_{[t-\varepsilon] \leq \gamma} \int_{[t-\varepsilon] \leq \gamma} N^2 \mathcal{L}_N (\pi_t^{N,\alpha}, \phi) dr \right] = 0 \quad (4.2)
\]

and

\[
\lim_{\gamma \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \sup_{[t-\varepsilon] \leq \gamma} \left| \mathcal{M}_t^{N,\alpha}(\phi) - \mathcal{M}_t^{N,\alpha}(\phi) \right| = 0. \quad (4.3)
\]

Since we are assuming that \( \phi \) has compact support in \( (0, 1) \), there exist \( N_0 \) such that, for all \( N \geq N_0 \), \( \phi(0) = \phi(1) = \phi(\frac{N-1}{N}) = 0 \), and so in (4.2) we can replace the full generator \( \mathcal{L}_N \) by the bulk generator \( \mathcal{L}_N^B \). Recall the function \( g^a_{x,x+1}(\eta) \) defined in (3.3). Since \( \xi^a_x(\eta) \in (0, 1) \) and \( |g^a_{x,x+1}(\eta)| \leq 1 \), by the mean value theorem, for any \( r \in [0, T] \), it holds

\[
\left| N^2 \mathcal{L}_N (\pi_t^{N,\alpha}, \phi) \right| \leq \frac{1}{N} \sum_{x=1}^{N-1} \Delta_N \phi(\frac{x}{N}) \xi^a_x(\eta_r) - \frac{\tilde{\beta}}{N} \sum_{x=1}^{N-2} \nabla^+_N \phi(\frac{x}{N}) g^a_{x,x+1}(\eta_r) \]

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\[\leq 2\|\phi''\|_{\infty} + \beta\|\phi'\|_{\infty}.\]

Therefore, there exists a constant \(C\) such that \(|N^{2}\mathcal{L}_{N}(\pi^{N,\alpha}_{t}, \phi)| < C\) for all \(N \geq N_0\), and this implies (4.2). Now let us verify (4.3). By the triangular, Jensen and Doob’s inequalities

\[
E_{\mu_{N}} \left[ \sup_{|t-s| \leq T} |\mathcal{M}^{N,\alpha}_{t}(\phi) - \mathcal{M}^{N,\alpha}_{s}(\phi)| \right] \leq 4E_{\mu_{N}} \left[ (\mathcal{M}^{N,\alpha}_{T}(\phi))^{2} \right]^{1/2}. \tag{4.4}
\]

To estimate (4.4), observe that

\[
\mathcal{M}^{N,\alpha}_{t}(\phi)^2 - \int_{0}^{t} N^{2}\mathcal{L}_{N}(\pi^{N,\alpha}_{s}, \phi)^2 - 2N^{2}(\pi^{N,\alpha}_{s}, \phi)\mathcal{L}_{N}(\pi^{N,\alpha}_{s}, \phi)ds
\]

is a mean zero martingale and a standard computation shows that the rightmost term in last display is, for all \(N \geq N_0\), equal to

\[
\int_{0}^{t} \frac{1}{N^{2}} \sum_{x=1}^{N-2} c_{x,x+1}(\eta_{s})(\nabla_{N} \phi(\frac{s}{N}))^{2}[\xi_{x+1}^{a}(\eta_{s}) - \xi_{x}^{a}(\eta_{s})]^{2}ds. \tag{4.5}
\]

From this and the fact that the rates \(c_{x,x+1}\) are bounded, we get

\[
E_{\mu_{N}} \left[ (\mathcal{M}^{N,\alpha}_{T}(\phi))^{2} \right] \leq \frac{C}{N}\|\phi'\|_{\infty}^{2}
\]

which vanishes as \(N \to \infty\). This concludes the proof. \(\square\)

### 5 Characterization of limit points

Let \(\pi^{N} = (\pi^{N,A}, \pi^{N,B}, \pi^{N,\emptyset}) : \Omega^{N} \to \mathcal{M}^{3}\), and consider \(Q^{N} = P_{\mu_{N}}(\pi^{N})^{-1}\), the probability measure on \(\mathcal{D}([0, T], \mathcal{M}^{3})\) induced by the process \(\{\pi^{N}_{t} : t \geq 0\}\) and by the initial measure \(\mu_{N}\). In this section we prove that the limit points \(Q\) of the sequence \(\{Q^{N}\}_{N \geq 1}\) are concentrated on trajectories, whose marginals are absolutely continuous with respect to the Lebesgue measure and whose densities \(\rho^{a}_{t}(u), a \in \{A,B,\emptyset\}\), are weak solutions of the corresponding hydrodynamic equation.

Recall the definitions of \(F_{\text{Dir}}\) and \(F_{\text{Rob}}\) in (2.8) and (2.10). Let \(C_{\text{Dir}} = C^{1,2}_{c}([0, T] \times [0, 1])\) and \(C_{\text{Rob}} = C^{1,2}_{c}([0, T] \times [0, 1])\).

**Proposition 5.1.** Let \(Q\) be any limit point of the sequence \(\{Q^{N}\}_{N \geq 1}\). Then, for \(a \in \{A,B,\emptyset\}\)

\[Q^{N}(\pi_{t} \in \mathcal{D}([0, T], \mathcal{M}^{3}) : F^{a}_{t}(t, \phi, \rho, g) = 0, \forall t \in [0, T], \forall \phi \in C_{T}) = 1,\]

where \(\Gamma = \text{Dir}\) or \(\Gamma = \text{Rob}\), depending on the values of \(\delta\) and \(\theta\), as in the cases a) and b) in Theorem 2.7.

**Proof.** We start with the case \(\Gamma = \text{Dir}\), i.e. the Dirichlet boundary conditions, the proof of the Robin case being almost identical. It is enough to show that, for any \(\phi \in C^{1,2}_{c}([0, T] \times [0, 1])\) and for any \(\tilde{\delta} > 0\),

\[Q(\pi_{t} \in \mathcal{D}([0, T], \mathcal{M}^{3}) : \sup_{0 \leq t \leq T} |F^{a}_{\text{Dir}}(t, \phi, \rho, g)| > \tilde{\delta} = 0,\]

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Using the definitions (2.8) and (3.6) and by summing and subtracting appropriate terms we can bound the probability from above by the sum of the following probabilities:

\[
\begin{align*}
Q & \left( \sup_{0 \leq t \leq T} \left| \langle \rho^a_t, \phi_t \rangle - \langle \rho^a_0, \phi_0 \rangle - \int_0^t \langle \rho^a_s, \partial_s \phi_s \rangle ds - \int_0^t \int_0^{1-\varepsilon} \langle \pi^a_s, \tau^a_e(u) \rangle \Delta \phi_s(u) duds \right| > \frac{\delta}{5} \right) \\
& + \frac{\beta}{2} \int_0^t \int_0^{1-\varepsilon} \nabla \phi_s(u)(\pi^a_s, \tau^a_e(u)) \left( \langle \pi^a_{s+1}, \tau^a_e(u) \rangle - \langle \pi^a_{s+2}, \tau^a_e(u) \rangle \right) duds \\
& + \frac{\beta}{2} \int_0^t \int_0^{1-\varepsilon} \nabla \phi_s(u)(\pi^a_s, \tau^a_e(u)) \left( \langle \pi^a_{s+1}, \tau^a_e(u) \rangle - \langle \pi^a_{s+2}, \tau^a_e(u) \rangle \right) duds \right) > \frac{\delta}{5} \right), \tag{5.1}
\end{align*}
\]

\[
\begin{align*}
Q & \left( \sup_{0 \leq t \leq T} \left| \langle \rho^a_0, \phi_0 \rangle - \langle \rho^a_0, \phi_0 \rangle \right| > \frac{\delta}{5} \right) \\
& + \frac{\beta}{2} \int_0^t \int_0^{1-\varepsilon} \nabla \phi_s(u)(\rho^a_s(u)\rho^a_{s+1}(u) - \rho^a_{s+2}(u)) \\
& - \langle \pi^a_s, \tau^a_e(u) \rangle \left( \langle \pi^a_{s+1}, \tau^a_e(u) \rangle - \langle \pi^a_{s+2}, \tau^a_e(u) \rangle \right) duds \right| > \frac{\delta}{5} \right), \tag{5.3}
\end{align*}
\]

\[
\begin{align*}
Q & \left( \sup_{0 \leq t \leq T} \left| \langle \rho^a_0, \phi_0 \rangle - \langle \rho^a_0, \phi_0 \rangle \right| > \frac{\delta}{5} \right) \\
& + \frac{\beta}{2} \int_0^t \int_0^{1-\varepsilon} \nabla \phi_s(u)(\pi^a_s, \tau^a_e(u)) \left( \langle \pi^a_{s+1}, \tau^a_e(u) \rangle - \langle \pi^a_{s+2}, \tau^a_e(u) \rangle \right) duds \right| > \frac{\delta}{5} \right), \tag{5.4}
\end{align*}
\]

\[
\begin{align*}
Q & \left( \sup_{0 \leq t \leq T} \left| \langle \rho^a_s, \Delta \phi_s \rangle ds - \int_0^t \int_0^{1-\varepsilon} \langle \pi^a_s, \tau^a_e(u) \rangle \Delta \phi_s(u) duds \right| > \frac{\delta}{5} \right). \tag{5.5}
\end{align*}
\]

Above we used the notation \( \tau^a_e(u)(v) = \frac{1}{2} 1_{[u,u+\varepsilon]}(v) \) and \( \tau^a_e(u)(v) = \frac{1}{2} 1_{[u-\varepsilon,u]}(v) \). Since \( \mu_N \) satisfies (2.11), (5.2) is equal to zero. For (5.5) we use (3.14) and we conclude that it goes to zero as \( \varepsilon \to 0 \). To show that (5.3) and (5.4) go to zero, we repeat exactly the previous argument together with (3.14).

Now we have to deal with (5.1). Since the set inside the probability is an open set, by Portmanteau’s theorem\(^1\) we can bound (5.1) by

\[
\liminf_{N \to \infty} Q^N \left( \sup_{0 \leq t \leq T} \left| \langle \rho^a_t, \phi_t \rangle - \langle \rho^a_0, \phi_0 \rangle - \int_0^t \langle \rho^a_s, \partial_s \phi_s \rangle ds - \int_0^t \int_0^{1-\varepsilon} \langle \pi^a_s, \tau^a_e(u) \rangle \Delta \phi_s(u) duds \right| > \frac{\delta}{5} \right) \\
+ \frac{\beta}{2} \int_0^t \int_0^{1-\varepsilon} \nabla \phi_s(u)(\pi^a_s, \tau^a_e(u)) \left( \langle \pi^a_{s+1}, \tau^a_e(u) \rangle - \langle \pi^a_{s+2}, \tau^a_e(u) \rangle \right) duds \\
+ \frac{\beta}{2} \int_0^t \int_0^{1-\varepsilon} \nabla \phi_s(u)(\pi^a_s, \tau^a_e(u)) \left( \langle \pi^a_{s+1}, \tau^a_e(u) \rangle - \langle \pi^a_{s+2}, \tau^a_e(u) \rangle \right) duds \right) > \frac{\delta}{5} \right). \tag{5.6}
\]

\(^1\)To use Portmanteau’s theorem we actually first have to approximate \( \tau^a_e(u) \) and \( \tau^a_e(u) \) by continuous functions, in such a way that the error vanishes as \( \varepsilon \to 0 \).
We sum and subtract inside the absolute value the term $\int_0^T N^2 \mathcal{L}_N (\pi^N, \phi_s) ds$, and by (3.5) we can bound the last probability from above by

$$
\mathbb{P}_{\mu_N} \left( \sup_{0 \leq t \leq T} |A_t^N| > \frac{\delta}{10} \right) + \mathbb{P}_{\mu_N} \left( \sup_{0 \leq t \leq T} \int_0^t N^2 \mathcal{L}_N (\pi^N, \phi_s) ds 
- \int_0^t \int_0^{1-\epsilon} (\pi^N_{s}, \tau^+_{e}(u)) \Delta \phi_s(u) duds 
+ \beta \int_0^t \int_0^{1-\epsilon} \nabla \phi_s(u)(\pi^N_{s}, \tau^+_{e}(u)) (\pi^N_{s+1}, \tau^-_{e}(u)) du ds \right) > \frac{\delta}{10}.
$$

From Doob’s inequality and (4.5), the first probability vanishes as $N \to \infty$. By making explicit the action of the generator, we can bound the second probability by the sum of the following terms:

$$
\mathbb{P}_{\mu_N} \left( \sup_{0 \leq t \leq T} \left| \int_0^t (\pi^N_{s+1}, \tau^-_{e}(u)) \right| \right) \Delta \phi_s(u) duds > \frac{\delta}{30},
$$

(5.7)

$$
\mathbb{P}_{\mu_N} \left( \sup_{0 \leq t \leq T} \left| \frac{\beta}{2} \int_0^t \left( \frac{1}{N} \sum_{x=1}^{N-2} \nabla^+_{N, x} \phi_s (\pi^N_{s}, \tau^+_{e}(u)) (\xi^a_{x+1}(\eta_s) - \xi^a_{x+2}(\eta_s)) \right) - \int_0^t \nabla \phi_s(u)(\pi^N_{s+1}, \tau^-_{e}(u)) (\pi^N_{s+1}, \tau^-_{e}(u)) du \right| > \frac{\delta}{30},
$$

(5.8)

To treat (5.7) it is enough to use Taylor expansion of $\phi_s$ and we see that (5.7) vanishes as $N \to \infty$ and $\epsilon \to 0$. For (5.8), we can consider only the first probability, since the second one is analogous. First, we split the term $\sum_{x=1}^{N-2} \nabla^+_{N, x} \phi_s (\pi^N_{s}, \tau^+_{e}(u)) (\xi^a_{x+1}(\eta_s) - \xi^a_{x+2}(\eta_s))$ and then we compare $\frac{1}{N} \sum_{x=1}^{N-2} \nabla^+_{N, x} \phi_s (\pi^N_{s}, \tau^+_{e}(u)) (\pi^N_{s+1}, \tau^-_{e}(u)) du$ for $i = 1, 2$. In order to do this, we sum and subtract $\xi^a_{x}(\eta_s) (\pi^N_{s+1}, \tau^-_{e}(u)) + \xi^a_{x+1}(\eta_s) (\pi^N_{s+1}, \tau^-_{e}(u))$ for $i = 1, 2$. Finally, we approximate $\nabla \phi$ with its discrete derivative $\nabla^+_{N, x} \phi_s (\pi^N_{s}, \tau^+_{e}(u))$ and we observe that, since $\sum_{x=1}^{N-2} \nabla^+_{N, x} \phi_s (\pi^N_{s}, \tau^+_{e}(u)) = (\xi^a_{x, s}(\eta_s) + \xi^a_{x+1, s}(\eta_s)) (\pi^N_{s+1}, \tau^-_{e}(u))$ if we reduce the sum to $\Lambda^e_N$, and we apply Lemma 6.6 to $\xi^a_{x}(\eta_s)$ and $\xi^a_{x+1}(\eta_s)$, we prove that (5.8) goes to zero as $N \to \infty$ and $\epsilon \to 0$.

For $\Gamma = \text{Rob}$, by (2.10) other boundary terms will appear in (5.1) and in addition to (5.2)-(5.5). We prove that the probability of one of these terms goes to zero as $N \to \infty$ and $\epsilon \to 0$. The rest is proved analogously. Let us consider, for example, $\int_0^t [\nabla \phi_s(0) - \nabla \phi_s(1)] duds$. We sum and subtract

$$
\int_0^t \left[ \nabla \phi_s(0) (\pi^a_{s}, \tau^a_{e}(0)) - \nabla \phi_s(1) (\pi^a_{s}, \tau^a_{e}(1)) \right] ds
$$
in \( F^a_{\text{Rob}}(t, \phi, \rho, g) \). As a consequence, we have to add in the absolute value of (5.1)

\[
\kappa_1 \int_0^t \left[ \nabla \phi_s(0)(\rho^a_s(0) - (\pi^a_s, \bar{t}^a_e(0))) - \nabla \phi_s(1)(\rho^a_s(1) - (\pi^a_s, \bar{t}^a_e(1))) \right] ds \tag{5.9}
\]

and by (3.14) the contribution from these terms, vanish as \( \epsilon \to 0 \). While (5.9) will results into extra terms in (5.6) and produce the terms

\[
\mathbb{P}_{\mu_N} \left( \sup_{0 \leq t \leq T} \left| \kappa_1 \int_0^t \left[ \nabla_N^+ \phi_s(\frac{N-1}{N}) \xi^a_s(\eta_s) - \nabla \phi_s(0)(\pi^a_s, \bar{t}^a_e(0)) \right] ds \right| > \bar{\epsilon} \right)
\]

\[
\mathbb{P}_{\mu_N} \left( \sup_{0 \leq t \leq T} \left| \kappa_1 \int_0^t \left[ \nabla_N^+ \phi_s(\frac{N-1}{N}) \xi^a_s(\eta_s) - \nabla \phi_s(1)(\pi^a_s, \bar{t}^a_e(1)) \right] ds \right| > \bar{\epsilon} \right).
\]

After approximating the continuous gradients with the discrete ones, these terms vanish as \( N \to \infty \) and \( \epsilon \to 0 \) as a consequence of Lemma 6.6 and (3.14).

\section{Replacement lemmas and energy estimate}

\subsection{Entropy bounds}

For a profile \( \rho = (\rho^A, \rho^B, \rho^\varnothing) : [0, 1] \to [0, 1]^3 \) with \( \rho^A + \rho^B + \rho^\varnothing = 1 \) let \( \nu^N_{\rho(.)} \) be the product measure on \( \Omega_N \) whose marginals are given by

\[
\nu^N_{\rho(.)}(\eta(x) = \alpha) = \rho^a\left(\frac{x}{N}\right), \quad \alpha \in \{A, B, \varnothing\}. \tag{6.1}
\]

The first estimate we need is an upper bound on the relative entropy \( H(\mu_N|\nu^N_{\rho(.)}) \) of the measure \( \mu_N \) with respect to \( \nu^N_{\rho(.)} \).

\textbf{Lemma 6.1.} Let \( \rho = (\rho^A, \rho^B, \rho^\varnothing) : [0, 1] \to [0, 1]^3 \) be a profile for which there exists some \( r_0 > 0 \) such that \( \rho^a(u) \geq r_0 \) for each \( u \in [0, 1] \) and \( \alpha \in \{A, B, \varnothing\} \). Then, there exists a constant \( C_0 \) such that \( H(\mu_N|\nu^N_{\rho(.)}) \leq C_0 N \) for every measure \( \mu_N \) on \( \Omega_N \).

\textbf{Proof.} Observe that \( \nu^N_{\rho(.)}(\eta) = \prod_{x} \sum_{\alpha} \left( \rho^a\left(\frac{x}{N}\right) \right)^{\xi^a_s(\eta)} \geq r_0^N \). Then, by the definition of relative entropy, we have that

\[
H(\mu_N|\nu^N_{\rho(.)}) = \sum_{\eta \in \Omega_N} \mu_N(\eta) \log \left( \frac{\mu_N(\eta)}{\nu^N_{\rho(.)}(\eta)} \right) \leq \sum_{\eta \in \Omega_N} \mu_N(\eta) \log \left( \frac{1}{\nu^N_{\rho(.)}(\eta)} \right) \leq N \log \frac{1}{r_0} \leq C_0 N. \tag{6.2}
\]

\subsection{Dirichlet forms}

Recall (6.1). We introduce the quadratic form, i.e. the non-negative functions associated to the bulk and the boundary generators given by:

\[
\mathcal{D}^B_N(\sqrt{f}, \nu^N_{\rho(.)}) = \sum_{x=1}^{N-2} \int_{\Omega_N} c_{x,x+1}(\eta)(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)})^2 d \nu^N_{\rho(.)} \tag{6.3}
\]

\[
\mathcal{D}^L_N(\sqrt{f}, \nu^N_{\rho(.)}) = \int_{\Omega_N} \left\{ c_1(\eta)(\sqrt{f(\eta^{1,+})} - \sqrt{f(\eta)})^2 + c_1(\eta)(\sqrt{f(\eta^{1,-})} - \sqrt{f(\eta)})^2 \right\} d \nu^N_{\rho(.)}. \tag{6.4}
\]
\[ \mathcal{D}_N^B(\sqrt{f}, \nu^N_{\rho(\cdot)}) = \int_{\Omega_N} \left\{ c_{N-1}^+(\eta)(\sqrt{f(\eta^{x,+})} - \sqrt{f(\eta)})^2 + c_{N-1}^-(\eta)(\sqrt{f(\eta^{-})} - \sqrt{f(\eta)})^2 \right\} d\nu^N_{\rho(\cdot)}. \]

(6.5)

We will make use of the following lemmas.

**Lemma 6.2** (Lemma 5.1 of [3]). Let \( T : \Omega_N \rightarrow \Omega_N \) be a transformation as the particle exchange between two sites \( \eta^{x,x+1} \) or the particle upgrade/downgrade \( \eta^{x,\pm} \). Let \( c : \Omega_N \rightarrow \mathbb{R} \) be a positive function and \( f \) a density with respect to \( \mu \). Then

\[
\langle c(\eta)[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}], \sqrt{f(\eta)} \rangle_{\mu} \leq \frac{1}{4} \int c(\eta)[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}]^2 d\mu
+
\frac{1}{16} \int \frac{1}{c(\eta)}[c(\eta) - c(T(\eta))] \left( \frac{\mu(T(\eta))}{\mu(\eta)} \right)^2 \left( \sqrt{f(T(\eta))} + \sqrt{f(\eta)} \right)^2 d\mu.
\]

**Lemma 6.3** (Lemma 5.2 of [3]). For \( \alpha \in \{A, B, \emptyset\} \), let \( \rho^\alpha : [0, 1] \rightarrow [0, 1] \) be a Lipschitz continuous functions such that

\[
0 < r_0 \leq \rho^\alpha(u) \leq r_1 < 1
\]

for \( u \in [0, 1] \). Then, there exists a constant \( C \) such that, for any \( N \in \mathbb{N} \) and any density \( f \) with respect to \( \nu^N_{\rho(\cdot)} \)

\[
\sup_{1 \leq x \leq N-2} \int_{\Omega_{u}} f(\eta^{x,x+1}) d\nu^N_{\rho(\cdot)} \leq C,
\]

\[
\sup_{x \in \{1, N-1\}} \int_{\Omega_{N}} f(\eta^{x,\pm}) d\nu^N_{\rho(\cdot)} \leq C.
\]

As a consequence of Lemmas 6.2 and 6.3 we obtain the following estimate on the Dirichlet form associated with the bulk generator.

**Corollary 6.4.** For \( \alpha \in \{A, B, \emptyset\} \), let \( \rho^\alpha : [0, 1] \rightarrow [0, 1] \) be a Lipschitz continuous function satisfying (6.6). There exists a constant \( C > 0 \) such that, for any density \( f \) and any \( N \in \mathbb{N} \),

\[
\langle \mathcal{D}_N^B(\sqrt{f}, \sqrt{f}) \rangle_{\nu^N_{\rho(\cdot)}} \leq \frac{1}{4} \mathcal{D}_N^B(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{C}{N}.
\]

**Proof.** By Lemma 6.2

\[
\langle \mathcal{D}_N^B(\sqrt{f}, \sqrt{f}) \rangle_{\nu^N_{\rho(\cdot)}} \leq \frac{1}{4} \mathcal{D}_N^B(\sqrt{f}, \nu^N_{\rho(\cdot)})
+
\frac{1}{16} \sum_{x=1}^{N-2} \int \frac{1}{c_{x,x+1}(\eta)} \left[ c_{x,x+1}(\eta) - c_{x,x+1}(\eta^{x,x+1}) \right] \left( \sqrt{f(\eta^{x,x+1})} + \sqrt{f(\eta)} \right)^2 d\nu^N_{\rho(\cdot)}.
\]

(6.7)

Since the exchange rate \( c_{x,x+1} \) depends on the species \( \alpha \in \{A, B, \emptyset\} \), we can write the integral on the right hand side of (6.7) as

\[
\sum_{\alpha \in \{A, B, \emptyset\}} \sum_{\zeta(x) = \alpha, \zeta(x+1) = \alpha+1} \frac{1}{c_{x,x+1}(\zeta)} \left[ c_{x,x+1}(\zeta) - c_{x,x+1}(\zeta^{x,x+1}) \right] \left( \sqrt{f(\zeta^{x,x+1})} + \sqrt{f(\zeta)} \right)^2 \nu^N_{\rho(\cdot)}(\zeta)
+
\sum_{\alpha \in \{A, B, \emptyset\}} \sum_{\zeta(x) = \alpha, \zeta(x+1) = \alpha+2} \frac{1}{c_{x,x+1}(\zeta)} \left[ c_{x,x+1}(\zeta) - c_{x,x+1}(\zeta^{x,x+1}) \right] \left( \sqrt{f(\zeta^{x,x+1})} + \sqrt{f(\zeta)} \right)^2 \nu^N_{\rho(\cdot)}(\zeta),
\]

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which equals

\[
\sum_{\alpha \in \{A, B, \emptyset\}} \sum_{\zeta: \zeta(x) = \alpha, \zeta(x+1) = \alpha+1} \frac{1}{\beta^2 N^2} \left[ 1 - \frac{\beta}{2N} - (1 + \frac{\beta}{2N}) \frac{\alpha^2(x+1)}{\rho^a(\frac{x}{N})} \rho^a(\frac{x+1}{N}) \right]^2 \langle f(x, x+1) \rangle + \sqrt{f(\zeta)} \nu_N^{\emptyset}(\zeta) \nonumber
\]

\[
+ \sum_{\alpha \in \{A, B, \emptyset\}} \sum_{\zeta: \zeta(x) = \alpha, \zeta(x+1) = \alpha+2} \frac{1}{1+\beta^2 N^2} \left[ 1 - \frac{\beta}{2N} - (1 - \frac{\beta}{2N}) \frac{\alpha^2(x+1)}{\rho^a(\frac{x}{N})} \rho^a(\frac{x+1}{N}) \right]^2 \langle f(x, x+1) \rangle + \sqrt{f(\zeta)} \nu_N^{\emptyset}(\zeta),
\]

where in the last passage we used the fact that \( \nu_N^{\emptyset}(\zeta) \) is a product measure. For fixed \( \alpha \), we rearrange the terms in the square brackets: in the first term we get

\[
\frac{1}{(\rho^a(\frac{x}{N})\rho^a(\frac{x+1}{N}))^2} \left[ \rho^a(\frac{x}{N}) \rho^a(\frac{x+1}{N}) - \rho^a(\frac{x}{N}) \rho^a(\frac{x+1}{N}) \right]^2
\]

\[
- \frac{\beta}{2N} \left( \rho^a(\frac{x+1}{N}) \rho^a(\frac{x+1}{N}) + \rho^a(\frac{x+1}{N}) \rho^a(\frac{x+1}{N}) \right) \tag{6.9}
\]

Note that

\[
\rho^a(\frac{x}{N})\rho^a(\frac{x+1}{N}) - \rho^a(\frac{x+1}{N})\rho^a(\frac{x+1}{N})
\]

\[
= \rho^a(\frac{x}{N})\rho^a(\frac{x+1}{N}) - \rho^a(\frac{x+1}{N})\rho^a(\frac{x+1}{N}) - \rho^a(\frac{x}{N})\rho^a(\frac{x+1}{N}) - \rho^a(\frac{x}{N})\rho^a(\frac{x+1}{N}).
\]

Since for each \( \alpha \), \( \rho^a(\cdot) \) is Lipschitz and satisfies (6.6), (6.9) is bounded from above by \( \frac{C}{\rho^a} \). For the second term of (6.8) we obtain the same bound. And so we can bound

\[
\sum_{x=1}^{N-2} \sum_{\alpha \in \{A, B, \emptyset\}} \sum_{\zeta: \zeta(x) = \alpha, \zeta(x+1) = \alpha+1} \frac{C}{N^2} \langle f(x, x+1) \rangle + \sqrt{f(\zeta)} \nu_N^{\emptyset}(\zeta)
\]

\[
+ \sum_{x=1}^{N-2} \sum_{\alpha \in \{A, B, \emptyset\}} \sum_{\zeta: \zeta(x) = \alpha, \zeta(x+1) = \alpha+2} \frac{C}{N^2} \langle f(x, x+1) \rangle + \sqrt{f(\zeta)} \nu_N^{\emptyset}(\zeta).
\]

Using the fact that \((a+b)^2 \leq 2(a^2 + b^2)\), since \( f \) is a density with respect to \( \nu_N^{\emptyset}(\cdot) \) we get, by Lemma 6.3, that

\[
\langle \mathcal{L}_N^B \sqrt{f}, \mathcal{L}_N^B \rangle \nu_N^{\emptyset}(\cdot) \leq -\frac{1}{4} \Theta_N^B(\sqrt{f}, \nu_N^{\emptyset}(\cdot) + \frac{C}{N}.
\]

\[
\square
\]

As a corollary of Lemma 6.3 we obtain an estimate on the boundary generators.

**Corollary 6.5.** For each \( \alpha \in \{A, B, \emptyset\} \), let \( \rho^a: [0, 1] \to [0, 1] \) be a Lipschitz continuous function satisfying (6.6) and also

\[
\rho^a(u) = r_\alpha, \forall u \in [0, \epsilon] \quad \text{and} \quad \rho^a(u) = \bar{r}_\alpha, \forall u \in [1 - \epsilon, 1],
\]

(6.10)

for some \( \epsilon > 0 \) small. Then, there exists a constant \( C > 0 \) such that, for any density \( f \) and any \( N \) sufficiently big, it holds

\[
\langle \mathcal{L}_N^B \sqrt{f}, \mathcal{L}_N^B \rangle \nu_N^{\emptyset}(\cdot) \leq -\frac{1}{2} \Theta_N^B(\sqrt{f}, \nu_N^{\emptyset}(\cdot) + \frac{C}{N}.
\]

\[
\langle \mathcal{L}_N^R \sqrt{f}, \mathcal{L}_N^R \rangle \nu_N^{\emptyset}(\cdot) \leq -\frac{1}{2} \Theta_N^R(\sqrt{f}, \nu_N^{\emptyset}(\cdot) + \frac{C}{N}.
\]
Proof. We prove only the inequality for the left boundary generator. The right boundary generator it follows identically. From the identity $(a - b)b = -\frac{1}{2}(a - b)^2 + \frac{1}{2}(a^2 - b^2)$ it follows that

$$\left(\int f^L \sqrt{f}, \sqrt{\tilde{f}} \right)_{\nu^N_{\rho(c)}}$$

$$= \int c^L_1(\eta) \sqrt{f(\eta)} \sqrt{\nu^N_{\rho(c)}} + \int c_1^-(\eta) \sqrt{f(\eta)} d\nu^N_{\rho(c)} + \int c_1^-(\eta) \sqrt{f(\eta)} d\nu^N_{\rho(c)}$$

$$= -\frac{1}{2} \int c_1^+(\eta) \sqrt{f(\eta)} d\nu^N_{\rho(c)} + \frac{1}{2} \int c_1^-(\eta) \sqrt{f(\eta)} d\nu^N_{\rho(c)}$$

The term in (6.11) is equal to $-\frac{1}{2} \Theta^N$ in (6.4), so we have to estimate (6.12). We can perform a change of variable in the second integral and we can rewrite (6.12) as

$$= \frac{1}{2} \sum_{\alpha} \sum_{\eta, \eta(1) = \alpha} [f(\eta^1) - f(\eta)] \left[ \left( \frac{1}{N^\delta} + \frac{\tilde{\beta}}{2N^\theta} \right) r_{\alpha+1} - \left( \frac{1}{N^\delta} - \frac{\tilde{\beta}}{2N^\theta} \right) r_{\alpha} \right] \nu^N_{\rho(c)}(\eta).$$

We observe that the terms proportional to $1/N^\delta$ are equal to zero for $N$ sufficiently big, since we choose a profile satisfying (6.10). The remaining terms are equal to

$$\frac{\tilde{\beta}}{2N^\theta} \sum_{\alpha} \sum_{\eta, \eta(1) = \alpha} [f(\eta^1) - f(\eta)] r_{\alpha+1} \nu^N_{\rho(c)}(\eta) \leq \frac{\tilde{\beta}}{2N^\theta} \int [f(\eta^1) - f(\eta)] d\nu^N_{\rho(c)} \leq \frac{C}{N^\theta},$$

by Lemma 6.3.

Now that we have the ingredients we need, in the next subsections we prove all the replacement lemmas that were used in the previous section on the characterization of limit points.

### 6.3 Replacement lemma at the bulk

#### Lemma 6.6 (Local replacement lemma).

For any $t \in [0, T]$ and $\theta \geq 1$, any $x \in \{1, \ldots, N - 1 - eN\}$ we have

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} E_{\mu_N} \left[ \int_0^T (\xi_x^a(\eta_s) - \xi_x^{a,eN}(\eta_s)) ds \right] = 0. \quad \text{(6.13)}$$

The same result above holds by replacing $\xi_x^{a,eN}(\eta_s)$ with $\xi_x^{a,eN}(\eta_s)$ for $x \in \{\epsilon N, \ldots, N - 1\}$.

Proof. Consider the product measure $\nu^N_{\rho(c)}$ as given in (6.1) where the profile $\rho(\cdot)$ is Lipschitz continuous and satisfying (6.6) and (6.10). By the entropy and Jensen’s inequality, we can bound the expected value in last display by

$$\frac{H(\mu_N | \nu^N_{\rho(c)})}{BN} + \frac{1}{NB} \log E_{\nu^N_{\rho(c)}} \left[ e^{NB | \int_0^T (\xi_x^a(\eta_s) - \xi_x^{a,eN}(\eta_s)) ds |} \right]. \quad \text{(6.14)}$$
where $B > 0$. We can remove the absolute value in the exponential, since $e^{|x|} \leq e^x + e^{-x}$ and
\[
\limsup_{N \to \infty} \frac{1}{N} \log(a_N + b_N) \leq \max \left\{ \limsup_{N \to \infty} \frac{1}{N} \log a_N, \limsup_{N \to \infty} \frac{1}{N} \log b_N \right\}.
\]
From Lemma 6.1 and from Feynman–Kac’s formula we can bound (6.14) from above by
\[
\frac{C_0}{B} + t \sup_f \left\{ \langle \xi^a_x(\eta) - \tilde{\xi}^a_x(\eta), f \rangle_{\rho(\cdot)} + \frac{N}{B} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\rho(\cdot)}} \right\}
\]
where the supremum is taken over all densities $f$ with respect to $\nu^N_{\rho(\cdot)}$. From Corollaries 6.4 and 6.5, this supremum is bounded from above by
\[
\sup_f \left\{ \langle \xi^a_x(\eta) - \tilde{\xi}^a_x(\eta), f \rangle_{\rho(\cdot)} - \frac{N}{4B} \mathcal{D}_N^B(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{CN}{BN^\delta} + \tilde{c}_1 \right\}.
\]
By writing the first term between brackets as a telescopic sum we get that
\[
\langle \xi^a_x(\eta) - \tilde{\xi}^a_x(\eta), f \rangle_{\rho(\cdot)} = \frac{1}{eN} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \left( \xi^a_x(\eta) - \xi^a_{x+\epsilon^a+1}(\eta) \right) f(\eta) d\nu^N_{\rho(\cdot)}.
\]
We split the above integral into two identical terms and we sum and subtract the term
\[
\frac{1}{2eN} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \left( \xi^a_x(\eta) - \xi^a_{x+\epsilon^a+1}(\eta) \right) f(\eta) d\nu^N_{\rho(\cdot)}.
\]
Rearranging the terms we have
\[
\langle \xi^a_x(\eta) - \tilde{\xi}^a_x(\eta), f \rangle_{\rho(\cdot)} = \frac{1}{2eN} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \left( \xi^a_x(\eta) - \xi^a_{x+\epsilon^a+1}(\eta) \right) \left[ f(\eta) - f(\eta^{\epsilon^a+1}) \right] d\nu^N_{\rho(\cdot)}
\]
(6.17)
\[
+ \frac{1}{2eN} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \left( \xi^a_x(\eta) - \xi^a_{x+\epsilon^a+1}(\eta) \right) \left[ f(\eta) + f(\eta^{\epsilon^a+1}) \right] d\nu^N_{\rho(\cdot)}.
\]
(6.18)
We start by estimating the first term. We multiply and divide it by $\sqrt{\xi^a_{x+\epsilon^a+1}(\eta)}$ and apply Young’s inequality with a constant $A > 0$ to bound it from above by
\[
\frac{1}{2eN} \frac{1}{2A} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \left( \xi^a_x(\eta) - \xi^a_{x+\epsilon^a+1}(\eta) \right)^2 \frac{1}{\xi^a_{x+\epsilon^a+1}(\eta)} \left[ \sqrt{f(\eta)} + \sqrt{f(\eta^{\epsilon^a+1})} \right]^2 d\nu^N_{\rho(\cdot)}
\]
\[
+ \frac{1}{2eN} \frac{A}{2} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \xi^a_{x+\epsilon^a+1}(\eta) \left[ \sqrt{f(\eta)} - \sqrt{f(\eta^{\epsilon^a+1})} \right]^2 d\nu^N_{\rho(\cdot)}.
\]
Choosing $A = N/B$ the first term goes to zero as $N \to \infty$ and $\epsilon \to 0$, while the second term cancels with the bulk Dirichlet form $-\frac{N}{4B} \mathcal{D}_N^B(\sqrt{f}, \nu^N_{\rho(\cdot)})$ in (6.16). Now we estimate (6.18). We split it as
\[
\frac{1}{2eN} \int_{\Omega_N} \sum_{y=x+2z=x+1}^{x+\epsilon N} \sum_{z=\epsilon^a+1}^{y-1} \left( \xi^a_x(\eta) - \xi^a_{x+\epsilon^a+1}(\eta) \right) f(\eta) d\nu^N_{\rho(\cdot)}
\]
(6.19)
In the second term we perform the change of variables \( \eta \rightarrow \eta^{x,z+1} \) and combine again the integrals to obtain

\[
\text{(6.18) } = \frac{1}{2e^N} \int_{\Omega_N} \sum_{y=x+2}^{x+iN} \sum_{z=x+1}^{y-1} \left( \xi_\eta^a(\eta) - \xi_\eta^{a+1}(\eta) \right)f(\eta)d\eta \rho_\eta^N. 
\]

Now we condition on having either one of the three species at \( z \) or at \( z + 1 \). We exclude the configurations \( \beta \beta \), for any \( \beta \), \((a+1)(a+2)\) and \((a+2)(a+1)\) that give contribution zero. The remaining configurations to check are \((a)(a+i)\) and \((a+i)(a)\), \(i = 1, 2\). So the previous term becomes equal to

\[
\frac{1}{2e^N} \sum_{i=1}^{2} \left( \int_{\Omega_N} \sum_{y=x+2}^{x+iN} \sum_{z=x+1}^{y-1} \right) \left[ \| \eta(z) = \alpha, \eta(z+1) = a+i \| \left( \xi_\eta^a(\eta) - \xi_\eta^{a+1}(\eta) \right)f(\eta)d\eta \rho_\eta^N \right]
\]

Let \( \eta = (\eta_1, \eta_2, \ldots, \eta_z, \eta_{z+1}, \ldots, \eta_N) \). Now the previous term can be written as

\[
\frac{1}{2e^N} \sum_{i=1}^{2} \left( \int_{\Omega_N} \sum_{y=x+2}^{x+iN} \sum_{z=x+1}^{y-1} \right) \left[ \| \eta(z) = a, \eta(z+1) = a+i \| \left( \xi_\eta^a(\eta) - \xi_\eta^{a+1}(\eta) \right)f(\eta)d\eta \rho_\eta^N \right]
\]

We have to estimate the contribution given by the difference of the measures:

\[
\rho_\eta^{N}(\eta_1, \eta_2, \ldots, \eta_z, \eta_{z+1}; \ldots, \eta_N) - \rho_\eta^{N}(\eta_1, \eta_{z+1}, \ldots, \eta_N)
\]

\[
= \prod_{x \neq z} \frac{1}{N} \left[ \frac{\rho_a^z}{N} \rho^{a+i}(\eta_{z+1}) - \rho^{a+i}(\eta_{z+0}) \right]
\]

\[
= \prod_{x \neq z} \frac{1}{N} \left[ \frac{\rho_a^z}{N} \left[ \rho^{a+i}(\eta_{z+1}) - \rho^{a+i}(\eta_{z+0}) \right] - \rho^{a+i}(\eta_{z+0}) \rho^{a+i}(\eta_{z+1}) \right] \approx \frac{1}{N} \rho_\eta^N(\eta_{z+1}),
\]

if we impose a Lipschitz condition on the density profiles of the three species. In the end, (6.21) is of order \( \frac{1}{N} (eN)^2 \frac{1}{N} \) which goes to zero as \( e \rightarrow 0 \).

Now, (6.15) can be bounded from above by

\[
\frac{C_\theta}{B^\theta} + \frac{CN}{BN^\theta} + \frac{C}{B} + \epsilon.
\]

Since \( \theta \geq 1 \), taking the limit in \( N \rightarrow +\infty \), then \( e \rightarrow 0 \) and finally \( B \rightarrow +\infty \), we are done.
By adapting the proof of last result, we can also derive the local replacement lemma which is useful to treat the contribution in Dynkin’s formula coming from the antisymmetric part of the dynamics.

**Lemma 6.7** (Global replacement lemma). Let \( G : [0, T] \times [0, 1] \rightarrow \mathbb{R} \) be a bounded function and let \( \psi : \Omega_N \rightarrow \mathbb{R} \) be a function such that for \( x \in \{eN, \cdots, N - 1 - eN\} \), \( \tau_x \psi(\eta) \) is invariant for the change of variables \( \eta \mapsto \eta^{x, x+1} \) for \( z \in \{x + 1, \cdots, x + eN\} \). Then, for any \( t \in [0, T] \) and \( \theta \geq 1 \), we have

\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \int_0^t \frac{1}{N} \sum_{x=eN}^{N-1-eN} G(s, \frac{x}{N}) \tau_x \psi(\eta_x(\xi^x_N(\eta_t)) - \xi^x_N(\eta_t)) ds \right] = 0.
\]

The same result holds with the right average replaced with the left average as long as for \( x \in \{eN, \cdots, N - 1 - eN\} \), \( \tau_x \psi(\eta) \) is invariant for the change of variables \( \eta \mapsto \eta^{x, x-1} \) for \( z \in \{x - eN, \cdots, x - 1\} \).

### 6.4 Replacement lemma at the boundary

We start this subsection by computing the adjoint \( \mathcal{L}_{\mu_N}^L \) of the left boundary generator with respect to the measure \( \nu_{\rho(\cdot)} \), given in (6.1). The computations for the right boundary generator are completely analogous and we omit them. Note that by the definition of \( \mathcal{L}_{\mu_N}^L \) in (2.6), for any \( f, g \in L^2(\nu_{\rho(\cdot)}) \) we have

\[
\int \mathcal{L}_{\mu_N}^L f(\eta) g(\eta) d \nu_N = \int \left( \frac{1}{N^{\delta}} + \frac{\bar{\beta}}{2N\theta} \right) \xi^A_{1}(\eta) r_B + \xi^B_{1}(\eta) r_A + \xi^\theta_{1}(\eta) r_A \left[ f(\eta^{1,1}) - f(\eta) \right] g(\eta) d \nu_N \tag{6.24}
\]

\[
+ \int \left( \frac{1}{N^{\delta}} - \frac{\bar{\beta}}{2N\theta} \right) \xi^A_{1}(\eta) r_B + \xi^B_{1}(\eta) r_A + \xi^\theta_{1}(\eta) r_A \left[ f(\eta^{1,1}) - f(\eta) \right] g(\eta) d \nu_N \tag{6.25}
\]

We look at (6.25) first. We perform the change of variables \( \zeta = \eta^{1,1} \) (equivalently \( \eta = \zeta^{1,-} \)) in the term with \( f(\eta^{1,1}) g(\eta) \) to rewrite that term as

\[
\int \left( \frac{1}{N^{\delta}} + \frac{\bar{\beta}}{2N\theta} \right) \left( \xi^A_{1}(\zeta^{1,-}) r_B + \xi^B_{1}(\zeta^{1,-}) r_A + \xi^\theta_{1}(\zeta^{1,-}) r_A \right) f(\zeta) g(\zeta^{1,-}) \frac{\nu_N}{\rho(\cdot)}(\zeta^{1,-}) d \nu_N = \int \left( \frac{1}{N^{\delta}} + \frac{\bar{\beta}}{2N\theta} \right) \left( \xi^A_{1}(\zeta^{1,-}) r_B + \xi^B_{1}(\zeta^{1,-}) r_A + \xi^\theta_{1}(\zeta^{1,-}) r_A \right) f(\zeta) g(\zeta^{1,-}) \frac{\nu_N}{\rho(\cdot)}(\zeta^{1,-}) d \nu_N = \int \left( \frac{1}{N^{\delta}} + \frac{\bar{\beta}}{2N\theta} \right) \left( \xi^A_{1}(\zeta^{1,-}) r_B + \xi^B_{1}(\zeta^{1,-}) r_A + \xi^\theta_{1}(\zeta^{1,-}) r_A \right) f(\zeta) g(\zeta^{1,-}) \frac{\nu_N}{\rho(\cdot)}(\zeta^{1,-}) d \nu_N.
\]

Now we split the integral by conditioning on having a particle of species \( A, B \) or a hole at position \( 1/N \), so that last display is equal to

\[
\sum_{a} \int \mathbb{1}_{(\zeta^{1,-}) = a} \left( \frac{1}{N^{\delta}} + \frac{\bar{\beta}}{2N\theta} \right) r_a f(\zeta) g(\zeta^{1,-}) \frac{\rho^{a+2}(\frac{1}{N})}{\rho_a(\frac{1}{N})} d \nu_N.
\]

In a similar way we rewrite the term \( f(\eta^{1,1}) g(\eta) \) in (6.25) as

\[
\sum_{a} \int \mathbb{1}_{(\zeta^{1,-}) = a} \left( \frac{1}{N^{\delta}} - \frac{\bar{\beta}}{2N\theta} \right) r_a f(\zeta) g(\zeta^{1,1}) \frac{\rho^{a+1}(\frac{1}{N})}{\rho_a(\frac{1}{N})} d \nu_N\]
Putting together the last two terms we can write them as

\[
\int \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) \frac{\rho^\beta}{\rho^\phi} r_A + \xi^B_1(\eta) \frac{\rho^\beta}{\rho^\phi} r_B + \xi^\phi_1(\eta) \frac{\rho^\beta}{\rho^\phi} \xi^\phi \right) f(\eta) g(\eta^{-}) d\nu^N_{\rho} \\
+ \int \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) \frac{\rho^\beta}{\rho^\phi} r_B + \xi^B_1(\eta) \frac{\rho^\beta}{\rho^\phi} r_A + \xi^\phi_1(\eta) \frac{\rho^\beta}{\rho^\phi} \xi^\phi \right) f(\eta) g(\eta^{+}) d\nu^N_{\rho}.
\]

Assuming (6.6) and (6.10) we can rewrite the last two terms, for $N$ sufficiently big as

\[
\int \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_A + \xi^\phi_1(\eta) r_B \right) f(\eta) g(\eta^{-}) d\nu^N_{\rho} \\
+ \int \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_B + \xi^\phi_1(\eta) r_A \right) f(\eta) g(\eta^{+}) d\nu^N_{\rho}.
\]

Putting now all the terms together we get

\[
\int \mathcal{L}_N^L f(\eta) g(\eta) d\nu^N_{\rho} \\
= \int \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_A + \xi^\phi_1(\eta) r_B \right) f(\eta) g(\eta^{-}) d\nu^N_{\rho} \\
+ \int \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_B + \xi^\phi_1(\eta) r_A \right) f(\eta) g(\eta^{+}) d\nu^N_{\rho} \\
- \int \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_B + \xi^\phi_1(\eta) r_A \right) f(\eta) g(\eta) d\nu^N_{\rho} \\
- \int \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_A + \xi^\phi_1(\eta) r_B \right) f(\eta) g(\eta) d\nu^N_{\rho}
\]

for $N$ sufficiently big. From this, we obtain the following expression for the adjoint of $\mathcal{L}_N^L$:

\[
\mathcal{L}_N^{L^*} g(\eta) = \left( \frac{1}{N^\delta} - \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_B + \xi^\phi_1(\eta) r_A \right) [g(\eta^{+}) - g(\eta)] \\
+ \left( \frac{1}{N^\delta} + \frac{\beta}{2N^\eta} \right) \left( \xi^A_1(\eta) r_B + \xi^B_1(\eta) r_A + \xi^\phi_1(\eta) r_B \right) [g(\eta^{-}) - g(\eta)] \\
+ \frac{\beta}{N^\eta} [\xi^A_1(\eta)(r_B - r_B) + \xi^B_1(\eta)(r_A - r_B) + \xi^\phi_1(\eta)(r_B - r_A)] g(\eta).
\] (6.27)

Now we are ready to prove the replacement needed for the Dirichlet regime, i.e. for item a) of Theorem 2.7.

**Lemma 6.8.** For $t \in [0, T]$, $\theta \geq 1 > \delta$ and for $\alpha \in \{A, B, \emptyset\}$, we have

\[
\lim_{N \to \infty} \mathbb{E}_{\nu^N} \left[ \int_0^t (r_\alpha - \xi^\alpha_1(\eta)) ds \right] = 0.
\] (6.28)

The same holds replacing $r_\alpha$ with $\tilde{r}_\alpha$ and $\xi^\alpha_1(\eta)$ with $\xi^\alpha_{N-1}(\eta)$.

**Proof.** We present the proof for the case $\alpha = A$ and for the left boundary, but for the other cases it is analogous. First we make the following key observation. By (6.27), we have

\[
\mathcal{L}_N^{L^*} \xi^A_1(\eta) = \frac{1}{N^\delta}(r_A - \xi^A_1(\eta)) + \frac{\beta}{2N^\eta}[r_A(\xi^B_1(\eta) - \xi^\phi_1(\eta)) + (r_B - r_B)\xi^A_1(\eta)].
\] (6.28)
As a consequence, and since we assumed $\theta \geq 1 > \delta$, the second term on the right hand side of last display, multiplied by $N^\delta$, goes to 0 as $N \to \infty$ and we are left with the estimate of
\[
\mathbb{E}_{\mu_N} \left[ \int_0^t N^\delta \mathcal{L}_N^{L,*} \xi_1^A(\eta) d\tau \right].
\]

Now we proceed as in the proof of the replacement lemma at the bulk: by the entropy and Jensen's inequalities, and Feynman–Kac formula the last expression is bounded above by
\[
C N + t \sup_f \left\{ \int N^\delta \mathcal{L}_N^{L,*} \xi_1^A(\eta) f(\eta) d\nu_N^{\rho(\cdot)} + \frac{N}{B} \mathcal{E}_N(\sqrt{f}, \sqrt{f}) \right\}.
\]

We apply Corollaries 6.4 and 6.5 to bound last display from above by
\[
C N + t \sup_f \left\{ \int N^\delta \mathcal{L}_N^{L,*} \xi_1^A(\eta) f(\eta) d\nu_N^{\rho(\cdot)} - \frac{N}{4B} \mathcal{E}_N(\sqrt{f}, \sqrt{f}) + \frac{CN}{N^\theta} \mathcal{H}_N(\sqrt{f}, \sqrt{f}) - \frac{N}{2B} \mathcal{H}_N(\sqrt{f}, \sqrt{f}) \right\}.
\]

Using the definition of the adjoint $\mathcal{L}_N^{L,*}$ we can bound the integral as follows
\[
\left| \int N^\delta \mathcal{L}_N^{L,*} \xi_1^A(\eta) f(\eta) d\nu_N^{\rho(\cdot)}(\eta) \right| = \left| \int N^\delta \mathcal{L}_N^{L} \xi_1^A(\eta) f(\eta) d\nu_N^{\rho(\cdot)}(\eta) \right| \leq \left| \int N^\delta c_1^+(\eta) \xi_1^A(\eta) \left[ f(\eta^{1,0}) - f(\eta) \right] d\nu_N^{\rho(\cdot)}(\eta) \right| + \left| \int N^\delta c_1^-(\eta) \xi_1^A(\eta) \left[ f(\eta^{1,0}) - f(\eta) \right] d\nu_N^{\rho(\cdot)}(\eta) \right|.
\]

The strategies to bound the terms (6.30) and (6.31) are identical, so we write down only the proof for the first term. As we did in the bulk, we bound using Young's inequality:
\[
(6.30) \leq \frac{N^\delta A}{2} \left( \int c_1^+(\eta) \left[ \sqrt{f}(\eta^{1,0}) - \sqrt{f}(\eta) \right]^2 d\nu_N^{\rho(\cdot)}(\eta) \right) + \frac{N^\delta}{2A} \left( \int c_1^+(\eta) \xi_1^A(\eta) \left[ \sqrt{f}(\eta^{1,0}) + \sqrt{f}(\eta) \right]^2 d\nu_N^{\rho(\cdot)}(\eta) \right).
\]

If we choose $A = N/BN^\delta$, (6.32) is killed by the Dirichlet form $\mathcal{H}_N^{L,*}(\sqrt{f}, \nu_N^{\rho(\cdot)})$ in the supremum, while (6.33) is bounded by $C \frac{BN^\delta}{2N} \max \{ \frac{1}{N^\theta}, \frac{1}{N} \} \leq CB \frac{N^\delta}{N^\theta}$, as long as $\int f(\eta^{1,0}) d\nu_N^{\rho(\cdot)}(\eta) < \infty$, which is true since $f$ is a density. Putting together all the estimates we get
\[
\mathbb{E}_{\mu_N} \left[ \int_0^t N^\delta \mathcal{L}_N^{L,*} \xi_1^A(\eta) d\tau \right] \leq \frac{1}{N} + \frac{1}{B} + \frac{BN^\delta}{N^\theta} + \frac{N}{N^\theta},
\]
which goes to zero as $N \to \infty$ and $B \to \infty$. 

\textbf{6.5 Energy estimate}

To prove that the boundary terms of the hydrodynamic equation are well defined and that the space derivatives are well defined in the weak sense, we need to show that the limiting densities of the empirical measure belong to the Sobolev Space $L^2([0, T] \times \mathbb{H})$ where $\mathbb{H} = \mathcal{H}(0, 1)^3$, or equivalently that the density profile $\rho_t$ belongs to $\mathbb{H}$, almost surely in $t \in [0, T]$. 25
Proposition 6.9. Let $\mathcal{Q}^*$ be a limit point of the sequence of measures $\{\mathcal{Q}_N\}_{N}$. Then, the measure $\mathcal{Q}^*$ is concentrated on paths $\rho(t,u)du$ such that $\rho \in L^2([0,T] \times \mathbb{H})$, i.e.

$$\mathcal{Q}^*(\pi \in \mathcal{D}([0,T],\mathcal{M}^3) : \rho \in L^2([0,T] \times \mathbb{H})) = 1.$$ 

This follows from the next two lemmas together with Riesz’s representation theorem. The strategy follows [12], [13] and [2].

Lemma 6.10. Fix $\alpha \in \{A,B,\emptyset\}$. For all $\theta \geq 1$ and $\theta \geq \delta$, there is a positive constant $\kappa > 0$ such that

$$\mathbb{E}_\mathcal{Q}^* \left[ \sup_H \left\{ \int_0^T \int_0^1 \partial_s H(s,u) \rho^\alpha_H(u) du ds - \kappa \int_0^T \int_0^1 H(s,u)^2 du ds \right\} \right] < \infty,$$

where the supremum is taken over all functions $H \in C^{0,2}_c([0,T] \times (0,1))$.

Proof. Here we want to show that the linear functional

$$\int_0^T \int_0^1 \partial_s H(t,u) \rho^\alpha_H(u) dt du$$

is $\mathcal{Q}^*$-almost surely continuous. If we consider a dense sequence $\{H_n\}_{n \geq 1} \subset C^{0,2}_c([0,T] \times (0,1))$, it is sufficient to prove that, for any $\ell \geq 1$,

$$\mathbb{E}_\mathcal{Q}^* \left[ \max_{1 \leq i \leq \ell} \left\{ \int_0^T \int_0^1 \partial_s H_i(s,u) \rho^\alpha_H(u) du ds - \kappa \int_0^T \int_0^1 H_i(s,u)^2 du ds \right\} \right] < C,$$

for some constant $C > 0$ independent from $\ell$. Since $\mathcal{Q}_N$ converges to $\mathcal{Q}^*$, this is equivalent to showing that

$$\lim_{N \to \infty} \mathbb{E}_{\mathcal{Q}^*_N} \left[ \max_{1 \leq i \leq \ell} \left\{ \int_0^T \langle \partial_s H_i(s,\cdot),\pi^N_s\rangle - \kappa \|H_i(s,\cdot)\|_{L^2(0,1)}^2 \right\} \right] < C. \quad (6.35)$$

As we did above in the replacement lemmas, we use entropy and Jensen’s inequality and the fact that $\exp\{\max_{1 \leq i \leq \ell} a_i\} \leq \sum_{1 \leq i \leq \ell} e^{a_i}$ to bound the expected value in (6.35) with

$$\frac{H(\mu_N|\mu(\cdot))}{N} + \frac{1}{N} \log \sum_{1 \leq i \leq \ell} \mathbb{E}_{\mu(\cdot)} \left[ \exp \left\{ N \int_0^T \langle \partial_s H_i(s,\cdot),\pi^N_s\rangle - \kappa \|H_i(s,\cdot)\|_{L^2(0,1)}^2 \right\} \right].$$

By Lemma 6.1, the first term is bounded by a constant $C_0$. For the second term it is enough to show that

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\mu(\cdot)} \left[ \exp \left\{ N \int_0^T \langle \partial_s H(s,\cdot),\pi^N_s\rangle - \kappa \|H(s,\cdot)\|_{L^2(0,1)}^2 \right\} \right] < \tilde{C} T,$$

for a fixed $H$ and for a constant $\tilde{C}$ independent of $H$. The definition of empirical measure leads us to the following lemma. \[\square\]

Lemma 6.11. Let $\rho$ be a profile satisfying (6.6) and (6.10). For all $\theta \geq 1$ and $\theta \geq \delta$, there exists a positive constant $\kappa > 0$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\mu(\cdot)} \left[ \exp \left\{ \int_0^T \sum_{x=1}^{N-1} \varepsilon^\rho_s(\eta_x) \partial_s H(s,\cdot) - \kappa N \|H(s,\cdot)\|_{L^2(0,1)}^2 \right\} \right] < CT.$$
Proof. By Feynman–Kac formula the expression in the statement is bounded above by

\[
\int_0^T \sup_f \left\{ \frac{1}{N} \int \sum_{x=1}^{N-1} \partial_x H(s, \frac{x}{N}) \xi_x^a(\eta) f(\eta) d \nu^N_{\rho(\cdot)} + N \langle \mathcal{E}_N \sqrt{f}, \sqrt{f} \rangle_{\rho(\cdot)} - \kappa ||H(s, \cdot)||_{L^2(0,1)}^2 \right\} ds.
\]

By Corollaries 6.4 and 6.5 we can bound the last display by

\[
\int_0^T \sup_f \left\{ \frac{1}{N} \int \sum_{x=1}^{N-1} \partial_x H(s, \frac{x}{N}) \xi_x^a(\eta) f(\eta) d \nu^N_{\rho(\cdot)} - \frac{N}{4} \mathcal{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{CN}{N^\theta} + C - \kappa ||H(s, \cdot)||_{L^2(0,1)}^2 \right\} ds.
\]

Using the fact that

\[
\partial_x H(s, \frac{x}{N}) = H(s, \frac{x+1}{N}) - H(s, \frac{x}{N}) + o(N^{-1}),
\]

after a summation by parts, we can write the integral inside the supremum as

\[
\sum_{x=1}^{N-2} \int H(s, \frac{x}{N}) (\xi_x^a(\eta) - \xi_{x+1}^a(\eta)) f(\eta) d \nu^N_{\rho(\cdot)} + o(N^{-1}).
\]

Now we split the integral as its half plus its half and we add and subtract the same term but replacing \( f(\eta) \) with \( f(\eta^{x+1}) \) to obtain the term

\[
\sum_{x=1}^{N-2} \int H(s, \frac{x}{N}) (\xi_x^a(\eta) - \xi_{x+1}^a(\eta)) [f(\eta) - f(\eta^{x+1})] d \nu^N_{\rho(\cdot)} \quad (6.36)
\]

\[
+ \sum_{x=1}^{N-2} \int H(s, \frac{x}{N}) (\xi_x^a(\eta) - \xi_{x+1}^a(\eta)) [f(\eta) + f(\eta^{x+1})] d \nu^N_{\rho(\cdot)} \quad (6.37)
\]

We perform the change of variables \( \eta \mapsto \eta^{x+1} \) in (6.37) to get

\[
\sum_{x=1}^{N-2} \int H(s, \frac{x}{N}) (\xi_x^a(\eta) - \xi_{x+1}^a(\eta)) [f(\eta) - f(\eta^{x+1})] d \nu^N_{\rho(\cdot)} \quad (6.38)
\]

\[
+ \sum_{x=1}^{N-2} \int H(s, \frac{x}{N}) (\xi_x^a(\eta) - \xi_{x+1}^a(\eta)) [f(\eta^{x+1})] \left( 1 - \frac{\nu^N_{\rho(\cdot)}(\eta^{x+1})}{\nu^N_{\rho(\cdot)}(\eta)} \right) d \nu^N_{\rho(\cdot)} \quad (6.39)
\]

We multiply and divide by \( c_{x,x+1}(\eta) \). Then, by Young’s inequality, for \( A > 0 \), we can bound (6.38) from above by

\[
\frac{A}{2} \sum_{x=1}^{N-2} \int (H(s, \frac{x}{N}))^2 \frac{1}{c_{x,x+1}(\eta)} [\sqrt{f(\eta)} + \sqrt{f(\eta^{x+1})}]^2 d \nu^N_{\rho(\cdot)} \quad (6.40)
\]

\[
+ \frac{1}{2A} \sum_{x=1}^{N-2} c_{x,x+1}(\eta) [\sqrt{f(\eta)} - \sqrt{f(\eta^{x+1})}] d \nu^N_{\rho(\cdot)} \quad (6.41)
\]

Choosing \( A = 1/2N \), last display is bounded from above by

\[
\frac{C}{N} \sum_{x=1}^{N-2} (H(s, \frac{x}{N}))^2 + \frac{N}{4} \mathcal{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}). \quad (6.42)
\]

For (6.39), we first recall that by (6.22)

\[
\left| 1 - \frac{\nu^N_{\rho(\cdot)}(\eta^{x+1})}{\nu^N_{\rho(\cdot)}(\eta)} \right| \leq \frac{C}{N}.
\]
Again by Young’s inequality, for $B > 0$, we can bound \((6.39)\) from above by
\[
\frac{B}{2} \sum_{x=1}^{N-2} (H(s, \frac{x}{N}))^2 d \nu^N_{\rho(\cdot)} + \frac{1}{2B} \sum_{x=1}^{N-2} f^2(\eta^x,x+1) \left( 1 - \frac{\nu^N_{\rho(\cdot)}(\eta^x,x+1)}{\nu^N_{\rho(\cdot)}(\eta)} \right)^2 d \nu^N_{\rho(\cdot)}
\]
\[
\leq \frac{C}{N} \sum_{x=1}^{N-2} (H(s, \frac{x}{N}))^2 + C,
\]
if we choose $B = 1/2N$. Putting together \((6.42)\) and \((6.43)\), we bound everything by
\[
\int_0^T \sup_f \left\{ \frac{C}{N} \sum_{x=1}^{N-2} (H(s, \frac{x}{N}))^2 - \kappa \int_0^1 (H(s,u))^2 du + C \right\} ds.
\]
And, we can conclude by observing that
\[
\frac{1}{N} \sum_{x=1}^{N-2} (H(s, \frac{x}{N}))^2 \to \int_0^1 (H(s,u))^2 du.
\]
\[
\square
\]
Now we show some consequences of the previous result.

Lemma 6.12. If $\rho^\alpha \in L^2([0, T] \times \mathcal{H}^1(0,1))$, then
\[
\lim_{\epsilon \to 0} \left| \rho^\alpha_s(u) - \frac{1}{\epsilon} \int_u^{u+\epsilon} \rho^\alpha_s(v) dv \right| = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \left| \rho^\alpha_s(u) - \frac{1}{\epsilon} \int_{u-\epsilon}^u \rho^\alpha_s(v) dv \right| = 0 \quad (6.44)
\]
for all $u \in [0,1]$ and for a.e. $s \in [0,T]$.

Proof. We present the proof for the limit on the left-hand side of last display but the other one is analogous. From the Cauchy-Schwarz inequality and the fact that $\rho^\alpha \in L^2([0, T] \times \mathcal{H}^1(0,1))$, we get that
\[
\left| \rho^\alpha_s(u) - \frac{1}{\epsilon} \int_u^{u+\epsilon} \rho^\alpha_s(v) dv \right| \leq \frac{1}{\epsilon} \int_u^{u+\epsilon} \left| \rho^\alpha_s(u) - \rho^\alpha_s(v) \right| dv \leq \frac{1}{\epsilon} \int_u^{u+\epsilon} \int_u^v \left| \partial_q \rho^\alpha_s(q) \right| dq dv
\]
\[
\leq \frac{\| \partial_q \rho^\alpha_s \|_2}{\epsilon} \int_u^{u+\epsilon} \sqrt{v-u} dv = \frac{2}{3} \| \partial_q \rho^\alpha_s \|_2 \sqrt{\epsilon}.
\]
Taking the limit as $\epsilon \to 0$ we are done. $\square$

6.6 Properties of the solution

Now we prove that the solution $\rho^\alpha_t$ satisfies item D2 of Definition 2.3 if $\delta < 1 \leq \theta$. Let $Q$ be a limit point of the sequence $\{Q_N\}_{N \geq 1}$ whose existence is a consequence of Proposition 4.1. In fact, we can assume that the whole sequence $\{Q_N\}_{N \geq 1}$ converges to $Q$. Since we are dealing with a process that has at most one particle per site, we can conclude easily (for details we refer the reader to [15]) that $Q$ is supported on trajectories of measures are absolutely continuous with respect to the Lebesgue measure, i.e. $\pi_t^\alpha(du) = \rho^\alpha_t(u) du$ for any $t \in [0,T]$. In last section we prove that the density $\rho^\alpha_t$ belongs to $L^2(0,T;\mathcal{H}^1)$ and we can identify the profile $\rho^\alpha_t$ with
a continuous function in \([0, 1]\). To show that the profile satisfies D2 of Definition 2.3, we use (3.11) and then, for any \(\delta > 0\),

\[
Q_N \left[ \left| \int_0^1 \langle \pi_s^a, \tilde{\iota}_\epsilon(0) \rangle - r_a \, ds \right| > \delta \right] \leq \delta^{-1} \mathbb{E}_{\mu_N} \left[ \left| \int_0^1 \tilde{\xi}_1^{a,\epsilon N}(\eta_s) - r_a \, ds \right| \right].
\]

Now we want to apply Portmanteau theorem but since \(\tilde{\iota}_\epsilon(0)\) is not a continuous function, the set \(\{ \pi; \left| \int_0^1 ((\pi_s^a, \tilde{\iota}_\epsilon(0)) - r_a) \, ds \right| > \delta \}\) is not an open set in the Skorohod topology. To overcome the problem, we use an \(L^1\)-approximation of \(\tilde{\iota}_\epsilon\) by continuous functions, to conclude that

\[
Q \left[ \left| \int_0^1 \langle \pi_s^a, \tilde{\iota}_\epsilon(0) \rangle - r_a \, ds \right| > \delta \right] \leq \delta^{-1} \liminf_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^1 \tilde{\xi}_1^{a,\epsilon N}(\eta_s) - r_a \, ds \right| \right].
\]

If the right-hand side of the previous inequality is zero, since \(Q\) a.s. \(\pi_s^a(du) = \rho^a_\epsilon(u) du\) with \(\rho^a_\epsilon\) a continuous function at 0 for a.e. \(s \in [0, T]\), by taking the limit \(\epsilon \to 0\), we conclude that \(Q\) a.s. \(\rho^a_s(0) = r_a\) for a.e. \(s \in [0, T]\). From the previous observations we are left to proving that for \(\delta < 1\) and for any \(t \in [0, T]\) it holds

\[
\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^1 \tilde{\xi}_1^{a,\epsilon N}(\eta_s) - r_a \, ds \right| \right] = 0.
\]

But this is an easy consequence of Lemmas 6.6 and 6.8.

A Uniqueness of weak solutions

In this appendix we show the uniqueness of solutions to the boundary value problems (2.7) and (2.9). We start by presenting an equivalent definition of a weak solution in the Dirichlet case, according to which we will show uniqueness.

A.1 Equivalence of two notions of weak solution for the Dirichlet boundary value problem

Recall (2.7). In Definition A.1 below, we give another notion of weak solutions to (2.7), and we prove that it is equivalent to Definition 2.3. The advantage, in the proof of uniqueness of solutions, is that the definition below has as input test functions in \(C_0^2([0, 1])\) i.e. twice continuously differentiable functions \(\phi\) that vanish at the boundary, \(\phi(0) = \phi(1) = 0\).

Definition A.1. We say that \(\rho = (\rho^A, \rho^B, \rho^\varnothing) : [0, T] \times [0, 1] \to [0, 1]^3\) is a solution of (2.7) if for each \(a \in \{A, B, \varnothing\}\) it holds:

- **D1.** \(\rho^a \in L^2([0, T] \times \mathcal{H}^1(0, 1)),\)

- **D2.’** for all \(t \in [0, T]\) and any \(\phi \in C_0^2([0, 1])\),

\[
\langle \rho^a_t, \phi \rangle - \langle \theta^a_t, \phi \rangle - \int_0^t \langle \rho^a_s, \Delta \phi \rangle - \beta \langle \rho^a_s (\rho^a_{s+1} - \rho^a_{s+2}), \nabla \phi \rangle \, ds
\]

\[
- \int_0^t \rho_s \nabla \phi(0) - \bar{r}_a \nabla \phi(1) \, ds = 0
\]

(A.1)

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We will show that these two definitions are equivalent. Clearly $D_2' \implies D_3$. It is not difficult to see that $D_2' \implies D_2$: to show $\rho_t^\alpha(0) = r_\alpha$ we consider in (A.1) smooth approximation of the test functions $\phi^{(n)}(u) = u_{[0,1/n]} \frac{1}{n}(u - 1)1_{[1/n,1]}$. We use integration by parts in the term with the Laplacian and then take the limit $n \to \infty$ to conclude. Now we will show that $\{D_1,D_2,D_3\} \implies \{D_1,D_2'\}$. Assume that $\rho$ satisfies $\{D_1,D_2,D_3\}$. Fix $\phi \in C_0^\infty([0,1])$ and take a sequence $\{\phi^{(n)}\}_{n \geq 1}$ in $C^2([0,1])$ converging to $\phi$ in $H^1_0$. Then $D_3$ holds for $\phi^{(n)}$ in place of $\phi$. To conclude $D_2'$ we observe that

$$\lim_{n \to \infty} \int_0^t \langle \rho_s^\alpha, \Delta \phi^{(n)} \rangle ds = \int_0^t \langle \rho_s^\alpha, \Delta \phi \rangle ds + \int_0^t r_\alpha \nabla \phi(0) - \bar{r}_\alpha \nabla \phi(1) ds,$$

(4.3)

$$\lim_{n \to \infty} \int_0^t \langle \rho_s^\alpha(\rho_{s+1}^\alpha - \rho_{s+2}^\alpha), \nabla \phi^{(n)} \rangle ds = \int_0^t \langle \rho_s^\alpha(\rho_{s+1}^\alpha - \rho_{s+2}^\alpha), \nabla \phi \rangle ds.$$

(4.4)

The limits in (4.2) and (4.4) follows from Cauchy-Schwarz inequality. To show (4.3), we integrate by parts twice and use $D_2$ to obtain

$$\lim_{n \to \infty} \langle \rho_s^\alpha, \Delta \phi^{(n)} \rangle = - \lim_{n \to \infty} \langle \nabla \rho_s^\alpha, \nabla \phi^{(n)} \rangle = - \langle \nabla \rho_s^\alpha, \nabla \phi \rangle = \langle \rho_s^\alpha, \Delta \phi \rangle + r_\alpha \nabla \phi(0) - \bar{r}_\alpha \nabla \phi(1).$$

This concludes the proof of the equivalence of the two notions of weak solution.

### A.2 Uniqueness for Dirichlet boundary condition

In this section we prove uniqueness of the solutions of the boundary value problem (2.7) (Dirichlet) following the strategy presented in Appendix 2.4 of [15].

Consider $\rho^1$ and $\rho^2$ two weak solutions of (2.7) according to Definition A.1 and with the same initial condition and call $\tilde{\rho} = \rho^1 - \rho^2$ their difference. We know that, for $\alpha \in \{A,B,\emptyset\}$, $\tilde{\rho}^\alpha \in L^2([0,T];H^1_0)$, where $H^1_0$ is the set of functions in $H^1$ vanishing at 0 and 1. In fact, since $\rho^1$ and $\rho^2$ have the same boundary values, for a.e. $t \in [0,T]$, we have $\tilde{\rho}^\alpha(0) = \tilde{\rho}^\alpha(1) = 0$. We consider the set $\{\psi_k\}$ of eigenfunctions of the Laplacian with Dirichlet boundary conditions, given by $\psi_k(u) = \sqrt{2} \sin(k\pi u)$, $k \geq 1$. These functions are in $C_0^\infty([0,1])$ and they form an orthonormal basis of $L^2([0,1])$. For $\alpha \in \{A,B,\emptyset\}$, let us define

$$V_\alpha(t) = \sum_{k \geq 1} \frac{1}{2a_k} \langle \tilde{\rho}_t^\alpha, \psi_k \rangle^2$$

with $a_k = (k\pi)^2 + 1$. Let $V(t) = \sum_\alpha V_\alpha(t)$. To conclude the uniqueness we will show that there exists a constant $C > 0$ such that $V'(t) = CV(t)$. Then, from Gronwall’s inequality it follows that $V(t) = 0$ for all $t \geq 0$. This implies that for any $\alpha \in \{A,B,\emptyset\}$ and any $t \in [0,T]$, $\tilde{\rho}_t^\alpha(u) = 0$ almost every $u \in [0,1]$, concluding the proof of uniqueness.

We have that

$$V'_\alpha(t) = \sum_{k \geq 1} \frac{1}{2a_k} \langle \tilde{\rho}_t^\alpha, \psi_k \rangle \frac{d}{dt} \langle \tilde{\rho}_t^\alpha, \psi_k \rangle,$$

and that from (A.1),

$$\frac{d}{dt} \langle \tilde{\rho}_t^\alpha, \psi_k \rangle = \langle \tilde{\rho}^\alpha_t, \Delta \psi_k \rangle - \beta \langle \Gamma^\alpha_t(\rho^1_t, \rho^2_t), \nabla \psi_k \rangle$$

$$= -k^2 \pi^2 \langle \tilde{\rho}^\alpha_t, \psi_k \rangle - \beta \langle \Gamma^\alpha_t(\rho^1_t, \rho^2_t), \nabla \psi_k \rangle,$$

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where
\[ \Gamma^\alpha(\rho^1, \rho^2) = \rho^{1,a}(\rho^{1,a+1} - \rho^{1,a+2}) - \rho^{2,a}(\rho^{2,a+1} - \rho^{2,a+2}). \] (A.5)

Then,
\[ V'_a(t) = -\sum_{k \geq 1} \frac{(k\pi)^2}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 - \sum_{k \geq 1} \frac{\beta}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle \langle \Gamma^\alpha(\rho^1_t, \rho^2_t), \nabla \psi_k \rangle. \]

From Young’s inequality applied to the last term,
\begin{align*}
V'_a(t) &\leq -\sum_{k \geq 1} \frac{(k\pi)^2}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 + \frac{1}{2A} \sum_{k \geq 1} \frac{\beta}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 + \frac{A}{2} \sum_{k \geq 1} \frac{\beta}{a_k} \langle \Gamma^\alpha(\rho^1_t, \rho^2_t), \nabla \psi_k \rangle^2 \\
&= -\sum_{k \geq 1} \frac{(k\pi)^2}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 + \frac{\beta}{2A} \sum_{k \geq 1} \frac{1}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 + \frac{A\beta}{2} \sum_{k \geq 1} \frac{(k\pi)^2}{a_k} \langle \Gamma^\alpha(\rho^1_t, \rho^2_t), \varphi_k \rangle^2 \\
&\leq -\sum_{k \geq 1} \frac{(k\pi)^2}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 + \frac{\beta}{2A} \sum_{k \geq 1} \frac{1}{a_k} \langle \hat{\rho}^a_t, \psi_k \rangle^2 + \frac{A\beta}{2} \sum_{k \geq 1} \langle \Gamma^\alpha(\rho^1_t, \rho^2_t), \varphi_k \rangle^2,
\end{align*}
(A.6)

where in the last two displays we used the facts that \( \nabla \psi_k(u) = k\pi \varphi_k(u) \) with \( \varphi_k(u) = \sqrt{2} \cos(k\pi u), \)
\( k \geq 1 \) and that \( a_k = (k\pi)^2 + 1 \). Since \( \{\varphi_k; k \geq 1\} \) is an orthonormal set in \( L^2([0,1]) \),
\[
\sum_{k \geq 1} \langle \Gamma^\alpha(\rho^1_t, \rho^2_t), \varphi_k \rangle^2 \leq \| \Gamma^\alpha(\rho^1_t, \rho^2_t) \|_2^2,
\] (A.7)

We can obtain an estimate on \( |\Gamma^\alpha(\rho^1, \rho^2)| \) in the following way. We sum and subtract appropriate terms in (A.5) to obtain
\[
\Gamma^\alpha(\rho^1, \rho^2) = (\rho^{1,a+1} - \rho^{1,a+2})(\rho^{1,a} - \rho^{2,a}) + \rho^{2,a}(\rho^{1,a+1} - \rho^{2,a+1}) - \rho^{2,a}(\rho^{1,a+2} - \rho^{2,a+2}),
\]
thus \( |\Gamma^\alpha(\rho^1_t, \rho^2_t)| \leq \sum_{\gamma} |\hat{\rho}^\gamma_t| \), and then from Cauchy-Schwarz inequality it follows that
\[
\| \Gamma^\alpha(\rho^1_t, \rho^2_t) \|_2^2 \leq 3 \sum_{\gamma} |\hat{\rho}^\gamma_t|^2.
\] (A.8)

Using the bounds (A.7) and (A.8) in the last term of (A.6) and then summing for \( \alpha \in \{A, B, \varnothing\} \) we get
\[
V'(t) \leq \sum_{\alpha} \sum_{k \geq 1} \left[ -\frac{(k\pi)^2}{a_k} \langle \hat{\rho}^\alpha_t, \psi_k \rangle^2 + \frac{\beta}{2A} \langle \hat{\rho}^\alpha_t, \psi_k \rangle^2 + \frac{9A\beta}{2} \| \hat{\rho}^\alpha_t \|_2^2 \right] \\
= \sum_{\alpha} \sum_{k \geq 1} \left( -\frac{(k\pi)^2}{a_k} + \frac{\beta}{2A} + \frac{9A\beta}{2} \right) \langle \hat{\rho}^\alpha_t, \psi_k \rangle^2.
\]

If we choose \( A = (9\beta/2)^{-1} \),
\[
V'(t) \leq \sum_{\alpha} \sum_{k \geq 1} \left( -\frac{(k\pi)^2}{a_k} + \frac{9\beta^2}{4a_k} + 1 \right) \langle \hat{\rho}^\alpha_t, \psi_k \rangle^2 = \left( \frac{9\beta^2}{2} + 2 \right) V(t),
\]
and this concludes the proof.
A.3 Uniqueness for Robin boundary condition

In the section we discuss the uniqueness of the solutions of the boundary value problems (2.9) (Robin), case b) in Theorem 2.7.

First we observe that following a strategy similar to the one presented for the Dirichlet case, we can prove uniqueness for the Robin case (2.9) when \( \kappa_1 = \kappa_2 = 0 \) (case b1)). We can adapt the strategy presented above by choosing \( \{ \varphi_k \} \) as orthonormal basis of \( L^2([0,1]) \). The weak formulation of equation (2.10) gives

\[
\frac{d}{dt} \langle \tilde{\rho}^\alpha, \varphi_k \rangle = \langle \tilde{\rho}^\alpha, \Delta \varphi_k \rangle - \beta \langle \Gamma^\alpha(\rho^1, \rho^2), \nabla \varphi_k \rangle + \nabla \varphi_k(0) \tilde{\rho}^\alpha(0) - \nabla \varphi_k(1) \tilde{\rho}^\alpha(1).
\]

Since \( \nabla \varphi_k(u) = -k \pi \psi_k(u) \), the proof is concluded after we observe that \( \nabla \varphi_k(0) = \nabla \varphi_k(1) = 0 \), for any \( k \).

Now we study the cases \( \kappa_1 = \beta/2, \kappa_2 = 1 \) (case b2)), and \( \kappa_1 = 0, \kappa_2 = 1 \) (case b3)). For the function \( \tilde{\rho} = \rho^1 - \rho^2 \), for each \( \alpha \in \{A, B, \emptyset\} \) by (2.10)

\[
0 = \langle \tilde{\rho}^\alpha, \phi_t \rangle - \int_0^t \langle \tilde{\rho}^\alpha_s, (\Delta + \partial_t)\phi_s \rangle ds + \beta \int_0^t \langle \Gamma^\alpha(\rho^1, \rho^2), \nabla \phi_s \rangle ds
- \int_0^t \nabla \phi_s(0) \tilde{\rho}^\alpha_s(0) - \nabla \phi_s(1) \tilde{\rho}^\alpha_s(1) ds
- \kappa_1 \int_0^t \phi_s(0) [(2r_{a+2} - 1) \tilde{\rho}^\alpha_s(0) - 2r_a \tilde{\rho}^{a+1}_s(0)] ds
- \kappa_1 \int_0^t \phi_s(1) [(1 - 2\bar{r}_{a+2}) \tilde{\rho}^{a}_s(1) + 2\bar{r}_a \tilde{\rho}^{a+1}_s(1)] ds
+ \kappa_2 \int_0^t \phi_s(0) \tilde{\rho}^{a}_s(0) + \phi_s(1) \tilde{\rho}^{a}_s(1)] ds.
\]

Approximating the function \( \tilde{\rho}^\alpha \) by a sequence \( \{ \phi(n) \} \) of functions in \( C^{1,2}([0,T] \times [0,1]) \), we will assume that (A.9) is true for \( \phi = \tilde{\rho}^\alpha \). This approximation could be formalized as in [11, Theorem B.7]. Then, after integration by part in the term with \( (\Delta + \partial_t)\phi_s \) we obtain

\[
\frac{1}{2} \| \tilde{\rho}^\alpha \|^2_2 = -\int_0^t \| \nabla \tilde{\rho}^\alpha_s \|^2_2 ds - \beta \int_0^t \langle \Gamma^\alpha(\rho^1, \rho^2), \nabla \tilde{\rho}^\alpha_s \rangle ds
+ \int_0^t [\kappa_1 (2r_{a+2} - 1) - \kappa_2] \tilde{\rho}^\alpha_s(0)^2 ds + \int_0^t [\kappa_1 (1 - 2\bar{r}_{a+2}) - \kappa_2] \tilde{\rho}^{a+1}_s(1)^2 ds
- \int_0^t 2\kappa_1 r_a \tilde{\rho}^{a}_s(0) \tilde{\rho}^{a+1}_s(0) ds + \int_0^t 2\kappa_1 \bar{r}_a \tilde{\rho}^{a}_s(1) \tilde{\rho}^{a+1}_s(1) ds.
\]

Define \( W(t) = \frac{1}{2} \sum_\alpha \| \tilde{\rho}^\alpha_t \|^2_2 \). We will show that, for some constant \( C \), \( W(t) \leq C \int_0^t W(s) ds \) and then by Gronwall’s inequality we will conclude that \( W(t) = 0 \), for \( t \in [0,T] \), which yields the uniqueness.

Using Young’s inequality in the second term of the right hand side of (A.10), summing in \( \alpha \)
and using (A.8) we obtain

\[
W(t) \leq 9\alpha \beta \int_0^t W(s)\,ds + \left(\frac{\beta}{2\alpha} - 1\right) \int_0^t \sum_a \|\nabla \tilde{\rho}_s^a\|^2\,ds \\
+ \sum_a \int_0^t \left[\kappa_1(2r_{a+2} - 1) - \kappa_2\right] \tilde{\rho}_s^a(0)^2\,ds + \sum_a \int_0^t \left[\kappa_1(1 - 2\tilde{r}_{a+2}) - \kappa_2\right] \tilde{\rho}_s^a(1)^2\,ds \\
- \sum_a \int_0^t 2\kappa_1 r_a \tilde{\rho}_s^a(0) \tilde{\rho}_s^{a+1}(0)\,ds + \sum_a \int_0^t 2\kappa_1 \tilde{r}_a \tilde{\rho}_s^a(1) \tilde{\rho}_s^{a+1}(1)\,ds.
\]

(A.11)

Note that, in the case \(\kappa_1 = 0\) and \(\kappa_2 = 1\) (case b3)) the terms in the second line are negative and the terms in the last line disappear. So, taking \(A > \beta/2\) we get \(W(t) \leq 9\alpha \beta \int_0^t W(s)\,ds\). Now let us consider the case \(\kappa_1 = 1/2\) and \(\kappa_2 = 1\) (case b2)). To conclude the proof it remains to show that the sum of last two lines in (A.11) is negative. Remember that, under the conditions of case b2), \(\theta = \bar{\delta}\) and the process is well defined only if \(\bar{\beta} < 2\). We will conclude the uniqueness under the condition \(\bar{\beta} < 4/3\). By Young’s inequality, for \(D > 0\)

\[
-2\tilde{\rho}_s^a(0)\tilde{\rho}_s^{a+1}(0) \leq D\tilde{\rho}_s^a(0)^2 + \frac{1}{D}\tilde{\rho}_s^{a+1}(0)^2
\]

\[
2\tilde{\rho}_s^a(1)\tilde{\rho}_s^{a+1}(1) \leq \frac{1}{D}\tilde{\rho}_s^a(1)^2 + D\tilde{\rho}_s^{a+1}(1)^2.
\]

Taking \(D = 2\), we get that the sum of the last two lines in (A.11) is bounded above by

\[
\sum_a \int_0^t \left[\frac{\beta}{2} \left(2(r_a + r_{a+2}) - 1 + \frac{r_{a+2}}{2}\right) - 1\right] \tilde{\rho}_s^a(0)^2\,ds + \sum_a \int_0^t \left[\frac{\beta}{2} \left(1 + \frac{\tilde{r}_a}{2}\right) - 1\right] \tilde{\rho}_s^a(1)^2\,ds.
\]

Using the bounds \(r_a + r_{a+2} \leq 1\) and \(\tilde{r}_a \leq 1\) we see that the terms in the brackets above are negative when \(\bar{\beta} < 4/3\), and this concludes the proof in this case.

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