Switching–GAS Copula Models for Systemic Risk Assessment

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Abstract

Recent financial disasters have emphasised the need to accurately predict extreme financial losses and their consequences for the institutions belonging to a given financial market. The ability of econometric models to predict extreme events strongly relies on their flexibility to account for the highly nonlinear and asymmetric dependence observed in financial returns. We develop a new class of flexible Copula models where the evolution of the dependence parameters follow a Markov–Switching Generalised Autoregressive Score (SGASC) dynamics. Maximum Likelihood estimation is consistently performed using the Inference Functions for Margins (IFM) approach and a version of the Expectation–Maximisation (EM) algorithm specifically tailored to this class of models. The SGASC models are then used to estimate the Conditional Value–at–Risk (CoVaR), which is defined as the VaR of a given asset conditional on another asset (or portfolio) being in financial distress, and the Conditional Expected Shortfall (CoES). Our empirical investigation shows that the proposed SGASC models are able to explain and predict the systemic risk contribution of several European countries. Moreover, we also find that the SGASC models outperform competitors using several CoVaR backtesting procedures.

Keywords: Markov–Switching, Generalised Autoregressive Score, Dynamic Conditional Score, Risk measures, Conditional Value–at–Risk, Conditional Expected Shortfall.

1. Introduction

Recent financial disasters have emphasised the need to accurately predict extreme financial losses and their consequences for the institutions’ financial health.
and, more generally, for the safety of the broader economy. Major financial crisis, such as the Global Financial Crisis (GFC) of 2007–2008, usually spread over the whole economy leading to sharp economic downturns and recessions. During huge crisis episodes the failure of banks and financial institutions is not rate and may trigger other non-financial institutions through the balance sheet and liquidity channels, threatening the stability of real economy, see, e.g., Adrian and Brunnermeier (2014), Adrian and Shin (2010), Brunnermeier and Pedersen (2009), Brunnermeier et al. (2009) and Brunnermeier (2009). The ability of econometric models to predict such extreme events strongly relies on their flexibility to model the highly nonlinear and asymmetric dependence observed in financial returns, see e.g., Nelsen (2007), Joe (2014) and Durante and Sempi (2015). Concerning the dependence structure of stock returns, it is well recognised that the simple linear correlation fails to capture the important tails behaviour of the joint probability distribution, see, e.g., McNeil et al. (2005) and Embrechts et al. (1999, 2002). Hence, modelling the tail dependence and the asymmetric dependence between pairs of assets becomes really important in a multivariate environment, especially after the recent GFC. The departure from the linear correlation as a measure of dependence usually implies to go beyond the multivariate Elliptical assumption for the joint distribution of asset returns. In this respect, the Copula functions allows to model a huge variety of dependence structures, see Cherubini et al. (2011).

This paper studies extreme tail co-movements among assets using the Copula methodology that overcomes the traditional limitations of linear correlation ensuring the identification of complicated nonlinear dynamics between financial assets. Another interesting feature of the dependence behaviour of stock returns, which is particularly crucial in measuring extreme co-movements, is that it usually evolves smoothly over time as a function of the past co-movements, see, e.g., Engle (2002a) and Tse and Tsui (2002). Due to the exposure of common shocks affecting all the market participants, the conditional correlations between asset returns increase during periods of financial instability, see, e.g., Kotkatvuori-Ornberg et al. (2013), Sandoval Junior and De Paula Franca (2012), Syllignakis and Kouretas (2011). Dynamic Copula models are referred to Patton (2006a) and Jondeau and Rockinger (2006), although the problem of modelling the joint co-movement of stock returns was already present in Bollerslev et al. (1988), Bollerslev (1990) and Engle et al. (1990), among others. Moreover, occasionally, we observe breaks into the dependence structure, which are more evident during crises periods and other infrequent events, as documented, for example, by Bernardi et al. (2013b) and Bernardi and Petrella (2015). Markov switching (MS) models have been proven to effectively capture breaks into the volatility and the dependence of stock returns. MS Copula with static regime-dependent parameters have been firstly employed to analyse financial contagion by Pelletier (2006), Chollete et al. (2009) and Rodriguez (2007).

We propose to model the regime dependent dynamic of the Copula parameters using the score driven framework recently proposed by Creal et al. (2013) and Harvey (2013). More precisely, we allow the Copula dependence parameters
to depend on the realisations of a Markovian process with a specific Generalised Autoregressive Score (GAS) dynamic in each regime, while retaining an appropriate GJR–GARCH model of Glosten et al. (1993) for the marginals’ volatility dynamics. In this way, we extend the GAS literature by introducing high nonlinearity and a stochastic behaviour of the dependence structure between financial indexes. We name this new class of models Switching Generalised Autoregressive Score Copula (SGASC) models. During last few years, many econometricians have been starting to use the conditional score approach to model the time varying behaviour of unobservable parameters in an observation driven environment. The use of the score has been justified in several ways by the literature. Harvey (2013), for example, represents the score process as a filter of an unobservable component model, while Creal et al. (2013) suggest that the use of the score as an updating mechanism for the parameter dynamic, can be interpreted as a steepest ascent direction for improving the model’s local fit given the current parameter position, as it usually happens into a Newton–Raphson algorithm. More recently, Blasques et al. (2015) and Blasques et al. (2014b) show some optimality criterion in favour of score driven process for a general class of nonlinear autoregressive dynamic. They argue that, only the GAS process is optimal in the sense of reducing the local Kullback–Leibler divergence between the true and the model implied conditional densities. They also argue that, this is true irrespective to the level of the possible model misspecification. Several theoretical results for maximum likelihood estimate of score process has been developed by Blasques et al. (2014c), Andres (2014), Blasques et al. (2014a) and Harvey (2013). Moreover, the high degree of flexibility of GAS models implies that the theoretical results recently obtained can be effectively used in contexts where dynamic models are special cases of GAS processes.

Score driven processes have been proved to be effectively used in many empirical applications. Most applications are focused on the volatility modelling, as for example in Harvey and Luati (2014), Harvey and Sucarrat (2014), Caivano and Harvey (2014) and Creal et al. (2011a). The use of score dynamics for volatility modelling helps in reducing the effect of outlier (or extreme event, like financial crisis) to the variance dynamic just assuming a fat tail distribution for the conditional innovation. This happens because the score driven process treats the new observation used to update the dynamic as coming from the hypothesised (fat tailed) distribution and not from a Gaussian distribution as usually happens in the GARCH context. Other empirical applications are in the systemic risk measurement as, for example, in Blasques et al. (2014d), Lucas et al. (2014a) and Oh and Patton (2013), in the credit risk analysis, Creal et al. (2011b), in macroeconomics, Massacci (2014) and Bazzi et al. (2014), and in dependence modelling field, as in Harvey and Thiele (2014), Janus et al. (2014), and De Lira Salvatierra and Patton (2015). More generally, Koopman et al. (2015) show that the GAS filter well approximate complicated nonlinear data generating processes in a straightforward and effective way. In the context of dependence modelling through Copulas, the use of score driven models really helps in cases where it is not clear how to update the parameter dynamics as for example in dynamic archemedian Copulas.
One of the main appealing characteristics of the Copula framework, with respect to standard distributions, relies on its ability to model the marginals’ dynamics separately from the joint dependence structure, see e.g., Nelsen (2007). The marginals and dependence separability has some additional advantages even from the econometric point of view, in the sense that it permits to employ a two-step procedure to estimate the marginal parameters and the dependence structure separately. This two-step procedure is known as Inference Function for margins (IFM) and is usually referred to Godambe (1960), Godambe (1976), Godambe (1991), and McLeish and Small (1988). To estimate the SGASC model parameters, we adapt the IFM two-step procedure of Patton (2006a) to the MS dynamics. The procedure allows us to estimate the marginals’ parameters for the volatility dynamics in the first step, while the Copula GAS parameters modelling the joint dependence are estimated by adapting the Expectation–Maximization algorithm of Dempster et al. (1977).

The empirical part of the work concerns about the comparison of several SGASC model specifications for the purpose of evaluating the systemic risk contributions of each European country and their evolutions during the recent financial crises of 2007–2008 and the European sovereign debt crisis of 2010. For the proposed SGASC model we introduce and estimate the Conditional Value–at–Risk (CoVaR) and the Conditional Expected Shortfall (CoES) risk measures, recently proposed by Adrian and Brunnermeier (2011, 2014) and Girardi and Ergün (2013). The CoVaR measure co–movements between any two distinct institutions by extending the Value–at–Risk (VaR) to a conditional approach. Following the CoVaR methodology, the risk of an institutions is evaluated as its VaR conditional to a relevant extreme event affecting the another institution. The appealing characteristic of the CoVaR risk measure is that it inherits the flexibility of the dynamic switching Copula framework here developed. The Copula approach naturally adapts to environments characterised by different kind of upper and lower tail dependence enabling the CoVaR as an effective measure of the extreme conditional co–movements among financial variables. The literature on co–movement risk measures has proliferated during the last few years, see, e.g., Bernardi et al. (2015), Bernal et al. (2014), Castro and Ferrari (2014), Girardi and Ergün (2013), Jäger-Ambroziewicz (2013), Sordo et al. (2015), Bisias et al. (2012) provide an extensive and up to date survey of the systemic risk measures that have been recently proposed. Our analysis confirm that the proposed SGASC model is able to explain and predict the systemic risk evolution of the considered countries in an effective way. In particular, we find results similar to those recently obtained by by Engle et al. (2015) using a modified version of the Marginal Expected Shortfall risk measure, Lucas et al. (2014b) who consider the European Sovereign debt CDS market. We also report some evidence on which Copula is more adequate to describe the systemic relation between the specific countries and the overall European economy. We find that the ability of the Copula function to reproduce negative tail dependence plays a fundamental role in analysing the systemic risk propagation.

The remainder of the paper is organised in the following manner. In Section 2 we present the model. We discuss the marginal models and then we detail
the general Markov–switching framework to model the joint behaviour of the series. We introduce the dynamic model for the Copula dependence parameters in Section 3 as well as several previously proposed alternative dynamics. Section 4 deals with the two–step estimation methodology and presents the EM algorithm to estimate the GAS parameters. Section 5 introduces the co–movement systemic risk measures. Section 6 presents data and discuss the main empirical results. Section 7 concludes.

2. The Model

In this section, we first introduce the general framework to model the univariate marginal return series and, then, we present the bivariate MS Dynamic Copula model. The introduction of the model for the dynamic evolution of the Copula dependence parameters conditional to the Markovian variable is instead postponed to Section 3. Multivariate MS Copula models have been previously considered by Jondeau and Rockinger (2006) and, subsequently, extended to the family of Vine Copula by Chollete et al. (2009). Jondeau and Rockinger (2006) propose several alternative dynamic model specifications allowing both the marginal distributions and the joint Copula function to vary over time, while, Chollete et al. (2009) only consider the case where the Copula dependence parameter switches among two different regimes. As documented in Manner and Segers (2011), accounting for different persistence regimes in the dependence structure of the observed series can provide benefits further beyond the simple improvement of the goodness–of–fit tests. Different persistence regimes can be obtained by allowing the dependence parameters corresponding to a given Markovian state to vary over time, as in Fei et al. (2013). In the same spirit of Jondeau and Rockinger (2006), in this paper, we allow for the Copula parameters to depend on an endogenous latent discrete Markovian state as well as on time, by adding a specific dynamic evolution. In this way we aim to capture the different persistence behaviours across the different regimes. The evolution over time of the dependence parameters conditional to a given Markovian state is modelled through a Generalised Autoregressive Score (GAS) dynamic, see e.g., Creal et al. (2013) and Harvey (2013). Recently, Oh and Patton (2013) and De Lira Salvatierra and Patton (2015) have proposed a GAS dynamic update for the dependence parameters of Copula models. They show that, such updating mechanism for the Copula parameters, overwhelming outperforms other competitive alternatives such as the static Copula model. In what follows, we present our MS Copula model, while in Section 3 we provide details on the competing models of Jondeau and Rockinger (2006), and Patton (2006b) along with the specification of the GAS dynamic for the MS Copula model.

2.1. Marginal models

We propose to estimate the parameters governing the Copula function and the associated dependence structure separately from those controlling for the dynamic evolution of the marginal distributions. This estimation technique, known
as Inference Function for Marginals (IFM) has been proved to be feasible in large parametric spaces and asymptotically consistent, see [Patton (2006a)]. One of the main advantages of using the IFM estimating procedure concerns its ability to separate the univariate financial returns stylised facts from those regarding the multivariate distribution. This latter aspect, allows the researcher to focus on multivariate dependence structure only in a second moment, when the most appropriate model to describe all the empirical regularities of the observed univariate series has been selected and estimated. Whenever the marginal models are able to capture all the univariate empirical regularities of the marginal series, the only interestingly phenomenon to study is the remaining dependence structure. It follows that, the choice of an appropriate model that acts like a filter for the univariate series plays an important role for the multivariate analysis. Moreover, the effect of misspecified marginals in a Copula framework are know to be able to strongly affect the resulting Copula estimates, especially in risk management applications as widely discussed by [Fantazzini (2009)].

In order to filter out each marginal time series \( \{y_{i,t}\}_{t=1} \), for \( i = 1, 2, \ldots, d \), taking into account the dynamic evolution of the volatility, we consider an autoregressive model of order 1, AR(1), with Skew Student–t innovations [Fernández and Steel (1998)] and asymmetric GJR–GARCH dynamics [Glosten et al. (1993)] for the conditional volatility. Specifically, our system for the \( i \)-th univariate \( i \)-th univariate series is expressed as

\[
y_{i,t} = \phi_{0,i} + \phi_{1,i}y_{i,t-1} + \sigma_{i,t} \varepsilon_{i,t}, \quad \forall t = 1, 2, \ldots, T, \ i = 1, 2, \ldots, d \tag{1}
\]

\[
\sigma_{i,t}^2 = \omega_1 + \vartheta_{1,i} \varepsilon_{i,t-1}^2 + \vartheta_{2,i} \mathbb{I}_{(-\infty,0)}(\varepsilon_{i,t-1}) \varepsilon_{i,t-1}^2 + \vartheta_{3,i} \sigma_{i,t-1}^2, \tag{2}
\]

\[
\varepsilon_{i,t} \sim sST(0, 1, \upsilon_i, \eta_i), \tag{3}
\]

where \( (\upsilon_i, \eta_i) \in (0, +\infty) \times (0, +\infty) \) represent the degrees–of–freedom and the skewness parameters, respectively. The standardised skew Student–t distribution \( sST(0, 1, \upsilon_i, \eta_i) \) for the innovation term \( \varepsilon_{i,t} \) has density given by

\[
\tilde{t}_{\upsilon_i}(\varepsilon_{i,t}, 0, 1, \eta_i) = \frac{2}{\eta_i + \frac{1}{\eta_i}} t_{\upsilon_i}(\frac{\varepsilon_{i,t}}{\eta_i}, 0, 1) \mathbb{I}_{[0, +\infty)}(\varepsilon_{i,t}) + \frac{2}{\eta_i + \frac{1}{\eta_i}} t_{\upsilon_i}(\eta_i \varepsilon_{i,t}, 0, 1) \mathbb{I}_{(-\infty, 0)}(\varepsilon_{i,t}), \tag{4}
\]

and \( t_{\upsilon_i}(\cdot, 0, 1) \) for \( i = 1, 2, \ldots, d \), denotes the standardised Student–t density with \( \upsilon_i \) degrees of freedom. To preserve the stationarity and to ensure the positiveness of the conditional volatility process \( \sigma_{i,t}^2 \) at any point in time, the following conditions are imposed to the GJR–GARCH(1, 1) dynamics in equation (2): \( \omega_1 > 0, \vartheta_{1,i} \geq 0, |\vartheta_{2,i}| < 1, 0 \leq \vartheta_{3,i} < 1, \) with \( \vartheta_{1,i} + \vartheta_{2,i} \zeta_i + \vartheta_{3,i} < 1 \), for \( i = 1, 2, \ldots, d \), where \( \zeta_i = \int_{-\infty}^{0} \tilde{t}_{\upsilon_i}(\varepsilon, 0, 1, \eta_i) d\varepsilon = \frac{\upsilon_i}{1 + \upsilon_i}. \) The skewing mechanism of [Fernández and Steel (1998)] results in a very flexible solution to obtain a skewed version from any arbitrary univariate unimodal symmetric density function such as the Student–t. Moreover, it is worth noting that for \( \eta_i = 1 \) the density defined in equation (4) coincides with the standard Student–t distribution.
with $\nu_i$ degrees of freedom. An asymmetry parameter $\eta_i$ smaller than one, corresponds to a left–skewed density which assigns more probability to negative returns than to positive returns. In line with the empirically observed stylised facts of financial time series, we expect to estimate an asymmetry parameter smaller than one in order to capture the skewness caused by large negative returns. The choice of modelling the conditional mean as a first order autoregressive process as in equation (1) is motivated by the need to account for the serial dependence sometimes displayed by the financial returns, as described by Embrechts et al. (2003). The GJR–GARCH conditional volatility structure in equation (2) is consistent with the goal of “cleaning” the univariate series from the highest number of empirical regularities such as the leverage effect described by Black (1976). The GJR–GARCH specification captures the asymmetric response of the conditional volatility dynamic after a negative shock in a simple and easy to interpret way. To increase the readability of the paper, we group all the marginal model parameters in a vector $\vartheta_i = (\phi_{0,i}, \phi_{1,i}, \varpi_i, \vartheta_{1,i}, \vartheta_{2,i}, \vartheta_{3,i}, \eta_i, \nu_i)$, for $i = 1, 2, \ldots, d$. The vector of marginal parameter is then estimated for each time series by maximum likelihood, see e.g., Francq and Zakoian (2010).

2.2. Switching Copula model

In order to model both the cross sectional and time series dependence of the bivariate sequence of observations $Y_t = (y_{1,t}, y_{2,t}, \ldots, y_{d,t}) \in \mathbb{R}^d, t = 1, 2, \ldots, T$, we use a Markov–Switching Copula model. Our dependence model can be thought of as a more general case of the static finite mixture of dynamic Copulas models as in Creal et al. (2013) or an improvement of the model of Jondeau and Rockinger (2006). Our specification differs from the former class of models since it introduces a GAS dynamic into a finite mixture of Copulas, while retaining a Markovian structure on the dependence parameters that switches among two or more endogenous states, as in Chollete et al. (2009), Pelletier (2006) and Rodriguez (2007). Furthermore, our specification is more general than those proposed by Jondeau and Rockinger (2006), because we allow for the Copula dependence parameters to depend on its past through the specification of a Generalised Autoregressive Score (GAS) dynamic. Similarly, in this context, Oh and Patton (2013) model the CDS data using a dynamic Copula where the dependence parameters follows a GAS specification. They also apply the IFM approach allowing the marginals to be estimated separately and prior to the joint Copula model. In the remainder of this section, we present the Markov–switching specification for the Copula dependence parameters.

As in Chollete et al. (2009), the joint cross section dependence of observations $Y_t$ conditional upon the past history of the process $Y_{1:t-1} = (Y_1, Y_2, \ldots, Y_{t-1})$ and the latent hidden Markov chain $S_t = l, l = 1, 2, \ldots, L$ having $L$ distinct regimes, is modelled using a bivariate Copula distribution, having the following
general form

\[
f (Y_t = y_t \mid Y_{1:t-1} = y_{1:t-1}, S_t = l) = c \left( u_{1:t}, u_{2:t}, \ldots, u_{d:t}, \kappa_l(t), \psi_l(t) \right)
\times \prod_{i=1}^{d} f_i (y_i, \theta_i),
\]

where \( u_{i,t} = F_i (y_{i,t}, \theta_i) \) is equal to the Probability Integral Transformation (PIT) of \( y_{i,t} \) according to \( F_i (\cdot) \) and \( c \left( u_{1:t}, u_{2:t}, \ldots, u_{d:t}, \kappa_l(t), \psi_l(t) \right) \) is the Copula density conditional to the regime \( l = 1, 2, \ldots, L \), with parameters \( \kappa_l(t) \) and \( \psi_l(t) \), \( f_i (\cdot) \) is the generic probability density function of the marginal distribution of \( y_i \), with parameters \( \theta_i \) and \( F_i (\cdot) \) for \( i = 1, 2, \ldots, d \) are the corresponding distribution functions. We assume the latent states are driven by a Markov process, \( S_t \), for \( t = 1, 2, \ldots, T \) defined on the discrete space \( \Omega = \{1, 2, \ldots, L\} \) with transition probability matrix \( Q = \{q_{l,k}\} \), where \( q_{l,k} = P (S_t = k \mid S_{t-1} = l) \), \( \forall l, k \in \Omega \) is the probability that state \( k \) is visited at time \( t \) given that at time \( t-1 \) the chain was in state \( l \), and initial probabilities vector \( \delta = (\delta_1, \ldots, \delta_L) \), \( \delta_l = P (S_1 = l) \), i.e., the probability of being in state \( l = 1, 2, \ldots, L \) at time 1. For an up to date review of HMMs, see e.g., [Cappé et al. (2005), Zucchini and MacDonald (2009) and Dymarski (2011)].

To complete the model specification we need to define a specific dynamics for the Copula dependence parameter \( \kappa_l(t) \) which depends on a given realisation of the hidden Markov chain \( S_t = l \), for each \( l = 1, 2, \ldots, L \). We assume that \( \kappa_l(t) \), for \( l = 1, 2, \ldots, L \) follow a Generalised Autoregressive Score (GAS) dynamic as in Creal et al. (2013) and Harvey (2013). In the next section, we first review the recent developments of time varying Copulas using score models, as reported by

3. Dynamic dependence modelling

We now introduce the dynamic model for the Copula dependence parameters denoted by \( \kappa_l(t) \), for \( l = 1, 2, \ldots, L \). The first work on time-varying Copula dates back to the paper of Patton (2006b) who introduced the basic theory to allows for time dependence of the Copula parameters. It must be noticed that the Patton (2006b)’s contribution permits to specify time varying dependence starting from time varying linear correlation models, such as the Dynamic Conditional Correlation (DCC), previously introduced by Engle (2002a), or the BEKK of Engle and Kroner (1995). The Patton’s specification allows to capture one of the most relevant stylised fact affecting multivariate financial time series which is the time varying behaviour of the tail dependence. In this section, we first briefly discuss the original Patton’s dynamic and a generalisation of the Time Varying Correlation (TVC) model of Tse and Tsui (2002) developed within the Copula framework by Jondeau and Rockinger (2006). Then, we discuss the recent developments of time varying Copulas using score models, as reported by
Creal et al. (2013), De Lira Salvatierra and Patton (2015), Harvey (2013), and Oh and Patton (2013). For notational convenience, in this section we will drop the state dependence index $S_t = l$, for $l = 1, 2, \ldots, L$. Moreover, to keep the notation as simple as possible, we denote the scalar Copula dependence parameter as $\kappa_t$. Furthermore, for the Patton’s and TVC dynamic we impose $d = 2$ for simplicity.

3.1. Patton’s dynamic

As previously reported, the work of Patton (2006b) is the first attempt to introduce time varying upper and lower tail dependence in a Copula framework. He models directly the upper and lower tail dependence using a Joe–Clayton Copula (named “BB7 Copula” in Joe 1997) and the linear correlation parameter in a Gaussian Copula. In what follows, we will use the generic notation $c(u_1,t,u_2,t,\kappa_t)$, with $u_{i,t} = F_i(y_{i,t},\vartheta_j)$ for $i = 1, 2$, to denote a Copula totally described by the time–varying dependence parameter $\kappa_t$. The extension to the case of Copula functions described by two parameters, such as the BB7 Copula, is straightforward.

The idea behind the Patton (2006b)’s model is to impose an autoregressive moving average dynamic to the Copula parameter $\kappa_t$. However, since the modelled phenomenon is not directly observable, the first relevant issue to face is the choice of an adequate forcing variable for the updating mechanism of the dependence parameter. While for standard ARMA and GARCH processes the use of the past residuals is somehow natural, the same choice does not seem to be obvious to describe the evolution of $\kappa_t$. Moreover, there is a problem related to the Copula parameter space. For example, it is obvious that the linear correlation parameter of the bivariate Student–t Copula is defined in the $(-1, 1)$ region, while the parameter of a Gumbel Copula should be in the $[1, \infty)$ space. Patton (2006a) solved the above mentioned problems by proposing the following dynamic for elliptical and archimedean Copulas

$$\kappa_t = \Lambda (\omega + \beta \kappa_{t-1} + \alpha \xi_{t-1}), \quad \forall t = 1, 2, \ldots, T, \quad (6)$$

where $\omega \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ are given parameters, $\xi_t$ is the forcing variable, and $\Lambda (\cdot)$ is a monotonic invertible function which maps the real line $\mathbb{R}$ into the natural parameter space of the chosen Copula. The forcing variable $\xi_t$ is defined as follows

$$\xi_t = \begin{cases} m^{-1} \sum_{j=1}^{m} D^{-1}(u_{1,t-j}) D^{-1}(u_{2,t-j}), & \text{elliptical Copula} \\ m^{-1} \sum_{j=1}^{m} |u_{1,t-j} - u_{2,t-j}|, & \text{archimedean Copula} \end{cases}, \quad (7)$$

where $m$ is a fixed integer and,

$$D = \begin{cases} \Phi (\cdot), & \text{for the Gaussian Copula} \\ T_\nu (\cdot), & \text{for the Student–t Copula with } \nu \text{ degree of freedom} \end{cases}, \quad (8)$$

1 Patton uses the modify logistic function $h_{(L,U)}(x) = L + (U - L) \left( 1 + e^{-x} \right)^{-1}$ which maps $\mathbb{R}$ into the open interval $(L, U)$. 

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with \( \Phi (\cdot) \) and \( T_\nu (\cdot) \) regarding as the standardised Gaussian and Student–t with \( \nu \) degree of freedom cdfs, respectively, while \( D^{-1} (u_{j,t}) \), for \( j = 1, 2 \) denotes the inverse of the cdfs.

The Patton’s dynamic has some disadvantages. First, due to the presence of the mapping function \( \Lambda (\cdot) \), equation (6) cannot be considered as a pure autoregression in \( \kappa_t \), and, second, the choice of the \( m \) parameter is somehow arbitrary. Concerning the first point, it should be noted that, the impact of \( \kappa_{t-1} \) is modified by the \( \Lambda (\cdot) \) function.\(^2\)

### 3.2. Time–Varying Correlation dynamic

The Time–Varying Correlation (TVC) approach of Jondeau and Rockinger (2006) is an application of the model of Tse and Tsui (2002) to the dynamic conditional linear correlation within the elliptical Copulas’ family. The main difference between the model of Jondeau and Rockinger (2006) and that of Patton (2006a) relies in the fact that the former avoids the use of the mapping function \( \Lambda (\cdot) \) for the dependence parameters. The dynamic specification of Jondeau and Rockinger (2006) instead imposes a mean–reversion structure on the conditional linear correlation dynamic.\(^3\) Moreover, in their specification, the forcing variable \( \xi_t \) is defined as the empirical correlation over the last \( m \) observations prior to the current one, estimated using a rolling window, i.e.

\[
\kappa_t = (1-\alpha-\beta) \omega + \beta \kappa_{t-1} + \alpha \xi_{t-1},
\]

where

\[
\xi_t = \frac{\sum_{i=0}^{m-1} y_{1,t-i} y_{2,t-i}}{\sqrt{\sum_{i=0}^{m-1} y_{1,t-i}^2 \sum_{i=0}^{m-1} y_{2,t-i}^2}},
\]

and the constraints \( 0 \leq \alpha, \beta \leq 1 \), \( \alpha + \beta \leq 1 \) and \( -1 \leq \omega \leq 1 \) are imposed to preserve stationarity of the process. From a risk management perspective, it should be noted that the Gaussian Copula does not help to capture the extreme tail dependence of the data. The TVC Student–t Copula, instead, permits to capture the symmetric upper and lower tail dependence of the same magnitude.

### 3.3. Generalised Autoregressive Score dynamic

The Generalised Autoregressive Score (GAS) framework recently developed by Creal et al. (2013) and Harvey (2013) is gaining a lot of consideration by econometricians in many field of time series analysis. Under the Cox et al. (1981)

\(^2\)In a note on pg. 543 of Patton (2006b), the author reports a referee comment which suggests that the use of \( \Lambda^{-1} (\kappa_{t-1}) \) instead of \( \kappa_{t-1} \) in equation (6) ensures a pure autoregressive process for the transformed dependence parameter \( \Lambda^{-1} (\kappa_t) \). However, we find better results with the specification originally proposed by Patton (2006b).

\(^3\)Hereafter, we will continue to use \( \kappa_t \) instead of the natural dependence parameter \( \rho_t \) to ensure parameter consistency throughout the paper. However, considering that this model is applicable only in the elliptical world (mainly for the chosen forcing variable), we have that \( \kappa_t \equiv \rho_t \).
classification, the GAS models can be considered as a class of observation driven models, which has some natural consequences such as the closed form of the likelihood function. The key feature of GAS models, which is particularly effective in our dynamic Copula context, is that the predictive score of the conditional density is used as forcing variable into the updating equation of a time–varying parameter. The econometric literature has advocated two main reasons for the adoption of such an updating procedure. Harvey (2013) points out that this specification can be seen as an approximation of a filter for a model driven by a stochastic latent parameter that is by definition “unobservable”. Creal et al. (2013), instead, interpret the conditional score as a steepest ascent direction for improving the model’s local fit given the current parameter position, as it usually happens into a Newton–Raphson algorithm. Moreover, the flexibility of the GAS class of models nests a huge amount of famous econometrics models, such as the ARCH–type models of Engle (1982) and Bollerslev (1986) for volatility modelling, and also the MEM models of Engle (2002b), the Autoregressive Conditional Duration model of Engle and Gallo (2006) and Engle and Russell (1998), and the ACI model of Russell (1999), among others. Finally, one of the practical implications of the score updating mechanism in order to update a time–varying parameter, is that it avoids the problem of using a non–adequate forcing variable when the choice of it is not so obvious. This is the case when the objective is to model the time–varying dynamic dependence within the Copula framework here considered.

Let us now briefly summarise the GAS model dynamics. Formally, let us assume that \( y_t \sim p(y_t | f_t, \psi) \), where \( p(\cdot) \) is a probability density and \( f_t \) is a vector of dimension \( d = \text{dim}(f_t) \) of time–varying parameters and \( \psi \) is a vector of time–independent parameters. For example, \( p(\cdot) \) can be chosen as a Student–t distribution with fixed degree of freedom (\( \psi = \nu \)) and time–varying volatility (\( f_t = \sigma_t \)). According to the GAS framework, the updating equation for the time–varying parameters is

\[
\begin{align*}
  f_{t+1} &= \omega + A s_t + B f_t \\
  s_t &= S_t (f_t | \psi) \nabla_t (f_t | \psi),
\end{align*}
\]

where \( \omega \in \mathbb{R}^d \), and \( A, B \) are matrices of parameters of dimension \((d \times d)\) such that the eigenvalues of \( B \) are in modulus strictly less than one to preserve the stationarity of the GAS dynamic. Here, \( \nabla_t (f_t | \psi) \) denotes the conditional score of the pdf \( p(\cdot) \), evaluated at \( f_t \)

\[
\nabla_t (f_t | \psi) = \frac{\partial \ln p(f_t | \psi)}{\partial f_t},
\]

and \( S_t (f_t | \psi) \) is a positive definite, possible parameter–dependent, scaling matrix. Convenient choices for \( S_t (f_t | \psi) \) are usually given by

\[
S_t (f_t | \psi) = [I (f_t | \psi)]^{-\zeta},
\]
where $\mathcal{I}(f_t|\psi)$ is the Fisher information matrix that, for well behaved densities, could be written as

$$
\mathcal{I}(f_t|\psi) = -\mathbb{E}_{t-1} \left[ \frac{\partial^2 \ln p(f_t|\psi)}{\partial f_t \partial f_t^\top} \right] = \mathbb{E}_{t-1} \left[ \nabla_t (f_t|\psi) \nabla_t (f_t|\psi)^\top \right],
$$

and $\zeta$ is usually set equal to $\{0, 1/2, 1\}$. Note that, for $\zeta = 0$ the scaling matrix $S_t(f_t|\psi)$ coincides with the identity matrix. Creal et al. (2013) suggest to use the inverse Fisher information matrix, which corresponds to $\zeta = 1$, or its pseudo-inverse square root, which corresponds to $\zeta = 1/2$, in order to scale the conditional score for a quantity that accounts for its curvature. In our empirical tests, we find this latter scaling mechanism much more efficient than using an identity scaling matrix. However, as it will be discussed later in the paper, sometimes, the Fisher information matrix is not available in closed form, and we need to resort to simulation or numerical evaluations methods, which should be traded-off with approximation degree and code efficiency. The GAS framework allows for another degree of freedom, in the sense that it adapts quite naturally to the parametrisation problem. This latter aspect becomes quite useful when the natural parameter’s space is constrained into a subset of the real hyperplane and hence a mapping function becomes necessary. The reparameterisation problem is heavily present into the dependence modelling framework. As previously discussed, this issue is solved in the ongoing literature, by imposing either a mean reverting dynamic to the time-varying correlation, as in the TVC model of Tse and Tsui (2002), or by rescaling the dependence parameters into the natural parameters space at each point in time $t$, as in the Patton (2006b)’s framework. As described below, the GAS approach adapts more naturally to the reparameterisation problem. Formally, let us define $F \in \mathbb{R}^d$ as the natural parameter space and $\tilde{\lambda} : \mathbb{R}^d \to F$ an absolutely continuous deterministic invertible function that maps $\mathbb{R}^d$ into the natural parameter space $F$. When $d > 1$ it is convenient to specify $\tilde{\lambda}$ as a vectorial function being able to map each component $f_t$ into the proper space. In general, for the $i$-th component of $\tilde{\lambda}$, we can consider the modified logistic function defined by

$$
\lambda^{(L, U)}_i(x) = L + \frac{U - L}{1 + e^{-x}},
$$

which maps $\mathbb{R}$ into the interval $(L, U)$. Moreover, let us define $\tilde{f}_t = \tilde{\lambda}^{-1}(f_t)$ the unmapped version of the dynamic parameter $f_t$, then the GAS model suited for the new time-varying parameter $\tilde{f}_t$ with $\zeta = 1$ can be easily defined as

$$
\tilde{f}_{t+1} = \omega + A\tilde{s}_t + B\tilde{f}_t,
$$

where $\tilde{s}_t = \tilde{\lambda} \odot s_t$, and $\tilde{\lambda} = \frac{\partial \tilde{\lambda}(f_t)}{\partial f_t}$ and “$\odot$” stands for the Hadamard product. For other possible choices of $\zeta$ we refer to Creal et al. (2013).
3.4. The SGASC Model

In this Section, we summarise the SGASC model. Using the Sklar (1959) theorem, the SGASC model can be represented as follow

\[ Y_t \mid (F_{t-1}, S_t = l) \sim C \left( F_1(y_{1,t}, \vartheta_1), F_2(y_{2,t}, \vartheta_2), \kappa_l^t, \psi_l^t \right), \]

(15)

for \( t = 1, 2, \ldots, T \), \( l = 1, 2, \ldots, L \), where \( F_j(y_j,t, \vartheta_j) \), \( j = 1, 2 \) is the cumulative density function of asset \( i \) at time \( t \) defined in Section 2.1, \( \kappa_l^t \) is a vector containing the time–varying parameters of the Copula function for \( l = 1, 2, \ldots, L \), and \( \psi_l^t \) is a vector containing other constant parameters. For example, \( C(\cdot) \) can be the cdf of a Student–t Copula with time varying correlation matrix \( (\kappa_l^t = R_l^t) \) and constant state dependent degree of freedom \( \psi_l^t = \nu_l^t \). To capture several dependence patterns, we consider several alternative Copula specifications such as the elliptical, Gaussian and Student–t, and the archimedean, Clayton, Gumbel, Frank and Plackett. Of course, the considered Copulas differ for their properties and, in particular, for their ability to capture different patterns of dependence in the tail. The tail dependence is of particular interest, since the main aim of the empirical part of this paper is to investigate the consequences of extreme “abnormal” events affecting the market participants, on the whole equity market, a concept known in the financial literature as tail interdependence. Regarding this aspect, it is worth noting that on the one hand, the Student–t and the Gumbel–Clayton Copulas describe situations of symmetric tail dependence and asymmetric tail dependence, respectively, while, on the other hand, the Gaussian, Frank and Plackett Copulas describe the reverse situation of tail independence. The general properties of the Copula functions are best described in Appendix D, while Appendix E reports the Copula densities for the bivariate case. For a catalog of Copulas and their properties see, e.g., Nelsen (2007), Joe (1997) and Durante and Sempi (2015). As detailed in Section 2.2, the specification of the SGASC model is completed by adding the stochastic variable \( S_t \) follow the first order Markov process. As previously discussed, we assume an updating mechanism for \( \kappa_l^t \) based on a Switching GAS dynamic. Formally, we have

\[ \kappa_l^t = \bar{\lambda} \left( \bar{\kappa}_l^t \right) \]
\[ \bar{\kappa}_l^t = \omega^t + A_l^t \bar{s}_{t-1}^l + B_l^t \bar{\kappa}_{t-1}^l, \]

(16)

where \( \bar{s}_{t-1}^l \) denotes the conditional score scaled by the inverse Fisher information matrix. Unfortunately, the Fisher information matrix in not analytically available for all the Copula specifications we consider here. Appendix F and Appendix G report the analytical formulae to compute the score and the Fisher information matrix for several bivariate Copulas specifications: Gaussian, Student–t, Gumbel, Clayton. The Frank and Plackett information matrices are evaluated using the grid approach proposed by Creal et al. (2013). Appendix G describes the procedure used to approximate the Hessian matrix for the Frank and Plackett Copulas. In this context, \( \omega = (\omega_1^t, \ldots, \omega_L^t) \) is a vector of parameters controlling for the level of the dynamics, while \( A_l^t = \{\alpha_l^t\} \) and \( B_l^t = \{\beta_l^t\} \)
are state dependent diagonal matrices controlling for the updating step and the persistence of the process, respectively. As previously discussed, given the diagonal specification of \( B_l = \{ \beta^l_i \} \), the only restriction needed to ensure stationarity of the considered process are \( |\beta^l_i| < 1 \) for \( i = 1, \ldots, p \) and \( l = 1, \ldots, L \) since \( s_t \) is a pure martingale difference.

Consider now the case when the Copula function is described by a scalar time varying dependence parameter i.e. when \( \kappa_t = \kappa_t, \omega = \omega, A = \alpha, B = \beta. \) Recently, for this subclass of models, Blasques et al. (2014c) and Blasques et al. (2013) show how to characterise the regions of the GAS parameter that ensure Stationarity and Ergodicity (SE) of the process. They use the results for a general stochastic recurrence equation given by Bougerol (1993) and Straumann et al. (2006). Given the general formulation of the GAS framework, here considered, the evaluation of the SE is nontrivial since a closed form for the SE region is not available. The difficulty to determine the SE region relies on the free specification of both the scaling matrix \( S_t(\kappa_t|\psi) \) and the mapping function \( \lambda(\cdot) \) because these choices strongly influences the parameters dynamics. The most difficult challenge is to find the SE space for the parameter \( \alpha \) since for the autoregressive coefficient \( \beta \) the constraint \( \beta \in (-1, 1) \) is imposed, while the \( \omega \) term is quite always free. Blasques et al. (2014c) show that the following relation must hold for every univariate GAS process in order to be stationary and ergodic

\[
E \sup_{f^* \in \mathcal{F}} |\frac{\partial s_t (f^*|\lambda)}{\partial f}| < 1 - |\beta|/|\alpha| ,
\]

where \( s_t \) is the scaled score that should include the mapping function, when employed. In a separate technical report we show the numerical evaluation of the SE for our model under different Copula functions and scaling assumptions.

4. Estimation and inference

In this section, we detail the technique employed to estimate the model parameters of the marginal and joint distributions. As mentioned in the Introduction, model parameters are estimated using a two step procedure that consists in a first step where the parameters involved in the marginals are estimated, followed by a second step where the dependence parameters are jointly estimated along with the latent Markovian states. The resulting Inference Functions for Margins (IMF) two step procedure has been proved to be asymptotically consistent by Patton (2006a) to which we refer for further details. The IMF procedure can be applied in this context as well as in general dynamic Copula models since the parameters of the marginals are separable from those of the Copula distribution in equation (5). In the specific case of the dynamic MS model here considered we also take advantage from the fact that the Markovian dynamics is imposed only on the dependence parameters of the Copula function and not on the marginals. To estimate the dependence parameters subject to the Markovian structure as well as to the GAS dynamics specified in the previous section we adapt the Expectation–Maximization algorithm of Dempster et al. (1977).
In this section we present the EM algorithm, assuming that, the parameters of the marginal distribution $\vartheta_i$ have been previously consistently estimated by maximum likelihood, see, e.g., Francq and Zakoian (2010).

The EM algorithm is a powerful and easy programmable tool for ML estimation on data having missing structures, such as finite mixtures and Markov–Switching models, and it releases a non–decreasing sequence of the log–likelihood function converging to the maximum. For a general and up–to–date reference on the EM algorithm see the book of McLachlan and Krishnan (2007). In what follows, we present the EM–algorithm for estimating the parameters of the SGASC model described in Section 2.

For the purpose of application of the EM algorithm the vector of observations $y_{t}, t = 1, 2, \ldots, T$ is regarded as being incomplete. Following the implementation described in McLachlan and Peel (2000) the following missing data are consequently introduced $z_{t} = (z_{t,1}, z_{t,2}, \ldots, z_{t,L})$ and $zz_{t} = (zz_{t,1,1}, zz_{t,1,2}, \ldots, zz_{t,l,k}, \ldots, zz_{t,L,L})$ being defined as:

$$z_{t,l} = \begin{cases} 
1 & \text{if } S_{t} = l, \\
0 & \text{otherwise}
\end{cases}$$

$$zz_{t,l,k} = \begin{cases} 
1 & \text{if } S_{t-1} = l, S_{t} = k, \\
0 & \text{otherwise}.
\end{cases}$$

Similarly to the latent class approaches, the class membership is unknown and conveniently treated as the value taken by a latent Multinomial variable with one trial and $L$ classes, where the temporal evolution of class membership is driven by the hidden Markov chain $S_{t}$ for $t = 1, 2, \ldots, T$. Augmenting the observations $\{Y_{t}, t = 1, 2, \ldots, T\}$ with the latent variables $\{z_{t}, zz_{t}, t = 1, 2, \ldots, T\}$ allows for replacing the log–likelihood function with the complete–data log–likelihood, which becomes

$$\log \mathcal{L}_{c}(\Xi) = \sum_{l=1}^{L} z_{1,l} \log(\delta_{l}) + \sum_{l=1}^{L} \sum_{k=1}^{L} \sum_{t=1}^{T} zz_{t,l,k} \log(q_{l,k})$$

$$+ \sum_{l=1}^{L} \sum_{t=1}^{T} z_{t,l} \log c(\hat{u}_{1}, \hat{u}_{2}, \ldots, \hat{u}_{d}, \Xi^l)$$

$$+ \sum_{t=1}^{T} \sum_{i=1}^{d} \log f_{i}(y_{i,t}, \hat{\vartheta}_{i}),$$

where $\hat{u}_{i} = \hat{F}_{i}\left(y_{i,t}, \hat{\vartheta}_{i}\right)$ for $i = 1, 2, \ldots, d$ are the pseudo–observations obtained by transforming $y_{i,t}$ for $i = 1, 2, \ldots, d$ and $t = 1, 2, \ldots, T$ using the estimated marginals cdf, and $\hat{\vartheta}_{i}$ for $i = 1, 2, \ldots, d$ are the maximum likelihood estimates of the marginal parameters. Here, we define $\Xi = \{\Xi^l\}_{l=1}^{L}$, where $\Xi^l = (\omega^l, A^l, B^l, \psi^l)$ is a vector containing the parameters of the GAS dynamics for the Copula dependence parameters $\kappa^l_i$ and $\psi^l_i$, for $l = 1, 2, \ldots, L$. The EM algorithm consists of two major steps, one for expectation (E–step)
and one for maximization (M–step), see McLachlan and Krishnan (2007) for a recent account of the EM algorithm in a general context. On the \((m+1)\)–th iteration the EM algorithm proceeds as follows:

**E–step:** computes the conditional expectation of the complete data log–likelihood given the observed data and the current parameters estimates, i.e.

\[
Q(\Xi, \Xi^{(m)}) = \mathbb{E}^{p(z_t|y_{1:T}, \Xi^{(m)})} \left[ \log L_c(\Xi) | \{y_t\}_T^T \right].
\] (18)

**M–step:** choose \(\Xi^{(m+1)}\) by maximizing the preceding expected values with respect to \(\Xi\), i.e.

\[
\Xi^{(m+1)} = \arg \max_\Xi Q(\Xi, \Xi^{(m)}).
\] (19)

The E–step in equation (18) requires the computation of the so–called \(Q\)–function, which calculates the conditional expectation of the complete–data log–likelihood given the observations and the current estimate of the parameter vector \(\Xi^{(m)}\) \(\forall t = 1, 2, \ldots, T\) and \(l = 1, 2, \ldots, L\). Exploiting the previous factorization we get the following representation of the function \(Q\):

\[
Q(\Xi, \Xi^{(m)}) = \mathbb{E}^{p(z_{1:l} = 1|y_{1:T}, \Xi^{(m)})} \left\{ \log L_c(\Xi) | \{y_t\}_T^T \right\} = \sum_{l=1}^L \hat{z}_{1,l} \log (\delta_l) + \sum_{l=1}^L \sum_{k=1}^L \sum_{t=1}^T \hat{z}_{t,l,k} \log (q_{l,k})
\]

\[
+ \sum_{l=1}^L \sum_{t=1}^T \hat{z}_{t,l} \log c(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_d, \Xi),
\]

where

\[
\hat{z}_{t,l} = \mathbb{P}\left(S_t = l | y_{1:T}, \Xi^{(m)}\right)
\]

\[
\hat{z}_{t,l,k} = \mathbb{P}\left(S_{t-1} = l, S_t = k | y_{1:T}, \Xi^{(m)}\right),
\]

for \(l, k = 1, 2, \ldots, L\), and \(\forall t = 1, 2, \ldots, T\), denote the current smoothed probabilities of the states evaluated using the well–known Forward–Filtering Backward–Smoothing (FFBS) algorithm detailed in Frühwirth-Schnatter (2006), and Cappé et al. (2005).

The M–step in equation (19) maximizes the function \(Q(\Xi, \Xi^{(m)})\) with respect to \(\Xi\) to determine the next set of parameters \(\Xi^{(m+1)}\). The updated estimates of the HMM parameters, i.e. the vector of initial probabilities \(\delta\) and the transition probability matrix of the hidden Markov chain \(Q\) are:

\[
\hat{\delta}_l^{(m+1)} = \hat{z}_{1,l}
\]

\[
\hat{q}_{l,k}^{(m+1)} = \frac{\sum_{t=2}^T \hat{z}_{t,l,k}}{\sum_{k=1}^L \sum_{t=2}^T \hat{z}_{t,l,k}},
\]
while the $\Xi^l$ parameters, for $l = 1, \ldots, L$ can be obtained as the solution of the following optimisation problem

$$
\Xi^{(m+1)} = \arg\max_{\Xi} \sum_{l=1}^{L} \sum_{t=1}^{T} \hat{z}_{t,l} \log c \left( \hat{u}_1, \hat{u}_2, \Xi^l \right).
$$

Convergence of the algorithm to the ML estimates is guaranteed since the last optimisation step delivers a parameter update that increases the log–likelihood function.

5. Systemic risk measures

In this section, we first introduce the two systemic risk measures we consider throughout the paper, namely the Conditional Value–at–Risk (CoVaR) and the Conditional Expected Shortfall, (CoES). Then, we describe how CoVaR and CoES can be calculated assuming the joint returns follow the SGASC model defined in the previous sections. CoVaR and CoES have been introduced in the systemic risk literature by Adrian and Brunnermeier (2011, 2014), and subsequently extended to a dynamic framework by Girardi and Ergün (2013). In their seminal paper, Adrian and Brunnermeier (2011) propose to estimate the CoVaR measure using a system of two quantile equations extending the traditional approach of direct Value–at–Risk estimate. Bernardi et al. (2013a, 2015) propose a Bayesian dynamic quantile model to estimate the co–movement risk measure where both the VaR and CoVaR equations are function of individual and macroeconomic observed risk factors, as in the original CoVaR approach, as well as unobserved components having their own stochastic dynamics. The Copula approach to the CoVaR has been recently proposed by Reboredo and Ugolini (2015) to evaluate the systemic risk in European sovereign debt markets. Here, the CoVaR and CoES are further extended to account for both the dynamic evolution of the Copula dependence parameters as well as the presence of Markovian regimes. The CoVaR measure co–movements between any two distinct institutions by extending the VaR approach to a conditional framework. Indeed, the CoVaR of one institution is measured conditional to a relevant extreme event affecting the other institution. The appealing characteristic of the CoVaR risk measure is that it inherits the flexibility and easy computability from the dynamic switching Copula framework here developed. The Copula approach naturally adapts to environments characterised by different kind of upper and lower tail dependence enabling the CoVaR as an effective measure of the extreme conditional co–movements among financial variables. Moreover, as we will see, the Copula–CoVaR can be easily evaluated.

The CoVaR measures the spillover between institutions by providing information on the Value–at–Risk of an institution or market, conditional on another institution’s distress event. Formally, given the return of institution $i$, $Y_{i,t}$, or the market portfolio, and the return of another institution $j$, $Y_{j,t}$, with $j \neq i$, belonging to that market, the CoVaR of institution $i$ at time $t = 1, 2, \ldots, T$,
denoted by $\text{CoVaR}_{ij,t}^{\tau_1,\tau_2}$, satisfies the following equation

$$
\mathbb{P} \left( Y_{i,t} \leq \text{CoVaR}_{ij,t}^{\tau_1,\tau_2} \mid Y_{j,t} \leq \text{VaR}_{j,t}^{\tau_2} \right) = \tau_1,
$$

(20)

where $\text{VaR}_{j,t}^{\tau_2}$ denotes the marginal Value–at–Risk (VaR) of institution $j$, i.e. $\mathbb{P} \left( Y_{j,t} \leq \text{VaR}_{j,t}^{\tau_2} \right) = \tau_2$ and $(\tau_1, \tau_2)$ are predetermined confidence levels, for the CoVaR of institution $i$ and for the marginal VaR of institution $j$, respectively.

Roughly speaking, the Conditional Value–at–Risk of institution $i$, is the quantile of the distribution of $Y_{i,t}$ conditional on a extreme event affecting institution $j$’s returns $Y_{j,t}$. As in Adrian and Brunnermeier (2011), we define such an extreme events as the marginal VaR at confidence value $\tau_2$ of $Y_{j,t}$, $t = 1, 2, \ldots, T$.

Remark 5.1. The definition of Conditional Value–at–Risk in equation (20) is substantially different from that originally presented in Adrian and Brunnermeier (2011) and coincides with that proposed in Girardi and Ergün (2013). As discussed in Girardi and Ergün (2013) and Mainik and Schaanning (2014) this definition essentially preserves the stochastic ordering introduced by the joint distribution.

Given the dynamic context introduced in the previous sections, the random variables we refer to for the calculation of the forward looking systemic risk measure $\text{CoVaR}_{ij,t+1}^{\tau_1,\tau_2}$ at each time point $t = 1, 2, \ldots, T$ are $(Y_{i,t+1}, Y_{j,t+1} \mid \mathcal{F}_t)$, where $\mathcal{F}_t$ is an appropriate filtration. Indeed, we suppose to know the past history of the observed process $Y_t$ as well as the hidden Markovian states characterising the dependence parameter dynamics up to time $t$, $S_{1:t}$, and we want to predict the distribution of $Y_{t+1}$ conditional to that information. The $\text{CoVaR}_{ij,t+1}^{\tau_1,\tau_2}$ and $\text{VaR}_{j,t+1}^{\tau_2}$ characterise the conditional and marginal quantiles of that predictive distribution.

Assuming a bivariate SGASC model for the assets return dynamics, i.e. setting $d = 2$, the one-step ahead predictive cumulative distribution function of the observed process $Y_t$ at time $t+1$, given information up to time $t$ is a mixture of component specific predictive cumulative distributions

$$
H \left( Y_{t+1} \mid y_{1:t} \right) = \sum_{l=1}^{L} \pi_{t+1|t}^{(l)} C \left( u_{1,t+1}, u_{2,t+1} \mid S_{t+1} = l, y_{1:t} \right),
$$

(21)

$\forall t = 1, 2, \ldots, T$, with mixing weights

$$
\pi_{t+1|t}^{(l)} = \sum_{m=1}^{L} q_{m,l} \mathbb{P} \left( S_t = m \mid y_{1:t} \right), \quad l = 1, 2, \ldots, L,
$$

(22)

where $u_i = F_Y (y_i, \vartheta_i)$, $i = 1, 2$ and $q_{j,l}$ is the $(j, l)$-th entry of the Markovian transition matrix $Q$, see e.g. Zucchini and MacDonald (2009). It is worth adding that equation (21) follows immediately from the fact that the cumulative predictive distribution of $Y_{t+1}$ given the past history of the process $Y_{1:t}$ is a finite mixture of Copulas and that mixtures of Copulas are Copulas themselves,
see, e.g., [Nelsen (2007)]. Based on this result we provide expressions for the one–step–ahead “predictive” systemic risk measures.

The calculation of CoVaR requires the prior evaluation of institution’s $j$ marginal VaR that can be done by inverting the marginal cdf of $Y_j$, i.e. $\text{VaR}_{j,t+1}^{\tau_2} = \frac{1}{\tau_2} F_{Y_j}^{-1}(\tau_2, \hat{\vartheta}_j)$. From a computational point of view, the Copula approach is more tractable since it does not require the inversion of the cdf to evaluate the marginal VaR since the PIT of the VaR coincide with the chosen quantile confidence level $\tau_2$. Conditional on $\text{VaR}_{j,t+1}^{\tau_2}$, the CoVaR of institution $i$ conditional on institution $j$’s distress $\text{CoVaR}_{i|j,t+1}^{\tau_1,\tau_2}$ is calculated as the value of $y_{i,t+1}$ such that

$$\Pr(Y_{i,t+1} \leq y_{i,t+1}, Y_{j,t+1} \leq \text{VaR}_{j,t+1}^{\tau_2}) = \tau_1 \tau_2,$$

or in terms of the bivariate SGASC predictive distribution we consider here

$$\sum_{l=1}^{L} \pi_{t+1|t}^{(l)} C \left( F_{Y_i} \left( y_{i,t+1}, \hat{\vartheta}_i \right), F_{Y_j} \left( \text{VaR}_{j,t+1}^{\tau_2}, \hat{\vartheta}_j \right), \kappa_{t, l}^{i}, \psi^{l} \right)$$

$$= \sum_{l=1}^{L} \pi_{t+1|t}^{(l)} C \left( F_{Y_i} \left( y_{i,t+1}, \hat{\vartheta}_i \right), \tau_2, \kappa_{t, l}^{i}, \psi^{l} \right) = \tau_1 \tau_2,$$

for $t = 1, 2, \ldots, T$. Adrian and Brunnermeier (2014) also proposes to extend the expected shortfall (ES) risk measure in a systemic framework by evaluating the marginal ES of each institution at the CoVaR level. Bernardi et al. (2013b) and Bernardi and Petrella (2015) instead consider a multiple co–movement risk measure alternative to the CoVaR proposed by Adrian and Brunnermeier (2011, 2014), namely the Multiple–CoVaR, and extend it to the expected shortfall measure. In what follows, we adapt the Multiple–CoES to the Copula framework where the conditioning events consists on the distress of a single institution. The CoES is the expected shortfall of $Y_{i,t+1}$ below its CoVaR $\text{CoVaR}_{i|j,t+1}^{\tau_1,\tau_2}$ level, given that $Y_{j,t+1}$ is below its VaR $\text{VaR}_{j,t+1}^{\tau_2}$ level, i.e.

$$\text{CoES}_{i|j,t+1}^{\tau_1,\tau_2} = \text{ES} \left( Y_{i,t+1} \leq \text{CoVaR}_{i|j,t+1}^{\tau_1,\tau_2} \mid Y_{j,t+1} \leq \text{VaR}_{j,t+1}^{\tau_2} \right)$$

$$= \mathbb{E} \left( Y_{i,t+1} \mid Y_{i,t+1} \leq \text{CoVaR}_{i|j,t+1}^{\tau_1,\tau_2}, Y_{j,t+1} \leq \text{VaR}_{j,t+1}^{\tau_2} \right),$$

for $t = 1, 2, \ldots, T$. Our definition of CoES coincides with the one given by Bernardi et al. (2013b), Bernardi and Petrella (2015) and Mainik and Schaanning (2014) and substantially differs from the one of Adrian and Brunnermeier (2011). In the Copula framework, the forward looking CoES$^{\tau_1,\tau_2}_{i|j,t+1}$ can be evaluated by numerically integrating the CoVaR$^{\tau_1,\tau_2}_{i|j,t+1}$ as follows:

$$\text{CoES}_{i|j,t+1}^{\tau_1,\tau_2} = \frac{1}{\tau_2} \int_{0}^{\tau_2} \text{CoVaR}_{i|j,t+1}^{\tau_1,\gamma} d\gamma,$$

for $t = 1, 2, \ldots, T$. As discussed by Bernardi et al. (2013b) and Mainik and Schaanning (2014), the CoES risk measures inherits the same properties of
the Expected Shortfall (ES) such as the sub–additivity with respect to linear combinations, see e.g. [Artzner et al. (1999)]. As a direct consequence of the sub–additivity property, the CoES can be effectively used in order to measure the total systemic risk contribution of different assets to the overall financial system. In a different context, [Engle et al. (2015)] suggest to employ a linear combination of individual Long Run Marginal Expected Shortfall (LRMES) in order to obtain an aggregate measure of the total market systemic risk for the European region. Using same arguments, [Brownlees and Engle (2015)] also rely on the sub–additivity property of the LRMES to get an aggregate version of the Systemic Risk (SRISK) indicator. From our definition of CoES in equations (25)–(26), it is easy to see that the LRMES can be obtained as a special case of the CoES by simply letting CoVaR$_{i\mid j,t+1}^{\tau_1,\tau_2}$ → ∞, i.e. by imposing $\tau_1 = 1$. To aggregate the individual systemic risk levels to get an overall indicator of total systemic risk for the whole economy, let $S$ denote a market index and assume to estimate the bivariate SGASC model to each pair of returns $(Y_S, Y_j)$, for $j = 1, 2, \ldots, N$, where $N$ denotes the number of institutions in the market, we can define the total market forward looking CoES as

$$\text{CoES}_{M,t+1}^{\tau_1,\tau_2} = \sum_{j=1}^{N} w_j \text{CoES}_{S\mid j,t+1}^{\tau_1,\tau_2},$$

(28)

for $t = 1, 2, \ldots, T$, where the scalars $w_j$, $j = 1, 2, \ldots, N$ denote the weights associated to each financial institution belonging to the market. The definition of CoES$_{M,t+1}^{\tau_1,\tau_2}$ is useful to estimate the total loss the overall market is going to face as a consequence of a crisis affecting a market participant, which is transmitted to the market by observing a realisation of $Y_i$ below the CoVaR level.

As discussed in [Adrian and Brunnermeier (2011, 2014), Girardi and Ergün (2013) and Mainik and Schanning (2014)], it is also useful to consider the difference between the CoVaR and the CoES from their “median”. Here, we consider the $\Delta\text{CoVaR}_{i\mid j,t+1}^{\tau_1,\tau_2}$ and the $\Delta\text{CoES}_{i\mid j,t+1}^{\tau_1,\tau_2}$ quantities defined as

$$\Delta\text{CoVaR}_{i\mid j,t+1}^{\tau_1,\tau_2} = 100 \times \frac{\text{CoVaR}_{i\mid j,t+1}^{\tau_1,\tau_2} - \text{CoVaR}_{i\mid j,t+1}^{b_j}}{\text{CoVaR}_{i\mid j,t+1}^{b_j}},$$

(29)

and

$$\Delta\text{CoES}_{i\mid j,t+1}^{\tau_1,\tau_2} = 100 \times \frac{\text{CoES}_{i\mid j,t+1}^{\tau_1,\tau_2} - \text{CoES}_{i\mid j,t+1}^{b_j}}{\text{CoES}_{i\mid j,t+1}^{b_j}},$$

(30)

where $b_j$ represents the benchmark state that we define as $P(Y_j,t+1 \leq \text{VaR}_{0.5}^{j,t+1}) = 0.5$, i.e. the CoVaR of the system when the country specific indexes are below their median value. The $\Delta\text{CoVaR}$ and the $\Delta\text{CoES}$ measure the percentage increase of the systemic risk conditional on a pre–specified distress event, being defined as the marginal $\tau_2$–VaR level of the conditioning institution $j$. It follows that the $\Delta\text{CoVaR}$ and the $\Delta\text{CoES}$ can be effectively used to measure how the
CoVaR and the CoES change when a particular institution become financially distressed. In other words, the \( \Delta \text{CoVaR} \) and the \( \Delta \text{CoES} \) estimate the dynamic evolution of the specific institution \( j \)'s contribution to the overall systemic risk. Furthermore, these two quantities can be employed for policy rules and for risk management. Adrian and Brunnermeier (2011, 2014) found strong evidence for the existence of a relation between the \( \Delta \text{CoVaR} \) and several macroeconomic indicators. In the empirical part of the paper, we will employ the \( \Delta \text{CoVaR} \) and the \( \Delta \text{CoES} \) risk measures to investigate the systemic risk contributions of specific countries to the overall risk of the European economic system.

6. Empirical study

In what follows, we apply the econometric framework and the methodology described in the previous sections to examine the European systemic risk evolution over the past decade. To this end we consider the systemic relation between nine different countries and the European economic system. The evaluation of systemic risk in the European financial system has been recently considered by Reboredo and Ugolini (2015) in order to assess the impact of the recent European sovereign debt crisis of 2010. Reboredo and Ugolini (2015) propose a dynamic Copula model as in Patton (2006b) to analyse the 10–year European sovereign bond market for the period 2000–2012. Engle et al. (2015) analyse the evolution of the European systemic risk by applying the Long Run Marginal Expected Shortfall (LRMES) that essentially aggregates the Marginal Expected Shortfall (MES) risk measure of Acharya et al. (2010a,b), over a given forecasting horizon conditioning to a predefined extreme event affecting the financial market. Lucas et al. (2014b) instead combines multivariate Generalised–Hyperbolic GAS model with the CoVaR to measure the Euro area sovereign default risk implied by the Credit Default Swaps. Billio et al. (2012) analyses both the US and Euro market using stock prices of a large panel of banks.

6.1. Data

We consider nine different equity indexes representing the major European economies. The chosen indexes are representative of the equity market of the country they belong to. In particular, we consider the Amsterdam Exchange Index (AEX) which is composed by the 25 most actively traded Dutch companies, the Austrian Traded Index (ATX) which is a free float weighted price index made up of the most liquid stocks traded on the Vienna Stock Exchange and consist of 20 firms, the CAC 40 (FCHI) which is a capitalization–weighted index representing the forty largest French–domiciled companies, the FTSE ALL–SHARE (FTAS) comprising 644 companies traded on the London Stock Exchange representing a net Market Cap of about 2,693,462 million of euro, the Deutscher Aktienindex 30 (GDAXI) also knows as DAX30 that is a blue chip stock market index consisting of the 30 major German companies trading on the Frankfurt Stock Exchange. We also consider the Indice Bursatil Español (IBEX-35), the Irish Stock Exchange Overall Index (ISEQ), the OMX Helsinki 25 (OMXH25)
and the OMX Stockholm 30 (OMXS30) as representative of the Spain, Irish, Finland and Sweden equity markets, respectively. As a proxy to the overall European equity market we use the the STOXX Europe 600 Index (STOXXE) which represents large, mid and small capitalisation companies across 18 countries of the European region. Table B.1 summarises the indexes’ name and country.

The quantity of interest for this empirical application are the first log differences log returns of the various equity indexes. Returns span period beginning on January 7, 1992 and ending on January 9, 2015, covering the recent Global Financial Crisis of 2007/2008 as well as the European sovereign debt crisis of 2010. A complete overview of all the crisis events we consider and their starting date is provided in Appendix A. Table B.2 reports descriptive statistics for all the considered series. In line with usual stylised facts of financial time series, the returns appear to be negatively skewed and leptokurtic, indicating that they are not normally distributed. In addition, the Jarque–Bera (JB) statistic confirms the departure from normality for all return series at the 1% level of significance. For all the considered country indexes’ returns, the Ljung–Box Q test and Engle’s ARCH LM test (not reported) provide strong evidence of serial dependence and heteroskedasticity, respectively. The presence of large volatility clusters followed by periods of low volatility is well documented by the data. These facts are coherent with the presence of different regimes of “bull” and “bear” market conditions, usually associated with “high” and “low” dependence, as for example discussed in Pelletier (2006).

Our goal is to analyse how the systemic risk spreads among the different European areas by inspecting the time evolution of the risk measures introduced in previous Sections. We would examine whether stock market co–movements have changed over time, with a focus on the crisis periods. On the one hand, we expect that the global nature of the financial crisis might imply that the co–movements become stronger, with an increase in the long–run risk. On the other hand, given the heterogeneous composition of the considered panel, where indexes strongly differ by market capitalisation and other country specific characteristics, we would expect that different countries were hit rather unequally by the 2007–2008 global financial crisis. For example, by looking at the Kendall tau coefficient in Table B.2 we observe that in the out of sample period all indexes are more correlated with the overall European index then they were before. This empirical finding suggests that the dependence structure of country specific indexes with the overall European stock market has been changing during time. We find that the 2007–2008 Global Financial Crises strongly characterises the dependence structure between the European equity markets.

### 6.2. Goodness–of–fit results

We apply the bivariate SGASC model presented in Sections 2 and 3 using the nine country specific indexes presented in the previous Section. The STOXX Europe 600 index is employed as proxy of the overall European equity market in order to evaluate the dependence of each country to the overall systemic risk. More specifically, we fit the the bivariate SGASC model to each European
regional index and the STOXX Europe 600 index described in the previous section. For model evaluation purposes the complete dataset has been divided into two sub–periods: an in–sample period, used for initial parameter estimation, beginning on January 7, 1992 and ending on September 17, 2006 and an out–of–sample period, used to evaluate the performance of the competing model, beginning on September 17, 2006 to the end of the sample period. In this section we discuss our empirical findings regarding parameter estimates of the marginal and joint model.

In order to capture all the stylised fact affecting the univariate marginal conditional distributions we estimate the AR(1)–GJR–GARCH(1,1) with $sST$ innovations described in Section 2.1. The autoregressive dynamic permits to capture the little serial correlation usually reported by financial time series, while the GJR–GARCH specification is used to efficacy describe the second order conditional moment behaviour of each series. Moreover, as can be see by inspecting Table B.2, the $sST$ innovation assumption results necessary in order to describe the negative skewness affecting the data.

The estimated coefficients of each marginal model are reported in Table B.3. Our results confirm those usually found in the financial econometric literature focusing on volatility modelling, such as strong persistence and positive reaction of the conditional variance to negative innovations. The skewness parameters $\hat{\eta}_i$ of the $sST$ distribution, reported in column 8 of Table B.3, are always statistically significant and smaller then one, justifying our choice of skew innovations. The estimated degree of freedom $\hat{\upsilon}_i$ strongly confirms the excess of kurtosis and the departure from normality. In particular, we note that for the ISEQ index the estimated value of the $\upsilon$ parameter is 5.6, which corresponds to a highly leptokurtic distribution. This finding is probably related to the post–2008 Irish economic downturn and the consequent financial crises that country experienced.

Before moving to the dependence structure analysis, we check if the AR(1)–GJR–GARCH(1,1)–$sST$ models estimated on each individual index series have been able to capture all the structure of up to the fourth conditional moments and if the Skew Student–t distributional assumption for the innovation terms is reliable. In particular, we test if the PIT of the estimated conditional densities are independently and identically uniformly distributed in the unit (0, 1). To this end, we perform the test employed by Vlaar and Palm (1993), Jondeau and Rockinger (2006), and Diebold et al. (1998) in density forecast evaluation. The iid uniform test is made of two parts. The “independent” assumption is tested by checking if all the conditional moments of the data up to the fourth have been captured by the model, while the second part of the test aims to verify if the $sST$ assumption is reliable by applying a Uniform (0, 1) test on the PIT transformations. In order to test if the PIT are independent and identically distributed, we define $\tilde{u}_{i,t} = T_i \left( \tilde{r}_{i,t}, \hat{\vartheta}_i \right)$, for $i = 1, 2, \ldots, M + 1$ as the PIT of return $i$ at time $t$, and $\tilde{u}_t = T^{-1} \sum_{j=1}^T \tilde{u}_{i,t}$ denotes the simple mean of the PIT series, for all $t = 1, 2, \ldots$ in the out–of–sample period. Here, $T_i \left( \tilde{r}_{i,t}, \hat{\vartheta}_i \right)$
denotes the estimated marginal cumulative density function for each marginal series $i = 1, 2, \ldots, 10$, and $\hat{\vartheta}_i$ denotes the correspondent marginal parameters estimate. The first test consists to examine the serial correlation of $(\hat{u}_{i,t} - \bar{\hat{u}}_i)^k$ for $k = 1, \ldots, 4$ by regressing $(\hat{u}_{i,t} - \bar{\hat{u}}_i)$ on 20 own lags. The null hypothesis of absence of serial correlation is tested using a Lagrange multiplier test defined by the statistics $(T - 20)R^2$ where $R^2$ is the coefficient of determination of the regressions of $(\hat{u}_{i,t} - \bar{\hat{u}}_i)^k$ for $k = 1, \ldots, 4$ on their lags. This test is distributed according to a $\chi^2(20)$ with a critic value of about 31.4 at confidence level of 5%.

The first four columns of Table B.4 reports the test for $k = 1, \ldots, 4$ denoted by DGT–AR$(k)$, respectively. We evince that, for almost every series and every conditional moment the marginal model is able to effectively represent the dynamic behaviour of each series. For what concern the Uniform $(0, 1)$ assumption, we employ again the test suggested in Diebold et al. (1998) by dividing the empirical distribution of $\hat{u}_{i,t}$ into $G$ bins and testing whether the empirical and the theoretical distribution significantly differ on each bin. More precisely, let us define $n_{i,q}$ as the number of $\hat{u}_{i,t}$ belonging to each bin $q = 1, 2, \ldots, G$, it can be shown that

$$\sum_{q=1}^{G} \left( \frac{n_{i,q} - \bar{n}_{i,q}}{\bar{n}_{i,q}} \right)^2 \sim \chi^2(G - 1), \quad i = 1, 2, \ldots, M + 1,$$

where $\bar{n}_{i,q}$ is the expected number of $\hat{u}_{i,t}$ belonging to the $q$–th bin. Moreover, for an estimated model the distribution is bracketed between $\chi^2(G - 1)$ and $\chi^2(G - b - 1)$ where $b$ is the number of estimated parameters. In order to be consistent with the results reported by Diebold et al. (1998) and Jondeau and Rockinger (2006) we employ $G = 20$ bins. For more information, we also refer to Vlaar and Palm (1993). The estimated statistics for the Uniform $(0, 1)$ test are reported in the last column of Table B.4 and are labelled as DGT–H(20). We can observe that, for each series the PITs are uniformly distributed over the interval $(0, 1)$ at a confidence level lower then 1%. Figure C.1 shows the empirical distribution of the PIT series with 5% approximated confidence levels reported in red. As it is possible to see, the model and the distributional assumption are able to describe almost all the bins of the real conditional distribution of returns. These findings definitely confirm the adequacy of our assumptions on the innovation term and the conditional returns’ dynamic.

After having checked the adequacy of the marginal models, we move to study the dependence structure of the series using the SGASC model. In particular, we assume two state of the world characterised by either strong or low dependence between the country specific index and the European equity index STOXXE 600. This choice is consistent with the data and the period analysed which are characterised by stable and turbulent phases as well as periods of financial crises in the equity market. Appendix A provides a detailed exposition of the financial crises that characterised the period covered by our data. Six different Copula specifications are employed in order to describe the dependence structure governing the European country. The selected Copulas differ for the dependence pattern they are able to reproduce. In particular, we consider, both
elliptical Copulas such as the Gaussian and Student–t, and Archimedean Copulas such as the Gumbel, Clayton, Frank and Plackett. Gaussian, Plackett and Frank Copulas are not able to represent any tail dependence, while the Clayton and the Gumbel Copulas consider only lower or upper tail dependence, respectively, and the Student–t Copula is able to reproduce simultaneously upper and lower tail dependence of the same magnitude. Table B.5 reports the in–sample estimated coefficients for the SGASC models. We find that almost all coefficients are strongly significant with high persistence in each state. Moreover, the positive and significantly different from zero impact of the scaled scores to the Copula parameter, suggests that the GAS dynamic effectively moves the Copula dependence parameters to the proper direction. We also find two well defined regimes that can be associated to high and low dependence. For what concerns the Markovian behaviour of the different dynamics across the various Copula specifications, we find evidence of two well defined persistent regimes. The only case when this is not true seems to be when the Plackett Copula is employed. As previously pointed out, we attribute these two different regimes to a high and low levels of dependence between the various countries and the overall European system.

6.3. Systemic risk contributions

In this section, we apply the estimated SGASC models and the systemic risk measures introduced in Section 5 to assess the systemic risk contribution of each country to the overall European equity market. We perform a one step ahead rolling forecast using the 2000 out–of–sample observations covering the period beginning on on September 17, 2006 to the end of the sample period. We update the parameter estimates every 100 observations for a total of 19 refit for the entire period. The estimated coefficients are quite stable during the forecast period suggesting that the underline structure of the economy is well described by our model. Tables with the estimated coefficients are not reported and are available upon request to the second author.

The SGASC model is employed to predict the CoVaR and the CoES of the overall European index, conditional to the distress event affecting each regional index, i.e. $\text{CoVaR}_{S|j}^{\tau_1,\tau_2}$ and $\text{CoES}_{S|j}^{\tau_1,\tau_2}$, where $j = 1, 2, \ldots, 9$ denotes the nine regional indexes and $S$ denotes the overall index of the European economic system. The confidence levels $(\tau_1, \tau_2)$ are fixed at 5%. This means that we use as conditional distress event the situation where the country specific index is below its 5% marginal Value–at–Risk level.

Figure C.2 reports, for the forecast period, the estimated comovement Value–at–Risk, $\text{CoVaR}_{S|j}$ of the European economic system, measured by the overall European index (STOXXE), with respect to each of the considered country specific indexes. Vertical dashed lines represent the major financial downturns experienced by the European economic system during the period 2006–2014. A timeline of the major European financial crisis is provided in Appendix A. The CoVaRs measure the impact of a distress event affecting each country to the overall health level of the system. Our definition implies that the lower the
CoVaR level, the higher the contribution of the individual country to the system failure. Figure C.2 gives insights about the dynamic evolution of the systemic risk contributions of individual countries during the different economic and financial phases the European system experienced since the mid 2006. Before the middle of 2007, the European systemic risk experienced a long period of small perturbations and stability ended shortly after the collapse of two Bear Stearns hedge funds in early August 2007. Starting from August 2007, the financial market experiences a huge fall, the subprime mortgage crisis that led to a financial crisis and subsequent recession that began in 2008. Several major financial institutions collapsed in September 2008, with significant disruption in the flow of credit to businesses and consumers and the onset of a severe global recession. The system hit the bottom in March 2009, and then started a slow recovery which culminated just before the European sovereign–debt crisis of April 2010.

Now we move to the evaluation of the CoVaR estimates goodness–of–fit. Since the CoVaR risk measure is essentially a modified version of the Value–at–Risk, we can employ the usual VaR backtesting procedure such as the unconditional (UC) and conditional coverage (CC) tests of Kupiec (1995) and Christoffersen (1998), the Actual over Expected (AE) ratio and the mean and maximum absolute deviation (ADmean, ADMax) considered by McAleer and da Veiga (2008). The only difference between our context and the usual VaR backtesting procedure, concerns the evaluation of the “hitting sequence” of returns exceeding the CoVaR levels. In line with our definition of CoVaR in equation (20), we follow the approach of Girardi and Ergün (2013) who consider the series of returns jointly exceeding both the CoVaR and VaR levels as the proper hitting sequences. Given our choice of the VaR and CoVaR probability levels (\( \tau_1 = \tau_2 = 5\% \)), on a forecast series of length 2000, we expect 5 CoVaR violations given that the conditioning distress event occurs. More precisely, we expect to observe only 5 times the STOXXE index exceeding its CoVaR level, given that the conditioning country specific index is below its 5% marginal VaR level. In Table B.6 we report, for each country, the AE violation ratios, as well as the p-values of the UC and CC tests. We observe that, irrespectively of the country, those Copulas which are not able to reproduce the negative tail dependence do not produce adequate predicted CoVaR levels. In fact, the Plackett, Frank and Gaussian Copulas, which have no tail dependence, display similar bad backtesting results as the Gumbel which has only upper tail dependence. Those Copulas reject the null hypothesis of correct unconditional and conditional coverage of the lower tail of the joint distribution. Clayton and Student–t Copulas instead report good CC and UC backtesting results for all the considered countries. Moreover, the Clayton and Student–t Copulas report AE ratios close to one, indicating a good forecast ability of extreme events. The ADmean and ADmax indexes also provide strong evidence in favour of the Clayton and Student–t SGASC model for capturing the dynamic evolution of the systemic risk in the European economic system.

In order to demonstrate the superior ability of the proposed model to deliver reliable systemic risk measures, we also perform a comparison between the SGASC and the Dynamic Conditional Correlation (DCC) model of Engle.
In particular, we perform the same forecast exercise described above using a DCC(1, 1) model assuming either multivariate Gaussian and Student–t innovations. We report the CC, UC p–values and the AE, ADMax, ADMean results in Table B.8. We can observe the cost of ignoring the fat–tailed shape of the joint distribution and the highly nonlinear behaviour of the dependence structure by looking at the AE ratio and the others measures that strongly support for the SGASC model.

The last part of our analysis consists to analyse the out–of–sample descriptive statistics of the $\Delta CoVaR_{S_j,t+1}^{\tau_1,\tau_2}$ and the $\Delta CoES_{S_j,t+1}^{\tau_1,\tau_2}$ for all the considered countries, $j = 1, 2, \ldots, 9$. As described in Section 5, the $\Delta CoVaR$ and the $\Delta CoES$ measure the systemic risk contribution of the country specific index to the overall European equity market represented by the STOXXE index.

We use the definition of $\Delta CoVaR_{S_j,t+1}^{\tau_1,\tau_2}$ and the $\Delta CoES_{S_j,t+1}^{\tau_1,\tau_2}$ given in Girardi and Ergün (2013) but we employ as the benchmark state of the economy the “median” CoVaR and CoES as suggested by Adrian and Brunnermeier (2011, 2014). Table B.9 reports the minimum, maximum, and average value as well as the standard deviation of the forecasted $\Delta CoVaR$ and $\Delta CoES$ over the entire out–of–sample forecasting period. To understand the meaning of these quantities, we can think of the average $\Delta CoVaR$ and $\Delta CoES$ as proxies for the average systemic risk contributions of each country specific index. Moreover, we can interpret their standard deviations as proxies for the volatility of the systemic risk contributions. According to Girardi and Ergün (2013), results, we observe an increase of the average of systemic risk contributions when the distributional assumption on the joint distributions is more reliable, as it is the case, for example, for the Clayton and Student–t Copulas. This means that, on average, the lack of an appropriate distributional assumption for the joint distribution results in an underestimate of the systemic risk contributions as measured by the CoVaR and CoES risk measures. Looking at the volatility of the systemic risk contributions, our results seem to contradict those reported by Girardi and Ergün (2013). In their work, Girardi and Ergün (2013) argue that the wrong distributional assumption is the cause for the large standard deviations observed in the systemic risk contributions, as well as for their large average values. We instead argue that, under a more reliable assumption for the joint distributions, the volatility of the systemic risk contributions decreases. These results are a direct consequence of a more accurate estimate of the systemic risk contributions in the distress situations as well as in the benchmark state.

The $\Delta CoVaR$ and $\Delta CoES$ risk measures are particularly useful in order to rank the countries in terms of their systemic risk contribution. Note that, the resulting ranking is not about the intrinsic riskiness of the particular country, but, it is instead about the role each country plays in the overall European system. This point is particularly relevant when systemic risk analyses are carried out. Indeed, the countries on the top of the rank are going to be those who play a central role into the overall European equity market. Figure C.7 reports the dynamic evolution of the out–of–sample monthly averages of the $\Delta CoVaR$ and the $\Delta CoES$ risk measures, as estimated by the Student–t SGASC model.
Figure C.7 summarises the evolution of systemic risk contributions and provides insights on the relative importance of each country to the overall European equity market. As it is possible to observe, according to the ranking generated by the Student–t SGASC model, over the last decade, France is the country with the highest systemic risk contribution, followed by Germany and Netherlands. Only in the last part of the sample, we observe an increase in the systemic risk contribution of the Spain. In the middle-ranking, we find United Kingdom, Sweden and Finland, while Austria and Ireland are at the bottom of our ranking. This unexpected result is in strict line with the finding reported by Engle et al. (2015) using the SRISK index. At a first glance, it could seem strange to observe the France on the top of the ranking, and the Ireland on its bottom. However, this finding is totally coherent with the definition of the systemic risk contribution, which, as stated before, strongly differs from the usual definition of idiosyncratic risk.

The SGASC model can be also effectively used in order to forecast several type of measures of association between the random variable such as the linear correlation, the two concordance measures Spearman’s rho and Kendall’s tau and the coefficient of tail dependence. However, given the Markovian nature of the SGASC model, and its flexibility to accommodate the marginals’ specification, the evaluation of such dependence measures can become cumbersome. The only exception is for the evaluation of upper and lower tail dependence. In fact, since this measure only depends on the Copula specification, the resulting tail dependence coefficients of the SGASC model are going to be equal to the convex linear combination those of each component conditional to the Markovian states. The linear correlation coefficient, the Spearman’s rho and the Kendall’s tau instead are not analytically available in the cases we consider in this empirical application, and the only achievable solution relies on simulation methods. Following the approach of Chollete et al. (2009), we simulate 10000 draws from the joint predictive distribution defined in equation (21), and we evaluate the linear correlation, the Spearman’s rho and the Kendall’s tau coefficients empirically on the simulated data. In order to have a prediction for the three considered dependence measure for the whole forecast period, we repeat this procedure for each one of the 2000 out of sample forecasts of the joint distribution. For each country, Figure C.6 reports the forecasted dependence measures for the Student–t SGASC model, along with the corresponding filtered marginal volatilities. Figure C.6 provides clear evidence of changing dependence structures during periods of financial turmoil. Moreover, Figure C.6 reveals that the SGASC model adequately predicts the upward shifts in the dependence measures at economically relevant dates represented by vertical dashed lines. The onset of the European sovereign debt crisis of 2010 coincides with a suddenly decrease of both the rank correlation coefficients for most of the countries. The only country that experienced an increase of only the linear correlation coefficient is Germany.

Although the dynamic patterns of contagion are evident for all country by comparing, for example, the linear correlation coefficients (green line in the main graphs) with the filtered volatilities (red line in the bottom graphs), there
is large huge cross country variability. For some countries, such as Ireland and Austria, there is no evidence of changing regimes in the linear correlation coefficient. Interestingly, those countries are the less linearly correlated with the market. Because the very well known of the correlation pitfalls, low correlations levels do not imply per se lack of dependence, see e.g., [Embretich et al. (1999)](1999) [Embretich et al. (2002)](2002). However, we observe that Austria and Ireland are those countries having the lower levels of tail dependence as well as that of the two concordance measures, Kendall’s tau and Spearman’s rho. The remaining countries show higher level of tail dependence even in the state of financial stability, with the only exception of Sweden which seems to be less affected by the 2010 European sovereign debt crisis. Interestingly, if we rank countries’ tail interdependence riskiness by their tail dependence coefficient, we observe a reversed order with respect that obtained by the co–movement Value–at–Risk and expected shortfall risk measures. The result deserves further investigation. A possible reason for this apparently counterintuitive result should be ascribed to the fact that tail dependence at penultimate levels may be significantly stronger than in the limit, see [Manner and Segers (2011)](2011).

7. Conclusion

Accurately predict the dynamic evolution of the tail interdependence between indexes returns aggregated at a country level is of great relevance for policy managers and regulators and can provide a valid instrument to understand the spread of contagion effect among countries belonging to the same financial system exposed to common risk factors. Historical evidence witnesses the presence of sudden upward shifts in the dependence as well as in the volatility patterns during periods of financial turmoil. In this paper, we propose a new MS dynamic Copula model named SGASC that explicitly accounts for the presence of different Markovian regimes as well as a smooth dynamic evolution of the dependence parameters in each state by means of a Generalised Autoregressive Score (GAS) dynamics. The major novelty of the SGASC model concerns the use of a score driven process in the dynamic of the Copula parameters when this is also subject to the realisation of a first order Markovian process. This choice allows us to introduce a stochastic behaviour in the GAS framework recently proposed by [Creal et al. (2013)](2013) and [Harvey (2013)](2013), in a natural and easy to implement way. The proposed Markovian dynamic allows for the Copula parameters to promptly react to important systemic shocks. We found the flexibility introduced by the SGASC specification of fundamental importance to describe the main empirical stylised fact of financial time series. In particular, the model is able to reproduce the highly nonlinear dynamic of the dependence structure affecting the multivariate financial time series. Moreover, as discussed by [Blasques et al. (2015)](2015), the GAS updating mechanism ensures that the resulting dependence dynamic is locally optimal in a Kullback-Leibler sense. The SGASC model is estimated by using the inference function for margins (IFM) two step maximum likelihood estimator, where the second step is performed using a version of the EM algorithm of [Dempster et al. (1977)](1977) specifically tailored to this
class of models. The proposed estimation methodology is consistently equivalent to a single step estimation as long as marginals do not depend on the latent Markovian states. The evaluation of model goodness–of–fit is conducted using both in–sample and out–of–the–sample tests and procedures. Out of sample performances of the model is then compared with those of the Dynamic Conditional Correlation (DCC) model with Gaussian and Student–t innovations. Our findings confirm that our model outperform the DCC specifications in terms of backtesting performances. By means of the alternative specifications of the proposed SGASC model which differ for the considered Copula, we provide a comprehensive study of the co–movements between European equity country indexes during the period 2006–2015 comprising the recent global financial crisis and the European sovereign debt crisis. The co–movements between indexes are evaluated by means of the CoVaR and CoES risk measures recently introduced in the econometric literature by [Adrian and Brunnermeier 2011, 2014] that have seen a huge amount of applications in particular to gauge systemic risk. We extend the definition of CoVaR to the specific framework here considered, where the predictive distribution at each time $t$ conditional to the past history of the observed process is a finite mixture of Copulas. Moreover, a new co–movement risk measures is introduced by adapting the CoES of [Adrian and Brunnermeier 2011], and analytical quasi–closed form expressions for both the CoVaR and the CoES under the SGASC specification are provided. Our definition of CoES includes the Long Run Marginal Expected Shortfall (LRMES) recently proposed by [Engle et al. 2015] as special case. Our empirical results confirm that, the proposed SGASC model is able to explain and predict the systemic risk evolution of the considered countries in an effective way. By means of our SGASC–CoVaR, France, Germany and Spain appears to be the most systemically important countries followed by, Netherlands, UK, Sweden and Finland, while, surprisingly, Austria and Ireland are at the bottom of the ranking. However, those results are in line with those obtained by [Engle et al. 2015] using a modified version of the Marginal Expected Shortfall risk measure and [Lucas et al. 2014b] who consider the European Sovereign debt CDS market.

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Appendix A. Financial crisis timeline

- August 5, 2007: the Bear Stearns hedge funds collapse.

- October 1, 2007: the UBS announcement of 3.4bn USD losses from sub-prime related investments.

- October 30, 2007: the Merrill Lynch announcement that 7.9 bn USD exposure to bad debt.

- January 19, 2008: the World Bank prediction of slowdown of global economic growth in 2008.

- February 17, 2008: UK government nationalisation of Northern Rock.

- March 16, 2008: the Bear Stearns acquisition by JP Morgan Chase.

- April 8, 2008: the IMF announcement that potential losses from the credit crunch could reach 1tn USD.

- September 7, 2008: large US mortgage lenders Fannie Mae and Freddie Mac are nationalised.

- September 15, 2008: the Lehman’s failure.

- October 10, 2008: the downgrading of Iceland Sovereign debt from A+ to BBB- by Fitch.

- November 6, 2008: Bank of England cutting of the base interest rate to lowest level since 1955.

- March 9, 2009: the peak of the onset of the recent global financial crisis.

- December 8, 2009: the downgrading of Greece’s credit rating from A- to BBB+ by Fitch ratings agency.

- April 23, 2010: the call for Eurozone–IMF rescue package by the Greek Prime Minister.

- May 18, 2010: the Greece achievement of 18bn USD bailout from EFSF, IMF and bilateral loans.

- November 29, 2010: Ireland achievement of 113bn USD bailout from EU, IMF and EFSF.

- January 5, 2011: downgrading of Iceland’s rating to junk grade by S&P.

- May 05, 2011: The ECB bails out Portugal.

- July 21, 2011: Having failed to get its house in order, Greece is bailed out for a second time.

- August 05, 2011: S&P downgrades US sovereign debt.
- February 12, 2012: Greece passes its most severe austerity package yet.

- March 12, 2012: The number of unemployed Europeans reaches its highest ever level.

- June 12, 2012: The level of Spanish borrowing reaches a record high.

- July 26, 2012: Unexpectedly, ECB president Mario Draghi, gives his strongest defence yet of the Euro, prompting markets to rally.

- March 16, 2013: Cyprus: Eurozone To Give Country 13 Billion dollars Bailout.

- April 07, 2013: Portugal: Prime Minister To Address Budget Block.

- April 30, 2013: Cyprus: Parliament Approves Bailout.

- August 23, 2013: Italy: Eurozone Crisis Leads To More Bankruptcies.

- September 17, 2013: EU: Car Sales Drop To Lowest Recorded Level.

- June 03, 2014: Eurozone Inflation Slows, Increasing Pressure On Central Bank.

- September 09, 2014: EU: Mediterranean Countries Prepare For Further Unrest.

- November 28, 2014: Italy: Unemployment Rate Reaches Record High Since 1977.

**Appendix B. Tables**

| Ticker | Index | Country          |
|--------|-------|------------------|
| AEX    | Amsterdam Exchange index | Netherlands |
| ATX    | Austrian Traded Index | Austria     |
| FCHI   | CAC 40 | France            |
| FTAS   | FTSE ALL–SHARE | UK            |
| GDAXI  | Deutsche Aktienindex 30, DAX30 | Germany |
| IBEX   | Índice Bursatil Español 35 | Spain       |
| ISEQ   | Irish Stock Exchange Overall Index | Ireland |
| OMXH25 | OMX Helsinki 25 | Finland     |
| OMXS30 | OMX Stockholm 30 | Sweden      |
| STOXXE | STOXX Europe 600 | Eurozone    |

Table B.1: Country, name and ticker of the considered European indexes.
January, 7th 1992 till September, 15th 2006. The apexes “a”, “b” and “c”, denote the GARCH(1,1) model, defined in equation (1). The in-sample period covers the period from

| Name          | Min   | Max   | Mean  | Std. Dev. | Skewness | Kurtosis | 1% Str. Lev. | JB   | Kendall's tau |
|---------------|-------|-------|-------|-----------|----------|----------|--------------|------|---------------|
| Netherlands   | -7.53 | 9.52  | 0.04  | 1.37      | -0.11    | 7.78     | -4.16        | 3270.54 | 0.69          |
| Austria       | -11.38| 7.51  | 0.04  | 1.10      | -0.71    | 10.13    | -3.22        | 7554.08 | 0.35          |
| France        | -7.68 | 7.00  | 0.03  | 1.39      | -0.07    | 5.55     | -3.92        | 932.35  | 0.73          |
| United Kingdom| -5.35 | 5.70  | 0.03  | 0.97      | -0.17    | 6.32     | -2.75        | 1592.48 | 0.59          |
| Germany       | -8.87 | 7.51  | 0.04  | 1.49      | -0.19    | 6.23     | -4.31        | 1515.19 | 0.70          |
| Spain         | -8.78 | 6.32  | 0.05  | 1.35      | -0.26    | 6.06     | -3.62        | 1378.94 | 0.62          |
| Ireland       | -7.57 | 5.84  | 0.05  | 1.02      | -0.42    | 8.17     | -2.96        | 3933.34 | 0.38          |
| Finland       | -11.54| 13.11 | 0.06  | 1.59      | -0.16    | 7.95     | -4.43        | 3521.76 | 0.48          |
| Sweden        | -8.81 | 10.74 | 0.05  | 1.53      | 0.13     | 6.40     | -4.02        | 1667.05 | 0.55          |
| Eurozone      | -9.18 | 7.25  | 0.04  | 1.22      | -0.24    | 7.11     | -3.57        | 2454.59 | -             |

Table B.2: Summary statistics of the panel of country specific indexes along with the Stoxx Europe 600 index, for the period form January, 7th 1992 till January, 6th 2015. The seventh column, denoted by “1% Str. Lev.” is the 1% empirical quantile of the returns distribution, while the eight column, denoted by “JB” is the value of the Jarque-Bera test-statistics. The last columns report the estimated Kendall tau with respect to the Eurozone Index.

| Asset          | $\phi_{0,i}$ | $\phi_{1,i}$ | $\phi_{2,i}$ | $\phi_{3,i}$ | $\phi_{4,i}$ | $\phi_{5,i}$ | $\phi_{6,i}$ | $\phi_{7,i}$ | $\sigma_{1,i}$ |
|----------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|---------------|
| Netherlands    | 0.0443       | 0.0074       | 0.0139       | 0.0271       | 0.0959       | 0.9151       | 0.8885       | 13.0670      |
| Austria        | 0.0521       | 0.0996       | 0.0495       | 0.0490       | 0.0726       | 0.9609       | 0.9633       | 6.7414       |
| France         | 0.0279      | -0.0006      | 0.0168       | 0.0112       | 0.0851       | 0.9364       | 0.9221       | 18.0409      |
| United Kingdom | 0.0265      | 0.0225       | 0.0096       | 0.0089       | 0.1006       | 0.9285       | 0.9699       | 10.7900      |
| Germany        | 0.0488      | -0.0283      | 0.0175       | 0.0276       | 0.0881       | 0.9193       | 0.9047       | 11.5742      |
| Spain          | 0.0623      | 0.0294       | 0.0179       | 0.0399       | 0.0744       | 0.9129       | 0.9230       | 9.1260       |
| Ireland        | 0.0548      | 0.1082       | 0.0152       | 0.0376       | 0.0587       | 0.9184       | 0.9504       | 5.6106       |
| Finland        | 0.0854      | 0.0803       | 0.0244       | 0.0609       | 0.0476       | 0.9069       | 0.9580       | 8.3926       |
| Sweden         | 0.0727      | 0.0242       | 0.0308       | 0.0305       | 0.1043       | 0.9035       | 0.9665       | 14.9206      |
| Eurozone       | 0.0422      | 0.0166       | 0.0123       | 0.0308       | 0.0843       | 0.9164       | 0.8956       | 11.0540      |

Table B.3: In sample parameters estimate of the marginal Skew–Student–t AR(1)–GJR–GARCH(1,1) model, defined in equation [4]. The in-sample period covers the period form January, 7th 1992 till September, 15th 2006. The apexes “a”, “b” and “c”, denote the rejection of the null hypothesis of not significance of the corresponding parameter, at different confidence levels 1%, 5% and 10%, while “d” denotes a p-value larger than 10%.
Table B.4: In sample Goodness–of–Fit test of Diebold et al. (1998). Significance is denoted by superscripts at the 1%(\textsuperscript{a}), 5%(\textsuperscript{b}), and 10%(\textsuperscript{c}) levels. See also Vlaar and Palm (1993) and Jondeau and Rockinger (2006).

| Asset         | DGT–AR(1) | DGT–AR(2) | DGT–AR(3) | DGT–AR(4) | DGT–H(20) |
|---------------|-----------|-----------|-----------|-----------|-----------|
| Netherlands   | 24.79     | 25.17     | 19.86     | 22.74     | 10.61     |
| Austria       | 22.92     | 19.70     | 20.12     | 22.01     | 12.95     |
| France        | 30.89\textsuperscript{c} | 24.44     | 26.41     | 27.61     | 14.96     |
| United Kingdom| 24.82     | 29.72\textsuperscript{c} | 23.30     | 31.76\textsuperscript{b} | 14.45     |
| Germany       | 21.49     | 26.40     | 19.97     | 25.39     | 26.56     |
| Spain         | 27.76     | 16.18     | 21.38     | 17.40     | 25.42     |
| Ireland       | 17.92     | 25.95     | 14.55     | 29.14\textsuperscript{c} | 16.88     |
| Finland       | 31.08\textsuperscript{c} | 11.48     | 34.74\textsuperscript{b} | 12.54     | 22.29     |
| Sweden        | 18.50     | 21.86     | 20.56     | 21.47     | 21.07     |
| Eurozone      | 28.86\textsuperscript{c} | 25.65     | 20.76     | 28.01     | 20.16     |
Table B.5: Parameters estimate of the Student-\( t\), Gaussian, Clayton, Plackett, Frank and Gumbel Switching-GAS Copula models. The apexes “a”, “b” and “c”, denote the rejection of the null hypothesis of not significance of the corresponding parameter, at different confidence levels 1\%, 5\% and 10\%, while “d” denotes a p-value larger than 10\%.
| Asset        | Clayton | Frank | Gumbel | Normal | Plackett | Student–t |
|--------------|---------|-------|--------|--------|----------|-----------|
| **Actual over Expected (AE)** |         |       |        |        |          |           |
| Netherlands  | 1.65    | 2.57  | 2.02   | 1.65   | 2.20     | 1.65      |
| Austria      | 1.25    | 2.92  | 1.81   | 1.67   | 2.08     | 1.39      |
| France       | 1.59    | 2.12  | 1.59   | 1.59   | 1.77     | 1.59      |
| United Kingdom | 1.37  | 2.56  | 1.71   | 1.37   | 1.71     | 1.37      |
| Germany      | 1.61    | 2.50  | 1.61   | 1.61   | 2.14     | 1.61      |
| Spain        | 1.49    | 1.98  | 1.82   | 1.65   | 1.82     | 1.49      |
| Ireland      | 1.36    | 3.22  | 3.22   | 1.86   | 2.54     | 1.69      |
| Finland      | 1.36    | 2.54  | 1.69   | 1.36   | 2.03     | 1.53      |
| Sweden       | 1.68    | 3.18  | 2.43   | 2.06   | 2.43     | 1.68      |
| **Unconditional Coverage (UC)** |         |       |        |        |          |           |
| Netherlands  | 15.20   | 0.15  | 3.10   | 15.20  | 1.23     | 15.20     |
| Austria      | 50.69   | 0.00  | 4.52   | 9.25   | 0.88     | 31.07     |
| France       | 18.14   | 1.64  | 18.14  | 18.14  | 8.87     | 18.14     |
| United Kingdom | 38.65 | 1.11  | 10.82  | 38.65  | 10.82    | 38.65     |
| Germany      | 17.38   | 0.20  | 17.38  | 17.38  | 1.53     | 17.38     |
| Spain        | 24.96   | 2.76  | 6.25   | 13.04  | 6.25     | 24.96     |
| Ireland      | 39.90   | 0.00  | 5.31   | 0.12   | 11.35    |           |
| Finland      | 39.90   | 0.12  | 11.35  | 39.9   | 2.29     | 22.26     |
| Sweden       | 13.85   | 0.00  | 0.38   | 2.72   | 0.38     | 13.85     |
| **Conditional Coverage (CC)** |         |       |        |        |          |           |
| Netherlands  | 34.24   | 0.42  | 9.68   | 34.24  | 1.51     | 34.24     |
| Austria      | 43.82   | 0.01  | 10.02  | 4.51   | 1.64     | 28.18     |
| France       | 38.68   | 1.80  | 38.68  | 38.68  | 23.31    | 38.68     |
| United Kingdom | 37.98 | 0.05  | 10.70  | 37.98  | 10.70    | 37.98     |
| Germany      | 37.56   | 0.52  | 37.56  | 37.56  | 4.26     | 37.56     |
| Spain        | 47.52   | 8.66  | 17.65  | 31.26  | 17.65    | 47.52     |
| Ireland      | 54.2    | 0.00  | 2.25   | 0.29   | 27.52    |           |
| Finland      | 58.56   | 0.36  | 28.19  | 58.56  | 7.30     | 44.23     |
| Sweden       | 32.05   | 0.02  | 1.29   | 8.63   | 1.29     | 32.05     |

Table B.6: Actual over expected (A/E) ratios and p-values of the Unconditional Coverage (UC) and Conditional Coverage (CC) tests of [Kupiec 1995](#) and [Christoffersen 1998](#) for the CoVaR$^{S_j}$, $j = 1, 2, \ldots$, for different Switching–GAS Copula specifications. The AE indicator is calculated as the ratio between the realised and expected CoVaR$^{S_j}$, $j = 1, 2, \ldots$ exceedances, while the hitting values of the UC and CC tests are calculated using the procedure suggested by [Girardi and Ergün 2013](#).
| Asset          | Clayton | Frank | Gumbel | Normal | Plackett | Student–t |
|---------------|---------|-------|--------|--------|----------|-----------|
| **Maximum Absolute Deviation (ADmax)** |         |       |        |        |          |           |
| Netherlands   | 2.35    | 2.15  | 2.33   | 2.34   | 2.29     | 2.35      |
| Austria       | 2.33    | 1.68  | 1.83   | 2.13   | 1.78     | 2.18      |
| France        | 2.35    | 2.24  | 2.34   | 2.35   | 2.32     | 2.35      |
| United Kingdom| 2.34    | 1.99  | 2.13   | 2.34   | 2.15     | 2.34      |
| Germany       | 2.35    | 2.17  | 2.34   | 2.34   | 2.31     | 2.35      |
| Spain         | 2.34    | 2.11  | 2.31   | 2.34   | 2.26     | 2.34      |
| Ireland       | 2.32    | 1.76  | 1.87   | 2.24   | 1.92     | 2.25      |
| Finland       | 2.34    | 1.92  | 2.24   | 2.31   | 2.15     | 2.33      |
| Sweden        | 2.34    | 1.93  | 2.05   | 2.33   | 2.07     | 2.33      |
| **Mean Absolute Deviation (ADmean)** |         |       |        |        |          |           |
| Netherlands   | 0.20    | 0.28  | 0.24   | 0.20   | 0.25     | 0.20      |
| Austria       | 0.15    | 0.25  | 0.17   | 0.18   | 0.19     | 0.15      |
| France        | 0.19    | 0.24  | 0.19   | 0.19   | 0.21     | 0.19      |
| United Kingdom| 0.16    | 0.26  | 0.18   | 0.16   | 0.19     | 0.16      |
| Germany       | 0.19    | 0.27  | 0.19   | 0.19   | 0.25     | 0.19      |
| Spain         | 0.18    | 0.21  | 0.21   | 0.20   | 0.21     | 0.18      |
| Ireland       | 0.16    | 0.29  | 0.30   | 0.21   | 0.25     | 0.19      |
| Finland       | 0.16    | 0.25  | 0.19   | 0.16   | 0.22     | 0.18      |
| Sweden        | 0.20    | 0.31  | 0.25   | 0.24   | 0.25     | 0.20      |

Table B.7: Maximum absolute deviation (ADmax) and Mean absolute deviation (ADmean) for the CoVaR$_{S_j}$, $j = 1, 2, \ldots$, for different Switching–GAS Copula specifications. The hitting values of the UC and CC tests are calculated using the procedure suggested by Girardi and Ergün (2013).

| Asset          | CC     | UC    | AE    | ADMax | ADMean  | CC     | UC    | AE    | ADMax | ADMean  |
|---------------|--------|-------|-------|-------|---------|--------|------|-------|-------|---------|
| **Gaussian**  |        |       |       |       |         |        |      |       |       |         |
| Netherlands   | 0.16   | 0.04  | 2.69  | 1.92  | 0.26    | 1.61   | 1.13 | 2.15  | 2.18  | 0.24    |
| Austria       | 0.14   | 0.03  | 2.57  | 1.76  | 0.23    | 12.71  | 6.37 | 1.72  | 2.07  | 0.18    |
| France        | 0.16   | 0.04  | 2.69  | 1.92  | 0.26    | 2.80   | 3.51 | 1.92  | 2.19  | 0.21    |
| United Kingdom| 0.09   | 0.18  | 2.44  | 1.91  | 0.23    | 7.16   | 7.69 | 1.76  | 2.21  | 0.20    |
| Germany       | 0.17   | 0.05  | 2.67  | 1.92  | 0.26    | 8.23   | 3.72 | 1.90  | 2.19  | 0.21    |
| Spain         | 3.34   | 1.03  | 2.11  | 1.92  | 0.20    | 21.75  | 8.07 | 1.70  | 2.26  | 0.19    |
| Ireland       | 0.12   | 0.04  | 2.71  | 1.82  | 0.25    | 13.54  | 8.91 | 1.72  | 2.21  | 0.19    |
| Finland       | 0.13   | 0.03  | 2.78  | 1.90  | 0.27    | 26.28  | 10.25| 1.68  | 2.19  | 0.19    |
| Sweden        | 0.03   | 0.01  | 3.01  | 1.90  | 0.29    | 0.51   | 0.73 | 2.26  | 2.16  | 0.25    |
| **Student–t** |        |       |       |       |         |        |      |       |       |         |

Table B.8: Backtesting results for the CoVaR$_{S_j}$, $j = 1, 2, \ldots$, under Gaussian and Student–t DCC specifications. “CC” and “UC” denotes the conditional coverage and unconditional coverage test of of Kupiec (1995) and Christoffersen (1998), respectively, “AE” stands for the actual vs expected exceedances, while “ADMean” and “ADmax”, denote the mean absolute deviation and the max absolute deviations, respectively. The hitting values of the UC and CC tests are calculated using the procedure suggested by Girardi and Ergün (2013).
## Table B.9: Descriptive statistics of the out-of-sample $\Delta$CoVaR$_{S1j}$ and $\Delta$CoES$_{S1j}$, for $j = 1, \ldots, 5$ of the Switching GAS Copula model specifications.

| Asset       | $\Delta$CoVaR | $\Delta$CoES |
|-------------|---------------|---------------|
|             | Clayton       | Frank         | Gumbel | Gaussian | Plackett | Student-t | Clayton       | Frank         | Gumbel | Gaussian | Plackett | Student-t |
| Netherlands | 57.62         | 33.07         | 33.54   | 53.57    | 45.69    | 54.16     | 45.56         | 26.04         | 28.54   | 44.90    | 36.03    | 43.76      |
| Austria     | 57.62         | 20.12         | 28.01   | 35.64    | 26.50    | 46.16     | 45.56         | 21.65         | 43.33    | 45.56    | 43.30    | 45.56      |
| France      | 57.62         | 27.62         | 55.06   | 57.62    | 57.62    | 57.62     | 57.62         | 57.62         | 57.62    | 57.62    | 57.62    | 57.62      |
| United Kingdom | 57.63        | 28.54         | 36.86   | 51.49    | 37.32    | 54.16     | 54.16         | 22.53         | 30.44    | 42.99    | 29.56    | 44.60      |
| Germany     | 52.39         | 38.17         | 44.01   | 57.57    | 57.57    | 57.57     | 57.57         | 57.57         | 57.57    | 57.57    | 57.57    | 57.57      |
| Spain       | 57.63         | 24.80         | 5.91    | 54.98    | 38.96    | 54.58     | 54.58         | 15.68         | 23.83    | 38.93    | 19.20    | 31.47      |
| Ireland     | 56.38         | 22.26         | 34.22   | 54.13    | 35.50    | -7.79     | 43.85         | 17.49         | 28.62    | 44.50    | 28.07    | 9.13       |
| Finland     | 56.53         | 20.39         | 4.17    | 21.42    | 32.88    | 10.69     | 43.90         | 15.88         | 3.54     | 18.16    | 25.82    | 15.04      |
| Mean        | 61.50         | 44.55         | 58.01   | 61.32    | 55.01    | 61.39     | 48.95         | 35.42         | 46.73    | 48.89    | 43.74    | 48.91      |
| Standard deviation | 1.63        | 3.67          | 3.04    | 1.61     | 2.85     | 1.65      | 1.39          | 2.97          | 2.32     | 1.37     | 2.34     | 1.40       |

*Table B.9: Descriptive statistics of the out-of-sample $\Delta$CoVaR$_{S1j}$ and $\Delta$CoES$_{S1j}$, for $j = 1, \ldots, 5$ of the Switching GAS Copula model specifications.*
Appendix C. Figures

Figure C.1: Marginal empirical PIT distributions. The histogram is divided into 20 bins, red lines represents 5% approximated confidence intervals. For more informations see Jondeau and Rockinger (2006) and Diebold et al. (1998).
Figure C.2: Predicted CoVaR \( S_{t+1} | Y_{1:t} \) for \( j = 1, 2, \ldots, 9 \) estimated using the Gaussian, Student-t, Frank, Gumbel, Clayton, and Plackett, SGASC model. The black line in the bottom figures represents the predicted estimates probabilities of the high systemic risk state, i.e. \( P (S_{t+1} | Y_{1:t}) \), for \( t = 1, 2, \ldots, T \). Vertical dashed lines represent major financial down-turns: for a detailed description see Figure C.2 and Appendix A.
Figure C.3: Predicted $\Delta \text{CoVaR}^{S_{jt}}_{t,t-1}$, for $j = 1, 2, \ldots, 9$ estimated using the Gaussian, Student-t, Frank, Gumbel, Clayton, and Plackett, SGASC model. The black line in the bottom figures represents the predicted estimates probabilities of the high systemic risk state, i.e. $\mathbb{P}(S_{t+1} \mid Y_{1:t})$, for $t = 1, 2, \ldots, T$. Vertical dashed lines represent major financial downturns: for a detailed description see Figure C.2 and Appendix A.
Figure C.4: Predicted CoES$^{S_{t+1}}_{t}$, for $j = 1, 2, \ldots, 9$ estimated using the Gaussian, Student–t, Frank, Gumbel, Clayton, and Plackett, SGASC model. The black line in the bottom figures represents the predicted estimates probabilities of the high systemic risk state, i.e. $\mathbb{P}(S_{t+1} \mid Y_{1:t})$, for $t = 1, 2, \ldots, T$. Vertical dashed lines represent major financial downturns: for a detailed description see Figure C.2 and Appendix A.
Figure C.5: Predicted $\Delta \text{CoES}^{(j)}_{t|t-1}$, for $j = 1, 2, \ldots, 9$ estimated using the Gaussian, Student-t, Frank, Gumbel, Clayton, and Plackett, SGASC model. The black line in the bottom figures represents the predicted estimates probabilities of the high systemic risk state, i.e. $P(S_{t+1} | \mathbf{Y}_{1:t})$, for $t = 1, 2, \ldots, T$. Vertical dashed lines represent major financial downturns; for a detailed description see Figure C.2 and Appendix A.
Figure C.6: Predicted dependence measures under the estimated Student-\(t\) SGAS Copula model: linear correlation (blue line), Kendall’s tau (red line), Spearman’s rho (black line), negative tail dependence (green line). The red line in the bottom figures represents the predicted marginal volatilities, i.e. \(\hat{\sigma}_{j,t+1}\), for \(t = 1, 2, \ldots, T\) and \(j = 1, 2, \ldots, 9\). Vertical dashed lines represent major financial downturns: for a detailed description see Figure C.2 and Appendix A.
Figure C.7: Out-of-sample evolution of the $\Delta$CoVaR (top graph) and $\Delta$CoES (bottom graph) weekly averages. Vertical dashed lines denote the Bear Stearns hedge funds collapse (August 5, 2007), the Lehmanns failure (September 15, 2008), the downgrading of Greece’s credit rating from A- to BBB+ by Fitch ratings agency (December 8, 2009), the protraction of the Greek austerity package (February 12, 2012). A complete timeline of the crisis events can be found in Appendix A.
Appendix D. Copulas

As previously discussed, the Copula functions are useful tools when dealing with dependence among random variables. In what follows we focus on the bivariate case with continuous marginal distributions. For more general cases and an up-to-date extensive introduction on Copulas we refer to Durante and Jaworski (2010), Chollete et al. (2011) and Embrechts et al. (2003).

Let be \( X, Y \) two continuous random variables on a complete probability space \((\mathbb{R} \times \mathbb{R}, \mathcal{X}, \mathbb{H})\), where \( \mathcal{X} \equiv \mathcal{B} (\mathbb{R} \times \mathbb{R}) \) is the Borel \( \sigma \)-field generated by \( \mathbb{R} \times \mathbb{R} \) and \( \mathbb{H} \) is a probability measure. Moreover, let \( F \) and \( G \) denote the conditional distribution function (cdf) of \( X \) and \( Y \), respectively, and let \( H \) be the cdf of the bivariate vector \((X, Y)\), so that \( F (x) = \mathbb{P} (X < x) \), \( G (y) = \mathbb{P} (Y < y) \) and \( H (x, y) = \mathbb{P} (X < x, Y < y) \). The Copula probability distribution of the bivariate random vector \((X, Y)\) is the joint probability distribution of \( U \equiv F (X) \) and \( V \equiv G (Y) \), where \( U \) and \( V \) are known as the probability integral transformation (PIT) of \( X \) and \( Y \). Fisher (1925) shows that \( V \) and \( U \) are uniformly distributed in Unif \((0,1)\). From the Fisher’s result it follows immediately that the bivariate Copula distribution function of \( X \) and \( Y \) has two uniform random variables as marginals. The bivariate Copula has the following properties:

1. The copula \( C \) is a function \( C : [0,1] \times [0,1] \rightarrow [0,1] \)
2. \( C (u, 0) = C (0, v) = 0 \), \( \forall (u, v) \in [0,1] \)
3. \( C (u_2, v_2) - C (u_1, v_2) - C (u_2, v_1) + C (u_1, v_1) \geq 0 \) \( \forall \) \( u_1, u_2, v_1, v_2 \in [0,1] \) such that \( u_1 \leq u_2 \) and \( v_1 \leq v_2 \)

One of the most important results in the copula theory regards the Sklar (1959)’s Theorem reported below.

**Theorem 1 (Sklar).** Let \( F \) be the distribution of \( X \), \( G \) be the distribution of \( Y \), and \( H \) be the joint distribution of \((X, Y)\). Assume that \( F \) and \( G \) are continuous.

Then there exists a unique Copula \( C \) such that:

\[
H (x, y) = C (F (x), G (y)), \quad \forall (x, y) \in \mathbb{R} \cup \{\pm \infty\} \times \mathbb{R} \cup \{\pm \infty\}.
\]

(D.1)

Conversely, if we let \( F \) and \( G \) be two distribution functions and \( C \) be a Copula function, then the function \( H \) defined by equation (D.1) is a bivariate distribution function with marginal distribution \( F \) and \( G \). The Sklar’s theorem allows for to write the probability density function \( h (\cdot) \) of the bivariate random vector \((X, Y)\) as:

\[
h (x, y) = c (F (x), G (y)) f (x) g (y)
\]

(D.2)

where

\[
c (x, y) = \frac{\partial^2 C (F (x), G (y))}{\partial x \partial y}
\]

and \( f (x), g (y) \) are the probability density function of \( X \) and \( Y \). From equation (D.2) it is clear the utility of Copulas in a multivariate framework. In fact,
using Sklar’s theorem it is possible to link any univariate distribution into a joint distribution using Copulas. From a financial econometrics point of view, Sklar’s theorem make possible to define several marginal distributions for the univariate time series of different assets, and then model their dependence using a specific Copula functions $C$.

**Appendix E. Copulae pdfs**

In this section we report the Copula densities used throughout the paper. The Gaussian Copula density is

$$c_{\text{Ga}}(u_1, u_2) = \frac{\exp \left( \frac{2\rho x_1 x_2 - x_1^2 - x_2^2}{2(1-\rho^2)} + \frac{1}{2} \left( x_1^2 + x_2^2 \right) \right)}{\sqrt{1-\rho^2}},$$

with $x_i = \Phi^{-1}(u_i)$, for $i = 1, 2$, where $\Phi(\cdot)$ denotes the univariate Gaussian cdf. The Student–t Copula density is

$$c_{\text{T}}(u_1, u_2) = \frac{\Gamma \left( \frac{\nu}{2} \right) \Gamma \left( \frac{\nu+2}{2} \right)}{\sqrt{1-\rho^2} \Gamma \left( \frac{\nu+1}{2} \right)^2} \times \left( 1 + \frac{x_1^2}{\nu} \right)^\frac{\nu-1}{\nu} \left( 1 + \frac{x_2^2}{\nu} \right)^\frac{\nu-1}{\nu} \left( 1 + \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\nu(1-\rho^2)} \right)^{-\frac{1}{2}(\nu+2)},$$

with $x_i = T^{-1}_\nu(u_i)$ for $i = 1, 2$, where $\Phi(\cdot)$ denotes the univariate Student–t cdf. The Gumbel Copula density is

$$c_{\text{Gu}}(u_1, u_2) = \frac{1}{u_1 u_2} e^{- \left( (-\log u_1)^\theta + (-\log u_2)^\theta \right)^{\frac{1}{\theta}}} \left[ (-\log u_1)^\theta + (-\log u_2)^\theta \right]^{\frac{1}{\theta}-2} \times \left[ 1 + (\theta - 1) \left( (-\log (u_1))^\theta + (-\log (u_2))^\theta \right)^{-1/\theta} \right] \times (\log (u_1) \log (u_2))^{\theta-1}.$$

The Clayton Copula density is

$$c_{\text{Cl}}(u_1, u_2) = (\theta + 1) (u_1 u_2)^{-\theta-1} \left( u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-\frac{1}{\theta} - 1}.$$

The Frank Copula density is

$$c_{\text{Fr}}(u_1, u_2) = -\frac{(e^{-\theta} - 1) \theta e^{-\theta(u_1 + u_2)}}{(e^{-\theta} + (e^{-\theta} u_1 - 1)(e^{-\theta} u_2 - 1)^2).}$$

The Plackett Copula density is

$$c_{\text{Pl}}(u_1, u_2) = \frac{\pi ((\pi - 1)(-2u_2 u_1 + u_1 + u_2) + 1)}{\left( ((\pi - 1)(u_1 + u_2) + 1)^2 - 4(\pi - 1)\pi u_1 u_2 \right)^{\frac{3}{2}}}.$$
Appendix F. Copulae scores

The Gaussian and Student–t Copula scores are

\[
\frac{\partial \log c_{\mathrm{Ga}}}{\partial \rho} = \frac{\rho}{1 - \rho^2} - \frac{\rho \left(-2\rho x_2 x_1 + x_1^2 + x_2^2\right) - (1 - \rho^2) x_1 x_2}{(1 - \rho^2)^2}
\]

\[
\frac{\partial \log c_{\mathrm{T}}}{\partial \rho} = \frac{x_2 x_1 (\nu + \nu \rho^2 + 2) - \rho \left(\nu (\rho^2 - 1) + (\nu + 1)x_2^2\right) + (\nu + 1)(-\rho)x_1^2}{(\rho^2 - 1)\left(\nu (\rho^2 - 1) + 2\rho x_2 x_1 - x_1^2 - x_2^2\right)},
\]

where \(x_i = \Phi^{-1}(u_i)\) and \(x_i = T_{\nu}^{-1}(u_i)\), for \(i = 1, 2\) are the PIT transformations in the Gaussian and Student–t case, respectively. The Gumbel Copula score is

\[
\frac{\partial \log c_{\mathrm{G}}}{\partial \rho} = \left(-\theta \log (-\log (u_1))\right) \left(-\log (u_1)\right)^\theta
\]

\[
- \theta \log (-\log (u_2)) \left(-\log (u_2)\right)^\theta \log \left(\Psi (u_1, u_2, \theta)\right) \Psi (u_1, u_2, \theta)
\]

\[
\times \frac{\Psi (u_1, u_2, \theta)^{\frac{1}{\theta}} - 2 \log \left(\Psi (u_1, u_2, \theta)\right)}{\theta^2}
\]

\[
- \frac{\theta - 1}{\theta^2} \log (-\log (u_1)) \left(-\log (u_1)\right)^\theta \Psi (u_1, u_2, \theta)
\]

\[
+ \frac{\theta - 1}{\theta^2} \log (-\log (u_1)) \left(-\log (u_1)\right)^\theta \Psi (u_1, u_2, \theta)
\]

\[
+ \frac{\theta - \theta \Psi (u_1, u_2, \theta) - (\theta - 1) \log (-\log (u_2)) \left(-\log (u_2)\right)^\theta}{\theta^2 \left(\theta + \Psi (u_1, u_2, \theta)^{\frac{1}{\theta}} - 1\right) \Psi (u_1, u_2, \theta)}
\]

\[
+ \frac{\left(\frac{\theta}{\theta - 2}\right) \left(-\log (u_1)\right)^\theta + \log (-\log (u_2)) \left(-\log (u_2)\right)^\theta}{\theta^2 \left(\theta + \Psi (u_1, u_2, \theta)^{\frac{1}{\theta}} - 1\right) \Psi (u_1, u_2, \theta)^\theta}
\]

where \(\Psi (u_1, u_2, \theta) = (-\log (u_1))^{\theta} + (-\log (u_2))^{\theta}\). The Clayton Copula score is

\[
\frac{\partial \log c_{\mathrm{C}}}{\partial \rho} = \frac{c_{\mathrm{C}}^N}{c_{\mathrm{C}}^D},
\]

where

\[
c_{\mathrm{C}}^N = -\theta \left(2\theta^2 + 3\theta + 1\right) u_1^{\theta} \log (u_2)
\]

\[- \theta \left(2\theta^2 + 3\theta + 1\right) u_2^{\theta} \log (u_1)
\]

\[- \left(u_1^{\theta} u_2^{\theta - 1} - u_2^{\theta}\right)
\]

\[
\times \left(-\theta^2 + (\theta + 1)^2 \theta \log (u_1 u_2) - (\theta + 1) \log (u_1^{\theta} + u_2^{\theta - 1})\right)
\]

\[
c_{\mathrm{C}}^D = \theta^2 (\theta + 1) \left(u_1^{\theta} (u_2^{\theta - 1} - u_2^{\theta})\right).
\]
The Frank Copula score is
\[
\frac{\partial \log c_{Fr}}{\partial \theta} = \frac{e^{\theta(u_2 + 2)} (\theta u_1 - \theta u_2 + 1) + e^{\theta + \theta u_1} (\theta + \theta u_1 - \theta u_2 - 1)}{(e^\theta - 1) \theta \left( -e^\theta + e^{\theta + \theta u_1} - e^{\theta(u_1 + u_2)} + e^{\theta + \theta u_2} \right)} \\
+ \frac{e^{\theta(u_1 + 2)} (-\theta u_1 + \theta u_2 + 1) + e^{\theta + \theta u_2} (\theta + \theta (-u_1) + \theta u_2 - 1)}{(e^\theta - 1) \theta \left( -e^\theta + e^{\theta + \theta u_1} - e^{\theta(u_1 + u_2)} + e^{\theta + \theta u_2} \right)} \\
- \frac{e^{2\theta} (\theta u_1 + \theta u_2 + 1) + e^{\theta(u_1 + u_2 + 1)} (\theta (u_1 + u_2 - 1) - 1)}{(e^\theta - 1) \theta \left( -e^\theta + e^{\theta + \theta u_1} - e^{\theta(u_1 + u_2)} + e^{\theta + \theta u_2} \right)} \\
+ \frac{e^{\theta(u_1 + u_2)} (1 - \theta (u_1 + u_2 - 1)) + e^{\theta} (\theta (u_1 + u_2 - 1) + 1)}{(e^\theta - 1) \theta \left( -e^\theta + e^{\theta + \theta u_1} - e^{\theta(u_1 + u_2)} + e^{\theta + \theta u_2} \right)}. 
\]

The Plackett Copula score is
\[
\frac{\partial \log c_{Pl}}{\partial \pi} = \frac{c_{Pl}^N}{c_{Pl}^D},
\]
where
\[
c_{Pl}^N = - (\pi - 1)^2 (1 + \pi) (2u_2 - 1) u_1^3 \\
+ (\pi - 1) (4 (1 - \pi + \pi^2) u_2^3 - (7 - 4\pi + \pi^2) u_2 + \pi + 3) u_1^2 \\
- 2 (\pi - 1)^2 (1 + \pi) u_2^3 - (7 - 11\pi + 5\pi^2 - \pi^3) u_1 u_2^2 \\
- 2 (4 - 5\pi + 3\pi^2) u_1 u_2 - (\pi - 3) u_1 \\
+ ((\pi - 1) u_2 + 1)^2 (\pi u_2 + u_2 - 1) \\
c_{Pl}^D = \pi (-\pi u_2 + u_2 + (\pi - 1) u_1 (2u_2 - 1) - 1) \\
\times \left( ((\pi - 1) (u_1 + u_2) + 1)^2 - 4(\pi - 1)\pi u_1 u_2 \right). \tag{F.1}
\]

Appendix G. Copulas Information Quantities

The Gaussian Copula information quantity with respect the correlation parameter \(\rho\) is
\[
I_{Ga} (\rho) = \frac{1 + \rho^2}{(1 - \rho^2)^2}. \tag{G.1}
\]
The Student–t information quantity with respect the correlation parameter \(\rho\) is
\[
I_{t} (\rho, \nu) = \frac{\nu + 2 + \nu \rho^2}{(\nu + 4)(1 - \rho^2)^2}. \tag{G.2}
\]
The Gumbel information quantity with respect to the \(\theta\) parameter is
\[
I_{Gu} (\theta, \nu) = \theta \left( \pi^2 - \frac{2}{3} \right) - 1 + \frac{2K_0}{\theta^2} \\
+ \left( \theta^2 + \theta^2 + (K_0 - 1) \theta - 2K_0 + \frac{K_0}{\theta} \right) \frac{E_1 (\theta - 1) e^{\theta - 1}}{\theta}. \tag{G.3}
\]
where \( K_0 = \left( \frac{5}{6} - \frac{\pi^2}{18} \right) \) and \( E_1(x) = \int_x^\infty v^{-1}e^{-v}dv \) is the Exponential Integral. The Clayton information quantity with respect to the \( \theta \) parameter is

\[
\mathcal{I}_{Cl}(\rho, \nu) = \frac{1}{\theta^2} + \frac{2}{\theta(\theta - 1)(2\theta - 1)} + \frac{4\theta}{(3\theta - 2)} - \frac{2(2\theta - 1)}{\theta - 1} \cdot q(\theta), \tag{G.4}
\]

where

\[
q(\theta) = \frac{1}{(3\theta - 2)(2\theta - 1)} \left( \frac{\theta}{2(\theta - 1)} - \frac{\theta}{2(\theta - 1)} \right)
+ \frac{1}{2\theta(3\theta - 2)(2\theta - 1)(\theta - 1)} \left[ \Psi_1 \left( \frac{1}{2(\theta - 1)} \right) - \Psi_1 \left( \frac{\theta}{2\theta(\theta - 1)} \right) \right]
+ \frac{1}{\theta(3\theta - 2)(2\theta - 1)(\theta - 1)} \left[ \Psi_1 \left( \frac{\theta}{2(\theta - 1)} \right) - \Psi_1 \left( \frac{2\theta - 1}{2(\theta - 1)} \right) \right], \tag{G.5}
\]

and \( \Psi_1(x) \) denotes the trigamma function. Regarding the Frank and Plackett Copulas, unfortunately, closed form solutions for the information quantities are not available. In our empirical work, these two quantities are evaluated using the same grid approach proposed by Creal et al. (2013). More precisely, we numerically evaluate the information quantities for such Copulas by using the well know relation between the Score and the Information Matrix of density functions reported in equation (12). For example, the information quantity for the Plackett Copula, evaluated at the parameter value \( \pi_0 \), is computed by solving integral

\[
\mathcal{I}_{Pl}(\pi_0) = \int_0^1 \int_0^1 \left( \frac{c_{Pl}^{N}}{c_{Pl}^{F} \mid_{\pi=\pi_0}} \right)^2 c_{Pl}(u_1, u_2, \pi_0) \, du_1 \, du_2, \tag{G.6}
\]

where \( \frac{c_{Pl}^{N}}{c_{Pl}^{F} \mid_{\pi=\pi_0}} \) is the Score of the Plackett Copula reported in equation (F.1) evaluated at \( \pi = \pi_0 \), and \( c_{Pl}(u_1, u_2, \pi_0) \) is the Plackett density function. Nu-
numerical solution of equation (G.6), using for example quadrature rules, results in a very precise evaluation of the information quantity for the Plackett Copula at $\pi_0$. In principle, the evaluation of (G.6) can be performed at each point in time when evaluating the parameter’s dynamic. However, this solution dramatically slows down the SGASC dynamic and consequently any optimisation procedure to get, for example, maximum likelihood parameter estimates. In order to avoid this computational burden, we solve equation (G.6) on a grid of 1000 values for $\pi$ and then we fit a B–spline on the resulting points. Hence, we use the predicted values coming from the fitted B–spline during the optimisation procedure as well as for the parameter’s dynamic. Figure G.8 reports the fitted B–splines for the Plackett and the Frank Copulas.
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