Stabilisation of non-diagonal infinite-dimensional systems with delay boundary control

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ABSTRACT
Here we deal with the stabilisation problem of non-diagonal systems by boundary control. In the studied setting, the boundary control input is subject to a constant delay. We use the spectral decomposition method and split the system into two components: an unstable and a stable one. To stabilise the unstable part of the system, we connect, for the first time in the literature, the famous backstepping control design technique with the direct-proportional control design. More precisely, we construct a proportional open-loop stabiliser, then, by means of the Artstein transformation we close the loop. At the end of the paper, an example is provided in order to illustrate the acquired results.

1. Introduction
In this work, we study the stabilisation problem of the following abstract boundary control system with delayed boundary control

\[
\begin{align*}
\frac{d}{dt}y(t) &= Ay(t), \quad t > 0, \\
By(t) &= u(t - \tau), \quad t > 0, y(0) = y_0.
\end{align*}
\]

A similar problem has been studied in the recent work (Lhachemi & Prieur, 2020), under the additional assumption of semi-simple eigenvalues. Here we drop this assumption and consider the general eigenvalues case.

Throughout the paper we assume that \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) and \((\mathcal{H}_0, \langle \cdot, \cdot \rangle_0)\) are separable Hilbert spaces over the field \(\mathbb{R}\), which is either \(\mathbb{R}\) or \(\mathbb{C}\). We denote by \(\| \cdot \|\) the induced norm in \(\mathcal{H}\). In this work, we make the following assumptions:

1. \(A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}\) is a linear unbounded operator.
2. \(B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}_0\), with \(\mathcal{D}(A) \subset \mathcal{D}(B)\), is a linear boundary operator.
3. \(u : [-\tau, +\infty) \to \mathcal{H}_0\), with a known constant delay \(\tau > 0\) and \(u|_{[-\tau, 0]} = 0\), is the boundary control

Besides this, we add the following assumptions:

Assumption 1.1: The operator \(A_0 := A|_{\mathcal{D}(A) \cap \ker(B)}\) is the generator of a \(C_0\)-analytic semigroup, \(\{e^{tA_0} : t \geq 0\}\) on \(\mathcal{H}\). Moreover, it has a countable set of eigenvalues \(\{\lambda_j\}_{j=1}^{\infty}\) (repeated accordingly to their multiplicity), which are not necessarily semi-simple, i.e. the operator \(A_0\) may be non-diagonal. (The system \(\{\lambda_j\}_{j=1}^{\infty}\) is arranged decreasingly with respect to the real part.) Furthermore, from the corresponding eigenfunctions set, one may construct a Riesz basis in \(\mathcal{H}\), \(\{\phi_j\}_{j=1}^{\infty}\).

Let us recall that the semi-simple eigenvalue assumption (i.e. self-adjoint, or equivalently, diagonal operator \(A_0\)) is related to the fact that the eigenfunction system forms a Riesz basis in \(\mathcal{H}\). We notice that this assumption was a key point in the proofs in Lhachemi and Prieur (2020). However, for non-self-adjoint operators, it may happen that the eigenfunctions do not form a Riesz basis, but it is possible to construct one. This is exactly the case of the example provided at the end of the paper.

Assumption 1.2: For each \(\rho > 0\), there exists \(N \in \mathbb{N}^*\) such that: \(\forall \lambda_j \leq -\rho\) for all \(j \geq N + 1\).

Let us denote by \(\{\psi_j\}_{j=1}^{\infty}\), the associated bi-orthogonal system, in \(\mathcal{H}\), of the basis \(\{\phi_j\}_{j=1}^{\infty}\). Namely, it holds \(\langle \phi_i, \psi_j \rangle = \delta_{ij}, i, j \in \mathbb{N}^*\), where \(\delta_{ij}\) is the Kronecker symbol.

Assumption 1.3: We have \(\langle A_0 \phi_j, \psi_j \rangle = 0\) for all \(j \geq N + 1, 1 \leq i \leq N\).

Assumption 1.3 assures that the space spanned by the first \(N\) functions \(\{\phi_i, 1 \leq i \leq N\}\) is invariant under \(A_0\).

Next, we introduce the matrix \(\Lambda = (\lambda_{ij})_{i,j=1}^{N}\), where

\[
\lambda_{ij} := \langle \psi_i, A_0 \phi_j \rangle, \quad 1 \leq i, j \leq N.
\]

\(\Lambda\) is the Jordan matrix representing the operator \(A_0\) restricted to the space spanned by the first \(N\) functions \(\phi_i\). The matrix \(\Lambda\) may be non-diagonal. For \(z \in \mathbb{C}\), we understand by \(\overline{z}\) the complex conjugate of \(z\).
**Assumption 1.4:** For \( \gamma > 0 \) large enough, and for each \( \beta \in R(B) \), there exists a unique solution, \( D \), to the equation

\[
-AD + 2 \sum_{i,j=1}^{N} \lambda_{ij} \langle D, \varphi_j \rangle \varphi_i + \gamma D = 0; \quad BD = \beta. \tag{2}
\]

This way, we may introduce the operator \( D_{\gamma} : R(B) \rightarrow \mathcal{H} \), \( D_{\gamma} \beta := D \), \( D \) solution to (2). Successively scalarly multiplying Equation (2) by \( \psi_1, \ldots, \psi_N \), we get

\[
(\Lambda + \gamma I) \begin{pmatrix} \langle D_{\gamma} \beta, \psi_1 \rangle \\ \langle D_{\gamma} \beta, \psi_2 \rangle \\ \vdots \\ \langle D_{\gamma} \beta, \psi_N \rangle \end{pmatrix} = \begin{pmatrix} \langle \beta, l_1 \rangle_0 \\ \langle \beta, l_2 \rangle_0 \\ \vdots \\ \langle \beta, l_N \rangle_0 \end{pmatrix}, \tag{3}
\]

where \( \{l_1, l_2, \ldots, l_N\} \) are functions in \( \mathcal{H}_0 \), which do not depend on \( \gamma \) or \( \beta \). Here, \( I \) is the identity matrix of order \( N \). We get

\[
L := \{l_1, l_2, \ldots, l_N\}^T.
\]

For a matrix \( A \), we set \( A^T \) for its transpose.

**Assumption 1.5:** The matrix \( (\Lambda L) = [L \Lambda L \Lambda^2 L \cdots \Lambda^{N-1} L] \) has full rank in a nonzero measure set.

Assumption 1.5 says, in fact, that the couple \( (\Lambda L) \) satisfies a Kalman-type controllability rank condition, which is not the classical Kalman condition associated to the system (1). In fact, it is weaker, as we shall see in the example below. In the diagonal case (Lhachemi & Prieur, 2020), the classical Kalman controllability rank condition is assumed. This provides the existence of a so-called ‘feedback gain’ matrix \( K \), which is involved in the definition of the stabilising controller.

Let us consider the following example:

\[
y_1 = \Delta y + cy, \quad t > 0, (x_1, x_2) \in (0, \pi) \times (0, \pi),
\]

\[
y(t, 0, x_2) = u(t, x_2), \quad t > 0,
\]

\[
x_2 \in (0, \pi), \quad y(t, x_1, x_2) = 0 \text{ in rest.}
\]

It is well known that the eigenvalues of the Dirichlet Laplace operator on the square \( (0, \pi) \times (0, \pi) \) are \( \{-k^2 + l^2\} \), \( k, l \in \mathbb{N} \setminus \{0\} \), with the corresponding eigenfunctions \( \varphi = \frac{2}{\pi} \sin(kx_1) \sin(lx_2), \) \( k, l \in \mathbb{N} \setminus \{0\} \) which form a Riesz basis in \( \mathcal{H} = L^2((0, \pi)^2) \). Let us project the equation on two eigenfunctions, namely on

\[
\text{span} \left\{ \frac{2}{\pi} \sin(x_1) \sin(x_2), \frac{2}{\pi} \sin(x_1) \sin(2x_2) \right\}.
\]

Also, consider the controller \( u \) of the form \( u(t, x_2) = v(t) \sin(x_2) \). In this case, the matrices \( A_{N_0} \) and \( B_{N_0} \) from Lhachemi and Prieur (2020, Equation (5)) are given by

\[
A_{N_0} = \text{diag}(-2 + c, -5 + c),
\]

\[
B_{N_0} = \begin{pmatrix} \frac{2}{\pi} \int_0^\pi \sin^2 x_2 \, dx_2 \\ \frac{2}{\pi} \int_0^\pi \sin(2x_2) \sin x_2 \, dx_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

So, the classical Kalman controllability matrix has the form

\[
(A_{N_0} B_{N_0}) = \begin{pmatrix} 1 \\ (-2 + c) \end{pmatrix}
\]

which, of course, does not have full rank. On the other hand, the new Kalman rank controllability condition, Assumption 1.5, reads as: the matrix

\[
(\Lambda L) = \frac{2}{\pi} \begin{pmatrix} \sin x_2 \\ \sin(2x_2) \end{pmatrix} \begin{pmatrix} (-2 + c) \sin x_2 \\ (-5 + c) \sin(2x_2) \end{pmatrix}
\]

has full rank for \( x_2 \in J \subset (0, \pi) \), where \( J \) is a nonempty interval. Easily seen, this holds true. Of course, one may suggest to look for a controller \( u \) not depending on the function \( \sin x_2 \), but on another function such that the resulting classical Kalman matrix \( (A_{N_0} B_{N_0}) \) has full rank. In fact, this is exactly the main idea behind the proportional-type controllers: one proves that there exists a functions set \( \{\chi_k : k = 1, 2, \ldots, N\} \) such that, once plugged the actuator

\[
u = \sum_{k=1}^{N} v_k(t) \chi_k,
\]

into the equations, it assures that the classical Kalman condition holds true. Clearly, this is not always possible, not even in the diagonal case. For the non-diagonal case, this task is even harder to realise. We refer to the recent work of Lasiecka and Triggiani (2015), where the authors study the problem of boundary stabilisation of the non-diagonal Navier–Stokes system, by proportional-type actuators (without delay). In order to ensure the existence of the set of functions \( \{\chi_k : k = 1, 2, \ldots, N\} \) which assure that the classical Kalman condition holds true, the authors are forced to plug into the equations an additional internal controller. So, for the stabilisation of the non-diagonal Navier–Stokes equations, both internal and boundary actuators are needed, while the exact form of the functions set \( \{\chi_k : k = 1, 2, \ldots, N\} \) and of the gain feedback matrix \( K \) are not known. In the present paper, based on the new Kalman-type rank condition, Assumption 1.5, and on improvements on the control design in Munteanu (2019), we stabilise non-diagonal abstract systems only by boundary proportional-type controllers with delay, where the exact form of the functions set \( \{\chi_k : k = 1, 2, \ldots, N\} \) and the feedback gain matrix \( K \) are given exactly. More precisely, \( \chi_k = \chi_k, \quad k = 1, 2, \ldots, N, \) and \( K = \sum_{k=1}^{N} (\Lambda^T + \gamma I)^{-1} A, \) see below for the notations.

At the end of the paper, in order to illustrate the results, we consider the problem of stabilisation of the heat equation, with nonlocal boundary conditions, by boundary delayed control. We show that all the above assumptions hold true, especially...
we show that the new Kalman rank condition, Assumption 1.5, holds true. Of course, it would be interesting to study the Navier–Stokes equations, but to show that the new Kalman type condition holds true for this case is not a simple task at all. Thus, it is left for a subsequent work.

Equations that take the form (1) and obey Assumptions 1.1–1.5 arise from physically meaningful problems, such as reaction–diffusion phenomena, phase turbulence phenomena and more. For details, see Lhachemi and Prieur (2020) and the references therein.

Here, our main objective is to design a feedback law $u$ such that, once plugged into (1), it ensures the exponential stability of the corresponding closed-loop system. Because we are only concerned in controlling the system from the starting time $t = 0$, we assume that the system is uncontrolled for $t < 0$. This is why it is imposed $u|_{[-\tau, 0]} = 0$. Therefore, due to the delay $\tau$ in the control input of (1), the system remains open-loop for $t < \tau$ while the effect of the control input has an impact on the system only at times $t \geq \tau$.

We should emphasise that, based on the ideas in the book (Krstic, 2009b), Prieur and his co-workers, in the recent papers (Lhachemi & Prieur, 2020; Prieur & Trelat, 2019), provide substantial results regarding the boundary stabilisation of parabolic-like equations by delayed controllers. They use the well-known backstepping technique and the diagonal assumption. Here, we aim to deal with the non-diagonal case. In order to do this, our approach is to combine the backstepping technique in Lhachemi and Prieur (2020) with the direct-proportional control design technique, developed by Barbu (2013), and improved latter in Munteanu (2019). In fact, here we improve further the results in Munteanu (2019). To be more precise, in Munteanu (2019) only the diagonal case was treated, here we improve the technique to deal with the non-diagonal case as well. To this aim, we generalise the definition of the perturbation of the Gram matrix $B_k$ (see (6) below) (which coincides with the one in Munteanu (2019) in the diagonal case). This allows us to introduce the more general Assumption 1.5 which is the homologous of nonzero determinant in Munteanu (2019, Equation (2.23)). It is clear that the two assumptions coincide in the diagonal case. However, Assumption 1.5 may hold true for more general cases (e.g. the non-diagonal case considered in the example provided at the end of the paper), while, clearly the nonzero-determinant requirement fails to hold. The second novel concept, we use here, is related to the controller’s feedback form. More exactly, while in Munteanu (2019, Equation (2.26)), the controller is given in feedback form in terms of the vector of the first $N$ modes of the solution, here (see (10) below) the controller’s form involves a generic vector $U$. Then, involving the backstepping technique, one recovers $U$ as an operator series applied to the vector of the first $N$ modes of the solution (see (16) below). This is in fact the main novelty of the present work: the combination of the direct-proportional technique with the backstepping technique. This was, in fact, the missing link which restricted the authors Krstic, Lhachemi, Prieur, of the aforementioned papers, to deal with the diagonal case only. Even if the two methods, backstepping and direct-proportional, have lots of common features, conceptually they are totally different. See the Appendix for more details about this.

2. The main result

In the spirit of proportional control design, the stabilising feedback form will be given directly, following latter to prove the stability of the corresponding closed-loop system. The main result of this work is stated below.

**Theorem 2.1:** The unique solution of the closed-loop parabolic equation with boundary input delay

\[
\frac{d}{dt}y(t) = Ay(t), \quad t > 0, \\
By(t) = -\sum_{k=1}^{N} \left( (A^T + \gamma_k l)^{-1} A \left[ \sum_{j=0}^{\infty} (T_t^j Y(t - \tau))^T \right] \right), \\
y(0) = y_0;
\]

is asymptotically exponentially converging to zero in $H$.

Here, $A$ is introduced in (8) below, $Y$ is the vector consisting of the first $N$ modes of $y$, i.e.

\[
Y(t) := \left( y(t), y_1, \ldots, y_N \right)^T; \\
0 < \gamma_1 < \gamma_2 < \cdots < \gamma_N, \text{ are } N \text{ large enough positive numbers;} \\
\text{and the operator } T_t \text{ is defined in (16) below: } \langle \cdot, \cdot \rangle_N \text{ stands for the standard scalar product in } \mathbb{R}^N.
\]

The first work dealing with input delayed unstable PDEs is Krstic (2009a), where a reaction–diffusion equation is considered, and a backstepping approach is developed to stabilise it. In Fridman and Orlov (2009), a stable PDE is controlled by means of a delayed bounded linear control operator (see also Solomon and Fridman (2015) for a semilinear case). In the present work, the control operator is unbounded (being a boundary control) and the open-loop system is unstable. Unbounded control operators have been considered in Nicaise et al. (2009), Nicaise and Valein (2007), and Nicaise and Pignotti (2008) for both wave and heat equations, where time-varying delays are allowed with a bound on the time-derivative of the delay function. For the finite-dimensional case we refer to the results by Mazenc and his co-workers (Mazenc et al., 2020, 2015) where they show the stability independent of how large the variation of the delay is. For other results on this subject we refer to Bresch-Pietri and Krstic (2014), Fridman et al. (2010), and Krstic (2008). From a practical perspective, it is worth noting that input delays are generally uncertain and possibly time varying. The case of a distributed actuation scheme is even more complex since spatially varying delays can arise due to network and transport effects that may vary among different spatial regions. A first example of this situation occurs in the context of biological systems and population dynamics (Schley & Gourley, 1999). An example of the latter can be found in the context of epidemic dynamics (Wang et al., 2020). The problem of stabilisation of reaction–diffusion PDEs with internal time- and spatially varying input delays is solved in the recent work (Lhachemi et al., 2021) via the backstepping technique. For future, we intend to design a boundary control with time- and spatially varying delay for such equations.

Before ending this section, let us mention that an important approach in the analysis of delay system is to consider approximate predicted feedback obtained using Pade approximations.
for linear systems. A standard control design approach is to develop prediction techniques to compute the future value of the state. We refer here to the well-known Smith Predictor Compensator in Smith (1957). The SPC solves the control problem for open-loop stable linear systems subject to time delay at the input signal. Palmor (1996) reports several improvements to the SPC approach. Thus, by developing state predictors, it is possible to synthesise causal state feedback for linear systems.

It is mandatory first to analyse the state prediction problem, to solve, using causal state feedback, control problems for nonlinear systems with delay at the input, which has led to developing approximate state predictors, for example, Mazenc and Bliman (2006). A sliding mode controller is presented in Roh and Oh (1999) for stabilisation of systems with delayed input. A state predictor and a sliding surface are designed to minimise the effect of the delay.

Another option, to tackle the control problem in time delay systems, is employing an infinite integral prediction strategy known as the finite spectrum assignment proposed in Manitius and Olbrot (1979) and its generalisations presented in Krstic (2009b). Infinite integral prediction strategies are analytical solutions that suffer from practical implementation problems. Zuo et al. (2016) propose truncated output feedback controllers for linear systems and a class of Lipschitz nonlinear systems, respectively, to avoid implementation problems. Truncated output feedback controllers safely ignore the infinite dimensional term. The work (Sanchez et al., 2017) addresses controllers for linear systems and a class of Lipschitz nonlinear systems, respectively, to avoid implementation problems. Which yields

\[ \langle (\Lambda + \gamma k I)^{-1} L, z \rangle_N = 0 \]

almost everywhere,

\[ k = 1, 2, \ldots, N. \]

Or, equivalently,

\[ \langle (\Lambda + \gamma k I)^{-1} L, z \rangle_N = 0 \]

almost everywhere,

\[ k = 1, 2, \ldots, N. \]

Recall that \( L := (l_1 l_2 \cdots l_N)^T \). The above can be viewed as a linear system in \( U \), of order \( N \), with the unknowns \( z_1, z_2, \ldots, z_N \), which are constants in \( U \). It has only the trivial solution if and only if the determinant of the matrix of the system

\[ \det \left[ (\Lambda + \gamma_1 I)^{-1} L \ (\Lambda + \gamma_2 I)^{-1} L \cdots (\Lambda + \gamma_N I)^{-1} L \right] \neq 0, \]

in a nonzero measure set. Performing elementary transformations in the determinant, namely subtracting from each column \( k = 2, 3, \ldots, N \) the first column, i.e.

\[ (\Lambda + \gamma_k I)^{-1} L - (\Lambda + \gamma_1 I)^{-1} L = (\gamma_k - \gamma_1) (\Lambda + \gamma_1 I)^{-1} (\Lambda + \gamma_k I)^{-1} L, \]

\[ k = 2, \ldots, N, \]

the above is equivalent with

\[ \det \left[ (\Lambda + \gamma_1 I)^{-1} L \ (\Lambda + \gamma_1 I)^{-1} (\Lambda + \gamma_2 I)^{-1} L \cdots (\Lambda + \gamma_1 I)^{-1} (\Lambda + \gamma_N I)^{-1} L \right] \]

is not equal zero. This holds true if and only if

\[ \det \left[ L \ (\Lambda + \gamma_2 I)^{-1} L \ (\Lambda + \gamma_3 I)^{-1} L \cdots (\Lambda + \gamma_N I)^{-1} L \right] \neq 0. \]

Similar actions as above, namely subtracting from each column \( k = 3, 4, \ldots, N \) the second column, then multiplying the result

\[ \langle (\Lambda + \gamma_k I)^{-1} L, z \rangle_N = 0 \]

almost everywhere,

\[ k = 1, 2, \ldots, N. \]

Recall that, in the semi-simple case in Munteanu (2019), we defined \( B_k \), see Munteanu (2019, Equation (2.21)), in terms of a diagonal matrix \( \Lambda \gamma_k \). If \( \Lambda \) is diagonal, then \((\Lambda + \gamma k I)^{-1} \) and \( \Lambda \gamma_k \) coincide.

The following result is essential in the definition of the feedback control.

**Proposition 3.1:** The sum of \( B_k \)'s, i.e. \( B_1 + B_2 + \cdots + B_N \) is an invertible matrix.

**Proof:** Let \( z = (z_1 \ z_2 \ \cdots \ z_N)^T \in \mathbb{R}^N \) such that \( \sum_{k=1}^{N} B_k(z) = 0 \). Hence,

\[ \langle B_1 z, z \rangle_N + \langle B_2 z, z \rangle_N + \cdots + \langle B_N z, z \rangle_N = 0. \]

Or, equivalently by the definition of \( B_k \)

\[ \sum_{k=1}^{N} \left\langle B \left( \Lambda^T + \gamma_k I \right)^{-1} z, \left( \Lambda^T + \gamma_k I \right)^{-1} z \right\rangle_N = 0. \]
by $\Lambda + \gamma_2 I$, lead to the equivalent condition
\[
\det \left[ (\Lambda + \gamma_2 I) L \ L \ (\Lambda + \gamma_3 I)^{-1} L \ \cdots \ (\Lambda + \gamma_N I)^{-1} L \right] \neq 0.
\]
We go on like this and arrive at
\[
\det \left[ (\Lambda + \gamma_N I) \ \cdots \ (\Lambda + \gamma_3 I) L \ \cdots \ (\Lambda + \gamma_2 I) L \right] \neq 0.
\]
Again elementary transformations: subtracting from the $(N - 1)$th column the $N$th column multiplied by $\gamma_N$, yield
\[
\det \left[ (\Lambda + \gamma_N I) \ \cdots \ (\Lambda + \gamma_2 I) L \ \cdots \ (\Lambda + \gamma_N I) \times (\Lambda + \gamma_{N-1} I) L \right] \neq 0.
\]
Then, subtracting from the $(N - 2)$th column the $(N - 1)$th column multiplied by $\gamma_{N-1} + \gamma_N - 1$ and the $N$th column multiplied by $\gamma_N \gamma_{N-1}$ we obtain that
\[
\det \left[ (\Lambda + \gamma_N I) \ \cdots \ (\Lambda + \gamma_2 I) L \ \cdots \ \Lambda^2 L \ \Lambda L \right] \neq 0.
\]
The procedure goes in a similar way until we get that the above is equivalent with the fact that the determinant
\[
\det \left[ \Lambda^{N-1} L \ \Lambda^{N-2} L \ \cdots \ \Lambda L \right] \neq 0.
\]
In virtue of Assumption 1.5, this holds true. We conclude that, $(\sum_{k=1}^N B_k)z = 0$ if and only if $z = 0$. Or, in other words, the matrix $\sum_{k=1}^N B_k$ is invertible. □

So, we may well define the matrix
\[
A := (B_1 + B_2 + \cdots + B_N)^{-1}. \quad (8)
\]
Next, for $U : [-\tau, \infty) \to \mathbb{K}^N$, we set
\[
[u_k(U)](t - \tau) := -\left( (\Lambda^T + \gamma_k I)^{-1} AU(t - \tau), L \right)_N,
\]
\[ t \geq -\tau, \quad (9)
\]
$k = 1, 2, \ldots, N$. Then, introduce $u$ as
\[
[u(U)](t - \tau) := [u_1(U) + u_2(U) + \cdots + u_N(U)](t - \tau) = -\sum_{k=1}^N \left( (\Lambda^T + \gamma_k I)^{-1} AU(t - \tau), L \right)_N.
\]
\[ t \geq -\tau, \quad (10)
\]
The vector $U(t)$ will be constructed below via the Artstein transform (Artstein, 2009), and, at the end of the day will be a function of $Y$ (this way we close the loop).

For latter purpose, let us show that
\[
\begin{pmatrix}
D\gamma_k u_k, \psi_1 \\
D\gamma_k u_k, \psi_2 \\
\vdots \\
D\gamma_k u_k, \psi_N
\end{pmatrix} = -B_k AU(t - \tau), \quad (11)
\]
for all $k = 1, \ldots, N$. This is indeed so. We have by (3) that
\[
\begin{pmatrix}
D\gamma_k u_k, \psi_1 \\
D\gamma_k u_k, \psi_2 \\
\vdots \\
D\gamma_k u_k, \psi_N
\end{pmatrix} = (\Lambda + \gamma_k I)^{-1} \begin{pmatrix}
\langle u_k, l_1 \rangle_0 \\
\langle u_k, l_2 \rangle_0 \\
\vdots \\
\langle u_k, l_N \rangle_0
\end{pmatrix}.
\]
Which, by taking into account (5) and (9), yields
\[
\begin{pmatrix}
D\gamma_k u_k, \psi_1 \\
D\gamma_k u_k, \psi_2 \\
\vdots \\
D\gamma_k u_k, \psi_N
\end{pmatrix} = -(\Lambda + \gamma_k I)^{-1} B(\Lambda^T + \gamma_k I)^{-1} AU(t - \tau),
\]
and so, by (6), (11) is proved.

Next, we plug this feedback into Equation (1), and argue similarly as in Munteanu (2019, Equations (2.27)–(2.29)). This way, we equivalently rewrite (1) as an internal-type control problem. More precisely, setting $z := y - \sum_{k=1}^N D\gamma_k u_k$, we have
\[
\frac{d}{dt} z = \frac{d}{dt} y - \sum_{k=1}^N D\gamma_k u_k
\]
\[
= A_0 z + \sum_{k=1}^N AD\gamma_k u_k - \frac{d}{dt} \sum_{k=1}^N D\gamma_k u_k
\]
(owing to (2))
\[
= A_0 z + 2 \sum_{k,l,i=1}^N \lambda_i \langle D\gamma_k u_k, \psi_i \rangle \phi_j + \sum_{k=1}^N \gamma_l D\gamma_l u_k
\]
\[
= A_0 z + [\mathbb{D}] - \frac{d}{dt} \sum_{k=1}^N D\gamma_k u_k, \quad t > 0,
\]
where $\mathbb{D} := 2 \sum_{k,l,i=1}^N \lambda_i \langle D\gamma_k u_k, \psi_i \rangle \phi_j + \sum_{k=1}^N \gamma_l D\gamma_l u_k$. Recall that $A_0$ generates a $C_0$-semigroup. So, the Datko formula gives
\[
z(t) = e^{tA_0} z(0) + \int_0^t e^{(t-s)A_0} \mathbb{D} ds + \sum_{k=1}^N \int_0^t e^{(t-s)A_0} D\gamma_k u_k ds.
\]
Integrating by parts the last term, yields
\[ y(t) = e^{tA_0}y(0) + \int_0^t e^{(t-s)A_0}D\gamma_0 u_k(s) ds + \sum_{k=1}^N \int_0^t e^{(t-s)A_0}A_0D\gamma_0^k u_k ds. \]

Here, \( A_0 \) stands for the extension of the operator \( A_0 \) to the whole \( \mathcal{H} \), i.e. \( A_0 : \mathcal{H} \rightarrow \mathcal{D}(A_0') \)
\[ D(A_0') \langle \tilde{A}_0 f, g \rangle_{D(A_0')} = \langle f, A_0'g \rangle, \quad \forall g \in D(A_0'). \]

Therefore, taking into account the above, we have that (1) can be equivalently re-written as
\[
\frac{d}{dt} y(t) = \tilde{A}_0 y(t) + \sum_{k=1}^N (\tilde{A}_0 + \gamma_k I)D\gamma_0 u_k(U(t - \tau)) - 2 \sum_{i,j,k=1}^N \gamma_{ij} D\gamma_0 u_k(U(t - \tau)), \psi_i \varphi_j, \quad t > 0. \tag{12}
\]

We apply the well-known projection method to (12), and split it into two systems: one of which is unstable but is finite-dimensional, and the other one which is infinite-dimensional but is stable. We will take care of the finite-dimensional unstable part, only.

Projecting Equation (12) on the space spanned by \{\psi_i\}_{i=1}^N\), and taking into account the relation (11), we arrive at
\[
\frac{d}{dt} Y(t) = \Lambda Y(t) - \left[ \Lambda + \sum_{k=1}^N \gamma_k B_k A \right] U(t - \tau), \quad t > 0. \tag{13}
\]

We denote by
\[ C := -\Lambda - \sum_{k=1}^N \gamma_k B_k A. \]

Thus, (13) becomes
\[
\frac{d}{dt} Y(t) = \Lambda Y(t) + CY(t) = -\sum_{k=1}^N \gamma_k B_k A Y(t), \quad t > 0. \tag{14}
\]

This is an equation of the same type as Krstic (2009b, Equation (2.1)). Hence, we may apply the backstepping design via a transport PDE technique described in Krstic (2009b). But, before this, let us notice that if there is no delay (\( \tau = 0 \)) in (14), we may take \( U \equiv Y \) to obtain an exponentially stable system. Indeed, in this case, (14) reads as
\[
\frac{d}{dt} Y(t) = \Lambda Y(t) + CY(t) = -\sum_{k=1}^N \gamma_k B_k A Y(t), \quad t > 0. \tag{15}
\]

We have
\[ \Lambda + C = -\sum_{k=1}^N \gamma_k B_k A = -\gamma_1 I + \sum_{k=2}^N (\gamma_1 - \gamma_k) B_k A, \]
by virtue of the fact that \( A = (B_1 + \cdots + B_N)^{-1} \). Then, for any \( z \in \mathbb{K}^N \), we have
\[ \langle (\Lambda + C)z, Az \rangle_N = -\gamma_1 \| A^{1/2} z \|^2_N + \sum_{k=2}^N (\gamma_1 - \gamma_k) \langle B_k Az, Az \rangle_N. \]

(Here, \( A^{1/2} \) is the square root matrix of \( A \), which can be defined because \( A \) is symmetric and positive definite.) Recalling the definition of \( B_k \), see also relation (7), we see that
\[ (\gamma_1 - \gamma_k) \langle B_k Az, Az \rangle_N \leq 0, \quad k = 2, 3, \ldots, N. \]

Consequently,
\[ \langle (\Lambda + C)z, Az \rangle_N \leq -\gamma_1 \| A^{1/2} z \|^2_N, \quad \forall z \in \mathbb{K}^N. \]

Using this, and taking into account that \( A \) is symmetric and positive definite, we get after scalarly multiplying Equation (15) by \( AY \) that
\[ \| Y(t) \|_N^2 \leq Ce^{-\gamma_1 t} \| Y(0) \|_N^2, \quad \forall t \geq 0. \]

We shall see below that this fact will eliminate the need of the matrix \( K \) from the pole shifting theorem, used in Krstic (2009b).

Following the ideas in Krstic (2009b, Section 2.2), we model the delay in (14) by the following first-order hyperbolic PDE
\[ \partial_t Z(s, t) = \partial_s Z(s, t); \quad Z(s, t) = U(t). \]

System (14) can now be written as
\[
\frac{d}{dt} Y(t) = \Lambda Y(t) + CZ(0, t).
\]

Then, we consider the backstepping transformation
\[ W(s, t) = Z(s, t) - \int_0^s Q(s, r)Z(r, t) dr - \Gamma(s) Y(t) \]
with which we want to map the above system into the target system
\[
\begin{cases}
\frac{d}{dt} Y(t) = (\Lambda + C) Y(t) + CW(0, t), \\
\partial_t W(s, t) = \partial_s W(s, t), \\
W(s, t) = 0.
\end{cases}
\]

Performing similar computations as in Krstic (2009b, Equations (2.26)–(2.39)), and recalling that \( \Lambda + C \) is Hurwitz, we get that
\[ Q(s, t) = e^{(s-t)\Lambda} \quad \text{and} \quad \Gamma(s) = e^{s\Lambda} C. \]

So, the stabilising control is given in an implicit form as
\[ U(t) = Y(t) + \int_0^t e^{(t-s-\tau)\Lambda} CU(s) ds, \quad t \geq 0, \]
and \( U(t) = 0, \quad t \in [-\tau, 0) \). We have to solve the above fixed point implicit equation. To this end, for any integrable vector
\( F \) on \( \mathbb{R} \), we define
\[
(\mathcal{T}_F)(t) := \int_{\max(t-r, \tau)}^{t} e^{(t-r-s)\Lambda} CF(s) \, ds. \tag{16}
\]
It follows that \( U(t) \) can be written as the Neumann series
\[
U(t) = \sum_{j=0}^{\infty} (\mathcal{T}_F Y)(t).
\]
We can show the convergence of this series in a similar manner as in Prieur and Trelat (2019, Lemma 3).

Finally, arguing as in Lhachemi and Prieur (2020, Section IV), we conclude that, once we plug the feedback
\[
u(t - \tau) = -\sum_{k=1}^{N} \left( \Lambda^T + y_k I \right)^{-1} \times A \left[ \sum_{j=0}^{\infty} (\mathcal{T}_F Y)(t - \tau), L \right]_N,
\]
into Equation (1) it yields the desired result of Theorem 2.1. The details are omitted.

4. Stabilisation of the heat equation with nonlocal boundary conditions and delay control

As an application, let us consider the following nonlocal boundary value problem:
\[
\begin{align*}
y(t, x) - y''(t, x) - cy(t, x) &= 0, \quad t > 0, \ x \in (0, \pi), \\
y(0, x) &= u(t - \tau), \\
y'(0, x) + y'(t, \pi) + c y(t, \pi) &= 0, \quad t > 0, \\
y(0, x) &= y_0(x), \quad x \in (0, \pi).
\end{align*}
\tag{17}
\]
Here, \( \tau \) stands for the spatial derivative, i.e. \( f' = \frac{\partial f}{\partial x} \). \( \alpha, c \) are some positive numbers. Boundary-value problems, with two, three, or multi-point nonlocal boundary conditions, arise naturally in thermal conduction, semiconductor or hydrodynamic problems. For details see Boucherif (2009) and the references therein. As far as we know, concerning the boundary stabilisation of PDEs with non-local boundary conditions, there exist only the result in Munteanu (2020), while for the case with delay in the control there is no result in the literature.

In this case \( \mathcal{H} = L^2(0, \pi) \) and \( \mathcal{H}_0 = \mathbb{R} \). The operator \( A : \mathcal{D}(A) \subset L^2(0, \pi) \rightarrow L^2(0, \pi) \) is given as
\[
Ay := y'' + cy, \quad \forall \ y \in \mathcal{D}(A),
\]
where \( \mathcal{D}(A) \) is the set
\[
\left\{ y \in H^2(0, \pi) : y(0) = 0, \ y'(0) + y'(\pi) + c y(\pi) = 0 \right\}.
\]
By Sadybekov et al. (2017), we know that \( A \) has a countable set of eigenvalues \( \{\lambda_j\}_{j=0}^{\infty} \) described as follows:
\[
\lambda_j = \begin{cases} 
-(2k+1)^2 + c & \text{if } j = 2k, \ k \in \mathbb{N}, \\
-(2\beta_k)^2 + c & \text{if } j = 2k+1, \ k \in \mathbb{N}.
\end{cases}
\]
Here, \( \beta_k, \ k \in \mathbb{N} \), are the roots of the equation
\[
\cot(\beta \pi) = -\frac{\alpha}{2\beta}.
\]
Easily seen, given \( \rho > 0 \), there exists \( N \in \mathbb{N} \) such that
\[
-\rho > \lambda_{2N+2} > \lambda_{2N+3} > \cdots.
\]
The corresponding eigenfunctions are precisely given in Sadybekov et al. (2017). More precisely they are \( \{w_{k_1}, w_{k_2}\}_{k=0}^{\infty} \), where
\[
w_{k_1} = \sin((2k+1)x), \quad \text{and} \quad w_{k_2} = \sin(2\beta_k x), \quad k \in \mathbb{N}.\]
As stated and proved in Sadybekov et al. (2017, Lemma 4.1), it happens that the above system does not form a basis in \( L^2(0, \pi) \). That is why, in Sadybekov et al. (2017), the authors introduced the following new set of functions:
\[
\varphi_j(x) = \begin{cases} 
w_{k_1}(x) & \text{if } j = 2k, \\
[w_{k_2}(x) - w_{k_1}(x)](2\delta_k)^{-1} & \text{if } j = 2k+1,
\end{cases}
\tag{18}
\]
Here, \( \delta_k := \beta_k - \frac{1}{2} \). Then, in Sadybekov et al. (2017, Lemma 5.1) they proved that the system \( \{\varphi_j\}_{j=0}^{\infty} \) forms a Riesz basis in \( L^2(0, \pi) \). Moreover, they do also precisely the bi-orthonormal system to \( \{\varphi_j\}_{j=0}^{\infty} \), which is given by
\[
\psi_j(x) = \begin{cases} 
v_{k_2}(x) + v_{k_1}(x) & \text{if } j = 2k, \\
2\delta_kv_{k_2}(x) & \text{if } j = 2k+1, \quad k \in \mathbb{N},
\end{cases}
\tag{19}
\]
Where
\[
v_{k_1}(x) = \frac{2}{\pi} \left( \sin((2k+1)x) - \frac{2k+1}{\alpha} \cos((2k+1)x) \right),
\]
and
\[
v_{k_2} = C_{k_2} \left( \sin(2\beta_k x) - \frac{2\beta_k}{\alpha} \cos(2\beta_k x) \right), \quad k = 0, 1, 2, \ldots
\]
Here, \( C_{k_2} \) is some constant which assures that the systems \( \{w_{k_1}, w_{k_2}\}_{k=0}^{\infty} \) and \( \{v_{k_1}, v_{k_2}\}_{k=0}^{\infty} \) are bi-orthonormal.

In the present case, the matrix \( \Lambda \) is given in Munteanu (2020, Equation (2.22)), as
\[
\begin{pmatrix}
\lambda_0 & 2\beta_0 + 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 2\beta_1 + 3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_{2N-1} & 0 & 0 \\
0 & 0 & 0 & \ldots & \lambda_{2N} & 2\beta_N & \lambda_{2N+1} + 2N + 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & \lambda_{2N+1}
\end{pmatrix}
\]

The lifting operator is defined as: for \( \gamma > 0 \), let \( D \) be the solution to the equation
\[
\begin{pmatrix}
-D''(x) - cD(x) + 2 \sum_{j=0}^{2N+1} \lambda_j \langle D, \psi_j \rangle \psi_j \\
+ 2 \sum_{j=0}^{2N+1} (2\beta_j + 2j + 1) \langle D, \psi_{2j+1} \rangle \psi_{2j+1} + \gamma D &= 0,
\end{pmatrix}
\tag{20}
\]
\[
x \in (0, \pi), \quad \lambda_j \psi_j \text{ is the } j\text{th eigenfunction of } \mathcal{H}, \quad D(0) = 1, \ D'(0) + D'(\pi) + \alpha D(\pi) = 0.
\]
In Munteanu (2020, Lemma 2.1), it is shown the well-posedness of this equation. So far, Assumptions 1.1–1.4 are verified. Let us show that Assumption 1.5 holds true as well. By Munteanu (2020, Equation (2.9)), we know that

\[
\begin{align*}
\{D, \psi_{j}\} &= \frac{1}{\gamma + \lambda_{j}} \left[ \psi_{j}(0) - \frac{2\beta_{j} + j + 1}{\gamma + \lambda_{j+1}} \psi_{j+1}(0) \right], \\
\{D, \psi_{j+1}\} &= \frac{1}{\gamma + \lambda_{j+1}} \psi_{j+1}(0),
\end{align*}
\]

for \(j = 0, 1, 2, \ldots, N\). Or, equivalently,

\[
\begin{pmatrix}
\{D, \psi_{0}\} \\
\{D, \psi_{1}\} \\
\vdots \\
\{D, \psi_{2N+1}\}
\end{pmatrix} = (\Lambda + \gamma I)^{-1}
\begin{pmatrix}
\psi_{0}'(0) \\
\psi_{1}'(0) \\
\vdots \\
\psi_{2N+1}'(0)
\end{pmatrix}
\]

(21)

Hence, in this case \(l_{j} = \psi_{j}(0), j = 0, 1, \ldots, 2N + 1\). Since \(l_{j} \neq 0\) for all \(j = 0, 1, \ldots, 2N + 1\), by the special form of \(\Lambda\), it is easy to see that the couple \((\Lambda, I)\) satisfies the Kalman rank condition. Therefore, Assumption 1.5 is fulfilled. Consequently, the following result holds true:

**Theorem 4.1:** The unique solution of the closed-loop parabolic equation with nonlocal boundary values and input delay

\[
\begin{align*}
\frac{\partial y(t, x)}{\partial t} &= y'(t, x) + cy(t, x), \quad t > 0, \quad x \in (0, \pi), \\
y(t, 0) &= -\sum_{k=0}^{2N+1} \left( (\Lambda^{T} + \gamma_{k} I)^{-1} \right) x_{k} \left[ \sum_{j=0}^{\infty} (T_{j} Y)(t - \tau) \right] L_{j} x_{2N+2}, \\
y'(t, 0) + y'(t, \pi) + \alpha y(t, \pi) &= 0, \quad t > 0, \\
y(0, x) &= y_{0}(x), \quad x \in (0, \pi).
\end{align*}
\]

is asymptotically exponentially converging to zero in \(L^{2}(0, \pi)\).

5. Conclusions

Merging the two techniques: backstepping design and direct-proportional design, we solved the stabilisation problem for non-diagonal systems with boundary delay control.

Time-delays are a delicate issue in engineering systems, which often involve either communications lags or physical dead-time which reveals troublesome in the design and tuning of feedback control laws. Therefore, a robust controller is needed. On this subject, in the finite-dimensional case, there are many important results obtained by Bresch-Pietri and co-workers, see, e.g. Bresch-Pietri and Petit (2014) and Bresch-Pietri (2012). Since the method we applied here consists of the split of the infinite-dimensional system into an unstable finite-dimensional one and an infinite-dimensional stable one, we can try to apply the complex robust control design methods in the aforementioned papers, to the finite-dimensional part, in order to construct a robust control with delay. But then, when returning to the initial infinite-dimensional system, the main problem would be to show that the robust controller assures its stability as well. This is not a simple task and is left for a subsequent work.

Numerical examples to show the effectiveness of a proportional-type controller were performed in Liu et al. (2016).

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### Appendix

#### Appendix. Backstepping vs. direct-proportional control design

The feedback law designed in Munteanu (2019) is given a priori in the form

\[ u(t, x) = \langle AY(t), \Phi(x) \rangle_N, \quad t > 0, \]

for \( x \) on the whole boundary or a part of it only. \( u \) is plugged into the equations, and is shown that it assures the stability of the system. Because of its special form it is called ‘proportional’. It can be equivalently rewritten as

\[ u(t, x) = \sum_{k=1}^{N} \int_{\Omega} y(t, \xi) \psi_k(\xi) \, d\xi \Phi_k(x), \]

where, of course, the boundary functions \( \Phi_k \) are known. Or, furthermore,

\[ u(t, x) = \int_{\Omega} \tilde{k}(x, \xi) y(t, \xi) \, d\xi, \]

where, \( \tilde{k}(x, \xi) = \sum_{k=1}^{N} \Phi_k(x) \psi(\xi) \). The kernel, \( k \), which defines the controller, is given a priori (directly) hence the terminology ‘direct-proportional’. On the other hand, the backstepping technique involves as well a kernel. Roughly speaking, the idea is to make the transformation \( w(t, x) = \int_{\Omega} \tilde{k}(x, \xi) y(t, \xi) \, d\xi \) which leads to a stable equation in terms of \( w \). The kernel \( \tilde{k} \) is deduced by imposing that \( w \) satisfies the targeted stable equation. After finding \( \tilde{k} \), one may express the backstepping control as

\[ u(t, x) = \int_{\Omega} \tilde{k}(x, \xi) y(t, \xi) \, d\xi. \]

So, the exact form of the controller is given post priori, i.e. in an indirect way since one has to solve first a hyperbolic equation in order to deduce \( \tilde{k} \).

In conclusion, both stabilising feedback forms involve some kernels, such that the Volterra transformation maps the original plant to a target stable system. In the backstepping case, the kernel is deduced by solving a PDE of hyperbolic type in the Goursat form; while, in the direct-proportional case the kernel is given a priori, based on the Riesz basis system.