We study random walks with stochastic resetting to the initial position on arbitrary networks. We obtain the stationary probability distribution as well as the mean and global first passage times, which allow us to characterize the effect of resetting on the capacity of a random walker to reach a particular target or to explore a finite network. We apply the results to rings, Cayley trees, random and complex networks. Our formalism holds for undirected networks and can be implemented from the spectral properties of the random walk without resetting, providing a tool to analyze the search efficiency in different structures with the small-world property or communities. In this way, we extend the study of resetting processes to the domain of networks.

When a stochastic process is occasionally reset, i.e., interrupted and restarted from the initial state, its occupation probability in the configuration space is strongly altered. Interestingly, the mean time needed to reach a given target state for the first time can often be minimized with respect to the resetting rate 1 3. Random search strategies can thus be improved by resetting, a fact that finds applications in statistical physics 4,5, computer science 6, enzymatic reactions 7, or foraging ecology 8,9,10. In recent years, different types of resetting protocols have been considered 11–14 on a variety of underlying processes, such as Brownian motion 1,2,15, processes with a drift 16,17, or models of anomalous diffusion 18–21. All these problems are more conveniently studied in simple search spaces, mainly, the semi-infinite line or 22, bounded 1D and 2D domains 23,24, or infinite regular lattices 8,26,27.

Nevertheless, random walks and diffusive processes on more complex structures such as networks 28–30 are relevant to a broad range of phenomena and applications 31,32 such as synchronization 33, epidemic spreading 34, and ranking and searching on the web 35–37, among others 38,39. Whereas lattice random walks have been explored for decades 40–42, the study of local random walks on complex networks is more recent and was introduced by Noh and Rieger 29. Network exploration by random walks is now better understood 29,33,43, including non-local strategies with long-range hops between distant nodes 44,50.

Despite their importance, random walks subject to resetting on arbitrary networks are little understood. Figure 1 illustrates a dynamics defined by the transition probabilities between adjacent nodes and a resetting probability 3. Three important properties of many complex and real-world networks are their finiteness, the small-world effect 51 and the presence of communities, i.e., subsets of nodes more densely connected to each other than to the other nodes 52. Both the network architecture and the choice of the resetting node should impact the mean first passage time (MFPT) to a target node, and more generally, the capacity of the walker to explore the whole network.

In this Letter, we develop an extension to arbitrary network topology of the diffusion problem with stochastic resetting of 1,18. We deduce general exact expressions for the stationary probability distribution and first passage times. The analytical results depend on the process without resetting and are written in terms of eigenvalues and eigenvectors of the transition matrix that generates the random walk. The methods introduced here can be used to study the effects of resetting on networks. We apply our findings to lattices, trees, random networks and several complex networks.

We study undirected connected networks with N nodes i = 1,…,N, described by an adjacency matrix A with elements Ajj = 1 if there is a link between the nodes i and j, and Aij = Aji = 0 otherwise. We set Aii = 0 to avoid self-loops. The degree of the node i is denoted as ki = ∑N l=1 Aii. On this structure, we con-
sider a random walker in discrete time and starting at \( t = 0 \) at \( i \). The walker performs at \( t = 1, 2, \ldots \) two types of steps: 1) a jump to one of the neighbors of the node currently occupied (all neighbors being equiprobable), and 2) a resetting to a fixed node \( r \). Actions 1) and 2) occur with probability \( 1 - \gamma \) and \( \gamma \), respectively. Without resetting (\( \gamma = 0 \)), the probability to hop from \( l \) to \( m \) is \( w_{l \to m} = A_{lm}/k_l \). This random walk is described by the transition matrix \( \mathbf{W} \) with elements \( w_{l \to m} \) for \( l, m = 1, \ldots, N \). With resetting, the occupation probability follows the master equation

\[
P_{ij}(t+1; r, \gamma) = (1 - \gamma) \sum_{l=1}^{N} P_{il}(t; r, \gamma) w_{l \to j} + \gamma \delta_{rj} , \quad (1)
\]

where \( P_{ij}(t; r, \gamma) \) is the probability to find the walker at \( j \) at time \( t \), having started from \( i \) at \( t = 0 \) (\( \delta_{rj} \) denotes the Kronecker delta). The first term on the right-hand side of Eq. (1) represents hops between adjacent nodes whereas the second term describes resetting to \( r \). Let us define the transition probability matrix \( \Pi(r, \gamma) \) with elements \( \pi_{l \to m}(r, \gamma) = (1 - \gamma) w_{l \to m} + \gamma \delta_{rm} \). Eq. (1) takes the simpler form

\[
P_{ij}(t+1; r, \gamma) = \sum_{l=1}^{N} P_{il}(t; r, \gamma) \pi_{l \to j}(r, \gamma) , \quad (2)
\]

where \( \pi_{l \to m}(r, \gamma) = 1 \). The matrix \( \Pi(r, \gamma) \) completely entails the dynamics with resetting. As we are considering connected undirected networks, the process defined by Eq. (2) is able to reach all the nodes of the network if the resetting probability \( \gamma \) is < 1. Like \( \mathbf{W} \), \( \Pi(r, \gamma) \) is a stochastic matrix: knowing its eigenvalues and eigenvectors allows calculating the stationary distribution and the mean first passage time to any node. We first analyze how the eigenvalues and eigenvectors of \( \Pi(r, \gamma) \) are related to the spectral form of \( \mathbf{W} \), which is recovered in the limit \( \gamma = 0 \) and can be readily computed numerically or analytically in some cases. We denote the eigenvalues of the matrix \( \mathbf{W} \), which is not symmetric in general, by \( \lambda_l \) (where \( \lambda_1 = 1 \)), and its right and left eigenvectors as \( |\phi_l\rangle \) and \( \langle \tilde{\phi}_l(\gamma) \rangle |\bar{\phi}_l(\gamma)\rangle \). Similarly, the eigenvalues of \( \Pi(r, \gamma) \) are denoted as \( \zeta_l(\gamma) \) and its eigenvectors as \( |\psi_l(\gamma)\rangle \) and \( \langle \bar{\psi}_l(\gamma) | \). Let us analyze the connection between the eigenvalues \( \lambda_l \) and \( \zeta_l(\gamma) \). We may use the identity \( \Pi(r, \gamma) = (1 - \gamma) \mathbf{W} + \gamma \Theta(r) \), where the elements of the matrix \( \Theta(r) \) are \( \Theta_{lm}(r) = \delta_{mr} \). Namely, \( \Theta(r) \) has entries 1 in the \( r^{th} \)-column and null entries everywhere else. We obtain

\[
\zeta_l(\gamma) = \begin{cases} 1 & \text{for } l = 1, \\ (1 - \gamma)\lambda_l & \text{for } l = 2, 3, \ldots, N. \end{cases} \quad (3)
\]

This result reveals that the eigenvalues are independent of the choice of the resetting node \( r \). The left eigenvectors of \( \Pi(r, \gamma) \) are further given by

\[
|\bar{\psi}_l(\gamma) | = |\bar{\phi}_l| + \sum_{m=2}^{N} \frac{\gamma}{1 - (1 - \gamma)\lambda_m} \langle r|\phi_m \rangle \langle \tilde{\phi}_m | \quad (4)
\]

whereas \( \langle \bar{\psi}_l(\gamma) | \) \( |\phi_l\rangle \) for \( l = 2, \ldots, N \). Similarly, the right eigenvectors are given by: \( |\psi_l(\gamma)\rangle \) \( |\phi_l\rangle \) and

\[
|\psi_l(\gamma)\rangle = |\phi_l\rangle - \gamma \frac{\langle r|\phi_l \rangle}{1 - (1 - \gamma)\lambda_l} \langle \bar{\phi}_l | \quad (5)
\]

for \( l = 2, \ldots, N \). With the left and right eigenvectors at hand, one can use the spectral representation \( \Pi(r, \gamma) = \sum_{l=1}^{N} \zeta_l(\gamma) |\psi_l(\gamma)\rangle \langle \bar{\psi}_l(\gamma) | \). On the other hand, the occupation probability of the process described by Eq. (2) is given by \( P_{ij}(t; r, \gamma) = \langle i|\Pi(r, \gamma)| j \rangle \). We deduce

\[
P_{ij}(t; r, \gamma) = \langle i|\psi_l(\gamma)\rangle \langle \bar{\psi}_l(\gamma) | j \rangle + \sum_{l=2}^{N} \langle i| \lambda_l \rangle \langle i|\psi_l(\gamma)\rangle \langle \bar{\psi}_l(\gamma) | j \rangle. \quad (6)
\]

The first term in Eq. (6) defines the long-time, stationary distribution \( P_j^{\infty}(r, \gamma) = \langle i|\psi_l(\gamma)\rangle \langle \bar{\psi}_l(\gamma) | j \rangle \). By using Eq. (3) and \( |\psi_l(\gamma)\rangle = |\phi_l\rangle \), we obtain

\[
P_j^{\infty}(r, \gamma) = \frac{k_j}{\sum_{m=1}^{N} k_m} + \gamma \sum_{l=2}^{N} \langle i|\phi_l \rangle \langle \bar{\phi}_l | j \rangle \frac{\langle r|\phi_l \rangle}{1 - (1 - \gamma)\lambda_l}. \quad (7)
\]

where we have used the identity \( \langle i|\phi_l \rangle \langle \bar{\phi}_l | j \rangle = \frac{k_j}{\sum_{m=1}^{N} k_m} \) for the equilibrium distribution of the usual random walk on networks. The second term of \( P_j^{\infty}(r, \gamma) \) in Eq. (7) is a non-equilibrium part, due to resetting. Similarly,

\[
P_{ij}(t; r, \gamma) = P_j^{\infty}(r, \gamma) + \sum_{l=2}^{N} \langle i| \lambda_l \rangle \langle i|\phi_l \rangle \langle \bar{\phi}_l | j \rangle \frac{\langle r|\phi_l \rangle}{1 - (1 - \gamma)\lambda_l}. \quad (8)
\]

The expression for the MFPT to the target \( j \) can be deduced from the general convolution property with \( P_{ij}(t; r, \gamma) \) and is given by \( T_{ij}(t; r, \gamma) = 1 \). Similarly, the MFPT \( T_{ij}(t; r, \gamma) \) is given by \( T_{ij}(t; r, \gamma) = \sum_{l=0}^{t} P_{ij}(t; r, \gamma) - P_j^{\infty}(r, \gamma) \). Using Eq. (8), one obtains in the case of resetting to the origin \( (i.e., r = i) \)

\[
T_{ij}(\gamma) = \frac{\delta_{ij}}{P_j^{\infty}(i, \gamma)} + \frac{1}{P_j^{\infty}(i, \gamma)} \sum_{l=2}^{N} \langle j|\phi_l \rangle \langle \bar{\phi}_l | j \rangle \frac{\langle r|\phi_l \rangle}{1 - (1 - \gamma)\lambda_l}. \quad (9)
\]

It is also useful to quantify the ability of a process to explore the whole network. To this purpose, we define
the global MFPT starting from node $i$, $T(i,γ)$, as

$$T(i,γ) = \frac{1}{N} \sum_{j=1}^{N} (T_{ij}(γ)).$$

(10)

The results in Eqs. (3)-(10) apply to random walks with resetting on any finite, connected and undirected network. The eigenvalues and eigenvectors of $W$ may be obtained by direct numerical calculation. We next explore the effects of resetting in different topologies.

**Rings.** We start our discussion with the analysis of the finite ring, i.e., the one-dimensional lattice with periodic boundary condition. In this case, $W$ is a circulant matrix with eigenvalues $λ_l = \cos \left( \frac{2πl(l-1)}{N} \right)$ and eigenvectors with components $\langle i | \phi_l \rangle = \frac{1}{\sqrt{N}} e^{2πi(l-1)l/N}$ and $\langle \phi_l | j \rangle = \frac{1}{\sqrt{N}} e^{2πi(l-1)j/N}$ (here $i$ is the central node. Keeping the distance $d_{ij}(=0,1,\ldots,N)$ between $i$ and the target node $j$ fixed, we see how resetting modifies the MFPT in comparison with the normal random walk ($γ = 0$). The mean first return time $T_{ii}(γ)$ is minimum, namely, that optimizes the capacity to reach a target at distance $d_{ij}$. Figure 3(b) displays a similar behavior for the global time $T(i,γ)$, see Eq. (11), in Cayley trees of varying $n$.

**Cayley trees.** We now consider finite Cayley trees of coordination number $z$ and composed of $n$ shells (see Fig. 3). The nodes of the last shell have degree 1, whereas the other nodes have degree $z$. We display the MFPT $\langle T_{ij}(γ) \rangle$ as a function of $γ$ in Fig. 3(a), where $n = 7$ and $z = 3$ ($N = 382$). The starting and resetting position $i$ is the central node. Keeping the distance $d_{ij}(=0,1,\ldots,n)$ between $i$ and the target node $j$ fixed, we see how resetting modifies the MFPT in comparison with the normal random walk ($γ = 0$). The mean first return time $T_{ii}(γ)$ (or $d_{ij} = 0$) decreases monotonically with $γ$, whereas for each positive distance there is a value $γ^*$ for which $\langle T_{ij}(γ^*) \rangle$ is minimum, namely, that optimizes the capacity to reach a target at distance $d_{ij}$. Figure 3(b) displays a similar behavior for the global time $T(i,γ)$, see Eq. (10), in Cayley trees of varying $n$.

The limit $n \to \infty$ can be solved analytically by using a general relation between the survival probabilities of discrete time processes with and without resetting [18], as well as known results on the simple random walk on infinite trees [56]. We obtain the MFPT $\langle T_{ii}(γ) \rangle$ to a target
FIG. 3. Random walks with stochastic resetting to the central node on Cayley trees with coordination number $z = 3$ and $n$ shells. (a) MFPT $\langle T_{ij}(\gamma) \rangle$ vs. $\gamma$ in a Cayley tree with $n = 7$ shells ($N = 382$ nodes). The results are presented as a family of curves defined by the distance $d_{ij}$ (shown in the color bar) between the central node where the resetting is produced and a target node $j$. (b) Global MFPT $T(i, \gamma)$ defined in Eq. (10) with $0 \leq \gamma \leq 0.99$ for different Cayley trees with $n$ shells. In each curve we include the number $n$ and circles indicate minima.

$\langle T_{d}(\gamma) \rangle = \frac{1}{\gamma} \left[ \left( \frac{2(1-\gamma)(z-1)}{z - \sqrt{z^2 - 4(1-\gamma)^2(z-1)}} \right)^d - 1 \right].$  

Setting $z = 2$ gives back Eq. (14) for the infinite ring $(i \neq j)$. Clearly, $\langle T_{d}(\gamma) \rangle \to \infty$ as $\gamma \to 0$, and $\langle T_{d}(\gamma) \rangle \approx z^d/(1-\gamma)^d \to \infty$ as $\gamma \to 1$. Thus a minimum is reached at some resetting probability $\gamma = \gamma^*_d$. For $z > 2$, solving $\partial \langle T_{d} \rangle / \partial \gamma = 0$ with $d \gg 1$ gives $\gamma^*_d \approx \frac{1}{d} \left( \frac{z-2}{z} \right)$. Thus the optimal probability behaves as $1/d$ at large $d$, instead of $1/d^2$ for Euclidean spaces \cite{22}. The optimal MFPT simply reads

$\langle T^*_d \rangle \approx \frac{dz}{z-2}(z-1)^d \approx dN_d,$  

where $N_d$ is the number of nodes at a distance $\leq d$ from the origin. Assume that a searcher is informed about the presence of a target at a distance $d$ and thus chooses $\gamma = \gamma^*_d$. Naturally, due to inevitable oversampling, $\langle T^*_d \rangle$ is larger than $N_d$, which is the mean time taken by the most efficient systematic exploration of every node at a distance $\leq d$; but it is larger only by a factor $d \approx \ln(N_d)/\ln(z-1)$ (see \cite{53}). On regular $D$-dimensional lattices, such factor is much bigger, of $O(N_d^{1/D})$ \cite{22}. Hence, random searches with optimized resetting are very efficient on Cayley trees, and possibly on other large networks where the number of nodes increases exponentially with the distance, which is the case of most complex networks.

Random and complex networks. With the help of Eqs. (7)-(11), we analyze different types of relatively small networks ($N = 100$) for clarity in the visualizations. In
Fig. 4, we focus on the global time $T(i, \gamma)$ as a function of $\gamma$ on Barbell graphs (constructed by connecting two fully connected networks with 45 nodes with a line of 10 nodes) [53], a Watts-Strogatz (WS) network [51] with rewiring probability $p = 0.01$, an Erdős-Rényi (ER) network [38] with average degree $\langle k \rangle = 2.72$, and a scale-free Barabási-Albert (BA) network with power-law distributed node degrees, generated with the preferential attachment rule [53].

Whereas network exploration depends remarkably little on the initial node $i$ for the simple random walk ($\gamma = 0$), it can become very sensitive to the closeness centrality $C_i$ of $i$ (more than to $k_i$) as soon as resetting is switched on (Figs. 4a,b). A moderate resetting can even reduce $T(i, \gamma)$ by orders of magnitude when the network has marked communities and $i$ is a central node, while resetting to a peripheral node of the same network has the opposite effect (Fig. 4b). A qualitatively similar behavior is observed in WS and BA networks with $m = 1$ (where the central nodes are those with shortcuts and the hubs, respectively), and to a lesser extent, in ER networks.

In conclusion, we have explored a stochastic process on networks that combines a random walk with hops to adjacent nodes and resetting to the initial node. Our formalism analyzes the dynamics in terms of the spectral representation of the transition matrix that defines the random walk strategy without resetting. In this way, we deduce exact results for the stationary distribution and the mean first passage time, which explicitly depend on the resetting probability and the resetting node. We apply these results to characterize the dynamics on rings, Cayley trees, and random networks, including complex and small-world networks. The methods introduced are general and pave the way to further extensions of the study of resetting processes to the domain of networks.

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I. EIGENVALUES AND EIGENVECTORS OF Π(r, γ)

We analyze the eigenvalues and eigenvectors of the matrix

$$\Pi(r, γ) = (1 - γ)W + γ Θ(r),$$

where \( r = 1, \ldots, N \) is the node where resetting is produced with probability \( 0 \leq γ < 1 \). The elements \( ℓ, m \) of the matrix \( Θ(r) \) satisfy \( Θ_{ℓm}(r) = δ_{mr} \).

### A. Left and right eigenvectors

We express the results in terms of the left and right eigenvectors \( \{ |φℓ⟩\}_{ℓ=1}^{N} \), \( \{ |φℓ⟩\}_{ℓ=1}^{N} \) of the transition matrix \( W \) with eigenvalues \( \{ λℓ\}_{ℓ=1}^{N} \). We have \( W |φℓ⟩ = λℓ |φℓ⟩ \) and \( ⟨φℓ|W = λℓ ⟨φℓ| \) for \( ℓ = 1, \ldots, N \), where the set of eigenvalues is ordered in the form \( λ1 = 1 \) and \( 1 > λℓ \geq -1 \) for \( ℓ = 2, 3, \ldots, N \). From right eigenvectors we define a matrix \( Z \) with elements \( Z_{ij} = ⟨i|φj⟩ \). The matrix \( Z \) is invertible, and a new set of vectors \( ⟨φi| \) is obtained by \( (Z^{-1})_{ij} = ⟨φi|j⟩ \), then

$$δ_{ij} = (Z^{-1}Z)_{ij} = \sum_{ℓ=1}^{N} ⟨φi|ℓ⟩ ⟨ℓ|φj⟩ = ⟨φi|φj⟩$$

and

$$1 = ZZ^{-1} = \sum_{ℓ=1}^{N} |φℓ⟩ ⟨φℓ|,$$

where \( 1 \) is the \( N \times N \) identity matrix. In addition, the normalization of the matrix \( W \), \( ∑_{ℓ=1}^{N} w_{i→ℓ} = 1 \), requires

$$|φ1⟩ ∝ \begin{pmatrix} 1 \\ 1 \\ ⋮ \\ 1 \end{pmatrix}.$$

In the following we denote as \( \{ |i⟩\}_{i=1}^{N} \) the canonical base of \( \mathbb{R}^{N} \).

By using \( ⟨φℓ|φ1⟩ = ∑_{i=1}^{N} ⟨φℓ|i⟩ ⟨i|φ1⟩ = δ_{ℓ1} \) and considering the vector \( ⟨i|φ1⟩ = ⟨r|φ1⟩ = constant \) for \( r = 1, \ldots, N \); we obtain

$$∑_{i=1}^{N} ⟨φℓ|i⟩ = δ_{ℓ1} / ⟨r|φ1⟩.$$

Then, from relations in Eqs. (2)-(4), we have

$$\Theta(r) = \sum_{ℓ=1}^{N} \sum_{m=1}^{N} |φℓ⟩ ⟨φℓ| Θ(r) |φm⟩ ⟨φm| = \sum_{ℓ=1}^{N} ∑_{m=1}^{N} ∑_{u=1}^{N} |φℓ⟩ ⟨φℓ| u⟩ Θ(r) v⟩ ⟨v|φm⟩ ⟨φm| \$$

$$= ∑_{ℓ=1}^{N} ∑_{m=1}^{N} ∑_{u=1}^{N} |φℓ⟩ ⟨φℓ| u⟩ δ_{υr} ⟨υ|φm⟩ ⟨φm| = ∑_{ℓ=1}^{N} ∑_{m=1}^{N} ∑_{u=1}^{N} |φℓ⟩ ⟨φℓ| u⟩ ⟨r|φm⟩ ⟨φm|$$

$$= ∑_{ℓ=1}^{N} ∑_{m=1}^{N} |φℓ⟩ \left[ ∑_{u=1}^{N} ⟨φℓ| u⟩ \right] ⟨r|φm⟩ ⟨φm| = ∑_{ℓ=1}^{N} ∑_{m=1}^{N} |φℓ⟩ δ_{ℓ1} / ⟨r|φ1⟩ ⟨r|φm⟩ ⟨φm|. \quad (5)$$
Therefore
\[
\Theta(r) = \sum_{m=1}^{N} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_1 \rangle \langle \phi_m |.
\]  

(6)

In the following, we explore the right and left eigenvectors of \( \Pi(r, \gamma) \), \(|\psi_\ell(r, \gamma)\rangle\) and \( \langle \psi_\ell(r, \gamma) | \) satisfying the relations

\[
\Pi(r, \gamma) |\psi_\ell(r, \gamma)\rangle = \zeta_\ell(r, \gamma) |\psi_\ell(r, \gamma)\rangle,
\]

\[
\langle \psi_\ell(r, \gamma) | \Pi(r, \gamma) = \zeta_\ell(r, \gamma) \langle \psi_\ell(r, \gamma) |
\]

for \( \ell = 1, 2, \ldots, N \) with eigenvalues \( \zeta_\ell(r, \gamma) \). From the result in Eq. (6), we see that \( \Theta(r) |\phi_1 \rangle = |\phi_1 \rangle \). Therefore, the vector \(|\psi_1(r, \gamma)\rangle = |\phi_1 \rangle \) satisfies

\[
\Pi(r, \gamma) |\psi_1(r, \gamma)\rangle = [(1 - \gamma) W + \gamma \Theta(r)] |\phi_1 \rangle = (1 - \gamma) |\phi_1 \rangle + \gamma \Theta(r) |\phi_1 \rangle
\]

\[
= (1 - \gamma) |\phi_1 \rangle + \gamma \sum_{m=1}^{N} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_1 \rangle \langle \phi_m |\phi_1 \rangle = (1 - \gamma) |\phi_1 \rangle + \gamma \sum_{m=1}^{N} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_1 \rangle \delta_{m1}
\]

\[
= (1 - \gamma) |\phi_1 \rangle + \gamma |\phi_1 \rangle = |\phi_1 \rangle = |\psi_1(r, \gamma) \rangle = \zeta_1(r, \gamma) |\psi_1(r, \gamma)\rangle
\]

(7)

where \( \zeta_1(r, \gamma) = 1 \).

In a similar way, we see that \( \langle \phi_\ell \ | \Theta(r) \rangle = 0 \) for \( \ell = 2, 3, \ldots, N \). As a consequence, we define the vectors \( \langle \psi_\ell(r, \gamma) | = \langle \phi_\ell | \) for \( \ell = 2, 3, \ldots, N \) that satisfy

\[
\langle \psi_\ell(r, \gamma) | \Pi(r, \gamma) = \langle \phi_\ell | [(1 - \gamma) W + \gamma \Theta(r)] = (1 - \gamma) \lambda_\ell |\phi_\ell \rangle + \gamma \langle \phi_\ell | \Theta(r)
\]

\[
= (1 - \gamma) \lambda_\ell |\phi_\ell \rangle + \gamma \sum_{m=1}^{N} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_\ell \rangle \langle \phi_m |\phi_1 \rangle = (1 - \gamma) \lambda_\ell |\phi_\ell \rangle + \gamma \sum_{m=1}^{N} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_1 \rangle \delta_{\ell1} \langle \phi_m |
\]

\[
= (1 - \gamma) \lambda_\ell |\phi_\ell \rangle = \zeta_\ell(r, \gamma) \langle \psi_\ell(r, \gamma) | \text{ for } \ell = 2, 3, \ldots, N.
\]

(8)

This result shows that \( \langle \psi_\ell(r, \gamma) | \Pi(r, \gamma) = \zeta_\ell(r, \gamma) \langle \psi_\ell(r, \gamma) | \) with eigenvalues \( \zeta(r, \gamma) = (1 - \gamma) \lambda_\ell \) for \( \ell = 2, 3, \ldots, N \).

Now, we deduce the rest of eigenvectors. For the case \( \langle \psi_1(r, \gamma) | \), we use the anzats

\[
\langle \psi_1(r, \gamma) | = \langle \phi_1 | + \sum_{m=2}^{N} a_m \langle \phi_m |
\]

(9)

This election is motivated by the structure of the matrix \( \Theta(r) \) in Eq. (6). Here, the goal is to deduce the values \( \{a_m\}_{m=2}^{N} \). We know that \( \langle \psi_1(r, \gamma) | \Pi(r, \gamma) = \langle \psi_1(r, \gamma) | \). Therefore:

\[
\langle \psi_1(r, \gamma) | \Pi(r, \gamma) = \left( \langle \phi_1 | + \sum_{m=2}^{N} a_m \langle \phi_m | \right) [(1 - \gamma) W + \gamma \Theta(r)]
\]

\[
= (1 - \gamma) |\phi_1 \rangle + \gamma \sum_{m=1}^{N} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_m \rangle + (1 - \gamma) \sum_{m=2}^{N} a_m \lambda_m |\phi_m \rangle
\]

\[
= \langle \phi_1 | + \sum_{m=2}^{N} \left[ \frac{\gamma |\phi_m \rangle}{\langle r | \phi_1 \rangle} + (1 - \gamma) a_m \lambda_m \right] |\phi_m \rangle.
\]

(10)

This result requires \( a_m = \gamma \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} + (1 - \gamma) a_m \lambda_m \). Therefore \( a_m = \frac{\gamma}{1 - (1 - \gamma) \lambda_m} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} \). In this way, we have

\[
\langle \psi_1(r, \gamma) | = \langle \phi_1 | + \sum_{m=2}^{N} \frac{\gamma}{1 - (1 - \gamma) \lambda_m} \frac{\langle r | \phi_m \rangle}{\langle r | \phi_1 \rangle} |\phi_m \rangle.
\]

(11)

Finally, we explore the eigenvectors \(|\psi_\ell(r, \gamma)\rangle\) for \( \ell = 2, 3, \ldots, N \). From Eq. (10) we know that \( \Theta(r) |\phi_\ell \rangle = \frac{\langle r | \phi_\ell \rangle}{\langle r | \phi_1 \rangle} |\phi_1 \rangle \). This result motivates the ansatz

\[
|\psi_\ell(r, \gamma)\rangle = |\phi_\ell \rangle + b_\ell |\phi_1 \rangle \text{ for } \ell = 2, 3, \ldots, N.
\]

(12)
By using $\Pi(r, \gamma) |\psi_\ell(r, \gamma)\rangle = (1 - \gamma) \lambda_\ell |\psi_\ell(r, \gamma)\rangle$, we have

$$\Pi(r, \gamma) |\psi_\ell(r, \gamma)\rangle = [(1 - \gamma) \mathbf{W} + \gamma \Theta(r)] (|r\rangle |\phi_\ell\rangle) = (1 - \gamma) \lambda_\ell |\phi_\ell\rangle + \gamma \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} |\phi_1\rangle + (1 - \gamma) b_\ell |\phi_1\rangle + \gamma b_\ell |\phi_1\rangle = (1 - \gamma) \lambda_\ell |\phi_\ell\rangle + \frac{1}{(1 - \gamma) \lambda_\ell} \left( b_\ell + \gamma \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} \right) |\phi_1\rangle . \quad (13)$$

In this way $b_\ell = \frac{1}{(1 - \gamma) \lambda_\ell} \left( b_\ell + \gamma \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} \right)$, therefore $b_\ell = -\frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle}$.

Thus, for $\ell = 2, 3, \ldots, N$, we have

$$|\psi_\ell(r, \gamma)\rangle = |\phi_\ell\rangle - \frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} |\phi_1\rangle . \quad (14)$$

In summary, for the transition matrix $\Pi(r, \gamma)$, we obtained the set of right eigenvectors

$$|\psi_\ell(r, \gamma)\rangle = \begin{cases} |\phi_1\rangle & \text{for } \ell = 1, \\ |\phi_\ell\rangle - \frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} |\phi_1\rangle & \text{for } \ell = 2, 3, \ldots, N, \end{cases} \quad (15)$$

and left eigenvectors

$$\langle \bar{\psi}_\ell(r, \gamma) | = \begin{cases} \langle \bar{\phi}_1 | + \gamma \sum_{m=2}^{N} \frac{1}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} \langle \bar{\phi}_m | & \text{for } \ell = 1, \\ \langle \bar{\phi}_\ell | & \text{for } \ell = 2, 3, \ldots, N, \end{cases} \quad (16)$$

with eigenvalues $\zeta_\ell(r, \gamma)$ given by

$$\zeta_\ell(r, \gamma) = \begin{cases} 1 & \text{for } \ell = 1, \\ (1 - \gamma) \lambda_\ell & \text{for } \ell = 2, 3, \ldots, N. \end{cases} \quad (17)$$

**B. Orthonormalization and completeness relation**

Now we check the properties of orthonormalization $\langle \bar{\psi}_\ell(r, \gamma) |\bar{\psi}_m(r, \gamma)\rangle = \delta_{\ell m}$ and the completeness relation $\sum_{\ell=1}^{N} |\psi_\ell\rangle \langle \psi_\ell | = 1$ satisfied by the eigenvectors of $\Pi(r, \gamma)$ defined in Eqs. (15) and (16).

We start with the completeness relation $\sum_{\ell=1}^{N} |\psi_\ell(r, \gamma)\rangle \langle \bar{\psi}_\ell(r, \gamma) | = 1$, we have

$$\sum_{\ell=1}^{N} |\psi_\ell(r, \gamma)\rangle \langle \bar{\psi}_\ell(r, \gamma) | = |\psi_1(r, \gamma)\rangle \langle \bar{\psi}_1(r, \gamma) | + \sum_{\ell=2}^{N} |\psi_\ell(r, \gamma)\rangle \langle \bar{\psi}_\ell(r, \gamma) |$$

$$= |\phi_1\rangle \langle \bar{\phi}_1 | + \sum_{m=2}^{N} \frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} |\bar{\phi}_m | + \sum_{\ell=2}^{N} |\phi_\ell\rangle \langle \bar{\phi}_\ell | - \sum_{\ell=2}^{N} \frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} \langle \bar{\phi}_\ell |$$

$$= |\phi_1\rangle \langle \bar{\phi}_1 | + \sum_{\ell=2}^{N} |\phi_\ell\rangle \langle \bar{\phi}_\ell | = \sum_{\ell=1}^{N} |\phi_\ell\rangle \langle \bar{\phi}_\ell | = 1. \quad (18)$$

Now, let us analyze the property $\langle \bar{\psi}_\ell(r, \gamma) |\psi_1(r, \gamma)\rangle = \delta_{\ell m}$, we have the following cases

- $\langle \bar{\psi}_1(r, \gamma) |\psi_1(r, \gamma)\rangle = 1$.

$$\langle \bar{\psi}_1(r, \gamma) |\psi_1(r, \gamma)\rangle = \langle \bar{\phi}_1 |\phi_1 \rangle + \sum_{m=2}^{N} \frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} \langle \bar{\phi}_m |\phi_1 \rangle = 1 + \sum_{m=2}^{N} \frac{\gamma}{1 - (1 - \gamma) \lambda_\ell} \frac{\langle r |\phi_\ell\rangle}{\langle r |\phi_1\rangle} \delta_{1 m} = 1. \quad (19)$$
However, since $\ell = 2, 3, \ldots, N$, we have
\[
\langle \tilde{\psi}_1(r, \gamma) | \psi_\ell(r, \gamma) \rangle = \frac{\gamma}{1 - (1 - \gamma) \lambda_{\ell}} \frac{\langle \varphi_{\ell} \rangle}{\langle \varphi_{\ell} \rangle} - \frac{\gamma}{1 - (1 - \gamma) \lambda_{\ell}} \frac{\langle \varphi_{\ell} \rangle}{\langle \varphi_{\ell} \rangle} = 0 \quad \text{for} \quad \ell = 2, 3, \ldots, N. \tag{21}
\]

- \(\langle \tilde{\psi}_1(r, \gamma) | \psi_\ell(r, \gamma) \rangle = 0\) for \(\ell = 2, 3, \ldots, N\).

- \(\langle \tilde{\psi}_\ell(r, \gamma) | \psi_r(r, \gamma) \rangle = \delta_{\ell m} \) for \(\ell, m = 2, 3, \ldots, N\).

\[
\langle \tilde{\psi}_\ell(r, \gamma) | \psi_m(r, \gamma) \rangle = \langle \tilde{\psi}_\ell(r, \gamma) | \psi_m(r, \gamma) \rangle = \delta_{\ell m} \quad \text{for} \quad \ell, m = 2, 3, \ldots, N. \tag{22}
\]

The results presented in this section prove that relations in Eqs. (15)-(17) define the eigenvalues and eigenvectors of the transition matrix $\Pi(r, \gamma)$ that describes the dynamics with resetting to the node $r$. The sets of left and right eigenvectors form an orthonormalized base, a result that allows us to deduce analytical expressions for different quantities that describe the dynamics of a random walker with resetting.

### II. Stationary Distribution and Mean First Passage Time

In this part we present general results for the mean first passage time valid for ergodic Markovian random walks. We center our discussion to the analysis of a Markovian process defined by the transition matrix $\Pi(r, \gamma)$. The occupation probability $P_{ij}(t; r, \gamma)$ can be expressed as [1, 2]

\[
P_{ij}(t; r, \gamma) = \delta_{00} \delta_{ij} + \sum_{t' = 0}^{t} P_{jj}(t - t'; r, \gamma) F_{ij}(t'; r, \gamma), \tag{24}
\]

where $F_{ij}(t; r, \gamma)$ is the first-passage probability to start in the node $i$ and finding the node $j$ for the first time after $t$ steps. Using the discrete Laplace transform $\hat{f}(s) = \sum_{t=0}^{\infty} e^{-st} f(t)$ in Eq. (24) we have [2]

\[
\hat{F}_{ij}(s; r, \gamma) = \frac{\hat{P}_{ij}(s; r, \gamma) - \delta_{ij}}{\hat{P}_{jj}(s; r, \gamma)}. \tag{25}
\]

The mean first passage time (MFPT) $\langle T_{ij}(r, \gamma) \rangle$, defined as the mean number of steps taken to reach the node $j$ for the first time, starting from node $i$ [1], can be obtained through the series expansion of $\hat{F}_{ij}(s; r, \gamma)$

\[
\hat{F}_{ij}(s; r, \gamma) = 1 - s \langle T_{ij}(r, \gamma) \rangle + \frac{s^2}{2} \langle T_{ij}^2(r, \gamma) \rangle + \ldots, \tag{26}
\]

where $\langle T_{ij}^2(r, \gamma) \rangle$ is the ensemble average of the squares of the first passages time between $i$ and $j$. In addition, we have the stationary distribution $P_{ij}^\infty(r, \gamma)$ of the Markovian process defined as

\[
P_{ij}^\infty(r, \gamma) \equiv \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} P_{ij}(t; r, \gamma). \tag{27}
\]
result that is independent of the initial condition and gives the probability to reach the node \( j \) for \( t \to \infty \). Now, in terms of \( P_j^\infty(r, \gamma) \), we define the moments \( R_{ij}^{(n)}(r, \gamma) \equiv \sum_{t=0}^{\infty} t^n \{ P_{ij}(t; r, \gamma) - P_j^\infty(r, \gamma) \} \). In this way, the expansion in series of \( \bar{P}_{ij}(s; r, \gamma) \) is

\[
\bar{P}_{ij}(s; r, \gamma) = \frac{1}{1 - e^{-s}} P_j^\infty(r, \gamma) + \sum_{n=0}^{\infty} (-1)^n R_{ij}^{(n)}(r, \gamma) \frac{s^n}{n!}.
\]  

(28)

Introducing this result into Eq. (25) and performing a series expansion of \( \bar{P}_{ij}(s; r, \gamma) \), we have

\[
\langle T_{ij}(r, \gamma) \rangle = \frac{R_{ij}^{(0)}(r, \gamma) - R_{ij}^{(0)}(r, \gamma) + \delta_{ij}}{P_j^\infty(r, \gamma)}.
\]  

(29)

Now, to calculate \( P_j^\infty(r, \gamma) \) and \( \langle T_{ij}(r, \gamma) \rangle \), we need to obtain the probability \( P_{ij}(t; r, \gamma) \). We start with the matricial form of the master equation \( \bar{P}(t; r, \gamma) = \bar{P}(0)\Pi(r, \gamma)^t \) where \( \bar{P}(t; r, \gamma) \) is the probability vector at time \( t \). Using Dirac’s notation

\[
P_{ij}(t; r, \gamma) = \langle i | \Pi(r, \gamma)^t | j \rangle,
\]  

(30)

where \( \{|m\rangle\}_{m=1}^{N} \) represents the canonical base of \( \mathbb{R}^N \). In terms of the eigenvectors and eigenvalues of \( \Pi(r, \gamma) \), we have the spectral representation

\[
\Pi(r, \gamma) = \sum_{\ell=1}^{N} \zeta_{\ell}(r, \gamma) |\psi_{\ell}(r, \gamma)\rangle \langle \bar{\psi}_{\ell}(r, \gamma)|.
\]  

(31)

In this way, the spectral form of the transition matrix allows to obtain for \( P_{ij}(t) \) in Eq. (31)

\[
P_{ij}(t; r, \gamma) = \sum_{\ell=1}^{N} (\zeta_{\ell}(r, \gamma))^t \langle i | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle.
\]  

(32)

Therefore, the stationary distribution \( P_j^\infty(r, \gamma) \) in Eq. (27) is

\[
P_j^\infty(r, \gamma) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \sum_{\ell=1}^{N} (\zeta_{\ell}(r, \gamma))^t \langle i | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle
\]

\[= \sum_{\ell=1}^{N} \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} (\zeta_{\ell}(r, \gamma))^t \langle i | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle \right]
\]

\[= \zeta_{1}(r, \gamma) \langle i | \psi_{1}(r, \gamma) \rangle \langle \bar{\psi}_{1}(r, \gamma)| j \rangle = \langle i | \psi_{1}(r, \gamma) \rangle \langle \bar{\psi}_{1}(r, \gamma)| j \rangle.
\]  

(33)

The result \( \langle i | \psi_{1}(r, \gamma) \rangle = \text{constant} \) makes the stationary distribution \( P_j^\infty(r, \gamma) \) independent of the initial position \( i \). In a similar way, by using the definition of \( R_{ij}^{(0)}(r, \gamma) \), we have

\[
R_{ij}^{(0)}(r, \gamma) = \sum_{t=0}^{\infty} (P_{ij}(t; r, \gamma) - P_j^\infty(r, \gamma)) = \sum_{t=0}^{\infty} \sum_{\ell=1}^{N} (\zeta_{\ell}(r, \gamma))^t \langle i | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle
\]

\[= \sum_{\ell=2}^{N} \frac{1}{1 - \zeta_{\ell}(r, \gamma)} \langle i | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle.
\]  

(34)

The introduction of Eq. (34) in (29) gives the result

\[
\langle T_{ij}(r, \gamma) \rangle = \frac{1}{P_j^\infty(r, \gamma)} \left[ \delta_{ij} + \sum_{\ell=2}^{N} \frac{1}{1 - \zeta_{\ell}(r, \gamma)} \left( \langle j | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle - \langle i | \psi_{\ell}(r, \gamma) \rangle \langle \bar{\psi}_{\ell}(r, \gamma)| j \rangle \right) \right].
\]  

(35)

Now, we use our previous findings, in Eqs. (15)-(17), that established a connection between the eigenvalues and eigenvectors of the matrix \( \Pi(r, \gamma) \) including the resetting and the matrix \( W \) that defines a random walker. We obtain
for the stationary distribution in Eq. (33)

\[ P_j^\infty(r, \gamma) = \langle i|\psi_1(r, \gamma)\rangle \langle \tilde{\psi}_1(r, \gamma)|j \rangle \]

\[ = \langle i|\phi_1\rangle \left[ \langle \phi_1|j\rangle + \gamma \sum_{m=2}^{N} \frac{1}{1-(1-\gamma)\lambda_m} \langle r|\phi_m\rangle \langle \tilde{\phi}_m|j\rangle \right] \]

\[ = \langle i|\phi_1\rangle \langle \tilde{\phi}_1|j\rangle + \gamma \sum_{m=2}^{N} \frac{\langle r|\phi_m\rangle \langle \tilde{\phi}_m|j\rangle}{1-(1-\gamma)\lambda_m}. \]

(36)

Here, \( \langle i|\phi_1\rangle \langle \tilde{\phi}_1|j\rangle \) is the stationary distribution of the random walker without resetting. In the particular case of a normal random walker with transition probabilities \( w_{i\rightarrow j} = \frac{\Delta_i}{k_r} \), \( \langle i|\phi_1\rangle = \frac{k_i}{\sum_{m=1}^N k_m} \) [2].

On the other hand, we have for \( \ell = 2, \ldots, N \)

\[ \langle i|\psi_\ell(r, \gamma)\rangle \langle \tilde{\psi}_\ell(r, \gamma)|j\rangle = \left( \langle i|\phi_\ell\rangle - \frac{\gamma}{1-(1-\gamma)\lambda_\ell} \langle r|\phi_\ell\rangle \right) \langle \tilde{\phi}_\ell|j\rangle = \langle i|\phi_\ell\rangle \langle \tilde{\phi}_\ell|j\rangle - \frac{\gamma}{1-(1-\gamma)\lambda_\ell} \langle r|\phi_\ell\rangle \langle \tilde{\phi}_\ell|j\rangle, \]

(38)

therefore

\[ \langle j|\psi_\ell(r, \gamma)\rangle \langle \tilde{\psi}_\ell(r, \gamma)|j\rangle - \langle i|\psi_\ell(r, \gamma)\rangle \langle \tilde{\psi}_\ell(r, \gamma)|j\rangle = \langle j|\phi_\ell\rangle \langle \tilde{\phi}_\ell|j\rangle - \langle i|\phi_\ell\rangle \langle \tilde{\phi}_\ell|j\rangle \]

\[ \ell = 2, \ldots, N, \]

(39)

result that is independent of the node \( r \) and the probability \( \gamma \). Then, introducing Eq. (39) in (35), we obtain for \( \langle T_{ij}(r, \gamma) \rangle \)

\[ \langle T_{ij}(r, \gamma) \rangle = \frac{\delta_{ij}}{P_j^\infty(r, \gamma)} + \frac{1}{P_j^\infty(r, \gamma)} \sum_{\ell=2}^{N} \frac{\langle j|\phi_\ell\rangle \langle \tilde{\phi}_\ell|j\rangle - \langle i|\phi_\ell\rangle \langle \tilde{\phi}_\ell|j\rangle}{1-(1-\gamma)\lambda_\ell}. \]

(40)

III. MEAN FIRST PASSAGE TIME FOR RINGS

Rings are one-dimensional lattices with periodic boundary conditions. In this case, \( W \) is a circulant matrix [3] with eigenvalues \( \lambda_l = \cos \left[ \frac{2\pi(l-1)}{N} \right] \) and eigenvectors with components \( \langle i|\phi_l\rangle = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi(l-1)(i-1)}{N}} \) and \( \langle \tilde{\phi}_l|j\rangle = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi(l-1)(j-1)}{N}} \) (here \( i \equiv \sqrt{-1} \)) for \( l = 1, \ldots, N \). Therefore, the stationary distribution in Eq. (37) for a ring takes the form

\[ P_j^\infty(i, \gamma) = \frac{1}{N} + \gamma \sum_{l=2}^{N} \frac{\langle i|\phi_l\rangle \langle \tilde{\phi}_l|j\rangle}{1-(1-\gamma)\lambda_l} = \frac{1}{N} + \gamma \sum_{l=2}^{N} \frac{e^{-i\frac{2\pi(l-1)(j-1)}{N}}}{1-(1-\gamma)\cos \left[ \frac{2\pi(l-1)}{N} \right]} \]

\[ = \frac{1}{N} + \gamma \sum_{l=2}^{N} \frac{\cos (\varphi_l d_{ij})}{1-(1-\gamma)\cos (\varphi_l d_{ij})} \]

(41)

with \( \varphi_l \equiv \frac{2\pi}{N}(l-1) \) and where \( d_{ij} \) is the distance between nodes \( i \) and \( j \) (we use \( \cos [\varphi_l(i-j)] = \cos (\varphi_l d_{ij}) \), see Ref. [3]). In a similar way, for the MFPT in Eq. (40) with resetting to the origin \( r = i \)

\[ \langle T_{ij}(\gamma) \rangle = \frac{1}{P_j^\infty(i, \gamma)} \left[ \delta_{ij} + \sum_{l=2}^{N} \frac{\langle j|\phi_l\rangle \langle \tilde{\phi}_l|i\rangle - \langle i|\phi_l\rangle \langle \tilde{\phi}_l|i\rangle}{1-(1-\gamma)\lambda_l} \right] \]

\[ = \frac{1}{P_j^\infty(i, \gamma)} \left[ \delta_{ij} + \sum_{l=2}^{N} \frac{1 - e^{-i\frac{2\pi(l-1)(j-1)}{N}}}{1-(1-\gamma)\cos \left[ \frac{2\pi(l-1)}{N} \right]} \right] = \frac{1}{P_j^\infty(i, \gamma)} \left[ \delta_{ij} + \sum_{l=2}^{N} \frac{1 - \cos (\varphi_l d_{ij})}{1-(1-\gamma)\cos (\varphi_l d_{ij})} \right]. \]

(42)

In the limit \( N \to \infty \), we have an infinite one-dimensional lattice with periodic boundary conditions and considering \( \varphi = \frac{2\pi}{N}(l-1) \) as a continuum variable with \( d\varphi = \frac{2\pi}{N} \), the stationary distribution \( P_j^\infty(i, \gamma) \) in Eq. (41) for the infinite ring takes the form

\[ P_j^\infty(i, \gamma) = \frac{\gamma}{2\pi} \int_0^{2\pi} \frac{\cos (d_{ij} \varphi)}{1-(1-\gamma)\cos (\varphi)} d\varphi. \]

(43)
To evaluate Eq. (43), we explore the integral
\[ I(x, b) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(x\theta)}{1 - b\cos(\theta)} d\theta \quad \text{for} \quad 0 \leq b < 1, \ x \geq 0 \] (44)
and, by using \( b = \frac{2a}{1+a^2} \), we have
\[ I(x, a) = \frac{a^2 + 1}{2\pi} \int_0^{2\pi} \frac{\cos(x\theta)}{1 + a^2 - 2a\cos(\theta)} d\theta = \frac{a^2 + 1}{(a^2 - 1)a^2}, \quad \text{for} \quad a^2 > 1. \] (45)
This expression can be performed thanks to the identity \( \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(x\theta)}{1 + a^2 - 2a\cos(\theta)} d\theta = \frac{1}{(a^2 - 1)a^2} \) (see Ref. [8]). Hence, using \( a = \frac{1}{b} + \sqrt{\frac{1}{b^2} - 1} \)
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(x\theta)}{1 - b\cos(\theta)} d\theta = \frac{1}{\sqrt{1 - b^2}} \left( \frac{1 + \sqrt{1-b^2}}{b} \right)^x. \] (46)
Combining this result with \( b = 1 - \gamma \) in Eq. (43), we obtain
\[ P^\infty_j(i, \gamma) = \sqrt{\frac{\gamma}{2 - \gamma}} \left( \frac{(2 - \gamma)\gamma + 1}{1 - \gamma} \right)^{-d_{ij}}. \] (47)
In addition, in the limit of small resetting \( 0 < \gamma \ll 1 \), \( \sqrt{\frac{\gamma}{2 - \gamma}} = \frac{\gamma}{\sqrt{2}} + O(\gamma^{3/2}) \) and \( \log \left( \frac{(2 - \gamma)\gamma + 1}{1 - \gamma} \right) = \sqrt{2\gamma} + O(\gamma^{3/2}) \). As a consequence, the stationary distribution satisfies
\[ P^\infty_j(i, \gamma) \approx \frac{\sqrt{\gamma}}{2} e^{-\sqrt{2\gamma}d_{ij}} \quad \text{for} \quad 0 < \gamma \ll 1. \] (48)
A result that reproduces the stationary distribution found in Ref. [7].

Now, we explore the MFPT for an infinite ring. In the case \( N \to \infty \), Eq. (42) takes the form
\[ \langle T_{ij}(\gamma) \rangle = \frac{1}{P^\infty_j(i, \gamma)} \left[ \delta_{ij} + \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \cos(d_{ij} \varphi)}{1 - (1 - \gamma) \cos(\varphi)} d\varphi \right] \] (49)
with \( P^\infty_j(i, \gamma) \) given by Eq. (47). In particular, if \( i = j \), we obtain the mean first return time
\[ \langle T_{ii}(\gamma) \rangle = \frac{1}{P^\infty_i(i, \gamma)} = \sqrt{\frac{2 - \gamma}{\gamma}}. \] (50)
On the other hand, if \( i \neq j \)
\[ \langle T_{ij}(\gamma) \rangle = \frac{1}{P^\infty_j(i, \gamma)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \cos(d_{ij} \varphi)}{1 - (1 - \gamma) \cos(\varphi)} d\varphi = \frac{1}{P^\infty_j(i, \gamma)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - (1 - \gamma) \cos(\varphi)} d\varphi - \frac{1}{\gamma}. \] (51)
However, from Eq. (47), we know that \( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - (1 - \gamma) \cos(\varphi)} d\varphi = \frac{1}{\gamma} \sqrt{\frac{2}{2 - \gamma}} \), therefore
\[ \langle T_{ij}(\gamma) \rangle = \frac{1}{\gamma} \left( \frac{(2 - \gamma)\gamma + 1}{1 - \gamma} \right)^{d_{ij}} - \frac{1}{\gamma} \quad \text{for} \quad i \neq j. \] (52)
Combining the results in Eqs. (50) and (52), we obtain for the MFPT
\[ \langle T_{ij}(\gamma) \rangle = \sqrt{\frac{2 - \gamma}{\gamma}} \delta_{ij} - \frac{1}{\gamma} + \frac{1}{\gamma} \left( \frac{(2 - \gamma)\gamma + 1}{1 - \gamma} \right)^{d_{ij}}. \] (53)
IV. MEAN FIRST PASSAGE TIMES ON INFINITE CAYLEY TREES

We recall some basic results on the first passage properties of simple random walks on Cayley trees, see e.g. [8, 9], as a preliminary step to further incorporate resetting. Let us consider a Cayley tree of coordination number \( z \) and a random walk initially at the origin node 0. The target node is located \( d \) links away from the origin. We define \( Q_d^{(0)}(t) \) as the probability that the walker has not reached the target site yet after \( t \) steps, in the absence of resetting. Owing to translational invariance and to the fact that all the sites are equivalent in the infinite lattice, one may write the “backward” equation

\[
Q_d^{(0)}(t) = \frac{z}{z-1} Q_{d+1}^{(0)}(t-1) + \frac{1}{z} Q_{d-1}^{(0)}(t-1),
\]

which asserts that, after the first step (thus with \( t-1 \) steps to go), with probability \( 1/z \), the walker can be one unit closer to the target (on the same branch), or with the complementary probability \( (z-1)/z \), one unit further away. The boundary and initial conditions are

\[
Q_0^{(0)}(t) = 0 \quad \text{and} \quad Q_{d>0}^{(0)}(t) = 1.
\]

We introduce the discrete Laplace transform \( \tilde{Q}_d^{(0)}(s) = \sum_{t=0}^{\infty} s^t Q_d^{(0)}(t) \), which from Eq. (55) must satisfy \( \tilde{Q}_0^{(0)}(s) = 0 \) and \( \tilde{Q}_{d>0}^{(0)}(s = 0) = 1 \). Transforming equation (54) gives, for non-zero \( d \)

\[
\tilde{Q}_d^{(0)}(s) = 1 + s \frac{z-1}{z} \tilde{Q}_{d+1}^{(0)}(s) + \frac{s}{z} \tilde{Q}_{d-1}^{(0)}(s).
\]

We look for solutions of the form \( \tilde{Q}_d^{(0)}(s) = a + Y_d \). By substitution we find \( a = 1/(1-s) \) and that \( Y_d \) obeys the recursion relation

\[
s \frac{z-1}{z} Y_{d+1} = Y_d + \frac{s}{z} Y_{d-1} = 0,
\]

which is easily solved as \( Y_d = C_1 \nu_1^d + C_2 \nu_2^d \), with

\[
\nu_1(s) = \frac{z - \sqrt{z^2 - 4(z-1)s^2}}{2s(z-1)}, \quad \nu_2(s) = \frac{z + \sqrt{z^2 - 4(z-1)s^2}}{2s(z-1)},
\]

and \( C_1, C_2 \) two constants. From \( Q_d^{(0)}(t = 0) = 1 \), the condition \( \lim_{s \to 0} \tilde{Q}_d^{(0)}(s) \to 1 \) must be fulfilled for all \( d > 0 \). Whereas \( \nu_1 \simeq s/z \to 0 \) at small \( s \), \( \nu_2 \simeq \frac{s}{z(z-1)} \to \infty \), which imposes \( C_2 = 0 \). The second condition \( \tilde{Q}_0^{(0)}(s = 0) = 0 \) is enforced by choosing \( C_1 = 1/(1-s) \). We deduce that

\[
\tilde{Q}_d^{(0)}(s) = \frac{1 - [\nu_1(s)]^d}{1-s}.
\]

The large time behavior of \( Q_d^{(0)}(t) \) is deduced from that of \( \tilde{Q}_d^{(0)}(s) \) as \( s \to 1 \). Noting that \( 1/(1-s) \) is the Laplace transform of the constant unity and that \( \lim_{s \to 1} \nu_1(s) < 1 \), we deduce from Eq. (59) that \( \lim_{t \to \infty} Q^{(0)}_d(t) = 1 - \nu_1^d(s = 1) \), or

\[
Q_d^{(0)}(t) \to 1 - (1-z)^{-d} \quad \text{as} \quad t \to \infty.
\]

We recover the well-known result that the probability that the walker ever reaches the target is \( (z-1)^{-d} \). The MFPT is readily deduced from the general relation \( \langle T_d \rangle = \sum_{t=0}^{\infty} Q_d(t) = \tilde{Q}_d(s = 1) \), which, from Eq. (59), is infinite.

For a random walk resetting to the origin with probability \( \gamma \) at each time step, we can use the renewal approach exposed in [10], which allows to derive for any process the survival probability in the Laplace domain, \( \tilde{Q}_d(s) \), as a function of the same quantity in the absence of resetting, \( \tilde{Q}_d^{(0)}(s) \). To do so, one notes that \( Q_d(t) \) can be decomposed as the sum of two contributions: (i) either the walker has never reset since \( t = 0 \), which happens with probability \( (1-\gamma)^t \), (ii) or the last reset occurred at some time \( 1 \leq \tau \leq t \), an eventuality that occurs with probability \( \gamma(1-\gamma)^{t-\tau} \). We write the relation [10]

\[
Q_d(t) = (1-\gamma)^t Q_d^{(0)}(t) + \sum_{\tau=1}^{t} \gamma(1-\gamma)^{t-\tau} Q_d(\tau-1) Q_d^{(0)}(t-\tau).
\]
The first term translates the fact that, if the walker has never reset, its survival is given by \( Q_{d}^{(0)}(t) \), whereas in the second term the walker has first survived \( \tau - 1 \) time steps following the dynamics with reset, and the last \( t - \tau \) steps following the reset-free process. Taking the discrete Laplace transform of Eq. (51) one obtains
\[
\tilde{Q}_d(s) = \frac{Q_{d}^{(0)}(s(1 - \gamma))}{1 - \gamma s Q_{d}^{(0)}(s(1 - \gamma))}.
\]
(62)

The MFPT, given by \( \langle T_d \rangle = \tilde{Q}_d(s = 1) \), is obtained from Eq. (62) by using Eq. (59)
\[
\langle T_d \rangle = \frac{1}{\gamma} \left[ \nu_1 (1 - \gamma)^{-d} - 1 \right],
\]
(63)

which is a finite quantity. Hence, with Eq. (58), the full expression for the MFPT is
\[
\langle T_d \rangle = \frac{1}{\gamma} \left[ \left( \frac{2(1 - \gamma)(z - 1)}{z - \sqrt{z^2 - 4(1 - \gamma)^2(z - 1)}} \right)^d - 1 \right].
\]
(64)

It is easy to check that \( \langle T_d \rangle \approx [\nu_1^{-d} (1 - \gamma)^{-d}] / \gamma \to \infty \) as \( \gamma \to 0 \), and that \( \langle T_d \rangle \approx z^d / (1 - \gamma)^d \to \infty \) as \( \gamma \to 1 \). Thus \( \langle T_d \rangle \) has a minimum for some optimal value \( \gamma^* \). The optimal resetting probability is obtained from solving \( \partial \langle T_d \rangle / \partial \gamma = 0 \), or
\[
1 - \nu_1^d (1 - \gamma^*) = d \gamma^* \nu_1^d (1 - \gamma^*),
\]
(65)

In the limit of large distances, \( d \gg 1 \), one can neglect \( \nu_1^d (1 - \gamma^*) \) in Eq. (65) to obtain
\[
\gamma^* \approx \frac{1}{d} \left( \frac{z - 2}{z} \right).
\]
(66)

Hence the optimal resetting rate tends to 0 at large \( d \) in a different way than on regular lattices, where \( \gamma^* \approx 1 / d^2 \) [7]. This is due to the fact that random walks on Cayley trees are effectively drifting away from their starting point [9] and thus travel a distance \( d \) during a time of order \( d \), instead of \( d^2 \). The MFPT at optimality is readily obtained by substituting Eq. (66) into (64) and neglecting the \(-1\) in the bracket
\[
\langle T_d^* \rangle \approx d \frac{z(z - 1)^d}{z - 2},
\]
(67)

This result can be interpreted as follows. The quantity \( \frac{z(z - 1)^d}{z - 2} \) in Eq. (67) represents the total number of nodes located at a distance \( d \) or smaller from the origin, that we denote as \( N_d \). It stems from the equality \( N_d = 1 + z \sum_{k=0}^{d-1} (z - 1)^k \approx \frac{z(z - 1)^d}{z - 2} \) at large \( d \). We deduce that \( d \approx \frac{\ln N_d}{\ln (z - 1)} \) and that Eq. (67) can be rewritten as
\[
\langle T_d^* \rangle \approx \frac{N_d \ln N_d}{\ln (z - 1)}.
\]
(68)

Hence, the optimized MFPT grows slightly faster than linearly with \( N_d \), the minimal size of the sub-network to be explored to find the target. It is instructive to compare this time with the average time \( \langle T_d^{(syst)} \rangle \) it would take to find the target by using a systematic search strategy, consisting in visiting only once each site located at a distance \( d \) from the origin, without going further than \( d \). This systematic search is the best possible strategy (if the searcher is informed that the target is located at a distance \( d \)). The minimal total number of steps necessary to visit all the sites at a distance \( d \) one by one with this strategy is twice the number of links \( L_d \) in the finite (but large) Cayley tree defined by the \( N_d \) nodes. This can be understood by noting that the walker needs to cross a link once on its way to the boundary and once on its way back toward the origin. Since on average, the target will be found after visiting half of the nodes at a distance \( d \), \( \langle T_d^{(syst)} \rangle = 2L_d/2 = L_d \). Noting that \( L_d \approx N_d \) for large Cayley trees, we obtain
\[
\langle T_d^{(syst)} \rangle \approx N_d.
\]
(69)
We hence come to the conclusion that the optimized resetting process will take only \( \ln N_d \) (or \( d \)) times longer than the best possible strategy:

\[
\frac{\langle T_d^* \rangle}{\langle T_d^{(syst)} \rangle} \simeq \frac{\ln N_d}{\ln(z - 1)} \simeq d.
\]  

(70)

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