Magnetosonic waves in the crust of a neutron star

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Abstract. The equations of magnetohydrodynamics are used to show that the energy released at the inner surface of the crust of a neutron star generates magnetosonic wave beams that propagate to the star’s surface. These equations can be linearized under the conditions appropriate for the matter in the crust of neutron stars and for the frequency range $10^7 - 10^{11}$ Hz. The solutions describe a beam of standing wave with approximately constant transverse cross-section (radius). The outer base of this beam on the star’s surface is a source of radio emission. Electrical currents are excited in this source and it becomes an antenna that emits radio waves into the circumstellar space. The intensity of the radio emission decreases at higher frequencies, so that the spectrum of emitted radiation by pulsars is limited from above ($\omega \leq 10^{11}$ Hz).

1. Introduction
Observations of more than 1700 pulsars have raised many questions about the structure of neutron stars and the processes taking place in them. One major unsolved problem is the mechanism of the radio emission from pulsars, whose radio luminosity is of the order of $10^{26} - 10^{30}$ erg/s which is a fraction $10^{-4} - 10^{-6}$ of the total loss of rotational energy of a pulsar. If one assumes that the source of the radioemission is the rotational energy of a pulsar, then one needs to identify the physical mechanism by which the rotational kinetic energy of the pulsar is converted into the energy of electromagnetic radiation in order to explain the features of this radioemission. In some earlier papers [1, 2, 3, 4] a possible source for the radioemission was taken to be the energy of the magnetic field drawn from the rotational energy of the star. The neutron star has a magnetic field on the order of $10^{12} - 10^{13}$ G. This field has a complicated structure and is produced by the superposition of a residual magnetic field and the magnetic field generated during the transition of the rotating nuclear component of a neutron star (neutrons and protons) into a superfluid state [5, 6, 7]. It has been shown in Refs. [3, 4] that a “magnetic spot” with a radius on the order of 300 m develops at the boundary of the superfluid core and crust of a neutron star. The magnetic spot forms due to the formation of a vortex free zone with a thickness on the order of 5 m at the boundary of the crust of the neutron star [8]. The energy released in this region excites a magnetosonic wave which propagates toward the star’s surface in the form of a cylindrical beam. In general, the radius of the beam can vary as it approaches the star’s surface. On reaching the surface, the wave beam is reflected, so that a transversely limited standing magnetosonic wave is formed over the width of the star’s crust. Magnetosonic waves transfer the released energy to the surface of the neutron star where it is subsequently emitted in the form of radiowaves which are generated on the star’s surface.
2. MHD equations and evolution equation

The behavior of the magnetoelastic plasma in the crust of a neutron star can be described by a system of MHD equations, which takes into account the viscous forces and the finite electrical and thermal conductivities of matter:

\[ \frac{\partial \rho}{\partial t} + \nabla ( \rho \vec{V} ) = 0, \]  
\[ \rho \frac{d\vec{V}}{dt} - \frac{1}{4\pi} ( \vec{H} \nabla ) \vec{H} = -\nabla \left( P + \frac{H^2}{8\pi} \right) + \eta \Delta \vec{V} + \left( \xi + \frac{\eta}{3} \right) \nabla \left( \nabla \vec{V} \right), \]  
\[ \frac{d\vec{H}}{dt} = -\left( \nabla \vec{q} \right) - \pi_{\alpha\beta} \frac{\partial V_\alpha}{\partial x_\beta} + \frac{J^2}{\sigma}, \]  
\[ \rho T \frac{ds}{dt} = -\left( \nabla T \right) - \pi_{\alpha\beta} \frac{\partial V_\alpha}{\partial x_\beta} + \frac{J^2}{\sigma}, \]  
\[ \rho T \frac{ds}{dt} = \frac{1}{\gamma - 1} \left( \frac{dP}{dt} - c_s^2 \frac{d\rho}{dt} \right). \]  

Here \( \rho, P \) and \( T \) are the density, pressure and temperature of matter, \( s \) is the entropy per unit mass, \( \vec{V} \) is the average velocity of matter, \( \vec{q} \) is the heat flux, \( \vec{H} \) is the magnetic field, \( \eta \) and \( \xi \) are the coefficients of the shear and bulk viscosities, \( \pi_{\alpha\beta} \) is the viscous stress tensor, \( \sigma \) is the electrical conductivity, \( \nu_m = c_A^2 / 4\pi \gamma \) is the magnetic viscosity, \( c_s \) is the speed of sound, \( \gamma = 4/3 \) is the adiabatic index for an ideal ultrarelativistic gas.

Let us consider the propagation of magnetosonic waves in the plasma of the crust of a neutron star of finite width \( l \). We assume that the magnetic field in the crust is directed along the \( y \) axis, and that the \( x \) axis is directed along the radius from the surface to the star’s center. The time variation in the magnetic field, density of matter and pressure are given in the form

\[ \vec{H}' = \vec{H}_0 + \vec{h}, \quad \rho = \rho_0 + \rho', \quad P = P_0 + P', \]  

where \( \vec{H}_0, \rho_0 \) and \( P_0 \) are the equilibrium values of these quantities, while \( \vec{h}, \rho' \) and \( P' \) are their perturbations. It is assumed that the magnetic field perturbation \( \vec{h} \) is also directed along the \( y \) axis. The solution of the MHD equations for the perturbations are quasimonochromatic waves \([9, 10, 11]\) and each perturbation can be written as the sum of two oppositely propagating wave beams. We choose as the unknown function the component of the velocity of matter, \( V_x \equiv u_x \). With this choice we obtain

\[ u(x, y, z, t) = u_1 (\tau_1, \tau'_1, y, z) + u_2 (\tau_2, \tau'_2, y, z), \]  

where

\[ \tau_{1,2} = \frac{t \mp x}{c_n}, \quad \tau_{1,2} = \tau'_{1,2} - t \]  

are the eikonals of waves moving to the right and left, and \( c_n \) is the velocity of the wave, which is given by

\[ c_n^2 = c_A^2 + c_s^2, \quad c_A^2 = \frac{H_0^2}{4\pi \rho}. \]

Here \( c_A \) is the Alfven speed. The perturbations in the unknown physical quantities are related to \( V_x \) by the following equations

\[ V_y = V_z = 0, \quad h_x = h_z = 0, \quad h_y = \mp \frac{H_0}{c_n} V_z, \quad \rho' = \mp \frac{\rho_0}{c_n} V_z, \quad P' = c_s^2 \rho'. \]
The equations for the \( u_{1,2} \) separate \([9, 10]\) and they can be found solving the following equation

\[
\frac{\partial^2 u}{\partial \tau' \partial \tau} + \frac{c_n^2}{2} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \frac{\partial u}{\partial \tau} \frac{d \ln \Phi}{d \tau'} = -\frac{1}{c_n} \frac{\partial}{\partial \tau} \left( \Gamma \frac{\partial u}{\partial \tau} + D \frac{\partial^2 u}{\partial \tau^2} \right),
\]

(11)

which is called the evolution equation. In Eq. (11) the subscripts 1 and 2 on the quantities \( u, \tau, \tau' \) and \( \Phi \) are omitted and \( r = \sqrt{y^2 + z^2} \) is the cylindrical coordinate. The coefficient \( \Gamma \) arises from nonlinear effects and \( D \) is the coefficient of dissipation. These coefficients are defined as

\[
\Gamma = \frac{\gamma + 1}{2} \frac{c_n^2}{c_n^2} + 3 \frac{c_A^2}{2 c_n^2},
\]

(12)

\[
D = -\frac{1}{2 c_n} \left\{ \frac{1}{\rho} \left( \xi + \frac{4}{3} \right) + \frac{c_A^2}{c_n^2} \nu_m + \left( \frac{\gamma - 1}{2} \kappa T \right) \rho c_n^2 \right\},
\]

(13)

where \( \kappa \) is the thermal conductivity. The function \( \Phi \) in the evolution equation is related to the inhomogeneity of the medium and represent the one-dimensional ray solution in the direction perpendicular to the wave front \([11]\). The nonlinear term in Eq. (11) causes expansion or contraction of the transverse dimensions of the wave beams that are the solutions of this equation. In addition, because of this term and the finite size of the crust of the neutron star, a monochromatic wave excited in this medium can excite waves at higer harmonics. But as a wave propagates in the crust of a neutron star, each successive harmonic is weaker in intensity than its predecessor. Therefore we can write the solution of the Eq. (11) in the form of two terms with first and second harmonics:

\[
u = \frac{1}{2} (v_1 e^{i \omega t} + v_2 e^{2i \omega t}) + c.c.
\]

(14)

Substituting this solution in Eq. (11) and equating the coefficients of the first and second harmonics, we obtain two equations for the unknowns \( v_1 \) and \( v_2 \):

\[
i \omega \frac{\partial v_1}{\partial \tau'} + \frac{c_n^2}{2} \left( \frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} \frac{\partial v_1}{\partial r} \right) - i \omega v_1 \frac{d \ln \Phi}{d \tau'} = \frac{\Gamma \omega^2}{2 c_n} v_1 v_2 + i \omega \frac{\omega^2 D}{c_n},
\]

(15)

\[
2 i \omega \frac{\partial v_2}{\partial \tau'} + \frac{c_n^2}{2} \left( \frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r} \frac{\partial v_2}{\partial r} \right) - 2 i \omega v_2 \frac{d \ln \Phi}{d \tau'} = \frac{\Gamma \omega^2}{2 c_n} v_1 v_1 + v_2 \frac{8 i \omega^3 D}{c_n}.
\]

(16)

Let us assume that the average value of \( \tau' \) satisfies the condition

\[
|v_2| \ll \frac{c_n}{\tau \Gamma \omega}.
\]

(17)

In this case we can neglect the nonlinear term in Eq. (15). On the other hand, since \( v_2 \) is a second order perturbation, it is natural to assume that the first term on the right hand side of Eq. (16) is of the order as the first term on the left, i.e.

\[
\frac{\omega |v_2|}{\tau'} \sim \frac{\Gamma \omega^2 |v_1|^2}{c_n}.
\]

(18)

On replacing \( \tau' \) by the characteristic value \( l/c_n \), instead of Eqs. (17) and (18) we obtain

\[
\frac{|v_2|}{|v_1|} \sim \frac{\Gamma \omega \mu}{c_n} |v_1| \ll 1.
\]

(19)
If this condition is met, then Eq. (15) can linearized and the second harmonic can neglected in Eq. (14). As we show below, condition (19) is well satisfied for the crust of a neutron star, so the contribution of the second and higher harmonics to the solution of Eq. (11) will also be small. This means that only the first term [which is determined using the linearized Eq. (15)] is retained in the desired solution.

The solution of Eq. (11) derived in Refs. [11, 12] and has the form

\[
\frac{\Phi(r^1_2)}{b_{1,2}} = \frac{\Phi(r^1_2)}{2 f(r^1_2)} \times \exp \left\{ - \frac{r^2}{2r_0^2 f^2(r^1_2)} + i \left[ \zeta(r^1_2) + \frac{r^2}{2Q(r^1_2)} \right] + i \omega \tau_{1,2} - \omega^2 \int_0^{\tau_{1,2}} \frac{D}{c_n} d\tau \right\} + c.c. \quad (20)
\]

where the additional factor \( \exp \left[ i \zeta(r^1_2) + i \frac{r^2}{2Q(r^1_2)} \right] \) takes into account the changes in the transverse dimensions of the wave beam, \( r_0 \) is the radius which defines the initial transverse cross-section of the beam, and \( f(\tau') \approx 1 \) is the dimensionless radius of the beam [11, 12]. Here we have restored the subscripts on \( u_{1,2} \) and the other functions. We have also written two subscripts on the function \( b_1 \); the relationship between \( b_1 \) and \( b_2 \) can be found from the boundary condition at the star’s surface, i.e., for \( x = 0 \). Because the outer boundary of the crust is a free surface, the appropriate boundary condition is \( \rho' = 0 \), which together with Eq. (10) yields

\[
\rho'|_{x=0} = \left( \frac{\rho_0}{c_n} u_1 + \frac{\rho_0}{c_n} u_2 \right) |_{x=0} = 0,
\]

i.e., \( u_1(0) = u_2(0) \). If we also note that \( \tau'|_{x=0} = \tau'_2 |_{x=0} = l/c_n \), then we obtain \( b_1 = b_2 = b \), and for the modulus of \( u \) we have

\[
|u| = \frac{2 \Phi(l/c_n) b}{f(l/c_n)} \exp \left\{ - \frac{r^2}{2r_0^2 f^2(l/c_n)} - \omega^2 \int_0^l \frac{D}{c_n} dx \right\} \quad (22)
\]

As magnetosonic wave propagates the magnetic field perturbation excites electric currents in the crust. Since \( j = (c/4\pi) \text{curl} \mathbf{H} \), while \( \mathbf{h} \) is directed along the \( y \) axes and depends only on \( x \), only the component \( j_z \) of the vector \( \mathbf{j} \) is nonzero. For the current density at the star’s surface we have

\[
 j_z(r, t) = \frac{c H_0}{4\pi} \frac{i \omega b \Phi(l/c_n)}{c_n f(l/c_n)} \times \exp \left\{ - \frac{r^2}{2r_0^2 f^2(r^1_2)} + i \omega \left( \frac{l}{c_n} - t \right) - \omega^2 \int_0^l \frac{D}{c_n} dx + i \left[ \zeta(l/c_n) + \frac{r^2}{2Q(l/c_n)} \right] \right\} + c.c., \quad (23)
\]

and, finally, for the modulus of the current density we have

\[
|j_z| = \frac{c}{4\pi} \frac{\omega H_0}{c_n^2} |u|, \quad (24)
\]

where \( |u| \) is given by Eq. (22).

The solutions for \( |u| \) and \( |j_z| \) show that thes quantities depend on the amplitude of the perturbation in the velocity of the matter \( b \) and, therefore, on the perturbation in the magnetic field at the boundary of the core and crust of a neutron star.
3. Estimating $b$ from observations of the pulsar radio emission

The electric current density excited on the star’s surface is proportional to $b$, which is the amplitude of the perturbation in the velocity of the plasma at the inner boundary of the crust. This quantity can be estimated using the observed total radio emission intensity of pulsars. Since the transverse dimensions of the wave beam in the crust do not vary essentially ($f \approx 1$), the transverse dimensions of the region where the currents are excited is also of the order of $r_0$. This region becomes the source of the radio emission from the pulsar. We assume that this source is a disk of radius $r_0$ and thickness $\lambda$, where $\lambda$ is the wavelength of the emitted wave. A volume on the order of $\lambda^3$ will radiate coherently. The intensity of the radio emission from this volume is given by the dipole radiation formula

$$I_1 = \frac{2}{3\pi^3} \left| \vec{\dot{d}} \right|^2, \quad (25)$$

where $\vec{\ddot{d}} = \sum e_k \vec{\dot{r}}_k$ is the dipole moment of this volume. Here $e_k$ and $\vec{\dot{r}}_k$ are the charge and radius-vector of the $k$-th particle and the sum is taken over the particles contained in this volume. Thus,

$$\vec{\ddot{d}} = \sum e_k \vec{\dot{V}}_k \approx \lambda^3 \sum e_\alpha n_\alpha \vec{\dot{V}}_\alpha = \vec{j}\lambda^3, \quad (26)$$

where the subscript $\alpha$ refers to the type of particles (electrons and ions) and $\vec{j}$ is the average current density in the volume $\lambda^3$. Differentiating Eq. (26) with respect to time, we finally obtain $\vec{\ddot{d}} = \vec{j}\lambda^3$. Substituting the expression for $\left| \vec{\ddot{d}} \right|$ in Eq. (25), we obtain

$$I_1 = \frac{2\lambda^6}{3\pi^3} \left| \vec{j} \right|^2. \quad (27)$$

The number of coherently radiating volumes in the source (the disk) will be $N \sim \left( r_0 / \lambda \right)^2$. The total intensity of the radio emission is the sum from the coherently radiating volumes, so that

$$I = I_1 N = \frac{2}{3\pi^3} \left| \vec{j} \right|^2 \lambda^4 r_0^2. \quad (28)$$

Here $\left| \vec{j} \right|$ is determined from Eq. (24), i.e.,

$$\left| \vec{j} \right| = \frac{c}{4\pi} \frac{\omega^2 H_0}{c_n} \left| u \left( l/c_n \right) \right|, \quad (29)$$

so that

$$I = \frac{2\pi^2 c}{3} \left( \frac{c H_0 r_0}{c_n^2} \right)^2 \left| u \left( l/c_n \right) \right|^2, \quad (30)$$

where $\lambda = 2\pi c/\omega$. With Eq. (30) we can estimate $\left| u \right|$ at the star’s surface for an observed total intensity of the radio emission from a pulsar of $I \sim 10^{30}$ erg/s:

$$\left| u \left( l/c_n \right) \right| \sim b\Phi \left( l/c_n \right) \sim 10^{-8} c_n. \quad (31)$$

From this, we determine the range of frequencies for which condition Eq. (19) is satisfied:

$$\omega \ll \frac{c_n^2}{\Gamma I \left| u \right|}. \quad (32)$$

Since $\Gamma \sim 1$, Eq. (31) implies that the condition (32) takes the form $\omega \ll 10^{12}$ Hz. We see that this condition is well satisfied within the range of frequencies of the radio emission from pulsars, $10^7 \lesssim \omega \lesssim 10^{11}$ Hz. Since $\left| u \right| = b\Phi$ has a maximum at the star’s surface ($x = 0$), the condition (32) will also be satisfied in the crust of the neutron star, where the wave beam propagates.
4. Conclusion
The MHD equations were used to show that the energy release on the interface between the core and the crust of a neutron star leads to generation of magnetosonic wave beams that propagate toward the star’s surface. It was shown that for radio frequencies \(10^7 – 10^{11}\) Hz and for the conditions corresponding to the matter in the crust these wave beams are linear. A wave beam which is reflected from the surface of the star forms a standing wave in the crust and a disk shaped region develops on the surface of the crust within which electric currents are excited by the magnetosonic wave. These currents convert a limited region of the star’s surface into an antenna that emits radio waves into the circumstellar space. The observed total intensity of the radio emission from a star has been used to estimate the initial amplitude of the magnetosonic wave. The obtained values of the amplitude show that our analysis is consistent with the linear solution of the evolution equation. This solution shows that the wave beams undergo dissipation in the star’s crust. The dissipation is greater at higher frequencies. As a result, the waves with frequencies \(\omega \geq 10^{11}\) Hz do not reach the star’s surface, so they cannot be emitted in the form of radio waves.

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