Stellar center is dynamical in Hořava-Lifshitz gravity

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In Hořava-Lifshitz gravity, regularity of a solution requires smoothness of not only the spacetime geometry but also the foliation. As a result, the regularity condition at the center of a star is more restrictive than in general relativity. Assuming that the energy density is a piecewise-continuous, non-negative function of the pressure and that the pressure at the center is positive, we prove that the momentum conservation law is incompatible with the regularity at the center for any spherically-symmetric, static configurations. The proof is totally insensitive to the structure of higher spatial curvature terms and, thus, holds for any values of the dynamical critical exponent \( z \). Therefore, we conclude that a spherically-symmetric star should include a time-dependent region near the center. This supports the picture that Hořava-Lifshitz gravity does not recover general relativity at low energy but can instead mimic general relativity plus cold dark matter: the “cold dark matter” accretes toward a star and thus makes the stellar center dynamical. We also comment on the condition under which linear instability of the scalar graviton does not show up.

I. INTRODUCTION

Recently, a power-counting renormalizable gravity theory was proposed by Hořava [1, 2]. The essential reason for the power-counting renormalizability is that in the ultraviolet (UV), the theory exhibits the Lifshitz-type anisotropic scaling

\[
t \to b^z t, \quad \vec{x} \to b \vec{x},
\]

with the dynamical critical exponent \( z \geq 3 \). Because of the Lifshitz scaling, this theory is often called Hořava-Lifshitz gravity. Although power-counting renormalizability does not necessarily imply renormalizability, there is a good possibility that the theory is unitary and renormalizable. Evidently, this is one of the driving forces behind recent enthusiasms for research on various aspects of the theory such as cosmology [3, 4], black holes [5], the solar system dynamics [6], and so on [7].

In Hořava-Lifshitz gravity a black hole is not “black” for high-energy probes even classically while it is indeed “black” for low-energy probes. The Lifshitz scaling implies that the dispersion relation is of the form \( \omega^2 \propto k^{2z} \) and the group velocity \( v_g \propto k^{z-1} \) is superluminal in the UV. This means that, although there are “black hole” solutions in this theory, waves with sufficiently high frequencies can probe the region deep inside the gravitational radius and the central singularity can be seen. On the other hand, waves with low frequencies follow the ordinary dispersion relation \( \omega^2 \propto k^2 \) and thus cannot probe the region inside the gravitational radius. In this sense a black hole horizon is only an emergent concept in the infrared (IR).

In order to investigate what happens inside a black hole, especially near the singularity and to see if the classical singularity can be resolved by quantum effects, we need to formulate a regular initial condition and to evolve it dynamically towards gravitational collapse.

The purpose of this paper is, as a first step towards this outstanding problem, to initiate investigation of stellar solutions in Hořava-Lifshitz gravity. This is important also for constraining the theory from astrophysical observations. A natural strategy would be to try to construct a globally static solution regular at the center. Surprisingly, we find a no-go result: we prove that a spherically-symmetric, globally static solution cannot be constructed in Hořava-Lifshitz gravity under the assumption that the energy density is a piecewise-continuous, non-negative function of the pressure and that the pressure at the center is positive. Therefore, under the assumption in the matter sector, a non-singular stellar solution should include a time-dependent region, presumably near the center. This supports the picture that Hořava-Lifshitz gravity does not recover general relativity at low energy but can instead mimic general relativity plus cold dark matter: the “cold dark matter” accretes toward a star and thus makes the stellar center dynamical.

The rest of this paper is organized as follows. In section II we shall describe the definition, properties and problems of Hořava-Lifshitz theory. In section III we present the main result of this paper, i.e. the proof of non-existence of spherically-symmetric, globally-static solution regular at the center under a set of reasonable assumptions in the matter sector. Section IV is devoted to a summary of this paper and discussion. In appendix A we present asymptotically-flat vacuum solutions with \( \lambda = 1 \). Those solutions presented in appendix A are used in appendix B where we show yet another no-go result for globally-static star solutions with \( \lambda = 1 \). In appendix C we derive the condition (24) under which linear instability of the scalar graviton does not show up. In appendix D we consider a simple stellar solution in general relativity in Painlevé-Gullstrand-like coordinate system to see how significant the regularity of time foliation is and to foresee what happens near a stellar center in Hořava-Lifshitz gravity.

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II. PROPERTIES OF HOŘAVA-LIFSHITZ GRAVITY

In this section we shall describe the definition, properties and problems of Hořava-Lifshitz gravity.

A. Definition and basic feature of Hořava-Lifshitz gravity

Hořava-Lifshitz theory is fundamentally non-relativistic since the Lifshitz scaling treats time and space differently. To be more precise, the fundamental symmetry of the theory is invariance under the foliation-preserving diffeomorphism:

\[ t \to \tilde{t}(t), \quad x^i \to \tilde{x}^i(t, x). \] (2)

The foliation-preserving diffeomorphism consists of the space-independent time reparametrization and the time-dependent spatial diffeomorphism. This means that the foliation of the spacetime manifold by constant-time hypersurfaces is not a gauge but has a physical meaning.

Basic quantities in the gravity sector are the lapse function \( N(t) \), the shift vector \( \gamma^i(t, \bar{x}) \) and the three-dimensional spatial metric \( g_{ij}(t, \bar{x}) \). These variables can be combined to form a four-dimensional metric in the Arnowitt-Deser-Misner (ADM) form:

\[ ds^2 = -N^2 dt^2 + g_{ij}(dx^i + \gamma^i dt)(dx^j + \gamma^j dt). \] (3)

Since the lapse function is roughly speaking a gauge freedom associated with the space-independent time reparametrization, it is rather natural to restrict the lapse function to be space-independent. This condition, called the projectability condition, is not only natural but also mandatory, as pointed out in Hořava’s original paper [1]. Indeed, if we abandoned the projectability condition then we would face phenomenological obstacles [8] and theoretical inconsistencies [9]. On the other hand, with the projectability condition (and without the detailed balance condition), the theory is free from those problems [1]. (See the discussion about strong coupling in the next subsection.) Therefore, throughout this paper, we impose the projectability condition and demand that the lapse function be space-independent. The Hamiltonian constraint is, as a result, not a local equation satisfied at each spatial point but an equation integrated over a whole space.

Under the infinitesimal transformation

\[ \delta t = f(t), \quad \delta x^i = \zeta^i(t, x), \] (4)

\( g_{ij}, N^i \) and \( N \) transform as

\[ \delta g_{ij} = f \partial_t g_{ij} + \mathcal{L}_\zeta g_{ij}, \]
\[ \delta N^i = \partial_t (N^i f) + \partial_t \zeta^i + \mathcal{L}_\zeta N^i, \]
\[ \delta (N_i) = \partial_t (N_i f) + g_{ij} \partial_t \zeta^j + \mathcal{L}_\zeta N_i, \]
\[ \delta N = \partial_t (N f). \] (5)

Thus, \( N \) remains independent of spatial coordinates after the transformation. In the IR, where \( dt \) and \( dx^i \) have the same scaling dimension, it makes sense to assemble \( g_{ij}, N^i \) and \( N \) into a 4-dimensional metric in the ADM form [3]. The action is

\[ I = I_g + I_m, \] (6)

\[ I_g = \frac{M_{Pl}^2}{2} \int dt dx^3 N \sqrt{g} (K^{ij} K_{ij} - \Lambda K^2 + \Lambda + R + L_{z>1}), \] (7)

where

\[ K_{ij} = \frac{1}{2N} (\partial_i g_{ij} - D_i N_j - D_j N_i), \quad K = g^{ij} K_{ij}, \] (8)

\( D_i \) is the covariant derivative compatible with \( g_{ij} \), \( \Lambda \) is a cosmological constant, \( R \) is the Ricci scalar of \( g_{ij} \), \( L_{z>1} \) represents higher spatial curvature terms and \( I_m \) is the matter action. Here, we have rescaled the time coordinate so that the coefficients of \( K^{ij} K_{ij} \) and \( R \) agree. Note that not only the gravitational action \( I_g \) but also the matter action \( I_m \) should be invariant under the foliation-preserving diffeomorphism.

By variation of the action with respect to \( N(t) \), we obtain the Hamiltonian constraint

\[ H_g + H_{m\perp} = 0, \] (9)

where

\[ H_g \equiv -\frac{\delta I_g}{\delta N} = \int dx^3 \sqrt{g} H_{g\perp}, \]
\[ H_{m\perp} \equiv -\frac{\delta I_m}{\delta N} = \int dx^3 \sqrt{g} T^\perp_{\mu\nu}, \] (10)

and

\[ H_{g\perp} = \frac{M_{Pl}^2}{2} (K^{ij} p_{ij} - \Lambda - R - L_{z>1}), \]

\[ T^\perp_{\mu\nu} = T_{\mu\nu} n^\mu n^\nu. \] (11)

Here, \( p_{ij} \) and \( n^\mu \) are defined as

\[ p_{ij} \equiv K_{ij} - \Lambda K g_{ij}, \] (12)

and

\[ n_\mu dx^\mu = -N dt, \quad n^\mu \partial_\mu = \frac{1}{N} (\partial_t - N^i \partial_i). \] (13)

Variation with respect to \( N^i(t, x) \) leads to the momentum constraint

\[ H_{g i} + H_{m i} = 0, \] (14)

where

\[ H_{g i} \equiv -\frac{1}{\sqrt{g}} \frac{\delta I_g}{\delta N^i} = -M_{Pl}^2 D^j p_{ij}, \]
\[ H_{m i} \equiv -\frac{1}{\sqrt{g}} \frac{\delta I_m}{\delta N^i} = T_{i\mu} n^\mu. \] (15)
Note that the momentum constraint is determined solely by the kinetic terms and thus is totally insensitive to the structure of higher spatial curvature terms. In particular, for \( \lambda = 1 \) the momentum constraint agrees with that in general relativity.

As in general relativity, the gravitational action can be written as the sum of kinetic terms and constraints up to boundary terms:

\[
I_g = \int dt dx^3 \left[ \pi^{ij} \partial_i g_{ij} - N^i \mathcal{H}_{pi} \right] - \int dt NH_{g\perp} + \text{(boundary terms)},
\]

(16)

where \( \pi^{ij} \) is momentum conjugate to \( g_{ij} \) given by

\[
\pi^{ij} = \frac{\delta I_g}{\delta (\partial_i g_{ij})} = M_P^2 \sqrt{g} \pi^{ij}, \quad p^{ij} \equiv g^{ik} g^{jl} p_{kl}.
\]

(17)

The Hamiltonian corresponding to the time \( t \) is the sum of constraints and boundary terms as

\[
H_g[\partial_t] = NH_{g\perp} + \int dx^3 N^i \mathcal{H}_{gi} + \text{(boundary terms)}.
\]

(18)

Finally, by variation with respect to \( g_{ij}(t, x) \), we obtain dynamical equation

\[
\mathcal{E}_{gij} + \mathcal{E}_{mij} = 0,
\]

(19)

\[
\mathcal{E}_{gij} \equiv g_{ik} g_{jl} \frac{2}{\sqrt{g}} \frac{\delta I_g}{\delta g_{kl}},
\]

\[
\mathcal{E}_{mij} \equiv g_{ik} g_{jl} \frac{2}{\sqrt{g}} \frac{\delta I_m}{\delta g_{kl}} = T_{ij}.
\]

(20)

The explicit expression for \( \mathcal{E}_{gij} \) is given by

\[
\mathcal{E}_{gij} = M_P^2 \left[ -\frac{1}{N}(\partial_t - N^k D_k)p_{ij} + \frac{1}{N}(p_{ik} D_j N^k + p_{jk} D_i N^k) - K p_{ij} + 2K_k p_{kj} + \frac{1}{2}g_{ij} K_{kl} p_{kl} + \frac{1}{2} \Lambda g_{ij} - G_{ij} \right] + \mathcal{E}_{z>1ij},
\]

(21)

where \( \mathcal{E}_{z>1ij} \) is the contribution from \( L_{z>1} \) and \( G_{ij} \) is Einstein tensor of \( g_{ij} \).

The invariance of \( I_\alpha \) under the infinitesimal transformation \( \mathcal{E}_\alpha \) leads to the following conservation equations, where \( \alpha \) represents \( g \) or \( m \).

\[
0 = N \partial_t H_{\alpha\perp} + \int dx^3 \left[ N^i \partial_i (\sqrt{g} \mathcal{H}_\alpha) + \frac{1}{2} N \sqrt{g} \mathcal{E}_\alpha^{ij} \partial_j g_{ij} \right],
\]

(22)

\[
0 = \frac{1}{N}(\partial_t - N^j D_j) \mathcal{H}_\alpha + K \mathcal{H}_\alpha
\]

\[-\frac{1}{N} \mathcal{H}_{aj} D_i N^j - D^i \mathcal{E}_{aij}.
\]

(23)

\[B. \, \text{Properties and problems of Hořava gravity}\]

Renormalization group (RG) flow of \( \lambda \)

The IR limit of the theory is characterized by the parameter \( \lambda \). When \( \lambda = 1 \), the gravitational action in the IR limit is identical to the ADM form of the Einstein-Hilbert action except that the lapse function is independent of spatial coordinates. Note, however, that setting \( \lambda = 1 \) at all scales would lead to a problem since deviation from \( \lambda = 1 \) in the UV is essential for avoidance of codimension-one caustics. While the (RG) flow of Hořava-Lifshitz gravity has not yet been analyzed, we therefore suppose that \( \lambda = 1 \) is an IR fixed point of the RG flow so that \( \lambda \) is sufficiently close to 1 in the IR but deviates from 1 in the UV.

Instabilities

In order to avoid ghost instability of the scalar graviton, the coefficient of the time kinetic term \((\lambda - 1)/(3\lambda - 1)\) must be positive. This makes the sound speed squared \( c_s^2 = -(\lambda - 1)/(3\lambda - 1) \) negative and leads to an IR instability in the linear level. As shown in Appendix C this type of instability does not show up if

\[|c_s| < \max \left[ H_L, (ML)^{-1}, |\Phi|^{1/2} \right], \]

(24)

where \( M \) is the energy scale suppressing higher-derivative terms, \( L (> 0.01 mm) \) is the length scale of interest, \( H \) is the Hubble expansion rate at the time of interest and \( \Phi \) is the Newtonian potential at the position of interest. This is a condition on the way how \( \lambda \) depends on \( L \), \( H \) and \( \Phi \) and, thus, validity of this condition can in principle be checked by analyzing the RG flow.

Strong coupling

The strong coupling between the scalar graviton and matter pointed out in \[3\] in the limit \( \lambda \to 1 \) is absent if the projectability condition is imposed. On the other hand, even with the projectability condition, there still remains the strong self-coupling of the scalar graviton in the limit \( \lambda \to 1 \) as is clear from Hořava’s original paper \[1\]. However, this is not a problem if the scalar graviton decouples from the rest of the world at the nonlinear level. In massive gravity theories this kind of decoupling phenomenon due to nonlinear dynamics is known as Vainstein effect \[11\]. A potential problem is that the strong self-coupling makes quantum corrections to the classical action very large. In non-renormalizable theories like massive gravity, this can be a fatal flaw since we really need to deal with an unknown quantum action including quantum gravity effects while the Vainstein effect has been shown only for a classical action with finite number of terms. On the other hand, in Hořava-Lifshitz gravity, if the theory is renormalizable then we can safely use the renormalizable action with finite number of terms to investigate whether the scalar graviton really decouples or not. Detailed investigation of decoupling of the scalar graviton is one of the most
important issues, but the present paper aims to shed light on a different aspect of the theory.

**Black hole**

With \( \lambda = 1 \) and vanishing cosmological constant, Schwarzschild spacetime is locally an exact vacuum solution of Hořava-Lifshitz gravity (See appendix [A]). In the Painlevé-Gullstrand coordinate system the Schwarzschild metric is

\[
 ds^2 = -dt^2 + \left( dr \pm \sqrt{\frac{2M}{r}} \, dt \right)^2 + r^2 d\Omega^2, \tag{25}
\]

and satisfies the projectability condition. Here, \( d\Omega^2 \) is the metric of the unit sphere. Since the spatial metric \( dr^2 + r^2 d\Omega^2 \) is flat, higher spatial curvature terms do not change the solution at all.

As we note in Section [I], in this theory a black hole is not "black" in the UV and we can see the central singularity of a Schwarzschild black hole if it exists. Since at the singular point the basic equations (14) and (19) are not satisfied, a Schwarzschild metric must be modified near the center. For the Schwarzschild metric (25), the extrinsic curvature of constant-time hypersurfaces becomes large near the center and the system enters the UV regime. Therefore, \( \lambda \) should deviate from 1 and the Schwarzschild spacetime is no more a valid description of the geometry near the center. For this reason, the apparent singularity at the center is not physical.

**III. NO SPHERICALLY SYMMETRIC AND STATIC SOLUTION**

As the last topic of Section [II] we touched upon the problem of the central singularity in a black hole. In order to investigate what happens inside the black hole, we need to study gravitational collapse and formation of a black hole. For this purpose we need to formulate a regular initial condition. As a first step, in the present paper we consider a static star. Surprisingly enough, we find a no-go result: a spherically-symmetric, globally static solution can not be constructed in Hořava-Lifshitz gravity under the assumption that the energy density is a piecewise-continuous, non-negative function of the pressure and that the pressure at the center is positive. This section provides a proof of this statement.

**A. Painlevé-Gullstrand coordinate**

We consider spherical-symmetric and static configurations. Since the lapse function does not depend on spatial coordinates, we can set it to unity by space-independent time reparametrization:

\[
 N = 1. \tag{26}
\]

For a spherically symmetric, static configuration, we can express the shift vector and the spatial metric as

\[
 N^i \partial_i = \beta(x) \partial_x, \quad g_{ij} dx^i dx^j = dx^2 + r^2(x) d\Omega^2, \tag{27}
\]

where \( d\Omega^2 \) is the metric of the unit sphere and \( x \) is the proper distance from the center. Non-vanishing components of the Ricci tensor and Ricci scalar for the three-dimensional geometry are

\[
 R^x_x = - \frac{2\rho''}{r}, \quad R^\theta_\theta = \frac{1}{r^2} \left[ 1 - r r'' - (r')^2 \right], \tag{28}
\]

\[
 R = \frac{2}{r^2} \left[ 1 - 2rr'' - (r')^2 \right], \tag{29}
\]

and the extrinsic curvature and its trace are

\[
 K_{ij} dx^i dx^j = -\beta' dx^2 - \beta rr' d\Omega^2, \quad K = -\frac{(r^2 \beta')'}{r^2}. \tag{30}
\]

where a prime denotes derivative w.r.t. \( x \). The corresponding ADM metric is

\[
 g^{(4)}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + [dx + \beta(x) dt]^2 + r^2(x) d\Omega^2, \tag{31}
\]

This is an analogue of the Painlevé-Gullstrand coordinate system. Note that

\[
 \xi^\mu = \left( \frac{\partial}{\partial t} \right)^\mu \tag{32}
\]

is a timelike Killing vector. Global staticity requires \( \xi^\mu \) to be globally timelike, i.e. \( 1 - \beta^2 > 0 \) everywhere. The unit vector \( n^\mu \) normal to the constant time hypersurface is given by

\[
 u_\mu dx^\mu = -dt, \quad n^\mu \partial_\mu = \partial_t - \beta \partial_x. \tag{33}
\]

It is instructive to see that the Newton potential \( \phi(r) \) and the Misner-Sharp energy \( m(r) \) are written as

\[
 e^{2\phi(r)} = -g^{(4)}_{\mu\nu} \xi^\mu = 1 - \beta^2, \tag{34}
\]

\[
 1 - \frac{2m(r)}{r} = g^{(4)\mu\nu} (dr)_\mu (dr)_\nu = (1 - \beta^2)(r')^2. \tag{35}
\]

**B. Matter sector**

As for (real) matter, for simplicity we consider the perfect-fluid form which is at rest w.r.t. the Killing vector \( \xi^\mu \):

\[
 T_{\mu\nu} = \rho(x) u_\mu u_\nu + P(x) \left[ g^{(4)}_{\mu\nu} + u_\mu u_\nu \right], \tag{36}
\]

\[
 u^\mu = \frac{\xi^\mu}{\sqrt{1 - \beta^2}}. \tag{37}
\]

Its components relevant for the ADM decomposition are

\[
 T_{\mu\nu} n^\mu n^\nu = (1 - \beta^2)^{-1} (\rho + P) - P, \tag{38}
\]

\[
 T_{\mu\nu} n^\mu dx^\nu = (1 - \beta^2)^{-1} (\rho + P) \beta dx_z, \tag{39}
\]

\[
 T_{ij} dx^i dx^j = [\beta^2 \rho + P] dx^2 + Pr^2 d\Omega^2. \tag{40}
\]
C. Inconsistency near the center

In this subsection, we show that a spherically-symmetric compact object cannot be globally-static under a set of reasonable assumptions in the matter sector. We assume that the energy density \( \rho \) is a piecewise-continuous \(^1\) non-negative function of the pressure \( P \) and that the pressure at the center \( P_c \) is positive. The three-dimensional spatial geometry and the extrinsic curvature must be regular at the center because the constant-time surfaces are physically embedded in Hořava-Lifshitz gravity. (See Appendix D on this point.) We prove that the regularity condition at the center is incompatible with the momentum conservation law of the matter under the above assumptions. Since the momentum conservation equation does not include higher spatial curvature terms in the gravity action, our proof is totally insensitive to the structure of higher spatial curvature terms and holds for any values of the dynamical critical exponent \( z \).

In the following argument, we make the proposition that there exists a spherically-symmetric, globally-static, regular solution, and show contradiction. As commented after eq. (42), the global-staticity implies that \((1 - \beta^2)\) is positive everywhere.

The momentum conservation equation, (23) with \( \alpha = m \), becomes

\[
P'(1 - \beta^2) + (\rho + P)(1 - \beta^2)' = 0.
\]

The regularity of the extrinsic curvature (30) implies that \( \beta' \) is finite. This and (41) imply that \( P' \) is also finite. As a corollary, \( \beta \) and \( P \) are continuous functions of \( x \). Since \( \rho \) is assumed to be a piecewise continuous function of \( P \), this means that \( \rho + P \) is a piecewise continuous function of \( x \).

Let \( x_c \) be the value of \( x \) at the center. Since we have assumed that \( \rho \) is non-negative everywhere and that \( P_c > 0 \), the continuity of \( P(x) \) implies that \( \rho + P \) is positive in a neighborhood of the center. Now let us define \( x_0 \) as the minimal value for which at least one of \((\rho + P)|_{x=x_0} \), \( \lim_{x \to x_0-0}(\rho + P) \) and \( \lim_{x \to x_0+0}(\rho + P) \) is non-positive.

Dividing eq. (41) by \((\rho + P)(1 - \beta^2)\) and integrating it over the interval \( x_c \leq x < x_0 \), we obtain

\[
\ln(1 - \beta_c^2) - \ln(1 - \beta_0^2) = - \int_{x_c}^{x_0-0} \frac{P'}{\rho + P} dx,
\]

where \( \beta_c \equiv \beta(x = x_c) \) and \( \beta_0 \equiv \beta(x = x_0) \). The regularity of the Ricci scalar (29) and the extrinsic curvature (30) at the center implies that \( \beta' = 1 \) and \( \beta_c = 0 \), where \( \beta_c \) is the value of \( \beta' \) at the center. Therefore, the left hand side of eq. (41) is non-positive.

Since \( P \) is a differentiable function of \( x \), the right hand side of eq. (41) can be transformed as

\[
- \int_{x_c}^{x_0-0} \frac{P'}{\rho + P} dx = - \int_{P_c}^{P_0} \frac{dP}{\rho(P) + P},
\]

where \( P_0 \equiv P(x = x_0) \). The definition of \( x_0 \) implies that at least one of \((\rho + P)|_{x=x_0} \), \( \lim_{x \to x_0-0}(\rho + P) \) and \( \lim_{x \to x_0+0}(\rho + P) \) is non-positive. Since we have assumed that \( \rho \) is non-negative everywhere, \( P_0 = \lim_{x \to x_0-0} P = \lim_{x \to x_0+0} P \) is non-positive. Thus, we have

\[
P_0 \leq 0 < P_c.
\]

This implies positivity of the right hand side of (43) since from the definition of \( x_0 \) the integrand is positive in the domain of integration. This leads to a contradiction with the previous statement that the left hand side of (41) should be non-positive.

IV. SUMMARY AND DISCUSSION

In Hořava-Lifshitz gravity, regularity of a solution requires smoothness of not only the spacetime geometry but also the foliation. As a result, the regularity condition at the center of a star is more restrictive than in general relativity. Under the assumptions that the energy density is piecewise-continuous non-negative function of the pressure and that the pressure at the center is positive, we have proved that the momentum conservation law is incompatible with the regularity at the center for any spherically-symmetric, globally-static configurations. The proof is totally insensitive to the structure of higher spatial curvature terms and, thus, holds for any values of the dynamical critical exponent \( z \). Therefore, under the assumption we made on the matter sector, we conclude that a spherically-symmetric star should include a time-dependent region, presumably near the center.

The assumptions we made are physically natural. For example, a polytropic fluid satisfies them. Note that our proof does not assume asymptotic flatness and that a cosmological constant \( \Lambda \) can be included in the gravity action (7). Shifting \( \rho, -P \) and \(-M_c^2 \) with the same amount does not change the physical system but may validate/invalidate some of the assumptions of the proof. In order to construct a static star solution, we need to violate at least one of the assumptions for all possible choices of such a shift. One possibility is to introduce an exotic matter such as a quintessence field. Introduction of an exotic matter is, however, not necessarily sufficient for the existence of a static star solution.

One must not consider our result, i.e. nonexistence of static star, as a serious problem of Hořava-Lifshitz theory. It is known that this theory does not recover general relativity at low energy but can instead mimic general

\(^1\) We assume piecewise-continuity instead of continuity, in order to allow \( dP/d\rho \) to vanish in a finite interval of \( \rho \).
relativity plus cold dark matter \footnote{The constraint algebra is smaller than in general relativity since the time slicing is synchronized with the rest frame of cold dark matter in the theory level.} \footnote{2} The existence of built-in “cold dark matter” is an inevitable prediction of the theory and might solve the mystery of dark matter in the universe. Our result in the present paper is totally consistent with this picture: as in the standard cold dark matter scenario, the “cold dark matter” accretes toward a star and thus inevitably makes the stellar center dynamical. This is the physical reason why there is no static star in Hořava-Lifshitz theory and, thus, our result strongly supports the “dark matter as an integration constant” scenario [3].

In subsection [11] we have commented on the Vainstein effect for the scalar graviton, and also derived the condition \footnote{[24]} under which linear instability of the scalar graviton does not show up. This condition should be considered as a phenomenological constraint on properties of the renormalization group (RG) flow. Detailed analysis of the RG flow and the fate of scalar graviton will be one of important future subjects.

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Appendix A: Asymptotically-flat vacuum solution with $\lambda = 1$ and without cosmological constant

For $T_{\mu\nu} = 0$, $\Lambda = 0$ and $\lambda = 1$, the momentum constraint \footnote{[14]} and the dynamical equation \footnote{[19]} become

\begin{align}
0 &= \beta \frac{\pi''}{r}, \\
0 &= \frac{1}{r^2} \left[ 2 \beta \pi r'' - (1 - \beta^2) (r')^2 \\
&\quad + 2 \beta \beta' r' + 1 \right] + \frac{1}{2} (r')^2 \mathcal{E}_{z>1}^{xx}, \tag{A1}
\end{align}

where $\mathcal{E}_{z>1}^{ij} \equiv g^{ik} g^{jl} \mathcal{E}_{z>1}^{kl}$.

On the other hand, we shall not impose the Hamiltonian constraint equation \footnote{[8]} to our solutions. The reason is as follows. In Hořava-Lifshitz gravity the Hamiltonian constraint is not a local equation satisfied at each spatial point but an equation integrated over a whole space. On the other hand, our assumption that $T_{\mu\nu} = 0$ is justified only if we consider physics at length scales sufficiently shorter than the cosmological horizon. Otherwise, the stress-energy tensor of cosmological fluids should be included. Also, our assumption of staticity completely neglects cosmological expansion of our universe. Therefore, static vacuum solutions would be valid only in a (large but) finite region. For this reason, it is not appropriate to substitute a static vacuum solution to the global Hamiltonian constraint, which is an integral over a whole space including the region outside the cosmological horizon. The contribution from the region within the regime of validity of a static vacuum solution to the integral can easily be canceled by the contribution from the region outside the regime of validity of the solution. Therefore, we must not impose the Hamiltonian constraint to our static vacuum solution.

There are two branches of solutions to the momentum constraint equation \footnote{[A1]}: $\beta = 0$ and $r' = r_1$ (constant).

In the following we shall consider each branch separately. Without loss of generality, we assume that $r_1 \geq 0$. (If $r_1 < 0$ then we can make redefinition $x \rightarrow -x$ so that $r_1 \rightarrow -r_1$.)

Let us consider the first branch, $\beta = 0$. In this case the dynamical equation \footnote{[A2]} is reduced to

\begin{align}
0 &= \frac{1}{r^2} \left[ 1 - (r')^2 \right] + \frac{1}{2} (r')^2 \mathcal{E}_{z>1}^{xx}. \tag{A3}
\end{align}

For asymptotically-flat solutions, the first term behaves as $\sim 1/r^2$ unless $r' = \pm 1$, while $\mathcal{E}_{z}$ decays at least as fast as $1/r^4$ in the asymptotic region. Therefore, asymptotically flat solutions with $\beta = 0$ has $r' = \pm 1$ and can be included in the second branch of solutions to (A1), i.e. $r' = r_1$ (constant).

In the second branch, $r' = r_1$ (constant), the Ricci tensor and the Ricci scalar become

\begin{align}
R_{x}^{x} &= 0, \quad R_{\theta}^{\theta} = \frac{1 - r_1^2}{r^2}, \quad R = \frac{2 - 2r_1^2}{r^2}. \tag{A4}
\end{align}

Evidently, the value $r_1 = 1$ is special. In this case, all components of the Ricci tensor vanish and, thus, $\mathcal{E}_{z>1}^{xx} = 0$. Thus, for $r_1 = 1$, (A2) is reduced to $(r_1^2)' = 0$ and leads to the Schwarzschild solution \footnote{[25]}.

For $r' = r_1$ (constant) with $r_1 \neq 1$, we can see from eq. \footnote{[A4]} that those terms in $\mathcal{E}_{z>1}^{xx}$ including $2n$ spatial derivatives are proportional to $1/r^{2n}$, where $n = 2, \cdots, z$. Thus, the dynamical equation \footnote{[A2]} is written as

\begin{align}
r_1 (\beta^2 r')' &= r_1^2 - 1 - \sum_{n=2}^{z} \alpha_n (r_1) \frac{1}{r^{2n}}, \tag{A5}
\end{align}

where $\alpha_n (r_1)$ ($n = 2, \cdots, z$) are constants depending on the constant $r_1$ and the structure of the higher spatial curvature terms $I_{z>1}$. Since all components of the Ricci tensor vanish for $r_1 = 1$, we have

\begin{align}
\alpha_n (1) = 0, \quad (n = 2, \cdots, z). \tag{A6}
\end{align}
Integrating the equation \((\text{A3})\), we obtain

\[
\beta^2 = \frac{r_1^2 - 1}{r_1^2} + \frac{2\mu}{r} + \sum_{n=2}^{\infty} \frac{\alpha_n(r_1)}{2n - 1} \frac{1}{2n - 1 r^{2n}},
\]

(A7)

where \(\mu\) is an integration constant.

Note that \(\beta\) must be real and thus \(\beta^2 \geq 0\). Since \((\text{A7})\) implies that \(\beta^2 \to (r_1^2 - 1)/r_1^2\) \((r \to \infty)\), it follows that \(r_1 \geq 1\). (Note that we have assumed that \(r_1 \geq 0\) without loss of generality.)

In summary, the asymptotically-flat vacuum solution is characterized by two integration constants \(r_1\) (\(\geq 1\)) and \(\mu\). The solution is given by \(r' = r_1\) and \((\text{A7})\). When \(r_1 = 1\), the solution is reduced to the Schwarzschild spacetime \([20]\) with \(M = \mu\) since \(\alpha_n(1)\) vanishes as shown in \((\text{A6})\).

Appendix B: Yet another no-go result

In this appendix we show yet another no-go result, using the momentum constraint equation. As in section \([11]\) we make the proposition that there exists a spherically-symmetric, globally-static, regular solution under a set of assumptions, and show contradiction. We set \(\lambda = 1\) and \(\beta\) (\(\geq 0\)) everywhere since \(\beta\) is assumed to be analytic. This corresponds to a trivial flat solution. We therefore suppose that

\[
\frac{\beta}{r} \left[ r'' + \frac{r}{2M_{pl}^2} \frac{\rho + P}{1 - \beta^2} \right] = 0
\]

(B1)

and is insensitive to the structure of higher spatial curvature terms. If \(\beta = 0\) in a finite interval of \(x\) then \(\beta = 0\) everywhere since \(\beta\) is assumed to be analytic. This corresponds to a trivial flat solution. We therefore suppose that

\[
r'' + \frac{r}{2M_{pl}^2} \frac{\rho + P}{1 - \beta^2} = 0
\]

(B2)

is satisfied everywhere. If the null energy condition \((\rho + P \geq 0)\) is satisfied then by integrating \((\text{B2})\) from the center towards outside we obtain

\[
r'|_{x=x_{out}} - r'|_{x=x_c} = \int_{x_c}^{x_{out}} \frac{r}{2M_{pl}^2} \frac{\rho + P}{1 - \beta^2} dx \leq 0,
\]

(B3)

where \(x = x_c\) is the center and \(x_{out} > x_c\). The equality holds if and only if \(\rho + P = 0\) everywhere in the region \(x_c < x < x_{out}\). Note that, as already stated just after \(\text{B2}\), the global staticity requires \(1 - \beta^2 > 0\). Since we are interested in a stellar solution, we suppose that \(\rho + P \neq 0\) somewhere. Therefore, we obtain

\[
r'|_{x=x_{out}} < r'|_{x=x_c}, \quad \text{for sufficiently large } x_{out}.
\]

(B4)

Now we can show that the regularity of the solution at the center is incompatible with the asymptotic flatness.

The regularity at the center requires that \(r'|_{x=x_c} = 1\). On the other hand, in Appendix \(\text{A}\) we see that the asymptotically-flat vacuum solution always has \(r' \geq 1\). Therefore, the stellar solution can be connected to the vacuum solution in the asymptotic region only if \(r'|_{x=x_{out}} \geq 1\), where we have chosen a sufficiently large \(x_{out}\). This is in conflict with \((\text{B4})\).

Appendix C: Stealth linear instability of scalar graviton

As shown in \([4]\), the lack of local Hamiltonian constraint leads to “dark matter as an integration constant”, a non-dynamical component which behaves like pressureless dust. As in the standard CDM scenario, the dust-like component exhibits Jeans instability and forms large-scale structures in the universe. The timescale of Jeans instability is

\[
t_J \sim (G_N \rho)^{-1/2},
\]

(C1)

where \(G_N\) is Newton constant and \(\rho\) is the energy density at the position of interest. Note that this instability is necessary for structure formation if we consider the dust-like component as an alternative to CDM.

As mentioned in the introduction, the scalar graviton exhibits linear instability due to the negative sound speed squared, \(c_s^2 = -(\lambda - 1)/(3\lambda - 1) < 0\). The corresponding time scale is

\[
t_L \sim \frac{L}{|c_s|},
\]

(C2)

where \(L\) is the length scale of interest. Thus, as far as

\[
t_L > t_J,
\]

(C3)

the linear instability does not show up.

Since the dispersion relation is of the form

\[
\omega^2 = k^2 \times \left[ c_s^2 + O(k^2/M^2) \right],
\]

(C4)

the linear instability is stabilized by higher derivative terms if

\[
|c_s| < \frac{1}{ML},
\]

(C5)

where \(M\) is the energy scale suppressing higher derivative terms and we have assumed that the coefficient of \(k^2/M^2\) in the square bracket in \((\text{C4})\) is positive and of order unity. Also, the linear instability is tamed by Hubble friction if

\[
t_L > H^{-1},
\]

(C6)

where \(H\) is the Hubble expansion rate at the time of interest.

In summary, if one or more of the three conditions \((\text{C5}), (\text{C6})\) and \((\text{C6})\) is satisfied then the linear instability
of the scalar graviton does not show up. By introducing
Newton potential $\Phi \sim -G_N \rho L^2$, these conditions are
summarized as

$$|c_s| < \max \left[ HL, (ML)^{-1}, |\Phi|^{1/2} \right]. \quad (C7)$$

As argued in [4], nonlinear extension of the linear in-
stability should be formation of would-be caustics and
bounce. For length scales shorter than $\sim 0.01 \text{mm}$, we
do not experimentally know how gravity behaves and,
thus, the existence of formation of would-be caustics and
bounce does not contradict with any experiments. In
some sense, this is similar to the so called spacetime foam.
Therefore, in (C7), we do not have to consider the length
scale $L$ shorter than $\sim 0.01 \text{mm}$.  

Appendix D: Non-triviality of regular foliation

Throughout this paper (eg. in subsection [III C]), we
demand the regularity of time foliation as a physical con-
dition. In this appendix we shall see how significant this
condition is. For this purpose, we consider a simple stel-
lar solution in general relativity in Painlevé-Gullstrand-
like coordinate system and show that the time foliation is
actually singular at the center. This makes it clear that
the regularity of time foliation is not a trivial but signifi-
cant condition. Based on this observation, we shall fore-
see what happens near a stellar center in Hořava-Lifshitz
gravity.

One of the simplest (and idealized) stellar solutions
in general relativity is that with uniform energy den-
sity. The explicit expression can be found in textbooks,
etg. [12]. By going to a Painlevé-Gullstrand-like coordi-
nate system, the solution is written as

$$ds^2 = -dt^2 + e^{-2\psi} (dr + \beta dt)^2 + r^2 d\Omega_2^2, \quad (D1)$$

$$\rho = \rho_0 = \text{constant}, \quad (D2)$$

$$\frac{P}{\rho_0} = \frac{(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}}{(1 - 2Mr^2/R^3)^{1/2} - 3(1 - 2M/R)^{1/2}}, \quad (D3)$$

where

$$\psi = \left\{ \begin{array}{ll}
\frac{1}{8\pi} \log \left( \frac{1}{2} \left( 1 - \sqrt{1 - 2Mr/R^3} \right) \right) & (r < R) \\
0 & (r > R)
\end{array} \right., \quad (D4)$$

$$\beta = \left\{ \begin{array}{ll}
\sqrt{\frac{3}{8}\pi^2 \rho_0 + e^{2\psi} - 1} & (r < R) \\
\sqrt{\frac{3}{2}M/r} & (r > R)
\end{array} \right., \quad (D5)$$

$$M = \frac{4\pi}{3} r^3 \rho_0, \quad (D6)$$

and $r = R$ is the surface of the star. For this solution, the
Ricci scalar of the three-dimensional spatial metric is

$$R = \frac{2}{r^2} \left( 1 - e^{-2\psi} + 2e^{-2\psi} \frac{d\psi}{dr} \right), \quad (D7)$$

and diverges at the center. It is also easy to see that the
extrinsic curvature of constant-time hypersurfaces
also diverges at the center. Physical reason for these
divergences is easy to understand. A metric with a con-
stant lapse function is characterized by a congruence of
geodesics orthogonal to constant-time hypersurfaces.
In general relativity, a contracting congruence of geodesics
forms caustics. This is the physical reason why the three-
dimensional spatial geometry and the extrinsic curvature
are singular at the center.

In general relativity, divergence of the extrinsic cur-
vature and the three-dimensional spatial curvature can
be just a coordinate singularity. Indeed, one can easily
see that the four-dimensional geometry is regular at the
center of the above solution. Therefore, divergence at
the center is just a coordinate singularity and is not a
problem for the above solution in general relativity.

On the other hand, in Hořava-Lifshitz gravity, divergence
of the three-dimensional spatial curvature or/and
the extrinsic curvature is a physical singularity. Thus,
the above solution with constant energy density is not
physically viable. Indeed, since the three-dimensional
spatial curvature diverges, higher spatial curvature terms
become important near the center and should change the
behavior of the solution. It is also expected that deviation
of $\lambda$ from 1 should also be important near the center.
Those effects should generate “dark matter as integration
constant”, which should develop would-be caustics and
bounce [4]. What we expect is occurring near the center
is, thus, a sequence of microscopic would-be caustics and
bounces.

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