Hamiltonian Theory of the Fractional Quantum Hall Effect: Effect of Landau Level Mixing

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We derive an effective hamiltonian in the Lowest Landau Level (LLL) that incorporates the effects of Landau-level mixing to all higher Landau levels to leading order in the ratio of interaction energy to the cyclotron energy, $\kappa = (e^2/\varepsilon l)/\omega_c$. We then transcribe the hamiltonian to the composite fermion basis using our hamiltonian approach and compute the effect of LL mixing on transport gaps.

Specifcally, the hamiltonian approach provides an expression for the LLL projected electron density $\bar{\rho}(q)$ in the CF basis $\bar{\rho}(q)$. Since the effective hamiltonian is expressed in terms $\bar{\rho}(q)$, this all we need.

In Section II, we derive the effective hamiltonian and in Section III we express it in the CF basis. Section IV contains the calculation of gaps and conclusions follow in Section V. Details are relegated to the two Appendices. While this paper focusses on the calculation of transport gaps, the approach presented here can be extended to calculations of other physical quantities in the Hamiltonian theory $\bar{\rho}(q)$.

II. THE EFFECTIVE HAMILTONIAN-ELECTRONIC BASIS

Our starting point will be the full electronic Hamiltonian, which is given by

$$H = \sum_{\alpha} a_{\alpha}^\dagger a_{\alpha} E_i + \frac{1}{2} \int \frac{d^2 q}{2\pi^2} \rho_{12}(q) \rho_{34}(-q) v(q) a_{\alpha}^\dagger a_{\beta} a_{\gamma} a_{\delta} \quad (1)$$

$$\equiv H_0 + V \quad (2)$$

where any subscripts $\alpha$ stands for pair of indices $(n_\alpha, m_\alpha)$ (in labels the LL index and $m$ the angular momentum within the LL). Both labels assume integral values from zero to infinity and

$$E_\alpha = n_\alpha \omega_0 \quad (3)$$

with $\omega_0$ being the cyclotron frequency. We have dropped the zero-point energy and kept only the normal-ordered part of the interaction. The operators $a$ and $a^\dagger$ are electron destruction and creation operators, $v(q)$ is the electron-electron interaction and $\rho_{\alpha\beta}(q)$ are the matrix elements of the charge density operator. These matrix elements can be determined using the standard decomposition of the electron coordinates and momenta into cyclotron ($\eta^e$) and guiding center ($R^e$) variables

$$\eta_e = l^2 \hat{z} \times \Pi_e \quad (4)$$

$$R_e = r_e - l^2 \hat{z} \times \Pi_e \quad (5)$$
where \( \mathbf{\Pi}_e = \mathbf{p}_e + e\mathbf{A} \) is the velocity operator of the electrons and \( l = \sqrt{1/eB} \) is the magnetic length. These coordinates obey the commutation relations
\[
[\eta_{ex}, \eta_{ey}] = il^2 \quad [R_{ex}, R_{ey}] = -il^2 \quad [\eta_e, R_e] = 0
\] (6)

Expressing the electron coordinate as \( R_e = R_e + \eta_e \) we have
\[
\rho_{\alpha\beta}(q) = (n_{\alpha}m_{\alpha}|e^{-i\mathbf{q} \cdot \mathbf{R}_e + \eta_e}|n_{\beta}m_{\beta})
\]
\[
= (m_{\alpha}|e^{-i\mathbf{q} \cdot \mathbf{R}_e}|m_{\beta}) \times (n_{\alpha}|e^{-i\mathbf{q} \cdot \eta_e}|n_{\beta})
\]
\[
= \rho^{(m)}_{\alpha\beta}(q) \times \rho^{(n)}_{\alpha\beta}(q)
\] (7)

where the superscripts \( m \) and \( n \) indicate that the matrix elements correspond to the guiding center and cyclotron coordinates respectively. These matrix elements can be expressed in terms of Laguerre polynomials. However, for this section we only need to know that
\[
\rho^{(n)}_{00}(q) = e^{-q^2i^2/4}
\]
\[
\sum_{\beta} \rho^{(m)}_{\alpha\beta}(q) \rho^{(n)}_{\beta\gamma}(q_2) = \rho^{(m)}_{\alpha\gamma}(q_1 + q_2) e^{i\mathbf{q}_1 \cdot \mathbf{q}_2 q^2i^2/2}
\] (10)

which reflects the magnetic translation algebra:
\[
e^{-i\mathbf{q}_1 \cdot \mathbf{R}_e} e^{-i\mathbf{q}_2 \cdot \mathbf{R}_e} = e^{-i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{R}_e} e^{i(\mathbf{q}_1 \cdot \mathbf{q}_2) q^2i^2/2}
\] (12)

It will prove convenient to define
\[
\int_q = \int \frac{d^2q}{4\pi^2} v(q).
\] (13)

where the potential is included as part of the measure.

The leading term in the effective hamiltonian is the obvious contribution from \( H_{00} \), the restriction of \( H \) to the LLL:
\[
H_{00} = \sum_m a_{0m}^\dagger a_{0m} \cdot 0 \cdot \omega_0
\]
\[
+ \frac{1}{2} \int_q \left[ a_{0m}^\dagger a_{0m} a_{0m_3} a_{0m_4} a_{0m_2}
\right.
\]
\[
\left. \rho^{(m)}_{m_1m_2}(q) \rho^{(m)}_{m_3m_4}(q) \rho^{(n)}_{00}(q) \rho^{(n)}_{00}(q) \right]
\]
\[
= \frac{1}{2} \int_q a_{0m}^\dagger a_{0m} a_{0m_2} \rho^{(m)}_{m_1m_2}(q) \rho^{(m)}_{m_3m_4}(q) \rho^{(n)}_{00}(q) \rho^{(n)}_{00}(q)
\]
\[
e^{-q^2i^2/2}
\] (14)

where we have labeled LLL operators with just an \( m \) index. In terms of \( \tilde{\rho}(q) \), the density restricted to the LLL and defined by
\[
\tilde{\rho}(q) = \sum_{ij} a_{ij}^\dagger a_{ij} \rho_{ij}(q)
\] (15)

we can write
\[
H_{00} = V = \frac{1}{2} \int_q e^{-(ql)^2/2} \tilde{\rho}(q) \tilde{\rho}(-q).
\] (16)

Here we have removed the normal-ordering, which leads to an unimportant additive constant in the energy. The operator \( \tilde{\rho}(q) \) has the same commutators as \( e^{i\mathbf{q} \cdot \mathbf{R}} \).

To extract the leading correction due to higher LL’s we write the Schrödinger equation as
\[
\begin{bmatrix}
H_{00} & H_{0n'}

H_{n'0} & H_{n'n'}
\end{bmatrix}
\begin{bmatrix}
\phi

\chi
\end{bmatrix}
= \begin{bmatrix}
E

\chi
\end{bmatrix}
\]
(17)

where \( \phi \) is restricted to the space spanned by Fock states composed of just the LLL states and \( \chi \) stands for everything else. The exact equation obeyed by \( \phi \) is
\[
\left( H_{00} + H_{0n'} \frac{1}{E - H_{n'n'}} H_{n'0} \right) \phi = E \phi
\]
(18)

which is not an eigenvalue problem since \( E \) appears on both sides. However we may approximate as follows:
\[
\frac{1}{E - H_{n'n'}} = -\frac{1}{H_{n'n'}} + O(v/\omega)
\]
(19)

since the eigenvalue \( E \) we are interested in is of order \( v \) and the eigenvalues of \( H_{n'n'} \) are of order \( \omega_0 \). To the same accuracy in \( \kappa = v/\omega_0 \) we can also replace
\[
H_{n'n'} \approx H_{n'n'}^0 = \sum_{\alpha} a_{\alpha}^\dagger a_{\alpha} n_{\alpha} \omega_0
\]
(20)

In fact, one can do a little better than this, and incorporate the interactions into \( H_{n'n'} \) by including the Hartree-Fock energy of the \( n^{th} \) level. This can be viewed as a perturbation theory in the non-HF part of the interaction. We will present results both with and without this HF energy in \( H_{n'n'} \). To maintain generality we define a reduced energy
\[
\varepsilon(n) = n + \frac{\nu}{\omega_0} \int \frac{d^2q}{2\pi^2} v(q) \left( |\rho^{(0)}_{00}(q)|^2 - |\rho^{(n)}_{00}(q)|^2 \right)
\]
(21)

where we have used the fact the the FQH states have uniform occupations of \( \nu \) in the LLL, and we have normalized the LLL energy to be zero. This leads us to the effective Hamiltonian
\[
H_{00}^{\text{eff}} = H_{00} + \delta H_{00}
\]
(22)

where
\[
\delta H = -H_{0n'} \frac{1}{H_{n'n'}} H_{n'0}
\]
(23)

\[
= -\frac{1}{4} \int_{q_1} \int_{q_2} \left[ \sum_{\alpha_1,\alpha_2} a_{\alpha_1}^\dagger a_{\alpha_2} \frac{1}{H_{n'n'}} a_{\alpha_2}^\dagger a_{\alpha_1} \right]
\]
\[
\rho_{12}(q_1) \rho_{34}(q_2) \rho_{57}(q_2) \rho_{56}(q_2)
\] (24)

The prime on \( \sum' \) denotes the fact that if \( \delta H_{00} \) is to act on the LLL sector, we need
\[
n_8 = n_6 = n_4 = n_2 = 0.
\]
(25)

This means we can have two kinds of terms:
\[ \delta H^{(2)}_{00} = -\frac{1}{4} \int_{q_1}^{q_2} \sum_{\text{1..8}} \rho_{12}(q_1) \rho_{34}(-q_1) \rho_{87}(q_2) \rho_{65}(-q_2) \]

\[ a^\dagger_8 a^\dagger_6 a_4 a_2 \left( \frac{1}{(\varepsilon(n'_1) + \varepsilon(n'_3))} \delta_{53,57} - \delta_{51,73} \right) \]

The contribution from the first delta function is

\[ -\frac{1}{4\omega_0} \int_{q_1}^{q_2} \sum_{m} \rho_{m_{2}m_{2}}(q_2) \rho_{m_{3}m_{3}}(-q_2) \rho_{m_{3}m_{3}}(q_1) \rho_{m_{3}m_{3}}(-q_1) \]

\[ F_1(q_1, q_2) a^\dagger_{0,m_{2}} a^\dagger_{0,m_{2}} a_{0,m_{2}} a_{0,m_{2}} \]

\[ F_1(q_1, q_2) = \sum_{n, m} \rho_{n_{1}m_{1}}(q_1) \rho_{n_{1}m_{1}}(q_2) \rho_{n_{2}m_{2}}(q_1) \rho_{n_{2}m_{2}}(q_2) \]

\[ \left( \frac{\rho_{n_{1}m_{1}}(q_1) \rho_{n_{1}m_{1}}(q_2)}{\varepsilon(n_1) + \varepsilon(n_1)} \right) \]

Using completeness to do the sum over \( m_1 \) and \( m_3 \), using the magnetic algebra Eqn. (11) and doubling the answer because of the other delta function which makes an equal contribution (upon relabeling of dummy indices) we arrive at

\[ \delta H^{(2)}_{00} = -\frac{1}{2\omega_0} \int_{q_1}^{q_2} e^{-i\mathbf{q}_1 \times \mathbf{q}_2 t^2/\hbar} \bar{\rho}(\mathbf{q}_1 + \mathbf{q}_2) \bar{\rho}(\mathbf{q}_1 - \mathbf{q}_2) F_1(q_1, q_2) \]

Now for the term with just one higher LL excitation:

\[ \delta H^{(1)}_{00} = -\frac{1}{4} \int_{q_1}^{q_2} \rho_{12}(q_1) \rho_{34}(-q_1) \rho_{87}(q_2) \rho_{65}(-q_2) \]

\[ a^\dagger_8 a^\dagger_6 a_5 a_7 \frac{1}{H_{n'\prime}} a^\dagger_5 a_4 a_2 \]

Consider the contribution when \( a^\dagger_7 \) takes the electron up to a higher LL and and \( a_7 \) brings it down. The other three possibilities obtained by permuting \( 1 \rightleftharpoons 3 \) and \( 5 \rightleftharpoons 7 \) make the same contribution upon relabeling of dummy indices. The net contribution due to these terms in which a single excitation to a higher LL occurs is

\[ \delta H^{(1)}_{00} = -\int_{q_1}^{q_2} \sum_{1..8} a^\dagger_{m_{2}} a^\dagger_{m_{3}} a_{m_{3}} a_{m_{3}} \delta_{71} a^\dagger_{m_{1}} a_{m_{3}} a_{m_{4}} a_{m_{2}} \]

\[ \left[ \rho_{12}(q_2) \rho_{12}(q_1) \right] \rho_{34}(-q_1) \rho_{65}(-q_2) \]

\[ = -\int_{q_1}^{q_2} \sum_{1..8} \rho_{m_{3}m_{1}}(q_2) \rho_{m_{1}m_{3}}(q_1) \times \]

\[ G(q_1, q_2) \sum_m \rho_{00}^{(m)}(q_1) \rho_{m_{3}m_{4}}^{(m)}(-q_1) \rho_{m_{2}m_{5}}^{(m)}(-q_2) \]

\[ = -\int_{q_1}^{q_2} \left[ e^{-i(q_1 \times q_2)x^2/2\epsilon} e^{-(q_1^2 + q_2^2)x^2/4} \times \right] \]

\[ G(q_1, q_2) \sum_m \rho_{m_{2}m_{3}}(q_2) \rho_{m_{3}m_{4}}(-q_1) \rho_{m_{1}m_{5}}(-q_2) \]

\[ a^\dagger_{m_{2}} a_{m_{3}} a_{m_{2}} a_{m_{4}} a_{m_{2}} \]

\[ G(q_1, q_2) = \sum_{n' > 0} \rho_{n' 0}^{(m)}(q_2) \rho_{n' 0}^{(m)}(q_1) \]

If we now rearrange the operators (whose subscripts now all describe \( m \)) so as to match those on \( \rho_{ij} \) as follows

\[ a^\dagger_8 a^\dagger_6 a_5 a_3 a_4 a_2 = \left( a^\dagger_8 a_2 a^\dagger_6 a_3 a_3 a_4 \right. \]

\[ - \delta_{124} a^\dagger_8 a_5 a_4 a_2 \]

\[ - \delta_{624} a^\dagger_8 a_5 a_3 a_3 a_4 \]

(27) we arrive at the following expression for \( \delta H^{(1)}_{00} \)

\[ \delta H^{(1)}_{00} = -\int_{q_1}^{q_2} G(q_1, q_2) e^{-i(q_1 \times q_2)x^2/2\epsilon} \bar{\rho}(q_1 + q_2) \bar{\rho}(q_1) \]

\[ -\bar{\rho}(q_2) \bar{\rho}(q_2) - e^{-i(q_1 \times q_2)x^2/2\epsilon} \bar{\rho}(q_1) \bar{\rho}(q_1) \]

In writing the answer in terms of \( \bar{\rho} \)'s we must pay attention to the order since they do not commute with each other. It should also be noted that the first term of Eqn. (33) produces a contribution to another effective two-body term via

\[ \bar{\rho}(q_1 + q_2) = \frac{\nu}{2\pi l^2} (2\pi)^2 \delta^2(q_1 + q_2) : \bar{\rho}(q_1 + q_2) : \]

where the first term contains the density \( \nu/2\pi l^2 \).

The final effective hamiltonian is given by

\[ H_{\text{eff}}^{(1)} = H_{\text{00}} + \delta H^{(2)}_{00} + \delta H^{(1)}_{00} \]

where \( H_{\text{00}}, \delta H^{(2)}_{00} \) and \( \delta H^{(1)}_{00} \) are given by Eqns. (12), (28) and (33).

### III. THE EFFECTIVE HAMILTONIAN IN THE CF BASIS

The preceding calculation of the effective LLL theory in terms of electrons could have been performed decades ago since it is not dependent on any particular representation of \( \bar{\rho} \). However, written in terms of electronic variables, the effective theory suffers from the same handicap as the original one: the noninteracting part of the
hamiltonian has a huge ground state degeneracy for filling fraction \( \nu = \frac{p}{2p+1} < 1 \). To get a nondegenerate starting point, we must transcribe \( H_{\text{eff}}(\phi(q)) \), calculated above in the electronic basis, to the CF basis. The CF description will be in terms of particles that see a weaker field of just the right strength that they fill exactly \( \rho \) LL’s, with no degeneracy in the ground state. Over the years we have developed a route that takes one from the electron-representation to the CF-representation. We will not describe it here since it is long and has been described before [10–12]. All we need here are the end results, which look a lot more attractive than the path that led to them, fraught as it was with approximations and inspired guesses. Instead, we will arrive at our final result armed with hindsight.

Consider a CF Hilbert space where each fermion is described by a coordinate \( r \) and momentum \( p \). (Hereafter, all unsubscripted variables will represent the CF degrees of freedom.) From these we construct the velocity operator

\[
\Pi = p + eA^* \quad A^* = \frac{A}{2ps + 1} \tag{38}
\]

where the weakened vector potential \( A^* \) is what the CF sees. In terms of these variables, the electron guiding center \( R_e \) takes the form

\[
R_e = r - \frac{l^2}{(1 + c)} \hat{z} \times \Pi, \tag{39}
\]

\[
[R_{ex}, R_{ey}] = -il^2, \tag{40}
\]

\[
e^2 = \frac{2ps}{2ps + 1} \tag{41}
\]

Note that \( R_e \) obeys the right commutation relations. Also, since it is written in terms of an object that sees a weaker field \( A^* = A/(2ps + 1) \), if we express \( H_{\text{eff}} \) in terms of these variables, we will not encounter the degeneracy problem at the Jain fractions. This is the motivation for switching to the new coordinates.

It is useful to write \( R_v \) in terms of the CF guiding center and cyclotron coordinates \( R \) and \( \eta \):

\[
R_v = R + e\eta. \tag{42}
\]

Since we have embedded \( R_e \) in a regular fermionic Hilbert space, we have room for another pair of guiding center-like coordinates. Let us call them \( R_v \). These will be of the form

\[
R_v = \alpha R + \beta \eta \tag{43}
\]

Demanding that \( R_e \) commutes with \( R_v \) gives us \( \beta = \alpha/c \). How about the overall scale of the operator? Here is where our previous work tells us to choose \( \alpha = 1 \), that is

\[
R_v = R + \eta/c. \tag{44}
\]

In terms of \( \mathbf{r} \) and \( \Pi \)

\[
\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1 + c)} \hat{z} \times \Pi \tag{45}
\]

\[
[R_{vx}, R_{vy}] = il^2/c^2, \tag{46}
\]

\[
[R_{xv}, R_{yv}] = 0. \tag{47}
\]

The merit of this choice is the following. These commutation relations correspond to the guiding center coordinates of a particle of charge \(-e^2 = -2ps/(2ps + 1) \). This is precisely the charge of an object that must pair with the electron to form the CF and we refer to it as the pseudovortex coordinate, since it has the same charge as a 2s-fold vortex in Laughlin states. It must be emphasized that \( R_v \) cannot be directly identified with the physical vortex that exists around an electron. There is, however, a connection between \( R_v \) and the physical vortex which arises upon the choice of a suitable state in the \( R_v \) variables, as explained below.

\( R_v \) is a cyclic coordinate that does not enter the hamiltonian, and so has no dynamics. The eigenfunctions of \( H \) will be of the form

\[
\Psi(R_{ex}, R_{ey}) = \Psi(R_{vx})\Psi(R_{vy}) \tag{48}
\]

where we have chosen as a commuting pair of coordinates \( R_{ex} \) and \( R_{vy} \) with conjugate momenta \( R_{vy} \) and \( R_{ey} \). (One can also use the Bargman representation and consider \( \Psi(z, w) \), where \( z \) and \( w \) are complex numbers, as we will find convenient to do later.) Whereas \( \Psi(R_{ex}) \) is an eigenfunction of \( H(R_e) \), \( \Psi(R_{vy}) \) is completely arbitrary. Thus each eigenfunction is infinitely degenerate. This is exactly the kind of degeneracy a gauge symmetry would introduce. We must therefore “fix our gauge” i.e., choose an arbitrary function \( \Psi_0(R_{vy}) \) to accompany \( \Psi(R_{ex}) \). No physical observable (function of \( R_e \)) will depend on this choice. Our previous derivation based on canonical transformations [10–12] naturally led to the following constraint on \( \Psi(R_{ex}) \)

\[
\left( \sum_j e^{-iq R_{vj}} \right) \text{(physical states)} = 0 \tag{49}
\]

The constraint arose because we had introduced additional oscillators at the cyclotron scale and these had to be paid for.

In hindsight we see that we are free to take an axiomatic view of \( R_v \) and \( R_v \) and to simply introduce them as the above functions of CF coordinates and momenta (Eqs. (43)). The hamiltonian depends on \( R_e \) alone. The constraint we were led to in our earlier work, Eqn. (11), is an acceptable choice for fixing the gauge. However, since all physical observables depend on just \( R_v \), and \( R_v \) commutes with \( R_v \), we may assign to \( R_v \) any dynamics we want without changing the physics. For example, we could add to \( H(R_e) \) a piece \( H(R_v) \) which is any generic repulsive interaction and demand that \( \Psi_0(R_{ex}) \) be its ground state. Now it turns out that for any generic repulsion there is essentially a unique answer, the \( \nu = 1/2s \) bosonic Laughlin wavefunction [3].
The reason is as follows. First, we must choose $R_v$ to be a bosonic coordinate (the wavefunctions have to be antisymmetric in $r_i$ and $R_v$, and hence symmetric in $R_v$). Next, since $R_v$ has a magnetic algebra charge of $-c^2$, and there is one pseudovortex per electron, it is easily seen that it is always at filling

$$\nu' = \frac{\nu}{c^2} = -\frac{p}{2ps + 1}, \quad (50)$$

which leads us to the wavefunction

$$\Psi^L_v = \prod_{i<j}(z_i - z_j)^{2s}e^{-\sum_i \frac{z_i^2 + |q^r|}{4}} \quad (51)$$

Note that the magnetic length appropriate to the bosons (with charge $c^2e$) has been used.

Now it might seem that all these virtues are moot since $R_v$ is a fictitious coordinate whose dynamics should make no difference to the actual physics. While this is certainly the case in any exact calculation, in the approximate HF calculations we will employ (where the wavefunction does not have the factorized form of Eqn. (48), different choices give different answers and the above choice gives the wave functions with the best correlations.

To see this, let us first define the CFHF wavefunction. Here we work in the CF single-particle basis in the effective field $B^e$ (defined in more detail in the next section), and fill the lowest single-particle states up to a filling of $\nu^* = \nu/(2s\nu + 1)$, which reduces to $\nu^* = p$ for $\nu = p/(2ps + 1)$. It is easy to show [13] that these states are indeed HF states of the Hamiltonian.

As mentioned above, the CFHF wavefunction cannot be a true eigenstate, since it is not of the required product form of Eqn. (48). However, we can “project” it in the following way

$$\mathcal{P}\Psi_{CFHF}([R_v,i], [R_v,i]) = \Psi_v([R_v,i]) \Psi_{CFHF}([R_v,i], [R_v,i]) > \Psi_v([R_v,i]) \Psi_v([R_v,i]) \quad (52)$$

The result is a wavefunction which is explicitly in the form of a product. Clearly, different choices of $\Psi_v$ will lead to different results for the electronic part of the wavefunction, and this dependence is indicated by the subscript on the electronic wavefunction. For a given CFHF wavefunction, say that of the ground state, we can imagine systematically varying $\Psi_v$ to obtain the electronic wavefunction with the best possible energy. It would then be reasonable to call this $\Psi_v$, the best possible pseudovortex wavefunction.

In Appendix A we present the details of how to carry out the “projection”. The result can be gleaned from an identity which we borrow from Appendix A:

$$\int d^2z_v f(\{z_v,i\}) e^{-\sum_i \frac{z_i^2 + k^r}{2s^2} + \sum_i \frac{z_i^2 + k^r}{2s^2}} = f(\{z_v,i\}) \quad (55)$$

which shows that correlations of the $\Psi_v$ wavefunction get transferred to the $\Psi_e$ wavefunction upon “projection”. Thus, the zeroes of the bosonic wavefunction for the pseudovortices induce Laughlin-Jastrow factors in the electronic wavefunction. This establishes the precise connection between $R_v$ and the physical vortices.

Furthermore, it is well-known that wavefunctions with 2s Laughlin-Jastrow zeroes attached to the electrons have very nearly the best energy for generic repulsive electronic interactions at $\nu = p/(2ps + 1)$. Thus, the $\Psi_v$ which produces these correlations upon “projection”, namely the $\nu = \frac{1}{2s}$ bosonic Laughlin wavefunction (Eqn. (51)), must therefore be very nearly the best choice for $\Psi_v$.

IV. DETAILS OF THE HARTREE-FOCK CALCULATION

Turning to the HF calculation of gaps we will once again use the preferred density [13] in $H$ as we have in the past:

$$\bar{\rho}(q) \rightarrow \bar{\rho}^p = \bar{\rho}(q) - c^2\bar{\chi}(q) \quad (56)$$

This should make no difference in any exact calculation of gaps, which can be seen as follows: First note that when we expand out $H^p = H(\bar{\rho}^p)$, there will be a $\bar{\rho}(q)\bar{\rho}(q)$ term, which is there to begin with, a $\bar{\chi}(q)\bar{\chi}(q)$ term which contributes a constant depending on our choice of $\Psi_v$ (equivalent to a choice of “gauge”), a constant which will drop out in the difference between the ground state and a state with a widely separated particle-hole pair, and finally a $\bar{\rho}(q)\bar{\chi}(q)$ part whose expectation value will vanish as along $\Psi_v$ is translationally invariant. The reason we make the replacement $\bar{\rho}(q) \rightarrow \bar{\rho}(q)^p$ is as in our earlier papers: in the HF calculation, this choice builds in Kohn’s theorem [15] (leading to $S(q) \approx q^4$) as well as the correct charge and dipole moment [17] of the CF at tree level, thus making it plausible that the results do not suffer strong vertex corrections [13].

It is now straightforward, though tedious, to compute the correction to the transport gap from the terms induced in the effective Hamiltonian. One follows the rules of standard first order perturbation theory and takes the average of the perturbation in the unperturbed states. In this context, note that the HF states of $H_{00}$ are not necessarily HF states of $H_{0f}^f$. However, this effect is of order $v(q)/\omega_0$, and will affect the gaps to second order in $v(q)/\omega_0$. In order to evaluate the averages in the HF states, one needs to do two two-dimensional integrations. While it is possible to do these numerically for an arbitrary potential, we have simplified the problem by choosing a particularly tractable form for the potential

$$v(q) = \frac{2\pi e^2}{q} e^{-\lambda^2 q^2} = \frac{2\pi e^2}{q} e^{-\lambda^2 q^2} \quad (57)$$
This potential enables us to perform all the integrations analytically. The relevant integrals are tabulated in Appendix B.

Let us briefly sketch how the process works. We first represent the projected preferred density in terms of CF annihilation and creation operators \((d, d^\dagger)\) in the Landau gauge:

\[
\psi_{CF}(\vec{r}) = \sum_{n,X} \phi_{n,X} d_{n,X}.
\]

\[
\phi_{n,X} = C e^{Xy/(l^*)^2} e^{-(x-X)^2/2(l^*)^2} H_n((x - X)/l^*)
\]

where \(C\) is a normalization constant, \(l^* = l\sqrt{2\rho\hbar + \Gamma}\) is the magnetic length in the effective field, and the \(H_n\) are the Hermite polynomials. In terms of \(d_{n,X}\) and \(d_{n,X}^\dagger\), the preferred density looks like

\[
\bar{\rho}(\vec{q}) = \sum_{\{n\}_X} e^{-iq_i X} d_{n_1 X}^\dagger d_{n_2 X} X^{\frac{q_2}{2}} \bar{\rho}_{n_1 n_2} (\vec{q})
\]

where the matrix element is defined by

\[
\bar{\rho}_{n_1 n_2} (\vec{q}) = (-1)^{n_2+n_2} \sqrt{\frac{n_1!}{n_2!}} e^{(n_1-n_2)/(2q_2)} \left( \frac{cQ}{\sqrt{2}} |n_1-n_2| e^{-c^2Q^2/4L_{n_1}^2} / (c^2Q^2/2) 
\]

\[
- c^2(Q/c\sqrt{2}) |n_1-n_2| e^{-c^2Q^2/4L_{n_1}^2} (c^2Q^2/2) \right) \] (61)

The two-body and three-body terms have to be treated differently. The two-body terms all involve the matrix elements of \(\bar{\rho}(\vec{q})\bar{\rho}(\vec{-q})\) integrated against different measures. One first computes the contribution to the gap due to \(\bar{\rho}(\vec{q})\bar{\rho}(\vec{-q})\). Recall that the transport gap for \(\nu = \frac{1}{2}\) is the energy difference between the \(n = 0\) CF-LL and the \(n = 1\) CF-LL. Thus,

\[
\delta(q) = \begin{cases} < n = 1 | \bar{\rho}(\vec{q})\bar{\rho}(\vec{-q}) | n = 1 > \\ < n = 0 | \bar{\rho}(\vec{q})\bar{\rho}(\vec{-q}) | n = 0 > \end{cases}
\]

This quantity is easily found to be

\[
\delta(q) = \begin{cases} \left( 2 - \frac{e^2}{1+c} (q_l)^2 \right) e^{-e^2(q_l)^2/(2(1-c^2))} \\ + \left( -4c^2 + \frac{2e^2}{1+c} (q_l)^2 \right) e^{-(c^2+e^2)(q_l)^2/(4(1-c^2))} \\ + \left( 2c^4 - \frac{e^2}{1+c} (q_l)^2 \right) e^{-(e^2)(q_l)^2/(2c^2(1-c^2))} \\ + (q_l)^2 (1 - c^2) e^{-(1-c^2)(q_l)^2/(4c^2)} \end{cases}
\] (63)

It is seen that the result contains only powers of \(q_l\) and gaussian factors. With the assumed expression for the interaction, Eqn. (57), the integrations can be carried out analytically.

The contribution of the three-body term to the gap is somewhat more involved. It can be seen that it involves the expectation value of the operator

\[
\bar{\rho}(\vec{q}_1 + \vec{q}_2) : \bar{\rho}(\vec{-q}_1)\bar{\rho}(\vec{-q}_2)
\]

in the \(n = 0\) and the \(n = 1\) CF-LL’s. After some effort, the expectation value of this operator in the state \(|m>\) can be computed to be

\[
|< m | : \bar{\rho}(\vec{q}_1 + \vec{q}_2) : \bar{\rho}(\vec{-q}_1)\bar{\rho}(\vec{-q}_2) |m> = e^{-\frac{\pi}{2} q_1 \times q_2 (l^*)^2} \sum_{n_1 n_2} (1 - N_F(n_1))(1 - N_F(n_2)) \times 
\]

\[
\bar{\rho}_{n_1 n_2} (\vec{q}_1 + \vec{q}_2 - \vec{q}_1 - \vec{q}_2) \bar{\rho}_{n_2 m} (-\vec{q}_1) - e^{-\frac{\pi}{2} q_1 \times q_2 (l^*)^2} \sum_{n_1 n_2} (1 - N_F(n_1))N_F(n_2) \times 
\]

\[
\bar{\rho}_{n_1 n_2} (\vec{q}_1 + \vec{q}_2) \bar{\rho}_{n_2 m} (-\vec{q}_2) - e^{-\frac{\pi}{2} q_1 \times q_2 (l^*)^2} \sum_{n_1 n_2} N_F(n_1)(1 - N_F(n_2)) \times 
\]

\[
\bar{\rho}_{n_1 n_2} (\vec{q}_1 - \vec{q}_2) \bar{\rho}_{n_2 m} (-\vec{q}_1 + \vec{q}_2) + e^{-\frac{\pi}{2} q_1 \times q_2 (l^*)^2} \sum_{n_1 n_2} N_F(n_1)N_F(n_2) \times 
\]

\[
\bar{\rho}_{n_1 n_2} (\vec{q}_1 - \vec{q}_2) \bar{\rho}_{n_2 m} (-\vec{q}_2) - e^{-\frac{\pi}{2} q_1 \times q_2 (l^*)^2} \sum_{n_1 n_2} N_F(n_1)(1 - N_F(n_2)) \times 
\]

\[
\bar{\rho}_{n_1 n_2} (\vec{-q}_2) \bar{\rho}_{n_2 m} (-\vec{q}_1 + \vec{q}_2) - e^{-\frac{\pi}{2} q_1 \times q_2 (l^*)^2} \sum_{n_1 n_2} N_F(n_1)(1 - N_F(n_2)) \times 
\]

\[
\bar{\rho}_{n_1 n_2} (\vec{-q}_2) \bar{\rho}_{n_2 m} (-\vec{q}_1 - \vec{q}_2) \] (65)

Only the \(n = 0\) CF-LL is occupied, and the infinite sums over \(n_1\) and \(n_2\) (where not truncated by \(N_F\)) can be carried out using the following expression for the matrix elements

\[
\bar{n}_{n_1 n_2} (bq) = < n_1 | e^{-iq \cdot \eta} - c^2 e^{-iq \cdot \eta}/n_2 > \] (66)

and using the completeness of the set of states \(|n>\). Sufficient to say that the results of this calculation can also be expressed entirely in terms of powers and gaussians. Appendix B describes an integral that can be used to find both the two-body and three-body contributions to the gap.

V. RESULTS

We have carried out this calculation for three cases, \(\frac{1}{2}\) and \(\frac{1}{2}\) (which are spin-polarized), and spin-singlet \(\frac{5}{2}\). The common factor in all these cases is that only the \(n = 0\) CF-LL is occupied. The only difference is in the value of \(c\), which is \(c = \sqrt{2}/3\) for \(\frac{1}{2}\) and \(\sqrt{4}/3\) for both \(\frac{1}{2}\) and singlet \(\frac{5}{2}\). This leads us to the interesting fact in our CF-HF approximation there is no difference between \(\frac{1}{2}\) and singlet \(\frac{5}{2}\). In computing the effect of LL-mixing excitations to all electronic LL’s must be included in principal. In practice, we found that the contributions drop rapidly for high \(n’\). We found that a cutoff of \(n’_{max} = 30\) was sufficient to capture all the contributions to five-figure accuracy.

In Figure 1 we present the variation of the spin-polarized and spin-reversed transport gaps for \(\frac{1}{2}\) as a function of \(\kappa = (e^2/\ell l)/(\omega_c\kappa)\) for a “thickness parameter”
FIG. 1. Variation of the spin-polarized and spin-reversed gaps with LL-mixing for $\nu = \frac{1}{3}$ at $\Lambda = 1.2l$. The solid and dotted lines have the electronic HF energy included (Eqn. (21)) while the other two do not. All energies are in units of $e^2/\varepsilon_l$.

FIG. 2. Variation of the spin-polarized and spin-reversed gaps with $\Lambda$ for $\nu = \frac{1}{3}$ at $\kappa = 2$. For reference, the results without LL-mixing are also presented. All energies are in units of $e^2/\varepsilon_l$.

$\Lambda = 1.2l$ (for which the HF gap in the limit of no LL-mixing has approximately the same value as for the true gap for Coulomb interaction). The electronic HF energy (Eqn. (21)) has not been added to the cyclotron energy in the dashed curve (which therefore shows a linear dependence on $\kappa$), while it has been included in the solid curve. It is seen that the gap decreases by a few percent for realistic $\kappa$, but there is no dramatic effect.

In Figure 2 we show the variation of the gaps with $\Lambda$ for fixed $\kappa = 1$, this time also plotting the result in the absence of LL-mixing. It is seen that high values of $\Lambda$ suppress Landau level mixing. These results are in agreement with the previous fixed-phase diffusion Monte Carlo (FPDMC) studies [2] at a semi-quantitative level. In general, our computation shows the gap to be more robust than the FPDMC calculation. For instance, at $\kappa = 4$, which roughly corresponds to $r_s = 10$, the FPDMC results for the pure Coulomb interaction show a reduction of about 30% in the spin-polarized gap for $\frac{1}{3}$. In contrast, our results for $\lambda = 1.2$, which has roughly the same gap in the absence of LL-mixing, shows a reduction of only about 15% (with the electronic HF energy included in $H_{n,n'}$). On the other hand, this reduction is about the same as the FPDMC results for a Coulomb interaction corrected for sample thickness (thickness parameter $\beta = 1.5l$) [3]. Our calculations for $\Lambda = 1.2l$ clearly do include the strong suppression of the interaction at large wavevectors, which is characteristic of the effects of sample thickness. However, in order to carry out a detailed quantitative comparison one would have to re-do our calculation with the same interaction potential as was used in the FPDMC work. In the present work, we have contented ourselves with a proof of principle, concentrating on the analytically tractable potential of Eqn. (57).

Figure 3 displays the variation of the gaps with $\kappa$ for $\frac{1}{3}/\text{singlet} \frac{1}{2}$ (note that the spin-reversed gap is only for $\frac{1}{2}$), this time for $\Lambda = 1.6l$ (which again gives a LLL gap roughly equal to that obtained from exact diagonalization for the Coulomb interaction for $\nu = \frac{1}{5}$).

Qualitatively, it is seen that the FQH states are very robust to LL-mixing, at least as far as the transport gaps are concerned (which has been pointed out before [6,8]). While the calculations which leave out the electronic HF energy in $H_{n,n'}$ show a linear dependence on $\kappa$, the ones which include the HF energy show a much more physical behavior, with results that are even more robust under LL-mixing.
VI. CONCLUSIONS

The FQHE has been a fertile source of new ideas in the last two decades. Much of the initial insight into the physics of the FQHE was obtained by considering wavefunctions. However, calculations of physical quantities based on the wavefunction approach were forced to resort to computationally demanding techniques. The advent of Chern-Simons field theories of the FQHE raised the possibility of approximate analytical schemes to calculate the desired properties. Our Hamiltonian approach lies squarely in this class of approaches. Building on previous results, we were able to obtain a LLL theory of CF’s. Simple approximations such as HF in this theory produce results in very good agreement with those obtained numerically on small systems, and with experiments in high fields. However, as the field is lowered, the question of LL-mixing becomes unavoidable, and must be accounted for theoretically. Previous approaches to this issue have been based on exact diagonalizations or the Quantum Monte Carlo method on finite systems.

The approach to LL-mixing presented in this paper is analytical and approximate. Despite the approximations, the results agree at a semi-quantitative level with previous results. For illustrative purposes, we have chosen to compute the effect of LL-mixing on the transport gap. In principle, one can calculate the effect of LL-mixing on any physical property in this approach. Some, such as dynamical response functions, or finite temperature spin polarization, would be prohibitive to calculate numerically, but are easily computed in our approach. One interesting question concerns instabilities of the FQH states to LL-mixing. This could be addressed in our approach by applying a conserving approximation to calculate the collective modes, and seeing whether any instabilities develop in these modes as LL-mixing increases. We intend to pursue this and other questions in the future.

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VII. APPENDIX A

In this appendix we will examine how “projection” of a CFHF wavefunction to a product form works in detail for \( \nu = \frac{1}{3} \). In this case the ground state CFHF wavefunction is particularly simple, being just the wavefunction of a fully filled CF-LLL. In order to carry out the projection we first have to reexpress this wavefunction in terms of \( \{ R_{e,i}, R_{v,i} \} \). Let us focus on the single-CF states in the lowest CF-LLL. These states have the property that they are annihilated by the CF-cyclotron destruction operator. The CF-cyclotron operators can be expressed as

\[
R_{CF,x}^c - i R_{CF,y}^c = l \sqrt{ \frac{2}{1 - \epsilon^2} } a_c
\]  

(67)

where

\[
[a_c, a_c^\dagger] = [a_g, a_g^\dagger] = 1
\]  

(71)

\[
[a_c, a_g^\dagger] = [a_g, a_c^\dagger] = 0
\]  

(72)

We can now express these operators in terms of electron and pseudovortex guiding center creation and annihilation operators, which are defined by

\[
R_{e,x} + i R_{e,y} = l \sqrt{2} A_e^\dagger = \frac{e}{2} - 2l^2 \partial_c
\]  

(73)

\[
R_{e,x} - i R_{e,y} = l \sqrt{2} A_e = \frac{e}{2} + 2l^2 \partial_c
\]  

(74)

\[
R_{v,x} + i R_{v,y} = l \sqrt{2} A_v = \frac{e}{2} + 2l^2 \partial_v
\]  

(75)

\[
R_{v,x} - i R_{v,y} = l \sqrt{2} A_v^\dagger = \frac{e}{2} - 2l^2 \partial_v
\]  

(76)

where \( \partial_c, \partial_v \) are shorthand for \( \frac{\partial}{\partial a_c}, \frac{\partial}{\partial a_c^\dagger}, \) etc. The resulting expressions are

\[
a_g = \frac{A_{v-c}^\dagger}{\sqrt{1-c^2}}
\]  

(77)

\[
a_g^\dagger = \frac{A_{v-c} A_v}{\sqrt{1-c^2}}
\]  

(78)

\[
a_c = \frac{A_{c-v}^\dagger}{\sqrt{1-c^2}}
\]  

(79)

\[
a_c^\dagger = \frac{A_{c-v} A_v^\dagger}{\sqrt{1-c^2}}
\]  

(80)

Consider now the CF state with CF-LLL index and angular momentum zero. This state will be labelled \( |0, 0 \rangle \) as satisfies

\[
a_v |0, 0 \rangle = (A_v - cA_v^\dagger) |0, 0 \rangle = 0
\]  

(81)

\[
a_c |0, 0 \rangle = (A_c - cA_c^\dagger) |0, 0 \rangle = 0
\]  

(82)

Since all the available states are tensor products of LLL states of electrons and pseudovortices we can express this state as

\[
|0, 0 \rangle = \sum_{k_e, k_v=0}^{\infty} A(k_e, k_v) |k_e \rangle \otimes |k_v \rangle
\]  

(83)

where

\[
|k_e \rangle \rangle = \frac{(A_v^\dagger)^{k_e}}{\sqrt{k_e!}} |0 \rangle = \frac{1}{\sqrt{2 \pi 2 \pi k_e}} \left( \frac{2 \pi}{k_e} \right)^{k_e} e^{-|z_e|^2/4k_e^2}
\]  

(84)

\[
|k_v \rangle \rangle = \frac{(A_v^\dagger)^{k_v}}{\sqrt{k_v!}} |0 \rangle = \frac{1}{\sqrt{2 \pi 2 \pi k_v}} \left( \frac{2 \pi}{k_v} \right)^{k_v} e^{-4|\bar{z}_v|^2/4k_v^2}
\]  

(85)

are the LLL states of electrons and pseudovortices.

Applying the conditions of Eqs(81,82) it can easily be found that in \( |0, 0 \rangle \), one is forced to have \( k_e = k_v \), and

\[
|0, 0 \rangle \rangle = \sqrt{1 - c^2} \sum_{k=0}^{\infty} c^k |k_e \rangle \otimes |k_v \rangle \rangle
\]  

(86)

\[
= \frac{e^{\epsilon \sqrt{2 \pi 2 \pi}}}{2 \pi \epsilon^2} \frac{e^{\frac{i}{2} |z_e|^2/4k_e^2}}{\epsilon^2 |z_e|^2/4k_e^2} + \frac{e^{2 |\bar{z}_v|^2/2k_v^2}}{2 \pi \epsilon^2}
\]  

(87)
By similarly using the expression for \( a_j^+ \) in terms of the electron and pseudovortex coordinates and their derivatives one can verify that
\[
|0, m > = \left( \frac{a_j^+}{\sqrt{m!}} \right)^m |0, 0 > = \frac{1}{\sqrt{m!}} \left( \sqrt{\frac{1 - e^2}{2}} \right)^m \left( \frac{z}{l} \right)^m |0, 0 >
\]
(88)

Thus, filling all the CF states in the CF-LLL leads to the wave function
\[
\Psi_{\psi_{e,c=1}}^{CF} = C \prod_{i<j} (z_{e,i} - z_{e,j}) |0, 0 >
\]
(89)
\[
= C \prod_{i<j} (z_{e,i} - z_{e,j}) e^{-\frac{z_{e,i}^2}{2} + \frac{z_{e,i}^2}{2} + \frac{z_{e,j}^2}{2}}
\]
(90)
where \( C \) is a normalization constant. It can be seen that this is not in the form of a product of functions of \( z_e \) alone.

Now consider “projecting” this wavefunction into a product form against an arbitrary wavefunction \( \Psi_\psi \) dependent on \( \{z_{e,i}\} \). Performing a few gaussian integrations one can easily show that
\[
\int d^2 z_e f(\{z_{e,i}\}) e^{-\sum_k \frac{z_{e,k}^2}{2} + \sum_k z_{e,k} z_{e,k}} = f(\{z_{e,i}\})
\]
(91)

Now if one takes \( \Psi_\psi \) to be the \( \nu = \frac{1}{2} \) bosonic Laughlin wavefunction, it can be seen from eq(31) that the zeroes of the pseudovortex wavefunction are translated into Laughlin-Jastrow factors in the electronic wavefunction. This establishes the precise connection between correlations in the pseudovortex coordinates and correlations in the projected electronic wavefunction.

**VIII. APPENDIX B**

In this appendix we present an integral that can be used to find the contributions of both the two-body and three-body terms. It is found that the contributions to the gap in both cases can be expressed as a product of powers of \((q_1 l)^2, (q_2 l)^2, (q_1 + q_2)^2 l^2\), and gaussians of the same argument. This has to be integrated with a complicated measure in \( q_1 \) and \( q_2 \). Calling \( q_1 l = y_1 \) and \( q_2 l = y_2 \), and using the specially chosen form of the interaction, it is easy to see that all the integrals can be reduced to special cases of the following integral
\[
Q_n(\beta, \alpha_1, \alpha_2, \alpha_3, m_1, m_2, m_3) = \int_0^{2\pi} \frac{d\theta_1 d\theta_2}{(2\pi)^2} \int d y_1 d y_2 \frac{(-1)^n}{n(n!)^2} y_1^{m_1} y_2^{m_2} e^{i m_3 (\theta_1 - \theta_2)} e^{i \frac{q y_1 y_2}{2} \theta} e^{-\alpha_3 y_1 y_2 \cos(\theta_1 - \theta_2)}
\]
(92)

One can perform the angular integration by choosing \( \theta = \theta_2 - \theta_1 \), and using the identity
\[
\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i n \theta} e^{i a \theta + be^{-i \theta}} = \left( \frac{b}{a} \right)^n I_n(2\sqrt{ab})
\]
(93)
where \( I_n \) are the Bessel functions with imaginary argument.

This identity holds even for negative and/or complex values of \( a \) and \( b \), which for our integral attain the values \( a = \frac{1}{2} + \beta - \frac{\alpha_3}{2} \) and \( b = -\frac{1}{2} - \beta - \frac{\alpha_3}{2} \).

For the radial integrals one uses the following two formulas \([1]\) (in which \( J_\nu \) are the Bessel functions, \( \Phi \) is the confluent hypergeometric function, and \( F \) is the hypergeometric function)
\[
\int_0^\infty e^{-a x^2} J_\nu(b x) = \frac{\beta^\nu \Gamma(\frac{\nu+\alpha+1}{2})}{2^{\nu+1} \Gamma(\nu+1)} \Phi\left(\frac{\nu + \frac{1}{2} \nu + 1; -\beta^2/4\alpha\right)
\]
(94)
\[
\int_0^\infty dte^{-st} b^{-1} \Phi(a; c; kt) = \Gamma(b)s^{-b} F(a; b; c; k/s)
\]
(95)
\[
|s| > |k|
\]
\[
= \Gamma(b)(s - k)^{-b} F(c - a, b; c; k/(k - s))
\]
\[
|s - k| > |k|
\]
to obtain, for the case \( ab > 0 \)
\[
Q_n(\beta, \alpha_1, \alpha_2, \alpha_3, m_1, m_2, m_3) = \frac{(-1)^n}{n} 2^{\text{m} - 1}(m_1 + m_2)/2 \times
\]
\[
\frac{(2b)m_3+n}{n/m+1} \frac{\Gamma(n+(m_1+m_3+1)/2)\Gamma(n+(m_2+m_3+1)/2)}{\Gamma(n+(m_1+m_3+1)/2)\Gamma(n+(m_3+1)/2)} \times
\]
\[
F(n+(m_1+m_3+1)/2, n+(m_2+m_3+1)/2, n+m_3+1, 1+2\lambda+\alpha_1, 1+2\lambda+\alpha_2, 1+2\lambda+\alpha_3)
\]
(96)

where
\[
z = \frac{4ab}{(1+2\lambda^2+2\alpha_1)(1+2\lambda^2+\alpha_2)}
\]
(97)

Similarly, for the case \( ab < 0 \) we obtain
\[
Q_n(\beta, \alpha_1, \alpha_2, \alpha_3, m_1, m_2, m_3) = \frac{(-1)^n}{n} 2^{m_3-n}(2b)m_3+n \times
\]
\[
(-4ab)^{-n-(m_2+m_3+1)/2} \frac{\Gamma(n+(m_1+m_3+1)/2)\Gamma(n+(m_3+1)/2)}{\Gamma(n+(m_1+m_3+1)/2)\Gamma(n+m_3+1)/2} \times
\]
\[
F(m_2+m_3+1/2, n+m_3+1, n+m_3+1, n+m_3+1, 1+2\lambda+\alpha_1, 1+2\lambda+\alpha_2, 1+2\lambda+\alpha_3)
\]
(98)

where
\[
z = \frac{-4ab}{-4ab + (1+2\lambda^2+2\alpha_1)(1+2\lambda^2+\alpha_2)}
\]
(99)

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