The directed plump ordering

Daniel Gratzer       Michael Shulman       Jonathan Sterling
March 21, 2022

Abstract

Based on Taylor’s hereditarily directed plump ordinals, we define the directed plump ordering on W-types in Martin-Löf type theory. This ordering is similar to the plump ordering but comes equipped with non-empty finite joins in addition to the usual properties of the plump ordering.

Acknowledgment. This research was supported by the United States Air Force Office of Scientific Research under award number FA9550-21-1-0009.

The theory of plump ordinals [Tay96] has been adapted to Martin-Löf type theory by Fiore, Pitts, and Steenkamp [FPS21] to produce directed well-founded orders suitable for certain transfinite constructions. Given a pair \( (A : U_1, B : A \to U_1) \), \( \text{op. cit.} \) defines the plump ordering: a pair of relations \( \preceq, < \) on a type \( W \) of well-founded trees satisfying the following conditions:

1) \( \preceq \) is reflexive and transitive
2) \( < \) is transitive and well-founded.
3) If \( u < v \) then \( u \preceq v \).
4) If \( u < v \leq w \) or \( u \leq v < w \) then \( u < w \).
5) \( (W, \preceq) \) has a least element.
6) For each \( a : A \), both \( \preceq \) and \( < \) have upper-bounds for all \( B(a) \)-families.

Following Taylor’s theory of hereditarily directed plump ordinals [Tay96], we refine this ordering to obtain well-behaved least upper-bounds:

7) Given \( u, v : W \) there exists \( u \sqcup v \) such that \( u \sqcup v \leq w \) if and only if \( u, v \leq w \).
8) If \( u, v < w \) then \( u \sqcup v < w \).

We have partially formalized our results in Martin-Löf type theory with the UIP principle in the Agda proof assistant [SG22].\(^1\) In particular, all results except the well-foundedness of the list ordering \( \sqsubseteq \) of Section 2 are formalized in Agda.

\(^1\)http://www.jonmsterling.com/agda-directed-plump-ordering/.
1 An ordering on W-types

(1\*1) Fix a U₁-container \( A \triangleright B \) in the sense of Abbott, Altenkirch, and Ghani [AAG05], i.e. a pair of a type \( A : U₁ \) together with a family of types \( B : A \rightarrow U₁ \). The extension of \( A \triangleright B \) is the endofunctor \( \| A \triangleright B \| : U₁ \rightarrow U₁ \) defined like so:

\[
\begin{align*}
\text{record } & \| A \triangleright B \| (X : U₁) : U₁ \text{ where} \\
\text{constructor } & (–, –) : \text{lbl} : A \\
& \text{sub} : B(\text{lbl}) \rightarrow X
\end{align*}
\]

The extension of a container is also known as the polynomial endofunctor associated to the corresponding morphism \( \sum_{x:A} B(x) \rightarrow A \).

(1\*2) The initial algebra for the extension \( \| A \triangleright B \| \) of a given container can be computed as a W-type in the sense of Martin-Löf [Mar84] consisting of well-founded trees labeled in \( a : A \) with subtrees of arity \( B(a) \), written \( W_{A}B : U₁ \). The structure map for this initial algebra is written \( \text{ub} : \| A \triangleright B \| (W_{A}B) \rightarrow W_{A}B \), which can be thought of as producing an upper-bound in the subtree order.

(1\*3) Suppose that the container \( A \triangleright B \) is closed under binary coproducts of shapes in the sense that we have an operation \( \triangleright : A \times A \rightarrow A \) such that \( B(a₁ \triangleright a₂) = B(a₁) + B(a₂) \). Given two trees \( u, v : W_{A}B \), we will write \( u \triangleright v \) for \( \text{ub}(u.\text{lbl} + v.\text{lbl}, [u.\text{sub} | v.\text{sub}]) \). For a non-empty finite set of trees \{\( u_i | i \leq n \}\) we will write \( \bigsqcup_i u_i \) for the corresponding n-ary instance of \( \triangleright \).

(1\*4) We may define the following two binary relations \( \leq, < \) on \( W_{A}B \) as the smallest ones closed under the following rules:

\[
\begin{align*}
\exists b₁, \ldots, b_n : B(v.\text{lbl}) \cdot u & \leq \bigsqcup_i v.\text{sub}(b_i) \\
\forall b : B(u.\text{lbl}) \cdot u.\text{sub}(b) & < v
\end{align*}
\]

Each of (1\*5) through (1\*8) has been formally verified in Agda.

(1\*5) The relation \( \leq \) is reflexive.

(1\*6) For any \( u, v, w : W_{A}B \) we have the following:

1) Transitivity. If \( u \leq v \leq w \) then \( u \leq w \); likewise if \( u < v < w \) then \( u < w \).

2) Left flex. If \( u \leq v \) and \( v < w \) then \( u < w \).

3) Right flex. If \( u < v \) and \( v \leq w \) then \( u < w \).

(1\*7) For any \( u, v : W_{A}B \), if \( u < v \) then \( u \leq v \).

(1\*8) Let \{\( u_i | i \leq n \}\) be a non-empty finite family of trees, and let \( v : W_{A}B \) be a tree; we have \( \bigsqcup_i u_i \leq v \) if and only if \( u_i \leq v \) for all \( i \leq n \). Moreover, we have \( \bigsqcup_i u_i < v \) if \( u_i < v \) for all \( i \leq n \).

2 An intermezzo on list orderings

(2\*1) Given a relation \( R : A \times A \rightarrow \Omega \), define the accessibility predicate as the following inductive type:

\[
\text{data } \text{Acc}(R) : A \rightarrow \Omega \text{ where} \\
\text{acc} : (a : A) \rightarrow ((b : A) \rightarrow R(b, a) \rightarrow \text{Acc}(R, b)) \rightarrow \text{Acc}(R, a)
\]
A relation is said to be well-founded when all its elements are accessible. Note that a well-founded relation need not be transitive.

(2*2) We eventually wish to show that $<$ is well-founded but prior to this we must introduce a supplementary well-founded ordering. The well-foundedness of $<$ will follow from well-founded induction on this secondary ordering.

Fix a type $X$ and a well-founded relation $\prec : X \times X \to \Omega$ for the remainder of this section. We define a new relation $\sqsubset$ on $\text{List}(X)$:

\[
m \geq 1 \quad \exists f : \{1 \ldots n\} \to \{1 \ldots m\} . \forall i \leq n . x_i < y_{f(i)} \quad [x_1, \ldots, x_n] \sqsubset [y_1, \ldots, y_m]
\]

We adapt a proof due to Wilfried Buchholz as described by Nipkow [Nip98] to prove that $\sqsubset$ is well-founded.

(2*3) The empty list is $\sqsubset$-accessible.

(2*4) If a list is $\sqsubset$-accessible, so too is any permutation.

(2*5) Fix $y : X$. Suppose for all accessible $l : \text{List}(X)$ and $x \prec y$, $\text{cons}(x, l)$ is accessible. Then for all accessible $l : \text{List}(X)$, $\text{cons}(y, l)$ is accessible.

Proof. Fix an accessible $l$ and suppose that $n \sqsubset \text{cons}(y, l)$. By definition, there exists a division of $n$ into $n_l$ and $n_y$ such that $n_l \sqsubset l$ and each element of $n_y$ is dominated by $y$. Because $l$ is accessible, so too is $n_l$. Therefore, $n_y + n_l$ is accessible by induction on the size of $n_y$ and repeated use of the assumption. Because $n$ is a permutation of $n_y + n_l$, we conclude that $n$ is accessible. \[\square\]

(2*6) If $l : \text{List}(X)$ is $\sqsubset$-accessible and $x : X$, then $\text{cons}(x, l)$ is accessible.

Proof. This follows immediately from the (2*5) and $\prec$-induction on $x$. \[\square\]

(2*7) If $\prec$ is well-founded, so too is $\sqsubset$.

Proof. Fix $l : \text{List}(X)$. We argue by induction on $l$ that $l$ is accessible. In the base case apply (2*3) and in the inductive step apply (2*6). \[\square\]

3 Well-foundedness of the directed plump ordering

(3*1) Write $\text{List}^+(X)$ for the type of non-empty lists. Given an non-empty list $l = [u_0, \ldots, u_n]$, write $\bigsqcup l$ for $\bigsqcup_{i \leq n} u_i$.

(3*2) Given $l : \text{List}^+(W_A B)$, if $u \leq \bigsqcup l$ then $u$ is $\prec$-accessible.

Proof. This follows by well-founded induction on the $\sqsubset$-accessibility of $l$; the details are formalized in Agda. \[\square\]

(3*3) The relation $\prec$ is well-founded.

Proof. We must prove that every $u : W_A B$ is $\prec$-accessible, but this is a consequence of (3*2) setting $l$ to be the singleton list $[u]$; the details are formalized in Agda. \[\square\]
Summarizing, given a pair \( (A : U_1, B : A \to U_1) \) together with an operation an operation \( + : A \times A \to A \) such that \( B(a_1 + a_2) = B(a_1) + B(a_2) \) there exists a type \( W_A B \) together with a pair of relations \( \leq, < : W_A B \times W_A B \to \Omega \) satisfying the following conditions:

1. \( \leq \) is transitive and reflexive.
2. \( < \) is transitive and well-founded.
3. If \( u < v \), then \( u \leq v \).
4. If \( u \leq v \leq w \) or \( u \leq v \), then \( u < w \).
5. If there exists \( a : A \) such that \( B(a) = 0 \) then \( (W_A B, \leq) \) has a least element.
6. For any \( a : A \), both \( \leq \) and \( < \) have upper-bounds for all \( B(a) \)-families.
7. Given \( u, v \) there exists an element \( u \sqcup v \) such that \( u \sqcup v \leq w \) if and only if \( u, v \leq w \).
8. If \( u, v < w \) then \( u \sqcup v < w \).

(3§5) Given a pair \( (A : U_1, B : A \to U_1) \), define a new pair \( (C, D) \) by setting \( C = \text{List}(A) \) and specifying \( D \) inductively:

\[
D([]) = 0 \quad D(\text{cons}(a, c)) = B(a) + D(c)
\]

Then (3§4) instantiated with this new family shows that \( (W_C D, \leq, <) \) satisfies the requirements outlined by (0§1).

References

[Mar84] Per Martin-Löf. *Intuitionistic type theory. Notes by Giovanni Sambin*. Vol. 1. Studies in Proof Theory. Bibliopolis, 1984, pp. iv+91. ISBN: 88-7088-105-9 (cit. on p. 2).

[Tay96] Paul Taylor. “Intuitionistic sets and ordinals”. In: *The Journal of Symbolic Logic* 61.3 (1996), pp. 705–744. DOI: 10.2307/2275781 (cit. on p. 1).

[Nip98] Tobias Nipkow. *An Inductive Proof of the Wellfoundedness of the Multiset Order*. Exposition of a proof due to Wilfried Buchholz. 1998. URL: https://www21.in.tum.de/~nipkow/Misc/multiset.ps (cit. on p. 3).

[AAG05] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. “Containers: Constructing strictly positive types”. In: *Theoretical Computer Science* 342.1 (2005). Applied Semantics: Selected Topics, pp. 3–27. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2005.06.002 (cit. on p. 2).

[FPS21] Marcelo P. Fiore, Andrew M. Pitts, and S. C. Steenkamp. *Quotients, inductive types, and quotient inductive types*. 2021. arXiv: 2101.02994 [cs.LO] (cit. on p. 1).

[SG22] Jonathan Sterling and Daniel Gratzer. *agda-directed-plump-ordering*. https://github.com/jonsterling/agda-directed-plump-ordering. 2022 (cit. on p. 1).