Research report:
State complexity of operations on two-way quantum finite automata

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Abstract
This paper deals with the size complexity of minimal two-way quantum finite automata (2qfa’s) necessary for operations to perform on all inputs of each fixed length. Such a complexity measure, known as state complexity of operations, is useful in measuring how much information is necessary to convert languages. We focus on intersection, union, reversals, and catenation operations and show some upper bounds of state complexity of operations on 2qfa’s. Also, we present a number of non-regular languages and prove that these languages can be accepted by 2qfa’s with one-sided error probabilities within linear time. Notably, these examples show that our bounds obtained for these operations are not tight, and therefore worth improving. We give an instance to show that the upper bound of the state number for the simulation of one-way deterministic finite automata by two-way reversible finite automata is not tight in general.

Keywords: Quantum finite automata; State complexity; Operations

1. Introduction

1.1. Background of this topic and relevant results
Finite (state) automata represent one of the most simple form of computation to date. The minimal number of inner states used in a finite automaton has been a focal point of a recent study of finite automata (for example, see [5,9,13,15,16,26,30] and the references therein). Such a number serves as a complexity measure and is generally referred to as state complexity. The importance of the study of state complexity arises from recent applications of finite automata to numerous practical fields, which include natural language and speech processing, software engineering, and image generation and encoding, etc [7,25]. In the literature, there are by and large three distinct streams of studies concerning state complexity of finite automata.

- State complexity of language recognition on a fixed automata model.
- State complexity of transformation of one automata model to another.
- State complexity of operations on a fixed automata model.

Indeed, these complexity measures have a long history of their own. In the 1950s, Rabin and Scott [24] showed that an n-state one-way nondeterministic finite automaton (1nfa) is transformed to its
equivalent one-way deterministic finite automaton (1dfa) of at most $2^n$ inner states. This is generally referred to as the state complexity of transformation. The notion of state complexity of operations, on the contrary, dates back to 1981 when Leiss [18] showed that the reversal of a language recognized by an $n$-state 1dfa requires $2^n$ inner states.

In this paper, we focus on the state complexity of operations on two-way quantum finite automata (2qfa’s), which is defined as the minimal number of inner states necessary for a finite automaton to “witness” a given operation on all inputs of each fixed length. An automaton that achieves such minimality is called a minimal automaton.

Since the state complexity dealt with in this report is concerning quantum computing models, we recall some background regarding quantum computing. Exactly, quantum mechanical computation—a new breed of nature-inspired computation—has recently drawn wide attention as an alternative computation paradigm [12,20]. The notion of quantum finite automata (qfa’s) was introduced by the pioneering work of Moore and Crutchfield [19] and Kondacs and Watrous [17]. The computational model of Kondacs and Watrous is a natural quantization of probabilistic finite automata, which have been studied for more than four decades since its introduction by Rabin [24]. Their model consists of an input tape, a read-only head, and a finite control unit holding an inner state and evolves by a rule of quantum mechanics. As two-way quantum finite automata, the read-only head moves along the input tape in two directions (also being allowed to stay still). A configuration of such a machine is in general in a so-called superposition. The models of Kondacs and Watrous and of Moore and Crutchfield differ in one point: the number of measurements performed during a computation. Specifically, Kondacs and Watrous’ models perform measurement each step of computing, whereas Moore and Crutchfield’s ones only measure at the end of a computation.

Note that, similar to probabilistic finite automata, 2qfa’s and 1qfa’s are quite different in power. We are particularly interested in the power of two-way quantum finite automata (2qfa’s). Kondacs and Watrous [17] showed that, by exploiting quantum interference, 2qfa’s can recognize even non-regular languages, in particular, $Upal = \{0^n1^n \mid n \in \mathbb{N}\}$ in worst-case linear time. This shows a sharp contrast with the fact that there is a regular language that cannot be recognized by any one-way quantum finite automaton (1qfa) with any constant error bound because of its reversibility constraint. Notably, Amano and Iwama [2] showed that the empty problem for certain restricted 2qfa’s (which they call 1.5qfa’s) is undecidable.

It is worth mentioning that Ambainis, etc. [1,3], Nayak [21], Brodsky and Pippenger [6], and the others have dealt with some operation properties and state size on quantum finite automata. For example, Ambainis etc. [3] proved that the union of the languages accepted by one-way quantum finite automata with bounded error is not closed.

1.2. Our goals and obtained results

A recent interest in practical fields makes state complexity a practically important and theoretically interesting entity. It is natural to discuss state complexity based on quantum finite automata. Is such complexity measure quite different from that classical one? In this paper, we focus our study on the state complexity of operations on fixed automata models, i.e., 2qfa’s, that were first proposed and studied by Kondacs and Watrous [17]. We attempt to demonstrate that this complexity measure indeed proves vital in quantum complexity theory.

For notational convenience, we introduce the notation $QSC_\epsilon[L](n)$ to denote the smallest number of inner states necessary to solve language $L$ on a 2qfa model $M$ with error probability bounded above by $\epsilon$ when inputs are exactly length $n$. To be more precise, for any $x \in L \cap \Sigma^n$, the probability of accepting $x$ is at least $1 - \epsilon$, and the probability of rejecting $x \in \overline{L} \cap \Sigma^n$ is also at least $1 - \epsilon$, where $\Sigma^n$ denotes the set of strings with length $n$, and $\overline{L}$ is the complement of $L$. For simplicity, we call $M$ accepting $L \cap \Sigma^n$ with error probability bounded by $\epsilon$.

In this paper, we mainly prove the following results.
Theorem 1. For any languages $L_1$ and $L_2$ over $\Sigma_1$ and $\Sigma_2$ respectively, and any $n \in N$, let $M_1$ and $M_2$ be the minimum 2qfa for $L_1 \cap \Sigma_1^n$ and $L_2 \cap \Sigma_2^n$ with error probabilities bounded by $\epsilon_1$ and $\epsilon_2$, respectively. If $M_2$ is non-recurrent, then

$$QSC_\epsilon[L_1 \cap L_2](n) \leq QSC_{\epsilon_1}[L_1](n) + QSC_{\epsilon_2}[L_2](n) \times (n+2) \times |Q_{acc,1}| - |Q_{acc,1}|$$

(1)

$$QSC_\epsilon[L_1 \cup L_2](n) \leq QSC_{\epsilon_1}[L_1](n) + QSC_{\epsilon_2}[L_2](n) \times (n+2) \times |Q_{rej,1}| - |Q_{rej,1}|$$

(2)

where $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2$, $|Q_{acc,1}|$ denotes the number of the rejecting states of $M_1$.

Theorem 2. For any language $L$ over $\Sigma$, let $M$ be the minimum 2qfa for $L$ with error probability bounded by $\epsilon$. If $M$ is non-recurrent, then

$$QSC_\epsilon[L](n) - 1 \leq QSC_{\epsilon[R]}[L](n) \leq QSC_\epsilon[L](n) + 1$$

(3)

for any $n \in N$, where $L^R$ denotes the reversal of $L$, i.e., $L^R = \{x^R | x \in L\}$ (for $x = \sigma_1 \sigma_2 \ldots \sigma_n$, then $x^R = \sigma_n \sigma_{n-1} \ldots \sigma_1$).

Theorem 3. Let $L_i$ be a language over alphabet $\Sigma_i$ with $\epsilon \notin L_i$ for $i = 1, 2$. If $\Sigma_1 \cap \Sigma_2 = \emptyset$, and the error probabilities of the minimum 2qfa’s $M_1$ and $M_2$ accepting $L_1$ and $L_2$ are respectively $\epsilon_1$ and $\epsilon_2$, then the catenation $L_1L_2$ is accepted by a 2qfa $M$ with error probability $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2$.

Proposition 1. For alphabet $\Sigma = \{a, b_1, b_2\}$, let $L = \{a^m b_1^m b_2^n : n, m \geq 1\}$. Then $L$ is accepted by 2qfa with one-sided error in linear time.

Proposition 2. For alphabet $\Sigma = \{a, b_1, b_2\}$, let $L_1 = \{a^m b_1^m a^m b_2^n : m \geq 1\}$ and $L_2 = \{a^m b_1^m a^m b_2^m : n \geq 1\}$. Then there exist 2qfa $M_1$ and $M_2$ accepting $L_1$ and $L_2$, respectively, with one-sided error in linear time.

Proposition 3. There exists regular language $L$ satisfying

$$QSC_\epsilon[L](2n) \leq \frac{\sqrt{DSC_\epsilon[L](2n)}}{6} - 4$$

(4)

for any $n \in N$, where $DSC_\epsilon[L](n)$ denotes the smallest number of inner states necessary to accept language $L$ on a one-way deterministic finite automaton when inputs are exactly length $n$.

From the above theorems and propositions it follows a number of corollaries which will be stated in Sections 3 and 4. Also, in order to prove the above results, we need verify some lemmas, that will be detailed in the sequel.

2. Basic notions and notation

We review some related notions and notation that will be used in this paper. In addition, some will be explained when they appear first.

2.1. General definitions

Let $N$ be the set of all natural numbers (that is, nonnegative integers), $Z$ be that of all integers, $R$ be that of all real numbers, and $C$ be that of all complex numbers. Let $N^+ = N \setminus \{0\}$. For any complex number $a$, $a^*$ denotes its conjugate. For any two numbers $m, n \in N$ with $m < n$, the notation $[m, n]$ denotes the set $\{m, m+1, m+2, \ldots, n\}$. For any finite set $Q$, $|Q|$ denotes the cardinality of $Q$.

We use the notation $\Sigma$ in general for a nonempty input alphabet (not necessarily be limited to $\{0, 1\}$). A string over $\Sigma$ is a finite sequence of symbols in $\Sigma$ and the length of a string is the number of occurrences of symbols in the string. In particular, the string of length 0 is called the empty string.
and often denoted $\lambda$. For each number $n \in \mathbb{N}$, $\Sigma^n$ denotes the set of all strings over $\Sigma$ that have length exactly $n$. Write $\Sigma^*$ for $\bigcup_{n \in \mathbb{N}} \Sigma^n$. A partial problem over alphabet $\Sigma$ is a pair $(A, B)$ such that $A, B \subseteq \Sigma^*$ and $A \cap B = \emptyset$. When $A \cup B = \Sigma^*$, $B$ becomes the complement of $A$ (denoted $\Sigma^* - A$) and thus we identify $(A, B)$ with $A$, which is simply called a language.

We assume the reader’s familiarity with classical finite automata. We use the notation REG to denote the collection of all regular languages.

Let $A$ be any $m \times n$ complex matrix. The notation $A^T$ denotes the transposed matrix of $A$. Moreover, $A^\dagger$ denotes the transposed conjugate of $A$. For any vector $x$, $\|x\|$ denotes the $\ell_2$-norm. Let $\|A\|$ be the operator norm defined as $\|A\| = \max\{\|Ax\| : \|x\| \neq 0\}$. Let $\|A\|_2$ be the Frobenius norm $(\sum_{i,j}|a_{i,j}|^2)^{1/2}$. Let $\|A\|_{tr}$ be the trace norm $\min\{|Tr(AX^\dagger)| : \|X\| \leq 1\}$.

2.2. Quantum finite automata

We briefly give the formal definition of a quantum finite automaton (qfa). Formally, a qfa is described as a sextuple $M = (Q, \Sigma, q_0, \delta, Q_{acc}, Q_{rej})$, where $Q$ is a finite set of inner states with $Q_{acc} \cup Q_{rej} \subseteq Q$, $\Sigma$ is a finite alphabet, $q_0$ is the initial inner state, and $\delta$ is a transition function mapping from $Q \times \Sigma$ to $Q \times \{-1, 0, +1\}$. The transition function $\delta$ is also expressed by a series of transitions defined by the complex number (called an amplitude) $\delta(p, \sigma, q, d)$ for $p, q \in Q$, $\sigma \in \Sigma$, and $d \in \{-1, 0, +1\}$. This means that, assuming that a machine is in inner state $p$ scanning a symbol $\sigma$, the machine at next step changes its inner state to $q$ moving its head in direction $d$. The set $Q$ is partitioned into three sets: $Q_{acc}$, $Q_{rej}$, and $Q_{non}$. Inner states in $Q_{acc}$ (in $Q_{rej}$, resp.) are called accepting states (rejecting states, resp.). A halting state refers to both an accepting state and a rejecting state. The rest $Q_{non}$ is known as a set of non-halting states. A configuration is a description of a single moment of $M$’s computation, including an inner state and a head position; we regard $Q \times [0, |x| + 1]\mathbb{Z}$ as the configuration space on input $x$. In general, $M$’s computation is a series of superpositions of configurations, each of which evolves by an application of $\delta$ to its predecessor (if not the initial configuration). More generally, we can view an application of $\delta$ as an application of a linear operator over a configuration space. More precisely, the operator $U_\delta$ with respect to inputs of length $n$ is defined as the linear operator acting on the configuration space $CON\mathcal{F}_n$, which is the Hilbert space spanned by $\{(p, i) \mid q \in Q, i \in [0, n + 1]\mathbb{Z}\}$, in the following way: for each $(p, i) \in Q \times [0, n + 1]\mathbb{Z}$, let $U_\delta(p, i) = \sum_{(q, d) \in Q \times [0, n + 1]} \delta(p, x_i, q, d)|q, i + d (\text{mod } n + 2))$, where $x_0 = \xi$, $x_{n+1} = \$ and $x_i$ is the $i$th symbol of $x$ for each $i \in [1, n]\mathbb{Z}$. Throughout this paper, we assume that $U_\delta$ is always unitary for any $n \in \mathbb{N}$.

We say that a 2qfa $M$ recognizes $(A, B)$ with error probability $\epsilon$ if (i) for every $x \in A$, $M$ accepts $c$ with probability $1 - \epsilon$ and (ii) for every $x \in B$, $M$ rejects $x$ with probability $1 - \epsilon$. When $(A, B)$ is identified with the language $A$, we simply say that $M$ recognizes $A$. For notational simplicity, we write $\text{Prob}_M[M(x) = 1]$ to denote the probability of $M$ accepting $x$. Similarly, $\text{Prob}_M[M(x) = 0]$ for the probability of $M$ rejecting $x$. We define the class 2QFA as the collection of all languages that can be recognized by 2qfa’s with error probability at most certain constant $\epsilon \in [0, 1/2]$. Similarly, 2QFA(poly-time) is defined by 2qfa’s which run in expected polynomial time.

3. State complexity by quantum finite automata

We formally introduce the notation of state complexity of language recognition on quantum finite automata.
3.1. Definition of state complexity

Roughly speaking, the state complexity of a language is the number of inner states of the minimal qfa that recognizes the language with designated error probability. More formally, we say that a language $L$ over alphabet $\Sigma$ has state complexity $s(n)$ with error probability $\epsilon$ and amplitude set $K$ if there exists a 2qfa $M_n = (Q_n, \Sigma, q_0, \delta_n, Q_{acc,n}, Q_{rej,n})$ such that (i) $|Q_n| \leq s(n)$ and (ii) for every $x \in L \cap \Sigma^n$, $\text{Prob}_{M_n}[M_n(x) = 1] = 1 - \epsilon$, and (iii) for every $x \in \overline{L} \cap \Sigma^n$, $\text{Prob}_{M_n}[M_n(x) = 0] = 1 - \epsilon$.

When $n$ is concerned, we say that $L$ has state complexity $s(n)$ at $n$ with error probability $\epsilon$. In the literature, qfa’s are sometimes specified by their partial transitions because it is easy to expand such partial transitions to standard transitions (see [27,28,29], for example). To discuss the number of inner states, it is therefore important to note that here we consider only “complete” qfa’s, which are equipped with transition functions defined completely on the domain $Q \times \Sigma$.

For notational convenience, we write $QSC_{K,\epsilon}[L](n)$ for the minimal number $m$ such that $L$ has state complexity $m$ at $n$ with error probability at most $\epsilon$ and with amplitude set $K$. In particular, when $K = C$, we omit the subscript $K$ from $QSC_{K,\epsilon}[L](n)$ for readability. For comparison, we also introduce the notations for classical state complexity measures. Write $DSC[L](n)$ for the minimal number of inner states of any 2dfa recognizing $L$. A minimal 2qfa refers to a 2qfa that witnesses $QSC_{K,\epsilon}[L](n)$. Moreover, we write $PSC_{\epsilon}[L](n)$ for the minimal number of inner states of any 2pfa recognizing $L$ with error probability at most $\epsilon$. Since any 2pfa is also a 2qfa, we clearly obtain the following relationships: Let $L$ be any language and let $\epsilon \in [0, 1/2)$. $DSC[L](n) \geq PSC_{\epsilon}[L](n)$ for all $n \in \mathbb{N}$. If the running time of automata are concerned, we write $QSC_{K,\epsilon}^{poly}[L](n)$ ($QSC_{K,\epsilon}^{lin}[L](n)$, resp.), for instance, to emphasize that we use only 2qfa’s which run in expected polynomial time (expected linear time, resp.).

In the subsequent subsection, we review basic properties of state complexity defined by 2qfa’s.

3.2. Fundamental properties

We show several fundamental properties of state complexity measures.

**Lemma 1.** Let $L$ be any language over alphabet $\Sigma$, let $\epsilon$ be any constant in $[0, 1/2)$, and let $K$ be any amplitude set.

1. $1 \leq QSC_{\epsilon}[L](n) \leq \sum_{i=0}^{n} |\Sigma|^i$ for any $n \in \mathbb{N}$.

2. $QSC_{\epsilon}^{lin}[L](n) \leq 2DSC[L](n) + 6$ for all $n \in \mathbb{N}$.

**Proof.** 1) The lower bound comes from the fact that every 2qfa requires the initial inner state and at least one accepting or rejecting inner state. Actually, we can set $Q_{acc} = \{q_0\}$ and $Q_{rej} = \emptyset$ in the case of $L = \Sigma^*$. In this case, $QSC_0[\Sigma^*](n) = 1$ for all $n \in \mathbb{N}$. The upper bound is shown as follows. After the machine reads the left end-marker symbol, the initial state evolves also into initial state. Then the machine needs at most $|\Sigma|$ states after scanning the first input symbol (since there are only $|\Sigma|$ symbols). For the second step, there are at most $|\Sigma|$ states and each evolves at most $|\Sigma|^2$ states. Proceeding with this method, we therefore get the above result.

2) By the proof of Proposition 4 in the reference [17] by Kondacs and Watrous, we directly obtain this result. \qed

A natural problem is whether or not the bound in Lemma 2 is tight. Indeed, from the following Lemma 2 we know that for some language $L$ we have Proposition 3 stated in Section 1. Therefore it says that it is not tight.

**Lemma 2.** Let $\Sigma = \{a, b\}$, and let $L = \{w \in \Sigma^*|\#_a(w) = \#_b(w) = n\}$, where $n \in \mathbb{N}$, and $\#_a(w)$ and $\#_b(w)$ denote respectively the number of $a$ and $b$ in the string $w$. Then there exists 2qfa (exactly
2rfa, i.e., two-way reversible finite automaton) $M$ with $6n + 4$ states to accept $L$.

**Proof.** We construct a 2rfa $M = (Q, \Sigma, \delta, q_0, Q_{\text{acc}}, Q_{\text{rej}})$ as follows.

\[
Q = \{q_0, q_1, \ldots, q_{2n+1}, p_1, p_2, \ldots, p_{n+1}, p_0, q_r^{(1)}, q_r^{(2)}, \ldots, q_r^{(2n)}, p_r^{(1)}, p_r^{(2)}, \ldots, p_r^{(n)}\},
\]

\[
Q_{\text{acc}} = \{p_0\}, \quad Q_{\text{rej}} = \{q_r^{(1)}, q_r^{(2)}, \ldots, q_r^{(2n)}, p_r^{(1)}, p_r^{(2)}, \ldots, p_r^{(n)}\},
\]

\[
\delta(p, \sigma, q, d) = \begin{cases} \langle q | V_\sigma | p \rangle, & D(q) = d, \\ 0, & \text{otherwise}, \end{cases}
\]

for any $p, q \in Q$ and any $\sigma \in \Sigma \cup \{\|, \}$, where unitary operators $V_\sigma : \ell_2(Q) \to \ell_2(Q)$ and mapping $D : Q \to \{0, \pm 1\}$ are defined as:

\[
V_{\|} | q_0 \rangle = | q_1 \rangle, \quad V_{\|} | p_{n+1} \rangle = | p_0 \rangle, \quad V_{\|} | p_i \rangle = | p^{(i)}_{\sigma} \rangle, \quad i = 1, 2, \ldots, n,
\]

\[
V_{\|} | q_{2n+1} \rangle = | p_{1} \rangle,
\]

for $\sigma \in \Sigma$.

$V_{\|}$ is readily extended unitarily to $\ell(Q)$.

$D(q_i) = 1$ for $i = 0, 1, \ldots, 2n + 1$, and $D(q) = 0$ for any $q \in Q_{\text{acc}} \cup Q_{\text{rej}}$.

One can readily check that the 2rfa constructed above can accept $L = \{w \in \Sigma^* | \#_a(w) = \#_b(w) = n\}$ with $6n + 4$ states.

However, in terms of the Myhill- Nerode Theorem [14], any deterministic finite automaton accepting $L$ needs at least $n^2$ states. Therefore, for language $L$ above we have that Proposition 3 holds.

**Proposition 3.** There exists regular language $L$ satisfying

\[
QSC_0|L|(2n) \leq \frac{\sqrt{DSC_0|L|(2n)}}{6} - 4
\]

for any $n \in \mathbb{N}$.

The following lemma is immediate from the definitions of 2QFA, 2QFA(poly-time), and state complexity.

**Lemma 3.** Let $L \subseteq \Sigma^*$.

1. $L \in$ 2QFA iff there exist constants $c' \in [0, 1/2)$ and $c \in \mathbb{N}$ such that $QSC_{c'}|L|(n) \leq c$ for all $n \in \mathbb{N}$.

2. $L \in$ 2QFA(poly-time) iff there exist constants $c' \in [0, 1/2)$ and $c \in \mathbb{N}$ such that $QSC_{c'}^{\text{poly}}|L|(n) \leq c$ for all $n \in \mathbb{N}$.

**4. State complexity of basic operations**

The main theme of this paper, as stated in Section 1, is to determine the upper bounds of state complexity of basic operations on 2qfa’s. First, we consider the operation called “complementation” on 2qfa’s. This is the easiest case. From the definition of 2qfa’s, we immediately obtain the exact bound as follows.

**Lemma 4.** For any language $L$ and any constant $c \in [0, 1/2)$, $QSC_c[\Sigma^* - L](n) = QSC_c[L](n)$ for all $n \in \mathbb{N}$.
Proof. This is obtained by replacing the roles of $Q_{acc}$ and $Q_{rej}$. \qed

Now, we further prove these results presented in Section 1. Firstly we need a definition that is used in the following results.

Definition 1. We call 2qfa $M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej})$ non-recurrent, if for any $q \in Q$ with $q \neq q_0$, $\delta(q, \sigma, q_0, d) = 0$ for any $\sigma \in \Sigma \cup \{\$, \}$.

Also, the well-formed conditions of 2qfa’s given by Kodacs and Watrous [17] for justifying the unitarity of evolution will be used in what follows. Therefore we recall these conditions here. A 2qfa $M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej})$ is well-formed if and only if for any $\sigma, \sigma_1, \sigma_2 \in \Sigma \cup \{\$, \}$, and any $q_1, q_2 \in Q$, the following hold.

1. $\sum_{q'} \delta(q_1, \sigma, q', d)\delta(q_2, \sigma, q', d) = \begin{cases} 1, & q_1 = q_2, \\ 0, & q_1 \neq q_2. \end{cases}$

2. $\sum_{q'} \delta(q_1, \sigma_1, q', 1)\delta(q_2, \sigma_2, q', 0) + \delta(q_1, \sigma_1, q', 1)\delta(q_2, \sigma_2, q', -1) = 0$.

3. $\sum_{q'} \delta(q_1, \sigma_1, q', 1)\delta(q_2, \sigma_2, q', -1) = 0$.

Theorem 1. For any languages $L_1$ and $L_2$ over $\Sigma_1$ and $\Sigma_2$ respectively, and any $n \in N$, let $M_1$ and $M_2$ be the minimum 2qfa for $L_1 \cap \Sigma_1^n$ and $L_2 \cap \Sigma_2^n$ with error probabilities bounded by $\epsilon_1$ and $\epsilon_2$, respectively. If $M_2$ is non-recurrent, then

$$QSC_e[L_1 \cap L_2](n) \leq QSC_{e_1}[L_1](n) + QSC_{e_2}[L_2](n) \times (n + 2) \times |Q_{acc}, 1| - |Q_{acc}, 1|$$ (6)

$$QSC_e[L_1 \cup L_2](n) \leq QSC_{e_1}[L_1](n) + QSC_{e_2}[L_2](n) \times (n + 2) \times |Q_{rej}, 1| - |Q_{rej}, 1|$$ (7)

where $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$, $|Q_{acc}, 1|$ denotes the number of the rejecting states of $M_1$.

Proof. The basic idea for constructing 2qfa $M$ that accepts $L_1 \cap L_2$ with the length of input strings being $n$ is as follows: Firstly let $M$ simulate $M_1$. If $M_1$ rejects the input then the computation ends with rejection; otherwise, $M$ continues to simulate $M_2$. Now we formally describe the process of proof.

Assume that 2qfa $M_i = (Q_i, \Sigma_i, \delta_i, q_{0i}, Q_{acc,i}, Q_{rej,i})$ accepts $L_i \cap \Sigma_i^n$ with error probability bounded at most $\epsilon_i$, $i = 1, 2$, and $0 \leq \epsilon_i < \frac{1}{2}$. We construct $M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej})$ as follows.

$$\Sigma = \Sigma_1 \cup \Sigma_2;$$

$$q_0 = q_{01};$$

$$Q = Q_1 \cup Q_2 \cup Q_A$$ where

$$Q_1 = \bigcup_{i \in [0, n+1]} Q_{i, q_{acc,i}} \in Q_{acc,1} Q_2^{(i, q_{acc,1})} = [0, n+1] \times Q_{acc,1} \times Q_2,$$ where

$$Q_{i, q_{acc,1}} = \{(i, q_{acc,1}, q)|q \in Q_2\};$$

$$Q_{acc} = \{(i, q_{acc,1}, q_{acc})|q_{acc} \in Q_{acc,2}, i = 0, 1, \ldots, |Q_{acc,1}|, q_{acc,1} \in Q_{acc,1}\};$$

$$Q_{rej} = Q_{rej,1} \cup \{(i, q_{acc,1}, q_{rej})|q_{rej} \in Q_{rej,2}, i = 0, 1, \ldots, |Q_{acc,1}|, q_{acc,1} \in Q_{acc,1}\};$$

$$Q_A = \{(i, q_{acc,1})|q_{acc,1} \in Q_{acc,1}, i = 0, 1, 2, \ldots, n+1\}$$

where $Q_A$ is the set of auxiliary states that make the tape head move back to the left end-marker $\$ when $M$ becomes a state in $Q_{acc,1}$, and $(0, q_{acc,1}) = q_{acc,1}$ for any $q_{acc,1} \in Q_{acc,1}$. Furthermore, $\delta$ is defined as follows: For any $\sigma \in \Sigma \cup \{\$, \}$,
\[ \delta(q_1, \sigma, q_2, d) = \begin{cases} \\
\delta_1(q_1, \sigma, q_2, d), & q_1, q_2 \in Q_1, q_1 \not\in Q_{acc,1}, \\
0, & q_1 \in Q_{acc,1}, q_2 \in Q_1, \\
\delta_2(p_1, \sigma, p_2, d), & q_1 = (i, q_{acc,1}, p_1), q_2 = (i, q_{acc,1}, p_2) \in Q^{(i,q_{acc,1})}_2, i = 0, 1, 2, \ldots, n + 1, \\
1, & \sigma \neq \xi, q_1 = (i, q_{acc,1}), q_2 = (i + 1, q_{acc,1}), d = -1, i = 0, 1, 2, \ldots, n, \\
1, & \sigma = \xi, q_1 = (i, q_{acc,1}), q_2 = (i, q_{acc,1}, q_{20}), d = 0, i = 0, 1, 2, \ldots, n + 1. \end{cases} \]

Then, in terms of \(\delta_1\) and \(\delta_2\) it is ready to extend \(\delta\) such that it satisfies the well-formed conditions of 2qfa’s. Using \(M\) to compute string \(x \in (\Sigma_1 \cup \Sigma_2)^n\), we obtain the following results:

(i) If \(x \in L_1 \cap L_2 \cap \Sigma^n\), then \(M\) accepts \(x\) with probability at least \((1 - \epsilon_1)(1 - \epsilon_2) = 1 - (\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2)\).

(ii) If \(x \in \overline{L_1} \cap L_2 \cap \Sigma^n\), or \(x \in L_1 \cap \overline{L_2} \cap \Sigma^n\), then \(M\) rejects \(x\) with probability at least \(1 - \epsilon_1\).

(iii) If \(x \in L_1 \cap L_2 \cap \Sigma^n\), then \(M\) rejects \(x\) with probability \((1 - \epsilon_1)(1 - \epsilon_2)\).

In addition, the number \(|Q|\) of states of \(M\) is \(QSC_{e_1}[L_1](n) + QSC_{e_2}[L_2](n) \times (n + 2) \times |Q_{acc,1}| - |Q_{acc,1}|\). Therefore, Eq. (6) is proved.

The proof of Eq. (7) has certain similarity to Eq. (6). The 2qfa \(M\) for \((L_1 \cup L_2) \cap \Sigma^n\) can be constructed according to the following process. For any \(x \in \Sigma^n\), firstly \(M\) simulates \(M_1\), and if \(M_1\) accepts \(x\) then \(M\) also accepts \(x\); otherwise \(M\) continues to simulate \(M_2\) and the rest computation is then completed in terms of \(M_2\). Therefore, with the analogous idea as above, we construct \(M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej})\) as follows.

\[ \Sigma = \Sigma_1 \cup \Sigma_2; \]

\[ q_0 = q_{00}; \]

\[ Q = Q_1 \cup Q'' \cup Q_B \text{ where} \]

\[ Q'' = \bigcup_{i \in [0,n+1]} \{ (i, q_{rej,1}) \} \times Q_{rej,1} \times Q_2, \]

\[ Q_{rej} = \{ (i, q_{rej,1}) : q_{rej,1} \in Q_{rej,1}, i = 0, 1, \ldots, |Q_{rej,1}|, q_{rej,1} \in Q_{rej,1} \}; \]

\[ Q_{acc} = \{ (i, q_{acc,1}) : q_{acc} \in Q_{acc,1}, i = 0, 1, \ldots, |Q_{acc,1}|, q_{rej,1} \in Q_{rej,1} \}; \]

\[ Q_B = \{ (i, q_{rej,1}) : q_{rej,1} \in Q_{rej,1}, i = 0, 1, 2, \ldots, n + 1 \} \]

where \(Q_B\) is the set of auxiliary states that make the tape head move back to the left end-marker \(\xi\) when \(M\) becomes a state in \(Q_{rej,1}\), and \((0, q_{rej,1}) = q_{rej,1}\) for any \(q_{rej,1} \in Q_{rej,1}\). Furthermore, \(\delta\) is defined as follows: For any \(\sigma \in \Sigma \cup \{\xi, \$\}, \)

\[ \delta(q_1, \sigma, q_2, d) = \begin{cases} \\
\delta_1(q_1, \sigma, q_2, d), & q_1, q_2 \in Q_1, q_1 \not\in Q_{rej,1}, \sigma \in \Sigma_1, \\
0, & q_1 \in Q_{rej,1}, q_2 \in Q_1, \sigma \in \Sigma_1, \\
\delta_2(p_1, \sigma, p_2, d), & q_1 = (i, q_{rej,1}, p_1), q_2 = (i, q_{rej,1}, p_2) \in Q^{(i,q_{rej,1})}_2, \\
i = 0, 1, 2, \ldots, n + 1, \sigma \in \Sigma_2, \\
1, & \sigma \neq \xi, q_1 = (i, q_{rej,1}), q_2 = (i + 1, q_{rej,1}), d = -1, i = 0, 1, 2, \ldots, n, \\
1, & \sigma = \xi, q_1 = (i, q_{rej,1}), q_2 = (i, q_{rej,1}, q_{20}), d = 0, i = 0, 1, 2, \ldots, n + 1. \end{cases} \]

Also, \(\delta\) can be extended to satisfy the well-formed conditions of 2qfa’s. Using \(M\) to compute string \(x \in (\Sigma_1 \cup \Sigma_2)^n\), we obtain the following results:

(i) If \(x \in L_1 \cap \Sigma^n\), then \(M\) accepts \(x\) with probability at least \(1 - \epsilon_1\).

(ii) If \(x \in \overline{L_1} \cap L_2 \cap \Sigma^n\), then \(M\) rejects \(x\) with probability at least \((1 - \epsilon_1)(1 - \epsilon_2)\).
(iii) If \( x \in \overline{L_1 \cap L_2} \cap \Sigma^n \), then \( M \) rejects \( x \) with probability at least \( (1 - \epsilon_1)(1 - \epsilon_2) \).

\( \square \)

From Theorem 1 it follows the following corollary.

**Corollary 1.** For any languages \( L_1 \) and \( L_2 \) over \( \Sigma_1 \) and \( \Sigma_2 \) respectively, and any \( n \in \mathbb{N} \), let \( M_1 \) and \( M_2 \) be the minimum 2qfa for \( L_1 \cap \Sigma_1^n \) and \( L_2 \cap \Sigma_2^n \) with error probabilities bounded by \( \epsilon_1 \) and \( \epsilon_2 \), respectively. If the tape head of \( M_1 \) stays always at the left end-marker when it enters accepting states, and \( M_2 \) is non-recurrent, then

\[
QSC_\epsilon[L_1 \cap L_2](n) \leq QSC_{\epsilon_1}[L_1](n) + QSC_{\epsilon_2}[L_2](n) \times |Q_{acc,1}| - |Q_{acc,1}|;
\]

if the tape head of \( M_1 \) stays always at the left end-marker when it enters rejecting states, and \( M_2 \) is non-recurrent, then

\[
QSC_\epsilon[L_1 \cup L_2](n) \leq QSC_{\epsilon_1}[L_1](n) + QSC_{\epsilon_2}[L_2](n) \times |Q_{rej,1}| - |Q_{rej,1}|
\]

where \( \epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2 \), \( |Q_{acc,1}| \) denotes the number of the rejecting states of \( M_1 \).

**Proof.** It is straightforward by the proof of Theorem 1. \( \square \)

To show that the above bounds are not tight, we verify the following propositions.

**Proposition 1.** For alphabet \( \Sigma = \{a, b_1, b_2\} \), let \( L = \{a^n b_1^m b_2^m : n, m \geq 1\} \). Then \( L \) is accepted by 2qfa with one-sided error in linear time.

**Proof.** The idea borrows Proposition 2 of [17] in which Kodacs and Watrous proved that non-regular language \( \{a^m b^m \mid m \geq 1\} \) can be accepted by 2qfa with one-sided error in linear time. Here, for any \( N \in \mathbb{N} \), we construct machine \( M_N \) in terms of the following idea. First we let machine \( M_N \) check whether the input is of form \( a^n b_1^m b_2^m \). If not, then \( M_N \) rejects it at once; otherwise, let the tape head of \( M_N \) stays at the right end-marker $, and then check whether or not the length of \( b_2 \) and \( a \) in the right side equals. If not, then \( M_N \) rejects it with probability at least \( 1 - \frac{1}{N} \); otherwise, with probability one \( M_N \) continues to check the equality of the length of \( b_1 \) and \( a \) in the left side. If not, then \( M_N \) rejects it with probability at least \( 1 - \frac{1}{N} \); otherwise, \( M_N \) accepts it with probability one. Now we give the formal description of \( M_N = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej}) \) where the state set \( Q \) consists of the all states appearing in the following,

\[
\delta(q, \sigma, p, d) = \begin{cases} \langle p | V_\sigma | q \rangle, & D(p) = d, \\ 0, & D(p) \neq d \end{cases}, \quad \text{and } Q_{acc} = \{s_{N}^{(2)}\};
\]

\[
Q_{rej} = \{q_r^{(0)}, q_r^{(1)}, q_r^{(2)}, q_r^{(3)}\} \cup \bigcup_{i=1}^{2} \{s_1^{(i)}, s_2^{(i)}, \ldots, s_{N-1}^{(i)}\};
\]

For all \( \sigma \in \Sigma \cup \{\sharp\} \), unitary operators \( V_\sigma \) on \( \ell_2(Q) \) are defined as follows:
Proof.

and give the detailed construction of $M$ on $Q \delta$:

$$V_\ell |q_0 \rangle = |q_0 \rangle,$$
$$V_\ell |r_j^{(2)} \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \exp(\frac{2\pi i j}{N}) s_i^{(2)}, 1 \leq j \leq N,$$

$$V_\delta |q_0 \rangle = |q_0^{(0)} \rangle,$$
$$V_\delta |q_2 \rangle = |q_2^{(2)} \rangle,$$
$$V_\delta |q_6 \rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |r_j^{(1)} \rangle,$$

$$V_\delta |q_1 \rangle = |q_1^{(1)} \rangle,$$
$$V_\delta |q_4 \rangle = |q_4^{(1)} \rangle,$$
$$V_\delta |q_6 \rangle = |q_6^{(1)} \rangle,$$

$$V_a |q_0 \rangle = |q_0 \rangle,$$
$$V_a |q_2 \rangle = |q_2 \rangle,$$
$$V_a |q_4 \rangle = |q_4 \rangle,$$
$$V_a |q_6 \rangle = |q_6^{(0)} \rangle,$$

$$V_a |r_j^{(2)} \rangle = |r_j^{(2)} \rangle, 1 \leq j \leq N,$$
$$V_a |r_j^{(1)} \rangle = |r_j^{(1)} \rangle, 1 \leq j \leq N, 1 \leq k \leq j,$$

$$V_b |q_0 \rangle = |q_1 \rangle,$$
$$V_b |q_3 \rangle = |q_4 \rangle,$$
$$V_b |q_0 \rangle = |q_0^{(1)} \rangle, 1 \leq j \leq N,$$
$$V_b |r_j^{(1)} \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \exp(\frac{2\pi i j}{N}) s_i^{(1)}, 1 \leq j \leq N,$$
$$V_b |r_j^{(2)} \rangle = |r_j^{(2)} \rangle, 1 \leq j \leq N, 1 \leq k \leq N - j + 1,$$

$$V_{b 2} |q_0 \rangle = |q^{(0)} r \rangle,$$
$$V_{b 2} |q_6 \rangle = |q_6 \rangle,$$
$$V_{b 2} |r_j^{(1)} \rangle = |r_j^{(1)} \rangle, 1 \leq j \leq N, 1 \leq k \leq N - j + 1,$$

$$D(q_0) = +1,$$
$$D(q_1) = -1,$$
$$D(q_2) = +1,$$
$$D(q_4) = +1,$$
$$D(q_6) = +1,$$
$$D(q^{(1)}_0) = 0, 1 \leq l \leq N,$$
$$D(q^{(1)}_i) = 0, i = 0, 1, 2, 3.$$

Proposition 2. For alphabet $\Sigma = \{a, b_1, b_2\}$, let $L_1 = \{a^i b_1^m a^m b_2^n : m \geq 1\}$ and $L_2 = \{a^m b_1^m a^m b_2^n : n \geq 1\}$. Then there exist 2qfa $M_1$ and $M_2$ accepting $L_1$ and $L_2$, respectively, with one-sided error in linear time.

Proof. By changing the construction of $M_N$ as the proof of Proposition 1, we can obtain 2qfa’s $M_1$ and $M_2$ to accept $L_1$ and $L_2$ with one-sided error, respectively. In the interest of completeness, we give the detailed construction of $M_1 = (Q_1, \Sigma, \delta_1, q_0, Q_{acc}, Q_{rej})$ and $M_2 = (Q_2, \Sigma, \delta_2, q_0, Q_{acc}, Q_{rej})$. For $M_1$,

$$\delta_1(q, \sigma, p, d) = \begin{cases} \langle p | V_{\sigma} | q \rangle, & D(p) = d, \\
0, & D(p) \neq d, \quad \text{and } Q_{acc} = \{s_N\}; 
\end{cases}$$

$$Q_{rej} = \{q_0^{(0)}, q_0^{(1)}, q_0^{(2)}, q_0^{(3)}\} \cup \{s_1, s_2, \ldots, s_{N-1}\};$$

$$D(q^{(i)}_0) = 0, 1 \leq l \leq N,$$
$$D(q^{(i)}_i) = 0, i = 0, 1, 2, 3.$$
For all $\sigma \in \Sigma \cup \{\dagger, \$\}$, unitary operators $V_\sigma$ on $\ell_2(Q)$ are defined as follows:

$$V_\dagger|q_0\rangle = |q_0\rangle, \quad V_\dagger|q_1\rangle = |q_r^{(1)}\rangle,$$

$$V_\dagger|q_2\rangle = |q_r^{(0)}\rangle, \quad V_\dagger|q_3\rangle = |q_r^{(1)}\rangle,$$

$$V_\dagger|q_4\rangle = |q_r^{(2)}\rangle, \quad V_\dagger|q_5\rangle = |q_r^{(3)}\rangle,$$

$$V_\dagger|r_{j,0}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |r_{j,0}\rangle, \quad V_\dagger|r_{j,1}\rangle = |r_{j,1}\rangle, 1 \leq j \leq N,$$

$$V_\dagger|r_{j,2}\rangle = |r_{j,2}\rangle, 1 \leq j \leq N,$$

$$V_\dagger|r_{j,3}\rangle = |r_{j,3}\rangle, 1 \leq j \leq N,$$

$$V_\dagger|r_{j,4}\rangle = |r_{j,4}\rangle, 1 \leq j \leq N,$$

$$V_\dagger|r_{j,5}\rangle = |r_{j,5}\rangle, 1 \leq j \leq N,$$

$$V_\dagger|r_{j,6}\rangle = |r_{j,6}\rangle, 1 \leq j \leq N,$$

$$V_\dagger|r_{j,7}\rangle = |r_{j,7}\rangle, 1 \leq j \leq N,$$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp(\frac{2\pi i}{N} j l)|s_l\rangle, 1 \leq l \leq N - 1,$$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp(\frac{2\pi i}{N} j l)|s_l\rangle, 1 \leq l \leq N - 1,$$

$$D(q_0) = +1, \quad D(q_1) = -1,$$

$$D(q_2) = +1, \quad D(q_3) = -1,$$

$$D(q_4) = +1, \quad D(q_5) = -1,$$

$$D(q_r^{(1)}) = +1, \quad D(q_r^{(2)}) = -1,$$

$$D(q_r^{(3)}) = 0, i = 0, 1, 2, 3, \quad D(s_l) = 0, 1 \leq l \leq N - 1,$$

$$D(s_N) = +1.$$

Therefore, $Q_1 = \{q_0, q_1, \ldots, q_6\} \cup \{s_1, \ldots, s_N\} \cup \{q_r^{(0)}, q_r^{(1)}, q_r^{(2)}, q_r^{(3)}\}$, in which $Q_{acc} = \{s_N\}$, $Q_{rej} = \{s_1, \ldots, s_{N-1}\} \cup \{q_r^{(0)}, q_r^{(1)}, q_r^{(2)}, q_r^{(3)}\}$.  

Next we construct $M_2 = (Q_2, \Sigma, \delta_2, q_0, Q_{acc}, Q_{rej})$ which is largely similar to $M_1$. We still present the detailed definitions of $V_\sigma$ for any $\sigma \in \Sigma \cup \{\dagger, \$\}$.
second inequality has been proved. The first inequality is only an inference from the second one, since the right end-marker $. Then $M$ state for constructing a 2qfa with the condition that $\varepsilon$ is bounded by $L$. Then it is clear that $3)$ $\delta$ is defined as follows: $D(q_0) = +1$, $D(q_1) = -1$, $D(q_2) = +1$, $D(q_3) = +1$, $D(q_4) = 0$, $i = 0, 1, 2, 3$, $D(q_5) = -1$, $D(q_6) = 0$, $i = 0, 1, 2, 3$, $D(q_7) = -1$.

**Remark 1.** From the above Propositions it follows that $QSC_{1/N}(L_1) \leq \frac{N(N+5)}{2} + 10$, $QSC_{1/N}(L_2) \leq \frac{N(N+5)}{2} + 11$, and $QSC_{1/N}(L_1 \cup L_2) \leq \frac{N(2N+8)}{2} + 10$, where $L_1 \cup L_2 = L$. This result shows that the bound in Eq. (6) is not tight. Since $L_1 \cup L_2 = \{a_i^+ b_i^1 a_i^+ b_i^2 \}$ is a regular language, the bound in Eq. (7) is not tight either.

Next we deal with the reversal of languages accepted by 2qfa's, by demonstrating Theorem 2.

**Theorem 2.** For any language $L$ over $\Sigma$, let $M$ be the minimum 2qfa for $L_1$ with error probability bounded by $\varepsilon$. If $M$ is non-recurrent, then

Proof. With the condition that $M$ is non-recurrent, we only need add a state $q_0'$ as starting state for constructing a 2qfa $M^R$ for $L^R$, and let $q_0'$ change to $q_0$ with its tape head moving to the right end-marker $. Then $M^R$ simulate $M$ in the reversal direction. Formally $M^R = (Q \cup \{q_0'\}, \Sigma, \delta^R, q_0', Q_{acc}, Q_{req})$ where $q_0' \not\in Q$, $\delta^R$ is defined as follows:

1) $\delta^R(q_0', q, q_0, -1) = 1$;
2) $\delta^R(q, Q, p, d) = \delta(q, Q, p, p, -d)$ for any $q, p \in Q$;
3) $\delta^R(q, Q, p, d) = \delta(q, Q, p, p, -d)$ for any $q, p \in Q$;
4) $\delta^R(q, Q, p, d) = \delta(q, Q, p, p, -d)$ for any $q, p \in Q$ and any $\sigma \in \Sigma$.

Then it is clear that $\delta^R$ satisfies the well-formed conditions of 2qfa's if $\delta$ does. Therefore, the second inequality has been proved. The first inequality is only an inference from the second one, since $L = (L^R)^R$.

Finally we deal with catenation operation of 2qfa’s. For technical reason, we restrict the 2qfa’s
to be non-circular, that is, when a machine’s tape head is scanning the left end-marker (or the right end-marker), the machine will not move its tape head left (right). Also, in the interest of simplicity, we consider only the languages without $\varepsilon$.

In the interest of simplicity, as in Corollary 1, we would like to assume that the tape head of the first machine $M_1$ stays at the right end-marker when it enters accepting states. Without this assumption one can also cope with it by virtue of the similar way used in the proof of Theorem 1.

**Theorem 3.** Let $L_i$ be a language over alphabet $\Sigma_i$ with $\varepsilon \notin L_i$ for $i = 1, 2$. If $\Sigma_1 \cap \Sigma_2 = \emptyset$, and the error probabilities of the minimum 2qfs’s $M_1$ and $M_2$ accepting $L_1$ and $L_2$ are respectively $\epsilon_1$ and $\epsilon_2$, then $L_1L_2$ is accepted by a 2qfa $M$ with error probability $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$.

**Proof.** Firstly we let $M$ check whether or not the input is of the form $\Sigma^+ \Sigma^+$. If not, $M$ rejects it immediately; otherwise, $M$ simulates $M_1$. If the input is rejected, then $M$ rejects it also; otherwise $M$ continues to simulate $M_2$ for the second part of the input, and therefore $M_2$ determines the accepting or rejecting result. Specifically, $M = (Q, \Sigma_1 \cup \Sigma_2, \delta, q_0, Q_{\text{acc}}, Q_{\text{rej}})$ where $\delta$ is defined as follows.

$$
\delta(q_0, \delta, q_0, 1) = 1, \quad \delta(q_0, \sigma, q_0, 1) = 1, \quad \delta(q_1, \delta, q_1, 1) = 1, \quad \delta(q_2, \delta, q_1, 1) = 1,
$$

$$
\delta(q_1, \sigma, q_1, 1) = 1, \quad \delta(q_2, \delta, q_2, 1) = 1, \quad \delta(q_1, \sigma, q_2, 1) = 1, \quad \delta(q_2, \sigma, q_1, 1) = 1,
$$

$$
\delta(q_1, \delta, q_2, 1) = 1, \quad \delta(q_2, \sigma, q_r, 0) = 1, \quad \delta(q_2, \sigma, q_r, 0) = 1, \quad \delta(q_2, \sigma, q_r, 0) = 1.
$$

The above process checks whether the input is the form of $\Sigma^+ \Sigma^+$. If it is, $M$ begins with simulating $M_1$. Therefore, $\delta$ is further defined as follows.

$$
\delta(p_1, \sigma, p_2, d) = \begin{cases} 
\delta_1(p_1, \sigma, p_2, d), & \sigma \neq \delta, \sigma \in \Sigma_1, p_1, p_2 \in Q_1, \\
\delta_1(p_1, \delta, p_2, d), & \sigma \in \Sigma_2, p_1, p_2 \in Q_1, \\
1, & \sigma \in \Sigma_2, p_1 \in Q_{\text{acc},1}, p_2 = p_2, d = -1, \\
\delta_2(p_1, \delta, p_2, d), & \sigma \in \Sigma_1, p_1 = p_2, p_2 \in Q_2, \\
\delta_2(p_1, \delta, p_2, d), & \sigma \in \Sigma_2 \cup \{\delta\}, p_1, p_2 \in Q_2.
\end{cases}
$$

It is seen that $Q = Q_1 \cup Q_2 \cup \{q_0, q_1, q_2, q_3, q_r\}$, where $Q_{\text{acc}} = Q_{\text{acc},1}$, and $Q_{\text{rej}} = \{q_r\} \cup Q_{\text{rej},1} \cup Q_{\text{rej},2}$. Then, $M$ accepts $x \in L_1L_2$ with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$, and rejects $x \notin L_1L_2$ with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$. \hfill \Box

Finally we present an example to show that the conditions such as $\Sigma_1 \cap \Sigma_2 = \emptyset$ in Theorem 3 are not necessary.

**Example.** For $\Sigma_1 = \{a, b_1\}$ and $\Sigma_2 = \{a, b_2\}$, languages $L_1$ over $\Sigma_1$ and $L_2$ over $\Sigma_2$ are respectively defined as:

$L_1 = \{a^m b_1^m | m \geq 1\}$ and

$L_2 = \{a^m b_2^m | m \geq 1\}$.

In terms of Proposition 2 of [17], $L_1$ and $L_2$ can be accepted by 2qfa’s with one-sided error in linear time. As well, by Proposition 1 above, the concatenation of $L_1$ and $L_2$ as the language over $\Sigma_1 \cup \Sigma_2$

$L_1L_2 = \{a^m b_1^m a^m b_2^m | m \geq 1\}$

is accepted by 2qfa with one-sided error in linear time. Therefore, this shows that the conditions such as $\Sigma_1 \cap \Sigma_2 = \emptyset$ in Theorem 3 are not necessary. \hfill \Box
5. Concluding remarks and future works

In this report, we dealt with the state complexity of some operations (including complementation, intersection, union, reversals, catenation) on two-way quantum finite automata. We proved a number of upper bounds of the size of states for these operations, and also obtained lower bound for reversal operation. Also, we provided in detail a number of non-regular languages and demonstrated that these languages can be accepted by two-way quantum finite automata with one-sided error probabilities in linear time. In terms of these examples we have seen that the bounds obtained for these operations are not tight. Therefore, this motivates to further consider related issues along this direction.

Therefore, the further work is how to improve these bounds to make them optimum, and how to verify related lower bounds for intersection and union. In particular, there are some restricted conditions in these theorems (such as non-recurrent), so, proving these theorems without these conditions is worth further exploring.

As is well-known, classical interactive proof systems [8,11] have played an important role in the study of computational complexity, and have been successfully applied to cryptography systems. Notably, by generalizing the classical interactive proof systems of Dwork and Stockmeyer [11] to quantum framework, Nishimura and Yamakami [22] recently have significantly dealt with quantum interactive proof systems by using 2qfa’s. Furthermore, to study zero-knowledge quantum interactive proof systems by using two-way quantum finite automata as verifiers is a significant issue for the future consideration.

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