The Generation of Compatible Jacobi Tensors via Gauge Transformations and its Applications

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We present a simple way of generating the infinite set of Jacobi tensors, compatible with a given one, via the "gauge transformations" of the functions on Jacobi manifold. We consider also some applications of this result to the construction of bi-Hamiltonian systems on Jacobi manifolds.

1 Introduction

It is well known that Poisson manifolds play important role in Hamiltonian mechanics and integrability [1]. The suitable analytic framework for these developments is usually referred as the Poisson calculus [2]. The Poisson structure on manifold $M$ endows the space of functions $C^\infty(M)$ by the structure of Lie algebra (via Poisson bracket), which respects the usual structure of commutative algebra on $C^\infty(M)$, i.e. the Leibnitz rule for the Poisson bracket takes place. Having rejected the Leibnitz rule, we may obtain more general structures of Lie algebra on $C^\infty(M)$, which are referred as local Lie algebras [3]. The manifolds, equipped with such structures, are usually called Jacobi manifolds. Namely, following Lichnerowicz [4], we may define Jacobi manifold as follows

**Definition 1** Let $M$ be the manifold, on which are defined a skew-symmetric 2-tensor $P$ and a vector $a$, which satisfy

\[
[P, P] = 2a \wedge P,
\]

\[
[P, a] = 0.
\]

Then the triple $(M, P, a)$ is called Jacobi manifold.

Here $[\cdot, \cdot]$ stands for the Schouten bracket (vide, e.g., [1, 4]) and $\wedge$ stands for the exterior product of differential forms.

The condition (2) means simply that the Lie derivative of $P$ with respect to $a$ vanishes.

Provided $(M, P, a)$ is Jacobi manifold, the space $C^\infty(M)$ becomes a Lie algebra (more precisely, a local Lie algebra [3]) with the commutator [4]

\[
\{f, g\} = i(P)(df \wedge dg) + f i(a)dg - g i(a)df.
\]

Note that in local coordinates $x^k, k = 1, \ldots, n = \dim M$ in each chart of $M$ (3) may be rewritten as

\[
\{f, g\} = P^{kl}(x) \partial f/\partial x^k \partial g/\partial x^l + a^k(x)(f \partial g/\partial x^k - g \partial f/\partial x^k).
\]

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Here $P^{kl} = -P^{lk}$ and the Einstein's summation convention is adopted with respect to the repeated indices $i, j, k, l, \ldots$, which are assumed to run from 1 to $n$ (unless otherwise stated).

The bracket (3) is skew-symmetric by construction and satisfies Jacobi identity in virtue of (1) and (2).

The pair $(P, a)$, satisfying (3) and (2), is usually called the Jacobi tensor (or Jacobi structure [4]). Two Jacobi tensors $(P_1, a_1)$ and $(P_2, a_2)$ are said to be compatible if $(P, a) = (P_1 + P_2, a_1 + a_2)$ also is Jacobi tensor.

2 The main result

The straightforward computation shows that the following assertion holds true:

**Theorem 1** Let $(P, a)$ be Jacobi tensor on $M$ and $\varphi$ be an arbitrary function from the space $C^\infty(M)$. Then

$$(\tilde{P}, \tilde{a}) \equiv (\exp(\varphi)P, \exp(\varphi)a - \exp(\varphi)i(P)d\varphi)$$

is Jacobi tensor, compatible with $(P, a)$, and the map

$$\Psi: f \mapsto f \exp(-\varphi), \quad f \in C^\infty(M), \quad (P, a) \mapsto (\tilde{P}, \tilde{a})$$

is an isomorphism of local Lie algebras.

Here $i(P)d\varphi$ stands for the vector with the components $P^{ij}(x)\partial\varphi/\partial x^j$.

As we see, the above isomorphism is constructed by means of the analog of the well known gauge transformation in physics, $f \mapsto f \exp(i\chi)$, with setting $\chi = i\varphi$, where $i \equiv \sqrt{-1}$.

Thus, the Jacobi tensors $(P, a)$ and $(\tilde{P}, \tilde{a})$ are equivalent and differ only by the "gauge" $\varphi$.

However, the compatibility of these Jacobi tensors allows one to proceed in almost standard manner and try to define bi-Hamiltonian systems on the Jacobi manifold in the following evident way:

Let $H \in C^\infty(M)$ be the Hamiltonian. Let us consider the corresponding equations of motion

$$\frac{df}{dt} = \{H, f\}_{(P, a)} \quad \text{for any} \ f \in C^\infty(M),$$

where we fix explicitly the Jacobi tensor, which participates in the definition of Jacobi bracket.

As we know, the Jacobi tensor $(\tilde{P}, \tilde{a})$ is compatible with $(P, a)$. Let us fix some non-constant function $\varphi$ in $\tilde{(P, a)}$. Then, if there exists the function $H_1 \in C^\infty(M)$ such that

$$\{H, f\}_{(P, a)} = \{H_1, f\}_{(\tilde{P}, \tilde{a})}, \quad \text{for any} \ f \in C^\infty(M),$$

2 If $\varphi$ is constant, the structure $(\tilde{P}, \tilde{a})$ coincides with $(P, a)$ up to multiplication by the constant and therefore is useless from the viewpoint of integrability of the system $\tilde{(P, a)}$. 

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the system (6) is bi-Hamiltonian, and one may construct the set of its integrals \( I_k \) (which will be in involution with respect to both brackets by construction (1)) in the standard way from the equation

\[
\{I_{k-1}, f\}(P, a) = \{I_k, f\}(\tilde{P}, \tilde{a}), \quad k = 1, 2, \ldots \text{ for any } f \in C^\infty(M). \quad (8)
\]

Two first terms in the sequence of \( I_k \)'s are \( I_0 = H \) and \( I_1 = H_1 \).

Thus, if for any \( k = 1, 2, \ldots, k_0 \) (in principle, the case \( k_0 = \infty \) is not excluded) there exists the solution \( I_k \) of (8), and among \( I_k, k = 0, 1, \ldots, k_0 \) one may choose a sufficient number of functionally independent ones, the system (6) will be completely integrable in Liouville’s sense.

3 Discussion

We would like to note that our approach is quite different from the usual scheme of proving the complete integrability in bi-Hamiltonian formalism, where the key problem was the construction of the Poisson structure, compatible with initial one. The key point is that we replace Poisson structure on manifold by more general Jacobi structure. Then Theorem 1 always guarantees the existence of the infinite set of Jacobi structures, compatible with initial one. Hence, we have a very simple sufficient condition for the Hamiltonian system (6) to be bi-Hamiltonian and completely integrable: if for some non-constant function \( \varphi \) in (5) one is able to find the sufficient number of functionally independent solutions of (8) for different \( k \), the system (6) will be completely integrable in Liouville’s sense, as it was already mentioned above.

The problem of existence of solutions of (8) is now under study and will be the subject of our following publications.

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References

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\(^3\)On each symplectic leaf \( S \) of \( M \) this number is \( r = (1/2) \dim S \). By analogy with the theory of Poisson structures, we mean here under the symplectic leaf a submanifold of \( M \), on which the bracket \( \{,\}(P, a) \) is non-degenerate.