Interfacial instability of ferrofluid flow under the influence of a vacuum magnetic field

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Abstract This study is to numerically test the interfacial instability of ferrofluid flow under the presence of a vacuum magnetic field. The ferrofluid parabolized stability equations (PSEs) are derived from the ferrofluid stability equations and the Rosensweig equations, and the characteristic values of the ferrofluid PSEs are given to describe the ellipticity of ferrofluid flow. Three numerical models representing specific cases considering with/without a vacuum magnetic field or viscosity are created to mathematically examine the interfacial instability by the computation of characteristic values. Numerical investigation shows strong dependence of the basic characteristic of ferrofluid Rayleigh-Taylor instability (RTI) on viscosity of ferrofluid and independence of the vacuum magnetic field. For the shock wave striking helium bubble, the magnetic field is not able to trigger the symmetry breaking of bubble but change the speed of the bubble movement. In the process of droplet formation from a submerged orifice, the collision between the droplet and the liquid surface causes symmetry breaking. Both the viscosity and the magnetic field exacerbate symmetry breaking. The computational results agree with the published experimental results.

Key words interfacial instability, ferrofluid, vacuum magnetic field, parabolized stability equation (PSE)

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1 Introduction

Ferrofluids (or magnetic fluids) are stable colloidal suspensions of magnetic nanoparticles when an external magnetic field is presented. A growing interest of the research on magnetic fluids is aroused due to its high economic potential. The magnetic materials are usually made from iron or cobalt particles, as well as compounds such as manganese zinc ferrite. The most common form of ferrofluids is made using particles of a type of iron oxide termed as magnetite

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Making a stable ferrofluid is not the simple mixture of micro-particles and liquid. The average size of micro-particles is around 5 nm–15 nm. Crushing or grinding raw material is not able to produce micro-particles. The magnetization field generated from the flow of ferrofluid plays a primary role to link the magnetic manifestation and hydrodynamics.

It is crucial to mathematically depict magnetics-related hydrodynamics. The parabolized stability equations (PSEs), which are first developed by Herbert\cite{1} based on the parabolized Navier-Stokes (PNS) equations\cite{2}, can be used to solve incompressible fluid problems. The non-linearized PSE approach has been widely used to study the stability and transition. This approach was initially applied to the incompressible Blasius boundary layer proposed by Bertolotti and Herbert, and the transition onset location agrees well with DNS results\cite{3}. The characteristics analysis of the PSEs has been investigated in some studies\cite{4–5}, in the sources of ellipticity are identified and suggestions are made for their suppression.

Interfacial instabilities are developed at the interface between two layers of fluid due to the unstable growth of interfacial perturbations\cite{6}. It was reported that instabilities were associated with surface tension (Rayleigh-Taylor instability (RTI), Richtmyer-Meshkov instability, and liquid bridge instability), magnetics (Rosensweig instability), as well as electrical properties (electro hydrodynamic instability). The RTI is an instability of an interface between light fluid and heavy fluid. There is asymmetry for its configuration due to gravity-induced interfacial destabilization. A stabilizing effect on the configuration would be presented if either viscosity tension or magnetic tension was added on two superposed fluids. The RTI of the interface between two superposed fluids has been investigated by many researches. Cowley and Rosensweig\cite{7} investigated the linear stability of two superposed magnetic fluids in the presence of an externally applied magnetic field. Zelazo and Melcher\cite{8} performed theoretical and experimental studies on the linear stability of an ideal magnetic fluid under the influence of a tangential magnetic field, which can stabilize the interface between two magnetic fluids. Ferrofluid has prominent character of the fluidity and also possesses ferromagnetic particles, that is to say, ferrofluid can be manipulated to position or forced to flow by means of magnetic force\cite{9}. Xu et al.\cite{10} studied the boundary layer flow over a smooth forward-facing stepped plate with particularly emphasis on stabilization and destabilization of the two-dimensional TS-waves. When magnetic stabilizing effect governs, the RTI only occurs in the long wave region and not only the permitted band. The linear analysis of the radiative RTI has been investigated by Yaghoobi and Shadmehri\cite{11} in the presence of magnetic field for both optically thin and thick regimes. Magnetic fields were not taken into account in most numerical simulations of massive star formation for simplified computation, whereas Yaghoobi and Shadmehri concluded a stabilizing effect of magnetic field on the radiative RTI\cite{11}. Recently, Carvalho and Gontijo\cite{12} found a non-uniform magnetoviscous effect. Further, they investigated spatial variations in the misalignment between the magnetization and the applied fields, and performed correlation to vorticity profiles.

To some extents, numerical simulations can mimic the flow field and interface formation in more detail. Numerical approaches, such as the volume of fluid, level set, and diffuse interface methods\cite{13–17}, have been developed to accurately capture the evolution of the two-phase interface. For instance, Balla et al.\cite{13} investigated the relative influence of surface tension force over the inertial force and the orientation of a droplet. Sun and Tao\cite{14} proposed a coupled volume-of-fluid and level set (VOSET) method for computing incompressible two-phase flows without heat transfer. Tripathi et al.\cite{15} reported an initially spherical bubble rising in liquid and provided a broad canvas of behavior patterns. Zhang et al.\cite{17} employed an improved diffuse interface method to reveal the dynamic mechanisms of bubbles and droplets moving in quiescent flows.

The goal of this study is to shed light on the mathematical features of the ferrofluid PSEs and further reveal the influence of interfacial instability of ferrofluid with involvement of a vacuum magnetic field.
2 Governing equations of ferrofluid under influence of vacuum magnetic field

Consider a laminar flow over a flat plate along the x-direction when the Reynolds number is high enough. Let the basic state be \( u_0 = (U(x, y, t), 0) \), \( \rho_0 = \rho(x, y, t) \) for velocity and density. The applied magnetic field \( \mathbf{M}_s \) of ferrofluid is given in terms of the particle dipole moment \( m \) as defined by

\[
\mathbf{M}_s = N m \mathbf{e}_s = (M_{sx}, M_{sy}, M_{sz}),
\]

where \( \mathbf{e}_s \) is the unit vector of the applied magnetic field \( \mathbf{M}_s \), and \( N \) is the number of particles per unit volume. The influence of the magnetic field on the heat transfer has been examined by several researchers\([18]\), and the thermal effect is ignored in this paper. Let \( (u, v, \rho) \) be small disturbances about velocity and density, respectively. By employing the fluid PSEs theory (see Eqs. (3a)–(3d) of Ref. [5]), then the ferrofluid stability equations can be expressed as follows:

\[
(U + u) \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + (\rho + \rho) \frac{\partial u}{\partial x} + (\rho + \rho) \frac{\partial v}{\partial y} = F_1,
\]

\[
(U + u) \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + 4 \mu \frac{\partial^2 u}{\partial x^2} + \frac{2 \mu}{3(\rho + \rho)Re} \frac{\partial^2 u}{\partial x \partial y} + \frac{2 \mu}{3(\rho + \rho)Re} \frac{\partial^2 v}{\partial x \partial y} = F_2,
\]

\[
(U + u) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + (\rho + \rho) \frac{\partial v}{\partial x} + (\rho + \rho) \frac{\partial v}{\partial y} + \frac{2 \mu}{3(\rho + \rho)Re} \frac{\partial^2 v}{\partial y^2} + \frac{2 \mu}{3(\rho + \rho)Re} \frac{\partial^2 u}{\partial y^2} = F_3,
\]

where \( U, u, \) and \( v \) are united by characteristic velocity \( U_c \), and \( x \) and \( y \) are united by characteristic length \( L \); \( \rho \) and \( \mu \) are united by characteristic density \( \rho_c \) and characteristic viscosity \( \mu_c \), respectively. Reynolds number \( Re = \rho_c U_c L / \mu_c \). \( F_1, F_2, \) and \( F_3 \) denote all of the other terms except for all of the terms of \( \frac{\partial \rho}{\partial y} \) and \( \frac{\partial \rho}{\partial x} \) in the equations. In this study, \( F_1, F_2, \) and \( F_3 \) denote the same meaning terms (see Refs. [4]–[5]). We also suppose the basic flow and the saturation magnetization satisfy the basic equations of magnetic fluid mechanics.

Let \( \mathbf{M} = \mathbf{M}_f + \mathbf{M}_s \). According to Maxwell’s equations theory, the applied magnetic field strength and magnetization field satisfy the following relations\([16]\):

\[
\begin{align*}
\frac{d\mathbf{M}}{dt} & = \mathbf{\Omega} \times \mathbf{M} - \frac{1}{\tau_B} (\mathbf{M} - \mathbf{M}_s) - \frac{1}{6\eta\Phi} \mathbf{M} \times (\mathbf{M} \times \mathbf{M}_s), \\
\nabla \cdot \mathbf{M} & = 0, \quad \nabla \times \mathbf{M}_s = 0.
\end{align*}
\]

In this paper we assumed that the initial portion of the magnetization versus field curve is linear. Thus, for small applied fields, a vacuum magnetic field is achieved almost instantaneously within a time much shorter than the characteristic time scale of a macroscopic process. Then, the magnetization field \( \mathbf{M} \) becomes paralleled to the vector of the applied magnetic field \( \mathbf{M}_s \) at a given instant \( \chi \) (for a ferromagnetic solid \( \chi = ||\mathbf{M}|/||\mathbf{M}_s|| \) is the magnetic susceptibility), i.e., \( \mathbf{M}/\mathbf{M}_s \). That is to say, there exists magnetic susceptibility \( \chi \) (see Eq. (5.70) in Ref. [18]) such that

\[
\mathbf{M} = \frac{1}{H} \int_0^H \mathbf{M} dH = \frac{1}{2} \chi \mathbf{M}_s = \frac{1}{2} \mathbf{M}.
\]
Let \( \|(u, v)\| \ll |U| \). Suppose magnetization field \( \mathbf{M} = (M_x, M_y, M_z) \) can be regarded as small disturbances about the applied magnetic field, that is to say \( \|\mathbf{M}\| \ll \|\mathbf{M}_a\| \), then the magnetization field equation (4) can be expressed as

\[
U \frac{\partial M_x}{\partial x} = g_1, \quad (6)
\]
\[
U \frac{\partial M_y}{\partial x} = g_2, \quad (7)
\]

where \( g_1 \) and \( g_2 \) denote the terms that cannot be added to \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \). Equations (6)–(7) are called the magnetization field stability equations.

Removing all of the viscous terms of the ferrofluid stability equations (1)–(3) with respect to the \( x \)-direction, the non-linearized ferrofluid PSEs are obtained as follows:

\[
(U + u) \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + (\bar{\rho} + \rho) \frac{\partial u}{\partial x} + (\bar{\rho} + \rho) \frac{\partial v}{\partial y} = F_1, \quad (8)
\]
\[
(U + u) \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\mu}{(\bar{\rho} + \rho) \text{Re}} \frac{\partial^2 u}{\partial y^2} + M_{sz} \frac{\partial M_x}{\partial x} = F_2, \quad (9)
\]
\[
(U + u) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{4\mu}{3(\bar{\rho} + \rho) \text{Re}} \frac{\partial^2 v}{\partial y^2} + M_{sz} \frac{\partial M_y}{\partial x} = F_3. \quad (10)
\]

Equations (11)–(13) are called the linearized ferrofluid PSEs.

In the next section, we will examine the characteristic and sub-characteristic of the ferrofluid PSEs under the influence of a vacuum magnetic field, and analyze the partial residual elliptical characteristic for the linearized ferrofluid PSEs with viscosity or magnetization field.

Further characteristic and sub-characteristic of the ferrofluid PSEs under the influence of a vacuum magnetic field will be solved numerically for some interface problems, and the partial residual elliptical characteristic for the linearized ferrofluid PSEs with viscosity or magnetization field would be carried out numerically.

3 Characteristic of ferrofluid PSEs

Interfacial instabilities are modeled using the classical linear stability analysis considering the momentum transport equations and the magnetization intensity equations. The models can test the stability of the interfacial perturbations to infinitesimal disturbances. By applying the theoretical approaches reported by published studies\cite{4}, two types of the transmission fashion of information in terms of the basic equations of fluid mechanics are investigated.
In terms of the characteristic theory for grade structure equations of ferrofluid flow under the influence of a vacuum magnetic field, let
\[ \frac{\partial u}{\partial y} = U(y), \quad \frac{\partial v}{\partial y} = V(y). \tag{14} \]

In order to analyze the characteristics of the ferrofluid PSEs using eigenvalue theory, the linearized ferrofluid PSEs (9)–(11) and the magnetization field stability equations (4)–(5) can be expressed as the following first-order quasi-linear partial differential equation:
\[ A \frac{\partial Z}{\partial x} + B \frac{\partial Z}{\partial y} = F, \tag{15} \]

where \( Z = (\rho, u, U(y), v, V(y), M_x, M_y) \). \( A \) and \( B \) are \( 7 \times 7 \) matrixes, \( Z \) and \( F \) are seventh-order vectors. Then, the characteristic equation of the linearized ferrofluid PSEs and the magnetization field stability equations is
\[ \det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = 0, \tag{16} \]

where
\[ \det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = \begin{vmatrix} U \sigma_1 & \rho \sigma_1 & 0 & \rho \sigma_2 & 0 & 0 & 0 \\ 0 & U \sigma_1 & -\frac{\mu}{(\rho + \overline{\rho})Re} \sigma_2 & 0 & 0 & M_x \sigma_1 & 0 \\ 0 & 0 & 0 & U \sigma_1 & -\frac{4 \mu}{3(\rho + \overline{\rho})Re} \sigma_2 & 0 & M_x \sigma_1 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U \sigma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U \sigma_1 \end{vmatrix} = \frac{4 U^3 \mu^2}{3(\rho + \overline{\rho})^2 Re^2 \sigma_1^3 \sigma_2^4}, \tag{17} \]

and then the characteristic values satisfy
\[ \sigma_1^3 = 0, \quad \sigma_2^4 = 0. \tag{18} \]

All the characteristic values are zero. Therefore, the linearized ferrofluid PSEs are parabolized. Furthermore, we find that the characteristics of the linearized ferrofluid PSEs are independent of the vacuum magnetic field.

Let us now turn to consider the relation of the sub-characteristic and Mach number of the linearized ferrofluid PSEs (5)–(8). Removing all the viscous terms of the linearized ferrofluid PSEs, then the governing equations become the following sub-characteristic ferrofluid PSEs,
\[ U \frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = F_1, \quad (19) \]
\[ U \frac{\partial u}{\partial x} + M_x \frac{\partial M_x}{\partial x} = F_2, \quad (20) \]
\[ U \frac{\partial v}{\partial x} + M_x \frac{\partial M_y}{\partial x} = F_3. \quad (21) \]

Then, the sub-characteristics equation of the equations (17)–(19) with the equations (4)–(5) is
\[ \det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = U^5 \sigma_1^5 = 0, \tag{22} \]
the sub-characteristic values satisfy
\[ \sigma_1^5 = 0. \] (23)

All the characteristic values are zero, and therefore the sub-characteristic linearized ferrofluid PSEs are parabolized.

Similarly, one can study the elliptical characteristic from the characteristic and sub-characteristic of the non-linearized ferrofluid PSEs. Let fluid flow \( u = (U + u, v) \) and magnetic field \( |M| \ll |M_x| \), then the magnetization field equations can be expressed as
\[
(U + u) \frac{\partial M_x}{\partial x} + v \frac{\partial M_x}{\partial y} = g_1, \tag{24}
\]
\[
(U + u) \frac{\partial M_y}{\partial x} + v \frac{\partial M_y}{\partial y} = g_2, \tag{25}
\]
where \( g_1 \) and \( g_2 \) denote the terms that cannot be added to \( \frac{\partial M_x}{\partial x} \) and \( \frac{\partial M_y}{\partial y} \). Then the characteristics equation of the non-linearized ferrofluid PSEs (8)–(10) with the non-linearized magnetization field equations (24)–(25) is
\[
\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = 0, \tag{26}
\]
where
\[
\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = \begin{vmatrix}
\sigma_{1,2} & (\rho + \rho)\sigma_1 & 0 & (\rho + \rho)\sigma_2 & 0 & 0 & 0 \\
0 & \sigma_{1,2} & -\frac{\mu}{(\rho + \rho)Re} \sigma_2 & 0 & 0 & M_{x2} \sigma_1 & 0 \\
0 & 0 & 0 & \sigma_{1,2} & -\frac{4\mu}{3(\rho + \rho)Re} \sigma_2 & 0 & M_{x2} \sigma_1 \\
0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_{1,2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{1,2}
\end{vmatrix}
= \frac{4\mu^2}{3(\rho + \rho)^2 Re^2}((U + u)\sigma_1 + v\sigma_2)^3 \sigma_2^4. \tag{27}
\]

Let \( \lambda = \sigma_2 / \sigma_1 \), then the sub-characteristic values satisfy
\[
\lambda_{1,2,3} = -\frac{U + u}{v}, \quad \sigma_2^4 = 0. \tag{28}
\]

\( M = (U + u)/v \) is called the local Mach number. All the characteristic values are real, and therefore the non-linearized ferrofluid PSEs (8)–(10) with the non-linearized magnetization field equations (24)–(25) are hyperbolic-parabolic.

4 Modified real-ghost fluid method to interfacial instability in multi-material flows

The mathematical link between the interface instability and fluid viscosity, as well as the correlation between the interface instability and magnetic field, is represented as ferrofluid RTI problem and solved by using the fifth-order CWENO-NZ scheme\[^{10}\]. A real-ghost fluid method is proposed for solving interfacial instability of shock wave striking helium bubble and droplet formation from a submerged orifice.
In 1980, Glimm et al.\[20\] first proposed the ghost fluid method (GFM) to mathematically depict multi-medium flows with immiscible material interface. This method can solve the problem of multi-medium fluid by defining virtual fluid and using the single medium fluid numerical scheme. By predicting the flow states for the real fluid nodes adjacent to the interface and the ghost fluid nodes using the Riemann problem solver, Wang et al.\[21\] developed a real-GFM. The method added a more accurate interface boundary condition. In the previous GFMs, the focus was on the definition of ghost fluid states while the pressure and velocity in the real fluid sides were taken at the real fluid nodes next to the interface to overcome the possible problems related to overheating.

![Fig. 1 Construction method of real-GFM in two-dimensional case (color online)](image)

In two-dimensional case, by assuming a limited diversity, real-GFM uses the Riemann solution to correct the points in the narrow band with the width of $2 \max(\Delta x, \Delta y)$ near the interface. For any point $A_+$ in the narrow band, in order to construct one-dimensional Riemann problem at point $A_+$, it is necessary to find a node $A_-$ on the other side of the interface to meet the minimum angle between the normal at node $A_+$ and node $A_-$. In order to better capture the interface, we present a modified real-ghost fluid method to interfacial instability in multi-material flows,

$$A_{\pm} = A_0 \pm (2 \max(\Delta x, \Delta y)) \mp \varphi(A_0) n,$$

$$n = \frac{\nabla \varphi}{|\nabla \varphi|},$$

where $n$ is the unit normal vector of the interface. The physical quantities of $A_+$ and $A_-$ are taken as the left and right initial states of the one-dimensional Riemann problem. By using the HLLC Riemann solver, the values of $u^*, p^*, \rho^*_l$, and $\rho^*_r$ at the interface are obtained.

$$I_t \pm n \cdot \nabla I = 0,$$

where $I$ is the physical quantity to be extrapolated, such as entropy, normal phase velocity, and tangential velocity. The formula takes “$+$” when pushing from area $\varphi < 0$ to area $\varphi > 0$ and “$-$” when pushing from area $\varphi > 0$ to area $\varphi < 0$. The extrapolation is employed to solve for the stable solution of the continuation equation by the first-order upwind difference scheme.

5 Numerical simulations

5.1 Ferrofluid RTI problem

The initial conditions of ferrofluid RTI problem are

$$(\rho, u, v, p) = \begin{cases} 
(2.0, -0.025\sqrt{\gamma p/\rho \cos(8\pi x)}, 2y + 1), & 0 \leq y < 0.5, \\
(1.0, -0.025\sqrt{\gamma p/\rho \cos(8\pi x)}, y + 1.5), & 0.5 \leq y < 1.
\end{cases}$$
As is well known, this problem describes the interface instability between fluids with different densities, when an acceleration is directed from the heavy fluid to the light one. The fact that the mushroom is principally grown in the heavy fluid can be considered from the perspective of stability theory. Usually, the computational domain is chosen as $[0, 0.25] \times [0, 1]$. The left and right boundaries of this case are defined as boundary conditions. The upper and bottom boundaries are $(\rho, u, v, p) = (1, 0, 0, 2.5)$ and $(\rho, u, v, p) = (2, 0, 0, 1)$, respectively. In order to test the effect of viscosity and applied magnetic field in the fluid flow, the vacuum magnetic field is given by $M_{sy} = \frac{1}{\sqrt{2\pi}} (4 \times 10^3 + 10^3y)$. Let the vacuum permeability $\chi = 0.1\rho$. The uniform mesh is created consisting of $240 \times 960$ elements. The computational time is $t = 1.95$.

Figure 2 displays the density contours of the ferrofluid for RTI, and the inviscid ferrofluid equations, the ferrofluid PSEs, and the inviscid Euler equations are solved, respectively. The mushroom unstable structures indicate that both viscosity and applied magnetic field can maintain symmetry of the flow structure well. Furthermore, it is found that the basic characteristic of the ferrofluid is independent of a vacuum magnetic field. The fluid viscosity is able to smooth out the interface shape of mushroom. That is to say, the viscosity is an important factor for the variation of ellipticity in ferrofluid RTI problem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Density contours of ferrofluid for RTI. (a) Inviscid ferrofluid equations without magnetic field, (b) ferrofluid PSEs with viscosity, and (c) inviscid ferrofluid equations under influence of vacuum magnetic field.}
\end{figure}

5.2 Shock wave striking helium bubble

Initial conditions\cite{24} are considered as

$$
\begin{cases}
(\rho, u, v, p) = \\
(1.3764, 0, 0.394, 1.5698), & y \geq 225.0, \\
(0.138, 0, 0, 1), & (y - 175)^2 + x^2 \leq 625.0, \\
(1, 0, 0, 1), & (y - 175)^2 + x^2 > 625.0, & y < 225.0.
\end{cases}
$$

(33)

The solution area is $[-44.5, 44.5] \times [0, 325]$. The uniform mesh is created with $80 \times 320$ elements. The computational time is $t = 140$. A tight support boundary condition is applied on the low boundary. An inflow boundary condition is used for the upper boundary. The reflection boundary conditions employ left boundary and right boundary (the value of Courant-Friedrichs-Lewy (CFL) conditions is 0.4, adiabatic coefficient $\gamma = 1.4$). The vacuum permeability is taken as $\chi = 0.1\rho$. In order to test the effect of a vacuum magnetic field on bubbles in the fluid flow, the vacuum magnetic fields are given by

Case 1: $M_{sy} = \frac{1}{\sqrt{2\pi}} (2 \times 10^3 + 3x)$,
Case 2: \[ M_{ss} = \frac{1}{\sqrt{2\pi}}(2 \times 10^3 - 3x). \]

The fifth-order centrally weighted essentially non-oscillatory (CWENO-NZ) scheme and level set method are used for numerical simulations. By predicting the flow states for the real fluid nodes adjacent to the interface, a modified real-ghost fluid method is used. The effects of a magnetic field and viscosity are examined as shown in Fig. 3, where the influence of a vacuum magnetic field is not present in the symmetry breaking of bubble. The speed of the bubble movement is faster under the action of a magnetic field than that in the fluid when the moving direction is the same as the external field. On the contrary, an external magnetic field with a reverse direction to the bubble motion would retard the bubble movement. Because the viscosity of gas is very small, the bubble motion at low speed is often regarded as inviscid fluid.

![Fig. 3 Density contours of shock wave striking helium bubble in magnetic field. (a) Without magnetic field, (b) Case 1, and (c) Case 2](image)

### 5.3 Dynamic mechanisms of droplet formation from submerged orifice

In general, the process of droplet formation from a submerged orifice can be divided into three stages: free fall, contact, and splash. In momentum equation, the surface tension is expressed as \( \sigma \kappa \nabla h \), where \( h \) is a smoothed Heaviside function\cite{16}, which is used to calculate the physical properties and surface tension in the momentum equation. The interface curvature is calculated by level set function

\[
h(\phi) = \begin{cases} 
0, & \phi < -\varepsilon, \\
\frac{1}{2} \left( 1 + \frac{\phi}{\varepsilon} + \frac{1}{\pi} \sin \left( \frac{\pi \phi}{\varepsilon} \right) \right), & |\phi| \leq \varepsilon, \\
1, & \phi > \varepsilon,
\end{cases}
\]

where \( \rho = h \rho_l + (1 - h) \rho_g \), \( \eta = h \eta_l + (1 - h) \eta_g \), and \( \kappa = \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \).

The solution area is \([0, 4] \times [0, 3]\) where liquid area is \([0, 4] \times [0, 0.5]\), and the diameter of the droplet is 0.6, which are shown in Fig. 4. The uniform mesh is created with 80 \times 320 elements. Before meshing the computational domain, the interface shapes and locations should be first confirmed by reconstructing the interfaces\cite{16}. For unsteady two-phase flow, firstly, the level set method is adopted to capture the phase interface. On the other hand, three cases of
the momentum equations were solved by using the CWENO scheme\cite{19}. The computational parameters of the droplet formation model are presented in Table 1.

| Parameter                        | Symbol | Value  |
|----------------------------------|--------|--------|
| Liquid density                   | $\rho_l$ | 10000  |
| Gas phase density                | $\rho_g$ | 10     |
| Acceleration of gravity          | $g$    | 9.8    |
| Viscosity of liquid phase        | $\nu_l$ | 1000   |
| Viscosity of gas phase           | $\nu_g$ | 3.1289 |
| Surface tension coefficient      | $\sigma$ | 2.0    |
| Termination time of calculation  | $T$    | 0.25   |

In Figs. 5 and 6 of Ref.\cite{23}, Li et al. measured velocity field in a horizontal plane at $Z/R = 1.4$ and carried out an experimental study for testing the influence of viscoelasticity on symmetry breaking of swirling cylinder flow with free surface driven by the constant rotation of bottom wall.

The collision between the droplet and the liquid surface induces symmetry breaking as clearly shown in Fig. 5. Along the contact edge of droplet and flat, some detached droplets of different sizes appeared, and the “symmetry breaking” liquid column is exacerbated by the comprehensive effect of viscosity and magnetic field. This finding is supported by the published experimental studies with focus on droplet formation in Figs. 5 and 6 of Ref.\cite{23}.

6 Conclusions and discussion

To summarize, we report on the ferrofluid PSEs which have been represented by ferrofluid stability equations and the Rosensweig equations under the influence of a vacuum magnetic field. The characteristic values of the two-dimensional compressible ferrofluid PSEs are solved to analyze elliptical characteristics of ferrofluid PSEs. The elliptical characteristic of the non-linearized ferrofluid PSEs is in agreement with that of the non-linearized fluid PSEs already in Refs.\cite{20}–\cite{21}. Effect of magnetization and viscosity on three mathematical model is excluded in this study. The following conclusions are drawn from this study.

(i) From the ferrofluid stability equations and the Rosensweig equations, we build ferrofluids PSEs with consideration of the influence of a vacuum magnetic field. As two simplified cases of PSEs, the non-linearized ferrofluid PSEs and the linearized ferrofluid PSEs are proposed. The sub-characteristic linearized ferrofluid PSEs are parabolized, and the non-linearized ferrofluid PSEs with the non-linearized magnetization field equations are hyperbolic-parabolic.
Fig. 5  Density contours of droplet formation model. (a) Inviscid ferrofluid equations, (b) ferrofluid PSEs with viscous, (c) inviscid ferrofluid equations with magnetic field, and (d) ferrofluid PSEs with magnetic field and viscosity

(ii) Three typical numerical examples are given in this paper, that is, the ferrofluid RTI problem, shock wave striking helium bubble, and the droplet formation from a submerged orifice. For the first two typical numerical examples, the basic characteristic of the ferrofluid depends on fluid viscosity rather than magnetization. The fluid viscosity can smooth out the interface shape of mushroom and retard a bubble movement if no magnetic field is present.

(iii) For droplet formation from a submerged orifice, both viscosity and magnetic field are able to exacerbate symmetry breaking of interface. Further analysis showed that the symmetry breaking caused by magnetic stress is less than that caused by viscosity.

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