A basic identity for Kolmogorov operators in the space of continuous functions related to RDEs with multiplicative noise*

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Abstract

We consider the Kolmogorov operator associated with a reaction-diffusion equation having polynomially growing reaction coefficient and perturbed by a noise of multiplicative type, in the Banach space $E$ of continuous functions. By analyzing the smoothing properties of the associated transition semigroup, we prove a modification of the classical identité du carré di champs that applies to the present non-Hilbertian setting. As an application of this identity, we construct the Sobolev space $W^{1,2}(E;\mu)$, where $\mu$ is an invariant measure for the system, and we prove the validity of the Poincaré inequality and of the spectral gap.

1 Introduction

In the present paper we are concerned with the analysis of the Kolmogorov operator associated with the following reaction-diffusion equation in the interval $(0,1)$, perturbed by a noise of multiplicative type

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t}(t,\xi) &= \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + f(\xi,u(t,\xi)) + g(\xi,u(t,\xi))\frac{\partial w}{\partial t}(t,\xi), \quad t \geq 0, \quad \xi \in [0,1], \\
u(t,0) &= u(t,1) = 0, \quad u(0,\xi) = x(\xi), \quad \xi \in [0,1].
\end{aligned}
\end{equation}

Here $\partial w/\partial t(t,\xi)$ is a space-time white noise. The non-linear terms $f,g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are both continuous, the mapping $g(\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, uniformly with respect to $\xi \in [0,1]$, without any restriction on its linear growth, and the mapping $f(\xi,\cdot)$ has polynomial growth, is locally Lipschitz-continuous and satisfies suitable dissipativity conditions, uniformly with respect to $\xi \in [0,1]$. The example of $f(\xi,\cdot)$ we have in mind is an odd-degree polynomial, having negative leading coefficient.

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In [4], the well-posedness of equation (1.1) has been studied and it has been proved that for any initial datum \( x \in E := C_0([0,1]) \) there exists a unique mild solution \( u^x \in L^p(\Omega; C([0,T];E)) \), for any \( T > 0 \) and \( p \geq 1 \). This allows us to introduce the Markov transition semigroup \( P_t \) associated with equation (1.1), by setting for any Borel measurable and bounded function \( \varphi : E \to \mathbb{R} \)

\[
P_t \varphi(x) = E \varphi(u^x(t)), \quad t \geq 0, \quad x \in E,
\]

where \( u^x(t) \) is the solution of (1.1) starting from \( x \) at time \( t = 0 \). Hence, by proceeding as in [2], we can define the Kolmogorov operator \( K \) associated with the semigroup \( P_t \) in terms of its Laplace transform

\[
(\lambda - K)^{-1} \varphi(x) = \int_0^\infty e^{-\lambda t} P_t \varphi(x) dt, \quad \varphi \in C_0(E).
\]  

(1.2)

In this paper we are going to study some important properties of the Kolmogorov operator \( K \) in \( C_0(E) \). If we write equation (1.1) in the abstract form

\[
du(t) = [Au(t) + F(u(t))] dt + G(u(t))dw(t),
\]

(see Section 2 below for all notations) then \( K \) reads formally as

\[
K \varphi = \frac{1}{2} \sum_{h=1}^{\infty} D^2 \varphi(G(x)e_k, G(x)e_k) + \langle Ax + F(x), D\varphi(x) \rangle_E
\]  

(1.3)

(here \( D\varphi \) and \( D^2 \varphi \) represent the first and the second derivatives of a twice differentiable function \( \varphi : E \to \mathbb{R} \) and \( \langle \cdot, \cdot \rangle_E \) is the duality between \( E \) and its topological dual \( E^* \)). Notice however that it is not easy to decide whether a given function belongs to the domain of \( K \) or not, as it is defined in an abstract way by formula (1.2). Our main concern here is studying some relevant properties of \( K \), such as the possibility to define the Sobolev space \( W^{1,2}(E, \mu) \), with respect to the invariant measure \( \mu \) for equation (1.1), or the validity of the Poincaré inequality and of the spectral gap, which, as well known, implies the exponential convergence to equilibrium.

In the case of additive noise, that is when \( G(x) \) is constant, it is possible to study equation (1.1) in the Hilbert space \( H = L^2(0,1) \) in a generalized sense, so that the associated transition semigroup and Kolmogorov operator can be introduced. In this case, it has been proved that the so-called identity du carré des champs

\[
K(\varphi^2) = 2\varphi K\varphi + |G^* D\varphi|^2_H
\]  

(1.4)

is valid for functions \( \varphi \) in a core of \( K \), that is a subset of \( D(K) \) which is dense in the graph norm of \( K \) (see [3]). Identity (1.4) has several important consequences. Actually, if there exists an invariant measure \( \mu \) for \( u^x(t) \), identity (1.4) provides the starting point to define the Sobolev space \( W^{1,2}(H, \mu) \). Moreover, under some additional conditions, it allows to prove the Poincaré inequality and the exponential convergence of \( P_t \varphi \) to equilibrium (spectral gap).
The case we are dealing with in the present paper is much more delicate, as we are considering a polynomial reaction term \( f \) combined with a multiplicative noise. Because of this, it seems better and more natural to work in the Banach space \( E \) of continuous functions vanishing at the boundary, instead of in \( H \). Moreover, the space \( C_b(E) \) is larger than the space \( C_b(H) \), and working in \( C_b(E) \) allows to estimate some interesting functions as, for example, the evaluation functional \( E_{\delta_0}(u) = E_u(\xi_0) \), for \( \delta_0 \in [0,1] \) fixed.

On the other hand, deciding to work in \( C_b(E) \) instead of \( C_b(H) \) has some relevant consequences and there is some price to pay. In our case it means in particular that formula (1.4) has to be changed in a suitable way, as for \( \varphi \in C_b^1(E) \) and \( x \in E \) we cannot say that \( D\varphi(x) \in H \) and hence the term \( |G^*(\cdot)D\varphi|_H \) is no more meaningful. In fact, it turns out that formula (1.4) has to be replaced by the formula

\[
K(\varphi^2) = 2\varphi K\varphi + \sum_{k=1}^{\infty} |\langle G(\cdot)e_k, D\varphi \rangle_E|^2,
\]

(1.5)

where \( \{e_k\}_{k \in \mathbb{N}} \) is the complete orthonormal system given by the eigenfunctions of the second derivative, endowed with Dirichlet boundary conditions.

Notice that, in order to give a meaning to (1.5), for \( \varphi \in D(K) \), we have to prove that:

(i) \( D(K) \) is included in \( C_b^1(E) \);

(ii) the series in (1.5) is convergent for any \( \varphi \in D(K) \);

(iii) \( \varphi^2 \in D(K) \), for any \( \varphi \in D(K) \) and (1.5) holds.

The proof of each one of these steps is very delicate in the framework we are considering here and requires the use of different arguments and techniques, compared to [8] and [3, Chapter 6 and 7].

In order to approach (i), we have proved that the solution \( u^x(t) \) of equation (1.1) is differentiable with respect to \( x \in E \). Moreover, we have proved that the second derivative equation is solvable and suitable bounds for the solution have been given. These results were not available in the existing literature and, in order to be proved, required some new arguments, based on positivity, as the classical techniques did not apply, due to the fact that \( f' \) is not globally bounded and the noise is multiplicative. Next, we had to prove that, as in the Hilbertian case, a Bismuth-Elworthy-Li formula holds for the derivative of the semigroup. Such a formula, as well known, provides the important gradient estimate

\[
\sup_{x \in E} |D(P_t\varphi)|_E^* \leq c (t \wedge 1)^{-\frac{d}{2}} \sup_{x \in E} |\varphi(x)|, \quad t > 0.
\]

In order to prove (ii), we couldn’t proceed directly as in [3, Chapter 5], by using the mild formulation of the first derivative equation and the fact that \( e^{tA} \) is Hilbert-Schmidt, for any \( t > 0 \), again because of the combination of the polynomial non-linearity \( f \) combined with the multiplicative noise. Nevertheless, by using a duality argument, we could prove that

\[
\sum_{k=1}^{\infty} |\langle G(x)e_k, D(P_t\varphi)(x) \rangle_E|^2 \leq c |G(x)|_E^2 \|\varphi\|_0^2 (t \wedge 1)^{-1}, \quad t > 0,
\]

and this allowed us to prove that the series in (1.5) is convergent, for any \( \varphi \in D(K) \). To this purpose, we would like to mention the fact that our duality argument does work because we
are dealing together with the two concrete spaces $E = C_0([0,1])$ and $H = L^2(0,1)$, where we can use some nice approximation and duality argument between the corresponding spaces of continuous functions $C_b(E)$ and $C_b(H)$ and the corresponding spaces of differentiable functions $C^1_0(E)$ and $C^1_b(H)$ (see Lemma 2.1).

Finally, in order to prove (iii), we had to use a suitable modification of the Itô formula that applies to Banach spaces and a suitable approximation argument based on the use of the Ornstein-Uhlenbeck semigroup in the Banach space $E$.

As mentioned before, as a consequence of the modified identity du carré des champs (1.5), we could construct the space $W^{1,2}(E;\mu)$ and we could prove the Poincaré inequality and the existence of a spectral gap. To this purpose, we would like to stress that in spite of the fact that the identity du carré des champs has to be modified and we have to replace $|G(\cdot)e_k,D\varphi|_E^2$ by the series

$$\sum_{k=1}^{\infty} |\langle G(\cdot)e_k,D\varphi \rangle_E|_E^2,$$

the Poincaré inequality proved is very similar to what we have in the case of the Hilbert space $H$, that is

$$\int_E |\varphi(x) - \overline{\varphi}|^2 d\mu(x) \leq \rho \int_E |D\varphi(x)|_{E^*}^2 d\mu(x).$$

## 2 Preliminaries

We shall denote by $H$ the Hilbert space $L^2(0,1)$, endowed with the usual scalar product $\langle \cdot, \cdot \rangle_H$ and the corresponding norm $|\cdot|_H$. Moreover, we shall denote by $E$ the Banach space $C_0([0,1])$ of continuous functions on $[0,1]$, vanishing at 0 and 1, endowed with the sup-norm $|\cdot|_E$ and the duality $\langle \cdot, \cdot \rangle_E$ between $E$ and its dual topological space $E^*$.

Now, if we fix $x \in E$ there exists $\xi_x \in [0,1]$ such that $|x(\xi_x)| = |x|_E$. Then, if $\delta$ is any element of $E^*$ having norm equal 1, the element $\delta_x \in E^*$ defined by

$$\langle y, \delta_x \rangle_E := \begin{cases} \frac{x(\xi_x)y(\xi_x)}{|x|_E} & \text{if } x \neq 0 \\
|y, \delta \rangle_E & \text{if } x = 0, \end{cases}$$

belongs to the subdifferential $\partial |x|_E := \{ h^* \in E^*; |h^*|_{E^*} = 1, \langle h, h^* \rangle_E = |h|_E \}$ (see e.g. Appendix A) for all definitions and details).

Next, let $X$ be a separable Banach space. $\mathcal{L}(X)$ shall denote the Banach algebra of all linear bounded operators in $X$ and $\mathcal{L}^1(X)$ shall denote the subspace of trace-class operators. We recall that

$$\|T\| = \sup_{|x|_X \leq 1} |Tx|_X, \quad T \in \mathcal{L}(X).$$

For any other Banach space $Y$, we denote by $B_b(X,Y)$ the linear space of all bounded and measurable mappings $\varphi : X \to Y$ and by $C_b(X,Y)$ the subspace of continuous functions. Endowed with the sup-norm

$$\|\varphi\|_0 = \sup_{x \in X} |\varphi(x)|_Y,$$
$C_b(X,Y)$ is a Banach space. Moreover, for any $k \geq 1$, $C^k_b(X,Y)$ shall denote the subspace of all functions which are $k$-times Fréchet differentiable. $C^k_b(X,Y)$, endowed with the norm

$$\|\varphi\|_k = \|\varphi\|_0 + \sum_{j=1}^{k} \sup_{x \in X} |D^j \varphi(x)| =: \|\varphi\|_0 + \sum_{j=1}^{k} [\varphi]_j,$$

is a Banach space. In the case $Y = \mathbb{R}$, we shall set $B_b(X,Y) = B_b(X)$ and $C^k_b(X,Y) = C^k_b(X)$, $k \geq 0$.

In what follows, we shall denote by $A$ the linear operator

$$Ax = \frac{\partial^2 x}{\partial \xi^2}, \quad x \in D(A) = H^2(0,1) \cap H^1_0(0,1).$$

$A$ is a non-positive and self-adjoint operator which generates an analytic semigroup $e^{tA}$, with dense domain in $L^2(0,1)$. The space $L^1(0,1) \cap L^\infty(0,1)$ is invariant under $e^{tA}$, so that $e^{tA}$ may be extended to a non-negative one-parameter contraction semigroup $e^{tA_p}$ on $L^p(0,1)$, for all $1 \leq p \leq \infty$. These semigroups are strongly continuous, for $1 \leq p < \infty$, and are consistent, in the sense that $e^{tA_p}x = e^{tA_q}(t)x$, for all $x \in L^p(0,1) \cap L^q(0,1)$. This is why we shall denote all $e^{tA_p}$ by $e^{tA}$. Finally, if we consider the part of $A$ in $E$, it generates a strongly continuous analytic semigroup.

For any $k \in \mathbb{N}$, we define

$$e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin k \pi \xi, \quad \xi \in [0,1]. \quad (2.2)$$

The family $\{e_k\}_{k \in \mathbb{N}}$ is a complete orthonormal system in $H$ which diagonalizes $A$, so that

$$Ae_k = -k^2 \pi^2 e_k, \quad k \in \mathbb{N}.$$ 

Notice that for any $t > 0$ the semigroup $e^{tA}$ maps $L^p(0,1)$ into $L^q(0,1)$, for any $1 \leq p \leq q \leq \infty$ and and for any $p \geq 1$ there exists $M_p > 0$ such that

$$\|e^{tA}\|_{L^p(0,1),L^q(0,1))} \leq M_{p,q} e^{-\omega_{p,q} t - \frac{2 \pi p}{2pq}}, \quad t > 0. \quad (2.3)$$

Here we are assuming that $\partial w(t)/\partial t$ is a space-time white noise defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Thus, $w(t)$ can be written formally as

$$w(t) := \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \geq 0,$$

where $\{e_k\}_{k \in \mathbb{N}}$ is the complete orthonormal system in $H$ which diagonalizes $A$ and $\{\beta_k(t)\}_{k \in \mathbb{N}}$ is a sequence of mutually independent standard real Brownian motions on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. As well known, the series above does not converge in $H$, but it does converge in any Hilbert space $U$ containing $H$, with Hilbert-Schmidt embedding.

Concerning the nonlinearities $f$ and $g$, we assume that they are both continuous. Moreover, they satisfy the following conditions.
Hypothesis 1.  
1. For any \( \xi \in [0, 1] \), both \( f(\xi, \cdot) \) and \( g(\xi, \cdot) \) belong to \( C^2(\mathbb{R}) \).

2. There exists \( m \geq 1 \) such that for \( j = 0, 1, 2 \)

\[
\sup_{\xi \in [0, 1]} |D^j f(\xi, \rho)| \leq c_j (1 + |\rho|^{m-j})^+.
\]  \( (2.4) \)

3. There exists \( \lambda \in \mathbb{R} \) such that

\[
\sup_{(\xi, \rho) \in [0, 1] \times \mathbb{R}} f'(\xi, \rho) \leq \lambda.
\]  \( (2.5) \)

4. The mapping \( g(\xi, \cdot) : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, uniformly with respect to \( \xi \in [0, 1] \), and

\[
\sup_{\xi \in [0, 1]} |g(\xi, \rho)| \leq c \left(1 + |\rho|^{\frac{1}{m}}\right).
\]  \( (2.6) \)

5. If there exist \( \alpha > 0 \) and \( \beta \geq 0 \) such that

\[
(f(\xi, \rho + \sigma) - f(\xi, \rho)) \sigma \leq -\alpha |\sigma|^{m+1} + \beta (1 + |\rho|^{m+1}),
\]  \( (2.7) \)

then no restriction is assumed on the linear growth of \( g(\xi, \cdot) \).

In what follows, for any \( x, y \in E \) and \( \xi \in [0, 1] \) we shall denote

\[
F(x)(\xi) = f(\xi, x(\xi)), \quad [G(x)y](\xi) = g(\xi, x(\xi))y(\xi).
\]

Due to \((2.4)\), \( F \) is well defined and continuous from \( L^p(0, 1) \) into \( L^q(0, 1) \), for any \( p, q \geq 1 \) such that \( p/q \geq m \). In particular, if \( m \neq 1 \) then \( F \) is not defined from \( H \) into itself. Moreover, due to \((2.5)\), for \( x, h \in L^{2m}(0, 1) \)

\[
\langle F(x + h) - F(x), h \rangle_H \leq \lambda |h|_H^2.
\]  \( (2.8) \)

Clearly, \( F \) is also well defined in \( E \) and it is possible to prove that it is twice continuously differentiable in \( E \), with

\[
[DF(x)y](\xi) = D_{\rho} f(\xi, x(\xi))y(\xi), \quad [D^2 F(x)(y_1, y_2)](\xi) = D^2_{\rho} f(\xi, x(\xi))y_1(\xi)y_2(\xi).
\]

In particular, for any \( x \in E \)

\[
|D^j F(x)|_{C^j(E)} \leq c (1 + |x|^{(m-j)+}_E).
\]  \( (2.9) \)

Moreover, for any \( x, h \in E \)

\[
\langle F(x + h) - F(x), \delta_h \rangle_E \leq \lambda |h|_E,
\]  \( (2.10) \)

where \( \delta_h \) is the element of \( \partial |h|_E \) defined above in \((2.1)\).

Finally, in the case \((2.7)\) holds, we have

\[
\langle F(x + h) - F(x), \delta_h \rangle_E \leq -\alpha |h|_E^\alpha + \beta (1 + |x|_E^\beta).
\]  \( (2.11) \)
Next, concerning the operator $G$, as the mapping $g(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, uniformly with respect to $\xi \in [0, 1]$, the operator $G$ is Lipschitz-continuous from $H$ into $\mathcal{L}(H; L^1(0, 1))$, that is
\[
\|G(x) - G(y)\|_{\mathcal{L}(H; L^1(0, 1))} \leq c|x - y|_H.  \tag{2.12}
\]
In the same way it is possible to show that the operator $G$ is Lipschitz-continuous from $H$ into $\mathcal{L}(L^\infty(0, 1); H)$ and
\[
\|G(x) - G(y)\|_{\mathcal{L}(L^\infty(0, 1); H)} \leq c|x - y|_H.  \tag{2.13}
\]

By proceeding similarly as in [3 Proposition 6.1.5], it is possible to prove the following result.

**Lemma 2.1.** For any $\varphi \in C_b(E)$ there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subset C_b(H)$ such that
\[
\begin{align*}
&\lim_{n \to \infty} \varphi_n(x) = \varphi(x), \quad x \in E, \\
&\sup_{x \in H} |\varphi_n(x)| \leq \sup_{x \in E} |\varphi(x)|, \quad n \in \mathbb{N}.
\end{align*}  \tag{2.14}
\]
Moreover, if $\varphi \in C^k_b(E)$, we have $\{\varphi_n\}_{n \in \mathbb{N}} \subset C^k_b(H)$ and for any $j \leq k$
\[
\begin{align*}
&\lim_{n \to \infty} D^j\varphi_n(x)(h_1, \ldots, h_j) = D^j\varphi(x)(h_1, \ldots, h_j), \quad x, h_1, \ldots, h_j \in E, \\
&\sup_{x \in H} |D^j\varphi_n(x)|_{\mathcal{L}_j(E)} \leq \sup_{x \in E} |D^j\varphi(x)|_{\mathcal{L}_j(E)}, \quad n \in \mathbb{N}.
\end{align*}  \tag{2.15}
\]

**Proof.** For any $x \in H$ we define
\[
x_n(\xi) = \frac{n}{2} \int_{\xi - \frac{1}{n}}^{\xi + \frac{1}{n}} \hat{x}(\eta) \, d\eta,
\]
where $\hat{x}(\xi)$ is the extension by oddness of $x(\xi)$, for $\xi \in (-1, 0)$ and $\xi \in (1, 2)$. Clearly, due to the boundary conditions, we have that $x_n \in E$, for any $n \in \mathbb{N}$, and
\[
|x_n|_E \leq \frac{n}{2} \sup_{\xi \in [0, 1]} \int_{\xi - \frac{1}{n}}^{\xi + \frac{1}{n}} |\hat{x}(\eta)| \, d\eta \leq \sqrt{\frac{n}{2}} |x|_H.  \tag{2.16}
\]

Now, for any $n \in \mathbb{N}$ we define
\[
\varphi_n(x) = \varphi(x), \quad x \in H.
\]
Due to (2.16), we have that if $\varphi \in C_b(E)$, then $\varphi_n \in C_b(H)$ and
\[
\sup_{x \in H} |\varphi_n(x)| = \sup_{x \in H} |\varphi(x)| = \sup_{x \in E} |\varphi(x)|.
\]
Finally, if $x \in E$, then we have that $x_n$ converges to $x$ in $E$, and then we can conclude that (2.14) holds.

Now, we prove (2.15) for $k = 1$. If $\varphi \in C^1_b(E)$, then for any $n \in \mathbb{N}$ and $x, h \in H$ we have
\[
\varphi_n(x + h) - \varphi_n(x) = \varphi(x_n + h_n) - \varphi(x_n) = \langle h_n, D\varphi(x_n) \rangle_E + o(|h_n|_E).
\]
Due to (2.16) we have that \( o(|h_n|_E) = o(|h|_H) \) and then we can conclude that \( \varphi_n \) is differentiable in \( H \) and
\[
\langle h, D\varphi_n(x) \rangle_H = \langle h_n, D\varphi(x_n) \rangle_E, \quad x, h \in H.
\]
As \( x_n \) and \( h_n \) converge respectively to \( x \) and \( h \) in \( E \), if \( x, h \in E \), this implies that (2.15) holds.

2.1 The approximating Nemytskii operators

Let \( \gamma \) be a function in \( C^\infty(\mathbb{R}) \) such that
\[
\gamma(x) = x, \; |x| \leq 1, \; |\gamma(x)| = 2, \; |x| \geq 2, \; |\gamma(x)| \leq |x|, \; x \in \mathbb{R}, \; \gamma'(x) \geq 0, \; x \in \mathbb{R}. \quad (2.17)
\]
For any \( n \in \mathbb{N} \), we define
\[
f_n(\xi, \rho) = f(\xi, n\gamma(\rho/n)), \quad (\xi, \rho) \in [0, 1] \times \mathbb{R}.
\]
It is immediate to check that all functions \( f_n \) are in \( C^2(\mathbb{R}) \) and satisfy (2.4) and (2.5), so that the corresponding composition operators \( F_n \) satisfy inequalities (2.9) and (2.10), for constants \( c \) and \( \lambda \) independent of \( n \). Namely
\[
|D^j F_n(x)|_E \leq c(1 + |x|^m)^j, \quad (2.18)
\]
and
\[
\langle F_n(x + h) - F_n(x), \delta h \rangle_E \leq \lambda |h|_E. \quad (2.19)
\]
Notice that all \( f_n(\xi, \cdot) \) are Lipschitz continuous, uniformly with respect to \( \xi \in [0, 1] \), so that all \( F_n \) are Lipschitz continuous in all \( L^p(0, 1) \) spaces and in \( E \).

According to (2.17), we can easily prove that for any \( j = 0, 1, 2 \) and \( R > 0 \)
\[
\lim_{n \to \infty} \sup_{(\xi, \rho) \in [0, 1] \times [-R, R]} |D^j f_n(\xi, \rho) - D^j f(\xi, \rho)| = 0,
\]
and then for any \( R > 0 \) and \( j = 1, 2 \)
\[
\begin{align*}
\lim_{n \to \infty} \sup_{|x|_E \leq R} |F_n(x) - F(x)|_E &= 0, \\
\lim_{n \to \infty} \sup_{|x|_E \leq R} |D^j F_n(x)(y_1, \ldots, y_j) - D^j F(x)(y_1, \ldots, y_j)|_E &= 0.
\end{align*}
\quad (2.20)
\]

We have already seen that the mappings \( F_n \) are Lipschitz-continuous in \( H \). The differentiability properties of \( F_n \) in \( H \) are a more delicate issue. Actually, even if \( f_n(\xi, \cdot) \) is assumed to be smooth, \( F_n : H \to H \) is only Gateaux differentiable and its Gateaux derivative at \( x \in H \) along the direction \( h \in H \) is given by
\[
[D F_n(x)h](\xi) = D_\rho f_n(\xi, x(\xi))h(\xi), \quad \xi \in [0, 1].
\]
Higher order differentiability is even more delicate, as the higher order derivatives do not exist along any direction in \( H \), but only along more regular directions. For example, the second order derivative of \( F_n \) exists only along directions in \( L^4(0, 1) \), and for any \( x \in H \) and \( h, k \in L^4(0, 1) \)
\[
[D^2 F_n(x)(h, k)](\xi) = D_\rho^2 f_n(\xi, x(\xi))h(\xi)k(\xi), \quad \xi \in [0, 1].
\]
3 The solution of (1.1)

With the notations introduced in Section 2, equation (1.1) can be rewritten as the following abstract evolution equation

\[ du(t) = [Au(t) + F(u(t))] \, dt + G(u(t)) \, dw(t), \quad u(0) = x. \]  

\[(3.1)\]

Definition 3.1. An adapted process \( u \in L^p(Ω; C([0, T]; E)) \) is a mild solution for equation (3.1) if

\[ u(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(u(s)) \, ds + \int_0^t e^{(t-s)A}G(u(s)) \, dw(s). \]

Let \( X = E \) or \( X = H \). In what follows, for any \( T > 0 \) and \( p \geq 1 \) we shall denote by \( C^w_{p,T}(X) \) the set of adapted processes in \( L^p(Ω; C([0, T]; E)) \). Endowed with the norm

\[ \| u \|_{C^w_{p,T}(X)} = \left( E \sup_{t \in [0,T]} |u(t)|^p_X \right)^{\frac{1}{p}}, \]

\( C^w_{p,T}(X) \) is a Banach space. Furthermore, we shall denote by \( L^w_{p,T}(X) \) the Banach space of adapted processes in \( C([0, T]; L^p(Ω; X)) \), endowed with the norm

\[ \| u \|_{L^w_{p,T}(X)} = \sup_{t \in [0,T]} \left( E |u(t)|^p_X \right)^{\frac{1}{p}}. \]

In [4] it has been proved that, under Hypothesis 1, for any \( T > 0 \) and \( p \geq 1 \) and for any \( x \in E \), equation (3.1) admits a unique mild solution \( u^x \) in \( C^w_{p,T}(E) \). Moreover

\[ \| u^x \|_{C^w_{p,T}(E)} \leq c_{p,T} \left( 1 + |x|_E \right). \]  

\[(3.2)\]

One of the key steps in the proof of such an existence and uniqueness result, is given in [4, Theorem 4.2], where it proved that the mapping

\[ u \in C^x_{p,T}(E) \mapsto \left( t \mapsto \Gamma(u)(t) := \int_0^t e^{(t-s)A}G(u(s)) \, dw(s) \right) \in C^x_{p,T}(E) \]

is well defined and Lipschitz continuous. By adapting the arguments used in the proof of [4, Theorem 4.2], it is also possible to show that

\[ E \sup_{s \in [0,t]} |\Gamma(u)(s) - \Gamma(v)(s)|_E^p \leq c_p(t) \int_0^t E|u(s) - v(s)|_E^p \, ds, \quad t \geq 0, \]  

\[(3.3)\]

so that, in particular, there exists \( T_p > 0 \) such that

\[ \| \Gamma(u) - \Gamma(v) \|_{L^w_{p,T}(E)} \leq \frac{1}{4} \| u - v \|_{L^w_{p,T}(E)}, \quad T \leq T_p. \]  

\[(3.4)\]

Now, for any \( n \in \mathbb{N} \), we consider the approximating problem

\[ du(t) = [Au(t) + F_n(u(t))] \, dt + G(u(t)) \, dw(t), \quad u(0) = x, \]  

\[(3.5)\]
and we denote by \( u^x_n \) its unique mild solution in \( C^w_{p,T}(E) \). As all \( F_n \) satisfy (2.18) and (2.19), we have that
\[
\|u^x_n\|_{C^w_{p,T}(E)} \leq c_p(T) (1 + |x|_E), \quad n \in \mathbb{N},
\] (3.6)
for a function \( c_p(T) \) independent of \( n \).

As proved in [3, Section 3], the mapping
\[
u \in C^w_{p,T}(H) \mapsto \left( t \mapsto \Gamma(u)(t) := \int_0^t e^{(t-s)A}G(u(s)) \, dw(s) \right) \in C^w_{p,T}(H)
\]
is well defined and Lipschitz continuous. Then, as the mapping \( F_n : H \rightarrow H \) is Lipschitz continuous, we have that for any \( x \in H \) and for any \( T > 0 \) and \( p \geq 1 \) problem (3.5) admits a unique mild solution \( u^x_n \in C^w_{p,T}(H) \) such that
\[
\|u^x_n\|_{C^w_{p,T}(H)} \leq c_{n,p}(T) (1 + |x|_H).
\] (3.7)

**Lemma 3.2.** Under Hypothesis [7] for any \( T, R > 0 \) and \( p \geq 1 \), we have
\[
\lim_{n \to \infty} \sup_{|x|_E \leq R} \|u^x_n - u^x\|_{C^w_{p,T}(E)} = 0.
\] (3.8)

**Proof.** For any \( n \in \mathbb{N} \) we consider the problem
\[
d\Gamma_n(t) = A\Gamma_n(t) \, dt + [G(u^x_n(t)) - G(u^x(t))] \, dw(t), \quad \Gamma_n(0) = 0.
\] (3.9)

Problem (3.9) admits a unique mild solution \( \Gamma_n \in C^w_{p,T}(E) \) which is given by
\[
\Gamma_n(t) = \Gamma(u^x_n)(t) - \Gamma(u^x)(t).
\]

Hence, due to (3.3) we have
\[
\mathbb{E} \sup_{t \in [0,T]} |\Gamma_n(t)|^p_E \leq c_p(T) \int_0^T \mathbb{E}|u^x_n(s) - u^x(s)|^p_E \, ds.
\] (3.10)

Now, if we define \( z_n = u^x_n - u^x - \Gamma_n \), we have that \( z_n \) solves the equation
\[
dz_n/dt(t) = Az_n(t) + F_n(u^x_n(t)) - F(u^x(t)), \quad z_n(0) = 0,
\]
so that, thanks to (2.19),
\[
d^-|z_n(t)|_E \leq \langle Az_n(t), \delta_{z_n(t)} \rangle_E + \langle F_n(u^x_n(t)) - F_n(u^x(t)), \delta_{z_n(t)} \rangle_E
\]
\[
+ \langle F_n(u^x(t)) - F(u^x(t)), \delta_{z_n(t)} \rangle_E \leq \lambda |z_n(t)|_E + |F_n(u^x(t)) - F(u^x(t))|_E.
\]

Due to (3.10), by comparison this yields for any \( t \in [0,T] \)
\[
\|u^x_n - u^x\|_{C^w_{p,T}(E)}^p \leq c_{p,T} \|F_n(u^x) - F(u^x)\|_{C^w_{p,T}(E)}^p + c_p(T) \int_0^t \|u^x_n - u^x\|_{C^w_{p,T}(E)}^p \, ds,
\]

\[\text{10}\]
and then, by using the Gronwall Lemma, we get
\[ \|u_n^x - u^x\|_{C^w_{p,T}(E)} \leq c_{p,T}\|F_n(u^x) - F(u^x)\|_{C^w_{p,T}(E)}. \]
This means that (3.8) follows once we prove that for any \( R > 0 \)
\[ \lim_{n \to \infty} \sup_{|x|E \leq R} \|F_n(u^x) - F(u^x)\|_{C^w_{p,T}(E)} = 0. \] (3.11)

For any \( K > 0 \), we have
\[ \|F_n(u^x) - F(u^x)\|_{C^w_{p,T}(E)} \]
\[ = E\left( \sup_{t \in [0,T]} |F_n(u^x(t)) - F(u^x(t))|_E^p; \sup_{t \in [0,T]} |u_n^x(t)|_E \lor \sup_{t \in [0,T]} |u^x(t)|_E \leq K \right) \]
\[ + E\left( \sup_{t \in [0,T]} |F_n(u^x(t)) - F(u^x(t))|_E^p; \sup_{t \in [0,T]} |u_n^x(t)|_E \lor \sup_{t \in [0,T]} |u^x(t)|_E > K \right), \]

hence, in view of (2.18), (3.2) and (3.6), we obtain
\[ \sup_{|x|E \leq R} \|F_n(u^x) - F(u^x)\|_{C^w_{p,T}(E)} \leq \frac{c_{R,p}(T)}{K} + \sup_{|y|E \leq K} |F_n(y) - F(y)|_E^p. \]
Thanks to (2.20), as \( K \) is arbitrary, this implies (3.11) and (3.8) follows.

\[ \square \]

4 The first derivative

For any \( x \in E, u \in L^w_{p,T}(E) \) and \( n \in \mathbb{N} \), we define
\[ \Lambda_n(x,u)(t) = e^{tA}x + \int_0^t e^{(t-s)A}F_n(u(s)) \, ds + \int_0^t e^{(t-s)A}G(u(s)) \, dw(s), \quad t \geq 0. \]
Clearly, for any \( x \in E \) the solution \( u_n^x \) of problem (3.5) is the unique fixed point of \( \Lambda_n(x, \cdot) \). The mapping \( F_n : E \to E \) is Lipschitz continuous, then due to (3.4), there exists \( T_p = T_p(n) > 0 \) such that
\[ \|\Lambda_n(x,u) - \Lambda_n(x,v)\|_{L^w_{p,T}(E)} \leq \frac{1}{2} \|u - v\|_{L^w_{p,T}(E)}, \quad T \leq T_p. \]
Therefore, if we show that the contraction mapping \( \Lambda_n \) is of class \( C^1 \), we get that the mapping \( x \in E \mapsto u_n^x \in L^w_{p,T}(E) \),
is differentiable and for any \( h \in E \)
\[ D_xu_n^xh = D_x\Lambda_n(x,u_n^x)h + D_u\Lambda_n(x,u_n^x)D_xu_n^xh \] (4.1)
Proof. To prove (4.4) we cannot use the Itô formula, due to presence of the white noise. Moreover we cannot use the same arguments used for example in [4] and [3], because of the unboundedness of $f$. Hence, for some constants $M$ and $p$, we have only to prove that (4.4) holds. To this purpose, the key remark here is that we can assume $h > 0$. Actually, in the general case we can decompose $h = h^+ - h^-$. As $h^+$ and $h^-$ are non-negative, both $\eta_n^{h^+}$ and $\eta_n^{h^-}$ verify the Lemma and then, since by linearity $\eta_n^h = \eta_n^{h^+} - \eta_n^{h^-}$, we can conclude that the Lemma is true also for $\eta_n^h$.

Let $\Gamma_n(t)$ be the mild solution of the problem

$$d\Gamma_n(t) = [A - I] \Gamma_n(t) \, dt + G'(u_n^x(t)) \eta_n^h(t) \, dw(t), \quad \Gamma_n(0) = 0.$$
Since we are assuming that $D_\rho g(\xi, \cdot)$ is bounded, uniformly with respect to $\xi \in [0,1]$, we have that the argument of [4] Theorem 4.2 and Proposition 4.5 can be adapted to the present situation and

$$
\mathbb{E} \sup_{s \in [0,t]} |\Gamma_n(s)|^p_E \leq c_p \int_0^t \mathbb{E}|\eta_n^h(s)|^p_E \, ds, \quad t \in [0, T],
$$

(4.5)

for some constant $c_p$ independent of $T > 0$.

Next, if we set $z_n = \eta_n^h - \Gamma_n$, we have that $z_n$ solves the equation

$$
\frac{dz_n}{dt}(t) = [A - I] z_n(t) + \left[F_n'(u_n^x(t)) + I\right] \eta_n^h(t), \quad z_n(0) = h.
$$

Now, since we are assuming that $h \geq 0$ and equation (4.3) is linear, we have that

$$
P \left( \eta_n^h(t) \geq 0, \ t \in [0, T] \right) = 1,
$$

(see [12] for a proof and see also [5] for an analogous result for equations with non-Lipschitz coefficients). Therefore, as $f_n'(\xi, \rho) \leq \lambda$, for any $(\xi, \rho) \in [0,1] \times \mathbb{R}$, recalling how $\delta_x$ is defined in (2.1) we obtain

$$
\frac{d^-}{dt}|z_n(t)|_E \leq \langle [A - I] z_n(t), \delta_{z_n(t)} \rangle_E + \left(\left[F_n'(u_n^x(t)) + I\right] \eta_n^h(t), \delta_{z_n(t)} \right)_E
$$

$$
\leq \left(f_n'(\xi_{z_n(t)}, u_n^x(t, \xi_{z_n(t)})) + 1\right) \eta_n^h(t, \xi_{z_n(t)}) \leq (\lambda + 1)^+ |\eta_n^h(t)|_E.
$$

Thanks to (4.5), this implies

$$
\mathbb{E} \sup_{s \in [0,t]} |\eta_n^h(s)|^p_E \leq c_p \mathbb{E} \sup_{s \in [0,t]} |z_n(s)|^p_E + c_p \mathbb{E} \sup_{s \in [0,t]} |\Gamma_n(s)|^p_E
$$

$$
\leq c_p |h|^p_E + c_p \int_0^t \mathbb{E} \sup_{r \in [0,s]} |\eta_n^h(r)|^p_E \, ds,
$$

so that from the Gronwall Lemma [1.4] follows.

\begin{lemma}
Under Hypothesis [1] there exists $\eta^h \in C^{w}_{p,T}(E)$ such that for any $R > 0$

$$
\lim_{n \to \infty} \sup_{x, h \in B_R(E)} \|\eta_n^h - \eta^h\|_{C^{w}_{p,T}(E)} = 0.
$$

(4.6)

Moreover, the limit $\eta^h$ solves the equation

$$
\frac{d\eta^h(t)}{dt} = \left[A\eta^h(t) + F'(u^x(t))\eta^h(t)\right] dt + G'(u^x(t))\eta^h(t) \, dw(t), \quad \eta^h(0) = h,
$$

(4.7)

and

$$
\|\eta^h\|_{C^{w}_{p,T}(E)} \leq M_p e^{c_p T} |h|_E.
$$

(4.8)
\end{lemma}
Proof. For any \( n, k \in \mathbb{N} \) we have

\[
\|\eta_{n+k}^h - \eta_n^h\|_{C^w_{p,T}(E)}^p = \mathbb{E} \left( \sup_{t \in [0,T]} |\eta_{n+k}^h(t) - \eta_n^h(t)|_E^p \right) \mathbb{E} \left( \sup_{t \in [0,T]} |u^x(t)|_E \leq n \right)
\]

\[
+ \mathbb{E} \left( \sup_{t \in [0,T]} |\eta_{n+k}^h(t) - \eta_n^h(t)|_E^p \right) \mathbb{P} \left( \sup_{t \in [0,T]} |u^x(t)|_E > n \right).
\]

Since

\[
\left\{ \sup_{t \in [0,T]} |u^x(t)|_E \leq n \right\} \subseteq \left\{ \eta_{n+k}^h(t) = \eta_n^h(t), \ t \in [0,T] \right\},
\]

thanks to (3.2) and (4.4) we get

\[
\|\eta_{n+k}^h - \eta_n^h\|_{C^w_{p,T}(E)}^p \leq \mathbb{E} \left( \sup_{t \in [0,T]} |\eta_{n+k}^h(t) - \eta_n^h(t)|_E^p \right) \mathbb{P} \left( \sup_{t \in [0,T]} |u^x(t)|_E \leq n \right) + \mathbb{E} \left( \sup_{t \in [0,T]} |\eta_{n+k}^h(t) - \eta_n^h(t)|_E^p \right) \mathbb{P} \left( \sup_{t \in [0,T]} |u^x(t)|_E > n \right)
\]

\[
\leq c_p(T) n^{2p} |h|_E^{2p} \left( 1 + |x|^{2p} \right),
\]

and this implies that \( \{\eta_n^h\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C^w_{p,T}(E) \).

Let \( \eta^h \) be its limit and let \( R > 0 \) and \( n \in \mathbb{N} \). For any \( m \geq n \) and \( x, h \in B_R(E) \), due to (4.9) we have

\[
\|\eta_{n+k}^h - \eta_m^h\|_{C^w_{p,T}(E)}^p \leq \|\eta_{n+k}^h - \eta_n^h\|_{C^w_{p,T}(E)}^p + \|\eta_n^h - \eta_m^h\|_{C^w_{p,T}(E)}^p
\]

\[
\leq \frac{c_p(T, R)}{n} + \|\eta_m^h - \eta^h\|_{C^w_{p,T}(E)}^p
\]

Therefore, if we fix \( \epsilon > 0 \) and \( \bar{m} = m(\epsilon, x, h, \rho, T, p) \geq n \) such that

\[
\|\eta_{\bar{m}}^h - \eta^h\|_{C^w_{p,T}(E)} < \epsilon,
\]

due to the arbitrariness of \( \epsilon > 0 \) we get (4.6).

Moreover, as

\[
\eta_{n,k}^h(t) = e^{tA}h + \int_0^t e^{(t-s)A} F_n^x(u_n^x(s)) \eta_n^h(s) \, ds + \int_0^t e^{(t-s)A} G_n^x(u_n^x(s)) \eta_n^h(s) \, dw(s),
\]

and, in addition to (4.6), also (3.8) holds, we can take the limit in both sides and we get that

the limit \( \eta^h \) is a mild solution of equation (4.7). \( \square \)

**Remark 4.3.** In [3, Chapter 4] the differentiability of the mapping

\[
x \in H \mapsto u_n^x \in L^p_{p,T}(H),
\]

\[
(4.10)
\]
has been studied, in the case \( g(\xi, \rho) = 1 \). The proof of [3, Proposition 4.2.1] can be adapted to the present situation of a general smooth \( g \) and we have that the mapping in (4.10) is differentiable and the derivative \( D_x u_n^x h \) satisfies equation (4.13). Moreover, by proceeding as in [3, Lemma 4.2.2], it is possible to prove that \( D_x u_n^x(t)h \in L^p(0,1) \) for any \( t > 0 \), \( \mathbb{P} \)-a.s., and for any \( p \geq 2 \) and \( q \geq 0 \\

\[
\sup_{x \in H} \mathbb{E} \left| D_x u_n^x(t)h \right|_{L^p(0,1)}^q \leq c_{p,q,n}(t \wedge 1) \frac{(p-2)q}{4p} |h|_H^q. \tag{4.11}
\]

Next, we show that we can estimate \( \eta^h \) in \( H \).

**Lemma 4.4.** Under Hypothesis [1] for any \( T > 0 \) and \( p \geq 1 \) we have

\[
\|\eta^h\|_{C_{\rho,T}(H)} \leq c_p(T) |h|_H. \tag{4.12}
\]

Moreover, for any \( p \geq 1 \) and \( q \in [2, +\infty] \) such that \( p(q - 2)/4q < 1 \), we have

\[
\mathbb{E} |\eta^h(t)|_{L^p(0,1)}^p \leq c_{p,q,T}(t) t^{-\frac{p(q-2)}{4q}} |h|_H^p, \quad t > 0. \tag{4.13}
\]

If we assume that the constant \( \lambda \) in (2.5) is strictly negative, then for any \( p \geq 1 \) there exists \( \delta_p > 0 \) such that

\[
\mathbb{E} |\eta^h(t)|_{L^p(0,1)}^p \leq c_p e^{-\delta_p t} |h|_H^p, \quad t \geq 0, \tag{4.14}
\]

and

\[
\mathbb{E} |\eta^h(t)|_{L^p(0,1)}^p \leq c_p e^{-\delta_p t} (t \wedge 1)^{-\frac{p}{q}} |h|_H^p. \tag{4.15}
\]

**Proof.** If we denote by \( \Gamma^h(t) \) the mild solution of

\[
d\gamma(t) = (A + \lambda)\gamma(t) dt + G'(u^x(t))\eta^h(t) dw(t), \quad \gamma(0) = 0,
\]

where \( \lambda \) is the constant introduced in (2.5), we have that \( \rho(t) := \eta^h(t) - \Gamma^h(t) \) solves the problem

\[
\frac{d\rho(t)}{dt} = (A + \lambda)\rho(t) + (F'(u^x(t)) - \lambda)\eta^h(t), \quad \rho(0) = h.
\]

As in the proof of Lemma [4.1], we decompose \( \rho(t) = \rho^+(t) - \rho^-(t) \), where

\[
\frac{d\rho^\pm(t)}{dt} = (A + \lambda)\rho^\pm(t) + (F'(u^x(t)) - \lambda)\eta^\pm h^\pm(t), \quad \rho(0) = h^\pm.
\]

As

\[
\mathbb{P} \left( \eta^\pm h^\pm(0) > 0, \quad t \in [0, T] \right) = 1,
\]

and \( f' - \lambda \leq 0 \), we have

\[
\rho^\pm(t) = e^{(A + \lambda)t} h^\pm + \int_0^t e^{(t-s)(A + \lambda)} (F'(u^x(s)) - \lambda)\eta^\pm h^\pm(s) ds \leq e^{t(A + \lambda)} h^\pm, \tag{4.16}
\]

and then

\[
|\rho(t)|_H \leq e^{\lambda t} |h|_H, \quad \mathbb{P} \text{- a.s.}
\]

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Therefore, since for any $q \in [2, +\infty]$ and $p \geq 0$
\[
\mathbb{E} \sup_{t \in [0,T]} |\Gamma(t)|^p_{L^q(0,1)} \leq c_{p,q}(t) \int_0^t \mathbb{E} |\eta^h(s)|^p_{L^q(0,1)} \, ds,
\]
we can conclude that
\[
\mathbb{E} \sup_{s \in [0,t]} |\eta^h(s)|^p_H \leq c_p e^{\lambda t} |h|^p_H + c_p(t) \int_0^t \mathbb{E} \sup_{r \in [0,s]} |\eta^h(r)|^p_H \, ds
\]
and (4.12) follows from the Gronwall Lemma.

In order to prove (4.13), we notice that, due to (4.16),
\[
|\eta^h(t)| \leq e^{t(A+\lambda)} |h| + |\Gamma(t)|,
\]
so that, in view of (2.3) and (4.17), we can conclude that
\[
\mathbb{E} |\eta^h(t)|^p_{L^q(0,1)} \leq c_p e^{t(A+\lambda)} |h|^p_{L^q(0,1)} + c_p \mathbb{E} |\Gamma(t)|^p_{L^q(0,1)}
\]
\[
\leq c_{p,q}(t) t^{\frac{p|q-2|}{4q}} |h|^p_H + c_{p,q}(t) \int_0^t \mathbb{E} |\eta^h(s)|^p_{L^q(0,1)} \, ds.
\]
If $p(q - 2)/4q < 1$, we can conclude by a comparison argument.

Finally, (4.14) and (4.15) can be proved by combining together the positivity arguments used above, with the exponential estimates proved in [6, Examples 4.4 and 4.5].

\[\square\]

5 The second derivative

Now, we investigate the second order differentiability of $u^x_n$ with respect to $x \in E$. For any process $z \in L^w_{p,T}(E)$ and $x \in E$ we define
\[
[T_n(x)z](t) = \int_0^t e^{(t-s)A} F_n^t(u^x_n(s)) z(s) \, ds + \int_0^t e^{(t-s)A} G_n^t(u^x_n(s)) z(s) \, dw(s),
\]
so that equation (4.3) can be rewritten as
\[
\eta^x_n(t) = e^{tA} h + T_n(x) \eta^x_n(t).
\]

Due to the boundedness of $D_p f_n(\xi, \cdot)$ and $D_p g(\xi, \cdot)$, we have that there exists $T_p = T_p(n) > 0$ such that for any $x \in E$
\[
\|T_n(x)\|_{\mathcal{L}(L^w_{p,T}(E))} \leq \frac{1}{2}, \quad T \leq T_p,
\]
so that
\[
\eta^h_n = [I - T_n(x)]^{-1} e^{tA} h.
\]

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Since $f_n$ and $g$ are twice differentiable with bounded derivatives and Lemma 4.4 holds, we have that the mapping
\[
x \in E \mapsto T_n(x)z \in L^w_{p,T}(E)
\]
is differentiable. Therefore, we can differentiate both sides in (5.1) with respect to $x \in E$ along the direction $k \in E$ and we obtain
\[
D_xT_n(x)z(k(t)) = \int_0^t e^{(t-s)A}F_n(u_n^z(s))(z(s), \eta_n(s)) \, ds + \int_0^t e^{(t-s)A}G_n(u_n^z(s))(\eta_n(s), \eta_n(s)) \, dw(s),
\]
and then $\zeta^{h,k}_n := D_x\eta_n^h = D^2_xu_n^z(h,k)$ satisfies the equation
\[
d_{s_n}^{h,k}(t) = \left[ A\zeta^{h,k}_n(t) + F_n(u_n^z(t))\zeta^{h,k}_n(t) + G_n(u_n^z(t))\eta_n(t), \eta_n(t) \right] dt
\]
\[
+ \left[ G'(u_n^z(t))\zeta^{h,k}_n(t) + G''(u_n^z(t))(\eta_n(t), \eta_n(t)) \right] dw(t), \quad \zeta^{h,k}_n(0) = 0. \tag{5.2}
\]

**Lemma 5.1.** Under Hypothesis 1, for any $T > 0$ and $p \geq 1$ the process $u^z_n$ is twice differentiable in $L^w_{p,T}(E)$ with respect to $x \in E$. Moreover the second derivative $D^2_xu_n^z(h,k) =: \zeta^{h,k}_n$ belongs to $C^p_{p,T}(E)$ and satisfies
\[
\|s_n^{h,k}\|_{C^p_{p,T}(E)} \leq c_p(T) \left( 1 + |x|^{(m-1)} \right) |h|_E |k|_E, \tag{5.3}
\]
for some continuous increasing function $c_p(T)$ independent of $n \in \mathbb{N}$.

**Proof.** We have already seen that $u^z_n$ is twice differentiable in $L^w_{p,T}(E)$ and $D^2_xu_n^z(h,k)$ satisfies equation (5.2). Hence, it only remains to prove estimate (5.3).

As proved in Lemma 4.4, for any $x \in E$ and any $h \in L^p(\Omega; E)$ which is $\mathcal{F}_s$-measurable, the equation
\[
d\eta(t) = \left[ A\eta(t) + F_n(u_n^z(t))\eta(t) \right] dt + G'(u_n^z(t))\eta(t) \, dw(t), \quad \eta(s) = h, \tag{5.4}
\]
adopts a unique solution $\eta_n^h(s, \cdot) \in L^p(\Omega; C([s,T]; E))$ such that
\[
\mathbb{E} \sup_{t \in [s,T]} |\eta_n^h(s,t)|_E^p \leq M_p e^{c_p(T-s)} \mathbb{E} |h|_E^p.
\]
Hence, we can associate to equation (5.4) a stochastic evolution operator $\Phi_n(t, s)$ such that
\[
\eta_n^h(s, t) = \Phi_n(t, s)h, \quad h \in L^p(\Omega; E),
\]
\[17\]
and such that
\[ \mathbb{E} \sup_{r \in [s, t]} |\Phi_n(r, s)h|_{L^p}^p \leq M_p e^{\omega_p(t-s)} \mathbb{E}|h|_{L^p}^p, \quad 0 \leq s \leq t. \quad (5.5) \]

We claim that \( \zeta_n^{h,k} \) can be represented in terms of the operator \( \Phi_n(t, s) \) as
\[ \zeta_n^{h,k}(t) = \Gamma_n^{h,k}(t) + \int_0^t \Phi_n(t, s)\Sigma_n^{h,k}(s) \, ds, \quad (5.6) \]
where \( \Gamma_n^{h,k} \) is the solution of the problem
\[ d\Gamma(t) = A\Gamma(t) \, dt + \left[ G'(u_n^{x}(t))\Gamma(t) + G''(u_n^{x}(t))(\eta_n^h(t), \eta_n^k(t)) \right] \, dw(t), \quad \Gamma(0) = 0, \quad (5.7) \]
and
\[ \Sigma_n^{h,k}(t) = F'_n(u_n^{x}(t))\Gamma_n^{h,k}(t) + F''_n(u_n^{x}(t))(\eta_n^h(t), \eta_n^k(t)). \]

Clearly, in order to prove \((5.6)\) we have to show that \( \int_0^t \Phi_n(t, s)\Sigma_n^{h,k}(s) \, ds \) solves the problem
\[ dz(t) = \left[ Az(t) + F'_n(u_n^{x}(t))z(t) + \Sigma_n^{h,k}(t) \right] \, dt + G'(u_n^{x}(t))z(t) \, dw(t), \quad z(0) = 0. \quad (5.8) \]
More in general, we have to prove that for any \( \Sigma \in C_{p,T}^w(E) \) the mild solution of the problem
\[ dz(t) = \left[ Az(t) + F'_n(u_n^{x}(t))z(t) + \Sigma(t) \right] \, dt + G'(u_n^{x}(t))z(t) \, dw(t), \quad z(0) = 0, \]
is given by
\[ \hat{z}(t) := \int_0^t \Phi_n(t, s)\Sigma(s) \, ds. \]

We have
\[ \int_0^t e^{(t-s)A}F'_n(u_n^{x}(s))\hat{z}(s) \, ds = \int_0^t e^{(t-s)A}F'_n(u_n^{x}(s)) \int_0^s \Phi_n(s, r)\Sigma(r) \, dr \, ds \]
\[ = \int_0^t \int_r^t e^{(t-s)A}F'_n(u_n^{x}(s))\Phi_n(s, r)\Sigma(r) \, ds \, dr \]
and analogously, by the stochastic Fubini theorem,
\[ \int_0^t e^{(t-s)A}G'(u_n^{x}(s))\hat{z}(s) \, dw(s) = \int_0^t e^{(t-s)A}G'(u_n^{x}(s)) \int_0^s \Phi_n(s, r)\Sigma(r) \, dr \, dw(s) \]
\[ = \int_0^t \int_r^t e^{(t-s)A}G'_n(u_n^{x}(s))\Phi_n(s, r)\Sigma(r) \, ds \, dr \]
Now, recalling the definition of \( \Phi_n(t, s)\Sigma \), we have
\[ \int_0^t \left[ \int_r^t e^{(t-s)A}F'_n(u_n^{x}(s))\Phi_n(s, r)\Sigma(r) \, ds \right] \, dr + \int_0^t e^{(t-s)A}G'_n(u_n^{x}(s))\Phi_n(s, r)\Sigma(r) \, dw(s) \, dr \]
\[ = \int_0^t \left[ \Phi_n(t, r)\Sigma(r) - e^{(t-r)A}\Sigma(r) \right] \, dr = \hat{z}(t) - \int_0^t e^{(t-r)A}\Sigma(r) \, dr, \]
so that \( \hat{\zeta} \) is the mild solution of equation (5.8).

Once we have representation (5.6) for \( \zeta_{h,k}^n \), we can proceed with the proof of estimate (5.3). As \( \Gamma_{h,k}^n \) solves equation (5.7), we have

\[
\Gamma_{h,k}^n(t) = \int_0^t e^{(t-s)A} \left[ G'(u^x(s))\Gamma_{h,k}^n(s) + G''(u^x(s))(\eta_1^h(s), \eta_1^k(s)) \right] \, dw(s).
\]

Therefore, due to the boundedness of \( D_\rho g(\xi, \rho) \) and \( D_\rho^2 g(\xi, \rho) \), from (3.4) and (3.3) we get

\[
\mathbb{E} \sup_{s \in [0,t]} |\Gamma_{h,k}^n(s)|_E^p \leq c_p(T) \int_0^t \mathbb{E} |\Gamma_{h,k}^n(s)|_E^p \, ds + c_p(T) |h|_E^p |k|_E^p,
\]

so that, from the Gronwall Lemma, we can conclude

\[
\mathbb{E} \sup_{s \in [0,t]} |\Gamma_{h,k}^n(s)|_E^p \leq c_p(T) |h|_E^p |k|_E^p. \tag{5.9}
\]

Next, as (3.0) and (4.4) hold and as the derivatives of \( F_n \) satisfy (2.18), due to (5.3) we have

\[
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \Phi_n(t, s) \Sigma_{h,k}^n(s) \, ds \right|_E^p \leq c_p(T) \left( \int_0^T \mathbb{E} \Sigma_{h,k}^n(s) \, ds \right)^{1/p} + c_p(T) \left( 1 + |x|_E^{(m-2)p} \right) |h|_E^p |k|_E^p,
\]

and then, thanks to (5.9), we get

\[
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \Phi_n(t, s) \Sigma_{h,k}^n(s) \, ds \right|_E^p \leq c_p(T) \left( 1 + |x|_E^{(m-1)p} \right) |h|_E^p |k|_E^p.
\]

Together with (5.9), this implies (5.3).

In view of the previous lemmas, by arguing as in the proof of Lemma 4.2, we get the following result.

**Lemma 5.2.** Under Hypothesis 7 there exists \( \zeta_{h,k} \in \mathcal{C}_{p,T}(E) \) such that for any \( R > 0 \)

\[
\lim_{n \to \infty} \sup_{x, h,k \in B_R(E)} \| \zeta_{h,k}^n - \zeta_{h,k} \|_{\mathcal{C}_{p,T}(E)} = 0. \tag{5.10}
\]

Moreover, the limit \( \zeta_{h,k} \) solves the equation

\[
d\zeta(t) = \left[ A\zeta(t) + F'(u^x(t))\zeta(t) + F''(u^x(t))(\eta_1^h(t), \eta_1^k(t)) \right] \, dt
\]

\[
+ \left[ G'(u^x(t))\zeta(t) + G''(u^x(t))(\eta_1^h(t), \eta_1^k(t)) \right] \, dw(t), \quad \zeta(0) = 0.
\]

In particular,

\[
\| \zeta_{h,k} \|_{\mathcal{C}_{p,T}(E)} \leq c_p(T) \left( 1 + |x|_E^{p-1} \right) |h|_E |k|_E. \tag{5.11}
\]
As a consequence of Lemmas 3.2, 4.1, 4.2, 5.1 and 5.2 we have the following fact.

**Theorem 5.3.** Under Hypothesis 1, the mapping

\[ x \in E \mapsto u^x \in L^w_{p,T}(E) \]

is differentiable and the derivative \( D_x u^x h \) along the direction \( h \in E \) solves the problem

\[
\frac{d\eta(t)}{dt} = \left[ A\eta(t) + F'(u^x(t))\eta(t) \right] dt + G'(u^x(t))\eta(t) dw(t), \quad \eta(0) = h.
\] (5.12)

**Proof.** For any \( n \in \mathbb{N} \) and \( x, h \in E \) we have

\[
u^{x+h}_n - \nu^x_n = D_x u^x_n h + \int_0^1 \int_0^1 D^2_x u^x_n + \rho \delta h(h, h) d\rho d\theta.
\]

Then, due to (3.8), (4.6) and (5.10), we can take the limit as \( n \to \infty \) and we get

\[
u^{x+h} - \nu^x = \eta h + \int_0^1 \int_0^1 \zeta^{h,h} d\theta d\rho.
\]

The mapping

\[ h \in E \mapsto \eta h \in L^w_{p,T}(E), \]

is clearly linear and according to (4.8) is bounded. Moreover, according to (5.11) we have

\[
\| \int_0^1 \int_0^1 \zeta^{h,h} d\theta d\rho \|_{L^w_{p,T}(E)} \leq c_p(T) \left( 1 + |x|_E^{m-1} \right) |h|^2_E,
\]

and then we can conclude that \( u^x \) is differentiable in \( L^w_{p,T}(E) \) with respect to \( x \in E \) and its derivative along the direction \( h \in E \) solves problem (5.12). \[\Box\]

In view of Lemma 4.4 and Theorem 5.3 for any \( T > 0, p \geq 1 \) and \( x, y \in E \) we have

\[
\mathbb{E} \sup_{t \in [0, T]} |u^x(t) - u^y(t)|^p_H \leq c_p(T) |x - y|^p_H.
\] (5.13)

Now, if \( x \in H \) and \( \{x_n\}_{n \in \mathbb{N}} \) is any sequence in \( E \), converging to \( x \) in \( H \), due to (5.13) we have that \( \{u^{x_n}\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C^w_{p,T}(H) \) and then there exists a limit \( u^x \in C^w_{p,T}(H) \), only depending on \( x \), such that

\[
\|u^x\|_{C^w_{p,T}(H)} \leq c_p(T) \left( 1 + |x|_H \right).
\] (5.14)

Such a solution will be called generalized solution.

**Theorem 5.4.** Under Hypothesis 1, for any \( x \in H \) equation (3.1) admits a unique generalized solution \( u^x \in C^w_{p,T}(H) \), for any \( T > 0 \) and \( p \geq 1 \). Moreover estimate (5.14) holds.

**Remark 5.5.** Concerning the second order differentiability of mapping (4.10), we can adapt again the arguments used in [3, Theorem 4.2.4] to the present situation and thanks to (4.11) we have that mapping (4.10) is twice differentiable with respect to \( x \in H \) and the derivative along the directions \( h, k \in H \) satisfies equation (5.2). Moreover,

\[
\|D^2_x u^x_n(h,k)\|_{C^w_{p,T}(H)} \leq c_{p,n}(T) \left( 1 + |x|_H \right) |h|_H |k|_H.
\] (5.15)
The transition semigroup

We define the transition semigroup associated with equation (3.1) as

\[ P_t \phi(x) = E \phi(u^x(t)), \quad x \in E, \quad t \geq 0, \]

for any \( \phi \in B_b(E) \). In view of Theorem 5.3, we have that

\[ \phi \in C^1_b(E) \implies P_t \phi \in C^1_b(E), \quad t \geq 0, \quad (6.1) \]

and there exist \( M > 0 \) and \( \omega \in \mathbb{R} \) such that

\[ \| P_t \phi \|_1 \leq M e^{\omega t} \| \phi \|_1, \quad t \geq 0. \]

We would like to stress that, in view of Theorem 5.4, the semigroup \( P_t \) can be restricted to \( C_b(H) \). Actually, for any \( \phi \in B_b(H) \) we can define

\[ P^H_t \phi(x) = E \phi(u^x(t)), \quad t \geq 0, \quad x \in H, \]

where \( u^x(t) \) is the unique generalized solution of (3.1) in \( C^w_{p,T}(H) \) introduced in Theorem 5.4.

Notice that if \( x \in E \) and \( \phi \in B_b(H) \), then \( P^H_t \phi(x) = P_t \phi(x) \).

Our first purpose here is to prove that the semigroup \( P_t \) has a smoothing effect in \( B_b(E) \). Namely, we want to prove that \( P_t \) maps \( B_b(E) \) into \( C^1_b(E) \), for any \( t > 0 \). To this purpose, we have to assume the following condition on the multiplication coefficient \( g \) in front of the noise.

**Hypothesis 2.** We have

\[ \inf_{(\xi, \rho) \in [0,1] \times \mathbb{R}} |g(\xi, \rho)| =: \beta > 0. \quad (6.2) \]

First of all, we introduce the transition semigroup \( P^n_t \) associated with the approximating equation (3.5), by setting

\[ P^n_t \phi(x) = E \phi(u^n_\phi(t)), \quad x \in E, \quad t \geq 0, \]

for any \( \phi \in B_b(E) \). It is important to stress that, according to Lemmas 4.1 and 5.1, we have

\[ \phi \in C^k(E) \implies P^n_t \phi \in C^k_b(E), \quad t \geq 0, \quad k = 0, 1, 2, \quad (6.3) \]

and

\[ \| P^n_t \phi \|_k \leq M e^{\omega t} \| \phi \|_k, \quad t \geq 0, \quad k = 0, 1, 2, \]

for some constants \( M > 0 \) and \( \omega \in \mathbb{R} \), which are independent of \( n \in \mathbb{N} \).

Notice that, as equation (3.5) is solvable in \( H \), we can also consider the restriction of \( P^n_t \) to \( B_b(H) \). In view of what we have seen in Remarks 4.3 and 5.5, we have that

\[ \phi \in C^k_b(H) \implies P^n_t \phi \in C^k_b(H), \quad t \geq 0, \quad k = 1, 2, \quad (6.4) \]

and there exist constant \( M_n > 0 \) and \( \omega_n \in \mathbb{R} \) such that

\[ \| P^n_t \phi \|_k \leq M_n e^{\omega_n t} \| \phi \|_k, \quad t \geq 0, \quad k = 1, 2. \quad (6.5) \]
Now, due to Hypothesis 2, for any \( x, y \in H \) we can define
\[
[G^{-1}(x)y](\xi) = \frac{y(\xi)}{g(\xi, x(\xi))}, \quad \xi \in [0, 1].
\]
It is immediate to check that for any \( p \in [1, +\infty] \)
\[
G^{-1} : H \to \mathcal{L}(L^p(0, 1), L^p(0, 1))
\]
and
\[
G^{-1}(x)G(x) = G(x)G^{-1}(x), \quad x \in H.
\]
Therefore, we can adapt the proof of [3, Proposition 4.4.3 and Theorem 4.4.5] to the present situation and we can prove that \( P^n_t \) has a smoothing effect. Namely, we have
\[
\varphi \in B_b(H) \implies P^n_t \varphi \in C^2_b(H), \quad t > 0,
\]
and the Bismut-Elworthy-Li formula holds
\[
\langle h, D(P^n_t \varphi)(x) \rangle_H = \frac{1}{t} \mathbb{E} \varphi(u^n_n(t)) \int_0^t \langle G^{-1}(u^n_n(s))D_xu^n_n(s)h, dw(s) \rangle_H, \quad t > 0, \quad (6.6)
\]
for any \( \varphi \in C^1_b(H) \) and \( x, h \in H \).

In view of all these results, by proceeding as in the proof of [3, Theorem 6.5.1], due to what we have proved in Sections 3, 4 and 5 we obtain the following fact.

**Theorem 6.1.** Under Hypotheses 1 and 2, we have
\[
\varphi \in B_b(E) \implies P_t \varphi \in C^1_b(E), \quad t > 0,
\]
and
\[
\langle h, D(P_t \varphi)(x) \rangle_E = \frac{1}{t} \mathbb{E} \varphi(u^n_n(t)) \int_0^t \langle G^{-1}(u^n_n(s))D_xu^n_n(s)h, dw(s) \rangle_H, \quad t > 0. \quad (6.7)
\]
In particular, for any \( \varphi \in B_b(E) \)
\[
\sup_{x \in E} |D(P_t \varphi)(x)|_{E^*} \leq c (t \wedge 1)^{-\frac{1}{2}} \| \varphi \|_0, \quad t > 0. \quad (6.8)
\]

Theorem 6.1 says that if \( \varphi \in B_b(E) \), then \( P_t \varphi \in C^1_b(E) \), for any \( t > 0 \). If we could prove that in fact \( P_t \varphi \in C^1_b(H) \), then we would have
\[
\sum_{i=1}^{\infty} |\langle (G(x)e_i, D(P_t \varphi)(x) \rangle_H|^2 = |G^* (x)D(P_t \varphi)(x)|^2_H < \infty.
\]
But in general we have only \( P_t \varphi \in C^1_b(E) \) and it is not clear in principle whether the sum
\[
\sum_{i=1}^{\infty} |\langle (G(x)e_i, D(P_t \varphi)(x) \rangle_E|^2
\]
is convergent or not. Next theorem provides a positive answer to this question, which will be of crucial importance for the statement and the proof of the egalité du carré des champs and for its application to the Poincaré inequality.
Theorem 6.2. Let \( \{e_i\}_{i \in \mathbb{N}} \) be the complete orthonormal basis of \( H \) defined in (2.22). Then, under Hypotheses 7 and 8 for any \( \varphi \in C_b(E) \) and \( x \in E \) we have

\[
\sum_{i=1}^{\infty} |\langle G(x)e_i, D(P_t \varphi)(x) \rangle_E|^2 \leq c |G(x)|_E^2 \|\varphi\|_0^2 (t \wedge 1)^{-1}, \quad t > 0.
\] (6.9)

Moreover, if \( \varphi \in C_b^1(E) \), for any \( x \in E \) we have

\[
\sum_{i=1}^{\infty} |\langle G(x)e_i, D(P_t \varphi)(x) \rangle_E|^2 \leq c(t) P_t \left( |D\varphi(\cdot)|_E^2 \right)(x) |G(x)|_E^2 t^{-\frac{1}{2}}, \quad t > 0,
\] (6.10)

for some continuous increasing function. If we also assume that the constant \( \lambda \) in (2.5) is strictly negative, then there exists \( \delta > 0 \) such that

\[
\sum_{i=1}^{\infty} |\langle G(x)e_i, D(P_t \varphi)(x) \rangle_E|^2 \leq ce^{-\delta t} P_t \left( |D\varphi(\cdot)|_E^2 \right)(x) |G(x)|_E^2 t^{-\frac{1}{2}}, \quad t > 0.
\] (6.11)

Proof. Assume \( \varphi \in C_b(E) \) and \( x, h \in E \). According to (4.12) and (6.7), for any \( t \in (0,1] \) we have

\[
|\langle h, D(P_t \varphi)(x) \rangle_E| = \frac{1}{t} \left| \mathbb{E} \varphi(u^x(t)) \int_0^t \langle G^{-1}(u^x(s)) D_x u^x(s) h, dw(s) \rangle_H \right|
\leq \frac{\|\varphi\|_0}{t} \left( \int_0^t |G^{-1}(u^x(s)) D_x u^x(s) h|_H^2 \, ds \right)^{\frac{1}{2}} \leq \frac{c}{t} \|\varphi\|_0 \left( \int_0^t c(s) \, ds \right)^{\frac{1}{2}} |h|_H \leq c \|\varphi\|_0 t^{-\frac{1}{2}} |h|_H.
\]

Due to the semigroup law, it follows that for any \( t > 0 \)

\[
|\langle G(x)h, D(P_t \varphi)(x) \rangle_E| \leq c \|\varphi\|_0 (t \wedge 1)^{-\frac{1}{2}} |G(x)|_E |h|_H.
\] (6.12)

This implies in particular that for any \( t > 0 \) and \( x \in E \) there exists \( \Lambda_\varphi(t,x) \in H \) such that

\[
\langle G(x)h, D(P_t \varphi)(x) \rangle_E = \langle \Lambda_\varphi(t,x), h \rangle_H, \quad h \in E.
\]

Therefore, in view of (6.12)

\[
\sum_{i=1}^{\infty} |\langle G(x)e_i, D(P_t \varphi)(x) \rangle_E|^2 = \sum_{i=1}^{\infty} |\langle \Lambda_\varphi(t,x), e_i \rangle_H|^2
\leq |\Lambda_\varphi(t,x)|_H^2 \leq c \|\varphi\|_0^2 (t \wedge 1)^{-1} |G(x)|_E^2,
\]

and (6.9) holds.

Next, in order to prove (6.10), we notice that if \( \varphi \in C_b^1(E) \), then

\[
\langle G(x)h, D(P_t \varphi)(x) \rangle_E = \mathbb{E} \langle Du^x(t)G(x)h, D\varphi(u^x(t)) \rangle_E.
\]
According to (4.13), with $p = 2$ and $q = +\infty$, for any $t > 0$ we have

$$\langle G(x)h, D(P_t\varphi)(x) \rangle_E^2 \leq P_t(|D\varphi(\cdot)|_\infty^2)(x)c_{2,\infty}(t) t^{-\frac{1}{2}} |G(x)|_E^2 |h|_H^2.$$ 

As above, this implies that for any $t > 0$ and $x \in E$ there exists $\hat{\Lambda}_\varphi(t, x) \in H$ such that

$$\langle G(x)h, D(P_t\varphi)(x) \rangle_E = \langle \hat{\Lambda}_\varphi(t, x), h \rangle_H$$

and as above we can conclude that (6.10) holds.

Finally, in order to get (6.11), we have to proceed exactly in the same way, by using (4.15) instead of (4.13).

7 Kolmogorov operator

We define the Kolmogorov operator $K$ in $C_b(E)$ associated with $P_t$, by proceeding as in [2] and [3]. The operator $K$ is defined through its resolvent by

$$(\lambda - K)^{-1} \varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t \varphi(x) dt, \quad x \in E,$$ 

for all $\lambda > 0$ and $\varphi \in C_b(E)$, see also [13].

We notice that, by Theorem 6.1, we have

$$D(K) \subset C^1_b(E),$$ 

where $D(K)$ is the domain of $K$. In fact, this stronger property holds.

**Theorem 7.1.** Let $\{e_i\}_{i \in \mathbb{N}}$ be the complete orthonormal basis of $H$ defined in (2.2). Then, under Hypotheses 2 and 3, for any $\varphi \in D(K)$ and $x \in E$ we have

$$\sum_{i=1}^{\infty} |\langle G(x)e_i, D\varphi(x) \rangle_E|^2 \leq c |G(x)|_E^2 (\|\varphi\|_0^2 + \|K\varphi\|_0^2).$$ 

**Proof.** Due to the Hölder inequality, for any $\epsilon \in (0, 1)$ and $\psi \in C_b(E)$ we have

$$|\langle G(x)e_i, D((1 - K)^{-1}\psi)(x) \rangle_E|^2 \leq \int_0^{+\infty} e^{-t}(t \wedge 1)^{-1-\epsilon} dt$$

$$\times \int_0^{+\infty} e^{-t}(t \wedge 1)^{1-\epsilon} |\langle G(x)e_i, D(P_t\psi)(x) \rangle_E|^2 dt$$

and then, according to (6.9), we get

$$\sum_{i=1}^{\infty} |\langle G(x)e_i, D((1 - K)^{-1}\psi)(x) \rangle_E|^2 \leq c \epsilon \int_0^{+\infty} e^{-t}(t \wedge 1)^{-t} dt |G(x)|_E^2 \|\psi\|_0^2.$$ 

Therefore, if we take $\psi = (1 - K)\varphi$, we get (7.3).
Our goal is to prove the following result

**Theorem 7.2.** Assume Hypotheses 1 and 2. Then, for any $\varphi \in D(K)$ we have $\varphi^2 \in D(K)$ and the following identity holds

$$K\varphi^2 = 2\varphi K\varphi + \sum_{i=1}^{\infty} |\langle G(\cdot)e_i, D\varphi \rangle_E|^2.$$  \hfill (7.4)

In order to prove identity (7.4), we need suitable approximations of problem (1.1) besides (3.5). For any $m \in \mathbb{N}$, we denote by $u_{n,m}^x$ the unique mild solution in $C^w_{p,T}(E)$ of the problem

$$\frac{du(t)}{dt} = [Au(t) + F_n(u(t))] dt + G(u(t)) P_m dw(t), \quad u(0) = x,$$

where $P_m x = \sum_{i=1}^{m} c_i \langle x, e_i \rangle e_i$, $x \in H$. Moreover for any $k \in \mathbb{N}$ we denote by $u_{n,m,k}^x$ the unique solution in $C^w_{p,T}(E)$ of the problem

$$\frac{du(t)}{dt} = [A_k u(t) + F_n(u(t))] dt + G(u(t)) P_m dw(t), \quad u(0) = x,$$

where $A_k = kA(k - A)^{-1}$ are the Yosida approximations of $A$. The following result is straightforward.

**Lemma 7.3.** Under Hypotheses 1 and 2 for any $x \in E$ and $T > 0$ we have

$$\lim_{m \to \infty} \left| u_{n,m}^x(t) - u_{n}^x(t) \right|_E = 0, \quad \text{uniformly on } [0, T].$$

Moreover for any $x \in E$, $m \in \mathbb{N}$ and $T > 0$ we have

$$\lim_{k \to \infty} \left| u_{n,m,k}^x(t) - u_{n,m}^x(t) \right|_E = 0, \quad \text{uniformly on } [0, T].$$

Let us introduce the approximating Kolmogorov operators. If $\varphi \in C_b(E)$ and $\lambda > 0$, they are defined as above throughout their resolvents

$$(\lambda - K_n)^{-1}\varphi(x) = \int_0^{\infty} e^{-t}\mathbb{E}\varphi(u_n^x(t)) dt,$$

$$(\lambda - K_{n,m})^{-1}\varphi(x) = \int_0^{\infty} e^{-t}\mathbb{E}\varphi(u_{n,m}^x(t)) dt,$$

and

$$(\lambda - K_{n,m,k})^{-1}\varphi(x) = \int_0^{\infty} e^{-t}\mathbb{E}\varphi(u_{n,m,k}^x(t)) dt,$$

for any $\varphi \in C_b(E)$ and $x \in E$. From Lemmas 3.2 and 7.3 we get the following approximation results.

**Lemma 7.4.** Assume Hypotheses 1 and 2. Then, for any $\lambda > 0$ and $x \in E$, we have

$$\lim_{n \to \infty} \left| (\lambda - K_n)^{-1}\varphi(x) - (\lambda - K)^{-1}\varphi(x) \right|_E = 0.$$

If moreover $m \in \mathbb{N}$,

$$\lim_{m \to \infty} \left| (\lambda - K_{n,m})^{-1}\varphi(x) - (\lambda - K_n)^{-1}\varphi(x) \right|_E = 0.$$

If finally $k \in \mathbb{N}$ we have

$$\lim_{k \to \infty} \left| (\lambda - K_{n,m,k})^{-1}\varphi(x) - (\lambda - K_{n,m})^{-1}\varphi(x) \right|_E = 0.$$
Lemma 7.5. Assume Hypotheses 1 and 2. Then, for any $n, m, k \in \mathbb{N}$ we have $C_b^2(E) \subset D(K_{n,m,k})$ and for any $\varphi \in C_b^2(E)$ we have

$$K_{n,m,k}\varphi^2 = 2\varphi K_{n,m,k}\varphi + \sum_{i=1}^m |\langle G(\cdot)e_i, D\varphi \rangle_E|^2.$$  \hfill (7.6)

Proof. Since the stochastic equation (7.5) has regular coefficients and a finite dimensional noise term, the conclusion follows from Itô’s formula in the Banach space $E$ (see Appendix A).

Corollary 7.6. Let $\varphi_{n,m,k} = (\lambda - K_{n,m,k})^{-1}\psi$, for $n, m, k \in \mathbb{N}$ and $\psi \in C_b^2(E)$. Then, under Hypotheses 1 and 2 the following identity holds

$$\varphi_{n,m,k}^2 = (2\lambda - K_{n,m,k})^{-1}(2\varphi_{n,m,k} \psi + \sum_{i=1}^m |\langle G(\cdot)e_i, D\varphi_{n,m,k} \rangle_E|^2).$$  \hfill (7.7)

Proof. As $\lambda \varphi_{n,m,k} - K_{n,m,k}\varphi_{n,m,k} = \psi$, since $\psi \in C_b^2(E)$ we have $\varphi_{n,m,k} \in C_b^2(E)$. Now, multiplying (7.8) by $\varphi_{n,m,k}$ and taking into account (7.6), we get

$$\lambda \varphi_{n,m,k}^2 - \frac{1}{2} K_{n,m,k}(\varphi_{n,m,k}^2) - \frac{1}{2} \sum_{i=1}^m |\langle G(\cdot)e_i, D\varphi_{n,m,k} \rangle_E|^2 = \psi \varphi_{n,m,k}$$

and the conclusion follows.

Lemma 7.7. Let $\varphi = (\lambda - K)^{-1}\psi$, for $\psi \in C_b^2(E)$ and $\lambda > 0$. Then, under Hypotheses 1 and 2 the following identity holds

$$\varphi^2 = (2\lambda - K)^{-1}(2\psi \psi + \sum_{i=1}^\infty |\langle G(\cdot)e_i, D\varphi \rangle_E|^2).$$  \hfill (7.9)

Consequently, $\varphi^2 \in D(K)$ and (7.4) holds.

Proof. The conclusion follows from Theorem 7.1, Lemma 7.4 and Corollary 7.6 by letting $n, m, k \to \infty$.

We are now in the position to prove Theorem 7.2.

Proof. Let $\varphi \in D(K)$, $\lambda > 0$ and $\psi = \lambda \varphi - K\varphi$. If we assume that $\psi \in C_b^2(E)$, then, due to Lemma 7.7 we know that (7.9) holds. Now assume $\psi \in C_b(E)$. It is well known that we cannot find an uniform approximation of $\psi$, because $C_b^2(E)$ is not dense in $C_b(E)$. Thus, we define

$$R_t\psi(x) = \int_H \psi(e^{tA}x + y)N_{Q_t}(dy),$$

where $N_{Q_t}$ is the Gaussian measure in $H$ with mean 0 and covariance $Q_t = -\frac{1}{2} A^{-1}(1 - e^{2tA})$ for $t \geq 0$. As $N_{Q_t}$ is the law of the solution of the linear equation

$$du(t) = Au(t)dt + dw(t), \quad u(0) = 0$$

we have

$$\int_0^t \int_H \psi(e^{sA}x + y)N_{Q_s}(dy)ds$$

is a Gaussian martingale. Since $\psi$ is bounded, we get

$$\int_0^t \int_H \psi(e^{sA}x + y)N_{Q_s}(dy)ds \to \psi(x) \quad \text{in} \quad L^2.$$
which takes values in $E$ and $e^{tA}x \in E$, for any $x \in H$ and $t > 0$, we have that $R_t \psi \in B_b(H)$. Moreover, as proved in [9], we have that for each $t > 0$, $R_t \psi$ belongs to $C_b^\infty(H)$ and consequently to $C_b^\infty(E)$.

Now let $\varphi_t = (\lambda - \kappa)^{-1} R_t \psi$. Since $R_t \psi \in C_b^2(E)$, we have by (7.9)

$$\varphi_t^2 = (2\lambda - \kappa)^{-1}(2\varphi_t R_t \psi + \sum_{i=1}^{\infty} |\langle G(\cdot)e_i, D\varphi_t \rangle_E|^2). \quad (7.10)$$

Therefore, the conclusion follows letting $t \to 0$. Actually, if for any $x \in E$ we have

$$\lim_{t \to 0} h_t(x) = h(x), \quad \sup_{t \in [0,1]} \sup_{x \in E} |h_t(x)| < \infty,$$

then it is immediate to check that

$$\lim_{t \to 0} (\lambda - \kappa)^{-1} h_t(x) = (\lambda - \kappa)^{-1} h(x), \quad x \in E.$$

Therefore, as for any $x \in E$

$$\lim_{t \to 0} \varphi_t^2(x) = \varphi^2(x), \quad \lim_{t \to 0} \varphi_t(x) R_t \psi(x) = \varphi^2(x) \psi(x),$$

we get (7.9) by taking the limit as $t \downarrow 0$ in both sides of (7.10) if we show that

$$\lim_{t \to 0} \sum_{i=1}^{\infty} |\langle G(x)e_i, D\varphi_t(x) \rangle_E|^2 = \sum_{i=1}^{\infty} |\langle G(x)e_i, D\varphi(x) \rangle_E|^2. \quad (7.11)$$

Now, since

$$\lim_{t \to 0} R_t \psi(x) = \psi(x), \quad \|R_t \psi\|_0 \leq \|\psi\|_0,$$

according to (6.7) we have

$$\lim_{t \to 0} \langle G(x)e_i, D\varphi_t(x) \rangle_E = \langle G(x)e_i, D\varphi(x) \rangle_E, \quad (7.12)$$

for any $i \in \mathbb{N}$. By proceeding as in the proof of Theorem 6.2 we see that for any $\varphi \in C_b(E)$

$$|\langle G(x)h, D(P_t\varphi)(x) \rangle_E| \leq \|\varphi\|_0 \frac{1}{t} \left| \mathbb{E} \int_0^t \langle G^{-1}(u^x(s)) Du^x(s) G(x)h, dw(s) \rangle_H \right|.$$

Now, as

$$\frac{1}{t} \left| \mathbb{E} \int_0^t \langle G^{-1}(u^x(s)) Du^x(s) G(x)h, dw(s) \rangle_H \right| \leq ct^{-\frac{1}{2}} |G(x)|_E |h|_H,$$

we have that there exists $\Lambda(t, x) \in H$ such that

$$\frac{1}{t} \left| \mathbb{E} \int_0^t \langle G^{-1}(u^x(s)) Du^x(s) G(x)h, dw(s) \rangle_H \right| = \langle \Lambda(t, x), h \rangle_H$$

and

$$|\Lambda(t, x)|_H \leq ct^{-\frac{1}{2}} |G(x)|_E. \quad (7.13)$$

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By arguing as in the proof of Theorem 7.1 with \( \epsilon = 1/2 \), this implies that
\[
|\langle G(x)e_i, D(\varphi_t(x)) \rangle_E |^2 \leq c \int_0^\infty e^{-\lambda s} (s \wedge 1)^{\frac{1}{2}} |\langle G(x)e_i, D(P_s\psi)(x) \rangle_E |^2 \, ds
\]
\[
\leq c \|R_t\| \|0\| \int_0^\infty e^{-\lambda s} (s \wedge 1)^{\frac{1}{2}} |\langle \Lambda(s,x), e_i \rangle_H |^2 \, ds \leq c \|\varphi\| \int_0^\infty e^{-\lambda s} (s \wedge 1)^{\frac{1}{2}} |\langle \Lambda(s,x), e_i \rangle_H |^2 \, ds.
\]

Therefore, as due to (7.13)
\[
\sum_{i=1}^\infty \int_0^\infty e^{-\lambda s} (s \wedge 1)^{\frac{1}{2}} |\langle \Lambda(s,x), e_i \rangle_H |^2 \, ds < \infty,
\]
and (7.12) holds, from the Fatou’s lemma we get (7.11) and (7.4) follows for a general \( \varphi \in D(K) \).

8 Invariant measures

In [4] it has been proved that there exists an invariant probability measure \( \mu \) on \( (E, \mathcal{B}(E)) \) for the semigroup \( P_t \). In particular, if \( \varphi \in D(K) \) we have
\[
\int_E \mathcal{K}\varphi \, d\mu = 0. \tag{8.1}
\]

From now on we shall assume that the following condition is satisfied.

**Hypothesis 3.** There exists \( \alpha > 0 \) such that \( A + F' \leq -\alpha I \), and \( g(\xi, \rho) \) is uniformly bounded on \( [0, 1] \times \mathbb{R} \).

In [6, Proposition 4.1], we have proved that under Hypothesis 3 there exists \( \delta > 0 \) such that for any \( p \geq 1 \) and \( x \in E \)
\[
\mathbb{E} |u^x(t)|_E^p \leq c_p \left( 1 + e^{-\delta pt} |x|_E^p \right), \quad t \geq 0. \tag{8.2}
\]
As a consequence of this, we have that for any \( p \geq 1 \)
\[
\int_E |x|_E^p \, \mu(dx) < \infty. \tag{8.3}
\]
Actually, due to the invariance of \( \mu \), for any \( t \geq 0 \) it holds
\[
\int_E |x|_E^p \, \mu(dx) = \int_E \mathbb{E} |u^x(t)|_E^p \, \mu(dx) \leq c_p \left( 1 + e^{-\delta pt} \int_E |x|_E^p \, \mu(dx) \right).
\]
Therefore, if we choose \( t_0 \) such that \( c_p e^{-\delta pt_0} < 1/2 \), we have that (8.3) follows.

**Remark 8.1.** In order to have (8.2), it is not necessary to assume that \( g \) is uniformly bounded. Actually, (2.6) is what we need to prove (8.2). In Hypothesis 3 we are assuming that \( g \) is bounded in view of the proof of the Poincaré inequality, where we need an estimate uniform with respect to \( x \in E \).
Now, as $\mu$ is invariant, it is well known that $P_t$ can be uniquely extended to a semigroup of contractions on $L^2(E,\mu)$ which we shall still denote by $P_t$, whereas we shall denote by $\mathcal{K}_2$ its infinitesimal generator.

**Lemma 8.2.** Assume Hypotheses 1, 2 and 3. Then, $D(\mathcal{K})$ is a core for $\mathcal{K}_2$.

**Proof.** Let $\psi := \varphi - \mathcal{K}_2\varphi$, for $\varphi \in D(\mathcal{K}_2)$ and $\lambda > 0$. Since $C_0(E)$ is dense in $L^2(E,\mu)$, there exists a sequence $(\psi_n) \subset C_0(E)$ convergent to $\psi$ in $L^2(E,\mu)$. If we set $\varphi_n := (\lambda - \mathcal{K}_2)^{-1}\psi_n$ then $\varphi_n \in D(\mathcal{K})$ and

$$\varphi_n \to \varphi, \quad \mathcal{K}_2\varphi_n \to \mathcal{K}\varphi \quad \text{in} \quad L^2(E,\mu),$$

which shows that $D(\mathcal{K})$ is a core for $\mathcal{K}_2$. \hfill $\blacksquare$

### 8.1 Consequences of the “Egalité du carré des champs”

Our first result is the so called *Egalité du carré des champs* (see [1]).

**Proposition 8.3.** Assume that Hypotheses 1, 2 and 3 hold. Then for any $\varphi \in D(\mathcal{K})$ we have

$$\int_E \mathcal{K}\varphi(x) \varphi(x) \, d\mu(x) = -\frac{1}{2} \sum_{i=1}^{\infty} |\langle G(x)e_i, D\varphi(x) \rangle_E|^2 \, d\mu(x). \quad (8.4)$$

**Proof.** Let $\varphi \in D(\mathcal{K})$. Then, by Theorem 7.2, $\varphi^2 \in D(\mathcal{K})$ and identity (7.4) holds. According to (7.3) and (8.3), we can integrate both sides of (7.4) with respect to $\mu$ and taking into account that, in view of (8.1), $\int_E \mathcal{K}(\varphi^2) \, d\mu = 0$, we get the conclusion. \hfill $\square$

Let us show a similar identity for the semigroup $P_t$.

**Proposition 8.4.** Let $\varphi \in C_0^1(E)$ and set $v(t,x) = P_t\varphi(x)$. Then, under Hypotheses 1, 2 and 3 we have

$$v \in L^\infty(0,T;L^2(E,\mu)), \quad \sum_{i=1}^{\infty} |\langle G(\cdot)e_i, D_x v(x) \rangle_E|^2 \in L^1(0,T;L^1(E,\mu)),$$

for any $T > 0$. Moreover

$$\int_E (P_t\varphi)^2 \mu(dx) + \int_0^t ds \sum_{i=1}^{\infty} |\langle G(x)e_i, D(P_s\varphi)(x) \rangle_E|^2 \mu(dx) = \int_H \varphi^2(x) \mu(dx). \quad (8.5)$$

**Proof.** If we assume that $\varphi \in D(\mathcal{K})$, we have $P_t\varphi \in D(\mathcal{K})$ and $\mathcal{K}P_t\varphi = P_t\mathcal{K}\varphi$ (for a proof see [3] Lemma B.2.1). According to (7.3), this yields

$$\sum_{i=1}^{\infty} |\langle G(x)e_i, D_x v(t,x) \rangle_E|^2 \leq c |G(x)|^2_E \left( \|P_t\varphi\|^2_0 + \|\mathcal{K}P_t\varphi\|^2_0 \right)$$

$$\leq c |G(x)|^2_E \left( \|\varphi\|^2_0 + \|\mathcal{K}\varphi\|^2_0 \right),$$

so that

$$\sum_{i=1}^{\infty} |\langle G(\cdot)e_i, D_x v(t,\cdot) \rangle_E|^2 \in L^1(0,T;L^1(E,\mu)), \quad (8.6)$$
for any $T > 0$. Now, as $D_{t}v(t, x) = Kw(t, x)$ (see [3 Proposition B.2.2]), multiplying both sides by $v(t, x)$ and integrating over $E$ with respect to $\mu$, due to (8.4) we get

$$
\frac{1}{2} \frac{d}{dt} \int_{E} v^{2}(t, x) \mu(dx) = \int_{E} Kv(t, x) v(t, x) \mu(dx) = -\frac{1}{2} \int_{E} \sum_{i=1}^{\infty} |(G(x)e_{i}, D_{x}v(t, x))_{E}|^{2} \mu(dx).
$$

Thus, integrating with respect to $t$, (8.5) follows when $\varphi \in D(\mathcal{K})$.

Now, assume $\varphi \in C^{1}_{b}(E)$. Clearly, the mapping $(t, x) \mapsto P_{t}\varphi(x)$ is in $L^{\infty}(0, T; L^{2}(E, \mu))$. Moreover, according to (8.5) holds for

$$
\sum_{i=1}^{\infty} |(G(x)e_{i}, D_{t}(P_{t}\varphi)(x))_{E}|^{2} \leq C(t) P_{t}(|D\varphi(\cdot)|^{2}_{E})(x)|G(x)|^{2}_{E} t^{-\frac{1}{2}}
$$

and then (8.6) holds. Next, for any $n \in \mathbb{N}$ we define $\varphi_{n} := n(n - \mathcal{K})^{-1} \varphi$. Clearly, $\varphi_{n} \in D(\mathcal{K})$ and for $x \in E$

$$
\lim_{n \to \infty} \varphi_{n}(x) = \varphi(x), \quad \|\varphi_{n}\|_{0} \leq \|\varphi\|_{0}, \quad n \in \mathbb{N}. \tag{8.8}
$$

Moreover, thanks to (8.8), we have

$$
\lim_{n \to \infty} |D\varphi_{n}(x) - D\varphi(x)|_{E^{*}} = 0, \quad \sup_{x \in E} |D\varphi_{n}(x)|_{E^{*}} \leq \sup_{x \in E} |D\varphi(x)|_{E^{*}}, \quad n \in \mathbb{N}. \tag{8.9}
$$

As (8.5) holds for $\varphi \in D(\mathcal{K})$, if we set $v_{n}(t, x) = P_{t}\varphi_{n}(x)$, we have for each $n \in \mathbb{N}$

$$
\int_{E} v_{n}^{2}(t, x) \mu(dx) + \int_{0}^{t} ds \int_{E} \sum_{i=1}^{\infty} |(G(x)e_{i}, D_{x}v_{n}(s, x))_{E}|^{2} \mu(dx) = \int_{H} \varphi_{n}(x) \mu(dx).
$$

Due to (8.7), (8.8) and (8.9), by arguing as in the proof of Lemma 7.7, we can take the limit in both sides above, as $n \to \infty$, and we get (8.5) for $\varphi \in C^{1}_{b}(E)$.

\begin{flushright}
$\square$
\end{flushright}

### 8.2 The Sobolev space $W^{1,2}(E, \mu)$

We are going to show that the derivative operator $D$ is closable in $C^{1}_{b}(E)$, so that we can introduce the Sobolev space $W^{1,2}(E, \mu)$.

**Proposition 8.5.** Assume Hypotheses [4][5][6] and [3] Then, the derivative operator

$$
D : C^{1}_{b}(E) \to L^{2}(E, \mu; E^{*}), \quad \varphi \mapsto D\varphi,
$$

is closable.

**Proof.** Let $(\varphi_{n}) \subset C^{1}_{b}(E)$ such that

$$
\varphi_{n} \to 0 \quad \text{in } L^{2}(E, \mu) \quad D\varphi_{n} \to F \quad \text{in } L^{2}(E, \mu; E^{*}).
$$

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We have to show that $F = 0$. We first prove that for any $t > 0$ we have
\[
\lim_{n \to \infty} D(P_t \varphi_n)(x) = \mathbb{E}[(Du^x(t))^* F(u^x(t))] \quad \text{in } L^2(E, \mu; E^*). \tag{8.10}
\]

In fact, recalling Theorem 5.3 and (4.8), we have
\[
\int_E |D(P_t \varphi_n)(x) - \mathbb{E}(Du^x(t))^* F(u^x(t))|^2 \mu(dx)
\]
\[
\leq \int_E \mathbb{E}(Du^x(t))^* (D\varphi_n(u^x(t)) - F(u^x(t)))^2 \mu(dx)
\]
\[
\leq M e^{\omega t} \int_E |D\varphi_n(u^x(t)) - F(u^x(t))|^2 \mu(dx) = M e^{\omega t} \int_E |D\varphi_n(x) - F(x)|^2 \mu(dx),
\]
last inequality following from the invariance of $\mu$. This implies (8.10).

Now, according to (8.5), we have
\[
\int_E (P_t \varphi_n)^2 \mu(dx) + \int_0^t ds \int_E \sum_{i=1}^{\infty} \langle (G(x)e_i, D(P_s \varphi_n)(x))_E \rangle^2 \mu(dx) = \int_H \varphi_n^2(x) \mu(dx).
\]
Then, we can take the limit as $n \to \infty$ in both sides and we get
\[
\lim_{n \to \infty} \int_0^t ds \int_E \sum_{i=1}^{\infty} \langle (G(x)e_i, D(P_s \varphi_n)(x))_E \rangle^2 \mu(dx) = 0.
\]
Due to (8.10), this implies that for any $i \in \mathbb{N}$
\[
\mathbb{E} \langle Du^x(t)G(x)e_i, F(u^x(t)) \rangle_E = 0,
\]
so that
\[
P_t(\langle G(x)e_i, F(x) \rangle_E) = \mathbb{E}(Du^x(t)e_i, F(u^x(t)))_E
\]
\[
= \mathbb{E}(Du^x(t)G(x)e_i, F(u^x(t)))_E + \mathbb{E}(G(x)e_i - Du^x(t)G(x)e_i, F(u^x(t)))_E
\]
\[
+ \mathbb{E}((G(u^x(t)) - G(x)) e_i, F(u^x(t)))_E
\]
\[
= \mathbb{E}(G(x)e_i - Du^x(t)G(x)e_i, F(u^x(t)))_E + \mathbb{E}((G(u^x(t)) - G(x)) e_i, F(u^x(t)))_E
\]
Consequently, due to the continuity at $t = 0$ of $u^x(t)$ and $Du^x(t)$, we get
\[
\lim_{t \to 0} P_t(\langle G(\cdot)e_i, F \rangle_E) = 0, \quad \text{in } L^1(E, \mu).
\]
Since $P_t$ is a strongly continuous semigroup in $L^1(E, \mu)$, we deduce $\langle G(\cdot)e_i, F \rangle_E = 0$ for all $i \in \mathbb{N}$. As $G(x)$ is invertible and for any $h \in E$
\[
\lim_{n \to \infty} \sum_{i \leq n} \langle h, e_i \rangle_E e_i = h, \quad \text{in } E, \tag{8.11}
\]
this implies $\langle h, F(x) \rangle_E = 0$, for any $x, h \in E$, and then $F = 0$. \qed
Since $D$ is closable in $L_2(E, \mu)$, we define as usual the Sobolev space $W^{1,2}(E, \mu)$ as the domain of the closure of $D$ endowed with its graph norm. Notice that if $\{\varphi_n\} \subset C^1_b(E)$ approximates some $\varphi \in W^{1,2}(E, \mu)$ in the graph norm of $D$, then, according to (8.7), the series
\[
\int_0^t ds \int_E \sum_{i=1}^{\infty} |(G(x)e_i, D(P_s\varphi_n)(x))_E|^2 \, d\mu(x)
\]
converges uniformly with respect to $n \in \mathbb{N}$, so that (8.3) holds for any $\varphi \in W^{1,2}(E, \mu)$.

**Proposition 8.6.** Under Hypotheses 1, 2 and 3 for any $\varphi \in D(K_2)$ we have
\[
\int_E K_2(\varphi)(x) \varphi(x) \, d\mu(x) = -\frac{1}{2} \int_E \sum_{i=1}^{\infty} |(G(x)e_i, D\varphi(x))_E|^2 \, d\mu(x).
\]

**Proof.** It follows from Lemma 8.2 and Proposition 8.5.

### 8.3 The Poincaré inequality

As a consequence of Hypothesis 3 if we take $\alpha$ large enough, by proceeding as in Lemma 4.4 we have that there exists some $\theta > 0$ such that
\[
|DP_t\varphi(x)|_E \leq e^{-\theta t} \sup_{x \in E} |D\varphi(x)|_{E^*}.
\]

By a standard argument this implies that for any $x \in E$
\[
\lim_{t \to \infty} P_t\varphi(x) = \overline{\varphi} = \int_E \varphi d\mu.
\]

**Proposition 8.7.** Under Hypotheses 1, 2 and 3 there exist $\rho > 0$ such that for all $\varphi \in W^{1,2}(E, \mu)$
\[
\int_E |\varphi(x) - \overline{\varphi}|^2 d\mu(x) \leq \rho \int_E |D\varphi(x)|^2_{E^*} d\mu(x).
\]

**Proof.** We start from (8.3) for $\varphi \in W^{1,2}(E, \mu)$,
\[
\int_E (P_t\varphi)^2(x) \mu(dx) + \int_0^t ds \int_E \sum_{i=1}^{\infty} |(G(x)e_i, D(P_s\varphi)(x))_E|^2 \, d\mu(dx) = \int_H \varphi^2(x) \mu(dx).
\]
Taking into account of (8.11), this yields
\[
\int_E (P_t\varphi)^2 \mu(dx) + c \int_0^t ds e^{-\delta s} s^{-\frac{1}{2}} \int_E P_s(|D\varphi(\cdot)|^2_{E^*})(x) \mu(dx) \geq \int_H \varphi^2(x) \mu(dx),
\]
which, by the invariance of $\mu$, yields
\[
\int_E (P_t\varphi)^2 \mu(dx) + c \int_0^t ds e^{-\delta s} s^{-\frac{1}{2}} \int_E |D\varphi(x)|^2_{E^*} \mu(dx) \geq \int_H \varphi^2(x) \mu(dx),
\]
Letting $t \to \infty$, and recalling (8.14), this implies that for some $\rho > 0$
\[
(\overline{\varphi})^2 + \rho \int_E |D\varphi(x)|^2_{E^*} \mu(dx) \geq \int_H \varphi^2(x) \mu(dx),
\]
which is equivalent to (8.15).
8.4 Spectral gap and convergence to equilibrium

Proposition 8.8. Under Hypotheses 1, 2 and 3 we have

\[ \sigma(K_2) \setminus \{ 0 \} \subset \left\{ \lambda \in \mathbb{C} : \Re \lambda \leq -\beta^2 / \rho \right\}, \]

where \( \sigma(K_2) \) denotes the spectrum of \( K_2 \).

Proof. Let us consider the space of all mean zero functions from \( L^2(E, \mu) \)

\[ L_\pi^2(E, \mu) := \{ \varphi \in L^2(E, \mu) : \varphi = 0 \}. \]

Clearly

\[ L^2(E, \mu) = L^2_\pi(E, \mu) \oplus \mathbb{R}. \]

Moreover if \( \varphi = 0 \) we have by the invariance of \( \mu \)

\[ \overline{(P_t \varphi)} = \int_H P_t \varphi(x) d\mu(x) = \int_H \varphi(x) d\mu(x) = 0, \]

so that \( L^2_\pi(E, \mu) \) is an invariant subspace of \( P_t \).

Denote by \( K_\pi \) the restriction of \( K_2 \) to \( L^2_\pi(E, \mu) \). Then we have clearly

\[ \sigma(K_2) = \{ 0 \} \cup \sigma(K_\pi). \]

Moreover, if \( \varphi \in L^2_\pi(E, \mu) \) we see, using (8.4), that

\[ \int_E K_\pi \varphi(x) \varphi(x) d\mu(x) = \int_E K_2 \varphi(x) \varphi(x) d\mu(x) = -\frac{1}{2} \int_E \sum_{i=1}^\infty |\langle G(x)e_i, D\varphi(x) \rangle_E|^2 d\mu(x), \tag{8.16} \]

Now, due to (8.11), for any \( x, h \in E \) we have

\[ |\langle G(x)h, D\varphi(x) \rangle_E|^2 = \left( \sum_{i=1}^\infty |\langle G(x)e_i, D\varphi(x) \rangle_E \langle h, e_i \rangle_H|^2 \right)^2 \]

\[ \leq \sum_{i=1}^\infty |\langle h, e_i \rangle_H|^2 \sum_{i=1}^\infty |\langle G(x)e_i, D\varphi(x) \rangle_E|^2 \]

so that, as \( |h|_E \leq 1 \) implies \( |h|_H \leq 1 \),

\[ |G^*(x)D\varphi(x)|_{L^2}^2 \leq \sum_{i=1}^\infty |\langle G(x)e_i, D\varphi(x) \rangle_E|^2. \]

Due to Hypothesis 2 according to (8.16) this yields

\[ \int_E K_\pi \varphi \varphi d\mu \leq -\frac{\beta^2}{2} \int_E |D\varphi(x)|_{E^*}^2 d\mu(x) \]

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and by the Poincaré’s inequality, we deduce
\[
\int_E K_\pi \varphi(x) \varphi(x) \, d\mu(x) \leq -\frac{\beta^2}{2} \int_E |D\varphi(x)|^2_E \, d\mu(x) \leq -\frac{\beta^2}{2\rho} \int_E \varphi^2(x) \, d\mu(x),
\]
which yields by the Hille–Yosida Theorem.
\[
\sigma(K_\pi) \subset \{ \lambda \in \mathbb{C} : \Re \lambda \leq -\beta^2/2\rho \}.
\]
\[\square\]

**Remark 8.9.** The spectral gap implies the exponential convergence of \( P_t \varphi \) to \( \overline{\varphi} \). In fact from
\[
\int_E K_\pi \varphi(x) \varphi(x) \, d\mu(x) \leq -\frac{\beta^2}{2} \int_E \varphi^2(x) \, d\mu(x),
\]
we deduce that \( K_\pi + \beta^2/2\rho I \) is \( m \)-dissipative, so that by the Hille–Yosida Theorem we have
\[
\int_E |P_t \psi(x)|^2 \, d\mu(x) \leq e^{-\frac{\beta^2 t}{\rho}} \int_E |\psi(x)|^2 \, d\mu(x), \quad \psi \in L^2_\pi(E, \mu).
\]
Now given \( \varphi \in L^2_\pi(E, \mu) \), setting in \(8.18\): \( \psi := \varphi - \overline{\varphi} \), we get
\[
\int_E |P_t \varphi(x) - \overline{\varphi}|^2 \, d\mu(x) \leq e^{-\frac{\beta^2 t}{\rho}} \int_E |\varphi(x) - \overline{\varphi}|^2 \, d\mu(x)
\]
\[
= e^{-\frac{\beta^2 t}{\rho}} \left( \int_H \varphi^2(x) \, d\mu(x) - |\overline{\varphi}|^2 \right) \leq e^{-\frac{\beta^2 t}{\rho}} \int_E |\varphi(x)|^2 \, d\mu(x).
\]
\[\square\]

### A An Itô formula in the space of continuous functions

Fix \( k \in \mathbb{N} \) and let \( b, \sigma_1, \ldots, \sigma_k \) be mappings from \( H \) into \( H \) and from \( E \) into \( E \), which are Lipschitz continuous both in \( H \) and in \( E \). Let \( X \) be the solution to the stochastic differential equation
\[
X(t) = x + \int_0^t b(X(s)) \, ds + \sum_{i=1}^k \int_0^t \sigma_i(X(s)) \, d\beta_i(s),
\]
where \( \beta_1, \ldots, \beta_n \) are independent real Brownian motions.

If \( \varphi \in C^2_0(H) \), then it is well known that the following Itô’s formula holds
\[
\mathbb{E} \varphi(X(t)) = \varphi(x) + \mathbb{E} \int_0^t \mathcal{L} \varphi(X(s)) \, ds,
\]
where \( \mathcal{L} \) is the Kolmogorov operator given by
\[
\mathcal{L} \varphi(x) = \frac{1}{2} \sum_{i=1}^k \langle D^2 \varphi(x) \sigma_i(x), \sigma_i(x) \rangle_H + \langle D \varphi(x), b(x) \rangle_H, \quad x \in H.
\]
Now we see what happens when dealing with \( A.2 \) for functions defined in \( E \).
**Proposition A.1.** If \( \varphi \in C^2_b(E) \), then it holds

\[
\mathbb{E}\varphi(X(t)) = \varphi(x) + \mathbb{E} \int_0^t \mathcal{L}_E \varphi(X(s))ds,
\]

where \( \mathcal{L}_E \) is given by

\[
\mathcal{L}_E \varphi(x) = \frac{1}{2} \sum_{i=1}^k \langle \sigma_i(x), D^2 \varphi(x) \sigma_i(x) \rangle_E + \langle b(x), D \varphi(x) \rangle_E,
\]

and \( D_E \) represents the Fréchet derivative in \( E \).

**Proof.** In view of Lemma 2.1, if \( \varphi \in C^2_b(E) \), there exists a sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \subset C^1_b(H) \) such that

\[
\lim_{n \to \infty} \varphi_n(x) = \varphi(x), \quad x \in E
\]

\[
\lim_{n \to \infty} \langle y, D \varphi_n(x) \rangle_H = \langle y, D \varphi(x) \rangle_E, \quad x, y \in E
\]

\[
\lim_{n \to \infty} \langle y, D^2 \varphi_n(x) y \rangle_H = \langle y, D^2_E \varphi(x) y \rangle_E, \quad x, y \in E.
\]

Consequently

\[
\lim_{n \to \infty} \mathcal{L}_E \varphi_n(x) = \mathcal{L}_E \varphi(x), \quad x \in E.
\]

Now, by Itô’s formula (A.2), we have for any \( n \in \mathbb{N} \)

\[
\mathbb{E}\varphi_n(X(t)) = \varphi_n(x) + \mathbb{E} \int_0^t \mathcal{L}_E \varphi_n(X(s))ds,
\]

and then, letting \( n \to \infty \), we get (A.4). \( \square \)

**Remark A.2.** Let \( \varphi \in C^2_b(E) \). Then \( \varphi^2 \in C^2_b(E) \) and we have

\[
\langle y, D_E \varphi^2(x) \rangle_E = 2 \varphi(x) \langle y, D_E \varphi(x) \rangle_E
\]

and

\[
\langle y, D^2_E \varphi^2(x) y \rangle_E = 2 \varphi(x) \langle y, D^2_E \varphi(x) y \rangle_E + 2 |\langle y, D_E \varphi(x) \rangle_E|^2.
\]

Consequently

\[
\mathcal{L}_E \varphi^2(x) = 2 \varphi(x) \mathcal{L}_E \varphi^2(x) + \sum_{k=1}^n |\langle \sigma_k(y), D_E \varphi(x) \rangle_E|^2.
\]

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