Bounded languages described by GF(2)-grammars

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Abstract
GF(2)-grammars are a (somewhat) recently introduced grammar family that have some unusual algebraic properties and are closely connected to unambiguous grammars. By using the method of formal power series, we establish strong conditions that are necessary for subsets of $a_1^*a_2^* \cdots a_k^*$ to be described by some GF(2)-grammar. By further applying the established results, we settle the long-standing open question of proving the inherent ambiguity of the language $\{a^n b^m c^\ell \mid n \neq m \text{ or } m \neq \ell \}$, as well as give a new, purely algebraic, proof of the inherent ambiguity of the language $\{a^n b^m c^\ell \mid n = m \text{ or } m = \ell \}$.

Keywords: Formal grammars, finite fields, bounded languages, unambiguous grammars, inherent ambiguity.

1 Introduction

GF(2)-grammars, recently introduced by Bakinova et al. [3], and further studied by Makarov and Okhotin [14], are a variant of ordinary context-free grammars (or just ordinary grammars, as I will call them later in the text), in which the disjunction is replaced by the exclusive OR, whereas the classical concatenation is replaced by a new operation called GF(2)-concatenation: $K \odot L$ is the set of all strings with an odd number of partitions into a concatenation of a string in $K$ and a string in $L$.

There are several reasons for studying GF(2)-grammars. Firstly, they are a class of grammars with better algebraic properties, compared to ordinary grammars and similar grammar families, because the underlying boolean semiring logic is replaced by the logic of the field with two elements. As we will see later in the paper, that makes GF(2)-grammars lend themselves very well to algebraic manipulations.

Secondly, GF(2)-grammars provide a new way of looking at unambiguous grammars. For example, instead of proving that some language is inherently ambiguous, one can prove that no GF(2)-grammar describes it. While the latter condition is, strictly speaking, stronger, it may turn out to be easier to prove, because the family of languages defined by GF(2)-grammars has good algebraic properties and is closed under symmetric difference.

Finally, GF(2)-grammars generalize the notion of parity nondeterminism to grammars. Recall that the most common types of nondeterminism that are considered in
complexity theory are classical nondeterminism, which corresponds to the existence of an accepting computation, unambiguous nondeterminism, which corresponds to the existence of a unique accepting computation and parity nondeterminism, which corresponds to the number of accepting computations being odd.

In a similar way, classical and parity nondeterminism can be seen as two different generalisations of unambiguous nondeterminism: if the number of accepting computations is in the set \( \{0, 1\} \), then it is positive (classical case) if and only if it is odd (parity case); the same is not true for larger numbers, of course.

The main result of this paper is Theorem 6, which establishes a strong necessary conditions on subsets of \( a_1^* a_2^* \cdots a_k^* \) that are described by GF(2)-grammars. Theorem 5, a special case of Theorem 6, implies that there are no GF(2)-grammars for the languages \( L_1 := \{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell \} \) and \( L_2 := \{ a^n b^m c^\ell \mid n \neq m \text{ or } m \neq \ell \} \).

As a consequence, both languages are inherently ambiguous. For \( L_1 \), all previously known arguments establishing its inherent ambiguity were combinatorial, mainly based on Ogden’s lemma.

Proving the inherent ambiguity of \( L_2 \) was a long-standing open question due to Autebert et al. [2, p. 375]. There is an interesting detail here: back in 1966, Ginsburg and Ullian fully characterized bounded languages described by unambiguous grammars in terms of semi-linear sets [11, Theorems 5.1 and 6.1]. However, most natural ways to apply this characterization suffer from the same limitation: they mainly rely on strings that are not in the language and much less on the strings that are. Hence, “dense” languages like \( L_2 \) leave them with almost nothing to work with. Moreover, \( L_2 \) has an algebraic generating function, meaning that a naive application of analytic methods cannot tackle it either. In fact, Flajolet [6], in his seminal work on analytic methods for proving grammar ambiguity, refers to the inherent ambiguity of \( L_2 \) as to a question that is still open (see page 286).

## 2 Basics

The proofs that we will see later make heavy use of algebraic methods. For the algebraic parts, the exposition strives to be as elementary and self-contained as possible. Hence, I will prove a lot of lemmas that are by no way original and may be considered trivial by someone with good knowledge of commutative algebra. This is the intended effect; if you consider something to be trivial, you can skip reading the proof. If, on the other hand, you have some basic knowledge of algebra, but still find some of the parts to be unclear, you may contact me and I will try to find a better wording. The intended “theoretical minimum” is being at least somewhat familiar with concepts of polynomials, rational functions and formal power series.

Let us recall the definition and the basic properties of GF(2)-grammars first. This section is completely based on already published work: the original paper about GF(2)-operations by Bakinova et al. [3] and the paper about basic properties of GF(2)-grammars by Makarov and Okhotin [14]. Hence, all the proofs are omitted; for proofs and more thorough commentary on definitions refer to the aforementioned papers. If you are already familiar with both of them, you may skip straight to the next section.

GF(2)-grammars are built upon GF(2)-operations [3]: symmetric difference and a new operation called GF(2)-concatenation:

\[
K \circ L = \{ w \mid \text{the number of partitions } w = uv, \text{ with } u \in K \text{ and } v \in L, \text{ is odd} \} \]
Syntactically, GF(2)-grammars do not differ from ordinary grammars. However, in the right-hand sides of the rules, the normal concatenation is replaced with GF(2)-concatenation, whereas multiple rules for the same nonterminal correspond to the symmetric difference of given conditions, instead of their disjunction.

**Definition 1 ([3]).** A GF(2)-grammar is a quadruple \( G = (\Sigma, N, R, S) \), where:

- \( \Sigma \) is the alphabet of the language;
- \( N \) is the set of nonterminal symbols;
- every rule in \( R \) is of the form \( A \to X_1 \oplus \cdots \oplus X_\ell \), with \( \ell \geq 0 \) and \( X_1, \ldots, X_\ell \in \Sigma \cup N \), which represents all strings that have an odd number of partitions into \( w_1 \cdots w_\ell \), with each \( w_i \) representable as \( X_i \);
- \( S \in N \) is the initial symbol.

The grammar must satisfy the following condition. Let \( \hat{G} = (\Sigma, N, \hat{R}, S) \) be the corresponding ordinary grammar, with \( \hat{R} = \{ A \to X_1 \cdots X_\ell \mid A \to X_1 \oplus \cdots \oplus X_\ell \in R \} \). It is assumed that, for every string \( w \in \Sigma^* \), the number of parse trees of \( w \) in \( \hat{G} \) is finite; if this is not the case, then \( G \) is considered ill-formed.

Then, for each \( A \in N \), the language \( L_G(A) \) is defined as the set of all strings with an odd number of parse trees as \( A \) in \( \hat{G} \).

**Theorem A ([3]).** Let \( G = (\Sigma, N, R, S) \) be a GF(2)-grammar. Then the substitution \( A = L_G(A) \) for all \( A \in N \) is a solution of the following system of language equations.

\[
A = \bigtriangleup_{A \to X_1 \oplus \cdots \oplus X_\ell \in R} X_1 \oplus \cdots \oplus X_\ell \quad (A \in N)
\]

Multiple rules for the same nonterminal symbol can be denoted by separating the alternatives with the “sum modulo two” symbol \( (\oplus) \), as in the following example.

**Example 1 ([3]).** The following GF(2)-linear grammar defines the language \( \{ a^\ell b^m c^n \mid \ell = m \text{ or } m = n, \text{ but not both} \} \).

\[
S \to A \oplus C \\
A \to aA \oplus B \\
B \to bBc \oplus \epsilon \\
C \toCc \oplus D \\
D \to aDb \oplus \epsilon 
\]

Indeed, each string \( a^\ell b^m c^n \) with \( \ell = m \) or with \( m = n \) has a parse tree, and if both equalities hold, then there are accordingly two parse trees, which cancel each other.

**Example 2 ([3]).** The following grammar describes the language \( \{ a^{2n} \mid n \geq 0 \} \).

\[
S \to (S \circ S) \oplus a
\]

The main idea behind this grammar is that the GF(2)-square \( S \circ S \) over a unary alphabet doubles the length of each string: \( L \circ L = \{ a^\ell \mid a^\ell \in L \} \). The grammar iterates this doubling to produce all powers of two.
As the previous example illustrates, GF(2)-grammars can describe non-regular unary languages, unlike ordinary grammars. We will need the classification of unary languages describable by GF(2)-grammars in the following Sections.

**Definition 2.** A set of nonnegative integers \( S \subseteq \mathbb{N}_0 \) is called \( q \)-automatic [1], if there is a finite automaton over the alphabet \( \Sigma_q = \{0, 1, \ldots, q - 1\} \) recognizing base-\( q \) representations of these numbers.

Let \( \mathbb{F}_q[t] \) be the ring of polynomials over the \( q \)-element field \( \mathbb{F}(q) \), and let \( \mathbb{F}_q[[t]] \) denote the ring of formal power series over the same field.

**Definition 3.** A formal power series \( f \in \mathbb{F}_q[[t]] \) is said to be algebraic, if there exists a non-zero polynomial \( P \) with coefficients from \( \mathbb{F}_q[t] \), such that \( P(f) = 0 \).

**Theorem B** (Christol’s theorem for \( \mathbb{F}(2) \) [7]). A formal power series \( \sum_{n=0}^{\infty} f_n t^n \in \mathbb{F}_2[[t]] \) is algebraic if and only if the set \( \{ n \in \mathbb{N}_0 \mid f_n = 1 \} \) is \( 2 \)-automatic.

**Theorem C** (Unary languages described by GF(2)-grammars [14]). For a unary alphabet, the class of all \( 2 \)-automatic languages coincides with the class of all languages described by GF(2)-grammars.

## 3 Subsets of \( a^*b^* \)

Suppose that some GF(2)-grammar over an alphabet \( \Sigma = \{a, b\} \) generates a language that is a subset of \( a^*b^* \). How does the resulting language look like?

It will prove convenient to associate subsets of \( a^*b^* \) with (commutative) formal power series in two variables \( a \) and \( b \) over the field \( \mathbb{F}_2 \). This correspondence is similar to the correspondence between languages over a unary alphabet with GF(2)-operations (\( \cdot, \triangle \)) and formal power series of one variable with multiplication and addition [14].

Formally speaking, for every set \( S \subseteq \mathbb{N}_2 \), the language \( \{ a^n b^m \mid (n, m) \in S \} \subseteq a^*b^* \) corresponds to the formal power series \( \sum_{(n,m) \in S} a^n b^m \) in variables \( a \) and \( b \). Let us denote this correspondence by \( \text{asSeries} : 2^* \rightarrow \mathbb{F}_2[[a, b]] \). Then, \( \text{asSeries}(L \triangle K) = \text{asSeries}(L) + \text{asSeries}(K) \), so the symmetric difference of languages corresponds to the addition of power series.

On the other hand, multiplication of formal power series does not always correspond to the GF(2)-concatenation of languages. Indeed, GF(2)-concatenation of subsets of \( a^*b^* \) does not have to be a subset of \( a^*b^* \). However, the correspondence does hold in the following important special case.

**Lemma 1.** If \( K \subseteq a^* \) and \( L \subseteq a^*b^* \), then \( \text{asSeries}(K \circ L) = \text{asSeries}(K) \cdot \text{asSeries}(L) \). The same conclusion holds when \( K \subseteq a^*b^* \) and \( L \subseteq b^* \).

**Sketch of the proof.** Follows from definitions. \( \square \)

Denote the set of all algebraic power series from \( \mathbb{F}_2[[a]] \) by \( A \). By Christol’s theorem [7], \( A \) corresponds to the set of all 2-automatic languages over \( \{a\} \). Similarly, denote the set of all algebraic power series from \( \mathbb{F}_2[[b]] \) by \( B \).

Recall that \( \mathbb{F}_2[a, b] \) denotes the set of all polynomials in variables \( a \) and \( b \) and \( \mathbb{F}_2(a, b) \) denotes the set of all rational functions in variables \( a \) and \( b \). It should be mentioned that \( \mathbb{F}_2[a, b] \) is a subset of \( \mathbb{F}_2[[a, b]] \), but \( \mathbb{F}_2(a, b) \) is not. Indeed, \( \frac{1}{a} \in \mathbb{F}_2(a, b) \), but not in \( \mathbb{F}_2[[a, b]] \). The following statement is true: \( \mathbb{F}_2(a, b) \subseteq \mathbb{F}_2((a, b)) \), where
\( \mathbb{F}_2((a, b)) \) denotes the set of all Laurent series in variables \( a \) and \( b \). Laurent series are defined as the fractions of formal power series with equality, addition and multiplication defined in the usual way.

We will allow Laurent series to appear in the intermediate results, because the intermediate calculations require division, and formal power series are not closed under division. However, there are no Laurent series (unless they are valid formal power series as well) in the statements of the main theorems, because they do not correspond to valid languages.

From now on, there are two possible roads this proof can take: the original argument that is more elementary, but requires lengthy manipulations with what I called algebraic expressions, and a more abstract, but much simpler approach suggested by an anonymous reviewer from MFCS 2020 conference, relying on well-known properties of rings and field extensions. The main body of the paper follows the latter approach. The former approach can be found in the appendix.

**Definition 4.** Denote by \( R_{a,b} \) the set of all Laurent series that can be represented as \( \sum_{i=1}^{n} A_i \frac{1}{B_i} p \), where \( n \) is a nonnegative integer, \( A_i \in A \) and \( B_i \in B \) for all \( i \) from 1 to \( n \), and \( p \in \mathbb{F}_2[a,b] \) is a non-zero polynomial.

It is not hard to see that \( R_{a,b} \) is a commutative ring. However (and we will use it later a lot), an even stronger statement is true:

**Lemma 2.** \( R_{a,b} \) is a field.

**Proof.** \( R_{a,b} \) is the result of adjoining the elements of \( A \cup B \), which are all algebraic over \( \mathbb{F}_2(a, b) \), to \( \mathbb{F}_2(a, b) \). It is known that the result of adjoining an arbitrary set of algebraic elements to a field is a larger field. \( \square \)

### 3.1 The main result for subsets of \( a^*b^* \)

Let us establish our main result about subsets of \( a^*b^* \).

**Theorem 1.** Assume that a language \( K \subset a^*b^* \) is described by a GF(2)-grammar. Then, the corresponding power series \( \text{asSeries}(K) \) is in the set \( R_{a,b} \).

**Proof.** Without loss of generality, the GF(2)-grammar that describes \( K \) is in the Chomsky normal form [3, Theorem 5]. Moreover, we can assume that \( K \) does not contain the empty string.

The language \( a^*b^* \) is accepted by the following DFA \( M \): \( M \) has two states \( q_a \) and \( q_b \), both accepting, and its transition function is \( \delta(q_a, a) = q_a, \delta(q_a, b) = q_b, \delta(q_b, b) = q_b \).

Let us formally intersect the GF(2)-grammar \( G \) with a regular language \( a^*b^* \), recognized by the automaton \( M \) (the construction of the intersection of an ordinary grammar with a regular expression by Bar-Hillel et al. [4] can be easily adapted to the case of GF(2)-grammars [14, Section 6]). The language described by the GF(2)-grammar will not change, because it was already a subset of \( a^*b^* \) before.

The grammar itself changes considerably, however. Every nonterminal \( C \) of the original GF(2)-grammar splits into three nonterminals: \( C_{a\to a}, C_{a\to b} \), and \( C_{b\to b} \). These nonterminals will satisfy the following conditions: \( L(C_{a\to a}) = L(C) \cap a^* \), \( L(C_{b\to b}) = L(C) \cap b^* \), and \( L(C_{a\to b}) = L(C) \cap (a^*b^*) \). Also, a new starting nonterminal \( S' \) appears.

Moreover, every “normal” rule \( C \to DE \) splits into four rules: \( C_{a\to a} \to D_{a\to a}E_{a\to a}, C_{a\to b} \to D_{a\to b}E_{a\to b}, C_{a\to b} \to D_{a\to b}E_{b\to b} \) and \( C_{b\to b} \to D_{b\to b}E_{b\to b} \).
The following happens with “final” rules: $C \rightarrow b$ turns into two rules $C_{a \rightarrow b} \rightarrow b$ and $C_{b \rightarrow b} \rightarrow b$, and $C \rightarrow a$ turns into one rule $C_{a \rightarrow a} \rightarrow a$. Finally, two more rules appear: $S' \rightarrow S_{a \rightarrow a}$ and $S' \rightarrow S_{a \rightarrow b}$.

For every nonterminal $C$ of the original GF(2)-grammar, the languages $L(C_{a \rightarrow a})$ and $L(C_{b \rightarrow b})$ are 2-automatic languages over unary alphabets $\{a\}$ and $\{b\}$ respectively. Indeed, every parse tree of $C_{a \rightarrow a}$ contains only nonterminals of type $a \rightarrow a$. Therefore, only character $a$ can occur as a terminal in a parse tree of $C_{a \rightarrow a}$. So, $L(C_{a \rightarrow a})$ is described by some GF(2)-grammar over an alphabet $\{a\}$, and is therefore 2-automatic. Similarly for $C_{b \rightarrow b}$.

By Theorem 5, the languages $L(C_{a \rightarrow b})$ for each nonterminal $C_{a \rightarrow b}$ of the new grammar satisfy the following system of language equations (System (1)).

Here, for each nonterminal $C$, the summation is over all rules $C \rightarrow DE$ of the original GF(2)-grammar. Also, $\text{end}(C_{a \rightarrow b})$ is either $\{b\}$ or $\emptyset$, depending on whether or not there is a rule $C_{a \rightarrow b} \rightarrow b$ in the new GF(2)-grammar.

$$L(C_{a \rightarrow b}) = \text{end}(C_{a \rightarrow b}) \oplus \bigoplus_{(C \rightarrow DE) \in R} (L(D_{a \rightarrow a}) \circ L(E_{a \rightarrow b})) \oplus (L(D_{a \rightarrow b}) \circ L(E_{b \rightarrow b})) \quad (1)$$

It is easy to see that all GF(2)-concatenations in the right-hand sides satisfy the conditions of Lemma 1. Denote asSeries($L(C_{a \rightarrow b})$) by Center($C$), asSeries($L(C_{a \rightarrow a})$) by Left($C$), asSeries($L(C_{b \rightarrow b})$) by Right($C$) and asSeries(\text{end}(C_{a \rightarrow b})) by final($C$) for brevity. Then, the algebraic equivalent of System (1) also holds:

$$\text{Center}(C) = \text{final}(C) + \sum_{(C \rightarrow DE) \in R} \text{Left}(D) \text{Center}(E) + \text{Center}(D) \text{Right}(E) \quad (2)$$

Let us look at this system as a system of $\mathbb{F}_2[[a,b]]$-linear equations over variables Center($C$) = asSeries($L(C_{a \rightarrow b})$) for every nonterminal $C$ of the original GF(2)-grammar.

We will consider final($C$), Left($C$) and Right($C$) to be the coefficients of the system. While we do not know their exact values, the following is known: final($C$) is 0 or $b$, Left($C$) $\in A$ as a formal power series that corresponds to a 2-automatic language over an alphabet $\{a\}$ and, similarly, Right($C$) $\in B$. That means that all coefficients of the system lie in $A \cup B$ and, therefore, in $R_{a,b}$. The latter is a field by Lemma 2.

Denote the number of nonterminals in the original GF(2)-grammar by $n$, (so there are $n$ nonterminals of type $a \rightarrow b$ in the new GF(2)-grammar), a column vector of values Center($C$) by $x$ and a column vector of values final($C$) in the same order by $f$. Let us fix the numeration of nonterminals $C$ of the old GF(2)-grammar. After that, we can use them as the “indices” of rows and columns of matrices.

Let $I$ be an identity matrix of dimension $n \times n$, $A$ be a $n \times n$ matrix with the sum of Left($D$) over all rules $C \rightarrow DE$ of the original grammar standing on the intersection of $C$-th row and $E$-th column:

$$A_{C,E} := \sum_{(C \rightarrow DE) \in R} \text{Left}(D) \quad (3)$$

Similarly, let $B$ be a $n \times n$ matrix with

$$B_{C,D} := \sum_{(C \rightarrow DE) \in R} \text{Right}(E) \quad (4)$$

Then, the equation System (2) can be rewritten as $x = f + (A + B)x$ in the matrix form. In other words, $(A + B + I)x = f$. Consider a homomorphism $h: \mathbb{F}_2[[a,b]] \rightarrow \mathbb{F}_2$
\( \mathbb{F}_2 \) that maps power series to their constant terms (coefficients before \( a^0b^0 \)). Then, \( h(\det(A + B + I)) = \det(h(A + B + I)) = \det(h(A) + h(B) + h(I)) \), where \( h \) is extended to the \( n \times n \) matrices with components from \( \mathbb{F}_2[[a,b]] \) in the natural way (replace each component of the matrix by its constant term).

Because the new \( \text{GF}(2) \)-grammar for \( K \) is also in Chomsky normal form, all languages \( L(C_{a \rightarrow a}) \) and \( L(C_{b \rightarrow b}) \) do not contain the empty string. Therefore, all series \( \text{Left}(C) = \text{asSeries}(L(C_{a \rightarrow a})) \) and \( \text{Right}(C) = \text{asSeries}(L(C_{b \rightarrow b})) \) have zero constant terms. Hence, \( h(A) = h(B) = 0 \), where by 0 we mean a zero \( n \times n \) matrix. On the other hand, \( h(I) = I \). Hence, \( h(\det(A + B + I)) = \det(h(A) + h(B) + h(I)) = \det(I) = 1 \). Therefore, \( \det(A + B + I) \neq 0 \), because \( h(0) = 0 \).

Hence, the System [2] has exactly one solution within the field \( \mathbb{F}_2((a,b)) \) — the actual values of \( \text{Center}(C) \). Moreover, we know that all coefficients of the system lie in the field \( R_{a,b} \subset \mathbb{F}_2((a,b)) \). Therefore, all components of the unique solution also lie within the field \( R_{a,b} \). Hence, \( \text{asSeries}(K) = \text{asSeries}(L(S')) = \text{asSeries}(L(S_{a \rightarrow a})) + \text{asSeries}(L(S_{a \rightarrow b})) \) also lies in \( R_{a,b} \).

**Remark 1.** Alternatively, one can prove the uniqueness of the solution to System [2] by some kind of fixed-point argument. However, I stick to proving that the determinant is non-zero, mainly because the proof of Theorem [3] still uses the existence of an inverse matrix regardless of how the uniqueness of the solution is established.

### 3.2 Using Theorem [1]

It is hard to use Theorem [1] directly. Hence, we will prove the following intermediate result:

**Theorem 2.** Suppose that \( L \subset a^*b^* \) is described by a \( \text{GF}(2) \)-grammar. Denote “the coefficient” of \( \text{asSeries}(L) \) before \( a^i \) by \( \ell(i) \in \mathbb{F}_2[[b]] \), in the sense that \( \text{asSeries}(L) = \sum_{i=0}^{+\infty} a^i \ell(i) \). Then, there exists a nonnegative integer \( d \) and polynomials \( p_0, p_1, \ldots, p_d \in \mathbb{F}_2[b] \), such that \( p_d \neq 0 \) and \( \sum_{i=0}^{d} p_i \ell(n-i) \) assumes only a finite number of distinct values, when \( n \) ranges over the set of all integers larger than \( d \).

**Example 3.** Suppose that \( \text{asSeries}(L) = \frac{A_1B_1 + A_2B_2}{1 + ab}, \) where \( A_1, A_2 \in A \) and \( B_1, B_2 \in B \). Then \( \text{asSeries}(L)(1 + ab) = A_1B_1 + A_2B_2 \). Denote the coefficient of \( \text{asSeries}(L) \) before \( a^n \) by \( \ell(n) \). Then \( (\sum_{n=0}^{+\infty} a^n \ell(n)) (1 + ab) = A_1B_1 + A_2B_2 \). Coefficients of the left-hand side before \( a^n \) are \( b \cdot \ell(n-1) + \ell(n) \) for \( n \geq 1 \). The corresponding coefficients of the right-hand side are always from the set \( \{0, B_1, B_2, B_1 + B_2\} \). Therefore, it is enough to choose \( d = 1, p_0 = 1, p_1 = b \) in this case.

**Remark 2.** Actually, \( \sum_{i=0}^{d} p_i \ell(n-i) \) is 2-automatic sequence [2] of elements of \( B \). That is, only elements from \( B \) appear in this sequence, only finite number of them actually appear, and every element appears on a 2-automatic set of positions. We just will not need the result in the maximum possible strength here.

**Proof of Theorem [2]** As we already know, \( \text{asSeries}(L) \in R_{a,b} \), meaning that \( \text{asSeries}(L) = \left( \sum_{k=1}^{K} A_k B_k \right) \left( \sum_{i=0}^{d} a^i p_i \right) \) for some nonnegative integers \( d \) and \( K \), \( A_1 \in A, B_1 \in B \) and \( p_i \in \mathbb{F}_2[b] \). Moreover, we can choose \( d \) in such a way, that \( p_d \neq 0 \): not all \( p_i \) are equal to zero, because otherwise the denominator of the fraction would be equal to zero. Also, \( \text{asSeries}(L) = \sum_{j=0}^{+\infty} a^j \ell(j) \) by definition of \( \ell(\cdot) \).
Therefore, \( \left( \sum_{i=0}^{d} a^i p_i \right) \cdot \left( \sum_{j=0}^{+\infty} a^j \ell(j) \right) = \sum_{k=1}^{K} A_k B_k \). The coefficients of the left-hand and the right-hand sides before \( a^n \) are \( \sum_{i=0}^{\min(n,d)} p_i \ell(n-i) \) and \( \sum_{k} a^n \in A_k B_k \) respectively. Here, the second sum is taken over all \( k \) from 1 to \( K \), such that the coefficient of \( A_k \) before \( a^n \) is one. When \( n \geq d \), the former of these two values is \( \sum_{i=0}^{d} p_i \ell(n-i) \) and the latter always takes one of \( 2^K \) possible values.

Let us consider a simple application of Theorem 2 to get the hang of how it can be used to prove something.

**Theorem 3.** The language \( K = \{ a^{2^n} b^{2^n} \mid n \in \mathbb{N} \} \) is not described by a GF(2)-grammar.

**Proof.** By contradiction. Let us use Theorem 2 on the language \( K \). The coefficient \( \ell(n) \) of asSeries(\( K \)) is \( b^n \), if \( n \) is a power of two and 0 otherwise. In any case, it is divisible by \( b^n \). On one hand, from the conclusion of Theorem 2 \( \sum_{i=0}^{d} p_i \ell(n-i) \) assumes only a finite number of values for some integer \( d \) and \( p_0, p_1, \ldots, p_d \in \mathbb{F}_2[b] \), satisfying the condition \( p_d \neq 0 \).

On the other hand, the sum \( \sum_{i=0}^{d} p_i \ell(n-i) \) is divisible by \( b^{n-d} \); every summand contains a factor \( \ell(n-i) \), which is divisible by \( b^{n-i} \). Therefore, \( \sum_{i=0}^{d} p_i \ell(n-i) \) is divisible by larger and larger powers of \( b \) as \( n \) grows. Therefore, the only value that this sum can assume infinitely often is 0: all other power series are not divisible by arbitrarily large powers of \( b \). Because the sum assumes only a finite number of values, 0 is obtained for large enough \( n \).

Therefore, some fixed linear combination of \( \ell(n-d), \ell(n-d+1), \ldots, \ell(n) \) is equal to 0 for large enough \( n \). However, non-zero values appear in the sequence \( \ell(i) \) extremely rarely: the gaps between them grow larger and larger. In particular, one can choose such \( n \geq d \), that \( \sum_{i=0}^{d} p_i \ell(n-i) = 0, \ell(n-d) \neq 0, \) but \( \ell(n-d+1) = \ldots = \ell(n) = 0 \). This is impossible, because \( p_d \neq 0 \) and, therefore, there is exactly one non-zero summand in a zero sum: \( p_d \ell(n-d) \).

To be exact, one can pick \( n = d + 2^m \) for large enough \( m \).

A more interesting application of the technique can be seen in the following theorem (a weaker result was earlier obtained by Makarov and Okhotin [14, Theorem 12] using elementary methods).

**Theorem 4.** Suppose that \( f : \mathbb{N}_0 \to \mathbb{N}_0 \) is a strictly increasing function. If a language \( L_f = \{ a^n b^{f(n)} \mid n \in \mathbb{N}_0 \} \) is generated by a GF(2)-grammar, then the set \( f(\mathbb{N}_0) \) is a finite union of arithmetic progressions.

**Proof.** Let us use Theorem 2 here. The proof is structured in the following way. The first step is to prove that polynomials \( b^{f(n)} \) satisfy some linear recurrence that has rational functions of \( b \) as coefficients. The second step is to prove that \( f(\mathbb{N}_0) \) is, indeed, a finite union of arithmetic progressions. Intuitively, it is hard to imagine a linear recurrence with all values looking like \( \ell^{\text{something}} \), but without strong regularity properties.

Let us use Theorem 2 on the language \( L_f \). The coefficient of asSeries(\( L_f \)) before \( a^n \) is \( b^{f(n)} \). Therefore, \( \sum_{i=0}^{d} p_i b^{f(n-i)} \) assumes only a finite number of values for some nonnegative integer \( d \) and some \( p_0, p_1, \ldots, p_d \in \mathbb{F}_2[b] \) satisfying the property \( p_d \neq 0 \). Notice that this sum starts being divisible by arbitrarily large powers of \( b \) when \( n \) increases (here we use the fact that \( f \) is an increasing function). Because the sum assumes only a finite number of values, it is equal to zero for large enough \( n \).
Now, we want to prove that \( \{ f(n) \mid n \in \mathbb{N}_0 \} \) is a finite union of arithmetic progressions. We have already established that \( \sum_{i=0}^{d} p_i b^{f(n-i)} = 0 \) for large enough \( n \).

Let \( j \) be the smallest index, such that \( p_j \neq 0 \): it exists, because \( p_d \neq 0 \). Moreover, \( j \neq d \), otherwise \( 0 = p_d b^{f(n-d)} \) for large enough \( n \), contradicting \( p_d \neq 0 \). Therefore, \( p_j b^{f(n-j)} = \sum_{i=j+1}^{d} p_i b^{f(n-i)} \). Let us rewrite the last statement in a slightly different way:

\[
b^{f(n-j)} = \sum_{i=j+1}^{d} \frac{p_i}{p_j} b^{f(n-i)}.
\]

Divide both sides of the last equation by \( b^{f(n-d)} \). Then,

\[
b^{f(n-j)} - b^{f(n-d)} = \sum_{i=j+1}^{d} \frac{p_i}{p_j} b^{f(n-i) - f(n-d)}.
\]

Therefore, the difference \( f(n-j) - f(n-d) \) depends only on differences \( f(n-j-1) - f(n-d), \ldots, f(n-d+1) - f(n-d) \), but not on \( f(n-d) \) itself. Now, let us prove that the difference \( f(n+1) - f(n) \) is bounded above (it is always positive, because \( f \) is increasing). Indeed, as we know, \( \sum_{i=0}^{d} p_i b^{f(m-i)} = 0 \) for large enough \( m \). Substitute \( n := m - d \) and \( k := d - i \), the result is \( \sum_{k=0}^{d} p_{d-k} b^{f(n+k)} = 0 \). Because \( f \) increases, all summands are divisible by \( b^{f(n+1)} \), with a possible exception of the first summand. Because the sum is equal to 0, the first summand should be divisible by \( b^{f(n+1)} \) as well. It is not equal to 0 (because \( p_d \neq 0 \) and \( b^{f(n)} \neq 0 \)) and its degree as a polynomial of \( b \) is equal to \( \deg p_d + \deg b^{f(n)} = \deg p_d + f(n) \). The degree of a non-zero polynomial divisible by \( b^{f(n+1)} \) is at least \( f(n+1) \), therefore \( f(n+1) - f(n) \leq \deg p_d \).

Because the differences \( f(n+1) - f(n) \) are bounded, then the differences \( f(n+k) - f(n) = (f(n+k) - f(n+k-1)) + (f(n+k-1) - f(n+k-2)) + \ldots + (f(n+1) - f(n)) \) are bounded as well for all \( k \leq d \). Therefore, the tuple of differences \( (f(n-j-1) - f(n-d), \ldots, f(n-d+1) - f(n-d)) \) assumes only a finite set of possible values as \( n \) goes towards infinity. As shown above, \( f(n-j) - f(n-d) \) can be uniquely restored from such a tuple. Therefore, the tuple for \( n+1 \) can be uniquely restored from the tuple for \( n \) indeed, it is enough to know the pairwise differences between the elements of \( \{ f(n-j), f(n-j-1), \ldots, f(n-d) \} \), and the current tuple along with the number \( f(n-j) - f(n-d) \) provide this information.

Because there is only a finite number of such tuples, and each tuple determines the next, they start “going in circles” at some moment. In particular, the differences \( f(n-d+1) - f(n-d) \) start going in circles. This fact, along with the function \( f \) being increasing, is enough to establish that \( \{ f(n) \mid n \in \mathbb{N}_0 \} \) is a finite union of arithmetic progressions. \( \square \)

4 Subsets of \( a^*b^*c^* \)

The language \( \{ a^n b^n c^n \mid n \geq 0 \} \) is, probably, the most famous example of a simple language that is not described by any ordinary grammar. It is reasonable to assume that it is not described by a GF(2)-grammar as well. Let us prove that.

We will do more than that and will actually establish some property that all subsets of \( a^*b^*c^* \) that can be described by a GF(2)-grammar have, but \( \{ a^n b^n c^n \mid n \geq 0 \} \) does not. Most steps of the proof will be analogous to the two-letter case.
There is a natural one-to-one correspondence between subsets of $a^*b^*c^*$ and formal power series in variables $a, b$ and $c$ over field $\mathbb{F}_2$. Indeed, for every set $S \subseteq \mathbb{N}_0^3$, we can identify the language $\{ a^n b^m c^n \mid (n, m, k) \in S \} \subseteq a^*b^*c^*$ with the formal power series $\sum_{(n,m,k) \in S} a^n b^m c^n$. Denote this correspondence by asSeries: $2^{a^*b^*c^*} \rightarrow \mathbb{F}_2[[a, b, c]]$. Then, asSeries($L \triangle K$) = asSeries($L$) + asSeries($K$). In other words, the symmetric difference of languages corresponds to the sum of formal power series.

Similarly to the Lemma 1, asSeries($K \odot L$) = asSeries($K$) · asSeries($L$) in the following important special cases: when $K$ is a subset of $a^*$, when $K$ is a subset of $a^*b^*$ and $L$ is a subset of $b^*c^*$, and, finally, when $L$ is a subset of $c^*$. Indeed, in each of these three cases, characters “are in the correct order”: if $u \in K$ and $v \in L$, then $uv \in a^*b^*c^*$.

However, we cannot insert character $b$ in the middle of the string: if $K$ is a subset of $b^*$ and $L$ is a subset of $a^*b^*c^*$, then $K \odot L$ does not even have to be a subset of $a^*b^*c^*$.

The “work plan” will remain the same as in the previous section: we will prove that there is a natural one-to-one correspondence between atomic functions and $a^*b^*c^*$-regular languages. Indeed, for every set $S \subseteq \mathbb{N}_0^3$,

1. Why is it logical to expect that the language $\{ a^n b^m c^n \mid n \geq 0 \}$ is not described by a GF(2)-grammar, but a similar language $\{ a^n b^n \mid n \geq 0 \}$ is?

2. Why will the proof work out for $\{ a^n b^m c^n \mid n \geq 0 \}$, but not for a regular language $\{ (abc)^n \mid n \geq 0 \}$, despite these languages having the same “commutative image”?

They can be answered in the following way:

1. Simply speaking, the reason is the same as for the ordinary grammars. On an intuitive level, both ordinary grammars and GF(2)-grammars permit a natural way to “capture” the events that happen with any two letters in subsets of $a^*b^*c^*$, but not all three letters at the same time. A rigorous result that corresponds to this intuitive limitation of ordinary grammars was proven by Ginsburg and Spanier [9, Theorem 2.1]. Theorem 5 is an analogue for GF(2)-grammars.

2. This argument only implies that any proof that relies solely on commutative images is going to fail. The real proof is more subtle. For example, it will also use the fact that $\{ a^n b^n c^n \mid n \geq 0 \}$ is a subset of $a^*b^*c^*$.

While the proof uses commutative images, it uses them very carefully, always making sure that the letters “appear in the correct order”. In particular, we will never consider GF(2)-concatenations $K \odot L$, where $K$ is a subset of $b^*$ and $L$ is an arbitrary subset of $a^*b^*c^*$, in the proof, because in this case $K \odot L$ is not a subset of $a^*b^*c^*$.

Avoiding this situation is impossible for language $\{ (abc)^n \mid n \geq 0 \}$, because in the string $abcabc$ from this language the letters “appear in the wrong order”.

Denote the set of algebraic power series in variable $c$ by $\mathcal{C}$. Similarly to Definition 4 define $R_{a,c} \subseteq \mathbb{F}_2((a, c))$ and $R_{b,c} \subseteq \mathbb{F}_2((b, c))$.

Finally, denote by $R_{a,b,c}$ the set of all Laurent series that can be represented as

\[
\sum_{i=1}^{n} A_i B_i C_i 
\]

\[
p_{a,b} \cdot p_{a,c} \cdot p_{b,c},
\]

where $n$ is a nonnegative integer, $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$, $C_i \in \mathcal{C}$ for all $i$ from 1 to $n$, and $p_{a,b} \in \mathbb{F}_2[a, b]$, $p_{a,c} \in \mathbb{F}_2[a, c]$, $p_{b,c} \in \mathbb{F}_2[b, c]$.
Lemma 3. \( R_{a,b,c} \) is a ring and a subset of \( \mathbb{F}_2((a,b,c)) \). Moreover, \( R_{a,b} \), \( R_{a,c} \) and \( R_{b,c} \) are subsets of \( R_{a,b,c} \).

Proof. It is easy to see that \( R_{a,b,c} \) is closed under addition and multiplication. Setting \( C_1 = C_2 = \ldots = C_n = p_{a,c} = p_{b,c} = 1 \) yields \( R_{a,b} \subset R_{a,b,c} \).

4.1 Main result

Unlike \( R_{a,b} \), \( R_{a,b,c} \) is not a field (in fact, Subsection [4.2] tells us that \( (1+abc)^{-1} \notin R_{a,b,c} \)), so a bit more involved argument will be necessary for the proof of the following theorem:

Theorem 5. Suppose that \( K \subset a^*b^*c^* \) is described by a GF(2)-grammar. Then the corresponding formal power series asSeries(K) is in the set \( R_{a,b,c} \).

Proof outline. The proof is mostly the same as the proof of Theorem 1. Let us focus on the differences. As before, we can assume that \( K \) does not contain the empty string.

In the same manner, we formally intersect our GF(2)-grammar in Chomsky’s normal form with the language \( a^*b^*c^* \). Now, all nonterminals \( C \) of the original GF(2)-grammar split into six nonterminals: \( C_{a\rightarrow a}, C_{a\rightarrow b}, C_{a\rightarrow c}, C_{b\rightarrow a}, C_{b\rightarrow b}, C_{b\rightarrow c}, C_{c\rightarrow c} \). However, their meanings stay the same. for example, \( L(C_{a\rightarrow b}) = L(C) \cap (a^*b^+) \) and \( L(C_{a\rightarrow c}) = L(C) \cap (a^*b^*c^+) \).

However, only the “central” nonterminals \( C_{a\rightarrow c} \) are important, similarly to the nonterminals of the type \( a \rightarrow b \) in the proof of Theorem 1. Why? Before, we had some a priori knowledge about the languages \( L(C_{a\rightarrow a}) \) and \( L(C_{b\rightarrow b}) \) from Christol’s theorem. But now, because of Theorem 1, we have a priori knowledge about the languages \( L(C_{a\rightarrow b}) \) and \( L(C_{b\rightarrow c}) \) as well, because they are subsets of \( a^*b^* \) and \( b^*c^* \) respectively.

Remark 3. In a sense, we used Theorem C as a stepping stone towards the proof of Theorem 1, and now we can use Theorem 7 as a stepping stone towards the proof of Theorem 5.

Denote by end(\( C \)) the language (\( \bigoplus_{(C \rightarrow DE) \in R} L(D_{a,b}) \odot L(E_{b,c}) \)) \( \oplus T_C \), where \( T_C \) is either \( \{c\} \) or \( \emptyset \), depending on whether or not there is a “final” rule \( C_{a\rightarrow c} \rightarrow c \) in the new GF(2)-grammar.

This means that we again can express the values asSeries(\( L(C_{a\rightarrow c}) \)) as a solution to a system of linear equations with relatively simple coefficients (denote asSeries(\( L(C_{a\rightarrow c}) \)) by Center(\( C \)), asSeries(\( L(C_{a\rightarrow a}) \)) by Left(\( C \)), asSeries(\( L(C_{b\rightarrow b}) \)) by Right(\( C \)) and asSeries(\( \text{end}(\( C \)) \)) by final(\( C \)):

\[
\text{Center}(\( C \)) = \text{final}(\( C \)) + \sum_{(C \rightarrow DE) \in R} \text{Left}(\( D \)) \text{Center}(\( E \)) + \text{Center}(\( D \)) \text{Right}(\( E \)) \quad (5)
\]

Here, the summation is over all rules \( C \rightarrow DE \) of the original GF(2)-grammar. Similarly to the proof of Theorem 1 this system can be rewritten as \( (A + B + I)x = f \), where \( x \) and \( f \) are column-vectors of Center(\( C \)) and final(\( C \)) respectively, while the matrices \( A \) and \( B \) are defined as follows:

\[
A_{C,E} := \sum_{C \rightarrow DE} \text{Left}(\( D \))
\]

\[
B_{C,E} := \sum_{C \rightarrow DE} \text{Right}(\( E \))
\]
Again, this system has a unique solution, because we can prove that \( \det(A + B + I) \neq 0 \). Moreover, said solution can be written down in the following way: \( x = (A + B + I)^{-1}f \). Recall that \( R_{a,c} \) is a field and all entries of \( A + B + I \) lie in \( R_{a,c} \). Hence, all entries of \( (A + B + I)^{-1} \) are elements of \( R_{a,c} \) as well. All entries of \( f \) are elements of \( R_{a,b,c} \). Therefore, all entries of \( x \) lie in \( R_{a,b,c} \) (here we use that \( R_{a,c} \) is a subring of \( R_{a,b,c} \)). In particular, \( \text{Center}(S) \in R_{a,b,c} \). Therefore, \( \text{asSeries}(K) = \text{asSeries}(L(S')) = \text{asSeries}(L(S_{a \rightarrow a})) + \text{asSeries}(L(S_{a \rightarrow b})) + \text{Center}(S) \) also lies in the set. \( \square \)

Consider the case of larger alphabets. Let \( A_i \) be the set of all algebraic formal power series in variable \( a_i \). Similarly to \( R_{a,b,c} \), denote by \( R_{a_1,a_2,...,a_k} \) the set of all Laurent series that can be represented as \( \sum_{i=1}^{n} A_{i,1} A_{i,2} \cdots A_{i,k} \prod_{1 \leq i < j \leq n} p_{i,j} \), for some \( n \geq 0 \), \( p_{i,j} \in \mathbb{F}_2(a_i, a_j) \) and \( A_{i,j} \in A_i \).

**Theorem 6.** If a language \( K \subset a_1^*a_2^* \ldots a_k^* \) is described by a GF(2)-grammar, then the corresponding power series \( \text{asSeries}(K) \) is in the set \( R_{a_1,a_2,...,a_k} \).

**Sketch of the proof.** Induction over \( k \), the induction step is analogous to the way we used Theorem 1 in the proof of Theorem 5. \( \square \)

4.2 The language \( \{ a^n b^n c^n \mid n \geq 0 \} \) and its relatives

In this subsection, we will use our recently obtained knowledge to prove that there is no GF(2)-grammar for the language \( \{ a^n b^n c^n \mid n \geq 0 \} \). It will almost immediately follow that the languages \( \{ a^n b^n c^n \mid n = m \text{ or } m = \ell \} \) and \( \{ a^n b^n c^n \mid n \neq m \text{ or } m \neq \ell \} \)

Consider the formal power series \( \text{asSeries}(\{ a^n b^n c^n \mid n \geq 0 \}) = \sum_{n=0}^{+\infty} a^n b^n c^n \). Denote these series by \( f \) for brevity. It is not hard to see that \( f = (1 + abc)^{-1} \). Indeed, \( f(1 + abc) = \sum_{n=0}^{+\infty} (a^n b^n c^n + a^{n+1} b^{n+1} c^{n+1}) = 1 \), because all summands except \( a^0 b^0 c^0 = 1 \) cancel out.

It sounds intuitive that \( (1 + abc)^{-1} \) “depends” on \( a \), \( b \) and \( c \) in a way that the \( R_{a,b,c} \) cannot capture: series in \( R_{a,b,c} \) should “split” nicely into functions that depend only on two variables out of three. Now, let us establish that \( f \not\in R_{a,b,c} \) formally.

Indeed, suppose that it is not true. In other words,

\[
 f = \frac{\sum_{i=1}^{n} A_i B_i C_i}{pq r}, \tag{6}
\]

where \( A_i \in \mathbb{A}, B_i \in \mathbb{B}, C_i \in \mathbb{C} \) for every \( i \) from 1 to \( n \) and, also, \( p \in \mathbb{F}_2[a, b], q \in \mathbb{F}_2[a, c] \) and \( r \in \mathbb{F}_2[b, c] \). Let us rewrite Equation (6) as \( pq rf = \sum_{i=1}^{n} A_i B_i C_i \) with an additional condition that neither of \( p, q \) and \( r \) is zero: otherwise the denominator of the right-hand side of Equation (6) is zero.

For every formal power series of three variables \( a, b \) and \( c \) we can define its *trace*: such subset of \( \mathbb{N}_0^3 \), that a triple \( (x, y, z) \) is in this subset if and only if the coefficient of the series before \( a^x b^y c^z \) is one. Traces of equal power series coincide.

How do the traces of left-hand and right-hand sides of equation \( pq rf = \sum_{i=1}^{n} A_i B_i C_i \) look like? Intuitively, the trace of the left-hand side should be near the diagonal \( x = y = z \) in its entirety; because \( pq rf \) is a polynomial \( pq r \), multiplied by \( f = \sum_{i=0}^{+\infty} a^i b^i c^i \). On the other hand, the trace of the right-hand side has a “block structure”: as we will establish later, it should be a finite union of disjoint sets with type \( X \times Y \times Z \)
Our goal is to prove that such traces can coincide only if they are both finite. This conclusion is quite natural: the trace of the left-hand side exhibits a “high dependency” between \(x, y\) and \(z\), while the coordinates “are almost independent” in the trace of the right-hand side (and they would be “fully independent” if there was only one set \(X \times Y \times Z\) in the disjoint union).

Let us proceed formally.

**Lemma 4.** The trace of the expression \(\sum_{i=1}^{n} A_i B_i C_i\) is a finite disjoint union of sets with type \(X \times Y \times Z\).

**Proof.** For \(x \in \mathbb{N}_0\), let us call the set of all such \(i\) from 1 to \(n\), that the coefficient of \(A_i\) before \(a^x\) is one, the \(a\)-type of \(x\). Similarly, define \(b\)-type and \(c\)-type.

Whether or not the triple \((x, y, z)\) is in the trace of \(\sum_{i=1}^{n} A_i B_i C_i\) depends only on the \(a\)-type of \(x\), \(b\)-type of \(y\) and \(c\)-type of \(z\). Indeed, the coefficient before \(a^x b^y c^z\) is one in exactly such summands \(A_i B_i C_i\), that the coefficient of \(A_i\) before \(a^x\) is one, the coefficient of \(B_i\) before \(b^y\) is one and the coefficient of \(C_i\) before \(c^z\) is one. Therefore the exact set of such summands depends only on types of \(x, y\) and \(z\).

Consequently, the trace of \(\sum_{i=1}^{n} A_i B_i C_i\) is a union of sets \(X \times Y \times Z\), where \(X\) is a set of numbers with some fixed \(a\)-type, \(Y\) is a set of numbers with some fixed \(b\)-type and \(Z\) is a set of numbers with some fixed \(c\)-type. There is only a finite number of such sets, because there is no more than \(2^n\) distinct \(a\)-types, no more than \(2^n\) distinct \(b\)-types and no more than \(2^n\) distinct \(c\)-types. \(\square\)

**Lemma 5.** There exists a such constant \(d\), that, for every triple \((x, y, z)\) from the trace of \(pqrf\), the conditions \(|x - y| \leq d, |x - z| \leq d\) and \(|y - z| \leq d\) hold.

**Proof.** Let \(d\) be the degree of \(pqrf\) as of a polynomial of three variables. Because \(pqrf = pqr \cdot \sum_{i=0}^{+\infty} a^i b^i c^i\), the trace of \(pqrf\) may only contain triples \((\ell + i, m + i, k + i)\) for monomials \(a^i b^i c^i\) from the polynomial \(pqrf\). For such triples, \(|x - y| = |\ell - m| \leq d\). Why? Because \(d\) is the total degree of \(pqrf\) and, therefore, \(0 \leq \ell \leq d\) and \(0 \leq m \leq d\). Similarly, \(|x - z| \leq d\) and \(|y - z| \leq d\). \(\square\)

**Lemma 6.** If the traces of \(\sum_{i=1}^{n} A_i B_i C_i\) and \(pqrf\) coincide, then they both are finite sets.

**Proof.** From Lemmata 4 and 5 a set that is close to the diagonal coincides with a disjoint union of sets of a type \(X \times Y \times Z\). Then, each of the sets \(X \times Y \times Z\) in the union is finite. Roughly speaking, infinite sets of such type should contain elements that are arbitrarily far from the diagonal \(x = y = z\).

Let us explain the previous paragraph more formally. Indeed, suppose that one of the \(X \times Y \times Z\) sets from the conclusion of Lemma 4 is infinite. Then, at least one of the sets \(X, Y\) and \(Z\) is infinite. Without loss of generality, \(X\) is infinite. Let \((x, y, z)\) be some element of \(X \times Y \times Z\): it exists, because every infinite set contains at least one element. Choose \(x_{\text{new}}\) so \(x_{\text{new}} > \max(y, z) + d\). Such \(x_{\text{new}}\) exists, because \(X\) is an infinite set of nonnegative integers. Then, \((x_{\text{new}}, y, z)\) \(\in X \times Y \times Z\). Therefore, \((x_{\text{new}}, y, z)\) is in the trace of \(\sum_{i=1}^{n} A_i B_i C_i\). However, by Lemma 5 \((x_{\text{new}}, y, z)\) cannot lie in the trace of \(pqrf\), because \(x_{\text{new}}\) differs from \(y\) and \(z\) too much. \(\square\)

**Lemma 7.** The polynomial \(1 + abc\) is irreducible as a polynomial over field \(\mathbb{F}_2\).
Proof. Proof by contradiction: suppose that $1 + abc$ is reducible. Because its total degree is 3, it should split into a product of two polynomials with the total degrees 1 and 2 respectively. In principle, enumerating all pairs of polynomials over $\mathbb{F}_2$ of total degree 1 and 2 on the computer does the job. For completeness, I will provide a proof that does not use computer search.

Because the degree of $1 + abc$ with respect to each variable is 1, each variable occurs in exactly one of two factors — if it occurs in both, the resulting degree is at least 2, if it occurs in neither, the resulting degree is 0. Hence, because the total degree of the second factor is 2, but its degree in every variable is only 1, exactly 2 variables occur in the second factor.

Therefore, only one variable occurs in the first factor. Because the polynomial $1 + abc$ is symmetric with respect to the permutation of variables, we may assume that the first factor depends only on $a$ and the second factor depends only on $b$ and $c$.

The first factor is invertible, because the product is invertible. Previously, we have shown that the first factor is a polynomial of degree 1, therefore the only one possibility for the first factor remains: $1 + a$. The second factor is also invertible and is of degree 2, therefore it is $1 + k_b b + k_c c + bc$ for some $k_b$ and $k_c$ from $\mathbb{F}_2$. Then, there is a summand $a \cdot 1 = a$ in their product, which does not have anything to cancel up with. But their product is $1 + abc$, contradiction.

Because $pqr f = \sum_{i=1}^n A_i B_i C_i$, the trace of $pqr f$ is finite. In other words, $pqr f$ is a polynomial. Recall that $f = (1 + abc)^{-1}$, so $\frac{pqr}{1 + abc}$ is a polynomial. Because the product of three polynomials $p$, $q$ and $r$ is divisible by an irreducible polynomial $1 + abc$, one of them is also divisible by $1 + abc$ (here we have used the fact that the ring $\mathbb{F}_2[a, b, c]$ of polynomials in three variables is a unique factorization domain). But this is impossible, because each of the polynomials $p$, $q$ and $r$ is non-zero (here we have finally used that condition from the statement of the lemma) and does not depend on one of the variables.

Finally, we have established the following theorem.

**Theorem 7.** The language $\{ a^n b^m c^n \mid n \geq 0 \}$ is not described by a GF(2)-grammar.

**Corollary 1.** The language $\{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell \}$ is not described by a GF(2)-grammar.

**Proof.** Suppose that $\{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell \}$ is described by a GF(2)-grammar. Then, $\{ a^n b^n c^n \mid n \geq 0 \}$ also is, as the symmetric difference of $\{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell \}$ and $\{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell$, but not both $\}$, where the latter is described by a GF(2)-grammar [14, Example 2]. Contradiction.

**Corollary 2.** The language $\{ a^n b^m c^\ell \mid n \neq m \text{ or } m \neq \ell \}$ is not described by a GF(2)-grammar.

**Proof.** Otherwise, $\{ a^n b^n c^n \mid n \geq 0 \} = (a^* b^* c^*) \triangle (a^n b^m c^\ell \mid n \neq m \text{ or } m \neq \ell)$ would be described by a GF(2)-grammar as well.

We have just proven that the language $\{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell \}$ is not described by a GF(2)-grammar. Hence, it is inherently ambiguous. Previous proofs of its inherent ambiguity were purely combinatorial, mainly based on Ogden’s lemma, while our approach is mostly algebraic.

More importantly, we have proven that the language $\{ a^n b^m c^\ell \mid n \neq m \text{ or } m \neq \ell \}$ is not described by a GF(2)-grammar. Therefore, it is inherently ambiguous. The inherent ambiguity of this language was a long-standing open question [2, p. 375].
4.3 Other applications

There are many other applications to our techniques as well. For example, consider the famous paper by Hibbard and Ullian about inherently ambiguous languages and languages that have a complement that can be recognized by an ordinary grammar [8]. Since the publication of that paper, a few improvements have been made. For example, Maurer [16] sharpened the statement of one of their theorems. Also, recently Martynova and Okhotin [15] found a language that is described by an unambiguous linear grammar, but has a complement that cannot be described by any ordinary grammar at all. I will suggest another potential improvement.

A significant part of Hibbard’s and Ullian’s paper is dedicated to proving that the language \( K := \{ a^p b^r c^s d^t e^f \mid (p = q \text{ and } r = s) \text{ or } (q = r \text{ and } s = t) \} \) is inherently ambiguous.

With our techniques, the proof is simple. Indeed, \( K \) is the symmetric difference of three languages \( \{ a^p b^q c^r d^s e^t \mid p = q \text{ and } r = s \} = \{ a^n b^m c^n d^m e^n \mid n, m, \ell \geq 0 \} \), \( \{ a^p b^q c^r d^s e^t \mid q = r \text{ and } s = t \} = \{ a^n b^m c^n d^m e^n \mid n, m, \ell \geq 0 \} \) and \( \{ a^p b^p c^s d^t e^f \mid p = q = r = s = t \} = \{ a^n b^n c^n d^n e^n \mid n \geq 0 \} \). The first two of these three languages are described by unambiguous grammars. Therefore, if there exists an unambiguous grammar for \( K \), then there exists a GF(2)-grammar for \( \{ a^n b^n c^n d^n e^n \mid n \geq 0 \} \). This is impossible by Theorem [7] because the language \( \{ a^n b^n c^n d^n e^n \mid n \geq 0 \} \) is “even harder” than \( \{ a^n b^n c^n \mid n \geq 0 \} \).

Intuitively, this is clear enough. Proving it formally with our current techniques can be a bit tricky. It is possible to prove this statement in an “ad-hoc” way, but let us use this opportunity to prove some useful closure results (they are easy to prove, but have not been formally proved yet anywhere, because they take quite a lot of space to write down).

**Definition 5.** For a language \( K \) over an alphabet \( \Sigma \) and a homomorphism \( h: \Sigma \rightarrow \Omega^* \) we can define the GF(2)-homomorphic image \( h_\oplus(K) \) of \( K \) under \( h \) in the following way. A string \( w \in \Sigma^* \) belongs to \( h_\oplus(K) \) if and only if the set \( h^{-1}(w) \cap K \) has an odd size. If this set is infinite for any \( w \in \Sigma^* \), then \( h_\oplus(K) \) is ill-defined.

**Lemma 8.** Assume that a GF(2)-grammar \( G \) describes a language \( K \) over an alphabet \( \Sigma \) and \( h: \Sigma \rightarrow \Omega^* \) is a homomorphism. Then, there exists a GF(2)-grammar \( G_h \) that describes the language \( h_\oplus(K) \), as long as the following technical condition holds: for every \( w \in \Omega^* \), the total number of parse trees in \( G \) for strings from \( h^{-1}(w) \) is finite (this is a non-trivial condition if the set \( h^{-1}(w) \) is infinite). In particular, \( h_\oplus(K) \) is well-defined in that case.

**Proof.** Informally, \( G_h \) is constructed by applying \( h \) to every rule. Formally speaking, \( G_h \) has the same set of nonterminals, the terminal alphabet \( \Omega \) and, for every rule \( A \rightarrow w_0 \circ X_1 \circ w_1 \circ X_2 \circ \cdots \circ w_{t-1} \circ X_t w_t \) in the original GF(2)-grammar \( G \), a corresponding rule \( A \rightarrow h(w_0) \circ X_1 \circ h(w_1) \circ X_2 \circ \cdots \circ h(w_{t-1}) \circ X_t \circ w_t \) and no other rules (here, \( w_i \) are strings over \( \Sigma \) and \( X_i \) are nonterminals of \( G \)).

Then, for every parse tree in \( G \) for a string \( w \in \Sigma^* \) we can construct a corresponding parse tree in \( G_h \) for string \( h(w) \) in the following way: when we apply some rule in the original parse tree, apply the corresponding rule in the corresponding parse tree. Moreover, the process is reversible. Therefore, for a parse tree in \( G_h \) for \( w \in \Omega^* \) corresponds to a parse tree in \( G \) for some string from \( h^{-1}(w) \). The exact string is uniquely determined by the sequence of used rules.
Hence, the number of parse trees in $G_h$ for a string $w \in \Omega^*$ is exactly the total number of parse trees in $G$ for strings from $h^{-1}(w)$. In particular, $G_h$ is a valid GF(2)-grammar only if this number is always finite.

If this number is finite, then there is only a finite number of strings in $K \cap h^{-1}(w)$, because each one has at least one parse tree (by definition, strings in $K$ have an odd number of parse trees in $G$, and every odd number is at least 1). Hence, $h \oplus (K)$ is well-defined. Finally, each string from $h^{-1}(w) \setminus K$ “contributes” an even number of parse trees in $G_h$ to $w$ and each string from $h^{-1}(w) \cap K$ “contributes” an odd number of parse trees in $G_h$ to $w$. Hence, $w$ is in $L(G_h)$ if and only if $h^{-1}(w) \cap K$ has odd size. Therefore, $L(G_h)$ is exactly the language $h \oplus (K)$. □

**Remark 4.** Lemma 8 is somewhat unconventional (and can be tricky to use) in the following way: usually we prove closure results for the languages themselves without any assumptions about the underlying grammar. In Lemma 8 on the other hand, we need to know some properties of the original GF(2)-grammar. Let us prove a weaker, but easier to use corollary that concerns the languages themselves.

**Lemma 9.** If a language $K$ over an alphabet $\Sigma$ is described by a GF(2)-grammar and $h: \Sigma \to \Omega^*$ is a non-erasing homomorphism, then $h \oplus (K)$ is described by a GF(2)-grammar as well.

**Proof.** Because $h$ is non-erasing, each string $w \in \Omega^*$ has only a finite number of preimages under $h$. Each of those preimages has a finite number of parse trees by the definition of a GF(2)-grammar. Hence, the technical condition from Lemma 8 is satisfied. □

**Corollary 3.** The set $M := \{a^n b^n c^n d^n e^n \mid n \geq 0\}$ is not described by a GF(2)-grammar.

**Proof.** Consider a homomorphism $h$, defined by $h(a) = aa, h(b) = b, h(c) = b, h(d) = c, h(e) = c$. It is a non-erasing homomorphism. Therefore, we can apply Lemma 9 and deduce that $h \oplus (M) = \{a^{2n} b^{2n} c^{2n} \mid n \geq 0\}$ is described by a GF(2)-grammar ($h \oplus (M)$ looks like this, because $h$ is injective on $M$). Now, let us apply an deterministic finite transducer that halves the lengths of all blocks of consequent equal letters and does not return anything at all if there are any such blocks of odd length. Any DFT is an injective NFT. Therefore, by closure under injective NFTs [14, Theorem 17], there is a GF(2)-grammar for $\{a^n b^n c^n \mid n \geq 0\}$. Contradiction. □

**Remark 5.** Lemma 9 generalizes the “method of unary image” from the original paper about expressive powers of GF(2)-grammars [14, Theorem 8]. It is possible to generalize even further, by defining GF(2)-transducers and proving the equivalents of Lemmata 8 and 9 for them, but we will avoid that for now, because both the statement and the proof are quite technical.

## 5 Are converse statements true?

It would be interesting to know whether the converse statements to Theorems 1, 5 and 6 are true. Of course, the exact converse statement to Theorem 1 is wrong on the technicality that not all elements of $R_{a,b}$ are power series. However, if we restrict ourselves to $R_{a,b} \cap \mathbb{F}_2[[a,b]]$, we get the following conjecture, which I believe in:

**Conjecture 1.** If $f \in R_{a,b} \cap \mathbb{F}_2[[a,b]]$, then the language $\text{asSeries}^{-1}(f) \subset a^*b^*$ can be described by a GF(2)-grammar.
The following Theorem 8 is an evidence in favor of Conjecture 1. We will need the following definition to state Theorem 8.

**Definition 6.** Denote by \( R_{a,b}^{int} \) the set of Laurent series from \( \mathbb{F}_2((a,b)) \) that can be represented as \( \sum_{i=1}^{n} A_i B_i p_i \), where \( n \geq 0 \), \( A_i \in \mathcal{A} \) and \( B_i \in \mathcal{B} \) for \( i \) in range from 1 to \( n \), and \( p \in \text{poly}(a,b) \) is a polynomial with constant term equal to 1.

**Remark 6.** The only difference between the definitions of \( R_{a,b} \) and \( R_{a,b}^{int} \) is that the denominator \( p \) of the fraction is required to be invertible as an element of \( \mathbb{F}_2[[a,b]] \supset \text{poly}(a,b) \). In particular, \( R_{a,b}^{int} \) is a subset of \( \mathbb{F}_2[[a,b]] \). Also, by definition, \( R_{a,b}^{int} \subset R_{a,b} \).

**Theorem 8.** If \( f \in R_{a,b}^{int} \), then \( \text{asSeries}^{-1}(f) \) can be described by a GF(2)-grammar.

**Sketch of the proof.** Suppose that \( f = (\sum_{i=1}^{n} A_i B_i) / p \), as in Definition 6 and \( p = 1 + \sum_{j=1}^{d} a^{k_j} b^{\ell_j} \), where \( k_j + \ell_j > 0 \). Let \( S \) be the starting symbol of some GF(2)-grammar that describes \( \text{asSeries}^{-1}(\sum_{i=1}^{n} A_i B_i) \) (such GF(2)-grammar exists by closure properties). Add a new starting symbol \( S_{\text{new}} \) and a new rule \( S_{\text{new}} \rightarrow (\oplus_{j=1}^{d} a^{k_j} S_{\text{new}} b^{\ell_j}) \oplus S \) to the GF(2)-grammar. The new GF(2)-grammar describes \( \text{asSeries}^{-1}(f) \). The condition \( k_j + \ell_j > 0 \) is important, because it assures that all strings still have only a finite number of parse trees.

Hence, the only thing that stops us from proving Conjecture 1 is the following conjecture, which I, personally, believe in:

**Conjecture 2.** \( R_{a,b}^{int} = R_{a,b} \cap F[[a,b]] \).

This is a purely algebraic statement. Moreover, it seems like either false statement, or one that should easily follow from well-known algebra. However, I could neither find an independent proof of Conjecture 2 nor derive it from some more general algebra, nor disprove it. Maybe the readers will be more successful?

As we have seen above, the converse to Theorem 1 is “almost correct”. On the other hand, the converse to Theorem 5 is clearly very far from being true, because it fails to take “overlapping requirements” into account. Indeed, the language \( \{ a^n b^m c^n d^m \mid n, m \geq 0 \} \) is most likely not described by a GF(2)-grammar, but does not contradict Theorem 5 because \( \{ a^n b^m c^n d^m \mid n, m \geq 0 \} = \frac{1}{(1 + ab)(1 + cd)} \).

### 6 Conclusion

Let us make some concluding remarks and discuss some possible future developments.

Firstly, note that it took us roughly the same effort to prove the inherent ambiguity of \( \{ a^n b^m c^\ell \mid n = m \text{ or } m = \ell \} \) and \( \{ a^n b^m c^\ell \mid n \neq m \text{ or } m \neq \ell \} \), despite the former being a textbook example of inherently ambiguous language and the latter not being known to be inherently ambiguous before. Intuitively, it is very difficult to capture weak conditions like inequality using Ogden’s lemma, while our approach can replace inequality with a strong condition (equality) by taking the complement.

Secondly, the proofs of Theorems 1 and 5 start similarly to the reasoning Ginsburg and Spanier used to characterize bounded languages described by ordinary grammars [9, 10], but diverge after taking some steps. This is not surprising; ordinary grammars have good monotonicity properties (a string needs only one parse tree to be in the
language), but bad algebraic properties (solving systems of language equations is much harder than solving systems of linear equations). In GF(2)-grammars, it is the other way around: there are no good monotonicity properties, but algebraic properties are quite remarkable.

**Most recent related work**  Since the publication of the conference version of this paper [13], Koechlin [12] presented a different proof of the inherent ambiguity of the language $L_2 = \{ a^nb^mc^\ell \mid n \neq m \text{ or } m \neq \ell \}$ by giving an algebraic characterization of bounded languages described by unambiguous grammars directly, without going through GF(2)-grammars first. Essentially, he directly transforms Ginsburg’s and Ullian’s result [11, Theorems 5.1 and 6.1] about the structure of bounded languages described by an unambiguous grammar into a very simple algebraic description of the underlying semilinear set. Therefore, his method avoids all the issues mentioned in the introduction (the “density” of $L_2$ and the fact that $L_2$ has an algebraic generating function). I recommend giving it a read, if you liked my paper. I think that it is very insightful and enlightening.

**Future research**  Perhaps, our methods could be used to make some progress on the equivalence problem for unambiguous grammars. Indeed, the equivalence problem for unambiguous grammars is closely related to the emptiness problem for GF(2)-grammars. If it is decidable, whether GF(2)-grammar describes an empty language or not, then the equivalence of unambiguous grammars is decidable as well. If it is not, the proof will most probably shed some light on the case of unambiguous grammars anyway. However, resolving the emptiness problem for GF(2)-grammars in one way or another still seems to be out of reach.

Understanding how our methods relate to the analytic methods of Flajolet [6], is another interesting question. One can see Theorem C as an alternative formulation of Christol’s theorem [7] for $F_2$ specifically, that involves GF(2)-grammars instead of 2-automatic sequences on the “combinatorial side”. Then, Christol’s theorem can be seen as a finite field analogue of Chomsky-Schutzenberger enumeration theorem, because both relate counting properties of different grammar families to algebraic power series over fields by “remembering” only the length of the string, but nothing else:

**Theorem D** (Chomsky-Schutzenberger enumeration theorem [5]). If $L$ is a language described by an unambiguous grammar, and $a_k$ is the number of strings of length $k$ in $L$, then the power series $\sum_{k=0}^{\infty} a_k x^k$ is algebraic over $\mathbb{Q}[x]$.

**Theorem E** (Christol’s theorem for $F_2$ [7, 14]; usually not stated this way). If $L$ is a language described by a GF(2)-grammar, and $a_k$ is the number of strings of length $k$ in $L$, then the power series $\sum_{k=0}^{\infty} (a_k \mod 2) \cdot x^k$ is algebraic over $F_2[x]$.

**Remark 7.** Technically speaking, Theorem E is much weaker than Christol’s theorem, because it is stated in only one direction (and the easier one to boot). I wanted to highlight the similarity between Theorems D and E, so I intentionally avoided stating the converse implication in Theorem E.

This similarity gives us some hope that our methods can be at least partially transferred to the analytic setting. Moreover, a lot (though not all) of the arguments used in our work can be modified to work over an arbitrary field.
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Appendices

A What is this part of the paper about?

In Sections A–C I present a longer, but a more elementary way to prove Theorems 9 and 10 through the use of algebraic expressions. In the end, it is no surprise that we will end up essentially reproving many of the algebraic statements I use in the main body of the paper, but in our particular special case.

So, was it all in vain? I believe that the answer is “No”. In a sense, this part of the paper “demystifies” its main body, because it more closely follows my original pattern of thinking. If you think that the new argument is too “magical”, then, probably, this part of the paper is for you. What follows is, at least in my opinion, a very intuitive line of thinking. And in the end, we will be just one simple, but brilliant observation away from the much simpler final version of the argument.

The credit for this observation goes to an anonymous reviewer from MFCS 2020 conference. In short, and very paraphrased, they said “You are essentially proving that \( \sum AB \) poly(\( a, b \)) describes a field, aren’t you? You can prove this in two sentences by using standard results about field extensions”. When pointed out, it sounds very simple, but, funny enough, I have never considered thinking about the equivalence between algebraic expressions \( \sum AB \) poly(\( a, b \)) and \( \sum AB \) in such a way before!

So, in the end, you can see the following part of the paper as both an explanation for more “magical” part of the final argument, and as a story with a “plot twist” that almost everything I did was unnecessary. Personally, I think that knowing the twist only makes it more interesting to observe.

On the other hand, this part of the paper has absolutely no new results that are not proved in the main body of the text. Hence, if you neither find abstract algebraic arguments too “magical” nor are interested in the “history” of the main results, you may safely skip it.

B Subsets of \( a^*b^* \) through algebraic expressions

B.1 Algebraic expressions

Let us define the meaning of words “Laurent series \( f \) match algebraic expression \( F \)”.

Informally, algebraic expressions are some formulas of symbols \( A, B, \) poly(\( a, b \)) and rat(\( a, b \)) that use additions, multiplications, divisions and “finite summation” operator, denoted by \( \sum \). Here, poly(\( a, b \)) denotes the set \( \mathbb{F}_2[\{a, b\}] \) of polynomials in variables \( a \) and \( b \). Similarly, rat(\( a, b \)) denotes the set \( \mathbb{F}_2(\{a, b\}) \) of rational functions in variables \( a \) and \( b \).

Several examples of algebraic expressions: rat(\( a, b \)), \( B, \sum A, \sum AB, \sum poly(\{a, b\}), \sum AB \) rat(\( a, b \)), \( \sum AB \) rat(\( a, b \)).

Definition 7. Laurent series \( f \) match algebraic expression \( F \) if and only if \( f \) can be obtained from \( F \) by substituting elements of poly(\( a, b \)), rat(\( a, b \)), \( A \) and \( B \) for the corresponding symbols (not necessarily the same elements for the same symbols). The
construct \( \sum G \) corresponds to a finite, possibly empty, sum of Laurent series, with every summand matching \( G \).

**Example 4.** The set of Laurent series matching \( \sum AB \) is exactly the set of all power series representable as \( A_1B_1 + \ldots + A_nB_n \), where \( n \) is any nonnegative integer, and \( A_i \in \mathcal{A}, B_i \in \mathcal{B} \) for every \( i \) from 1 to \( n \) inclusive.

**Example 5.** All rational functions in variables \( a \) and \( b \), and only them, match algebraic expressions \( \frac{\text{rat}(a,b)}{\text{poly}(a,b)} \).

**Example 6.** Laurent series match \( \sum \frac{AB}{\text{rat}(a,b)} \) if and only if it can be represented as a finite sum, where each summand can be represented as \( a^{1+n}b^{2m-10} + a^{2n+10}b^{2m+15} \), for some non-negative integer \( n \) and some \( A_1, \ldots, A_n \in \mathcal{A}, B_1, \ldots, B_n \in \mathcal{B}, p \in \text{rat}(a,b) \), with an additional condition \( p \neq 0 \). The last condition is necessary because otherwise the denominator would be equal to zero and the fraction would not make sense.

**Example 7.** Laurent series \( f := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a^{2n}b^{2m-10} + a^{2n+10}b^{2m+15}) \) match \( \sum \frac{AB}{\text{poly}(a,b)} \), because \( f = \frac{A_1B_1 + A_2B_2}{b^{10}} \), where \( b^{10} \in \text{poly}(a,b) \), \( A_1 = \sum_{n=0}^{+\infty} a^{2n} \in \mathcal{A} \), \( B_1 = \sum_{m=0}^{+\infty} b^{2m} \in \mathcal{B} \), \( A_2 = \sum_{n=0}^{+\infty} a^{2n+10} \in \mathcal{A} \), \( B_2 = \sum_{m=0}^{+\infty} b^{2m+25} \in \mathcal{B} \).

**Definition 8.** Algebraic expressions \( F \) and \( G \) are equivalent (denoted by \( F = G \)), if they define the same subset of the whole field \( \mathbb{F}_2((a,b)) \) of Laurent series in variables \( a \) and \( b \).

Some equivalencies follow directly from definitions and properties of classes \( \text{poly}(a,b), \text{rat}(a,b), \mathcal{A} \) and \( \mathcal{B} \). For example, \( \sum \sum \frac{AB}{\text{rat}(a,b)} = \sum \frac{AB}{\text{poly}(a,b)} \), aforementioned \( \text{rat}(a,b) = \frac{\text{poly}(a,b)}{\text{poly}(a,b)}, \sum \mathcal{A} \mathcal{A} = \mathcal{A} \) and \( \sum \frac{AB}{\text{poly}(a,b)} \).

On the other hand, some equivalences are not so trivial, like the equivalence \( \frac{\sum \frac{AB}{\text{poly}(a,b)}}{\sum \frac{AB}{\text{poly}(a,b)}} = \frac{\sum \frac{AB}{\text{poly}(a,b)}}{\sum \frac{AB}{\text{poly}(a,b)}} \), which I shall establish later.

### B.2 Switching to the algebraic track

The purpose of this section is to prove the following intermediate result:

**Lemma 10.** Assume that language \( K \subset a^*b^* \) is described by a GF(2)-grammar. Then, the corresponding power series \( \text{asSeries}(K) \) match \( \sum \frac{AB}{\sum AB} \).

**Proof.** Without loss of generality, the GF(2)-grammar that describes \( K \) is in the Chomsky normal form \([3 \text{ Theorem 5}] \). Moreover, let us assume that \( K \) does not contain the empty string.

The language \( a^*b^* \) is recognized by the following incomplete deterministic finite automaton \( M \): \( M \) has two states \( q_a \) and \( q_b \), both accepting, and its transition function is \( \delta(q_a, a) = q_a, \delta(q_a, b) = q_b, \delta(q_b, b) = q_b \).

Let us formally intersect the GF(2)-grammar \( G \) with a regular language \( a^*b^* \), recognized by the automaton \( M \), using the construction of Bar-Hillel et al. \([4 \text{ (the construction of the intersection of an ordinary grammar with a regular expression by Bar-Hillel} \)](\text{[4 \text{ (the construction of the intersection of an ordinary grammar with a regular expression by Bar-Hillel}])} \).
et al. [4] can be easily adapted to the case of GF(2)-grammars [14, Section 6]). The language described by the GF(2)-grammar will not change, because it was already a subset of \(a^*b^*\).

The grammar will change considerably, however. Every nonterminal \(C\) of the original grammar splits into three nonterminals: \(C_{a\to a}, C_{a\to b}, C_{b\to b}\). Also a new starting nonterminal \(S'\) appears.

Every “normal” rule \(C \to DE\) splits into four rules: \(C_{a\to a} \to D_{a\to a}E_{a\to a}\), \(C_{a\to b} \to D_{a\to a}E_{a\to b}\), \(C_{b\to b} \to D_{b\to b}E_{b\to b}\).

The following happens with “final” rules: \(C \to b\) turns into two rules \(C_{a\to b} \to b\) and \(C_{b\to b} \to b\), and \(C \to a\) turns into one rule \(C_{a\to a} \to a\). Finally, two more rules appear: \(S' \to S_{a\to a}\) and \(S' \to S_{a\to b}\).

What do the nonterminals of the new GF(2)-grammar correspond to? The state \(C_{a\to a}\) corresponds to the strings \(w\in \{a, b\}^*\) that are derived from the nonterminal \(C\) of the original GF(2)-grammar and make \(M\) go from the state \(q_a\) to itself. Formally speaking, \(w\in L(C_{a\to a})\) if and only if \(w\in L(C)\) and \(\delta(q_a, w) = q_a\). Similarly, \(w\in L(C_{a\to b})\) if and only if \(w\in L(C)\) and \(\delta(q_a, w) = q_b\). Finally, \(w\in L(C_{b\to b})\) if and only if \(w\in L(C)\) and \(\delta(q_b, w) = q_b\).

By looking more closely on the transitions of \(M\), we can see that \(\delta(q_a, w) = q_a\) if and only if \(w\) consists only of letters \(a\), in other words, if and only if \(w\in a^*\). Similarly, \(\delta(q_b, w) = q_b\) if and only if \(w\in b^*\).

Every language \(L(C_{a\to a})\) is a 2-automatic language over a unary alphabet \(\{a\}\). Indeed, every parse tree of \(C_{a\to a}\) contains only nonterminals of type \(a\to a\). Therefore, only character \(a\) can occur as a terminal in a parse tree of \(C_{a\to a}\). So, \(L(C_{a\to a})\) is described by some GF(2)-grammar over an alphabet \(\{a\}\), and is therefore 2-automatic. Similarly, all languages \(L(C_{b\to b})\) are 2-automatic over the alphabet \(\{b\}\). Then, by Christol’s theorem, \(\text{asSeries}(L(C_{a\to a}))\in \mathcal{A}\) and \(\text{asSeries}(L(C_{b\to b}))\in \mathcal{B}\).

How do the languages \(L(C_{a\to b})\) look like? Let us look at the rules \(C_{a\to b} \to D_{a\to a}E_{a\to b}\) and \(C_{a\to b} \to D_{a\to b}E_{b\to b}\). These rules can be interpreted in the following way: when starting a parse from nonterminal \(C_{a\to b}\), we can append a language from \(\mathcal{A}\) from the left and go to \(E_{a\to b}\) or append a language from \(\mathcal{B}\) from the right and go to \(D_{a\to b}\).

What can we say about \(K\)? By definition, \(K = L(S) = L(S') = L(S_{a\to a})\Delta L(S_{a\to b})\). We can forget about the language \(L(S_{a\to a})\): it is from the class \(\mathcal{A}\), and \(L(S_{a\to b})\) is from much more complicated class, that will “absorb” \(\mathcal{A}\) in the end.

The languages \(L(C_{a\to b})\) for each nonterminal \(C_{a\to b}\) of the new grammar satisfy the following system of language equations:

\[
L(C_{a\to b}) = \text{end}(C_{a\to b})\Delta \bigcup_{C\to DE} (L(D_{a\to a}) \circ L(E_{a\to b}))\Delta L(D_{a\to b}) \circ L(E_{b\to b})
\]

Here, the summation happens over all rules \(C \to DE\) for each nonterminal \(C\) of the original grammar, and \(\text{end}(C_{a\to b})\) is either \(\{b\}\) or \(\emptyset\), depending on whether or not there is a rule \(C_{a\to b} \to b\) in the new grammar.

Look more closely at the system (7). In all GF(2)-concatenations that appear in its right-hand side either the first language is a subset of \(a^*\), or the second language is a subset of \(b^*\). Hence, we can apply the Lemma 1.

Denote \(\text{asSeries}(L(C_{a\to b}))\) by \(\text{Center}(C)\), \(\text{asSeries}(L(C_{a\to a}))\) by \(\text{Left}(C)\), \(\text{asSeries}(L(C_{b\to b}))\) by \(\text{Right}(C)\) and \(\text{asSeries}(\text{end}(C_{a\to b}))\) by \(\text{final}(C)\) for brevity.

Applying \(\text{asSeries}\) to the both sides of (7) gives us the following system of equations over formal power series:

\[
\text{asSeries}(L(C_{a\to b})) = \text{Center}(C) + \text{asSeries}(L(C_{a\to a}))\text{Left}(C) + \text{asSeries}(L(C_{b\to b}))\text{Right}(C) + \text{asSeries}(\text{end}(C_{a\to b}))\text{final}(C).
\]
Center(\(C\)) = final(\(C\)) + \(\sum_{C \rightarrow DE} \) Left(\(D\)) Center(\(E\)) + Center(\(D\)) Right(\(E\)) \hspace{1cm} (8)

Let us look at this system as a system of \(\mathbb{F}_2[[a, b]]\)-linear equations over variables Center(\(C\)) = asSeries(L(\(C_{a \rightarrow b}\))) for every nonterminal \(C\) of the original GF(2)-grammar.

We will consider final(\(C\)), Left(\(C\)) and Right(\(C\)) to be the coefficients of the system. While we do not know their exact values, the following is known: final(\(C\)) is 0 or \(b\), Left(\(C\)) \(\in A\) as a formal power series that corresponds to a 2-automatic language over an alphabet \(\{a\}\) and, similarly, Right(\(C\)) \(\in B\).

Denote the number of nonterminals in the original GF(2)-grammar by \(n\), (so there are \(n\) nonterminals of type \(a \rightarrow b\) in the new GF(2)-grammar), a column vector of values Center(\(C\)) by \(x\) and a column vector of values final(\(C\)) in the same order by \(f\).

Let us fix the numeration of nonterminals \(C\) of the old GF(2)-grammar. After that, we can use them as the “indices” of rows and columns of matrices.

Let \(I\) be an identity matrix of dimension \(n \times n\), \(A\) be a \(n \times n\) matrix with the sum of Left(\(D\)) over all rules \(C \rightarrow DE\) of the original grammar standing on the intersection of \(C\)-th row and \(E\)-th column:

\[ A_{C,E} := \sum_{C \rightarrow DE} \text{Left}(D) \hspace{1cm} (9) \]

Similarly, let \(B\) be a \(n \times n\) matrix with \(B_{C,D} := \sum_{C \rightarrow DE} \text{Right}(E) \hspace{1cm} (10) \)

Then the equation system (8) can be rewritten as \(x = f + (A + B)x\) in the matrix form. In other words, \((A + B + I)x = f\).

We have already proven earlier that Center(\(C\)) is a solution of this system. Our plan is to prove that there is exactly one solution to this system and express it in some form. Then, in particular, we will find some expression for asSeries(L(\(S_{a \rightarrow b}\))) = Center(\(S\)).

This system has exactly one solution if and only if \(\det(A + B + I) \neq 0\). If \(\det(A + B + I) \neq 0\), then, by Cramer’s formula, every entry of the solution, including Center(\(S\)) can be written as

\[
\frac{\det(A + B + I, \text{but with one of the columns replaced by } f)}{\det(A + B + I)}
\]

It remains to establish three things: that \(\det(A + B + I)\) matches \(\sum AB\), that \(\det(A + B + I, \text{but with one of the columns replaced by } f)\) matches \(\sum AB\), independently of the replaced column and that \(\det(A + B + I) \neq 0\).

Let us prove the first two statements at the same time. Every entry of \(A + B + I\) matches \(A + B\) because of the equations (9)–(10). Indeed, every entry of \(A\) matches \(A = A_{C,E}\), every entry of \(B\) matches \(B\), and entries of \(I\) are ones and zeroes that lie in both \(A\) and \(B\). This property will not disappear, if you replace every column of the matrix by \(f\): all entries of \(f\) are equal to 0 or \(b\), so they match \(B\), let alone \(A + B\).

Now, let us prove that the determinant of the matrix with every entry matching \(A + B\) matches \(\sum AB\). Indeed, by expressing the determinant through the explicit formula with \(n!\) summands, we get that the determinant matches

\[
\sum_{n \text{ times}} (A + B) \cdot \ldots \cdot (A + B).
\]
By expanding the brackets and using the fact that $\mathcal{A}\mathcal{A} = \mathcal{A}$ and $\mathcal{B}\mathcal{B} = \mathcal{B}$, we see that the determinant match $\sum \mathcal{A}\mathcal{B}$.

It remains to prove that $\det(A + B + I) \neq 0$. Let us prove a stronger statement: that the power series $\det(A + B + I) \in \mathbb{F}_2[[a, b]]$ is invertible, that is, its coefficient at $a^0b^0$ is equal to 1.

Notice that finite product of power series is invertible if and only if each factor is invertible. Also a finite sum of power series with exactly one invertible summand is invertible.

Because the new GF(2)-grammar is also in Chomsky’s normal form, all languages $L(C_{a\rightarrow a})$ and $L(C_{b\rightarrow b})$ do not contain the empty string. Therefore, all series $\text{Left}(C)$ and $\text{Right}(C)$ are invertible. Therefore, by equations (9)–(10), all entries of $A + B$ are invertible. It follows that exactly the diagonal entries of $A + B + I$ are invertible: they are obtained by adding one to invertible series, and other entries of $A + B + I$ coincide with the same entries of $A + B$.

Let us use the formula for $\det(A + B + I)$ with $n!$ summands again. Exactly one summand is invertible: the one that corresponds to the identity permutation. Indeed, all other summands have at least one nondiagonal, therefore, non-invertible, element. And the summand that corresponds to the identity permutation is a product of the diagonal entries of $A + B + I$. Hence, said summand is invertible as power series.

We have just proved that $\det(A + B + I)$ is invertible. In particular, $\det(A + B + I) \neq 0$.

Now we can use the Cramer’s formula and conclude that $\text{asSeries}(L(S_{a\rightarrow b}))$ match $\sum \mathcal{A}\mathcal{B}$. Then $\text{asSeries}(K) = \text{asSeries}(L(S')) = \text{asSeries}(L(S_{a\rightarrow a}) + \text{asSeries}(L(S_{a\rightarrow b}))$ match $A + \sum \mathcal{A}\mathcal{B} = \sum \mathcal{A}\mathcal{B}$. The last equivalence holds, because we can find a common denominator for the summands and obtain $A\sum \mathcal{A}\mathcal{B} + \sum \mathcal{A}\mathcal{B} = \sum \mathcal{A}\mathcal{B}$ in the numerator.

\section*{B.3 Algebraic manipulations}

The purpose of this section is to prove the following theorem:

**Theorem 9.** If $L \subset a^*b^*$ is described by a GF(2)-grammar. Then, the corresponding power series $\text{asSeries}(L)$ match $\frac{\sum \mathcal{A}\mathcal{B}}{\text{poly}(a, b)}$.

In the previous section, we have already moved to this goal, by dealing with the language-theoretic details. Now, we want to use some algebraic manipulations. Theorem 9 would follow from the Lemma 10 and the following lemma:

**Lemma 11.** Algebraic expressions $\frac{\sum \mathcal{A}\mathcal{B}}{\text{poly}(a, b)}$ and $\frac{\sum \mathcal{A}\mathcal{B}}{\text{poly}(a, b)}$ are equivalent.

**Remark 8.** It is immediately apparent that Theorem 9 is exactly Theorem 1, but stated in terms of algebraic expressions. What can be more difficult to see is the fact that Lemma 11 is also, essentially, a restatement of Lemma 8. Indeed, representing $\frac{\sum \mathcal{A}\mathcal{B}}{\text{poly}(a, b)}$ as $\frac{\sum \mathcal{A}\mathcal{B}}{\text{poly}(a, b)}$ is exactly the same as proving that a ratio of two elements of $\mathbb{R}_{a,b}$ is still an element of $\mathbb{R}_{a,b}$. And the proof of Lemma 11 is, essentially, a roundabout way to prove
Indeed, by conditions of the lemma, by \( p_d \neq 0 \), because \( \sum \mathcal{A} \mathcal{B} \) is not weaker than \( \text{poly}(a, b) \). Indeed, every polynomial is a finite sum of monomials of type \( a^n b^m \), and each such monomial match \( \mathcal{A} \mathcal{B} \), because \( a^n \in \mathcal{A} \) and \( b^m \in \mathcal{B} \).

Moreover, suppose that Laurent series match \( \mathcal{A} \mathcal{B} \). Then, by definition of “matching algebraic expression”, these series are of the type \( \sum a_i b_i \), where \( n \) is a positive integer, and \( A_i \in \mathcal{A}, B_i \in \mathcal{B} \) for every \( i \) from 0 to \( n \) inclusive. Moreover, this expression makes sense, meaning that \( A_1 B_1 + \ldots + A_n B_n \neq 0 \).

We still have not used that \( \mathcal{A} \) and \( \mathcal{B} \) are exactly the sets of algebraic power series, and not just some subsets of \( \mathbb{F}_2[[a]] \) and \( \mathbb{F}_2[[b]] \) that are closed under addition. Let us use that.

More exactly, we want to get rid of difficult expression in the numerator by rewriting \( \frac{1}{A_1 B_1 + \ldots + A_n B_n} \), that is, \( (A_1 B_1 + \ldots + A_n B_n)^{-1} \), as a finite \( \text{rat}(a, b) \)-linear combination of nonnegative powers of \( A_1 B_1 + \ldots + A_n B_n \).

The least painful way to do so is to find a nontrivial \( \text{rat}(a, b) \)-linear dependence between nonnegative powers of \( A_1 B_1 + \ldots + A_n B_n \) and then get the required expression from it. It still is not very easy, see below for details.

Because every \( A_i \) is an algebraic power series in variable \( a \) over the ring \( \mathbb{F}_2[a] \), it also is an algebraic power series in variables \( a \) and \( b \) over the field \( \text{rat}(a, b) \): the same polynomial equation will suffice to show that.

We will need a few technical lemmas:

**Lemma 12.** Suppose that Laurent series \( f \in \mathbb{F}_2((a, b)) \) is a solution to a polynomial equation of degree \( d \) with coefficients from \( \text{rat}(a, b) \). Then, for every \( m \geq d \), the power series \( f^m \) can be represented as a \( \text{rat}(a, b) \)-linear combination of \( f^{m-1}, f^{m-2}, \ldots, f^{m-d} \).

**Proof.** Indeed, by conditions of the lemma, \( \sum_{i=0}^{d} p_i f^i = 0 \) for some \( p_i \in \text{rat}(a, b) \). Moreover, \( p_d \neq 0 \), because the degree of the equation is exactly \( d \). Divide both sides by \( p_d \) and move \( f^d \) to the right-hand side: \( \sum_{i=0}^{d-1} \frac{p_i}{p_d} f^i = f^d \). Multiply both sides by \( f^{m-d} \): \( \sum_{j=m-d}^{m-1} \frac{p_i}{p_d} f^{j} = f^m \), exactly a representation of \( f^m \) as a \( \text{rat}(a, b) \)-linear combination of \( f^{m-1}, f^{m-2}, \ldots, f^{m-d} \).

**Lemma 13.** Suppose that Laurent series \( f \in \mathbb{F}_2((a, b)) \) is a root of a polynomial equation of degree \( d \) with coefficients from \( \text{rat}(a, b) \). Then, for every \( m \geq 0 \), \( f^m \) can be
represented as a rat\((a, b)\)-linear combination of \(f^{d-1}, f^{d-2}, \ldots, f^0\). In other words, all nonnegative powers of \(f\) are in the rat\((a, b)\)-linear space generated by \(f^0, f^1, \ldots, f^{d-1}\).

Proof. Induction over \(m\). Denote the rat\((a, b)\)-linear space, generated by \(f^0, f^1, \ldots, f^{d-1}\) by \(L\). The statement is trivially true for \(m < d\), because \(f^m\) is one of generators of \(L\).

Now, suppose that we want to prove the statement of the lemma for some \(m \geq d\). By induction hypothesis, \(f^{m-1}, f^{m-2}, \ldots, f^{d-1}\) all lie in \(L\). By the previous lemma, \(f^m\) can be represented as a rat\((a, b)\)-linear combination of \(f^{m-1}, f^{m-2}, \ldots, f^{d-1}\). Therefore, \(f^m\) lies in \(L\) as a finite rat\((a, b)\)-linear combination of elements of \(L\).

**Lemma 14.** There is no infinite subset of monomials in variables \(A_i\) and \(B_i\), that is linearly independent over rat\((a, b)\). In other words, the rat\((a, b)\)-linear space generated by all values of polynomials in variables \(A_1, A_2, \ldots, A_n\) and \(B_1, B_2, \ldots, B_n\) with coefficients from rat\((a, b)\) is finite-dimensional.

Proof. Because \(A_i \in \mathcal{A}\) for every \(i\) from 1 to \(n\) inclusive, there are some \(\ell_i\) such that \(A_i\) is a root of degree-\(d\) polynomial equation with coefficients from rat\((a, b)\). Similarly, denote by \(r_i\) the degrees of polynomial equations for \(B_i\).

Let us try to represent expression \(A_{i_1}^{j_1} \ldots A_{i_n}^{j_n} \cdot B_{1_1}^{k_1} \ldots B_{s_n}^{k_n}\) for some nonnegative \(j_s\) and \(k_s\) as a rat\((a, b)\)-linear combination of similar expressions with small degrees.

Indeed, by previous lemma, every \(A_{i_s}^{j_s}\) is a rat\((a, b)\)-linear combination of \(A_0^{s_s}, A_1^{s_s}, \ldots, A_{d_s}^{s_s}-1\). Similarly, every \(B_{s}^{k_s}\) is a rat\((a, b)\)-linear combination of \(B_0^s, B_1^s, \ldots, B_{d_s}^s-1\). Represent \(A_{i_1}^{j_1} \ldots A_{i_n}^{j_n} \cdot B_{1_1}^{k_1} \ldots B_{s_n}^{k_n}\) as a product of such linear combination and expand all brackets. The result is some rat\((a, b)\)-linear combination of expressions \(A_{i_1}^{j_1} \ldots A_{i_n}^{j_n} \cdot B_{1_1}^{k_1} \ldots B_{s_n}^{k_n}\), but with \(0 \leq x_s < \ell_s\) and \(0 \leq y_s < r_s\).

Let \(L\) be the rat\((a, b)\)-linear space generated by all products of type \(A_{i_1}^{j_1} A_{i_2}^{j_2} \ldots A_{i_n}^{j_n} \cdot B_{1_1}^{k_1} \ldots B_{s_n}^{k_n}\), where \(0 \leq x_s < \ell_s\) and \(0 \leq y_s < r_s\) for all \(s\) from 1 to \(n\) inclusive. This space is generated by \(\ell_1 \ell_2 \ldots \ell_n \cdot r_1 r_2 \ldots r_n\) elements and therefore is finite-dimensional.

We have already established that every monomial \(A_{i_1}^{j_1} A_{i_2}^{j_2} \ldots A_{i_n}^{j_n} \cdot B_{1_1}^{k_1} \ldots B_{s_n}^{k_n}\) is a rat\((a, b)\)-linear combination of elements of \(L\) (moreover, exactly the elements that were \(L\’\)s generators), therefore it lies in \(L\). Then, every polynomial expression in variables \(A_s\) and \(B_s\) lies in \(L\), as a linear combination of monomials that lie in \(L\).

Because the space of all polynomial expression of \(A_s\) and \(B_s\) is finite-dimensional, the space generated by nonnegative powers of \(A_1 B_1 + \ldots + A_n B_n\) also is. Therefore, there exists a nontrivial rat\((a, b)\)-linear dependence between nonnegative powers of \(A_1 B_1 + \ldots + A_n B_n\). In other words, there is some nonnegative integer \(d\) and rational functions \(p_0, p_1, \ldots, p_d \in \text{rat}(a, b)\), not all equal to zero, such that \(\sum_{i=0}^{d} p_i (A_1 B_1 + \ldots + A_n B_n)^i = 0\).

Let us find an expression of \((A_1 B_1 + \ldots + A_n B_n)^{-1}\) through nonnegative powers of \(A_1 B_1 + \ldots + A_n B_n\) with that knowledge.

Indeed, let us take the smallest such \(j\) that \(p_j \neq 0\). It exists, because not all \(p_i\) are equal to zero. Then, our equation can be rewritten as \(\sum_{i=j}^{d} p_i (A_1 B_1 + \ldots + A_n B_n)^i = 0\), because \(p_0 = p_1 = \ldots = p_{j-1} = 0\) anyways. By dividing both sides by \(p_j (A_1 B_1 + \ldots + A_n B_n)^{j+1} \neq 0\), we obtain

\[
\sum_{i=j}^{d} \frac{p_i}{p_j} (A_1 B_1 + \ldots + A_n B_n)^{i-j-1} = 0.
\]

All powers of \(A_1 B_1 + \ldots + A_n B_n\) from \((-1)\)-st to \((d-j-1)\)-st are here with some coefficients, the coefficient before \((-1)\)-st power is \(p_j/p_j = 1\). By moving all powers,
except \((-1)^{-1}\)-st to the right-hand side, we obtain

\[
(A_1 B_1 + \ldots + A_n B_n)^{-1} = \sum_{i=0}^{d} \frac{p_i}{p_j} (A_1 B_1 + \ldots + A_n B_n)^{i+1} = \sum_{i=0}^{d-j} \frac{p_{i+j+1}}{p_j} (A_1 B_1 + \ldots + A_n B_n)^{i+1}.
\]

Therefore, \((A_1 B_1 + \ldots + A_n B_n)^{-1}\) match \(\sum \text{rat}(a, b) AB\) (to understand that, expand all brackets in the right-hand side). Therefore, \(A_1 B_1 + \ldots + A_n B_n\) match \(AB \cdot \sum \text{rat}(a, b) AB = \sum \text{rat}(a, b) AB\). We are almost done!

**Remark 9.** Generally speaking, \(\text{rat}(a, b)\) cannot be split into two parts with the first being “absorbed” by \(A\) and the second being “absorbed” by \(B\). Keep the following example in the head: \(1 + ab\). It is not hard to prove that \(1 + ab\) is not a product of a factor depending only on \(a\) and a factor depending only on \(b\).

As we understood earlier, every Laurent series matching \(\frac{AB}{\sum AB}\) also match \(\sum \text{rat}(a, b) AB\). Then all Laurent series matching \(\sum \frac{AB}{\sum AB}\) also match \(\sum \sum \text{rat}(a, b) AB = \sum \text{rat}(a, b) AB\). Finally, by adding the fractions up, every Laurent series matching \(\sum \text{rat}(a, b) AB\) match \(\sum \frac{\text{poly}(a, b) AB}{\text{poly}(a, b)} = \frac{\sum AB}{\text{poly}(a, b)}\).

\[\square\]

### C Subsets of \(a^* b^* c^*\) through algebraic expressions

The language \(\{a^n b^n c^n \mid n \geq 0\}\) is, probably, the most famous example of a simple language that is not described by any ordinary grammar. It is reasonable to assume that it is not described by a GF(2)-grammar as well. Let us prove that.

We will do more than that and will actually establish some property that all GF(2)-grammatical subsets of \(a^* b^* c^*\) have, but \(\{a^n b^n c^n \mid n \geq 0\}\) does not. Most steps of the proof will be analogous to the two-letter case.

There is a natural one-to-one correspondence between subsets of \(a^* b^* c^*\) and formal power series in variables \(a, b\) and \(c\) over field \(\mathbb{F}_2\). Indeed, for every set \(S \subset \mathbb{N}^3\), we can identify the language \(\{a^m b^n c^k \mid (n, m, k) \in S\} \subset a^* b^* c^*\) with the formal power series \(\sum_{(n,m,k)\in S} a^m b^n c^k\). Denote this correspondence by \(\text{asSeries}: 2^{a^* b^* c^*} \to \mathbb{F}_2[[a, b, c]]\). Then, \(\text{asSeries}(L \triangle K) = \text{asSeries}(L) + \text{asSeries}(K)\). In other words, the symmetric difference of languages corresponds to the sum of formal power series.

Similarly to the Lemma[1] \(\text{asSeries}(K \cap L) = \text{asSeries}(K) \cdot \text{asSeries}(L)\) in the following important special cases: when \(K\) is a subset of \(a^*\), when \(K\) is a subset of \(a^* b^*\) and \(L\) is a subset of \(b^* c^*\), and, finally, when \(L\) is a subset of \(c^*\). Indeed, in each of these three cases, characters “are in the correct order”: if \(u \in K\) and \(v \in L\), then \(uv \in a^* b^* c^*\).

However, we cannot insert character \(b\) in the middle of the string: if \(K\) is a subset of \(b^*\) and \(L\) is a subset of \(a^* b^* c^*\), then \(K \cap L\) and \(K \cap \text{comm} L\) do not have to coincide, because \(K \cap L\) does not have to be a subset of \(a^* b^* c^*\).

The “work plan” will remain the same as in the previous section: we will switch to algebraic track first and then we simplify the expression obtained.

An attentive reader may ask two questions:

1. Why is it logical to expect that the language \(\{a^n b^n c^n \mid n \geq 0\}\) is not described by a GF(2)-grammar, but a similar language \(\{a^n b^n \mid n \geq 0\}\) is?
2. Why will the proof work out for \( \{ a^n b^n c^n \mid n \geq 0 \} \), but not for a regular language \( \{ (abc)^n \mid n \geq 0 \} \), despite these languages having the same “commutative image”?

They can be answered in the following way:

1. Simply speaking, the reason is the same as for the ordinary grammars. On an intuitive level, both ordinary grammars and GF(2)-grammars permit a natural way to “capture” the events that happen with any two letters in subsets of \( a^* b^* c^* \), but not all three letters at the same time. A rigorous result that corresponds to this intuitive limitation of ordinary grammars was proven by Ginsburg and Spanier \([9, \text{Theorem 2.1}]\). Theorem 5 is an analogue for GF(2)-grammars.

2. This argument only implies that any proof that relies solely on commutative images is going to fail. The real proof is more subtle. For example, it will also use the fact that \( \{ a^n b^n c^n \mid n \geq 0 \} \) is a subset of \( a^* b^* c^* \).

While the proof uses commutative images, it uses them very carefully, always making sure that the letters “appear in the correct order”. In particular, we will never consider GF(2)-concatenations \( K \otimes L \), where \( K \) is a subset of \( b^* \) and \( L \) is an arbitrary subset of \( a^* b^* c^* \), in the proof, because in this case \( K \otimes L \) is not a subset of \( a^* b^* c^* \).

Avoiding this situation is impossible for language \( \{ (abc)^n \mid n \geq 0 \} \), because in the string \( abcabc \) from this language the letters “appear in the wrong order”.

Denote the set of algebraic power series of variable \( c \) by \( C \), the set of polynomials in variables \( a \) and \( c \) by \( \text{poly}(a,c) \), et cetera. The definition of an algebraic expression stays the same for the most part, but now the new symbols \( C \), \( \text{poly}(a,c) \), \( \text{poly}(b,c) \), \( \text{poly}(a,b,c) \), \( \text{rat}(a,c) \), \( \text{rat}(b,c) \) and \( \text{rat}(a,b,c) \) may appear alongside the old symbols \( A \), \( B \) and \( \text{poly}(a,b) \).

### C.1 Switching to the algebraic track

Our goal for this subsection is to establish the following lemma:

**Lemma 15.** Suppose that \( K \subset a^* b^* c^* \) is described by a GF(2)-grammar. Then the corresponding formal power series \( \text{asSeries}(K) \) match algebraic expression

\[
\sum \text{ABC} \quad \text{poly}(a,b) \text{poly}(b,c) \cdot \sum \text{AC}
\]

**Proof.** The proof is mostly the same as the proof of Lemma \([10]\).

Without loss of generality, GF(2)-grammar \( G \) that describes \( K \) is in Chomsky’s normal form. Also we can assume that \( K \) does not contain the empty string.

The language \( a^* b^* c^* \) is accepted by the following incomplete deterministic finite automaton \( M \). Firstly, \( M \) has three states \( q_a, q_b \) and \( q_c \), all accepting. Secondly, its transition function \( \delta \) is defined as \( \delta(q_a, a) = q_a, \delta(q_b, b) = q_c, \delta(q_a, c) = q_c, \delta(q_b, b) = q_a, \delta(q_b, c) = q_a, \delta(q_c, c) = q_c \).

Intersect the GF(2)-grammar \( G \) formally with regular language \( a^* b^* c^* \), recognized by \( M \). Because \( L(G) = K \) was a subset of \( a^* b^* c^* \) anyway, the described language will not change. Each nonterminal \( C \) of the original grammar will split into six nonterminals \( C_{a\rightarrow a}, C_{a\rightarrow b}, C_{a\rightarrow c}, C_{b\rightarrow b}, C_{b\rightarrow c}, C_{c\rightarrow c} \). Also, a new starting nonterminal \( S' \) will appear.
Every “normal” rule \( C \to DE \) will split into rules \( C_{a\to a} \to D_{a\to a}E_{a\to a}, \) \( C_{a\to b} \to D_{a\to b}E_{b\to b}, \) \( C_{a\to c} \to D_{a\to c}E_{c\to c}, \) \( C_{a\to c} \to D_{a\to b}E_{b\to c}, \) \( C_{a\to c} \to D_{a\to c}E_{c\to c}, \) and \( C_{a\to c} \to D_{a\to c}E_{c\to c}. \) Less horrifying than it looks, because most of these rules will not be interesting to us in the slightest.

A “final” rule \( C \to a \) will turn into a rule \( C_{a\to a} \to a. \) Similarly, rule \( C \to b \) will split into two rules \( C_{a\to b} \to b \) and \( C_{b\to b} \to b, \) and rule \( C \to c \) will split into three rules \( C_{a\to c} \to c, \) \( C_{b\to c} \to c \) and \( C_{c\to c} \to c. \)

Finally, three new rules will appear: \( S' \to S_{a\to a}, \) \( S' \to S_{b\to b}, \) \( S' \to S_{c\to c}. \)

The nonterminal \( C_{x\to y} \) of the new GF(2)-grammar, where \( x, y \in \{a, b, c\}, \) corresponds to exactly such strings from \( L(C) \) that move the automaton \( M \) from the state \( q_x \) to the state \( q_y. \)

By looking more closely at the transitions of the automaton \( M, \) we can see that any string that makes \( M \) go from \( q_a \) to \( q_c \) is from \( a^*b^c+, \) any string that makes \( M \) go from \( q_b \) to \( q_c \) is from \( b^c+, \) etcetera. In particular, in all new “normal” rules GF(2)-concatenations happen “in the correct order”.

As already mentioned, most of the new rules are not interesting, because we already know, how the languages \( L(C_{a\to a}), \ L(C_{a\to b}), \ L(C_{b\to b}), \) and \( L(C_{c\to c}). \) look like.

More specifically, the corresponding formal power series match algebraic expressions

\[
\sum \frac{AB}{\text{poly}(a,b)}, \sum \frac{BC}{\text{poly}(b,c)} \text{ and } C \text{ respectively.}
\]

Therefore, we are only interested in nonterminals of the type \( a \to c. \) The rule \( X \to YZ \) of the original GF(2)-grammar \( G \) produces three rules for \( X_{a\to c}: \ X_{a\to c} \to Y_{a\to c}Z_{c\to c}, \ X_{a\to c} \to Y_{a\to b}Z_{b\to c} \text{ and } X_{a\to c} \to Y_{a\to c}Z_{a\to c}. \) The first and last rule relate \( L(X_{a\to c}) \) to other nonterminals of type \( a \to c, \) and the second rule just outright tells us that we can replace \( X_{a\to c} \) with a language matching \( \sum \frac{AB}{\text{poly}(a,b)} \cdot \sum \frac{BC}{\text{poly}(b,c)} \). Finally, there may be a final rule \( X_{a\to c} \to c \) for nonterminal \( X_{a\to c}. \)

We can conclude that the languages \( L(C_{a\to c}) \) satisfy the following system of language equations.

\[
L(C_{a\to c}) = \text{end}(C_{a\to c}) \Delta \bigcup_{C \to DE} (L(D_{a\to a}) \odot L(E_{a\to c})) \Delta (L(D_{a\to c}) \odot L(E_{c\to c})) \tag{11}
\]

Here, the summation happens over all rules \( C \to DE \) for the nonterminal \( C \) of the original GF(2)-grammar, and \( \text{end}(C_{a\to c}) \) is defined as:

\[
\text{end}(C_{a\to c}) = (\{c\} \text{ or } \emptyset) \Delta \bigcup_{C \to DE} L(D_{a\to b}) \odot L(E_{b\to c}) \tag{12}
\]

Here, the first “summand” depends on whether or not there is a rule \( C_{a\to c} \to c \) in the new GF(2)-grammar.

Consider the equations from System [11] more closely. For all GF(2)-concatenations that appear in their right-hand sides, either the first factor is a subset of \( a^*, \) or the second is a subset of \( c^*. \) Therefore, we can replace all GF(2)-concatenations here satisfy the conditions of Lemma [1]

\[
L(C_{a\to c}) = \text{end}(C_{a\to c}) \Delta \bigcup_{C \to DE} (L(D_{a\to a}) \odot_{\text{comm}} L(E_{a\to c})) \Delta (L(D_{a\to c}) \odot_{\text{comm}} L(E_{c\to c})) \tag{13}
\]
Denote asSeries\((L(C_{a→c}))\) by Center\((C)\), asSeries\((L(C_a→a))\) by Left\((C)\), asSeries\((L(C_{c→c}))\) by Right\((C)\) and asSeries\((\text{end}(C_{a→c}))\) by final\((C)\). By applying the correspondence asSeries to the both sides of each equation of the System (13),

\[
\text{Center}(C) = \text{final}(C) + \sum_{C→DE} \text{Left}(D) \text{Center}(E) + \text{Center}(D) \text{Right}(E) \quad (14)
\]

This system of equation can be interpreted as a system \(\mathbb{F}_2[[a, b, c]]\)-linear equations over the variables Center\((C)\) = asSeries\((L(C_{a→c}))\) for every nonterminal \(C\) of the original GF(2)-grammar.

We will consider final\((C)\), Left\((C)\) and Right\((C)\) to be the coefficients of said system. While we do not know their exact values, we know that final\((C)\) match the expression 

\[
\sum (\sum AB \text{poly}(a, b)) \cdot (\sum BC \text{poly}(b, c)) = \sum ABC \text{poly}(a, b) \text{poly}(b, c)
\]

by formula (12) and Theorem 9, Left\((C)\) is in \(A\), because it corresponds to a 2-automatic language over an alphabet \(\{a\}\) and, similarly, Right\((C)\) is in \(C\).

Let us say that the original GF(2)-grammar has \(n\) nonterminals. Then, the new GF(2)-grammar has \(n\) nonterminals of type \(a → c\). Denote the column-vector of values Center\((C)\) by \(x\), and the column-vector of values of final\((C)\), listed in the same order, by \(f\). Fix such numeration of nonterminals of the original GF(2)-grammar. Now, we can indice both rows and columns of \(n \times n\) matrices by the nonterminals of the original GF(2)-grammar.

Let \(I\) be an identity \(n \times n\) matrix and \(A\) be a \(n \times n\) matrix, where the cell on the intersection of \(C\)-th row and \(E\)-th column contains the sum Left\((D)\) over all rules \(C → DE\) of the original grammar:

\[
A_{C,E} := \sum_{C→DE} \text{Left}(D) \quad (15)
\]

Similarly, let \(B\) be \(n \times n\) matrix with sum of Right\((E)\) over all rules \(C → DE\) of the original grammar standing on the intersection of \(C\)-th row and \(D\)-th column (it would make more sense to call this matrix \(C\) rather than \(B\), but we have already used the letter \(C\) for a different purpose):

\[
B_{C,E} := \sum_{C→DE} \text{Right}(E) \quad (16)
\]

Then, System (13) can be stated in the following compact matrix form: \(x = f + (A + B)x\), or \((A + B + I)x = f\), which is the same.

As we have shown above, the column-vector of Center\((C)\) values indeed is a solution to such a system. If we somehow establish that this system has only one solution, and the said solution can be expressed in relatively simple algebraic terms, we will get an expression for for asSeries\((L(S_{a→c})) = \text{Center}(S)\).

This system has exactly one solution if and only if \(\det(A + B + I) \neq 0\). If \(\det(A + B + I) \neq 0\), then, by Cramer’s formula, each component of the solution, \(\text{Center}(S)\), in particular, can be represented in the following form:

\[
\frac{\det(A + B + I, \text{ but one of the columns was replaced by } f)}{\det(A + B + I)}\]

Now, we still need to prove three things: that \(\det(A + B + I)\) matches \(\sum A\mathcal{C}\), that \(\det(A + B + I, \text{ but one of the columns was replaced by } f)\)
matches \( \frac{\sum ABC}{\text{poly}(a, b) \text{poly}(b, c)} \), independently of the replaced column, and, finally, that \( \det(A + B + I) \) is not zero.

Each entry of \( A + B + I \) matches \( A + C \), because of Equations (15) and (16). Indeed, each entry of \( A \) match \( \sum A = A \), similarly, each component of \( B \) matches \( C \), and entry of \( I \) are zeroes and ones, which match both \( A \) and \( C \).

The determinant of the matrix with all entries matching \( A + C \), matches \( \sum AC \). We have already proven that during the proof of Lemma 10. The proof of \( \det(A + B + I) \) being non-zero also comes from the same place verbatim.

Finally, determinant of \( A + B + I \) with one column replaced by \( f \) is something new, because entries of \( f \) match rather complicated expression \( \sum (\sum AB \text{poly}(a, b)) \cdot (\sum B \text{poly}(b, c)) = \sum ABC \). By using the formula for determinant with \( n! \) summands, we see that the determinant matches \( \sum \frac{\sum ABC}{\text{poly}(a, b) \text{poly}(b, c)} \).

Here, in each summand, exactly one factor is complicated and the others are very simple. By expanding all brackets, the determinant matches \( \sum \frac{\sum ABC}{\text{poly}(a, b) \text{poly}(b, c)} \). By taking the lowest common denominator of all fractions in the sum, the determinant matches \( \sum \frac{\sum ABC}{\text{poly}(a, b) \text{poly}(b, c)} \).

Now, \( \text{Center}(S) = L(S_{a \rightarrow c}) \) is not exactly the language described by the new \( \text{GF}(2) \)-grammar, \( L(S') = L(S_{a \rightarrow a}) \triangle L(S_{a \rightarrow b}) \triangle L(S_{a \rightarrow c}) \) is. However, \( \text{asSeries}(L(S')) = \text{asSeries}(L(S_{a \rightarrow a})) + \text{asSeries}(L(S_{a \rightarrow b})) + \text{asSeries}(L(S_{a \rightarrow c})) \). Therefore, series \( \text{asSeries}(K) = \text{asSeries}(L(S')) \) match \( A + \sum \frac{AB}{\text{poly}(a, b)} + \sum \frac{ABC}{\text{poly}(a, b) \text{poly}(b, c) \sum AC} \).

The last equivalence holds, because the first two summands are simple and are “absorbed” by complicated third summand.

### C.2 Algebraic manipulations

We will establish the following theorem:

**Theorem 10.** Let \( L \) be a subset of \( a^*b^*c^* \) described by a \( \text{GF}(2) \)-grammar. Then, the formal power series \( \text{asSeries}(L) \) match algebraic expression \( \sum \frac{ABC}{\text{poly}(a, b) \text{poly}(b, c) \sum AC} \).

By Lemma 15 it is enough to prove the following lemma:

**Lemma 16.** The algebraic expressions \( \sum \frac{ABC}{\text{poly}(a, b) \text{poly}(b, c) \text{poly}(a, c)} \) and \( \sum \frac{ABC}{\text{poly}(a, b) \text{poly}(b, c) \sum AC} \) are equivalent.

**Proof.** The proof is much simpler than the proof of Theorem 9 because we can use it now.
The second expression is not stronger than the first, because \( \text{poly}(a, c) \) is not stronger than \( \sum AC \).

On the other hand, we already know that the expression \( \frac{\sum AC}{\text{poly}(a, c)} \) is not stronger than \( \sum AC \), because we needed that to prove Theorem 9. Therefore, the expression \( \frac{\sum ABC}{\text{poly}(a, b) \cdot \text{poly}(b, c) \cdot \sum AC} \) is not stronger than \( \frac{\sum ABC \cdot \sum AC}{\text{poly}(a, b) \cdot \text{poly}(b, c) \cdot \text{poly}(a, c)} \). In the last expression, the factor \( \sum AC \) can be “absorbed” into \( \sum ABC \), giving us exactly the expression \( \frac{\sum ABC}{\text{poly}(a, b) \cdot \text{poly}(b, c) \cdot \text{poly}(a, c)} \).

\( \square \)

Remark 10. In a very similar way, with induction over the number \( k \) of letters in the alphabet, we can prove the following result: for \( K \subset a_1^* a_2^* \ldots a_k^* \), the corresponding power series \( \text{asSeries}(K) \) match the expression \( \sum \prod_{i=1}^{n} A_i \prod_{1 \leq i < j \leq n} \text{poly}(a_i, a_j) \), where \( A_i \) is the set of algebraic formal power series of variable \( a_i \).