\( \mathcal{N} = 4 \) Supersymmetric Quantum Mechanical Model: Novel Symmetries

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**Abstract:** We discuss a set of novel discrete symmetry transformations of the \( \mathcal{N} = 4 \) supersymmetric quantum mechanical model of a charged particle moving on a sphere in the background of Dirac magnetic monopole. The usual *five* continuous symmetries (and their conserved Noether charges) and *two* discrete symmetries *together* provide the physical realizations of the de Rham cohomological operators of differential geometry. We have also exploited the supervariable approach to derive the nilpotent \( \mathcal{N} = 4 \) SUSY transformations and provided the geometrical interpretation in the language of translational generators along the Grassmannian directions \( \theta^a \) and \( \bar{\theta}^a \) onto (1, 4)-dimensional supermanifold.

PACS numbers: 11.30.Pb, 03.65.-w, 02.40.-k

*Keywords:* \( \mathcal{N} = 4 \) SUSY QM algebra; continuous and discrete symmetries; de Rham cohomological operators; Hodge theory; supervariable approach; nilpotency property
1 Introduction

It is a well-known fact that three out of four fundamental interactions of nature are theoretically described by the gauge theories. These theories are characterized by the local gauge symmetries at the classical level which are generated by the first-class constraints in the language of Dirac’s prescription for the classification scheme [1,2]. Some of the gauge theories provide the physical examples of the Hodge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism where the local gauge symmetries of the classical theory are traded with the nilpotent (anti-) BRST and (anti-)co-BRST symmetries at the quantum level. In a recent paper (see, e.g. [3] for more details), we have shown that any Abelian p-form ($p = 1, 2, 3, ...$) gauge theory is a tractable model for the Hodge theory in $D = 2p$ dimensions of spacetime within the framework of BRST formalism.

In an earlier work [4], the 2D free (non-)Abelian 1-form gauge theories (without any interaction with matter fields) have been studied and they have been shown to be a new class of topological field theories TFTs which capture some salient feature of Witten type and a few key feature of Schwarz-type TFTs (see, e.g. [4-7] for more details). Furthermore, it has been shown that the 2D Abelian $U(1)$ gauge theory, interacting with Dirac fields [8,9], is a perfect model for the Hodge theory within framework of BRST formalism. In such kind of studies, we have shown that the 2D modified version of Proca theory and 6D Abelian 3-form gauge theory [10,11] are also the perfect examples of the Hodge theory. In a very recent set of papers (see, e.g. [12-14] for more details), a collection of $\mathcal{N} = 2$ supersymmetric quantum mechanical models have also been shown to represent the models for the Hodge theory.

In our earlier works [15-17], we have applied supervariable approach for the derivation of supersymmetric (SUSY) transformations for the $\mathcal{N} = 2$ SUSY quantum mechanical models (QMMs) which is a novel approach in the context of SUSY theories. We have established that the $\mathcal{N} = 2$ SUSY QMMs also provide a set of tractable physical examples of the Hodge theory because their continuous symmetries (and conserved Noether charges) provide the physical realizations of the de Rham cohomological operators of differential geometry [18-22] and the discrete symmetry of the theory turns out to be the analogue of Hodge duality operation. It has been demonstrated that the algebra of the continuous symmetries (and their conserved Noether charges) for the $\mathcal{N} = 2$ SUSY QMMs is exactly similar to the Hodge algebra obeyed by the de Rham cohomological operators of differential geometry.

In a very recent set of our works [23,24], we have shown that the free version as well as interacting $\mathcal{N} = 2$ SUSY QM model of a charged particle moving on a sphere (in the background Dirac magnetic monopole) provide a set of physical examples of the Hodge theory. This model has also been studied by others [25,26] in a different context. We have also shown in our works [23,24] that one can provide the geometrical meaning to $\mathcal{N} = 2$ SUSY transformations in the language of translational generators ($\partial_\theta, \partial_{\bar{\theta}}$) along the Grassmannian directions ($\theta, \bar{\theta}$) of the (1, 2)-dimensional super-submanifolds on which the
differential operators ($d, \delta, \Delta$) which are called cohomological operators of differential geometry. These three de Rham cohomological operators obey the following algebra: $d^2 = \delta^2 = 0, \Delta = (d + \delta)^2 = \{d, \delta\}, [\Delta, d] = [\Delta, \delta] = 0$ where ($\delta$) is the (co-)exterior derivatives and $\Delta$ is the Laplacian operator. The exterior and co-exterior derivatives together satisfy an interesting relationship: $d = \pm \ast \delta \ast$ where $\ast$ is the Hodge duality operation.
ordinary $\mathcal{N} = 2$ SUSY quantum theory [25] is generalized. We have also shown that our $\mathcal{N} = 2$ SUSY QMMs are the physical examples of the Hodge theory.

The main motivations of our present investigation are as follows. First, we shall prove that the $\mathcal{N} = 4$ SUSY quantum mechanical model of a charged particle on a sphere, in the background of Dirac magnetic monopole is a perfect model for the Hodge theory. Second, we shall provide the physical realizations of the de Rham cohomological operators of differential geometry in the language continuous symmetries (and their conserved Noether charges) and a set of novel discrete symmetries of our present theory. Finally, we shall apply supervariable approach for the derivation of SUSY transformations by exploiting chiral and anti-chiral SUSY invariant restrictions and show the invariance of the Lagrangian (and its geometrical interpretation in the language translational generators $(\partial_{g_{\alpha}}, \partial_{\bar{g}_{\alpha}})$ along the Grassmannian directions $(\theta^\alpha, \bar{\theta}^\alpha)$ of the (anti-)chiral super-submanifolds, in our present SUSY theory).

The contents of our present endeavor are as follows. In section 2, we discuss the continuous symmetries (and their conserved Noether charges) of the Lagrangian for the $\mathcal{N} = 4$ SUSY QM model of a charged particle moving on a sphere in the background Dirac magnetic monopole. Our section 3 is devoted to the discussion of a set of novel discrete symmetry transformations. In section 4, we lay emphasis on the algebraic structure of the $\mathcal{N} = 4$ SUSY symmetries and corresponding conserved charges. Our section 5 is devoted to the derivation for the $\mathcal{N} = 4$ SUSY transformations $(s_{\alpha}, \bar{s}_{\alpha})$ by exploiting SUSY invariant restrictions within the framework of supervariable approach. Finally, we make some concluding remarks in our section 6.

In our Appendix A, we discuss the explicit computations of $\mathcal{N} = 4$ SUSY QM algebra for the generators $(Q_{\alpha}, \bar{Q}_{\alpha})$ and corresponding Hamiltonian $H$ by exploiting the symmetry transformations in our present SUSY theory. We provide the key difference between the (anti-)BRST symmetries and the $\mathcal{N} = 4$ SUSY transformations in our Appendix B.

**Notations and Convention:** We adopt the following notations and convention of the Grassmannian variables $\theta^\alpha$ and $\bar{\theta}^\alpha$ such as: $\{\theta^\alpha, \theta^\beta\} = 0 = (\theta^\alpha)^2 \Rightarrow (\theta^1)^2 = (\theta^2)^2 = 0$, $\{\bar{\theta}^\alpha, \bar{\theta}^\beta\} = 0 = (\bar{\theta}^\alpha)^2 \Rightarrow (\bar{\theta}_1)^2 = (\bar{\theta}_2)^2 = 0$, $\{\theta^\alpha, \bar{\theta}^\beta\} = 0$. Similarly, $\{\partial_{g_{\alpha}}, \partial_{\bar{g}_{\beta}}\} = 0 = (\partial_{g_{\alpha}})^2 \Rightarrow (\partial_{g_1})^2 = (\partial_{g_2})^2 = 0$, $\{\partial_{\bar{g}_{\alpha}}, \partial_{g_{\beta}}\} = 0 = (\partial_{\bar{g}_{\alpha}})^2 \Rightarrow (\partial_{\bar{g}_1})^2 = (\partial_{\bar{g}_2})^2 = 0$, $\{\partial_{\bar{g}_{\alpha}}, \partial_{\bar{g}_{\beta}}\} = 0$ where $\partial_{g_{\alpha}} = \frac{\partial}{\partial g^\alpha}$, $\partial_{\bar{g}_{\alpha}} = \frac{\partial}{\partial \bar{g}^\alpha}$.

## 2 Preliminaries: $\mathcal{N} = 4$ SUSY Symmetries

Let us begin with the Lagrangian for the $\mathcal{N} = 4$ SUSY quantum mechanical model of the motion of an electron on a sphere in the background of Dirac magnetic monopole based on the $CP^{(1)}$-model approach (see, e.g. [26] for more details)

$$L = 2(D_t \bar{z} \cdot (D_t z) + \frac{i}{2} \left[ \bar{\psi}_\alpha \cdot (D_t \psi_\alpha) - (D_t \bar{\psi}_\alpha) \cdot \psi_\alpha \right]$$

$$\quad - \frac{1}{4} \left[ (\epsilon_{\alpha\beta} \bar{\psi}_\alpha \cdot \psi_\beta)^2 + (\bar{\psi}_\alpha \cdot \psi_\alpha)^2 \right] - 2 g a,$$  \hspace{1cm} (1)

where the covariant derivatives are defined as: $D_t \bar{z} = (\partial_t + i a) \bar{z}$, $D_t z = (\partial_t - i a) z$, $D_t \bar{\psi}_\alpha = (\partial_t + i a) \bar{\psi}_\alpha$, $D_t \psi_\alpha = (\partial_t - i a) \psi_\alpha$. Here $a$ is the “gauge” variable and $t$ is the evolution...
parameter (with \( \partial_t = \partial / \partial t \)) of our present SUSY theory. The dynamical variables \( z \) and \( \bar{z} \) are bosonic in nature and the variables \( \psi_\alpha \) and \( \bar{\psi}_\alpha \) are the fermionic in nature (i.e. \( \psi_\alpha^2 = \bar{\psi}_\alpha^2 = 0, \psi_\alpha \cdot \bar{\psi}_\beta + \bar{\psi}_\beta \cdot \psi_\alpha = 0 \)) at the classical level (with \( \alpha, \beta, \ldots = 1, 2 \)). The parameter \( g \) stands for the charge on the magnetic monopole (with mass \( m = 1 \)) and charge of the electron is taken to be \( e = -1 \). We adopt the following conventions of the dot product between two bosonic variables \((\bar{z}_i, z_j)\) and two fermionic variables \(\bar{\psi}_{\alpha i}, \psi_{\alpha j}\) \((i, j = 1, 2; \alpha, \beta, \ldots = 1, 2)\) are as follows:

\[
\begin{align*}
\bar{z}_i &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, & \bar{z}_j &= \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \implies \bar{z} \cdot z &= \bar{z}_1 z_1 + \bar{z}_2 z_2, \\
\bar{\psi}_{\alpha i} \psi_{\alpha j} &\implies \bar{\psi}_\alpha \cdot \psi_\alpha = \bar{\psi}_1 \cdot \psi_1 + \bar{\psi}_2 \cdot \psi_2.
\end{align*}
\]

(2)

Here the fermionic variables \( \bar{\psi}_{\alpha i} \) and \( \psi_{\alpha j} \) satisfy the following properties: \((\bar{\psi}_\alpha \cdot \psi_\alpha)^2 = (\bar{\psi}_1 \cdot \psi_1 + \bar{\psi}_2 \cdot \psi_2)^2 \equiv 2(\bar{\psi}_1 \cdot \psi_1)(\bar{\psi}_2 \cdot \psi_2)\) and \((\epsilon_{\alpha \beta} \bar{\psi}_\alpha \cdot \psi_\beta)^2 = (\bar{\psi}_1 \cdot \psi_2 - \bar{\psi}_2 \cdot \psi_1)^2 \equiv -2(\bar{\psi}_1 \cdot \psi_2)(\bar{\psi}_2 \cdot \psi_1)\) (with \((\bar{\psi}_1 \cdot \psi_1)^2 = (\bar{\psi}_2 \cdot \psi_2)^2 = 0\) \((\bar{\psi}_1 \cdot \psi_2)^2 = (\bar{\psi}_2 \cdot \psi_1)^2 = 0\)) at the classical level.

The infinitesimal, continuous and nilpotent \((s_\alpha^2 = \bar{s}_\alpha^2 = 0)\) \(\mathcal{N} = 4\) SUSY transformations \((s_\alpha, \bar{s}_\alpha)\) of the Lagrangian (1) are

\[
\begin{align*}
s_\alpha z &= \frac{\psi_\alpha}{\sqrt{2}}, & s_\alpha \psi_\beta &= 0, & s_\alpha \bar{\psi}_\beta &= \frac{2i}{\sqrt{2}} \nabla_{\alpha \beta} \bar{z}, & s_\alpha \bar{z} &= 0, \\
s_\alpha (D_t z) &= \frac{D_t \psi_\alpha}{\sqrt{2}}, & s_\alpha (D_t \bar{z}) &= 0, & s_\alpha a &= 0, \end{align*}
\]

\[
\begin{align*}
\bar{s}_\alpha \bar{z} &= \frac{\bar{\psi}_\alpha}{\sqrt{2}}, & \bar{s}_\alpha \bar{\psi}_\beta &= 0, & \bar{s}_\alpha \psi_\beta &= \frac{2i}{\sqrt{2}} \nabla_{\alpha \beta} \bar{z}, & \bar{s}_\alpha z &= 0, \\
\bar{s}_\alpha (D_t \bar{z}) &= \frac{D_t \bar{\psi}_\alpha}{\sqrt{2}}, & \bar{s}_\alpha (D_t z) &= 0, & \bar{s}_\alpha a &= 0,
\end{align*}
\]

(3)

where,

\[
\begin{align*}
\nabla_{\alpha \beta} \bar{z} &= \delta_{\alpha \beta} D_t \bar{z} - \frac{i}{2} (\bar{\psi}_\beta \cdot \psi_\alpha - \delta_{\alpha \beta} \bar{\psi}_\gamma \cdot \psi_\gamma) \bar{z}, \\
\nabla_{\alpha \beta} z &= \delta_{\alpha \beta} D_t z + \frac{i}{2} (\psi_\beta \cdot \bar{\psi}_\alpha - \delta_{\alpha \beta} \psi_\gamma \cdot \bar{\psi}_\gamma) z.
\end{align*}
\]

(4)

The gauge variable \( a \) is defined as

\[
a = -\frac{i}{2} (\bar{z} \cdot \bar{z} - \bar{z} \cdot z) - \frac{1}{2} (\bar{\psi}_\alpha \cdot \psi_\alpha),
\]

(5)

which is invariant under the \( \mathcal{N} = 4\) SUSY transformations \( s_\alpha \) and \( \bar{s}_\alpha \) (i.e. \( s_\alpha a = \bar{s}_\alpha a = 0 \)) due to the following constraints (see, e.g. \cite{26} for more details), namely;

\[
\begin{align*}
\bar{z} \cdot z &= 1, & \bar{z} \cdot \psi_\alpha &= 0, & \bar{\psi}_\alpha \cdot z &= 0.
\end{align*}
\]

(6)

The Lagrangian (1) transforms to the total time derivatives as

\[
\begin{align*}
s_\alpha L &= \frac{d}{dt} \left[ \frac{(D_t \bar{z}) \cdot \psi_\alpha}{\sqrt{2}} \right], & \bar{s}_\alpha L &= \frac{d}{dt} \left[ \frac{\bar{\psi}_\alpha \cdot (D_t z)}{\sqrt{2}} \right].
\end{align*}
\]

(7)
which demonstrates the invariance of the action integral $S = \int dt \, L$.

We obtain the bosonic symmetry ($s_{\alpha \beta}^\omega$) for the $\mathcal{N} = 4$ SUSY transformations $s_\alpha$ and $s_\alpha$ which is nothing but the anticommutator of $s_\alpha$ and $s_\alpha$ (i.e. $s_{\alpha \beta}^\omega = \{s_\alpha, s_\beta\}$). The bosonic symmetry of the dynamical variables $z, \bar{z}, \psi_\alpha$ and $\bar{\psi}_\alpha$ are

$$s_{\alpha \beta}^\omega z = \nabla_{\alpha \beta} z \equiv \delta_{\alpha \beta} D_t z + \frac{i}{2} (\bar{\psi}_\alpha \cdot \psi_\beta - \delta_{\alpha \beta} \bar{\psi}_\gamma \cdot \psi_\gamma) z,$$

$$s_{\alpha \beta}^\omega \bar{z} = \nabla_{\alpha \beta} \bar{z} \equiv \delta_{\alpha \beta} D_t \bar{z} - \frac{i}{2} (\bar{\psi}_\beta \cdot \psi_\alpha - \delta_{\alpha \beta} \bar{\psi}_\gamma \cdot \psi_\gamma) \bar{z},$$

$$s_{\alpha \beta}^\omega \psi_\gamma = \nabla_{\alpha \beta} \psi_\gamma \equiv \delta_{\alpha \beta} D_t \psi_\gamma + \frac{i}{2} (\bar{\psi}_\alpha \cdot \psi_\beta - \delta_{\alpha \beta} \bar{\psi}_\rho \cdot \psi_\rho) \psi_\gamma,$$

$$s_{\alpha \beta}^\omega \bar{\psi}_\gamma = \nabla_{\alpha \beta} \bar{\psi}_\gamma \equiv \delta_{\alpha \beta} D_t \bar{\psi}_\gamma - \frac{i}{2} (\bar{\psi}_\beta \cdot \psi_\alpha - \delta_{\alpha \beta} \bar{\psi}_\rho \cdot \psi_\rho) \bar{\psi}_\gamma,$$

modulo a factor of $(i)$. Under the above $\mathcal{N} = 4$ SUSY transformations $(8)$, the starting Lagrangian $L$ transforms to a total time derivative as:

$$s_{\alpha \beta}^\omega L = \frac{d}{dt} [\delta_{\alpha \beta}(L + 2ga)] \equiv \delta_{\alpha \beta} \frac{d}{dt} \left[ 2(D_t \bar{z}) \cdot (D_t z) + \frac{i}{2} \{\bar{\psi}_\gamma \cdot (D_t \psi_\gamma) - (D_t \bar{\psi}_\gamma) \cdot \psi_\gamma \} - \frac{1}{4} \{(\epsilon_{\gamma \rho} \bar{\psi}_\gamma \cdot \psi_\rho)^2 + (\bar{\psi}_\gamma \cdot \psi_\gamma)^2 \} \right].$$

As a consequence, the corresponding action ($S = \int dt \, L$) remains invariant under the above bosonic symmetry transformations ($s_{\alpha \beta}^\omega$) of the $\mathcal{N} = 4$ SUSY QM model.

According to Noether’s theorem, the above continuous symmetry transformations ($s_\alpha, s_\alpha, s_{\alpha \beta}^\omega$) lead to the derivation of the following conserved charges

$$Q_\alpha = \frac{\Pi_z \cdot \psi_\alpha}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left[ 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) \bar{z} + 2 i a \bar{z} (1 - \bar{z} \cdot z) \right] \cdot \psi_\alpha \equiv \frac{1}{\sqrt{2}} \left[ 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) \bar{z} \right] \cdot \psi_\alpha,$$

$$Q_\alpha = \frac{\bar{\psi}_\alpha \cdot \Pi_{\bar{z}}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \bar{\psi}_\alpha \cdot \left[ 2 D_t z - \frac{i}{2} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) z - 2 i a z (1 - \bar{z} \cdot z) \right] \equiv \frac{1}{\sqrt{2}} \bar{\psi}_\alpha \cdot \left[ 2 D_t z - \frac{i}{2} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) z \right],$$

$$Q_{\alpha \beta}^\omega = \delta_{\alpha \beta} \left[ 2(D_t \bar{z}) \cdot (D_t z) - g(\bar{\psi}_\gamma \cdot \psi_\gamma) + \frac{1}{4} \{(\epsilon_{\gamma \rho} \bar{\psi}_\gamma \cdot \psi_\rho)^2 - (\bar{\psi}_\gamma \cdot \psi_\gamma)^2} \right] \equiv \delta_{\alpha \beta} H,$$

where $H$ is the Hamiltonian of our $\mathcal{N} = 4$ SUSY QM model. The above Hamiltonian can be derived from the Legendre transformations as:

$$H = \Pi_{\bar{z}} \cdot \dot{\bar{z}} + \Pi_z \cdot \dot{z} - \Pi_{\psi_\alpha} \cdot \dot{\psi}_\alpha + \Pi_{\bar{\psi}_\alpha} \cdot \dot{\bar{\psi}_\alpha} - L \equiv 2(D_t \bar{z}) \cdot (D_t z) - g(\bar{\psi}_\gamma \cdot \psi_\gamma) + \frac{1}{4} \{(\epsilon_{\alpha \beta} \bar{\psi}_\alpha \cdot \psi_\beta)^2 - (\bar{\psi}_\alpha \cdot \psi_\alpha)^2} \right].$$
It is elementary to check that the above canonical momenta $\Pi_z, \Pi_{\bar{z}}, \Pi_{\psi_\alpha}$ and $\Pi_{\bar{\psi}_\alpha}$ w.r.t. the dynamical variables $z, \bar{z}, \psi_\alpha$ and $\bar{\psi}_\alpha$ of the Lagrangian (1) turn out to be

$$
\Pi_z = \frac{\partial L}{\partial \dot{z}} = 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi}_\alpha \cdot \psi_\alpha + 2 g) \bar{z} + 2 i a \bar{z} (1 - \bar{z} \cdot z),
$$

$$
\Pi_{\bar{z}} = \frac{\partial L}{\partial \dot{\bar{z}}} = 2 D_t z - \frac{i}{2} (\bar{\psi}_\alpha \cdot \psi_\alpha + 2 g) z - 2 i a z (1 - \bar{z} \cdot z),
$$

$$
\Pi_{\psi_\alpha} = \frac{\partial L}{\partial \dot{\psi}_\alpha} = -\frac{i}{2} \bar{\psi}_\alpha, \quad \Pi_{\bar{\psi}_\alpha} = \frac{\partial L}{\partial \dot{\bar{\psi}}_\alpha} = -\frac{i}{2} \psi_\alpha,
$$

where we have adopted the convention of left derivative w.r.t. the fermionic variables $\psi_\alpha$ and $\bar{\psi}_\alpha$ in the computation of $\Pi_{\psi_\alpha}$ and $\Pi_{\bar{\psi}_\alpha}$, respectively.

The conservation law (i.e. $Q = \dot{Q} = Q_{\alpha \beta} = 0$) can be proven by exploiting the following equations of motion (with the constraints $\bar{z} \cdot z = 1, \bar{z} \cdot \psi_\alpha = 0, \bar{\psi}_\alpha \cdot z = 0$) that emerge from the Lagrangian (1) of our $\mathcal{N} = 4$ SUSY theory, namely:

$$
\frac{d\Pi_{\bar{z}}}{dt} - i \left[ 2 a D_t z + \frac{\bar{z}}{2} (\bar{\psi}_\alpha \cdot \psi_\alpha + 2 g) \right] = 0,
$$

$$
\frac{d\Pi_{\bar{z}}}{dt} + i \left[ 2 a D_t \bar{z} + \frac{\bar{z}}{2} (\bar{\psi}_\alpha \cdot \psi_\alpha + 2 g) \right] = 0,
$$

$$
D_t \psi_\alpha + \frac{i}{2} (\epsilon_{\gamma \rho} \bar{\psi}_\gamma \cdot \psi_\rho) (\epsilon_{\alpha \beta} \psi_\beta) - i g \psi_\alpha = 0,
$$

$$
D_t \bar{\psi}_\alpha + \frac{i}{2} (\epsilon_{\gamma \rho} \bar{\psi}_\gamma \cdot \psi_\rho) (\epsilon_{\alpha \beta} \bar{\psi}_\beta) + i g \bar{\psi}_\alpha = 0,
$$

where canonical conjugate momenta $\Pi_{\bar{z}}$ and $\Pi_{\bar{z}}$ in the above equation are from (12) of the $\mathcal{N} = 4$ SUSY quantum mechanical system.

## 3 Novel Discrete Symmetry Transformations

It is straightforward to check that under the following discrete symmetry transformations

$$
z \rightarrow \mp \bar{z}, \quad \bar{z} \rightarrow \mp z, \quad \psi_\alpha \rightarrow \mp \bar{\psi}_\alpha, \quad \bar{\psi}_\alpha \rightarrow \pm \psi_\alpha,
$$

$$
t \rightarrow - t, \quad a \rightarrow - a, \quad g \rightarrow g,
$$

the Lagrangian (1) remains invariant. The time-reversal (i.e. $t \rightarrow - t$) symmetry implies:

$$
z \rightarrow \mp \bar{z} \Rightarrow z(t) \rightarrow z(-t) = \mp \bar{z}^T(t), \quad \bar{z} \rightarrow \mp z \Rightarrow \bar{z}(t) \rightarrow \bar{z}(-t) = \mp z^T(t), \quad \psi_\alpha \rightarrow \mp \bar{\psi}_\alpha \Rightarrow \psi_\alpha(t) \rightarrow \psi_\alpha(-t) = \mp \bar{\psi}_\alpha^T(t), \quad \bar{\psi}_\alpha \rightarrow \psi_\alpha \Rightarrow \bar{\psi}_\alpha(t) \rightarrow \bar{\psi}_\alpha(-t) = \pm \psi_\alpha^T(t), \quad a(t) \rightarrow a(-t) = -a(t),$$

where the superscript $T$ denotes the transpose operations on the dynamical variables.

The above set of discrete symmetry transformations are the novel useful symmetries because they establish a set of connections between the $\mathcal{N} = 4$ SUSY symmetry transformations $s_\alpha$ and $\bar{s}_\alpha$ as

$$
\bar{s}_\alpha = \pm * s_\alpha *.
$$
where \( \ast \) is the discrete symmetry transformations. The \((\pm)\) signs in the above equation are governed by two successive operations on the generic variable \( \Phi = z, \bar{z}, \psi_\alpha, \bar{\psi}_\alpha \)

\[
(\ast (\ast \Phi)) = \pm \Phi. 
\]

(16)

It can be explicitly checked that

\[
(\ast (\ast \Phi_1)) = + \Phi_1, \quad \Phi_1 = z, \bar{z}, \\
(\ast (\ast \Phi_2)) = - \Phi_2, \quad \Phi_2 = \psi_\alpha, \bar{\psi}_\alpha. 
\]

(17)

Thus, we obtain the following relationships among the continuous symmetry transformations \((s_\alpha, \bar{s}_\alpha)\) and the discrete symmetry \((\ast)\) for the \(\mathcal{N} = 4\) SUSY quantum mechanical system are

\[
\bar{s}_\alpha \Phi_1 = + \ast s_\alpha \ast \Phi_1 \Rightarrow \bar{s}_\alpha = + \ast s_\alpha *, \quad \Phi_1 = z, \bar{z}, \\
\bar{s}_\alpha \Phi_2 = - \ast s_\alpha \ast \Phi_2 \Rightarrow \bar{s}_\alpha = - \ast s_\alpha *, \quad \Phi_2 = \psi_\alpha, \bar{\psi}_\alpha, 
\]

(18)

and it can be easily checked that their reciprocal relationships are also true, namely;

\[
s_\alpha \Phi_1 = - \ast \bar{s}_\alpha \ast \Phi_1 \Rightarrow s_\alpha = - \ast \bar{s}_\alpha *, \quad \Phi_1 = z, \bar{z}, \\
s_\alpha \Phi_2 = + \ast \bar{s}_\alpha \ast \Phi_2 \Rightarrow s_\alpha = + \ast \bar{s}_\alpha *, \quad \Phi_2 = \psi_\alpha, \bar{\psi}_\alpha. 
\]

(19)

The above relationships (15), (18) and (19) are the analogues of the relationship \(\delta = \pm \ast d\ast\) of differential geometry where \(d = dt \partial_t (d^2 = 0)\) is the exterior derivative, \(\delta\) (with \(\delta^2 = 0\)) is the co-exterior derivative and \((\ast)\) is the Hodge duality operation on a given compact manifold. In our \(\mathcal{N} = 4\) SUSY QM model, the discrete symmetry \((\ast)\) transformation is the analogue of the the Hodge duality operation \((\ast)\).

In addition to the discrete symmetry transformations (14), the Lagrangian (1) remains invariant under the following discrete symmetry transformations

\[
z \rightarrow \pm i \bar{z}, \quad \bar{z} \rightarrow \mp i z, \quad \psi_\alpha \rightarrow \pm i \bar{\psi}_\alpha, \quad \bar{\psi}_\alpha \rightarrow \pm i \psi_\alpha, \\
t \rightarrow - t, \quad a \rightarrow + a, \quad g \rightarrow g, 
\]

(20)

which obey all the conditions that have been satisfied by (14). Thus, these discrete symmetries are also useful, in our present theory.

Furthermore, under another discrete symmetry transformations

\[
z \rightarrow \pm \bar{z}, \quad \bar{z} \rightarrow \pm z, \quad \psi_\alpha \rightarrow \pm \bar{\psi}_\alpha, \quad \bar{\psi}_\alpha \rightarrow \pm \psi_\alpha, \\
t \rightarrow + t, \quad a \rightarrow - a, \quad g \rightarrow - g, 
\]

(21)

the Lagrangian (1) remains unchanged. But, these symmetries are not acceptable to us because they do not comply with the strictures laid down by the duality invariant theories [27].

In the above discrete symmetry (21), it can be checked that

\[
(\ast (\ast z)) = z, \quad (\ast (\ast \bar{z})) = \bar{z}, \quad (\ast (\ast \psi_\alpha)) = \bar{\psi}_\alpha, \quad (\ast (\ast \bar{\psi}_\alpha)) = \psi_\alpha. 
\]

(22)
In view of the above equation (22), we can verify that the following is true:

\[ s_2 \Phi = + * s_1 * \Phi, \quad \Phi = z, \bar{z}, \psi_\alpha, \bar{\psi}_\alpha. \]  

(23)

However, we note that the reciprocal relation

\[ s_1 \Phi = - * s_2 * \Phi, \quad \Phi = z, \bar{z}, \psi_\alpha, \bar{\psi}_\alpha. \]  

(24)

is not satisfied at all by the above discrete symmetries. Thus, the discrete symmetry transformations (21) of the Lagrangian (1) are not acceptable because they do not satisfy all the conditions (e.g. reciprocal relationship (24)) laid down by the duality invariant theories [27].

The conserved charges \((Q_\alpha, \bar{Q}_\alpha, Q^{\alpha\beta} \equiv \delta_{\alpha\beta} H)\) under the discrete symmetry transformations transform as:

\[
\begin{align*}
* Q_\alpha &= - \bar{Q}_\alpha, \\
* (\bar{Q}_\alpha) &= - Q_\alpha, \\
* H &= H,
\end{align*}
\]

(25)

As a consequence, under the discrete symmetries the conserved charges \(Q_\alpha\) and \(\bar{Q}_\alpha\) transform as: \(Q_\alpha \to -\bar{Q}_\alpha, \bar{Q}_\alpha \to +Q_\alpha\) which is like the duality transformations in the electrodynamics where we have: \(B \to -E, E \to +B\) for the electric and magnetic fields present in source free Maxwell’s equations. Furthermore, these charges \((Q_\alpha, \bar{Q}_\alpha)\) and the corresponding Hamiltonian \((H)\) remain invariant under two successive \((*)\) operations corresponding to the discrete symmetry transformations (14) and (20). The above charges \(Q_\alpha\) and \(\bar{Q}_\alpha\) are the fermionic in nature and they obey the following \(\mathcal{N} = 4\) SUSY QM algebra:

\[
\begin{align*}
\{Q_\alpha, Q_\beta\} &\equiv Q^2_\alpha = 0 \quad \Rightarrow Q^2_1 = 0, \quad Q^2_2 = 0, \\
\{\bar{Q}_\alpha, \bar{Q}_\beta\} &\equiv \bar{Q}^2_\alpha = 0 \quad \Rightarrow \bar{Q}^2_1 = 0, \quad \bar{Q}^2_2 = 0, \\
\{Q_\alpha, \bar{Q}_\beta\} &\equiv \delta_{\alpha\beta} H, \quad [H, Q_\alpha] = [H, \bar{Q}_\alpha] = 0.
\end{align*}
\]

(26)

It is the well-known algebra for the \(\mathcal{N} = 4\) supersymmetric quantum mechanical models. The above equation (26) shows that the Hamiltonian \((H)\) commutes with the charges \(Q_\alpha\) and \(\bar{Q}_\alpha\) (i.e. \([H, Q_\alpha] = [H, \bar{Q}_\alpha] = 0\)). The anticommutator of these charges also give rise to the Hamiltonian (i.e. \(\{Q_\alpha, \bar{Q}_\beta\} = \delta_{\alpha\beta} H\)) and they are fermionic in nature (i.e. \(Q^2_\alpha = 0, \bar{Q}^2_\alpha = 0\)) which show the nilpotency property of our \(\mathcal{N} = 4\) SUSY QM system.

4 Algebraic structure: Towards Cohomological Aspects for the \(\mathcal{N} = 4\) SUSY Symmetries

The continuous symmetry transformations \((s_\alpha, \bar{s}_\alpha, s^{\alpha\beta}_\gamma)\) satisfy the following algebraic structure:

\[
\begin{align*}
\{s_\alpha, s_\beta\} &\equiv s^2_\alpha = 0 \quad \Rightarrow s^2_1 = 0, \quad s^2_2 = 0, \\
\{\bar{s}_\alpha, \bar{s}_\beta\} &\equiv \bar{s}^2_\alpha = 0 \quad \Rightarrow \bar{s}^2_1 = 0, \quad \bar{s}^2_2 = 0, \\
\{s_\alpha, \bar{s}_\beta\} &\equiv s^{\alpha\beta}_\gamma, \quad \left[s^{\alpha\beta}_\gamma, s_\gamma\right] = 0, \quad \left[s^{\alpha\beta}_\gamma, \bar{s}_\gamma\right] = 0, \quad \{s_\alpha, \bar{s}_\beta\} \neq 0.
\end{align*}
\]

(27)
Here the bosonic symmetry transformations \( (s^\omega_{\alpha\beta}) \) is just like as a Casimir operator of our present theory (because it commutes with all other SUSY transformations \( s_\alpha \) and \( \bar{s}_\alpha \)).

We note that the conserved charges \( (Q_{\alpha}, \bar{Q}_{\alpha}, Q^\omega_{\alpha\beta}) \) can also be expressed in language five continuous symmetries \( (s_1, s_2) \Rightarrow s_\alpha, (\bar{s}_1, \bar{s}_2) \Rightarrow \bar{s}_\alpha, s^\omega_{\alpha\beta} \) as:

\[
\begin{align*}
  s_\alpha Q_{\beta} &= i \{ Q_{\beta}, Q_{\alpha} \} = 0, \\
  \bar{s}_\alpha \bar{Q}_{\beta} &= i \{ \bar{Q}_{\beta}, \bar{Q}_{\alpha} \} = 0, \\
  s^\omega_{\alpha\beta} Q_{\gamma} &= -i [ Q_{\gamma}, \delta_{\alpha\beta} H ] = 0, \\
  \bar{s}^\omega_{\alpha\beta} \bar{Q}_{\gamma} &= -i [ \bar{Q}_{\gamma}, \delta_{\alpha\beta} H ] = 0.
\end{align*}
\]  

(28)

At the algebraic level, the equations (26), (27) and (28) are reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry, namely;

\[
d^2 = 0, \quad \delta^2 = 0, \quad \Delta = \{ d, \delta \}, \quad [\Delta, d] = 0, \quad [\Delta, \delta] = 0,
\]

(29)

where \((\delta)d\) are the (co-)exterior derivatives and \(\Delta\) is the Laplacian operator. We note that the Laplacian operator \(\Delta\) is the Casimir operator, because it commutes with all rest of the de Rham cohomological operators. Thus, ultimately, we observe that our \( N = 4 \) SUSY quantum mechanical model provides the physical realizations of the de Rham cohomological operators of differential geometry in the language of symmetries and their conserved Noether charges. Hence, our \( N = 4 \) SUSY quantum mechanical model is the perfect model for the Hodge theory.

5 Derivation of \( N = 4 \) SUSY Transformations: Super-variable Approach

We derive the \( N = 4 \) SUSY transformations \( s_\alpha \) and \( \bar{s}_\alpha \) within the framework of the super-variable approach. First, we focus on the derivation of the SUSY transformations \( s_\alpha \) by exploiting the chiral supervariable approach which is defined on the \( (1, 2) \)-dimensional super submanifold\(^1\). Thus, the chiral supervariables\(^2\) expansions in terms of the ordinary variables \((z(t), \bar{z}(t), \psi_\alpha(t), \bar{\psi}_\alpha(t))\) are

\[
\begin{align*}
  z(t) &\longrightarrow Z(t, \theta) = z(t) + \theta^\alpha f^1_\alpha(t), \\
  \bar{z}(t) &\longrightarrow \bar{Z}(t, \theta) = \bar{z}(t) + \theta^\alpha f^2_\alpha(t), \\
  \psi_\alpha(t) &\longrightarrow \Psi_\alpha(t, \theta) = \psi_\alpha(t) + i \theta^\beta b^1_{\alpha\beta}(t), \\
  \bar{\psi}_\alpha(t) &\longrightarrow \bar{\Psi}_\alpha(t, \theta) = \bar{\psi}_\alpha(t) + i \theta^\beta b^2_{\alpha\beta}(t),
\end{align*}
\]

(30)

where the secondary variables \((b^1_{\alpha\beta}(t), b^2_{\alpha\beta}(t))\) and \((f^1_\alpha(t), f^2_\alpha(t))\) are bosonic and fermionic in nature, respectively.

---

\(^1\)Here the ordinary 1D manifold characterized by \( t \) has been generalized to \((1, 2)\)-dimensional super-submanifold. The latter is characterized by the superspace variables \((t, \theta) \equiv (t, \theta^\alpha)\) (with \( \alpha = 1, 2 \)). The Grassmannian variables \( \theta^\alpha \) satisfy the following properties such as: \((\theta^1)^2 = (\theta^2)^2 = 0\).

\(^2\)We have chosen here the (anti-)chiral supervariables because the nilpotent \( N = 4 \) SUSY transformations do not anticommute (i.e. \( \{ s_\alpha, \bar{s}_\beta \} \neq 0 \)). This should be different from the nilpotent (anti-)BRST symmetry transformations because (anti-)BRST symmetries are nilpotent as well as absolutely anticommuting (see, e.g., [30-33]). Within the framework of superfield approach to (anti-)BRST symmetries, the superfields are expanded along both the Grassmannian directions \((\theta, \bar{\theta})\) (see, e.g., Appendix B).
For the derivation of these secondary variables \((b^1_{\alpha\beta}(t), b^2_{\alpha\beta}(t))\) and \((f^1_{\alpha}(t), f^2_{\alpha}(t))\) in terms of the basic variables, we have to impose the SUSY invariant restrictions (SUSYIRs). The following quantities are invariant under \(s_{\alpha}\) such that

\[
s_{\alpha}(\psi_{\beta}) = 0, \quad s_{\alpha}(\bar{z}) = 0, \quad s_{\alpha}(z^T \cdot \psi_{\beta}) = 0, \quad s_{\alpha}\left[2D_t \bar{z} \cdot z + i \bar{\psi}_{\beta} \cdot \psi_{\beta}\right] = 0, \tag{31}
\]

where \(z^T(t) \cdot \psi_{\beta}(t) = z_1 \psi_1 + z_2 \psi_2\). The above SUSYIRs can be generalized onto \((1, 2)\)-dimensional chiral super-submanifold. In this context, we obtain the following relationships:

\[
\Psi_{\alpha}(t, \theta) = \psi_{\alpha}(t) \implies b^1_{\alpha\beta}(t) = 0, \quad Z(t, \theta) = \bar{z}(t) \implies f^2_{\alpha}(t) = 0,
\]

\[
Z^T(t, \theta) \cdot \Psi_{\alpha}(t, \theta) = z^T(t) \cdot \psi_{\alpha}(t),
\]

\[
2D_t \bar{Z}(t, \theta) \cdot Z(t, \theta) + i \Psi_{\alpha}(t, \theta) \cdot \Psi_{\alpha}(t, \theta) = 2D_t \bar{z}(t) \cdot z(t) + i \bar{\psi}_{\alpha}(t) \cdot \psi_{\alpha}(t). \tag{32}
\]

The non-trivial solution of the above restrictions is \(f^1_{\alpha}(t) \propto \psi_{\alpha}(t)\). For the algebraic convenience, however, we choose \(f^1_{\alpha}(t) = \psi_{\alpha}(t)/\sqrt{2}\). For instance, we obtain \(b^2_{\alpha\beta}(t) = 2\nabla_{\alpha\beta} \bar{z}(t)/\sqrt{2}\) from last entity of equation (32).

Plugging in the value \(b^1_{\alpha\beta}(t) = 0, f^2_{\alpha}(t) = 0, f^1_{\alpha}(t) = \psi_{\alpha}(t)/\sqrt{2}\) and \(b^2_{\alpha\beta}(t) = 2\nabla_{\alpha\beta} \bar{z}(t)/\sqrt{2}\) into the chiral supervariable expansions (30), we obtain the following

\[
Z^{(1)}(t, \theta) = z(t) + \theta^\alpha \left(\frac{\psi_{\alpha}(t)}{\sqrt{2}}\right) \equiv z(t) + \theta^\alpha \left(s_{\alpha} z(t)\right),
\]

\[
\bar{Z}^{(1)}(t, \theta) = \bar{z}(t) + \bar{\theta}^\alpha (0) \equiv \bar{z}(t) + \theta^\alpha \left(s_{\alpha} \bar{z}(t)\right),
\]

\[
\Psi^{(1)}_{\alpha}(t, \theta) = \psi_{\alpha}(t) + \theta^\beta (0) \equiv \psi_{\alpha}(t) + \theta^\beta \left(s_{\alpha} \psi_{\beta}(t)\right),
\]

\[
\bar{\Psi}^{(1)}_{\alpha}(t, \theta) = \bar{\psi}_{\alpha}(t) + \bar{\theta}^\beta \left(\frac{2i \nabla_{\alpha\beta} \bar{z}(t)}{\sqrt{2}}\right) \equiv \bar{\psi}_{\alpha}(t) + \bar{\theta}^\beta \left(s_{\alpha} \bar{\psi}_{\beta}(t)\right). \tag{33}
\]

Here the superscript (1) denotes the expansions of the supervariables obtained after the application of the SUSYIRs.

Geometrically, the above chiral expansions of the supervariables obey the following mapping in terms of \(s_{\alpha}\) and Grassmanian derivative \(\partial/\partial \theta^\alpha\) such as

\[
\frac{\partial}{\partial \theta^\alpha} \Omega^{(1)}(t, \theta) = s_{\alpha} \omega(t) \implies s_{\alpha} \leftrightarrow \frac{\partial}{\partial \theta^\alpha} \tag{34}
\]

where \(\Omega^{(1)}(t, \theta)\) is the generic chiral supervariable which stands for \(Z^{(1)}(t, \theta), \bar{Z}^{(1)}(t, \theta), \Psi^{(1)}_{\alpha}(t, \theta), \bar{\Psi}^{(1)}_{\alpha}(t, \theta)\) and \(\omega(t) = z(t), \bar{z}(t), \psi_{\alpha}(t), \bar{\psi}_{\alpha}(t)\) is the generic ordinary variable of our present theory.

To derive the SUSY transformations \(\bar{s}_{\alpha}\) by exploiting SUSY invariant restrictions (SUSYIRs) on the anti-chiral supervariables of the basic variables \((z(t), \bar{z}(t), \psi_{\alpha}(t), \bar{\psi}_{\alpha}(t))\) onto \((1, 2)\)-dimensional anti-chiral super submanifold. The anti-chiral supervariable expansions of these basic variables are

\[
z(t) \rightarrow Z(t, \bar{\theta}) = z(t) + \bar{\theta}^\alpha f^3_{\alpha}(t),
\]

\[
\bar{z}(t) \rightarrow \bar{Z}(t, \bar{\theta}) = \bar{z}(t) + \bar{\theta}^\alpha f^4_{\alpha}(t),
\]

\[
\psi_{\alpha}(t) \rightarrow \Psi_{\alpha}(t, \bar{\theta}) = \psi_{\alpha}(t) + i \bar{\theta}^\beta b^3_{\alpha\beta}(t),
\]

\[
\bar{\psi}_{\alpha}(t) \rightarrow \bar{\Psi}_{\alpha}(t, \bar{\theta}) = \bar{\psi}_{\alpha}(t) + i \theta^\beta b^4_{\alpha\beta}(t). \tag{35}
\]
where \((f^3_\alpha(t), f^4_\alpha(t))\) and \((b^3_{\alpha\beta}(t), b^4_{\alpha\beta}(t))\) are the pair of fermionic and bosonic secondary variables, respectively, on the r.h.s. of anti-chiral supervariable expansions (35). Here the anti-chiral super-submanifold is parametrized by the superspace variables \((t, \bar{\theta}^\alpha)\) (where \(\bar{\theta}^\alpha = \bar{\theta}^1, \bar{\theta}^2\)). We obtain the following SUSYIRs under \(\bar{s}_\alpha\)

\[
\bar{s}_\alpha ( \bar{\psi}_\beta ) = 0, \quad \bar{s}_\alpha ( z ) = 0, \quad \bar{s}_\alpha ( \bar{z} \cdot \bar{\psi}_\beta^T ) = 0, \quad \bar{s}_\alpha \left[ 2 \bar{z} \cdot D_t z - i \bar{\psi}_\beta \cdot \psi_\beta \right] = 0. \tag{36}
\]

We demand the SUSY invariant quantities would be independent of the Grassmannian variable \(\bar{\theta}^\alpha\) on the \((1, 2)\)-dimensional anti-chiral super submanifold. The above secondary variables \((f^3_\alpha, f^4_\alpha, b^3_{\alpha\beta}, b^4_{\alpha\beta})\) can be obtained in terms of the basic variables if we impose SUSYIRs on the anti-chiral supervariables. Thus, we impose the following SUSYIRs

\[
\begin{align*}
Z(t, \bar{\theta}) &= z(t), \quad \bar{\Psi}_\alpha(t, \bar{\theta}) = \bar{\psi}_\alpha(t), \quad 
\bar{Z}(t, \bar{\theta}) \cdot \bar{\Psi}_\alpha^T(t, \bar{\theta}) &= \bar{z}(t) \cdot \bar{\psi}_\alpha^T(t), \\
2 \bar{Z}(t, \bar{\theta}) \cdot D_t Z(t, \bar{\theta}) - i \bar{\Psi}_\alpha(t, \bar{\theta}) \cdot \Psi_\alpha(t, \bar{\theta}) &= 2 \bar{z}(t) \cdot D_t z(t) - i \bar{\psi}_\alpha(t) \cdot \psi_\alpha(t), \quad \tag{37}
\end{align*}
\]

which imply the following results after the substitution of the proper supervariable expansions (35), namely;

\[
\begin{align*}
f^3_\alpha(t) &= 0, \quad b^4_{\alpha\beta}(t) = 0, \quad f^4_\alpha(t) = \frac{\bar{\psi}_\alpha(t)}{\sqrt{2}}, \quad b^3_{\alpha\beta}(t) = \frac{2 \nabla_{\alpha\beta} z(t)}{\sqrt{2}}. \tag{38}
\end{align*}
\]

The substitution of the above secondary variables (38) into the supervariable expansions (35) after SUSYIRs lead to the following anti-chiral supervariable expansions

\[
\begin{align*}
Z^{(2)}(t, \bar{\theta}) &= z(t) + \bar{\theta}^\alpha (0) \equiv z(t) + \bar{\theta}^\alpha (\bar{s}_\alpha z), \\
\bar{Z}^{(2)}(t, \bar{\theta}) &= \bar{z}(t) + \bar{\theta}^\alpha \left( \frac{\bar{\psi}_\alpha}{\sqrt{2}} \right) \equiv \bar{z}(t) + \bar{\theta}^\alpha (\bar{s}_\alpha \bar{z}), \\
\Psi^{(2)}_\alpha(t, \bar{\theta}) &= \psi_\alpha(t) + \bar{\theta}^\beta \left( \frac{2i \nabla_{\alpha\beta} z}{\sqrt{2}} \right) \equiv \psi_\alpha(t) + \bar{\theta}^\beta (\bar{s}_\alpha \psi_\beta), \\
\bar{\Psi}^{(2)}_\alpha(t, \bar{\theta}) &= \bar{\psi}_\alpha(t) + \bar{\theta}^\beta (0) \equiv \bar{\psi}_\alpha(t) + \bar{\theta}^\beta (\bar{s}_\alpha \bar{\psi}_\beta), \quad \tag{39}
\end{align*}
\]

where the superscript \(2\) denotes the expansions of the supervariables after the application of SUSYIRs in (38).

The conserved charges \(Q_\alpha\) and \(\bar{Q}_\alpha\) can be expressed in terms of the (anti-)chiral supervariable expansions after the application of SUSYIRs and can be expressed in two different
ways:

\[
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} \left[ 2 D_t \tilde{Z}^{(1)}(t, \theta) \cdot Z^{(1)}(t, \theta) \right] = \frac{\partial}{\partial \theta^\alpha} \left[ 2 D_t \tilde{z}(t) \cdot Z^{(1)}(t, \theta) \right],
\]

\[
= \int d\theta^\alpha \left[ 2 D_t \tilde{Z}^{(1)}(t, \theta) \cdot Z^{(1)}(t, \theta) \right] = \int d\theta^\alpha \left[ 2 D_t \tilde{z}(t) \cdot Z^{(1)}(t, \theta) \right],
\]

\[
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} \left[ -i \bar{\Psi}_\beta^{(1)}(t, \theta) \cdot \Psi_\beta^{(1)}(t, \theta) \right] = \frac{\partial}{\partial \theta^\alpha} \left[ -i \bar{\Psi}_\beta^{(1)}(t, \theta) \cdot \psi_\beta(t) \right],
\]

\[
= \int d\theta^\alpha \left[ -i \bar{\Psi}_\beta^{(1)}(t, \theta) \cdot \Psi_\beta^{(1)}(t, \theta) \right] = \int d\theta^\alpha \left[ -i \bar{\Psi}_\beta^{(1)}(t, \theta) \cdot \psi_\beta(t) \right],
\]

\[
\bar{Q}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} \left[ 2 \tilde{Z}^{(2)}(t, \bar{\theta}) \cdot D_t \tilde{Z}^{(2)}(t, \bar{\theta}) \right] = \frac{\partial}{\partial \bar{\theta}^\alpha} \left[ 2 \tilde{Z}^{(2)}(t, \bar{\theta}) \cdot D_t \tilde{z}(t) \right],
\]

\[
= \int d\bar{\theta}^\alpha \left[ 2 \tilde{Z}^{(2)}(t, \bar{\theta}) \cdot D_t \tilde{Z}^{(2)}(t, \bar{\theta}) \right] = \int d\bar{\theta}^\alpha \left[ 2 \tilde{Z}^{(2)}(t, \bar{\theta}) \cdot D_t \tilde{z}(t) \right],
\]

\[
\bar{Q}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} \left[ +i \bar{\Psi}_\beta^{(2)}(t, \bar{\theta}) \cdot \Psi_\beta^{(2)}(t, \bar{\theta}) \right] = \frac{\partial}{\partial \bar{\theta}^\alpha} \left[ +i \bar{\psi}_\beta(t) \cdot \Psi_\beta^{(2)}(t, \bar{\theta}) \right],
\]

\[
= \int d\bar{\theta}^\alpha \left[ +i \bar{\Psi}_\beta^{(2)}(t, \bar{\theta}) \cdot \Psi_\beta^{(2)}(t, \bar{\theta}) \right] = \int d\bar{\theta}^\alpha \left[ +i \bar{\psi}_\beta(t) \cdot \Psi_\beta^{(2)}(t, \bar{\theta}) \right].
\]

The nilpotency of \(\partial/\partial \theta^\alpha\) and \(\partial/\partial \bar{\theta}^\alpha\) (i.e. \(\{\partial_{\theta^\alpha}, \partial_{\bar{\theta}^\alpha}\} \Rightarrow \partial_{\theta^\alpha}^2 = \partial_{\bar{\theta}^\alpha}^2 = 0\)) implies that \(\partial_{\theta^\alpha} Q_\alpha = 0, \partial_{\bar{\theta}^\alpha} \bar{Q}_\alpha = 0\). The above charges \(Q_\alpha\) and \(\bar{Q}_\alpha\) can be written in terms of the symmetry transformations \((s_\alpha, \bar{s}_\alpha)\) and basic variables \((z, \tilde{z}, \psi_\alpha, \bar{\psi}_\alpha)\) in the following manner, namely:

\[
Q_\alpha = s_\alpha \left( 2 D_t \tilde{z} \cdot z \right) \equiv s_\alpha \left( -i \bar{\psi}_\beta \cdot \psi_\beta \right),
\]

\[
\bar{Q}_\alpha = \bar{s}_\alpha \left( 2 \tilde{z} \cdot D_t z \right) \equiv \bar{s}_\alpha \left( +i \bar{\psi}_\beta \cdot \psi_\beta \right).
\]

Thus, the above charges \(Q_\alpha\) and \(\bar{Q}_\alpha\) are nilpotent of order two, i.e. \(Q_\alpha^2 = \frac{1}{2} \{Q_\alpha, Q_\alpha\} = 0, \bar{Q}_\alpha^2 = \frac{1}{2} \{\bar{Q}_\alpha, \bar{Q}_\alpha\} = 0\) (because \(s_\alpha^2 = 0, \bar{s}_\alpha^2 = 0\) when we use the constraints \(\tilde{z} \cdot z - 1 = 0, \tilde{z} \cdot \psi_\alpha = 0, \bar{\psi}_\alpha \cdot z = 0\)).

It is straightforward to check that the invariance of the Lagrangian (1) in terms of the (anti-)chiral supervariables obtained after the application of SUSYIRs as given below

\[
L \quad \Rightarrow \quad \tilde{L}^{(ac)} = 2 D_t Z^{(2)} \cdot D_t Z^{(2)} + \frac{i}{2} \left[ \Psi_\alpha^{(2)} \cdot D_t \Psi_\alpha^{(2)} - D_t \Psi_\alpha^{(2)} \cdot \Psi_\alpha^{(2)} \right] - 2 g a
\]

\[
\equiv L + \bar{\theta}^\alpha \left[ \frac{d}{dt} \left( \frac{D_t \tilde{z} \cdot \psi_\alpha}{\sqrt{2}} \right) \right],
\]

\[
L \quad \Rightarrow \quad \tilde{L}^{(c)} = 2 D_t Z^{(1)} \cdot D_t Z^{(1)} + \frac{i}{2} \left[ \Psi_\alpha^{(1)} \cdot D_t \Psi_\alpha^{(1)} - D_t \Psi_\alpha^{(1)} \cdot \Psi_\alpha^{(1)} \right] - 2 g a
\]

\[
\equiv L + \theta^\alpha \left[ \frac{d}{dt} \left( \frac{\bar{\psi}_\alpha \cdot D_t \tilde{z}}{\sqrt{2}} \right) \right],
\]

where the superscripts \((c)\) and \((ac)\) denote the chiral and anti-chiral nature of the La-
Thus, the above relationships provide the geometrical meaning for the SUSY invariances of the Lagrangian (1) in the language of the translational generators $\partial_{\theta^a}$ and $\partial_{\bar{\theta}^a}$ along the Grassmannian discretions $\theta^a$ and $\bar{\theta}^a$ onto (1, 4)-dimensional (anti-)chiral super-submanifolds, respectively, to produce the ordinary time derivatives [cf. (43)] in ordinary 1D space thereby leading the symmetry invariance of our $\mathcal{N} = 4$ SUSY theory.

6 Conclusions

In our present endeavor, we have demonstrated that the $\mathcal{N} = 4$ SUSY QMM of the motion of a charged particle on a sphere in the background of Dirac magnetic monopole is a perfect model for the Hodge theory. In this paper, we have shown that the physical realizations of the de Rham cohomological operators ($d, \delta, \Delta$) of differential geometry in the language of continuous symmetries (and their conserved Noether charges) and a set of novel discrete symmetries. In addition, the discrete symmetries (14) and (20) play the key role in establishing the relationships ($\bar{s}_\alpha = \pm \ast s_\alpha \ast$, $s_\alpha = \mp \ast \bar{s}_\alpha$) between the continuous symmetry transformations ($s_\alpha, \bar{s}_\alpha$). These relations are exactly same as the relation ($\delta = \pm \ast d\ast$) between the differential operators $d$ and $\delta$ where $\ast$ is the Hodge duality operation. Here the discrete symmetry is the analogue of Hodge duality operation. Thus, we have shown that the perfect analogy between the de Rham cohomological operators of differential geometry and the $\mathcal{N} = 4$ SUSY transformations (and their conserved Noether charges and Hamiltonian of the system) exists at the algebraic level, in our present investigation.

In our present endeavor, we have applied supervariable approach (we have already applied this approach for different $\mathcal{N} = 2$ SUSY QMMs in [15-17,23,24]) for the derivation of the SUSY transformations for the $\mathcal{N} = 4$ SUSY QMM of a charged particle (i.e. an electron) moving on a sphere in the background of Dirac magnetic monopole [26] within the framework of (anti-)chiral super-submanifolds. Similarly, we have established SUSY invariance of the Lagrangian in the language translational of generators ($\partial_{\theta^a}, \partial_{\bar{\theta}^a}$) along the directions of the Grassmannian variables ($\theta^a, \bar{\theta}^a$) within the framework of chiral and anti-chiral supervariable expansions (33) and (39), respectively, after imposing the SUSYIRs.

Our future endeavor is to find out the physical realizations of the de Rham cohomological operators of differential geometry for different $\mathcal{N} = 4$ and $\mathcal{N} = 8$ SUSY quantum mechanical models in the language of symmetries and conserved Noether charges. Furthermore, we shall apply this idea in our future investigations of the nonlinear superconformal symmetry of a fermion in the field of a Dirac monopole [28,29]. Our main goal is to apply the supervariable/superfield approach to BRST formalism [30-34] for the study of $\mathcal{N} = 2, 4, 8$ SUSY gauge theories (because of their relevance to the recent developments in the superstring theories), in our future publications [35].
Acknowledgements: We would like to gratefully acknowledge financial support from DST, Government of India, New Delhi, under grant No. DST-15-0081. We would like to thank BHU for the local hospitality during the visit. Fruitful suggestions by Prof. C. S. Aulakh and Mr. T. Bhanja on the preparation of this present paper are also thankfully acknowledged. Enlightening comments by our esteemed Reviewer are thankfully acknowledged. Aulakh and Mr. T. Bhanja on the preparation of this present paper are also thankfully acknowledged. Enlightening comments by our esteemed Reviewer are thankfully acknowledged, too.

Appendix A: Symmetries and Algebraic Structure

In this Appendix A, we shall be showing explicitly the $\mathcal{N} = 4$ SUSY quantum mechanical algebra (26) amongst the conserved charges $(Q_\alpha, \bar{Q}_{\alpha}, Q^\alpha_{\alpha\beta} \equiv \delta_{\alpha\beta} H)$ from the symmetry principle. It can be explicitly checked that

$$s_\alpha Q_\beta = i \{Q_\beta, Q_\alpha\} = 0, \quad \bar{s}_\alpha \bar{Q}_\beta = i \{\bar{Q}_\beta, \bar{Q}_\alpha\} = 0,$$  \hspace{1cm} (A.1)

the l.h.s. of above equation (A.1) by using the expression for the generators $(Q_\alpha, \bar{Q}_\alpha)$ from (10) and the symmetry transformations from (3), we obtain the following:

$$s_\alpha Q_\beta = -\frac{1}{2} (\nabla_{\alpha\gamma} \bar{z} \cdot \psi_\gamma) (\bar{z} \cdot \psi_\beta), \quad \bar{s}_\alpha \bar{Q}_\beta = \frac{1}{2} (\bar{\psi}_\gamma \cdot \nabla_{\alpha\gamma} z) (\bar{\psi}_\beta \cdot z),$$  \hspace{1cm} (A.2)

which turn out to be zero on the constrained surface defined by the constraint conditions $\bar{z} \cdot \psi_\beta = 0$ and $\bar{\psi}_\beta \cdot z = 0$, respectively. Similarly, we compute the l.h.s. of the following relationships:

$$s_\alpha \bar{Q}_\beta = i \{\bar{Q}_\beta, Q_\alpha\} = i \delta_{\alpha\beta} H, \quad \bar{s}_\alpha Q_\beta = i \{Q_\beta, \bar{Q}_\alpha\} = i \delta_{\alpha\beta} H,$$  \hspace{1cm} (A.3)

by using the equations (10) and (3) in the above equation (A.3), we obtain the following:

$$\bar{s}_\alpha Q_\beta = i \left[ 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) \bar{z} \right] \cdot \nabla_{\alpha\beta} z + D_t \bar{\psi}_\alpha \cdot \psi_\beta$$

$$+ \frac{i}{4} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) (\bar{\psi}_\alpha \cdot \psi_\beta) + \frac{1}{2} (\bar{\psi}_\gamma \cdot \nabla_{\alpha\gamma} z) (\bar{\psi}_\beta \cdot \psi_\beta),$$

$$s_\alpha \bar{Q}_\beta = i \nabla_{\alpha\beta} \bar{z} \left[ 2 i D_t \bar{z} - \frac{i}{2} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) \bar{z} \right] - (\bar{\psi}_\beta \cdot D_t \psi_\alpha)$$

$$+ \frac{i}{4} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) (\bar{\psi}_\beta \cdot \psi_\alpha) - \frac{1}{2} (\bar{\psi}_\beta \cdot z) (\nabla_{\alpha\gamma} z \cdot \psi_\gamma).$$  \hspace{1cm} (A.4)

Substituting the constraints $\bar{\psi}_\beta \cdot z = 0, \bar{z} \cdot \psi_\beta = 0$ and using the definitions of $\nabla_{\alpha\beta} z, \nabla_{\alpha\beta} \bar{z}, D_t z, D_t \bar{z}$ plus the equations of motion w.r.t. $\psi_\alpha$ and $\bar{\psi}_\alpha$ from (10) in the above equation, we obtain the following:

$$s_\alpha \bar{Q}_\beta = 2 i \delta_{\alpha\beta} D_t \bar{z} \cdot D_t z + (\bar{\psi}_\beta \cdot \psi_\alpha - \delta_{\alpha\beta} \bar{\psi}_\gamma \cdot \psi_\gamma) (\bar{z} \cdot \bar{z} - i a \bar{z} \cdot z)$$

$$+ \frac{1}{2} \delta_{\alpha\beta} (\bar{z} \cdot z + i a \bar{z} \cdot z) (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) + \frac{i}{4} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) (\bar{\psi}_\beta \cdot \psi_\alpha),$$

$$\bar{s}_\alpha Q_\beta = 2 i \delta_{\alpha\beta} D_t z \cdot D_t \bar{z} + (\bar{\psi}_\beta \cdot \psi_\alpha - \delta_{\alpha\beta} \bar{\psi}_\gamma \cdot \psi_\gamma) (\bar{z} \cdot \bar{z} - i a \bar{z} \cdot z)$$

$$+ \frac{1}{2} \delta_{\alpha\beta} (\bar{z} \cdot z + i a \bar{z} \cdot z) (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) + \frac{i}{4} (\bar{\psi}_\gamma \cdot \psi_\gamma + 2g) (\bar{\psi}_\beta \cdot \psi_\alpha),$$
Furthermore, we use the definition of $\sigma$ and constraint $\bar{z} \cdot z = 1$ and $\frac{d}{dt}(\bar{z} \cdot z - 1) = 0$ in the above equation. We obtain the following results

$$s_{\alpha} \bar{Q}_{\beta} = i \delta_{\alpha \beta} \left[ 2(D_t \bar{z}) \cdot (D_t z) - g (\bar{\psi}_{\gamma} \cdot \psi_{\gamma}) + \frac{1}{4} \left\{ (\epsilon_{\gamma \rho} \bar{\psi}_{\gamma} \cdot \psi_{\rho})^2 - (\bar{\psi}_{\gamma} \cdot \psi_{\gamma})^2 \right\} \right] \equiv i \delta_{\alpha \beta} H,$$

$$\bar{s}_{\alpha} Q_{\beta} = i \delta_{\alpha \beta} \left[ 2(D_t \bar{z}) \cdot (D_t z) - g (\bar{\psi}_{\gamma} \cdot \psi_{\gamma}) + \frac{1}{4} \left\{ (\epsilon_{\gamma \rho} \bar{\psi}_{\gamma} \cdot \psi_{\rho})^2 - (\bar{\psi}_{\gamma} \cdot \psi_{\gamma})^2 \right\} \right] \equiv i \delta_{\alpha \beta} H. \quad (A.6)$$

We note that, in the computation of $s_{\alpha} \bar{Q}_{\beta} = i \{ \bar{Q}_{\beta}, Q_{\alpha} \} = i \delta_{\alpha \beta} H$ we have used the constraint conditions $\bar{\psi}_{\beta} \cdot z = 0, \bar{z} \cdot z = 1, \frac{d}{dt}(\bar{z} \cdot z - 1) = 0$ and the definitions of $a, \nabla_{\alpha \beta} \bar{z}, D_t \bar{z}, D_t z \bar{z}$ plus equation of motion for $\bar{\psi}_{\alpha}$ (i.e. $D_t \bar{\psi}_{\alpha} + \frac{1}{2} (\epsilon_{\gamma \rho} \bar{\psi}_{\gamma} \cdot \psi_{\rho}) (\epsilon_{\alpha \beta} \bar{\psi}_{\beta}) - ig \psi_{\alpha} = 0$). On the other hand, in the explicit composition of $\bar{s}_{\alpha} Q_{\beta} = i \{ Q_{\beta}, \bar{Q}_{\alpha} \} = i \delta_{\alpha \beta} H$ we have exploited the constraint conditions $\bar{\psi}_{\beta} \cdot z = 0, \bar{z} \cdot z = 1, \frac{d}{dt}(\bar{z} \cdot z - 1) = 0$ and the definitions of $a, \nabla_{\alpha \beta} \bar{z}, D_t \bar{z}, D_t z \bar{z}$ with using equation of motion for $\psi$ (i.e. $D_t \psi_{\alpha} + \frac{1}{2} (\epsilon_{\gamma \rho} \bar{\psi}_{\gamma} \cdot \psi_{\rho}) (\epsilon_{\alpha \beta} \bar{\psi}_{\beta}) + ig \psi_{\alpha} = 0$), respectively.

**Appendix A: On the Choice of (Anti-)Chiral Supervariables for the Description of SUSY Model**

In this Appendix B, we would like to emphasize the key difference between the (anti-)chiral supervariables in the context of the derivation of nilpotent SUSY transformations for the $\mathcal{N} = 4$ SUSY quantum mechanical model and the (anti-)BRST symmetry transformations for a gauge theory. In the literature, it is well-known that the (anti-)BRST symmetries ($s_{(a)b}$) for a given gauge theory are nilpotent and absolutely anticommuting in nature whereas the $\mathcal{N} = 4$ SUSY symmetries are nilpotent but not absolutely anticommuting in nature. Within the framework of BT-superfield approach [30-33] to BRST symmetries, a bosonic field $\sigma(x)$ for D-dimensional gauge theory, one has to generalize it onto a (D, 2)-dimensional supermanifold along the Grassmannian directions ($\theta$ and $\bar{\theta}$) (with $\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0$) in the following manner:

$$\Sigma(x, \theta, \bar{\theta}) = \sigma(x) + \theta \bar{R}(x) + \bar{\theta} R(x) + i\theta \bar{\theta} S(x), \quad (B.1)$$
where $R(x), \tilde{R}(x)$ are the fermionic secondary fields and $S(x)$ is a bosonic secondary field and $\Sigma(x, \theta, \bar{\theta})$ is the corresponding superfield which is defined on the $(D, 2)$-dimensional supermanifold. In the above equation (B.1), the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ are found to correspond to the (anti-)BRST symmetry transformations $s_{(a)b}$ which are nilpotent of order two due to $(\partial_{\theta}^2 = \partial_{\bar{\theta}}^2 = 0)$ and they are absolutely anticommuting because it is straightforward to check that:

$$\partial_\theta \partial_{\bar{\theta}} \Sigma(x, \theta, \bar{\theta}) = i S(x) \iff s_b s_{ab} \sigma(x), \quad (B.2)$$

$$\partial_{\bar{\theta}} \partial_\theta \Sigma(x, \theta, \bar{\theta}) = -i S(x) \iff s_{ab} s_b \sigma(x). \quad (B.3)$$

It is clear from the above relationships (B.2) and (B.3), we obtain the following

$$(\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta) \Sigma(x, \theta, \bar{\theta}) = 0 \iff (s_b s_{ab} + s_{ab} s_b) \sigma(x) = 0, \quad (B.4)$$

which shows the absolute anticommutativity property of the (anti-)BRST symmetry transformations. In our present $\mathcal{N} = 4$ SUSY QM model, we are compelled to avoid the relation (B.4) so that our nilpotent SUSY symmetries could not become absolutely anticommuting in nature. The anticommutator of our present investigation (i.e. SUSY theory), is nothing but the bosonic symmetry (i.e. $s_\alpha \bar{s}_\beta + s_\beta \bar{s}_\alpha = s_{\alpha \beta}^\omega$ with $\alpha, \beta = 1, 2$) [cf. (8)].

Geometrically, $\mathcal{N} = 4$ SUSY symmetry transformations are identified with the translational generators $(\partial_{\theta_\alpha}, \partial_{\bar{\theta}_\alpha})$ along the Grassmannian directions $(\theta_\alpha, \bar{\theta}_\alpha)$ of the (anti-)chiral super-submanifolds which encapsulate only the nilpotency property (not absolute anticommutativity property). The purpose of this Appendix B is to develop the theoretical tools and techniques so that we could derive the complete structure of the SUSY symmetries for the $\mathcal{N} = 4$ SUSY QM model.

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