Nonmonotone slip problem for miscible liquids *

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Abstract. In this paper we prove the existence and uniqueness of a solution to
the nonstationary two dimensional system of equations describing miscible liquids
with nonsmooth, multivalued and nonmonotone boundary conditions of subdifferen-
tial type. We employ the regularized Galerkin method combined with results from
the theory of hemivariational inequalities.

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1 Introduction

In this paper we consider the mathematical model for two dimensional miscible liquids and provide a result on existence and uniqueness of weak solution under a nonmonotone slip boundary condition. The model is a system of partial differential equations which consists of Navier-Stokes equations with Korteweg stress terms for the velocity and pressure of the fluid coupled with the reaction-diffusion equation for the concentration of the fluid.

Miscibility is the property of substances to fully dissolve in each other at any concentration forming a homogeneous solution. This notion is mostly applied to liquids, but applies also to solids and gases. Two liquids are miscible if the molecules of the one liquid can mix freely with the molecules of the other liquid forming a uniform blend. For historical reasons the substance less abundant in the mixture is called a solute, while the most abundant one a solvent. There is no sharp interface between miscible liquids, but rather a transition zone. Examples of such phenomenon is the mixing of water and glycerin, and water and ethanol. The study of miscible liquids is motivated by problems in oil recovery, hydrology, polymer blends, groundwater pollution and filtration [1, 2, 3, 6, 15].

It was experimentally confirmed that between two miscible liquids there exists a transient capillary phenomena since the change of concentration gradients near the transition zone causes capillary forces between two liquids, see [5]. For this reason due to the concentration inhomogeneities, we need to take into account additional terms in the equation of motion. These terms introduced first in the work by Korteweg [14] represent additional volume forces in the equations of motion called now Korteweg stresses.

Results on the unique weak solvability of models describing miscible liquids can be found [1, 15] where the problem was studied in two dimensional case, in the absence of external and source forces, with no subdifferential boundary conditions, and in [2, 3] who treated the three dimensional case with the homogeneous Dirichlet boundary condition on the whole boundary. A result on existence of the global weak solution for a multiphasic incompressible fluid model with Korteweg stress can be found in [8] where the Galerkin method combined with a fixed point argument have been employed.

The remainder of the paper is as follows. In Section 2 we recall some preliminary material and the functional setup of the problem. The classical and variational formulations of a model of miscible liquids are described in Sections 3 and 4 respectively. Section 5 is devoted to the proof of Theorem 8 which is the main result of the paper on existence and uniqueness of weak solution to the model.

2 Notation and preliminaries

In this section we introduce notation and recall some preliminary material.

Let Ω be a bounded open subset of \(\mathbb{R}^2\) with boundary \(\Gamma\) of class \(C^2\) composed of two disjoint measurable parts \(\Gamma_0\) and \(\Gamma_1\), i.e., \(\overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma\) and \(\Gamma_0 \cap \Gamma_1 = \emptyset\) with
meas (\(\Gamma_0\)) > 0. Given a vector \(\xi \in \mathbb{R}^2\) on the boundary \(\Gamma\), we denote by \(\xi_\nu\) and \(\xi_\tau\) its normal and tangential components, respectively, i.e., \(\xi_\nu = \xi \cdot \nu\) and \(\xi_\tau = \xi - \xi_\nu \nu\), where \(\nu\) denotes the outward normal unit vector to the boundary. The notation \(S^2\) represents the class of second order symmetric \(2 \times 2\) tensors. The inner products and norms in \(\mathbb{R}^2\) and \(S^2\) are denoted by

\[
\langle u, v \rangle = u_i v_i, \quad \|v\| = (v \cdot v)^{1/2} \quad \text{for all } u = (u_i), v = (v_i) \in \mathbb{R}^2,
\]

\[
\sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\| = (\tau : \tau)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in S^2,
\]

respectively. We introduce the following function spaces

\[
E = \{ v \in H^1(\Omega)^2 \mid v = 0 \text{ on } \Gamma_0, \ v_\nu = 0 \text{ on } \Gamma_1, \ \text{div} \ v = 0 \},
\]

\[
H = \{ v \in L^2(\Omega)^2 \mid v = 0 \text{ on } \Gamma_0, \ v_\nu = 0 \text{ on } \Gamma_1, \ \text{div} \ v = 0 \},
\]

\[
V = \{ v \in H^2(\Omega) \mid \frac{\partial v}{\partial \nu} = 0 \}.
\]

We denote by \(X^*\) the dual space to a Banach space \(X\). The notation

\[
\text{div} \ v = \nabla \cdot v = v_{i,i}, \quad \text{Div} \ \sigma = \nabla \cdot \sigma = \sigma_{ij,j}
\]

stand for the divergence operators of the vector field \(v \in L^2(\Omega)^2\) and of the tensor field \(\sigma \in L^2(\Omega, S^d)\). An index that follows a comma indicates a derivative with respect to the corresponding component of the variable, and the summation convention over repeated indices is used. For the scalar field \(C \in H^1(\Omega)\), its gradient is denoted by \(\nabla C = (C_1, C_2)\) and if \(C \in H^2(\Omega)\) its conormal derivative is defined by

\[
\frac{\partial C}{\partial \nu} = \nabla C \cdot \nu.
\]

Recall, see [18, Theorem 2.15], that the embedding \(i : E \rightarrow H^{1-\delta}(\Omega)^2\) is compact for \(\delta \in (0, \frac{1}{2})\). By \(\gamma_1 : H^{1-\delta}(\Omega)^2 \rightarrow L^2(\Gamma)^2\), we denote the trace operator, which is known to be continuous, see [18, Theorem 2.21]. Hence, the trace operator \(\gamma = \gamma_1 i : E \rightarrow L^2(\Gamma)^2\) is compact. In what follows, the norm of \(\gamma\) in \(L(E, L^2(\Gamma)^2)\) (the space of linear and bounded operators from \(E\) into \(L^2(\Gamma)^2\)) is denoted by \(\|\gamma\|\), and instead of \(\gamma v\) we often write simply \(v\). We will also use the following special case of the Gagliardo–Nirenberg interpolation inequality, proof of which can be found in [10, Theorem 10.1].

**Lemma 1.** If \(\Omega \subset \mathbb{R}^2\) is a domain with \(C^1\) boundary, then there exists a constant \(M > 0\) such that

\[
\|u\|_{L^4(\Omega)} \leq M \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \quad \text{for all } u \in H^1(\Omega).
\]

For a finite number \(T > 0\), we introduce the Bochner-Lebesque spaces

\[
\mathbb{E} = \{ v \in L^2(0, T; E) \mid v' \in L^2(0, T; E^*) \}
\]
and
\[ W = \{ C \in L^2(0,T;V) \mid C' \in L^2(0,T;V^*) \}, \]
where \( v' \) and \( C' \) denote the time derivatives in the sense of distributions.

We recall two useful results on evolution triples, proofs of which can be found in \[21\] Lemma 2.1 and \[19\] Corollary 4, respectively.

**Lemma 2 (Erhling).** Let \( X, Y \) and \( Z \) be Banach spaces such that \( X \) is compactly embedded in \( Y \), and \( Y \) is continuously embedded in \( Z \). Then, for every \( \varepsilon > 0 \), there exists a constant \( C(\varepsilon) > 0 \) such that
\[ \| x \|_Y \leq \varepsilon \| x \|_X + C(\varepsilon) \| x \|_Z \]
for all \( x \in X \).

**Lemma 3 (Aubin-Lions).** Let \( X, Y \) and \( Z \) be reflexive Banach spaces and \( X \subset Y \subset Z \) continuously with compact embedding \( X \subset Y \), and \( p, q \in (1, \infty) \). Then, for any \( T > 0 \), the space
\[ \{ u \in L^p(0,T;X) \mid u' \in L^q(0,T;Z) \} \]
is compactly embedded into \( L^p(0,T;Y) \).

In what follows, we denote by \( \langle \cdot, \cdot \rangle_{X^*,X} \) the duality pairing between a Banach space \( X \) and its dual.

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function \( \varphi : X \to \mathbb{R} \), where \( X \) is a Banach space, see \[9\]. The generalized directional derivative of \( \varphi \) at \( x \in X \) in the direction \( v \in X \), denoted by \( \varphi^0(x; v) \), is defined by
\[ \varphi^0(x; v) = \limsup_{y \to x, \lambda \to 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}. \]
The generalized gradient of \( \varphi \) at \( x \), denoted by \( \partial \varphi(x) \), is a subset of a dual space \( X^* \) given by
\[ \partial \varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^*,X} \text{ for all } v \in X \}. \]

Finally, we recall the Green formula, proof of which can be found in e.g. \[18\] Theorem 2.25.

**Lemma 4.** Let \( \Omega \) be an bounded domain in \( \mathbb{R}^d \), \( d = 2,3 \) with Lipschitz boundary. Then, the following formula holds
\[ \int_{\Omega} \varepsilon(v) \, dx + \int_{\Omega} \text{Div} \, v \, dx = \int_{\Gamma} \sigma \nu \cdot v \, d\Gamma \]
for all \( v \in H^1(\Omega)^d \) and \( \sigma \in C^1(\overline{\Omega}; \mathbb{S}^d) \), where \( \varepsilon(u) = (\varepsilon_{ij}(u)) = (\frac{1}{2}(u_{i,j} + u_{j,i})) \), \( i, j = 1, 2 \).

Throughout the paper, we denote by \( M \) a generic constant whose value may change from line to line.
3 Classical formulation

In this section we provide the classical formulation of a model for miscible liquids which describe evolution of the velocity \( u : \Omega \times (0, T) \to \mathbb{R}^2 \), pressure \( p : \Omega \times (0, T) \to \mathbb{R} \) and concentration \( C : \Omega \times (0, T) \to \mathbb{R} \) of a viscous incompressible fluid filling domain \( \Omega \) with the time interval \((0, T)\).

The model consists with the incompressible Navier-Stokes equation modified by the (additional) Korteweg tensor. The classical stress tensor \( \sigma \) for incompressible fluids is given by

\[
\sigma = -pI + 2 \nu_0 \varepsilon(u) \quad \text{in} \quad \Omega \times (0, T),
\]

where \( I \) denotes the identity matrix and \( \nu_0 \) is the kinetic viscosity coefficient. We suppose that the fluid is incompressible

\[
\text{div} \, u = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

and governed by the Navier-Stokes equation for miscible fluids

\[
\frac{\partial u}{\partial t} - \nu_0 \Delta u + (u \cdot \nabla)u + \nabla p = \text{Div} \, K(C) + f \quad \text{in} \quad \Omega \times (0, T),
\]

where \( f : \Omega \times (0, T) \to \mathbb{R}^2 \) denotes external forces field such as gravity and buoyancy, and \( K(C) = (K_{ij}(C)) \) is the Korteweg stress tensor given by the following relations

\[
K_{11}(C) = k \frac{\partial C}{\partial x_2} \frac{\partial C}{\partial x_2}, \quad K_{22}(C) = k \frac{\partial C}{\partial x_1} \frac{\partial C}{\partial x_1}, \quad K_{12}(C) = K_{21}(C) = -k \frac{\partial C}{\partial x_1} \frac{\partial C}{\partial x_2}
\]

where \( k \) is a nonnegative constant.

We use a concentration function \( C \) to represent and track the interface between liquids. The concentration function is transported by the velocity field \( u \)

\[
\frac{\partial C}{\partial t} - d \Delta C + u \cdot \nabla C = g C \quad \text{in} \quad \Omega \times (0, T),
\]

where \( d > 0 \) is the coefficient of mass diffusion and \( g \) represents the source term. We assume also the homogeneous Neumann boundary condition on the boundary \( \Gamma \) for the concentration function

\[
\frac{\partial C}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0, T).
\]

We supplement the system with boundary and initial conditions. On the part \( \Gamma_0 \), we suppose adhesive boundary condition

\[
u = 0 \quad \text{on} \quad \Gamma_0 \times (0, T).
\]

The following nonmonotone slip boundary condition of frictional type with no leak is assumed on the part \( \Gamma_1 \)

\[
u = 0, \quad -\sigma_\tau \in \partial j(u_\tau) \quad \text{on} \quad \Gamma_1 \times (0, T),
\]
where $\partial j$ denotes the generalized gradient of a prescribed locally Lipschitz function $j$. The boundary friction law (8) has been considered for the Navier-Stokes problems in [16, 13, 17, 20]. Finally, the initial conditions for the velocity and concentration are prescribed

$$u(0) = u_0, \quad C(0) = C_0 \quad \text{in} \quad \Omega. \quad (9)$$

The classical formulation of the problem for miscible liquids is the following.

**Problem 5.** Find $u: \Omega \times (0,T) \to \mathbb{R}^2$, $p: \Omega \times (0,T) \to \mathbb{R}$ and $C: \Omega \times (0,T) \to \mathbb{R}$ such that (4)–(9) are satisfied.

In the next section we will study the weak formulation of Problem 5.

We conclude this section with remarks on the Korteweg stress tensor which will be useful in next sections. Using the notation

$$\nabla C = \left( \frac{\partial C}{\partial x_1}, \frac{\partial C}{\partial x_2} \right), \quad \Delta C = \sum_{i=1}^2 \frac{\partial^2 C}{\partial x_i^2}$$

and formula (3), we calculate the first component of $\text{Div} \ K(C)$ by

$$\frac{\partial K_{11}}{\partial x_1} + \frac{\partial K_{12}}{\partial x_2} = 2k \frac{\partial C}{\partial x_2} \frac{\partial^2 C}{\partial x_1 \partial x_2} - k \frac{\partial^2 C}{\partial x_1 \partial x_2} \frac{\partial C}{\partial x_2} - k \frac{\partial^2 C}{\partial x_2^2} \frac{\partial C}{\partial x_1}$$

$$= k \left( \frac{\partial C}{\partial x_2} \frac{\partial^2 C}{\partial x_1 \partial x_2} - \frac{\partial^2 C}{\partial x_2^2} \frac{\partial C}{\partial x_1} \right)$$

$$= k \left( \frac{\partial C}{\partial x_2} \frac{\partial^2 C}{\partial x_1 \partial x_2} + \frac{\partial C}{\partial x_1} \frac{\partial^2 C}{\partial x_2^2} \right) - k \frac{\partial C}{\partial x_1} \left( \frac{\partial^2 C}{\partial x_1^2} + \frac{\partial^2 C}{\partial x_2^2} \right)$$

$$= \frac{k}{2} \frac{\partial}{\partial x_1} \|\nabla C\|^2 - k \frac{\partial C}{\partial x_1} \Delta C. \quad (10)$$

Calculating, in the analogous way, the second component, we get

$$\frac{\partial K_{21}}{\partial x_1} + \frac{\partial K_{22}}{\partial x_2} = k \frac{\partial}{\partial x_2} \|\nabla C\|^2 - k \frac{\partial C}{\partial x_2} \Delta C.$$ 

Hence, we have

$$\text{Div} \ K(C) = \left( \frac{\partial K_{11}}{\partial x_1} + \frac{\partial K_{12}}{\partial x_2}, \frac{\partial K_{21}}{\partial x_1} + \frac{\partial K_{22}}{\partial x_2} \right) = \frac{k}{2} \nabla \|\nabla C\|^2 - k \Delta C \nabla C. \quad (11)$$

Using (10), we easily obtain

$$\int_{\Omega} \text{Div} \ K(C) \cdot v \, dx = \int_{\Omega} \left( \frac{k}{2} \nabla \|\nabla C\|^2 - k \Delta C \nabla C \right) \cdot v \, dx$$

$$= - \int_{\Omega} \frac{k}{2} \|\nabla C\|^2 \text{div} \ v \, dx - \int_{\Omega} k \Delta C \nabla C \cdot v \, dx = - \langle k \Delta C \nabla C, v \rangle_{L^2(\Omega)}$$

for all $v \in E$ and $C \in V$. Hence, we conclude

$$\langle \text{Div} \ K(C), v \rangle_{E^* \times E} = - \langle k \Delta C \nabla C, v \rangle_{L^2(\Omega)} \quad \text{for all} \quad v \in E, \ C \in V. \quad (11)$$
4 Variational formulation

In this section we provide a variational formulation of Problem 5 and state the main result of this paper.

We start by introducing the following forms and formulating their properties. The bilinear forms \(a_0: H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{R}\), \(b_0: H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}\) and \(c: H^1(\Omega)^2 \times L^2(\Omega) \to \mathbb{R}\) are given by

\[
a_0(u,v) = \frac{\nu_0}{2} \sum_{i,j=1}^{2} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \, dx \quad \text{for } u, v \in H^1(\Omega)^2,
\]

\[
b_0(\xi, \eta) = d \int_{\Omega} \nabla \xi \cdot \nabla \eta \, dx \quad \text{for } \xi, \eta \in H^1(\Omega),
\]

\[
c(v,q) = -\int_{\Omega} (\text{div} \, v) \, q \, dx \quad \text{for } v \in H^1(\Omega)^2, \, q \in L^2(\Omega).
\]

We also define the trilinear forms \(a_1: H^1(\Omega)^2 \times H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{R}\) and \(b_1: H^1(\Omega)^2 \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}\) by

\[
a_1(u,v,w) = \int_{\Omega} ((u \cdot \nabla) \cdot w) \, dx \quad \text{for } u, v, w \in H^1(\Omega)^2,
\]

\[
b_1(v,\xi,\eta) = \int_{\Omega} (v \cdot \nabla \xi) \, \eta \, dx \quad \text{for } v \in H^1(\Omega)^2, \, \xi, \eta \in H^1(\Omega).
\]

There exists \(\alpha = \frac{1}{2} \nu_0 M_k > 0\), where \(M_k\) is a constant arising from Korn inequality, such that

\[
a_0(u,u) \geq \alpha \|u\|_{E}^2, \quad b_0(\xi,\xi) = d \|\nabla \xi\|_{L^2(\Omega)}^2
\]

for all \(u \in H^1(\Omega)^2\) and \(\xi \in H^1(\Omega)\). Also, by the definition of space \(E\), we have

\[
c(v,q) = 0 \quad \text{for } v \in E, \, q \in L^2(\Omega).
\]

Moreover, we recall the properties of forms \(a_1\) and \(b_1\). They follow from Lemma 1 and Lemma 1.3(II) in [21].

**Lemma 6.** (a) For all \(u, v, w \in E\), we have

\[
a_1(u,v,w) = -a_1(u,w,v),
\]

\[
|a_1(u,v,w)| \leq M \|u\|_{L^2(\Omega)^2}^{1/2} \|v\|_{E}^{1/2} \|w\|_{L^2(\Omega)^2} \|w\|_{E}^{1/2} \quad \text{with } M > 0,
\]

\[
a_1(u,v,w) = 0.
\]

(b) For all \(u \in E, \eta, \zeta \in H^1(\Omega)\), we have

\[
b_1(u,\eta,\zeta) = -b_1(u,\zeta,\eta),
\]

\[
|b_1(u,\eta,\zeta)| \leq M \|u\|_{L^2(\Omega)^2}^{1/2} \|u\|_{E}^{1/2} \|\eta\|_{H^1(\Omega)} \|\zeta\|_{L^2(\Omega)^2} \|\zeta\|_{H^1(\Omega)} \|\eta\|_{H^1(\Omega)} \|\zeta\|_{H^1(\Omega)} \quad \text{with } M > 0,
\]

\[
b_1(u,\eta,\eta) = 0.
\]
Furthermore, we introduce operators $A_0, A_1 : E \to E^*$, $B_0 : V \to V^*$ and $B_1 : E \times V \to V^*$ defined by

\[ \langle A_0 u, v \rangle_{E^* \times E} = a_0(u, v), \quad \langle A_1 u, v \rangle_{E^* \times E} = a_1(u, u, v) \quad \text{for} \quad u, v \in E, \]
\[ \langle B_0 C, \eta \rangle_{V^* \times V} = b_0(C, \eta), \quad \langle B_1 (u, C), \eta \rangle_{V^* \times V} = b_1(u, C, \eta) \quad \text{for} \quad u \in E, \; C, \eta \in V. \]

Assume now that $u, \; p$ and $C$ are sufficiently smooth functions which solve Problem 5. Let $v \in E$ be sufficiently smooth and $t \in (0, T)$. Using the Green formula of Lemma 4 combined with the definition of (1), similarly as in [20], we obtain the following equality

\[ \int_{\Omega} p \; \text{div} \; v \; dx + \nu_0 \int_{\Omega} \varepsilon(u) : \varepsilon(v) \; dx + \int_{\Omega} (-\nabla p + \nu_0 \Delta u) \cdot v \; dx = \int_{\Gamma} \sigma \nu \cdot v \; d\Gamma. \]

From the definition of forms $a_0$ and $c$, we have

\[ \int_{\Omega} (\nabla p - \alpha \Delta u) \cdot v \; dx = a_0(u, v) + c(v, p) - \int_{\Gamma} \sigma \nu \cdot v \; d\Gamma. \quad (14) \]

We now multiply (3) by $v \in E$. Exploiting definitions of operators $A_0, \; A_1$, using (13) and (14) we deduce

\[ \langle u'(t) + A_0 u(t) + A_1 u(t), v \rangle_{E^* \times E} - \int_{\Gamma} \sigma \nu \cdot v \; d\Gamma \]
\[ = \langle \text{Div} K(C), v \rangle_{E^* \times E} + \langle f(t), v \rangle_{E^* \times E}. \quad (15) \]

Next, we use the orthogonality relation \( \sigma \nu \cdot v = \sigma_{\nu} \nu_{\nu} + \sigma_{\tau} \nu_{\tau} = \sigma_{\tau} \cdot \nu_{\tau} \) and (8) to arrive at the equality

\[ \langle u'(t) + A_0 u(t) + A_1 u(t), v \rangle_{E^* \times E} + \langle \xi(t), v \rangle_{L^2(\Gamma_1)^2} = \langle \text{Div} K(C(t)) + f(t), v \rangle_{E^* \times E}, \quad (16) \]

where $\xi \in L^2(0, T; L^2(\Gamma_1)^2)$, $\xi(t) \in \partial j(u_r(t))$ for a.e. $t \in (0, T)$. On the other hand, we multiply (5) by $\eta \in V$, using (3) we find

\[ \langle C'(t) + B_0 C(t) + B_1 (u(t), C(t)), \eta \rangle_{V^* \times V} = \langle g C(t), \eta \rangle_{L^2(\Omega)}. \quad (17) \]

Summarizing, we obtain the following system of equations and inclusion which is the variational formulation of Problem 5.

**Problem 7.** Find $u \in \mathbb{E}$ and $C \in \mathbb{W}$ such that there exists $\xi \in L^2(0, T; L^2(\Gamma_1)^2)$ and

\[ \langle u'(t) + A_0 u(t) + A_1 u(t), v \rangle_{E^* \times E} + \langle \xi(t), v \rangle_{L^2(\Gamma_1)^2} \]
\[ = \langle \text{Div} K(C(t)) + f(t), v \rangle_{E^* \times E} \quad \text{for all} \quad v \in E, \; \text{a.e.} \; t \in (0, T), \]
\[ \xi(t) \in \partial j(u_r(t)) \quad \text{for a.e.} \; t \in (0, T), \]
\[ \langle C'(t) + B_0 C(t) + B_1 (u(t), C(t)), \eta \rangle_{V^* \times V} = \langle g C(t), \eta \rangle_{L^2(\Omega)} \]
\[ \text{for all} \quad \eta \in V, \; \text{a.e.} \; t \in (0, T), \]
\[ u(0) = u_0, \; C(0) = C_0. \]
We need the following hypotheses.

\[
H(j): \quad j: \Gamma_1 \times \mathbb{R}^2 \to \mathbb{R} \text{ is such that}
\]

(a) \( j(\cdot, \xi) \) is measurable for all \( \xi \in \mathbb{R}^2 \), \( j(\cdot, 0) \in L^2(\Gamma_1) \),
(b) \( j(x, \cdot) \) is locally Lipschitz for a.e. \( x \in \Gamma_1 \),
(c) \( \eta \cdot s \geq 0 \) for all \( \eta \in \partial j(x, s) \), \( s \in \mathbb{R}^2 \), a.e. \( x \in \Gamma_1 \),
(d) \( \| \xi \| \leq m_0 (1 + \| \xi \|) \) for all \( \xi \in \mathbb{R}^2 \), \( \xi \in \partial j(x, \xi) \), a.e. \( x \in \Gamma_1 \) with \( m_0 > 0 \),
(e) \( (\xi_1 - \xi_2) \cdot (\xi_1 - \xi_2) \geq -m_1 \| \xi_1 - \xi_2 \|^2 \) for all \( \xi_i \in \partial j(x, \xi_i) \), \( \xi_i \in \mathbb{R}^2 \),
\[ i = 1, 2, \text{ a.e. } x \in \Gamma_1 \text{ with } m_1 \geq 0. \]

We observe that since \( \eta \cdot s \geq 0 \), \( s \in \mathbb{R}^2 \), a.e. \( x \in \Gamma_1 \),
\( \eta \cdot \xi \geq 0 \) for all \( \eta \in \partial j(x, \xi) \), \( \xi \in \mathbb{R}^2 \), a.e. \( x \in \Gamma_1 \)
with \( m_0 > 0 \),
\( (\xi_1 - \xi_2) \cdot (\xi_1 - \xi_2) \geq -m_1 \| \xi_1 - \xi_2 \|^2 \) for all \( \xi_i \in \partial j(x, \xi_i) \), \( \xi_i \in \mathbb{R}^2 \),
\[ i = 1, 2, \text{ a.e. } x \in \Gamma_1 \text{ with } m_1 \geq 0. \]

(Ho): \( d, k, \nu_0 > 0 \), \( g \in L^\infty(\Omega) \), \( g \geq 0 \), \( f \in L^2(0, T; E^*) \), \( u_0 \in E \), \( C_0 \in V \).

Our main result of this paper on a unique solvability of Problem 7 reads as follows.

**Theorem 8.** Under hypotheses \( H(j)(a)-(d) \) and \( (H_0) \), Problem 7 has a solution
such that \( C \in L^\infty(0, T; H^1(\Omega)) \). If, in addition, \( H(j)(e) \) holds, then
the solution to Problem 7 is unique.

## 5 Proof of the main result

In this section we provide the proof of Theorem 8. For the existence, we use the
regularized Galerkin method. To this end, we define the regularization of the multivalued
term as follows.

Let \( \rho \in C^\infty_0(\mathbb{R}^2) \) be the mollifier such that \( \rho \geq 0 \) on \( \mathbb{R}^2 \), \( \text{supp } \rho \subset [-1, 1]^2 \) and
\( \int_{\mathbb{R}^2} \rho \, dx = 1 \). We define \( \rho_m(x) = m^2 \rho(mx) \) for \( m \in \mathbb{N} \). Then \( \text{supp } \rho_m \subset [-\frac{1}{m}, \frac{1}{m}]^2 \) for
all \( m \in \mathbb{N} \). Consider functions \( j_m: \Gamma_1 \times \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
j_m(x, \xi) = \int_{\text{supp } \rho_m} \rho_m(z) j(x, \xi - z) \, dz \text{ for } (x, \xi) \in \Gamma_1 \times \mathbb{R}^2.
\]

We observe that since \( j_m(x, \cdot) \in C^\infty(\mathbb{R}^2) \) for all \( x \in \Gamma_1 \), therefore \( \partial j_m(x, \xi) \) reduces
to a single element. We write \( \partial j_m(x, \xi(t)) = \{ D_u j_m(x, \xi(t)) \} \) for all \( \xi(t) \in E \), where
\( D_u j_m \) represents the derivative of \( j_m(x, \cdot) \). Moreover, it is easy to see that \( j_m \)
satisfies the growth condition \( H(j)(d) \).

Using the separability of the space \( E \), we may write a basis of \( E \) as \( \{ \varphi_1, \varphi_2, \ldots \} \). We
choose in \( V \) a special basis \( \{ \psi_1, \psi_2, \ldots \} \) of eigenvectors of the \(-\Delta\) eigenvalue problem
associated with zero Neumann boundary condition, see [11] Theorem 6.1.31.

We define finite dimensional subspaces \( E^m = \text{span}\{ \varphi_1, \ldots, \varphi_m \} \) of \( E \), and \( V^m = \text{span}\{ \psi_1, \ldots, \psi_m \} \) of \( V \) for \( m \geq 1 \). Let \( u_0m, C_0m \) be such that \( u_0m \to u_0 \) in \( H \) and
\( C_0m \to C_0 \) in \( L^2(\Omega) \) with \( u_0m \in E^m \) and \( C_0m \in V^m \) for \( m \geq 1 \). Next, for a fixed
\( m \geq 1 \), consider the following problem in finite dimensional spaces.

**Problem 9.** Find \( u_m \in L^2(0, T; E^m) \) with \( u'_m \in L^2(0, T; E^m) \) and \( C_m \in L^2(0, T; V^m) \)
with \( C'_m \in L^2(0, T; V^m) \) such that
\[
\langle u'_m(t) + A_0 u_m(t) + A_1 u_m(t), v_m \rangle_{E^m \times E^m} + \langle D_u j_m(u_m(t)), v_m \rangle_{L^2(\Gamma_1)^2}.
\]
that

Carathéodory existence theorem. We now show a priori estimates to extend the so-
and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$

Using the Gronwall lemma, from the last inequality, we deduce that

Now, we take $v$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$ and rewrite Problem 9 as follows: find $z$

Integrating (21) over $(0, T)$, we get

Using the Gronwall lemma, from the last inequality, we deduce that

and putting (23) in (24), we obtain

Now, we take $v$ in equality (18). Using coercivity of $A_0$ stated in (12), condition $H(j)(d)$ and Lemma (4)(a), we find

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for a.e. $t \in (0, T)$. Using (18) and the Cauchy inequality with $\varepsilon > 0$ in (25), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| u_m(t) \|^2_H + \alpha \| u_m(t) \|^2_E \leq M \| u_m(t) \|^2_{L^2(\Gamma)} + \langle \Delta C_m(t) \nabla C_m(t), u_m(t) \rangle_{L^2(\Omega)}^2 + \varepsilon \| u_m(t) \|^2_V + M
\]
for a.e. $t \in (0, T)$. From Lemma 2 there exists $M(\varepsilon) > 0$ such that
\[
\| u_m(t) \|^2_{L^2(\Gamma)} \leq 2 \varepsilon \| u_m \|^2_E + M(\varepsilon) \| u_m(t) \|^2_H.
\]
Using this inequality in (26), we have
\[
\frac{1}{2} \frac{d}{dt} \| u_m(t) \|^2_H + (\alpha - 3\varepsilon) \| u_m(t) \|^2_E \leq \langle k \Delta C_m(t) \nabla C_m(t), u_m(t) \rangle_H + M(\varepsilon) \| u_m(t) \|^2_H + M.
\]
Subsequently, we take $\eta_m = -k \Delta C_m(t)$ in (19) to find
\[
\frac{1}{2} \frac{d}{dt} \| \nabla C_m(t) \|^2_{L^2(\Omega)}^2 + d k \| \Delta C_m(t) \|^2_{L^2(\Omega)} - k \langle \Delta C_m(t), \nabla C_m(t) u_m(t) \rangle_{L^2(\Omega)} \leq k \| g \| \| \nabla C_m(t) \|^2_{L^2(\Omega)}.
\]
Next, adding (27) and (28), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \| u_m(t) \|^2_H + \| \nabla C_m(t) \|^2_{L^2(\Omega)}^2 \right) + (\alpha - 3\varepsilon) \| u_m(t) \|^2_E + \| \Delta C_m(t) \|^2_{L^2(\Omega)} \leq M \| \nabla C_m(t) \|^2_{L^2(\Omega)} + M(\varepsilon) \| u_m(t) \|^2_H + M.
\]
Choosing $\varepsilon > 0$ sufficiently small, integrating (29) over $(0, t)$ for $t \in (0, T)$, from the Gronwall lemma, we deduce
\[
\| u_m \|_{L^\infty(0,T;H)} \leq M, \quad \| u_m \|_{L^2(0,T;E)} \leq M, \quad \| C_m \|_{L^\infty(0,T;H^1(\Omega))} \leq M, \quad \| C_m \|_{L^2(0,T;H^2(\Omega))} \leq M,
\]
where (33) holds due to Theorem 3.1.2.3. Now, we estimate $\| C'_m(t) \|^2_{L^2(\Omega)}$. To this end, using Lemmata 3(b) and 1 we take $\eta_m = C'_m(t)$ in (19) to find
\[
\| C'_m(t) \|^2_{L^2(\Omega)} \leq d \| \Delta C_m(t) \|^2_{L^2(\Omega)} \| C'_m(t) \|_{L^2(\Omega)}
\]
\[
+ \| u_m(t) \|^2_{L^4(\Omega)} \| \nabla C_m(t) \|^2_{L^2(\Omega)} \| C'_m(t) \|_{L^2(\Omega)} + \| g(t) \|^2 \| C_m(t) \|^2_{L^2(\Omega)} \| C'_m(t) \|_{L^2(\Omega)}
\]
\[
\leq M \| C_m(t) \|^2_{H^2(\Omega)} \| C'_m(t) \|^2_{L^2(\Omega)} + M \| u_m(t) \|^2_{H^2} \| u_m(t) \|^2_{L^2} \| C_m(t) \|^2_{H^1(\Omega)} \| C'_m(t) \|^2_{L^2(\Omega)}.
\]
for a.e. \( t \in (0, T) \). From (34), we infer that
\[
\| C_m' \|_{L^2(0,T;L^2(\Omega))} \leq M. \tag{35}
\]

Next, we estimate the term \( \text{Div} K(C_m) \). We observe that
\[
\text{Div} K(C_m) = \sum_{i,j,k=1}^2 a_{ijk} D_i (D_j C_m(t) D_k C_m(t)),
\tag{36}
\]
where \( a_{ijk} \) are constants for \( i, j, k = 1, 2 \) and \( D_l = \frac{\partial}{\partial x_l} \) for \( l = 1, 2 \). We estimate one term in (36) and find
\[
\| D_i (D_j C_m(t) D_k C_m(t)) \|_{E^*} = \sup_{\| v \|_{E^*} = 1} | \langle D_i (D_j C_m(t) D_k C_m(t), v \rangle_{E^* \times E} |
\leq \| D_j C_m(t) D_k C_m(t) \|_{L^2(\Omega)} \leq \| D_j C_m(t) \|_{L^4(\Omega)} \| D_k C_m(t) \|_{L^4(\Omega)}
\leq M \| D_j C_m(t) \|_{H^{1/2}(\Omega)} \| D_k C_m(t) \|_{H^{1/2}(\Omega)} \| D_i C_m(t) \|_{H^{1/2}(\Omega)}.
\tag{37}
\]

From bounds (30)–(33), (36) and (37), we have
\[
\| \text{Div} K(C_m) \|_{L^2(0,T;E^*)} \leq M. \tag{38}
\]

Furthermore, from (24), (30)–(33) and (37), we find elements \( u \in L^\infty(0,T;H) \cap L^2(0,T;E) \) and \( C \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \) such that, up to a subsequence, we get
\[
\begin{align*}
&u_m \to u \text{ weakly* in } L^\infty(0,T;H), \tag{39} \\
&u_m \to u \text{ weakly in } L^2(0,T;E), \tag{40} \\
&C_m \to C \text{ weakly* in } L^\infty(0,T;H^1(\Omega)), \tag{41} \\
&C_m \to C \text{ weakly in } L^2(0,T;V), \tag{42}
\end{align*}
\]
as \( m \to \infty \). By the definition of operator \( A_1 \) and [21, Lemma 3.4], we have
\[
\| A_1(u_m) \|_{L^2(0,T;E^*)} \leq M \| u_m \|_{L^\infty(0,T;H)} \| u_m \|_{L^2(0,T;E)}.
\tag{43}
\]

From (30), (31), (38), (43) and the definition of operator \( A_0 \), we infer that
\[
\| u_m' \|_{L^2(0,T;E^*)} \leq M, \tag{44}
\]
and hence
\[
\begin{align*}
&u_m' \to u' \text{ weakly in } L^2(0,T;E^*), \tag{45} \text{ as } m \to \infty. \\
&\text{By Lemma 3, we know that the embedding } E \subset L^2(0,T;H) \text{ is compact, so from (31) and (44), we have} \\
&u_m \to u \text{ in } L^2(0,T;H), \tag{46} \text{ as } m \to \infty.
\end{align*}
\]
Since the operator \(A_0: E \to E^*\) is linear and continuous, so is its Nemytskii operator which is denoted in the same way. Therefore, we find that
\[
A_0 u_m \to A_0 u \text{ weakly in } L^2(0, T; E^*), \quad m \to \infty. \tag{47}
\]
From (30) and (31) and the technique used in [21, Lemma III.3.2], we have
\[
A_1(u_m) \to A_1(u) \text{ weakly in } L^2(0, T; E^*), \quad m \to \infty. \tag{48}
\]
We use the fact that the embedding \(W \subset L^2(0, T; H^1(\Omega))\) is compact. From (35), (24) and Lemma 3, we deduce
\[
C_m \to C \text{ in } L^2(0, T; H^1(\Omega)), \quad m \to \infty. \tag{49}
\]
Moreover, from (38) and (49), we see that
\[
\text{Div } K(C_m) \to \text{Div } K(C) \text{ weakly in } L^2(0, T; E^*), \quad m \to \infty. \tag{50}
\]
Next, by the compactness of the trace operator from \(E\) to \(L^2(0, T; L^2(\Gamma_1)^2)\), it follows
\[
u_{mr} \to u_r \text{ in } L^2(0, T; L^2(\Gamma_1)^2), \quad m \to \infty.
\]
Hence, by passing to a next subsequence, if necessary, we have
\[
u_{mr}(t) \to u_r(t) \text{ in } L^2(\Gamma_1)^2 \text{ for a.e. } t \in (0, T), \quad m \to \infty. \tag{51}
\]
On the other hand, by hypothesis \(H(j)(d)\) and (31), we may suppose that
\[
Dj_m(u_{mr}(\cdot)) \to \xi \text{ weakly in } L^2(0, T; L^2(\Gamma_1)^2), \quad m \to \infty \tag{52}
\]
with \(\xi \in L^2(0, T; L^2(\Gamma_1)^2)\). Now, we are in a position to use convergences (51) and (52), and apply the Aubin-Cellina convergence theorem, see [1] Theorem 1, p.60 to the inclusion
\[
Dj_m(u_{mr}(t)) \in \partial j_m(u_{mr}(t)) \text{ for a.e. } t \in (0, T).
\]
We deduce that
\[
\xi(t) \in \text{co } \partial j(u_r(t)) = \partial j(u_r(t)) \text{ for a.e. } t \in (0, T),
\]
where \(\text{co}\) denotes the closure of the convex hull of a set. The last equality follows from the fact that the values of the generalized subgradient are closed and convex sets, see [18, Proposition 3.23].

In a similar way, as in (47), by linearity and continuity of operator \(B_0\), by using (49), we have
\[
B_0 C_m \to B_0 C \text{ weakly in } L^2(0, T; V^*), \quad m \to \infty. \tag{53}
\]
Also, from (46) and (49), we obtain
\[
B_1(u_m, C_m) \to B_1(u, C) \text{ weakly in } L^2(0, T; V^*), \quad m \to \infty. \tag{54}
\]
From (35) we infer that
\[ C'_m \rightarrow C \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \] (55)

Thus, using convergences (15), (17), (18), (50) and (52), we pass to the limit in (18) and using standard techniques, see [17, p.739] we obtain
\[
\langle u'(t) + A_0 u(t) + A_1 u(t), v \rangle_{E^* \times E} + \langle \xi(t), v \rangle_{L^2(\Gamma_1)}^2
= \langle \text{Div} K(C(t)) + f(t), v \rangle_{E^* \times E} \quad \text{for all} \quad v \in E, \ a.e. \ t \in (0,T).
\]

Moreover, using (53)–(55), we pass to limit in (19) and get
\[
\langle C'(t) + B_0 C(t) + B_1(u(t), C(t)), \eta \rangle_{V^* \times V} = \langle g C(t), \eta \rangle_{L^2(\Omega)}
\]
for all \( \eta \in V, \ a.e. \ t \in (0,T). \)

Since the mapping \( E \ni w \rightarrow w(0) \in H \) is linear and continuous, from (10) and (13), we have \( u_m(0) \rightarrow u(0) \) weakly in \( H \), which together with \( u_{0m} \rightarrow u_0 \) in \( H \) entails \( u(0) = u_0 \). Similarly, since \( W \ni \zeta \rightarrow \zeta(0) \in L^2(\Omega) \) is linear and continuous, we obtain \( C(0) = C_0 \). Finally, taking into account that \( \xi(t) \in \partial j(u_\tau(t)) \) for a.e. \( t \in (0,T) \), we conclude that \( u \in E \) and \( C \in W \) is a solution to Problem 7. Observe, that by (11), we have the additional regularity \( C \in L^\infty(0,T;H^1(\Omega)) \). This concludes the existence proof.

We pass to the proof of uniqueness of solution to Problem 7. To show uniqueness of solution, we assume additionally the regularity of function \( j \) stated in \( H(j)(\varepsilon) \).

Let \( (u_1, C_1) \) and \( (u_2, C_2) \) be two solutions of Problem 7. Set \( u = u_1 - u_2 \) and \( C = C_1 - C_2 \). Using property (10), we obtain that \( (u, C) \) is a solution to the following problem.

\[
\langle u'(t) + A_0 u(t) + A_1 u_1(t) - A_1 u_2(t), v \rangle + \langle \xi^1(t) - \xi^2(t), v \rangle
= -k \langle \Delta C_1(t) \nabla C_1(t) - \Delta C_2(t) \nabla C_2(t), v \rangle \quad \text{for all} \quad v \in E, \ a.e. \ t \in (0,T), \quad (56)
\]
\[
\langle C'(t) + B_0 C(t) + B_1(u_1(t),C_1(t)) - B_1(u_2(t),C_2(t), \eta \rangle = \langle g C(t), \eta \rangle
\]
for all \( \eta \in V, \ a.e. \ t \in (0,T), \quad (57)
\]
\[
u(0) = 0, \quad C(0) = 0.
\]
(58)

Since \( C \in L^2(0,T;V) \) equation (57) is equivalent to the following
\[
\langle C'(t) - d \Delta C(t) + B_1(u_1(t),C_1(t)) - B_1(u_2(t),C_2(t)), \eta \rangle_{L^2(\Omega)} = \langle g C(t), \eta \rangle_{L^2(\Omega)}
\]
for all \( \eta \in L^2(\Omega), \ a.e. \ t \in (0,T). \quad (59)
\]
\[
u(0) = 0, \quad C(0) = 0.
\]
First, observe that from Lemma 6 we have for \( u_1, u_2, u = u_1 - u_2 \in E \)
\[
a_1(u_1, u_1 - u_2) - a_1(u_2, u_1 - u_2) = a_1(u_1, u_1, -u_2) - a_1(u_2, u_2, u_1)
\]
Moreover, from Lemma 2 and $H(\xi)(\epsilon)$ we have

$$-\langle \xi_1(t) - \xi_2(t), u_1(t) - u_2(t) \rangle \leq m_1 \|u(t)\|_{H^1(\Gamma)}^2 \leq \epsilon \|u(t)\|_E^2 + M(\epsilon) \|u(t)\|_H^2 \quad (61)$$

for $\epsilon > 0$ and a.e. $t \in (0, T)$. Finally, choosing $v = u(t)$ and $\eta = -k\Delta C(t)$ in (54) and (55), respectively and adding resulting equations gives, using $g \geq 0$, (12), (60) and (61) we calculate

$$\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_H^2 + k \|\nabla C(t)\|_{L^2(\Omega)}^2 \right) + (\alpha - \epsilon) \|u(t)\|_E^2 + k d \|\Delta C(t)\|_{H^2(\Omega)}^2$$

for $\epsilon > 0$ and a.e. $t \in (0, T)$. Hence, finally

$$\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_H^2 + k \|\nabla C(t)\|_{L^2(\Omega)}^2 \right) + (\alpha - \epsilon) \|u(t)\|_E^2 + k d \|\Delta C(t)\|_{L^2(\Omega)}^2$$

for $\epsilon > 0$ and a.e. $t \in (0, T)$. We now estimate terms on the right-hand side of (62). From Lemma 3 and the Cauchy inequality with $\epsilon > 0$, we have

$$|a_1(u(t), u_2(t), u(t))| \leq \epsilon \|u(t)\|_E^2 + M(\epsilon) \|u(t)\|_H^2 \|u_2(t)\|_E^2, \quad (63)$$

$$|\langle \Delta C_1(t) \Delta C(t), u(t) \rangle| \leq M \|\Delta C_1(t)\|_{L^2} \|\nabla C(t)\|_{L^4} \|u(t)\|_{L^4}$$

$$\leq \|\Delta C_1(t)\|_{L^2} \|\nabla C(t)\|_{L^2} \|\Delta C(t)\|_{L^2} \|u(t)\|_{L^2}$$

$$\leq \epsilon \left( \|\Delta C(t)\|_{L^2}^2 + \|u(t)\|_E^2 \right) + M(\epsilon) \|\Delta C_1(t)\|_{L^2}^2 \|u(t)\|_H^2 \|\nabla C(t)\|_{L^2}^2, \quad (64)$$

$$|\langle \Delta C(t) \nabla C(t), u_1(t) \rangle| \leq M \|\Delta C(t)\|_{L^2} \|\nabla C(t)\|_{L^2} \|u_1(t)\|_{L^4}$$

$$\leq \epsilon \|\Delta C(t)\|_{L^2}^2 + M(\epsilon) \|\nabla C(t)\|_{L^2}^2 \|u_1(t)\|_E^2 \quad (65)$$
for a.e. \( t \in (0, T) \). We now choose \( \varepsilon < \min\{ \frac{a}{3}, \frac{kd}{2} \} \). From (63)–(65) applied to the right hand side of (62), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{H}^{2} + k \| \nabla C(t)\|_{L^2(\Omega)}^{2} \right) + (\alpha - 3\varepsilon)\|u(t)\|_{E}^{2} + (kd - 2\varepsilon)\|\Delta C(t)\|_{L^2}^{2} + \| \nabla C(t)\|_{E}^{2} \leq M(\|u(t)\|_{H}^{2} + k \|u(t)\|_{E}^{2} + (\alpha - 3\varepsilon)\|u(t)\|_{E}^{2} + (kd - 2\varepsilon)\|\Delta C(t)\|_{L^2}^{2}) (66)
\]

for a.e. \( t \in (0, T) \). By estimates (30)–(33), it is clear that functions \( u_1 \) and \( u_2 \) belong to \( L^2(0, T; E) \cap L^\infty(0, T; H) \), and \( C_1 \) belongs to \( L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; V) \). Integrating (65) over \((0, t)\) for \( t \in (0, T) \) and applying the Gronwall lemma, we obtain

\[
\|u(t)\|_{H}^{2} + k \| \nabla C(t)\|_{L^2(\Omega)}^{2} \leq M(\|u(0)\|^{2} + \|C(0)\|^{2}) \quad (67)
\]

for a.e. \( t \in (0, T) \). Finally, from conditions (58) and (67), we conclude that \( u \equiv 0 \) and \( C \equiv 0 \). This proves the uniqueness of solution to Problem 7.

In this paper we have studied a two-dimensional fluid flow and we have left three-dimensional problem for a future work since this problem would be more difficult and the solution would be probably defined only on a smaller time interval. Moreover, it would be interesting to consider in the future work a problem not with the slip boundary condition, but with a leak boundary condition, see [13] for Navier-Stokes problem with the leak condition.

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