Pluripotential theory on quaternionic manifolds.

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Abstract

On any quaternionic manifold of dimension greater than 4 a class of plurisubharmonic functions (or, rather, sections of an appropriate line bundle) is introduced. Then a Monge-Ampère operator is defined. It is shown that it satisfies a version of theorems of A. D. Alexandrov and Chern-Levine-Nirenberg. For more special classes of manifolds analogous results were previously obtained in [2] for the flat quaternionic space $\mathbb{H}^n$ and in [6] for hypercomplex manifolds. One of the new technical aspects of the present paper is the systematic use of the Baston differential operators, for which we also prove a new multiplicativity property.

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0 Introduction.

In the recent years the classical theory of plurisubharmonic functions of complex variables has been generalized in several directions. These generalizations are in some respects analogous to the complex case and the real one (i.e. the theory of convex functions), but nevertheless reflect rather different geometry behind.

The author [2] and independently at the same time G. Henkin [22] have introduced and studied a class of plurisubharmonic functions of quaternionic variables on the flat quaternionic space \( \mathbb{H}^n \). This class was studied further in [4] where the author has also obtained applications to the theory of valuations on convex sets. Analogous, though geometrically different, results with applications to the valuations theory were obtained in [5] for the case of plurisubharmonic functions of two octonionic variables.

A class of plurisubharmonic functions on hypercomplex manifolds was introduced by M. Verbitsky and the author [6]; in the special case of the flat hypercomplex manifold \( \mathbb{H}^n \) this class coincides with the above mentioned one. Also in [4] an interesting geometric interpretation of strictly plurisubharmonic functions on hypercomplex manifolds was obtained: locally they are precisely the potentials for a special class of Riemannian metrics called Hyper-Kähler with Torsion (HKT). This is obviously analogous to the well known interpretation of strictly plurisubharmonic functions of complex variables as local potentials of Kähler metrics.

In the above mentioned papers an important role was played by quaternionic (and octonionic in [5]) versions of the Hessian and the Monge-Ampère operator. They were applied further for the theory of quaternionic Monge-Ampère equations in [3] in the flat case and in [7], [32] on hypercomplex manifolds in the context of HKT-geometry.

In the recent series of articles [18]-[20] Harvey and Lawson have developed another approach to pluripotential theory in the context of calibrated geometries. In various special cases it partly overlaps with the above mentioned approach. For example on the flat space \( \mathbb{H}^n \) the two approaches lead to the same class of plurisubharmonic functions.

In this paper we introduce a class of plurisubharmonic functions (or more precisely, sections of certain specific line bundle) on an arbitrary quaternionic manifold. Quaternionic manifolds were introduced independently by S. Salamon [29] and L. Bérard-Bergery (see Ch. 14 in the book [12]). They carry quite rich structures, for instance admit the twistor space \( \mathbb{H}^n \), the quaternionic projective space \( \mathbb{HP}^n \), hypercomplex manifolds (in particular, hyper-Kähler manifolds), and quaternionic Kähler manifolds. For more examples of quaternionic manifolds we refer to Ch. 14 of the book [12] and to [24].

We introduce a Monge-Ampère operator on quaternionic manifolds and prove a version of theorems of A.D. Aleksandrov [1] and Chern-Levine-Nirenberg [15]. In the special cases of the flat space and hypercomplex manifolds analogous results were obtained in [2] and [6] respectively. Formally speaking, the theory in the flat case in [2] is indeed a special case of the theory developed in this paper (see Section 7), as well as it is a special case of the theory [6] in the hypercomplex case. However for a general hypercomplex manifold the theory of
is not a special case of the theory of this paper, at least in the case when the holonomy of the Obata connection is not contained in $SL_n(\mathbb{H})$: for example plurisubharmonic sections belong to different line bundles in the two theories. It would be of interest to make a more detailed comparison of the two approaches, in particular when the holonomy of the Obata connection of a hypercomplex manifold is contained in $SL_n(\mathbb{H})$.

The new step of the present paper is the use of the Baston differential operators; one of them turns out to be the right quaternionic analogue of the Hessian on a general quaternionic manifold.

0.1 Remark. One new special case covered by this paper in comparison to [2], [6] is the case of quaternionic Kähler manifolds. Since such manifolds can be considered from the point of view of calibrated geometry, they are also covered by the Harvey-Lawson theory. It turns out that in general the Harvey-Lawson class of plurisubharmonic functions is different from that introduced here, e.g. in the case of quaternionic projective space $\mathbb{H}P^n$ equipped with the Fubini-Study metric. Nevertheless when the metric is flat, e.g. on $\mathbb{H}^n$, the two classes do coincide.

Let us describe the main results in greater detail. The first step is to introduce the right version of of quaternionic Hessian on a general quaternionic manifold $M^{4n}$. We assume throughout the article that $n > 1$; the case $n = 1$ is more elementary but somewhat exceptional for quaternionic manifolds. We claim that this is a differential operator $\Delta$ of second order which was introduced for general quaternionic manifolds by Baston [10] for completely different reasons. The operator $\Delta$ is defined on smooth sections of a real vector bundle over $M$ which is denoted by $(\det \mathcal{H}_0^*)^0 \mathbb{R}$ (for the moment, take it as a single notation), and takes values in a vector bundle denoted by $\wedge^2 \mathcal{E}^*[-2]_\mathbb{R}$. It is analogous to the operator $dd^c$ on a complex manifold. The Baston’s construction is discussed in Section 3. It is rather involved and uses the twistor space of $M$ and the Penrose transform. Notice that in the flat case essentially the same construction was invented much earlier by Gindikin and Henkin [16]. It is shown in Section 7 that on the flat space $\mathbb{H}^n$ the operator $\Delta$ coincides with the quaternionic Hessian introduced in [2] whose construction was elementary.

Next we define the notion of positivity in the fibers of the bundle $\wedge^2 \mathcal{E}^*[-2]_\mathbb{R}$ (Section 5). Then we call a $C^2$-smooth section $h$ of $(\det \mathcal{H}_0^*)^0 \mathbb{R}$ to be plurisubharmonic if $\Delta h$ is non-negative.  

Now let us state the main theorem of the paper. For the sake of simplicity of exposition we do it here in a weaker form; we refer to Section 6 for a complete statement. We have the natural wedge product

$$\wedge^p \mathcal{E}^*[i]_\mathbb{R} \otimes \wedge^q \mathcal{E}^*[j]_\mathbb{R} \to \wedge^{p+q} \mathcal{E}^*[i+j]_\mathbb{R}.$$ 

Using this product we define the Monge-Ampère operator on sections of $(\det \mathcal{H}_0^*)^0 \mathbb{R}$ by

$$h \mapsto (\Delta h)^n$$

\footnote{In fact we define plurisubharmonicity in a slightly greater generality of just continuous sections.}
where $4n$ is the real dimension of the manifold $M$ as previously. Thus the Monge-Ampère operator takes values in the real line bundle $\det \mathcal{E}^*_0[-2n]_{\mathbb{R}}$. Then we show that if $\{h_N\}$ is a sequence of $C^2$-smooth plurisubharmonic sections which converges to a $C^2$-section $h$ in the $C^0$-topology (not necessarily in the $C^2$-topology!) then

$$(\Delta h_N)^n \to (\Delta h)^n$$

in the sense of measures.\footnote{A bit more precisely, we can trivialize the line bundle $\det \mathcal{E}^*_0[-2n]_{\mathbb{R}}$ somehow and consider $(\Delta h_N)^n$ and $(\Delta h)^n$ as measures on $M$. For the very precise statement we refer to Section 6.}

The proof of this result uses, among other things, the following new multiplicativity property of the Baston operators:

$$\Delta(\omega \wedge \Delta \eta) = \Delta \omega \wedge \Delta \eta.$$ \hfill (0.1)

Here the operator $\Delta$ is understood on more general bundles $\wedge^k \mathcal{E}^*_0[-k]_{\mathbb{R}}$ for various $k$'s, and $\omega, \eta$ are sections of such more general bundles. This identity should be compared with its obvious analogue from the case of complex manifolds: for any differential forms $\omega, \eta$ one has

$$dd^c(\omega \wedge dd^c \eta) = dd^c \omega \wedge dd^c \eta.$$ 

The equality (0.1) is proved in Section 4. The proof uses general multiplicative structure in spectral sequences; this is due to the fact that $\Delta$ itself is defined with the use of spectral sequences.

0.2 Remark. The operators $\Delta$ on various bundles over $M$ mentioned above fit into complexes of differential operators whose cohomology can be interpreted as cohomology of the twistor space of $M$. On the flat manifolds of any dimension some of these complexes with such an interpretation were first constructed by Gindikin and Henkin [16], and the remaining ones by Henkin and Polyakov [21].

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1 Quaternionic manifolds.

In this section we remind the definition and basic properties of a quaternionic manifolds due to S. Salamon [29]. First let us fix notation. Let $GL_n(\mathbb{H})$ denote the group of invertible $n \times n$-matrices with quaternionic entries. The group $GL_n(\mathbb{H}) \times GL_1(\mathbb{H})$ acts on $\mathbb{H}^n$ by

$$(A, q)(x) = Axq^{-1}$$ \hfill (1.1)
where \((A, q) \in GL_n(\mathbb{H}) \times GL_1(\mathbb{H})\), and \(x \in \mathbb{H}^n\) is a column of quaternions. Thus we get a group homomorphism

\[
GL_n(\mathbb{H}) \times GL_1(\mathbb{H}) \to GL_{4n}(\mathbb{R}).
\]

(1.2)

The image of it is a closed subgroup denoted by \(GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H})\). Clearly the kernel is the multiplicative group of real numbers \(\mathbb{R}^*\) imbedded diagonally into \(GL_n(\mathbb{H}) \times GL_1(\mathbb{H})\). Let us denote by \(\mathcal{G}\) the subgroup of \(GL_n(\mathbb{H}) \times GL_1(\mathbb{H})\) defined by

\[
\mathcal{G} := \{(A, q) \in GL_n(\mathbb{H}) \times GL_1(\mathbb{H})| \det A \cdot |q| = 1\}
\]

(1.3)

where \(\det\) of a quaternionic matrix is taken in the sense of Diedonné, and \(|q|\) denotes the absolute value of the quaternion \(q\). Alternatively \(\mathcal{G}\) can be characterized as a connected subgroup whose Lie algebra is equal to \(\{(X, Y) \in gl_n(\mathbb{H}) \times gl_1(\mathbb{H})| \sum_{i=1}^n \text{Re}(X_{ii}) + \text{Re}(Y) = 0\}\). The homomorphism

\[
\mathcal{G} \to GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H})
\]

given by the composition of the imbedding \(\mathcal{G} \hookrightarrow GL_n(\mathbb{H}) \times GL_1(\mathbb{H})\) with the homomorphism (1.2) is onto, and the kernel is equal to \(\{\pm 1\}\) imbedded diagonally.

Let \(M\) be a real smooth manifold of dimension \(4n\). A quaternionic structure on \(M\) is a reduction of the structure group of the tangent bundle of \(M\) to \(GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H})\) such that there exists a torsion free \(GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H})\)-connection on the tangent bundle \(TM\). The manifold \(M\) with a quaternionic structure is called a quaternionic manifold. Through the rest of the paper we assume that \(n > 1\) unless otherwise stated; the case \(n = 1\) is rather exceptional and requires a separate treatment.

### 1.1 Remark.

On a quaternionic manifold a torsion free connection as above is not unique.

Let us make an additional assumption about the quaternionic manifold \(M\) that the structure group of \(TM\) has been lifted from \(GL_n(\mathbb{H}) \cdot GL_1(\mathbb{H})\) to \(\mathcal{G}\) (this is possible provided e.g. \(H^2(M, \mathbb{Z}/2\mathbb{Z}) = 0\); see [30]). Let us fix such a lifting. This assumption and this choice will influence none of the main results of the paper since locally this always can be done, but then it can be shown that all the main results are independent of these local choices and make sense globally. Let \(G_0 \to M\) be the corresponding principal \(\mathcal{G}\)-bundle. Then the tangent bundle \(TM\) is isomorphic to \(G_0 \times_\mathcal{G} \mathbb{H}^n\) where the group \(\mathcal{G}\) acts on \(\mathbb{H}^n\) by (1.1). Let us define two other bundles over \(M\)

\[
\mathcal{E}_0 = G_0 \times_\mathcal{G} \mathbb{H}^n
\]

(1.4)

where now the action of \(\mathcal{G}\) on \(\mathbb{H}^n\) is given by \((A, q)(x) = Ax\); and the bundle

\[
\mathcal{H}_0 = G_0 \times_\mathcal{G} \mathbb{H}
\]

(1.5)

where the action of \(\mathcal{G}\) on \(\mathbb{H}\) is given by \((A, q)(x) = xq^{-1}\).

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3See e.g. [8], [2]; let us only mention that it is the only homomorphism of groups \(GL_n(\mathbb{H}) \rightarrow \mathbb{R}^*_0\) whose restriction to the subgroup of complex \(n \times n\) matrices is equal to the absolute value of the usual determinant of complex matrices.
It is easy to see that $E_0$ is a bundle of right $\mathbb{H}$-vector spaces of rank $n$ over $\mathbb{H}$, while $H_0$ is a bundle of left $\mathbb{H}$-vector spaces of rank 1 over $\mathbb{H}$. Moreover we have a natural isomorphism

$$TM \simeq E_0 \otimes_\mathbb{H} H_0$$  \hspace{1cm} (1.6)$$

(notice that the tensor product is over $\mathbb{H}$).

Let us study now the complexified tangent bundle $^C TM := TM \otimes_\mathbb{R} \mathbb{C}$. By (1.6)

$$^C TM = (E_0 \otimes_\mathbb{C} \mathbb{C}) \otimes_\mathbb{H} (H_0 \otimes_\mathbb{R} \mathbb{C})$$  \hspace{1cm} (1.7)$$

where $^C \mathbb{H} := \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ is a $\mathbb{C}$-algebra (non-canonically) isomorphic to the algebra $M_2(\mathbb{C})$ of $2 \times 2$ complex matrices.

1.2 Lemma. There exists an isomorphism of vector bundles

$$^C TM \simeq E_0 \otimes_\mathbb{C} H_0$$

where the tensor product is over $\mathbb{C}$, and $E_0$, $H_0$ are considered as $\mathbb{C}$-vector bundles via the imbedding of fields $\mathbb{C} \hookrightarrow \mathbb{H}$ given by $\sqrt{-1} \mapsto I$.

1.3 Remark. The isomorphism constructed in the proof of the lemma will be used in the rest of the paper. This isomorphism is almost canonical: it depends on some non-canonical universal choice on the level of linear algebra.

Proof of Lemma 1.2. Recall that $^C \mathbb{H} \simeq M_2(\mathbb{C})$ is a central simple $\mathbb{C}$-algebra. It has a unique up to (non-canonical) isomorphism simple right $^C \mathbb{H}$-module $T$. We choose $T$ to be the right $^C \mathbb{H}$-submodule of $^C \mathbb{H}$ as follows

$$T := \{ x - \sqrt{-1}I \cdot x \mid x \in \mathbb{H} \} = \{ z \in ^C \mathbb{H} \mid I \cdot z = \sqrt{-1}z \}.$$ 

Let $E$ be a right $\mathbb{H}$-vector space. We have a functorial isomorphism of right $^C \mathbb{H}$-modules

$$E \otimes_\mathbb{C} T \rightarrow E \otimes_\mathbb{R} \mathbb{C}$$  \hspace{1cm} (1.8)$$

given by

$$\xi \otimes (x - \sqrt{-1}Ix) \mapsto \xi \cdot x - \sqrt{-1}\xi \cdot (Ix).$$  \hspace{1cm} (1.9)$$

It is an easy exercise to check that this defines a well defined morphism of right $^C \mathbb{H}$-modules $E \otimes_\mathbb{C} T \rightarrow E \otimes_\mathbb{R} \mathbb{C}$, and it is an isomorphism.

Similarly let $T'$ be the simple left $^C \mathbb{H}$-submodule of $^C \mathbb{H}$

$$T' = \{ x - \sqrt{-1}xI \mid x \in \mathbb{H} \} = \{ z \in ^C \mathbb{H} \mid z \cdot I = \sqrt{-1}z \}.$$ 

For any left $\mathbb{H}$-module $H$ we have the functorial isomorphism of left $^C \mathbb{H}$-modules

$$T' \otimes_\mathbb{C} H \rightarrow H \otimes_\mathbb{R} \mathbb{C}$$  \hspace{1cm} (1.10)$$
given by
\[(x - \sqrt{-1}xI) \otimes h \mapsto xh - \sqrt{-1}xIh.\] (1.11)

Then by (1.7), (1.8), (1.10) we have
\[\mathbb{C}T M \simeq (\mathcal{E}_0 \otimes_\mathbb{C} T) \otimes_\mathbb{H} (T' \otimes_\mathbb{C} \mathcal{H}_0) \simeq (\mathcal{E}_0 \otimes_\mathbb{C} \mathcal{H}_0) \otimes_\mathbb{C} (T \otimes_\mathbb{H} T').\]

It remains to observe that \(T \otimes_\mathbb{C} H T' \simeq \mathbb{C}\). Q.E.D.

It is well known that for \(n > 1\) the manifold \(M^{4n}\) and the bundles \(\mathcal{E}_0, \mathcal{H}_0\) are real analytic. We denote by \(X\) a small enough complexification of \(M\). This \(X\) is a complex manifold of complex dimension \(4n\). We denote by \(\mathcal{E}, \mathcal{H}\) the holomorphic vector bundles over \(X\) extending the above \(\mathcal{E}_0, \mathcal{H}_0\), i.e. \(\mathcal{E}|_M = \mathcal{E}_0, \mathcal{H}|_M = \mathcal{H}_0\). Then there is an isomorphism of holomorphic vector bundles \(TX \simeq \mathcal{E} \otimes_\mathbb{C} \mathcal{H}\) extending the isomorphism \(\mathbb{C}T M \simeq \mathcal{E}_0 \otimes_\mathbb{C} \mathcal{H}_0\). Clearly \(X\) is equipped with the complex conjugation diffeomorphism
\[\sigma: X \to X\]
which is an anti-holomorphic involution, and the set of fixed points is \(X^\sigma = M\).

Recall that \(\mathcal{E}_0\) is a bundle of right \(\mathbb{H}\)-vector spaces, in particular the multiplication by \(J \in \mathbb{H}\) on the right is an \(I\)-anti-linear operator on fibers of \(\mathcal{E}_0\). By analytic continuation (since everything is real analytic) we get the following structure on \(\mathcal{E}\): on the total space of \(\mathcal{E}\), which is a complex manifold, we have an anti-holomorphic map
\[\hat{J}: \mathcal{E} \to \mathcal{E}\]
such that for any \(x \in M \subset X\) the restriction of \(\hat{J}\) to the fiber \(\mathcal{E}|_x\) coincides with the right multiplication by \(J\), and for any \(z \in X\)
\[\hat{J}: \mathcal{E}|_z \to \mathcal{E}|_{\sigma(z)}\]
is anti-linear,
\[\hat{J}^2 = -Id_{\mathcal{E}|_z}.
\]
Moreover \(\hat{J}\) preserves the class of holomorphic sections of \(\mathcal{E}\). This action on holomorphic sections of \(\mathcal{E}\) will be also denoted by \(\hat{J}\). Now we remind a definition (see e.g. [25], Ch. 2, §1.7).

1.4 Definition. Let \(X\) be a complex manifold.

(1) A real structure on \(X\) is an antiholomorphic involution \(\sigma: X \to X\) (i.e. \(\sigma^2 = Id_X\)). Clearly \(\sigma\) defines the anti-linear map on sections of the sheaf \(\mathcal{O}_X\) of holomorphic functions given by
\[f \mapsto \sigma(f) := \overline{f} \circ \sigma.\]

(2) Let \(\mathcal{F}\) be a coherent sheaf on \(X\). Let \(\sigma\) be a real structure on \(X\). A real structure on \(\mathcal{F}\) is an antilinear map \(\rho\) on sections of \(\mathcal{F}\) extending \(\sigma\) from \(X\) and \(\mathcal{O}_X\) (in particular \(\rho(f\xi) = \sigma(f)\rho(\xi)\) for any section \(f\) of \(\mathcal{O}_X\) and any section \(\xi\) of \(\mathcal{F}\)) such that \(\rho^2 = Id\).

A quaternionic structure on \(\mathcal{F}\) is an antilinear map \(\rho\) on sections of \(\mathcal{F}\) extending \(\sigma\) from \(X\) and \(\mathcal{O}_X\) such that \(\rho^2 = -Id\).
The previous discussion implies that in our situation of a quaternionic manifold the sheaf of holomorphic sections of $E$ is equipped with the quaternionic structure $\hat{J}$.

Similarly the left multiplication by $J$ on $H_0$ induces an anti-holomorphic diffeomorphism $\hat{J} : H \to H$ with the similar properties. It follows that $\sigma, \hat{J}$ induce a quaternionic structure on $H$. Taking the dual bundle we get a quaternionic structure on $H^*$ which we will denote again by $\hat{J}$. Notice also that it induces a real structure on the projectivization $\mathbb{P}(H^*)$, and the natural projection $\mathbb{P}(H^*) \to X$ commutes with the real structures on the two spaces.

Let us return back again to a general complex manifold $X$ with a real structure $\sigma$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are coherent sheaves on $X$ with either real or quaternionic structures $\rho_1$ and $\rho_2$ respectively extending a real structure $\sigma$ on $X$, then $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$ is equipped with $\rho_1 \otimes \rho_2$ (also extending $\sigma$) which is real if $\rho_1, \rho_2$ have the same type (i.e. real or quaternionic simultaneously), and quaternionic if $\rho_1, \rho_2$ have different type. In particular it follows that if $\mathcal{F}$ is the sheaf of holomorphic sections of a holomorphic vector bundle, and $\mathcal{F}$ is equipped with a quaternionic structure $\rho$, then $\wedge^k \mathcal{F}$ is equipped with $\wedge^k \rho$ which is real for even $k$, and quaternionic for odd $k$.

Next, let $\mathcal{F}$ be the sheaf of holomorphic sections of a holomorphic vector bundle $\mathcal{V}$ over $X$ with a real structure $\rho$ extending $\sigma$. Let $X^\sigma$ be the set of fixed points of $\sigma$. Then $X^\sigma$ is a real analytic submanifold of $X$. Moreover if it is non-empty then $\dim_{\mathbb{R}} X^\sigma = \dim_{\mathbb{C}} X$, and $X$ is a complexification of $X^\sigma$. Let us denote by $\mathcal{V}_0$ the restriction of the vector bundle $\mathcal{V}$ to $X^\sigma$. This complex vector bundle $\mathcal{V}_0$ is equipped with the fiberwise real structure. The latter condition means that $\mathcal{V}_0 = (\mathcal{V}_0)_\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ where $(\mathcal{V}_0)_\mathbb{R}$ is the real vector bundle whose fiber over any point from $X^\sigma$ consists of $\rho$-invariant vectors in the fiber of $\mathcal{V}_0$ over that point.

The above discussion implies that for a quaternionic manifold $M$ with a complexification $X$ and vector bundles $E, H$ as previously, we obtain that each vector bundle $\wedge^{2k} E^* \otimes (\det H^*)^{\otimes 2k}$, which also will be denoted briefly as $\wedge^{2k} E^*[-2k]$, is equipped with a real structure. Consequently its restriction $\wedge^{2k} E^*_0 \otimes (\det H^*_0)^{\otimes 2k} =: \wedge^{2k} E^*_0[-2k]$ to $M$ has a pointwise real structure, namely

$$\wedge^{2k} E^*_0[-2k] = \wedge^{2k} E^*_0[-2k]_\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C},$$

where $\wedge^{2k} E^*_0[-2k]_\mathbb{R}$ is a real analytic vector bundle over $M$ defined by the previous general construction.

These real vector bundles $\wedge^{2k} E^*_0[-2k]_\mathbb{R}$ will be used below, in particular in Section 5 where we will introduce the notion of positivity of sections of these bundles necessary to develop quaternionic pluripotential theory.

2 The twistor space.

For a quaternionic manifold $M^{4n}, n > 1$, S. Salamon [29] has constructed a complex manifold $Z$ of complex dimension $2n+1$ called the twistor space of $M$. We remind now this construction since it will be important in Section 5 for the description of Baston’s differential operators.

We will assume that the vector bundles $H_0, E_0$ from Section 1 are defined globally over $M$ though this assumption is not necessary; the whole construction can be done first locally on $M$, and then it can be easily shown that it is independent of local choices.

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As a smooth manifold the twistor space

\[ Z := \mathbb{P}(\mathcal{H}_0^*) \]

is the (complex) projectivization of \( \mathcal{H}_0^* \). To describe the complex structure on \( Z \) it will be convenient to describe first the complex structure on the total space of \( \mathcal{H}_0^* \) with the zero section removed (in fact this space carries not only complex structure, but a hypercomplex structure, see [28]).

Let us choose a torsion free \( \mathcal{G} \)-connection \( \nabla \). For any point \( z \in \mathcal{H}_0^*\{0\} \) the connection \( \nabla \) induces a decomposition of the tangent space \( T_z\mathcal{H}_0 \) to the direct sum of the vertical \( V_z \) and the horizontal \( L_z \) subspaces with respect to the natural projection \( p: \mathcal{H}_0^* \rightarrow M \):

\[ T_z\mathcal{H}_0^* = V_z \oplus L_z. \]

Clearly \( V_z = \mathcal{H}_0^*|_{p(z)} \). Next the differential \( dp: \mathcal{L}_z \rightarrow T_{p(z)}M \) is an isomorphism of \( \mathbb{R} \)-vector spaces. One equips \( V_z = \mathcal{H}_0^*|_{p(z)} \) with the complex structure \( I \) (recall that \( \mathcal{H}_0^* \) is a quaternionic vector bundle). Let us describe the complex structure on \( L_z \rightarrow T_{p(z)}M \). We follow [30]. Let \( l \subset \mathcal{H}_0^*|_{p(z)} \) be the complex line spanned by \( z \). The complex structure on \( T_{p(z)}M \) is uniquely characterized by the property that the space of (1, 0)-forms at \( p(z) \) is equal to the subspace

\[ \mathcal{E}_0^*|_{p(z)} \otimes \mathbb{C} l \subset \mathcal{E}_0^*|_{p(z)} \otimes \mathbb{C} \mathcal{H}_0^*|_{p(z)} \overset{\text{Lemma 12}}{\simeq} T_{p(z)}^*M \otimes_{\mathbb{R}} \mathbb{C}. \]

Thus we got an almost complex structure on \( \mathcal{H}_0^* \). It is was shown in [28], Theorem 3.2, that it is integrable and is independent of a choice of a torsion free connection \( \nabla \).

Now the non-zero complex numbers \( \mathbb{C}^* \) act holomorphically by the product on \( \mathcal{H}_0^*\{0\} \), and the quotient is equal to the twistor space \( Z = \mathbb{P}(\mathcal{H}_0^*) \). Hence \( Z \) carries a complex structure.

Observe moreover that we have constructed also a holomorphic principal \( \mathbb{C}^* \)-bundle \( \mathcal{H}_0^*\{0\} \rightarrow Z \). Let \( \mathcal{O}_Z(-1) \) be the holomorphic line bundle over \( Z \) defined by

\[ \mathcal{O}_Z(-1) := \mathcal{H}_0^*\{0\} \times_{\mathbb{C}^*} \mathbb{C} \]

where \( \mathbb{C}^* \) acts on \( \mathbb{C} \) by the usual multiplication. The dual line bundle is denoted by \( \mathcal{O}_Z(1) \); this holomorphic line bundle is called sometimes the Swann bundle. Notice also that as a smooth complex line bundle \( \mathcal{O}_Z(1) \) is easily described as follows: it is usual dual Hopf line bundle over \( Z = \mathbb{P}(\mathcal{H}_0^*) \). Let us emphasize that \( \mathcal{O}_Z(1) \) is defined globally only under the assumption that the structure group of \( TM \) is lifted to \( \mathcal{G} \). However \( Z \) itself and \( \mathcal{O}_Z(2) := (\mathcal{O}_Z(1))^\otimes 2 \) are defined globally independently of any local choices. In this paper we will really use only \( \mathcal{O}_Z(2) \) and its tensor powers.

As in Section 1 we denote by \( X \) a small enough complexification of \( M \), and by \( \mathcal{H} \) the vector bundle over \( X \) which is the (unique) holomorphic extension of the real analytic vector bundle \( \mathcal{H}_0 \) from \( M \) to \( X \). Let us consider the complex analytic manifold \( \mathbb{P}(\mathcal{H}^*) \). We have the obvious holomorphic map \( \tau: \mathbb{P}(\mathcal{H}^*) \rightarrow X \). It is obvious that

\[ \tau^{-1}(M) = \mathbb{P}(\mathcal{H}_0^*) = Z. \]
(Warning: \( Z = \tau^{-1}(M) \) is not a complex submanifold of \( \mathbb{P}(\mathcal{H}^*) \).)

The next non-trivial claim is that there exists a holomorphic submersion

\[ \eta: \mathbb{P}(\mathcal{H}^*) \to Z \]

which is uniquely characterized by the property that \( \eta|_{\tau^{-1}(M)=Z} = Id_Z \) (see [10], [14]).

Thus we get a diagram of holomorphic submersions

\[ Z \xleftarrow{\eta} \mathbb{P}(\mathcal{H}^*) \xrightarrow{\tau} X. \]

(2.1)

Let us assume again that the structure group of \( T M \) is lifted to \( G \). Over the complex manifold \( \mathbb{P}(\mathcal{H}^*) \) we have the usual holomorphic dual Hopf line bundle which will be denoted by \( \tilde{O}(1) \). Observe that its restriction to \( Z = \tau^{-1}(M) \) is equal to \( O_Z(1) \).

2.1 Lemma. There exists a unique isomorphism of holomorphic line bundles

\[ \Phi: \eta^*(O_Z(1)) \xrightarrow{\sim} \tilde{O}(1) \]

(2.2)

such that the restriction of \( \Phi \) to \( Z = \tau^{-1}(M) \) is equal to the identity.

Proof. First let us prove the uniqueness; we will show that in fact \( \Phi \) is unique locally on \( \mathbb{P}(\mathcal{H}^*) \), more precisely \( \Phi \) is unique in a neighborhood of any point \( p \in \tau^{-1}(M) \). Any such point \( p \) has a neighborhood and holomorphic coordinates \((z_1, \ldots, z_{4n}, w)\) such that

\[ \tau^{-1}(M) = \{(z_1, \ldots, z_{4n}, w) | \text{Im}(z_1) = \cdots = \text{Im}(z_{4n}) = 0\}. \]

Assume that we have two such isomorphisms \( \Phi \) and \( \Phi' \). Then there exists a holomorphic function \( f \) such that

\[ \Phi' = f \cdot \Phi \]

and \( f|_{\tau^{-1}(M)} = 1 \). In order to show that \( f = 1 \) identically, let us decompose \( f \) into the holomorphic Taylor series:

\[ f = \sum_{\alpha,b} f_{\alpha} z^{\alpha} w^{b} \]

where \( \alpha = (\alpha_1, \ldots, \alpha_{4n}) \in \mathbb{Z}_{\geq 0}^{4n}, b \in \mathbb{Z}_{\geq 0}, z = (z_1, \ldots, z_{4n}) \). We know that \( f = 1 \) whenever \( \text{Im}(z_1) = \cdots = \text{Im}(z_{4n}) = 0 \). It follows that all partial derivatives of \( f \) of positive degree vanish. Hence \( f = 1 \) identically, and \( \Phi' = \Phi \).

Now let us prove the existence of \( \Phi \). Due to the uniqueness of such an isomorphism, which was proved even locally on \( \mathbb{P}(\mathcal{H}^*) \), it suffices to prove the existence locally in a neighborhood of an arbitrary point \( p \in \tau^{-1}(M) \); here one should make \( X \) smaller if necessary and use the properness of the map \( \tau \).

The obvious (identity) isomorphism between the vector bundles \( \eta^*(O_Z(1)|_{\tau^{-1}(M)}) \) and \( \tilde{O}(1)|_{\tau^{-1}(M)} \) over \( \tau^{-1}(M) \) is real analytic (since all the manifolds, morphisms and vector bundles are real analytic). Let us fix holomorphic trivializations of \( \eta^*(O_Z(1)) \) and \( \tilde{O}(1) \) in a neighborhood \( U \subset X \) of \( p \). Then our real analytic isomorphism over \( \tau^{-1}(M) \) between the trivialized line bundles is given by a non-vanishing real analytic function

\[ g: \tau^{-1}(M) \cap U \to \mathbb{C}. \]
Let us decompose $g$ into (real) Taylor series converging in $\tau^{-1}(M) \cap U$

$$g = \sum_{\alpha,b,c} g_{\alpha,b,c} x^\alpha w^b \bar{w}^c$$

(2.3)

where $x = (x_1, \ldots, x_{4n}) \in \mathbb{R}^{4n}$ with $x_i = \text{Re}(z_i)$, $\alpha \in \mathbb{Z}_{\geq 0}^{4n}, b, c \in \mathbb{Z}_{\geq 0}$.

Next it is clear that the restriction of $g$ to any complex curve $\tau^{-1}(m) \cap U$, with $m \in M$ being an arbitrary point, is a holomorphic function (this is because $\eta^* \mathcal{O}_Z(1)|_{\tau^{-1}(m)} \simeq \mathcal{O}(1)|_{\tau^{-1}(m)}$ is the Hopf bundle over $\tau^{-1}(m) \simeq \mathbb{C}P^1$).

This is equivalent to say that for any fixed $x \in \mathbb{R}^{4n}$, $g$ is a holomorphic function in $w$. This implies that actually $\bar{w}$ does not appear in (2.3):

$$g = \sum_{\alpha,b} g_{\alpha,b} x^\alpha w^b.$$ 

Since the Taylor series converges in $U$, it is a restriction to $\tau^{-1}(M)$ of the holomorphic function

$$\tilde{g} := \sum_{\alpha,b} g_{\alpha,b} z^\alpha w^b.$$ 

Then $\tilde{g}$ induces a local isomorphism

$$\eta^* \mathcal{O}_Z(1)|_U \xrightarrow{\sim} \mathcal{O}(1)|_U$$

which is equal to the identity on $\tau^{-1}(M) \cap U$. Q.E.D.

Let us discuss now quaternionic structures on the above line bundles. In Section I we have defined a real structure on $X$ which is just the complex conjugation $\sigma: X \to X$. Also we have defined a quaternionic structure $\hat{J}: \mathcal{H}^* \to \mathcal{H}^*$. It induces a real structure $\tilde{\sigma}$ on the projectivization of $\mathcal{H}^*$:

$$\tilde{\sigma}: \mathbb{P}(\mathcal{H}^*) \to \mathbb{P}(\mathcal{H}^*).$$

Moreover $\hat{J}$ induces a quaternionic structure $\tilde{J}$ on the line bundle $\mathcal{O}(1)$ which extends $\tilde{\sigma}$ (see Definition 1.4):

$$\tilde{J}(\xi) = \xi \cdot \hat{J}$$

(notice that $\hat{J}$ acts on the right since $\mathcal{H}^*_0$ is a bundle of right $\mathbb{H}$-modules).

The restriction of $\tilde{\sigma}$ to $Z = \tau^{-1}(M) \subset \mathbb{P}(\mathcal{H}^*)$ is antiholomorphic, and hence it can be considered as a real structure on $Z$ which will be denoted by $\sigma_Z$. The map $\eta: \mathbb{P}(\mathcal{H}^*) \to Z$ intertwines $\tilde{\sigma}$ and $\sigma_Z$:

$$\eta \circ \tilde{\sigma} = \sigma_Z \circ \eta.$$ 

(2.4)

Indeed $\sigma_Z^{-1} \circ \eta \circ \tilde{\sigma}: \mathbb{P}(\mathcal{H}^*) \to Z$ is a holomorphic map whose restriction to $Z = \tau^{-1}(M)$ is equal to $\text{Id}_Z$. Hence this map must be equal to $\eta$.

Next, the restriction of $\tilde{J}$ to $\mathcal{O}_Z(1)$ induces a quaternionic structure on $\mathcal{O}_Z(1)$ which we denote by $J_Z$. 

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The holomorphic line bundle \( \eta^*(\mathcal{O}_Z(1)) \) has an induced quaternionic structure \( \eta^*J_Z \) which extends the real structure \( \tilde{\sigma} \) because of (2.4). The isomorphism \( \Phi: \eta^*(\mathcal{O}_Z(1)) \tilde{\to} \tilde{\mathcal{O}}(1) \) from (2.2) is compatible with quaternionic structures:

\[
\Phi \circ \eta^*J_Z = \tilde{J} \circ \Phi.
\]

Indeed \( \tilde{J}^{-1} \circ \Phi \circ \eta^*J_Z : \eta^*(\mathcal{O}_Z(1)) \tilde{\to} \tilde{\mathcal{O}}(1) \) is a holomorphic isomorphism of holomorphic vector bundles whose restriction to \( Z = \tau^{-1}(M) \) is equal to identity.

3 The Penrose transform and the Baston complexes.

The goal of this section is to describe a construction due to Baston [10] of certain complexes of vector bundles over a quaternionic manifold \( M^{4n}, n > 1 \), where the differentials are differential operators of either first or second order. These complexes depend only on the quaternionic structure of \( M \), in particular they are equivariant under quaternionic automorphisms of \( M \). One of the differential operators of the second order in one of these complexes will be necessary in our definition of plurisubharmonic functions in Section 6 below. Some of the other operators will be useful for technical reasons, e.g. in the proof of a quaternionic generalization of theorems of Aleksandrov and Chern-Levine-Nirenberg.

The Baston approach [10] to construct these complexes is based on the use of the Penrose transform. Thus we will have to remind this notion in this section. It is convenient to assume existence and fix a lifting of the structure group of \( TM \) to \( \mathcal{G} \subset GL_n(\mathbb{H}) \times GL_1(\mathbb{H}) \) defined in Section 1 (in applications to the pluripotential theory a lifting always exists locally). Global results are independent of such local liftings.

We keep the notation of Sections 1, 2. But for brevity we will denote in this section \( F := \mathbb{P}(\mathcal{H}^*) \). Consider again as in Section 2 the holomorphic maps

\[
Z \xleftarrow{\eta} F \xrightarrow{\tau} X
\]

(recall that \( X \) is a small enough complexification of \( M \), and \( Z \) is the twistor space). Let \( \mathcal{L} \) be a holomorphic vector bundle over \( Z \); by the abuse of notation we will denote also by \( \mathcal{L} \) the sheaf of holomorphic sections of it. Let \( \eta^{-1}\mathcal{L} \) denote the pull-back of \( \mathcal{L} \) under the map \( \eta \) in the category of abstract sheaves (thus in particular \( \eta^{-1}\mathcal{L} \) is not a sheaf of holomorphic sections of anything). The Penrose transform of \( \mathcal{L} \), by the definition, is \( R\tau_*(\eta^{-1}\mathcal{L}) \) where \( R\tau_* \) is the push-forward morphism under \( \tau \) between the derived categories of sheaves:

\[
R\tau_* : D^+(\mathbb{P}F) \to D^+(\mathbb{P}X)
\]

where \( D^+(\mathbb{P}X) \) denotes the bounded from below derived category of sheaves of \( \mathbb{C} \)-vector spaces on \( X \).

Let \( (\Omega^*_{F/Z}, d) \) denote the complex of holomorphic relative differential forms with respect to the map \( \eta: F \to Z \). We have the resolution \( \Omega^*_{F/Z} \) of the sheaf \( \eta^{-1}\mathcal{L} \) by coherent sheaves

\[
0 \to \eta^{-1}\mathcal{L} \to \eta^*\mathcal{L} \xrightarrow{d} \eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^1_{F/Z} \xrightarrow{d} \eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^2_{F/Z} \xrightarrow{d} \ldots \tag{3.1}
\]
where $\eta^*\mathcal{L}$ denotes the pull-back of $\mathcal{L}$ in the category of quasi-coherent sheaves, all the tensor products are over the sheaf $\mathcal{O}_F$ of holomorphic functions on $F$, and the differentials are the holomorphic de Rham differentials. Hence the Penrose transform $R\tau_*(\eta^{-1}\mathcal{L})$ is equal to

$$R\tau_*(\Omega^*(\mathcal{L})) = R\tau_*\left(0 \to \eta^*\mathcal{L} \xrightarrow{d} \eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^1_{F/Z} \xrightarrow{d} \eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^2_{F/Z} \xrightarrow{d} \ldots\right). \quad (3.2)$$

Let us consider the hypercohomology spectral sequence for (3.2) such that its first terms, denoted by $E_{p,q}^1(\mathcal{L})$, are

$$E_{p,q}^1(\mathcal{L}) = R^q\tau_*(\eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^p_{F/Z}).$$

Since the fibers of $\tau$ are complex projective lines and all the sheaves $\eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^p_{F/Z}$ are coherent, we have

$$E_{p,q}^1(\mathcal{L}) = 0 \text{ for } q \neq 0, 1.$$

Next assume that $\mathcal{L}$ has real (resp. quaternionic) structure $\rho$ extending the real structure $\sigma_Z$ on $Z$ (see Definition 1.4 in Section 1). Then clearly $\eta^*\mathcal{L}$ has real (resp. quaternionic) structure $\eta^*\rho$ extending the real structure $\tilde{\sigma}$ on $F = \mathbb{P}(\mathcal{H}^*)$. The sheaves $\eta^*\mathcal{L} \otimes_{\mathcal{O}_F} \Omega^p_{F/Z}$ are equipped with a real (resp. quaternionic) structure $\rho$ as follows:

$$\rho^p(\xi \otimes \omega) = (\eta^*\rho)(\xi) \otimes \tilde{\sigma}^*(\omega)$$

where the bar denotes the complex conjugation on differential forms. It is easy to see that the holomorphic relative de Rham differential (3.1) is compatible with $\rho$'s:

$$d \circ \rho^p = \rho^{p+1} \circ d.$$ 

Since $\tau: F \to X$ intertwines the real structures $\tilde{\sigma}$ and $\sigma$, it follows that all the terms $E_{p,q}^r(\mathcal{L})$ of our spectral sequence are equipped with real (resp. quaternionic) structure, and the differentials $d_{p,q}^r$ commute with it.

In order to construct the Baston complexes, Baston has chosen

$$\mathcal{L} = \mathcal{O}_Z(-k) := \mathcal{O}_Z(-1)^{\otimes k}, \quad 2 \leq k \leq 2n,$$

computed $E_{p,q}^1(\mathcal{O}_Z(-k))$, and appropriate differentials $d_{1}^{p,q}, d_{2}^{p,q}$. We will use these computations, so let us describe them. For a sheaf $\mathcal{S}$ on $X$ and an integer $l$ let us denote for brevity

$$\mathcal{S}[l] := \mathcal{S} \otimes_{\mathcal{O}_X} (\det \mathcal{H})^\otimes l.$$

3.1 Proposition (Baston [10]). Let $k \geq 2$. Then one has

$$E_{p,q}^1(\mathcal{O}_Z(-k)) = \begin{cases} 
Sym^{p-k}\mathcal{H} \otimes \wedge^p \mathcal{E}^*[-p] & \text{if } p-k \geq 0, \\
0 & \text{if } p-k < 0
\end{cases};$$

$$E_{p,1}^1(\mathcal{O}_Z(-k)) = \begin{cases} 
Sym^{k-p-2}\mathcal{H}^* \otimes \wedge^p \mathcal{E}^*[-p-1] & \text{if } k-p-2 \geq 0, \\
0 & \text{if } k-p-2 < 0
\end{cases};$$

where all the tensor products are over $\mathcal{O}_X$.

Notice that in particular

$$E_{k-1,q}^1(\mathcal{O}_Z(-k)) = 0 \text{ for any } q.$$

The proposition implies that the first terms of the spectral sequence look as follows:
Figure 1: The first term $E_1^{p,q}(\mathcal{O}_Z(-k))$ of the spectral sequence.

The first differentials in this spectral sequence

$$d_1^{p,q}: E_1^{p,q} \rightarrow E_1^{p+1,q}$$

are differential operators of the first order (see [10]).

The only non-zero differential in the second term spectral sequence is

$$d_{2}^{k-2,1}: E_2^{k-2,1} \rightarrow E_2^{k,0}.$$

This clearly induces a morphism of sheaves

$$\Delta: \wedge^{k-2} \mathcal{E}^*[-k + 1] \rightarrow \wedge^{k} \mathcal{E}^*[-k].$$

This $\Delta$ is a differential operator of the second order [10]. Putting together all $d_1^{p,q}$ and $\Delta$ one gets the Baston complex of coherent sheaves on $X$ for any $2 \leq k \leq 2n$:

$$0 \rightarrow \text{Sym}^{k-2} \mathcal{H}^*[-1] \xrightarrow{d_1^{0,1}} \text{Sym}^{k-3} \mathcal{H}^* \otimes \mathcal{E}^*[-2] \xrightarrow{d_1^{1,1}} \ldots$$

$$\ldots \xrightarrow{d_1^{k-3,1}} \wedge^{k-2} \mathcal{E}^*[-k + 1] \xrightarrow{\Delta} \wedge^{k} \mathcal{E}^*[-k] \xrightarrow{d_1^{k,0}}$$

$$d_1^{k,0} \rightarrow \mathcal{H} \otimes \wedge^{k+1} \mathcal{E}^*[-k - 1] \xrightarrow{d_1^{1,k+1,0}} \text{Sym}^2 \mathcal{H} \otimes \wedge^{k+2} \mathcal{E}^*[-k - 2] \xrightarrow{d_1^{1,k+2,0}} \ldots$$

$$\ldots \xrightarrow{d_1^{2n-1,0}} \text{Sym}^{2n-k} \mathcal{H} \otimes \wedge^{2n} \mathcal{E}^*[-2n] \rightarrow 0$$

where all the tensor products are over $\mathcal{O}_X$. We call this complex of coherent sheaves the holomorphic Baston complex. Baston shows [10], Proposition 10, that this complex of holomorphic sheaves is acyclic except at the left. Moreover when the quaternionic manifold $M$ is a small enough ball (not necessarily flat as a quaternionic manifold), the complex of the global sections of the Baston complex is a resolution of $H^1(Z, \mathcal{O}_Z(-k))$. 

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Let us discuss now the quaternionic and the real structures on the Baston complex. On each term of it let us take just the tensor product of the quaternionic structures on $\mathcal{H}, \mathcal{E}$ and their duals defined in Section 1. Then for each even $k$ we get a real structure on each term of the complex (this case will be particularly important for pluripotential theory), and for odd $k$ we get a quaternionic structure. The differentials of the Baston complex commute with these real (resp. quaternionic) structures.

Next passing to a germ of $M$ inside of $X$ we get the complex of differential operators between the sheaves on $M$ of real analytic sections of the corresponding vector bundles over $M$ (in the above holomorphic Baston complex, one just replaces $\mathcal{H}, \mathcal{E}$ with $\mathcal{H}_0, \mathcal{E}_0$ respectively everywhere). The differential operators in that complex have real analytic coefficients; we will denote them by the same letters as in the holomorphic Baston complex. They extend uniquely by continuity to morphisms of sheaves of infinitely smooth sections of these vector bundles. In particular we get the following complex:

We will call this complex the smooth Baston complex.

3.2 Remark. Baston claims that the smooth Baston complex is also acyclic except at the left. Moreover when $M$ is a small enough ball, the complex of its global sections computes $H^1(Z, O_Z(-k))$. We will not use however these facts in this paper.

The case $k = 2$ will be particularly important; let us write the beginning of the smooth Baston complex in this case:

We will call this complex the smooth Baston complex.

3.3 Remark. Later on in Section 7 we will change slightly the definitions of $\Delta$ by multiplying it by an appropriate constant in order to make it compatible with conventions in the flat case.

As a side remark let us mention that some other complexes on quaternionic manifolds were considered in [33].

4 Multiplicative properties of the Baston operators.

Recall that on a quaternionic manifold $M^{4n}, n > 1$, we have the Baston differential operators for any $2 \leq k \leq 2n$

$$\Delta: C^\infty(M, \wedge^{k-2} \mathcal{E}_0^*[-k+1]) \to C^\infty(M, \wedge^k \mathcal{E}_0^*[-k]).$$
In this section we discuss the relation of these operators to the obvious wedge product
\[ \wedge^p E^*_0[i] \otimes \wedge^q E^*_0[j] \to \wedge^{p+q} E^*_0[i + j]. \]
This relation will be important in the proof of the quaternionic generalization of the theorems of Aleksandrov and Chern-Levine-Nirenburg. The main result of this section is the following proposition which is a new result, to the best of our knowledge.

4.1 Proposition. Let \( k, l \geq 2 \), \( k + l \leq 2n \). Then for any
\[ \omega \in C^\infty(M, \wedge^{k-2} E^*_0[-k + 1]), \eta \in C^\infty(M, \wedge^{l-2} E^*_0[-l + 1]) \]
one has
\[ \Delta(\omega \wedge \Delta \eta) = \Delta \omega \wedge \Delta \eta. \] (4.1)

Proof. We may and will assume that \( \omega \) and \( \eta \) are real analytic. Hence they can be considered as holomorphic sections over \( X \) of the bundles \( \wedge^{k-2} E^*_0[-k + 1] \) and \( \wedge^{l-2} E^*_0[-l + 1] \) respectively. The result follows from rather general properties of the multiplicative structure in spectral sequences.

Recall that the (holomorphic) Baston operator was defined as the only non-zero second differential in the hypercohomology spectral sequence for the complex
\[ \Omega^\bullet(O_Z(-k)) = \left( 0 \to \mathcal{O}(-k) \xrightarrow{d} \mathcal{O}(-k) \otimes O_F \Omega^1_{F/Z} \xrightarrow{d} \mathcal{O}(-k) \otimes O_F \Omega^2_{F/Z} \xrightarrow{d} \ldots \right) \]
and similarly with \( l \) instead of \( k \).

First obviously \( \mathcal{O}_Z(-k) \otimes O_Z(-l) = \mathcal{O}_Z(-(k + l)) \). Hence we have the obvious natural morphism of complexes of sheaves
\[ \Omega^\bullet(O_Z(-k)) \otimes \mathcal{O}^\bullet(O(-l)) \to \Omega^\bullet(O(-k - l)) \] (4.2)
where the tensor product is taken in the sense of complexes of sheaves over the sheaf \( \mathbb{C} \) of locally constant \( \mathbb{C} \)-valued functions. We will need few basic general facts on the multiplicative structure in spectral sequences. We failed to find a precise reference to these facts, but they seem to be a common knowledge among experts in homological algebra. The author has learned them from A. Beilinson [11].

First let us equip the complexes \( \Omega^\bullet(O_Z(-k)) \) and \( \Omega^\bullet(O_Z(-l)) \) with the stupid filtration, and to replace them with a resolution in the bounded from below filtered derived category \( D^+ F \) of sheaves of \( \mathbb{C} \)-modules (see e.g. [23], Ch. V). The need to work with the filtered derived category comes from the fact that in \( D^+ F \) we have both the derived push-forward \( R\tau_* \), tensor product, and the spectral sequence related to \( R\tau_* \). The usual derived category is not sufficient since the spectral sequence is not defined on elements of it; on the other hand in the category of actual complexes \( R\tau_* \) is not well defined.

There is a general notion of tensor product of abstract spectral sequences [26]. Without giving all the formal details and definitions, let us only explain the main idea. Given two
spectral sequences with terms and differentials \((E'_r^{p,q}, d'_r^{p,q})\) and \((E''_r^{p,q}, d''_r^{p,q})\) respectively, the \(E^p,q_r\)-term of their tensor product is defined by

\[
E^p,q_r := \bigoplus_{p' + p'' = p, \quad q' + q'' = q} E'_r^{p',q'} \otimes \mathbb{C} E''_r^{p'',q''}.
\]

The differentials in the tensor product of the spectral sequences, which will be denoted by \(d_{\otimes,r}^{p,q}\), are defined as the differential in the tensor product of complexes:

\[
d_{\otimes,r}^{p,q} := \bigoplus_{p' + p'' = p, \quad q' + q'' = q} \left( d'_r^{p',q'} \otimes 1 + (-1)^{p'+q'} \cdot 1 \otimes d''_r^{p'',q''} \right).
\]

Then \((E^p,q_r, d_{\otimes,r}^{p,q})\) is a new spectral sequence, i.e. the cohomology of \(d_{\otimes,r}^{p,q}\) is naturally isomorphic to \(E^{r+1}\). This is due to the fact that we work in the category of sheaves over a field.

The key property of this tensor product of spectral sequences computing \(R \tau_*\) is that there exists a canonical morphism from the tensor product of spectral sequences of two complexes of sheaves of \(\mathbb{C}\)-modules to the spectral sequence (again computing \(R \tau_*\)) of the tensor product over \(\mathbb{C}\) of these complexes. This fact and the morphism (4.2) lead to a canonical morphism of spectral sequences

\[
E(\mathcal{O}_Z(-k)) \otimes E(\mathcal{O}_Z(-l)) \to E(\mathcal{O}_Z(-k-l)) \tag{4.3}
\]

where the first expression denotes the tensor product of spectral sequences mentioned above. In particular for \(r = 1\) we get morphism

\[
E^{k-2,1}_1(\mathcal{O}_Z(-k)) \otimes \mathbb{C} E^{l,0}_1(\mathcal{O}_Z(-l)) \to E^{k+l-2,1}_1(\mathcal{O}_Z(-k-l)).
\]

Computing these terms by Proposition 3.1 we get a morphism of sheaves of \(\mathbb{C}\)-modules

\[
\wedge^{k-2}\mathcal{E}^*[k-1] \otimes \mathbb{C} \wedge^l \mathcal{E}^*[-l] \to \wedge^{k+l-2}\mathcal{E}^*[-k-l+1].
\]

It is easy to see that this morphism coincides with the composition of the natural morphism of sheaves

\[
\wedge^{k-2}\mathcal{E}^*[k-1] \otimes \mathbb{C} \wedge^l \mathcal{E}^*[-l] \to \wedge^{k-2}\mathcal{E}^*[k-1] \otimes_{\mathcal{O}_X} \wedge^l \mathcal{E}^*[-l]
\]

with the wedge product.

By the definition of morphism of spectral sequences, the morphism (4.3) commutes with the differentials in spectral sequences. We will apply this to the second differential \(d_{\otimes,2}(\omega \otimes \Delta \eta)\).

First observe that \(d_2(\Delta \eta)\), and hence \(d_{\otimes,2}(\omega \otimes \Delta \eta)\), is well defined, in other words \(d_1^{0,0}(\Delta \eta) = 0\). Indeed \(d_1^{1,0}(\Delta \eta) = d_1^{0,0}(d_2 \eta)\), and, by the definition of spectral sequence, \(d_2 \eta\) takes values in the cohomology space of \(d_1^{0,0}\) which in our case is equal to the kernel of \(d_1^{0,0}\), namely \(d_1^{0,0} \circ d_2 = 0\).
Thus by the definition of $d_{\otimes,2}$ we get

$$d_{\otimes,2}(\omega \otimes \Delta \eta) = d^{k-2,1}_2 \omega \otimes \Delta \eta \pm \omega \otimes d^{0,0}_2(\Delta \eta).$$

But $d^{0,0}_2 \equiv 0$. Hence

$$d_{\otimes,2}(\omega \otimes \Delta \eta) = d^{k-2,1}_2 \omega \otimes \Delta \eta = \Delta \omega \otimes \Delta \eta.$$

The last expression is mapped to $\Delta \omega \wedge \Delta \eta$ under the morphism (4.3). Q.E.D.

We will need few more differential operators on $M$ depending only on the quaternionic structure. They are defined as dual operators to $\Delta$ with respect to a very general notion of duality which we remind now.

For the moment we assume that $M$ is a smooth oriented manifold without any additional structure. Let $\mathcal{S}$ and $\mathcal{T}$ be two finite dimensional vector bundles over $M$, both either real or complex simultaneously. Let $D: C^\infty(M, \mathcal{S}) \to C^\infty(M, \mathcal{T})$ be a linear differential operator with $C^\infty$-smooth coefficients. Let us denote by $\omega_M$ the complex line bundle of differential forms of top degree (either real or complex valued, depending whether $\mathcal{T}$ and $\mathcal{S}$ are real or complex). Let us consider the operator $D^*: (C^\infty_c(M, \mathcal{T}^* \otimes \omega_M))^* \to (C^\infty_c(M, \mathcal{S}^* \otimes \omega_M))^*$

where the subscript $c$ stays for compactly supported sections.

Next we have the canonical map

$$C^\infty(M, \mathcal{T}^* \otimes \omega_M) \to (C^\infty_c(M, \mathcal{T}))^*$$

given by $f \mapsto [\phi \mapsto \int_M f, \phi]$. This map is injective and has dense image in the weak topology. Thus we will just identify the image of this map with the source space itself:

$$C^\infty(M, \mathcal{T}^* \otimes \omega_M) \subset (C^\infty_c(M, \mathcal{T}))^*$$

$$C^\infty(M, \mathcal{S}^* \otimes \omega_M) \subset (C^\infty_c(M, \mathcal{S}))^*.$$ 

It is easy to see that $D^*$ preserves the class of $C^\infty$-smooth sections. Actually

$$D^*: C^\infty(M, \mathcal{T}^* \otimes \omega_M) \to C^\infty(M, \mathcal{S}^* \otimes \omega_M)$$

is a differential operator of the same order as $D$ with $C^\infty$-smooth coefficients.

Let us apply this construction to our quaternionic manifold $M$ and the Baston operator

$$\Delta: C^\infty(M, \wedge^{k-2} \mathcal{E}_0^*[-k+1]) \to C^\infty(M, \wedge^k \mathcal{E}_0^*[-k]), \quad 2 \leq k \leq 2n.$$ 

Then we get a second order differential operator which will be used later

$$\Delta^*: C^\infty(M, \wedge^k \mathcal{E}_0[k] \otimes \omega_M) \to C^\infty(M, \wedge^{k-2} \mathcal{E}_0[k-1] \otimes \omega_M).$$
Since the line bundle $\omega_M$ has a canonical real structure and orientation, the bundles $\wedge^p E_0[i] \otimes \omega_M$ are equipped with a real structure for even $p$, and with a quaternionic structure for odd $p$. The operator $\Delta^*$ commutes with these structures, since $\Delta$ does. Notice also that

$$\omega_M \simeq (\det E_0^*)^{\otimes 2}[-2n].$$

Thus by the definition of the dual operator we have

$$\int_M < f, \Delta \xi > = \int_M < \Delta^* f, \xi > \tag{4.4}$$

for any $\xi \in C^\infty(M, \wedge^k E_0[-k+1]), f \in C^\infty(M, \wedge^k E_0[k] \otimes \omega_M)$.

### 5 Positive currents on quaternionic manifolds.

Let $M^{4n}$, $n > 1$, be a quaternionic manifold. We introduce in this section the notion of positive (generalized) sections of the bundles $\wedge^{2k} E_0[-2k], 0 \leq k \leq n$. This is analogous to the notion of positive current from the complex analysis. The case $k = 1$ will be necessary for the definition of quaternionic plurisubharmonic function. In fact the other $k$'s will be important too, e.g. in the statement (and the proof) of the Aleksandrov and the Chern-Levine-Nirenberg type theorems.

Most of the discussion is actually purely linear algebraic. Let $E$ be a right $\mathbb{H}$-module of rank $n$, and let $H$ be a left $\mathbb{H}$-module of rank 1. As in Section 1 we will consider $E$ and $H$ as $\mathbb{C}$-vector spaces via the imbedding of fields $\mathbb{C} \hookrightarrow \mathbb{H}$ given by $\sqrt{-1} \mapsto I$. Denote

$$E^* := Hom_\mathbb{R}(E, \mathbb{R}), \quad H^* := Hom_\mathbb{R}(H, \mathbb{R});$$

they are left and right $\mathbb{H}$-modules respectively.

Let $0 \leq k \leq n$. The space $\wedge^{2k} E^*[-2k] := \wedge_\mathbb{C}^{2k} E^* \otimes_\mathbb{C} (\det H^*)^{\otimes 2k}$ has the real structure defined as follows (compare with the discussion at the end of Section 1). Let $\rho_E$ be the operator on $E$ of the right multiplication by $j \in \mathbb{H}$, and let $\rho_H$ be the operator on $H$ of the right multiplication by $j \in \mathbb{H}$. Then $\wedge^{2k} \rho_E \otimes (\wedge^2 \rho_H)^{\otimes 2k}$ is the real structure on $\wedge^{2k} E^*[-2k]$. The subspace of real elements with respect to this real structure will be denoted by $\wedge^{2k} E^*[-2k]_\mathbb{R}$. In this space $\wedge^{2k} E^*[-2k]_\mathbb{R}$ we are going to define convex cones $K^k(E)$ and $C^k(E)$ of weakly positive and strongly positive elements respectively (the notation might look a bit misleading: these sets depend of course on the space $H$ too). The cones satisfy

$$C^k(E) \subset K^k(E) \subset \wedge^{2k} E^*[-2k]_\mathbb{R},$$

and for $k = 0, 1, n - 1, n$, $C^k(E) = K^k(E)$. The cones are essentially the same as in \cite{4}, Section 2.2, though the construction presented here is simpler.

The definition in the case $k = 0$ is obvious: in this case $\wedge^{2k} E^*[-2k]_\mathbb{R} = \mathbb{R}$ and the positive elements are the usual ones.

Consider the other easy case $k = n$. Clearly $\dim_\mathbb{C} \wedge^{2n} E^*[-2n] = 1$. Let us fix an $\mathbb{H}$-bases $e_1, \ldots, e_n$ in $E^*$ and $h$ in $H^*$. Then one can easily see that $(\wedge_{i=1}^n (e_i \wedge Je_i)) \otimes (h \wedge hJ)^{\otimes 2n}$
is a real element of $\wedge^{2n}E^*[-2n]$ and spans $\wedge^{2n}E^*[-2n]_R$. We will call this element positive. We define the cones $C^n(E) = K^n(E)$ to be the half-line of non-negative multiples of this element. It is easy to see that this half-line is independent of choice of bases.

Next assume that $1 \leq k \leq n - 1$. Let us notice that an $\mathbb{H}$-linear map $f: E \to U$ to another right $\mathbb{H}$-module induces a $\mathbb{C}$-linear map

$$f^*: \wedge^{2k} U^*[-2k] \to \wedge^{2k} E^*[-2k]$$

which preserves the real structure. (More precisely $f^*$ is defined to be $\wedge^{2k} f^* \otimes Id_{(\det H^*)^\otimes 2k}$.)

**5.1 Definition.** (1) An element $\eta \in \wedge^{2k} E^*[-2k]_R$, $1 \leq k \leq n - 1$, is called strongly positive if it can be presented as a finite sum of elements of the form $f^*(\xi)$ where $f: E \to U$ is a morphism of right $\mathbb{H}$-modules, $\dim_{\mathbb{H}} U = k$, and $\xi \in \wedge^{2k} U^*[-2k]_R$, $\xi \in C^k(U) = K^k(U)$. (Notice that in particular $\eta = 0$ is strongly positive.)

(2) An element $\eta \in \wedge^{2k} E^*[-2k]_R$ is called weakly positive if for any strongly positive $\xi \in \wedge^{2(n-k)} E^*[-2(n-k)]_R$ the wedge product $\eta \wedge \xi$ is strongly (=weakly) positive element of $\wedge^{2n} E^*[-2n]_R$.

We have the following properties which are proved in [4], Section 2.2.

**5.2 Proposition.** (1) The cones $C^k(E)$ and $K^k(E)$ are $Aut_\mathbb{H}(E) \times Aut_\mathbb{H}(H) \simeq GL_n(\mathbb{H}) \times GL_1(\mathbb{H})$-invariant, both have non-empty interior in $\wedge^{2k} E^*[-2k]$, and their closures contain no $\mathbb{R}$-linear non-zero subspaces (notice also that $K^k(E)$ is closed).

(2) $C^k(E) \subset K^k(E)$.

(3) $C^k(E) \wedge C^l(E) \subset C^{k+l}(E)$.

(4) For $k = 0, 1, n-1, n$

$$C^k(E) = K^k(E).$$

For a quaternionic manifold $M$ we denote naturally by $\wedge^{2k} E^*_0[-2k]_R$ the vector bundle over $M$ whose fiber over $p \in M$ is equal to $\left((\wedge^{2k} E^*_0|_p) \otimes (\det H^*_0|_p)^\otimes 2k\right)_R$.

**5.3 Definition.** A continuous section of $\wedge^{2k} E^*_0[-2k]_R$ is weakly (resp. strongly) positive if its value at every point is weakly (resp. strongly) positive.

Let us discuss now positive currents. The discussion is analogous to the complex case. First let us remind the definition of generalized section of a vector bundle. Let us discuss this only in the case of complex vector bundles; the real case is practically the same. Let $M$ be a smooth manifold which we will assume to be oriented for simplicity. Let $\mathcal{L}$ be a (finite dimensional) complex vector bundle over $M$. Let $C_c^\infty(M, \mathcal{L})$ denote the space of smooth compactly supported sections of $\mathcal{L}$. This space has a natural standard linear topology of inductive limit of Fréchet spaces. One denotes by $\omega_M$ the complex line bundle of complex valued differential forms of top degree. One denotes

$$C^{-\infty}(M, \mathcal{L}) := (C_c^\infty(M, \mathcal{L}^* \otimes \omega_M))^*.$$
be the continuous dual. As in Section 4 we have the natural injective imbedding
\[ C^\infty(M, \mathcal{L}) \hookrightarrow C^{-\infty}(M, \mathcal{L}) \]
given by \( f \mapsto [\phi \mapsto \int_M <f, \phi>] \). Elements of \( C^{-\infty}(M, \mathcal{L}) \) are called \textit{generalized sections} of \( \mathcal{L} \). The image of this map is dense in the weak topology in the space of generalized sections.

Let us return back to a quaternionic manifold \( M \). One has
\[ \omega_M \simeq (\det \mathcal{E}_0^*) \otimes^2 [-2n]. \]
The wedge product gives a linear map of vector bundles
\[ \Lambda^{2k} \mathcal{E}_0^*[-2k] \otimes \Lambda^{2(n-k)} \mathcal{E}_0^*[-2(n-k)] \to \Lambda^{2n} \mathcal{E}_0^*[-2n] = (\det \mathcal{E}_0^*)[-2n]. \]
By duality this map induces a map on vector bundles which is an isomorphism
\[ (\Lambda^{2k} \mathcal{E}_0^*[-2k])^* \to \Lambda^{2(n-k)} \mathcal{E}_0^*[2n] \otimes \det \mathcal{E}_0[2n] = \Lambda^{2(n-k)} \mathcal{E}_0^* \otimes (\det \mathcal{E}_0)[2k]. \]
Hence we get an isomorphism
\[ (\Lambda^{2k} \mathcal{E}_0^*[-2k])^* \otimes \omega_M \simeq \Lambda^{2(n-k)} \mathcal{E}_0^*[-2(n-k)] \otimes \det \mathcal{E}_0^*. \] (5.1)

All these spaces are equipped with the real structures which are preserved under the isomorphism \((5.1)\). The real line bundle of real elements of \( \det \mathcal{E}_0^* \) is canonically oriented. Hence we can define the convex cones of weakly (resp. strongly) positive elements in fibers of \( \Lambda^{2(n-k)} \mathcal{E}_0^*[-2(n-k)] \otimes \det \mathcal{E}_0^* \) by taking tensor products of weakly (resp. strongly) positive elements in fibers of \( \Lambda^{2(n-k)} \mathcal{E}_0^*[-2(n-k)] \) (in the sense of Definition \(5.1\)) with a positive generator of \( \det \mathcal{E}_0^* \). Via the isomorphism \((5.1)\) this defines the convex cones of weakly and strongly positive elements in fibers of \( (\Lambda^{2k} \mathcal{E}_0^*[-2k])^* \otimes \omega_M \). A continuous section of \( (\Lambda^{2k} \mathcal{E}_0^*[-2k])^* \otimes \omega_M \) is called weakly (resp. strongly) positive if it is weakly (resp. strongly) positive at every point.

5.4 Definition. A generalized section \( \xi \in C^{-\infty}(M, \Lambda^{2k} \mathcal{E}_0^*[-2k]) \) is called \textit{weakly positive} (or just \textit{positive}) if for any strongly positive smooth compactly supported section \( \phi \in C^\infty_c(M, (\Lambda^{2k} \mathcal{E}_0^*[-2k])^* \otimes \omega_M) \) one has \( \langle \xi(\phi) \rangle \geq 0 \). We also write in this case: \( \xi \geq 0 \).

5.5 Remark. It is easy to see that if \( \xi \) is a continuous section, then the positivity of \( \xi \) in the sense of Definition 5.4 is equivalent to the weak positivity of \( \xi \) in the sense of Definition 5.3.

6  Pluripotential theory.

In this section we introduce plurisubharmonic functions on a quaternionic manifold \( M^{4n} \), \( n > 1 \). To be more precise these are not functions but sections of a real line bundle over \( M \) defined as follows.

Let us denote by \( (\det \mathcal{H}_0^*)_R \) the real line bundle of real elements in the bundle \( \det \mathcal{H}_0^* \) which was discussed at the end of Section 4. From now on we denote by \( \Delta \) the Baston operator multiplied by an appropriate normalizing constant (to be chosen in Section 7 below) in order to satisfy the compatibility conventions for the flat manifolds.
6.1 Definition. (i) A continuous section \( f \in C(M, (\det H_0^*)_\mathbb{R}) \) is called plurisubharmonic if for the Baston operator \( \Delta f \in C^{-\infty}(M, \wedge^2 \mathcal{E}_0^*[-2]_\mathbb{R}) \) is positive in the sense of Definition 5.4 \( \Delta f \geq 0 \).

(ii) A \( C^2 \)-smooth section \( f \in C^2(M, (\det H_0^*)_\mathbb{R}) \) is called strictly plurisubharmonic if at every point \( x \in M, \Delta f(x) \) belongs to the interior of the cone of strongly (=weakly) positive elements of \( \wedge^2 \mathcal{E}_0^*[-2]_\mathbb{R} \).

(iii) A generalized section \( f \) of \( (\det H_0^*)_\mathbb{R} \) is called pluriharmonic if \( \Delta f = 0 \).

6.2 Remark. One can show that any pluriharmonic generalized section \( f \) of \( (\det H_0^*)_\mathbb{R} \) is infinitely smooth. This follows from general elliptic regularity results (see e.g. [9], Ch. 3, §6.2, Theorem 3.54).

In order to state the Chern-Levine-Nirenberg type estimate let us fix an auxiliary Riemannian metric on \( M \) and metrics on the bundles \( \mathcal{E}_0, H_0 \) (or on their relevant tensor products in case \( \mathcal{E}_0, H_0 \) are not defined globally).

For any vector bundle \( \mathcal{L} \) with a fixed metric we have norms on the spaces of (say, continuous) sections of \( \mathcal{L} \):
\[
||\phi||_{L^1(M)} := \int_M |\phi(x)| d\text{vol}(x),
||\phi||_{L^\infty(M)} := \sup_{x \in M} |\phi(x)|.
\]

6.3 Proposition (Chern-Levine-Nirenberg type estimate). Let \( M^{4n}, n > 1, \) be a quaternionic manifold. Let \( 1 \leq k \leq n \). Let \( K, L \) be compact subsets of \( M \) such that \( K \) is contained in the interior of \( L \). Then there exists a constant \( C \) depending only on auxiliary metrics and \( K, L \) such that for any \( C^2 \)-smooth plurisubharmonic sections \( f_1, \ldots, f_k \in C^2(M, (\det H_0^*)_\mathbb{R}) \), one has
\[
||\Delta f_1 \wedge \cdots \wedge \Delta f_k||_{L^1(K)} \leq C \prod_{i=1}^k ||f_i||_{L^\infty(L)}.
\]

Proof. We prove the statement by the induction in \( k \). The case \( k = 0 \) is trivial. Now let us assume the result for \( k \) functions and let us prove it for \( k + 1 \). Let us fix a compact subset \( K_1 \) containing \( K \) and contained in the interior of \( L \). Let us choose \( \gamma \in C^\infty(M, (\wedge^2(k+1) \mathcal{E}_0^*[−2(k+1)])^* \otimes \omega_M) \) to be strongly positive (as in Section 5), supported in \( K_1 \), and at every point \( x \in K \) the value \( \gamma(x) \) belongs to the interior of the cone of strongly positive elements. Then there exists a constant \( C_1 \) such that for any weakly positive continuous section \( \xi \in C(M, \wedge^2(k+1) \mathcal{E}_0^*[−2(k+1)]) \) one has
\[
||\xi||_{L^1(K)} \leq C_1 \int_K < \gamma, \xi >.
\]

For any continuous plurisubharmonic sections \( f_1, \ldots, f_{k+1} \) we have
\[
||\Delta f_1 \wedge \cdots \wedge \Delta f_{k+1}||_{L^1(K)} \leq C_1 \int_K < \gamma, \Delta f_1 \wedge \cdots \wedge \Delta f_{k+1} > \leq C_1 \int_{K_1} < \gamma, \Delta f_1 \wedge \cdots \wedge \Delta f_{k+1} > \overset{\text{Prop.} 6.3}{=} C_1 \int_{K_1} < \gamma, (\Delta f_1 \wedge \cdots \wedge \Delta f_k \wedge \Delta f_{k+1}) > = C_1 \int_{K_1} < \Delta^* \gamma, \Delta f_1 \wedge \cdots \wedge \Delta f_k \wedge \Delta f_{k+1} > \leq C_2 ||f_{k+1}||_{L^\infty(K_1)} \cdot ||\Delta f_1 \wedge \cdots \wedge \Delta f_k||_{L^1(K_1)}
\]
where \( C_2 \) depends on \( C_1 \) and \( ||\Delta^* \gamma||_{L^\infty(K_1)} \). Notice that the expression \( \Delta(\Delta f_1 \wedge \cdots \wedge \Delta f_k \wedge f_{k+1}) \) in the second line is understood in the generalized sense, i.e. as a \( C^{-\infty} \)-section of an appropriate vector bundle. Now the rest follows by the induction assumption. Q.E.D.

Generalizing the notation from [6] in the hypercomplex case, we denote by \( P'(M) \) the set of all continuous plurisubharmonic sections of \( (\det H_0^*)_R \). We denote by \( P''(M) \) the subset of \( P'(M) \) consisting of continuous plurisubharmonic sections \( h \) with the following additional property: for any point \( x \in M \) there exist a neighborhood \( U \) and a sequence \( \{h_N\} \) of \( C^2 \)-smooth strictly plurisubharmonic sections over \( U \) such that \( \{h_N\} \) converges to \( h \) uniformly on compact subsets of \( U \) (i.e. in the \( C^0 \)-topology).

**6.4 Remark.** (1) If \( M \) is a locally flat quaternionic manifold, i.e. locally isomorphic to \( \mathbb{H}^n \), then \( P'(M) = P''(M) \). This can be proved easily by considering convolutions with smooth non-negative functions.

(2) It is natural to expect that \( P'(M) = P''(M) \) for any quaternionic manifold \( M \).

(3) It is easy to see that every section \( h \in P''(M) \) locally can be approximated in the \( C^0 \)-topology by \( C^\infty \)-smooth strictly plurisubharmonic sections.

The following result is an analogue of Proposition 7.8 in [6] proved in the hypercomplex case.

**6.5 Proposition.** Let \( \{h_N\} \subset C(M, (\det H_0^*)_R) \). Let \( h_N \xrightarrow{C^0} h \). Then

(1) if for any \( N \), \( h_N \in P'(X) \) then \( h \in P'(X) \);

(2) if for any \( N \), \( h_N \in P''(X) \) then \( h \in P''(X) \).

**Proof.** Part (2) easily follows from part (1). Thus let us prove part (1). We have to show that \( h \) is plurisubharmonic. Let \( \phi \in C^\infty_0(M, (\wedge^2 E_0^*[-2])^* \otimes \omega_M) \) be an arbitrary strongly (=weakly) positive section. We have to show that \( \langle \phi, \Delta h \rangle \geq 0 \). We have

\[
\langle \phi, \Delta h \rangle = \int_M \Delta^* \phi \cdot h = \lim_{N \to \infty} \int_M \Delta^* \phi \cdot h_N = \lim_{N \to \infty} \int_M \phi \cdot \Delta h_N \geq 0.
\]

Q.E.D.

Let us introduce one more notation. We denote by \( \tilde{C}(M, \wedge^{2k} E_0^*[-2k]) \) the continuous dual space to the space of continuous compactly supported sections \( C_0(M, (\wedge^{2k} E_0^*[-2k])^* \otimes \omega_M) \). The last space is equipped with the topology of inductive limit of Banach spaces; that means that a sequence \( \{\xi_i\} \) of continuous compactly supported sections converges in \( C_0(M, (\wedge^{2k} E_0^*[-2k])^* \otimes \omega_M) \) to another such section if and only if all their supports are contained in some compact subset and \( \xi_i \to \xi \) uniformly. Then \( \tilde{C}(M, \wedge^{2k} E_0^*[-2k]) \) is equipped with the weak topology. We have a natural continuous map

\[
\tilde{C}(M, \wedge^{2k} E_0^*[-2k]) \to C^{-\infty}(M, \wedge^{2k} E_0^*[-2k])
\]

which is injective and has dense image. We will identify

\[
\tilde{C}(M, \wedge^{2k} E_0^*[-k]) \subset C^{-\infty}(M, \wedge^{2k} E_0^*[-k]).
\]

The following lemma is essentially well known even in a greater generality, see e.g. Proposition 5.4 in [6].
6.6 Lemma. If $\xi \in C^{-\infty}(M, \wedge^{2k} \mathcal{E}_0^*[-2k])$ is positive then it belongs to $\tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k])$.

A section $\xi \in \tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k])$ is called positive if its image in $C^{-\infty}(M, \wedge^{2k} \mathcal{E}_0^*[-2k])$ is positive; or equivalently for any continuous strongly positive $\phi \in C_0(M, (\wedge^{2k} \mathcal{E}_0^*[-2k])^* \otimes \omega_M)$ one has $\langle \xi, \phi \rangle \geq 0$.

6.7 Theorem. Let $1 \leq k \leq n$. For any $h_1, \ldots, h_k \in P''(M)$ one can define a positive element denoted by $\Delta h_1 \wedge \cdots \wedge \Delta h_k \in \tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k]_R)$ which is uniquely characterized by the following two properties:

(1) if $h_1, \ldots, h_k$ are $C^2$-smooth, then the definition is straightforward, i.e. pointwise;

(2) if $\{h_i^N\} \subset C^2(M, \wedge^2 \mathcal{E}_0^*[-2]_R)$ are plurisubharmonic, $h_i^N \xrightarrow{c} h_i$ as $N \to \infty$, $i = 1, \ldots, k$, then

$$\Delta h_1^N \wedge \cdots \wedge \Delta h_k^N \to \Delta h_1 \wedge \cdots \wedge \Delta h_k, \quad N \to \infty,$$

in the weak topology on $\tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k]_R)$.

Proof. Let $h_1, \ldots, h_k \in P''(M)$. Replacing $M$ by a smaller open subset if necessary let us choose sequences

$$\{h_i^N\}_{N=1}^\infty \subset P'(M) \cap C^2(M, \det \mathcal{H}_0^*), \quad i = 1, \ldots, k,$$

such that $h_i^N \xrightarrow{c} h_i$ for any $i = 1, \ldots, k$. Let us show that $\prod_{i=1}^k \Delta h_i^N$ converges weakly in $\tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k])$ to a positive element $\mu \in \tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k]_R)$. Since the sequence $\{\prod_{i=1}^k \Delta h_i^N\}_{N=1}^\infty$ is positive and locally bounded in $L^1$ by Proposition 6.3, there exists a subsequence $\{N_i\}$ which converges weakly to a positive element $\mu \in \tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k]_R)$, and the weak convergence is understood in the sense of the last space. This fact is a straightforward generalization of the classical fact that the set of non-negative measures with bounded integral is compact in the weak topology. Let us show that $\mu$ does not depend on a choice of convergent subsequence. This will be shown by induction in $k$. For $k = 0$ this is obvious. Let us assume the statement for $k - 1$ and prove for $k$. Let $\nu$ be a weak limit of another subsequence. It suffices to check that for any $\phi \in C_0^\infty(M, (\wedge^{2k} \mathcal{E}_0^*[-2k])^* \otimes \omega_M)$ one has

$$\langle \mu, \phi \rangle = \langle \nu, \phi \rangle.$$

We have

$$\langle \mu, \phi \rangle = \lim_{l \to \infty} \int_M \bigg\langle \prod_{i=1}^k \Delta h_i^{N_i}, \phi \bigg\rangle \quad \text{Prop. 4.1}$$

$$= \lim_{l \to \infty} \int_M \bigg\langle \Delta \left( \prod_{i=1}^{k-1} \Delta h_i^{N_i} \right) \wedge h_k^{N_k}, \phi \bigg\rangle \quad \text{Prop. 4.1}$$

$$= \lim_{l \to \infty} \int_M \bigg\langle \prod_{i=1}^{k-1} \Delta h_i^{N_i}, h_k^{N_k} \cdot \Delta^* \phi \bigg\rangle.$$

Notice that in (6.3) the expression $\Delta \left( \prod_{i=1}^{k-1} \Delta h_i^{N_i} \right)$ is understood in the generalized sense, i.e. as a $C^{-\infty}$-section of an appropriate vector bundle.
By the induction assumption the sequence \(g_N := \prod_{i=1}^{k-1} \Delta a_i^N\) weakly converges to an element \(g\). The sequence \(\{f_N := h_i^N \wedge \Delta^* \phi\}\) has uniformly bounded support and converges uniformly (i.e. in the \(C^0\)-topology) to \(f := h_i \wedge \Delta^* \phi\). Thus existence of the limit (6.5) follows from the following well known lemma (see e.g. Lemma 7.12 in [6]).

6.8 Lemma. Let \(X\) be a compact topological space. Let \(E \to X\) be a finite dimensional vector bundle. Let \(\{f_N\} \subseteq C(X, E)\) be a sequence of continuous sections which converges to \(f \in C(X, E)\) uniformly on \(X\). Let \(\{g_N\} \subseteq C(X, E)^*\) be a sequence in the dual space which weakly converges to \(g \in C(X, E)^*\). Then \(< g_N, f_N > \to < g, f >\) as \(N \to \infty\).

Let us only notice that this lemma is an easy consequence of the Banach-Steinhauss theorem. This lemma implies that the limit \(\lim_{N \to \infty} \int_M < \prod_{i=1}^{k-1} \Delta a_i^N, h_i^N \wedge \Delta^* \phi >\) does exist, and the same argument shows that it is equal also to \(< \nu, \phi >\). Hence the equality (6.2) is proved. Hence there exists a weak limit \(\mu\) of the sequence \(\prod_{i=1}^{k-1} \Delta a_i^N\).

Let us show that if \(h_1, \ldots, h_k\) are \(C^2\)-smooth then the limit \(\mu\) is equal to \(\prod_{i=1}^{k-1} \Delta a_i\) defined just pointwise. The proof is again by induction on \(k\). The case \(k = 0\) is trivial. To make the induction step, let us fix \(\phi \in C_0^\infty (M, \wedge^2 \mathcal{E}_0^*[-2k]) \otimes \omega_M\). We have

\[
< \mu, \phi > = \lim_{N \to \infty} \int_M < \prod_{i=1}^{k} \Delta a_i^N, \phi > \quad \text{Prop. 4.1} = (6.6)
\]

\[
\lim_{N \to \infty} \int_M < \Delta \left( \prod_{i=1}^{k-1} \Delta a_i^N \right) \wedge h_k^N, \phi > = (6.7)
\]

\[
\lim_{N \to \infty} \int_M < \prod_{i=1}^{k-1} \Delta a_i^N, h_k^N \cdot \Delta^* \phi >. \quad (6.8)
\]

By Lemma 6.8 and the induction assumption the last limit exists and is equal to

\[
\int_M < \prod_{i=1}^{k-1} \Delta a_i, h_k \cdot \Delta^* \phi >. \quad (6.9)
\]

Let us show that (6.9) is equal to \(\int \langle \prod_{i=1}^{k} \Delta a_i, \phi \rangle\). But this is proved exactly in the same way as the equality (6.6)= (6.8). To complete the proof of the theorem, it remains to prove the following result which we would like to formulate as a separate statement.

6.9 Theorem. Let \(\{h_i^N\}_{N=1}^{\infty} \subset P^m(M), i = 1, \ldots, k\), be a sequence such that for any \(i = 1, \ldots, k\)

\[
h_i^N \sim_{C^0} h_i.
\]

Then \(h_i \in P^m(M)\) and

\[
\prod_{i=1}^{k} \Delta a_i^N \to \prod_{i=1}^{k} \Delta a_i
\]

in the space \(\tilde{C}(M, \wedge^{2k} \mathcal{E}_0^*[-2k])\), and the products are defined in the sense of Theorem 6.7.
This theorem is proved by inspection of the proof of Theorem 6.7. Q.E.D.

6.10 Remark. Theorems 6.7 and 6.9 have classical real and complex analogs for convex and complex plurisubharmonic functions respectively. The real analogue is due to A.D. Aleksandrov [1], and the complex one is due to Chern-Levine-Nirenberg [15]. Notice also that Theorems 6.7 and 6.9 were proved in the special case of flat quaternionic space in [2], and their analogue for hypercomplex manifolds in [5].

6.11 Definition. Let us define the Monge-Ampère operator on sections of $(\det \mathcal{H}_0^*)_\mathbb{R}$ by

$$h \mapsto (\Delta h)^n.$$ 

The Monge-Ampère operator is naturally defined on $C^2$-smooth sections of $(\det \mathcal{H}_0^*)_\mathbb{R}$. But by Theorem 6.7 it can be defined for sections from $P^n(M)$. Using this operator one can easily introduce a Monge-Ampère equation on any quaternionic manifold, but we do not pursue this point here.

7 The case of flat manifolds.

A basic theory of plurisubharmonic functions on the flat space $\mathbb{H}^n$ was developed by the author in [2] (see also [3], [4]), and on more general class of hypercomplex manifolds by M. Verbitsky and the author [6]. The theory of [2] is a special case of [6]. In this section we show that the theory of the flat case in [2] is a special case of the theory of this paper too. Since hypercomplex manifolds form a subclass of quaternionic manifolds, it is natural to compare the classes of plurisubharmonic functions on hypercomplex manifolds from [6] and of the present paper. If the holonomy of the Obata connection is not contained in $SL_n(\mathbb{H})$ then the theories are formally different since plurisubharmonic sections belong to different line bundles. For the moment we do not know whether the two classes do coincide when the holonomy of the Obata connection is contained in $SL_n(\mathbb{H})$. It would be interesting to give a sufficient condition on a hypercomplex manifold under which the two theories can be identified.

We need some preparations. Let us make some identifications in the special case of $M = \mathbb{H}^n$. Recall that by (1.6)

$$TM \cong \mathcal{E}_0 \otimes_{\mathbb{H}} \mathcal{H}_0.$$  \hspace{1cm} (7.1)

Let $X$ denote a complexification of $M = \mathbb{H}^n$, as previously. As previously, we denote by $\mathcal{E}, \mathcal{H}$ the holomorphic vector bundles over $X$ extending the real analytic vector bundles $\mathcal{E}_0, \mathcal{H}_0$ respectively. $\mathcal{E}_0, \mathcal{H}_0$ are topologically trivial quaternionic vector bundles, while $\mathcal{E}, \mathcal{H}$ are holomorphically trivial (complex) vector bundles. Moreover $\mathcal{E}_0, \mathcal{H}_0$ are equivariant under the group of affine transformations $\mathcal{A} := \mathbb{H}^n \rtimes (SL_n(\mathbb{H}) \times SL_1(\mathbb{H}))$, i.e. the semi-direct product of the group of translations $\mathbb{H}^n$ and linear transformations $SL_n(\mathbb{H}) \times SL_1(\mathbb{H})$ (here $SL_1(\mathbb{H}) = \{ q \in \mathbb{H} | |q| = 1 \}$); notice that the factor $SL_1(\mathbb{H})$ acts trivially on $\mathcal{E}_0$ while the factor $SL_n(\mathbb{H})$ acts trivially on $\mathcal{H}_0$. Moreover $\mathcal{A}$ acts by automorphisms of $\mathbb{H}^n$ as a quaternionic manifold, and the induced action on the tangent bundle is compatible with the isomorphism (7.1). Hence all the Baston operators become equivariant under this group...
This property will be crucial in the proof of the main result of this section. Thus in particular the operator
\[ \Delta : C^\infty (M, \det H_0^*) \to C^\infty (M, \wedge^2 E_0^*[-2]) \]
is \( A \)-equivariant.

It is easy to see that the complex line bundle \( \det H_0^* \) is isomorphic to the trivial bundle in the category of \( A \)-equivariant vector bundles. Moreover the isomorphism can be chosen in such a way that the real structure on fibers of \( \det H_0^* \) (which is induced by the quaternionic structure on fibers on \( H_0 \)) is mapped to the real structure on the trivial line bundle, and positive half line is mapped to the positive half line.

Next the complex vector bundle \( \wedge^2 E_0^*[-2] \) is \( A \)-equivariantly isomorphic to the vector bundle over \( H^n \) whose fiber over a point \( p \in H^n \) is equal to the space of \( \mathbb{C} \)-valued quaternionic Hermitian forms on \( T_p H^n \) (equivalently, to \( \mathbb{C} \)-valued quadratic forms on the real space \( T_p H^n \) which are invariant under group of norm one quaternions); we refer to [6] for this linear algebra. Moreover this isomorphism can be chosen to preserve the real structures on both spaces and the cones of positive elements (the positive cone in the latter space was defined in [2], and an equivalent description in the more general case of hypercomplex manifolds was given in [6]; the definitions of positivity in the present paper just a direct generalization of the definitions from [6]). Notice that the quaternionic Hessian defined in [2] in the flat case took values exactly in this vector bundle for quaternionic Hermitian forms. With these identifications, the Baston operators \( \Delta \) and the quaternionic Hessian introduced in [2] act between the same vector bundles. We will denote the quaternionic Hessian from [2] by \( \Delta' \). By [2], the operator \( \Delta' \) is also \( A \)-equivariant.

**7.1 Proposition.** Let \( M = H^n \). With the above identifications
(i) the Baston operator \( \Delta \) (appropriately normalized) coincides with \( \Delta' \) from [2];
(ii) the Monge-Ampère operator \( h \mapsto (\Delta h)^n \) coincides with the Monge-Ampère operator from [2];
(iii) the class of plurisubharmonic functions in the sense of Definition 6.1 of this paper coincides with the class of plurisubharmonic functions introduced in [2].

**Proof.** Parts (ii),(iii) follow from part (i) and the definitions of the real structures, positive cones, and the wedge product (which is equivalent by [6] to the Moore determinant of quaternionic matrices used in [2]). Part (i) is the main one, and we are going to prove it. We have to show that \( \Delta' = \Delta \) when \( \Delta \) is appropriately normalized.

First let us prove the vanishing of the symbol of \( \Delta' - \Delta \) considered as a differential operator of second order. By the translation invariance it suffices to show that the symbol of \( \Delta' - \Delta \) vanishes at 0. Since \( \Delta' - \Delta \) is \( A \)-equivariant, its symbol belongs to

\[ \text{Hom}_{\mathbb{R}, \mathcal{B}} \left( \text{Sym}^2_{\mathbb{R}} (\mathbb{H}^n) \otimes \det (H_0^*)_{\mathbb{R}}|_0, (\wedge^2 E_0^*[-2])_{\mathbb{R}}|_0 \right) \]  
(7.2)

where we have denoted for brevity \( \mathcal{B} := SL_n(\mathbb{H}) \times SL_1(\mathbb{H}) \), and \( \text{Hom}_{\mathbb{R}, \mathcal{B}} \) denotes the space of \( \mathbb{R} \)-linear maps commuting with the group \( \mathcal{B} \). Now we are going to show that the space \( (7.2) \)
is at most one dimensional. It suffices to show the one dimensionality of the $\text{Hom}$ between complexified representations, namely that

$$\text{Hom}_{\mathbb{C}B} \left( \text{Sym}^2_{\mathbb{C}}(\mathbb{C}^{2n} \otimes \mathbb{C}^2)^* \otimes \det(\mathbb{C}^{2*}), \wedge^2_{\mathbb{C}}(\mathbb{C}^{2n})^* \otimes (\det \mathbb{C}^{2*}) \otimes 2 \right)$$ (7.3)

is one dimensional. Here $\mathbb{C}B = SL_{2n}(\mathbb{C}) \times SL_2(\mathbb{C})$ is the complexification of $B$. Next in (7.3) the action of $\mathbb{C}B$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$ is as follows: $SL_{2n}(\mathbb{C})$ and $SL_2(\mathbb{C})$ act on $\mathbb{C}^{2n}$ and respectively $\mathbb{C}^2$ in the standard way, $SL_{2n}(\mathbb{C})$ acts trivially on $\mathbb{C}^2$, and $SL_2(\mathbb{C})$ acts trivially on $\mathbb{C}^{2n}$.

The representation of $SL_{2n}(\mathbb{C})$ on $\wedge^2_{\mathbb{C}}(\mathbb{C}^{2n})^*$ is irreducible (see e.g. [17], Corollary 5.5.3). The representation of $SL_{2n}(\mathbb{C}) \times SL_2(\mathbb{C})$, and hence of $\mathbb{C}B$, in $\text{Sym}^2(\mathbb{C}^{2n} \otimes \mathbb{C}^2)$ is multiplicity free (see e.g. [17], Corollary 5.6.6); hence the representation of $\mathbb{C}B$ in $\text{Sym}^2_{\mathbb{C}}(\mathbb{C}^{2n} \otimes \mathbb{C}^2)^* \otimes \det(\mathbb{C}^{2*})$ is also multiplicity free. Then the Schur’s lemma implies that the $\text{Hom}$-space (7.3) is at most one dimensional. This implies that the symbols of $\Delta'$ and $\Delta$ must be proportional. Hence $\Delta$ can be normalized in such a way that the symbols just coincide.

Thus the differential operator $\Delta' - \Delta$ has order at most one. Let us consider the symbol of this first order differential operator. It is an element of

$$\text{Hom}_{\mathbb{R},B} \left( \mathbb{H}_{n*} \otimes \det(\mathcal{H}^*_0)|_0, (\wedge^2 \mathcal{E}^*_0[-2]|_0) \right) .$$ (7.4)

But the two representations under the $\text{Hom}$ are irreducible and non-isomorphic. Hence (7.4) vanishes, and the symbol of $\Delta' - \Delta$ vanishes. Hence $\Delta' - \Delta$ has order zero. But then $\Delta' - \Delta$ defines an element of

$$\text{Hom}_{\mathbb{R},B} \left( \left( \det \mathcal{H}^*_0 \right)|_0, \wedge^2 \mathcal{E}^*_0[-2]|_0 \right) .$$

Obviously the last space vanishes. This implies that $\Delta' - \Delta = 0$. Q.E.D.

References

[1] Aleksandrov, A. D.; Dirichlet’s problem for the equation $\det ||z_{ij}|| = \phi(z_1, \ldots, z_n, z, x_1, \ldots, x_n)$. I. (Russian) Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 13 1958 no. 1, 5-24.

[2] Alesker, Semyon; Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables. Bull. Sci. Math. 127 (2003), no. 1, 1–35.

[3] Alesker, Semyon; Quaternionic Monge-Ampère equations. J. Geom. Anal. 13 (2003), no. 2, 205–238.

[4] Alesker, Semyon; Valuations on convex sets, non-commutative determinants, and pluripotential theory. Adv. Math. 195 (2005), no. 2, 561–595.

[5] Alesker, Semyon; Plurisubharmonic functions on the octonionic plane and $Spin(9)$-invariant valuations on convex sets. J. Geom. Anal. 18 (2008), no. 3, 651–686.
[6] Alesker, Semyon; Verbitsky, Misha; Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry. J. Geom. Anal. 16 (2006), no. 3, 375–399. Also: arXiv:math/0510140.

[7] Alesker, Semyon; Verbitsky, Misha; Quaternionic Monge-Ampère equation and Calabi problem for HKT-manifolds. Israel J.Math. 176 (2010), 109-138. Also: arXiv:0802.4202.

[8] Aslaksen, Helmer; Quaternionic determinants. Math. Intelligencer 18 (1996), no. 3, 57–65.

[9] Aubin, Thierry; Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[10] Baston, R. J.; Quaternionic complexes. J. Geom. Phys. 8 (1992), no. 1-4, 29–52.

[11] Beilinson, Alexander; Private communication, April 2005.

[12] Besse, Arthur L.; Einstein manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 10. Springer-Verlag, Berlin, 1987.

[13] Boyer, Charles P.; A note on hyper-Hermitian four-manifolds. Proc. Amer. Math. Soc. 102 (1988), no. 1, 157–164.

[14] Čap, A.; Slovák, J.; Souček, V.; Invariant operators on manifolds with almost Hermitian symmetric structures. II. Normal Cartan connections. Acta Math. Univ. Comenian. (N.S.) 66 (1997), no. 2, 203–220.

[15] Chern, Shiing-shen; Levine, Harold I.; Nirenberg, Louis; Intrinsic norms on a complex manifold. 1969 Global Analysis (Papers in Honor of K. Kodaira) pp. 119-139 Univ. Tokyo Press, Tokyo.

[16] Gindikin, Simon; Henkin, Gennadi; Integral geometry for $\bar{\partial}$-cohomology in $q$-linearly concave domains in $\mathbb{C}P^n$. (Russian) Funktsional. Anal. i Prilozhen. 12 (1978), no. 4, 6–23.

[17] Goodman, Roe; Wallach, Nolan R.; Symmetry, representations, and invariants. Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009.

[18] Harvey, F. Reese; Lawson, H. Blaine, Jr.; An introduction to potential theory in calibrated geometry. Amer. J. Math. 131 (2009), no. 4, 893–944.

[19] Harvey, F. Reese; Lawson, H. Blaine, Jr.; Duality of positive currents and plurisubharmonic functions in calibrated geometry. Amer. J. Math. 131 (2009), no. 5, 1211–1239.

[20] Harvey, F. Reese; Lawson, H. Blaine, Jr.; Plurisubharmonicity in a General Geometric Context. Preprint arXiv:0804.1316
[21] Henkin, Gennadi; Polyakov, Pierre; Homotopy formulas for the $\bar{D}$-operator on $\mathbb{CP}^n$ and the Radon-Penrose transform. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no. 3, 566–597, 639.

[22] Henkin, Gennadi; Private communications, 1999-2002.

[23] Illusie, Luc; *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[24] Joyce, Dominic; Compact hypercomplex and quaternionic manifolds. J. Differential Geom. 35 (1992), no. 3, 743761.

[25] Manin, Yuri I.; *Gauge field theory and complex geometry.* Translated from the Russian by N. Koblitz and J. R. King. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 289. Springer-Verlag, Berlin, 1988.

[26] McCleary, John; A user’s guide to spectral sequences. Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[27] Obata, Morio; Affine connections on manifolds with almost complex, quaternion or Hermitian structure. Jap. J. Math. 26, 1956 43–77.

[28] Pedersen, H.; Poon, Y. S.; Swann, A. F.; Hypercomplex structures associated to quaternionic manifolds. Differential Geom. Appl. 9 (1998), no. 3, 273–292.

[29] Salamon, Simon; Quaternionic Kähler manifolds. Invent. Math. 67 (1982), no. 1, 143–171.

[30] Salamon, Simon; Quaternionic structures and twistor spaces. *Global Riemannian geometry.* (Durham, 1983), 65–74, Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1984.

[31] Verbitsky, Misha; HyperKähler manifolds with torsion, supersymmetry and Hodge theory. Asian J. Math. 6 (2002), no. 4, 679–712.

[32] Verbitsky, Misha; Balanced HKT metrics and strong HKT metrics on hypercomplex manifolds. Math. Res. Lett. 16 (2009), no. 4, 735–752.

[33] Widdows, Dominic; A Dolbeault-type double complex on quaternionic manifolds. Asian J. Math. 6 (2002), no. 2, 253275.