SUPREMUM, INFIMUM AND HYPERLIMITS IN THE 
NON-ARCHIMEDEAN RING OF COLOMBEAU GENERALIZED 
NUMBERS

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Abstract. It is well-known that the notion of limit in the sharp topology of 
sequences of Colombeau generalized numbers \( \mathbb{R} \) does not generalize classical 
results. E.g. the sequence \( \frac{1}{n} \not\to 0 \) and a sequence \((x_n)_{n \in \mathbb{N}}\) converges if and 
only if \(x_{n+1} - x_n \to 0\). This has several deep consequences, e.g. in the study 
of series, analytic generalized functions, or sigma-additivity and classical limit 
theorems in integration of generalized functions. The lacking of these results 
is also connected to the fact that \( \mathbb{R} \) is necessarily not a complete ordered 
set, e.g. the set of all the infinitesimals has neither supremum nor infimum. 
We present a solution of these problems with the introduction of the notions 
of hypernatural number, hypersequence, close supremum and infimum. In 
this way, we can generalize all the classical theorems for the hyperlimit of a 
hypersequence. The paper explores ideas that can be applied to other non-
Archimedean settings.

1. Introduction

A key concept of non-Archimedean analysis is that extending the real field \( \mathbb{R} \) into 
a ring containing infinitesimals and infinite numbers could eventually lead to the 
solution of non trivial problems. This is the case, e.g., of Colombeau theory, where 
nonlinear generalized functions can be viewed as set-theoretical maps on domains 
consisting of generalized points of the non-Archimedean ring \( \mathbb{R} \). This orientation 
has become increasingly important in recent years and hence it has led to the study 
of preliminary notions of \( \mathbb{R} \) (cf., e.g., [17, 3, 1, 18, 2, 5, 26, 16, 12]; see below for a 
self-contained introduction to the ring of Colombeau generalized numbers \( \mathbb{R} \)). 

In particular, the sharp topology on \( \mathbb{R} \) (cf., e.g., [10, 22, 23] and below) is the 
appropriate notion to deal with continuity of this class of generalized functions and 
for a suitable concept of well-posedness. This topology necessarily has to deal with 
balls having infinitesimal radius \( r \in \mathbb{R} \), and thus \( \frac{1}{n} \not\to 0 \) if \( n \to +\infty, n \in \mathbb{N} \), because 
we never have \( \mathbb{R}_{>0} \ni \frac{1}{n} < r \) if \( r \) is infinitesimal. Another unusual property related to 
the sharp topology can be derived from the following inequalities (where \( m \in \mathbb{N} \),

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As used in Colombeau theory, into a more general $\rho$ gauge
In order to settle this problem, it is important to generalize the role of the net (properties so as to arrive at useful notions for the theory of generalized functions. (actually, this is a well-known property of every ultrametric space, cf., e.g., [15, 22]). Naturally, this has several counter-intuitive consequences (arising from differences with the classical theory) when we have to deal with the study of series, analytic generalized functions, or sigma-additivity and classical limit theorems in integration of generalized functions (cf., e.g., [20, 25, 13]).

One of the aims of the present article is to solve this kind of counter-intuitive properties so as to arrive at useful notions for the theory of generalized functions. In order to settle this problem, it is important to generalize the role of the net ($\varepsilon$), as used in Colombeau theory, into a more general $\rho = (\rho_\varepsilon) \to 0$ (which is called a gauge), and hence to generalize $\widehat{R}$ into some $^*\widehat{R}$ (see Def. 1). We then introduce the set of hypernatural numbers as

$$^*\mathbb{N} := \left\{ [n_\varepsilon] \in ^*\widehat{R} \mid n_\varepsilon \in \mathbb{N} \quad \forall \varepsilon \right\},$$

so that it is natural to expect that $\frac{1}{n} \to 0$ in the sharp topology if $n \to +\infty$ with $n \in \mathbb{N}$, because now $n$ can also take infinite values. The notion of sequence is therefore substituted with that of hypersequence, as a map $(x_\varepsilon)_n : ^*\mathbb{N} \to ^*\widehat{R}$, where $\sigma$ is, generally speaking, another gauge. As we will see, (cf. example 36) only in this way we are able to prove e.g. that $\frac{1}{\log n} \to 0$ in $^*\widehat{R}$ as $n \in ^*\mathbb{N}$ but only for a suitable gauge $\sigma$ (depending on $\rho$), whereas this limit does not exist if $\sigma = \rho$.

Finally, the notions of supremum and infimum are naturally linked to the notion of limit of a monotonic (hyper)sequence. Being an ordered set, $^*\widehat{R}$ already has a definition of, let us say, supremum as least upper bound. However, as already preliminary studied and proved by [9], this definition does not fit well with topological properties of $^*\widehat{R}$ because generalized numbers $[x_\varepsilon] \in ^*\widehat{R}$ can actually jump as $\varepsilon \to 0^+$ (see Sec. 4.1). It is well known that in $\widehat{R}$ we have $m = \sup(S)$ if and only if $m$ is an upper bound of $S$ and

$$\forall r \in \mathbb{R}_{>0} \exists s \in S : m - r \leq s. \quad (1.1)$$

This could be generalized into the notion of close supremum in $^*\widehat{R}$, generalizing [9], that results into better topological properties, see Sec. 4. The ideas presented in the present article, which is self-contained, can surely be useful to explore similar ideas in other non-Archimedean settings, such as [7, 6, 24, 19, 15].

2. The Ring of Robinson Colombeau and the Hypernatural Numbers

In this section, we introduce our non-Archimedean ring of scalars and its subset of hypernatural numbers. For more details and proofs about the basic notions introduced here, the reader can refer e.g. to [8, 13, 14].

As we mentioned above, in order to accomplish the theory of hyperlimits, it is important to generalize Colombeau generalized numbers by taking an arbitrary asymptotic scale instead of the usual $\rho_\varepsilon = \varepsilon$:

**Definition 1.** Let $\rho = (\rho_\varepsilon) \in (0, 1]^{I}$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a gauge), then

(i) $\mathcal{I}(\rho) := \{ ([\rho_\varepsilon^{-a}] \mid a \in \mathbb{R}_{>0} \}$ is called the asymptotic gauge generated by $\rho$. 


(ii) If \( P(\varepsilon) \) is a property of \( \varepsilon \in I \), we use the notation \( \forall^0 \varepsilon : P(\varepsilon) \) to denote 
\[ \exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0] : P(\varepsilon). \]
We can read \( \forall^0 \varepsilon \) as for \( \varepsilon \) small.

(iii) We say that a net \((x_\varepsilon) \in \mathbb{R}^I\) is \( \rho \)-moderate, and we write \((x_\varepsilon) \in \mathbb{R}_\rho\) if 
\[ \exists (J_\varepsilon) \in I(\rho) : x_\varepsilon = O(J_\varepsilon) \text{ as } \varepsilon \to 0^+, \]
i.e., if 
\[ \exists N \in \mathbb{N} \forall^0 \varepsilon : |x_\varepsilon| \leq \rho_\varepsilon^{-N}. \]

(iv) Let \((x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I\), then we say that \((x_\varepsilon) \sim_\rho (y_\varepsilon)\) if 
\[ \forall (J_\varepsilon) \in I(\rho) : x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1}) \text{ as } \varepsilon \to 0^+, \]
that is if 
\[ \forall n \in \mathbb{N} \forall^0 \varepsilon : |x_\varepsilon - y_\varepsilon| \leq \rho_\varepsilon^n. \]
This is a congruence relation on the ring \( \mathbb{R}_\rho \) of moderate nets with respect to pointwise operations, and we can hence define 
\[ \mathbb{R}_\rho^* := \mathbb{R}_\rho / \sim_\rho, \]
which we call Robinson-Colombeau ring of generalized numbers. This name is justified by [21, 8]: Indeed, in [21] A. Robinson introduced the notion of moderate and negligible nets depending on an arbitrary fixed infinitesimal \( \rho \) (in the framework of nonstandard analysis); independently, J.F. Colombeau, cf. e.g. [8] and references therein, studied the same concepts without using nonstandard analysis, but considering only the particular infinitesimal \( \varepsilon \).

(v) In particular, if the gauge \( \rho = (\rho_\varepsilon) \) is non-decreasing, then we say that \( \rho \) is a monotonic gauge. Clearly, considering a monotonic gauge narrows the class of moderate nets: e.g. if \( \lim_{\varepsilon \to 0^+} x_\varepsilon = +\infty \) for all \( k \in \mathbb{N}_{>0} \), then \((x_\varepsilon) \notin \mathbb{R}_\rho\) for any monotonic gauge \( \rho \).

In the following, \( \rho \) will always denote a net as in Def. 1, even if we will sometimes omit the dependence on the infinitesimal \( \rho \), when this is clear from the context. We will also use other directed sets instead of \( I \): e.g. \( J \subseteq I \) such that 0 is a closure point of \( J \), or \( I \times \mathbb{N} \). The reader can easily check that all our constructions can be repeated in these cases.

We also recall that we write \([x_\varepsilon] \leq [y_\varepsilon]\) if there exists \((z_\varepsilon) \in \mathbb{R}^I\) such that \((z_\varepsilon) \sim_\rho 0\) (we then say that \((z_\varepsilon)\) is \( \rho \)-negligible) and \(x_\varepsilon \leq y_\varepsilon + z_\varepsilon\) for \( \varepsilon \) small. Equivalently, we have that \( x \leq y \) if and only if there exist representatives \([x_\varepsilon] = x\) and \([y_\varepsilon] = y\) such that \(x_\varepsilon \leq y_\varepsilon\) for all \( \varepsilon \).

Although the order \( \leq \) is not total, we still have the possibility to define the infimum \([x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]\), the supremum \([x_\varepsilon] \vee [y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]\) of a finite number of generalized numbers. Henceforth, we will also use the customary notation \( \mathbb{R}_\rho^* \) for the set of invertible generalized numbers, and we write \( x < y \) to say that \( x \leq y \) and \( x - y \in \mathbb{R}_\rho^* \). Our notations for intervals are: \([a, b] := \{ x \in \mathbb{R}_\rho | a \leq x \leq b\}\), \([a, b]_\rho := [a, b] \cap \mathbb{R}_\rho\). Finally, we set \( d\rho := [\rho_\varepsilon] \in \mathbb{R}_\rho^*\), which is a positive invertible infinitesimal, whose reciprocal is \( d\rho^{-1} = [\rho_\varepsilon^{-1}]\), which is necessarily a strictly positive infinite number.

The following result is useful to deal with positive and invertible generalized numbers. For its proof, see e.g. [3, 2, 13, 14].

**Lemma 2.** Let \( x \in \mathbb{R}_\rho^* \). Then the following are equivalent:

(i) \( x \) is invertible and \( x \geq 0 \), i.e. \( x > 0 \).
(ii) For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of $x$ we have $\forall \varepsilon > 0$.

(iii) For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of $x$ we have $\exists m \in \mathbb{N} \forall \varepsilon > 0$.

(iv) There exists a representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of $x$ such that $\exists m \in \mathbb{N} \forall \varepsilon > 0$.

2.1. The language of subpoints. The following simple language allows us to simplify some proofs using steps recalling the classical real field $\mathbb{R}$. We first introduce the notion of subpoint:

**Definition 3.** For subsets $J, K \subseteq I$ we write $K \subseteq_0 J$ if 0 is an accumulation point of $K$ and $K \subseteq J$ (we read it as: $K$ is co-final in $J$). Note that for any $J \subseteq_0 I$, the constructions introduced so far in Def. 1 can be repeated using nets $(x_\varepsilon)_{\varepsilon \in J}$. We indicate the resulting ring with the symbol $\mathbb{R}^n|_J$. More generally, no peculiar property of $I = (0, 1]$ will ever be used in the following, and hence all the presented results can be easily generalized considering any other directed set. If $K \subseteq_0 J$, $x \in \mathbb{R}^n|_J$ and $x' \in \mathbb{R}^n|_K$, then $x'$ is called a subpoint of $x$, denoted as $x' \subseteq x$, if there exist representatives $(x_\varepsilon)_{\varepsilon \in J}, (x'_\varepsilon)_{\varepsilon \in K}$ of $x, x'$ such that $x'_\varepsilon = x_\varepsilon$ for all $\varepsilon \in K$. In this case we write $x' = x|_K, \text{dom}(x') := K$, and the restriction $(\neg)|_K : \mathbb{R}^n \to \mathbb{R}^n|_K$ is a well defined operation. In general, for $X \subseteq \mathbb{R}^n$ we set $X|_J := \{x|_J \in \mathbb{R}^n|_J \mid x \in X\}$.

In the next definition, we introduce binary relations that hold only on subpoints.

Clearly, this idea is inherited from nonstandard analysis, where cofinal subsets are always taken in a fixed ultrafilter.

**Definition 4.** Let $x, y \in \mathbb{R}$, $L \subseteq_0 I$, then we say

(i) $x \lessdot_L y : \iff x|_L < y|_L$ (the latter inequality has to be meant in the ordered ring $\mathbb{R}|_L$). We read $x \lessdot_L y$ as “$x$ is less than $y$ on $L$’’.

(ii) $x \lessdot_s y : \iff \exists L \subseteq_0 I : x \lessdot_L y$. We read $x \lessdot_s y$ as “$x$ is less than $y$ on subpoints”.

Analogously, we can define other relations holding only on subpoints such as e.g.: $\in_s, \leq_s, =_s, \gtrdot_s, \leq_s$, etc.

For example, we have

$$x \leq y \iff \forall L \subseteq_0 I : x \leq_L y$$

$$x < y \iff \forall L \subseteq_0 I : x <_L y,$$

the former following from the definition of $\leq$, whereas the latter following from Lem. 2. Moreover, if $\mathcal{P} \{x_\varepsilon\}$ is an arbitrary property of $x_\varepsilon$, then

$$\neg(\forall \varepsilon \in : \mathcal{P} \{x_\varepsilon\}) \iff \exists L \subseteq_0 I \forall \varepsilon \in L : \neg \mathcal{P} \{x_\varepsilon\}. \quad (2.1)$$

Note explicitly that, generally speaking, relations on subpoints, such as $\leq_s$ or $=_s$, do not inherit the same properties of the corresponding relations for points. So, e.g., both $=_s$ and $\leq_s$ are not transitive relations.

The next result clarifies how to equivalently write a negation of an inequality or of an equality using the language of subpoints.

**Lemma 5.** Let $x, y \in \mathbb{R}$, then

(i) $x \not\lessdot y \iff x \gtrdot y$

(ii) $x \not\leq y \iff x \gtrdot y$

(iii) $x \not\gtrdot y \iff x \gtrdot y$ or $x \lessdot y$
Proof. (i) ⇐: The relation \( x \succ y \) means \( x_L > y_L \) for some \( L \subseteq I \). By Lem. 2 for the ring \( \tilde{\mathbb{R}} \), we get \( \forall \varepsilon \in L : x_\varepsilon > y_\varepsilon \), where \( x = [x_\varepsilon], y = [y_\varepsilon] \) are any representatives of \( x, y \) resp. The conclusion follows by (2.1)

(ii) ⇒: Take any representatives \( x = [x_\varepsilon], y = [y_\varepsilon] \). The property
\[
\forall \varepsilon \in \mathbb{R} > 0 \forall \varepsilon : x_\varepsilon \leq y_\varepsilon + \varepsilon^2
\]
for \( \varepsilon \to +\infty \) implies \( x \leq y \). We therefore have
\[
\exists \varepsilon \in \mathbb{R} > 0 \exists L \subseteq I \forall \varepsilon \in L : x_\varepsilon > y_\varepsilon + \varepsilon^2,
\]
i.e. \( x \succ_L y \).

(ii) ⇔: We have two cases: either \( x - y \) is not invertible or \( x \preceq y \). In the former case, the conclusion follows from [14, Thm. 1.2.39]. In the latter one, it follows from (i).

(iii) ⇒: By contradiction, assume that \( x \succ y \) and \( x \preceq y \). Then (i) would yield \( x \leq y \) and \( y \leq x \), and hence \( x = y \). The opposite implication directly follows by contradiction.

Using the language of subpoints, we can write different forms of dichotomy or trichotomy laws for inequality. The first form is the following

Lemma 6. Let \( x, y \in \tilde{\mathbb{R}} \), then
\[
(i) \quad x \leq y \text{ or } x > y
(ii) \quad \neg(x > y \text{ and } x \leq y)
(iii) \quad x = y \text{ or } x < y \text{ or } x > y
(iv) \quad x \leq y \Rightarrow x \leq y \text{ or } x = y
(v) \quad x \leq y \iff x < y \text{ or } x = y.
\]

Proof. (i) and (ii) follows directly from Lem. 5. To prove (iii), we can consider that \( x > y \) or \( x \not\succ y \). In the second case, Lem. 5 implies \( x \leq y \). If \( y \leq x \) then \( x = y \); otherwise, once again by Lem. 5, we get \( x < y \). To prove (iv), assume that \( x \leq y \) but \( x \not\leq y \), then \( x \geq y \) by Lem. 5.(i) and hence \( x = y \). The implication \( \iff \) is trivial. On the other hand, if \( x \leq y \) and \( x \not\leq y \), then \( y \leq x \) from Lem. 5.(i), and hence \( x = y \).

As usual, we note that these results can also be trivially repeated for the ring \( \tilde{\mathbb{R}}_L \).

So, e.g., we have \( x \not\leq_L y \) if and only if \( \exists J \subseteq L : x >_J y \), which is the analog of Lem. 5.(i) for the ring \( \tilde{\mathbb{R}}_L \).

The second form of trichotomy (which for \( \tilde{\mathbb{R}} \) can be more correctly named as quadrichotomy) is stated as follows:

Lemma 7. Let \( x = [x_\varepsilon], y = [y_\varepsilon] \in \tilde{\mathbb{R}} \), then
\[
(i) \quad x \leq y \text{ or } x \geq y \text{ or } \exists L \subseteq I : L \subseteq L \leq I, x \geq L \text{ and } x \leq L \leq y
(ii) \quad \text{If for all } L \subseteq I \text{ the following implication holds}
\[
\forall \varepsilon \in L : \mathcal{P} \{x_\varepsilon, y_\varepsilon\},
\]
then \( \forall \varepsilon \in L : \mathcal{P} \{x_\varepsilon, y_\varepsilon\} \),
\[\tag{2.2}\]

(iii) \quad \text{If for all } L \subseteq I \text{ the following implication holds}
\[
\forall \varepsilon \in L : \mathcal{P} \{x_\varepsilon, y_\varepsilon\},
\]
then \( \forall \varepsilon \in L : \mathcal{P} \{x_\varepsilon, y_\varepsilon\} \).
\[\tag{2.3}\]
Proof. (i): if $x \leq y$, then $x > s y$ from Lem. 5.(i). Let $[x_e] = x$ and $[y_e] = y$ be two representatives, and set $L := \{ \varepsilon \in I \mid x_{\varepsilon} \geq y_{\varepsilon} \}$. The relation $x > s y$ implies that $L \subseteq 0 \ I$. Clearly, $x \geq L y$ (but note that in general we cannot prove $x > L y$). If $L^c \subseteq 0 \ I$, then $(0, \varepsilon_0) \subseteq L$ for some $\varepsilon_0$, i.e. $x \geq y$. On the contrary, if $L^c \subseteq 0 \ I$, then $x \leq L^c y$.

(ii): Property (i) states that we have three cases. If $x_{\varepsilon} \leq y_{\varepsilon}$ for all $\varepsilon \leq \varepsilon_0$, then it suffices to set $L := (0, \varepsilon_0)$ in (2.2) to get the claim. Similarly, we can proceed if $x \geq y$. Finally, if $x \geq L y$ and $x \leq L^c y$, then we can apply (2.2) both with $L$ and $L^c$ to obtain

$$\forall \varepsilon \in L : \mathcal{P} \{x_{\varepsilon}, y_{\varepsilon}\}$$

$$\forall \varepsilon \in L^c : \mathcal{P} \{x_{\varepsilon}, y_{\varepsilon}\},$$

from which the claim directly follows.

(iii): By contradiction, assume

$$\forall \varepsilon \in L : \neg \mathcal{P} \{x_{\varepsilon}, y_{\varepsilon}\},$$

for some $L \subseteq 0 \ I$. We apply (i) to the ring $\mathcal{R}|_L$ to obtain the following three cases:

$$x \leq L y \quad \text{or} \quad x \geq L y \quad \text{or} \quad \exists L \subseteq 0 \ L : \mathcal{J} \subseteq 0 \ L, \quad x \geq J \ y \quad \text{and} \quad x \leq J \ y.$$ 

(2.5)

If $x \leq L y$, by Lem. 6.(iv) for the ring $\mathcal{R}|_L$, this case splits into two sub-cases:

$x = L y \quad \text{or} \quad \exists K \subseteq 0 \ L : x < K y$. If the former holds, using (2.3) we get $\mathcal{P} \{x_{\varepsilon}, y_{\varepsilon}\} \forall \varepsilon \in L$, which contradicts (2.4). If $x < K y$, then $K \subseteq 0 \ I$ and we can apply (2.4) with $K$ to get $\mathcal{P} \{x_{\varepsilon}, y_{\varepsilon}\} \forall \varepsilon \in K$, which again contradicts (2.4) because $K \subseteq 0 \ L$. Similarly we can proceed with the other three cases stated in (2.5).

Property Lem. 7.(ii) represents a typical replacement of the usual dichotomy law in $\mathcal{R}$: for arbitrary $L \subseteq 0 \ I$, we can assume to have two cases: either $x \leq L y$ or $x \geq L y$.

If in both cases we are able to prove $\mathcal{P} \{x_{\varepsilon}, y_{\varepsilon}\}$ for $\varepsilon \in L$ small, then we always get that this property holds for all $\varepsilon$ small. Similarly, we can use the strict trichotomy law stated in (iii).

2.2. Inferior and superior parts. Other simple tools that we can use to study generalized numbers of $\mathcal{R}$ are the inferior and superior parts of a number. Only in this section of the article, we assume that $\rho$ is a monotonic gauge.

Definition 8. Let $x = [x_e] \in \mathcal{R}$ be a generalized number, then:

(i) If $\exists L \in \mathcal{R}: L \leq x$, then $x_i := \inf_{e \in (0, e]} x_e$ is called the inferior part of $x$.

(ii) If $\exists U \in \mathcal{R}: x \leq U$, then $x_s := \sup_{e \in (0, e]} x_e$ is called the superior part of $x$.

Moreover, we set:

(iii) $x_{i}^o := \lim_{e \to 0^+} x_{e} \in \mathcal{R} \cup \{\pm \infty\}$, where $[x_e] = x$ is any representative of $x$, is called the inferior standard part of $x$. Note that if $\exists x_i$, i.e. if $x$ is finitely bounded from below, then $(x_i)^o = x_{i}^o \in \mathcal{R}$ and $x_{i}^o \geq x_i$.

(iv) $x_{s}^o := \lim_{e \to 0^+} x_{e} \in \mathcal{R} \cup \{\pm \infty\}$, where $[x_e] = x$ is any representative of $x$, is called the superior standard part of $x$. Note that if $\exists x_s$, i.e. if $x$ is finitely bounded from above, then $(x_s)^o = x_{s}^o \in \mathcal{R}$ and $x_{s}^o \leq x_s$.

Note that, since $\rho = (\rho_e)$ is non-decreasing, if $[x_e] = x$ is another representative, then for all $e \in (0, e]$, we have $x_{e}' \leq x_e + \rho_e \leq x_e + \rho_{c} \leq \rho_{e} + \sup_{e \in (0, e]} x_e$ and hence $\sup_{e \in (0, e]} x_{e}' \leq \rho_{c} + \sup_{e \in (0, e]} x_e$. This shows that inferior and superior parts, when they exist, are well-defined. Moreover, if $(z_e)$ is negligible, then
lim sup \(_{x \to 0^+} (x + z) \leq \lim sup \_{x \to 0^+} x + 0\), which shows that \(x^+\) is well-defined (similarly for \(x^-\) using super-additivity of \(\lim\inf\)).

Clearly, \(x_i \leq x \leq x_s\) and \(x^+ \leq x^-\). We have that the generalized number \(x\) is near-standard if and only if \(x^+ = x^- =: x^= \in \mathbb{R}\); it is infinitely small if and only if \(\exists x^= = 0\); it is a positive infinite number if and only if \(x^+ = x^- =: x^= = +\infty\) (the same for negative infinite numbers); it is a finite number if and only if \(x^+ = x^- \in \mathbb{R}\).

Finally, there always exist \(x\), \(x' \leq x\) such that \(x' \approx x^=\) and \(x'' \approx x^=\), where \(x \approx y\) means that \(x - y\) is infinitesimal (i.e. \(|x - y| \leq r\) for all \(r \in \mathbb{R}_{>0}\) or, equivalently, \(\lim_{x \to 0^+} x - y = 0\) for all \([x]\ = x, |y| = y\). Therefore, any generalized number in \(^*\mathbb{R}\) is either finite or some of its subpoints is infinite; in the former case, some of its subpoints is near standard.

### 2.3. Topologies on \(^*\mathbb{R}^n\).

On the \(^*\mathbb{R}\)-module \(^*\mathbb{R}^n\) we can consider the natural extension of the Euclidean norm, i.e. \(|[x]| := |[x]| \in ^*\mathbb{R}\), where \([x] \in ^*\mathbb{R}^n\). Even if this generalized norm takes values in \(^*\mathbb{R}\), it shares some essential properties with classical norms:

\[
\begin{align*}
|x| &= x \vee (-x) \\
|x| &\geq 0 \\
|x| = 0 &\Rightarrow x = 0 \\
|y \cdot x| &= |y| \cdot |x| \\
|x + y| &\leq |x| + |y| \\
||x| - |y|| &\leq |x - y|.
\end{align*}
\]

It is therefore natural to consider on \(^*\mathbb{R}^n\) topologies generated by balls defined by this generalized norm and a set of radii:

**Definition 9.** We say that \(\mathcal{R}\) is a set of radii if

(i) \(\mathcal{R} \subseteq ^*\mathbb{R}_{>0}\) is a non-empty subset of positive invertible generalized numbers.
(ii) For all \(r, s \in \mathcal{R}\) the infimum \(r \wedge s \in \mathcal{R}\).
(iii) \(k \cdot r \in \mathcal{R}\) for all \(r \in \mathcal{R}\) and all \(k \in \mathbb{R}_{>0}\).

Moreover, if \(\mathcal{R}\) is a set of radii and \(x, y \in ^*\mathbb{R}\), then:

(i) We write \(x <_{\mathcal{R}} y\) if \(\exists r \in \mathcal{R}: r \leq y - x\).
(ii) \(B^\mathcal{R}_r(x) := \{y \in ^*\mathbb{R}^n | |y - x| <_{\mathcal{R}} r\}\) for any \(r \in \mathcal{R}\).
(iii) \(B^\mathcal{R}_\rho(x) := \{y \in \mathbb{R}^n | |y - x| < \rho\}\), for any \(\rho \in \mathbb{R}_{>0}\), denotes an ordinary Euclidean ball in \(\mathbb{R}^n\).

For example, \(^*\mathbb{R}_{>0}\) and \(\mathbb{R}_{>0}\) are sets of radii.

**Lemma 10.** Let \(\mathcal{R}\) be a set of radii and \(x, y, z \in ^*\mathbb{R}\), then

(i) \( \neg (x <_{\mathcal{R}} x)\).
(ii) \( x <_{\mathcal{R}} y \ and \ y <_{\mathcal{R}} z \ imply \ x <_{\mathcal{R}} z\).
(iii) \( \forall r \in \mathcal{R}: 0 <_{\mathcal{R}} r\).

The relation \(<_{\mathcal{R}}\) has better topological properties as compared to the usual strict order relation \(x \leq y\) and \(x \neq y\) (a relation that we will therefore never use) because of the following result:
Theorem 11. The set of balls \( \{ B_2^R(x) \mid r \in R, \ x \in \mathbb{R}^n \} \) generated by a set of radii \( R \) is a base for a topology on \( \mathbb{R}^n \).

Henceforth, we will only consider the sets of radii \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{>0} \) and will use the simplified notation \( B_r(x) := B_r^R(x) \) if \( R = \mathbb{R}_{>0} \). The topology generated in the former case is called \( \text{sharp topology} \), whereas the latter is called \( \text{Fermat topology} \).

We will call \( \text{sharply open set} \) any open set in the sharp topology, \( \text{and large open set} \) any open set in the Fermat topology; clearly, the latter is coarser than the former. It is well-known (see e.g. [2, 10, 11, 13] and references therein) that this is an equivalent way to define the sharp topology usually considered in the ring of Colombeau generalized numbers. We therefore recall that the sharp topology on \( \mathbb{R}^n \) is Hausdorff and Cauchy complete, see e.g. [2, 11].

2.4. Open, closed and bounded sets generated by nets. A natural way to obtain sharply open, closed and bounded sets in \( \mathbb{R}^n \) is by using a net \((A_x)\) of subsets \( A_x \subseteq \mathbb{R}^n \). We have two ways of extending the membership relation \( x \in A_x \) to generalized points \( x_1 \in A_x \) to generalized points \( x_1 \in \mathbb{R}^n \) (cf. [18, 12]):

Definition 12. Let \((A_x)\) be a net of subsets of \( \mathbb{R}^n \), then

(i) \( [A_x] := \{x_1 \in \mathbb{R}^n \mid \forall \epsilon > 0, x_1 \in A_x \} \) is called the \textit{internal set} generated by the net \((A_x)\).

(ii) Let \((x_\epsilon)\) be a net of points of \( \mathbb{R}^n \), then we say that \( x_\epsilon \in A_x \), and we read it as \( x_\epsilon \text{ strongly belongs to } A_x \), if

(i) \( \forall \epsilon > 0, x_\epsilon \in A_x \).

(ii) If \( (x'_\epsilon) \sim \rho (x_\epsilon) \), then also \( x'_\epsilon \in A_x \) for \( \epsilon \) small.

Moreover, we set \([A_x] := \{x_\epsilon = \{x_\epsilon \in \mathbb{R}^n \mid x_\epsilon \in A_x \} \}, \) and we call it the \textit{strongly internal set} generated by the net \((A_x)\).

(iii) We say that the internal set \( K = [A_x] \) is \textit{sharply bounded} if there exists \( M \in \mathbb{R}_{>0} \) such that \( K \subseteq B_M(0) \).

(iv) Finally, we say that the net \((A_x)\) is \textit{sharply bounded} if there exists \( N \in \mathbb{R}_{>0} \) such that \( \forall \epsilon > 0 \forall x \in A_x : |x| \leq \rho_x^{-N} \).

Therefore, \( x \in [A_x] \) if there exists a representative \( [x_\epsilon] = x \) such that \( x_\epsilon \in A_x \) for \( \epsilon \) small, whereas this membership is independent from the chosen representative in case of strongly internal sets. An internal set generated by a constant net \( A_x = A \subseteq \mathbb{R}^n \) will simply be denoted by \([A]\).

The following theorem (cf. [18, 12] for the case \( \rho_x = \epsilon \), and [13] for an arbitrary gauge) shows that internal and strongly internal sets have dual topological properties:

Theorem 13. For \( \epsilon \in I \), let \( A_x \subseteq \mathbb{R}^n \) and let \( x_\epsilon \in \mathbb{R}^n \). Then we have

(i) \( x_\epsilon \in [A_x] \) if and only if \( \forall q \in \mathbb{R}_{>0} \forall \epsilon > 0 : d(x_\epsilon, A_x) \leq \rho_x \). Therefore \( [x_\epsilon] \in [A_x] \) if and only if \( d(x_\epsilon, A_x) = 0 \in \mathbb{R} \).

(ii) \( x_\epsilon \in (A_x) \) if and only if \( \exists q \in \mathbb{R}_{>0} \forall \epsilon > 0 : d(x_\epsilon, A_x^c) > \rho_x \), where \( A_x^c := \mathbb{R}^n \setminus A_x \).

Therefore, if \( d(x_\epsilon, A_x^c) > \rho_x \), then \( x_\epsilon \in (A_x) \) if and only if \( d(x_\epsilon, A_x^c) > 0 \).

(iii) \( [A_x] \) is sharply closed.

(iv) \( (A_x) \) is sharply open.

(v) \( [A_x] = \text{cl}(A_x) \), where \( \text{cl}(S) \) is the closure of \( S \subseteq \mathbb{R}^n \).

(vi) \( (A_x) = \text{int}(A_x) \), where \( \text{int}(S) \) is the interior of \( S \subseteq \mathbb{R}^n \).
For example, it is not hard to show that the closure in the sharp topology of a ball of center \( c = [x_\varepsilon] \) and radius \( r = [r_\varepsilon] > 0 \) is
\[
\overline{B_r}(c) = \left\{ x \in \mathbb{R}^d \mid |x - c| \leq r \right\} = \left[ B^E_r(c_\varepsilon) \right],
\]
whereas
\[
\overline{B_r}(c) = \left\{ x \in \mathbb{R}^d \mid |x - c| < r \right\} = (B^E_r(c_\varepsilon)).
\]

Using internal sets and adopting ideas similar to those used in proving Lem. 7, we also have the following form of dichotomy law:

**Lemma 14.** For \( \varepsilon \in I \), let \( A_\varepsilon \subseteq \mathbb{R}^n \) and let \( x = [x_\varepsilon] \in \mathbb{R}^n \). Then we have:

(i) \( x \in [A_\varepsilon] \) or \( x \in [A_\varepsilon^c] \) or \( \exists L \subseteq_0 I : L^c \subseteq_0 I, L \not\subseteq_0 I \), \( x \in L^c \) \( A_\varepsilon \)

(ii) If for all \( L \subseteq_0 I \) the following implication holds
\[
x \in L \quad [A_\varepsilon] \quad or \quad x \in L \ [A_\varepsilon^c] \quad \Rightarrow \quad \forall \varepsilon \in L : \mathcal{P}\{x_\varepsilon\},
\]
then \( \forall \varepsilon : \mathcal{P}\{x_\varepsilon\} \).

**Proof.** (i): If \( x \notin [A_\varepsilon^c] \), then \( x_\varepsilon \in A_\varepsilon \) for all \( \varepsilon \in K \) and for some \( K \subseteq_0 I \). Set \( L := \{ \varepsilon \in I \mid x_\varepsilon \in A_\varepsilon \} \), so that \( K \subseteq L \subseteq_0 I \). We have \( x \in L \ [A_\varepsilon] \). If \( L^c \subseteq_0 I \), then \( (0, \varepsilon_0] \subseteq L \) for some \( \varepsilon_0 \), i.e. \( x \in [A_\varepsilon] \). On the contrary, if \( L^c \subseteq_0 I \), then \( x \in L^c \ [A_\varepsilon^c] \).

(ii): We can proceed as in the proof of Lem. 7.(ii) using (i). \( \square \)

### 3. Hypernatural numbers

We start by defining the set of hypernatural numbers in \( \mathbb{R}^* \) and the set of \( \rho \)-moderate nets of natural numbers:

**Definition 15.** We set

(i) \( \mathbb{N}^* := \left\{ [n_\varepsilon] \in \mathbb{R}^* \mid n_\varepsilon \in \mathbb{N} \ \forall \varepsilon \right\} \)

(ii) \( \mathbb{N}_\rho := \left\{ (n_\varepsilon) \in \mathbb{R}_\rho \mid n_\varepsilon \in \mathbb{N} \ \forall \varepsilon \right\} \).

Therefore, \( n \in \mathbb{N}^* \) if and only if there exists \( (x_\varepsilon) \in \mathbb{R}_\rho \) such that \( n = [\text{int}(x_\varepsilon)] \).

Clearly, \( \mathbb{N} \subseteq \mathbb{N}^* \). Note that the integer part function \( \text{int}(\cdot) \) is not well-defined on \( \mathbb{R}^* \). In fact, if \( x = 1 = \left[ 1 - \rho_\varepsilon \right] = \left[ 1 + \rho_\varepsilon \right] \), then \( \text{int}(1 - \rho_\varepsilon) = 0 \) whereas \( \text{int}(1 + \rho_\varepsilon) = 1 \), for \( \varepsilon \) sufficiently small. Similar counter examples can be set for floor and ceiling functions. However, the nearest integer function is well defined on \( \mathbb{N}^* \), as proved in the following

**Lemma 16.** Let \( (n_\varepsilon) \in \mathbb{N}_\rho \) and \( (x_\varepsilon) \in \mathbb{R}_\rho \) be such that \( [n_\varepsilon] = [x_\varepsilon] \). Let \( \text{rpi} : \mathbb{R} \rightarrow \mathbb{N} \) be the function rounding to the nearest integer with tie breaking towards positive infinity, i.e. \( \text{rpi}(x) = \lfloor x + \frac{1}{2} \rfloor \). Then \( \text{rpi}(x_\varepsilon) = n_\varepsilon \) for \( \varepsilon \) small. The same result holds using \( \text{rni} : \mathbb{R} \rightarrow \mathbb{N} \), the function rounding half towards \( -\infty \).

**Proof.** We have \( \text{rpi}(x) = \lfloor x + \frac{1}{2} \rfloor \), where \( \lfloor \cdot \rfloor \) is the floor function. For \( \varepsilon \) small, \( \rho_\varepsilon < \frac{1}{2} \) and, since \( [n_\varepsilon] = [x_\varepsilon] \), always for \( \varepsilon \) small, we also have \( n_\varepsilon - \rho_\varepsilon + \frac{1}{2} < x_\varepsilon + \frac{1}{2} < n_\varepsilon + \rho_\varepsilon + \frac{1}{2} \). But \( n_\varepsilon \leq n_\varepsilon - \rho_\varepsilon + \frac{1}{2} \) and \( n_\varepsilon + \rho_\varepsilon + \frac{1}{2} < n_\varepsilon + 1 \). Therefore \( [x_\varepsilon + \frac{1}{2}] = n_\varepsilon \). An analogous argument can be applied to \( \text{rni}(\cdot) \). \( \square \)

Actually, this lemma does not allow us to define a nearest integer function \( \text{ni} : \mathbb{N}^* \rightarrow \mathbb{N}_\rho \) as \( \text{ni}((x_\varepsilon)) := \text{rpi}(x_\varepsilon) \) because if \( [x_\varepsilon] = [n_\varepsilon] \), the equality \( n_\varepsilon = \text{rpi}(x_\varepsilon) \) holds only for \( \varepsilon \) small. A simpler approach is to choose a representative \( (n_\varepsilon) \in \mathbb{N}_\rho \).
for each \( x \in \mathbb{N} \) and to define \( n_i(x) := (n_x) \). Clearly, we must consider the net \((n_i(x))\) only for \( \varepsilon \) small, such as in equalities of the form \( x = \lceil n_i(x) \rceil \). This is what we do in the following

**Definition 17.** The nearest integer function \( n_i(\cdot) \) is defined by:

(i) \( n_i : \mathbb{N} \to \mathbb{N} \)

(ii) If \([x]_\varepsilon \in \mathbb{N}\) and \( n_i([x]) = (n_x) \) then \( \forall \varepsilon : n_x = \rho(x) \).

In other words, if \( x \in \mathbb{N} \), then \( x = \lceil n_i(x) \rceil \) and \( n_i(x) \in \mathbb{N} \) for all \( \varepsilon \). Another possibility is to formulate Lem. \( \mathbf{16} \) as

\[
[x]_\varepsilon \in \mathbb{N} \iff [x] = \rho(x).
\]

Therefore, without loss of generality we may always suppose that \( x \in \mathbb{N} \) whenever \( [x]_\varepsilon \in \mathbb{N} \).

**Remark 18.**

(i) \( \mathbb{N} \), with the order \( \leq \) induced by \( \mathbb{R} \), is a directed set; it is closed with respect to sum and product although recursive definitions using \( \mathbb{N} \) are not possible.

(ii) In \( \mathbb{N} \) we can find several chains (totally ordered subsets) such as: \( N, \mathbb{N} \cdot \lceil \text{int}(\rho_{-k}) \rceil \) for a fixed \( k \in \mathbb{N} \), \( \{\lceil \text{int}(\rho_{-k}) \rceil \mid k \in \mathbb{N} \} \).

(iii) Generally speaking, if \( m, n \in \mathbb{N} \), \( m^n \neq \mathbb{N} \) because the net \((m^{\sigma})\) can grow faster than any power \((\rho_{-k})\). However, if we take two gauges \( \sigma, \rho \) satisfying \( \sigma \leq \rho \), using the net \( (\sigma_{-1}) \) we can measure infinite nets that grow faster than \((\rho_{-k})\) because \( \sigma_{-1} \geq \rho_{-1} \) for \( \varepsilon \) small. Therefore, we can take \( m, n \in \mathbb{N} \) such that \((n(m))\), \((n(n)) \in \mathbb{R}_\sigma\); we think at \( m, n \) as \( \sigma \)-hypernatural numbers growing at most polynomially with respect to \( \rho \). Then, it is not hard to prove that if \( \rho \) is an arbitrary gauge, and we consider the auxiliary gauge \( \sigma_{-1} := \rho_{-\varepsilon}^{1\varepsilon} \), then \( m^n \in \mathbb{N} \).

(iv) If \( m \in \mathbb{N} \), then \( 1^m := [(1 + z\varepsilon)^{m\varepsilon}] \), where \( (z\varepsilon) \) is \( \rho \)-negligible, is well defined and \( 1^m = 1 \). In fact, \( \log(1 + z\varepsilon)^{m\varepsilon} \) is asymptotically equal to \( m\varepsilon z\varepsilon \to 0 \), and this shows that \((1 + z\varepsilon)^{m\varepsilon} \) is moderate. Finally, \( |(1 + z\varepsilon)^{m\varepsilon} - 1| \leq |z\varepsilon| m\varepsilon (1 + z\varepsilon)^{m\varepsilon - 1} \) by the mean value theorem.

### 4. Supremum and Infimum in \( \mathbb{R} \)

To solve the problems we explained in the introduction of this article, it is important to generalize at least two main existence theorems for limits: the Cauchy criterion and the existence of a limit of a bounded monotone sequence. The latter is clearly related to the existence of supremum and infimum, which cannot be always guaranteed in the non-Archimedean ring \( \mathbb{R} \). As we will see more clearly later (see also \( \mathbf{9} \)), to arrive at these existence theorems, the notion of supremum, i.e. the least upper bound, is not the correct one. More appropriately, we can associate a notion of close supremum (and close infimum) to every topology generated by a set of radii (see Def. \( \mathbf{9} \)).

**Definition 19.** Let \( \mathbb{R} \) be a set of radii and let \( \tau \) be the topology on \( \mathbb{R} \) generated by \( \mathbb{R} \). Let \( P \subseteq \mathbb{R} \), then we say that \( \tau \) separates points of \( P \) if

\[
\forall p, q \in P : p \neq q \Rightarrow \exists A, B \in \tau : p \in A, q \in B, A \cap B = \emptyset,
\]

i.e. if \( P \) with the topology induced by \( \tau \) is Hausdorff.
Definition 20. Let \( \tau \) be a topology on \( ^{\ast}R \) generated by a set of radii \( \mathfrak{R} \) that separates points of \( P \subseteq ^{\ast}R \) and let \( S \subseteq ^{\ast}R \). Then, we say that \( \sigma \) is \((\tau, P)\)-supremum of \( S \) if

(i) \( \sigma \in P \);

(ii) \( \forall s \in S : s \leq \sigma \);

(iii) \( \sigma \) is a point of closure of \( S \) in the topology \( \tau \), i.e. if \( \forall A \in \tau : \sigma \in A \Rightarrow \exists \bar{s} \in S \cap A \).

Similarly, we say that \( \iota \) is \((\tau, P)\)-infimum of \( S \) if

(i) \( \iota \in P \);

(ii) \( \forall s \in S : \iota \leq s \);

(iii) \( \iota \) is a point of closure of \( S \) in the topology \( \tau \), i.e. if \( \forall A \in \tau : \iota \in A \Rightarrow \exists \bar{s} \in S \cap A \).

In particular, if \( \tau \) is the sharp topology and \( P = ^{\ast}R \), then following [9], we simply call the \((\tau, P)\)-supremum, the close supremum (the adjective close will be omitted if it will be clear from the context) or the sharp supremum if we want to underline the dependency on the topology. Analogously, if \( \tau \) is the Fermat topology and \( P = R \), then we call the \((\tau, P)\)-supremum the Fermat supremum. Note that (iii) implies that if \( \sigma \) is \((\tau, P)\)-supremum of \( S \), then necessarily \( S \neq \emptyset \).

Remark 21.

(i) Let \( S \subseteq ^{\ast}R \), then from Def. 9 and Thm. 11 we can prove that \( \sigma \) is the \((\tau, P)\)-supremum of \( S \) if and only if

(a) \( \forall s \in S : s \leq \sigma \);

(b) \( \forall r \in \mathfrak{R} \exists \bar{s} \in S : \sigma - r \leq \bar{s} \).

In particular, for the sharp supremum, (b) is equivalent to

\[ \forall q \in \mathbb{N} \exists \bar{s} \in S : \sigma - d_{\rho^q} \leq \bar{s}. \] (4.1)

In the following of this article, we will also mainly consider the sharp topology and the corresponding notions of sharp supremum and infimum.

(ii) If there exists the sharp supremum \( \sigma \) of \( S \subseteq ^{\ast}R \) and \( \sigma \notin S \), then from (4.1) it follows that \( S \) is necessarily an infinite set. In fact, applying (4.1) with \( q_1 := 1 \) we get the existence of \( \bar{s}_1 \in S \) such that \( \sigma - d_{\rho^{q_1}} \leq \bar{s}_1 \). We have \( \bar{s}_1 \neq \sigma \) because \( \sigma \notin S \). Hence, Lem. 5.(iii) and Def. 20.(ii) yield that \( \bar{s}_1 < \sigma \). Therefore, \( \sigma - \bar{s}_1 \geq d_{\rho^{q_2}} \) for some \( q_2 > q_1 \). Applying again (4.1) we get \( \sigma - d_{\rho^{q_2}} \leq \bar{s}_2 \) for some \( \bar{s}_2 \in S \setminus \{\bar{s}_1\} \). Recursively, this process proves that \( S \) is infinite. On the other hand, if \( S = \{s_1, \ldots, s_n\} \) and \( s_i = [s_{i\varepsilon}] \), then 

\[ \text{sup} ([\{s_{i\varepsilon}, \ldots, s_{n\varepsilon}\}]) = s_1 \lor \ldots \lor s_n. \]

In fact, \( s_1 \lor \ldots \lor s_n = [\max_{i=1,\ldots,n}s_{i\varepsilon}] \in [\{s_{i\varepsilon}, \ldots, s_{n\varepsilon}\}] \).

(iii) If \( \exists \text{sup}(S) = \sigma \), then there also exists the \( \text{sup}(\text{interl}(S)) = \sigma \), where (see [18]) we recall that

\[ \text{interl}(S) := \left\{ \sum_{j=1}^{m} e_{S_j} s_j \mid m \in \mathbb{N}, S_j \subseteq I, s_j \in S \forall j \right\}, \quad e_S := [1_S] \in ^{\ast}R \]

(\( 1_S \) is the characteristic function of \( S \subseteq I \)). This follows from \( S \subseteq \text{interl}(S) \). Vice versa, if \( \exists \text{sup}(\text{interl}(S)) = \sigma \) and \( \text{interl}(S) \subseteq S \) (e.g. if \( S \) is an internal or strongly internal set), then also \( \exists \text{sup}(S) = \sigma \).

Theorem 22. There is at most one sharp supremum of \( S \), which is denoted by \( \text{sup}(S) \).
Proof. Assume that $\sigma_1$ and $\sigma_2$ are supremum of $S$. That is Def. 20.(ii) and (4.1) hold both for $\sigma_1, \sigma_2$. Then, for all fixed $q \in \mathbb{N}$, there exists $\bar{s}_2 \in S$ such that $\sigma_2 - d\rho^q \leq \bar{s}_2$. Hence $\bar{s}_2 \leq \sigma_1$ because $\bar{s}_2 \in S$. Analogously, we have that $\sigma_1 - d\rho^q \leq \bar{s}_1 \leq \sigma_2$ for some $\bar{s}_1 \in S$. Therefore, $\sigma_2 - d\rho^q \leq \sigma_1 \leq \sigma_2 + d\rho^q$, and this implies $\sigma_1 = \sigma_2$ since $q \in \mathbb{N}$ is arbitrary.

In [9], the notation $\sup_+(S)$ is used for the close supremum. On the other hand, we will never use the notion of supremum as least upper bound. For these reasons, we prefer to use the simpler notation $\sup(S)$. Similarly, we use the notation $\inf(S)$ for the close (or sharp) infimum. From Rem. 21.(a) and (b) it follows that

$$\inf(S) = - \sup(-S)$$

(4.2)

in the sense that the former exists if and only if the latter exists and in that case they are equal. For this reason, in the following we only study the supremum.

Example 23.

(i) Let $K = [K_0] \in \mathcal{F}$ be a functionally compact set (cf. [11]), i.e. $K \subseteq B_M(0)$ for some $M \in \mathcal{F}_{>0}$ and $K_0 \subseteq \mathbb{R}$ for all $\varepsilon$. We can then define $\sigma_\varepsilon := \sup(K_\varepsilon) \in K_0$. From $K \subseteq B_M(0)$, we get $\sigma := \lceil \sigma_\varepsilon \rceil \in K$. It is not hard to prove that $\sigma = \sup(K) = \max(K)$. Analogously, we can prove the existence of the sharp minimum of $K$.

(ii) If $S = (a, b)$, where $a, b \in \mathcal{F}$ and $a \leq b$, then $\sup(S) = b$ and $\inf(S) = a$.

(iii) If $S = \left\{ \frac{1}{n} \mid n \in \mathcal{F} \right\}$, then $\inf(S) = 0$.

(iv) Like in several other non-Archimedean rings, both sharp supremum and infimum of the set $D_\infty$ of all infinitesimals do not exist. In fact, by contradiction, if $\sigma$ were the sharp supremum of $D_\infty$, then from (4.1) for $q = 1$ we would get the existence of $\bar{h} \in D_\infty$ such that $\sigma \leq \bar{h} + d\rho$. But then $\sigma \in D_\infty$, so also $2\sigma \in D_\infty$. Therefore, we get $2\sigma \leq \sigma$ because $\sigma$ is an upper bound of $D_\infty$, and hence $\sigma = 0 \geq d\rho$, a contradiction. Similarly, one can prove that there does not exist the infimum of this set.

(v) Let $S = (0, 1)_{\mathbb{R}} = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$, then clearly $\sigma = 1$ is the Fermat supremum of $S$ whereas there does not exist the sharp supremum of $S$. Indeed, if $\sigma = \sup(S)$, then $\bar{s} \leq \sigma \leq \bar{s} + d\rho$ for all $\bar{s} \in S$ and for some $\bar{s} \in S$. Taking any $s \in (\bar{s}, 1] \subseteq S$ we get $s \leq \sigma \leq \bar{s} + d\rho$, which, for $\varepsilon \rightarrow 0$, implies $s \leq \bar{s}$ because $s, \bar{s} \in \mathbb{R}$. This contradicts $s \in (\bar{s}, 1]$. In particular, 1 is not the sharp supremum. This example shows the importance of Def. 20, i.e. that the best notion of supremum in a non-Archimedean setting depends on a fixed topology.

(vi) Let $S = (0, 1) \cup \{ \bar{s} \}$ where $\bar{s}_{|L} = 2, \bar{s}_{|L^c} = \frac{1}{2}, L \subseteq_0 I, L^c \subseteq_0 I$, then $\bar{s}$ is not a supremum. In fact, if $\exists \sigma := \sup(S)$, then $\sigma_{|L} \geq \bar{s}_{|L} = 2$ and $\sigma_{|L^c} = 1$. Assume that $\exists s \in S : \sigma - d\rho \leq s$, then $2 - d\rho_{|L} \leq \sigma_{|L} - d\rho_{|L} \leq \bar{s}_{|L}$. Thereby, $\bar{s}_{|L} > \frac{3}{2}$ and hence $s \not\in (0, 1)$ and $s = \bar{s}$. We hence get $\sigma_{|L^c} - d\rho_{|L^c} \leq \bar{s}_{|L^c}$, i.e. $1 - d\rho_{|L^c} \leq \frac{1}{2}$, which is impossible. We can intuitively say that the subpoint $\bar{s}_{|L}$ creates a “$\varepsilon$-hole” (i.e. a “hole” only for some $\varepsilon$) on the right of $S$ and hence $S$ is not an $\varepsilon$-continuum” on this side. Finally note that the point $u_{|L} := 2$ and $u_{|L^c} := 1$ is the least upper bound of $S$.

Lemma 24. Let $A, B \subseteq \mathcal{F}$, then
(i) \( \forall \lambda \in \bar{\mathbb{R}}_{>0} : \sup(\lambda A) = \lambda \sup(A) \), in the sense that one supremum exists if and only if the other one exists, and in that case they coincide;

(ii) \( \forall \lambda \in \bar{\mathbb{R}}_{<0} : \sup(\lambda A) = \lambda \inf(A) \), in the sense that one supremum/infimum exists if and only if the other one exists, and in that case they coincide;

Moreover, if \( \exists \sup(A) \), \( \sup(B) \), then:

(iii) If \( A \subseteq B \), then \( \sup(A) \leq \sup(B) \);

(iv) \( \sup(A + B) = \sup(A) + \sup(B) \);

(v) If \( A, B \subseteq \mathbb{R}_{\geq 0} \), then \( \sup(A \cdot B) = \sup(A) \cdot \sup(B) \).

Proof. (i): If \( \exists \sup(\lambda A) \), then we have \( a \leq \frac{1}{\lambda} \sup(\lambda A) \) for all \( a \in A \). For all \( q \in \mathbb{N} \), we can find \( \bar{a} \in A \) such that \( \sup(\lambda A) - \lambda \bar{a} \leq \frac{1}{\lambda}d\rho^q \). Thereby, \( \frac{1}{\lambda} \sup(\lambda A) - \bar{a} \leq \frac{1}{\lambda}d\rho^q \to 0 \) as \( q \to +\infty \) because \( \lambda \) is moderate. This proves that \( \exists \sup(A) = \frac{1}{\lambda} \sup(\lambda A) \).

Similarly, we can prove the opposite implication.

(ii): From (i) and (4.2) we get: \( \sup(\lambda A) = \sup(-\lambda(-A)) = -\lambda \sup(-A) = \lambda \inf(A) \).

(iii): By contradiction, using Lem. 5.(i), if \( \sup(A) >_L \sup(B) \) for some \( L \subseteq \mathbb{R} \), then \( \sup(A) - \sup(B) >_L d\rho^q \) for some \( q \in \mathbb{R} \) by Lem. 2 for the ring \( \mathbb{R}_{L} \). Property (4.1) yields \( \sup(A) - d\rho^q \leq \bar{a} \) for some \( \bar{a} \in A \), and \( \bar{a} \leq \sup(B) \) because \( A \subseteq B \). Thereby, \( \sup(A) - \sup(B) \leq d\rho^q \), which implies \( d\rho^q <_L d\rho^q \), a contradiction.

(iv) and (v) follow easily from Def. 20.(ii) and (4.1). \( \square \)

In the next section, we introduce in the non-Archimedean framework \( \mathbb{R} \) how to approximate \( \sup(S) \) of \( S \subseteq \mathbb{R} \) using points of \( S \) and upper bounds, and the non-Archimedean analogues of the notion of upper bound.

4.1. Approximations of \( \sup \), completeness from above and Archimedean upper bounds. In the real field, we have the following peculiar properties:

(i) The notion of least upper bound coincides with that of close supremum, i.e. it satisfies property (1.1). We can hence question when these two notions coincide also in \( \mathbb{R} \). Example 23.(vi) shows that the answer is not trivial. A first solution of this problem is already contained in [9, Prop. 1.4], where it is shown that the close supremum, assuming that it exists, coincides with the least upper bound.

(ii) The notion of upper bound in \( \mathbb{R} \) is very useful because it entails the existence of the supremum. Clearly, since there are infinite upper bounds but only one supremum, the notion of upper bound results to be really useful in estimates with inequalities. Moreover, in the ring \( \mathbb{R} \), the presence of infinite numbers (of different magnitudes) allows one to have trivial upper bounds, such as in the case \( S = (0, 1) \) and \( M = d\rho^{-1} \), or \( S = (0, d\rho^{-1}) \) and \( M = d\rho^{-2} \). Thereby, we can also investigate whether we can consider non trivial upper bounds, i.e. numbers which are, intuitively, of the same order of magnitude of the elements of \( S \subseteq \mathbb{R} \). On the other hand, example 23.(vi) shows that with respect to any reasonable definition of “same order of magnitude”, the upper bound \( m = 3 \) must be of the same order of any point in \( S \), although \( \not\exists \sup(S) \).

We will solve this problem by introducing the definition of Archimedean upper bound.

(iii) If \( \emptyset \neq S \subseteq \mathbb{R} \) admits an upper bound, then \( \sup(S) \) can be arbitrarily approximated using upper bounds and points of \( S \). When is this possible if \( \emptyset \neq S \subseteq \mathbb{R} ? \)
Example 23.(vi) shows that these problems cannot be solved in general, and we
are hence searching for a useful sufficient condition on $S$. As we will see more
clearly below, we could also say that we are searching for a practical notion or
procedure “at the $\varepsilon$-level” (i.e. working on representatives) to determine whether a
set has the supremum or the least upper bound. However, we are actually far from
a real solution of this non trivial problem, and the present section presents only
preliminary steps in this direction.

We first prove the following useful characterization of the existence of $\sup(S)$,
which also solves problem (iii):

**Theorem 25.** Let $S \subseteq \bar{\mathbb{R}}$, and let $U \subseteq \bar{\mathbb{R}}$ denote the set of upper bounds of $S$.
Then $S$ has supremum if and only if

$$\forall q \in \mathbb{N} \exists u_q \in U \exists s_q \in S : u_q - s_q \leq d\rho^q.$$  \hspace{1cm} (4.3)

**Proof.** If $\sigma = \sup(S)$, then (4.3) simply follows by setting $u_q := \sigma$ and $s_q \in S$ from
(4.1). Vice versa, if (4.3) holds, then

$$-d\rho^q \leq s_q - u_q \leq u_{q+1} - u_q \leq u_{q+1} - s_{q+1} \leq d\rho^{q+1} \quad \forall q \in \mathbb{N}.$$  

Thereby, $-(p-q)d\rho^q \leq u_p - u_q \leq (p-q)d\rho^q$ for all $p > q$, and hence $-d\rho^{\min(p,q)} \leq u_p - u_q \leq d\rho^{\min(p,q)}$ for all $p, q \in \mathbb{N}_>0$. This shows that $(u_q)_{q \in \mathbb{N}}$ is a Cauchy
sequence which thus converges to some $\sigma \in \bar{\mathbb{R}}$. Property (4.3) yields that also
$(s_q)_{q \in \mathbb{N}} \rightarrow \sigma$, and this implies condition (4.1). Since each $u_q$ is an upper bound, for
all $s \in S$ we have $s \leq u_q$, which gives $s \leq \sigma$ for $q \rightarrow +\infty$. \hspace{1cm} $\square$

To solve problem (iii), assume that $u \in \bar{\mathbb{R}}$ is an upper bound of a non empty $S \subseteq \bar{\mathbb{R}}$.
Let $[u_\varepsilon] = u$, and for all $s \in S$ choose a representative $[s_\varepsilon(u)] = s$ such that

$$\forall \varepsilon \in I : s_\varepsilon(u) \leq u.$$  \hspace{1cm} (4.4)

We first note that setting

$$\sigma_\varepsilon := \sup \{ s_\varepsilon(u) \mid s \in S \} \quad \forall \varepsilon \in I$$

(4.5)
do not work to define a representative of the supremum, e.g. if $S = (0,1)$. Assume,
e.g., that $u_\varepsilon = 3$ and take any sequence $(s_n)_{n \in \mathbb{N}}$ of different points of $S$: $s_i \neq s_j$
if $i \neq j$. Change representatives of $s_n = [s_{n\varepsilon}]$ satisfying (4.4) by setting $\bar{s}_{n\varepsilon} := [s_{n\varepsilon}(u)] = s_{n\varepsilon}(3)$ if $\varepsilon = \frac{1}{n}$ and $s_{n\varepsilon} := 3$. These new representatives still satisfy (4.4),
but defining $\sigma_\varepsilon$ with them as in (4.5), we would get $\sigma_\varepsilon \geq \sup \{ \bar{s}_{n\varepsilon} \mid n \in \mathbb{N}_>0 \} = 3,$
and hence $[\sigma_\varepsilon] \neq 1 = \sup(S)$. We want to refine this idea by considering suitable
representatives $[s_\varepsilon(u)] = s$ satisfying (4.4), and setting

$$\sigma_\varepsilon(S) := \sigma_\varepsilon := \inf \{ \sup \{ s_\varepsilon(u) \mid s \in S \} \mid u \geq S \} \quad \forall \varepsilon \in I,$$

(4.6)

$$(\sigma_\varepsilon(S)) \in \bar{\mathbb{R}} \Rightarrow \sigma(S) := [\sigma_\varepsilon(S)] \in \bar{\mathbb{R}},$$

(4.7)

where $u \geq S$ means that $u$ is an upper bound of $S$, and where the representatives
are chosen as follows: set $\bar{\mathbb{R}} := \mathbb{R} \cup \{ +\infty \}$, and for all $(u_\varepsilon) \in \bar{\mathbb{R}}$ and $s \in S$:

$$\begin{cases} s \leq [u_\varepsilon] \in \bar{\mathbb{R}} \Rightarrow \exists [s_\varepsilon(u)] = s \forall \varepsilon \in I : s_\varepsilon(u) \leq u_\varepsilon \\ (u_\varepsilon) \notin \bar{\mathbb{R}} \text{ or } s \notin [u_\varepsilon] \Rightarrow [s_\varepsilon(u)] = s \text{ is any representative of } s. \end{cases}$$

(4.8)

Note that definition (4.6) depends on the chosen representatives $(s_\varepsilon(u))$ for $s \in S$
and $(u_\varepsilon)$ for $u \geq S$; trivially, if $(\bar{s}_\varepsilon(S))$ is defined using different representatives $(\bar{s}_\varepsilon(u))$ and $(\bar{u}_\varepsilon)$, and both $(\sigma_\varepsilon(S))$ and $(\sigma_\varepsilon(S))$ well-define the supremum $\sup(S)$
For the least upper bound \( \lub(S) \) of \( S \), then \( \bar{s} = [s] \) = \([\sigma(S)]\). On the other hand, if we calculate \((\sigma(S))\) using a certain choice of representatives, and we notice that \((\sigma(S))\) is not an upper bound of \( S \), we do not know whether another choice of representatives can give an upper bound. This is one of the weaknesses of the present solution. To highlight this dependence, we will also sometimes use the following notations for our choice functions (their existence depends on the axiom of choice):

\[
e(s, u, \varepsilon) := s_\varepsilon(u) \quad \forall s \in S \forall u \geq S
\]

\[
b(u, \varepsilon) := u_\varepsilon \quad \forall u \geq S.
\]

We first observe that, for all \( \varepsilon \in I \):

\[
\exists u \geq S \Rightarrow \sigma_\varepsilon = +\infty
\]

\[
\exists u \geq S \Rightarrow \sigma_\varepsilon \leq \sup \{s_\varepsilon(u) \mid s \in S\} \leq u_\varepsilon
\]

\[
S = \emptyset \Rightarrow \sigma_\varepsilon = \sup \{s_\varepsilon(u) \mid s \in S\} = -\infty.
\]

We therefore have:

**Lemma 26.** Assume that \( S \subseteq \mathbb{R}\), \((\sigma_\varepsilon(S))\) \( \in \mathbb{R}_\rho \) and \( \sigma(S) \geq S \). Then the following properties hold:

(i) \( \sigma(S) = \lub(S) \).

(ii) If \( b(\sigma(S), \varepsilon) = \sigma_\varepsilon(S) \), then \( \sigma_\varepsilon(S) = \sup \{s_\varepsilon(\sigma(S)) \mid s \in S\} = \inf \{u_\varepsilon \mid u \geq S\} \) for all \( \varepsilon \in I \).

**Proof.** If \( \sigma := \sigma(S) \geq S \), inequality (4.10) shows that \( \sigma \) is the least upper bound of \( S \). From (4.6) and (4.10), we have \( \sigma_\varepsilon \leq \sup \{s_\varepsilon(s) \mid s \in S\} \leq \sigma_\varepsilon \) because \( \sigma \geq S \) and \( b(\sigma, \varepsilon) = \sigma_\varepsilon \) (i.e. the chosen representative \( u_\varepsilon \)) for the upper bound \( \sigma \geq S \) is exactly \( \sigma_\varepsilon \) as defined in (4.6)). Finally, the inequality \( \sigma_\varepsilon \leq \inf \{u_\varepsilon \mid u \geq S\} \) follows from (4.10). The other inequality follows from \( \sigma = \sigma(S) \geq S \) and from \( b(\sigma, \varepsilon) = \sigma_\varepsilon \). \( \square \)

In general, the net \((\sigma_\varepsilon(S))\) is not \( \rho \)-moderate. In fact, if \((u_n)_{n \in \mathbb{N}}\) is a sequence of different upper bounds and we set \( s_{n, 1}(u_n) = -\rho_{1/n}^{-1} \), this yields \( \sigma_{1/n} \leq -\rho_{1/n}^{-1}/n \).

On the other hand, we have:

**Lemma 27.** Let \( u \in \mathbb{R}, S \subseteq \mathbb{R}\) with \( S \leq u \). Assume that for some \( \bar{s} \in S \) we have

\[
\forall \varepsilon \in I : \quad \sigma_\varepsilon(S) \geq \bar{s}(u).
\]

Then \((\sigma_\varepsilon(S))\) \( \in \mathbb{R}_\rho \) and \( \bar{s} \leq \sigma(S) \leq u \).

**Proof.** From (4.10), we get \( \sigma_\varepsilon \leq u_\varepsilon \). The conclusion thus follows from (4.11) and \( \bar{s}, u \in \mathbb{R} \). \( \square \)

Since the set of all infinitesimals \( S = D_\infty \) has no least upper bound, the previous two results imply that \( \sigma(D_\infty) \not\subseteq D_\infty \). Using Lem. 27 with \( l = -r, u = r \in \mathbb{R}_{>0} \), we have that \( \sigma(D_\infty) \) is always an infinitesimal (that actually depends on the chosen representatives \((s_\varepsilon(u))\) and \((u_\varepsilon))\).

The following condition solves problem (ii):

**Definition 28.** Let \( S \subseteq \mathbb{R}\) and for simplicity use \( \sigma_\varepsilon = \sigma_\varepsilon(S) \), then we say that \( S \) is complete from above if the following conditions hold:

(i) \( \forall s \in S \exists \varepsilon : s_\varepsilon = s \forall \varepsilon \in I : s_\varepsilon \leq \sigma_\varepsilon \).
(ii) If \((s^e)_{e \in I}\) is a family of \(S\) which satisfies:

\[
\exists [u_\varepsilon] \in \bar{S} \quad \forall e \in I \forall \varepsilon \colon \; s^e_\varepsilon(\sigma) \leq u_\varepsilon
\]  

(4.12)

then

\[
\exists [\bar{s}_\varepsilon] \in \bar{S} \forall \varepsilon \colon \; s^\varepsilon(\sigma) \leq \bar{s}_\varepsilon,
\]  

(4.13)

where \(\bar{S}\) is the closure of \(S\) in the sharp topology.

Moreover, if \(\exists s \in S : \; s > 0\), then we say that \(M\) is a complete from above if and only if the following condition holds

\[
M \in \bar{\mathbb{R}} \quad \text{and} \quad \forall s \in S : \; s \leq M;
\]

(a) \(M \in \bar{\mathbb{R}}\) and \(\forall s \in S : \; s \leq M\);

(b) \(\exists n \in \mathbb{N} \exists \bar{s} \in S : \; M < n\bar{s}\). The minimum \(n \in \mathbb{N}\) that satisfies this property is called the order of \(M\) (clearly, \(n \geq 2\)). Note that this condition, using an Archimedean-like property, formalizes the idea that \(M\) and \(\bar{s}\) are of the same order of magnitude.

Dually, we can define the notion of completeness from below by reverting the inequalities in (i) and (ii). If \(\exists s \in S : \; s < 0\), then \(N\) is an Archimedean lower bound (ALB) of \(S\) if it is a lower bound such that \(\exists n \in \mathbb{N} \exists \bar{s} \in S : \; \bar{s}n < N\).

Note that \(\sigma = \sup(S)\) is always an AUB of order 2. In fact, from the existence of \(s \in S_{>0}\), we have \(s > d\rho^q\) for some \(q \in \mathbb{N}\) and the existence of \(\bar{s} \in S\) with \(\bar{s} \geq \sigma - d\rho^{q+1}\). Thereby, \(\bar{s} \geq s - d\rho^{q+1} > d\rho^q - d\rho^{q+1} > d\rho^{q+1}\) and thus \(\sigma \leq \bar{s} + d\rho^{q+1} < 2\bar{s}\). We also note that \(S = \bar{\mathbb{R}}\) is trivially complete from above (because \(\sigma_\varepsilon = +\infty\) from (4.10), and by setting \(\bar{s}_\varepsilon = u_\varepsilon\)) but \(\not\exists \sup(\bar{\mathbb{R}})\). Looking at Lem. 27, in the case of a non empty subset \(S \subseteq \bar{\mathbb{R}}\) bounded from above, the condition of being complete from above can be intuitively described as follows:

(a) Choose representatives \([u_\varepsilon] = u\) for each \(u \geq S\) and \([s_\varepsilon(u)] = s\) for each \(s \in S\) satisfying (4.8);

(b) Define \(\sigma_\varepsilon(S) =: \sigma_\varepsilon \in \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}\) as in 4.6.

(c) Check if the inequality \(s_\varepsilon(\sigma) \leq \sigma_\varepsilon\) holds (in this case, for the chosen representatives satisfying (4.8), without loss of generality, we can assume that \(b(\sigma_\varepsilon, \varepsilon) = \sigma_\varepsilon\) for all \(\varepsilon \in I\));

(d) From any family \((s^e)_{e \in I}\) of \(S\) (which is therefore bounded from above, so that (4.12) always holds) pick the diagonal net \((s^\varepsilon(\sigma))\) from its representatives (depending on \(\sigma \geq S\)) and check if \(s^\varepsilon(\sigma) \leq \bar{s}_\varepsilon\) for some \(\bar{s}\) in the sharp closure \(\bar{S}\).

(e) If any of the two previous steps do not hold, consider a different set of representatives in the first step (a).

We therefore have the following simplified case:

**Lemma 29.** Assume that \(\emptyset \neq S \subseteq \bar{\mathbb{R}}\) is sharply bounded from above, then \(S\) is complete from above if and only if the following condition holds

(i) \(\sigma(S) := \sigma \geq S\)

(ii) If \((s^e)_{e \in I}\) is a family of \(S\), then \(\exists [\bar{s}_\varepsilon] \in \bar{S} \forall \varepsilon : \; s^\varepsilon_\varepsilon(\sigma) \leq \bar{s}_\varepsilon\).

Note that example 23.(vi) satisfies the first one of these conditions (so that \(\sigma(S)\) is its least upper bound) but not the second one because it does not admit supremum (see the following theorem). Cases which remain excluded from the previous lemma are e.g. intervals \((a, +\infty)\), with \(-\infty \leq a \in \bar{\mathbb{R}}\) which are complete from above even if they do not admit supremum nor least upper bound.
The following results solve the remaining problems (i) and (ii) we set at the beginning of this section.

**Theorem 30.** Assume that $\emptyset \neq S \subseteq \bar{\mathbb{R}}$, then

(i) If $S$ is complete and bounded from above and $b(\sigma(S), \varepsilon) = \sigma(\varepsilon(S))$, then $\exists \sup(S) = \sigma(S)$;

Let $(s_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$ be two sequences as in Thm. 25, then

(ii) If $\exists s \in S : s > 0$, and if there exists $C \in \mathbb{R}_{>0}$ such that $s_n \geq C d_{\rho^p}$ for all $q \in \mathbb{N}$ large, then $u_q$ is an AUB of $S$ for all $q$ sufficiently large;

(iii) If $\exists s \in S : s > 0$, then $u_q$ is an AUB of $S$ of order 2 for all $q$ sufficiently large.

**Proof.** (i): From Lem. 27 we get that $\sigma(S) =: \sigma$ is well-defined because $\sigma \geq S$ by definition of completeness from above, i.e. Def. 28.(i). Therefore, Lem. 26 and the assumption $b(\sigma(S), \varepsilon) = \sigma(\varepsilon(S))$, yield that $\sigma(\varepsilon(S)) = \sup \{ s(\varepsilon) \mid s \in S \}$ for all $\varepsilon$. For arbitrary $q \in \mathbb{N}$ and $e \in I$, this yields

$$\sigma(e) - \rho_{\varepsilon}^{p+2} < s(\varepsilon) =: s(\varepsilon)$$

(4.14)

for some $s(\varepsilon) \in S$ (that depends on both $q$ and $e$). By definition of completeness from above, we get the existence of $s = [s(e)] \in \overline{S}$ such that $s(\varepsilon) \leq s(\varepsilon)$ for $\varepsilon$ mall. Setting $e = \varepsilon$ in (4.14), we get $\sigma(e) - \rho_{\varepsilon}^{p+2} < s(\varepsilon) \leq s(\varepsilon)$ for $\varepsilon$ small, i.e. $\sigma - d_{\rho^p}^{p+2} < s$. Since $s \in \overline{S}$, there exists $s \in S(\varepsilon - d_{\rho^p}^{p+1}, s + d_{\rho^p}^{p+1})$. Thereby, $\sigma - d_{\rho^p}^{p+1} < s - d_{\rho^p}^{p+1} < s$, and hence $\sigma - d_{\rho^p}^{p+1} < s - d_{\rho^p}^{p+1} < s$, which proves our claim (i).

Now, assume that $s_0 \geq C d_{\rho^p}$ for some $C \in \mathbb{R}_{>0}$ and for all $q \in \mathbb{N}$ sufficiently large. Then, for these $q$ we have $\frac{s_{e}}{s_{e} + d_{\rho^p}} \leq 1 + \frac{1}{C} \leq [1 + \frac{1}{C}] =: n \in \mathbb{N}$. This yields $u_q < s_q + d_{\rho^p} < n s_q$, i.e. $u_q$ is an AUB of $S$. Finally, from the existence of at least one $s \in S_{>0}$, we get the existence of $p \in \mathbb{N}$ such that $s > d_{\rho^p}$. Therefore, also $d_{\rho^p} < s \leq \sigma$. From (i), we hence get that for $q \in \mathbb{N}$ sufficiently large $d_{\rho^p} < q \leq \sigma$, i.e. $\frac{1}{\sigma^p} < d_{\rho^p}$ and $\frac{u_q + \rho^p}{s_q} \leq 1 + d_{\rho^p} \leq 2$ for all $q > p$. Proceeding as above we can prove the claim. \qed

Example 23.(vi) shows the necessity of the assumption of completeness from above in this theorem.

Directly from Thm. 30.(i), we obtain:

**Corollary 31.** Let $\emptyset \neq S \subseteq \bar{\mathbb{R}}$. Assume that $S$ is complete from above and $b(\sigma(S), \varepsilon) = \sigma(\varepsilon(S))$, then $\exists \sup(S)$ if and only if $S$ admits an upper bound.

Now, we can also complete the relationships between close supremum and least upper bound (see also [9, Prop. 1.4]) and study what happens if we consider only the upper bounds $u$ lower than a fixed upper bound $\bar{u}$ in (4.6).

**Corollary 32.** Let $\emptyset \neq S \subseteq \bar{\mathbb{R}}$, then the following properties hold:

(i) If $\exists \sup(S) = \sigma$, then $\exists \sup(S) = \sigma$.

(ii) If $S$ is complete and bounded from above, then

$$\exists \sup(S) = \sigma \iff \exists \sup(S) = \sigma.$$  

(iii) Assume that $\bar{u} \geq S$ and define $\sigma(\bar{u}) := \inf \{ \sup \{ s(\varepsilon) \mid s \in S \} \mid \bar{u} \geq u \geq S \}$. Then $\sigma(\bar{u})$ is well-defined and $\sigma(\bar{u}) \geq \sigma(S)$. If $\sigma(S) \geq S$, then $\sigma(\bar{u}) = \sigma(S)$. If $\sigma(S) \geq S$, then $\sigma(\bar{u})$ is the least upper bound of $S$, thus $\sigma(\bar{u}) = \sigma(S)$ if $S$ is complete from above.
(iv) Assume that $\exists \sigma(S) \geq S$, $b(\sigma(S), \varepsilon) = \sigma_{\varepsilon}(S)$ and $\exists \sup(S)$. Then $S$ is complete from above.

Proof. (i) and (ii): Assume that $\exists \sup(S) = \sigma$, and let $u$ be an upper bound of $S$; by condition (4.1) we get $\sigma - d\rho^q \leq s_q \leq u$ for all $q \in \mathbb{N}$ and for some $s_q \in S$. For $q \to +\infty$, we get $\sigma \leq u$. Vice versa, if $S \neq \emptyset$ is complete from above and $\sigma$ is the least upper bound of $S$, then the conclusion follows from Cor. 31.

(iii): If $\bar{s} \in S \leq u$, we can prove that $(\bar{\sigma}_e(S)) \in \mathbb{R}_\rho$ and $\bar{s} \leq \bar{\sigma}(S) \leq u$ as in the proof of Lem. 27. We always have that $\sigma(S) \leq \bar{u}$ because $\bar{u} \geq S$. Therefore, if $\sigma(S) \geq S$, then $\bar{u} \geq \sigma(S) \geq S$ and hence $\sigma(S) \leq \bar{\sigma}(S) \leq \bar{\sigma}(S)$. Finally, if we assume that $\sigma(S) \geq S$ and we consider an arbitrary upper bound $u \geq S$, then either $u \geq \bar{u}$ or $u \leq L \leq \bar{u}$ for some $L \subseteq \sigma$. Therefore, $\sigma(S) \leq u$ or $\sigma(S) \leq L u$, and hence $\bar{\sigma}(S) \leq u$. Therefore, $\bar{\sigma}(S)$ is the least upper bound of $S$, and the final claim follows from (ii).

(iv): From Lem. 26, we have $\sigma(S) = \inf(S) =: \sigma$ and hence $\sup(S) = \sigma \in S$ from (i). From $b(\sigma(S), \varepsilon) = \sigma_{\varepsilon}(S)$ and (4.8) we have $s_{\varepsilon}(\sigma) \leq \sigma_{\varepsilon}$ for all $s \in S$ and all $\varepsilon \in I$. In particular, if $(s^e)_{e \in I}$ is a family of $S$, we have $s_{e}(\sigma) \leq \sigma_{e}$ for all $e \in I$ and all $\varepsilon \in I$. Taking $e = \varepsilon$, we get that (9.2) holds. \hfill $\square$

Example 33.

(i) Example 33.(v) shows that the assumption of being complete from above is necessary in Cor. 32. On the other hand, using the notation of this example, one can prove that $\sigma_{\varepsilon}(S) = 2$ if $\varepsilon \in L$ and $\sigma_{\varepsilon}(S) = 1$ if $\varepsilon \in L^c$. From Lem. 26 it follows that $\sigma(S)$ is the least upper bound of $S$. This underscores the differences between the order theoretical definition of supremum as least upper bound and the topological definition of closed supremum.

(ii) Any set having a maximum is trivially complete from above: set $[\bar{s}] := \max(S)$ in (4.13) and consider that $\sigma(S) = \max(S).

(iii) $S = (0, 1)$ is complete from above for $[\bar{s}] = 1$ and because $\sigma_e(S) =: \sigma_e = 1$. In fact, $\sigma_e \leq 1$ from (4.10). Now, take any $u = [u_e] \geq S$, so that $u_e \geq 1$ for all $\varepsilon \geq \varepsilon_0$. For $\varepsilon \geq \varepsilon_0$, by contradiction assume that $1 > \sup \{s_{\varepsilon}(u) \mid s \in S\}$, and hence $1 > r > \sup \{s_{\varepsilon}(u) \mid s \in S\}$ for some $r \in (0, 1)_{\infty} \subseteq S$. Take $r_e = r$ as representative of $r$ in (4.8); we have two cases: If $r_e(u) = u_e \geq 1$, then $1 > \sup \{s_{\varepsilon}(u) \mid s \in S\} \geq r_e(u) = u_e \geq 1$; if $r_e(u) = r$, then $\sup \{s_{\varepsilon}(u) \mid s \in S\} \geq r_e(u) = r > \sup \{s_{\varepsilon}(u) \mid s \in S\}$. In any case, we get a contradiction, and this proves that $1 \leq \sup \{s_{\varepsilon}(u) \mid s \in S\}$ for all $\varepsilon \leq \varepsilon_0$, and hence $\sigma_e \geq 1$.

(iv) There do not exist neither the supremum nor the least upper bound of $S = 1 + D_{\infty}$. On the other hand, 2 is an AUB of $S$ and hence $S$ is not complete from above.

(v) $D_{\infty}$ has neither AUB nor ALB; $\bar{\mathbb{R}}$ has neither AUB nor ALB; $\{d\rho^q \mid r \in \mathbb{R}_{>0}\}$ has no supremum and no AUB and hence it is not complete from above.

(vi) Assume that there does not exist and upper bound of $S$. This means that

$$\forall u \in \mathbb{R}_{>0} \exists s \in S : s > u.$$

Thereby, there exists a sequence $(s_q)_{q \in \mathbb{N}}$ of $S$ such that $s_q > s$. Based on this, we could set $\sup(S) := +\infty$.

5. The hyperlimit of a hypersequence

5.1. Definition and examples.
**Definition 34.** A map \( x : ^\ast \mathbb{N} \to \tilde{\mathbb{R}} \), whose domain is the set of hypernatural numbers \(^\ast \mathbb{N}\) is called a \((\sigma-)\) hypersequence (of elements of \(\tilde{\mathbb{R}}\)). The values \( x(n) \in \tilde{\mathbb{R}} \) at \( n \in ^\ast \mathbb{N} \) of the function \( x \) are called terms of the hypersequence and, as usual, denoted using an index as argument: \( x_n = x(n) \). The hypersequence itself is denoted by \((x_n)_{n \in ^\ast \mathbb{N}}\), or simply \((x_n)\) if the gauge on the domain is clear from the context. Let \( \sigma, \rho \) be two gauges, \( x : ^\ast \mathbb{N} \to \tilde{\mathbb{R}} \) be a hypersequence and \( l \in \tilde{\mathbb{R}} \). We say that \( l \) is a hyperlimit of \((x_n)\) as \( n \to \infty \) and \( n \in ^\ast \mathbb{N} \), if

\[
\forall q \in \mathbb{N} \exists M \in ^\ast \mathbb{N} \forall n \in ^\ast \mathbb{N} \geq M : |x_n - l| < d\rho^q.
\]

In the following, if not differently stated, \( \rho \) and \( \sigma \) will always denote two gauges and \((x_n)\) a \(\sigma\)-hypersequence of elements of \(\tilde{\mathbb{R}}\). Finally, if \( \sigma \geq \rho \), at least for all \( \varepsilon \) small, we simply write \( \sigma \geq \rho \).

**Remark 35.** In the assumption of Def. 34, let \( k \in \tilde{\mathbb{R}}_{>0}, N \in \mathbb{N} \), then the following are equivalent:

(i) \( l \in \tilde{\mathbb{R}} \) is the hyperlimit of \((x_n)\) as \( n \in ^\ast \mathbb{N} \).

(ii) \( \forall q \in \tilde{\mathbb{R}}_{>0} \exists M \in ^\ast \mathbb{N} \forall n \in ^\ast \mathbb{N} \geq M : |x_n - l| < q \).

(iii) Let \( U \subseteq \tilde{\mathbb{R}} \) be a sharply open set, if \( l \in U \) then \( \exists M \in ^\ast \mathbb{N} \forall n \in ^\ast \mathbb{N} \geq M : x_n \in U \).

(iv) \( \forall q \in \mathbb{N} \exists M \in ^\ast \mathbb{N} \forall n \in ^\ast \mathbb{N} \geq M : |x_n - l| < k \cdot d\rho^q \).

(v) \( \forall q \in \mathbb{N} \exists M \in ^\ast \mathbb{N} \forall n \in ^\ast \mathbb{N} \geq M : |x_n - l| < d\rho^{q-N} \).

Directly by the inequality \(|l_1 - l_2| \leq |l_1 - x_n| + |l_2 - x_n| \leq 2d\rho^{q+1} < d\rho^q \) (or by using that the sharp topology on \(\tilde{\mathbb{R}}\) is Hausdorff) it follows that there exists at most one hyperlimit, so that we can use the notation

\[
^\rho \lim_{n \in ^\ast \mathbb{N}} x_n := l.
\]

As usual, a hypersequence (not) having a hyperlimit is said to be (non-)convergent. We can also similarly say that \((x_n) : ^\ast \mathbb{N} \to \tilde{\mathbb{R}}\) is divergent to \(+\infty\) \((-\infty\)) if

\[
\forall q \in \mathbb{N} \exists M \in ^\ast \mathbb{N} \forall n \in ^\ast \mathbb{N} \geq M : x_n > d\rho^{-q} \quad (x < -d\rho^{-q}).
\]

**Example 36.**

(i) If \( \sigma \leq \rho^R \) for some \( R \in \mathbb{R}_{>0} \), we have \( ^\rho \lim_{n \in ^\ast \mathbb{N}} \frac{1}{n} = 0 \). In fact, \( \frac{1}{n} < d\rho^q \) holds e.g. if \( n > [\text{int}(\rho^{-q}) + 1] \in ^\ast \mathbb{N} \) because \( \rho^{-q} \leq \sigma^{-q/R} \) for \( \varepsilon \) small.

(ii) Let \( \rho \) be a gauge and set \( \sigma \varepsilon := \exp \left(-\rho \frac{\varepsilon}{\rho} \right) \), so that \( \sigma \) is also a gauge. We have

\[
^\rho \lim_{n \in ^\ast \mathbb{N}} \frac{1}{\log n} = 0 \in \tilde{\mathbb{R}} \quad \text{wheras} \quad ^\rho \lim_{n \in ^\ast \mathbb{N}} \frac{1}{\log n} = 0 \in \tilde{\mathbb{R}}\]

In fact, if \( n > 1 \), we have \( 0 < \frac{1}{\log n} < d\rho^q \) if and only if \( \log n > d\rho^{-q} \), i.e. \( n > e^{d\rho^{-q}} \) (in \( \tilde{\mathbb{R}} \)). We can thus take \( M := \left[\text{int} \left( e^{d\rho^{-q}} \right) + 1 \right] \in ^\ast \mathbb{N} \) because \( e^{d\rho^{-q}} < \exp \left( \rho\frac{\varepsilon}{\rho} \right) = \sigma^{-1} \varepsilon \) for \( \varepsilon \) small. Vice versa, by contradiction, if
\[ \exists \lim_{n \in \mathbb{N}} \frac{1}{\log n} =: l \in \mathbb{R}, \text{ then by the definition of hyperlimit from } \mathbb{N}^* \text{ to } \mathbb{R}, \]
we would get the existence of \( M \in \mathbb{N}^* \) such that
\[ \forall n \in \mathbb{N}^* : n \geq M \Rightarrow \frac{1}{\log n} - \delta < l < \frac{1}{\log n} + \delta \]
(5.1)
We have to explore two possibilities: if \( l \) is not invertible, then \( l_{\varepsilon_k} = 0 \) for some sequence \( (\varepsilon_k) \downarrow 0 \) and some representative \( |l| = l \). Therefore from 34, we get
\[ \frac{1}{\log M_{\varepsilon_k}} < l_{\varepsilon_k} + \rho_{\varepsilon_k} = \rho_{\varepsilon_k} \]
hence \( M_{\varepsilon_k} > e^{-\frac{1}{\rho_{\varepsilon_k}}} \) \( \forall k \in \mathbb{N} \), in contradiction with \( M \in \mathbb{R}^* \). If \( l \) is invertible, then \( d\rho^p < |l| \) for some \( p \in \mathbb{N} \). Setting \( q := \min\{p \in \mathbb{N} | d\rho^p < |l|\} + 1 \), we get that \( l_{\varepsilon_k} < \rho_{\varepsilon_k}^q \) for some sequence \( (\varepsilon_k) \downarrow 0 \). Therefore
\[ \frac{1}{\log M_{\varepsilon_k}} < l_{\varepsilon_k} + \rho_{\varepsilon_k} \leq |l_{\varepsilon_k}| + \rho_{\varepsilon_k} < \rho_{\varepsilon_k}^q + \rho_{\varepsilon_k} \]
and hence \( M_{\varepsilon_k} \) gets (\( \frac{1}{\rho_{\varepsilon_k}^q + \rho_{\varepsilon_k}} \)) for all \( k \in \mathbb{N} \), which is in contradiction with \( M \in \mathbb{R}^* \) because \( q \geq 1 \).
Analogously, we can prove that \( \mathbb{N}^* \lim_{n \in \mathbb{N}} \frac{1}{\log(\log n)} = 0 \) if \( \sigma = [\sigma_\varepsilon] = \left[ e^{-e^{-\frac{1}{\varepsilon}} \varepsilon} \right] \)
whereas \( \# \mathbb{N}^* \lim_{n \in \mathbb{N}} \frac{1}{\log(\log n)} \) (and similarly using \( \log(\log(\ldots k \ldots (\log n) \ldots)) \).
(iii) Set \( x_n := d\rho^{-n} \) if \( n \in \mathbb{N} \), and \( x_n := \frac{1}{n} \) if \( n \in \mathbb{N} \setminus \mathbb{N} \), then \( \{x_n | n \in \mathbb{N}\} \) is unbounded in \( \mathbb{R}^* \) even if \( \mathbb{N}^* \lim_{n \in \mathbb{N}} x_n = 0 \). Similarly, if \( x_n := d\rho^n \) if \( n \in \mathbb{N} \) and \( x_n := \sin(n) \) otherwise, then \( \lim_{n \in \mathbb{N}} x_n = 0 \) whereas \( \# \mathbb{N}^* \lim_{n \in \mathbb{N}} x_n \).
In general, we can hence only state that convergent hypersequence are eventually bounded:
\[ \exists \mathbb{N}^* \lim_{n \in \mathbb{N}} x_n = 0 \forall M \in \mathbb{R}^* \forall N \in \mathbb{N}^* \forall n \in \mathbb{N}^* : |x_n| \leq M. \]
(iv) If \( k < s \) and \( k > s, \) then \( \mathbb{N}^* \lim_{n \in \mathbb{N}} k^n = s 0 \) and \( \mathbb{N}^* \lim_{n \in \mathbb{N}} k^n = s + \infty, \) hence \( \# \mathbb{N}^* \lim_{n \in \mathbb{N}} k^n \).
(v) Since for \( n \in \mathbb{N} \) we have \((1 - d\rho)^n = 1 - nd\rho + O_n(d\rho^2), \) it is not hard to prove that \((1 - d\rho)^n\) is not a Cauchy sequence. Therefore, \( \# \mathbb{N}^* \lim_{n \in \mathbb{N}} (1 - d\rho)^n, \)
whereas \( \mathbb{N}^* \lim_{n \in \mathbb{N}} (1 - d\rho)^n = 0. \)

A sufficient condition to extend an ordinary sequence \((a_n)_{n \in \mathbb{N}} : \mathbb{N} \to \mathbb{R}^* \) of \( \rho \)-generalized numbers to the whole \( \mathbb{N}^* \) is
\[ \forall n \in \mathbb{N}^* : \left( a_{n(n)} \right)_n \in \mathbb{R}_\rho. \]
(5.2)
In fact, in this way \( a_n := [a_{n(n)}]_n \in \mathbb{R}^* \) for all \( n \in \mathbb{N}^* \), is well-defined because of Lem. 16: on the other hand, we have defined an extension of the old sequence \((a_n)_{n \in \mathbb{N}} \) because if \( n \in \mathbb{N} \), then \( n_{\varepsilon}(n) = n \) for \( \varepsilon \) small and hence \( a_n = [a_n]. \) For example, the sequence of infinities \( a_n = \frac{1}{n} + d\rho^{-1} \) for all \( n \in \mathbb{N} \) can be extended to any \( \mathbb{N}^* \), whereas \( a_n = d\sigma^{-n} \) can be extended as \( a : \mathbb{N} \to \mathbb{R}^* \) only for some gauges \( \rho, \) e.g. if the gauges satisfy
\[ \exists N \in \mathbb{N} \forall n \in \mathbb{N}^* \exists \varepsilon : \sigma_n^\varepsilon \geq \rho_n^N, \]
(5.3)
(e.g. \( \sigma_\varepsilon = \varepsilon \) and \( \rho_\varepsilon = \varepsilon^{1/\varepsilon} \)).

The following result allows us to obtain hyperlimits by proceeding \( \varepsilon \)-wise

**Theorem 37.** Let \( (a_{n,\varepsilon})_{n,\varepsilon} : \mathbb{N} \times I \rightarrow \mathbb{R} \). Assume that for all \( \varepsilon \)
\[
\exists \lim_{n \to +\infty} a_{n,\varepsilon} =: l_\varepsilon, 
\]
and that \( l := [l_\varepsilon] \in \overline{\mathbb{R}} \). Then there exists a gauge \( \sigma \) (not necessarily a monotonic one) such that

(i) There exists \( M \in \mathbb{N} \) and a hypersequence \( (a_n)_{n} : \mathbb{N} \rightarrow \overline{\mathbb{R}} \) such that
\[
a_n = [a_{n(n),\varepsilon}] \in \overline{\mathbb{R}} \text{ for all } n \in \mathbb{N} \geq M;
\]

(ii) \( l = \tilde{\lim}_{n \in \mathbb{N}} a_n \).

**Proof.** From (5.4), we have
\[
\forall \varepsilon \forall q \exists M_{eq} \in \mathbb{N}_{>0} \forall n \geq M_{eq} : \rho_\varepsilon^2 - l_\varepsilon < a_{n,\varepsilon} < \rho_\varepsilon^2 + l_\varepsilon. 
\]

Without loss of generality, we can assume to have recursively chosen \( M_{eq} \) so that
\[
M_{eq} \leq M_{eq+1} \quad \forall \varepsilon \forall q. 
\]

Set \( \bar{M}_\varepsilon := M_\varepsilon + \frac{q}{\varepsilon} > 0 \); since \( \forall q \in \mathbb{N} \forall \varepsilon : q \leq \left\lfloor \frac{1}{\varepsilon} \right\rfloor \), (5.6) implies
\[
\forall q \in \mathbb{N} \forall \varepsilon : \bar{M}_\varepsilon \geq M_{eq}.
\]

If the net \( (\bar{M}_\varepsilon) \) is \( \rho \)-moderate, set \( \sigma := \rho \); otherwise set \( \sigma_\varepsilon := \min \{ \rho_\varepsilon, \bar{M}_\varepsilon^{-1} \} \in (0, 1] \). Thereby, the net \( \sigma_\varepsilon \to 0 \) as \( \varepsilon \to 0^+ \) (note that not necessarily \( \sigma \) is non-decreasing, e.g., \( \lim_{\varepsilon \to 0} \bar{M}_\varepsilon = +\infty \) for all \( k \in \mathbb{N}_{>0} \) and \( \bar{M}_\varepsilon \geq \rho_\varepsilon^{-1} \), i.e., it is a gauge. Now set \( \bar{M} := [\bar{M}_\varepsilon] \in \mathbb{N} \) because our definition of \( \sigma \) yields \( \bar{M}_\varepsilon \leq \sigma_\varepsilon^{-1} \), \( M_q := [M_{eq}] \in \mathbb{N} \) because of (5.7), and
\[
a_n := \begin{cases} [a_{n(n),\varepsilon}] & \text{if } n \geq M_1 \text{ in } \mathbb{N} \geq M_q \\ 1 & \text{otherwise} \end{cases} \forall n \in \mathbb{N}. 
\]

We have to prove that this well-defines a hypersequence \( (a_n)_{\varepsilon} : \mathbb{N} \rightarrow \overline{\mathbb{R}} \). First of all, the sequence is well-defined with respect to the equality in \( \mathbb{N} \) because of Lem. 16. Moreover, setting \( q = 1 \) in (5.5), we get \( \rho_\varepsilon - l_\varepsilon < a_{n,\varepsilon} < \rho_\varepsilon + l_\varepsilon \) for all \( \varepsilon \) and for all \( n \geq M_{eq} \). If \( n \geq M_1 \) in \( \mathbb{N} \), then \( n(n)_{\varepsilon} \geq M_{eq} \) for \( \varepsilon \) small, and hence \( \rho_\varepsilon - l_\varepsilon < a_{n(n),\varepsilon} < \rho_\varepsilon + l_\varepsilon \). This shows that \( a_n \in \overline{\mathbb{R}} \) because we assumed that \( l := [l_\varepsilon] \in \overline{\mathbb{R}} \). Finally, (5.5) and (5.6) yield that if \( n \geq M_q \) then \( n \geq M_1 \) and hence \( |a_n - l| < d\rho^q \).

From the proof it also follows, more generally, that if \( (M_{eq})_{\varepsilon,q} \) satisfies (5.5) and if
\[
\exists(q_\varepsilon) \to +\infty : (M_{\varepsilon,q\varepsilon}) \in \mathbb{R}_\rho,
\]
then we can repeat the proof with \( q_\varepsilon \) instead of \( \left\lfloor \frac{1}{\varepsilon} \right\rfloor \) and setting \( \sigma := \rho \).

5.2. Operations with hyperlimits and inequalities. Thanks to Def. 9 of sharp topology and our notation for \( x < y \) (and of the consequent Lem. 2), some results about hyperlimits can be proved by trivially generalizing classical proofs. For example, if \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) are two convergent hypersequences then their sum \( (x_n + y_n)_{n \in \mathbb{N}} \), product \( (x_n \cdot y_n)_{n \in \mathbb{N}} \) and quotient \( \left( \frac{x_n}{y_n} \right)_{n \in \mathbb{N}} \) (the last one being defined only when \( y_n \) is invertible for all \( n \in \mathbb{N} \)) are convergent hypersequences and
the corresponding hyperlimits are sum, product and quotient of the corresponding hyperlimits.

The following results generalize the classical relations between limits and inequalities.

**Theorem 38.** Let \( x, y, z : ^\sim N \to ^\sim R \) be hypersequences, then we have:

(i) If \( \lim_{n \in ^\sim N} x_n < \lim_{n \in ^\sim N} y_n \), then \( \exists M \in ^\sim N \) such that \( x_n < y_n \) for all \( n \geq M \), \( n \in ^\sim N \).

(ii) If \( x_n \leq y_n \leq z_n \) for all \( n \in ^\sim N \) and \( \lim_{n \in ^\sim N} x_n = \lim_{n \in ^\sim N} z_n =: l \), then \( \exists \lim_{n \in ^\sim N} y_n = l \).

**Proof.** (i) follows from Lem. 2 and the Def. 34 of hyperlimit. For (ii), the proof is analogous to the classical one. In fact, since \( \lim_{n \in ^\sim N} x_n = \lim_{n \in ^\sim N} z_n =: l \) given \( q \in \mathbb{N} \), there exist \( M', M'' \in ^\sim N \) such that \( l - |d\rho^q| < x_n \) and \( z_n < l + |d\rho^q| \) for all \( n > M', n > M'', n \in ^\sim N \), then for \( n > M := M' \lor M'' \), we have \( l - |d\rho^q| < x_n \leq y_n \leq z_n < l + |d\rho^q| \).

**Theorem 39.** Assume that \( C \) is a sharply closed subset of \( ^\sim R \), that \( \exists \lim_{n \in ^\sim N} x_n =: l \) and that \( x_n \) eventually lies in \( C \), i.e. \( \exists N \in ^\sim N \forall n \in ^\sim N_{\geq N} : x_n \in C \). Then also \( l \in C \). In particular, if \( (y_n)_n \) is another hypersequence such that \( \exists \lim_{n \in ^\sim N} y_n =: k \), then \( \exists N \in ^\sim N \forall n \in ^\sim N_{\geq N} : x_n \geq y_n \) implies \( l \geq k \).

**Proof.** A reformulation of the usual proof applies. In fact, let us suppose that \( l \in ^\sim R \setminus C \). Since \( ^\sim R \setminus C \) is sharply open, there is an \( \eta > 0 \), for which \( B_\eta(l) \subseteq ^\sim R \setminus C \). Let \( \bar{n} \in ^\sim N_{\geq N} \) be such that \( |x_{\bar{n}} - l| < \eta \) when \( n > \bar{n} \). Then we have \( x_n \in C \) and \( x_n \in B_\eta(l) \subseteq ^\sim R \setminus C \), a contradiction.

The following result applies to all generalized smooth functions (and hence to all Colombeau generalized functions, see e.g. [12, 13]; see also [1] for a more general class of functions) because of their continuity in the sharp topology.

**Theorem 40.** Suppose that \( f : U \rightarrow ^\sim R \). Then \( f \) is sharply continuous function at \( x = c \) if and only if it is hyper-sequentially continuous, i.e. for any hypersequence \( (x_n)_n \) in \( U \) converging to \( c \), the hypersequence \( (f(x_n))_n \) converges to \( f(c) \), i.e. \( f(\lim_{n \in ^\sim N} x_n) = \lim_{n \in ^\sim N} f(x_n) \).

**Proof.** We only prove that the hyper-sequential continuity is a sufficient condition, because the other implication is a trivial generalization of the classical one. By contradiction, assume that for some \( Q \in \mathbb{N} \)

\[
\forall n \in \mathbb{N} \exists x_n \in U : |x_n - c| < d\rho^n, |f(x_n) - f(c)| >_x d\rho^Q. \tag{5.9}
\]

For \( n \in \mathbb{N} \) set \( \omega_n := n \) and for \( n \in ^\sim N \setminus \mathbb{N} \) set \( \omega_n := \min \{ N \in \mathbb{N} | n \leq d\rho^{-N} \} \) and \( x_n := x_{\omega_n}. \) Then for all \( n \in ^\sim N \), from (5.9) we get \( |x_n - c| < d\rho^{|\omega_n|} \to 0 \) because \( \omega_n \to +\infty \) as \( n \to +\infty \) in \( n \in ^\sim N \). Therefore, \( (x_n)_n \) is a hypersequence of \( U \) that converges to \( c \), which yields \( f(x_n) \to f(c) \), in contradiction with (5.9).

**Example 41.** Let \( \sigma \leq \rho^R \) for some \( R \in \mathbb{R}_{>0} \). The following inequalities hold for all generalized numbers because they also hold for all real numbers:

\[
\ln(x) \leq x, \quad e \left( \frac{n}{e} \right)^n \leq n! \leq e\left( \frac{n}{e} \right)^n. \tag{5.10}
\]
From the first one it follows $0 \leq \lim_{n \to \infty} \frac{\ln(n)}{n} = 2 \ln \sqrt{n} \leq 2 \sqrt{n}$, so that $\lim_{n \in \mathbb{N}} \frac{\ln(n)}{n} = 0$ from Thm. 38 and $\lim_{n \in \mathbb{N}} n^{1/n} = 1$ from Thm. 40 and hence $\lim_{n \in \mathbb{N}} (n!)^{1/n} = +\infty$ by (5.10). Similarly, we have $\lim_{n \in \mathbb{N}} \left(1 + \frac{1}{n}\right)^n = e$ because $n \log \left(1 + \frac{1}{n}\right) = 1 - \frac{1}{2n} + O \left(\frac{1}{n^2}\right) \to 1$ and because of Thm. 40.

A little more involved proof concerns L’Hôpital rule for generalized smooth functions. For the sake of completeness, here we only recall the equivalent definition:

**Definition 42.** Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^d$. We say that $f : X \to Y$ is a generalized smooth function (GSF) if

(i) $f : X \to Y$ is a set-theoretical function.

(ii) There exists a net $(f_x) \in C^\infty([0,1])$ such that for all $[x] \in X$:

(a) $f(x) = [f_x(x)]$

(b) $\forall \alpha \in \mathbb{N}^n : (\partial^\alpha f_x(x))$ is $\rho$-moderate.

For generalized smooth functions lots of results hold: closure with respect to composition, embedding of Schwartz’s distributions, differential calculus, one-dimensional integral calculus using primitives, classical theorems (intermediate value, mean value, Taylor, extreme value, inverse and implicit function), multidimensional integration, Banach fixed point theorem, a Picard-Lindelöf theorem for both ODE and PDE, several results of calculus of variations, etc.

In particular, we have the following (see also [10] for the particular case of Colombeau generalized functions):

**Theorem 43.** Let $U \subseteq \mathbb{R}$ be a sharply open set and let $f : U \to \mathbb{R}$ be a GSF defined by the net of smooth functions $f_x \in C^\infty([0,1])$. Then

(i) There exists an open neighbourhood $T$ of $U \times \{0\}$ and a GSF $R_f : T \to \mathbb{R}$, called the generalized incremental ratio of $f$, such that

$$f(x + h) = f(x) + h \cdot R_f(x,h) \quad \forall (x,h) \in T. \quad (5.11)$$

Moreover $R_f(x,0) = [f'(x)] = f'(x)$ is another GSF and we can hence recursively define $f^{(k)}(x)$.

(ii) Any two generalized incremental ratios of $f$ coincide on the intersection of their domains.

(iii) More generally, for all $k \in \mathbb{N}_{>0}$ there exists an open neighbourhood $T$ of $U \times \{0\}$ and a GSF $R^k_f : T \to \mathbb{R}$, called $k$-th order Taylor ratio of $f$, such that

$$f(x + h) = \sum_{j=0}^{k-1} \frac{f^{(j)}(x)}{j!} h^j + R^k_f(x,h) \cdot h^k \quad \forall (x,h) \in T. \quad (5.12)$$

Any two ratios of $f$ of the same order coincide on the intersection of their domains.

We can now prove the following generalization of one of L’Hôpital rule:

**Theorem 44.** Let $U \subseteq \mathbb{R}$ be a sharply open set $(x_n)_n, (y_n)_n : \mathbb{N} \to U$ be hypersequences converging to $l \in U$ and $m \in U$ respectively and such that

$$\lim_{n \in \mathbb{N}} \frac{x_n - l}{y_n - m} = C \in \mathbb{R}.$$
Definition 45. Let $k \in \mathbb{N}_{>0}$ and $f, g : U \to \mathcal{O}$ be GSF such that for all $n \in \tilde{\mathbb{N}}$ and all $j = 0, \ldots, k - 1$
\[ g^{(j)}(y_n) \in \tilde{\mathbb{R}}^* \]
\[ f^{(j)}(l) = g^{(j)}(m) = 0 \] (5.13)
Then for all $j = 0, \ldots, k - 1$
\[ \exists \tilde{\rho} \lim_{n \in \mathbb{N}} \frac{f^{(j)}(x_n)}{g^{(j)}(y_n)} = C^k \cdot \tilde{\rho}^k \lim_{n \in \mathbb{N}} \frac{f^{(k)}(x_n)}{g^{(k)}(y_n)} \]
Proof. Using (5.12) and (5.13), we can write
\[ \frac{f(x_n)}{g(y_n)} = \frac{\sum_{j=0}^{k-1} f^{(j)}(l) (x_n - l)^j + (x_n - l)^k R^k_f(l, x_n - l)}{\sum_{j=0}^{k-1} g^{(j)}(m) (y_n - m)^j + (y_n - m)^k R^k_f(m, y_n - m)} \]
Since $R^k_f$ and $R^k_g$ are GSF, they are sharply continuous. Therefore, the right hand side of the previous equality tends to $C^k \cdot \frac{R^k_f(l, 0)}{R^k_f(m, 0)} = C^k \cdot \frac{f^{(k)}(l)}{g^{(k)}(m)}$. At the same limit converges the quotient $C^k \cdot \frac{f^{(k)}(x_n)}{g^{(k)}(y_n)}$ because $f^{(k)}$ and $g^{(k)}$ are also GSF and hence they are sharply continuous. The claim for $j = 1, \ldots, k - 1$ follows by applying the conclusion for $j = 0$ with $f^{(j)}$ and $g^{(j)}$ instead of $f$ and $g$. \hfill \Box

Note that for $x_n = y_n$, $l = m$, we have $C = 1$ and we get the usual L'Hôpital rule (formulated using hypersequences). Note that a similar theorem can also be proved without hypersequences and using the same Taylor expansion argument as in the previous proof.

5.3. Cauchy criterion and monotonic hypersequences. In this section, we deal with classical criteria implying the existence of a hyperlimit.

Definition 45. We say that $(x_n)_{n \in \tilde{\mathbb{N}}}$ is a Cauchy hypersequence if
\[ \forall q \in \mathbb{N} \exists M \in \tilde{\mathbb{N}} \forall n, m \in \tilde{\mathbb{N}}_{\geq M} : |x_n - x_m| < \rho^q. \]

Theorem 46. A hypersequence converges if and only if it is a Cauchy hypersequence.

Proof. To prove that the Cauchy criterion is a necessary condition it suffices to consider the inequalities:
\[ |x_n - x_m| \leq |x_n - l| + |x_m - l| < \rho^q + 1 + \rho^q + 1 < \rho^q \]

Vice versa, assume that
\[ \forall q \in \mathbb{N} \exists M_q \in \tilde{\mathbb{N}} \forall n, m \in \tilde{\mathbb{N}}_{\geq M_q} : |x_n - x_m| < \rho^q. \] (5.14)

The idea is to use Cauchy completeness of $\tilde{\mathbb{R}}$. In fact, set $h_1 := M_1$ and $h_{q+1} := M_{q+1} \vee h_q$. We claim that $(x_{h_q})_{q \in \mathbb{N}}$ is a standard Cauchy sequence converging to the same limit of $(x_n)_{n \in \tilde{\mathbb{N}}}$. From (5.14) it follows that $(x_{h_q})_{q \in \mathbb{N}}$ is a standard
A hypersequence converges if and only if
\[ \forall q \in \mathbb{N} \exists M \in \bar{\mathbb{N}} \forall n, m \in \bar{\mathbb{N}} : m \geq n \Rightarrow |x_n - x_m| < d\rho^q. \]

Proof. It suffices to apply the inequality \[ |x_n - x_m| \leq |x_n - x_{n\vee m}| + |x_{n\vee m} - x_m|. \]

The second classical criterion for the existence of a hyperlimit is related to the notion of monotonic hypersequence. The existence of several chains in \( \bar{\mathbb{N}} \) does not allow to arrive at any \( M \in \bar{\mathbb{N}} \) starting from any other lower \( N \in \bar{\mathbb{N}} \) and using the successor operation only a finite number of times. For this reason, the following is the most natural notion of monotonic hypersequence:

**Definition 48.** We say that \( (x_n)_{n \in \bar{\mathbb{N}}} \) is a non-decreasing (or increasing) hypersequence if
\[ \forall n, m \in \bar{\mathbb{N}} : n \geq m \Rightarrow x_n \geq x_m. \]

Similarly, we can define the notion of non-increasing (decreasing) hypersequence.

**Theorem 49.** Let \( (x_n) : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{R}} \) be a non-decreasing hypersequence. Then
\[ \exists \lim_{n \in \bar{\mathbb{N}}} x_n \iff \exists \sup \left\{ x_n \mid n \in \bar{\mathbb{N}} \right\}, \]

and in that case they are equal. In particular, if \( \left\{ x_n \mid n \in \bar{\mathbb{N}} \right\} \) is complete from above for all the upper bounds, then
\[ \exists \lim_{n \in \bar{\mathbb{N}}} x_n \iff \exists U \in \bar{\mathbb{R}} \forall n \in \bar{\mathbb{N}} : x_n \leq U. \]

Proof. Assume that \( (x_n)_{n \in \bar{\mathbb{N}}} \) converges to \( l \) and set \( S := \{ x_n \mid n \in \bar{\mathbb{N}} \} \), we will show that \( l = \sup(S) \). Now, using Def. 34, we have that \( \forall n \in \bar{\mathbb{N}} : x_n < l + d\rho^q \) for some \( N \in \bar{\mathbb{N}} \). But from Def. 48 \( \forall n \in \bar{\mathbb{N}} : x_n \leq x_{n\vee N} < l + d\rho^q \). Therefore \( x_n \leq l + d\rho^q \) for all \( n \in \bar{\mathbb{N}} \), and the conclusion \( x_n \leq l \) follows since \( q \in \mathbb{N} \) is arbitrary. Finally, from Def. 34 of hyperlimit, for all \( q \in \mathbb{N} \) we have the existence of \( L \in \bar{\mathbb{N}} \) such that \( l - d\rho^q < x_L \in S \) which completes the necessity part of the proof. Now, assume that \( \exists \sup(S) := l \). We have to prove that \( \lim_{n \in \bar{\mathbb{N}}} x_n = l \). In fact, using Rem. (i), we get
\[ \forall q \in \mathbb{N} \exists x_N \in S : l - d\rho^q < x_N, \]
and \( x_N \leq x_n \leq l < l + d\rho^q \) for all \( n \in \bar{\mathbb{N}} \) by Def. 48 of monotonicity. That is, \( |l - x_n| = x_n - l < d\rho^q \).
\[ \square \]
Example 50. The hypersequence \( x_n := d \rho^{1/n} \) is non-decreasing. Assume that \( (x_n)_n \) converges to \( l \) and that \( \sigma \leq \rho^R \) for some \( R \in \mathbb{R}_{>0} \). Since \( x_n \geq d \rho \), by Thm. 39, we get \( l \geq d \rho \). Therefore, applying the logarithm and the exponential functions, from Thm. 40 we obtain that \( l = 1 \) because from \( \sigma \leq \rho^R \) it follows that 
\[
\rho \left( \log(d \rho) \right) = 0.
\]
But this is impossible since \( 1 \approx 1 - d \rho \neq d \rho^{1/n} \). Therefore, \( \not\exists \sup \left\{ d \rho^{1/n} \mid n \in \bar{\mathbb{N}}_{>0} \right\} \) and this set is also not complete from above.

6. Limit superior and inferior

We have two possibilities to define the notions of limit superior and inferior in a non-Archimedean setting such as \( \bar{\mathbb{R}} \): the first one is to assume that both \( \alpha_m := \sup \{ x_n \mid n \in \bar{\mathbb{N}}_{\geq m} \} \) and \( \inf \{ \alpha_m \mid m \in \bar{\mathbb{N}} \} \) exist (the former for all \( m \in \bar{\mathbb{N}} \)); the second possibility is to use inequalities to avoid the use of supremum and infimum. In fact, in the real case we have \( \forall n \geq m : x_n \leq \tau + \varepsilon \) if and only if
\[
\forall n \geq m : x_n \leq \tau + \varepsilon \leq x_n.
\]

Definition 51. Let \( (x_n)_n : \bar{\mathbb{N}} \to \bar{\mathbb{R}} \) be a hypersequence, then we say that \( \tau \in \bar{\mathbb{R}} \) is the limit superior of \( (x_n)_n \) if
\[
(i) \quad \forall q \in \bar{\mathbb{N}} \exists \bar{N} \in \bar{\mathbb{N}} \forall n \geq N : \ x_n \leq \tau + d \rho^q;
(ii) \quad \forall q \in \bar{\mathbb{N}} \forall \bar{N} \exists \bar{n} \geq \bar{N} : \ \tau - d \rho^q \leq x_n.
\]
Similarly, we say that \( \sigma \in \bar{\mathbb{R}} \) is the limit inferior of \( (x_n)_n \) if
\[
(iii) \quad \forall q \in \bar{\mathbb{N}} \exists \bar{N} \in \bar{\mathbb{N}} \forall n \geq N : \ x_n \geq \sigma - d \rho^q;
(iv) \quad \forall q \in \bar{\mathbb{N}} \forall \bar{N} \exists \bar{n} \geq \bar{N} : \ \sigma + d \rho^q \geq x_n.
\]

We have the following results (clearly, dual results hold for the limit inferior):

Theorem 52. Let \( (x_n)_n, (y_n)_n : \bar{\mathbb{N}} \to \bar{\mathbb{R}} \) be hypersequences, then
\[
(i) \quad \text{There exists at most one limit superior and at most one limit inferior. They are denoted with } \bar{\limsup}_{n \in \bar{\mathbb{N}}} x_n \text{ and } \bar{\liminf}_{n \in \bar{\mathbb{N}}} x_n.
(ii) \quad \text{If } \exists \sup \left\{ x_n \mid n \in \bar{\mathbb{N}}_{\geq m} \right\} =: \alpha_m \text{ for all } m \in \bar{\mathbb{N}}, \text{ then } \exists \bar{\limsup}_{n \in \bar{\mathbb{N}}} x_n \text{ if and only if } \exists \inf \left\{ \alpha_m \mid m \in \bar{\mathbb{N}} \right\}, \text{ and in that case}
\]
\[
\bar{\limsup}_{n \in \bar{\mathbb{N}}} x_n = \bar{\limsup}_{m \in \bar{\mathbb{N}}} \alpha_m = \inf \left\{ \alpha_m \mid m \in \bar{\mathbb{N}} \right\}.
\]
\[
(iii) \quad \bar{\limsup}_{n \in \bar{\mathbb{N}}} (-x_n) = -\bar{\liminf}_{n \in \bar{\mathbb{N}}} x_n \text{ in the sense that if one of them exists, then also the other one exists and in that case they are equal.}
(iv) \quad \exists \lim_{n \in \bar{\mathbb{N}}} x_n \text{ if and only if } \exists \bar{\limsup}_{n \in \bar{\mathbb{N}}} x_n = \bar{\liminf}_{n \in \bar{\mathbb{N}}} x_n.
(v) \quad \text{If } \exists \lim_{n \in \bar{\mathbb{N}}} x_n, \bar{\limsup}_{n \in \bar{\mathbb{N}}} y_n, \lim_{n \in \bar{\mathbb{N}}} (x_n + y_n), \text{ then}
\]
\[
\bar{\lim}_{n \in \bar{\mathbb{N}}} (x_n + y_n) \leq \bar{\limsup}_{n \in \bar{\mathbb{N}}} x_n + \bar{\limsup}_{n \in \bar{\mathbb{N}}} y_n.
\]
In particular, if \( \forall N \in \bar{\mathbb{N}} \forall \bar{n} \geq N \exists \bar{n} \geq N : x_n + y_n \leq x_n + y_n \), then the existence of the single limit superiors implies the existence of the limit superior of the sum.
(vi) If \( x_n, \ y_n \geq 0 \) for all \( n \in \mathbb{N} \) and if \( \exists \lim_{n \to \mathbb{N}} x_n, \ \limsup_{n \to \mathbb{N}} y_n, \ \limsup_{n \to \mathbb{N}} (x_n \cdot y_n) \), then
\[
\limsup_{n \to \mathbb{N}} (x_n \cdot y_n) \leq \limsup_{n \to \mathbb{N}} x_n \cdot \limsup_{n \to \mathbb{N}} y_n.
\]
In particular, if \( \forall N \in \mathbb{N}, \exists n \geq N : x_n \cdot y_n \leq x_n \cdot y_n, \) then the existence of the single limit superiors implies the existence of the limit superior of the product.

(vii) If \( \exists \limsup_{n \to \mathbb{N}} x_n =: \ell \), then there exists a sequence \( (\vec{n}_q)_{q \in \mathbb{N}} \) of \( \mathbb{N} \) such that
(a) \( \vec{n}_{q+1} > \vec{n}_q \) for all \( q \in \mathbb{N} \);
(b) \( \lim_{q \to +\infty} \vec{n}_q = +\infty \) in \( \mathbb{N} \);
(c) \( \exists \lim_{q \to +\infty} x_{\vec{n}_q} = \ell \).

(viii) Assume to have a sequence \( (\vec{n}_q)_{q \in \mathbb{N}} \) satisfying the previous conditions (a), (b), (c) and
\[
\forall n \in \mathbb{N}, \exists p \in \mathbb{N} : \vec{n}_p \geq n, \ x_n \leq x_{\vec{n}_p}.
\]
Then \( \exists \limsup_{n \to \mathbb{N}} x_n =: \ell \).

Proof. (i): Let \( \ell_1, \ell_2 \) be both limit superior of \( (x_n)_n \). Based on Lem. 6.(iii), without loss of generality we can assume that \( \ell_1 \leq \ell_2 \). According to Lem. 2, there exists \( m \in \mathbb{N} \) such that \( \ell_1 + d\rho^m \leq \ell_2 \). Take \( q_1, q_2 \) large enough so that \( d\rho^{q_1} + d\rho^{q_2} < d\rho^m \). Using the last two inequalities, we obtain
\[
\ell_1 + d\rho^m < \ell_2 - d\rho^{q_2}.
\]
Using Def. 51.(i), we can find \( N_1 \in \mathbb{N} \) such that
\[
\forall n \in [\vec{n}, N_1] : x_n \leq \ell_1 + d\rho^m.
\]
Using Def. 51.(ii) with \( q = q_1 \) and \( N = N_1 \), we get
\[
\exists n \in [\vec{n}, N_1] : \ell_2 - d\rho^{q_2} \leq x_n.
\]
We now use (6.2), (6.4) and (6.3) for \( n = \vec{n} \) and we obtain \( x_n \leq \ell_1 + d\rho^{q_1} < \ell_2 - d\rho^{q_2} \leq x_n \), which is a contradiction.

(ii): Lem. 24.(iii) implies that \( (\alpha_m)_m \) is non-increasing. Therefore, we have \( \lim_{n \to \mathbb{N}} \alpha_m = \inf \{ \alpha_m \mid m \in \mathbb{N} \} \) if these terms exist from Thm. 49. But Cor. 32 and Def. 51.(i) imply \( \alpha_m \leq \ell + d\rho^q \). Finally, Def. 51.(ii) yields \( \ell - d\rho^q \leq \ell \leq \alpha_m \), which proves that \( \exists \lim_{n \to \mathbb{N}} \alpha_m = \limsup_{n \to \mathbb{N}} x_m = \ell \).

(iii): Directly from Def. 51.

(iv): Assume that hyperlimit superior and inferior exist and are equal to \( \ell \). From Def. 51.(i) and Def. 51.(iii) we get \( \ell - d\rho^q \leq x_n \leq \ell + d\rho^q \) for all \( n \geq N \). Vice versa, assume that the hyperlimit exists and equals \( \ell \), so that \( \ell - d\rho^q \leq x_n \leq \ell + d\rho^q \) for all \( n \geq N \). Then both Def. 51.(i) and Def. 51.(iii) trivially hold. Finally, Def. 51.(ii) and Def. 51.(iv) hold taking e.g. \( \vec{n} = N \).

(v): Setting
\[
\ell := \limsup_{n \to \mathbb{N}} x_n \\
j := \limsup_{n \to \mathbb{N}} y_n \\
l := \limsup_{n \to \mathbb{N}} (x_n + y_n),
\]
from Def. 51 we get \( l - d\rho^q \leq x_{\bar{n}} + y_{\bar{n}} \leq l + j + 2d\rho^q \), which implies \( l \leq l + j \) for \( q \to +\infty \). Adding Def. 51.(ii) we obtain \( l + j - 2d\rho^q \leq x_{\bar{n}} + y_{\bar{n}} \) for some \( \bar{n} \), \( \bar{n} \geq N \in \mathbb{N} \). Therefore, if \( x_{\bar{n}} + y_{\bar{n}} \leq x_n + y_n \) for some \( n \geq N \), this yields the second claim. Similarly, one can prove (vi).

(vii): From Def. 51.(i), choose an \( N_q = N \) for each \( q \in \mathbb{N} \), i.e.
\[
\forall q \in \mathbb{N} \exists N_q \in \mathbb{N} \forall n \geq N_q : x_n \leq l + d\rho^q.
\] (6.5)

Applying Def. 51.(ii) with \( q > 0 \) and \( N = N_q \lor (\bar{n}_{q-1} + 1) \lor [\text{int}(\sigma^{-q})] \in \mathbb{N} \), we get the existence of \( \bar{n}_q \geq N_q \) such that both (a) and (b) hold and \( l - d\rho^q \leq x_{\bar{n}_q} \). Thereby, from (6.5) we also get (c).

(viii): Write (c) as
\[
\forall q \in \mathbb{N} \exists Q_q \in \mathbb{N} \forall p \in \mathbb{N} \geq Q_q : \lceil \tau - d\rho^p \rceil \leq x_{\bar{n}_p} \leq \lceil \tau + d\rho^p \rceil.
\] (6.6)

Set \( N := \bar{n}_{Q_q} \in \mathbb{N} \). For \( n \geq N \), from (6.1) we get the existence of \( p \in \mathbb{N} \) such that \( \bar{n}_p \geq n \) and \( x_n \leq x_{\bar{n}_p} \). Thereby, \( \bar{n}_p \geq \bar{n}_{Q_q} \) and hence \( p \geq Q_q \) because of (a) and thus \( x_n \leq x_{\bar{n}_p} \leq \tau + d\rho^p \). Finally, condition (ii) of Def. 51 follows from (6.6) and (b).

It remains an open problem to show an example that proves as necessary the assumption of Thm. 52.(ii), i.e. that the previous definition of limit superior and inferior is strictly more general than the simple transposition of the classical one.

Example 53.

(i) Directly from Def. 51, we have that
\[
\limsup_{n \in \mathbb{N}} (-1)^n = 1, \quad \liminf_{n \in \mathbb{N}} (-1)^n = -1
\]

(ii) Let \( \mu \in \mathbb{R}^\mathbb{R} \) be such that \( \mu|_L = 1 \) and \( \mu|_{L^c} = -1 \), where \( L, L^c \subset \mathbb{R} I \). Then \( \mu^n \leq 1 \) and \( 1 - d\rho^q \leq \mu^n \) if \( n(\bar{n}) \) is even for all \( \varepsilon \) small. Therefore \( \limsup_{\tau, n \in \mathbb{N}} \mu^n = 1, \sup_{n \geq m} \mu^n = 1 \), whereas \( \limsup_{\tau, n \in \mathbb{N}} \mu^n \).

(iii) From (vii) and (viii) of Thm. 52 it follows that for an increasing hypersequence \((x_n)_{\infty} \), \( \limsup_{\tau, n \in \mathbb{N}} x_n \) if and only if \( \limsup_{\tau, n \in \mathbb{N}} x_n \). Therefore, example 50 implies that \( \limsup_{\tau, n \in \mathbb{N}} d\rho^{l/n} \).

7. Conclusions

In this work we showed how to deal with several deficiencies of the ring of Robinson-Colombeau generalized numbers \( \mathbb{R}^\mathbb{R} \): trichotomy law for the order relations \( \leq \) and \( < \), existence of supremum and infimum and limits of sequences with a topology generated by infinitesimal radii. In each case, we obtain a faithful generalization of the classical case of real numbers. We think that some of the ideas we presented in this article can inspire similar works in other non-Archimedean settings such as (constructive) nonstandard analysis, p-adic analysis, the Levi-Civita field, surreal numbers, etc. Clearly, the notions introduced here open the possibility to extend classical proofs in dealing with series, analytic generalized functions, sigma-additivity in integration of generalized functions, non-Archimedean functional analysis, just to mention a few.
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