HYPERBOLIC GEOMETRY AND HOMOTOPIC HOMEOMORPHISMS OF SURFACES

JOHN CANTWELL AND LAWRENCE CONLON

Abstract. The Epstein-Baer theory of curve isotopies is basic to the remarkable theorem that homotopic homeomorphisms of surfaces are isotopic. The groundbreaking work of R. Baer was carried out on closed, orientable surfaces and extended by D. B. A. Epstein to arbitrary surfaces, compact or not, with or without boundary and orientable or not. We give a new method of deducing the theorem about homotopic homeomorphisms from the results about homotopic curves via the hyperbolic geometry of surfaces. This works on all but 13 surfaces where ad hoc proofs are needed.

1. Introduction

On arbitrary connected surfaces \( L \), compact or not, orientable or not and with or without boundary, D. B. A. Epstein proves that (with some qualifications) properly homotopic homeomorphisms of surfaces \( L \) are isotopic \([4]\). For this, he extends a result of R. Baer \([1]\) to prove that homotopic, essential, simple, 2-sided closed curves on \( L \) are ambient isotopic. To accommodate the case that \( \partial L \neq \emptyset \), he further proves that properly imbedded arcs which are homotopic mod the endpoints are ambient isotopic mod the endpoints. These are Theorems 2.1 and 3.1 in \([4]\), the proofs of which in the PL category are quite elegant. Their truth in the topological category follows from results in \([4\text{ Appendix}]\) about approximating topological curves and homeomorphisms by their PL counterparts in dimension two. We restate these results as Theorem 4.3 and Theorem 4.4 and refer the reader to \([4]\) for the proofs.

The method in \([4]\) of deducing from these two results the theorem about homotopic homeomorphisms is a bit involved and requires that the homotopy be proper if \( \partial L \) has any noncompact components. In this note we propose a new method of deducing the theorem from the same two results using a hyperbolic metric on \( L \) with certain reasonable properties (which we will call a standard metric). There are exactly 13 exceptional surfaces, which do not admit a standard metric (see Section 6). Our proof, therefore, works for all surfaces except the "unlucky" 13 and nowhere requires the homotopy to be proper.

In the following definition and throughout the paper we use the open unit disk \( \Delta \) as our model of the hyperbolic plane and denote the circle at infinity by \( S_\infty^1 \).

Definition 1.1. A hyperbolic half-plane \( H \) is a hyperbolic surface isometric to the subsurface of \( \Delta \) which is the union of a geodesic \( \gamma \subset \Delta \) and one of the components of \( \Delta \setminus \gamma \).

Remark. Generally, we just refer to these as half planes. Many results fail for hyperbolic metrics with imbedded half-planes on non-compact surfaces. Any non-compact surface can be given a complete hyperbolic metric with geodesic boundary and imbedded half-planes (see the example below).
Definition 1.2. A hyperbolic metric on a surface $L$ is called “standard” if it is complete, makes $\partial L$ geodesic, and admits no isometrically imbedded half planes. We call a surface equipped with such a standard metric a standard hyperbolic surface. If a surface is homeomorphic to a standard hyperbolic surface, it will simply be called a standard surface.

In this paper we prove for standard hyperbolic surfaces several results that are well-known for complete, hyperbolic surfaces with geodesic boundary and finite area. We believe that our proofs of two of these results (Theorems 3.1 and 4.1) are new, even for compact surfaces.

Example. We give an easy example showing how half planes can occur in an arbitrary noncompact surface. In Figure 1 we depict a 2-ended surface $L$ of infinite genus and a shaded subsurface $H'$ homeomorphic to $\mathbb{R} \times [0, \infty)$. Excise the interior of $H'$ and on the remaining surface put a complete hyperbolic metric making the one boundary component a geodesic. Now glue on a hyperbolic half plane by an isometry along the boundary to obtain a complete hyperbolic metric on $L$ with an isometrically imbedded hyperbolic half plane $H$, as pictured in Figure 2.

2. Limit Points and the Ideal Boundary

We will let $\Delta$ denote the open unit disk with the Poincaré metric. The closed unit disk will be denoted by $\mathbb{D}^2 = \Delta \cup S^1_\infty$, where $S^1_\infty$ is the unit circle, called the circle at infinity. If $L$ is a complete hyperbolic surface without boundary, the universal covering space $\tilde{L}$ is $\Delta$. If $\partial L \neq \emptyset$ and is geodesic, the double $2L$ has a canonical hyperbolic metric which is the double of the metric on $L$ and $2L = \Delta$. If one fixes a lift $\tilde{L} \subset \Delta$ of $L$, this serves as the universal covering space. The group of deck transformations is the restriction to $\tilde{L}$ of the subgroup of deck transformations for $p : \Delta \to 2L$ which leaves $\tilde{L}$ invariant.
We will write  for the closure of  in , and call this the completion of . Thus  is a compact subset of  and is equal to the unit circle precisely when .

Many results are more easily proven for surfaces without boundary. One often proves a result for  by proving the corresponding result for  and remarking that its truth there implies its truth in  for this, the following will be needed.

**Lemma 2.1.** If , then  is a standard hyperbolic surface if and only if  is a standard hyperbolic surface.

*Proof.* The “if” direction is trivial. For the converse, suppose  has a standard hyperbolic metric. Write  and note that, in the universal cover , the lifts of  and  form a countable set  which completely fills  with no proper overlaps. Let  denote the ideal boundary of  Thus,  is dense in . If  contains a nondegenerate subarc , then some  must contain a maximal nondegenerate subarc . Otherwise,  would be a countable union of nowhere dense sets.

Let  be the unique half plane in  subtended by the arc . Note that, if , hence  and we must have . Without loss of generality, suppose that  covers  and that a deck transformation  for  identifies distinct points of . Then, replacing  with its inverse, if necessary, there are three possibilities: (1) , or (2) , or (3) . The only possibility, then, is case (1), where the geodesic bounding  is the axis of a hyperbolic deck transformation. In this case, we can assume that  generates the infinite cyclic group of deck transformations having this axis, hence that  is a half open annulus with
one geodesic boundary component and end a hyperbolic trumpet. It is well known that this contains imbedded half planes.

We have proven that $E_1$ is a dense subset of $S^1_{\infty}$ with empty interior. One then sees that, given an arbitrary point $a \in S^1_{\infty}$ and an arbitrary Euclidean neighborhood $U$ of $a$ in the closed unit disk $D^2$, $L_j \subset U$ for some index $j$. If $2L$ contains an imbedded half plane $H$, fix a lift $\tilde{H} \subset \Delta$ and note that its closure is a neighborhood of points on $S^1_{\infty}$. Thus, some $L_j \subset \tilde{H}$ implying that $H$ contains closed geodesics, a contradiction.

**Definition 2.2.** The limit points of $L$ are the accumulation points in $S^1_{\infty}$ of the set $\{\gamma(x_0) \mid \gamma$ a deck transformation of $\tilde{L}\}$ for fixed $x_0 \in \tilde{L}$. The union $Y$ of these points is the limit set of $L$.

The following is well known and elementary.

**Lemma 2.3.** The limit set of $L$ is independent of $x_0 \in \tilde{L}$.

Let $X \subset S^1_{\infty}$ be the set of fixed points of the (extensions to $\tilde{L}$ of the) deck transformations of $\tilde{L}$. Then $X \subset Y \subset E$.

**Definition 2.4.** We will call an end of $L$ a simply connected end if it has a simply connected neighborhood in $L$.

Simply connected ends can occur only if $\partial L \neq \emptyset$. For instance, an isolated simply connected end has a neighborhood homeomorphic to $[0, 1] \times [0, \infty)$. But the simply connected ends can form a very complicated subset of the endset, even a Cantor set of such ends being possible.

**Theorem 2.5.** If $L$ is a standard hyperbolic surface with no simply connected ends, then $X$ is dense in the ideal boundary $E$ and $Y = E$.

**Proof.** If $L$ is nonorientable, its orientation cover is intermediate between $L$ and $\tilde{L}$, hence we can assume that $L$ is orientable. Now suppose the contrary, i.e. that there is a point $e \in E \setminus X$ not approached by points in $X$. If $X = \emptyset$, our surface is simply connected and has been excluded by hypothesis. Let $A$ be the maximal open interval in $S^1_{\infty} \setminus X$ containing the point $e \in E$, letting $a, b \in \tilde{X}$ denote the endpoints of $A$. We consider the cases $a = b$ and $a \neq b$.

If $a = b$, then all deck transformations would be parabolics fixing this point. In this case, the covering group $G$ is infinite cyclic and $L = \Delta/G$. If $\partial L = \emptyset$, this is the open annulus with one end a cusp and the other a “flaring” annular end (an infinite hyperbolic trumpet). This surface contains an imbedded half-plane and has been excluded by our ongoing hypothesis. If $\partial L \neq \emptyset$, then $L$ has one cusp and at least one simply connected end.

Let $C \subset \Delta$ be the closed circular arc perpendicular to $S^1_{\infty}$ with endpoints $a, b$, let $H'$ be the portion of $D^2$ bounded by $C \cup A$, and set $H = H' \cap \tilde{L}$. Since $A$ contains $e \in E$, $H \neq \emptyset$. Remark that, if $g$ is a deck transformation, either $g(H) \cap H = \emptyset$, or $g(H) = H$ and $g$ is hyperbolic with axis $C$. Otherwise, $A \cap X \neq \emptyset$.

We consider two cases: (a) $C$ is nondegenerate and is not the axis of a deck transformation, or (b) $C$ is the axis of a deck transformation $g$.

In case (a), the above remark assures us that no deck transformation identifies distinct points of $H$ or of $\text{int} \ C$. Therefore, under the covering projection, $C$ projects to a geodesic $C' \subset L$ homeomorphic to the reals which cuts off the image of $H$.
under the covering projection, which is a homeomorphism on $H$. If $\partial L = \emptyset$, the image of $H$ is a half-plane, contradicting the fact that $L$ contains no half-planes. If $\partial L \neq \emptyset$, the image of $H$ contains at least one simply connected end of $L$, again a contradiction.

In case (b), the geodesic $C$ projects to an essential closed curve $C' \subset L$ which cuts off the image of $H$ in $L$. Thus, if $\partial L = \emptyset$, $H$ projects to a neighborhood $H/g$ of a flaring annular end of $L$, contrary to hypothesis. If $\partial L \neq \emptyset$, $H$ projects to a neighborhood $H/g$ of at least one simply connected end of $L$, contrary to hypothesis.

Thus, $E = \overline{X} \subseteq Y \subseteq E$, so we have also proven that $Y = E$. □

As this proof reveals, simply connected ends cause essentially the same obstruction to this theorem as imbedded half planes.

Corollary 2.6. If $L$ is a standard hyperbolic surface with no simply connected ends and $e \in E$, then the orbit of $e$ under the group of covering transformations is dense in $E$.

Proof. We will show that the orbit clusters at every point of the dense set $X$. Let $x \in X$ and $g$ a covering transformation fixing $x$. Choose $e'$ in the orbit of $e$ not fixed by $g$. Since $g$ is either hyperbolic or parabolic, applying the positive and negative iterates of $g$ to $e'$ produces a subset of the orbit of $e$ clustering at $x$. □

Corollary 2.7. If $L$ is a standard hyperbolic surface with no simply connected ends and $\tilde{L} \neq \Delta$, then the ideal boundary $E$ is a Cantor set.

Proof. By Corollary 2.6, every point of $E$ is a limit point of $E$. If $E$ is not a Cantor set, it contains a closed, nondegenerate interval. Let $A$, with endpoints $a, b \in E$, be a maximal such interval. Let the open interval $U$ be any component of $S^1_\infty \setminus E$ and let $x \in \text{int} A$ be fixed by a deck transformation $\gamma$. Then, under the action of $\gamma$ or $\gamma^{-1}$, the interval $U$ is pulled towards $x$. Thus there is an integer $k \geq 1$ so that $\gamma^k(U) \subset A \subset E$. This contradicts the fact that $U \cap E = \emptyset$ and that $E$ is $\gamma$-invariant, proving our assertion. □

Remark. Theorem 2.5 and its corollaries are standard for complete hyperbolic surfaces of finite area.

Remark. Remark that, even if $L$ has simply connected ends, its double $2L$ does not and so the conclusion of Theorem 2.5 holds for $2L$. In applications of this theorem in what follows, we will always be working either in $2L$ or in $L$ if $\partial L = \emptyset$.

3. Extensions of Homeomorphisms to the Ideal Boundary

Theorem 3.1. If $L$ is a standard hyperbolic surface and $h : L \to L$ is a homeomorphism, then any lift $\tilde{h} : \tilde{L} \to \tilde{L}$ extends canonically to a homeomorphism $\hat{h} : \hat{L} \to \hat{L}$.

If the assertion holds for the double $2L$, it holds for $L$. Thus, we may assume that $\partial L = \emptyset$. If $L$ is nonorientable, $h$ admits orientation reversing lifts $\tilde{h}$ as well as orientation preserving ones. If $L$ is orientable, then $h$ is orientation preserving (respectively, reversing) if and only if all of its lifts $\tilde{h}$ are orientation preserving (respectively, reversing).
Definition 3.2. If $\gamma \subset L$ is a curve such that some (hence every) lift $\tilde{\gamma}$ has well defined endpoints on $S^1_{\infty}$, $\gamma$ is a pseudo-geodesic. We denote the extension of $\tilde{\gamma}$ to $D^2 = \Delta \cup S^1_{\infty}$ by $\tilde{\gamma}$ and call it a completed lift of $\gamma$.

Suppose $\sigma : S^1 \to L$ is an essential closed loop. It is well known that $\sigma$ is a pseudo-geodesic, oriented by the counterclockwise orientation of $S^1$. Remark that the endpoints of completed lifts $\tilde{\sigma}$ depend only on the free homotopy class of $\sigma$.

We regularly write $h(\sigma)$ for $h \circ \sigma$, where $\sigma : S^1 \to L$ or $\sigma : S^1 \to \tilde{L}$. We always consider $\sigma$ to be an oriented loop with the orientation induced by the counterclockwise orientation of $S^1$. Thus, whether $h$ preserves or reverses orientation on the surface, $h(\sigma)$ has a well defined orientation.

Definition 3.3. $Z = \{ z \in S^1_{\infty} \mid z \text{ is an endpoint of } \tilde{\sigma} , \sigma \text{ an essential closed loop}\}$.

Lemma 3.4. Any lift $\tilde{h} : \Delta \to \Delta$ of $h$ induces a natural bijection $\overline{h} : Z \to Z$.

Proof. Let $z \in Z$ and let $\tilde{\sigma}$ be the lift of an essential closed curve $\sigma$ with endpoints $z_{\pm}$, where $z_+$ is the positive end determined by the lifted orientation of $\sigma$, and $z_-$ is the negative end. Then $\tilde{h}(\tilde{\sigma})$ is a lift of $h(\sigma)$ having endpoints $y_{\pm}$. We set $\overline{h}(z_{\pm}) = y_{\pm}$, obtaining a well defined map $\overline{h} : Z \to Z$. Applying this same argument to the lift $\tilde{h}^{-1}$ of $h^{-1}$, we obtain an inverse to $\overline{h}$, proving that this map is a bijection. $\square$

Definition 3.5. Let $\mathcal{G}$ be the set of pseudo-geodesics in $L$ such that some, hence every, completed lift $\tilde{\gamma}$ has both endpoints in $Z$.

Lemma 3.6. If $\gamma \in \mathcal{G}$, then $h(\gamma) \in \mathcal{G}$.

Proof. Fix a lift $\tilde{h} : \Delta \to \Delta$ and let $\overline{h} : Z \to Z$ be the bijection given by Lemma 3.4. If $\tilde{\gamma}$ has an endpoint $z \in Z$, we will show that $\overline{h}(\tilde{\gamma})$ limits on the point $\overline{h}(z) \in Z$. Applying this to both endpoints of $\tilde{\gamma}$, $\gamma \in \mathcal{G}$, will prove the lemma.

Let $\sigma$ be a closed geodesic with a lift $\tilde{\sigma}$ such that one endpoint of $\tilde{\sigma}$ is $z$. Let $\tau$ be a closed geodesic intersecting $\sigma$ transversely and let $a$ be one of these intersection points. Let $\{a_n\}_{n \in \mathbb{Z}}$ be the lifts of $a$ in $\tilde{\sigma}$, indexed so that $\lim_{n \to \infty} a_n = z$. Let $\tau_n$ be the lift of $\tau$ through $a_n$. Each $\tilde{\gamma}_n$ has endpoints $u_n, w_n$ bounding a subarc $\alpha_n \subset S^1_{\infty}$ containing the point $z$. The sequences $u_n \to u$ and $w_n \to w$ as $n \to \infty$. We claim that $u = z = w$. Otherwise, the geodesics $\tau_n$ accumulate uniformly on the geodesic $\tau'$ with endpoints $u, w$. Projecting downstairs, we would have that the constant geodesic $\tau$ accumulates on a distinct geodesic, which is nonsense. Thus, $\tau_n \cup \alpha_n$ bounds a closed neighborhood $V_n$ of $z$ in $D^2$ and $\bigcap_{n=1}^{\infty} V_n = \{ z \}$. In particular, if we set $U_n = \Delta \cap V_n$, $\bigcap_{n=1}^{\infty} U_n = \emptyset$. Note that, since $\tilde{\gamma}$ has endpoint $z$, $\tilde{\gamma}$ crosses each $\tau_n$ into $U_n$ a last time.

Applying $\overline{h}$ to this picture gives a descending nest $U'_n = \overline{h}(U_n)$ with empty intersection, $\partial U'_n = \tau'_n = \overline{h}(\tau_n)$. The endpoints of $\tau'_n$ are $\overline{h}(u_n)$ and $\overline{h}(w_n)$. We claim that these points converge to $\overline{h}(z)$ from both sides. Otherwise, apply the above argument to the geodesics $\tau''_n$ having the same endpoints as $\tau'_n$ and get a contradiction. Since $\overline{h}(\tilde{\gamma})$ crosses each $\tau'_n$ into $U'_n$ a last time, since $\overline{h}(u_n)$ and $\overline{h}(w_n)$ converge to $\overline{h}(z)$ from both sides, and since $\bigcap_{n=1}^{\infty} U'_n = \emptyset$, it follows that $\overline{h}(\tilde{\gamma})$ limits on the point $\overline{h}(z) \in Z$. $\square$

Since $Z$ is a union of orbits in $S^1_{\infty}$ of the group of (extended) covering transformations and since $\partial L = \emptyset$, we can apply Corollary 2.6 to obtain the following.


Lemma 3.7. The set $Z$ is dense in $S^1_\infty$.

Lemma 3.8. If $\tilde{h}$ is orientation orientation preserving (respectively, reversing), there is a bijection $\tilde{h} : \mathbb{D}^2 \to \mathbb{D}^2$ which extends $h : \Delta \to \Delta$ and restricts to an orientation preserving (respectively, reversing) homeomorphism $\tilde{h} : S^1_\infty \to S^1_\infty$.

Proof. On $Z$ fix the cyclic order induced by the counterclockwise orientation of $S^1_\infty$. If $\tilde{h}$ is orientation preserving, it is easy to see that the map $\tilde{h} : Z \to Z$ of Lemma 3.4 preserves the cyclic order. If $\tilde{h}$ is orientation reversing, $\tilde{h} : Z \to Z$ reverses the cyclic order. It is well known and elementary that, in the first case, $\tilde{h}$ extends uniquely to an orientation preserving homeomorphism $\tilde{h} : S^1_\infty \to S^1_\infty$, and, in the second case, to an orientation reversing one. Then $\tilde{h} = \tilde{h} \cup \tilde{h}$.

Definition 3.9. For $x \in S^1_\infty$, $S_x$ consists of those $\gamma \in S$ such that no completed lift $\tilde{\gamma}$ has $x$ as an endpoint.

Definition 3.10. For $x \in S^1_\infty$ and $\gamma \in S_x$, let $H^s_{\hat{\gamma}}$ be the closed disk in $\mathbb{D}^2$ with $\hat{\gamma} \cup \alpha$ as boundary, where $\alpha$ is a subarc of $S^1_\infty$ containing $x$ in its interior and having its endpoints in common with $\hat{\gamma}$.

Proof of Theorem 3.1. We want to show that $\tilde{h} : \mathbb{D}^2 \to \mathbb{D}^2$ is continuous at $z$. This is clear if $z \in \Delta$, so we assume $z \in S^1_\infty$. Let $U$ be an open neighborhood of $\tilde{h}(z)$. Then $U \supset H = H^s_{\Lambda}(z)$ for a suitable choice of $\Lambda \in S_{\tilde{h}(z)}$. Now $H$ subtends an arc $\alpha \subset S^1_\infty$ with $\tilde{h}(z) \in \text{int} \alpha$ and so $\tilde{h}^{-1}(\alpha)$ is an arc having $z$ in its interior. Furthermore, $\tilde{h}^{-1}(\gamma)$ is a curve $\beta$ in $\mathbb{D}^2$ with the same endpoints as $\tilde{h}^{-1}(\alpha)$ and meeting $S^1_\infty$ exactly in these endpoints. Finally, if $w \in \text{int} H$, there is an arc $s$ in $\mathbb{D}^2$, not meeting $\gamma$ and with one endpoint $w$ and the other endpoint in $\alpha \cap Z$. Clearly $\tilde{h}^{-1}(s)$ is an arc in $\mathbb{D}^2$, not meeting $\tilde{h}^{-1}(\gamma)$ and having $\tilde{h}^{-1}(w)$ as one end and the other end in $\tilde{h}^{-1}(\alpha \cap Z)$. Thus, $\tilde{h}^{-1}(w)$ lies in the interior of the disk bounded by $\tilde{h}^{-1}(\gamma \cup \alpha)$, proving that $\tilde{h}^{-1}(H)$ is a closed neighborhood of $z$ in $\mathbb{D}^2$ contained in $\tilde{h}^{-1}(U)$. This proves continuity at arbitrary $z \in S^1_\infty$.

Corollary 3.11. If $\gamma$ is a pseudo-geodesic in $L$ and $h : L \to L$ is a homeomorphism, then $h(\gamma)$ is a pseudo-geodesic.

Remark. Our proof of Theorem 3.1 includes the case in which $L$ is compact, hence gives a fundamentally different proof in that case from the ones given by Casson and Bleiler [2, Lemma 3.7] and Handel and Thurston [4, Corollary 1.2]. These proofs make use of compactness, whereas we do not. The analogous result holds for higher dimensional, compact, hyperbolic manifolds [4, Proposition C.1.2], [7, Theorem 11.6.2], where compactness is only used to guarantee that $\tilde{h}$ is a pseudo-isometry. In [2], compactness is only used to guarantee that $\tilde{h}$ is uniformly continuous.

4. The Basic Isotopy Theorem.

The following is well known when $L$ is compact.

Theorem 4.1. If $L$ is a standard hyperbolic surface and $h : L \to L$ is a homeomorphism, then $h$ is isotopic to the identity if and only if it has a lift to $\hat{L}$ such that $\hat{h}|E$ is the identity.
The “only if” part of this theorem is elementary and is left to the reader.

**Corollary 4.2.** If \(L\) is a standard hyperbolic surface and \(f, g : L \rightarrow L\) are homeomorphisms, then \(f\) is isotopic to \(g\) if and only if there are lifts to \(\tilde{L}\) such that \(\tilde{f}, \tilde{g}\) agree on the ideal boundary \(E\).

Indeed, set \(h = g^{-1} \circ f\).

We cannot find a proof of Theorem 4.1 in the literature for surfaces of infinite Euler characteristic, so we give a detailed sketch here. We do not need orientability nor empty boundary. Basic to our proof are the following two Epstein-Baer theorems. They are, respectively, Theorem 2.1 and Theorem 3.1 in [4]. While the proofs are carried out in the PL category, it is shown in the Appendix of [4] that these and other results in the paper remain true in the topological category. There is no hyperbolic metric assumed on \(L\) in these two theorems.

**Theorem 4.3 (Epstein-Baer).** Let \(\alpha, \beta : S^1 \rightarrow \text{int } L\) be freely homotopic, imbedded, 2-sided, essential circles. Then there is a compactly supported homeomorphism \(\varphi : L \rightarrow L\) and an ambient isotopy \(\Phi : L \times I \rightarrow L\) with \(\Phi(\cdot, 0) = \text{id}_L\) and \(\Phi(\cdot, 1) = \varphi\), compactly supported in \((\text{int } L) \times I\), such that \(\varphi \circ \beta = \alpha\).

**Theorem 4.4 (Epstein).** Let \(\alpha, \beta : [0, 1] \rightarrow L\) be properly imbedded arcs with the same endpoints which are homotopic modulo the endpoints. Then there is a compactly supported homeomorphism \(\varphi : L \rightarrow L\) and an ambient isotopy \(\Phi : L \times I \rightarrow L\) with \(\Phi(\cdot, 0) = \text{id}_L\) and \(\Phi(\cdot, 1) = \varphi\), compactly supported in \(L \times I\) and carrying \(\partial L \times I \rightarrow \partial L\) by \(\Phi(x, t) = x\), such that \(\varphi \circ \beta = \alpha\).

We will generally call \(\varphi\) itself a compactly supported ambient isotopy. As noted earlier, we sometimes abuse terminology by identifying a curve \(\alpha\) with its image. Thus, instead of writing \(g \circ \alpha = \alpha\), we might write \(g|\alpha = \text{id}_{\alpha}\) (or \(= \text{id}\)).

4.1. Preliminaries. The following is elementary and well known.

**Lemma 4.5.** Orientation preserving homeomorphisms \(h : S^1 \rightarrow S^1\) and \(h : \mathbb{R} \rightarrow \mathbb{R}\) are isotopic to the identity.

**Corollary 4.6.** The homeomorphism \(h : L \rightarrow L\) of Theorem 4.1 admits an ambient isotopy \(\varphi\) supported near \(\partial L\), such that \(\varphi \circ h\) restricts to the identity on \(\partial L\).

**Proof.** Since \(h\) fixes \(E\) pointwise, it is clear that \(h\) preserves the components of \(\partial L\) and is orientation preserving on each. In a collar neighborhood of each boundary component, one extends the isotopy of Lemma 4.5 to an ambient isotopy supported in the collar.

From now on, therefore, we will assume that \(h|\partial L = \text{id}_{\partial L}\).

**Lemma 4.7.** If \(h\) admits a lift \(\tilde{h}\) such that \(\tilde{h}|E = \text{id}_E\), then it admits a unique such lift.

**Proof.** Let \(g, f : \tilde{L} \rightarrow \tilde{L}\) be lifts of \(h\) so that \(\tilde{f}|E = \tilde{g}|E = \text{id}_E\). Then \(f \circ g^{-1}\) is a covering transformation \(\psi\) which is the identity on \(E\), hence \(\psi = \text{id}\). □

**Remark.** Hereafter, \(h^*\) will denote this unique lift. By abuse, its extension to \(\tilde{L} = L \cup E\) will also be denoted by \(h^*\). Remark that, if \(h\) is varied by an ambient isotopy \(\varphi\), then \(\varphi \circ h\) also admits a lift \((\varphi \circ h)^*\). Indeed, \(\varphi\) itself has such a lift, since it preserves the free homotopy classes of loops, and so we set \((\varphi \circ h)^* = \varphi^* \circ h^*\).
Corollary 4.8. The canonical lift \( h^* \) commutes with all covering transformations.

Proof. Indeed, if \( \psi \) is a covering transformation, \( \psi \circ h^* \circ \psi^{-1} \) is a lift of \( h \) which fixes \( E \) pointwise. By Lemma 4.7, \( \psi \circ h^* \circ \psi^{-1} = h^* \). \( \square \)

The following is a key lemma.

Lemma 4.9. Let \( \sigma : S^1 \to L \) be an essential imbedded circle such that \( h|\sigma = \text{id}_\sigma \).

Then there is an ambient isotopy \( \varphi \), supported in an arbitrarily small neighborhood of \( \sigma \), such that \( (\varphi \circ h)^* \) is the identity on any lift \( \tilde{\sigma} \) of \( \sigma \).

Proof. Since \( h(\sigma) = \sigma \) and since \( h^* \) preserves the endpoints of \( \tilde{\sigma} \) in \( E \), \( h^*(\tilde{\sigma}) = \tilde{\sigma} \).

If we set \( S^1 = \mathbb{R}/\mathbb{Z} \), this parametrizes \( \sigma \) and so parametrizes \( \tilde{\sigma} \) as \( \mathbb{R} \). The fact that \( h|\sigma = \text{id} \) then implies that \( h^*|\tilde{\sigma} \) is just translation of \( \mathbb{R} \) by an integer \( n \).

Fix a normal neighborhood \( N = \mathbb{R} \times [-1,1] \) of \( \sigma \), with \( \tilde{\sigma} = \mathbb{R} \times \{0\} \). Do this so that \( N \) is a lift of an annular neighborhood \( A \) of \( \sigma \). If \( \sigma \) is a component of \( \partial L \), make \( N \) a 1-sided normal neighborhood \( \mathbb{R} \times [0,1] \). Now translation on \( \mathbb{R} \) by \(-n\) is isotopic to the identity through translations and this easily defines an ambient isotopy \( \varphi^* \), supported in \( N \), such that \( \varphi^* \circ h^* \) is the identity along \( \tilde{\sigma} \).

As the notation suggests, we have taken care that \( \varphi^* \) is a lift of an ambient isotopy \( \varphi \) supported in \( A \) which leaves \( \sigma \) (as a point set) invariant. If \( \tilde{\sigma}_1 \) is another lift of \( \sigma \), it is the image of \( \tilde{\sigma} \) under a deck transformation \( \psi \). The parametrization of \( \tilde{\sigma}_1 \) corresponds to that of \( \tilde{\sigma} \) under \( \psi \) up to translation by an integer. Since \( h^* \circ \psi = \psi \circ h^* \) by Corollary 4.8, \( h^* \) is translation by \( n \) on \( \tilde{\sigma}_1 \) and the isotopy \( \varphi \) lifts to a neighborhood \( N_1 \) of \( \tilde{\sigma}_1 \) to give the same proof that the lift \( (\varphi \circ h)^* \) fixes \( \tilde{\sigma}_1 \) pointwise. \( \square \)

Corollary 4.10. There is an ambient isotopy \( \varphi \), supported in any preassigned neighborhood of \( \partial L \), such that \( (\varphi \circ h)^*|\partial L = \text{id}_{\partial L} \).

Proof. For each compact component of \( \partial L \), apply Lemma 4.9. The lifts of noncompact components \( \ell \) project one-to-one onto \( \ell \) and the assertion is trivial. \( \square \)

From now on, therefore, we will assume both that \( h|\partial L = \text{id}_{\partial L} \) and \( h^*|\partial L = \text{id}_{\partial L} \).

4.2. The proof of Theorem 4.1 We will concentrate on the case in which \( L \) is noncompact, remarking at the end on how our methods adapt easily to the compact case.

There is an exhaustion \( K_0 \subset K_1 \subset \cdots \subset K_i \subset \cdots \) of \( L \) by compact, connected surfaces \( K_i \), with \( K_i \subset \text{int} K_{i+1} \), \( i \geq 0 \). We can require that the finitely many connected components of \( L \setminus K_i \) are neighborhoods of ends, the frontier of each being a finite, disjoint union of properly imbedded arcs and 2-sided, essential circles.

Let the collection of all of these arcs and circles be denoted by \( \mathfrak{A} = \{ \alpha_i \}_{i \geq 1} \). Note that \( \mathfrak{A} \) partitions \( L \) into a family of compact submanifolds with boundary composed of elements of \( \mathfrak{A} \) and arcs and/or circles in \( \partial L \). Enumerate these submanifolds as \( \{ B_i \}_{i \geq 0} \), in such a way that \( B_0 = K_0 \), \( B_1, B_2, \ldots, B_k \) are the components of \( K_1 \setminus \text{int} K_0 \), \( B_{k+1}, B_{k+2}, \ldots, B_{k+2} \) are the components of \( K_2 \setminus \text{int} K_1 \), etc.

Lemma 4.11. There is an ambient isotopy \( \psi : L \to L \) such that \( \psi \circ h|\alpha_i = \text{id} \) and \( (\psi \circ h)^*|\tilde{\alpha}_i = \text{id} \), for each \( \alpha_i \in \mathfrak{A} \) and each lift \( \tilde{\alpha}_i \).

Proof. If \( \alpha = \alpha_i \in \mathfrak{A} \) is a properly imbedded arc in \( L \), then \( h(\alpha) \) is properly imbedded with the same endpoints as \( \alpha \), since \( h|\partial L = \text{id}_{\partial L} \). Since \( h^*|\partial L = \text{id}_{\partial L} \), the lifts \( \tilde{\alpha} \) and \( h^*(\tilde{\alpha}) \) have the same endpoints. It follows that \( \alpha \) and \( h(\alpha) \) are homotopic.
modifies their endpoints. By Proposition 4.4 there is a compactly supported ambient isotopy \( \varphi \), keeping \( \partial L \) pointwise fixed, such that \( \varphi \circ h|\alpha = \text{id} \). Note that \( \varphi \) perturbs only finitely many \( h \)-images of elements of \( \mathfrak{A} \). Note that each lift \( \tilde{\alpha} \) has endpoints in \( \partial \tilde{L} \), fixed by \( (\varphi \circ h)^* \). It is evident, then, that \( (\varphi \circ h)^*|\tilde{\alpha} = \text{id} \).

Similarly, if \( \alpha \in \mathfrak{A} \) is a circle imbedded in \( \text{int} L \), any lift \( \tilde{\alpha} \) is an imbedded copy of \( R \) in \( \tilde{L} \) having well defined ideal endpoints in \( E \). Since \( h^* \) fixes these endpoints, the lift \( h^*(\tilde{\alpha}) \) of \( h(\alpha) \) has these same ideal endpoints and \( \alpha \) is freely homotopic to \( h(\alpha) \) in \( \text{int} L \). By Proposition 4.4, there is an ambient isotopy \( \varphi \), compactly supported in \( \text{int} L \), such that \( \varphi \circ h|\alpha = \text{id} \). Again, \( \varphi \) perturbs only finitely many \( h \)-images of elements of \( \mathfrak{A} \). An application of Lemma 4.9 shows that the ambient isotopy \( \varphi \) can be modified so that \( (\varphi \circ h)^*|\tilde{\alpha} = \text{id} \).

Let \( \alpha_1 \in \mathfrak{A} \) be contained in \( \partial B_0 \). Construct \( \varphi_1 \) as above. If \( \alpha_2 \subset \partial B_0 \), we must choose \( \varphi_2 \) so \( \varphi_2 \circ \alpha_1 = \alpha_2 \). The trick is to temporarily cut \( L \) apart along (the image of) \( \alpha_1 \) and apply the above argument in the resulting component \( L' \) containing \( \alpha_2 \). Suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_n \) is the full list of elements of \( \mathfrak{A} \) in \( \partial B_0 \). Then continuing in this way, we find \( \psi_0 = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1 \), compactly supported, the identity on \( \partial L \), such that \( \psi_0 \circ h \) restricts to the inclusion map on \( \partial B_0 \) and \( (\psi_0 \circ h)^* \) restricts to the inclusion map on \( \partial \tilde{B}_0 \), for each lift \( \tilde{B}_0 \subset \tilde{L} \). Now let \( L' \) be the component of \( L \setminus \text{int} B_0 \) containing \( B_1 \). Work in \( L' \) to produce \( \psi_1 \), a compactly supported isotopy on \( L' \), the identity on \( \partial L' \), such that \( \psi_1 \circ h \) restricts to the inclusion map on \( \partial B_1 \) and \( (\psi_1 \circ h)^* \) restricts to the inclusion map on \( \partial B_1 \). It is important to note that \( \psi_1 \) can be viewed as a compactly supported isotopy on \( L \) which is the identity on \( B_0 \) throughout the isotopy. Proceeding in this way, produce an isotopy \( \psi = \cdots \circ \psi_j \circ \psi_{j-1} \circ \cdots \circ \psi_0 \) such that \( \psi \circ h|\alpha \) and \( (\psi \circ h)^*|\tilde{\alpha} \) are the inclusions, for each \( \alpha \in \mathfrak{A} \) and each lift \( \tilde{\alpha} \). The isotopy is well defined since, for each \( \ell \geq 0 \), all but finitely many \( \psi_j \) are the identity isotopy on \( B_0 \cup B_1 \cup \cdots \cup B_{\ell} \). □

Hereafter, we replace \( h \) by \( \varphi \circ h \), assuming that \( h \) is the identity on \( \partial L \cup \bigcup_{\alpha \in \mathfrak{A}} \alpha \) and that \( h^* \) is the identity on the entire lift of this set.

**Proof of Theorem 4.1.** Each \( B_k \) is a compact surface with nonempty boundary. As such it can be viewed as the result of attaching finitely many bands (twisted and/or untwisted) to a disk (see [8] pp. 43-45]). Thus, there are finitely many disjoint, properly imbedded arcs \( \tau_1, \ldots, \tau_r \) in \( B_k \) which decompose this surface into a disk. The properly imbedded arcs \( \tau_i \) and \( h \circ \tau_i \) have the same endpoints since \( h \) fixes \( \partial B_k \) pointwise. Furthermore, since \( h^* \) fixes \( \partial B_k \) pointwise, they have lifts in \( \tilde{B}_k \) with common endpoints. This implies that they are homotopic in \( B_k \) by a homotopy keeping their endpoints fixed. Applying Proposition 4.4 in the usual way, allows us to assume that \( h \) fixes \( \tau_i \) pointwise, \( 1 \leq i \leq r \). Since cutting \( B_k \) apart along these arcs gives a disk \( D \) and the homeomorphism \( h' : D \to D \) induced by \( h \) is the identity on \( \partial D \), we apply Alexander’s trick to find an isotopy of \( h' \) to the identity which is constant on \( \partial B \). Regluing gives an isotopy of \( h : B_k \to \tilde{B}_k \) to the identity which is constant on the boundary. Carrying this out for each \( B_k \), we obtain isotopies that fit together to an isotopy on \( L \) since they leave all boundary components pointwise fixed. □

**Remark.** While the isotopy in the above proof is constant on \( \partial L \), this is only after altering the original \( h \) by an ambient isotopy that is not generally constant on \( \partial L \).
Thus the isotopy in Theorem 4.1 is not generally constant on the boundary. A simple example is a Dehn twist on the closed annulus.

Remark. If $L$ is compact, the above proof easily adapts. Indeed, if $\partial L \neq \emptyset$, use our methods to isotope $h$ to be the identity on $\partial L$ and so that $h^*$ is the identity on $\partial \tilde{L}$. Now introduce the arcs $\tau_i$ as above and use Alexander’s trick to complete the isotopy. If $\partial L = \emptyset$, find a simple, closed, 2-sided curve $\sigma \subset L$ that does not disconnect ($\chi(L) < 0$). First do the isotopy of $h$ that makes it the identity on $\sigma$ and makes $h^*$ the identity on all lifts of $\sigma$. Now cut apart along $\sigma$ and apply Alexander’s trick. Since this last isotopy is constant on the boundary, we reglue to obtain the desired isotopy on $L$.

5. Homotopic Homeomorphisms

In [4], Epstein states and proves the following theorem.

**Theorem 5.1 (Epstein-Baer).** Let $h : (L, \partial L) \to (L, \partial L)$ be a homeomorphism where $L$ is an arbitrary surface. If $h$ is properly homotopic to the identity, then $h$ is isotopic to the identity.

**Remarks.** The homotopy and isotopy are, of course, through maps carrying $\partial L$ into itself. Epstein removes the condition that the homotopy be proper in the case that all components of $\partial L$ are compact. The proof is carried out in the PL category, but by PL approximation theorems proven in the Appendix of [4], remain true in the topological category. Finally, the statement of the theorem needs to be amended for four surfaces: the open disk, the open annulus, the half plane $\mathbb{R} \times [0, \infty)$ and the strip $[0, 1] \times \mathbb{R}$. For these surfaces, there are orientation reversing homeomorphisms $h : (L, \partial L) \to (L, \partial L)$ which are homotopic to the identity but, of course, not isotopic to the identity. For those surfaces, Theorem 5.1 remains true if it is stipulated that $h$ be orientation preserving.

Epstein deduces this result by a hands-on construction using Theorem 4.3 and Theorem 4.4. Using these same two basic results, we deduce the following by the hyperbolic methods of this paper.

**Theorem 5.2.** If $L$ is a standard surface and $h : L \to L$ is a homeomorphism, then $h$ is isotopic to the identity if and only if $h$ is homotopic to the identity.

**Proof.** Since $L$ is standard, it has a standard hyperbolic metric. Then any homeomorphism $h$ of $L$ which is homotopic to the identity preserves the free homotopy classes of closed loops, hence has a lift to $\tilde{L}$ whose extension $\hat{h}$ is the identity on $E$. Thus, by Theorem 4.1, the homeomorphism $h$ is isotopic to the identity. The converse is trivial. □

We note that the requirement that homotopies respect the boundary is dropped, as is the requirement that the homotopy be proper. But the restriction to standard surfaces might cause concern. As the following shows, this restriction is very mild. It also shows that the notion of a standard surface is strictly topological without requiring reference to a metric.

**Theorem 5.3.** Up to homeomorphism there are exactly 13 nonstandard surfaces. They are: the open disk, the closed disk, the open annulus, the half open annulus, the closed annulus, the open Möbius band, the closed Möbius band, the half plane
$\mathbb{R} \times [0, \infty)$, the doubly infinite strip $[0, 1] \times \mathbb{R}$, the sphere, the projective plane, the torus, the Klein bottle.

For these 13 surfaces, there are ad hoc proofs of Theorem 5.1 which can be gleaned from [4]. Some are sufficiently elementary to be posed as exercises.

The next section will be devoted to the proof of Theorem 5.3.

6. The Nonstandard Surfaces

The following is quite elementary.

**Lemma 6.1.** The 13 surfaces listed in Theorem 5.3 are nonstandard.

For instance, the closed disk has a hyperbolic metric, but the boundary cannot be geodesic. The open disk has a complete hyperbolic metric, necessarily isometric to the canonical one on $\Delta$. This has imbedded half planes. The open annulus has a complete hyperbolic metric and either one end is a cusp and the other a hyperbolic trumpet, or both ends are trumpets. In any case there are imbedded half planes. And so forth. Checking out the rest can be left to the reader.

We need the notion of Euler characteristic for possibly noncompact surfaces. For any space $X$ with finite dimensional real homology, the Euler characteristic is the alternating sum of the betti numbers $b_i(X)$. For a closed, orientable surface $L$, this is $\chi(L) = 2 - b_1(L)$. For every other surface, $\chi(L) = 1 - b_1(L)$. If $b_1(L)$ is infinite, we set $\chi(L) = -\infty$. In particular, nonnegative Euler characteristic is finite.

Recall that, if $L$ is a surface with $g$ handles, $c$ crosscaps, $b$ compact boundary components, $a$ annular ends (all finite integers) and no other boundary components or ends, then the Euler characteristic of $L$ is given by the following formula:

$$\chi(L) = 2 - 2g - c - b - a.$$  

We fix the meaning of these letters. Remark that the values of $g$ and $c$ are not individually well defined by $S$. For example, if $g = 1 = c$, we can also take $g = 0$ and $c = 3$ (cf. [5, p. 26, Lemma 7.1]). In fact, 3 crosscaps equals 1 handle and 1 crosscap.

**Remark.** In particular, all of this applies to complete hyperbolic surfaces with finite area and compact geodesic boundary, where one also has the formula

$$2\pi\chi(L) = -\text{area } L,$$

computed by integrating the constant curvature $-1$. In such a surface, the annular ends are cusps. (See [3] pp. 31 - 37 for a discussion of such surfaces).

**Definition 6.2.** A complete hyperbolic surface $S$ with geodesic boundary and finite area is called a generalized pair of pants if $g = c = 0$ and $b + a = 3$. We will refer to a cusp of a generalized pair of pants as a boundary component of length 0.

The following lemma is well known.

**Lemma 6.3.** Given a triple of numbers $x_i \geq 0$, $1 \leq i \leq 3$, there exists a generalized pair of pants whose three boundary components have length $x_i$.

**Proposition 6.4.** Given a surface $S$ with finite Euler characteristic $\chi < 0$ and with boundary $b$ closed curves $\gamma_1, \ldots, \gamma_b$ and endset consisting of $a \geq 0$ annular ends, and given positive real numbers $x_1, \ldots, x_b$, there exists a complete hyperbolic metric on $S$ such that $\gamma_i$ is a geodesic of length $x_i$, $1 \leq i \leq b$, and the annular ends are cusps. In particular, this metric is standard.
Proof. Cut the handles of $S$ apart along $g$ curves and the crosscaps apart along $c$ curves to yield a surface $S'$ homeomorphic to a sphere with $2g + c + b$ boundary components and $a$ punctures. Since $\chi = 2 - 2g - c - b - a < 0$ it follows that $2 < 2g + c + b + a$. Thus $b + a \geq 3$ and $S'$ has a generalized pair of pants decomposition. By Lemma 6.3, each generalized pair of pants can be given a hyperbolic metric so that the annular ends are cusps and the boundary lengths are such that the regluing can be done to give a hyperbolic metric on $S$ with the $\gamma_i$ having the correct lengths. □

Consider a surface $L$ with boundary and its double $2L$. The Mayer-Vietoris sequence (with real coefficients) gives

$$H_2(2L) \to H_1(\partial L) \to H_1(L) \oplus H_1(2L) \to H_1(\partial L) \to H_0(L) \oplus H_0(2L) \to H_0(2L).$$

Remark that the dimension of $H_1(\partial L)$ is the number $b$ of compact boundary components and the dimension of $H_0(\partial L)$ is the number $d$ of boundary components. Also, since $L$ is assumed connected, the dimension of $H_0(L)$ and $H_0(2L)$ is 1. The sequence becomes

$$\mathbb{R} \text{ or } 0 \to \mathbb{R}^b \to \mathbb{R}^{2b_1(L)} \to \mathbb{R}^{b_1(2L)} \to \mathbb{R}^d \to \mathbb{R}^2 \to \mathbb{R},$$

where the first term injects, the last term surjects, and the first term is nonzero if and only if $2L$ is closed and orientable. Set

$$r = \begin{cases} 
    d - b, & \text{if } b < \infty, \\
    \infty, & \text{if } b = \infty.
\end{cases}$$

Thus, if $b$ is finite $r$ is the number (possibly infinite) of noncompact boundary components. One easily deduces the following.

**Lemma 6.5.** The Euler characteristic of $L$ and $2L$ are related by the formula

$$\chi(2L) = 2\chi(L) - r.$$  

Thus, $2L$ has finite Euler characteristic if and only if $L$ has finite Euler characteristic and finitely many boundary components.

**Proposition 6.6.** If $L$ is a nonstandard surface with finite Euler characteristic and finitely many boundary components, then it is homeomorphic to one of the 13 surfaces in Lemma 6.1.

Proof. Let $L$, as above, have compact boundary and finite Euler characteristic. If $\chi(L) \geq 0$, then $2g + c + b + a \leq 2$. The possibilities are:

1. $g \leq 1$ and $c = b = a = 0$, so $L$ is the sphere or torus.
2. $0 < c \leq 2$ and $g = b = a = 0$, so $L$ is the projective plane or Klein bottle.
3. $0 < b \leq 2$ and $g = c = a = 0$, so $L$ is the closed disk or closed annulus.
4. $0 < a \leq 2$ and $g = c = b = 0$, so $L$ is the open disk or open annulus.
5. $c = b = 1$ and $g = a = 0$, so $L$ is the closed Möbius strip.
6. $c = a = 1$ and $g = b = 0$, so $L$ is the open Möbius strip.
7. $b = a = 1$ and $g = c = 0$, so $L$ is the half open annulus.

Thus, any surface $L$ with finite Euler characteristic and only compact boundary components, other than the 11 surfaces just listed, has negative Euler characteristic and so, by Proposition 6.4, has a standard hyperbolic metric. By Lemma 6.1 the above 11 are nonstandard.
Let $L$ have finite Euler characteristic and finitely many boundary components, not all of which are compact. Since $L$ is nonstandard, so is $2L$ (Lemma 2.1). By Lemma 6.5, $2L$ has finite Euler characteristic. Since the boundary is empty, $2L$ must be in the list of 11. Since $L$ cannot be compact, $2L$ is either the open disk, the open annulus or the open Möbius strip. The only surface whose double is the open disk is the half plane. The doubly infinite strip is the only surface with noncompact boundary whose double is the open annulus, and no surface has double equal to the open Möbius strip. This completes the proof.

The following completes the proof of Theorem 5.3.

**Proposition 6.7.** If $L$ is a surface with infinitely many boundary components and/or infinite Euler characteristic, then $L$ is standard.

**Proof.** If $L$ has nonempty boundary, then $L$ is standard if and only if $2L$ is standard (Lemma 2.1). Also, under our hypothesis, $2L$ has infinite Euler characteristic (Lemma 6.5). Thus we may limit our attention to surfaces without boundary which have infinite Euler characteristic.

There is at least one end that is not annular, hence there is an exhaustion $K_0 \subset K_1 \subset \cdots \subset K_i \subset \cdots$ of $L$ by connected surfaces $K_i$ with finite Euler characteristic such that the finitely many connected components of $L \setminus K_i$ are neighborhoods of non-annular ends, the frontier of each being a simple closed curve. Let the collection of all of these simple closed curves be denoted by $\mathcal{A} = \{\alpha_i\}_{i=1}^\infty$. Note that $\mathcal{A}$ partitions $L$ into a countable family $\{B_k\}$ of submanifolds with finite Euler characteristic and with boundary composed of elements of $\mathcal{A}$. We can assume the exhaustion is chosen so that all the $B_k$ have negative Euler characteristic. Then by Proposition 6.4, each $B_k$ has a standard hyperbolic metric such that every simple closed curve in $\mathcal{A}$ is a geodesic of length 1. Thus, these hyperbolic metrics can be fitted together to give a hyperbolic metric on $L$. Evidently, an imbedded half plane $H \subset L$ would have to meet closed geodesics in $\partial B_k$, for infinitely many indices $k$. Since closed geodesics are nullhomotopic, the intersections of these geodesics with $H$ will be arcs which, together with an arc in $\partial H$ would form a geodesic digon. Such digons are forbidden in hyperbolic geometry. □

**References**

[1] R. Baer, *Isotopien von Kurven auf orientierbaren, geschlossenen Fächen*, Journal für die Reine und Angewandte Mathematik, 159 (1928), 101–116.

[2] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Springer-Verlag, Berlin, 1991.

[3] S. A. Bleiler and A. J. Casson, *Automorphisms of surfaces after Nielsen and Thurston*, Cambridge Univ. Press, Cambridge, 1988.

[4] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. 115 (1966), 83–107.

[5] M. Handel and W. Thurston, *New proofs of some results of Nielsen*, Adv. in Math. 56 (1985), 173–191.

[6] W. S. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, New York, NY, 1967.

[7] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer-Verlag, New York, 1994.