Differential invariants on symplectic spinors in contact projective geometry
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Abstract
We present a complete classification and the construction of $\text{Mp}(2n + 2, \mathbb{R})$-equivariant differential operators acting on the principal series representations, associated to the contact projective geometry on $\mathbb{RP}^{2n+1}$ and induced from the irreducible $\text{Mp}(2n, \mathbb{R})$-submodules of the Segal–Shale–Weil representation twisted by a one-parameter family of characters. The proof is based on the classification of homomorphisms of generalized Verma modules for the Segal–Shale–Weil representation twisted by a one-parameter family of characters, together with a generalization of the well-known duality between homomorphisms of generalized Verma modules and equivariant differential operators in the category of inducing smooth admissible modules.

Keywords: Contact projective geometry, generalized Verma module, Segal–Shale–Weil representation.

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Introduction
The idea of a symplectic analogue of spinor fields in Riemannian (or conformal) geometry is based on the use of the Segal–Shale–Weil representation and originated in the work by B. Kostant, [17]. The geometrical and functional theoretical properties of the associated first order differential invariant, the symplectic Dirac operator $D_s$, were consequently studied on symplectic manifolds, cf. [10] and the references therein. However, there is another useful realization of symplectic differential invariants like $D_s$, which is based on an extension of their symplectic symmetry group.
In particular, following Cartan’s approach to generalized geometry there is a minimal (i.e., locally described by a one dimensional family of symplectic leaves) model in this particular case: the contact projective geometry, characterized by the maximally non-integrable symplectic distribution of codimension one. This program was initiated in [12], and it immediately raised the question of a classification of flat (homogeneous) invariants for (irreducible metaplectic submodules of) the Segal–Shale–Weil representation twisted by a one-parameter family of characters.

Relying on the recently developed approach to the classification of homomorphisms between generalized Verma modules called F-method, [15], [16], [12], the present article shows a convenient setting given by extending the category of induced modules to smooth admissible representations. As an example we give a complete classification result in the case of irreducible metaplectic submodules of the Segal–Shale–Weil representation twisted by a family of characters as the inducing representations, thereby supplying the missing structural information (well known for finite-dimensional inducing representations.) We recall that the idea of the F-method is to characterize the vector space of all singular vectors in a generalized Verma module by a system of partial differential equations acting on its geometric realization. The detailed explanation of this idea is given in Section 1. We also remark that for characters as inducing representations in the case of the $C_n$-series of simple finite-dimensional Lie algebras, the composition structure for scalar generalized Verma modules is rather poor (as we shall discuss in the forthcoming article [14]) when compared to the classical series $A_n, B_n, D_n$, see [13]. The vector-valued representation given by the irreducible metaplectic submodules of the Segal–Shale–Weil representation is a prominent example from the perspective of potential applications.

We emphasize that the restriction to smooth inducing modules stems from our main reason for the applicability, namely the construction of equivariant differential operators on generalized flag manifolds. Without this purpose, the F-method itself can be treated for any inducing representation.

The content of our article goes as follows. In Section 1 we briefly introduce the formalism of the F-method (cf. [15], [13]) and extend it to a convenient category of smooth admissible inducing modules covering the finite-dimensional representations. For this purpose we use a few standard arguments related to smooth globalization of infinite-dimensional representations, cf. [2], which in the same way as for finite-dimensional representations allows us to reformulate the task of finding geometrical invariants in the analytical framework of solving a system of PDEs. In Section 2 we describe the case of contact projective geometry in detail, and introduce the family of representations induced from the irreducible metaplectic submodules of the Segal–Shale–Weil representation twisted by characters. It is described in the non-compact model on the open Schubert cell of the generalized flag manifold given by the projective space. Section 3 presents in Theorem 3.6 a complete list of solutions of the former problem in terms of singular vectors for the inducing representations of our interest. In Section 4 we prove the duality between equivariant differential operators acting on induced representations and homomorphisms of generalized Verma modules for smooth irreducible inducing representations, a result well-known for finite-dimensional inducing representations, cf. [6], [22]. Then we dualize the (inducing) representations and in Theorem 4.3 construct, following the complete set of singular vectors, all equivariant differential operators on the irreducible metaplectic submodules of the Segal–Shale–Weil representation twisted by a character. Roughly speaking, our result produces two types of equivariant differential operators. The first class are the twistor-like operators landing in more complicated metaplectic modules, and we expect their appearance in a not-yet-developed Bernstein–Gelfand–Gelfand sequences associated with infinite-dimensional representations. The second class of equivariant operators is valued in the irreducible metaplectic submodules of the Segal–Shale–Weil representation twisted by a character: the operators have a right to be called “contact powers of the contact symplectic Dirac operator”, in complete analogy with the equivariant differential operators termed “conformal powers of the Laplace operator”, “conformal powers of the Dirac operator” and “CR invariant powers of the sub-Laplacian” studied in the realm of the conformal and CR geometries, respectively, cf. [9], [11], [8]. The existence of first order invariant differential operators for the contact projective geometry and the Segal–Shale–Weil representation was proved in [7].

Throughout the article, the sets $\mathbb{N}$ and $\mathbb{N}_0$ denote $\{1, 2, \ldots \}$ and $\{0, 1, 2, \ldots \}$, respectively.
1 The structure of generalized Verma modules and equivariant differential operators

We consider the pair \((G, P)\) consisting of a connected real semisimple Lie group \(G\) and its parabolic subgroup \(P\). In the Levi decomposition \(P = LU\), \(L\) denotes the Levi subgroup and \(U\) the unipotent radical of \(P\). We write \(g(\mathbb{R})\), \(p(\mathbb{R})\), \(l(\mathbb{R})\), \(u(\mathbb{R})\) for the real Lie algebras and \(g, p, l, u\) for the complexified Lie algebras of \(G, P, L, U\), respectively. The symbols \(U\) and \(S\) applied to a Lie algebra denote its universal enveloping algebra and symmetric algebra, respectively.

It is well-known that the \(G\)-equivariant differential operators acting on principal series representations for \(G\) can be recognized in the study of homomorphisms between generalized Verma modules for the Lie algebra \(g\). In the classical case of finite-dimensional inducing representations, the latter homomorphisms are determined by the image of inducing representation and its vectors referred to as the singular vectors. They are characterized as the vectors in generalized Verma module annihilated by the positive nilradical \(u\).

As a starting point we use the F-method in order to classify and find precise positions of singular vectors in a given representation space, see [16], [15], [13] for a detailed exposition. In the present paper we generalize this technique in the sense that we prove its natural extension from the category of finite-dimensional inducing representations to a wider class of smooth admissible inducing representations. We briefly recall that a representation \(\pi\) of \(G\) on a Hausdorff complete locally convex topological (real or complex) vector space \(V\) is smooth if all vectors in \(V\) are smooth, and \(v \in V\) is a smooth vector provided the mapping from \(P\) to \(V\) given by \(p \mapsto \sigma(p)v\) is smooth. A \(U(p)\)-module \(V\) is admissible if it is \(\mathfrak{z}(l)\)-finite, where \(\mathfrak{z}(l)\) is the center of \(U(l)\) and \(\mathfrak{z}(l)\)-finite means that the annihilator ideal of \(V\) in \(\mathfrak{z}(l)\) is of finite codimension. We refer to Section 4 for a more detailed discussion of these notions.

Let \((\sigma, V)\) be a smooth admissible \(P\)-module. Let us assume that \(\lambda \in \text{Hom}_P(p, \mathbb{C})\) defines a group character \(e^\lambda: P \to \text{GL}(1, \mathbb{C})\) of \(P\) and define \(\rho \in \text{Hom}_P(p, \mathbb{C})\) by

\[
\rho(X) = \frac{1}{2} \text{tr}_u \text{ad}(X)
\]

for all \(X \in p\), where \(\text{tr}_u \text{ad}(X)\) denotes the trace of \(\text{ad}(X): u \to u\). Then we can define a twisted \(P\)-module \((\sigma_{\lambda + \rho}, V_{\lambda + \rho})\) with a twist \(\lambda + \rho \in \text{Hom}_P(p, \mathbb{C})\) such that \(p \in P\) acts as \(e^{\lambda + \rho}(p)\sigma(p)v\) instead of \(\sigma(p)v\) for all \(v \in V_{\lambda + \rho}\) smooth, and \(v \in V\) is a smooth vector provided the mapping from \(P\) to \(V\) given by \(p \mapsto \sigma(p)v\) is smooth. A \(U(p)\)-module \(V\) is admissible if it is \(\mathfrak{z}(l)\)-finite, where \(\mathfrak{z}(l)\) is the center of \(U(l)\) and \(\mathfrak{z}(l)\)-finite means that the annihilator ideal of \(V\) in \(\mathfrak{z}(l)\) is of finite codimension. We refer to Section 4 for a more detailed discussion of these notions.

For a chosen smooth principal series representation of \(G\) on the vector space \(\text{Ind}_P^G(V_{\lambda + \rho})\) of smooth sections of the homogeneous vector bundle \(G \times P \to V_{\lambda + \rho} \to G/P\) associated to a complex smooth admissible \(P\)-module \(V_{\lambda + \rho}\) (cf. [19]), due to the smoothness of \(V\) we obtain the infinitesimal action

\[
\pi_{\lambda}: g \to \mathcal{D}(U_c) \otimes \mathbb{C} \text{End} V_{\lambda + \rho}
\]

on the vector space \(\mathcal{C}^\infty(U_c) \otimes \mathbb{C} V_{\lambda + \rho}\) of \(V_{\lambda + \rho}\)-valued smooth functions on \(U_c\) in the non-compact picture of the induced representation. Here \(\mathcal{D}(U_c)\) denotes the \(\mathbb{C}\)-algebra of smooth complex linear differential operators on \(U_c = UP \subset G/P\) \((\mathfrak{U}\) is the Lie group whose Lie algebra is the opposite nilradical \(\mathfrak{n}(\mathbb{R})\) to \(u(\mathbb{R})\)).

Since the vector space \(\mathcal{D}_0'(U_c) \otimes \mathbb{C} V_{\lambda + \rho}\) of \(V_{\lambda + \rho}\)-valued distributions on \(U_c\) supported in the unit coset \(o = eP \in G/P\) is a \(\mathcal{D}(U_c) \otimes \mathbb{C} \text{End} V_{\lambda + \rho}\)-module, we get also the infinitesimal action of \(\pi_{\lambda}(X)\) for \(X \in g\) on \(\mathcal{D}_0'(U_c) \otimes \mathbb{C} V_{\lambda + \rho}\). The exponential map allows us to identify \(U_c\) with \(\text{Nil} g\) for the nilpotent Lie algebra \(\mathfrak{n}(\mathbb{R})\). If we denote by \(A^g_{\mathfrak{n}}\) the Weyl algebra of the complex vector space \(\mathfrak{n}\), then the vector space \(\mathcal{D}_0'(U_c)\) can be conveniently analyzed by identifying it as an \(A^g_{\mathfrak{n}}\)-module with \(\text{Nil} g\) by the left ideal \(I_c\) generated by all polynomials on \(\mathfrak{n}\) vanishing at the origin. Moreover, there is a \(U(g)\)-module isomorphism

\[
\Phi_{\lambda}: M^g_{\mathfrak{n}}(V_{\lambda + \rho}) \cong U(g) \otimes_{U(p)} V_{\lambda + \rho} \xrightarrow{\sim} \mathcal{D}_0'(U_c) \otimes \mathbb{C} V_{\lambda + \rho} \xrightarrow{\sim} A^g_{\mathfrak{n}}/I_c \otimes \mathbb{C} V_{\lambda + \rho}.
\]
Let \((x_1, x_2, \ldots, x_n)\) be the linear coordinate functions on \(\mathfrak{g}\) and let \((y_1, y_2, \ldots, y_n)\) be the dual linear coordinate functions on \(\mathfrak{g}^*\). Then the algebraic Fourier transform

\[
\mathcal{F}: A^0_{\mathfrak{g}} \rightarrow A^0_{\mathfrak{g}}
\]

is given by

\[
\mathcal{F}(x_i) = -\partial_{y_i}, \quad \mathcal{F}(\partial_{x_i}) = y_i
\]

for \(i = 1, 2, \ldots, n\), and leads to a vector space isomorphism

\[
\tau: A^0_{\mathfrak{g}}/I_e \xrightarrow{\sim} A^0_{\mathfrak{g}}/\mathcal{F}(I_e) \simeq \mathbb{C}[\mathfrak{g}^*]
\]

defined by

\[
Q \mod I_e \mapsto \mathcal{F}(Q) \mod \mathcal{F}(I_e)
\]

for \(Q \in A^0_{\mathfrak{g}}\). The composition of the previous mappings (1.3) and (1.6) gives the vector space isomorphism

\[
\tau \circ \Phi_\lambda: U(\mathfrak{g}) \otimes_{U(p)} V_{\lambda - \rho} \xrightarrow{\sim} A^0_{\mathfrak{g}}/I_e \otimes_{\mathbb{C}} V_{\lambda - \rho} \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}} V_{\lambda + \rho} \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}} V_{\lambda - \rho},
\]

where the last conventional isomorphism accounts for the identification between the left and right \(\mathcal{D}\)-module structure, and thereby it induces an action

\[
\hat{\pi}_\lambda: \mathfrak{g} \rightarrow A^0_{\mathfrak{g}} \otimes_{\mathbb{C}} \text{End} V_{\lambda - \rho}
\]

of \(\mathfrak{g}\) on \(\mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}} V_{\lambda - \rho}\).

In order to find an explicit form of \(G\)-equivariant differential operators, discussed in more detail in Section 4 it is necessary to have an explicit formula for the inverse of the mapping \(\tau \circ \Phi_\lambda - \rho: M^0_{\mathfrak{g}}(V_{\lambda - \rho}) \rightarrow \mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}} V_{\lambda - \rho}\). Let us introduce the symmetrization map \(\beta: S(\mathfrak{g}) \rightarrow U(\mathfrak{g})\), which is a \(\mathbb{C}\)-linear isomorphism of filtered vector spaces defined on monomials by

\[
\beta(a_1a_2\ldots a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)}a_{\sigma(2)}\ldots a_{\sigma(k)}
\]

(1.10)

for all \(k \in \mathbb{N}\) and \(a_1, a_2, \ldots, a_k \in \mathfrak{g}\), where \(S_k\) is the permutation group of the set \(\{1, 2, \ldots, k\}\). If we denote by \((f_1, f_2, \ldots, f_n)\) a basis of \(\mathfrak{g}\) satisfying \(\pi_\lambda(f_i) = -y_i\) mod \(\mathcal{F}(I_e)\) for \(i = 1, 2, \ldots, n\), then we have

\[
(\tau \circ \Phi_\lambda)(\beta(f_{i_1}f_{i_2}\ldots f_{i_k}) \otimes v) = (-1)^k y_{i_1}y_{i_2}\ldots y_{i_k} \otimes v
\]

(1.11)

for all \(v \in V_{\lambda - \rho}\), \(k \in \mathbb{N}\) and \(i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}\).

**Definition 1.1.** Let \(V\) be a smooth admissible \(L\)-module which extends to a \(P\)-module by \(U\) acting trivially. Then the generalized Verma module induced from \(V\) is the \((\mathfrak{g}, P)\)-module

\[
M^0_P(V) = \text{Ind}^0_P V \equiv U(\mathfrak{g}) \otimes_{U(p)} V \simeq U(\mathfrak{g}) \otimes_{\mathbb{C}} V,
\]

(1.12)

where the last isomorphism of \(U(\mathfrak{g})\)-modules follows from the Poincaré–Birkhoff–Witt theorem.

By abuse of notation we call the modules \(M^0_P(V)\) generalized Verma modules, although this terminology is usually reserved for modules induced from finite-dimensional representations (cf. [20] for the terminology in the case of inducing l-modules in the BGG category \(\mathcal{O}\) associated to \(U(l)\)). The generalized Verma module \(M^0_P(V)\) has a \(P\)-module structure defined as follows. As \(M^0_P(V)\) is locally \(u\)-finite and \(U\) is simply connected, a \(U\)-module structure on \(M^0_P(V)\) is given by the exponential mapping \(\exp: u \rightarrow U\). Since \(U(\mathfrak{g})\) is an \(L\)-module by the adjoint action of \(L\) and \(V\) is by definition a \(L\)-module, we have a \(L\)-module structure on \(U(\mathfrak{g}) \otimes_{\mathbb{C}} V\). The Levi decomposition \(P = LU\) then results in a \(P\)-module structure on \(M^0_P(V)\).
As $U(\mathfrak{g})$ is a direct sum of finite-dimensional $L$-modules, the tensor product $U(\mathfrak{g}) \otimes_{C} V$ is a direct sum of the tensor products of finite-dimensional $L$-modules with $V$. We notice that basic properties of $L$-module composition structure of such tensor products can be found for example in [1].

Based on the language of globalizations of infinite-dimensional representations it is well-known that the tensor product of a smooth module (smooth globalization of a Harish-Chandra module) with a finite-dimensional representation is a smooth module, and a direct sum of smooth modules is a smooth module. Moreover, quotients of smooth modules by smooth submodules are smooth modules, cf. [2] for all these results. In particular, $M^g\mathfrak{g}(V)$ is a smooth $(g, P)$-module which is a direct sum of smooth admissible $L$-modules because $U(\mathfrak{g})$ is a direct sum of finite-dimensional $L$-modules.

**Definition 1.2.** Let $V$ be a smooth admissible $L$-module which extends to a $P$-module by $U$ acting trivially. We define an $L$-module

$$M^g\mathfrak{g}(V)^u = \{ v \in M^g\mathfrak{g}(V); Xv = 0 \text{ for all } X \in u \},$$

and call it the vector space of singular vectors.

Let us note that the $L$-module structure on $M^g\mathfrak{g}(V)^u$ is induced by the $P$-module structure on the generalized Verma module $M^g\mathfrak{g}(V)$. Since $Ad(l^{-1})(X) \in u$ for all $X \in u$ and $l \in L$, where $Ad(l^{-1})$ denotes the adjoint action on $g$, we obtain $X(lv) = lAd(l^{-1})(X)v = 0$ for all $X \in u$, $l \in L$, $v \in M^g\mathfrak{g}(V)^u$ and so $lv \in M^g\mathfrak{g}(V)^u$ for all $l \in L$, $v \in M^g\mathfrak{g}(V)^u$.

We also notice that for a pair of smooth admissible $L$-modules $W$ and $V$, any $(g, P)$-module homomorphism

$$\varphi: M^g\mathfrak{g}(W) \longrightarrow M^g\mathfrak{g}(V)$$

is determined by the image $\varphi(1 \otimes W)$, because $M^g\mathfrak{g}(W)$ is a free $U(\mathfrak{g})$-module by (1.12).

The main difference between [15] and its generalization considered here is that $M^g\mathfrak{g}(V)^u$ does not decompose into direct sum of simple $L$-modules in general. Let us denote by $W$ an $L$-submodule of $M^g\mathfrak{g}(V)^u$. By the characterizing property of $(g, P)$-module homomorphism in the category of modules $M^g\mathfrak{g}(V)$ mentioned in the previous paragraph, we have

$$\text{Hom}_{(g, P)}(M^g\mathfrak{g}(W), M^g\mathfrak{g}(V)) \simeq \text{Hom}_{L}(W, M^g\mathfrak{g}(V)^u),$$

where $M^g\mathfrak{g}(W)$ has the composition series given by smooth admissible $(g, P)$-modules.

In the main application treated in Section 2 and Section 3 we apply the previous considerations to the infinite-dimensional smooth (highest weight unitarizable) $L$-module given by the simple $L$-submodules of the Segal–Shale–Weil representation, see [3]. The space of singular vectors has in this case an unexpected feature, namely, it is a direct sum of infinite-dimensional simple highest weight $L$-modules.

The rest of the $F$-method as present in [15] works without any modification for smooth admissible inducing modules $V$. In particular, the isomorphism $\tau \circ \Phi_{\lambda, \rho}: M^g\mathfrak{g}(\mathcal{V}_{\lambda, \rho}) \rightarrow \mathbb{C}[\mathfrak{g}^*] \otimes_{C} \mathcal{V}_{\lambda, \rho}$ of $\mathfrak{g}$-modules ensures that $\mathbb{C}[\mathfrak{g}^*] \otimes_{C} \mathcal{V}_{\lambda, \rho}$ is a $(g, P)$-module as well. We introduce an $L$-module

$$\text{Sol}(g, p; \mathbb{C}[\mathfrak{g}^*] \otimes_{C} \mathcal{V}_{\lambda, \rho})^F = \{ f \in \mathbb{C}[\mathfrak{g}^*] \otimes_{C} \mathcal{V}_{\lambda, \rho}; \hat{\pi}_\lambda(X)f = 0 \text{ for all } X \in u \},$$

which is in fact the vector space of singular vectors in $\mathbb{C}[\mathfrak{g}^*] \otimes_{C} \mathcal{V}_{\lambda, \rho}$, and by (1.18) there is an $L$-equivariant isomorphism

$$\tau \circ \Phi_{\lambda}: M^g\mathfrak{g}(\mathcal{V}_{\lambda, \rho})^u \simeq \text{Sol}(g, p; \mathbb{C}[\mathfrak{g}^*] \otimes_{C} \mathcal{V}_{\lambda, \rho})^F.$$
converts the algebraic problem of finding singular vectors in generalized Verma modules into an analytic problem of solving the systems of partial differential equations.

The formulation above has a classical dual statement (cf. [6], [22] for the formulation in the category of finite-dimensional inducing P-modules), which explains the relationship between the geometrical problem of finding G-equivariant differential operators between induced representations and the algebraic problem of finding homomorphisms between generalized Verma modules. We refer to Section [1] for a detailed exposition of this duality for an arbitrary inducing smooth admissible P-module.

2 The contact projective structure

In the present section we describe a few basic geometrical and representation theoretical aspects of the flat (or homogeneous) contact projective structure needed in our analysis. For the purposes of applications, we review the real homogeneous model of the contact projective geometry.

2.1 The geometry of flat contact projective structure

The generalized flag manifold corresponding to the flat model of real contact projective structure is the homogeneous space $G/P \simeq \mathbb{R}^{2n+1}$, where the group of symplectic automorphisms $G = \text{Sp}(2n+2, \mathbb{R})$ of the standard symplectic vector space $(\mathbb{R}^{2n+2}, \Omega)$, $n \in \mathbb{N}$, acts transitively on the space of lines in $\mathbb{R}^{2n+2}$ by $(g,[v]) \mapsto [g.v]$ for $0 \neq v \in \mathbb{R}^{2n+2}$. The stabilizer of a line is a parabolic subgroup conjugate to the standard parabolic subgroup $P = (\text{GL}(1, \mathbb{R}) \times \text{Sp}(2n, \mathbb{R})) \times \text{H}(n, \mathbb{R})$, where $\text{H}(n, \mathbb{R})$ is the real Heisenberg group of dimension $2n + 1$. In the Dynkin-diagrammatic notation, the Lie algebra of $P$ is given by omitting the first simple root in the $C_{n+1}$-series of simple Lie algebras.

The geometry of real contact projective structure can be described as follows. We denote by $V^1 \subset \mathbb{R}^{2n+2}$ the 1-dimensional subspace whose stabilizer is the group $P$, so that its $\Omega$-complement $V^2 = (V^1)^ot$ yields the $P$-invariant filtration $V^1 \subset V^2 \subset \mathbb{R}^{2n+2}$. The choice $0 \neq v_\infty \in V^1$ and $v_0 \in \mathbb{R}^{2n+2}$ such that $\Omega(v_\infty, v_0) = 1$ determines a splitting $\mathbb{R}^{2n+2} = V_0 \oplus V_{-1} \oplus V_{-2}$ with $V_0 = V^1$ and $v_0$ spans $V_{-2}$. This grading on $\mathbb{R}^{2n+2}$ induces a Z-grading on the Lie algebra $\mathfrak{g}(\mathbb{R})$ of $G$, $\mathfrak{g}(\mathbb{R}) = \oplus_{i=-2}^2 \mathfrak{g}(\mathbb{R})_i$, by

$$\mathfrak{g}(\mathbb{R})_i = \{ X \in \mathfrak{g}(\mathbb{R}) \mid X(V_k) \subset V_{k+i} \text{ for } -2 \leq k \leq 0 \}.$$ 

The subalgebra $\mathfrak{g}(\mathbb{R})_{-2} \oplus \mathfrak{g}(\mathbb{R})_{-1}$ is the Heisenberg algebra, $\dim \mathfrak{g}(\mathbb{R})_{-2} = 1$, $\dim \mathfrak{g}(\mathbb{R})_{-1} = 2n$, and $\mathfrak{g}(\mathbb{R})_0$ is the conformal symplectic Lie algebra. The vectors $v_0$ and $v_\infty$ together with a basis $v_1, \ldots, v_{2n}$ of $V_{-1}$ then determine linear coordinate functions $x^i, x^\infty, x^1, \ldots, x^{2n}$ on $\mathbb{R}^{2n+2}$, such that $\Omega = \frac{1}{2} \Omega_{IJ} dx^I \wedge dx^J$, $I, J \in \{ 0, \infty, 1, \ldots, 2n \}$, with

$$\Omega_{00} = 1, \quad \Omega_{0i} = 0, \quad \Omega_{\infty i} = 0, \quad \Omega_{ij} = \omega_{ij},$$

where $\omega_{ij} = \Omega(v_i, v_j)$. The group of automorphisms of $(\mathbb{R}^{2n+2}, \Omega)$ is $G$, the Lie algebra of $P$ is $\mathfrak{g}(\mathbb{R})_0 \oplus \mathfrak{g}(\mathbb{R})_1 \oplus \mathfrak{g}(\mathbb{R})_2$. We denote the stabilizer of the vector $v_\infty$ by $P^\circ \subset P$, its Lie algebra by $\mathfrak{p}(\mathbb{R})^\circ$. Since we have $P/P^\circ \simeq \text{GL}(1, \mathbb{R})$, the principal $\text{GL}(1, \mathbb{R})$-bundle $G/P^\circ \rightarrow G/P$ determines the tautological vector bundle $\mathbb{R}^{2n+2} \setminus \{ 0 \} \rightarrow \mathbb{R}^{2n+1}$, and the Maurer-Cartan form on $G$ is the Cartan connection for the flat contact projective structure.

The $P$-invariant filtration on $\mathfrak{g}(\mathbb{R})$ allows us to define on $\mathfrak{g}(\mathbb{R})_{-1}$ the conformal symplectic structure, which further determines a symplectic structure on $\mathfrak{g}(\mathbb{R})/\mathfrak{p}(\mathbb{R})^\circ$ isomorphic to $(\mathbb{R}^{2n+2}, \Omega)$. The basis $v_0, v_1, \ldots, v_{2n}$ of $\mathfrak{g}(\mathbb{R})_{-2} \oplus \mathfrak{g}(\mathbb{R})_{-1}$ can be exponentiated to the basis of left-invariant vector fields $E_0, E_1, \ldots, E_{2n}$ on $G/P$, with the Lie bracket $[E_\alpha, E_\beta] = -\Omega_{\alpha\beta} E_0$, $\alpha, \beta \in \{ 0, 1, \ldots, 2n \}$. The canonical contact structure (or, the contact distribution) on $T(G/P)$ is the left $G$-invariant subbundle of $T(G/P)$, generated by $\mathfrak{g}(\mathbb{R})_{-1}$ and spanned at each point by $E_1, \ldots, E_{2n}$. A left-invariant section $\theta \in T^*(G/P)$ of the annihilator of the contact distribution is then determined by the requirement that $E_0$ is its Reeb vector field. The covariant derivative $\nabla$, defined by the
requirement that the vector fields $E_\alpha$ are $\nabla$-parallel, preserves $\theta$ and $d\theta$ and determines a model for any contact projective geometry.

From the representation-theoretical point of view, we need the double cover of the symplectic group $G = Sp(2n + 2, \mathbb{R})$ called the metaplectic group $\tilde{G} = Mp(2n + 2, \mathbb{R})$. Thus we have a double cover homomorphism $\psi: G \rightarrow \tilde{G}$ of Lie groups, and if we define the parabolic subgroup $P$ of $\tilde{G}$ by $\tilde{P} = \psi^{-1}(P)$, then the generalized flag manifold $G/P$ is isomorphic to $G/P$ induced by the mapping $\psi$.

For more detailed exposition with a view towards the description of a general (curved) contact projective structure, cf. [7] and the references therein.

### 2.2 Representation theory of contact projective structure

Let us consider the connected complex simple Lie group $G_\mathbb{C} = Sp(2n + 2, \mathbb{C})$, $n \geq 1$, defined by

$$Sp(2n + 2, \mathbb{C}) = \{ g \in GL(2n + 2, \mathbb{C}); g^T J_{2n+2} g = J_{2n+2} \}, \quad J_{2n+2} = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}, \quad (2.1)$$

and its Lie algebra $g = sp(2n + 2, \mathbb{C})$ given by

$$sp(2n + 2, \mathbb{C}) = \{ X \in M_{2n+2,2n+2}(\mathbb{C}); X^T J_{2n+2} + J_{2n+2} X = 0 \} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}; A, B, C \in M_{n+1,n+1}(\mathbb{C}), B^T = B, C^T = C \right\}. \quad (2.2)$$

A Cartan subalgebra $h$ of $g$ is defined by the diagonal matrices

$$h = \{ \text{diag}(a_1, \ldots, a_{n+1}, -a_1, \ldots, -a_{n+1}); a_1, a_2, \ldots, a_{n+1} \in \mathbb{C} \}. \quad (2.3)$$

For $i = 1, 2, \ldots, n + 1$ we define $\varepsilon_i \in h^*$ by $\varepsilon_i(\text{diag}(a_1, \ldots, a_{n+1}, -a_1, \ldots, -a_{n+1})) = a_i$. Then the root system of $g$ with respect to $h$ is $\Delta = \{ \pm \varepsilon_i \pm \varepsilon_j; 1 \leq i < j \leq n + 1 \} \cup \{ \pm 2\varepsilon_i; 1 \leq i \leq n + 1 \}$. A positive root system is $\Delta^+ = \{ \varepsilon_i \pm \varepsilon_j; 1 \leq i < j \leq n + 1 \} \cup \{ 2\varepsilon_i; 1 \leq i \leq n + 1 \}$ with the set of simple roots $\Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_{n+1} \}$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, \ldots, n$ and $\alpha_{n+1} = 2\varepsilon_{n+1}$. Finally, the fundamental weights are $\omega_i = \sum_{j=1}^{n} \varepsilon_j$, $i = 1, 2, \ldots, n + 1$.

The Lie subalgebras $b$ and $\tilde{b}$ defined as the direct sum of positive and negative root spaces together with the Cartan subalgebra are called the standard Borel and the opposite standard Borel subalgebras of $g$, respectively. The subset $\Sigma = \{ \alpha_2, \alpha_3, \ldots, \alpha_{n+1} \}$ of $\Pi$ generates the root subsystem $\Delta_\Sigma$ in $h^*$, and we associate to $\Sigma$ the standard parabolic subalgebra $p$ of $g$ by $p = l \oplus u$.

The reductive Levi subalgebra $l$ of $p$ is defined through

$$l = h \oplus \bigoplus_{\alpha \in \Delta_\Sigma} g_{\alpha}, \quad (2.4)$$

and the nilradical $u$ of $p$ and the opposite nilradical $\tilde{u}$ are

$$u = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma^+} g_{\alpha} \quad \text{and} \quad \tilde{u} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma^+} g_{-\alpha}, \quad (2.5)$$

respectively. We define the $\Sigma$-height $ht_\Sigma(\alpha)$ of $\alpha \in \Delta$ by

$$ht_\Sigma\left(\sum_{i=1}^{n+1} a_i \alpha_i \right) = a_i, \quad (2.6)$$

so $g$ is a $|2|$-graded Lie algebra with respect to the grading given by $g_i = \bigoplus_{\alpha \in \Delta, ht_\Sigma(\alpha)=i} g_{\alpha}$ for $0 \neq i \in \mathbb{Z}$, and $g_0 = h \oplus \bigoplus_{\alpha \in \Delta, ht_\Sigma(\alpha)=0} g_{\alpha}$. Moreover, we have $u = g_1 \oplus g_2$, $\tilde{u} = g_{-2} \oplus g_{-1}$ and $l = g_0 \cong \mathbb{C} \oplus sp(2n, \mathbb{C})$.

Let us denote by $1_i$, $i = 1, 2, \ldots, n$, the $(n \times 1)$-matrix having $1$ at the $i$-th row and $0$ elsewhere. The basis $(f_1, \ldots, f_n, g_1, \ldots, g_n, c)$ of the root spaces in the opposite nilradical $\tilde{u}$ is given by

$$f_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1_i & 0 & 0 & 0 \\ 0 & 0 & -1_i^T & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1_i^T & 0 & 0 \\ 1_i & 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.7)$$
where the only non-trivial Lie brackets are \([f_i, g_i] = -c \) for all \(i = 1, 2, \ldots, n\). Analogously, the basis \((d_1, \ldots, d_n, e_1, \ldots, e_n, a)\) of the root spaces in \(u\) is given by

\[
d_i = \begin{pmatrix} 0 & 1^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_i = \begin{pmatrix} 0 & 0 & 0 & 1^T \\ 0 & 0 & 1_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

where \([d_i, e_i] = a\) for all \(i = 1, 2, \ldots, n\). The Levi subalgebra \(l\) of \(p\) is the linear span of

\[
h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_{A,B,C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & C & 0 & -A^T \end{pmatrix},
\]

where \(A, B, C \in M_{n,n}(\mathbb{C})\) satisfy \(B^T = B\) and \(C^T = C\). Moreover, the element \(h\) is a basis of the center \(z(l)\) of \(l\).

Finally, the parabolic subgroup \(P_C\) of \(G_C\) with the Lie algebra \(p\) is defined by \(P_C = N_C(p)\). The parabolic subalgebra \(p \subset \mathfrak{g}\) is given by

\[
p = \left\{ \begin{pmatrix} h & d^T & a & e^T \\ 0 & A & e & B \\ 0 & 0 & -h & 0 \\ 0 & C & -d & -A^T \end{pmatrix} : h, a \in \mathbb{C}, d, e \in \mathbb{C}^n, A, B, C \in M_{n,n}(\mathbb{C}), B^T = B, C^T = C \right\}.\]

The real connected simple Lie group \(G\) and its parabolic subgroup \(P\) is defined by \(G = G_C \cap \text{GL}(2n + 2, \mathbb{R})\) and \(P = P_C \cap \text{GL}(2n + 2, \mathbb{R})\).

Any character \(\lambda \in \text{Hom}_P(p, \mathbb{C})\) is given by

\[
\lambda = \lambda_1 \tilde{\omega}_1
\]

for some \(\lambda_1 \in \mathbb{C}\), where \(\tilde{\omega}_1 \in \text{Hom}_P(p, \mathbb{C})\) is equal to \(\omega_1 \in \mathfrak{h}^*\) regarded as trivially extended to \(p = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_E} \mathfrak{g}_\alpha) \oplus u\). The vector \(\rho \in \text{Hom}_P(p, \mathbb{C})\) defined by \((1.1)\) is

\[
\rho = (n + 1) \tilde{\omega}_1.
\]

By abuse of notation, for the character \(\lambda \tilde{\omega}_1 \in \text{Hom}_P(p, \mathbb{C})\), \(\lambda \in \mathbb{C}\), we use the simplified notation \(\lambda \in \text{Hom}_P(p, \mathbb{C})\).

2.3 Description of the representation

Here we describe the representations of \(\mathfrak{g}\) on the space of sections of vector bundles on \(G/P\) associated to the simple metaplectic submodules of the Segal–Shale–Weil representation \(S_{\lambda+\rho}\) of \(P\) twisted by characters \(\lambda + \rho \in \text{Hom}_P(p, \mathbb{C})\).

The induced representations in question are described in the non-compact picture, given by restricting sections on \(G/P\) to the open Schubert cell \(U_c\) isomorphic by the exponential map to the opposite nilradical \(\mathfrak{u}(\mathbb{R})\). Let us denote by \((\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n, \hat{z})\) the linear coordinate functions on \(\mathfrak{u}\) with respect to the basis \((f_1, \ldots, f_n, g_1, \ldots, g_n, c)\) of the opposite nilradical \(\mathfrak{u}\), and by \((x_1, \ldots, x_n, y_1, \ldots, y_n, z)\) the dual linear coordinate functions on \(\mathfrak{u}^\vee\). Then the Weyl algebra \(A^\mathfrak{g}_{\mathfrak{u}}\) is generated by

\[
\{\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n, \hat{z}, \partial_{\hat{x}_1}, \ldots, \partial_{\hat{x}_n}, \partial_{\hat{y}_1}, \ldots, \partial_{\hat{y}_n}, \partial_{\hat{z}}\}
\]

and the Weyl algebra \(A^\mathfrak{g}_{\mathfrak{u}^\vee}\) is generated by

\[
\{x_1, \ldots, x_n, y_1, \ldots, y_n, z, \partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}, \partial_{z}\}.
\]
The local coordinate chart \( u_c : x \in U_c \mapsto u_c(x) \in \mathfrak{n}(\mathbb{R}) \subset \mathfrak{n} \) for the open subset \( U_c \subset G/P \), in coordinates with respect to the basis \((f_1, \ldots, f_n, g_1, \ldots, g_n, c)\) of \( \mathfrak{n} \), is given by

\[
  u_c(x) = \sum_{i=1}^{n} u_i^c(x) f_i + \sum_{i=1}^{n} u_i^c(x) g_i + u_c(x)c
\]

for all \( x \in U_c \).

Let \( (\sigma, \mathcal{V}), \sigma : \mathfrak{p} \to \mathfrak{gl}(\mathcal{V}) \), be a \( \mathfrak{p} \)-module. Then a twisted \( \mathfrak{p} \)-module \((\sigma_\lambda, \mathcal{V}_\lambda), \sigma_\lambda : \mathfrak{p} \to \mathfrak{gl}(\mathcal{V}_\lambda)\), with a twist \( \lambda \in \text{Hom}_\mathfrak{p}(\mathfrak{p}, \mathbb{C}) \), is defined by

\[
  \sigma_\lambda(X)v = \sigma(X)v + \lambda(X)v
\]

for all \( X \in \mathfrak{p} \) and \( v \in \mathcal{V}_\lambda \simeq \mathcal{V} \) (as vector spaces).

Let us introduce the notation

\[
  E_x = \sum_{j=1}^{n} x_j \partial_{x_j}, \quad E_z = z \partial_z, \quad E_y = \sum_{j=1}^{n} y_j \partial_{y_j}
\]

and

\[
  E_\hat{x} = \sum_{j=1}^{n} \hat{x}_j \partial_{\hat{x}_j}, \quad E_\hat{z} = \hat{z} \partial_{\hat{z}}, \quad E_{\hat{y}} = \sum_{j=1}^{n} \hat{y}_j \partial_{\hat{y}_j}
\]

for the Euler homogeneity operators.

**Theorem 2.1.** Let \( \lambda \in \text{Hom}_\mathfrak{p}(\mathfrak{p}, \mathbb{C}) \) and let \((\sigma, \mathcal{V})\) be a \( \mathfrak{p} \)-module. Then the embedding of \( \mathfrak{g} \) into \( A_\mathbb{R}^p \otimes \mathbb{C} \text{End} \mathcal{V}_\lambda \) and \( A_\mathbb{C}^\mathbb{C} \otimes \mathbb{C} \text{End} \mathcal{V}_\lambda \) is given by

1) \[
  \pi_\lambda(f_i) = -\partial_{x_i} + \frac{1}{2} \hat{y}_i \partial_{\hat{z}}, \\
  \pi_\lambda(g_i) = -\partial_{\hat{y}_i} - \frac{1}{2} \hat{x}_i \partial_{\hat{z}}, \\
  \pi_\lambda(c) = -\partial_{\hat{z}},
\]

2) \[
  \tilde{\pi}_\lambda(f_i) = -x_i - \frac{1}{2} z \partial_{y_i}, \\
  \tilde{\pi}_\lambda(g_i) = -y_i + \frac{1}{2} z \partial_{x_i}, \\
  \tilde{\pi}_\lambda(c) = -z
\]

for \( i = 1, 2, \ldots, n \);

2) \[
  \pi_\lambda(h) = E_{\hat{x}} + E_{\hat{y}} + 2E_{\hat{z}} + \sigma_{\lambda+\rho}(h), \]

\[
  \pi_\lambda(h_{A,B,C}) = - \sum_{i,j=1}^{n} a_{ij} (x_i \partial_{x_j} - \hat{y}_j \partial_{\hat{y}_i}) - \sum_{i,j=1}^{n} (b_{ij} \hat{y}_j \partial_{\hat{x}_i} + c_{ij} x_i \partial_{x_j}) + \sigma_{\lambda+\rho}(h_{A,B,C}),
\]

\[
  \tilde{\pi}_\lambda(h) = -E_{\hat{x}} - E_y - 2E_{\hat{z}} + \sigma_{\lambda-\rho}(h), \]

\[
  \tilde{\pi}_\lambda(h_{A,B,C}) = \sum_{i,j=1}^{n} a_{ij} (x_i \partial_{x_j} - \hat{y}_j \partial_{\hat{y}_i}) + \sum_{i,j=1}^{n} (b_{ij} x_i \partial_{\hat{y}_j} + c_{ij} y_i \partial_{x_j}) + \sigma_{\lambda-\rho}(h_{A,B,C})
\]

for \( A, B, C \in M_{n,n}(\mathbb{C}) \) satisfying \( B^T = B, C^T = C \);
relations in the Weyl algebra

Proof.\[\text{S}\text{and odd polynomials, respectively. The action of generator } s \text{ of the metaplectic Lie algebra is}\]

\[\pi_s(d_i) = 2\hat{z}\partial_{y_i} + \hat{x}_i(E_x + E_y + E_z) + \hat{x}_i \sigma_{\lambda + \rho}(h) - \sum_{j=1}^n \hat{x}_j \sigma_{\lambda + \rho}(h_{E_{ij},0}),\]

\[\pi_s(e_i) = -2\hat{z}\partial_{y_i} + \hat{y}_i(E_x + E_y + E_z) + \hat{y}_i \sigma_{\lambda + \rho}(h) + \sum_{j=1}^n \hat{y}_j \sigma_{\lambda + \rho}(h_{E_{ij},0})\]

\[\pi_s(a) = 4\hat{z}(E_x + E_y + E_z) + 4\hat{z} \sigma_{\lambda + \rho}(h) - 2 \sum_{i,j=1}^n \hat{x}_i \hat{y}_j \sigma_{\lambda + \rho}(h_{E_{ij},0})\]

\[
\hat{\pi}_s(d_i) = -2y_i \partial_z + \partial_{x_i}(E_x + E_y + E_z) - \partial_x \sigma_{\lambda - \rho}(h) + \sum_{j=1}^n \partial_{y_j} \sigma_{\lambda - \rho}(h_{E_{ij},0})
\]

\[
\hat{\pi}_s(e_i) = 2x_i \partial_z + \partial_{y_i}(E_x + E_y + E_z) - \partial_y \sigma_{\lambda - \rho}(h) + \sum_{j=1}^n \partial_{y_j} \sigma_{\lambda - \rho}(h_{E_{ij},0})
\]

\[
\hat{\pi}_s(a) = 4\partial_z(E_x + E_y + E_z) - 4\partial_z \sigma_{\lambda - \rho}(h) - 2 \sum_{i,j=1}^n \partial_{x_i} \partial_{y_j} \sigma_{\lambda - \rho}(h_{E_{ij},0})\]

for \(i = 1, 2, \ldots, n\).

The proof is a straightforward but tedious computation based on the use of commutation relations in the Weyl algebra \(A^\mathbb{C}_\mathbb{C}\) and of the fact that \(\sigma_{\lambda + \rho}: p \rightarrow gl((\mathbb{V}_{\lambda + \rho})\) is a homomorphism of Lie algebras. Let us note that a more elegant proof for a 1-dimensional representation of \(p\) easily follows from Theorem 1.3 and the proof of Theorem 3.1 in [13]. A proof of a similar formula for an arbitrary representation of \(p\) is a subject of a forthcoming work.

Now, we shall fix a realization of the simple metaplectic submodules of the Segal–Shale–Weil smooth admissible (unitarizable highest weight) representation of \(sp(2n, \mathbb{C})\) on the Schwartz space \(S(\mathbb{R}^n, \mathbb{C})\). Here we restrict to its dense subspace of \(K\)-finite vectors for \(K = U(n)\) (the underlying Harish-Chandra module), realized on the vector space \(S = \mathbb{C}[\mathbb{R}^n] \simeq \mathbb{C}[q_1, q_2, \ldots, q_n]\) with the simple metaplectic submodules of the Segal–Shale–Weil representation given by subspaces of even and odd polynomials, respectively. The action of generators of the metaplectic Lie algebra is

\[\sigma(h_{E_{ij},0}) = -q_j \partial_{q_i} - \frac{1}{2} \delta_{ij},\]

\[\sigma(h_{0, E_{ij}+E_{ij},0}) = i\partial_{q_i} \partial_{q_j},\]

\[\sigma(h_{0,0, E_{ij}+E_{ij},0}) = iq_i q_j\]

for \(i, j = 1, 2, \ldots, n\), where \(i \in \mathbb{C}\) denotes the imaginary unit. The scalar product \(\langle \cdot, \cdot \rangle: S \otimes \mathbb{C}S \rightarrow \mathbb{C}\) on \(S\) is defined through the \(\Gamma\)-equivariant embedding into the space of Schwartz functions \(\iota: S \rightarrow S(\mathbb{R}^n, \mathbb{C})\), i.e. we have

\[\langle p_1, p_2 \rangle = \int_{\mathbb{R}^n} \iota(p_1) \iota(p_2) \, dq.\]
We remark that (2.25) also defines the Segal–Shale–Weil representation on the Schwartz space and vice versa, the action (2.25) preserves the subspace of $K$-finite vectors in its smooth globalization on the Schwartz space. In order to have a more efficient weight structure for our choice of positive roots and associated Borel subalgebra, we shall work with the Fock space realization (2.25). This realization is also the most convenient one from the computational point of view and indeed, the final results are independent on the choice of a model: by exactness of globalization functors are our results independent of the realization we are working with.

We extend this representation of $\mathfrak{p} = \mathfrak{sp}(2n, \mathbb{C})$ to a representation of $\mathfrak{p}$ by the trivial action of the center $\mathfrak{z}(l)$ of $\mathfrak{l}$ and by the trivial action of the nilradical $\mathfrak{u}$ of $\mathfrak{p}$, and retain the same notation $\sigma: \mathfrak{p} \to \mathfrak{gl}(\mathbb{S})$ for the extended action of the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. In what follows, we are interested in the twisted $\mathfrak{p}$-module $\sigma_\lambda: \mathfrak{p} \to \mathfrak{gl}(\mathbb{S}_\lambda)$ with a twist $\lambda \in \text{Hom}_P(\mathfrak{p}, \mathbb{C})$.

Let us define the differential operators

$$D_s = \sum_{j=1}^n (i\eta_j \partial_{y_j} - \partial_z \partial_{q_j}), \quad E = \sum_{j=1}^n (x_j \partial_{x_j} + y_j \partial_{y_j}), \quad X_s = \sum_{j=1}^n (ix_j q_j + y_j \partial_{q_j}),$$

(2.27)

which satisfy the following commutation relations

$$[E + n, D_s] = -D_s, \quad [X_s, D_s] = i(E + n), \quad [E + n, X_s] = X_s$$

(2.28)

in $A^p_{\mathbb{R}_\mathbb{C}} \otimes \text{End} \mathbb{S}_\lambda^{-\rho}$. Hence, the complex Lie algebra generated by $D_s$, $E + n$ and $X_s$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

**Theorem 2.2.** Let $\lambda \in \text{Hom}_P(\mathfrak{p}, \mathbb{C})$. Then the embedding of $\mathfrak{g}$ into $A^p_{\mathbb{R}_\mathbb{C}} \otimes \text{End} \mathbb{S}_\lambda^{-\rho}$ is given by

$$\hat{\pi}_\lambda(f_i) = -x_i - \frac{1}{2} z \partial_{y_i},$$

$$\hat{\pi}_\lambda(g_i) = -y_i + \frac{1}{2} z \partial_{x_i},$$

$$\hat{\pi}_\lambda(c) = -z$$

(2.29)

for $i = 1, 2, \ldots, n$;

$$\hat{\pi}_\lambda(h) = -E_x - E_y - 2E_z + \lambda - (n + 1),$$

$$\hat{\pi}_\lambda(h_{A,B,C}) = \sum_{i,j=1}^n a_{ij} (x_i \partial_{x_j} - y_j \partial_{y_i}) + \sum_{i,j=1}^n (b_{ij} x_i \partial_{y_j} + c_{ij} y_i \partial_{x_j}) + \sigma(h_{A,B,C})$$

(2.30)

for $A, B, C \in M_{n,n}(\mathbb{C})$ satisfying $B^T = B$, $C^T = C$;

$$\hat{\pi}_\lambda(d_i) = -2y_i \partial_z + \partial_x (E_x + E_y + E_z - \lambda + n - \frac{1}{2}) + q_i D_s,$$

$$\hat{\pi}_\lambda(e_i) = 2x_i \partial_z + \partial_y (E_x + E_y + E_z - \lambda + n - \frac{1}{2}) - i \partial_{q_i} D_s,$$

$$\hat{\pi}_\lambda(a) = 4\partial_z (E_x + E_y + E_z - \lambda + n) + i D_s^2$$

(2.31)

for $i = 1, 2, \ldots, n$.

**Proof.** The proof is a straightforward combination of Theorem 2.1 and the Segal–Shale–Weil representation (2.25) twisted by a character $\lambda \in \text{Hom}_P(\mathfrak{p}, \mathbb{C})$.

### 3 Generalized Verma modules and singular vectors

In what follows the generators $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ of the graded commutative $\mathbb{C}$-algebra $\mathbb{C}[\mathfrak{u}^+]$ have the grading defined by $\deg(x_i) = \deg(y_i) = 1$ for $i = 1, 2, \ldots, n$ and $\deg(z) = 2$. Let us note that the choice of the grading on $(\mathfrak{u}^+)^*$ is uniquely determined by the grading on $\mathfrak{u}$ through the canonical isomorphism $(\mathfrak{u}^+)^* \simeq \mathfrak{u}$. As there is a canonical isomorphism of left $A^p_{\mathbb{R}_\mathbb{C}}$-modules

$$\mathbb{C}[\mathfrak{u}^+] \cong A^p_{\mathbb{R}_\mathbb{C}} / \mathcal{F}(I_c),$$

(3.1)
we obtain the isomorphism
\[
\tau \circ \Phi_\lambda : M^0_p(S_\lambda^\rho) \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho}, \tag{3.2}
\]
where the action of $g$ on $\mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho}$ is given by Theorem 2.2. Since $\mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho}$ and $\text{Sol}(\mathfrak{g}, \mu; \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu \subset \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho}$ are semisimple $\mathfrak{l}$-modules, we denote by $(\mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu$ and $\text{Sol}(\mathfrak{g}, \mu; \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu$ the $\mathfrak{l}$-isotypical components of highest weight $\mu \in \mathfrak{h}^\ast$.

Let us assume $R \in \text{Sol}(\mathfrak{g}, \mu; \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu$ for some $\mu \in \mathfrak{h}^\ast$. Then we have
\[
\hat{\pi}_\lambda(h)R = \mu(h)R, \tag{3.3}
\]
which is by Theorem 2.2 equivalent to
\[
(E_x + E_y + 2E_z)R = (\lambda - (n + 1) - \mu(h))R. \tag{3.4}
\]
This restricts the values of $\mu$ for which $\text{Sol}(\mathfrak{g}, \mu; \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu$ is a non-trivial vector space, because the Euler homogeneity operators $E_x$, $E_y$ and $E_z$ acting on $\mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho}$ have eigenvalues in $\mathbb{N}_0$. Consequently,
\[
(E_x + E_y + 2E_z)R = mR \tag{3.5}
\]
for some $m \in \mathbb{N}_0$, and so we may apply the solution operators
\[
\hat{\pi}_\lambda(d_i)R = 0 \quad \text{and} \quad \hat{\pi}_\lambda(e_i)R = 0 \tag{3.6}
\]
for $i = 1, 2, \ldots, n$ to polynomials $R$ of the form
\[
R = \sum_{k=0}^{[\frac{m}{2}]} z^k R_{m-2k}, \tag{3.7}
\]
where $R_{m-2k} \in \mathbb{C}[(\mathfrak{g})^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho}$ and satisfy
\[
(E_x + E_y)R_{m-2k} = (m - 2k)R_{m-2k}. \tag{3.8}
\]
for $k = 0, 1, \ldots, [\frac{m}{2}]$. Hence we obtain the recurrence relations
\[
-2(k + 1)y_x R_{m-2k-2} + (m - k + \lambda + n - \frac{1}{2})\partial_x R_{m-2k} + q_x D_x R_{m-2k} = 0, \tag{3.9}
\]
\[
2(k + 1)x_x R_{m-2k-2} + (m - k + \lambda + n - \frac{1}{2})\partial_y R_{m-2k} - i\partial_q D_x R_{m-2k} = 0 \tag{3.10}
\]
for $k = 0, 1, \ldots, [\frac{m}{2}]$, where $R_{m-2k} = 0$ for $k < 0$ and for $k > [\frac{m}{2}]$. In particular, for $k = 0$ we get
\[
-2y_x R_{m-2} + (m - \lambda + n - \frac{1}{2})\partial_x R_m + q_x D_x R_m = 0, \tag{3.11}
\]
\[
2x_x R_{m-2} + (m - \lambda + n - \frac{1}{2})\partial_y R_m - i\partial_q D_x R_m = 0. \tag{3.12}
\]

In light of the structure of the recurrence relations (3.9) and (3.10) we see that $R$ is uniquely determined by $R_m$. Therefore, we can define a linear mapping
\[
\pi_{\text{top}} : \text{Sol}(\mathfrak{g}, \mu; \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu \rightarrow (\mathbb{C}[(\mathfrak{g})^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho})^\mu \tag{3.13}
\]
by
\[
R \mapsto R_m, \tag{3.14}
\]
which is injective and $\mathfrak{l}$-equivariant.

Furthermore, the Fischer decomposition (cf. Appendix A) implies the isomorphism of vector spaces
\[
\varphi : \mathbb{C}[\mathfrak{g}^\ast] \otimes_{\mathbb{C}} S_{\lambda^\rho} \cong \bigoplus_{a,b \in \mathbb{N}_0} \mathbb{C}[z] \otimes_{\mathbb{C}} X_b^a M_a, \tag{3.15}
\]
where \( M_a = M^+_a \oplus M^-_a \) is the subspace of \( \ker D_s \) of \( a \)-homogeneous polynomials in the variables \((x_1, \ldots, x_m, y_1, \ldots, y_n)\).

**Lemma 3.1.** Let us assume \( n \geq 1 \). Then the isotypical components of the \( 1 \)-module \( \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C} S_{\lambda - \rho} \) are of the form

\[
\bigoplus_{k=0}^{|\lambda|} \mathbb{C}[z]^k \otimes \mathbb{C} X_s^{-2k} M^+_b \quad \text{and} \quad \bigoplus_{k=0}^{|\lambda|} \mathbb{C}[z]^k \otimes \mathbb{C} X_s^{-2k} M^-_b \quad (3.16)
\]

for \( a, b \in \mathbb{N}_0 \), and they are of highest weights \((\lambda - (n + 1) - a - 2b + \frac{1}{2}) \omega_1 + b \omega_2 - \frac{1}{2} \omega_{n+1} \) and \((\lambda - (n + 1) - a - 2b + \frac{1}{2}) \omega_1 + b \omega_2 + \omega_n - \frac{3}{2} \omega_{n+1} \), respectively.

**Proof.** It follows from the fact that \( \mathbb{C}[z] \otimes \mathbb{C} X_s^{-2k} M^+_b \) and \( \mathbb{C}[z] \otimes \mathbb{C} X_s^{-2k} M^-_b \) are the irreducible \( \mathfrak{h} \)-modules with highest weights \( b \omega_2 - \frac{1}{2} \omega_{n+1} \) and \( b \omega_2 + \omega_n - \frac{3}{2} \omega_{n+1} \) for \( n > 1 \), and \( b \omega_2 - \frac{1}{2} \omega_2 \) and \( b \omega_2 - \frac{1}{2} \omega_2 \) for \( n = 1 \), respectively, cf. Appendix A for the review of the results in \( \mathfrak{A} \). The rest of the proof follows from (3.8, 3.15) and Appendix A

**Lemma 3.2.** Let \( m, r \in \mathbb{N}_0 \). Then we have

\[
D_s X^r v_m = -i \frac{r(2m + 2n + r - 1)}{2} X^r X_s^{-1} v_m. 
\quad (3.17)
\]

for all \( v_m \in M_m \).

**Proof.** By (A.3), we have

\[
D_s X^r v_m = [D_s, X^r] v_m = \sum_{j=0}^{r-1} X^r X_s^{-j-1} [D_s, X_s^j] v_m = -i \sum_{j=0}^{r-1} X^r X_s^{-j-1} (E + n) X_s^j v_m
\]

\[
= -i \sum_{j=0}^{r-1} (j + m + n) X^r X_s^{-j-1} v_m = -i \frac{r(2m + 2n + r - 1)}{2} X^r X_s^{-1} v_m
\]

for all \( r, m \in \mathbb{N}_0 \) and \( v_m \in M_m \).

**Lemma 3.3.** Let \( m, r \in \mathbb{N}_0 \). Then we have

\[
\partial_s X^r v_m = i r q X^r - 1 v_m + i \frac{r(r - 1)}{2} y_i X^r - 2 v_m + X^r \partial_s v_m, 
\quad (3.18)
\]

\[
\partial_j X^r v_m = r \partial_j X^r - 1 v_m - i \frac{r(r - 1)}{2} x_i X^r - 2 v_m + X^r \partial_j v_m 
\quad (3.19)
\]

for all \( i = 1, 2, \ldots, n \) and \( v_m \in M_m \).

**Proof.** A direct computation gives \([\partial_s, X_s] = i q_i\) and \([X^r, q_i] = r y_i X^r - 1\) for all \( r \in \mathbb{N}_0 \) and \( i = 1, 2, \ldots, n \). Then

\[
\partial_s X^r v_m = \sum_{j=0}^{r-1} X^j _s \partial_s X_s^r X_s^{-j-1} v_m + X^r \partial_s v_m
\]

\[
= \sum_{j=0}^{r-1} (i q_i X^r - 1 + i [X^j, q_i] X^r - j - 1) v_m + X^r \partial_s v_m
\]

\[
= i r q_i X^r - 1 v_m + i \frac{r(r - 1)}{2} y_i X^r - 2 v_m + X^r \partial_s v_m
\]

for all \( v_m \in M_m \).

The formula (3.19) follows from \([\partial_j, X_s] = \partial_j\) and \([X^r, \partial_j] = -i r x_i X^r - 1\) for \( r \in \mathbb{N}_0 \), by analogous computation as for (3.18). The proof is complete.

Let us introduce the differential operators \( P_1, P_2 \in \mathcal{A}_0^\rho \otimes \mathbb{C} S_{\lambda - \rho} \) by

\[
P_1 = \sum_{j=1}^n (x_j \tilde{\pi}_\lambda (d_j) + y_j \tilde{\pi}_\lambda (e_j)), \quad P_2 = \sum_{j=1}^n (\partial_{x_j} \tilde{\pi}_\lambda (e_j) - \partial_{y_j} \tilde{\pi}_\lambda (d_j)). 
\quad (3.20)
\]
Lemma 3.4. The operators $P_1$ and $P_2$ have the following explicit form
\begin{align}
P_1 &= (E_x + E_y)(E_x + E_y + E_z - \lambda + n - \frac{1}{2}) - iX_sD_s, \quad (3.21) \\
P_2 &= 2\partial_x(E_x + E_y + 2n) + iD_s^2. \quad (3.22)
\end{align}

Proof. It follows from (2.31)
\begin{align*}
\sum_{j=1}^{n} x_j \hat{\pi}_\lambda(d_j) &= -\sum_{j=1}^{n} 2x_j y_j \partial_z + E_x(E_x + E_y + E_z - \lambda + n - \frac{1}{2}) + \sum_{j=1}^{n} x_j q_j D_s, \\
\sum_{j=1}^{n} y_j \hat{\pi}_\lambda(e_j) &= \sum_{j=1}^{n} 2x_j y_j \partial_z + E_y(E_x + E_y + E_z - \lambda + n - \frac{1}{2}) - i\sum_{j=1}^{n} y_j \partial_q D_s,
\end{align*}
and this implies (3.21). Similarly, by (2.31) we get
\begin{align*}
-\sum_{j=1}^{n} \partial_d \hat{\pi}_\lambda(d_j) &= 2(E_x + n)\partial_z - \sum_{j=1}^{n} \partial_x \partial_y (E_x + E_y + E_z - \lambda + n - \frac{1}{2}) - \sum_{j=1}^{n} \partial_y q_j D_s, \\
\sum_{j=1}^{n} \partial_e \hat{\pi}_\lambda(e_j) &= 2(E_x + n)\partial_z + \sum_{j=1}^{n} \partial_x \partial_y (E_x + E_y + E_z - \lambda + n - \frac{1}{2}) - i\sum_{j=1}^{n} \partial_x \partial_q D_s,
\end{align*}
and this gives (3.22). \hfill \Box

Our next step is to describe the vector space of singular vectors $\text{Sol}(g, p; \mathbb{C}[\mathfrak{u}^*] \otimes \mathbb{C}S_{\lambda-p})^F$. Since $\text{Sol}(g, p; \mathbb{C}[\mathfrak{u}^*] \otimes \mathbb{C}S_{\lambda-p})^F$ is a semisimple $I$-module, first of all we decompose $\mathbb{C}[\mathfrak{u}^*] \otimes \mathbb{C}S_{\lambda-p}$ into the isotypical components for $I$, and then we find the space of solutions of $\hat{\pi}_\lambda(d_i)$ and $\hat{\pi}_\lambda(e_i)$ for $i = 1, 2, \ldots, n$ in a given isotypical component for $I$.

By Lemma 3.4 we know the form of the isotypical components of the $I$-module $\mathbb{C}[\mathfrak{u}^*] \otimes \mathbb{C}S_{\lambda-p}$. Let us assume $R \in \text{Sol}(g, p; \mathbb{C}[\mathfrak{u}^*] \otimes \mathbb{C}S_{\lambda-p})^F_\mu$ for $\mu = (\lambda - (n + 1) - r - 2m + \frac{1}{2})/2 \omega_1 + m \omega_2 - \frac{1}{2} \omega_{n+1}$ or $\mu = (\lambda - (n + 1) - r - 2m + \frac{1}{2})/2 \omega_1 + m \omega_2 - \frac{1}{2} \omega_{n+1}$ and $r, m \in \mathbb{N}_0$, i.e.
\begin{equation}
R = \sum_{k=0}^{[\frac{|\mu|}{2}]} a_k z^k X_s^{-2k} v_m, \quad (3.23)
\end{equation}
where $a_k \in \mathbb{C}$ for $k = 0, 1, \ldots, [\frac{|\mu|}{2}]$ and each $v_m \in M_m^+$ or $v_m \in M_m$. As $R \in \text{Sol}(g, p; \mathbb{C}[\mathfrak{u}^*] \otimes \mathbb{C}S_{\lambda-p})^F$, we have $R \in \ker P_1 \cap \ker P_2$ by the construction of $P_1$ and $P_2$. Using (3.17), it easily follows from (3.21) and (3.22) that $R \in \ker P_1 \cap \ker P_2$ if and only if
\begin{equation}
((r - 2k + m)(r - k + m - \lambda + n - \frac{1}{2}) - \frac{1}{2}(r - 2k)(2m + 2n - r - 2k - 1)) a_k = 0 \quad (3.24)
\end{equation}
and
\begin{align*}
2(k + 1)(2n + m - r - 2k - 2)a_{k+1} \\
- \frac{1}{2}(r - 2k)(r - 2k - 1)(2m + 2n - r - 2k - 1)(2m + 2n - r - 2k - 2)a_k = 0 \quad (3.25)
\end{align*}
hold for all $k = 0, 1, \ldots, [\frac{|\mu|}{2}]$. Let us note that the coefficients $a_k \in \mathbb{C}$ for $k = 0, 1, \ldots, [\frac{|\mu|}{2}]$ do not depend on the vector $v_m$.

The next step is to solve the recurrence relations (3.24) and (3.25). If $a_0 = 0$, then from (3.24) we obtain $a_k = 0$ for all $k = 0, 1, \ldots, [\frac{|\mu|}{2}]$. Therefore, we may assume that $a_0 \neq 0$, and by (3.25) we have $a_k \neq 0$ for all $k = 0, 1, \ldots, [\frac{|\mu|}{2}]$.

Now, if $m = 0$ and $r = 0$, there is just one equation (3.24) for $k = 0$ and it is satisfied for any $\lambda \in \mathbb{C}$. On the other hand, if $m \neq 0$ or $r \neq 0$, then from (3.24) for $k = 0$ we obtain
\begin{equation}
\lambda = \frac{1}{2} \frac{(m + r)^2 + m(m + 2n - 1)}{m + r}. \quad (3.26)
\end{equation}
The substitution for $\lambda$ from (3.24) reduces the system of equations (3.24) for $k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$ to
\[
\frac{m(m + 2n - 1)k}{m + r} a_k = 0.
\] (3.27)

Therefore, four mutually exclusive cases have to be considered:

1) $m = 0, r = 0$ and $\lambda \in \mathbb{C}$;
2) $m = 0, r \in \mathbb{N}$ and $\lambda = \frac{1}{2} r$;
3) $m \neq 0, r = 0$ and $\lambda = m + n - \frac{1}{2}$;
4) $m \neq 0, r = 1$ and $\lambda = \frac{1}{2} (m+1)^2 + m(m+2n-1)$.

We shall work out each case separately, which is a content of the following lemma.

**Lemma 3.5.**

1) If $m = 0, r = 0$ and $\lambda \in \mathbb{C}$, then we have
\[
M_0 \subset \text{Sol}(g, p, \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho})^F
\] (3.28)
for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$.

2) If $m = 0, r \in \mathbb{N}$ and $\lambda = \frac{1}{2} r$, then we have
\[
T_{\frac{1}{2}} M_0 \subset \text{Sol}(g, p, \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho})^F
\] (3.29)
for $\lambda = \frac{1}{2} r$ and $n \in \mathbb{N}$, where the differential operator $T_{\frac{1}{2}} : \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho} \to \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho}$ for $a \in \mathbb{N}$ is defined by
\[
T_{\frac{1}{2}} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!} \left( \frac{a}{2} - \frac{1}{2} \right) \left( \frac{a}{2} - \frac{1}{2} + n \right) z^k X_s^{2k}.
\] (3.30)

3) If $m \neq 0, r = 0$ and $\lambda = m + n - \frac{1}{2}$, then we have
\[
M_m \subset \text{Sol}(g, p, \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho})^F
\] (3.31)
for $\lambda = m + n - \frac{1}{2}$ and $m, n \in \mathbb{N}$.

4) If $m \neq 0, r = 1$ and $\lambda = \frac{1}{2} (m+1)^2 + m(m+2n-1)$, then we have
\[
X_s M_m \subset \text{Sol}(g, p, \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho})^F
\] (3.32)
for $\lambda = m + \frac{1}{2}, m \in \mathbb{N}$ and $n = 1$.

**Proof.** 1) Let us assume $m = 0, r = 0$ and $\lambda \in \mathbb{C}$. Then we have
\[
R = v_0,
\]
where either $v_0 \in M_0^+$ or $v_0 \in M_0^-$. Since $v_0 \in \ker D_s$ and $\partial_x v_0 = 0$, $\partial_y v_0 = 0$ for $i = 1, 2, \ldots, n$, we obtain $\hat{\pi}_\lambda (d) R = 0$ and $\hat{\pi}_\lambda (e) R = 0$ for $i = 1, 2, \ldots, n$. Therefore, we have
\[
M_0 \subset \text{Sol}(g, p, \mathbb{C}[\mathbb{P}^1] \otimes \mathbb{S}_{\lambda - \rho})^F
\]
for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$.
2) Let us assume $m = 0$, $r \in \mathbb{N}$ and $\lambda = \frac{1}{2} r$. Then we have
\[ R = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_k z^k X_s^{r-2k} v_0, \]
where either $v_0 \in M^+_0$ or $v_0 \in M^-_0$ and $a_k \in \mathbb{C}$ for $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ satisfy the recurrence relation
\[ 2(k+1) a_{k+1} - \frac{1}{r}(r-2k)(r-2k-1) (2n+r-2k-1) a_k = 0. \quad (3.33) \]
By the recurrence relations (3.9) and (3.10), it follows that the condition $R \in \text{Sol}(g, p; \mathbb{C}[\mathbb{P}^1] \otimes_{\mathbb{C}} S_{\lambda - \rho})^F$ is equivalent to
\[
-2(k+1) a_{k+1} y_i X_s^{r-2k} v_0 + \left( \frac{1}{2} r - k + n - \frac{1}{2} \right) a_k \partial_{x_i} X_s^{r-2k} v_0 + a_k q_i D_s X_s^{r-2k} v_0 = 0,
\]
\[ 2(k+1) a_{k+1} x_i X_s^{r-2k} v_0 + \left( \frac{1}{2} r - k + n - \frac{1}{2} \right) a_k \partial_{y_i} X_s^{r-2k} v_0 - i a_k \partial_{q_i} D_s X_s^{r-2k} v_0 = 0 \]
for $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Using the formulas (3.17), (3.18) and (3.19), they reduce into
\[
( - 2(k+1) a_{k+1} + \frac{1}{4} (r-2k) (r-2k-1) (2n+r-2k-1) a_k ) y_i X_s^{r-2k} v_0 = 0,
\]
\[ (2(k+1) a_{k+1} - \frac{1}{4} (r-2k) (r-2k-1) (2n+r-2k-1) a_k ) x_i X_s^{r-2k} v_0 = 0, \]
which are satisfied due to the recurrence relation (3.33). Let us define the differential operator $T^m_r : \mathbb{C}[\mathbb{P}^1] \otimes_{\mathbb{C}} S_{\lambda - \rho} \rightarrow \mathbb{C}[\mathbb{P}^1] \otimes_{\mathbb{C}} S_{\lambda - \rho}$ for $r \in \mathbb{N}$ by
\[ T^m_r = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_k z^k X_s^{r-2k} v_0, \]
where $a_k \in \mathbb{C}$ for $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ satisfy the recurrence relation (3.33). Therefore, we have
\[ T^m_r M_0 \subset \text{Sol}(g, p; \mathbb{C}[\mathbb{P}^1] \otimes_{\mathbb{C}} S_{\lambda - \rho})^F \]
for $\lambda = \frac{1}{2} r$ and $n \in \mathbb{N}$.

3) Let us assume $m \neq 0$, $r = 0$ and $\lambda = m + n - \frac{1}{2}$. Then we have
\[ R = v_m, \]
where either $v_m \in M^+_m$ or $v_m \in M^-_m$. Since $v_m \in \ker D_s$ and $(E_x + E_y) v_m = m v_m$, we obtain that $\hat{\pi}_\lambda(d_i) R = 0$ and $\hat{\pi}_\lambda(e_i) R = 0$ for $i = 1, 2, \ldots, n$. Therefore, we have
\[ M_m \subset \text{Sol}(g, p; \mathbb{C}[\mathbb{P}^1] \otimes_{\mathbb{C}} S_{\lambda - \rho})^F \]
for $\lambda = m + n - \frac{1}{2}$ and $m, n \in \mathbb{N}$.

4) Let us assume $m \neq 0$, $r = 1$ and $\lambda = \frac{1}{2} \frac{(m+1)^2 + m(m+2n-1)}{m+1}$. Then we have
\[ R = X_s v_m, \]
where either $v_m \in M^+_m$ or $v_m \in M^-_m$. From (3.17) we have $D_s X_s v_m = -i (m + n) v_m$. By (3.18) and (3.19), we obtain $\partial_{x_i} X_s v_m = i q_i v_m + X_s \partial_{x_i} v_m$ and $\partial_{y_i} X_s v_m = \partial_{q_i} v_m + X_s \partial_{y_i} v_m$. Hence, we may write
\[
\hat{\pi}_\lambda(d_i) R = (m + 1 - \lambda + n - \frac{1}{2}) \partial_{x_i} + q_i D_s) X_s v_m = \left( \frac{m+n}{m+1} \partial_{x_i} + q_i D_s \right) X_s v_m
\]
and
\[
\hat{\pi}_\lambda(e_i) R = ((m + 1 - \lambda + n - \frac{1}{2}) \partial_{y_i} - i \partial_{q_i} D_s) X_s v_m = \left( \frac{m+n}{m+1} \partial_{y_i} - i \partial_{q_i} D_s \right) X_s v_m
\]
Therefore, we have
\[
\hat{\pi}_\lambda(d_1)R = (-imq_1 + X_s \partial_{x_1})v_m = (-imq_1 + i q_1 x_1 \partial_{x_1} + y_1 \partial_{x_1} \partial_{q_1})v_m
\]
\[
= (-imq_1 + i q_1 x_1 \partial_{x_1} + y_1 \partial_{q_1} \partial_{q_1})v_m = (-imq_1 + i q_1 (E_x + E_y))v_m = 0
\]
and
\[
\hat{\pi}_\lambda(e_1)R = (-m \partial_{q_1} + X_s \partial_{q_1})v_m = (-m \partial_{q_1} + \partial_{q_1} y_1 \partial_{q_1} + i x_1 q_1 \partial_{q_1})v_m
\]
\[
= (-m \partial_{q_1} + \partial_{q_1} y_1 \partial_{q_1} + \partial_{q_1} x_1 \partial_{x_1})v_m = (-m \partial_{q_1} + \partial_{q_1} (E_x + E_y))v_m = 0,
\]
where we used the fact that \(D_s v_m = (iq_1 \partial_{q_1} - \partial_{x_1} \partial_{q_1})v_m = 0\).

Now, if \(n \neq 1\), then we may write
\[
\partial_{q_1} \hat{\pi}_\lambda(d_1) = \frac{m+n}{m+1} (-imq_1 + i x_1 \partial_{x_1} + X_s \partial_{x_1} \partial_{q_1})v_m,
\]
\[
q_1 \hat{\pi}_\lambda(e_1) = \frac{m+n}{m+1} (-mq_1 \partial_{q_1} - y_1 \partial_{q_1} + X_s q_1 \partial_{q_1})v_m.
\]
Putting all partial results together, we get
\[
\frac{m+1}{m+n} \sum_{i=1}^{n} (\partial_{q_i} \hat{\pi}_\lambda(d_i) - i q_i \hat{\pi}_\lambda(e_i))R = (-imn + i(E_x + E_y) - X_s D_s)v_m = -i(n-1)mv_m \neq 0.
\]

Therefore, we have
\[
X_s M_m \subset \text{Sol}(g, p, C[\pi^\star] \otimes \mathbb{C} S_{\lambda - \rho})^R
\]
for \(\lambda = m + \frac{1}{2}, m \in \mathbb{N}\) and \(n = 1\). \(\square\)

**Theorem 3.6.**

1) If \(n = 1\), then we have
\[
\tau \circ \Phi_{\lambda + \rho}: M_p^g(S_\lambda)^u \rightarrow \begin{cases}
M_0, & \text{if } \lambda + 2 \notin \frac{1}{2}\mathbb{N}, \\
M_0 + T_{1(\lambda + 2)}^1 M_0, & \text{if } \lambda + 1 \in \mathbb{N}_0, \\
M_0 + T_{2(\lambda + 2)}^1 M_0, & \text{if } \lambda = -\frac{3}{2}, \\
M_0 + M_{\lambda + \frac{1}{2}} + X_s M_{\lambda + \frac{1}{2}} + T_{2(\lambda + 2)}^1 M_0, & \text{if } \lambda + \frac{1}{2} \in \mathbb{N}_0.
\end{cases}
\]

2) If \(n \geq 2\), then we have
\[
\tau \circ \Phi_{\lambda + \rho}: M_p^g(S_\lambda)^u \rightarrow \begin{cases}
M_0, & \text{if } \lambda + n + 1 \notin \frac{1}{2}\mathbb{N}, \\
M_0 + T_{2(\lambda + n + 1)}^1 M_0, & \text{if } \lambda + n \in \mathbb{N}_0, \\
M_0 + T_{2(\lambda + n + 1)}^n M_0, & \text{if } \lambda + n + \frac{1}{2} \in \mathbb{N}_0, \lambda + \frac{1}{2} \notin \mathbb{N}_0, \\
M_0 + M_{\lambda + \frac{1}{2}} + T_{2(\lambda + n + 1)}^1 M_0, & \text{if } \lambda + \frac{1}{2} \in \mathbb{N}_0.
\end{cases}
\]

**Proof.** The decomposition of the space of singular vectors \(M_p^g(\mathcal{V})^u\) is an easy consequence of Lemma 3.5. \(\square\)

Now we illustrate the general results given in Theorem 3.6, and write down explicit formulas for the corresponding homomorphisms between generalized Verma modules in several examples.

Let us assume \(\lambda = -(n+1 - \frac{a}{2})\omega_1\) and \(\mu = -(n+1 + \frac{a}{2})\omega_1\) for \(a \in \mathbb{N}\). Then Theorem 3.6 implies that
\[
T_{\sigma}^n(S_\lambda) \subset (\tau \circ \Phi_{\lambda + \rho})(M_p^g(S_\lambda)^u)
\]
(3.34)
for \( n \in \mathbb{N} \), and moreover \( T_e^n(S_\lambda) \cong S_\mu \) as \( \mathfrak{p} \)-modules as follows from Lemma 3.1. Therefore, there exists a homomorphism

\[
\varphi: M_\mathfrak{p}^\theta(S_\mu) \rightarrow M_\mathfrak{p}^\theta(S_\lambda)
\]

(3.35)
of generalized Verma modules, uniquely determined by a \( \mathfrak{p} \)-homomorphism \( \varphi_0: S_\mu \rightarrow M_\mathfrak{p}^\theta(S_\lambda) \) through the formula

\[
\varphi(u \otimes v) = u\varphi_0(v)
\]

(3.36)
for all \( u \in U(\mathfrak{g}) \) and \( v \in S_\mu \). Since \( S_\mu \) is isomorphic to \( S_\lambda \) as \( \mathfrak{p} \)-modules, we may set

\[
\varphi_0(v) = (\tau \circ \Phi_{\mathfrak{p}+\rho})^{-1}(T_e^n v)
\]

(3.37)
for all \( v \in S_\mu \cong S_\lambda \) (as \( \mathfrak{p} \)-modules) and we obtain the required homomorphism of \( \mathfrak{p} \)-modules. Through the formula (3.30) we have

\[
T_e^n = \sum_{k=0}^{n} \frac{1}{k!} \alpha k^2 X_s^{a-2k} = \sum_{k=0}^{n} \frac{1}{k!} \left( \frac{n}{k} \right) \left( \frac{n-1}{k} \right) \left( \frac{n-2}{k} \right) \ldots \frac{n-k}{k} X_s^{a-2k}.
\]

(3.38)
We denote by

\[
Q_s = -\sum_{j=1}^{n}(if_j q_j + g_j \partial_{q_j})
\]

(3.39)
an element in \( S(\mathfrak{g}) \otimes \mathbb{C} \) \( \text{End} S_\lambda \). Then from (1.11) and Theorem 2.2 we immediately obtain

\[
(\tau \circ \Phi_{\mathfrak{p}+\rho})(\beta \otimes \text{id}_{S_\lambda})(e^m Q_s^k v) = (-1)^m z^m X_s^k v
\]

(3.40)
for all \( m, k \in \mathbb{N}_0 \) and \( v \in S_\lambda \), which gives

\[
(\tau \circ \Phi_{\mathfrak{p}+\rho})^{-1}(T_e^n v) = \sum_{k=0}^{n} (-1)^k \alpha_k (\beta \otimes \text{id}_{S_\lambda})(e^k Q_s^{a-2k} v)
\]

(3.41)
\[
= \sum_{k=0}^{n} (-1)^k \alpha_k e^k (\beta \otimes \text{id}_{S_\lambda})(Q_s^{a-2k} v)
\]

for all \( v \in S_\lambda \), where we used that \( c \in \mathfrak{h}(\mathfrak{g}) \) in the last equality. Let us denote by

\[
P_s = \sum_{j=1}^{n}(if_j q_j + g_j \partial_{q_j})
\]

(3.42)
an element in \( U(\mathfrak{g}) \otimes \mathbb{C} \) \( \text{End} S_\lambda \). Then a straightforward computation gives

\[
(\beta \otimes \text{id}_{S_\lambda})(Q_s v) = P_s v,
\]

(3.43)
\[
(\beta \otimes \text{id}_{S_\lambda})(Q_s^2 v) = \left( P_s^2 - \frac{1}{2} n c \right) v,
\]

(3.44)
\[
(\beta \otimes \text{id}_{S_\lambda})(Q_s^3 v) = \left( P_s^3 - \frac{1}{6} (3n + 1) P_s c \right) v,
\]

(3.45)
\[
(\beta \otimes \text{id}_{S_\lambda})(Q_s^4 v) = \left( P_s^4 - \frac{1}{2} (6n + 4) P_s^2 c - \frac{1}{4} (3n^2 + 3n) c^2 \right) v
\]

(3.46)
for all \( v \in S_\lambda \). Finally, using (3.37), (3.41) and (3.38) we obtain the following explicit formulas for the homomorphisms \( \varphi_0: S_\mu \rightarrow M_\mathfrak{p}^\theta(S_\lambda) \) of \( \mathfrak{p} \)-modules.

1) If \( a = 1 \), then

\[
\varphi_0(v) = P_s v, \quad v \in S_\mu.
\]

(3.47)
2) If \( a = 2 \), then
\[
\varphi_0(v) = (P_s^2 - i(n + \frac{1}{2})c)v, \quad v \in S_\mu. \tag{3.48}
\]

3) If \( a = 3 \), then
\[
\varphi_0(v) = (P_s^3 - i(3n + 2)cP_s)v, \quad v \in S_\mu. \tag{3.49}
\]

4) If \( a = 4 \), then
\[
\varphi_0(v) = (P_s^4 - i(6n + \frac{13}{2})cP_s^2 - (3n^2 + \frac{9n}{4} + \frac{9}{16})c^2)v, \quad v \in S_\mu. \tag{3.50}
\]

4 Equivariant differential operators on generalized flag manifolds

In the present section we retain the notation introduced in Section 1. Let us recall a few basic facts concerning infinite-dimensional complex representations of finite-dimensional real Lie groups, see [5]. For a complex continuous representation \((\sigma, \mathcal{V})\) of \( P \), a vector \( v \in \mathcal{V} \) is called smooth if the mapping \( p \mapsto \sigma(p)v \) from \( P \) to \( \mathcal{V} \) is smooth. A continuous representation \((\sigma, \mathcal{V})\) of \( P \) is smooth provided all vectors in \( \mathcal{V} \) are smooth, and in particular the vector subspace \( \mathcal{V}^{\infty} \subset \mathcal{V} \) of all smooth vectors is a continuous representation both for \( P \) and \( p \). The topological dual of a smooth representation \((\sigma, \mathcal{V})\) of \( P \) is a representation on the space of tempered distributional vectors \((\mathcal{V}^*)^{-\infty}\) in the weak linear dual \( \mathcal{V}^* \) of \( \mathcal{V} \). We define the contragredient representation of \( P \) as the smooth representation \((\sigma^*, \mathcal{V}^*)\) on the subspace of smooth vectors \( \mathcal{V}^* = (\mathcal{V}^*)^{\infty} \subset (\mathcal{V}^*)^{-\infty} \), where the canonical pairing \( \langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V}^* \to \mathbb{C} \) is non-degenerate. The previous discussion is a summary of the standard results by Casselman, Wallach, and Schmid on the globalization of representations, cf. [4], [21].

Given a smooth \( P \)-module \((\sigma, \mathcal{V})\), we consider the induced representation of \( G \) on the space \( \text{Ind}_P^G(\mathcal{V}) \) of smooth sections of the homogeneous vector bundle \( \mathcal{V} = G \times_P \mathcal{V} \to G/P \) identified with
\[
\mathcal{C}^{\infty}(G, \mathcal{V})^P = \{ f \in \mathcal{C}^{\infty}(G, \mathcal{V}) : f(gp) = \sigma(p^{-1})f(g) \text{ for all } g \in G, p \in P \}. \tag{4.1}
\]
We denote by \( J^k_e(G, \mathcal{V})^P \) the space of \( k \)-jets supported at \( e \in G \) of \( P \)-equivariant smooth mappings \( f \in \mathcal{C}^{\infty}(G, \mathcal{V})^P \) for \( k \in \mathbb{N}_0 \), and by \( J^\infty_e(G, \mathcal{V})^P \) its projective limit
\[
J^\infty_e(G, \mathcal{V})^P = \lim_{\rightarrow k} J^k_e(G, \mathcal{V})^P. \tag{4.2}
\]
A well-known fact, usually stated for a complex finite-dimensional \( P \)-module \((\sigma, \mathcal{V})\), is the existence of a non-degenerate \((g, P)\)-equivariant pairing between \( J^\infty_e(G, \mathcal{V})^P \) and \( M^*_P(\mathcal{V}^*) \), see [6], [22] for independent proofs. This pairing identifies \( M^*_P(\mathcal{V}^*) \) with the vector space of all \( \mathbb{C} \)-linear mappings \( J^\infty_e(G, \mathcal{V})^P \to \mathbb{C} \) that factor through \( J^k_e(G, \mathcal{V})^P \) for some \( k \in \mathbb{N}_0 \).

In the following proposition we give a straightforward generalization of the statement made in the previous paragraph for finite-dimensional inducing \( P \)-modules. For the reader’s convenience, we give a proof of this fact. Let us note that the proof goes along the same lines as the proof for finite-dimensional inducing \( P \)-modules. However, some of the specific results used in the proof of the claim are non-trivial for infinite-dimensional inducing \( P \)-modules, and in particular, hold only for smooth inducing \( P \)-modules, cf. the manuscript [5].

We denote by \( R_g \) and \( L_g \) the right and left regular action of \( g \in G \) on \( \mathcal{C}^{\infty}(G, \mathcal{V}) \) defined by
\[
(R_g(f))(h) = f(hg) \quad \text{and} \quad (L_g(f))(h) = f(g^{-1}h) \tag{4.3}
\]
for all $h \in G$, respectively. The right and left regular representation of $G$ on $C^\infty(G, V)$ induces the right and left regular representation of the universal enveloping algebra $U(g)$ on $C^\infty(G, V)$ defined by

$$ (R_X(f))(g) = \frac{d}{dt} \big|_{t=0} f(g \exp(tX)) \quad \text{and} \quad (L_X(f))(g) = \frac{d}{dt} \big|_{t=0} f(\exp(-tX)g) \quad (4.4) $$

for $X \in g(\mathbb{R})$, where $g \in G$, $f \in C^\infty(G, V)$, and extended to $U(g)$ as representations of associative $\mathbb{C}$-algebras, respectively. We will denote $R_u$ and $L_u$ for $u \in U(g)$. Moreover, we have a $\mathbb{C}$-bilinear mapping

$$ \langle \cdot, \cdot \rangle : C^\infty(G, V) \times \mathcal{V}' \to C^\infty(G, \mathbb{C}) \quad (4.5) $$

given by

$$ \langle f, \alpha \rangle (g) = \alpha(f(g)) \quad (4.6) $$

for all $g \in G$.

**Proposition 4.1.** Let $\mathcal{V}$ be a smooth $P$-module. The bilinear mapping

$$ \Phi_\mathcal{V} : C^\infty(G, \mathcal{V}) \times (U(g) \otimes \mathcal{V}') \to C^\infty(G, \mathbb{C}) \quad (4.7) $$

defined by

$$ \Phi_\mathcal{V}(f, u \otimes \alpha) = (R_u(f), \alpha) = R_u((f, \alpha)), \quad (4.8) $$

where $f \in C^\infty(G, \mathcal{V})$, $\alpha \in \mathcal{V}'$ and $u \in U(g)$, induces a bilinear mapping

$$ \Phi_\mathcal{V} : C^\infty(G, \mathcal{V})^P \times M^2_\mathbb{C}(\mathcal{V}') \to C^\infty(G, \mathbb{C}). \quad (4.9) $$

Moreover, the composition of $\Phi_\mathcal{V}$ with the evaluation map at $e \in G$

$$ C^\infty(G, \mathcal{V})^P \times M^2_\mathbb{C}(\mathcal{V}') \to \mathbb{C} \quad (4.10) $$

$$(f, u \otimes \alpha) \mapsto \Phi_\mathcal{V}(f, u \otimes \alpha)(e)$$

is $(g, P)$-equivariant.

**Proof.** Let us denote by $I(g, p, \mathcal{V}')$ the $(g, P)$-submodule of $U(g) \otimes \mathcal{V}'$, generated by complex subspace

$$ (X \otimes \alpha - 1 \otimes \sigma(X)\alpha; X \in p, \alpha \in \mathcal{V}') \subset U(g) \otimes \mathcal{V}' $$

If $f \in C^\infty(G, \mathcal{V})^P$ and $X \in p$, then we have $R_X(f) = -\sigma(X)f$. Hence, we may write

$$ \Phi_\mathcal{V}(f, uX \otimes \alpha) = R_u((R_X(f), \alpha)) = R_u((-\sigma(X)f, \alpha)) 
= R_u((f, \sigma^*(X)\alpha)) = \Phi_\mathcal{V}(f, u \otimes \sigma^*(X)\alpha) $$

for all $u \in U(g)$ and $\alpha \in \mathcal{V}'$, which implies that $\Phi_\mathcal{V}(f, I(g, p, \mathcal{V}')) = 0$ for all $f \in C^\infty(G, \mathcal{V})^P$.

Further, we prove that the bilinear mapping $\Phi_\mathcal{V}$ is $(g, P)$-equivariant. Let $f \in C^\infty(G, \mathcal{V})^P$, $u \in U(g)$ and $\alpha \in \mathcal{V}'$. Then we have

$$ \Phi_\mathcal{V}(L_X(f), u \otimes \alpha) = L_X \Phi_\mathcal{V}(f, u \otimes \alpha), $$

$$ \Phi_\mathcal{V}(f, Xu \otimes \alpha) = R_X \Phi_\mathcal{V}(f, u \otimes \alpha) $$

for all $X \in g$, where we used $[L_X, R_u] = 0$. Since we have

$$ (R_X(f) + L_X(f))(e) = 0, \quad (4.11) $$

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we get

$$\Phi_V(L_X(f), u \otimes \alpha)(e) + \Phi_V(f, Xu \otimes \alpha)(e) = 0,$$

which means that (4.10) is g-equivariant. Furthermore, we may write

$$\Phi_V(f, \text{Ad}(p)u \otimes \sigma^*(p)\alpha) = R_{\text{Ad}(p)u}(\langle f, \sigma^*(p)\alpha \rangle) = R_{\text{Ad}(p)u}(\langle \sigma(p^{-1})f, \alpha \rangle)$$

$$= R_{\text{Ad}(p)u}(\langle R_p(f), \alpha \rangle) = (R_{\text{Ad}(p)u} \circ R_p)(\langle f, \alpha \rangle)$$

$$= (R_p \circ R_u)(\langle f, \alpha \rangle) = R_p\Phi_V(f, u \otimes \alpha)$$

for $p \in P$, where we used $R_p(f) = \sigma(p^{-1})f$ and $R_{\text{Ad}(p)u} \circ R_p = R_p \circ R_u$. Finally, since we have

$$\Phi_V(L_{p^{-1}}(f), u \otimes \alpha) = L_{p^{-1}}\Phi_V(f, u \otimes \alpha),$$

we obtain

$$\Phi_V(L_{p^{-1}}(f), u \otimes \alpha)(e) - \Phi_V(f, \text{Ad}(p)u \otimes \sigma^*(p)\alpha)(e) = \Phi_V(f, u \otimes \alpha)(p) - \Phi_V(f, u \otimes \alpha)(p) = 0,$$

which means that (4.10) is $P$-equivariant. The proof is complete. 

Let us note that the $(g, P)$-equivariant bilinear mapping (4.10) identifies the generalized Verma module $M_p^0(\mathbb{V}^\vee)$ with the vector space of all $\mathbb{C}$-linear mappings $J^\infty_0(G, \mathbb{V})^P \to \mathbb{C}$ that factor through $J^\infty_0(G, \mathbb{V})^P$ for some $k \in \mathbb{N}_0$. Another remark is that the duality in question is equivalent to a more geometric construction of invariant differential jet operator.

**Corollary 4.2.** Let $\mathbb{V}$ and $\mathbb{W}$ be smooth $P$-modules. Then there is a bijection between $(g, P)$-equivariant homomorphisms of generalized Verma modules

$$\varphi: M_p^0(\mathbb{W}^\vee) \to M_p^0(\mathbb{V}^\vee),$$

(4.11)

and $G$-equivariant differential operators

$$D: C^\infty(G, \mathbb{V})^P \to C^\infty(G, \mathbb{W})^P$$

(4.12)

given by

$$\langle D(f), \beta \rangle = \Phi_V(f, \varphi(1 \otimes \beta))$$

(4.13)

for $f \in C^\infty(G, \mathbb{V})^P$ and $\beta \in \mathbb{W}^\vee$.

We note that the proof of both injectivity and surjectivity is completely parallel to the case of finite-dimensional inducing representations. Now we apply this result to the case of our interest studied from the algebraic perspective in Section 2 and Section 3.

Let us notice that we could have defined an action of equivariant differential operators on sections valued in the representations given by tempered distributional vectors rather than its dense subspace of smooth vectors. However, the subspace of smooth vectors is sufficient for the construction of the required duality in Corollary 4.2 and it is also more natural to work with in various differential geometrical application. Moreover, for the Dirac operator in symplectic geometry, the virtue of this step was stressed in [17].

### 4.1 Equivariant differential operators on symplectic spinors in the contact projective geometry

By [23], there is a double cover of the maximal parabolic subgroup of the symplectic group $Sp(2n + 2, \mathbb{R})$ to the metaplectic group $Mp(2n + 2, \mathbb{R})$, which splits over the unipotent radical $H(n, \mathbb{R})$ in the Langlands–Wassawa decomposition of this parabolic subgroup. The group $Mp(2n + 2, \mathbb{R})$ has the maximal parabolic subgroup with the Levi subgroup isomorphic to $GL(1, \mathbb{R}) \times Mp(2n, \mathbb{R})$. We note that the extension cocycle splits over the field of complex numbers.
Let us consider \( \tilde{G} = \text{Mp}(2n + 2, \mathbb{R}) \) for \( n \in \mathbb{N} \), and \( \tilde{P} \subset \tilde{G} \) its maximal parabolic subgroup with the Levi subgroup isomorphic to \( \text{GL}(1, \mathbb{R}) \times \text{Mp}(2n, \mathbb{R}) \) and the unipotent radical in the Langlands–Iwasawa decomposition of \( \tilde{P} \) isomorphic to the Heisenberg group \( H(n, \mathbb{R}) \). As for the representation we take the simple representations of the metaplectic group \( \text{Mp}(2n, \mathbb{R}) \) of the Segal–Shale–Weil representation \( S \), extended to a representation of \( \tilde{P} \) by the trivial action of \( \text{GL}(1, \mathbb{R}) \) and \( H(n, \mathbb{R}) \). Assuming that \( \lambda \in \text{Hom}_P(p, \mathbb{C}) \) defines a group character \( e^\lambda : P \to \text{GL}(1, \mathbb{C}) \), we denote by \( S_\lambda \) the corresponding twisted representation of \( \tilde{P} \) with a twist \( \hat{\lambda} \in \text{Hom}_P(p, \mathbb{C}) \). Here \( S_\lambda \) is understood as a smooth globalization of the corresponding Harish-Chandra module.

A non-degenerate pairing \( \langle \cdot, \cdot \rangle : S(\mathbb{R}^n, \mathbb{C}) \otimes_\mathbb{C} S \to \mathbb{C} \) defined by the formula

\[
(f(q), p(q)) = \int_{\mathbb{R}^n} f(q)p(q) \, dq,
\]

where \( q = (q_1, q_2, \ldots, q_n) \), identifies \( S^\vee \) with \( S(\mathbb{R}^n, \mathbb{C}) \). The generators of \( \mathfrak{p} \) act in the contragredient \( \mathfrak{p} \)-module on \( S^\vee \cong S(\mathbb{R}^n, \mathbb{C}) \)

\[
\sigma^*(h_{E_{ij},0,0}) = -q_j \partial_{q_i} - \frac{1}{2} \delta_{ij},
\]

\[
\sigma^*(h_{0,E_{ij}+E_{ji},0}) = -i\partial_{q_i} \partial_{q_j},
\]

\[
\sigma^*(h_{0,0,E_{ij}+E_{ji}}) = -iq_jq_j
\]

for \( i, j = 1, 2, \ldots, n \), and the generators of the center \( j(1) \) of \( \mathfrak{l} \) and of the nilradical \( \mathfrak{u} \) of \( \mathfrak{p} \) act trivially. Then for \( V = S_\lambda \) we take \( V^\vee = S(\mathbb{R}^n, \mathbb{C})_{-\lambda} \), and denote by \( \pi^\lambda_\mathfrak{s} \) the embedding of \( \mathfrak{g} \) into \( \mathfrak{a}_{\mathbb{R}}^0 \otimes_\mathbb{C} \text{End} S^\vee_{\lambda+\rho} \) (see Theorem 2.1) for the \( \mathfrak{p} \)-module \( (\sigma^\lambda_{\lambda+\rho}, S^\vee_{\lambda+\rho}) \).

**Theorem 4.3.** The singular vectors in Theorem 3.3 (2) correspond to the \( \tilde{G} \)-equivariant differential operators, given in the non-compact picture of the induced representations as follows.

1) Let \( \lambda = -(n + 1 + \frac{a}{2})\tilde{a}_1 \) and \( \mu = -(n + 1 + \frac{a}{2})\tilde{a}_1 \) for \( a \in \mathbb{N} \). Then there are equivariant differential operators

\[
D_a : C^\infty(\mathfrak{g}(\mathbb{R}), S^\vee_{-\lambda}) \to C^\infty(\mathfrak{g}(\mathbb{R}), S^\vee_{-\mu})
\]

of order \( a \in \mathbb{N} \). The infinitesimal intertwining property of \( D_a \) is

\[
D_a \pi^\lambda_{-\frac{a}{2}}(X) = \pi^\mu_{\frac{a}{2}}(X)D_a
\]

for all \( X \in \mathfrak{g} \). We call these operators contact powers of the contact symplectic Dirac operator.

2) Let \( \lambda = -\left(\frac{a}{2} - a\right)\tilde{a}_1 \) for \( a \in \mathbb{N} \). Then there are equivariant differential operators

\[
T_a : C^\infty(\mathfrak{a}(\mathbb{R}), S^\vee_{-\lambda}) \to C^\infty(\mathfrak{a}(\mathbb{R}), (M_{\lambda+\frac{a}{2}})^\vee)
\]

of order \( a \in \mathbb{N} \). We call these operators twistor operators on symplectic spinors.

**Proof.** The existence of \( \tilde{G} \)-equivariant differential operators easily follows from Theorem 3.3 and Corollary 1.2.

The remaining collection of singular vectors corresponds to the \( \tilde{G} \)-equivariant differential operator given by a multiple of the identity map. Analogous statement as the last theorem can be made in the case \( n = 1 \) using Theorem 3.3 (1).

Let us define

\[
D_x = \sum_{j=1}^n (iq_j \partial\hat{x}_j - \partial q_j \partial \hat{x}_j) \quad \text{and} \quad X = \sum_{j=1}^n (i\hat{x}_j q_j + \hat{x}_j \partial q_j),
\]

(4.19)
then we can write down explicit formulas for the operators $D_1, D_2, D_3, D_4$ in the form

\[
\begin{align*}
D_1 &= D_s + \frac{i}{2} X_s \partial_z, \\
D_2 &= (D_s + \frac{i}{2} X_s \partial_z)^2 - \frac{i}{4} \partial_z, \\
D_3 &= (D_s + \frac{i}{2} X_s \partial_z)((D_s + \frac{i}{2} X_s \partial_z)^2 - i \partial_z), \\
D_4 &= ((D_s + \frac{i}{2} X_s \partial_z)^2 - \frac{i}{4} \partial_z)((D_s + \frac{i}{2} X_s \partial_z)^2 - i \frac{9}{4} \partial_z)
\end{align*}
\]

(4.20)

in the case $n \geq 1$.

The examples of $G$-equivariant differential operators $D_a$ for $a = 1, 2, 3, 4$ suggest the existence of the following remarkable conjectural factorization property of $D_a$, $a \in \mathbb{N}$,

\[
\begin{align*}
D_{2k+2} &= \prod_{j=0}^{k} \left( (D_s + \frac{i}{2} X_s \partial_z)^2 - i \left( \frac{2(j+1)}{4} \right) \partial_z \right), \\
D_{2k+1} &= (D_s + \frac{i}{2} X_s \partial_z) \prod_{j=1}^{k} \left( (D_s + \frac{i}{2} X_s \partial_z)^2 - i j^2 \partial_z \right)
\end{align*}
\]

(4.21) (4.22)

for $k \in \mathbb{N}_0$. We do not know a direct proof of this observation.

## Appendix A  The Fischer decomposition for $\mathfrak{sp}(2n, \mathbb{C})$

Throughout the article, we needed a rather precise information about the simple part $\mathfrak{t}^* \simeq \mathfrak{sp}(2n, \mathbb{C})$ of the Levi subalgebra $\mathfrak{l}$-module structure on the generalized Verma module $M_{\lambda}^\mathbb{H}(\mathfrak{S}_{\lambda-\rho})$ for $\lambda \in \text{Hom}_P(\mathfrak{p}, \mathbb{C})$. Let us first mention that as discussed beyond Definition 1 in Section II the present decomposition holds for both the Harish-Chandra module $\mathfrak{S}_{\lambda-\rho}$ and its smooth globalization. It can be described in terms of the Howe duality for the pair $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$ acting on $\mathbb{C}[\mathfrak{f}^*] \otimes_{\mathbb{C}} \mathfrak{S}_{\lambda-\rho}$ by (2.30) for $\mathfrak{sp}(2n, \mathbb{C})$ and by

\[
\begin{align*}
D_s &= \sum_{j=1}^{n} (i q_j \partial_{y_j} - \partial x_j \partial_{q_j}), \\
E &= \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j}), \\
X_s &= \sum_{j=1}^{n} (i x_j q_j + y_j \partial_{q_j})
\end{align*}
\]

(A.1)

for $\mathfrak{sl}(2, \mathbb{C})$. For a detailed exposition, see [3]. This decomposition can be schematically represented as

\[
\mathbb{C}[\mathfrak{g}^{-1}] \otimes \mathfrak{S}_{\lambda-\rho} \simeq \bigoplus_{a,b \in \mathbb{N}_0} X^a_s M_b, \quad M_b = (\mathbb{C}[\mathfrak{g}^{-1}] \otimes \mathfrak{S}_{\lambda-\rho}) \cap \ker D_s.
\]

(A.2)

\[
\begin{array}{cccccccc}
P_0 \otimes \mathbb{S} & P_1 \otimes \mathbb{S} & P_2 \otimes \mathbb{S} & P_3 \otimes \mathbb{S} & P_4 \otimes \mathbb{S} & P_5 \otimes \mathbb{S} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
M_0 \longrightarrow X_s M_0 \longrightarrow X^2_s M_0 \longrightarrow X^3_s M_0 \longrightarrow X^4_s M_0 \longrightarrow X^5_s M_0 \\
\oplus & \oplus & \oplus & \oplus & \oplus & \\
M_1 \longrightarrow X_s M_1 \longrightarrow X^2_s M_1 \longrightarrow X^3_s M_1 \longrightarrow X^4_s M_1 \\
\oplus & \oplus & \oplus & \oplus & \\
M_2 \longrightarrow X_s M_2 \longrightarrow X^2_s M_2 \longrightarrow X^3_s M_2 \\
\oplus & \oplus & \\
M_3 \longrightarrow X_s M_3 \longrightarrow X^2_s M_3 \\
\oplus & \\
M_4 \longrightarrow X_s M_4 \oplus \\
\oplus & \\
M_5
\end{array}
\]
We used the notation $P_b$ for the $b$-homogeneous polynomials instead of $\mathbb{C}[(g_{-1})^*]_b$ in the last picture. The operators $D_s$ and $X_s$ act in the previous picture horizontally, but in the opposite direction, $E$ preserves each simple symplectic module in the decomposition and the $\mathfrak{sl}(2,\mathbb{C})$-commutation relations in $A^q_{\mathbb{R}} \otimes \mathbb{C} \text{End} S_{\lambda-\rho}$ are

$$\left[ E + n, D_s \right] = -D_s, \quad \left[ X_s, D_s \right] = i(E + n), \quad \left[ E + n, X_s \right] = X_s. \quad (A.3)$$

For the operator $a \otimes \text{id}_{S_{\lambda-\rho}} \in A^q_{\mathbb{R}} \otimes \mathbb{C} \text{End} S_{\lambda-\rho}$ with $a \in \mathbb{C}$, we use the shorthand notation $a$. Let us note that the operators $X_s, D_s$ and $E$ commute with the action of $\mathfrak{sp}(2n,\mathbb{C})$ on $\mathbb{C}[\mathfrak{H}] \otimes \mathbb{C} S_{\lambda-\rho}$.

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