On specific log integrals, polylog integrals and alternating Euler sums

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Abstract

The main purpose of this article is the evaluation of 85 specific logarithmic integrals, 89 alternating Euler sums and 263 polylogarithmic generalizations with their weights \( \leq 5 \). By establishing linear relations between 3 kinds of values, we discover the common pattern on their closed-forms and present a systematic proof. Among these results, 7 weight 5 sums and over 200 polylog integrals are new. Based on previous results, we solved several problems on related integrals and series which are also unknown in literatures.

Keywords. Logarithmic integrals, polylogarithmic integrals, Euler sums, multiple zeta values.

0. Introduction

Recall that polylogarithm function is defined as \( \text{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} \) in the unit circle and its unique analytic continuation (via inversion formula) outside. We also have the definition of Riemann Zeta function \( \zeta(s) = \text{Li}_s(1) \) for \( \Re(s) > 1 \) and its analytic continuation (via reflection formula) otherwise, and similarly Dirichlet Eta function \( \eta(s) = -\text{Li}_s(-1) = (1-2^{1-s})\zeta(s) \) for \( \Re(s) > 0 \). Based on this, we introduce some basic notations about natural logarithms, polylogarithms (abbr. log and polylog), generalized harmonic numbers and its alternating analogue:

\[
\begin{align*}
\text{f}(0; x) &= \frac{1}{1-x}, \text{f}(1; x) = \frac{1}{x}, \text{f}(2; x) = \frac{1}{1+x} \quad (0.1) \\
\text{l}(0; x) &= \log(1-x), \text{l}(1; x) = \log(x), \text{l}(2; x) = \log(x+1) \quad (0.2) \\
L(n, 1; x) &= \text{Li}_n(x), L(n, 2; x) = \text{Li}_n(-x), L(n, 3; x) = \text{Li}_n(\frac{1-x}{2}), n \geq 2 \quad (0.3a) \\
L(n, 4; x) &= \text{Li}_n(\frac{1+x}{2}), L(n, 5; x) = \text{Li}_n(\frac{1-x}{1+x}), L(n, 6; x) = \text{Li}_n(\frac{x-1}{1+x}), n \geq 2 \quad (0.3b)
\end{align*}
\]
\[ H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k}, \quad \widetilde{H}_n^{(k)} = \frac{1}{\prod_{j=1}^{n} j^k}, \quad H_n = H_n^{(1)}, \quad \widetilde{H}_n = \widetilde{H}_n^{(1)} \]  

Now we define log, polylog integrals (abbr. LIs and PLIs) and Euler sums (abbr. ESs) in this article as follow:

\[ LI(a(0), a(1), a(2); p) = \int_{0}^{1} \prod_{m=0}^{l} \frac{l(m; x)^{a(m)}}{f(p; x)} \, dx \]  

[Equation 5]

\[ PLI(a(0), a(1), a(2); b(2, 1), \ldots, b(2, 6); \ldots; b(N, 1), \ldots, b(N, 6); p) = \int_{0}^{1} \prod_{m=0}^{l} \frac{l(m; x)^{a(m)}}{f(p; x)} \prod_{k=1}^{6} \prod_{n=2}^{N} L(n, k; x)^{b(n,k)} \, dx \]  

[Equation 6]

\[ ES(a(1), \ldots, a(M), -b(1), \ldots, -b(N); \pm p) = \sum_{n=1}^{\infty} \frac{\pm 1}{\prod_{k=1}^{M} H_n^{(a(k))} \prod_{j=1}^{N} \widetilde{H}_n^{(b(j))}} \]  

[Equation 7]

Parameters in above expressions are natural numbers or 0. The notations of ESs come from Flajolet & Salvy [6] but differ a little, and we may assume the sequence \( a(1), \ldots, a(M) \) and \( b(1), \ldots, b(N) \) are non-decreasing since a permutation doesn’t change the ES value. If their exists no \( b(k) \), we call an ES non-alternating, otherwise alternating. We will also use same notations on multiple zeta values (abbr. MZVs) in their articles below and trivially extended it to multiple zeta star values (abbr. MZSVs) and alternating analogues. For instance, \( \zeta(1, -2, 3) = \sum_{l=1}^{\infty} \sum_{k=1}^{l} \sum_{j=1}^{k} \frac{(-1)^{k-j}}{j^{l+1}k^{j-1}} \) and \( \zeta^*(1, -2, 3) = \sum_{l=1}^{\infty} \sum_{k=1}^{l} \sum_{j=1}^{k} \frac{(-1)^{k-j}}{j^{l+1}k^{j-1}} \).

In later discussion, the number \( N \) and parameters \( b(n,k) \) will be small, and many of them equal to 0. In this case, we omit those zero parameters and simply repeat the word ‘\( nk \)’ \( b(n,k) \) times for abbreviation. For instance, \( LI(1, 2, 3; 0) = \int_{0}^{1} \frac{\log(1-x)\log^2(x)\log(x+1)}{1-x} \, dx \), \( PLI(0, 0, 1; 24, 24; 32; 2) = \int_{0}^{1} \frac{\log(1-x)^2\log^2(x+1)}{x+1} \, dx \), and \( ES(2, -3, -3; -4) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}H_n^{(2)}\widetilde{H}_n^{(5)}}{n^4} \).

We define the weight (denoted by \( W \)) of an LI/PLI/ES above as an integer \( \sum_{m=0}^{2} a(m) + 1 \), \( \sum_{k=1}^{N} nb(n,k) + \sum_{m=0}^{2} a(m) + 1 \) and \( \sum_{k=1}^{M} a(k) + \sum_{j=1}^{N} b(j) + p \) respectively. MZVs’ weight is simply the sum of the absolute value of all indexes. Constants are also given weights; precisely, rational numbers have weight 0, \( \pi, \log(2) \) have weight 1, \( \zeta(n), Li_n(\frac{1}{2}) \) have weight \( n \), and the weight given to product of these constants equals to sum of each components’ weight. For example, the LI, PLI and ES mentioned above have weight 7, 9 and 12, \( \zeta^*(2, -3, -1) \) has weight 6, while the constant \( \pi^2 Li_4 \left( \frac{1}{2} \right) \zeta(3) \log(2) \) has weight 10. Moreover, the depth of an MZV is defined as the quantity of indexes; similarly we define the depth of an ES as \( M + N + 1 \). By this definition \( ES(2, -3, -3; -4) \) is weight 12 depth 4 and \( \zeta^*(2, -3, -1) \) weight 6 depth 3.
A linear relation is an equality in form ‘rational combination of LIs/PLIs/ESs = log/polylog constants’. If all items in LHS have the same weight, we call it a homogeneous relation; otherwise we call it a non-homogeneous relation. According to Au [1], by combining various methods that offer linear relations between LIs, we are able to express all LIs with weight $W$ no more than 5, in rational combination of at most $t(W)$ weight $W$ polylog constants, which are product of log(2), $\pi$, $\zeta(m)$ and $\text{Li}_n\left(\frac{1}{2}\right)$ that conjectured to be linearly independent over $\mathbb{Q}$. Specifically, $t(1) = 1$, $t(2) = 2$, $t(3) = 3$, $t(4) = 5$, $t(5) = 8$, and the set of these homogeneous constants generating LIs are known as Fibonacci basis, due to similarity between $t(n)$ and Fibonacci numbers $F_n$.

For LIs with weight higher than 5, we need MZVs (or ESs) for closed-form expressions. In other words, some of the LIs are irreducible. The counting of $t(W)$ is much more complicated here, and $t(W)$ is not proved to have reached its possible minimum as in the case $W \leq 5$. See Au for details on higher weight LIs.

For need in this article, we denote $A_W$ the weight $W$ Fibonacci basis containing $t(W)$ full-simplified elements. By full-simplified we mean $\zeta(2n)$, $n > 0$ and $\text{Li}_n\left(\frac{1}{2}\right)$, $n < 4$ won’t appear, due to simplification formulas $\text{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$, $\text{Li}_3\left(\frac{1}{2}\right) = \frac{7\zeta(3)}{8} + \frac{\log^3(2)}{6} - \frac{1}{12}\pi^2\log(2)$ and $\zeta(2n) = \frac{2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}$, where $B_n$ are Bernoulli numbers (apparently these simplifications keep the weight unchanged). Thus, we may write down $A_W$ explicitly for $W \leq 5$:

\[
A_1 = \{\log(2)\}, A_2 = \{\pi^2, \log^2(2)\}, A_3 = \{\pi^2 \log(2), \log^3(2), \zeta(3)\}
\]

\[
A_4 = \left\{\pi^4, \pi^2 \log^2(2), \log^4(2), \zeta(3) \log(2), \text{Li}_4\left(\frac{1}{2}\right)\right\}
\]

\[
A_5 = \left\{\pi^4 \log(2), \pi^2 \log^3(2), \log^5(2), \pi^2 \zeta(3), \zeta(3) \log^2(2), \zeta(5), \text{Li}_4\left(\frac{1}{2}\right) \log(2), \text{Li}_5\left(\frac{1}{2}\right)\right\}
\]

For $W = 2, 3, 4, 5$, we abbreviate Au’s remarkable result as: all weight $W$ LIs can be generated by $A_W$ over $\mathbb{Q}$. For $W = 6$, a possible representation is:

\[
A_6 = \left\{\pi^6, \pi^4 \log^2(2), \pi^2 \log^4(2), \log^6(2), \pi^2 \zeta(3) \log(2), \zeta(3) \log^3(2), \zeta(5) \log(2), \right.
\]

\[
\left.\zeta(3)^2, \text{Li}_4\left(\frac{1}{2}\right) \log^2(2), \pi^2 \text{Li}_4\left(\frac{1}{2}\right), \text{Li}_5\left(\frac{1}{2}\right) \log(2), \text{Li}_6\left(\frac{1}{2}\right), S\right\}
\]

Where the constant $S = ES(1; -5) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} H_k}{k^5}$ is very likely to be irreducible. Due to purposes of this article, we omit higher weights’ representations.

Now we state the main result of this article:

**Theorem 1.** All ESs and PLIs with weight $W \leq 5$ can be generated by $A_W$ over $\mathbb{Q}$.

**1. LI evaluation**
First of all, we summarize known methods generating linear relations between LIs based on our discoveries and other references, a plenty of which can be modified to apply in PLIs later. By solving the linear system given by these methods, explicit closed-forms of LIs are deduced and tabulated in appendix 1. It’s a little strange that few ES/PLI-related literature pay attention to the evaluation of LIs which is apparently powerful, thus we hold the view that a thorough summary is urgently needed.

Before the summary, we’d like to point out that all LIs with weight no more than 3 can be evaluated by brute force, that is, calculating their polylog primitives directly, which can be verified using CAS like Mathematica. This is due to the fact that \( \int \log(a+x) \log(b+x) \log(c+x) \, dx \) and \( \int \frac{\log(a+x) \log(b+x)}{c+x} \, dx \) are solvable via polylog identities. Some weight 4 integrals can be solved similarly if we apply formulas for \( \int \log^2(a+x) \log(b+x) \, dx \), \( \int \frac{\log(a+x) \log^2(b+x)}{a+x} \, dx \), etc, see Lewin [9] for more. Nevertheless, this method cannot be generalized to high weight cases, therefore we won’t discuss about it later.

1-1. General formulas

**Proposition 1.** The following formulas hold:

\[
\int_0^1 x^m \log^n(x) \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \quad (1.1.1)
\]

\[
LI(0,0;2) = \frac{\log^{n+1}(2)}{n+1} \quad (1.1.2)
\]

\[
LI(0,n,0;2) = (-1)^n n! \zeta(n+1) \quad (1.1.3)
\]

\[
LI(n,0,0;2) = (-1)^n n! Li_{n+1} \left( \frac{1}{2} \right) \quad (1.1.4)
\]

\[
LI(0,0,1;1) = \sum_{k=0}^{n-1} \log^k(2) Li_{-k+n+1} \left( \frac{1}{2} \right) + n! \zeta(n+1) - \frac{n \log^{n+1}(2)}{n+1} \quad (1.1.5)
\]

\[
LI(0,0,1;2) = \sum_{k=0}^{n} \log^k(2) Li_{-k+n+2} \left( \frac{1}{2} \right) - n! \zeta(n+2) + \frac{\log^{n+2}(2)}{n+2} \quad (1.1.6)
\]
\[ LI(1, 0, n; 2) = n! \sum_{k=2}^{n+2} \frac{(-1)^{n-k} \zeta(k) \log(-k+n+2) (2)}{(-k+n+2)!} + (-1)^n n! Li_{n+2} \left( \frac{1}{2} \right) + \frac{\log^{n+2} (2)}{n+1} \] (1.1.10)

\[ LI(1, n, 0; 0) = LI(n, 1, 0; 1) = \frac{1}{2} (-1)^{n-1} n! \left( (n+1) \zeta(n+2) - \sum_{k=2}^{n} \zeta(k) \zeta(-k+n+2) \right) \] (1.1.11)

\[ LI(0, n, 1; 0) = \frac{1}{2} (-1)^n n! \sum_{k=2}^{n} (1 - 2^{1-k}) (1 - 2^{k-n-1}) \zeta(k) \zeta(-k+n+2) \]

\[-(-1)^n n! \left( \frac{1}{2} (n+1) \zeta(n+2) - 2 \left( 1 - 2^{-(n+1)} \right) \log(2) \zeta(n+1) \right) \] (1.1.12)

\[ LI(0, 2n-1, 1; 2) = -(2n-1)! \sum_{k=1}^{n-1} (1 - 2^{1-2k}) \zeta(2k) \zeta(-2k+2n+1) \]

\[-(2n-1)! \left( 2^{-(2n+1)} \zeta(2n+1) - (n-1) (1 - 2^{-2n}) \zeta(2n+1) \right) \] (1.1.13)

\[ LI(1, 2n-1, 0; 2) = -(2n-1)! \sum_{k=1}^{n-1} (1 - 2^{2k-2n}) \zeta(2k) \zeta(-2k+n+1) \]

\[+(2n-1)! \left( (n+1) (1 - 2^{-2n}) + 2^{(2n+1)} \right) \zeta(2n+1) - 2 \left( 1 - 2^{-2n} \right) \log(2) \zeta(2n) \] (1.1.14)

\[ LI(2, n, 0; 1) = (-1)^n n! \left( (n+2) \zeta(n+3) - \sum_{k=2}^{n+1} \zeta(k) \zeta(-k+n+3) \right) \] (1.1.15)

\[ LI(0, 2n, 2; 1) = 2(2n)! \sum_{k=1}^{n} (1 - 2^{1-2k}) \zeta(2k) \zeta(-2k+2n+3) \]

\[-2(2n)! \left( 1 - 2^{-2n+2} \right) \zeta(2n+3) - 2^{-(2n+3)} \zeta(2n+3) \] (1.1.16)

\[ LI(1, 2n, 1; 1) = -(2n)! 2^{-(2n+1)} \sum_{k=1}^{n} (1 - 2^{-2k+2n+2}) \zeta(2k) \zeta(-2k+2n+3) \]

\[+(2n)! \left( (2n+3) 2^{-(2n+3)} - 1 \right) \zeta(2n+3) \] (1.1.17)

Proof. Using \( \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx \) and \( \Gamma(n+1) = n! \), one immediately comes to (1.1.1). (1.1.2) is elementary via Newton-Leibniz formula. Noticing the geometric series identity \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \) holds on (0,1), exchange the order, invoke (1.1.1) and a reflection \( t = 1-x \) gives (1.1.3). Similarly, \( \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-x} \) yields (1.1.4), where we’ve used \( \beta(s) = (1 - 2^{1-s}) \zeta(s) \) for Dirichlet Beta function. A reflection \( t = 1-x \) and scaling \( \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^n = \frac{1}{1-x/2} \) yields (1.1.5). For (1.1.6)
and (1.1.7), apply \( \sum_{n=1}^{\infty} \frac{(\pm x)^n}{n} = -\log(1 \mp x) \) and do the same as above. (1.1.8) is the celebrated Nielsen-Ramanujan integral, and can be shown via substitution \( t = \frac{1}{1+x} \), and the formula \( \int_{-\frac{1}{2}}^{\frac{1}{2}} x^n \log^n(x) \, dx = \frac{\partial^n}{\partial m^n} \frac{1-2^{-(m+1)}}{m+1}, \) or directly calculating polylog primitives by induction. (1.1.9) is deduced easily from (1.1.8) and integration by parts, or calculating primitives as above. Note that this also works for (1.1.10). (1.1.11) together with (1.1.12) can be proved by using (1.1.9) is deduced easily from (1.1.8) and integration by parts, or calculating primitives as above. See also Valean [11].

For (1.1.13)-(1.1.17) their methods in (1.1.12) (that is, expanding some log terms, exchange the order, and use general formulas for ESs) still work, although they didn’t generalize their ES result to LIs which we proposed here. We refer readers to their article for more (the statement of their ES-theorems are in section 2-1 and corresponding series expansion results are tabulated in section 2-5). See also Valean [11].

\[ \square \]

1-2. Integration by parts

By using Newton-Leibniz formula \( \int_a^b f(x) \, dx = F(b) - F(a) \) (where \( F'(x) = f(x) \)) on differentiable function \( F(x) = (l(2; x) - \log(2))^k \prod_{m=0}^{\infty} l(m; x)^{a(m)} \), such that \( |F(0^+)|, |F(1^-)| < \infty \) to obtain linear a linear relation between LIs. If \( k = 0 \), the relation is homogeneous. Otherwise, we may have to apply integration by parts to reduce integrals containing \( (l(2; x) - \log(2)) \) term to ordinary LIs. As an example, let \( k = 0 \) we have:

\[ -aLI(a-1, b, c; 0) + bLI(a-, b-1, c; 1) + cLI(a, b, c-1; 2) = 0 \quad (1.2.1) \]

Whenever \( a, b, c > 0 \), and when some parameters are 0 the formula is simpler. Fix the weight \( W \), it’s natural to consider all multiple index \((k, a(0), a(1), a(2))\) such that \( k + a(0) + a(1) + a(2) = W \) to obtain relations.

Now we explain why we introduce an extra index \( k \). From another point of view, we pay attention to every single LI. Integrate by parts to lift up every LI’s denominator to a log term and differentiate the numerator; this offer relations equivalent to the process above (in this article, by lifting up \( u \) we mean transforming the LHS to RHS in equality \( \int u'v = uv - \int uv' \)). If an LI contains \( l(0; x) \) and no \( l(1; x) \) in numerator, moreover the denominator is \( 1 + x \), due to convergence issue we cannot lift up the denominator to \( l(2; x) \) but \( l(2; x) - \log(2) \) instead, which shows the necessity of \( k \). What’s more, a little analysis shows that we don’t need to modify \( l(0; x), l(1; x) \) similarly, so that \( F \) is in simplest form.

1-3. Fractional linear transformation

Consider \( F(x) = \prod_{m=0}^{\infty} l(m; x)^{a(m)} / f(p; x) \) such that the corresponding LI is convergent, by substitution \( t = \frac{1-x}{x+1} \) it’s direct that \( \int_0^1 F(x) \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{F\left(\frac{1-x}{x+1}\right)}{(x+1)^2} \, dx \) holds. Fix the weight \( W \), apply this substitution to all weight \( W \) LIs. For every LI, expanding \( F\left(\frac{1-x}{x+1}\right) \) into log monomials using

\[ l \left( 0, \frac{1-x}{x+1} \right) = l(1; x) - l(2; x) + \log(2), l \left( 1, \frac{1-x}{x+1} \right) = l(0; x) - l(2; x), l \left( 2, \frac{1-x}{x+1} \right) = \log(2) - l(2; x), \]

we
may obtain a homo/non-homo linear relation. Note that only a part of them is linear independent and they’re usually non-homogeneous, therefore it is essential to evaluate LI s with lower weights first for later use.

1.4. Beta derivatives

**Proposition 2.** The following formulas hold:

\[ LI(n, m, 0; 1) = LI(m, n, 0; 0) = \lim_{\{a, b\} \to \{0, 1\}} \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} \quad (1.4.1) \]

\[ \sum_{k=0}^{n} \binom{n}{k} LI(k, m, n-k; 1) = 2^{-(m+1)} \lim_{\{a, b\} \to \{0, 1\}} \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} \quad (1.4.2) \]

\[ \sum_{k=0}^{n} \binom{n}{k} LI(k, m, n-k; 0) = 2^{-(m+1)} \left( \lim_{\{a, b\} \to \{1, 0\}} \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} + \lim_{\{a, b\} \to \{\frac{1}{2}, 0\}} \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} \right) \quad (1.4.3) \]

\[ \sum_{k=0}^{n} \binom{n}{k} LI(k, m, n-k; 2) = 2^{-(m+1)} \left( \lim_{\{a, b\} \to \{0, \frac{1}{2}\}} \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} - \lim_{\{a, b\} \to \{1, 0\}} \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} \right) \quad (1.4.4) \]

\[ LI(0, m, 1; 0) = \left( 2^{-(m+1)} - 1 \right) \lim_{\{a, b\} \to \{1, 0\}} \frac{\partial^{m+1} B(a, b)}{\partial a^m \partial b^1} + 2^{-(m+1)} \lim_{\{a, b\} \to \{\frac{1}{2}, 0\}} \frac{\partial^{m+1} B(a, b)}{\partial a^m \partial b^1} \quad (1.4.5) \]

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} LI(k, n-k, 0; 2) = \lim_{s \to 1} \frac{\partial^m \left( 2^{1-s} - 1 \right) \pi}{\sin(\pi s)} \quad (1.4.6) \]

\[ \sum_{j=0}^{n} \sum_{k=0}^{n+1} \binom{n}{j} \binom{n+1}{k} \left( -\log(2) \right)^{j+k} LI(-k + n + 1, 0, n - j; 2) = \frac{1}{\partial \{x, y\} \to \{0, 1\}} \frac{\partial^{2n+1} B(a, b)}{\partial a^n \partial b^{n+1}} - \frac{\log(2)^{2n+2}}{2n+2} \quad (1.4.7) \]

**Proof.** By differentiating Beta function we have: \( \int_{0}^{1} x^{s-1}(1-x)^{b-1} \log^m(x) \log^n(1-x) \, dx = \frac{\partial^{m+n} B(a, b)}{\partial a^m \partial b^n} \).

Now plug in \( a = 0, b = 1 \) and apply reflection \( t = 1 - x \), we arrive at (1.4.1). Moreover, consider \( \int_{0}^{1} \frac{\log^m(x) \log^n(1-x)}{x^{1+s}} \, dx \). Noticing \( \frac{1}{1-x} = \frac{1-x}{1-x^2}, \frac{1}{1-x^2} = \frac{1-x}{1-x^2}^2 \) and the substitution \( t = x^2 \), we may transform these integrals into RHSs of next three equalities with the help of (1.4.1). On the other hand, by using \( \log(1-x^2) = \log(1-x) + \log(1+x) \) and applying the binomial theorem on \( \log^n(1-x^2) \), they are also equal to corresponding LHSs, and this yields (1.4.2)-(1.4.4). Furthermore, start with \( \frac{\log(x+1)}{1-x} = \frac{x \log(1-x)}{1-x^2} + \frac{\log(1-x^2)}{1-x^2} - \frac{\log(1-x)}{1-x} \),
we multiply both sides of this equality with \( \log^n(x) \) and integrate it on interval \((0, 1)\). This together with substitution \( t = x^2 \) gives \((1.4.5)\). For the next one consider \( F(x) = \frac{\log^n\left(\frac{x}{1-1}ight)}{x+1} \).

By substitution \( t = x/(1 - x) \) we have \( \int_0^1 F(x) \, dx = \int_0^\infty \frac{\log^n(x)}{(x+1)(2x+1)} \, dx \). Then, using the another well-known formula \( \int_0^\infty \frac{x^{n-1}}{(x+1)^{n+1}} \, dx = B(a, b) \), the fact that \( \frac{1}{(x+1)(2x+1)} = \frac{2}{2x+1} - \frac{1}{x+1} \) and scaling \( t = 2x \), we have

\[
\int_0^\infty \frac{x^{n-1}}{(x+1)(2x+1)} \, dx = \frac{\pi}{\sin(\pi s)} \frac{\pi(2^{1-s}-1)}{\sin(\pi s)} \cdot \text{Differentiating w.r.t } s \text{ and setting } s = 1 \text{ gives the RHS of } (1.4.6),
\]

and binomial theorem on \((\log(x) - \log(1 - x))^n\) leads us to its LHS. Finally, for \((1.4.7)\) we consider integral \( \int_0^1 \log^n\left(\frac{1-x}{1-x}\right) \frac{\log^n(x)}{1-x} \, dx \). Integration by parts (lift up a \( \frac{\log^{n+1}(1-x)}{n+1} \)), a reflection \( t = 1 - x \) and Beta's definition yields the RHS. On the other hand, substitute \( x = \frac{1-y}{2} \) and use binomial theorem twice yields the corresponding LHS and completes the proof. \( \square \)

Now we explain how Beta derivatives are calculated analytically. Recall the definition of polygamma \( \psi^{(n)}(x) = \frac{\partial^{n+1} \log\Gamma(x)}{\partial x^{n+1}} \) and that \( B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \), it’s not difficult to see that all order \( N \) mixed partial derivative of Beta function are expressible as:

\[
B(a, b)R(\psi(0)(a), \psi(0)(b), \psi(0)(a+b), \ldots, \psi(N-1)(a), \psi(N-1)(b), \psi(N-1)(a+b))
\]

Where \( R(\ldots) \) is a polynomial with \( 3N \) variables. Now, suppose \( a \) will go to 0 and \( b \) won’t, since all formulas above are of this form. Plug in \( b \)'s value directly to get a function \( g(a) \), then expand all polygamma terms of \( g \) in Laurent series w.r.t \( a \) at the origin. By calculating the asymptotic expansion, we will see that origin is a movable singularity of \( g \), hence the double limit, i.e. \( g(0) \) is uniquely determined. Furthermore, based on \( \psi^{(n)}(z) = (-1)^{n-1}n!\zeta(n+1, z) \) and the basic properties of Hurwitz Zeta \( \zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \), we have \( \psi^{(n)}(k+1) = (-1)^{n-1}n! \left( \zeta(n+1) - H_k^{(n+1)} \right) \).

Due to this fact, all coefficients of Taylor expansion of \( \psi^{(n)}(1+a) \) at origin is expressible in \( \pi, \log(2) \) and zeta values modulo \( \mathbb{Q} \), for arbitrary integer \( n \). Moreover, the case \( \psi^{(n)}(\frac{1}{2}+a) \) is similar if we make use of the relation between zeta and eta function cleverly (left to the readers), whose definitions are recalled at the beginning of the article. Now, all values at integers or half-integers of Beta derivatives can be generated by the Fibonacci basis over \( \mathbb{Q} \), hence also all limits of mixed derivatives in the proposition above. Because of this, we are able to conclude that formulas in the above proposition offers linear relations between LIs. This procedure is easily carried out with the help of CAS. Fix the weight \( W \), for every formula, consider all possibilities of multi-indexes that the maximum weight of LIs appear do not exceed \( W \) to obtain relations.

1-5. Contour integration

Consider the function defined on complex plane, in which all logarithms are in their principal branches (i.e. the absolute value of the argument do not exceed \( \pi \)):

\[
F(z) = \frac{\log^n\left(\frac{0(z)l(1; z)\cdot m_1 l(1; -z)\cdot m_2 l(2; z)}{z(1 \pm z)}\right)}{\log^n\left(\frac{1(z)l(1; z)\cdot m_1 l(1; -z)\cdot m_2 l(2; z)}{z(1 \pm z)}\right)}
\]

We have ten types of contour to get linear relations between LIs. The most important one is the large semicircle contour on the upper half plane (with 3 small semicircle indents at singularities
0, 1, −1). Integral on large semicircle trivially goes to 0 as the radius tends to infinity due to the quadratic denominator of $F$. Moreover, if $k + n > 0$ and $m_1 > 0$ (resp. $m_2 > 0$) for the case the denominator is $z(1 - z)$ (resp. $z(1 + z)$), all 3 singularities are movable, hence the integral along small semicircle vanishes as the radius tends to 0 either. Therefore, by Cauchy’s theorem we can say that $\int_{-\infty}^{\infty} F(x) \, dx = 0$ where the contour is actually the upper side of the real axis and the argument jumps are clear. Now, separate the axis into 4 parts by 3 singularities, by using substitution $t = -x$ and $t = \frac{1}{z}$, we can map the other 3 parts bijectively into $(0, 1)$. Therefore after simplification we arrive at a formula $\int_0^1 (A(x) + B(x) - C(x) - D(x)) \, dx = 0$, where:

$$A(x) = \frac{l(0; x) l(1; x)^{m_1} (l(1; x) - i\pi)^{m_2} l(2; x)^n}{x(1 \pm x)}$$

$$B(x) = (-1)^{m_1 + m_2} \frac{l(0; x) - l(1; x) - i\pi) l(1; x)^{m_1} (l(1; x) + i\pi)^{m_2} (l(2; x) - l(1; x))^n}{x(1 \mp x)}$$

$$C(x) = \frac{l(2; x) l(1; x)^{m_2} l(1; x) + i\pi)^{m_1} l(0; x)^n}{x(1 + x)}$$

$$D(x) = (-1)^{m_1 + m_2} \frac{l(0; x) - l(1; x) + i\pi) n l(1; x)^{m_2} l(1; x) - i\pi)^{m_1} l(2; x) - l(1; x))^k}{x(1 \mp 1)}$$

Expanding this formula just like in method 3, decompose the denominator into partial fractions and take real/imaginary part, we obtain 2 linear relations. They are non-homogeneous and not necessarily of same weight.

Now consider the chain $-\infty < -1 < 0 < 1 < \infty$ and all ordered pair $(a, b)$ where $a < b$ are 2 of the 5 elements. Excluding the case $(-\infty, \infty)$ (which is in fact the case above), we have 9 pairs in total. Each pair defines a unique keyhole contour that’s composed of two large semicircles on upper/lower plane and two keyhole encircling the interval $(-\infty, a)$ and $(b, \infty)$ with possible indents at singularities, and if one of $a, b$ is not finite the corresponding contour vanishes. What’s more, by choosing the interval of arguments of $l(0; z), l(1; \pm z), l(2; z)$ as $(0, 2\pi)$ or $(-\pi, \pi)$ respectively, we can choose the direction of their branch cuts so that all cuts lies in $X = (-\infty, a) \cup (b, \infty)$, therefore $F$ as product of above logarithms is meromorphic inside the contour (if both 2 directions are invalid, we simply restrict the degree of this term to be 0, for instance, if $(a, b) = (-\infty, 0)$ no $l(2; z) = \log(1 + z)$ term should exist). Now, we restrict some of the parameters $k, m_1, m_2, n$ to be non-zero (dependent on the pair chosen), in order to ensure singularities of $F$ that belong to $X$ is movable. Now apply residue theorem to get an integral equality on the real line, and the remaining steps are exactly the same as the upper semicircle case and we’ll omit the details. Using all kinds of these contours with all possible multiple index $(k, m_1, m_2, n)$ in $F$, we are able to obtain much relations as the weight become higher.

1-6. Double integration

According to Au [1], by considering two ways of evaluating $\int_0^1 \int_0^1 \log^2(\frac{x+y}{x+y}) \, dx \, dy$ (one using binomial theorem and (1.1.5), another using substitution $u = x, v = \frac{1-x}{1-y}$), it can be shown that
\[ LI(2n, 0, 1; 1) = (2n)! \left( \sum_{k=0}^{n-1} (-1)^k \Li_{k+1} \left( \frac{1}{2} \right) \Li_{-k+2n+1} \left( \frac{1}{2} \right) + \frac{1}{2} (-1)^n \Li_{n+1} \left( \frac{1}{2} \right)^2 + \Li_{2n+2} \left( \frac{1}{2} \right) \right) \]  

Another formula from Au that offers one relation between LIs arises from evaluating the integral 
\[ \int_0^1 \int_0^1 \log(1 - x) \log(1 - y) \frac{1+x}{1+y} \, dx \, dy. \]  
It can’t be rewritten into an explicit general formula and will not be used later, thus we omitted it here. However, some similar operations will be considered in section 4.

1-7. Hypergeometric identities

It’s also worthy to mention that Au [1] derived an ingenious method that resembles method 4. Using an classical hypergeometric identity, the Beta integral representation of hypergeometric function and differentiation of parameters, we may obtain further relations between high weight LIs. Same as the reason above, we omit it and refer the readers to Au’s original paper.

1-8. Series expansion/Iterated integral

Expanding some part of the LI integrand (for instance \( \log(1 - x) \log(x + 1) \), see section 2-5 for more) into several power series, and using identity (1.1.1) or more complicated results on \( \int_0^1 x^m \log^n(1 - x) \, dx \), \( \int_0^1 x^m \log^k(x) \log^n(1 - x) \, dx \), we may convert an LI into single/multiple alternating sums. Some of them can be evaluated successfully via ES or MZV identities. Another thought is to write the LI into iterated integral which resembles the canonical form of MZV iterated integral representation (see Zagier [16]), then use various shuffle relations and MZV identities to obtain a closed-form. However, we won’t display much about evaluating LIs via ESs or MZVs according to the route of this article (other than those in section 1-1), that we use LI results as a weapon that assist us to carry out the mutual transformation between ESs and PLIs, instead of deducing LI values based on more complicated results. Recording this method is merely for the sake of completeness of LI reduction; on the contrary, the method of series expansion is an important step on establishing PLI relations in section 3.

1-9. Obtaining closed-forms for LIs

For each weight \( W \), a simple counting shows that there are exactly \( \frac{1}{2} (3W^2 + W - 2) \) convergent LIs. Thus, the total number of LIs with weight \( W \leq 5 \) (i.e. low weight LIs) is 85\( (= 1 + 6 + 14 + 25 + 39) \). Now we restate Au’s result [1] here:

Lemma 1. All LIs with weight \( W \leq 5 \) can be generated by \( A_W \) over \( \mathbb{Q} \).

Proof. All linear relations deduced from methods in section 1-1 to 1-7 (in fact 1-1 to 1-6 is enough), satisfying that the maximum weight in the relation does not exceed 5, yields a system containing 85 linear independent equations of 85 variables (that is, the exact value of 85 low weight LIs). Solving
this system completes the proof. □

We don’t claim much originality on this result but a verification of Au instead. We derived section 1-1 to 1-5, the former part of 1-6 and 1-8 independently, but the latter part of section 1-6 and 1-7 are new to us. All 85 low weight LIs are listed in appendix 1.

2. ES evaluation

Comparing to LIs, ESs receive much more attention in literatures. Therefore, we won’t summarize thoroughly like in previous section, but simply review some typical methods and refer readers to corresponding references for others.

2-1. General formulas

All classical general formulas for ESs are recorded in Flajolet & Salvy [6]. We summarize their statements below:

Lemma 2. $ES(\pm 1; p), ES(\pm 1; -p)$ ($p$ even), $ES(p; p), ES(p; q)$ ($p + q$ odd or smaller than 7), $ES(1, 1; p)$ ($p$ odd or smaller than 5) are reducible to zeta values (directly or via Dirichlet eta function).

See their article for proofs using contour integration and Borwein’s alternative solutions [2,3]. Note that these formulas together with those in section 2-5 are applied in (1.1.13)-(1.1.17).

2-2. Symmetry

To some extent, this is an ES-analogue of shuffle relations of MZVs. Rewrite an ES into restricted multiple sums, where the restriction is that one index is always the biggest, for example $ES(2, 3; 4) = \sum_{0<j,k,l}$. By inclusion-exclusion (abbr. I-E) principle, we have:

\[
\sum_{0<j,k,l} = \sum_{0<j,k\leq l} + \sum_{0<j,l\leq k} + \sum_{0<k,l\leq j} - \sum_{0<j\leq k=l} - \sum_{0<k\leq j=k=l} - \sum_{0<l\leq j=k} + \sum_{0<j=k=l}.
\]

Apply this identity as an operator, on all triple summands $\frac{\pm 1}{x+y+z}$ to obtain linear relations between ESs. Notice that when 2 indexes are equal the triple summand reduce to double and similarly single when 3 indexes meets together. If we set all 3 ‘±’ to be ‘+’ and assume $a, b, c > 1$, we obtain:

\[
\zeta(a)\zeta(b)\zeta(c) = ES(a, b; c) + ES(a, c; b) + ES(b, c; a) - ES(a; b + c) - ES(b; a + c) - ES(c; a + b) + ES(a + b + c).
\]

(2.1.1)

Formulas of many other choices of 3 ‘±’ and $a, b, c$ are similar and we omit the details (for alternating summands some of $a, b, c$ can be 1). This method can be extended to $n$-ple sums (i.e. weight $n$)
ESs) easily, namely \( n = 2 \) yields symmetry formulas of ESs immediately, and applications of higher depth version can be found in Xu [14,15]. Also see Borwein [5] for similar contents on weight 3 MZVs.

2-3. Partial fraction decomposition

Consider Cauchy product of the double sum \( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j^k} = \zeta(a) \zeta(b) \). By partial fraction decomposition we know that all sums of the form \( \sum_{k=1}^{n-1} \frac{1}{k^a(n-k)^b} \) are expressible in a combination of homogeneous \( H_k(n) \) for \( a, b = 1, \ldots, a+b-1 \), we found that \( \zeta(a) \zeta(b) \) is expressed in a corresponding combination of ESs, which is a linear relation. The consideration on \( \frac{(\pm 1)^{j-1}(\pm 1)^{k-1}}{j^k} \) gives similar results, but extension to weight 3 is more subtle. In fact in this case we need MZV again, please see Borwein [5] for details.

2-4. Contour integration

Flajolet & Salvy [6] used residue theorem on large circular contour and specific functions to obtain more independent relations for ESs. These functions are of the form \( FG \), where \( F = \frac{1}{z^p} \) and \( G \) is a product of cotangent(resp. cosecant, depending on whether ESs in the relation alternating) and polygamma (resp. poly-Nielsen Beta). Please see their article for further reference.

2-5 Obtaining closed-forms for ESs

Similar to LI case, we simply state that there are 2,9,24,54 different ESs for weight 2,3,4,5(the simple weight 1 case \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) excluded), thus there are 89 (non-alter/alter) ESs with weight \( W \leq 5 \) in total. While the total number of weight \( W \) non-alternating ESs is apparently \( \sum_{k=1}^{W-2} p(k) \) (\( p(k) \) partition numbers), there doesn’t seem to be a general formula for alternating case.

Xu [14,15] had applied various methods (some of them listed above) that generate linear relations to evaluate these ESs. As a result, he remarkably gave explicit closed-forms to all but 1 weight 3 ES, 2 weight 4 ESs and 7 weight 5 ESs. Based on tools above together with what he applied, we verified much of his results successfully. As to solve the remaining problems, we succeeded at weight 3, 4 but partially failed at weight 5. After a considerable time of trying, by the method of ES-PLI mutual transformation, we successfully obtained values of seven remaining high-depth weight 5 ESs, namely:

\[
ES(-1, -1, -1; -2), ES(-1, -1, -1, -1; -1), ES(1, 1, -2; -1), ES(1, 1, -1, -1; -1) \\
ES(1, -1, -1; -2), ES(1, -1, -2; -1), ES(1, -1, -1; -1)
\]

These 7 evaluations are not found in known literatures about ESs, therefore we consider them to be new. Now we will sketch the route of proving them. All LIIs deduced from section 1, low weight ESs together with 49 weight 5 ESs recorded in [6, 14, 15], are supposed to be known and will be used in the proof. Also, several 2-Euler Sum (that is, adding a \( \frac{1}{2^p} \) in the summand of an non-alternating...
ES, e.g. \( \sum_{n=1}^{\infty} \frac{H_n}{n^{2m}} \) evaluations derived by Xu [13] will be applied in later calculations. We abbreviate 2-Euler Sums as 2-ESs.

**Proposition 3.** The following formulas hold where \(|x| < 1\):

\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-x)^n, \quad -\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \tag{2.5.1}
\]

\[
- \frac{\log(1 + x)}{1 + x} = \sum_{n=1}^{\infty} H_n (-x)^n, \quad \log(1 + x) = \sum_{n=1}^{\infty} \tilde{H}_n (-x)^n \tag{2.5.2}
\]

\[
\log^2(1 + x) = \sum_{n=1}^{\infty} 2(-1)^{n-1} \left( \frac{1}{n^2} - \frac{H_n}{n} \right) (-x)^n \tag{2.5.3}
\]

\[
\log(1 - x) \log(x + 1) = - \sum_{n=1}^{\infty} \left( \frac{\tilde{H}_{2n}}{n} + \frac{1}{2n^2} \right) x^{2n} \tag{2.5.4}
\]

\[
\sum_{n=1}^{\infty} \frac{(-x)^n}{n^k} = \text{Li}_k(-x), \quad \sum_{n=1}^{\infty} H_n^{(k)} (-x)^n = \frac{\text{Li}_k(-x)}{1 + x}, \quad \sum_{n=1}^{\infty} \tilde{H}_n^{(k)} (-x)^n = - \frac{\text{Li}_k(-x)}{1 + x} \tag{2.5.5}
\]

\[
\sum_{n=1}^{\infty} \frac{H_n x^n}{n} = \text{Li}_2(x) + \frac{1}{2} \log^2(1 - x) \tag{2.5.6}
\]

\[
\sum_{n=1}^{\infty} \frac{\tilde{H}_n x^n}{n} = \text{Li}_2 \left( \frac{1 - x}{2} \right) - \text{Li}_2(-x) - \text{Li}_2 \left( \frac{1}{2} \right) - \log(2) \log(1 - x) \tag{2.5.7}
\]

\[
\sum_{n=1}^{\infty} (H_n)^2 x^n = \frac{\text{Li}_2(x)}{1 - x} + \frac{\log^2(1 - x)}{1 - x} \tag{2.5.8}
\]

\[
\sum_{n=1}^{\infty} (\tilde{H}_n)^2 x^n = \frac{2 \left( \text{Li}_2 \left( \frac{1 - x}{2} \right) - \text{Li}_2 \left( \frac{1}{2} \right) + \log \left( \frac{x + 1}{2} \right) \log(1 - x) \right) + \text{Li}_2(x)}{1 - x} \tag{2.5.9}
\]

\[
\sum_{n=1}^{\infty} H_n \tilde{H}_n x^n = \frac{\text{Li}_2 \left( \frac{1 - x}{2} \right) - \text{Li}_2(-x) - \text{Li}_2 \left( \frac{1}{2} \right) - \frac{1}{2} \log^2(x + 1) - \log(2) \log(1 - x)}{1 - x} \tag{2.5.10}
\]

\[
\sum_{n=1}^{\infty} H_n H_n^{(2)} x^n = \frac{\text{Li}_3(1 - x) + \text{Li}_3(x) + \frac{3}{2} \log(x) \log^2(1 - x) - \frac{1}{6} \pi^2 \log(1 - x) - 
\]

\[
\zeta(3)}{1 - x} \tag{2.5.11}
\]

\[
\sum_{n=1}^{\infty} (H_n)^3 x^n = \frac{1}{1 - x} \left( 3\text{Li}_3(1 - x) + \text{Li}_3(x) - \log^3(1 - x) + \frac{3}{2} \log(x) \log^2(1 - x) - \frac{1}{2} \pi^2 \log(1 - x) - 3\zeta(3) \right) \tag{2.5.12}
\]
\[ \sum_{n=1}^{\infty} \frac{(H_n)^2 x^n}{n} = \text{Li}_3(x) - \text{Li}_2(x) \log(1 - x) - \frac{1}{3} \log^3(1 - x) \quad (2.5.13) \]

\[ \sum_{n=1}^{\infty} \frac{H_n^{(2)} x^n}{n} = 2\text{Li}_3(1 - x) + \text{Li}_3(x) - 2\text{Li}_2(1 - x) \log(1 - x) - \text{Li}_2(x) \log(1 - x) - \log^2(1 - x) - 2\zeta(3) \quad (2.5.14) \]

\[ \sum_{n=1}^{\infty} \frac{H_n x^n}{n^2} = -\text{Li}_3(1 - x) + \frac{1}{2} \log(x) \log^2(1 - x) + \log(2) \log^2(1 - x) + \zeta(3) \quad (2.5.15) \]

Proof. (2.5.1) is elementary. By considering Cauchy products of formulas in (2.5.1) and use a little partial fraction decomposition, we easily come to (2.5.2)-(2.5.4). Note that (2.5.5) follows similarly using the definition of polylogarithms. By applying the operator \( L(f) = \int_0^x f(t) \, dt \) on specific equalities in (2.5.5), we readily deduce (2.5.6) and (2.5.7). For the next one, we need the technique of calculating differences. Denote LHS of (2.5.8) as \( F(x) \) and suppose it’s of form \( F \sim \frac{H_n}{n} \). By Cauchy product we have \( \frac{H_n}{n} \sim \sum_{j=1}^{\infty} \frac{(H_n)^2 - (H_{n-1})^2}{j(j+n)} \) where \( H_n = H_{n-1} - \frac{1}{n} \). Expanding RHS of \( G \) and making use of (2.5.5), (2.5.6) gives (2.5.8). (2.5.9)-(2.5.12) can be proved using the same technique as in (2.5.8) if we notice that \( \tilde{H}_n = \tilde{H}_{n-1} + (-1)^{n-1} \). Finally (2.5.13), (2.5.14) and (2.5.15) are consequences of (2.5.8), (2.5.5), (2.5.6) separately as we make use of \( L(f) \) again. All (indefinite) integration operations can be carry out by CAS, or manually using \( \int \frac{\text{Li}_n(x)}{x} \, dx = \text{Li}_n(x) \) and basic integration methods flexibly, therefore we omit the details. \( \square \)

**Proposition 4.** The following formulas hold:

\[ \int_0^1 x^{n-1} \log^k(x) \, dx = \frac{(-1)^k k!}{n^{k+1}} \quad (2.5.16) \]

\[ \int_0^1 x^{n-1} \log(1 - x) \, dx = -\sum_{j=1}^{\infty} \frac{1}{j(j+n)} = -\frac{H_n}{n} \quad (2.5.17) \]

\[ \int_0^1 x^{n-1} \log^2(1 - x) \, dx = \sum_{j=1}^{\infty} \frac{1}{j(j+n)^2} = \frac{(H_n)^2 + H_n^{(2)}}{n} \quad (2.5.18) \]

\[ \int_0^1 x^{n-1} \log(1 - x) \log(1 - x) \, dx = \frac{H_n}{n^2} + \frac{H_n^{(2)}}{n} - \frac{\pi^2}{6n} \quad (2.5.19) \]

\[ \int_0^1 x^{n-1} \log^3(1 - x) \, dx = -\frac{3H_n H_n^{(2)} + (H_n)^3 + 2H_n^{(3)}}{n} \quad (2.5.20) \]

\[ \sum_{j=1}^{\infty} \frac{(-1)^j}{j(j+n)} = \frac{(-1)^{n-1} \tilde{H}_n + (-1)^n \log(2) - \log(2)}{n} \quad (2.5.21) \]
\[
\sum_{j=1}^{\infty} \frac{(-1)^j}{j(j+n)^2} = \frac{(-1)^{n-1} H_n^{(2)}}{n} + \frac{(-1)^{n-1} H_n}{n^2} - \frac{(-1)^{n-1} \log(2)}{n^2} - \frac{\log(2)}{n^2} - \frac{\pi^2 (-1)^{n-1}}{12 n}
\] (2.5.22)

Proof. (2.5.16) is just (1.1.1) with indexes rewritten. (2.5.17)-(2.5.20) follows differentiating Beta functions and use relationships between polygamma and generalized harmonic numbers (see section 1-4), possibly with a expansion of log terms recorded in proposition 3. (2.5.21) and (2.5.22) are direct consequences of partial fraction decomposition and \( \eta(1) = \log(2), \eta(2) = \frac{\pi^2}{12}. \) \( \Box \)

Now we are able to obtain new closed-forms for 7 weight 5 ESs mentioned above. 1 weight 3 ES and 2 weight 4 ESs, which are not contained in literatures either, are relatively trivial and we simply omit their proofs.

**Proposition 5.** The following formulas hold:

\[
ES(-1, -1, -1; -2) = -18 \text{Li}_5 \left( \frac{1}{2} \right) - \frac{\pi^2 \zeta(3)}{8} + \frac{1229 \zeta(5)}{64} \tag{2.5.23}
\]

\[
ES(-1, -1, -1, -1; -1) = -48 \text{Li}_5 \left( \frac{1}{2} \right) - 4 \text{Li}_4 \left( \frac{1}{2} \right) \log(2)
+ \frac{5\pi^2 \zeta(3)}{48} + \frac{733 \zeta(5)}{16} + \frac{13 \log^5(2)}{30} + \frac{3}{2} \pi^2 \log^3(2) - \frac{71}{180} \pi^4 \log(2) \tag{2.5.24}
\]

\[
ES(1, 1, -2; -1) = 6 \text{Li}_5 \left( \frac{1}{2} \right) - \frac{\pi^2 \zeta(3)}{48} - \frac{93 \zeta(5)}{64}
- \frac{1}{20} \log^5(2) + \frac{1}{36} \pi^2 \log^3(2) - \frac{13 \pi^4 \log(2)}{1440} \tag{2.5.25}
\]

\[
ES(1, 1, -1, -1; -1) = 12 \text{Li}_5 \left( \frac{1}{2} \right) + \frac{\pi^2 \zeta(3)}{16} - \frac{201 \zeta(5)}{16} \tag{2.5.26}
\]

\[
- \frac{3}{8} \zeta(3) \log^2(2) + \frac{\log^5(2)}{10} + \frac{1}{12} \pi^2 \log^3(2) + \frac{97}{720} \pi^4 \log(2)
\]

\[
ES(1, -1, -1; -2) = 10 \text{Li}_5 \left( \frac{1}{2} \right) - 2 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) + \frac{\pi^2 \zeta(3)}{96} \tag{2.5.27}
\]

\[
-10 \zeta(5) - \frac{7}{8} \zeta(3) \log^2(2) - \frac{1}{6} \log^5(2) + \frac{2}{9} \pi^2 \log^3(2) + \frac{1}{10} \pi^4 \log(2)
\]

\[
ES(1, -1, -2; -1) = 4 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) + \frac{7}{16} \zeta(3) \log^2(2)
+ \frac{\log^5(2)}{6} - \frac{5}{72} \pi^2 \log^3(2) - \frac{1}{480} \pi^4 \log(2) \tag{2.5.28}
\]
\[ ES(1, -1, -1, -1; -1) = 2\text{Li}_5 \left( \frac{1}{2} \right) + \frac{13\pi^2 \zeta(3)}{96} - \frac{83\zeta(5)}{32} \]

\[ + \frac{9}{16} \zeta(3) \log^2(2) - \frac{13}{60} \log^5(2) + \frac{41}{72} \pi^2 \log^3(2) - \frac{1}{180} \pi^4 \log(2) \quad (2.5.29) \]

Proof. (2.5.23) and (2.5.24) follows immediately by symmetry, we take the latter one as an example. Following section 2-2, we have:

\[ ES(-1, -1, -1, -1; -1) = \sum_{0<j,k,l,m \leq n} (-1)^{j-1}(-1)^{k-1}(-1)^{l-1}(-1)^{m-1}(-1)^{n-1} \]

Consider the analogous sum without restrictions, by I-E principle we have:

\[ \log(2) = 5ES(-1, -1, -1, -1; -1) - 10ES(-1, -1, -1; 2) + 10ES(-1, -1, -1; -3) - 5ES(-1, 4) + \eta(5) \]

In this equality, everything but \( ES(-1, -1, -1, -1; -1) \) are known, therefore we’ve readily deduced (2.5.24). (2.5.23) is similar if we consider the I-E equality for \( \sum (-1)^{j-1}(-1)^{k-1}(-1)^{l-1}(-1)^{m-1} \). This method won’t work for the other 5 ESs, for if we write one of them as multiple sums like (2.5.24), the corresponding I-E equality will contain a harmonic factor \( \sum \frac{1}{k} \) one side and \( \sum \frac{H(k)}{k} \) (product of alter/non-alter harmonic numbers) another, which are both divergent apparently.

Now we deal with (2.5.25). By multiplying both sides of (2.5.22) with \((H_n)^2\) and sum it for all positive integer \( n \), we have:

\[ -\frac{\pi^2}{12} ES(1, 1; -1) - \log(2) ES(1, 1; 2) - \log(2) ES(1, 1; -2) \]

\[ + ES(1, 1, -1; -2) + ES(1, 1, -2; -1) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j (H_n)^2}{j(j+n)^2} \]

But by (2.5.16), Fubini theorem (conditions satisfied), (2.5.1) and (2.5.8) we have:

\[ \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j (H_n)^2}{j(j+n)^2} = \int_0^1 \log(x) \left( \sum_{j=1}^{\infty} \frac{(-1)^j x^j}{j} \right) \left( \sum_{n=1}^{\infty} (H_n)^2 x^n \right) dx \]

\[ = \int_0^1 \left( \frac{1}{x} + \frac{1}{1-x} \right) \log(x) \log(x+1) \left( \text{Li}_2(x) + \log^2(1-x) \right) dx \]

After expanding the RHS, we obtain a relation between 5 ESs, 2 LIs and 2 PLIs, where 4 ESs other than the desired one and 2 LIs are already known. Therefore, we only need to get the value for two PLIs, namely \( PLI(0, 1, 1; 21; 0), PLI(0, 1, 1; 21; 1) \) to finish (2.5.25) (i.e. this ES depends on two PLIs modulo known LIs and ESs). This leads us to one theme of this article: mutual transformations between PLIs and ESs. Although we present our solution on ESs via solving particular problems and LI/PLIs via offering general patterns, the idea behind is the same: find enough relations, solve...
them all. Since we would like to solve all ESs before investigating PLIs, it’s clear that we should apply double series expansion to go back to ESs, using (2.5.1), (2.5.2) and (2.5.5). That is:

\[ PLI(0, 1, 1; 21; 1) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \int_0^1 \log(x)x^{j+n-1} \, dx}{jn^2} = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j}{jn^2(j+n)^2} \]

\[ PLI(0, 1, 1; 21; 0) = ... = -\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{\widetilde{H}_j}{n^2(j+n+1)^2} \]

For the first one, sum either \( n \) or \( j \) first elementarily and the PLI reduces to ESs already known. For the second one, sum \( n \) first and after a index change we get:

\[ -\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{\widetilde{H}_j}{n^2(j+n+1)^2} = -\sum_{j=1}^{\infty} \left( \widetilde{H}_j - \frac{(-1)^{j-1}}{j} \right) \left( -\frac{2H_j}{j^3} - \frac{H_j^{(2)}}{j^2} + \frac{\pi^2}{3j^2} \right) \]

Expand the RHS and use known ES values again, we find the second PLI and the desired (2.5.25) is established, and we add (2.5.23)-(2.5.25) to the set of known ESs. Furthermore, solution of (2.5.25) can be modified to fit for (2.5.26)-(2.5.28). We won’t present a step-by-step solution like in (2.5.25) but sketch the route and point out the important steps:

For (2.5.26), start from (2.5.18), multiply both sides with \((-1)^{n-1} \widetilde{H}_n^2\) and sum \( n \), using (2.5.9) we know that the desired ES follows from 4 PLIs – \( PLI(2, 0, 0; 22; 1/2) \), \( PLI(2, 0, 0; 24; 1/2) \) modulo known ESs and LIIs. For \( PLI(2, 0, 0; 22; 1/2) \), use (2.5.5) for single expansion to reduce them to known ESs. For \( PLI(2, 0, 0; 24; 1/2) \), recall reflection formula of dilogarithm to obtain:

\[ \text{Li}_2\left(\frac{1-x}{2}\right) + \text{Li}_2\left(\frac{x+1}{2}\right) = \log(2) \log(1-x) + \log(2) \log(x+1) - \log(1-x) \log(x+1) + \frac{\pi^2}{6} - \log^2(2). \]

This allow us to reduce them to \( PLI(2, 0, 0; 23; 1/2) \) modulo LIIs again. Now, by substitution \( t = 1-x \), single/double expansion for \( \frac{\text{Li}_2\left(\frac{x}{1-x}\right)}{x} \) (see (2.5.1) and (2.5.5)) and Xu’s result on 2-Euler Sums [13] reduce them to known ESs.

Techniques used in (2.5.27) are exactly the same (e.g. multiplying \((-1)^{n-1} \widetilde{H}_n^2\), reflection formula and double expansion), except that we start from (2.5.19) instead. Readers may verify that all ESs appeared during the reduction are known and all PLIs can be reduced to known ESs. Thus, we add (2.5.26) and (2.5.27) to the set of known ESs. (2.5.28) is similar to (2.5.25): also start from (2.5.22), choose \( H_n \widehat{H}_n \) as the multiplier, the remaining steps are alike to (2.5.25)-(2.5.27). If readers carry out the whole process, they will find that all unknown PLIs reduces to known ESs satisfactorily because among them the result of (2.5.27) is used, which had been shown just now.

Finally we prove (2.5.29). Denote \( f(x) = \sum_{n=1}^{\infty} \left( \widehat{H}_n \right)^3 x^n \), by calculating differences of the generating function (see proposition 3) and a change of variable, one can show that \( f(-x) = \frac{-3P(x)-3Q(x)-\text{Li}_3(x)}{x+1} \), where \( P(x) = \sum_{n=1}^{\infty} \frac{\left( \widehat{H}_n \right)^2 x^n}{n} \), \( Q(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \widehat{H}_n x^n}{n^4} \). By (2.5.17) we have:

\[ ES(1, -1, -1, -1; -1) = \int_0^1 \frac{f(-x) \log(1-x)}{x} \, dx = \int_0^1 \frac{\log(1-x)(-3P(x)-\text{Li}_3(x)-3Q(x))}{x(x+1)} \, dx \]
Now integrate by parts with the help of
\[
\int \frac{\log(1-x)}{x(x+1)} \, dx = u(x) = C - \text{Li}_2 \left( \frac{1-x}{2} \right) - \text{Li}_2(x) - \log(1-x) \log \left( \frac{x+1}{2} \right)
\]
(where we take \( C = \frac{\pi^2}{6} \) for convergence issues), we know that the desired ES value equals to the following integral:

\[
\int_0^1 u(x) \left( 3 \sum_{n=1}^{\infty} \frac{(-1)^n H_n x^n}{n} + 3 \sum_{n=1}^{\infty} \left( H_n \right)^2 x^n + \text{Li}_2(x) \right) \, dx
\]

Using (2.5.7) and (2.5.9), the integral can be furtherly decomposed into 3 parts:

\[
3 \int_0^1 u(x) \left( 2 \text{Li}_2 \left( \frac{1-x}{2} \right) + \text{Li}_2(x) - 2 \text{Li}_2 \left( \frac{1}{2} \right) + 2 \log \left( \frac{x+1}{2} \right) \log(1-x) \right) \frac{1}{x(1-x)} \, dx
\]

\[
+ 3 \int_0^1 u(x) \left( \text{Li}_2 \left( \frac{x+1}{2} \right) - \text{Li}_2(x) - \text{Li}_2 \left( \frac{1}{2} \right) - \log(2) \log(x+1) \right) \frac{1}{x} \, dx + \int_0^1 \text{Li}_2(x) u(x) \, dx
\]

Now we expand all parentheses (including \( u(x) \)) in the above expression. Note that some parts of the expanded integrand are divergent and we have to put them together again, such as \( \int_0^1 \text{Li}_2 \left( \frac{x+1}{2} \right) - \text{Li}_2 \left( \frac{1}{2} \right) - \log(2) \log(x+1) \frac{1}{x} \, dx \). Apply integration by parts on all these combined integrals (in the example lift up a \( \text{Li}_3(x) \)) to convert it into convergent integrals. After this we can reduce the monstrous expression to combination of plenty of LIs and PLIs. Now integrate by parts and use techniques in (2.5.25)-(2.5.28) over and over again, we’ve verified manually that all PLIs can be reduced to lower weight ESs and 53 weight 5 ESs already deduced, hence we get (2.5.29) and the proof of proposition is finished.

In proof of (2.5.27)-(2.5.29) (especially the last one) much details are omitted and left to interested readers. One may verify these proofs more easily after reading section 3, that is, the systematic approach of establishing relations between PLIs and reducing them to ESs. In fact, the 54th ES in (2.5.29) is one of the most complicated results in this article and will not appear in all reductions of PLIs, which confirms that we are not arguing in a circle. All 89 ESs with weight \( \leq 5 \) are tabulated in appendix 2.

3. PLI Evaluation

It’s straightforward to see that a PLI’s weight is at least 3 due to the existence of a polylog term. After a counting we know that there are 11, 55, 197 distinct PLIs for weight 3, 4, 5. As we are going to show, they all lies in the \( \mathbb{Q} \)-linear space spanned by the Fibonacci basis with corresponding weights. Parallel to LI case, we point out that some of the PLIs are evaluated directly by calculating polylog primitives, for instance, \( \text{PLI}(1, 0, 0; 24; 2) = \int_0^1 \frac{\log(1-x)}{x+1} \, dx \) has primitive \( \log(2) \text{Li}_3 \left( \frac{x+1}{2} \right) - \frac{1}{2} \text{Li}_2 \left( \frac{x+1}{2} \right)^2 \). However, this only works for a few PLIs with low weight, so we exclude it from the systematic approach below.

Most results obtained in this section are completely new.
3.1 General formulas

Similar to ES case, we refer the reader to Freitas [7] and Valean [10] on for 3 well-known PLI formulas. We summarize their statements as

\[ \int_0^1 \frac{\text{Li}_m(x)\text{Li}_n(x)}{x} \, dx, \quad \int_0^1 \frac{\log^m(x)\text{Li}_n(x)}{1-x} \, dx \quad \text{(m + n even)} \] and \[ \int_0^1 \frac{\text{Li}_{m+1}(x)\log^n(x)}{x+1} \, dx \] are reducible to zeta values.

Note that using proposition 3 we can modify these appropriately, to obtain general formulas for other PLIs such as \[ \int_0^1 \frac{\text{Li}_m(x)\text{Li}_n(-x)}{x} \, dx \] and \[ \int_0^1 \frac{\log^m(x)\text{Li}_n(x)}{x+1} \, dx. \] The crucial idea here is (multiple) series expansion, which will be explained more clearly in section 3-5, so we left the simple generalization to readers.

3-2. Integration by parts

Completely analogous to section 1-2, apply N-L formula where the differentiable \( F \) is of form \((l(2; x) - \log(2))\frac{x}{x+1} \int_0^1 \prod_{m=0}^2 l(m; x)^{u(m)} \prod_{k=1}^6 \Pi_{n=2} L(n, k; x)^{b(n, k)}\), such that \(|F(0^+)|, |F(1^-)| < \infty \) as before. For instance, taking \( F \) to be \( \text{Li}_2(x) \log(1-x) \log(x+1) \log(\frac{x+1}{2}) \) gives a homogeneous relation of weight 5, and generally we consider all \( F \) with \( r + \sum_{k=1}^6 \sum_{n=2}^N n b(n, k) + \sum_{m=0}^2 a(m) = W \) to obtain weight \( W \) relations. Note that polylog terms can appear more than once, so there’re plenty of choices on the kernel \( F \). From the view of solving single PLIs, the above process can be explained as doing integration by parts flexibly, that is, choosing different parts of the PLI integrand to lift up. As an example, we may lift up either \( \frac{1}{2} \log^2(x+1) = \int \frac{\log(x+1)}{x+1} \, dx \) or \( \text{Li}_3(\frac{x+1}{2}) = \int \frac{\text{Li}_2(\frac{x+1}{2})}{x+1} \, dx \) in \( \text{PLI}(0, 1, 1; 24; 2) = \int_0^1 \frac{\text{Li}_2(\frac{x+1}{2})\log(x)\log(x+1)}{x+1} \, dx \), which leads to different relations.

3-3. Fractional transformations

Similar to section 1-3, take \( F = \prod_{m=0}^2 l(m; x)^{u(m)} \prod_{k=1}^6 \Pi_{n=2} L(n, k; x)^{b(n, k)} \) with \( \int_0^1 F(x) \, dx \) convergent and \( b(n, 3) = b(n, 4) = 0 \) for all \( n \) (equivalently, \( \text{Li}_n(\frac{1+z}{2}) \) terms won’t appear). Fix the weight \( W \), use the same substitution \( t = \frac{x+1}{2} \) on every suitable \( F \), then expand all log terms (via formulas in section 1-3) and plug in LI values to obtain weight \( W \) relations. Also, \( L(n, 1; x) \) (resp. \( L(n, 2; x) \)) and \( L(n, 5; x) \) (resp. \( L(n, 6; x) \)) are transformed to each other under this substitution.

Here is another way to obtain relations. Denote \( P(x) = \prod_{n=2}^N \text{Li}_n(x)^{a(n)} \), by splitting \((0, 1)\) and substituting \( t = \frac{1+x}{2} \) for 2 parts respectively we have:

\[
\int_0^1 \left( \frac{P(\frac{1-x}{2}) \log^m(\frac{1-x}{2}) \log^n(\frac{x+1}{2})}{1+x} + \frac{P(\frac{x+1}{2}) \log^m(\frac{x+1}{2}) \log^n(\frac{1-x}{2})}{1-x} \right) \, dx.
\]

\[
= \int_0^1 \frac{P(x) \log^m(x) \log^n(1-x)}{f_+(x)} \, dx
\]

Where \( f_+(x) = x, f_-(x) = 1-x \). Vary \( a(n) \), this formula offer relations that connect PLIs involving...
$L(n, 1; x), L(n, 3; x)$ and $L(n, 4; x)$. 

3-4. Polylogarithmic functional equations

Proposition 6. The following formulas hold as $0 < x < 1$, where all logs/polylogs are on their principle branches:

\[ \text{Li}_n(-x) + \text{Li}_n(x) = \frac{\text{Li}_n(x^2)}{2 - 1} \quad (3.4.1) \]

\[ \text{Li}_2(1 - x) + \text{Li}_2(x) = \frac{\pi^2}{6} - \log(x) \log(1 - x) \quad (3.4.2) \]

\[ \text{Li}_2 \left( \frac{x}{x - 1} \right) + \text{Li}_2(x) = -\frac{1}{2} \log^2(1 - x) \quad (3.4.3) \]

\[ \text{Li}_2 \left( \frac{-1}{x} \right) + \text{Li}_2(-x) = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(x) \quad (3.4.4) \]

\[ \text{Li}_2 \left( \frac{1 - x}{2} \right) + \text{Li}_2 \left( \frac{x + 1}{x + 1} \right) = \log(2) \log(x + 1) + \log(2) \log(1 - x) \]

\[ -\log(1 - x) \log(x + 1) + \frac{\pi^2}{6} - \log^2(2) \quad (3.4.5) \]

\[ \text{Li}_2 \left( \frac{1 - x}{2} \right) + \text{Li}_2 \left( \frac{x - 1}{x + 1} \right) = -\frac{1}{2} \log^2(x + 1) + \log(2) \log(x + 1) - \frac{1}{2} \log^2(2) \quad (3.4.6) \]

\[ \text{Li}_3(-x) - \text{Li}_3 \left( \frac{-1}{x} \right) = -\frac{1}{6} \log^3(x) - \frac{1}{6} \pi^2 \log(x) \quad (3.4.7) \]

\[ \text{Li}_3 \left( \frac{x}{x - 1} \right) + \text{Li}_3(1 - x) + \text{Li}_3(x) = \frac{1}{6} \log^3(1 - x) - \frac{1}{2} \log(x) \log^2(1 - x) + \frac{1}{6} \pi^2 \log(1 - x) + \zeta(3) \quad (3.4.8) \]

\[ \text{Li}_3 \left( \frac{1 - x}{2} \right) + \text{Li}_3 \left( \frac{x + 1}{2} \right) + \text{Li}_3 \left( \frac{x - 1}{x + 1} \right) = \frac{1}{6} \log^3(x + 1) - \frac{1}{2} \log^2(2) \log(1 - x) \]

\[ -\frac{1}{2} \log^2(x + 1) \log(1 - x) + \left( \frac{\pi^2}{6} - \frac{\log^2(2)}{2} \right) \log(x + 1) \quad (3.4.9) \]

\[ + \log(2) \log(1 - x) \log(x + 1) + \zeta(3) + \frac{\log^3(2)}{3} - \frac{1}{6} \pi^2 \log(2) \]

Proof. (3.4.1)-(3.4.4), (3.4.7)-(3.4.8) are classical formulas for polylogarithms and can be found in Lewin [9]. (3.4.5) is easily deduced from (3.4.2) if we apply the substitution $t = \frac{1 - x}{2}$. The same substitution in (3.4.3) gives (3.4.6), and (3.4.8) gives (3.4.9). For the last one a little simplification
is needed.

Now we can obtain relations via proposition 6. First of all, fix \( m,n \), multiply both sides of (3.4.1) with \( \log^m (1-x^2) \) and integrate it on \( (0,1) \). Substitute \( t = x^2 \) on RHS and expand \( \log(1-x)+\log(x+1))^m \) on LHS using binomial theorem, we obtain a relation. Moreover, (3.4.5), (3.4.6) and (3.4.9) can offer many nontrivial relations either. For instance, multiply both sides of (3.4.9) with a log kernel \( F(x) = \prod_{m=0}^{l} l(m;x) \) such that \( \int_0^1 F(x)L(3,k;x) \) on LHS and all LIs on RHS are convergent. It gives a relation between 3 PLIs involving trigonometric functions already, since the value of LIs on RHS are already known. For (3.4.5) and (3.4.6) the process is similar, except that we can broaden the range of \( F \), that is, allowing \( F \) to contain polylog terms. For our case (weight \( \leq 5 \)), they are single dilog terms \( L(2,k;x) \). Fix the weight \( W \), consider all \( F \) such that \( \int_0^1 LHS \cdot F(x) \) and \( \int_0^1 RHS \cdot F(x) \) are combinations of weight \( W \) LI/PLIs to obtain relations. Generally these relations connect PLIs involving \( L(n,3;x), L(n,4;x) \) and \( L(n,6;x) \).

3-5. Series expansion

Now we come to the last part of building the PLI relation system—reduce some of PLIs to combination of ESs to obtain their value directly. Because all PLI we deal with have weight no more than 5, we may classify them clearly. For further explanation, we have 4 kinds of weight 5 PLIs in total, named as (2,1,1) class, (3,1) class, (4) class and (2,2) class separately. The index shows the quantity as well as order of polylog terms, for instance, \( PLI(1,1,0;22;2) \) belongs to (2,1,1) class because we have one dilog term and two log terms in the numerator, and the classification for other weight 5 PLIs is similar. Note that class (1,1,1,1,1) is excluded for integrals belong to this class are nothing but LIs determined in section 1.

From now on we only discuss about weight 5 PLIs, for lower weight PLIs are relevantly trivial. Moreover, we restrict the use of this method to (2,1,1), (3,1) and (4) type PLIs only, for relations generated by these PLIs via series expansion, together with all relations given by secticons above, have lead to closed-forms of all 197 weight 5 PLIs already. Furthermore, we require the only polylog term \( L(n,k;x) \) \( n = 2,3 \) contained in the PLI to satisfy \( 1 \leq k \leq 3 \) as other polylog terms are not convenient to apply series expansion. Denote the set of these PLIs as \( Q \).

We’ll only present general procedure of converting the PLI into combination of ESs. Note that in this section ‘ES’ represents all ordinary ESs together with all Xu’s 2-ES results [13].

Step 1. Reflection. If the only polylog term involved is \( \text{Li}_n \left( \frac{1-x}{x} \right) \), substitute \( t = 1-x \) in the integral. If the only polylog term involved is \( \text{Li}_n(-x) \), keep it unchanged. If the only polylog term involved is \( \text{Li}_n(x) \), keep it unchanged, or reduce the integral to the corresponding one involving \( \text{Li}_n(1-x) \) via (3.4.2) modulo known LIs, then substitute \( t = 1-x \). After this operation, the argument in polylog term is either \( x \) or \( \frac{x}{2} \). In this step we have 2 choices for case \( \text{Li}_n(x) \) and 1 choice for other 2 cases.
Step 2. Determine the invariant part of integrand. Since we will use (2.5.16)-(2.5.19) to convert the integral to ESs, we choose a part of the integrand to be 'invariant', which means we won’t expand them into series later. This part should be of form \( F_1(x) = \log^m(x) \log^n(1-x) \), with \( m \) equal to the degree of \( \log(x) \) in the original integrand, \( n \) do not exceed the degree of \( \log(1-x) \) in the original integrand. In this step we have at most 3 choices on determining \( n \), namely \( n = 0, 1, 2 \).

Step 3. Multiple expansion. Denote \( F = F_1F_2 \) where \( F \) is the whole integrand (after possible manipulations in step 1), \( F_1 \) the invariant part determined in step 2. We have \( F_2 \) a combination of products of functional terms belong to the set:

\[
\left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{x+1}, \frac{1}{1-\frac{x}{2}}, \log(1-x), \log(x+1), \log\left(1-\frac{1}{2}\right), \text{Li}_n(x), \text{Li}_n\left(\frac{x}{2}\right) \right\}
\]

Where by combination we mean the integral is composed of several integrals with different weights (see examples below). Separate \( F_2 \), expand all terms (other than \( \frac{1}{x} \)) for each into elementary power series using (2.5.1)/(2.5.5) and exchange the order of evaluation, we get a multiple sum (at most quadruple) with the summand an integral consistent with (2.5.16)-(2.5.19). In this step we have only 1 choice for the process of expansion is definite.

Step 4. Converting into multiple number series. This is straightforward if we use (2.5.16)-(2.5.19) to evaluate the integral in rational/harmonic terms. In this step we have only 1 choice for the process of integration is definite.

Step 5. Cauchy’s simplification. After step 4 we arrive at a multiple sum with at most 4 indexes. If there’s only 1 index, it’s already a combination of ESs hence the procedure is finished. If there are more than 2 indexes, sum w.r.t all indexes with rational summands to reduce the weight, and go to the next step if the order reduces to 2. Otherwise (order greater than 2 and summand harmonic to all indexes), consider all index pairs \((j, k)\) and check whether their Cauchy product is expressible in harmonic terms with a new index \( m \). If so, replace \((j, k)\) with \( m \) in the summation, so that the order of the sum is reduced by 1, and we should consider index pairs again for Cauchy simplification until the order of summation reduce to 2, or still larger than 2 but cannot be Cauchy-simplified any more (this can be done equivalently in step 3, where we should consider Cauchy type generating functions instead, namely (2.5.2), (2.5.3) and (2.5.5)). For the first case go to step 6, and for the second discard this sum and stop the procedure. In this step we have a great deal of choices to simplify the sum, but much of them will be discarded.

Step 6. Reduction to ESs. If we go through step 5, we will have a double series in hand. Denote the index of summation as \((j, k)\). If the summand is rational functions of \( j, k \), sum w.r.t either of the indexes first to finish the procedure; if the summand is rational to \( j \) but harmonic to \( k \), sum w.r.t \( j \) also finishes; see proof of proposition 5 for examples. If we have all harmonic terms with index \( j + k \) and other parts rational, invoke Cauchy product again to finish the procedure. For other cases (e.g. the sum involves \( H_kH_{j+k} \)), discard it. In this step we have 1 or 2 choices for using
iterated summation or Cauchy simplification.

Fix a weight 5 PLI that belongs to $Q$, consider the tree of choices that we would make through step 1-6 of the procedure above. If there exist a branch of this tree that is not ended with 'discarded' in step 5-6, we have reduced the PLI to ESs successfully. Since $[6, 14, 15]$ together with section 2 already cover all weight 5 ESs, we readily obtained the closed-form of this PLI. If all branches are discarded, we get nothing new from the procedure. Now carry out the procedure for every PLI belong to $Q$ to obtain independent relations which give PLI values directly. The whole process is extremely lengthy and we only show some typical examples:

A successful example: Consider $PLI(1, 0, 1; 21; 2) = \int_0^1 \frac{Li_2(x) \log(1-x) \log(x+1)}{x+1} \, dx$. Step 1: Modulo known LIs, we only need to evaluate $\int_0^1 \frac{Li_2(x) \log(1-x) \log(x+1)}{x+1} \, dx$ then plug in (3.4.2), so substitute $t = 1 - x$. Step 2: After reflection, the problem boils down to evaluating $\int_0^1 \log(2) \frac{Li_2(x) \log(x)}{1-x^2} \, dx$ and $\int_0^1 \frac{Li_2(x) \log(x) \log(1-x^2)}{1-x^2} \, dx$. Set $\log(x)$ to be invariant in both 2 integrals. Making use of double expansion, (2.5.16) and [13]'s results, we come to the closed-form of the former weight 4 integral. Now we deal with the latter one. Step 3: Expand $\frac{Li_2(x) \log(x)}{1-x^2}$ with 3 independent index $j, k, l$. Step 4: Make use of (2.5.16). Step 5: Consider Cauchy product of the summation w.r.t $k, l$. The convolution term is harmonic hence the triple sum is reduced to double, which can be deduced either if we expand $\frac{\log(1-x^2)}{1-x^2}$ directly via (2.5.2) in step 3. Step 6: Denote the single Cauchy index replacing $j, k$ as $m$. The double sum we have in hand is $\sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{H_m}{j(m+j+1)}$. Sum w.r.t $j$ (this is nothing but the elementary sum occured in proof of (2.5.25), proposition 5), we reduced this sum to a combination of three 2-ESs, which are also solved in [13]. The result is:

$$PLI(1, 0, 1; 21; 2) = -2Li_5 \left( \frac{1}{2} \right) - Li_4 \left( \frac{1}{2} \right) \log(2) + \frac{23\pi^2\zeta(3)}{96} - \frac{15\zeta(5)}{64} - \frac{21}{16} \zeta(3) \log^2(2) - \frac{1}{40} \log^5(2) + \frac{5}{72} \pi^2 \log^3(2) - \frac{11}{720} \pi^4 \log(2)$$

(3.5.1)

Unsuccessful examples: Readers may verify that $PLI(1, 0, 1; 22; 2), PLI(2, 0, 0; 22; 2)$ are not solvable using this procedure. At first glance they are not much more difficult than the PLI above, but it’s impossible to make choices in the procedure above to avoid the multiple sum containing summand of form $H_k H_{j+k}$ or its alternating analogue, hence in step 6 they must be discarded. Actually, these two integrals are evaluated nontrivially when we put all relations generated from section 3 together. From the viewpoint of solving single problems, we have to integrate by parts repeatedly, then use fractional transformation and connection formulas in section 3-4, finally evaluate those PLIs involving $Li_3(-x), Li_4 \left( \frac{1-x^2}{2} \right)$ by multiple expansions. This is a highly circuitous process, which reveals the necessity of a systematic approach providing linear relations, that is, the whole section 3.

3-6. Obtaining closed-forms for PLIs
Finally we are able to prove the main result:

Proof (of Theorem 1). In section 2 we showed that the theorem hold for all ESs with weight \( \leq 5 \). Moreover, all linear relations deduced from methods in section 3-1 to 3-5, satisfying that the maximum weight of PLIs in the relation does not exceed 5, forms a linear equation system from which all PLIs with weight \( \leq 5 \) are explicitly solved. \( \square \)

All closed-forms of PLIs with weight \( \leq 5 \), most of which are not discussed in literatures, are tabulated in appendix 3.

4. Applications

We can apply results obtained in section 1 to section 3 on many topics. In section 4-1 to 4-5 we present some systematic applications, and in 4-6 10 specific problems are solved. Furthermore, some typical pattern-displaying results, instead of all closed-form integrals and series we deduced, are tabulated in appendix 4.

Most results obtained in this section are completely new.

4-1. Logsine integrals, poly-logsine integrals and others

The first natural application of LIs is the evaluation of a special kind of logsine integrals (abbr. LSIs).

\[ LSI(a, b, c; d) = \int_0^{\frac{\pi}{2}} x^a \log^b(2 \sin(x)) \log^c(2 \cos(x)) \cot^d(x) \, dx \quad (4.1.1) \]

For 'logsine' we mean the interval is \((0, \frac{\pi}{2})\) and integrand a product of functions belong to set

\[ \{x, \log(2 \sin(x)), \log(2 \cos(x)), \cot(x), \tan(x)\} \]

(where \(\cot, \tan\) do not appear together), and the weight of the logsine integral \((d = 0 \text{ or } \pm 1)\) is defined as \(W = a + b + c + 1\). We have the following result:

**Proposition 7.** All convergent LSIs with weight \( W \leq 5 \) can be generated by \( A_W \cup \pi A_{W-1} \) over \( \mathbb{Q} \). Here \( \pi S = \{\pi s | s \in S\} \), \( A_0 = \{1\} \).

Proof. Consider LI kernel \( F_0(x) = \frac{i(0; x)^s((1; x)^m((2; x)^n})}{f(p; x)} \), with all logarithms on their principle branches. Integrate \( F \) along a semicircle contour whose diameter is interval \((-1, 1)\). More precisely, separate the contour into 3 parts by 0, \( \pm 1\), substitute \( t = -x \) for the \((-1, 0)\) part and parametrize the semicircle part by \( t = e^{2ix} \), we obtain a relation between LIs and LSIs. Noticing that we have

\[ \log(-x) = \log(x) + i\pi \text{ on } (-1, 0) \text{ and } \log(1 + \exp(2ix)) = \log(2 \cos(x)) + ix, \log(1 - \exp(2ix)) = \log(2 \sin(x)) + i(x - \frac{\pi}{2}) \text{ on } (0, \frac{\pi}{2}) \] due to definition of logs, we can deduce that:
\[ \int_0^\pi F(x)G_1(x) \, dx = LI(k, m, n; l) + \sum_{j=0}^{m} \binom{m}{j} (i\pi)^j LI(n, m - j, k; 2 - l) \quad (4.1.2) \]

Where \( F(x) = (2ix)^m (\log(2\sin(x)) + i(x - \frac{\pi}{2}))^k (\log(2\cos(x)) + ix)^n, G_0(x) = \cot(x) + i, G_1(x) = -2i \) and \( G_2(x) = \tan(x) - i \), whenever all integrals on both sides are convergent. Now consider all LI kernel \( F_0 \) with \( k + m + n + 1 \leq 5 \), apply this formula (and take real or imaginary part of the identity) to obtain relations between LI/LSIs. Solving the whole system yields the result. In fact, according to LSI-LI dependence and Au’s results on higher weight LIs, we are able to express higher weight LSIs via ESs or MZVs using the connection formula above. \( \Box \)

We also dealt with poly-log sine integrals (abbr. PLSIs), that is, the integrand \( F_0G_0 \) with \( F_0 \) defined above, and \( G_0 \) a product whose terms belong to the set

\[ \{ \text{Cl}_2(2x), \text{Cl}_2(4x), \text{Li}_2(-\tan^2(x)), \text{Ti}_2(\tan(x)) \} \]

Where \( \text{Cl}_2(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{nx^2} \) Clausen function and \( \text{Ti}_2(x) = \Im(\text{Li}_2(ix)) \) inverse tangent integral. These actually arise in parametrizations of \( L(2, k; x), k = 1, 2, 5, 6 \) via substitution \( t = e^{2ix} \) in some PLIs, see Lewin [9]. Many of these PLIs with weight no more than 5 can be evaluated similarly as LSIs (we omit the details). Nevertheless, because the general structure of high weight PLIs are not quite known, numerous high weight PLIs remain unknown.

Furthermore, by substitution \( x = \tan(t) \) and repeated integration by parts, it’s possible to convert the family (with suitable parameters ensuring convergence) \( \int_0^\pi \frac{\tan^{-1}(x)^k \log^m(x^2+1)}{x(x^2+1)^n} \, dx \) to combination of LSIs hence obtain their closed-forms. See appendix 4 for some of these results, together with typical LSIs and PLIs and other integrals using tricks on trigonometric functions.

### 4-2. Quadratic log integrals and trigonometric analogue

Another way to generalize LI evaluations is considering ’quadratic log integrals’ (abbr. QLIs):

\[ QLI(a, b, c, d; e; p) = \int_0^1 \frac{\log^a(1-x) \log^b(x) \log^c(x+1) \log^d(x^2+1) \tan^{-1}(x)^e}{f(p; x)} \, dx \quad (4.2.1) \]

Here the range of \( p \) is extended to 0, 1, 2, 3, 4, where \( f(3; x) = \frac{1}{x^2+1}, f(4; x) = \frac{x}{x^2+1} \). The weight of a QLI is \( W = a + b + c + d + e + 1 \). Notice that taking real/imaginary part of principal \( \log(1+ix) \) gives the ’quadratic’ terms \( \frac{1}{2} \log(x^2+1), \tan^{-1}(x) \) separately. This is a far more general topic than LI but few investigations are found in literatures. Based on experiences in section 1, we tried to modify all those methods to make them suitable for solving QLIs. Some general formulas are trivially generalized. Moreover, integration by parts, fractional transformation, Beta derivatives and residue method also works here (with possible slight modifications, e.g. adding \( t = \frac{(1+i)x^2}{2(x^2+1)} \) in...
the fractional transformation part). However, generalizing section 1-6 to 1-8 seems not practical. Based on calculation, we discovered the pattern fully for QLIs with weight \( \leq 3 \), and partially for higher weights, that is:

**Proposition 8.** All QLIs with weight \( W = 2, 3 \), together with some QLIs with weight \( W = 4 \), can be generated by \( B_W \) over \( \mathbb{Q} \), where:

\[
B_2 = A_2 \cup \{ C, \pi \log(2) \} \\
B_3 = A_3 \cup \{ C \log(2), \pi C, \pi \log^2(2), \pi^3, \Im(Li_3(1 + i)) \} \\
B_4 = A_4 \cup \{ C \log^2(2), \pi^2 C, \pi \log^3(2), \pi^3 \log(2), \log(2) \Im(Li_3(1 + i)), \\
\pi \zeta(3), \pi \Im(Li_3(1 + i)), \Im(Li_4(1 + i)), \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \}
\]

Where \( C = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \) denotes the Catalan constant and \( \psi = \psi^{(0)} \) digamma function (section 1-4).

Analog of QLIs on trigonometric case is logsine integrals on the interval \( (0, \frac{\pi}{4}) \) with integrands unchanged, i.e. QLSIs. Since QLIs with weight \( \geq 4 \) receives no complete solution in neither literatures nor here, we are not able to manipulate much on QLSIs. However, we deduced closed-forms for several low weight QLSIs by brute force, that is, deform contour like in section 4-1 to convert them to complex log integrals, then calculate polylog primitives with the help of CAS. Note that we may consider more general cases, for instance LIs involving cubic logs and LSI with interval \( (0, \frac{\pi}{6}) \), which leads to general logsine functions and Apery-type central binomial series, but these are off-topic. See appendix 4 for some typical QLIs and QLSIs.

4-3. Non-homogeneous patterns

Now we present some non-homogeneous patterns. We define non-homogeneous log integrals (abbr. NLIs) as

\[
NLI(a(0), a(1), a(2); k) = \int_0^1 x^k \prod_{m=0}^{2} l(m; x)^{a(m)} \, dx 
\]

(4.3.1)

Where \( k \neq -1 \) integer and the integral convergent. We have:

**Proposition 9.** All NLIs with weight \( W = a(0) + a(1) + a(2) \leq 5 \) can be generated by \( \bigcup_{w=0}^{W} A_w \) over \( \mathbb{Q} \).

Proof. It elementary by using integration by parts repeatedly. Suppose \( k \) a positive integer, lift up \( x^k \) to \( \frac{x^{k+1}}{k+1} \) (or \( \frac{x^{k+1-1}}{k+1} \) sometimes, for convergence issues) and differentiate the log terms, after partial fraction decomposition on \( \frac{x^{k+1}}{1+x} \) we arrive at a combination of some weight \( W \) LIs and some
Moreover, we get non-homo quadratic log integrals (abbr. NQLIs) from QLIs, if we replace the denominator \( f(p; x) \) in QLI definitions with a moment function \( x^k \). They have shown a pattern similar to NPLIs, that weight \( W \) NQLIs can be generated by \( \bigcup_{w=0}^{W} A_w \) for \( W = 2, 3 \). Last but not least, this kind of variation is also valid for non-homo polylog integrals (abbr. NPLIs). For instance, we state that all 219 NPLIs (count it) of form \( \int_0^1 \prod_{m=0}^{4} l(m; x) s^{(m)} \prod_{k=1}^{10} \prod_{n=2}^{5} L(n, k; x) \) \( d\xi \) with weight \( W = \sum_{k=1}^{6} \sum_{n=2}^{N} n b(n, k) + \sum_{m=0}^{2} a(m) = 5 \) are generated by \( \bigcup_{w=0}^{5} A_w \) over \( \mathbb{Q} \). Readers may find the process of reducing them to PLIs (exactly the same as NLI reduction, i.e. integrate by parts) boring but results attractive, since most of them are generated by over 10 polylog constants in \( \bigcup_{w=0}^{5} A_w \), and some of them 20\((=1+1+2+3+5+8)\). See appendix 4 for some typical NLIs, NQLIs, NPLIs and variations.

4-4. PLIs and ESs with other arguments

We may extend the range of PLIs to, for example, \( \int_0^1 \prod_{m=0}^{L} l(m; x) s^{(m)} \prod_{k=1}^{10} \prod_{n=2}^{5} L(n, k; x) \) \( d\xi \), where 4 new polylog term \( L(n, 7; x) \) to \( L(n, 10; x) \) stand for \( \text{Li}_n \left( \frac{x^2}{x} \right), \text{Li}_n \left( \frac{x}{x-1} \right), \text{Li}_n \left( \frac{x}{x+1} \right) \) respectively. With the definition of weight of PLIs trivially extended, Theorem 1 also hold for all these new PLIs with weight \( \leq 5 \). This is apparent since we can apply formula (3.4.1)-(3.4.3) (possibly a substitution \( t = -x \) in the last one) to reduce these new PLIs to ordinary ones, which are solved completely in section 3. Moreover, by applying series expansion on these new terms, we may obtain closed-forms of some new ESs. For instance, by expanding \( \text{Li}_2 \left( x^2 \right) \) in some new PLIs, plenty of new ESs containing \( H_{2n} \) and its alternating analogue can be evaluated. Readers may give further variations.

4-5. Restricted sums and multiple log integrals

The evaluation on MZVs is the central topic in numerous literatures [5, 6, 16]. Shuffle relations, duality, partial fraction decomposition and residue methods are frequently used as obtaining their closed-forms. Nevertheless, we won’t discuss about systematic evaluation of MZVs, but only show some examples on connection between restricted MZV-like sums and known ESs.

Consider a general \( n \)-ple sum \( \sum_n \prod_{i=1}^{n} f_i(k_i) \), where for all \( i \), \( f_i(k) = \frac{(\pm 1)^{s_i - 1}}{k^{r_i}}, s_i \) natural numbers, and \( R \) some restrictions on \( n \) positive summation indexes. If \( R \) is empty (i.e. no restrictions between indexes given), the sum is simply a Zeta–Eta product. If \( R \): indexes are completely ordered in a ’\(<\)' (resp. ’\(\leq\)’) chain, we get MZVs (resp. MZSVs). If \( R \): one of the indexes is always not smaller than all others, we get ESs. For other restrictions \( R \), we get some restricted MZV-like sums (abbr. RSs), for instance:

\[
RS_1 = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \frac{H_m}{\prod_{n=1}^{m} H_n} = \sum_{1 \leq n \leq m, 1 \leq q \leq p \leq m} \frac{(-1)^{m-1}}{mnp^2q}
\]
Some of PLIs that are not able to be calculated directly by multiple expansion (i.e. discarded in the algorithm in section 3-5), but nontrivially evaluated by solving the system of all relations obtained in section 3, can be used here to give closed-forms to several RSs. For instance, by expansion via (2.5.2), (2.5.5), (2.5.17) and Cauchy simplification, we have:

$$\int_0^1 \frac{\text{Li}_2(-x) \log^2(1-x)}{x(x+1)} \, dx = \sum_{m=1}^{\infty} \sum_{n=1}^{m} (-1)^{m-1} \frac{H_m}{m} \sum_{k=1}^{n} \frac{H_k}{k^2} \sum_{j=1}^{k} (j+k)$$

The LHS integral is easily reduced to 2 PLIs via partial fraction decomposition, hence the value of sum on RHS is also known. To simplify \(\sum_{k=1}^{m} \frac{H_m}{m} \sum_{n=1}^{m} \frac{H_k}{k^2} \sum_{j=1}^{k} (j+k)\), rewrite it into a double sum, do finite Cauchy simplification again. Using the fact that \(\sum_{k=1}^{m} \frac{(-1)^{m-k}}{k^2(m-k)}\) is expressible in alternating harmonic numbers (carry it out), and the symmetric identity: \(\sum_{n=1}^{m} (-1)^{n-1} \left( \frac{H_n}{n^2} + \frac{H_n^{(2)}}{n} \right) = H_n H_n^{(2)} + H_n^{(3)}\), we know that \(\sum_{n=1}^{m} \frac{H_n}{n^2} = \sum_{n=1}^{m} \frac{H_n}{n^2} - H_n H_n^{(2)} - H_n^{(3)}\). Now the known RHS is decomposed into 1 RS and 3 weight 5 ESs, namely \(\text{ES}(1, \pm 3; -1), \text{ES}(1, -1, -2; -1)\). Plug in ES values, we obtain the closed-form of this RS:

$$RS_2 = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_m}{m} \sum_{n=1}^{m} \frac{H_n}{n^2} = -10 \text{Li}_5 \left( \frac{1}{2} \right) + \frac{\pi^2 \zeta(3)}{96} + \frac{27 \zeta(5)}{32} \tag{4.5.1}$$

The sum \(RS_1\) can be evaluated similarly if we consider expanding \(\int_0^1 \frac{\text{Li}_2(-x) \log(1-x) \log(1+x)}{x(x+1)} \, dx\) instead.

Symmetric relations are also helpful in solving restricted sums. Recall that for ESs we have I-E relations (section 2-2) and for MZVs shuffle and duality formulas [6]. Analogously, plenty of relations between RSs are deduced from outer and inner symmetry. Generally we have:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} f(n) + \sum_{m=1}^{\infty} \sum_{n=1}^{m} g(n) = \sum_{m=1}^{\infty} f(m) \sum_{m=1}^{\infty} g(m) + \sum_{m=1}^{\infty} f(m) g(m) \tag{4.5.2}$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} f(n) g(n) h(k) + \sum_{m=1}^{\infty} \sum_{n=1}^{m} h(n) g(k) = \sum_{m=1}^{\infty} f(m) \left( \sum_{n=1}^{m} g(n) \sum_{n=1}^{m} h(n) + \sum_{n=1}^{m} g(n) h(n) \right) \tag{4.5.3}$$

Both of them are apparent according to section 2-2. Now apply 2 formulas and \(n\)-ple sum generalizations with all \(f, g, h, \ldots\) arbitrary harmonic terms and repeat alternatively, we obtain relations between RSs, for example, all weight 5 RSs of form \(\sum_{m=1}^{\infty} f(m) \sum_{n=1}^{m} H_n H_n^{(2)} \sum_{n=1}^{m} (-1)^{n-1} \frac{H_n}{n}\)
with $f$ harmonic term can be reduced to ESs directly via inner symmetric identity
\[ \sum_{m=1}^{\infty} \frac{H_n m}{m^2} = \frac{1}{2} (H_m)^2 + H_m^{(2)} \] and its alternating analogue, while \[ \sum_{m=1}^{\infty} \frac{H_n^{(2)}}{m^2} \] can be evaluated via outer symmetry identity, with the help of value of weight 5 ESs and RSs above. However, based upon our observation the inner structure of RSs is far more complicated than ESs, which means expanding PLIs and using symmetry cannot provide us enough relations to evaluate all weight 5 RSs in closed-forms (by weight we mean $\sum_{i=1}^{\infty} s_i$ in the restricted sum definition). We only present some result on RSs obtained by methods above in appendix 4.

Armed with ESs and RSs we already known, we can evaluate plentiful multiple integrals using simple formulas, namely (2.5.16), (2.5.17). We take $\text{RS}_2$ as an example. Since
\[ \sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_n m}{m^2} \] can be evaluated via outer symmetry identity, with the help of value of weight 5 ESs and RSs above. However, based upon our observation the inner structure of RSs is far more complicated than ESs, which means expanding PLIs and using symmetry cannot provide us enough relations to evaluate all weight 5 RSs in closed-forms (by weight we mean $\sum_{i=1}^{\infty} s_i$ in the restricted sum definition). We only present some result on RSs obtained by methods above in appendix 4.

4-6. Some exotic applications

Based on closed-forms of LI/ES/PLIs we obtained, some exotic problems can be solved. We sketch their proofs briefly, details and possible generalizations are left to interested readers.

(1) Extended LI

Using the same technique in section 1-6, consider $\int_0^1 \int_0^1 \log^n((1-x)(1-y)) \frac{dx dy}{(x+1)(y+1)}$. By binomial theorem and (1.1.5) we may express this integral in terms of polylogs. On the other hand, substitute $u = (1-x)(1-y)$ for $y$, integrate w.r.t $x$ first, we find that the integral also equals to \[ \int_0^1 (2 \log(2-u) - \log(u)) \frac{\log^n(u)}{4-u} \] $du$. Now a reflection $x = 1-u$ for the former part and \[ \int_0^1 \frac{\log^{n+1}(x)}{4-u} \] for the latter yields the formula:

\[ \int_0^1 \log^n(1-x) \frac{dx}{x+3} = (-1)^n (n+1)! \left( \frac{1}{n+1} \sum_{k=0}^{n} \text{Li}_{k+1} \left( \frac{1}{2} \right) \text{Li}_{k+n+1} \left( \frac{1}{2} \right) - \text{Li}_{n+2} \left( \frac{1}{4} \right) \right) \]

(4.6.1)

Formula (1.1.5) plays an important role in the proof above. Also, similar approaches based on (1.1.8) give some complicated but not so beautiful results. By using (1.1.8) we mean starting from
\[ \int_0^1 \int_0^1 \frac{\log(x+1) \log(y+1) \log(x+1) \log(y+1)}{xy} \, dx \, dy \] or \[ \int_0^1 \int_0^1 \frac{\log(x+1) \log(x+1) \log(y+1)}{x(y+1)} \, dx \, dy. \]

(2) A PLI master formula

Formula (3.4.1) is used on \( L(n, 1; x) \), \( L(n, 2; x) \) for extending PLIs in section 4-4. Noticing that \( L(n, 5; x), L(n, 6; x) \) have opposite argument either, we have the following general formula:

\[ \int_0^1 F(x) \left( \text{Li}_2 \left( \frac{x-1}{x+1} \right) + \text{Li}_2 \left( \frac{1-x}{x+1} \right) \right) \, dx = \int_0^1 \left( 1 - \sqrt{1-x^2} \right) F \left( \frac{1-\sqrt{1-x^2}}{x} \right) \text{Li}_2 \left( \frac{1-x}{x+1} \right) \, dx \]

(4.6.2)

Where \( F \) is chosen so that both sides are convergent. The proof is simple: LHS can be rewritten as \( \frac{1}{2} \int_0^1 F(x) \text{Li}_2 \left( \frac{x^2-2x+1}{x^2+1} \right) \, dx \) via (3.4.1), then a substitution \( \frac{1}{2} = \frac{x^2+1}{2x} \) transforms this to RHS. By choosing appropriate \( F \) we can deduce some beautiful formulas using PLI/NPLI values on LHS, for instance, set \( F = \text{Li}_2 \left( \frac{x^2+1}{x+1} \right) \) then plug in closed-forms of PLI \((0, 0; 25, 25; 2), PLI(0, 0, 0; 25, 26; 2)\) recorded in appendix 3 gives:

\[ \int_0^1 \frac{\left( 1 - \sqrt{1-x^2} \right) \text{Li}_2 \left( \frac{1-x}{x+1} \right) \text{Li}_2 \left( \frac{x+\sqrt{1-x^2}-1}{x-\sqrt{1-x^2}+1} \right)}{x \sqrt{1-x^2} \left( -\sqrt{1-x^2} + x + 1 \right)} \, dx = 8 \text{Li}_5 \left( \frac{1}{2} \right) + 8 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) \]

(4.6.3)

\[ - \frac{13\pi^2 \zeta(3)}{16} - \frac{\zeta(5)}{4} + \frac{7}{2} \zeta(3) \log^2(2) + \frac{4 \log^5(2)}{15} - \frac{2}{9} \pi^2 \log^3(2) + \frac{1}{36} \pi^4 \log(2) \]

(3) Exotic multiple sum

In section 3-5 we only expand PLIs containing \( L(n, k; x), k = 1, 2, 3 \) into multiple series. If we deal with expansions on other \( L(n, k; x) \), some exotic sums can be evaluated. For example:

\[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{n=0}^{j} \frac{(-1)^n(j-1)!}{2^k k^2 (k+n)(j-n)! (2n)!} \] = \[ 3 \text{Li}_5 \left( \frac{1}{2} \right) + 3 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{81 \zeta(5)}{64} \]

(4.6.4)

\[ - \frac{7 \pi^2 \zeta(3)}{96} + \frac{21}{16} \zeta(3) \log^2(2) + \frac{\log^5(2)}{20} - \frac{1}{18} \pi^2 \log^3(2) - \frac{1}{144} \pi^4 \log(2) \]

By expanding 2 dilog terms simultaneously in \( PLI(0, 0, 0; 23, 24; 0) = \int_0^1 \frac{\text{Li}_4 \left( \frac{x^2+1}{x+1} \right) \text{Li}_2 \left( \frac{x+1}{x} \right)}{x} \, dx \), we have a double sum with the summand a rational integral. Now substitute \( t = 1 - x \), use binomial theorem on \((2 - t)^k\) then simplify to finish the proof. Many more interesting exotic identity can be obtained if we apply direct expansion on PLIs.

(4) High weight PLI

Consider 4 integrals

\[ \int_0^1 \frac{\text{Li}_3(x) \log(x) \log(1-x)}{x} \, dx, \quad \int_0^1 \frac{\text{Li}_3(x) \log(x) \log(1-x)}{1-x} \, dx \]
Using methods introduced in section 3-5 (or MZV theory), it’s not difficult to check that they are all reducible to zeta values. By manipulating (3.4.7) and (3.4.8) we know that \( \text{Li}_3 \left(1 - \frac{1}{2} \right) \) is expressible in \( \text{Li}_3 \) and \( \text{Li}_4 \). Similarly to section 3-4 we have 4 corresponding integrals on \((0, \infty)\) with integrands composed of \( \text{Li}_3 \) and logs. Using methods introduced in section 3-5 (or MZV theory), it’s not difficult to check that they are all reducible to zeta values. By manipulating (3.4.7) and (3.4.8) we know that \( \text{Li}_3 \left(1 - \frac{1}{2} \right) \) is expressible in \( \text{Li}_3(x), \text{Li}_3(1 - x) \) and log terms. Similar to section 3-4 we have 4 corresponding integrals on \((0, \infty)\) with integrands composed of \( \text{Li}_3 \) and logs. Combining 2 of them we get that \( \int_0^\infty \text{Li}_3(-x) \log(x+1) \log(\frac{1}{x} + 1) \, dx \) also zeta expressible. Substitute \( \xi = x^{\frac{1}{2}} \) in the latter part, use (3.4.7) again and plug in weight 6 LI values \[1\], we readily arrive at:

\[
\int_0^1 \frac{\text{Li}_3(x) \log^2(x)}{1-x} \, dx + \int_0^1 \frac{\text{Li}_3(x) \log^2(1-x)}{x} \, dx
\]

(4.6.5)

(5) Mixed harmonic sum

Consider the integral \( \int_0^1 \frac{\log(x) \log(1-x^2) \sin^{-1}(x)^4}{\sqrt{1-x^2}} \, dx \). Expand the power of \( \sin^{-1}(x)^4 \) into Taylor series using Borwein’s formula \( \sin^{-1}(x)^4 = \frac{3}{2} \sum_{n=1}^\infty \frac{H_n^{(2)}(2x)^{2n}}{n^2 \binom{2n}{n}} \) [4], exchange the order of integration and summation, substitute \( t = x^2 \). To evaluate the inner integral in closed-form, apply identities of Beta derivatives (see section 1-4 for general case), namely \( \psi(0)(n+1) = H_n - \gamma \) and \( \psi(0)(n+\frac{1}{2}) = 2(-H_n + H_{2n} - \log(2)) \). Therefore we’ve completely transformed the integral to a mixed harmonic sum. On the other hand, by a simple substitution \( x = \sin(t) \) we have the integral also equals to \( 2 \int_0^\pi t^4(\log(2 \sin(t)) - \log(2)) (\log(2 \cos(t)) - \log(2)) \, dt \), which can be reduced to several LSIs with weight at most 6. Now general method in section 4-1 and Au’s results \[1\] lead to the following formula:

\[
\sum_{n=1}^\infty \frac{H_n^{(2)}}{n^2} \left( 2(H_n - H_{2n} + \log(2)) (H_n + 2 \log(2)) + H_n^{(2)} - \frac{\pi^2}{6} \right) = -4 ES(1; -5) - \frac{4}{3} \pi^2 \text{Li}_4 \left(\frac{1}{2}\right)
\]

(4.6.6)

(6) Inverse ES and multiple integral

Some ‘inverse ESs’ admit a special pattern, where by inverse we mean the summand contains something like \( \zeta(k) - H_n^{(k)} \) or \( \eta(k) - H_n^{(k)} \), that is, the tail of generalized harmonic sums.
Consider the convergent inverse ES \( \sum_{n=1}^{\infty} (-1)^n \left( \log(2) - \tilde{H}_n \right)^5 \). By Abel’s summation by parts (lift up \((-1)^n\) by partial summation and take difference of the other part) we have it equals to:

\[
\sum_{n=1}^{\infty} \frac{1}{2} \left( (-1)^{n-1} - 1 \right) \left( \frac{(-1)^{n-1}}{n} + \log(2) - \tilde{H}_n \right)^5 - \left( \log(2) - \tilde{H}_n \right)^5 \]

Expand the RHS, we find that all terms are weight 5 ESs except another inverse ES \( \sum_{n=1}^{\infty} \left( \log(2) - \tilde{H}_n \right)^5 \).

Sum by parts again and split the summand, the new ES can be reduced to ESs either. Therefore we have:

\[
\sum_{n=1}^{\infty} (-1)^n \left( \log(2) - \tilde{H}_n \right)^5 = 10 \text{Li}_5 \left( \frac{1}{2} \right) + 10 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) + \frac{157 \zeta(5)}{16} + \frac{35}{8} \zeta(3) \log^2(2) - \frac{2 \log^5(2)}{3} - \frac{5}{18} \pi^2 \log^3(2) \quad (4.6.7)
\]

Using the same method we are able to deduce that:

\[
\sum_{n=0}^{\infty} \left( \log(2) - \tilde{H}_n \right)^5 = -20 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{45}{4} \zeta(3) \log(2) + \frac{259 \pi^4}{1440} + \frac{5 \log^4(2)}{3} + \frac{5}{12} \pi^2 \log^2(2) \quad (4.6.8)
\]

The reason why a quintic inverse ES yields a weight 4 value is that the existence of \( n \) after summation by parts once. Note that \( \int_0^1 \frac{x^n}{x+1} \, dx = (-1)^n \left( \log(2) - \tilde{H}_n \right) \), combine 2 results above and apply an expansion, we may conclude:

\[
\sum_{n=0}^{\infty} \left( \log(2) - \tilde{H}_n \right)^5 = \int_{(0,1)^5} \frac{dudvdxudydz}{(u+1)(v+1)(x+1)(y+1)(z+1)(1-(uvwxyz)^2)}
\]

\[
= -10 \text{Li}_4 \left( \frac{1}{2} \right) + 5 \text{Li}_5 \left( \frac{1}{2} \right) + 5 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{157 \zeta(5)}{32} + \frac{35}{16} \zeta(3) \log^2(2) - \frac{45}{8} \zeta(3) \log(2) + \frac{259 \pi^4}{2880} + \frac{5 \log^4(2)}{6} + \frac{5 \log^4(2)}{6} - \frac{5}{36} \pi^2 \log^3(2) + \frac{5}{24} \pi^2 \log^2(2) \quad (4.6.9)
\]

(7) Parseval LSI

Generating functions recorded in section 2.5 may take part in some surprising identities. For instance we have:

\[
\int_0^{2\pi} \left( \frac{1}{12} \pi^2 f(x) - \frac{1}{3} f(x)^3 + \frac{1}{2} (x - \pi) g(x) + h(x) \right)^2 \, dx + \left( \frac{(\pi - x)}{2} f(x)^2 - f(x) g(x) + \frac{\pi^2 (\pi - x)}{24} \right)^2 \, dx = 6\pi \zeta(3)^2 + \frac{979 \pi^7}{11340} \quad (4.6.10)
\]

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Where \( f(x) = \log(2\sin(\frac{x}{2})) \), \( g(x) = Cl_2(x) \), \( h(x) = Cl_3(x) \) are logsine and Clausen functions. This is a direct consequence of Parseval identity applied to \( f(x) = \sum_{n=1}^{\infty} \frac{(H_n)_{2}^{\pi}x^{n}}{n} \) on \((0, 2\pi)\), formula (2.5.13) and the value of \( ES(1, 1, 1, 1; 2) \) (easily deduced from section 3).

(8) Valean PLI

We generalize Valean’s elegant formulas of PLIs with argument \( \pm \frac{2x}{\pi+1} \) [12]. Denote

\[
X = (-1)^k (1 - 2^{k-2}) \zeta(-k + 2n + 1) + \sum_{j=0}^{-k+2n-1} \frac{(-1)^{j-1} \zeta(j + 2)(i\pi)^{-j-k+2n-1}}{(-j - k + 2n - 1)!}
\]

\[
Y = \Re \left( \frac{X}{(2i)^{2n-k}} \right) - \log(2) \frac{(\pi)^{2n-k}}{(2n-k)!}
\]

Then we have:

\[
\int_{0}^{1} \frac{\text{Li}_2 \left( \frac{2x}{\pi+1} \right) \log^{2n-2}(x)}{x} \, dx = (-1)^{n-1}(2n-2)! \sum_{k=0}^{2n-1} \frac{B_k(2\pi)^k Y}{k!} \tag{4.6.11}
\]

\[
\int_{0}^{1} \frac{\text{Li}_2 \left( -\frac{2x}{\pi+1} \right) \log^{2n-2}(x)}{x} \, dx = (-1)^{n}(2n-2)! \sum_{k=0}^{2n-1} \frac{B_k(2\pi)^k (1 - 2^{1-k}) Y}{k!} \tag{4.6.12}
\]

Here is the proof. Following Valean [12], by applying double integration, together with fourier expansion of Poisson kernel and logsine functions, we have the formula:

\[
\int_{0}^{1} \frac{\text{Li}_2 \left( \frac{2x}{\pi+1} \right) \log^{2n-2}(x)}{x} \, dx = 2(-1)^{2n-1}(2n-1)! \int_{0}^{2} \log(\cos(t)) \left( \sum_{k=1}^{\infty} \frac{\sin(kt)}{k^{2n-1}} \right) \, dt
\]

Gradstheyn & Ryzhik [8] offers the Fourier expansion \( \sum_{k=1}^{\infty} \frac{\sin(kt)}{k^{2n-1}} = \frac{(2\pi)^{2n-1}(1-\eta)B_{2n-1}(\frac{1}{4})}{2(2n-1)!} \), where \( B_n(x) = \sum_{k=0}^{n} B_k(\frac{n}{k}) x^{n-k} \) are Bernoulli polynomials and \( B_n \) Bernoulli numbers. Plugging in these results we find that:

\[
\int_{0}^{1} \frac{\text{Li}_2 \left( \frac{2x}{\pi+1} \right) \log^{2n-2}(x)}{x} \, dx = \frac{(2\pi)^{2n-1}(1-\eta)^{n-2}}{2n-1} \sum_{k=0}^{2n-1} \frac{B_k(2\pi)^{k} \int_{0}^{\frac{\pi}{2}} t^{-k+2n-1} \log(\cos(t)) \, dt}{(2\pi)^{-k+2n-1}}
\]

Finally, by simple induction we have \( \int \frac{\log(z+1) \log^{n}(z)}{z} \, dz = \sum_{k=0}^{n} \frac{(-1)^{k-1} n! \text{Li}_{k+2}(z) \log^{n-k}(z)}{(n-k)!} \). Integrate \( \frac{\log(z+1) \log^{n}(z)}{z} \) along the upper half of unit circle (counter clockwise), parametrize \( z = e^{2ix} \) just like section 4-1, it is direct that \( \int_{0}^{\frac{\pi}{2}} x^{n} \log(2\cos(x)) \, dx = \Re \left( \frac{z^{n}}{(2i)^{n+1}} \right) \), where

\[
Z = (-1)^{n-1} (1 - 2^{-n-1}) n! \zeta(n + 2) + \sum_{j=0}^{n} \frac{(-1)^{j-1} n! (i\pi)^{n-j} \zeta(j + 2)}{(n-j)!}
\]
Finally make use of this general LSI result to complete the proof of (4.6.11). (4.6.12)'s proof is exactly the same and we omit it.

(9) Hypergeometric sum

Recall the definition of generalized hypergeometric function

\[ _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_{n} \cdots (a_p)_n z^n}{(b_1)_n \cdots (b_q)_n n!} \]

For \(|z| < 1\) and its unique analytic continuation outside, where \((a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\) is the Pochhammer symbol. Now consider \(\int_0^1 \frac{\log^2(x) \sin^{-1}(\frac{x}{\sqrt{2}})}{\sqrt{2-x^2}} \, dx = \int_0^\pi \left( \log(2 \sin(t)) - \frac{\log(2)}{2} \right)^3 \, dt \). The RHS is clearly combination of QLSIs which are known according to section 4-2 and appendix 4. Meanwhile, expanding \(\frac{1}{\sqrt{2-x^2}}\) into power series and using (1.1.1) transforms the LHS into a generalized hypergeometric series. After simplification we arrive at:

\[ _5F_4\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2} \right) = \frac{1}{2} \sqrt{2} \Im(Li_3(1 + i)) + \frac{1}{16} \pi \sqrt{2} \zeta(3) + \frac{1}{768} \pi^3 \sqrt{2} \log(2) - \frac{\sqrt{2} \left( \psi(3) \left( \frac{1}{4} \right) - \psi(3) \left( \frac{3}{4} \right) \right)}{12288} \]  

(4.6.13)

If we consider \(\int_0^1 \frac{\log^2(x) \sin^{-1}(\frac{x}{\sqrt{2}})}{\sqrt{2-x^2}} \, dx = \int_0^\pi t \left( \log(2 \sin(t)) - \frac{\log(2)}{2} \right)^2 \, dt\) instead and apply the same method above, we will have the following analogous result (note that \(\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\) also has a simple hypergeometric power series by differentiating the well known \(\sin^{-1}(x)^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2(2n)!}\)):

\[ _5F_4\left( 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2; \frac{1}{2} \right) = 2 \pi \Im(Li_3(1 + i)) + \frac{5}{2} Li_4 \left( \frac{1}{2} \right) - \frac{11 \pi^4}{144} + \frac{5 \log^4(2)}{48} - \frac{1}{6} \pi^2 \log^2(2) \]  

(4.6.14)

Finally, make an imaginary substitution in the \(\sin^{-1}(x)^2\) series above we have \(\sinh^{-1}\left( \frac{x}{\sqrt{2}} \right)^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^2(2n)!}\). Multiply both sides by \(\frac{\log(x)}{x}\) then integrate it on \((0, 1)\). For LHS substitute \(t = \sinh^{-1}\left( \frac{x}{\sqrt{2}} \right)\), for RHS exchange the order to use (1.1.1), we deduce \(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 2^{2k} \log\left( \frac{k}{4} \right)} = 2 \int_0^{\log(2)} x^2 \coth(x) \log^k(\sinh(x)) \, dx\). Therefore we only need to evaluate log-hyperbolic integrals \(\int_0^{\log(2)} x^2 \coth(x) \log^k(\sinh(x)) \, dx \) for \(k = 0, 1, 2\), which is direct if we substitute \(x = \frac{1}{2} \log(y + 1)\) to reduce them to known LIs. Rewrite the binomial sum into hypergeometric form and simplify, we have:

\[ _6F_5\left( 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2; \frac{1}{8} \right) = 40 Li_5 \left( \frac{1}{2} \right) + 24 Li_4 \left( \frac{1}{2} \right) \log(2) - 38 \zeta(5) + 4 \zeta(3) \log^2(2) + \frac{19}{30} \log^5(2) - \frac{4}{9} \pi^2 \log^3(2) + \frac{7}{45} \pi^4 \log(2) \]  

(4.6.15)
We conclude by a simple but fascinating integral. Denote $D$ the subset of $\mathbb{R}^2$ where $0 < x < 1, 0 < y < \frac{1-x}{x+1}$. Because the fractional transformation $f(x) = \frac{1-x}{x+1}$ is self-dual (i.e. $f(f(x)) = x$), we have the region $D$ symmetric about the line $y = x$. Let $D_1$ denote the part of $D$ that’s above $y = x$, by symmetry and direct integration w.r.t $y$, we have
\[
\int_{D} \frac{\log(1-x) \log(1-y)}{xy} \, dxdy = 2 \int_{D_1} \frac{\log(1-x) \log(1-y)}{xy} \, dxdy = 2 \int_{0}^{\sqrt{2}-1} \frac{\left(\text{Li}_2(x) - \frac{x}{1-x}\right) \log(1-x)}{x} \, dx.
\]
Hence by using $\int \text{Li}_2(x) \log(1-x) \, dx = -\frac{\text{Li}_2(x)^2}{2}$ and the value of known $\text{PLI}(1, 0, 0; 25; 1)$ we come to the result:
\[
\int_{0}^{\sqrt{2}-1} \frac{\text{Li}_2\left(\frac{1-x}{x+1}\right) \log(1-x)}{x} \, dx = -\frac{1}{2} \frac{\text{Li}_2\left(\sqrt{2} - 1\right)^2}{2} - \frac{\text{Li}_4\left(\frac{1}{2}\right)}{2} - \frac{7}{16} \zeta(3) \log(2) + \frac{7\pi^4}{2880} - \frac{1}{48} \log^4(2) + \frac{1}{48} \pi^2 \log^2(2) \tag{4.6.16}
\]
Note that a similar consideration for integrand $\frac{\text{Li}_n(x) \text{Li}_n(y)}{xy}$ yields the general formula:
\[
\left(\int_{0}^{\sqrt{2}-1} - \int_{\sqrt{2}-1}^{1}\right) \frac{\text{Li}_n(x) \text{Li}_{n+1}\left(\frac{1-x}{x+1}\right)}{x} = \text{Li}_{n+1}\left(\sqrt{2} - 1\right)^2 \tag{4.6.17}
\]

Appendix 1. LI closed-forms

We tabulate all LI with weight $\leq 5$ here. These integrals are independently solved by Au and the author. For each row’s LI, multiply every column’s rational coefficient with the element in Fibonacci basis on the top, then sum up to get the closed-form of this LI. As an example, we find this row in table of weight 3 LIs:

| Weight 3 LI | $\pi^2 \log(2)$ | $\log^3(2)$ | $\zeta(3)$ |
|-------------|----------------|------------|-------------|
| (1,0,1,2)   | $-\frac{1}{18}$ | $\frac{1}{3}$ | $\frac{5}{6}$ |

Which implies that $\text{LI}(1, 0, 1; 2) = \int_{0}^{1} \frac{\log(1-x) \log(x+1)}{x+1} \, dx = \frac{\zeta(3)}{8} + \frac{\log^3(2)}{3} - \frac{1}{18} \pi^2 \log(2)$. We can read tables in appendix 2, 3 in a similar manner to find the closed-form for a certain ES/PLI.

Weight 1 LI:
\[
\text{LI}(0, 0, 0; 1) = \log(2)
\]
| Weight 2 LI | \( \pi^2 \) | \( \log^2(2) \) |
|------------|------------|------------|
| (0,1,0:0)  | \(-\frac{1}{6}\) | 0          |
| (1,0,0:1)  | \(-\frac{1}{6}\) | 0          |
| (0,0,1:1)  | \(\frac{1}{12}\)  | 0          |
| (1,0,0:2)  | \(-\frac{1}{12}\) | \(\frac{1}{7}\) |
| (0,1,0:2)  | \(-\frac{1}{12}\) | 0          |
| (0,0,1:2)  | 0          | \(\frac{1}{7}\) |

| Weight 3 LI | \( \pi^2 \log(2) \) | \( \log^3(2) \) | \( \zeta(3) \) |
|------------|----------------|----------------|----------------|
| (0,2,0:0)  | 0              | 0              | 2              |
| (1,1,0:0)  | 0              | 0              | 1              |
| (0,1,1:0)  | \(-\frac{1}{4}\) | 0              | 1              |
| (2,0,0:1)  | 0              | 0              | 2              |
| (0,0,2:1)  | 0              | 0              | \(\frac{1}{2}\) |
| (1,1,0:1)  | 0              | 0              | 1              |
| (0,1,1:1)  | 0              | 0              | \(-\frac{1}{4}\) |
| (1,0,1:1)  | 0              | 0              | \(-\frac{1}{8}\) |
| (2,0,0:2)  | \(-\frac{1}{8}\) | \(\frac{1}{3}\) | \(\frac{4}{3}\) |
| (0,2,0:2)  | 0              | 0              | \(\frac{1}{3}\) |
| (0,0,2:2)  | 0              | \(\frac{1}{3}\) | 0              |
| (1,1,0:2)  | \(-\frac{1}{8}\) | 0              | \(\frac{13}{8}\) |
| (0,1,1:2)  | 0              | 0              | \(-\frac{1}{8}\) |
| (1,0,1:2)  | \(-\frac{1}{12}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) |
| Weight 4 LI | $\pi^4$ | $\pi^2 \log^2(2)$ | $\log^3(2)$ | $\zeta(3) \log(2)$ | $\text{Li}_4\left(\frac{1}{2}\right)$ |
|------------|--------|-----------------|-------------|-----------------|-----------------|
| (0,3,0:0)  | $-\frac{1}{15}$ | 0               | 0           | 0               | 0               |
| (2,1,0:0)  | $-\frac{1}{15}$ | 0               | 0           | 0               | 0               |
| (1,2,0:0)  | $-\frac{1}{180}$ | 0               | 0           | 0               | 0               |
| (0,2,1:0)  | $-\frac{7}{120}$ | 0               | $\frac{7}{6}$ | 0               | 0               |
| (0,1,2:0)  | $\frac{17}{360}$ | $-\frac{1}{24}$ | $-\frac{1}{12}$ | $\frac{7}{12}$ | $\frac{4}{3}$   |
| (1,1,1:0)  | $\frac{17}{360}$ | $-\frac{1}{24}$ | $-\frac{1}{12}$ | $\frac{7}{12}$ | $\frac{4}{3}$   |
| (3,0,0:1)  | $-\frac{1}{15}$ | 0               | 0           | 0               | 0               |
| (0,0,3:1)  | $\frac{1}{15}$  | $\frac{1}{3}$  | $-\frac{1}{3}$ | $-\frac{7}{4}$ | $-4$            |
| (2,1,0:1)  | $-\frac{1}{180}$ | 0               | 0           | 0               | 0               |
| (2,0,1:1)  | $-\frac{1}{120}$ | $-\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{7}{3}$  | $2$             |
| (1,2,0:1)  | $-\frac{1}{35}$  | 0               | 0           | 0               | 0               |
| (0,2,1:1)  | $\frac{7}{180}$  | 0               | 0           | 0               | 0               |
| (1,0,2:1)  | $-\frac{1}{120}$ | 0               | 0           | 0               | 0               |
| (0,1,2:1)  | $\frac{1}{24}$   | $\frac{1}{6}$  | $-\frac{1}{6}$ | $\frac{7}{6}$  | $4$             |
| (1,1,1:1)  | $-\frac{1}{180}$ | $-\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{7}{12}$ | $\frac{4}{3}$   |
| (3,0,0:2)  | 0             | 0               | 0           | 0               | $-6$            |
| (0,3,0:2)  | $-\frac{7}{120}$ | 0               | 0           | 0               | 0               |
| (0,0,3:2)  | 0             | 0               | $\frac{1}{3}$ | 0               | 0               |
| (2,1,0:2)  | $\frac{1}{1080}$ | 0               | $-\frac{1}{4}$ | 0               | $-6$            |
| (2,0,1:2)  | $-\frac{7}{360}$ | $-\frac{1}{6}$  | $\frac{1}{4}$ | 2               | 0               |
| (1,2,0:2)  | $\frac{7}{180}$  | $\frac{1}{6}$  | $-\frac{1}{6}$ | 0               | $-4$            |
| (0,2,1:2)  | $-\frac{1}{24}$  | $-\frac{1}{6}$  | $\frac{1}{6}$ | $\frac{7}{6}$  | $4$             |
| (1,0,2:2)  | $-\frac{1}{35}$  | $-\frac{1}{6}$  | $\frac{1}{6}$ | $\frac{7}{6}$  | $2$             |
| (0,1,2:2)  | $\frac{7}{45}$   | $-\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{7}{12}$ | $\frac{4}{3}$   |
| (1,1,2:2)  | $-\frac{7}{45}$  | $-\frac{1}{24}$ | $\frac{1}{12}$ | $\frac{7}{12}$ | $\frac{4}{3}$   |

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| Weight 5 LI | $\pi^4 \log(2)$ | $\pi^2 \log^2(2)$ | $\log^2(2)$ | $\pi^2 \zeta(3)$ | $\zeta(3) \log^2(2)$ | $\zeta(5)$ | $\text{Li}_4(\frac{1}{2}) \log(2)$ | $\text{Li}_5(\frac{1}{2})$ |
|-------------|----------------|----------------|-----------|----------------|----------------|--------|----------------|----------------|
| (0,4,0,0)   | 0              | 0              | 0        | 0              | 0              | 24     | 0              | 0              |
| (3,1,0,0)   | 0              | 0              | 0        | 0              | 0              | 6      | 0              | 0              |
| (1,3,0,0)   | 0              | 0              | 0        | -1             | 0              | 12     | 0              | 0              |
| (0,3,1,0)   | -\frac{1}{2}  | 0              | 0        | -\frac{1}{2}  | 0              | 12     | 0              | 0              |
| (0,1,3,0)   | -\frac{17}{28} | -\frac{1}{3}  | -\frac{1}{20} | -\frac{7}{16} | \frac{23}{3}  | \frac{117}{4} | -6             | -24            |
| (2,2,0,0)   | 0              | 0              | 0        | -\frac{1}{4}  | 0              | 8      | 0              | 0              |
| (0,2,2,0)   | -\frac{33}{280} | -\frac{1}{5}  | -\frac{1}{5} | -\frac{11}{40} | \frac{7}{8}   | \frac{27}{4}  | -8             | -16            |
| (2,1,1,0)   | \frac{1}{120} | -\frac{1}{18} | \frac{1}{40} | -\frac{11}{16} | \frac{7}{8}   | \frac{27}{4}  | 2              | 8              |
| (1,2,1,0)   | \frac{1}{180} | -\frac{1}{18} | \frac{1}{10} | -\frac{11}{16} | \frac{7}{8}   | \frac{27}{4}  | 4              | 8              |
| (1,1,2,0)   | -\frac{1}{36} | -\frac{1}{6}  | 0         | -\frac{1}{6}  | \frac{7}{8}   | 6      | 0              | 0              |
| (4,0,0,1)   | 0              | 0              | 0        | 0              | 0              | 24     | 0              | 0              |
| (0,0,4,1)   | 0              | -\frac{2}{5}  | -\frac{1}{5} | 0              | -\frac{21}{12} | 24     | -24            | -24            |
| (3,1,0,1)   | 0              | 0              | 0        | -1             | 0              | 12     | 0              | 0              |
| (3,0,1,1)   | 0              | -\frac{1}{6}  | \frac{1}{6} | -\frac{11}{16} | \frac{7}{8}   | -\frac{21}{12} | 6 | 6 |
| (1,3,0,1)   | 0              | 0              | 0        | 0              | 0              | 6      | 0              | 0              |
| (3,1,1,1)   | 0              | 0              | 0        | 0              | 0              | -\frac{41}{8} | 0              | 0              |
| (1,0,3,1)   | 0              | \frac{1}{5}   | -\frac{1}{5} | \frac{7}{16}  | -\frac{21}{8} | \frac{7}{8} | -6             | -6             |
| (0,1,3,1)   | 0              | \frac{1}{5}   | -\frac{1}{5} | \frac{7}{16}  | -\frac{21}{8} | \frac{7}{8} | -6             | -6             |
| (2,0,1,1)   | 0              | 0              | 0        | -\frac{1}{2}  | 0              | 8      | 0              | 0              |
| (2,0,2,1)   | 0              | -\frac{1}{5}  | \frac{1}{15} | 0              | \frac{7}{10}  | -\frac{21}{8} | 4 | 4 |
| (0,2,2,1)   | 0              | 0              | 0        | \frac{1}{3}   | 0              | -\frac{21}{8} | 0 | 0 |
| (2,1,1,1)   | 0              | -\frac{1}{5}  | \frac{1}{15} | -\frac{11}{16} | \frac{7}{8}   | -\frac{21}{8} | 4 | 4 |
| (1,2,1,1)   | 0              | 0              | 0        | \frac{1}{5}   | 0              | -\frac{21}{8} | 0 | 0 |
| (1,1,2,1)   | 0              | 0              | 0        | \frac{1}{38}  | 0              | -\frac{21}{8} | 0 | 0 |
| (4,0,0,2)   | 0              | 0              | 0        | 0              | 0              | 24     | 0              | 0              |
| (0,4,0,2)   | 0              | 0              | 0        | 0              | 0              | \frac{24}{5} | 0              | 0              |
| (0,0,4,2)   | 0              | 0              | 0        | \frac{1}{5}   | 0              | 0      | 0              | 0              |
| (3,1,0,2)   | \frac{1}{36}  | 0              | -\frac{1}{36} | -\frac{1}{8} | 0              | -\frac{3}{16} | 18            |
| (3,0,1,2)   | -\frac{1}{120} | -\frac{1}{6}  | \frac{1}{40} | -\frac{1}{8} | 3              | \frac{1}{16} | 0 | 6 |
| (1,3,0,2)   | -\frac{1}{36} | 0              | 0        | -\frac{1}{36} | 0              | \frac{23}{8} | 0 | 0 |
| (3,1,2,0)   | 0              | 0              | 0        | -\frac{1}{2}  | 0              | \frac{47}{8} | 0 | 0 |
| (1,0,3,2)   | -\frac{1}{15} | -\frac{1}{5}  | \frac{1}{4} | 0              | 3              | 6      | 0              | -6            |
| (0,1,3,2)   | 0              | -\frac{1}{6}  | \frac{5}{6} | 0              | \frac{21}{8}  | -6     | 6              | 6 |
| (2,2,0,2)   | \frac{1}{72}  | -\frac{1}{6}  | -\frac{1}{15} | -\frac{11}{16} | 0              | \frac{15}{8} | 0 | 8 |
| (2,0,2,2)   | -\frac{1}{72} | -\frac{1}{6}  | \frac{7}{30} | -\frac{1}{3}  | 4              | \frac{23}{8} | 0 | -4 |
| (0,2,2,2)   | 0              | -\frac{1}{6}  | \frac{1}{15} | -\frac{1}{3}  | \frac{7}{8}   | -\frac{3}{16} | 8 | 8 |
| (2,1,1,2)   | -\frac{1}{120} | -\frac{1}{6}  | \frac{1}{15} | -\frac{1}{3}  | \frac{7}{8}   | -\frac{3}{16} | 8 | 8 |
| (1,2,1,2)   | -\frac{1}{120} | \frac{1}{15} | -\frac{1}{15} | -\frac{1}{3}  | \frac{7}{8}   | -\frac{3}{16} | 8 | 8 |
| (1,1,2,2)   | -\frac{1}{240} | -\frac{1}{6}  | \frac{1}{20} | -\frac{1}{23} | \frac{7}{8}   | -\frac{3}{16} | 8 | 8 |
Appendix 2. ES closed-forms

We tabulate all ES with weight \( \leq 5 \) here. For weight \( \leq 4 \) we solved all ESs by using ES-PLI transformation independently. For weight 5, based on 17 ESs recorded in (or easily deduced from) Flajolet&Salvy, 30 ESs recorded in Xu, we obtain closed-forms of the remaining 7 ESs (section 2-5) using various methods.

Weight 2 ESs:

\[ ES(1; -1) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}, \quad ES(-1; -1) = \frac{\pi^2}{12} + \frac{\log^2(2)}{2} \]

| Weight 3 ES | \( \pi^2 \log(2) \) | \( \log^4(2) \) | \( \zeta(3) \) |
|-------------|----------------|----------------|----------|
| (1; 2)      | 0              | 0              | 2        |
| (-1; 2)     | 1/3            | 0              | -1/3     |
| (1; -2)     | 0              | 0              | 2/5      |
| (-1; -2)    | 1/4            | 0              | -3/2     |
| (2; -1)     | -1/12          | 0              | 1        |
| (-2; -1)    | -1/5           | 0              | 13/3     |
| (1, 1; -1)  | -1/12          | 1/3            | 2/3      |
| (1, -1; -1) | 1/5            | -1/3           | 1/4      |
| (-1, -1; -1)| 1/4            | 1/3            | -1/2     |
| Weight 4 ES | $\pi^4$ | $\pi^2 \log^2(2)$ | $\log^3(2)$ | $\zeta(3) \log(2)$ | $\text{Li}_4(\frac{1}{2})$ |
|-------------|--------|---------------------|--------|-----------------|-----------------|
| (1; 3) | $\frac{1}{72}$ | 0 | 0 | 0 | 0 |
| (-1; 3) | $\frac{1}{288}$ | 0 | 0 | $\frac{7}{4}$ | 0 |
| (1; -3) | $\frac{1}{360}$ | $\frac{1}{12}$ | $-\frac{1}{12}$ | $-\frac{7}{4}$ | $-2$ |
| (-1; -3) | $\frac{1}{72}$ | $\frac{1}{12}$ | $-\frac{1}{12}$ | 0 | $-2$ |
| (2; 2) | $\frac{1}{360}$ | 0 | 0 | 0 | 0 |
| (-2; 2) | $\frac{1}{288}$ | $\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{7}{4}$ | $-4$ |
| (2; -2) | $\frac{1}{480}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{7}{4}$ | 4 |
| (-2; -2) | $\frac{1}{1440}$ | 0 | 0 | 0 | 0 |
| (1; 1; 2) | $\frac{1}{360}$ | 0 | 0 | 0 | 0 |
| (1, -1; 2) | $\frac{1}{4320}$ | $\frac{1}{8}$ | $-\frac{1}{8}$ | 0 | $-3$ |
| (-1, -1; 2) | $-\frac{13}{720}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | 0 | 2 |
| (1, 1; -2) | $\frac{1}{4320}$ | $\frac{1}{12}$ | $-\frac{1}{12}$ | $-\frac{7}{4}$ | $-2$ |
| (1, -1; -2) | $\frac{29}{1440}$ | $\frac{1}{8}$ | $-\frac{1}{8}$ | 0 | $-3$ |
| (-1, -1; -2) | $-\frac{61}{1440}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{7}{4}$ | 4 |
| (3; -1) | $\frac{19}{1440}$ | 0 | 0 | $-\frac{3}{4}$ | 0 |
| (-3; -1) | $\frac{1}{1344}$ | $-\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{3}{4}$ | 2 |
| (1, 2; -1) | $-\frac{1}{96}$ | $-\frac{1}{24}$ | $\frac{1}{24}$ | $\frac{1}{4}$ | 2 |
| (-1, 2; -1) | $\frac{43}{1440}$ | $-\frac{1}{24}$ | $-\frac{1}{24}$ | 0 | $-3$ |
| (1, -2; -1) | $\frac{1}{1440}$ | $-\frac{1}{24}$ | $\frac{1}{24}$ | 0 | 3 |
| (-1, -2; -1) | $\frac{61}{1440}$ | $-\frac{1}{24}$ | $\frac{1}{24}$ | $-\frac{7}{8}$ | $-4$ |
| (1, 1; 1; -1) | $\frac{1}{1440}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $-\frac{7}{8}$ | 0 |
| (1, 1, -1; -1) | $\frac{1}{1440}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | 0 | $-3$ |
| (1, -1, -1; -1) | 0 | $\frac{7}{24}$ | $-\frac{7}{8}$ | $\frac{5}{8}$ | 0 |
| (-1, -1, -1; -1) | $-\frac{59}{1440}$ | $\frac{13}{24}$ | $\frac{11}{24}$ | 0 | 5 |
| Weight 5 ES | $\pi^4 \log(2)$ | $\pi^2 \log^2(2)$ | $\log^3(2)$ | $\pi^2 \zeta(3)$ | $\log^3(2)$ | $\zeta(5)$ | $\text{Li}_4(\frac{1}{2}) \log(2)$ | $\text{Li}_5(\frac{1}{2})$ |
|------------|-----------------|----------------------|-------------|-----------------|----------------|-----------|-----------------|-----------------|
| (1; 4)     | 0               | 0                    | 0           | $-\frac{1}{6}$  | 0              | 3         | 0               | 0               |
| (-1; 4)    | $\frac{1}{16}$ | 0                    | 0           | $\frac{1}{16}$ | 0              | $-\frac{17}{16}$ | 0               | 0               |
| (1; -4)    | 0               | 0                    | 0           | $-\frac{1}{12}$ | 0              | $\frac{9}{2}$ | 0               | 0               |
| (-1; -4)   | $\frac{1}{16}$ | 0                    | 0           | $\frac{1}{16}$ | 0              | $-\frac{19}{16}$ | 0               | 0               |
| (2; 3)     | 0               | 0                    | 0           | $\frac{1}{3}$   | 0              | $9$        | 0               | 0               |
| (-2; 3)    | 0               | 0                    | 0           | $-\frac{1}{33}$ | 0              | $\frac{21}{32}$ | 0               | 0               |
| (2; -3)    | 0               | 0                    | 0           | $\frac{1}{35}$  | 0              | $-\frac{1}{32}$ | 0               | 0               |
| (-2; -3)   | 0               | 0                    | 0           | $-\frac{1}{35}$ | 0              | $\frac{23}{36}$ | 0               | 0               |
| (1; 3)     | 0               | 0                    | 0           | $-\frac{1}{6}$  | 0              | $\frac{5}{6}$ | 0               | 0               |
| (1, -1; 3) | $\frac{5}{15}$ | $\frac{1}{30}$      | $\frac{1}{6}$ | $-\frac{5}{3}$  | $\frac{5}{5}$  | $-\frac{12}{5}$ | 0               | 2               |
| (-1, -1; 3)| $\frac{19}{720}$ | $\frac{1}{35}$      | $\frac{1}{30}$ | $\frac{3}{5}$   | $\frac{1}{5}$  | $-\frac{19}{5}$ | 0               | 4               |
| (1; 1; -3) | 0               | $-\frac{1}{5}$      | $\frac{1}{3}$ | $-\frac{1}{11}$ | $\frac{1}{11}$ | $-\frac{19}{11}$ | 0               | 4               |
| (1, -1; -3)| $\frac{2}{15}$ | $\frac{1}{30}$      | $\frac{1}{6}$ | $-\frac{5}{3}$  | $\frac{5}{3}$  | $-\frac{12}{3}$ | 0               | 2               |
| (-1, -1; -3)| $\frac{19}{720}$ | $\frac{1}{35}$      | $\frac{1}{30}$ | $\frac{3}{5}$   | $\frac{1}{5}$  | $-\frac{19}{5}$ | 0               | 4               |
| (3; 2)     | 0               | 0                    | 0           | $-\frac{1}{7}$  | 0              | $\frac{14}{7}$ | 0               | 0               |
| (-3; 2)    | 0               | 0                    | 0           | $\frac{1}{35}$  | 0              | $\frac{2}{35}$ | 0               | 0               |
| (3; -2)    | 0               | 0                    | 0           | $-\frac{1}{35}$ | 0              | $\frac{2}{35}$ | 0               | 0               |
| (-3; -2)   | 0               | 0                    | 0           | $\frac{1}{35}$  | 0              | $\frac{2}{35}$ | 0               | 0               |
| (1; 2; 2)  | 0               | 0                    | 0           | $\frac{1}{1}$   | 0              | 1          | 0               | 0               |
| (-1; 2; 2) | $-\frac{23}{1440}$ | $\frac{1}{18}$      | $\frac{1}{10}$ | $\frac{1}{12}$ | 0              | $\frac{27}{4}$ | $-4$            | $-8$             |
| (1, -2; 2) | 0               | $-\frac{1}{5}$      | $\frac{2}{7}$ | $-\frac{3}{7}$  | $\frac{3}{7}$  | $-\frac{19}{7}$ | 0               | 4               |
| (-1, -2; 2)| $\frac{29}{720}$ | $\frac{1}{15}$      | $\frac{1}{30}$ | $-\frac{2}{5}$  | $-\frac{2}{5}$ | $\frac{19}{5}$ | 0               | 4               |
| (1, 2; -2) | 0               | $\frac{5}{7}$       | $-\frac{2}{7}$ | $\frac{4}{7}$   | $\frac{4}{7}$  | $\frac{2}{7}$  | $\frac{19}{7}$ | $-4$            | $-4$             |
| (-1, 2; -2) | $-\frac{23}{1440}$ | $\frac{1}{18}$      | $\frac{1}{30}$ | $\frac{3}{5}$   | $\frac{3}{5}$  | $-\frac{29}{5}$ | 0               | $-4$             |
| (1, -2; -2)| 0               | 0                    | 0           | $\frac{1}{35}$  | 0              | $\frac{14}{35}$ | 0               | 0               |
| (-1, -2; -2)| $\frac{19}{720}$ | $\frac{1}{35}$      | $\frac{1}{30}$ | $\frac{3}{5}$   | $\frac{3}{5}$  | $-\frac{19}{5}$ | 0               | 4               |
| (1, 1, 1; 2)| 0               | 0                    | 0           | $\frac{1}{5}$   | 0              | 10         | 0               | 0               |
| (1, 1, -1; 2)| $\frac{109}{1440}$ | $\frac{1}{36}$      | $\frac{1}{20}$ | $\frac{1}{17}$ | 0              | $-\frac{109}{32}$ | 2               | 4               |
| (1, -1, -1; 2)| $\frac{1}{10}$ | $\frac{1}{6}$      | $\frac{1}{10}$ | 0              | $-\frac{1}{10}$ | 0          | 12              |                 |
| (-1, -1, -1; 2)| $-\frac{29}{1920}$ | $\frac{1}{11}$      | $\frac{1}{35}$ | $\frac{1}{17}$ | 0              | $\frac{39}{17}$ | -6              | -24              |
| (1, 1, 1; -2)| 0               | $-\frac{1}{5}$      | $\frac{1}{9}$ | $-\frac{1}{21}$ | $\frac{1}{21}$ | $-\frac{9}{21}$ | 0               | 6               |
| (1, 1, -1; -2)| $\frac{109}{1440}$ | $\frac{1}{36}$      | $\frac{1}{60}$ | $-\frac{1}{8}$  | $-\frac{1}{8}$ | $-\frac{9}{8}$ | $\frac{19}{8}$ | 0               | 2               |
| (1, -1, -1; -2)| $\frac{1}{10}$ | $\frac{1}{6}$      | $\frac{1}{9}$ | $-\frac{1}{8}$  | $-\frac{1}{8}$ | $-\frac{9}{8}$ | $-\frac{10}{8}$ | 0               | 10              |
| (-1, -1, -1; -2)| $-\frac{29}{1920}$ | $\frac{1}{11}$      | $\frac{1}{35}$ | $-\frac{1}{9}$  | $-\frac{1}{8}$ | $-\frac{9}{8}$ | $\frac{19}{8}$ | 0               | -18             |
| (4; -1)    | $-\frac{7}{260}$ | 0                    | 0           | $-\frac{1}{15}$ | 0              | 2          | 0               | 0               |
| (-4; -1)   | $-\frac{1}{90}$ | 0                    | 0           | $-\frac{1}{15}$ | 0              | $\frac{1}{15}$ | 0               | 0               |
| (1, 3; -1) | $-\frac{1}{1440}$ | $\frac{1}{36}$      | $-\frac{1}{60}$ | $\frac{1}{6}$  | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{19}{3}$ | 0               | -2              |
| (-1, 3; -1)| $\frac{11}{1440}$ | $\frac{1}{36}$      | $-\frac{1}{60}$ | $\frac{1}{6}$  | $-\frac{1}{3}$ | $-\frac{19}{3}$ | 0               | -2              |
| (1, -3; -1)| $\frac{1}{360}$ | $\frac{1}{36}$      | $-\frac{1}{60}$ | $\frac{1}{6}$  | $-\frac{1}{3}$ | $-\frac{19}{3}$ | 0               | 2               |
## Appendix 3. PLI closed-forms

We tabulate the main result of this article here: 263 (=11+55+197) convergent PLIs with weight ≤ 5. These integrals are independently solved by the author. Note that there are few literatures dealing with PLIs systematically.

For the sake of simplicity, we leave out 2 commas between $a(0), a(1)$ and $a(2)$, in the definition of PLI. For instance, we have \( PLI(100; 24; 1) = \int_0^1 \frac{\text{Li}_2(z^2)}{x} \log(1-x) \, dx \), \( PLI(200; 25; 26; 0) = \int_0^1 \frac{\text{Li}_2(z_1 z^2)}{1-x} \, dx \). Moreover, by noticing this row in weight 4 PLI table:

| Weight 4 PLI | $\pi^2 log(2)$ | $\pi^2 log^2(2)$ | $\log^4(2)$ | $\zeta(3)log^2(2)$ | $\zeta(5)$ | $\text{Li}_4(\frac{1}{2}) log^2(2)$ | $\text{Li}_5(\frac{1}{2})$ |
|--------------|----------------|-----------------|-----------|------------------|----------|-------------------------------|-----------------|
| (100; 24; 1) | $\frac{11}{1440}$ | $-\frac{1}{12}$ | $\frac{1}{24}$ | $\frac{3}{8}$ | $-\frac{19}{40}$ | 2 | 0 |
| (99; 25; 26; 0) | $\frac{29}{120}$ | $-\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{3}{10}$ | 0 | $-\frac{299}{36}$ | 8 | 16 |
| (2, 2; -1) | $-\frac{1}{25}$ | 0 | 0 | $\frac{1}{16}$ | 0 | 0 | 0 |
| (2, -2; 1) | $-\frac{13}{120}$ | $\frac{1}{5}$ | $-\frac{1}{3}$ | $\frac{1}{16}$ | 0 | $\frac{299}{36}$ | -8 | -16 |
| (1, 1, 2; -1) | $\frac{7}{240}$ | $\frac{1}{10}$ | $-\frac{1}{12}$ | $\frac{1}{16}$ | $\frac{7}{8}$ | 0 | -2 | 0 |
| (1, -1, 2; -1) | $-\frac{1}{45}$ | $-\frac{1}{30}$ | $\frac{1}{27}$ | $\frac{1}{16}$ | $\frac{155}{36}$ | 0 | -6 | 0 |
| (1, -1, 2; -1) | $\frac{1}{240}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | 0 | $-\frac{299}{36}$ | 0 | 6 |
| (1, -1, -2; -1) | $-\frac{1}{240}$ | $-\frac{1}{20}$ | $\frac{1}{5}$ | 0 | $\frac{7}{16}$ | 0 | 4 | 0 |
| (1, 1, 1; -1) | $\frac{7}{240}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{24}$ | $\frac{159}{36}$ | 0 | 14 |
| (1, 1, 1; -1) | $\frac{7}{240}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{24}$ | $\frac{159}{36}$ | 0 | 4 | 8 |
| (1, 1, 1; -1) | $\frac{7}{240}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{24}$ | $\frac{159}{36}$ | 0 | -2 |
| (1, -1, -1; -1) | $-\frac{1}{240}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{24}$ | $\frac{159}{36}$ | 0 | 12 |
| (1, -1, -1; -1) | $\frac{11}{120}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{24}$ | $\frac{159}{36}$ | 0 | 2 |
| (1, -1, -1; -1) | $-\frac{11}{120}$ | $\frac{1}{5}$ | $-\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{24}$ | $\frac{159}{36}$ | 0 | -4 | -48 |

We have \( \int_0^1 \frac{\text{Li}_2(z^2 log(1-x))}{x} \, dx = -3\text{Li}_4\left(\frac{1}{2}\right) - \frac{5}{8} \zeta(3) log(2) - \frac{11\pi^4}{1440} - \frac{1}{8} log^4(2) + \frac{1}{2\pi^2} log^2(2) \).
| Weight 3 PLI | $\pi^2 \log(2)$ | $\log^3(2)$ | $\zeta(3)$ |
|----------------|----------------|-------------|------------|
| (000; 21; 1)  | 0              | 0           | 1          |
| (000; 21; 2)  | $\frac{1}{8}$  | 0           | $-\frac{5}{8}$ |
| (000; 22; 1)  | 0              | 0           | $-\frac{1}{2}$ |
| (000; 22; 2)  | $-\frac{1}{12}$| 0           | $\frac{1}{3}$ |
| (000; 23; 0)  | $-\frac{1}{12}$| $\frac{1}{6}$| $\frac{1}{8}$ |
| (000; 23; 2)  | $\frac{1}{12}$ | $-\frac{1}{6}$| $-\frac{1}{4}$ |
| (000; 24; 2)  | $\frac{1}{12}$ | $-\frac{1}{6}$| $\frac{1}{8}$ |
| (000; 25; 0)  | $-\frac{1}{8}$ | 0           | $\frac{13}{8}$ |
| (000; 25; 2)  | $\frac{1}{8}$ | 0           | $-\frac{2}{8}$ |
| (000; 26; 0)  | $\frac{1}{12}$ | 0           | $-1$        |
| (000; 26; 2)  | $-\frac{1}{12}$| 0           | $\frac{1}{4}$ |

| Weight 4 PLI | $\pi^4$ | $\pi^2 \log^2(2)$ | $\log^4(2)$ | $\zeta(3) \log(2)$ | $\text{Li}_4\left(\frac{1}{2}\right)$ |
|----------------|---------|-------------------|-------------|-------------------|----------------------------|
| (010; 21; 0)   | $-\frac{1}{120}$ | 0                 | 0           | 0                 | 0                          |
| (010; 21; 1)   | $-\frac{1}{60}$  | 0                 | 0           | 0                 | 0                          |
| (010; 21; 2)   | $-\frac{1}{480}$ | 0                 | 0           | 0                 | 0                          |
| (001; 21; 1)   | $-\frac{1}{120}$ | $-\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ | 2                          |
| (001; 21; 2)   | $-\frac{1}{480}$ | $\frac{1}{12}$  | 0           | 0                 | 0                          |
| (100; 21; 1)   | $-\frac{1}{24}$  | 0                 | 0           | 0                 | 0                          |
| (100; 21; 2)   | $-\frac{1}{120}$ | $-\frac{1}{24}$ | $\frac{1}{5}$ | 0                 | 3                          |
| (010; 22; 0)   | $\frac{1}{1340}$ | $\frac{1}{5}$    | $-\frac{1}{5}$ | $-\frac{1}{3}$ | $-4$                      |
| (010; 22; 1)   | $\frac{1}{250}$  | 0                 | 0           | 0                 | 0                          |
| (010; 22; 2)   | $\frac{1}{288}$  | $\frac{1}{5}$    | $-\frac{1}{5}$ | $-\frac{1}{3}$ | $-4$                      |
| (001; 22; 1)   | $-\frac{1}{228}$ | 0                 | 0           | 0                 | 0                          |
| (001; 22; 2)   | $\frac{1}{60}$   | $\frac{1}{12}$   | $-\frac{1}{5}$ | $-\frac{21}{8}$ | $-3$                      |
| (100; 22; 1)   | $\frac{1}{360}$  | $\frac{1}{12}$   | $-\frac{1}{12}$ | $-\frac{5}{8}$ | $-2$                      |
| (100; 22; 2)   | $\frac{1}{12}$   | $\frac{1}{5}$    | $-\frac{1}{5}$ | $-\frac{21}{8}$ | $-4$                      |
| (010; 23; 0)   | $-\frac{1}{120}$ | 0                 | $-\frac{1}{24}$ | $\frac{1}{5}$   | $-1$                      |
| (010; 23; 2)   | $\frac{1}{360}$  | $\frac{1}{24}$   | 0           | $-1$              | 0                          |
| Weight 4 PLI  | \(\pi^4\) | \(\pi^2 \log^2(2)\) | \(\log^4(2)\) | \(\zeta(3) \log(2)\) | \(\text{Li}_4(\frac{1}{2})\) |
|--------------|----------|----------------|-----------|----------------|----------------|
| (001; 23; 0) | \(-\frac{1}{258}\) | \(-\frac{3}{21}\) | \(-\frac{1}{23}\) | \(-\frac{3}{8}\) | 0 |
| (001; 23; 1) | \(\frac{14}{23}\) | \(\frac{1}{5}\) | \(-\frac{1}{15}\) | \(-\frac{3}{7}\) | \(-4\) |
| (001; 23; 2) | \(\frac{1}{8}\) | \(-\frac{1}{15}\) | \(-\frac{1}{3}\) | \(-\frac{3}{5}\) | \(-3\) |
| (100; 23; 0) | 0 | 0 | 0 | 0 | \(-1\) |
| (100; 23; 1) | \(-\frac{19}{1440}\) | \(\frac{1}{21}\) | \(-\frac{1}{23}\) | \(\frac{1}{4}\) | 1 |
| (100; 23; 2) | \(-\frac{1}{1440}\) | \(\frac{1}{23}\) | \(-\frac{1}{3}\) | \(\frac{3}{4}\) | 0 |
| (010; 24; 0) | \(-\frac{1}{288}\) | \(-\frac{1}{21}\) | \(\frac{1}{5}\) | 1 | 3 |
| (010; 24; 2) | \(\frac{1}{1440}\) | 0 | \(-\frac{1}{12}\) | \(-\frac{1}{5}\) | \(-2\) |
| (001; 24; 1) | \(-\frac{1}{36}\) | \(-\frac{1}{21}\) | \(\frac{1}{5}\) | \(\frac{23}{8}\) | 4 |
| (001; 24; 2) | \(-\frac{1}{96}\) | 0 | 0 | 1 | 1 |
| (100; 24; 1) | \(-\frac{1}{1440}\) | \(\frac{1}{21}\) | \(-\frac{1}{5}\) | \(-\frac{3}{8}\) | \(-3\) |
| (100; 24; 2) | \(-\frac{1}{96}\) | \(\frac{1}{21}\) | \(-\frac{1}{3}\) | 0 | 0 |
| (100; 25; 0) | \(\frac{1}{1440}\) | \(\frac{3}{21}\) | \(-\frac{3}{21}\) | \(-\frac{3}{3}\) | \(-5\) |
| (100; 25; 2) | \(-\frac{1}{1440}\) | \(-\frac{1}{21}\) | \(\frac{1}{5}\) | 0 | 3 |
| (001; 25; 0) | \(\frac{1}{1440}\) | 0 | \(-\frac{1}{12}\) | \(-\frac{1}{5}\) | \(-2\) |
| (001; 25; 1) | \(\frac{1}{1440}\) | \(\frac{1}{21}\) | \(-\frac{1}{21}\) | \(-\frac{1}{5}\) | \(-1\) |
| (001; 25; 2) | \(\frac{1}{1440}\) | \(\frac{1}{12}\) | 0 | \(-\frac{1}{5}\) | 0 |
| (100; 25; 0) | \(\frac{1}{1440}\) | 0 | \(-\frac{1}{12}\) | \(-\frac{1}{5}\) | \(-2\) |
| (100; 25; 1) | \(\frac{1}{24}\) | \(\frac{1}{21}\) | \(-\frac{1}{21}\) | \(\frac{1}{5}\) | 1 |
| (100; 25; 2) | 0 | \(\frac{1}{12}\) | 0 | \(-\frac{1}{5}\) | 0 |
| (010; 26; 0) | \(\frac{14}{1440}\) | \(\frac{1}{21}\) | \(-\frac{1}{23}\) | \(-\frac{1}{3}\) | \(-1\) |
| (010; 26; 2) | \(\frac{1}{1440}\) | \(\frac{1}{21}\) | \(-\frac{1}{23}\) | 0 | \(-1\) |
| (010; 26; 0) | \(\frac{1}{1440}\) | \(\frac{1}{6}\) | \(-\frac{1}{8}\) | \(-\frac{20}{3}\) | \(-3\) |
| (001; 26; 1) | \(-\frac{19}{240}\) | \(-\frac{1}{21}\) | \(\frac{1}{21}\) | \(\frac{29}{8}\) | 7 |
| (001; 26; 2) | \(-\frac{1}{96}\) | \(-\frac{1}{5}\) | \(-\frac{1}{3}\) | \(\frac{23}{8}\) | 3 |
| (100; 26; 0) | \(\frac{1}{12}\) | \(\frac{1}{23}\) | \(-\frac{1}{23}\) | \(-\frac{1}{3}\) | \(-1\) |
| (100; 26; 1) | \(\frac{17}{1440}\) | 0 | \(-\frac{1}{12}\) | \(-\frac{1}{3}\) | \(-2\) |
| (000; 31; 1) | \(\frac{1}{256}\) | 0 | 0 | 0 | 0 |
| (000; 31; 2) | \(\frac{1}{64}\) | \(\frac{1}{12}\) | \(-\frac{1}{12}\) | \(-\frac{1}{3}\) | \(-2\) |
| (000; 32; 1) | \(-\frac{1}{256}\) | 0 | 0 | 0 | 0 |
| (000; 32; 2) | \(\frac{1}{256}\) | 0 | 0 | \(-\frac{1}{3}\) | 0 |
| (000; 33; 0) | 0 | 0 | 0 | 0 | 0 |
| (000; 33; 2) | \(-\frac{1}{256}\) | \(-\frac{1}{23}\) | \(-\frac{1}{23}\) | \(-\frac{1}{3}\) | 0 |
| (000; 34; 2) | \(\frac{1}{256}\) | 0 | 0 | 0 | \(-1\) |
| (000; 35; 0) | \(-\frac{1}{128}\) | \(-\frac{1}{12}\) | \(-\frac{1}{12}\) | \(-\frac{1}{3}\) | \(-2\) |
| (000; 35; 2) | \(\frac{1}{128}\) | \(-\frac{1}{12}\) | \(-\frac{1}{12}\) | \(-\frac{1}{3}\) | \(-2\) |
| (000; 36; 0) | \(-\frac{1}{128}\) | 0 | 0 | \(-\frac{1}{3}\) | 0 |
| (000; 36; 2) | \(\frac{1}{256}\) | 0 | 0 | \(-\frac{1}{3}\) | 0 |
| Weight 5 PLI | \( \pi^4 \log(2) \) | \( \pi^4 \log^3(2) \) | \( \log^3(2) \) | \( \pi^4(3) \) | \( \zeta(3) \log^2(2) \) | \( \zeta(5) \) | \( \text{Li}_4(\frac{1}{2}) \log(2) \) | \( \text{Li}_5(\frac{1}{2}) \) |
|-----------|----------------|----------------|----------|----------|----------------|--------|----------------|--------|
| (020; 21; 0) | 0 | 0 | 0 | 1 | 0 | -11 | 0 | 0 |
| (020; 21; 1) | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 |
| (020; 21; 2) | 0 | 0 | 0 | \(-\frac{71}{25}\) | \(-\frac{2}{5}\) | \(-\frac{7}{2}\) | -2 | -2 |
| (002; 21; 1) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{7}{2}\) | \(-\frac{3}{2}\) | 8 | 8 |
| (002; 21; 2) | 0 | \(-\frac{3}{2}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{7}{2}\) | \(-\frac{3}{2}\) | 8 | 8 |
| (200; 21; 1) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | -2 | 0 | 0 |
| (200; 21; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (110; 21; 0) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | 0 | 0 | 0 |
| (110; 21; 1) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | 0 | 0 | 0 |
| (110; 21; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (011; 21; 0) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (011; 21; 1) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (011; 21; 2) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (101; 21; 1) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (101; 21; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (020; 22; 0) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (020; 22; 1) | 0 | 0 | 0 | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (020; 22; 2) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (002; 22; 1) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (002; 22; 2) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (200; 22; 1) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (200; 22; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (110; 22; 0) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (110; 22; 1) | 0 | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 |
| (110; 22; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (011; 22; 0) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (011; 22; 1) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (011; 22; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (101; 22; 0) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (101; 22; 1) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (101; 22; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (020; 23; 0) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (020; 23; 1) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (020; 23; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (002; 23; 1) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (002; 23; 2) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| (110; 23; 0) | \(-\frac{71}{25}\) | \(-\frac{1}{12} - \frac{1}{19}\) | \(-\frac{71}{25}\) | 0 | -2 | -6 | 0 | 0 |
| Weight 5 PLI | $\pi^4 \log(2)$ | $\pi^4 \log^3(2)$ | $\log^5(2)$ | $\pi^4 \log^2(2)$ | $\zeta(3) \log^3(2)$ | $\zeta(5)$ | $\text{Li}_4(\frac{1}{2}) \log(2)$ | $\text{Li}_5(\frac{1}{2})$ |
|-------------|--------------|--------------|-------------|----------------|----------------|-----------|----------------|-----------|
| (110; 23; 1) | $\frac{-3}{100}$ | $\frac{1}{12}$ | $\frac{-7}{60}$ | $\frac{3}{8}$ | $\frac{-11}{8}$ | $\frac{133}{64}$ | $-4$ | $-6$ |
| (110; 23; 2) | $\frac{-1}{80}$ | $\frac{1}{10}$ | $\frac{-5}{60}$ | $\frac{1}{12}$ | $\frac{-7}{12}$ | $\frac{137}{64}$ | $-4$ | $-5$ |
| (011; 23; 0) | $\frac{-1}{12}$ | $\frac{-5}{72}$ | $\frac{-1}{120}$ | $\frac{1}{12}$ | $\frac{-5}{12}$ | $\frac{247}{64}$ | $-5$ | $-8$ |
| (011; 23; 1) | $\frac{7}{144}$ | $\frac{-1}{12}$ | $\frac{1}{12}$ | $\frac{3}{24}$ | $\frac{1}{16}$ | $\frac{262}{64}$ | $4$ | $8$ |
| (011; 23; 2) | $\frac{7}{144}$ | $\frac{-1}{12}$ | $\frac{1}{12}$ | $\frac{3}{24}$ | $\frac{1}{16}$ | $\frac{262}{64}$ | $4$ | $9$ |
| (101; 23; 1) | $\frac{-3}{88}$ | $\frac{-1}{27}$ | $\frac{-7}{120}$ | $\frac{5}{12}$ | $\frac{-7}{12}$ | $\frac{247}{64}$ | $-2$ | $-1$ |
| (101; 23; 2) | $\frac{7}{144}$ | $\frac{-1}{12}$ | $\frac{-1}{10}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{247}{64}$ | $-3$ | $-3$ |
| (020; 24; 0) | $\frac{-1}{360}$ | $\frac{-1}{18}$ | $\frac{1}{30}$ | $\frac{1}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $0$ | $3$ |
| (020; 24; 2) | $\frac{-1}{240}$ | $\frac{-1}{18}$ | $\frac{1}{30}$ | $\frac{1}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $0$ | $4$ |
| (020; 24; 2) | $\frac{-1}{240}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $0$ | $8$ |
| (200; 24; 1) | $\frac{-3}{132}$ | $\frac{1}{12}$ | $\frac{-7}{120}$ | $\frac{5}{12}$ | $\frac{-7}{12}$ | $\frac{247}{64}$ | $-6$ | $-8$ |
| (200; 24; 2) | $\frac{-1}{80}$ | $\frac{1}{10}$ | $\frac{-5}{60}$ | $\frac{1}{12}$ | $\frac{-7}{12}$ | $\frac{247}{64}$ | $-6$ | $-8$ |
| (110; 24; 0) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $2$ | $7$ |
| (110; 24; 1) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (110; 24; 2) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (011; 24; 1) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (011; 24; 2) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (020; 25; 0) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (020; 25; 2) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (002; 25; 0) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (002; 25; 1) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (002; 25; 2) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (110; 25; 0) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (110; 25; 1) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (110; 25; 2) | $\frac{-1}{180}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (111; 25; 0) | $\frac{-1}{360}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (111; 25; 1) | $\frac{-1}{360}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (111; 25; 2) | $\frac{-1}{360}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (101; 25; 1) | $\frac{-1}{360}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |
| (101; 25; 2) | $\frac{-1}{360}$ | $\frac{-1}{18}$ | $\frac{-1}{10}$ | $\frac{3}{24}$ | $\frac{1}{12}$ | $\frac{1}{8}$ | $\frac{48}{64}$ | $7$ | $8$ |

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| Weight 5 PLI | $\pi^4 \log(2)$ | $\pi^2 \log^3(2)$ | $\log^2(2)$ | $\pi^2 \zeta(3)$ | $\zeta(3) \log^2(2)$ | $\zeta(5)$ | $\text{Li}_4 \left( \frac{1}{3} \right) \log(2)$ | $\text{Li}_5 \left( \frac{1}{3} \right)$ |
|-------------|----------------|----------------|-------------|----------------|----------------|-------------|----------------|----------------|
| (002; 26:0) | $\frac{19}{720}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{12}$ | $-\frac{22}{3}$ | 8 | 12 |
| (002; 26:2) | $-\frac{19}{120}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{12}$ | $-\frac{22}{3}$ | 0 | -4 |
| (002; 26:0) | $\frac{19}{720}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{12}$ | $-\frac{22}{3}$ | 6 | 12 |
| (002; 26:1) | 0 | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{12}$ | $-\frac{22}{3}$ | 6 | 6 |
| (002; 26:2) | $-\frac{1}{15}$ | $-\frac{1}{18}$ | $-\frac{1}{60}$ | 0 | 2 | 8 | $-\frac{2}{3}$ | $-8 \pm 2$ | $-8 \pm 4$ |
| (200; 26:0) | $\frac{1}{360}$ | $\frac{1}{18}$ | $-\frac{1}{30}$ | $\frac{1}{6}$ | $-1$ | $-\frac{1}{10}$ | $-2 \pm 2$ | 6 | 8 |
| (200; 26:1) | 0 | $-\frac{1}{6}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{21}{30}$ | $\frac{69}{30}$ | $-\frac{1}{3}$ | 6 | 6 |
| (200; 26:2) | $\frac{17}{720}$ | $-\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{9}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 6 | 8 |
| (110; 26:0) | $\frac{1}{95}$ | 0 | 0 | $\frac{23}{95}$ | $-\frac{7}{30}$ | $-\frac{7}{30}$ | 0 | 0 | 0 |
| (110; 26:1) | 0 | $-\frac{1}{6}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{21}{30}$ | $\frac{69}{30}$ | $-\frac{1}{3}$ | 6 | 6 |
| (110; 26:2) | $\frac{17}{720}$ | $-\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{9}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 6 | 6 |
| (011; 26:0) | $\frac{1}{365}$ | $\frac{1}{18}$ | $-\frac{1}{30}$ | $\frac{1}{6}$ | $-1$ | $-\frac{1}{10}$ | $-2 \pm 2$ | 12 | 20 |
| (011; 26:1) | 0 | $-\frac{1}{6}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{21}{30}$ | $\frac{69}{30}$ | $-\frac{1}{3}$ | 6 | 6 |
| (011; 26:2) | $-\frac{17}{720}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{9}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 6 | 6 |
| (101; 26:0) | $\frac{11}{230}$ | 0 | $\frac{1}{30}$ | $\frac{17}{30}$ | $1$ | $-1$ | $-\frac{205}{64}$ | 2 | 4 |
| (101; 26:1) | 0 | $-\frac{1}{6}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{21}{30}$ | $\frac{69}{30}$ | $-\frac{1}{3}$ | 6 | 6 |
| (101; 26:2) | $\frac{31}{720}$ | $\frac{1}{18}$ | $-\frac{1}{30}$ | $\frac{1}{6}$ | $-1$ | $-\frac{1}{10}$ | $-2 \pm 2$ | 6 | 12 |
| (010; 31:0) | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{3}$ | 0 | 0 |
| (010; 31:1) | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{3}$ | 0 | 0 |
| (010; 31:2) | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{3}$ | 0 | 0 |
| (100; 31:1) | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{3}$ | 0 | 0 |
| (100; 31:2) | $\frac{1}{365}$ | $\frac{1}{18}$ | $\frac{1}{30}$ | $\frac{1}{6}$ | $\frac{23}{30}$ | $\frac{69}{30}$ | $-\frac{1}{3}$ | 0 | 2 |
| (001; 31:1) | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{3}$ | 0 | 0 |
| (001; 31:2) | 0 | $\frac{1}{6}$ | $-\frac{2}{15}$ | $\frac{1}{3} \pm \frac{1}{15}$ | $-\frac{5}{2}$ | $\frac{5}{2}$ | $\frac{35}{16}$ | -4 | -4 |
| (010; 32:0) | 0 | 0 | 0 | $\frac{31}{30}$ | $\frac{1}{19}$ | $\frac{1}{19}$ | 0 | 0 | 0 |
| (010; 32:1) | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{31}{30}$ | 0 | 0 |
| (010; 32:2) | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{31}{30}$ | 0 | 0 |
| (100; 32:0) | $\frac{19}{1240}$ | $-\frac{1}{36}$ | $\frac{1}{33}$ | $-\frac{1}{3}$ | $-\frac{3}{3}$ | $-\frac{3}{3}$ | 2 | 4 |
| (001; 32:1) | 0 | 0 | 0 | $\frac{31}{30}$ | $\frac{1}{19}$ | $\frac{1}{19}$ | 0 | 0 | 0 |
| (001; 32:2) | 0 | $\frac{1}{120}$ | $\frac{1}{30}$ | $\frac{1}{6}$ | $\frac{31}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | -2 | -2 |
| (010; 33:0) | $\frac{1}{720}$ | $\frac{1}{36}$ | $-\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{31}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | -2 | -2 |
| (010; 33:1) | $\frac{1}{1240}$ | $\frac{1}{36}$ | $-\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{31}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | 0 |
| (001; 33:0) | 0 | $\frac{1}{30}$ | $-\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{31}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | -1 | -2 |
| (001; 33:1) | $\frac{1}{720}$ | $\frac{1}{36}$ | $-\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{31}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | -3 | -7 |
| (001; 33:2) | $\frac{1}{1240}$ | $\frac{1}{36}$ | $-\frac{1}{15}$ | $\frac{1}{6}$ | $\frac{31}{30}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | 0 |
| Weight 5 PLI | $\pi^1 \log(2)$ | $\pi^2 \log^2(2)$ | $\pi^3 \log^3(2)$ | $\pi^4 \zeta(3)$ | $\zeta(3) \log^2(2)$ | $\zeta(5)$ | $L_i(\frac{1}{2}) \log(2)$ | $L_i(\frac{3}{2})$ |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| (010; 34; 0) | $\frac{1}{180}$ | $- \frac{1}{18}$ | $\frac{1}{20}$ | $\frac{5}{32}$ | $\frac{13}{16}$ | $- \frac{205}{64}$ | 2 | 4 |
| (010; 34; 2) | $- \frac{1}{150}$ | $- \frac{1}{36}$ | $\frac{1}{48}$ | $\frac{1}{16}$ | $\frac{1}{2}$ | $- \frac{20}{10}$ | 1 | -1 |
| (100; 34; 1) | $\frac{1}{1440}$ | $\frac{1}{2}$ | $\frac{1}{40}$ | $\frac{1}{8}$ | $\frac{1}{10}$ | $\frac{29}{16}$ | 0 | -3 |
| (100; 34; 2) | $\frac{1}{90}$ | $- \frac{1}{36}$ | $\frac{1}{12}$ | $\frac{1}{32}$ | $\frac{1}{16}$ | $- \frac{123}{32}$ | 1 | 2 |
| (001; 34; 1) | $\frac{1}{180}$ | $\frac{1}{72}$ | $- \frac{1}{30}$ | $- \frac{5}{8}$ | $- \frac{23}{16}$ | $- \frac{35}{32}$ | 0 | 4 |
| (001; 34; 2) | $\frac{1}{90}$ | 0 | 0 | 0 | 0 | $- \frac{1}{10}$ | 0 | 1 |
| (010; 35; 0) | $- \frac{1}{360}$ | $\frac{1}{72}$ | $- \frac{1}{60}$ | $\frac{1}{12}$ | $- \frac{5}{32}$ | $- \frac{205}{32}$ | -4 | -6 |
| (010; 35; 2) | $\frac{1}{360}$ | $- \frac{1}{72}$ | $\frac{1}{60}$ | $\frac{5}{32}$ | 2 | $- \frac{205}{32}$ | 4 | 6 |
| (100; 35; 0) | $- \frac{1}{360}$ | $\frac{1}{36}$ | $- \frac{1}{20}$ | $- \frac{5}{16}$ | $- \frac{5}{32}$ | $2$ | -2 | -4 |
| (100; 35; 1) | 0 | $- \frac{1}{16}$ | $\frac{1}{15}$ | $\frac{5}{12}$ | 2 | $- \frac{5}{32}$ | 2 | 2 |
| (100; 35; 2) | $\frac{1}{90}$ | $- \frac{1}{36}$ | $\frac{1}{20}$ | $\frac{1}{16}$ | 2 | $- \frac{5}{32}$ | 2 | 4 |
| (001; 35; 0) | $- \frac{1}{180}$ | $\frac{1}{72}$ | $- \frac{1}{30}$ | $- \frac{1}{2}$ | $- \frac{19}{28}$ | $2$ | -2 | -4 |
| (001; 35; 1) | 0 | $- \frac{1}{18}$ | $\frac{1}{15}$ | $- \frac{1}{16}$ | $\frac{7}{5}$ | $108$ | 2 | 2 |
| (001; 35; 2) | $\frac{1}{90}$ | $- \frac{1}{36}$ | $\frac{1}{20}$ | $\frac{1}{16}$ | 2 | $- \frac{5}{32}$ | 2 | 4 |
| (010; 36; 0) | $- \frac{1}{1350}$ | $\frac{1}{72}$ | $- \frac{1}{20}$ | $- \frac{1}{16}$ | $- \frac{5}{32}$ | $2$ | -4 | -6 |
| (010; 36; 2) | $\frac{1}{1350}$ | $- \frac{1}{12}$ | $\frac{7}{60}$ | $\frac{1}{12}$ | $\frac{2}{5}$ | $- \frac{31}{64}$ | 4 | 6 |
| (100; 36; 0) | $- \frac{1}{1350}$ | $\frac{1}{72}$ | $- \frac{1}{60}$ | $- \frac{1}{16}$ | $- \frac{5}{32}$ | $2$ | -2 | -2 |
| (100; 36; 1) | 0 | $- \frac{1}{18}$ | $\frac{1}{15}$ | $- \frac{1}{32}$ | 2 | $- \frac{5}{32}$ | 2 | 2 |
| (100; 36; 2) | $\frac{1}{1350}$ | $- \frac{1}{72}$ | $\frac{1}{60}$ | $\frac{1}{32}$ | 2 | $- \frac{5}{32}$ | 2 | 2 |
| (001; 36; 0) | $- \frac{1}{1350}$ | $\frac{1}{72}$ | $- \frac{1}{60}$ | $- \frac{1}{16}$ | $- \frac{5}{32}$ | $2$ | -2 | -2 |
| (001; 36; 1) | 0 | $- \frac{1}{18}$ | $\frac{1}{15}$ | $- \frac{1}{16}$ | $\frac{7}{5}$ | $108$ | 2 | 2 |
| (001; 36; 2) | $\frac{1}{1350}$ | $- \frac{1}{72}$ | $\frac{1}{60}$ | $\frac{1}{16}$ | 2 | $- \frac{5}{32}$ | 2 | 2 |
| (000; 41; 1) | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| (000; 41; 2) | $\frac{1}{90}$ | 0 | 0 | 0 | $\frac{1}{8}$ | 0 | $- \frac{5}{32}$ | 0 | 0 |
| (000; 42; 1) | 0 | 0 | 0 | 0 | 0 | $- \frac{5}{32}$ | 0 | 0 |
| (000; 42; 2) | $- \frac{1}{720}$ | 0 | 0 | $- \frac{1}{36}$ | 0 | $\frac{29}{32}$ | 0 | 0 |
| (000; 43; 0) | 0 | 0 | 0 | 0 | 0 | $- \frac{5}{32}$ | 0 | 0 |
| (000; 43; 2) | $\frac{1}{36}$ | $- \frac{1}{36}$ | $\frac{1}{20}$ | $- \frac{1}{16}$ | $- \frac{7}{32}$ | $- \frac{37}{32}$ | -1 | -2 |
| (000; 44; 2) | 0 | 0 | 0 | 0 | 0 | 1 | 0 | -1 |
| (000; 45; 0) | $- \frac{1}{90}$ | 0 | 0 | $- \frac{1}{36}$ | 0 | $\frac{29}{32}$ | 0 | 0 |
| (000; 45; 2) | $\frac{1}{90}$ | 0 | 0 | $\frac{1}{36}$ | 0 | $- \frac{29}{32}$ | 0 | 0 |
| (000; 46; 0) | $\frac{1}{720}$ | 0 | 0 | $\frac{1}{16}$ | 0 | $- \frac{7}{32}$ | 0 | 0 |
| (000; 46; 2) | $- \frac{1}{720}$ | 0 | 0 | $- \frac{1}{16}$ | 0 | $\frac{7}{32}$ | 0 | 0 |
| (000; 21, 21; 1) | 0 | 0 | 0 | $\frac{1}{36}$ | 0 | $- \frac{5}{32}$ | 0 | 0 |
| (000; 21, 21; 2) | $\frac{1}{36}$ | 0 | 0 | $- \frac{1}{24}$ | 0 | $\frac{29}{32}$ | 0 | 0 |
| (000; 22, 22; 1) | 0 | 0 | 0 | $\frac{1}{36}$ | 0 | $- \frac{5}{32}$ | 0 | 0 |
| (000; 22, 22; 2) | $\frac{1}{36}$ | $- \frac{1}{36}$ | $\frac{1}{20}$ | $- \frac{1}{24}$ | 0 | $\frac{7}{32}$ | 0 | 0 |
| (000; 23, 23; 0) | $- \frac{1}{144}$ | $\frac{1}{72}$ | $- \frac{1}{20}$ | $\frac{1}{24}$ | 0 | $- \frac{7}{32}$ | -2 | -2 |
| (000; 23, 23; 2) | $\frac{1}{144}$ | $- \frac{1}{36}$ | $\frac{1}{20}$ | $- \frac{1}{24}$ | 0 | $\frac{7}{32}$ | 0 | 0 |
\[
\begin{array}{cccccccc}
\text{Weight 5 PLI} & \pi^1 \log(2) & \pi^2 \log^2(2) & \log^3(2) & \pi^2 \zeta(3) & \zeta(3) \log^2(2) & \zeta(5) & \text{Li}_4(\frac{1}{2}) \log(2) & \text{Li}_5(\frac{1}{2}) \\
(000; 24, 24; 2) & -\frac{1}{144} & -\frac{1}{12} & \frac{7}{60} & -\frac{3}{16} & \frac{5}{8} & -\frac{123}{32} & 2 & 2 \\
(000; 25, 25; 2) & -\frac{1}{144} & 0 & 0 & -\frac{3}{24} & 0 & -\frac{79}{32} & 0 & 0 \\
(000; 25, 25; 0) & -\frac{1}{144} & 0 & 0 & -\frac{3}{24} & 0 & -\frac{79}{32} & 0 & 0 \\
(000; 26, 26; 2) & -\frac{1}{144} & -\frac{3}{6} & \frac{4}{15} & -\frac{4}{13} & \frac{2}{3} & -\frac{125}{32} & 8 & 8 \\
(000; 26, 26; 0) & -\frac{1}{144} & -\frac{3}{6} & -\frac{4}{13} & -\frac{1}{6} & -\frac{3}{2} & -\frac{125}{32} & -8 & -8 \\
(000; 21, 22; 1) & 0 & 0 & 0 & -\frac{3}{24} & 0 & -\frac{59}{32} & 0 & 0 \\
(000; 21, 22; 2) & -\frac{1}{144} & -\frac{1}{6} & \frac{4}{15} & -\frac{2}{3} & \frac{2}{3} & -\frac{125}{32} & 4 & 4 \\
(000; 21, 23; 0) & -\frac{1}{144} & -\frac{1}{6} & \frac{4}{15} & -\frac{2}{3} & \frac{2}{3} & -\frac{125}{32} & 3 & 2 \\
(000; 21, 23; 1) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 0 & -2 \\
(000; 21, 23; 2) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 4 & 5 \\
(000; 21, 24; 1) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 2 & 2 \\
(000; 21, 24; 2) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 0 & -3 \\
(000; 21, 25; 1) & 0 & 0 & 0 & -\frac{3}{24} & 0 & -\frac{59}{32} & 0 & 0 \\
(000; 21, 25; 2) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 2 & 4 \\
(000; 21, 25; 0) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & -3 & -4 \\
(000; 21, 26; 1) & 0 & 0 & 0 & -\frac{3}{24} & 0 & -\frac{59}{32} & 0 & 0 \\
(000; 21, 26; 2) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 2 & 4 \\
(000; 21, 26; 0) & -\frac{1}{144} & -\frac{1}{36} & -\frac{1}{10} & -\frac{27}{16} & 0 & -\frac{39}{32} & 3 & 7 \\
(000; 22, 23; 0) & -\frac{1}{288} & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 0 & 0 \\
(000; 22, 23; 1) & -\frac{1}{288} & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -4 & -8 \\
(000; 22, 23; 2) & -\frac{1}{288} & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 2 & 4 \\
(000; 22, 24; 1) & -\frac{1}{288} & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -3 & -4 \\
(000; 22, 24; 2) & -\frac{1}{288} & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 0 & 4 \\
(000; 22, 25; 2) & -\frac{1}{288} & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 3 & 7 \\
(000; 22, 25; 1) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -2 & -2 \\
(000; 22, 26; 1) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -2 & -2 \\
(000; 22, 26; 2) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 5 & 12 \\
(000; 22, 26; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -7 & -14 \\
(000; 23, 24; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 3 & 3 \\
(000; 23, 24; 2) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -3 & -3 \\
(000; 23, 25; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -1 & -1 \\
(000; 23, 25; 2) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -3 & -3 \\
(000; 23, 26; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 2 & 2 \\
(000; 23, 26; 2) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -4 & -4 \\
(000; 24, 23; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -5 & -5 \\
(000; 24, 25; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 1 & 1 \\
(000; 24, 26; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -6 & -6 \\
(000; 24, 26; 2) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -4 & -4 \\
(000; 25, 26; 0) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & -4 & -4 \\
(000; 25, 26; 2) & 0 & 0 & 0 & -\frac{1}{18} & \frac{3}{8} & -\frac{15}{32} & 4 & 4 \\
\end{array}
\]
Appendix 4. Other closed-forms

We only present some new and typical examples. Interested readers may derive similar results by themselves using methods recorded in section 4.

(1) LSIs, PLSIs and Arctan integrals

\[
\int_0^{\pi/2} x \log^2(2 \sin(x)) \, dx = \text{Li}_4 \left( \frac{1}{2} \right) + \frac{7}{8} \zeta(3) \log(2) - \frac{19\pi^4}{2880} + \frac{\log^4(2)}{24} - \frac{1}{24} \pi^2 \log^2(2)
\]

\[
\int_0^{\pi/2} x \log^3(2 \sin(x)) \, dx = 3 \text{Li}_5 \left( \frac{1}{2} \right) + 3 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{3\pi^2 \zeta(3)}{16}
\]

\[
\int_0^{\pi/2} x \log^3(2 \sin(x)) \log(2 \cos(x)) \, dx = \text{Li}_5 \left( \frac{1}{2} \right) + \text{Li}_4 \left( \frac{1}{2} \right) \log(2)
\]

\[
\int_0^{\pi/2} x^2 \log^2(2 \sin(x)) \, dx = \pi \text{Li}_4 \left( \frac{1}{2} \right) + \frac{7}{8} \pi \zeta(3) \log(2) - \frac{3\pi^5}{320} + \frac{1}{24} \pi \log^4(2) - \frac{1}{24} \pi^3 \log^2(2)
\]

\[
\int_0^{\pi/2} x^3 \log^2(2 \sin(x)) \, dx = \frac{3}{4} \text{ES}(1; -5) + \frac{3}{4} \pi^2 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{3\zeta(3)^2}{8}
\]

\[
\int_0^{\pi/2} x^2 \log^2(2 \sin(x)) \log(2 \cos(x)) \, dx = -\pi \text{Li}_5 \left( \frac{1}{2} \right) - \pi \text{Li}_4 \left( \frac{1}{2} \right) \log(2)
\]

\[
\int_0^{\pi/2} x \log^4(2 \sin(x)) \, dx = \frac{3}{2} \text{ES}(1; -5) + 12 \text{Li}_6 \left( \frac{1}{2} \right) + 6 \text{Li}_4 \left( \frac{1}{2} \right) \log^2(2)
\]

\[
\int_0^{\pi/2} x \log^3(2 \sin(x)) \log(2 \cos(x)) \, dx = \frac{9}{4} \text{ES}(1; -5) - \frac{1}{8} \pi^2 \text{Li}_4 \left( \frac{1}{2} \right) + 6 \text{Li}_6 \left( \frac{1}{2} \right)
\]

\[
\int_0^{\pi/2} x \log^3(2 \sin(x)) \log^2(2 \cos(x)) \, dx = \frac{1579\pi^6}{161280} - \frac{13}{192} \pi^2 \log^4(2) + \frac{1}{192} \pi^4 \log^2(2)
\]
\[ \int_0^\frac{\pi}{2} x^5 \log(2 \sin(x)) \, dx = \frac{15\pi^4 \zeta(3)}{256} - \frac{225\pi^2 \zeta(5)}{256} + \frac{1905\zeta(7)}{512} \]

\[ \int_0^\frac{\pi}{2} x^4 \log^2(2 \sin(x)) \, dx = \frac{3}{2} \pi E_1(1; -5) + \frac{1}{2} \pi^3 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{7}{16} \pi^3 \zeta(3) \log(2) - \frac{269\pi^7}{40320} + \frac{1}{48} \pi^3 \log^4(2) - \frac{1}{48} \pi^5 \log^2(2) \]

\[ \int_0^\frac{\pi}{2} x^3 \cot(x) \log(2 \sin(x)) \, dx = -\frac{3}{2} \pi \text{Li}_4 \left( \frac{1}{2} \right) - \frac{21}{16} \pi \zeta(3) \log(2) + \frac{9\pi^5}{640} - \frac{1}{16} \pi \log^4(2) + \frac{1}{8} \pi^3 \log^2(2) \]

\[ \int_0^\frac{\pi}{2} x^2 \cot(x) \log(2 \sin(x)) \log(2 \cos(x)) \, dx = \frac{\pi^2 \zeta(3)}{12} - \frac{31\zeta(5)}{64} - \frac{7}{8} \zeta(3) \log^2(2) + \frac{1}{12} \pi^2 \log^3(2) - \frac{1}{192} \pi^4 \log(2) \]

\[ \int_0^\frac{\pi}{2} \text{Cl}_2(2x) \log^2(2 \sin(x)) \, dx = -2 \text{Li}_5 \left( \frac{1}{2} \right) - 2 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) + \frac{155\zeta(5)}{64} - \frac{7}{8} \zeta(3) \log^2(2) - \frac{1}{15} \log^5(2) + \frac{1}{18} \pi^2 \log^3(2) \]

\[ \int_0^\frac{\pi}{2} \text{Cl}_2(2x) \log(2 \sin(x)) \log(2 \cos(x)) \, dx = 2 \text{Li}_5 \left( \frac{1}{2} \right) + 2 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{279\zeta(5)}{128} + \frac{7}{8} \zeta(3) \log^2(2) + \frac{\log^5(2)}{15} - \frac{1}{18} \pi^2 \log^3(2) \]

\[ \int_0^\frac{\pi}{2} \text{Cl}_2(4x) \log^2(2 \cos(x)) \, dx = 16 \text{Li}_5 \left( \frac{1}{2} \right) + 16 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{465\zeta(5)}{32} - \frac{7}{48} \pi^2 \zeta(3) \log^2(2) + \frac{8 \log^5(2)}{15} - \frac{4}{9} \pi^2 \log^3(2) \]

\[ \int_0^\frac{\pi}{2} x \text{T}_{12}(\tan(x)) \log(2 \sin(x)) \, dx = \frac{7}{4} \pi \text{Li}_4 \left( \frac{1}{2} \right) + \frac{49}{32} \pi \zeta(3) \log(2) - \frac{163\pi^5}{11520} + \frac{7}{96} \pi \log^4(2) - \frac{7}{96} \pi^3 \log^2(2) \]

\[ \int_0^\frac{\pi}{2} x \text{Cl}_2(2x) \cot(x) \log(2 \sin(2x)) \, dx = 4 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{65\pi^2 \zeta(3)}{192} + \frac{341\zeta(5)}{128} + \frac{7}{2} \zeta(3) \log^2(2) + \frac{\log^5(2)}{6} - \frac{1}{6} \pi^2 \log^3(2) - \frac{61\pi^4 \log(2)}{2880} \]

\[ \int_0^\frac{\pi}{2} x \text{Cl}_2(4x) \cot(x) \log(2 \sin(2x)) \, dx = 16 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{23\pi^2 \zeta(3)}{48} + \frac{31\zeta(5)}{4} + 14 \zeta(3) \log^2(2) + \frac{2 \log^5(2)}{3} - \frac{2}{3} \pi^2 \log^3(2) - \frac{151}{720} \pi^4 \log(2) \]
\[
\int_0^\infty \frac{\log^3 (x^2 + 1) \tan^{-1}(x)}{x^2} \, dx = -24 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{83\pi^4}{120} - \log^4(2) + 4\pi^2 \log^2(2)
\]

\[
\int_0^\infty \frac{\log^2 (x^2 + 1) \tan^{-1}(x)^2}{x^4} \, dx = -\pi \zeta(3) + \frac{\pi^3}{18} - \frac{4}{9} \pi \log^3(2) + \frac{4}{3}\pi^2 \log^2(2) - \frac{2}{9} \pi^3 \log(2) + \frac{4}{3} \pi \log(2)
\]

\[
\int_0^\infty \frac{\log (x^2 + 1) \tan^{-1}(x)^3}{x^2} \, dx = 24 \text{Li}_5 \left( \frac{1}{2} \right) + \frac{3\pi^2 \zeta(3)}{8}
\]

\[
\int_0^\infty \frac{93\zeta(5)}{2} - \frac{1}{5} \log^5(2) + \frac{4}{3} \pi^2 \log^3(2) + \frac{47}{60} \pi^4 \log(2)
\]

\[
\int_0^\infty \frac{\log (x^2 + 1) \tan^{-1}(x)^2}{x^2 + 1} \, dx = 2\pi \zeta(3) \log(2) + \frac{11\pi^5}{360} + \frac{1}{6} \pi^3 \log^2(2)
\]

\[
\int_0^\infty \frac{\log (x^2 + 1) \tan^{-1}(x)^2}{x^3 + x} \, dx = 8 \text{Li}_5 \left( \frac{1}{2} \right) - \frac{3\pi^2 \zeta(3)}{8}
\]

\[
-\frac{217\zeta(5)}{16} - \frac{1}{15} \log^5(2) + \frac{4}{9} \pi^2 \log^3(2) + \frac{79}{360} \pi^4 \log(2)
\]

(2) QLIs and QLSIs

\[
\int_0^1 \frac{\log x + \log (x^2 + 1)}{x^2 + 1} \, dx = \frac{5}{2} C \log(2) - 4\Im \left( \text{Li}_3 \left( \frac{1}{2} \right) \right) + \frac{7\pi^3}{64} + \frac{3}{8} \pi \log^2(2)
\]

\[
\int_0^1 \frac{\log(1 - x) \log(x^2 + 1)}{x + 1} \, dx = -\pi C + \frac{7\zeta(3)}{4} + \frac{11 \log^3(2)}{24} - \frac{5}{32} \pi^2 \log(2)
\]

\[
\int_0^1 \frac{\log(x) \log^2(x^2 + 1)}{x^2 + 1} \, dx = -C \log^2(2) - 8\Im(\text{Li}_4(1 + i)) - 4 \log(2) \Im(\text{Li}_3(1 + i)) + \frac{7\pi\zeta(3)}{4} + \frac{1}{3} \pi \log^3(2) + \frac{3}{8} \pi^3 \log(2) - \frac{1}{768} \left( \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \right)
\]

\[
\int_0^1 \frac{\log(x) \log(1 - x) \log(x + 1)}{x^2 + 1} \, dx = \frac{\pi^2 C}{6} - C \log^2(2) - 2\Im(\text{Li}_4(1 + i)) + 2 \log(2) \Im(\text{Li}_3(1 + i)) - \frac{5}{48} \pi \log^3(2) - \frac{1}{64} \pi^3 \log(2)
\]

\[
\int_0^1 \frac{\log(1 - x) \log(x + 1) \log(x^2 + 1)}{x^2 + 1} \, dx = \frac{11\pi^2 C}{48} + 2\Im(\text{Li}_4(1 + i)) + \log(2) \Im(\text{Li}_3(1 + i)) - 2C \log^2(2) + \frac{7\pi\zeta(3)}{64} + \frac{1}{24} \pi \log^3(2) - \frac{3}{64} \pi^3 \log(2) - \frac{1}{384} \left( \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \right)
\]

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\[ \int_0^1 \frac{x \log(1 - x) \log^2(x + 1)}{x^2 + 1} \, dx = \frac{\text{Li}_4 \left( \frac{1}{2} \right)}{8} - \frac{77\pi^4}{46080} + \frac{\log^4(2)}{48} - \frac{5}{384}\pi^2 \log^2(2) \]

\[ \int_0^1 \frac{\log(x^2 + 1) \tan^{-1}(x)}{x^2 + 1} \, dx = -\frac{1}{4}\pi C + \frac{21\zeta(3)}{64} + \frac{1}{16}\pi^2 \log(2) \]

\[ \int_0^1 \frac{\log(1 - x) \tan^{-1}(x)}{x + 1} \, dx = \frac{1}{2}C \log(2) + 23 \left( \text{Li}_3 \left( \frac{1 + i}{2} \right) \right) - \frac{23\pi^3}{384} + \frac{1}{32}\pi \log^2(2) \]

\[ \int_0^1 \frac{\log^2 \left( x^2 + 1 \right) \tan^{-1}(x)}{x} \, dx = 83 \left( \text{Li}_4 \left( \frac{1 + i}{2} \right) \right) + 4 \log(2) \cdot 3 \left( \text{Li}_3 \left( \frac{1 + i}{2} \right) \right) + C \log^2(2) - \frac{35\pi \zeta(3)}{32} - \frac{1}{12}\pi \log^3(2) - \frac{1}{768} \left( \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \right) \]

\[ \int_0^1 \frac{\log^3(x) \tan^{-1}(x)}{x^2 + 1} \, dx = \frac{7\pi^2 \zeta(3)}{64} + \frac{93\zeta(5)}{32} - \frac{\pi}{1024} \left( \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \right) \]

\[ \int_0^1 \frac{x \log^3(x) \tan^{-1}(x)}{x^2 + 1} \, dx = 3\pi^3 \zeta(3) \left( \frac{3}{32} \right) + \frac{93\pi \zeta(5)}{256} - \frac{5}{64}\pi^5 \log(2) + \frac{\psi^{(5)} \left( \frac{1}{4} \right) - \psi^{(5)} \left( \frac{3}{4} \right)}{40960} \]

\[ \int_0^\pi \log^3(2 \cos(x)) \, dx = 33 \left( \text{Li}_4(1 + i) \right) - \frac{3}{2} \log(2) \cdot \Im(\text{Li}_3(1 + i)) + \frac{3}{8}C \log^2(2) - \frac{3\pi \zeta(3)}{8} + \frac{1}{16}\pi \log^3(2) - \frac{\psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right)}{2048} \]

\[ \int_0^\pi \log(2 \sin(x)) \log^2(2 \cos(x)) \, dx = \frac{1}{8}C \log^2(2) + 3 \left( \text{Li}_4(1 + i) \right) - \frac{\pi \zeta(3)}{16} - \frac{1}{2} \log(2) \cdot \Im(\text{Li}_3(1 + i)) + \frac{1}{48}\pi \log^3(2) - \frac{5 \left( \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \right)}{6144} \]

\[ \int_0^\pi x^3 \log(2 \sin(x)) \, dx = -\frac{1}{128}\pi^3 C + \frac{9\pi^2 \zeta(3)}{2048} - \frac{1581\zeta(5)}{4096} + \frac{5 \left( \psi^{(3)} \left( \frac{1}{4} \right) - \psi^{(3)} \left( \frac{3}{4} \right) \right)}{8192} \]

Apart from these integrals, a pair of surprising closed-form polylog special values arise from evaluating weight 4 QLIs by 2 different ways. We have:

\[ \Re(\text{Li}_4(1 + i)) = -\frac{5\text{Li}_4 \left( \frac{1}{2} \right)}{16} + \frac{97\pi^4}{9216} - \frac{5 \log^4(2)}{384} + \frac{1}{48}\pi^2 \log^2(2) \]

\[ \Re \left( \text{Li}_4 \left( \frac{1}{2} + \frac{i}{2} \right) \right) = \sum_{k=1}^{\infty} \frac{\left( -1 \right)^{k} \zeta(2k)}{k^4 2^k} = \frac{5\text{Li}_4 \left( \frac{1}{2} \right)}{16} + \frac{343\pi^4}{92160} + \frac{\log^4(2)}{96} - \frac{5}{768}\pi^2 \log^2(2) \]
Where \([a]\) denote the integer part of \(a\). These are weight 4 generalizations of the well-known result
\[
\Re(Li_3(1+i)) = \frac{35\zeta(3)}{64} + \frac{1}{32} \pi^2 \log(2),
\]
\[
\Re(Li_3\left(\frac{1+i}{2}\right)) = \frac{35\zeta(3)}{64} + \frac{\log^2(2)}{48} - \frac{5}{192} \pi^2 \log(2).
\]
Also note that we can write some of the results in alternate forms, by using \(\mathbb{I}\) and \(\mathbb{R}\).

(3) NLIs, NPLIs and NQLIs

\[
\int_0^1 \log(1-x) \log(x) \log(x+1) dx = \frac{21\zeta(3)}{8} + \frac{5\pi^2}{12} - 6 - \log^2(2) - \frac{1}{2} \pi^2 \log(2) + 4 \log(2)
\]

\[
\int_0^1 \log(x) \log^3(x+1) dx = 6 Li_4 \left(\frac{1}{2}\right) + \frac{3\zeta(3)}{4} + \frac{21}{4} \zeta(3) \log(2) - \frac{\pi^2}{2}
\]

\[-\frac{\pi^4}{15} + 24 + \frac{\log^4(2)}{4} - 2 \log^3(2) - \frac{1}{4} \pi^2 \log^2(2) + 12 \log^2(2) - 36 \log(2)
\]

\[
\int_0^1 \log^3(1-x) \log^2(x+1) dx = 24 Li_4 \left(\frac{1}{2}\right) + 24 Li_5 \left(\frac{1}{2}\right) - 2 \pi^2 \zeta(3) + 45 \zeta(3) + \frac{3\zeta(5)}{4}
\]

\[+ 12 \zeta(3) \log^2(2) - 24 \zeta(3) \log(2) + \frac{\pi^4}{30} + 6 \pi^2 - 120 + \frac{3 \log^5(2)}{5} - 3 \log^4(2) - \frac{2}{3} \pi^2 \log^3(2)
\]

\[+ 16 \log^3(2) + 2 \pi^2 \log^2(2) - 48 \log^2(2) - \frac{1}{30} \pi^4 \log(2) - 6 \pi^2 \log(2) + 96 \log(2)
\]

And the symmetric but monstrous one:

\[
\int_0^1 \log^2(1-x) \log^2(x) \log^2(x+1) dx = -4 ES(1; -5) - \frac{16}{3} \pi^2 Li_4 \left(\frac{1}{2}\right) - 32 Li_4 \left(\frac{1}{2}\right)
\]

\[+ 24 Li_5 \left(\frac{1}{2}\right) - 16 Li_6 \left(\frac{1}{2}\right) - 8 Li_4 \left(\frac{1}{2}\right) \log^2(2) + 24 Li_4 \left(\frac{1}{2}\right) \log(2) - 16 Li_5 \left(\frac{1}{2}\right) \log(2)
\]

\[+ \frac{275 \zeta(3)^2}{8} + \frac{34 \pi^2 \zeta(3)}{3} - 237 \zeta(3) - \frac{341 \zeta(5)}{2} + 7 \zeta(3) \log^3(2) - \frac{91}{2} \zeta(3) \log^2(2)
\]

\[+ \frac{77}{6} \pi^2 \zeta(3) \log(2) + 156 \zeta(3) \log(2) + \frac{217}{2} \zeta(5) \log(2) + \frac{73 \pi ^6}{3024} - 40 \pi^2 - \frac{29 \pi^4}{45} + 720
\]

\[- \frac{2}{9} \log^6(2) + \frac{4 \log^5(2)}{5} - \frac{1}{18} \pi^2 \log^4(2) + \frac{2 \log^4(2)}{3} + \frac{2}{5} \pi^2 \log^3(2) - 24 \log^3(2)
\]

\[- \frac{1}{36} \pi^4 \log^2(2) - 8 \pi^2 \log^2(2) + 144 \log^2(2) + \frac{2}{3} \pi^4 \log(2) + 32 \pi^2 \log(2) - 480 \log(2)
\]

\[
\int_0^1 Li_2 \left(\frac{1-x}{2}\right) \log(x+1) dx = -\frac{\zeta(3)}{2} - \frac{\pi^2}{12} + 3 - \frac{1}{3} \log^3(2) + \frac{\log^2(2)}{2} - 4 \log(2) + \frac{1}{6} \pi^2 \log(2)
\]
\[\int_0^1 \text{Li}_2 \left( \frac{x + 1}{2} \right)^2 \, dx = -7\zeta(3) + \frac{7\pi^4}{144} - \frac{\pi^2}{2} + 6 - \frac{1}{4} \log^4(2)\]
\[\quad - \frac{\log^3(2)}{3} + \frac{1}{12} \pi^2 \log^2(2) + \log^2(2) - \frac{1}{6} \pi^2 \log(2) + 6 \log(2)\]

\[\int_0^1 \text{Li}_3 \left( \frac{x + 1}{2} \right) \log(x) \log(1 - x) \, dx = -7 \text{Li}_4 \left( \frac{1}{2} \right) + 11 \text{Li}_5 \left( \frac{1}{2} \right) + 4 \text{Li}_4 \left( \frac{1}{2} \right) \log(2) + \frac{49\pi^2 \zeta(3)}{96}\]
\[-10\zeta(3) - \frac{463\zeta(5)}{32} + \frac{21}{16} \zeta(3) \log^2(2) - \frac{11}{4} \zeta(3) \log(2) + \frac{121\pi^4}{1440} - \frac{13\pi^2}{6} + 20 + \frac{3 \log^5(2)}{40} - \frac{5 \log^4(2)}{24}\]
\[-\frac{1}{24} \pi^2 \log^3(2) + \frac{\log^3(2)}{3} + \frac{5}{24} \pi^2 \log^2(2) - 3 \log^2(2) + \frac{1}{288} \pi^4 \log(2) + \frac{2}{3} \pi^2 \log(2) + 12 \log(2)\]

\[\int_0^1 \log(x) \log(x + 1) \log(x^2 + 1) \, dx = -\frac{\pi C}{2} + 2C - 4C \log(2) - 63 \left( \text{Li}_3 \left( \frac{1}{2} + \frac{i}{2} \right) \right) + \frac{3\zeta(3)}{2}\]
\[+ \frac{11\pi^3}{64} + \frac{3\pi^2}{16} + \pi - 12 - \frac{1}{4} \pi \log^2(2) + \frac{3}{16} \pi \log^2(2) \quad \frac{1}{16} \pi^2 \log(2) - \frac{1}{4} \pi \log(2) + 10 \log(2)\]

\[\int_0^1 \log(x) \log(1 - x) \tan^{-1} \left( x^2 - x + 1 \right) \, dx = -2C - C \log(2) - 43 \left( \text{Li}_3 \left( \frac{1}{2} + \frac{i}{2} \right) \right)\]
\[+ \frac{41\zeta(3)}{32} + \frac{\pi^3}{24} + \frac{7\pi^2}{48} - \frac{1}{4} \log^2(2) + \frac{\pi^2}{8} \log^2(2) - \frac{3}{16} \pi^2 \log(2) + \frac{1}{4} \pi \log(2) + 2 \log(2)\]

\[\int_0^1 \log^4 \left( x^2 + 1 \right) \, dx = -192C - 24\pi \log^2(2) + 96C \log(2) + 1923 \text{Li}_3(1 + i) - 1923 \text{Li}_4(1 + i)\]
\[-96 \log(2) \text{Li}_3(1 + i) + 24\pi \zeta(3) - 7\pi^3 - 96\pi + 384 + \log^4(2) + 24 \pi \log^3(2) - 8 \log^3(2)\]
\[-60 \pi \log^2(2) + 48 \log^2(2) + 7 \pi^3 \log(2) + 96 \pi \log(2) - 192 \log(2) + \frac{1}{32} \left( \psi^{(3)}(1) - \psi^{(3)}(3) \right)\]

(4) RSs and multiple integrals

\[\sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_m}{m} \sum_{n=1}^{m} \frac{(-1)^{n-1} H_n^{(2)}}{n} = \int_0^1 \int_0^1 \frac{\log(x) \log(1 - y) \log \left( \frac{1 - xy}{1 - y} \right)}{(1 - x)y(y + 1)} \, dx \, dy\]
\[= 6 \text{Li}_5 \left( \frac{1}{2} \right) + \frac{11 \pi^2 \zeta(3)}{48} - \frac{49 \zeta(5)}{8} - \frac{1}{2} \zeta(3) \log^2(2) - \frac{1}{20} \log^5(2) + \frac{1}{36} \pi^2 \log^3(2) + \frac{13}{360} \pi^4 \log(2)\]

\[\sum_{m=1}^{\infty} \frac{(-1)^{m-1} (\overline{H}_m)}{m} \sum_{n=1}^{m} \frac{(-1)^{n-1} H_n}{n} = \int_0^1 \int_0^1 \int_0^1 \frac{\log(1 - z) \log \left( \frac{(1 - x)(1 - y)(1 - z)(1 - xy)}{2(xy + 1)(x + 1)(xyz + 1)} \right)}{(x + 1)(y + 1)(z + 1)} \, dx \, dy \, dz\]
\[
= 22 \text{Li}_5 \left( \frac{1}{2} \right) + \frac{\pi^2 \zeta(3)}{8} - \frac{349 \zeta(5)}{16} - \frac{5 \zeta(3) \log^2(2)}{8} - \frac{17 \log^5(2)}{60} - \frac{7}{72} \pi^2 \log^3(2) + \frac{43}{240} \pi^4 \log(2)
\]

\[
\sum_{m=1}^{\infty} \left( -1 \right)^{m-1} \bar{H}_m \sum_{n=1}^{m} \frac{(-1)^{n-1} H_n H_n}{n^2}.
\]

\[
\int_0^1 \int_0^1 \int_0^1 \frac{\log(1-y) \log(1-x) \log(1-z) \log(1-w)}{(x+1)(y+1)(z+1)} dxdydz - \int_0^1 \int_0^1 \int_0^1 \frac{x \log(1-y) \log(1-z) \log(1-w)}{(x+1)(y+1)(z+1)} dxdydz
\]

\[
= 14 \text{Li}_5 \left( \frac{1}{2} \right) + \frac{\pi^2 \zeta(3)}{12} - \frac{115 \zeta(5)}{8} - \frac{3 \zeta(3) \log^2(2)}{8} - \frac{11 \log^5(2)}{60} - \frac{1}{18} \pi^2 \log^3(2) + \frac{23}{180} \pi^4 \log(2)
\]

Finally, we use a high-depth symmetric RS (also an MZSV) to conclude the appendix. This can be shown by using inner and outer symmetry relations repeatedly:

\[
\zeta^* \left( \{-1\}^5 \right) = \zeta^* \left( -1, -1, -1, -1, -1 \right) = \sum_{1 \leq j \leq k \leq l \leq m \leq n} \left( -1 \right)^{i-1} \left( -1 \right)^{k-1} \left( -1 \right)^{l-1} \left( -1 \right)^{m-1} \left( -1 \right)^{n-1} \frac{(-1)^{j-1} (-1)^{k-1} (-1)^{l-1} (-1)^{m-1} (-1)^{n-1}}{jklmn}
\]

\[
\int_{(0,1)^4} \frac{p_1 q_1 - p_2 q_2 + p_3 q_3}{x+1} d\mu = \frac{\pi^2 \zeta(3)}{48} + \frac{3 \zeta(5)}{16} + \frac{1}{8} \zeta(3) \log^2(2) + \frac{\log^5(2)}{120} + \frac{1}{72} \pi^2 \log^3(2) + \frac{1}{160} \pi^4 \log(2)
\]

Where \( d\mu = dx dy dz dw \), \( p_1 = \frac{x}{(x+1)(1-x)} \), \( p_2 = \frac{1}{(y+1)(z+1)} \), \( p_3 = \frac{y^2}{(1-xy)(1-xyw)} \), \( q_1 = f(w) + zf(-zw) \), \( q_2 = f(w) + xzf(-xyzw) \), \( q_3 = f(w) - yzg(yzw) \), \( f(t) = \frac{\log(2) - \log(1-t)}{1+t} \).

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