A definability criterion for linear Lie groups

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Abstract

It is known since [7] that any group definable in an o-minimal expansion of the real field can be equipped with a Lie group structure. It is then natural to ask when does a Lie group is Lie isomorphic to a group definable in such expansion. Conversano, Starchenko and the first author answered this question in [2] in the case where the group is solvable. We give here a criterion in the case where the group is linear. More precisely if $G$ is a linear Lie group it is isomorphic to a group definable in an o-minimal expansion of the reals if and only if its solvable radical is isomorphic to such group.

Section 1: Preliminaries

We start here with some definitions. We shall use the usual notation for classical o-minimal structures: $\mathbb{R}_{\exp}$ for the structure $(\mathbb{R}, 0, +, 1, \cdot, <, \exp)$ and $\mathbb{R}_{\text{an,exp}}$ for the structure $(\mathbb{R}, 0, +, 1, \cdot, <, \exp, f_{f \in F})$ where $F$ is the set of analytic functions $f : \mathbb{R} \to \mathbb{R}$ with compact support.

By definable group we will mean a group definable in an o-minimal extension of the reals. We will say that a group is definably linear if it is a definable group that it acts definably, faithfully and linearly on $\mathbb{R}^n$, for some integer $n$. The main goal of this paper is to prove:

**Theorem 1**

Let $G$ be a connected linear Lie group whose solvable radical has a normal, connected, torsion-free and super solvable subgroup $T$ such that $R/T$ is compact. Then $G$ is Lie-isomorphic to a definably linear group.

It is worth mentioning that we cannot avoid the “Lie-isomorphism” in this theorem as there are some presentations of groups that are not definable in certain expansions of the reals. For example

$$G = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix} : t \in \mathbb{R} \right\}$$

is not definable in $\mathbb{R}_{\text{an,exp}}$ but it is isomorphic to $(\mathbb{R}, +)$.

This theorem is actually the linear version (without assumption on the solvability of the group) of:
Fact 1 (Solvable case [2])

Let $R$ be a solvable Lie group. Then the following are equivalent:

- $R$ has a normal, connected, torsion-free and supersolvable subgroup $T$ such that $R/T$ is compact.
- $R$ is Lie isomorphic to a group definable in an $o$-minimal expansion of the reals.

We recall that a connected Lie group is said supersolvable (sometimes called triangular) if the eigenvalues of the operator $\text{Ad}(g)$ of the adjoint representation are real for all $g \in G$. This is equivalent to say that $g = \text{Lie}(G)$ is supersolvable as described in [2, Section 2].

In the next section we will show that a group satisfying the conditions above is actually Lie isomorphic to a definably linear group.

Section 2: Definable linearity of definable solvable Lie groups

A theorem of Malcev says the following:

Fact 2 ([3])

Let $R$ be a solvable Lie group. Then $R$ admits a faithful finite-dimensional representation if and only if $R$ can be decomposed as a semidirect product $T \rtimes K$ where $T$ is simply connected and $K$ is a torus (maximal compact group).

But the conditions for definability of a solvable Lie group given in Fact 1 imply that the group is a semidirect product and hence linear. In this section we show that it is actually definably linear. More precisely we prove:

Theorem 2 (Definable representation of solvable Lie groups)

Let $G$ be a solvable Lie group that contains a normal connected torsion-free completely solvable subgroup $H$ such that $G/H$ is compact. Then $G$ is Lie-isomorphic to a group definably linear in $\mathbb{R}_{an, \exp}$.

The general approach is to follow the last chapter of [3] and adapt it to a definable context.

Recall that if $\rho : G \to GL(V)$ is a representation of $G$ over a finite-dimensional vector space $V$ and $(0) = V_0 \leq \cdots \leq V_k = V$ is a composition series for the $G$-module $V$ then the direct sum of the $V_i/V_{i+1}$ is a semi-simple $G$-module $V'$ (two such $G$-modules differ by a $G$-isomorphism by Jordan-Hölder’s theorem). We denote by $\rho'$ the representation associated to $\rho$ over $V'$ and we say that $\rho$ is unipotent if $\rho'$ is trivial (this is equivalent to nilpotency of $N = \{\rho(g) - \text{Id}_V : g \in G\}$). Now if we fix a representation $\rho$ of $G$ we call representative function associated to $\rho$ the space $S(\rho) = \{\varphi \circ \rho : \varphi \in \text{End}(V^*)\}$.

The following lemma will ensure that the representations we build in the proof of Theorem 2 is over a finite dimensional vector space.

Lemma 1 ([3, Chap. XVIII, Lemma 2.1])

Let $G$ be a solvable Lie group and $\rho : G \to GL(V)$ be a faithful continuous representation in the finite-dimensional $\mathbb{R}$-vector space $V$. Let $A$ be a set of automorphisms of $G$ such that $\rho'(\alpha(x)x^{-1}) = \text{Id}$ for all $x \in G$ and $\alpha \in A$. Then
if \( f \in S(\rho) \) is a representative function associated to \( \rho \) then the vector space generated by \( \{ f \circ \alpha \}_{\alpha \in A} \) is finite-dimensional.

This lemma is the main brick in the proof of the following theorem (this is what makes the representation finite-dimensional).

**Lemma 2 (Extension Lemma)**

Let \( G = K \ltimes H \) be a group definable in an o-minimal expansion of the reals with \( H \) and \( K \) definable subgroups, \( H \) normal and solvable. Suppose that \( H \) admits a faithful definable representation \( \rho : H \to \text{GL}(V) \) where \( V \) is a finite-dimensional vector space over \( \mathbb{R} \). Suppose moreover that \( \rho \) satisfies that for all \( x \in G \) and \( y \in H \) we have \( \rho(xyx^{-1}y^{-1}) = \text{id} \). Then there is a definable representation \( \sigma \) on \( G \) that is faithful on \( H \) and extends \( \rho \).

**Proof:**

Let us consider the action of \( H \) on the space \( C^0(H) \) of continuous functions from \( H \) to \( \mathbb{R} \) as follow: \( \forall h \in H, f \in C^0(H) \) we define \( h \cdot f : x \mapsto f(xh) \). This action is faithful.

We can extend the action to \( G \):

If \( h \in H, k \in K \) with \( g = kh \) and \( f \in C^0(H) \) define \( g \cdot f : x \mapsto f(k^{-1}xh) \).

Remark that we had to consider the right action of \( K \) on \( H \) to get a left action of \( G \) on \( C^0(H) \). Extending the action to \( G \) we preserve faithfulness on \( H \).

We now consider the subspace of representative functions of \( C^0(H) \):

\[
\mathcal{R}(H) = \{ f \in C^0(H) : \dim(\text{span}\{h \cdot f \mid h \in H\}) < \omega \}
\]

More concisely, the elements of \( \mathcal{R}(H) \) are functions \( f \) such that \( H \cdot f \) lies in a finite-dimensional subspace of \( C^0(H) \). We are going to show that not only \( H \) but all of \( G \) acts on \( \mathcal{R}(H) \). Indeed \( g \in H, kh \in G \) with \( h \in H, k \in K \) and \( f \in \mathcal{R}(H) \) then:

\[
g \cdot ((kh) \cdot f) = (gh) \cdot f = (kk^{-1}gh) \cdot f = k \cdot (k^{-1}gh) \cdot f \in k \cdot (H \cdot f)
\]

As \((H \cdot f)\) is finite-dimensional \( hh \cdot f \in \mathcal{R}(H) \).

Let \( c_k : x \mapsto k^{-1}xk \) be the conjugation in \( H \) by \( k \in K \). We notice that for all \( f \in C^0(H) \) we have \( k \cdot f = f \circ c_k \). So let us fix \( f \in S(\rho) \). The hypothesis of the theorem allows us to apply [Lemma 1](#) to \( f \): the vector space generated by the \( k \cdot f = f \circ c_k \) for \( k \in K \) is finite-dimensional. But we also know that \( S(\rho) \) has finite dimension so the vector subspace \( U \subseteq \mathcal{R}(H) \) generated by \( G \cdot S(\rho) \) has finite dimension. \( G \) acts on this space \( U \) and as noticed earlier the restriction of the action of \( H \) is faithful. If \( (f_1, f_2, \ldots, f_k) \) is a basis for \( U \) we know that each \( f_i = k_i \cdot (\rho \circ \varphi_i) \) for some \( k_i \in K \) and \( \varphi_i \in \text{End}(V)^* \). Since all of those functions are definable and \( \rho \) is also definable we get a definable finite dimensional representation of \( G \).

We want to apply this Extension Lemma to the decomposition of a solvable Lie group into its supersolvable and compact parts. To do so we need to find a representation of the supersolvable part that is unipotent in some sense:
Proposition 1
Let $G$ be a simply connected, connected, supersolvable Lie group and $N$ its nilradical (i.e. its maximal normal nilpotent subgroup). Then $G$ is Lie isomorphic to a definably linear group $G_1$ whose nilradical $N_1$ is unipotent (upper-triangular with 1’s on the diagonal).

Proof:
We first notice that since $G$ is simply connected and solvable the exponential map gives us a diffeomorphism between $G$ and its Lie algebra $\mathfrak{g}$. If we can get a representation of $\mathfrak{g}$ that is strictly upper triangular on the nilradical $\mathfrak{n}$ then the matrix image of the exponential will be a linear Lie group $G_1$ Lie-isomorphic to $G$ whose nilradical $N_1 = \exp(\mathfrak{n})$ is unipotent. But since $G$ is supersolvable, so is $\mathfrak{g}$. Supersolvable Lie algebras have upper triangular representations (see [2, Lemma 3.1]) and $\mathfrak{n}$ is an upper triangular nilpotent subalgebra, it must be strictly upper triangular.

We recall the following fact:

Fact 3 ([5])
Let $G$ be a connected solvable Lie group of dimension $n$. The following are equivalent:

- $G$ is torsion-free,
- $G$ is simply connected,
- $G$ is diffeomorphic to $\mathbb{R}^n$.

We are now ready to apply all these results and prove:

Theorem 2 (Definable representation of solvable Lie groups)
Let $G$ be a solvable Lie group that contains a normal connected torsion-free completely solvable subgroup $H$ such that $G/H$ is compact. Then $G$ is Lie-isomorphic to a group definably linear in $\mathbb{R}_{an,\exp}$.

Proof:
Basically we just add linearity to [2, Theorem 5.4]. So let us start with $G_1$ the definable Lie group isomorphic to $G$ given by this theorem. We know that we can decompose $G_1$ as a definable semidirect product of a supersolvable subgroup $H_1$ and a compact group $K_1$. We want to apply the extension theorem to $G_1$ so we consider the definable representation of $H_1$ given by the previous theorem (it is simply connected because solvable and torsion-free as in Fact 3). To apply the Extension Lemma we have to check that this representation satisfies the commutator condition, this can be checked on it’s Lie algebra. Since $[\mathfrak{g}_1, \mathfrak{h}_1]$ is nilpotent (the commutator algebra of a solvable Lie algebra is nilpotent) it is included in $\mathfrak{n} = Lie(N)$ (where $N$ is the nilradical of $H_1$). But the representation we chose was unipotent on $N$ so it does satisfy the condition. Using the Extension Lemma we get a definable representation of $G_1$ which is faithful on $H_1$. Consider its direct sum with any faithful continuous representation of $K_1$ (we can find such representation with the Peter-Weyl theorem for example) such representation is algebraic as we will see in Appendix A and hence definable. The resulting representation is definable and faithful which concludes the proof.

\[ \square \]
Section 3: From linearity to definability

The main idea to generalize the solvable case to the general case is to use the so called Levi decomposition of a linear Lie group $G$:

**Theorem 3 ([4])**

Let $G$ be a connected linear Lie group and $R$ its solvable radical (i.e. its maximal solvable normal and connected Lie subgroup). There is a unique (up to conjugacy) maximal connected and semisimple Lie subgroup $S$ of $G$. Moreover $G = R(\ltimes)S$ i.e. $G$ is the almost semi-direct product of those subgroups.

At this point we should mention that all the results on the solvable case will articulate well with the rest of the group since the definable solvable radical and solvable radical coincide for definable groups:

**Fact 4 ([1])**

Let $G$ be a definable group and $R(G)$ the group generated by all the normal and solvable subgroups of $G$ (not only the definable ones). Then $R(G)$ is a normal, definable and solvable subgroup of $G$.

The last part we should worry about is the semisimple part but since the group is supposed to be linear we are actually working with a semialgebraic group:

**Fact 5 ([6])**

If $S$ is a connected semisimple linear Lie group, then $S$ is semialgebraic (i.e. definable in $(\mathbb{R}, 0, +, \cdot, <)$).

We finally give the proof of Theorem 1:

**Theorem 1**

Let $G$ be a connected linear Lie group whose solvable radical has a normal, connected, torsion-free and super solvable subgroup $T$ such that $R/T$ is compact. Then $G$ is Lie-isomorphic to a definably linear group.

Proof:

The solvable radical $R$ is, by Theorem 2, Lie-isomorphic to a definably linear group $R_1$. We would like to invoke the Extension Lemma but the decomposition of $G$ into its solvable radical $R$ and a Levi subgroup $S$ is actually not fine enough. We will need to refine the decomposition. Consider the decomposition of $R$ into a simply connected (torsion free) normal subgroup $T$ and a compact subgroup $K$ as in [2].

We first show that $S$ can be chosen such that $SK$ is actually a subgroup of $G$. Consider the adjoint action of $K$ on $g$: we can see $g$ as a semisimple $K$-module (since $K$ is compact). Looking at the simple case one can easily see that $g = (\mathfrak{k}, \mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{k})$ where $\mathfrak{k} = Lie(K)$. Let us use Levi decomposition for $\mathfrak{z}(\mathfrak{k})$ as a semidirect sum $\mathfrak{z}(\mathfrak{k}) = u_R + u_S$ where $u_R$ is the solvable radical and $u_S$ is a maximal semisimple subalgebra. But $\mathfrak{r} + u_R$ is a solvable ideal of $g$ (here $\mathfrak{r} = Lie(R)$) and since $\mathfrak{r}$ is the maximal solvable ideal of $g$ we must have $\mathfrak{r} + u_R \subseteq \mathfrak{r}$ so $u_R \subseteq \mathfrak{r}$. So the Levi decomposition of $g$ can be written $g = \mathfrak{r} + u_S$ and $u_S$ is a maximal semisimple subalgebra. Let $S_1$ be the subgroup of $G$ whose Lie algebra is $u_S$, it is also a Levi factor for $G$ (it is actually conjugated to $S$) but since $[u_S, \mathfrak{r}] = (0)$ we get that $S_1$ commutes with $K$. So we can assume
$S = S_1$ and $SK$ is a group.

We will show that $G$ is a semidirect product of $T$ and $H := SK$. We already know that $G = SKT$ we are left to show that $H \cap T$ is trivial. Indeed by linearity of $G$ we get that $T \cap S$ is finite (Levi decomposition) and since $R = T \times K$ we get $T \cap K = 1$ hence $H \cap T$ is finite. But $T$ has no torsion so we get triviality of the intersection.

We will now make use of Proposition 1 to start with a representation of $T$ whose image is definable. We continue by extending the representation thanks to [3, Chap. XVII, Theorem 2.2] (which is the “Lie”-version of the Extension Lemma) and we obtain a representation of $G$ that is faithful on $T$ and where $T$ is definable. We just need to realize that both images of $K$ and $S$ are actually definable so that the whole image is definable. Here comes the magic of compact and semisimple matrix groups: they are definable! First $S$ is semialgebraic by Fact 5 and by Theorem 4 (see Appendix A) we know that $K$ is algebraic so the whole image of the representation is definable.

To conclude we need a faithful and definable representation of $H = SK$. But since $S$ and $K$ commute we have $H \cong (S \times K)/F$ where $F = S \cap K$ is a central and finite (because it is connected and $S$ is linear). Since both $K$ and $S$ have definable and faithful representations so does $S \times K$. Using Lemma 4 we get a definable and faithful representation of $H$. When we take the direct sum of those two representations we get a faithful representation of $G$ whose image is definable.
Appendix A: Algebraicity of linear compact groups

We think the following is known but could not find any clear reference so we provide a proof.

Theorem 4
Let $K$ be a compact subgroup of $GL_n(\mathbb{R})$. Then $K$ is definable in $(\mathbb{R},+,\cdot,0,1)$, more precisely it is an algebraic subgroup of $GL_n(\mathbb{R})$.

Proof:
First note that the somewhat usual theorem say that a compact real Lie group is Lie isomorphic to an algebraic group. This theorem is quite interesting but it does not tell us anything about the definable structure of our group since we get algebraicity under isomorphism. We will follow a proof from [5] that uses classical analytic argument and start with a useful lemma.

Lemma 3
Let $K$ be a compact group acting linearly and continuously on a real vector space $V$. Let $v \in V$ such that the subspace $W := \text{Span}(K \cdot v)$ is finite dimensional. Then there is a $K$-fixed point $w$ in $W$.

Proof:
The proof is largely inspired by a similar result on amenable groups that can be found in [3] Chapter 12.4.

Since the orbit $K \cdot v$ is compact (here we see $W = \mathbb{R}^l$ equipped with the coordinate topology) we consider all of the finite open coverings of it. Let $\mathcal{O} := \{\text{finite coverings of } K \cdot v\}$ assuming that there is only necessary open sets in each covering. For each $\pi \in \mathcal{O}$ we pick a point in each open set, that is we have a collection $\{v_U^\pi\}_{U \in \pi}$ with $v_U^\pi \in U \in \pi$. Now we will define the following net:

$$\Phi: \mathcal{O} \to W$$

$$\pi \mapsto v_\pi = \sum_{U \in \pi} \lambda_U v_U^\pi$$

where $\lambda_U := \mu(\{k \in K : k \cdot v \in U\})$ and $\mu$ is the normalized Haar measure on $K$.

Note that for all $V$ neighborhood of 0 in $W$ there is a covering $\pi \in \mathcal{O}$ such that $\pi$ is $V$-fine, that is every $U \in \pi$ can be translated to fit inside $V$. The reason is that $W$ is metrizable hence subsets have diameter).

Claim 1 ($\Phi$ is a Cauchy net):
To prove it we show that if $\pi$ is $V/2$-fine (with $V$ convex and symmetric since $\mathbb{R}^l$ is locally convex) and $\pi'$ refines $\pi$ then $v_\pi - v_\pi' \in V$. For the sake of simplicity assume that $\pi = \{U_1, U_2\}$ and $\pi' = \{W_1, W_2, Z_1, Z_2\}$ with designated points $v_1 \in U_1$ and $v_1' \in W_1$ or $Z_1$. Then $\Phi(\pi) = \alpha_1 v_1 + \alpha_2 v_2$ and $\Phi(\pi') = \beta_1 v_1' + \beta_2 v_2' + \beta_3 v_3' + \beta_4 v_4'$ with $1 = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 + \beta_3 + \beta_4$. But containment gives us $\beta_1 + \beta_2 \leq \alpha_1$ and $\beta_2 + \beta_4 \leq \alpha_2$ and hence equality. We use $u$ for a generic vector in $V/2$ that could stand for different elements of $V/2$. Because $\pi$ is $V/2$-fine we get $v_1 = t_1 + u$ and $v_2 = t_2 + u$ and since $\pi' \subset \pi$ we get $v_1' = t_1 + u$, $v_2' = t_1 + u$, $v_3' = t_2 + u$ and $v_4' = t_2 + u$. Finally we get
\[ \beta v'_1 + \beta_2 v'_2 = \alpha_1 t_1 + u \] (V is convex) and similarly \[ \beta_3 v'_3 + \beta_4 v'_4 = \alpha_2 t_2 + u. \] In the end we have \[ \Phi(\pi) - \Phi(\pi') = (\alpha_1 t_1 + \alpha_2 t_2 + u) - (\alpha_1 t_1 + \alpha_2 t_2 + u) \in V. \]

Claim 2 (\( \Phi \) is convergent and the limit does not depend of the choice of the points):

Since the net takes values in a complete Hausdorff space \((\mathbb{R}^d)\) the net converges to a point \(\tilde{v}\) and this limit is unique. We now need to show that this limit does not depend of the choice we made of the \(v_U\)'s. Let \( \Psi \) be a net defined as \( \Phi \) but with different points. By definition of convergence, for any convex neighborhood \(V \) of \(0\) in \(W\) we can find a finite covering \( \pi' \subseteq \pi \) we have \( \Phi(\pi') \in \tilde{v} + V \). Given such \( V \) we refine \( \pi \) to get a \(V/4\)-fine covering. If \( \pi' \) refines \( \pi \) then by the previous claim we know that \( \Phi(\pi') \in \tilde{v} + V/2 \). Suppose that \( \Phi(\pi') \) is defined using \( v_i \) and that \( \Psi(\pi') \) uses \( w_i \), there are points \( t_i \) in \( W \) such that \( v_i, w_i \in t_i + V/4 \). Now since \( V = -V \), \( v_i - w_i \in V/2 \) and since \( V/2 \) is convex we get \( \Phi(\pi') - \Psi(\pi') = \sum \alpha_i v_i - \sum \alpha_i w_i \in V/2 \). We finally get \( \Psi(\pi') - \tilde{v} = (\Phi(\pi') - \Psi(\pi')) + (\Psi(\pi') - \tilde{v}) \in V/2 + V/2 \subseteq V \).

Claim 3 (\( \tilde{v} \) is a fixed point):

We consider here the action of \( K \) on \( O \) and realize that it defines for each \( k \in K \) bijections \( \pi \mapsto k^{-1} \pi := \{ k^{-1} U \setminus U \in \pi \} \) on \( O \). We define an other net as follow:

\[
\Phi_k: O \rightarrow W \\
\pi \mapsto \sum_{U \in \pi} \lambda_{k^{-1} U} w_U
\]

where \( \lambda_U \) is defined as before and we chose the points \( w_U = k^{-1} \cdot v_U \in k^{-1} U \).

Note that since we are just changing the points in the open sets of the covering, the previous claim tell us that \( \Phi_k \) also converges to \( \tilde{v} \). Now a little calculus and \( K \)-invariance of the Haar measure gives us:

\[
k \cdot \Phi_k(\pi) = k \cdot \sum_{U \in \pi} \lambda_{k^{-1} U} (k^{-1} \cdot v_U) = \sum_{U \in \pi} \lambda_{k^{-1} U} (k \cdot k^{-1} \cdot v_U) = \sum_{U \in \pi} \lambda_U v_U = \Phi(\pi)
\]

On the other side, since multiplication by element of \( K \) is continous, is \( \Psi \) is a convergent net with limit \( w \) the limit of \( k \cdot \Psi \) is \( k \cdot w \). So when we pass to the limit on both end of the previous equation we get that \( k \cdot \tilde{v} = \tilde{v} \).

\( \square \)

Now let us consider the vector space of \( n \)-squared matrix \( M_n(\mathbb{R}) \) on which \( K \) acts (linearly) by multiplication.

Remark 1:

Note that we can build from a finite dimensional \( \mathbb{K} \)-vector space \( V \) its algebra of polynomials denoted by \( \mathbb{K}[V] \) and defined as follow: if \( (e_1, \ldots, e_n) \) is a basis of \( V \) and \( (e'_1, \ldots, e'_n) \) a dual basis then \( \mathbb{K}[V] := \mathbb{K}[e'_1, \ldots, e'_n] \). If a group \( G \) acts linearly on \( V \) then it also acts linearly on its polynomial algebra. If \( g \in G, P \in \mathbb{K}[V] \) and \( v \in V \) then \( g \cdot P(v) := P(g^{-1} \cdot v) \).

We consider the action of \( K \) on \( \mathbb{R}[M_n(\mathbb{R})] \) and remark that for any \( P \in \mathbb{R}[M_n(\mathbb{R})] \) the vector space spanned by \( K \cdot P \) is finite dimensional (the degree of \( P \) stays bounded under the action since \( K \) acts linearly on \( M_n(\mathbb{R}) \)). So for any \( P \) we can find \( \tilde{P} \in \text{Span}(K \cdot P) \) that is \( K \)-invariant.
Let’s go back to the action of $K$ on $\mathcal{M}_n(\mathbb{R})$. Thanks to Weierstrass approximation theorem for any two orbits we can find a polynomial $P$ in $\mathbb{R}[\mathcal{M}_n(\mathbb{R})]$ that separates them. Indeed there is an analytic function $f$ whose value on one orbit is 0 and 1 on the other one. We find a polynomial $P$ that is $\epsilon$-close to $f$ (value less than $\epsilon$ on the first orbit and $P - 1$ takes values less than $\epsilon$ on the other one).

Remark 2:
Note that if we apply the process of the previous lemma to a polynomial that separates two orbits then the limit is a $K$-invariant polynomial that also separates those orbits. This is because convergence for the coordinate topology implies punctual convergence as well. We can easily modify this limit so that we get a $K$-invariant polynomial whose value is 0 on the first orbit and 1 on the other orbit.

So we consider the orbit of $I_n \in \mathcal{M}_n(\mathbb{R})$ which is actually all of $K$ and for any other orbit we find $K$-invariant polynomials that separate $K$ from it (always picking the value 0 for the orbit $K = K \cdot I_n$). Now let $I$ be the ideal generated by all of those polynomials. Since $\mathbb{R}[\mathcal{M}_n(\mathbb{R})]$ is noetherian $I$ is finitely generated, say by $P_1, \ldots, P_h$. Let $X = (x_{i,j}) \in \mathcal{M}_n(\mathbb{R})$, and the system of equations in $x_{i,j}$ given by all of the coefficient equalities of $P(X) = 0$. We have achieved a definition for $K$:

$$K = \{X \in \mathcal{M}_n(\mathbb{R}) : P_i(X) = 0 \forall 1 \leq i \leq h\}$$

Since the $P_i$ are $K$-invariant members of $K$ do satisfy this equation and if a matrix is not in $K$ then there is a polynomial that separates its orbits with $K \cdot I_n$ and this should be reflected on at least one of the generators. Those equations are purely algebraic giving us the algebraic representation we were looking for.
Appendix B: Definable representation of a quotient

Lemma 4
Let $G$ be a definable Lie group with a definable and faithful representation over a finite dimensional vector space $V$. Let $F$ be a central and finite subgroup of $G$. Then there is a definable and faithful representation of the quotient $G/F$ on a finite dimensional vector space $W$.

Proof:
First we may assume that $F$ has prime order $q$ (by induction on its order and using that $F$ is abelian and finite). We will need pass to a complex representation on $\tilde{V} = V \otimes \mathbb{C}$ and decompose $\tilde{V}$ into $V_1 \oplus \cdots \oplus V_h$ where the $V_i$ are $F$-stable and $F$-irreducible subspaces (here we used Maschke’s Theorem for finite groups). By Schur’s lemma the action of $F$ on each $V_i$ is exactly the kernel, taking the quotient by $F$ will give us a faithful representation on $W$. Note that we are a priori acting on a complex vector space but we can consider the real action on each of its components (the real one and the imaginary one) so that we obtain a real action.

Let us check that if $g$ is in the kernel of the action it must be in $F$. Let $v_i \in V_i$ be a non zero element such that $g \cdot v_i = \lambda v_i$ for some complex $\lambda$. Since $g$ fixes $(v_1 \otimes \cdots \otimes v_i) \in V_1 \otimes \cdots \otimes V_i$, we get $\lambda^q = 1$ and hence $\lambda = e^{2\pi i q}$. We will now show that $g$ acts by multiplication by $\lambda$ on $V_i$. Let $w_i \in V_i$ be linearly independant from $v_i$. Invariance of $(w_i \otimes v_1 \otimes \cdots \otimes v_i) \in V_1 \otimes \cdots \otimes V_i$ gives us:

$$w_i \otimes v_1 \otimes \cdots \otimes v_i = \lambda^{q-1} g \cdot w_i \otimes v_1 \otimes \cdots \otimes v_i$$

and hence by linear independance $g \cdot w_i = \lambda w_i$ and $g$ acts by multiplication by $\lambda$ on $V_i$.

Now fix $r \in \{[1, h-1]\}$, since $g$ acts by multiplication by $e^{2\pi i q}$ on each $V_i$, the action of $g$ on $V_1 \otimes \cdots \otimes V_h \otimes \cdots \otimes V_{h-1}$ is a multiplication by $\mu^{K_r}$, where $K_r = \sum_{s=1}^h a_{r,s} x_s$. But since the action is trivial we get $K_r = 0$ for all $r$ which means that $(x_1, \ldots, x_h)$ is in the line defined earlier and $(x_1, \ldots, x_h) = m(k_1, \ldots, k_h)$. Finally we have $g = x^m \in F$. 

\[\square\]
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