Conformal infinitesimal bendings of Euclidean hypersurfaces

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Abstract

In the realm of conformal geometry, we give a classification of the Euclidean hypersurfaces that admit a non-trivial conformal infinitesimal bending. In the restricted case of conformal bendings, such a classification was obtained by E. Cartan in 1917. The case of infinitesimal isometric bendings was done by U. Sbrana in 1908. In particular, we show that the class of hypersurfaces that allow a conformal infinitesimal bending is much larger than the one considered by Cartan.

Classifying Euclidean hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, that admit isometric or, more generally, conformal bendings is a classical subject that was considered, among others, by Sbrana [11] and Cartan [1, 2]. The case of bendings that are only infinitesimally isometric was studied by Sbrana [12]. Modern presentations of their results, as well as a large amount of additional information, can be found in [4], [5], [6], [7], [8], [9] and [10].

Cartan [2] gave a local classification of the hypersurfaces $M^n$ in $\mathbb{R}^{n+1}$ of dimension $n \geq 5$ that admit non-trivial conformal bendings in a long and rather difficult paper. They are conformally surface-like, conformally flat, conformally ruled or certain two-parameter congruences of hyperspheres. That $f: M^n \to \mathbb{R}^{n+1}$ is conformally surface-like means that it is conformally congruent to either a cylinder or a rotation hypersurface over a surface in $\mathbb{R}^3$ or a cylinder over a three-dimensional hypersurface of $\mathbb{R}^4$ that is a cone over a surface in the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. The hypersurface is conformally ruled if $M^n$ carries an integrable ($n-1$)-dimensional distribution such that the restriction of $f$ to any leaf is an umbilical submanifold of $\mathbb{R}^{n+1}$. Cartan in [2] also proved that conformally flat hypersurfaces are characterized by possessing at any point a principal curvature of multiplicity at least $n-1$, and that they are
highly conformally deformable. Hence, the really interesting class in Cartan’s classification and target of much of the work is the latter one.

A conformal bending of an Euclidean hypersurface $f: M^n \rightarrow \mathbb{R}^{n+1}$ is a smooth variation $F: I \times M^n \rightarrow \mathbb{R}^{n+1}$, with $0 \in I \subset \mathbb{R}$ an open interval and $F(0, \cdot) = f$, such that $f_t = F(t, \cdot)$ for any $t \in I$ is a conformal immersion with respect to the metric induced by $f$. The bending is said to be trivial if each $f_t$ is conformally congruent to $f$, that is, congruent by a conformal transformation of $\mathbb{R}^{n+1}$; recall that the latter are characterized by Liouville’s theorem.

If $F$ is a conformal bending, there is a positive function $\gamma \in C^\infty(I \times M)$ with $\gamma(0, x) = 1$ such that

$$\gamma(t, x) \langle f_t^* X, f_t^* Y \rangle = \langle X, Y \rangle$$

for any tangent vector fields $X, Y \in \mathfrak{X}(M)$. The derivative of the above equation with respect to $t$ computed at $t = 0$ yields that the variational vector field $\mathcal{T} = F_* \partial/\partial t|_{t=0}$ of $F$ has to satisfy the condition

$$\langle \nabla_X \mathcal{T}, f_0^* Y \rangle + \langle f_0^* X, \nabla_Y \mathcal{T} \rangle = 2\rho(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$, where $\rho \in C^\infty(M)$ given by $\rho(x) = -(1/2)\partial \gamma/\partial t(0, x)$ is called the conformal factor of $\mathcal{T}$. Here and elsewhere we use the same notation for the inner products in $M^n$ and $\mathbb{R}^{n+1}$. We denote by $\nabla$ and $\nabla$, respectively, the Levi-Civita connections associated to the induced metric in $M^n$ and the flat metric of the ambient space.

A smooth variation $F: I \times M^n \rightarrow \mathbb{R}^{n+1}$ is called an infinitesimal conformal variation if there is $\gamma \in C^\infty(I \times M)$ satisfying $\gamma(0, x) = 1$ and

$$\frac{\partial}{\partial t}|_{t=0} (\gamma(t, x) \langle f_{t*} X, f_{t*} Y \rangle) = 0$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. This concept is just the infinitesimal analogue to a conformal bending. It is already known from classical differential geometry that the appropriate approach to study infinitesimal variations is to deal with the variational vector field. This leads to the following concept to which this paper is devoted.

A conformal infinitesimal bending with conformal factor $\rho \in C^\infty(M)$ of an isometric immersion $f: M^n \rightarrow \mathbb{R}^{n+1}$ of a Riemannian manifold $M^n$ into
Euclidean space is a smooth section \( T \in \Gamma(f^*\mathbb{R}^{n+1}) \) that satisfies condition (1). Notice that the associated variation \( F: \mathbb{R} \times M^n \to \mathbb{R}^{n+1} \) given by

\[
F(t, x) = f(x) + t\mathcal{T}(x)
\]

is an infinitesimal conformal variation with variational vector field \( \mathcal{T} \) since (2) is satisfied for \( \gamma(t, x) = e^{-2t\rho(x)} \). The bending is called trivial if it is locally the restriction of a conformal Killing vector field of the Euclidean ambient space to the hypersurface. It is well-known that any conformal Killing field on an open connected subset of \( \mathbb{R}^n, n \geq 3 \), has the form

\[
X(x) = (\langle x, v \rangle + \lambda)x - (1/2)\|x\|^2v + Cx + w
\]

where \( \lambda \in \mathbb{R}, v, w \in \mathbb{R}^n, C \in \text{End}(\mathbb{R}^n) \) is skew-symmetric and the conformal factor is \( \rho = \langle x, v \rangle + \lambda \); cf. [13] for details.

An isometric infinitesimal bending is a conformal infinitesimal bending whose conformal factor is \( \rho = 0 \). It is called trivial if it is locally the restriction to the hypersurface of a Killing vector field of \( \mathbb{R}^{n+1} \). Let \( \mathcal{T}_1 \) be a conformal infinitesimal bending of \( f \) with conformal factor \( \rho \) and let \( \mathcal{T}_0 \) be an isometric infinitesimal bending of \( f \). Then \( \mathcal{T}_2 = \mathcal{T}_1 + \mathcal{T}_0 \) satisfies (1), and thus is also a conformal infinitesimal bending of \( f \) with conformal factor \( \rho \). In this paper, we always identify two conformal infinitesimal bendings of \( f \) if they have the same conformal factor and differ by a trivial isometric infinitesimal bending. We also identify a conformal infinitesimal bending \( \mathcal{T} \) with any of its constant multiples \( c\mathcal{T}, 0 \neq c \in \mathbb{R} \).

The purpose of this paper is to parametrically classify the hypersurfaces in Euclidean space \( f: M^n \to \mathbb{R}^{n+1}, n \geq 5 \), that are conformally infinitesimally bendable, that is, they admit a non-trivial conformal infinitesimal bending. Such result belongs to the realm of conformal geometry since being a conformal infinitesimal bending is invariant by conformal transformations of the ambient Euclidean space. We recall that in order to be conformally infinitesimally bendable, the hypersurface \( f \) must possess a principal curvature of multiplicity at least \( n - 2 \); see [5].

We first deal with the interesting class that includes, but is much larger than, the interesting class in Cartan’s classification of hypersurfaces that admit conformal bendings; see Remark [14].

**Theorem 1.** Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 5 \), be a conformally infinitesimally bendable hypersurface that is neither conformally surface-like nor conformally flat nor conformally ruled on any open subset of \( M^n \). Then, on each
connected component of an open dense subset of $M^n$, the hypersurface is parametrized in terms of the conformal Gauss parametrization by a special hyperbolic or a special elliptic pair.

Conversely, any hypersurface that is given in terms of the conformal Gauss parametrization by a special hyperbolic or special elliptic pair admits locally a conformal infinitesimal bending, unique up to trivial conformal infinitesimal bendings.

Special hyperbolic and elliptic pairs are the object of next section. As for the conformal Gauss parametrization, it goes as follows: Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, be a hypersurface with Gauss map $N: M^n \to S^n(1) \subset \mathbb{R}^{n+1}$ that possesses a principal curvature $\lambda > 0$ of multiplicity $n-2$. It is well-known that the corresponding eigenspaces form an integral distribution and that $\lambda$ is constant along the leaves. Then, the focal map $f + rN: M^n \to \mathbb{R}^{n+1}$, where $r = 1/\lambda$, induces an isometric immersion $h: L^2 \to \mathbb{R}^{n+1}$. Here $L^2$ is the quotient space of leaves and $r \in C^\infty(L^2)$ satisfies $\|\nabla^h r\| < 1$.

Then $f$ can be locally parametrized along the unit normal bundle $N_1L$ of $h$ by
\[ X(\xi) = h - r \left( h_* \nabla^h r + \sqrt{1 - \|\nabla^h r\|^2} \xi \right). \]

Conversely, given a surface $h: L^2 \to \mathbb{R}^{n+1}$ and $r \in C^\infty(L^2)$ positive whose gradient satisfies $\|\nabla^h r\| < 1$, then on the open subset of regular points, the parametrized hypersurface determined as above by the pair $(h, r)$ has, with respect to the Gauss map $N = h_* \nabla^h r + \sqrt{1 - \|\nabla^h r\|^2} \xi$, the principal curvature $\lambda = 1/r$ of multiplicity $n-2$.

We conclude with the case of ruled hypersurfaces, a class for which the bendings being infinitesimal or not makes no difference.

**Theorem 2.** Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a conformally ruled hypersurface that is neither conformally surface-like nor conformally flat on any open subset of $M^n$. Then $f$ admits on each connected component of an open dense subset of $M^n$ a family of conformal infinitesimal bendings that are in one-to-one correspondence with the set of smooth functions on an interval. Moreover, any such bending is the variational vector field of a conformal bending.
1 Special hyperbolic and elliptic pairs

The purpose of this section is to introduce the notions of special hyperbolic and special elliptic pairs and show how they can be parametrically generated in terms of a set of solutions of a second order hyperbolic or elliptic PDE.

Let \( g: L^2 \rightarrow S_1^{n+2} \) be a surface in the unit Lorentzian sphere (de Sitter space) considered as a hypersurface of the Lorentzian space \( \mathbb{L}^{n+3} \), that is, \( S_1^{n+2} = \{ x \in \mathbb{L}^{n+3}: \langle x, x \rangle = 1 \} \).

We fix a pseudo-orthonormal basis \( e_1, \ldots, e_{n+3} \) of \( \mathbb{L}^{n+3} \), that is, \( \| e_1 \| = 0 = \| e_{n+3} \|, \langle e_1, e_{n+3} \rangle = -1/2 \) and \( \langle e_i, e_j \rangle = \delta_{ij} \) if \( i \neq 1, n+3 \), and set \( g = (g_1, g_2, \ldots, g_{n+3}): L^2 \rightarrow S_1^{n+2} \subset \mathbb{L}^{n+3} \) in terms of this basis. We assume that \( g_1 \neq 0 \) everywhere, and let the map \( h: L^2 \rightarrow \mathbb{R}^{n+1} \) and the function \( r \in C^\infty(L) \) be given by

\[
    h = r(g_2, \ldots, g_{n+2}) \quad \text{and} \quad r = 1/g_1. \tag{3}
\]

Notice that \( g \) can be recovered from the pair \((h, r)\) by taking

\[
    g = r^{-1}(1, h, \| h \|^2 - r^2). \tag{4}
\]

**Proposition 3.** We have that \( L^2 \) is a Riemannian manifold with the metric induced by \( g \) if and only if \( h \) is an immersion and the gradient of \( r \) in the metric induced by \( h \) satisfies \( \| \nabla^h r \| < 1 \).

**Proof:** See Lemma 12 of [7]. \( \blacksquare \)

A Riemannian surface \( g: L^2 \rightarrow S_1^m, m \geq 4 \), is said to be a **hyperbolic surface** (respectively, **elliptic surface**) if there is a tensor \( I \neq J \in \text{End}(TL) \) that verifies \( J^2 = I \) (respectively, \( J^2 = -I \)) such that the second fundamental form \( \alpha^g: TL \times TL \rightarrow N_g L \) of \( g \) satisfies

\[
    \alpha^g(JX, Y) = \alpha^g(X, JY)
\]

for all \( X, Y \in \mathfrak{X}(L) \). It is easily seen that \( J \) is unique up to sign.

A local system of coordinates \((u, v)\) on \( L^2 \) is said to be **real conjugate** for the surface \( g: L^2 \rightarrow S_1^m \) if the condition

\[
    \alpha^g(\partial_u, \partial_v) = 0
\]
holds for the coordinate vector fields $\partial_u = \partial/\partial u$ and $\partial_v = \partial/\partial v$. The coordinate system is said to be \textit{complex conjugate} for $g$ if

$$\alpha^g(\partial_z, \partial_z) = 0$$

where $z = u + iv$ and $\partial_z = (1/2)(\partial_u - i\partial_v)$, that is, if

$$\alpha^g(\partial_u, \partial_u) + \alpha^g(\partial_v, \partial_v) = 0.$$

For a system of real conjugate coordinates let $\Gamma^1, \Gamma^2$ be the Christoffel symbols defined by

$$\nabla_{\partial_u} \partial_v = \Gamma^1 \partial_u + \Gamma^2 \partial_v. \tag{5}$$

For a system of complex conjugate coordinates let $\Gamma$ be defined by

$$\nabla_{\partial_z} \partial_z = \Gamma \partial_z + \bar{\Gamma} \partial_{\bar{z}}, \tag{6}$$

where $\nabla$ also denotes the $\mathbb{C}$-bilinear extension of $\nabla$.

An elementary argument gives the following result.

\textbf{Proposition 4.} If $g: L^2 \to \mathbb{S}^m_1$ is a hyperbolic (respectively, elliptic) surface, then there exist local real conjugate (respectively, complex conjugate) coordinates on $L^2$ for $g$. Conversely, if there exist real conjugate (respectively, complex conjugate) coordinates on $L^2$, then $g: L^2 \to \mathbb{S}^m_1$ is hyperbolic (respectively, elliptic).

\textit{Proof:} See Proposition 11.10 of [7].

We call a hyperbolic surface $g: L^2 \to \mathbb{S}^m_1$ endowed with a system of real conjugate coordinates as in Proposition 4 a \textit{special hyperbolic surface} if the Christoffel symbols $\Gamma^1, \Gamma^2$ given by (5) satisfy the condition

$$\Gamma^1_u = \Gamma^2_v. \tag{7}$$

\textbf{Proposition 5.} Let $g: L^2 \to \mathbb{S}^m_1$ be a simply connected special hyperbolic surface and let $\mu \in C^\infty(L)$ be the unique (up to a constant factor) positive solution of

$$d\mu + 2\mu \omega = 0 \tag{8}$$

where $\omega = \Gamma^2 du + \Gamma^1 dv$. Then $\varphi \in C^\infty(U)$ is a solution of the equation

$$\varphi_{uv} - \Gamma^1 \varphi_u - \Gamma^2 \varphi_v + F \varphi = 0 \text{ where } F = \langle \partial_u, \partial_v \rangle \tag{9}$$
if and only if \( \psi = \sqrt{\mu} \varphi \) satisfies

\[
\psi_{uv} + M \psi = 0
\]  
(10)

where

\[
M = F - \frac{\mu_{uv}}{2\mu} + \frac{\mu_u \mu_v}{4\mu^2}.
\]  
(11)

In particular, the map \( k = \sqrt{\mu} h: L^2 \to L^{m+1} \), where \( h \) is the composition \( h = i \circ g \) of \( g \) with the inclusion \( i: \mathbb{S}_1^m \to \mathbb{L}^{m+1} \), satisfies

\[
k_{uv} + Mk = 0.
\]  
(12)

Conversely, for a system of coordinates \((u,v)\) on an open subset \( U \subset \mathbb{R}^2 \) let \( \{k_1, \ldots, k_{m+1}\} \) be a set of solutions of (10) for \( M \in \mathcal{C}^\infty(U) \). Assume that the map \( k = (k_1, \ldots, k_{m+1}): U \to L^{m+1} \) satisfies \( \mu = \|k\|^2 > 0 \) and that the map \( h = (1/\sqrt{\mu}) k: U \to L^{m+1} \) is an immersed surface with induced Riemannian metric. Then \( g: U \to \mathbb{S}_1^m \) defined by \( h = i \circ g \) is a special hyperbolic surface.

Proof: Notice that (7) is the integrability condition of (8). Since \( \mu \in \mathcal{C}^\infty(L) \) is a solution of (8), it satisfies

\[
\Gamma^1 = -\frac{\mu_v}{2\mu} \quad \text{and} \quad \Gamma^2 = -\frac{\mu_u}{2\mu}.
\]

Hence (9) becomes

\[
\varphi_{uv} + \frac{\mu_v}{2\mu} \varphi_u + \frac{\mu_u}{2\mu} \varphi_v + F \varphi = 0.
\]  
(13)

It follows easily that (13) takes the form (10) for \( \psi = \sqrt{\mu} \varphi \), where \( M \) is given by (11).

We prove the converse. It is easily seen that \( h = (1/\sqrt{\mu}) k: U \to L^{m+1} \) satisfies

\[
h_{uv} + \frac{\mu_v}{2\mu} h_u + \frac{\mu_u}{2\mu} h_v + Fh = 0
\]  
(14)

where \( F = M + \frac{\mu_{uv}}{2\mu} - \frac{\mu_u \mu_v}{4\mu^2} \). If \( h \) is an immersed Riemannian surface and \( g: U \to \mathbb{S}_1^m \) is the surface defined by \( h = i \circ g \), then (14) implies that \((u,v)\) are real conjugate coordinates for \( g \) and that the Christoffel symbols of the metric induced by \( g \) are

\[
\Gamma^1 = -\frac{\mu_v}{2\mu} \quad \text{and} \quad \Gamma^2 = -\frac{\mu_u}{2\mu}.
\]
It follows that (7) is satisfied and that \( \mu \) is a positive solution of (8).

We call an elliptic surface \( g: L^2 \to S^m_1 \) endowed with a system of complex conjugate coordinates as in Proposition 4 a \textit{special elliptic surface} if the Christoffel symbol \( \Gamma \) given by (6) satisfies the condition

\[\Gamma_z = \bar{\Gamma}_{\bar{z}},\]  

that is, \( \Gamma_z \) is real-valued.

**Proposition 6.** Let \( g: L^2 \to S^m_1 \) be a simply connected special elliptic surface and let \( \mu \in C^\infty(L) \) be the unique (up to a constant factor) real-valued positive solution of

\[\mu \bar{\mu} + 2\mu \Gamma = 0.\]  

Then \( \varphi \in C^\infty(L) \) is a solution of

\[\varphi_{zz} - \Gamma \varphi_z - \bar{\Gamma} \bar{\varphi}_z + F \varphi = 0\]  

if and only if \( \psi = \sqrt{\mu} \varphi \) satisfies

\[\psi_{zz} + M \psi = 0,\]  

where

\[M = F - \frac{\mu \bar{\mu}}{2\mu} + \frac{\mu_z \mu_{\bar{z}}}{4\mu^2}.\]  

In particular, the map \( k = \sqrt{\mu} h: L^2 \to \mathbb{L}^{m+1} \), where \( h \) is the composition \( h = i \circ g \) of \( g \) with the inclusion \( i: S^m_1 \to \mathbb{L}^{m+1} \), satisfies

\[k_{zz} + Mk = 0.\]  

Conversely, for a system of coordinates \((u,v)\) on an open subset \( U \subset \mathbb{R}^2 \) let \( \{k_1, \ldots, k_{m+1}\} \) be a set of solutions of (18) for \( M \in C^\infty(U) \). Assume that the map \( k = (k_1, \ldots, k_{m+1}): U \to \mathbb{L}^{m+1} \) satisfies \( \mu = \|k\|^2 > 0 \) and that the map \( h = (1/\sqrt{\mu}) k: U \to \mathbb{L}^{m+1} \) is an immersed surface with induced Riemannian metric. Then \( g: U \to S^m_1 \) defined by \( h = i \circ g \) is a special elliptic surface.

**Proof:** Notice that (15) is the integrability condition of equation (16). Since \( \mu \in C^\infty(L) \) is a real-valued solution of (16) then \( \Gamma = -(1/2\mu)\mu_z \). Hence (17) becomes

\[\varphi_{zz} + \frac{\mu_z}{2\mu} \varphi_z + \frac{\mu_z}{2\mu} \varphi_z + F \varphi = 0.\]  

Conversely, for a system of coordinates \((u,v)\) on an open subset \( U \subset \mathbb{R}^2 \) let \( \{k_1, \ldots, k_{m+1}\} \) be a set of solutions of (18) for \( M \in C^\infty(U) \). Assume that the map \( k = (k_1, \ldots, k_{m+1}): U \to \mathbb{L}^{m+1} \) satisfies \( \mu = \|k\|^2 > 0 \) and that the map \( h = (1/\sqrt{\mu}) k: U \to \mathbb{L}^{m+1} \) is an immersed surface with induced Riemannian metric. Then \( g: U \to S^m_1 \) defined by \( h = i \circ g \) is a special elliptic surface.
It follows easily that (21) has the form (20) for \( k = \sqrt{\mu} \varphi \) where \( M \) is given by (19).

We prove the converse. It is easily seen that \( h = (1/\sqrt{\mu}) k: U \to \mathbb{L}^{m+1} \) satisfies

\[
h_{\bar{z}z} + \frac{\mu}{2\mu} h_{z\bar{z}} + \frac{\mu}{2\mu} h_z + F h = 0
\]

where \( F = M + \frac{\mu z}{2\mu} - \frac{\mu z}{4\mu^2} \). If \( h \) is an immersed Riemannian surface and \( g: U \to \mathbb{S}^m \) is the surface defined by \( h = i \circ g \), then (22) implies that \((u, v)\) are complex conjugate coordinates for \( g \) and that the complex Christoffel symbol of the metric induced by \( g \) is \( \Gamma = -(1/2\mu) \mu_z \). It follows that (15) is satisfied and that \( \mu \) is a positive solution of (16).

We call the pair \((h, r)\) formed by a surface \( h: L^2 \to \mathbb{R}^m \) and a function \( r \in C^\infty(L) \) a special hyperbolic pair (respectively, special elliptic pair) if there exists a special hyperbolic surface (respectively, special elliptic surface) \( g: L^2 \to \mathbb{S}^{m+1} \) such that \((h, r)\) are given by (3).

We conclude this section with the following result that will be used for the proof of Theorem 1.

**Proposition 7.** For a simply connected surface \( g: L^2 \to \mathbb{S}^m \subset \mathbb{L}^{m+1} \) the following assertions are equivalent:

(i) The surface \( g \) is special hyperbolic (respectively, special elliptic).

(ii) The surface is hyperbolic (respectively, elliptic) with respect to a tensor \( J \) on \( L^2 \) satisfying \( J^2 = I \) (respectively, \( J^2 = -I \)) and there exists \( \mu \in C^\infty(L) \) nowhere vanishing such that \( D = \mu J \) is a Codazzi tensor on \( L^2 \).

**Proof:** Let \( g \) be a hyperbolic surface as in part (ii) and let \((u, v)\) be local real conjugate coordinates on \( L^2 \) given by Proposition 4. Then the equation

\[
(\nabla_{\partial_u} D) \partial_v - (\nabla_{\partial_v} D) \partial_u = 0
\]

is easily seen to be equivalent to (8).

Conversely, if \( g \) is special hyperbolic with real conjugate coordinates \((u, v)\), \( J \in \Gamma(\text{End}(T)) \) is given by \( J \partial_u = \partial_u \) and \( J \partial_v = -\partial_v \), and \( \mu \in C^\infty(L) \) satisfies (8), then \( D = \mu J \) satisfies (23) in view of (8), and hence is a Codazzi tensor on \( L^2 \). The proof for the elliptic case is similar. \( \blacksquare \)
2 The proofs

Let \( f: M^n \to \mathbb{R}^{n+1} \) be a hypersurface with second fundamental form \( A \) with respect to the Gauss map \( N \in \Gamma(N_f M) \). Associated to a conformal infinitesimal bending \( \mathcal{T} \) with conformal factor \( \rho \) there is a symmetric tensor \( \mathcal{B} \in \Gamma(\text{End}(TM)) \) defined as follows: Let \( L \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^{n+1})) \) be the tensor defined by

\[
LX = \tilde{\nabla}_X T - \rho f_* X = \mathcal{T}_* X - \rho f_* X
\]

for any \( X \in \mathfrak{X}(M) \). Then let \( B: TM \times TM \to f^*T\mathbb{R}^{n+1} \) be given by

\[
B(X,Y) = (\tilde{\nabla}_X L)Y = \tilde{\nabla}_X LY - L\nabla_X Y
\]

for any \( X,Y \in \mathfrak{X}(M) \). We define \( \mathcal{B} \in \Gamma(\text{End}(TM)) \) by

\[
\langle \mathcal{B}X,Y \rangle = B(X,Y), N\rangle
\]

for any \( X,Y \in \mathfrak{X}(M) \). Notice that flatness of the ambient space and

\[
B(X,Y)_{N_f M} = (\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{T} - \tilde{\nabla}_{\nabla_X Y} \mathcal{T})_{N_f M} - \rho\langle AX,Y \rangle N
\]

give that \( \mathcal{B} \) is symmetric.

By Proposition 5 in [5] the tensor \( \mathcal{B} \) satisfies the following fundamental system of equations:

\[
\mathcal{B}X \wedge AY - \mathcal{B}Y \wedge AX + X \wedge HY - Y \wedge HX = 0 \quad (24)
\]

and

\[
(\nabla_X \mathcal{B})Y - (\nabla_Y \mathcal{B})X + (X \wedge Y)A\nabla \rho = 0 \quad (25)
\]

where \( H \in \text{End}(TM) \) is defined by \( HY = \nabla_Y \nabla \rho \) and \( X,Y \in \mathfrak{X}(M) \). The equations are called fundamental because they are the integrability condition for the system of equations that determines a conformal infinitesimal bending; see Corollary 2 in [5].

Trivial conformal infinitesimal bendings can be characterized in terms of their associated tensors as follows.

**Proposition 8.** A conformal infinitesimal bending \( \mathcal{T} \) of \( f: M^n \to \mathbb{R}^{n+1} \), \( n \geq 3 \), is trivial if and only if its associated tensor \( \mathcal{B} \) has the form \( \mathcal{B} = \varphi I \) for \( \varphi \in C^\infty(M) \).
Proof: See Corollary 9 in [5]. □

Let $\mathcal{T}$ be a conformal infinitesimal bending of $f: M^n \to \mathbb{R}^{n+1}$ with conformal factor $\rho$. At any point of $M^n$ we obtain from (24) that the associated bilinear form $\theta: TM \times TM \to \mathbb{R}^4$ defined by

$$\theta(X, Y) = \langle (A + B)X, Y \rangle, \langle (I + H)X, Y \rangle, \langle (A - B)X, Y \rangle, \langle (I - H)X, Y \rangle \rangle \ (26)$$

is flat with respect to the inner product $\langle \langle \cdot, \cdot \rangle \rangle$ of signature $(1, 1, -1, -1)$. That $\theta$ is flat means that

$$\langle \langle \theta(X, Y), \theta(Z, W) \rangle \rangle - \langle \langle \theta(X, W), \theta(Z, Y) \rangle \rangle = 0$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

**Proposition 9.** Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an isometric immersion with no umbilical points. If $\mathcal{T}$ is a conformal infinitesimal bending of $f$ such that the associated flat bilinear form $\theta$ given by (26) is null at any point of $M^n$ then $\mathcal{T}$ is trivial.

**Proof:** That $\theta$ is null means that

$$\langle \langle \theta(X, Y), \theta(Z, W) \rangle \rangle = 0$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. Equivalently,

$$\langle AX, Y \rangle B + \langle B X, Y \rangle A + \langle X, Y \rangle H + \langle HX, Y \rangle I = 0$$

for any $X, Y \in \mathfrak{X}(M)$. Fix a point $x \in M^n$. By the above $A(x), B(x)$ and $H(x)$ commute, that is, there exists an orthonormal basis $\{X_i\}_{1 \leq i \leq n}$ of $T_x M$ that diagonalizes them simultaneously. If $\lambda_i, b_i$ and $h_i$ are the respective eigenvalues of $A(x), B(x)$ and $H(x)$ corresponding to $X_i$, $1 \leq i \leq n$, then

$$\lambda_i B + b_i A + H + h_i I = 0.$$ 

Let $i, j$ be such that $\lambda_i \neq \lambda_j$. Then

$$(\lambda_i - \lambda_j)B + (b_i - b_j)A + (h_i - h_j)I = 0.$$ 

Hence

$$(\lambda_i - \lambda_j)b_i + (b_i - b_j)\lambda_i + (h_i - h_j) = (\lambda_i - \lambda_j)b_j + (b_i - b_j)\lambda_j + (h_i - h_j) = 0,$$
and therefore
\[(\lambda_i - \lambda_j)(b_i - b_j) = 0\]
showing that \(b_i = b_j\). If \(i \neq j\) are such that \(\lambda_i = \lambda_j\) then
\[(b_i - b_j)A + (h_i - h_j)I = 0.\]

But since \(f\) has no umbilical points, we necessarily have \(b_i = b_j\) and hence \(B = bI\) at any \(x \in M^n\). We conclude from Proposition 8 that \(T\) is trivial.

**Lemma 10.** Let \(f: M^n \to \mathbb{R}^{n+1}\) be an isometric immersion free of umbilical points and let \(\mathcal{T}\) be a non-trivial conformal infinitesimal bending of \(f\). Then \(A, B\) and \(H\) share on connected components of an open and dense subset a common eigenbundle \(\Delta\) with \(\dim \Delta \geq n - 2\).

**Proof:** By Proposition 9 the bilinear form \(\theta\) is not null. Theorem 3 in [3] or Lemma 4.22 in [8] yield an orthogonal decomposition \(\mathbb{R}^4 = \mathbb{R}^{2,2} = \mathbb{R}^{\ell,\ell} \oplus \mathbb{R}^{2-\ell,2-\ell}, 1 \leq \ell \leq 2\), such that the \(\mathbb{R}^{\ell,\ell}\)-component \(\theta_1\) of \(\theta\) is nonzero but is null since we have \(S(\theta_1) = S(\theta) \cap S(\theta)^\perp\), and the \(\mathbb{R}^{2-\ell,2-\ell}\)-component \(\theta_2\) is flat and satisfies \(\dim N(\theta_2) \geq n - 4 + 2\ell\). Moreover, since \(\theta\) is not null then \(\ell = 1\).

We denote \(\Delta = N(\theta_2)\) and restrict ourselves to connected components of an open and dense subset where \(\dim \Delta \geq n - 2\) is constant. Since we have that \(\theta(T, X) = \theta_1(T, X)\) for any \(T \in \Gamma(\Delta)\) and \(X \in \mathfrak{X}(M)\), then
\[\langle \theta(T, X), \theta(Y, Z) \rangle = 0\]
for any \(T \in \Gamma(\Delta)\) and \(X, Y, Z \in \mathfrak{X}(M)\). Equivalently,
\[\langle AT, X \rangle B + \langle BT, X \rangle A + \langle T, X \rangle H + \langle HT, X \rangle I = 0\]
(27)
for any \(T \in \Gamma(\Delta)\) and \(X \in \mathfrak{X}(M)\). Taking \(X\) orthogonal to \(T\) we see that
\[\langle AT, X \rangle B + \langle BT, X \rangle A + \langle HT, X \rangle I = 0.\]
(28)
Fix \(x \in M^n\) and assume that there exists \(T \in \Delta(x)\) and \(X \in T_xM\) such that \(\langle X, T \rangle = 0\) and \(\langle BT, X \rangle \neq 0\). From (28) and since \(f\) is free of umbilic points we have that \(A\) commutes with \(B\). Hence also does \(H\). Let \(\{X_i\}_{1 \leq i \leq n}\) be an orthonormal basis of \(T_xM\) of common eigenvectors of \(A, B\) and \(H\) with corresponding eigenvalues \(\lambda_i, b_i\) and \(h_i\). Since \(\langle BT, X \rangle \neq 0\) with \(\langle X, T \rangle = 0\),
then $T$ is not an eigenvector. Hence, there are two eigenvalues $b_1 \neq b_2$ such that $\langle T, X_1 \rangle \neq 0 \neq \langle T, X_2 \rangle$. Thus we have from (27) that

$$\lambda_1 B + b_1 A + H + h_1 I = 0 \text{ and } \lambda_2 B + b_2 A + H + h_2 I = 0.$$  

Hence

$$(\lambda_1 - \lambda_2) B + (b_1 - b_2) A + (h_1 - h_2) I = 0,$$  

from where we obtain

$$(\lambda_1 - \lambda_2) b_j + (b_1 - b_2) \lambda_j + h_1 - h_2 = 0, \quad 1 \leq j \leq n.$$  

Taking the difference between the cases $j = 1$ and $j = 2$ we have

$$\lambda_1 - \lambda_2 = 0,$$  

and hence $\lambda_1 = \lambda_2$. It follows from (27) that $A$ is a multiple of the identity which is a contradiction.

Therefore $\langle B T, X \rangle = 0$ for any $T \in \Delta(x)$ and $X \in T_x M$ with $\langle X, T \rangle = 0$. This implies that $\Delta$ is an eigenspace of $B$. If $\langle A T, X \rangle \neq 0$, for some $T \in \Delta(x)$ and $X \in T_x M$ with $\langle T, X \rangle = 0$, then we have from (28) that $B$ is a multiple of the identity and this is contradiction. Hence $\Delta$ is also an eigenspace of $A$, and consequently of $H$. 

Let $f: M^n \to \mathbb{R}^{n+1}$ be a hypersurface that carries a principal curvature of constant multiplicity $n - 2$ with corresponding eigenbundle $\Delta$. Recall that the splitting tensor $C: \Gamma(\Delta) \to \Gamma(\text{End}(\Delta^\perp))$ of $\Delta$ is defined by

$$C_T X = -\nabla^h_T X = (\nabla_T X)_{\Delta^\perp}$$  

for any $T \in \Gamma(\Delta)$ and $X \in \Gamma(\Delta^\perp)$. If $f$ is not conformally surface-like on any open subset of $M^n$ we say that $f$ is hyperbolic (respectively, parabolic or elliptic) if there exists $J \in \Gamma(\text{End}(\Delta^\perp))$ satisfying the following conditions:

(i) $J^2 = I$ and $J \neq I$ (respectively, $J^2 = 0$, with $J \neq 0$, and $J^2 = -I$),

(ii) $\nabla^h_T J = 0$ for all $T \in \Gamma(\Delta)$,

(iii) $C_T \in \text{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.
Let \( V^{n+2} \subset \mathbb{L}^{n+3} \) be the light cone of \( \mathbb{L}^{n+3} \), that is,

\[
V^{n+2} = \{ v \in \mathbb{L}^{n+3} : \langle v, v \rangle = 0, v \neq 0 \}.
\]

Given \( w \in V^{n+2} \) we have that

\[
E^{n+1} = \{ v \in V^{n+2} : \langle v, w \rangle = 1 \}
\]

is a model of \( \mathbb{R}^{n+1} \) in \( \mathbb{L}^{n+3} \). In fact, fix \( v \in E^{n+1} \) and a linear isometry

\[
C: \mathbb{R}^{n+1} \to (\text{span}\{v, w\})^\perp \subset \mathbb{L}^{n+3}.
\]

The map \( \Psi: \mathbb{R}^{n+1} \to V^{n+2} \subset \mathbb{L}^{n+3} \) given by

\[
\Psi(x) = v + Cx - \frac{1}{2} \|x\|^2 w
\]

is an isometric embedding such that \( \Psi(\mathbb{R}^{n+1}) = E^{n+1} \).

The normal bundle of \( \Psi \) is \( N_{\Psi} \mathbb{R}^{n+1} = \text{span}\{\Psi, w\} \) and its second fundamental form is given by

\[
\alpha^\Psi(U, V) = -\langle U, V \rangle w
\]

for any \( U, V \in T\mathbb{R}^{n+1} \). For further details we refer to Section 9.1 in [8].

Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 5, \) be an oriented hypersurface with a principal curvature \( \lambda \) of constant multiplicity \( n-2 \). By composing with an appropriate inversion, if necessary, and given that \( f \) is orientable, we can always assume that \( \lambda > 0 \) at any point of \( M^n \). Let \( A \) be the second fundamental form associated to the Gauss map \( N \) of \( f \) and let \( \Delta(x) \subset T_x M \) be the eigenspace corresponding to \( \lambda(x) \) at \( x \in M^n \). Fix an embedding \( \Psi \) as in (30) and let

\[
S: M^n \to \mathbb{L}^{n+3} \text{ be the map given by}
\]

\[
S(x) = \lambda(x)\Psi(f(x)) + \Psi_*N(x).
\]

Then \( S(x) \in S_1^{n+2} \subset \mathbb{L}^{n+3} \) and

\[
S_*X = X(\lambda)\Psi(f(x)) - \Psi_*f_*(A - \lambda I)X
\]

for any \( X \in \mathfrak{X}(M) \). From (33) it follows that \( S \) is constant along the leaves of \( \Delta \). Let \( L^2 \) be the quotient space of leaves of \( \Delta \) and let \( \pi: M^n \to L^2 \) be the canonical projection. Thus \( S \) induces an immersion \( s: L^2 \to S_1^{n+2} \subset \mathbb{L}^{n+3} \) such that \( S = s \circ \pi \). Moreover, the metric \( \langle \cdot, \cdot \rangle \) on \( L^2 \) induced by \( s \) satisfies

\[
\langle X, Y \rangle' = \langle (A - \lambda I)X, (A - \lambda I)Y \rangle
\]

where \( X, Y \in \mathfrak{X}(M) \) are the horizontal lifts of \( \bar{X}, \bar{Y} \in \mathfrak{X}(L^2) \).
Proposition 11. Let \( f : M^n \to \mathbb{R}^{n+1}, n \geq 5 \), be an oriented hypersurface and let \( \mathcal{T} \) be a non-trivial conformal infinitesimal bending of \( f \). Assume that the principal curvature \( \lambda \) of \( A \) determined by \( \Delta \) from Lemma 10 is positive and has constant multiplicity \( \dim \Delta = n - 2 \). Then, on each connected component of an open dense subset of \( M^n \) either \( f \) is conformally surface-like or is hyperbolic, parabolic or elliptic with respect to \( J \in \Gamma(\text{End}(\Delta^\perp)) \) and there exists \( \mu \in C^\infty(M) \) nowhere vanishing and constant along the leaves of \( \Delta \) such that \( D = \mu J \in \Gamma(\text{End}(\Delta^\perp)) \) satisfies:

(i) \( (A - \lambda I)D \) is symmetric,

(ii) \( \nabla_1 D = 0 \),

(iii) \( (\nabla_X(A - \lambda I)D)Y - (\nabla_Y(A - \lambda I)D)X = X \wedge Y (D^1 \nabla \lambda) \),

(iv) \( \langle (\nabla_Y D)X - (\nabla_X D)Y, \nabla \lambda \rangle + \text{Hess} \lambda(DX, Y) - \text{Hess} \lambda(X, DY) \\
\quad = \lambda(\langle AX, (A - \lambda I)DY \rangle - \langle (A - \lambda I)DX, AY \rangle) \),

(v) \( (A - \lambda I)DX \wedge (A - \lambda I)Y - (A - \lambda I)DY \wedge (A - \lambda I)X = 0 \)

for any \( T \in \Gamma(\Delta) \) and \( X, Y \in \Gamma(\Delta^\perp) \).

Conversely, assume that \( f \) as above is either hyperbolic, parabolic or elliptic with respect to \( J \in \Gamma(\text{End}(\Delta^\perp)) \) and there is \( 0 \neq D = \mu J \in \Gamma(\text{End}(\Delta^\perp)) \) that satisfies conditions (i) to (v). If \( M^n \) is simply connected there exists a non-trivial conformal infinitesimal bending \( \mathcal{T} \) of \( f \) determined by \( D \), unique up to trivial conformal infinitesimal bendings.

Proof: We have from Lemma 10 that \( \Delta \) is a common eigenbundle for \( A, B \) and \( H \). Thus \( B|\Delta = bI \) and \( H|\Delta = hI \) where \( b, h \in C^\infty(M) \). We obtain from (27) that

\[
bA + \lambda B + H + hI = 0.
\]

In particular \( \lambda b + h = 0 \), and thus locally

\[
bA + \lambda(B - bI) + H = 0. \tag{35}
\]

From (25) we have

\[
T(b - \lambda \rho) = T(b) - \lambda T(\rho) = 0 \tag{36}
\]

for any \( T \in \Gamma(\Delta) \). Then (25) is equivalent to

\[
(\nabla_X(B - bI))Y - (\nabla_Y(B - bI))X + (X \wedge Y)(A \nabla \rho - \nabla b) = 0. \tag{37}
\]
It follows from (36) and (37) that

\[(\nabla^h_T(\mathcal{B} - bI))X = (\mathcal{B} - bI)C_TX\]  

(38)

for any \(X \in \Gamma(\Delta^\perp)\) and \(T \in \Gamma(\Delta)\).

We regard \(A - \lambda I\) and \(\mathcal{B} - bI\) as tensors on \(\Delta^\perp\). We obtain from (38) and the Codazzi equation \(\nabla^h_T A = (A - \lambda I)C_T\) that

\[(\mathcal{B} - bI)C_T = C^t_T(\mathcal{B} - bI)\quad\text{and}\quad(A - \lambda I)C_T = C^t_T(A - \lambda I).\]

We have that \(D \in \Gamma(\text{End}(\Delta^\perp))\) defined by

\[D = (A - \lambda I)^{-1}(\mathcal{B} - bI)\]

satisfies \(D \neq 0\) since \(T\) is non-trivial. Hence

\[(A - \lambda I)DC_T = (\mathcal{B} - bI)C_T = C^t_T(\mathcal{B} - bI) = C^t_T(A - \lambda I)D = (A - \lambda I)C_TD,\]

and therefore

\[[D, C_T] = 0.\]  

(39)

We also have

\[(A - \lambda I)C_TD = (\nabla^h_T A)D\]

and

\[(A - \lambda I)DC_T = (\mathcal{B} - bI)C_T = \nabla^h_T(\mathcal{B} - bI) = \nabla^h_T((A - \lambda I)D) = \nabla^h_T(AD) - \lambda \nabla^h_T D.\]

Thus

\[(A - \lambda I)\nabla^h_T D = (A - \lambda I)[D, C_T],\]

and hence

\[\nabla^h_T D = 0\]  

(40)

for any \(T \in \Gamma(\Delta)\).

It follows from (39), (40) and Corollary 11.7 of [8] that \(D\) is projectable with respect to \(\pi: M^n \to L^2\), that is, \(D\) is the horizontal lift of a tensor \(\bar{D}\) on \(L^2\). Hence

\[\pi_*DX = \bar{D}\pi_*X = \bar{D}\tilde{X} \circ \pi \quad\text{if}\quad \pi_*X = \tilde{X} \circ \pi.\]
We have that (24) reads as
\[ B_X \wedge AY - B_Y \wedge AX + X \wedge HY - Y \wedgeHX = 0. \]
Since \( H = \lambda(bI - B) - bA \) from (35), then
\[ (B - bI)X \wedge (A - \lambda I)Y - (B - bI)Y \wedge (A - \lambda I)X = 0 \] (41)
for any \( X, Y \in \mathfrak{X}(M) \). From (41) and the definition of \( D \) we have
\[ \langle (A - \lambda I)DX \wedge (A - \lambda I)Y - (A - \lambda I)DY \wedge (A - \lambda I)X \rangle = 0 \]
for any \( X, Y, Z, W \in \Gamma(\Delta) \). This implies that
\[ \langle \tilde{D} \tilde{X} \wedge \tilde{Y} - \tilde{D} \tilde{Y} \wedge \tilde{X} \rangle = 0 \]
for any \( \tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in \mathfrak{X}(L) \). In other words, we have
\[ \tilde{D} \tilde{X} \wedge \tilde{Y} - \tilde{D} \tilde{Y} \wedge \tilde{X} = 0 \]
with respect to the metric \( \langle , \rangle' \). Thus \( \text{tr} \tilde{D} = 0 \).

We have that \( \tilde{D} \) has either two smooth distinct real eigenvalues, a single real eigenvalue of multiplicity two or a pair of smooth complex conjugate eigenvalues. Thus there is \( \mu \in C^\infty(L) \) such that \( \tilde{D} = \mu J, J \neq I \), where the tensor \( J \in \Gamma(\text{End}(TL)) \) satisfies \( J^2 = \epsilon I \), for \( \epsilon = 1, 0 \) or \(-1\). Hence \( D = \mu J \) where \( J \) is the lifting of \( \tilde{J} \) and \( \bar{\mu} = \mu \circ \pi \). In particular \( \text{tr} D = 0 \).

If \( \text{span}\{C_T : T \in \Delta\} \subset \text{span}\{I\} \) we have from Corollary 9.33 in [3] that \( f \) is conformally surface-like. Hence, we assume \( \text{span}\{C_T : T \in \Delta\} \nsubseteq \text{span}\{I\} \) and obtain from (39) that \( C_T \in \text{span}\{I, J\} \) for any \( T \in \Gamma(\Delta) \).

We have from (40) that
\[ T(\mu)J + \mu \nabla^h_T J = 0 \]
for any \( T \in \Gamma(\Delta) \). Therefore
\[ \epsilon T(\mu)I + \mu J \nabla^h_T J = 0 \quad \text{and} \quad \epsilon T(\mu)I + \mu (\nabla^h_T J) = 0. \]
Since \( J^2 = \epsilon I \) we obtain that \( T(\mu) = 0 \), and hence \( \nabla^h_T J = 0 \). Thus, the hypersurface \( f \) is either hyperbolic, parabolic or elliptic.

We have from (35) that
\[ X(b)AY + X(\lambda)BY - X(\lambda b)Y + b(\nabla_X A)Y + \lambda(\nabla_X B)Y + (\nabla_X H)Y = 0. \] (42)
On the other hand, the Gauss equation yields
\[(\nabla_X H)Y - (\nabla_Y H)X = R(X,Y)\nabla \rho = \langle AY, \nabla \rho \rangle AX - \langle AX, \nabla \rho \rangle AY. \tag{43}\]

Then (25), (42), (43) and the Codazzi equation imply that
\[
X(b)AY + X(\lambda)BY - X(\lambda b)Y - Y(b)AX - Y(\lambda)BX + Y(\lambda b)X \\
- \lambda(X \wedge Y)A\nabla \rho + \langle AY, \nabla \rho \rangle AX - \langle AX, \nabla \rho \rangle AY = 0
\]
for any \(X,Y \in \mathfrak{X}(M)\). Then
\[
\langle X, \nabla b - A\nabla \rho \rangle (A - \lambda I)Y - \langle Y, \nabla b - A\nabla \rho \rangle (A - \lambda I)X \\
+ \langle X, \nabla \lambda \rangle (B - bI)Y - \langle Y, \nabla \lambda \rangle (B - bI)X = 0
\]
for any \(X,Y \in \mathfrak{X}(M)\). For \(X,Y \in \Gamma(\Delta^\perp)\) we obtain
\[
\langle X, \nabla b - A\nabla \rho \rangle Y - \langle Y, \nabla b - A\nabla \rho \rangle X + \langle X, \nabla \lambda \rangle DY - \langle Y, \nabla \lambda \rangle DX = 0.
\]

Taking \(X\) and \(Y\) orthonormal, we obtain
\[
\langle X, \nabla b - A\nabla \rho \rangle - \langle X, \nabla \lambda \rangle (DY, X) + \langle Y, \nabla \lambda \rangle (DX, X) = 0
\]
and
\[
\langle X, \nabla b - A\nabla \rho \rangle + \langle X, \nabla \lambda \rangle (DY, Y) - \langle Y, \nabla \lambda \rangle (DX, Y) = 0.
\]

Using that \(\text{tr} \, D = 0\) this gives
\[
D^t \nabla \lambda = \nabla b - A\nabla \rho \tag{44}
\]
where \(D^t\) denotes the transpose of \(D\).

So far we have that \((i)\) holds from the definition of \(D\), \((ii)\) is (10), \((iii)\) follows from (37) and (44), and \((iv)\) is (41). It remains to prove that \((iv)\) holds. To do this, fix a pseudo-orthonormal basis \(e_1, \ldots, e_{n+3}\) of \(\mathbb{L}^{n+3}\) and set \(v = e_1\) and \(w = -2e_{n+3}\). Let \(\Psi: \mathbb{R}^{n+1} \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}\) and \(S: M^n \to \mathbb{L}^{n+3}\) be given by (30) and (32) respectively. We see next that the immersion \(s: L^2 \to \mathbb{S}^{n+2}_{\perp} \subset \mathbb{L}^{n+3}\) induced by \(S\) satisfies \(s = g\), where \(g\) is given by (4). \(h: L^2 \to \mathbb{R}^{n+1}\) is induced by \(f + rN\) and \(r = \lambda^{-1}\). In fact, we have that \(\Psi(y) = (1, y, \|y\|^2)\). Then
\[
S(x) = \lambda(1, f(x), \|f(x)\|^2) + (0, N(x), 2\langle f(x), N(x) \rangle) \\
= \lambda(1, f(x) + rN, \|f(x)\|^2 + 2r\langle f(x), N(x) \rangle).
\]
Since \( h \circ \pi = f + rN \), it follows that
\[
s = r^{-1}(1, h, \|h\|^2 - r^2) = g.
\]

Let \( X, Y \in \Gamma(\Delta^\perp) \) be the horizontal lifts of \( \tilde{X}, \tilde{Y} \in \mathfrak{X}(L) \). We have
\[
\hat{\nabla}'_X S_*\bar{D}Y = \hat{\nabla}'_{\pi_*X} g_* \pi_* \bar{D}Y = \hat{\nabla}'_X g_* \bar{D}Y = g_* \hat{\nabla}'_X D\bar{Y} + \alpha^g(X, D\bar{Y}) - \langle X, D\bar{Y} \rangle g \circ \pi
\]
where \( \hat{\nabla}' \) and \( \nabla' \) denote the connections in \( \mathbb{L}^{n+3} \) and \( L^2 \), respectively. We obtain from (33) that
\[
\hat{\nabla}'_X \Psi_* f_*(A - \lambda I) D\bar{Y} = X \langle D\bar{Y}, \nabla \lambda \rangle \Psi \circ f + \langle D\bar{Y}, \nabla \lambda \rangle \Psi_* f_* X
\]
\[- g_* \hat{\nabla}'_X D\bar{Y} - \alpha^g(X, D\bar{Y}) + \langle (A - \lambda I)X, (A - \lambda I) D\bar{Y} \rangle (\lambda \Psi \circ f + \Psi_* N).
\]

On the other hand, using (31) and (33) it follows that
\[
\hat{\nabla}'_X \Psi_* f_*(A - \lambda I) D\bar{Y} = \Psi_* \hat{\nabla}'_X f_*(A - \lambda I) D\bar{Y} + \alpha^\Psi(f_* X, f_*(A - \lambda I) D\bar{Y})
\]
\[= \Psi_* f_* \hat{\nabla}_X (A - \lambda I) D\bar{Y} + \langle AX, (A - \lambda I) D\bar{Y} \rangle \Psi_* N - \langle X, (A - \lambda I) D\bar{Y} \rangle w
\]
\[= \Psi_* f_* \hat{\nabla}_X (A - \lambda I) D\bar{Y} + \langle AX, (A - \lambda I) D\bar{Y} \rangle \Psi_* N - \langle X, (A - \lambda I) D\bar{Y} \rangle w
\]
\[= \Psi_* f_* \hat{\nabla}_X (A - \lambda I) D\bar{Y} + \langle D\nabla_X Y, \nabla \lambda \rangle \Psi \circ f - g_* D\pi_* \nabla_X Y
\]
\[+ \langle AX, (A - \lambda I) D\bar{Y} \rangle \Psi_* N - \langle X, (A - \lambda I) D\bar{Y} \rangle w.
\]

We obtain from the last two equations and \( \pi_* [X, Y] = [\tilde{X}, \tilde{Y}] \) that
\[
g_*((\nabla'_Y D)X - (\nabla'_X D)Y) + \alpha^g(Y, DX) - \alpha^g(X, DY)
\]
\[= \Psi_* f_* \Omega(X, Y) - \lambda \psi(X, Y) \Psi_* N + \varphi(X, Y) \Psi \circ f + \psi(X, Y) w
\]
where
\[
\Omega(X, Y) = \langle \nabla_X (A - \lambda I) D\bar{Y} \rangle X - \langle \nabla_Y (A - \lambda I) D\bar{Y} \rangle X - X \wedge Y (D^t \nabla \lambda), \tag{45}
\]
\[\psi(X, Y) = \langle Y, (A - \lambda I) DX \rangle - \langle X, (A - \lambda I) DY \rangle, \tag{46}
\]
\[\varphi(X, Y) = \langle (\nabla_Y D)X - (\nabla_X D)Y, \nabla \lambda \rangle + \text{Hess } \lambda(DX, Y) - \text{Hess } \lambda(X, DY)
\]
\[- \lambda ((A - \lambda I)X, (A - \lambda I)DY) - \langle (A - \lambda I)DX, (A - \lambda I)DY \rangle. \tag{47}
\]

It follows from (37) and (44) that \( \Omega \) vanishes. The symmetry of \( \mathfrak{B} \) yields \( \psi = 0 \). Hence
\[
g_*((\nabla'_Y D)X - (\nabla'_X D)Y) + \alpha^g(Y, DX) - \alpha^g(X, DY) = \varphi(X, Y) \Psi \circ f.
\]
Since the term on the left-hand side is constant along the leaves of $\Delta$ then $
abla$ has to vanish, which proves \[ (iv) \]

We prove the converse. Let $D = \mu J \in \Gamma(\text{End}(\Delta^\perp))$ verify conditions \[ (i) \] to \[ (v) \]. In the sequel, we extend $D$ to an element of $\text{End}(TM)$ defining $DT = 0$ for any $T \in \Gamma(\Delta)$. Then \[ (v) \] holds for any $X,Y \in \mathcal{X}(M)$.

Set $F = \Psi \circ f : M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$. Then let $\beta : TM \times TM \to N_F M$ be the symmetric tensor defined by

$$\beta(X,Y) = \langle (A - \lambda I)DX,Y \rangle (\Psi_* N + \lambda F)$$

where $N$ is a Gauss map of $f$. Let $B_\eta \in \Gamma(\text{End}(TM))$ be given by

$$\langle B_\eta X,Y \rangle = \langle \beta(X,Y) , \eta \rangle$$

for any $\eta \in \Gamma(N_F M)$. For simplicity we write $N = \Psi_* N$. Observe that $B_N = (A - \lambda I)D$ and $B_w = \lambda B_N$. Since

$$\alpha^F(X,Y) = \langle AX,Y \rangle N - \langle X,Y \rangle w,$$

we have from \[ (v) \] and $A|_\Delta = \lambda I$ that

$$A^F_{\beta(Y,Z)}X + B_{\alpha^F(Y,Z)}X - A^F_{\beta(X,Z)}Y - B_{\alpha^F(X,Z)}Y = 0$$

for any $X,Y,Z \in \mathcal{X}(M)$, where $A^F_\eta$ is the shape operator of $F$ with respect to $\eta \in \Gamma(N_F M)$.

We define $\mathcal{E} : TM \times N_F M \to N_F M$ by

$$\mathcal{E}(X,N) = \langle DX, \nabla \lambda \rangle F, \quad \mathcal{E}(X,w) = -\langle DX, \nabla \lambda \rangle N \quad \text{and} \quad \mathcal{E}(X,F) = 0$$

for any $X \in \mathcal{X}(M)$. Observe that $\mathcal{E}$ satisfies the condition

$$\langle \mathcal{E}(X,\eta), \xi \rangle = -\langle \mathcal{E}(X,\xi), \eta \rangle$$

for any $X \in \mathcal{X}(M)$ and $\eta, \xi \in \Gamma(N_F M)$.

It follows from \[ (iii) \] that

$$(\nabla_X B_N)Y - (\nabla_Y B_N)X = \langle DY, \nabla \lambda \rangle X - \langle DX, \nabla \lambda \rangle Y$$

for any $X,Y \in \Gamma(\Delta^\perp)$. Using \[ (ii) \] and $[D,C_T] = 0$ we obtain

$$\nabla_X B_N T - (\nabla_T B_N)X = B_N C_T X - (\nabla_T (A - \lambda I))DX - (A - \lambda I)(\nabla_T D)X = (A - \lambda I)C_T DX - (\nabla_T (A - \lambda I))DX$$
for any $T \in \Gamma(\Delta)$. Now using the Codazzi equation, we have
\[
(\nabla_{X}B_{N})T - (\nabla_{T}B_{N})X = (A - \lambda I)C_{T}DX - (\nabla_{DX}A)T
= (A - \lambda I)C_{T}DX - (DX, \nabla\lambda)T - (A - \lambda I)C_{T}DX
= - (DX, \nabla\lambda)T. \tag{54}
\]

Since $\Delta$ is integrable, we obtain
\[
(\nabla_{T}B_{N})S - (\nabla_{S}B_{N})T = 0 \tag{55}
\]
for any $T, S \in \Gamma(\Delta)$. It follows from (53), (54) and (55) that
\[
(\nabla_{X}B_{N})Y - (\nabla_{Y}B_{N})X = A^{F}_{e(X,N)}Y - A^{F}_{e(Y,N)}X \tag{56}
\]
for any $X, Y \in \mathcal{X}(M)$.

We have from (53) that
\[
(\nabla_{X}B_{w})Y - (\nabla_{Y}B_{w})X = (X, \nabla\lambda)B_{N}Y - (Y, \nabla\lambda)B_{N}X
+ \lambda(DY, \nabla\lambda)X - \lambda(DX, \nabla\lambda)Y
\]
for any $X, Y \in \Gamma(\Delta^\perp)$. Let $\sigma \in \Gamma(\Delta^\perp)$ be given by $\nabla\lambda = (A - \lambda I)\sigma$. Using (v) we obtain
\[
(\nabla_{X}B_{w})Y - (\nabla_{Y}B_{w})X = (X, \nabla\lambda)B_{N}Y - (Y, \nabla\lambda)B_{N}X
+ \lambda(DY, \nabla\lambda)X - \lambda(DX, \nabla\lambda)Y
= (DX, \nabla\lambda)(A - \lambda I)X - (DX, \nabla\lambda)Y + \lambda(DY, \nabla\lambda)X - \lambda(DX, \nabla\lambda)Y
= (DX, \nabla\lambda)(A - \lambda I)X - (DX, \nabla\lambda)AY. \tag{57}
\]

Using (54) it follows that
\[
(\nabla_{X}B_{w})T - (\nabla_{T}B_{w})X = \lambda((\nabla_{X}B_{N})T - (\nabla_{T}B_{N})X)
= - (DX, \nabla\lambda)AT \tag{58}
\]
for any $T \in \Gamma(\Delta)$. As before, we have that
\[
(\nabla_{T}B_{w})S - (\nabla_{S}B_{w})T = 0 \tag{59}
\]
for any $S, T \in \Gamma(\Delta)$. We conclude from (57), (58) and (59) that
\[
(\nabla_{X}B_{w})Y - (\nabla_{Y}B_{w})X = A^{F}_{e(X,w)}Y - A^{F}_{e(Y,w)}X \tag{60}
\]
for any $X, Y \in \mathfrak{X}(M)$.

We have that $B_F = 0 = \mathcal{E}(X, F)$, and hence it holds trivially that

$$
(\nabla_X B_F)Y - (\nabla_Y B_F)X = A^F_{\mathcal{E}(X,F)}Y - A^F_{\mathcal{E}(Y,F)}X
$$

(61)

for any $X, Y \in \mathfrak{X}(M)$.

Next we focus on the covariant derivative of $\mathcal{E}$. Let $\nabla'\perp$ denote the normal connection on $N_F M$. We have

$$
(\nabla'\perp_X E)(Y, N) = \nabla'\perp_X \mathcal{E}(Y, N) - \mathcal{E}(\nabla_X Y, N) \\
= X \langle DY, \nabla \lambda \rangle F - \langle D\nabla_X Y, \nabla \lambda \rangle F \\
= (\langle \nabla_X D Y, \nabla \lambda \rangle + \text{Hess } \lambda(DY, X)) F
$$

for any $X, Y \in \mathfrak{X}(M)$. Then

$$
(\nabla'\perp_X \mathcal{E})(Y, N) - (\nabla'\perp_Y \mathcal{E})(X, N) = \lambda(\langle B_N X, AY \rangle - \langle AX, B_N Y \rangle) F
$$

(62)

for all $X, Y \in \mathfrak{X}(M)$. From \([iv]\) we have

$$
(\nabla'\perp_X \mathcal{E})(Y, N) - (\nabla'\perp_Y \mathcal{E})(X, N) = \langle \nabla_X D Y - \nabla_Y D X, \nabla \lambda \rangle \\
+ \text{Hess } \lambda(DY, X) - \text{Hess } \lambda(DX, Y) F
$$

for all $X, Y \in \mathfrak{X}(M)$. From \([iv]\) we have

$$
(\nabla'\perp_X \mathcal{E})(Y, N) - (\nabla'\perp_Y \mathcal{E})(X, N) = \lambda(\langle B_N X, AY \rangle - \langle AX, B_N Y \rangle) F
$$

(62)

for all $X, Y \in \Gamma(\Delta \perp)$. Using \([ii]\) and $[D, C_T] = 0$ we obtain

$$
(\nabla'\perp_X \mathcal{E})(T, N) - (\nabla'\perp_T \mathcal{E})(X, N) = \mathcal{E}([T, X], N) - \nabla'\perp_T \mathcal{E}(X, N) \\
= ((DC_T X - \nabla_T D X, \nabla \lambda) - \text{Hess } \lambda(DX, T)) F \\
= ((C_T DX, \nabla \lambda) - \text{Hess } \lambda(DX, T)) F \\
= (\langle T, \nabla DX \nabla \lambda \rangle - \text{Hess } \lambda(DX, T)) F \\
= 0
$$

(63)

for any $X \in \Gamma(\Delta \perp)$ and $T \in \Gamma(\Delta)$. We also have

$$
(\nabla'\perp_T \mathcal{E})(S, N) - (\nabla'\perp_S \mathcal{E})(T, N) = 0
$$

(64)

for any $S, T \in \Gamma(\Delta)$.

On the other hand, from \([48]\) and \([49]\) we obtain

$$
\beta(X, AY) - \beta(AX, Y) + \alpha^F(X, B_N Y) - \alpha^F(B_N X, Y) \\
= \lambda(\langle B_N X, AY \rangle - \langle AX, B_N Y \rangle) F
$$

(65)
for all $X,Y \in \mathfrak{X}(M)$. From (62), (63), (64) and (65) we conclude that
\[
(\nabla^\perp_X E)(Y,N) - (\nabla^\perp_Y E)(X,N) = \beta(X,AY) - \beta(AX,Y) + \alpha^F(X,B_N Y) - \alpha^F(B_N X,Y)
\] (66)
for any $X,Y \in \mathfrak{X}(M)$.

Similarly as above, we obtain
\[
(\nabla^\perp_X E)(Y,w) - (\nabla^\perp_Y E)(X,w) = \langle (\nabla_Y D)X - (\nabla_X D)Y, \nabla \lambda \rangle N + (\text{Hess } \lambda(D X,Y) - \text{Hess } \lambda(D Y,X)) N
\]
for all $X,Y \in \mathfrak{X}(M)$. From (iv) it follows that
\[
(\nabla^\perp_X E)(Y,w) - (\nabla^\perp_Y E)(X,w) = \lambda(\langle AX, (A - \lambda I)DY \rangle - \langle (A - \lambda I)DX, AY \rangle) N
\]
for $X,Y \in \Gamma(\Delta^\perp)$. As before, we have from (ii) and $[D,C_T] = 0$ that
\[
(\nabla^\perp_X E)(T,w) - (\nabla^\perp_Y E)(X,w) = (-\langle C_T DX, \nabla \lambda \rangle + \text{Hess } \lambda(D X,T)) N
\]
\[= (-\langle T, \nabla_{DX} \lambda \rangle + \text{Hess } \lambda(D X,T)) N = 0
\]
and
\[
(\nabla^\perp_X E)(S,w) - (\nabla^\perp_Y E)(T,w) = 0
\]
for any $T,S \in \Gamma(\Delta)$. It holds that
\[
\beta(X,A^F w Y) - \beta(A^F w X,Y) + \alpha^F(X,B_w Y) - \alpha^F(B_w X,Y) = \lambda(\langle AX, B_N Y \rangle - \langle B_N X, AY \rangle) N
\]
for all $X,Y \in \mathfrak{X}(M)$. Thus
\[
(\nabla^\perp_X E)(Y,w) - (\nabla^\perp_Y E)(X,w) = \beta(X,A^F w Y) - \beta(A^F w X,Y) + \alpha^F(X,B_w Y) - \alpha^F(B_w X,Y)
\] (67)
for all $X,Y \in \mathfrak{X}(M)$. Finally, we have
\[
\beta(X,A^F Y) - \beta(A^F X,Y) + \alpha^F(X,B_{F X} Y) - \alpha^F(B_{F X} X,Y) = 0
\]
for all $X,Y \in \mathfrak{X}(M)$, and since $E(X,F) = 0$, then
\[
(\nabla^\perp_X E)(Y,F) - (\nabla^\perp_Y E)(X,F) = \beta(X,A^F Y) - \beta(A^F X,Y) + \alpha^F(X,B_{F Y} Y) - \alpha^F(B_{F Y} X,Y)
\] (68)
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for all $X,Y \in \mathfrak{X}(M)$ holds trivially.

We have that $\beta$ is symmetric and that the tensor $\mathcal{E}$ satisfies condition (52). Moreover, the pair $(\mathcal{E}, \beta)$ also satisfies (50), (56), (60), (61), (66), (67) and (68). In this situation, the Fundamental theorem for isometric infinitesimal bendings, namely, Theorem 6 in [4], applies. Notice that in the introduction of [4] it was observed that Theorem 6 holds for ambient spaces of any signature, in particular, for the Lorentzian space considered here. Making also use of Proposition 5 of [4], we conclude that there is an isometric infinitesimal bending $\tilde{T} \in \Gamma(F^*(T\mathbb{L}^{n+3}))$ of $F$ whose associated pair of tensors $(\tilde{\beta}, \tilde{\mathcal{E}})$ satisfies
\[
\tilde{\beta} = \beta + \alpha F \quad \text{and} \quad \tilde{\mathcal{E}} = \mathcal{E} - \nabla^\perp C
\] (69)
where $C \in \Gamma(\text{End}(N_F M))$ is skew-symmetric. Moreover, we have that $\tilde{T}$ is unique up to trivial isometric infinitesimal bendings.

Write $\tilde{T}$ as
\[
\tilde{T} = \Psi_* T + \langle \tilde{T}, w \rangle F + \langle \tilde{T}, F \rangle w.
\]
Being $\tilde{T}$ an isometric infinitesimal bending of $F$, we have
\[
\langle \tilde{\nabla}'_X \tilde{T}, F_* Y \rangle + \langle \tilde{\nabla}'_Y \tilde{T}, F_* X \rangle = 0
\]
for all $X,Y \in \mathfrak{X}(M)$. Then
\[
\langle \nabla X \tilde{T}, f_* Y \rangle + \langle \nabla Y \tilde{T}, f_* X \rangle + 2 \langle \tilde{T}, w \rangle \langle X, Y \rangle = 0
\]
for all $X,Y \in \mathfrak{X}(M)$. Hence, setting $\rho = -\langle \tilde{T}, w \rangle$ we have that $T$ is a conformal infinitesimal bending of $f$ with conformal factor $\rho$. It follows from (69) that the symmetric tensor $B \in \Gamma(\text{End}(TM))$ associated to $T$ has the form $B = B_N + cI$ where $c = -\langle Cw, N \rangle$. And given that $B_N|_{\Delta^\perp} \neq 0$ we conclude that $T$ is not trivial.

Any other conformal infinitesimal bending $T'$ arising in this manner has the associated tensor $B' = B_N + c'I$. Proposition 8 now gives that $T - T'$ is trivial, and this concludes the proof. \square

**Proposition 12.** Any parabolic conformally infinitesimally bendable hypersurface $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, is conformally ruled.

Conversely, let $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a simply connected conformally ruled hypersurface free of points with a principal curvature of multiplicity at least $n - 1$ and that is not conformally surface-like on any open subset of $M^n$. 24
Then \( f \) is parabolic and admits a family of conformal infinitesimal bendings that are in one-to-one correspondence with the set of smooth functions on an interval.

**Proof:** We have that \( D = \mu J \) where \( J^2 = 0 \). Let \( Y \in \Gamma(\Delta^\perp) \) be of unit-length such that \( JY = 0 \) and let \( X \in \Gamma(\Delta^\perp) \) be orthogonal to \( Y \) satisfying \( JX = Y \). That \( \nabla_T^h J = 0 \) for any \( T \in \Gamma(\Delta) \) is equivalent to

\[
\nabla_T^h Y = 0 = \nabla_T^h X
\]

for all \( T \in \Gamma(\Delta) \). Hence, replacing \( J \) by \( \|X\|J \), one can assume that also \( X \) is of unit-length.

For the sequel, we extend \( J \) to \( TM \) as being zero on \( \Delta \). Recall that

\[
\mathcal{B} - bI = (A - \lambda I)D = \mu(A - \lambda I)J
\]

is symmetric. Then

\[
\langle(A - \lambda I)Y, Y\rangle = \langle(A - \lambda I)JX, Y\rangle = 0.
\]

Hence \( (A - \lambda I)Y = \nu X \) where \( \nu = \langle AX, Y\rangle \neq 0 \) by assumption. Then

\[
(\nabla_X \mu(A - \lambda I)J)Y - (\nabla_Y \mu(A - \lambda I)J)X = -\mu(A - \lambda I)J\nabla_X Y - \nabla_Y(\mu \nu X).
\]

On the other hand, we obtain from [iii] that

\[
(\nabla_X \mu(A - \lambda I)J)Y - (\nabla_Y \mu(A - \lambda I)J)X = -\mu Y(\lambda)Y.
\]

Hence

\[
\mu(A - \lambda I)J\nabla_X Y + \nabla_Y(\mu \nu X) = \mu Y(\lambda)Y.
\]

Taking the inner product with \( X \) and \( Y \), respectively, gives

\[
Y(\mu \nu) = \mu \nu \langle \nabla_X X, Y \rangle
\]

and

\[
Y(\lambda) = -\nu \langle \nabla_Y Y, X \rangle.
\]

Since \( C_T \in \text{span}\{I, J\} \), we have

\[
\langle \nabla_Y T, X \rangle = -\langle C_T Y, X \rangle = 0
\]
for any $T \in \Gamma(\Delta)$. Let $T \in \Gamma(\Delta)$ be of unit length. The inner product of the Codazzi equation $(\nabla_T A)Y - (\nabla_Y A)T = 0$ with $T$ easily gives

$$Y(\lambda) = -\nu\langle \nabla_T T, X \rangle.$$  \hfill (76)

It follows from (70), (74), (75) and (76) that the subspaces $\Delta \oplus \text{span}\{Y\}$ form an umbilical distribution. Moreover, we have from (71) that $f$ restricted to any leaf of $\Delta \oplus \text{span}\{Y\}$ is umbilical in $\mathbb{R}^{n+1}$. Thus $f$ is conformally ruled.

We now prove the converse. Let $L$ be an $(n-1)$-dimensional umbilical distribution of $M^n$ such that the restriction of $f$ to any leaf is also umbilical. Therefore, there is $\lambda \in C^\infty(M)$ such that $L \subset \ker((A - \lambda I)_L)$, that is, $(A - \lambda I)(L) \subset L^\perp$. By assumption, we have that $\Delta = \ker(A - \lambda I)$ satisfies $\dim \Delta = n - 2$.

Let $X, Y$ be an orthonormal frame of $\Delta^\perp$ with $X$ orthogonal to $L$. Hence

$$\langle (A - \lambda I)Y, Y \rangle = 0.$$

(77)

We have that $J \in \Gamma(\text{End}(\Delta^\perp))$ defined by $JX = Y$ and $JY = 0$ verifies $J^2 = 0$. It follows from (77) that $(A - \lambda I)J$ is symmetric. Now, since $L$ is umbilical, we have

$$\nabla^h_T Y = 0,$$

(78)

and this is equivalent to $\nabla^h_T J = 0$ for any $T \in \Gamma(\Delta)$. To verify that $C(\Gamma(\Delta)) \subset \text{span}\{I, J\}$ it suffices to prove that $C_T \circ J = J \circ C_T$ for any $T \in \Gamma(\Delta)$. This is equivalent to

$$\langle \nabla_Y T, X \rangle = 0 \text{ and } \langle \nabla_X X, T \rangle = \langle \nabla_Y Y, T \rangle$$

(79)

for all $T \in \Gamma(\Delta)$. The first equation holds since $L$ is umbilical. We have from (77) that

$$(A - \lambda I)Y = \nu X$$

(80)

where $\nu = \langle AX, Y \rangle \neq 0$. From the Codazzi equation we easily obtain

$$\nabla^h_T A = (A - \lambda I)C_T,$$

and hence the right-hand side is symmetric. We have

$$\langle (A - \lambda I)C_T X, Y \rangle = \nu \langle \nabla_X X, T \rangle \text{ and } \langle (A - \lambda I)C_T Y, X \rangle = \nu \langle \nabla_Y Y, T \rangle,$$

from where we obtain

$$\langle \nabla_X X, T \rangle = \langle \nabla_Y Y, T \rangle.$$
for any $T \in \Gamma(\Delta)$. Thus $f$ is parabolic with respect to $J$.

To show that $f$ admits a non-trivial conformal infinitesimal bending it suffices to prove that there is a smooth function $\mu$ such that the tensor $D = \mu J \in \Gamma(\text{End}(\Delta^\perp))$ satisfies all conditions in Proposition 11. We already know that $(A - \lambda I) J$ is symmetric, hence condition $(i)$ is satisfied for any function $\mu$. We assume that $\mu$ is constant along the leaves of $\Delta$, and now condition $(ii)$ follows from (78). From the definition of $D$ it is easy to see that also $(v)$ holds.

Condition $(iii)$ is just (72). We know that (76) holds for any $T \in \Gamma(\Delta)$ of unit-length. Hence and given that $L = \Delta \oplus \text{span}\{Y\}$ is an umbilical distribution, we obtain that (74) holds. But (74) is just the $Y$-component of (72). The $X$-component of (72) is (73), which can be stated as

$$Y(\log \mu \nu) = \langle \nabla_X X, Y \rangle.$$  

Choosing an arbitrary function as initial condition along one maximal integral curve of $X$, there exists a unique function $\mu$ such that $T(\mu) = 0$ for all $T \in \Gamma(\Delta)$ and $\mu \nu$ is a solution of the preceding equation. Therefore, we have as many tensors $D$ satisfying $(iii)$ as smooth functions on an open interval.

We have that

$$\langle (\nabla_Y \mu J) X - (\nabla_X \mu J) Y, \nabla \lambda \rangle = (Y(\mu) - \mu \langle \nabla_X X, Y \rangle) Y(\lambda) + \mu \langle \nabla_Y Y, X \rangle X(\lambda).$$

Choose any $D$ satisfying condition $(iii)$. Then (73) and (74) yield

$$\langle (\nabla_Y \mu J) X - (\nabla_X \mu J) Y, \nabla \lambda \rangle = -\frac{\mu}{\nu} Y(\lambda)(Y(\nu) + X(\lambda)).$$

We have using (74) that

$$\text{Hess} \lambda(\mu J X, Y) - \text{Hess} \lambda(X, \mu J Y) = \mu(Y Y(\lambda) - (\nabla_Y Y, X) X(\lambda))$$

$$= \mu(Y Y(\lambda) + \frac{1}{\nu} Y(\lambda) X(\lambda))$$

and using (81) that

$$\lambda((A - \lambda I) \mu J X, AY) - \langle A X, (A - \lambda I) \mu J Y \rangle = \lambda \mu \nu^2.$$  

The last three equations give that condition $(iv)$ is equivalent to

$$YY(Y) - \frac{1}{\nu} Y(\lambda) Y(\nu) = -\lambda \nu^2.$$  

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that can also be written as

\[ Y((1/\nu)Y(\lambda)) = -\lambda\nu. \]  

(81)

To conclude, we show that (81) is just the Gauss equation

\[ \langle R(Y,T)T,X \rangle = \langle AT,T \rangle \langle AY,X \rangle - \langle AY,T \rangle \langle AT,X \rangle = \lambda\nu. \]

In fact, we have using (76) and (79) that

\[ \langle \nabla_Y \nabla_T T,X \rangle = Y\langle \nabla_T T,X \rangle + \langle \nabla_T Y \rangle \langle \nabla_Y X \rangle \]

\[ = -Y((1/\nu)Y(\lambda)) + \langle \nabla_T Y \rangle \langle \nabla_Y X \rangle. \]

Also

\[ \langle \nabla_T \nabla_Y T,X \rangle = -\langle \nabla_T T,\nabla_T X \rangle = 0. \]

Using (78) we obtain

\[ \langle \nabla_{[Y,T]} T,X \rangle = -\langle \nabla_T \nabla_Y T,X \rangle = \langle \nabla_T Y \rangle \langle \nabla_T T,X \rangle. \]

The last three equations yield

\[ \langle R(Y,T)T,X \rangle = -Y((1/\nu)Y(\lambda)). \]

Now the proof follows from Proposition 11.

Proposition 13. Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 5, \) be a simply connected conformally ruled hypersurface free of points with a principal curvature of multiplicity at least \( n-1 \) and that is not conformally surface-like on any open subset of \( M^n \). Then any conformal infinitesimal bending of \( f \) is the variational vector field of a conformal bending.

Proof: We have seen that the conformal infinitesimal bendings of \( f \) are in one-to-one correspondence with the tensors \( D \) given in the proof of Proposition 12. Take such a \( D \) and let \( F: M^n \to \mathbb{R}^{n+2} \subset \mathbb{L}^{n+3} \) be the immersion \( F = \Psi \circ f \), where \( \Psi \) was given in (30). Let \( \beta: TM \times TM \to N_F M \) and \( \varepsilon: TM \times N_F M \to N_F M \) be given by (48) and (51), respectively. The tensors \( \beta \) and \( \varepsilon \) are associated to an isometric infinitesimal bending of \( F \), say \( \mathcal{T} \), which determines a conformal infinitesimal bending \( \mathcal{T} \) of \( f \). Let \( \alpha^t: TM \times TM \to N_F M, t \in (-\epsilon, \epsilon) \), be the symmetric tensor defined by

\[ \alpha^t(X,Y) = \alpha^F(X,Y) + t\beta(X,Y) \]
for any $X, Y \in \mathfrak{X}(M)$. Since $\mathcal{E}$ satisfies (52) then $\nabla^t_X \eta = \nabla^t_X \eta + t \mathcal{E}(X, \eta)$ is a connection on $N_F M$ that is compatible with the induced metric, where $X \in \mathfrak{X}(M)$, $\eta \in \Gamma(N_F M)$ and $\nabla^t$ denotes the normal connection of $F$.

It follows from (50), (56), (60), (61), (66), (67), (68) together with the Gauss, Codazzi and Ricci equations for $F$ that $\alpha^t$ and $\nabla^t$ verify the Gauss, Codazzi and Ricci equations. Therefore, there is a family of isometric immersions $F_t : M^n \to \mathbb{L}^{n+3}$ with $F_0 = F$ together with vector bundle isometries $\Phi_t : N_F M \to N_{F_t} M$ satisfying

$$\alpha^{F_t} = \Phi_t \alpha^t \quad \text{and} \quad \nabla^t \Phi_t = \Phi_t (\nabla^t)$$

where $\alpha^{F_t}$ and $\nabla^t$ are the second fundamental form and normal connection of $F_t$, respectively. Then, we have

$$A^t_{\Phi_t F} X = -X \quad \text{and} \quad \nabla^t_{\Phi_t F} F = \Phi_t (\nabla^t F) = 0$$

where $A^t_{\eta}$ is the shape operator of $F_t$ in the direction of $\eta \in \Gamma(N_{F_t} M)$. Hence $F_t - \Phi_t F = v_t$ is a constant vector field along $F_t$ for any $t$. Given that $\langle F_t - v_t, F_t - v_t \rangle = 0$, we obtain that $F_t - v_t$ determines an isometric variation of $F_0 = F$ in $\mathcal{V}^{n+2} \subset \mathbb{L}^{n+3}$. Hence, we assume that $F_t(x) \in \mathcal{V}^{n+2}$ for all $x \in M^n$. The variational vector field $T' = \partial/\partial t|_{t=0} F_t$ is clearly an infinitesimal bending of $F$ and the tensor $\beta'$ associated to $T'$ satisfies

$$\beta' = (\partial/\partial t|_{t=0} \alpha^{F_t})_{N_F M}$$

(see the proof of Proposition 7 in [6]). Since $\alpha^{F_t} = \Phi_t (\alpha + t \beta)$ then

$$\beta' = \beta + \Phi' \alpha^F$$

where $\Phi' = \partial/\partial t|_{t=0} \Phi_t \in \Gamma(\text{End}(N_F M))$ is skew symmetric.

Let $\Pi : \mathcal{V}^{n+2} \setminus \mathbb{R}^n \to \mathbb{E}^{m+1} = \Psi(\mathbb{R}^{n+1})$ be the map $\Pi(u) = (1/\langle u, v \rangle) u$. Then each $F_t$ induces an immersion $f_t : M^n \to \mathbb{R}^{n+1}$ such that $\Psi \circ f_t = \Pi \circ F_t$ for any $t$. Observe that the metric induced by $f_t$ satisfies

$$\langle f_t \ast X, f_t \ast Y \rangle(x) = \langle (\Pi \circ F_t) \ast X, (\Pi \circ F_t) \ast Y \rangle(x) = \langle F_t(x), w \rangle^{-2} \langle X, Y \rangle(x)$$

at any $x \in M^n$. Hence, the variation $f_t$ determines a conformal bending of $f$ in $\mathbb{R}^{n+1}$. The variational vector field $T'$ is a conformal infinitesimal bending of $f$ with associated tensor $B' = B_N - \langle \Phi' w, N \rangle I$. Hence $T - T'$ is trivial, and this concludes the proof.
Proof of Theorem. If \( f \) is conformally infinitesimally bendable and not conformally surface-like, we have from Proposition 11 and Proposition 12 that \( f \) is either hyperbolic or elliptic. The proof of Proposition 11 gives that \( D = \mu J \) is the lifting of a tensor \( \bar{D} = \bar{\mu} \bar{J} \) on \( L^2 \). Also from that proof we obtain that

\[
g_*((\nabla'_Y \bar{D})\bar{X} - (\nabla'_X \bar{D})\bar{Y}) + \alpha^g(\bar{Y}, \bar{D}\bar{X}) - \alpha^g(\bar{X}, \bar{D}\bar{Y}) = \Psi_*\mathcal{f}_*\mathcal{\Omega}(X, Y) - \lambda \psi(X, Y)\Psi \circ f + \psi(X, Y)w
\]

(82)

where \( X, Y \in \Gamma(\Delta^\perp) \) are the liftings of \( \bar{X}, \bar{Y} \in \mathcal{X}(L^2) \) and \( \Omega, \psi \) are given by (45), (46) and (47) respectively. Recall that \( D \) satisfies the conditions (i) to (iv). Therefore, we have

\[
(\nabla'_X \bar{D})\bar{Y} = (\nabla'_Y \bar{D})\bar{X}
\]

(83)

and, since \( \bar{D} = \bar{\mu} \bar{J} \), that

\[
\alpha^g(\bar{X}, \bar{J}\bar{Y}) = \alpha^g(\bar{J}\bar{X}, \bar{Y}).
\]

Finally, that \( g \) is a special hyperbolic or elliptic surface follows from Proposition 7 and the integrability condition of \( \bar{\mu} \) in (83).

Conversely, let \( f: M^n \to \mathbb{R}^{n+1} \) be parametrized by the conformal Gauss parametrization in terms of a special hyperbolic or a special elliptic pair. Then \( f \) has a nowhere vanishing principal curvature \( \lambda(x) \) at \( x \in M^n \) of constant multiplicity \( n - 2 \) and corresponding eigenspace \( \Delta(x) \). Take \( v = e_1, w = -2e_{n+3} \) and let \( \Psi: \mathbb{R}^{n+1} \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3} \) be the embedding given by (30). Then \( S: M^n \to S^{n+2} \) given by (32) induces a map \( s: L^2 \to S^{n+2} \) on the (local) space of leaves \( L^2 \) of \( \Delta \). Moreover, by the choice of \( v \) and \( w \) we have that \( s = g \).

We obtain from Proposition 7 that, at least locally, there is a nowhere vanishing function \( \bar{\mu} \in C^\infty(L^2) \) such that \( \bar{D} = \bar{\mu} \bar{J} \) is a Codazzi tensor. Let \( X, Y \in \Gamma(\Delta^\perp) \) be the liftings of \( \bar{X}, \bar{Y} \in \mathcal{X}(L) \). If \( D = \mu J \) is the lifting of \( \bar{D} \) we have as before that (82) holds. Given that \( g \) is special hyperbolic or special elliptic, we have that \( \Omega = \psi = \varphi = 0 \). In other words, we obtain that conditions (i) (iii) and (iv) are satisfied.

We recall that

\[
\langle (\bar{D}\bar{X} \wedge \bar{Y} - \bar{D}\bar{X} \wedge \bar{Y})\bar{Z}, \bar{W} \rangle' = 0
\]
for any $\bar{X}, \bar{Y}, \bar{Z}, \bar{W} \in \mathfrak{X}(L)$. It follows from (34) that

$$\langle (A - \lambda I)DX \wedge (A - \lambda I)Y - (A - \lambda I)DY \wedge (A - \lambda I)X,(A - \lambda I)Z,(A - \lambda I)W \rangle = 0$$

where $X, Y, Z, W \in \Gamma(\Delta^\perp)$ are the liftings of $\bar{X}, \bar{Y}, \bar{Z}$ and $\bar{W}$. Then

$$(A - \lambda I)DX \wedge (A - \lambda I)Y - (A - \lambda I)DY \wedge (A - \lambda I)X = 0$$

for any $X, Y \in \Gamma(\Delta^\perp)$, and hence (v) holds. Given that $D$ is projectable it follows from Corollary 11.7 of [8] that $\nabla^h_T D = [D, C_T] = 0$ for all $T \in \Gamma(\Delta)$. Hence (ii) holds. Now the proof follows from Proposition 11.

**Remark 14.** In order to obtain in terms of the conformal Gauss parametrization that a non-trivial infinitesimal conformal bending is, in fact, a conformal bending one has to require the special hyperbolic or special elliptic surface to satisfy a strong additional condition, namely, that $\Gamma^1_u = \Gamma^2_v = 2\Gamma^1 \Gamma^2$ in the former case and $\Gamma_z = 2\Gamma \bar{\Gamma}$ in the latter case, see [7] or [8].

**Proof of Theorem 2:** The proof follows from Propositions 12 and 13.

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