Algebraic Bethe ansatz for $\mathfrak{o}_{2n+1}$-invariant integrable models

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Abstract

A class of $\mathfrak{o}_{2n+1}$-invariant quantum integrable models is investigated in the framework of algebraic Bethe ansatz method. A construction of the $\mathfrak{o}_{2n+1}$-invariant Bethe vector is proposed in terms of the Drinfeld currents for the double of Yangian $\mathcal{D}Y(\mathfrak{o}_{2n+1})$. Action of the monodromy matrix entries onto off-shell Bethe vectors for these models is calculated. Recursion relations for these vectors were obtained. The action formulas can be used to investigate structure of the scalar products of Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant models.

1 Introduction

Quantum integrable systems where the algebraic Bethe ansatz may be applied are described by the monodromy matrices that satisfy quadratic relations defined by some $R$-matrix. Main goal of the algebraic Bethe ansatz is to construct state vectors or Bethe vectors for integrable models from matrix elements of the corresponding monodromies. Eigenvalue property of the Bethe vectors with respect to the transfer matrix uses these quadratic commutation relations between monodromy matrix entries and leads to the Bethe equations on the parameters of Bethe vectors.

In this paper, we will consider quantum integrable systems whose monodromy matrices satisfy the commutation relations with the $\mathfrak{o}_{2n+1}$-invariant $R$-matrix proposed in [1,2] and solved in [3]. This $R$-matrix is associated with an infinite-dimensional algebra, that is double of Yangian $\mathcal{D}Y(\mathfrak{o}_{2n+1})$ [4,5]. Generating series of generators of this algebra

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can be assembled into a matrix or $T$-operator that will satisfy the same quadratic commutation relations as the monodromy matrix of $\mathfrak{o}_{2n+1}$-invariant integrable system. The coincidence of these commutation relations allows to construct state vectors in terms of generators of the algebra $DY(\mathfrak{o}_{2n+1})$ and use the properties of this algebra to study the space of states in $\mathfrak{o}_{2n+1}$-invariant integrable system.

The advantages of this approach to the algebraic Bethe ansatz are that doubles of Yangians and quantum affine algebras, in addition to the $R$-matrix formulation, also have a 'new' realization in terms of formal series or 'currents' proposed by [5]. For quantum affine algebra $U_q(\hat{\mathfrak{gl}}_n)$ the equivalence of the two formulations was proved in [6], and for similar infinite-dimensional algebras associated with algebras of the series $B$, $C$ and $D$ analogous equivalences were recently proved in [7–10].

The main tools in proving the equivalence of various realizations of quantum affine algebras and doubles of Yangians are the so-called Gaussian coordinates of $T$-operators. These objects turned out to be very useful in research of the space of states in quantum integrable models associated with trigonometric and rational deformations of affine algebra $\mathfrak{gl}_n$ and their super-symmetric extensions [11–13]. Gaussian coordinates may be related to the currents in the 'new' realization of the Yangians and quantum affine algebras [5] using projections to the intersections of the different types Borel subalgebras. The Bethe vectors itself can be constructed as projections from the products of currents [14].

Until recently the description of the space of states in the quantum integrable models using the current realization of the corresponding infinite-dimensional algebras was available only for the models associated with serie $A$ algebras. Recent results published in [7–10] open the possibility to develop tools for the investigation of the quantum integrable models associated with infinite-dimensional algebras for $B$, $C$ and $D$ series. In particular, the relations between Gaussian coordinates for the Yangian doubles of $B$, $C$ and $D$ series and projections of the corresponding currents were established in [15]. Note also the papers [16–18], where the different methods of investigations of the orthogonal and symplectic quantum integrable models were developed.

The present paper is a continuation of the research started in [19] where a description of the off-shell Bethe vectors for the integrable models associated with the algebra $\mathfrak{o}_3$ was obtained. In many cases, to investigate the space of states of quantum integrable model one does not need the explicit formulas for the Bethe vectors in terms of monodromy entries. It is sufficient to explore the explicit formulas of the actions of the monodromy matrix elements onto off-shell Bethe vectors. In [20] such action formulas were presented in case of the supersymmetric $\mathfrak{gl}(m|n)$-invariant integrable models. In this paper we obtain the explicit formulas for such an action for the Bethe vectors in the $\mathfrak{o}_{2n+1}$-invariant quantum integrable models.

The paper is composed as follows. Section 2 is devoted to definition of the class of $\mathfrak{o}_{2n+1}$-invariant quantum integrable models based on $R$-matrix found by A.B. Zamolod-
Main results of our research is presented in the section together with demonstration of the different reductions of the main action formula to already known cases. Last section contains the proof of two technical statements necessary for the calculation of the action formulas. These proofs use the projection method formulated for this case in [13].

2 $\mathfrak{so}_{2n+1}$-invariant $R$-matrix and $RTT$ algebra

Let $N = 2n + 1$ for positive integer $n \geq 1$ and $e_{i,j}$ are $N \times N$ matrices with the only nonzero entry equal to 1 at the intersection of the $i$-th row and $j$-th column. We will use integers to number matrix entries of the operators in $\text{End}(\mathbb{C}^N)$: $-n \leq i, j \leq n$. Let $P$ (permutation operator) and $Q$ be operators acting in $\mathbb{C}^N \otimes \mathbb{C}^N$:

$$P = \sum_{i,j=-n}^{n} e_{i,j} \otimes e_{j,i}, \quad Q = \sum_{i,j=-n}^{n} e_{i,j} \otimes e_{-i,-j}.$$  

We denote by $R(u, v)$ $\mathfrak{so}_{2n+1}$-invariant $R$-matrix [1]

$$R(u, v) = I \otimes I + \frac{cP}{u-v} - \frac{cQ}{u-v + c\kappa_n},$$  

(2.1)

where $I = \sum_{i=-n}^{n} e_{i,i}$ is the identity operator in $\mathbb{C}^N$, $c$ is a constant, $u$ and $v$ are arbitrary complex parameters called spectral parameters and

$$\kappa = N/2 - 1 = n - 1/2.$$  

Due to the properties

$$P Q = Q P = Q, \quad P^2 = I \otimes I, \quad Q^2 = NQ$$

this $R$-matrix obeys the Yang-Baxter equation

$$R_{1,2}(u_1, u_2) \cdot R_{1,3}(u_1, u_3) \cdot R_{2,3}(u_2, u_3) = R_{2,3}(u_2, u_3) \cdot R_{1,3}(u_1, u_3) \cdot R_{1,2}(u_1, u_2)$$

in $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ (here subscripts of $R$ denote spaces $\mathbb{C}^N$ in which $R$-matrix acts) and satisfies the unitarity condition

$$R(u, v) R(v, u) = \left(1 - \frac{c^2}{(u-v)^2}\right) I \otimes I.$$  

(2.2)

Let $T(u)$ be an operator valued $N \times N$ matrix satisfying the quadratic commutation relations

$$R(u, v) (T(u) \otimes I) (I \otimes T(v)) = (I \otimes T(v)) (T(u) \otimes I) R(u, v).$$  

(2.3)
These commutation relations can be written in terms of the matrix entries $T_{i,j}(u)$

$$T(u) = \sum_{i,j=-n}^{n} e_{i,j} T_{i,j}(u),$$

$$[T_{i,j}(u), T_{k,l}(v)] = \frac{c}{u-v} (T_{k,j}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v)) +$$

$$+ \frac{c}{u-v + c\kappa} \left( \delta_{k,-i} \sum_{p=-n}^{n} T_{p,j}(u)T_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n}^{n} T_{k,-p}(v)T_{i,p}(u) \right). \tag{2.4}$$

As a direct consequence of the commutation relations (2.4) the matrix elements should satisfy the relations

$$\sum_{m=-n}^{n} T_{-m,-i}(u - c\kappa)T_{m,j}(u) = \sum_{m=-n}^{n} T_{i,m}(u)T_{-j,-m}(u - c\kappa) = z(u)\delta_{i,j}, \tag{2.5}$$

where $z(u)$ is a central elements in the RTT algebra (2.4). In what follows we set this central element to 1. The quadratic relations (2.5) imposes nontrivial relations between entries $T_{i,j}(u)$.

### 2.1 $\mathfrak{o}_{2n+1}$-invariant integrable models

We denote the RTT algebra (2.3) as $B$ algebra. The algebraically dependent elements of this algebra are operators $T_{i,j}[\ell]$, $\ell \geq 0$, $-n \leq i, j \leq n$ gathered into generating series

$$T_{i,j}(u) = \delta_{ij} + \sum_{\ell \geq 0} T_{i,j}[\ell](u/c)^{-\ell-1}. \tag{2.6}$$

The zero modes operators $T_{i,j}[0] \equiv T_{i,j}$ will play a special role below.

The generating series (2.6) can be used to construct $\mathfrak{o}_{2n+1}$-invariant integrable models. Let $\mathcal{H}$ be a Hilbert space of states of such a model, which can be considered as representation space for the algebra $B$. In order to treat this model by the algebraic Bethe ansatz method, the physical space of the model $\mathcal{H}$ must include a special vector or reference state $|0\rangle$ such that

$$T_{i,j}(u)|0\rangle = 0, \quad -n \leq j < i \leq n,$$

$$T_{i,i}(u)|0\rangle = \lambda_i(u)|0\rangle, \quad -n \leq i \leq n, \tag{2.7}$$

where $\lambda_i(u)$ characterize the concrete model. They are free functional parameters modulo the relations (2.17) which follow from (2.5). As a result of the commutation relations (2.3) the transfer matrix

$$\mathcal{E}(z) = \sum_{i=-n}^{n} T_{i,i}(z) \tag{2.8}$$
generates a commuting set of integrals of the model: \([T(u), T(v)] = 0\).

Let \(t_i^\ell \in \mathbb{C}, i = 0, 1, \ldots, n - 1, \ell = 1, \ldots, r_i\) be generic complex parameters, where the positive integers \(r_i \geq 0\) count the number of parameters of the type \(i\). These numbers are cardinalities of the sets \(t^\ell_i\): \(r_i = \# t^\ell_i\) and if some \(r_i = 0\) then the corresponding set \(t^\ell_i\) is empty: \(t^\ell_i = \emptyset\). We will gather these parameters into sets \(\bar{t}_i = \{\bar{t}^0_i, \ldots, \bar{t}^{n-1}_i\}\), \(\bar{t}^\ell_i = \{t^\ell_i, \ldots, t^\ell_{r_i}\}\), \(i = 0, 1, \ldots, n - 1\). (2.9)

For \(\ell = 1, \ldots, r_i\), the notation \(\bar{t}^\ell_i = \{\bar{t}^\ell_i \setminus t^\ell_i\}\) stands for the set \(\bar{t}^\ell_i\) with the parameter \(t^\ell_i\) omitted. Then, \(\bar{t}^\ell_i\) has cardinality \(\# \bar{t}^\ell_i = r_i - 1\). We will call the parameters \(t^\ell_i\) as the Bethe parameters.

In what follows we will need the following rational functions

\[
g(u, v) = \frac{c}{u - v}, \quad f(u, v) = \frac{u - v + c}{u - v}, \quad h(u, v) = \frac{f(u, v)}{g(u, v)} = \frac{u - v + c}{c}. \tag{2.10}
\]

To simplify presentation of our results we introduce the set of functions

\[
f_s(u, v) = \begin{cases} f(u, v) = \frac{u - v + c/2}{u - v}, & s = 0, \\ f(u, v), & s = 1, \ldots, n - 1. \end{cases} \tag{2.11}
\]

There is an equality

\[
f(u, v) = f(u + c/2, v) f(u, v). \tag{2.12}
\]

We will use a shorthand notation for the products of functions of one or two variables.

For example,

\[
\lambda_k(\bar{u}) = \prod_{u_i \in \bar{u}} \lambda_k(u_i), \quad f(u, \bar{v}) = \prod_{v_j \in \bar{v}} f(u, v_j), \quad f(\bar{u}, \bar{v}) = \prod_{u_i \in \bar{u}} \prod_{v_j \in \bar{v}} f(u_i, v_j). \tag{2.13}
\]

If any set in these formulas is empty, then the corresponding product is equal to 1.

The algebraic Bethe ansatz allows to describe the physical states of the model in terms of Bethe vectors. The vectors \(B(\bar{t})\) are called on-shell Bethe vectors if the set \(\bar{t}\) of the Bethe parameters satisfies the so-called Bethe equations

\[
\alpha_s(t^\ell_i) = \frac{f_s(t^\ell_i, \bar{t}^\ell_i) f(\bar{t}^{\ell+1}_i, t^{\ell+1}_i)}{f_s(t^{\ell-1}_i, t^\ell_i) f(t^{\ell-1}_i, \bar{t}^{\ell-1}_i)}, \quad \ell = 1, \ldots, r_s, \quad s = 0, 1, \ldots, n - 1, \tag{2.12}
\]

where we assume \(\bar{t}^{-1} = \bar{t}^n = \emptyset\) and define functions

\[
\alpha_s(z) = \frac{\lambda_s(z)}{\lambda_{s+1}(z)}. \tag{2.13}
\]
On-shell Bethe vectors are eigenstates of the transfer matrix
\[ \mathcal{T}(z) \mathbb{B}(\tilde{t}) = \tau(z; \tilde{t}) \mathbb{B}(\tilde{t}). \] (2.14)

To describe the eigenvalue \( \tau(z; \tilde{t}) \) it is convenient to introduce the notation
\[ z_s = z - c \left( s - \frac{1}{2} \right), \quad s = 0, 1, \ldots, n - 1, n. \] (2.15)

Note that \( z_n = z - c \kappa \). The eigenvalue \( \tau(z; \tilde{t}) \) in (2.14) is equal to
\[ \tau(z; \tilde{t}) = \lambda_0(z) f(\tilde{t}^0, z_0) f(z, \tilde{t}^0) + \sum_{s=1}^{n} \left( \lambda_s(z) f(\tilde{t}^s, z) f(z, \tilde{t}^s) + \lambda_{-s}(z) f(\tilde{t}^{s-1}, z_{s-1}) f(z_s, \tilde{t}^s) \right), \] (2.16)

where
\[ \lambda_{-j}(z) = \frac{1}{\lambda_n(z_n)} \prod_{s=j}^{n-1} \lambda_{s+1}(z_s) \lambda_s(z_s) = \frac{1}{\lambda_j(z_j)} \prod_{s=j+1}^{n} \lambda_s(z_{s-1}) \lambda_s(z_s) \] (2.17)

for \( j = 0, 1, \ldots, n \).

The Bethe equations (2.12) can be obtained as vanishing of the residues of the eigenvalue \( \tau(z; \tilde{t}) \) (2.16) at the values \( z = t^0_i \) or \( z_s = t^s_i \). If the Bethe parameters \( \tilde{t} \) are free, such vectors are called off-shell Bethe vectors. There exist different methods to describe the off-shell Bethe vectors \( \mathbb{B}(\tilde{t}) \in \mathcal{H} \) in terms of polynomials of the non-commuting monodromy matrix elements \( T_{i,j}(u), i < j \) acting onto reference vector \( |0\rangle \). But it appears that in order to calculate physically interesting quantities in such integrable models, one does not need to have fully explicit formulas for these vectors. In many cases it is sufficient to know explicit formulas for the action of the monodromy matrix elements \( T_{i,j}(z) \) onto off-shell Bethe vectors. One can prove that the set of these vectors is closed under this action, which means that the product \( T_{i,j}(z) \cdot \mathbb{B}(\tilde{t}) \) maybe written as a linear combination of the vectors of the same structure.

The search of the explicit formulas for such an action in \( \mathfrak{so}_{2n+1} \)-invariant integrable models is the main goal of this paper.

### 3 Monodromy matrix elements action

In this section we present the main result of this paper. Besides collections of the Bethe parameters \( \tilde{t} \) defined by (2.9) we introduce the collection of sets \( \tilde{w} \)
\[ \tilde{w} = \{ \tilde{w}^0, \tilde{w}^1, \ldots, \tilde{w}^{n-1} \}, \quad \tilde{w}^s = \{ t^s_1, \ldots, t^s_{r_s}, z, z_s \}, \] (3.1)

with \( z_s \) defined by (2.15). Let \( \tilde{w}^n = \{ z, z_n \} \) be an auxiliary set of the cardinality 2.
The action of the monodromy matrix element $T_{i,j}(z), -n \leq i, j \leq n$ onto off-shell Bethe vector $B(t)$ is given by the sum over partitions of the sets $\{\bar{w}_1^s, \bar{w}_n^s, \bar{w}_m^s\} \vdash \bar{w}^s$, $s = 0, 1, \ldots, n$ such that cardinalities of the sets $\bar{w}_1^s$ and $\bar{w}_m^s$ are

$$\#\bar{w}_1^s = \Theta(i + s) + \Theta(i - s - 1),$$
$$\#\bar{w}_m^s = \Theta(s - j) + \Theta(-j - s - 1).$$

Here $\Theta(p)$ is a Heaviside step function defined for integers $p \in \mathbb{Z}$

$$\Theta(p) = \begin{cases} 1, & p \geq 0, \\ 0, & p < 0. \end{cases}$$

The cardinalities of the sets $\bar{w}_1^s$ and $\bar{w}_m^s$ may be equal to 0, 1 or 2 depending on the interrelations between integers $i$, $j$ and $s$. Let us define

$$\sigma_i = 2\Theta(i - 1) - 1.$$  

(3.3)

It is clear that $\sigma_i = -1$ for $i \leq 0$ and $\sigma_i = 1$ for $i > 0$. Note also, that according to (3.2),

$$\#\bar{w}_1^n = \#\bar{w}_m^n = 1 \text{ for all values of the indices } -n \leq i, j \leq n.$$  

Let us define the rational functions of two variables

$$\gamma_s(u, v) = \begin{cases} \hat{f}(u, v) = \frac{u - v + c/2}{u - v}, & s = 0, \\ f(u, v) = \frac{c^2}{(u - v)(v - u + c)}, & s = 1, \ldots, n - 1. \end{cases}$$

(3.4)

The main result of this paper may be formulated as following

**Theorem 3.1.** The action of the monodromy matrix element $T_{i,j}(z), -n \leq i, j \leq n$ onto off-shell Bethe vector $B(t)$ is given by the explicit formula

$$T_{i,j}(z) \cdot B(t) = -\frac{\sigma_i\sigma_j}{\kappa} \lambda_n(z) g(z, t^0) \sum_{\text{part}} \mathbb{B}(\bar{w}_m) \prod_{s=0}^{n-1} \alpha_s(\bar{w}_m^s)$$

$$\times \prod_{s=0}^{n-1} \gamma_s(\bar{w}_1^s, \bar{w}_m^s) \gamma_s(\bar{w}_n^s, \bar{w}_m^s) \frac{h(\bar{w}_1^{s+1}, \bar{w}_1^s)h(\bar{w}_m^{s+1}, \bar{w}_m^s)h(\bar{w}_1^s, \bar{w}_m^s)}{g(\bar{w}_1^{s+1}, \bar{w}_1^s)g(\bar{w}_n^{s+1}, \bar{w}_n^s)g(\bar{w}_1^s, \bar{w}_m^s)}.$$  

(3.5)

Here we use notation $\bar{w}_m = \{\bar{w}_n^0, \bar{w}_n^1, \ldots, \bar{w}_n^{n-1}\}$. The sets $\bar{w}^s$ for $s = 0, 1, \ldots, n$ are divided into subsets $\{\bar{w}_1^s, \bar{w}_n^s, \bar{w}_m^s\} \vdash \bar{w}^s$ and summation in (3.5) goes over these partitions with cardinalities described by (3.2).

Recall that functions $\alpha_s(z)$ are defined by (2.13).
Remark 3.1. It follows from the action formula that effectively the sum over partitions of the set $\hat{w}^n$ reduces only to one term when $\hat{w}_i^n = \{z_n\}$ and $\hat{w}_m^n = \{z\}$. Indeed, let us assume that $\hat{w}_i^n = \{z\}$ since $\#\hat{w}_i^n = 1$. Then the product $g(z, \hat{w}_i^n)^{-1}g(z, \hat{w}_m^n)^{-1}$ in denominator of the second line of (3.5) yields $z \in \hat{w}_i^n$ otherwise we got zero term. Considering other factors $g(\hat{w}_i^{s+1}, \hat{w}_m^s)^{-1}g(\hat{w}_i^{s+1}, \hat{w}_m^s)^{-1}$ for $s = 0, n-2$ we prove analogously that in order to have non-zero terms $z$ should be in the set $\hat{w}_i^s$ for all $s$. Analogously, let us assume that $\hat{w}_m^n = \{z_n\}$. The product $h(z_n, \hat{w}_i^n)^{-1}h(z_n, \hat{w}_m^n)$ yields $z_n \in \hat{w}_m^n$ since otherwise we got zero contribution because $h(z_n, z_{n-1}) = 0$. Continuing we prove that assumption $\hat{w}_m^n = \{z_n\}$ yields that $z_s \in \hat{w}_m^n$ for all $s$. But (3.5) has a factor $f(\hat{w}_i^0, \hat{w}_m^0) \sim f(z, z_0) \equiv 0$. This proves that our initial assumptions that $\hat{w}_i^n = \{z\}$ and $\hat{w}_m^n = \{z_n\}$ lead to the zero contribution into sum over partitions.

Remark 3.2. (Reduction to $\g_{2n-1}$-invariant Bethe vectors.) By the same arguments as in the previous remark we may observe that the action formula respect the hierarchical embedding of the $\g_{2n-1}$-invariant RTT algebra into $\g_{2n+1}$-invariant RTT algebra described in lemma (3.6) of the paper [8]. It means, that if $\int_{n-1} = \emptyset$ then we can prove that $\hat{w}_i^n = \{z_{n-1}\}$ and $\hat{w}_m^n = \{z\}$ and that the action formula (3.5) for the values of the indices $-n+1 \leq i, j \leq n-1$ becomes the action formula for the $\g_{2n-1}$-invariant integrable model.

## 3.1 Zero-modes method

To prove the statement of the theorem 3.1 we will use so called zero-modes method. To describe it we modify slightly the definition of the algebra $\B$ given in the section 2.1. Let

$$\K = \text{diag}(\chi_{-n}, \ldots, \chi_{-1}, 1, \chi_1, \ldots, \chi_n)$$

be a diagonal matrix with non-zero entries $\chi_i \in \C$. Sometimes we will use in the formulas below the notation $\chi_0 = 1$. The diagonal matrix $\K$ satisfies the relation

$$R(u, v) \cdot \K_1 \otimes \K_2 = \K_1 \otimes \K_2 \cdot R(u, v)$$

(3.6)

if

$$\chi_{-i} = \chi_i^{-1}, \quad i = 1, \ldots, n.\quad (3.7)$$

So it has only $n$ independent parameters $\chi_i$ for $i = 1, \ldots, n$. Due to (3.6), the monodromy matrix $T^\K(u) = \K \cdot T(u)$ satisfies the same commutation relations (2.4). The only difference will be that instead of the expansion (2.6)\footnote{Recall that we denoted zero modes $T_{i,j}[0]$ as $T_{i,j}$ for $-n \leq i, j \leq n.$} we will have

$$T^\K_{i,j}(u) = \chi_i \delta_{ij} + \frac{c}{u} T_{i,j} + O(u^{-2}).$$

(3.8)
The commutation relations (2.4) yield the commutation of the zero modes and monodromy matrix elements:

\[
[T_{i,j}(u), T_{k,l}] = \delta_{i,l} \chi_l T_{k,j}(u) - \delta_{k,j} \chi_j T_{i,l}(u) - \delta_{i,-k} \chi_l T_{-l,j} + \delta_{-l,j} \chi_k T_{i,-k}.
\]  
(3.9)

Proof of the theorem 3.1 is based on the commutation relations (3.9) and two propositions.

**Proposition 3.1.** The action of the zero mode operators $T_{j,i}$ onto off-shell Bethe vectors $\mathcal{B}(\hat{t})$ for $0 \leq i < j \leq n$ is given by the equality

\[
T_{j,i} \cdot \mathcal{B}(\hat{t}) = \sum_{\text{part}} \mathcal{B}(\bar{t}_a) \prod_{s=1}^{n-1} \frac{1}{g(t^s, \hat{t}_i^{s-1})g(t^s, \hat{t}_j^{s-1})} \frac{1}{h(t^s, \hat{t}_i^s)} \frac{1}{h(t^s, \hat{t}_j^s)} \left( \sum_{s=i}^{j-1} \alpha_s(\hat{t}_i^s) f(t^s, \hat{t}_i^{s-1}) f(t^s, \hat{t}_j^{s-1}) - \chi_{j-1} \prod_{s=i}^{j-1} \frac{f(t^s+1, \hat{t}_i^s) f(t^s, \hat{t}_j^s)}{h(t^s+1, \hat{t}_i^s) h(t^s, \hat{t}_j^s)} \right),
\]  
(3.10)

where the functions $\alpha_s(t)$ are defined by (2.12) and the sum goes over partitions $\{\bar{t}_i^s, \bar{t}_j^s\}$ of $\hat{t}$ with cardinalities $\#\bar{t}_i^s = 1$ for $s = i, \ldots, j - 1$ and $\#\bar{t}_i^s = 0$ for other $s$.

**Remark 3.3.** If $\chi_j = 1$ and the Bethe parameters $\hat{t}$ satisfy the Bethe equations (2.12) the on-shell Bethe vectors become highest weight vectors for the algebra $\mathfrak{e}_{2n+1}$. Literally, it means that in this case $T_{j,i} \cdot \mathcal{B}(\hat{t}) = 0$.

Note that the product in the first line of (3.10) is going effectively from $s = i$ to $s = j$.

**Proposition 3.2.** The action of monodromy matrix element $T_{-n,n}(z)$ onto off-shell Bethe vector $\mathcal{B}(\hat{t})$ (3.9) is regular and given by the relation

\[
T_{-n,n}(z) \cdot \mathcal{B}(\hat{t}) = -\kappa \frac{g(z, \bar{t}_0^0) h(z, \bar{t}_0^{-n})}{h(z, \bar{t}_0^{-n}) g(z, \bar{t}_0^{-n})} \lambda_n(z) \mathcal{B}(\bar{w}),
\]  
(3.11)

and the collection of sets $\bar{w}$ is defined by (5.1).

Proofs of the propositions 3.1 and 3.2 will be given in section 4 using identification of RTT algebra $\mathcal{B}$ with the standard Borel subalgebra of the Yangian double $DY(\mathfrak{e}_{2n+1})$ [4]. This infinite-dimensional algebra was investigated in [7, 15] and we will use certain projections onto intersections of the different types Borel subalgebras studied in [14] for the quantum affine algebras.

As a direct consequence of (3.9) the zero-modes operators obey the relations

\[
[T_{i,j}, T_{k,l}] = \delta_{i,l} \chi_l T_{k,j} - \delta_{k,j} \chi_j T_{i,l} - \delta_{i,-k} \chi_l T_{-l,j} + \delta_{-l,j} \chi_k T_{i,-k}.
\]  
(3.12)
As well as for the generating series $T_{i,j}(u)$ the quadratic relation (2.5) impose several relations to the zero-modes operators. Substituting expansion (3.8) into (2.5) and equating terms at $u^0$ and $u^{-1}$ we obtain (3.7) and

$$\chi_{-i} T_{i,j} + \chi_i T_{-j,-i} = 0.$$  \hfill (3.13)

It follows from (3.7) and (3.13) that

$$T_{-j,-i} = -\chi_i^{-1} \chi_j^{-1} T_{i,j}, \quad -n \leq i, j \leq n$$  \hfill (3.14)

and this equality for $j = -i$ implies that

$$T_{i,-i} = -T_{i,-i} = 0, \quad -n \leq i \leq n.$$  

Equality (3.14) and commutation relations (3.12) allow to express all zero-modes through the algebraically independent set of generators $T_{i,i}, T_{i,-i}, T_{i,i-1}$ for $i = 1, \ldots, n$.

Let us sketch the proof of the theorem 3.1 using zero-modes method. We start from (3.11) and commutation relation

$$[T_{-n,n}(z), T_{n,i}] = -\chi_n T_{-n,i}(z) - \chi_i T_{-i,n}(z)$$  \hfill (3.15)

for $0 \leq i \leq n-1$ to obtain from (3.10) the action formula for the operators $T_{-i,n}(z)$. This can be achieved by applying the equality (3.15) to the off-shell Bethe vector $\mathcal{B}(\bar{t})$ and equating the coefficients at $\chi_i$ in left and right hand side of this equality. This can be done because $\chi_i$ are independent parameters.

Next we start from the action of the entry $T_{0,n}(z)$ and applying the commutation relations

$$[T_{0,n}(z), T_{i,0}] = T_{i,n}(z) - \delta_{i,n} \chi_n T_{0,0}(z)$$

to $\mathcal{B}(\bar{t})$ for $1 \leq i \leq n$ we obtain the action of the rest entries $T_{i,n}(z)$ from the last column of monodromy matrix $T(z)$.

At the next step we explore already calculated actions of the entries $T_{i,n}(z), -n \leq i \leq n$ and the commutation relation

$$[T_{i,n}(z), T_{n,j}] = -\chi_n T_{i,j}(z) - \delta_{i,-n} \chi_j T_{-j,n}(z) + \delta_{i,j} \chi_i T_{n,n}(z)$$

to obtain the action formulas of the entries $T_{i,j}(z)$ for all $i$ and $0 \leq j \leq n-1$.

Finally, we start from the action of the entries $T_{i,0}(z)$ for all $i$ and use the commutation relation

$$[T_{i,0}(z), T_{j,0}] = \chi_j T_{i,-j}(z) - \delta_{i,-j} T_{0,0}(z) + \delta_{i,0} \chi_i T_{j,0}(z)$$

to obtain action formulas for remaining entries $T_{i,-j}(z), -n \leq i \leq n$ and $1 \leq j \leq n$. Performing these calculations we can always equate the terms at the independent parameters $\chi_j, 0 \leq j \leq n$. \hfill $\square$
3.2 Recurrence relations

The action formulas (3.5) allows to obtain the recurrence relations for the off-shell Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant integrable models. For $\mathfrak{gl}(m|n)$-invariant supersymmetric models such recurrence relations were obtained in [21].

**Proposition 3.3.** For $n \geq 2$ one has following recurrence relations for the off-shell Bethe vector $\mathbb{B}(\bar{t})$

$$
\mathbb{B}(\bar{t}^0, \ldots, \bar{t}^{n-2}, \{\bar{t}^{n-1}, z\}) = \frac{1}{h(z, \bar{t}^{n-1})\lambda_n(z)} \times \sum_{i=-n}^{n-1} \sum_{\text{part}} \sigma_{i+1} T_{i,n}(z) \cdot \mathbb{B}(\bar{t}^0, \ldots, \bar{t}^{s-1}) \prod_{s=0}^{n-1} \gamma_s(\bar{t}^s, \bar{t}^{s-1}) \prod_{s=1}^{n-1} h(\bar{t}^s, \bar{t}^{s-1}),
$$

(3.16)

where sum in (3.16) goes over partitions $\{\bar{t}^s, \bar{t}^{s-1}\} \vdash \bar{t}^s$ with different cardinalities of $\bar{t}^s$ for different terms depending on $i$ for $s \leq n - 2$

$$
\#\bar{t}^s = \begin{cases} 
0, & 0 \leq s \leq i - 1, \\
1, & i \leq s \leq n - 2
\end{cases}
$$

(3.17)

for $0 \leq i \leq n - 1$,

$$
\#\bar{t}^s = \begin{cases} 
2, & 0 \leq s \leq -i - 1, \\
1, & -i \leq s \leq n - 2
\end{cases}
$$

for $-n + 1 \leq i \leq -1$ and for $s = n - 1$

$$
\#\bar{t}^{n-1} = \begin{cases} 
0, & -n + 1 \leq i \leq n - 1, \\
1, & i = -n
\end{cases}
$$

(3.18)

Proof of this proposition can be performed in a similar way as in [21]. We have to use the action formulas (3.5) in the right hand side of (3.16) and prove that the right hand side identically coincide with the left hand side in this equality. The recurrence relations in case of $n = 1$ were proved in [19].

The statement of the proposition 3.3 is a recursion procedure which allows to reduce the set of the Bethe parameters $\bar{t}^{n-1}$ until this set becomes empty. This yields the presentation of the off-shell $\mathfrak{o}_{2n+1}$-invariant Bethe vector $\mathbb{B}(\bar{t}^0, \ldots, \bar{t}^{n-1})$ as a linear combinations of the products of the matrix entries $T_{i,n}(\bar{t}^{n-1})$, $-n \leq i \leq n - 1$ with rational coefficients acting onto $\mathfrak{o}_{2n-1}$-invariant Bethe vector $\mathbb{B}(\bar{t}^0, \ldots, \bar{t}^{n-2})$ (see remark 3.2). Then we repeat this procedure to express $\mathfrak{o}_{2n-1}$-invariant Bethe vector $\mathbb{B}(\bar{t}^0, \ldots, \bar{t}^{n-2})$ as a linear combinations of the products of the matrix entries $T_{i,n-1}(\bar{t}^{n-2})$, $-n + 1 \leq i \leq n - 2$ acting onto $\mathfrak{o}_{2n-3}$-invariant Bethe vector $\mathbb{B}(\bar{t}^0, \ldots, \bar{t}^{n-3})$ and so on. Finally, one may obtain a
representation of the off-shell Bethe vector as polynomial of the matrix entries $T_{i,j} (t_{i}^{-1})$ with rational coefficients acting on the reference vector $|0\rangle$ with $-j \leq i \leq j-1$ for $1 \leq j \leq n$.

Note that at each step this recurrence procedure is in accordance with embedding of the smaller algebra $\mathcal{B}_{n-1}$ into the bigger algebra $\mathcal{B}_{n}$ described in [8] (see remark 3.2).

For example, first nontrivial Bethe vectors in $\sigma_{5}$-invariant model

$$\mathbb{B}(t^{0}; t^{1}) = \frac{T_{0,2}(t^{1})|0\rangle}{\lambda_{2}(t^{1})} + \frac{1}{g(t^{1}, t^{0})} \frac{T_{1,2}(t^{1})T_{0,1}(t^{0})|0\rangle}{\lambda_{2}(t^{1})\lambda_{1}(t^{0})}$$

$$\mathbb{B}(t^{0}; \{t_{1}^{0}, t_{2}^{0}\}) = \frac{h(t_{1}^{0}, t_{2}^{0}) T_{0,2}(t_{1}^{0})T_{1,2}(t_{2}^{0})|0\rangle}{h(t_{1}^{0}, t_{1}^{0}) \lambda_{2}(t_{1}^{0})\lambda_{2}(t_{2}^{0})} + \frac{1}{g(t_{2}^{0}, t^{0})h(t_{1}^{0}, t^{1})} \frac{T_{1,2}(t_{2}^{0})T_{0,2}(t_{1}^{0})|0\rangle}{\lambda_{2}(t_{1}^{0})\lambda_{2}(t_{2}^{0})}$$

$$\mathbb{B}(\{t_{1}^{0}, t_{2}^{0}\}; t^{1}) = -\frac{T_{1,2}(t^{1})|0\rangle}{\lambda_{2}(t^{1})} + \frac{1}{g(t^{1}, t^{0})h(t_{2}^{0}, t_{1}^{0})} \frac{T_{0,2}(t^{1})T_{1,1}(t_{2}^{0})|0\rangle}{\lambda_{2}(t^{1})\lambda_{1}(t_{2}^{0})} + \frac{1}{g(t^{1}, t^{0})h(t_{2}^{0} + c/2, t_{1}^{0})} \frac{T_{1,2}(t^{1})T_{0,1}(t_{2}^{0})|0\rangle}{\lambda_{2}(t^{1})\lambda_{1}(t_{2}^{0})\lambda_{1}(t_{1}^{0})}.$$ 

First nontrivial off-shell Bethe vector in $\sigma_{7}$-invariant model is

$$\mathbb{B}(t^{0}; t^{1}; t^{2}) = \frac{T_{0,3}(t^{2})|0\rangle}{\lambda_{3}(t^{2})} + \frac{1}{g(t^{1}, t^{0})} \frac{T_{1,3}(t^{2})T_{0,1}(t^{0})|0\rangle}{\lambda_{3}(t^{2})\lambda_{1}(t^{0})} + \frac{1}{g(t^{2}, t^{1})} \frac{T_{2,3}(t^{2})T_{0,2}(t^{1})|0\rangle}{\lambda_{3}(t^{2})\lambda_{2}(t^{1})}$$

$$+ \frac{1}{g(t^{2}, t^{1})g(t^{1}, t^{0})} \frac{T_{2,3}(t^{2})T_{1,2}(t^{1})T_{0,1}(t^{0})|0\rangle}{\lambda_{2}(t^{1})\lambda_{1}(t^{0})\lambda_{1}(t^{0})}.$$ 

To obtain these formulas one has to apply recurrence relations (5.16) and similar relations for the $\sigma_{3}$-invariant Bethe vectors presented in [19].

Besides recurrence relations given by the proposition 5.3 corresponding to the last column of monodromy matrix one can obtain from the action formulas the recurrence relation with respect to the first row of the monodromy matrix. We formulate this relation as the following

**Proposition 3.4.** For $n \geq 2$ one has an alternative recurrence relations for the off-shell Bethe vector $\mathbb{B}(\tilde{t})$

$$\mathbb{B}(\tilde{t}^{0}, \ldots, \tilde{t}^{n-2}, \{\tilde{t}^{n-1}, \tilde{z}_{n-1}\}) = \frac{1}{h(t^{n-1}, \tilde{z}_{n-1})\lambda_{-n+1}(\tilde{z})} \times \sum_{j=-n+1}^{n} \sum_{\text{part}} (-1)^{\delta_{j,n}} \frac{\sigma_{j} T_{-n,j}(z_{j}) \cdot \mathbb{B}(\tilde{t}_{n})}{h(z_{n-1}, \tilde{t}_{n-1}^{n-1})} \prod_{s=0}^{n-1} \alpha_{s}(\tilde{t}_{n}^{n}) \gamma_{s}(\tilde{t}_{n}^{s}, \tilde{t}_{n}^{s+1}) \prod_{s=1}^{n-1} \frac{h(\tilde{t}_{n}^{s}, \tilde{t}_{n}^{s-1})}{g(\tilde{t}_{n}^{s}, \tilde{t}_{n}^{s-1})},$$

(3.19)
where sum in (3.19) goes over partitions \( \{ \tilde{t}_m^s, \tilde{t}_m^s \} \) with cardinalities \((s < n - 2)\)

\[
\# \tilde{t}_m^s = \begin{cases} 
0, & 0 \leq s \leq -j - 1, \\
1, & -j \leq s \leq n - 2 
\end{cases}
\]

for \(-n + 1 \leq j \leq 0\),

\[
\# \tilde{t}_m^s = \begin{cases} 
2, & 0 \leq s \leq j - 1, \\
1, & j \leq s \leq n - 2 
\end{cases}
\]

for \(1 \leq j \leq n - 1\) and for \(s = n - 1\)

\[
\# \tilde{t}_m^{n-1} = \begin{cases} 
0, & -n + 1 \leq j \leq n - 1, \\
1, & j = n. 
\end{cases}
\]

In (3.16) and (3.19) the sign factor \(\sigma_i\) is defined by (3.3).

### 3.3 Eigenvalue property of the Bethe vectors \(\mathbb{B}(\tilde{t})\)

Let us demonstrate in this section how the action formulas (3.5) reproduce the eigenvalue (2.16). We formulate it as following

**Proposition 3.5.** The action of the transfer matrix (2.8) onto off-shell Bethe vectors \(\mathbb{B}(\tilde{t})\) which follows from (3.5) is

\[
T(z) \cdot \mathbb{B}(\tilde{t}) = \tau(z; \tilde{t}) \mathbb{B}(\tilde{t}) + \cdots, \tag{3.20}
\]

where eigenvalue \(\tau(z; \tilde{t})\) is given by the equality (2.10) and \(\cdots\) stands for the terms which are vanishing if Bethe equations (2.12) are satisfied.

It is clear that first term in the right hand side of (3.20) will correspond to the so called 'wanted' terms in the right hand side of the diagonal elements actions \(T_{\ell, \ell}(z)\) which corresponds to the partitions such that \(\tilde{w}_n = \tilde{t}\). It is convenient to consider separately the action of \(T_{\ell, \ell-\ell}(z)\) and \(T_{\ell, \ell}(z)\) for \(\ell = 0, 1, \ldots, n-1, n\). According to (3.2) the cardinalities of the sets \(\tilde{w}_i^s\) and \(\tilde{w}_m^s\) for the actions of \(T_{\ell, \ell-\ell}(z)\) are

\[
s = 0, \ldots, \ell - 1, \quad \# \tilde{w}_i^s = 0 \quad \text{and} \quad \# \tilde{w}_m^s = 2, \\
s = \ell, \ldots, n - 1, \quad \# \tilde{w}_i^s = 1 \quad \text{and} \quad \# \tilde{w}_m^s = 1, 
\]

and for the action of \(T_{\ell, \ell}(z)\)

\[
s = 0, \ldots, \ell - 1, \quad \# \tilde{w}_i^s = 2 \quad \text{and} \quad \# \tilde{w}_m^s = 0, \\
s = \ell, \ldots, n - 1, \quad \# \tilde{w}_i^s = 1 \quad \text{and} \quad \# \tilde{w}_m^s = 1. 
\]
In the first case of the action $T_{-\ell,-\ell}(z)$ the sets $\bar{w}_i^s$ and $\bar{w}_m^s$ which will correspond to the wanted terms are for
\[
s = 0, \ldots, \ell - 1, \quad \bar{w}_i^s = \emptyset, \quad \bar{w}_m^s = \{z, z_s\},
\]
\[
s = \ell, \ldots, n - 1, \quad \bar{w}_i^s = \{z\}, \quad \bar{w}_m^s = \{z_s\}, \quad \text{or} \quad \bar{w}_i^s = \{z_s\}, \quad \bar{w}_m^s = \{z\}. \tag{3.21}
\]
In the second case of the action $T_{\ell,\ell}(z)$ the sets $\bar{w}_i^s$ and $\bar{w}_m^s$ for the wanted terms are
\[
s = 0, \ldots, \ell - 1, \quad \bar{w}_i^s = \{z, z_s\}, \quad \bar{w}_m^s = \emptyset,
\]
\[
s = \ell, \ldots, n - 1, \quad \bar{w}_i^s = \{z\}, \quad \bar{w}_m^s = \{z_s\}, \quad \text{or} \quad \bar{w}_i^s = \{z_s\}, \quad \bar{w}_m^s = \{z\}. \tag{3.22}
\]
The cases $\ell = 0$ in (3.21) and in (3.22) both correspond to the action of $T_{0,0}(z)$ and in this case all sets $\bar{w}_i^s$ and $\bar{w}_m^s$ have cardinalities 1. On the other hand the cases $\ell = 1, \ldots, n$ in (3.21) and in (3.22) correspond to the action of $T_{-\ell,-\ell}(z)$ and $T_{\ell,\ell}(z)$ respectively. For the action of $T_{-\ell,-\ell}(z)$ the term in (3.5) which produces the wanted term corresponds to the partition $\bar{w}_m^s = \{z, z_s\}$ for $s = 0, \ldots, \ell - 1$ and $\bar{w}_m^s = \{z\}$ for $s = \ell, \ldots, n - 1$. Analogously, for the action of $T_{\ell,\ell}(z)$ we have $\bar{w}_i^s = \{z, z_s\}$ for $s = 0, \ldots, \ell - 1$ and $\bar{w}_i^s = \{z\}$, $\bar{w}_i^s = \{z_s\}$ for $s = \ell, \ldots, n - 1$.

Let us calculate the contribution of the action $T_{-\ell,-\ell}(z)$ to the eigenvalue (2.16). For the factors depending on the functional parameters $\lambda_j(z)$ we have
\[
\lambda_n(z) \prod_{s=0}^{n-1} \alpha_s(z) \prod_{s=1}^{\ell-1} \alpha_s(z_s) = \lambda_n(z) \prod_{s=0}^{n-1} \frac{\lambda_s(z)}{\lambda_{s+1}(z)} \prod_{s=1}^{\ell-1} \frac{\lambda_s(z_s)}{\lambda_{s+1}(z_s)} = \frac{\lambda_0(z_0)}{\lambda_0(z_{\ell-1})} \prod_{s=1}^{\ell-1} \frac{\lambda_s(z_s)}{\lambda_s(z_{s-1})} = \frac{1}{\lambda_0(z_0) \lambda_0(z_{\ell-1})} \prod_{s=1}^{\ell-1} \frac{\lambda_s(z_{s-1})}{\lambda_s(z_s)} = \lambda_{-\ell}(z),
\]
where we have used the relations (2.17) for the functional parameters. One may further check that wanted terms of (3.5) for the action $T_{-\ell,-\ell}(z)$ reduces to the function $\lambda_{-\ell}(z)f(\bar{t}^{\ell-1}, z_{\ell-1})f(z_\ell, \bar{t}_\ell)$ restoring part of the eigenvalue (2.16). Analogously, one may check that the wanted terms of (3.5) for the action of $T_{\ell,\ell}(z)$ produces the term $\lambda_\ell(z)f(\bar{t}^{\ell}, z)f(z, \bar{t}^{\ell+1})$ of the eigenvalue (2.16).

Unfortunately, the unwanted terms marked by dots in (3.20) cannot be presented in the nice form. One can investigate all these terms case by case and verify that all these terms will be proportional to the differences of the left and right hands sides of the equalities (2.12) and will disappear if the Bethe equations for the parameters $\bar{t}$ are satisfied.

\[\square\]

### 3.4 Reduction to $\mathfrak{gl}_n$-invariant Bethe vectors

Let us consider formula (3.5) for $i = 1, j = n$ and $\bar{t}^0 = \emptyset$. For these values of indices $\sigma_1 = 1, \sigma_{-n} = -1$ and (3.2) yields following cardinalities $\#\bar{w}_m^s = 0$ for $s = 0, \ldots, n - 1$;
Lemma 4.2 of the paper [20]. Indeed, if we renormalize the Bethe vectors of this paper where \( \bar{w}_n^0 = \emptyset \) for all \( s = 0, \ldots, n - 1 \) and \( \bar{t}^0 = \emptyset \), the set \( \bar{w}_n^0 = \emptyset \) is empty and the action formula (3.5) simplifies to (recall that there is no summation over partition of the set \( \bar{w}^n = \{ z, z_n \} \) and \( \bar{w}_i^1 = \{ z, z_1 \} \), \( \bar{w}_i^m = \{ z \} \), see remark [3.1]

\[
T_{1,n}(z) \cdot \mathcal{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}^{n-1}) = -\lambda_n(z) \frac{h(z, \bar{t}^{n-1})}{g(z_n, \bar{t}^{n-1})} \times \sum_{\text{part}} \mathcal{B}(\emptyset, \bar{w}_i^1, \ldots, \bar{w}_n^{n-1}) \frac{1}{h(\bar{w}_i^{n-1}, z_{n-1})} \prod_{s=1}^{n-1} \frac{h(\bar{w}_n^s, \bar{w}_i^{s-1})g(\bar{w}_n^s, \bar{w}_i^{s-1})}{h(\bar{w}_n^s, \bar{w}_n^{s-1})g(\bar{w}_n^s, \bar{w}_i^{s-1})}. \tag{3.23}
\]

The set \( \bar{w}_i^0 \) is equal to \( \{ z, z_0 = z + c/2 \} \). Then because of the factor \( h(\bar{w}_i^1, \bar{w}_i^0) \) the sum over partitions of the set \( \bar{w}^1 \) reduces to the single partition \( \bar{w}_n^1 = \{ \bar{t}^1, z \} \) and \( \bar{w}_i^1 = \{ z_1 = z - c/2 \} \). Next the factor

\[
\frac{h(\bar{w}_n^2, \bar{w}_i^1)}{g(\bar{w}_n^2, \bar{w}_i^1)} = \frac{h(\bar{w}_n^2, z - c/2)}{g(\bar{w}_n^2, z)g(\bar{w}_i^1, \bar{t}^1)}
\]

reduces summation over partitions of the set \( \bar{w}^2 \) to a single partition \( \bar{w}_n^2 = \{ \bar{t}^2, z \} \) and \( \bar{w}_i^2 = \{ z_2 = z - 3c/2 \} \). Continuing we find that sum over partitions of all sets \( \bar{w}_s^s \), \( s = 1, 2, \ldots, n - 1 \) disappear and reduces to a single partition when

\[
\bar{w}_n^s = \{ \bar{t}^s \} \quad \text{and} \quad \bar{w}_i^s = \{ z_s \} \quad s = 1, 2, \ldots, n - 1.
\]

For these sets and \( \bar{w}_i^0 = \{ z, z_0 = z + c/2 \} \), \( \bar{w}_i^0 = \emptyset \) the product in (3.23) is equal to

\[
\frac{1}{h(\bar{w}_i^{n-1}, z_{n-1})} \prod_{s=1}^{n-1} \frac{h(\bar{w}_n^s, \bar{w}_i^{s-1})g(\bar{w}_n^s, \bar{w}_i^{s-1})}{h(\bar{w}_n^s, \bar{w}_i^{s-1})g(\bar{w}_i^{s-1}, \bar{w}_n^s)} = -\frac{1}{\kappa} \frac{h(\bar{t}^1, z)g(z_n, \bar{t}^{n-1})}{h(\bar{t}^1, \bar{t}^{n-1})}.
\]

Summarizing we can write the action (3.23) in the form

\[
T_{1,n}(z) \cdot \mathcal{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}^{n-1}) = \lambda_n(z) h(\bar{t}^1, z)h(z, \bar{t}^{n-1}) \mathcal{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}^{n-1}), \tag{3.24}
\]

where \( \bar{t}^s = \{ \bar{t}^s, z \}, s = 1, \ldots, n - 1 \). This action coincides with the action given by lemma 4.2 of the paper [20]. Indeed, if we renormalize the Bethe vectors of this paper

\[
\tilde{\mathcal{B}}(\bar{t}^1, \ldots, \bar{t}^{n-1}) = \frac{\prod_{s=1}^{n-1} h(\bar{t}^s, \bar{t}^{s-1})}{h(\bar{t}^s, \bar{t}^{s-1})} \mathcal{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}^{n-1}) \tag{3.25}
\]

we obtain from (3.24)

\[
T_{1,n}(z) \cdot \tilde{\mathcal{B}}(\bar{t}^1, \ldots, \bar{t}^{n-1}) = \lambda_n(z) \tilde{\mathcal{B}}(\{ \bar{t}^1, z \}, \ldots, \{ \bar{t}^{n-1}, z \}), \tag{3.26}
\]
which literally coincide with the action of monodromy matrix entry $T_{1,n}(z)$ onto off-shell Bethe vectors $\tilde{B}(\bar{t})$ in $\mathfrak{gl}_n$-invariant integrable models.

Let us note that formulas (4.14) at $\bar{t}^0 = \emptyset$ for $i = 1, \ldots, n - 1$ yields the equality (4.3) of the paper [20]

$$T_{i+1,i} \cdot \tilde{B}(\bar{t}^1, \ldots, \bar{t}^{n-1}) = \sum_{t \in t_i} \tilde{B}(\bar{t}^1, \ldots, \bar{t}^{i-1}, t_{i+1}, \bar{t}^{i+1}, \ldots, \bar{t}^{n-1})$$

$$\times \left( \chi_{i+1} \frac{\lambda_i(t_i)}{\lambda_{i+1}(t_i)} \frac{f(t_i, t_i)}{f(t_{i+1}, t_i)} - \chi_i \frac{f(t_i, t_i)}{f(t_{i+1}, t_{i+1})} \right)$$

(3.27)

for renormalized vectors $\tilde{B}(\bar{t})$ (3.25).

Since the action formulas (3.26) and (3.27) coincide with the statements of the lemma 4.2 of the paper [20] they can be used to restore the action of entries $T_{i,j}(z)$ onto Bethe vectors $\tilde{B}(\bar{t}^1, \ldots, \bar{t}^{n-1})$ in the framework of zero-modes method. So far we have demonstrated that the action formulas (3.5) of $T_{i,j}(z)$ for $1 \leq i, j \leq n$ onto off-shell Bethe vectors $\tilde{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}^{n-1})$ with empty set $\bar{t}^0 = \emptyset$ leads to the action formulas in $\mathfrak{gl}_n$-invariant models which was calculated in [20].

Let us also check that recurrence relations given by the proposition 3.3 yield the corresponding relations found in [21]. Formula (3.16) in case of $\bar{t}^0 = \emptyset$ becomes

$$\tilde{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}^{n-2}, \{\bar{t}^{n-1}, z\}) = \sum_{i=1}^{n-2} \sum_{\text{part}} T_{i,n}(z) \cdot \tilde{B}(\emptyset, \bar{t}^1, \ldots, \bar{t}_{i-1}, \bar{t}_i, \ldots, \bar{t}_{n-2}, \bar{t}^{n-1})$$

$$\times \left( \lambda_n(z) h(z, \bar{t}^{n-1}, z) g(z, \bar{t}^{n-2}) \right)$$

$$\times \prod_{s=1}^{n-2} \left( h(\bar{t}_{s+1}, \bar{t}_s) h(\bar{t}_s, \bar{t}_{s+1}) \right)$$

The sum over $i$ in (3.16) reduces to the interval $1 \leq i \leq n - 1$ and we have to take into account (3.17) which states that $\bar{t}_{i-1} = \emptyset$ for $1 \leq s \leq i - 1$ and $\bar{t}_{n-1} = \emptyset$ according to (3.18). This recurrence relation for the renormalized Bethe vector $\tilde{B}$ (3.25)

$$\tilde{B}(\bar{t}^1, \ldots, \bar{t}^{n-2}, \{\bar{t}^{n-1}, z\}) = \sum_{i=1}^{n} \frac{T_{i,n}(z)}{\lambda_n(z)} \sum_{\text{part}} \tilde{B}(\bar{t}^1, \ldots, \bar{t}_{i-1}, \bar{t}_i, \ldots, \bar{t}_{n-2}, \bar{t}^{n-1})$$

$$\times \frac{g(z, \bar{t}^{n-2})}{f(z, \bar{t}^{n-2})} \prod_{s=1}^{n-2} f(\bar{t}_{s+1}, \bar{t}_s) \prod_{s=1}^{n-2} g(\bar{t}_s, \bar{t}_{s+1})$$

coincides identically with equation (4.4) of the paper [21] in case $m = 0$.

### 3.5 Action formulas for $\mathfrak{o}_3$-invariant models

In [22] we have calculated the action formulas for $\mathfrak{o}_3$-integrable model. Let us verify that general formula (3.5) reduces to these action formulas at $n = 1$. Cardinalities of the sets
\( \bar{w}^0_i \) and \( \bar{w}^0_m \) according to (3.2) will be in this case \((-1 \leq i, j \leq 1)\)

\[
\# \bar{w}^0_i = \Theta(i) + \Theta(i - 1) = i + 1, \\
\# \bar{w}^0_m = \Theta(-j) + \Theta(-j - 1) = 1 - j.
\]

Equality (3.5) reduces to

\[
T_{i,j}(z) \cdot B(\bar{t}^0) = (-1)^{\delta_{i,-1}\delta_{j,-1}} \frac{\lambda_1(z)}{2} \sum_{\text{part}} \mathbb{B}(\bar{w}_m^0) \frac{\alpha_0(\bar{w}_m^0) f(\bar{w}_m^0, \bar{w}_n^0) f(\bar{w}_n^0, \bar{w}_m^0)}{h(z, \bar{w}_m^0)h(\bar{w}_n^0, z + c/2)}
\]

which coincides exactly with the action calculated in the paper [22] and \( \bar{w}^0 = \{ \bar{t}^0, z, z + c/2 \} \). Here we used the fact that for \(-1 \leq i \leq 1, \sigma_i(-1) = \sigma_{-i} \) and \( \sigma_i = (-1)^{\delta_i,1} + 1 \).

### 4 Bethe vectors and algebra \( \mathcal{D}Y(\mathfrak{o}_{2n+1}) \)

As we already mentioned above in order to prove the propositions 3.1 and 3.2 we identify the \( \mathcal{R}T \mathcal{T} \) algebra \( \mathcal{B} \) with the Borel subalgebra in the Yangian double \( \mathcal{D}Y(\mathfrak{o}_{2n+1}) \). This infinite dimensional algebra is generated by two \( T \)-operators \( T^\pm(u) \) which satisfy the commutation relations

\[
R(u, v) (T^\mu(u) \otimes \mathbb{I}) (\mathbb{I} \otimes T^\nu(v)) = (\mathbb{I} \otimes T^\nu(v)) (T^\mu(u) \otimes \mathbb{I}) R(u, v)
\]

with the same \( R \)-matrix (2.1) for \( \mu, \nu = \pm \). We set the corresponding central elements given by the equality (2.5) for \( T \)-operators \( T^\pm(u) \) equal to 1. We assume following series expansion of the generating series \( T_{i,j}^\pm(u) \)

\[
T_{i,j}^\pm(u) = \chi_i \delta_{ij} \pm \sum_{\ell \geq 0} T_{i,j}[\ell](u/c)^{-\ell-1}.
\]

This expansion allows to identify monodromy matrix \( T(u) \) of \( \mathfrak{o}_{2n+1} \)-invariant model with \( T \)-operator \( T^+(u) \).

Algebra \( \mathcal{D}Y(\mathfrak{o}_{2n+1}) \) has also so called 'current' realization [5] described in details in [7, 15]. The link between \( \mathcal{R}T \mathcal{T} \) and 'current' realizations is established through \textit{Gauss coordinates} which can be defined as follows

\[
T_{i,j}^\pm(u) = \sum_{\max(i,j) \leq s \leq n} F_{s,i}^\pm(u) k_s^\pm(u) E_{j,s}^\pm(u),
\]

where we assume that \( F_{i,j}^\pm(u) = E_{i,j}^\pm(u) = 0 \) for \( i < j \) and \( F_{i,i}^\pm(u) = E_{i,i}^\pm(u) = 1 \) for \( i = -n, \ldots, n \). It was shown in [7, 15] that the set of the Gauss coordinates

\[
F_{i+1,i}^\pm(u), \quad E_{i,i+1}^\pm(u), \quad 1 \leq i \leq n - 1, \quad k_j^\pm(u), \quad 1 \leq j \leq n
\]
can be chosen as the algebraically independent set of the generators of the Yangian double $DY(o_{2n+1})$. Commutation relations of this algebra contains also currents $k^\pm_0(u)$. The modes of these currents $k_0[\ell]$, $\ell \in \mathbb{Z}$ can be expressed through modes $k_s[\ell]$, $\ell \in \mathbb{Z}$, $s = 1, \ldots, n$ of the algebraically independent currents by the relations

$$k_0^+(u + c/2)k_0^-(u) = \prod_{s=1}^n \frac{k_s^+(u - c(s - 3/2))}{k_s^-(u - c(s - 1/2))}.$$ 

RTT commutation relations (4.1) for the Yangian double $DY(o_{2n+1})$ can be presented in terms of the formal generating series or currents

$$F_i(u) = F^+_{i,i+1}(u) - F^-_{i,i+1}(u) = \sum_{\ell \in \mathbb{Z}} F_i[\ell]u^{-\ell - 1},$$
$$E_i(u) = E^+_{i,i+1}(u) - E^-_{i,i+1}(u) = \sum_{\ell \in \mathbb{Z}} E_i[\ell]u^{-\ell - 1}$$

(4.4)

for $0 \leq i \leq n - 1$. We will write explicitly only those commutation relations in terms of the currents which will be relevant for the calculations (see, for example, (4.6), (4.13) and (4.1) below). One can find the whole set of the relations between currents in [7, 15].

RTT realization of the Yangian double and its ’current’ realization correspond to the different choices of Borel subalgebras. Following the ideas of the paper [14] one can define projections onto intersections of these different type Borel subalgebras. As it was shown in several papers (see, for example, [13] for the super-symmetric Yangian double $DY(gl(m|n))$ and references therein) these projections being applied to the products of the currents $F_i(u)$ (4.4) can be identified with off-shell Bethe vectors in the corresponding $g$-invariant integrable models.

In [15] one can find a formal definition of the projections onto intersections of the different type Borel subalgebras in the Yangian doubles. Very often one can use following approach to calculate these projections of the product of the currents $F_i(u)$. One should replace each current by the difference of the Gauss coordinates according to the Ding-Frenkel formulas (4.4), expand all brackets and use commutation relations between Gauss coordinates which follow from (4.1) to order all monomials in such a way that all ’positive’ Gauss coordinates $F^+_{i,j}(u)$, $i < j$ are on the right of all ’negative’ Gauss coordinates $F^-_{j,i}(u)$ in each monomial. Although we start from the Gauss coordinates $F^+_{j,i}(u)$ for $j > i + 1$ the higher Gauss coordinates for $j > i + 1$ will appear during the process of the ordering. Then application of the projection $P^+_j$ amounts to delete all ordered monomials composed from the Gauss coordinates which have at least one ’negative’ Gauss coordinate $F^-_{j,i}(u)$ on the left. Analogously, application of the projection $P^-_j$ amounts to delete all monomials which have at least one ’positive’ Gauss coordinate $F^+_{j,i}(u)$ on the right. Analogously, one can define the applications of the projections $P^+_{ij}$ to the products of the currents $E_i(u)$.

Let $\bar{t}$ be a set of the generic parameters described in (2.9). For any scalar function $x(t, t')$ of two variables and a set $\bar{t}^s = \{t^s_1, \ldots, t^s_{r_s}\}$, $r_s > 1$ we introduce the ’triangular’
products of these functions
\[
\delta_x(\tilde{t}^s) = \prod_{t > \tilde{t}^s} x(t_t^s, t_r^s). \tag{4.5}
\]

For any \(s = 0, \ldots, n - 1\) and the set \(\tilde{t}^s\) of the Bethe parameters we define the normalized ordered products of the currents
\[
\mathcal{F}_s(\tilde{t}^s) = \delta_{f_s}(\tilde{t}^s) F_s(t_s^s) \cdots F_s(t_1^s) \quad \text{for} \quad s = 0, 1, \ldots, n - 1,
\]
where \(f_s\) are defined by (2.7). By definition we set \(\mathcal{F}_s(\emptyset) = 1\) for all \(s\). It is clear from the commutation relation
\[
f_s(t, t') F_s(t) F_s(t') = f_s(t', t) F_s(t') F_s(t), \quad 0 \leq s \leq n - 1 \tag{4.6}
\]
that these ordered products are symmetric with respect to permutations of the elements in any of the set \(\tilde{t}^s\).

Let \(\tilde{t}\) be a set of generic parameters described by (2.9). Let us define the ordered products of the currents
\[
\mathbb{F}_s(\tilde{t}) = \mathcal{F}_{n-1}(\tilde{t}^{n-1}) \mathcal{F}_{n-2}(\tilde{t}^{n-2}) \cdots \mathcal{F}_1(\tilde{t}) \mathcal{F}_0(\tilde{0}) \tag{4.7}
\]
and
\[
\mathbb{F}(\tilde{t}) = \prod_{s=1}^{n-1} \frac{1}{g(\tilde{t}^s, \tilde{t}^{s-1})} h(\tilde{t}^s, \tilde{t}^s) \mathbb{F}_s(\tilde{t}) \tag{4.8}
\]

**Definition 4.1.** The \(\mathfrak{o}_{2n+1}\)-invariant off-shell Bethe vector \(\mathbb{B}(\tilde{t})\) is defined by the action of the projection \(P_j^+ (\mathbb{F}(\tilde{t}))\) onto reference vector \(|0\rangle\) (2.7)
\[
\mathbb{B}(\tilde{t}) = P_j^+ (\mathbb{F}(\tilde{t})) |0\rangle. \tag{4.9}
\]

Products of the currents (4.7) and (4.8) can be written in the ordered form [15]
\[
\mathbb{F}_s(\tilde{t}) = \sum_{\text{part}} f(\tilde{t}_1^s, \tilde{t}_u^s) \prod_{s=1}^{n-1} f(\tilde{t}_s^s, \tilde{t}_s^{s-1}) P_j^- (\mathbb{F}_s(\tilde{t}_1^s)) \cdot P_j^+ (\mathbb{F}_s(\tilde{t}_u^s)) \tag{4.10}
\]
and
\[
\mathbb{F}(\tilde{t}) = \sum_{\text{part}} f(\tilde{t}_1^0, \tilde{t}_u^0) \prod_{s=1}^{n-1} g(\tilde{t}_s^s, \tilde{t}_u^s) h(\tilde{t}_s^s, \tilde{t}_s^{s-1}) P_j^- (\mathbb{F}(\tilde{t}_1)) \cdot P_j^+ (\mathbb{F}(\tilde{t}_u)) \tag{4.11}
\]
\[
= \sum_{\text{part}} \prod_{s=0}^{n-1} \gamma_s(\tilde{t}_1^s, \tilde{t}_u^s) \prod_{s=1}^{n-1} h(\tilde{t}_s^s, \tilde{t}_s^{s-1}) P_j^- (\mathbb{F}(\tilde{t}_1)) \cdot P_j^+ (\mathbb{F}(\tilde{t}_u)),
\]
\[19\]
where functions $\gamma_s(u, v)$ are defined by (3.4) and summation goes over all possible partitions of the sets $\bar{t}^+$ onto nonintersecting subsets $\bar{t}^+_1$ and $\bar{t}^+_s$ such that $\{\bar{t}^+_1, \bar{t}^+_s\} \equiv \bar{t}^+$. Cardinalities of the sets $\bar{t}^+_1$ and $\bar{t}^+_s$ can be equal to zero. Notations $\bar{t}$ and $\bar{t}_n$ means the collections of the subsets $\bar{t}^+_1$ and $\bar{t}^+_n$

$$\bar{t} = \{\bar{t}^+_0, \bar{t}^+_1, \ldots, \bar{t}^+_n\} \quad \text{and} \quad \bar{t}_n = \{\bar{t}^+_0, \bar{t}^+_1, \ldots, \bar{t}^+_n\}. $$

We assume that $\mathbb{F}^0(\emptyset) \equiv \mathbb{F}(\emptyset) \equiv 1$ and $P_f^+(1) = 1$.

### 4.1 Proof of the Proposition 3.1

Equations (4.2) and (4.3) yield the following series expansion

$$k_f^+(u) = \chi_f \pm \sum_{\ell \geq 0} k_f[\ell](u/c)^{-\ell-1}. $$

Because of this expansion the zero-modes of $T_{j+1,j}^+(u)$ for $0 \leq j \leq n - 1$ is

$$T_{j+1,j}^+ = \chi_{j+1}E_j[0], $$

where $E_j[0]$ are the zero-modes of the currents $E_j(z)$ for $j = 0, \ldots, n - 1$ corresponding to the simple roots of the algebra $\mathfrak{so}_{2n+1}$.

To calculate the action of the operator $E_j[0]$ onto off-shell Bethe vector we use a special presentation for the ordered product of the currents $\mathbb{F}(\bar{t})$ (1.8) which follows from (4.11)

$$P_f^+(\mathbb{F}(\bar{t})) = \mathbb{F}(\bar{t}) + \sum_{\ell=1}^{r_0} f(t^0_\ell, t^0_\ell) f(t^1_\ell, t^0_\ell) F_{1,0}(t^0_\ell) \mathbb{F}(t^0_\ell, t^1_\ell, \ldots, t^{n-1}_\ell) + $$

$$+ \sum_{s=1}^{n-1} \sum_{\ell=1}^{r_s} f(t^s_\ell, t^s_\ell) f(t^{s+1}_\ell, t^s_\ell) g(t^{s+1}_\ell, t^s_\ell) h(t^s_\ell, t^{s-1}_\ell) h(t^s_\ell, t^s_\ell) F_{s+1,s}(t^0_\ell, \ldots, t^s_\ell, \ldots, t^{n-1}_\ell) + \cdots. $$

(4.12)

where $\cdots$ stands for terms which are annihilated by the projection $P_f^+$ after the adjoint action of the zero mode $E_j[0]$.

Using the commutation relations

$$k_0^+(u)F_0(v)k_0^+(u)^{-1} = f(u, v)F(v, v + c/2)F_0(v), $$

$$k_i^+(u)F_i(v)k_i^+(u)^{-1} = f(u, v)F_i(v), \quad 1 \leq i \leq n - 1, $$

$$k_{i+1}^+(u)F_i(v)k_{i+1}^+(u)^{-1} = f(u, v)F_i(v), \quad 0 \leq i \leq n - 1, $$

$$k_j^+(u)F_j(v)k_i^+(u)^{-1} = F_j(v), \quad j \neq i, i + 1, \quad 0 \leq j \leq n - 1, \quad 0 \leq i \leq n $$

(4.13)
and formulas
\[E_j[0]F_l(v) = F_l(v)E_j[0] + \delta_{jl} \left( k_j^+(v)k_{j+1}^+(v)^{-1} - k_j^-(v)k_{j+1}^-(v)^{-1} \right),\]
\[E_j[0]F_{l+1,l}(v) = F_{l+1,l}(v)E_j[0] + \delta_{jl} \left( k_j^-(v)k_{j+1}^-(v)^{-1} - \chi_j\chi_{j+1} \right),\]
where \(j, l = 0, \ldots, n - 1\) we may calculate for \(i = 0, \ldots, n - 1\)
\[
T_{i+1,i} \cdot B(\bar{t}) = \sum_{t=1}^{n} B(\bar{t}^0; \ldots; \bar{t}^{i-1}; \bar{t}^i; \bar{t}^{i+1}; \ldots; \bar{t}^{n-1}) \times \left( \frac{\chi_{t+1}(t^i)}{\chi_{i+1}(t^i)} - \chi_i \frac{f_i(t^i_t, t^i_t)f(\bar{t}^{i+1}, t^i)}{f_i(t^i_t, t^i_t)f(t^i_t, t^{i-1})} \right) \frac{f_i(t^i_t, t^i_t)}{g(t^{i+1}, t^i_t)} \frac{h(t^i_t, t^i)}{h(t^i_t, t^i_t)}, \tag{4.14}
\]
In (4.12) and (4.14) we set \(\bar{t}^{-1} = \bar{t}^n = \emptyset\) and the products of the rational functions depending on these sets are equal to 1.
If \(\chi_j = 1\) and the Bethe parameters \(\bar{t}\) satisfy the Bethe equations (2.12) the on-shell Bethe vectors become highest weight vectors for the algebra \(\mathfrak{o}_{2n+1}\). Performing calculations of the zero modes actions (4.14) we omit the terms which are annihilated by the action of the projection \(P^\mu_j\). In particular, the terms containing the ratio of \(k_j^-(v)/k_{j+1}^-(v)\) do not contribute to these actions.
To prove proposition 3.1 it is sufficient to use the action (4.14) of the \(\mathfrak{o}_{2n+1}\) simple roots monodromy matrix elements zero modes and the commutation relations (3.12). \(\square\)

### 4.2 Proof of the proposition 3.2
According to Gauss decomposition (4.3) the monodromy matrix element \(T_{-n,n}^+(z)\) is
\[T_{-n,n}^+(z) = F_{n,-n}^+(z)k_n^+(z). \tag{4.15}\]
We will calculate the action of this element \(T_{-n,n}^+(z)\) onto ordered product of the currents \(P_j^+ (\mathbb{F}^0(\bar{t}))\). We need following

**Lemma 4.1.** In the subalgebra \(\mathcal{B}\) and for \(-n \leq i < j \leq n\) we have the equality
\[P_j^+ \left( T_{-n,n}^+(v)F_{j,i}^-(u) \right) = 0. \tag{4.16}\]

Let us consider the equality (4.1) for the values \(\mu = -\) and \(\nu = +\) and multiply it from the right by \(R(v, u)\). Using unitarity condition (2.2) we obtain
\[
\left(1 - \frac{c^2}{(u - v)^2}\right) (I \otimes T^+(v)) (T^-(u) \otimes I) = R(u, v) (T^-(u) \otimes I) (I \otimes T^+(v)) R(v, u).
\]
If we consider the element \((i, j; -n, n)\) with \(-n \leq i < j \leq n\) in this matrix equation we obtain

\[
\left(1 - \frac{c^2}{(v - u)^2}\right) T_{-n,n}^+(v)T_{i,j}^-(u) = T_{-n,n}^-(u)T_{i,j}^+(v) + \frac{c}{u-v} T_{-n,j}^-(u)T_{i,n}^+(v) + \frac{c}{v-u} T_{i,n}^-(u)T_{-n,j}^+(v).
\] (4.17)

Multiplying both sides of the equality (4.17) on the right by \(k_j^-(u)^{-1}\) and ordering all terms according to the circular ordering in the Yangian double \(\mathcal{DY}(\mathfrak{g}_{2n+1})\) described in [15] we obtain the statement of the lemma. All terms in RHS of (4.17) start with some Gauss coordinates \(F_{-i}^-(u)\), so they annihilated by the action of projection \(P_j^+\).

\[\square\]

The total currents \(F_i(u)\) (4.14) are defined for the values \(i = 0, \ldots, n - 1\). It was shown in [15] using results of the paper [22] that the same differences of the Gauss coordinates defines the currents for the values \(i = -n, \ldots, -1\)

\[F_i(u) = -F_{-i-1}(u + c(i + 3/2)).\]

It was proved further in [15] that the Gauss coordinates of \(T\)-operators \(T_j^\pm(u)\) are related to the currents \(F_i(u)\), for \(i = -n, \ldots, n - 1\)

\[F_{j,i}^+(v) = P_j^+(F_i(v) \cdot F_{i+1}(v) \cdots F_{j-2}(v) \cdot F_{j-1}(v))\], (4.18)

\[F_{j,i}^-(v) = P_j^-(F_i(v) \cdot F_{i+1}(v) \cdots F_{j-2}(v) \cdot F_{j-1}(v))\], (4.19)

where \(-n \leq i < j \leq n\) and

\[\hat{F}_{j,i}^\pm(u) = \sum_{\ell=0}^{j-i-1} (-1)^{\ell+1} \sum_{j > i_2 > \cdots > i_1 > i} F_{i_1,i_2}^\pm(u)F_{i_2,i_1}^\pm(u) \cdots F_{i_\ell,i_{\ell-1}}^\pm(u)F_{i_{\ell},i_\ell}^\pm(u).\]

Let us introduce the set \(\{z^0, z^1, \ldots, z^{n-1}\}\) of the generic complex parameters such that \(z^s \neq z^{s'}\) if \(s \neq s'\) for all \(s, s' = 0, 1, \ldots, n - 1\) and

\[\hat{z}^i = z^i - c \left( i - \frac{1}{2} \right).\] (4.20)

Then using (4.18) we may express monodromy element entry (4.15) through the currents as follows

\[T_{-n,n}^+(z) = (-1)^n P_j^+(F_{n-1}(\hat{z}^{n-1}) \cdots F_0(z^0) \cdot F_0(z^0) \cdots F_{n-1}(z^{n-1})) k_n^+(z^{n-1}) \big|_{z = \hat{z}} ;\]

where parameters \(\hat{z}^i\) are defined by (4.20) for \(i = 0, \ldots, n - 1\).
Using the statement of the lemma 4.1 the equality 4.10 and the fact that 'negative' projection \( P^+_f (\mathcal{F}^0(\bar{t})) \) in this equality can be expressed in terms of the 'negative' Gauss coordinates \( \mathcal{F}^-_{j,i}(u) \) due to (4.19) we can calculate the action of matrix entries \( T^+_{-n,n}(z) \) onto ordered product of the currents \( P^+_f (\mathcal{F}^0(\bar{t})) \) as follows
\[
T^+_{-n,n}(z) \cdot P^+_f (\mathcal{F}^0(\bar{t})) = P^+_f \left( T^+_{-n,n}(z) \cdot \mathcal{F}^0(\bar{t}) \right) .
\] (4.21)

Let us define an element \( T^0_{-n,n}(z^0, \ldots, z^{n-1}) \)
\[
T^0_{-n,n}(z^0, \ldots, z^{n-1}) = (-1)^n F_{n-1}(z^{n-1}) \cdots F_0(z^0) \cdot F_0(z^0) \cdots F_{n-1}(z^{n-1}) k^+_n(z^{n-1}).
\]
Then the action (4.21) can be written as
\[
T^+_{-n,n}(z) \cdot P^+_f (\mathcal{F}^0(\bar{t})) = P^+_f \left( T^0_{-n,n}(z^0, \ldots, z^{n-1}) \cdot \mathcal{F}^0(\bar{t}) \right) |_{z_i = \bar{z}} .
\]

Using the commutation relations between currents (4.6), (4.13) and (4.9) we can reorder the currents in the element \( T^0_{-n,n}(z^0, \ldots, z^{n-1}) \cdot \mathcal{F}^0(\bar{t}) \) as follows
\[
T^0_{-n,n}(z^0, \ldots, z^{n-1}) \cdot \mathcal{F}^0(\bar{t}) = \frac{(-1)^n}{2} \prod_{i=1}^{n-1} f(z^s, z^{s-1}) f(\bar{t}^s, \bar{t}^{s-1}) k^+_n(z^{n-1}) + \cdots ,
\] (4.22)
where \( \cdots \) stands for the terms which have higher zeros when \( z^\ell = z^{\ell'} = z \) and do not contribute into (4.22). The collection of sets \( \bar{w} \) in (4.22) is
\[
\bar{w} = \{ \bar{w}^0, \ldots, \bar{w}^{n-1} \} \quad \text{and} \quad \bar{w}^{\ell} = \{ \bar{t}^{\ell}, z^{\ell}, \bar{z}^{\ell} \} .
\]
To obtain (4.22) we used a trivial identity
\[
f(u - c, v) f(v, u) = 1
\]
and the fact that we may replace any \( z^{\ell} \) by any \( z^{\ell'} \) since at the end we will set \( z^{\ell} = z^{\ell'} = z \).

It is obvious that we cannot set \( z^{\ell} = z \) in (4.22) since \( f(z, z)^{-1} = 0 \). But the action of \( T^0_{-n,n}(z^0, \ldots, z^{n-1}) \) onto renormalized ordered product of currents \( \mathcal{F}(\bar{t}) \) (4.8) is non-singular and yields the action of \( T^+_{-n,n}(z) \) onto off-shell Bethe vector \( \mathbb{B}(\bar{t}) \)
\[
T^+_{-n,n}(z) \cdot \mathbb{B}(\bar{t}) = T^+_{-n,n}(z) \cdot P^+_f (\mathcal{F}(\bar{t})) |0\rangle = P^+_f \left( T^+_{-n,n}(z) \cdot \mathcal{F}(\bar{t}) \right) |0\rangle = \]
\[
P^+_f \left( T^0_{-n,n}(z^0, \ldots, z^{n-1}) \cdot \mathcal{F}(\bar{t}) \right) |_{z_i = \bar{z}} |0\rangle = -k \frac{g(z_1, \bar{t}^0) h(z, \bar{t}^{n-1})}{h(z, t^0) g(z_n, \bar{t}^{n-1})} \lambda_n(z) \mathbb{B}(\bar{w}) ,
\] (4.23)
where \( z_1, z_n \) are defined in (2.15). This proves the proposition 5.2 \( \square \)
5 Conclusion

This paper is a continuation of our work \cite{19} to investigate the $\mathfrak{o}_{2n+1}$-invariant quantum integrable models which are defined by the $(2n + 1) \times (2n + 1)$ monodromy matrices satisfying the commutation relations (2.3) with $\mathfrak{o}_{2n+1}$-invariant $R$-matrix (2.1). To describe the space of states in these models we are using the method introduced in \cite{14} and developed in \cite{13, 20} for the supersymmetric integrable models associated with Yangian double $DY(\mathfrak{gl}(m|n))$. In these papers the action of the upper-triangular monodromy matrix elements were used to describe the recurrent relations for the corresponding off-shell Bethe vectors and the action of the lower-triangular matrix elements were exploited to find recurrence relations for the higher coefficients in the summation formula for the Bethe vectors scalar products.

Analogous program in case of $\mathfrak{o}_{2n+1}$-invariant integrable models is far from completion. The arguments of the paper \cite{20} cannot be directly repeated since we yet do not have clear picture on the structure of the summation formulas for the scalar products and properties of the higher coefficients. We hope to investigate this picture in our forthcoming publications using the identification of the higher coefficients with the kernels in the integral presentation of the Bethe vectors through the currents \cite{23}.

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