Antinormally-Ordered Quantizations, phase space path integrals and the Olshanski semigroup of a symplectic group.

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Abstract
The main aim of this article is to show some intimate relations among the following three notions: (1) the metaplectic representation of $Sp(2n, \mathbb{R})$ and its extension to some semigroups, called the Olshanski semigroup for $Sp(2n, \mathbb{R})$ or Howe’s oscillator semigroup, (2) antinormally-ordered quantizations on the phase space $\mathbb{R}^{2m} \cong \mathbb{C}^m$, (3) path integral quantizations where the paths are on the phase space $\mathbb{R}^{2m} \cong \mathbb{C}^m$. In the Main Theorem, the metaplectic representation $\rho(e^X)$ ($X \in sp(2n, \mathbb{R})$) is expressed in terms of generalized Feynman–Kac–(Itô) formulas, but in real-time (not imaginary-time) path integral form. Olshanski semigroups play the leading role in the proof of it.

1 Introduction
The main aim of this article is to show some intimate relations among the following three notions:

(1) the metaplectic representation of $Sp(2n, \mathbb{R})$ and its extension to some semigroups, called e.g. the Olshanski semigroup for $Sp(2n, \mathbb{R})$ [4], the symplectic category [5] or Howe’s oscillator semigroup [1].

(2) antinormally-ordered quantizations on the phase space $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

(3) path integral quantizations where the paths are on the phase space $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

A very rough idea of the theory of general Olshanski semigroups is as follows [4]. Let $G$ be a real Lie group, $\mathfrak{g}$ the Lie algebra of $G$, and $G_\mathbb{C}$, $\mathfrak{g}_\mathbb{C}$ be their complexifications, respectively. Let $\pi$ be a continuous unitary representation of $G$ on a Hilbert space $\mathcal{H}$, and $\pi(X)$ the corresponding representation of $\mathfrak{g}_\mathbb{C}$ on $\mathcal{H}_\infty$ (the space of smooth vectors in $\mathcal{H}$), where $i \cdot \pi(X)$ is essentially self-adjoint for all $X \in \mathfrak{g}$. Define the subsemigroup $\Gamma_{G,\pi}$ of $G_\mathbb{C}$ by

$$\Gamma_{G,\pi} := \{ge^X | g \in G, X \in i \mathfrak{g}, \pi(X) \text{ is bounded from above}\},$$

which is called an Olshanski semigroup. Then the unitary representation $\pi$ of $G$ is extended to the bounded representation $\hat{\pi}$ of $\Gamma_{G,\pi}$ by

$$\hat{\pi}(ge^X) = \pi(g)e^{i\pi(X)}, \quad g \in G, X \in i \mathfrak{g}.$$ 

If $\pi$ is a highest weight representations of $G$, then $\hat{\pi}$ turns out to be a holomorphic representation of $\Gamma_{G,\pi}$. Furthermore, $\hat{\pi}$ turns out to be “natural”: $\hat{\pi}$ can be seen as the maximal analytic continuation of $\pi$ in a sense. Thus it is expected that the problems on highest weight representations of $G$ is translated to those on holomorphic representations of $\Gamma_{G,\pi}$. However, since the above definition of $\Gamma_{G,\pi}$ depends on the representation $\pi$, we need to know $\pi$ to know $\Gamma_{G,\pi}$. Thus we need some redefinition of Olshanski semigroups so as not to refer to any representation of $G$ to do that translation. In fact, such
a redefinition is possible: The definition of an Olshanski semigroup \( \Gamma_G(W) \) in [4] does not depend on \( \pi \), but on an invariant cone \( W \) in \( g \).

Although our Main Theorem has no reference to the Olshanski semigroups, they play the leading role in the proof of it.

The Main Theorem is outlined as follows. Let \( m \in \mathbb{N} \). Define \( \alpha_k : \mathbb{R}^{2m} \to \mathbb{R} \) for \( k = 1, ..., 2m \) by

\[
\alpha_k(\vec{x}) := x_{m+k}, \quad \alpha_{m+k}(\vec{x}) := -x_k, \quad k = 1, ..., m, \quad \vec{x} := (x_1, ..., x_{2m}),
\]

and the (positive-definite) magnetic Laplacian \( \Delta^\alpha \) on \( \mathbb{R}^{2m} \) by

\[
\Delta^\alpha := -\sum_{k=1}^{2m} \left( \frac{\partial}{\partial x_k} + i\alpha_k \right)^2.
\]

The symplectic group \( Sp(2m, \mathbb{R}) \) acts on the phase space (symplectic vector space) \( \mathbb{R}^{2m} \cong \mathbb{C}^m \) as linear symplectic transformations. Hence each \( A \in \mathfrak{sp}(2m, \mathbb{R}) \) (the Lie algebra of \( Sp(2m, \mathbb{R}) \)) corresponds to a Hamiltonian function \( H_A : \mathbb{C}^m \to \mathbb{R} \) which generates a one-parameter group of linear symplectic transformations on \( \mathbb{C}^m \). It is explicitly given by

\[
H_A(\vec{z}) := \frac{i}{2} \vec{z}^\dagger IA_c \vec{z}, \quad I := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad A_c := \mathcal{W}AW^{-1}, \quad \mathcal{W} := \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}, \quad (1.1)
\]

where \( \vec{z} := (z_1, ..., z_m, \bar{z}_1, ..., \bar{z}_m)^T \), \( \vec{z}^\dagger := (z_1, ..., z_m, \bar{z}_1, ..., \bar{z}_m) \).

For any Hamiltonian function \( H : \mathbb{C}^m \to \mathbb{R} \) which satisfies some regularity conditions, let \( \mathcal{E}_\delta(H) \) denote the antinormally ordered quantization of \( H \), which is a self-adjoint operator on a Hilbert subspace \( \mathcal{H} \) of \( L^2(\mathbb{C}^m) \) (see Sec. 4 and Sec. 8).

Then we have the following relation between the classical Hamiltonian \( H_A \) and the quantum Hamiltonian \( \mathcal{E}_\delta(H_A) \):

**Theorem 1.1.** *(Cor. 8.1)* For any \( A \in \mathfrak{sp}(2m, \mathbb{R}) \) and \( t \in \mathbb{R} \),

\[
\lim_{\nu \to \infty} \nu e^{\nu m} \exp \left( -itH_A - \frac{\nu}{2}\Delta^\alpha \right) = e^{-it\mathcal{E}_\delta(H_A)} \quad \text{on } \mathcal{H}, \quad (1.2)
\]

where \( H_A \) in the l.h.s. is viewed as a multiplication operator on \( L^2(\mathbb{C}^m) \).

Possibly this theorem itself is not a new result, because the l.h.s. can be explicitly realized as a Gaussian integral operator, which has been extensively studied in various fields; see e.g. [1, 5] and references therein. However, (hopefully) it has some new aspects:

1. Our proof does not refer to any specific Gaussian integral operator realizations; Instead we use more algebraic method of Olshanski semigroups for symplectic groups. In fact, the main part of the proof in Sec. 7 is on the finite-dimensional matrices. Thus it is expected that this method can be extended to “non-Gaussian” cases, in terms of more general Lie groups and their highest weight representations.

2. Notice that the form of the l.h.s. of (1.2) (without limit, i.e. with each fixed \( \nu > 0 \)) is suitable to the application of the Feynman–Kac–Itô formula (e.g. [9, 3, 2]). Thus (1.2) leads to a “real-time” (not “imaginary-time”) path-integral representation of the “quantum time evolution” in the r.h.s.

3. Also note that we can interpret the magnetic Laplacian geometrically: \( \Delta^\alpha = \nabla^* \nabla \), where \( \nabla \) is a connection on a complex line bundle over the phase space \( \mathbb{C}^m \). For further geometric implication, see [10].

There has been extensive literature in rigorous justification of the path-integral methods in quantum physics. However, it appears that there are only few rigorous studies on real-time geometric path integrals. (Note that the general Feynman–Kac formula for vector bundles on Riemannian manifolds is formulated and proven relatively recently; see Güneysu [2].) One of our “real-time geometric path integral formulas” is stated as follows:
Theorem 1.2. (a rough outline of Thm. 9.1 and Cor. 9.6) There exists a sequence of probability measures $\mu_\nu, \nu \in \mathbb{N}$, (in fact, Brownian bridge measures) on the loop space

$$\text{loop}_0 := \{ \varphi \in C([0, 1], \mathbb{R}^{2m}) | \varphi(0) = \varphi(1) = 0 \}$$

such that for any $A \in \mathfrak{sp}(2m, \mathbb{R})$, the “vacuum expectation value” of $e^{-iE_b(H_A)}$ is expressed as follows:

$$\langle \Omega_0 | e^{-iE_b(H_A)} \Omega_0 \rangle = \lim_{\nu \to \infty} e^{\nu \rho} \int_{\text{loop}_0} e^{iS_A(\varphi)} d\mu_\nu(\varphi) = \lim_{\nu \to \infty} e^{\nu \rho} \mathbb{E}_\nu [e^{iS_A(\varphi)}]$$

(1.3)

where

$$S_A(\varphi) := \int_0^1 \alpha^b + \int_0^1 H_A(\varphi(t)) dt, \quad \alpha^b := \sum_{k=1}^{2m} a_k dx_k.$$ 

Here $\mathbb{E}_\nu[\cdot]$ denotes the (classical) expectation value w.r.t. the probability measure $\mu_\nu$.

In [10], we proved some similar results for the cases where the Hamiltonian is bounded, and for more general phase spaces: (possibly non-compact) complete Kähler manifolds satisfying some technical conditions. Thus the above theorem is an extension of the result of [10].

Remark 1.3. Let $G$ be a locally compact group, and $\pi$ a unitary representation of $G$ on $\mathcal{H}$. For $\nu \in \mathcal{H} \setminus \{0\}$, define $\phi_\nu : G \to \mathbb{C}$ by $\phi_\nu(g) := \langle \nu | \pi(g) \nu \rangle$. Then $(\pi, \mathcal{H})$ can be reconstructed from $\phi_\nu$ by the GNS construction. If $G$ is a connected Lie group and $\phi_\nu$ is real analytic, then the restriction $\phi_\nu \restriction U$ of $\phi_\nu$ for some neighborhood $U$ of the unit suffices to reconstruct $(\pi, \mathcal{H})$, in terms of the GNS together with the analytic continuation.

Remark 1.4. Eq. (1.3) may have some conceptual implications because it says that a quantum expectation value can be approximated by some expectation values in the classical sense, not the converse. It also suggests the Monte Carlo methods for numerical computations of quantum expectation values, while they do not seem very effective.

Remark 1.5. In the above theorem, the “quantum Hamiltonian” $E_b(H_A)$ is a generator of a projective unitary representation of $\text{Sp}(2m, \mathbb{R})$, rather than a quantum Hamiltonian of some realistic physical system. Hence the physical meaning of this path-integral representation is not so clear, but mathematically this may be intriguing since $E_b(H_A)$ is non-semibounded, i.e. unbounded both from below and from above; in such cases, most of the conventional Feynman–Kac methods of imaginary-time path integrals will not be applicable since both $e^{-tE_b(H_A)}$ and $etE_b(H_A) (t > 0)$ are unbounded.

2 Symplectic groups and Olshanski semigroups

In this section, we give basic definitions on symplectic groups and related topics [5, 1]; For the general theory of Olshanski semigroups, see [4]. The symbols defined in this section are mainly taken from Neretin [5], and partially from Folland [1]. However note that the term “Olshanski semigroup” is not found in [5, 1], but in Neeb [4], while precisely he spells “Ol’shanskii”; Neretin [5] use the term “contraction semigroup”, which seems slightly vague and misleading.

Define $J \in \text{Mat}(2n, \mathbb{R})$ by

$$J = J_{2n} := \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix},$$

where $O_n$ and $I_n$ are the $n \times n$ zero matrix and the identity matrix, respectively. The symplectic form on $\mathbb{R}^{2n}$ determined by $J$ is written by $\omega(u, v) := u^T J v$ for $u, v \in \mathbb{R}^{2n}$ (as column vectors). Let $K = \mathbb{R}, \mathbb{C}$, and consider the symplectic group over $K$:

$$\text{Sp}(2n, K) := \{ A \in \text{GL}(2n, K) | A^T J A = J \} = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}(2n, K) | AB^T = BA^T, CD^T = DC^T, AD^T - BC^T = I_n \}.$$
The group $\text{Sp}(n, \mathbb{K}) = \{ A \in \text{Mat}(n, \mathbb{K}) \mid JA + A^T J = 0 \}$

$$= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mid A, B, C \in \text{Mat}(n, \mathbb{K}), \ B^T = B, \ C^T = C \right\}. $$

Let

$$\mathcal{W} \equiv \mathcal{W}_{2n} := \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix},$$

and for any $A \in \text{Mat}(2n, \mathbb{C})$, let

$$A_c := \mathcal{W}AW^{-1}.$$  

Then we see $J_c := \mathcal{W}J\mathcal{W}^{-1} = \begin{pmatrix} -iI_n & 0 \\ 0 & iI_n \end{pmatrix}$. Let

$$\mathcal{I} \equiv \mathcal{I}_{2n} := \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} = -iJ_c,$$

and define the Hermitian form (indefinite inner product)

$$\langle u|v \rangle_{\mathcal{I}} := \langle u|\mathcal{I}v \rangle = -iu^*J_cv, \quad u, v \in \mathbb{C}^{2n}.$$  

Then the group $U(n, n)$ is defined by

$$U(n, n) = \left\{ g \in \text{Mat}(2n, \mathbb{C}) : \langle gv|gv \rangle_{\mathcal{I}} = \langle u|v \rangle_{\mathcal{I}} \text{ for all } u, v \in \mathbb{C}^{2n} \right\}$$

$$= \left\{ g \in \text{Mat}(2n, \mathbb{C}) : g^*\mathcal{I}g = \mathcal{I} \right\}.$$

The group $\text{Sp}_c(2n, \mathbb{R}) (\cong \text{Sp}(2n, \mathbb{R}))$ has several equivalent definitions:

$$\text{Sp}_c(2n, \mathbb{R}) : = \{ \mathcal{W}AW^{-1} \mid A \in \text{Sp}(2n, \mathbb{R}) \}$$

$$= \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \mid P, Q \in \text{Mat}(n, \mathbb{C}), \ PP^* - QQ^* = I_n, \ PQ^T = QP^T \right\}$$

$$= U(n, n) \cap \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \mid P, Q \in \text{Mat}(n, \mathbb{C}) \right\}$$

$$= U(n, n) \cap \text{Sp}(2n, \mathbb{C}).$$

Thus any element of $\text{Sp}_c(2n, \mathbb{R})$ preserves both the bilinear form $(v, w) \mapsto v^TJw$ and the hermitian form $(v, w) \mapsto v^*\mathcal{I}w$ on $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$. Its Lie algebra is given by

$$\text{sp}_c(2n, \mathbb{R}) = u(n, n) \cap \text{sp}(2n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \mid A, B \in \text{Mat}(n, \mathbb{C}), \ A^* = -A, \ B^T = B \right\}. \quad (2.1)$$

Let

$$\Gamma U(n, n) : = \{ g \in \text{GL}(2n, \mathbb{C}) : \quad \forall v \in \mathbb{C}^{2n}, \ \langle gv|gv \rangle_{\mathcal{I}} \leq \langle v|v \rangle_{\mathcal{I}} \}$$

$$= \{ g \in \text{GL}(2n, \mathbb{C}) : \mathcal{I} - g^*\mathcal{I}g \geq 0 \}.$$

$$\Gamma \text{Sp}_c(2n) : = \text{Sp}(2n, \mathbb{C}) \cap \Gamma U(n, n).$$

Evidently $\Gamma U(n, n)$ (resp. $\Gamma \text{Sp}_c(2n)$) is a semigroup. This is called a contraction semigroup or Olshanski semigroup for $U(n, n)$ (resp. $\text{Sp}_c(2n, \mathbb{R})$).

Note that the Lie algebra of $U(n, n)$ is given by

$$u(n, n) := \{ X \in \text{Mat}(2n, \mathbb{C}) \mid \langle v|Xv \rangle_{\mathcal{I}} \in i\mathbb{R} \text{ for all } v \}.$$

Let

$$\text{Diss}(n, n) := \{ X \in \text{Mat}(2n, \mathbb{C}) \mid \Re \langle v|Xv \rangle_{\mathcal{I}} \leq 0 \text{ for all } v \}.$$
SDiss\( (n,n) \) := \{ X \in \text{Mat}(2n, \mathbb{C}) | \langle v | X v \rangle \leq 0 \text{ for all } v \}
\quad = \{ X \in i \cdot u(n,n) | \langle v | X v \rangle \leq 0 \text{ for all } v \}
\quad = \{ X \in i \cdot u(n,n) | \mathbb{I} X \leq 0 \}
\quad = 1 \cdot u(n,n) \cap \text{Diss}(n,n)

\text{Diss}(n,n) \text{ is called the cone of } \mathcal{I} \text{-dissipative operators. } \text{SDiss}(n,n) \text{ is called the cone of } \mathcal{I} \text{-self-adjoint dissipative operators. Let }

\text{Diss}_{sp_c}(2n) := \text{sp}_c(2n, \mathbb{C}) \cap \text{SDiss}(n,n).
\quad = \{ X \in \text{sp}_c(2n, \mathbb{C}) | \Re \langle v | X v \rangle \leq 0 \text{ for all } v \}

\text{SDiss}_{sp_c}(2n) := i \cdot \text{sp}_c(2n, \mathbb{R}) \cap \text{Diss}(n,n)
\quad = \{ X \in i \cdot \text{sp}_c(2n, \mathbb{R}) | \forall v, \Re \langle v | X v \rangle \leq 0 \}
\quad = \{ X \in i \cdot \text{sp}_c(2n, \mathbb{R}) | \mathbb{I} X \leq 0 \}.

\text{We have}
\text{Diss}(n,n) = u(n,n) \oplus \text{SDiss}(n,n).
\text{Diss}_{sp_c}(2n) = \text{sp}(2n, \mathbb{R}) \oplus \text{SDiss}_{sp_c}(2n).

**Proposition 2.1.** [5, Theorem 7.5 in Ch.2.] The following conditions are equivalent:

1. \( e^{tX} \in \Gamma U(n,n) \) for all \( t > 0 \),
2. \( X \in \text{Diss}(n,n) \).

**Proposition 2.2** (The Potapov-Olshanski decomposition). ([5]: Theorem 7.7 in Ch.2 and Theorem 5.2 in Ch.3) Each element \( g \in \Gamma \text{Sp}_c(2n) \) admits a unique decomposition

\( g = he^{X}, \quad h \in \text{Sp}_c(2n, \mathbb{R}), \ X \in \text{SDiss}_{sp_c}(2n) \).

### 3 Metaplectic representation

Let \( a_k, a_k^* (k = 1, ..., n) \) be annihilation/creation operators on \( \mathcal{H} \) s.t.

\begin{align}
[a_k, a_l] = 0, \quad [a_k, a_l^*] = \delta_{kl}.
\end{align}

For \( A \in \text{sp}_c(2n, \mathbb{R}) \), define the (essentially) skew-self-adjoint operator \( d\rho(A) \) by

\begin{align}
d\rho(A) := \frac{1}{2} a^* \mathcal{I} A a
\end{align}

\( a := (a_1^*, ..., a_n^*, a_1, ..., a_n)^\top = (a^*, \bar{a}) \),

\( a^* := (a_1, ..., a_n, a_1^*, ..., a_n^*) = (a^T, a^* \bar{a}^\top) \).

For example, we have

\( \text{sp}_c(2, \mathbb{R}) = \left\{ \begin{pmatrix} i r & z \\ \bar{z} & -i r \end{pmatrix} | r \in \mathbb{R}, \ z \in \mathbb{C} \right\} \)

and

\begin{align}
d\rho \left( \begin{pmatrix} i r & z \\ \bar{z} & -i r \end{pmatrix} \right) &= i \frac{1}{2} \left( r (a_1 a_1^* + a_1^* a_1) - i (za_1^2 - \bar{z}a_1^2) \right).
\end{align}

Especially,

\begin{align}
d\rho \left( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) &= i \frac{1}{2} (a_1 a_1^* + a_1^* a_1) = i \left( a_1^* a_1 + \frac{1}{2} \right).
\end{align}
We find that $d\rho$ is faithful representation of $\mathfrak{sp}_c(2n, \mathbb{R})$ on the space $\mathcal{H}^\infty$ of smooth vectors of $\mathcal{H}$. Roughly speaking, the metaplectic representation of $\mathfrak{sp}_c(2n, \mathbb{R})$ is determined by this representation of $\mathfrak{sp}_c(2n, \mathbb{R})$: The set of unitary operators $\{e^{d\rho(A)}|A \in \mathfrak{sp}_c(2n, \mathbb{R})\}$ generates a group $\text{Mp}(2n, \mathbb{R}) \subset \text{U}(\mathcal{H})$ which satisfies $\text{Mp}(2n, \mathbb{R})/\{\pm 1\} \cong \text{Sp}(2n, \mathbb{R})$. Hence it determines a projective unitary representation $\rho$ of $\text{Sp}(2n, \mathbb{R}) \cong \text{Sp}(2n, \mathbb{R})$, called the metaplectic representation (also called the oscillator, harmonic, or Segal–Shale–Weil representation).

The double-valuedness of the metaplectic representation is not very important in this article; Here we take account only of the projectivity of it. Hence we will consider as follows. Let $\mathcal{B}(\mathcal{H})$ denote the semigroup of bounded operators on $\mathcal{H}$. Then $\mathcal{B}(\mathcal{H})/\mathbb{C}^\times \ (\mathbb{C}^\times := \mathbb{C} \setminus \{0\})$ also is a semigroup. More precisely we consider the semigroup $(\mathcal{B}(\mathcal{H}) \setminus \{0\})/\mathbb{C}^\times$, which has no zero element. For each $A \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$, $\hat{A} := \mathbb{C}^\times A \in \mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ is called an operator ray (of $A$).

The norm (resp. strong, weak) topology on $\mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ is naturally determined by the norm (resp. strong, weak) topology on $\mathcal{B}(\mathcal{H})$. Let $A_k \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$ ($k = 1, 2, \ldots, \infty$), and $\hat{A}_k := \mathbb{C}^\times A_k \in \mathcal{B}(\mathcal{H})/\mathbb{C}^\times$. Then, $A_k \rightarrow A_\infty$ as $k \rightarrow \infty$ in norm (resp. strong, weak) topology iff there exists a sequence $z_1, z_2, \ldots \in \mathbb{C}^\times$ s.t. $z_k A_k \rightarrow A_\infty$ in norm (resp. strong, weak) topology.

Let $G$ be a topological semigroup. A strongly continuous homomorphism $\pi : G \rightarrow \mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ is called a (continuous) projective representation of $G$. If $U \in \mathcal{B}(\mathcal{H})$ is unitary, we also call $\mathbb{C}^\times U \in \mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ unitary. The set of unitary element of $\mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ is denoted by $\text{U}(\mathcal{H})/\mathbb{C}^\times$, while a more precise notation is $(\mathbb{C}^\times \text{U}(\mathcal{H}))/\mathbb{C}^\times \times$. If $G$ is a group and $\pi : G \rightarrow \text{U}(\mathcal{H})/\mathbb{C}^\times$ is a homomorphism, we call $\pi$ a projective unitary representation (PUR). We regard the metaplectic representation $\rho$ as the projective unitary representation $\rho : \text{Sp}(2n, \mathbb{R}) \rightarrow \text{U}(\mathcal{H})/\mathbb{C}^\times$ s.t. $\rho(e^X) = e^{d\rho(X)}$, $X \in \mathfrak{sp}_c(2n, \mathbb{R})$.

The definition (3.2) of $d\rho(X)$ is naturally generalized. First, if $X \in \text{SDiss}_{\text{sp}}(2n)$, we find that $d\rho(X)$ is defined as an essentially self-adjoint operator bounded from above. Hence $e^{d\rho(X)}$ is also defined as a bounded operator.

By Theorem 2.2, the metaplectic representation $\rho$ is extended to a projective representation $\hat{\rho}$ of the Olshanski semigroup $\Gamma\text{Sp}_c(2n)$ as follows. Let $g \in \Gamma\text{Sp}_c(2n)$, and $g = he^X$, $h \in \text{Sp}(2n, \mathbb{R})$, $X \in \text{SDiss}_{\text{sp}}(2n)$. Then we define $\hat{\rho}(g)$ by

$$\hat{\rho}(g) := \rho(h)e^{d\rho(X)}.$$ 

In the following, we write $\rho(g)$ for $\hat{\rho}(g)$.

Remark 3.1. Roughly speaking, Howe’s oscillator semigroup [1] is given by $\rho(\Gamma\text{Sp}_c(2n))$, the range of the metaplectic representation of $\Gamma\text{Sp}_c(2n)$. However, it is not precise: According to the definition in [1], any element of the oscillator semigroup is a Hilbert–Schmidt operator. If $n \geq 2$, the operator $e^{-t\alpha_1^a_1} (t \geq 0)$ is not Hilbert–Schmidt, and hence not in the oscillator semigroup in the sense of [1].

On the other hand, we see $e^{-t\alpha_1^a_1} \in \rho(\Gamma\text{Sp}_c(2n))$.

4 Antinormally-ordered quantization

Assume that $n$ is even, $n = 2m$, and

$$b_k := a_{m+k}, \quad k = 1, \ldots, m,$$

$$Z_k := a_k^* + b_k \quad k = 1, \ldots, m.$$ 

Then we see

$$[Z_k, Z_l] = [Z_k, Z_l^*] = 0, \quad k, l = 1, \ldots, m,$$

i.e. $\{Z_k\}_{k=1}^m$ are (unbounded) commuting normal operators. Thus, for any Borel function $f : \mathbb{C}^m \rightarrow \mathbb{C}$, the operator $f(Z_1, \ldots, Z_m)$ is well-defined.

Let

$$N_a := \sum_{k=1}^m a_k^* a_k, \quad N_b := \sum_{k=1}^m b_k^* b_k.$$
and \( E_a \) (resp. \( E_b \)) be the orthogonal projection onto the subspace \( \ker N_a \subset \mathcal{H} \) (resp. \( \ker N_b \subset \mathcal{H} \)). Let
\[
\mathcal{E}_a(X) := E_a X E_a, \quad \mathcal{E}_b(X) := E_b X E_b.
\]
Then we see
\[
\mathcal{E}_a(Z^p_k Z^q_l) = \mathcal{E}_a(b^p_k b^q_l E_a), \quad \mathcal{E}_b(Z^p_k Z^q_l) = \mathcal{E}_b(a^p_k a^q_l E_b), \quad k, l = 1, \ldots, m, \ p, q = 0, 1, \ldots
\]
This means that the map
\[
 f \mapsto \mathcal{E}_a(f(\tilde{Z})), \quad \text{resp.} \ f \mapsto \mathcal{E}_b(f(\tilde{Z})), \quad \tilde{Z} := (Z_1, \ldots, Z_m)
\]
is the antinormally-ordered quantization of the function \( f \) on the phase space \( \mathbb{C}^m \cong \mathbb{R}^n \) in terms of \( \{b_k\}_{k=1,\ldots,m} \) (resp. \( \{a_k\}_{k=1,\ldots,m} \)).

We expect that this “quantization by projection \( E_a/b \)” viewpoint makes the notion of quantization more transparent. However, the nature of the operation \( \mathcal{E}_a \) (or \( \mathcal{E}_b \)) is not so clear, contrary to the appearance. To examine the projections \( \mathcal{E}_a, \mathcal{E}_b \) further, we express them by the creation/annihilation operators as follows
\[
\mathcal{E}_a = \lim_{\nu \to \infty} e^{-\nu N_a}, \quad \mathcal{E}_b = \lim_{\nu \to \infty} e^{-\nu N_b}.
\]

Furthermore, we wish to describe them in terms of the symplectic groups and its metaplectic representations.

In the following, we consider \( \mathcal{E}_b \) and \( \mathcal{N}_b \) only. Define \( \mathcal{N}_b \in \text{Mat}(2n, \mathbb{C}) \) by
\[
\mathcal{N}_b := O_m \oplus I_m \oplus O_m \oplus (-I_m) = \text{diag}(0, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1).
\]
We see \( -\mathcal{N}_b \in \text{SDiss}_{\mathfrak{sp}_e}(2n) \subset i \cdot \mathfrak{sp}_e(2n, \mathbb{R}) \), and
\[
\mathfrak{sp}(\mathcal{N}_b) = - \left( \mathcal{N}_b + \frac{m}{2} \right).
\]
Hence
\[
\mathcal{E}_b = \lim_{\nu \to \infty} \mathbb{C}^\times e^{\nu \mathfrak{sp}(\mathcal{N}_b)} = \lim_{\nu \to \infty} \rho(e^{-\nu \mathcal{N}_b}),
\]
in norm (and also strong or weak) topology on \( \mathcal{B}(\mathcal{H})/\mathbb{C}^\times \). (Note that for \( A \in \mathcal{B}(\mathcal{H}) \setminus \{0\} \) and \( z \in \mathbb{C} \setminus \{0\} \), we find \( A + zI \neq \tilde{A} \) but \( e^{A+zI} = e^A \), where \( \tilde{X} := \mathbb{C}^\times X \).)

Thus our quantization procedure is related with the limit \( \lim_{\nu \to \infty} \rho(g_\nu) \) where \( \{g_\nu\}_\nu \subset \mathbb{R} \) is some sequence in the Olshanski semigroup \( \Gamma \text{Sp}_e(2n) \). If \( g_\nu := \lim_{\nu \to \infty} g_\nu \) exists in \( \Gamma \text{Sp}_e(2n) \), we have \( \text{s-limit} \rho(g_\nu) = \rho(g_\infty) \), and hence it suffices to deal with the finite-dimensional matrix \( g_\infty \). However, for \( g_\nu := e^{-\nu \mathcal{N}_b} \), we see \( g_\nu \) diverges as \( \nu \to \infty \) in \( \text{Mat}(2n, \mathbb{C}) \). We shall see that such divergent cases are of our main concern, in the context of quantizations. An elegant solution of this problem is given by the graph viewpoint [5], which will be explained in Sec.6.

5 Number operator representation of quantized time evolution

Let \( f : \mathbb{C}^m \to \mathbb{R} \) (a “classical Hamiltonian”). The quantum time evolution w.r.t. the quantized Hamiltonian \( \mathcal{E}_b(f(\tilde{Z})) \) is given by the unitary operators \( U_{f,t} := \exp[-it\mathcal{E}_b(f(\tilde{Z}))], \ t \in \mathbb{R} \). The following conjecture expresses \( U_{f,t} \) without the projection \( \mathcal{E}_b \) but with \( \mathcal{N}_b \), the number operator which is more familiar and usable for calculations.

**Conjecture 5.1** (Number operator representation of quantized time evolution). If the classical Hamiltonian \( f \) is “physically reasonable,” then \( \mathcal{E}_b(f(\tilde{Z})) \) is essentially self-adjoint, and the following holds:
\[
\lim_{\nu \to \infty} \exp[-itf(\tilde{Z}) - \nu \mathcal{N}_b]\varphi = U_{f,t}\varphi, \quad t \in \mathbb{R}, \ \varphi \in \text{ran} \mathcal{E}_b.
\] (5.1)
in other words,
\[
\text{s-limit} \exp[-itf(\tilde{Z}) - \nu \mathcal{N}_b] = U_{f,t} \mathcal{E}_b, \quad t \in \mathbb{R}.
\]
However, it does not seem easy to formulate the “physical reasonability” rightly and to prove (5.1). In fact, even the essential self-adjointness of $\mathcal{E}_b(f(\tilde{Z}))$ is not assured in general. If $f$ is a classical Hamiltonian in the narrow sense, i.e. if $f$ can be interpreted as the energy of the system, $f$ is bounded from below typically, and hence the essential self-adjointness is widely assured. However, we want to consider more general Hamiltonian functions $f$, which are generators of various continuous physical transformations, not only time translations; in such cases $f$ is often unbounded both from below and from above.

We verify Conjecture 5.1 in the case where $f : \mathbb{C}^m \cong \mathbb{R}^{2m} \to \mathbb{R}$ is the Hamiltonian function of $\mathfrak{sp}_c(2m, \mathbb{R})$.

For $\mathcal{A} \in \mathfrak{sp}_c(2m, \mathbb{R})$, the corresponding Hamiltonian function $h_{\mathcal{A}} : \mathbb{C}^m \to \mathbb{C}$ is given by

$$h_{\mathcal{A}}(\tilde{z}) := \frac{1}{2} \tilde{z}^T \mathcal{I} \mathcal{A} \tilde{z}, \quad \tilde{z} := (z_1, \ldots, z_m),$$

where

$$z := (\overline{z}_1, \ldots, \overline{z}_m, z_1, \ldots, z_m)^T, \quad \tilde{z}^* := (z_1, \ldots, z_m, \overline{z}_1, \ldots, \overline{z}_m).$$

Indeed, we see the value of $h_{\mathcal{A}}(\tilde{z})$ is pure imaginary. This seems strange since a classical-mechanical Hamiltonian function on a phase space is usually real-valued. However we prefer to retain the similarity to (3.2). If one prefers the consistency with classical mechanics, instead take $h'_{\mathcal{A}}(\tilde{z}) := ih_{\mathcal{A}}(\tilde{z})$.

The following lemma is shown by a straightforward calculation.

**Lemma 5.2.** Let $\mathcal{A} = \begin{pmatrix} A & B \\ B & \overline{A} \end{pmatrix} \in \mathfrak{sp}_c(2m, \mathbb{R})$, so that $A, B \in \text{Mat}(m, \mathbb{C})$, $A^* = -A$, $B^T = B$. Then

$$h_{\mathcal{A}}(\tilde{Z}) = \frac{1}{2} \tilde{Z}^* \mathcal{I}_{2m} \mathcal{A} \tilde{Z} = \frac{1}{2} \tilde{a}^* \mathcal{I}_{2m} \tilde{A} a = dp(\tilde{A}), \quad \tilde{Z} := (Z_1, \ldots, Z_m),$$

where

$$Z := (Z_1, \ldots, Z_m, \overline{Z}_1, \ldots, \overline{Z}_m)^T, \quad \tilde{Z}^* := (Z_1, \ldots, Z_m, \overline{Z}_1, \ldots, \overline{Z}_m).$$

$\tilde{A} := \begin{pmatrix} -A & -B & -\overline{B} & -\overline{A} \\ B & A & \overline{A} & \overline{B} \\ -B & -A & \overline{B} & \overline{A} \\ \overline{A} & \overline{B} & B & A \end{pmatrix} \in \mathfrak{sp}_c(2n, \mathbb{R}).$

Substitute $-if := h_{\mathcal{A}}$ in the conjecture (5.1), then we have

$$s-lim_{\nu \to \infty} \exp[h_{\mathcal{A}}(\tilde{Z}) - \nu N_b] = \exp[\mathcal{E}_b(h_{\mathcal{A}}(\tilde{Z}))] E_b.$$  

(5.4)

Notice that (5.3) implies

$$h_{\mathcal{A}}(\tilde{Z}) - \nu N_b = dp \left( \tilde{A} - \nu N_b \right) + \frac{\nu m}{2}.$$  

Furthermore, consider the convergence as operator rays instead of operators, i.e. the topology of $\mathcal{B}(\mathcal{H})/\mathbb{C}^\times$, instead of $\mathcal{B}(\mathcal{H})$, then (5.4) becomes

$$s-lim_{\nu \to \infty} \rho \left( \exp(\tilde{A} - \nu N_b) \right) = \mathbb{C}^\times \exp \left[ \mathcal{E}_b(h_{\mathcal{A}}(\tilde{Z})) \right] E_b.$$  

(5.5)

where the term $\nu m/2$ vanishes. The first goal of this article is to prove this. Note that let $g_\nu := \exp(\tilde{A} - \nu N_b)$, then we see $g_\nu \in \Gamma\text{Sp}_c(2n)$. Here we again encounter the problem of convergence of the sequence $(g_\nu)$ in the Olshanski semigroup $\Gamma\text{Sp}_c(2n)$.

## 6 Extended Olshanski semigroup

To deal with the convergence problem in the Olshanski semigroup $\Gamma\text{Sp}_c(2n)$, we introduce the notion of extended Olshanski semigroup. Since this notion does not seem common in the community of mathematical physics, we outline the theory here. This section is based on Neretin [5]: Especially, see Sec. 8 and 9 in Ch. 1, Sec. 9 in Ch. 2, and Sec. 1 in Ch. 5 of [5]. Again note that the term “(extended)
Olsaniski semigroup” is not found in [5], but in Neeb [4]; Our notion of extended Olsaniski semigroup for \( \text{Sp}_c(2n, \mathbb{R}) \) amounts to the semigroup of morphisms in the “symplectic category” in [5].

Let \( T \in \text{Mat}(2n, \mathbb{C}) \) and regard it as a linear operator \( T : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \). The graph of \( T \)
\[
\text{graph}(T) := \{ v + T v | v \in \mathbb{C}^{2n} \} \subset \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}
\]
is a \( 2n \)-dimensional linear subspace of \( \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \cong \mathbb{C}^{4n} \). Thus \( T \) is identified with a single point in the Grassmann manifold \( G_{4n,2n}(\mathbb{C}) \).

Observe that \( \text{graph}(e^{-\nu N_b}) \) converges in \( G_{4n,2n}(\mathbb{C}) \) as \( \nu \to \infty \), although \( \lim_{\nu \to \infty} e^{-\nu N_b} \) does not converge to any matrix. If we can reformulate the metaplectic representation \( \rho(g) \) as \( \rho(\text{graph}(g)) \) \( (g \in \Gamma \text{Sp}_c(2n)) \), it is expected that
\[
\lim_{\nu \to \infty} \rho(e^{-\nu N_b}) = \rho\left( \lim_{\nu \to \infty} \text{graph}(e^{-\nu N_b}) \right).
\]
holds. However, the rhs is not defined yet. It will be found in Theorem 6.11 below.

For any subspaces \( A, B \subseteq \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \), define the product \( AB := \{ x + y \in \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} | \exists w \in \mathbb{C}^{2n}, x + w \in P, w + y \in Q \} \),
and ker \( A \), indef \( A \) \( \subset \mathbb{C}^{2n} \)
\[
\ker A \ := \{ x \in \mathbb{C}^{2n} | x \oplus 0 \in A \}, \quad \indef A \ := \{ y \in \mathbb{C}^{2n} | 0 \oplus y \in A \}
\]

**Definition 6.1.** Define \( U_{n,n} \) to be the set of \( 2n \)-dimensional subspaces \( P \) of \( \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \) such that

1. \( P \) is \( \mathcal{I} \)-contractive, i.e. for any \( v \oplus w \in P \), \( \langle v | w \rangle_{\mathcal{I}} \geq \langle w | w \rangle_{\mathcal{I}} \).
2. for any nonzero \( v \in \text{indef} \) \( A \), \( \langle v | v \rangle_{\mathcal{I}} < 0 \).
3. for any nonzero \( v \in \text{ker} \) \( A \), \( \langle v | v \rangle_{\mathcal{I}} > 0 \).

**Definition 6.2.** Let \( P \) be a subspace of \( \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \). \( P \) is said to be symplectic if for any \( v \oplus w, v' \oplus w' \in P \), \( v^T J w' = w^T J w' \). Let
\[
\text{Sp}_{2n} := \{ P \in U_{n,n} | P \text{ is symplectic} \},
\]

Since \( \text{Sp}_{2n} \subset U_{n,n} \subset G_{4n,2n}(\mathbb{C}) \), \( U_{n,n} \) and \( \text{Sp}_{2n} \) are given the topology induced by that of \( G_{4n,2n}(\mathbb{C}) \).

**Proposition 6.3.** Let \( G \) be \( U_{n,n} \) or \( \text{Sp}_{2n} \). For any \( A, B \in G \), we have \( AB \in G \); that is, \( G \) is a semigroup. The product in \( G \) is continuous.

**Proof.** See Sec.9.1 in Ch.1 of [5]. However, the continuity of \( (A, B) \mapsto AB \) does not appear to be stated explicitly in [5]. It follows from Proposition 6.10 below. \( \square \)

**Proposition 6.4.**
1. If \( g \in \Gamma U(n, n) \), then \( \text{graph}(g) \in U_{n,n} \). Furthermore the semigroup \( \Gamma U(n, n) \) is embedded continuously and densely into the semigroup \( U_{n,n} \) by \( g \mapsto \text{graph}(g) \).
2. If \( g \in \Gamma \text{Sp}_c(2n) \), then \( \text{graph}(g) \in \text{Sp}_{2n} \). Furthermore the semigroup \( \Gamma \text{Sp}_c(2n) \) is embedded continuously and densely into the semigroup \( \text{Sp}_{2n} \) by \( g \mapsto \text{graph}(g) \).

**Proof.** Recall that \( g \in \Gamma U(n, n) \) iff \( \langle gv | gv \rangle_{\mathcal{I}} \leq \langle v | v \rangle_{\mathcal{I}} \), and \( \Gamma \text{Sp}_c(2n) = \text{Sp}(2n, \mathbb{C}) \cap \Gamma U(n, n) \). Then the above is evident except the density, which is not stated explicitly in [5]. The density follows from Proposition 6.9 below. \( \square \)

We call \( \text{Sp}_{2n} \) the extended Olsaniski semigroup, or the Krein–Shmul’yan–Olsaniski (KSO) semigroup for \( \text{Sp}_c(2n, \mathbb{R}) \).

Let \( V_\pm := \ker(\mathcal{I} \mp 1) \subset \mathbb{C}^{2n} \), so that \( \mathbb{C}^{2n} = V_- \oplus V_+ \). Define the linear operator \( \Pi : \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \to \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \) as follows. Let \( \Pi \) act on
\[
(v_- \oplus v_+) \oplus (w_- \oplus w_+) \in (V_- \oplus V_+) \oplus (V_- \oplus V_+) = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n},
\]
Then where the conditions are (1)–(4) are equivalent, and so one of them suffices. Also note that if 
expressed by (1)–(4) is w.r.t. the usual norm $\| \|$ on $\mathbb{C}^n$. Thus (1)–(4)
for any $g \in \text{Mat}(2n, \mathbb{C})$, then $d - ca^{-1}b = a^T b$.

**Example 6.8.** Let $X := \left( \begin{array}{ccc} 1 & 0 \\ 0 & -1 \end{array} \right)$ and consider $e^{tX}$ for $t \geq 0$. Since $e^{tX} \in \text{Sp}(2, \mathbb{C})$ and $I_2 - (e^{tX})^* I_2 e^{tX} \geq 0$, we have $e^{tX} \in \Gamma \text{Sp}_e(2)$. We see

$$
\lim_{t \to +\infty} \Pi(e^{tX}) = \lim_{t \to +\infty} \begin{pmatrix} 0 & e^{-t} \\ e^t & 0 \end{pmatrix} = O.
$$

Thus $P := \lim_{t \to +\infty} \text{graph}(e^{tX}) = \Pi^{-1}(\text{graph}(O)) = \{(z, 0, 0, z')^T \in \mathbb{C}^4 : z, z' \in \mathbb{C}\}$, $\ker P = \{(0, 0, 0, 0)^T \in \mathbb{C}^4\}$, and $\text{ind} P = \{(0, 0, 0, 0)^T \in \mathbb{C}^4\}$. Therefore we conclude that $P \in \text{Sp}_2$, and that $\lim_{t \to +\infty} e^{tX}$ does not converge in $\Gamma \text{Sp}_e(2)$, but does in $\text{Sp}_2$.

**Proposition 6.9.** ([5]: Theorem 8.1, 8.2, 9.3 in Ch.2) We have

$$
\Pi(U(n, n)) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2n) \mid \text{conditions (1)-(4)} \right\},
$$

$$
\Pi(\Gamma U(n, n)) = \left\{ r := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Mat}(2n, \mathbb{C}) \mid \| r \| \leq 1, \text{ conditions (1)-(4)} \right\},
$$

$$
\Pi(U(n, n)) = \left\{ r := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Mat}(2n, \mathbb{C}) \mid \| r \| \leq 1, \text{ conditions (3), (4)} \right\},
$$

where the conditions are (1) $\det \beta = 0$, (2) $\det \gamma = 0$, (3) $\| \alpha \| < 1$, (4) $\| \delta \| < 1$; and the operator norm $\| r \|$ is w.r.t. the usual norm $\| v \|^2 := \langle v | v \rangle = v^* v$ on $\mathbb{C}^n$. As to (6.2), in fact, the conditions (1)-(4) are equivalent, and so one of them suffices. Also note that if $\| r \| \leq 1$, then (1) $\Rightarrow$ (3)$\&$(4) and (2) $\Rightarrow$ (3)$\&$(4).

**Proposition 6.10.** ([5]: Theorem 9.4 in Ch.2) Let $P_1, P_2 \in U(n, n)$, and $\Pi(P_1), \Pi(P_2) \in \text{Mat}(2n, \mathbb{C})$ be expressed by

$$
\Pi(P_1) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \Pi(P_2) = \begin{pmatrix} \varphi & \psi \\ \theta & \kappa \end{pmatrix}.
$$

Then

$$
\Pi(P_1 P_2) = \begin{pmatrix} \alpha + \beta(1 - \varphi \delta)^{-1} \varphi \gamma & \beta(1 - \varphi \delta)^{-1} \psi \\ \theta(1 - \delta \varphi)^{-1} \gamma & \theta \delta(1 - \varphi \delta)^{-1} \psi \end{pmatrix}.
$$

10
Let $\rho : \Gamma \text{Sp}_e(2n) \to \mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ be the metaplectic representation.

**Theorem 6.11.** ([5]: Theorem 8.2, 8.5 in Ch.1, Theorem 1.5 in Ch.5) There exists a strongly continuous projective representation $\rho' : \text{Sp}_{2n} \to \mathcal{B}(\mathcal{H})/\mathbb{C}^\times$ which extends $\rho$; That is,

\[
\rho'(\text{graph}(g)) = \rho(g), \quad g \in \Gamma \text{Sp}_e(2n)
\]

We call $\rho'$ the **extended metaplectic representation**, and we write $\rho$ for $\rho'$ in the following. To obtain the explicit formula for calculating $\rho(P)$ for any $P \in \text{Sp}_{2n}$, one may consider the Schrödinger representation on $L^2(\mathbb{R}^{2n})$ or the Fock–Bargmann representation on the boson Fock space $\mathcal{F}(\mathbb{C}^n)$. In both cases $\rho(P)$ is expressed as an integral operator whose kernel is a Gaussian function. (Precisely, in the former case, the kernel can be a tempered distribution, not a function.) The Gaussian kernel is written in terms of the Potapov transform of $P$ in an explicit form.

**7 Results for $\text{Sp}_e(2m, \mathbb{R})$**

Our first goal (5.5) is achieved by Corollary 7.3 and Theorem 7.4 below, which follows from the following theorem. Recall $\mathcal{I}_{2n} = (-I_m) \oplus (-I_n) \oplus I_m \oplus I_n$ and $\mathcal{N}_b = O_m \oplus I_m \oplus O_m \oplus (-I_m)$.

**Theorem 7.1.** Let $n = 2m$ and $A \in \text{sp}_e(2n, \mathbb{R})$. Then

\[
\lim_{\nu \to \infty} \text{graph} e^{A - \nu\mathcal{N}_b}
\]

converges to some $P \in \text{Sp}_{2n}$, explicitly written as follows. Let $V^{(\lambda)} \subset \mathbb{C}^{4m}$ be the eigenspaces of $\mathcal{N}_b$, corresponding to the eigenvalues $\lambda = 0, \pm 1$ of $\mathcal{N}_b$. Then

\[
P = \{ (v_1 + v_0) + (e^{\nu A} v_0 + v_1) | v_{\lambda} \in V^{(\lambda)}, \lambda = 0, \pm 1 \},
\]

where $A_0 := (I - N_b^2) A (I - N_b^2)$. Hence we have

\[
\ker P = V^{(-1)} \subset \ker(\mathcal{I}_{2n} - I), \quad \text{indef } P = V^{(1)} \subset \ker(\mathcal{I}_{2n} + I),
\]

which verifies $P \in \text{Sp}_{2n}$.

**Proof.** There exists $r > 0$ and smooth maps $(-r, r) \ni \epsilon \mapsto V^{(\lambda)} (\lambda = 0, \pm 1)$ s.t. $V^{(\lambda)}$ is an invariant subspace of $\epsilon A - \mathcal{N}_b$ for each $\epsilon \in (-r, r)$ and $\lambda$, and that $V^{(0)}_0 = V^{(\lambda)}$. Let $P^{(\lambda)}_\epsilon$ be the corresponding projection onto $V^{(\lambda)}_{\epsilon}$. Precisely, $P^{(\lambda)}_\epsilon$ is determined by

\[
P^{(\lambda)}_\epsilon : = \begin{cases} v & \text{if } v \in V^{(\lambda)}, \\ 0 & \text{if } v \in V^{(\lambda)}, \lambda' \neq \lambda \end{cases}.
\]

Let

\[
A^{(\lambda)}_\epsilon : = (\epsilon A - \mathcal{N}_b) P^{(\lambda)}_\epsilon,
\]

Set $\epsilon = 1/\nu$. We have

\[
\text{graph} e^{A - \nu \mathcal{N}_b} = \text{graph} e^{\nu (A - \mathcal{N}_b)} = \text{graph} \left( \nu \sum_{\lambda} A^{(\lambda)}_\epsilon \right) = \bigoplus_{\lambda} \text{graph} \left( \exp \left( \nu A^{(\lambda)}_\epsilon |_{V^{(\lambda)}} \right) \right).
\]

Since $\lim_{\nu \to \infty} A^{(\lambda)}_\epsilon = A^{(\lambda)}_0 = \lambda P^{(\lambda)}_0$, we have

\[
\lim_{\nu \to \infty} \exp \nu A^{(-1)}_\epsilon = P^{(1)}_0 + P^{(0)}_0,
\]

\[
\lim_{\nu \to \infty} \exp (-\nu A^{(1)}_\epsilon) = P^{(-1)}_0 + P^{(0)}_0,
\]

\[
\lim_{\nu \to \infty} \exp \nu A^{(0)}_\epsilon = \exp A^{(0)}_0,
\]

11
where \( \dot A^{(0)}_0 := \frac{d}{de} A^{(0)}_e \bigg|_{e=0} \). Thus the following hold:

\[
\begin{align*}
\lim_{\nu \to \infty} \text{graph exp} \left( \nu A^{-1}_{\nu} \right) &= \left\{ v \oplus \left( F^{(0)}_0 + P^{(1)}_0 \right) | v \in V^{-1} \right\} = \left\{ v \oplus 0 | v \in V^{-1} \right\}, \\
\lim_{\nu \to \infty} \text{graph exp} \left( \nu A^{1}_{\nu} \right) &= \left\{ \left( P^{(0)}_0 + P^{(1)}_0 \right) v \oplus v | v \in V^{(1)} \right\} = \left\{ 0 \oplus v | v \in V^{(1)} \right\}, \\
\lim_{\nu \to \infty} \text{graph exp} \left( \nu A^{(0)}_\nu \right) &= \left\{ v \oplus \left( \exp \dot A^{(0)}_0 \right) | v \in V^{(0)} \right\}.
\end{align*}
\]

Therefore we find that graph \( e^{A-\nu N_b} \) converges as \( \nu \to \infty \), and that

\[
\lim_{\nu \to \infty} \text{graph} \left( e^{A-\nu N_b} \right) = \left\{ v \oplus 0 | v \in V^{-1} \right\} \oplus \left\{ v \oplus \left( \exp \dot A^{(0)}_0 \right) | v \in V^{(0)} \right\} \oplus \left\{ 0 \oplus v | v \in V^{(1)} \right\} \quad (7.3)
\]

Thus by Lemma 7.2 below, we find (7.1) and (7.2).

\[
\text{Lemma 7.2.}
\]

\[
\frac{d}{de} \bigg|_{e=0} P^{(0)}_e = N_b A \left( I - N_b^2 \right) + \left( I - N_b^2 \right) A N_b,
\]

and

\[
\dot A^{(0)}_0 = A_0 := \left( I - N_b^2 \right) A \left( I - N_b^2 \right).
\]

\[
\text{Lemma 7.2.}
\]

Proof. To prove (7.5), observe that \( P^{(0)}_e \) is explicitly written as

\[
P^{(0)}_e = \lim_{j \to \infty} P^{(0)}_{e,j}, \quad P^{(0)}_{e,j} := \left[ \left( 2 (\epsilon A - N_b) \right)^{2j} + I \right]^{-1},
\]

and that if \( t \to X \) is any matrix-valued smooth function \( \mathbb{R} \to \text{GL}(n, \mathbb{C}) \), then

\[
\frac{d}{dt} \bigg|_{t=0} X^{-1} = -X^{-1} \left( \frac{d}{dt} \bigg|_{t=0} X \right) X^{-1}. \quad (7.7)
\]

Since

\[
P^{(0)}_{0,j} = \left( (2N_b)^{2j} + I \right)^{-1} = \left( I - N_b^2 \right) + (2^{2j} + 1)^{-1} N_b^2 = I + \left[ -1 + (2^{2j} + 1)^{-1} \right] N_b^2
\]

and

\[
\frac{d}{de} \bigg|_{e=0} P^{(0)}_{e,j} = -2^{2j} \left[ A N_b + N_b A + (j - 1) N_b^2 A N_b + (j - 1) N_b^2 A N_b \right],
\]

we find from (7.7) that

\[
\frac{d}{de} \bigg|_{e=0} P^{(0)}_{e,j} = -P^{(0)}_{0,j} \left[ \frac{d}{de} \bigg|_{e=0} \left( P^{(0)}_{e,j} \right)^{-1} \right] P^{(0)}_{0,j}.
\]

Substituting (7.8) and (7.9), a straightforward calculation shows that

\[
\frac{d}{de} \bigg|_{e=0} P^{(0)}_{e,j} = \left[ -\frac{2^{2j}}{2^{2j} + 1} N_b A \left( I - N_b^2 \right) + 2^{2j} (2^{2j} + 1)^{-1} \right.
\]

\[
\times \left[ -\left( I - N_b^2 \right) A N_b - \frac{1}{2^{2j} + 1} N_b^2 \left[ A N_b + N_b A + (j - 1) N_b^2 A N_b + (j - 1) N_b^2 A N_b \right] N_b^2 \right].
\]

Therefore we find

\[
\lim_{j \to \infty} \frac{d}{de} \bigg|_{e=0} P^{(0)}_{e,j} = N_b A \left( I - N_b^2 \right) + \left( I - N_b^2 \right) A N_b
\]

and hence (5.1) holds. We can verify (7.5) by

\[
\dot A^{(0)}_0 = \frac{d}{de} \bigg|_{e=0} A^{(0)} \bigg|_{e=0} = \frac{d}{de} \left( \epsilon A - N_b \right) P^{(0)}_e \bigg|_{e=0} = A P^{(0)}_0 - N_b \frac{d}{de} P^{(0)}_e \bigg|_{e=0}
\]

\[
= A P^{(0)}_0 - N_b \left[ N_b A \left( I - N_b^2 \right) + \left( I - N_b^2 \right) A N_b \right] \quad \text{(by (7.5))}
\]

\[
= A \left( I - N_b^2 \right) - N_b^2 \left( I - N_b^2 \right) - N_b \left( I - N_b^2 \right) A N_b
\]

\[
= \left( I - N_b^2 \right) A \left( I - N_b^2 \right) = A_0.
\]
**Corollary 7.3.** \( s\text{-lim}_{\nu \to \infty} \rho(e^{t(A - \nu N_b)}) \) converges to some \( C^\times T_{A,t} \in \mathcal{B}(\mathcal{H})/C^\times \) for each \( t > 0 \), where

1. \( T_{A,t} \in \mathcal{B}(\mathcal{H}) \) is strongly continuous w.r.t. \( t > 0 \),
2. \( \|T_{A,t}\| = 1 \) for all \( t > 0 \),
3. \( s\text{-lim}_{\nu \to 0} T_{A,t} = E_b \).

**Proof.** Clear from Theorem 6.11 and 7.1. \( \square \)

The following theorem is a special case of Conjecture 5.1.

**Theorem 7.4.** Let \( A \in \text{sp}_c(2m, \mathbb{R}) \). Then we have

\[
\text{s-lim}_{\nu \to \infty} \exp(h_A(\bar{Z}) - \nu N_b) = \exp[\xi(h_A(\bar{Z}))]E_b. \tag{7.10}
\]

**Proof.** By Lemma 5.2, we find

\[
\text{s-lim}_{\nu \to \infty} \exp(h_A(\bar{Z}) - \nu N_b) = \text{s-lim}_{\nu \to \infty} \exp(A - \nu N_b) \tag{7.11}
\]

Hence by Corollary 7.3,

\[
\text{s-lim}_{\nu \to \infty} \xi_{t,\nu} \exp t(h_A(\bar{Z}) - \nu N_b) = T_{A,t} \in \mathcal{B}(\mathcal{H}), \tag{7.12}
\]

for some sequence \( \xi_{t,1}, \xi_{t,2}, \ldots \in C^\times \), for each \( t > 0 \). We can assume that \( \xi_{t,\nu} \) depends on \( t \) as \( \xi_{t,\nu} = e^{t\xi_\nu} \), so that

\[
T_{A,t} = \text{s-lim}_{\nu \to \infty} T_{A,t}^{(\nu)} \quad \text{and} \quad T_{A,t}^{(\nu)} := \exp t(h_A(\bar{Z}) - \nu N_b + \xi_\nu) \quad \text{and} \quad T_{A,s}T_{A,t} = T_{A,s+t}.
\]

Let \( T_{A,0} := s\text{-lim}_{\nu \to 0} T_{A,t} = E_b \). Then \( \{T_{A,t} | t \geq 0\} \) is a strongly continuous one-parameter semigroup, but this is not a contraction semigroup in the usual sense (e.g. [6, X.8]), since \( T_{A,0} \neq I \). Noticing

\[
T_{A,\nu}\psi = \lim_{\nu \to \infty} T_{A,t+t}\psi = \lim_{\nu \to \infty} E_b T_{A,t}\psi = E_b \psi \quad \psi \in \mathcal{K},
\]

instead we consider \( \{T_{A,t} | \mathcal{K} | t \geq 0\} \), which is a strongly continuous contraction semigroup of bounded operators on \( \mathcal{K} \) in the usual sense. Hence there exists a closed operator \( X_A \) densely defined on \( \mathcal{K} \) s.t. \( T_{A,t} | \mathcal{K} \) is formally expressed as \( e^{tX_A} \). Let \( \psi \in \mathcal{K} \cap \text{dom}(h_A(\bar{Z})) \), then

\[
\lim_{\nu \to \infty} E_b \frac{d}{dt} T_{A,t}^{(\nu)} \psi \bigg|_{t=0} = \lim_{\nu \to \infty} E_b (h_A(\bar{Z}) - \nu N_b + \xi_\nu) \psi = E_b h_A(\bar{Z}) \psi + \lim_{\nu \to \infty} \xi_\nu \psi.
\]

Hence the above l.h.s. converges if \( \lim_{\nu \to \infty} \xi_\nu \) converges. If we set \( \xi_\nu \equiv 0 (\xi_{t,\nu} \equiv 1) \), then we have

\[
X_A \psi = \frac{d}{dt} T_{A,t} \psi \bigg|_{t=0} = \lim_{\nu \to \infty} E_b \frac{d}{dt} T_{A,t}^{(\nu)} \psi \bigg|_{t=0} = E_b h_A(\bar{Z}) \psi.
\]

Eq. (7.10) follows from this. \( \square \)

### 8 Differential operators

Let \( m \in \mathbb{N} \), \( n := 2m \). Consider the representations of annihilation/creation operators \( a_k, b_k, a_k^*, b_k^* \) \((k = 1, \ldots, m)\) on \( L^2(\mathbb{C}^m) \) where \( Z_k = a_k^* + b_k \) is represented by \( z_k \) \((\text{the } k\text{th coordinate of } \mathbb{C}^m, \text{viewed as a multiplication operator on } L^2(\mathbb{C}^m))\) for all \( k \). An example of such representation is given by

\[
a_k := \frac{\partial}{\partial z_k} + \frac{z_k}{2}, \quad b_k := \frac{\partial}{\partial z_k^*} + \frac{z_k}{2}.
\]
and so
\[ a_k^* = -\frac{\partial}{\partial z_k} + \frac{z_k}{2}, \quad b_k^* = -\frac{\partial}{\partial z_k} + \frac{\overline{z}_k}{2}. \]

Let
\[ \Re b_k := \frac{1}{2} (b_k + b_k^*) = \frac{1}{2} \left( i \frac{\partial}{\partial y_k} + x_k \right), \quad \Im b_k := \frac{1}{2} (b_k - b_k^*) = \frac{1}{2} \left( -i \frac{\partial}{\partial x_k} + y_k \right), \]
where \( z_k = x_k + iy_k, x_k, y_k \in \mathbb{R} \). Then we have the following:
\[ [\Re b_k, \Im b_k] = \frac{i}{2}, \quad b_k^* b_k = (\Re b_k)^2 + (\Im b_k)^2 = \frac{1}{2}, \]
and hence
\[ N_b = \sum_{k=1}^m \left( (\Re b_k)^2 + (\Im b_k)^2 \right) - \frac{m}{2} \]
\[ = \frac{1}{4} \sum_{k=1}^m \left[ \left( i \frac{\partial}{\partial y_k} + x_k \right)^2 + \left( -i \frac{\partial}{\partial x_k} + y_k \right)^2 \right] - \frac{m}{2}. \]

Let \( \alpha = (\alpha_1, ..., \alpha_{2m}) : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) be a smooth map, where \( \alpha_k : \mathbb{R}^{2m} \to \mathbb{R} \) (\( k = 1, ..., 2m \)). Then \( \alpha \) is seen as a smooth vector field on \( \mathbb{R}^{2m} \cong \mathbb{C}^m \). The (positive-definite) magnetic Laplacian \( \Delta^\alpha \) is defined by
\[ \Delta^\alpha := -\sum_{k=1}^{2m} \left( \frac{\partial}{\partial x_k} + i\alpha_k \right)^2, \]
where we set \( x_{m+k} := y_k \) for \( k = 1, ..., m \). (For more general, geometric and rigorous definitions of magnetic Laplacians, see e.g. [8].) Set
\[ \alpha_k(\vec{x}) := x_{m+k}, \quad \alpha_{m+k}(\vec{x}) := -x_k, \quad k = 1, ..., m, \quad \vec{x} := (x_1, ..., x_{2m}), \]
then we have
\[ N_b = \frac{1}{4} \Delta^\alpha - \frac{m}{2}. \]

In this representation, the operator \( h_{\mathcal{A}}(\vec{Z}) \) is nothing other than \( h_{\mathcal{A}} \) as a multiplication operator on \( L^2(\mathbb{C}^m) \). Thus by Theorem 7.4, we have the following:

**Corollary 8.1.** For any \( \mathcal{A} \in \mathfrak{sp}_{c}(2m, \mathbb{R}) \),
\[ \lim_{\nu \to \infty} \exp \left( h_{\mathcal{A}} - \nu \left( \frac{1}{4} \Delta^\alpha - \frac{m}{2} \right) \right) = \exp \left[ \mathcal{E}_b(h_{\mathcal{A}}) \right] E_b. \]  

### 9 Path integral representation

Let \( \mathcal{H} := L^2(\mathbb{C}^m) \) and \( \mathcal{K} := \text{ran } E_b = \ker N_b \subset \mathcal{H} \). Let \( \Omega_0 \in \mathcal{K} \) be a unit vector such that \( N_b \Omega_0 = 0 \), which is unique up to scalar multiples; equivalently, \( a_k \Omega_0 = 0 \) for all \( k = 1, ..., m \). We call it the vacuum vector. We also fix a unit vector \( \Omega_\vec{z} \in \mathcal{K} \) for each \( \vec{z} = (z_1, ..., z_m) \in \mathbb{C}^m \) such that
\[ a_k \Omega_\vec{z} = z_k \Omega_\vec{z}, \quad k = 1, ..., m. \]
These are called coherent vectors.
\[ p_{\vec{z}} := |\Omega_\vec{z}\rangle \langle \Omega_\vec{z}| \]
\[ d\mu(\vec{z}) := \frac{1}{\pi^m} dx_1 dy_1 \cdots dx_m dy_m, \quad z_k = x_k + iy_k, \]
\[ \int_{\mathbb{C}^m} p_{\vec{z}} d\mu(\vec{z}) = E_b \text{ on } L^2(\mathbb{C}^m). \]
We find that the manifold $\mathbf{M} := \{ \mathbf{p} \in \mathbb{C}^m \} \cong \mathbb{C}^m$, where each $\mathbf{p}$ is viewed as an operator on $\mathcal{K}$, not on $L^2(\mathbb{C}^m)$, is an example of more general notion of the family of coherent states defined in [10]. Then we find that the antinormally ordered quantization $\mathcal{E}_h(\phi)$ of a function $f : \mathbb{C}^m \to \mathbb{R}$ is expressed as

$$\mathcal{E}_h(f) = \mathcal{Q}(f) := \int_{\mathbb{C}^m} f(\zeta) \mathbf{p}_d\mu(\zeta).$$

From Sec. 7 of [10], we know that Corollary 5.8 in [10] holds for this case. This implies the following.

For $T > 0$, let $\mu_{[0, T]}$ be the standard Brownian bridge measure on the path space

$$\text{loop}_{[0, T], 0} := \{ \varphi \in C([0, T], \mathbb{R}^m) | \varphi(0) = \varphi(T) = 0 \}.$$

For $\nu > 0$ and $\varphi \in \text{loop}_{[0, T], 0}$, define $\varphi(\nu) \in \text{loop}_{0, \nu} := \text{loop}_{[0, T], 0}$ by $\varphi(\nu)(t) := \varphi(\nu t)$.

The measure $\mu_{[0, \nu]}$ and the bijection $\text{loop}_{[0, \nu], 0} \to \text{loop}_{0, \nu}$, $\varphi \mapsto \varphi(\nu)$, induce the measure $\mu_{\nu}$ on $\text{loop}_{0, \nu}$. Let

$$\alpha^b := m \sum_{k=1}^m (y_k d x_k - x_k d y_k), \quad z_k = x_k + iy_k.$$

This is denoted by $\theta_{\text{nor}}$ in Sec. 7 of [10].

**Theorem 9.1.** If $H : \mathbb{C}^m \to \mathbb{R}$ is smooth and bounded, then

$$\langle \Omega_0 | e^{-i \mathcal{E}_h(H)} \Omega_0 \rangle = \lim_{\nu \to \infty} e^{\nu m} \int_{\text{loop}_{0, \nu}} e^{i S_H(\varphi)} d \mu_{\nu}(\varphi) = \lim_{\nu \to \infty} e^{\nu m} \mathbb{E}_\nu \left[ e^{i S_H(\varphi)} \right]$$

where

$$S_H(\varphi) := \int_{\varphi} \alpha^b + \int_0^1 H(\varphi(t)) dt.$$

Here, the line integral $\int_{\varphi} \alpha^b$ of the 1-form $\alpha^b$ along the curve $\varphi$ is understood as a stochastic integral in the sense of Stratonovich w.r.t. the path measure $\mu_{\nu}$, $\mathbb{E}_\nu[\cdot]$ denotes the (classical) expectation value w.r.t. the probability measure $\mu_{\nu}$.

Recall that $h_A$ is defined by (5.2), and pure imaginary. For nonzero $A \in \mathfrak{sp}(2m, \mathbb{R})$, we see $h_A$ is unbounded. Hence we cannot set $H = ih_A$ in the above theorem. Instead we take a simple workaround: For $\tau > 0$, define the cutoff Hamiltonian $h_A^{\tau} : \mathbb{C}^m \to \mathbb{R}$ by

$$h_A^{\tau}(\zeta) := \frac{1}{i} \max \{ \min \{ ih_A(\zeta), \tau \}, -\tau \}.$$

This is bounded but not smooth. Although the above theorem assumes the smoothness of $H$, it is not used in its proof in [10]; we assumed it simply because it usually holds for classical mechanical systems. Hence we can set $H = ih_A^{\tau}$ in the above theorem.

Here we recall the following three theorems:

**Theorem 9.2.** [6, Theorem VIII.21] Let $\{ A_n \}$ and $A$ be self-adjoint operators. Then $A_n \to A$ in the strong resolvent sense if and only if $e^{t A_n}$ converges strongly to $e^{t A}$ for each $t$.

**Theorem 9.3.** [6, Theorem VIII.25] Let $\{ A_n \}_{n=1}^\infty$ and $A$ be self-adjoint operators and suppose that $D$ is a common core for all $A_n, A$; in other words, that they are essentially self-adjoint on $D$. If $A_n \varphi \to A \varphi$ for each $\varphi \in D$, then $A_n \to A$ in the strong resolvent sense.

**Theorem 9.4.** [7, Theorem X.39] (Nelson’s analytic vector theorem) Let $A$ be a symmetric operator on a Hilbert space $\mathcal{H}$. If $\text{dom}(A)$ contains a total set of analytic vectors, then $A$ is essentially self-adjoint.

**Proposition 9.5.** For each $A \in \mathfrak{sp}_c(2m, \mathbb{R})$ and $t \in \mathbb{R}$, we have

$$\lim_{\tau \to \infty} e^{i h_A^{\tau}} = e^{it h_A^{\tau}}.$$ (9.2)
Proof. We know that the vectors in the dense subspace \( D \) of “finite particle states” in \( K \) w.r.t. the creation/annihilation operators \( a_k^*, a_k \) are analytic vectors of \( i\mathcal{E}_b(h_A) \); See e.g. [1, p.190]. (Alternatively we could take the space of finite linear combinations of coherent vectors.) Hence \( i\mathcal{E}_b(h_A) \) is essentially self-adjoint on \( D \) by Theorem 9.4. \( (i\mathcal{E}_b(h_A)^{\tau}) \) is also essentially self-adjoint on \( D \) since it is bounded.) For any \( v \in D \), we easily find that
\[
\lim_{\tau \to \infty} \mathcal{E}_b(h_A^{\tau})v = \lim_{\tau \to \infty} E_b h_A^{\tau} v = E_b h_A v = \mathcal{E}_b(h_A)v.
\]
Thus by Theorem 9.2 and 9.3, we have (9.2).

**Corollary 9.6.** Let \( \mathcal{A} \in \mathfrak{sp}_c(2m, \mathbb{R}) \). With the notations of Theorem 9.1, we have
\[
\langle \Omega_0 | e^{-\mathcal{E}_b(h_A)} \Omega_0 \rangle = \lim_{\tau \to \infty} \lim_{\nu \to \infty} e^{\nu_m} \mathbb{E}_\nu \left[ e^{iS_A\tau(\varphi)} \right] \quad (9.3)
\]
where
\[
S_A\tau(\varphi) := \int_{\mathbb{R}} \alpha_\varphi + \int_0^t i h_A^{\tau}(\varphi(t)) dt \in \mathbb{R}.
\]
Here, the line integral \( \int_{\mathbb{R}} \alpha_\varphi \) is understood as a stochastic integral in the sense of Stratonovich w.r.t. the path measure \( \mu_\nu \).

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