In [1], we describe the design and development of a task taxonomy for temporal graph visualisation. This paper details the full instantiation of that task taxonomy. Our task taxonomy is based on the Andrienko framework [2], which uses a systematic approach to develop a formal task framework for visual tasks specifically associated with Exploratory Data Analysis. The Andrienko framework is intended to be applicable to all types of data, however, it does not consider relational (graph) data. We therefore extended both their data model and task framework for temporal graph data, and instantiated the extended version to produce a comprehensive list of tasks of interest during exploratory analysis of temporal graph data. As expected, our instantiation of the framework resulted in a very large task list; with more than 144 variations of attribute based tasks alone, it is too large to fit in a standard journal paper, hence we provide the detailed listing in this document.

This paper is organised as follows: in section 1 we give a short overview of the task categories of the taxonomy, a more detailed explanation of which is provided in [1]. A key notion in the Andrienko framework is that of behaviours: in section 2 we briefly summarise the partial and aspectual behaviours of interest when analysing temporal graphs. In section 3 we provide a short guide to the formal notation used in the task definitions of the original framework. Finally, in section 4, we give the complete task listings for both structural and attribute based graph visualisation tasks.

1 Overview of task categories

Under the Andrienko framework, there are two components to every task: the target, or unknown information to be obtained, and the constraints, or the known conditions that information needs to fulfil. These targets and constraints distinguish the tasks and determine the shape of the model. We provide an overview of the task categories in Figure 1, which is based on Aigner et al.’s [3] representation of the original Andrienko task model organised into a taxonomy. We have redrawn and extended their figure to include our addition of structural tasks to the taxonomy. A detailed description of the task categories is provided in [1].

![Figure 1 The task model. Based on Aigner et al.'s drawing [2, p74] of the Andrienko task model organised into a taxonomy, redrawn and extended to include structural tasks for graph visualisation.](image)
2 Behaviours
Behaviours are a key notion in the Andrienko framework: they are representative of real-world phenomena, describing the configurations of sets of attribute values over the independent (referential) component of the data, as determined by the data function (the mapping between the independent and dependent data components), and the relations which exist between the elements of the independent data component e.g. distance and order. A pattern results from an observation of a behaviour and provides a descriptive summary of its essential features. Four main types of pattern are distinguished under the Andrienko framework: association, differentiation, arrangement and distribution. Where there are multiple independent components of the data (as in the case of temporal graphs), they distinguish between overall behaviours (which consider all behaviours over the entire dataset) and aspectual behaviours (which consider only certain aspects of the overall behaviour).

In addition to the attribute based behaviours of the Andrienko framework, we introduce the notion of structural behaviours in temporal graphs. We here outline the two partial and two aspectual attribute-based behaviours applicable to temporal graphs, along with the four analogous structural behaviours.

2.1 Attribute based behaviours
Partial behaviours:
1. The behaviour of an attribute of a single graph element (a node, edge, or graph object) over time (the whole time period or a subset of time) e.g. a temporal trend in the attribute of a node.
2. The behaviour of an attribute over a set of nodes (or subset of nodes/graph object) at a single time e.g. the distribution of the attribute values over the set.

Aspectual behaviours:
1. The behaviour of the temporal trends (1, above) over the graph (i.e. the distribution of temporal behaviours over the graph).
2. The behaviour over time of the behaviours of attribute values over the set of graph objects (2, above) (i.e. the temporal trend in the distribution of the attribute values over the graph).

2.2 Structural behaviours
(1) The behaviour of association relations between two graph objects over time e.g. the pattern of change in connectivity between two nodes over time.

(2) The behaviour, or configuration, of association relations within a set of nodes at a single time e.g. clusters, cliques, motifs etc.

(3) The behaviour of the collection of patterns in (1) i.e. the aggregate pattern of all association relations between pairs of graph objects over time, or the distribution of individual temporal behaviours over the graph.

(4) The behaviour of the configurations of association relations over the set of nodes (i.e. (2)), over time.

3 Formal Notation
This section provides a brief summary of the formal notation used to represent variations in tasks in the framework.

3.1 Data function applied to temporal graphs
In the case of temporal graphs, we use the following formalism to represent the Andrienko data function which maps a graph element at a particular time point to the corresponding values of the attributes in the data set:

\[ f(t, g) = (y_1, y_2, ..., y_n) \]
Where:

- \( t \) represents a time point
- \( g \) represents a graph element (node, edge, graph object)
- \( y_1, y_2, \ldots, y_N \) represents the \( N \) attributes in the data set

### 3.2 Key to formal notation

| **Bold** | **a specified value (constant)** |
|----------|----------------------------------|
| **Italics** | **an unknown value (variable)** |
| **t** | **a time point** |
| **T'** | **a (sub)set of time points/a time interval** |
| **g** | **a graph element (node, edge, graph object)** |
| **G', G''** | **a (sub)set of graph elements** |
| **\( \gamma \)** | **the value of an unknown characteristic** |
| **c** | **a specified characteristic** |
| **C'** | **a subset of characteristics** |
| **\( \Lambda, \Psi, \Phi, \lambda, \psi, \phi \)** | **a relation (e.g. \( y_1 \lambda y_2 \) can be read as ‘the relation between’ \( y_1 \) and \( y_2 \))** |
| **\( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 \in T') \)** | **the behaviour \( \beta \) of a data function \( f \) over the set of graph objects \( G' \), and time interval \( T' \), where \( x_1 \) is a graph object in the set of graph objects \( (G') \) and \( x_2 \) is a time point in the time interval \( (T') \)** |
| **\( \beta_0[\beta_1(f(x_1, x_2) \mid x_1 \in G') \mid x_2 \in T] \)** | **formulae representing the two aspectual behaviours: the behaviour of the temporal behaviours (trends) over the graph (i.e. the distribution of temporal behaviours over the graph), and the behaviour over time of the behaviours (distributions) of attribute values over the set of graph elements (i.e. the temporal trend in the distribution of the attribute values)** |
| **P** | **a known pattern** |
| **p** | **An unknown pattern** |
| **\( \approx \)** | **‘approximates’** |

### 4 Tasks

In this section we list the tasks associated with analysis of temporal graph data. In order to systematically specify all possible permutations of tasks under the framework, we used a series of task matrices when generating the tasks. The comparison and relation seeking matrices can also be found in their complete form at [http://www.iidi.napier.ac.uk/c/downloads/downloadid/13377254](http://www.iidi.napier.ac.uk/c/downloads/downloadid/13377254) for easier reading and printing.

#### 4.1 Attribute based tasks

##### 4.1.1 Lookup tasks

**4.1.1.1 Elementary lookup**

In the temporal graph case, elementary lookup tasks involve the correspondence between a graph element (a node, edge, or graph object) at a particular time point, and its associated attribute value. In *direct lookup*, given a graph element at a given time point, we seek to find the corresponding attribute value. In *inverse lookup*, we seek to find the graph element(s) and/or time point(s) associated with a given attribute value. There are a number of possible variations of inverse lookup, depending on the additional constraints involved: we may specify just the attribute value, or in addition specify either a time point or graph object. These variations are shown in the lookup task matrix (Figure 3).
Additional task variations which are not shown in the matrix, but can be formulated based on the tasks in the matrix, include:

- The case where the attribute value is imprecisely specified, and we allow a set of attribute values e.g. where the value is ‘greater than 50’ or in the set {red, green, blue}: $\exists t, g: f(t, g) \in C'$

- Specifying a subset of graph elements or time points (e.g. an interval) (in place of a single graph element or time point) as an additional constraint e.g.
  - Find the time(s) at which any graph object in the specified subset have the given attribute value: $\exists t: f(t, G') = c$
  - Find the graph object(s) which have the given attribute value at any time during the given time interval: $\exists g: f(T', g) = c$

- Where the values of either time or graph are of no importance, we allow the whole set of time points or graph elements to be specified:
  - Find the time(s) at which any graph object had the given attribute value: $\exists t: f(t, G) = c$
  - Find the graph object(s) which had the given attribute value at any time: $\exists g: f(T, g) = c$

### 4.1.1.2 Synoptic lookup

Behaviour characterisation involves finding the pattern which approximates the behaviour of an attribute over a reference set (or subset). Pattern search is the opposite: given a pattern, we find the subset of references over which the behaviour corresponds to the specified pattern. In the temporal graph case, these tasks involve the aspectual and partial behaviours described in section 2, and the corresponding graph and temporal references. This results in three task variations, depending on the referential components involved. We outline these in quadrants 2-4 of Figure 2. Further variations in the pattern search task depend on which referential components are specified, and these are detailed in the full lookup task matrix (Figure 3).
Figure 2 Quadrant-level overview of the lookup task matrix

| Time Intervals | Graph Elements (nodes, edges, graph objects) | Graph subsets |
|----------------|---------------------------------------------|---------------|
| Q1 Elementary  | **Task components:**
|                | Referrers are graph elements and time points; characteristics are attribute values. |
|                | **Direct lookup**
|                | \(? y: f(t, g) = y\) Involves finding the attribute value of a given graph element at a given time point. |
|                | **Inverse lookup**
|                | \(? t, g: f(t, g) = c\) Involves finding the graph element(s)/time point(s) associated with a given attribute value |
| Q2 Synoptic    | **Task components:**
|                | The referential component involves the whole graph (or a subset of the graph) and a single time point; behaviour is that of an attribute over the graph (at a single time). |
|                | **Behaviour characterisation**
|                | \(? p: \beta(f(x_1, x_2)) | x_1 \in G, x_2 = t) = p\)
|                | Involves finding the pattern which approximates the behaviour of an attribute over the graph (or a specified subset of the graph) at the given time point. |
|                | **Pattern search**
|                | \(? G, t: \beta(f(x_1, x_2)) | x_1 \in G, x_2 = t) = P\)
|                | Involves finding the time point(s) and/or subset(s) of graph elements over which a given pattern of attributes occur. |
| Q3 Synoptic    | **Task components:**
|                | The referential component involves the whole time period (or a time interval) and a single graph element; behaviour is that of an attribute of a single graph element over time. |
|                | **Behaviour characterisation**
|                | \(? p: \beta(f(x_1, x_2)) | x_2 \in T) \approx p\)
|                | Involves finding the pattern which approximates the behaviour of an attribute of a given graph element over the whole time period (or a specified time interval). |
|                | **Pattern search**
|                | \(? g, T: \beta(f(x_1, x_2)) | x_1 = g, x_2 \in T) = P\)
|                | Involves finding the graph element(s) and/or time interval(s) over which a given pattern of attributes occurs. |
| Q4 Synoptic    | **Task components:**
|                | The referential component involves the whole time period (or a time interval) and the whole graph (or a subset of the graph); behaviour is either of the two aspectual behaviours: the distribution of temporal trends over the graph or the distributions of an attribute over the graph, over time. |
|                | **Behaviour characterisation**
|                | \(? p: \beta_2(\beta_1(f(x_1, x_2)) | x_1 \in G) | x_2 \in T) = p\)
|                | Involves finding the pattern that approximates the aspectual behaviours: the behaviour of the temporal behaviours (trends) over the graph (i.e. the distribution of temporal behaviours over the graph) or the behaviour over time of the behaviours (distributions) of attribute values over the set of graph objects (i.e. the temporal trend in the distribution of the attribute values); in both cases we may be interested in the behaviour associated with a given subset of the time period or the graph. |
|                | **Pattern search**
|                | \(? T, G: \beta_2(\beta_1(f(x_1, x_2)) | x_1 \in G, x_2 \in T) = P\) or \(? G, T: \beta_2(\beta_1(f(x_1, x_2)) | x_1 \in G) | x_2 \in T) = P\)
|                | Involves finding the subset(s) of time and/or graph elements over which a (sub)pattern of an aspectual behaviour occurs. |
| Time point | Target | Graph Elements | Constraint | Target | Graph subsets | Constraint | Target |
|------------|--------|----------------|------------|--------|--------------|------------|--------|
|            |        | **Direct lookup** | given a graph object and time, find the attribute value | inverse lookup | given an attribute value and a time point, find the graph object(s) which have this value | behaviour characterisation | find the pattern that approximates (i.e. characterise) the behaviour of an attribute over the graph (or a subset of the graph) at the given time point |
|            |        | $y: f(t, g) = y$ | $y: g(t, g) = c$ | $\beta$: $\theta(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ |
|            |        | inverse lookup | $y: g(t, g) = c$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\theta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ |
|            |        | behaviour characterisation | characterise the behaviour of a single node over time. | pattern search | find the node(s) over which a particular pattern of attribute values occurs, over the given time interval. | behaviour characterisation | (i) characterise the behaviour of the temporal trends over the graph (i.e. the distribution of temporal behaviours over the graph) |
|            |        | $\beta(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(g(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ |
|            |        | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup |
|            |        | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs |
|            |        | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ |
|            |        | behaviour characterisation | characterise the behaviour of the attribute values over the graph, over time | pattern search | find the node(s) and time interval(s) over which a given pattern in the pattern of attribute values over the graph occurs | pattern search | (i) Find the subset(s) of graph elements over which a given pattern in the collection of temporal trends occurs, over the given time interval |
|            |        | $\beta(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(g(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ |
|            |        | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup |
|            |        | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs |
|            |        | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ |
|            |        | behaviour characterisation | characterise the behaviour of the attribute values over the graph, over time | pattern search | find the time interval(s) over which a given pattern in the pattern of attribute values over the graph occurs | pattern search | (ii) find the subset(s) of the graph over which a given (temporal) pattern in the pattern of attribute values over the graph occurs |
|            |        | $\beta(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(g(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ |
|            |        | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup |
|            |        | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs |
|            |        | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ |
| Target     |        | pattern search | find the time interval over which a given pattern of attribute values occurs for a given node. | pattern search | find the time interval(s) over which the specified pattern of attribute values occurs | pattern search | (i) Find the subset(s) of graph elements and time interval(s) over which a given pattern in the collection of temporal trends occurs |
|            |        | $\beta(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(g(f(x_1, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ | $\beta(f(\{x_i\}, x_2) | x_1 \notin G, x_2 = t) = \rho$ |
|            |        | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup | inverse lookup |
|            |        | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs | given an attribute value, find the graph object(s), and the time point(s), at which the value occurs |
|            |        | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ | $t: f(t, g) = c$ | $t: g(t, g) = c$ |
4.1.2 Comparison
Comparison tasks are compound tasks, which consist of lookup tasks to find the elements to be compared, and comparison of these elements to find the relation between them. Direct and inverse comparison are distinguished based on the lookup tasks and resulting elements involved in the comparison subtask. In the elementary case, in direct comparison, we use direct lookup and compare the found attribute values; in inverse comparison, we use inverse lookup and compare references (time points and/or graph elements). In the synoptic case, direct comparison involves behaviour characterisation subtasks and comparison of patterns, while inverse comparison involves pattern search subtasks and comparison of the associated graph subsets and/or time intervals.

4.1.2.1 Direct comparison
In both the elementary and synoptic direct comparison case, four subtasks are distinguished in the Andrienko framework, depending on whether:

- One of the attribute values/patterns involved in the comparison is specified or two lookup/behaviour characterisation subtasks are required
- the attributes involved in each lookup/behaviour characterisation task are the same or different
- the references involved in each lookup task are the same or different

As temporal graphs have two referrers, “the same reference” implies the same graph element or graph subset at the same point in time or \( t_1 = t_2, g_1 = g_2 \); we use just \( t \) and \( g \) to indicate this in the task listings); there are three possible variations of what could be meant by “different references” (note that the same applies to subsets of the graph and time intervals):

a. The same graph object at different time points \( t_1 \neq t_2, g_1 = g_2 \)

b. Different graph objects at the same time point \( t_1 = t_2, g_1 \neq g_2 \)

c. Two different graph objects at two different time points \( t_1 \neq t_2, g_1 \neq g_2 \)

Task a. is the typical temporal graph scenario: we refer to this type of task as an evolutionary task, as we are interested in how the properties of a graph element have changed or evolved between time points. We refer to tasks b and c as contextual tasks, as we often carry out such tasks in order to put the properties of one graph object in the context of another. Task b is also applicable to static graphs, as this is equivalent to considering the graph at a single time point.

Note that we do not show the variations of tasks involving the same/different attributes in the task matrix, but all tasks (with the exception of direct comparisons involving the same time point/interval and graph element/subset) could potentially be formulated to consider comparison involving the same attributes or two different attributes in the lookup subtask.

4.1.2.2 Inverse comparison
Three variations of the inverse comparison task are identified in the Andrienko framework based on:

- Whether two inverse lookup subtasks are involved, or one of the references (time point/graph element) or reference subsets (time interval/graph subset) is specified.
• Whether the attribute involved in each subtask is the same or two different attributes are considered

As noted above, we do not show variations of tasks involving the same/different attributes, but these can potentially be formulated for each task. The large number of tasks in the task matrix are derived from the permutations of references (time point/interval, graph element/subset) and to what degree the references in the subtasks are specified.

We summarise the comparison tasks at the quadrant level, based on the references involved, in Figure 4.

Figure 4 Quadrant-level overview of the comparison task matrix

| Graph Elements (nodes, edges, graph objects) | Graph subsets |
|-------------------------------------------|--------------|
| **Q1 Elementary** Direct comparison | Q2 Synoptic Direct comparison |
| ? y, y, λ: f_1(t_1, g_1) = y; f_2(t_2, g_2) = y; y, y, λ | ? p_1, p_2, λ: β(f(x_1, x_3), x_2) \mid x_1 \in G', x_2 = t_1 = p_1; β(f(x_1, x_3), x_2) \mid x_2 \in G''; x_2 = t_2 = p_2; p_1, p_2 |
| - of attribute values associated with a given graph element at a given time (the attribute involved in the lookup tasks may be the same or different, hence the data functions f_1(x) and f_2(x)). Relations: | - of two patterns of an attribute(s) over the graph (or a subset of the graph elements) at given time point(s) Relations: |
| • between attribute values are domain dependent. | • between patterns: same(similar)/different/opposite² |
| **Inverse comparison** | **Inverse comparison** |
| ? t_1, t_2, g_2, λ: f(t_1, g_1) \in C'; f(t_2, g_2) \in C''; (t_1, g_1) λ(t_2, g_2) | ? G', G'', t_1, t_2, λ, ψ: β(f(x_1, x_3), x_2) \mid x_1 \in G'; x_2 = t_1 = P_1; β(f(x_1, x_3), x_2) \mid x_2 \in G''; x_2 = t_2 = P_2; (G', t_1) λ (G'', t_2); t_1, t_2 |
| - of two graph elements and/or two time points associated with given attribute values Relations: | - of the time points at which the given patterns occur - of the graph subsets over which a given pattern occurs; - comparison of both time points and graph subsets. Relations: |
| • between graph elements: equality (same/different element); set relations (between the sets of elements belonging to graph objects); equality of configuration (in graph objects); association (between nodes/graph objects, at a single time point only); • between two time points: happens before(after), happens at the same time [4]. |

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1 i.e. each pattern may correspond to a different attribute
2 In descriptive synoptic tasks (in connectional synoptic tasks, patterns of “mutual” behaviours include correlation, dependency, and structural connection.
Notes on comparison task matrix:

- Due to issues of space on the printed page, we here show each quadrant of the comparison task matrix separately. The compiled task matrix can be found at http://www.iidi.napier.ac.uk/c/downloads/downloadid/13377254.
- In the following tasks we use \( G', t_j \) to specify a graph subset at a given time (as opposed to just \( G' \)). This is due to the nature of the graph referrer: as association relations in the graph referrer may change over time, a graph object at \( t_1 \) may be quite different from “the same” graph object at \( t_2 \).
- Where both graph elements/subsets and/or both time points/intervals are unspecified, we can add an additional constraint to the task i.e. that the components in question have a specified relation between them e.g. in the case of the graph referrer, that they are the same, connected, a certain distance from one another etc. or in the case of time that they are the same, overlapping, a given distance from one another etc. Where we restrict graph elements/subsets to being the same, and the temporal component is different, these become evolutionary tasks e.g. compare the time intervals over which two patterns occur over two time intervals for the same graph object:

\[ g, T', T'', \lambda, \psi: \theta(f(x_1, x_2) \mid x_1 = g, x_2 \in T') = P_1; \theta(f(x_1, x_2) \mid x_1 = g, x_2 \in T'') = P_2; T' \psi T'' \]
Figure 5 Comparison task matrix, quadrant 1: considers comparisons involving graph elements (nodes, edges, graph objects) and time points (i.e. the elementary comparison tasks)

| Time points | Single/same element | Two different elements | One element specified | Neither element specified |
|-------------|---------------------|------------------------|-----------------------|--------------------------|
| Both constraints | Direct comparison | Direct comparison | Inverse comparison | Inverse comparison |
| Same time | Compare the values of different attributes for a given node at a given time point. | Compare the attribute values associated with two different nodes at the same time point. | This task reduces to comparison with a specified reference. Find and compare with a given node, the node(s) associated with the given attribute value at the given time. | Find and compare the nodes associated with two different attribute values at the given time. |
| ? y₁, y₂, λ: | f₁(t, g₁) = y₁; f₂(t, g₂) = y₂; y₂ ∈ C₂ | ? y₁, y₂, λ: | f₁(t, g₁) = y₁; f₂(t, g₂) = y₂; y₂ ∈ C₂ | ? g₁, g₂, λ: |
| | y₁, y₂ | | y₁, y₂ | f₁(t, g₁) ∈ C; f₂(t, g₂) ∈ C’; (t, g₁) λ (t, g₂) |
| Different times | Direct comparison | Direct comparison | Inverse comparison | Inverse comparison |
| | Compare the attribute values associated with a single node at two different times. | Compare the attribute values associated with two different nodes at two different times. | As above but involving two different time points. Find and compare with a given node, the node(s) associated with the given attribute value at the given time. | As above, but involving two different time points. Find and compare the nodes associated with two different attribute values at the given times. |
| ? y₁, y₂, λ: | f₁(t₁, g₁) = y₁; f₁(t₂, g₂) = y₂; y₂ ∈ C₂ | ? y₁, y₂, λ: | f₁(t₁, g₁) = y₁; f₁(t₂, g₂) = y₂; y₂ ∈ C₂ | ? g₁, g₂, λ: |
| | y₁, y₂ | | y₁, y₂ | f₁(t₁, g₁) ∈ C; f₁(t₂, g₂) ∈ C’; (t₁, g₁) λ (t₂, g₂) |
| One time point specified | Inverse comparison | Inverse comparison | Inverse comparison | Inverse comparison |
| | This task reduces to comparison with a specified reference. Find the time point(s) associated with the given attribute value for the given node, and compare it with a given time point. | As left, this task reduces to comparison with a specified reference. | Either: A task reduced to comparison with a specified reference. Find the node(s) and time point(s) at which it has a given attribute value, and compare this with a given node at a given time point. | Find the node(s) having a specified attribute value at a given time, and the node(s) and time point(s) having a given attribute value, and compare the nodes and time points. |
| | ? t₂, λ: | f(t₂, g) ∈ C; | t₁, t₂ | ? t₂, g₂, λ: |
| | | t₁, t₂ | | f(t₁, g₁) ∈ C; f(t₂, g₂) ∈ C’; (t₁, g₁) λ (t₂, g₂) |
| Graph elements (nodes, edges, graph objects) | Both constraints | One element specified | Neither element specified |
|---------------------------------------------|------------------|----------------------|--------------------------|
|                                             |                  |                      |                          |
| **Single/same element**                     |                  |                      |                          |
| **Two different elements**                 |                  |                      |                          |
| **Neither element specified**               |                  |                      |                          |
| **Inverse comparison**                      |                  |                      |                          |
| Find and compare the times at which the given node had the given attribute values. |                  |                      |                          |
| $? t_1, t_2, \lambda$:                     |                  |                      |                          |
| $f(t_1, g) \in C'$; $f(t_2, g) \in C''$;  |                  |                      |                          |
| $t_2 \lambda t_2$                         |                  |                      |                          |

**Inverse comparison** Find and compare the times at which two given nodes had the given attribute values.

$? t_1, t_2, \lambda$:  
$f(t_1, g_1) \in C'$; $f(t_2, g_2) \in C''$;  
$t_2 \lambda t_1$  

**Inverse comparison** Find the time point(s) at which a given node had a given attribute value, and the time point(s) and node(s) having a second given attribute value, and compare the nodes and time points.

$? t_1, t_2, g_2, \lambda$:  
$f(t_1, g_1) \in C'$; $f(t_2, g_2) \in C''$;  
$(t_1, g_1) \lambda (t_2, g_2)$  

**Inverse comparison** Find the time point at which a given node has a given attribute value, and the node which has a given attribute value at a given time, and compare the nodes and time points.

$? t_1, g_2, \lambda, \Psi$:  
$f(t_1, g_1) \in C'$;  
$(t_1, g_1) \lambda (t_2, g_2)$;  
$t_1 \Psi t_2$  

OR  
Find the time point at which a given node has a given attribute value, and the node which has a given attribute value at a given time, and compare the nodes and time points.

$? t_1, g_2, \lambda, \Psi$:  
$f(t_1, g_1) \in C'$; $f(t_2, g_2) \in C''$;  
$(t_1, g_1) \lambda (t_2, g_2)$;  
$t_1 \Psi t_2$  

**Inverse comparison** Find the time points associated with two given attribute values and compare them.

$? t_1, t_2, g_2, \lambda$:  
$f(t_1, g_1) \in C'$; $f(t_2, g_2) \in C''$;  
$(t_1, g_1) \lambda (t_2, g_2)$  

**Inverse comparison** Find the time points at which a given node had a given attribute value, and the node which has a given attribute value at a given time, and compare the nodes and time points.

$? t_1, t_2, g_2, \lambda$:  
$f(t_1, g_1) \in C'$; $f(t_2, g_2) \in C''$;  
$(t_1, g_1) \lambda (t_2, g_2)$
**Figure 6 Comparison quadrant 2: considers comparisons involving the behaviour of an attribute over the graph (or a graph subset)**

| Time points | Both constraints | Graph subsets | Both are targets |
|-------------|-----------------|---------------|-----------------|
| Same time   | Both constraints | **Same subset** | **Different subsets** | **One constraint, one target** | **Both are targets** |
| Direct comparison of the attribute patterns of two different attributes over the same subset of the graph at the same time point. | | | Direct comparison of the attribute patterns over two different subsets of the graph at the same time point. | Direct comparison of the given graph subset with the graph subset associated with a given pattern at a given time. | Direct comparison of two graph subsets associated with two given patterns at the same specified time. |
| \( ? p_1, p_2, \lambda; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t) = p_1; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_2) = p_2; \) \( p_1 \lambda p_2 \) | Direct comparison of the attribute patterns over two different subsets of the graph at the same time point. | Direct comparison of the given graph subset with the graph subset associated with a given pattern at a given time. | Direct comparison of two graph subsets associated with two given patterns at the same specified time. | \( ? G', G'', \lambda; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t) = p_1; \) \( \beta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t) = p_2; \) \( G', t) \lambda (G'', t) \) |
| Different times | Direct comparison of the attribute patterns over the same subset of the graph at two different time points. | Direct comparison of the attribute patterns over two different subsets of the graph at two different time points. | Direct comparison as above, but the specified subset of graph elements is associated with a different time point: | Direct comparison of two graph subsets associated with two given patterns at two different, specified time points. | \( ? G', G'', \lambda; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_1) = p_1; \) \( \beta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t_2) = p_2; \) \( (G', t_1) \lambda (G'', t_2) \) |
| \( ? p_1, p_2, \lambda; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_3) = p_1; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_2) = p_2; \) \( p_1 \lambda p_2 \) | | | | |

\( ^3 \) Reduced from: \( t_2, \lambda; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_1) = p_1; \) \( \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_2) = p_2; \) \( t_1, t_2 \) i.e. all information (the graph subset, time point and pattern) is known in the first lookup subtask.

\( ^4 \) NB in this case, the formula from which this task is reduced would have involved two different graph subsets.
Graph subsets

| Both constraints | One constraint, one target | Both are targets |
|------------------|---------------------------|-----------------|
| **Same subset**  | **Different subsets**     |                 |
|                  |                           |                 |
| ? \( t_2 \lambda \): \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_2 \lambda t_2 \) | pattern. This may involve only one lookup subtask\(^5\) or two: | ? \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_2 \psi t_2 \) |
| \( \theta(f(x, y) \mid x \in G', x_2 = t_3) = P_1; t_3 \lambda t_2 \) | ? \( G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G'' \), \( x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 | \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_3 \psi t_2 \) |
|                  | \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_2 \lambda t_2 \) | or | \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 |
| Inverse comparison of the time points at which two different patterns occur, over the same graph subset. | Inverse comparison of the graph subsets associated with given patterns, where one of the graph subsets is specified, but the time at which it occurs is unknown, the other graph subset and time at which the pattern occurs is not specified. In addition, we may wish to compare the time points at which the patterns occurred. | Inverse comparison of the graph subsets and time points associated with two given patterns. |
| ? \( t_1 \lambda t_2 \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; t_2 \lambda t_2 \) | ? \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 | ? \( G', G'' \), \( t_1 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 |
| \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_2 \lambda t_2 \) | ? \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 | \( G', G'' \), \( t_1 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 |
| \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_2 \lambda t_2 \) | ? \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 | \( G', G'' \), \( t_1 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 |
| \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; t_2 \lambda t_2 \) | ? \( G', G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 | \( G', G'' \), \( t_1 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_1) = P_1; \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 |

\(^5\) The first task is reduced from: ? \( G'' \), \( t_2 \lambda \), \( \psi \): \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_1; \) \( \theta(f(x, y) \mid x \in G', x_2 = t_2) = P_2; \) (G', t_3) \( \lambda \) (G'', t_2); t_3 \psi t_2 i.e. all information (the graph subset, time point and pattern) is known in the first lookup subtask.
**Figure 7 Comparison quadrant 3: considers comparisons involving the behaviour of an attribute of a single graph element over time (i.e. a temporal trend)**

| Time intervals | Both Constraints | One graph element specified | Neither graph element specified |
|----------------|-----------------|----------------------------|-------------------------------|
| Same interval  | Direct comparison of the patterns of two different graph elements over the same time interval. | Inverse comparison of a graph element associated with a given pattern over a given time interval, with a given graph element. | Inverse comparison of two graph elements associated with given patterns over the same given time interval. |
| ?p₁, p₂, λ; \( \theta(f(x₁, x₂) \mid x₁ = g₁, x₂ ∈ T') = p₂ \); \( \theta(f(x₂, x₃) \mid x₂ = g₂, x₃ ∈ T') = p₂ \); p₁ \( \lambda \) p₂ | ?p₁, p₂, λ; \( \theta(f(x₁, x₂) \mid x₁ = g₁, x₂ ∈ T') = p₂ \); \( \theta(f(x₂, x₃) \mid x₂ = g₂, x₃ ∈ T') = p₂ \); p₁ \( \lambda \) p₂ | ?g₂, λ; \( \theta(f(x₃, x₄) \mid x₃ = g₃, x₄ ∈ T''') ≈ P; T'' \); \( g₁ \lambda g₂ \) | \( ?g₁, g₂ \lambda; \) \( \theta(f(x₅, x₆) \mid x₅ = g₅, x₆ ∈ T') = P₁; \theta(f(x₆, x₇) \mid x₆ = g₆, x₇ ∈ T') = P₂; T₁ \( \lambda \) T₂ |
| Different intervals | Direct comparison of the patterns of two different graph elements over two different time intervals. | Inverse comparison as above. | Inverse comparison of two graph elements associated with given patterns over the two different given time intervals. |
| ?p₁, p₂, λ; \( \theta(f(x₁, x₂) \mid x₁ = g₁, x₂ ∈ T') = p₂ \); \( \theta(f(x₂, x₃) \mid x₂ = g₂, x₃ ∈ T') = p₂ \); p₁ \( \lambda \) p₂ | ?p₁, p₂, λ; \( \theta(f(x₁, x₂) \mid x₁ = g₁, x₂ ∈ T') = p₂ \); \( \theta(f(x₂, x₃) \mid x₂ = g₂, x₃ ∈ T') = p₂ \); p₁ \( \lambda \) p₂ | ?g₂, λ; \( \theta(f(x₃, x₄) \mid x₃ = g₃, x₄ ∈ T''') ≈ P; T'' \); \( g₁ \lambda g₂ \) | \( ?g₁, g₂ \lambda; \) \( \theta(f(x₅, x₆) \mid x₅ = g₅, x₆ ∈ T') = P₁; \theta(f(x₆, x₇) \mid x₆ = g₆, x₇ ∈ T') = P₂; T₁ \( \lambda \) T₂ |
| One constraint, one target | Inverse comparison of a time interval associated over which a given pattern occurs for a given graph element, with a specified time interval. | Inverse comparison as left. | Inverse comparison of two graph elements associated with given patterns (one of which is a pattern over a specified time interval) and comparison of the time intervals over which the patterns occur. |
| ?T''', λ; | | | ?g₁, g₂, T'''; λ, ψ: |

6 Reduced from: \( ?g₂, λ; \) \( \theta(f(x₅, x₆) \mid x₅ = g₅, x₆ ∈ T') = P₁; \theta(f(x₆, x₇) \mid x₆ = g₆, x₇ ∈ T') = P₂; g₁ \( \lambda \) g₂
7 Reduced from: ?g₂, λ; \( \theta(f(x₅, x₆) \mid x₅ = g₅, x₆ ∈ T') = P₁; \theta(f(x₆, x₇) \mid x₆ = g₆, x₇ ∈ T') = P₂; g₁ \( \lambda \) g₂
8 Reduced from: ?T''', λ; \( \theta(f(x₅, x₆) \mid x₅ = g₅, x₆ ∈ T') = P₁; \theta(f(x₆, x₇) \mid x₆ = g₆, x₇ ∈ T') = P₂; T₁ \( \lambda \) T'''
9 Reduced from: ?T''', λ; \( \theta(f(x₅, x₆) \mid x₅ = g₅, x₆ ∈ T') = P₁; \theta(f(x₆, x₇) \mid x₆ = g₆, x₇ ∈ T') = P₂; T₁ \( \lambda \) T'''
| Graph elements (nodes, edges, graph objects) | Both graph elements specified | One graph element specified | Neither graph element specified |
|--------------------------------------------|------------------------------|-----------------------------|--------------------------------|
| Single/same graph element                  | Two different graph elements |                             |                                |

| **Both are targets** | **Inverse comparison** of the time intervals over which the given patterns occur for a single given graph element. | **Inverse comparison** of the time intervals over which the given patterns occur for two different graph elements. | **Inverse comparison** of a specified graph element and a graph element associated with a given pattern (over an unspecified time interval) and comparison of the time intervals over which the patterns occur. | **Inverse comparison** of graph elements and time intervals associated with two given patterns. |
|----------------------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| ?T', T'', λ:          | \( \theta(f(x_1, x_2) \mid x_1 = g, x_2 \in T') = P_1; \) T' \( \lambda \) T'' | \( \theta(f(x_1, x_2) \mid x_1 = g_1, x_2 \in T') = P_1; \) \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_2; \) T' \( \lambda \) T'' | \( \theta(f(x_1, x_2) \mid x_1 = g_1, x_2 \in T') = P_1; \) \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_2; \) \( \theta(f(x_1, x_2) \mid x_1 = g_3, x_2 \in T') = P_3; \) T' \( \lambda \) T'' | \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_1; \) \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_2; \) \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_3; \) T' \( \lambda \) T'' |

\(^{10}\) Reduced from: \( ?g_2, T', \lambda, \psi: \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_1; \) \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_2; \) \( \theta(f(x_1, x_2) \mid x_1 = g_2, x_2 \in T') = P_3; \) T' \( \lambda \) T''
Figure 8 Comparison quadrant 4: considers comparisons involving aspectual behaviours (i) the behaviour of temporal trends for all graph elements, over the graph (ii) the behaviour of an attribute over the graph, over time

| Time intervals | Single/same subset | Two different subsets | One graph subset specified | Neither graph subset specified |
|----------------|--------------------|-----------------------|---------------------------|-------------------------------|
| Same time      |                    |                       |                           |                               |
| Direct comparison of distributions of temporal trends over the graph for two different attributes over the same time interval and for the same graph subset: | | | | |
| ? \( p_1, p_2, \lambda \): \( \theta_0[\theta_1[f_1(x_1, x_2) | x_2 \in T'] | x_2 \in G'] = p_1 \); \( \theta_0[\theta_1[f_2(x_1, x_2) | x_2 \in T'] | x_2 \in G'] = p_2 \); \( p_1 \lambda p_2 \) | | | |

Or

| Time intervals | Single/same subset | Two different subsets | One graph subset specified | Neither graph subset specified |
|----------------|--------------------|-----------------------|---------------------------|-------------------------------|
| Same time      |                    |                       |                           |                               |
| Direct comparison of distributions of temporal trends over two different graph subsets over the same time interval: | | | | |
| ? \( p_1, p_2, \lambda \): \( \theta_0[\theta_1[f_1(x_1, x_2) | x_2 \in T'] | x_2 \in G'] = p_1 \); \( \theta_0[\theta_1[f_2(x_1, x_2) | x_2 \in T'] | x_2 \in G'] = p_2 \); \( p_1 \lambda p_2 \) | | | |

Or

| Time intervals | Single/same subset | Two different subsets | One graph subset specified | Neither graph subset specified |
|----------------|--------------------|-----------------------|---------------------------|-------------------------------|
| Same time      |                    |                       |                           |                               |
| Direct comparison of distributions of temporal trends in distributions of an attribute over the graph for two different attributes for the same graph subset and over the same time interval: | | | | |
| ? \( p_1, p_2, \lambda \): \( \theta_0[\theta_1[f_1(x_1, x_2) | x_1 \in G'] | x_2 \in T'] = p_1 \); \( \theta_0[\theta_1[f_2(x_1, x_2) | x_1 \in G'] | x_2 \in T'] = p_2 \); \( p_1 \lambda p_2 \) | | | |

Reduced from: ? \( G^*, \lambda \): \( \theta_0[\theta_1[f_1(x_1, x_2) | x_2 \in T'] | x_2 \in G'] = P_1 \); \( \theta_0[\theta_1[f_2(x_1, x_2) | x_2 \in T'] | x_2 \in G'] = P_2 \); \( G' \lambda G'' \); OR

? \( G^*, \lambda \): \( \theta_0[\theta_1[f_1(x_1, x_2) | x_1 \in G'] | x_2 \in T'] = P_1 \); \( \theta_0[\theta_1[f_2(x_1, x_2) | x_1 \in G'] | x_2 \in T'] = P_2 \); \( G' \lambda G'' \);
### Graph subsets

|                     | Both graph subsets specified | One graph subset specified | Neither graph subset specified |
|---------------------|-----------------------------|---------------------------|-------------------------------|
| **Single/same subset** | Direct comparison of distributions of temporal trends over the graph for the same graph subset during two different time intervals: | Direct comparison of distributions of temporal trends over two different graph subsets over two different time intervals: | Inverse comparison as above\(^{12}\) |
|                     | \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) | \(p_1 \neq p_2\); \(p_2 \neq p_1\) | Inverse comparison of two graph subsets associated with two given patterns involving two different time intervals |
| **Two different subsets** | Direct comparison of distributions of temporal trends over two different graph subsets over two different time intervals: | Or \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) | or \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) |
|                     | \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) | Or \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) | \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) |

\(^{12}\) Reduced from: \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\) OR \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\); OR \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\); OR \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\); OR \(\beta\{f(x_1, x_2) \mid x_1 \in T \land x_2 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_2 \in T\} \approx p_2\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_1\); \(\beta\{f(x_1, x_2) \mid x_1 \in G\} \approx p_2\); \(p_1 \neq p_2\); \(p_2 \neq p_1\); OR
Graph subsets

| Both graph subsets specified | One graph subset specified | Neither graph subset specified |
|------------------------------|---------------------------|-------------------------------|
| Single/same subset           | Two different subsets     |                               |

**Inverse comparison** of a time interval associated with a given pattern and graph subset, and a given time interval:\n
\[ ? T″, λ: \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P; \]
\[ T′ λ T″; \]

or\n
\[ ? T″, λ: \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P; \]
\[ T′ λ T″; \]

**Inverse comparison** of a time interval and graph subset associated with a given pattern, with a given time interval and graph subset:\n
\[ ? G″, T″, λ, ψ: \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P; \]
\[ T′ λ T″; \]
\[ G′ \psi G″; \]

or\n
\[ ? G″, T″, λ, ψ: \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P; \]
\[ T′ λ T″; \]
\[ G′ \psi G″; \]

OR

**Inverse comparison** of a graph object associated with a pattern involving a given time interval, and a given graph object and a time interval associated with a pattern:

**Inverse comparison** of graph subsets and time intervals associated with two given patterns, where one of the patterns involves a given time interval:

\[ ? G′, G″, T″, λ, ψ: \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_1; \]
\[ \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_2; \]
\[ T′ λ T″; \]
\[ G′ \psi G″; \]

or

\[ ? G′, G″, T″, λ, ψ: \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_1; \]
\[ \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_2; \]
\[ T′ λ T″; \]
\[ G′ \psi G″; \]

---

13 Reduced from: \[ ? T″, λ: \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_1; \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_2; T′ λ T″; \]
\[ \text{or} \]
\[ ? T″, λ: \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_1; \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_2; T′ λ T″; \]

14 Reduced from: \[ ? T″, λ: \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_1; \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_2; T′ λ T″; \]
\[ \text{or} \]
\[ ? T″, λ: \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_1; \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_2; T′ λ T″; \]

15 Reduced from: \[ ? G″, T″, λ, ψ: \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_1; \theta_0(\theta_1[f(x_1, x_2) | x_2 \in T″]) \mid x_1 \in G″ = P_2; T′ λ T″; G′ \psi G″; \]
\[ \text{or} \]
\[ ? G″, T″, λ, ψ: \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_1; \theta_1(\theta_0[f(x_1, x_2) | x_1 \in G″]) \mid x_2 \in T″ = P_2; T′ λ T″; G′ \psi G″; \]
### Graph subsets

| Both graph subsets specified | One graph subset specified | Neither graph subset specified |
|-----------------------------|---------------------------|-------------------------------|
| **Single/same subset**      | **Two different subsets** |                               |

**Inverse comparison** of two time intervals associated with the same graph subset:

- $T', T''$, $\lambda$:
  - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in G'] \mid x_1 \in G'= P_1$;
  - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in G'] \mid x_1 \in G'= P_2$; $T' \lambda T''$;
  - or
  - $T', T''$, $\lambda$:
    - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in G'] \mid x_1 \in G'= P_1$;
    - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in G'] \mid x_1 \in G'= P_2$; $T' \lambda T''$;
    - or

**Inverse comparison** of two time intervals associated with two given patterns involving two different graph subsets:

- $T', T''$, $\lambda$:
  - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in T'] \mid x_1 \in G'= P_1$;
  - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in T'] \mid x_1 \in G'= P_2$; $T' \lambda T''$;
  - or

**Inverse comparison** of two time intervals associated with two given patterns involving a given graph subset, and a given time interval.

- $G'$, $T'$, $T''$, $\lambda, \psi$:
  - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in T'] \mid x_1 \in G'= P_1$;
  - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in T'] \mid x_1 \in G'= P_2$; $T' \lambda T''$; $G' \psi G''$;
  - or

**Inverse comparison** of two time intervals associated with given patterns, where one of the patterns involves a given graph subset:

- $G'$, $G''$, $T'$, $T''$, $\lambda, \psi$:
  - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in T'] \mid x_1 \in G'= P_1$;
  - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in T'] \mid x_1 \in G'= P_2$; $T' \lambda T''$; $G' \psi G''$;
  - or

**Inverse comparison** of graph subsets and time intervals associated with given patterns:

- $G'$, $G''$, $T'$, $T''$, $\lambda, \psi$:
  - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in T'] \mid x_1 \in G'= P_1$;
  - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in T'] \mid x_1 \in G'= P_2$; $T' \lambda T''$; $G' \psi G''$;
  - or

- $G'$, $G''$, $T'$, $T''$, $\lambda, \psi$:
  - $\theta_0[\theta_f(x_1, x_2) \mid x_2 \in T'] \mid x_1 \in G'= P_1$;
  - $\theta_0[\theta_f(x_1, x_3) \mid x_3 \in T'] \mid x_1 \in G'= P_2$; $T' \lambda T''$; $G' \psi G''$;
  - or
## Graph subsets

| Both graph subsets specified | One graph subset specified | Neither graph subset specified |
|-----------------------------|----------------------------|--------------------------------|
| Single/same subset          | Two different subsets      |                                |
| $T' \lambda T''$;           | $\Theta(z(x_1, x_2) \mid x_1 \in G'') \mid x_2 \in T'') = P_2$; $T' \lambda T''$; $G' \psi G''$; | $T' \lambda T''$; $G' \psi G''$; |
4.1.3 Relation seeking

In relation seeking tasks, a relation between elements is given and the task is to find the elements related in the specified manner. In elementary relation seeking this generally involves finding attribute values related in a specified way, but may also involve a specified relation on time points and/or graph elements. Similarly in synoptic tasks, this involves finding patterns related in a specified way, but may also involve a specified relation on time intervals and/or graph subsets. The Andrienko framework makes an additional distinction between tasks involving the same or different attributes in the subtasks. The tasks in the matrices have been formulated to show the same attribute, but each task could also be formulated for the case where two different attributes are involved.

Note also that we do not show in the matrix tasks where attribute values or patterns are specified. These tasks can be formulated to produce tasks where either:

i. Both attribute values or patterns are specified. In this case, the relation seeking task will involve a specified relation on time points/intervals and/or graph elements/subsets. Taking an example from quadrant 2, we could have:

Find graph subsets and time points associated with given patterns, where the graph subsets/time points are related in the specified way.

? \( G', \ G'', \ t_1, \ t_2; \)
\[ \theta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_1) \approx P_1; \]
\[ \theta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t_2) \approx P_2; \]
\[ t_1 \Psi t_2; \]
\[ G' \Phi G''; \]

ii. One attribute value or pattern is specified. In this case, the specified relation may be between attribute values or patterns, graph elements or subsets and/or time points or intervals (as appropriate to the specified/unspecified elements in the task). Again, we give an example from quadrant 2:

Find patterns related to a given pattern in the given way. Find also the graph subsets and time points over/at which the related patterns occur. A relation between graph subsets and/or time points may also be specified.

? \( G', \ G'', \ t_1, \ t_2, \ P_2; \)
\[ \theta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_1) \approx P_1; \]
\[ \theta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t_2) \approx P_2; \]
\[ t_1 \Psi t_2; \]
\[ G' \Phi G''; \]
\[ P_1 \Lambda P_2 \]

Note that in the case where one of the subtasks is completely specified, we reduce the task to relation seeking involving a specified pattern or graph subset e.g.

? \( G'', \ t_2, \ P_2; \)
\[ \theta(f(x_1, x_2) \mid x_1 \in G', x_2 = t_1) \approx P_1; \]
\[ \theta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t_2) \approx P_2; \]
Can be reduced to:

Find patterns/graph elements related in the given way to given patterns/graph elements. A relation on time points may also be specified.

\[ ? \ G'', \ t_2, \ P_2 : \]
\[ \beta (f(x_1, x_2) \mid x_1 \in G'', \ x_2 = t_2) = P_2; \]
\[ t_1 \psi t_2; \]
\[ G' \Phi G''; \]
\[ P_1 \Lambda P_2 \]

Again, we provide an overview of the relation seeking tasks based on the referential components involved. The permutations of tasks involving the same/different/specified/unspecified elements are given in the full task matrix.

**Figure 9 Relation seeking quadrant-level overview**

| Time points | Graph elements (nodes, edges, graph objects) | Graph subsets |
|-------------|---------------------------------------------|---------------|
| **Elementary** | \( ? t_1, t_2, g_1, g_2, y_1, y_2 : \)
\[ f(t_1, g_1) = y_1; \]
\[ f(t_2, g_2) = y_2; \]
\[ t_1 \psi t_2; \]
\[ g_1 \Phi g_2; \]
\[ y_1 \Lambda y_2; \] |
| **Synoptic** | \( ? G', G'', t_2, P_2 : \)
\[ \beta (f(x_1, x_2) \mid x_1 \in G', \ x_2 = t_2) = P_2; \]
\[ t_1 \psi t_2; \]
\[ G' \Phi G''; \]
\[ P_1 \Lambda P_2 \] |

**Relation seeking** – find the attribute values related in the given manner (and possibly the corresponding graph element(s)/time point(s)). In this case the possible relation specified is domain dependent. Variations of this task depend on the number of time points and graph elements specified in the lookup sub tasks.

Additional constraints on the relations between graph elements and/or time points may also be specified. Depending on the elements involved in the lookup tasks (i.e. whether they are specified/unspecified, same or different), constraints may be any of the relations noted in the comparison matrix e.g.:

- between time points: equality (same/different time point), that time points are consecutive, occur before/after a given time point, that a certain distance exists between them etc.
- between graph elements: equality (same/different element); set relations (between the sets of elements belonging to graph objects); equality of configuration (in graph objects); association (where a single time point is specified in the lookup task or a constraint of equality is added on unspecified time points).

**Time intervals**

| Synoptic | \( ? g_1, g_2; T, T', P_1, P_2 : \)
\[ \beta (f(x_1, x_2) \mid x_1 \in g_1, x_2 \in T) = P_1; \]
\[ \beta (f(x_1, x_2) \mid x_1 \in g_2, x_2 \in T') = P_2; \]
\[ g_1 \Phi g_2; T \psi T'; P_1 \Lambda P_2 \] |

**Synoptic**

\( ? G', G'', T, T', P_1, P_2 : \)
\[ \beta (f(x_1, x_2) \mid x_1 \in G', x_2 \in T) = P_1; \]
\[ \beta (f(x_1, x_2) \mid x_1 \in G'', x_2 \in T') = P_2; \]
\[ G' \Phi G''; P_1 \Lambda P_2 \]
**Relation seeking** – find the patterns of attribute(s) over time which are related in the given manner (and possibly find the graph element(s) to which they correspond/the time period(s) over which they occur). The possible specified relations between patterns are the same (similar)/different/opposite. Variations depend on the number of graph elements and time intervals specified in the lookup subtasks.

Additional constraints on relations between graph elements and/or time intervals may also be included in the task specification, depending on the elements involved in the lookup tasks. These are similar to the relations noted in the comparison matrix e.g.

- between graph elements: equality (same/different element); set relations (between the sets of elements belonging to graph objects); equality of configuration (in graph objects).
- Between the time intervals (over which the pattern occurs): happens before/after, happens at the same time; between two intervals, or an instant and an interval: happens before/after, starts, finishes, happens during; between intervals only: overlaps, meets [4].

or

\[
\begin{align*}
\text{? } G', G'', T', T'', P_1, P_2; \\
\beta_t(\beta_d[f(x_1, x_2) \mid x_1 \in G']) \{ x_2 \in T' \} & = P_1; \\
\beta_t(\beta_d[f(x_1, x_2) \mid x_1 \in G'']) \{ x_2 \in T'' \} & = P_2; \\
T' & \Psi T''; G' & \Phi G''; P_1 & \Lambda P_2
\end{align*}
\]

**Relation seeking** – find (sub)patterns of either of the aspectual behaviours which are related in the given manner (and possibly find the graph subset/time interval associated with the found patterns). The possible specified relations between patterns are the same (similar)/different/opposite. Variations depend on the number of graph subsets and time intervals specified in the lookup subtasks.

Additional constraints on relations between time points/or graph subsets may also be included in the task specification, depending on the elements involved in the lookup tasks. These are similar to the relations noted in the comparison matrix e.g.

- between two graph subsets: equality (same/different subset); set relations (between the sets of nodes/edges belonging to the subset); equality of configuration of the subset, association (between nodes/graph objects, at a single time point only).
- Between the time intervals (over which the pattern occurs): happens before/after, happens at the same time; between two intervals, or an instant and an interval: happens before/after, starts, finishes, happens during; between intervals only: overlaps, meets [4].
Figure 10 Relation seeking, quadrant 1: considers elementary relation seeking involving graph elements (nodes, edges, graph objects) and time points (i.e. the elementary comparison tasks)

| Time points | Both constraints | Graph elements (nodes, edges, graph objects) | Both are targets |
|-------------|------------------|---------------------------------------------|------------------|
| Same time   | Both constraints | Find the attribute value (and associated node) at a given time, which is related in the given way to an attribute value associated with a given graph object at the same given time point. A relation between graph elements may also be specified. | Find attribute values (and the nodes associated with them) at the same given time, which are related in the given way. A relation between graph elements may also be specified. |
|             | Same element     | $f(t, g_1) = y_1; f(t, g_2) = y_2; y_1 \land y_2; g_1 \Phi g_2$ | $g_1, g_2, y_1, y_2$: $f(t, g_1) = y_1; f(t, g_2) = y_2; y_1 \land y_2; g_1 \Phi g_2$ |
|             | Different elements | Find the attribute value (and associated node) at a given time, which is related in the given way to an attribute value associated with a given graph object at a different given time point. | Find attribute values (and the nodes associated with them) at two given times, which are related in the given way. A relation between graph elements may also be specified. |
|             | Different time    | $f(t_1, g_1) = y_1; f(t_2, g_2) = y_2; y_1 \land y_2; g_1 \Phi g_2$ | $g_1, g_2, y_{t_1}, y_{t_2}$: $f(t_1, g_1) = y_{t_1}; f(t_2, g_2) = y_{t_2}; y_{t_1} \land y_{t_2}; g_1 \Phi g_2$ |
|             | Not applicable    |                                                            |
# Graph elements (nodes, edges, graph objects)

| Same element | Different elements | One constraint, one target | Both are targets |
|--------------|--------------------|---------------------------|-----------------|
| Find an attribute value (and the time point at which it occurs) associated with a given graph element, which is related in the given way to an attribute value associated with a different given graph element at a given time. A relation between time points may also be specified. | Find an attribute value (and the time point at which it occurs) associated with a given graph element, which is related in the given way to an attribute value associated with a different given graph element at a given time. A relation between time points may also be specified. | Find an attribute value (and the time point and graph element for which it occurs) related in the given way to an attribute value which is associated with a given graph element at a given time point. Relations between time points and/or graph elements may also be specified. | Find attribute values related in the given way where one of the values occurs at the given time. Relations between time points and graph elements may also be specified. |
| \(t_2, y_1, y_2; f(t_1, g_1) = y_1; f(t_2, g_2) = y_2; t_1 \Psi t_2; y_1 \Lambda y_2\) | \(t_2, y_1, y_2; f(t_1, g_1) = y_1; f(t_2, g_2) = y_2; t_1 \Psi t_2; y_1 \Lambda y_2\) | \(t_2, g_1, g_2, y_1, y_2; f(t_1, g_1) = y_1; f(t_2, g_2) = y_2; t_1 \Psi t_2; g_1 \Phi g_2; y_1 \Lambda y_2\) |
## Graph elements (nodes, edges, graph objects)

| Both constraints | Different elements | One constraint, one target | Both are targets |
|------------------|--------------------|---------------------------|------------------|
| **Same element** | Find attribute values (and the time points at which they occur) associated with the same given graph element, which are related in the given way. A relation between time points may also be specified. | Find attribute values (and the time points at which they occur) associated with two given graph elements, which are related in the given way. A relation between time points may also be specified. | Find attribute values related in the given way, where one of the attribute values is associated with a given graph element. Relations between time points and/or graph elements may also be specified. |
| **Different elements** | Find attribute values related in the given way. Relations between time points and/or graph elements may also be specified. | | |
| ? \( t_1, t_2, y_1, y_2 \); \( f(t_1, g_1) = y_1; f(t_2, g_2) = y_2 \); \( y_1 \land y_2; t_1 \bowtie t_2 \); | ? \( t_3, t_2, y_3, y_2 \); \( f(t_3, g_3) = y_3; f(t_2, g_2) = y_2 \); \( t_3 \bowtie t_2; y_3 \land y_2 \); | ? \( t_1, t_2, g_1, g_2, y_1, y_2 \); \( f(t_1, g_1) = y_1; f(t_2, g_2) = y_2 \); \( t_1 \bowtie g_1; g_1 \bowtie g_2; y_1 \land y_2 \); |
### Figure 11 Relation seeking quadrant 2: considers synoptic relation seeking involving the behaviour of an attribute over the graph (or a graph subset)

| Graph subsets | Both constraints | One constraint, one target | Both are targets |
|---------------|-----------------|----------------------------|-----------------|
| **Same subset** | Find a pattern and the graph subset over which it occurs at a given time point, which is related in the given way to a pattern over a given graph subset at the same time point. A relation between graph subsets may also be specified. | Find patterns related in the given way at the same time point. A relation between graph subsets may also be specified. | Find patterns related in the given way where one of the patterns occurs at the given time. Relations between time points and graph subsets may also be specified. |
| **Different subsets** | | | |
| **Same time** | | | |
| **Not applicable** | | | |
| **Different times** | Tasks as above, but involving two different time points. | Tasks as above, but involving two different time points. | |

- $G'^*$, $P_1$, $P_2$:
- $\beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t) = P_1$;
- $\beta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t) = P_2$;
- $G' \Phi G''$;
- $P_1 \land P_2$

- $G'^*$, $P_1$, $P_2$:
- $\beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t) = P_1$;
- $\beta(f(x_1, x_2) \mid x_1 \in G'', x_2 = t) = P_2$;
- $G' \Phi G''$;
- $P_1 \land P_2$
| **Graph subsets** | **Both constraints** | **One constraint, one target** | **Both are targets** |
|-------------------|----------------------|--------------------------------|---------------------|
| **Same subset**   |                      |                                |                     |
| $\theta(f(x_t, x_{t_2}) \mid x_2 \in G', x_2 = t_1) = P_1; \theta(f(x_t, x_{t_2}) \mid x_2 \in G', x_2 = t_2) = P_2; \ t_1 \psi t_2; \ P_1 \land P_2$ | $t_1 \psi t_2; \ G' \Phi G''; \ P_1 \land P_2$ | $P_1 \land P_2$ |
| **Different subsets** |                      |                                |                     |
| $\theta(f(x_t, x_{t_2}) \mid x_2 \in G', x_2 = t_1) = P_1; \theta(f(x_t, x_{t_2}) \mid x_2 \in G'', x_2 = t_2) = P_2; \ t_1 \psi t_2; \ P_1 \land P_2$ | $t_1 \psi t_2; \ G' \Phi G''; \ P_1 \land P_2$ | $P_1 \land P_2$ |

Find patterns related in the given way where one of the patterns occurs at a given time, and the other occurs over a given graph subset. Also find the unspecified graph subset and time point over which/ at the patterns occur. Relations between time points and graph subsets may also be specified.

$\Psi t_1 = G''; \ t_1, P_1, P_2$;
$\theta(f(x_t, x_{t_2}) \mid x_2 \in G', x_2 = t_1) = P_1; \theta(f(x_t, x_{t_2}) \mid x_2 \in G'', x_2 = t_2) = P_2; \ t_1 \psi t_2; \ G' \Phi G''; \ P_1 \land P_2$
Figure 12 Relation seeking quadrant 3: considers relation seeking tasks involving the behaviour of an attribute of a single graph element over time (i.e. a temporal trend)

Graph elements (nodes, edges, graph objects)

| Time intervals | Both constraints | One constraint, one target | Both are targets |
|----------------|------------------|---------------------------|-----------------|
|                | Same element     | Different elements        |                 |
| Same time      | Find a pattern (and the graph element associated with it) which occurs over a given time interval and is related in the given way to a pattern associated with a given graph element over the same time interval. A relation between graph elements may also be specified. | Find patterns (and their associated graph elements) which occur over the same given time interval and are related in the given way. A relation between graph elements may also be specified. |
|                | ? g₁, g₂, P₁, P₂;  
β(f(x₁, x₂) | x₁ = g₁, x₂ ∈ T') = P₁;  
β(f(x₁, x₂) | x₁ = g₂, x₂ ∈ T') = P₂;  
P₁ ∧ P₂;  
g₁ Φ g₂ | ? g₁, g₂, P₁, P₂;  
β(f(x₁, x₂) | x₁ = g₁, x₂ ∈ T') = P₁;  
β(f(x₁, x₂) | x₁ = g₂, x₂ ∈ T') = P₂;  
P₁ ∧ P₂;  
g₁ Φ g₂ |
| Different times | Find a pattern (and the graph element associated with it) which occurs over a given time interval and is related in the given way to a pattern associated with a given graph element over a given time interval. A relation between graph elements may also be specified | Find patterns (and their associated graph elements) which occur over two given time intervals and are related in the given way. A relation between graph elements may also be specified. |
|                | ? g₁, g₂, P₁, P₂;  
β(f(x₁, x₂) | x₁ = g₁, x₂ ∈ T') = P₁;  
β(f(x₁, x₂) | x₁ = g₂, x₂ ∈ T'') = P₂;  
P₁ ∧ P₂;  
g₁ Φ g₂ | ? g₁, g₂, P₁, P₂;  
β(f(x₁, x₂) | x₁ = g₁, x₂ ∈ T') = P₁;  
β(f(x₁, x₂) | x₁ = g₂, x₂ ∈ T'') = P₂;  
P₁ ∧ P₂;  
g₁ Φ g₂ |

In all cases in this table, if we wish to specify an association relation between the graph elements, we must also specify a time at which the association relation occurs i.e. (g₁, t) Φ (g₂, t): ‘*a given association relation exists between the graph elements at time t*’.
### Graph elements (nodes, edges, graph objects)

| Both constraints | One constraint, one target | Both are targets |
|------------------|----------------------------|------------------|
| **Same element** | **Different elements** | **Find patterns related in the given way where one of the patterns occurs over a given time interval. A relation between time intervals and/or graph elements may also be specified.** |
| Find a pattern (and the time interval over which it occurs) for a given graph element, which is related in the given way to a pattern associated with the same graph element over a given time interval. A relation between time intervals may also be specified. | Find a pattern, and the graph element and time interval over which it occurs, which is related in the given way to a pattern associated with a given graph element over a given time interval. A relation between time intervals and/or graph elements may also be specified. | ? g₁, g₂, T″, P₁, P₂; |
| T°, P₁, P₂; | ? ? T°, P₁, P₂; | β(f(x₁, x₂) | x₁= g₁, x₂∈ T') = P₁; |
| ? T°, P₁, P₂; | β(f(x₁, x₂) | x₁= g₂, x₂∈ T") = P₂; | T° Ψ T"; |
| β(f(x₁, x₂) | x₁= g₁, x₂∈ T') = P₁; | P₁ ∨ P₂; | g₁ ⊕ g₂; |
| β(f(x₁, x₂) | x₁= g₂, x₂∈ T") = P₂; | g₁ Φ g₂ | Or |
| **One constraint, one target** | **Find patterns related in the given way where one of the patterns occurs over a given time interval, and the other is associated with a given graph element. Also find the unspecified graph element and time interval associated with the patterns. A relation between time intervals and/or graph elements may also be specified.** |
| Find patterns related in the given way where one of the patterns occurs over a given time interval, and the other is associated with a given graph element. Also find the unspecified graph element and time interval associated with the patterns. A relation between time intervals and/or graph elements may also be specified. | ? g₂, T°, P₁, P₂; | β(f(x₁, x₂) | x₁= g₁, x₂∈ T') = P₁; |
| ? g₂, T°, P₁, P₂; | ? ? g₂, T°, P₁, P₂; | β(f(x₁, x₂) | x₁= g₂, x₂∈ T") = P₂; |
| β(f(x₁, x₂) | x₁= g₁, x₂∈ T') = P₁; | T° Ψ T"; | P₁ ∨ P₂; |
| β(f(x₁, x₂) | x₁= g₂, x₂∈ T") = P₂; | g₁ ⊕ g₂; | g₁ Φ g₂ |

Note: The above table outlines the constraints and targets for different scenarios involving graph elements, such as nodes and edges, and how to find patterns related to these elements over specific time intervals. The symbols used in the table represent different operations and relations between graph elements and time intervals.
| Graph elements (nodes, edges, graph objects) | Both constraints | One constraint, one target | Both are targets |
|--------------------------------------------|------------------|---------------------------|------------------|
| **Same element**                           |                  |                           |                  |
| Find patterns (and the time intervals over  | Find patterns (and | Find patterns related in | Find patterns related in |
| which they occur) associated with a single  | the time intervals | the given way, where one  | the given way. A |
| graph element, which are related in the     | over which they    | of the patterns is        | relation between    |
| given way. A relation between time          | occur) associated  | associated with a given   | time intervals and/or |
| intervals may also be specified.            | with two given     | graph element. A relation  | graph elements may |
|                                            | graph elements,    | between time intervals    | also be specified. |
|                                            | which are related  | may also be specified.    |                  |
| ? T′, T″, P1, P2:                          | ? T′, T″, P1, P2:  | g2, T′, T″, P1, P2:       |                  |
| θ(afx1, x2) | x1 = g1, x2 ∈ T′) = P1; | θ(afx1, x2) | x1 = g1, x2 ∈ T′) = P1; |                  |
| θ(afx1, x2) | x1 = g2, x2 ∈ T′) = P2; | θ(afx1, x2) | x1 = g2, x2 ∈ T′) = P2; |                  |
| T′ ∪ T″;                                           | T′ ∪ T″;                  | T′ ∪ T″;                  |                  |
| P1 ∨ P2                                           | P1 ∨ P2                  | P1 ∨ P2                  |                  |
|                                               | g1 ∨ g2                 | g1 ∨ g2                 |                  |
| **Different elements**                       |                  |                           |                  |
| Find patterns (and the time intervals over  |                  |                           |                  |
| which they occur) associated with two       |                  |                           |                  |
| given graph elements, which are related    |                  |                           |                  |
| in the given way. A relation between time   |                  |                           |                  |
| intervals may also be specified.            |                  |                           |                  |
| ? T′, T″, P1, P2:                          | ? T′, T″, P1, P2:       | g2, T′, T″, P1, P2:       |                  |
| θ(afx1, x2) | x1 = g1, x2 ∈ T′) = P1; | θ(afx1, x2) | x1 = g1, x2 ∈ T′) = P1; |                  |
| θ(afx1, x2) | x1 = g2, x2 ∈ T′) = P2; | θ(afx1, x2) | x1 = g2, x2 ∈ T′) = P2; |                  |
| T′ ∪ T″;                                           | T′ ∪ T″;                  | T′ ∪ T″;                  |                  |
| P1 ∨ P2                                           | P1 ∨ P2                  | P1 ∨ P2                  |                  |
|                                               | g1 ∨ g2                 | g1 ∨ g2                 |                  |
Figure 13 Relation seeking quadrant 4: considers relation seeking tasks involving aspectual behaviours (i) the behaviour of temporal trends for all graph elements, over the graph (ii) the behaviour of an attribute over the graph, over time

| Graph subsets | Both constraints | One constraint, one target | Both are targets |
|---------------|-----------------|----------------------------|-----------------|
|               | Same subset     | Different subsets          |                 |
| Both constraints |                 |                            |                 |
| Same time      | Find a pattern (and the graph subset associated with it) which is associated with a given time interval and is related in the given way to a pattern associated with a given graph subset and the same time interval. A relation between graph subsets may also be specified. | Find patterns (and their associated graph subsets) which are associated with a single given time interval and are related in the given way. A relation between graph subsets may also be specified. | ? G′, G″, P1, P2; β1\{β1[f(x1, x2) | x1 ∈ T′]| x1 ∈ G′]= P1; β1\{β1[f(x1, x2) | x2 ∈ T′]| x2 ∈ G′]= P1; G′ ⊗ G″; P1 Λ P2 | ? G′, G″, P1, P2; β1\{β1[f(x1, x2) | x2 ∈ T′]| x2 ∈ G′]= P1; β1\{β1[f(x1, x2) | x2 ∈ T′]| x2 ∈ G′]= P1; G′ ⊗ G″; P1 Λ P2 |
| Different times | Find a pattern (and the graph subset associated with it) which is associated with a given time interval and is related in the given way to a pattern associated with a given graph subset and a given time interval. A relation between graph subsets may also be specified. | Find patterns (and their associated graph subsets) which are associated with two given time intervals and are related in the given way. A relation between graph subsets may also be specified. | ? G′, G″, P1, P2; β1\{β1[f(x1, x2) | x1 ∈ T′]| x1 ∈ G′]= P1; β1\{β1[f(x1, x2) | x2 ∈ T′]| x2 ∈ G′]= P1; G′ ⊗ G″; P1 Λ P2 | ? G′, G″, P1, P2; β1\{β1[f(x1, x2) | x2 ∈ T′]| x2 ∈ G′]= P1; β1\{β1[f(x1, x2) | x2 ∈ T′]| x2 ∈ G′]= P1; G′ ⊗ G″; P1 Λ P2 |
### Graph subsets

| Both constraints | One constraint, one target | Both are targets |
|------------------|---------------------------|------------------|
|                  | Same subset               | Different subsets |
|                  | G' $\cup$ G'';           |                  |
|                  | $P_1 \land P_2$          |                  |
| or               | ? G', P_3, P_2;          |                  |
|                  | $\theta_1\{\theta_0[\psi(x_i, x_j) \mid x_i \in G']\} \cup x_j \in T'\} = P_3;$ |                  |
| or               | ? G', P_3, P_2;          |                  |
|                  | $\theta_1\{\theta_0[\psi(x_i, x_j) \mid x_i \in G']\} \cup x_j \in T''\} = P_3;$ |                  |
| or               | $G' \cup G''$;           |                  |
|                  | $P_1 \land P_2$          |                  |
| or               | ? G', P_3, P_2;          |                  |
|                  | $\theta_1\{\theta_0[\psi(x_i, x_j) \mid x_i \in G']\} \cup x_j \in T'\} = P_3;$ |                  |
| or               | ? G', P_3, P_2;          |                  |
|                  | $\theta_1\{\theta_0[\psi(x_i, x_j) \mid x_i \in G']\} \cup x_j \in T''\} = P_3;$ |                  |

---

### One constraint, one target

- Find a pattern (and the time interval with which it is associated) for a given graph subset, which is related in the given way to a pattern associated with the same graph subset and a given time interval. A relation between time intervals may also be specified.

- Find patterns (and the time intervals over which they occur) associated with two given graph subsets, which are related in the given way. A relation between time intervals may also be specified.

- Find a pattern, and the graph subset and time interval with which it is associated, which is related in the given way to a pattern associated with a given graph subset and a given time interval. Relations between time intervals and/or graph subsets may also be specified.

- Find patterns related in the given way where one of the patterns involves a given time interval. Relations between time intervals and/or graph subsets may also be specified.
Both constraints

| Same subset | Different subsets |
|-------------|------------------|

One constraint, one target

| Find patterns related in the given way where one of the patterns is associated with a given time interval, and the other is associated with a given graph subset. Also find the unspecified graph subset and time interval associated with the patterns). Relations between time intervals and/or graph subsets may also be specified. |
| ? $G', T', P_1, P_2$: |
| $\theta_d(\theta_f(x_1, x_2) \mid x_2 \in T') \mid x_1 \in G') = P_1$; |
| $\theta_d(\theta_f(x_1, x_2) \mid x_2 \in T'' \mid x_1 \in G'') = P_2$; |
| $T' \Psi T''$; |
| $G' \Phi G''$; |
| $P_1 \Lambda P_2$ |

| or |
| ? $G'', T', T'', P_1, P_2$: |
| $\theta_d(\theta_f(x_1, x_2) \mid x_1 \in G'') \mid x_2 \in T' = P_1$; |
| $\theta_d(\theta_f(x_1, x_2) \mid x_2 \in G'') \mid x_1 \in T'' = P_2$; |
| $T' \Psi T''$; |
| $G' \Phi G''$; |
| $P_1 \Lambda P_2$ |

Both are targets

| Find patterns related in the given way where one pattern is associated with a given graph subset. Relations between time intervals and/or graph subsets may also be specified. |
| ? $G', G'', T', T'', P_1, P_2$: |
| $\theta_d(\theta_f(x_1, x_2) \mid x_2 \in T') \mid x_1 \in G') = P_1$; |
| Graph subsets | Both constraints | One constraint, one target | Both are targets |
|---------------|----------------|----------------------------|-----------------|
| **Same subset** | T’, T”, P1, P2; | T’(T[x2], x2) | θ(T[x2], x2) |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | Ψ T’; P1, P2 | or | or |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | Ψ T’; P1, P2 | or | or |
| **Different subsets** | T’(T[x2], x2) | x2 E T’) | x1 E G’= P2; |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | Ψ T’; P1, P2 | or | or |
|                | θ2[T[x1], x3] | x2 E T’) | x1 E G’= P2; |
|                | Ψ T’; P1, P2 | or | or |
| Ψ T’; P1, P2 | or | or | or |
4.2 Structural tasks

As for the attribute based tasks, we divide the structural task space based on the referential components involved (Figure 14). Quadrant 1 contains the elementary tasks, while the other three quadrants contain the synoptic tasks involving the partial and aspextual structural patterns.

Figure 14 Structural tasks based on referential components involved

| Time Points | Pair of nodes | Graph subsets |
|-------------|---------------|---------------|
| Q1 Elementary | Tasks involving connection between a pair of nodes at a single time point (inverse comparison and relation seeking only). | Q2 Synoptic | Tasks involving the behaviour, or configuration, of association relations within a set of nodes at a single time e.g. clusters, cliques, motifs etc.: |
| Q3 Synoptic | Tasks involving the behaviour of association relations between two graph objects over time e.g. the pattern of change in connectivity between two nodes over time. | Q4 Synoptic | Tasks involving: The behaviour of the collection of patterns in (Q3) i.e. the aggregate pattern of all association relations between pairs of graph objects over time, or the distribution of individual temporal behaviours over the graph. OR The behaviour of the configurations of association relations over the set of nodes (i.e. (Q2)), over time. |

4.2.1 Elementary structural tasks

We distinguish two main elementary tasks: finding association relations (connections) between elements and finding elements connected in the given way. Note that association relations may exist between individual graph elements (nodes) or subsets of the graph treated as individual elements (graph objects). We may also wish to compare and find relations between association relations. We discuss these tasks below.

4.2.1.1 Find connections between elements (comparison)

This is the “comparison” subtask: the term is rather unintuitive when dealing with questions of connectivity between graph objects, however, its definition of ‘finding the relation between given elements’ is exactly what we are attempting to do here. This reflects scheme 1 of Andrienko’s pure relational tasks ([2] p.62-63) (how are the elements p and q (or the subsets P and Q) of the set S related?):

(How) is $g_1$ connected to $g_2$ at the given time, $t$?

$\lambda: (g_1, t) \lambda (g_2, t)$

e.g. (how) are graph objects a and b connected at time 4?

4.2.1.2 Find elements connected in the given way (relation seeking)

This is the relation seeking subtask, and there are two versions reflecting Andrienko’s pure relational schemes 2 and 3, plus a third, hybrid version. Temporal variations of each task are also given in the summary table, below. The association relation can be specified as generally as stating whether or
not any connection exists between two objects, or as specifically as including the distance, direction, and/or domain attributes.

(i) where the connection and one of the graph objects is specified (i.e. scheme 2: what element (or subset) of the set \( S \) is related to the element \( p \) (or subset \( P \)) in the way \( \rho \)?)

Find the graph object(s) to which graph object \( g_1 \) is connected in the given way at time \( t \):

\[ ? g_2 : (g_1, t) \land (g_2, t) \]

*e.g. find the node(s) directly connected to node \( a \) at time \( t_4 \); find node(s) connected to node \( a \) at a distance of less than 3 hops at time \( t_2 \); find nodes with a strong direct connection to node \( a \); find nodes directly connected to node \( a \) with link type “friend”; find clusters connected to cluster \( B \) at time \( t_3 \).*

(ii) where only the connection is specified (i.e. scheme 3: what elements (or subsets) of the set \( S \) are related in the way \( \rho \)?)

Find graph objects that are connected in the given way at the given time

\[ ? g_1, g_2, t : (g_1, t) \land (g_2, t) \]

*e.g. find nodes directly connected with a weight greater than 4 at time \( t_6 \); find closely connected clusters at time \( t_9 \).*

(iii) In addition to the tasks constructed according to the pure relational schemes, there is a hybrid variation where we have two graph objects and a given connection relation, and we want to find the time(s) at which the objects were connected in the given way:

Find the time points at which two given graph objects were connected in the given way

\[ ? t : (g_1, t) \land (g_2, t) \]

*e.g. find the times at which a direct connection exists between graph objects \( a \) and \( b \).*

**Figure 15** summary of comparison and relation seeking graph structural tasks concerning graph elements, including variations involving specified and unspecified time points

| Time Points | Graph elements (nodes, graph objects) | Both constraints | One constraint, one target | Both are targets |
|-------------|---------------------------------------|------------------|-----------------------------|-----------------|
| Both constraints | Find connections between elements (comparison) (How) is graph element \( g_1 \) connected to graph element \( g_2 \) at the given time, \( t \)? | \( ? \lambda : (g_1, t) \land (g_2, t) \) | Find elements connected in the given way (relation seeking) Find the graph element(s) to which graph element \( g_1 \) is connected in the given way at time \( t \): | \( ? g_2 : (g_1, t) \land (g_2, t) \) | Find elements connected in the given way (relation seeking) Find graph objects which are connected in the given way at the given time \( ? g_1, g_2 : (g_1, t) \land (g_2, t) \) |
Both are targets | Hybrid | Find the time points at which two given graph objects were connected in the given way | Find the graph element(s) to which graph element $g_1$ is connected and the time(s) at which the connection(s) occur | Find graph objects (and their associated time points) at any time that are connected in the given way

| $t : (g_1, t) \Lambda (g_2, t)$ | $g_2, t : (g_1, t) \Lambda (g_2, t)$ |

### 4.2.1.3 Compare and find relations between association relations (connections)

The previous tasks deal with finding association relations and finding elements connected in a given way. It is conceivable that we may also wish to compare connections. Comparison of relations does not exist within the original Andrienko framework, and we do not wish to add more task categories than necessary. Therefore, while we use the tasks outlined above to find connection relations, when comparing association relations we treat them as references: either edges in the case of direct connection, or paths (a graph object) in indirect connection. This allows us to use the attribute based tasks of the original Andrienko framework (see section 4.1): direct comparison to compare the attributes associated with the edge or path (e.g. weight, domain attributes, path length), and inverse comparison when we are interested in the (structural) equality of the graph objects themselves. We suggest handling relation seeking tasks involving association relations in a similar manner (e.g. ‘find dyads whose edge weight increased between times 3 and 4’ would involve relation seeking where the edge/dyad is the reference and weight is the attribute).

### 4.2.2 Synoptic structural tasks

#### 4.2.2.1 Structural pattern characterisation

Structural pattern characterisation involves describing the pattern associated with one of the four structural behaviours outlined in section 2.2.

#### 4.2.2.2 Structural pattern search

This is the opposite of the above task in that we seek to find the set of graph elements associated with a given pattern or configuration of connections, and/or the time at which they occur.

#### 4.2.2.3 Comparison and relation seeking involving structural patterns

As is the case with the attribute based tasks, we may also wish to compare or find relations between structural patterns, and the graph elements/subsets and time points/intervals associated with these patterns. The many possible permutations of these tasks mirror those of the original attribute based comparison and relation seeking tasks, which are outlined in the task matrices described in section 4.1.

### 4.3 Connection Discovery tasks

We here discuss some examples of the three variations of relational behaviours involved in connection discovery tasks in temporal graphs. We also discuss the possible case of connectional behaviours between graph structural patterns.

#### 4.3.1 Heterogeneous behaviours

(1) Relational behaviour involving two (or more) different attributes of the same reference set
The formal notation for this behaviour is given in the Andrienko framework as:

\[(3.43) \quad \rho(f_1(x), f_2(x) \mid x \in R)\]

Where \(f_1(x)\) and \(f_2(x)\) are two attributes defined on the same reference set \(R\).

Applied to temporal graphs, this task considers a relational behaviour between two different attributes and the same graph and temporal components. Note that the Andrienko framework does not explicitly discuss how to handle multiple referrers in relational behaviours. We therefore draw on the partial behaviours to guide us in our discussion.

One example this relational behaviour applied to temporal graphs might be the (partial) relational behaviour between two different attributes of the elements of the graph (nodes or edges) at a single timepoint:

\[\rho(f_1(g, t), f_2(g, t) \mid g \in G, t = t)\]

We can imagine representing such a behaviour using a scatterplot, as given in the Andrienko examples. For example, we might consider the relationship between the indegree and out degree of all nodes in the graph at a given time point.

Further, we might consider the relational behaviour between two attribute values for the same graph object over time:

\[\rho(f_1(g, t), f_2(g, t) \mid g = g, t \in T)\]

In this case we might look for some correlation or dependency in the two temporal trends, for example, in node indegree and outdegree (e.g. an increase in outdegree followed by an increase in indegree).

Finally, we consider such relations over all graph elements of the time points. The general formula applied to temporal graphs could be written:

\[\rho(f_1(g, t), f_2(g, t) \mid g \in G, t \in T)\]

This would consider the relational behaviour between the two attributes for all graph objects at all time points. Perhaps a scatterplot matrix would help us here, or a cumulative view of the temporal trends, depending on the aspect of interest.

\(2\) Relational behaviour involving two (or more) different attributes of different reference sets

The formal notation for relational behaviour involving different attributes of different reference sets is given as

\[(3.44) \quad \rho(f_1(x), f_2(z) \mid x \in R, z \in Z)\]

Where \(f_1(x)\) is an attribute defined on reference set \(R\), and \(f_2(z)\) is an attribute defined on a different reference set, \(Z\). (However, it is noted that it is highly unlikely that the two reference sets would be completely unrelated.)

We could apply this to temporal graphs in two ways:

1. Where the reference sets are a graph over time and external events
In this case, we may wish to investigate the relational behaviour between an attribute of a graph object and external events (in time):

\[ \rho(f_1(g, t), f_2(z) \mid g \in G, x_2 \in T_2, z \in Z) \]

In such a case we may be looking at how the attribute values in the graph are influenced by outside events over time. This may be of particular interest, for example where some form of external intervention in the network is under observation, such as vaccination in a public health network.

**B. Where the reference sets are two different temporal graphs**

Investigate the relations between two (possibly different) attributes of two different graphs over possibly different time periods:

\[ \rho(f_1(g_1, t_1), f_2(g_2, t_2) \mid g_1 \in G_1, t_1 \in T_1, g_2 \in G_2, t_2 \in T_2) \]

This behaviour may be of interest where we are investigating two different networks which are related in some way, for example, co-authorship networks from different domains, or the energy grid and a computer network. Moreover, we might not only be interested in attribute based behaviours, but also the relation between structural patterns. For example, like Gloo and Zhao [5], we might be interested in the relationship between networks constructed to reflect different communication mediums e.g. face-face, telephone, email. In this case we might also wish to find some correlation between the structural patterns of the network itself, for example, we can imagine that the times at which the email network is densely connected, the face-to-face network may be less so.

### 4.3.2 Homogenous behaviours

**3) Relational behaviour involving the same attributes of different reference subsets**

The formal notation is given as:

\[ \rho(f(x), f(x') \mid x \in R_1, x' \in R_2) \]

Applied to temporal graphs, we might consider investigating the relations between different parts of the graph or the graph over different subsets of time. We can formulate these to consider whether there is some correlation or influence between attribute values of...

- **a. different parts of the graph over the same time interval:**
  \[ \rho(f(x_1, x_2), f(x'_1, x'_2) \mid x_1 \in O_1, x_2 \in T_1, x'_1 \in O_2, x'_2 \in T_1) \]

- **b. the graph or same graph subset during different subsets of time:**
  \[ \rho(f(x_1, x_2), f(x'_1, x'_2) \mid x_1 \in O_1, x_2 \in T_1, x'_1 \in O_2, x'_2 \in T_2) \]

- **c. different parts of the graph during different subsets of time:**
  \[ \rho(f(x_1, x_2), f(x'_1, x'_2) \mid x_1 \in O_1, x_2 \in T_1, x'_1 \in O_2, x'_2 \in T_2) \]

In the graph case, the role of graph structure in relation to attribute values is also very much of interest, for example, we may wish to investigate whether particular structures influence attribute values or vice versa (e.g. Christakis and Fowler’s [6] investigation into the influence of network structure on obesity). Moreover we may be interested in relational behaviour between graph
structures. To take a structural example which would reflect formulation (b) above, in social network analysis a number of theories surround tie formation e.g. Yi et al. [7] discuss examples of these including preferential attachment, accumulative advantage (actors with many ties gain more ties), homophily (the theory that those with similar traits connect to one another), follow-the-trend (i.e. the dominant choices of others), and multiconnectivity (a pursuit for diversity and multiplexity). In all cases, we would look for how structural patterns in the graph at one point in time influence the structural patterns at another.

4.4 Summary of temporal graph tasks

Our extension to, and instantiation of, the Andrienko task framework has resulted in two main types of temporal graph tasks: structural tasks and attribute-based tasks (note that most attribute-based tasks describe attributes in the context of the graph structure.) We summarise these tasks below:

Structural tasks include:

- Find connections between graph elements.
- Find graph elements connected in a given way.
- Comparison and relation seeking between connections.
- Structural pattern characterisation: describe one of four structural patterns associated with given graph elements/subsets and time points/intervals.
- Structural pattern search: find the temporal and graph references associated with one of the above specified patterns.
- Comparison and relation seeking involving structural patterns.
- Connection discovery tasks involving graph structure.

Attribute based tasks include:

- Elementary direct lookup: find the value of an attribute of a graph element (node, edge, graph object) at a given time point.
- Elementary inverse lookup: find the graph element(s) and/or time point(s)
- Behaviour characterisation: find the pattern approximating a partial or aspectual behaviour.
- Pattern search: given a pattern describing one of the partial or aspectual behaviours, find the corresponding graph element/subsets and time point/interval over which it occurs.
- Direct comparison: of attribute values associated with a given time point and graph element (elementary) or of partial/aspectual behaviours associated with given time points/intervals and graph elements/subsets (synoptic).
- Inverse comparison: of time points and/or graph elements associated with the occurrence of particular attribute values (elementary), or comparison of time points/intervals and/or graph elements/subsets over which particular patterns of attribute values occur (synoptic).
- Relation seeking: (elementary) find attribute values related in a specified way, possibly with a specified relation on time points and/or graph elements; (synoptic) find patterns related in a specified way, possibly with a specified relation on time intervals and/or graph subsets.
- Connection discovery tasks involving ‘mutual’ behaviours.

Further variations of attribute-based tasks involving the same or different, specified or unspecified temporal and/or graph components are specified in the task matrices.
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\[ Reduced from: g_2, \lambda: f(t, g_1) \in C'; f(t, g_2) \in C''; (t, g_1) \lambda (t, g_2) \]

\[ Reduced from: \lambda: f(t_1, g_1) \in C'; f(t_2, g_2) \in C''; t_1 \lambda t_2 \]

\[ Reduced from: t_2, \lambda: f(t_1, g) \in C'; f(t_2, g) \in C''; t_1 \lambda t_2 \]

\[ Reduced from: t_2, \lambda: f(t_1, g_1) \in C'; f(t_2, g_2) \in C''; (t_1, g_1) \lambda (t_2, g_2) \]

\[ Reduced from: t_2, g_2, \lambda: f(t_2, g_1) \in C'; f(t_2, g_2) \in C''; (t_2, g_1) \lambda (t_2, g_2) \]

\[ This is reduced from: \lambda: \beta(f(x_1, x_2) \mid x_1 \in G', x_2 = t) \approx P_1; \beta(f(x_1, x_2) \mid x_2 = t) \approx P_2; (G', t) \lambda (G'', t); i.e. all information (the graph subset, timepoint and pattern) is known in the second lookup subtask. \]