SOME ASPECTS OF \((r, k)\)-PARKING FUNCTIONS

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Abstract. An \((r, k)\)-parking function of length \(n\) may be defined as a sequence \((a_1, \ldots, a_n)\) of positive integers whose increasing rearrangement \(b_1 \leq \cdots \leq b_n\) satisfies \(b_i \leq k + (i-1)r\). The case \(r = k = 1\) corresponds to ordinary parking functions. We develop numerous properties of \((r, k)\)-parking functions. In particular, if \(F_n^{(r,k)}\) denotes the Frobenius characteristic of the action of the symmetric group \(S_n\) on the set of all \((r, k)\)-parking functions of length \(n\), then we find a combinatorial interpretation of the coefficients of the power series \(\left( \sum_{n \geq 0} F_n^{(r,1)} t^n \right)^k\) for any \(k \in \mathbb{Z}\). For instance, when \(k > 0\) this power series is just \(\sum_{n \geq 0} F_n^{(r,k)} t^n\). We also give a \(q\)-analogue of this result. For fixed \(r\), we can use the symmetric functions \(F_n^{(r,1)}\) to define a multiplicative basis for the ring \(\Lambda\) of symmetric functions. We investigate some of the properties of this basis.

1. Introduction

Parking functions were first defined by Konheim and Weiss as follows. We have \(n\) cars \(C_1, \ldots, C_n\) and \(n\) parking spaces \(1, 2, \ldots, n\). Each car \(C_i\) has a preferred space \(a_i\). The cars go one at a time in order to their preferred space. If it is empty they park there; otherwise they park at the next available space (in increasing order). If all the car s are able to park, then the sequence \(\alpha = (a_1, \ldots, a_n)\) is called a parking function of length \(\ell(\alpha) = n\). For instance, \((3,1,4,3)\) is not a parking function since the last car will go to space 3, but spaces 3 and 4 are already occupied. It is easy to see that \((a_1, \ldots, a_n) \in [n]^n\) (where \([n] = \{1, 2, \ldots, n\}\)) is a parking function if and only if its increasing rearrangement \(b_1 \leq b_2 \leq \cdots \leq b_n\) satisfies \(b_i \leq i\).

Let \(PF_n\) denote the set of all parking functions of length \(n\). A fundamental result of Konheim and Weiss [2] (earlier proved in an equivalent form by Steck [7]—see Yan [8, §1.4] for a discussion) states that \(#PF_n = (n + 1)^{n-1}\). An elegant proof of this result was given by Pollak (reported in [3]), which we now sketch since it will be generalized later. Suppose that we have the same \(n\) cars, but now there are \(n + 1\) spaces \(1, 2, \ldots, n + 1\). The spaces are arranged on a circle. The cars follow the same algorithm as before, but once a car reaches space \(n + 1\) and is unable to park, it can continue around the circle to spaces \(1, 2, \ldots\) until it can finally park. Of course all the cars can park this way, so at the end there will be one empty space. Note that their preferences \((a_1, \ldots, a_n) \in [n + 1]^n\) will be a parking function if and only if the empty space is \(n + 1\). If the empty space is \(e\) and the preferences are changed to \((a_1 + i, \ldots, a_n + i)\) for some \(i\), where \(a_j + i\) is taken modulo \(n + 1\) so that \(a_j + i \in [n + 1]\), then the empty space becomes \(e + i\). Hence given \((a_1, \ldots, a_n) \in [n + 1]^n\), exactly one of the vectors \((a_1 + i, \ldots, a_n + i)\) will be a parking function. It follows that \(#PF_n = \frac{1}{n+1}(n + 1)^n = (n + 1)^{n-1}\).

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We will use notation and terminology on symmetric functions from [6, Chap. 7]. The symmetric group $\mathfrak{S}_n$ acts on $PF_n$ by permuting coordinates. Let $F_n := \text{ch} PF_n$ denote the Frobenius characteristic of this action of $\mathfrak{S}_n$, as defined in [6, §7.18]. Hence $F_n$ is a homogeneous symmetric function of degree $n$, called the parking function symmetric function.

If $\alpha = (a_1, \ldots, a_n)$ is a sequence of positive integers with $m_i$'s (so $\sum m_i = n$), then the Frobenius characteristic of the action of $\mathfrak{S}_n$ on the set of permutations of the terms of $\alpha$ is the complete symmetric function $h_{m_1}h_{m_2}\cdots$ (with $h_0 = 1$). Hence to compute $F_n$, take all vectors $(b_1, \ldots, b_n) \in PF_n$ with $b_1 \leq b_2 \leq \cdots \leq b_n$ (the number of such vectors is the Catalan number $C_n$) and add the corresponding $h_\lambda$ for each. For instance, when $n = 3$ the weakly increasing parking functions are 111, 112, 113, 122, 123, so $F_3 = h_3 + 3h_2h_1 + h_1^3$.

The symmetric function $F_n$ has many remarkable properties, summarized (in a dual form, and with equation (1.2) below not included) in [6, Exer. 7.48(f)].

**Proposition 1.1.** We have

\[
F_n = \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)} \zeta(\lambda)^{-1} \zeta(\lambda-1)^{-1} p_\lambda
\]

\[
= \frac{1}{n+1} \sum_{\lambda \vdash n} s_\lambda(1^{n+1}) \bar{s}_\lambda
\]

\[
= \frac{1}{n+1} \sum_{\lambda \vdash n} \left[ \prod_i \left( \begin{array}{c} \lambda_i + n \\ \lambda_i \end{array} \right) \right] m_\lambda
\]

(1.1)

\[
= \sum_{\lambda \vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{d_1(\lambda)! \cdots d_n(\lambda)!} h_\lambda
\]

(1.2)

\[
\omega F_n = \frac{1}{n+1} \left[ \prod_i \left( \begin{array}{c} n+1 \\ \lambda_i \end{array} \right) \right] m_\lambda,
\]

where $d_i(\lambda)$ denotes the number of parts of $\lambda$ equal to $i$ and $\varepsilon_\lambda = (-1)^{n-\ell(\lambda)}$. Moreover,

(1.3)

\[
F_n = \frac{1}{n+1} [t^n] H(t)^{n+1},
\]

where $[t^n] f(t)$ denotes the coefficient of $t^n$ in the power series $f(t)$, and

\[
H(t) = \sum_{n \geq 0} h_n t^n = \frac{1}{(1 - x_1 t)(1 - x_2 t) \cdots}.
\]

Note in particular that the coefficient of $h_\lambda$ in equation (1.3) is the number of weakly increasing parking functions of length $n$ whose entries occur with multiplicities $\lambda_1, \lambda_2, \ldots$.

A further important property of $F_n$, an immediate consequence of equation (1.3) and the Lagrange inversion formula, is the following. Let

(1.4)

\[
E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t),
\]
and let \( G(t)^{(-1)} \) denote the compositional inverse of the power series \( G(t) \) (which will exist as a formal power series if \( G(t) = a_1 t + a_2 t^2 + \cdots \), where \( a_1 \neq 0 \)). Then

\[
(1.5) \quad \sum_{n \geq 1} F_n t^n = (tE(-t))^{(-1)}.
\]

There are several known generalizations of parking functions. In particular, if \( u = (u_1, \ldots, u_n) \) is a weakly increasing sequence of positive integers, then a \( u \)-parking function is a sequence \( (a_1, \ldots, a_n) \in \mathbb{P}^n \) such that its increasing rearrangement \( b_1 \leq b_2 \leq \cdots \leq b_n \) satisfies \( b_i \leq u_i \). Thus an ordinary parking function corresponds to \( u = (1, 2, \ldots, n) \). For the general theory of \( u \)-parking functions, see the survey [8, §13.4]. We will be interested here in the special case \( u = (k, r + k, 2r + k, \ldots, (n - 1)r + k) \), where \( r, k \geq 1 \). We call such a \( u \)-parking function an \((r, k)\)-parking function. With this terminology, an ordinary parking function is a \((1, 1)\)-parking function. We call an \((r, 1)\)-parking function simply an \( r \)-parking function.

**Note.** Our terminology is not universally used. For instance, if \( (a_1, \ldots, a_n) \) is what we call an \((r, r)\)-parking function, then Bergeron [1] would call \((a_1 - 1, \ldots, a_n - 1)\) an \( r \)-parking function.

Pollak’s proof that \( \#PF_n = (n + 1)^{n-1} \) extends easily to \((r, k)\)-parking functions. Namely, we now have \( rn \) cars \( C_1, \ldots, C_{rn} \) and \( rn + k - 1 \) spaces \( 1, 2, \ldots, rn + k - 1 \). We consider preferences \( \alpha = (a_1, \ldots, a_n) \), \( 1 \leq a_i \leq rn + k - 1 \), where cars \( C_{r(i-1)+1}, \ldots, C_{ri} \) all prefer \( a_i \). The cars use the same parking algorithm as before. It is not hard to check that all the cars can park if and only if \( \alpha \) is an \((r, k)\)-parking function. Now arrange \( rn + k \) spaces on a circle, allow the preferences \( 1 \leq a_i \leq rn + k \), and park as in Pollak’s proof. Then \( \alpha \) is an \((r, k)\)-parking function if and only if the space \( rn + k \) is empty. Reasoning as in Pollak’s proof gives the following result, which in an equivalent form is due to Steck [7].

**Theorem 1.2.** Let \( PF_{n}^{(r,k)} \) denote the set of \((r, k)\)-parking functions of length \( n \). Then

\[
\#PF_n^{(r,k)} = k(rn + k)^{n-1}.
\]

The results in Proposition 1.1 can be extended to \((r, k)\)-parking functions (Theorem 2.1). Most of them appear in Bergeron [1, Prop. 1] for the case \( k = r \). (Bergeron and his collaborators have gone on to generalize their results in a series of papers on rectangular parking functions.) One of our key results (Theorem 3.1) connects \( r \)-parking functions to \((r, k)\)-parking functions as follows.

Let \( PF_n^{(r,k)} \) denote the set of all \((r, k)\)-parking functions of length \( n \), and let \( F_n^{(r,k)} \) denote the Frobenius characteristic \( \chi \) \( PF_n^{(r)} \) of the action of \( \mathfrak{S}_n \) on \( PF_n^{(r,k)} \) by permuting coordinates. Define

\[
\mathcal{P}^{(r,k)}(t) = \sum_{n \geq 0} F_n^{(r,k)} t^n,
\]

\[
\mathcal{P}^{(r)}(t) = \mathcal{P}^{(r,1)}(t),
\]

Then (Theorem 3.1)

\[
(1.6) \quad \mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t).
\]

Equation (1.6) suggests looking at \( \mathcal{P}^{(r)}(t)^k \) for negative integers \( k \). We obtain parking function interpretations of the coefficients of such power series in Section 4. As some motivation
for what to expect, consider two power series $A(t), B(t)$, with $B(0) = 0$, that are related by

$$A(t) = \frac{1}{1 - B(t)} = 1 + B(t) + B(t)^2 + \cdots.$$ 

Thus

$$B(t) = 1 - \frac{1}{A(t)},$$

and often $B(t)$ will be a generating function for certain “prime” objects, while $A(t)$ will be a generating function for all objects, i.e., products of primes. See for instance [5, Prop. 4.7.11]. We will see examples of this relationship with our generating functions for parking functions.

For instance, if we set

$$P^{(r,k)}(t) - 1 = 1 - \sum_{n \geq 1} G_{n}^{(r,k)} t^n,$$

then $G_{n}^{(1,1)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on prime parking functions of length $n$, i.e., parking functions that remain parking functions when some term equal to 1 is deleted (a concept due to Gessel [6, Exer. 5.49(f)]). An increasing parking function $b_1 b_2 \cdots b_n$ can be uniquely factored $\beta_1 \cdots \beta_k$, such that (1) if $b_j$ is the first term of $\beta_i$ then $b_j = j$, and (2) if we subtract from each term of $\beta_i$ one less than its first element (so it now begins with a 1), then we obtain a prime parking function.

As a direct generalization of the previous example, $G_{n}^{(r,1)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on sequences $a_1 a_2 \cdots a_n$ such that some $a_i = 1$, and if remove this term then we obtain an $(r, r)$-parking function. More generally, if $1 \leq k \leq r$ then $G_{n}^{(r,k)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on sequences $a_1 a_2 \cdots a_n$ such that we can remove some term less than $k + 1$ and obtain an $(r, r)$ parking function (Theorem 4.3). For instance, when $r = 2$ and $n = 3$ the increasing sequences with this property are 111, 112, 113, 114, 122, 123, 124, 222, 223, 224. Hence $G_{3}^{(2,2)} = 2 h_1^2 + 6 h_2 h_1 + 2 h_3$. The situation for $P^{(r,k)}(t)^{-j}$ when $j > r$ is more complicated (Theorem 4.1).

2. Expansions of $F_{n}^{(r,k)}$

In this section we consider the expansion of $F_{n}^{(r,k)}$ into the six classical bases for symmetric functions. These expressions are defined even when $k$ is an indeterminate, so we can use any of them to define $F_{n}^{(r,k)}$ in this situation. For later combinatorial applications we will only consider the case when $k$ is an integer. We use notation from [6, Ch. 7] regarding symmetric functions. We also use multinomial coefficient notation such as

$$\binom{k}{d_1, \ldots, d_n, k - \sum d_i} = \frac{k(k - 1) \cdots (k - \sum d_i + 1)}{d_1! \cdots d_n!},$$

where $d_1, \ldots, d_n$ are nonnegative integers and $k$ may be an indeterminate. As usual we abbreviate $\binom{k}{d, k - d}$ as $\binom{k}{d}$. 

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Theorem 2.1. Recall that $d_i(\lambda)$ denotes the number of parts of $\lambda$ equal to $i$. Then $F_0^{(r,k)} = 1$, and for $n \geq 1$ we have

\[(2.1) \quad F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \left( \frac{rn+k}{d_1(\lambda), \ldots, d_n(\lambda), rn+k-\ell(\lambda)} \right) h_\lambda\]

\[(2.2) \quad = \frac{k}{rn+k} \sum_{\lambda \vdash n} \varepsilon_\lambda \left( \frac{rn+k+\ell(\lambda)-1}{d_1(\lambda), \ldots, d_n(\lambda), rn+k-1} \right) e_\lambda\]

\[(2.3) \quad = \frac{k}{rn+k} \sum_{\lambda \vdash n} \left[ \prod_i \left( \frac{\lambda_i+rn+k-1}{\lambda_i} \right) \right] m_\lambda\]

Moreover,

\[(2.4) \quad = \frac{k}{rn+k} \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda\]

\[\omega F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \left[ \prod_i \left( \frac{rn+k}{\lambda_i} \right) \right] m_\lambda,\]

Proof. Define two elements $\alpha$ and $\beta$ of $[rn+k]^n$ to be equivalent if their difference is a multiple of $(1,1,\ldots,1)$ mod $rn+k$. This defines an equivalence relation on $[rn+k]^n$, and each equivalence class contains $rn+k$ elements. It follows from the proof of Theorem 1.2 that each equivalence class contains exactly $k$ $(r,k)$-parking functions. Moreover, all the elements $\alpha$ in each equivalence class have the same multiset of part multiplicities, i.e., the multiset \{d_1, \ldots, d_{rn+k}\}, where $d_i$ is the number of $i$'s in $\alpha$.

For $n \geq 1$ let $D_n^{(r,k)}$ denote the Frobenius characteristic of the action of $\mathfrak{S}_n$ on $[rn+k]^n$ by permuting coordinates. It follows that

\[F_n^{(r,k)} = \frac{k}{rn+k} D_n^{(r,k)} (t^n) H(t)^{rn+k}.\]

Hence if we set $q = 1, k = n, and n = rn+k$ in Exercise 7.75(a) of [6] then we get

\[D_n^{(r,k)} = \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda.\]

(Exercise 7.75 deals with $\mathfrak{S}_k$ acting on submultisets $M$ of $\{1^n, \ldots, k^n\}$. Replace $M$ with the vector $(d_1, \ldots, d_k)$, where $d_i$ is the multiplicity of $i$ in $M$, to get our formulation.) Therefore

\[F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda.\]

The remainder of the proof is routine symmetric function manipulation. "
Let $E(t)$ be given by equation (1.4). Then
\begin{equation}
\sum_{n \geq 0} F_n^{(r,r)} t^{n+1} = (tE(-t)^r)^{-1}
\end{equation}

3. A RELATION BETWEEN $r$-PARKING FUNCTIONS AND $(r,k)$-PARKING FUNCTIONS

In this section we give a combinatorial proof of the following result.

**Theorem 3.1.** Let $k, r \in \mathbb{P}$. Then $P^{(r)}(t)^k = P^{(r,k)}(t)$.

**Proof.** We need to give a bijection $\psi: (PF_{n}^{(r,1)})^k \rightarrow PF_{n}^{(r,k)}$ such that if $\psi(\alpha_1, \ldots, \alpha_k) = \beta$, then $\ell(\alpha_1) + \cdots + \ell(\alpha_k) = \ell(\beta)$. Note that we consider the empty sequence $\emptyset$ to be an $(r,j)$-parking function for any $r$ and $j$.

Given $(\alpha_1, \ldots, \alpha_k) \in (PF_{n}^{(r,1)})^k$, define $\alpha'_i$ to be the sequence obtained by adding $r(\ell(\alpha_1) + \cdots + \ell(\alpha_{i-1}))+ i - 1$ to every term of $\alpha_i$. For instance, if $r = 2$ and 
\[ (\alpha_1, \ldots, \alpha_5) = ((1,2), \emptyset, (1), (1,3,4)), \]
then $\alpha'_1 = (1,2)$, $\alpha'_2 = \alpha'_3 = \emptyset$, $\alpha'_4 = (8)$, and $\alpha'_5 = (11,13,14)$.

It is easily seen that $\psi$ is the desired bijection. In particular, the inverse $\psi^{-1}$ has the following description. Given $\beta = (b_1, \ldots, b_n) \in PF_{n}^{(r,k)}$, let $c_i = b_i - ri + r - 1$. (The term $r-1$ could be replaced by any constant independent from $i$; we made the choice so $c_1 = 0$.) Let $c_1 < \cdots < c_n$ be the left-to-right maxima of the sequence $c_1, \ldots, c_n$, so $j_1 = 1$. Factor $\beta$ (regarded as a word $b_1 \cdots b_n$) as $\beta_1 \cdots \beta_r$, where $\beta_i$ begins with $b_{j_i}$. Subtract a constant $t_i$ from each term of $\beta_i$ so that we obtain a sequence (or word) $\beta'_i$ beginning with a 1. Insert $c_{j_i+1} - c_{j_i} - 1$ empty words $\emptyset$ between $\beta'_i$ and $\beta'_{i+1}$, and place empty words at the end so that there are $k$ words in all. These words $\alpha_1, \ldots, \alpha_k$ then satisfy $\psi^{-1}(\beta) = (\alpha_1, \ldots, \alpha_k)$. $\square$

**Example 3.2.** Suppose that $r = 2, k = 7$, and 
\[ \beta = (1,2,2,10,12,14,15,19,22). \]
Then $(c_1, \ldots, c_9) = (0,-1,-3,3,3,3,3,4,5)$. The left-to-right maxima are $c_1 = 0$, $c_4 = 3$, $c_8 = 4$, $c_9 = 5$. Thus $\beta_1 = (1,2,2)$, $\beta_2 = (10,12,14,15)$, $\beta_3 = (19)$, and $\beta_4 = (22)$. Hence $\beta'_1 = (1,2,2)$, $\beta'_2 = (1,3,5,6)$, $\beta'_3 = \beta'_4 = (1)$. Between $\beta'_1$ and $\beta'_2$ we insert $c_4 - c_1 - 1 = c_9 - c_8 - 1 = 0$ we insert no further copies of $\emptyset$ between remaining $\beta'_i$'s. We now have the six words $\beta'_1, \emptyset, \emptyset, \beta'_2, \emptyset, \beta'_3, \beta'_4$. Since $k = 7$ we insert one $\emptyset$ at the end, finally obtaining 
\[ \psi^{-1}(\beta) = ((1,2,2), \emptyset, (1,3,5,6), (1), (1), \emptyset). \]

Theorem 3.1 has a natural $q$-analogue. We simply state the relevant result since the bijection in the proof of Theorem 3.1 is compatible with our $q$-analogue, so the proof carries over. More specifically, using the notation of equation (3.1) below it is easy to check that if $\beta \in PF_{n}^{(r,k)}$ and $\psi^{-1}(\beta) = (\alpha_1, \ldots, \alpha_k)$, then
\begin{equation}
s^{(r,k)}(\beta) = \sum_{j=1}^{k} (s^{(r,1)}(\alpha_j) + (k - j)\ell(\alpha_j)).
\end{equation}
Given an \((r,k)\)-parking function \(\alpha = (a_1, \ldots, a_n)\) of length \(n\), note that the largest possible value of \(\sum a_i\) is \(k + (k + r) + \cdots + (k + (n - 1)r) = kn + \binom{n}{2}r\). Define
\[
(3.1) \quad s^{(r,k)}(\alpha) = kn + \binom{n}{2}r - \sum_{i=1}^{n} a_i.
\]
When \(k = r\) this is a well-known statistic on parking functions, sometimes used in the variant form \(\sum a_i\). See for instance [4][8, §§1.2.2, 1.3.3]. Note that the action of \(S_n\) on \((r,k)\)-parking functions \(\alpha\) of length \(n\) is compatible with this statistic, i.e., if \(w \in S_n\) then \(s^{(r,k)}(w \cdot \alpha) = w \cdot s^{(r,k)}(\alpha)\).

Given a sequence \(\beta = (b_1, \ldots, b_n) \in \mathbb{P}^n\), let \(U_\beta\) denote the Frobenius characteristic of the action by permuting coordinates of \(S_n\) on all permutations of the terms of \(\beta\). Hence if \(m_i\) is the number of \(i\)'s in \(\beta\) then \(U_\beta = h_{m_1} h_{m_2} \cdots\). Given \(r, k, n \geq 1\), define
\[
F_n^{(r,k)}(q) = \sum_\beta q^{s^{(r,k)}(\beta)} U_\beta,
\]
where \(\beta\) runs over all increasing \((r,k)\)-parking functions of length \(n\). Write
\[
\mathcal{P}^{(r,k)}(q, t) = \sum_{n \geq 0} F_n^{(r,k)}(q) t^n,
\]
\[
\mathcal{P}^{(r)}(q, t) = \mathcal{P}^{(r,1)}(q, t).
\]
Thus \(\mathcal{P}^{(r,k)}(1, t) = \mathcal{P}^{(r,k)}(t)\).

Theorem 3.3. We have
\[
\mathcal{P}^{(r,k)}(q, t) = \prod_{i=0}^{k-1} \mathcal{P}^{(r)}(q, q^i t).
\]

Equation (1.7) gives a relationship between a generating function \(A(t)\) for all objects and \(B(t)\) for prime objects. There is another basic relationship of this nature between exponential generating functions \(A(t)\) for all objects and \(B(t)\) for “connected” objects, namely, the exponential formula \(A(t) = \exp B(t)\) or \(B(t) = \log A(t)\). See [6, §5.1]. Thus we can ask whether there is a combinatorial interpretation of the coefficients of \(\log \mathcal{P}^{(r,k)}(t)\). Recall that \(D_n^{(r,k)}\) denotes the Frobenius characteristic of the action of \(S_n\) on \([rn + k]^n\) by permuting coordinates, as in the proof of Theorem 2.1. The case \(k = r\) is handled by the following result.

Proposition 3.4. We have
\[
\log \mathcal{P}^{(r,r)}(t) = \sum_{n \geq 1} D_n^{(r,r)} \frac{t^n}{n}.
\]

Proof. The proof is a simple consequence of the following variant of the Lagrange inversion formula appearing in [6, Exer. 5.56]: for any power series \(F(t) = a_1 t + a_2 t^2 + \cdots \in \mathbb{C}[[t]]\) with \(a_1 \neq 0\) we have
\[
(3.2) \quad n [t^n] \log \frac{F(t)}{t} = [t^n] \left( \frac{t}{F(t)} \right)^n.
\]
Choose $F(t) = tE(-t)^r$, where $E(t)$ is given by equation (1.4). Now
\[
\frac{1}{E(-t)} = H(t) = \sum_{n \geq 0} h_n t^n.
\]
Hence by equation (2.5), we see that equation (3.2) becomes
\[
n[t^n] \log P^{(r,r)}(t) = [t^n]H(t)^{nr}.
\]
It is clear that $[t^n]H(t)^{nr} = D_n^{(r,r)}$, so the proof follows. \qed

4. A dual to $(r,k)$-parking functions

Equation (1.6) suggests looking at $P^{(r)}(t)^k$ for negative integers $k$. We obtain an object “dual” (in the sense of combinatorial reciprocity) to $(r,k)$-parking functions.

We define $F_n^{(r,k)}$ for $k \leq 0$ by (2.1) (therefore all the equations in Theorem 2.1 hold for $k \leq 0$). It follows from the definition of $P^{(r,k)}(t)$ and equation (1.6) that
\[
P^{(r)}(t)^k = P^{(r,k)}(t) = \sum_{n \geq 0} F_n^{(r,k)} t^n
\]
holds for all $k > 0$. Thus it also holds for all $k \leq 0$. Comparing the coefficients of $t^n$ with those in equation (1.8), namely,
\[
P^{(r)}(t)^{−k} = 1 - \sum_{n \geq 1} G_n^{(r,k)} t^n, \quad \text{for all } k \geq 0,
\]
and combining with (2.1), we see that
\[
G_n^{(r,k)} = -F_n^{(r,k)} = \frac{k}{rn-k} \sum_{\lambda \vdash n} \left( \frac{rn-k}{d_1(\lambda), \ldots, d_n(\lambda)} \right) b_{\lambda}, \quad \text{for all } k \geq 0, n \geq 1.
\]

We then have the following combinatorial interpretation of $G_n^{(r,k)}$.

**Theorem 4.1.** If $rn - k > 0$, then $G_n^{(r,k)}$ is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on the set $S$ of $n$-tuples whose increasing rearrangements have the following form:
\[
(w, \ldots, w, q(w)+1, q(w)+2, \ldots, b_n),
\]
where $w \in [k]$ and $q(w)$ is the smallest integer such that $w \leq q(w)r$, and
\[
b_j \leq \min\{(j-1)r, w-1+rn-k\} \quad \text{for } j = q(w) + 1, q(w) + 2, \ldots, n.
\]
Note that $w \leq \min\{(j-1)r, w-1+rn-k\}$ for all $j \geq q(w) + 1$; therefore (4.3) is equivalent to
\[
b_j \leq \min\{(j-1)r, w-1+rn-k\} \quad \text{whenever } b_j > w.
\]
In other words, a weakly increasing integer sequence $b$ is in $S$ if and only it satisfies the following properties.

**I.** $b_1 = w$ for some $w \in [k]$, and $b_n - b_1 < rn - k$.

**II.** $q(w) = w$.

**III.** $b_j \leq (j-1)r$ for all $j \in [n]$ whenever $b_j > w$.\]
Proof of Proposition 4.4. The case.

A special case. When

Example 4.2. Let \( r = 1, k = 2, \) and \( n = 5 \). The coefficient of \( t^5 \) in \( -P(1)(t)^{-2} \) is

\[
2h_3h_1^2 + 2h_2^2h_1 + 4h_3h_2 + 4h_4h_1 + 2h_5.
\]

This symmetric function is the Frobenius characteristic of the action of \( S_5 \) on all sequences \((a_1, \ldots, a_5) \in \mathbb{P}^5\) whose increasing rearrangement \( b_1 \geq \cdots \geq b_5 \) satisfies either of the conditions (1) \( b_1 = 1, b_2 \leq 1 \) (so in fact \( b_2 = 1 \)), \( b_3 \leq 2, b_4 \leq 3, b_5 \leq 3 \), or (2) \( b_1 = b_2 = 2, b_3 \leq 2 \) (so in fact \( b_3 = 2 \)), \( b_4 \leq 3, b_5 \leq 4 \). We get the fourteen increasing sequences (orbit representatives) \( 11111, 11112, 11113, 11122, 11123, 11133, 11222, 11233, 11223, 22222, 22223, 22224, 22233, 22234. \)

Remark. If \( k \in \{1, \ldots, r\} \), for all \( w \in [k] \) we have \( q(w) = 1 \) and \( (n - 1)r \leq rn - k \leq w - 1 + rn - k \). Therefore (4.4) becomes \( b_j \leq (j - 1)r \) for all \( j > 1 \), so \( b \) having the form (4.2) is equivalent to \( b_1 \in [k] \) and \((b_2, b_3, \ldots, b_n)\) is a weakly increasing \((r, r)\)-parking functions of length \( n - 1 \). Thus Theorem 4.1 becomes the following result.

Theorem 4.3. If \( k \in \{1, \ldots, r\} \), then \( G_a^{(r,k)} \) is the Frobenius characteristic of the action of \( S_n \) on the distinct \( n\)-tuples we get by adjoining \( 1, 2, \ldots, r \) to \((r, r)\)-parking functions of length \( n - 1 \); or equivalently, the \( n\)-tuples whose increasing rearrangements start with \( 1, 2, \ldots, r \), or \( k \) and followed by weakly increasing \((r, r)\)-parking functions of length \( n - 1 \).

Then the number of increasing vectors \( b \in S \) such that the increasing rearrangement of \((b \mod rn - k)\) is in \( R_a \) is \( \frac{pk}{rn-k} \).

A special case. When \( k \in \{1, \ldots, r\} \), it follows from (4.3) that \( b_n \leq (n - 1)r \leq rn - k \) for all \( b \in S \); therefore \( b \mod rn - k = b \). In other words, we only need to consider \( b \) instead of \( b \mod rn - k \). Thus, combined with Theorem 4.1, Proposition 4.4 becomes as follows.

Proposition 4.4. Suppose that \( rn - k > 0 \). Given \( a = (a_1, \ldots, a_n) \in [rn - k]^n \), let \( p \in [rn - k] \) be the smallest positive integer \( i \) such that the increasing rearrangements of \( a \) and \((a + p) \mod rn - k \) coincide, where \( a + i := (a_1 + i, \ldots, a_n + i) \) and \( a_j + i \mod rn - k \) is the \( a_j + i \) taken modulo \( rn - k \) so that \( a_j + i \in [rn - k] \); equivalently, \( p = \#R_a \), where \( R_a \) is the set of increasing rearrangements of vectors \( a + i \mod rn - k \) \((i \in \mathbb{Z})\).

Then the number of increasing vectors \( b \in S \) such that the increasing rearrangement of \((b \mod rn - k)\) is in \( R_a \) is \( \frac{pk}{rn-k} \).

Remark 4.5. The reason why we need the vector \( b \mod rn - k \) is that we may have \( b \in S\backslash[rn - k]^n \) and \( b \mod rn - k \in S \). For instance, when \( r = 2, n = 4, k = 3, rn - k = 5 \), we have \((6, 2, 2, 4) \in S\backslash[rn - k]^n \) and \((1, 2, 2, 4) \in S \).

A special case. When \( k \in \{1, \ldots, r\} \), it follows from (4.3) that \( b_n \leq (n - 1)r \leq rn - k \) for all \( b \in S \); therefore \( b \mod rn - k = b \). In other words, we only need to consider \( b \) instead of \( b \mod rn - k \). Thus, combined with Theorem 4.1, Proposition 4.4 becomes as follows.

Proposition 4.6. If \( k \in \{1, \ldots, r\} \), then for any given \((a_1, \ldots, a_n) \in [rn - k]^n \), there are exactly \( k \) \( i \)'s \((\mod rn - k)\) such that the vector \((a_1 + i, \ldots, a_n + i) \mod rn - k \) is an \((r, r)\)-parking function of length \( n - 1 \) adjoined by \((1, 2, \ldots)\), or \( k \), where \( a_j + i \mod rn - k \) is the \( a_j + i \) taken modulo \( rn - k \) so that \( a_j + i \in [rn - k] \).

Proof of Proposition 4.4. The case \( k = 0 \) is trivial. Assume that \( k \geq 1 \). It suffices to prove the proposition for a weakly increasing sequence \( a = (a_1, \ldots, a_n) \) with \( a_1 = 1 \). For convenience, let \( N := rn - k > 0 \) and denote the increasing rearrangement of a sequence \( x \) by \( x_\uparrow \).
We have two cases: \( p < N \) and \( p = N \).

**Case 1.** \( p < N \).

Then \( a \) has the form:

\[
(4.5) \quad \left( \begin{array}{c} 1, \ldots, 1, 1 + p, \ldots, 1 + p, 1 + 2p, \ldots, 1 + 2p, \ldots, 1 + (\ell - 1)p, \ldots, 1 + (\ell - 1)p \end{array} \right) \]

where \( d, \ell \in \mathbb{P} \) with \( \ell > 1 \) such that \( \ell d = n \) and \( \ell p = N \). Thus \( k = rn - N = (rd - p)\ell \).

The following fact can be verified immediately from the definition of \( S \) and \( R_a \).

**Lemma 4.7.** If \( b \in S \), then \( b + i \in S \) for all \( i \in \{0, -1, \ldots, -b_1 + 1\} \). Further, if \( (b \mod N)_r \in R_a \), then \( (b + i \mod N)_r \in R_a \).

In particular, when \( i = -b_1 + 1 \), the smallest coordinate of \( b + i \) is 1. According to (4.4), we have \( b + i \in \mathcal{M} = [N] \), and therefore \( b + i \mod N = b + i \). If \( (b \mod N)_r \in R_a \), then \( (b + i)_r = (b + i \mod N)_r \in R_a \).

We also need the following lemma.

**Lemma 4.8.** We have \( a + i \in S \) if and only if \( i \in [0, 1, \ldots, rd - p - 1] \).

On the strength of Lemmas 4.7 and 4.8 and the fact that \( R_a = \{a + i : 0 \leq i \leq p - 1\} \), the number of vectors \( b \in S \) such that \( (b \mod rn - k)_r \in R_a \) is \( rd - p = \frac{n}{\ell} \), as desired.

**Proof of Lemma 4.8.** If \( a + i \in S \) with the form (4.2), then applying (4.3) to \( a + i \) and \( d + 1 \) yields \( 1 + p + i \leq rd \), and therefore \( i \leq rd - p - 1 \).

On the other hand, for any \( i \in \{0, 1, \ldots, rd - p - 1\} \), we have \( a + i \in S \). In fact, the vector

\[
a + i = \left( \begin{array}{c} w, \ldots, w, w + p, \ldots, w + p, w + 2p, \ldots, w + 2p, \ldots, w + (\ell - 1)p, \ldots, w + (\ell - 1)p \end{array} \right)
\]

where \( w = 1 + i \leq rd - p \leq (rd - p)\ell = k \). Property I then follows from \( (a + i)_n - (a + i)_1 = (\ell - 1)p < \ell p = N \). Property II holds since \( q(\cdot) \) is weakly increasing and \( q(w) \leq q(rd) = d \). Finally, Property III is satisfied because \( (a + i)_{jd+1} = \cdots = (a + i)_{(j+1)d} = w +jp \leq j(w + p) \leq (jd)r \) for all \( j \in [\ell - 1] \).

**Case 2.** \( p = N \).

Namely, the vectors \((a + i \mod N)_r, i \in [N]\) are distinct. We will determine explicitly the \( \frac{pk}{r_{n-k}} \) vectors in \( S \) desired in Proposition 4.4.

For convenience, we denote \( x_j = a_{j+1} (\leq N), j = 0, \ldots, n - 1 \), and consider the weakly increasing sequence \( x = (x_0, \ldots, x_{n-1}) \) with \( x_0 = 1 \). Then \( x \in S \) if and only if \( x_j \leq r_j \) for all \( j \in [n - 1] \). In general, a weakly increasing integer sequence \( y \) is in \( S \) if and only if

\begin{enumerate}
  \item \( y_0 = w \) for some \( w \in [k] \), and \( y_{n-1} - y_0 < N \).
  \item \( y_{q(w)-1} = w \).
  \item \( y_j \leq jr \) for all \( j \in [n - 1] \) whenever \( y_j > w \).
\end{enumerate}

In the rest of the proof, all variables are integers, and for a vector \( y \), we denote by \( y_j \) its \( (j + 1) \)-th coordinate.

Let \( \Delta_j := r_j - x_j, j = 0, 1, \ldots, n - 1 \). Then \( \Delta_0 = -1 \), and \( x \in S \) if and only if \( \Delta_j \geq 0 \) for all \( j \in [n - 1] \).

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Lemma 4.9. There exists \( i \in \mathbb{Z} \) such that the vector \( (x + i \mod N)_{1} \in S \), with the smallest coordinate equal to 1. More precisely, if \( x \in S \), then we can take \( i = 0 \); otherwise, take \( i = 1 - x_{j} \), where \( j \) is the largest number in \([n - 1]\) such that \( \Delta_{j} = \min_{j' \in [n-1]} \Delta_{j'} \).

Proof. Assume that \( x \notin S \), then \( \Delta_{j} \leq -1 \) and \( j \in [n - 1] \) for the \( j \) taken in the lemma. Taking \( i = 1 - x_{j} \), we get

\[
x + i \mod N = (2 - x_{j} + N, x_{1} - x_{j} + 1 + N, \ldots, x_{j-1} - x_{j} + 1 + N, 1, x_{j+1} - x_{j} + 1, \ldots, x_{n-1} - x_{j} + 1).
\]

and thus

\[
\alpha := (x + i \mod N)_{1} = (x_{j+1} - x_{j} + 1, \ldots, x_{n-1} - x_{j} + 1, x_{1} - x_{j} + 1 + N, \ldots, x_{j-1} - x_{j} + 1 + N).
\]

It follows from the definition of \( j \) that \( x_{j} \geq r_{j+1} \), and for \( j' > j \) we have \( \Delta_{j'} \geq \Delta_{j} + 1 \), and therefore \( x_{j'} - x_{j} \leq r(j' - j) - 1 \); for \( j' < j \) we have \( \Delta_{j'} \geq \Delta_{j} \), and therefore \( x_{j'} - x_{j} \leq r(j' - j) \). Thus

\[
\begin{align*}
\alpha_{u} &= x_{j+u} - x_{j} + 1 \leq r(j + u - j) - 1 + 1 = ru, \quad u \in [n - 1 - j], \\
\alpha_{n-j} &= 2 - x_{j} + rn - k \leq 2 - rj - 1 + rn - 1 = r(n - j), \\
\alpha_{n-j+u} &= x_{u} + 1 - x_{j} + rn - k \leq r(u - j + n), \quad u \in [j - 1].
\end{align*}
\]

Hence \( \alpha \in S \).

On the strength of Lemma 4.9, we can assume that \( x \in S \) with \( x_{0} = 1 \). The following result determines the \( k \) vectors in \( S \) desired in Proposition 4.4.

Lemma 4.10. Let \( 0 = j_{0} < j_{1} < j_{2} < \cdots \) be the elements of the subset

\[ J^* := \{ j \in J : \Delta_{j} > \Delta_{j} \text{ for all } n - 1 \geq j' > j \} \subseteq J := \{0\} \cup \{j \in [n - 1] : x_{j} > x_{j-1}\} \]

and \( m \) be the nonnegative integer determined by

\[
-1 = \Delta_{j_{0}} < \Delta_{j_{1}} < \cdots < \Delta_{j_{m}} \leq k - 2 < \Delta_{j_{m+1}} < \cdots
\]

(if \( j_{m+1} \) does not exist, then set \( j_{m+1} = \Delta_{j_{m+1}} \) to be infinity). In particular, \( j_{1} \) is the largest number in \([n - 1]\) such that \( \Delta_{j_{1}} = \min_{j \in [n-1]} \Delta_{j} \geq 0 \).

Then \( y \) is a weakly increasing sequence in \( S \) such that \( (y \mod N)_{1} \in R_{x} \) if and only if

(1) \( y = x + i \) with \( 0 \leq i \leq \Delta_{j_{1}} \land (k - 1) \), where \( \land \) represents the minimum function; or

(2) \( y = (x + i_{1} \mod N)_{1} + i_{2} \) with

(i) \( i_{1} = 1 - x_{j_{v}} \) for some \( v \in [m] \), and

(ii) \( 0 \leq i_{2} = y_{0} - 1 \leq \Delta_{j_{v+1}} \land (k - 1) - \Delta_{j_{v}} - 1 < k - 1 \).

Further, the \( k \) vectors given in (1) and (2) are distinct.

Remark 4.11. Note that (1) is the special case of (2) with \( i_{1} = 0 = v \) and \( i_{2} = i \).

Proof. As a consequence of \( p = N \), the vectors \( (x + i_{1} \mod N)_{1} \) with \( i_{1} \) given in (1) \( (i_{1} = i) \) and (2), whose smallest coordinates are all 1, are distinct. Thus the \( k \) vectors given in (1) and (2) are distinct.
(1) If \( y = x + i \in S \), then by definition we have \( 1 \leq (x+i)_0 \leq k \) and \( (x+i)_{j+1} \leq r_{j+1} \). Thus \( 0 \leq i \leq \Delta_{j+1} \) \( \land \) \((k-1)\).

Conversely, for any \( y = x + i \) with \( 0 \leq i \leq \Delta_{j+1} \) \( \land \) \((k-1)\), we have \((y \mod N)\) \( \in \) \( R_x \), \( y_{n-1} - y_0 = x_{n-1} - x_0 < N \), and \( 1 \leq w := y_0 = (x+i)_0 \leq (1 + \Delta_{j+1}) \) \( \land \) \( k \leq k \), and Property I follows.

For Property II, notice that for any \( j \in [n-1] \) such that \( x_j \geq 2 \), since \( r_j - x_j = \Delta_j \geq \Delta_{j+1} \), we have \( j \geq (2 + \Delta_{j+1})/r > w/r \), and therefore \( j \geq q(w) \).

Finally for Property III, for all \( j \in [n-1] \), \( \Delta_{j+1} \leq \Delta_j \), we have \( x_j - x_{j+1} \leq r(j - j_1) \), and therefore \( y_j = (x+i)j = x_j + i \leq r(j - j_1) + \Delta_{j+1} = r_j \).

(2) If \( y \) is a weakly increasing sequence in \( S \) such that \((y \mod N)\) \( \in \) \( R_x \), then applying \((4.6)\) to \( \alpha \)

Recall that

\[ (1) \] If \( y = x + i \in S \), then by definition we have \( 1 \leq (x+i)_0 \leq k \) and \( (x+i)_{j+1} \leq r_{j+1} \). Thus \( 0 \leq i \leq \Delta_{j+1} \) \( \land \) \((k-1)\).

Conversely, for any \( y = x + i \) with \( 0 \leq i \leq \Delta_{j+1} \) \( \land \) \((k-1)\), we have \((y \mod N)\) \( \in \) \( R_x \), \( y_{n-1} - y_0 = x_{n-1} - x_0 < N \), and \( 1 \leq w := y_0 = (x+i)_0 \leq (1 + \Delta_{j+1}) \) \( \land \) \( k \leq k \), and Property I follows.

For Property II, notice that for any \( j \in [n-1] \) such that \( x_j \geq 2 \), since \( r_j - x_j = \Delta_j \geq \Delta_{j+1} \), we have \( j \geq (2 + \Delta_{j+1})/r > w/r \), and therefore \( j \geq q(w) \).

Finally for Property III, for all \( j \in [n-1] \), \( \Delta_{j+1} \leq \Delta_j \), we get \( x_j - x_{j+1} \leq r(j - j_1) \), and therefore \( y_j = (x+i)j = x_j + i \leq r(j - j_1) + \Delta_{j+1} = r_j \).

Recall that \( \alpha \in S \) if and only if

(4.6) \( \alpha_u \leq ru \), for all \( u \in [n-1] \).

Applying to \( u = 1, \ldots, n - 1 - j \) leads to

\[ x_{j'} - x_j + 1 \leq r(j' - j), \] i.e., \( \Delta_j \leq \Delta_{j'} \), for all \( j' < j \leq n - 1 \);

applying to \( u = n - j \) leads to

\[ 2 - x_j + rn - k \leq r(n - j) \] i.e., \( \Delta_j \leq k - 2 \).

Therefore \( j = j_v \) for some \( v \in [m] \).

Conversely, from the above argument we see that if \( i_1 = 1 - x_j \) with \( j = j_v \) for some \( v \in [m] \), then we have \( \alpha_u \leq ru \) for all \( u \in [n - j] \). Further, we have

\[ \alpha_{n-j+u} = x_u - x_j + 1 + N \leq ru + \Delta_j - r_{j+1} + 1 + rn - k < r(n - j + u) \]

for all \( u \in [j - 1] \). Hence \( \alpha \in S \).

It remains to show that \( \alpha + i_2 \in S \) if only if \( i_2 \) satisfies the inequality in (ii).

If \( \alpha + i_2 \in S \), then applying (4.6) to \( \alpha' := \alpha + i_2 \) and \( u = j_v + 1 - j_v \) (if exists) leads to

\[ x_{j_v+1} - x_{j_v} + 1 + i_2 \leq r(j_{v+1} - j_v) \] i.e., \( i_2 \leq \Delta_{j_v+1} - \Delta_{j_v} - 1 \);

applying (4.6) to \( \alpha' := \alpha + i_2 \) and \( u = n - j_v \) leads to

\[ 2 - x_{j_v} + rn - k + i_2 \leq r(n - j_v) \] i.e., \( i_2 \leq k - 2 - \Delta_{j_v} \).

Recall that \( i_2 = y_0 - 1 \geq 0 \), and thus \( i_2 \) satisfies the inequality in (ii).
Conversely, if \( i_2 \) satisfies the inequality in (ii), then \( \alpha' \in S \). In fact, we have \( 1 \leq w := 1 + i_2 \leq k \) and \( \alpha'_{n-1} - \alpha'_{n-2} = \alpha_{n-1} - \alpha_0 < N \), and Property I’ then follows.

For Property II’, by the definition of \( j_{v+1} \), we have \( \Delta_u \geq \Delta_{j_{v+1}} \) for any \( j_v < u \in J \), and hence for any \( j_v < u \leq n-1 \) such that \( x_u > x_{j_v} \). Thus \( ru - x_u \geq \Delta_{j_{v+1}} \). It follows that

\[
ru \geq \Delta_{j_{v+1}} + x_u > \Delta_{j_{v+1}} + x_{j_v} = \Delta_{j_{v+1}} + r j_v - \Delta_{j_v}
\]

and

\[
u - j_v > (\Delta_{j_{v+1}} - \Delta_{j_v})/r \geq w/r, \text{ i.e., } u \geq q(w) + j_v.
\]

Hence \( \alpha'_{q(w)-1} = x_{q(w)-1+j_v} - x_{j_v} + w = w \).

Finally for Property III’, from the above argument we see that \( \alpha'_u \leq ru \) for \( u = j_{v+1} - j_v, n - j_v \). Further, we have

\[
\alpha'_{u-\nu} = x_u - x_{j_v} + 1 + N + i_2 \leq ru - x_{j_v} + 1 + rn - k + k - 2 - \Delta_{j_v} < r(n - j_v + u)
\]

for all \( u \in [j_v-1] \). For \( j_v + 1 \leq u \leq n - 1 \) such that \( x_u > x_{j_v} \) and \( u \in J \), we have \( \Delta_u \geq \Delta_{j_{v+1}} \) by the definition of \( j_{v+1} \), and therefore

\[
\alpha'_{u-j_v} = x_u - x_{j_v} + w \leq (ru - \Delta_{j_{v+1}}) - x_{j_v} + (\Delta_{j_{v+1}} - \Delta_{j_v}) = r(u - j_v).
\]

Hence \( \alpha' \in S \), as desired.

**Note.** We have been unable to find a satisfactory \( q \)-analogue of Theorem 4.1, generalizing Theorem 3.3.

5. The \( r \)-Parking Function Basis

Equation (1.6) and other considerations suggest looking at products of the symmetric functions \( F_n^{(r)} \) for various values of \( n \). Thus for any partition \( \lambda \) define

\[
F^{(r)}_\lambda = F^{(r)}_{\lambda_1} F^{(r)}_{\lambda_2} \ldots,
\]

where \( F_0 = 1 \).

Recall that \( \Lambda \) denotes the ring of all symmetric functions that can be written as an integer linear combination of the monomial symmetric functions \( m_\lambda \) (or equivalently, \( s_\lambda, h_\lambda, \text{ or } e_\lambda \)).

**Proposition 5.1.** Fix \( r \geq 1 \). Then the symmetric functions \( F^{(r)}_\lambda \), where \( \lambda \) ranges over all partitions of all \( n \geq 0 \), form an integral basis for the ring \( \Lambda \).

**Proof.** We need to show that for each \( n \), the set \( \{ F^{(r)}_\lambda : \lambda \vdash n \} \) is an integral basis for the (additive) group \( \Lambda^n \) of all homogeneous symmetric functions of degree \( n \) contained in \( \Lambda \). Let \( \lambda^1, \lambda^2, \ldots \) be any ordering of the partitions of \( n \) that is compatible with refinement, that is, if \( \lambda^i \) is a refinement of \( \lambda^j \) then \( i \leq j \). Now \( F^{(r)}_n = h_n + \cdots \in \Lambda^n \). Hence \( F^{(r)}_\lambda = h_\lambda \) terms involving \( h_\mu \) where \( \mu \) refines \( \lambda \). Hence the transition matrix for expressing the \( F^{(r)}_\lambda \)'s in terms of the \( h_\lambda \)'s is lower triangular with 1's on the main diagonal. Since the \( h_\lambda \)'s form an integral basis, the same is true of the \( F^{(r)}_\lambda \)'s. \( \square \)

Now that for each \( r \geq 1 \) we have this “parking function basis” \( \{ F^{(r)}_\lambda \} \), we can ask about its expansion in terms of other bases and vice versa. If we restrict ourselves to the six “standard” bases (where the power sums \( p_\lambda \) are a basis over \( \mathbb{Q} \) but not \( \mathbb{Z} \)), we thus have
twelve transition matrices to consider. We can also ask about various scalar products such as \( \langle F^{(r)}_{\lambda}, F^{(r)}_{\mu} \rangle \). Moreover, we could also consider the basis \( \{ F^{(r)}_{\lambda} \} \) dual to \( \{ F^{(r)}_{\lambda} \} \), i.e.,
\[
\langle F^{(r)}_{\lambda}, F^{(r)}_{\mu} \rangle = \delta_{\lambda \mu}.
\]
However, these dual bases will not yield any new coefficients since the dual basis to a standard basis is also a standard basis (up to a normalizing factor in the case of \( p_{\lambda} \)). We have not systematically investigated these problems. Some miscellaneous results are below.

We first consider scalar products \( \langle F^{(r,k)}_{\mu}, F^{(r,k)}_{\lambda} \rangle \). We can give an explicit formula when \( \mu = (n) \). In fact, we can give a more general result where \( F^{(r,k)}_{\lambda} \) is replaced with a “mixed” product.

**Theorem 5.2.** Let \( \lambda \vdash n \), and let \( r, r_1, r_2, \ldots \) be positive integers. Let \( k, k_1, k_2, \ldots \) be integers or even indeterminates. Then
\[
\left\langle F^{(r,k)}_n, \prod_i F^{r_i,k_i}_{\lambda_i} \right\rangle = \frac{k}{rn + k} \prod_{i \geq 1} \frac{k_i}{r_i \lambda_i + k_i} \binom{(rn + k)(r_i \lambda_i + k_i) + \lambda_i - 1}{\lambda_i}.
\]

**First proof.** If \( \lambda = (\lambda_1, \lambda_2, \ldots) \) then write \([t^{\lambda}] \) for the operator that takes the coefficient of \( t^{\lambda_1} t^{\lambda_2} \cdot \cdot \cdot \). By equation (2.4) we have
\[
\left\langle F^{(r,k)}_n, \prod_i F^{r_i,k_i}_{\lambda_i} \right\rangle = \frac{k}{rn + k} \prod_{i} \frac{k_i}{\lambda_i + k_i} \langle [t^{\lambda}] (H(1)^{rn+k}, H(t_1)^{r_1 \lambda_1+k_1} H(t_2)^{r_2 \lambda_2+k_2} \cdot \cdot \cdot) \rangle.
\]
Writing \( H(u)^b = \prod (1 - x_i u)^{b_i} \), taking logarithms, expanding in terms of the power sums \( p_k \), and then exponentiating, we get the well-known result
\[
H(u)^b = \sum_{\mu} z_{\mu}^{-1} b^{\ell(\mu)} p_{\mu} u^{1|\mu|},
\]
where \( \mu \) ranges over all partitions of all integers \( j \geq 0 \). (For the case \( b = 1 \), see [6, (7.22)].)

Since \( \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda \mu} \), we get
\[
\left\langle F^{(r,k)}_n, \prod_i F^{r_i,k_i}_{\lambda_i} \right\rangle = \frac{k}{rn + k} \prod_{i} \frac{k_i}{r_i \lambda_i + k_i} \langle [t^{\lambda}] \left( \sum_{\nu \vdash n} z_{\mu}^{-1} (rn + k)^{\ell(\mu)} p_{\mu} \prod_{i \geq 1} \left( \sum_{\nu \vdash \lambda_i} z_{\nu}^{-1} (r_i \lambda_i + k_i) \right)^{\ell(\nu)} \right) \rangle.
\]
(5.1)
\[
\sum_{\nu \vdash m} z_{\nu}^{-1} u^{\ell(\nu)} = \binom{u + m - 1}{m}.
\]
Hence the proof follows immediately from equation (5.1).
Second proof. From equation (2.4) we see that

\[
\langle F^{(r,k)} n, \prod_i F_{\lambda_i}^{r_i,k_i} \rangle = \frac{k}{rn+k} \left( \prod_{i \geq 1} \frac{k_i}{r_i \lambda_i + k_i} \right)
\]

\[
\cdot \left\langle \sum_{a_1 + \ldots + a_{rn+k} = n} h_{a_1} \cdots h_{a_{rn+k}}, \prod_i \left( \sum_{b_{i,1} + \ldots + b_{i,r_i n+k_i} = \lambda_i} h_{b_{i,1}} \cdots h_{b_{i,r_i n+k_i}} \right) \right\rangle,
\]

where \(a_i, b_{i,j} \geq 0\). Let

\[
Z = \frac{k}{rn+k} \prod_{i \geq 1} \frac{k_i}{r_i \lambda_i + k_i}.
\]

Now \(\langle h_\lambda, h_\mu \rangle\) is equal to the number of matrices \((a_{ij})_{i,j \geq 1}\) of nonnegative integers with row sum vector \(\lambda\) and column sum vector \(\mu\) \([6, (7.31)]\). Hence \(\frac{1}{Z} \left\langle F^{(r,k)} n, \prod_i F_{\lambda_i}^{r_i,k_i} \right\rangle\) is equal to the total number of \((rn+j) \times (\sum_i (r_i + n + k_i))\) matrices of nonnegative integers whose entries sum to \(n\), such that the first \(r_1 \lambda_1 + k_1\) columns sum to \(\lambda_1\), the next \(r_2 \lambda_2 + k_2\) columns sum to \(\lambda_2\), etc. Since \(\sum \lambda_i = n\), if the conditions on the columns is satisfied then the entries will automatically sum to \(n\). By elementary and well-known reasoning, the number of ways to write \(\lambda_i\) as an ordered sum of \((rn+k)(r_i n+k_i)\) nonnegative integers is \(\binom{(rn+k)(r_\lambda+\lambda_k)+\lambda_i-1}{\lambda_i}\), and the proof follows.

We now consider the expansion of the symmetric functions \(p_\lambda, h_\lambda,\) and \(e_\lambda\) in terms of the basis \(F^{(r)}_n\) (for fixed \(r\), which we may even regard as an indeterminate).

**Proposition 5.3.** For \(n \geq 1\) we have

\[
F^{(r,-rn-1)}\Big|_n = (-1)^n (rn + 1) e_n,
\]

\[
F^{(r,-rn)}\Big|_n = -rp_n,
\]

\[
F^{(r,-rn+1)}\Big|_n = (1 - rn) h_n.
\]

**Proof.** Putting \(k = -rn - 1\) in equation (2.3) gives \((-1)^n (rn + 1) \sum_{\lambda \vdash n} z_\lambda^{-1} (-1)^{n-\ell(\lambda)} p_\lambda\). It is well-known that this sum is just \(e_n\), and the proof of the first equation follows. (We could also substitute \(k = -rn - 1\) in equation (2.2) and simplify.) The other two equations are similar. \(\Box\)
Now by Proposition 5.3 we have (writing $d_i = d_i(\lambda)$)

\((-1)^n(rn + 1)e_n = F_n^{(r, -rn - 1)}\)

\[= \left[\ell^n\right] \left(\sum_{i \geq 0} F_i^{(r)} t^i\right)^{-rn - 1}\]

\[= \left[\ell^n\right] \sum_{j \geq 0} (-1)^j \binom{rn + j}{j} \left(\sum_{i \geq 1} F_i^{(r)} t^i\right)^j\]

\[= \sum_{a_1 + \cdots + a_j = n} (-1)^j \binom{rn + j}{j} F_{a_1}^{(r)} \cdots F_{a_j}^{(r)}\]

\[= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \binom{rn + \ell(\lambda) - 1}{d_1, d_2, \ldots, rn - 1} F_{\lambda}^{(r)}\]

where the penultimate sum is over all $2^{n-1}$ compositions of $n$. We have therefore expressed $e_n$ as a linear combination of $F_{\lambda}^{(r)}$’s. In exactly the same way we obtain

\[-rp_n = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \binom{rn + \ell(\lambda) - 1}{d_1, d_2, \ldots, rn - 1} F_{\lambda}^{(r)}\]

\[-(rn - 1)h_n = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \binom{rn + \ell(\lambda) - 2}{d_1, d_2, \ldots, rn - 2} F_{\lambda}^{(r)}\]

(For $r = n = 1$, the last equation becomes $0 = 0$, but it is clear that $h_1 = F_{1}^{(r)}$. ) Since \{${e}_\mu$, ${p}_\mu$, ${h}_\mu$\} and \{${F}_{\lambda}^{(r)}$\} are multiplicative bases, we have in principle expressed each $e_\mu$, $p_\mu$, and $h_\mu$ as a linear combination of $F_{\lambda}^{(r)}$’s. We leave open, however, whether there is some more elegant form of these expansions, e.g., a simple combinatorial interpretation of the coefficients.

Similarly, since Theorem 2.1 in the case $k = 1$ gives the expansion of $F_n^{(r)}$ in terms of the multiplicative bases $p_\mu$, $h_\mu$, and $e_\mu$, we in principle also have an expansion of $F_{\lambda}^{(r)}$ in terms of these bases, but perhaps a better description is available. We cannot expect a simple product formula for the coefficients in general since for instance the coefficient of $p_3 p_6$ in the power sum expansion of $F_{(3,2,1,1,1)}^{(1)}$ is equal to $2 \cdot 7 \cdot 157/3$.

References

[1] F. Bergeron, Combinatorics of r-Dyck paths, r-parking functions, and the r-Tamari lattices, arXiv:1202.6269.
[2] A. G. Konheim and B. Weiss, An occupancy discipline and applications, SIAM J. Applied Math. 14 (1966), 1266–1274.
[3] J. Riordan, Ballots and trees, J. Combinatorial Theory 4 (1969), 408–411.
[4] R. P. Stanley, Hyperplane arrangements, parking functions, and tree inversions, in Mathematical Essays in Honor of Gian-Carlo Rota (B. Sagan and R. Stanley, eds.), Birkhäuser, Boston/Basel/Berlin, 1998, pp. 359–375.
[5] R. P. Stanley, Enumerative Combinatorics, vol. 1, second ed., Cambridge University Press, 2012.
[6] R. P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
[7] G. P. Steck, The Smirnov two sample tests as rank tests, Ann. Math. Statist. 40 (1969), 1449–1466.
[8] C. H. Yan, Parking functions, in *Handbook of Enumerative Combinatorics* (M. Bóna, ed.), CRC Press, Boca Raton, FL, 2015, pp. 835–893.

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