CATEGORICAL ANALYSIS

ZBIGNIEW OZIEWICZ AND GUILLERMO ARNULFO VÁZQUEZ COUTIÑO

ABSTRACT. We propose the categorification of algebraic analysis in terms of a specific 2-category, called here the Leibniz 2-category given by generators and relations which include the Leibniz-like relation (strict 3-cell) among extended 2-cells. The Leibniz 2-category offers the ‘most general’ notion of a (co)derivation’, as a strict 3-cell, for a general (al- co-)gebra, not necessarily (co-)associative, not necessarily (co-)unital, nor necessarily (co-)commutative.

We outline a program in which every 2-cell related to a partial (co-)derivation (called a Leibniz strict 3-cell) is translated into an appropriate 2-cell related to a Cartan’s-like (co-)derivation (called a Cartan strict 3-cell), and vice versa. We found also that a non-Leibniz component (2-cell related to a Leibniz 3-cell), responsible for the stochastic calculus, must be a kind of ternary operation.

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1. Introduction

A general algebra \( A \), not necessarily unital, associative nor commutative, with one partial derivation \( D \in \text{der}(A,A) \subset A^A \) is said to be a differential algebra [Ritt 1950] or a Leibniz algebra.

An algebra \( A \) together with a set (eventually a Lie algebra) \( L \) acting as partial derivations on an algebra \( A, L \rightarrow \text{der}(A,A) \), do possess even a richer historical terminology showing the importance of this concept in algebraic analysis, and in commutative and non-commutative differential geometry. Here is the partial list of a multitude of names, however, the corresponding references are mostly omitted because this historical remark is irrelevant for what follows. A pair of algebras \((A,L)\) is said to be a pseudo-algèbre de Lie [Herz 1953], a differential Lie algebra [Palais 1961], an \( A \)-Lie algebra [Rinehart 1963], a Lie module [Nelson 1967], a Lie-Cartan pair [Kastler & Stora 1985], a Leibniz pair [Flato, Gerstenhaber and Voronov 1995], a Cartan pair [Borowiec 1996], a Lie-Rinehart algebra [Huebschmann 1999].

What is known as the Leibniz ‘rule’, or as the Leibniz condition defining a partial or a Cartan’s derivation, we prefer to call rather as the Leibniz relation, Leibniz’s axiom, in analogy to the presentation in terms of ‘generators and relations’ in universal and linear algebras. Algebraic analysis deals exactly with Leibniz-like relations, \( i.e. \) with various deviations from the strict Leibniz relation in terms of the quasi-Leibniz and non-Leibniz components, or such deviation as there is in a stochastic calculus [Arnold 1974, §5.3-5.4, pp. 89-91; Sobczyk 1991]. It is not our intention, however, to go into stochastic calculus in the present paper. In the standard set theory, if \( A \) is a binary algebra and \( x, y \in A \), then the Leibniz relation for a partial quasi derivation \( D \in q\text{der}(A,A) \) is presented in terms of irrelevant elements in the
Because this holds for all elements of $A$ and for any quasi derivation in $qder(A,A)$, therefore the three symbols, $x$, $y$ and $D$ in (1), are in fact totally irrelevant for this relation. If we drop them what is left? The genuine Leibniz-like relation (among what species, quantities?) is hidden in expression (1). Even more, an entire algebra $A$ must be considered as a variable object (and not only the elements of $A$) because the Leibniz-like relation holds probably for many algebras. We wish to express the Leibniz relation not only as element-free, but also as independent of the choice of an algebra, as object-free, i.e. as a model independent abstract relation, exhibiting exactly what this relation is, and among what species. We see analysis as a part of universal algebra, rather than a part of just linear algebra. If we wish to consider an algebra (a ring) as an irrelevant variable object then we must move from functions and operations to categories and functors. Essentially, we do propose the categorification of analysis, a categorical analog of algebraic analysis, i.e. the categorical analysis. A guiding motto for the passing XX century by Gian Carlo Rota [1998, p. 3] ‘analysis play second fiddle to algebra’, must be supplemented for the next XXI century with ‘algebra play second fiddle to categories’. Categories are an aid to understanding.

1.1. Pedagogical Example. We wish to illustrate the general idea on an example of the bilinear term $xy \in A \in \text{obj cat}$ in (1). First, $xy$ is the value of the element-free multiplication $m_A : A \otimes A \to A$, i.e. $m_A(x \otimes y) = xy$. Next, we do not need to be restricted to some specific bifunctor $\otimes : \text{cat} \times \text{cat} \to \text{cat}$, therefore we will use a symbol $\Upsilon$ denoting the name of an arbitrary bifunctor (with a place on the top for two inputs and a place at the root for one output). In an expression for a value of a multiplication $m_A$, a bifunctor $\Upsilon$ is restricted to (or composed with) a diagonal cofunctor $\lambda : A \mapsto (A, A)$, and this composition (we prefer the name grafting) gives a composed endofunctor denoted pictorially by $\Upsilon \equiv \Upsilon \circ \lambda$. Therefore $m_A$ is a map from an object $\Upsilon A$ to an object $(id) A$ and finally an object-free multiplication $m$ (which we wish that substitute the $xy$ term in (1)) is a (natural) transformation from an endofunctor $\Upsilon$ to an identity functor $\equiv id \equiv id_{\text{cat}}$.

1.2. More History (More History). Hausdorff proposed for noncommutative algebra to keep the Leibniz axiom still in the same form (1) as for a commutative algebra. Gian-Carlo Rota, Sagan and Stein found in 1980 that the Leibniz axiom for non-commutative algebra in
the Hausdorff form \( \Box \) break down the chain rule for partial derivation (derivative), and they proposed an altogether different notion of derivative, which is not a derivation, called the cyclic derivative with the cyclic axiom instead of the Leibniz axiom \( \Box \). The cyclic axiom implies that the chain rule still holds for non-commutative algebra.

Altogether different partial derivations for a not necessarily commutative algebra were proposed by Woronowicz [1989], Majid [993, 1995 §10.4], Oziewicz, Paal & Różański [1995], Borowiec [1996, 1997], see Motivation 5.4 below and the last Section 12.

Our purpose is to describe some examples of the 2-categories given by generating 1-cells and by generating 2-cells. A Leibniz 2-category includes the Leibniz-like binary relation (axiom) on expanded 2-cells, and such axiom is nothing more than a strict 3-cell. Still weaker is a Leibniz 3-category which includes the not strict 3-cells of the modifications of the Leibniz axiom. Another purpose of this paper is to suggest some directions for the study of 2-and 3-categories with Leibniz-like axioms.

2. 2-CATEGORY

The classical works on 2-categories are [Bénabou 1967, Gray 1974; Kelly & Street 1974]. For a recent account we refer to [Baez & Dolan 1998, Batanin 1998].

A set of natural numbers with zero (non negative integers) is denoted by \( \mathbb{N} \ni 0 \). For \( n \in \mathbb{N} \), let \( G_n \) be a collection of ‘\( n \)-cells’, such that \( G_{n+1} \) is a disjoint sum of sets indexed by \( G_n \times G_n \). If \( x, y \in G_n \times G_n \) then the corresponding subset in \( G_{n+1} \) is denoted by \( G_n(x, y) \subset G_{n+1} \). Let \( B \rightrightarrows A \) be a convenient notation for a ‘bundle’ \( (B \times B) \leftarrow A \). An \( n \)-graph is \( G_0 \rightrightarrows G_1 \rightrightarrows G_2 \rightrightarrows \ldots \rightrightarrows G_n \), etc, and similarly for an \( \infty \)-graph.

2.1. Definition. An \( n \)-category is an \( n \)-graph with (strictly or weakly) associative and unital compositions for all \( i \in \{0, 1, \ldots, n-1\} \),

\[
(2) \quad \text{for all } x, y, z \in G_i, \quad G_i(x, y) \times G_i(y, z) \to G_i(x, z).
\]

This means that for each fixed \( x, y \in G_i \times G_i \), a collection \( G_i(x, y) \subset G_{i+1} \) with

\[
\forall \alpha, \beta \in G_i(x, y) \subset G_{i+1}, \quad [G_i(x, y)](\alpha, \beta) \subset G_{i+2}
\]

is a 1-category \( \{\text{objects } = G_i(x, y), \text{morphisms } = \ldots\} \). In particular

\[
\text{cat} \equiv \{\text{objects } = G_0 \rightrightarrows G_1 = \text{morphisms}\}
\]

is a 1-category and a 2-category \( G_{0,1,2} \) is a collection of 1-categories \( \{G_0(x, y)\} \) indexed by \( G_0 \times G_0 \).
For the case $G_i \equiv \mathbb{N}$, the only one considered in this paper, we found convenient for an (i+1)-cell to adopt the name *plant* from botany, because plants will be *grafted*, and not only composed \(2\), (see Section 4). An (i+2)-cell is called a ‘formal instance’ in \cite[§4, p. 199]{KellyLaplaza1980}, or an arrow, or the name of a natural transformation, because models (with respect to n-functors) of the (i+2)-cells are natural transformations of (i+1)-cells. In this paper, for an (i+2)-cell we adopt the name *footpath* (among plants).

A 2-category of plants and his footpaths, with grafting of plants, but with the usual (strict) associative compositions of footpaths \(2\), can be called an *algebraic 2-category with grafting* \(\{\mathbb{N}, \text{plants}, \text{footpaths}\}\). However, we adopt the shorter name ‘grafted club’, where the name ‘club’ was introduced by Kelly \cite[pp. 113–123]{Kelly1972}, although Kelly’s formal definition of a club is not exactly the same as our’s.

3. Plants

If for some $i \in \mathbb{N}$, a collection of i-cells $G_i$ has an extra structure of ‘an algebraic sketch’, *i.e.* is a (strict or weak) monoidal category whose monoid of object is $\mathbb{N}$, with $+$ for the binary monoidal operation and with terminal $0 \in \mathbb{N}$, then an $(i + 1)$-cell is said to be a *plant*, and the collection of all plants $P \equiv G_{i+1}$ is a disjoint collection of 1-categories indexed by $\mathbb{N} \times \mathbb{N}$. Every such 1-category is denoted by $(m \mapsto n)$,

$$\text{Obj } (m \mapsto n) \subset P.$$

In this case, a monoid $\mathbb{N}$ with a collection of $(i+1)$-morphisms $G_{i+1} \equiv \{n \mapsto m\}$, where $n \mapsto m$ is a 1-category, is a monoidal small 2-category. Every plant is an object of a unique 1-category $m \mapsto n$ for some $m, n \in \mathbb{N} \times \mathbb{N}$ and denoted by (Figure 1 suppose strict associativity of $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$)

**Figure 1.**

```
```

All plants (as graphs) are implicitly directed from the top (input: leaves from $\mathbb{N}$) to the bottom (output: roots from $\mathbb{N}$). Plants do not have outer nodes, every node is inner. One can think that a plant $\in P$ is the *name* of a functor, because models of plants are many variable and many-valued multi-functors often of mixed variances. Our plant is almost the same as a ‘shape’ introduced by Kelly and MacLane \cite[p. 102]{KellyMacLane1971}.
In this paper we can rather restrict ourself to the case of one sort (or one color), \textit{i.e.} in the realization to just one (but any) category \textbf{cat} with structure (one exception will be discussed separately late on). In such cases of one sort, the edges need not to be colored, and in a model category \textbf{cat}, all edges of plants are the same identity functor \( \text{id} \equiv \text{id}_{\text{cat}} \). A plant \( \text{id} \in (1 \mapsto 1) \) given by a vertical dash on Figures 2 & 4 (or any other edge without vertices) in a model category \textbf{cat} is mapped to \( \text{id} \equiv \text{id}_{\text{cat}} \).

A plant on Figure 1, given by a directed tree with vertices, with \( m \) free leaves and \( n \) free roots, \textit{i.e.} an element of a category \( n \mapsto m \), in a model is mapped to a multi-variable and multi-valued functor (or morphism): input \( \rightarrow \) output, \( \text{cat}^{x_n} \rightarrow \text{cat}^{x_m} \). In particular, any endofunctor on \textbf{cat}, different from \( \text{id} \), like the adjoint in a compact closed category does represent a plant from \( 1 \mapsto 1 \) with exactly one input and one output and with exactly one vertex (labelled if needed) in the case of not being composed.

A category with one object and one (identity) morphism is said to be the zero category \( \text{zero} \equiv \text{cat}^{x_0} \) (some authors do prefer the name unit category, as in \cite{Kelly 1972, p. 74}). For example, if \( \mathbb{k} \) is a field, then in the category \( \mathbb{k}\text{-vec} \) of the vector \( \mathbb{k} \)-spaces, we have \( (\mathbb{k}\text{-vec})^{x_0} \equiv \mathbb{k} \), whereas in the category of sets \( \text{set}^{x_0} \equiv \emptyset \). A category \( 1 \mapsto 0 \) possesses the unique plant which we call \textit{killer}. The killer in a model is represented by the functor from \textbf{cat} to the terminal \textbf{zero}.

\[
\begin{array}{cccccccc}
\text{killer} & \text{creator} & \text{const} & \text{id} & \text{nilpotent} & \text{unipotent} & \text{idempotent} \\
\bullet & & & & & & \\
1 \mapsto 0 & 0 \mapsto 1 & 1 \mapsto 1 & & & & \\
\Uparrow & & & \Downarrow & & & \Downarrow \\
2 \mapsto 1 & & & & & \Downarrow & \Downarrow \\
& & & & 1 \mapsto 1 & & \\
\end{array}
\]

\textbf{Figure 2.} Some plants and some axioms.

A plant from the category \( 0 \mapsto 1 \) we call a \textit{creator}. In a model, a creator is represented by a functor from initial \textbf{zero} to \textbf{cat} and is identified with some object of \textbf{cat}. These plants are given on Figure 2 by edges with one vertex, they are objects (elements) of categories \( 1 \mapsto 0 \) and \( 0 \mapsto 1 \) respectively, what for the reader convenience is indicated on the bottom of the plants. The same convention of indicating a category, to which a given plant belongs, at the bottom of a plant, is used on Figures 3 & 4.
A plant on the right in the second row on Figure 2 in realization is a bifunctor \( \mathbf{cat}^\times \mathbf{2} \to \mathbf{cat} \), and a plant on the left in realization is a ‘cobifunctor’ \( \mathbf{cat} \to \mathbf{cat}^\times \mathbf{2} \). We wish to have a functorial calculus with explicit functors whose co-domain is a product \( \mathbf{cat} \times \mathbf{cat} \), like the diagonal functor \( \mathbf{cat} \to \mathbf{cat} \times \mathbf{cat} \) given by \( \mathbf{cat} \ni S \mapsto (S,S) \).

A functor \( \mathbf{cat} \times m \to \mathbf{cat} \times n \) is the same as an \( n \)-tuple of functors \( \mathbf{cat} \times m \to \mathbf{cat} \). A functor \( \mathbf{cat} \to \mathbf{cat} \times n \) is an ordered set (of cardinality \( n \)) of \( n \) endofunctors, therefore there is a bijection of functor categories,

\[
(\mathbf{cat}_A \times \mathbf{cat}_A)^{\mathbf{cat}_B} \simeq (\mathbf{cat}_A)^{(\mathbf{cat}_B + \mathbf{cat}_B)}.
\]

Here + is the disjoint sum (coproduct). This means that we can get by with a calculus that has no explicit place for a disjoint sum of categories - and that is precisely what we wish.

4. Grafting

In the case, when \( G_i = \mathbb{N} \) is a monoid, we wish to generalize the composition \( \otimes \) to a multivalued operation for which we adopt the name grafting from botany. Consequently, the objects of a 1-category \( n \mapsto m \) \(((i+1)\text{-cells}, (i+1)\text{-morphisms})\), we call plants (processes, etc), because plants are going to be allowed to be ‘composed’ in a more sophisticated way than compositions of morphisms. Let \( P \) be a collection of plants, \textit{i.e.} a collection of 1-categories \( \{m \mapsto n\} \) indexed by \( \mathbb{N} \times \mathbb{N} \).

A multivalued operation of grafting generalize and unify composition, substitution and concatenation. The grafting of plants is a specific symmetric map,

\[
\text{grafting in an orchard: } P \times P \longrightarrow 2^P.
\]

The above map generates an expanded grafting, \textit{i.e.} an associative, unital and commutative binary operation \( 2^P \times 2^P \longrightarrow 2^P \). One can graft any two plants and they do not need to be necessarily of the same arities. Grafting a plant from a category \( n \mapsto m \) with a plant from \( p \mapsto q \) (with no restrictions like \( m = p \) and so on at all!) gives no more than \( n + p + m + q \) different derived plants. However, these derived plants are scattered among different categories \( \{n+p \geq r \mapsto s \leq m+q\} \).

The grafting is a collection of \( n \)-graftings, say, components of grafting. However, in general, any specific \( n \)-grafting can be still multivalued. In particular a 0-grafting is the same as a 2-valued concatenation.

Instead of a formal definition of grafting we will explain what we have in mind on two examples.

4.1. Example. Grafting a plant \( \Upsilon \in (2 \mapsto 1) \) with a plant \( \lambda \in (1 \mapsto 2) \) (in any order, because grafting is the commutative ‘operation’) gives
six derived plants as shown on Figure 3. All these derived plants are two-letter words without repetition of the same letter twice.

\[ \begin{align*}
3 &\mapsto 3 \\
2 &\rightarrow 2 \\
3 &\mapsto 3 \\
2 &\mapsto 2 \\
1 &\mapsto 1 \\
2 &\mapsto 2
\end{align*} \]

\textbf{Figure 3.} Two-letters derived plants in a garden.

To get the above six derived plants one must ‘rotate and graft’ one plant around another (without rotating the plants itself) up to isotopy (of the kind of Reidemeister moves). Therefore, concatenation of plants giving two composed plants from \(3 \mapsto 3\) on Figure 3 (the only not connected graphs or 0-grafted), can be considered as a particular case of a grafting, namely this is a 0-grafting. In Figure 3 we have three composed plants of type \(2 \mapsto 2\) derived by 1-grafting and one composed plant from \(1 \mapsto 1\) derived by 2-grafting. The derived plants on Figure 3 are ordered according to rotation.

\textbf{4.2. Example.} The grafting of a plant from \(1 \mapsto 1\) with itself gives only two new plants, one from \(1 \mapsto 1\) (composition = 1-grafting) and another from \(2 \mapsto 2\) (concatenation = 0-grafting).

The (free) generators of all plants are said to be fundamental (or basic) plants (alphabet of plants), and similarly the generators of all footpaths are called fundamental or basic footpaths (alphabet of footpaths). An alphabet of plants generates a free club of derived plants, derived 1-cells [Kelly 1972, §3, p. 116, 1974] (or an operad, garden, forest, orchard), made from fundamental plants by iterated grafting, generalizing and unifying substitutions, compositions and concatenations.

An alphabet of plants does freely generate by iterated graftings an infinite garden-forest of derived plants-words (\(\mathbb{N}\)-graded by the number of letters, \(\mathbb{N}\)-graded by the number of graftings and \(\mathbb{N} \times \mathbb{N}\)-graded by input & output). The set of all plants derived in this way, with the multi-valued operation of grafting, is said to be a free club (or operad) presented by this alphabet. (We feel that this formalism needs also co-grafting, a split plant \(2^p \rightarrow 2^p \times 2^p\); however, this is another story.)

\textbf{4.3. Definition.} An \(n\)-category \(G\) (or an \(\infty\)-category \(G\)) is said to be a \textit{plant-like} if the following two conditions holds:
(i): For some $i \in \mathbb{N}$, $G_i \equiv \mathbb{N}$, i.e. $G \equiv \{\ldots, \triangleright G_{i-1} \triangleright \mathbb{N} \triangleright (P \equiv G_{i+1}) \triangleright G_{i+2} \triangleright \ldots\}$.

(ii): The usual partial compositions (2) for plants $P \equiv G_{i+1}$ are extended to multivalued grafting $P \times P \rightarrow 2^P$ (3).

If $G$ is a plant-like $n$-category with $G_i \equiv \mathbb{N}$, then a $(i + 1)$-cell in $P \equiv G_{i+1}$ is said to be a plant.

5. Why?

Why [should one] use graphs (tangles, hieroglyphics) jointly with the techniques of naming objects by letters from the Greek or Latin alphabets?

Graphs convey more information than these letters. The techniques of naming objects and arrows (elements, functors, morphisms, functions, etc) do allow a large amount of the most essential but routine detail to be hidden, like changing street’s names after a political revolution.

Graphs lead to drawing pictures as in this paper, which display the relationships between various operations, functors etc. This must be contrasted with the usual lists of Greek or English letters and equations which frequently convey nothing at all to the reader. Compare for example graphs from Figure 11 below, with a Table of the letters after Motivation 6.4. If there is nothing else, we do like graphs and it is easier to spot errors in a graph than in a list. We agree with Dieudonné: ‘half the success (in mathematics) depends on a proper choice of notation’. Unfortunate notation might kill fortunate ideas.

6. The Leibniz alphabets: plants and footpaths

In what follows we will need an alphabet consisting of four plants: $id \in (1 \mapsto 1)$, a killer, a binary tree and a two-rooted tree as shown on Figure 4. It is convenient to call this alphabet the Leibniz alphabet - this choice is motivated at the end of this Section.

![Figure 4. Alphabet of plants.](image)

The Leibniz alphabet of plants given by Figure 4, generates by iterated graftings a free algebra (club) on four plants, the garden of derived plants (terms), and a small fragment of this garden, relevant for what follows, namely the fragment of type $2 \mapsto 1$ is shown next.
Among the seven plant-words on Figure 5, five of them are grafted from three letters.

6.1. Definition. If $G$ is plant-like $n$-category with $G_i \equiv \mathbb{N}$, then an $(i + 2)$-cell $\in G_{i+2}$ is said to be a footpath. Related names: formal instance or arrow [Kelly & Laplaza 1980, p. 199]. A footpath is a map which does not change the arity of a plant,

$$(m \mapsto n) \ni \text{plant}_1 \mapsto \text{plant}_2 \in (m \mapsto n).$$

A not invertible footpath is said to be ‘lax’, a twosided invertible footpath also is said to be a ‘pseudo relation’, and a strict (equivalence) relation is the same as an identity $\approx$.

In what follows we use the convention that not named paths are necessarily twosided invertible. A model of a footpath is a natural transformation of functors, more about models is in Section 11.

We need an alphabet of footpaths. We wish first to discuss the following footpaths as a fundamental one (a motivation for this choice will be explained later on),

The above footpaths we call appropriately: a binary algebra $m$ (a multiplication), a binary cogebrda $\triangle$ (a comultiplication), the left and the right actions $l \& r$, the left and the right co-actions $L \& R$. A path $u$ is said to be a left unit for an algebra $m$ (or an algebra $m$ is said to be a left $u$-unital) if a composed 2-cell $m \circ (u \times \text{id})$ is twosided invertible. We do not yet wish to impose this pair of two strict 3-cells-axioms, because exists the competing possibility of a weaker left unit in the case that this composed 2-cell is only one-sided invertible. We are going to motivate these names next. Note that, disregarding
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universality, $l \ & r$ looks like a product, i.e. as a pair of ‘projections’. Similarly $L \ & R$ looks like a coproduct i.e. a sum.

We will explain an interpretation of these footpaths on an example of the particular functorial realization of the plants $\gamma$ and $\lambda$. This example will give a feeling, what these footpaths could mean in the usual life of.

A plant $\gamma$ can be realized as some bifunctor $\mathbf{cat}^2 \rightarrow \mathbf{cat}$. For example in a category of sets $\mathbf{set}$, for $A, B \in \text{obj } \mathbf{set}$, we could have for example

$\cd$:

- a cartesian product, $\gamma(A, B) = A \times B$.

In a category of bimodules this is the bifunctor of the tensor product $\otimes$.

$\cd$:

- an exponentiation, $\gamma(A, B) = B^A$,

  or more generally $\gamma = \text{hom}$ bifunctor, $\gamma(A, B) = \text{hom}(A, B)$.

$\cd$:

- any derived bifunctor as $\gamma(A, B) = 2^A \times B$, etc.

Let for simplicity $\lambda$ be the diagonal, $\lambda : \text{obj } \mathbf{cat} \ni S \mapsto \lambda(S) = (S, S)$. A 2-grafting $\lambda$ with $\gamma$ gives the unique endoplant $\check{\phi} : \mathbf{cat} \rightarrow \mathbf{cat}$. For a choice in a $\mathbf{set}$, $\check{\gamma} = \times$, an endoplant $\check{\phi}$ is given by the value $\check{\phi}(S) = S \times S$ (the obvious values on morphisms are omitted). The value of the plant ‘$|$ ≡ $\text{id}$’ on $S \in \text{obj } \mathbf{cat}$, (or the evaluation of $S$ on the plant ‘$|$’) must be $S$. Therefore, in this case, the value of $S$ on a footpath $m$ (or vice versa) is a map $m_S : S \times S \rightarrow S$, i.e. $m_S$ is a binary multiplication. A domain of a footpath $m$ is said to be a type of algebra $(S, m_S)$, and in the above example, this is a binary algebra.

6.2. Note. A footpath can act on a grafted plant independently of his action on an alphabet of plants. In the example above, a footpath $m$ do not change neither $\gamma$ nor $\lambda$ whereas is transforming their composition $\check{\phi}$.

Besides of the above six fundamental arrows $\{m, \triangle, l, L, r, R\}$, we will need in what follows, also two more footpaths, not necessarily invertible, as shown on Figure 6,

\[
\begin{align*}
\text{exp} &
\end{align*}
\]

\begin{center}
\text{Figure 6. Two more footpaths needed.}
\end{center}

An expansion on Figure 6 (and further on Figures 8-11, 13, 15 etc.) is a particular case of a pasting introduced by Bénabou in 1967.

The footpaths on Figure 6 might be consequences of the following stronger conditions: the strict mitosis for the diagonal functor (Figure
8) and the strict associativity for a bifunctor $\gamma$. The lax associativity means commutativity of the MacLane pentagon on Figure 9, where a footpath ‘a’ is stronger than that on Figure 6. However, in what follows we will need explicitly only the footpaths from Figure 6.

![Figure 7](image1.png)

**Figure 7.** Footpaths ‘m’ and ‘c ◦ l’ almost coincide on a plant $\Phi$, $c \circ l \approx m$ & $L \approx \Delta \circ c$.

![Figure 8](image2.png)

**Figure 8.** Strict mitosis.

All arrows are expandable to the appropriate derived plants. An alphabet of footpaths provide generators (of a partial monoid) of expanded footpaths (‘multi-arrows’, expanded arrows). This includes in particular a pasting introduced by Bénabou [1967]. Footpaths need not to be always composable.

Grafting on plants and the pasting and the expansion on footpaths is a functor from a 2-category of alphabets to a free 2-category,

\[
\{\mathbb{N}, \text{alphabets of plants & footpaths} \} \rightarrow \{\mathbb{N}, \text{derived plants & pasted footpaths} \},
\]

i.e. 0-cells are the same, but 1-cells are to be derived and 2-cells are to be expanded.

### 6.3. Bigebra (Bigebra)

A famous expansion of an associativity path $a$ provide the MacLane and Stasheff pentagon [MacLane 1963].

![Figure 9](image3.png)

**Figure 9.** The MacLane & Stasheff pentagon.

A particular example of an expansion of an algebra and cogebras paths $m$ & $\triangle$ is shown on Figure 10. This expansion with an appropriate axiom for 2-cells is said to be a $\sigma$-braided bigebra.
The fundamental footpaths \( \{m, l, r\} \) can be expanded as follows (an expanded footpath from Figure 6 is not shown for simplicity on Figure 11 where the ‘heptagon’ must possess 11 edges).

6.4. Motivation (Motivation). The above choices of the Leibniz alphabets of plants and of footpaths have the following motivation.

Borowiec [1996, 1997] proposed an altogether different notion of a partial derivation for a not necessarily commutative algebra. Borowiec postulated the classical Leibniz strict 3-cell-axiom for the Cartan bimodule-valued derivation \( d \in \text{der} (m, m_l \& m_r) \) and for a bi-module-sourced co-derivation \( \delta \in \text{der} (\Delta_l \& \Delta_r, \Delta) \) [Borowiec & Vázquez Coutiño], in the framework of the calculus of the differential forms for a not necessarily (co- bi-)commutative (al- co- bi-)gebra \( m \& \Delta \) [Woronowicz 1989]. Then Borowiec derived the ‘correct’, but still an altogether different notion of a partial derivation (which in our terminology is an example of a 3-cell), such that in the case of a commutative algebra the
Borowiec partial derivation coincides with the classical Leibniz partial derivation.

Please note that the third term in the expression \( \Box \) needs a permutation of letters \( D \) with \( x \), (or sometimes a cyclic permutation as in [Przeworska-Rolewicz 1995 [36], p. 779, formula (3.9), where \( c \) must read \( x \)]. This is reflected in the functorial calculus in the necessity of introducing a braid plant - functor \( \times : \text{cat}^{x^2} \rightarrow \text{cat}^{x^2} \). Contrary to this, the Borowiec partial derivation do not need permutation, at the cost that in the first two terms in \( \Box \) \( D(xy), Dx \in A \) as usual; however, the third term, contrary to the habit, must be \( (Dx)y \) with \( Dx \in L \). Then Borowiec is proving that the classical Leibniz relation \( \Box \) appears to be the consequence of the commutativity in an algebra. The quasi-Leibniz term in \( \Box \) needs besides the braid \( \times \), and a \( D \)-dependent family of multiplications, also duplication of the letter \( D \). In order to understand the Borowiec results we do not need braid plant \( \times \) and in this paper we restrict the attention to the Leibniz-like axioms which do not involve \( \times \).

In order to categorificate the Leibniz-like 3-terms relation for a partial derivation as invented by Borowiec [1996, 1997] and investigated in [Borowiec & Vázquez Coutiño] for a not necessarily commutative algebra, it appears that we need exactly an alphabet of plants given by Figure 4, and the fundamental paths as presented above in this Section.

Altogether we have the following seven paths from plant \( \downarrow \) to plant \( \downarrow \), and one must read them from left to right,

| Leibniz’s paths | \( l \circ (\text{id} \times m) \circ a \) | \( m \circ (l \times \text{id}) \) | \( c \circ l \circ (l \times \text{id}) \) |
|------------------|----------------------------------|-------------------------------|----------------------------------|
| Borowiec’s paths | \( l \circ (\text{id} \times c) \circ (r \times \text{id}) \) | \( c \circ l \circ (r \times \text{id}) \) |
| Stochastic paths | \( m \circ l \circ a \) | \( c \circ l \circ l \circ a \) |

On Figure 11 there are two paths, through the dashed vector, involving the footpath \( l \) represented as a ternary operation ‘of a type \( 3 \mapsto 2 \)’. We call them the stochastic paths (or non-Leibniz paths), because they are absent in the standard Leibniz axiom and because they are responsible for the stochastic differential calculus [Arnold 1974, §5.3-5.4, pp. 89-91; Sobczyk 1991].
In what follows, for simplicity, we identify on a plant \( q \), the footpath \( m \) with \( c \circ r \), see Figure 7. The footpaths from Figure 11 in a model category are represented as operations whose graphs are shown on Figure 12.

\[
\begin{align*}
&\text{The Leibniz operations.} \\
&\text{Borowiec’s operations.} \\
&\text{A stochastic operation is ternary.}
\end{align*}
\]

**Figure 12.**

7. **The Cartan Club**

An extended algebra \( m \mapsto m_l \& m_r \), is said to be a \( m \)-bimodule. An extended cogebera \( \Delta \mapsto \Delta_l \& \Delta_r \) is said to be a \( \Delta \)-bicomodule. An extension of a 2-cell is given by a set of 3-cells (axioms if they are strict) [Eilenberg 1948, Gugenheim 1962, Kelly 1972, p. 94, Cuntz & Quillen 1995]. A Cartan’s-like club (the Cartan differential calculus of the differential forms) consists, roughly speaking, of a ‘derivation’ \( d \in \text{der} (m, m_l \& m_r) \) with target in a \( m \)-bimodule \( m_r \& m_l \), or/and of a ‘coderivation’ \( \delta \in \text{der} (\Delta_l \& \Delta_r, \Delta) \) with source in a \( \Delta \)-bicomodule \( \Delta_r \& \Delta_l \).

Here we use the symbol ‘der’ to denote a Leibniz-like (quasi) axiom for both directions, either for a ‘derivation’ as well as for a ‘coderivation’, although Borowiec & Vázquez COUTINO proposed a longer symbol ‘coder’.

The following statements are equivalent:

(i) \( d \in \text{der} (m, m_l \& m_r) \)

(ii) the set of 2-cells \( \{d, m, m_l, m_r\} \) is related by a 3-cell ‘der’.

A 3-cell ‘der’ is just a particular Leibniz-like axiom. A categorification of analysis is an interpretation of ‘der’ as a 3-cell (or as a set of the 3-cells), *i.e.* as a modification(s) in the terminology of Bénabou [1967].

Limited space does not allow to discuss 3-cells in this paper. Therefore our purpose is limited to the discussion of 2-cells only.

The Cartan club does need, besides the Leibniz alphabet given on Figure 4, just one more plant which must be a creator. Altogether, we need five fundamental plants. The most important fragment of the
derived garden of type \(1 \mapsto 1\), what we call the Cartan garden, is shown on Figure 13, together with the all involved now fundamental footpaths \(\{d, m, m_t, m_r, \ldots\}\).

A fundamental footpath denoted by \(d\), subject to the appropriate axiom, is going to be the Cartan bimodule-valued derivation with respect to the triple \(\{m, m_r, m_l\}\). The Cartan footpath \(d\) looks like a ‘gluing slot’, however the right interpretation must be that \(d\) is ‘transforming’ a constant functor into the identity functor.

In the Cartan garden on Figure 13 we have, among other, three classical Cartan’s paths, and also a path which we call ‘the stochastic differential’ - this is non trivially expanded from the Cartan fundamental footpath \(d\).

The two plants are named sink and source, and these names correspond to a ‘derivation’ arrow \(d\). For a coderivation \(\delta\) these names must be interchanged: sink ↔ source.

7.1. **Adjoint plant.** The Cartan-Leibniz and Leibniz-Cartan dictionaries, which will be given in the next Sections, need two extra fundamental plants, which, strictly speaking belongs neither to the Leibniz garden, nor to the Cartan garden, however must be in a dictionary. The first extra plant is \(\uparrow \in (1 \mapsto 1)\), to be refered as an adjoint with respect to a unital binary plant and some suitable footpaths.
A unit & counit for the binary plants \( Y \& \lambda \) is the same as an invertible (co-)modul paths, as for example the appropriate 2-cells on Figure 13 in case they are invertible. We keep the convention that not named paths are necessarily twosided invertible, as for example the invertible paths on ‘cyclic pentagons’ with the expanded evaluation & coevaluation on Figure 15.

The following definition was essentially invented by Kelly & Laplaza [1980].

Let a creator \( 1 \in (0 \mapsto 1) \) be a unit for a binary plant \( Y \). A plant \( 1 \in (1 \mapsto 1) \) is said to be left adjoint with respect to a unital \( Y \& \lambda \), given jointly with \( \lambda \), if there exist a pair of fundamental footpaths called left evaluation \( \text{ev}_l \), and a left coevaluation \( \text{coev}_l \), such as shown on Figure 14.

![Figure 14. Two distinct closed structures: a plant \( 1 \) is said to be a left or a right adjoint wrt to unit creator \( \in (0 \mapsto 1) \).](image)

The two fragments of the garden \( 1 \mapsto 1 \) on Figures 14 & 15 with evaluation and coevaluation, and with not named paths being invertible unit & counit is known as the closed structure [Kelly & Laplaza 1980, p. 193]. For evaluation & coevaluation paths is necessary that a binary plant \( Y \) be unital.

In [Kelly & Laplaza 1980] the evaluation is called a counit and denoted by \( e \), and the coevaluation is called a unit and denoted by \( d \) (what can not be accepted here because we wish to keep the symbol \( d \) for a footpath which under an additional axiom play the role of the traditional bimodule-valued Cartan derivation). However, a twosided ‘counit’ for a plant \( \lambda \) is a pair of identities given by mitosis on Figure 8.
7.1. **Geometry** (Geometry). In the most interesting applications in functorial realization appears to be that plants \(\uparrow\) and \(\mid \in (1 \mapsto 1)\) do have different variance. In this case, a path between them must be modeled by, what is called, a Barr dinatural transformation [Paré & Román 1998] (see Section 10 for more details). We use for this footpath, on Figures 15 & 16, the name ‘geometry’, because in a particular important example of different variance, the geometry footpath is responsible for a Riemann-like structure (a scalar-like product) in the riemannian or euclidean geometries. A geometry footpath would be necessary if we would like to have the concepts of gradient and rotation in the differential calculus. However, the riemannian differential geometry is outside the scope of the present paper and a geometry footpath will not be considered in what follows.

An unipotent plant \(\uparrow\) (Figure 2) is also said to be reflexive. When drawing the right evaluation pentagon (Figure 15) we assumed implicitly, just for simplicity, that an adjoint plant \(\uparrow\) is reflexive.

7.2. **Translators.** The second extra plant, besides of an adjoint \(\uparrow\), must be another creator \(\star \in (0 \mapsto 1)\) labelled by a star to be distinguished from the first creator labelled by bullet \(\bullet \in (0 \mapsto 1)\). We are showing the fragment of a garden expanded by two extra ‘dictionary’ plants, together with suitable new fundamental ‘dictionary’ footpaths, on Figure 16.

**Figure 15.** The pentagons of evaluation for a reflexive left adjoint.

**Figure 16.** The Cartan paths on the left and the translators.

We need also to give an excuse to the reader for denoting ‘translators’ paths by \(d^* \& \delta^*\). These arrows correspond to the case when a model for a plant \(\uparrow\) is a contra-variant functor, however, such choice was
not yet made. Moreover, such notation suggests that the translators paths must be related to the Cartan fundamental paths $d$ & $\delta$. At this moment we prefer to consider such notation to be nothing more than a convenient one. Possible not trivial interconnections might appear after we introduce a Cartan-Leibniz & Leibniz-Cartan dictionary.

8. Dictionary: from Cartan 2-cells to Leibniz 2-cells

In order to translate the Cartan (‘bimodule valued’) fundamental path $d$ (as well as $\delta$ with source in a cobimodule) into the Leibniz-like derivation, we need the translator paths $\delta^*$ and $d^*$ from Figure 16.

\[ \delta^* \times d \quad \text{ev}_l \quad \text{ev}_r \quad \text{coev}_r \quad \delta \times d^* \]

\[ \begin{array}{ccc}
\text{l} & \text{L} & \text{r} \\
\circ & \circ & \circ \\
\text{coev}_l & \text{ev}_l & \text{coev}_r \\
\end{array} \]

**Figure 17.** The Leibniz 2-cells from Cartan.

Figure 17 contain a subtle point, not shown for simplicity. The evaluation & coevaluation paths can hold for a unital binary plant $\gamma$ only. Therefore a creator $\uparrow \in (0 \rightarrow 1)$ on Figure 17 must be unit for evaluation & coevaluation paths (as on Figures 14 & 15). However $\uparrow$ needs not to be a unit for other paths on Figure 17. Therefore in fact, what is missing on the top of Figure 17 is a path ‘ch’ between two different binary plants $\gamma$, which must be labelled (alternatively this can be traced to the change of the categories),

$$( \gamma \text{ for which } \uparrow \text{ is not unit}) \xrightarrow{\text{ch}} (\text{unital } \gamma \text{ with unit } \uparrow).$$

The Figure 17 is known in the literature as ‘the Cartan formula’. For example, we can recover a left action $l$ and a left coaction $L$ from the Cartan garden, formally as follows

$$l \equiv \text{ev}_l \circ \text{ch} \circ (\delta^* \times d), \quad L \equiv (d^* \times \delta) \circ \text{ch}^{-1} \circ \text{coev}_r.$$

However, it is not clear here what could mean $\text{ch}^{-1}$?

Figure 17 is a categorification of a naive definition of a partial ‘derivation’ $\partial_\mu$ from the Cartan ‘derivation’ (we do not yet use any Leibniz-like axiom), viz.,

$$df \mapsto \partial_\mu f = (df) \partial_\mu \quad (= \text{ev}_l[(\delta^* \partial_\mu) \otimes (df)])).$$

In the classical differential geometry a translator $\delta^*$ on Figure 16 is denoted by $i \equiv \delta^*$, and moreover the following strict 3-cells-axioms
holds (i.e. $d^* \& \delta^*$ are pseudo relations),

\[
d^* \circ \delta^* \approx \text{id} \quad \text{and} \quad \delta^* \circ d^* \approx \text{id}.
\]

(4)

An unsolved problem in [Borowiec & Vázquez Coutiño] can be reformulated equivalently as the word-like problem of a ‘compatibility’ of the strict 3-cells (II) with a Leibniz 3-cell ‘der’.

9. Dictionary: from Leibniz 2-cells to Cartan bimodule

In order to translate the Leibniz-like (right or left) (co)derivations \{r, l, R, L\} into the Cartan fundamental paths \(d\) (target in a bimodule \(m_r, m_l\)) & \(\delta\) (source in a cobimodule \(\Delta_r, \Delta_l\)), we need the translator paths \(d^* \& \delta^*\) from Figure 16.

\[
\text{Figure 18. Heptagon: from Leibniz to Cartan.}
\]

In Figure 18, for abbreviation, we let a ‘unit’ \(u\) stand for the composition of a left genuine unit \(-1\) for a bifunctor \(\Upsilon\) with a virtual unit \(v\) for an algebra \(m\) (virtual because we are not yet sure that we need a compatibility axiom with a path \(m\): invertibility of the composed 2-cells \(m \circ (v \times \text{id}) \& m \circ (\text{id} \times v)\), i.e. we have the following abbreviations:

\[
u = (\text{virtual unit for an algebra } m) \circ (\text{a left unit } ^{-1} \text{ for a plant } \Upsilon),
\]

\[
\varepsilon = (\text{left unit for a plant } \Upsilon) \circ (\text{a virtual counit for a cogebera } \Delta).
\]

Figure 18 is the categorification of a well known naive ‘definition’ of the Cartan derivation \(d \in \text{der}(A,M)\) in terms of the given partial derivations, viz.,

\[
\partial_u \mapsto df \equiv (dx^\mu)(\partial_\mu f).
\]

The above ‘definition’ is like a perpetum mobile because the differential \(df\) is ‘defined’ again in terms of the differentials \(dx^\mu\) and a cycle is closed.
9.1. **Partial braid.** On the way, from the given 2-cells in the garden for the partial derivation 3-cell, to the Cartan club, there is the need of the direct constructions of an extended gebra bi(co-)module 2-cells-paths \( \{m_r, m_l, \triangle_r, \triangle_l\} \) from the actions \( \{r, l, R, L\} \). The crucial step for this is the construction of the derived 2-cells-paths of ‘partial braiding’ of \( \uparrow \) with \( | \), where a creator \( \uparrow \) is *not* a unit for \( \Upsilon \), as shown here

\[
\begin{array}{c}
\bullet \\
\uparrow
\end{array} \quad \xrightarrow{b} \quad \begin{array}{c}
\bullet \\
\downarrow
\end{array}
\]

One can show that \( b \) can be given by the right action \( r \), and \( b^{-1} \) analogously by the right coaction \( R \). However, limited space does not allow to present this construction here in detail. Also, limited space does not allow to present the deeper consequences of our assumptions.

10. **Axioms, word problem, coherence**

Axioms generate binary relations in the set of all expanded foot-paths. The determination of the category-congruence generated by axioms relating footpaths is almost the Thue classical word problem [or the Birkhoff problem in universal algebra: to determine all other ‘equations, or identities’ as the consequence of the given axioms], except that footpaths (our letters) are not always composable. These two word problems, for plants and for footpaths, are called jointly the coherence problem for a free club [Kelly & Laplaza 1980, §4 p. 198 and §10, p. 211].

Let \( \approx \) be a binary relation among derived plants. As is usual for the word problem, we use ‘relation’ also for an element of an actual binary equivalence relation, *i.e.* also for an element \( \in \approx \), like the examples of generating relations on Figure 2. An equivalence binary relation \( \approx \) on derived plants is said to be a congruence if \( \approx \) is compatible with grafting, *i.e.* if a grafting of a related pair from \( \approx \) gives again the related pair from \( \approx \). A category-congruence is generated from the given relations by means of the Birkhoff’s rules of derivations, see e.g. in [Graczyńska 1998, p. 13]. The determination of this congruence is the classical word problem of Thue and this is a first part of the coherence problem for a club. A free club factored by a congruence \( \approx \) gives the quotient category.

Axioms among plants (axioms generating the binary equivalence relations \( \approx \) on plants, axiom \( \in \approx \)), by definition, are allowed within the same category only

\[
\approx \subset (n \mapsto m) \times (n \mapsto m).
\]
No relations are allowed between plants of the different arities. Axioms among plants, as the three examples on Figure 2 within a category 1 → 1, do express the properties of the involved plants in terms of the derived plants.

An axiom is a particular case of the more general concept of a modification, introduced by Bénabou in 1967. According to this notion, an axiom is the same as a strict modification. A weaker version is a quasi-axiom = a quasi-modification, which is an invertible modification, and still weaker is the so called ‘lax’ version (most general) which is said just to be a modification.

10.1. Example of a Leibniz 3-cell. The Leibniz relation is a 3-cell in an abelian operad generated by an alphabet of footpaths. Because of very limited space, we will give an illustrative example only. Let on Figure 12, \(D\) be a composition of a fixed element of \(L\) (a creator in a model) with a left action \(l\). Then, a classical example of a strict 3-cell for the set of 2-cells \(\{D, m, \psi\}\) is illustrated on Figure 19 in a model, \(m \equiv m_A, D \equiv D_A, \psi \equiv \psi_A, \)

\[
\begin{array}{c}
\xymatrix{A \\
D \\
A}
\end{array} = \begin{array}{c}
\xymatrix{D \\
\quad m \\
\quad m}
\end{array} + \begin{array}{c}
\xymatrix{\psi \circ \bullet \\
\quad \quad \quad m}
\end{array}
\]

Figure 19. \(\psi\)-skew algebra derivation \(D_A \in \text{der} (m_A)\)

The first two terms (composed 2-cells) on Figure 19 are precisely the Leibniz operations from Figure 12. However, the last term needs a braiding of objects \(L\) with \(A\), not included into our simplest alphabet of footpaths in the present paper.

10.1. Problem (Problem). One of the open ‘higher order’ problems in a categorical & algebraic analysis is the systematic investigations of a weaker (lax) version of the Leibniz-like axioms (lax axioms for 2-cells, either for partial derivatives as well as for the Cartan bimodule-valued derivations), and the determination of the new strict axioms for the Leibniz modifications, i.e. the strict axioms for the Leibniz’s 3-cells. This would be the alternative, even the best, way to understand the non-Leibniz component in [Przeworska-Rolewicz 1995].

11. Functorial realization: functorial models

Eilenberg and Mac Lane in 1945 introduced the notions of category, functor and natural transformation of functors. In a functor category \((\text{cat}_B)^{\text{cat}_A}\), the set-valued hom bifunctor is denoted by \(\text{nat}\). Let \(f, g\)
be two functors from $\text{cat}_A$ to $\text{cat}_B$, $f, g \in \text{obj} \{(\text{cat}_B)\text{cat}_A\}$. A natural transformation $t \in \text{nat}(f, g)$ is an application $t : \text{obj cat}_A \rightarrow \text{cat}_B(f \cdot, g \cdot)$, i.e. a family of morphisms in $\text{cat}_B$, such that every morphism $\phi \in \text{cat}_A(\cdot, \cdot)$ gives rise to a commutative diagram [Mac Lane 1963, 1965, 1971; Eilenberg and Kelly 1966; Paré and Román 1998].

11.1. Remark. We wish that natural transformations must be always closed under composition in

$$\text{nat}[\text{nat}(f, g) \times \text{nat}(g, h), \text{nat}(f, h)].$$

This implies that in the case of bivariant (i.e. of mixed variances) (multi)functors the definition of a natural transformation, as given by Eilenberg & Kelly in 1966, and by Kelly in [1972, §4 pp.93-94], must be modified. A ‘generalized’ natural transformation is known under the names: dinatural transformation, or Barr dinatural transformation [Paré & Román 1998].

A bicategory [Bénabou 1967] is the same as a not strict 2-category (also called pseudo or more general lax). A typical example of a 2-category is a 1-category, or, in the case of many sorts, the collection of 1-categories, with a structure, according to the following definition.

11.2. Definition (Kelly 1972, §3, p. 116). A structure on a category consists of a set of functors, and of various natural transformations, subject to equational axioms (like, for example, the Leibniz-like axiom).

Lawvere in his Thesis in 1963 gave a program of a categorification: a program of replacing standard ‘theories’, i.e. a language with rules of deduction and axioms, by n-categories with structures, ‘models’ by n-functors and morphisms by natural transformations. A theory, a mathematical theory, such as the algebraic theory [Lawvere 1963] based on Birkhoff’s equational class, or in particular an algebraic analysis (based on the Leibniz-like axiom), consists of an n-category (or n-categories = sorts in computer science) with various amounts of structure imposed on them. For the most elementary introduction, see [Lawvere & Schanuel 1997]. For a most recent account we can refer to [Baez & Dolan 1998, Batanin 1998].

An n-functor from an algebraic n-category $N$ with grafting, into an n-category $\text{Cat}$ of small functor categories (with grafting), is said to be the functorial model of $N$ (a realization or an $N$-algebra). Such n-functor is a model of plants as 1-functors, and a model of footpaths as natural transformations of 1-functors.
12. Conclusions and some directions for further studies

The authors see an algebraic analysis as just another instance of an equational theory (an instance of the Birkhoff universal algebra) with a Leibniz-like axiom as a strict or a weak 3-cell. In this paper we do propose to apply Lawvere’s categorification program to analysis. Very limited space does not allow to discuss here neither the specific Leibniz 3-categories with weak 3-cells nor many other important issues. Therefore our purpose was limited to an elementary introduction into the basic notions of the categorical analysis in terms of the general theory of n-categories. There are many important directions which deserve further detailed studies, besides of these mentioned in the main body of this paper. A logical order of these directions, from the most fundamental to the less relevant, is contrary to the historical developments.

12.1. Extension of gebra. We believe that the most fundamental for any analysis (algebraic, categorical, braided) is an extension of gebra i.e. an extension of 2-cells in terms of 3-cells denoted below by ‘\( \mapsto \)’, [Eilenberg 1948, Gugenheim 1962, Kelly 1972, p. 94, Cuntz & Quillen 1995], viz.,

\[ m \mapsto m_l \& m_r \quad \text{or/and} \quad \triangle \mapsto \triangle_l \& \triangle_r. \]

The extension 3-cells are said to be axioms if they are strict, and have nothing to do with a grafting dependent expansion of 2-cells or with a Bénabou’s pasting of 2-cells. The strict Eilenberg-like extension 3-cells are known in the non-commutative differential geometry under the name of gebra bimodule, bicomodule for cogebra, double dimodule [Pareigis 1996], quadruple comodule [Borowiec & Vázquez Coutiño], etc. A gebra extension must precede algebraic analysis understood as a Leibniz-like 3-cell. We see that the logical order of differential calculus is as follows: firstly the 3-cells of an Eilenberg-like gebra extension, and after a Leibniz-like 3-cell.

The extension 3-cells need an abelian category, i.e. a pair of binary plants (bifunctors) and moreover a distributive law between them [Beck 1969; Kelly 1972, p. 94].

12.2. A Leibniz-like axiom as a 3-cell. Classical differential geometry deals with the Cartan derivative from an algebra \( m \) to an \( m \)-bimodule \( m_l \& m_r \) of the differential one-forms, \( d \in \text{der}(m, m_l \& m_r) \). Therefore \( \text{der} \) is interpreted usually as a bifunctor

\[ (\text{algebra,bimodule}) \mapsto \text{set}. \]
We believe that it would be desirable to identify \( \text{der} \) also as a strict 3-cell in an appropriate Leibniz 3-category and to investigate a weak version in terms of the Bénabou modification.

12.3. **Braided analysis.** The classical work on a braided analysis was done by Woronowicz [1989], who located partial braided derivations \( \text{der} (m, \sigma, \ldots) \) within a braided Lie algebra. Majid in [1993, 1995 §10.4] introduced another braided version of the Leibniz rule, that is, another braided derivation. From the point of view of the categorical analysis, which we do propose here, Majid’s braided analysis seems to be not so much an important example. A slightly different approach was proposed in [Oziewicz, Paal & Róźański 1995]. The main unsolved problem here is to locate partial derivations (braided or not braided), as a 3-cell \( \text{der} (m) \equiv \text{der} (m, m & m) \), within an extra structure of a braided Lie algebra which we do propose to identify as some strict 4-cell, *i.e.* as an axiom on 3-cells. In particular, it would be very interesting to identify possible strict 4-cells for the Borowiec partial (co)derivations [Borowiec 1996, 1997; Borowiec & Vázquez Coutiño].

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Universidad Nacional Autónoma de México, Facultad de Estudios Superiores Cuautitlán, Apartado Postal #25, C.P. 54700 Cuautitlán Izcalli, Estado de México, and Uniwersytet Wrocławski, Poland

E-mail address: oziewicz@servidor.unam.mx

Universidad Autónoma Metropolitana, Unidad Iztapalapa, Avenida Michoacán y la Purísima, Colonia Vicentina, Apartado Postal 55-534, C.P. 09340 México D.F.

E-mail address: gavc@xanum.uam.mx