A CAHN-HILLIARD-NAVIER-STOKES MODEL WITH DELAYS

T. Tachim Medjo
Department of Mathematics, Florida International University, DM413B
University Park
Miami, Florida 33199, USA
(Communicated by Shouhong Wang)

Abstract. In this article, we study a coupled Cahn-Hilliard-Navier-Stokes model with delays in a two-dimensional domain. The model consists of the Navier-Stokes equations for the velocity, coupled with an Cahn-Hilliard model for the order (phase) parameter. We prove the existence and uniqueness of the weak and strong solution when the external force contains some delays. We also discuss the asymptotic behavior of the weak solutions and the stability of the stationary solutions.

1. Introduction. It is well accepted that the incompressible Navier-Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [9]. For instance, this approach is used in [1] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, [2, 9, 8, 10]. In the isothermal compressible case, the existence of a global weak solution is proved in [7]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity $v$ and the order parameter $\phi$. This system can be written as a NS equation coupled with a convective Allen-Cahn equation, [9]. The associated initial and boundary value problem was studied in [9] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [9] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [15]. In the case of binary fluids, the analysis is even more complicate and the mathematical study is still at its infancy as noted in [9]. As noted in [8], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then a shear stage in which these patters organize themselves into parallel layers (see, e.g. [13] for experimental snapshots). This model has to take

2010 Mathematics Subject Classification. 35Q30, 35Q35, 35Q72.
Key words and phrases. Cahn-Hilliard-Navier-Stokes, delays, strong solutions, stability.
into account the chemical interactions between the two phases at the interface, achieved using a Cahn-Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier-Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called “Model H” (cf. [11]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity $v$ is coupled with a convective Cahn-Hilliard equation for the order parameter $\phi$, which represents the relative concentration of one of the fluids.

In [4, 5, 6], the authors studied the NS equations in which the forcing term contains some hereditary features. The model can be used for instance to control a system by applying a force which takes into account not only the present state of the system, but also the history of the solutions. The existence and uniqueness of solutions to the 2D NS equations with delays was investigated in [4] and the asymptotic behavior of the solutions is studied in [5]. The existence of attractors for the 2D NS equations with delays is proved in [6]. In [3], the authors studied the existence of an attractor for the 3D Lagrangian averaged Navier-Stokes $\alpha$ – (3D LAN-$\alpha$) model with delays. Instead of working directly with the 3D LAN-$\alpha$ model, they proved the existence of attractors for an abstract delay model and then applied the result to the 3D LAN-$\alpha$ model.

In this article, we study an CH-NS model with delays. We prove the existence and uniqueness of a weak and a strong solutions when the external force contains some delays. Let us note that the coupling between the Navier-Stokes and the Cahn-Hilliard systems makes the analysis more involved.

The article is divided as follows. In the next section, we introduce the CH-NS model with delays and its mathematical setting. The third section studies the existence of solutions when the delay term satisfies some hypothesis similar to that of [4, 5, 6]. In the fourth section, we study the asymptotic behavior of the weak solutions when the delay term satisfies some hypothesis used in [14]. The stability of the stationary solutions is analyzed in the fifth section.

2. The CH-NS model and its mathematical setting.

2.1. Governing equations. In this article, we study a 2D non-autonomous Cahn-Hilliard-Navier-Stokes system. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\mathbb{R}^2$. Then, we consider the system

$$
\begin{align*}
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla p - K\mu \nabla \phi &= Q(t - \tau(t), (v, \phi)(t - \tau(t))), \\
\text{div } v &= 0, \\
\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi - \Delta \mu &= 0, \\
\mu &= -\epsilon \Delta \phi + \alpha f(\phi),
\end{align*}
$$

in $\mathcal{M} \times (0, +\infty)$.

In (2.1), the unknown functions are the velocity $v = (v_1, v_2)$ of the fluid, its pressure $p$ and the order (phase) parameter $\phi$. The quantity $\mu$ is the variational
derivative of the following free energy functional

\[ F(\phi) = \int_M \left( \frac{\epsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds, \]

(2.2)

where, e.g., \( F(r) = \int_0^r f(\zeta) d\zeta \). Here, the constants \( \nu > 0, \epsilon > 0 \) and \( K > 0 \) correspond to the kinematic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here \( \epsilon, \alpha > 0 \) are two physical parameters describing the interaction between the two phases. In particular, \( \epsilon \) is related with the thickness of the interface separating the two fluids, \[8\].

A typical example of potential \( F \) is that of logarithmic type (see \[8\]). However, this potential is often replaced by a polynomial approximation of the type \( F(r) = \gamma_1 r^4 - \gamma_2 r^2 \), \( \gamma_1, \gamma_2 \) being positive constants. As noted in \[8\], (2.1) can be replaced by

\[ \frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla \tilde{p} = -K \text{div} (\nabla \phi \otimes \nabla \phi) + Q(\tau(t), (v, \phi)(t - \tau(t))), \]

(2.3)

where \( \tilde{p} = p - K(\frac{\epsilon}{2} |\nabla \phi|^2 + \alpha F(\phi)) \), since \( K \mu \nabla \phi = K(\frac{\epsilon}{2} |\nabla \phi|^2 + \alpha F(\phi)) - \text{Kdiv} (\nabla \phi \otimes \nabla \phi) \). The stress tensor \( \nabla \phi \otimes \nabla \phi \) is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for these models, as in \[8\] we assume that the boundary conditions for \( \phi \) are the natural no-flux condition

\[ \partial_n \phi = \partial_n \mu = 0, \text{ on } \partial M \times (0, \infty), \]

(2.4)

where \( \partial M \) is the boundary of \( M \) and \( \eta \) is the outward normal to \( \partial M \). These conditions ensure the mass conservation. In fact, from \( \partial_n \mu = 0 \) on \( \partial M \times (0, \infty) \), we have the conservation of the following quantity

\[ \langle \phi(t) \rangle = \frac{1}{|M|} \int_M \phi(x, t) dx, \]

(2.5)

where \( |M| \) stands for the Lebesgue measure of \( M \). More precisely, we have

\[ \langle \phi(t) \rangle = \langle \phi(0) \rangle, \forall t \geq 0. \]

(2.6)

Concerning the boundary condition for \( v \), we assume the Dirichlet (no-slip) boundary condition

\[ v = 0, \text{ on } \partial M \times (0, \infty). \]

(2.7)

Therefore we assume that there is no relative motion at the fluid-solid interface.

The initial condition is given by

\[ (v, \phi)(s) = \theta(s) = (\theta_1, \theta_2)(s) s \in [-r, 0]. \]

(2.8)

The external forcing \( Q \) takes into account not only the present state of the system, but also the history of the solutions. We assume that the function \( \tau(t) \) is differentiable and there exists constants \( \tau^* \) and \( r > 0 \) such that

\[ \tau : [0, \infty) \to [0, r]; \frac{d\tau(t)}{dt} \leq \tau^* < 1. \]

(2.9)

Throughout this article we also suppose that the forcing \( Q \) satisfies the local Lipschitz condition.

(2.10)
2.2. Mathematical setting. We first recall from [8] a weak formulation of (2.1), (2.4), (2.7)-(2.8). Without loss of generality, we set \(\epsilon = 1\). Hereafter, we assume that the domain \(\mathcal{M}\) is bounded with a smooth boundary \(\partial \mathcal{M}\) (e.g., of class \(C^3\)). We also assume that \(f \in C^2(\mathbb{R})\) satisfies

\[
\begin{cases}
\lim_{|r| \to +\infty} f'(r) > 0, \\
|f''(r)| \leq c_f (1 + |r|^{k-1}), \forall r \in \mathbb{R},
\end{cases}
\tag{2.11}
\]

where \(c_f\) is some positive constant and \(k \in [1, +\infty)\) is fixed. It follows from (2.11) that

\[
|f'(r)| \leq c_f (1 + |r|^k), \quad |f(r)| \leq c_f (1 + |r|^{k+1}), \forall r \in \mathbb{R}.
\tag{2.12}
\]

Note that the derivative of the typical double-well potential \(f\) satisfies (2.11) with \(k = 2\). Let us now recall from [8] the functional set up of the model (2.1), (2.4), (2.7), (2.8).

If \(X\) is a real Hilbert space with inner product \((\cdot, \cdot)_X\), we will denote the induced norm by \(|\cdot|_X\), while \(X^*\) will indicate its dual. We set

\[V_1 = \{u \in C_0^\infty(\mathcal{M}) : \text{div } u = 0 \text{ in } \mathcal{M}\}.\]

We denote by \(H_1\) and \(V_1\) the closure of \(V_1\) in \((L^2(\mathcal{M}))^2\) and \((H_0^1(\mathcal{M}))^2\) respectively. The scalar product in \(H_1\) is denoted by \((\cdot, \cdot)_{L^2}\) and the associated norm by \(|\cdot|_{L^2}\).

Moreover, the space \(V_1\) is endowed with the scalar product

\[((u, v)) = 2 \sum_{i=1}^2 (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u, u))^{1/2}.\]

We now define the operator \(A_0\) by

\[A_0 v = \mathcal{P} \Delta v, \quad \forall v \in D(A_0) = H^2(\mathcal{M}) \cap V_1,\]

where \(\mathcal{P}\) is the Leray-Helmholtz projector in \(L^2(\mathcal{M})\) onto \(H_1\). Then, \(A_0\) is a self-adjoint positive unbounded operator in \(H_1\), which is associated with the scalar product defined above. Furthermore, \(A_0^{-1}\) is a compact linear operator on \(H_1\) and \(|A_0|_{L^2}\) is a norm on \(D(A_0)\) that is equivalent to the \(H^2\)-norm.

Hereafter, we set

\[H_2 = L^2(\mathcal{M}), \quad V_2 = H^1(\mathcal{M}), \quad H = H_1 \times H_2, \quad V = V_1 \times V_2.\tag{2.13}\]

Then we introduce the linear nonnegative unbounded operator on \(L^2(\mathcal{M})\)

\[A_N \phi = -\Delta \phi, \quad \forall \phi \in D(A_N) = \{\phi \in H^2(\mathcal{M}), \partial_{\eta} \phi = 0, \text{ on } \partial \mathcal{M}\}, \tag{2.14}\]

and we endow \(D(A_N)\) with the norm \(|A_N|_{L^2} + |\cdot|_{L^2}\), which is equivalent to the \(H^2\)-norm. Also we define the linear positive unbounded operator on the Hilbert space \(L^2_0(\mathcal{M})\) of the \(L^2\) functions with null mean

\[B_N \phi = -\Delta \phi, \quad \forall \phi \in D(B_N) = D(A_N) \cap L^2_0(\mathcal{M}).\tag{2.15}\]

Note that \(B_N^{-1}\) is a compact linear operator on \(L^2_0(\mathcal{M})\). More generally, we can define \(B_N^s\) for any \(s \in \mathbb{R}\), noting that \(|B_N^{s/2}|_{L^2}, s > 0\), is an equivalent norm to the canonical \(H^s\)-norm on \(D(B_N^{s/2}) \subset H^s(\mathcal{M}) \cap L^2_0(\mathcal{M})\). Also note that \(A_N = B_N\) on \(D(B_N)\). If \(\phi\) is such that \(\phi - \langle \phi \rangle \in D(B_N^{s/2})\), we have that \(|B_N^{s/2}(\phi - \langle \phi \rangle)|_{L^2} + |\langle \phi \rangle|_{L^2}\) is equivalent to the \(H^s\)-norm. Moreover, we set \(H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*\), whenever \(s < 0\).
We will denote by $\lambda_1 > 0$ a positive constant such that
\begin{equation}
\lambda_1 |w|^2_{L^2} \leq \|w\|^2 \quad \forall w \in V_1, \quad \lambda_1 \|\psi\|^2 \leq |B_N \psi|^2_{L^2} \quad \forall \psi \in D(B_N). \tag{2.16}
\end{equation}

We introduce the bilinear operators $B_0$, $B_1$ (and their associated trilinear forms $b_0$, $b_1$) as well as the coupling mapping $R_0$, which are defined from $D(A_0) \times D(A_0)$ into $H$, $D(A_0) \times D(A_N)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times (D(A_N) \cap H^3(\mathcal{M}))$ into $H_1$, respectively. More precisely, we set

\begin{equation}
(B_0(u, v), w) = \int_\mathcal{M} [(u \cdot \nabla)v] \cdot w dx = b_0(u, v, w), \quad \forall u, v, w \in D(A_0),
\end{equation}

\begin{equation}
(B_1(u, \phi), \rho) = \int_\mathcal{M} [(u \cdot \nabla)\phi] \rho dx = b_1(u, \phi, \rho),
\end{equation}

$\forall u \in D(A_0), \phi, \rho \in D(A_N)$, \tag{2.17}

\begin{equation}
(R_0(\mu, \phi), w) = \int_\mathcal{M} \mu [\nabla \phi \cdot w] dx = b_1(w, \phi, \mu),
\end{equation}

$\forall w \in D(A_0), \phi \in D(A_N) \cap H^3(\mathcal{M}), \mu \in L^2(\mathcal{M})$.

Note that
\begin{equation}
R_0(\mu, \phi) = \mathcal{P} \mu \nabla \phi.
\end{equation}

We recall that $B_0$, $B_1$ and $R_0$ satisfy the following estimates
\begin{equation}
|B_0(u, v)|_{V_1^*} \leq c |u|_{L^2}^{1/2} |v|_{L^2}^{1/2}, \quad \forall u, v \in V_1,
\end{equation}

\begin{equation}
|B_0(u, v)|_{L^2} \leq c |u|_{L^2}^{1/2} |v|_{L^2}^{1/2}|A_0v|_{L^2}^{1/2}, \quad \forall u \in V_1, \quad v \in D(A_0),
\end{equation}

\begin{equation}
|B_1(u, \phi)|_{V_2^*} \leq c |u|_{L^2}^{1/2} |\phi|_{L^2}^{1/2}, \quad \forall u \in V_1, \phi \in V_2,
\end{equation}

\begin{equation}
|B_1(u, \phi)|_{L^2} \leq c |u|_{L^2}^{1/2} |\phi|_{L^2}^{1/2}|A_N \phi|_{L^2}^{1/2}, \quad \forall u \in V_1, \phi \in D(A_N),
\end{equation}

\begin{equation}
|R_0(A_N \phi, \rho)|_{V_1^*} \leq c |A_N \phi|_{L^2}^{1/2} |\rho|_{H^3}^{1/2}, \quad \forall \phi \in D(A_N), \rho \in V_2,
\end{equation}

\begin{equation}
|R_0(A_N \phi, \rho)|_{L^2} \leq c |\rho|_{L^2}^{1/2} |A_N \phi|_{L^2}^{1/2} |\phi|_{H^3}^{1/2}, \tag{2.20}
\end{equation}

$\forall \phi \in D(A_N), \rho \in D(A_N^{1/2})$.

We recall that (due to the mass conservation) we have
\begin{equation}
\langle \phi(t) \rangle = \langle \phi(0) \rangle = M_0, \quad \forall t > 0. \tag{2.21}
\end{equation}

Thus, up to a shift of the order parameter field, we can always assume that the mean of $\phi$ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that
\begin{equation}
\langle \phi(t) \rangle = \langle \phi(0) \rangle = 0, \quad \forall t > 0. \tag{2.22}
\end{equation}

We set
\begin{equation}
\mathbb{Y} = H_1 \times D(B_N^{1/2}). \tag{2.23}
\end{equation}

The space $Y$ is a complete metric space with respect to the norm
\begin{equation}
|\langle v, \phi \rangle|_{\mathbb{Y}}^2 = \mathcal{K}^{-1} |v|_{L^2}^2 + \epsilon |\nabla \phi|_{L^2}^2 - |v|_{L^2}^2. \tag{2.24}
\end{equation}
We define the Hilbert space $\mathbb{V}$ by
\[
\mathbb{V} = V_1 \times D(B_N),
\]
endowed with the scalar products whose associated norm is
\[
\|(v, \phi)\|_\mathbb{V}^2 = \|v\|^2 + |B_N\phi|_{L^2}^2.
\] (2.26)

Throughout this article, we shall denote by $c_1, K_1, K$ several positive constants that depend on the data $(v_0, \phi_0)$ and $Q$. We will also denote by $c$ a generic positive constant that depends on the domain $\mathcal{M}$. To simplify the notations, we set (without loss of generality) $\alpha = K = 1$.

Using the notations above, we rewrite (2.1), (2.4), (2.7)-(2.5) as (see [8] for the details)
\[
\begin{align*}
\frac{dv}{dt} + \nu A_0 v + B_0(v, v) - R_0(\epsilon A_N \phi, \phi) &= Q(t - \tau(t), (v, \phi)(t - \tau(t))), \\
\frac{d\phi}{dt} + A_N \mu + B_1(v, \phi) &= 0, \quad \mu = \epsilon A_N \phi + \alpha f(\phi), \\
(v, \phi)(s) &= \vartheta(s) = (\vartheta_1, \vartheta_2)(s), \quad s \in [-r, 0],
\end{align*}
\] (2.27)

where $Q : [-r, \infty) \times \mathcal{Y} \to H_1$ and $\vartheta : [-r, 0] \to \mathcal{Y}$ are continuous functions satisfying some additional conditions (see (2.10)). Hereafter, we will also use the notation $(v_0, \phi_0) = \vartheta(0) = (\vartheta_1, \vartheta_2)(0)$.

**Remark 1.** In the weak formulation (2.27), the term $\mu \nabla \phi$ is replaced by $\epsilon A_N \nabla \phi$. This is justified since $f'(\phi) \nabla \phi$ is the gradient $F(\phi)$ and can be incorporated into the pressure gradient, see [8] for details. For the sake of convenience, as in [8] we will replace $\mu$ in (2.27) by $\bar{\mu} = \mu - \langle \mu \rangle$, that is $\bar{\mu} = \epsilon A_N \phi + \alpha f(\phi) - \alpha \langle f(\phi) \rangle$, a.e. in $\mathcal{M} \times (0, +\infty)$. Obviously we have $\langle \bar{\mu}(t) \rangle = 0$ for all $t > 0$.

**Definition 2.1.** Suppose that $(v_0, \phi_0) \in \mathcal{Y}$ and $T > 0$. A pair $(v, \phi)$ is called a weak solution to (2.27) and (2.8) on $[0, T]$ if it satisfies (2.27) and (2.8) in a weak sense on $[0, T]$ and
\[
(v, \phi) \in C([0, T]; \mathcal{Y}) \cap L^2([0, T]; \mathcal{Y}), \quad \frac{dv}{dt} \in L^2([0, T]; \mathcal{Y}'), \quad \frac{d\phi}{dt} \in L^2([0, T]; H^{-1}(*)).
\] (2.28)

If $(v_0, \phi_0) \in \mathcal{V}$, a weak solution $(v, \phi)$ is called a strong solution on the time interval $[0, T]$ if in addition to (3.34), it satisfies
\[
(v, \phi) \in C([0, T]; \mathcal{V}) \cap L^2([0, T]; D(A_0) \times D(A_N) \cap H^3(\mathcal{M})).
\] (2.29)

In the case when the delay $r$ is zero, the weak formulation of (2.27) and (2.8) was proposed and studied in [8], and the existence and uniqueness of solution was proved. In this article, we study the pullback asymptotic behavior of solutions (2.27) and (2.8).

3. **Existence of solution.** In this section we discuss the existence of the weak solution and the strong solution for the CH-NS system with delay (2.27). Since the injection of $\mathcal{V} \subset \mathcal{Y}$ is compact, let $\{(w_i, \psi_i), i = 1, 2, 3, \cdots\} \subset \mathcal{V}$ be an orthonormal basis of $\mathcal{Y}$, where $\{w_i, i = 1, 2, \cdots\}$, $\{\psi_i, i = 1, 2, \cdots\}$ are eigenvectors of $A_0$ and $A_N$ respectively. We set $\mathcal{V}_m = \mathcal{Y}_m = \text{span}\{(w_1, \psi_1), \cdots, (w_m, \psi_m)\}$.
Let

\[ \text{Theorem 3.1.} \]

the following result and we assume that the inverse \( G \) such that

\[ (2.27) \text{ has a unique weak solution} \]

By taking the scalar product in \( L^2 \), we derive that (see \([9]\) for the details) there exists a monotone non-decreasing function \( C_0 \) (independent on time and the initial condition) such that

\[ |(w, \psi)|^2_{\mathcal{V}} \leq \mathcal{E}(w, \psi) \leq C_0(|(w, \psi)|^2_{\mathcal{V}}), \forall (w, \psi) \in \mathcal{Y}. \]  

Let \( g \) be a continuous nonnegative scalar function defined on \([-r, +\infty)\) and let \( R \) be a continuous positive monotone non-decreasing function defined on \([0, +\infty)\). As in \([14, 4, 5, 6, 3]\), we set

\[ G = \int \frac{1}{R(x)} \, dx \]  

and we assume that the inverse \( G^{-1} \) of \( G \) is well-defined on \([0, +\infty)\). Then we have the following result

**Theorem 3.1.** Let \( T > 0 \) fixed. We assume that there exists a constant \( b_f \geq 0 \) such that

\[ |Q(t(v, \phi))|_{L^2}^2 \leq g(t)R(|(v, \phi)|^2_{\mathcal{V}}) + b_f, \forall v \in H_1, t \geq -r. \]  

Then \((2.27)\) has a unique weak solution \((v, \phi)\) for any initial value \((v_0, \phi_0) \in \mathcal{Y}, \vartheta \in L^2(-r, 0; \mathcal{Y})\).

**Proof.** By taking the scalar product in \( H_1 \) of \((3.1)_1 \) with \( v_m \), then taking the scalar product in \( L^2(\mathcal{M}) \) of \((3.1)_2 \) with \( \mu_m \), we derive that (see \([9]\) for the details)

\[ \frac{dE}{dt} + 2\nu\|v_m\|^2 + 2\|\mu_m\|^2 = 2Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t)))v_m, \]  

where \( E = E(t) = \mathcal{E}(v_m(t), \phi_m(t)) \).

It follows from \((3.5)\) and \((3.6)\) that

\[ \frac{dE}{dt} + \nu\|v_m\|^2 + 2\|\mu_m\|^2 \leq c_2(\|Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t)))\|_{L^2}^2 + c_2g(t - \tau(t))R(|(v_m, \phi_m)(t - \tau(t))|^2_{\mathcal{V}}) + c_2b_f \]  

\[ \leq c_2g(t - \tau(t))R(E(t - \tau(t))) + c_2b_f. \]
Let
\[ K = E(0) + \frac{c_2}{(1 - \tau^*)} \int_{-\tau}^{0} g(s) R(|\varphi|^2) ds. \] (3.8)

Then for \( T \geq t \geq 0 \), we have
\[ E(t) \leq K + \frac{c_2}{(1 - \tau^*)} \int_{-\tau}^{0} g(s) R(E(t)) ds + c_2 b f T. \] (3.9)

Therefore we have by the Bihari inequality (see [14])
\[ E(t) \leq G^{-1} \left\{ G(K + c_2 T b f) + \frac{c_2}{1 - \tau^*} \int_{0}^{t} g(s) ds \right\} \equiv M_1, \ 0 \leq t \leq T. \] (3.10)

This proves that \((v_m, \phi_m)\) is uniformly bounded in \( L^\infty(0, T; \mathcal{Y}) \cap L^2(0, T; \mathcal{V})\).

Note that from
\[ \mu_m = \epsilon A_N \phi_m + \alpha f(\phi_m), \]
we can also check as in [8] that
\[
\begin{align*}
|A_N \phi_m|_{L^2} &\leq c \|\mu_m\| + Q_1(\|\phi_m\|), \\
|A_N^{3/2} \phi_m|_{L^2} &\leq c \|\mu_m\| + Q_1(\|\phi_m\|),
\end{align*}
\] (3.11)

where \( Q_1 \) is a monotone non-decreasing function independent on time, the initial condition and \( m \). It follows from (3.11) and (3.10) that \( A_N^{3/2} \phi_m \) is bounded in \( L^2(0, T; H^2) \).

Using (3.9), (3.11) and (2.18)-(2.20), we can check that
\[ \frac{d}{dt}(v_m, \phi_m) \text{ is bounded in } L^2(0, T; V_1^*) \times L^2(0, T; \dot{H}^{-1}). \] (3.12)

Therefore, we can take a subsequence (still) denoted by \((v_m, \phi_m)\) such that
\[
\begin{align*}
(v_m, \phi_m) &\to (v, \phi) \text{ weakly-star in } L^\infty(0, T; \mathcal{Y}), \\
(v_m, \phi_m) &\to (v, \phi) \text{ weakly in } L^2(0, T; \mathcal{V}), \\
\frac{d}{dt}(v_m, \phi_m) &\to \frac{d}{dt}(v, \phi) \text{ weakly in } L^2(0, T; V_1^*) \times L^2(0, T; \dot{H}^{-1}).
\end{align*}
\] (3.13)

Thus we have by the compactness theorem (see [15]) that
\[ (v_m, \phi_m) \to \text{ strongly in } L^2(0, T; \mathcal{Y}). \] (3.14)

Using (3.13)-(3.14) and standard methods as in [15], we can pass to the limit in (3.1) as \( m \to \infty \), and we obtain that \((v, \phi)\) is a weak solution to (2.27). Let us recall that the passage to the limit in the delay force is obtained as in [14, 4].

For the uniqueness of weak solutions and their continuous dependence (from \( \mathcal{Y} \times L^2(-r, 0, \mathcal{Y}) \) into \( \mathcal{Y} \)) with respect to the initial data, we proceed as in [14]. Let \( u_1 = (v_1, \phi_1) \) and \( u_2 = (v_2, \phi_2) \) be two weak solutions to (2.27). Let \( w = v_1 - v_2 \), \( \psi = \phi_1 - \phi_2 \) and
\[ t_1 = \sup\{\rho > 0, |u_1(s) - u_2(s)|_{\mathcal{Y}} = 0, \ \forall s \in [0, \rho]\}. \] (3.15)

Let \( \tilde{u} = \epsilon A_N \psi + \alpha(f(\phi_1) - f(\phi_2)) \).
We recall that \((w, \psi)\) satisfies
\[
\frac{dw}{dt} + \nu A_0 w + B_0(v_1, w) + B_0(w, v_2) - R_0(\epsilon A_N \psi, \phi_1) - R_0(\epsilon A_N \phi_2, \psi) \\
= Q(t - \tau(t), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t))),
\]
\[\text{(3.16)}\]
\[
\frac{d\bar{\mu}}{dt} + A_N \bar{\mu} + B_1(w, \phi_1) + B_1(v_2, \psi) = 0,
\]
\[
\bar{\mu} = \epsilon A_N \psi + \alpha(f(\phi_1) - \alpha f(\phi_2)) - (\bar{\mu}).
\]

We set \(w_1 = (v_1, \phi_1)(t_1)\). Since \(Q(t, (v, \phi))\) satisfies a local Lipschitz condition, for any positive constant \(\epsilon_0 > 0\), there exists \(L(\epsilon_0) > 0\) such that
\[
|Q(t, \zeta) - Q(t, \varsigma)|_{L^2} \leq L(\epsilon_0)|\zeta - \varsigma|_{\mathcal{Y}}, \forall (t, \zeta), (t, \varsigma) \in \Omega_1(t_1, \epsilon_0),
\]
where
\[
\Omega_1(t_1, \epsilon_0) = \{(t, \zeta) : |t - t_1| < \epsilon_0, |\zeta - w_1|_Y < \epsilon_0\}. \tag{3.18}\]

We can assume without loss of generality that \((t - \tau(t), (v_1, \phi_1)(t - \tau(t))), (t - \tau(t), (v_2, \phi_2)(t - \tau(t))) \in \Omega_1(t_1, \epsilon_0)\).

Multiplying (3.16)_1 by \(w\), (3.16)_3 and (3.16)_2 respectively by \(A_N \bar{\mu} + \epsilon \xi A_N \psi\) (with \(\xi > 0\) sufficiently small to be selected in the sequel) and \(\epsilon A_N \psi\), respectively, we derive as in [12, 9] that
\[
\frac{dw}{dt} + \nu \|w\|^2 + (1 - c\xi)\|\bar{\mu}\|^2 + \frac{\epsilon^2 \xi}{2}|A_N \psi|_{L^2}^2 \leq \Upsilon(t)y(t) \\
+ c|Q(t - \tau(t), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t))))|_{L^2}^2 \\
\leq \Upsilon(t)y(t) + cL(\epsilon_0)^2|(w, \psi)(t - \tau(t))|_{Y}^2 \\
\leq \Upsilon(t)y(t) + cL(\epsilon_0)^2y(t - \tau(t)),
\]
where \(c = c_M\) is a constant that depends only on \(M\) and
\[
y(t) = \|(w, \psi)\|_{Y}^2, \\
\Upsilon(t) = \|v_1\|^2 + c(1 + \|\phi_1\|^2)|A_N \phi_1|_{L^2}^2 + |v_2|_{L^2}^2 \|v_2\|^2 \\
+ Q_1(|\phi_1|_{H^1}, |\phi_2|_{H^1})(1 + |A_N \phi_1|_{L^2}^2 + |A_N \phi_2|_{L^2}^2).
\]

It follows that (with \(\xi\) small enough such that \(1 - c\xi > 0\))
\[
y(t) \leq y(0) + \int_0^t \left(\Upsilon(s) + \frac{cL(\epsilon_0)^2}{1 - \tau^*}\right) ds. \tag{3.21}\]

By the Gronwall lemma, we obtain that
\[
y(t) \leq y(0) \exp \left[\int_0^t \left(\Upsilon(s) + \frac{cL(\epsilon_0)^2}{1 - \tau^*}\right) ds\right]. \tag{3.22}\]

This proves the uniqueness of weak solutions and the continuous dependence (in the \(\mathcal{Y}\)-norm) on the initial data follow.

As a corollary, we have...
Corollary 1. We suppose that there exist constants \(a_f > 0\) and \(b_f \geq 0\) such that
\[
|Q(t, (v, \phi))(\nu)|_{L^2}^2 \leq a_f |(v, \phi)|_{L^2}^2 + b_f, \quad \forall (v, \phi) \in \mathcal{Y}, t \geq -r.
\] (3.23)

Then (2.27) has a unique weak solution \((v, \phi) \in L^\infty(0, T; \mathcal{Y}) \cap L^2(0, T; \mathcal{V})\) for any initial value \((v_0, \phi_0) \in \mathcal{Y}, \mathcal{V} \in L^2(-r, 0; \mathcal{Y})\).

3.1. Strong solution. In this part we discuss the existence and uniqueness of the strong solution to (2.27).

Theorem 3.2. Let \((v_0, \phi_0) \in \mathcal{V} \) Then there exists a unique strong solution \((v, \phi)\) to (2.27) such that
\[
(v, \phi) \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; D(A_0) \times D(A_N^{3/2})).
\] (3.24)

Proof. Let \((v_m, \phi_m)\) be any fixed approximation to the solution to (2.27). It follows from the proof of Theorem 3.1 that there exists a constant \(M_1 = M_1(T) > 0\) such that
\[
|(v_m, \phi_m)|_{L^2}^2 \leq M_1,
\]
uniformly in \(m\). Let \(\mu_m = \mu_m - \langle \mu_m \rangle\).

Now taking the inner product in \(H_1\) of (3.1) with \(2A_0v_m\), the inner product in \(L^2(\mathcal{M})\) of (3.1) with \(2\epsilon A_N^2 \phi_m + 2A_N^2 \mu_m + 2\zeta A_N^2 \phi_m\) respectively (where \(\zeta > 0\) is a small parameter to be chosen later). By adding the resulting inequalities gives (see [12, 9] for the details)
\[
\begin{align*}
\frac{d\|v\|_{L^2}}{dt} + \nu |A_0v_m|_{L^2}^2 + \frac{\epsilon}{4} |A_N^2 \phi_m|_{L^2}^2 &+ |2 - (4\epsilon^{-1} + \alpha)| |B_N \mu_m|_{L^2}^2 \\
\leq &\quad G_1(t)\|v\| + G_2(t) + 2\|Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t)), A_0v_m),
\end{align*}
\] (3.25)

where
\[
\begin{align*}
G_1(t) &= c(M_1^2) + |A_N \phi_m|_{L^2}^2, \\
G_2(t) &= c(1 + |v_m|_{L^2}^2 + |\phi_m|^2).
\end{align*}
\] (3.26)

Let us set
\[
Z(t) = \|v(t)\|_{L^2}^2 + \frac{1}{(1 - \tau^*\nu)} \int_{t-\tau}^t Q(s, (v_m, \phi_m)(s))|_{L^2}^2 ds.
\]

Then
\[
\int_0^T Z(t) dt < \infty.
\] (3.27)

Choosing \(\zeta = (4\epsilon^{-1} + \alpha)^{-1}\), it follows from (3.25) that (with \(2\nu = \nu, \bar{\epsilon}_1 = \frac{\epsilon}{4}, \bar{\epsilon}_2 = 2 - (4\epsilon^{-1} + \alpha)\) and
\[
\begin{align*}
\frac{dZ}{dt} &\leq -2\nu |A_0v_m|_{L^2}^2 - \bar{\epsilon}_1 |A_N^2 \phi_m|_{L^2}^2 - \bar{\epsilon}_2 |B_N \mu_m|_{L^2}^2 \\
&+ (\frac{1}{1 - \tau^*\nu}) |Q(t, (v_m, \phi_m)(t))|_{L^2}^2 - \frac{1}{\nu} |Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t))|_{L^2}^2 \\
&+ G_1(t)\|v\| + G_2(t) + 2\|Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t)), A_0v_m,\}
\]
Using the estimates (3.31)-(3.32), we can prove as \[9\] the existence of strong solution
Therefore, we can take a subsequence (still) denoted (v)
Thus we have by the compactness theorem (see \[15\]) that
\[\text{Theorem 3.3.}\]
Let
3.1.1.
Proof.
Let
where
Using (3.29) and (2.18)-(2.20), we can check that
\[\epsilon \text{ any positive constant}\]
\[w\]
\[|Q(t, (v_m, \phi_m)(t))|_{L^2}^2 + G_1(t)Y(t) + G_2(t)\]
\[\leq -\nu|A_0v_m|_{L^2}^2 - \epsilon_1|A_N^2\phi_m|_{L^2}^2 - \epsilon_2 |B_N \mu_m|_{L^2}^2\]
\[+ \frac{1}{(1 - r^2)}|Q(t, (v_m, \phi_m)(t))|_{L^2}^2 + G_1(t)Y(t) + G_2(t)\]
\[(3.28)\]
It follows from the Gronwall lemma that
\[\|(v_m, \phi_m)\|_{Y}^2 \leq Z(t) \leq C,\]
\[\int_0^T (|A_0v_m|_{L^2}^2 + |A_N^2\phi_m|_{L^2}^2 + |B_N \mu_m|_{L^2}^2) ds \leq C.\]
Using (3.29) and (2.18)-(2.20), we can check that
diff (v_m, \phi_m) is bounded in \(L^2(0, T; Y).\)
\[(3.30)\]
Therefore, we can take a subsequence (still) denoted (v_m, \phi_m) such that
\[(v_m, \phi_m) \to (v, \phi) \text{ weakly in } L^\infty(0, T; V),\]
\[(v_m, \phi_m) \to (v, \phi) \text{ weakly in } L^2(0, T; D(A_0) \times D(A^2_N)),\]
\[(3.31)\]
diff (v_m, \phi_m) \to diff (v, \phi) weakly in \(L^2(0, T; Y).\)
Thus we have by the compactness theorem (see [15]) that
\[(v_m, \phi_m) \to \text{ strongly in } L^2(0, T; V).\]
\[(3.32)\]
Using the estimates (3.31)- (3.32), we can prove as \[9\] the existence of strong solution to (2.27). The uniqueness of strong solution follows from Theorem 3.3 below. \[\square\]
3.1.1. Continuity in \(V\) with respect to the initial data.
\[\textbf{Theorem 3.3.}\]
Let \((v_i, \phi_i), i = 1, 2\) be two strong solutions corresponding to the initial conditions \((v_{i0}^0, \phi_{i0}^0)\). Let the initial data on the time interval \((-r, 0)\) be denoted \(\Phi_1, \Phi_2\) respectively. Then we have
\[\|(v_1, \phi_1) - (v_2, \phi_2)\|_V^2\]
\[\leq \left(\|(v_{10}^0, \phi_{10}^0) - (v_{20}^0, \phi_{20}^0)\|_V^2 + \frac{cL(\epsilon_0)^2}{1 - r} \int_{-r}^0 \|\Phi_1 - \Phi_2\|_V^2 ds\right) \times\]
\[\exp\left[ c \int_{-r}^t \left(\Psi(s) + \frac{cL(\epsilon_0)^2}{1 - r^2}\right) ds \right],\]
where \(L(\epsilon_0)\) and \(\Psi\) are defined below.
\[\text{Proof.}\]
Let
\[t_2 = \sup\{\rho > 0, \|(v_1, \phi_1)(s) - (v_2, \phi_2)(s)\|_V = 0, \forall s \in [0, \rho]\}.\]
\[(3.34)\]
We set \(w_1 = (v_1, \phi_1)(t_2)\). Since \(Q(t, v, \phi)\) satisfies a local Lipschitz condition, for any positive constant \(\epsilon_0 > 0\), there exists \(L(\epsilon_0) > 0\) such that
\[|Q(t, \zeta) - Q(t, w)|_{L^2} \leq L(\epsilon_0)|\zeta - w|_V, \forall (t, \zeta), (t, w) \in \Omega_1(t_2, \epsilon_0),\]
\[(3.35)\]
where $\Omega_1(t_2, \epsilon_0)$ is defined in (3.18). We can assume without loss of generality that 
$(t - \tau(t), (v_1, \phi_1)(t - \tau(t))), (t - \tau(t), (v_2, \phi_2)(t - \tau(t))) \in \Omega_1(t_2, \epsilon_0)$.

Let $\bar{\mu} = \mu - (\mu)$ and $(\bar{w}, \bar{\psi}) = (v_1, \phi_1) - (v_2, \phi_2)$. Then $(\bar{w}, \bar{\psi})$ satisfies (3.16). We multiply (3.16) by $A_0 w$ and (5.4) by $\epsilon A_N^2 \bar{\psi}$ and (3.16) by $A_N \bar{\mu} + \epsilon A_N \psi$ (with $\xi$ small enough to be selected later). Adding the resulting equations, we derive that

$$\frac{1}{2}\frac{d}{dt} [\|A_0 w\|_2^2 + \epsilon^2 \xi |A_N^{3/2} \psi|_2^2 + |A_N \bar{\mu}|_2^2] = A_3(t),$$

(3.36)

where

$$A_3(t) = -b_0(v_2, w, A_0 w) - b_1(w, v_1, A_0 w) + \langle R_0(\epsilon A_N \psi, \phi_2, A_0 w) \rangle$$

$$+ \langle R_0(\epsilon A_N \phi_1, \psi, A_0 w) \rangle - b_1(w, \phi_1, A_N^2 \psi) - c(v_2, \psi, A_N^2 \psi)$$

$$+ \langle \xi \alpha(\bar{w}, \bar{\psi}) \rangle + \langle \epsilon(\bar{w}, \bar{\psi}) \rangle (A_N \bar{\mu}) + \langle \epsilon(\bar{w}, \bar{\psi}) \rangle (A_N \bar{\mu})$$

$$+ (Q(t - \tau(t), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t)))) A_0 w,$$

and

$$y(t) = \| (w, \psi)(t) \|_2^2.$$
\[c\xi |A_N\psi, A_N\bar{\mu}|^2 \leq \frac{\xi}{2} |A_N\bar{\mu}|_{L^2}^2 + c\xi |A_N\psi|_{L^2}^2. \quad (3.44)\]

Let us set
\[\Psi(t) = c(\|y_2\|^2 + \|v_2\|^2 + \|v_1\| A_0 v_1 + \|\phi_2\|^2) + c(\|\phi_1\| A_N \phi_1 + |A_N \phi_1|_{L^2})^2 + |A_N \phi_1|_{L^2}^2 + |\phi_1||A_N \phi_1|_{L^2} + \xi)\]
\[+ (\xi + \xi^{-1}) Q_2(|A_N \phi_1|_{L^2}, |A_N \phi_1|_{L^2}).\] (3.45)

Since \((v_i, \phi_i), i = 1, 2\) are strong solutions, then \(\int_0^t \Psi(s) ds < \infty, \forall t > 0\).

From (3.37)-(3.45), we have (with \(2\bar{v} = \nu\))
\[\frac{dy}{dt} + 2\bar{v} A_0 w + c\xi |A_N^2 \psi|_{L^2}^2 + |A_N \bar{\mu}|_{L^2}^2 \leq \Psi(t) y(t)\]
\[+ 2(Q(t (t), (v_1, \phi_1)(t - \tau(t))) - Q(t (t), (v_2, \phi_2)(t - \tau(t)), A_0 w)\] (3.46)

Let
\[Z(t) = y(t) + \frac{1}{(1 - \tau^\nu)} \int_{t-\tau(t)}^t |Q(s, (v_1, \phi_1)(s)) - Q(s, (v_2, \phi_2)(s))|_{L^2}^2.\] (3.47)

It follows from (3.37)-(3.47) that
\[\frac{dZ}{dt} \leq \Psi(t) y(t) + \frac{1}{(1 - \tau^\nu)} |Q(t, (v_1, \phi_1)(t)) - Q(t, (v_2, \phi_2)(t))|_{L^2}^2.\] (3.48)

Note that
\[\int_0^T Z(s) ds < \infty,\]
\[|Q(t, (v_1, \phi_1)(t)) - Q(t, (v_2, \phi_2)(t))|_{L^2}^2 \leq L(\epsilon_0)^2 |(w, \psi)(t)|_{L^2}^2\]
\[\leq L(\epsilon_0)^2 Z(t),\] (3.49)

\[y(t) \leq Z(t),\]
\[Z(0) \leq y(0) + \frac{c L(\epsilon_0)^2}{1 - \tau^\nu} \int_{-\tau}^0 |\Phi_1 - \Phi_2|_{L^2}^2 ds.\]

It follows from (3.48)-(3.49) that
\[Z(t) \leq Z(0) \exp \left[ c \int_0^t \left( \Psi(s) + \frac{L(\epsilon_0)^2}{1 - \tau^\nu} \right) ds \right]\] (3.50)
and
\[y(t) \leq \left( y(0) + \frac{c L(\epsilon_0)^2}{1 - \tau^\nu} \int_{-\tau}^0 |\Phi_1 - \Phi_2|_{L^2}^2 ds \right) \exp \left[ \int_0^t \left( \Psi(s) + \frac{L(\epsilon_0)^2}{1 - \tau^\nu} \right) ds \right].\] (3.51)

Then for a fixed time \(t > 0\), the Lipschitz continuous dependence (in the \(V\)-norm) on the initial data follow.

As a corollary, we have:
Corollary 2. We suppose that there exist constants \( a_f > 0 \) and \( b_f \geq 0 \) such that
\[
|Q(t, (v, \phi))|^2_{L^2} \leq a_f |(v, \phi)|^2_{L^2} + b_f, \forall (v, \phi) \in \mathcal{V}, t \geq -r.
\] (3.52)
Then for every \((v_0, \phi_0) \in \mathcal{V}\) and \( \varphi \in L^2(-r, 0; \mathcal{V}) \), there exists a unique strong solution of the system (5.4), which depends continuously (from \( \mathcal{V} \times L^2(-r, 0; \mathcal{V}) \) into \( \mathcal{V} \)) on the initial data.

4. Exponential behavior of weak solutions. In this part we discuss the exponential behavior of weak solutions to the CH-NS with delay (2.27). In this section we assume that there exist two constants \( a_f > 0 \) and \( b_f \geq 0 \) such that the forcing term \( Q \) satisfies
\[
|Q(t, (w, \psi))|^2_{L^2} \leq a_f |(w, \psi)|^2_{L^2} + b_f, \forall (w, \psi) \in \mathcal{V}, t \geq -r.
\] (4.53)

Theorem 4.1. Let \( \kappa, \alpha_1, \lambda_1, c_1 \) be given respectively by (4.64), (4.65), (2.16) and (4.63). We assume that
\[
-k + \frac{a_f}{(1 - \tau_*) \alpha_1 \lambda_1} < 0.
\] (4.54)
Then we have the following asymptotic behavior of weak solutions.
\[
[(v, \phi)(t)]^2_{L^2} \leq (M_0 |(v_0, \phi_0)|^2_{L^2}) + K e^{-\rho t} + \frac{b_f e^{\rho r}}{(1 - \tau^*) \rho} + c_1 \rho,
\] (4.55)
where
\[
K = \frac{1}{(1 - \tau^*) \alpha_1 \lambda_1} \int_0^\infty e^{\rho s} e^{\rho r} (a_f |\varphi(s)|^2_{L^2} + b_f) ds,
\] (4.56)
\( \rho > 0 \) is a positive number such that
\[
\rho - k + \frac{a_f e^{\rho r}}{(1 - \tau^*) \alpha_1 \lambda_1} = 0,
\] (4.57)
and hereafter \( M_0 \) denotes a suitable monotone non-decreasing function independent of time.

Proof. We take the scalar product in \( H_1 \) of (2.27) with \( v \) and the scalar product in \( L^2(\mathcal{M}) \) of (2.27) with \( 2\xi \phi \), \( \xi > 0 \) and adding the resulting relationships to derive as in [9] that
\[
dE \frac{dt}{dt} + \kappa E(t) = \w_1(t),
\] (4.58)
where
\[
E(t) = [(v, \phi)(t)]^2_{L^2} + 2\alpha (F(\phi(t)), 1)_{L^2} + C_e,
\] (4.59)
and
\[
\w_1(t) = -2\nu \|v\|^2 + \kappa \|v\|^2_{L^2} - 2\|\nabla \mu\|^2_{L^2} - (2\xi - \kappa)\|\nabla \phi\|^2_{L^2} + 2\alpha [\kappa (F(\phi) - f(\phi, \phi, 1)_{L^2} - (\xi - \kappa)(\phi \phi, 1)_{L^2}) + 2\xi (\mu, \phi)_{L^2}]
\] + \(2(v, Q(t - \tau(t), (v, \phi)(t - \tau(t)))] + \kappa |\phi(t)|^2_{L^2} + \kappa \alpha |\mathcal{M}|.\) (4.60)

From (2.11), we have
\[
c_1 |f(y)|(1 + |y|) \leq 2f(y)y + c_f,
\]
\[
F(y) - f(y)y \leq c_f' y^2 + c_f'',
\] (4.61)
\[2\xi (\mu, \phi)_{L^2} \leq \|\nabla \mu\|^2_{L^2} + \xi^2 |\mathcal{M}| |\nabla \phi|^2_{L^2}.
\]
Note that if (4.54) is satisfied, we can find where
\[ \theta > f. \]

From (4.61), we derive that
\[ \wedge_1(t) \leq -(\nu - \kappa C_m|\mathcal{M}|)\|v(t)\|^2 - |\nabla \mu(t)|^2_{L^2} \]
\[ -c_{1}\alpha(\xi - \kappa)\langle [f(\phi(t)), 1 + |\phi(t)|]_{L^2} + 2\langle v, Q(t - \tau(t), (v, \phi)(t - \tau(t))) \rangle \rangle \quad (4.62) \]
\[ -[\xi(2 - \xi C_m|\mathcal{M}|\epsilon^{-1}) - \kappa(1 + 2\alpha \epsilon^{-1}c_f'|\mathcal{M}|)]c_1|\nabla \phi|^2_{L^2} + c_1, \]
where \( C_m \) depends on the shape of \( \mathcal{M} \), but not its size and \( c_1 \) is given by
\[ c_1 = 2\kappa \alpha C_f|\mathcal{M}| + 2\alpha \epsilon^{-1}C_m|\mathcal{M}|c_f(\xi - \kappa)|\mathcal{M}|. \quad (4.63) \]

Let us choose \( \kappa \in (0, 1) \) as
\[ \kappa = \min \left\{ \nu(2C_m|\mathcal{M}|)^{-1}, \xi(1 + 2\alpha \epsilon^{-1}C_m|\mathcal{M}|c_f')^{-1} \right\}. \quad (4.64) \]

From now on, \( c_i \) will denote a positive constant independent on the initial data and on time. Let us set
\[ 2\alpha_1 = \nu - \kappa C_m|\mathcal{M}|, \quad \alpha_2 = [\xi(2 - \xi C_m|\mathcal{M}|\epsilon^{-1}) - \kappa(1 + 2\alpha \epsilon^{-1}c_f'|\mathcal{M}|)]\epsilon. \quad (4.65) \]

It follows from (4.59)-(4.64) that
\[ \frac{dE}{dt} + \kappa E(t) + 2\alpha_1\|v(t)\|^2 + \alpha_2|\nabla \phi(t)|^2_{L^2} + |\nabla \mu(t)|^2_{L^2} \]
\[ + c_3\langle [f(\phi(t)), 1 + |\phi(t)|]_{L^2} \leq 2\langle v, Q(t - \tau(t), (v, \phi)(t - \tau(t))) \rangle + c_1, \]
which gives
\[ \frac{dE}{dt} + \kappa E(t) + 2\alpha_1\|v(t)\|^2 + \alpha_2|\phi(t)|^2 + 2|\nabla \mu(t)|^2_{L^2} \]
\[ + c_3\langle [f(\phi(t)), 1 + |\phi(t)|]_{L^2} \leq 2\langle v, Q(t - \tau(t), (v, \phi)(t - \tau(t))) \rangle + c_1. \quad (4.67) \]

Let
\[ Z(t) = e^{\theta t}E(t) + \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} \int_{t-\tau(t)}^t e^{\theta s} e^{\theta r} |Q(s, (v, \phi)(s))|^2_{L^2} ds, \quad (4.68) \]
where \( \theta > 0 \) is a positive number such that
\[ \theta - \kappa + \frac{a_f e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1} < 0. \quad (4.69) \]

Note that if (4.54) is satisfied, we can find \( \theta > 0 \) small enough such that (4.69) holds.

Then from (4.66)-(4.68), we have
\[ \frac{dZ}{dt} = \theta e^{\theta t}E + e^{\theta t} \frac{dE}{dt} + \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|^2_{L^2} \]
\[ - \frac{1}{\alpha_1 \lambda_1} e^{\theta t}|Q(t - \tau(t), (v, \phi)(t - \tau(t)))|^2_{L^2} \]
and (4.55) follows. Recall that $E$ which yields (letting $\theta$)

Moreover, (4.69) is satisfied for $\theta > 0$ and small enough provided that (4.54) holds.

\[ E \leq \theta e^{\theta t} E(t) - \kappa e^{\theta t} E + e^{\theta t} (-2\alpha_2 \| \phi \|^2 - 2\alpha_1 \| v \|^2) + 2\langle v, Q(t - \tau(t)), (v, \phi)(t - \tau(t)) \rangle + c_1 \]

\[ + \frac{1}{(1 - \tau^*) \alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|_L^2 
- \frac{1}{\alpha_1 \lambda_1} e^{\|Q(t - \tau(t), (v, \phi)(t - \tau(t))\|_L^2} \leq (\theta - \kappa) e^{\theta t} E(t) + e^{\theta t} (-\alpha_1 \| v \|^2 - 2\alpha_2 \| \phi \|^2) \]

\[ + \frac{1}{(1 - \tau^*) \alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|_L^2 + e^{\theta t} c_1, \]

which gives (see (4.69))

\[ \frac{dZ}{dt} \leq (\theta - \kappa) e^{\theta t} E + \frac{1}{(1 - \tau^*) \alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|_L^2 + c_1 e^{\theta t} \]

\[ \leq (\theta - \kappa) e^{\theta t} E + \frac{1}{(1 - \tau^*) \alpha_1 \lambda_1} e^{\theta t} e^{\theta r} (a_f |(v, \phi)|_2^2 + b_f) + c_1 e^{\theta t} \]

\[ \leq \left( \theta - \kappa + \frac{a_f e^{\theta r}}{(1 - \tau^*) \alpha_1 \lambda_1} \right) e^{\theta t} E + \frac{b_f e^{\theta t} e^{\theta r}}{(1 - \tau^*) \alpha_1 \lambda_1} + c_1 e^{\theta t} \]

\[ \leq \frac{b_f e^{\theta t} e^{\theta r}}{(1 - \tau^*) \alpha_1 \lambda_1} + c_1 e^{\theta t}. \]

It follows that

\[ e^{\theta t} E(t) \leq E(0) + \int_{-r}^{0} e^{\theta s} e^{\theta r} |Q(s, (v, \phi)(s))|_L^2 ds \]

\[ + c_1 \int_{0}^{t} e^{\theta s} ds + \frac{b_f}{(1 - \tau^*) \alpha_1 \lambda_1} \int_{0}^{t} e^{\theta s} e^{\theta r} ds \]

\[ \leq E(0) + K + c_1 \frac{b_f}{\theta} e^{\theta t} + \frac{b_f e^{\theta t} e^{\theta r}}{(1 - \tau^*) \alpha_1 \lambda_1} \theta, \]

which yields (letting $\theta \to \rho$)

\[ E(t) \leq (E(0) + K) e^{-\rho t} + \frac{b_f e^{\rho r}}{(1 - \tau^*) \alpha_1 \lambda_1 \rho} + c_1 \frac{1}{\rho} \]

and (4.55) follows. Recall that $E(0) \leq M_0((v_0, \phi_0)|_Y^2)$.

Note that if (4.54) is satisfied, there exists $\rho > 0$ such that (4.57) holds true. Moreover, (4.69) is satisfied for $\theta > 0$ and small enough provided that (4.54) holds.

5. **Stability of stationary solutions.** Hereafter, we study the stability of stationary solutions to (2.27). We first prove the existence of stationary solution to (2.27) when the delay has a special form, provided that viscosity $\nu$ and the physical parameter $\epsilon$ are large enough. Then we prove that all weak solutions to (2.27) converge exponentially to this unique stationary solution. In this section, we assume that the delay term is given by

\[ Q(t, (v_t, \phi_t)) = Q_0((v, \phi)(t - \tau(t)), \]
where
\[ Q_0 : \mathcal{V} \to V_1^* \]
satisfies
\[ \| Q_0(v_1, \phi_1) - Q_0(v_2, \phi_2) \|_{V_1^*} \leq L_1 \| (v_1, \phi_1) - (v_2, \phi_2) \|_{V}, \quad \forall (v_1, \phi_1), (v_2, \phi_2) \in \mathcal{V}, \]
for some fixed constant \( L_1 > 0 \).

A stationary solution to (2.27) is a \((v^*, \phi^*)\) such that
\[
\begin{cases}
\nu A_0 v^* + B_0(v^*, v^*) - R_0(\epsilon A_N \phi^*, \phi^*) = Q_0(v^*, \phi^*), \\
\epsilon A_N^2 \phi^* + \alpha A_N f(\phi^*) + B_1(v^*, \phi^*) = 0.
\end{cases}
\]

5.1. Existence and uniqueness of stationary solution. Let \( \{ (w_i, \psi_i), i = 1, 2, 3, \ldots \} \subset \mathcal{V} \) be an orthonormal basis of \( \mathcal{Y} \), where \( \{ w_i, i = 1, 2, \ldots \} \) and \( \{ \psi_i, i = 1, 2, \ldots \} \) are eigenvectors of \( A_0 \) and \( A_N \) respectively. We set \( \mathcal{V}_m = \text{span}\{ (w_1, \psi_1), \ldots, (w_m, \psi_m) \} \). For fixed \((U, \Phi) \in \mathcal{V}_m\), We consider the following approximating problems: find \((v_m, \phi_m) \in \mathcal{V}_m\) such that
\[
\begin{cases}
\nu A_0 v_m + B_0(U, v_m) - R_0(\epsilon A_N \phi_m, \Phi) = Q_0(U, \Phi), \\
\epsilon A_N^2 \phi_m + \alpha A_N f(\Phi) + B_1(v_m, \Phi) = 0.
\end{cases}
\]

It is clear (using the Lax-Milgram theorem) that for \((U, \Phi) \in \mathcal{V}_m\), there exists a unique solution \((v_m, \phi_m)\) to (5.5). Define \( T_m : \mathcal{V}_m \to \mathcal{V}_m \) the linear operator given by \( T_m(U, \Phi) = (v_m, \phi_m) \).

We will see that for each \( m \) we may apply a fixed point theorem to the map \( T_m \) (restricted to a suitable subset \( \wedge_m \subset \mathcal{V}_m \)) to ensure that we can obtain the existence of a solution \((v_m, \phi_m)\) to (5.5).

Lemma 5.1. We assume that \( Q_0 \) satisfies (5.2). Then any solution \((v_m, \phi_m)\) to (5.5) satisfies the estimate
\[
\| (v_m, \phi_m) \|_{\mathcal{V}} \leq \kappa_1^{-1} \left( \| Q_0(0, 0) \|_{V_1^*} + L_1 \| (U, \Phi) \|_{\mathcal{V}} + M_1(\| A_N \Phi \|_{L^2}) \right),
\]
where \( \kappa_1 \) is given by (5.10) below and \( M_1(\cdot) \) is a suitable monotone non-decreasing function independent of \( m \).

Proof. If \((v_m, \phi_m) \in \mathcal{V}\) is a solution to (5.5), we can easily check that
\[
\nu \| v_m \|^2 + \epsilon^2 \| A_N^{3/2} \phi_m \|_{L^2}^2 + \alpha \epsilon \langle A_N f(\Phi), A_N \phi_m \rangle = \langle Q_0(U, \Phi), v_m \rangle,
\]
which gives
\[
\nu \| v_m \|^2 + \epsilon^2 \| A_N^{3/2} \phi_m \|_{L^2}^2 \leq L_1 \| v_m \| \| (U, \Phi) \|_{\mathcal{V}} + \alpha \epsilon \| A_N^{1/2} f(\Phi) \|_{L^2} \| A_N^{3/2} \phi_m \|_{L^2}
\]
\[+ \| Q_0(0, 0) \|_{V_1^*} \| v_m \|,
\]
and
\[
\nu \| v_m \|^2 + \epsilon^2 \| A_N^{3/2} \phi_m \|_{L^2}^2 \leq \left( \| Q_0(0, 0) \|_{V_1^*} + L_1 \| (U, \Phi) \|_{\mathcal{V}} \right) + \alpha \epsilon \| A_N^{1/2} f(\Phi) \|_{L^2} \| (v_m, \phi_m) \|_{\mathcal{V}}.
\]

Let
\[ \kappa_1 \equiv \min(\nu, \epsilon^2). \]
It follows that
\[ \kappa_1 \|(v_m, \phi_m)\|_V^2 \leq (\|Q_0(0,0)\|_V^* + L_1\|(U, \Phi)\|_V^* + L_1\|\|\Phi\||_V)\|_V^* + M_1(|A_N\Phi|_{L^2}), \] (5.11)
which gives
\[ \|(v_m, \phi_m)\|_V \leq \kappa_1^{-1}(\|Q_0(0,0)\|_V^* + L_1\|(U, \Phi)\|_V^* + M_1(|A_N\Phi|_{L^2})); \] (5.12)
for a suitable monotone non-decreasing function independent of \( m \). Note that from (2.12), \(|A_N^{1/2}f(\Phi)|_{L^2} \leq M_1(|A_N\Phi|_{L^2})\).

**Theorem 5.2.** Suppose that \( Q_0 \) satisfies (5.2)-(5.3). We also assume that
\[ \kappa_1 - L_1 > 0. \] (5.13)
Then there exists at least one solution to (5.5).

**Proof.** Recall that \((v_m, \phi_m)\) satisfies the a priori estimates (5.6). Let \( K_0 > 0 \) such that \( K_0(\kappa_1 - L_1) \geq \|Q_0(0,0)\|_V^* + M_1(K_0) \). Then from (5.12) we can check that \( \|(v_m, \phi_m)\|_V \leq \kappa_0 \) provided that \( \|(U, \Phi)\|_V \leq \kappa_0 \).

Now let
\[ \wedge_m = \{(U, \Phi) \in V_m, \|(U, \Phi)\|_V \leq K_0 \}. \] (5.14)
Then \( \wedge_m \) is a compact and convex subset of \( V_m \). It is also clear that \( T_m \) maps \( \wedge_m \) into itself. To prove the existence of solution, we apply the Brouwer fixed point theorem to the restriction of \( T_m \) to \( \wedge_m \). Therefore it remains to check that \( T_m \) is continuous.

For the continuity of \( T_m \), we proceed as follows. Let \((v_1, \phi_1) = T_m(U_1, \Phi_1) \) and \((v_2, \phi_2) = T_m(U_2, \Phi_2) \), where \((U_1, \Phi_1), (U_2, \Phi_2) \in V_m \). Let \((w, \psi) = (v_1, \phi_1) - (v_2, \phi_2), (U, \Phi) = (U_1, \Phi_1) - (U_2, \Phi_2) \). Then from (5.5) can check that we \((w, \psi)\) satisfies
\[
\begin{align*}
\nu A_0w + B_0(U, v_1) + B_0(U, w) - R_0(\epsilon A_N\phi_1, \Phi) - R_0(\epsilon A_N\psi, \Phi_2) \\
= Q_0(U_1, \Phi_1) - Q_0(U_2, \Phi_2), \\
\epsilon A_N^2\psi + \alpha A_N f(\Phi_1) - \alpha A_N f(\Phi_2) + B_1(v_1, \Phi) + B_1(w, \Phi_2) = 0.
\end{align*}
\] (5.15)

Note that
\[
\begin{align*}
\langle R_0(\epsilon A_N\psi, \Phi_2), w \rangle &= \langle B_1(w, \Phi_2), \epsilon A_N\psi \rangle, \\
\|\langle Q_0(U_1, \Phi_1) - Q_0(U_2, \Phi_2), w \rangle \| &\leq L_1\|(U, \Phi)\|_V w, \\
\|\langle B_0(U, v_1), w \rangle \| &\leq \epsilon\|(U)\|_V v_1 \|w\|, \\
\|\langle R_0(\epsilon A_N\phi_1, \Phi), w \rangle \| &\leq \epsilon\epsilon\|A_N\Phi\|_{L^2} |A_N\phi_1|_{L^2} \|w\|, \\
\|\langle B_1(v_1, \Phi), \epsilon A_N\psi \rangle \| &\leq \epsilon\|A_N\Phi\|_{L^2} |A_N\psi|_{L^2} \|v_1\|, \\
\|\langle \alpha A_N f(\Phi_1) - \alpha A_N f(\Phi_2), \epsilon A_N\psi \rangle \| &\leq M_2(|A_N\Phi_1|_{L^2}, |A_N\Phi_2|_{L^2}, |A_N^{3/2}\Phi|_{L^2} |A_N^{3/2}\psi|_{L^2}. 
\end{align*}
\] (5.16)
Multiplying (5.15)\textsubscript{1} and (5.15)\textsubscript{2} by \(w\) and \(\epsilon A_N \psi\) respectively and using (5.16), we derive that
\[
\nu \|w\|^2 + \epsilon^2 |A_N^{3/2} \psi|^2_{L^2} \leq c \|U\| \|v\| + c \epsilon |A_N \Phi|_{L^2} |A_N \phi_1|_{L^2} \|w\|
+ c \epsilon |A_N \Phi|_{L^2} |A_N \psi|_{L^2} \|v\| + M_2(|A_N \Phi_1|_{L^2}, |A_N \Phi_2|_{L^2}) |A_N^{3/2} \Phi|_{L^2} |A_N^{3/2} \psi|_{L^2}
+ L_1 \|(U, \Phi)\| \|v\|, \tag{5.17}
\]
which gives
\[
\kappa_1 \|(w, \psi)\| \|v\| \leq (\|v\| + \epsilon |A_N \phi_1|_{L^2} + L_1) \|(U, \Phi)\| \|v\| \tag{5.18}
+ M_2(|A_N \Phi_1|_{L^2}, |A_N \Phi_2|_{L^2}) \|(U, \Phi)\| \|v\|,
\]
and the continuity of the mapping \(T_m\) follows. Note that \(M_2(\cdot, \cdot)\) denotes a suitable monotone non-decreasing function independent of \(m\).

It follows that there exists a fixed point \((v_m, \phi_m)\) of \(T_m\) in \(\mathcal{A}_m\). Therefore we can extract a subsequence (still denoted \((v_m, \phi_m)\)) that converges to \((v^*, \phi^*)\) strongly in \(V\). Using the same argument as in \cite{9}, we can prove that \((v^*, \phi^*)\) is a weak solution to (5.4).

\[\Box\]

5.2. Some a priori estimates on \((v^*, \phi^*)\). Hereafter, we assume that \(f\) satisfies the additional condition
\[
\alpha (A_N^{1/2} f(\psi), A_N^{3/2} \psi) \geq -\kappa_0 |A_N^{3/2} \psi|_{L^2}^2 \forall \psi \in D(A_N^{3/2}), \tag{5.19}
\]
where \(\kappa_0 > 0\) is a fixed constant. We derive some explicit a priori estimates in the \(V\)–norm and under some additional assumptions, we prove the uniqueness of solutions. In particular, we assume that \(\epsilon > 0\) is larger enough such that
\[
\epsilon > \kappa_0. \tag{5.20}
\]

**Theorem 5.3.** We assume that (5.20) is satisfied. Then any solution \((v^*, \phi^*)\) to (5.4) satisfies the following estimate:
\[
\|(v^*, \phi^*)\|_{V} \leq c \|Q_0(0,0)\|_{V^*_1} \equiv K_1. \tag{5.21}
\]
Moreover if
\[
\kappa_1 - c(2K_1 + M_2(K_1, K_1)) > 0, \tag{5.22}
\]
then the solution to (5.4) is unique.

**Proof.** To prove (5.21), by multiplying (5.4)\textsubscript{1} by \(v^*\) and (5.4)\textsubscript{2} by \(\epsilon A_N \phi^*\), to derive that
\[
\nu \|v^*\|^2 + \epsilon^2 |A_N^{3/2} \phi^*|^2_{L^2} + (\alpha A_N^{1/2} f(\phi^*), \epsilon A_N^{3/2} \phi^*) = (Q_0(v^*, \phi^*), v^*)
\leq \|Q_0(0,0)\|_{V^*_1} \|v^*\| + L_1 \|(v^*, \phi^*)\| \|v^*\| \tag{5.23}
\leq \|Q_0(0,0)\|_{V^*_1} \|(v^*, \phi^*)\| + L_1 \|(v^*, \phi^*)\|^2_{V^*_1}
\]
which gives (assuming (5.20))
\[
(\alpha_1 - L_1) \|(v^*, \phi^*)\|^2_{V^*_1} \leq \|Q_0(0,0)\|_{V^*_1} \|(v^*, \phi^*)\|_{V^*_1}, \tag{5.24}
\]
where
\[
\alpha_1 = \min(\nu, \epsilon^2 - \epsilon \kappa_0) > 0.
\]
We derive that
\[ \| (v^*, \phi^*) \|_V \leq (\alpha_1 - L_1)^{-1} \| Q_0(0, 0) \|_{V_1}, \]
and (5.21) is proved.

For the uniqueness, let \((v_1^*, \phi_1^*), (v_2^*, \phi_2^*)\) be two solutions and \((w, \psi) = (v_1^*, \phi_1^*) - (v_2^*, \phi_2^*)\). Then \((w, \psi)\) satisfies
\[
\begin{cases}
\nu A_0 w + B_0(w, v_1^*) + B_0(v_2^*, w) - R_0(\epsilon A_N \phi_2^*, \psi) - R_0(\epsilon A_N \psi, \phi_1^*) \\
\epsilon A_N^{3/2} \psi + \alpha A_N f(\phi_1^*) - \alpha A_N f(\phi_2^*) + B_1(v_2^*, \psi) + B_1(w, \phi_1^*) = 0.
\end{cases}
\]
Note that
\[ \langle R_0(\epsilon A_N \psi, \phi_1^*), w \rangle = \langle B_1(w, \phi_1^*), \epsilon A_N \psi \rangle, \]
\[ \| (B_0(w, v_1^*), w) \| \leq c\| v_1^* \| \| w \|^2, \]
\[ \| R_0(\epsilon A_N \phi_2^*, \psi), w \| \leq c\| A_N \psi \|_{L^2} \| A_N \phi_2^* \|_{L^2} \| w \|, \]
\[ \| B_1(v_2^*, \phi), A_N \psi \| \leq c\| A_N \psi \|_{L^2}^{3/2} \| v_2^* \|, \]
\[ \| \alpha A_N f(\phi_1^*) - \alpha A_N f(\phi_2^*), \epsilon A_N \psi \| \leq M_2(\| A_N \phi_1^* \|_{L^2}, \| A_N \phi_2^* \|_{L^2}) A_N^{3/2} \| \psi \|_{L^2} \]
\[ \leq M_2(K_1, K_1) A_N^{3/2} \| \psi \|_{L^2}^2. \]
Multiplying (5.26) and (5.26) by \(w\) and \(\epsilon A_N \psi\) respectively and using (5.27) yields
\[
\begin{align*}
\nu \| w \|^2 + \epsilon^2 A_N^{3/2} \psi_{L^2}^2 & \leq c(\| v_1^* \| + A_N \phi_2^* \|_{L^2} + \| v_2^* \| + M_2(K_1, K_1)) \| (w, \psi) \|_{V}^2 \\
& \leq c(2K_1 + M_2(K_1, K_1)) \| (w, \psi) \|_{V}^2.
\end{align*}
\]
which gives
\[ (\kappa_1 - c(2K_1 + M_2(K_1, K_1))) \| (w, \psi) \|_{V}^2 \leq 0, \]
and \(\| (w, \psi) \|_{V} = 0\) assuming (5.22), where \(\kappa_1 = \min(\nu, \epsilon^2)\), and the theorem is proved.

5.3. **Asymptotic behavior.** Hereafter we assume that
\[ Q_0 : \mathbb{Y} \to H_1 \]
satisfies
\[ |Q_0(v_1, \phi_1) - Q_0(v_2, \phi_2)|_{L^2} \leq L_1 |(v_1, \phi_1) - (v_2, \phi_2)|_V, \]
\[ \forall (v_1, \phi_1), (v_2, \phi_2) \in \mathbb{Y}, \]
for some fixed constant \(L_1 > 0\). We also assume that \(f\) satisfies
\[ \alpha(A_N^{1/2} f(\psi_1) - A_N^{1/2} f(\psi_2), A_N^{3/2} (\psi_1 - \psi_2)) \geq -\kappa_0 A_N^{3/2} (\psi_1 - \psi_2)^2 \]
\[ \forall \psi_1, \psi_2 \in D(A_N^{3/2}), \]
Theorem 5.4. We suppose that $\nu, \epsilon$ are large enough such that (5.22) and (5.20) are satisfied. We also assume that
\[-2\nu + K_1 + K_1^2 + L_1 \lambda^{-1} + \frac{L_1 \lambda^{-1}}{1 - \tau} < 0, \quad -\epsilon^2 + \kappa_0 \epsilon + K_1^2 + \frac{L_1 \lambda^{-1}}{1 - \tau} < 0. \tag{5.33}\]
Then any weak solution $(v, \phi)$ to (2.27) converges to the unique solution $(v^*, \phi^*)$ to (5.4) exponentially as $t$ goes to $\infty$. More precisely, we have the following estimate
\[|(v, \phi)(t) - (v^*, \phi^*)|^2 \leq Ce^{-\theta t} \left( |(v, \phi)(t) - (v^*, \phi^*)|^2 + \int_{-t}^0 |(\vartheta_1, \vartheta_2) - (v^*, \phi^*)|^2 ds \right), \tag{5.34}\]
for all $t \geq 0$, where $C > 0$ is a constant and $\theta > 0$ is given by (5.44).

Proof. Let $(w, \psi) = (v, \phi) - (v^*, \phi^*)$. Then $(w, \psi)$ satisfies
\[
\begin{aligned}
\frac{dw}{dt} + \nu A_0 w + B_0(w, v^*) - R_0(\epsilon A_N \psi, \phi) - R_0(\epsilon A_N \phi^*, \psi) \\
= Q_0((v, \phi)(t - \tau(t))) - Q_0((v^*, \phi^*)) ,
\end{aligned}
\tag{5.35}
\]
Let
\[\mathcal{Y}(t) = |(w, \psi)|^2 = |w|_{L^2}^2 + \epsilon \|\psi\|^2 . \]
Recall that
\[|b_0(w, v^*, w)| \leq c\|w\|^2\|v^*\| \leq K_1\|w\|^2, \tag{5.36}\]
\[\langle R_0(\epsilon A_N \phi^*, \psi), w \rangle \leq c\|w\|\|\psi\|^1/2\|A_N \psi\|_{L^2}^{1/2}\|A_N \phi^*\|_{L^2} \]
\[\leq \frac{c^2}{2} |A_N \psi|^2_{L^2} + c\|w\|^2\|A_N \phi^*\|_{L^2}^2, \tag{5.37}\]
\[\langle (B_1(v^*, \psi), \epsilon A_N \psi) \rangle \leq c\|v^*\|\|\psi\|^1/2\|A_N \psi\|_{L^2}^{1/2} \]
\[\leq \frac{c^2}{2} |A_N \psi|^2_{L^2} + c\|v^*\|^2\|A_N \psi\|_{L^2}^2, \tag{5.38}\]
\[\langle \alpha A_N f(\phi) - \alpha A_N f(\phi^*), \epsilon A_N \psi \rangle \geq -\kappa_0 \epsilon |A_N \psi|^2_{L^2}, \quad \text{ (see } (5.32) \text{) } \tag{5.39}\]
\[2|\langle -Q_0((v^*, \phi^*)) + Q_0((v, \phi)(t - \tau(t)), w) \rangle| \leq 2L_1 |(w, \psi)(t - \tau(t))| \|w\|_{L^2} \]
\[\leq L_1 \lambda^{-1}\|w\|^2 + L_1 |(w, \psi)(t - \tau(t))|_{L^2}^2. \tag{5.40}\]
Multiplying (5.35) by $w$ and (5.35) by $\epsilon A_N \psi$ and using (5.36)-(5.40) gives
\[
\begin{aligned}
\frac{d\mathcal{Y}}{dt} + (2\nu - K_1 - K_1^2 - L_1 \lambda^{-1})\|w\|^2 + (\epsilon^2 - \kappa_0 \epsilon - K_1^2)\|A_N \psi\|_{L^2}^2 \\
\leq L_1 |(w, \psi)(t - \tau(t))|_{L^2}^2. \tag{5.41}\n\end{aligned}
\]
Let $\gamma(t) = t - \tau(t)$ and $\mu > 0$ such that $\gamma^{-1}(t) \leq t + \mu$ for all $t \geq -\tau(0)$. Then setting $\zeta = s - \tau(s) = \gamma(s)$, we obtain that for $\theta > 0$

$$
\int_0^t e^{\theta s} |w(t - \tau(t))|^2_Y ds = \int_{-\tau(0)}^{t-\tau(t)} e^{\theta \gamma^{-1}(\zeta)} |w(\gamma^{-1}(\zeta))|^2_Y \frac{1}{\gamma'(\gamma^{-1}(\zeta))} d\zeta
$$

$$
\leq \frac{e^{\theta \mu}}{1 - \tau^*} \int_{-\tau}^t e^{\theta \zeta} |w(\gamma^{-1}(\zeta))|^2_Y d\zeta
$$

$$
\leq \frac{e^{\theta \mu}}{1 - \tau^*} \int_{-\tau}^t e^{\theta \zeta} Y(d\zeta).
$$

It follows from (5.41) and (5.42) that

$$
\frac{d}{dt}(e^{\theta t} Y) = \theta e^{\theta t} Y + e^{\theta t} \frac{dY}{dt}
$$

$$
\leq \theta e^{\theta t} |w|_{L_2}^2 + e^{\theta t} \|\psi\|^2 + e^{\theta t} (-2\nu + K_1 + K_1^2 + L_1 \lambda^{-1}) \|w\|^2 + e^{\theta t}(-2 + \kappa_0 \epsilon + K_1^2) |A_N \psi|_{L_2}^2 + e^{\theta t} L_1 |(w, \psi)(t - \tau(t))|^2_Y.
$$

If we choose $\theta$ such that

$$
 \theta \lambda_1^{-1} - 2\nu + K_1 + K_1^2 + L_1 \lambda^{-1} + \frac{L_1 \lambda^{-1}}{1 - \tau^*} e^{\theta \mu} < 0,
$$

$$
 -2 + \kappa_0 \epsilon + K_1^2 + \theta \lambda_1^{-1} \epsilon + \frac{L_1 \lambda^{-1}}{1 - \tau^*} e^{\theta \mu} < 0,
$$

then we derive that

$$
e^{\theta t} Y(t) \leq Y(0) + \frac{L_1 e^{\theta \mu}}{1 - \tau^*} \int_{-\tau}^0 e^{\theta s} |w(\gamma^{-1}(s))|^2_Y ds
$$

$$
\leq Y(0) + \frac{L_1 e^{\theta \mu}}{1 - \tau^*} \int_{-\tau}^0 e^{\theta s} Y(s) ds,
$$

and (5.34) follows. It is clear that (5.44) is satisfied for $\theta > 0$ small enough provided that (5.33) holds true.

**Conclusion.** In this article, we study a CH-NS model with delays in a two-dimensional domain. We prove the existence and uniqueness of the weak and strong solution when the external force contains some delays. We also discuss the asymptotic behavior of the weak solutions and the stability of the stationary solutions. There are still some interesting topics related to the CH-NS with delays. For instance, one can study the existence of a pullback attractor. We can also consider the case when the delay appears not only in the forcing terms, but also in the convective and the diffusion terms.

**Acknowledgments.** The author would like to thank the anonymous referees whose comments help to improve the contain of this article.
REFERENCES

[1] T. Blesgen, A generalization of the Navier-Stokes equation to two-phase flow, Physica D (Applied Physics), 32 (1999), 1119–1123.
[2] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Rational Mech. Anal., 92 (1986), 205–245.
[3] T. Caraballo, A. M. Marquez-Duran and J. Real, Pullback and forward attractors for a 3D LANS-α model with delay, Discrete Contin. Dyn. Syst., 15 (2006), 559–578.
[4] T. Caraballo and J. Real, Navier-Stokes equations with delays, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), 2441–2453.
[5] T. Caraballo and J. Real, Asymptotic behavior of two-dimensional Navier-Stokes equations with delays, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 459 (2003), 3181–3194.
[6] T. Caraballo and J. Real, Attractors for 2D Navier-Stokes models with delays, J. Differential Equations, 205 (2004), 271–297.
[7] E. Feireisl, H. Petzeltová, E. Rocca and G. Schimperna, Analysis of a phase-field model for two-phase compressible fluids, Math. Models Methods Appl. Sci., 20 (2010), 1129–1160.
[8] C. G. Gal and M. Grasselli, Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 401–436.
[9] C. G. Gal and M. Grasselli, Longtime behavior for a model of homogeneous incompressible two-phase flows, Discrete Contin. Dyn. Syst., 28 (2010), 1–39.
[10] C. G. Gal and M. Grasselli, Trajectory attractors for binary fluid mixtures in 3D, Chin. Ann. Math. Ser. B, 31 (2010), 655–678.
[11] P.C. Hohenberg and B. I. Halperin, Theory of dynamical critical phenomena, Rev. Modern Phys., 49 (1977), 435–479.
[12] T. Tachim Medjo, Pullback attractors for a non-autonomous homogeneous two-phase flow model, J. Diff. Equa., 253 (2012), 1779–1806.
[13] A. Omuku, Phase transition of fluids in shear flow, J. Phys. Condens. Matter, (2009), 641–709.
[14] T. Taniuchi, The exponential behavior of Navier-Stokes equations with time delay external force, Discrete Contin. Dyn. Syst., 12 (2005), 997–1018.
[15] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, volume 68. Appl. Math. Sci., Springer-Verlag, New York, second edition, 1997.

Received November 2015; revised July 2016.

E-mail address: tachimt@fiu.edu