Strong double coverings of groups

Ana Breda, Antonio Breda d’Azevedo, Domenico Catalano

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Abstract

By a covering of a group $G$ we mean an epimorphism from a group $\hat{G}$ to $G$. Introducing the notion of strong covering as a covering $\pi : \hat{G} \to G$ such that every automorphism of $\hat{G}$ is a projection via $\pi$ of an automorphism of $G$, the main aim of this paper is to characterise double coverings which are strong. This is done in details for metacyclic groups, rotary platonic groups and some finite simple groups.

1 Preliminaries

In map/hypermap theory, or in polytopes theory, it is sometimes desirable to know not only the coverings $\hat{G}$ of a given group $G$ but also whether the automorphism group $\text{Aut}(\hat{G})$ of the covering also covers the automorphism group of $G$. Such coverings will be called “strong coverings”. In this paper we address this problem to double coverings. The characterisation of strong double coverings will be done in details for metacyclic groups, rotary platonic groups and some finite simple groups.

For simplicity when dealing with presentations of groups we need to establish some conventions. We will generally see the set of generators and the set of relators of a given group presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ as ordered sets and, as such, as tuples. Hence, setting $X = \langle x_1, \ldots, x_n \rangle$ and $R = \langle r_1, \ldots, r_m \rangle$ we will write $\langle X \mid R \rangle$ to mean $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ and call $n$ the rank of the presentation $\langle X \mid R \rangle$. Moreover by $r_k(X)$ and $R(X)$ we mean $r_k(x_1, \ldots, x_n)$ and $r_k(x_1, \ldots, r_m(X))$, respectively. Seeing the set of generators and the set of relators of a group presentation as ordered sets in the form of tuples, we can bring the cartesian product notations to presentations. For instance, if $R = \langle r_1, \ldots, r_m \rangle$ and $S = \langle s_1, \ldots, s_m \rangle$ are ordered set of words on $x_1, \ldots, x_n$ then $R = S$ has the usual “cartesian” meaning $r_k = s_k, k = 1, \ldots, m$. For convenience, especially when dealing with equations, we may use relations instead of relators in presentations. We reserve the notation $[a, b]$ for the commutator $a^{-1}b^{-1}ab$ of elements $a, b$ of a group $G$ and the natural simplification $[a, X]$ to mean $[a, x_1], \ldots, [a, x_n]$ for the $n$-tuple $X = (x_1, \ldots, x_n)$ of elements of $G$. Finally, given two presentations $P_1$ and $P_2$ we write $P_1 \sim P_2$ if they are presentations of the same group.

Let $\Gamma$ be a group with presentation $\langle X \mid W \rangle$ where $X = \langle x_1, \ldots, x_n \rangle$ and $W$ is an ordered set of words on $x_1, \ldots, x_n$. By a $\Gamma$-base of a finite group $G$ we mean a $n$-tuple $(a_1, \ldots, a_n)$ of elements of $G$ for which there is an epimorphism from $\Gamma$ to $G$ mapping $x_i$ to $a_i$. If $\Gamma$ is the free group of rank $n$ we say $n$-base instead of $\Gamma$-base. Hence a $n$-base of a group $G$ is just a $n$-tuple $(a_1, \ldots, a_n)$ of elements of $G$ such that $\{a_1, \ldots, a_n\}$ generates $G$. We will denote by $B_n(G)$ the set of $n$-bases of $G$ and say that $G$ is $n$-generated if $B_n(G) \neq \emptyset$.

Given a presentation $P = \langle X \mid R \rangle$ of rank $n$ of a group $G$ we denote by $S_P$ the set of $n$-bases $(a_1, \ldots, a_n)$ of $G$ satisfying $R(a_1, \ldots, a_n) = 1$. The elements of $S_P$ will be called presentation $n$-tuples of $P$. Obviously every $n$-base of $G$ belongs to a presentation set $S_P$ for some presentation $P$ of $G$ of rank $n$.

Theorem 1 $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in B_n(G)$ belong to the same presentation set if and only if there is an automorphism of $G$ mapping $a_i$ to $b_i$.

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Proof. If \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in B_n(G)\) belong to the same presentation set \(S_P\) then by the Substitution Test the functions \(x_i \mapsto a_i\) and \(x_i \mapsto b_i\) extend to isomorphisms \(\alpha\) and \(\beta\) from \(P\) to \(G\), respectively. Then \(\alpha^{-1}\beta\) is an automorphism of \(G\) mapping \(a_i\) to \(b_i\). The converse is straightforward. ■

Corollary 2 \(|\text{Aut}(G)| = |S_P|\) for every presentation \(P\) of finite rank of the group \(G\).

Each function \(f : A \to B\) extends, in a natural way, to a function \(f^* : A^n \to B^n\) defined by \((a_1, \ldots, a_n) \mapsto (a_1f, \ldots, a_nf)\). As usual, we will denote \(f^*\) also by \(f\), provided no confusion arises from such simplification. Hence, for each \(n\)-tuple \(b \in B^n\), \(bf^{-1}\) will denote the set \(\{a \in A^n \mid a \beta = b\}\).

2 Presentations of double coverings

Let \(p \in \mathbb{N}\). An epimorphism \(\pi : \hat{G} \to G\) will be called a \(p\)-covering (of \(G\)) if it has kernel of size \(p\) and a central covering (of \(G\)) if its kernel is in the center of \(G\).

Remark 1 A \(p\)-covering of \(G\) is determined by a pair \((\hat{G}, N)\) consisting of a group \(\hat{G}\) and a normal subgroup \(N\) of \(\hat{G}\) of cardinality \(p\) such that \(\hat{G}/N\) is isomorphic to \(G\); best known in the literature as an extension of \(G\) by \(N\).

The proof of the following Lemma is straightforward.

Lemma 3 Every double covering (2-covering) is a central covering.

From now on, let \(P = \langle X \mid R \rangle\) be a presentation of a group \(G\), where \(X = \langle x_1, \ldots, x_n \rangle\) and \(R = \langle r_1, \ldots, r_m \rangle\) and \(\pi : \hat{G} \to G\) be a double covering. Since \(\langle i \mid i^2 \rangle\) is a presentation of \(\text{Kern}(\pi) \cong C_2\), which lies in the center of \(G\) (Lemma 3), applying Johnson [9], Chapter 10, to this case we get the following result.

Lemma 4 For every presentation \(n\)-tuple \(a \in S_P\) and every \(\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n) \in a\pi^{-1}\) there is \(J \in \{1, i\}^m = C_2^m\) such that

\[
P_J = \langle X, i \mid R = J, i^2 = 1, [i, X] = 1 \rangle\]

is a presentation of \(\hat{G}\) with \((\hat{a}, i) = (\hat{a}_1, \ldots, \hat{a}_n, i) \in S_P\).

For simplicity we will omit in the expression \(\pi\) the obvious relations \(i^2 = 1\) and \([i, X] = 1\) that the central involution \(i\) naturally satisfies, and write

\[
P_J = \langle X, i \mid R = J \rangle.
\]

If \(J = 1\) then \(P_J\) is a presentation of \(G \times C_2\) of rank \(n + 1\). If \(J \neq 1\) then there is \(k \in \{1, \ldots, m\}\) such that \(r_k = i\). Then \(i \in \langle x_1, \ldots, x_n \rangle\) and therefore we write

\[
P_J = \langle X \mid R = J \rangle
\]

and regard \(P_J\) as a presentation of rank \(n\) with \(\hat{a} \in S_P\).

Remark 2 For every \(J \in C_2^m\) we have \(P_J/\langle i \rangle \cong G\) and therefore \((P_J, \langle i \rangle)\) is, according to Remark 1, an extension of \(G\) by the subgroup of \(P_J\) generated by \(i\). Two cases may occur: either \(i\) is in the normal closure \(K\) generated by the relators of \(P_J\) in the free group \(F(X, i)\) or not. In the first case \(P_J\) is a presentation of \(G\). In the second case \(i \notin K\) and \((P_J, \langle i \rangle)\) is an extension of \(G\) by \(C_2 \cong \langle i \rangle\) i.e. \(P_J\) is a double covering of \(G\). We call \(P_{(i, \ldots, i)}\) the binary of \(G\), and denote it by \(\hat{G}\), if it is a double covering of \(G\). Not every such presentation gives always a binary group, there are cases in which \(P_{(i, \ldots, i)}\) collapses. For example, \(G = PSL(2, 2f)\), for \(f > 2\), and some metacyclic groups \(G\) (see Table 2 where \([P_{(i, i, i)}] = [P_{(1, 1, i)}] \in \#1\), 5 and \([P_{(i, i, i)}] = [P_{(1, 1, i)}] \in \#7\) the binary presentation collapses to a presentation of \(G\). In the case of \(G = D_n, A_4, S_4\) and \(A_5\) the binary presentation \(P_{(i, i, i)}\) coincides with the usual binary group \(\hat{G}\) (see [10]).
Let $I = (i_1, \ldots, i_n)$ be an element of the elementary 2-group $Kern(\pi)^n \cong C_2^n$. Seeing $X$ as an element of $\hat{G}$ we may write $IX = (i_1 x_1, \ldots, i_n x_n)$. Since $i$ is in the centre of $G$ we have
\[
r_k(IX) = r_k(i_1 x_1, \ldots, i_n x_n) = r_k(i_1, \ldots, i_n) r_k(x_1, \ldots, x_n) = r_k(I) r_k(X)
\]
and therefore $R(IX) = R(I) R(X)$ in the direct product $F(X, i)^m$.

**Lemma 5** Let $J \in C_2^m$ and $P_J$ be a presentation of $\hat{G}$ (a double covering of $G$). For any $I \in C_2^m$, $P_{R(I),J}$ is another presentation of $\hat{G}$; that is, $P_{R(I),J} \sim P_J$.

**Proof.** By changing generators $(X, i)$ to $(Y, i) = (IX, i)$ we get $P_J = \langle X, i \mid R = J \rangle \sim \langle Y, i \mid R = R(I)J \rangle = P_{R(I),J}$. ■

Let $R(C_2^m) = \{ R(I) \mid I \in C_2^m \} \subset C_2^m$. It is easy to check that $R(C_2^m)$ is a normal subgroup of $C_2^m$. The following corollaries are easy consequence of Lemma 5.

**Corollary 6** Given a group $G$ with presentation $\langle X \mid R \rangle$ the number of presentation classes is given by the index of $R(C_2^m)$ in $C_2^m$.

**Corollary 7** Given a group $G$ with presentation $\langle X \mid R \rangle$ the number of (non-isomorphic) double coverings of $G$ is less or equal than the index $[C_2^m : R(C_2^m)]$.

Equality is not always reached as the examples in \[\] will show. However, according to Lemma 5 we may cluster the $2^m$ presentations $P_J$, $J \in C_2^m$, in equivalence classes given by the equivalence relation
\[
P_J \approx P_L \iff L \in R(C_2^m)J.
\]
Each equivalence class $[P_J]$ contains $|R(C_2^m)|$ presentations of the same group $\hat{G}$ and therefore $[P_J]$ will be called a presentation class of $\hat{G}$. As remarked above, if $P_J$ is not a presentation of a double covering of $G$ then it is a presentation of $G$. We observe that this can not happen if $P_J \in [P_1]$ since $P_1$ (and therefore $P_J$) is a presentation of the direct product $\hat{G} = G \times C_2$. Two distinct presentation classes may be presentation classes of the same double covering $G$. However, in the particular case of $\hat{G} = G \times C_2$ this can not happen as the following theorem shows.

**Theorem 8** Let $P$ be a presentation of the group $G$. Then $G \times C_2$ has only one presentation class, namely $[P_1]$.

**Proof.** Let $\pi : G \times C_2 \to G$ be the canonical projection. Suppose that $J \in C_2^m$ gives rise to a presentation $P_J \notin [P_1]$ of $G \times C_2$. In this case each presentation in $[P_J]$ has rank $n$. By Lemma 4 there are $a \in S_P$ and $\hat{a} \in a^{-1}$ such that $\hat{a} \in S_{P_J}$. Then by Lemma 4 and Lemma 5 for every $\hat{x} \in a^{-1}$ there is $P_L \in [P_J]$ such that $\hat{x} \in S_{P_L}$. This shows that $a^{-1} \subset B_n(G \times C_2)$, which is a contradiction since $\hat{x} = ((a_1,1), \ldots, (a_n,1)) \in a^{-1} \setminus B_n(G \times C_2)$. Hence $P_J \in [P_1]$. ■

For practical reasons, it is convenient to simplify, whenever possible, the presentation $P$ of $G$ in order to emphasize the number of cosets of $R(C_2^m)$ in $C_2^m$. An obvious simplification is given by choosing an “efficient” presentation of $G$, i.e. a presentation with maximal deficiency, where the deficiency of a presentation is the non-positive integer given by the difference between the rank and the number of relators of the presentation. Another possibility is given by presentations of “simple type” defined below.

Let $w$ be an element of the free group $F(n) = F(x_1, \ldots, x_n)$. By the $i^{th}$-exponent-sum of $w$ we mean the image $w_i$ of $w$ under the epimorphism $F(n) \to \mathbb{Z}$, $x_j \to \delta_{ij}$, $j = 1, \ldots, n$, where $i \in \{1, \ldots, n\}$ and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

If a relator $w = r_k$ of $P$ has even exponent-sum $w_i$, for all $i$, we say that $w$ is even, otherwise we say that $w$ is odd. Let $E_P$ and $O_P$ be respectively the subset of even and odd relators of $P$. Of course $E_P \cup O_P$ is the set of all relators of $P$. We say that $P$ is of simple type if:

- Each $w \in O_P$ gives rise to only one odd $i^{th}$-exponent-sum $w_i$;
For each \( i \in \{1, \ldots, n\} \) there is at most one \( w \in O_P \) such that \( w_i \) is odd.

In this case we call the number \( d = |O_P| \) the degree of the presentation \( P \).

**Theorem 9** If \( P \) is a presentation of simple type of \( G \) of rank \( n \) with \( m \) relators and degree \( d \) then \( d \leq n \) and the index of \( R(C^m_2) \) in \( C^m_2 \) is \( 2^{|E_P|} = 2^{m-d} \).

**Proof.** That \( d = |O_P| \leq |X| = n \) follows from the definition. Suppose, without loss of generality, that \( O_P = (r_1, \ldots, r_d) \) and \( E_P = (r_{d+1}, \ldots, r_m) \). Then \( R(C^m_2) = O_P(C^m_2) \times E_P(C^m_2) \) is a normal subgroup of \( C^m_2 = C_2^{O_P|+E_P|} = C_2^{O_P|} \times C_2^{E_P} \). Now, since \( O_P(C^m_2) \cong C_2^{O_P|} \) and \( E_P(C^m_2) = \{(1, \ldots, 1)\} < C_2^{E_P|} \), we have that \( |C^m_2 : R(C^m_2)| = |C_2^{E_P|} : E_P(C^m_2)| = 2^{|E_P|} = 2^{m-d} \). \( \blacksquare \)

Let \( P \) be a presentation of simple type of \( G \) of degree \( d \). By Theorem 9 and Corollary 10 \( P \) gives rise to \( 2^{|E_P|} \) presentation classes. Analysing the structure of the elements of each presentation class we get the following useful result for presentation of simple type.

**Corollary 10** Each presentation class \( [P_L] \) contains \( 2^d \) presentations and is represented by a unique presentation of the form \( P_L = \langle X, i \mid O_P = 1, E_P = L \rangle \) with \( L = (i_{d+1}, \ldots, i_m) \in \{1, i\}^{m-d} \), where \( m \) is the number of relators of \( P \).

### 3 Strong double coverings

A \( p \)-covering \( \pi: \hat{G} \to G \) induces an equivalence relation \( \sim_{\pi} \) on \( \hat{G} \) defined by \( \hat{g}_1 \sim_{\pi} \hat{g}_2 \iff \hat{g}_1 \pi = \hat{g}_2 \pi \). The equivalence classes are the cosets \( [\hat{g}]_{\pi} = \hat{g} \text{Kern}(\pi) \) and the class partition is

\[
\hat{G}/\pi = \hat{G}/\sim_{\pi} = \hat{G}/\text{Kern}(\pi) \cong G.
\]

If \( \psi \) is an automorphism of \( \hat{G} \) satisfying \( \text{Kern}(\pi) \psi \subset \text{Kern}(\pi) \) then

\[
\psi^\pi: \hat{G}/\pi \to \hat{G}/\pi, [\hat{g}]_{\pi} \mapsto [\hat{g} \psi]_{\pi}
\]

is a well-defined function satisfying \( \hat{g} \pi \psi^\pi = \hat{g} \psi \pi \) for every \( \hat{g} \in \hat{G} \) i.e. the following diagram commutes

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{\psi} & \hat{G} \\
\pi \downarrow & & \pi \downarrow \\
G & \xrightarrow{\psi^\pi} & G
\end{array}
\]

It is straightforward to show that \( \psi^\pi \) is an automorphism. Since \( \psi_1^\pi \psi_2^\pi = (\psi_1 \psi_2)^\pi \) for every \( \psi_1, \psi_2 \in \text{Aut}(\hat{G}) \) we have the following statement.

**Lemma 11** If \( \text{Kern}(\pi) \) is a characteristic subgroup of \( \hat{G} \), then \( \zeta^\pi : \text{Aut}(\hat{G}) \to \text{Aut}(G) , \psi \mapsto \psi^\pi \), is a well-defined homomorphism.

When well-defined, \( \zeta^\pi \) maps inner automorphisms of \( \hat{G} \) to inner automorphisms of \( G \), but in general it may be not onto. If \( \pi: \hat{G} \to G \) is a \( p \)-covering with characteristic kernel and \( \zeta^\pi : \text{Aut}(\hat{G}) \to \text{Aut}(G) \) is an epimorphism we say that \( \pi \) is a strong covering, or a \( q \)-strong \( p \)-covering if \( \zeta^\pi \) is a \( q \)-covering.

Let \( \pi: \hat{G} \to G \) be a \( p \)-covering and let \( \hat{a} \in S_{\hat{P}} \) where \( \hat{P} \) is a presentation of \( \hat{G} \). Further let \( P \) be a presentation of \( G \) with \( a = \hat{a} \pi \in S_P \).

**Theorem 12** If \( \text{Kern}(\pi) \) is a characteristic subgroup of \( \hat{G} \), then

(a) \( \hat{b} \pi \in S_P \) for every \( \hat{b} \in S_{\hat{P}} \);

(b) \( |\text{Kern}(\zeta^\pi)| = |a \pi^{-1} \cap S_P| \);
Proof. (a) Let \( \hat{b} \in S_P \) and, according to Theorem 1, let \( \psi \in Aut(\hat{G}) \) such that \( \hat{a} \psi = \hat{b} \). Then \( \hat{b} \pi = \hat{a} \psi \pi = \hat{a} \pi \psi = a \psi \pi = a \pi \psi = a \pi \psi^{-1} \). As \( a \in S_P \) the statement follows from Theorem 1 and the fact that for every \( \psi \in \pi \), \( \hat{a} \pi \psi \) is an epimorphism and by Theorem 1 \( \hat{b} \pi \in S_P \) since \( \psi \pi = \psi \). \( \hat{a} \psi \pi = \hat{a} \pi \psi = a \pi \psi = a \pi \psi^{-1} \).

(b) \( \psi \in Ker(\pi) \) if and only if \( \hat{a} \pi \psi = a \pi \psi^{-1} \). Then \( \hat{a} \pi \psi \) is an epimorphism if and only if \( \hat{a} \pi \psi^{-1} \). According to Theorem 1 this is equivalent to

\[
\forall b \in S_P, \exists \hat{b} \in S_P, \hat{b} \pi = b
\]

proving that \( \pi \) is onto if and only if \( b \pi^{-1} \cap S_P \neq \emptyset \) for every \( b \in S_P \).

Theorem 13 Let \( \pi : \hat{G} \to G \) be a double covering with characteristic kernel and let \( P \) be a presentation of \( G \). If \( G \) has a presentation \( P_J \) with \( J \neq 1 \) then \( \pi \) is a strong double covering of \( G \) if and only if \( [P_J] \) is the only presentation class of \( \hat{G} \).

Proof. Let \([P_L]\) be a presentation class of \( \hat{G} \) with \( L \in C_2^m \), where \( m \) is the number of relators of the presentation \( P \). Then \([P_L] \neq [P_I] \) since otherwise \( \hat{G} \cong G \times C_2 \) and Theorem 5 will contradict the hypothesis that there is a presentation \( P_J \) of \( \hat{G} \) with \( J \neq 1 \). Hence we can assume \( L \neq 1 \) and regard \( P_L \) as a presentation of rank \( n \), where \( n \) is the rank of the presentation \( P \).

Suppose that \( \pi \) is a strong double covering of \( G \) and let \( \hat{b} \in S_P \). Then \( b = \hat{b} \pi \in S_P \) and \( b \pi^{-1} = \{ I b | I \in C_2^n \} \). As \( \pi \) is an epimorphism then, by Theorem 12 (c), there is \( I \in C_2^n \) such that \( I \hat{b} \in S_P \). This implies \( J = R(I) \hat{b} = R(I)R(b) = R(I)L \). Hence \([P_J] = [P_L]\).

Reciprocally, suppose that \([P_J]\) is the only presentation class of \( \hat{G} \). Let \( b \in S_P \) and \( \hat{b} \in b \pi^{-1} \). Then \( R(\hat{b}) = L \) for some \( L \in C_2^n \). As \([P_J]\) is the only presentation class of \( \hat{G} \) then \( L = R(I)J \) for some \( I \in C_2^n \). Hence \( J = R(I) \hat{b} = R(I)R(b) = R(I) \hat{b} \) and therefore \( I \hat{b} \in b \pi^{-1} \cap S_P \). By Theorem 12 (c) we conclude that \( \pi \) is an epimorphism and therefore \( \pi \) is a strong double covering of \( G \).

Corollary 14 Let \( \pi : \hat{G} \to G \) be a q-strong double covering of \( G \) and let \( P = \langle X | R \rangle \) be a presentation of rank \( n \) of \( G \) with \( m \) relators. If \( \hat{G} \) has a presentation \( P_J \) for some \( J \neq 1 \) then \( q = |Ker(\rho)| \), where \( \rho : C_2^n \to C_2^m \), \( I \to R(I) \).

Proof. Let \( \pi : \hat{G} \to G \) be a strong double covering of \( G \) and let \( \hat{a} \in S_P \) where \( \hat{P} = P_J \) is a presentation of \( \hat{G} \) with \( J \neq 1 \). Then \( \hat{P} \) is a presentation of rank \( n \) of \( \hat{G} \) and, by Theorem 12 (a), \( a = \hat{a} \pi \in S_P \). According to Lemma 1 for every \( \hat{a} \in \pi^{-1} \) there is \( L \in C_2^n \) such that \( P_L \) is a presentation of \( \hat{G} \) with \( b \in S_P \). By Theorem 13 \( P_L \in [P_J] \), i.e. \( L = R(I)J \) for some \( I \in C_2^n \). Hence

\[
a \pi^{-1} \cap S_P = \{ I | I \in C_2^n \} \cap S_P = \{ I \hat{a} | I \in C_2^n, R(I) = 1 \}
\]

and the statement follows from Theorem 12 (b).

Under the assumptions of Corollary 14 if \( P \) is a presentation of simple type we may deduce from Theorem 4 the following corollary.

Corollary 15 Let \( \pi : \hat{G} \to G \) be a q-strong double covering of \( G \). If \( P \) is a presentation of simple type of degree \( d \) (and rank \( n \)) of \( G \) and \( \hat{G} \) has presentation \( P_J \) for some \( J \neq 1 \), then \( q = 2^n - d \).

The direct product \( G \times C_2 \)

Remark that if \( \pi : \hat{G} \to G \) is a double covering and \( G \) has no central involution, then \( Ker(\pi) \) is a characteristic subgroup of \( \hat{G} \). The following Theorem shows that the converse is true for \( \hat{G} = G \times C_2 \).

Theorem 16 The kernel of the canonical epimorphism \( \pi : G \times C_2 \to G \) is a characteristic subgroup of \( \hat{G} \) if and only if \( G \) has no central involution.
Moreover, every metacyclic group has a presentation of the form $M$ where we can obviously assume $r,s$ satisfy $rs \equiv r \mod m$. Thus

We conclude this section with some results about the special case when $G$ is a simple group without central involutions, i.e. not isomorphic to $C_2$.

**Theorem 18** If $G$ is a simple group not isomorphic to $C_2$ then $G \times \{1\}$ and $\{1\} \times C_2$ are the only proper normal subgroups of $G \times C_2$.

**Proof.** Let $\pi$ be the canonical epimorphism from $G \times C_2$ to the simple group $G \neq C_2$ and let $N$ be a proper normal subgroup of $G \times C_2$. Then $N \pi$ is a normal subgroup of $G$ and the simplicity of $G$ implies $N \pi = \{1\}$ or $G$. If $N \pi = \{1\}$ then $N = \{1\} \times C_2$. If $N \pi = G$ then $N$ has index 2 in $G \times C_2$. As $N \cap (G \times \{1\})$ is a normal subgroup of the simple group $G \times \{1\}$ we conclude that $N = G \times \{1\}$. ■

**Corollary 19** If $G$ is a simple group not isomorphic to $C_2$ then $|\text{Aut}(G \times C_2)| = |\text{Aut}(G)|$.

Since a simple group not isomorphic to $C_2$ has no central involution we have the following corollary.

**Corollary 20** If $G$ is a simple group not isomorphic to $C_2$, then the canonical epimorphism $G \times C_2 \to G$ is a 1-strong double covering.

## 4 Double coverings of metacyclic groups

A group $G$ is called *metacyclic* if it has a normal subgroup $N$ such that both $N$ and $G/N$ are cyclic. It is widely known [2] that the group $M(m,n,r,s) = \langle x, y \mid x^m = 1, y^n = x^r, xy = x^s \rangle$ with parameters $m, n, r, s \in \mathbb{N}$ is a metacyclic group of order less or equal to $mn$ with equality if and only if the parameters satisfy

$$s^n \equiv 1 \mod m \quad \text{and} \quad rs \equiv r \mod m. \quad (2)$$

Moreover, every metacyclic group has a presentation of the form $M(m,n,r,s)$ with $m, n, r, s$ satisfying (2), where we can obviously assume $r, s \leq m$. Cyclic groups, dihedral groups and dicyclic groups are just particular cases of metacyclic groups. So, as an introduction to the general case, we will first handle the cyclic, dihedral and dicyclic groups, in this order. But before that, we give an observation, compute the central involutions and give the presentations pairs for $M(m,n,r,s)$.

### 4.1 Observation

Unless otherwise clearly specified, $M(m,n,r,s)$ is a metacyclic group of order $mn$. Directly from the presentation $M(m,n,r,s)$ we get that for all $\alpha, \beta \in \mathbb{N}$

$$x^\alpha y^\beta = y^\beta x^{\alpha s^\beta}. \quad (3)$$

Thus $M(m,n,r,s) = \{y^px^q \mid 0 \leq p < n, 0 \leq q < m\}$ and so, as a set, $M(m,n,r,s) = \mathbb{Z}_n \times \mathbb{Z}_m$. From (3) we get the following power formula

$$(y^px^q)^k = y^{pk} x^{q \sigma(s^p,k)}, \quad (4)$$

where $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(h,k) \mapsto \sum_{i=0}^{k-1} h^i = \begin{cases} k \frac{h^{k-1}}{h-1} & \text{if } h = 1 \\ \frac{h^{k-1}}{h-1} & \text{otherwise} \end{cases}$. 


4.2 The central involutions of $M(m,n,r,s)$

By (4) if $m$ and $n$ are both odd, then $G = M(m,n,r,s)$ has no involution at all. If $m$ is even, then $x^{m/2}$ is an involution. According to (2), $m$ even implies $s$ odd and from (4) we get

$$x^{m/2} y = y x^{m/2} = y x^{m},$$

which shows that $x^{m/2}$ belongs to the center. If in addition $n$ is odd then $x^{m/2}$ is the unique involution in $G$. If $n$ is even and $p \neq 0$ then $y^p x^q$ is an involution if and only if $p = n/2$ and

$$q(s^{2} - 1) + r \equiv 0 \mod m.$$

Moreover, $y^{n/2} x^q$ is in the center if and only if

$$\begin{cases} (y^{n/2} x^q) x = x (y^{n/2} x^q) & \iff x = x^{s^2} \\ (y^{n/2} x^q) y = y (y^{n/2} x^q) & \iff x^{q^2} = x^q \end{cases} \iff \begin{cases} q(s^{2} + 1) + r \equiv 0 \mod m \\ s^{2} \equiv 1 \mod m \\ q(s - 1) \equiv 0 \mod m \end{cases}$$

Hence $y^{n/2} x^q$ is a central involution if and only if

$$\begin{cases} q(s^{2} + 1) + r \equiv 0 \mod m \\ s^{2} \equiv 1 \mod m \\ q(s - 1) \equiv 0 \mod m \end{cases} \iff \begin{cases} 2q + r \equiv 0 \mod m \end{cases}$$

where $(\ast)$ has solutions $q = \frac{m-r}{2}$ if $m$, $r$ have the same parity and $q = m - \frac{r}{2}$ if $r$ is even.

If $m$ and $r$ are both odd, then $q = \frac{m-r}{2}$ is the only solution of $(\ast)$. In this case $(\circ)$ is equivalent to $(m - r)(s - 1) \equiv 0 \mod 2m$; but since $(m - r)(s - 1) \equiv 0 \mod m$ by (2), and $m$ is odd, then $(\circ)$ is equivalent to $(m - r)(s - 1) \equiv 0 \mod 2$, which is true. Hence $(\circ)$ is superfluous in this case.

If $m$ is odd and $r$ is even, then $q = m - \frac{r}{2}$ is the only solution of $(\ast)$. With a similar argument we see that $(\circ)$ is superfluous in this case too.

If $m$ and $r$ are both even, then $(\ast)$ has two solutions $q_1 = \frac{m-r}{2}$ and $q_2 = m - \frac{r}{2}$. In this case $s$ is odd and $(\circ)$ is equivalent to $\frac{r}{2}(s - 1) \equiv 0 \mod m$. Now the congruence $(\ast)$ says that $M(m, \frac{r}{2}, \frac{r}{2}, s)$ is a metacyclic group.

If $m$ is even and $r$ is odd, then $(\ast)$ has no solution.

In the following table we list the central involutions of $G$ according to the parity of the parameters $m, n, r$ and $s$.

| # | m     | n     | r     | s     | central involutions                                  |
|---|-------|-------|-------|-------|-----------------------------------------------------|
| 1 | odd   | odd   | odd   | odd   | —                                                   |
| 2 | odd   | odd   | odd   | even  | —                                                   |
| 3 | odd   | odd   | even  | odd   | —                                                   |
| 4 | odd   | odd   | even  | odd   | —                                                   |
| 5 | odd   | even  | odd   | odd   | $y^{\frac{r}{2}} x^{\frac{m-r}{2}}$ if $(\ast)$   |
| 6 | odd   | even  | odd   | even  | $y^{\frac{r}{2}} x^{\frac{m-r}{2}}$ if $(\ast)$   |
| 7 | odd   | even  | odd   | odd   | $y^{\frac{r}{2}} x^{\frac{m-r}{2}}$ if $(\ast)$   |
| 8 | odd   | even  | even  | odd   | $y^{\frac{r}{2}} x^{\frac{m-r}{2}}$ if $(\ast)$   |
| 9 | even  | odd   | odd   | odd   | $x^{m/2}$                                           |
| 10| even  | odd   | even  | odd   | $x^{m/2}$                                           |
| 11| even  | odd   | odd   | odd   | $x^{m/2}$                                           |
| 12| even  | even  | odd   | odd   | $x^{m/2}$ and $y^{\frac{r}{2}} x^{\frac{m-r}{2}}, y^{\frac{r}{2}} x^{\frac{m-r}{2}}$ if $(\ast)$ and $\frac{r}{2}(s - 1) \equiv 0 \mod m$ |

Table 1: The central involutions of a metacyclic group.
4.3 Presentation pairs

Let \( a = y^p x^q, b = y^r x^s \in G = M(m, n, r, s) \). From \( \Box \) and \( \mathbb{1} \), the pair \((a, b)\) is a presentation pair for \( G \) if and only if \( \langle a, b \rangle = G \) and

\[
\begin{align*}
pm &\equiv 0 \mod n, \\
pr &\equiv 0 \mod n, \\
p(s - 1) &\equiv 0 \mod n, \\
q^m(s^m, r) &\equiv r^m \mod m,
\end{align*}
\]

By Theorem \( \mathbb{1} \) and Corollary \( \mathbb{2} \) these equations determine the automorphism group of \( G \).

4.4 Double coverings of cyclic groups

Every double covering of the cyclic group \( C_m = M(m, 1, m, 1) \) with presentation \( P = \langle x \mid x^m = 1 \rangle \) of simple type has presentation

\[
P_1 = \langle x, i \mid x^m \rangle \quad \text{or} \quad P_4 = \langle x \mid x^m = i \rangle \sim \langle x \mid x^{2m} \rangle.
\]

Case \( m \) even. In this case \([P_1]\) and \([P_4]\) are distinct presentation classes of two non-isomorphic double coverings of \( C_m \), namely \( C_m \times C_2 \) and \( C_{2m} \). The first double covering is not strong since \( x^{m/2} \) is a central involution in \( C_m \) (Corollary \( \mathbb{17} \)). On the other hand \( C_{2m} \) is a 2-strong double covering of \( C_m \) since \([P_4]\) is the only presentation class of \( C_{2m} \), and \( C_{2m} \) has only one central involution (Theorem \( \mathbb{13} \) and Corollary \( \mathbb{16} \)).

Case \( m \) odd. In this case there is only one presentation class, namely \([P_1] = \{P_1, P_4\}\), and therefore we have only one double covering \( C_m \times C_2 \cong C_{2m} \). Since this has only one central involution, the double covering is 1-strong (Theorem \( \mathbb{13} \) and Corollary \( \mathbb{16} \)).

Diagram 1: The double coverings of cyclic groups.

4.5 Double coverings of dihedral groups

Consider the dihedral group \( D_m = M(m, 2, m, m - 1) \) with presentation \( P = \langle x, y \mid x^2, y^2, (xy)^m \rangle \), where \( m > 1 \). The transposition of the two generators \( x, y \) of \( P \) gives rise to an automorphism of \( D_m \). This implies that \( P_{1,1,1} \sim P_{1,1,1} \) though \( P_{1,1,1} \neq P_{1,1,1} \), where \( \varepsilon = 1, i \).

Case \( m \) even. In this case \( P \) is a presentation of simple type of degree \( d = 0 \). It lifts to 8 presentation classes, each containing a single presentation. As seen above, \( P_{1,1,1} \sim P_{1,1,1} \) and hence we have at most 6 potentially distinct double coverings of \( D_m \).

D1) \( P_1 \) is a presentation of \( D_m \times C_2 \).

D2) \( P_{1,1,1} \sim \langle x, y \mid x^2, y^2, (xy)^{2m} \rangle \) which is a presentation of \( D_{2m} \).

D3) \( P_{1,1,1} \sim P_{1,1,1} \sim \langle y, z \mid y^4, z^2, (yz)^m, [y^2, z] \rangle \). Adding \( x = (yz)^2 \leftrightarrow y^2 = y^{-1}x \) to the presentation we get

\[
\langle y, z \mid y^4, z^2, (yz)^m, [y^2, z] \rangle \sim \langle x, y, z \mid x^2, y^4, xy, z^2, x^2z, x^{-1}yy^2 \rangle \equiv \langle (x \times (y)) \times (z) \rangle.
\]

Note that \( x^{-1} = (zy)^{-1} = (zy)^2 = x^2 = x^y \). Hence \( P_{1,1,1} \) is a presentation of \( (C_4 \rtimes C_4) \times C_2 \).
D4) \( P_{(i,i)} \sim P_{(i,i,i)} \sim \langle y, z \mid (yz)^m y^2, z^2, (yz)^2m, [y^2, z] \rangle \). Adding \( x = (yz)^2 \Rightarrow y^z = y^{-1}x \) to the presentation (note that \( x^{-1} = (yz^{-1})^2 = (zy)^2 = x^z = x^y \)) we get

\[
\langle a, b \mid a^n b^2, a^2 b^2, a^3 b^2, a b, a b \rangle \sim \langle a, b \mid a^n b, a a b \rangle.
\]

This is a presentation of \( Q_m \times C_2 \), where \( Q_{2n} \) is the generalized quaternion group of order \( 4n \) with presentation

\[
\langle a, b \mid a^n b^2, a^2 b^2, a^3 b^2, a b, a b \rangle.
\]

D5) \( P_{(i,i,1)} \sim \langle x, y \mid x^4, x^2 y^2, (xy)^m, [x^2, y] \rangle \). Changing generators \( a = xy \) and \( b = x \) we get \( P_{(i,i,1)} \sim \langle a, b \mid a_n b, a b \rangle \), which is a presentation of \( C_m \times C_4 \).

D6) \( P_{(i,i,i)} \sim \langle x, y \mid x^2 y^2, (xy)^m y^2, y^4 \rangle \). Changing generators \( a = xy \) and \( b = y \) we get \( P_{(i,i,i)} \sim \langle a, b \mid a^m b^2, b^4, a b \rangle \) which is a presentation of \( Q_{2m} \).

**Case \( m \) odd.** In this case we have \( P_{(j_1,j_2,1)} \sim P_{(j_2,j_1,1)} \approx P_{(j_1,j_2,j_2),i} \sim P_{(j_1,j_2,j_2),i} \), for every \( j_1, j_2 \in \{1, i\} \).

Moreover, for every \( (j_1, j_2, j_3) \in \{1, i\}^3 \) we have

\[
\begin{align*}
j_3 &= (xy)^m = (j_1x^{-1}j_2y^{-1})^m = j_1j_2^m = j_1j_2^m (xy)^y^{-1} = \frac{j_1j_2^m}{j_1j_2^m} = j_1j_2j_3,
\end{align*}
\]

that is \( j_1j_2j_3 = 1 \). If \( j_1 \neq j_2 \) then \( i = 1 \) and \( P_{(j_1,j_2,j_3)} \) is a presentation of \( D_m \), hence not a double covering. Thus \( j_1 = j_2 \) and we have two cases. Namely,

D1) \( P_{(1,1,1)} \approx P_{(1,1,i)} \) are presentations of \( D_m \times C_2 \cong D_{2m} \).

D2) \( P_{(i,i,1)} \approx P_{(i,i,i)} \) are presentations of \( Q_{2m} \). This can be easily seen by changing generators \( a = xy \) and \( b = y \).

If \( m = 2 \) then \( D_{2m} \cong (C_m \times C_4) \times C_2 \cong Q_m \times C_2 \cong C_m \times C_4 \) in which case we have 3 non-isomorphic double coverings of \( D_2 = V_4 \), namely \( D_2 \times C_2, D_4 \) and \( Q_4 \). While the first two are not strong, \( Q_4 \) is a 4-strong double covering.

For \( m > 2 \) we have 6 or 2 non-isomorphic double coverings of \( D_m \), according as \( m \) is even or odd. As both \( D_{2m} = M(2m, 2, 2m, 2m - 1) \) and \( Q_{2m} = M(2m, 2, 2m, 2m - 1) \) have only one central involution (Table 1, lines 11 and 12) and only one presentation class, they are \( q \)-strong double coverings of \( D_m \), where \( q = 2 \) if \( m \) is odd and \( q = 4 \) if \( m \) is even (Theorem 13 and Corollary 14). If \( m \) is odd, these are the only double coverings of \( D_m \).

Let \( m \) be even. Then \( D_m \) has a central involution and therefore \( D_m \times C_2 \) is not a strong double covering of \( D_m \) (Corollary 17). The double coverings \( (C_m \times C_4) \times C_2 \) and \( Q_m \times C_2 \) are not strong since both have more than one presentation class (Theorem 13). The double covering \( \hat{G} = C_m \times C_4 \), having presentation

\[
\langle x, y \mid x^m, y^4, xy^2 \rangle \sim \langle x, y \mid x^m = 1, y^4 = x^m, xy^{-1} = M(4, m, m - 1) \rangle.
\]

is a metacyclic group with central involutions \( x^{m/2}, y^2, x^{m/2} \) and \( y^2 \) (Table 1, #12). Now \( [P_{(i,i,1)}] \) is the only presentation class of \( \hat{G} \). Then \( \hat{G} \) is strong (hence 4-strong by Corollary 14) if and only if \( i = y^2 \) is fixed by every automorphism of \( \hat{G} \) (Theorem 13). If \( \psi \in Aut(\hat{G}) \) maps \( \langle x, y \rangle \) to \( \langle a, b \rangle = (y^u x^a, y^u x^b) \) then \( \langle a, b \rangle \) is another presentation pair for \( M(4, m, m - 1) \). By 13,

\[
\begin{align*}
p(m - 2) &= 0 \quad \text{mod} \ 4 \quad \text{and} \quad p(m - 2) = 0 \quad \text{mod} \ 4
\end{align*}
\]

which is equivalent to \( p \) even (0 or 2). Then \( \langle a, b \rangle = \hat{G} \) implies \( u \) odd (1 or 3). Thus

\[
\begin{align*}
i\psi &= y^2 \psi = b^2 = (y^u x^a)^2 = y^{2u} x^{(1 + (1)^u)} = y^{2u} = y^2 = i,
\end{align*}
\]

proving that \( \hat{G} = C_m \times C_4 \) is a strong double covering of \( D_m \).
4.6 Double coverings of dicyclic groups

A dicyclic group, often called “generalised quaternion group”, is a group \( Q_{2m} = M(2m, 2, m, 2m - 1) \) of order \( 4m \) with presentation

\[
P = \langle x, y \mid x^m y^{-2}, xy \rangle.
\]

This presentation has deficiency zero and is of simple type of degree 0 or 1 according as \( m \) is even or odd.

**Case \( m \) even.** In this case \( P \) lifts to 4 singular presentation classes \([P_1], [P_{(i,1)}], [P_{(i,2)}], [P_{(i,3)}] \) and \([P_{(i,4)}] \).

Q1) \( P_1 \) is a presentation of \( Q_{2m} \times C_2 \).

Q2) \( P_{(i,1)} \sim \langle x, y \mid x^m y^4, xy \rangle \) which is a presentation of \( C_{2m} \times C_4 \).

Q3) \( P_{(i,2)} \sim P_{(i,3)}, \) by changing generators \( a = x, b = xy \) and \( i = i \), which is a presentation of the binary group \( \tilde{Q}_{2m} = P_{(i,1)} \).

**Case \( m \) odd.** In this case \( P \) lifts to 2 presentation classes.

Q1) \( P_1 \approx P_{(i,1)} \) are presentations of \( Q_{2m} \times C_2 \).

Q2) \( P_{(i,2)} \approx P_{(i,3)} \) are presentation of the binary group \( \tilde{Q}_{2m} = P_{(i,1)} \). Since

\[
P_{(i,1)} = \langle x, y \mid x^m = y^2, xy = i \rangle
\]

\[
\sim \langle x, y \mid x^m = y^2, xy = i, x^{2m} = i \rangle
\]

\[
\sim \langle x, y \mid x^{4m} = 1, y^2 = x^m, xy = x^{2m-1} \rangle
\]

we see that the binary \( \tilde{Q}_{2m} \) is a metacyclic group \( M(4m, 2, m, 2m - 1) \). Since \( \tilde{Q}_{2m} \) has a central involution (Table 1, \#11 and 12), \( Q_{2m} \times C_2 \) is not strong (Corollary 17). If \( m \) is even then \( \tilde{Q}_{2m} \) is not strong as well since it has two presentation classes (Theorem 13). If \( m \) is odd then \( \tilde{Q}_{2m} \) is a metacyclic group \( M(4m, 2, m, 2m - 1) \) which has a unique central involution (Table 1, \#11) and is therefore 2-strong (Theorem 13 and Corollary 15). Finally, if \( m \) is even, \( C_{2m} \times C_4 \not\cong \tilde{Q}_{2m} \) is a 4-strong double covering of \( Q_{2m} \) (Remark 3, Theorem 13 and Corollary 15).
4.7 Double coverings of metacyclic groups: The general case

Let $G$ be a metacyclic group with presentation $P = \langle m, n, r, s \rangle$ where the parameters $m, n, r, s \in \mathbb{N}$ satisfy (2) and assume $r, s \leq m$. In the following, for $i \in \{0, 1\}$, let $i^r \in C_2$ have the usual meaning $i^0 = 1$ and $i^1 = i$. For every $J = (i^r, i^s, i^t) \in C_2^3$, where $C_2 = \{1, i\}$, $P_J = \langle x, y \mid x^{m} = i^r, y^n = x^s i^t, x^y = x^s i^t \rangle$ with $m, n, r$ and $s$ satisfy (2). This is either a presentation of a double covering of $G$ or a presentation of $G$. In the following theorems we give necessary and sufficient conditions that assures when $(a), (b)$ and $(d)$ implies $(c)$ ($r, s$ is even). It also comes from (a) and (c) that

\[ r \equiv n \equiv 1 \mod 2. \]

Theorem 21 Let $J = (i, i^r, i^t) \in C_2^3$, where $\epsilon, \tau \in \{0, 1\}$. The following statements are equivalent:

(a) $P_J$ is a double covering of $G$.

(b) $P_J$ is a metacyclic group $M(2m, n, r + m\epsilon, s + m\tau)$.

(c) $(s + m\tau)^n \equiv 1 \mod 2m$ and $(r + m\epsilon)(s + m\tau) \equiv r + m\epsilon \mod 2m$.

Proof. It is enough to prove that (a) implies (b). Since $P_J \sim \langle x, y \mid x^{2m} = 1, y^n = x^{r + m\epsilon}, x^y = x^{s + m\tau}, [x^m, y] = 1 \rangle$, $P_J$ is clearly a metacyclic group of order $2mn$. Hence the relation $[x^m, y] = 1$ is superfluous and $P_J = M(2m, n, r + m\epsilon, s + m\tau)$.

Corollary 22 Let $J = (i, 1, i^\tau)$ and $K = (i, i^\epsilon, i^\tau)$, where $\tau \in \{0, 1\}$. Then $P_J$ is a double covering of $G$ if and only if $P_K$ is a double covering of $G$.

Proof. This follows from statement (b) of previous theorem. Assuming $(s + m\tau)^n \equiv 1 \mod 2m$, which implies $s + m\tau$ odd, we have that $r(s + m\tau) \equiv r \mod 2m$ is equivalent to $(r + m)(s + m\tau) \equiv r + m \mod 2m$.

Corollary 23 Let $J = (i, i^\epsilon, 1)$ and $K = (i, i^\epsilon, i^\tau)$ where $\epsilon \in \{0, 1\}$. Then $P_J$ and $P_K$ are both double coverings of $G$ if and only if $s^n \equiv 1 \mod 2m$, $rs \equiv r \mod 2m$ and $m, n, r$ are even.

Proof. From previous theorem we have that $P_J$ and $P_K$ are both double coverings of $G$ if and only if

(a) $s^n \equiv 1 \mod 2m$,

(b) $(r + m\epsilon)s \equiv r + m\epsilon \mod 2m$,

(c) $(s + m)^n \equiv 1 \mod 2m$,

(d) $(r + m\epsilon)(s + m) \equiv r + m\epsilon \mod 2m$.

Assuming (a), which implies $s$ odd, then (b) is equivalent to $rs \equiv r \mod 2m$. As (c) implies $s + m$ odd, it follows from (a) and (c) that $m$ is even. It also comes from (a) and (c) that $n$ is even, since $n > 1$ and

\[(s + m)^n = s^n + mn s^{n-1} + m^2 \kappa, \quad \text{for some } \kappa \in \mathbb{N}.\]

On the other hand, (b) and (d) imply $r + m\epsilon$ even which together with $m$ even implies $r$ even. Reciprocally, (a), (b) and $m, n, r$ even imply (c) and (d).

Theorem 24 Let $J = (1, i^\epsilon, i^\tau) \in C_2^3$ where $\epsilon \in \{0, 1\}$. Then $P_J$ is a double covering of $G$ if and only if $m, n$ and $r$ are even, in which case $P_J$ is a metabelian group of order $2mn$.

Proof. ($\Rightarrow$): Suppose that $P_J = \langle x, y \mid x^m = 1, y^n = x^i i^r, x^y = x^i i^r \rangle$ is a double covering of $G$. Then $i \neq 1$ and from condition (2) we get

\[ 1 = (x^y)^m = (x^s i^r)^m = i^m, \]

\[ x = x^y = x^{s^n} i^{\sigma(s, n)} = x^i \sigma(s, n), \quad \text{where } \sigma(s, n) = \sum_{k=0}^{n-1} s^k, \]

\[ x^r = (x^y)^r = (x^s i^r)^r = x^{r s i^r} = x^r i^r. \]
Then $m, \sigma(s, n)$ and $r$ are even. Hence, by the condition $\mathcal{E}, \mathcal{s}$ is odd and so $i^{\sigma(s, n)} = 1$ is equivalent to $i^n = 1$ which implies that $n$ is even.

($\Leftarrow$): Since $N = \langle x, i \rangle$ is a normal subgroup of $P_J$ and $P_J/N \cong C_n$, we have that $|P_J| = |N||n|$. Taking into account that $m, n, r$ even implies $\sigma(s, n)$ odd, since by condition $\mathcal{E}$ $s$ odd, by the Reidemeister-Schreier rewriting process we get that $\langle x, i \mid x^m, i^2, [x, i] \rangle$ is a presentation for $N$. Hence $N \cong C_m \times C_2$. This proves that $P_J$ is a metabilian group of order $2mn$ and hence a double covering of $G$. ■

**Theorem 25** $P_{(1,i,1)}$ is the metacyclic group $M(m, 2n, 2r, s)$ and hence a double covering of $G$.

**Proof.** Taking into account $\mathcal{E}$ we get $P_{(1,i,1)} \sim \langle x, y \mid x^m = x^{2r}, x^y = x^s \rangle$. As $s^n \equiv 1 \mod m$ implies $s^{2n} \equiv 1 \mod m$ and as $rs \equiv r \mod m$ implies $2rs \equiv 2r \mod m$, $P_{(1,i,1)}$ is the metacyclic group $M(m, 2n, 2r, s)$ of order $2mn$ and therefore a double covering of $G$. ■

**Corollary 26** The following statements are equivalent:

(a) For all $J \in C_2^3$, $P_J$ is a presentation of a double covering of $G$.

(b) $m, n, r$ are even and $s^n \equiv 1 \mod m$, $rs \equiv r \mod 2m$.

**Proof.** The statement follows from Corollary $\mathcal{E}$ and Theorems $\mathcal{E}$ and $\mathcal{F}$ ■

**Lemma 27** If $(s + mt)^n \equiv 1 \mod m$ holds for $\tau = 0, 1$ and $m$ is even, then $\frac{s^{n-1}}{m}$ and $\frac{(s + m)^{n-1}}{m}$ have the same parity if and only if $n$ is even.

**Proof.** Since $(s + m)^n = s^n + ns^{n-1} + \kappa m$ for some $\kappa \in \mathbb{N}$ we can write $\frac{(s + m)^{n-1}}{m} = \frac{s^{n-1}}{m} + ns^{n-1} + \kappa m$. Now, since $\kappa m$ is even and $ns^{n-1}$ has the same parity as $n$ (as $s$ is odd), then $ns^{n-1} + \kappa m$ has the same parity as $n$. Hence $\frac{(s + m)^{n-1}}{m}$ and $\frac{s^{n-1}}{m}$ have the same parity if and only if $n$ is even. ■

This lemma together with Theorem $\mathcal{A}$ gives the following corollary.

**Corollary 28** If $m, n$ are even and $\frac{s^{n-1}}{m}$ is odd, then for any $J = (i, i', i'' \rangle$, where $\epsilon, \tau \in \{0, 1\}$, $P_J$ is not a double covering of $G$.

Table below displays the presentation classes $[P_J]$ of possible double coverings of the metacyclic group $G$ with presentation $P = M(m, n, r, s)$ according to the parity of the parameters. The number of such presentation classes is an upper bound of the number of (non-isomorphic) double coverings of $G$ (Theorem $\mathcal{G}$). On the right column we display a better upper bound $\delta$ based on Theorems $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and Corollary $\mathcal{D}$. The erased presentation classes are, according to Theorem $\mathcal{D}$ presentation classes of $G$ and therefore not presentation classes of a double covering of $G$.

| # | $m$ | $n$ | $r$ | $s$ | $[P_J]$ | $\delta$ |
|---|---|---|---|---|---|---|
| 1 | odd | odd | odd | odd | $[P_1] = [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 1 |
| 2 | odd | odd | odd | even | $[P_1] = [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 1 |
| 3 | odd | odd | even | odd | $[P_1] = [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 1 |
| 4 | odd | odd | even | even | $[P_1] = [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 1 |
| 5 | odd | even | odd | odd | $[P_1], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}, \{P_{(1, i'', 1)}\}$ | 2 |
| 6 | odd | even | odd | even | $[P_1], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}, \{P_{(1, i'', 1)}\}$ | 2 |
| 7 | odd | even | even | odd | $[P_1], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}, \{P_{(1, i'', 1)}\}$ | 2 |
| 8 | odd | even | even | even | $[P_1], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}, \{P_{(1, i'', 1)}\}$ | 2 |
| 9 | even | odd | odd | odd | $[P_1] = [P_{(1, i, 1)}], [P_{(1, i, 1)}], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 2 |
| 10 | even | odd | even | odd | $[P_1] = [P_{(1, i, 1)}], [P_{(1, i, 1)}], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 2 |
| 11 | even | odd | even | even | $[P_1] = [P_{(1, i, 1)}], [P_{(1, i, 1)}], [P_{(1, i, 1)}], \{P_{(1, i', 1)}\}$ | 2 |
| 12 | even | odd | even | even | $[P_1], [P_{(1, 1)}, \ldots, P_{(1, 1)}], J_1 \in C_3 \backslash \{1\}$ | 8 |

Table 2: The double coverings of metacyclic groups.
Remark 4 In #1-4 there is exactly one double covering, $G \times C_2$. In #5-8 there are exactly two double coverings, namely $G \times C_2$ and the metacyclic group $M(m, 2n, 2r, s)$ (Theorem 26) which are non-isomorphic according to Theorem 27. Hence the upper bound $\delta = 2$ is always attained.

In #9-11 Corollary 28 justifies the upper bound $\delta = 2$ which is also always attained. In fact, $m$ even implies

$$\frac{(s + m)^n - 1}{m} = \frac{s^n - 1}{m} + ns^{n-1} + 2\kappa,$$

for some non-negative integer $\kappa$. Multiplying $5$ by $r$ we get

$$r\frac{(s + m)^n - 1}{m} = \frac{r(s + m - 1)}{n} \mod 2,$$

since $r\frac{s^n - 1}{m} = \sum_{k=0}^{n-1} s^k \equiv \frac{r(s-1)n}{m} \mod 2$. Now, $5$ and $6$ together with the following equality

$$r\frac{(s + m - 1)}{m} = \frac{r(s-1)}{m} + r$$

guarantee in #9-11 that $(s + m + 1) \equiv 0 \mod 2m$ and $r(s + m + 1) \equiv r \mod 2m$, for some $r \in \{0, 1\}$. According to Theorem 21 this shows that $\delta = 2$ is always attained.

We have just seen that the upper bound $\delta$ in #1-11 is the exact number of non-isomorphic double coverings of $G$. The same cannot be said in #12. For example the dihedral groups $D_m$ and the dicyclic groups $Q_{2m}$ for $m$ even falls in this line and the number of their double coverings is 6 and 3 respectively.

The direct product $G \times C_2$ is the only double covering of $G$ in #1-4 and since it has presentation $P_{(1,1,1)}$ it is a metacyclic group $M(m, 2n, 2r, s)$ (Theorem 26). As $G$ has no central involutions (Table 1, #1-4), $G \times C_2$ is $1$-strong (Corollaries 17 and 14).

In #5-8, $G \times C_2$ has presentation $P_{1} \approx P_{(s', r')}$ for some $\epsilon, \tau \in \{0, 1\}$ with $r + m \epsilon$ even, hence $G \times C_2$ is the metacyclic group $M(2m, n, r + m \epsilon, s + m \tau)$ (Theorem 21). Now $G$ has a central involution if and only if $s^2 \equiv 1 \mod m$ (Table 1, #5-8), hence $G \times C_2$ is a strong double covering of $G$ if and only if $s^2 \not\equiv 1 \mod m$ (Corollary 17), being $2$-strong in such case (Corollary 14). The other double covering of $G$ has presentation $P_{(1,1,1)}$, which is a metacyclic group $M(m, 2n, 2r, s)$ (Theorem 26) with one central involution only (Table 1, #7 and #8) and therefore a $2$-strong double covering (Theorem 8, Theorem 15 and Corollary 14).

In #9-11, $G \times C_2$ is the metacyclic group $M(m, 2n, 2r, s)$ with presentation $P_{(1,1,1)}$. As $G$ has a central involution (Table 1, #9-11), $G \times C_2$ is not strong (Corollary 17). There is one more double covering $\hat{G}$ of $G$ other than $G \times C_2$. This has presentation either $P_{(1,1,1)}$ or $P_{(1,1,1)}$, being $\hat{G}$ a metacyclic group, either $M(2m, n, r, s)$ or $M(2m, n, r, s + m)$ (Theorem 21). In either cases, $\hat{G}$ has only one central involution (Table 1, #9-11) which makes it a $2$-strong double covering of $G$ (Theorem 8, Theorem 15 and Corollary 14).

In #12, $G$ has one or three central involutions (Table 1, #12) and so $G \times C_2$ is not strong. The others 7 (at most) double coverings $\hat{G} \not\cong G \times C_2$ have 1 or more central involutions depending on the parameters. As remarked below, they may have one or more presentation classes. In this case the strongness of $\hat{G}$ depends sharply on the choice of the parameters.

Remark 5 In #12 a little more can be said. According to Corollary 28 if $s^{n-1}$ is odd then $\delta = 4$. The value of $\delta$ cannot be always reduced. Using GAP 12 we see that for $(m, n, r, s) = (10, 4, 10, 3)$ and $(10, 4, 10, 7)$ the eight resulting groups are all non-isomorphic and so the upper bound $\delta = 8$ is reached in these two cases. Observe yet that $M(m, n, r, s)$ has, according to [11], zero deficiency if and only if

$$\gcd \left( m, r, s - 1, \frac{s^n - 1}{m}, \frac{r(s-1)}{m}, \frac{s^n - 1}{s-1} \right) = 1.$$  

In the particular case when $r = \frac{m}{(m,s-1)}$, the metacyclic group $M(m, n, r, s)$ has the following presentation of zero deficiency (see Johnson [9, pg 91]

$$\langle x, y \mid y^n = x^r, [y, x^{-1}] = x^{(m,s-1)} \rangle,$$
for a certain integer \( t \). In this case Table 2 reduces to

| # | \( n \) | \( r \) | \( (m, s - 1) \) | \( \delta \) |
|---|---|---|---|---|
| 1 | odd | odd | odd | 1 |
| 2 | odd | odd | even | 2 |
| 3 | odd | even | odd | 1 |
| 4 | odd | even | even | 2 |
| 5 | even | odd | odd | 2 |
| 6 | even | odd | even | 2 |
| 7 | even | even | odd | 2 |
| 8 | even | even | even | 4 |

Table 3: A particular case.

As we can see, \( \#12 \) of Table 2, which corresponds to \( \#8 \) in Table 3, has the upper bound \( \delta \) dropped to 4.

5 Double coverings of rotary platonic groups

By a rotary platonic group we mean the rotation group of some platonic solid. Thus the rotary platonic groups are \( A_4, S_4 \) and \( A_5 \). For \( n = 3, 4, 5 \),

\[
P = \langle x, y \mid x^3, y^n, (xy)^2 \rangle
\]

is a presentation of simple type of \( A_4, S_4 \) and \( A_5 \), respectively.

1. Double coverings of \( G = A_4, A_5 \) (\( n = 3, 5 \)).

   Being \( n \) odd there are 2 non-isomorphic double coverings (Corollary 7 and Theorems 8 and 9), the direct product \( G \times C_2 \) with presentations \( P_{(j_1,j_2,1)} \) and the binary group \( \hat{G} \) with presentations \( P_{(j_1,j_2,i)} \), where \( j_1, j_2 \in \{1, i\} \).

2. Double coverings of \( S_4 \) (\( n = 4 \)).

   In this case we have \( 4 = 8/2 \) presentation classes:
   1) \( [P_{(1,1,1)}] = [P_{(1,1,1)}] \), giving presentations of \( S_4 \times C_2 \).
   2) \( [P_{(1,i,1)}] = [P_{(i,i,1)}] \), where \( P_{(i,i,1)} \sim \langle x, y \mid x^6 = (xy)^2 = 1, x^3 = y^4 \rangle \) giving presentations of \( GL(2,3) \).
   3) \( [P_{(1,1,0)}] = [P_{(1,1,0)}] \), giving presentations of a group \( B \) described below.
   4) \( [P_{(1,i,0)}] = [P_{(i,i,0)}] \), giving presentations of the binary \( \hat{S}_4 \).

The group \( B \)

In contrast to \( GL(2,3) \) and the binary octahedral group \( \hat{S}_4 \) that are double coverings \( \hat{G} \) of \( S_4 \) with \( C_2 \) in the derived group \( \hat{G}' \), the group \( B \) has \( C_2 \) not in \( B' \); and while for \( GL(2,3) \) and \( \hat{S}_4 \) we have \( G/G' \cong C_2 \) and for \( S_4 \times C_2 \) we have \( G/G' \cong V_4 \), for \( B \) we have \( B/B' \cong C_4 \). The normal subgroups of \( B = \langle x, y \mid x^6, y^4, (xy)^2 x^3, [x^3, y] \rangle \) are: \( A_4 \times C_2 = \langle x, y^{-1}xy^{-1} \rangle \), \( A_4 = B' = \langle x^4, y^{-1}xy^{-1} \rangle \), \( V_4 \times C_2 = \)
\( (x^3y^2, x^2y^2, (xy)^2) \), \( V_4 = (x^3y^2, x^2y^2x) = (x^3y^2)^B \) and \( C_2 = \langle (xy)^2 \rangle \).

\[
\begin{align*}
    &B/C_2 \cong S_4 \\
    &B/C_2 \cong S_4 \\
    &V_4/C_2 \cong S_4 \\
    &A_4 = B'
\end{align*}
\]

Diagram 4: The normal subgroups of \( B \).

Since \( A_4, A_5 \) and \( S_4 \) are centerless, every double covering has exactly one central involution which is obviously fixed by every automorphism. By Theorem 13, every double covering of \( A_4 \), \( A_5 \) and \( S_4 \) is \( q \)-strong, where by Corollary 15, \( q = 1 \) for double coverings of \( A_4 \) and \( A_5 \), and \( q = 2 \) for double coverings of \( S_4 \).

\[
\begin{align*}
    A_4 \times C_2 &\to \tilde{A}_4 \\
    S_4 \times C_2 &\to \tilde{S}_4 \\
    GL(2,3) &\to B \to \tilde{S}_4 \\
    A_5 \times C_2 &\to \tilde{A}_5 \\
    \end{align*}
\]

Diagram 5: The double coverings of the rotary platonic groups.

6 Double coverings of some simple groups

6.1 Double coverings of projective linear groups over odd prime fields

Let \( p \) be an odd prime and \( k = \frac{p+1}{2} \). Then

\[
P = \langle x, y \mid x^2, (xy)^3, (xy^4xy^k)^2 y^p \rangle
\]

is a presentation of the projective special linear group \( PSL(2, p) \) (see 3). There are 2 presentation classes \( [P] \) and for each choice of \( j_2, j_3 \in \{1, i\} \) we have:

1) \( P_{(1,j_2,j_3)} \) is a presentation of \( PSL(2, p) \times C_2 \).

2) \( P_{(i,j_2,j_3)} \) is a presentation of \( SL(2, p) \).

\[
\begin{align*}
    PSL(2, p) \times C_2 &\to SL(2, p) \\
    1- &\to 1-
\end{align*}
\]

Diagram 6: Double coverings of \( PSL(2, p) \).

Both double coverings are 1-strong, i.e. \( Aut(SL(2, p)) \cong Aut(PSL(2, p) \times C_2) \cong Aut(PSL(2, p)) \). Note that \( A_4 \cong PSL(2, 3) \) and \( A_5 \cong PSL(2, 5) \) were treated before as rotary platonic groups.
The existence of only two double coverings is not a surprise. If fact, if $G$ is simple and non-abelian then $G$ is perfect ($G' = G$) and centerless. Let $\hat{G}$ be a double covering of $G$. Then $\hat{G}$ is perfect or $\hat{G}' \triangleleft_2 \hat{G}$. Since $G$ has no central involution then the center $Z$ of $\hat{G}$ is $C_2$.

(i) The central extension $(\hat{G}, Z)$ is not irreducible. This means that $\hat{G} = NZ$ for some $N \triangleleft_2 \hat{G}$. But then $N \cap Z = 1$ and so $\hat{G} = N \times Z \cong N \times C_2 \cong G \times C_2$. Reciprocally, if $\hat{G} = G \times C_2$ then clearly $\hat{G}$ is not irreducible. Thus $(\hat{G}, Z)$ is not irreducible if and only if $\hat{G} = G \times Z \cong G \times C_2$.

(ii) The extension $(\hat{G}, Z)$ is irreducible (that is, $\hat{G}$ is not a direct product $G \times C_2$). Since $G$ is perfect, if $\hat{G}' \triangleleft_2 \hat{G}$ then $\hat{G}' \cap Z = 1$ and $\hat{G} = \hat{G}' \times Z \cong G \times Z$ is not irreducible. Hence $\hat{G}$ must be also perfect. Consequently $\hat{G}' \cap Z = \hat{G} \cap Z = Z$ and as $M(G) = Z \cap \hat{G}'$ (Theorem 9.9 [13] (pg 251)) then $|M(G)| = |\hat{G}' \cap Z| = 2$, where $M(G)$ is the multiplier of $G$. Hence $(\hat{G}, Z)$ is a primitive central extension (by definition). From Theorem 9.18(4) [13] (pg 257) $\hat{G}$ is a representation group of $G$. Thus, if $(\hat{G}, Z)$ is an irreducible double extension of $G$ then $\hat{G}$ is a representation group of $G$.

For $G = PSL(2, q)$ with $q = p^f > 4$ and $q \neq 9$, $SL(2, q)$ is the only representation group of $PSL(2, q)$ (Theorem 25.7, pg 646 of [7]). Hence $PSL(2, q)$ with $p \neq 2$ has only two double coverings $PSL(2, q) \times C_2$ and $SL(2, q)$. If the field has characteristic 2 then $PSL(2, q) = SL(2, q)$ and hence $PSL(2, 2^f)$ for $f > 2$ has only one double covering which is the direct product $PSL(2, 2^f) \times C_2$.

### 6.2 Other simple groups

It is a known fact that simple groups are 2-generated [1]. Some simple groups $G$ have presentations of simple type with only two odd relators. They have only one double covering, the direct product $G \times C_2$. The unitary groups $U_3(3)$ and $U_3(4)$, the projective special linear group $PSL(3, 3)$ and the Mathieu group $M_{11}$, as can be observed in Table 4, are such examples.

| $G$          | Relators of $P$                                | type of $P$ | $\delta$ | Ref. |
|--------------|-----------------------------------------------|-------------|----------|------|
| $A_5$        | $a^4$, $b^3$, $abab^3abaBABB$                 | simple      | 1        | 2    |
| $A_7$        | $A_3(ab^2)^4$, $b^5$, $(b^2Ab)^2$              | simple      | 2        | 3    |
| $A_8$        | $(a^2b)^2$, $a^7$, $(abAB^2)abAB^2$           | simple      | 2        | 2    |
| $PSL(3, 3)$  | $a_2B^3$, $BA(ba)^3(BA)^7(bA)^3(Ba)^2a^2B^2a(Ba)^2(ba)^2$ | simple      | 1        | 4    |
| $PSL(2, 8)$  | $ababaB$, $a^4(b^2ab)^2b$                     | simple      | 1        | 5    |
| $PSL(2, 16)$ | $a^3b^2A_2B^2$, $abAB^2AbB$                   | simple      | 1        | 2    |
| $PSL(2, 25)$ | $a_5^2$, $a_2b^2a_2b^2$, $ab^3a^6b^1aB$       | simple      | 2        | 2    |
| $PSL(2, 27)$ | $(ab)^2$, $a^7$, $a^2ababaBAb^4$              | simple      | 2        | 2    |
| $PSL(2, 32)$ | $ababaB$, $(ab)^3b_3a_5b^3$                   | simple      | 1        | 5    |
| $PSL(2, 49)$ | $a^4$, $b^5$, $a^2b^4a^2BababaB$              | simple      | 2        | 2    |
| $U_3(3)$     | $B_2Ab^4aBAb^3B$, $b_2AB^2Ab_2$Ba $aB$        | simple      | 1        | 6    |
| $U_3(4)$     | $a^2B^3$, $a^5b(bA)^2(ab)^2aBab(BA)^2Bb(BA)^2$ | simple      | 1        | 4    |
| $M_{11}$     | $abaB^3$, babABAbaBAbA                      | simple      | 1        | 7    |
| $M_{12}$     | $a^2B^3$, $(ab)^{(10)}(b, a)^6$, $(ab)^4BababaB^3$ | 2        | 1    |
| $Sz(8)$      | $a^7$, $b_3$, $AB^2a^3Ba^2B$, $BaBABa^3B^2a^2ba$ | simple      | 4        | 2    |

Table 4: Presentations of some simple groups. Here $A = a^{-1}$ and $B = b^{-1}$.

As we saw before, the only double covering of $PSL(2, 2^f)$, for $f \geq 3$, is $PSL(2, 2^f) \times C_2$. As it can be observed in Table 4, $PSL(2, 8)$, $PSL(2, 16)$ and $PSL(2, 32)$ have deficiency zero. Do all $PSL(2, 2^f)$ have a presentation of deficiency zero?

Being $G$ simple, the double covering $G \times C_2$ is always 1-strong (Corollary 20). In the case of $\delta = 2$, besides $G \times C_2$ there is another double covering $\hat{G} \not\cong G \times C_2$ (Theorem 8) which is strong by Theorem 13.

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Ana Breda  Antonio Breda d’Azevedo  Domenico Catalano
ambreda@mat.ua.pt  breda@mat.ua.pt  domenico@mat.ua.pt

Departamento de Matemática
Universidade de Aveiro
Campus Universitário de Santiago
PT-3810-193 Aveiro
Portugal