Group of $L$-homeomorphisms and $L_f$-representability of Permutation Groups

Sini P

Department of Mathematics, University of Calicut, Kerala, India

ABSTRACT
In this paper we investigate the group of $L$-homeomorphisms of an $L$-topological space. If there exists an $L$-topology $\delta$ on a set $X$ such that the group of $L$-homeomorphisms of the $L$-topological space $(X, \delta)$ is a permutation group $K$ on $X$, then $K$ is $L_f$-representable on $X$. It is proved that the direct sum of finite $L_f$-representable permutation groups is $L_f$-representable on $X$. We also investigate $L_f$-representability of some cyclic subgroups of the group $S_X$.

1. Introduction
The classical mathematical theories have limitations for solving complicated problems that include uncertain data in many areas namely engineering, social science, medical science etc. One of the tools for dealing with these uncertainties is fuzzy sets. In 1965, Zadeh [1] introduced the theory of fuzzy sets. Later Gougen defined the concept of an $L$-fuzzy set or an $L$-set where $L$ is a semigroup, a partially ordered set, a lattice or a boolean ring [2]. We can extend most of the mathematical theories using the concept of an $L$-set since $L$-set is a generalisation of the fundamental mathematical concept of a set. Now many generalisations of fuzzy sets are presented and it gives a framework for generalising the classical mathematical theories [3–12].

Based on the notion introduced by Zadeh [1], Chang introduced fuzzy topology and studied its properties [13]. Researchers modified several concepts of classical topological spaces to include in fuzzy topological spaces. Several authors studied the group of homeomorphisms of topological spaces and group of $L$-homeomorphisms of $L$-topological spaces [14–21]. Johnson [22–24] and Ramachandran [15,16] considered the problem of representing a subgroup $K$ of the symmetric group $S_X$ as the group of $L$-homeomorphisms of some $L$-topological space $(X, \delta)$. The theory of permutation groups is well developed.
and using this theory we can study \( L \)-topological spaces. The order of the group of \( L \)-homeomorphisms of an \( L \)-topological space depends upon the structure of the space. If the group of \( L \)-homeomorphisms of a topological space is a transitive permutation group then the \( L \)-topological space is homogeneous. We can study the action of the group of \( L \)-homeomorphisms of an \( L \)-topological space on the space itself.

In [22–24] Johnson proved that the subgroups generated by a finite cycle and some proper non-trivial normal subgroups can be represented as group of \( L \)-homeomorphisms for some \( L \)-topology \( \delta \) on \( X \), when \(|X| \leq |L| \). Ramachandran proved that the group of permutations on a set \( X \) generated by a finite cycle and the group generated by an arbitrary product of infinite cycles can be represented as the group of \( L \)-homeomorphisms for some \( L \)-topology, if the membership lattice \( L \neq \{0, 1\} \) [15,16].

A subgroup \( K \) of the group of all permutations of a set \( X \) is called t-representable [17] on \( X \) if there exists a topology \( \tau \) on \( X \) such that the group of \( L \)-homeomorphisms of \((X, \tau) = K \). Analogous to t-representability in topology, in [19] we defined \( L_f \)-representability of permutation groups and we determined the \( L_f \)-representability of some subgroups of \( S_X \). Here also we investigate the same problem.

The paper is organised as follows. In section two, we recall the most essential concepts that are needed for our study. In section three, we study some properties of permutation groups that can be represented as the group of \( L \)-homeomorphisms of topological spaces. The \( L_f \)-representability of direct sum of finite \( L_f \)-representable permutation groups are investigated in section four. In section five, \( L_f \)-representability of cyclic subgroups of the symmetric group is studied.

### 2. Preliminaries

Here we recall the most essential concepts that are needed for our study. Throughout this paper \( X \) stands for a non empty set, \( S_X \) for symmetric group on \( X \), \( I_X \) for an identity permutation on \( X \) and \( L \) for an \( F \)-lattice.

Let \( L \) be a lattice. Then \( L \) is an \( F \)-lattice [25] if it is completely distributive and there is an order reversing involution \( ^t : L \rightarrow L \). Let \( X \) be a non empty set and \( L \) be an \( F \)-lattice. The set of all \( L \)-subsets or \( L \)-fuzzy subsets of \( X \) is denoted by \( L^X \).

**Definition 2.1:** [25] Let \( A \) and \( B \) be two sets and \( h : A \rightarrow B \) be a function. Then for any \( L \)-set \( f \) in \( A \), \( h(f) \) is an \( L \)-set in \( B \) defined by

\[
h(f)(b) = \begin{cases} \bigvee \{f(a) : a \in A, h(a) = b\} & h^{-1}(b) \neq \emptyset \\ 0; & h^{-1}(b) = \emptyset \end{cases}
\]

For an \( L \)-set \( g \) in \( B \), we define \( h^{-1}(g)(a) = g(h(a)) \) for all \( a \in B \).

**Definition 2.2:** [25] Let \( X \) be a nonempty set and \( \delta \subseteq L^X \). Then \( \delta \) is called an \( L \)-topology on \( X \), and \((X, \delta)\) is called an \( L \)-topological space if \( \delta \) satisfies the following three conditions.

1. \( 0, 1 \in \delta \);
2. \( f \land g \in \delta \) for all \( f, g \in \delta \);
3. \( \forall A \in \delta \mbox{ for all } A \subseteq \delta \).
Every element in $\delta$ is called an $L$-open subset of $X$.

Let $(X, \delta)$ and $(Y, \delta')$ be any two $L$-topological spaces and $h$ be a mapping from $(X, \delta)$ to $(Y, \delta')$. Then $h$ is said to be an $L$-continuous map from $X$ to $Y$, if $h^{-1}(f') \in \delta$ for every $f'$ in $\delta'$ where $h^{-1}(f')$ means $f' oh$ and $h$ is said to be $L$-open if it maps every $L$-open subset of $X$ as an $L$-open one in $Y$. Now $(X, \delta)$ and $(Y, \delta')$ are $L$-homeomorphic if (i) there is an $L$-continuous bijection $h : X \to Y$, (ii) there is an $L$-continuous bijection $k : Y \to X$ and (iii) $hok = I_Y$ and $koh = I_X$. An equivalent condition for a permutation $h$ of a set $X$ to be an $L$-homeomorphism of $(X, \delta)$ on to itself is that $f \in \delta$ if and only if $f oh \in \delta$. Let $GLH(X, \delta) = \{ h \in SX : h$ is a homeomorphism on $(X, \delta)$ onto itself $\}$. The $GLH(X, \delta)$ is a group under composition and is called the group of $L$-homeomorphisms of $(X, \delta)$. Clearly $GLH(X, \delta)$ is a subgroup of $SX$.

Now we recall two definitions in permutation group theory.

**Definition 2.3:** [26] Let $X$ and $Y$ be two disjoint sets and $G$ and $H$ be subgroups of $SX$ and $SY$ respectively. Then direct product $G \times H$ is a subgroup of $X \cup Y$ by the rule

$$(g, h)(x) = \begin{cases} g(x) & \text{if } x \in X \\ h(x) & \text{if } x \in Y. \end{cases}$$

**Definition 2.4:** [17] Let $\{X_i : i \in I\}$ be an arbitrary family of mutually disjoint sets and $K_i$ be a subgroup of $SX_i$ for every $i \in I$. Then the direct product of permutation groups $\{K_i : i \in I\}$ is the permutation group $\times_{i \in I} K_i$ on $X = \bigcup_{i \in I} X_i$ whose elements are $\times_{i \in I} k_i$ where $k_i \in K_i$ and the action of $\times_{i \in I} k_i$ is given by $\times_{i \in I} k_i(x) = k_i(x)$ if $x \in X_i$, $i \in I$.

We need following theorems taken from [15,18].

**Theorem 2.5:** [15] Let $X$ be any set and $L$ be any complete distributive lattice containing more than two elements. Then the group of permutations of $X$ generated by any finite cycle on $X$ can be represented as the group of homeomorphisms of the $L$-topological space $(X, \delta)$ for some $L$-topology $\delta$ on $X$.

**Theorem 2.6:** [18] Let $X$ be any set and $K$ be a subgroup of $SX$ generated by $\tau = \prod_{i \in I} C_i$ where $\{C_i, i \in I\}$ be an indexed family of disjoint cycles with equal length $m$. Then

1. $K$ is $t$-representable if $|I| > 2$ or $m < 3$.
2. $K$ is not $t$-representable if $|I| \leq 2$ and $m \geq 3$.

### 3. $L_f$-representability of Permutation Groups

A subgroup $K$ of the group $SX$ of all permutations of a set $X$ is called $L_f$-representable [19] on $X$ if there exists an $L$-topology $\delta$ on $X$ such that the group of $L$-homeomorphisms of $(X, \delta) = K$. If we take $L$ as the lattice containing only two elements 0 and 1, then in view point of lattice theory $L^X$ is isomorphic to the power set of $X$ and hence topologies and topological spaces become special cases of $L$-topologies and $L$-topological spaces. So every $t$-representable permutation group on a set $X$ is also $L_f$-representable on $X$. But an $L_f$-representable permutation group need not be $t$-representable on $X$. 
The following theorem plays a major role in proving results related to $L_f$-representability of permutation groups.

**Theorem 3.1:** [15] Let $L$ and $L'$ be two complete and distributive lattices such that $L$ is isomorphic to a sublattice of $L'$. Then if $K$ is a subgroup of $S_X$ which can be represented as the group of $L$-homeomorphisms of an $L$-topological space $(X, \delta)$ for some $L$-topology $\delta$ on $X$, then $K$ can also be represented as the group of $L'$-homeomorphisms of the $L'$-topological space $(X, \delta')$ for some $L'$-topology $\delta'$ on $X$.

From Theorem 3.1, we can easily deduce the following.

**Remark 3.2:** If $L$ and $L'$ are two $F$-lattices such that $L$ is isomorphic to a sublattice of $L'$ and a permutation group $K$ is $L_f$-representable on an arbitrary set $X$, then $K$ is also $L_{f'}$-representable on $X$.

So if we prove a permutation group $K$ is $L_f$-representable on a set $X$ by taking $L = \{0, a, 1\}$ with the usual order, then $K$ is $L_f$-representable on $X$ for any $F$-lattice $L \neq \{0, 1\}$.

Let $X$ be any set and $H$ be a subgroup of the symmetric group $S_X$. In [19] it is proved that $H$ is $L_f$-representable on $X$ if and only if its conjugate is also $L_f$-representable on $X$. So it suffices to determine the conjugacy classes of subgroups of $S_X$ which are $L_f$-representable on $X$. Here we discuss some more properties of $L_f$-representable permutation groups.

Our next Theorem is a generalisation Theorem 2 of [15] and Theorem 2.2 of [21].

**Theorem 3.3:** Let $X$ be any set and $A$ be a non-empty subset of $X$. If a subgroup $K$ of $S_A$ is $L_f$-representable on $A$, then the subgroup $K \times \{l_X|A\}$ of $S_X$ is $L_f$-representable on $X$.

**Proof:** Let $\delta$ be an $L$-topology on $A$ such that $GLH(A, \delta) = K$. The result is obvious if $X \setminus A = \emptyset$. We assume that $X \setminus A \neq \emptyset$. Now for any $f \in \delta$, let $f' : X \to L$ defined by

$$f'(x) = \begin{cases} 1 & \text{if } x \in X \setminus A \\ f(x) & \text{if } x \in A. \end{cases}$$

By using well-ordering Theorem, well-order the set $X \setminus A$ with the order relation '$<$. Now for $a \in X \setminus A$, define $f_a : X \to L$ as

$$f_a(x) = \begin{cases} 1 & \text{if } x \in X \setminus A \text{ and } x < a \\ 0 & \text{otherwise.} \end{cases}$$

Let $\delta_1 = \{f' : f \in \delta\}$ and $\delta_2 = \{f_a : a \in X \setminus A\}$. Using $\delta_1$ and $\delta_2$ we define $\delta'$ on $X$ as $\delta' = \delta_1 \cup \delta_2$. It is easy to see that $\delta'$ is an $L$-topology on $X$. Now we prove that $GLH(X, \delta') = K \times \{l_X|A\}$.

Let $k_1 \in K \times \{l_X|A\}$. This implies that $k_1 = (k, l_X|A)$ for some $k$ in $K$. Let $g \in \delta'$. If $g = f_a$ for some $a \in X \setminus A$, then $(k, l_X|A)^{-1}(f_a)(x) = f_a o(k, l_X|A)(x)$.

Now

$$(k, l_X|A)^{-1}(f_a)(x) = f_a o(k, l_X|A)(x) = \begin{cases} f_a(x) & \text{if } x \in X \setminus A \\ f_a(k(x)) & \text{if } x \in A \end{cases}$$

Let $k_2 \in K \times \{l_X|A\}$. This implies that $k_2 = (k, l_X|A)$ for some $k$ in $K$. Let $g \in \delta'$. If $g = f_a$ for some $a \in X \setminus A$, then $(k, l_X|A)^{-1}(f_a)(x) = f_a o(k, l_X|A)(x)$.
Observe that

\[
= \begin{cases} 
  f_a(x) & \text{if } x \in X \setminus A \\
  0 & \text{if } x \in A \\
  1 & \text{if } x \in X \setminus A \text{ and } x < a \\
  0 & \text{otherwise}
\end{cases}
= f_a(x).
\]

Thus if \( g = f_a \), then \((k, l_{X \setminus A})^{-1}(g) = g\), which belongs to \( \delta' \).

If \( g = f' \), then \((k, l_{X \setminus A})^{-1}(f') = f' \circ (k, l_{X \setminus A})\) and

\[
(k, l_{X \setminus A})^{-1}(f')(x) = \begin{cases} 
  (f \circ k)(x) & \text{if } x \in A \\
  1 & \text{if } x \in X \setminus A
\end{cases}
= (f \circ k)'(x)
\]

Observe that \( f \in \delta \) and \( k \in K \). So \( f \circ k \in \delta \) and hence \((k, l_{X \setminus A})^{-1}(f') = (f \circ k)' \in \delta' \). So \((k, l_{X \setminus A})^{-1}(g) \in \delta' \) for all \( g \in \delta' \). Thus \((k, l_{X \setminus A})\) is an \( L \)-continuous map on \( X \) onto itself. Similarly we can prove that \((k, l_{X \setminus A})(f') \in \delta' \) for all \( f' \in \delta' \) and hence \((k, l_{X \setminus A})^{-1}\) is also an \( L \)-continuous map on \( X \). Thus \((k, l_{X \setminus A})\) is an \( L \)-homeomorphism on \((X, \delta')\) for all \( k \in K \). So

\[ K \times \{l_{X \setminus A}\} \subseteq GLH(X, \delta'). \]

Conversely assume that \( k \) is an \( L \)-homeomorphism on \((X, \delta')\) onto itself. First we show that \( k(x) = x \) for all \( x \in X \setminus A \). We consider the case \(|X \setminus A| = 1\). Let \( X \setminus A = \{x_0\} \). Since \( 0 \in \delta, f = 0' \in \delta' \). Now

\[
f(x) = \begin{cases} 
  1 & \text{if } x = x_0 \\
  0 & \text{otherwise}
\end{cases}
\]

We have \( k^{-1}(f) = f \circ k \) and \( f \circ k \) takes the value 1 only at \( k^{-1}(x_0) \) and 0 for all other values of \( x \). Since \( k \in GLH(X, \delta') \), \( k^{-1}(f) \in \delta' \). By the definition of \( \delta' \), the only possibility is \( k(x_0) = x_0 \).

Now we assume that \(|X \setminus A| \geq 2\). Let \( x_0 \) be the first element of the set \( X \setminus A \) and \( x_1 \) be the first element of the set \((X \setminus A) \setminus \{x_0\}\). Now consider \( k^{-1}(f_{x_1}) \), which takes the value 1 at exactly one point \( k^{-1}(x_0) \) and 0 elsewhere. Since \( f_{x_1} \in \delta' \), \( k^{-1}(f_{x_1}) \in \delta' \). This implies that \( k^{-1}(f_{x_1}) = f_{x_1} \) and so \( k(x_0) = x_0 \). Let \( x_\alpha \) be any element of \( X \setminus A \) such that \( k(x) = x \) for all \( x \) in \( X \setminus A, x < x_\alpha \). Now we claim that \( k(x_\alpha) = x_\alpha \).

If \( x_\alpha \) has no immediate successor in \( X \setminus A, x_\alpha \) be the last element of the set \( X \setminus A \). Since \( f = 0 \in \delta, f' : X \to L \) defined by

\[
f'(x) = \begin{cases} 
  0 & \text{if } x \in A \\
  1 & \text{if } x \in X \setminus A
\end{cases}
\]

belongs to \( \delta' \). So

\[
k^{-1}(f')(x) = (f' \circ k)(x)
= \begin{cases} 
  1 & \text{for all } x \in X \setminus A \text{ such that } x < x_\alpha \text{ and } k^{-1}(x_\alpha) \\
  0 & \text{otherwise}
\end{cases}
\]

and \( k^{-1}(f') \in \delta' \). This implies that \( k^{-1}(f') = f' \) and hence \( k(x_\alpha) = x_\alpha \).
If \( x_\alpha \) has an immediate successor \( x_\beta \) in \( X \setminus A \), then consider \( f_\beta \) in \( \delta' \). We have that \( k^{-1}(f_\beta) \in \delta' \) and

\[
k^{-1}(f_\beta)(x) = (f_\beta \circ k)(x) = \begin{cases} 1 & \text{for all } x < x_\alpha \text{ and } k^{-1}(x_\alpha) \\ 0 & \text{otherwise.} \end{cases}
\]

This gives that \( k^{-1}(f_\beta) = f_\beta \) and hence \( k(x_\alpha) = x_\alpha \). It follows that \( k(x) = x \) for all \( x \in X \setminus A \) and \( k(A) = A \). So \( k|A \) is a homeomorphism on \( (A, \delta) \). Hence \( k \in K \times \{ l_{X \setminus A} \} \). Since \( k \) is arbitrary, we have

\[
GLH(X, \delta') \subseteq K \times \{ l_{X \setminus A} \} \quad (2)
\]

From equations 1 and 2, we get \( GLH(X, \delta') = K \times \{ l_{X \setminus A} \} \).

**Remark 3.4:** Let \( K \) be a non-trivial permutation group on a set \( X \). Let \( A = X \setminus \{ x \in X : k(x) = x \text{ for all } k \in K \} \). Define \( K' = \{ k|A : k \in K \} \), which is a permutation group on \( A \). Note that \( K' \) moves all the elements of \( A \) and \( K = K' \times \{ l_{X \setminus A} \} \). By Theorem 3.3, it follows that, if \( K' \) is \( L_f \)-representable on \( A \), then \( K \) is \( L_f \)-representable on \( X \). So if \( (X, \delta) \) is an \( L \)-topological space which is not rigid and \( K = GLH(X, \delta) \) then without loss of generality, we can assume that \( K \) moves all the elements of \( X \).

### 4. \( L_f \)-representability of Direct Product of \( L_f \)-representable Permutation Groups

It is easy to see that the intersection of \( L_f \)-representable subgroups of the symmetric group \( S_X \) and the group generated by union of \( L_f \)-representable subgroups of \( S_X \) are need not be \( L_f \)-representable on \( X \). Now we turn our attention to the \( L_f \)-representability of direct product of \( L_f \)-representable subgroups of symmetric groups. In [17], it is proved that the direct product of finite \( t \)-representable permutation groups is \( t \)-representable. Here we prove analogous result in the case of \( L \)-topological spaces.

**Theorem 4.1:** Let \( \{ X_i \}_{i \in I} \) be an arbitrary family of mutually disjoint finite sets and \( K_i \) be an \( L_f \)-representable subgroup of \( S_{X_i} \) for \( i \in I \). Then \( \times_{i \in I} K_i \) is \( L_f \)-representable on \( X = \bigcup_{i \in I} X_i \).

**Proof:** Since \( K_i \) is \( L_f \)-representable on \( X_i \) for all \( i \in I \), there exists an \( L \)-topology \( \delta_i \) on \( X_i \) such that the group of \( L \)-homeomorphisms, \( GLH(X_i, \delta_i) = K_i \). By the well-ordering Theorem, we can choose a well-order \( < \) on \( I \). For each \( f \in \delta_i, i \in I \) define \( f' : X \to L \) as follows

\[
f'(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{j < i} X_j \\ f(x) & \text{if } x \in X_i \\ 0 & \text{otherwise} \end{cases}
\]

and let

\[
\delta'_i = \{ f'_i : f_i \in \delta_i \}.
\]
Using this \( \delta'_i \), we can define \( \delta \) on \( X \) as follows.

\[
\delta = \{1\} \cup \bigcup_{i \in I} \delta'_i.
\]

Then \( \delta \) is an \( L \)-topology on \( X \). We claim that \( GLH(X, \delta) = K \) where \( K = \times_{i \in I} K_i \).

Let \( k_i \in K_i \) for all \( i \in I \) and \( k = \times_{i \in I} k_i \). Clearly \( k \) is a bijection of \( X \) onto itself. Let \( f \in \delta \).

If \( f = 1 \), then \( k^{-1}(f) = fok = f \). Suppose \( f \neq 1 \), then \( f = f'_i \) for some \( i \in I \). Consider \( k^{-1}(f'_i) \).

Now

\[
k^{-1}(f'_i)(x) = (f'_iok)(x)
\]

\[
= \begin{cases} 
1 & \text{if } k(x) \in \bigcup_{j < i} X_j \\
 f_i(k(x)) & \text{if } k(x) \in X_i \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } k(x) \in \bigcup_{j < i} X_j \\
f_iok(x) & \text{if } k(x) \in X_i \\
0 & \text{otherwise}
\end{cases}
\]

\[
= (f'_iok_i)'(x).
\]

Since \( f_i \in \delta_i \) and \( k_i \in K_i \), we have that \( f_iok_i \in \delta_i \) and hence \( (f_iok_i)' \in \delta \). This implies that \( k^{-1}(f'_i) \) belongs to \( \delta \). Similarly we can prove that \( f'_iok^{-1} \in \delta \). So \( K \subseteq GLH(X, \delta) \). Conversely suppose that \( k \in GLH(X, \delta) \). Let \( i_0 \) be the smallest element of \( I \). Since \( 1 \in \delta_{i_0} \), consider \( f_{i_0}' \) defined by

\[
f_{i_0}'(x) = \begin{cases} 
1 & \text{if } x \in X_{i_0} \\
0 & \text{otherwise}
\end{cases}
\]

which takes the value 1 at exactly \( |X_{i_0}| \) points and belongs to \( \delta \). Since \( k \in GLH(X, \delta) \), \( k(f_{i_0}') \in \delta \). Also we have that

\[
k(f_{i_0}') = (f_{i_0}'ok^{-1})(x)
\]

\[
= \begin{cases} 
1 & \text{if } k^{-1}(x) \in X_{i_0} \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } x \in k(X_{i_0}) \\
0 & \text{otherwise}
\end{cases}
\]

Note that \( k(f') \) takes the value 1 at \( |X_{i_0}| \) points and hence the only possibility is \( k(f_{i_0}') = f_{i_0}' \). Thus we get \( k(X_{i_0}) = X_{i_0} \).

Now assume that \( j \in I \) and \( k(X_j) = X_j \) for all \( j \in I \) and \( j < i \).

We prove that \( k(X_i) = X_i \). Suppose \( k(X_i) \neq X_i \), then there exists \( x \in X_i \) such that \( k(x) \notin X_i \) or there exists \( x \notin X_i \) such that \( (x) \in X_i \). In the second case also we can see that, there exists \( x \in X_i \) such that \( (x) \notin X_i \), since \( X_i \) is finite. Thus without loss of generality, we can assume that there exists \( x \in X_i \) such that \( k(x) \notin X_i \). Then \( k(x) \in X_k \) for some \( k > i \). Consider \( f = f'_i \)
where

\[ f_i'(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{j \leq i} X_j \\ 0 & \text{otherwise} \end{cases} \]

which is an L-open set in \( \delta \). Therefore \( k(f) \) is also an L-open set in \( X \). Let \( x \in X \). Then

\[
k(f)(x) = (f \circ k^{-1})(x)
= f(k^{-1}(x))
= \begin{cases} 1 & \text{if } k^{-1}(x) \in \bigcup_{j \leq i} X_j \\ 0 & \text{otherwise} \end{cases}
= \begin{cases} 1 & \text{if } x \in k \left( \bigcup_{j \leq i} X_j \right) \\ 0 & \text{otherwise} \end{cases}
= \begin{cases} 1 & \text{if } x \in \bigcup_{j < i} X_j \cup k(X_i) \\ 0 & \text{otherwise}. \end{cases}
\]

This is true for all \( x \in X \). Since \( k(x) \in X_k \) for some \( x \in X_i \) and \( k(f) \in \delta \), we get \( |k(X_i)| > |X_i| \), which is a contradiction. Hence \( k(X_j) = X_j \) for all \( j \in I \). Thus \( k|_{X_j} = k_j \) is an L-homeomorphism of \( X_j \) for all \( j \in I \) and \( k = \times_{i \in I} k_i \). So \( k \in K = \times_{i \in I} K_i \). Thus \( GLH(X, \delta) = K \) and hence \( \times_{i \in I} K_i \) is an Lf-representable permutation group on \( X \).

\[ \blacksquare \]

**Remark 4.2:** The following example ensures that finiteness can not be dropped in Theorem 4.1 even when \( I \) is finite and \( L = \{0, 1\} \).

**Example 4.3:** Let \( X_1 \) be the set of all non negative integers and \( \tau_1 \) be the topology on \( X_1 \) defined by

\[ \tau_1 = \{X_1, \emptyset\} \cup \{\{a \in X_1 : a \leq n\} : n \in X_1\}. \]

Let \( X_2 \) be the set of all negative integers and \( \tau_2 \) be the topology on \( X_2 \) defined by

\[ \tau_2 = \{X_2, \emptyset\} \cup \{\{a \in X_2 : a \leq n\} : n \in X_2\}. \]

Here \( GLH(X_1, \tau_1) = \{k_1\} \) and \( GLH(X_2, \tau_2) = \{k_2\} \). Let \( X = X_1 \cup X_2 \). Now define the topology \( \tau \) on \( X \) as in the Theorem 4.1. That is, \( \tau = \{X, \emptyset\} \cup \{\{a \in X : a \leq n\} : n \in X\} \) and hence \( GLH(X, \tau) \) is the group generated by the infinite cycle

\[ (\ldots, -2, -1, 0, 1, 2, \ldots) \]

which is not equal to the direct product of \( GLH(X_1, \tau_1) \) and \( GLH(X_2, \tau_2) \).
5. $L_f$-representability of Cyclic Group of Permutations

Now we investigate the $L_f$-representability of some cyclic permutation groups.

In [18], it is proved that the permutation group generated by a permutation which is a product of two disjoint cycles having equal length $m$, $m \geq 3$ is not $t$-representable. Here we prove such groups are $L_f$-representable on $X$ when $|L| \neq 2$.

**Theorem 5.1:** Let $X$ be any set and $\tau$ be a permutation on $X$ such that $\tau$ can be written as a product of two disjoint cycles having equal length $m$. If $|L| > 2$, then the permutation group generated by $\tau$ is $L_f$-representable on $X$.

**Proof:** Let $\tau = \tau_1 \tau_2$ where

$$\tau_1 = (x_{11}, x_{12}, \ldots, x_{1m})\text{and } \tau_2 = (x_{21}, x_{22}, \ldots, x_{2m})$$

respectively. Take $L = \{1, .5, 0\}$ with the usual order. By Remark 3.4, without loss of generality we assume that $X = X_1 \cup X_2$ where $X_i$ is the set of all elements in the cycle $\tau_i$ for $i = 1, 2$.

Define $\delta = \{f \in L^X : f(x_{ij}^{\oplus 1}) \geq f(x_{ij}) - .5 \text{ for every } j = 1, 2, \ldots, m \text{ and } i = 1, 2 \text{ and if } f(x_{2j}) = 1, \text{ then } f(x_{1j}) = 1 \text{ for all } j = 1, 2, \ldots, m\}$, where $\oplus$ denote addition modulo $n$. Then $\delta$ is an $L$-topology on $X$.

The constant functions 0 and 1 belongs to $\delta$. Let $f_1, f_2 \in \delta$. Then

$$(f_1 \vee f_2)(x_{ij}^{\oplus 1}) = f_1(x_{ij}^{\oplus 1}) \vee f_2(x_{ij}^{\oplus 1})$$

$$\geq (f_1(x_{ij}) - .5) \vee (f_2(x_{ij}) - .5)$$

$$\geq (f_1 \vee f_2)(x_{ij}) - .5$$

and $(f_1 \vee f_2)(x_{2j}) = 1$ gives $f_1(x_{2j}) = 1$ or $f_2(x_{2j}) = 1$. If $f_1(x_{2j}) = 1$, then $f_1(x_{1j}) = 1$. Similarly if $f_2(x_{2j}) = 1$, then $f_2(x_{1j}) = 1$. Thus if $f_1 \vee f_2)(x_{2j}) = 1$, then $(f_1 \vee f_2)(x_{1j}) = 1$. Now

$$(f_1 \wedge f_2)(x_{ij}^{\oplus 1}) = f_1(x_{ij}^{\oplus 1}) \wedge f_2(x_{ij}^{\oplus 1})$$

$$\geq (f_1(x_{ij}) - .5) \wedge (f_2(x_{ij}) - .5)$$

$$\geq (f_1 \wedge f_2)(x_{ij}) - .5.$$

Note that $(f_1 \wedge f_2)(x_{2j}) = 1$ gives $f_1(x_{2j}) = 1$ and $f_2(x_{2j}) = 1$. Thus $(f_1 \wedge f_2)(x_{1j}) = 1$. So $\delta$ is an $L$-topology on $X$.

Now we prove that $\tau$ is an $L$-homeomorphism on $X$. Let $f \in \delta$. Then $\tau^{-1}(f) = f_0 \tau$ and for $i = 1, 2$,

$$f_0 \tau(x_{ij}^{\oplus 1}) = f(x_{ij}^{\oplus 1})$$

$$\geq f(x_{ij}^{\oplus 1}) - .5$$

$$= f_0 \tau(x_{ij}) - .5$$

and

$$f_0 \tau(x_{2j}) = 1$$

$$\Rightarrow f(x_{2j}^{\oplus 1}) = 1$$

$$\Rightarrow f(x_{1j}^{\oplus 1}) = 1$$

$$\Rightarrow f_0 \tau(x_{1j}) = 1.$$
Hence for $\tau \in \delta$. Similarly $\tau(f) = f\tau^{-1} \in \delta$. Hence $\tau$ is an $L$-homeomorphism on $(X, \delta)$ and consequently all the powers of $\tau$ are also $L$-homeomorphisms on $X$. Thus
\[ K \leq GLH(X, \delta). \] (3)

Conversely let $h \in GLH(X, \delta)$. For $i = 1, 2$ and $j = 1, 2, \ldots, m$, Define $f_{ij}: X \to L$ as follows
\[ f_{ij}(x) = \begin{cases} 1 & \text{if } x = x_{kj}, \ k \leq i \\ .5 & \text{if } x = x_{kj \oplus 1}, \ k \leq i \\ 0 & \text{otherwise.} \end{cases} \]

Then $f_{ij} \in \delta$ for all $i = 1, 2$ and $j = 1, 2, \ldots, m$. Since $h$ is an $L$-homeomorphism, $h(f_{ij}) = f_{ij} o h^{-1} \in \delta$ for all $i = 1, 2$ and $j = 1, 2, \ldots, m$. Consider $h(f_{11})$. We have $f_{11} o h^{-1}(x) = 1$ for an unique element of $X$ namely $h(x_{11})$. Then $h(x_{11}) = x_{1k}$ for some $k$. Now $f_{11} o h^{-1}(x_{1k}) = 1$ gives $f_{11} o h^{-1}(x_{1k \oplus 1}) = .5$ and $f_{11} o h^{-1}(x) = 0$ for all other values of $X$. Thus $f_{11} o h^{-1} = f_{1k}$.

Also $f_{11} o h^{-1}(x_{1k \oplus 1}) = f_{1k}(x_{1k \oplus 1}) = .5$ and hence $h^{-1}(x_{1k \oplus 1}) = x_{12}$ or $h(x_{12}) = x_{1k \oplus 1}$.

Now we prove that if $h(x_{1\alpha}) = x_{1\beta}$ for some $\alpha$ and $\beta$, then $h(x_{1\alpha \oplus 1}) = x_{1\beta \oplus 1}$. Note that $h(f_{1\alpha}) = f_{1\alpha} o h^{-1} \in \delta$. Now $f_{1\alpha} o h^{-1}(x_{1\beta}) = 1$ and this gives $f_{1\alpha} o h^{-1}(x_{1\beta \oplus 1}) = .5$. So $h^{-1}(x_{1\beta \oplus 1}) = x_{1\alpha \oplus 1}$ or $h(x_{1\alpha \oplus 1}) = x_{1\beta \oplus 1}$. Thus $h(x_{1j}) = x_{1j \oplus (k-1)}$ for all $j = 1, 2, \ldots, m$. So $h(x_{1j}) = \tau^{k-1}(x_{1j})$ and maps $(x_{11}, x_{12}, \ldots, x_{1m})$ on to itself cyclically.

Let $h(x_{2i}) = x_{2\rho}$. We claim that $\rho = i \oplus (k-1)$. Suppose $\rho \neq i \oplus (k-1)$. Now $f_{2i} o h^{-1}(x_{2\rho}) = f_{2i}(x_{2\rho}) = 1$ and this gives $f_{2i} o h^{-1}(x_{1\rho}) = 1$. Now $h^{-1}(x_{1\rho}) = x_{1\rho \oplus (n-k) \oplus 1}$ and so, $f_{2i}(x_{1\rho \oplus (n-k) \oplus 1}) = 1$, which is a contradiction to the fact that $f_{2i} \in \delta$. So $\rho = i \oplus (k-1)$ and hence $h(x_{2i}) = x_{2i \oplus (k-1)} = \tau^{k-1}(x_{2i})$ and it follows that $h = \tau^{k-1}$. Thus
\[ GLH(X, \delta) \subseteq K. \] (4)

From equations 3 and 4 we get $GLH(X, \delta) = K$. This completes the proof.

So if $X$ is any set and $L$ be any $F$-lattice containing more than two elements, then the group generated by a permutation which is a product of two disjoint cycles having equal length is $L_l$-representable on $X$. Combining Theorems 5.1 and 2.6 and Remark 3.2, we have the following Theorem.

**Theorem 5.2:** Let $X$ be any set and $\tau$ be a permutation on $X$ such that $\tau$ can be written as a product of two disjoint cycles having equal length $n$ where $n \geq 3$. Then the group generated by $\tau$ is $L_l$-representable on $X$ if and only if $L \neq \{0, 1\}$.

Let $X$ be an infinite set and $\tau \in S_X$. If $\tau$ is an arbitrary product of disjoint cycles having equal lengths then the permutation group generated by $\tau$ is $L_l$-representable on $X$.

**Corollary 5.3:** Let $X$ be any set and $K$ be a subgroup of $S_X$ such that $|K| = p$, a prime number. Then $K$ is $L_l$-representable on $X$.

**Proof:** By Remark 3.4, without loss of generality we can assume that $K$ moves all the elements of $X$. Since $K$ is of order $p$, $K$ is a cyclic group generated by a permutation $\tau$ which is of order $p$. This implies that $\tau$ is a product of disjoint cycles having equal length $p$. So by Theorem 2.6 and Theorem 5.2, $K$ is $L_l$-representable on $X$.  

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Now using Theorems 3.3, 4.1 and 2.5, we can easily prove the following Theorem.

**Theorem 5.4:** Let $X$ be any set and $L$ be an $F$-lattice containing more than two elements. If $\tau$ is any permutation on $X$, which is a product of two disjoint cycles having lengths $m_1, m_2$ where $(m_1, m_2) = 1$, then the permutation group generated by $\tau$ is $L_f$-representable on $X$.

**Proof:** Let $\tau = \tau_1 \tau_2$ where the length of $\tau_1 = m_1$ and length of $\tau_2 = m_2$. Since $(m_1, m_2) = 1$, we have the cyclic group generated by $\tau$ is the direct sum of the cyclic groups generated by $\tau_1$ and $\tau_2$. By Remark 3.4, without loss of generality we can assume that $X = X_1 \cup X_2$ where $X_i$ is the set of all elements of the cycle $\tau_i$ for $i = 1, 2$. Then by Theorem 2.5, the cyclic group generated by $\tau_i$ is $L_f$-representable on $X_i$ for all $i = 1, 2$ when $L \neq \{0, 1\}$. So from Theorem 4.1, it follows that the cyclic group generated by $\tau$ is $L_f$-representable on $X$.

This theorem can be extended to more than two groups and we state this as a corollary.

**Corollary 5.5:** If $X$ is any set and $\tau$ is any permutation on $X$, which is a product of $n$ disjoint cycles having lengths $m_1, m_2, \ldots, m_n$ where $(m_i, m_j) = 1$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. Then the permutation group generated by $\tau$ is $L_f$-representable on $X$.

**Proof:** Let $\tau = \tau_1 \tau_2 \ldots \tau_n$ where the length of $\tau_i = m_i$, $i = 1, 2, \ldots, n$. Since $(m_i, m_j) = 1$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$, $\tau = \tau_1 \times \tau_2 \times \ldots \times \tau_n$. Now take $X = \bigcup_{i=1}^{n} X_i$ where $X_i$ is the set of all elements of the cycle $\tau_i$ for $i = 1, 2, \ldots, n$. Then by Theorem 2.5, the cyclic group generated by $\tau_i$ is $L_f$-representable on $X_i$ for all $i = 1, 2, \ldots, n$ when $L \neq \{0, 1\}$. So from Theorem 4.1, it follows that the cyclic group generated by $\tau$ is $L_f$-representable on $X$.

### 6. Conclusion

In this paper we studied $L_f$-representability of subgroups of symmetric groups. We proved that the direct sum of finite $L_f$-representable permutation groups is $L_f$-representable. But the condition under which the direct sum of infinite $L_f$-representable permutation groups become $L_f$-representable is an open problem. We investigated $L_f$-representability of some cyclic permutation groups. Characterisation of $L_f$-representable cyclic permutation groups is not yet obtained.

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**Notes on contributor**

**Sini P** works as Assistant Professor in Department of Mathematics, University of Calicut. She received her PhD degree from University of Calicut, Kerala, India. Her research interests are in Topology and Fuzzy mathematics.
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