INVASION WAVES FOR A NONLOCAL DISPERSAL PREDATOR-PREY MODEL WITH TWO PREDATORS AND ONE PREY

FEIYING YANG, WANTONG LI* AND RENHU WANG

School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

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ABSTRACT. This paper is concerned with the propagation dynamics of a nonlocal dispersal predator-prey model with two predators and one prey. Precisely, our main concern is the invasion process of the two predators into the habitat of one prey, when the two predators are weak competitors in the absence of prey. This invasion process is characterized by the spreading speed of the predators as well as the minimal wave speed of traveling waves connecting the predator-free state to the coexistence state. Particularly, the right-hand tail limit of wave profile is derived by the idea of contracting rectangle.

1. Introduction. In population dynamics, predator-prey systems have been widely studied due to their importance. Traveling wave, as a special solution maintaining its shape and moving at a constant speed, is a very important dynamical issue in the field of reaction-diffusion equations and is a useful tool to predict some ecological phenomena of different predator-prey systems, one can see Ai et al. [2], Du et al. [9], Hsu and Lin [14], Lin [19], Wang and Lin [25], Zhang and Jin [34], Zhang et al. [35] for further understanding. Moreover, Lin and Ruan [20] used the idea of contracting rectangles to verify the limit behavior of a traveling wave solution. But in [20] they needed strictly contracting rectangles. Huang and Lin [15] obtained the minimal wave speed of non-negative traveling wave solutions connecting trivial equilibrium with positive equilibrium, where the right-hand tail limit of wave profile is shown by the abstract results in Lin and Ruan [20]. Then, applying the idea in [20], Bi and Pan [4] proved the limit behavior of traveling wave solutions by general contracting rectangles. Recently, Guo et al. [13] considered the traveling wave solutions in a predator-prey system with two alien predators and one aborigine prey in which the growth rates of both predators are negative. They characterized the invading speed of these two predators by the minimal wave speed of traveling wave solutions connecting the predator-free state to the coexistence state, where the right-hand tail limit of wave profile is obtained by a new form of shrinking rectangles.

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* Corresponding author.
It is known that the long distance dispersal events play a key role in the spread of many natural populations, and the nonlocal term (spatial convolution with a dispersal kernel) is fit for modelling the diffusion phenomenon, see Andreu-Vaillo et al. [1], Fife [11], Murray [21] and so on. Nowadays, there are many results regarding the nonlocal dispersal problems, see Bates et al. [3], Chen [6], Shen and Zhang [22], Sun et al. [24] and Zhang et al. [32] for the monotone scalar equations, and Bao et al. [5], Li et al. [18] and Wang and Lv [26] for competition/cooperative systems. In addition, we refer the readers to Yang and Li [30], and Yang et al. [31] concerning the wave phenomena for some special SIR epidemic models with nonlocal dispersal. For traveling waves of nonlocal dispersal predator-prey type, there are very limited results so far (see, e.g., Sherratt [23]). Zhang et al. [33] considered traveling waves in delayed predator-prey systems with nonlocal diffusion and stage structure. Dong et al. [10] characterized the minimal speed of traveling waves for a Leslie-Gower predator-prey model with nonlocal dispersal. Ducrot et al. [8] obtained the invasion of the predator into the habitat of the aborigine prey by the asymptotic spreading of a predator-prey system with nonlocal dispersal. Very recently, Yang et al. [29] considered the traveling waves for a class of nonlocal dispersal non-cooperative system of the form

\[
\begin{align*}
    u_t &= d_1(J_1 * u - u) + g(u) - f(u, v)v, \\
    v_t &= d_2(J_2 * v - v) + \alpha f(u, v)v - \delta v - \gamma v^2,
\end{align*}
\]

which can model the predator-prey and disease-transmission mechanism.

From the above introduction, we know that the known results are about 2-species predator-prey models with nonlocal dispersal. In this paper, we are concerned with the following 3-species predator-prey model

\[
\begin{align*}
    (u_1)_t &= d_1(J_1 * u_1 - u_1) + r_1 u_1 (-1 - u_1 - k u_2 + a v), \\
    (u_2)_t &= d_2(J_2 * u_2 - u_2) + r_2 u_2 (-1 - h u_1 - u_2 + a v), \\
    (v)_t &= d_3(J_3 * v - v) + r_3 v (1 - b u_1 - b u_2 - v),
\end{align*}
\]

in which \(d_i, (i = 1, 2, 3)\) are diffusion coefficients of species \(u_1, u_2\) and \(v\), respectively. The positive parameters \(r_1, r_2\) denote the death rates of predators \(u, v\), respectively, and \(r_3 > 0\) denotes the intrinsic growth rate. \(h, k\) are interspecific competition coefficients. Both predation rates of predators \(u, v\) are equal to \(r_3 b\), and their biomass conversion rates are assumed to be \(r_1 a\) and \(r_2 a\) for simplicity. Throughout the whole paper, we always assume that the kernel functions \(J_i(x) (i = 1, 2, 3)\) satisfy

(J1) \(J_i \in C^1(\mathbb{R}), J_i(x) = J_i(-x) \geq 0, \int_{\mathbb{R}} J_i(x) dx = 1.\)

(J2) \(J_i\) satisfy the decay bounds:

\[
\int_{\mathbb{R}} J_i(x)e^{\lambda x} dx < \infty \text{ for any } \lambda \in (0, \lambda_0) \text{ and } \lim_{\lambda \to \lambda_0^-} \int_{\mathbb{R}} J_i(x)e^{\lambda x} dx = \infty
\]

for some \(\lambda_0 \in (0, +\infty)\) and \(\int_{\mathbb{R}} |J_i'(x)| dx < \infty.\)

Further, the parameters satisfy

(H) \(a > 1, 0 < h, k < 1\) and \(0 < b < \frac{1}{2(a-1)}.\)

Under the condition (H), there is a unique co-existence state \((u_1^*, u_2^*, v^*)\), where

\[
\begin{align*}
    u_1^* &= \frac{1 - k}{1 - h k} (av^* - 1), \\
    u_2^* &= \frac{1 - h}{1 - h k} (av^* - 1),
\end{align*}
\]
\[ v^\ast = \frac{(1 - hk) + b(2 - h - k)}{(1 - hk) + ab(2 - h - k)}. \]

Meanwhile, there is an unstable predator-free state \((0, 0, 1)\). Here, \(0 < h, k < 1\) implies that two predators are weak competitors. Particularly, it should be pointed out that there is no traveling wave solution connecting the predator-free state and the co-existence state of (1.1) if the condition (J2) is replaced by \(\int_R J_i(x)e^{\lambda x}dx = \infty\) for all \(\lambda > 0\) with \(i = 1\) or \(i = 2\) or \(i = 3\), see Theorem 2.3 in Section 2.

In the present paper, we characterized the features of the predator invasion process by the asymptotic spreading and traveling wave solutions for system (1.1). First, we want to study the existence and nonexistence of traveling waves connecting the predator-free state \((0, 0, 1)\) and the co-existence state \((u_1^\ast, u_2^\ast, v^\ast)\). It is known that the method of applying Schauder’s fixed point theorem with the help of upper-lower-solution has been successful to prove the existence of traveling waves about predator-prey type model with nonlocal dispersal, see [29, 31]. We first construct a suitable pairs of upper-lower-solution for the 3-species predator-prey model (1.1) whether at super-critical speed or critical speed on an idea from [13]. Then, combining Schauder’s fixed point theorem, we can get the existence of traveling waves. From a biological point of view, the invasion of predators is successful if the traveling waves are persistent at the end. That is, it is important to verify the convergence to the co-existence state at the end, which is very difficult and challenging to estimate because system (1.1) is non-monotonic and nonlocal. In [29], Yang et al. only obtained the weak traveling waves to avoid this difficult, in which it is enough if we only want to know whether the invasion is successful. In this paper, inspired by the idea from [7, 15, 13], we drive the convergence to the co-existence state at the end by the method of contracting rectangle, which strongly depends on the boundedness of traveling waves. Further, the nonexistence of traveling waves is obtained by the delicate analysis. Finally, we consider the initial value problem for system (1.1) with initial values

\[ u_i(x, 0) = u_{i0}(x), \quad v(x, 0) = 1, \quad x \in \mathbb{R}, \quad i = 1, 2, \]

wherein \(u_{i0}(x)\) \((i = 1, 2)\) is some nonnegative continuous function with nonempty compact support. Inspired by the idea from [8, 27], we are interested in the persistence and spatial invasion of the predator population by using the notion of spreading speed for system (1.1) with (1.2). Combining comparison principle and the known results on the scalar logistic equation with nonlocal dispersal ([16]), we are able to characterize the asymptotic spreading speed for system (1.1) with (1.2).

The rest of this paper is organized as follows. In section 2, we discuss the characteristic equation which is obtained by the linearization of the wave equation corresponding to system (1.1) on the predator-free state, and give the main results in this paper. In section 3, the upper-lower-solution are constructed. And applying Schauder’s fixed point theorem, we get the existence of traveling waves in the super-critical speed case and the critical speed case. Then, the contracting rectangle is used to prove the asymptotic behavior of traveling waves at \(+\infty\). In section 4, the nonexistence of traveling waves is obtained. In section 5, we give the spreading speed for the predators. Finally, some discussion appears in section 6.

**2. Preliminaries and main results.** Let \(u_1(x, t) = U_1(x + ct), \ u_2(x, t) = U_2(x + ct), \ v(x, t) = V(x + ct)\) and \(\xi = x + ct\). Then, the wave equation corresponding to
(1.1) is as follows:
\[
\begin{cases}
  c U_1'(\xi) = d_1(J_1 \ast U_1(\xi) - U_1(\xi)) + r_1 U_1(-1 - U_1 - k U_2 + a V), \\
  c U_2'(\xi) = d_2(J_2 \ast U_2(\xi) - U_2(\xi)) + r_2 U_2(-1 - h U_1 - U_2 + a V), \\
  c V'(\xi) = d_3(J_3 \ast V(\xi) - V(\xi)) + r_3 V(1 - b U_1 - b U_2 - V).
\end{cases}
\]

We intend to find traveling waves \((U_1(\xi), U_2(\xi), V(\xi))\) connecting two points \((0, 0, 1)\) and \((u_1^*, u_2^*, v^*)\). That is,
\[
\lim_{\xi \to -\infty} (U_1(\xi), U_2(\xi), V(\xi)) = (0, 0, 1), \quad \lim_{\xi \to +\infty} (U_1(\xi), U_2(\xi), V(\xi)) = (u_1^*, u_2^*, v^*).
\]

Define
\[
f_i(\lambda, c) = d_i \left( \int_{\mathbb{R}} J_i(y) e^{-\lambda y} dy - 1 \right) - c \lambda + r_i(a - 1), \quad i = 1, 2.
\]

By direct calculation, it is easy to get
\[
f_i(0, c) = r_i(a - 1) > 0, \quad \lim_{\lambda \to +\infty} f_i(\lambda, c) = +\infty \text{ for all } c,
\]
\[
\frac{\partial f_i(\lambda, c)}{\partial c} = -\lambda < 0 \text{ and } \lim_{c \to +\infty} f_i(\lambda, c) = -\infty \text{ for } \lambda > 0,
\]
\[
\frac{\partial f_i(\lambda, c)}{\partial \lambda} \bigg|_{\lambda = 0} = -c < 0 \text{ for all } c > 0,
\]
\[
\frac{\partial^2 f_i(\lambda, c)}{\partial \lambda^2} = d_i \int_{\mathbb{R}} J_i(y) y^2 e^{-\lambda y} dy > 0 \text{ for all } \lambda \text{ and } c.
\]

Thus, we have the following results.

**Lemma 2.1.** Some \(c_i^* > 0\) and \(\lambda_i > 0\) exist such that
\[
\frac{\partial f_i(\lambda, c)}{\partial \lambda} \bigg|_{(\lambda_i, c_i^*)} = 0 \quad \text{and} \quad f_i(\lambda_i, c_i^*) = 0, \quad i = 1, 2.
\]

Furthermore, the following alternatives hold.

(i) For each \(i = 1, 2\) and \(c > c_i^*\), the equation \(f_i(\lambda, c) = 0\) has two positive roots, denote \(\lambda_i\) be the smaller one and \(\lambda_i\) be the larger one, respectively. Particularly,
\[
c_i^* = d_i \int_{\mathbb{R}} J_i(y)(-y) e^{-\lambda_i y} dy.
\]

(ii) If \(0 < c < c_i^*\), we have \(f_i(\lambda, c) > 0\) for all \(\lambda > 0\).

**Remark 1.** We note that \(c_i^*\) \((i = 1, 2)\) can be defined as follows:
\[
c_i^* = \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \left[ d_i \int_{\mathbb{R}} J_i(y) e^{-\lambda y} dy - d_i + r_i(a - 1) \right] \right\}.
\]

According to (J1), (J2) and (H), it is easy to see that \(c_i^*\) is well defined and \(c_i^* > 0\).

Additionally, define
\[
f_3(\lambda, c) = d_3 \left( \int_{\mathbb{R}} J_3(y) e^{-\lambda y} dy - 1 \right) - c \lambda - r_3.
\]

The direct computation gives that the equation \(f_3(\lambda, c) = 0\) has a unique positive root, denoted by \(\lambda_3\).

Let \(c^* = \max\{c_1^*, c_2^*\}\). The main results in this paper are as follows.
Theorem 2.2. Suppose (J1), (J2) and (H) hold. If $c \geq c^*$, system (1.1) admits a positive traveling wave solution $(U_1(\xi), U_2(\xi), V(\xi))$ satisfying

$$
\lim_{\xi \to -\infty} U_1(\xi) = \lim_{\xi \to -\infty} U_2(\xi) = 0 \text{ and } \lim_{\xi \to -\infty} V(\xi) = 1.
$$

Moreover, if we further assume

$$
0 < b < \frac{1}{2a} \min\{1 - h, 1 - k\},
$$

then $(U_1(\xi), U_2(\xi), V(\xi))$ satisfies

$$
\lim_{\xi \to +\infty} U_1(\xi) = u_1^*, \lim_{\xi \to +\infty} U_2(\xi) = u_2^* \text{ and } \lim_{\xi \to +\infty} V(\xi) = v^*.
$$

Additionally, there is no bounded positive traveling wave solution connecting the predator-free state and the co-existence state for any $0 < c < c^*$.

Theorem 2.3. Suppose (J1) and (H) hold. We further assume that one of the following conditions

$$
\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy = +\infty \text{ for all } \lambda > 0, \ i = 1, 2, 3
$$

holds. Then, there is no bounded positive traveling wave solution of (1.1) connecting the predator-free state and the co-existence state.

Theorem 2.4. (Predator’s spreading) Assume (J1), (J2), (H) and (2.2) hold. Then $c^*$ is the spreading speed of the predator for system (1.1) with initial data (1.2) as long as $u_{i0}(x) (i = 1, 2)$ is a nonnegative continuous function with nonempty compact support and $u_{i0} \leq a - 1$ in $\mathbb{R}$. That is, $u_i(x, t) (i = 1, 2)$ satisfies

$$
\lim_{t \to \infty} \sup_{|x| > ct} u_i(x, t) = 0 \text{ for any } c > c^*,
$$

$$
\lim_{t \to \infty} \inf_{|x| < ct} u_i(x, t) > 0 \text{ for any } 0 < c < c^*.
$$

3. Existence of traveling waves. In this section, we discuss the existence of traveling waves when $c \geq c^*$, where $c^*$ is defined as in Section 2.

3.1. Lower and upper solution.

- $c > c^*$

Denote $\alpha = a - 1$ and define functions as follows.

$$
\overline{v}_1(\xi) := \min\{\alpha e^{\lambda_1 \xi}, \alpha\}, \quad \underline{v}_1(\xi) := \alpha \max\{e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi}, 0\},
$$

$$
\overline{v}_2(\xi) := \min\{\alpha e^{\lambda_2 \xi}, \alpha\}, \quad \underline{v}_2(\xi) := \alpha \max\{e^{\lambda_2 \xi} - q_2 e^{\mu_2 \lambda_2 \xi}, 0\},
$$

$$
\overline{v}(\xi) := 1, \quad \underline{v}(\xi) := \max\{1 - (e^{\lambda_2 \xi} + p e^{\beta \lambda_2 \xi}), 1 - a \gamma\},
$$

where $\gamma \in \left(\frac{2b(a-1)}{a}, \frac{1}{2}\right)$, $\mu_i > 1$, $q_i > 1$, $i = 1, 2$, $p > 0$ and $\beta \in (0, 1)$. Below, we intend to prove that $(\overline{u}_1(\xi), \overline{u}_2(\xi), \underline{v}(\xi))$ and $(\overline{v}_1(\xi), \overline{v}_2(\xi), \overline{v}(\xi))$ are lower and upper solutions of system (2.1).

Lemma 3.1. The function $(\overline{v}_1(\xi), \overline{v}_2(\xi), \overline{v}(\xi))$ satisfies

$$
c \overline{v}_1(\xi) \geq d_1 (J_1 * \overline{u}_1 - \overline{v}_1(\xi)) + r_1 \overline{u}_1(\xi)(-1 - \overline{v}_1(\xi) - k \overline{u}_2(\xi) + a \overline{v}(\xi)), \quad \xi \neq 0, \quad (3.1)
$$

$$
c \overline{v}_2(\xi) \geq d_2 (J_2 * \overline{u}_2 - \overline{v}_2(\xi)) + r_2 \overline{u}_2(\xi)(-1 - h \overline{u}_1(\xi) - \overline{v}_2(\xi) + a \overline{v}(\xi)), \quad \xi \neq 0, \quad (3.2)
$$

$$
c \overline{v}(\xi) \geq d_3 (J_3 * \overline{v} - r_3 \overline{v}(\xi))(1 - b \overline{u}_1(\xi) - b \overline{u}_2(\xi) - \overline{v}(\xi)). \quad (3.3)
$$
Proof. For $\xi > 0$, there is $\xi_1(\xi) = \alpha$, $\xi_2(\xi) = 0$, $\xi(\xi) = 1$. Since

$$J_1 * \xi_1(\xi) = \alpha \min \left\{ e^{\lambda_1 \xi} \int_\mathbb{R} J_1(y) e^{-\lambda_1 y} dy, 1 \right\},$$

and

$$r_1 \xi_1(\xi)(-1 - \xi_1(\xi) - k \xi_2(\xi) + a \xi(\xi)) = r_1 \alpha (-1 - \alpha + a) = 0.$$

For $\xi < 0$, $\xi_1(\xi) = \alpha e^{\lambda_1 \xi}$, $\xi_2(\xi) > 0$ and $\xi(\xi) = 1$. Thus, we have

$$e^\xi \xi_1(\xi) - d_1 (J_1 * \xi_1 - \xi_1)(\xi) - r_1 \xi_1(\xi)(-1 - \xi_1(\xi) - k \xi_2(\xi) + a \xi(\xi))$$

$$\geq \alpha \xi_1 e^{\lambda_1 \xi} - \alpha e^{\lambda_1 \xi} \left( d_1 \int_\mathbb{R} J_1(y) e^{-\lambda_1 y} dy - d_1 \right) - r_1 \alpha e^{\lambda_1 \xi} (-1 - \alpha e^{\lambda_1 \xi} - k \xi_2(\xi) + a)$$

$$= - \alpha f_1(\lambda_1, c) e^{\lambda_1 \xi} - r_1 \alpha e^{\lambda_1 \xi} (-ae^{\lambda_1 \xi} - k \xi_2(\xi)) \geq 0.$$

Then, (3.1) holds. Similarly, (3.2) is true. Additionally, for all $\xi$, $\xi(\xi) = 1$. It follows from $(J_3 * \xi - \xi)(\xi) = 0$ and

$$r_3 \xi(1 - b \xi_1(\xi) - b \xi_2(\xi) - \xi(\xi)) = -r_3 (b \xi_1(\xi) + b \xi_2(\xi)) \leq 0$$

that (3.3) holds.

Lemma 3.2. The function $(\xi_1(\xi), \xi_2(\xi), \xi(\xi))$ satisfies

$$c_2 \xi_1(\xi) \leq d_1 (J_1 * \xi_1 - \xi_1)(\xi) + r_1 \xi_1(\xi)(-1 - \xi_1(\xi) - k \xi_2(\xi) + a \xi(\xi)), \xi \neq \xi_1,$$  

$$c_2 \xi_2(\xi) \leq d_2 (J_2 * \xi_2 - \xi_2)(\xi) + r_2 \xi_2(\xi)(-1 - h \xi_1(\xi) - \xi_2(\xi) + a \xi(\xi)), \xi \neq \xi_2,$$  

$$c_2 \xi(\xi) \leq d_3 (J_3 * \xi - \xi)(\xi) + r_3 \xi(\xi)(1 - b \xi_1(\xi) - b \xi_2(\xi) - \xi(\xi)), \xi \neq \xi_3,$$  

in which $\xi_i = \frac{1}{(\mu_i - 1) \lambda_i} \ln \frac{1}{\mu_i}$ (i = 1, 2), $\xi_3 := \xi_3(\alpha, \beta, \lambda_3) < 0$.

Proof. Note that $\xi_i < 0$ if $\mu_i > 1$ and $q_i > 1$ (i = 1, 2). When $\xi > \xi_1$, $u_1(\xi) = 0$. Then, (3.4) holds naturally. When $\xi < \xi_1$, $u_1(\xi) = \alpha(\xi - \xi_1 e^{\mu_1 \lambda_1 \xi})$. In view of $\xi_1 < 0$, $\xi_2(\xi) = \alpha e^{\lambda_2 \xi}$ and $\xi(\xi) \geq 1 - (e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi})$. Meanwhile,

$$J_1 * \xi_1(\xi) \geq \max \left\{ 0, \alpha \left( e^{\lambda_1 \xi} \int_\mathbb{R} J_1(y) e^{-\lambda_1 y} dy - q_1 e^{\mu_1 \lambda_1 \xi} \int_\mathbb{R} J_1(y) e^{-\lambda_1 y} dy \right) \right\}.$$  

By direct computation, we have

$$c_2 \xi_1(\xi) - d_1 (J_1 * \xi_1 - \xi_1)(\xi) - r_1 \xi_1(\xi)(-1 - \xi_1(\xi) - k \xi_2(\xi) + a \xi(\xi))$$

$$\leq \alpha \xi_1 e^{\lambda_1 \xi} - q_1 \mu_1 \xi_1 e^{\mu_1 \lambda_1 \xi} - d_1 \alpha e^{\lambda_1 \xi} \int_\mathbb{R} J_1(y) e^{-\lambda_1 y} dy$$

$$+ d_1 \alpha q_1 e^{\mu_1 \lambda_1 \xi} \int_\mathbb{R} J_1(y) e^{-\mu_1 \lambda_1 y} dy + d_1 \alpha e^{\lambda_1 \xi} - d_1 \alpha e^{\lambda_1 \xi} e^{\mu_1 \lambda_1 \xi}$$

$$- r_1 \alpha e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi} [-1 - \alpha e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi}]$$

$$- kae^{\lambda_1 \xi} + a - (ae^{\lambda_1 \xi} + pe^{\beta \lambda_3 \xi})$$

$$\leq - \alpha e^{\lambda_1 \xi} f_1(\lambda_1, c) + \alpha q_1 e^{\mu_1 \lambda_1 \xi} f_1(\mu_1 \lambda_1, c)$$

$$+ r_1 \alpha e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi} [\alpha e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi}]$$

$$+ kae^{\lambda_1 \xi} + a e^{\lambda_1 \xi} + pe^{\beta \lambda_3 \xi}]$$

$$\leq q_1 e^{\mu_1 \lambda_1 \xi} f_1(\mu_1 \lambda_1, c) + r_1 \alpha e^{\lambda_1 \xi} [\alpha e^{\lambda_1 \xi} + kae^{\lambda_1 \xi} + a (e^{\lambda_1 \xi} + pe^{\beta \lambda_3 \xi})]$$

$$= \alpha e^{\mu_1 \lambda_1 \xi} [q_1 f_1(\mu_1 \lambda_1, c) + r_1 [\alpha e^{\lambda_1 (2 - \mu_1) \xi} + kae^{(\lambda_1 + \lambda_2 - \mu_1 \lambda_1) \xi}$$
\[ + a\{e^{(\lambda_3 + \lambda_1 - \mu_1 \lambda_1)\xi} + pe^{(\beta \lambda_3 + \lambda_1 - \mu_1 \lambda_1)\xi}\}. \]

Now, we choose
\[ 1 < \mu_1 < \min \left\{ \frac{\lambda_1}{\lambda_1}, 2, 1 + \frac{\lambda_2}{\lambda_1}, 1 + \frac{\lambda_3}{\lambda_1}, 1 + \frac{\beta \lambda_3}{\lambda_1} \right\} \]
and
\[ q_1 > \frac{r_1(\alpha + k\alpha + a + ap)}{-f_1(\mu_1 \lambda_1, c)} \text{ large enough.} \]

Then, the inequality (3.4) holds.

By the same discussion, if we take
\[ 1 < \mu_2 < \min \left\{ \frac{\lambda_2}{\lambda_2}, 2, 1 + \frac{\lambda_1}{\lambda_2}, 1 + \frac{\lambda_3}{\lambda_2}, 1 + \frac{\beta \lambda_3}{\lambda_2} \right\} \]
and
\[ q_2 > \frac{r_2(\alpha + h\alpha + a + ap)}{-f_2(\mu_2 \lambda_2, c)} \text{ large enough,} \]
then (3.5) is true.

To get (3.6), we first consider the equation \( e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi} = a\gamma \) and let \( \xi < \theta_3 \) be the solution of this equation. By the choice of \( \gamma \), we know \( a\gamma < 1 \). Thus, \( \xi < 0 \). If \( \xi > \xi_3 \), then \( \xi_3 = 1 - a\gamma \). And \( \pi_1(\xi) < \alpha \), \( \pi_2(\xi) < \alpha \). In these cases, one can get
\[ c\xi' \xi - d_3(J_3 * \nu - \psi)(\xi) - r_3\psi(\xi)(1 - b\pi_1(\xi) - b\pi_2(\xi) - \nu(\xi)) \leq -r_3(1 - a\gamma)(1 - b\alpha - b_a - 1 + a\gamma) \]
\[ = -r_3(1 - a\gamma)(a\gamma - 2b\alpha) \leq 0 \]
with \( \gamma > \frac{2b\alpha}{a} = \frac{2(a-1)}{a} \). If \( \xi < \xi_3 \), then \( \psi(\xi) = 1 - (e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi}) \) and \( \pi_1(\xi) = a\xi, \)
\( \pi_2(\xi) = a\beta \xi \). Meanwhile,
\[ J_3 * \nu(\xi) \geq \max \left\{ 1 - e^{\lambda_3 \xi} \int_R J_3(y)e^{-\lambda_3 \xi} dy - pe^{\beta \lambda_3 \xi} \int_R J_3(y)e^{-\beta \lambda_3 \xi} dy, 1 - a\gamma \right\}. \]
Consequently,
\[ c\xi' \xi - d_3(J_3 * \nu - \psi)(\xi) - r_3\psi(\xi)(1 - b\pi_1(\xi) - b\pi_2(\xi) - \nu(\xi)) \leq -c\lambda_3 e^{\lambda_3 \xi} - cp\beta \lambda_3 e^{\beta \lambda_3 \xi} + d_3 e^{\lambda_3 \xi} \int_R J_3(y)e^{-\lambda_3 \xi} dy \]
\[ + d_3 pe^{\beta \lambda_3 \xi} \int_R J_3(y)e^{-\beta \lambda_3 \xi} dy - d_3 e^{\lambda_3 \xi} - d_3 pe^{\beta \lambda_3 \xi} \]
\[ - r_3[1 - (e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi})][e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi}] - b\alpha e^{\lambda_3 \xi} - b\alpha e^{\beta \lambda_3 \xi}] \]
\[ = le^{\lambda_3 \xi} f_3(\lambda_3, c) + pe^{\beta \lambda_3 \xi} f_3(\beta \lambda_3, c) + r_3b\alpha(e^{\lambda_3 \xi} + e^{\lambda_2 \xi}) \]
\[ + r_3(e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi})[e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi}] - b\alpha e^{\lambda_3 \xi} - b\alpha e^{\lambda_2 \xi}] \]
\[ \leq e^{\beta \lambda_3 \xi} f_3(\beta \lambda_3, c) + r_3[e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi}]^2 + b\alpha e^{\lambda_3 \xi} + e^{\lambda_2 \xi}] \]
Since \( \xi < \xi_3 \), we have \( e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi} \leq a\gamma \). Hence,
\[ pe^{\beta \lambda_3 \xi} f_3(\beta \lambda_3, c) + r_3[e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi}]^2 + b\alpha e^{\lambda_3 \xi} + e^{\lambda_2 \xi}] \]
\[ \leq e^{\beta \lambda_3 \xi} f_3(\beta \lambda_3, c) + r_3[a\gamma(e^{\lambda_3 \xi} + pe^{\beta \lambda_3 \xi}) + b\alpha(e^{\lambda_3 \xi} + e^{\lambda_2 \xi})] \]
\[ = e^{\beta \lambda_3 \xi} \{ pf_3(\beta \lambda_3, c) + r_3[a\gamma(e^{(1-\beta) \lambda_3 \xi} + p) + b\alpha(e^{(\lambda_1 - \beta \lambda_3) \xi} + e^{(\lambda_2 - \beta \lambda_3) \xi})]. \]
Due to $0 < \beta < 1$, there is $f_3(\beta \lambda_3, c) < 0$. Some $\beta_0 > 0$ exists such that $f_3(\beta \lambda_3, c) + \alpha \gamma r_3 < 0$ if $\beta < \beta_0$. Therefore, it is noticed that if we take

$$0 < \beta < \min \left\{ 1, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}, \beta_0 \right\}$$

and

$$p > \frac{r_3(a \gamma + 2ba)}{-(f_3(\beta \lambda_3, c) + \alpha \gamma r_3)}$$

large enough, then (3.6) holds.

$\bullet \ c = c^*$

Without loss of generality, we may assume that $c^*_1 > c^*_2$. Thus, $c^* = c^*_1$. In this case, we have $c = c^*_1 > c^*_2$. Hence, $f_1(\lambda, c) = 0$ has a positive double root $\lambda_1$ and $f_2(\lambda, c) = 0$ has two positive roots $\lambda_2, \lambda_3$ with $\lambda_2 < \lambda_3$. Define

$$\bar{u}_1(\xi) = \begin{cases} e\alpha \lambda_1(-\xi) e^{\lambda_1 \xi}, & \xi < \overline{\xi}_1, \\ \alpha, & \xi > \overline{\xi}_1, \end{cases}$$

and

$$\bar{u}_2(\xi) = \begin{cases} [\alpha \lambda_1 e(-\xi) - q_1(-\xi)^{\frac{1}{2}}] e^{\lambda_1 \xi}, & \xi < \overline{\xi}_1, \\ 0, & \xi > \overline{\xi}_1, \end{cases}$$

where $\overline{\xi}_1 = -\frac{1}{\lambda_1}$ and $\overline{\xi}_2 = -(\frac{q_1}{\alpha \lambda_1^2})^2$ for $q_1 > 1$ large enough. Let $\bar{v}_2(\xi), \bar{u}_2(\xi), \bar{v}(\xi)$ and $\bar{u}(\xi)$ be defined as $c > c^*$ above. We can verify that $(\bar{u}_1(\xi), \bar{u}_2(\xi), \bar{v}(\xi))$ and $(\bar{u}_1(\xi), \bar{u}_2(\xi), \bar{v}(\xi))$ are upper and lower solutions of (2.1). In fact, for all $\xi > \overline{\xi}_1$, we have $\bar{u}_1(\xi) = \alpha, \bar{u}_2(\xi) > 0$ and $\bar{v}(\xi) = 1$. Obviously, (3.1) holds naturally. For all $\xi < \overline{\xi}_1$, $\bar{u}_1(\xi) = e\alpha \lambda_1(-\xi) e^{\lambda_1 \xi}, \bar{u}_2(\xi) > 0$ and $\bar{v}(\xi) = 1$. One can get

$$c^*_1 \bar{u}_1'(\xi) - d_1(J_1 * \bar{u}_1 - \bar{u}_1) + (1 - \bar{u}_1) (-q_1(\xi)^{\frac{1}{2}}) e^{\lambda_1 \xi} \geq -c^*_1 e\alpha \lambda_1 e^{\lambda_1 \xi} - c^*_1 e\alpha \lambda_1^2 e^{\lambda_1 \xi} + d_1 e\alpha \lambda_1 \xi \int \frac{J_1(y)(-y) e^{-\lambda_1 y} dy e^{\lambda_1 \xi}}{\int \frac{J_1(y)(-y) e^{-\lambda_1 y} dy e^{\lambda_1 \xi}}{d_1 e\alpha \lambda_1(\xi) - d_1 e\alpha \lambda_1 \xi e^{\lambda_1 \xi}}$$

$$+ r_d e\alpha \lambda_1 \xi e^{\lambda_1 \xi} (a - 1 - \bar{u}_1(\xi) - k\bar{u}_2(\xi))$$

$$= c^*_1 e\alpha \lambda_1 \xi [-c^*_1 + d_1 \int \frac{J_1(y)(-y) e^{-\lambda_1 y} dy e^{\lambda_1 \xi}}{d_1 e\alpha \lambda_1(\xi) - d_1 e\alpha \lambda_1 \xi e^{\lambda_1 \xi}}]$$

$$+ r_d e\alpha \lambda_1 \xi f_1(\lambda_1, c^*_1) - r_1 \bar{u}_1(\xi) \geq 0$$

Thus, (3.1) is true. By the same discussion as before, we also can prove that (3.2) and (3.3) are true.

Next, we show $(\bar{u}_1(\xi), \bar{u}_2(\xi), \bar{v}(\xi))$ is a lower solution of (2.1). To obtain this goal, we first give one fact which is always used below. That is,

$$\sup_{\xi \leq 0} \{(-\xi)^{\nu} e^{\sigma \xi}\} = \left(\frac{\nu}{\sigma e}\right)^\nu$$

for any positive constants $\sigma$ and $\nu$.

Note that for all $\xi > \overline{\xi}_1$, $\bar{u}_1(\xi) = 0$. So (3.4) holds naturally. For all $\xi < \overline{\xi}_1$, we have $\bar{u}_1(\xi) = [\alpha \lambda_1 e(-\xi) - q_1(-\xi)^{\frac{1}{2}}] e^{\lambda_1 \xi}$. Since $\overline{\xi}_1 < 0$, $\bar{v}_2(\xi) = \alpha e^{\lambda_1 \xi}$ and
\( v \geq 1 - (e^{\lambda^2} + pe^{\beta\lambda^2}). \) Thus, one can get

\[
c_1^* u_1^* (\xi) - d_1 (J_1 * u_1 - u_1)(\xi) - r_1 u_1 (\xi)(-1 - u_1(\xi) - k\pi^2 + aq(\xi)) \leq e^{\lambda^2} \left\{ \right. \\
\frac{1}{2} q_1 c_1^* (\xi)^{-\frac{1}{2}} + c_1^* \lambda_1 (\alpha \lambda_1 e^{\lambda^2} - q_1(\xi)^{-\frac{1}{2}}) \\
+ q_1 \int_{\mathbb{R}} J_1(y)e^{-\lambda^2} dy - d_1 \alpha \lambda_1 \int_{\mathbb{R}} J_1(y)ye^{-\lambda^2} dy \\
+ q_1 \int_{\mathbb{R}} J_1(y)(y - \xi)^{-\frac{1}{2}} e^{-\lambda^2} dy - d_1 \alpha \lambda_1 e^{\lambda^2} - d_1 q_1(\xi)^{-\frac{1}{2}} \\
- r_1 (\alpha \lambda_1 e^{\lambda^2} - q_1(\xi)^{-\frac{1}{2}})[-1 - kae^{\lambda^2} + a - a(e^{\lambda^2} + pe^{\beta\lambda^2})] \left. \right\}
\]

\[
e^{\lambda^2} \left\{ \right. \\
\frac{1}{2} q_1 c_1^* (\xi)^{-\frac{1}{2}} + f_1(c_1^*, \lambda_1) \cdot \alpha \lambda_1 e^{\lambda^2} - c_1^* \lambda_1 q_1(\xi)^{-\frac{1}{2}} \\
+ q_1 \int_{\mathbb{R}} J_1(y)(y - \xi)^{-\frac{1}{2}} e^{-\lambda^2} dy - d_1 q_1(\xi)^{-\frac{1}{2}} + r_1(a-1)q_1(\xi)^{-\frac{1}{2}} \\
- r_1 (\alpha \lambda_1 e^{\lambda^2} - q_1(\xi)^{-\frac{1}{2}})[-1 - kae^{\lambda^2} - a(e^{\lambda^2} + pe^{\beta\lambda^2})] \left. \right\}
\]

\[
e^{\lambda^2} \left\{ \right. \\
\frac{1}{2} q_1 c_1^* (\xi)^{-\frac{1}{2}} + q_1 \int_{\mathbb{R}} J_1(y)[(y - \xi)^{-\frac{1}{2}} - (\xi)^{-\frac{1}{2}}] e^{-\lambda^2} dy \\
+ r_1 (\alpha \lambda_1 e^{\lambda^2} - q_1(\xi)^{-\frac{1}{2}})^2 e^{\lambda^2} - r_1 \alpha \lambda_1 e^{\lambda^2} + r_1(a-1) \alpha \lambda_1 e^{\lambda^2} \\
+ r_1 a(\alpha \lambda_1 e^{\lambda^2} + pe^{\beta\lambda^2}) \left. \right\}
\]

\[
e^{\lambda^2} \left\{ \right. \\
\frac{1}{2} d_1 q_1 \int_{\mathbb{R}} J_1(y)(y - \xi)^{-\frac{1}{2}} e^{-\lambda^2} dy \\
+ d_1 q_1 \int_{\mathbb{R}} J_1(y)(\xi)^{-\frac{1}{2}} + \frac{1}{2} (\xi)^{-\frac{1}{2}} y - \frac{1}{8} (\xi)^{-\frac{3}{2}} y^2 e^{-\lambda^2} dy \\
- d_1 q_1 \int_{\mathbb{R}} J_1(y)e^{-\lambda^2} dy(\xi)^{-\frac{1}{2}} + r_1 \alpha \lambda_2 e^{\lambda^2} e^{-\lambda^2} \\
+ r_1 \alpha \lambda_2 e(\xi)^{-\frac{1}{2}} e^{\lambda^2} + r_1 a\alpha \lambda_1 e(\xi)^{-\frac{1}{2}} e^{\lambda^2} + r_1 a\alpha \lambda_1 ep(\xi)^{-\frac{1}{2}} e^{\beta\lambda^2} \left. \right\}
\]

\[
\leq (-\xi)^{-\frac{1}{2}} e^{\lambda^2} \left\{ \right. \\
- \frac{1}{8} d_1 q_1 \int_{\mathbb{R}} J_1(y)y^2 e^{-\lambda^2} dy + r_1 \alpha \lambda_2 e(\xi)^{-\frac{1}{2}} e^{\lambda^2} \left( \frac{7}{2\lambda_1 e} \right)^{\frac{7}{2}} \\
+ r_1 \alpha \lambda_2 e \left( \frac{5}{2\lambda_2 e} \right)^{\frac{5}{2}} + r_1 a\alpha \lambda_1 e \left( \frac{5}{2\lambda_3 e} \right)^{\frac{5}{2}} + r_1 a\alpha \lambda_1 ep \left( \frac{5}{2\beta\lambda_3 e} \right)^{\frac{5}{2}} \left. \right\}
\]

Taking

\[
q_1 > \frac{8r_1 \alpha \lambda_1 e \left[ \alpha \lambda_1 e \left( \frac{7}{2\lambda_1 e} \right)^{\frac{7}{2}} + \alpha k \left( \frac{5}{2\lambda_2 e} \right)^{\frac{5}{2}} + a \left( \frac{5}{2\lambda_3 e} \right)^{\frac{5}{2}} + ap \left( \frac{5}{2\beta\lambda_3 e} \right)^{\frac{5}{2}} \right]}{d_1 \int_{\mathbb{R}} J_1(y)y^2 e^{-\lambda^2} dy} + 1
\]
large enough, one can get (3.4) holds.
When \( \xi > \xi_2, \bar{u}_2(\xi) = 0 \). Then, (3.5) holds naturally. When \( \xi < \xi_2, \bar{u}_2(\xi) = \alpha(e^{\lambda_2\xi} - q_2e^{\mu_2\lambda_2\xi}), \bar{u}_1(\xi) = e\alpha\lambda_1(-\xi)e^{\lambda_1\xi} \) and \( \bar{v}(\xi) \geq 1 - (e^{\lambda_3\xi} + pe^{\beta\lambda_3\xi}) \). The direct calculation yields that

\[
\begin{align*}
&c_1^*\bar{u}_2(\xi) - d_2(J_2 * \bar{u}_2 - \bar{u}_2)(\xi) - r_2\bar{u}_2(\xi)(-1 - h\bar{u}_1(\xi) - \bar{u}_2(\xi) + a_2(\xi)) \\
\leq &\alpha e^{\lambda_2\xi}[-f_2(\lambda_2, c_1^*)] + \alpha\log f_2(\mu_2\lambda_2, c_1^*)e^{\mu_2\lambda_2\xi} + r_2\alpha(e^{\lambda_2\xi} - q_2e^{\mu_2\lambda_2\xi}) \\
&[he\alpha\lambda_1(-\xi)e^{\lambda_1\xi} + \alpha(e^{\lambda_2\xi} - q_2e^{\mu_2\lambda_2\xi}) + a(e^{\lambda_3\xi} + pe^{\beta\lambda_3\xi})] \\
\leq &\alpha\log f_2(\mu_2\lambda_2, c_1^*)e^{\mu_2\lambda_2\xi} + r_2\alpha e^{\lambda_2\xi}[he\alpha\lambda_1(-\xi)e^{\lambda_1\xi} \\
&+ \alpha e^{\lambda_2\xi} + a(e^{\lambda_3\xi} + e^{\beta\lambda_3\xi})] \\
\leq &\alpha\log f_2(\mu_2\lambda_2, c_1^*) + r_2\alpha e^{\lambda_1\lambda_2\xi} + r_2\alpha e^{\lambda_3\lambda_2\xi} + r_2\alpha p \log(\beta_3 + \lambda_2 - \mu_2\lambda_2\xi)] \\
&+ r_2\alpha e^{(2\lambda_2 - \mu_2\lambda_2\xi)} + r_2\alpha e^{(\lambda_3 + \lambda_2 - \mu_2\lambda_2\xi)} + r_2\alpha \log(\beta_3 + \lambda_2 - \mu_2\lambda_2\xi)] \\
\end{align*}
\]

Now, taking

\[
1 < \mu_2 < \min\left\{\frac{\lambda_2}{\lambda_2'}, 1 + \frac{\lambda_1}{\lambda_2}, 2 + \frac{\lambda_3}{\lambda_2}, 1 + \frac{\beta\lambda_3}{\lambda_2}\right\}
\]

and

\[
q_2 > \frac{r_2\left[he\alpha\lambda_1(-\xi)e^{\lambda_1\xi} + \alpha + a + ap\right]}{-f_2(\mu_2\lambda_2, c_1^*)} + 1
\]

large enough, one can get (3.5). Similarly, if we choose

\[
1 < \beta < \min\left\{1, \beta_0, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}\right\}
\]

and

\[
p > \max\left\{\frac{r_3a\gamma + r_3ba\left(\frac{\lambda_1}{\lambda_3 - \beta_3}\right)}{-f_3(\beta\lambda_3, c_1^*)}, a\gamma e\right\}
\]

then (3.6) holds.

Below, we assume \( c_1 = c_2 \). That is \( c = c_1 = c_2 \) \((i = 1, 2)\). Hence, \( f_i(\lambda, c) = 0 \) has a positive double root \( \lambda_i \) \((i = 1, 2)\). Define

\[
\bar{u}_2(\xi) = \begin{cases} 
\alpha\lambda_2(-\xi)e^{\lambda_2\xi}, & \xi < \xi_2, \\
\alpha, & \xi > \xi_2,
\end{cases}
\]

and

\[
\bar{v}_2(\xi) = \begin{cases} 
[a\lambda_2e(-\xi) - q_2(\xi)^2]e^{\lambda_2\xi}, & \xi < \xi_2, \\
0, & \xi > \xi_2,
\end{cases}
\]

where \( \xi_2 = -\frac{1}{\alpha_2\lambda_2} \) and \( \xi_2 = -\frac{a_2\lambda_2^2}{\alpha_2\lambda_2^{\lambda_2}} \) for \( q_2 > 1 \) large enough. Meanwhile, let \( \bar{v}(\xi) \) and \( \bar{v}(\xi) \) be defined as \( c > c^* \). By the analogous argument, if we take

\[
0 < \beta < \min\left\{1, \beta_0, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}\right\},
\]

\[
p > \max\left\{\frac{r_3a\gamma + r_3ba\left(\frac{\lambda_1}{\lambda_3 - \beta_3} + \frac{\lambda_2}{\lambda_2 - \beta_3}\right)}{-f_3(\beta\lambda_3, c^*) + r_3a\gamma}, a\gamma e\right\}
\]
\begin{align*}
q_1 > & \frac{8r_1\alpha\lambda_1e}{a_1 \int_{\mathbb{R}} J_1(y)y^2e^{-\lambda_1y}dy} \left\{ \alpha\lambda_1e \left( \frac{7}{2\lambda_1e} \right)^\frac{k}{2} + k\alpha e\lambda_2 \left( \frac{7}{2\lambda_2e} \right)^\frac{k}{2} \right\}, \\
q_2 > & \frac{8r_2\alpha\lambda_2e}{a_2 \int_{\mathbb{R}} J_2(y)y^2e^{-\lambda_2y}dy} \left\{ \alpha\lambda_2e \left( \frac{7}{2\lambda_2e} \right)^\frac{k}{2} + hae\lambda_1 \left( \frac{7}{2\lambda_1e} \right)^\frac{k}{2} \right\},
\end{align*}

and \( q_i > 1 \ (i = 1, 2) \) large enough, we can prove that \((u_1, u_2, v)\) and \((\bar{u}, \bar{v}, \bar{r})\) are lower and upper solutions of (2.1) with \( c = c^* \).

### 3.2. Existence of traveling waves

Under the construction of lower and upper solutions, we can get the existence of traveling waves when \( c \geq c^* \) with the Schauder’s fixed point theorem. Let \( D \) satisfy

\[
D > \max \{r_1 + r_1(a - 1)(2 + k), r_2 + r_2(a - 1)(2 + h), r_3(a - 1)(2h + 1) \}
\]

dependent such that

\[
\begin{align*}
H_1(\varphi_1, \varphi_2, \varphi_3) &= D\varphi_1 + r_1\varphi_1(-1 - \varphi_1 - k\varphi_2 + a\varphi_3) \text{ is increasing on } \varphi_1, \\
H_2(\varphi_1, \varphi_2, \varphi_3) &= D\varphi_2 + r_2\varphi_2(-1 - h\varphi_1 - \varphi_2 + a\varphi_3) \text{ is increasing on } \varphi_2, \\
H_3(\varphi_1, \varphi_2, \varphi_3) &= D\varphi_3 + r_3\varphi_3(1 - \varphi_1 - b\varphi_2 - \varphi_3) \text{ is increasing on } \varphi_3.
\end{align*}
\]

Take \( 0 < \mu < \min_{i=1,2,3} \left\{ \frac{D + d_i}{c} \right\} \) and define

\[
B_\mu(\mathbb{R}, \mathbb{R}^3) := \left\{ u(\xi) : u(\xi) \in C(\mathbb{R}, \mathbb{R}^3) \text{ and } \sup_{\xi \in \mathbb{R}} |u(\xi)|e^{-\mu|\xi|} < \infty \right\},
\]

with \(|\cdot|\) denotes the super norm in \( \mathbb{R}^3 \). It is clear that \( B_\mu(\mathbb{R}, \mathbb{R}^3) \) is a Banach space equipped with the norm \(|\cdot|_\mu\) defined by

\[
|u|_\mu = \sup_{\xi \in \mathbb{R}} |u(\xi)|e^{-\mu|\xi|} \text{ for } u \in B_\mu(\mathbb{R}, \mathbb{R}^3).
\]

Set \( \Pi := [0, a - 1] \times [0, a - 1] \times [0, 1] \). For any \((\varphi_1, \varphi_2, \varphi_3)\) in \( \Pi \), we define

\[
\begin{align*}
P_1(\varphi_1, \varphi_2, \varphi_3) := (P_1(\varphi_1, \varphi_2, \varphi_3), P_2(\varphi_1, \varphi_2, \varphi_3), P_3(\varphi_1, \varphi_2, \varphi_3))
\end{align*}
\]

with

\[
\begin{align*}
P_1(\varphi_1, \varphi_2, \varphi_3) &= \frac{1}{c} \int_{-\infty}^{\xi} \left[ d_1 J_1 * \varphi_1 + D\varphi_1 + r_1\varphi_1(-1 - \varphi_1 - k\varphi_2 + a\varphi_3) \right] e^{\mu_1(y-\xi)}dy, \\
P_2(\varphi_1, \varphi_2, \varphi_3) &= \frac{1}{c} \int_{-\infty}^{\xi} \left[ d_2 J_2 * \varphi_2 + D\varphi_2 + r_2\varphi_2(-1 - h\varphi_1 - \varphi_2 + a\varphi_3) \right] e^{\mu_2(y-\xi)}dy, \\
\end{align*}
\]

and

\[
\begin{align*}
P_3(\varphi_1, \varphi_2, \varphi_3) &= \frac{1}{c} \int_{-\infty}^{\xi} \left[ d_3 J_3 * \varphi_3 + D\varphi_3 + r_3\varphi_3(1 - \varphi_1 - b\varphi_2 - \varphi_3) \right] e^{\mu_3(y-\xi)}dy,
\end{align*}
\]

\[
(\varphi_1, \varphi_2, \varphi_3) \in \Pi.
\]
here \( \mu_1 = \frac{d_1 + D}{e} \), \( \mu_2 = \frac{d_2 + D}{e} \), \( \mu_3 = \frac{d_3 + D}{e} \). Meanwhile, let
\[
\Gamma = \left\{ (\varphi_1(\cdot), \varphi_2(\cdot), \varphi_3(\cdot)) \in C(\mathbb{R}, \mathbb{R}^3) \mid \begin{array}{l}
\varphi_1(\xi) \leq \varphi_1(\xi) \leq \pi_1(\xi), \\
\varphi_2(\xi) \leq \varphi_2(\xi) \leq \pi_2(\xi), \\
\varphi(\xi) \leq \varphi_3(\xi) \leq \pi(\xi), \xi \in \mathbb{R}
\end{array} \right\}.
\]
Thus, it is easy to verify that \( \Gamma \) is a bounded, closed, nonempty and convex subset of \( C(\mathbb{R}, \mathbb{R}^3) \) with respect to the decay norm \( |\cdot|_\mu \).

**Lemma 3.3.** \( P : C(\Pi, \mathbb{R}^3) \to C(\mathbb{R}, \mathbb{R}^3) \) is continuous with respect to the norm \( |\cdot|_\mu \).

**Proof.** For any \((\varphi_1(\cdot), \varphi_2(\cdot), \varphi_3(\cdot))\) and \((\tilde{\varphi}_1(\cdot), \tilde{\varphi}_2(\cdot), \tilde{\varphi}_3(\cdot))\) in \( \Pi \), we have
\[
|P_1(\varphi_1, \varphi_2, \varphi_3) - P_1(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)| \cdot e^{-\mu |\xi|} \\
= \frac{1}{c} \left[ \int_{-\infty}^\xi [d_1 J_1 \ast \varphi_1 + \varphi_1(-1 - \varphi_1 - k \varphi_2 + a \varphi_3) + \frac{d_1 + D}{e} (y - \xi)] dy \right] e^{-\mu |\xi|} \\
- \left[ \int_{-\infty}^\xi [d_1 J_1 \ast \tilde{\varphi}_1 + \tilde{\varphi}_1(-1 - \tilde{\varphi}_1 - k \tilde{\varphi}_2 + a \tilde{\varphi}_3) + \frac{d_1 + D}{e} (y - \xi)] dy \right] e^{-\mu |\xi|} \\
\leq \frac{d_1}{c} \left[ \int_{-\infty}^\xi (J_1 \ast \varphi_1(y) - J_1 \ast \tilde{\varphi}_1(y)) e^{\frac{d_1 + D}{e} (y - \xi)} dy \right] e^{-\mu |\xi|} \\
+ \frac{D + r_1(1 + (2 + k + a)(a - 1))}{c} \left[ \int_{-\infty}^\xi (\varphi_1(y) - \tilde{\varphi}_1(y)) e^{\frac{d_1 + D}{e} (y - \xi)} dy \right] e^{-\mu |\xi|} \\
+ \frac{r_1 k(a - 1)}{c} \left[ \int_{-\infty}^\xi (\varphi_2(y) - \tilde{\varphi}_2(y)) e^{\frac{d_1 + D}{e} (y - \xi)} dy \right] e^{-\mu |\xi|} \\
+ \frac{r_1 a(a - 1)}{c} \left[ \int_{-\infty}^\xi (\varphi_3(y) - \tilde{\varphi}_3(y)) e^{\frac{d_1 + D}{e} (y - \xi)} dy \right] e^{-\mu |\xi|} \\
\leq L_1 |\varphi_1 - \varphi_1|_\mu + L_2 |\varphi_2 - \tilde{\varphi}_2|_\mu + L_3 |\varphi_3 - \tilde{\varphi}_3|_\mu
\]
with
\[
L_1 := \frac{1}{D + d_1 - c_\mu D + d_1 + c_\mu J_\mathbb{R}}, \quad L_2 := \frac{r_1 k(a - 1)}{D + d_1 - c_\mu}, \quad L_3 := \frac{r_1 a(a - 1)}{D + d_1 - c_\mu}.
\]
Similarly, one can get
\[
|P_2(\varphi_1, \varphi_2, \varphi_3) - P_2(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)|_\mu \leq \tilde{L}_1 |\varphi_1 - \tilde{\varphi}_1|_\mu + \tilde{L}_2 |\varphi_2 - \tilde{\varphi}_2|_\mu + \tilde{L}_3 |\varphi_3 - \tilde{\varphi}_3|_\mu
\]
and
\[
|P_3(\varphi_1, \varphi_2, \varphi_3) - P_3(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)|_\mu \leq \tilde{L}_1 |\varphi_1 - \tilde{\varphi}_1|_\mu + \tilde{L}_2 |\varphi_2 - \tilde{\varphi}_2|_\mu + \tilde{L}_3 |\varphi_3 - \tilde{\varphi}_3|_\mu
\]
for some positive constants \( \tilde{L}_i \) and \( \tilde{L}_i \) (\( i = 1, 2, 3 \)). Consequently, there is
\[
|P(\varphi_1, \varphi_2, \varphi_3) - P(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)|_\mu \leq C_1 |\varphi_1 - \tilde{\varphi}_1|_\mu + C_2 |\varphi_2 - \tilde{\varphi}_2|_\mu + C_3 |\varphi_3 - \tilde{\varphi}_3|_\mu
\]
for some positive constants \( C_i \) (\( i = 1, 2, 3 \)).

**Lemma 3.4.** The operator \( P \) maps \( \Gamma \) into \( \Gamma \).
Proof. For any \((\varphi_1, \varphi_2, \varphi_3) \in \Gamma\), we need to prove that
\[ u_1 \leq P_1(\varphi_1, \varphi_2, \varphi_3) \leq \bar{u}_1, \quad u_2 \leq P_2(\varphi_1, \varphi_2, \varphi_3) \leq \bar{u}_2, \quad \underline{v} \leq P_3(\varphi_1, \varphi_2, \varphi_3) \leq \bar{v}. \] \tag{3.7}

To get the first inequality of (3.7), it is sufficient to show that
\[ u_1 \leq P_3(\varphi_1, \bar{u}_2, \underline{v}) \text{ and } P_3(\varphi_1, \underline{u}_2, \bar{v}) \leq \bar{u}_1. \]

The direct calculation yields that
\[
P_3(\varphi_1, \bar{u}_2, \underline{v}) = \frac{1}{c} \int_{-\infty}^{\xi} [d_1 J_1 \ast \varphi_1 + D\varphi_1 + r_1 \varphi_1(-1 - \varphi_1 - k\bar{u}_2 + \underline{v})]e^{\frac{d_1 + D}{c}(y - \xi)} dy 
\geq \frac{1}{c} \int_{-\infty}^{\xi} [d_1 J_1 \ast \varphi_1 + D\bar{u}_1 + r_1 \varphi_1(-1 - \varphi_1 - k\bar{u}_2 + \underline{v})]e^{\frac{d_1 + D}{c}(y - \xi)} dy 
\geq \frac{1}{c} \int_{-\infty}^{\xi} [c\bar{u}_1(y) + d_1 \varphi_1(y) + D\bar{u}_1(y)]e^{\frac{d_1 + D}{c}(y - \xi)} dy 
= \bar{u}_1(\xi).
\]

On the other hand, we have
\[
P_3(\varphi_1, \underline{u}_2, \bar{v}) = \frac{1}{c} \int_{-\infty}^{\xi} [d_1 J_1 \ast \varphi_1 + D\varphi_1 + r_1 \varphi_1(-1 - \varphi_1 - k\underline{u}_2 + \bar{v})]e^{\frac{d_1 + D}{c}(y - \xi)} dy 
\leq \frac{1}{c} \int_{-\infty}^{\xi} [d_1 J_1 \ast \bar{u}_1 + D\bar{u}_1 + r_1 \varphi_1(-1 - \bar{u}_1 - k\underline{u}_2 + \bar{v})]e^{\frac{d_1 + D}{c}(y - \xi)} dy 
\leq \frac{1}{c} \int_{-\infty}^{\xi} [c\bar{u}_1(y) + d_1 \varphi_1(y) + D\bar{u}_1(y)]e^{\frac{d_1 + D}{c}(y - \xi)} dy 
= \bar{u}_1(\xi).
\]

Similarly, we can prove that
\[ u_2 \leq P_2(\bar{u}_1, \varphi_2, \underline{v}), \quad P_2(\underline{u}_1, \varphi_2, \bar{v}) \leq \bar{u}_2, \]
\[ \underline{v} \leq P_3(\bar{u}_1, \bar{u}_2, \varphi_3), \quad P_3(\underline{u}_1, \underline{u}_2, \varphi_3) \leq \underline{v}. \]

Hence, (3.7) holds. \(\square\)

**Lemma 3.5.** \(P\) is compact on \(\Gamma\) with respect to the decay norm \(| \cdot |_\mu\).

*Proof.* For any \((\varphi_1, \varphi_2, \varphi_3) \in \Gamma\) and \(n \in \mathbb{N}\), define
\[
P^n(\varphi_1, \varphi_2, \varphi_3)(\xi) = \begin{cases} 
P(\varphi_1, \varphi_2, \varphi_3)(-n), & \xi < -n, \\
P(\varphi_1, \varphi_2, \varphi_3)(\xi), & \xi \in [-n, n], \\
P(\varphi_1, \varphi_2, \varphi_3)(n), & \xi > n. \end{cases}
\]

Thus, \(P^n(\varphi_1, \varphi_2, \varphi_3)(\xi)\) is compact once \(P(\varphi_1, \varphi_2, \varphi_3)(\xi)\) on \([-n, n]\) is compact. It is clear that \(P^n(\varphi_1, \varphi_2, \varphi_3)(\xi)\) is equicontinuous and uniform bounded in \(B_\mu(\mathbb{R}, \mathbb{R}^3)\). Therefore, \(P^n(\varphi_1, \varphi_2, \varphi_3)(\xi)\) is a precompact subset with respect to the decay norm since the Ascoli-Arzela Lemma implies that \(P(\varphi_1, \varphi_2, \varphi_3)(\xi)\) is compact in the sense of decay norm. Furthermore,
\[
|P^n(\varphi_1, \varphi_2, \varphi_3)(\xi) - P(\varphi_1, \varphi_2, \varphi_3)(\xi)| e^{-\mu|\xi|} \leq Me^{-\mu n} \to 0 \text{ as } n \to +\infty,
\]
in which \(M\) depends on \(a, d_i, D, r_i\) \((i = 1, 2, 3)\). Then, \(P^n(\varphi_1, \varphi_2, \varphi_3)(\xi)\) converges to \(P(\varphi_1, \varphi_2, \varphi_3)(\xi)\) in the sense of decay norm. Consequently, the compactness
of $P^n(\varphi_1, \varphi_2, \varphi_3)(\xi)$ implies that $P(\varphi_1, \varphi_2, \varphi_3)(\xi)$ is precompact. This ends the proof. □

Now, applying Schauder’s fixed point theorem on $\Gamma$, there exists $(\tilde{u}_1, \tilde{u}_2, \tilde{v})$ such that

$$P(\tilde{u}_1, \tilde{u}_2, \tilde{v})(\xi) = (\tilde{u}_1(\xi), \tilde{u}_2(\xi), \tilde{v}(\xi)),$$

which implies that $(\tilde{u}_1, \tilde{u}_2, \tilde{v})$ is a fixed point of $P$ in $\Gamma$.

**Theorem 3.6.** Assume $c \geq c^*$. Then, system (2.1) admits a positive solution $(\tilde{u}_1(\xi), \tilde{u}_2(\xi), \tilde{v}(\xi))$. Moreover, $(\tilde{u}_1(\xi), \tilde{u}_2(\xi), \tilde{v}(\xi))$ satisfies

$$u_1(\xi) \leq \tilde{u}_1(\xi) \leq u_1(\xi), \quad u_2(\xi) \leq \tilde{u}_2(\xi) \leq u_2(\xi), \quad v(\xi) \leq \tilde{v}(\xi) \leq v(\xi)$$

for any $\xi \in \mathbb{R}$ and

$$\lim_{\xi \to -\infty} \tilde{u}_1(\xi) = 0, \quad \lim_{\xi \to -\infty} \tilde{u}_2(\xi) = 0, \quad \lim_{\xi \to -\infty} \tilde{v}(\xi) = 1.$$

4. **Asymptotic behavior.** To finish the proof of Theorem 2.2, we must show the boundary asymptotic behavior

$$\lim_{\xi \to +\infty} \tilde{u}_1(\xi) = u_1^*, \quad \lim_{\xi \to +\infty} \tilde{u}_2(\xi) = u_2^*, \quad \lim_{\xi \to +\infty} \tilde{v}(\xi) = v^*.$$ (4.1)

To derive this result, we need to make the further restriction (H) on $b$ that

$$0 < b < \frac{1}{2a} \min\{1-h, 1-k\}. \quad (4.2)$$

Moreover, we change the range of $\gamma$ from $[\frac{2b(a-1)}{a}, \frac{1}{a}]$ to

$$\left[\frac{2b(a-1)}{a}, (a-1) \min\{1-h, 1-k\}\right] \subset \left[\frac{2b(a-1)}{a}, \frac{1}{a}\right]. \quad (4.3)$$

Under the additional conditions (4.2) and (4.3), it is noted that $(\tilde{u}_1, \tilde{u}_2, \tilde{v})$ constructed before are also upper-lower solutions of (2.1). Set

$$\tilde{u}_i^- := \liminf_{\xi \to +\infty} \tilde{u}_i(\xi), \quad \tilde{u}_i^+ := \limsup_{\xi \to +\infty} \tilde{u}_i(\xi), \quad i = 1, 2,$$

$$\tilde{v}^- := \liminf_{\xi \to +\infty} \tilde{v}(\xi), \quad \tilde{v}^+ := \limsup_{\xi \to +\infty} \tilde{v}(\xi).$$

First, we have the following result.

**Lemma 4.1.** Let

$$b_1 := (a-1)(1-k) - a^2 \gamma, \quad b_2 := (a-1)(1-h) - a^2 \gamma.$$

Then $\tilde{u}_i^- \geq b_i > 0, \quad i = 1, 2$.

**Proof.** We first consider (1.1) with $(u_1, u_2, v)(x, t) = (\tilde{u}_1, \tilde{u}_2, \tilde{v})(x + ct)$. Note that $\tilde{u}_2 = u_2(x, t) \leq \alpha$ and $\tilde{v} = v(x, t) \geq 1 - a\gamma$. Thus, $u_1(x, t)$ satisfies

$$\begin{cases}
\frac{\partial u_1}{\partial t} \geq d_1(J_1 * u_1 - u_1)(x, t) + r_1 u_1[-1 - u_1 - k\alpha + a(1 - a\gamma)], & x \in \mathbb{R}, \quad t > 0, \\
u_1(x, 0) = \tilde{u}_1(x), & x \in \mathbb{R}.
\end{cases}$$

We compute that

$$-1 - k\alpha + a(a - a\gamma) = (a-1)(1-k) - a^2 \gamma := b_1,$$

where $b_1 > 0$ due to the choice of $\gamma$. So, it follows from the comparison principle that

$$\tilde{u}_1^- := \liminf_{\xi \to +\infty} \tilde{u}_1(\xi) = \liminf_{\xi \to +\infty} u_1(0, \xi/c) \geq b_1 > 0.$$
Similarly, we can derive that $\tilde{u}_i^- \geq b_2 > 0$ and the lemma follows.  

Below, we will get (4.1) through constructing the sequence of shrinking rectangles, which is inspired by the ideas how to choose the left and right endpoints of rectangle from [7, 15, 13]. That is, for $\theta \in [0, 1]$, we define

$$
\phi_1(\theta) := \theta u_1^* + (1 - \theta)(b_1 - \varepsilon), \quad \chi_1(\theta) := (1 - \theta)(\alpha + \varepsilon) + \theta u_1^*, \\
\phi_2(\theta) := \theta u_2^* + (1 - \theta)(b_2 - \varepsilon), \quad \chi_2(\theta) := (1 - \theta)(\alpha + \varepsilon) + \theta u_2^*, \\
\phi_3(\theta) := \theta v^* + (1 - \theta)(b_3 - \varepsilon^2), \quad \chi_3(\theta) := (1 - \theta)(1 + \varepsilon^2) + \theta v^*.
$$

where $\alpha = a - 1$, $b_1$, $b_2$ are defined as above, $b_3 = 1 - a\gamma$ and $\varepsilon$ is a small positive constant such that

$$
\varepsilon < \min \left\{ \frac{1 - h}{a}, \frac{1 - k}{a}, \frac{a\gamma - 2b(a - 1)}{2b}, \frac{1}{2} \right\}.
$$

It is clear that $u_1^*, u_2^* < \alpha$ and $v^* < 1$. By choosing $b_i$ ($i = 1, 2, 3$) smaller if it is necessary, we can ensure that Lemma 4.1 holds such that

$$
0 < b_i < u_1^*, \quad 0 < b_2 < u_2^* \quad \text{and} \quad 0 < b_3 < v^*.
$$

Thus, we see that $\phi_i(\theta)$ is a monotone increasing and $\chi_i(\theta)$ is a monotone decreasing function of $\theta \in [0, 1]$ ($i = 1, 2, 3$) respectively such that

$$(\phi_1, \phi_2, \phi_3)(1) = (\chi_1, \chi_2, \chi_3)(1) = (u_1^*, u_2^*, v^*).
$$

Then, it suffices to show that the set

$$
A := \left\{ \theta \in [0, 1] \mid \phi_i(\theta) < \tilde{u}_i^- \leq \tilde{u}_i^+ < \chi_i(\theta), \quad i = 1, 2, \right\}
$$

is nonempty and $\sup A = 1$.

**Step 1.** $A$ is nonempty.

Indeed, it follows from Lemma 3.2 and the definitions of upper-lower solutions that

$$
\phi_1(0) = b_1 - \varepsilon < b_1 \leq \tilde{u}_1^- \leq \tilde{u}_1^+ \leq \alpha < \alpha + \varepsilon = \chi_1(0), \\
\phi_2(0) = b_2 - \varepsilon < b_2 \leq \tilde{u}_2^- \leq \tilde{u}_2^+ \leq \alpha < \alpha + \varepsilon = \chi_2(0), \\
\phi_3(0) = b_3 - \varepsilon^2 < b_3 \leq \tilde{v}^- \leq \tilde{v}^+ \leq 1 < 1 + \varepsilon^2 = \chi_3(0).
$$

This implies that $0 \in A$ and $A \neq \emptyset$.

**Step 2.** $\sup A = 1$. On the contrary, we assume that $\sup A = \theta^* \in (0, 1)$. Set

$$
l_1(\theta) := -1 - \phi_1(\theta) - k\chi_2(\theta) + a\phi_3(\theta), \quad g_1(\theta) := -1 - \chi_1(\theta) - k\phi_2(\theta) + a\chi_3(\theta), \\
l_2(\theta) := -1 - h\chi_1(\theta) - \phi_2(\theta) + a\phi_3(\theta), \quad g_2(\theta) := -1 - h\phi_1(\theta) - \chi_2(\theta) + a\chi_3(\theta), \\
l_3(\theta) := 1 - b\chi_1(\theta) - b\chi_2(\theta) - \phi_3(\theta), \quad g_3(\theta) := 1 - b\phi_1(\theta) - b\phi_2(\theta) - \chi_3(\theta).
$$

By direct computation, we have

$$
l_1(\theta) := (1 - \theta)[\varepsilon(1 - k - a\varepsilon)] > 0, \quad g_1(\theta) := -(1 - \theta)[kb_2 + \varepsilon(1 - k - a\varepsilon)] < 0, \\
l_2(\theta) := (1 - \theta)[\varepsilon(1 - h - a\varepsilon)] > 0, \quad g_2(\theta) := -(1 - \theta)[kb_1 + \varepsilon(1 - h - a\varepsilon)] < 0, \\
l_3(\theta) := (1 - \theta)[a\gamma - 2b(a - 1)] - \varepsilon(2b - \varepsilon) > 0, \\
g_3(\theta) := -(1 - \theta)[b(b_1 + b_2 - 2\varepsilon) + \varepsilon^2] < 0.
$$

Now, let $\theta \to \theta^*$, one can get

$$
\phi_i(\theta^*) \leq \tilde{u}_i^- \leq \tilde{u}_i^+ \leq \chi_i(\theta^*), \quad i = 1, 2, \\
\phi_3(\theta^*) \leq \tilde{v}^- \leq \tilde{v}^+ \leq \chi_3(\theta^*).
$$
Thus, one of the following equalities must hold
\[ \tilde{u}_i = \phi_i(\theta^*), \quad \tilde{u}_i^+ = \chi_i(\theta^*) \ (i = 1, 2), \quad \tilde{v}^- = \phi_3(\theta^*), \quad \tilde{v}^+ = \chi_3(\theta^*). \]

At last, we only treat the case that \( \tilde{u}_1^- = \phi_1(\theta^*) \) to derive a contradiction. The other five cases are similar. To get this goal, we will discuss it in two parts.

(i) \( \tilde{u}_1(\xi) \) is ultimately monotone.

In this case, \( \tilde{u}_1(+\infty) = \phi_1(\theta^*) \). Integrating the first equation of (2.1) from 0 to \( n \) for any \( n \in \mathbb{N} \), we obtain
\[
\begin{align*}
    c[\tilde{u}_1(n) - \tilde{u}_1(0)] &= d_1 \int_0^n (J_1 * \tilde{u}_1 - \tilde{u}_1)(\xi)d\xi + r_1 \int_0^n \tilde{u}_1(-1 - \tilde{u}_1 - k\tilde{u}_2 + a\tilde{v})d\xi. \\
    &\quad (4.4)
\end{align*}
\]
Due to
\[
\begin{align*}
    \int_0^n (J_1 * \tilde{u}_1 - \tilde{u}_1)(\xi)d\xi &= \int_{-\infty}^{\infty} J_1(y) \int_0^n (\tilde{u}_1(\xi - y) - \tilde{u}_1(\xi))d\xi dy \\
    &= \int_{-\infty}^{\infty} J_1(y)(-y) \int_0^1 [\tilde{u}_1(n - \tau y) - \tilde{u}_1(-\tau y)]d\tau dy.
\end{align*}
\]
Thus, it follows from (4.4) that
\[
\begin{align*}
    c[\tilde{u}_1(n) - \tilde{u}_1(0)] + d_1 \int_{-\infty}^{\infty} J_1(y) y \int_0^1 [\tilde{u}_1(n - \tau y) - \tilde{u}_1(-\tau y)]d\tau dy \\
    = r_1 \int_0^n \tilde{u}_1(-1 - \tilde{u}_1 - k\tilde{u}_2 + a\tilde{v})d\xi. \\
    &\quad (4.5)
\end{align*}
\]
Note that
\[
\liminf_{\xi \to +\infty} \tilde{u}_1(\xi)(-1 - \tilde{u}_1(\xi) - k\tilde{u}_2(\xi) + a\tilde{v}(\xi)) \\
\geq \phi_1(\theta^*)(-1 - \phi_1(\theta^*) - k\chi_2(\theta^*) + a\phi_3(\theta^*)) = \phi_1(\theta^*)l_1(\theta^*) > 0.
\]
Hence,
\[
\lim_{n \to +\infty} \int_0^n \tilde{u}_1(\xi)(-1 - \tilde{u}_1(\xi) - k\tilde{u}_2(\xi) + a\tilde{v}(\xi))d\xi = \infty,
\]
which contradicts to the boundedness of the left hand part of (4.5).

(ii) \( \tilde{u}_1(\xi) \) is oscillatory as \( \xi \to +\infty \).

Since \( \tilde{u}_1(\xi) \) is bounded in \( \mathbb{R} \), we can choose a sequence of local minimal points \( \{\xi_n\} \) of \( \tilde{u}_1(\xi) \) such that \( \xi_n \to +\infty \) and \( \tilde{u}_1(\xi_n) \to \phi_1(\theta^*) \) as \( n \to +\infty \). Following from the first equation of (2.1), we have
\[
0 = c\tilde{u}_1^\prime(\xi_n) \\
= d_1 \int_{\mathbb{R}} J_1(y)(\tilde{u}_1(\xi_n) - y) - \tilde{u}_1(\xi_n))dy \\
+ r_1 \tilde{u}_1(\xi_n)(-1 - \tilde{u}_1(\xi_n) - k\tilde{u}_2(\xi_n) + a\tilde{v}(\xi_n)).
\]
Letting \( n \to +\infty \), one can get that
\[
\lim_{n \to +\infty} d_1 \int_{\mathbb{R}} J_1(y)(\tilde{u}_1(\xi_n) - y) - \tilde{u}_1(\xi_n))dy > 0
\]
and
\[
\liminf_{n \to +\infty} r_1 \tilde{u}_1(\xi_n)(-1 - \tilde{u}_1(\xi_n) - k\tilde{u}_2(\xi_n) + a\tilde{v}(\xi_n)) \geq r_1 \phi_1(\theta^*)l_1(\theta^*) > 0,
\]
a contradiction happens. Therefore, \( \sup A = 1 \) and
\[
(\tilde{u}_1(\infty), \tilde{u}_2(\infty), \tilde{v}(\infty)) = (u^*_1, u^*_2, v^*).
\]
Now, we have the following result.

**Theorem 4.2.** Suppose (4.2) and (4.3) hold. Then, for $c \geq c^*$, there are traveling wave solutions connecting $(0, 0, 1)$ and $(u^*_1, u^*_2, v^*)$ of system (1.1).

5. **Non-existence of traveling waves.** In this section, we show that there are no bounded positive traveling wave solutions with speed $c < c^*$ connecting $(0, 0, 1)$ to $(u^*_1, u^*_2, v^*)$ of system (1.1). In order to get this fact, we first list the following results obtained by Zhang et al. [32] and Yang et al. [29].

**Lemma 5.1** (Zhang et al. [32]). Assume $c > 0$ and $B(\cdot)$ is a continuous function with $B(\pm \infty) := \lim_{\xi \to \pm \infty} B(\xi)$. Let $Z(\xi)$ be a measurable function satisfying

$$cZ(\xi) = \int_{\mathbb{R}} J_i(y) e^{-\xi y} Z(\xi + y) dy + B(\xi) \text{ in } \mathbb{R}.$$ 

Then, $Z$ is uniformly continuous and bounded. Moreover, $\mu^\pm := \lim_{\xi \to \pm \infty} Z(\xi)$ exist and are real roots of the characteristic equation

$$c\mu = \int_{\mathbb{R}} J_i(y) e^{-\mu y} dy + B(\pm \infty) \quad (i = 1, 2, 3).$$

**Lemma 5.2** (Yang et al. [29]). Let $Z \in C^1(\mathbb{R})$ satisfy

$$Z'(\xi) \geq \int_{\mathbb{R}} J_i(y) Z(\xi - y) dy + b(\xi)Z(\xi) \text{ in } \mathbb{R},$$

(5.1)

where $b(\xi)$ is continuous and $b(\xi) \geq -\tilde{M}$ on $\mathbb{R}$ for some $\tilde{M} > 0$. Then there exists some constant $C = C(M) > 0$ such that

$$C^{-1} < \int_{\mathbb{R}} J_i(y) \frac{Z(\xi - y)}{Z(\xi)} dy < C \text{ in } \mathbb{R}, \quad i = 1, 2, 3.$$ 

**Theorem 5.3.** For any $0 < c < c^*$, there are no bounded positive traveling wave solutions connecting $(0, 0, 1)$ to $(u^*_1, u^*_2, v^*)$ of system (1.1).

**Proof.** Without loss of generality, we assume $c^* = c^*_1$ and $(u_1(\xi), u_2(\xi), v(\xi))$ is one bounded traveling wave solution of (1.1) with $(u_1, u_2, v)(-\infty) = (0, 0, 1)$ for some $c < c^*_1$. It follows from the first equation of (2.1) that

$$cu_1' = d_1 \int_{\mathbb{R}} J_1(y) u_1(\xi - y) u_1(\xi) dy - d_1 - r_1(-1 - u_1 - ku_2 + av). \quad (5.2)$$

Applying the boundedness of $(u_1(\xi), u_2(\xi), v(\xi))$ and Lemma 5.2, we have $|u_1'|$ is also bounded in $\mathbb{R}$. Now, take some sequence $\{\xi_n\}$ with $\xi_n \to -\infty$ as $n \to +\infty$ and define

$$u_{1n}(\xi) := \frac{u_1(\xi_n + \xi)}{u_1(\xi_n)}, \quad u_{2n}(\xi) := u_2(\xi_n + \xi), \quad v_{n}(\xi) := v(\xi_n + \xi).$$

Since $\lim_{\xi \to -\infty} u_i(\xi) = 0$ $(i = 1, 2)$ and $\lim_{\xi \to -\infty} v(\xi) = 1$, we have

$$\lim_{n \to +\infty} u_1(\xi_n + \xi) = 0, \quad \lim_{n \to +\infty} u_{2n}(\xi) = 0 \quad \text{and} \quad \lim_{n \to +\infty} v_{n}(\xi) = 1$$

locally uniformly in $\mathbb{R}$. Following from the first equation of (2.1), there is

$$cu_{1n}' = d_1 \int_{\mathbb{R}} J_1(y) u_{1n}(\xi - y) u_1(\xi) dy - d_1 u_{1n}(\xi)$$

$$+ r_1 u_{1n}(\xi)(-1 - u_1(\xi_n + \xi) - ku_{2n}(\xi) + av_{n}(\xi)).$$
We only prove that and the proof in Theorem 5.3 give that.

Obviously, Theorem 5.2 exists such that one of the following inequalities holds.

Remark 2. Obviously, Theorem 4.2 together with Theorem 5.3 give that $c^*$ is the minimal wave speed of traveling wave solutions connecting the predator-free state to a nontrivial state.

Below, we prove Theorem 2.3. To get this goal, we first give the following necessary condition for the existence of traveling wave solutions.

Corollary 1. Suppose there exist $c_0 \in \mathbb{R}$ and $(u_1, u_2, v) \in \Pi$ with

$$(u_1(-\infty), u_2(-\infty), v(-\infty)) = (0, 0, 1), \ (u_1(+\infty), u_2(+\infty), v(+\infty)) = (u_1^*, u_2^*, v^*)$$

such that $(u_1(\xi), u_2(\xi), v(\xi))$ is a solution of (2.1). Then, some positive constant $\lambda$ exists such that one of the following inequalities holds.

$$\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy < +\infty, \ i = 1, 2, 3$$

Proof. We only prove that

$$\int_{\mathbb{R}} J_1(y)e^{\lambda y}dy < +\infty \text{ for some } \lambda > 0$$

and the other two cases can be showed by the same argument. Let $\Phi(\xi) = \frac{u_1(\xi)}{u_1'(\xi)}$. In view of Lemma 5.2 and the proof in Theorem 5.3, we know that

$$|\Phi(\xi)| < \infty \text{ and } \int_{\mathbb{R}} J_1(y)\frac{u_1(\xi - y)}{u_1(\xi)}dy < \infty \text{ in } \mathbb{R}.$$
Following from (5.2) that \( \Phi(\xi) \) satisfies
\[
c_0 \Phi(\xi) = d_1 \int_{\mathbb{R}} J_1(y)e^{\xi y \Phi(\tau)}dy - d_1 + r_1(-1 - u_1 - ku_2 + v).
\]
Since \( u_1 \) (\( i = 1, 2 \)) and \( v \) are bounded, and \((u_1(\pm \infty), u_2(\pm \infty), v(\pm \infty))\) exist, according to Lemma 5.1, we have \( \lim_{\xi \to \pm \infty} \Phi(\xi) \) exist, denoted by \( \Phi^\pm \) respectively. Moreover, \( \Phi^\pm \) satisfy
\[
c_0 \Phi^\pm = d_1 \int_{\mathbb{R}} J_1(y)e^{-\Phi^\pm y}dy - d_1 + r_1(a - 1).
\] (5.4)
Additionally, due to \( u_1(\xi) > 0 \) in \( \mathbb{R} \) and \( \lim_{\xi \to -\infty} u_1(\xi) = 0 \), some \( \xi_0 < 0 \) small enough exists such that \( u_1(\xi) \geq 0 \) for all \( \xi < \xi_0 \). Hence, one can get that \( \Phi^- \geq 0 \).
Since \( r_1 > 0 \) and \( a > 1 \), it follows from (5.4) that \( \Phi^- > 0 \) and \( \Phi^- \) satisfies
\[
\int_{\mathbb{R}} J_1(y)e^{-\Phi^- y}dy = \frac{1}{d_1} \left[c_0 \Phi^- + d_1 - r_1(a - 1)\right] < +\infty.
\]
This ends the proof.

Now, it follows from Corollary 1 that Theorem 2.3 holds.

6. Spreading speed. In this section, we shall prove Theorem 2.4. To get this goal, we denote by \( X \) the space of all uniformly continuous bounded functions defined in \( \mathbb{R} \), and it is a Banach space when endowed with the sup-norm. Meanwhile, define
\[
X^+ = \{ u \in X : u(x) \geq 0, \forall x \in \mathbb{R} \},
\]
which is the positive cone of \( X \) and
\[
X_L = \{ u \in X : 0 \leq u(x) \leq L, \forall x \in \mathbb{R} \}
\]
for any constant \( L > 0 \).
In order to get the spreading speed for (1.1) and show Theorem 2.4, we want to recall some known results on the scalar logistic equation with nonlocal dispersal. For the following nonlocal logistic equation
\[
\begin{align*}
\frac{\partial u}{\partial t} &= d \int_{\mathbb{R}} J(x-y)(u(y,t) - u(x,t))dy + ru(x,t)(m - u(x,t)), \quad x, t > 0, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\] (6.1)
wherein the initial data \( u_0 \in X_m \) admit a nonempty compact support. Here, the kernel function \( J \) satisfies (J1) and (J2). \( d, r \) and \( m \) are all positive constants. Now, it follows from the results in [16] that the following results hold.

**Lemma 6.1.** (Comparison principle) Let \( u \) be a solution of (6.1) with \( u_0 \in X_m \) and \( u(\cdot, t) \in X_m \) for all \( t > 0 \). If \( w(\cdot, t) \in X_m \) and \( w(x, t) \) satisfies
\[
\begin{align*}
\frac{\partial w(x, t)}{\partial t} &\geq d \int_{\mathbb{R}} J(x-y)(w(y, t) - w(x, t))dy + rw(m - w), \quad x, t > 0, \\
w(x, 0) &\geq u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]
then \( w(x, t) \geq u(x, t) \) for all \( x \in \mathbb{R} \) and \( t > 0 \). Similar result holds for the reverse inequality.

Define
\[
\hat{\alpha} := \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \left( d \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - d + rm \right) \right\}.
\]
Since \( rm > 0 \), \( \hat{\alpha} \) is well defined and \( \hat{\alpha} > 0 \). Then, the solution \( u(x, t) \) of (6.1) satisfies the following result.
Lemma 6.2. (Spreading speed) Let \( u \) be a solution of (6.1) with \( u_0 \in X_m \) and \( u(\cdot, t) \in X_m \) for all \( t > 0 \). Assume that \( u_0(x) \) has a nonempty compact support. Then, we have

\[
\lim_{t \to \infty} \inf_{|x| < ct} u(x, t) = m \text{ for any } 0 < c < \hat{c},
\]

\[
\lim_{t \to \infty} \sup_{|x| > ct} u(x, t) = 0 \text{ for any } c > \hat{c}.
\]

By the same discussion as in [8], system (6.1) can generate a maximal positive nonlinear semiflow, denoted by \( T(t) \) on \( X^+ \times X^+ \times X^+ \). Using the positivity of \( T(t) \), it follows from (4.2) that the set \( X^3_B \) defined by

\[
X^3_B := X_\alpha \times X_\alpha \times X_1, \quad \alpha = a - 1 > 0
\]

is positively invariant with respect to the semiflow \( T(t) \). Particularly, this implies that the initial value problem (1.1) and (1.2) in which \( u_{i0} \in X_\alpha \) \((i = 1, 2)\) admits a unique globally defined solution \( u(x, t), u_2(x, t), v(x, t)) \) with

\[
(u_1, u_2, v) \in C^1([0, \infty), X^3) \text{ and } (u_1, u_2, v)(\cdot, t) \in X^3_B, \quad \forall t \geq 0.
\]

Additionally, since \( (u_1, u_2, v) \) is bounded from \([0, \infty)\) into \( X^3 \), it follows from (1.1) that the time derivative of \( (u_1, u_2, v) \) is also bounded from \([0, \infty)\) into \( X^3 \). Furthermore, if \( u_{i0} \in X_\alpha \) \((i = 1, 2)\) has a nonempty compact support, then

\[
u_i(x, t) > 0 \text{ for all } t > 0 \text{ and } x \in \mathbb{R}, \quad i = 1, 2.
\]

Below, we fix an initial data \( u_{i0} \in X_\alpha \setminus \{0\} \) and consider the corresponding solution \( (u_1, u_2, v) \) of system (1.1) with (1.2). That is, we shall prove Theorem 2.4 in two steps.

**Step 1.** \( \lim_{t \to \infty} \sup_{|x| \geq ct} u_i(x, t) = 0 \) for any \( c > c^* \). By the above discussion, we know that \( u_i(x, t) > 0 \) and \( v(x, t) \leq 1 \) for all \( x \in \mathbb{R} \) and \( t > 0 \). Thus, it is easy to verify that

\[
\frac{\partial u_1(x, t)}{\partial t} \leq d_1(J_1 \ast u_1 - u_1)(x, t) + r_1 u_1(\alpha - u_1)
\]

and

\[
\frac{\partial u_2(x, t)}{\partial t} \leq d_2(J_2 \ast u_2 - u_2)(x, t) + r_2 u_2(\alpha - u_2)
\]

for all \( x \in \mathbb{R} \) and \( t > 0 \). Now, we consider the following initial problem

\[
\begin{cases}
\frac{\partial U_i}{\partial t} = d_i(J_i \ast U_i - U_i)(x, t) + r_i U_i(\alpha - U_i), & x \in \mathbb{R}, t > 0, \\
U_i(x, 0) = u_{i0}(x), & x \in \mathbb{R},
\end{cases}
\]

in which \( i = 1, 2 \). Therefore, following from Lemmas 6.1 and 6.2 yields that

\[
\lim_{t \to \infty} \left\{ \sup_{|x| > ct} u_i(x, t) \right\} \leq \lim_{t \to \infty} \left\{ \sup_{|x| > ct} U_i(x, t) \right\} = 0 \text{ for any } c > c^*, \quad i = 1, 2.
\]

**Step 2.** \( \lim_{t \to \infty} \inf_{|x| < ct} u_i(x, t) > 0 \) for any \( 0 < c < c^* \). Note that \( u_i(x, t) \leq \alpha \) for all \( x \in \mathbb{R} \) and \( t > 0 \). Hence, we have

\[
\frac{\partial v(x, t)}{\partial t} \geq d_3(J_3 \ast v - v)(x, t) + r_3 v(1 - 2b_\alpha - v), \quad x \in \mathbb{R}, \quad t > 0.
\]

Recalling the condition (H) and \( v_0(x) = 1 \), it follows from Lemma 6.1 that \( v(x, t) \geq 1 - 2b_\alpha = 1 - 2b(a - 1) > 0 \), \( \forall t \geq 0, \quad x \in \mathbb{R} \).
Given some constant $\tau > 0$. We know that $u_i(x, \tau) > 0$ ($i = 1, 2$) for all $x \in \mathbb{R}$ by the above argument. Furthermore, it is easy to verify that $u_i(x, t)$ ($i = 1, 2$) satisfies
\[
\frac{\partial u_1(x, t)}{\partial t} \geq d_1(J_1 * u_1 - u_1)(x, t) + r_1 u_1[-1 - u_1 - k\alpha + a(1 - 2b\alpha)], \quad x \in \mathbb{R}, t > \tau
\]
and
\[
\frac{\partial u_2(x, t)}{\partial t} \geq d_2(J_2 * u_2 - u_2)(x, t) + r_2 u_2[-1 - h\alpha - u_2 + a(1 - 2b\alpha)], \quad x \in \mathbb{R}, t > \tau,
\]
respectively. Additionally, since $b$ satisfies the condition (2.2), we have
\[-1 - k\alpha + a(1 - 2b\alpha) = a[1 - (k + 2ab)] > 0\]
and
\[-1 - h\alpha + a(1 - 2b\alpha) = a[1 - (h + 2ab)] > 0.\]
Now, let us consider the problems as follows:
\[
\begin{cases}
\frac{\partial U_1(x, t)}{\partial t} = d_1(J_1 * U_1 - U_1)(x, t) + r_1 U_1\{a[1 - (k + 2ab)] - U_1\}, \quad x \in \mathbb{R}, t > 0, \\
U_1(x, 0) = U_{10}(x) \leq u_1(x, \tau), \quad x \in \mathbb{R}
\end{cases}
\]
and
\[
\begin{cases}
\frac{\partial U_2(x, t)}{\partial t} = d_2(J_2 * U_2 - U_2)(x, t) + r_2 U_2\{a[1 - (h + 2ab)] - U_2\}, \quad x \in \mathbb{R}, t > 0, \\
U_2(x, 0) = U_{20}(x) \leq u_2(x, \tau), \quad x \in \mathbb{R},
\end{cases}
\]
in which $U_{10}(x)$ ($i = 1, 2$) is a nontrivial compactly supported function in $\mathbb{R}$. Thus, applying Lemma 6.1 yields that
\[u_i(x, t + \tau) \geq U_i(x, t)\] for all $x \in \mathbb{R}, t \geq 0, i = 1, 2$.
Then, using Lemma 6.2, we have
\[
\liminf_{t \to \infty} \left\{ \inf_{|x| \leq (c^* - \varepsilon)t} u_1(x, t) \right\} \geq \liminf_{t \to \infty} \left\{ \inf_{|x| \leq (c^* - \varepsilon)t} U_1(x, t) \right\} = a[1 - (k + 2ab)] > 0
\]
and
\[
\liminf_{t \to \infty} \left\{ \inf_{|x| \leq (c^* - \varepsilon)t} u_2(x, t) \right\} \geq \liminf_{t \to \infty} \left\{ \inf_{|x| \leq (c^* - \varepsilon)t} U_2(x, t) \right\} = a[1 - (h + 2ab)] > 0.
\]
We can choose $\varepsilon > 0$ small enough to complete the proof.
Consequently, Theorem 2.4 is derived by steps 1 and 2.

7. Discussion. In this paper, we mainly study the invasion process of the two predators into the habitat of one prey. Under the assumptions (J1), (J2), (H), and (2.2), the invasion process is characterized by the spreading speed of the predators as well as the minimal wave speed of traveling waves connecting the predator-free state to the co-existence state, in which the convergence to the co-existence state is derived by the idea of contracting rectangle.

Note that if the kernel functions $J_i$ ($i = 1, 2, 3$) satisfy the condition (J2), then $J_i$ is called the thin-tailed kernel. In this case, Theorems 2.2 and 2.4 imply that $c^*$ is finite spreading speed. Furthermore, we have proved that there is no bounded positive traveling wave solution connecting the predator-free state and the co-existence state if one of $J_i$ satisfies
\[
\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy = +\infty \text{ for all } \lambda > 0, \quad i = 1, 2, 3. \tag{7.1}
\]
In fact, if the kernel function $J_i$ satisfies (7.1), then $J_i$ is called the fat-tailed kernel. For nonlocal dispersal equations with fat-tailed kernel, the phenomenon of acceleration propagation occurs, we refer [12, 28], and so on for understanding the recent results about the acceleration propagation. The known results imply that there is a higher opportunity for the nonlocal dispersal with a fat-tailed kernel to travel a long distance than the thin-tailed case, which gives that the non-locality of the former is stronger than the later case. To our knowledge, if the kernel function is fat-tailed, there are no results for predator-prey model with nonlocal dispersal so far. The main reason is that the solutions of predator-prey model with nonlocal dispersal have low regularity and do not have the comparison principle in general, which can lead some difficulties to consider the corresponding problem. Naturally, we want to know how to overcome the difficulties brought by the nonlocal dispersal, and whether there is the acceleration propagation of system (1.1) when all the kernel functions $J_i$ ($i = 1, 2, 3$) or one of the kernel functions $J_i$ ($i = 1, 2, 3$) satisfies (7.1). This is an interesting problem and we leave it for further study. In addition, $J_i$ ($i = 1, 2, 3$) are symmetric kernel functions in the present paper. Owing to some technical difficulties, we do not know whether there are traveling waves and what the spreading speed is when the kernel function is asymmetric. We also leave this problem for further study.

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