SPARSITY LIKELIHOOD FOR SPARSE SIGNAL AND CHANGE-POINT DETECTION

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Abstract

We propose here sparsity likelihood scores for sparse signal detection and segmentation of multiple sequences with sparse change-points. The scores achieve the same optimality as the higher-criticism and Berk-Jones test statistics in sparse signal detection. We extend its optimality to sparse change-point estimation and detection, for both normal and Poisson models, with asymptotics that differ for the two models. We illustrate its application on simulated datasets as well as on a single-cell copy number dataset taken from a mixture of normal and tumor cells of the same cancer patient.

1 Introduction

The higher-criticism (HC) test statistic, proposed by Tukey (1976) to check for significantly large number of small p-values, was shown by Donoho and Jin (2004) to be optimal in the detection of a sparse normal mixture. Cai, Jeng and Jin (2011) extended it to detect intervals in multiple sequences where the means of a sparse fraction of the sequences deviate from a known baseline, and showed that the HC test statistic is optimal. Chan and Walther (2015) considered sequence length much larger than number of sequences, with detection boundaries that are more complex. They showed that the HC test statistic achieves detection at these boundaries and is thus optimal in a more general setting. They also showed that the Berk and Jones (1979) test statistic achieves the same optimality.

A related problem is where there is no baseline, and of interest are the number and locations of change-points, where the mean changes in a sparse fraction of the sequences. A typical strategy is to sum up scores across sequence after subjecting the scores to thresholding or penalizations. Algorithms employing this strategy include Sparsified Binary Segmentation (SBS) (Cho and Fryzlewicz, 2015), double CUSUM (DC) (Cho, 2016), Informative Sparse Projection (INSPECT) (Wang and Samworth, 2018) and the scan algorithm of Enikeeva and Harachaoui (2019). The strategy was also employed by Mei (2010) and Xie and Siegmund (2014) in sequential change-point detection on multiple sequences, and Zhang, Siegmund, Ji and Li (2010) to detect deviations of means from known baselines on multiple
sequences. Thresholding and penalization suppress noise by removing small and moderate scores, mostly from the majority of sequences without mean change, thus enhancing the signals from the sparse sequences with mean changes.

Instead of thresholding we consider a sum of sparsity likelihood scores. These scores are carefully designed transformations of p-values that come from the testing done to evaluate change-points. The procedure is computationally efficient and optimal at all levels of change-point sparsity, for both normal and Poisson observations.

SBS, DC and INSPECT adopt the Wild Binary Segmentation (WBS) scheme of Fryzlewicz (2014) in generating large number of random intervals for the evaluation of change-points. We adopt a more systematic two-step approach, with a quick detection first step that makes use of the screening idea behind the Screening and Ranking algorithm (SaRa) of Niu and Zhang (2012), followed by an estimation second step for more precise location of change-points. Instead of applying the BIC criterion of Zhang and Siegmund (2007) to determine the number of change-points, we apply fixed critical values and, like SBS, DC and INSPECT, rely on binary segmentation (Olshen et al., 2003) to detect the change-points one at a time.

An alternative to binary segmentation is to estimate the full set of change-points in one go by applying global optimization criteria, and making use of dynamic programming to manage computational complexity. This was employed by the HMM algorithms of Yao (1984) and Lai and Xing (2011), the multi-scale SMUCE algorithm of Frank, Munk and Sieling (2014) and the Bayesian Likelihood algorithm of Du, Kao and Kou (2016). These methods are however designed for single sequence segmentation. For a good overview, see Niu, Hao and Zhang (2016).

The outline of this paper is as follows. In Section 2 we introduce the sparsity likelihood (SL) scores and show that they achieve the same optimality as the higher-criticism and Berk-Jones test statistics in the detection of sparse normal mixtures. In Section 3 we extend SL scores to detect change-points in multiple sequences. In Section 4 we show that SL scores are optimal for change-point detection when the observations are normal or Poisson. In Section 5 we apply SL scores on simulated datasets as well as on a single-cell copy number dataset taken from a mixture of normal and tumor cells of the same patient. In the Appendix we provide the proofs to show that SL scores are optimal.
2 Detection of a sparse mixture

Let \( p = (p_1, \ldots, p_N) \) be the p-values of \( N \) independent tests, and let \( p^{(1)} \leq \cdots \leq p^{(N)} \) be the sorted p-values. Tukey (1976) proposed the higher-criticism test statistic

\[
HC(p) = \max_{n : Np^{(n)} \leq n} \frac{n - Np^{(n)}}{\sqrt{Np^{(n)}(1 - p^{(n)})}},
\]

with \( HC(p) = 0 \) if \( Np^{(n)} > n \) for all \( n \), to test whether all the null hypotheses are true.

Donoho and Jin (2004) showed that the HC test statistic is optimal for the detection of a sparse fraction of false null hypotheses. Consider test scores \( Z_n \sim N(0, 1) \) when the \( n \)th null hypothesis is true and \( Z_n \sim N(\mu_N, 1) \) for some \( \mu_N > 0 \) when the \( n \)th null hypothesis is false. Define

\[
\rho_Z(\beta) = \begin{cases} 
\beta - \frac{1}{2} & \text{if } \frac{1}{2} < \beta < \frac{3}{4}, \\
(1 - \sqrt{1 - \beta})^2 & \text{if } \frac{3}{4} \leq \beta < 1.
\end{cases}
\]

Donoho and Jin (2014) showed that on the sparse mixture \( (1 - \epsilon)N(0, 1) + \epsilon N(\mu_N, 1) \), no algorithm is able to achieve

\[
P(\text{Type I error}) + P(\text{Type II error}) \to 0 \quad \text{as } N \to \infty,
\]

for the test of \( H_0: \epsilon = 0 \) versus \( H_1: \epsilon = N^{-\beta} \), if \( \mu_N = \sqrt{2\nu \log N} \) for \( \nu < \rho_Z(\beta) \), whereas the HC test statistic achieves (2.2) when \( \nu > \rho_Z(\beta) \).

Ingster (1997, 1998) established the detection lower bound, showing that (2.2) cannot be achieved when \( \nu < \rho_Z(\beta) \).

Like the HC test statistic, the Berk and Jones (1979) test statistic

\[
BJ(p) = \max_{n : Np^{(n)} \leq n} \left[ n \log \left( \frac{n}{Np^{(n)}} \right) + (N - n) \log \left( \frac{N-n}{N(1-p^{(n)})} \right) \right]
\]

achieves (2.2) when \( \nu > \rho_Z(\beta) \). We introduce the sparsity likelihood scores in Section 2.1 and show that they achieve (2.2) in the detection of sparse mixtures in Section 2.2.

2.1 Sparsity likelihood

Let \( f_1(p) = \frac{1}{p(2 - \log p)^2} - \frac{1}{2} \) and \( f_2(p) = \frac{1}{\sqrt{p}} - 2 \). Note that \( \int_0^1 f_i(p) dp = 0 \) for \( i = 1 \) and 2. Let \( \lambda_1 \geq 0 \) and \( \lambda_2 > 0 \). Define the sparsity likelihood score

\[
\ell(p) = \sum_{n=1}^N \ell(p^n),
\]

where

\[
\ell(p) = \log \left( 1 + \frac{\lambda_1 \log N}{N} f_1(p) + \frac{\lambda_2}{\sqrt{N \log N}} f_2(p) \right).
\]
We conclude that the p-values are from a sparse mixture when $\ell(p) \geq c_N$ for a selected critical value $c_N$.

The sparsity likelihood is the likelihood of testing

$$H_0 : p^n \overset{i.i.d.}{\sim} \text{Uniform}(0, 1) \text{ vs } H_1 : p^n \overset{i.i.d.}{\sim} f = 1 + \frac{\lambda_1 \log N}{N} f_1 + \frac{\lambda_2}{\sqrt{N \log N}} f_2.$$  

The essential idea behind this likelihood for sparse signal detection is as follows. Consider first $\lambda_1 = 0$. The distribution function of $f$ is

$$F(p) = p + \frac{2\lambda_2}{\sqrt{N \log N}} (\sqrt{p} - p).$$

Consider the situation in which $H_0$ does not hold and the true distribution of $p^n$ is $G$. The fraction of p-values less than some small $p_0$ has mean $G(p_0)$ and standard deviation approximately $\sqrt{G(p_0)/N}$. Hence we need

$$G(p_0) \geq p_0 + C \sqrt{p_0/N}$$

for some $C > 0$ in order to achieve reasonable detection power for a test using $p_0$ as critical value. Setting $C = \frac{-2\lambda_2}{\sqrt{N \log N}}$ to the right-hand side of (2.4) gives us a function close to $F$.

Taking into consideration $\sqrt{\log N}$ small compared to $\sqrt{N}$, we can view $F$ as lying near the boundary at which detection is possible, at each small $p_0$. This suggests that for any $G$ that is greater than $F$ at some $p_0$, we are able to differentiate it from the uniform distribution using the optimal likelihood test for $F$.

When working with p-values small compared to $\frac{1}{\sqrt{N \log N}}$, the selection of $\lambda_1 > 0$ is advantageous because $\frac{\log N}{N} f_1(p)$ dominates $\frac{1}{\sqrt{N \log N}} f_2(p)$ for $p$ small compared to $\frac{1}{\sqrt{N \log N}}$. This is relevant in the extension of sparsity likelihood scores to detect change-points on long sequences, due to the large number of likelihood comparisons.

To visualize the sparsity likelihood score we compare, in Figure 1, the plot of $\ell(p(z))$ for $p(z) = 2\Phi(-z)$, $N = 500$, $\lambda_1 = 1$ and $\lambda_2 = 1.84$, with that of $(z - 2)_{+}^2/2$. For $0 \leq z \leq 5$, the two functions are close to each other, however within $0 \leq z \leq 2$, $\ell(p(z))$ is not constant but has a gentle upward curve. The sparsity likelihood score is negative for $z \leq 1.18$ and for $Z$ standard normal, $\ell(p(Z))$ has a mean of $-0.00397$. The negative mean helps in controlling the sum of scores when $N$ is large and $p^n \overset{i.i.d.}{\sim} \text{Uniform}(0, 1)$.

### 2.2 Optimal detection

We show here that the sparsity likelihood score is optimal at all levels of sparsity. Let $E_0$ and $P_0$ denote expectation and probability respectively
Figure 1: Graphs of $\ell(p(z))$ (black, —) and $(z - 2)^2/2$ (red, —), with $p(z) = 2\Phi(-z)$, for $0 \leq z \leq 5$ (left) and $0 \leq z \leq 2$ (right). The parameters of $\ell$ are $N = 500$, $\lambda_1 = 1$ and $\lambda_2 = 1.84\left(\sqrt{\frac{\log T}{\log \log T}}\right)$ for $T = 500$.

with respect to $p^n \overset{i.i.d.}{\sim}$ Uniform$(0, 1)$. Since

$$E_0 \exp(\ell(p)) = \prod_{n=1}^{N} E_0[1 + \frac{\lambda_1 \log N}{N} f_1(p^n) + \frac{\lambda_2}{\sqrt{N \log N}} f_2(p^n)] = 1,$$

it follows from Markov’s inequality that

$$P_0(\ell(p) \geq c) \leq e^{-c}.$$  \hfill (2.5)

This exponential bound makes the sparsity likelihood score easy to work with when there are large number of likelihood comparisons, as critical values satisfying a required level of Type I error control can have a simple expression not depending on $N$. We show in Theorem 1 that by selecting

$$c_N \to \infty \text{ with } c_N N^{-\delta} \to 0 \text{ for all } \delta > 0,$$  \hfill (2.6)

both Type I and II error probabilities goes to zero near the detection boundary.

**Theorem 1.** Consider the test of $H_0$: $Z^n \overset{i.i.d.}{\sim} N(0,1)$ versus $H_1$: $Z^n \overset{i.i.d.}{\sim} (1-\epsilon)N(0,1) + \epsilon N(\mu_N, 1)$, for $1 \leq n \leq N$, with $\epsilon = N^{-\beta}$ for some $\frac{1}{2} < \beta < 1$. 
If \( \mu_N = \sqrt{2 \nu \log N} \) for \( \nu > \rho_Z(\beta) \), then the sparsity likelihood score \( (2.3) \), with \( \lambda_1 \geq 0 \) and \( \lambda_2 > 0 \), achieves
\[
P(\text{Type I error}) + P(\text{Type II error}) \to 0
\]
for critical values \( c_N \) satisfying \( (2.0) \).

## 3 Change-point detection

Consider \( N \) sequences of length \( T \). Let \( X^n_t \) denote the \( t \)th observation of the \( n \)th sequence. We assume first that \( X^n_t \) are independent normal with unit variances. Let \( \mu^n_t \) denote the mean of \( X^n(t) \). Hence
\[
X^n_t \sim N(\mu^n_t, 1). \tag{3.1}
\]

We are interested in the detection and estimation of the change-point set
\[
\tau = \{ t : \mu^n_t \neq \mu^n_{t+1} \text{ for some } n \}.
\]

For \( s < t \), let \( \bar{X}^n_{st} = (t-s)^{-1} \sum_{u=s+1}^t X^n_u \). To check for a mean change on the \( n \)th sequence at location \( t \), we can select \( s < t < u \) and consider the p-value
\[
p^n_{stu} = 2\Phi(-|Z^n_{stu}|), \text{ where } Z^n_{stu} = \frac{\bar{X}^n_{tu} - \bar{X}^n_{st}}{\sqrt{(u-t)^{-1} + (t-s)^{-1}}},
\]
with \( \Phi \) denoting the distribution function of the standard normal. In the sparsity likelihood algorithm, we combine these p-values using the score \( \ell(p^n_{stu}) \), where \( p^n_{stu} = (p^n_{stu}, \ldots, p^n_{stu}) \).

The use of a sparsity likelihood score is motivated by the need to detect well when only a small fraction of the sequences undergo mean changes at \( t \). For \( T \) large, computing the sparsity likelihood score for all \( (s, t, u) \) is expensive. Instead we integrate the approximating set idea of Walther (2010) to identify quickly a local region where a change-point lies, with the CUSUM-type scores used in WBS to estimate more accurately the change-point location within this region.

In addition to computational savings, through this two-step approach we are able to incorporate multi-scale penalization terms in \( (3.2) \), similar to those used in Dümbgen and Spokoiny (2001) and in the SMUCE algorithm of Frick, Munk and Sieling (2014), to ensure optimality not only at all levels of sparsity, but also at all orders of mean changes.
Let \( 1 \leq h_1 < h_2 < \cdots \) and \( 1 \leq d_1 < d_2 < \cdots \) be integer-valued sequences with \( h_i \geq d_i \) for all \( i \). Let \( K_i(g) = \lfloor \frac{g-1}{d_i} \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the greatest integer function. Define

\[
\mathcal{A}_i(g) = \{(s(ik), t(ik), u(ik)) : 1 \leq k \leq K_i(g)\},
\]

with \( s(ik) = \max(0, kd_i - h_i) \), \( t(ik) = kd_i \) and \( u(ik) = \min(kd_i + h_i, g) \).

The elements of \( \mathcal{A}_i(g) \) are the indices where sparsity scores for windows of length \( h_i \) are to be computed. Initially we have the full dataset \( X_{1:T} = (X^n_t : 1 \leq t \leq T, 1 \leq n \leq N) \), and after one or more change-points have been estimated, it is split into sub-datasets \( X_{b:e} = (X^n_t : b \leq t \leq e, 1 \leq n \leq N) \), with length \( g = e - b + 1 \). We check for change-points in \( X_{b:e} \) using windows specified by \( \mathcal{A}_i(g) \).

Let the penalized sparsity likelihood scores

\[
p^\ell(p_{stu}) = \ell(p_{stu}) - \log(\frac{T}{T-s+1} + \frac{1}{n-1})). \tag{3.2}
\]

For \( g \geq 1 \), let \( i_g = \max\{i : h_i + d_i \leq g\} \). The detection of change-points within \( X_{b:e} \), with window lengths at least \( h_{i_0} \), is as follows.

\begin{function}
\text{function SL-estimate}(c, i_0, b, e) \n\text{X} \leftarrow X_{b:e} \n g \leftarrow e - b + 1 \n \text{for } i = i_0, \ldots, i_g \n\text{if } \max_{1 \leq k \leq K_i(g)} p^\ell(p_{s(ik), t(ik), u(ik)}) \geq c \text{ then} \n\text{j} \leftarrow \arg\max_{k:1 \leq k \leq K_i(g)} p^\ell(p_{s(ik), t(ik), u(ik)}) \n\hat{\tau} \leftarrow \left[ \arg\max_{t:s(ij) < t < u(ij)} p^\ell(p_{s(ij), t, u(ij)}) \right] + b - 1 \n\text{output } (\hat{\tau}, i) \n\text{stop} \n\text{end if} \n\text{end for} \n\text{output } (0,0) \n\text{end function}
\end{function}

There are two steps in SL-estimate in the estimation of a change-point, when the largest penalized score exceeds critical value \( c \). The first is the identification of an interval \((s(ij), u(ij))\), associated with the largest penalized score, within which a change-point lies. The second is the estimation of the change-point location within this interval. In the approximating set \( \mathcal{A}_i(g) \), neighboring windows are located \( d_i \) apart, hence we are unable to estimate the change-points accurately in the first step.
Figure 2: Graphs of Type I error probability against critical value for the sparsity likelihood detection algorithm, for independent unit variance normal observations. We consider parameters $d_i$, $h_i$, $\lambda_1$ and $\lambda_2$ as applied in the numerical studies in Section 5, with $T = 2000$ (left), $T = 20,000$ (right), and $N = 50$ (black), $N = 100$ (red), $N = 200$ (green), $N = 500$ (blue), $N = 1000$ (orange).

is carried out, with more intensive computations within $(s(ij), u(ij))$, in the second step. Since the second step is performed only after an interval has been identified as containing a change-point, performing this two-step procedure saves computations in regions where scores are generally small and the likelihood of change-points is low. In contrast when applying the WBS scheme, the computations that we perform in the second step are applied on each randomly generated interval.

After a change-point has been identified, we split the dataset into two and execute the same algorithm on each split dataset. To avoid repetitive computations, we start from the window length $h_{i0}$ used in the evaluation of the change-point splitting the dataset, instead of starting from the smallest window length $h_1$, on the split datasets. The recursive segmentation algorithm for the computation of the estimated change-point set $\hat{\tau}$ is as follows. We initialize with parameters $(c, 1, 1, T, \emptyset)$, where $\emptyset$ denotes the empty set.

```plaintext
function SL-detect(c, i0, b, e, \hat{\tau})
    (\tilde{\tau}, i) \leftarrow SL-estimate(c, i0, b, e)
    if \tilde{\tau} > 0 then
```
\[ \hat{\tau} \leftarrow \hat{\tau} \cup \{ \hat{\tau} \} \]
\[ \hat{\tau} \leftarrow \text{SL-detect}(c, i, b, \hat{\tau}, \hat{\tau}) \]
\[ \hat{\tau} \leftarrow \text{SL-detect}(c, i, \hat{\tau}, e, \hat{\tau}) \]
end if

output \( \hat{\tau} \)
end function

The critical value of this sparsity likelihood algorithm, for a specified Type I error probability, is stable over \( N \), see Figure 2. Contributing factors include \( \ell(p) \) having a close to zero mean and \( \ell(p) \) having exponential tail probabilities not depending on \( N \), see \((2.5)\), when \( p \) and \( p^n \) are uniformly distributed.

4 Optimal detection

Let \( \mu = (\mu^t : 1 \leq t \leq T, 1 \leq n \leq N) \) and \( J = \#\tau \), where \( \#A \) denotes the number of elements in a set \( A \). We show that the sparsity likelihood algorithm is optimal for normal observations in Section 4.1, and for Poisson observations in Section 4.2.

4.1 Normal model

Let \( m_{j\Delta} = \#\{ n : |\mu_{\tau_j+1} - \mu_{\tau_j}| \geq \Delta \} \)
be the number of sequences undergoing mean change of at least \( \Delta \) at the \( j \)th change-point. Let

\[
\Omega_0 = \{ \mu : J = 0 \}, \\
\Omega_1(\Delta, V, h) = \{ \mu : \text{there exists } j \text{ such that } \\
\min(\tau_j - \tau_{j-1}, \tau_{j+1} - \tau_j) \geq h \text{ and } m_{j\Delta} \geq V \},
\]
with the convention that \( \tau_0 = 0 \) and \( \tau_{J+1} = T \). We consider here the test of \( H_0: \mu \in \Omega_0 \) versus \( H_1: \mu \in \Omega_1(\Delta, h, V) \).

Consider \( T \to \infty \) and \( N \to \infty \) such that

\[ \log T \sim N^\zeta \text{ for some } 0 < \zeta \leq 1, \]
where \( a_n \sim b_n \) if \( \lim_{n \to \infty} (a_n/b_n) = 1 \). Let \( a_n = o(b_n) \) if \( \lim_{n \to \infty} (a_n/b_n) = 0 \).

Define

\[
\rho_Z(\beta, \zeta) = \begin{cases} 
\beta - \frac{1-\zeta}{2} & \text{if } \frac{1-\zeta}{2} \leq \beta \leq \frac{3(1-\zeta)}{4}, \\
\frac{1-\zeta}{2} & \text{if } \frac{3(1-\zeta)}{4} < \beta \leq 1 - \zeta.
\end{cases}
\]

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The detection boundary, of the region for which asymptotically zero Type I and II error probabilities is not possible for any algorithm, is specified in Theorem 2. Analogous detection boundaries for a single sequence is given in Arias-Castro, Donoho and Huo (2005, 2006).

**Theorem 2.** Assume (3.1) and (4.1). No algorithm is able to achieve

\[
\sup_{\mu \in \Omega_0} P_{\mu}(\text{Type I error}) + \sup_{\mu \in \Omega_1(\Delta, V, h)} P_{\mu}(\text{Type II error}) \to 0 \quad (4.3)
\]

under any of the following conditions.

(a) Consider \( \Delta > 0 \) fixed.
   i. When \( V = o\left(\frac{N^\zeta}{\log N}\right) \) and \( h V \Delta^2 = 4(1-\epsilon) \log T \) for some \( 0 < \epsilon < 1 \).
   ii. When \( V \sim N^{1-\beta} \) for some \( \frac{1-\zeta}{2} < \beta < 1 - \zeta \) and \( h \Delta^2 = 4(1-\epsilon) \rho_Z(\beta, \zeta) \log N \) for some \( 0 < \epsilon < 1 \).

(b) Consider \( \Delta = T^{-\eta} \) for some \( 0 < \eta < \frac{1}{2} \).
   i. When \( V = o\left(\frac{N^\zeta}{\log N}\right) \) and \( h V \Delta^2 = 4(1-2\eta-\epsilon) \log T \) for some \( 0 < \epsilon < 1 - 2\eta \).
   ii. When \( V \sim N^{1-\beta} \) for some \( \frac{1-\zeta}{2} < \beta < 1 - \zeta \) and \( h \Delta^2 = 4(1-\epsilon) \rho_Z(\beta, \zeta) \log N \) for some \( 0 < \epsilon < 1 \).

In Theorem 3 below we show that the Type I and II error probabilities of the sparsity likelihood algorithm go to zero at the detection boundary. Recall that \( i_T = \max\{i : h_i + d_i \leq T\} \). Select \( d_i \) and \( h_i \) such that

\[
\frac{h_{i+1}}{h_i} \to 1 \text{ and } d_i = o(h_i) \text{ as } i \to \infty, \quad (4.4)
\]

\[
\log \left( \sum_{i=1}^{i_T} \frac{h_i}{d_i} \right) = o(\log T) \text{ as } T \to \infty, \quad (4.5)
\]

and critical values \( c_T \) such that

\[
c_T = o(\log T) \text{ and } c_T - \log \left( \sum_{i=1}^{i_T} \frac{h_i}{d_i} \right) \to \infty \text{ as } T \to \infty. \quad (4.6)
\]

Select parameters

\[
\lambda_1 > 0 \text{ and } \lambda_2 = \sqrt{\frac{\log T}{\log \log T}}. \quad (4.7)
\]
Theorem 3. Assume (3.1) and (4.1). The sparsity likelihood algorithm with parameters satisfying (4.4)–(4.7) achieves (4.3) under any of the following conditions.

(a) Consider $\Delta > 0$ fixed.
   i. When $V = o\left(\frac{N^\epsilon}{\log N}\right)$ and $h V^{2\epsilon} = 4(1 + \epsilon) \log T$ for some $\epsilon > 0$.
   ii. When $V \sim N^{1-\beta}$ for some $\frac{1-\zeta}{2} < \beta < 1 - \zeta$ and $h^{2\epsilon} = 4(1 + \epsilon) \rho Z(\beta, \zeta) \log N$ for some $\epsilon > 0$.

(b) Consider $\Delta = T^{-\eta}$ for some $0 < \eta < \frac{1}{2}$.
   i. When $V = o\left(\frac{N^\epsilon}{\log N}\right)$ and $h V^{2\epsilon} = 4(1 - 2\eta + \epsilon) \log T$ for some $\epsilon > 0$.
   ii. When $V \sim N^{1-\beta}$ for some $\frac{1-\zeta}{2} < \beta < 1 - \zeta$ and $h^{2\epsilon} = 4(1 + \epsilon) \rho Z(\beta, \zeta) \log N$ for some $\epsilon > 0$.

Remarks 1. The optimality in Theorem 3 still holds, in the case of extreme sparsity with $V = o\left(\frac{N^\epsilon}{\log N}\right)$ when $\lambda_2 = 0$, and in the case of moderate sparsity with $V \sim N^{1-\beta}$ for some $\frac{1-\zeta}{2} < \beta < 1 - \zeta$ when $\lambda_1 = 0$. The optimality also holds more generally for $\lambda_2 = o\left(\sqrt{\log T}\right)$ with $\log \lambda_2 \sim \frac{1}{2} \log \log T$ as $T \to \infty$.

2. We satisfy (4.4) when $h_i \sim \exp\left(\frac{i}{\log i}\right)$ and $d_i \sim \frac{h_i}{i}$ as $i \to \infty$. Moreover

$$\log \left(\sum_{i=1}^{iT} \frac{h_i}{d_i}\right) \sim 2 \log iT \sim 2 \log \log T,$$

and thus (4.5) is also satisfied.

3. Let $C > 8$. Under the conditions of Theorem 3(a)i. and (b)i.,

$$\inf_{\mu \in \Omega_1(\Delta, V, h)} P_{\mu}(\hat{\tau}_1 - \tau) \leq \frac{C \log T}{\Delta^2 V T} \to 1,$$

where $\hat{\tau}_1$ is the first change-point estimated by the sparsity likelihood algorithm and $|\tau_1 - A| = \min_{a \in A} |\tau_1 - a|$. Hence the algorithm is consistent with convergence rate of $\frac{\log T}{\Delta^2 V T}$. For moderate sparsity with $V \sim N^{1-\beta}$ for some $\frac{1-\zeta}{2} < \beta < 1 - \zeta$, the convergence rate is $\frac{\log N}{\Delta^2 V T}$.

4.2 Poisson model

We show here the optimality of the sparsity likelihood detection algorithm for Poisson random variables with, instead of (3.1),

$$X_i^p \sim \text{Poisson}(\mu_i^p).$$

(4.8)
For optimal detection on a single Poisson sequence, see Rivera and Walther (2013). Let $Y_{st}^n = \sum_{u=s+1}^{t} X_u^n$.

Consider $s < t < u$. Under the hypothesis that there are no change-points in the interval $(s, u)$, conditioned on $Y_{su}^n = y$, $Y_{st}^n$ is binomial distributed with $y$ trials and success probability $\frac{t-s}{u-s}$. Let $p_{stu}^n$ be the two-sided p-value of this conditional binomial test, with continuity adjustments so that it is distributed as Uniform(0,1) under the hypothesis of no change-points.

Let
\[ m_{j\Delta} = \#\{n : |\log(\mu_{\tau_j+1}/\mu_{\tau_j})| \geq \Delta\}, \]
and for a given $\mu_0 > 0$, let
\[ \Lambda = \{\mu : \mu_t^n \geq \mu_0 \text{ for all } n \text{ and } t\}, \]
\[ \Lambda_0 = \{\mu \in \Lambda : J = 0\}, \]
\[ \Lambda_1(\Delta, V, h) = \{\mu \in \Lambda : \text{ there exists } j \text{ such that } \min(\tau_{j+1} - \tau_j, \tau_j - \tau_{j-1}) \geq h \text{ and } m_{j\Delta} \geq V\}. \]

For a given $r > 1$, let the large deviations constant
\[ I_r = r \log\left(\frac{2r}{r+1}\right) + \log\left(\frac{2}{r+1}\right). \tag{4.9} \]
Let $g_r(\omega) = (\frac{1+\omega^n}{2})^{1/\omega}$ and for $\frac{1-\zeta}{2} < \beta < 1 - \zeta$, define
\[ \rho_r(\beta, \zeta) = \max_{\frac{1-\zeta}{2} < \omega \leq 2} \left(\frac{2 - \omega^{-1}(1-\zeta)}{2g_r(\omega) - 1 - r}\right). \tag{4.10} \]

In Theorem 4 we show that (4.10) is the asymptotic constant in the detection boundary of Poisson random variables, analogous to (4.2) for normal random variables. In Theorem 5 we show that the sparsity likelihood algorithm achieves detection at this boundary for all levels of change-point sparsity.

**Theorem 4.** Assume (4.1), (4.8) and consider $r = \epsilon \Delta$ for some $\Delta > 0$. No algorithm is able to achieve
\[ \sup_{\mu \in \Lambda_0} P_\mu(\text{Type I error}) + \sup_{\mu \in \Lambda_1(\Delta, V, h)} P_\mu(\text{Type II error}) \to 0 \tag{4.11} \]
under any of the following conditions.

(a) When $V = o\left(\frac{N^c}{\log N}\right)$ and $hV I_r \mu_0 = (1 - \epsilon) \log T$ for some $0 < \epsilon < 1$.

(b) When $V \sim N^{1-\beta}$ for some $\frac{1-\zeta}{2} < \beta < 1 - \zeta$ and $h \mu_0 = (1 - \epsilon)\rho_r(\beta, \zeta) \log N$ for some $0 < \epsilon < 1$. 

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Theorem 5. Assume (4.1), (4.8) and consider \( r < e^\Delta \) for some \( \Delta > 0 \) and \( r > 1 \). The sparsity likelihood algorithm with parameters satisfying (4.4)–(4.7) achieves (4.11) under any of the following conditions.

(a) When \( V = o\left(\frac{N\epsilon}{\log N}\right) \) and \( h V I_r \mu_0 = (1 + \epsilon) \log T \) for some \( \epsilon > 0 \).

(b) When \( V \sim N^{1-\beta} \) for some \( \frac{1-\epsilon}{2} < \beta < 1 - \zeta \) and \( h \mu_0 = (1 + \epsilon) \rho_r(\beta, \zeta) \log N \) for some \( \epsilon > 0 \).

5 Numerical studies

We conduct studies on normal random variables (Section 5.1) and on a copy-number aberration dataset with count data (Section 5.2).

5.1 Simulation studies

We consider here the simulation set-ups in Sections 5.1 and 5.3 of Wang and Samworth (2018). It is assumed there that the random variables are normal with variances that are unknown but equal within sequence. These variances are estimated using median absolute differences of adjacent observations and after normalization, the random variables are treated like unit variance normal.

In the first study there is exactly one change-point \( \tau_1 \). Consider (3.1) with \( \mu^0_t = 0 \) for \( t \leq \tau_1 \) and all \( n \). For \( t > \tau_1 \),

\[
\mu^t_n = \begin{cases} 
0.8\sqrt{n \sum_{m=1}^{V} m^{-1}} & \text{if } n \leq V, \\
0 & \text{if } n > V.
\end{cases}
\]

The objective is to estimate \( \tau_1 \) assuming that it is known that there is exactly one change-point. We estimate \( \tau_1 \) by

\[
\hat{\tau}_1 = \arg \max_{0 < t < T} p^\ell(p_0(t)),
\]

where \( p^\ell \) is the penalized sparsity score (3.2), with \( \lambda_1 = 1 \) and \( \lambda_2 = \sqrt{\frac{\log T}{\log \log T}} \).

We simulate the probabilities that \( |\hat{\tau}_1 - \tau_1| \leq k \) for \( k = 3 \) and 10, and compare against the INSPECT algorithm and the scan algorithm of Enikeeva and Harchaoui (2019). These two algorithms have the best numerical performances in Wang and Samworth (2018). The comparisons in Table 1 show that sparsity likelihood performs well. The computation time of scan
Table 1: The fraction of simulation runs (out of 1000) for which $\hat{\tau}_1$ is within distance $k$ from $\tau_1$ for $k = 3$ and 10. The same datasets are used to compare sparsity likelihood (SL), INSPECT and the scan test, with $\tau_1 = 200$ for $T = 500$ and $\tau_1 = 800$ for $T = 2000.$

| $T$ | $N$ | $V$ | SL  | INSPECT | scan |
|-----|-----|-----|-----|---------|------|
|     |     |     | $k = 3$ | $k = 10$ | $k = 3$ | $k = 10$ |
| 500 | 500 | 3   | 0.511 | 0.801 | 0.478 | 0.785 | 0.520 | 0.804 |
|     |     | 5   | 0.466 | 0.740 | 0.427 | 0.718 | 0.463 | 0.722 |
|     |     | 10  | 0.393 | 0.645 | 0.370 | 0.637 | 0.362 | 0.599 |
|     |     | 22  | 0.319 | 0.553 | 0.282 | 0.547 | 0.256 | 0.465 |
|     |     | 50  | 0.244 | 0.462 | 0.211 | 0.453 | 0.197 | 0.378 |
|     |     | 500 | 0.177 | 0.339 | 0.148 | 0.335 | 0.112 | 0.240 |
| 500 | 2000| 3   | 0.481 | 0.748 | 0.410 | 0.667 | 0.480 | 0.730 |
|     |     | 5   | 0.423 | 0.673 | 0.344 | 0.584 | 0.394 | 0.633 |
|     |     | 10  | 0.320 | 0.546 | 0.246 | 0.480 | 0.261 | 0.456 |
|     |     | 20  | 0.237 | 0.431 | 0.198 | 0.403 | 0.188 | 0.332 |
|     |     | 45  | 0.186 | 0.344 | 0.136 | 0.311 | 0.130 | 0.242 |
|     |     | 200 | 0.114 | 0.227 | 0.095 | 0.235 | 0.074 | 0.153 |
|     |     | 2000| 0.068 | 0.160 | 0.078 | 0.189 | 0.042 | 0.096 |
| 2000| 500 | 3   | 0.603 | 0.859 | 0.587 | 0.855 | 0.589 | 0.854 |
|     |     | 5   | 0.604 | 0.865 | 0.595 | 0.855 | 0.558 | 0.832 |
|     |     | 10  | 0.565 | 0.827 | 0.569 | 0.833 | 0.487 | 0.764 |
|     |     | 22  | 0.522 | 0.789 | 0.522 | 0.795 | 0.438 | 0.714 |
|     |     | 50  | 0.472 | 0.748 | 0.468 | 0.745 | 0.384 | 0.652 |
|     |     | 500 | 0.378 | 0.643 | 0.336 | 0.609 | 0.273 | 0.524 |
| 2000| 2000| 3   | 0.607 | 0.866 | 0.608 | 0.861 | 0.599 | 0.858 |
|     |     | 5   | 0.594 | 0.864 | 0.586 | 0.857 | 0.557 | 0.829 |
|     |     | 10  | 0.553 | 0.847 | 0.558 | 0.847 | 0.476 | 0.780 |
|     |     | 20  | 0.494 | 0.807 | 0.498 | 0.789 | 0.435 | 0.726 |
|     |     | 45  | 0.447 | 0.747 | 0.451 | 0.746 | 0.377 | 0.657 |
|     |     | 200 | 0.362 | 0.649 | 0.342 | 0.604 | 0.297 | 0.554 |
|     |     | 2000| 0.274 | 0.532 | 0.241 | 0.471 | 0.225 | 0.457 |
and sparsity likelihood increases roughly linearly with $N$ whereas for INSPECT the computation time increases at a faster than linear rate.

In the second study there are three change-points within $N = 200$ sequences of length $T = 2000$, at $\tau_1 = 500$, $\tau_2 = 1000$ and $\tau_3 = 1500$. At each change-point exactly 40 sequences undergo mean changes. Six scenarios are considered, corresponding to

$$\mu_{\tau_{j+1}}^{k(j-1)+n} - \mu_{\tau_{j}}^{k(j-1)+n} = r\sqrt{\sum_{m=1}^{40} m^{-1}}$$

for $1 \leq j \leq 3$ and $1 \leq n \leq 40$, for $r = 0.4, 0.6$ and $k = 0, 20, 40$. For $k = 0$, the mean changes are within the same 40 sequences at all three change-points, whereas for $k = 40$ the mean changes at all three change-points are on distinct sequences. For $k = 20$, there is partial overlap of the sequences having mean changes at adjacent change-points. The number of estimated change-points over 100 simulated datasets on each sequence is recorded, as well as the adjusted Rand index (ARI), see Rand (1971) and Hubert and Arabie (1985), to measure the quality of the change-point estimation.

In the application of the sparse likelihood algorithm, we select $h_1 = 1$ and $h_{i+1} = [1.1h_i]$ for $i \geq 1$, and $d_i = [h_i/i]$, for a total of $i_T = 61$ window lengths. We select critical value $c_T = 5$ and parameters $\lambda_1 = 1$, $\lambda_2 = \sqrt{\frac{\log T}{\log \log T}} \approx 1.94$.

Wang and Samworth (2018) showed that INSPECT achieves average ARI of 0.90 when $r = 0.6$ and either 0.73 (for $k = 20$) or 0.74 (for $k = 0$ and 40) when $r = 0.4$, comparable to sparsity likelihood, see Table 2. The sparsity likelihood algorithm is faster due to the simple computation of the sparsity likelihood, as well as the use of a two-stage procedure to save computational resources in regions with low likelihoods of change-points.

In addition to INSPECT, Wang and Samworth (2018) considered DC, SBS and scan, as well as the CUSUM aggregation algorithms of Jirak (2015) and Horváth and Hušková (2012), with average ARI of between 0.77–0.87 when $r = 0.6$ and 0.68–0.72 when $r = 0.4$.

5.2 Copy-number dataset

Everyone inherits two copies of the genome, one each from his or her father and mother. Tumor cells often undergo somatic structural mutations that delete or amplify certain chromosomal segments in one or both copies. These somatic copy number aberration (CNA) play critical roles in cancer progression, and their accurate detection and characterization is important for disease prognosis and treatment. Methods for quantifying CNA have
Table 2: Number of change-points estimated by the sparsity likelihood detection algorithm and the average ARI over 100 simulated datasets.

| r  | k  | # change-points | ARI |
|----|----|-----------------|-----|
| 0.6| 0  | 11 80 8 1       | 0.91|
| 0.4| 0  | 61 35 4 0       | 0.74|
| 0.6| 20 | 12 80 8 0       | 0.91|
| 0.4| 20 | 66 31 2 1       | 0.74|
| 0.6| 40 | 10 78 12 0      | 0.91|
| 0.4| 40 | 68 26 6 0       | 0.75|

evolved with the advance of technology, from traditional spectral karyotyping to array-based CGH, to next-generation DNA sequencing, and more recently whole genome single cell DNA sequencing (scDNA-seq).

ScDNA-seq for the first time made possible characterization of copy number profiling at the cellular level. Unlike bulk-tissue DNA sequencing that is confounded by cell subpopulation, scDNA-seq can deconvolve cancer subclones and subsequently the cancer evolutionary history with much less ambiguity. In a typical scDNA-seq dataset for a patient, it contains cells at different stages of the tumor (from normal cells to early stage tumor cells and later stage tumor cells). It is common that CNA events are shared by only a subset of cells. Our newly proposed method is a good fit for this area of applications.

We illustrate our method through an analysis of a scDNA-seq dataset of a breast cancer patient from Navin et al. (2011). This dataset contains sequencing results from 100 single cells. We used SCOPE (Wang, Lin and Jiang, 2020) to preprocess the data. In particular, 2 cells were removed due to low proportion of mapped reads. Among the remaining 98 cells, 39 cells are identified as normal cells by cell-specific Gini coefficients provided in SCOPE. We set 20 cells with the lowest cell-specific Gini coefficients as normal cells and use them as controls. It is well known that the reads are inhomogeneous across the chromosome even in normal cells due to various reasons, such as mappability. Here, we use these 20 normal cells to construct bins such that each bin contains 20 reads in total for these 20 cells.

For the remaining \( N = 78 \) cells, we model \( X_t^n \sim \text{Poisson}(\mu_t^n) \), where \( X_t^n \) denotes the number of reads in bin \( t \) and cell \( n \). The high-resolution binning results in low number of reads used to estimate the change-points, hence p-values are based on the conditional binomial tests elaborated in
| Cell | L  | R  | P-val | SL | Cell | L  | R  | P-val | SL | Cell | L  | R  | P-val | SL |
|------|----|----|-------|----|------|----|----|-------|----|------|----|----|-------|----|
| 1    | 26 | 14 | 0.081 | 0.1| 27  | 2  | 3  | > 0.1 | -0.1| 53  | 1  | 0  | > 0.1 | -0.1|
| 2    | 4  | 14 | 0.031 | 0.3| 28  | 1  | 12 | 0.003 | 1.0| 54  | 58 | 4  | < 10^{-3} | 19.2|
| 3    | 34 | 2  | < 10^{-3} | 9.0| 29  | 2  | 9  | 0.065 | 0.2| 55  | 6  | 3  | > 0.1 | -0.1|
| 4    | 4  | 13 | 0.049 | 0.2| 30  | 3  | 0  | > 0.1 | 0.0| 56  | 0  | 0  | > 0.1 | -0.1|
| 5    | 1  | 2  | > 0.1 | -0.1| 31  | 8  | 22 | 0.016 | 0.5| 57  | 21 | 11 | > 0.1 | 0.1 |
| 6    | 2  | 11 | 0.022 | 0.4| 32  | 11 | 7  | > 0.1 | -0.1| 58  | 2  | 2  | > 0.1 | -0.1|
| 7    | 0  | 8  | 0.008 | 0.7| 33  | 2  | 9  | 0.065 | 0.2| 59  | 3  | 2  | > 0.1 | -0.1|
| 8    | 6  | 4  | > 0.1 | -0.1| 34  | 2  | 3  | > 0.1 | -0.1| 60  | 2  | 3  | > 0.1 | -0.1|
| 9    | 0  | 1  | > 0.1 | -0.1| 35  | 5  | 1  | > 0.1 | 0.0| 61  | 2  | 0  | > 0.1 | -0.1|
| 10   | 1  | 1  | > 0.1 | -0.1| 36  | 5  | 0  | 0.063 | 0.2| 62  | 11 | 6  | > 0.1 | 0.0 |
| 11   | 9  | 15 | > 0.1 | 0.0| 37  | 2  | 6  | > 0.1 | 0.0| 63  | 13 | 7  | > 0.1 | 0.0 |
| 12   | 0  | 0  | > 0.1 | -0.1| 38  | 2  | 3  | > 0.1 | -0.1| 64  | 13 | 10 | > 0.1 | -0.1|
| 13   | 6  | 2  | > 0.1 | 0.0| 39  | 49 | 3  | < 10^{-3} | 15.8| 65  | 20 | 5  | 0.004 | 1.0 |
| 14   | 9  | 0  | 0.004 | 1.0| 40  | 0  | 3  | > 0.1 | 0.0| 66  | 4  | 0  | > 0.1 | 0.1 |
| 15   | 0  | 0  | > 0.1 | -0.1| 41  | 3  | 2  | > 0.1 | -0.1| 67  | 12 | 13 | > 0.1 | -0.1|
| 16   | 2  | 14 | 0.004 | 0.9| 42  | 4  | 1  | > 0.1 | 0.0| 68  | 13 | 14 | > 0.1 | -0.1|
| 17   | 0  | 4  | > 0.1 | 0.1| 43  | 0  | 5  | 0.063 | 0.2| 69  | 10 | 7  | > 0.1 | -0.1|
| 18   | 5  | 14 | 0.064 | 0.2| 44  | 1  | 4  | > 0.1 | 0.0| 70  | 48 | 37 | > 0.1 | 0.0 |
| 19   | 1  | 0  | > 0.1 | -0.1| 45  | 0  | 2  | > 0.1 | -0.1| 71  | 13 | 5  | 0.096 | 0.1 |
| 20   | 5  | 1  | > 0.1 | 0.0| 46  | 5  | 13 | 0.096 | 0.1| 72  | 6  | 0  | 0.031 | 0.3 |
| 21   | 4  | 4  | > 0.1 | -0.1| 47  | 3  | 0  | > 0.1 | 0.0| 73  | 6  | 3  | > 0.1 | -0.1|
| 22   | 1  | 0  | > 0.1 | -0.1| 48  | 0  | 4  | > 0.1 | 0.1| 74  | 6  | 15 | 0.078 | 0.2 |
| 23   | 2  | 3  | > 0.1 | -0.1| 49  | 1  | 3  | > 0.1 | -0.1| 75  | 13 | 8  | > 0.1 | -0.1|
| 24   | 2  | 11 | 0.022 | 0.4| 50  | 8  | 6  | > 0.1 | -0.1| 76  | 13 | 18 | > 0.1 | -0.1|
| 25   | 1  | 0  | 0.012 | 0.6| 51  | 5  | 2  | > 0.1 | -0.1| 77  | 10 | 10 | > 0.1 | -0.1|
| 26   | 2  | 12 | 0.013 | 0.6| 52  | 2  | 6  | > 0.1 | 0.0| 78  | 38 | 25 | > 0.1 | 0.1 |

Table 3: The number of location reads and corresponding p-values at a change-point estimate at base pair 772,759, for \( N = 78 \) cells of the same patient. Both L and R refer to the number of reads in 5 bins, with L corresponding to base pairs 766,660–772,759 and R to base pairs 772,760–775,778. SL refers to \( \ell(p^n) \), where \( p^n \) is the p-value of cell \( n \), with parameters \( \lambda_1 = 1 \) and \( \lambda_2 = 1.86 \).
Section 4.2. The sparsity likelihood algorithm is applied to boost detection power by summing appropriately transformed p-value scores.

Table 3 lists the p-values at a typical change-point, estimated using a window length of 5 bins. Eight of them have p-values less than 0.01 and three of them p-values less than $10^{-3}$. The sum of p-value scores is 50.1, with the 3 smallest p-values (cells 3, 39 and 54) contributing 44.0 of the total score.

A Proof of Theorem 1

By (2.5), $P(\text{Type I error}) \leq e^{-CN} \to 0$. The proof that $P(\text{Type II error}) \to 0$ applies the lemma.

Lemma 1. Let $q = (q^1, \ldots, q^N)$, with $q^n \overset{i.i.d.}{\sim} \text{Uniform}(0,1)$. For fixed $\lambda_1 \geq 0$ and $\delta > 0$,

$$\sup_{\delta \leq \lambda_2 \leq \sqrt{N}} P(\ell(q) \leq -C\lambda_2^2) \to 0 \text{ as } C \to \infty \text{ and } N \to \infty.$$  

Proof. Let

$$x(p) = \frac{\lambda_1 \log N}{N} f_1(p) + \frac{\lambda_2}{\sqrt{N \log N}} f_2(p),$$

where $f_1(p) = \frac{1}{p(2-\log p)} - \frac{1}{2}$ and $f_2(p) = \frac{1}{\sqrt{p}} - 2$. Consider $N$ large such that $x(\frac{1}{N \log N}) \geq 0$, $x(1) \geq -\frac{1}{2}$ and $\frac{1}{\log N} \leq 1$. Hence

$$x(q) \geq \begin{cases} 0 & \text{if } q < \frac{1}{N \log N}, \\ -\frac{1}{2} & \text{if } q \geq \frac{1}{N \log N}. \end{cases}$$

Since $\log(1 + x) \geq x - x^2$ for $x \geq -\frac{1}{2}$, it follows that

$$\ell(q) = \sum_{n=1}^{N} \log(1 + x(q^n)) \geq \sum_{n=1}^{N} [x(q^n) - x^2(q^n)] 1_{\{q^n \geq (N \log N)^{-1}\}}. \quad (A.1)$$
Since $q^n \sim \text{Uniform}(0,1)$ and $\lambda_1 \leq a\lambda_2$ for $a = \lambda_1/\delta$,

$$
E[x(q^n)1_{\{q^n \geq (N \log N)^{-1}\}}]
\geq \frac{\lambda_1 \log N}{N} \left[ \frac{1}{2 - \log p} \right] (N \log N)^{-1} - \frac{1}{2} \left( 1 - \frac{1}{N \log N} \right)
+ \frac{\lambda_2}{\sqrt{N \log N}} \left[ 2 \sqrt{p} \right] (N \log N)^{-1} - 2 \left( 1 - \frac{1}{N \log N} \right)
\geq -2\lambda_1 - \frac{3\lambda_2}{N \log N} \geq -\frac{\lambda_2}{N} (2a + 3),
$$

$$
\text{Var}(x(q^n)1_{\{q^n \geq (N \log N)^{-1}\}})
\leq E[x^2(q^n)1_{\{q^n \geq (N \log N)^{-1}\}}]
\leq \left[ \frac{2\lambda_2^2 \log p}{N \log N} - \frac{3\lambda_2^2 (\log N)^2}{N \log N} \right] (N \log N)^{-1}
\leq \frac{3\lambda_2^2}{N} + \frac{4\lambda_2^4}{N \log N} \leq \frac{\lambda_2^2}{N} (3 + 4a^2),
$$

$$
\text{Var}(x^2(q^n)1_{\{q^n \geq (N \log N)^{-1}\}})
\leq E[x^4(q^n)1_{\{q^n \geq (N \log N)^{-1}\}}]
\leq [-\frac{4\lambda_4^2}{N^2 (\log N)^2} - \frac{2\lambda_4^2 (\log N)^4}{N^2 (\log N)^2}] (N \log N)^{-1}
\leq \frac{5\lambda_4^2 + 3\lambda_4^2}{N \log N} \leq \frac{\lambda_4^2}{N} (5 + 3a^4).
$$

Lemma follows from (A.1) and

$$
P\left( \sum_{n=1}^{N} x(q^n)1_{\{q^n \geq (N \log N)^{-1}\}} \leq -\frac{C\lambda_2^2}{2} \right)
\leq \frac{N \text{Var}(x(q^n)1_{\{q^n \geq (N \log N)^{-1}\}})}{N \text{E}[x(q^n)1_{\{q^n \geq (N \log N)^{-1}\}}]} \leq \frac{3 + 4a^2}{(\frac{4\lambda_4}{2} - 2a - 3)^2} \rightarrow 0,
$$

$$
P\left( \sum_{n=1}^{N} x^2(q^n)1_{\{q^n \geq (N \log N)^{-1}\}} \geq \frac{C\lambda_2^2}{2} \right)
\leq \frac{N \text{Var}(x^2(q^n)1_{\{q^n \geq (N \log N)^{-1}\}})}{C\lambda_2^2 \text{E}[x^2(q^n)1_{\{q^n \geq (N \log N)^{-1}\}}]} \leq \frac{5 + 3a^4}{(\frac{4\lambda_4}{2} - 3 - 4a^2)^2} \rightarrow 0,
$$

as $C \rightarrow \infty$. □

**Proof of Theorem** Let $q^n = \Phi(-Z^n + \mu_N Q^n)$, where $\mu_N = \sqrt{2\nu \log N}$, $Q^n$ is Bernoulli with mean $N^{-\beta}$ and $Z^n \sim N(\mu_N Q^n, 1)$ conditioned on $Q^n$. If $Q^n = 0$ then $p^n = q^n$. If $Q^n = 1$ and $Z^n \geq \mu_N$, then

$$
p^n = \int_{Z^n}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy
= \int_{Z^n - \mu_N}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z + \mu_N)^2/2} dz \leq q^n e^{-\frac{\mu_N^2}{2}} = o(q^n).
$$
Since \( \ell(p) \) is stochastically monotone with respect to \( \mu_N \), we may assume without loss of generality that \( \nu \leq 1 \) in case 1 and \( \nu \leq 4\beta - 2 \) in case 2 below.

Case 1: \( \frac{3}{4} \leq \beta < 1 \). Let

\[
\Gamma = \{ n : Q^n = 1, Z^n \geq \sqrt{2 \log N} \}.
\]

Hence \( \#\Gamma \sim \text{Binomial}(N, r_N) \), where
\[
r_N = N^{-\beta} \Phi(-\sqrt{2 \log N} + \sqrt{2\nu \log N}) \geq N^{\delta - 1}, \tag{A.3}
\]
for \( 0 < \delta < 1 - \beta - (1 - \sqrt{\nu})^2 \). For \( Z_n \geq 0 \),
\[
p^n = \Phi(-Z_n) \leq e^{-Z_n^2/2}. \tag{A.4}
\]

Hence \( p^n \leq N^{-1} \) for \( n \in \Gamma \) and by (A.2) and \( \log(1 + x) \sim x \) as \( x \to 0 \),
\[
\sum_{n \in \Gamma} [\ell(p^n) - \ell(q^n)] \geq [1 + o(1)] \frac{\lambda_2(\#\Gamma)}{\sqrt{\log N}}. \tag{A.5}
\]

Since \( \ell(p^n) \geq \ell(q^n) \) for all \( n \), \( P(\text{Type II error}) \to 0 \) follows from Lemma 1, (A.3), (A.5) and \( c_N = o(N^{1/2}) \).

Case 2: \( \frac{1}{2} < \beta < \frac{3}{4} \). Let

\[
\Gamma = \{ n : Q^n = 1, Z^n \geq 2\sqrt{(2\beta - 1) \log N} \}.
\]

Hence \( \#\Gamma \sim \text{Binomial}(N, r_N) \), where
\[
r_N = N^{-\beta} \Phi(-2\sqrt{(2\beta - 1) \log N} + \sqrt{2\nu \log N}) \geq N^{-2\beta + \frac{1}{2} + \delta}, \tag{A.6}
\]
for \( 0 < \delta < \beta - \frac{1}{2} - (2\sqrt{\beta - \frac{1}{2} - \sqrt{\nu}})^2 \). By (A.4), for \( n \in \Gamma \), \( p^n \leq N^{2-4\beta} \) and therefore by (A.2) and \( \log(1 + x) \sim x \) as \( x \to 0 \),
\[
\sum_{n \in \Gamma} [\ell(p^n) - \ell(q^n)] \geq [1 + o(1)] \frac{\lambda_2 N^{-\frac{3}{2} + 2\beta(\#\Gamma)}}{\sqrt{\log N}}. \tag{A.7}
\]

Since \( \ell(p^n) \geq \ell(q^n) \) for all \( n \), \( P(\text{Type II error}) \to 0 \) follows from Lemma 1, (A.6), (A.7) and \( c_N = o(N^{1/2}) \). \( \square \)
B Proof of Theorem 2

Let $\lfloor \cdot \rfloor$ ($\lceil \cdot \rceil$) denote the greatest (least) integer function.

Proof of Theorem 2 (a)i. Let $h = \lfloor \frac{4(1-\epsilon)\log T}{\Delta^2V} \rfloor$ for some $0 < \epsilon < 1$. Let $P_0$ denote probability with respect to $\mu_t^n = 0$ for all $n$ and $t$. Let $t_k = (2k-1)h$ and let $P_k$, $1 \leq k \leq K := \lceil \frac{T}{2h} \rceil$, denote probability under which, for $n \leq V$,

$$\mu_{t_k-h}^n = \cdots = \mu_t^n = -\frac{\Delta}{2},$$

$$\mu_{t_k}^n = \cdots = \mu_{t_k+h}^n = \frac{\Delta}{2},$$

$$\mu_t^n = 0 \text{ for } t \leq t_k - h \text{ and } t > t_k + h,$$

and $\mu_1^n = \cdots = \mu_T^n = 0$ for $n > V$. Let $E_k$ denote expectation with respect to $P_k$.

Let $P = \frac{1}{K} \sum_{k=1}^K P_k$ and let $L = \frac{1}{K} \sum_{k=1}^K L_k$, where $L_k = \frac{dP_k}{dP_0}(X)$ with $X = (X^n_t : 1 \leq n \leq N, 1 \leq t \leq T)$. Hence

$$\log L_1 = \frac{h\Delta}{2} \sum_{n=1}^V \left( \bar{X}_{n,2h}^n - \bar{X}_{0h}^n \right) - \frac{hV\Delta^2}{4}. \quad (B.2)$$

Let $A_i = \{L \leq 3\} \cap \{\text{conclude } H_i\}$. Since $P(A_1) = E_0(L1_{A_1}) \leq 3P_0(A_1)$,

$$P_0(\text{Type I error}) + P(\text{Type II error}) \geq P_0(A_1) + P(A_0) \geq \frac{1}{3}P(L \leq 3) = \frac{1}{3}P_1(L \leq 3). \quad (B.3)$$

Since $E_1L_k = 1$ for $k \geq 2$, it follows that $P_1(\frac{1}{K} \sum_{k=2}^K L_k \leq 2) \geq \frac{1}{2}$, and hence by (B.3), to show $P_0(\text{Type I error}) + P(\text{Type II error}) \to 0$ is not possible, it suffices to show that

$$P_1(L_1 \leq K) \to 1 \text{ as } T \to \infty. \quad (B.4)$$

By (B.2), $\log L_1 \sim N(\frac{hV\Delta^2}{4}, \frac{hV\Delta^2}{2})$, and indeed

$$P_1(L_1 \leq K) = \Phi\left( \frac{\log K - \frac{1}{2}hV\Delta^2}{\sqrt{\frac{1}{2}hV\Delta^2}} \right) \to 1. \quad \Box$$

Proof of Theorem 2 (a)ii. Proceed as in the proof of Theorem 2 (a)i., but with $h = \lfloor \frac{4(1-\rho)2\beta(\beta_1)\log N}{\Delta^2V} \rfloor$, and $P_k$ probability under which, independently for $1 \leq n \leq N$, $Q^n = 1$ with probability $2N^{-\beta}$ and $Q^n = 0$ otherwise, with (B.1) satisfied when $Q^n = 1$ and $\mu_1^n = \cdots = \mu_T^n = 0$ when $Q^n = 0$. 

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By the law of large numbers, \( P_1(\mu \in \Omega_1(h, \Delta, V)) = P_1(\sum_{n=1}^{N} Q^n \geq V) \rightarrow 1 \). Hence by (B.3) it suffices to show (B.4) with

\[
L_1 = \prod_{n=1}^{N} \left( 1 + 2N^{-\beta} \left[ \exp \left( Z^n \Delta \sqrt{\frac{h}{2}} - \frac{h\Delta^2}{4} \right) - 1 \right] \right),
\]

(B.5)

\[
Z^n = \sqrt{\frac{h}{2}} \left( X^n_{h,2n} - X^n_{0,h} \right) \sim N \left( Q^n \Delta \sqrt{\frac{h}{2}}, 1 \right).
\]

(B.6)

Case 1: \( \frac{1-\zeta}{2} < \beta < \frac{3(1-\zeta)}{4} \), \( \rho Z(\beta, \zeta) = \beta - \frac{1-\zeta}{2} \). By (B.5) and (B.6),

\[
E_1 L_1 = \left( 1 + 4N^{-2\beta}[\exp(\frac{h\Delta^2}{4}) - 1] \right)^N \leq \exp(4N^{1-2\beta+2(1-\epsilon)\rho_N(\beta, \zeta)}) = \exp(4N^{\zeta-2\rho_N(\beta, \zeta)}).
\]

Hence \( P_1(L_1 \leq K) \geq 1 - K^{-1}E_1 L_1 \rightarrow 1 \) and (B.4) is shown.

Case 2: \( \frac{3(1-\zeta)}{4} \leq \beta < 1 - \zeta \), \( \rho Z(\beta, \zeta) = (\sqrt{1-\zeta} - \sqrt{1-\zeta - \beta})^2 \). Express \( \log L_1 = \sum_{i=0}^{3} R_i \), where

\[
R_i = \sum_{n \in \Gamma_i} \log \left( 1 + 2N^{-\beta} \left[ \exp \left( Z^n \Delta \sqrt{\frac{h}{2}} - \frac{h\Delta^2}{4} \right) - 1 \right] \right),
\]

\[
\Gamma_0 = \{ n : Q^n = 0 \},
\]

\[
\Gamma_1 = \{ n : Q^n = 1, Z^n \leq \sqrt{2(1-\zeta)} \log N \},
\]

\[
\Gamma_2 = \{ n : Q^n = 1, \sqrt{2(1-\zeta)} \log N < Z^n \leq 2\sqrt{2 \log N} \},
\]

\[
\Gamma_3 = \{ n : Q^n = 1, Z^n > 2\sqrt{2 \log N} \}.
\]

To show (B.4), it suffices to show that

\[
P_1(R_i \geq \frac{1}{4} \log K) \rightarrow 0 \text{ for } 0 \leq i \leq 3.
\]

(B.7)

Since \( \Delta \sqrt{\frac{h}{2}} \leq \sqrt{2 \log N} \),

\[
P_1(R_3 > 0) \leq 2N^{1-\beta} \Phi(-\sqrt{2 \log N}) \rightarrow 0,
\]

and (B.7) holds for \( i = 3 \).

Moreover,

\[
E_1 R_2 \leq E_1(\#\Gamma_2) \log(1 + 2N^{4-\beta}).
\]

(B.8)

Since \( \Delta \sqrt{\frac{h}{2}} \leq \sqrt{1-\epsilon}(\sqrt{2(1-\zeta)} \log N - \sqrt{2(1-\zeta - \beta)} \log N) \), it follows that for \( N \) large,

\[
\Phi \left( \Delta \sqrt{\frac{h}{2}} - \sqrt{2(1-\zeta)} \log N \right) \leq \frac{1}{\log N} \Phi \left( -\sqrt{2(1-\zeta - \beta)} \log N \right)
\]

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and therefore

$$E_1(\#\Gamma_2) \leq 2N^{1-\beta} \Phi\left(\Delta \sqrt{\frac{h}{2}} - \sqrt{2(1 - \zeta) \log N}\right) = o\left(\frac{N^\zeta}{\log N}\right). \quad (B.9)$$

We conclude (B.7) for $i = 2$ from (B.8) and (B.9).

Since $\log(1 + x) \leq x$,

$$E_1R_1 \leq 4N^{1-2\beta}e^{-h\Delta^2/4}$$

$$\times \int_{-\infty}^{\sqrt{2(1-\zeta)\log N}} \frac{1}{\sqrt{2\pi}} e^{-\left(z-\Delta\sqrt{\frac{h}{2}}\right)^2/2} z \, dz$$

$$= 4N^{1-2\beta} \Phi\left(\sqrt{2(1 - \zeta) \log N} - 2\Delta \sqrt{\frac{h}{2}}\right) e^{h\Delta^2/2}$$

$$\leq 4N^{1-2\beta}\left(\sqrt{1-\zeta} - 2\sqrt{(1-\zeta)\rho_2(\beta,\zeta)} + 2(1-\zeta)\rho_2(\beta,\zeta)\right).$$

Let $m(\rho) = -(\sqrt{1-\zeta} - 2\sqrt{\rho})^2 + 2\rho$. Check that

$$m(\rho_2(\beta,\zeta)) = -(\sqrt{1-\zeta} - 2\sqrt{\rho_2(\beta,\zeta)}) + 2\rho_2(\beta,\zeta)$$

$$= -2\sqrt{1-\zeta - \beta - \sqrt{1-\zeta}} + 2(\sqrt{1-\zeta} - \sqrt{1-\zeta - \beta})^2$$

$$= 1 - \zeta - 2(1 - \zeta - \beta) = \zeta - 1 + 2\beta.$$  

Moreover,

$$\frac{d}{d\rho} m(\rho) = 2\rho^{-\frac{1}{2}}(\sqrt{1-\zeta} - 2\sqrt{\rho}) + 2$$

$$= 2\rho^{-\frac{1}{2}} \sqrt{1-\zeta} - 2 > 0 \text{ for } \rho < 1 - \zeta.$$  

Hence by (B.10), there exists $\delta > 0$ such that

$$E_1R_1 \leq 4N^{1-2\beta-m(\rho_2(\beta,\zeta))-\delta} = 4N^{\zeta-\delta} = o(\log K),$$

and (B.7) for $i = 1$ follows from

$$R_1 \geq (\#\Gamma_1)\log(1 - 2N^{-\beta}) \overset{p}{\sim} -4N^{1-2\beta} = o(N^\zeta) = o(\log K).$$

Since $E_1e^{R_0} = 1$,

$$P_1(R_0 \geq \frac{1}{4}\log K) \leq K^{-\frac{1}{4}} \to 0,$$

and (B.7) is shown for $i = 0$. ☐

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Proof of Theorem 2(b). The proof is similar to that of Theorem 2(a), with \( h = \left\lfloor \frac{4(1-2\eta - \epsilon) T^2 \log T}{T^2 V^2} \right\rfloor \) for i. and \( h = \left\lfloor \frac{4(1-\epsilon) \rho Z(\beta, \zeta) T^2 \log N}{T^2 V} \right\rfloor \) for ii. In Theorem 2(a), \( \log K \sim \log T \) whereas here

\[
\log K \sim (1 - 2\eta) \log T. \tag{B.11}
\]

In the case of i., the constant \((1 - 2\eta)\) in (B.11) lowers the limits of \( h \), under which (4.3) cannot be achieved, by a factor of \((1 - 2\eta)\). \( \square \)

C Proof of Theorem 3

Since \( t - s \leq h_i \) and \( u - t \leq h_i \) for all \((s, t, u) \in A_i(T)\), by (2.5) and \( cT - \log(\sum_{i=1}^{i_T} h_i^2) \to \infty \),

\[
\sup_{\mu \in \Omega_0} P_{\mu} \text{(Type I error)} \leq \sum_{i=1}^{i_T} \frac{T}{d_i} \exp(-c_T - \log(\frac{T}{2d_i})) \tag{C.1}
\]

\[
= 2e^{-c_T \sum_{i=1}^{i_T} h_i^2} \to 0.
\]

Consider \( \mu \in \Omega_1(\Delta, h, V) \) and let \( \tau_j \) be the change-point satisfying the conditions in the definition of \( \Omega_1(\Delta, h, V) \). Let \( Q^n = 1 \) if \( |\mu^n_{\tau_{j+1}} - \mu^n_{\tau_j}| \geq \Delta \) and \( Q^n = 0 \) otherwise. We may assume without loss of generality that \( 0 < \epsilon < 1 \).

Proof of Theorem 3(a)i. Consider \( V = o(\frac{N\rho}{\log N}) = o(\frac{\log T}{\log N}) \). Since \( hV \Delta^2 = 4(1+\epsilon) \log T, \frac{h_{i+1}}{h_i} \to 1 \) and \( d_i = o(h_i) \), for large \( T \) there exists

\[
h_i \geq 4(1+\epsilon)\frac{\Delta}{2} \Delta^{-2} V^{-1} \log T
\]

such that for all \( \mu \in \Omega_1(\Delta, h, V) \), there exists \( k \) satisfying

\[
\tau_{j-1} < s(ik) < u(ik) < \tau_{j+1} \text{ and } |t(ik) - \tau_j| \leq \frac{d_j}{2}. \tag{C.2}
\]

Hence when \( Q^n = 1 \),

\[
|E_{\mu Z^n}| \geq \Delta (1 - \frac{d_j^2}{2d_i^2}) \sqrt{\frac{h_j}{2}} \geq \sqrt{2(1+\epsilon)\frac{\Delta}{2} V^{-1} \log T}, \tag{C.3}
\]

where \( Z^n = Z^n_{s(ik), t(ik), u(ik)} \).
Let \( m = \sum_{n=1}^{N} Q^n \) and \( \Gamma = \{ n : Q^n = 1, |Z^n| \geq \sqrt{2(1 + \epsilon) V^{-1} \log T} \}. \)

By (C.3) and the law of large numbers,

\[
P_{\mu}(\# \Gamma \geq (1 + \epsilon)^{-\frac{1}{4}} m) \to 1. \tag{C.4}
\]

Let \( p^n = p^n_{s(i(k), t(i(k), u(i(k)))}. \) Let \( q^n \) be the upper tail probability of \( |Z^n| \)
under \( P_{\mu}. \) Since \( q^n \) i.i.d. Uniform(0,1) and \( \lambda_2 = o(\sqrt{\log T}), \) by Lemma 1,

\[
P_{\mu}\left( \sum_{n: Q^n = 1} \ell(q^n) \geq -C \log T \right) \to 1 \text{ for any } C > 0. \tag{C.5}
\]

If \( Q^n = 1 \) and \( |Z^n| \geq z_0 \) for \( z_0 \leq |E_\mu Z^n|, \) then

\[
p^n = 2 \int_{|Z^n|}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \tag{C.6}
= 2 \int_{|Z^n| - z_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z+z_0)^2/2} dz \leq 2e^{-z_0^2/2} q^n.
\]

In particular by (C.3), for \( n \in \Gamma, \)

\[p^n \leq 2 \exp(- (1 + \epsilon)^{\frac{1}{4}} V^{-1} \log T). \tag{C.7}\]

Since \( V^{-1} \log T \) is large compared to \( \log N, \) by (C.7),

\[\ell(p^n) \geq (1 + \epsilon)^{\frac{1}{4}} V^{-1} \log T \text{ for } n \in \Gamma. \]

Moreover \( \ell(p) \geq -1 \) for \( N \) large and therefore

\[
\sum_{n: Q^n = 1} \ell(p^n) \geq (\# \Gamma)(1 + \epsilon)^{\frac{1}{4}} V^{-1} \log T - m. \tag{C.8}
\]

We conclude

\[
\sup_{\mu \in \Omega_1(\theta, \Delta, V)} P_{\mu} \text{ (Type II error)} \to 0 \tag{C.9}
\]

from (C.4), (C.5), (C.8), \( m \geq V, \)

\[
c_T + \log(\frac{T}{2m}) \sim \log T \sim N^\zeta \tag{C.10}
\]

and \( V^{-1} \log T \to \infty. \) □
Proof of Theorem 3(a)ii. Case 1: \( V \sim N^{1-\beta} \) for \( \frac{3(1-\xi)}{4} \leq \beta < 1 - \zeta \). Since \( h\Delta^2 = 4(1+\epsilon)(\sqrt{1-\zeta} - \sqrt{1-\zeta-\beta})^2 \log N \) and \( d_i = o(h_i) \), for large \( N \) there exists \( i \) satisfying \( h_i \geq (1+\epsilon)^{-\frac{1}{2}}h \) such that whenever \( Q^n = 1 \),

\[
|E_\mu Z^n| \geq \Delta (1 - \frac{d_i}{2h_i}) \sqrt{\frac{h_i}{2}} \geq \sqrt{2\nu \log N},
\]

(C.11)

with \( Z^n = Z^n_{s(ik),t(ik),u(ik)} \) and \( k \) satisfying (C.2).

Let \( \Gamma = \{ n: Q^n = 1, |Z^n| \geq \sqrt{2(1-\zeta) \log N} \} \). Let \( \delta \) be such that \( 0 < \delta < 1 - \zeta - \beta - (\sqrt{1-\zeta} - \sqrt{\nu})^2 \). By (C.11), for \( N \) large,

\[
E_\mu(\#\Gamma) \geq V\Phi\left(-\sqrt{2(1-\zeta) \log N} + \sqrt{2\nu \log N}\right) \\
\geq 2N^{1-\beta - (\sqrt{1-\zeta} - \sqrt{\nu})^2 - \delta} = 2N^{\zeta+\delta},
\]

and therefore by the law of large numbers,

\[
P_\mu(\#\Gamma \geq N^{\zeta+\delta}) \to 1.
\]

(C.12)

Let \( p^n \) and \( q^n \) be defined as in the proof of Theorem 3(a)i. By (C.6), for \( n \in \Gamma \),

\[
p^n \leq 2N^{\zeta-1} \text{ and } p^n = o(q^n).
\]

(C.13)

By (C.13), \( \lambda_2 \sim \sqrt{\frac{N^\zeta}{\log N}} \) and \( \log(1+x) \sim x \) as \( x \to 0 \),

\[
\sum_{n \in \Gamma} \left[ \ell(p^n) - \ell(q^n) \right] \geq [1 + o(1)] \frac{\lambda_2(\#\Gamma)}{\sqrt{2N^\zeta \log N}} \sim \frac{\#\Gamma}{\sqrt{2\log N}}.
\]

(C.14)

Since \( \ell(p^n) \geq \ell(q^n) \) for all \( n \), (C.9) follows from (C.5), (C.10), (C.12) and (C.14).

Case 2: \( V \sim N^{1-\beta} \) for \( \frac{1-\zeta}{2} < \beta < \frac{3(1-\xi)}{4} \). Since \( h\Delta^2 = 4(1+\epsilon)(\beta - \frac{1-\zeta}{2}) \log N \), for large \( N \) there exists \( h_i \geq (1+\epsilon)^{-\frac{1}{2}}h \) such that whenever \( Q^n = 1 \),

\[
|E_\mu Z^n| \geq \Delta (1 - \frac{d_i}{2h_i}) \sqrt{\frac{h_i}{2}} \geq \sqrt{2\nu \log N},
\]

(C.15)

with \( Z^n = Z^n_{s(ik),t(ik),u(ik)} \) and \( k \) satisfying (C.2).

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Let $\Gamma = \{ n : Q^n = 1, |Z^n| \geq 2\sqrt{(2\beta - 1 + \zeta)\log N} \}$. Let $\delta$ be such that $(0 <) 2\delta = \beta - \frac{1-\zeta}{2} - (2\sqrt{\beta - \frac{1-\zeta}{2} - \sqrt{\nu})^2}$. By (C.15), for $N$ large,

$$
E_\mu(\#\Gamma) \geq V\Phi\left( -2\sqrt{(2\beta - 1 + \zeta)\log N + \sqrt{2\nu\log N}} \right) \geq 2N^{1-\beta - (2\sqrt{\beta - \frac{1-\zeta}{2} - \sqrt{\nu})^2 - \delta} = 2N^{\frac{3}{2} - 2\beta - \frac{\zeta}{2} + \delta},
$$

and therefore by the law of large numbers,

$$
P_\mu(\#\Gamma \geq N^{\frac{3}{2} - 2\beta - \frac{\zeta}{2} + \delta}) \to 1. \quad (C.16)
$$

Let $p^n$ and $q^n$ be defined as in the proof of Theorem 3(a)i. By (C.6), for $n \in \Gamma$,

$$
p^n \leq 2N^{-4\beta + 2 - 2\zeta} \text{ and } p^n = o(q^n). \quad (C.17)
$$

By (C.17) and $\log(1 + x) \sim x$ as $x \to 0$,

$$
\sum_{n \in \Gamma} [\ell(p^n) - \ell(q^n)] \geq [1 + o(1)]\frac{N^{2\beta - 1 + \zeta}(\#\Gamma)}{\sqrt{2N\log N}} \sim \frac{N^{2\beta - \frac{3}{2}(1-\zeta)(\#\Gamma)}}{\sqrt{2N\log N}}. \quad (C.18)
$$

Since $\ell(p^n) \geq \ell(q^n)$ for all $n$, (C.9) follows from (C.5), (C.10), (C.16) and (C.18). □

Proof of Theorem 3(b)i. Let $h_i \geq (1 - 2\eta + \epsilon)^{-\frac{1}{4}}h$ be such that for all $\mu \in \Omega_1(h, \Delta, V)$, (C.2) holds for some $k$. Since $h_i \geq T^{2\eta}$ for $T$ large, 

$$
c_T + \log(T^{2\epsilon}) \leq [1 - 2\eta + o(1)]\log T. \quad (C.19)
$$

In contrast in (C.11) the right-hand side is $\log T$. The additional factor $(1 - 2\eta)$ allows us to follow the proof of Theorem 3(a)i. to show (C.9) with an additional $(1 - 2\eta)$ in the expression of $h$.

ii. When $V \sim N^{1-\beta}$ for $\frac{1-\zeta}{2} < \beta < 1 - \zeta$, the same proof of Theorem 3(a)ii. applies. □

D Proof of Theorem 4

Proof of Theorem 4(a). Let $h \equiv \frac{(1-\epsilon)\log T}{\mu_0}$ for some $0 < \epsilon < 1$. Let $P_0$ denote probability with respect to $\mu_0^n = (\frac{1+\epsilon}{2})^t \mu_0$ for all $n$ and $t$. Let
\( t_k = (2k - 1)h \). Let \( P_k, 1 \leq k \leq K := \left\lfloor \frac{T}{2h} \right\rfloor \), denote probability under which for \( n \leq V \),

\[
\mu^n_t = \begin{cases} 
\mu_0 & \text{for } t_k - h < t \leq t_k, \\
r\mu_0 & \text{for } t_k < t \leq t_k + h, \\
\left(\frac{1+r}{2}\right)\mu_0 & \text{for } t \leq t_k - h \text{ and } t > t_k + h,
\end{cases}
\]  

(D.1)

and \( \mu^n_1 = \cdots = \mu^n_T = \left(\frac{1+r}{2}\right)\mu_0 \) for \( n > V \). Let \( E_k \) and \( \text{Var}_k \) denote expectation and variance respectively with respect to \( P_k \). Let \( L_1 \) = \( \frac{dP_1}{dP_0}(X) = \prod_{n=1}^{V} \exp(U^n) \),

(D.2)

\( U^n \) = \( S^n_{0h} \log(\frac{2}{1+r}) + S^n_{h} \log(\frac{2r}{r+1}) \).

(D.3)

By (B.3)–(B.4), it suffices to show that \( P_1(L_1 \leq K) \to 1 \) as \( T \to \infty \).

(D.4)

Since \( E_1(\log L_1) = h\mu_0 V I_r \) and \( \text{Var}_1(\log L_1) = h\mu_0 V C_r \), where \( C_r = r[\log(\frac{2r}{r+1})]^2 + [\log(\frac{2}{1+r})]^2 \), by Chebyshev’s inequality,

\[ P_1(L_1 \leq K) \geq 1 - \frac{hV\mu_0 C_r}{(\log K - h\mu_0 C_r)^2} \to 1, \]

and (D.4) is shown. \( \Box \)

We preface the proof of Theorem 4(b) with Lemma 2, which provides an alternative representation of \( \rho_r(\beta, \zeta) \). Let

\[ D(\omega) = \frac{1}{1+r} \log(\frac{2}{1+r}) + \frac{\omega}{1+r} \log(\frac{2\omega}{1+r}) \text{ and } g(\omega) = \left(\frac{1+\omega}{2}\right)^{\frac{1}{2}}. \]  

(D.5)

By (4.10), for \( \frac{1-\zeta}{2} < \beta < 1 - \zeta \),

\[ \rho_r(\beta, \zeta) = \max_{\frac{1-\zeta}{2} < \omega \leq 2} \xi(\omega), \text{ where } \xi(\omega) = \frac{\beta - \omega^{-1}(1-\zeta)}{2g(\omega) - 1 - r}. \]

(D.6)

**Lemma 2.** For \( \frac{1}{2} < \frac{\beta}{1-\zeta} \leq \frac{1}{2}[1 + \frac{2g(2)-1-r}{g(2)D(2)}] \), \( \xi \) achieves its maximum at \( \omega = 2 \) and

\[ \rho_r(\beta, \zeta) = \frac{\beta - \frac{1}{2}(1-\zeta)}{2g(2)-1-r}. \]

(D.7)

For \( \frac{1}{2}[1 + \frac{2g(2)-1-r}{g(2)D(2)}] < \frac{\beta}{1-\zeta} < 1 \), \( \xi \) achieves its maximum at some \( \omega < 2 \) and

\[ \rho_r(\beta, \zeta) = \frac{1-\zeta}{2g(\omega)D(\omega)}. \]

(D.8)
\[ \frac{d}{d\omega} \log \xi(\omega) = \frac{\omega^{-2}(1-\zeta)}{\beta - \omega^{-1}(1-\zeta)} - \frac{2g(\omega)}{2g(\omega) - 1 - r}, \]
\[ \frac{d}{d\omega} g(\omega) = \frac{\partial}{\partial \omega} \exp \left[ \frac{1}{2} \log \left( \frac{1 + \omega}{2} \right) \right] \]
\[ = \frac{\left( \frac{\omega}{\omega(1+r)} \right) - \frac{1}{\omega^2} \log \left( \frac{1 + \omega}{2} \right) }{D(\omega)g(\omega)}, \]

it follows that \( \frac{d}{d\omega} \log \xi(\omega) = 0 \) when
\[ \omega^{-2}(1-\zeta)[2g(\omega) - 1 - r] = 2[\beta - \omega^{-1}(1-\zeta)] \frac{D(\omega)g(\omega)}{\omega^2}, \]

that is when
\[ \frac{\beta}{1-\zeta} = \omega^{-1} + \frac{2g(\omega) - 1 - r}{2g(\omega)D(\omega)}. \]

For \( \frac{1}{2} < \frac{\beta}{1-\zeta} \leq \frac{1}{2} \left[ 1 + \frac{2g(2)-1-r}{g(2)D(2)} \right] \), the solution of \( \omega \) to (D.10) is at least 2 and the maximum in (D.6) is attained at \( \omega = 2 \). We conclude (D.7). For \( \frac{1}{2} \left[ 1 + \frac{2g(2)-1-r}{g(2)D(2)} \right] < \frac{\beta}{1-\zeta} < 1 \), the solution of \( \omega \) to (D.10) lies in the interval \( (\frac{1-\zeta}{\beta}, 2) \). We conclude (D.8) from (D.6) and a rearrangement of (D.9). \( \square \)

**Proof of Theorem 4.** For \( \frac{1-\zeta}{\beta} < \beta < 1-\zeta \), let \( \omega \) be the maximizer in
\[ \rho_r(\beta, \zeta) = \max_{1/2 < \omega \leq 2} \left( \frac{\beta - \omega^{-1}(1-\zeta)}{2g(\omega) - 1 - r} \right). \]

Let \( h = \left( \frac{(1-\zeta)\rho_r(\beta, \zeta)\log N}{\mu_0} \right) \) for some \( \epsilon > 0 \). Let \( P_0 \) denote probability with respect to \( \mu^n_1 = g(\omega)\mu_0 \) for all \( n \) and \( t \). Let \( t_k = (2k-1)h \). Let \( P_k, 1 \leq k \leq K : = \lceil \frac{1}{h} \rceil \), denote probability under which, independently for \( 1 \leq n \leq N, Q^n = 1 \) with probability \( 2N^{-\beta} \), and \( Q^n = 0 \) otherwise, such that when \( Q^n = 1 \),
\[ \mu^n_k = \begin{cases} 
\mu_0 & \text{for } t_k - h < t \leq t_k, \\
r\mu_0 & \text{for } t_k < t \leq t_k + h, \\
g(\omega)\mu_0 & \text{for } t \leq t_k - h \text{ and } t > t_k + h, 
\end{cases} \]

whereas when \( Q^n = 0, \mu^n_1 = \cdots = \mu^n_K = g(\omega)\mu_0 \). Let \( E_1 \) denote expectation with respect to \( P_1 \). Let \( P_Q = P_1(\cdot | Q^1 = 1) \) and let \( E_Q \) denote expectation with respect to \( P_Q \).

By (B.3)–(B.4), it suffices to show (D.4) for
\[ L_1 = \frac{dP_0}{d\mu_0}(X) = \prod_{n=1}^{N} \left( 1 + 2N^{-\beta}[\exp(U^n) - 1] \right), \]
\[ U^n = S_{0n}^0 \log \left( \frac{1}{g(\omega)} \right) + S_{h,2n}^n \log \left( \frac{r}{g(\omega)} \right) - h\mu_0 [1 + r - 2g(\omega)]. \] (D.13)

Let \( S_{0n} = S_{0n}^1 \) and \( S_{h,2n} = S_{h,2n}^1 \). We apply in (D.15), (D.18) and (D.19) the identity, for \( X \sim \text{Poisson}(\lambda) \) and constant \( C > 0 \),
\[
E(C^X) = \sum_{x=0}^{\infty} e^{-\lambda(C\lambda)^r_x} = e^{\lambda(C-1)}. \] (D.14)

Case 1: \( \frac{1}{2} < \frac{\beta}{1-\zeta} \leq \frac{1}{2} \left[ 1 + \frac{2\beta(2)-1-r}{g(2)D(2)} \right], \omega = 2 \). By Lemma 2, (D.12)–(D.14) and \([g(2)]^2 = \frac{1+r^2}{2}\),
\[
E_Q \exp(U^1) = E_Q \left[ (\frac{1}{g(2)})^{S_{0n}^h} \left( \frac{r}{g(2)} \right)^{S_{h,2n}^n} \right] \exp(-h\mu_0[1+r-2g(2)])
\]
\[
= \exp(h\mu_0 \left[ \frac{1}{g(2)} - 1 + \frac{r}{g(2)} - r \right] - h\mu_0 [1 + r - 2g(2)])
\]
\[
= \exp(2h\mu_0 [2g(2) - 1 - r])
\]
\[
= \exp \left( \frac{h\mu_0 (2\beta - 1 + \zeta)}{\rho \nu (\beta, \zeta)} \right) \leq N^{(1-\epsilon)(2\beta - 1 + \zeta)}. \] (D.15)

Hence
\[
E_1 L_1 = (1 + 4N^{-2\beta} [E_Q \exp(U^1) - 1])^N \leq \exp(4N^{\zeta - \delta}) = o(K),
\]
where \( \delta = \epsilon(2\beta - 1 + \zeta) \), and (D.3) holds.

Case 2: \( \frac{1}{2} \left[ 1 + \frac{2\beta(2)-1-r}{g(2)D(2)} \right] < \frac{\beta}{1-\zeta} < 1 \). Express
\[
\log L_1 = R_0 + R_1, \tag{D.16}
\]
where \( R_i = \sum_{n \in \Gamma_i} \log(1 + 2N^{-\beta} \{ \exp(U^n) - 1 \}) \),
\[
\Gamma_0 = \{ n : Q^n = 0 \} \cup \{ n : Q^n = 1, \exp(U^n) \leq N^\beta \},
\]
\[
\Gamma_1 = \{ n : Q^n = 1, \exp(U^n) > N^\beta \}.
\]

We show (D.1) by showing that
\[
P_i(R_i \leq \frac{1}{2} \log K) \to 1 \text{ for } i = 0 \text{ and } 1. \] (D.17)

Let \( a = \omega - 1 \) with \( \omega \) the maximizer in (D.11). By (D.11), (D.13), (D.14) and \( g(\omega) = \frac{1+r^{\alpha+1}}{2g(\omega)} \),
\[
E_Q[\exp(U^1) \mathbf{1}_{(1 \in \Gamma_0)}] \leq N^{\beta(1-a)} E_Q \exp(aU^1) \tag{D.18}
\]
\[
\leq N^{\beta(1-a)} \exp(h\mu_0 \left[ \frac{1}{g(\omega)} - 1 + \frac{r^{\alpha+1}}{g(\omega)} - r \right] - ah\mu_0 [1 + r - 2g(\omega)])
\]
\[
= N^{\beta(1-a)} \exp(\omega h\mu_0 [2g(\omega) - 1 - r])
\]
\[
= N^{\beta(1-a)} \exp \left( \frac{h\mu_0 (\beta - 1 + \zeta)}{\rho \nu (\beta, \zeta)} \right) \leq N^{2\beta - 1 + \zeta - \delta},
\]

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where \( \delta = \epsilon (\beta \omega - 1 + \zeta) \). Since \( E_0 \exp(U^n) = 1 \), it follows from (D.18) that
\[
E_1 \exp(R_0) \leq (1 + 4N^{-2\beta}E_Q[\exp(U^1)1_{\{1 \in \Gamma_0\}}])^N \leq \exp(4N^{\zeta - \delta}),
\]
and (D.17) holds for \( i = 0 \).

Express \( U^1 = v_1S_{0h} + v_2S_{h,2h} - z \), where \( v_1 = \log(\frac{1}{g(\omega)}) \), \( v_2 = \log(\frac{1}{g(\omega)}) \) and \( z = h\mu_0[1 + r - 2g(\omega)] \). By Markov’s inequality, (D.14) and \( g(\omega) = 1 + ra + 1 + \frac{1}{2}g(\omega) \),
\[
E_1(\#\Gamma_1) = 2N^{1-\beta}P_Q(e^{aU^1} > N^{\alpha\beta}) \leq 2N^{1-\beta-a\beta} e^{-az}E_Q(e^{\frac{aS_{0h}}{2} + \frac{aS_{h,2h}}{2}}) = 2N^{1-\omega\beta} \exp(wh\mu_0[2g(\omega) - 1 - r]) = 2N^{1-\omega\beta} \exp\left(\frac{h\mu_0(\beta \omega - 1 + \zeta)}{r(\beta, \zeta)}\right) \leq N^{\zeta - \delta},
\]
where \( \delta = \epsilon (\beta \omega - 1 + \zeta) \). Since \( R_1 \leq (\#\Gamma_1) \max_{n \in \Gamma_1} U^n \) and \( P_1(\max_n U^n \geq N^{\frac{\delta}{4}}) \to 0 \),
we conclude (D.17) for \( i = 1 \) from (D.19) and Markov’s inequality. □

E Proof of Theorem 5

By the arguments in (C.3), \( \sup_{\mu \in \Lambda_0} P_\mu(\text{Type I error}) \to 0 \).

Consider \( \mu \in \Lambda_1(h, \Delta, V) \) and let \( \tau_j \) be the change-point satisfying the conditions in the definition of \( \Lambda_1(h, \Delta, V) \). Let \( Q^n = 1 \) if \( |\log(\mu_n / \mu_0)| \geq \Delta \) and \( Q^n = 0 \) otherwise.

**Proof of Theorem 5.** Consider \( V = o\left(\frac{N^{\zeta}}{\log N}\right) = o\left(\frac{\log T}{\log N}\right) \) and recall from (4.9) that \( I_r = r \log\left(\frac{2e}{r+1}\right) + \log\left(\frac{2}{r+1}\right) \). Let \( \tau_1 \) and \( \mu_1 \) be such that \( e^\Delta > r_1 > r \) and \( \mu_0 / (1 + e)^{\frac{1}{2}} < \mu_1 < \mu_0 \). Since \( hVI \mu_0 = (1 + e) \log T \), \( \frac{h_{i+1}}{h_i} \to 1 \) and \( d_i = o(h_i) \), for \( T \) large there exists
\[
h_i \geq (1 + e)^{\frac{1}{2}}(V I \mu_1)^{-1} \log T,
\]
such that for all \( \mu \in \Lambda_1(h, \Delta, V) \), there exists \( k \) such that
\[
\tau_{j-1} < s(ik) < u(ik) < \tau_{j+1}, \quad |t(ik) - \tau_j| \leq \frac{d_j}{2},
\]
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and that when \( Q^n = 1 \),
\[
|\log(E_\mu Y^n_{t(ik),u(ik)}/E_\mu Y^n_{s(ik),t(ik)})| \geq \log r_1. \tag{E.2}
\]

Let
\[
\Gamma = \{ n : Q^n = 1, Y^n_{s(ik),u(ik)} \geq (1 + r)h_i\mu_1, \\
|\log(Y^n_{t(ik),u(ik)}/Y^n_{s(ik),t(ik)})| \geq \log r \}.
\]

Let \( m = \sum_{n=1}^{N} Q^n (\geq V) \). By the law of large numbers,
\[
P_\mu(\#\Gamma \geq (1 + \epsilon)^{-\frac{1}{2}}m) \to 1. \tag{E.3}
\]

Let \( p^n = p^n_{s(ik),t(ik),u(ik)} \). By \((E.1)\), for \( n \in \Gamma \),
\[
p^n \leq 2\exp(-\mu_1 h_i I_\nu) \leq 2\exp(-(1 + \epsilon)\frac{3}{2}V^{-1}\log T). \tag{E.4}
\]

Since \( V^{-1}\log T \) is large compared to \( \log N \), it follows from \((E.4)\) that for \( n \in \Gamma \) with \( N \) large,
\[
\ell(p^n) \geq (1 + \epsilon)\frac{3}{2}V^{-1}\log T.
\]

Since \( \ell(p) \geq -1 \) for \( N \) large,
\[
\sum_{n:Q^n=1} \ell(p^n) \geq (\#\Gamma)(1 + \epsilon)\frac{3}{2}V^{-1}\log T - m. \tag{E.5}
\]

Since \( \lambda_2 = o(\sqrt{\log T}) \), by Lemma \([4]\)
\[
P_\mu\left( \sum_{n:Q^n=0} \ell(p^n) \geq -C \log T \right) \to 1 \text{ for all } C > 0. \tag{E.6}
\]

We conclude
\[
\sup_{\mu \in \Lambda_1(h, \Delta, V)} P_\mu(\text{Type II error}) \to 0 \tag{E.7}
\]
from \((C.10)\), \((E.3)\), \((E.5)\) and \((E.6)\). \(\square\)

**Proof of Theorem 5(b).** Consider \( V \sim N^{1-\beta} \) for \( \frac{1-\zeta}{2} < \beta < 1 - \zeta \). For \( N \) large, there exists
\[
h_i \geq (1 + \epsilon)^\frac{3}{2}\mu_0^{-1}\rho_r(\zeta, \zeta) \log N, \tag{E.8}
\]
such that for all \( \mu \in \Lambda_1(h, \Delta, V) \), there exists \( k \) such that
\[
\tau_{j-1} < s(ik) < u(ik) < \tau_{j+1}, \quad |t(ik) - \tau_j| \leq \frac{d_i}{2},
\]

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and conditioned on $Q^n = 1$, either
\[ E_{\mu}Y^n_{t(i),u(i)} \leq rE_{\mu}s^n_{t}(i),t(i) \text{ or } E_{\mu}s^n_{t}(i),t(i) \geq rE_{\mu}Y^n_{t(i),u(i)}. \tag{E.9} \]

We apply Stirling’s approximation $x! \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ in (E.12) and (E.18), so that for $X \sim \text{Poisson}(\eta)$ and $x \to \infty$,
\[ P(X = x) = e^{-\eta}\frac{x^\eta}{x!} \sim \frac{1}{\sqrt{2\pi x}} \exp[-\eta + x - x \log(\frac{e}{\eta})]. \tag{E.10} \]

Case 1: \( \frac{1}{2} \leq \frac{\beta}{1-\zeta} \leq \frac{1}{2}[1 + \frac{2g(2)-1-\rho}{2g(2)}] \) and \( \rho(r, \zeta) = \frac{\beta - \frac{1}{2}(1-\zeta)}{2g(2)-1-\rho} \). Let
\[ \Gamma = \{ n : Q^n = 1, Y^n_{s(i),u(i)} \geq \sqrt{2(1+r^2)}h_i\mu_0 - 1, \log(Y^n_{t(i),u(i)}) \geq 2\log r \}. \tag{E.11} \]
Let $Y_1 \sim \text{Poisson}(h_i\mu_0)$ and $Y_2 \sim \text{Poisson}(r^2h_i\mu_0)$. By (E.10),
\[ P(Y_1 = \lfloor (\frac{2}{1+r^2})^{\frac{1}{2}}h_i\mu_0 \rfloor) \tag{E.12} \]
\[ \geq \frac{1}{h_i\mu_0} \exp(h_i\mu_0[-1 + (\frac{2}{1+r^2})^{\frac{1}{2}} - (\frac{2}{1+r^2})^{\frac{1}{2}} \log((\frac{2}{1+r^2})^{\frac{1}{2}}))], \]
\[ P(Y_2 = \lfloor (\frac{2}{1+r^2})^{\frac{1}{2}}r^2h_i\mu_0 \rfloor) \geq \frac{1}{h_i\mu_0} \exp(h_i\mu_0[-r + r^2(\frac{2}{1+r^2})^{\frac{1}{2}} - r^2(\frac{2}{1+r^2})^{\frac{1}{2}} \log(r(\frac{2}{1+r^2})^{\frac{1}{2}}))]. \]
By $g(2) = (\frac{1+r^2}{2})^{\frac{1}{2}}$, $D(2) = \frac{2}{1+r^2} \log((\frac{2}{1+r^2})^{\frac{1}{2}}) + \frac{r^2}{1+r^2} \log((\frac{2}{1+r^2})^{\frac{1}{2}})$ [see (D.5) and (E.12),
\[ E_{\mu}(\#\Gamma) \geq VP(Y_1 = \lfloor (\frac{2}{1+r^2})^{\frac{1}{2}}h_i\mu_0 \rfloor)P(Y_2 = \lfloor (\frac{2}{1+r^2})^{\frac{1}{2}}r^2h_i\mu_0 \rfloor) \geq \frac{N^{1-\beta}}{2(h_i\mu_0)^2} \exp(h_i\mu_0[2g(2) - 1 - r - g(2)D(2)]). \tag{E.13} \]
By (E.11), for $n \in \Gamma$,
\[ p^n \leq 2 \exp(-Y^n_{s(i),u(i)}D(2)) \leq C_2 \exp(-2h_i\mu_0g(2)D(2)) \text{ for } C_2 = 2e^{D(2)}. \tag{E.14} \]
Let $q^n = F_n(p^n)$, where $F_n$ is the distribution function of $p^n$. Since $p^n = o(q^n)$ for $n \in \Gamma$ and $\lambda_2 \sim \frac{\sqrt{2}}{N^{\frac{1}{2}}\log N}$, by (E.14) and $\log(1+x) \sim x$ as $x \to 0$,
\[ \sum_{n \in \Gamma} [\ell(p^n) - \ell(q^n)] \geq [1 + o(1)](\#\Gamma) \exp(h_i\mu_0g(2)D(2)) \frac{\lambda_2 C_2^{-\frac{1}{2}}}{\sqrt{N^{1-\frac{1}{2}}\log N}} \tag{E.15} \]
\[ = [1 + o(1)](\#\Gamma) \exp(h_i\mu_0g(2)D(2))(C_2^{-\frac{1}{2}}N^{-\frac{1}{2}}\log N). \]

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By (E.8) and \( \rho_r(\beta, \zeta) = \frac{\beta - \frac{1}{2}(1 - \zeta)}{2g(\omega) - 1 - r} \),

\[
h_i \mu_0 [2g(2) - 1 - r] \geq (1 + e) \left( \beta - \frac{1}{2} (1 - \zeta) \right) \log N. \tag{E.16}
\]

Since \( \ell(p^n) \geq \ell(q^n) \) for all \( n \), (E.7) follows from (C.5), (C.10), (E.13), (E.15) and (E.16).

Case 2: \( \frac{1}{2} \left[ 1 + \frac{2g(2) - 1 - r}{g(2)D(2)} \right] < \frac{\beta}{1 - \zeta} < 1 \) and \( \rho_r(\beta, \zeta) = \frac{1 - \zeta}{2g(\omega)D(\omega)} = \frac{\beta - \omega^{-1}(1 - \zeta)}{2g(\omega) - 1 - r} \)

with \( \omega \) achieving the maximum in (D.6). Let

\[
\Gamma = \{ n : Q^n = 1, Y^n_s(ik), u(ik) \geq 2g(\omega)h_i \mu_0 - 1, | \log(Y^n_s(t(ik), u(ik))/Y^n_s(ik), t(ik)) | \geq \omega \log r \}. \tag{E.17}
\]

Let \( Y_1 \sim \text{Poisson}(h_i \mu_0) \) and \( Y_2 \sim \text{Poisson}(rh_i \mu_0) \). By (E.10),

\[
P(Y_1 = \left[ \frac{2g(\omega)}{r+1} h_i \mu_0 \right]) \tag{E.18}
\]

\[
\geq \frac{1}{h_i \mu_0} \exp(h_i \mu_0[-1 + \frac{2g(\omega)}{r+1} - \frac{2g(\omega)}{r+1} \log(\frac{2g(\omega)}{r+1})]),
\]

\[
P(Y_2 = \left[ \frac{2r^\omega g(\omega)}{r+1} h_i \mu_0 \right]) \tag{E.19}
\]

\[
\geq \frac{1}{h_i \mu_0} \exp(h_i \mu_0[-r + \frac{2r^\omega g(\omega)}{r+1} - \frac{2r^\omega g(\omega)}{r+1} \log(\frac{2r^\omega - 1 g(\omega)}{r+1})]).
\]

By \( g(\omega) = \left( \frac{1 + r^\omega}{2} \right)^{\frac{1}{r}}, D(\omega) = \frac{1}{1 + r^\omega} \log(\frac{2}{1 + r^\omega}) + \frac{r^\omega}{1 + r^\omega} \log(\frac{2r^\omega}{1 + r^\omega}) \) [see (D.5)] and (E.18),

\[
E_{\mu}(\#\Gamma) \geq V P(Y_1 = \left[ \frac{2g(\omega)}{r+1} h_i \mu_0 \right]) P(Y_2 \geq \left[ \frac{2r^\omega g(\omega)}{r+1} h_i \mu_0 \right]) \tag{E.20}
\]

\[
\geq \frac{N^{1 - \beta}}{2h_i \mu_0} \exp(h_i \mu_0[2g(\omega) - 1 - r - 2(\frac{r^\omega}{r+1})g(\omega)D(\omega)])
\]

By (E.17), for \( n \in \Gamma \),

\[
p^n \leq 2 \exp(-Y^n_s(ik), u(ik)D(\omega)) \tag{E.21}
\]

\[
\leq C_{\omega} \exp(-2h_i \mu_0 g(\omega)D(\omega)) \text{ for } C_{\omega} = 2e^{D(\omega)}.
\]

Let \( q^n = F_n(p^n) \), where \( F_n \) is the distribution function of \( p^n \). Since \( p^n = o(q^n) \) for \( n \in \Gamma \) and \( \lambda_2 \sim \frac{N^{\frac{1}{\zeta}}}{\sqrt{N \log N}} \), by (E.20) and \( \log(1 + x) \sim x \) as \( x \to 0 \),

\[
\sum_{n \in \Gamma} [\ell(p^n) - \ell(q^n)] \geq [1 + o(1)](\#\Gamma) \exp(h_i \mu_0 g(\omega)D(\omega)) \frac{\lambda_2 C_{\omega} \frac{1}{2} N^{\frac{1}{\zeta}}}{\sqrt{N \log N}} \tag{E.22}
\]

\[
= [1 + o(1)](\#\Gamma) \exp(h_i \mu_0 g(\omega)D(\omega))(C_{\omega} \zeta)^{-\frac{1}{2}} N^{\frac{1}{\zeta}} \log N.
\]
By (E.8) and $\rho_r(\beta, \zeta) = \frac{\beta - \omega^{-1}(1-\zeta)}{2g(\omega) - 1 - r} = \frac{1 - \zeta}{2g(\omega)D(\omega)}$,

$$h_i\mu_0(2g(\omega) - 1 - r + [1 - 2(\omega - 1)]g(\omega)D(\omega))$$

$$= h_i\mu_0(\beta - \omega^{-1}(1-\zeta) + \frac{1 - \zeta}{\rho_r(\beta, \zeta)}(1-\zeta))$$

$$\geq (1 + \epsilon)^{1/2}(\beta - 1 - \zeta) \log N.$$ (E.22)

Since $\ell(p^n) \geq \ell(q^n)$ for all $n$, (E.7) follows from (C.5), (C.10), (E.19), (E.21) and (E.22). □

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