Carleman estimate of solutions to equations adjoint to Navier–Stokes compressible-medium equations

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Abstract. Analyzing the controllability problem for the Navier–Stokes equations of a compressible medium requires Carleman estimates of a corresponding adjoint system. The paper presents a Carleman estimate of solutions to two 2D equations, which are adjoint to continuity and momentum equations in compressible-solid mechanics at the start time. The paper proposes a method for generating Carleman estimates for a hyperbolic-parabolic system with Dirichlet and Neumann conditions featuring the same weight function.

1. Introduction
The controllability problems relating to Navier–Stokes equations of a constrained domain are soluble by [1, 2] building a solution upon an extremal problem. From the uniqueness of the Cauchy problem for an adjoint equation, one can derive results relating to the controllability of the original equation. Carleman estimates are used to prove the Cauchy problem unique. A Carleman estimate is an estimate of a singular weight, the formulation of which depends on the differential equation type as well as on the boundary condition.

In general, the controllability of Navier–Stokes systems for compressible media has been covered for 1D cases only. Paper [3] proves the exact local controllability from the boundary of the domain, while [4] covers the controllability and stabilization of the Navier–Stokes system to the permanent steady state \( (Q_0, 0) \). Paper [5] covers a Navier–Stokes system, the control of which is concentrated in a fixed sub-domain provided that the initial density is already on the target trajectory. Paper [6] proves the null controllability of a homogeneous compressible system of Navier–Stokes equations linearized around a permanent steady state with periodic boundary conditions. Paper [7] covers a Navier–Stokes system on a torus ring for a compressible medium. Paper [8] proves that a 2D compressible Navier–Stokes system is controllable by local internal control.

In this paper, we derive the Carleman estimate for the adjoint system of the Navier–Stokes equations for a compressible medium without taking thermal processes into account, with small density changes in the equation of motion. In this case, in the equation of motion, the density is replaced by the average value of \( \bar{\rho} \), which, without loss of generality, is assumed to be unity. When studying exact local controllability of viscous gas motions, the optimality system consists of differential equations of different types: a continuity, or transport equation; an elliptic equation; and two equations of thermal conductivity. Paper [9] presents a Carleman estimate...
for a parabolic equation with a Dirichlet boundary condition. Paper [10] proposes a method for generating Carleman estimates for a hyperbolic-parabolic system with the same weight function. The method is based on generating a priori estimates in the space of weight functions for a regularized system of differential equations [11] at the monotony intervals of a singular weight function, followed by a passage to the limit. Carleman estimates of the Neumann problem for parabolic and elliptic equations are obtained in [12].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the boundary $\partial \Omega \in C^\infty$, $T > 0$, $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \partial \Omega$ be the lateral surface of the cylinder $Q$. $H^k(\Omega)$, with $k$ being a natural number, denotes the Sobolev space. Assume $W^{1,1}(Q) = \{ u \in L^2(0, T; H^1(\Omega)) : \partial_t u \in L^2(Q) \}$, $W^{1,2}(Q) = \{ u \in L^2(0, T; H^{k+2}(\Omega)) : \partial_t u \in L^2(0, T; H^k(\Omega)) \}$. Denote the norm in the space $L^2(\Omega)$ as $\| \cdot \|$, $(\cdot, \cdot)$, which is scalar product in $L^2(\Omega)$.

Let $\omega' \subset \subset \omega \subset \subset \Omega$. Now we need an auxiliary lemma:

**Lemma 1.** (3) There is a function $\beta(x) \in C^2(\Omega)$ with no critical points in $\Omega \setminus \omega'$ such that $(\nabla \beta(x) \cdot n(x)) \leq 0$, $\forall x \in \Omega$, where $n(x)$ is the vector field of external normals to $\partial \Omega$.

Since $\beta(x) \in \Omega \setminus \omega'$ does not have any critical points, we obtain $\min_{x \in \Omega \setminus \omega'} (\nabla \beta(x)) > 0$. Besides, assume that $\beta(x) \geq \ln 3$, $\min_{x \in \Omega} \beta(x) > \frac{3}{4} \max_{x \in \Omega} \beta(x)$.

Let $\gamma(t) \in C^\infty(0, T)$ be a function that meets the conditions

$$0 < \gamma(t) \leq 1, \quad \gamma(t) = \begin{cases} t, & t \in (0, T_0), \\ T - t, & t \in (T - T_0, T), \end{cases} \quad T_0 = \min \left\{ \frac{T}{3}, \frac{1}{2} \right\}.$$ Introduce the functions

$$\varphi(t, x) = \frac{e^{\lambda \beta(x)}}{\gamma(t)}, \quad \alpha_{\lambda}(t, x) = \frac{e^{\frac{4}{3} \| \beta \| C(\Omega)}}{\gamma(t)} - e^{\lambda \beta(x)}, \quad \eta_{\lambda}(t, x) = \frac{e^{\frac{4}{3} \| \beta \| C(\Omega)}}{(T - t)} - e^{\lambda \beta(x)}.$$

2. **Adjoint System**

Write a system of equations adjoint to continuity and momentum equations in solid mechanics neglecting the thermal processes [13], where density changes in the motion equation and those set at $t = 0$ are small. In this case, the density in the motion equation is replaced by the mean value of $\bar{\rho}$, which might be assumed to equal one without loss of generality.

Let $\bar{u}, \bar{\rho}$ denote the vector of velocity and the function of density for the solid. There occurs the expansion $\bar{u} = \nabla \bar{\phi} + \text{Rot } \psi$.

**Adjoint State of the Current Function:**

$$\partial_t \Delta \xi(t, x) + \Delta^2 \xi + B_1^* (\hat{\psi}, \xi) + B_1^* (\xi, \hat{\psi}) + \Delta (\nabla \hat{\phi} \cdot \nabla \xi) + \Delta (\bar{u} \cdot \text{Rot } \theta) - \text{rot} \ (\bar{\rho} \nabla \tau) - \text{div} \ (\Delta \hat{\phi} \nabla \theta) = - \frac{e^{2\eta_{\lambda}}}{(T - t)^6} w, \quad \xi|_{\Sigma} = \Delta \xi|_{\Sigma} = 0,$$ (1)

where

$$B_1^* (h, \hat{\psi}) = \partial_{x_1} (\Delta \hat{\phi} \partial_{x_2} h) - \partial_{x_2} (\Delta \hat{\phi} \partial_{x_1} h), \quad B_2^* (\hat{\psi}, h) = \Delta (\partial_{x_1} h \partial_{x_2} \hat{\psi} - \partial_{x_2} h \partial_{x_1} \hat{\psi}).$$ (2)

**Adjoint State of the Potential Velocity Function:**

$$\partial_t z + \Delta z - \text{div} \ (\bar{\rho} \nabla \tau) + \text{div} \ (\Delta \hat{\phi} \nabla \xi) + \text{div} \ (\bar{u} \cdot \nabla) - \text{rot} \ (\Delta \hat{\phi} \nabla \theta) = - \frac{e^{2\eta_{\lambda}}}{(T - t)^4} \tau,$$

$$(z, 1) = 0, \quad (\nabla \cdot z) |_{\Sigma} = 0.$$ (3)

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Adjoint State of Density:

\[ \partial_t \tau_1 + (\hat{u} \cdot \nabla \tau_1) - k \tau_1 \text{div} \hat{u} = \gamma z, \quad \tau|_{t=T} = 0; \]  

(4)

where

\[ \tau_1 = \hat{\rho}^{1-\gamma} \tau, \quad k = (\gamma - 1)^{-1}, \quad \hat{u} = \nabla \hat{\varphi} + \text{Rot} \hat{\psi}. \]

The function \( \theta(t, x) \), which is part of the equation (1), (2), satisfies the following boundary problem:

\[ \Delta \theta = z, \quad (\theta, 1) = 0, \quad (\nabla \theta \cdot n)|_{\Sigma} = 0. \]  

(5)

This research has produced the following theorem.

**Theorem 1.** Let \( (\hat{\psi}, \hat{\varphi}, \hat{\rho}) \in W^{1,2}(Q) \times W^{1,2}(Q) \times W^{1,2}(Q) \) such that \( \hat{\rho} \in L^\infty(0,T;W^1_\infty(\Omega)), \ (\hat{\psi}, \hat{\varphi}) \in L^\infty(0,T;W^3_\infty(\Omega)) \times L^\infty(0,T;W^3_\infty(\Omega)). \) Then there is the \( \lambda > 0 \) such that at any \( \lambda > \lambda \) for functions \( \xi, z, \tau, w \in L^2(Q,e^{\eta_0}/(T-t)^2), r \in L^2(Q,e^{\eta_0}/(T-t)^2), \) Carleman evaluation takes place:

\[
\int_Q \left( (T-t)^7 |\partial_t \Delta \xi|^2 + \sum_{|\alpha| \leq 2} (T-t)^{3+2|\alpha|} |D^2_{x} \xi|^2 \right) e^{-2\eta_\lambda} \, dxdt + \\
\int_Q \left( (T-t)^7 |\partial_t|^2 + \sum_{|\alpha| \leq 2} (T-t)^{3+2|\alpha|} |D^2_{x} z|^2 \right) e^{-2\eta_\lambda} \, dxdt + \\
\int_Q \left( (T-t)^6 |\partial_t \tau|^2 + \sum_{|\alpha| \leq 2} (T-t)^{3+2|\alpha|} |D^2_{x} \tau|^2 \right) e^{-2\eta_\lambda} \, dxdt \leq \ \\
c \int_Q (T-t)^{-6} |w|^2 e^{2\eta_\lambda} \, dxdt + c \int_Q (T-t)^{-4} |r|^2 e^{2\eta_\lambda} \, dxdt + \\
+ C \int_{Q^\omega} (T-t)^{-1} |\xi|^2 e^{-(18/10)\eta_\lambda} \, dxdt + c \int_{Q^\omega} |z|^2 e^{-(18/10)\eta_\lambda} \, dxdt. \]  

(6)

Proof Procedure. Consider a parabolic equation with a Dirichlet boundary condition

\[ \partial_t \Delta \xi(t, x) + \Delta^2 \xi = f_\xi - B^*_2(\hat{\psi}, \xi) - B^*_4(\xi, \hat{\psi}), \quad \xi|_{\Sigma} = \Delta \xi|_{\Sigma} = 0, \]  

(7)

where \( \hat{\psi} \in W^{1,2}(Q), f_\xi \) are the defined functions, and \( B^*_2, B^*_4 \) are defined by the equations (2). The book [3, Theorem 7.3, p. 334] proves the following theorem

**Theorem 2.** [3] There exists \( \lambda > 0 \), such that at any \( \lambda > \lambda \) for functions \( \xi, f_\xi \in L^2(Q) \) satisfying (7), the following estimate holds true:

\[
\int_Q \left( \varphi^{-7} |\partial_t \Delta \xi|^2 + \varphi^{-5} |\nabla \Delta \xi|^2 + \varphi^{-3} |\Delta \xi|^2 + \sum_{k=0}^4 \sum_{|\alpha|=k} |D^2_{x} \xi|^2 \varphi^{-2k} \right) e^{-2\alpha \lambda} \, dxdt \leq \\
\leq c e^{-2\alpha \lambda} |f_\xi|^2 \varphi^{-6} \, dxdt + c \int_{Q^\omega} (1 + \lambda^{13}) e^{-2\alpha \lambda} |\xi|^2 \varphi \, dxdt, \]

where \( Q^\omega = (0,T) \times \omega, c > 0 \) is a constant independent of \( f_\xi, \xi \) and \( \lambda \).

Consider a parabolic equation with a Neumann boundary condition

\[ \partial_t z(t, x) + \Delta z = f_z, \quad (t, x) \in Q, \quad (\nabla z \cdot n) = 0, \quad x \in \Sigma, \]  

(8)
where \( f_z \) is the set function.

**Theorem 3.** [5] There exists \( \tilde{\lambda} > 0 \) such that at any \( \lambda > \tilde{\lambda} \) and \( s \geq -3 \), for functions \( z, f_z \in L^2(Q) \) satisfying (8) the following estimate holds true:

\[
\int_Q \varphi^{2s-1}(\lambda^{-1}|\partial_t z|^2 + \lambda^{-1} \sum_{i,j=1}^{2} |\partial^2_{x_i,x_j} z|^2 + \lambda \varphi^2 |\nabla z|^2 + \lambda^4 \varphi^4 |z|^2) e^{-2\alpha \lambda} \, dx \, dt \leq c \int_Q \varphi^{2s}|f_z|^2 e^{-2\alpha \lambda} \, dx \, dt + c \int_Q \lambda^4 \varphi^{2s+3}|z|^2 e^{-2\alpha \lambda} \, dx \, dt,
\]

where \( Q^{w'} = (0, T) \times \omega' \), \( c > 0 \) is a constant independent of \( f_z, z \) and \( \lambda \).

For the quasi-stationary elliptic problem

\[
\Delta \theta = f_\theta(t,x), \quad (\nabla \theta \cdot n) = 0, \quad x \in \Sigma,
\]

the following estimate is shown in the same source

\[
\int_Q \varphi^{2s-1}(\lambda^{-1} \sum_{i,j=1}^{2} |\partial^2_{x_i,x_j} \theta|^2 + \lambda \varphi^2 |\nabla \theta|^2 + \lambda^4 \varphi^4 |\theta|^2) e^{-2\alpha \lambda} \, dx \, dt \leq c \int_Q \varphi^{2s}|f_\theta|^2 e^{-2\alpha \lambda} \, dx \, dt + c \int_Q \lambda^4 \varphi^{2s+3}|\theta|^2 e^{-2\alpha \lambda} \, dx \, dt,
\]

where \( Q^{w'} = (0, T) \times \omega' \), \( c > 0 \) is a constant independent of \( \theta, f_\theta \) and \( \lambda \).

In the domain \( Q = (0, T) \times \Omega \), consider a problem for a hyperbolic-parabolic system with Cauchy data for advection equation:

\[
\partial_t \tau + (u \cdot \nabla \tau) - k \text{div} u = f, \quad \tau|_{t=T} = 0, \quad (10)
\]

\[
\partial_t f + \Delta f = g + \Delta \tau, \quad (\nabla f \cdot n)|_{\Sigma} = 0; \quad \int_{\Omega} f \, dx = 0. \quad (11)
\]

Here \( k \geq 0 = \text{const} \), \( u = (u_1, u_2) \), \( g \) are set, \( n = (n_1, n_2) \) is the vector field of external normals to \( \partial \Omega \).

**Theorem 4.** [4] Let \( s \geq -5/2, u \in W^{1,2}(Q) \cap L^\infty(0, T; W^2_\infty(\Omega)), (u \cdot n)|_{\Sigma} = 0, g \in L^2(Q) \).

There exist \( \tilde{\lambda} > 0 \) such that at any \( \lambda > \tilde{\lambda} \) for the functions \( \tau, f, g \) satisfying (10), (11), the following estimate holds true

\[
\int_Q \varphi^{2s-2} \left( \lambda^{-1} |\partial_t f|^2 + |\partial^2_{x_i,x_j} f|^2 + \lambda \varphi^2 |\nabla f|^2 + \lambda^4 \varphi^4 |f|^2 \right) e^{-2\alpha \lambda} \, dx \, dt +
\]

\[
+ \int_Q \varphi^{2s} \left( (\lambda \varphi)^{-2} |\partial^2_{x_i,x_j} \tau|^2 + \lambda^{-1} |\partial_t \tau|^2 + \lambda^{-1} |\Delta \tau|^2 + |\nabla \tau|^2 + \lambda^3 \varphi^3 |\tau|^2 \right) e^{-2\alpha \lambda} \, dx \, dt \leq c \int_Q \varphi^{2s}|g|^2 e^{-2\alpha \lambda} \, dx \, dt + c \int_{Q^{w'}} \lambda^4 \varphi^{2s+2}|f|^2 e^{-2\alpha \lambda} \, dx \, dt, \quad i, j = 1, 2,
\]

where, \( Q^{w'} = (0, T) \times \omega', \omega' \subset \omega \), the constant \( c \) do not depend on \( \lambda, f, \tau \) or \( g \).
Using the theorems 2 to 4, generate Carleman estimates for the solutions of the system (1)--(5). Denote $g_k = e^{2q_k} w_j (T - t)^{6}$, $g_z = e^{2q_z} r_j (T - t)^{4}$. Compare (7),(8) with (1),(3) and choose $s = -5/2$ (9), find the estimate

$$
\int_Q \varphi^2 \left( |f|_x^2 + \varphi |f_z|^2 \right) e^{-2 \alpha} \ dx dt \leq c \int_Q \varphi^2 \left( |g_k|_x^2 + \varphi |g_z|^2 \right) e^{-2 \alpha} \ dx dt + cF_{\lambda}(\xi, z),
$$

(12)

where

$$
F_{\lambda}(\xi, z) \leq \int_Q \varphi^{-6} \left( \sum_{i,j=1}^2 |\partial_{x_i x_j} \xi|^2 + |\nabla \Delta \xi|^2 + \lambda^2 |\partial^2 \xi|^2 \right) e^{-2 \alpha} \ dx dt +
$$

$$
+ c \int_Q \varphi^{-6} \left( |\nabla z|^2 + \lambda^2 |\partial^2 z|^2 \right) e^{-2 \alpha} \ dx dt + c \int_Q \varphi^{-6} \left( |\nabla \tau_1|^2 + |\Delta \tau_1|^2 \right) e^{-2 \alpha} \ dx dt +
$$

$$
+ c \int_{Q'} \lambda^5 \varphi^{-2} |\theta|^2 e^{-2 \alpha} \ dx dt.
$$

(13)

Let $\alpha_{\lambda}(t) = \max_{x \in \Omega} \alpha_{\lambda}(t, x)$, $\tilde{\alpha}_{\lambda}(t) = \min_{x \in \Omega} \alpha_{\lambda}(t, x)$, then $\alpha_{\lambda}(t) < (10/9) \tilde{\alpha}_{\lambda}(t)$. Find the estimate

$$
\int_{Q'} \lambda^5 \varphi^{-2} |\theta|^2 e^{-2 \alpha} \ dx dt \leq \int_0^T \lambda^3 e^{-((18/10)\tilde{\alpha}_{\lambda}(t) \max_{x \in \Omega} |\theta|^2) \ dx dt \
$$

$$
\leq c \int_0^T \lambda^3 e^{-((18/10)\tilde{\alpha}_{\lambda}(t) \max_{x \in \Omega} |\theta|^2) \ dx dt \leq c \int_0^T \lambda^3 e^{-((18/10)\tilde{\alpha}_{\lambda}(t) \max_{x \in \Omega} |\theta|^2) \ dx dt =
$$

$$
= c \int_0^T \lambda^3 e^{-((18/10)\tilde{\alpha}_{\lambda}(t) \max_{x \in \Omega} |\theta|^2) \ dx dt =
$$

$$
= c \int_0^T \lambda^3 e^{-((18/10)\tilde{\alpha}_{\lambda}(t) \max_{x \in \Omega} |\theta|^2) \ dx dt =
$$

(14)

Here the constant $c$ depends only on $mes \omega'$, $\omega' \subset \subset \omega \subset \subset \Omega$.

Substitute (14) in the right side of (13) to obtain

$$
F_{\lambda}(\xi, z) \leq \int_Q \varphi^{-6} \left( \sum_{i,j=1}^2 |\partial_{x_i x_j} \xi|^2 + |\nabla \Delta \xi|^2 + \lambda^2 |\partial^2 \xi|^2 \right) e^{-2 \alpha} \ dx dt +
$$

$$
+ c \int_Q \varphi^{-6} \left( |\nabla z|^2 + \lambda^2 |\partial^2 z|^2 \right) e^{-2 \alpha} \ dx dt + c \int_Q \varphi^{-6} \left( |\nabla \tau_1|^2 + |\Delta \tau_1|^2 \right) e^{-2 \alpha} \ dx dt +
$$

$$
+ c \int_{Q'} \lambda^5 e^{-((18/10)\tilde{\alpha}_{\lambda}) \max_{x \in \Omega} |\theta|^2) \ dx dt.
$$

(15)

Denote

$$
J_{\lambda}(\xi) = \int_Q \left( \varphi^{-7} |\partial \Delta \xi|^2 + \varphi^{-5} |\nabla \Delta \xi|^2 + \varphi^{-3} |\Delta \xi|^2 + \sum_{k=0}^4 \sum_{|\alpha| = k} |D_{\alpha}^k \xi|^2 \varphi^{-2k} \right) e^{-2 \alpha} \ dx dt,
$$

$$
J_{\lambda}(z, \tau) = \int_Q \varphi^{-5} \left( \lambda^{-1} |\partial \tau z|^2 + \varphi^2 |\partial^2 z|^2 + \lambda \varphi^2 |\nabla \tau z|^2 + \lambda^4 \varphi^2 |\nabla \tau z|^2 \right) e^{-2 \alpha} \ dx dt +
$$

(16)

Additionally, 
\[ \xi, \tau \in \Omega \]

necessary, increase \( \lambda \), to find the estimate

\[ J_{\lambda}(\xi) + J_{\lambda}(z, \tau) \leq c \int_{Q} \varphi^{-6} \left( |g_{\xi}|^2 + \varphi |g_{z}|^2 \right) e^{-2\alpha \lambda} \, dxdt + c \int_{Q'} \lambda^1 \varphi^{-6} |z|^2 e^{-2\alpha \lambda} \, dxdt + c \int_{Q''} \lambda^1 \varphi^{-6} |z|^2 e^{-2\alpha \lambda} \, dxdt. \]  

(17)

For the system (1)–(5), set the initial condition at \( t_0 \in (0, T) \):

\[ \xi|_{t=t_0} = \xi_0, \quad z|_{t=t_0} = z_0, \quad \tau_1|_{t=t_0} = \tau_0. \]  

(18)

Consider the problem (1)–(5), (18) in the cylinder \((0, t_0) \times \Omega\).

**Lemma 2.** Let the conditions of Theorem 1, \( \xi_0 \in H^3(\Omega), \quad z_0 \in H^1(\Omega) \), be fulfilled. Additionally, \( \xi_0|_{\partial \Omega} = \Delta \xi_0|_{\partial \Omega} = 0, \quad (\nabla \xi_0 \cdot n)|_{\partial \Omega} = 0 \). Then for any \( g_{\xi} \in L^2(Q), \quad g_z \in L^2(Q) \), the problem (1)–(5), (20) has a unique solution \( \xi \in W^{1,2}(0, t_0) \times \Omega), \quad z \in W^{1,2}(0, t_0) \times \Omega), \quad \tau \in W^{1,2}(0, t_0) \times \Omega), \quad \theta \in L^2((0, t_0); H^2(\Omega)). \) The estimate holds true:

\[ \| \xi \|^2_{W^{1,2}(0, t_0) \times \Omega}) + \| z \|^2_{W^{1,2}(0, t_0) \times \Omega}) + \| \tau \|^2_{H^1(\Omega)} + \| \theta \|^2_{H^1(\Omega)} \leq c \| \xi_0 \|^2_{H^3(\Omega)} + \| z_0 \|^2_{H^1(\Omega)} + \| \tau_0 \|^2_{H^1(\Omega)} + \| g_{\xi} \|^2_{L^2((0, t_0) \times \Omega)} + \| g_z \|^2_{L^2((0, t_0) \times \Omega)}. \]  

(19)

where the constant \( c \) is independent of \( \xi_0, \quad z_0, \quad \tau_0, \quad g_{\xi}, \quad g_z \).

Rewrite (16) as

\[ J_{\lambda}(\xi) + J_{\lambda}(z, \tau_1) = \int_{Q} (R_{1}(\xi) + R_{2}(z, \tau_1)) e^{-2\alpha \lambda} \, dxdt. \]  

(20)

(19) means that given the known results on the solvability of boundary elliptic problems, the inequality holds: \( \| \xi_0 \|_{H^3(\Omega)} \leq c \| \Delta \xi_0 \|_{H^1(\Omega)} \). From the trace theorem and (20), obtain

\[ \| \xi(T - T_0) \|^2_{H^3(\Omega)} + \| z(T - T_0) \|^2_{H^1(\Omega)} + \| \tau_1(T - T_0) \|^2_{H^1(\Omega)} \leq c \| \xi_0 \|^2_{W^{1,2}(0, T_0 - \epsilon, T_0) \times \Omega}) + c \| z \|^2_{W^{1,2}(0, T_0 - \epsilon, T_0) \times \Omega}) + c \| \tau_1 \|^2_{W^{1,2}(0, T_0 - \epsilon, T_0) \times \Omega}) \leq c \int_{0}^{T - T_0} \int_{\Omega} (R_{1}(\xi, z) + R_{2}(\tau)) e^{-2\alpha \lambda} \, dxdt. \]  

(21)

Consider the expression:

\[ I_{\lambda}(\xi, z, \tau_1) = \int_{Q} \left( (T - t)^{7} |\partial_{\xi} \Delta \xi|^2 + \sum_{|\alpha| \leq 2} (T - t)^{3+2|\alpha|} |D_{x}^{\alpha} \xi|^2 + \sum_{|\alpha| \leq 4} (T - t)^{2|\alpha|} |D_{x}^{\alpha} \xi|^2 \right) e^{-2\alpha \lambda} \, dxdt + \

\int_{Q} \left( (T - t)^{7} |\partial_{\xi} z|^2 + \sum_{|\alpha| \leq 2} (T - t)^{3+2|\alpha|} |D_{x}^{\alpha} z|^2 \right) e^{-2\alpha \lambda} \, dxdt. \]  

(6)
\[ + \int_{Q} \left( (T - t)^{6} |\partial_t \tau_1|^2 + \sum_{|\alpha| \leq 2} (T - t)^{3+2|\alpha|} |D_x^\alpha \tau_1|^2 \right) e^{-2\eta \lambda} \, dx \, dt. \]  

(22)

Denote the integrand expression (22) as \( \varrho_1(\xi) e^{-2\eta \lambda} + \varrho_2(z, \tau_1) e^{-2\eta \lambda} \), with the estimates (20), (21) meaning that

\[ I_{\lambda}(\xi, z, \tau_1) \leq c \int_{T-T_0}^{T} \left( \varrho_1(\xi) + \varrho_2(z, \tau_1) \right) e^{-2\eta \lambda} \, dx \, dt + 
\]
\[ + c \|\xi\|_{W^{1,2}}^2((0,T-T_0) \times \Omega) + c \|z\|_{W^{1,2}}^2((0,T-T_0) \times \Omega) + c \|\tau_1\|_{W^{1,2}}^2((0,T-T_0) \times \Omega) \leq 
\]
\[ \leq c \int_{Q} \left( R_1(\xi) + R_2(z, \tau_1) \right) e^{-2\alpha \lambda} \, dx \, dt + 
\]
\[ + c \|\varphi^{-3} g_x e^{-\eta \lambda}\|_{L^2((0,T-T_0) \times \Omega)}^{2} + c \|\varphi^{-2} g_x e^{-\eta \lambda}\|_{L^2((0,T-T_0) \times \Omega)}^{2}. \]  

(23)

Apply the estimate (17) to the right side of (23) in the context of (20), obtain (6).

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