Matching the heavy-quark fields in QCD and HQET at four loops

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The QCD/HQET matching coefficient for the heavy-quark field is calculated up to four loops. It must be finite; this requirement produces analytical results for some terms in the four-loop on-shell heavy-quark field renormalization constant which were previously only known numerically. The effect of a non-zero lighter-flavor mass is calculated up to three loops. A class of on-shell integrals with two masses is analyzed in detail. By specifying our result to QED, we obtain the relation between the electron field and the Bloch–Nordsieck field with four-loop accuracy.

I. INTRODUCTION

Some classes of QCD problems with a single heavy quark can be examined in a simpler effective theory, the so-called heavy quark effective theory (HQET, see, e.g., [1,3]). Let us consider QCD with a single heavy flavor and light flavors \( n_l = n_l + n_h, n_h = 1 \). The heavy-quark momentum can be decomposed as \( p = M v + k \), where \( M \) is the on-shell Q mass, and \( v \) is some reference 4-velocity \((v^2 = 1)\). In the case of QED, it is called Bloch–Nordsieck effective theory [4].

In the effective theory, the heavy quark (respectively lepton) is represented by the field \( h_v \). The \( \overline{\text{MS}} \) renormalized fields \( Q(\mu) \) and \( h_v(\mu) \) are related by [5]

\[
Q(\mu) = e^{-i M v \varepsilon} \left[ \sqrt{z(\mu)} \left( 1 + \frac{D_\mu}{2M} \right) h_v(\mu) + O \left( \frac{1}{M^2} \right) \right],
\]

where \( D_\mu = D^\mu - \nu^\mu v \cdot D \), and the matching coefficient is given by

\[
z(\mu) = \frac{Z_h(\alpha_s(n_l)\mu), \epsilon(n_l)(\mu)) Z_Q^{n_l}(\alpha_s(n_l)\mu), \xi(n_l)(\mu))}{Z_Q(\alpha_s(n_l)\mu), \epsilon(n_l)(\mu)) Z_h^{n_l}(\alpha_s(n_l)\mu), \xi(n_l)(\mu))}.
\]

Here \( Z_Q^{n_l} \) and \( Z_h^{n_l} \) are the on-shell field renormalization constants (they depend on the corresponding bare couplings and bare gauge-fixing parameters), and \( Z_Q \) and \( Z_h \) are the \( \overline{\text{MS}} \) renormalization constants. The covariant-gauge fixing parameter is defined in such a way that the bare gluon propagator is given by \((g_{\mu\nu} - \xi_0 p_{\mu} p_{\nu}/p^2)\); it is renormalized by the gluon-field renormalization constant: \( 1 - \xi_0 = Z_A(\alpha_s(\mu), \xi(\mu))(1 - \xi(\mu)) \). The \( 1/M \) correction in [1] is fixed by reparametrization invariance [6].

\[\varepsilon\to0\]

\[\overline{\text{MS}}\] renormalized matching coefficient is obviously finite at \( \varepsilon \to 0 \), because it relates the off-shell renormalized propagators in the two theories, which are both finite. The ultraviolet divergences cancel in the ratios \( Z_Q/Z_Q^{n_l} \) and \( Z_h/Z_h^{n_l} \), because they relate renormalized fields; the infrared divergences cancel in \( Z_Q^{n_l}/Z_h^{n_l} \), because HQET is constructed to reproduce the infrared behavior of QCD; the \( \overline{\text{MS}} \) renormalization constants \( Z_Q \) and \( Z_h \) (purely off-shell quantities) are infrared finite. If we assume that all light flavors are massless we have \( Z_h^{n_l} = 1 \): all loop corrections vanish because they contain no scale, ultraviolet and infrared divergences of \( Z_h^{n_l} \) mutually cancel. Taking light-quark masses \( m_i \) into account produces corrections suppressed by powers of \( m_i/M \), see Sect. III.

The matching coefficient satisfies the renormalization-group equation

\[
d \log z(\mu) \overline{d \log \mu} = \gamma_h(\alpha_s(n_l)\mu), \epsilon(n_l)(\mu)) - \gamma_Q(\alpha_s(n_l)\mu), \epsilon(n_l)(\mu)),
\]

where the anomalous dimensions are defined as \( \gamma_i = d \log Z_i / d \log \mu \) (\( i = Q, h \)). It is sufficient to obtain the initial condition \( z(\mu_0) \) for some scale \( \mu_0 \sim M \); \( z(\mu) \) for other renormalization scales \( \mu \) can be found by solving Eq. [3]. We choose to present the result for \( \mu_0 = M \).

The heavy-quark field matching coefficient \( z(\mu) \) has been calculated up to three loops [5]. When the matching coefficient is used within a quantity containing \( 1/e \) divergences, terms with positive powers of \( e \) in \( z(\mu) \) are needed; such terms were not given in [5]. We present the four-loop result in Sect. IV. Power corrections due to lighter-flavor masses up to three loops are obtained in Sect. III. The QED result, i.e. the four-loop relation between the lepton field and the Bloch–Nordsieck field, is discussed in Sect. V. In Appendix A we provide analytic results for the decoupling coefficients for the strong coupling constant and the gluon field up to three-loop order including linear \( \varepsilon \) terms. Appendix B contains a detailed analysis of a class of on-shell integrals with two masses. It al-

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lows us, in particular, to obtain exact results for the three-loop term in the MS–on-shell mass relation with a closed massless and a closed lighter-flavor massive fermion loop (previously this term was only known as a truncated series in this mass ratio).

II. THE QCD AND HQET HEAVY-QUARK FIELDS

If we assume that all light flavors are massless, then the on-shell mass relation with a closed massless
\[ z(\mu) = \log Z_Q^{\text{MS}}(\mu, \xi_0^{(n)}) + \log Z_h(\alpha_s^{(n)}(\mu), \xi^{(n)}(\mu)). \] (4)
The on-shell heavy-quark field renormalization constant \( Z_Q^{\text{MS}} \) depends on the bare coupling \( \alpha_s^{(n)}(\mu) \), the bare gauge parameter \( \xi_0^{(n)} \) and the on-shell mass \( M \):
\[ Z_Q^{\text{MS}} = 1 + \sum_{L=1}^{\infty} \left( \frac{4(\mu_0^{(n)})^2 M^{2\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma_E \varepsilon} \right)^L Z_L, \]
\[ Z_L = \sum_{n=0}^{\infty} Z_{L,n}(\xi_0^{(n)}) \varepsilon^{n-L}. \] (5)
The two-loop expression is known exactly in \( \varepsilon [7] \); it contains a single non-trivial master integral, further terms of its expansion are presented in \( [8, 9] \). The three-loop term has been calculated in \( [10, 11] \). At four loops, the terms with \( n_1^3 \) and \( n_1^2 \) are known analytically \( [12] \), and the remaining ones numerically \( [13] \). Recently the QED-like color structures \( C_F \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( d_{FF}\mu \) have been calculated numerically \( [13] \). Here and below we use the notation
\[ d_{\mu} = \frac{d_{\mu}^{abcd} d_{\mu}^{abcd}}{N_F}, \quad d_{\mu} = \frac{d_{\mu}^{abcd} d_{\mu}^{abcd}}{N_F}. \] (6)

where \( N_R = \text{Tr} \mathbf{1}_R \) (with \( R = F \)), \( d_{\mu}^{abcd} = \text{Tr} t^a_R t^b_R t^c_R t^d_R \) (with \( R = F \) or \( A \)), and the round brackets mean symmetrization (for \( SU(N_c) \) gauge group \( d_{FF} = (N_c^2 - 1)(N_c^2 - 6N_c^2 + 18)/(96N_c^3) \), \( d_{FA} = (N_c^2 - 1)(N_c^2 + 6)/48 \). This result contains the same master integrals as the electron \( g = 2 \) \( [15, 16] \). In \( [15] \) they have been calculated numerically to 1100 digits, and analytical expressions have been reconstructed using PSLQ. In the case of the light-by-light contribution \( d_{FF}\mu \) the results contain \( \varepsilon^0 \) terms of 6 master integrals (known numerically to 1100 digits); all the remaining constants are completely expressed via known transcendental numbers (Note that the definition of the constant \( t_{63} \) is missing in the journal article \( [14] \); it is included in the version v3 of the arXiv publication.).

The MS quark-field anomalous dimension \( \gamma_q \) (and hence \( \log Z_Q \)) is well known \( [17, 20] \). The HQET field anomalous dimension \( \gamma_h \) (and hence \( \log Z_h \)) is known analytically \( [10, 21] \). At four loops, some color structures are known analytically: \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( d_{FF}\mu \) have been calculated numerically \( [13] \). Recently the QED-like color structures \( C_F \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( d_{FF}\mu \) have been calculated numerically \( [13] \). We need to express the three terms in \( [6] \) in terms of the same set of variables, for which we choose \( \alpha_s^{(n)}(\mu) \) and \( \xi^{(n)}(\mu) \). Expressing \( \alpha_s^{(n)}(\mu) \) and \( \xi^{(n)}(\mu) \) via these variables is straightforward, since the three-loop renormalization constants in QCD are well known. Expressing \( \alpha_s^{(n)}(\mu) \) and \( \xi^{(n)}(\mu) \) via the \( n_f \)-flavor quantities requires decoupling relations up to \( O(\varepsilon) \) at three loops. For convenience we present explicit results in Appendix A.

The resulting matching coefficient \( z(M) \) must be finite at \( \varepsilon \rightarrow 0 \). This requirement together with the known results for \( Z_Q \) and \( Z_h \) leads to analytical expressions for the four-loop coefficients \( Z_{4,0}, Z_{4,1}, \) and \( Z_{4,2} \) in \( [5] \) as well as for \( Z_{4,3} \), except two color structures \( C_F C_A \) and \( d_{FA} \) where the corresponding terms in \( \gamma_h \) are not known analytically. The analytical results are presented in the tables I and II. We refrain from showing results for the \( n_1^2 \) and \( n_1^3 \) terms, which are already known since a few years \( [12] \). Furthermore, we have introduced \( \alpha_n = \text{Li}_n(1/2) \) (in particular \( \alpha_1 = \log 2 \); \( \xi_0 \) denotes the Riemann zeta function and \( \xi_0 = \xi^{(n)}(\mu) \). Analytical results for the color structures \( C_F \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( C_F(T_{FF})^3 \), \( d_{FF}\mu \) were recently obtained \( [13] \). They agree with the expressions given in tables I and II. Numerical results for these coefficients are given in the tables VI, VII, and VIII of Ref. [13]. Good agreement is found.

Using the matching coefficient \( z(\mu) \) together with quantities which contains \( 1/\varepsilon \) divergences, terms with positive powers of \( \varepsilon \) are needed. In order to get the finite four-loop contribution, we need the \( \alpha_s^{(n)}(\mu) \) term in \( z(\mu) \) expanded up to \( \varepsilon^{4-L} \). Our result for \( \mu = M \) is given by

\[ z(M) = 1 - \frac{\alpha_s}{\pi} C_F \left[ 1 + \varepsilon \left( \frac{\pi^2}{16} + 2 \right) - \varepsilon^2 \left( \frac{\xi_3}{4} - \frac{\pi^2}{12} - 4 \right) - \varepsilon^3 \left( \frac{\xi_3}{3} - \frac{3}{640} \frac{\pi^4}{6} - \frac{\pi^4}{6} - 8 \right) + O(\varepsilon^4) \right] \]
\[ + \left( \frac{\alpha_s}{\pi} \right)^2 C_F \left[ C_F \left( \frac{\pi^2}{16} + 2 \right) - \frac{3}{2} \xi_3 - \frac{13}{16} \pi^2 + \frac{241}{128} - \frac{C_A}{2} \right] \left( \frac{\pi^2}{16} + 2 \right) \]
\[ + \frac{1}{2} \left( T_{FFh}^2 \right) \left( \frac{\pi^2}{16} + 2 \right) \]
\[ + \varepsilon \left( -C_F \left( 24a_4 + a_4 + 2\pi^2 a_1 - \frac{23}{4} \pi^2 a_1 + 147 \frac{\pi^2}{8} - \frac{7}{20} \xi_3 - \frac{7}{20} \pi^4 + \frac{347}{128} \pi^2 + \frac{557}{256} \right) \right]. \]
TABLE I. Coefficients \(Z_{\alpha,n}\) of the \(1/\varepsilon^{4,3,2}\) terms entering the four-loop result \(Z_4\) in Eq. [5]. Note that the color structures \(d_{FFm}, d_{FFnh}, d_{FA}\) have zero coefficients.

| Color | \(\varepsilon^{-4}\) | \(\varepsilon^{-3}\) | \(\varepsilon^{-2}\) |
|-------|----------------|----------------|----------------|
| \(C_F\) | \(\frac{27}{2004}\) | \(\frac{171}{2004}\) | \(\frac{3}{22}\) |
| \(C_F C_A\) | \(-\frac{99}{1024}\) | \(-\frac{177}{1024}\) | \(-\frac{1}{16}\) |
| \(C_F C_A^2\) | \(-\frac{1311}{6744}\) | \(\frac{203}{96}\) | \(\frac{1}{16}\) |
| \(C_F C_A^3\) | \(-\frac{1311}{9256}\) | \(-\frac{6569}{9256}\) | \(-\frac{1}{16}\) |
| \(C_F^2 C_T F n_h\) | \(\frac{9}{128}\) | \(\frac{131}{1532}\) | \(\frac{1}{4}\) |
| \(C_F^2 C_A C_T F n_h\) | \(-\frac{67}{128}\) | \(-\frac{256}{128}\) | \(\frac{1}{8}\) |
| \(C_F^2 C_A^2 C_T F n_h\) | \(-\frac{256}{2048}\) | \(\frac{256}{2048}\) | \(-\frac{1}{16}\) |
| \(C_F^2 C_A (T F n_h)^2\) | \(\frac{3}{32}\) | \(\frac{27}{128}\) | \(\frac{1}{4}\) |
| \(C_F C_A (T F n_h)^2\) | \(-\frac{15}{32}\) | \(-\frac{141}{128}\) | \(\frac{1}{4}\) |
| \(C_F (T F n_h)^3\) | \(\frac{1}{64}\) | \(\frac{41}{32}\) | \(\frac{1}{16}\) |
| \(C_F C_A T F n_l\) | \(\frac{9}{256}\) | \(\frac{1}{4}\) | \(\frac{1}{16}\) |
| \(C_F^2 C_A T F n_l\) | \(-\frac{121}{6744}\) | \(-\frac{373}{2048}\) | \(\frac{1}{16}\) |
| \(C_F C_A^2 T F n_l\) | \(\frac{121}{6744}\) | \(\frac{256}{2048}\) | \(\frac{1}{16}\) |
| \(C_A T F n_h/\) | \(-\frac{649+50}{576}\) | \(-\frac{19555+50}{192}\) | \(-\frac{1}{2}\) |
| \(C_F T F n_l/\) | \(\frac{1}{24}\) | \(\frac{19}{96}\) | \(-\frac{1}{12}\) |

\[
\begin{align*}
+ C_A \left(12a_4 + a_1^4 + 2\pi^2 a_1^3 - \frac{23}{8} a_1^4 - 23 \pi^2 a_1^3 + \frac{129}{16} \pi^3 a_1^2 - \frac{7}{40} \pi^4 + \frac{769}{152} \pi^2 - 9907 \right) \\
+ T_F n_h \left(2 a^2 a_1 - 7 \zeta_3 - \frac{445}{288} \pi^2 + \frac{17971}{1728} \pi^2 \right) + C_F \left(\zeta_3 + \frac{127}{288} \pi^2 - 851 \right) \\
+C_F \left(144 a_5 + 138 a_4 - \frac{6}{5} a_5^3 + \frac{23}{4} a_4^3 - 4 a_2 a_1^3 - \frac{23}{2} \pi^2 a_1^2 + \frac{13}{15} \pi^3 a_1 - \frac{41}{2} \pi^2 a_1 \right) \\
- \frac{609}{4} \zeta_5 - \frac{11}{4} \pi^2 \zeta_3 - \frac{2061}{32} \pi^3 \zeta_3 - \frac{1555}{1536} \pi^4 + \frac{8947}{768} \pi^2 - \frac{1817}{512} \pi^2 \\
+ C_A \left(72 a_5 + 69 a_4 - \frac{3}{5} a_5^3 + \frac{23}{8} a_4^3 - 2 a_2 a_1^3 + \frac{23}{4} \pi^2 a_1^2 + \frac{13}{30} \pi^3 a_1 - \frac{41}{4} \pi^2 a_1 \right) \\
- \frac{609}{8} \zeta_5 - \frac{11}{8} \pi^2 \zeta_3 - \frac{7595}{288} \pi^3 \zeta_3 - \frac{14359}{2304} \pi^4 + \frac{6367}{2304} \pi^2 - \frac{79225}{1536} \pi^2 \\
- T_F n_h \left(48 a_4 + 2 a_1^4 + 4 a_2 a_1^3 - \frac{19}{2} \pi^2 a_1^3 + \frac{2405}{72} \zeta_3 - \frac{93}{320} \pi^2 + \frac{8605}{1728} \pi^2 - \frac{422747}{10368} \right) \\
+ T_F n_l \left(305 \zeta_3 + \frac{199}{80} \pi^2 + \frac{853}{24} \pi^2 + \frac{5753}{16} \pi^2 \right) + O(\varepsilon^3) \}
\end{align*}
\]

\[\left(\frac{\alpha}{\pi}\right)^3 C_F \left(-C_F \left(28 a_4 + \frac{7}{6} a_1^4 - \frac{3}{2} \pi^2 a_1^3 - \frac{223}{12} \pi^2 a_1^3 + \frac{5}{16} \zeta_5 - \frac{\pi^2}{8} \zeta_3 - \frac{157}{8} \zeta_3 + \frac{19}{240} \pi^4 + \frac{4801}{576} \pi^2 - \frac{3023}{768} \right) \right.\]

\[\left.\left.- C_F \left(\frac{a_1}{6} a_1^4 + \frac{a_1 a_1}{144} + \frac{181}{72} \pi^2 a_1^3 + \frac{43}{9} \pi^2 a_1^3 - \frac{145}{16} \zeta_5 + \frac{45}{16} \pi^2 \zeta_3 + 289 \zeta_3 - \frac{17280}{576} \pi^4 - \frac{2137}{4608} \pi^2 \right) \right.\]

\[\left.\left.+ C_A^2 \left(\frac{1}{2} \left(\frac{85}{2} a_1^4 + \frac{85}{48} a_1^4 + \frac{127}{24} \pi^2 a_1^3 - \frac{325}{24} \pi^2 a_1^3 - \frac{375}{12} \pi^2 \zeta_3 + \frac{5857}{96} \zeta_3 - \frac{3419}{384} \pi^4 - \frac{4339}{576} \pi^2 - \frac{165717}{20736} \right) \right.\right.\]
TABLE II. Coefficients $Z_{4,5}$ of the $1/\varepsilon$ term entering the four-loop result $Z_4$ in Eq. (5). Note that the color structures $C_F C_A^3$ and $d_F A$ are not known analytically.
$$- 949 \pi^4 a_1 + \frac{30803}{432} \pi^2 a_1 - \frac{125473}{288} \zeta_3 + \frac{143}{4} \zeta_3^2 + \frac{2703}{128} \pi^2 \zeta_3 - \frac{16339}{181440} \pi^6 - \frac{496741}{103680} \pi^4 + \frac{17665}{55296} \pi^2$$

$$- 861659$$

$$+ C_A^2 \left[ \frac{707}{6} a_5 - 7 \pi^2 a_4 + \frac{935}{9} a_4 - \frac{707}{720} a_5^3 - \frac{7}{24} \pi^2 a_1 + \frac{935}{216} a_4^3 - \frac{905}{216} \pi^2 a_1^3 + \frac{7}{24} \pi^4 a_2^3 + \frac{7081}{216} \pi^2 a_1^3 - \frac{49}{8} \pi^2 \zeta_3 a_1 \right]$$

$$- 41 \pi^4 a_1 - \frac{8833}{864} \pi^2 a_1 - \frac{11569}{256} \zeta_5 + \frac{745153}{384} \zeta_3^2 + \frac{149153}{6680} \pi^2 \zeta_3 + \frac{67807}{3456} \zeta_4 + \frac{45047}{36280} \pi^4 - \frac{1263911}{51840} \pi^4 - \frac{150229}{41472} \pi^2$$

$$- 72476083 \left[ \frac{1495}{3} - \frac{25}{3} \zeta_3 - \frac{77}{72} \pi^2 \zeta_3 + \frac{63}{2} \zeta_3^2 - \frac{49}{405} \pi^6 - \frac{383}{1080} \pi^4 + \frac{15}{8} \pi^2 + \frac{35}{2} \pi \right]$$

$$+ C_{TF_{nh}} \left[ \frac{72 a_5 - \frac{229}{6} a_4 + \frac{3}{5} a_5^3 - \frac{229}{144} a_4^3 + \pi^2 a_3^3 - \frac{2291}{144} \pi^2 a_1^3 + \frac{143}{180} \pi^4 a_1 + \frac{293}{6} \pi^2 a_1 - \frac{87}{8} \zeta_5 - \frac{81}{8} \pi^2 \zeta_3 \right]$$

$$- \frac{10913}{192} \zeta_3 + \frac{3649}{8640} \pi^4 - \frac{818069}{41472} \pi^2 + \frac{164069}{6912}$$

$$- C_{AT_{F_{nh}}} \left[ \frac{48 a_5 - 8 \pi^2 a_4 + \frac{4247}{12} a_4 - \frac{2}{5} a_5^3 - \frac{\pi^2}{3} a_1 + \frac{4247}{288} a_4^3 + \frac{2}{3} \pi^2 a_1^3 + \frac{\pi^4}{3} a_1^3 + \frac{18133}{288} \pi^2 a_1^3 - 7 \pi^2 \zeta_3 a_1 \right]$$

$$+ \frac{97}{180} \pi^4 a_1 - \frac{775}{9} \pi^2 a_1^3 + \frac{5515}{64} \zeta_5 - \frac{181}{32} \zeta_3^2 - \frac{549}{64} \pi^2 \zeta_3 + \frac{88855}{384} \zeta_3 + \frac{1501}{15120} \pi^6 - \frac{12607}{5760} \pi^4 + \frac{286961}{13824} \pi^2$$

$$- \frac{35801821}{248832} \left[ \frac{121}{1728} \pi^2 - \frac{7367}{1152} \right]$$

$$+ C_{TF_{hl}} \left[ \frac{224 a_5 + \frac{1028}{3} a_4 + \frac{28}{15} a_5^3 + \frac{257}{18} a_4^3 - \frac{56}{9} \pi^2 a_3^3 + \frac{257}{9} \pi^2 a_1^3 - \frac{17}{9} \pi^2 a_1^3 - \frac{539}{9} \pi^2 a_1^3 - \frac{1027}{4} \zeta_5 \right]$$

$$- \frac{119}{16} \pi^2 \zeta_3 + \frac{1081}{6} \zeta_3 - \frac{18599}{8640} \pi^4 + \frac{160081}{4608} \pi^2 + \frac{3103}{72}$$

$$- C_{AT_{F_{nl}}} \left[ \frac{112}{3} a_5 + \frac{514}{9} a_4 - \frac{14}{45} \pi^2 a_1 + \frac{28}{27} \pi^2 a_3^3 + \frac{547}{54} \pi^2 a_1^3 - \frac{17}{54} \pi^2 a_1^3 - \frac{539}{54} \pi^2 a_1^3 - \frac{859}{24} \zeta_5 \right]$$

$$- \frac{11}{16} \pi^2 \zeta_3 + \frac{1229}{432} \zeta_3 - \frac{3691}{4680} \pi^4 + \frac{1991}{648} \pi^2 - \frac{4500377}{93312}$$

$$+ \left( \frac{T_{F_{nh}}}{3} \right)^2 \left[ \frac{56 a_4 + \frac{7}{3} a_1^4 - \frac{7}{3} \pi^2 a_1^3 - \frac{4}{5} \pi^2 a_1^3 + \frac{3221}{80} \zeta_3 - \frac{31}{72} \pi^4 + \frac{39661}{7200} \pi^2 - \frac{636911}{8640} \right]$$

$$+ T_{F_{nl}} \left[ \frac{32}{3} a_4 + \frac{4}{9} a_1^4 + \frac{8}{9} \pi^2 a_1^3 - \frac{35}{9} \pi^2 a_1^3 + \frac{27}{2} \zeta_3 + \frac{179}{1080} \pi^4 + \frac{2245}{1290} \pi^2 - \frac{264817}{7776} \right]$$

$$- \left( \frac{T_{F_{nl}}}{54} \right)^2 \left[ \frac{275 \zeta_3 + \frac{23}{5} \pi^4 + \frac{1081}{16} \pi^2 + \frac{253783}{864} \right] + O(\varepsilon^2) \right)$$

$$+ \left( \frac{\alpha}{\pi} \right)^4 \left\{ C_F^4 \left[ \frac{L_0 - \frac{139}{2} a_5 + 12 \pi^2 a_4 - \frac{9137}{16} a_4 + \frac{139}{240} a_5^3 + \frac{\pi^2}{2} a_1^3 - \frac{9137}{384} a_1^4 - \frac{311}{72} \pi^2 a_3^3 + \frac{\pi^4}{2} a_1^3 - \frac{8597}{192} \pi^2 a_1^3 \right] \right.$$

$$+ \frac{21}{2} \pi^2 \zeta_3 a_1 - \frac{2783}{2880} \pi^4 a_1 + \frac{33687}{256} \pi^2 a_1^3 - \frac{2937}{128} \pi^2 \zeta_3 + \frac{87}{128} \zeta_3^2 + \frac{2755}{192} \pi^2 \zeta_3 - \frac{113181}{512} \zeta_3 - \frac{899}{7560} \pi^6 + \frac{18553}{23040} \pi^4$$

$$- \frac{24129}{1024} \pi^2 - \frac{90577}{8192} \right.$$
\[ + C_F^2 C_A T_F n_h \left[ 14.893 \pm 0.083 - \left( 0.657352 \pm 0.00024 \right) \xi \right] \]
\[ - C_F C_A^2 T_F n_h \left[ 3.1601 \pm 0.056 - \left( 0.198984 \pm 0.00013 \right) \xi + 0.0244254 \xi^2 \right] \]
\[ + C_F^3 (T_F n_h)^2 \left[ L_2 + 120 a_5 + \frac{2749}{4} a_4 - a_1^5 + \frac{2749}{1152} a_1^4 - \frac{\pi^2}{3} a_1^3 + \frac{10525}{1152} \pi^2 a_1^2 + \frac{43}{36} \pi^4 a_1 + \frac{711}{20} \pi^2 a_1 - 493 \frac{\xi^5}{8} \right] \]
\[ - 269 \frac{\xi^3}{24} \zeta_3 - 10127 \frac{\xi^3}{2560} \zeta_3 - 5513 \frac{\xi^3}{13824} \pi^4 \left( 678719 \frac{\xi^3}{64800} - 8452817 \frac{\xi^3}{414720} \right) \]
\[ - C_F C_A (T_F n_h)^2 \left[ 0.01995 \pm 0.0062 - 0.10436 \xi \right] \]
\[ + C_F (T_F n_h)^3 \left[ L_3 + \frac{1}{3} \left( 104 a_4 + \frac{13}{3} a_4^2 + \frac{5}{3} a_2^3 - \frac{103}{10} \pi^2 a_1 + 5881 \frac{\xi^3}{80} - 299 \frac{\pi^4}{360} + 31451 \frac{\xi^3}{2700} - 5981281 \frac{\xi^3}{51840} \right) \right] \]
\[ + d_{F F n h} L_4 - C_F^2 T_F n_l (4.92605 \pm 0.0067) + C_F^2 C_A T_F n_l (15.0599 \pm 0.012) \]
\[ + C_F C_A^2 T_F n_l (166.421 \pm 0.031 - 0.134051 \xi) - C_F^2 T_F n_h (5.08715 \pm 0.00074) \]
\[ + C_F C_A T_F^3 n_h n_l (0.53235 \pm 0.0015 + 0.0910988 \xi) + 0.0138079 C_F T_F^3 n_h^2 n_l - d_{F F n} (2.18 \pm 0.8) \]
\[ - C_F^2 (T_F n_l)^2 \left[ \frac{32}{3} a_5 + \frac{188}{9} a_4 - \frac{4}{45} a_5^4 + \frac{8}{27} \pi^2 a_1^6 + \frac{47}{27} \pi^2 a_1^6 - \frac{31}{270} \pi^4 a_1 - \frac{239}{54} \pi^2 a_1 - \frac{601}{48} \pi^2 - \frac{\xi^3}{2} \right] \]
\[ + \frac{6925}{576} \zeta_3 - \frac{1181}{10368} \pi^4 + \frac{1043}{384} \pi^4 + \frac{31469964}{497664} \]
\[ + \frac{C_F C_A (T_F n_l)^2}{3} \left[ 16 a_5 + \frac{94}{3} a_4 - \frac{2}{15} a_5^4 + \frac{47}{36} a_4^4 - \frac{1}{9} \pi^2 a_1^3 + \frac{47}{18} \pi^2 a_1^3 - \frac{31}{180} \pi^4 a_1 - \frac{239}{36} \pi^2 a_1 - \frac{365}{32} \zeta_3 - \frac{11}{12} \zeta_3 \right] \]
\[ - \frac{4333}{64} \zeta_3 - \frac{6815}{17280} \pi^4 - \frac{4676085}{165888} \]
\[ + \frac{C_F T_F^3 n_h n_l^2}{3} \left[ \frac{53}{16} - \frac{4}{15} \pi^4 + \frac{19}{27} \pi^4 + \frac{199325}{20736} \right] + \frac{C_F (T_F n_l)^3}{216} \left[ \frac{467}{2} \zeta_3 + \frac{71}{20} \pi^4 + \frac{16793}{3} \pi^4 + \frac{103933}{846} \right] + \mathcal{O}(\varepsilon) \}
+ \mathcal{O}(\alpha_s^5),
\]

where \( \alpha_s = \alpha_s^{(n_f)}(M) \), \( \xi = \xi^{(n_f)}(M) \). \( L_{0,1,2,3} \) are the \( \varepsilon^0 \) parts of the quantities \( Z_2^{(4,0)}, Z_2^{(4,1)}, Z_2^{(4,2)}, Z_2^{(4,3)} \) given in Eqs. (28–32) of [14]. Their numerical values are given in Eqs. (5–9) of that paper. The finite four-loop terms of Eq. (7) are equal to the corresponding finite four-loop terms in \( Z_Q^{(0)} \) plus products of lower-loop quantities which are all known analytically. For 14 out of 23 color structures these coefficients in \( Z_Q^{(0)} \) are only known numerically [13]. We use these numerical values, together with their uncertainty estimates, from the tables V, VI, and VII of that paper. Note that in Ref. [13] \( Z_Q^{(0)} \) has been computed in an expansion in \( \xi \) up to the second order; 9 out of these 19 color structures are obviously gauge invariant, and 7 more seem to be either gauge-invariant or have at most 12 \( \xi \) terms (though we know no explicit proof). The remaining 3 structures \( (C_F C_A, d_{FA}, C_F C_A^2 T_F n_h) \) may contain terms with higher powers of \( \xi \), which are not known. The same is true for the corresponding terms in \( z(\mu) \) in Eq. (7).

If we re-express \( z(M) \) in Eq. (7) via \( \alpha_s^{(n_f)}(M) \), the terms up to three loops agree with [5]. (Note that positive powers of \( \varepsilon \) are not presented [5].) The \( \alpha_s^2 n_l^3 \) term also agrees with [5].

After specifying the color factors to QCD with \( N_c = 3 \) we obtain for \( \varepsilon = 0 \)
\[ z(M) = 1 - \frac{4}{3} \alpha_s + \left( \frac{\alpha_s}{\pi} \right)^2 (17.45 - 1.33 n_l) \]

\[ - \left( \frac{\alpha_s}{\pi} \right)^3 (262.42 - 0.78 \xi - 35.81 n_l + 0.98 n_l^2) \]
\[ - \left( \frac{\alpha_s}{\pi} \right)^4 [5137.52 - 15.67 \xi + 1.07 \xi^2 \]
\[ - (1030.82 - 0.71 \xi) n_l + 60.30 n_l^2 - 1.00 n_l^3] \]
\[ + \mathcal{O}(\alpha_s^5). \]

In Landau gauge \( (\xi^{(n_f)} = 1) \) at \( n_f = 4 \) this gives
\[ z(M) = 1 - \frac{4}{3} \alpha_s - 12.12 \left( \frac{\alpha_s}{\pi} \right)^2 + 134.11 \left( \frac{\alpha_s}{\pi} \right)^3 \]
\[ - 1903.22 \left( \frac{\alpha_s}{\pi} \right)^4 + \mathcal{O}(\alpha_s^5), \]

while the naive nonabelianization [22] (large \( \beta_0 \) limit) predicts [5]
\[ 1 - \frac{4}{3} \alpha_s - 16.66 \left( \frac{\alpha_s}{\pi} \right)^2 - 153.41 \left( \frac{\alpha_s}{\pi} \right)^3 \]
\[ - 1953.40 \left( \frac{\alpha_s}{\pi} \right)^4 + \mathcal{O}(\alpha_s^5). \]

The comparison to Eq. (9) shows that up to four loops these predictions are rather good. The coefficients are all negative and grow very fast, which can be explained by the infrared renormalon at \( u = 1/2 \) [5]. This is the closest possible position of a renormalon singularity in the Borel plane \( u \) to the
III. EFFECT OF A LIGHTER-FLAVOR MASS

Now we suppose that \( n_m \) light flavors have a non-zero mass \( m \), while the remaining \( n_0 = n_l - n_m \) light flavors are massless. In practice, \( n_m = 1 \), e. g. c in b-quark HQET. In this case the massless result (7) for the matching coefficient should be multiplied by the additional factor

\[
\zeta' = \frac{Z_{Q}^{0}(g_0^{(n_f)}; \xi_0^{(n_f)}, m_0^{(n_f)})}{Z_{Q}^{0}(g_0^{(n_f)}; \xi_0^{(n_f)}, 0)} \times \frac{Z_{h}^{0}(g_0^{(n_l)}; \xi_0^{(n_l)}, 0)}{Z_{h}^{0}(g_0^{(n_l)}; \xi_0^{(n_l)}, m_0^{(n_l)})}
\]

where \( Z_{Q,h}^{0}(\ldots, 0) \equiv Z_{Q,h}^{0}(\ldots) \) in Eq. (2) and \( Z_{h}^{0}(g_0^{(n_l)}; \xi_0^{(n_l)}, 0) = 1 \). This factor does not depend on the renormalization scale \( \mu \). In the expression

\[
\log \zeta' = \log Z_{Q}^{0}(g_0^{(n_f)}; \xi_0^{(n_f)}, m_0^{(n_f)}) - \log Z_{Q}^{0}(g_0^{(n_f)}; \xi_0^{(n_f)}, 0) - \log Z_{h}^{0}(g_0^{(n_l)}; \xi_0^{(n_l)}, m_0^{(n_l)}) - \log Z_{h}^{0}(g_0^{(n_l)}; \xi_0^{(n_l)}, 0)
\]

we re-express all terms via \( \alpha_s^{(n_f)}(M) \), \( \xi^{(n_f)}(M) \) and the on-shell lighter-flavor mass \( m \) (it is the same in both \( n_f \) and \( n_l \) flavor theories). The result depends on the dimensionless ratio

\[
x = \frac{m}{M}.
\]

If we express \( z' \) via \( \alpha_s^{(n_f)}(\mu) \), \( \xi^{(n_f)}(\mu) \), the coefficients will depend on \( \mu \). This dependence is determined by the renormalization-group equation

\[
\frac{d \log \zeta'}{d \log \mu} = 0
\]

together with

\[
\frac{d \log \alpha_s^{(n_f)}(\mu)}{d \log \mu} = -2\varepsilon - 2\beta^{(n_f)}(\alpha_s^{(n_f)}(\mu)),
\]

\[
\frac{d \log(1 - \xi^{(n_f)}(\mu))}{d \log \mu} = -\alpha_s^{(n_f)}(\mu) \xi^{(n_f)}(\mu) - \alpha_s^{(n_f)}(\mu) \xi^{(n_f)}(\mu).
\]

Ultraviolet divergences cancel in each fraction in (11). On the other hand, the on-shell wave-function renormalization factors have extra infrared divergences at \( m = 0 \). However, \( z' \) in Eq. (11) has a smooth limit for \( x \to 0 \). In the following we illustrate the cancellation for infrared divergences at two-loop order. Similar mechanisms are also at work at higher loop orders. For dimensional reasons the two-loop corrections in Fig. 1 lead to \( \log Z_{h}^{0}(m) \sim g_0^{(n_l)} m^{-4\varepsilon} \). Furthermore, we have \( \log Z_{h}^{0}(0) = 0 \). Thus, the limit \( x \to 0 \) is discontinuous. In QCD (Fig. 1b) we have \( \log Z_{h}^{0}(0) \sim g_0^{4} M^{-2 \varepsilon} \) for dimensional reasons. For \( m \ll M \) there are 3 regions (see [28]):

- Hard (all momenta \( \sim M \)): a regular series in \( m^2 \), \( \log Z_{h}^{0}(m) \) has a smooth limit 1 at \( x \to 0 \).
- Soft-hard (momentum of one \( m \)-line \( \sim m \), all the remaining momenta \( \sim M \)). If we take the term \( m \) from the numerator \( k + m \) of the soft propagator, there is another factor \( m \) in the numerator of the hard mass-\( m \) propagator, and the soft-loop integral is \( \sim m^{2 - 2 \varepsilon} \); if we take \( k \) instead, we have to expand the hard subdiagram in \( k \) up to the linear term, and the soft loop is \( \sim m^{4 - 2 \varepsilon} \).
- Soft (all momenta \( \sim m \)): the leading term is the HQET one, the Taylor series is in \( x \) (not in \( x^2 \)), \( \log Z_{h}^{0}(m) \) has the same discontinuity as \( \log Z_{h}^{0}(0) \), hence smooth.

As a result, \( \log Z_{h}^{0}(m) \) has a smooth limit 1 at \( x \to 0 \).
\[ + C_F T_F \left( \frac{\alpha_s(n_f)}{\pi} \right)^3 \left( C_F A_F + C_A A_A + T_{F0n0}A_t + T_{FmAm} + T_{FnhAn} + O(\varepsilon) \right) + O(\alpha_s^4), \]  

(15)

where

\[ A_0 = \frac{1}{4} \left[ (1 - x)(2 - x - x^2 - 6x^3)H_{1,0}(x) - (1 + x)(2 + x - x^2 + 6x^3)H_{-1,0}(x) \right. \]

\[ - \left. \frac{3}{2} \pi^2 x + (4 \log x + 7)x^2 - \frac{5}{2} \pi^2 x^3 + (6 \log^2 x + \pi^2)x^4 \right]. \]  

(16)

The expansion of this function in \( x \) reads

\[ A_0 = \frac{1}{4} \left[ -\frac{3}{2} \pi^2 x + 12x^2 - \frac{5}{2} \pi^2 x^3 + \left( 6 \log^2 x - 11 \log x + \pi^2 + \frac{125}{12} \right) x^4 + \sum_{n=3}^{\infty} \left( 2g(2n) \log x + \frac{dg(2n)}{d n} \right) x^{2n} \right]. \]

\[ g(x) = \frac{2}{x} - \frac{3}{x-1} - \frac{5}{x-3} + \frac{6}{x-4}. \]  

(17)

Note that the only terms with odd powers of \( x \) are \( x^4 \) and \( x^3 \). The expansion in \( x^{-1} \) is given by

\[ A_0 = \frac{1}{4} \left[ -2 \log^2 x^{-1} - \frac{19}{3} \log x^{-1} - \frac{\pi^2}{3} - \frac{299}{36} + \sum_{n=1}^{\infty} \left( 2g(-2n) \log x^{-1} + \frac{dg(-2n)}{d n} \right) x^{-2n} \right]. \]  

(18)

For illustration we show in Fig. 3 \( A_0(x) \) for \( x \in [0, 1] \). The \( O(\varepsilon) \) term at two loops reads

\[ A_1 = \frac{1}{4} \left[ (1 - x)(2 - x - x^2 - 6x^3)(2H_{1,1,0}(x) - 4H_{1,-1,0}(x)) \right. \]

\[ + (1 + x)(2 + x - x^2 + 6x^3)(2H_{-1,1,0}(x) - 4H_{-1,-1,0}(x) - \pi^2 H_{-1}(x)) \]

\[ + (1 - x)(9 - 6x + 6x^2 - 17x^3)H_{1,0}(x) - (1 + x)(9 + 6x + 6x^2 + 17x^3)H_{-1,0}(x) \]

\[ + 4x(3 + 5x^2)(H_{0,1,0}(x) + H_{0,-1,0}(x)) + 6\pi^2 \left( L + 2a_1 - \frac{5}{4} \right) x + \left( L + 2\pi^2 + \frac{53}{2} \right) x^2 \]

\[ + 10\pi^2 \left( L + 2a_1 - \frac{23}{20} \right) x^3 - 12 \left( \frac{L^3 - \frac{17}{12} L^2 - 2}{L^3 - \frac{17}{12} L^2 - 2} - \frac{17}{12} \right) x^4 \]

\[ = \pi^2 \left( \frac{3}{2} L + 3a_1 - 1 \right) x^5 - \frac{5}{2} \pi^2 x^2 + \pi^2 \left( \frac{5}{2} L + 5a_1 - \frac{8}{3} \right) x^3 - \left( 3L^3 - \frac{17}{4} L^2 - \frac{3}{8} L - 3\pi^2 + \frac{2827}{288} \right) x^4 \]

\[ - \frac{63}{80} \pi^2 x^5 - \frac{2}{15} \left( \frac{61}{5} \right) L - 2\pi^2 + \frac{4243}{225} \right) x^6 - \frac{15}{112} \pi^2 x^7 \right] \]

\[ + \left( \frac{53}{35} L - 3 \pi^2 - \frac{5900}{1960} \right) x^8 + O(x^9), \]  

(19)

where \( L = \log x \).

At three-loop order the \( C_F T_F n_m n_0 \alpha_s^4 \) term is known exactly via harmonic polylogarithms of \( x \):

\[ A_t = \frac{1}{3} \left[ (1 - x)(2 - x - x^2 - 6x^3) \left( H_{1,-1,0}(x) + \frac{\pi^2}{12} H_{1}(x) \right) + (1 + x)(2 + x - x^2 + 6x^3) \left( H_{-1,1,0}(x) + \frac{5}{12} \pi^2 H_{-1}(x) \right) \right. \]

\[ - \frac{1}{6} (1 - x)(19 - 11x + x^2 - 39x^3)H_{1,0}(x) + \frac{1}{6} (1 + x)(19 + 11x + x^2 + 39x^3)H_{-1,0}(x) \]

\[ - x(3 + 5x^2)(H_{0,1,0}(x) + H_{0,-1,0}(x)) - \pi^2 \left( \frac{3}{2} - 3a_1 - \frac{5}{2} \right) x - \left( \frac{17}{2} L + 2\pi^2 + \frac{91}{3} \right) x^2 - \frac{5\pi^2}{3} \left( \frac{L}{2} + a_1 - \frac{2}{3} \right) x^3 \]

\[ + \left( 2L^3 - \frac{13}{12} \pi^2 L^2 - 9\pi^2 + \frac{13}{12} \pi^2 \right) x^4 \]

\[ = -\pi^2 \left( \frac{L^3}{2} + a_1 - \frac{7}{6} \right) x^5 - \frac{7}{3} \pi^2 \left( \frac{L}{2} + a_1 - \frac{7}{12} \right) x^6 + \left( \frac{2}{3} L^3 - \frac{13}{6} \pi^2 L^2 - \pi^2 \left( \frac{3}{4} \right) L - 3\pi^2 + \frac{3}{4} \right) x^7 \]

\[ + \frac{21}{40} \pi^2 x^5 + \frac{4}{45} \left( \frac{13}{5} L - 4 \pi^2 + \frac{2441}{225} \right) x^6 + \frac{5}{56} \pi^2 x^7 \right] + \left( \frac{4}{35} L - \pi^2 \left( \frac{4}{4} - \frac{40489}{29400} \right) x^8 + O(x^9), \right. \]  

(20)

where after the second equality sign we show the expansion in \( x \). In principle, it is straightforward to obtain exact results in \( x \) also the four-loop \( C_F T_F n_m n_0 \alpha_s^4 \) term. However, we refrain from presenting such results because the remaining four-loop color structures are not known.
The remaining three-loop terms can be obtained in a series expansion in \( x \) with the help of the result from \([31]\). Including terms up to order \( x^5 \) gives

\[
A_F = \frac{\pi^2}{3} \left( 8a_1 + \frac{13}{4} \pi - \frac{343}{24} \right) x - \left( L^2 - \frac{67}{6} \pi - \frac{17}{8} \pi^2 + \frac{229}{18} \right) x^2 + \frac{\pi^2}{3} \left( \frac{11}{3} L + \frac{44}{3} a_1 + \frac{35}{8} \pi - \frac{157}{8} \right) x^3 \\
+ \left[ \frac{19}{6} L^3 - \frac{911}{120} L^2 - \left( 3\pi^2 a_1 - \frac{3}{2} \zeta_3 - \frac{45}{16} \pi^2 - \frac{40567}{3600} \right) L + 20a_4 + \frac{5}{6} a_1^2 + \frac{2}{3} \pi^2 a_1^2 + \frac{11}{16} \pi^2 a_1 + \frac{387}{32} \zeta_3 \right. \\
- \frac{43}{144} \pi^4 - \frac{155}{72} \pi^2 - \frac{2534579}{216000} \right] x^4 + \frac{7}{5} \pi^2 \left( \frac{3}{32} \pi + \frac{1}{5} \right) x^5 \\
+ \left[ \frac{1579}{70} L^2 + \left( \frac{77}{16} \pi^2 - \frac{328067}{11025} \right) L - \frac{1}{16} \left( 77\pi^2 a_1 - \frac{539}{2} \zeta_3 - \frac{83}{15} \pi^2 - \frac{126231437}{1157625} \right) \right] \frac{x^6}{9} - \frac{\pi^2}{28} \left( \frac{25}{16} \pi + \frac{7}{4} \right) x^7 \\
+ \left[ \frac{2843}{105} L^2 + \left( \frac{21}{2} \pi^2 - \frac{718639}{33075} \right) L - \frac{1}{4} \left( 21\pi^2 a_1 - \frac{147}{2} \zeta_3 - \frac{4379}{240} \pi^2 + \frac{1213332979}{83349000} \right) \right] \frac{x^8}{32} + \mathcal{O}(x^9),
\]

\[
A_A = \frac{\pi^2}{8} \left( \frac{25}{2} L + \frac{313}{3} a_1 - \frac{13}{3} \pi - \frac{2473}{36} \right) x \\
+ \left[ \frac{L^2}{2} + \left( \frac{3}{2} \zeta_3 - \frac{31}{90} \pi + 7\pi^2 - \frac{7}{3} \right) L - 5\zeta_5 - \frac{7}{2} \pi^2 \zeta_3 + 79 \frac{\pi}{4} \zeta_3 - \frac{17}{180} \pi^4 + \frac{35}{18} \pi^2 + \frac{517}{9} \right] \frac{x^2}{4} \\
+ \frac{\pi^2}{24} \left[ \frac{269}{6} L + \frac{1291}{3} a_1 - \frac{35}{2} \pi - \frac{865}{3} \right] x^3 - \frac{83}{48} L^3 + \left( 3\pi^2 - \frac{3977}{60} \right) \frac{L^2}{8} - \frac{\pi^2}{2} a_1 - 3\zeta_3 - \frac{13}{24} \pi^2 - \frac{230293}{28800} L \\
+ 10a_4 + \frac{5}{12} a_1^2 + \frac{\pi^2}{3} a_1^2 + \frac{11}{32} \pi^2 a_1^2 + \frac{111}{64} \zeta_3 - \frac{161}{1440} \pi^4 - \frac{631}{1152} \pi^2 - \frac{452033}{864000} \right] \frac{x^4}{24} \\
+ \left[ \frac{5}{3} L^3 + \frac{9911}{840} L^2 - \frac{\pi^2}{2} + \frac{8394157}{5292000} \right] + \frac{1}{12} \left( 77\pi^2 a_1 - \frac{509}{2} \zeta_3 - \frac{3607}{60} \pi^2 + \frac{8471770063}{18522000} \right] \frac{x^5}{24} \\
+ \left[ \frac{\pi^2}{257} \left( 57 \pi^2 + \frac{25}{32} \pi - \frac{11549}{14000} \right) L^2 + \frac{43}{27} L^3 + \frac{209}{20} L^2 + \left( \frac{125}{3} - \frac{12327647}{14700} \right) L \\
+ \frac{1}{8} \left( 21\pi^2 a_1 - \frac{1435}{18} \zeta_3 - \frac{1213519}{45360} \pi^2 + \frac{103012907}{2058000} \right) \frac{x^8}{32} + \mathcal{O}(x^9),
\]

\[
A_h = -\left( \frac{2L + \frac{13}{5}}{5} \right) \frac{x^2}{5} + \frac{2}{15} \pi^2 x^3 + \left[ \frac{3}{70} L^2 + \left( \frac{\pi^2}{2} - \frac{35887}{4900} \right) \frac{L}{3} - \frac{1}{36} \left( 13\pi^2 - \frac{59985349}{514500} \right) \right] \frac{x^4}{4} \\
- \left( \frac{244L^2}{135} + \frac{92779}{793800} \right) \frac{x^6}{945} - \left( \frac{47L^2}{13860} + \frac{925823}{384199200} \right) \frac{x^8}{770} + \mathcal{O}(x^9),
\]

\[
A_m = -\pi^2 \left( \frac{L^2}{2} - \frac{2}{25} \pi^2 - \frac{7}{3} \zeta_3 - \frac{5}{6} \pi^2 L^3 + \pi^2 \right) - \frac{1}{36} \left( 13\pi^2 - \frac{15}{4} \right) L + \frac{1}{4} \left( \pi^2 + \frac{203}{108} \right) \frac{x^4}{4} \\
- \left( \frac{308}{5} L + \frac{16}{3} \pi^2 - \frac{13159}{225} \right) \frac{x^6}{45} + \left( 3L^2 - \frac{751}{70} L + \frac{2095}{336} \right) \frac{x^8}{14} + \mathcal{O}(x^9).
\]

Starting from three loops the individual terms in Eq. (12) are gauge parameter dependent. However, \( \xi \) cancels in the three-loop expression for \( z' \). It might be that \( z' \) is gauge invariant to all orders, but we have no proof of this conjecture.

IV. THE QED AND BLOCH–NORDSIIECK HEAVY-LEPTON FIELDS

In QED the matching coefficient \( z(\mu) \) is gauge invariant to all orders in \( \alpha \). The proof given in this paper is literally valid only for \( n_f = 1 \) lepton flavor, but can be easily generalized for any \( n_f \), as we demonstrate in the following.

The QED on-shell renormalization constant \( Z_\psi \) is gauge invariant to all orders \([10] [35] [36]\). Gauge dependence of the MS \( Z_\psi \) can be found using the so-called LKF transformation \([37] [38]\) for arbitrary \( n_f \). In the gauge where the free photon propagator is

\[
D_{\mu \nu}^0(k) = \frac{1}{k^2} (g_{\mu \nu} - k_\mu k_\nu) + \Delta(k) k_\mu k_\nu,
\]

the full bare lepton propagator reads

\[
S(x) = S_L(x) e^{\frac{i}{\Delta}(\lambda(x) - \hat{\Delta}(0))},
\]

\[
\hat{\Delta}(x) = \int \frac{d^4k}{(2\pi)^4} \Delta(k) e^{-ikx},
\]

where \( S_L(x) \) is the Landau-gauge propagator. In the covariant...
gauge \( \Delta(k) = (1 - \xi_0)/(k^2)^2 \), and \( \tilde{\Delta}(0) = 0 \) in dimensional regularization. The lepton fields renormalization does not depend on their masses, so, let us assume that all \( n_f \) flavors are massless. The propagator has a single Dirac structure

\[
S(x) = S_0(x)e^{\sigma(x)},
\]

where \( S_0(x) \) is the \( d \)-dimensional free propagator. Then

\[
\sigma(x) = \sigma_L(x) + (1 - \xi_0) \frac{\epsilon_0^2}{(4\pi)^{d/2}} \left( -\frac{x^2}{4} \right)^\varepsilon \Gamma(-\varepsilon);
\]

re-expressing this result via the renormalized quantities, we obtain

\[
\log Z_\psi(\alpha, \xi) = \log Z_L(\alpha) - (1 - \xi) \frac{\alpha}{4\pi^\varepsilon} - 2\varepsilon \]

exactly, and the anomalous dimension

\[
\gamma_\psi(\alpha, \xi) = \gamma_L(\alpha) + 2(1 - \xi) \frac{\alpha}{4\pi^\varepsilon}.
\]

In QED \( Z_A Z_\alpha = 1 \) due to Ward identities, hence

\[
\frac{d \log((1 - \xi(\mu))\alpha(\mu))}{d \log \mu} = -2\varepsilon
\]

contains \( \xi \) only in the one-loop term.

In the Bloch-Nordsieck EFT with \( n_f \) light lepton flavors \( Z_{\mu}^{\text{rev}} \) is gauge-invariant (even if some of these flavors have non-zero masses). Gauge dependence of the \( \overline{\text{MS}} \) \( Z_h \) can be found using exponentiation. The full bare propagator is

\[
S_h(t) = S_{h0}(t) \exp \left( \sum_i w_i \right),
\]

where \( w_i \) are webs \(^{[39, 40]} \). In QED all webs have even numbers of photon legs; all webs with \( \geq 2 \) legs are gauge invariant; all 2-leg webs except the trivial one (the free photon propagator) are gauge invariant, too. Therefore,

\[
\log \frac{S_h(t)}{S_{hL}(t)} = (1 - \xi_0) \frac{\epsilon_0^2}{(4\pi)^{d/2}} \left( \frac{st}{2} \right)^{2\varepsilon} \Gamma(-\varepsilon);
\]

re-expressing this result via the renormalized quantities, we obtain

\[
\log Z_h(\alpha, \xi) = \log Z_{hL}(\alpha) - (1 - \xi) \frac{\alpha}{4\pi^\varepsilon}, \quad (25)
\]

\[
\gamma_h(\alpha, \xi) = \gamma_{hL}(\alpha) + 2(1 - \xi) \frac{\alpha}{4\pi^\varepsilon}. \quad (26)
\]

Finally, in the abelian case \( \zeta_\alpha(\mu) = \zeta_A(\mu)^{-1} \) due to Ward identities, hence \((1 - \xi(\mu))\alpha(\mu)) = (1 - \xi(\mu))\alpha(\mu))\), and we arrive at the conclusion that \( z(\mu) \) is gauge invariant (some light flavors may be massive, this does not matter).

Let us in the following specify \( z(M) \) from Eq (7) to QED. Setting \( C_F = T_F = d_{FF} = 1 \) and \( C_A = d_{FA} = 0 \) we see that our four-loop result is indeed gauge invariant and is given by

\[
z(M) = 1 - \frac{\alpha}{\pi} \left[ 1 + \epsilon \left( \frac{\pi^2}{16} + 2 - \epsilon^2 \left( \frac{\zeta_3}{4} - \frac{\pi^2}{12} - 4 \right) - \epsilon^3 \left( \frac{\zeta_3}{3} - \frac{3}{640} \pi^4 - \frac{\pi^2}{6} - 8 \right) + \mathcal{O}(\epsilon^4) \right) \right.
\]

\[
\left. + \frac{\alpha}{\pi} \left\{ \frac{\pi^2 a_1}{3} - \frac{3}{2} \zeta_3 - \frac{55}{48} \pi^2 + \frac{5957}{1152} + \frac{n_l}{12} \left( \pi^2 + \frac{113}{8} \right) \right. 
\right]
\]

\[
\left. \left. + \epsilon \left[ -24a_4 - a_4^2 - 2\pi^2 a_1 + \frac{31}{4} \pi^2 a_1 - \frac{203}{8} \zeta_3 + \frac{7}{20} \pi^4 - \frac{4903}{1152} \pi^2 + \frac{56845}{6912} + n_l \left( \frac{305}{3} \zeta_3 + \frac{199}{80} \pi^4 + \frac{853}{24} \pi^2 + \frac{5753}{16} \right) \right] + \mathcal{O}(\epsilon^3) \right\} \right]
\]

\[
\left. + \frac{\alpha}{\pi} \left\{ -16a_4 + \frac{2}{3} a_4^2 + \frac{737}{36} \pi^2 a_1 - \frac{5}{16} \zeta_3 + \frac{\pi^2}{8} \zeta_3 - \frac{4747}{288} \zeta_3 - \frac{13}{360} \pi^4 - \frac{259133}{25920} \pi^2 - \frac{230447}{20736} 
\right.
\]

\[
\left. + \frac{n_l}{18} \left( \frac{16a_4 + \frac{2}{3} a_4^2 + \frac{4}{3} \pi^2 a_1 + \frac{47}{6} \pi^2 a_1 + \frac{137}{8} \zeta_3 + \frac{229}{720} \pi^4 + \frac{139}{24} \pi^2 + \frac{2201}{432} \right) - \frac{n_l^2}{18} \left( 7\zeta_3 + \frac{19}{6} \pi^2 + \frac{5767}{432} \right) 
\right]
\]

\[
\left. + \epsilon \left[ -\frac{224}{3} a_4 + \frac{5005}{6} a_4 + \frac{28}{45} a_4 + \frac{2}{3} \pi^2 a_1 + \frac{5005}{144} a_4^2 + \frac{2}{3} \pi^2 a_1 + \frac{2}{3} \pi^2 a_1 + \frac{5}{144} a_4^2 + \frac{2}{3} \pi^2 a_1 + \frac{11567}{144} \pi^2 a_1 + \frac{14\pi^2 a_1}{3} \right] \right].
\]
MS QED coupling with one active α flavor at \( \mu \mu \) for lattice only if \( M a \) is the ε⁰ term in \( Z_2^{(4)} \) of Eq. (26) in [14]. Its numerical value is given in Eq. (15) in this paper.

Numerically, in pure QED \( (n_l = 0) \) at \( \varepsilon = 0 \) we have

\[
z(M) = 1 - \alpha - 1.09991 \left( \frac{\alpha}{\pi} \right)^2 + 4.40502 \left( \frac{\alpha}{\pi} \right)^3 \nonumber - 2.16215 \left( \frac{\alpha}{\pi} \right)^4 + O(\alpha^5),
\]

(28)

where \( \alpha = \alpha^{(1)}(M) \), the \( \overline{\text{MS}} \) QED coupling with one active flavor at \( \mu = M \), the on-shell electron mass. In contrast to the QCD case [7] the coefficients are numerically smaller and have different signs.

V. CONCLUSION

We have calculated the (finite) matching coefficient between the QCD heavy-quark field \( Q \) and the corresponding HQET field \( h_v \) up to four loops. Explicit results are presented for \( \mu = M \); results for different values of \( \mu \) can be obtained with the help of (known) renormalization group equations. The effect of a non-zero light-flavor mass (e.g., \( c \) in \( b \)-quark HQET) is calculated up to three loops. We also present results for the matching constant in QED.

As a possible application of our results we want to mention the possibility to obtain the QCD heavy-quark propagator (say, in Landau gauge) from lattice QCD results for the HQET propagator. A heavy-quark field can be put onto the lattice only if \( M a \ll 1 \), where \( a \) is the lattice spacing. On the other hand, in HQET simulations there is no lattice \( h_v \) field at all. The HQET propagator is just a straight Wilson line, i.e., a product of lattice gauge links. It is therefore much easier to obtain the HQET propagator from lattice simulations.

After taking the continuum limit, one can get the continuum coordinate-space HQET propagator. Then the QCD heavy-quark propagator can be obtained with the help of the matching coefficient \( z(\mu) \), provided that \( 1/M^\alpha \) corrections can be neglected. Note that this can be done for arbitrarily heavy QCD quark, including the case when the use of the dynamic heavy-quark field on the lattice is impossible.

ACKNOWLEDGMENTS

We are grateful to R.N. Lee for discussions of the Appendix [9]. This research was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant 396021762 — TRR 257 “Particle Physics Phenomenology after the Higgs Discovery”. This work was supported in part by the EU TMR network SAGEX Marie Skłodowska-Curie grant agreement No. 764850 and COST action CA16201: Unraveling new physics at the LHC through the precision frontier. The work of A. G. was supported by the Russian Ministry of Science and Higher Education.

Appendix A: The coupling and gluon-field decoupling coefficients

The \( n_f \)-flavor QCD strong coupling constant and gauge parameter are related to the corresponding quantities in the \( n_f \)-flavor theory by the decoupling relations

\[
\alpha_s^{(n_f)}(\mu) = \zeta_\alpha(\mu) \alpha_s^{(n_f)}(\mu),
\]

\[
1 - \xi^{(n_f)}(\mu) = \zeta_A(\mu) \left[ 1 - \xi^{(n_f)}(\mu) \right].
\]

(A1)
The decoupling coefficients satisfy the renormalization group equations

\[
\frac{d \log \zeta_n(\mu)}{d \log \mu} = 2 \left[ \beta^{(n)}(\alpha_s^{(n)}(\mu)) - \beta^{(n)}(\alpha_s^{(n)}(\mu)) \right],
\]
\[
\frac{d \log \zeta_A(\mu)}{d \log \mu} = \gamma_A^{(n)}(\alpha_s^{(n)}(\mu), \xi^{(n)}(\mu)) - \gamma_A^{(n)}(\alpha_s^{(n)}(\mu), \xi^{(n)}(\mu)).
\]

(A2)

It is sufficient to have initial conditions, say, at \( \mu = M \) for solving these equations. For the computation of \( z(M) \) we need the decoupling coefficients up to \( \alpha_s^3 \varepsilon \). Up to the order \( \alpha_s^2 \) expression exact in \( \varepsilon \) can be found in [41]. The finite three-loop results have been obtained in [42] in term of \( N_c \) and in [43] for an arbitrary color group. The \( \alpha_s^2 \varepsilon \) terms were derived in the course of four-loop calculations [43,44]. However, results for an arbitrary color group, including positive powers of \( \varepsilon \), are not explicitly presented in these publications. Therefore, we present them here:
where \( \alpha_s = \alpha_s^{(n_f)}(M) \).

**Appendix B: On-shell diagrams with two masses**

Light-quark mass effects in the heavy-quark on-shell propagator diagrams arise for the first time at two loops, see Fig. 1b. The corresponding integral family can be defined as

\[
I_{n_1 n_2 n_3 n_4} = \frac{C}{(i\pi^{d/2})^2} \int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}} \ ,
\]

\[
C = \frac{1}{\Gamma^2(1+\varepsilon)} , \\
D_1 = M^2 - (p + k_1)^2 , \quad D_2 = -k_1^2 , \\
D_3 = m^2 - k_2^2 , \quad D_4 = m^2 - (k_1 - k_2)^2 ,
\]

with \( p^2 = M^2 \). If there are insertions to gluon lines in Fig. 1b containing only massless lines, such diagrams are expressed via the integrals \((B1)\) with \( n_2 = n + \varepsilon \), where \( I \) is the total number of loops in these insertions and \( n \) is integer \((n_{1,3,4} \text{ are always integer})\). These integrals have been studied in \([30]\). The IBP algorithm obtained there reduces them to four master integrals

\[
I_{0,1,\varepsilon,1,1} = C \quad I_{1,1,\varepsilon,1,1} = C \quad I_{1,\varepsilon,1,0} = C \quad I_{1,\varepsilon,1,1} = C
\]

\[
I_{1,1,\varepsilon,1,1} = C \quad \frac{l_\varepsilon}{\varepsilon}
\]

We set \( M = 1 \) and \( m = x \).

It is more convenient to use the column vector

\[
j = (I_{0,\varepsilon,2,2,2}, I_{2,\varepsilon,2,0,2}, I_{2,\varepsilon,2,1,1,2,1,2})^T
\]

as master integrals instead of \((B2)\). Differentiating them in \( m \) and reducing the results back to \( j \) \([46]\), we obtain the differential equations

\[
\frac{dj}{dx} = M(\varepsilon, x) j ,
\]

\[
j = T(\varepsilon, x) J , \quad \frac{dJ}{dx} = \varepsilon M(x) J
\]

This makes their iterative solution to any order in \( \varepsilon \) almost trivial.

Differential equations for on-shell sunsets \( I_{n_{1,0,n_3,n_4}} \) were considered in \([33, 48]\), but they were not in \( \varepsilon \)-form. Several terms of small-\( x \) expansions were obtained from differential equations in \([49]\): it is easier to obtain them by calculating the corresponding residues in the Mellin–Barnes representation \([38]\).

We use the Mathematica package Libra \([50]\) which implements the algorithm of \([51]\) to reduce the master integrals \( j \) in Eq. (B3) to a canonical basis \( J \):

\[
j_1 = I_{0,\varepsilon,2,2} = CV_{2,2,2,2,2,2} x^{-2(l+2)\varepsilon} = \frac{2(1 - (l + 1)\varepsilon)}{(l + 2)(1 - \varepsilon)} J_1
\]
\[ j_2 = I_{0,1,2,0} = CV_2 M_{2,1} x^{-2\varepsilon} = \frac{1 - 2(l + 1)\varepsilon}{1 - (l + 2)\varepsilon} j_2, \]
\[ \frac{1}{2} (j_3 + J_4), \]
\[ j_4 = I_{1,1,2,2} = \frac{1}{2x} \left( \frac{1}{1 - 2x - 2l(1 - x)} \right) J_3 + \left[ 1 + 2x - \frac{2l\varepsilon(1 + x)}{1 - 2\varepsilon} \right] J_4 \right \}, \quad \text{(B6)} \]

where

\[ V_{n_1} = \frac{1}{\Gamma(\frac{d}{2} - n_1)} \Gamma(n_1) \]
\[ V_{n_1 n_2 n_3} = \frac{2}{\Gamma(n_1 + n_2 + n_3)} \Gamma(d - n_1 - 2n_2) \]
\[ M_{n_1 n_2} = \frac{\Gamma(n_1 + n_2 - \frac{d}{2}) \Gamma(d - n_1 - 2n_2)}{\Gamma(n_1) \Gamma(d - n_1 - n_2)} \], \quad \text{(B9)}

and \[ H_{n_1 n_2 n_3 n_4} = \frac{2}{\Gamma(n_1 + n_2 + n_3 + n_4 - d)} \Gamma(d - n_1 - 2n_2) \]
\[ \frac{\Gamma(n_1/2) \Gamma((n_1 - d)/2 + n_2 + n_3) \Gamma((n_1 - d)/2 + n_2 + n_4)}{2 \Gamma(n_1) \Gamma(n_3) \Gamma(n_4)} \]
\[ \times \frac{\Gamma(n_1/2 + n_2 + n_3 + n_4 - d - \varepsilon) \Gamma((d - n_1)/2 - n_2)}{\Gamma(n_1 + 2n_2 + n_3 + n_4 - d) \Gamma((d - n_1)/2)} \] \quad \text{(B10)}

The integrals \( J \) satisfy the \( \varepsilon \)-form differential equations

\[ \frac{dJ}{dx} = \varepsilon \left( \frac{M_0}{x} + \frac{M_{1+}}{1 - x} + \frac{M_{-1}}{1 + x} \right) J, \]
\[ \text{(B11)} \]

where

\[ M_0 = \begin{pmatrix} -2(l + 2) & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & -1 & -(l + 2) & l + 2 \\ 1 & -1 & l + 2 & -(l + 2) \end{pmatrix}, \]
\[ M_{1+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2(l + 2) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ M_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -2(l + 2) & -2 \end{pmatrix}. \]
\[ \text{(B12)} \]

The first two are, of course, known exactly:

\[ J_1 = x^{-2(l + 2)\varepsilon} \left( \frac{1 - 2(l + 1)\varepsilon}{1 - (l + 2)\varepsilon} \right)^2 \]
\[ J_2 = \frac{\Gamma(1 - (l + 1)\varepsilon) \Gamma(1 - (l + 2)\varepsilon) \Gamma(1 + (l + 1)\varepsilon)}{\Gamma(1 - (l + 1)\varepsilon) \Gamma(1 - (l + 2)\varepsilon) \Gamma(1 + (l + 1)\varepsilon)} \]
\[ \times \frac{x^{-2\varepsilon} \Gamma(1 - (l + 1)\varepsilon) \Gamma(1 + (l + 1)\varepsilon)}{(1 + (l + 1)\varepsilon)^2} \] \quad \text{(B13)}

The equations for \( j_3, j_4 \) can be solved iteratively in terms of harmonic polylogarithms \([52]\) of \( x \). However, we need initial conditions. They can be fixed using the asymptotics of \( I_{n_1 n_2 n_3 n_4} \) at \( x \to 0 \). It is given by contributions of three regions (Sect. \( \text{III} \)) corresponding to residues of the Mellin–Barnes representation \([50]\) at three series of poles:

- Hard: the poles \( s = -(n_3 - n_4 + d/2) \) (\( n \geq 0 \) is integer), the result is a regular series in \( x^2 \). The leading term is \( CG_{n_1 n_2} M_{n_1 n_2 n_3 + n_4 - d/2} \), where

\[ G_{n_1 n_2} = \frac{\Gamma(n_1 + n_2 - \frac{d}{2}) \Gamma\left(\frac{d}{2} - n_1\right) \Gamma\left(\frac{d}{2} - n_2\right)}{\Gamma(n_1) \Gamma(n_2) \Gamma(d - n_1 - n_2)} \]
\[ \text{(B14)} \]

- Sort-hard: \( s = -n - n_3, n_4 \). All these poles are double except the first \( n_3 - n_4 \) ones (and hence, the representation of \( I_{n_1 n_2 n_3 n_4} \) via hypergeometric functions of \( x \) is awkward). We assume \( n_3 \geq n_4 \), then the result is \( x^{(d-2)n_3} \) times a regular series in \( x^2 \). If \( n_3 > n_4 \) then the leading term is \( CV_{n_3} M_{n_1 n_2 n_3 + n_4 - d/2} \); if \( n_3 = n_4 \) there is an extra factor 2 because each of the lines 3, 4 can be soft.

- Soft: \( s = -(n_1 - d - n)/2 + n_2 \), the result is \( x^{2(d-n_3-n_4)-n_1} \) times a regular series in \( x \). The leading term is \( H_{n_1 n_2 n_3 n_4} x^{2(d-n_3-n_4)-n_1} \).

For example,

\[ I_{2,1,2,1} \to \frac{1}{2\varepsilon^2} \times \]
\[ \left[ \frac{1}{l + 1} \Gamma(1 - \varepsilon) \Gamma(1 - (l + 2)\varepsilon) \Gamma(1 + (l + 1)\varepsilon) \right] \]
\[ \times \left[ \frac{1}{l + 1} \Gamma(1 - 2(l + 1)\varepsilon) \Gamma(1 + (l + 1)\varepsilon) \right] \]
\[ \times \frac{x^{-2\varepsilon} \Gamma(1 - (l + 1)\varepsilon) \Gamma(1 + (l + 1)\varepsilon)}{(1 + (l + 1)\varepsilon)^2} \]
\[ \text{(B15)} \]

where the 3 contributions are the hard one \( CG_{21} M_{2,1} x^{(l+1)\varepsilon} \) (the pole \( s = -1 - \varepsilon \)), the soft-3 one \( CV_{21} M_{2,1} x^{(l+1)\varepsilon} \) (the pole \( s = -1 \)), and the soft one (the formula \( \text{B10} \)) or \( s = -1 - (l + 1)\varepsilon \). The leading asymptotics of \( I_{1,1,2} \) is given by the soft contribution (the formula \( \text{B10} \)) or the pole \( s = -3/2 + (l + 1)\varepsilon \):

\[ I_{1,1,2,2} \to 2^{-1-4(l+2)\varepsilon} n^2 x^{1-2(l+2)\varepsilon} \frac{1-2(l+1)\varepsilon}{1-2\varepsilon} \]
\[ J_3 = -2 \left\{ H_{1,0}(x) + H_{0,0}(x) + \frac{\pi^2}{3} + \left[ -(l + 2)(2H_{1,-1,0}(x) + H_{0,1,0}(x) + H_{0,-1,0}(x)) + 2H_{1,1,0}(x) \right. \right.
\[ \left. \quad - 2(l + 3)H_{0,0,0}(x) - l\frac{\pi^2}{6}H_1(x) - (l + 3)\frac{\pi^2}{3}H_0(x) + \frac{1}{2}(3l + 2)\epsilon_3 - (l + 2)\pi^2a_1 \right] \epsilon \right. \]
\[ + 2 \left[ (l + 2)^2(-2H_{1,-1,1,0}(x) + H_{0,1,-1,0}(x) - H_{0,-1,1,0}(x) + H_{0,0,1,0}(x) + H_{0,0,-1,0}(x)) \right. \]
\[ + (l + 1)(l + 2)(H_{1,0,1,0}(x) + H_{1,-1,0,0}(x)) \]
\[ + (l + 2)(-2H_{1,1,-1,0}(x) + 2H_{1,-1,1,0}(x) - H_{0,1,1,0}(x) + H_{0,1,-1,0}(x) - 2H_{1,1,0,0}(x)) \]
\[ + 2(l^2 + 5l + 7)H_{0,0,0,0}(x) + \frac{\pi^2}{12} - 2lH_{1,1}(x) - (l + 2)(5l + 6)(2H_{1,-1,1}(x) + H_{0,1,0}(x)) \]
\[ + (8l^2 + 23l + 12)H_{1,0}(x) + l(l + 2)H_{0,1}(x) + (6l^2 + 23l + 24)H_{0,0}(x) + \frac{1}{2}(l + 3)(3l + 2)\epsilon_3H_1(x) \]
\[ + (l + 2)\pi^2a_1 \left[ (l + 1)H_1(x) + (l + 2)H_0(x) \right] + \left( l + 2 \right)^2\pi^2a_1 \left( 84l^2 + 227l + 122 \right) \frac{\pi^4}{720} \epsilon^2 \right\} + O(\epsilon^3), \]
\[ J_4 = -2 \left\{ -H_{1,0}(x) + H_{0,0}(x) - \frac{\pi^2}{6} + \left[ -(l + 2)(2H_{1,-1,1,0}(x) - H_{0,-1,0}(x) - H_{0,1,0}(x)) + 2H_{1,-1,0}(x) \right. \right.
\[ \left. \quad - 2(l + 3)H_{0,0,0}(x) - (5l + 6)\frac{\pi^2}{6}H_1(x) + (2l + 3)\frac{\pi^2}{3}H_0(x) + \frac{1}{2}(3l + 2)\epsilon_3 + (l + 2)\pi^2a_1 \right] \epsilon \right. \]
\[ + 2 \left[ (l + 2)^2(2H_{1,-1,1,0}(x) + H_{0,1,1,0}(x) - H_{0,-1,1,0}(x) - H_{0,0,1,0}(x)) \right. \]
\[ + (l + 1)(l + 2)(H_{1,0,1,0}(x) + H_{1,-1,0,0}(x)) \]
\[ + (l + 2)(2H_{1,-1,1,0}(x) - 2H_{1,-1,1,0}(x) - H_{0,1,1,0}(x) + H_{0,1,1,0}(x) + 2H_{1,0,0,0}(x)) \]
\[ - 2H_{1,-1,1,0}(x) \]
\[ + 2(l^2 + 5l + 7)H_{0,0,0,0}(x) + \frac{\pi^2}{12}(5l + 6)H_{1,-1,1}(x) + l(l + 2)(2H_{1,-1,1}(x) - H_{0,1}(x)) \]
\[ + (4l^2 + 13l + 12)H_{1,0}(x) + (l + 2)(5l + 6)H_{0,1}(x) - (2l + 3)(3l + 8)H_{0,0}(x) \]
\[ + \frac{1}{2}(l + 3)(3l + 2)\epsilon_3H_{1,0}(x) \]
\[ + (l + 2)\pi^2a_1 \left[ (l + 1)H_{1,-1}(x) - (l + 2)H_0(x) \right] - (l + 2)^2\pi^2a_1 - (36l^2 + 103l + 58) \frac{\pi^4}{720} \epsilon^2 \right\} + O(\epsilon^3). \]
\[ J_3 = -2 \left\{ -H_{1,0}(x^{-1}) + \left[ -(l + 4)H_{0,1,0}(x^{-1}) - 2(l + 3)H_{1,0,0}(x^{-1}) + (l + 2)(2H_{1,-1,0}(x^{-1}) + H_{0,-1,0}(x^{-1})) \right] \varepsilon \\
+ 2 \left[ (l + 2)^2(2H_{1,-1,0}(x^{-1}) + H_{0,-1,0}(x^{-1})) + (l + 2)(l + 5)H_{1,0,-1,0}(x^{-1}) \right] \varepsilon \right. \\
+ (l + 2)(l + 4)(H_{0,1,-1,0}(x^{-1}) + H_{0,0,1,0}(x^{-1})) + (l + 2)(l + 3)(2H_{1,-1,0,0}(x^{-1}) + H_{0,-1,0,0}(x^{-1})) \\
- (l + 3)(l + 4)H_{0,1,0,0}(x^{-1}) - (l^2 + 6l + 10)H_{0,0,1,0}(x^{-1}) - (l^2 + 5l + 8)H_{1,0,1,0}(x^{-1}) \\
- 2(l^2 + 5l + 7)H_{1,0,0,0}(x^{-1}) - (l + 4)H_{0,1,1,0}(x^{-1}) - 2(l + 3)H_{1,1,0,0}(x^{-1}) \\
+ (l + 2)\left(2H_{1,1,-1,0}(x^{-1}) - 2H_{1,-1,1,0}(x^{-1}) - H_{0,-1,1,0}(x^{-1}) \right) - 2H_{1,1,1,0}(x^{-1}) \\
+ \frac{\pi^2}{12} \left[ (l + 4)H_{0,1}(x^{-1}) - (l + 2)H_{1,-1}(x^{-1}) + H_{0,1}(x^{-1}) + 2H_{1,1}(x^{-1}) + H_{1,0}(x^{-1}) \right] \varepsilon^2 \right\} + O(\varepsilon^3), \]

\[ J_4 = -2 \left\{ H_{1,0}(x^{-1}) + \left[ (l + 4)H_{0,1,0}(x^{-1}) + 2(l + 3)H_{-1,1,0}(x^{-1}) + (l + 2)(2H_{-1,1,0}(x^{-1}) - H_{0,1,0}(x^{-1})) \right] \varepsilon \right. \\
- 2H_{0,-1,1,0}(x^{-1}) - \frac{\pi^2}{6} H_{-1,0}(x^{-1}) \varepsilon \right. \\
+ 2 \left[ (l + 2)^2(-2H_{1,-1,1,0}(x^{-1}) + H_{0,1,-1,0}(x^{-1})) + (l + 2)(l + 5)H_{-1,0,1,0}(x^{-1}) \right] \varepsilon \right. \\
+ (l + 2)(l + 4)(H_{0,1,-1,0}(x^{-1}) - H_{0,0,1,0}(x^{-1})) + (l + 2)(l + 3)(2H_{-1,0,1,0}(x^{-1}) - H_{0,0,1,0}(x^{-1})) \\
+ (l + 3)(l + 4)H_{0,1,0,0}(x^{-1}) + (l^2 + 6l + 10)H_{0,0,1,0}(x^{-1}) - (l^2 + 5l + 8)H_{-1,0,1,0}(x^{-1}) \\
+ 2(l^2 + 5l + 7)H_{1,0,0,0}(x^{-1}) - (l + 4)H_{0,1,1,0}(x^{-1}) - 2(l + 3)H_{1,1,0,0}(x^{-1}) \\
- (l + 2)\left(2H_{1,1,-1,0}(x^{-1}) - 2H_{-1,1,0}(x^{-1}) + H_{0,1,0}(x^{-1}) + 2H_{1,-1,0}(x^{-1}) - H_{0,0}(x^{-1}) \right) \varepsilon \right. \\
+ \frac{\pi^2}{12} \left[ - (l + 4)H_{0,1}(x^{-1}) - (l + 2)H_{1,1}(x^{-1}) - H_{0,1}(x^{-1}) + 2H_{1,1}(x^{-1}) + H_{1,0}(x^{-1}) \right] \varepsilon^2 \right\} + O(\varepsilon^3). \]

(B19)

This is, of course, the analytical continuation of (B17) to \( x > 1 \). The same results (B19) can be obtained if we express \( J_{3,4} \) via \( I_{2,\varepsilon,2,1} \) and \( J_{1,\varepsilon,2,2} \) using (B6) and expand the hypergeometric representations (see Eq. (A1) in [30]) of these two integrals in \( \varepsilon \) using the Mathematica package HypExp [55, 56]. However, solving the differential equations (B18) up to higher orders in \( \varepsilon \) is simpler than expanding hypergeometric functions.

Both (B17) and (B19) lead to identical results at \( x = 1 \):

\[ J_3(1) = -\frac{\pi^2}{3} + \frac{1}{2} (l + 2)(2\pi^2 a_1 - 7\zeta_3) \varepsilon \\
- \left[ (l + 2)(l + 3) \left( 8a_4 + \frac{1}{3} a_1^4 + \frac{2}{3} \pi^2 a_1^2 \right) \\
+ (17l^2 - 36l - 124) \frac{\pi^4}{360} \right] \varepsilon^2 + O(\varepsilon^3), \]

\[ J_4(1) = \frac{\pi^2}{6} - \frac{1}{2} (2\pi^2 a_1 - 7\zeta_3) \varepsilon \\
+ \left[ (l + 3) \left( 8a_4 + \frac{1}{3} a_1^4 + \frac{2}{3} \pi^2 a_1^2 \right) \\
+ (24l^2 + 27l - 62) \frac{\pi^4}{360} \right] \varepsilon^2 + O(\varepsilon^3). \]

(B20)

If \( l = 0 \) and \( x = 1 \), we obviously have \( I_{1022}(1) = I_{2021}(1) \), and hence

\[ J_3(1) = -2J_4(1) = -4I_{2021}(1). \]

(B21)

Expanding the hypergeometric representation (30) of \( I_{2021}(1) \) (or \( I_{1022}(1) \)) at \( x = 1 \) in \( \varepsilon \) we get (B22) with \( l = 0 \). Alternatively, we can use another hypergeometric representation in [8, 9]. Using integration by parts we obtain

\[ I_{2021}(1) = \frac{7}{32\varepsilon^2} \left[ \frac{\Gamma(1 - \varepsilon)\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{\Gamma^2(1 + 4\varepsilon)} - 1 \right] \\
+ \frac{2^{-2-6\varepsilon} \pi^2}{3} \frac{\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{\Gamma^2(1 + 2\varepsilon)} + \frac{3}{4} \varepsilon^2 B_4(\varepsilon), \]

(B22)

where \( B_4(\varepsilon) \) is given by the formulas (41), (43) in [9]. This leads to the same result.

The functions \( L_+(x) = -\frac{1}{2}J_{3,4}(l = 0, \varepsilon = 0) \) were used in [7, 8, 22, 30]. In addition to the two expressions for these functions in (B17) and (B19), several additional representations can be found in [30].

The results (B17) and (B19) are expansions in \( \varepsilon \) where the coefficients are exact functions of \( x \). On the other hand, it is straightforward to obtain expansions of \( J_{3,4} \) in \( x \) (or \( x^{-1} \)) to any finite order using residues of left (or right) poles in the
Mellin–Barnes representations of the integrals \( j_{3,4} \) \([\text{B3}]\), the coefficients being exact functions of \( \varepsilon \). If we expand them in \( \varepsilon \), they should agree with expansions of \([\text{B17}]\) in \( x \) and of \([\text{B19}]\) in \( x^{-1} \). We have checked this up to rather high degrees of \( x \) and \( x^{-1} \).

Now we can find all contributions to \( Z_{M}^{\alpha} (j = M, Q) \) with the maximum number of quark loops, at most one of which is massive, to all orders exactly in \( \varepsilon \):

\[
Z_{M}^{\alpha} = 1 + C_{F} \sum_{l=1}^{4} T_{E}^{l-1} (n_{0} P)^{l-2} [n_{0} P B_{j_{0}}^{(l)}] \quad (B23)
\]

where

\[
P = -4 \frac{1 - \varepsilon}{(1 - 2 \varepsilon)(3 - 2 \varepsilon)} \Gamma^{2}(1 - \varepsilon),
\]

\[
B_{M}^{(l)} (x) = 2 p_{0} \left\{- \frac{2 \varepsilon}{l(1 - \varepsilon)(1 + (l - 1) \varepsilon)(3 + 2(l - 1) \varepsilon)} \right. \\
\times \left[ (1 + (l - 1) \varepsilon)(19 l - 3 - (11 l^{2} + 50 l - 11) \varepsilon) - 2(4 l^{3} - 20 l^{2} - 15 l + 6) \varepsilon^{2} + 4(4 l^{2} - 11 l^{2} + 2 l + 1) \varepsilon^{3} - 8 l(l - 1) \varepsilon^{4} \\
+ l(1 - \varepsilon)(1 + (l - 1) \varepsilon)(3 + 2(l - 1) \varepsilon) \frac{4 - (3 l + 1) \varepsilon - (l - 1)(l - 7) \varepsilon^{2} - 4(l - 1) \varepsilon^{3} x^{2}}{1 + (l - 1) \varepsilon} J_{1}^{(l-2)} (x) \\
+ 2 \left[ 2(1 - \varepsilon)(1 + (l - 1) \varepsilon)(1 - l \varepsilon) + \varepsilon - 4(l + 1) \varepsilon - (l - 1)(l - 17) \varepsilon^{2} - 2(l - 1)(l^{2} - 3) \varepsilon^{3} x^{2} \right] J_{2}^{(l-2)} (x) \\
+ (p_{3} + p_{4} x + p_{5} x^{2} + p_{6} x^{3})(1 - x) J_{3}^{(l-2)} (x) + (p_{3} - p_{4} x + p_{5} x^{2} - p_{6} x^{3})(1 + x) J_{4}^{(l-2)} (x) \right\}, \quad (B24)
\]

The contribution of these color structures to the ratio of the MS mass and the on-shell one \( z_{m}(\mu) = M(\mu)/M \) can be written as

\[
z_{m}(M) = z_{m}^{(\beta_{0})} + \sum_{i} \Delta_{m}(x_{i}) + \cdots, \quad (B26)
\]

where \( z_{m}^{(\beta_{0})} \) is the well-known large-\( \beta_{0} \) result \([\text{S7}]\).

The results \([\text{B24}]\) at \( l = 2 \) agree with \([\text{S0}]\) exactly in \( \varepsilon \). Note that

\[
\lim_{x \to 0} B_{M}^{(l)} (x) = B_{M0}^{(l)} P, \quad (B25)
\]

so that \( Z_{M}^{\alpha} \) has a smooth limit \( x \to 0 \); this is not so for \( Z_{Q}^{\alpha} \).
Note that we first expand \( S(u) \) in \( u \), then integrate term-by-term assuming \( \beta_0 > 0 \), and at the very end substitute
\[
\beta_0 \to -(4/3)T_F n_f .
\]
\( \Delta_m(x) \) comes from the differences of diagrams with a single massive quark loop and corresponding diagrams with all quark loops being massless and is given by
\[
\Delta_m(x) = C_F T_F \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ \frac{1}{2} \left( 1 - x \right)^2 (1 + x + x^2) H_{1,0}(x) - (1 + x)^2 (1 - x + x^2) H_{-1,0}(x) + 2x^4 H_{0,0}(x) \right. \\
+ x^2 H_{0}(x) - x (3 + 3x^2 - x^3) \frac{\pi^2}{6} + \left. \frac{3}{2} x^2 \right\} \\
+ T_F n_0 \frac{\alpha_s}{\pi} \frac{2}{3} \left[ (1 - x)^2 (1 + x + x^2) \left( H_{1,1,0}(x) + \frac{\pi^2}{12} H_1(x) \right) \\
+ (1 + x)^2 (1 - x + x^2) \left( H_{-1,1,0}(x) + \frac{5\pi^2}{12} H_{-1}(x) \right) - x (1 + x^2) \left( H_{0,1,0}(x) + H_{0,-1,0}(x) + \pi^2 a_1 \right) \\
+ x^4 \left( 2H_{0,0,0}(x) - \frac{13}{6} H_{0,0}(x) - \frac{3}{2} \zeta_3 \right) - x (3 + 3x^2 + x^3) \frac{\pi^2}{6} H_{0}(x) - \frac{1}{12} (1 - x)^2 (13 + 10x + 13x^2) H_{1,0}(x) \\
+ \frac{1}{12} (1 + x)^2 (13 - 10x + 13x^2) H_{-1,0}(x) + x (48 - 12x + 48x^2 - 13x^3) \frac{\pi^2}{72} - \frac{7}{12} x^2 H_{0}(x) - \frac{11}{8} x^2 \right\} \\
+ \left( T_F n_0 \frac{\alpha_s}{\pi} \right)^2 \frac{2}{3} \left( (1 - x)^2 (1 + x + x^2) \right. \\
\left. \times \left( -2H_{1,1,1,0}(x) + H_{1,0,1,0}(x) + H_{1,0,-1,0}(x) - \frac{\pi^2}{6} (5H_{1,1,0}(x) - 4H_{1,0}(x)) \right) + \left( \frac{\pi^2}{3} \zeta_3 \right) H_1(x) \right) \\
+ (1 + x)^2 (1 - x + x^2) \\
\times \left( 2H_{-1,1,1,0}(x) + H_{-1,0,1,0}(x) + H_{-1,0,-1,0}(x) + \frac{\pi^2}{6} (H_{-1,1}(x) + 2H_{-1,0}(x)) \right) + \left( \frac{\pi^2}{3} \zeta_3 \right) H_{-1}(x) \\
+ 2x (1 + x^2) \left( -H_{0,1,1,0}(x) + H_{0,0,1,0}(x) - H_{0,0,1,0}(x) - H_{0,0,-1,0}(x) + \frac{4}{3} \left( H_{0,1,0}(x) + H_{0,-1,0}(x) + \pi^2 a_1 \right) \right) \\
- \frac{\pi^2}{12} \left( 6H_{0,1}(x) - 5H_{0,-1}(x) + 6H_{0,0}(x) \right) - \pi^2 a_1 H_0(x) - \pi^2 a_1^2 \right) \\
+ x^4 \left( 4H_{0,0,0,0}(x) - \frac{13}{3} H_{0,0,0}(x) + \frac{89}{36} H_{0,0}(x) \right) \\
- \frac{1}{6} (1 - x)^2 (13 + 10x + 13x^2) \left( H_{1,1,0}(x) + \frac{\pi^2}{12} H_1(x) \right) - \frac{1}{6} (1 + x)^2 (13 - 10x + 13x^2) \left( H_{-1,1,0}(x) + \frac{5\pi^2}{12} H_{-1}(x) \right) \\
+ \frac{1}{72} (1 - x)^2 (89 + 68x + 89x^2) H_{1,0}(x) - \frac{1}{72} (1 + x)^2 (89 - 68x + 89x^2) H_{-1,0}(x) \\
+ x (48 + 6x + 48x^2 + 13x^3) \frac{\pi^2}{36} H_0(x) + \frac{47}{72} x^2 H_{0}(x) + \frac{1}{4} x^2 (6 + 13x^2) \zeta_3 - x (5 + 5x^2 - 2x^3) \frac{\pi^2}{30} \\
- x (330 - 192x + 330x^2 - 89x^3) \frac{\pi^2}{432} + \frac{33}{16} x^2 \right\} + O(\alpha_s^3) \right). \tag{B28}
\]

Note that \( \Delta_m(0) = 0 \). Expanding the three-loop term in \( x \) we reproduce the series (up to \( x^3 \)) obtained in \( (11) \). The three-loop coefficient exact in \( x \) (Fig. 5) and well as the four-loop one are new. The contribution of the external flavor \( (m = M) \) is given by
\[
\Delta_m(1) = C_F T_F \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ - \frac{\pi^2}{3} - \frac{3}{4} \right\} \\
+ T_F n_0 \frac{\alpha_s}{\pi} \left( \frac{\pi^2}{3} \zeta_3 + \frac{13}{36} \pi^2 - \frac{11}{12} \right)
\]
\[
- \left( T_F n_0 \frac{\alpha_s}{\pi} \right)^2 \left( \frac{13}{6} \zeta_3 + \frac{4}{45} \pi^4 + \frac{53}{216} \pi^2 - \frac{11}{8} \right) \\
+ O(\alpha_s^3) \right]. \tag{B29}
\]
The three- and four-loop terms here agree with \( (11) \) and \( (12) \). We do not present lower-loop terms of \( z_m \) with positive powers of \( \varepsilon \) which may be needed when this ratio is used within calculations containing \( 1/\varepsilon \) divergences; these terms can be easily obtained from Eqs. \( (B23) \) and \( (B24) \).
The results read

\[ Z_h^{\alpha} = 1 + C_F \sum_{l=2}^{\infty} T_F^{-1}(n_0 P)^{l-2} (l-1) B_h^{(l)} \times \prod_i \left( \frac{g_0 m_i^{-2\epsilon}}{(4\pi)^{d/2} \Gamma(\epsilon)} \right)^{l-1} \]

where \( g_0 \equiv g_{0}^{(n)} \) and dots denote other color structures. The \( (l = 2) \)-loop term agrees with \([22]\), and the three-loop one with the corresponding color structure in \([33]\). According to the regions-based argument in Sect. III

\[ \lim_{x \to 0} \left[ B_Q^{(l)}(x) - B_Q^{(0)} P - B_h^{(l)} x^{-2\epsilon} \right] = 0. \]
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