Rigged Hilbert space of the free coherent states
and $p$–adic numbers

S.V.Kozyrev

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Abstract

Rigged Hilbert space of the free coherent states is investigated. We prove
that this rigged Hilbert space is isomorphous to the space of generalized func-
tions over $p$–adic disk. We discuss the relation of the described isomorphism
of rigged Hilbert spaces and noncommutative geometry and show, that the
considered example realises the isomorphism of the noncommutative line and
$p$–adic disk.

In the present paper, continuing the investigations of [1], [2], we investigate the
free coherent states (or shortly FCS), which are (unbounded) eigenvectors of the
linear combination of annihilators in the free Fock space. In [1], [2] it was shown
that the space of the free coherent states is highly degenerate for the fixed eigenvalue
$\lambda$ (and infinite dimensional), and this degeneracy is naturally described by the space
$D'(\mathbb{Z}_p)$ of generalized functions on $p$–adic disk ($p$ is a number of independent creators
in the free Fock space). In the present paper we reformulate the results of [1], [2]
using the language of rigged Hilbert spaces and propose an interpretation of
the relation between the free coherent states and $p$–adics using noncommutative
geometry. We speculate that the isomorphism between the space of FCS and the
space of generalized function on $p$–adic disk in the language of noncommutative
geometry reduces to the isomorphism between the noncommutative (or quantum)
line and $p$–adic disk.

$p$–Adic mathematical physics studies the problems of mathematical physics with
the help of $p$–adic analysis. $p$–Adic mathematical physics was studied in [3]–[15].
For instance in the book [3] the analysis of $p$–adic pseudodifferential operators was
developed. In [4] a $p$–adic approach in the string theory was proposed. In [5] a
theory of $p$–adic valued distributions was investigated. In [6], [7] it was shown
that the Parisi matrix used in the replica method is equivalent, in the simplest case,
to a $p$–adic pseudodifferential operator. In [8] it was shown that the wavelet basis
in $L^2(R)$ after the $p$–adic change of variable (the continuous map of $p$–adic numbers
onto real numbers conserving the measure) maps onto the basis of eigenvectors of the
Vladimirov operator of $p$–adic fractional derivation. In [9] a procedure to generate
the ultrametric space used in the replica approach was proposed.
The Free (or quantum Boltzmann) Fock space has been considered in some works on quantum chromodynamics [16]–[18] and noncommutative probability [19]–[25].

The free Fock space \( \mathcal{F} \) over a Hilbert space \( \mathcal{H} \) is the completion of the tensor algebra
\[
\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n.
\]
Creation and annihilation operators act as follows:
\[
A^\dagger(f)f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n; \quad f,f_i \in \mathcal{H}
\]
\[
A(f)f_1 \otimes \ldots \otimes f_n = \langle f,f_1 \rangle f_2 \otimes \ldots \otimes f_n; \quad f,f_i \in \mathcal{H}
\]
where \( \langle \cdot, \cdot \rangle \) is the scalar product in the Hilbert space \( \mathcal{H} \). Scalar product in the free Fock space (which we also denote \( \langle \cdot, \cdot \rangle \)) is defined in the standard way.

In the case when \( \mathcal{H} \) is the \( p \)-dimensional complex Euclidean space we have \( p \) creation operators \( A^\dagger_i, i = 0, \ldots, p-1 \); \( p \) annihilation operators \( A_i, i = 0, \ldots, p-1 \) with the relations
\[
A_i A^\dagger_j = \delta_{ij}.
\]
and the vacuum vector \( \Omega \) in the free Fock space satisfies
\[
A_i \Omega = 0.
\]

The free coherent states (or shortly FCS) were introduced in [1], [2] as the formal eigenvectors of the annihilation operator \( A = \sum_{i=0}^{p-1} A_i \) in the free Fock space \( \mathcal{F} \) for some eigenvalue \( \lambda \),
\[
A \Psi = \lambda \Psi.
\]
The formal solution of (3) is
\[
\Psi = \sum_{I} \lambda^{|I|} \Psi_I A^\dagger_I \Omega.
\]
Here the multiindex \( I = i_0 \ldots i_{k-1}, i_j \in \{0, \ldots, p-1\} \) and
\[
A^\dagger_I = A^\dagger_{i_k-1} \ldots A^\dagger_{i_0}
\]
\( \Psi_I \) are complex numbers which satisfy
\[
\Psi_I = \sum_{i=0}^{p-1} \Psi_{Ii}.
\]
The summation in the formula (4) runs on all sequences \( I \) with finite length. The length of the sequence \( I \) is denoted by \( |I| \) (for instance in the formula above \( |I| = k \)). The formal series (4) defines the functional with a dense domain in the free Fock space. For instance the domain of each free coherent state for \( \lambda \in (0, \sqrt{p}) \) contains the dense space \( X \) introduced below.

We define the free coherent state \( X_I \) of the form
\[
X_I = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^k \lambda^{|I|} A^\dagger_I \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I|} A^\dagger_I \Omega
\]
The sum on \( l \) in fact contains \(|I|\) terms. For \( \lambda \in (0, \sqrt{p}) \) the coherent state \( X_I \) lies in the Hilbert space (the correspondent functional is bounded).

We denote by \( X \) the linear span of free coherent states of the form (7) and by \( X' \) we denote the space of all the free coherent states (given by (4)).

Definition 1 was proposed and lemmas 2 and 7, corollary 4 and example 6 below were proven in [1], [2].

**Definition 1.** We define the renormalized pairing of the spaces \( X \) and \( X' \) as follows:

\[
(\Psi, \Phi) = \lim_{\lambda \to \sqrt{p} - 0} \left( 1 - \frac{\lambda^2}{p} \right) \langle \Psi, \Phi \rangle
\]

(8)

Here \( \Psi \in X' \), \( \Phi \in X \).

Note that the coherent states \( \Psi \) and \( \Phi \) defined by (4), (7) depend on \( \lambda \) and the product \( (\Psi, \Phi) \) does not.

The correctness of the definition above is justified by the following lemma.

**Lemma 2.** Vectors \( X_I \in X \) lie in the domain of the functional \( \Psi \) for \( \lambda \in (0, \sqrt{p}) \) for an arbitrary free coherent state \( \Psi \) defined by (4). Moreover, the following limit exists and is equal to

\[
(\Psi, X_I) = \lim_{\lambda \to \sqrt{p} - 0} \left( 1 - \frac{\lambda^2}{p} \right) \langle \Psi, X_I \rangle = p^{|I|} \Psi_I
\]

(9)

**Proof** The pairing of the functional \( \Psi \in X' \) given by (4) and the state (7) is given by the following series

\[
\langle \Psi, X_I \rangle = \sum_{k=0}^{\infty} \lambda^{2k} \langle \Psi^k, X_I^k \rangle
\]

(10)

Here \( \Psi^k \) and \( X_I^k \) are the coefficients of \( \lambda^k \) in the series for \( \Psi \) and \( X_I \). \( \Psi^k \) is defined by the formula

\[
\Psi^k = \sum_{|J|=k} \Psi_J A_J^\dagger \Omega;
\]

and \( X_I^k \) for \( k > |I| \) have the form

\[
X_I^k = \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^{k-|I|} A_I^\dagger \Omega.
\]

We obtain for \( k > 0 \)

\[
\langle \Psi^{|I|+k}, X_I^{|I|+k} \rangle = \langle X_I^{|I|+k}, \Psi^{|I|+k} \rangle^* = \langle X_I^{|I|+k-1}, \frac{1}{p} \sum_{i=0}^{p-1} A_i \Psi^{|I|+k} \rangle^* = \langle X_I^{|I|+k-1}, \frac{1}{p} \sum_{i=0}^{p-1} A_i \sum_{|J|=|I|+k} \Psi_J A_J^\dagger \Omega \rangle^*
\]

\[
= \langle X_I^{|I|+k-1}, \frac{1}{p} \sum_{i=0}^{p-1} A_i \sum_{|J|=|I|+k} \Psi_J A_J^\dagger \Omega \rangle^*
\]
This implies
\[ \frac{1}{p} \sum_{i=0}^{p-1} A_i \sum_{|J|=|I|+k} \Psi_J A_J^\dagger \Omega = \sum_{|J|=|I|+k-1} \frac{1}{p} \sum_{i=0}^{p-1} \Psi_{J_i} A_{J_i}^\dagger \Omega = \frac{1}{p} \Psi_{|I|+k-1}. \]

Therefore
\[ \langle \Psi_{|I|+k}, X_{I}^{|I|+k} \rangle = \frac{1}{p} \langle \Psi_{|I|+k-1}, X_{I}^{|I|+k-1} \rangle = p^{-k} \langle \Psi_{|I|}, X_{I}^{|I|} \rangle = p^{-k} \Psi_{I} \quad (11) \]

By (11) the series (10) takes the form
\[ \langle \Psi, X_{I} \rangle = \sum_{k=0}^{|I|} \lambda^{2k} \langle \Psi^k, X_{I}^{k} \rangle + \sum_{k=|I|+1}^{\infty} \lambda^{2k} p^{|I|-k} \Psi_{I} \]

Since for \( \frac{\lambda^2}{p} < 1 \) the series above is majorized by the geometric series, the series (10) converges and the corresponding renormalized pairing takes the form
\[ \langle \Psi, X_{I} \rangle = p^{|I|} \Psi_{I} \]

This finishes the proof of the lemma.

Define the characteristic functions of \( p \)-adic disks
\[ \theta_k(x - x_0) = \theta(p^k |x - x_0|_p); \quad \theta(t) = 0, t > 1; \quad \theta(t) = 1, t \leq 1. \quad (12) \]

Here \( x, x_0 \in Z_p \) lie in the ring of integer \( p \)-adic numbers and the function \( \theta_k(x - x_0) \) equals to 1 on the disk \( D(x_0, p^{-k}) \) of radius \( p^{-k} \) with the center in \( x_0 \) and equals to 0 outside this disk.

Identify multiindex \( I = i_0 \ldots i_k \) with \( p \)-adic number \( I = \sum_{j=0}^{k} i_j p^j \).

The following lemma shows the relation between the renormalized pairing of the free coherent states and the scalar product of square integrable functions on \( p \)-adic disk.

**Lemma 3.** The space \( X \) with the renormalized scalar product is isomorphous, as a Euclidean space, to the space \( D(Z_p) \) of test functions on \( p \)-adic disk with the scalar product in \( L^2 \) with the isomorphism given by
\[ \phi : X \to D(Z_p) \]
\[ \phi : X_{I} \mapsto p^{|I|} \theta_{|I|} (x - I) \quad (14) \]

**Proof** The space \( X \) is a filtrated space with the filtration
\[ X = \bigcup_k X^{(k)}, \quad X^{(k+1)} \supset X^{(k)} \]
where $X^{(k)}$ is generated by $X_I$ with $|I| = k$. The spaces $X^{(k)}$ are finite dimensional.

Analogously, for $D(Z_p)$ there is the filtration by the finite dimensional subspaces

$$D(Z_p) = \bigcup_k D_k(Z_p), \quad D_{k+1}(Z_p) \supset D_k(Z_p)$$

where $D_k(Z_p)$ is generated by $\theta_k(x - I)$ with $|I| = k$.

To prove the lemma it is enough to prove that $\phi$ is the isomorphism for the maps

$$\phi : X^{(k)} \to D_k(Z_p)$$

By the definition $D_k(Z_p)$ is a finite dimensional Euclidean space, generated by the functions $\theta_{|I|}(x - I)$ with $|I| = k$. The functions $\theta_{|I|}(x - I)$ obey the relation

$$\sum_{i=0}^{p-1} \theta_{|I|}(x - I) = \theta_{|I|}(x - I)$$

and have the scalar products

$$(\theta_k(x - I), \theta_k(x - J)) = p^{-|I|}\delta_{IJ}$$

The space $X^{(k)}$ is generated by $X_I$ with $|I| = k$. Vectors $X_I$ obey the following relation

$$X_I = p^{-1} \sum_{j=0}^{p-1} X_{I_j}$$

which we derive from (7) as follows:

$$X_I = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^k \lambda^{|I|} A_I^\dagger \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I|} A_I^\dagger \Omega =$$

$$= p^{-1} \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^k \lambda^{|I_j|} A_{I_j}^\dagger \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I|} A_I^\dagger \Omega =$$

$$= p^{-1} \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^k \lambda^{|I_j|} A_{I_j}^\dagger \Omega + p^{-1} \sum_{j=0}^{p-1} \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I_j|} A_{I_j}^\dagger \Omega =$$

$$= p^{-1} \sum_{j=0}^{p-1} X_{I_j}$$

Vectors $X_I$ have the following scalar products for $|I| = |J|$:

$$(X_I, X_J) = p^{|I|}\delta_{IJ}$$

Comparing (15), (16) with (17), (18) we obtain the statement of the lemma.

As a corollary, we obtain the following:
Corollary 4. The renormalized scalar product \((X_I, X_J)\) of the free coherent states \(X_I, X_J \in X\) equals to the integral over \(p\)-adic disk with respect to the Haar measure

\[
(X_I, X_J) = \mathbb{p}^{|I|+|J|} \int_{Z_p} \theta_{|I|}(x-I) \theta_{|J|}(x-J) \mu(dx) = \\
\left( \frac{\theta_{|I|}(x-I)}{||\theta_{|I|}(x-I)||^2}, \frac{\theta_{|J|}(x-J)}{||\theta_{|J|}(x-J)||^2} \right)_{L^2} \quad (19)
\]

Lemma 5. The isomorphism \(\phi\) induces the injection \(\phi'\) of the space \(X'\) of free coherent states into the space \(D'(Z_p)\) of generalized functions over \(p\)-adic disk:

\[
\phi'({\Psi}) = {\Psi} \circ \phi^{-1}
\]

Proof. We have to prove that for an arbitrary non–zero free coherent state \(\Psi\) the functional \(\Psi \circ \phi^{-1}\) is a non–zero continuous linear functional over \(D(Z_p)\).

By (9) and (14) we have

\[
(\Psi \circ \phi^{-1}, \theta_{|I|}(x-I)) = \Psi_I \quad (20)
\]

Since for any non–zero coherent state \(\Psi\) at least one coefficient \(\Psi_I\) is non–zero, this proves that the functional \(\Psi \circ \phi^{-1}\) is non–zero.

Topology in \(D(Z_p)\) is defined as follows, see for instance [3]. The space \(D(Z_p)\) is

\[
D(Z_p) = \bigcup_k D_k(Z_p)
\]

where \(D_k(Z_p)\) is a linear span of \(\theta_j(x-I), j \leq k\) and the sequence \(\{\phi_j\}\) in \(D(Z_p)\) is convergent when all \(\phi_j \in D_k(Z_p)\) for all \(j\) and some \(k\) and the functions \(\phi_j \to 0\) homogeneously.

Each of the spaces \(D_k(Z_p)\) is the normed space (with the \(C\)-norm, equal to the supremum over the \(p\)-adic disk of the modulus of the function).

Formula (9) implies that \(\Psi \circ \phi^{-1}\) is a bounded functional on \(D_k(Z_p)\) with the norm

\[
\|\Psi \circ \phi^{-1}\|_{D_k(Z_p)} = \max_{|I| \leq k} |\Psi_I|
\]

Therefore the functional \(\Psi \circ \phi^{-1}\) is a continuous functional on \(D(Z_p)\), which finishes the proof of the lemma.

The important example of a generalized function is the \(\delta\)-function. Let us introduce a coherent state that corresponds to the \(\delta\)-function. Consider an infinite sequence \(I = i_0 \ldots i_k \ldots, i_j = 0, \ldots, p-1\) and the corresponding \(p\)-adic number \(I = \sum_{k=0}^{\infty} i_k p^k\). Let us denote \(I_k = i_0 \ldots i_{k-1}\). We introduce the free coherent state \(\delta_I\) of the form

\[
\delta_I = \sum_{k=0}^{\infty} \lambda^k A^\dagger_{I_k} \Omega.
\]
Example 6. The map $\phi'$ maps the free coherent state $\delta_I$ onto the $\delta$–function:

$$\phi'(\delta_I) = \delta(x - I)$$

Proof Follows from (9) and (19).

Lemma 7. The injection of the space $X$ of free coherent states into the space $D'(\mathbb{Z}_p)$ of generalized functions over $p$–adic disk constructed in lemma 5 is surjective (and therefore is an isomorphism of linear spaces).

Proof To prove the lemma it is sufficient to construct the free coherent state which, applied to the inverse image of the indicator of an arbitrary $p$–adic disk in $\mathbb{Z}_p$, will give an arbitrary complex number (arbitrary up to the relation which follows from the linearity of the functional and the fact that the indicator of the disk is equal to the sum of the indicators of the subdisks).

This follows from the formula (20):

$$(\Psi, \phi^{-1}|_{I}(x - I)) = \Psi_I$$

where $\Psi_I$ is an arbitrary set of complex numbers satisfying the relation (6):

$$\Psi_I = \sum_{i=0}^{p-1} \Psi_{I_i}$$

which is exactly the property of linearity applied to the indicators of the subdisks:

$$(\Psi, \phi^{-1}|_{I}(x - I)) = \sum_{i=0}^{p-1} (\Psi, \phi^{-1}|_{I_i}(x - I_i))$$

This finishes the proof of the lemma.

Corollary 8. The maps $\phi$, $\phi'$ (which are the isomorphisms of the linear spaces) allow to map the topology of $D(\mathbb{Z}_p)$ and $D'(\mathbb{Z}_p)$ onto $X$ and $X'$ correspondingly. This procedure makes $\phi$ and $\phi'$ the isomorphisms of vector topological spaces.

Lemmas 3, 5, 7 suggest the following definition.

Definition 9. We denote $\hat{F}$ the completion of the space $X$ of the free coherent states with respect to the norm defined by the renormalized scalar product.

The space $\hat{F}$ is a Hilbert space with respect to the renormalized scalar product.

Lemma 10. The Hilbert space $\hat{F}$ lies in the space of the free coherent states $X'$:

$$\hat{F} \subset X'$$
Proof  For $\Psi \in \tilde{F}$, $\Psi = \lim_{n \to \infty} \Psi^{(n)}$, $\Psi^{(n)} \in X$ consider the product $(\Psi, X_I)$ which we denote:

$$(\Psi, X_I) = p^{|I|} \Psi_I$$

Taking into account (17) we have

$$(\Psi, X_I) = p^{-1} \sum_{i=0}^{p-1} (\Psi, X_{I_i})$$

which implies that $\Psi_I$ satisfies (3):

$$\Psi_I = \sum_{i=0}^{p-1} \Psi_{I_i}$$

Therefore

$$\tilde{\Psi} = \sum_I \lambda^{|I|} \Psi_I A_I^\dagger \Omega$$

is the free coherent state in $X'$ with

$$(\tilde{\Psi}, X_I) = (\Psi, X_I) = p^{|I|} \Psi_I$$

which implies that $\tilde{\Psi} = \Psi$. This finishes the proof of the lemma.

Lemma 11. The map

$$j : X \to \tilde{F}$$

in (21) is a continuous injection with the dense range.

Proof  Formula (3) implies that a non–zero coherent state in $X$ has a non–zero norm in $\tilde{F}$. Therefore $j$ is an injection.

Assume that $\{\Phi^{(n)}\}$ is a convergent (in the topology induced from $D(Z_p)$) sequence in $X$. To prove the continuity of the injection $j$ we have to prove that the sequence $\{\Phi^{(n)}\}$ is fundamental in $\tilde{F}$. By definition of the topology in $D(Z_p)$ there exists $k$ such that $\{\Phi^{(n)}\} \subset X^{(k)}$. Therefore each $\Phi^{(n)}$ is a finite linear combination of the functions $X_I$, $|I| = k$, and the convergence of the sequence $\{\Phi^{(n)}\}$ reduces to the convergence of a finite number of coefficients in the decomposition over $X_I$.

By (3) this implies that $\{\Phi^{(n)}\}$ is a fundamental sequence in $\tilde{F}$, which proves the continuity of $j$.

This finishes the proof of the lemma.

Summing up the lemmas 10 and 11, we obtain the following:

Theorem 12. The space of the free coherent states

$$X \xrightarrow{i} \tilde{F} \xrightarrow{j} X'$$

is a rigged Hilbert space.
Remind that a rigged Hilbert space is a triple of space

\[ A \xrightarrow{i} \mathcal{H} \xrightarrow{j} A^* \]

where \( \mathcal{H} \) is a Hilbert space, \( A \) and \( A^* \) are mutually conjugated topological vector spaces, the maps \( i : A \to \mathcal{H} \) and \( j : \mathcal{H} \to A^* \) are continuous injections, the image of \( i \) is dense in \( \mathcal{H} \), and the maps \( i \) and \( j \) are conjugated in the following sense

\[(ia, h) = (a, jh), \quad a \in A, h \in \mathcal{H} \]

We compare the rigged Hilbert spaces of the free coherent states \((21)\) and of generalized functions over \( p \)-adic disk:

\[ D(Z_p) \xrightarrow{i'} L^2(Z_p) \xrightarrow{j'} D'(Z_p) \]

We arrive to the following theorem which is an extension of the theorem proven in [2].

**Theorem 13.** The map \( \phi \) defined by

\[ \phi : X_I \mapsto p^{|I|/|I|}(x - I); \]

extends to an isomorphism \( \phi \) of the rigged Hilbert spaces:

\[ X \xrightarrow{i} \tilde{F} \xrightarrow{j} X' \]

\[ D(Z_p) \xrightarrow{i'} L^2(Z_p) \xrightarrow{j'} D'(Z_p) \]

between the rigged Hilbert space of the free coherent states (with the pairing given by the renormalized scalar product) and the rigged Hilbert space of generalized functions over \( p \)-adic disk.

**Proof** The proof is by lemmas 3, 5, 7 and corollary 8.

**Remark 14.** Definition \( (3) \) of the space of FCS:

\[(A - \lambda)\Psi = 0\]

may be interpreted as the equation of the noncommutative (or quantum) plane \( A = \lambda \). The free coherent state \( \Psi \) in this picture corresponds to a generalized function on a non–commutative space (with non–commutative coordinates \( A_i, A_i^\dagger \)) with support on the non–commutative plane \( A = \lambda, A = \sum_{i=0}^{p-1} A_i \).

The theorem 13 means that the space of generalized functions over the non–commutative plane is isomorphic as a rigged Hilbert space to the space of generalized functions over a \( p \)-adic disk, or roughly speaking the non–commutative plane is equivalent to a \( p \)-adic disk.

Let us note that

\[ \lambda = \sqrt{p} \]
is the maximal possible value of $\lambda$ (the threshold). For $\lambda > \sqrt{p}$ any vector [4] has an infinite norm and therefore does not lie in the Hilbert space.

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