Research Article

A Coupled Torsional-Transition Nonlinear Vibration and Dynamic Model of a Two-Stage Helical Gearbox Reducer for Electric Vehicles

Abdulhameed M. Y. Al-Tayari\textsuperscript{1}, Siyu Chen, and Zhou Sun

School of Mechanical and Electrical Engineering, Central South University, Changsha 410083, China

Correspondence should be addressed to Abdulhameed M. Y. Al-Tayari; abdulhameed161@csu.edu.cn

Received 4 June 2020; Revised 22 July 2020; Accepted 7 August 2020; Published 30 August 2020

Academic Editor: Wahyu Caesarendra

Copyright © 2020 Abdulhameed M. Y. Al-Tayari et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A coupled torsional-transition nonlinear dynamic model of a two-stage helical gear (TSHG) reduction system for electric vehicles (EVs) is presented in this paper. The model consists of 16 degrees of freedom (DOF), which includes factors such as the nonlinearity of backlash, time-varying mesh stiffness (TVMS), mesh damping, supporting bearings, static transmission error (STE), and the torsional damping and stiffness of the intermediate shaft, in which the fourth-order Runge-Kutta numerical integration method was applied to solve the differential equations. With the help of bifurcation diagrams, time-domain histories diagrams, amplitude-frequency spectrums, phase plane diagrams, Poincaré maps, root-mean-square (RMS) curves, peak-peak values (PPVs), and Lyapunov exponents, the effects of pinion rotational speed, backlash, torsional stiffness, and torque fluctuation on the dynamic behavior of TSHG system are investigated. The stability properties of steady-state responses are investigated using Lyapunov exponents. The results reveal various types of dynamic evolution mechanisms and nonlinear phenomena such as periodic-one responses, quasiperiodic responses, jumps phenomena, and chaotic responses. The research presents useful results and information to vibration control and dynamic design of the TSHG transmission system used in EVs.

1. Introduction

It is known that the power mechanism of EVs is composed of a power coupling system (transmission shafts, gear pairs, bearings, etc.), electrical control systems, motor, and other subsystems. As gear pairs are one of the main parts of EV power coupling system, the dynamic behavior of the gear pairs mechanism has a significant influence on power transmission and even the whole vehicle [1]. Therefore, the analysis of the dynamic characteristics of gear pairs is becoming essential in power transmission systems.

Numbers of dynamical models for both helical and spur gear pairs systems have been presented; a mathematical model and computer simulation of a two-stage gearbox including translational-torsional vibration responses of gear pairs system were presented by Walter Bartelmus [2]. Based on ANSYS software, Yan and Liu [3] developed a new finite-element modeling method for helical gear transmission with multiple shafts considering the effects of bearing flexibilities and shafts. A dynamic model of a helical gear multiple-shafts reduction box was presented by Kubur et al. in [4, 5], which consists of a finite-element model of shaft structures combined with another three-dimensional discrete model of gear pairs. Moreover, Zhang et al. [6] presented an effective dynamic model of a multiple-shafts helical geared-rotor system with geometric eccentricity, bearing flexibility, and gear mesh, in which the eigenvalue solution and summation method were applied to estimate the forced responses and the natural frequencies of the geared system.

Since the nonlinear characteristics of gear-pair system have been the most significant research fields, many studies on the dynamic model and response of two-stage gear systems have been published. For instance, Walha et al. [7] proposed a purely torsional spur gear system with three
shocks connected by two spur gear pairs, in which the Newton–Raphson algorithm was used to examine the influence of backlash on two-stage spur gear (TSSG) system. Based on the variation of frequency response characteristics, Baek et al. [8] proposed a new technique to predict the contribution ratio of each stage backlash of the TSSG reducer; the validity of the used method was validated in a seeker gimbal with adequate results achieved. Since the dynamic coupling and TVMS between the helical gear pairs have a significant effect on the vibration characteristics of the helical gear system, Wang and Shi [9] proposed a systematic model to analyze the influence of TVMS and helix angle on gear pair system. In [10], Wang and Zhang developed a dynamics model of tensional-bending-swing-axial coupled motion for (TSHG) transmission considering the TVMS, backlash, and deviation, where a fifth-order Runge–Kutta method was used to solve the differential equations. An eighteen-DOF model considering the effects of the loading system, driving motor, and supporting bearings was proposed by Brethee et al. [11] to investigate the dynamic response of surface wear on gear including tooth friction and TVMS based on electrohydrodynamic lubrication (EHL) principles.

Considering the effects of TVMS and nonlinear characteristics of bearing, a dynamical model of a TSHG box including the motion of housing was established by Xu et al. [12], and steady-state vibration response of the gearbox has been obtained based on the coupling gear-rotor-bearing dynamic model. Furthermore, experimental modal analysis was used by Patel and Pathan [13] to derive the natural frequencies of TSHG reducer considering the effect of TVMS on natural frequency, and the modal parameters were obtained by applying the Frequency Response Function (FRF) method. Wallha et al. [14] developed a 12-DOF dynamic model to study the nonlinear dynamic responses of a TSSG system including mesh stiffness fluctuation, backlash, and bearing flexibility, in which the technique of linearization was used to decompose the system from nonlinear to linear. Ma et al. [15, 16] presented a 14-DOF dynamic model with an experimental study of TSSG to investigate the nonlinear dynamic response analysis of the TSSG space driving mechanism under large inertia load taking into consideration TVMS, damping, backlash, and transmission error. And Bin [17] used the model to analyze the effects of the profile modification parameters on the load transmission error, dynamic load coefficient, tooth profile error, and modification. Another 26 DOF of TSSG was established by Jia et al. [18] to study the dynamical modeling of multiple pairs of spur gears in mesh considering the effect of variable tooth stiffness, friction, geometrical errors, localized tooth crack, and pitch and profile errors on one gear.

Systematic modeling and analysis of a TSSG box model with 12 and 26 DOF were used to describe the gear fault features when processed with harmonic wavelet transform (HWT) in [19, 20]. Furthermore, He et al. [21] proposed a TSSG with a twelve-DOF dynamic model to study the influences of gear eccentricity on transverse and torsional dynamic responses and the dynamic transmission errors. Dynamic behavior of a three-dimensional model TSHG system considering the effect of manufacturing defects was formulated by Wallha et al. [22], in which the dynamic response was performed by the Newmark method. Dyk [23] examined the models of TSSG used by the discrete models and considering the effects of the intermediate shaft on dynamic loads in both two stages. Abboudi et al. [24] developed a lumped-mass dynamic model of TSHG with twelve DOF used in wind turbines, excited by the variability in wind resources and TVMS fluctuation; the differential equations’ motion of system was solved by the implicit Newmark algorithm. Beyaoui et al. [25] proposed a new methodology considering uncertainties in a gear transmission system of a horizontal-axis wind turbine, in which the dynamic equations of 12 DOF were solved by applying the polynomial chaos method and the ODE-45 MATLAB solver. A dynamic model for an automotive train system with 22 DOF was established by Ghorbel et al. [26] to study the kinetic, the vibration mode, and strain modal energies distributions taking into consideration the engine excitation, clutch, gearbox, and disc brake. In order to investigate the nonlinear dynamic response of the TSHG system coupled with an automotive clutch, Walha et al. [27] proposed a dynamic model of 27 DOF considering spline clearance, double-stage stiffness, and dry friction path, in which the equation of motion was solved by Runge–Kutta method.

According to the literature reviews mentioned in Section 1, it is indicated that various dynamical models have been presented to analyze the dynamics of two-stage gears pairs. Nevertheless, limited studies have addressed the effect of gear nonlinear dynamic response of TSHG box reduction used in EVs. For this reason, a nonlinear torsional-translational dynamic model of a TSHG box reduction with a 16 DOF is derived based on Zheng’s method [6], in which the TVMS, mesh damping, support bearings, STE, backlash, and torsional stiffness and damping of the intermediate shaft are considered in this paper.

Consequently, the dynamic model in Section 2 and numerical experiments in Section 3 were carried out to demonstrate the nonlinear dynamic characteristics, where the equation of motion is solved by the Runge–Kutta method. Moreover, the effects of pinion rotational speed, backlash, torsional stiffness, and torque fluctuation on the dynamic behavior of TSHG reduction were also studied, which provide an essential understanding of the nonlinear dynamic features of TSHG reduction, as concluded in Section 4.

2. The Dynamic Model of TSHG System

A 16-DOF nonlinear dynamic model is presented to simulate the dynamic behavior of a TSHG reduction system, as shown in Figure 1. As revealed by the geometric description in Table 1, the chosen gearbox system is a speed reducer. The system is formed by four helical gears \( p_1, g_1, p_2, \) and \( g_2 \); pinion and gear are denoted with the subscripts \( p \) and \( g \) as in \( (i = p, g) \); the two stages of the system are denoted with subscripts 1 and 2 as in \( j = 1, 2 \), respectively. Each gear is represented as rigid blocks with 4 DOF (one rotation and three translations), in which \( \Omega_{ij}, I_{ij}, \) and \( r_{ij} \) represent the rotating speed, the moment of inertia, and the base circle
radius of gears \(i\) and \(j\), respectively. \(T_D\) and \(T_L\) represent the torque values applied to the pinion \(p_1\) and the gear \(g_2\), respectively. The nonlinear backlash function \(f(x_{mj})\), STE \(e_j(t)\), TVMS \(k_{mj}(t)\), and mesh damping \(c_{mj}\) are combined to describe the gear deformation during the meshing process [28]. The resilient elements of bearing supports are represented by the damping \(c_{vw}\) and stiffness \(k_{vw}\) coefficients, where \(v = 1, 2, 3, 4\) indicates the four bearings in \(w = x, y, z\) directions (LOA, OLOA, and axial direction), respectively. The intermediate shaft of the TSHG component is described by torsional stiffness \(K_1\) and damping \(C_1\) components.

The generalized coordinates vectors of the nonlinear dynamic model including 16 DOF can be defined as

\[
\begin{align*}
\mathbf{e}_1(t) & \quad \mathbf{e}_2(t) \\
\begin{bmatrix}
\Omega_{p1} \\
\Omega_{g1} \\
\theta_{g1} \\
\mathbf{p}_1 \end{bmatrix} & \quad \begin{bmatrix}
\Omega_{p2} \\
\Omega_{g2} \\
\theta_{g2} \\
\mathbf{p}_2 \end{bmatrix} \\
TD & \quad TL
\end{align*}
\]

\[
\begin{bmatrix}
I_{p1} \theta_{p1} \\
I_{g1} \theta_{g1} \\
I_{p2} \theta_{p2} \\
I_{g2} \theta_{g2} \\
\end{bmatrix}
\]

**Figure 1:** A simplified dynamic model of a TSHG system.

**Table 1:** The geometric properties of the TSHG system.

| Properties                  | Pinion (\(p_1\)) | Gear (\(g_1\)) | Pinion (\(p_2\)) | Gear (\(g_2\)) |
|-----------------------------|-------------------|----------------|-------------------|----------------|
| Teeth number                | \(z_{p1} = 25\)  | \(z_{g1} = 77\) | \(z_{p2} = 29\)  | \(z_{g2} = 91\) |
| Base circle radius (mm)     | \(r_{p1} = 21.4\) | \(r_{g1} = 65.8\) | \(r_{p2} = 30.6\) | \(r_{g2} = 96.0\) |
| Mass (kg)                   | \(m_{p1} = 0.286\) | \(m_{g1} = 2.496\) | \(m_{p2} = 0.797\) | \(m_{g2} = 7.068\) |
| Working face width (mm)     | \(b_{p1} = 22.5\) | \(b_{g1} = 21.0\) | \(b_{p2} = 30.0\) | \(b_{g2} = 28.0\) |
| Helix angle (°)             | \(\beta_{p1} = 21.25\) | \(\beta_{g1} = 16.65\) | \(\beta_{p2} = 16.5\) | \(\beta_{g2} = 16.5\) |
| Pressure angle (°)          | \(\phi_{p1} = 18.5\) | \(\phi_{g1} = 16.5\) | \(\phi_{p2} = 16.5\) | \(\phi_{g2} = 16.5\) |
| Modulus (mm)                | \(M_1 = 1.69\) | \(M_2 = 2.12\) | \(M_4 = 2.12\) | \(M_5 = 2.12\) |
follows: spreads out into Fourier series [31]; the nonlinear TVMS as periodic waveforms under the meshing frequency and teeth number.

The STE \(e_j(t)\) is generally considered to be a periodic function of displacement, usually spreads out in Fourier series as fundamental frequency part of the harmonics [29], and can be expressed as

\[
e_j(t) = e_{0j} + E_{hj} \sin(\omega_{hj}t + \phi_{hj}),
\]

where \(e_{0j}, E_{hj}, \) and \(\phi_{hj}\) are the constant amplitude of STE, the amplitude of STE, and phase angle, respectively. The excitation meshing frequency \(\omega_{hj}\) of gear pair is determined as shown in [30]

\[
\omega_{hj} = \frac{\Omega_p Z_{pj} j}{60},
\]

where \(\Omega_p\) and \(Z_{pj}\) are the rotational speed of pinion and teeth number.

According to Ishikawa’s method, the TVMS is simplified as periodic waveforms under the meshing frequency and spreads out into Fourier series [31]; the nonlinear TVMS \(k_{mj}(t)\) obtained using the Fourier expansion is written as follows:

\[
k_{mj}(t) = k_{oij} + A_{kij} \sin(\omega_{kij}t + \phi_{kij}),
\]

where \(k_{oij}, A_{kij}, \) and \(\phi_{kij}\) are the mean value of meshing stiffness, the stiffness fluctuation amplitude, which equals \(k_{oij} \times 0.2, \) and phase angle, respectively. The mean value of the mesh damping \(c_{mj}\) is expressed as

\[
c_{mj} = 2\zeta_i \sqrt{\frac{2 k_{mj}^2 I_{pj} I_{gj}^3}{(r_{pj}^2 I_{pj} + r_{gj}^2 I_{gj})}}
\]

Here, the damping \(\zeta_i\) ratio is calculated as Rayleigh damping [32, 33], in which the value generally ranges from 0.03 to 0.17; in this model, the ratio is \(\zeta_i = 0.1, \) where \(j = 1, 2\) demonstrates the first and second stages.

When \(x_{mj}\) presents the relative meshing displacement of the gear under the influence of the gear backlash \(s_{ij}\) [34], the nonlinear backlash function is expressed as follows:

\[
\{q_m\} = \{\theta_{p1}, x_{p1}, y_{p1}, z_{p1}, \theta_{g1}, x_{g1}, y_{g1}, z_{g1}, \theta_{p2}, x_{p2}, y_{p2}, z_{p2}, \theta_{g2}, x_{g2}, y_{g2}, z_{g2}\}^T,
\]

\[
f(x_{mj}) = \begin{cases} x_{mj} - s_j, & x_{mj} > s_j, \\ 0, & -s_j \leq x_{mj} \leq s_j, \\ x_{mj} + s_j, & x_{mj} < s_j. \end{cases}
\]

Here, \(2s_j\) represents the total backlash.

The governing motion equations of the TSHG system shown in Figure 1 are expressed in the state space form, which is solved by MATLAB ODE-45 solver and derived considering the following assumptions:

1. Pinions and gears are modeled as rigid disks
2. Both input \(T_D\) and load \(T_L\) torque are applied to a system with constant values
3. The gear teeth are considered to be fully involute; assembly and manufacturing errors are ignored
4. Spring and damper are used to represent the torsional stiffness of the intermediate shafts in the midsection of the shaft

It is noted that the modeling method of Zhang in [6] takes into account the effects of geometric eccentricity on gears, though, in this study, the influence of eccentricity is neglected. According to the proposed concept, the rotational and translational motion equations of the TSHG reduction gear pairs are expressed by the following equations, respectively:

\[
\begin{align*}
I_{p1} \ddot{\theta}_{p1} + F_{m1}r_{p1} \cos(\beta_1) &= T_D, \\
I_{g1} \ddot{\theta}_{g1} + C_1(\dot{\theta}_{g1} - \dot{\theta}_{p1}) + K_1(\theta_{g1} - \theta_{p1}) + F_{m1}r_{g1} \cos(\beta_1) &= 0, \\
I_{p2} \ddot{\theta}_{p2} - C_1(\dot{\theta}_{g1} - \dot{\theta}_{p2}) - K_1(\theta_{g1} - \theta_{p2}) - F_{m2}r_{p2} \cos(-\beta_1) &= 0, \\
I_{g2} \ddot{\theta}_{g2} - F_{m2}r_{g2} \cos(-\beta_2) &= -T_L,
\end{align*}
\]
With those four DOF for each gear, the two-stage gear pair \(ij\) has a total of 16 DOF that define the coupling among the TSHG system, where \(m_{ij}\) indicates the mass of \((i = p, g)\) pinion \(p\) and gear \(g\) in the first and second stages \((j = 1, 2)\), respectively.

The relative displacement of the first-stage \(x_{m1}\) and second-stage \(x_{m2}\) gear mesh along the LOA is defined as

\[
\begin{align*}
\dot{x}_{m1} &= \left[ \left(-\ddot{x}_{p1} + \ddot{x}_{g1}\right) \sin(-\alpha_{n1}) + \left(\ddot{y}_{p1} - \ddot{y}_{g1}\right) \cos(-\alpha_{n1}) \right] \cos(\beta_{1}) + \left(\ddot{z}_{p1} - \ddot{z}_{g1}\right) \sin(\beta_{1}) - \ddot{e}_1(t), \\
\dot{x}_{m2} &= \left[ \left(-\ddot{x}_{p2} + \ddot{x}_{g2}\right) \sin(\alpha_{n2} - \pi) + \left(\ddot{y}_{p2} - \ddot{y}_{g2}\right) \cos(\alpha_{n2} - \pi) \right] \cos(-\beta_{2}) - \left(\ddot{z}_{p2} - \ddot{z}_{g2}\right) \sin(-\beta_{2}) - \ddot{e}_2(t),
\end{align*}
\]

Equation (8) can be combined and substituted into equation (10); thus, the relative displacement of the first and second stages became as follows:

\[
\begin{align*}
\dot{x}_b &= \left[ \left(\dddot{x}_{p1} + \dddot{x}_{p2}\right) \sin(-\alpha_{n1}) + \left(\dddot{y}_{p1} + \dddot{y}_{p2}\right) \cos(-\alpha_{n1}) \right] \cos(\beta_{1}) + \left(\dddot{z}_{p1} + \dddot{z}_{p2}\right) \sin(\beta_{1}) - \dddot{e}_1(t), \\
\dot{x}_b &= \left[ \left(\dddot{x}_{g1} + \dddot{x}_{g2}\right) \sin(\alpha_{n2} - \pi) + \left(\dddot{y}_{g1} + \dddot{y}_{g2}\right) \cos(\alpha_{n2} - \pi) \right] \cos(-\beta_{2}) - \left(\dddot{z}_{g1} + \dddot{z}_{g2}\right) \sin(-\beta_{2}) - \dddot{e}_2(t),
\end{align*}
\]

where \(x_b\) is the relative displacement of the first gear and second pinion and can be written as follows:

\[
\dot{x}_b = \left(\frac{C_1D}{I_{g1}} + \frac{C_1H}{I_{p2}}\right) \dot{x}_b - \left(\frac{K_{11}D}{I_{g1}} + \frac{K_{11}H}{I_{p2}}\right) \dot{x}_b - \frac{F_{m1}r_{p1}^2 \cos(\beta_{1})}{I_{g1}} - \frac{F_{m2}r_{g1}^2 \cos(\beta_{1})}{I_{p2}},
\]

\[
\dot{x}_b = \left(\frac{C_1D}{I_{g1}} + \frac{C_1H}{I_{p2}}\right) \dot{x}_b - \left(\frac{K_{11}D}{I_{g1}} + \frac{K_{11}H}{I_{p2}}\right) \dot{x}_b - \frac{F_{m1}r_{p1}^2 \cos(\beta_{1})}{I_{g1}} - \frac{F_{m2}r_{g1}^2 \cos(\beta_{1})}{I_{p2}}.
\]
with \( B = (r_{g1} + r_{p2})/2 \), \( D = r_{g1}/B \), and \( H = r_{p2}/B \).

For analytical convenience, the dimensionless form of equations (9), (11), and (12) is obtained by assuming the following nondimensional parameters as

\[
X_{ij} = \frac{x_{ij}}{s_j}, \quad Y_{ij} = \frac{y_{ij}}{s_j}, \quad Z_{ij} = \frac{z_{ij}}{s_j}, \quad X_b = \frac{x_b}{s_j},
\]

\[
E_j(\tau) = \frac{e_j(t)}{s_j}, \quad \tau = \omega_n t.
\]

Here, \( \omega_n = \sqrt{\frac{k_{nj}}{m_{ei}}} \) represents the natural frequency of gear pairs and \( m_{ei} = \frac{(I_{g1}I_{g2})/(I_{g1}r_{g2}^2 + I_{g2}r_{g1}^2)} \) is the equivalent mass of gear pairs. The small letter of each variable represents the derivative to time \( t \), while the big letter represents the derivative with respect to dimensionless time \( \tau \).

Therefore, the dimensionless form of the nonlinear backlash function in equation (7) becomes

\[
f(X_{mj}) = \begin{cases} X_{mj}(\tau) - 1, & X_{mj} > 1, \\ 0, & -1 \leq X_{mj} \leq 1, \\ X_{mj}(\tau) + 1, & X_{mj} < 1. \end{cases}
\]

The equations of motion of the entire system in the dimensionless form can be expressed in matrix forms as

\[
[M][\ddot{Q}_m] + [C][\dot{Q}_m] + [K][Q_m] = \{FQ\}.
\]

The dimensionless mass matrix, damping matrix, stiffness matrix, and the external excitation force vectors of the system are represented by \([M], [C], [K],\) and \([FQ]\), respectively. Consequently, the simplified coordinates vectors of the dimensional nonlinear dynamic model can be defined as \([Q_m]\):

\[
\{Q_m\} = \begin{bmatrix} X_{p1}, Y_{p1}, Z_{p1}, X_{g1}, Y_{g1}, Z_{g1}, X_{p2}, Y_{p2}, Z_{p2}, X_{g2}, Y_{g2}, Z_{g2}, X_b, X_m1, X_m2 \end{bmatrix}^T.
\]

The mass matrix \([M]\) can be expressed by

\[
[M] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & m_7 & m_8 & m_9 & m_{10} & m_{11} & m_{12} & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

Here, \( m_i \) represents the mass of the block \((i = 1, \cdots, 12)\) which can be expressed as follows:
\[ m_1 = -\cos(\beta_1)\sin(-\alpha_{n1}), \]
\[ m_2 = \cos(\beta_1)\cos(-\alpha_{n1}), \]
\[ m_3 = \sin(\beta_1), \]
\[ m_4 = \cos(\beta_1)\sin(-\alpha_{n1}), \]
\[ m_5 = -\cos(\beta_1)\cos(\alpha_{n1}), \]
\[ m_6 = -\sin(\beta_1), \]
\[ m_7 = -\cos(-\beta_2)\sin(\alpha_{n2} - \pi), \]
\[ m_8 = \cos(-\beta_2)\cos(\alpha_{n2} - \pi), \]
\[ m_9 = -\sin(-\beta_2), \]
\[ m_{10} = \cos(-\beta_2)\sin(\alpha_{n2} - \pi), \]
\[ m_{11} = -\cos(-\beta_2)\cos(\alpha_{n2} - \pi), \]
\[ m_{12} = \sin(-\beta_2). \]

(18)

For the sake of simplicity, the stiffness matrix \([K]\) and damping matrix \([C]\) are expressed by \([\Lambda_{CK}]\) as follows:

\[
[\Lambda_{CK}] = \begin{bmatrix}
\Lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_2 & 0 \\
0 & \Lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_4 & 0 \\
0 & 0 & \Lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_6 & 0 \\
0 & 0 & 0 & \Lambda_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_8 & 0 \\
0 & 0 & 0 & 0 & \Lambda_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & \Lambda_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{14} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{16} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{18} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{20} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{24} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{25} & \Lambda_{26} & \Lambda_{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{28} & \Lambda_{29} & \Lambda_{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{31} & \Lambda_{131} & \Lambda_{121} & \Lambda_{111} & \Lambda_{101} & \Lambda_{91} & \Lambda_{81} & \Lambda_{71} & \Lambda_{61} & \Lambda_{51} & \Lambda_{41} & \Lambda_{31} & \Lambda_{21} & \Lambda_{11} & \Lambda_0 & \Lambda_{19} & \Lambda_{18} & \Lambda_{17} & \Lambda_{16} & \Lambda_{15} & \Lambda_{14} & \Lambda_{13} & \Lambda_{12} & \Lambda_{11} & \Lambda_2 & \Lambda_1 & \Lambda_0
\end{bmatrix}
\]

(19)

where the damping matrix \([C]\) and stiffness matrix \([K]\) are denoted in \([\Lambda_{CK}]\) with subscripts \(C\) and \(K\), respectively. Here, \(\Lambda_i\) is the damping and stiffness of the block \((i = 1, \ldots, 31)\), which can be expressed by equations (20) and (21) as follows:

\[ k_5 = \frac{-k_{x1}}{m_{p1} \omega_{n1}^2}, \]
\[ k_6 = \frac{-K_{m1} (\tau) \sin(\beta_1)}{m_{p1} \omega_{n1}^2}, \]
\[ k_7 = \frac{-k_{y1}}{m_{g1} \omega_{n1}^2}, \]
\[ k_8 = \frac{-K_{m1} (\tau) \cos(\beta_1) \sin(-\alpha_{n1})}{m_{g1} \omega_{n1}^2}, \]
\[ k_9 = \frac{-k_{y2}}{m_{g1} \omega_{n1}^2}, \]
\[ k_{10} = \frac{K_{m1} (\tau) \cos(\beta_1) \cos(-\alpha_{n1})}{m_{g1} \omega_{n1}^2}. \]
\[ k_{11} = \frac{-k_{22}}{m_{g1}\omega_{n1}^2}, \]
\[ k_{12} = \frac{K_{m1}(\tau)\sin(-\beta_1)}{m_{g1}\omega_{n1}^2}, \]
\[ k_{13} = \frac{-k_{33}}{m_{p2}\omega_{n2}^2}, \]
\[ k_{14} = \frac{K_{m2}(\tau)\cos(-\beta_2)\sin(\alpha_{n2} - \pi)}{m_{p2}\omega_{n2}^2}, \]
\[ k_{15} = \frac{-k_{y3}}{m_{p2}\omega_{n2}^2}, \]
\[ k_{16} = \frac{-K_{m2}(\tau)\cos(-\beta_2)\cos(\alpha_{n2} - \pi)}{m_{p2}\omega_{n2}^2}, \]
\[ k_{17} = \frac{-k_{33}}{m_{p2}\omega_{n2}^2}, \]
\[ k_{18} = \frac{K_{m2}(\tau)\sin(-\beta_2)}{m_{p2}\omega_{n2}^2}, \]
\[ k_{19} = \frac{-k_{44}}{m_{g2}\omega_{n2}^2}, \]
\[ k_{20} = \frac{-K_{m2}(\tau)\cos(-\beta_2)\sin(\alpha_{n2} - \pi)}{m_{g2}\omega_{n2}^2}, \]
\[ k_{21} = \frac{-k_{y4}}{m_{g2}\omega_{n2}^2}, \]
\[ k_{22} = \frac{K_{m2}(\tau)\cos(-\beta_2)\cos(\alpha_{n2} - \pi)}{m_{g2}\omega_{n2}^2}, \]
\[ k_{23} = \frac{-k_{44}}{m_{g2}\omega_{n2}^2}, \]
\[ k_{24} = \frac{-K_{m2}(\tau)\sin(-\beta_2)}{m_{g2}\omega_{n2}^2}, \]
\[ k_{25} = \left( \frac{K_{1}D}{l_{g1}\omega_{n1}^2} + \frac{K_{1}H}{l_{p2}\omega_{n2}^2} \right), \]
\[ k_{26} = \frac{-K_{m1}(\tau)l_{g1}^2\cos(\beta_2)}{l_{g1}\omega_{n1}^2}, \]
\[ k_{27} = \frac{-K_{m2}(\tau)l_{p2}^2\cos(-\beta_2)}{l_{p2}\omega_{n2}^2}, \]
\[ k_{28} = \left( \frac{K_{1}D\cos(\beta_1)}{l_{g1}\omega_{n1}^2} \right), \]
\[ k_{29} = \frac{-K_{m1}(\tau)\cos^2(\beta_1)}{\omega_{n1}^2}, \]
\[ k_{30} = \left( \frac{l_{p1}^2 + l_{g1}^2}{l_{g1}^2 + l_{p1}^2} \right), \]
\[ k_{31} = \frac{-K_{m2}(\tau)\cos^2(-\beta_2)}{\omega_{n2}^2}, \]
\[ \frac{-K_{m1}(\tau)\cos^2(\beta_1)}{\omega_{n1}^2}, \]
\[ \frac{-K_{m1}(\tau)\cos^2(-\beta_2)}{\omega_{n2}^2}, \]
\[ c_{1} = \frac{-e_{x1}}{m_{p1}\omega_{n1}}, \]
\[ c_{2} = \frac{C_{m1}\cos(\beta_1)\sin(-\alpha_{n1})}{m_{p1}\omega_{n1}}, \]
\[ c_{3} = \frac{-e_{y1}}{m_{p1}\omega_{n1}}, \]
\[ c_{4} = \frac{-C_{m1}\cos(\beta_1)\cos(-\alpha_{n1})}{m_{p1}\omega_{n1}}, \]
\[ c_{5} = \frac{-e_{z1}}{m_{p1}\omega_{n1}}, \]
\[ c_{6} = \frac{-C_{m1}\sin(\beta_1)}{m_{p1}\omega_{n1}}, \]
\[ c_{7} = \frac{-e_{x2}}{m_{g1}\omega_{n1}}, \]
\[ c_{8} = \frac{-C_{m1}\cos(\beta_1)\sin(-\alpha_{n1})}{m_{g1}\omega_{n1}}, \]
\[ c_{9} = \frac{-e_{y2}}{m_{g1}\omega_{n1}}, \]
\[ c_{10} = \frac{C_{m1}\cos(\beta_1)\cos(-\alpha_{n1})}{m_{g1}\omega_{n1}}, \]
\[ c_{11} = \frac{-e_{x2}}{m_{g1}\omega_{n1}}, \]
\[ c_{12} = \frac{C_{m1}\sin(\beta_1)}{m_{g1}\omega_{n1}}, \]
\[ c_{13} = \frac{-e_{x3}}{m_{p2}\omega_{n2}}, \]
\[ c_{14} = \frac{-C_{m2}\cos(-\beta_2)\sin(\alpha_{n2} - \pi)}{m_{p2}\omega_{n2}}, \]
\[ c_{15} = \frac{-e_{y3}}{m_{p2}\omega_{n2}}, \]
3. Results and Discussion

The basic parameters of the TSHG reduction gear pairs studied in this paper are shown in Table 2. With those parameters, the set of the second-order differential equations in equation (16) are solved by the Runge–Kutta method. With the help of the FFT spectrum, phase portrait, Poincaré point section, time-domain history, RMS, PPVs curves, and Lyapunov exponents, the effects of the rotational speed of pinion \( \Omega_p \), backlash, torsional stiffness, and torque fluctuation on the dynamic behavior of TSHG components were studied by applying the numerical integration method considering different conditions. Generally, the dynamic responses are divided into five categories: chaotic, quasi-periodic, subharmonic, periodic harmonic, and periodic nonharmonic response. When the time history is non-periodic and has unlimited nonrepeating points in the Poincaré section, the response is chaotic. While the response is quasiperiodic, the phase portrait has nonperiodic circles and the points of the Poincaré section formed a closed orbit. When multiple discrete points are formed in the Poincaré section and repeat themselves at the excitation frequencies, the responses become subharmonic response. The system is harmonic period when the phase portrait is circular and repeats itself in Poincaré section. At last, the response is nonharmonic periodic if the phase portrait is noncircular and repeats itself in Poincaré section [35]. The results in this section provide some optimization suggestions for the TSHG reduction gear-pair design.

3.1. Effect of the Rotational Speed on the TSHG Dynamic Response. The rotational speed of pinion \( \Omega_p \) is considered as one of the main parameters that affect the dynamical behavior of system transmission. In this section, the half backlash of gears is set to \( s_1 = s_2 = 40 \mu m \); the other parameters of TSHG transmission system are given in Table 2. Figure 2 presents the forward bifurcation characteristic of the first stage \( X_{m1} \) and second stage \( X_{m2} \) in dimensionless displacement with regard to the rotating speed of pinion \( \Omega_p \) as the control parameter, which increases gradually from 1000 rpm to 20000 rpm, while Figure 3 presents the backward bifurcation characteristic with regard to the decreasing of \( \Omega_p \) from 20000 rpm to 1000 rpm. As shown in Figures 2 and 3, the system responses of both \( X_{m1} \) and \( X_{m2} \) under the variation of \( \Omega_p \) contain types of motion forms such as periodic-one motion marked with letter \( P \), jumps phenomena, quasi-periodic motion marked with letter \( Q \), and chaotic motion marked with letter \( C \).
In the case of the foreword bifurcation, the system $X_m$ behaves as a steady period-one motion at low speed and persists until $\Omega_{p1}$ reaches 7110.5528 rpm. The response of time-domain history shows a sine wave, one main peak amplitude is found in FFT spectrum, the phase plane has a single closed circle, a single point is found in Poincaré section, and these characteristics indicate that the response of $X_m$ is harmonic periodic-one motion, as shown in Figure 4.

The system keeps transforming between periodic motion and quasiperiodic motion several times until $\Omega_{p1}$ reaches 17040 rpm. However, the system is periodic-one motion except when $\Omega_{p1}$ is in the ranges of 7110–7493 rpm, 9974–10452 rpm, 11407–13221 rpm, and 15703–17040 rpm; the system experiences the quasiperiodic motion response. As illustrated in Figure 5, the phase plane diagram does not repeat itself, which forms closed orbit points in Poincaré section and the FFT spectrum is continuous; this means that

---

Table 2: The basic parameters of the TSHG reduction gear pairs.

| Parameters                  | First stage |        | Second stage |        |
|-----------------------------|------------|--------|--------------|--------|
| Moment of inertia (kg·mm²)  | $I_{p1} = 76.19$ | $I_{g1} = 6000$ | $I_{p2} = 437.2$ | $I_{g2} = 36000$ |
| Meshing damping (N·s/mm)   | $c_{m1} = 0.8064 \times 10^{-3}$ | $c_{m2} = 3.2918 \times 10^{-3}$ |
| Meshing stiffness (N/mm)    | $k_{m1} = 4.676 \times 10^{3}$ | $k_{m2} = 6.623 \times 10^{3}$ |
| Torque (Nm)                | $T_D = 250 \times 10^3$ | $T_L = 245.70 \times 10^3$ |
| STE (μm)                   | $E_{ij} = 20$ |        |              |        |
| Bearing damping (N·s/mm)   | $c_{ij} = 0.5$, $(i = x, y, z)$, $(j = 1, 2, 3, 4)$ |            |
| Bearing stiffness (N/mm)   | $k_{ij} = 3.5 \times 10^4$, $(i = x, y, z)$, $(j = 1, 2, 3, 4)$ |        |
| Torsional damping (N·mm/s/rad) | $C_1 = 2091.6$ | $E_{ij} = 20$ |
| Torsional stiffness (N·mm/rad) | $K_1 = 3.3134 \times 10^{10}$ | $E_{ij} = 20$ |

---

Figure 2: Forward bifurcation diagrams of system using $\Omega_{p1}$ as control parameter; $s_1 = s_2 = 40 \mu m$. (a) $X_{m1}$. (b) $X_{m2}$.
the system response is in quasiperiodic motion. Meanwhile, a jump phenomenon can be observed when $\Omega_{p1}$ is around the resonant regions at 12744 rpm. When $\Omega_{p1}$ increases to 17040 rpm, the system response changes from quasiperiodic motion to chaotic and remains until reaching 18758 rpm. From Figure 6, it can be seen that system response does not repeat itself in any pattern, the FFT spectrum has continuous broadband, and the phase plane shows disorder circles with many discrete points found in Poincaré section. It is revealed from these properties that the system responses as chaotic motion at the range of 17040–18758 rpm. However, with the increase of pinion speed, the system abandons the chaotic

**Figure 3:** Backward bifurcation diagrams of system using $\Omega_{p1}$ as control parameter; $s_1 = s_2 = 40 \mu$m. (a) $X_{m1}$. (b) $X_{m2}$.

**Figure 4:** The dynamic response of system $X_{m1}$ at 2500 rpm. (a) Time-domain response. (b) FFT spectrum. (c) Phase plane and Poincaré point.
region and returns to quasiperiodic motion response again which preserves from 18758 rpm to 20000 rpm.

As for the backward bifurcation, considering the decreasing of $\Omega_{p1}$ from 20000 rpm to 1000 rpm continuously, a chaotic motion is determined at 20000–18759 rpm and 17613–17040 rpm. As illustrated in Figure 7, the phase plane diagram shows nonrepeated circles with many clustered points of Poincaré map and the diagram of time-domain history also shows a nonperiodic motion. Figure 8 reveals that the system transforms into quasiperiodic response at 18759–17613 rpm, 17040–15703 rpm, and 13221–15512 rpm, where a jump phenomenon occurs at 17613 rpm. (“_he system behaves as periodic-one motion response at 15703–13221 rpm and undergoes the chaotic region at 13221–11979 rpm. (“_he chaotic phenomenon causes the vibration and noise issues in the system, which should be avoided during the design. Meanwhile, the system leaves the chaotic region with a jump phenomenon occurring at 11979 rpm. However, the system characteristic responses are the same as the responses in the forwarding bifurcation when $\Omega_{p1}$ is less than 11979 rpm, which is verified by PPVs and RMS curves, as shown in Figures 9 and 10. At this range of speed, the system response of $X_{m1}$ is mostly periodic-one motion, as marked in bifurcation diagram with letter P and proved in Figure 11.

It is observed that, under the variation of $\Omega_{p1}$ between 1000 and 20000 rpm, the characteristic of system $X_{m1}$ in forward bifurcation shown in Figure 2(b) is similar to the one in backward bifurcation shown in Figure 3(b), which is verified by RMS and PPVs curves in Figures 9(b) and 10(b). For the sake of simplicity, both bifurcations are discussed in detail as one bifurcation. As illustrated in Figures 2(b) and 3(b), the system $X_{m2}$ behaves as a periodic-one motion response at 1000–1763 rpm, as illustrated in Figure 12. (“_he system is in quasiperiodic response at 1763–2241 rpm and returns into periodic-one motion again at 2214–5869 rpm. Consequently, the system behaves as quasiperiodic motion and remains in the range of 5869–17040 rpm, where the quasiperiodic motion is indicated by the closed orbit formed in Poincaré section, as shown in Figure 13. And jumps phenomena occur around 3577 rpm and 17040 rpm. In the range from 17135 rpm to 17899 rpm, the system becomes irregular and turns rapidly into chaotic motion, as verified by the disorder circles of phase plane diagram and the discrete points of Poincaré map in Figure 14. In the meantime, with the increasing of the speed, the displacement
range of chaotic motion narrows gradually. Eventually, the system behaves as a quasiperiodic motion at 17899–2000 rpm, as illustrated in Figure 15. The results mentioned above reveal that the dynamic behavior of $X_{m1}$ and $X_{m2}$ is steady at low-speed range. Therefore, it is concluded that the dynamic behavior transforms from linear to nonlinear response with the increases of $\Omega_{pl}$.

From the corresponding PPVs and RMS curves, the conclusions of the relative displacement are obtained that there are two apparent bistable response regions in system $X_{m1}$ at the range of 11979–12839 rpm and 17613–20000 rpm, respectively. The chaotic motion ranges of $X_{m1}$ are indicated at 17040–18758 rpm as for forward case, but at 20000–18758 rpm, 17040–17613 rpm, and 11979–12457 rpm as for backward case, as shown in Figure 9(a), while the chaos response of $X_{m2}$ is indicated at 17135–17899 rpm, as illustrated in Figure 9(b). The response of the system $X_{m1}$ experiences a jump down phenomenon at 12744 rpm in the acceleration process and a jump up phenomenon at 11979 rpm and 17613 rpm in the deceleration process, as illustrated in Figure 10(a), while in system $X_{m2}$ jump up and down phenomena occur around 3577 rpm and 17040 rpm, as shown in Figure 10(b).

### 3.2. Effect of the Nonlinear Backlash on the TSHG Dynamic Response

According to the manufacturing accuracy level of gear pair, half of the backlash values of systems $X_{m1}$ and $X_{m2}$ are set to increase from $s_1 = s_2 = 40 \mu m, 60 \mu m$ to 100 $\mu m$. For the sake of simplicity, the backward bifurcation diagrams of systems under different backlash values are investigated, as illustrated in Figures 3, 16, and 17, respectively.

By comparing the bifurcation diagrams of both systems $X_{m1}$ and $X_{m2}$ under different backlash values, it is found that the range of chaotic motion in the rotational speed of pinion enlarges; the critical speed goes frontward along with the increasing of backlash value. In the meantime, the characteristics of chaotic motion are increased in both systems, which increases the difficulty of predicting the vibration of the gear pairs and leads to more noise possibility.

The corresponding RMS and PPVs curves of both systems $X_{m1}$ and $X_{m2}$ under different backlash values are illustrated in Figures 9, 10, and 18–21. It is seen that the displacement amplitudes of the PPV and RMS of both systems $X_{m1}$ and $X_{m2}$ decrease along with the increase of the backlash, and the range of $\Omega_{pl}$ expands as the chaos range extends.
Figure 9: PPVs of forward and backward behaviors of system; $s_1 = s_2 = 40 \mu m$. (a) $X_{m1}$. (b) $X_{m2}$.

Figure 10: Continued.
Considering the bifurcation diagrams, PPVs, and RMS curves of both systems, we conclude that, under the increases of backlash value, the characteristics of chaotic response could be enhanced which expand the chaotic motion range and enlarge the rotational speed range; as a result, this will lead to more possibility of noise vibration. Nevertheless, the
backlash could also lower the amplitudes of RMS and PPVs and deescalate the vibration extension of the system. Consequently, a suitable value of backlash should be selected to meet the requirements of vibration amplitude and reduce the possibility of chaotic behavior extension [32–34].

3.3. Effect of the Torsional Stiffness on the TSHG Dynamic Response. The torsional stiffness of the intermediate shaft is one of the key parameters that affect the dynamic behavior of the transmission system. Therefore, it is essential to investigate the effect of the torsional stiffness of the intermediate
Figure 16: Backward bifurcation diagrams of system; $s_1 = s_2 = 60 \mu m$. (a) $X_{m1}$. (b) $X_{m2}$.

Figure 17: Continued.
Shaft on the dynamic characteristics of the TSHG system. In this section, the backlash values of both systems $X_{m1}$ and $X_{m2}$ are set to $s_1 = s_2 = 40 \, \mu m$; the torsional stiffness $K_1$ is set to increase as $3.3134 \times 10^7 \, N \cdot mm/\text{rad}$, $3.3134 \times 10^8 \, N \cdot mm/\text{rad}$, and $3.3134 \times 10^{10} \, N \cdot mm/\text{rad}$, respectively.

Figures 22, 23, and 2 illustrate the corresponding forward bifurcation diagrams of the systems $X_{m1}$ and $X_{m2}$ with respect to the rotational speed of pinion under different torsional stiffness values. According to the comparison of the bifurcation diagrams under different torsional stiffness values, the substantial region of periodic, quasiperiodic, and chaotic responses...
Figure 19: RMS curves of forward and backward bifurcation system; $s_1 = s_2 = 60\, \mu m$. (a) $X_{m1}$. (b) $X_{m2}$.

Figure 20: Continued.
including jump phenomena are observed. At high-speed range of 17000–19000 rpm, the system is stable and behaves as a periodic motion when $K_1 \approx 3.3134 \times 10^7$ N·mm/rad. The dynamic response transits from periodic to quasiperiodic motion when $K_1 = 3.3134 \times 10^7$ N·mm/rad and eventually turns into chaotic response as $K_1$ increases to $3.3134 \times 10^{10}$ N·mm/rad. Therefore, the nonlinear system response becomes unstable and the chaotic behavior consequently expands as $K_1$ increases.

3.4. Effect of the Torque Fluctuation on the TSHG Dynamic Response. According to the BMW i8 electric motor characteristics [36], the ideal torque fluctuation $T_D$ is set to vary
Figure 22: Forward bifurcation diagrams of system when $K_1 = 3.3134 \times 10^7$ N·mm/rad. (a) $X_{m1}$, (b) $X_{m2}$.

- P: periodic motion
- Q: quasiperiodic motion

Figure 23: Forward bifurcation diagrams of system when $K_1 = 3.3134 \times 10^8$ N·mm/rad. (a) $X_{m1}$, (b) $X_{m2}$. 
from $44 \times 10^3$ Nmm to $250 \times 10^3$ Nmm under the rotational speed range of 1000–20000 rpm. In this section, the backlash values of both systems $X_{m1}$ and $X_{m2}$ are set to $s_1 = s_2 = 100 \mu m$, the torsional stiffness $K_1$ is set to $3.3134 \times 10^{10}$ N·mm/rad, and the torque fluctuation $T_D$ is considered as a control parameter that varies along with the changing of rotational speed. From the corresponding backward bifurcation diagrams of both systems $X_{m1}$ and $X_{m2}$ illustrated in Figure 24, it is observed that the dynamic response and the displacement of both systems at the speed range of 1000–4000 rpm are stable when the torque fluctuation is $250 \times 10^3$ Nmm. As the torque varies among $[240 – 70] \times 10^3$ Nmm, the corresponding displacement changes clearly at the speed range of 4000–13000 rpm. Then, the displacement is stable again under the torque variation among $[70 – 44] \times 10^3$ Nmm until the speed reaches 20000 rpm.

However, it is observed that chaotic motion for both systems exists at the speed range of 13000–20000 rpm when the torque fluctuates at low values between $[52 – 44] \times 10^3$ Nmm and $[68 – 66] \times 10^3$ Nmm. Quasiperiodic motions are observed at the speed range of 7111–17000 rpm when the torque fluctuates between $[131 – 124]$, $[92 – 88]$, $[80 – 70]$, and $[67 – 52] \times 10^3$ Nmm as for the system $X_{m1}$ and at the speed range of 6000–17000 rpm when the torque fluctuation range is around $[153 – 68] \times 10^3$ Nmm and $[65 – 61] \times 10^3$ Nmm for the system $X_{m2}$. When the torque fluctuation is fixed at $250 \times 10^3$ Nmm, the responses of system are stable one-periodic motion at the speed range of 1000–7000 rpm. As a result, comparing with Figure 17, the torque fluctuation at high-speed range extends the chaotic and quasiperiodic regions corresponding to the speed and expands their ranges corresponding to the displacement. The corresponding displacement response is stable at $250 \times 10^3$ Nmm and at low range $[70 – 44] \times 10^3$ Nmm, but it has the greatest influence when the torque fluctuates among $[240 – 70] \times 10^3$ Nmm.

The Lyapunov exponents shown in Figure 25 are calculated based on the Gram–Schmidt QR-decomposition method [37]; corresponding to the chaotic motions of $X_{m1}$ in Figure 24(a), the maximum Lyapunov exponents are larger than zero around the range of 16000–20000 rpm. Besides, the largest Lyapunov exponents are closed to zero, around the range of 7110–7492 rpm, 9900–10400 rpm, and 11400–16800 rpm corresponding to the quasiperiodic motion in Figure 24(a). Similarly, corresponding to the chaotic motions of $X_{m2}$ in Figure 24(b), the maximal Lyapunov exponents are larger than zero around the range of...
16000–20000 rpm, and the largest Lyapunov exponents are closed to zero around the range of 8000–16000 rpm, indicating the quasiperiodic motions in Figure 24(b).

4. Conclusion

The torsional-translational model consists of 16 degrees of freedom (DOF), which includes factors such as the non-linearity backlash, support bearings, STE, TVMS, and the torsional damping and stiffness of the intermediate shaft. According to the motion equations of system, the dynamical response of $X_{m1}$ and $X_{m2}$ is examined by means of the Runge–Kutta method; the effects of pinion rotational speed, backlash, torsional stiffness, and torque fluctuation are also studied on the dynamic responses.

It was shown that the dynamic behavior of $X_{m1}$ and $X_{m2}$ is steady at low-speed range. Under the increases of $\Omega_{p1}$, the dynamic behavior transforms from linear to nonlinear and becomes unstable. Besides, the dynamic response of $X_{m2}$ under the variations of the $\Omega_{p1}$ has the least effect when the forward and backward bifurcations characteristics are compared, which tend to have the same dynamic responses and behaviors. With the help of bifurcation diagrams, PPVs, and RMS curves, the observed periodic-one responses, quasiperiodic responses, chaotic responses, and jumps phenomena were confirmed, which furthermore verifies the dynamic characteristics of system obtained from previous results according to FFT spectrum, phase portrait, time-domain history, and Poincaré section.

The increase of the backlash value could change the dynamic responses significantly, enhance the chaotic characteristics, enlarge the rotational speed range, and widen chaos motion range, which leads to more possibility of noise vibration [32–34]. Moreover, the backlash could also lower the amplitudes of the RMS and PPVs and weaken the vibration extension of both systems. The choice of the dynamic parameters plays such a critical role in the controlling and designing of the gear-pair system. Hence, an appropriate value of backlash should be selected to meet the requirements of vibration amplitude and reduce the possibility of chaotic behavior extension.

At high-speed range, the torsional stiffness of the intermediate shaft $K_1$ has a great influence on the TSHG dynamic characteristic. As the torsional stiffness increases, the dynamic response transits from periodic to quasiperiodic motion and then eventually turns into chaotic response. Therefore, the nonlinear system response becomes unstable and the chaotic behavior consequently expands as $K_1$ increases.

The torque fluctuation at high-speed range could affect the dynamic responses, extend the chaotic and quasiperiodic regions corresponding to the speed, and expand their ranges corresponding to the displacement. The corresponding displacement response is stable at the fixed torque value.
250 \times 10^3 \text{ Nmm} \text{ and at low torque range value} 
\left[70 - 44\right] \times 10^3 \text{ Nmm}, \text{ but it has the greatest influence when} \text{ the torque fluctuates among} \left[240 - 70\right] \times 10^3 \text{ Nmm}. \text{ At high-speed range and at low torque fluctuation, both systems are unstable and in chaotic motions according to the maximum Lyapunov exponents.}

This paper mainly focuses on the theoretical study of the nonlinear responses of the TSHG transmission system for EVs. In order to improve the transmission performance and verify the theoretical results of TSHG pairs for EVs, the relevant experimental verification with more particular model will be carried out to accurately describe the TSHG reduction system, and deep study must be done in the future.

Data Availability

The data generated or analyzed to support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (NSFC) (Grant nos. 51535012 and U1604255) and Key Research and Development Project of Hunan Province (Grant no. 2016JC2001).

References

[1] S. Lekshmi and P. S. Lal Priya, "Mathematical modeling of Electric vehicles—a survey," Control Engineering Practice, vol. 92, Article ID 104138, 2019.
[2] W. Bartelmus, "Mathematical modelling and computer simulations as an aid to gearbox diagnostics," Mechanical Systems and Signal Processing, vol. 15, no. 5, pp. 855–871, 2001.
[3] M. Yan and H.-q. Liu, "A dynamic modeling method for helical gear systems," Journal of Vibroengineering, vol. 19, no. 1, pp. 111–124, 2017.
[4] M. Kubur, A. Kahraman, D. Zini, and K. Kienzle, "Dynamic analysis of multi-mesh helical gear sets by finite elements," in Proceedings of the ASME 2003 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, American Society of Mechanical Engineers Digital Collection, Chicago, IL, USA, pp. 333–342, September 2003.
[5] M. Kubur, A. Kahraman, D. M. Zini, and K. Kienzle, "Dynamic analysis of a multi-shaft helical gear transmission by finite elements: model and experiment," Journal of Vibration and Acoustics, vol. 126, no. 3, pp. 398–406, 2004.
[6] Y. Zhang, Q. Wang, H. Ma, J. Huang, and C. Zhao, "Dynamic analysis of three-dimensional helical geared rotor system with geometric eccentricity," Journal of Mechanical Science and Technology, vol. 27, no. 11, pp. 3231–3242, 2013.
[7] L. Walha, T. Fakhfakh, and M. Haddar, "Backlash effect on dynamic analysis of a two-stage spur gear system," Journal of Failure Analysis & Prevention, vol. 6, no. 3, pp. 60–68.
[8] J. H. Baek, Y. K. Kwak, and S. H. Kim, "Backlash estimation of a seeker gimbal with two-stage gear reducers," The International Journal of Advanced Manufacturing Technology, vol. 21, no. 8, pp. 604–611, 2003.
[9] C. Wang and Z. Shi, "A dynamic calculation method of sliding friction losses for a helical gear pair," Journal of the Brazilian Society of Mechanical Sciences and Engineering, vol. 39, no. 5, pp. 1521–1528, 2017.
[10] Q. Wang and Y. Zhang, "Coupled analysis based dynamic response of two-stage helical gear transmission system," Journal of Vibration and Shock, vol. 31, no. 10, pp. 87–91, 2012.
[11] K. F. Brethee, D. Zhen, F. Gu, A. D. Ball, and M. Theory, "Helical gear wear monitoring: modelling and experimental validation," Mechanism and Machine Theory, vol. 117, pp. 210–229, 2017.
[12] G. Xu, X. Xiong, X. Yan, and C. Shen, "Dynamic modelling of a two-stage helical gearbox with the consideration of non-linear rolling bearing contact and time-varying meshing stiffness," in Proceedings of the Prognostics and System Health Management Conference (PHM-Qingdao), IEEE, Qingdao, China, pp. 1–6, October 2019.
[13] R. Patel and S. Pathan, "Effect of time varying mesh stiffness on natural frequencies of two stage spur gear speed reducer," in Proceedings of the 5th International Conference on Advances in Mechanical Engineering (ICAME-2011), Gujarat, India, June 2011.
[14] L. Walha, T. Fakhfakh, and M. Haddar, "Nonlinear dynamics of a two-stage gear system with mesh stiffness fluctuation, bearing flexibility and backlash," Mechanism and Machine Theory, vol. 44, no. 5, pp. 1058–1069, 2009.
[15] J. Ma, C. Li, and L. Cui, "Transmission error analysis and disturbance optimization of two-stage spur gear space driven mechanism with large inertia load," Shock and Vibration, vol. 2018, Article ID 6863176, 14 pages, 2018.
[16] J. Ma, C. Li, J. Liu, D. Cao, and J. Huang, "Experimental study on transmission error of space-driven gear system under large inertia load," Advances in Mechanical Engineering, vol. 11, no. 6, Article ID 1687814019856952, 2019.
[17] Z. Bin, "Research on tooth profile modification of two spur gear driven system," IOP Conference Series: Materials Science and Engineering, vol. 563, no. 3, Article ID 032019, 2019.
[18] S. Jia, I. Howard, and J. Wang, "The dynamic modeling of multiple pairs of spur gears in mesh, including friction and geometrical errors," The International Journal of Rotating Machinery, vol. 9, no. 6, pp. 437–442, 2003.
[19] E. J. Diehl and J. Tang, "Predictive modeling of a two-stage gearbox towards fault detection," Shock and Vibration, vol. 2016, Article ID 9638325, 13 pages, 2016.
[20] E. J. Diehl, J. Tang, and H. DeSmedt, "Gear fault modeling and vibration response analysis," in Proceedings of the ASME 2012 5th Annual Dynamic Systems and Control Conference Joint with the JSME 2012 11th Motion and Vibration Conference, American Society of Mechanical Engineers Digital Collection, Lauderdale, FL, USA, pp. 709–718, October 2012.
[21] X. He, X. Zhou, Z. Xue, Y. Hou, Q. Liu, and R. Wang, "Effects of gear eccentricity on time-varying mesh stiffness and dynamic behavior of a two-stage gear system," Journal of Mechanical Science and Technology, vol. 33, no. 3, pp. 1019–1032, 2019.
[22] L. Walha, Y. Driss, T. Fakhfakh, and M. Haddar, "Effect of manufacturing defects on the dynamic behaviour for an helical two-stage gear system," Mécanique & Industries, vol. 10, no. 5, pp. 365–376, 2009.
[23] J. Dyk, "Dynamic loads in a model of the two-stage gears by varying stiffness of the countershaft," Solid State Phenomena, vol. 210, pp. 65–76, 2014.
[24] K. Abboudi, L. Walha, Y. Driss, M. Maatar, T. Fakhfakh, and M. Haddar, "Dynamic behavior of a two-stage gear train used in a fixed-speed wind turbine," Mechanism and Machine Theory, vol. 46, no. 12, pp. 1888–1900, 2011.

[25] M. Beyaoui, M. Tounsi, K. Abboudi, N. Feki, L. Walha, and M. Haddar, "Dynamic behaviour of a wind turbine gear system with uncertainties," Comptes Rendus Mécánique, vol. 344, no. 6, pp. 375–387, 2016.

[26] A. Ghorbel, M. Abdennadher, B. Zghal, L. Walha, and M. Haddar, "Modal analysis and dynamic behavior for analytical drivetrain model," Journal of Mechanics, vol. 34, no. 4, pp. 399–415, 2018.

[27] L. Walha, Y. Driss, M. T. Khabou, T. Fakhfakh, and M. Haddar, "Effects of eccentricity defect on the nonlinear dynamic behavior of the mechanism clutch-helical two stage gear," Mechanism and Machine Theory, vol. 46, no. 7, pp. 986–997, 2011.

[28] A. Kahraman and R. Singh, "Non-linear dynamics of a spur gear pair," Journal of Sound and Vibration, vol. 142, no. 1, pp. 49–75, 1990.

[29] A. Raghothama and S. Narayanan, "Bifurcation and chaos in geared rotor bearing system by incremental harmonic balance method," Journal of Sound and Vibration, vol. 226, no. 3, pp. 469–492, 1999.

[30] M. Zhao and J. C. Ji, "Nonlinear torsional vibrations of a wind turbine gearbox," Applied Mathematical Modelling, vol. 39, no. 16, pp. 4928–4950, 2015.

[31] A. Kahraman and R. Singh, "Interactions between time-varying mesh stiffness and clearance non-linearities in a geared system," Journal of Sound and Vibration, vol. 146, no. 1, pp. 135–156, 1991.

[32] M. Bozca, "Torsional vibration model based optimization of gearbox geometric design parameters to reduce rattle noise in an automotive transmission," Mechanism and Machine Theory, vol. 45, no. 11, pp. 1583–1598, 2010.

[33] M. Bozca, "Transmission error model-based optimisation of the geometric design parameters of an automotive transmission gearbox to reduce gear-rattle noise," Applied Acoustics, vol. 130, pp. 247–259, 2018.

[34] M. Bozca and P. Fietkau, "Empirical model based optimization of gearbox geometric design parameters to reduce rattle noise in an automotive transmission," Mechanism and Machine Theory, vol. 45, no. 11, pp. 1599–1612, 2010.

[35] R. Singh, D. Houser, and A. Kahraman, "Non-linear dynamic analysis of geared systems," Technical Report, NASA, Washington, DC, USA, 1990.

[36] Admin, "The BMW i8," 2020, http://www.carfab.com/the-bmw-i8.

[37] W. Chen, S. Chen, J. Tang, and H. Li, "Stability and bifurcation analysis of a bevel gear system supported by finite-length squeeze film dampers," Nonlinear Dynamics, vol. 100, pp. 3321–3345.