On the ramification of non-abelian Galois coverings of degree $p^3$

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Abstract

The refined Swan conductor is defined by K. Kato [2], and generalized by T. Saito [5]. In this part, we consider some smooth $l$-adic étale sheaves of rank $p$ such that we can be define the $rsw$ following T. Saito, on some smooth dense open subscheme $U$ of a smooth separated scheme $X$ of finite type over a perfect fields $\kappa$ of characteristic $p > 0$. We give an explicit expression of $rsw(F)$ in some situation. As a consequence, we show that it is integral.

1 Introduction

The classical Swan conductor is defined in the case where the extension of residue field is separable.

Kato in [2], gives a natural definition of the Swan conductor for a character of degree one, which works without the separability assumption of the extension of the residue field. In the same paper, he gives the definition of the refined Swan conductor $rsw$.

Kato in [3], redefines the refined Swan conductor as an element in the stalk of the sheaf $\omega^1_X(sw(\chi))$, where $\omega^1_X$ is the logarithmic cotangent sheaf, and extend it to a global section. We call the property of $rsw$ that can extend to a global section “ integral ”. He defines a 0-cycle $c_\chi$ to generalize the Swan conductor, by using the refined Swan conductor.

Along with the generalization of the Swan conductor, Grothendieck-Ogg-Shafarevich formula is generated also.

Kato and Saito in [4], define the Swan class as a 0-cycle class supported on the ramification locus as a refinement of Swan conductor, and generalize the G-O-S formula to arbitrary rank. In the situation that the scheme have dimension $\leq 2$ and the sheave have rank 1, the Swan class is consistent with the 0-cycle class $c_\chi$ defined in [3].

Abbes and Saito in [1], define a characteristic class of an $l$-adic sheaf as a refinement of the Euler-Poincaré characteristic, and get a refinement of the G-O-S formula for $l$-adic sheaf of rank 1, in term characteristic class and the 0-cycle class defined for rank 1 sheaves in [3], by explain the refine Swan conductor by a linear form on some vector bundle on $D$.

Saito in [5], by redefine $rsw$, defines the characteristic cycle of an $l$-adic sheaf, as a cycle on the logarithmic cotangent bundle. Under some condition ([5], p53, the condition (R) and (C)), he proves that the intersection of characteristic cycle with the 0-section computes the characteristic class defined in [1] for $l$-adic sheaf of arbitrary rank, and hence the Euler number. The property “ integral ” of $rsw$ is a part of the condition.
In this paper, we consider some smooth $l$-adic étale sheaves of rank $p$, that can be defined the $rsw$ follow [5], on some smooth dense open subscheme $U$ of a smooth separated scheme $X$ of finite type over a perfect fields $\kappa$ of characteristic $p > 0$. We give a explicit expression of $rsw(F)$ in some situation. In fact, we would see, that it is integral, namely

$$rsw(F) \in \Gamma(D, i_D^* \Omega^1_{\Delta X}(\log D)(kD))$$

where $i_D : D \rightarrow X$ is the closed immersion.

In §2, we study the ramification of the Artin-Schreier covering. In §3, we study the ramification of some Galois coverings of degree $p^3$. In §4, we study the ramification of some $l$-adic sheaves.

\section{Artin-Schreier covering}

Let $X$ be a separated smooth scheme of finite type over a perfect field $\kappa$ of characteristic $p > 0$, $D = X \setminus U$ be a smooth divisor, $U = X \setminus D$.

Let $(X \times X)^\sim$ and $(X \times X)^{(kD)}$ be the log blow-up and the log diagram blow-up of $X \times X$ defined in [5] respectively, $E^0 := (X \times X)^\sim \setminus U \times U$, $E := ((X \times X)^{(kD)} \setminus U \times U)_{red}$. We denote the ideal sheaves of $\Delta X$ in $(X \times X)^\sim$ and $(X \times X)^{(kD)}$ by $I_{\Delta X}$ and $J_{\Delta X}$ respectively. Let $\xi, \eta$ be the generic point of $D$ and $E$ respectively, then there exists only valuation $v_E$ of $O((X \times X)^{(kD)}, \eta)$, such that $v_E(f \otimes 1) = v_D(f)$ for any $f \in O_{X, \xi}$.

\textbf{Lemma 2.1.} If $f \in \Gamma(X, O(nD))$, then $1 \otimes f - f \otimes 1 \in \Gamma((X \times X)^{(kD)}, J_{\Delta X}((n - k)E))$.

\textbf{Proof.} Let us consider the commutative diagram

\begin{center}
\begin{tikzcd}
E \ar[r] \ar[d] & (X \times X)^{(kD)} \ar[l] \ar[d] \ar[r] \ar[d] & \Delta X \cup kE \\
E^0 \ar[r] & (X \times X)^\sim \ar[l] \ar[r] & \Delta X \\
D \ar[r] & X
\end{tikzcd}
\end{center}

We have

$$I_{\Delta X} \rightarrow \phi^k_* J_{\Delta X}((-kE)), O_{(X \times X)^\sim}(nE^0) \rightarrow \phi^k_* O_{(X \times X)^{(kD)}}(nE)$$

hence

$$I_{\Delta X}(nE^0) \rightarrow \phi^k_* J_{\Delta X}((n - k)E)$$

Therefore

$$1 \otimes f - f \otimes 1 = f \otimes 1(f^{-1} \otimes f - 1) \in \Gamma((X \times X)^\sim, I_{\Delta X}(nE^0))$$

$$\rightarrow \Gamma((X \times X)^\sim, \phi^k_* J_{\Delta X}(n - k)E) \rightarrow \Gamma((X \times X)^{(kD)}, J_{\Delta X}(n - k)E)$$

$\Box$
We have an exact sequence
\[
H^0(U, \mathcal{O}_U) \rightarrow H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathbb{Z}/p\mathbb{Z}) \rightarrow 0
\]
Therefore any Galois covering of \( U \) with Galois group \( \mathbb{Z}/p\mathbb{Z} \) can be defined by equation \( T^p - T - f \), where \( f \in H^0(U, \mathcal{O}_U) \). \( T^p - T - f \) and \( T^p - T - f_0 \) define same covering if and only if \( f = f_0 + f_1 - f_1^p \) for some \( f_1 \in H^0(U, \mathcal{O}_U) \). It easy to see, for any Galois covering \( \tilde{V} \) of \( U \) with Galois group \( \mathbb{Z}/p\mathbb{Z} \), that wild ramified on \( D \), there exists a \( n \in \mathbb{N} \) such, that \( \tilde{V} \) can be defined by equation \( T^p - T - f \) for some \( f \in \Gamma(X, \mathcal{O}_X(nD)) \) and \( df \in \Omega^1_{\Delta_X}((\log D)(nD))\xi \setminus \Omega^1_{\Delta_N}((\log D)(nD^-))\xi \).

**Proposition 2.2.** If \( f \in \Gamma(X, \mathcal{O}_X(nD)) \) and \( df \in \Omega^1_{\Delta_X}((\log D)(nD))\xi \setminus \Omega^1_{\Delta_N}((\log D)(nD^-))\xi \), then as the element of \( K((X \times X)_{(R)}), v_E(1 \otimes f - f \otimes 1) = k - n \).

**Proof.** Consider the commutative diagram
\[
\begin{array}{ccc}
(X \times X)^{(kD)} & \xrightarrow{\delta} & \Delta X \\
\downarrow i & & \downarrow i_D & \downarrow \delta \\
E & \xrightarrow{i_D} & D \\
\downarrow \xi & & \downarrow \xi & \downarrow \xi
\end{array}
\]
Then we have a commutative diagram
\[
\begin{array}{ccc}
\Gamma((X \times X)^{(kD)}, J_{\Delta X}((n-k)E)) & \rightarrow & \Gamma(\Delta X, \delta^* J_{\Delta X}((n-k)E)) \\
\downarrow & & \downarrow \\
\Gamma(E, i^* J_{\Delta X}((n-k)E)) & \rightarrow & \Gamma(D, i_D^* \delta^* J_{\Delta X}((n-k)E))
\end{array}
\]
but
\[
\delta^* J_{\Delta X}((n-k)E) = \delta^* J_{\Delta X} \cdot \delta^* O_{(X \times X)^{(kD)}}((n-k)E) = \delta^* J_{\Delta X} \cdot O_{\Delta X}((n-k)D) \\
\rightarrow \Omega^1_{\Delta X}((\log D)(kD)) \cdot O_{\Delta X}((n-k)D) = \Omega^1_{\Delta X}((\log D)(nD))
\]
and
\[
i^* J_{\Delta X}((n-k)E) = \mathcal{I}_D((n-k)E)
\]
Therefore we have
\[
\begin{array}{ccc}
\Gamma((X \times X)^{(kD)}, J_{\Delta X}((n-k)E)) & \rightarrow & \Gamma(\Delta X, \delta^* J_{\Delta X}((n-k)E)) \\
\downarrow & & \downarrow \\
\Gamma(E, \mathcal{I}_D((n-k)E)) & \rightarrow & \Gamma(D, i_D^* \delta^* J_{\Delta X}((n-k)E)) \\
\downarrow & & \downarrow \\
\Gamma(D, i_D^* \Omega^1_{\Delta X}((\log D)(nD))) & \rightarrow & \Gamma(D, i_D^* \Omega^1_{\Delta X}((\log D)(nD)))
\end{array}
\]
The image of \( 1 \otimes f - f \otimes 1 \) in \( \Gamma(D, i_D^* \Omega^1_{\Delta X}((\log D)(nD))) \) is \( df \).

However
\[
\begin{align*}
i_D^* \Omega^1_{\Delta X}((\log D)(kD)) &= i_D^* \Omega^1_{\Delta X}((\log D)(kD)) \otimes i_D^* O_{\Delta X} O_D \\
&= i_D^* \Omega^1_{\Delta X}((\log D)(kD)) \otimes i_D^* O_{\Delta X} i_D^* \Omega^1_{\Delta X} \\
&= i_D^* \Omega^1_{\Delta X}((\log D)(kD)) \otimes i_D^* O_{\Delta X} i_D^* \Omega^1_{\Delta X} \\
&= i_D^* \Omega^1_{\Delta X}((\log D)(kD)) \\
&= i_D^* \Omega^1_{\Delta X}((\log D)(kD^-))
\end{align*}
\]
hence

\[
\begin{align*}
(i_D^* \Omega^1_{\Delta_X}(log D)(kD))_{\xi} & = (i_D^* \Omega^1_{\Delta_X}(log D)(kD^\prime))_{\xi} \\
(i_D^* \Omega^1_{\Delta_X}(log D)(kD))_{\xi} & = (i_D^* \Omega^1_{\Delta_X}(log D)(kD^\prime))_{\xi}
\end{align*}
\]

\(df\) is not 0 in \((i_D^* \Omega^1_{\Delta_X}(log D)(kD))_{\xi}\), so is not 0 in \(\Gamma(D, i_D^* \Omega^1_{\Delta_X}(log D)(nD))\). Hence \(1 \otimes f - f \otimes 1\) is not 0 in \(\Gamma(E, D_U((n-k)E))\). Therefore \(v_E(1 \otimes f - f \otimes 1) = k - n\).

3 Galois coverings of degree \(p^3\)

Now, let us consider the Galois covering \(V\) defined by equations \(T^p - T - f, S^p - S - g, U^p - U - fS - h\) on \(U\), where \(f, g, h\in \Gamma(U, O_U)\). The Galois group \(G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}\).

\(V\) has two quotient coverings \(V_T\) and \(V_S\), which are defined on \(U\) by equation \(T^p - T - f\) and \(S^p - S - g\) respectively. In the following part of this paper, we always suppose

\[
\begin{align*}
df & \in \Omega^1_X(log D)(nD)_{\xi} \setminus \Omega^1_X(log D)(nD^-)_{\xi} \\
dg & \in \Omega^1_X(log D)(mD)_{\xi} \setminus \Omega^1_X(log D)(mD^-)_{\xi} \\
dh & \in \Omega^1_X(log D)(rD)_{\xi} \setminus \Omega^1_X(log D)(rD^-)_{\xi}
\end{align*}
\]

where \(n = -v_D(f), m = -v_D(g), r = -v_D(h)\).

We have a filter of subgroups of \(G \times G\) as follows:

\[
G \times G \supset N \supset \Delta G \supset 1
\]

where \(N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & c' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c, c' \in \mathbb{F}_p \right\}, \Delta G = \{(g, g) | g \in G\} \).

Let \(Z_1 := (V \times V)_{\Delta G}, W_1 := (V \times V)_N\) be the quotient of \(V \times V\) under the action of \(\Delta G\) and \(N\) respectively; \(Z_0, W_0\) be the normalization of \(Z_1, W_1\) over \((X \times X)^{(kD)}\) respectively.

**Lemma 3.1.** (1). If \(k \geq \max\{m, n\}\), then \(W_0 \to (X \times X)^{(kD)}\) is Galois with Galois group isomorphic to \(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\).

(2). If \(k > \max\{m, n\}\), then \(Z_0\) and \(W_0\) is splitting to \(p^2\)’s connected components over \(E \cup \Delta X\).

**Proof.**

(1).

\[
V \times V = \text{Spec}\frac{O_{U} \otimes O_{U}[T, S, U, T', S', U']}{T^p - T - f \otimes 1, S^p - S - g \otimes 1, U^p - U - (f \otimes 1S + h \otimes 1), T^p - T' - 1 \otimes f, S^p - S' - 1 \otimes g, U^p - U' - (1 \otimes fS' + 1 \otimes h)}
\]

\[
= \text{Spec}\frac{O_{U} \otimes O_{U}[T, S, U, T', S', U']}{T^p - T - f \otimes 1, S^p - S - g \otimes 1, U^p - U - (f \otimes 1S + h), T' - T' - K_1, S^p - S - K_2, U^p - U' - (1 \otimes fS' - f \otimes 1S + K_4)}
\]

Where \(K_1 = 1 \otimes f - f \otimes 1, K_2 = 1 \otimes g - g \otimes 1, K_4 = 1 \otimes h - h \otimes 1 \in O_{U \times U}\).
Therefore

\[ W_1 = \text{Spec} \frac{\mathcal{O}_U \otimes \mathcal{O}_U[\check{T}, \check{S}]}{(T^p - \check{T} - K_1, S^p - \check{S} - K_2)} \]

If \( k \geq \max(m, n) \), then by lemma (2.2), we know \( K_1, K_2 \in \Gamma((X \times X)^{(kD)}, \mathcal{O}_{(X \times X)^{(kD)}}) \).
Then

\[ W_0 = \text{Spec} \frac{\mathcal{O}_{(X \times X)^{(R)}}[\check{T}, \check{S}]}{(T^p - \check{T} - K_1, S^p - \check{S} - K_2)} \]

(2). If \( k > \max(m, n) \), then by lemma (2.2), we know \( v_E(K_1) > 0, v_E(K_2) > 0 \). On the other hand, we know \( K_1\Delta = 0, K_2\Delta = 0 \) because \( K_1\Delta = 0, K_2\Delta = 0 \). Then

\[
W_0 \times_{(X \times X)^{(kD)}} (E \cup \Delta X) = \text{Spec} \frac{\mathcal{O}_{(X \times X)^{(R)}}[\check{T}, \check{S}] \otimes \mathcal{O}_{(E \cup \Delta X)}}{(T^p - \check{T} - K_1, S^p - \check{S} - K_2)} \]

\[
= \text{Spec} \frac{\mathcal{O}_{(E \cup \Delta X)}[\check{T}, \check{S}]}{(T^p - \check{T} - K_1, S^p - \check{S} - K_2)}
\]

\[
= \prod_{\check{t}, \check{s}} (E \cup \Delta X)_{\check{t}, \check{s}}
\]

Therefore

\[ Z_0 \times_{(X \times X)^{(kD)}} (E \cup \Delta X) = Z_0 \times W_0 \prod_{\check{t}, \check{s}} (E \cup \Delta X)_{\check{t}, \check{s}} = \prod_{\check{t}, \check{s}} \tilde{F}_{\check{t}, \check{s}} \]

\[ \square \]

**Theorem 3.2.** Take a suitable \( k \), then we have

(1). Over \( E_{0,0} \), \( \tilde{F} \) is a Galois covering defined by an Artin-Schreier equation, or splits to \( p \)'s connected components; over \( \Delta X_{0,0} \), it splits \( p \)'s connected components.

(2). In the commutative diagram

\[
\begin{array}{ccc}
Z_1 & \longrightarrow & Z_0 \\
\downarrow & & \downarrow \\
W_1 & \longleftarrow & W_0 \\
\phi_1 & & \phi_0 \\
U \times U & \xrightarrow{j} & (X \times X)^{(R)} \quad i^+ \quad E \cup \Delta X \quad \leftarrow (E \cup \Delta X)
\end{array}
\]

the base change map \( \psi_{31}^{32*} \psi_{11}^{31*} \psi_{11}^{01*} \longrightarrow \psi_{12}^{02*} \psi_{12}^{01*} \) is a isomorphism for constructible sheaf. Where we use \( \psi_{k_1}^{i_1} \) to denote the unique morphism from the object at site \( (i, j) \) to the object at site \( (k, l) \) (if it exists). For example \( \psi_{10}^{00} = j, \psi_{10}^{20} = i^+ \).

**Proof.**

(1). We can see

\[ Z_1 = \text{Spec} \frac{\mathcal{O}_{W_1}[\check{U} - \check{ST}]}{((U - ST)^p - (U - ST)^q - K_3)} \]

where \( K_3 = (K_1 + f \otimes 1)\check{S} - g \otimes 1(\check{T} + K_1) + K_4 \).

Let \( \check{W}_0 = W_0 \setminus \prod_{\check{t}, \check{s} \neq (0, 0)} (E \cup \Delta X)_{\check{t}, \check{s}} \). Then \( \check{W}_0 = Z_0 \times W_0 = Z_0 \setminus \prod_{\check{t}, \check{s} \neq (0, 0)} \tilde{F}_{\check{t}, \check{s}} \). Then \( \check{Z}_0 \) is the normalization of \( Z_1 \) over \( \check{W}_0 \). Let \( \eta \) and \( \eta_{0,0} \) be the generic point of \( E \) and \( E_{0,0} \)
respectively, then there is a unique extension $v_{E_0,0}$ on the stalk $\mathcal{O}_{\tilde{W}_0,\eta}$ of valuation $v_E$, such that the restriction of $v_{E_0,0}$ on $\mathcal{O}_{(X \times X)(\mathcal{O}),\eta}$ is just $v_E$.

We have

$$K_3 = (K_1 + f \otimes 1)(\hat{S}^p - K_2) - g \otimes 1(\hat{T} + K_1) + K_4$$

$$= K_1\hat{S}^p + f \otimes 1\hat{S}^p - K_1K_2 - g \otimes 1\hat{T}^p + K_4 - f \otimes 1K_2$$

where $v_{E_0,0}(f \otimes 1) = -n$, $v_{E_0,0}(g \otimes 1) = -m$, $v_{E_0,0}(K_4) \geq k - r$ by lemma 2.2, and $v_{E_0,0}(K_1) = k - n$, $v_{E_0,0}(K_2) = k - m$ by proposition 2.2. Moreover, we have $v_{E_0,0}(\hat{T}) = k - n$, and $v_{E_0,0}(\hat{S}) = k - m$, because $\hat{T}^p - \hat{T} - K_1 = 0$ and $\hat{S}^p - \hat{S} - K_2 = 0$. Therefore we can take enough big $k$ such, that $v_{E_0,0}(K_3) \geq 0$, then $K_3 \in \Gamma(\tilde{W}_0,\mathcal{O}_{\tilde{W}_0})$. Therefore

$$\tilde{Z}_0 = \text{Spec } \mathcal{O}_{\tilde{W}_0}[\hat{U} - \hat{ST}] / ((\hat{U} - \hat{ST})^p - (U - ST) - K_3)$$

We can see $\tilde{Z}_0$ is a Galois covering of $\tilde{W}_0$ and

$$F_{0,0} = \tilde{Z}_0 \times \tilde{W}_0 E_{0,0} = \text{Spec } \mathcal{O}_{E_{0,0}}[R] / (R^p - R - K_3)$$

is a Galois covering of $E_{0,0}$.

On the other hands, it easy to see $K_3$ vanish on $\Delta U_{0,0}$, then on $\Delta X_{0,0}$. Therefore

$$\tilde{Z}_0 \times \tilde{W}_0 \Delta X_{0,0} = \text{Spec } \mathcal{O}_{\Delta X_{0,0}}[R] / (R^p - R) = \prod_r \Delta X_{0,0,r}$$

(2). We have

$$\psi_{31}^{32} \psi_{11}^{31} \psi_{11}^{01} = \psi_{31}^{32} \psi_{11}^* \psi_{11}^* \psi_{11}^*$$

$$= \psi_{31}^{32} \psi_{11}^* \psi_{11}^* \psi_{11}^* \quad (\psi_c \text{ is a open immersion})$$

$$= \psi_{12}^{32} \psi_{12}^{02} \psi_{01}^{02} \quad \text{(smooth base change)}$$

$$= \psi_{12}^{32} \psi_{12}^{02} \psi_{01}^{02} \quad (\psi_c \text{ is a open immersion})$$
Corollary 3.3. For constructible sheaf, The base change map $\phi_{51}^{\psi_{11}^*} \phi_{11}^{\psi_{111}^*} \phi_{111}^{\psi_{1111}^*} \rightarrow \phi_{12}^{\psi_{12}^*} \phi_{12}^{\psi_{1212}^*}$ $\phi_{1212}^{\psi_{121212}^*}$ is an isomorphism.

\[ (1) \]

Corollary 3.4. Take a suitable k, we have a commutative diagram
Denote. Let \( \phi_{i,j,k}^{p,q,r} \) denote the unique map from the object at site \((i,j,k)\) to the object at site \((p,q,r)\) (if it exists) of the diagram \( \mathcal{D} \). For example \( \phi_{000}^{000} = i^*, \phi_{000}^{010} = j, \phi_{000}^{011} = \phi_1, \) and \( \phi_{000}^{410} = j_U. \)

4 \text{ } l\text{-adic sheaves of rank } p

\( G \) have a normal subgroup \( H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\} \). For any character \( \chi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Q}_l^* \), we define a character

\[
\tilde{\chi} : H \rightarrow \mathbb{Q}_l^* \quad \mapsto \quad \chi(c)
\]

We denote the induced representation \( \text{Ind}_H^G \tilde{\chi} \) by \( \rho \). In fact, we can see

\[
\rho : G \rightarrow GL(\mathbb{Q}_l^p) \quad \text{ maps } \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \chi(c) & & & \\ & \chi((p-b)a+c) & & \\ & & \ddots & \\ & & & \chi((p-1)a+c) \end{pmatrix}
\]

This defines a locally constant \( \mathbb{Q}_l \)-sheaf \( \mathcal{F} \) of rank \( p \) on \( U \).

**Proposition 4.1.** Under the setting of the diagram

\[
\begin{align*}
U \times U & \xrightarrow{j} E \\
E & \xrightarrow{j_u} \Delta U
\end{align*}
\]

There exists a sheaf \( \mathcal{L} \) satisfying the following conditions

1. \( \mathcal{L} \) is embedding in \( i^* \circ j_* \mathcal{H} \).
2. \( \mathcal{L}|_E \) is a constant sheaf or a locally constant sheaf defined by an Artin-Schreier equation which constant term a linear form on \( E \).
3. \( \mathcal{L}|_{\Delta X} = j_U \circ \mathbb{Q}_l \text{id}. \)

**Proof.**
(1). \( \mathcal{H} \) is determined by the action of \( \pi_1(U \times U) \) on \( \text{Hom}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}'_l) \) as follow:

\[
\pi_1(U \times U) \quad \rightarrow \quad \text{Gal}(V \times V/U \times U) \quad \rightarrow \quad \text{GL}(\text{Hom}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}'_l)) \quad \rightarrow \quad P' \times \times P^{-1}
\]

where

\[
P = \left[ \begin{array}{ccc}
\chi((p-b)a+c) \\
\chi(c) \\
\vdots \\
\chi((p-b-1)a+c)
\end{array} \right], \\
P' = \left[ \begin{array}{ccc}
\chi((p-b')a'+c') \\
\chi(c') \\
\vdots \\
\chi((p-b'-1)a'+c')
\end{array} \right]
\]

For any \( \sigma \in \pi_1(W_1) \), the image of \( \sigma \) in \( \text{Gal}(V \times V/U \times U) \) is in fact contained in \( \text{Gal}(V \times V/W_1) \). Therefore we can write the image by \( \left( \begin{array}{ccc} 1 & a & c \\
0 & 1 & b \\
0 & 0 & 1 \end{array} \right), \quad \left( \begin{array}{ccc} 1 & a' & c' \\
0 & 1 & b' \\
0 & 0 & 1 \end{array} \right) \).

We can see the image of the identity element \( I \in \text{Hom}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}'_l) \) under the action of \( \sigma \) is \( \chi(c - c)I \). It follow that \( \overline{\mathbb{Q}}_lI \) is a \( \overline{\mathbb{Q}}_l[\pi_1(W_1)] \)-submodule of \( \text{Hom}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}'_l) \). Therefore we got a smooth subsheaf \( D \) of \( \phi_1^* \mathcal{H} \) which is trivialized by \( Z_1 \).

Let functor \( \phi_{01}^{201*} \phi_{001*}^{011} \) act on \( 0 \rightarrow D \rightarrow \phi_1^* \mathcal{H} \), we get

\[
0 \rightarrow \phi_{001}^{201*} \phi_{001}^{011} D \rightarrow \phi_{001}^{201*} \phi_{001}^{011} \phi_1^* \mathcal{H}
\]

But we know the base change map \( \phi_{000}^{011} j, \mathcal{H} \rightarrow \phi_{001}^{011} \phi_1^* \mathcal{H} \) is a isomorphism, because \( \phi_{000}^{011} \) is Galois, so is smooth. Therefore we have

\[
\phi_{001}^{201*} \phi_{001}^{011} \phi_1^* \mathcal{H} = \phi_{001}^{201*} \phi_{001}^{011} \phi_{000}^{011} j, \mathcal{H} = \phi_{000}^{201*} \phi_{000}^{011} j, \mathcal{H} = i^{++} j, \mathcal{H}
\]

Let \( \mathcal{L} = \phi_{001}^{201*} \phi_{001}^{011} \mathcal{D} \), then \( \mathcal{L} \) is embedded to \( i^{++} j, \mathcal{H} \).

(2). Consider the diagram \( (1) \). We have

\[
\mathcal{L}_{|E_{0,0}|F_{0,0}} = (\psi_{11}^{34*} \psi_{11}^{01*} \mathcal{D})_{|E_{0,0}|F_{0,0}} = (\psi_{11}^{34*} \psi_{11}^{01*} \mathcal{D})_{|F_{0,0}} = (\psi_{12}^{32*} \psi_{12}^{02*} \mathcal{D})_{|F_{0,0}} \quad \text{(by theorem \( (2) \))}
\]

But \( \psi_{01}^{02*} \mathcal{D} \) is a constant sheaf, then \( \mathcal{L}_{|E_{0,0}|F_{0,0}} \) is a constant sheaf also.

(3). Refer to the diagram \( (2) \), we have

\[
\phi_{01}^{402*} \phi_{001}^{011} \mathcal{D} = \phi_{002}^{402*} \phi_{002}^{012*} \phi_{011}^{012*} \mathcal{D}
\]
by corollary 3.3. On the other hand, it easy to see,  
\[ \phi_{401}^* \phi_{401}^* \phi_{011}^* D = \phi_{402}^* \phi_{012}^* \phi_{011}^* D \]

But \( \phi_{011}^* D \) is a constant sheaf, then  
\[ \phi_{402}^* \phi_{002}^* \phi_{012}^* \phi_{011}^* D = \phi_{402}^* \phi_{012}^* \phi_{011}^* D \]

Therefore  
\[ \phi_{402}^* \phi_{401}^* \phi_{001}^* \phi_{011}^* D = \phi_{402}^* \phi_{401}^* \phi_{011}^* D \]

i.e
\[ L_{|\Delta X} = j_{U!} \overline{\mathbb{Q}_l} \text{id} \]

\[ \square \]

Remark 4.2. In [9], the rsw is defined to a injective ([9], Corollary 1.3.4):
\[ rsw : \text{Hom}(G_{r, K}^a G_K, \mathbb{F}_p) \rightarrow \Omega^1_F(\log) \otimes F \text{m}_K^{(-k)}/\text{m}_K^{(-k)+} \]

where \( K \) is the henselization of the stalk of \( \mathcal{O}_X \) at the generic point of \( D \). \( F \) is its residue fields, \( \overline{K} \) is a separable closure of \( K \). \( \text{m}_K^{(-k)} = \{ a \in \overline{K} | v_K(a) \geq -k \} \) and \( \text{m}_K^{(-k)+} = \{ a \in \overline{K} | v_K(a) > -k \} \). There is a filtration \( (G_{r, K}^a)_{a \in \mathbb{Q}_{\geq 0}} \) of \( G_K = \text{Gal}(\overline{K}/K) \), and \( Gr_{r, K}^a G_K \) is the graded pieces \( (G_{K, \log}^a/G_{K, \log}^{a+}). \)

In our case, we have a morphism of filter
\[
\begin{align*}
G_{r, K, \log}^0 & \supset G_{r, K, \log}^k & \supset G_{r, K, \log}^{k+} \\
G & \supset G^k & \supset 1
\end{align*}
\]

and \( Gr_{r, K, \log}^k G_K = G^k \), where
\[
G^k = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bigg| c \in \mathbb{Z}/p\mathbb{Z} \right\} \simeq \mathbb{Z}/p\mathbb{Z}
\]

\( \rho \) is a Galois representation of dimension \( p \), whose unramified on \( U \). Its restriction on \( Gr_{r, K, \log}^k G_K \) factors by
\[
\begin{align*}
Gr_{r, K, \log}^k G_K & \rightarrow GL(\mathbb{Q}_l^p) \\
\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mapsto \chi(c)I_p
\end{align*}
\]

This is a direct sum of \( p \)'s character \( \chi : Gr_{r, K, \log}^r G_K \rightarrow \mathbb{Q}_l^p \). \( \chi \) can be regarded as a element of \( \text{Hom}(Gr_{r, K, \log}^r G_K, \mathbb{F}_p) \), therefore we can define the refined Swan conductor of \( \mathcal{F} \) by \( rsw(\chi) \).
Now, let us consider the commutative diagram following:

\[ \tilde{W}_0 \] \quad \tilde{\psi} \quad \delta \\
\tilde{i} \quad (X \times X)^{(kD)} \quad \delta \\
\tilde{i} \quad \Delta X \\
E \quad \theta \quad D

where \( \tilde{W}_0 = W_0 \setminus \bigsqcup_{(i,s) \neq (0,0)} (E \cup \Delta X)_{(i,s)} \), and \( \tilde{i}, \tilde{\delta} \) are the only liftings of \( i, \delta \) respectively.

Let the ideal sheaf of \( \Delta X \) in \( \tilde{W}_0 \) is \( \tilde{J}_{\Delta X} \), then we have a commutative diagram

\[ \Gamma(\tilde{W}, \tilde{J}_{\Delta X}) \rightarrow \Gamma(\Delta X, \delta^* \tilde{J}_{\Delta X}) \]
\[ \Gamma(E, i^* \tilde{J}_{\Delta X}) \rightarrow \Gamma(D, i^*_D \delta^* \tilde{J}_{\Delta X}) \]

However, \( \tilde{J}_{\Delta X} = \tilde{\psi}^* J_{\Delta X} \). Hence

\[ \tilde{i}^* J_{\Delta X} = i^* J_{\Delta X}, \quad \tilde{\delta}^* J_{\Delta X} = \delta^* J_{\Delta X} \]

Therefore, we have a commutative diagram

\[ \Gamma(\tilde{W}, \tilde{J}_{\Delta X}) \rightarrow \Gamma(\Delta X, \delta^* J_{\Delta X}) \rightarrow \Gamma(\Delta X, \Omega^1_{\Delta X}(logD)(kD)) \]
\[ \Gamma(E, I_D) \rightarrow \Gamma(D, i^*_D \delta^* J_{\Delta X}) \rightarrow \Gamma(D, i^*_D \Omega^1_{\Delta X}(logD)(kD)) \]

By Corollary 1.3.4 and the proof of Theorem 1.3.3 in [5], when \( \mathcal{L}|_E \) is a locally constant sheaf defined by an Artin-Schreier equation \( R^n - R - K_3 \), the image of \( K_3 \) in \( \Gamma(D, i^*_D \Omega^1_{\Delta X}(logD)(kD)) \) is the \( rs \mathcal{W}(F) \).

**Theorem 4.3.** We can compute \( rs \mathcal{W}(F) \) as follow:

1. If \( r > m + n \), then \( rs \mathcal{W}(F) = dh \in \Gamma(D, i^*_D \Omega^1_{\Delta X}(logD)(kD)) \), where \( k = r \).
2. If \( r < m + n \), then \( rs \mathcal{W}(F) = -fdg \in \Gamma(D, i^*_D \Omega^1_{\Delta X}(logD)(kD)) \), where \( k = m + n \).
(3). If \( r = m + n, \) and \( n \geq m, \) and \( dh - f dg \in \Omega^1_X((\log D)(kD))_\xi \setminus \Omega^1_X((\log D)(kD^{-1}))_\xi \) for some \( k > n + \frac{p}{e} \), then \( rsw(\xi) = dh - f dg \in \Gamma(D, i_D^*\Omega^1_{\Delta X}(\log D)(kD)). \)

(4). If \( r = m + n, \) and \( m \geq n, \) and \( dh - f dg \in \Omega^1_X((\log D)(kD))_\xi \setminus \Omega^1_X((\log D)(kD^{-1}))_\xi \) for some \( k > m + \frac{p}{e} \), then \( rsw(\xi) = dh - f dg \in \Gamma(D, i_D^*\Omega^1_{\Delta X}(\log D)(kD)). \)

**Proof.** We have

\[
K_3 = (K_1 + f \otimes 1)(\check{S}^p - K_2) - g \otimes 1(\check{T} + K_1) + K_4
\]

\[
= K_1 \check{S}^p + f \otimes 1\check{S}^p - K_1 K_2 - g \otimes 1\check{T}^p + K_4 - f \otimes 1K_2
\]

where \( v_{E_0,0}(f \otimes 1) = -n, v_{E_0,0}(g \otimes 1) = -m, \) and \( v_{E_0,0}(K_1) = k - r, v_{E_0,0}(K_2) = k - n, v_{E_0,0}(K_3) = k - m \) by proposition (2.2). Moreover, we have \( v_{E_0,0}(\check{T}) = k - n, \) and \( v_{E_0,0}(\check{S}) = k - m, \) because \( \check{T}^p - \check{T} - K_1 = 0 \) and \( \check{S}^p - \check{S} - K_2 = 0. \) We can abuse \( v_{E_0,0} \) and \( v_E, \) and write the valuation of terms of \( K_3 \) in the following table

| term \( v_E \)                  | \( K_1 \check{S}^p \) | \( f \otimes 1\check{S}^p \) | \( K_1 K_2 \) | \( g \otimes 1\check{T}^p \) | \( K_4 \) | \( f \otimes 1K_2 \) |
|-------------------------------|------------------------|-----------------------------|--------------|--------------------------|--------|---------------------|
| \( k - n + p(k - m) \)        | \( p(k - m) - n \)      | \( 2k - m - n \)            | \( p(k - n) - m \) | \( k - r \)  | \( k - m - n \) |

Therefore,

(1). In this situation, \( v_E(K_4) = 0, \) and \( v_E(K_1 \check{S}^p + f \otimes 1\check{S}^p - K_1 K_2 - g \otimes 1\check{T}^p - f \otimes 1K_2) > 0. \) Hence \( K_3 \in \Gamma(E, \mathcal{L}_D) \). The image of \( K_3 \) in \( \Gamma(D, i_D^*\Omega^1_{\Delta X}(\log D)(kD)) \) is \( dh \).

(2). In this situation, \( v_E(-f \otimes 1K_2) = 0, \) and \( v_E(K_1 \check{S}^p + f \otimes 1\check{S}^p - K_1 K_2 - g \otimes 1\check{T}^p + K_4) > 0. \) Hence \( K_3 = -f \otimes 1K_2 \in \Gamma(E, \mathcal{L}_D) \). The image of \( -f \otimes 1K_2 \) in \( \Gamma(D, i_D^*\Omega^1_{\Delta X}(\log D)(kD)) \) is \( -fdg \).

(3) By the following lemma, we know \( v_E(K_4 - f \otimes 1K_2) \geq 0. \) In other words, we know also

\[
v_E(K_1 \check{S}^p + f \otimes 1\check{S}^p - K_1 K_2 - g \otimes 1\check{T}^p) > 0
\]

Hence \( K_3 = K_4 - f \otimes 1K_2 \in \Gamma(E, \mathcal{L}_D) \). The image of \( K_4 - f \otimes 1K_2 \) in \( \Gamma(D, i_D^*\Omega^1_{\Delta X}(\log D)(kD)) \) is \( dh - f dg. \) \( dh - f dg \) is not 0 as an element in \( \Gamma(D, i_D^*\Omega^1_{\Delta X}(\log D)(kD)) \), because it is not in \( \Omega^1_X((\log D)(kD^{-1}))_\xi \). Therefore \( K_4 \) is not 0 in \( \Gamma(E, \mathcal{L}_D) \).

(4). It is similar to (3).  

**Lemma 4.4.** Under the condition in (3) of previous theorem, we have \( v_E(K_4 - f \otimes 1K_2) \geq 0. \)

**Proof.** Let us consider the diagram

\[
\begin{array}{ccc}
(X \times X)^{(kD)} & \xleftarrow{(\Delta X \cup kE)} & kE \\
\phi^k & \downarrow & \delta^k \\
(X \times X)^{\Delta X} & \xleftarrow{\delta^k} & \Delta X
\end{array}
\]

Let \( i^k \) be the morphism from \( kE \) to \( (X \times X)^{(kD)} \), then we have a commutative diagram

\[
\begin{array}{cccc}
0 & \xrightarrow{\mathcal{O}_{(X \times X)^{(kD)}}(-kE)} & \mathcal{O}_{(X \times X)^{(kD)}} & \xrightarrow{i^k_\mathcal{O}} \mathcal{O}_{kE} & 0 \\
0 & \xrightarrow{\mathcal{O}_{(X \times X)^{(kD)}}((r-2k)E)} & \mathcal{O}_{(X \times X)^{(kD)}} & \xrightarrow{i^k_\mathcal{O}} \mathcal{O}_{kE} & 0
\end{array}
\]
of sheaves of $\mathcal{O}_{(X \times X)^{(k, D)}}$ module, and a commutative diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & T_{\Delta X}^2(kE^0) & \rightarrow & T_{\Delta X}(kE^0) & \rightarrow & \delta^0_*(\Omega^1_{\Delta X}(\log D)(kD)) & \rightarrow & 0 \\
0 & \rightarrow & T_{\Delta X}^2(rE^0) & \rightarrow & T_{\Delta X}(rE^0) & \rightarrow & \delta^0_*(\Omega^1_{\Delta X}(\log D)(rD)) & \rightarrow & 0 \\
\end{array}
$$

of sheaves of $\mathcal{O}_{(X \times X)^{\sim}}$ module. Where where $E^0$ is the complement of $U \times U$ in $(X \times X)^{\sim}$ as a reduced scheme, and $T_{\Delta X}$ is the ideal sheaf of $\Delta X$ in $(X \times X)^{\sim}$. Let $J_{\Delta X}$ be the ideal sheaf of $\Delta X$ in $(X \times X)^{(k, D)}$, then we have a morphism

$$
T_{\Delta X} \rightarrow \phi^*_{\Delta X}J_{\Delta X}(-kE) \rightarrow \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(k, D)}}(-kE)
$$

of sheaves of $\mathcal{O}_{(X \times X)^{\sim}}$ module. Therefore we have a commutative diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & T_{\Delta X}^2(kE^0) & \rightarrow & T_{\Delta X}(kE^0) & \rightarrow & \delta^0_*(\Omega^1_{\Delta X}(\log D)(kD)) & \rightarrow & 0 \\
0 & \rightarrow & \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(k, D)}}(-kE) & \rightarrow & \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(k, D)}}(-kE) & \rightarrow & \phi^*_{\Delta X}\mathcal{O}_{kE} \\
0 & \rightarrow & T_{\Delta X}^2(rE^0) & \rightarrow & T_{\Delta X}(rE^0) & \rightarrow & \delta^0_*(\Omega^1_{\Delta X}(\log D)(rD)) & \rightarrow & 0 \\
0 & \rightarrow & \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(r, D)}}((r - 2k)E) & \rightarrow & \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(r, D)}}((r - k)E) & \rightarrow & \phi^*_{\Delta X}\mathcal{O}_{kE}((r - k)E) \\
\end{array}
$$

of sheaves of $\mathcal{O}_{(X \times X)^{\sim}}$ module, where the morphisms

$$
\delta^0_*(\Omega^1_{\Delta X}(\log D)(kD)) \rightarrow \phi^*_{\Delta X}\mathcal{O}_{kE}, \quad \delta^0_*(\Omega^1_{\Delta X}(\log D)(rD)) \rightarrow \phi^*_{\Delta X}\mathcal{O}_{kE}((r - k)E)
$$

are induced by

$$
T_{\Delta X}(kE^0) \rightarrow \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(r)}}, \quad T_{\Delta X}(rE^0) \rightarrow \phi^*_{\Delta X}\mathcal{O}_{(X \times X)^{(r)}((r - k)E)}
$$

respectively.
Take the global sections, we have

\[ \Gamma((X \times X)_{\Delta X}(kE^0)) \rightarrow \Gamma(\Delta X, \Omega^1_{\Delta X}(\log D)(kD)) ](\Delta X, \Omega^1_{\Delta X}(\log D)(rD)) \rightarrow \Gamma(kE, \mathcal{O}_{kE}) \\
\Gamma((X \times X)^{(kD)}_{(X \times X)^{(kD)}}) \rightarrow \Gamma(kE, \mathcal{O}_{kE}(rE_0)) \rightarrow \Gamma(kE, \mathcal{O}_{kE}((r - k)E)) \\
\Gamma((X \times X)^{(kD)}_{(X \times X)^{(kD)}}) \rightarrow \Gamma(kE, \mathcal{O}_{kE}((r - k)E)) \\
\Gamma((X \times X)\sim_{\Delta X}(rE_0)) \rightarrow \Gamma(\Delta X, \Omega^1_{\Delta X}(\log D)(rD)) \rightarrow \Gamma(kE, \mathcal{O}_{kE}(rE_0)) \\
\Gamma((X \times X)^{(kD)}_{(X \times X)^{(kD)}}) \rightarrow \Gamma(kE, \mathcal{O}_{kE}((r - k)E)) 

Now, \( K_4 \) and \( f \otimes 1K_2 \) is in \( \Gamma((X \times X)^\sim_{\Delta X}(rE_0)) \), the image \( dh - fdg \) of \( K_4 - f \otimes 1K_2 \) in \( \Gamma(\Delta X, \Omega^1_{\Delta X}(\log D)(rD)) \) is in fact inside \( \Gamma(\Delta X, \Omega^1_{\Delta X}(\log D)(kD)) \). Hence its image \( \overline{K}_4 - \overline{f} \otimes \overline{1K}_2 \) in \( \Gamma(kE, \mathcal{O}_{kE}(rE_0)) \) is in fact inside \( \Gamma(kE, \mathcal{O}_{kE}((r - k)E)) \). Therefore \( v_E(K_4 - f \otimes 1K_2) \geq 0 \).

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References

[1] Ahmed Abbes, Takeshi Saito, The characteristic class and ramification of an l-adic étale sheaf. Inven. math. 168(2007), 567-612.

[2] Kazuya Kato, Swan conductors for characters of degree one in the imperfect residue field case. Contemp. Math. 83(1989), 101-132.

[3] Kazuya Kato, Class field theory, \( \mathcal{D} \)-modules, and ramification on higher dimensional schemes, part I. American Journal of Mathematics 116 (1994), 757-784.

[4] Kazuya Kato, Takeshi Saito, Ramification theory for varieties over a perfect field. [math.AG/0402010](http://arxiv.org/abs/math.AG/0402010) revised on 05/05/16 to appear at Annals of Mathematics.

[5] Takeshi Saito, Wild ramification and the characteristic cycle of an l-adic sheaf. [arXiv:0705.2799](http://arxiv.org/abs/0705.2799)