On univoque and strongly univoque sets

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Abstract

Much has been written about expansions of real numbers in noninteger bases. Particularly, for a finite alphabet \(\{0, 1, \ldots, \alpha\}\) and a real number (base) \(1 < \beta < \alpha + 1\), the so-called univoque set of numbers which have a unique expansion in base \(\beta\) has garnered a great deal of attention in recent years. Motivated by recent applications of \(\beta\)-expansions to Bernoulli convolutions and a certain class of self-affine functions, we introduce the notion of a strongly univoque set. We study in detail the set \(D_\beta\) of numbers which are univoque but not strongly univoque. Our main result is that \(D_\beta\) is nonempty if and only if the number 1 has a unique nonterminating expansion in base \(\beta\), and in that case, \(D_\beta\) is uncountable. We give a sufficient condition for \(D_\beta\) to have positive Hausdorff dimension, and show that, on the other hand, there are infinitely many values of \(\beta\) for which \(D_\beta\) is uncountable but of Hausdorff dimension zero.

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1 Introduction and main results

In the last 25 years or so, there has been significant interest in expansions of real numbers in noninteger bases; see [12] for an excellent survey of the pre-2011 literature. In the general setting considered here, we have a finite alphabet \(A := \{0, 1, \ldots, \alpha\}\),

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where $\alpha \in \mathbb{N}$ is fixed, and a real number $\beta$ with $1 < \beta < \alpha + 1$. For $x \in \mathbb{R}$, we call an expression of the form

$$x = \sum_{j=1}^{\infty} \frac{\omega_j}{\beta^j}, \quad \text{where } \omega_1, \omega_2, \cdots \in A$$

(1.1)

an expansion of $x$ in base $\beta$ (or simply, a $\beta$-expansion). By a classical result of Rényi [22] such an expansion of $x$ exists if and only if $x \in J_{\beta} := [0, \alpha/(\beta - 1)]$. Observe that we allow for the possibility that $\beta \leq \alpha$, in which case digits larger than the base $\beta$ may be used. It is well known (see [23]) that almost every $x \in J_{\beta}$ has a continuum of $\beta$-expansions, but there are also many points having a unique expansion. These include, trivially, the endpoints 0 and $\alpha/(\beta - 1)$ of $J_{\beta}$. We will exclude these points from consideration here and define the set

$$A'_{\beta} := \{x \in \text{int}(J_{\beta}) : x \text{ has a unique expansion of the form } (1.1)\},$$

(1.2)

which has been called a univoque set and is known to have a complicated structure. Specifically (see [10, 17, 18]), there is a number $G := G(\alpha) > 1$ (introduced in [4] and called a generalized golden ratio) and a critical value $\beta_c := \beta_c(\alpha) \in (G, \alpha + 1)$, called the Komornik-Loreti constant, such that $A'_{\beta}$ is empty for $\beta \leq G$; nonempty but countable for $G < \beta < \beta_c$; uncountable but with Hausdorff dimension zero for $\beta = \beta_c$; and of positive Hausdorff dimension for $\beta > \beta_c$. The Komornik-Loreti constant was introduced in the papers [14, 15].

The set $A'_{\beta}$ is symmetric (i.e. invariant under the map $x \mapsto \bar{x} := \frac{\alpha}{\beta - 1} - x$), as can be seen by replacing each $\omega_j$ with $\alpha - \omega_j$ in (1.1). We will be interested in a subset of $A'_{\beta}$ defined by a slightly stronger condition, explained below, which is motivated by some recent applications of $\beta$-expansions. For ease of presentation, we initially restrict ourselves to the interval $J_{\beta} := [(\alpha - \beta + 1)/(\beta - 1), 1]$, and write $A_{\beta} := A'_{\beta} \cap \text{int}(J_{\beta})$. Note that this set is again invariant under the map $x \mapsto \bar{x}$. Later we will show that all our results hold also when $A_{\beta}$ is replaced with the larger set $A'_{\beta}$.

Observe that the interval $J_{\beta}$ is nonempty if and only if $\beta \geq 1 + \alpha/2$. However, as shown in [4], $G(\alpha) \geq 1 + \alpha/2$ for each $\alpha$, and therefore $A'_{\beta} = \emptyset$ when $\beta \leq 1 + \alpha/2$, so this range of bases is not of interest for us here.

The set $A_{\beta}$ is most easily studied by considering the set of all corresponding sequences $\omega_1\omega_2\cdots$. Let $\Omega := A^\mathbb{N}$. For $1 < \beta < \alpha + 1$, let $\Pi_{\beta} : \Omega \to \mathbb{R}$ denote the projection map given by

$$\Pi_{\beta}(\omega) = \sum_{j=1}^{\infty} \frac{\omega_j}{\beta^j}, \quad \omega = \omega_1\omega_2\cdots \in \Omega,$$
so that (1.1) can be written compactly as \( x = \Pi_\beta(\omega) \). Define the set
\[
U_\beta := \Pi_\beta^{-1}(A_\beta).
\] (1.3)
Let \( \sigma \) denote the left shift map on \( \Omega \); that is, \( \sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots \). For a number \( a \in A \), let \( \bar{a} := \alpha - a \) denote the reflection of \( a \). For \( \omega = \omega_1\omega_2\cdots \in \Omega \), likewise denote \( \bar{\omega} = \bar{\omega}_1\bar{\omega}_2\cdots \), and similarly define the reflection for finite words. It is a straightforward consequence of the definitions that
\[
\omega \in U_\beta \iff \Pi_\beta(\sigma^n(\omega)) < 1 \text{ and } \Pi_\beta(\sigma^n(\bar{\omega})) < 1 \text{ for all } n \geq 0.
\] (1.4)
(See [5] or [1, Lemma 5.1] for the case \( \alpha = 1 \).) In some recent applications of \( \beta \)-expansions [11, 1], it was necessary to consider the (conceivably smaller) set
\[
\tilde{U}_\beta := \{ \omega \in U_\beta : \limsup_{n \to \infty} \Pi_\beta(\sigma^n(\omega)) < 1 \text{ and } \limsup_{n \to \infty} \Pi_\beta(\sigma^n(\bar{\omega})) < 1 \}.
\] (1.5)
We shall call elements of \( \tilde{U}_\beta \) and their projections under \( \Pi_\beta \) strongly univoque, and refer to \( \tilde{U}_\beta \) and its projection \( \Pi_\beta(\tilde{U}_\beta) \subset A_\beta \) as strongly univoque sets. In [11], the sets \( \tilde{U}_\beta \) were used to study the multifractal spectra of asymmetric Bernoulli convolutions. In [1], the present author studied a one-parameter family \( \{ F_a : 0 < a < 1 \} \) of self-affine functions introduced by Okamoto [20], and showed that for \( a > 1/2 \), the set of points where \( F_a \) has an infinite derivative is intimately connected to the sets \( U_\beta \) and \( \tilde{U}_\beta \), where \( \beta = 1/a \). The characterization of these infinite derivatives is most elegant in the case where \( \tilde{U}_\beta = U_\beta \).

Motivated by this second application, we define the sets
\[
W_\beta := U_\beta \setminus \tilde{U}_\beta, \quad 1 < \beta < \alpha + 1,
\]
and put \( D_\beta := \Pi_\beta(W_\beta) \). Thus, \( D_\beta \) is the set of numbers in \( J_\beta \) which are univoque but not strongly univoque. We aim to characterize the values of \( \beta \) for which \( D_\beta \) is nonempty, and, if it is, to investigate its cardinality and Hausdorff dimension. Since we will see in Section 2 that the restriction of the map \( \Pi_\beta \) to \( U_\beta \) is bi-Lipschitz continuous (with respect to the metric \( \rho_\beta \) defined in (1.10) below), the sets \( D_\beta \) and \( W_\beta \) have the same cardinality and Hausdorff dimension. Thus, all statements given below about \( W_\beta \) hold for \( D_\beta \) as well.

Our first result, an easy consequence of known facts from the literature, says that for most \( \beta \), all univoque numbers are in fact strongly univoque.

**Proposition 1.1.** The set \( \{ \beta \in (1, \alpha + 1) : W_\beta \neq \emptyset \} \) is nowhere dense and of Lebesgue measure zero. Moreover, \( W_\beta = \emptyset \) when \( \beta < \beta_c \).
In order to state the main theorems of this paper, some more notation is needed. First, we introduce the lexicographic ordering on $\Omega$: For $\omega = \omega_1\omega_2 \cdots$ and $\eta = \eta_1\eta_2 \cdots$ in $\Omega$, we write $\omega < \eta$ if there is an index $n \in \mathbb{N}$ such that $\omega_i = \eta_i$ for $i = 1, \ldots, n-1$, and $\omega_n < \eta_n$. Likewise, for finite words $\omega = \omega_1 \cdots \omega_m$ and $\eta = \eta_1 \cdots \eta_m$ of the same length, we write $\omega < \eta$ if there is an index $n \leq m$ such that $\omega_i = \eta_i$ for $i = 1, \ldots, n-1$, and $\omega_n < \eta_n$. For finite or infinite sequences $\omega$ and $\eta$, we write $\omega \leq \eta$ if $\omega < \eta$ or $\omega = \eta$.

Call an expansion $\omega$ of a number $x$ infinite if $\omega_i > 0$ for infinitely many $i$. The lexicographically largest infinite expansion of $x$ is called the quasi-greedy expansion of $x$. Quasi-greedy expansions were introduced formally in [6] (where they are called quasi-regular expansions), but are already implicit in the work of Parry [21]. A critical role in the study of $\beta$-expansions is played by the quasi-greedy expansion of $x = 1$, which we shall denote by $d = d_1 d_2 \cdots$. We shall often write $d = d(\beta)$ to express that $d$ is the quasi-greedy expansion of 1 in base $\beta$. It is well known that $d$ satisfies the inequalities

$$\sigma^n(d) \leq d, \quad n \in \mathbb{N},$$

and unless $d$ is periodic, the inequality is strict for each $n$. There is also a simple lexicographic characterization of $U_\beta$, namely

$$\omega \in U_\beta \iff \sigma^n(\omega) < d \text{ and } \sigma^n(\bar{\omega}) < d \text{ for all } n \geq 0.$$  \hspace{1cm} (1.6)

This is essentially due to Parry [21]; see also [7, 8, 10, 17]. The corresponding characterization of the larger set $\Pi_{\beta}^{-1}(A_\beta')$ is slightly more subtle; see [17, 25] and (1.12) below.

Some of our results below have an interpretation in terms of the set

$$A(\alpha) := \{ \beta \in (1, \alpha + 1) : 1 \text{ has a unique expansion in base } \beta \}. \hspace{1cm} (1.7)$$

This set is akin to the set $U$ studied in [16], but we emphasize that $A(\alpha)$ depends on the size of the alphabet $A$, which is fixed, whereas the set $U$ in [16] consists of those bases $\beta$ for which 1 has a unique expansion in base $\beta$ using only digits strictly smaller than $\beta$. Nonetheless, the sets $U$ and $A(\alpha)$ have very similar properties. As usual, we denote by $\overline{A(\alpha)}$ the topological closure of $A(\alpha)$.

The following theorem is the main result of this paper. The equivalences (iii) $\iff$ (iv) and (iii) $\iff$ (v) were proved in [25] and [16], respectively, but are included here to put our result in historical perspective.

**Theorem 1.2.** For $1 < \beta < 1 + \alpha$, the following statements are equivalent:

(i) $W_\beta \neq \emptyset$;

(ii) $W_\beta$ is uncountable;
(iii) \(\sigma^n(d) < d\) for every \(n \in \mathbb{N}\), where \(d = d(\beta)\);

(iv) The number 1 has a unique infinite expansion in base \(\beta\).

(v) \(\beta \in \mathcal{A}(\alpha)\).

It has been shown, in [6] for the case \(\alpha = 1\) and in [13] for the general case, that \(\mathcal{A}(\alpha)\) has Hausdorff dimension 1. Thus, we immediately get the following consequence of Theorem 1.2 which the reader may contrast with Proposition 1.1.

**Corollary 1.3.** The set \(\{\beta \in (1, \alpha + 1) : W_\beta \neq \emptyset\}\) has Hausdorff dimension 1.

In [1], where \(\alpha = 1\) and hence \(1 < \beta < 2\), it is shown that Okamoto’s function \(F_a\) with \(a = 1/\beta\) has an infinite derivative at a point \(x\) if and only if

\[
\lim_{n \to \infty} (3/\beta)^n (1 - \Pi_\beta(\sigma^n(\omega))) = \lim_{n \to \infty} (3/\beta)^n (1 - \Pi_\beta(\sigma^n(\bar{\omega}))) = \infty,
\]

where \(\omega\) is derived from the ternary expansion of \(x\). In the follow-up paper [2], a sequence of functions is identified whose infinite derivatives correspond similarly to \(\beta\)-expansions over the more general alphabet considered here. The limits in (1.8) hold automatically when \(\omega \in \mathcal{U}_\beta\), clearly fail when \(\omega \notin \mathcal{U}_\beta\), but may or may not hold when \(\omega \in W_\beta = \mathcal{U}_\beta \setminus \mathcal{U}_\beta\). This suggests that it is meaningful to study the rate of convergence to 1 for subsequences of \(\Pi_\beta(\sigma^n(\omega))\). As the next theorem indicates, any desired rate of convergence can be achieved by a suitable choice of \(\omega\), provided of course that \(W_\beta\) is nonempty.

**Theorem 1.4.** (i) Assume \(W_\beta \neq \emptyset\). Then for any sequence \((\theta_n)_{n \in \mathbb{N}}\) of positive real numbers there exist uncountably many \(\omega \in W_\beta\) such that

\[
\liminf_{n \to \infty} \theta_n (1 - \Pi_\beta(\sigma^n(\omega))) < \infty.
\]

(ii) For each \(\beta \geq \beta_c\) and for any increasing sequence \((\theta_n)_{n \in \mathbb{N}}\) of positive real numbers with \(\lim_{n \to \infty} \theta_n = \infty\), there exist uncountably many \(\omega \in \mathcal{U}_\beta\) such that

\[
\lim_{n \to \infty} \theta_n (1 - \Pi_\beta(\sigma^n(\omega))) = \lim_{n \to \infty} \theta_n (1 - \Pi_\beta(\sigma^n(\bar{\omega}))) = \infty.
\]

The set \(D_\beta\), being a subset of \(A_\beta\), is Lebesgue null, and therefore it is interesting to ask about its Hausdorff dimension (or equivalently, that of \(W_\beta\)). In order to formulate precise results, we first introduce a metric \(\rho_\beta\) on \(\Omega\), defined by

\[
\rho_\beta(\omega, \eta) := \beta^{1-\inf\{n : \omega_n \neq \eta_n\}}.
\]
(The reason for the extra ‘1’ is that this way, \( \text{diam}(\Omega) = 1 \).) Equipped with this metric and the lexicographic order “\(<\)”, \( \Omega \) is a linearly ordered metric space, and so statements such as \( \limsup_{n \to \infty} \sigma^n(\omega) = d \) are meaningful. Note by (1.6) that if \( \omega \in U_\beta \), then \( \limsup \sigma^n(\omega) = d \) if and only if for each \( k \in \mathbb{N} \), there is an index \( n \) such that \( \omega_{n+1} \cdots \omega_{n+k} = d_1 \cdots d_k \).

For a Borel set \( E \subset \Omega \), we denote by \( \dim_H E \) the Hausdorff dimension of \( E \) induced by the metric \( \rho_\beta \), where the value of \( \beta \) will usually be clear form the context. Note that the value of \( \dim_H E \) depends on the choice of \( \beta \), but whether \( \dim_H E \) is positive or zero is independent of \( \beta \).

**Theorem 1.5.**

(i) \( \dim_H U_\beta = \dim_H \tilde{U}_\beta \) for every \( \beta \).

(ii) Let \( d = d(\beta) \). If \( \limsup_{n \to \infty} \sigma^n(\bar{d}) < d \), then \( \dim_H W_\beta > 0 \).

(iii) There are infinitely many values of \( \beta \) in \( (1, \alpha + 1) \) such that \( W_\beta \) is uncountable but \( \dim_H W_\beta = 0 \). The smallest such \( \beta \) is \( \beta_c \).

The above theorem leaves several interesting questions.

**Question 1.6.** Does the converse of (ii) hold? In other words, does

\[
\limsup_{n \to \infty} \sigma^n(\bar{d}) = d
\]  

imply that \( \dim_H W_\beta = 0 \)? (Note that if \( \limsup_{n \to \infty} \sigma^n(\bar{d}) > d \), then \( W_\beta = \emptyset \) by Theorem 1.2.)

**Question 1.7.** For which values of \( \beta \) (other than those specified in Theorem 2.9 below) does \( d := d(\beta) \) satisfy (1.11)?

**Question 1.8.** One would expect that \( W_\beta \) is always a small subset of \( U_\beta \). Is it true that \( \dim_H W_\beta < \dim_H U_\beta \) for all \( \beta \)?

### 1.1 Extension to the interval \( J_\beta' \)

We now show that all of the above results remain valid if the set \( A_\beta \) is replaced with the larger set \( A'_\beta \) defined in (1.2). By analogy with (1.3), define the set

\[
U'_\beta := \Pi^{-1}_\beta(A'_\beta).
\]

Instead of the simple characterization (1.6), we have the following (see 1.7 Theorem 3.4):
Lemma 1.9. Let $\omega = (\omega_n)_n \in \Omega \setminus \{000 \cdots, \alpha\alpha\alpha \cdots\}$, and let

$$m^+(\omega) := \min \{n : \omega_n < \alpha\}, \quad m^- (\omega) := \min \{n : \omega_n > 0\}.$$  

Then $\omega \in U'_\beta$ if and only if

$$\sigma^n(\omega) < d \text{ for all } n \geq m^+(\omega) \quad \text{and} \quad \sigma^n(\bar{\omega}) < d \text{ for all } n \geq m^-(\omega),$$

where $d = d(\beta)$.

(A precursor of this lemma was proved in [8].) One checks easily that (1.12) is equivalent to the condition

$$\Pi_\beta(\sigma^n(\omega)) < 1 \text{ for all } n \geq m^+(\omega) \quad \text{and} \quad \Pi_\beta(\sigma^n(\bar{\omega})) < 1 \text{ for all } n \geq m^-(\omega).$$

It is therefore natural to define a set $\tilde{U}'_{\beta}$ by complete analogy with the definition of $U_{\beta}$, i.e.

$$\tilde{U}'_{\beta} := \{\omega \in U'_{\beta} : \limsup_{n \to \infty} \Pi_\beta(\sigma^n(\omega)) < 1 \text{ and } \limsup_{n \to \infty} \Pi_\beta(\sigma^n(\bar{\omega})) < 1\}.$$  

Let $W'_\beta := U'_\beta \setminus \tilde{U}'_{\beta}$. It follows immediately that $W_{\beta} \subseteq W'_\beta$. Vice versa, if $\omega \in W'_\beta$, then putting $N := \max\{m^+(\omega), m^- (\omega)\}$ we have by (1.6) and (1.12) that $\sigma^N(\omega) \in U_{\beta}$. But since $\omega \notin \tilde{U}'_{\beta}$, it follows from the definitions of $\tilde{U}_{\beta}$ and $\tilde{U}'_{\beta}$ that $\sigma^N(\omega) \notin \tilde{U}_{\beta}$, and so $\sigma^N(\omega) \in W_{\beta}$. As a result,

$$W'_\beta \subseteq \bigcup_{N=1}^{\infty} \bigcup_{\eta_1 \cdots \eta_N \in \mathbb{A}^N} \{\eta_1 \cdots \eta_N \omega : \omega \in W_{\beta}\}.$$  

Since the right hand side is a countable union of similar copies of $W_{\beta}$, and by Theorem 1.2 $W_{\beta}$ is either empty or uncountable, it follows that $W_{\beta}$ and $W'_\beta$ have the same cardinality and the same Hausdorff dimension. Therefore, we may replace $W_{\beta}$ with $W'_\beta$ in each of the above results without affecting their validity.

The remainder of this article is organized as follows. Section 2 reviews necessary background from the literature on $\beta$-expansions, including the important notion of a $p$-mirror sequence, and includes a proof of Proposition 1.1. Theorems 1.2 and 1.4 are proved in Section 3 and Theorem 1.5 is proved in Section 4.
2 Preliminaries

The first two lemmas below were proved in [11] for the case \( \alpha = 1 \). We include short proofs here for completeness.

**Lemma 2.1.** If \( \beta_1 > \beta_2 \), then \( \tilde{U}_{\beta_1} \supset U_{\beta_2} \).

*Proof.* It suffices to find \( \delta > 0 \) such that for \( \omega \in \Omega \) the implication

\[
\Pi_{\beta_2}(\omega) < 1 \implies \Pi_{\beta_1}(\omega) \leq 1 - \delta \tag{2.1}
\]

holds; applying this to \( \sigma^n(\omega) \) and \( \sigma^n(\tilde{\omega}) \) in place of \( \omega \) then gives the lemma.

Choose \( M \in \mathbb{N} \) large enough so that \( \alpha \beta_1 - M < \beta_1 - 1 \), and let

\[
\delta := \min \left\{ 1 - \frac{\alpha \beta_1 - M}{\beta_1 - 1}, \min_{1 \leq j \leq M} (\beta_2^{-j} - \beta_1^{-j}) \right\}.
\]

If \( \omega_j = 0 \) for \( j = 1, \ldots, M \), then

\[
\Pi_{\beta_1}(\omega) = \sum_{j=M+1}^{\infty} \frac{\omega_j}{\beta_1^j} \leq \frac{\alpha \beta_1 - M}{\beta_1 - 1} \leq 1 - \delta.
\]

On the other hand, if \( \omega_{j_0} > 0 \) for some \( j_0 \in \{1, \ldots, M\} \), then

\[
\Pi_{\beta_2}(\omega) - \Pi_{\beta_1}(\omega) \geq \omega_{j_0} (\beta_2^{-j_0} - \beta_1^{-j_0}) \geq \min_{1 \leq j \leq M} (\beta_2^{-j} - \beta_1^{-j}) \geq \delta.
\]

In both cases, (2.1) follows. \( \square \)

**Lemma 2.2.** The map \( \Pi_{\beta}|_{U_\beta} \) is bi-Lipschitz from \( (U_\beta, \rho_\beta) \) onto \( \Pi_\beta(U_\beta) \).

*Proof.* Let \( \omega = \omega_1 \omega_2 \cdots \) and \( \eta = \eta_1 \eta_2 \cdots \) be in \( U_\beta \). If \( \min\{n : \omega_n \neq \eta_n\} = k + 1 \), then \( \rho_\beta(\omega, \eta) = \beta^{-k} \), and

\[
|\Pi_\beta(\omega) - \Pi_\beta(\eta)| = \beta^{-k} |\Pi_\beta(\sigma^k(\omega)) - \Pi_\beta(\sigma^k(\eta))|.
\]

Hence, it suffices to show that there are constants \( 0 < C_1 < C_2 \) such that

\[
\omega_1 \neq \eta_1 \implies C_1 \leq |\Pi_\beta(\omega) - \Pi_\beta(\eta)| \leq C_2. \tag{2.2}
\]

Assume without loss of generality that \( \omega_1 > \eta_1 \). On the one hand,

\[
|\Pi_\beta(\omega) - \Pi_\beta(\eta)| \leq \sum_{j=1}^{\infty} \frac{|\omega_j - \eta_j|}{\beta^j} \leq \sum_{j=1}^{\infty} \frac{\alpha}{\beta^j} = \frac{\alpha}{\beta - 1} =: C_2.
\]
On the other hand, $\omega \in \mathcal{U}_\beta$ implies $\Pi_\beta(\sigma(\omega)) < 1$, so $\sum_{j=2}^{\infty} \bar{\omega}_j \beta^{-j} < \beta^{-1}$, and hence,

$$\sum_{j=2}^{\infty} \frac{\omega_j}{\beta^j} = \sum_{j=2}^{\infty} \frac{\alpha - \bar{\omega}_j}{\beta^j} > \sum_{j=2}^{\infty} \frac{\alpha}{\beta^j} - \frac{1}{\beta} = \frac{1}{\beta} \left( \frac{\alpha + 1 - \beta}{\beta - 1} \right),$$

while $\eta \in \mathcal{U}_\beta$ implies $\Pi_\beta(\sigma(\eta)) < 1$, so $\sum_{j=2}^{\infty} \eta_j \beta^{-j} < \beta^{-1}$. Thus,

$$\Pi_\beta(\omega) - \Pi_\beta(\eta) = \frac{\omega_1 - \eta_1}{\beta} + \sum_{j=2}^{\infty} \frac{\omega_j - \eta_j}{\beta^j} > \frac{1}{\beta} + \frac{1}{\beta} \left( \frac{\alpha + 1 - \beta}{\beta - 1} - 1 \right) = \frac{\alpha + 1 - \beta}{\beta(\beta - 1)} =: C_1.$$

Hence, we have (2.2). □

**Lemma 2.3** ([25], Proposition 2.4). The projection map $\Pi_\beta$ is strictly increasing on $\mathcal{U}_\beta$.

In view of the last two lemmas, any subset of $\mathcal{U}_\beta$ has the same Hausdorff dimension and cardinality as its projection under $\Pi_\beta$, and the restriction of $\Pi_\beta$ to $\mathcal{U}_\beta$ is an order-preserving homeomorphism. This allows us to work entirely within the symbol space $\Omega$, as illustrated by the following important characterization.

**Proposition 2.4.** Let $d = d(\beta)$. For $\omega \in \mathcal{U}_\beta$, we have

$$\limsup_{n \to \infty} \Pi_\beta(\sigma^n(\omega)) = 1 \iff \limsup_{n \to \infty} \sigma^n(\omega) = d.$$

*Proof.* This is a direct consequence of Lemmas 2.2 and 2.3. □

### 2.1 De Vries-Komornik numbers and $p$-mirror sequences

An important role in the theory of $\beta$-expansions is played by a certain countably infinite set of bases, which we now introduce. For $a \in A$ with $a < \alpha$, we denote $a + 1$ more compactly by $a^+$; and for $a \in A$ with $a > 0$ we denote $a - 1$ by $a^-$. For a word $t_1 \cdots t_n$ in $A^n$, let $t_1 \cdots t_n^+ := t_1 \cdots t_{n-1}t_n^+$ if $t_n < \alpha$, and $t_1 \cdots t_n^- := t_1 \cdots t_{n-1}t_n^-$ if $t_n > 0$. The following definition is taken from [17], with a minor modification – see Remark 2.7 below.

**Definition 2.5.** Let $p \in \mathbb{N}$. A word $t_1 \cdots t_p$ in $A^p$ is admissible if $t_p < \alpha$ and for each $1 \leq i \leq p$ we have

$$t_i \cdots t_p t_1 \cdots t_{i-1} \leq t_1 \cdots t_p^+ \quad \text{and} \quad t_i \cdots t_p^+ t_1 \cdots t_{i-1} \leq t_1 \cdots t_p^+. \quad (2.3)$$

...
Definition 2.6. A sequence \( d = (d_i) \) in \( \Omega \) is called a \( p \)-mirror sequence if there is \( p \in \mathbb{N} \) and an admissible word \( t_1 \cdots t_p \) such that
\[
d_1 \cdots d_p = t_1 \cdots t_p^+,
\]
and
\[
d_{2m+1} \cdots d_{2m+p} = \overline{d_1 \cdots d_{2m}^+}, \quad \text{for all } m \geq 0.
\]
In this case we call \( t_1 \cdots t_p \) a generating word of \( d \).

Observe that there are infinitely many admissible words. For instance, when \( \alpha = 1 \), any word of the form \( 1^k0^l \) with \( k \geq l \geq 1 \) is admissible. As a result, there are infinitely many \( p \)-mirror sequences.

Remark 2.7. The term “\( p \)-mirror sequence” was introduced by Allouche and Cosnard \([3]\). Kong and Li \([17]\) use the term “generalized Thue-Morse sequence”, since the special case \( \alpha = 1, \ p = 1 \) and \( t_1 = 0 \) results in the (truncated) Thue-Morse sequence \( 110100110010110 \cdots \). Note that, in order for this special case to be included, we slightly weakened the definition from \([17]\), where the right-hand side of the first inequality in (2.3) is given as \( t_1 \cdots t_p \) (and the inequality is written in its reflected form). It also seems aesthetically pleasing that the two inequalities in (2.3) have the same right-hand side. In any case, the two definitions lead to the same set of \( p \)-mirror sequences, since it is easy to see that, if \( t_1 \cdots t_p \) is admissible in the sense of Definition 2.5, then the word \( s_1 \cdots s_{2p} := t_1 \cdots t_p^+ \overline{t_1 \cdots t_p^+} \) is admissible in the sense of Kong and Li, and generates the same \( p \)-mirror sequence. This shows in addition that every \( p \)-mirror sequence has infinitely many generating words.

Kong and Li \([17]\) show that any \( p \)-mirror sequence \( d \) is the quasi-greedy expansion of 1 in some base \( \beta \), and that this \( \beta \) is necessarily transcendental. Slightly modifying their terminology, we shall call such a base \( \beta \) a de Vries-Komornik number. It is clear that there are infinitely many such numbers; the smallest is the Komornik-Loreti constant \( \beta_c(\alpha) \), which corresponds to the 1-mirror sequence generated by \( t_1 = \lceil \alpha/2 \rceil \). See \([17]\) for further details.

Observe that any \( p \)-mirror sequence \( d \) satisfies
\[
\sigma^n(d) < d \quad \text{and} \quad \sigma^n(\overline{d}) < d \quad \text{for all } n \in \mathbb{N},
\]
so the corresponding bases lie in \( \mathcal{A}(\alpha) \) by a result in \([16]\). In fact, it follows from \([16\) Lemma 4.2] and its proof that the set of de Vries-Komornik numbers is dense in \( \mathcal{A}(\alpha) \). In \([25]\), they are shown to be exactly the right endpoints of the connected components of \((1, \alpha + 1) \setminus \mathcal{A}(\alpha)\).

The significance of \( p \)-mirror sequences for the purposes of this article lies in the following.
Lemma 2.8. Let \( d \) be a \( p \)-mirror sequence with corresponding de Vries-Komornik number \( \beta \). Define \( b_m := d_1 \cdots d_{2^m-1} \), \( m \in \mathbb{N} \). Suppose \( \omega = \omega_1 \omega_2 \cdots \in \mathcal{U}_\beta \). Then for every \( m \in \mathbb{N} \) and \( k \geq 0 \), the following statements hold:

(i) If \( \omega_{k+1} \cdots \omega_{k+2^m-1} = b_m \), then \( \omega_{k+1} \cdots \omega_{k+2^m} = b_m \bar{b}_m \) or \( b_{m+1} \).

(ii) If \( \omega_{k+1} \cdots \omega_{k+2^m-1} = \bar{b}_m \), then \( \omega_{k+1} \cdots \omega_{k+2^m} = \bar{b}_m b_m \) or \( \bar{b}_{m+1} \).

Vice versa, for any sequence \( (j_1, j_2, \ldots) \) with \( 0 \leq j_\nu < \infty \) for each \( \nu \) and \( \sum_{\nu=1}^{\infty} j_\nu = \infty \), the sequence \( \omega \) lies in \( \mathcal{U}_\beta \), where

\[
\omega = (b_1 \bar{b}_1)^{j_1} (b_2 \bar{b}_2)^{j_2} \cdots (b_\nu \bar{b}_\nu)^{j_\nu} \cdots. \tag{2.4}
\]

Proof. That any \( \omega \in \mathcal{U}_\beta \) satisfies (i) and (ii) was proved in \([18\text{ Lemma } 4.11]\) for the case \( p = 1 \). Since the argument for general \( p \) is practically the same, we omit it here.

As for the second statement, let \( \omega \) be as in (2.4). Given \( n \geq 0 \), there are integers \( k \geq 0 \) and \( m \in \mathbb{N} \) such that \( k \leq n < k + 2^m \) and \( \omega_{k+1} \cdots \omega_{k+2^m} = b_m \) or \( \bar{b}_m \). Without loss of generality assume the former, and assume \( m \) is the largest integer with this property, so that \( \omega_{k+2^m+1} \cdots \omega_{k+2^{m+1}} = \bar{b}_m \). Since \( b_m \bar{b}_m < b_{m+1} \) and these two words differ only in their last digit, it follows that \( \sigma^n(\omega) < \sigma^{n-k}(d) \leq d \). Furthermore, it is a consequence of \([17\text{ Lemma } 4.2]\) that \( \sigma^n(\bar{\omega}) < d \). Hence, by (1.6), \( \omega \in \mathcal{U}_\beta \).

It is easy to see that any \( p \)-mirror sequence \( d \) satisfies (1.11). This leads us to a more precise version of Theorem 1.5(iii).

Theorem 2.9. If \( \beta \) is a de Vries-Komornik number, then \( \mathcal{W}_\beta \) is uncountable but \( \dim_H \mathcal{W}_\beta = 0 \).

The proof of Theorem 2.9 is given at the end of the paper. It implies the statements of Theorem 1.5(iii) since there are infinitely many de Vries-Komornik numbers, and \( \beta_c \) is the smallest such number.

Proof of Proposition 1.7. For \( 1 < \beta < \beta_c \), \( \mathcal{U}_\beta \) contains only eventually periodic sequences \([18\text{ Lemma } 4.12]\), and it is clear that an eventually periodic sequence lies in \( \mathcal{U}_\beta \) whenever it lies in \( \mathcal{U}_\beta \). Hence, \( \mathcal{W}_\beta = \emptyset \) for all \( 1 < \beta < \beta_c \).

On the other hand, according to \([17\text{ Theorem } 2.5]\), the interval \((\beta_c, \alpha)\) is covered, with the exception of a Lebesgue-null set, by the collection of intervals \((\beta_L, \beta_R) := (\beta_L(t_1 \cdots t_p), \beta_R(t_1 \cdots t_p))\), where \( t_1 \cdots t_p \) ranges over all admissible words (see Definition 2.5) of all lengths \( p \geq 2 \); \( d(\beta_L) = (t_1 \cdots t_p)^\infty \); and \( d(\beta_R) \) is the \( p \)-mirror sequence with generating word \( t_1 \cdots t_p \). It is implicit in Theorem 1.7 of \([25]\) that any such interval \((\beta_L, \beta_R)\) can be written as a countable union of half open intervals \((\beta_i, \beta_{i+1}]\), \( i = 1, 2, \ldots \), so-called stability intervals, with the property that
for each $i$, $U_{\beta}$ is constant on $(\beta_i, \beta_{i+1}]$. Combined with Lemma 2.1, this yields that $W_{\beta} = \emptyset$ for $\beta \in (\beta_i, \beta_{i+1}]$, and hence for $\beta \in (\beta_L, \beta_R)$. As a result, the set 
\{ $\beta \in (1, \alpha + 1) : W_{\beta} = \emptyset$\} contains a dense open set of full Lebesgue measure. Both properties stated in the proposition now follow. \hfill $\square$

**Remark 2.10.** One can say a bit more about the intervals $(\beta_L, \beta_R)$ in the above proof: By [17, Theorem 2.6], the Hausdorff dimension of $U_{\beta}$ with respect to a fixed metric $\rho_{\beta_0}$ is constant on each such interval, although $U_{\beta}$ itself is not.

### 3 Proofs of Theorems 1.2 and 1.4

We begin by introducing the basic building blocks for constructing sequences in $W_{\beta}$.

**Definition 3.1.** Let $d = (d_i)_i \in \Omega$. A sequence $M = (m_k)_k \in \mathbb{N}$ of positive integers is $d$-positive if $d_{m_k} > 0$ for every $k \in \mathbb{N}$.

**Definition 3.2.** For a $d$-positive sequence $M = (m_k)_k \in \mathbb{N}$, we define a sequence $\omega(M) \in \Omega$ by

$$
\omega(M) := \prod_{k=1}^{\infty} d_1 \cdots d_{m_k} := d_1 \cdots d_{m_1} d_1 \cdots d_{m_2} \cdots
$$

It follows immediately that, if $d$ is the quasi-greedy expansion of 1 in some base $\beta$, then

$$
\sigma^n(\omega(M)) < d \quad \text{for every } n \geq 0. \quad (3.1)
$$

Moreover,

$$
\limsup_{n \to \infty} \sigma^n(\omega(M)) = d \iff \lim_{k \to \infty} m_k = \infty. \quad (3.2)
$$

Observe also that the mapping $M \mapsto \omega(M)$ is an injection: If $M_1 = (m_{k}^{(1)})$ and $M_2 = (m_{k}^{(2)})$ are different $d$-positive sequences, then $\omega(M_1) \neq \omega(M_2)$.

The critical step in the proofs below is to choose a $d$-positive sequence $M = (m_k)$ that satisfies the properties in (3.2) and

$$
\sigma^n(\omega(M)) < d \quad \text{for every } n \geq 0, \quad (3.3)
$$

so that $\omega(M) \in U_{\beta}$ in view of (1.6) and (3.1). Doing so entails slightly different subtleties in the two cases (i) $\limsup_{n \to \infty} \sigma^n(\bar{d}) < d$ and (ii) $\limsup_{n \to \infty} \sigma^n(\bar{d}) = d$. We deal with these two cases in Theorems 3.3 and 3.4, respectively.
Theorem 3.3. Let \( d = d(\beta) \), and suppose

\[
\sup_{n \in \mathbb{N}} \sigma^n(\bar{d}) < d. \tag{3.4}
\]

Then there is a strictly increasing \( d \)-positive sequence \( \mathcal{M}_0 = (m_k) \) such that, whenever \( \mathcal{M} = (\mu_j) \) is a sequence such that \( \mu_j \) is a term of \( \mathcal{M}_0 \) for each \( j \), we have \( \sigma^n(\omega(\mathcal{M})) < d \) for all \( n \geq 0 \).

Proof. Let \( \mathcal{K} \) be the set of those positive integers \( k \) for which the sequence \( \bar{d} \) contains the word \( d_1 \cdots d_k \) infinitely many times. By the hypothesis (3.4), \( \mathcal{K} \) is finite. Let \( k_0 := 0 \) if \( \mathcal{K} = \emptyset \), and \( k_0 := \max \mathcal{K} \) otherwise. Let \( l_0 \) be the largest integer \( l \) for which \( \bar{d} \) contains the word \( d_1 \cdots d_l \) (or \( l_0 := 0 \) if no such \( l \) exists), so \( l_0 \geq k_0 \). If \( l_0 = k_0 \), set \( N := 0 \); otherwise, set

\[
N_k := \max\{n \geq 0 : \overline{d_{n+1} \cdots d_{n+k}} = d_1 \cdots d_k\} + k, \quad k = k_0 + 1, \ldots, l_0,
\]

and

\[
N := \max\{N_k : k = k_0 + 1, \ldots, l_0\}.
\]

Now set \( m_0 := N \), and define recursively, for \( k = 1, 2, \ldots, \)

\[
n_k := \begin{cases} \min\{n > m_{k-1} : \overline{d_{n+1} \cdots d_{n+k_0}} = d_1 \cdots d_{k_0}\} & \text{if } k_0 > 0, \\ m_{k-1} + 1 & \text{if } k_0 = 0, \end{cases}
\]

\[
m_k := n_k + k_0 + 1.
\]

It is clear that the sequence \( \mathcal{M}_0 := (m_k) \) is strictly increasing. Furthermore, we claim that \( \mathcal{M}_0 \) is \( d \)-positive. To see this, fix \( k \in \mathbb{N} \), and note that on the one hand,

\[
\overline{d_{n_k+1} \cdots d_{n_k+k_0}} = d_1 \cdots d_{k_0}, \tag{3.5}
\]

by definition of \( n_k \); while on the other hand,

\[
\overline{d_{n_k+1} \cdots d_{n_k+k_0+1}} < d_1 \cdots d_{k_0+1}. \tag{3.6}
\]

The last inequality is immediate if \( l_0 = k_0 \), and otherwise follows since \( n_k \geq n_1 > N \geq N_{k_0+1} \). Together, (3.5) and (3.6) yield

\[
\bar{d}_{m_k} < d_{k_0+1} \leq \alpha, \tag{3.7}
\]

and as a result, \( d_{m_k} > 0 \).

Now let \( \mathcal{M} = (\mu_j) \) be a sequence such that for each \( j \), there is a \( k \) such that \( \mu_j = m_k \). For any such pair \((j, k)\), we also write \( \nu_j := n_k \). Consider the sequence
\( \omega := \omega(\mathcal{M}) \); our goal is to show that \( \sigma^n(\bar{\omega}) < d \) for all \( n \geq 0 \). Set \( s_0 := 0 \) and \( s_j := \mu_1 + \cdots + \mu_j \), \( j \geq 1 \). Then the sequence \((s_j)\) is strictly increasing, and

\[
\begin{align*}
\omega_{s_j} &= d_{\mu_j}, \quad j \in \mathbb{N}, \\
\omega_{s_j + i} &= d_i, \quad i = 1, 2, \ldots, \mu_{j+1} - 1, \quad j \geq 0.
\end{align*}
\] (3.8)

Fix \( n \geq 0 \), and let \( j \) be the integer such that \( s_j \leq n < s_{j+1} \). We consider two cases.

**Case 1.** Suppose \( s_j \leq n < s_{j+1} - k_0 \), or equivalently, \( s_j \leq n < s_{j+1} - k_0 \). Let \( l := s_{j+1} - n - 1 \). Then \( l > k_0 \). If in fact \( l > k_0 \), then we have immediately that

\[
\omega_{n+1} \cdots \omega_{s_{j+1}-1} = d_{s_{j+1}-s_j-1} < d_1 \cdots d_l,
\] (3.9)

by definition of \( l \). Suppose \( k_0 < l \leq l_0 \). Then

\[
N_l \leq N < n_1 \leq \nu_{j+1} \leq \nu_{j+1} + k_0 = \mu_{j+1} - 1 = s_{j+1} - s_j - 1 = n - s_j + l;
\]

that is, \( N_l < n - s_j + l \). Thus, by definition of \( N_l \), (3.9) holds also in this case. But (3.9) and (3.8) together imply

\[
\omega_{s_{j+1}-1} = d_{s_{j+1}-s_j-1} < d_1 \cdots d_l,
\]

and so \( \sigma^n(\bar{\omega}) < d \).

**Case 2.** Suppose \( s_j + \nu_{j+1} \leq n < s_{j+1} \). Note that we have

\[
\sigma^n(\bar{\omega}) = d_{n-s_{j+1}} \cdots d_{\mu_{j+1}} + \prod_{r=j+2}^{\infty} d_1 \cdots d_{\mu_r} +.\] (3.10)

We first claim that

\[
d_{k_0+2} \cdots d_{k_0+\mu_{j+2}} < d_1 \cdots d_{\mu_{j+2}-1}.
\] (3.11)

Since \( \sigma^{k_0+1}(\bar{d}) < d \), we have “\( \leq \)”. And equality cannot hold by the definition of \( l_0 \), since

\[
\mu_{j+2} - 1 \geq m_1 - 1 = n_1 + k_0 \geq l_0,
\]

where the last inequality is obvious if \( l_0 = k_0 \), and follows otherwise since \( n_1 + k_0 \geq n_1 > N \geq N_{l_0} \geq l_0 \).

Reflecting both sides of (3.11) gives

\[
d_1 \cdots d_{\mu_{j+2}-1} < d_{k_0+2} \cdots d_{k_0+\mu_{j+2}}.
\]
and therefore,
\[ \prod_{r=j+2}^{\infty} d_1 \cdots d_{\mu_r}^+ < d_{k_0+2} d_{k_0+3} \cdots. \tag{3.12} \]

On the other hand, with \( i := n - s_j - \nu_{j+1}, \) we have
\[
\begin{align*}
&d_{n-s_j+1} \cdots d_{\mu_{j+1}}^+ = d_{\nu_{j+1}+i+1} \cdots d_{\nu_{j+1}+k_0} d_{\mu_{j+1}}^+ \\
&= d_{i+1} \cdots d_{k_0} d_{\mu_{j+1}}^+ \\
&\leq d_{i+1} \cdots d_{k_0} d_{k_{0+1}},
\end{align*}
\]

where the second equality follows by definition of \( \nu_{j+1}, \) and the inequality follows by \( (3.7). \) Putting this last development together with \( (3.12) \) and substituting in \( (3.10) \) gives
\[ \sigma^n(\bar{\omega}) < \sigma^i(d) \leq d. \]

In both cases, the conclusion of the theorem follows.

**Theorem 3.4.** Let \( d = d(\beta), \) and suppose that \( \sigma^n(d) < d \) for every \( n \in \mathbb{N}, \) but
\[ \limsup_{n \to \infty} \sigma^n(\bar{d}) = d. \tag{3.13} \]

Then there is a strictly increasing \( d \)-positive sequence \( M_0 = (m_k) \) such that, for any subsequence \( \mathcal{M} = (m_{k(j)}) \) of \( M_0, \) we have \( \sigma^n(\omega(\mathcal{M})) < d \) for all \( n \geq 0. \)

**Proof.** We first define the run lengths
\[ t_n := \max \{ j \in \mathbb{N} : d_{n+1} \cdots d_{n+j} = d_1 \cdots d_j \}, \quad n \in \mathbb{N}, \]

where we set \( \max \emptyset := 0. \) Note that \( (3.13) \) implies \( \limsup_{n \to \infty} t_n = \infty. \) Set \( l_1 := 1, \) and recursively, for \( k = 1, 2, \ldots, \) define:
\[
\begin{align*}
n_k &:= \min \{ n \geq 0 : d_{n+1} \cdots d_{n+l_k} = d_1 \cdots d_l \}, \\
m_k &:= n_k + \min \{ i \in \mathbb{N} : d_{n_k+i} < d_i \}, \\
l_{k+1} &:= \max \{ t_n : n \leq m_k \} + 1.
\end{align*}
\]

Then \( m_k > n_k > m_{k-1}, \) and \( \bar{d}_{m_k} < d_{m_k-n_k}, \) so \( d_{m_k} > 0. \) Hence \( M_0 := (m_k) \) is a strictly increasing \( d \)-positive sequence. For the sake of readability, we will prove the theorem only for the sequence \( \mathcal{M}_0 \) itself. The proof for arbitrary subsequences, while notationally more cumbersome, is essentially the same.
Let \( \omega := \omega(M_0); \) we will show that \( \sigma^n(\bar{\omega}) < d \) for all \( n \geq 0. \) Set \( s_0 := 0, \) and \( s_k := m_1 + \cdots + m_k, \ k \in \mathbb{N}. \) As in the proof of Theorem 3.3, the sequence \((s_k)\) is strictly increasing, and
\[
\omega_{s_k} = d_1^{m_k}, \quad k \in \mathbb{N},
\omega_{s_k+i} = d_i, \quad i = 1, 2, \ldots, m_{k+1} - 1, \ k \geq 0.
\tag{3.14}
\]
Fix \( n \geq 0, \) and let \( k \) be the integer such that \( s_k - 1 \leq n < s_k. \) We consider two cases.

**Case 1.** Suppose \( s_k - 1 \leq n < s_k + n_k. \) We claim that
\[
d_{n-s_k-1+1} \cdots d_{m_k-1} < d_1 \cdots d_{m_k - (n-s_k-1)-1}.
\tag{3.15}
\]
Since \( \sigma^{n-s_k-1}(\bar{d}) < d \) we have “\( \leq \)”, and equality cannot hold by the choice of \( n_k, \) because \( n - s_k - 1 < n_k \) and \( m_k - (n - s_k - 1) - 1 > m_k - n_k - 1 \geq l_k. \)

Using (3.14), (3.15) can be written as
\[
\omega_{s_k+1} \cdots \omega_{s_k-1} > d_1 \cdots d_{m_k - (n-s_k-1)-1},
\]
and so \( \sigma^n(\bar{\omega}) < d. \)

**Case 2.** Suppose \( s_k - 1 + n_k \leq n < s_k. \) Let \( i := n - s_k - 1 - n_k, \ so 0 \leq i < m_k - n_k. \) We claim first that
\[
d_{m_k-n_k-i+1} \cdots d_{m_k-n_k-i+m_{k+1}-1} < d_1 \cdots d_{m_{k+1}-1}.
\tag{3.16}
\]
Since \( \sigma^{m_k-n_k-i}(\bar{d}) < d \) we have “\( \leq \)”, and equality cannot hold by the choice of \( n_{k+1}, \) since \( m_{k+1} - 1 \geq m_{k+1} - n_{k+1} > l_{k+1}, \) and \( m_k - n_k - i < m_k < n_{k+1}. \)

Taking complements in (3.16) gives
\[
d_{1} \cdots d_{m_{k+1}-1} < d_{m_k-n_k-i+1} \cdots d_{m_k-n_k-i+m_{k+1}-1},
\]
and hence
\[
\prod_{r=k+1}^{\infty} d_1 \cdots d_{m_r} < \sigma^{m_k-n_k-i}(d).
\tag{3.17}
\]
On the other hand,
\[
d_{n-s_k-1+1} \cdots d_{m_k} = d_{n_k+i+1} \cdots d_{m_k} + \\
= d_{i+1} \cdots d_{m_k-n_k-1} + d_{m_k} \\
< d_{i+1} \cdots d_{m_k-n_k-1} + d_{m_k-n_k} \\
\leq d_{1} \cdots d_{m_k-n_k-i},
\]
\[16\]
where the second equality follows by the definitions of \( n_k \) and \( m_k \); the first inequality follows since by the choice of \( m_k \), \( \tilde{d}_{m_k} < d_{m_k-n_k} \); and the last inequality follows since \( \sigma^i(d) \leq d \). Combining this last development with (3.17), we obtain (compare with (3.10))

\[
\sigma^n(\bar{\omega}) = d_{n-s_{k-1}} + \cdots + d_{m_k} + \prod_{r=k+1}^{\infty} d_1 \cdots d_{m_r}^+ > d_1 \cdots d_{m_k-n_k-i} \sigma^{m_k-n_k-i}(d) = d.
\]

This completes the proof. \( \square \)

**Proof of Theorem 1.2.** (i) \( \Rightarrow \) (iii): If \( \sigma^{n_0}(\tilde{d}) \geq d \) for some \( n_0 \), then \( \mathcal{W}_\beta = \emptyset \). To see this, let \( \omega \in \mathcal{U}_\beta \). We claim that

\[
\omega_{n+1} \cdots \omega_{n+n_0} < d_1 \cdots d_{n_0} \quad \text{for all } n \geq 0.
\] (3.18)

Let \( n \geq 0 \). Since \( \omega \in \mathcal{U}_\beta \) we have \( \sigma^{n+n_0}(\omega) < d \), and so \( \sigma^{n_0}(\tilde{d}) \geq d > \sigma^{n+n_0}(\omega) \). Taking complements this gives \( \sigma^{n_0}(d) < \sigma^{n+n_0}(\omega) \). Writing

\[
\sigma^n(\omega) = \omega_{n+1} \cdots \omega_{n+n_0} \sigma^{n+n_0}(\omega), \quad d = d_1 \cdots d_{n_0} \sigma^{n_0}(d),
\]

and using that \( \sigma^n(\omega) < d \) since \( \omega \in \mathcal{U}_\beta \), (3.18) follows. But (3.18) in turn implies that \( \limsup \sigma^n(\omega) < d \). Interchanging the roles of \( \omega \) and \( \bar{\omega} \) (and using that \( \omega \in \mathcal{U}_\beta \) if and only if \( \bar{\omega} \in \mathcal{U}_\beta \)), the same argument gives \( \limsup \sigma^n(\bar{\omega}) < d \). Thus, by Proposition 2.3 and (1.5), \( \omega \in \bar{\mathcal{U}}_\beta \).

(iii) \( \Rightarrow \) (ii): Since any strictly increasing sequence \( (m_k) \) has uncountably many different subsequences, this implication follows from the injectivity of the map \( \mathcal{M} \mapsto \omega(\mathcal{M}) \), together with Theorem 3.3 in case \( \limsup \sigma^n(\tilde{d}) < d \), or Theorem 3.4 in case \( \limsup \sigma^n(\tilde{d}) = d \).

Finally, (ii) \( \Rightarrow \) (i) is obvious, and the equivalence (iii) \( \iff \) (iv) was proved in [25], as pointed out earlier. \( \square \)

**Proof of Theorem 1.4.** (i) Assume that \( \mathcal{W}_\beta \neq \emptyset \), and let a sequence \( (\theta_n) \) of positive numbers be given. Let \( \mathcal{M}_0 = (m_k) \) be the sequence from Theorem 3.3 or Theorem 3.4 as appropriate, and extract from \( \mathcal{M}_0 \) a subsequence \( \mathcal{M} := (m_{k(j)}) \) which grows sufficiently fast so that

\[
\beta^{-m_{k(j)+1}} \leq \theta_{s_j}^{-1} \quad \text{for every } j \in \mathbb{N},
\] (3.19)

where \( s_j := m_{k(1)} + \cdots + m_{k(j)} \). This can always be done, since the sequence \( (m_k) \) increases to \( +\infty \). Let \( \omega := \omega(\mathcal{M}) \). Theorem 3.3 or 3.4 guarantees \( \omega \in \mathcal{U}_\beta \). By Lemma 2.2 there is a constant \( K > 0 \) such that

\[
|1 - \Pi_\beta(\sigma^n(\omega))| = |\Pi_\beta(d) - \Pi_\beta(\sigma^n(\omega))| \leq K \rho_\beta(d, \sigma^n(\omega)), \quad n \in \mathbb{N}.
\]
Taking \( n = s_j \) and using (3.19) and the definition of the metric \( \rho_\beta \), we obtain

\[
1 - \Pi_\beta(\sigma^{s_j}(\omega)) \leq K \beta^{-m_k(j+1)} \leq K \theta_1^{-1}, \quad j \in \mathbb{N},
\]

and hence,

\[
\liminf_{n \to \infty} \theta_n (1 - \Pi_\beta(\sigma^n(\omega))) \leq K < \infty. \tag{3.20}
\]

Finally, since there are clearly uncountably many subsequences \((m_k(j))\) satisfying (3.19) and the map \( M \mapsto \omega(M) \) is injective, there are uncountably many sequences \( \omega \in \mathcal{W}_\beta \) satisfying (3.20).

(ii) Since (1.9) is satisfied for all \( \omega \in \tilde{U}_\beta \) and \( U_\beta \) is uncountable for every \( \beta > \beta_c \), the second statement follows immediately from Lemma 2.1 for \( \beta > \beta_c \). Assume \( \beta = \beta_c \), and recall that \( d := d(\beta) \) is a \( p \)-mirror sequence with \( p = 1 \). Let \( b_i := d_1 \cdots d_{2^{i-1}} \), \( i \in \mathbb{N} \). Choose a sequence \( (k_i)_{i \in \mathbb{N}} \) so that

\[
\theta_{k_i} \geq \beta^{2^{i+1}}, \quad i \in \mathbb{N}, \tag{3.21}
\]

and let

\[
\omega = (b_1 \overline{b_1})^{k_1} (b_2 \overline{b_2})^{k_2} (b_3 \overline{b_3})^{k_3} \cdots. \tag{3.22}
\]

By the second part of Lemma 2.8, \( \omega \in \mathcal{U}_\beta \). For any finite word \( u \), let \( |u| \) denote the length (number of symbols) of \( u \). Then \( |b_i| = 2^{i-1} \) and so \( |(b_i \overline{b_i})^{k_i}| = k_i 2^i \). Given \( n \in \mathbb{N} \), let \( j \) be the integer such that \( \sum_{i=1}^j k_i 2^i \leq n < \sum_{i=1}^{j+1} k_i 2^i \). Then \( \sigma^n(\omega) \) coincides with \( d \) for at most the first \( |b_{j+1}| \) digits, and the same is true for \( \sigma^n(\overline{\omega}) \). Thus, \( \rho_\beta(\sigma^n(\omega), d) \geq \beta^{-2^j} \), and it follows from Lemma 2.2 that there is a constant \( C > 0 \) for which

\[
1 - \Pi_\beta(\sigma^n(\omega)) = |\Pi_\beta(d) - \Pi_\beta(\sigma^n(\omega))| \geq C \beta^{-2^j}, \quad j \in \mathbb{N},
\]

and the same with \( \overline{\omega} \) in place of \( \omega \). On the other hand, \( n > k_j \), so \( \theta_n \geq \theta_{k_j} \) and (3.21) implies

\[
\theta_n (1 - \Pi_\beta(\sigma^n(\omega))) \geq C \beta^{2^j} \to \infty,
\]

and the same for \( \overline{\omega} \). Finally, since for each \( i \) we could replace \( k_i \) with \( k_i + 1 \) and \( b_i \overline{b_i} = b_{i+1} \), there are uncountably many \( \omega \) of the form (3.22) with this property. \( \square \)

4 Proofs of Theorems 1.5 and 2.9

Instead of proving Theorem 1.5 directly, it is notationally simpler to prove a more general and slightly stronger result. To this end, we slightly extend the notation from earlier sections. Let \( A \) be any finite alphabet, write \( A^* := \bigcup_{n=1}^\infty A^n \), and let \( \Omega := A^\mathbb{N} \). Let \( 0 < \lambda < 1 \) be a constant, and equip \( \Omega \) with the metric \( \rho(\omega, \eta) = \lambda^{\min\{n : \omega_n = \eta_n\}} - 1 \),
for $\omega = \omega_1\omega_2\cdots$ and $\eta = \eta_1\eta_2\cdots$. For any subset $V$ of $A^*$, let $V^\mathbb{N}$ denote the set of all infinite concatenations $v_1v_2v_3\cdots$, where $v_i \in V$ for each $i$. For $v \in A^*$, let $|v|$ denote the length of $v$; that is, $|v| = n$ when $v \in A^n$. If $u, v \in A^*$ and $|u| < |v|$, we say $v$ extends $u$ if $v_1\cdots v_{|u|} = u$. The following fact is well known; see, for instance, [24, Proposition 3].

**Theorem 4.1.** Let $V \subset A^*$ such that no word in $V$ extends any other word in $V$. Then $\dim_H V^\mathbb{N}$ is the unique real number $s$ such that

$$\sum_{v \in V} \lambda^{s|v|} = 1.$$  

Now let $V = \{v_1, v_2, \ldots\}$ be an infinite subset of $A^*$ such that no word in $V$ extends any other. Let

$$E := V^\mathbb{N} \setminus \bigcup_{i=1}^{\infty} (V \setminus \{v_i\})^\mathbb{N}.$$  

In other words, $E$ is the set of all infinite concatenations $v_1v_2\cdots$ of words from $V$ with the property that each $i \in \mathbb{N}$ occurs at least once in the sequence $(i_1, i_2, \ldots)$. The following theorem may well be known, but since the author could not find it in the literature, a proof is included for completeness.

**Theorem 4.2.** It holds that $\dim_H E = \dim_H V^\mathbb{N}$.

**Proof.** Assume the elements of $V$ are ordered such that $|v_1| \leq |v_2| \leq |v_3| \leq \ldots$. Since Theorem 4.1 easily implies that $\dim_H V^\mathbb{N} = \sup \{\dim_H W^\mathbb{N} : W \subset V, \#W < \infty\}$, it suffices to show that $\dim_H E \geq \dim_H W^\mathbb{N}$ for any finite subset $W$ of $V$. Fix such a set $W$, and let $s := \dim_H W^\mathbb{N}$, so that

$$\sum_{v \in W} \lambda^{s|v|} = 1.$$  

(4.1)

Fix $0 < t < s$, and let $K = (k_l)_{l \in \mathbb{N}}$ be a strictly increasing sequence of positive integers such that

$$k_l \geq l \left(1 + \frac{t|v_l|}{(s-t)|v_1|}\right), \quad l \in \mathbb{N}.$$  

(4.2)

Slightly abusing notation, define, for $k \in \mathbb{N}$, the index set

$$I_k := \begin{cases} 
\{i \in \mathbb{N} : v_i \in W\}, & \text{if } k \not\in K, \\
\{l\}, & \text{if } k = k_l \in K.
\end{cases}$$

Define the set

$$E_{W,K} := \{v_{i_1}v_{i_2}v_{i_3}\cdots : i_k \in I_k \text{ for all } k \in \mathbb{N}\}.$$
Clearly $E \supset E_{W,K}$. We will show that $\dim_H E_{W,K} \geq t$.

For a word $v \in A^*$, define the cylinder set $[v]$ by $[v] := v \cdot A^\infty$, the set of all sequences in $\Omega$ that begin with $v$. Write

$$C_{i_1 \cdots i_n} := [v_{i_1} \cdots v_{i_n}], \quad i_1, \ldots, i_n \in \mathbb{N}.$$ 

Observe that the above cylinder set has diameter

$$|C_{i_1 \cdots i_n}| = \lambda |v_{i_1}| + \cdots + |v_{i_n}|.$$ 

Put

$$I_{W,K}^n := \bigotimes_{k=1}^n I_k \quad (n \in \mathbb{N}), \quad I_{W,K}^\ast := \bigcup_{n=1}^\infty I_{W,K}^n.$$ 

Further, denote

$$S'_n := \sum_{t \in \mathbb{N} : k_t \leq n} |v_t|, \quad N_n := \# \{1 \leq k \leq n : k \notin K \}, \quad \text{for } n \in \mathbb{N}.$$ 

We will define a mass distribution $\mu$ on $E_{W,K}$ as follows. For $(i_1, \ldots, i_n) \in \mathbb{N}^n$, set

$$\mu(C_{i_1 \cdots i_n}) := \begin{cases} |C_{i_1 \cdots i_n}| s \lambda^{-s} S_n' & \text{if } (i_1, \ldots, i_n) \in I_{W,K}^\ast, \\ 0 & \text{otherwise}. \end{cases}$$ 

If $[\omega_1 \cdots \omega_m]$ is an arbitrary cylinder set in $\Omega$ which is not of the form $C_{i_1 \cdots i_n}$, let $n$ be the largest integer such that $[\omega_1 \cdots \omega_m] \subset C_{i_1 \cdots i_n}$ for some $i_1, \ldots, i_n$, and set

$$\mu([\omega_1 \cdots \omega_m]) := \sum \{ \mu(C_{i_1 \cdots i_n}) : C_{i_1 \cdots i_n} \subset [\omega_1 \cdots \omega_m] \}, \quad (4.3)$$ 

where the summation is over all $j \in \mathbb{N}$ satisfying the given condition. Or, if $\omega_1 \cdots \omega_m$ cannot be extended to a sequence in $E_{W,K}$, set $\mu([\omega_1 \cdots \omega_m]) = 0$. We first check the consistency condition

$$\sum_{j=1}^\infty \mu(C_{i_1 \cdots i_n}) = \mu(C_{i_1 \cdots i_n}), \quad n \in \mathbb{N}, \quad i_1, \ldots, i_n \in \mathbb{N}. \quad (4.4)$$ 

Assume $(i_1, \ldots, i_n) \notin I_{W,K}^\ast$, as otherwise (4.4) is trivial. If $n + 1 = k_t \in K$, then $S'_{n+1} = S'_n + |v_t|$, and so

$$\mu(C_{i_1 \cdots i_n}) = |C_{i_1 \cdots i_n}| s \lambda^{-s} S'_n + (|C_{i_1 \cdots i_n}| s \lambda^{-s} S'_n - |v_t|)$$ 

$$= |C_{i_1 \cdots i_n}| s \lambda^{-s} S'_n = \mu(C_{i_1 \cdots i_n}).$$
while for $j \neq l$, $\mu(C_{i_1 \ldots i_n}) = 0$. Thus, (4.4) holds when $n + 1 \in K$. If $n + 1 \not\in K$, then $S_{n+1} = S_n$, so for $j$ such that $v_j \in W$, we have
\[ \mu(C_{i_1 \ldots i_{n+1}}) = |C_{i_1 \ldots i_n}| \lambda^{-s} S_{n+1} = |C_{i_1 \ldots i_n}| \lambda^{s} \lambda^{-s} S_n = \mu(C_{i_1 \ldots i_n}) \lambda^{s} |v_j|, \]
and in this case, (4.4) follows from (4.1).

Now (4.4), together with (4.3) and Kolmogorov’s consistency theorem, implies that $\mu$ can be extended to a unique Borel probability measure on $\Omega$, and $\mu(E_{W,K}) = 1$. We claim that
\[ \mu(C_{i_1 \ldots i_n}) \leq |C_{i_1 \ldots i_n}|^l, \tag{4.5} \]
for all $(i_1, \ldots, i_n) \in I_{W,K}$. Fix such an $n$-tuple $(i_1, \ldots, i_n)$, and set $S_n := |v_1| + \cdots + |v_n|$. Observe that
\[ S_n \geq S_n' + N_n |v_1|. \tag{4.6} \]
Let $l \in \mathbb{N}$ be such that $k_l \leq n < k_{l+1}$. Then (4.2) implies
\[ (s-t)(n-l)|v_1| \geq (s-t)(k_l - l)|v_1| \geq tl|v_1|. \]
Since $S_n' = S_l' \leq l|v_l|$ and $N_n = n - l$, this gives
\[ (s-t)N_n|v_1| = (s-t)(n-l)|v_1| \geq tl|v_1| \geq tS_n', \]
which by (4.6) implies $(s-t)S_n \geq sS_n'$. Hence,
\[ |C_{i_1 \ldots i_n}|^{s-t} = \lambda^{(s-t)S_n} \leq \lambda^{sS_n'}, \]
which yields (4.5).

Now let $[\omega_1 \cdots \omega_m]$ be an arbitrary cylinder set in $\Omega$. We may assume this set intersects $E_{W,K}$, for otherwise its $\mu$-measure is zero. Let $n$ be the largest integer such that $[\omega_1 \cdots \omega_m] \subset C_{i_1 \ldots i_n}$ for some $(i_1, \ldots, i_n) \in I_{W,K}$. If $n + 1 \not\in K$, then
\[ \mu([\omega_1 \cdots \omega_m]) \mu(C_{i_1 \ldots i_n}) \leq |C_{i_1 \ldots i_n}|^{l} \leq \lambda^{-Ll} |[\omega_1 \cdots \omega_m]|^{l}, \]
where $L := \max\{|v| : v \in W\}$. On the other hand, if $n + 1 = k_l \in K$, then
\[ \mu([\omega_1 \cdots \omega_m]) = \mu(C_{i_1 \ldots i_{n+1}}) \leq |C_{i_1 \ldots i_{n+1}}|^{l} \leq |[\omega_1 \cdots \omega_m]|^{l}. \]
In both cases,
\[ \mu([\omega_1 \cdots \omega_m]) \leq \lambda^{-Ll} |[\omega_1 \cdots \omega_m]|^{l} = \lambda^{-Ll} \lambda^{mt}. \]
Finally, let $U$ be any subset of $\Omega$. Let $m$ be the integer such that $\lambda^m < |U| \leq \lambda^{m-1}$. Then $U$ intersects at most $\#A$ cylinders $[\omega_1 \cdots \omega_m]$, and so
\[ \mu(U) \leq \#A \lambda^{-Ll} \lambda^{mt} < \#A \lambda^{-Ll} |U|^{l}. \]
Hence, by the distribution of mass principle (see [2]), $\dim_H E_{W,K} \geq t$. But then also $\dim_H E \geq t$, and letting now $t \uparrow s$ yields $\dim_H E \geq s = \dim_H W^N$, as desired. \qed
It is worth noting that the analog of Theorem 4.2 for finite $V$ is nearly trivial. To see why, let $s = \dim_H V$, and let $\mathcal{H}^s$ denote $s$-dimensional Hausdorff measure on $\Omega$. To avoid the trivial case where $\dim_H E = \dim_H V = 0$, assume $#V \geq 2$. Then $0 < \mathcal{H}^s(V) < \infty$, since $V$ is the attractor of a finite iterated function system. Write $V = \{v_1, \ldots, v_n\}$, and let $F_i := (V \setminus \{v_i\})^\delta$, $i = 1, \ldots, n$. Then $\dim_H F_i < s$ and so $\mathcal{H}^s(F_i) = 0$, for each $i$, and hence $\mathcal{H}^s(V \setminus E) = \mathcal{H}^s\left(\bigcup_{i=1}^n F_i\right) = 0$. But this implies $\mathcal{H}^s(E) > 0$, so that $\dim_H E \geq s$. I thank Kenneth Falconer for pointing this out. The above method does not work for infinite $V$, since in general it is possible that $\mathcal{H}^s(V) = 0$. Various sufficient conditions are known under which the attractor of an infinite iterated function system has positive Hausdorff measure in its dimension; see, for instance, Mauldin and Urbanski [19] or Staiger [24]. But our theorem above holds equally whether or not this is the case.

Proof of Theorem 1.5. Statement (i) is immediate from Lemma 2.1 and the fact, proved recently by Komornik et al. [13], that the function $\beta \mapsto \dim_H U_\beta$ is continuous. For (ii), let $M_0 = (m_k)$ be the sequence used in the proof of Theorem 1.2, and apply Theorem 4.2 to the set $V = \{v_1, v_2, \ldots\}$, where $v_k = d_1 \cdot \cdots \cdot d_{m_k}$ for $k \in \mathbb{N}$, and $\lambda := 1/\beta$. Statement (iii) follows from Theorem 2.9, which is proved below. □

In preparation for the proof of Theorem 2.9 we need the following lemma.

Lemma 4.3. Let $\{a_{i_1, \ldots, i_n} : n \in \mathbb{N}, \ i_1, i_2, \ldots \in \{1, 2\}\}$ be a binary tree of positive integers defined by $a_1 = 1$, $a_2 = 2$, and for $n \in \mathbb{N}$,

$$a_{i_1, \ldots, i_n, j} = ja_{i_1, \ldots, i_n}, \quad n \in \mathbb{N}, \ i_1, \ldots, i_n, j \in \{1, 2\}. $$

Let $\sigma_{i_1, \ldots, i_n} := 1 + a_{i_1} + a_{i_1, i_2} + \cdots + a_{i_1, i_2, \ldots, i_n}$. Let $0 < \gamma < 1$ be a constant, and define the sum

$$S_n := \sum_{i_1, \ldots, i_n \in \{1, 2\}} \gamma^{\sigma_{i_1, \ldots, i_n}}. $$

Then $S_n \to 0$ as $n \to \infty$.

Proof. Define the sums

$$S_n^L := \sum_{i_2, \ldots, i_n \in \{1, 2\}} \gamma^{\sigma_{1, i_2, \ldots, i_n}}, \quad S_n^R := \sum_{i_2, \ldots, i_n \in \{1, 2\}} \gamma^{\sigma_{2, i_2, \ldots, i_n}}, $$

so that $S_n = S_n^L + S_n^R$. Since $a_{1, i_2, \ldots, i_n} = a_{2, \ldots, i_n}$, we have $\sigma_{1, i_2, \ldots, i_n} = 1 + \sigma_{2, \ldots, i_n}$ and this shows that

$$S_{n+1}^L = \gamma S_n. \quad (4.7)$$
Furthermore, $a_{i_1,i_2,...,i_n} = 2a_{1,i_2,...,i_n}$, so $\sigma_{i_1,i_2,...,i_n} = 2\sigma_{1,i_2,...,i_n} - 1$. Clearly $\sigma_{1,i_2,...,i_n} \geq n + 1$, and so $\sigma_{i_1,i_2,...,i_n} \geq \sigma_{1,i_2,...,i_n} + n$. Hence,

$$S^R_n \leq \gamma^n S^L_n.$$  \hfill (4.8)

From (4.7) and (4.8) (with $n + 1$ in place of $n$ in the latter), it follows that

$$S_{n+1} = S_{n+1}^L + S_{n+1}^R \leq (1 + \gamma^{n+1})S_{n+1}^L = \gamma(1 + \gamma^{n+1})S_n = (\gamma + \gamma^{n+2})S_n.$$

Choose $n_0$ large enough so that $\delta := \gamma + \gamma^{n_0+2} < 1$. Then for each $n \geq n_0$, $S_{n+1} \leq \delta S_n$, from which the result follows.

**Proof of Theorem 2.3.** Let $\beta$ be a de Vries-Komornik number, and $d := d(\beta)$. If $\omega \in \mathcal{W}_\beta$, then $\limsup_{n \to \infty} \sigma_n(\omega) = d$, so certainly there exists $k \in \mathbb{N}$ such that $\omega_k \cdots \omega_{k+p} = d_1 \cdots d_p$. Thus, $\mathcal{W}_\beta$ is covered by countably many similar copies of the set

$$\mathcal{Y}_\beta := \{\omega : \omega_1 \omega_2 \cdots \in U_\beta : \omega_1 \cdots \omega_p = d_1 \cdots d_p\},$$

and it suffices to show that $\dim_H \mathcal{Y}_\beta = 0$.

Let $b_m := d_1 \cdots d_{2m-1}$, $m \in \mathbb{N}$. Define numbers $\sigma_{i_1,...,i_n}$ for $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, 2\}$ as in Lemma 4.3. By Lemma 2.8, $\mathcal{Y}_\beta$ is covered at “level 1” by the two cylinders $C_1 := [b_1 \overline{b_1}]$ and $C_2 := [b_2]$. Applying Lemma 2.8 again, we see that at “level 2”, $\mathcal{Y}_\beta$ is covered by the four cylinders $C_{11} := [b_1b_1b_1]$, $C_{12} := [b_1b_2]$, $C_{21} := [b_2b_1]$ and $C_{22} := [b_1]$. Continuing this way, we find that at “level $n$”, $\mathcal{Y}_\beta$ is covered by $2^n$ cylinders $C_{i_1,...,i_n}$, $i_1, \ldots, i_n \in \{1, 2\}$, where $C_{i_1,...,i_n}$ has depth $p(1 + \sigma_{i_1,...,i_n-1})$, so that

$$|C_{i_1,...,i_n}| = \beta^{-p(1+\sigma_{i_1,...,i_n-1})}.$$ 

Now let $s > 0$ be given, and apply Lemma 4.3 with $\gamma := \beta^{-p} s \to \infty$. Thus, $\dim_H \mathcal{Y}_\beta = 0$, as desired.

**Remark 4.3.** (a) A slight extension of the above argument shows also that $\dim_H \mathcal{U}_{\beta_c} = 0$, since $d := d(\beta_c)$ is a $p$-mirror sequence. This well-known result was stated for the case $\alpha = 1$ in [10] without a detailed proof, and later for the general case in [18], where Lemma 2.8 is used but the further details are omitted.

(b) It would be of interest to determine the correct gauge function $h$ for which $0 < \mathcal{H}^h(\mathcal{U}_{\beta_c}) < \infty$, where $\mathcal{H}^h$ is the generalized Hausdorff measure induced by $h$. It is clear that an answer to this question would require a substantial refinement of the analysis carried out in the proof of Lemma 4.3.
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