Variation of GIT and Variation of Lagrangian skeletons.

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based on

Peng Zhou. Variation of GIT and variation of Lagrangian skeletons I: flip and flop. arXiv preprint arXiv:2011.03719, 2020

Jesse Huang and Peng Zhou. Variation of GIT and variation of Lagrangian skeletons II: Quasi-symmetric case. arXiv preprint arXiv:2011.06114, 2020
Outline

§0. Background on toric mirror symmetry.
§1. Motivations
§2. VGIT and Window subcategories
§3. VLag and Window subskeletons.
cheat sheet for constructible sheaves on $\mathbb{R}^2$

(1) $\Lambda = \begin{array}{c}
\text{half conormal to the line } \delta x_1 = 0^3 \\
+ \text{zero section}
\end{array}
\left\{ (x_1, x_2, s_1, s_2) \in T^* \mathbb{R}^2 \mid \begin{array}{l}
\delta x_1 = 0, \\
\delta s_2 = 0, \\
\delta s_1 > 0, \\
\delta x_2 \text{ free}
\end{array} \right\}$

$\text{Sh}(\cdot \leftarrow \cdot) \cong \text{Rep}(\cdot \leftarrow \cdot)$

(2) $\Lambda = \begin{array}{c}
\text{(zero section)} \\
+ \text{half conormal to } \delta x_1 = 0^3 \\
+ \text{half conormal to } \delta x_2 = 0^3 \\
+ \text{quadrant conormal to } \delta x_1 = 0, \delta x_2 = 0^3
\end{array}
\left\{ (0, 0 ; \delta s_1 > 0, \delta s_2 \geq 0)^3 \right\}$

$\text{Sh}(\Lambda) = \text{Rep}(\cdot \downarrow \cdot \downarrow \cdot)
\leftarrow
\begin{array}{cccc}
B \\
A \\
C \\
D
\end{array}$

(3) $\Lambda = \begin{array}{c}
\text{sh}(\Lambda) = \text{Rep}(\cdot \downarrow \cdot \downarrow \cdot)
\end{array}$

Cartesian diagram.
Toric Coherent Constructible Correspondence (CCC)

- $N \cong \mathbb{Z}^d$, $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^d$, $N_{\mathbb{C}^*} \cong N \otimes \mathbb{C}^*$, $T = \mathbb{R} / \mathbb{Z} = S'$, $M = \text{Hom}(N, \mathbb{Z})$, $M_T = M \otimes \mathbb{T} = (S')^d$

- $\Sigma \subset N_{\mathbb{R}}$: smooth toric fan. (cone are simplicial)

- $X_\Sigma$: a smooth toric DM stack.

- $\Lambda \subset T^*(S')^d = T^*M_T$: FLTZ Lagrangian skeleton.

- $\tilde{\Lambda} \subset T^*\mathbb{R}^d = T^*M_{\mathbb{R}}$: FLTZ equivariant skeleton.

Theorem (Bondal, Fang-Liu-Treumann-Zaslow, ----, Kuwagaki)

We have equivalence of categories:

- non-equiv CCC: $\text{Coh}(X_\Sigma) \cong \text{Sh}^w(M_T, \Lambda \Sigma)$

- equivariant CCC: $\text{Coh}(\mathbb{C}^*)^d(X_\Sigma) \cong \text{Sh}(M_{\mathbb{R}}, \tilde{\Lambda}_{\Sigma})$
\[ (1) \quad X_\Sigma = \mathbb{C}^*, \quad \Lambda_\Sigma = S' \subset T^*S' \]

\[ \text{Coh}(\mathbb{C}^*) \cong Sh^w( S', \Lambda_\Sigma) \cong \text{Loc}(S') \]

\[ \forall \lambda \in \mathbb{C}^* : \quad \mathcal{O}_{\xi \lambda^2} \longmapsto \text{rank } 1 \text{ local system with monodromy } \lambda. \]

\[ \mathcal{O}_{\mathbb{C}^*} \longmapsto \pi_* \mathcal{C}_\mathbb{R} \]

\[ \text{an infinite rank locally constant sheaf.} \]

\[ (\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}) \]

\[ (2) \quad X_\Sigma = \mathbb{C} \quad , \quad \Lambda_\Sigma = \]

\[ \text{Coh}(\mathbb{C}) \longmapsto Sh^w( S', \Lambda_\Sigma) \]

\[ 0_{\xi \lambda^2} \longmapsto \pi_* \mathcal{C}_{[-1,0]} [1] \]

\[ 0_{\mathbb{C}} \longmapsto \pi_* \mathcal{C}_{(-\infty,0]} [1] \]
(3) \( X = \mathbb{P}^1, \) 

\[ \Lambda = \begin{array}{c}
\end{array}, \hspace{1cm} \Sigma = \begin{array}{c}
\end{array} \]

\[ D\xi_0 = \begin{array}{c}
\end{array}, \hspace{1cm} D\xi_1 = \begin{array}{c}
\end{array} \]

\[ D\xi_{0,1} = \begin{array}{c}
\end{array}, \hspace{1cm} D\xi_{-1,0} = \begin{array}{c}
\end{array} \]

\[ \pi: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \]
Microlocal Sheaf Theory and Fukaya Category.

- Microlocal sheaf theory can be used to compute Fukaya category of a Weinstein mfd (e.g. cotangent bundle).

**Thm (Kontsevich, Nadler, Zaslow, Ganatra-Pardon-Shende)**

(a) Let $(M,\omega = dx)$ be a Weinstein domain, let $\Lambda = \text{Skel}(M,\omega)$ be the Liouville skeleton of $M$. Then

$$\text{msh}^\omega(\Lambda) \simeq \text{Fuk}^\omega(M)$$

$$\text{Hom}(L_1, L_2) := \text{Hom}(\text{R}^0 L_1, L_2)$$

(b) If $M$ is a Weinstein domain, $H \subset \partial M$ is a Weinstein hypersurface, $\Lambda = \text{skel}(M) \cup \{R_{>0} \cdot \text{skel}(H)\}$, then

$$\text{msh}^\omega(\Lambda) \simeq \text{Fuk}^\omega(M, \text{stop} = H)$$
Toric Homological Mirror Symmetry

- Using previous "microlocal sheaf ↔ Lagrangian" correspondence,
  
  \[ \text{Coh}(X_\Sigma) \cong \text{Sh}^w(\mathcal{M}_T, \Lambda_\Sigma) \cong \text{Fuk}^w(T^*\mathcal{M}_T, \text{stop} = \Lambda^\infty_\Sigma) \]

- The traditional toric HMS mirror uses LG A-model.

  \[ \text{Coh}(X_\Sigma) \cong \text{FS}(\mathbb{C}^n, W_\Sigma). \]

  \[ \text{e.g.,} \quad \text{Coh}(\mathbb{P}^n) \cong \text{FS}(\mathbb{C}^n, W = z_1 + \ldots + z_n + \frac{1}{z_1 \cdots z_n}) \]

\[ \textbf{Thm} \] (Ruddat-Sibilia-Treuman-Zaslow, Gammage-Shende, Zhou)

Let \( X_\Sigma \) be a smooth toric Fano, \( W_\Sigma : (\mathbb{C}^*)^d \to \mathbb{C}, \ \Lambda_\Sigma \subset T^*(\mathbb{C}^*)^d \), then

\[ \Lambda^\infty_\Sigma = \text{skel}(W_\Sigma^{-1}(R)) \] for \( R > 0 \).

\[ \mathbb{P}^2, \quad W = x + y + \frac{1}{xy}, \quad \Lambda^\infty_\Sigma = \text{skel} \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \]
§1 Motivations.

(1.1) Let $A \subset \mathbb{Z}^d$, $\Delta = \text{Convex Hull} (A) \subset \mathbb{R}^d$, and let

$$W = \sum_{\alpha \in A} C_\alpha \cdot Z^\alpha$$

be a Laurent polynomial with generic coefficients $C_\alpha \in \mathbb{C}^*$. Then, $Fuk^w ((\mathbb{C}^*)^d, W)$ only depends on $\Delta$.

- One can choose "tropicalization" for $W$,

$$W = \sum_{\alpha \in A} e^{i\Theta_\alpha} \cdot R^{h_\alpha} \cdot Z^\alpha \quad \Theta_\alpha \in \mathbb{R}/2\pi \mathbb{Z} \quad h_\alpha \in \mathbb{R} \quad R \to \infty$$

Then different choices results in different $\Lambda^\infty_w = \text{skel} (W^{-1}(+\infty))$, however

$$\text{Sh}^w ((S^1)^d, \Lambda_w) \quad \Lambda_w = (S^1)^d \cup \mathbb{R}_0 \cdot \Lambda^\infty_w$$

should be invariant. How?
Ex: (a). \( W = z + \frac{e^{i\theta}}{z} \) (c.f. Hanlon's thesis)

\[ W^{-1}(R) = \{ z \in R \; \exists \; u \frac{e^{i\theta}}{z} \in R \; \exists \; \theta_z = 0 \} \]

\[ \theta_z = \Theta \]

\( \Lambda W = \)

\[ T^*S^1: \omega = d\theta \wedge d\overline{z} \]

\[ C^* : \omega = \frac{i}{2} \frac{dz \wedge d\overline{z}}{|z|^2} = d\rho \wedge d\theta \]

\[ z = e^{\rho + i\theta} \]

\[ z = R \]

\[ z = \infty \]

\[ -\log|z| = -\rho \]

\[ \arg(z) \]

(b) \( W = y(x^r + x) \)

\[ \Delta = \]

\[ W^{-1}(R) = \Lambda_w \]

\[ W = y(x^r + R + x) \]

\[ \Delta = \]

\[ W^{-1}(R) = \Lambda_w \]

\[ \Lambda_w = \]
In the comparison of $\text{FS}(\mathbb{C}^n, W)$ and $\text{Sh}(T^n, \Lambda)$, one natural question is, where does Lagrangian thimbles go?

Ex: mirror for $\mathbb{P}^1$:

$W: \mathbb{C}^* \rightarrow \mathbb{C}, \quad W(z) = z + \frac{1}{z}$.

In general, the critical values of $W$ has no pattern, and choice of vanishing path is very ad hoc (straight line paths is complicated).
Semi-orthogonal decomposition

- If we tropicalize $W$ (in a generic way), then $\text{Crit}(W)$ comes in circle clusters.

$$\text{Fuk}(\mathbb{A}^4, W) = \langle A^{(n)}, \ldots, A^{(2)}, A^{(1)} \rangle$$

- On the mirror side, we have birational transformation.

$$X = X^{(n)} \to X^{(n-1)} \to \cdots \to X^1$$

$$\text{Coh}(X^{(k)}) = \langle B^{(k)}, \text{Coh}(X^{(k-1)}) \rangle \quad \forall k=1, \ldots, n.$$ 

- $B^{(k)}$ is supported on the exceptional loci.

- The loci in $X^{(k)} \to X^{(k-1)}$.

- Conjugation: $\text{Coh}(X^{(k)}) \cong \langle A^{(k)}, \ldots, A^{(1)} \rangle$

- One trouble that this is still a conj is that Fukaya cat is hard.
The bridge between $A$ and $B$ sides is $\text{Sh}(T^d, \Lambda)$.

- Let $\Lambda^{(k)} \subset T^*M_T$ such that $\text{Coh}(X^{(k)}) \cong \text{Sh}(M_T, \Lambda^{(k)})$.

Then, if we can "measure the difference" between $\Lambda^{(k-1)}$ and $\Lambda^{(k)}$, we can see where the thimbles in SOD component $A^{(k)}$ go.

\[ (1.2 \text{ continue}) \]

**Example:**

\[ W = x + y + \frac{1}{xy} + t \cdot xy \]

- $\text{Crit}(W) = \bullet \circ$

**Triangulation of Newton Polytope**

\[ W, \Delta_W \]

**Tropical hyper surfaces:**

\[ t^* = x + y + \frac{1}{xy} + t \cdot xy, \ (k \to 0) \]

\[ \text{Bl}_c\mathbb{P}^2 \]

\[ X = \text{Bl}_c\mathbb{P}^2 \]

- $\phi \to X^{(c)} \rightarrow X^{(c)}$

\[ \phi \to \Delta X^{(0)} \rightarrow \Delta X^{(0)} \quad \text{moment polytope} \]

\[ \phi \to \Lambda^{(c), 00} \rightarrow \Lambda^{(c), 00} \]

\[ \phi \to \text{Bl}_c\mathbb{P}^2 \rightarrow \text{Bl}_c\mathbb{P}^2 \]
Goal:

We want to find a family of Lagrangian skeletons
\( \{ \Lambda_t \}_{t \in [0,1]} \), interpolating \( \Lambda_0 \) and \( \Lambda_1 \).

- If \( \text{Sh}(\Lambda_0) \approx \text{Sh}(\Lambda_1) \), we want \( \Lambda_t \) variation to
  be "non-characteristic", i.e. \( \text{Sh}(\Lambda_t) \) remain constant
  thimbles.

- If \( \text{Sh}(\Lambda_1) \approx \langle \mathcal{T}, \text{Sh}(\Lambda_0) \rangle \)
  SOD,
  we want to have critical moments,

\[
\begin{array}{cccccc}
\Lambda_0 & x & x & x & x & \Lambda_1 \\
0 & t_1 & t_2 & \cdots & t_n & 1 \\n\end{array}
\]

\( \text{Sh}(\Lambda_t) \) constant over \( (t_i, t_{i+1}) \)
Ex: (1) \[
\begin{align*}
\Lambda_- & \quad \Lambda_? \quad \Lambda_+
\end{align*}
\]

(2) \[
\begin{align*}
\text{Coh}(\mathbb{C}/\mathbb{Z}_2) & \sim \text{Coh}(T^*P') \\
\mathbb{C}^2 & \longrightarrow \text{Blo } \mathbb{C}^2 \\
\Lambda & \rightarrow \Lambda
\end{align*}
\]

\[
\begin{align*}
\text{Coh}(\mathbb{C}^2) & \leftrightarrow \text{Coh}(\text{Blo } \mathbb{C}^2) \\
\text{Sh}^w(\Lambda_0) & \leftrightarrow \text{Sh}^w(\Lambda_i)
\end{align*}
\]
\section{Variation of GIT and Window subcategory}

(Herbst - Hori - Page, E. Segal, Halpern-Leistner, Ballard-Favero-Katzarkov)

\subsection{Idea}

When we study transitions between toric varieties $X_-$ and $X_+$ (e.g. $X_- = \mathbb{C}^2$, $X_+ = \text{Bl}_0 \mathbb{C}^2$), they often come from different "phases" of GIT quotients.

\[ X_\pm = \left[ \tilde{X}_\pm / \mathbb{C}^* \right] = \left[ \tilde{X}_- \sim \tilde{X}_{\theta \pm}^u / \mathbb{C}^* \right] \xrightarrow{l_\pm} \left[ \tilde{X} / \mathbb{C}^* \right] \]

\[ \text{Coh} \left( [ \tilde{X} / \mathbb{C}^* ] \right) = \text{Coh}_{\mathbb{C}^*}(\tilde{X}) \]

\[ \text{Coh}(X_+) \quad \text{Coh}(X_-) \]

\[ \text{Coh}(X_\pm) = \text{Coh} \left( [ \tilde{X} / \mathbb{C}^* ] \right) \left/ \langle \text{sheaves supported on unstable loci } \tilde{X}_{\pm}^u \rangle \right. \]
Def: A window subcategory for a GIT quotient $X_\pm$ is a subcategory $W_\pm \subset \text{Coh}([X/\mathbb{C}^*])$, such that

$$l_i^*|_{W_i} : W_i \sim \text{Coh}(X_i) \quad i = +, -$$

is an equivalence.

Rmk:
- We can compare $\text{Coh}(X_\pm)$ via comparing $W_\pm$ in $\text{Coh}([\tilde{X}/\mathbb{C}^*])$ now.

- Choices of $W_\pm$ are far from unique.

- Window subcategories exist for general GIT quotients by algebraic group $[\tilde{X}/G] \rightarrow [X/G]$. [BFK, HL]
Example:

1. \( \mathbb{C}^3 / \mathbb{C}^* \), \( \mathbb{C}^* \hookrightarrow \mathbb{C}^3 \) with weight \((1,1,-1)\).

\[
X_+ = \left[ \mathbb{C}^3 - \left\{ (0,0,z) \big| z \in \mathbb{C}^3 \big/ \mathbb{C}^* \right\} \right] = \text{Bl}_{\mathbb{Z}_+} \mathbb{C}^2 = \text{Tot} \left[ \mathcal{O}_{\mathbb{P}^1}(1) \right]
\]

\[
X_- = \left[ \mathbb{C}^3 - \left\{ (z_1,z_2,0) \big| z \in \mathbb{C}^3 \big/ \mathbb{C}^* \right\} \right] \cong \mathbb{C}^2
\]
Coherent sheaves on $Z_+ = \{ (0, 0, z) \}$ is generated by $O_{Z_+}$.

$$0 \rightarrow O_{\Delta^3}(-Z_2, Z_1) \rightarrow O_{\Delta^3}^2(Z_1, Z_2) \rightarrow O_{\Delta^3}^{303} \rightarrow O_{Z_+}^{303} \rightarrow 0 \quad (\text{Koszul resolution})$$

$\Delta^3$ means $C^*$ equivariant degree.

Thus, when restricted to $X_+$, $O_{Z_+}$ become 0, hence we have exact seq.

$$0 \rightarrow O^{Z-2}\Delta \rightarrow O^{Z-1}\Delta \oplus O \rightarrow O_{Z_+} \rightarrow 0$$

$\Rightarrow \{ \cdot O_{k3} \}$ can be expressed using $O_{9k-13}$ and $O_{9k-23}$

$\Rightarrow \{ \cdot O_{k23} \}$ can be expressed using $O_{9k-13}$ and $O_{9k-23}$.

$\Rightarrow \text{Coh}(C^3-Z_+ / C^*)$ can be generated by

$l^+_t \langle O_{k3}, O_{k+13} \rangle$

$\Rightarrow \forall k \in \mathbb{Z}$, we can choose $W_+ = \langle O_{k3}, O_{k+13} \rangle$

still need to prove $l^+_t |_{W_+}$ is fully-faithful
Coherent sheaves on $Z_\ell = \{(z_1, z_2, 0)\}$ is generated by $O_{Z_\ell}$

$$0 \to O_{\mathbb{C}^3} \xrightarrow{z_3} O_{\mathbb{C}^3} \xrightarrow{z_3^{\otimes 3}} O_{Z_\ell} \xrightarrow{z_3^{\otimes 3}} 0$$

Same arguments shows $\forall k \in \mathbb{Z}$, we can choose $W_- = \langle O_{\mathbb{C}^3} \rangle \subset \text{Coh}(\mathbb{C}^3/\mathbb{C}^*)$.

$W_+$: as lattice points in an interval of length $d_+ = 2$

$W_- = \{0\}$

$d_- = 1$

$\text{SOD} : \quad \text{Coh}(\text{Bl}_0 \mathbb{C}^3) \cong \langle O_{\mathbb{C}^3}(-1), \pi^* \text{Coh} \mathbb{C}^3 \rangle$

$l : \text{Coh}(\mathbb{C}^3) \xrightarrow{\sim} W_- \hookrightarrow W_+ \cong \text{Coh}(\text{Bl}_0 \mathbb{C}^3)$.
§2.3 Magic windows  (HL-Sam, Spenko-van den Bergh)

How about general \([\mathbb{C}^N/\mathbb{C}^*]^k\) ?

- any smooth projective toric variety arises from Cox construct (GIT quotient)

- If \((\mathbb{C}^*)^k \subset \mathbb{C}^N\) preserves \(d\zeta_1 \cdots d\zeta_N\), then different smooth quotients are all toric CY, and derived equivalent. (in a non-canonical way).

Q: Can we find a universal window

\[
W = \langle \text{some line} \rangle \subset \text{Coh}[\mathbb{C}^N/(\mathbb{C}^*)^k]
\]

such that,

\[
l^*_\Theta : W \xrightarrow{\sim} \text{Coh} \left[ \mathbb{C}^N/_{\Theta}(\mathbb{C}^*)^k \right]
\]

for all GIT param \(\Theta\) in stable chambers?  

(in general, I cannot)
One can achieve this under stronger assumption than toric CY,

**Def:** (quasi-symmetric condition)

Let $\beta_1, \ldots, \beta_N \in \mathbb{Z}^k$ denote the collection of weights for $(\mathbb{C}^*)^k \hookrightarrow \mathbb{C}^n$. If for any line (passing through 0) $L \subset \mathbb{R}^k$, the sum of weights on $L$ is zero, then we say $(\mathbb{C}^*)^k \hookrightarrow \mathbb{C}^n$ is quasi-symmetric.

**Ex:** • $k = 1$, toric CY $\iff$ quasi-symmetric.

• If $\{\beta_1, \ldots, \beta_N\}$ is invariant under $(-1): \mathbb{Z}^k \rightarrow \mathbb{Z}^k$, i.e. symmetric under inversion, then it is quasi-symmetric.
\cdot \text{Let } \Delta = \frac{1}{2} \sum_{i=1}^{n} [0, \beta_i] \quad \text{Minkowski sum of line segments.}

\cdot \text{for any generic } \eta \in \mathbb{R}^k, \text{ we have lattice points.}

\quad A_\eta = (\eta + \Delta) \cap \mathbb{Z}^k

\text{and corresponding windows}

\quad W_\eta = \left\langle \bigoplus \mathcal{O}_{\{d \in \eta \}} \right\rangle \subset \text{Coh } \mathbb{C}^N / \mathbb{C}^*^k

\text{Thm (HL-Sam, Svåb)}

\text{For any stable GIT parameter } \Theta \in \mathbb{R}^k, \text{ any window param } \eta, \text{ we have equivalence}

\quad i^*_\Theta : W_\eta \xrightarrow{\sim} \text{Coh } \mathbb{C}^N / \Theta \mathbb{C}^*^k
§3 Variation of Lagrangian Skeleton and windows

§3.1 General $\mathcal{VLag}$:

1. Some variations of Lagrangians induces equivalences of categories.

Ex: $\land \subset T^*\mathbb{R}$

- equivalence from invariance of its Weinstein tubular nbhd

- constructible sheaves in $\text{Sh}(\mathbb{R}, \land)$ deform along:

\[
\begin{array}{ccc}
C_{[-1,1]} & \longrightarrow & C_{\text{fov}} \\
\downarrow & & \downarrow \\
C_{(-1,1)} & \longrightarrow & C_{(-1,1)}[1]
\end{array}
\]
(2) Some Vlag are not equivalences:

\[ \Lambda \subset T^* \mathbb{R}^2 \]

front projection

the new Reeb chords ending on \( \Lambda \) causes trouble.

(3). In general, given a family of skeletons \( \{ \Lambda_b \}_{b \in B} \), we can construct a universal skeleton \( \Lambda_b \subset T^*(M \times B) \),

Q: when is restriction

\[ l_b^*: Sh(M \times B, \Lambda_B) \rightarrow Sh(M, \Lambda_b) \]

an equivalence of category?
\[\Lambda B = \downarrow \quad \Lambda_{t_1} \quad \Lambda_{t_2} = \downarrow \quad \Lambda_{t} = \downarrow \quad \nabla T^* \mathbb{R} \]

\(\text{Sh} (\mathbb{R}, \Lambda B) \rightarrow \text{Sh} (\mathbb{R}, \Lambda_{t})\) is not an equivalence

\(C_{[0,1]}\) cannot be produced.

\(\text{Sh} (\Lambda_{-}) \leftarrow \text{Sh} (\Lambda_{B}) \sim \rightarrow \text{Sh} (\Lambda_{t})\)
§3.2 Window subskeleton from window subcategory.

Given a window

- \textbf{idea:} \quad W = \langle L_1, \ldots, L_m \rangle \rightarrow \text{Coh} \left[ \mathbb{C}^n / (\mathbb{C}^*)^k \right], \quad L_i = O \{ d_i \} \\

\tau \downarrow \quad \tau(W) = \langle F_1, \ldots, F_m \rangle \rightarrow \text{Sh} \left( \mathbb{R}^k \times T^{n-k}, \Lambda_{\text{full}} \right) \quad \alpha ; \in \mathbb{Z}^k.

- Define \( \Lambda_W := \bigcup_{i=1}^{m} \text{SS}(F_i) \subset \Lambda_{\text{full}} \)

- Let \( \Pi: \mathbb{R}^k \times T^{n-k} \rightarrow \mathbb{R}^k \), for any \( b \in \mathbb{R}^k \), let \( \Lambda_{W,b} \subset T^* \mathbb{R}^k \) be the restriction of \( \Lambda_W \).

- For \( b \) deep in the GIT chamber \( C \subset \mathbb{R}^k \), \( \Lambda_{W,b} \cong \Lambda_C^{T^* T^{n-k}}, \Lambda_C \) mirror to \( \left[ \mathbb{C}^n / C_{(\mathbb{C}^*)^k} \right] \)

\[
\begin{align*}
\tau(W) & \rightarrow \text{Sh}(\Lambda_W) \\
\tau(W) & \rightarrow \text{Sh}(\Lambda_W) \\
\tau(W) & \rightarrow \text{Sh}(\Lambda_W) \\
\tau(W) & \rightarrow \text{Sh}(\Lambda_W) \\
\end{align*}
\]

\[
\begin{align*}
k = 1 & \quad \text{C-} \quad \text{window} \quad \text{C+} \\
R = 2 & \quad \text{C2} \quad \text{C} \\
\end{align*}
\]
Thm A. Let $C^* \otimes C^N$ with weight $(a_1, \ldots, a_N)$, s.t. $a_i$ coprime, nonzero. Let $d_+ = \sum_{a_i > 0} a_i$.

Assume $d_+ > d_-$. Then for any $k \in \mathbb{Z}$, define

$$W := \langle 0 \otimes k, \ldots, 0 \otimes k + d_+ - 1 \rangle \subset \text{Coh } C^* (C^N)$$

(1) $W \simeq \text{sh} (R \times T^{N-1}, \Lambda w)$

(2) $\text{sh} (T^{N-1}, \Lambda w, t)$ is locally constant as $t$ vary in $\mathbb{R}$ except at $t \in \{ k, k+1, \ldots, k+\eta-1 \}$ \(\eta\) many.

\(\forall t < 0\) $\text{sh} (T^{N-1}, \Lambda w, t) \simeq \text{Coh } (\mathbb{C}^N / \mathbb{C}^* T)$

\(\forall t > 0\) $\text{sh} (T^{N-1}, \Lambda w, t) \simeq \text{Coh } (\mathbb{C}^N / t \mathbb{C}^*)$. 
Ex: (1) $\mathbb{C}^* \cong \mathbb{C} \times \mathbb{C}$ with weight $(1,1)$.

$X_0 = \phi, \quad X_+ = \mathbb{P}^1$

$W = \langle 0.503, 0.813 \rangle$

$W = \mathbb{C} \cdot t \cdot \frac{e^{i\Theta}}{z}$

$\Lambda_t = \frac{z}{x}$

$\Lambda_{\mathbb{P}^1}$

$\mathbb{R} \times S^1 = \mathbb{R} \times [0,1] / \sim$

no extra "hair" on the point

quadrat hairs.
**Example 1.3.** Consider $C^*$ acting on $\mathbb{C}^2$ with weight $(3, -1)$. The window skeleton is shown as below, living over $S^1 \times \mathbb{R}$ (drawn as $\mathbb{R} \times [0, 1]$ with top and bottom edge identified).

The window skeleton is the union of three skeleton $\Lambda(0), \Lambda(1), \Lambda(2)$, whose vertices are marked in black nodes. The window region is marked in shadow. Take a vertical slice on the right of the window region, we get the skeleton $\Lambda_+$ for $[\mathbb{C}/\mathbb{Z}_3]$; and the vertical slice on the left of the window region gives skeleton $\Lambda_-$ for $\mathbb{C}$.
Thm B [Huang–Z]. Suppose $(C^*)^k \subseteq C^N$ satisfies quasi-symmetric condition, then for any $s \in \mathbb{R}^k$, we have $B$-side window subcat $W_s \subset \text{Coh}(\mathbb{C}/(C^*)^k)$, and $A$-side window skeleton $\Lambda_s \subset T^*(\mathbb{R}^k \times T^{N-k})$, and

1) $W_s = \text{Sh}(\mathbb{R}^k \times T^{N-k}, \Lambda_s)$

2) For any $\eta$ deep in GIT chamber $C \subset \mathbb{R}^k$, $\text{Sh}(T^{N-k}, \Lambda_s, \eta) = \text{Sh}(T^{N-k}, \Lambda_C) = \text{Coh}(\mathbb{C}/C(C^*)^k)$.

3) For generic $s$, $\Lambda_s$ defines a non-characteristic $k$-parameter variation of skeleton $\frac{1}{2} \Lambda_s, \eta \in \mathbb{R}^k$. 

(a) More generally, $\pi^* (\mathfrak{sh} \wedge \delta)$ defines a sheaf of categories over $\mathbb{R}^k$, with singular support along some thickened hyperplanes.

**Example 1.10** ($N = 6$, $k = 2$). Consider the example of $(\mathbb{C}^*)^2$ acting on $\mathbb{C}^6$ with weight vectors $\beta_i$ (as column vectors) given by

$$ (\beta_1, \beta_2, \ldots, \beta_6) = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} $$

There are 6 GKZ chambers, separated by the 6 rays generated by $\beta_i$.

The stratification of the shift parameter space $\mathbb{R}_\delta^k$ (subscript is used to indicate the name of the coordinate) is shown in Figure 1. We consider three sample choices of $\delta$ as shown above, with $\delta_1$ being the most non-generic and $\delta_3$ being generic. For each $\delta$, we illustrate in Figure 2 the zonotope, window points, and the singular support of $C_\delta$. Note that in the first figure, over the vertices the zonotope, we have Lagrangian cones in the cotangent fiber, marked by the blue arcs, and over other intersections of the blue hairy lines, we don’t have anything extra in the cotangent fiber.

![Figure 1. Stratification of the shift parameter space $\mathbb{R}_\delta^k$.](image)

**Figure 1.** Stratification of the shift parameter space $\mathbb{R}_\delta^k$. 

![Figure 2.](image)

(A) $\delta = \delta_1$ 

(B) $\delta = \delta_2$ 

(C) $\delta = \delta_3$
\[ W = \sum c_i z^i \]

\[ \mathbb{Z}_{d\alpha_c} \subset \{ \text{space of } C_w \} \]

\[ \{ \text{space of } W \} \]