Rainbow connectivity of the non-commuting graph of a finite group

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Abstract

Let $G$ be a finite non-abelian group. The non-commuting graph $\Gamma_G$ of $G$ has the vertex set $G \setminus Z(G)$ and two distinct vertices $x$ and $y$ are adjacent if $xy \neq yx$, where $Z(G)$ is the center of $G$. We prove that the rainbow 2-connectivity of $\Gamma_G$ is 2. In particular, the rainbow connection number of $\Gamma_G$ is 2. Moreover, for any positive integer $k$, we prove that there exist infinitely many non-abelian groups $G$ such that the rainbow $k$-connectivity of $\Gamma_G$ is 2.

Key words: Non-commuting graph; non-abelian group; rainbow connectivity; rainbow path.

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1 Introduction

Let $\Gamma$ be a connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Given an edge coloring of $\Gamma$. A path $P$ is rainbow if no two edges of $P$ are colored the same. The vertex connectivity of $\Gamma$, denoted by $\kappa(\Gamma)$, is the smallest number of vertices whose deletion from $\Gamma$ disconnects it. For any positive integer $k \leq \kappa(\Gamma)$, an edge-colored graph is called rainbow-$k$-connected if any two distinct vertices of $\Gamma$ are connected by at least $k$ internally disjoint rainbow paths. The rainbow-$k$-connectivity of $\Gamma$, denoted by $rc_k(\Gamma)$, is the minimum number of colors required to color the edges of $\Gamma$ to make it rainbow-$k$-connected. We usually denote $rc_1(\Gamma)$ by $rc(\Gamma)$, which is called the rainbow connection number of $\Gamma$.

In [5] and [6], Chartrand et al. first introduced the concept of rainbow $k$-connectivity for $k = 1$ and $k \geq 2$, respectively. Rainbow $k$-connectivity has application in transferring information of high security in communication networks. For details we refer to [6] and [8]. The NP-hardness of determining $rc(\Gamma)$ was shown by Chakraborty et al. [4]. Recently, the rainbow connectivity of some special classes of graphs have been studied; see [12] for complete graphs, [11] for regular complete bipartite graphs, [10, 14, 15] for Cayley graphs and [16] for power graphs. For more information, see [13].

For a non-abelian group $G$, the non-commuting graph $\Gamma_G$ of $G$ has the vertex set $G \setminus Z(G)$ and two distinct vertices $x$ and $y$ are adjacent if $xy \neq yx$, where $Z(G)$
is the center of \( G \). According to \cite{17} non-commuting graphs were first considered by Erdős in 1975. Over the past decade, non-commuting graphs have received considerable attention. For example, Abdollahi et al. \cite{1} proved that the diameter of any non-commuting graph is 2. For two non-abelian groups with isomorphic non-commuting graphs, the sufficient conditions that guarantee their orders are equal were provided by Abdollahi and Shahverdi \cite{2} and Darafsheh \cite{7}. Akbari and Moghaddamfar \cite{3} studied strongly regular non-commuting graphs. Solomon and Woldar \cite{18} characterized some simple groups by their non-commuting graphs.

In this paper we study the rainbow \( k \)-connectivity of non-commuting graphs and obtain the following results.

**Theorem 1.1** Let \( G \) be a finite non-abelian group. Then \( \text{rc}_2(\Gamma_G) = 2 \). In particular, \( \text{rc}(\Gamma_G) = 2 \).

**Theorem 1.2** For any positive integer \( k \), there exist infinitely many non-abelian groups \( G \) such that \( \text{rc}_k(\Gamma_G) = 2 \).

## 2 Preliminaries

In this section we present some lemmas which we need in the sequel.

For vertices \( x, y \) of a graph \( \Gamma \), let \( \tau(x, y) \) be the number of the common neighbors of \( x \) and \( y \).

**Lemma 2.1** Let \( G \) be a finite non-abelian group, and let \( x \) and \( y \) be two distinct vertices of \( \Gamma_G \). Then \( \tau(x, y) \geq \frac{1}{6}|G| \).

**Proof.** For each \( g \in G \), \( C_G(g) \) denotes the centralizer of \( g \) in \( G \). By the principle of inclusion and exclusion,

\[
\tau(x, y) = |G| - |C_G(x) \cup C_G(y)|
\]

\[
= |G| - |C_G(x)| - |C_G(y)| + |C_G(x) \cap C_G(y)|
\]

\[
\geq |G| - |C_G(x)| - |C_G(y)| + \frac{|C_G(x)| \cdot |C_G(y)|}{|G|}.
\]

If \( |C_G(x)| = |C_G(y)| = \frac{1}{2}|G| \), then

\[
\tau(x, y) \geq \frac{|C_G(x)| \cdot |C_G(y)|}{|G|} = \frac{1}{4}|G|;
\]

if not,

\[
\tau(x, y) \geq |G| - |C_G(x)| - |C_G(y)| \geq \frac{1}{6}|G|.
\]

The lexicographic product \( \Gamma \circ \Lambda \) of graphs \( \Gamma \) and \( \Lambda \) has the vertex set \( V(\Gamma) \times V(\Lambda) \), and two vertices \((\gamma, \lambda), (\gamma', \lambda')\) are adjacent if \( \{\gamma, \gamma'\} \in E(\Gamma) \), or if \( \gamma = \gamma' \) and \( \{\lambda, \lambda'\} \in E(\Lambda) \).
Lemma 2.2 Let $G$ be a non-abelian group and $A$ be an abelian group of order $n$. Then
\[ \Gamma_{G \times A} \cong \Gamma_G \circ \overline{K}_n, \]
where $\overline{K}_n$ is the complement of the complete graph $K_n$.

For positive integers $l,r$ and $t$, let $K_{[l]}$ denote a complete $l$-partite graph with each part of order $r$, and let $K_{[l],t}$ denote a complete $(l+1)$-partite graph with $l$ parts of order $r$ and a part of order $t$.

Lemma 2.3 Let $D_{2n}$ and $Q_{4m}$ be respectively the dihedral group of order $2n$ and the generalized quaternion group of order $4m$, where $n \geq 3$ and $m \geq 2$. Then
(i) If $n$ is odd, then $\Gamma_{D_{2n}} \cong K_{n[1],n-1}$.
(ii) If $n$ is even, then $\Gamma_{D_{2n}} \cong K_{2[2],n-2}$.
(iii) $\Gamma_{Q_{4m}} \cong K_{m[2],2m-2}$.

Li and Sun [12] studied the rainbow $k$-connectivity of some families of complete multipartite graphs. Now we compute the rainbow $k$-connectivity of another family.

Proposition 2.4 Let $m \geq n + 1$, $lmn \neq 2$. Then $rc_2(K_{m[l],ln}) = 2$.

Proof. Write $\Gamma = K_{m[l],ln}$. Let $\{a_{j,i} : 1 \leq j \leq l\}$ and $\{a_{j,m+1} : 1 \leq j \leq ln\}$ be all parts of $\Gamma$, where $i = 1, \ldots, m$.

Case 1. $n = 1$.

Case 1.1. $m = 2$.

If $l = 2r$, then we assign a color to the edges
\[ \{a_{2j-1,1},a_{2j-1,2}\}, \{a_{2j-1,1},a_{2j,2}\}, \{a_{2j-1,1},a_{2j-1,3}\}, \{a_{2j,1},a_{2j-1,2}\}, \{a_{2j,1},a_{2j,2}\}, \{a_{2j,1},a_{2j,3}\}, \{a_{2j-1,2},a_{2j-1,3}\}, \{a_{2j-1,2},a_{2j,3}\}, \quad 1 \leq j \leq r \] (1)
and another color to the remaining edges.

If $l = 2r + 1$, then we assign a color to the edges
\[ \{a_{l,2},a_{l,3}\}, \{a_{l,1},a_{2j-1,2}\}, \{a_{l,1},a_{2j-1,3}\}, \{a_{l,2},a_{2j-1,1}\}, \{a_{l,2},a_{2j,1}\}, \{a_{l,2},a_{2j-1,3}\}, \{a_{l,2},a_{2j,3}\}, \{a_{l,3},a_{2j,2}\}, \quad 1 \leq j \leq r \]
and the edges in (1), and another color to all other edges.

Case 1.2. $m = 3$.

The edges
\[ \{a_{j,1},a_{j,2}\}, \{a_{j,2},a_{j,4}\}, \{a_{j,3},a_{j,4}\}, \quad 1 \leq j \leq l \]
are assigned by a color and all other edges are assigned by another color.

Case 1.3. $m \geq 4$.

The edges
\[ \{a_{j,i},a_{j,i+1}\}, \{a_{j,m+1},a_{j,1}\}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq l \]
are assigned by a color and all other edges are assigned by another color.
Note that all the above colorings make $\Gamma$ rainbow-2-connected. Hence $rc_2(\Gamma) = 2$.

Case 2. $n \geq 2$.

We assign a color to $\{a_{j,i}, a_{j,i+1}\}, \{a_{j,1}, a_{j,m}\}, \{a_{j,i'}, a_{j-1+n+i', m+1}\}, 1 \leq i \leq m - 1, 1 \leq i' \leq n, 1 \leq j \leq l$

and another color to the remaining edges. Note that this coloring makes $\Gamma$ rainbow-2-connected. This implies that $rc_2(\Gamma) = 2$. $\square$

3 Proof of main results

In this section, we shall prove Theorems 1.1 and 1.2.

Proposition 3.1 Let $G$ be a finite non-abelian group with $|G| \geq 114$. Then $rc_2(\Gamma_G) = 2$.

Proof. We randomly color the edges of $\Gamma_G$ with two colors. Denote by $P_G$ the probability that such a random coloring makes it not rainbow-2-connected. It suffices to prove that $P_G < 1$.

Let $x$ and $y$ be two distinct vertices of $\Gamma_G$. If $x$ and $y$ are adjacent, then the probability that there exist no rainbow paths of length 2 from $x$ to $y$ is $(1/2)^{\tau(x,y)}$. If $x$ and $y$ are non-adjacent, then the probability that $\Gamma_G$ has precisely a rainbow path of length 2 from $x$ to $y$ is $\tau(x,y)(1/2)^{\tau(x,y)}$, and the probability that $\Gamma_G$ has no rainbow paths of length 2 from $x$ to $y$ is $(1/2)^{\tau(x,y)}$. Note that

$$|E(\Gamma_G)| = \frac{1}{2} \sum_{x \in V(\Gamma_G)} (|G| - |C_G(x)|) \geq \frac{1}{4}|G|(|G| - |Z(G)|).$$

(2)

Write

$$P = \sum_{x \sim y} \left( \frac{1}{2} \right)^{\tau(x,y)} + \sum_{x \not\sim y} \left( \frac{1}{2} \right)^{\tau(x,y)} \tau(x,y) \left( \frac{1}{2} \right)^{\tau(x,y)}$$

(3)
where $x \sim y$ denotes that $x$ and $y$ are adjacent. Now

$$\mathcal{P}_G \leq \mathcal{P}$$

$$\leq \sum_{x \sim y} \left(\frac{1}{2}\right)^{\frac{1}{6}|G|} + \sum_{x \sim y} \left|G\right| \left(\frac{1}{2}\right)^{\frac{1}{6}|G|} \quad \text{(by Lemma 2.1)}$$

$$= \left(\frac{|V(\Gamma_G)|}{2}\right) \left(\frac{1}{2}\right)^{\frac{1}{6}|G|} + \sum_{x \sim y} \left|G\right| \left(\frac{1}{2}\right)^{\frac{1}{6}|G|}$$

$$\leq \left(\frac{|G| - |Z(G)|}{2}\right) \left(\frac{1}{2}\right)^{\frac{1}{6}|G|}$$

$$+ |G| \left(\frac{1}{2}\right)^{\frac{1}{6}|G|} \left(\frac{|G| - |Z(G)|}{2}\right) - \frac{1}{4} (|G| - |Z(G)|)|G| \quad \text{(by (2))}$$

$$= \frac{1}{4} (|G| - |Z(G)|)(|G|^2 - 2|Z(G)| - |Z(G)||G| - 2) \left(\frac{1}{2}\right)^{\frac{1}{6}|G|}$$

$$< \left(\frac{1}{2}\right)^{\frac{1}{6}|G|+2} |G|^3.$$  

It follows that if $|G| \geq 114$, then $\mathcal{P}_G < 1$. \qed

**Proposition 3.2** Let $G$ be a finite non-abelian group with $|G| < 114$. Then $r_{c_2}(\Gamma_G) = 2$.

**Proof.** Let $\mathcal{P}_G$ be the probability that such a random coloring makes $\Gamma_G$ not rainbow-2-connected. Thus if $\mathcal{P}_G \leq 1$, then $r_{c_2}(\Gamma_G) = 2$. Using GAP [9], we compute $\mathcal{P}$ (see (3)) by the following code.

```gap
M:=Elements(G);
k:=0;
s:=0;
for i in [1..Size(M)] do
if Centralizer(G,M[i])<>G then
  for j in [1..Size(M)] do
    if Centralizer(G,M[j])<>G and IsAbelian(Group(M[i],M[j]))=false
      and M[i]<>M[j] then
      t:=Order(G)-Size(Union(Elements(Centralizer(G,M[i])),
      Elements(Centralizer(G,M[j]))));
      k:=k+(1/2)^t;
    fi;
  od;
fi;
for i in [1..Size(M)] do
if Centralizer(G,M[i])<>G then
  for j in [1..Size(M)] do
    if Centralizer(G,M[j])<>G and IsAbelian(Group(M[i],M[j]))=false
      and M[i]<>M[j] then
      t:=Order(G)-Size(Union(Elements(Centralizer(G,M[i])),
      Elements(Centralizer(G,M[j]))));
      k:=k+(1/2)^t;
    fi;
  od;
fi;
```

It follows that if $|G| \geq 114$, then $\mathcal{P}_G < 1$. \qed

**Proposition 3.2** Let $G$ be a finite non-abelian group with $|G| < 114$. Then $r_{c_2}(\Gamma_G) = 2$.

**Proof.** Let $\mathcal{P}_G$ be the probability that such a random coloring makes $\Gamma_G$ not rainbow-2-connected. Thus if $\mathcal{P}_G \leq 1$, then $r_{c_2}(\Gamma_G) = 2$. Using GAP [9], we compute $\mathcal{P}$ (see (3)) by the following code.

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      and M[i]<>M[j] then
      t:=Order(G)-Size(Union(Elements(Centralizer(G,M[i])),
      Elements(Centralizer(G,M[j]))));
      k:=k+(1/2)^t;
    fi;
  od;
fi;
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    if Centralizer(G,M[j])<>G and IsAbelian(Group(M[i],M[j]))=false
      and M[i]<>M[j] then
      t:=Order(G)-Size(Union(Elements(Centralizer(G,M[i])),
      Elements(Centralizer(G,M[j]))));
      k:=k+(1/2)^t;
    fi;
  od;
fi;
```
if Centralizer(G,M[j])<>G and IsAbelian(\text{Group}(M[i],M[j]))
\text{ and } M[i]<M[j] \text{ then }
\begin{align*}
t &:= \text{Order}(G) - \text{Size}(\text{Union}(\text{Elements}(\text{Centralizer}(G,M[i]),
\text{Elements}(\text{Centralizer}(G,M[j])))));
\end{align*}
s := s + (1/2)^t + t \cdot (1/2)^t;
fi;
\text{od; }
fi;
\text{od; }
P := (s+k)/2;
\end{verbatim}

By this code, one gets that $P_G \leq P < 1$ with the following exceptions:

(i) $D_6, D_8, Q_8, D_{10}, Q_{12}, D_{14}, D_6 \times \mathbb{Z}_3, D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$, and all non-abelian groups of order 16.

(ii) $G_1$ and $G_2$, where $G_1 = \text{SmallGroup}(32,49)$ and $G_2 = \text{SmallGroup}(32,50)$ in GAP.

Note that $\Gamma_H \cong K_4[2], K_6[4]$ for any non-abelian group $H$ of order 16. By Lemmas 2.2, 2.3 and Proposition 2.4, the rainbow 2-connectivity of the non-commuting graph of each group in (i) is 2.

Next we shall prove that $\text{rc}_2(\Gamma_{G_1}) = \text{rc}_2(\Gamma_{G_2}) = 2$. Note that $\Gamma_{G_1} \cong J(6,2) \circ K_2$.

For convenience, in the following we use $ab$ to denote the set $\{a, b\}$ for two distinct letters $a, b$. Write $\Gamma = J(6,2) \circ K_2$ and $V(\Gamma) = \{a_i a_j, b_i b_j : 1 \leq i, j \leq 6, i \neq j\}$.

$E(\Gamma) = \{a_i a_j, a_i a_k, \{b_i b_j, b_i b_k\}, \{a_i a_j, b_i b_k\} : 1 \leq i, j, k \leq 6, i \neq j, i \neq k, j \neq k\}.$

We assign the edges in $\{\{a_i a_j, a_i a_k\} : i > \max\{j, k\}\}$, $\{\{b_i b_j, b_i b_k\} : i > \max\{j, k\}\}$ and $\{\{a_i a_j, b_i b_k\} : i < \min\{j, k\}\}$ a color and all other edges another color. It follows that $\text{rc}_2(\Gamma) = 2$. Hence $\text{rc}_2(\Gamma_{G_1}) = \text{rc}_2(\Gamma_{G_2}) = 2$.

Combining Propositions 3.1 and 3.2, we complete the proof of Theorem 1.1.

Proof of Theorem 1.2: By Theorem 1.1, we may assume $k \geq 3$. Note that for any non-abelian group $H$, $\kappa(\Gamma_H)$ is divisible by $|\text{Z}(H)|$ by [1], Proposition 2.4]. Therefore, we may choose a non-abelian group $G$ with $k \leq \kappa(\Gamma_G)$. Next we prove that $\text{rc}_k(\Gamma_G) = 2$ if $|G|$ is large enough.

We randomly color the edges of $\Gamma_G$ with two colors. Denote by $P_G$ the probability that such a random coloring makes it not rainbow-2-connected. It suffices to show that $P_G < 1$. Let $x$ and $y$ be distinct vertices of $\Gamma_G$. If $x$ and $y$ are adjacent, then the probability that there exist no $k$ rainbow paths of length 2 from $x$ to $y$ is

$$
\sum_{i=0}^{k-2} \binom{\tau(x,y)}{i} \left(\frac{1}{2}\right)^{\tau(x,y)}.
$$

If $x$ is not adjacent to $y$, then the probability that there are no $k$ rainbow paths of length 2 from $x$ to $y$ is

$$
\sum_{i=0}^{k-1} \binom{\tau(x,y)}{i} \left(\frac{1}{2}\right)^{\tau(x,y)}.
$$
Write $|G| = n$. Then we have

$$
\mathcal{P}_G \leq \sum_{x \sim y} \sum_{i=0}^{k-2} \binom{\tau(x, y)}{i} \left( \frac{1}{2} \right)^i \tau(x, y) + \sum_{x \sim y} \sum_{i=0}^{k-1} \binom{\tau(x, y)}{i} \left( \frac{1}{2} \right)^i \tau(x, y)
$$

$$
\leq \sum_{x \sim y} \sum_{i=0}^{k-2} \tau(x, y)^i \left( \frac{1}{2} \right)^i \tau(x, y) + \sum_{x \sim y} \sum_{i=0}^{k-1} \tau(x, y)^i \left( \frac{1}{2} \right)^i \tau(x, y)
$$

$$
\leq \left( \frac{1}{2} \right)^n \left( \sum_{x \sim y} \sum_{i=0}^{k-2} n^i + \sum_{x \sim y} \sum_{i=0}^{k-1} n^i \right) \quad \text{(by Lemma 2.1)}
$$

$$
= \left( \frac{1}{2} \right)^n \left( \sum_{i=0}^{k-2} \binom{|V(\Gamma_G)|}{2} n^i + \left( \left( \binom{|V(\Gamma_G)|}{2} - |E(\Gamma_G)| \right)n^{k-1} \right) \right)
$$

$$
< \left( \frac{1}{2} \right)^n \left( \sum_{i=0}^{k-2} n^{i+2} + n^{k+1} \right)
$$

$$
= \frac{\sum_{i=2}^{k+1} n^i}{2^n}.
$$

This implies that $\mathcal{P}_G < 1$ if $n$ is large enough. \qed

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