Reduced submodules of finite dimensional polynomial modules

Tilahun Abebaw, Nega Arega, Teklemichael Worku Bihonegn ¹ and David Ssevviiri ²

Abstract

Let $k$ be a field with characteristic zero, $R$ be the ring $k[x_1, \ldots, x_n]$ and $I$ be a monomial ideal of $R$. We study the Artinian local algebra $R/I$ when considered as an $R$-module $M$. We show that the largest reduced submodule of $M$ coincides with both the socle of $M$ and the $k$-submodule of $M$ generated by all outside corner elements of the Young diagram associated with $M$. Interpretations of different reduced modules is given in terms of Macaulay inverse systems. It is further shown that these reduced submodules are examples of modules in a torsion-torsionfree class, together with their duals; coreduced modules, exhibit symmetries in regard to Matlis duality and torsion theories. Lastly, we show that any $R$-module $M$ of the kind described here satisfies the radical formula.

Keywords: Reduced modules, socle, Macaulay inverse systems, local Artinian algebras, modules that satisfy the radical formula

MSC 2010 : 13E10, 16D60, 16D80

1 Introduction

Reduced rings play an important role in algebra. A module analogue of reduced rings was defined by Lee and Zhou in [20]. Reduced modules have since been studied in [17, 18, 25, 29, 30] among others. Let $R$ be a commutative unital ring and $I$ be an ideal of $R$. Reduced modules form a full subcategory of $R$-Mod on which the $I$-torsion functor $\Gamma_I$ is representable, i.e., if $M$ is an $I$-reduced $R$-module, then $\Gamma_I(M) \cong \text{Hom}(R/I, M)$, see [29]. It was shown in [29] that reduced modules and their dual, coreduced modules, provide a setting in which both the Matlis-Greenlees-May Equivalence and Greenlees-May Duality hold. In [16], reduced modules were used to characterize regular

¹This work forms part of the third author’s PhD thesis.
²Corresponding author.
modules. In [30], it was shown that $I$-reduced and $I$-coreduced modules provide the necessary and sufficient conditions for the functor $\text{Hom}_R(R/I, -)$ to be a radical. The same conditions unify and subsume different conditions which were proved on a case-by-case basis for the $I$-torsion functor $\Gamma_I$ to be a radical. A more general version of reduced modules was studied in [17]. We note that this is what Rohrer and Yekutieli studied in [26] and [35] respectively, although called them modules with bounded torsion. The same modules are called modules whose submodules \((0 :_M a^t)\) with \(a \in R\) and \(t \in \mathbb{Z}^+\) are stationary, by Schenzel and Simon in [27, Proposition 3.1.10].

A general version of reduced modules relates to prisms which belong to the groundbreaking theory of perfectoid rings, [5, 35].

A module \(M\) over a commutative ring \(R\) is reduced if for all \(a \in R\) and \(m \in M\), \(a^2m = 0\) implies that \(am = 0\). We wish to study for an arbitrary \(R\)-module \(M\) in a suitable full subcategory of the category of \(R\)-modules, the set

\[ \mathfrak{R}(M) := \{m \in M : a^2m = 0 \Rightarrow am = 0 \text{ for all } a \in R\}. \]

By definition, \(M\) is a reduced \(R\)-module if and only if \(\mathfrak{R}(M) = M\). It is easy to see that in general, \(\mathfrak{R}(M)\) is not a submodule of \(M\), see Example 2.2. Let \(k\) be a field, \(R := k[x_1, \ldots, x_n]\) and \(I\) a monomial ideal of \(R\). In this paper, we characterize reduced submodules, \(\mathfrak{R}(M)\) of \(R\)-modules \(M\) in the full subcategory \(\mathfrak{C}\) of \(R\)-Mod consisting of \(R\)-modules of the form \(R/I\), with \(\dim_k(R/I) < \infty\). A notion which has been so useful in characterizing \(\mathfrak{R}(M)\) is that of socle of \(M\). For an Artinian local algebra, \(M := R/I\) and a maximal ideal \(\mathfrak{m}\) of \(R\), the socle of \(M\), \(\text{Soc}(M)\) is the submodule of \(M\) given by \((0 :_M \mathfrak{m})\); the collection of all elements of \(M\) annihilated by \(\mathfrak{m}\). This definition is equivalent to saying that \(\text{Soc}(M)\) is the direct sum of all simple submodules of \(M\). Socle of Artinian local algebras has been widely studied, see [1, 2, 6, 7, 32, 33] among others. It is well known, for instance that a local Artinian algebra \(R/I\) is Gorenstein if and only if \(\dim_k(\text{Soc}(R/I)) = 1\).

We show in Section 2 that for any \(M \in \mathfrak{C}\), \(\mathfrak{R}(M)\) is a submodule of \(M\) which coincides with both \(\text{Soc}(M)\), and with the submodule of \(M\) generated by all outside corner elements of the Young diagram associated with \(M\), Theorem 2.1. We hope that this coincidence will increase the versatility of both the largest reduced submodule of \(M\) and the socle of \(M\), which are already widely studied. In Section 3, we exploit the Macaulay inverse systems to give a correspondence between different reduced submodules of \(M\) in \(\mathfrak{C}\) and their associated Macaulay inverse duals. The correspondences are summarized in Figure 3. In Section 4, we exhibit using a diagram, see
Figure 4, symmetries from the following notions: $I$-reduced, $I$-coreduced, $I$-torsion and $I$-complete together with their Matlis duals and associated torsion theories. The work in this section is mainly complementary to that done in papers [29, 30]. Lastly, in Section 5, we show that modules $M$ in the subcategory $\mathcal{C}$ satisfy the radical formula and their semiprime radicals coincide with the Jacobson radical.

2 For any $M \in \mathcal{C}$, $\mathcal{R}(M)$ coincides with $\text{Soc}(M)$

In this section, we prove the coincidence of $\mathcal{R}(M)$ with both $\text{Soc}(M)$ and the submodule of $M$ generated by all outside corner elements of $M$, for any $M \in \mathcal{C}$.

Definition 2.1 Let $R := k[x_1, \ldots, x_n]$ and $M \in \mathcal{C}$. An element $m \in M$ is an outside corner element if $x_im = 0$ for all $1 \leq i \leq n$. $m \in M$ is inner if it is not an outside corner element.

In Definition 2.1, if $n = 2$, then this definition is exactly what appears in [11, page 8] and given combinatorially as the boxes which are at the outside corners of the Young diagram. Note that any $M \in \mathcal{C}$ can be viewed as a $k$-vector space or an $R$-module. When we say, “generating set of $M$”, we will always mean the monomial basis of $M$ seen as a $k$-vector space.

Lemma 2.1 Every simple module is reduced.

Proof: We prove first that a simple module is prime. An $R$-module $M$ is prime if for any $a \in R$ and $m \in M, am = 0$ implies that either $m = 0$ or $aM = 0$. Now, suppose that $M$ is simple, $a \in R$ and $m \in M$ such that $am = 0$. Then $aRm = 0$. $M$ being simple implies that either $Rm = 0$ or $Rm = M$. If $Rm = 0$, then $m = 0$. Suppose $Rm = M$, then $aM = 0$. So, $M$ is prime. We now prove that a prime module is reduced. Let $M$ be a prime $R$-module, $a \in R$ and $m \in M$ such that $a^2m = 0$. Then we have $a(am) = 0$. By definition of a prime module, we have $am = 0$ or $aM = 0$. In both cases $am = 0$, since $am \in aM$. This shows that $M$ is reduced. ■

Theorem 2.1 Let $M \in \mathcal{C}$, $S := \{m_1, m_2, \cdots, m_n\}$ be the collection of all outside corner elements of $M$ and $\langle S \rangle_k$ be the $k$-submodule of $M$ generated by $S$.

$$\langle S \rangle_k = \mathcal{R}(M) = \text{Soc}(M).$$
Proof: Let \( m \in \langle S \rangle_k \), i.e., \( m = \sum_{i=1}^{n} r_i m_i \) where \( r_i \in k \) and \( m_i \in S \). Suppose that \( a^2m = 0 \) for some \( a \in R \). If \( a \in \langle x_1, \ldots, x_n \rangle \), then by definition of outside corner elements, \( am = 0 \). Now suppose that \( a \in R \setminus \langle x_1, \ldots, x_n \rangle \). If \( a = a_0 + a_1 \), where \( a_0 \in k \) and \( a_1 \in \langle x_1, \ldots, x_n \rangle \), then \( am = a_0m \neq 0 \). So, \( a^2m = a_0^2m \neq 0 \). Thus, in all cases \( m \in \mathcal{R}(M) \) and \( \langle S \rangle_k \subseteq \mathcal{R}(M) \). Suppose that \( \langle S \rangle_k \nsubseteq \mathcal{R}(M) \). Then there exists \( m \in \mathcal{R}(M) \) and \( m \notin \langle S \rangle_k \). This implies that \( m \) is not an outside corner element of \( M \). So, there exists \( x_i \) for some \( i \in \{1, \ldots, n\} \) such that \( x_im \neq 0 \). However, since \( a \in \langle x_1, \ldots, x_n \rangle \) for sufficiently large \( t \in \mathbb{Z}^+ \), \( a^tm = 0 \). This shows that \( m \notin \mathcal{R}(M) \) which is a contradiction. It is therefore impossible for the inclusion, \( \langle S \rangle_k \subseteq \mathcal{R}(M) \) to be strict. Thus \( \langle S \rangle_k = \mathcal{R}(M) \). Any simple module is reduced, Lemma 2.1 and a direct sum of reduced modules is reduced, [20]. So, \( \text{Soc}(M) \subseteq \mathcal{R}(M) \).

Let \( m \in \mathcal{R}(M) \). Since \( \mathcal{R}(M) = \langle S \rangle_k \) and \( m \notin \langle S \rangle_k \) for each \( 1 \leq i \leq n \). So, \( \langle x_1, \ldots, x_n \rangle m = 0 \), by definition of socle it follows that \( m \in \text{Soc}(M) \). Thus, \( \mathcal{R}(M) \subseteq \text{Soc}(M) \).

Corollary 2.1 If \( \mathcal{C}_{\text{red}} \) is the collection of all reduced \( R \)-modules \( N \) such that \( N \) is a submodule of \( M \in \mathcal{C} \), then \( \mathcal{C}_{\text{red}} \) coincides with all semisimple submodules \( N \) of \( M \in \mathcal{C} \).

Proof: By Theorem 2.1, for any \( M \in \mathcal{C} \), \( \mathcal{R}(M) = \text{Soc}(M) \). If \( N \) is a submodule of \( M \in \mathcal{C} \) which is reduced, then \( N \subseteq \mathcal{R}(M) \) the largest reduced submodule of \( M \). So, \( N \) is semisimple. Similarly, if \( N \) is a semisimple submodule of \( M \in \mathcal{C} \), then \( N \subseteq \text{Soc}(M) \) and therefore, \( N \) is a reduced submodule of \( M \).

Example 2.1 The monomial basis of the \( k \)-algebra \( R := k[x, y] \) takes the form of Figure 1 when represented on the grid. If the monomial ideal \( I \) of \( k[x, y] \) is given by \( I = \langle x^4, x^3y, x^2y^2, xy^3, y^4 \rangle \), then the quotient \( k \)-module \( M = k[x, y]/I \) is 11 dimensional and is generated by all elements in the Young diagram given in Figure 1. The outside corner elements, which are circled red, generate \( \mathcal{R}(M) \), i.e., \( \mathcal{R}(M) = \langle x^3, x^2y, xy^2, y^4 \rangle_k \) mod \( I \) and this is the socle of \( M \).
Example 2.2 In general, $\mathcal{R}(M)$ for $M \notin \mathcal{C}$ need not be a submodule of $M$. Consider the $\mathbb{Z}$-module $M := \mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$, where $p$ is a prime number. The elements $(1, \bar{1})$ and $(1, \bar{0})$ of $M$ are torsion-free and therefore belong to $\mathcal{R}(M)$. However, the element $(0, \bar{1}) = (1, \bar{1}) - (1, \bar{0})$ of $M$ does not belong to $\mathcal{R}(M)$ since $p^2(0, \bar{1}) = (0, \bar{0})$ but $p(0, \bar{1}) \neq (0, \bar{0})$. This shows that in this case $\mathcal{R}(M)$ is not a submodule of $M$.

Remark 2.1 The elements in $S$ as given in Theorem 2.1 were described in [34], as the truly isolated monomials of a survival complex of a semigroup.

An $R$-module $M$ is coreduced if for all $a \in R$, $aM = a^2M$.

Corollary 2.2 For any $M \in \mathcal{C}$, $\mathcal{R}(M)$ is coreduced.

Proof: For all $a \in \langle x_1, \ldots, x_n \rangle$, $a\mathcal{R}(M) = a^2\mathcal{R}(M) = 0$, since $\mathcal{R}(M) = \text{Soc}(M) = (0 : M \langle x_1, \ldots, x_n \rangle)$. If $a \in R \setminus \langle x_1, \ldots, x_n \rangle$, then $a = a_0 + f(x_1, \ldots, x_n)$ where $a_0 \in k$. Now, $a\mathcal{R}(M) = a_0\mathcal{R}(M) = \mathcal{R}(M)$ and $a^2\mathcal{R}(M) = a_0^2\mathcal{R}(M) = \mathcal{R}(M)$.

3 Reduced modules via inverse systems

The Macaulay inverse system is a powerful method for solving problems about Artinian local algebras of the form $R/I$. It was for instance used in problems such as weak Lefschetz property [13], in Waring’s problem [12], and in the classification of Artinian Gorenstein rings, [9]. In this section, we establish Macaulay inverse correspondences between reduced modules.
Let $k$ be an algebraically closed field of characteristic zero. Let $R = k[x_1, \cdots, x_n]$, $V$ be the $k$-vector space $\langle x_1, \cdots, x_n \rangle$ and $P = \bigoplus_{i \geq 0} \text{Sym}^i V$, the standard graded polynomial in $n$-indeterminates over $k$. If $V^*$ is the $k$-vector space dual to $V$, then we have $V^* = \langle X_1, \cdots, X_n \rangle$ and $\Gamma = D^k(V^*) = \bigoplus_{i \geq 0} \text{Hom}_k(P, k)$ the graded $P$-module of graded $k$-linear homomorphisms from $P$ to $k$. It is well-known that; $\Gamma \cong k[X_1, \cdots, X_n]$ the power divided ring and $\Gamma$ is an $R$-module via the following action which is called apolarity action.

$$R \circ \Gamma \longrightarrow \Gamma$$

$$(x^\alpha, X^\beta) \mapsto x^\alpha \circ X^\beta := \begin{cases} 
\frac{\beta!}{(\beta - \alpha)!} X^{\beta - \alpha} & \text{if } \beta_i \geq \alpha_i, \text{ for all } i = 1, \cdots, n \\
0 & \text{otherwise}
\end{cases}$$

where $x^\alpha = x_1^{\alpha_1}, \cdots, x_n^{\alpha_n}$ and $X^\beta = X_1^{\beta_1}, \cdots, X_n^{\beta_n}$.

Macaulay’s correspondence is a special case of Matlis duality, which gives a one-to-one correspondence between ideals of $R$ and finitely generated submodules of $\Gamma$. For any ideal $I \subseteq R$, the dual of $I$

$$I^\perp := \{ m \in \Gamma \mid I \circ m = 0 \}$$

is a finitely generated submodule of $\Gamma$ called the Macaulay’s inverse system of $I$. Conversely, if $W$ is a finitely generated submodule of $\Gamma$, then

$$W^\perp = \{ r \in R \mid r \circ W = 0 \}$$

is an ideal of $R$. So, to each Artinian local algebra $R/I$, we associate a finitely generated submodule $I^\perp$ of $\Gamma$. Conversely, if $W$ is a finitely generated submodule of $\Gamma$, then $R/W^\perp$ is a local Artinian algebra. We write $(R/I)^\vee = I^\perp$ and $W^\vee = R/W^\perp$ respectively. For more details about inverse systems, see for instance [8, 9, 10, 12, 13, 15, 22]. Just like in Section 2, we consider the Artinian local algebra $R/I$ as an $R$-module $M \in \mathfrak{C}$.

**Definition 3.1** Let $I$ be an ideal of $R$. An $R$-module $M$ is $I$-reduced if for all $m \in M$, $I^2m = 0$ implies that $Im = 0$. Note that $M$ is reduced if it is $I$-reduced for all ideals $I$ of $R$.

We denote the largest $I$-reduced submodule of $M$ by $\mathfrak{R}_I(M)$.

**Lemma 3.1** Let $R$ be a ring. For any $R$-module $M$, $\mathfrak{R}(M) = \bigcap_{I \subseteq R} \mathfrak{R}_I(M)$.
**Theorem 3.1** Let $R = k[x_1, \ldots, x_n]$ and $\Gamma = k[X_1, \ldots, X_n]$. $\Gamma$ is an $R$-module via apolarity action. Define $\Gamma_i := k[X_1, \ldots, X_i]$ a submodule of $\Gamma$ and $J_i := \langle x_{i+1}, x_{i+2}, \ldots, x_n \rangle$ an ideal of $R$.

1. If $J$ is an ideal of $R$ contained in $J_i$, then $\Gamma_i$ is $J$-reduced and $\Gamma_i \subseteq J_i^\perp$.

2. If $W$ is a submodule of $\Gamma$ contained in $\Gamma_i$, then $W$ is $J_i$-reduced and $J_i \subseteq W^\perp$.

3. $\mathfrak{R}_{J_i}(\Gamma) = \Gamma_i \subseteq J_i^\perp$.

4. $\mathfrak{R}(\Gamma) = k$.

**Proof:**

1. Let $a \in J \subseteq J_i = \langle x_{i+1}, \ldots, x_n \rangle$. For any $a \in J_i$, $a \circ \Gamma_i = 0$. This is because $a$ consists of indeterminates $x_t$ such that $t > i$ for all indeterminates $X_i$ of $\Gamma_i$. In this case, $x_i \circ X_i = 0$. It is also true that $x_i \circ k = 0$, for all $t \geq 1$. So, $J \circ \Gamma_i = 0$. Therefore $\Gamma_i$ is $J$-reduced and $\Gamma_i \subseteq (0 : J) = J_i^\perp$.

2. If $W$ is a submodule of $\Gamma_i \subseteq \Gamma$ and $J_i = \langle x_{i+1}, \ldots, x_n \rangle$. $W$ has the form $\sum_k kX_j^k$ for some $j$ between 0 and $i$. Just like in 1 above $J_i \circ W = 0$.

So $W$ is $J_i$-reduced and $J_i \subseteq (0 : W) = W^\perp$.

3. Let $m$ be a monomial in the $R$-module $\Gamma$ and $a$ be a monomial in the ideal $J_i$ of $R$. If $a \circ m = 0$, then either $a = x_t^i$ and $m = X_s^n$ such that $t > i$ or $a$ involves a term $x_j^i$ and $m$ involves a term $X_s^j$ such that $t > s$. However, $m \in \mathfrak{R}_{J_i}(\Gamma)$ if and only if $m$ is of the former type. To be of the former type is equivalent to having $m \in \Gamma_i$. $m$ in the latter case cannot be in $\mathfrak{R}_{J_i}(\Gamma)$. For if $t > s \neq 0$, then $x_j^i \circ X_j^s = 0$ but $x_j \circ X_j^s \neq 0$. So, $\mathfrak{R}_{J_i}(\Gamma) = \Gamma_i$. Since for all monomials $m \in \Gamma_i$ we have some $a \in J_i$ such that $a \circ m = 0$, $\Gamma_i \subseteq (0 : J_i) = J_i^\perp$.

4. Every nonzero $J_i$-reduced submodule of $\Gamma$ contains $k$. So,

$$k \subseteq \bigcap_{I \subseteq R} \{ \Gamma_i \mid \Gamma_i \text{ is an } J_i \text{-reduced submodule of } \Gamma \}.$$
Suppose $T := \bigcap_{I \subseteq R} \{ \Gamma_i \mid \Gamma_i \text{ is an } I\text{-reduced submodule of } \Gamma \}$ and $k$ is strictly contained in $T$. Then, by Lemma 3.1 $T$ is a reduced submodule of $M$ and takes the form $k \oplus kX_1 \oplus kX_2 \oplus \cdots \oplus kX_n \oplus \cdots \oplus kX_{t_1}^{t_1} \oplus \cdots \oplus kX_{t_n}^{t_n}$, for some $t_i \in \mathbb{Z}^+$. Let $a = (x_1, \ldots, x_n) \in R$. $a^{t+1} \circ T = 0$, but $a \circ T \neq 0$, contradicting the fact that $T$ is reduced. So, $k = T = \mathcal{R}(\Gamma)$.

Proposition 3.1 For any $M := R/I \in \mathcal{C}$ and a maximal ideal $m$ of $R$,

$$\mathcal{R}(M)^\vee = \frac{I^\perp}{m \circ I^\perp}.$$ 

Proof: Follows from Theorem 2.1 and [15, Proposition 2.4.3].

Example 3.1 Let $R = k[x_1, x_2]$ and $I = \langle x_1^2, x_1x_2, x_2^3 \rangle$. If $M = R/I$, then $\mathcal{R}(M) = \langle x_1, x_2 \rangle_k$ mod $I$ and $I^\perp = k \oplus (kX_1 \oplus kX_2) \oplus kX_2^2$. $\mathcal{R}(M)^\vee = \frac{I^\perp}{m \circ I^\perp} = \langle X_1, X_2 \rangle_k$ mod $k \oplus kX_2$, which is a quotient of $I^\perp$.

Definition 3.2 Let $M = \bigoplus_{d \in \mathbb{N}} M_d$ be a graded $R$-module. We define the Hilbert function of $M$ $HF(M, -) : \mathbb{N} \rightarrow \mathbb{N}$ to be $HF(M, d) := \dim_k M_d$ for all $d \in \mathbb{N}$. Furthermore, we define the Hilbert series of $M$ as

$$HS(M, t) := \sum_{d \in \mathbb{N}} HF(M, d)t^d.$$ 

Definition 3.3 Let $m$ be a maximal ideal of $R$. $X \in I^\perp$ is called an outside corner element if $x_i \circ X \in m \circ I^\perp$ for all $1 \leq i \leq n$. An element $X \in I^\perp$ is inner if it is not an outside corner element.

Theorem 3.2 For any $M \in \mathcal{C}$,

1. a submodule $\mathcal{R}(M)$ of $M$ is generated by monomials $x_1^{a_1} \cdots x_k^{a_k}$ mod $I$, $0 \leq k \leq n$ if and only if $\mathcal{R}(M)^\vee$, the quotient of $I^\perp$ is generated by the monomials $X_1^{a_1} \cdots X_k^{a_k}$ mod $m \circ I^\perp$, where $a_1, \ldots, a_k$ are nonnegative integers;
2. $HS(\mathcal{R}(M), t) = HS(\mathcal{R}(M)^\vee, t)$;
3. $\mathcal{R}(M)^\vee$ is the largest reduced quotient of $I^\perp$.

Proof:
1. It is well known that there is a one-to-one correspondence between the $R$-module $M = R/I$ and $I^\perp$, the Macaulay inverse system of $I$, [22, IV]. It is also known that $HS(R/I, t) = HS(I^\perp, t)$, [15, Proposition 2.3.3, Proposition 2.2.19]. Combinatorially, the generators of $R/I$ and $I^\perp$ can be represented in a Young diagram (YD). To distinguish between the two Young diagrams, we name the one for the former YD$_1$ and for the latter YD$_2$. It is easy to check that $m \circ I^\perp$ is generated by all the inner elements of YD$_2$. It follows that $R(M)^\vee$, the quotient of $I^\perp$ by $m \circ I^\perp$ is generated by all the elements at outside corners of the YD$_2$. However, this is in a one-to-one correspondence with elements at the outside corners of YD$_1$, which generate $R(M)$, see Theorem 2.1. So, the generators of $R(M)$ are in one-to-one correspondence with the generators of $R(M)^\vee$.

2. It is clear from 1.

3. Since $R(M)^\vee = I^\perp$ mod $m \circ I^\perp = \langle$outside corner elements of YD$_2\rangle$, $x_i \circ R(M)^\vee \in m \circ I^\perp$, whenever $x_i^2 \circ R(M)^\vee \in m \circ I^\perp$ for all integers $1 \leq i \leq n$. The reduced quotients of $I^\perp$ are generated by elements at the outside corners of YD$_2$. Since $R(M)^\vee$ is generated by all the elements at the outside corners of YD$_2$, it is the largest reduced quotient of $I^\perp$.

\[ \text{Example 3.2} \] Consider the $R$-module $M = \frac{k[x,y]}{(x^4, x^3y, y^2)}$. $R(M) = \langle x^3, x^2y \rangle_k$ mod $\langle x^4, x^3y, y^2 \rangle$ and $I^\perp = k \oplus kX \oplus kY \oplus kX^2 \oplus kXY \oplus kX^3 \oplus kX^2Y$. So, $R(M)^\vee = kX^3 \oplus kX^2Y$ mod $(k \oplus kX \oplus kY \oplus kX^2 \oplus kXY)$. The YD$_1$ and YD$_2$ associated to $M$ and $I^\perp$ are respectively given in Figure 2.

![Figure 2: YD$_1$ and YD$_2$ of $M$ and $I^\perp$ respectively.](image)

Define $\mathfrak{D} := \left\{ I^\perp \leq \Gamma : \text{for an ideal } I \text{ of } R \text{ such that } R/I \in \mathfrak{C} \right\}$.

\[ \text{Corollary 3.1} \] The following statements hold for any $M \in \mathfrak{C}$ and $I^\perp \in \mathfrak{D}$.
1. $\mathcal{R}(M)^\vee$ is both a reduced and a coreduced quotient of $I^\perp$.

2. $\mathcal{R}(M^\vee) = \mathcal{R}(I^\perp) = k$ (the only reduced submodule of $\Gamma$).

3. $\mathcal{R}(I^\perp)^\vee \cong k$ (the only reduced quotient of $M$).

4. $\mathcal{R}((I^\perp)^\vee) = \mathcal{R}(M)$ (is both a reduced and a coreduced submodule of $M$).

**Proof:**

1. Since $\mathcal{R}(M)^\vee = I^\perp \mod \mathfrak{m} \circ I^\perp = \langle \text{outside corner elements of } \text{YD}_2 \rangle$, $x_i \circ \mathcal{R}(M)^\vee \in \mathfrak{m} \circ I^\perp$ and $x_i^2 \circ \mathcal{R}(M)^\vee \in \mathfrak{m} \circ I^\perp$, for all $i \in \mathbb{Z}^+$ which implies $x_i \circ \mathcal{R}(M)^\vee = x_i^2 \circ \mathcal{R}(M)^\vee$ for all $i \in \mathbb{Z}^+$. This shows that $\mathcal{R}(M)^\vee$ is a coreduced $R$-module. For the reduced part see, Theorem 3.1 (3).

2. $\mathcal{R}(M^\vee) = \mathcal{R}(I^\perp) = I^\perp \cap \mathcal{R}(\Gamma) = I^\perp \cap k = k$.

3. $\mathcal{R}(I^\perp)^\vee = k^\vee = R/k^\perp = R/\mathfrak{m} \cong k$.

4. Follows by definition.

**Proposition 3.2** Let $R = k[x_1, \cdots, x_n]$ and $I$ be an ideal of $R$ which is homogeneous of degree $n + 1$. If $M := R/I$, then

1. $\mathcal{R}(M) = (x_1, \cdots, x_n)^n M = M_n$ (the homogeneous part of $M$ with the highest degree).

2. $\mathcal{R}(\Gamma) = (x_1, \cdots, x_n)^n \circ I^\perp = k$ (the homogeneous part of $I^\perp$ with the least degree).

**Proof:**

1. Successive multiplication of an element $(x_1, \cdots, x_n)$ with $M := R/I$ a graded $R$-module eliminates on each multiplication the homogeneous part of $M$ with the lowest degree. Therefore multiplying $n$-times leaves only the homogeneous part of $R/I$ with the highest degree. This is however, the submodule of $M$ generated by the outside corner elements of $M$. So, $\mathcal{R}(M) = (x_1, \cdots, x_n)^n M$.

2. $I^\perp$ is a finite dimensional graded $R$-module. Successive multiplication of $I^\perp$ by the element $(x_1, \cdots, x_n)$ via apolarity action eliminates the homogeneous part with the highest degree. Multiplying $n$ times leaves only the homogeneous part of $I^\perp$ with degree 0, which is $k$. By Theorem 3.1, $\mathcal{R}(\Gamma) = k$. So, $\mathcal{R}(\Gamma) = (x_1, \cdots, x_n)^n \circ I^\perp = k$. 

10
Corollary 3.2 For any \( I^\perp \in \mathcal{D} \), \( \mathcal{R}(I^\perp) = \text{Soc}(I^\perp) \).

\[ \text{Proof:} \quad \text{For any monomial ideal } I \text{ of } R, \text{the corresponding submodule } I^\perp \text{ of } \Gamma \text{ contains } k \text{ and this is the only submodule of } \Gamma \text{ which is simple, hence } \text{Soc}(I^\perp) = k. \text{ Moreover, by Corollary 3.1 } k \text{ is the largest reduced submodule of } I^\perp. \text{ Thus, } \mathcal{R}(I^\perp) = \text{Soc}(I^\perp). \] 

Let \( \mathcal{D}_{\text{red}} \) be the full subcategory of \( R\text{-Mod} \) consisting of all reduced \( R \)-modules \( N \) defined under apolarity action such that there exists a surjection map \( I^\perp \to N \) for some \( I^\perp \in \mathcal{D} \). Then \( \mathcal{D}_{\text{red}} \) is dual to \( \mathcal{C}_{\text{red}} \) and coincides with all semisimple modules \( T \) defined under apolarity action for which the surjection \( I^\perp \to T \) exists for some \( I^\perp \in \mathcal{D} \).

4 The symmetries exhibited

This section is mostly complementary to papers [29, 30]. In particular, it serves three purposes: 1) it demonstrates that \( \mathcal{C}_{\text{red}} \), provides a concrete and accessible example of modules: a) in the TTF class \( T_I \) constructed in [30, Theorem 3.1] for all ideals \( I \) of \( R \), b) for which the Matlis-Greenlees-May Equivalence proved in [29, Theorem 4.3] and later showed to be an equality in [30, Proposition 4.2] holds for all ideals \( I \) of \( R \); 2) it exhibits: a) commutativity between taking torsion theories and taking Matlis duality, b) symmetries between the \( R \)-modules \( M, \Gamma_I(M), \Lambda_I(M), M/IM, \) and \( (0 :_M I) \) and their associated Matlis duals; 3) gives new results and a summary of several results studied in [29, 30].

For an \( R \)-module \( M \), define \( \Gamma_I(M) := \lim_{\leftarrow k} \text{Hom}(R/I^k, M) \) and \( \Lambda_I(M) := \lim_{\rightarrow k} M/I^kM \).

Figure 3 summarizes the Macaulay inverse correspondences of reduced modules studied in Section 3.
| Modules under usual action | Modules under apolarity action | Remarks |
|----------------------------|--------------------------------|---------|
| $1. M = R/I \in \mathcal{C} \iff I^\perp \in \mathcal{D}$ | | the two have the same dimension |
| $2. \mathcal{R}(M) \iff \frac{I^\perp}{\mathfrak{m} \mathfrak{o} l^+}$ | | both are reduced, coreduced and generated by outside corner elements |
| $3. k[x_1, \ldots, x_t] \cong k \in \mathcal{C} \iff k \in \mathcal{D}$ | | the only reduced modules in $\mathcal{C}$ and $\mathcal{D}$ respectively |
| $4. M/\mathfrak{r}(M) \iff \mathfrak{m} \circ I^\perp$ | | generated by inner elements |
| $5. \mathcal{R}(M) \hookrightarrow M \iff I^\perp \rightarrow \frac{I^\perp}{\mathfrak{m} \mathfrak{o} l^+}$ | | an embedding and a surjection respectively |
| $6. \frac{M}{\mathfrak{m}}, \mathfrak{m} = \left(\frac{x_1, \ldots, x_t}{I} \right) \iff \mathcal{R}(I^\perp) \hookrightarrow I^\perp$ | | a surjection and an embedding respectively |
| $7. \mathcal{C}_{\text{red}} \iff \mathcal{D}_{\text{red}}$ | | both consist of semisimple modules |
| $8. \mathcal{R}(M) = \text{Soc}(M) \iff \mathcal{R}(I^\perp) = \text{Soc}(I^\perp)$ | | in both cases the reduced submodule and the socle coincide |
| $9. \text{Soc}(M) = (0 :_M \mathfrak{m}) \iff \text{Soc}(I^\perp) = (0 :_\Gamma \mathfrak{m})$ | | Socle is the annihilating submodule by $\mathfrak{m}$ of $M$ and $\Gamma$ respectively |

**Figure 3:** The summary of Macaulay inverse correspondences about reduced modules.

**Definition 4.1** An $R$-module $M$ is $I$-torsion (resp. $I$-complete, $I$-reduced and $I$-coreduced) if $\Gamma_I(M) \cong M$ (resp. $\Lambda_I(M) \cong M, \Gamma_I(M) \cong \text{Hom}(R/I, M)$ and $\Lambda_I(M) \cong R/I \otimes M$).

**Definition 4.2** A torsion theory of an abelian category $\mathcal{C}$, is a pair $(T, F)$ of full subcategories of $\mathcal{C}$ such that $\text{Hom}(T, F) = 0$ and for all $M \in \mathcal{C}$, there exists a short exact sequence

$$0 \rightarrow M_T \rightarrow M \rightarrow M_F \rightarrow 0$$

with $M_T \in T$ and $M_F \in F$. In this case, we call $T$ a torsion class and $F$ a torsionfree class. A class $\mathcal{L}$ of an abelian category $\mathcal{C}$ is a torsion-torsionfree (TTF) class if it is both a torsion and a torsion-free class.
Let $\mathcal{A}_I$ (resp. $\mathcal{B}_I$) be an abelian full subcategory of $R$-$\text{Mod}$ consisting of $I$-reduced (resp. $I$-coreduced) $R$-modules. In [30, Theorem 3.1]; it was shown that the class

$$T_I = \{ M \in \mathcal{A}_I \mid \Gamma_I(M) = M \}$$

is a TTF with the associated torsion-torsionfree triple $(\mathcal{S}_I, T_I, \mathcal{F}_I)$, where $\mathcal{F}_I = \{ M \in \mathcal{A}_I : \Gamma_I(M) = 0 \}$ and $\mathcal{S}_I = \{ M \in \mathcal{B}_I : IM = M \}$.

**Example 4.1** $\mathfrak{C}_{\text{red}} \subseteq T_I$, for all ideals $I$ of $R$. Secondly, any $M = R/I \in \mathfrak{C}$ is also contained in $T_I$. In these two cases the modules are $I$-reduced and $I$-torsion if and only if they are $I$-coreduced and $I$-complete, [30, Proposition 4.2].

**Proposition 4.1** Let $M$ be an $R$-module which is both $I$-reduced and $I$-coreduced. Consider the TTF triple $(\mathcal{S}_I, T_I, \mathcal{F}_I)$ and the Matlis dual $\text{Hom}_R(-, E)$, then

1. $M \in T_I$ if and only if $\text{Hom}_R(M, E) \in T_I$;
2. let in addition $I$ be finitely generated, $M \in \mathcal{F}_I$ if and only if $\text{Hom}_R(M, E) \in \mathcal{S}_I$;
3. $M \in \mathcal{S}_I$ if and only if $\text{Hom}_R(M, E) \in \mathcal{F}_I$.

**Proof:**

1. $\Gamma_I(M) \cong M \iff \Lambda_I(M) \cong M \iff \text{Hom}_R(\Lambda_I(M), E) \cong \text{Hom}_R(M, E) \iff \text{Hom}_R(M, E) \cong \text{Hom}_R(M, E)$. The first equivalence is due to Proposition [30, Proposition 4.2]. The second equivalence holds because the functor $\text{Hom}_R(-, E)$ preserves and reflects isomorphisms. The third equivalence is due to [29, Proposition 5.3].

2. $\Gamma_I(M) = 0 \iff \text{Hom}_R(\Gamma_I(M), E) \cong 0 \iff \Lambda_I(\text{Hom}_R(M, E)) \cong 0$. The part of the first equivalence is trivial; its reverse is due to the fact that $\text{Hom}_R(-, E)$ reflects zero since $E$ is an injective cogenerator. The second equivalence is due to [29, Proposition 5.5].

3. First note that $IM = M$ if and only if $\Lambda_I(M) \cong 0 \iff \text{Hom}_R(\Lambda_I(M), E) \cong 0 \iff \Gamma_I(\text{Hom}_R(M, E)) \cong 0$. The first equivalence holds because $\text{Hom}_R(-, E)$ reflects and preserves zeros. The second equivalence is due to [29, Proposition 5.3].
For any $M \in \mathfrak{C}_{\text{red}}$, levels 1, 2 and 3 of Figure 4 collapse to one level, i.e.,
$\Gamma_I(M) \cong \text{Hom}_R(R/I, M) \cong M$ and $\Lambda_I(M) \cong R/I \otimes M \cong M$. In this case,
$M$ belongs to the torsion class $T_I$ (in the red ellipse). By Matlis duality levels 5, 6 and 7 also collapse to one level and $\text{Hom}(M, E)$ also belongs to $T_I$.

If $\Gamma_I(M) = 0$, then the left hand side of levels 2, 3 and 4 collapse to one level. In this case $M \in \mathcal{F}_I$ (the green ellipse). However, on taking the Matlis dual of the module $\Gamma_I(M)$, $\text{Hom}(R/I, M)$ and 0 we get the left hand side of
levels 6, 7 and 8 collapse to one level in which case \( \text{Hom}(M, E) \in \mathcal{T}_I \) (the blue ellipse).

If \( \Lambda_I(M) = 0 \), the right hand side of levels 2, 3 and 4 collapse to one level. So, \( M \in \mathcal{T}_I \) (the blue ellipse). Taking the Matlis dual of the modules \( \Lambda_I(M), R/I \otimes M \) and 0, one gets the right hand side of levels 6, 7 and 8 also collapse to just one level and \( \text{Hom}(M, E) \in \mathcal{F}_I \) (the green ellipse).

The following results can also be interpreted using Figure 4.

1. Let \( I \) be finitely generated ideal of a ring \( R \). If \( M \) is an \( I \)-reduced and \( I \)-torsion \( R \)-module then \( \text{Hom}_R(M, E) \) is \( I \)-complete, [29, Corollary 5.6 (2)].

2. \( M \) is \( I \)-coreduced, then \( \text{Hom}_R(M, E) \) is \( I \)-reduced, [29, Proposition 2.6 (1)].

3. Let \( R \) be a Noetherian ring. \( M \) is \( I \)-reduced if and only if \( \text{Hom}_R(M, E) \) is \( I \)-coreduced, [29, Proposition 5.1].

4. Let \( I \) be any ideal of a ring \( R \) and \( N \) any \( R \)-module. If \( M \) is \( I \)-coreduced and \( I \)-complete, then \( \text{Hom}_R(M, N) \) is \( I \)-torsion, [29, Corollary 5.4 (2)].

5. Let \( R \) be a Noetherian ring, and \( \mathcal{A}_I \) be an abelian full subcategory of \( R\text{-Mod} \) consisting of \( I \)-reduced \( R \)-modules. The \( I \)-torsion free modules in \( \mathcal{A}_I \) coincide with the \( I \)-coreduced \( R \)-modules \( M \) for which \( IM = M \), i.e., \( \mathcal{T}_I = \mathcal{F}_I \) (the green and the blue ellipses coincide), [30, Proposition 3.3 (2)].

The two processes namely; (1) of taking a TTF and then dualising and (2) of dualising and then taking the TTF are commutative. The arrow between the ellipses summarises Proposition 4.1.

5 Modules in \( \mathcal{C} \) satisfy the radical formula

A submodule \( P \) of an \( R \)-module \( M \) is prime (resp. semiprime) if the \( R \)-module \( M/P \) is prime (resp. reduced). The prime radical \( \beta(M) \) (resp. semiprime radical, \( \mathcal{S}(M) \)) of \( M \) is the intersection of all prime (resp. semiprime) submodules of \( M \). If \( N \) is a submodule of \( M \), then \( \beta(N) \) (resp. \( \mathcal{S}(N) \)) denotes the intersection of all prime (resp. semiprime) submodules of \( M \) containing
For a submodule $N$ of an $R$-module $M$, we define the envelope $E_M(N)$ of $N$ as the set

$$E_M(N) := \{ rm \mid r \in R, m \in M \text{ and } r^km \in N \text{ for some } k \in \mathbb{Z}^+\}.$$

Denote the submodule of $M$ generated by $E_M(N)$ by $\langle E_M(N) \rangle$. A submodule $N$ of $M$ is said to satisfy the radical formula (or s.t.r.f for short) if $\langle E_M(N) \rangle = \beta(N)$. A module $M$ is said to satisfy the radical formula if every submodule $N$ of $M$ s.t.r.f. Modules that s.t.r.f have been studied in [3, 4, 14, 19, 21, 23, 24, 28, 31] among others. Lastly, we define the Jacobson radical $\mathcal{J}(M)$ of $M$ as the intersection of all maximal submodules of $M$.

$\mathcal{S}(M)$ (resp. $\mathcal{J}(M)$) is dual to $\mathfrak{R}(M)$ (resp. $\text{Soc}(M)$). As expected, (since $\mathfrak{R}(M) = \text{Soc}(M)$ for $M \in \mathfrak{C}$) the submodules $\mathcal{S}(M)$ and $\mathcal{J}(M)$ also coincide whenever $M \in \mathfrak{C}$.

**Theorem 5.1** For any $M \in \mathfrak{C}$,

1. $\langle E_M(0) \rangle = \mathcal{S}(M) = \beta(M) = \mathcal{J}(M)$,

2. $M$ s.t.r.f.

**Proof:**

1. The inclusion $\langle E_M(0) \rangle \subseteq \mathcal{S}(M) \subseteq \beta(M) \subseteq \mathcal{J}(M)$ is well-known but also easy to see, since we have the following implications between submodules,

   maximal $\Rightarrow$ prime $\Rightarrow$ semiprime. For $\langle E_M(0) \rangle \subseteq \mathcal{S}(M)$, see for instance [19, section 3]. Since $M$ has only one maximal submodule; namely $\bar{m} = \langle x_1, \ldots, x_n \rangle/I$, $\mathcal{J}(M) = \bar{m}$. Note also that $\langle E_M(0) \rangle = \langle x_1, \ldots, x_n \rangle M = \bar{m}$, so $\langle E_M(0) \rangle = \mathcal{J}(M)$.

2. From part 1) $\langle E_M(0) \rangle = \beta(M)$, i.e., the zero submodule of $M$ s.t.r.f. By [23, Theorem 1.5] given an epimorphism $f : M \to M/N$ for any submodule $N$ of $M$, the zero submodule s.t.r.f if and only if every submodule $N$ of $M$ s.t.r.f.

**Corollary 5.1** Any module $M \in \mathfrak{C}$, has exactly one semiprime submodule; namely, $\langle x_1, \ldots, x_n \rangle/I$. 

16
Acknowledgment

The authors acknowledge support from the International Science Program, EMS-Simons for Africa and the Eastern Africa Algebra Research Group. The authors are grateful to Rikard Bøgvad for pointing out Example 2.2 to them and to Dominic Bunnett, Alexandru Constantinescu and Balazs Szendroi for the comments. The third author is grateful to the fourth author for facilitating his visit to Makerere University.

References

[1] G. Agnarsson and N. Epstein, On monomial ideals and their socles, Order 37 (2) (2020), 341–369.

[2] G. Agnarsson and N. Epstein, On posets, monomial ideals, Gorenstein ideals and their combinatorics, arXiv preprint arXiv:2302.10068, (2023).

[3] A. Azizi, Radical formula and prime submodules, J. Algebra 307 (1) (2007), 454–460.

[4] A. Azizi, Radical formula and weakly prime submodules, Glasgow Math. J. 51 (2) (2009), 405–412.

[5] B. Bhatt and P. Scholze, Prisms and prismatic cohomology, Ann. of Math. 196(3) (2022), 1135-1275.

[6] W. Bruns and H. J. Winfried, Cohen-macaulay rings, Cambridge university press, Cambridge (39) (1998).

[7] D. Eisenbud, Commutative algebra with a view towards algebraic geometry, Graduate Texts in Mathematics, Springer-verlag, New York, 150, (1995).

[8] J. Elias and M. Rossi, A constructive approach to one-dimensional Gorenstein k-algebras, Trans. Amer. Math. Soc. 374 (7) (2021), 4953–4971.

[9] J. Elias and M. E. Rossi, Isomorphism classes of short Gorenstein local rings via Macaulay’s inverse system, Trans. Amer. Math. Soc. 364 (9) (2012), 4589–4604.

[10] J. Emsalem and A. Iarrobino, Inverse system of a symbolic power, I, J. Algebra 174 (3) (1995), 1080–1090.
[11] W. Fulton, Young tableaux: with applications to representation theory and geometry, Mathematical Society Student Texts, Cambridge University Press, Cambridge, 35 (1997).

[12] A. V. Geramita, Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals., The Curves Seminar at Queen’s 40 (13) (1996).

[13] B. Harbourne, H. Schenck, and A. Seceleanu, Inverse systems, Gelfand-Tsetlin patterns and the weak Lefschetz property, J. Lond. Math. Soc. 84 (3) (2011), 712–730.

[14] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, Comm. Algebra 20 (12) (1992), 3593–3602.

[15] E. Juan, Inverse systems of local rings, Comm. Algebra and its Interactions to Algebraic Geometry: VIAM 2013–2014, 2210 (2018), 119–163.

[16] P. I. Kimuli and D. Ssevviiri, Characterizations of regular modules, Int. Electron. J. Algebra 33 (2023), 54–76.

[17] A. Kyomuhangi and D. Ssevviiri, Generalized reduced modules, Rend. Circ. Mat. Palermo(2) 72 (1) (2023), 421–431.

[18] A. Kyomuhangi and D. Ssevviiri, The locally nilradical for modules over commutative rings, Beitr. Algebra Geom. 61(4) (2020), 759–769.

[19] S. C. Lee and R. Varmazyar, Semiprime submodules of a module and related concepts, J. Algebra Appl. 18 (08) (2019).

[20] T. K. Lee and Y. Zhou, Reduced modules, rings, modules, algebras and abelian groups, Lecture Notes in Pure and Appl. Math. 236 (2004), 365–377.

[21] K. H. Leung and S. H. Man, On commutative Noetherian rings which satisfy the radical formula, Glasgow Math. J. 39 (1997), 285–293.

[22] F. S. Macaulay, The algebraic theory of modular systems, Cambridge Math. Lib. 19 (1994).

[23] R. L. McCasland and M. E. Moore, On radicals of submodules, Comm. Algebra 19 (5) (1991), 1327–1341.

[24] D. Pusat-Yilmaz and P. F. Smith, Modules which satisfy the radical formula, Acta. Math. Hungar. 95 (2002), 155-167.
[25] M. B. Rege and A. M. Buhphang, On reduced modules and rings, Int. Electron. J. Algebra 3 (2008), 58–74.

[26] F. Rohrer Torsion functors, small or large, Beitr. Algebra Geom. 60(2) (2019), 233–256.

[27] P. Schenzel and A. M. Simon, Completion, Čech and local homology and cohomology, interactions between them, Springer Monographs in Mathematics, Springer, Cham (2018).

[28] H. Sharif, Y. Sharifi and S. Namazi, Rings satisfying the radical formula, Acta. Math. Hungar. 71 (1996), 103–108.

[29] D. Ssevviiri, Applications of reduced and coreduced modules I, Int. Electron. J. Algebra (2023), DOI: 10.24330/ieja.1299587.

[30] D. Ssevviiri, Applications of reduced and coreduced modules II, arXiv preprint arXiv:2205.13241, (2023).

[31] D. Ssevviiri, A relashionship between 2-primal modules and modules that satisfy the radical formula, Int. Electron. J. Algebra 18 (2015), 34–45.

[32] A. Van Tuyl and F. Zanello, Simplicial complexes and Macaulay’s inverse systems, Math. Z. 265 (2010), 151–160.

[33] R. H. Villareal Monomial Algebras, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 238 (2001).

[34] A. R. G. Wolff, The survival complex, arXiv preprint arXiv:1602.08998, (2016).

[35] A. Yekutieli, Weak proregularity, derived completion, adic flatness, and prisms, J. Algebra 583 (2021), 126–152.

- Tilahun Abebaw, Department of Mathematics, Addis Ababa University, P. O. BOX 1176, Addis Ababa, Ethiopia, tilahun.abebaw@aau.edu.et
- Nega Arega, Department of Mathematics and Statistics, The Namibia University of Science and Technology, Namibia, nechere@must.na
- Teklemichael Worku Bihoneg, Department of Mathematics, Addis Ababa University, P. O. BOX 1176, Addis Ababa, Ethiopia, teklemichael.worku@aau.edu.et
- David Ssevviiri, Department of Mathematics, Makerere University, P. O. BOX 7062, Kampala, Uganda, david.ssevviiri@mak.ac.ug