Matrix tree theorem analog for the net Laplacian matrix of a signed graph

Sudipta Mallik

Department of Mathematics and Statistics, Northern Arizona University, 801 S. Osborne Dr.
PO Box: 5717, Flagstaff, AZ 86011, USA
sudipta.mallik@nau.edu

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Abstract

For a simple signed graph $G$ with the adjacency matrix $A$ and net degree matrix $D^\pm$, the net Laplacian matrix is $L^\pm = D^\pm - A$. We introduce a new oriented incidence matrix $N^\pm$ which can keep track of the sign as well as the orientation of each edge of $G$. Also $L^\pm = N^\pm (N^\pm)^T$. Using this decomposition, we find the numbers of positive and negative spanning trees of $G$ in terms of the principal minors of $L^\pm$ generalizing Matrix Tree Theorem for an unsigned graph. We present similar results for the signless net Laplacian matrix $Q^\pm = D^\pm + A$ along with a combinatorial formula for its determinant.

1 Introduction

A signed graph $G = \{V, E, \sigma\}$ is defined to be a graph on vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$, where each edge $e_\ell$ has a sign $\sigma(e_\ell) \in \{1, -1\}$. An edge with positive sign is a positive edge and negative sign is a negative edge. Throughout this article, we consider only simple signed graphs, i.e., signed graphs that do not have loops and multiple edges. The adjacency matrix of signed graph $G$ is an $n \times n$ binary matrix $A = [a_{ij}]$ where $a_{ij}$ is $\sigma\{i, j\}$ if $\{i, j\}$ is an edge and 0 otherwise. Note that $|A|$, obtained from $A$ by taking absolute values of its entries, is the adjacency matrix of the underlying unsigned graph $|G|$ of the signed graph $G$. Note that $|G|$ can be thought as the signed graph obtained from $G$ by making the sign of each edge of $G$ positive. The degree matrix of $G$ and $|G|$, denoted by $D$, is an $n \times n$ diagonal matrix with the degrees of vertices of $G$ as diagonals. The net degree of a vertex $v$, denoted by $d^\pm(v)$, in signed graph $G$ is the number of positive edges minus that of negative edges incident with $v$. The net degree matrix of $G$, denoted by $D^\pm$, is an $n \times n$ diagonal matrix with the net-degrees of vertices of $G$ as diagonals. Note that $|D^\pm| = D$. The Laplacian, signless Laplacian, net Laplacian, and signless net Laplacian of signed graph $G$ are defined as $L = D - A$, $Q = D + A$, $L^\pm = D^\pm - A$, and $Q^\pm = D^\pm + A$ respectively.
Example 1.1. For the signed paw $G$ in Figure 1, we have

$$L = D - A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix}, \quad Q = D + A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix},$$

$$L^\pm = D^\pm - A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad Q^\pm = D^\pm + A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$ 

Observation 1.2. Suppose $-G$ is the signed graph obtained from the signed graph $G$ by reversing the sign of each of its edges. The following show the relations among matrices associated with $G$ and $-G$:

1. $L_{-G} = D_{-G} - A_{-G} = D_G + A_G = Q_G.$
2. $Q_{-G} = D_{-G} + A_{-G} = D_G - A_G = L_G.$
3. $L^\pm_{-G} = D^\pm_{-G} - A_{-G} = -D_G^\pm + A_G = -L_G^\pm.$
4. $Q^\pm_{-G} = D^\pm_{-G} + A_{-G} = -D_G^\pm - A_G = -Q_G^\pm.$

We define a new incidence matrix of signed graph $G$, called the net incidence matrix and denoted by $M^\pm = [m_{ij}]$, as follows: $M^\pm$ is an $n \times m$ binary matrix with rows indexed by vertices and columns indexed by edges where each column of $M^\pm$ has exactly two nonzero entries $m_{i\ell} = m_{j\ell}$ equaling to 1 for positive edge $e_\ell = \{i, j\}$ and the imaginary number $i$ for negative edge $e_\ell = \{i, j\}$.

Example 1.3. For the signed paw $G$ in Figure 1, we have

$$M^\pm = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & i & 0 & 1 \\ 0 & i & i & 0 \\ 0 & 0 & i & 1 \end{bmatrix}.$$
Similarly we define a new oriented incidence matrix of signed graph $G$, called the oriented net incidence matrix and denoted by $N^\pm = [n_{ij}]$, as follows: $N^\pm$ is obtained from the net incidence matrix of $G$ by multiplying one of two nonzero entries of each column by $-1$. An oriented net incidence matrix $N^\pm = [n_{ij}]$ of signed graph $G$ induces the following orientation of edges: edge $e_\ell = \{i, j\}$ is oriented as $(i, j)$ (i.e., vertex $i$ to vertex $j$) if $m_{i\ell} > m_{j\ell}$ on the real or imaginary axis. In the literature of signed graph, edges are oriented with two arrows giving a bidirected graph $[1]$. When each edge of a signed graph is singly oriented with one arrow, the oriented net incidence matrix can keep track of the sign as well as the orientation of each edge.

**Example 1.4.** For the oriented signed paw $G$ in Figure $G$ we have

$$N^\pm = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -i & 0 & 1 \\ 0 & i & i & 0 \\ 0 & 0 & -i & -1 \end{bmatrix}.$$ 

For the literature of signed graphs and associated matrices, see $[10, 11, 2]$. In $[1]$, the coefficients of the characteristic polynomial of the Laplacian matrix of a signed graph were studied. The net Laplacian of a signed graph has recently started getting more attention. For example, it was used in the study of controllability of undirected signed graphs $[4]$. Stanić studied the spectrum of the net Laplacian matrix of a signed graph in $[9]$. He mentioned the lack of a decomposition for the net Laplacian of a signed graph similar to that of Laplacian as the product of an incidence matrix and its transpose. The net incidence matrix defined above provides such a decomposition which is explained in section 2. As a consequence of this decomposition, we obtain a matrix tree theorem analog for the net Laplacian of a signed graph in section 3. In recent years, different combinatorial aspects of signless Laplacian matrices of graphs emerge to be active areas of research $[3, 6, 7, 5, 8]$. In section 4 we present a matrix tree theorem analog for the signless net Laplacian of a signed graph. In section 5 we find a combinatorial formula for the determinant of the signless net Laplacian.

## 2 Basic results

In this section we present some basic results regarding net incidence and net Laplacian matrices similar to that of incidence and Laplacian matrices.

**Proposition 2.1.** Let $D$, $D^\pm$, $A$, $M^\pm$, $L^\pm$, $Q^\pm$ and $Q$ be the degree, net degree, adjacency, net incidence, net Laplacian, signless net Laplacian, and net Laplacian matrices of a simple signed graph $G$ respectively. Let $L_{|G|}$ be the Laplacian matrix of the underlying unsigned graph $|G|$ of the signed graph $G$.

(a) $Q^\pm = D^\pm + A = M^\pm (M^\pm)^T$.

(b) $L^\pm = D^\pm - A = N^\pm (N^\pm)^T$ for any oriented net incidence matrix $N^\pm$ of $G$.

(c) $Q = D + A = M^\pm (M^\pm)^*$.
Proof. Suppose $G$ has $n$ vertices. To prove (a), suppose the rows of $M$ are $M_1, M_2, \ldots, M_n$. Then the $(i,j)$-entry of $M^\pm (M^\pm)^T$ is $(M^\pm (M^\pm)^T)_{ij} = M_i M_j^T$. This implies $(M^\pm (M^\pm)^T)_{ii} = d^\pm (v_i)$ and for $i \neq j$, $(M^\pm (M^\pm)^T)_{ij} = 0$ when $\{i, j\}$ is not an edge of $G$ and $(M^\pm (M^\pm)^T)_{ij} = \sigma\{i, j\}$ when $\{i, j\}$ is an edge of $G$. Thus $Q^\pm = D^\pm + A = M^\pm (M^\pm)^T$.

The proof is similar for (b) by noting that $(N^\pm (N^\pm)^T)_{ij} = -\sigma\{i, j\}$ when $\{i, j\}$ is an edge of $G$. The proofs for (c) and (d) are similar. □

**Proposition 2.2.** Let $G$ be a simple signed graph on $n$ vertices with the positive edge set $E^+$, negative edge set $E^-$, net Laplacian $L^\pm$, and signless net Laplacian $Q^\pm$. Then for all $x \in \mathbb{R}^n$,

$$x^T L^\pm x = \sum_{\{u,v\} \in E^+} (x_u - x_v)^2 - \sum_{\{i,j\} \in E^-} (x_i - x_j)^2$$

and

$$x^T Q^\pm x = \sum_{\{u,v\} \in E^+} (x_u + x_v)^2 - \sum_{\{i,j\} \in E^-} (x_i + x_j)^2.$$

**Proof.** Suppose $L^\pm = N^\pm (N^\pm)^T$ for some oriented net incidence matrix $N^\pm$ of $G$. For any $x \in \mathbb{R}^n$, $(N^\pm)^T x \in \mathbb{C}^m$. Each entry of $(N^\pm)^T x$ corresponds to an edge of $G$ and has the form $\pm(x_u - x_v)$ if $\{u, v\} \in E^+$ and $\pm i(x_i - x_j)$ if $\{i, j\} \in E^-$. Then

$$x^T L x = x^T N^\pm (N^\pm)^T x$$

$$= ((N^\pm)^T x)^T ((N^\pm)^T x)$$

$$= \sum_{\{u,v\} \in E^+} [\pm(x_u - x_v)]^2 + \sum_{\{i,j\} \in E^-} [\pm i(x_i - x_j)]^2$$

$$= \sum_{\{u,v\} \in E^+} (x_u - x_v)^2 - \sum_{\{i,j\} \in E^-} (x_i - x_j)^2.$$

A similar proof can derive the result for $Q^\pm = M^\pm (M^\pm)^T$ where $M^\pm$ is the net incidence matrix of $G$. □

**Observation 2.3.** If $x \in \mathbb{R}^n$ is an eigenvector corresponding to the eigenvalue 0 of the net Laplacian matrix $L^\pm$ of a simple signed graph, then

$$\sum_{\{u,v\} \in E^+} (x_u - x_v)^2 = \sum_{\{i,j\} \in E^-} (x_i - x_j)^2.$$

If 0 is an eigenvalue of the signless net Laplacian matrix $Q^\pm$ of a simple signed graph with an eigenvector $x \in \mathbb{R}^n$, then

$$\sum_{\{u,v\} \in E^+} (x_u + x_v)^2 = \sum_{\{i,j\} \in E^-} (x_i + x_j)^2.$$
Proposition 2.4. Let $M^\pm$ and $N^\pm$ be the net incidence matrix and an oriented net incidence matrix of a simple signed graph $G$ respectively. Let $M_{[G]}$ and $N_{[G]}$ be the incidence matrix and an oriented incidence matrix of the underlying unsigned graph $|G|$ of the signed graph $G$ respectively. Then

$$\text{rank}(M^\pm) = \text{rank}(M_{[G]}) \text{ and } \text{rank}(N^\pm) = \text{rank}(N_{[G]}).$$

Proof. The proof follows from that facts that the left null spaces of $M^\pm$ and $M_{[G]}$ are the same and the left null spaces of $N^\pm$ and $N_{[G]}$ are the same. \qed

Observation 2.5.

(a) $\text{rank}(M^\pm) = \text{rank}(M^\pm(M^\pm)^*) = \text{rank}(Q)$.

(b) $\text{rank}(M^\pm) \geq \text{rank}(M^\pm(M^\pm)^T) = \text{rank}(Q^\pm)$.

(c) $\text{rank}(N^\pm) \geq \text{rank}(N^\pm(N^\pm)^T) = \text{rank}(L^\pm)$.

Lemma 2.6. Suppose $G$ is a signed cycle on $n$ vertices. Then the determinant of an oriented net incidence matrix of $G$ is 0. Also the determinant of the net incidence matrix of $G$ is $\pm 2e^-$ if $n$ is odd and zero otherwise, where $e^-$ is the number of negative edges of $G$.

Proof. Suppose $M^\pm$ is the net incidence matrix of $G$. Then we have

$$M^\pm = PM^\pm Q = P \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & a_n \\ a_1 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & a_3 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & a_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-1} & a_n \end{bmatrix} Q,$$

for some permutation matrices $P$ and $Q$ and for some $a_1, a_2, \ldots, a_n \in \{1, i\}$. By a cofactor expansion across the first row, we have

$$\det(M^\pm) = \det(P) \det(M^\pm_{[G]}) \det(Q)$$

$$= (\pm 1)[a_1a_2\cdots a_n + (-1)^{n+1}a_na_1\cdots a_{n-1}](-1)^{\pm 1}$$

$$= (\pm 1)a_1a_2\cdots a_n[1 + (-1)^{n+1}](-1)^{\pm 1}$$

$$= \pm i^e[1 + (-1)^{n+1}].$$

Then $\det(M^\pm) = \pm 2i^e$ if $n$ is odd and zero otherwise.

A similar proof can derive the result for any oriented net incidence matrix $N^\pm$:

$$N^\pm = PN^\pm Q = P \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & -a_n \\ -a_1 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & a_3 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & a_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & -a_{n-1} & a_n \end{bmatrix} Q,$$
for some permutation matrices $P$ and $Q$ and for some $a_1, a_2, \ldots, a_n \in \{1, i\}$. By a cofactor expansion across the first row, we have

$$\det(N) = \det(P) \det(N') \det(Q)$$

$$= (\pm 1)[a_1 a_2 \cdots a_n + (-1)^{n+1}(-a_n)(-a_1)(-a_2)\cdots(-a_{n-1})](\pm 1)$$

$$= (\pm 1)a_1 a_2 \cdots a_n [1 + (-1)^{2n+1}](\pm 1)$$

$$= 0.$$

Lemma 2.7. Suppose $G$ is a signed unicyclic graph on $n$ vertices with a cycle $C_k$, $k \leq n$. Then the determinant of an oriented net incidence matrix $N$ of $G$ is zero. Also the determinant of the net incidence matrix $M$ of $G$ is $\pm 2i e$ if $k$ is odd and zero otherwise, where $e$ is the number of negative edges of $G$.

Proof. The proof follows from the preceding lemma when $G$ is a signed cycle. Now suppose $G$ is not a signed cycle, i.e., $G$ has pendant vertices. By successive cofactor expansions along rows corresponding to the pendant vertices, the determinant becomes $\pm i e$ times the determinant of the signed cycle, where $e$ is the number of negative edges in $C_k$. The rest follows from the preceding Lemma.

3 The numbers of positive and negative spanning trees in a signed graph

The matrix tree theorem can be proved by the Cauchy-Binet formula:

**Theorem 3.1 (Cauchy-Binet).** Let $m \leq n$. For $m \times n$ matrices $A$ and $B$, we have

$$\det(AB^T) = \sum S \det(A(S)) \det(B(S)),$$

where the summation runs over $\binom{n}{m}$ $m$-subsets $S$ of $\{1, 2, \ldots, n\}$.

In the preceding theorem, $A(S)$ is the submatrix of $A$ with the columns indexed by $S \subseteq \{1, 2, \ldots, n\}$ and all the rows of $A$ (i.e., no rows deleted). In the next observation and the rest of this article, $L(i)$ is the submatrix of $L$ with row $i$ and column $i$ deleted, $N(i)$ is the submatrix of $N$ with row $i$ deleted, and $N(i; S)$ is the submatrix of $N$ with row $i$ deleted keeping the columns indexed by $S \subseteq \{1, 2, \ldots, m\}$.

Observation 3.2. Let $G$ be a signed graph on $n \geq 2$ vertices with $m$ edges and $m \geq n - 1$. Suppose $L^\pm$ is the net Laplacian matrix and $N^\pm$ is an oriented net incidence matrix of $G$. Then

(a) $L^\pm(i) = N^\pm(i)N^\pm(i)^T$, $i = 1, 2, \ldots, n$, and
(b) \( \det(L^\pm(i)) = \det(N^\pm(i;i)N^\pm(i;i)^T) = \sum_S \det(N^\pm(i;S))^2 \), where the summation runs over all \((n - 1)\)-subsets \(S\) of \(\{1, 2, \ldots, m\}\) (by Theorem 3.1).

A subgraph \(H\) of a signed graph \(G\) is \textit{positive} if \(H\) has even number of negative edges (i.e., the product of the signs of edges of \(H\) is positive). Similarly \(H\) is \textit{negative} if \(H\) has odd number of negative edges (i.e., the product of the signs of edges of \(H\) is negative). By this definition, \(K_1\) is a positive tree.

\textbf{Lemma 3.3.} Let \(T\) be a signed tree with at least one edge and \(N^\pm\) be an oriented net incidence matrix of \(T\). Then for all vertices \(j\) of \(T\),

\[
\det(N^\pm(j;)) = \begin{cases} 
\pm1 & \text{if } T \text{ is positive} \\
\pm i & \text{if } T \text{ is negative}.
\end{cases}
\]

\textit{Proof.} We prove the statement by induction on \(n\), the number of vertices of \(T\). For \(n = 2\), \(T\) is \(P_2\) and then \(\det(N^\pm(j;)) = \pm1\) when \(T\) is positive and \(\det(N^\pm(j;)) = \pm i\) when \(T\) is negative. Assume the statement holds for some \(n \geq 2\). Let \(T\) be a tree with \(n + 1\) vertices. Suppose vertex \(v\) is a pendant vertex incident with the unique edge \(e_\ell = \{v, k\}\).

Case 1. \(T\) is a negative tree

Subcase (a). \(e_\ell = \{v, k\}\) is a negative edge

In this case, \(T(v)\) is a positive tree. Then by the induction hypothesis, \(\det(N^\pm(v, j; \ell)) = \pm1\) for any \(j \neq v\). The \(v\)th row of \(N^\pm\) has only one nonzero entry which is the \((v, \ell)\)th entry and it is equal to \(\pm i\). To find \(\det(N^\pm(j;)), j \neq v\), we have a cofactor expansion across the \(v\)th row and get

\[
\det(N^\pm(j;)) = \pm i \cdot (\pm \det(N(v, j; \ell))) = \pm i(\pm1) = \pm i.
\]

Note that the \(\ell\)th column of \(N^\pm(v;\) has only one nonzero entry which is the \((k, \ell)\)th entry and it is equal to \(\pm i\). To find \(\det(N^\pm(v;))\), we have a cofactor expansion across the \(\ell\)th column and get

\[
\det(N^\pm(v;)) = \pm i \cdot (\pm \det(N(v, k; \ell))) = \pm i(\pm1) = \pm i.
\]

Subcase (b). \(e_\ell = \{v, k\}\) is a positive edge

In this case, \(T(v)\) is a negative tree. Then by the induction hypothesis, \(\det(N^\pm(v, j; \ell)) = \pm i\) for any \(j \neq v\). The \(v\)th row of \(N^\pm\) has only one nonzero entry which is the \((v, \ell)\)th entry and it is equal to \(\pm1\). To find \(\det(N^\pm(j;)), j \neq v\) we have a cofactor expansion across the \(v\)th row and get

\[
\det(N^\pm(j;)) = \pm1 \cdot (\pm \det(N(v, j; \ell))) = \pm1(\pm i) = \pm i.
\]

Note that the \(\ell\)th column of \(N^\pm(v;\) has only one nonzero entry which is the \((k, \ell)\)th entry and it is equal to \(\pm1\). To find \(\det(N^\pm(v;))\), we have a cofactor expansion across the \(\ell\)th column and get

\[
\det(N^\pm(v;)) = \pm1 \cdot (\pm \det(N(v, k; \ell))) = \pm1(\pm i) = \pm i.
\]
Case 2. $T$ is a positive tree
Subcase (a). $e_\ell = \{v, k\}$ is a negative edge
In this case, $T(v)$ is a negative tree. Then by the induction hypothesis, \( \det(N^\pm(v, j; \ell)) = \pm i \) for any \( j \neq v \). The \( v \)th row of \( N^\pm \) has only one nonzero entry which is the \((v, \ell)\)th entry and it is equal to \( \pm i \). To find \( \det(N^\pm(j;)) \), \( j \neq v \), we have a cofactor expansion across the \( v \)th row and get
\[
\det(N^\pm(j;)) = \pm i \cdot (\pm \det(N(v, j; \ell))) = \pm i(\pm i) = \pm 1.
\]
Note that the \( \ell \)th column of \( N^\pm(v;\ell) \) has only one nonzero entry which is the \((k, \ell)\)th entry and it is equal to \( \pm 1 \). To find \( \det(N^\pm(v;\ell)) \), we have a cofactor expansion across the \( \ell \)th column and get
\[
\det(N^\pm(v;\ell)) = \pm 1 \cdot (\pm \det(N(v, k; \ell))) = \pm 1(\pm 1) = \pm 1.
\]

Subcase (b). $e_\ell = \{v, k\}$ is a positive edge
In this case, $T(v)$ is a positive tree. Then by the induction hypothesis, \( \det(N^\pm(v, j; \ell)) = 1 \) for any \( j \neq v \). The \( v \)th row of \( N^\pm \) has only one nonzero entry which is the \((v, \ell)\)th entry and it is equal to \( \pm 1 \). To find \( \det(N^\pm(j;)) \), \( j \neq v \), we have a cofactor expansion across the \( v \)th row and get
\[
\det(N^\pm(j;)) = \pm 1 \cdot (\pm \det(N(v, j; \ell))) = \pm 1(\pm 1) = \pm 1.
\]
Note that the \( \ell \)th column of \( N^\pm(v;\ell) \) has only one nonzero entry which is the \((k, \ell)\)th entry and it is equal to \( \pm 1 \). To find \( \det(N^\pm(v;\ell)) \), we have a cofactor expansion across the \( \ell \)th column and get
\[
\det(N^\pm(v;\ell)) = \pm 1 \cdot (\pm \det(N(v, k; \ell))) = \pm 1(\pm 1) = \pm 1.
\]

Lemma 3.4. Let $T$ be a signed tree with at least one edge and $M^\pm$ be the net incidence matrix of $T$. Then for all vertices $j$ of $T$,
\[
\det(M^\pm(j;)) = \begin{cases} 
\pm 1 & \text{if } T \text{ is positive} \\
\pm i & \text{if } T \text{ is negative}.
\end{cases}
\]

Proof. The proof is similar to that for an oriented net incidence matrix.

A positive spanning tree of a simple signed graph $G$ is a spanning tree of $G$ that is positive (i.e., it contains an even number of negative edges). A negative spanning tree of $G$ is a spanning tree of $G$ that is negative (i.e., it contains an odd number of negative edges).

Example 3.5. The signed paw $G$ in Figure 1 has one positive spanning tree $T_1$ and two negative spanning trees $T_2$ and $T_3$ (see in Figure 2).

The following is the matrix tree theorem analog for the net Laplacian of a signed graph.
Theorem 3.6. Let $G$ be a simple connected signed graph on $n \geq 2$ vertices $1, 2, \ldots, n$ with the net Laplacian matrix $L^\pm$. Then for each $i = 1, 2, \ldots, n$, $\det(L^\pm(i))$ is the number of positive spanning trees minus the number of negative spanning trees of $G$.

Proof. By Observation 3.2, we have,

$$\det(L^\pm(i)) = \sum_S \det(N^\pm(i; S))^2,$$

where the summation runs over all $(n - 1)$-subsets $S$ of $\{1, 2, \ldots, m\}$. Note that each such $S$ corresponds to a spanning subgraph $H_S$ of $G$ consisting of $n - 1$ edges indexed by $S$ and possibly with some isolated vertices. Suppose $H_S$ is not a spanning tree of $G$. Then we show $\det(N^\pm(i; S)) = 0$. Since $H_S$ is not a spanning tree of $G$, $H_S$ contains a tree component $T$ and a connected component $H'$ containing a cycle. If $i$ is not in $T$, then the rows of $N^\pm(i; S)$ corresponding to the vertices of $T$ are linearly dependent and consequently $\det(N^\pm(i; S)) = 0$. Suppose $i$ is in $T$ and $S' \subseteq S$ corresponds to the edges of a cycle in $H'$ on vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$, $t = |S'|$. Then $\det(N^\pm[i_1, i_2, \ldots, i_t; S']) = 0$ by Lemma 2.6. Thus the columns of $N^\pm(i; S)$ corresponding to $S'$ are linearly dependent and consequently $\det(N^\pm(i; S)) = 0$.

Suppose $\mathcal{T}^+$ and $\mathcal{T}^-$ are the sets of all positive and negative spanning trees of $G$ respectively. Then

$$\det(L^\pm(i)) = \sum_S \det(N^\pm(i; S))^2 = \sum_{H_S \in \mathcal{T}^+} (\det(N^\pm(i; S))^2 + \sum_{H_S \in \mathcal{T}^-} \det(N^\pm(i; S))^2$$

$$= \sum_{H \in \mathcal{T}^+} (\pm 1)^2 + \sum_{H \in \mathcal{T}^-} (\pm i)^2 \quad \text{(by Lemma 3.3)}$$

$$= |\mathcal{T}^+| - |\mathcal{T}^-|.$$

Observation 3.7. The preceding theorem implies the matrix tree theorem for a graph when it is considered as a signed graph with all positive edges.
Corollary 3.8. Let $G$ be a simple connected signed graph on $n \geq 2$ vertices with the net Laplacian matrix $L^\pm$. Let $L_{|G|}$ be the Laplacian matrix of the underlying unsigned graph $|G|$ of the signed graph $G$. Then for each $i = 1, 2, \ldots, n$,

(a) $\frac{1}{2} [\det(L_{|G|}(i)) + \det(L^\pm(i))]$ is the number of positive spanning trees of $G$, and

(b) $\frac{1}{2} [\det(L_{|G|}(i)) - \det(L^\pm(i))]$ is the number of negative spanning trees of $G$.

Proof. Suppose $\mathcal{T}^+$ and $\mathcal{T}^-$ are the sets of all positive and negative spanning trees of $G$ respectively. Then the proof follows from the following:

$$\det(L_{|G|}(i)) = |\mathcal{T}^+| + |\mathcal{T}^-| \quad \text{and} \quad \det(L^\pm(i)) = |\mathcal{T}^+| - |\mathcal{T}^-|$$

Example 3.9. The signed paw $G$ in Figure 1 has one positive spanning tree $T_1$ and two negative spanning trees $T_2$ and $T_3$ (see in Figure 2). For each $i = 1, 2, 3, 4$,

$$\det(L^\pm(i)) = 1 - 2 = -1.$$  

The number of positive spanning trees of $G$ is

$$\frac{1}{2} [\det(L_{|G|}(i)) + \det(L^\pm(i))] = \frac{1}{2} (3 - 1) = 1.$$  

The number of negative spanning trees of $G$ is

$$\frac{1}{2} [\det(L_{|G|}(i)) - \det(L^\pm(i))] = \frac{1}{2} (3 + 1) = 2.$$  

4 Signless net Laplacian

A $TU$-graph is a graph whose connected components are trees or odd-unicyclic graphs. A $TU$-subgraph of $G$ is a subgraph of $G$ that is a $TU$-graph. We are using the same definitions for signed $TU$-graphs and signed $TU$-subgraphs of a signed graph (which is different from the definition given in [1]). According to our definition before, a signed unicyclic graph is negative if it has an odd number of negative edges. A negative component of a signed $TU$-graph $G$ is a negative tree or a negative odd-unicyclic graph. The number of negative components in a signed $TU$-graph $G$ is denoted by $b^-(G)$. If $G$ is an odd-unicyclic graph, then the determinant of its net incidence matrix is $\pm 2^{b^-(G)}$ by Lemma 2.7.

Example 4.1. Consider the signed graph $G$ in Figure 3. All the spanning $TU$-subgraphs of $G$ with 4 edges consisting of a unique tree on vertex 1 are $H_1$, $H_2$, $H_3$, $H_4$ and $H_5$ (see Figure 4). Note that $b^-(H_3) = b^-(H_5) = 1$ and $b^-(H_1) = b^-(H_2) = b^-(H_4) = 0$.

Theorem 4.2. Let $G$ be a simple connected signed graph on $n \geq 2$ vertices and $m$ edges with the net incidence matrix $M^\pm$. Let $i$ be an integer from $\{1, 2, \ldots, n\}$. Let $S$ be an $(n - 1)$-subset of $\{1, 2, \ldots, m\}$ and $H$ be a spanning subgraph of $G$ with edges indexed by $S$ and possibly with some isolated vertices.
Figure 3: A signed graph $G$

Figure 4: Spanning TU-subgraphs of $G$ (in Figure 3) with 4 edges containing a unique tree on vertex 1

(a) If $H$ is not a spanning TU-subgraph of $G$, then $\det(M^\pm(i; S)) = 0$.

(b) Suppose $H$ is a spanning TU-subgraph of $G$ consisting of a unique tree $T$ and $c$ odd-unicyclic graphs $U_1, U_2, \ldots, U_c$.

(i) If $i$ is a vertex of $U_j$ for some $j = 1, 2, \ldots, c$, then $\det(M^\pm(i; S)) = 0$.

(ii) If $i$ is a vertex of $T$, then $\det(M^\pm(i; S)) = \pm 2^c i^b(H)$.

Proof. (a) Suppose $H$ is not a TU-graph on $n$ vertices and $n - 1$ edges. Then $n \geq 5$ and $H$ has at least two tree components and a component containing at least two cycles. Suppose $T$ is a tree component of $H$ that does not contain vertex $i$. If $T$ consists of just one vertex, then the corresponding row in $M^\pm(i; S)$ is a zero row giving $\det(M^\pm(i; S)) = 0$. Now suppose $T$ has at least two vertices. Consider the square submatrix $M'$ of $M^\pm(i; S)$ with rows corresponding to the vertices of $T$ and columns corresponding to the edges of $T$. Since the row rank of $M'$ is less than or equal to the number of columns of $M'$, the rows of $M'$ and consequently of $M^\pm(i; S)$ corresponding to the vertices of $T$ are linearly dependent. Thus $\det(M^\pm(i; S)) = 0$. 

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(b) If $i$ is a vertex of $U_j$ for some $j = 1, 2, \ldots, c$, then the columns of the submatrix of $M^\pm(i; S)$ corresponding to $U_j$ are linearly dependent which implies $\det(M^\pm(i; S)) = 0$. If $i$ is a vertex of the positive (respectively negative) tree $T$, then $M^\pm(i; S)$ is a direct sum of incidence matrices of odd-unicyclic graphs $U_1, U_2, \ldots, U_c$ and the incidence matrix of the tree $T$ with one row deleted and consequently $\det(M^\pm(i; S)) = (\pm 2i^{b^-}(H)) \cdot (\pm 1) = \pm 2i^{b^-}(H)$ (respectively $\det(M^\pm(i; S)) = (\pm 2i^{-1+b^-}(H)) \cdot (\pm i) = \pm 2i^{b^-}(H)$) by Lemmas 2.7 and 3.4.

The following is the matrix tree theorem analog for the signless net Laplacian of a signed graph.

**Theorem 4.3.** Let $G$ be a simple connected signed graph on $n \geq 2$ vertices $1, 2, \ldots, n$ with the signless net Laplacian matrix $Q^\pm$. Then for each $i = 1, 2, \ldots, n$,

$$\det(Q^\pm(i)) = \sum_{H \in U_e} 4^{c(H)} - \sum_{H \in U_o} 4^{c(H)},$$

where $U_e$ is the set of all spanning $TU$-subgraphs $H$ of $G$ with $n-1$ edges and an even number of negative components consisting of a unique tree on vertex $i$ and $c(H)$ odd-unicyclic graphs and $U_o$ is the set of all spanning $TU$-subgraphs $H$ of $G$ with $n-1$ edges and an odd number of negative components consisting of a unique tree on vertex $i$ and $c(H)$ odd-unicyclic graphs.

**Proof.** First note that

$$\det(Q^\pm(i)) = \sum_S \det(M^\pm(i; S))^2,$$

where the summation runs over all $(n-1)$-subsets $S$ of $\{1, 2, \ldots, m\}$. Note that each such $S$ corresponds to a spanning subgraph $H_S$ of $G$ consisting of $n-1$ edges indexed by $S$ and possibly with some isolated vertices. Note by Theorem 4.2 $\det(M^\pm(i; S)) \neq 0$ only when $H_S$ is a spanning $TU$-subgraph of $G$ consisting of a unique tree $T$ containing vertex $i$ and $c(H_S)$ odd-unicyclic graphs. Then

$$\det(Q^\pm(i)) = \sum_S \det(M^\pm(i; S))^2 = \sum_{H_S \in U_e} (\det(M^\pm(i; S))^2 + \sum_{H_S \in U_o} \det(M^\pm(i; S))^2$$

$$= \sum_{H_S \in U_e} (\pm 2^{c(H_S)}i^{b^-(H_S)})^2 + \sum_{H_S \in U_o} (\pm 2^{c(H_S)}i^{b^-(H_S)})^2 \quad (\text{by Theorem 4.2})$$

$$= \sum_{H_S \in U_e} (\pm 2^{c(H_S)})^2 + \sum_{H_S \in U_o} (\pm i2^{c(H_S)})^2$$

$$= \sum_{H \in U_e} 4^{c(H)} - \sum_{H \in U_o} 4^{c(H)}.$$

**Observation 4.4.** The preceding theorem implies the matrix tree theorem for the signless Laplacian of a graph (see [5, Theorem 2.9]) when it is considered as a signed graph with all positive edges.
Example 4.5. For the signed graph $G$ in Figure 3, $U_e = \{H_1, H_2, H_4\}$ and $U_o = \{H_3, H_5\}$ corresponding to vertex 1 (see Figure 4).

$$\det(Q^\pm(1)) = \sum_{H \in U_e} 4^{c(H)} - \sum_{H \in U_o} 4^{c(H)} = (4^{c(H_1)} + 4^{c(H_2)} + 4^{c(H_4)}) - (4^{c(H_3)} + 4^{c(H_5)})$$
$$= (4^0 + 4^0 + 4^1) - (4^0 + 4^1)$$
$$= 6 - 5$$
$$= 1$$

## 5 Determinant of a signless net Laplacian

Since the row-sums of the net Laplacian $L^\pm$ of a signed graph $G$ are zeros, 0 is an eigenvalue of $L^\pm$ and consequently $\det(L^\pm) = 0$. In this section we investigate $\det(Q^\pm)$, the determinant of the signless net Laplacian $Q^\pm$ of a signed graph $G$.

**Lemma 5.1.** Let $G$ be a simple signed graph on $n$ vertices and $m \geq n$ edges with the net incidence matrix $M^\pm$. Let $S$ be a $n$-subset of $\{1, 2, \ldots, m\}$ and $H$ be a spanning subgraph of $G$ with edges indexed by $S$ and possibly some isolated vertices.

(a) If one of the connected components of $H$ is not an odd-unicyclic graph, then $\det(M^\pm[S]) = 0.$

(b) If $H$ consists of $k$ odd-unicyclic connected components and even of them are negative, then $\det(M^\pm[S]) = \pm 2^k.$ If $H$ consists of $k$ odd-unicyclic connected components and odd of them are negative, then $\det(M^\pm[S]) = \pm 2^{k_i}.$

**Proof.** (a) Suppose one of the connected components of $H$ is not an odd-unicyclic graph. Then $H$ has a connected component that is a tree or an even-unicyclic graph. Then by Lemma 2.7, $\det(M^\pm[S]) = 0$ when $H$ has an even-unicyclic component. When $H$ has a tree component $T$, the rows of $M^\pm[S]$ corresponding to the vertices of $T$ are linearly dependent and consequently $\det(M^\pm[S]) = 0.$

(b) Suppose $H$ consists of $k$ odd-unicyclic components. Since $M^\pm[S]$ is a direct sum of the net incidence matrices of $k$ odd-unicyclic graphs, then $\det(M^\pm[S]) = \pm 2^{k_i}$ by Lemma 2.7. Thus $\det(M^\pm[S])$ is $\pm 2^{k_i}$ when $b^-(H)$ is odd and $\pm 2^k$ otherwise.

By Theorem 3.1 and 5.1 we have the following theorem.

**Theorem 5.2.** Let $G$ be a simple signed graph on $n$ vertices with signless net Laplacian matrix $Q^\pm$. Then

$$\det(Q^\pm) = \sum_{H \in U^\pm_e} 4^{c(H)} - \sum_{H \in U^\pm_o} 4^{c(H)},$$

where $U^\pm_e$ is the set of all spanning $TU$-subgraphs $H$ of $G$ with $n$ edges consisting of $c(H)$ odd-unicyclic graphs including an even number of negative odd-unicyclic graphs and $U^\pm_o$ is the set of all spanning $TU$-subgraphs $H$ of $G$ with $n$ edges consisting of $c(H)$ odd-unicyclic graphs including an odd number of negative odd-unicyclic graphs.
Proof. By Theorem 3.1

\[ \det(Q^\pm) = \det(M^\pm (M^\pm)^T) = \sum_S \det(M^\pm (; S))^2, \]

where the summation runs over all \( n \)-subsets \( S \) of \( \{1, 2, \ldots, m\} \). By Lemma 5.1 we have

\[ \det(Q^\pm) = \sum_S \det(M^\pm (; S))^2 = \sum_{H \in \mathcal{U}_e} (\pm 2^{e(H)})^2 - \sum_{H \in \mathcal{U}_o} (\pm 2^{e(H)})^2 \]
\[ = \sum_{H \in \mathcal{U}_e} 4^{e(H)} - \sum_{H \in \mathcal{U}_o} 4^{e(H)}. \]

Example 5.3. The signed graph \( G \) in Figure 3 is a \( TU \)-graph, in particular, an odd-unicyclic graph that is positive. Then \( \mathcal{U}_e = \{G\} \) and \( \mathcal{U}_o = \emptyset \). Thus

\[ \det(Q^\pm) = \sum_{H \in \mathcal{U}_e} 4^{e(H)} - \sum_{H \in \mathcal{U}_o} 4^{e(H)} = 4^{e(G)} - 0 = 4^1 = 4. \]

6 Open problems

We end this article by posing some relevant open problems.

Question 6.1. Find combinatorial formulas of the coefficients of the characteristic polynomial of the signless net Laplacian matrix \( Q^\pm \) of a signed graph.

Suppose the characteristic polynomial of signless net Laplacian \( Q^\pm \) of a simple connected signed graph \( G \) on \( n \) vertices and \( m \geq n \) edges is

\[ P_{Q^\pm}(x) = \det(xI_n - Q^\pm) = x^n + \sum_{i=1}^{n} a_i x^{n-i}. \]

For \( i = 1, 2, \ldots, n \), \( a_i \) will be obtained in terms of spanning \( TU \)-subgraphs of \( G \) with \( i \) edges.

Question 6.2. Characterize the signed graphs whose signless net Laplacian matrix is not invertible.

One of such graphs is a bipartite graph with all positive edges. Using \( \det(Q^\pm) = 0 \), Theorem 5.2 gives the following necessary and sufficient condition:

\[ \sum_{H \in \mathcal{U}_e} 4^{e(H)} = \sum_{H \in \mathcal{U}_o} 4^{e(H)}. \]

Question 6.3. Find the multiplicity of the eigenvalue 0 of the signless net Laplacian matrix of a signed graph when it is not invertible.
Examples suggest that the answer may have a connection with the balancedness of cycles in spanning $TU$-subgraphs with $n$ edges. Observation 2.3 regarding eigenvectors corresponding to the eigenvalue 0 may be helpful.

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