A Semi-Lagrangian Scheme with Radial Basis Approximation for Surface Reconstruction

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Abstract

We propose a Semi-Lagrangian scheme coupled with Radial Basis Function interpolation for approximating a curvature-related level set model, which has been proposed by Zhao et al. in [19] to reconstruct unknown surfaces from sparse, possibly noisy data sets. The main advantages of the proposed scheme are the possibility to solve the level set method on unstructured grids, as well as to concentrate the reconstruction points in the neighbourhood of the data set, with a consequent reduction of the computational effort. Moreover, the scheme is explicit. Numerical tests show the accuracy and robustness of our approach to reconstruct curves and surfaces from relatively sparse data sets.

1 Introduction

Curvature-related level set models have been proposed in many problems of image and surface processing, due to their well understood analytical features and nice smoothing properties. In this paper, we consider one of such models, related to the problem of reconstructing a surface from a discrete set of (possibly noisy) points on it.

For $n \in \{2, 3\}$, let $S = \{x_1, \ldots, x_N\}$ be a discrete set of points in $\mathbb{R}^n$, which are to be understood as points on a surface (or a curve if $n = 2$), from which the surface itself has to be reconstructed. The set $S$ will be also termed as “data set” in what follows, and we assume that its points can be affected by some form of noise. Thus, the reconstructed surface is not expected to pass through each of the data set points, but rather (depending on the criterion used to perform the reconstruction) to provide some trade-off between an exact interpolation and a smooth behaviour.

A level set model for this problem has been proposed by Zhao et al. in [19], and leads to the following evolutive problem

\[
\begin{cases}
    u_t(x, t) = d(x) \text{ div} \left( \frac{Du(x, t)}{|Du(x, t)|} \right) |Du(x, t)| + Dd(x) \cdot Du(x, t) \\
    u(x, 0) = u_0(x)
\end{cases}
\]  

(1.1)

where $d(x)$ is the euclidean distance from the set $S$, and $D$ denotes the gradient operator. As customary in level set methods, the reconstructed surface at a given time $t$ is represented as the zero-level set of the solution $u(x, t)$, that is,

\[ \Gamma_t = \{ x \in \mathbb{R}^n : u(x, t) = 0 \}. \]

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In particular, we are interested with the regime solution for $t \to \infty$. In fact, \( (1.1) \) may be interpreted as the gradient flow for an energy functional in which the $L^1$ norm of the distance from $S$ is integrated on the whole surface (see [19]), and its regime solution represents a local minimum of the energy. Well-posedness of \( (1.1) \) can be proved in the framework of viscosity solutions (see [8]), which in particular requires minimal regularity assumptions on the solution.

We also quote that similar techniques, retaining the regularizing effect of curvature-like terms, but possibly based on different evolution operators, are proposed for example in [11, 13].

Of course, level set methods are not the only strategy proposed to solve the problem of surface reconstruction – for example, segmentation techniques have been successfully proposed for this problem in recent years (see [12, 14]). On the other hand, the largest amount of literature on the topic is probably devoted to least squares techniques. While we give up a complete review of this line of research, we mention that a relatively recent and successful technique in this framework makes use of Radial Basis Functions (RBF) space reconstructions, whose application to surface reconstruction stems from pioneering works published in the late 90s [16, 6, 5] (see also [15, 17] for a general review).

The aim of the present work is to investigate the application of semi-Lagrangian (SL) numerical techniques to \( (1.1) \), focusing in particular on their implementation with RBF space reconstructions. The application of RBF techniques to SL schemes has gained a certain popularity, although it has been restricted so far to the case of hyperbolic problems (see, e.g., [1] and the literature therein). On the other hand, Semi-Lagrangian schemes for curvature-related equations have been first proposed in [9] and has gone through a number of improvements and applications (see, in particular, [2] for an in-depth convergence analysis and [3, 4] for two applications to image processing, along with [10] for a general review on SL schemes). In this framework, RBF would be expected to provide a more flexible tool to construct a sparse space reconstruction. In fact, since their construction is not based on a space grid, Radial Basis Functions allow in principle for a sparse implementation, as well as for local refinements, although we are not aware of general strategies which could effectively handle RBFs in very disordered geometries. We propose therefore a structured, but local RBF space reconstruction in which one could better focus on the region close to the data set, instead of working on a full (and computationally more expensive) grid. This will be the final goal of the paper.

We finally remark that, while a model like \( (1.1) \) might not be rated as the cutting-edge technique for surface reconstruction, yet the coupling of SL schemes with localized RBF space reconstructions shows a good potential in terms of accuracy and computational cost, and seems to be valuably applicable to a wider class of problems (in particular, level set models as the one under consideration).

The outline of the paper is as follows. Section 2 will review the basic principles of construction of a SL scheme for \( (1.1) \), as well as the underlying ideas of RBF interpolation. Section 3 will show numerical tests of increasing difficulty in two and three space dimensions, with both full and reduced grids.

## 2 Numerical scheme

We sketch in this section the construction of a Semi-Lagrangian (SL) approximation to \( (1.1) \). We start by sketching the basic ideas on the two dimensional case, then give the three-dimensional version of the scheme, and last describe the main improvements and modification for the case of RBF space interpolations.

The main feature of the SL scheme under consideration is to be explicit, yet not constrained by the classic “parabolic CFL” condition, typical in the explicit treatment of diffusion terms.
The degenerate diffusion performed by the curvature operator is treated by means of a convex combination of (interpolated) values of the numerical solution at the previous step, as we will soon show.

2.1 The Mean Curvature Operator

Before introducing our Semi-Lagrangian scheme, we rewrite the mean curvature operator in such a way that the derivation of the scheme will be more natural. Let us recall that

\[
|Du(x,t)| \text{div} \left( \frac{Du(x,t)}{|Du(x,t)|} \right) = \text{tr}(P(Du)D^2u) = \text{tr} \left( \left( I - \frac{Du \otimes Du}{|Du|^2} \right) D^2u \right)
\]

where \( P(Du) \) is the projection matrix on the tangent plane to the surface, it is of rank \( n-1 \) with \( n-1 \) eigenvectors corresponding to the eigenvalue \( \lambda = 1 \). In the case \( n = 2 \) there is only one eigenvector orthogonal to the gradient

\[
\sigma(Du) = \frac{1}{|Du|} \begin{pmatrix} u_{x2} \\ -u_{x1} \end{pmatrix}.
\]

In this case the operator can be rewritten as

\[
|Du(x,t)| \text{div} \left( \frac{Du(x,t)}{|Du(x,t)|} \right) = \sigma(Du)^\top D^2u \sigma(Du).
\] (2.1)

In the case \( n = 3 \), the projection matrix \( P \) is a \( 3 \times 2 \) matrix spanning the space orthogonal to the gradient of \( u \). The two orthonormal eigenvectors of \( P \) are:

\[
\nu_1(Du) = \begin{pmatrix} -u_{x2} \\ \sqrt{u_{x1}^2 + u_{x3}^2} \end{pmatrix}, \quad \nu_2(Du) = \frac{1}{|Du|} \begin{pmatrix} u_{x1} \\ \sqrt{u_{x1}^2 + u_{x3}^2} \end{pmatrix},
\] (2.2)

and the mean curvature operator can be rewritten as

\[
|Du(x,t)| \text{div} \left( \frac{Du(x,t)}{|Du(x,t)|} \right) = \sigma(Du)^\top D^2u \sigma(Du) = \frac{1}{2}(\nu_1 + \nu_2)^\top D^2u (\nu_1 + \nu_2) + \frac{1}{2}(\nu_1 - \nu_2)^\top D^2u (\nu_1 - \nu_2),
\] (2.3)

where \( \sigma(Du) = (\nu_1(Du), \nu_2(Du)) \). We observe that the last formula may also be recast in the form

\[
\frac{1}{2}(\nu_1 + \nu_2)^\top D^2u (\nu_1 + \nu_2) + \frac{1}{2}(\nu_1 - \nu_2)^\top D^2u (\nu_1 - \nu_2) = \nu_1^\top D^2u \nu_1 + \nu_2^\top D^2u \nu_2,
\]

although we will rather derive the scheme from \[2.3\], which corresponds to the probabilistic interpretation described in [2].
2.2 The basic 2D form

In the two-dimensional case, the scheme has the form

\[
\begin{align*}
    u_j^{n+1} &= \frac{1}{2} I[u^n](x_j + \Delta t Dd(x_j) + \sqrt{2\Delta t d(x_j)} \sigma^n_j) + \\
    &+ \frac{1}{2} I[u^n](x_j + \Delta t Dd(x_j) - \sqrt{2\Delta t d(x_j)} \sigma^n_j) \\
    u_j^n &= u^0(x_j)
\end{align*}
\]

for some consistent approximations \(D^n_{1,j}, D^n_{2,j}\) (e.g., centered partial incremental ratios of the numerical solution) for the partial derivatives \(u_{x_1}(x_j, t_n), u_{x_2}(x_j, t_n)\), with \(|D^n_j|\) denoting the euclidean norm of the approximate gradient at \((x_j, t_n)\), i.e.,

\[
|D^n_j| = \left( D^n_{1,j}^2 + D^n_{2,j}^2 \right)^{1/2}.
\]

In addition, \(I[u^n](x)\) denotes a numerical interpolation of the solution \(u^n\) computed at the point \(x\).

Note that, at each node, (2.4) performs an average of the solution at the two points \(x_j + \Delta t Dd(x_j) \pm \sqrt{2\Delta t d(x_j)} \sigma^n_j\). The term \(\Delta t Dd(x_j)\) clearly represents an upwinding along the advection field \(Dd(x_j)\), as usual in SL schemes. On the other hand, the degenerate operator related to curvature is treated by adding a further displacement \(\pm \sqrt{2\Delta t d(x_j)} \sigma^n_j\), which has the effect of generating a diffusion along the tangent space of the level sets, which is consistent with the curvature operator (2.1). More explicitly, (2.4) can be rewritten via minor algebraic manipulations as

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{d(x_j)}{|h_j|^2} \left( I[u^n](x_j + \Delta t Dd(x_j) + h_j) - 2I[u^n](x_j + \Delta t Dd(x_j)) + \\
+ I[u^n](x_j + \Delta t Dd(x_j) - h_j) \right) + \frac{1}{\Delta t} \left( I[u^n](x_j + \Delta t Dd(x_j)) - u_j^n \right)
\]

where \(h_j = \sqrt{2\Delta t d(x_j)} \sigma^n_j\) is a finite increment along the tangent space (approximately) spanned by \(\sigma^n_j\). In the right-hand side of (2.6), it is easy to recognize that the two terms are consistent with respectively the second and the first directional derivatives which appear in (1.1). A more detailed consistency analysis for the pure curvature case is carried out in [9, 2].

Singular points at which \(D^n_j \approx 0\) (where the definition (2.5) would not make sense) are to be treated by suitable techniques. A typical choice (see [2]) is to replace the degenerate diffusion with an isotropic one (e.g., the heat operator) whenever the approximate gradient is below a given threshold, say, \(|D^n_j| < C \Delta t^{\alpha}\), for a proper choice of the constant \(C\) and the exponent \(\alpha > 0\). More explicitly, in a form parallel to (2.6), by computing the diffusion term as a 5-point laplacian we obtain

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{d(x_j)}{2|h_j|^2} \left( I[u^n](x_j + \Delta t Dd(x_j) + |h_j|e_1) + I[u^n](x_j + \Delta t Dd(x_j) - |h_j|e_1) + \\
+ I[u^n](x_j + \Delta t Dd(x_j) + |h_j|e_2) + I[u^n](x_j + \Delta t Dd(x_j) - |h_j|e_2) - \\
- 4I[u^n](x_j + \Delta t Dd(x_j)) \right) + \frac{1}{\Delta t} \left( I[u^n](x_j + \Delta t Dd(x_j)) - u_j^n \right)
\]

(2.7)

(where \(e_1, e_2\) are the base vectors of \(\mathbb{R}^2\)), which results in a convex combination of the values \(I[u^n](x_j + \Delta t Dd(x_j) \pm |h_j|e_i)\) and is therefore explicit and stable. This technique of treating singularities can be shown to be compatible with the definition of viscosity solution for (1.1) (see [2]).
2.3 The 3D case

Following [2], we can also write a three-dimensional version of the scheme (which will be used in the section on numerical tests to recover surfaces). The new form of \( \sigma_j^n \) will then be written as

\[
\sigma_j^n = \begin{cases} 
(\nu_1(D^n_j), \nu_2(D^n_j)) & \text{if } \sqrt{D_{1,j}^2 + D_{2,j}^2} \neq 0 \\
(d_1, d_2) & \text{if } \sqrt{D_{1,j}^2 + D_{2,j}^2} = 0,
\end{cases}
\]

where \( d_1 = (1,0,0)^\top \) and \( d_2 = (0,0,1)^\top \). Using this matrix to perform an average of 4 points on the tangent plane, we write the scheme as

\[
v_j^{n+1} = \frac{1}{4} \sum_{i=1}^{4} I[u^n](x_j + \Delta t \, Dd(x_j) + \sqrt{2\Delta t \, d(x_j) \, \sigma_j^n \delta_i})
\]

in which the vectors \( \delta_i \) \( (i = 1, \ldots, 4) \) are defined as

\[
\delta_i = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}
\]

for all combinations of the signs. More explicitly, (2.8) can be rewritten via minor algebraic manipulations as

\[
v_j^{n+1} - v_j^n = \frac{1}{\Delta t} \left(I[u^n](x_j + \Delta t \, Dd(x_j)) - u_j^n\right) + \frac{1}{\Delta t} \left((I[u^n](x_j + \Delta t \, Dd(x_j) + h_j^1) - 2I[u^n](x_j + \Delta t \, Dd(x_j)) + I[u^n](x_j + \Delta t \, Dd(x_j) - h_j^1)) + (I[u^n](x_j + \Delta t \, Dd(x_j) + h_j^2) - 2I[u^n](x_j + \Delta t \, Dd(x_j)) + I[u^n](x_j + \Delta t \, Dd(x_j) - h_j^2))\right)
\]

where \( h_j^1 = \sqrt{2\Delta t \, d(x_j) \, (\nu_1(D_j^n) + \nu_2(D_j^n))} \) and \( h_j^2 = \sqrt{2\Delta t \, d(x_j) \, (\nu_1(D_j^n) - \nu_2(D_j^n))} \). In the second and third line of (2.9), it is possible to recognize second finite difference along the directions \( \nu_1(Du) + \nu_2(Du) \) and \( \nu_1(Du) - \nu_2(Du) \), which has the effect of generating a diffusion along the tangent space of the level sets, which is consistent with the curvature operator (2.3). In the first line of (2.9), an up-wind approximation of the transport term appears.

In (approximately) singular conditions, i.e., when \( |D_j^n| < C \Delta t^n \), the diffusion term is switched to a 7-point laplacian by analogy with (2.7).

2.4 RBF implementation

While the SL scheme (2.4) has proved to be robust and relatively accurate in a variety of applications, we study in this work an adaptation to this specific case. In (1.1), the interest is in following the zero-level set of the solution, which is in turn supposed to stay in the neighbourhood of the data set \( S \). The computational effort has therefore to be concentrated in this latter region – computing an accurate solution away from the data set is useless.

Keeping this idea in mind, we implement (2.4) with a space reconstruction in the form of a Radial Basis Function (RBF) interpolation, which lends itself to a sparse implementation (see [15]). In particular, we have used here the Matlab RBF interpolation toolbox described in [7].

The general structure of the space reconstruction under consideration is

\[
I[u](x) = c_0 + c \cdot x + \sum_i \lambda_i \phi(|x - x_i|)
\]

where the constant \( c_0 \), the vector \( c \) and the coefficients \( \lambda_i \) (which depend on \( u \)) are determined by imposing interpolation conditions, and a suitable closure of the system (see [7]). Note that in
this case, since $I[u]$ has a globally defined form, the computation of the $\sigma_j^n$ in (2.4)–(2.8) might be carried out explicitly.

In (2.10), the function $\phi_\rho : \mathbb{R}^+ \to \mathbb{R}$ provides the radial term, and is expected to depend in general on a shape parameter $\rho$ (e.g., the variance in a Gaussian function). It is generally recognized that the choice of this parameter has a critical effect on the accuracy whenever the set $G$ of the nodes $x_i$ is genuinely scattered, and, in addition, instabilities may occur when $x$ is far from the region covered by the nodes. On the other hand, a structured implementation of RBF space reconstruction give essentially no advantage with respect to a more conventional interpolation. For all these reasons, we will implement the following strategy:

- In order to have a clear guideline for choosing the scale parameter $\rho$, nodes are distributed over a reduced, but structured set, i.e., we start from a structured orthogonal grid and keep only the nodes belonging to some given neighbourhood of the data set. For example, we can use nodes for which
  \[
  d(x_j, S) < \delta_S, \tag{2.11}
  \]
  for some suitable threshold $\delta_S$. The nodes of the data set are also used as RBF nodes. As an example, Fig. 3 shows data set points as circles in red, and RBF nodes as asterisks in blue. Note that taking $\delta_S = O(k\Delta x)$ results in creating a reduced grid containing about $k$ layers of nodes both inside and outside $S$.

- In order to avoid instabilities when far from this set, we introduce a set $A$ of anchor nodes in which the numerical solution has a prescribed value, i.e., which work as boundary conditions, possibly both inside and outside the curve/surface to be reconstructed. Referring again to Fig. 3 anchor points appear as the outer crosses in black.

We will show in the section on numerical examples that this framework results in a considerable reduction of the computational grid, still preserving robustness of the algorithm.

**Initial condition** In (2.4), a natural choice for the initial condition $u^0$ would be the signed distance from the initial position $\Gamma_0 = \partial \Omega_0$ of the evolving surface, that is

\[
  d_s(x, \Gamma_0) := \begin{cases} 
  d(x, \Omega_0) & \text{if } x \notin \Omega_0 \\
  -d(x, \Omega_0) & \text{if } x \in \Omega_0.
  \end{cases}
\]

In order to speed up convergence, the general idea would be to use as initial condition an approximation of the signed distance from the surface (or curve) to be reconstructed, although, of course, we cannot define it in a rigorous way, the surface being unknown and $S$ being a discrete set. However, since we look for an asymptotic solution for (1.1), it will suffice in practice to have a criterion to decide whether a node is likely to be inside or outside of the curve – clearly, a better choice of the initial surface results in a faster solution. In lack of a sharper information, a relatively conservative choice of the initial condition might be set as

\[
  u_j^0 = \begin{cases} 
  |x_j|^2 - R^2 & \text{if } x_j \in G \\
  C & \text{if } x_j \in A
  \end{cases} \tag{2.12}
\]

with a suitable choice of the constants $R, C > 0$ in order to start from a zero-level set enclosing the data set, but not too far from it. This choice will be used in the numerical examples of the next section.

Note that, strictly speaking, (2.12) does not correspond to a signed distance. However, we did not experience a critical behaviour of the scheme with respect to this choice, nor the need for a re-initialization of the level set function. In fact, the effect of the advection term is to push the solution towards the data set, and this causes the solution to maintain a steep transition in the neighbourhood of the interface.
3 Numerical tests

We present in this section some numerical examples of shape reconstructions with the algorithm described above. In the two-dimensional case, we will also compare the reduced grid with the full grid scheme. Note that, in two dimensions, it is known (see [19]) that a polygonal line joining the points of the data set is a local minimizer of the energy, so this should be regarded as an “exact solution” for the 2D case. In all the test, we compute the normalized $L_1$-norm of the update between two successive iterates:

$$E_1^n = \frac{\sum_{j \in G} |u_{j}^{n+1} - u_{j}^{n}|}{\sum_{j \in G} |u_{j}^{n}|},$$

(3.1)

this being an indication on the speed of convergence to the regime solution. Note that, when plotted on a semi-log scale (as it will be done in what follows), a linear decrease of $E_1^n$ suggests a contraction-like behaviour of the scheme.

In the following examples, we will keep the data set in a quite sparse form. We found it discriminating to work with a relatively low number of points, and this will be our strategy in order to check the robustness of the numerical algorithm.

3.1 Heart-shaped 2D data set

**Full grid** We first consider a data set $S$ made of 24 points uniformly chosen on a heart-shaped curve and a grid $G$ of $20 \times 20$ evenly spaced nodes on a domain $[-2, 2] \times [-2, 2]$, as shown in the left plot of Fig. 1. We apply (2.4) on the grid $G$ with time step $\Delta t = 0.01$ and show (in the right plot of Fig. 1) the trend of the norm of the update between two successive iterations, (3.1). The algorithm is stopped after 300 iteration, showing the initial condition and the final solution in the two plots of Fig. 2. The reconstructed curve is very close to the theoretical forecast of a polygonal line, except for a slight smoothing of the non-convex, upper section of the shape (see Fig. 2). Regardless of the number of points in the data set, this gives the natural indication that the computational grid is too coarse. Note also that convergence of iterates tends to become quite slow.

**Reduced grid** Then, the same test is performed on a reduced grid $G$ made of 106 point distributed around the data set according to (2.11) with $\delta_S = 0.2$, as shown in the left plots of Fig. 3, which also shows the anchor points (marked as the outer black crosses). The smaller area covered by the grid allows to improve the resolution by reducing the space step, still decreasing the total number of nodes to a fraction of about 25% of the original grid. Compared with the full (but coarser) grid, we obtain now better results (Fig. 4). Note that, comparing the right plots of Fig. 1 and 3, convergence of the iterative solver seems to be comparable in terms of iteration number, but with a lower cost for a single iteration.

3.2 3D data sets

**Heart-shaped surface** We consider a data set $S$ made of 1586 points chosen on a 3D heart shape. We build a grid $G$ made by the dataset and by 554 additional nodes uniformly distributed on a narrow band of the data set, computed according to (2.11) with $\delta_S = 0.1$ from a full grid $20 \times 20 \times 20$. The dataset and the grid are shown in the left plot of Fig. 5, in which it is also possible to locate the outer anchor nodes.

We apply (2.8) on the grid $G$ with time step $\Delta t = 0.005$ and show (in the right plot of Fig. 5) the norm of the update between two successive iterations. The algorithm is stopped after 80 iteration, showing the initial condition and the final solution in the two plots of Fig. 6.
Figure 1: Left: Grid nodes $\mathcal{G}$ (blue rhombus) and data set $S$ (red circles). Right: relative update between two successive iterations.

Figure 2: Left: Initial condition (continuous line) and data set (red circles). Right: Solution at the final time (continuous line) and data set (red circles).
Figure 3: Left: Grid nodes $G \setminus S$ (blue rhombus) and data set $S$ (red circles). Right: relative update between two successive iterations.

Figure 4: Left: Initial condition (continuous line) and data set (red circles). Right: Solution at the final time (continuous line) and data set (red circles).
Figure 5: Left: Grid nodes $G \setminus S$ (red rhombus markers) and data set $S$ (blue circle markers). Right: relative update between two successive iterations.

Figure 6: Top left: Zero-level set of the initial condition. Top right: initial condition and data set (blue circle markers). Bottom left: Zero-level set of the solution at the final time. Bottom right: solution at the final time and data set (blue circle markers).
Two intersecting cubes As a third test, we consider a data set $S$ made of 4020 points uniformly chosen on a shape formed by two intersecting cubes, whose axes are not aligned with the grid. We consider a grid $G$ containing the data set and 1076 additional nodes uniformly chosen on a narrow band, computed according to (2.11) with $\delta_S = 0.01$ from a full grid $81 \times 81 \times 81$. All these points are shown in the left plot of Fig. 7 together with the frame of anchor points outside the shape. We apply (2.8) on the grid $G$ with time step $\Delta t = 0.01$. The algorithm is stopped after 100 iteration, showing the initial condition and the final solution in the two plots of Fig. 8.

Teapot Finally, we test our algorithm on the more complex benchmark of a teapot-shaped surface, whose data set has been taken from http://www.itl.nist.gov/iad/vug/sharp/benchmark/3DInterestPoint/ We consider a data set $S$ made of 2602 points (about 25% of the original dataset) and a grid $G$ containing the data set and 3109 additional nodes chosen on a narrow band of the data set, computed according to (2.11) with $\delta_S = 0.1$ from a full grid $50 \times 50 \times 50$. Moreover, anchor points are added outside the shape, as shown in the left plot of Fig. 9. We apply (2.8) on the grid $G$ with time step $\Delta t = 0.001$. The algorithm is stopped after 150 iterations, showing the initial condition and the final solution in the two plots of Fig. 10.

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Figure 7: Left: grid nodes $\mathcal{G} \setminus \mathcal{S}$ (blue rhombus markers) and data set $\mathcal{S}$ (red circle markers). Right: relative update between two successive iterations.

Figure 8: Top left: zero-level set of the initial condition. Top right: initial condition and data set (blue circle markers). Bottom left: zero-level set of the solution at the final time. Bottom right: solution at the final time and data set (blue circle markers).
Figure 9: Left: grid nodes $\mathcal{G} \setminus S$ (red rhombus markers) and data set $S$ (blue circle markers). Right: relative update between two successive iterations.

Figure 10: Top left: zero-level set of the initial condition. Top right: initial condition and data set (blue circle markers). Bottom left: zero-level set of the solution at the final time. Bottom right: solution at the final time and data set (blue circle markers).
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