A Note on Local Mode-in-State Participation Factors for Nonlinear Systems

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Abstract

The paper studies an extension to nonlinear systems of a recently proposed approach to the concept of modal participation factors. First, a definition is given for local mode-in-state participation factors for smooth nonlinear autonomous systems. The definition is general, and, unlike in the more traditional approach, the resulting participation measures depend on the assumed uncertainty law governing the system initial condition. The work follows Hashlamoun, Hassouneh and Abed (2009) in taking a mathematical expectation (or set-theoretic average) of a modal contribution measure with respect to an assumed uncertain initial state. As in the linear case, it is found that a symmetry assumption on the distribution of the initial state results in a tractable calculation and an explicit and simple formula for mode-in-state participation factors.

1 Introduction

Analysis of modal content of the response of dynamic systems is of interest in many application areas, ranging from electric power networks to vibration of structures. Many approaches to modal analysis occur in the literature. For linear time invariant systems, modal content consists of the eigenmodes, and can be studied analytically. For nonlinear systems, the possibility of global oscillations gives rise to global oscillatory modes that might not be connected to the eigenmodes of the system’s linearization at an equilibrium. In this paper, we focus on local modal analysis of nonlinear autonomous systems near an equilibrium, paying particular attention to what can be viewed as eigenmodes in a neighborhood of the equilibrium point of interest. The aim of the paper is to explore the possibility of extending to the nonlinear setting the modal participation analysis pursued by Hashlamoun, Hassouneh and Abed [8] for linear systems. This analysis attempts to systematically quantify the relative contributions of system modes to system states, and of system states to system modes. Here, system states refers to the scalar elements of the system state vector.

In the early 1980s, Verghese, Perez-Arriaga and Schweppe [21, 22] introduced quantities they referred to as modal participation factors. These quantities have been used widely, especially in the electric power systems field. In 2009, the authors of [8] presented a new approach to the fundamental definition of modal participation factors. The idea of modal participation factors, which will be reviewed further in the next section, is to give measures of the relative contribution of system modes in system states, and of system states in system modes. In [8], such measures are developed by taking an average of relative contribution measures over an uncertain set of system initial conditions. The idea is that fixing the system initial condition affects the modal participations, and that initial conditions are in reality uncertain, indeed possibly random due to inherent noise. Indeed, if one takes a view that the initial time also isn’t fixed, noise can be viewed as having the effect of allowing the initial condition to be re-set over time, effectively allowing the initial condition to explore a neighborhood of an equilibrium point over a short time interval. By taking an averaging approach, the authors of [8] find that a dichotomy arises in this new view of modal participation factors. In this dichotomy, participation factors measuring mode-in-state participation need to be viewed as distinct from participation factors measuring state-in-mode participation. This dichotomy was not recognized prior to [8], and a single formula was previously used to quantify both types of modal participation.

In [8], it was found that analytical formulas for mode-in-state participation factors fell out of the analysis very nicely, under basic symmetry assumptions on the distribution of the initial state. The same symmetry assumptions

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did not allow for a similarly simple derivation of state-in-mode participation factors, and when a formula was obtained in a particular scenario on the initial state, that formula was more complicated than for the mode-in-state case and didn’t share the desirable property of being independent to rescaling of the system state variables (i.e., the formula wasn’t invariant under changes of state variable units). This issue is now better understood by the authors, and will be reported on elsewhere.

Here, we explore extension of the work in [8], especially for the analysis of mode-in-state participation, to the nonlinear setting, for local system behavior near an equilibrium point. We are able to give an analysis and derivation of formulas for mode-in-state participation factors (under basic symmetry assumptions as in the linear case). This work follows a different approach to defining participation factors for nonlinear models than that pursued in [10], where modal participation was studied from a fixed initial state using Taylor series methods.

Before proceeding to the development of the paper, it is perhaps useful to provide a brief discussion of studies on modal participation, addressing motivation of researchers on this topic, the various approaches taken in different disciplines, and applications that have been pursued.

The present work is motivated by the original work of Verghese, Perez-Arriaga and Schewpe [21, 22] that was mentioned above. The authors of [21, 22] introduced their notion of modal participation factors as a tool to aid in modal analysis of large power grids, with benefits anticipated in tasks such as model order reduction and control design. Oscillatory modes are common in power systems, and it is important to have systematic tools for their analysis. Since power grids consist of interconnected areas and can cover large expenses of territory (indeed entire continents), engineers are naturally interested in obtaining reduced models that capture modes of special interest. The modal participation factors of [21, 22] were employed for this purpose, in an overarching framework that the authors referred to as Selective Modal Analysis (SMA) (a recent review of SMA is [3]). In addition, modal analysis in a power grid should provide tools for determining the best sites for insertion of actuators to control modes that may be troubling or dangerous, or for determining the best locations for placing measurement devices that allow system operators to monitor such modes in real time. An example of a recent application of the concepts in [21, 22] to power systems is [5], which focuses on power grids with significant levels of renewable generation. Early examples of work on actuator placement in power networks using the original modal participation concept include [6, 15]. Recent examples of modal participation studies in power systems motivated by the more recent approach of Hashlamoun, Hassouneh and Abed [8] include [17, 18, 9, 16]. The approach has also been applied in power electronics [24] and electromagnetic devices [4].

The term “modal participation factors” is also commonly used in the field of structural analysis, with applications in mechanical, aerospace, and civil engineering. The concept of modal participation factors introduced in electric power engineering in [21, 22] was developed independently of the notion used in structural analysis. Modal participation factors as studied in structural analysis have been used, for example, to study vibrations of tall buildings [20] and rotorcraft dynamics [13]. The concepts of modal participation factors in electric power engineering and in structural analysis are distinct. In the structural analysis framework, the focus has been on the impact of forcing functions on modal response. In contrast, in the electric power engineering concept, a large ostensibly autonomous dynamic system is considered (motivated by the driving power grid application). Bridging between these frameworks could be a fruitful area for future investigation. The two types of modal participation factors (in electric power engineering and related control theory literature, and in structural analysis) are not absolute by any means. These concepts are definitions deemed suitable for various purposes by their authors and employed over many years by practitioners in the respective fields. Later researchers have at times proposed modifications to address a perceived need for improvements. For example, in structural analysis, Chopra [14] introduced a new notion of modal participation factor aiming to make major improvements to the standard definition used in that field, including providing a more clear measure of modal contribution of an external forcing function and removing unit dependence from the standard notion. Similarly, the original concept of [21, 22] in electric power engineering has been revisited in [8], as noted above. In the remainder of the paper, we focus on the modal participation factors concepts that have been used in electric power engineering, beginning with the work of [21, 22] and continuing with the work of [8]. The concepts from structural analysis were mentioned above to provide context for this work in the larger literature, but will not be addressed in the technical work in this paper.

The remainder of the paper is organized as follows. In Section 2, needed background material is recalled. In Section 3, mode-in-state participation factors are defined for nonlinear systems in the vicinity of an equilibrium point, under a symmetry assumption on the uncertainty in the system initial condition. Conclusions and issues for further research are discussed in Section 4. A preliminary version of this paper appeared in [7].
2 Background

In this section, we give background material that is relevant to our investigation. In particular, we recall modal participation factors (the original definition as well as the more recent work as described above). We also sample some of the applications of modal participation factors, including very recent references to the literature. Finally, we recall two fundamental theorems on local representations of nonlinear autonomous systems. As noted above, the remainder of the paper focuses on modal participation analysis as pursued in the electric power engineering literature and related work in control theory, and on extending the concepts given in [8] to a nonlinear setting.

2.1 Modal Participation Factors for Linear Systems: Original Definition [21, 22]

Let \( \Sigma_L \) denote the linear time-invariant system

\[
\dot{x} = Ax
\]

where \( x \in \mathbb{R}^n \) and the state dynamics matrix \( A \in \mathbb{R}^{n \times n} \) has \( n \) distinct eigenvalues \( \lambda_i, i = 1, \ldots, n \).

The system state \( x(t) \) of course consists of a linear combination of exponential functions

\[
x^i(t) = e^{\lambda_i t} c^i,
\]

where the vectors \( c^i \) are determined by the system’s initial condition \( x(0) \). These functions are the system modes and are useful in modal analysis of linear systems.

Let \( r^i \) be the right eigenvector of the matrix \( A \) associated with eigenvalue \( \lambda_i, i = 1, \ldots, n \), and let \( \ell^i \) be the left(row) eigenvector of \( A \) associated with the eigenvalue \( \lambda_i, i = 1, \ldots, n \). The right and left eigenvectors are taken to satisfy the normalization

\[
\ell^i r^j = \delta_{ij},
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

Given a linear system \( \dot{x} = Ax \) with initial condition \( x(0) = x^0 \), its solution can be written as

\[
x(t) = e^{At} x^0 = \sum_{i=1}^{n} (\ell^i x^0) e^{\lambda_i t} r^i.
\]

The \( k \)-th state variable evolves according to

\[
x_k(t) = (e^{At} x^0)_k = \sum_{i=1}^{n} (\ell^i x^0) e^{\lambda_i t} r^i_k.
\]

Using these facts and taking two scenarios with rather special initial conditions, Verghese, Perez-Arriaga and Schweppe [21, 22] motivated the following definition of quantities \( p_{ki} \) which they named modal participation factors:

\[
p_{ki} := \ell^i_k r^i_k
\]

Choosing the initial condition to be \( x^0 = e_k \), the unit vector along the \( k \)-th coordinate axis, the authors of [21, 22] gave reasoning for considering the quantities \( p_{ki} \) as mode-in-state participation factors. The scalars \( p_{ki} \) are dimensionless. Next, employing a coordinate transformation to focus on the system modes and considering instead an initial condition \( x^0 = r^i \), the right eigenvector corresponding to \( \lambda_i \), the quantities \( p_{ki} \) were also given an interpretation as state-in-mode participation factors. Thus, it has been very common in papers and books using modal participation factor analysis to interchangeably refer to participation of modes in states and participation of states in modes, always using the same formula (6) for both types of participation measure.

2.2 Modal Participation Factors for Linear Systems: Recent Approach ([8])

In [8], it was argued that a deeper analysis of modal participation would not necessarily lead to identical measures for mode in state and state in mode participation factors. This issue continues to deserve the attention of the control, dynamics, and power systems research communities, largely because of the importance of modal analysis in many complex systems, in power engineering and in other application areas. Simple examples were used in [8] to motivate the need for a new approach to defining modal participation factors. In fact, the examples indicated that it would be desirable to achieve definitions that gave different measures for mode-in-state participation factors and
state-in-mode participation factors. Indeed, the examples showed that, especially when quantifying the contribution of system states in system modes, the formula (6) could well fall short of giving an intuitively acceptable result. Thus, new fundamental definitions were given based on averaging over the system initial condition, taken to be uncertain.

The linear system
\[ \dot{x} = Ax \] (7)
usually represents the small perturbation dynamics near an equilibrium. The initial condition for such a perturbation is usually viewed as being an uncertain vector of small norm. In [8], new definitions of mode-in-state and state-in-mode participation factors were given using deterministic (i.e., set-theoretic) and probabilistic uncertainty models for the initial condition.

**Definition 2.1.** In the set-theoretic formulation, the participation factor measuring relative influence of the mode associated with \( \lambda_i \) on state component \( x_k \) is
\[ p_{ki} := \text{avg}_{x^0 \in S} \frac{(\ell_i x^0)r_k^i}{x_k^0} \] (8)
whenever this quantity exists. Here, \( x_k^0 = \sum_{i=1}^{n}(\ell_i x^0)r_k^i \) is the value of \( x_k(t) \) at \( t = 0 \), and “\( \text{avg}_{x^0 \in S} \)” is an operator that computes the average of a scalar function over a set \( S \subseteq \mathbb{R}^n \) (representing the set of possible values of the initial condition \( x^0 \)).

With a probabilistic description of the uncertainty in the initial condition \( x^0 \), the average in (8) is replaced in [8] by a mathematical expectation:

**Definition 2.2.** The general formula for the participation factor \( p_{ki} \) measuring participation of mode \( i \) in state \( x_k \) becomes
\[ p_{ki} := E\left\{ \frac{(\ell_i x^0)r_k^i}{x_k^0} \right\} \] (9)
where the expectation is evaluated using some assumed joint probability density function \( f(x^0) \) for the initial condition uncertainty. (Of course, this definition applies only when the expectation exists.)

In [8], it was found that both Definition 2.1 (Eq. (8)) and Definition 2.2 (Eq. (9)) lead to a simple result that agrees with Eq. (6) under a symmetry assumption on the uncertain initial condition. In the set-theoretic definition, the symmetry assumption is that the initial condition uncertainty set \( S \) is symmetric with respect to each of the hyperplanes \( \{x_k = 0\} \), \( k = 1, \ldots, n \). In the probabilistic setting of Definition 2.2, the assumption is that the the initial condition components \( x_0^1, x_0^2, \ldots, x_0^n \) are independent random variables with marginal density functions which are symmetric with respect to \( x_k^0 = 0, k = 1, 2, \ldots, n \), or are jointly uniformly distributed over a sphere centered at the origin. Under either the set-theoretic or probabilistic symmetry assumption, it was found in [8] that the same expression originally introduced by Perez-Arriaga, Verghese and Schweppe [21, 22] results as a measure of mode-in-state participation factors:
\[ p_{ki} = \ell_k r_k^i. \] (10)

### 2.3 State-in-Mode Participation Factors

Hashlamoun, Hassouneh and Abed [8] also gave similar set-theoretic and probabilistic definitions for state-in-mode participation factors for linear systems. The calculations were found to be less straightforward than for the mode-in-state participation factors setting, even under the same symmetry assumption on the initial condition as used in the mode-in-state participation factor calculation. We will not recall the details of the development of state-in-mode participation factors for linear systems from [8]. We will simply recall from [8] the general definition and an associated result for the case of distinct real eigenvalues to have an idea of the nature of the results.

**Definition 2.3.** The participation factor of state \( x_k \) in mode \( i \) is
\[ \pi_{ki} := E\left\{ \frac{\ell_k x_k^0}{\sum_{j=1}^{n}(P_j x_j^0)} \right\} = E\left\{ \frac{\ell_i x_k^0}{z_i^0} \right\}, \] (11)
whenever this expectation exists, where \( z_i^0 = z_i(0) = \ell^i x^0 \), and where \( z_i(t) \) is the \( i \)th system mode

\[
z_i(t) = e^{\lambda_i t} \ell^i x^0 = e^{\lambda_i t} \sum_{j=1}^n (\ell^i_j x^0_j).
\] (12)

It was shown in [8] that

\[
\pi_{ki} = E \left\{ \frac{\ell^i_k x^0_k}{\sum_{j=1}^n (\ell^i_j x^0_j)} \right\}
= \ell^i_k r^i_k + \sum_{j=1, j \neq i}^n \ell^i_k r^j_k E \left\{ \frac{z^0_j}{z^0_i} \right\}.
\]

Note that the first term in the expression for \( \pi_{ki} \) coincides with \( p_{ki} \), the original participation factors formula. However, the second term does not vanish in general. This is true even when the components \( x^0_1, x^0_2, \ldots, x^0_n \) representing the initial conditions of the state are assumed to be independent. Assuming that the units of the state variables have been scaled to ensure that the probability density function \( f(x^0) \) is such that the components \( x^0_1, x^0_2, \ldots, x^0_n \) are jointly uniformly distributed over the unit sphere in \( \mathbb{R}^n \) centered at the origin, modal participation factors were evaluated in [8] using Definition 2.3, yielding the following explicit formula that is applicable under the foregoing uncertainty model for the system initial state.

**Proposition 2.1.** ([8]) Under the assumption that the initial condition has a uniform probability density on a sphere centered at the origin, the participation factor of state \( x_k \) in mode \( i \) is

\[
\pi_{ki} = \ell^i_k r^i_k + \sum_{j=1, j \neq i}^n \ell^i_k r^j_k E \left\{ \frac{z^0_j}{z^0_i} \right\}. \tag{13}
\]

### 2.4 Poincaré Linearization

Poincaré linearization is a well known technique for transforming an autonomous nonlinear system into a locally equivalent linear system via diffeomorphism. The technique is useful in this paper for extending the definitions of mode-in-state participation factors proposed in [8] to the nonlinear setting. In the following, we review the technique.

Consider a nonlinear system of ordinary differential equations

\[
\dot{x} = f(x), \tag{14}
\]

where \( x \in \mathbb{R}^n \) and \( f \) is an analytic vector field on \( \mathbb{R}^n \). Let \( A = \frac{\partial f}{\partial x} |_{x=0} \) be the Jacobian of \( f \) at the origin.

**Definition 2.4.** ([11]) Given a matrix \( A \in \mathbb{R}^{n \times n} \) with eigenvalues \( \lambda_i \), \( i = 1, \ldots, n \), we say that the \( n \)–tuple \( \lambda = (\lambda_1, \cdots, \lambda_n) \) is resonant if among the eigenvalues there exists a relation of the form

\[
(m, \lambda) = \sum_{k=1}^n m_k \lambda_k = \lambda_s,
\] (15)

where \( m = (m_1, \cdots, m_n) \), \( m_k \geq 0 \), \( \sum_k m_k \geq 2 \). Such a relation is called a resonance. The number \( |m| = \sum_{k=1}^n m_k \) is called the order of the resonance.

**Example 2.1.** ([11]) The relation \( \lambda_1 = 2\lambda_2 \) is a resonance of order 2; the relation \( 2\lambda_1 = 3\lambda_2 \) is not a resonance; the relation \( \lambda_1 + \lambda_2 = 0 \), or equivalently \( \lambda_1 = 2\lambda_1 + \lambda_2 \), is a resonance of order 3.

**Theorem 2.1** (Poincaré’s Theorem [11]). If the eigenvalues of the matrix \( A \) are nonresonant, then the nonlinear ODE

\[
\dot{x} = Ax + O(||x||^2)
\] (16)

can be reduced to the linear ODE

\[
\dot{y} = Ay
\] (17)

by a formal change of variable \( x = y + \cdots \) (the dots denote series starting with terms of degree two or higher).
If the $n$-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ is resonant, we will say that
\[
x^m := x_1^{m_1} \cdots x_n^{m_n} e_s
\]
is resonant if $\lambda_s = (m, \lambda)$, $|m| \geq 2$ with $e_s$ a vector in the eigenbasis of $A$ and $x_i$ are the coordinates with respect to the basis $e_s$. For example, for the resonance $\lambda_1 = 2\lambda_2$, the unique resonant monomial is $x_2^3 e_1$. For the resonance $\lambda_1 + \lambda_2 = 0$, all monomials $(x_1 x_2)^k x_s e_s$ are resonant $\mathbb{I}$.

**Theorem 2.2** (Poincaré-Dulac Theorem $\mathbb{I}$). If the eigenvalues of the matrix $A$ are resonant, then the nonlinear ODE
\[
\dot{x} = Ax + \cdots
\]
can be reduced to the ODE
\[
\dot{y} = Ay + w(y)
\]
by a formal change of variable $x = y + \cdots$ (the dots denote series starting with terms of degree two or higher), where all monomials in the series $w$ are resonant.

There are also several convergence results associated with Poincaré linearization, of which the following is the most well known.

**Theorem 2.3** (Poincaré-Siegel). Suppose the eigenvalues $\{\lambda_i\}$, $i = 1, \ldots, n$, of the linear part of an analytic vector field at an equilibrium point are nonresonant and either $\Re(\lambda_i) > 0$, $i = 1, \ldots, n$ or $\Re(\lambda_i) < 0$, $i = \ldots, n$, or the $(\lambda_i)$ satisfy the Siegel condition, i.e. are such that there exists $C > 0$ and $\nu$ such that for all $i = 1, \ldots, n$
\[
|\lambda_i - (m, \lambda)| \geq \frac{C}{|m|^\nu}
\]
for all $m = (m_1, \ldots, m_n)$, where $(m_i)$ are nonnegative integers with $|m| = \sum_{i=1}^n m_i \geq 2$. Then the power series in Poincaré’s Theorem converges in some neighbourhood of the equilibrium point.

**Remark.** There are also some convergence results in the case of resonant eigenvalues; the reader is encouraged to consult $\mathbb{I}$ for further details on Poincaré linearization.

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### 2.5 Hartman-Grobman Theorem

Another very important result in the local qualitative theory of nonlinear ordinary differential equations is the Hartman-Grobman Theorem, which says that near a hyperbolic equilibrium point $x^e$, the nonlinear system $\mathbb{I}$ has the same qualitative structure as the linear system $\mathbb{I}$.

**Theorem 2.4.** $\mathbb{I}$ Let $E$ be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let $\phi_t$ be the flow of the nonlinear system $\mathbb{I}$. Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism $\varphi$ of an open set $U$ containing the origin onto an open set $V$ containing the origin such that for each $x^0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x^0 \in U$ and $t \in I_0$
\[
\varphi \circ \phi_t(x^0) = e^{At}(x^0),
\]
i.e., $\varphi$ maps trajectories of $\mathbb{I}$ near the origin onto trajectories of $\mathbb{I}$ near the origin.

### 3 Mode-in-State Participation Factors for Nonlinear Systems

Consider a nonlinear ODE
\[
\dot{x} = f(x)
\]
with $f \in C(\mathbb{R}^n; \mathbb{R}^n)$, $f(0) = 0$, and consider the Taylor expansion of $f$ around the origin
\[
\dot{x} = Ax + \tilde{f}^{[2]}(x) + O(||x||^3)
\]
where $A = \frac{\partial f}{\partial x} |_{x=0}$ and $\tilde{f}^{[2]}$ represents terms of order 2. We have the following result.
Theorem 3.1. If the eigenvalues of $A$ are nonresonant (resp. satisfy one of the conditions of the Poincaré-Siegel Theorem) then there exists a diffeomorphism that formally (resp. analytically) transforms the nonlinear ODE (23) into a linear ODE. In this case, the mode-in-state participation factors of (22) are the same as those of the linearized system $\dot{x} = Ax$.

Proof. First, we normalize $A$ using the change of coordinates

$$z = V^{-1}x,$$  \hspace{1cm} (24)

where $V = [v^1 \, v^2 \, \cdots \, v^n]$ represents the matrix of right eigenvectors of $A$. Under the change of coordinates (21) the ODE (23) becomes

$$\dot{z} = \Lambda z + V^{-1} f^{[2]}(V^{-1}z) + O(||z||^3) := \Lambda z + f^{[2]}(z) + O(||z||^3)$$  \hspace{1cm} (25)

Next, we normalize the higher order terms through the change of coordinates

$$\tilde{z} = \phi(z) = z + \phi^{[2]}(z) + O(||z||^3) = z + z^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} z + O(||z||^3)$$  \hspace{1cm} (26)

where $\phi \in C(\mathbb{R}^n; \mathbb{R}^n)$. Using Poincaré linearization, we know that if the eigenvalues of $A$ are nonresonant, then there is a formal change of coordinates $\phi$ such that the trajectories of (22) are locally diffeomorphic to the trajectories of

$$\dot{\tilde{z}} = \Lambda \tilde{z}$$  \hspace{1cm} (27)

If $\Lambda = \text{diag}(\lambda_i)_{i=1}^n$, then

$$\tilde{z}(t) = e^{\Lambda t} \tilde{z}(0),$$  \hspace{1cm} (28)

whose $i$-th component is

$$\tilde{z}_i(t) = e^{\lambda_i t} \tilde{z}_i(0).$$  \hspace{1cm} (29)

Using (26), we get $z(t) = \phi^{-1}(e^{\Lambda t} \phi(z(0)))$, which can be rewritten as

$$z(t) = e^{\Lambda t} \phi(z(0)) - \phi(z(0))^T e^{\Lambda t} \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} e^{\Lambda t} \phi(z(0)) + O(||z||^3),$$  \hspace{1cm} (30)

and

$$z_i(t) = e^{\lambda_i t} \phi_i(z(0)) - \phi^T(z(0)) e^{\lambda_i t} P_i e^{\lambda_i t} \phi(z(0)) + \cdots$$  \hspace{1cm} (31)

Using (24), we get

$$x_k(t) = \begin{bmatrix} r^1 \cdots r^n \end{bmatrix}_{k\text{-th row}} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sum_i r^1_k z_i(t)$$  \hspace{1cm} (32)

$$= \sum_i r^1_k \left( e^{\lambda_i t} \phi_i(z(0)) - \phi^T(z(0)) P_i \phi(z(0)) \right) + \cdots$$  \hspace{1cm} (33)

It is instructive to consider the linear case first. We set $P_i = 0$ and the higher order terms are also set to zero in (26). This gives

$$x_k(t) = \sum_{i=1}^n r^i_k e^{\lambda_i t} \phi_i(z(0)) = \sum_{i=1}^n r^i_k e^{\lambda_i t} \ell^i x^0$$  \hspace{1cm} (34)

Then the participation of the $e^{\lambda_i t}$ mode in the state $x_k(t)$ is

$$p_{ki} := \frac{e^{\lambda_i t} r^i_k \ell^i x^0}{x_k(t)} \bigg|_{t=0} = \ell^i r^i_k$$  \hspace{1cm} (35)

(agreeing, of course, with the previous calculation of $p_i$ in the linear case $p_i$).
Next, we consider the nonlinear setting, where we assume that $P_i \neq 0$. The participation of $e^{\lambda_i t}$ in $x_k(t)$ is obtained using the set-theoretic definition as follows (quantities are evaluated at time $t = 0$):

\[
\text{avg} e^{\lambda_i t} r_k \phi_i(z^0) = \text{avg} e^{\lambda_i t} r_k \phi_i(z^0) \frac{1}{x_k(t)} \bigg|_{t=0} = \text{avg} e^{\lambda_i t} r_k \phi_i(z^0) - \sum_{j,m} \theta_{j,m} e^{(\lambda_j + \lambda_m) t} \bigg|_{t=0} = r_k \ell_k.
\]

Since $\phi_i(z^0) = \ell_i x_0 + \cdots$, then

\[
\sum_{j,m} \theta_{j,m} = \sum_{j,m} \phi_j(z^0) \phi_m(z^0) p_{j,m} = \sum_{j,m} (\ell_j x^0) (\ell_m x^0) p_{j,m}
\]

Hence, the participation of the mode $e^{\lambda_i t}$ in $x_k(t)$ is

\[
p_{ki} := \text{avg} e^{\lambda_i t} r_k \phi_i(z^0) = \text{avg} e^{\lambda_i t} r_k \phi_i(z^0) \frac{1}{x_k(t)} \bigg|_{t=0} = \text{avg} e^{\lambda_i t} r_k \phi_i(z^0) - \sum_{j,m} \theta_{j,m} e^{(\lambda_j + \lambda_m) t} \bigg|_{t=0} = r_k \ell_k.
\]

Perhaps somewhat surprisingly, under the assumptions made, the mode-in-state participation factors are seen to agree with those of the linearized system.

**Example** Consider a nonlinear system whose linear part is from an example in [8]:

\[
\dot{x} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} x + \Psi(x),
\]

with $\Psi$ a polynomial of order $N \geq 2$. If $a \neq m \cdot d$ for any $m \in \mathbb{N}$, then the eigenvalues of the matrix $A_1$ are nonresonant and, therefore, by the Poincaré’s theorem there exists a formal transformation that transforms (37) to $\dot{z} = Az$.

Furthermore, if $\lambda_1 = a$ and $\lambda_2 = d$ satisfy one of the conditions of the Poincaré-Siegel Theorem, then the transformation is analytic. In both cases, the mode-in-state participation factors of (37) are locally equal to the mode-in-state participation factors of the linear system (38).

A similar result holds for the following nonlinear system, whose linear part is from another example of [8]:

\[
\dot{x} = \begin{bmatrix} 1 & 1 \\ -d & -d \end{bmatrix} x + \Psi(x),
\]

with $d \neq 1$ (nonresonance condition) and $\Psi$ is a polynomial of order $N \geq 2$.

If the eigenvalues are resonant, and the origin is hyperbolic, we can still say something on the mode-in-state participation factors.

**Theorem 3.2.** [23] If the origin is a hyperbolic point then there exists a homeomorphism that transforms the nonlinear ODE (22) into the linear ODE (27). In this case, the mode-in-state participation factors of (27) are the same as those of the linearized system $\dot{x} = Ax$.

**Proof.** First, we normalize $A$ using the change of coordinates (24) where $V = [r^1 r^2 \cdots r^n]$ represents the matrix of right eigenvectors of $A$. Under the homeomorphism in the Hartman-Grobman theorem the ODE (22) becomes

\[
\dot{z} = \Lambda z
\]

The proof regarding mode-in-state participation factors comes directly from applying the result for the linear case in Section 3. 

\[\square\]
Example Consider the system

\[
\begin{align*}
\dot{y} &= -y \\
\dot{z} &= z + y^2
\end{align*}
\] (41) (42)

It can be shown that with the homeomorphism

\[
\phi(y, z) = \begin{bmatrix} y \\ z + \frac{y^2}{3} \end{bmatrix}
\] (43)

the solution of (41)-(42) is homeomorphic to the solution of

\[
\begin{align*}
\dot{y} &= -y \\
\dot{z} &= z
\end{align*}
\] (44) (45)

and, therefore, the mode-in-state participation factors of the nonlinear system are the same as those of the linearized system.

4 Conclusion

There is a dichotomy in modal participation for linear systems. Hence we expect a similar dichotomy for nonlinear systems. Participation of modes in states is relatively easy to evaluate using averaging over an uncertain set of initial conditions assuming symmetric uncertainty. Somewhat surprisingly, the mode-in-state participation formulas under these circumstances were found to be the same for a nonlinear system as for its linearization, assuming the nonresonance condition. Participation of states in modes for nonlinear systems is an open question, and its distinction from mode-in-state participation factors is part of the dichotomy in modal participation seen in the linear case. Besides calculation of state-in-mode participation factors, some other issues that could be considered in future work are: computing modal participation factors for nonlinear systems from data; using the Frobenius-Perron operator to compute these measures. Another possible extension is to use the recently introduced “nonlinear eigenvalues” and “nonlinear eigenvectors” for nonlinear systems to introduce possibly “more nonlinear” notions of modal participation factors for nonlinear systems. Moreover, as mentioned in Section 1, bridging between modal participation concepts that have been proposed and used in different engineering fields would be worthwhile.

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