THE KERNEL OF THE RARITA-SCHWINGER OPERATOR ON RIEMANNIAN SPIN MANIFOLDS

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ABSTRACT. We study the Rarita-Schwinger operator on compact Riemannian spin manifolds. In particular, we find examples of compact Einstein manifolds with positive scalar curvature where the Rarita-Schwinger operator has a non-trivial kernel. For positive quaternion Kähler manifolds and symmetric spaces with spin structure we give a complete classification of manifolds admitting Rarita-Schwinger fields. In the case of Calabi-Yau, hyperkähler, G$_2$ and Spin(7) manifolds we find an identification of the kernel of the Rarita-Schwinger operator with certain spaces of harmonic forms. We also give a classification of compact irreducible spin manifolds admitting parallel Rarita-Schwinger fields.

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1. Introduction

Rarita-Schwinger fields are the solutions of the classical field equation for spin $\frac{3}{2}$ fields proposed by Rarita and Schwinger in [31]. They can be considered as sections in the kernel of the Rarita-Schwinger operator, a generalization of the classical Dirac operator acting on spinor-valued 1-forms. Rarita-Schwinger fields are important in supergravity and superstring theories. The Rarita-Schwinger equation on a product $M^4 \times B$ with a space-time $M$ and a compact Riemannian manifold $B$ decouples into one equation on $M$ and one on $B$ after introducing a suitable gauge fixing condition. It can be seen that zero modes of the (internal) Rarita-Schwinger operator on $B$ become massless spin $\frac{1}{2}$ fermions on $M$ (cf. [43]). It is important to note that the existence of these zero modes is less restricted than for the Dirac operator. Thus it is interesting to study the kernel of the Rarita-Schwinger operator on compact Riemannian manifolds. The Rarita-Schwinger operator and in particular its index also played an important role in connection with gravitational anomalies and a miraculous anomaly cancellation described in [11] and [44]. An other motivation to study Rarita-Schwinger fields in physics came from a proposal of Penrose to use these fields for a twistorial description of curved space-time, i.e. to formulate and to solve the Einstein vacuum equations ([25], [29]).

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There is a vast physics literature on Rarita-Schwinger fields but there are only comparatively few articles in mathematics directly investigating properties of the Rarita-Schwinger operator. Among the few relevant articles we want to mention the work of M.Y. Wang who studied in [41] the relation between Rarita-Schwinger fields and deformations of Einstein metrics. Then, in the work of Hitchin [18] Rarita-Schwinger fields surprisingly appeared in connection with stable forms and special geometries in dimension 8. Moreover, Branson and Hijazi considered in [3] the Rarita-Schwinger operator as an important example of a conformally invariant 1st order differential operator and established for the first time Weitzenböck formulas for it. The Rarita-Schwinger operator was also much studied in Clifford analysis (on flat spaces) as a generalization of the Dirac operator (cf. [5]). Finally, the Rarita-Schwinger operator, or rather the twisted Dirac operator $D_{TM}$, was important in connection with the elliptic genus. Here its index appeared as the second term in the development of the elliptic genus in its $\hat{A}$-cusp (cf. [17] and [45]).

In the present paper we study the Rarita-Schwinger operator on compact Riemannian spin manifolds. We are mainly interested in its kernel, i.e. the existence or non-existence of Rarita-Schwinger fields. For the classical Dirac operator the well-known argument of Lichnerowicz implies that on a compact spin manifold of positive scalar curvature the Dirac operator has a trivial kernel. However, for the Rarita-Schwinger operator this argument does not work, the formula for its square is more complicated. Hence, it is interesting to see to which extend Weitzenböck formulas for the Rarita-Schwinger operator can be applied and to find examples of compact spin manifolds of positive scalar curvature admitting non-trivial Rarita-Schwinger fields.

In the first part of our article we give a precise definition of the Rarita-Schwinger operator and of Rarita-Schwinger fields. We derive several interesting Weitzenböck formulas using the approach of M.Y. Wang in [41]. These formulas simplify a lot for Einstein manifolds which we usually will assume for our applications. Using index calculations in dimensions 8 and 12 we find Rarita-Schwinger fields on certain complete intersections, in particular Fermat surfaces, with Kähler-Einstein metrics. We discuss the examples of Rarita-Schwinger fields on 8-dimensional manifolds with a PSU(3)- or $\text{Sp}(1) \cdot \text{Sp}(2)$-structure. Moreover, we completely determine the kernel of the Rarita-Schwinger operator on quaternion Kähler manifolds of positive scalar curvature, on irreducible symmetric spaces with spin structure, on Calabi-Yau and Hyperkähler manifolds and on manifolds with $G_2$ or Spin(7) holonomy. For the last two cases we reprove results of M.Y. Wang from [41] using simpler methods. In the Calabi-Yau and hyperkähler case we extend results from [41] in dimension 6 and 8 to arbitrary dimensions. Our main tool are Weitzenböck formulas which give a relation between the square of the Rarita-Schwinger operator and the standard Laplace operator. The standard Laplace operator is a natural Laplace type operator on geometric vector bundles (cf. [35]). It coincides with the Hodge-Laplace operator on parallel subbundles of the form bundle and thus reveals an interesting relation between harmonic forms and Rarita-Schwinger fields. A consequence of our discussion is the classification of compact irreducible Riemannian spin
manifolds admitting parallel Rarita-Schwinger fields. A similar classification for spinor fields was done by M.Y. Wang in [41]. However, the situation for the Rarita-Schwinger operator turns out to be much more restrictive. We show that parallel Rarita-Schwinger fields only exist on hyperkähler manifolds and on the five symmetric spaces in dimension 8 listed in Theorem 4.5.

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2. Preliminaries

Let \((M^n, g)\) be a Riemannian spin manifold with spinor bundle \(S_{1/2}\). Then there is the well-known decomposition \(S_{1/2} \otimes TM^C \cong S_{1/2} \oplus S_{3/2}\), where \(S_{3/2}\) is the kernel of the Clifford multiplication \(\mu : S_{1/2} \otimes TM^C \rightarrow S_{1/2}, \varphi \otimes X \mapsto X \cdot \varphi\). The spinor bundle \(S_{1/2}\) is embedded into the tensor product \(S_{1/2} \otimes TM^C\) by the map \(\iota : S_{1/2} \rightarrow S_{1/2} \otimes TM^C, \varphi \mapsto -\frac{i}{n} \sum e_i \cdot \varphi \otimes e_i\), where \(\{e_i\}\) is a locally defined orthonormal frame. Then \(S_{1/2} \otimes TM^C = i(S_{1/2}) \oplus S_{3/2}\) and the projection onto the second summand \(pr_{S_{3/2}} : S_{1/2} \otimes TM^C \rightarrow S_{3/2}\) is given by \(pr_{S_{3/2}} = id - i \circ \mu\).

On spinors, i.e. sections of the spinor bundle \(S_{1/2}\), there are two natural 1st order differential operators. The Dirac operator \(D : \Gamma(S_{1/2}) \rightarrow \Gamma(S_{1/2})\) defined by \(D = \mu \circ \nabla\) and the twistor or Penrose operator \(P : \Gamma(S_{1/2}) \rightarrow \Gamma(S_{3/2})\) defined by \(P = pr_{S_{3/2}} \circ \nabla\). Considering sections of \(S_{3/2}\) as spinor-valued 1-forms the adjoint operator \(P^* : \Gamma(S_{3/2}) \rightarrow \Gamma(S_{1/2})\) can be written as

\[
P^* \psi = \delta \psi = - \sum_i (\nabla_{e_i} \psi)(e_i).
\]

Next we consider the twisted Dirac operator \(D_{TM} : \Gamma(S_{1/2} \otimes TM^C) \rightarrow \Gamma(S_{1/2} \otimes TM^C)\) defined by \(D_{TM} = \mu \circ \nabla_{S_{1/2} \otimes TM}\), i.e. locally we have \(D_{TM}(\varphi \otimes X) = D\varphi \otimes X + \sum e_i \cdot \varphi \otimes \nabla_{e_i} X\). With respect to the decomposition \(S_{1/2} \otimes TM^C = i(S_{1/2}) \oplus S_{3/2}\) the twisted Dirac operator \(D_{TM}\) takes the matrix form (cf. [11], Prop. 2.7)

\[
D_{TM} = \left(\begin{array}{cc}
\frac{2-n}{n} i \circ D \circ i^{-1} & 2 i \circ P^* \\
\frac{2}{n} P \circ i^{-1} & Q
\end{array}\right).
\]

The operator \(Q : \Gamma(S_{3/2}) \rightarrow \Gamma(S_{3/2})\) with \(Q = pr_{S_{3/2}} \circ D_{TM}|_{\Gamma(S_{3/2})}\) is called Rarita-Schwinger operator. In the physics literature a Rarita-Schwinger field is a section \(\psi\) of \(S_{3/2}\) satisfying the equations \(P^* \psi = 0\) and \(Q \psi = 0\), or equivalently \(D_{TM} \psi = 0\) and \(\mu(\psi) = 0\), i.e. \(\psi \in \Gamma(S_{3/2})\). Note, that the operators \(D, D_{TM}\) and in particular the Rarita-Schwinger operator \(Q\) are all formally self adjoint.
Computing $D^2_{TM}$ using the matrix presentation given in (1) one has three immediate consequences (cf. [11], Prop. 2.9): the well-known Weitzenböck formula

\begin{equation}
\frac{1}{2} P^* P - \frac{n-1}{2n} D^2 = -\frac{\text{scal}}{8},
\end{equation}

a formula for the action of $D^2_{TM}$ on sections of $S_{3/2}$

\begin{equation}
\text{pr}_{S_{3/2}} \circ D^2_{TM}|_{\Gamma(S_{3/2})} = \frac{4}{n} P P^* + Q^2
\end{equation}

and the equation

\begin{equation}
\frac{2-n}{n} P \circ D + Q \circ P = \frac{1}{2} (\text{Ric} - \text{scal})
\end{equation}

where we consider an endomorphism $F \in \text{End} TM$ as a map $F : S_{1/2} \rightarrow S_{3/2}$ by defining $F(\varphi)(X) = F(X) \cdot \varphi$ for any spinor $\varphi$ and any vector field $X$. In particular, if the metric $g$ is Einstein the right-hand side vanishes and we obtain the two equations

\begin{equation}
Q \circ P = \frac{n-2}{n} P \circ D \quad \text{and} \quad P^* \circ Q = \frac{n-2}{n} D \circ P^*.
\end{equation}

We remark that a direct consequence of (4) is the well-known integrability condition for the Rarita-Schwinger equation (cf. [18], [22]): if $\psi = \sum \varphi_k \otimes e_k \in \Gamma(S_{3/2})$ is a Rarita-Schwinger field, i.e. $P^* \psi = 0 = Q \psi$, then the Ricci tensor satisfies the condition $\sum \text{Ric}(e_k) \cdot \varphi_k = 0$. Indeed, it follows from (4) that $(\text{Ric} - \text{scal})^* \psi = 0$ for a Rarita-Schwinger field $\psi$. But if $F$ is a symmetric endomorphism of $TM$ the adjoint map $F^* : S_{3/2} \rightarrow S_{1/2}$ is given by $F^*(\psi) = \sum F(e_k) \cdot \varphi_k$ and the formula for the Ricci tensor follows since $\psi$ is assumed to be in the kernel of the Clifford multiplication.

Let $M$ be a compact spin manifold. Since $P^* P$ is elliptic we have the decomposition $\Gamma(S_{3/2}) = \text{Ker} P^* \oplus \text{Im} P$ and it immediately follows from (5) that the Rarita-Schwinger operator $Q$ preserves this decomposition in the case of Einstein manifolds.

On any vector bundle $VM$ associated to the frame bundle, or the $\text{Spin}(n)$-principle bundle of the fixed spin structure, we have a natural 2nd order differential operator: the standard Laplace operator $\Delta_V$ as introduced in [35]. If $\nabla$ denotes the covariant derivative on $VM$ induced by the Levi-Civita connection then $\Delta_V := \nabla^* \nabla + q(R)$, where $q(R) \in \text{End} VM$ is a curvature term defined by $q(R) = \frac{1}{2} \sum (e_i \wedge e_j)_* \circ R(e_i \wedge e_j)_*$. Here $\Lambda^2 T$ is identified with $\mathfrak{so}(n)$ and $(X \wedge Y)_*$ denotes the action of $X \wedge Y \in \Lambda^2 T$ via the differential of the representation defining the bundle $VM$. For details we refer to [35]. The definition of $\Delta_V$ is a generalization of the classical Weitzenböck formula $\Delta = d^* d + d d^* = \nabla^* \nabla + q(R)$ for the Hodge-Laplace operator on differential forms. An important property of $\Delta_V$ is that it depends only on the defining representation $V$ and not on the particular embedding in some larger bundle, e.g. if $VM$ is a parallel subbundle of the form bundle $\Lambda^* T M$, then the restriction of the Laplace operator $\Delta$ to sections of $VM$ can be identified with $\Delta_V$.

We will need the following two Weitzenböck formulas:

\begin{equation}
D^2 = \Delta_{S_{1/2}} + \frac{\text{scal}}{8} \quad \text{and} \quad D^2_{TM} = \Delta_{S_{1/2} \otimes T} + \frac{\text{scal}}{8} - \text{id} \otimes \text{Ric}.
\end{equation}
Both formulas can be proved by an easy local calculation (cf. [33]). Note that as for the Laplace operator we have that the restriction of $\Delta_{S_{3/2} \otimes TM}$ to sections of the parallel subbundle $S_{3/2}$ coincides with the standard Laplacian $\Delta_{S_{3/2}}$. From this remark, the second Weitzenböck formula in (6) and the expression in (3) for the action of $D^2_M$ on sections of $S_{3/2}$ we obtain

$$Q^2 + \frac{4}{n} PP^* = \Delta_{S_{3/2}} + \frac{\text{scal}}{8} - \text{Ric}^{3/2}$$

where the endomorphism $\text{Ric}^{3/2}$ is defined on sections $\psi = \sum \varphi_i \otimes e_i$ by the local formula

$$\text{Ric}^{3/2}(\psi) = (\text{pr}_2 \circ (\text{id} \otimes \text{Ric}))(\psi) = \sum_{i,k} (\text{Ric}_{ik} \varphi_i + \frac{1}{n} \sum_j \text{Ric}_{ij} e_k \cdot e_j \cdot \varphi_i) \otimes e_k .$$

In particular we see that $\text{Ric}^{3/2} = \frac{\text{scal}}{n} \text{id}$ on Einstein manifolds.

For the rest of this article we will restrict to compact Einstein manifolds. Here we already know that the Rarita-Schwinger operator $Q$ preserves the splitting $\Gamma(S_{3/2}) = \text{Ker} P^* \oplus \text{Im} P$. More precisely we have

**Proposition 2.1.** Let $(M^n, g)$ be an Einstein spin manifold, then:

(i) $Q^2 = \Delta_{S_{3/2}} + \frac{n-8}{8n} \text{scal} \quad \text{on sections of } \text{Ker} P^*$

(ii) $Q^2 = (\frac{n-2}{n})^2 (\Delta_{S_{3/2}} + \frac{\text{scal}}{8}) \quad \text{on sections of } \text{Im} P.$

**Proof.** Equation (i) directly follows from the Weitzenböck formula (7) restricted to sections of $\text{Ker} P^*$ in the case of Einstein manifolds. For proving (ii) we apply $Q$ to the first formula in (5) and obtain $Q^2 \circ P = \frac{n-2}{n} Q \circ P \circ D$ using the same formula again we conclude

$$Q^2 \circ P = (\frac{n-2}{n})^2 P \circ D^2 = (\frac{n-2}{n})^2 P \circ (\Delta_{S_{3/2}} + \frac{\text{scal}}{8}) = (\frac{n-2}{n})^2 (\Delta_{S_{3/2}} + \frac{\text{scal}}{8}) \circ P .$$

Here the second equality is obtained by the first Weitzenböck formula in (4) and the third equality is implied by the commutator formula $P \circ \Delta_{S_{1/2}} = \Delta_{S_{3/2}} \circ P$. This formula is contained in [20] and also follows from the general commutator formula in [33]. However, it is easily checked directly as we will show now. \qed

**Lemma 2.2.** Let $(M^n, g)$ be an Einstein spin manifold then $\Delta_{S_{3/2}} \circ P = P \circ \Delta_{S_{1/2}}$.

**Proof.** Using Weitzenböck formula (7) in the case of Einstein manifolds, i.e. with $\text{Ric}^{3/2} = \frac{\text{scal}}{n}$, we can replace $\Delta_{S_{3/2}}$ and obtain $\Delta_{S_{3/2}} \circ P = Q^2 \circ P + \frac{4}{n} PP^* P + \frac{(8-n)\text{scal}}{8n} P$ . Then we use the first equation in (5) and the Weitzenböck formula (2) to get

$$\Delta_{S_{3/2}} \circ P = ((\frac{n-2}{n})^2 + \frac{4}{n} \frac{n-1}{n}) P \circ D^2 - \frac{4}{n} \frac{\text{scal}}{4} P + \frac{(8-n)\text{scal}}{8n} P = P \circ D^2 - \frac{\text{scal}}{8} P .$$

Hence, it follows $\Delta_{S_{3/2}} \circ P = P \circ \Delta_{S_{1/2}}$ from the first Weitzenböck formula in (6). \qed

As a corollary to Proposition 2.1 and to Weitzenböck formula (7) we also have an interesting product formula for the square of the Rarita-Schwinger operator. Our formula is a generalization to curved manifolds of a similar result in the flat case (cf. [33]).
Corollary 2.3. Let \((M^n, g)\) be an Einstein spin manifold then
\[
\left(Q^2 - \left(\frac{n-2}{n}\right)^2((\Delta_{S^3/2} + \text{scal}) \right) \circ \left(Q^2 - \left(\frac{n-8}{8n}\right)\text{scal}\right) = 0 .
\]

Proof. Since \((M, g)\) is Einstein we can write (7) as: \(Q^2 - (\Delta_{S^3/2} + \frac{n-8}{8n}\text{scal}) = -\frac{A}{n} P P^*\) and we can write (ii) of Proposition 2.1 as: \((Q^2 - \left(\frac{n-2}{n}\right)^2(\Delta_{S^3/2} + \frac{\text{scal}}{8})) \circ P = 0.\) Combining these two equations proves the corollary. \(\Box\)

3. The Index of the Rarita-Schwinger Operator

Let \((M^n, g)\) be an even-dimensional spin manifold. In this section we do not have to assume the metric \(g\) to be Einstein. The splitting of the spin representation \(\Sigma_n\) into the half-spinor spaces \(\Sigma^\pm_n\), induces the corresponding splittings \(S_{1/2}^+ = S_{1/2}^+ \oplus S_{1/2}^-\) and \(S_{3/2}^\pm = S_{3/2}^+ \oplus S_{3/2}^-,\) with \(S_{3/2}^\pm \subset S_{1/2}^\pm \otimes TM\). With the notation \(Q^\pm := Q|_{\Sigma^\pm_{3/2}}\) the index of the Rarita-Schwinger operator \(Q\) is defined as \(\text{ind } Q = \dim \ker Q^+ - \dim \ker Q^-\). For calculating the index of \(Q\) we use Theorem 13.13 in [24] (cf. [2], Prop. 2.17). In our situation with \(n = 2m\) it implies
\[
\text{ind } Q = (-1)^m \left(\frac{\text{ch}(S_{3/2}^+) - \text{ch}(S_{3/2}^-)}{e(TM)} \hat{A}(TM)^2\right) [M] ,
\]
where \(\hat{A}(TM)\) and \(e(TM)\) are the \(\hat{A}\)-class and the Euler class of \(TM\), and \(\text{ch}(S_{3/2}^\pm)\) is the Chern character of \(S_{3/2}^\pm\). Using the properties of the Chern character and the splittings above we obtain:
\[
\text{ch}(S_{1/2}^+) \text{ch}(TM^C) = \text{ch}(S_{1/2}^+ \otimes TM^C) = \text{ch}(S_{3/2}^+) + \text{ch}(S_{3/2}^-) .
\]
Subtracting these two equations gives
\[
(\text{ch}(S_{3/2}^+) - \text{ch}(S_{3/2}^-)) + (\text{ch}(S_{1/2}^+) - \text{ch}(S_{1/2}^-)) = (\text{ch}(S_{1/2}^+) - \text{ch}(S_{1/2}^-)) \text{ch}(TM^C)
\]
and thus we arrive at
\[
\text{ch}(S_{3/2}^+) - \text{ch}(S_{3/2}^-) = (\text{ch}(TM^C) + 1) (\text{ch}(S_{1/2}^+) - \text{ch}(S_{1/2}^-)) .
\]
On the other hand an easy calculation (using the weights of the spinor representation) gives \(\hat{A}(TM) = (-1)^m \frac{\text{ch}(S_{1/2}^+) - \text{ch}(S_{1/2}^-)}{e(TM)} \hat{A}(TM)^2\). This also follows from Theorem 13.13 of [24] for the classical Dirac operator \(D\). Substituting this formula for \(\hat{A}(TM)\) and equation (10) into (11) we finally obtain the relation
\[
\text{ind } Q = \hat{A}(TM) (\text{ch}(TM^C) + 1)[M] = \text{ind } D_{TM} + \text{ind } D .
\]

This formula can be used in small dimensions to relate the index of the Rarita-Schwinger operator to other topological invariants such as the Euler characteristic \(\chi(M)\), the signature \(\sigma(M)\) and the \(\hat{A}\)-genus \(\hat{A}(M) = \hat{A}(TM)[M]\). For \(n = 4, 8\) and \(12\) we have (cf. [4])
Proposition 3.1. Let $(M^n, g)$ be a compact spin manifold, then

(i) $n = 4$ : \[ \text{ind} Q = -19 \hat{A}(M) = \frac{41}{8} \sigma(M) \]

(ii) $n = 8$ : \[ \text{ind} Q = 25 \hat{A}(M) - \sigma(M) \]

(iii) $n = 12$ : \[ \text{ind} Q = 5 \hat{A}(M) + \frac{1}{8} \sigma(M) . \]

Remark 3.2. On any 8-dimensional manifold with a structure group reduction to one of the groups $\text{Sp}(1) \cdot \text{Sp}(2)$, $\text{Spin}(7)$, $\text{SU}(4)$ or $\text{Sp}(2)$ one has the formula \[ \chi = -\frac{1}{8} (p_2 - 4p_2) \] (cf. [32], p. 166). Comparing this with the corresponding expressions for the $\hat{A}$ and $L$ genus we obtain \[ 16 \hat{A}(M) - \sigma(M) = -\frac{1}{3} \chi(M) \] and thus for all these manifolds we can rewrite the index as \[ \text{ind} Q = 25 \hat{A}(M) - \sigma(M) = 9 \hat{A}(M) - \frac{1}{3} \chi(M). \]

Remark 3.3. Recall that $\hat{A}(M)$ is zero on compact spin manifolds admitting a metric with positive scalar curvature. We also note that the signature $\sigma(M)$ of a compact spin manifold in dimensions $8k + 4$ is divisible by 16 (cf. [28]). Moreover in these dimensions the bundles $S_{1/2}$ and $S_{3/2}$ have a quaternionic structure and thus the index of the Rarita-Schwinger operator $Q$ and the index of the Dirac operator $D$ are even numbers.

Remark 3.4. Let $M \times N$ be a Riemannian product of two spin manifolds. Then a direct consequence of (11) is the following formula for index of the Rarita-Schwinger operator on $M \times N$: \[ \text{ind} Q^{M \times N} = \text{ind} Q^M \text{ind} D^N - \text{ind} D^M \text{ind} D^N + \text{ind} D^M \text{ind} Q^N. \] In particular, the index of $Q^{M \times N}$ vanishes if the factors $M$ and $N$ do not admit harmonic spinors. Recall that a product manifold is spin if and only if all its factors are spin.

Remark 3.5. If $M^n$ is a compact, homogeneous spin manifold with $n \not\equiv 0 \mod 8$, then $\text{ind} Q$, as well as $\hat{A}(M)$ and $\sigma(M)$ all vanish. Moreover, since the elliptic genus on compact homogeneous spin manifolds of dimension $n = 8k$ is the constant modular function $\Phi(M) = \sigma(M)$ it follows that $\text{ind} D$ and $\text{ind} D_{TM}$, and thus also $\text{ind} Q$, vanish in all dimensions $n \geq 12$ (cf. [16], Th. 2.3). There are many results for general spin manifolds stating the vanishing of $\text{ind} D$ and $\text{ind} D_{TM}$ under certain assumptions, e.g. for manifolds $M^n, n \neq 8$, with positive sectional curvature, $b_2(M) = 0$ and an effective isometric $S^1$ action or for manifolds $M$ admitting a smooth non-trivial $S^1$-action and with $b_4(M) = 0$ ([11], [12]).

Remark 3.6. In applications of the Rarita-Schwinger operator in supergravity and superstring theory, the index of the Rarita-Schwinger operator is calculated as $\text{ind} D_{TM} - \text{ind} D$. This is motivated by the necessity of "discarding zero modes that can be gauged away or that violate gauge conditions". These are "cancelled by zero modes of the spin $\frac{1}{2}$ ghost fields". Thus in physics one has to "subtract from index of the Rarita-Schwinger field the corresponding index of the spin $\frac{1}{2}$ ghosts" (cf. [43], p. 252). This modification of the index can also be explained with a structure group reduction from $\text{SO}(1, D - 1)$ to $\text{SO}(D - 2)$ and by considering the twisted Dirac operator between suitable virtual vector bundle (cf. [14]).
4. The kernel of the Rarita-Schwinger operator

In this section we want to show the existence of Rarita-Schwinger fields on various types of Riemannian manifolds, in particular on compact Einstein manifolds of positive scalar curvature.

4.1. Rarita-Schwinger operator on compact Einstein manifolds. Note that for compact manifolds we have $\text{Ker} \, Q^2 = \text{Ker} \, Q$, because $Q$ is formally self-adjoint. Hence, Proposition 2.1 enables us to identify the kernel of $Q$ with the eigenspace of $\Delta_{S_{3/2}}$ for the eigenvalue $-\frac{n-8}{n}$ on $\text{Ker} \, P^*$ and for the eigenvalue $-\frac{\text{scal}}{8}$ on $\text{Im} \, P$. In particular we see in case (i), i.e. on $\text{Ker} \, P^*$, that for $n = 8$ the kernels of $Q$ and $\Delta_{S_{3/2}}$ coincide. The same is true if $\text{scal} = 0$, e.g. for Ricci-flat manifolds. The case (ii) in Proposition 2.1 turns out to be more restrictive. Here we have

**Proposition 4.1.** If $(M^n, g), n \geq 3$, is a compact Einstein manifold with non-negative scalar curvature, then $Q^2 \geq \frac{(n-2)^2 \text{scal}}{n(n-1)}$ holds on $\text{Im} \, P$ and in particular the kernel of $Q$ is trivial.

**Proof.** We multiply Weitzenböck formula (2) with $\frac{(n-2)^2}{n^2} P D^2 = \frac{(n-2)^2}{n(n-1)}(P P^* P + \frac{\text{scal}}{4} P)$, then we can use the first equation of (8) to obtain $Q^2 P \varphi = \frac{(n-2)^2}{n(n-1)}(P P^* P \varphi + \frac{\text{scal}}{4} P \varphi)$ for any section $\varphi$ of the spinor bundle $S_{1/2}$. Taking the $L^2$-scalar product with $P \varphi$ in the last equation implies

$$(Q^2 P \varphi, P \varphi) = \frac{(n-2)^2}{n(n-1)} \left( |P^* P \varphi|^2 + \frac{\text{scal}}{4} |P \varphi|^2 \right) \geq \frac{(n-2)^2}{n(n-1)} \frac{\text{scal}}{4} |P \varphi|^2.$$ 

This proves the eigenvalue estimate for the operator $Q^2$ on $\text{Im} \, P$. Now, let $P \varphi$ be in the kernel of $Q$, then we obtain $|P^* P \varphi|^2 = 0$ from (12), thus $P^* P \varphi = 0$ and taking a scalar product with $\varphi$ it follows that $|P \varphi|^2 = 0$ and finally $P \varphi = 0$. Hence $Q$ has no non-trivial kernel on $\text{Im} \, P$ if the scalar curvature is non-negative. 

**Remark 4.2.** In the case of Einstein spin manifolds with negative scalar curvature the Dirac operator can have a non-trivial kernel and if $\varphi$ is any harmonic spinor then by (5) we have $Q(P \varphi) = 0$, i.e. the Rarita-Schwinger operator has a non-trivial kernel on $\text{Im} \, P$. Note that by (2) the map $\varphi \mapsto P \varphi$ is injective on $\ker \, D$ if the scalar curvature is non-zero.

Examples for Einstein spin manifold with negative scalar curvature admitting harmonic spinors can be found as complete intersections. Let $X_n(d_1, \ldots, d_r)$ be the complete intersection of hypersurfaces defined by homogeneous polynomials of degree $d_1, \ldots, d_r$ in $\mathbb{C}P^{n+r}$ and denote with $d = d_1 + \ldots + d_r$ the sum of the degrees. Then it is well-known that the 1st Chern class is given by $c_1(X_n(d_1, \ldots, d_r)) = (n+r+1-d) h$, where $h$ is the pull-back of the standard generator of $H^2(\mathbb{C}P^{n+r})$. Hence, $X_n(d_1, \ldots, d_r)$ is spin for $n+r-d$ odd and by the Calabi-Yau Theorem it has a Kähler-Einstein metric of negative scalar curvature if $n+r+1 < d$. On the other hand there is a very easy formula for the $\hat{A}$-genus of $X_{2n}(d_1, \ldots, d_r)$ and it turns out that for $r-d$ odd $\hat{A}(X_{2n}(d_1, \ldots, d_r))$ is different from zero precisely if $2n + r + 1 < d$ (cf.
Hence, for \( r - d \) odd and \( 2n + r + 1 < d \) any complete intersection \( X_{2n}(d_1, \ldots, d_r) \) carries non-trivial harmonic spinors and thus the Rarita-Schwinger operator has a non-trivial kernel on \( \text{Im} P \).

### 4.2. Application of the index calculation.

Let \((M^n, g)\) be a compact Riemannian spin manifold. If the index of the Rarita-Schwinger operator is non-zero then its kernel is automatically non-trivial. This can be used to produce many examples of manifolds with Rarita-Schwinger fields. We are particularly interested in Einstein spin manifolds with positive scalar curvature. By applying Proposition 3.1 we find examples in dimensions 8 and 12.

As an example we consider the Fermat surface \( X_m(d) \subset \mathbb{C}P^{m+1} \). It is a special case of a complete intersection defined by one homogeneous polynomial of degree \( d \). The 1st Chern class is \( c_1(X_m(d)) = (m + 2 - d)h \). Thus \( X_m(d) \) is spin if and only if \( m - d \) is even and \( c_1(X_m(d)) \) is positive if \( d \leq m + 1 \). In this situation the existence of a Kähler-Einstein metric with positive scalar curvature was shown under the condition \( m+1 \leq d \leq m+1 \) (cf. [39], [27]), e.g. for \( X_4(4) \) in real dimension 8, or \( X_6(4) \) and \( X_6(6) \) in real dimension 12.

The signature \( \sigma(X_m(d)) \) can be calculated as the coefficient of \( z^{m+1} \) in the power series expansion of \( \frac{1}{1 - z^2} \frac{(1+z)^d-(1-z)^d}{(1+z)^{d/2}+(1-z)^{d/2}} \) (cf. [15], Section 22), e.g. the signature is 100 for \( X_4(4) \), \(-576 \) for \( X_6(4) \) and \(-12544 \) for \( X_6(6) \). Since all three manifolds are spin manifolds with a positive scalar curvature metric their \( \hat{A} \) genus vanishes and by Proposition 3.1 the Rarita-Schwinger operator has non-vanishing index and thus also non-trivial kernel. Hence, we have our first examples of compact Einstein manifolds of positive scalar curvature admitting Rarita-Schwinger fields.

Other examples are the Fermat surfaces \( X_2(6) \), \( X_4(8) \) and \( X_6(10) \). They are all spin and have a negative 1st Chern class. Hence they admit a Kähler-Einstein metric of negative scalar curvature. The computation of the \( \hat{A} \) genus gives 8, 12 and 16. It follows that all three spaces carry harmonic spinors and, according to Remark 4.2, the kernel of \( Q \) on \( \text{Im} P \) is non-trivial. Moreover, since \( P \) is injective on \( \ker D \) and maps sections of \( S_{1/2}^+ \) to sections of \( S_{3/2}^\pm \), we have:

\[
\text{ind } Q = \text{ind } D + \dim \ker Q^+|_{\ker P^*} - \dim \ker Q^-|_{\ker P^*}.
\]

Using Proposition 3.1 we see that in all three cases \( \text{ind } Q \neq \text{ind } D \). This is clear in the 4-dimensional case. For the other two cases we have to calculate the signature which is 4040 in the 8-dimensional and \(-505088 \) in the 12-dimensional case. In consequence we obtain three examples of Kähler-Einstein manifolds with negative scalar curvature admitting non-trivial Rarita-Schwinger fields.

A special case of the Fermat surfaces is the complex quadric \( Q_m = X_m(2) \). It is spin if and only if \( m \) is even and in this case the signature of \( Q_m \) is 2 for \( m = 4k \) and 0 for \( m = 4k + 2 \). The complex quadric \( Q_m \) can also be written as the Riemannian symmetric space \( Q_m = \text{SO}(m+2)/\text{SO}(m) \times \text{SO}(2) \) and in particular it is Einstein with positive scalar curvature. Thus the \( \hat{A} \)-genus vanishes for even \( n \) and Proposition 3.1 implies for \( n = 8 \) that the index of the Rarita-Schwinger operator is \( \text{ind } Q = -\sigma(M) = -2 \). Hence, on the 8-dimensional
complex quadric $Q$, the Rarita-Schwinger operator $Q$ has at least a two-dimensional kernel. Later we will see that the kernel is exactly 2-dimensional.

4.3. Harmonic PSU(3)- and Sp(1) · Sp(2)-structures. We will consider a special class of 8-dimensional manifolds with a structure group reduction to PSU(3) and Sp(1) · Sp(2), respectively. Both structures induce a Riemannian metric, they are automatically spin and they admit a Rarita-Schwinger operator with a non-trivial kernel (cf. [18], [42]).

The reduction to PSU(3) is defined by a stable 3-form $\rho$, i.e. in any point $x \in M$ the form $\rho_x$ lies in an open orbit of the GL(8)-action on $\Lambda^3T^*_x M$. The structure is called harmonic if $\rho$ is harmonic form, i.e. $\Delta \rho = 0$. This is equivalent to the existence of a Rarita-Schwinger field, cf. [18], Theorem 3 and [42], Theorem 30. The simplest example is the group SU(3) itself with the canonical invariant 3-form $\rho(X,Y,Z) = B([X,Y],Z)$, where $B$ is the Killing form of SU(3). The metric $g$ induced by the Killing form is the symmetric metric on the Riemann symmetric space $SU(3) = SU(3) \times SU(3)/SU(3)$. In particular, $g$ is Einstein with positive scalar curvature. The 3-form $\rho$ is parallel with respect to the Levi-Civita connection of $g$.

There are no other known examples of Einstein PSU(3)-manifolds. Other compact examples are given in the form $M = T^2 \times N^6$, where $N^6$ is a certain 6-dimensional nilmanifold. The manifold $M$ in this example has constant negative scalar curvature (cf. [12]).

The reduction to Sp(1) · Sp(2) is defined by a stable 4-form $\Omega$ and again the structure is called harmonic if $\Omega$ is harmonic, which is equivalent to $\Omega$ being closed. Note that this is a speciality of dimension 8, in all other dimensions 4$m$ the 4-form defining the structure group reduction to Sp(1) · Sp($m$) is closed if and only if it is parallel and then the manifold is by definition quaternion Kähler (cf. [38], Theorem A.3). Thus the simplest examples of a harmonic Sp(1) · Sp(2)-structure are the three quaternion Kähler manifolds in dimension 8 (cf. Theorem 4.3). These are Einstein manifolds with positive scalar curvature and parallel 4-form $\Omega$. Similar to the PSU(3)-case, one does not know any compact Einstein examples with non-parallel $\Omega$.

The first compact example of a harmonic Sp(1) · Sp(2)-manifold with non-parallel 4-form $\Omega$ was given by S. Salamon in [33]. Again it is of the form $T^2 \times N^6$, with a certain compact nilmanifold $N$. The manifold $M$ has constant negative scalar curvature but it is not Einstein. Other compact manifolds were recently constructed by D. Conti and T.B. Madsen in [8] as a family of nilmanifolds and by D. Conti, T.B. Madsen and S. Salamon in [9] as a family of metrics on $G_2/\text{SO}(4)$ obtained by a perturbation of the quaternion Kähler metric. The parameter can be chosen to obtain a metric of constant positive scalar curvature.

4.4. The Rarita-Schwinger operator on quaternion Kähler manifolds. A Riemannian manifold $(M^{4m}, g)$, $m \geq 2$, is quaternion Kähler if its holonomy group is contained in the group Sp(1) · Sp($m$). Quaternion Kähler manifolds are automatically Einstein. They are spin in even quaternion dimensions, with the exception of the quaternion projective spaces $\mathbb{H}P^m$ which are spin in all dimensions. Using the $H, E$ formalism of S. Salamon we can write
the spinor bundle as $S_{3/2} = \bigoplus_{k=0}^{m} \text{Sym}^{m-k} H \otimes \Lambda_{b}^{k} E$ and the complexified tangent bundle as $TM^{c} = H \otimes E$. Here $H = \mathbb{H}$ and $E = \mathbb{H}^{m}$ denote the standard representations of $\text{Sp}(1)$ and $\text{Sp}(m)$, respectively. We use the same notation for the corresponding locally defined associated vector bundles on $M$. Recall that a tensor product of $H$ and $E$ factors gives rise to a globally defined vector bundle if the number of factors is even or if $M = \mathbb{HP}^{m}$.

**Theorem 4.3.** Let $(M^{4m}, g)$ be a complete quaternion Kähler spin manifold of positive scalar curvature. Then the Rarita-Schwinger operator $Q$ is positive on $\text{Im} P$ and it has a non-trivial kernel on $\text{Ker} P^{*}$ only for $m = 2$. In this case the kernel is 2-dimensional for $M = \text{Gr}_{2}(\mathbb{C}^{4})$ and 1-dimensional for $M = \mathbb{HP}^{2}$ and $M = G_{2}/\text{SO}(4)$.

**Proof.** We first note that the operator $\Delta_{S_{3/2}}$ is non-negative on the bundle $S_{3/2}$. Indeed, since $S_{3/2}$ is a subbundle of the tensor product $S_{1/2} \otimes TM^{c}$ it decomposes into a sum of bundles defined by representations of the type $\text{Sym}^{d} H \otimes \Lambda_{a}^{b} E$, where $0 \leq d \leq m$ and $0 \leq b \leq a \leq m$. Here $\Lambda_{a}^{b} E$ is the $\text{Sp}(m)$-representation given by the Cartan summand in $\Lambda_{a}^{b} E \otimes \Lambda_{b}^{a} E$. On these bundles $V$ one has the following lower bound for the operator $\Delta_{V}$:

$$\Delta_{V} \geq \frac{\text{scal}}{8m(m+2)}(d + a - b)(d - a - b + 2m + 2).$$

Hence, the standard Laplace operator $\Delta_{V}$ is non-negative and it can have a non-trivial kernel only for representations $V$ with $d = 0$ and $a = b$. The estimates follows from Theorem 8.5 in [19] and also from Proposition 3.5 and Theorem 4.4 in [34].

Then, since the scalar curvature of $M^{4m}$ is positive, Proposition 2.1 implies that the Rarita-Schwinger operator $Q$ is positive on $\text{Im} P$ for all $m$ and on $\text{Ker} P^{*}$ for $m > 2$. Moreover, for $m = 2$ it follows that $\text{Ker} Q^{2} = \text{Ker} \Delta_{S_{3/2}}$ holds on $\text{Ker} P^{*}$.

In order to prove Theorem 4.3 we have to consider the operator $\Delta_{S_{3/2}}$ on complete quaternion Kähler manifold $M$ of positive scalar curvature in real dimension 8. It is well known (cf. [30]) that these manifolds are isometric to one of the three symmetric spaces to $\mathbb{HP}^{2}, G_{2}/\text{SO}(4)$ (with $b_{2}(M) = 0$) or to the complex Grassmannian $\text{Gr}_{2}(\mathbb{C}^{4})$ (with $b_{2}(M) = 1$). In quaternion dimension $m = 2$ the spinor bundle is defined by the representation $\text{Sym}^{2} H \oplus (H \otimes E) \oplus \Lambda_{b}^{2} E$. It is then easy to check that the only representations $V$ of the form $\Lambda_{a}^{b} E$ in $S_{3/2}$ are the trivial representation $\mathbb{C}$ and the representation $\Lambda_{1,1}^{1} E = \text{Sym}^{2} E$. These representations define parallel subbundles in $\Lambda^{2} TM$ and the operator $\Delta_{S_{3/2}}$ restricted to these subbundles coincides with the Hodge-Laplace operator $\Delta = dd^{*} + d^{*} d$, thus $\dim \text{Ker} Q = b_{2}(M) + 1$. The last statement of the theorem follows from the remark above about the values of the Betti numbers. □

**Remark 4.4.** The proof shows that on 8-dimensional quaternion Kähler manifolds there is a one-dimensional summand in the kernel of the Rarita-Schwinger operator corresponding to the trivial representation and hence to a parallel section of $S_{3/2}$. This is a special case of the harmonic $\text{Sp}(1) \cdot \text{Sp}(m)$- structures discussed in [32], Theorem 28. In our situation the defining 4-form is the parallel Kraines form corresponding to the parallel Rarita-Schwinger field. However, also the additional Rarita-Schwinger field on $\text{Gr}_{2}(\mathbb{C}^{4})$ is parallel, since in general harmonic forms on compact Riemannian symmetric spaces are parallel.
4.5. The Rarita-Schwinger operator on symmetric spaces. Let \((M^n, g)\) be a non-flat irreducible Riemannian symmetric space of compact type, admitting a spin structure. Then the metric \(g\) is Einstein with positive scalar curvature. If \(VM = G \times_\rho V\) is a homogeneous vector bundle over a symmetric space \(M = G/K\), defined by a \(K\)-representation \(\rho\) then the action of the standard Laplace operator \(\Delta_V\) on sections of \(VM\) coincides with the action of the Casimir operator of \(G\). In particular, it is a non-negative operator (cf. [26], Lemma 5.2), i.e. \(\Delta S_{3/2} \geq 0\) for Riemannian symmetric spaces of compact type, with spin structure.

From Proposition 4.1 we see that we only have to study the kernel of the Rarita-Schwinger operator on sections of \(\text{Ker}P^*\). We consider \(Q\) on \(\text{Ker}P^*\) and assume that it has a non-trivial kernel. Then, according to Proposition 2.1 there are three cases: either \(n > 8\), which is only possible with \(\text{scal} = 0\), but then the symmetric space \(M\) is Ricci-flat and thus flat, or \(n = 8\) and the kernel of \(Q\) coincides with the kernel of \(\Delta S_{3/2}\), or \(n < 8\), in which case the kernel of \(Q\) is the eigenspace of \(\Delta S_{3/2}\) for the eigenvalue \(\frac{8-n}{8n}\) \(\text{scal}\). Note that by Proposition 2.1 the calculation of the spectrum of the Rarita-Schwinger operator on compact symmetric spaces reduces to the application of branching rules and the calculation of Casimir eigenvalues.

First, we study the kernel of \(\Delta S_{3/2}\) on 8-dimensional irreducible Riemannian symmetric spaces of compact type admitting a spin structure. Since \(\Delta S_{3/2}\) can be identified with the Casimir operator of \(G\) which acts trivially only on the trivial representation we conclude in the case \(n = 8\) that the kernel of \(Q\) consists of parallel sections of \(\text{Ker}P^*\).

In dimension 8 we have besides the three quaternion Kähler symmetric spaces, considered in Theorem 4.3 only the complex quadric \(Q_4\) and the symmetric space \(\text{SU}(3)\) written as \(\text{SU}(3) = \text{SU}(3) \times \text{SU}(3)/\text{SU}(3)\). This follows from checking the Cahen-Gutt list of compact simply connected symmetric spaces \(G/K\) with spin structure and simple \(G\) (cf. [7] or [13]).

We have already seen in Subsection 4.2 that on \(Q_4\) the Rarita-Schwinger operator has an at least 2-dimensional kernel. In fact, it is easy to check that the representation defining the bundle \(S_{3/2}^{-}\) contains a 2-dimensional trivial summand (cf. [37]). Hence the kernel of the Rarita-Schwinger operator on \(Q_4\) is exactly 2-dimensional and its index is \(-2\). The symmetric space \(\text{SU}(3)\) carries a Rarita-Schwinger field (cf. [15], Theorem 3). Here the canonical (parallel) 3-form of \(\text{SU}(3)\) defines a harmonic \(\text{PSU}(3)\) structure on \(\text{SU}(3)\) as already considered in Subsection 4.3 Using the fact that the tangent representation of \(\text{SU}(3)\) is isomorphic to the half-spin representations \(\Sigma_{1/2}^{\pm}\) it is easy to check that there is a 1-dimensional trivial summand in the \(\text{SU}(3)\)-representations \(\Sigma_{3/2}^{\pm}\). Hence on \(\text{SU}(3)\) there is a 2-dimensional space of (parallel) Rarita-Schwinger fields.

Next, we have to study the kernel of \(\Delta S_{3/2}\) on compact irreducible Riemannian symmetric spaces with spin structure and dimension \(n < 8\). By checking once again the list of [7] we see that there are up to isometries only spheres and the complex projective space \(\mathbb{C}P^3\).

First, we will show that the standard sphere admits no Rarita-Schwinger fields. In fact we will prove an eigenvalue estimate which shows that \(Q^2\) is a positive operator on the standard
sphere. We only have to consider \( Q \) on \( \text{Ker} P^* \) and here \( Q^2 = \Delta_{S_{3/2}} + \frac{2}{3n} \text{scal} \). By definition \( \Delta_{S_{3/2}} = \nabla^* \nabla + q(R) \) and thus a lower bound of the spectrum of \( \Delta_{S_{3/2}} \) is given by the smallest eigenvalue of the symmetric endomorphism \( q(R) \). We consider the standard metric on the sphere \( S^n \) with scalar curvature \( \text{scal} = n(n-1) \). In this normalization the curvature operator \( R : \Lambda^2 T M \rightarrow \Lambda^2 T M \) is minus the identity and it follows that \( q(R) \) is in any point the Casimir operator of the representation \( \Sigma_{3/2} \) of highest weight \( \lambda = (\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \). On any irreducible representation with highest weight \( \lambda \) the Casimir operator acts by Freudenthal’s formula as \( \langle \lambda + 2\delta, \lambda \rangle \text{id} \), where \( \delta \) is the half-sum of positive roots and \( \langle \cdot, \cdot \rangle \) is in our situation the euclidean standard scalar product. An easy calculation for the representation \( \Sigma_{3/2} \) then gives that \( q(R) = \frac{1}{8} n(n+7) \text{id} \). Thus \( \Delta_{S_{3/2}} \geq \frac{1}{8} n(n+7) > - \frac{n-8}{8n} n(n-1) \) and it follows that \( Q^2 \) is a positive operator on \( S_{3/2} \) in case of the standard sphere.

The spectrum of the Rarita-Schwinger operator on complex projective spaces was computed in [30]. It is easy to check that \( \frac{8-n}{8n} \text{scal} \) for \( n = 6 \) is not an eigenvalue of \( Q \) on \( \mathbb{C} P^3 \). Hence, there are also no Rarita-Schwinger fields on complex projective spaces. Summarizing we have

**Theorem 4.5.** The only irreducible Riemannian symmetric spaces of compact type admitting a spin structure, such that the Rarita-Schwinger has a non-trivial kernel are the 8-dimensional symmetric spaces: \( \text{Gr}_2(\mathbb{C}^4) \), \( \mathbb{H} P^2 \), \( G_2/\text{SO}(4) \), \( \text{SU}_3 \) and \( Q_4 = \text{SO}(6)/\text{SO}(2) \times \text{SO}(4) \). In these cases all Rarita-Schwinger fields are parallel.

### 4.6. Rarita-Schwinger fields on Calabi-Yau manifolds

Let \( (M^{2n}, g, J) \) be a compact Calabi-Yau manifold of complex dimension \( n \), i.e. we have \( \text{Hol}(M,g) = \text{SU}(n) \). Calabi-Yau manifolds are automatically spin and Ricci-flat. From Proposition [21] it follows that the operator \( Q^2 \) coincides with the standard Laplacian \( \Delta_{S_{3/2}} \) and the space of Rarita-Schwinger fields is given by the kernel of \( \Delta_{S_{3/2}} \). We will see that all \( \text{SU}(n) \)-representations appearing as summands in \( \Sigma_{3/2} \) are components of the form representation, i.e. Rarita-Schwinger fields can be described by certain harmonic forms.

Let \( E = \mathbb{C}^n \) be the standard representation of \( \text{SU}(n) \), then \( \Lambda^n E \cong \mathbb{C} \) and \( \Lambda^p E \cong \Lambda^{n-p} \bar{E} \). The complexified tangent bundle \( TM^C \) is associated to the \( \text{SU}(n) \)-representation \( E \oplus \bar{E} \) and the space of \( (p,q) \)-forms to the representation \( \Lambda^{p,q} := \Lambda^p E \otimes \Lambda^q \bar{E} \). We will need the following decomposition: \( \Lambda^0,1 = \Lambda^p E \otimes E = \Lambda^{n-p} \bar{E} \otimes E = \Lambda^{n-p},1 \).

Let \( h^{p,q} \) denote the Hodge numbers of \( (M, g, J) \). Then the k-th Betti number can be written as \( b_k(M) = \sum_{p+q=k} h^{p,q} \). As for any Kähler manifolds we have \( h^{p,q} = h^{q,p} \) and, since \( (M, g, J) \) is Calabi-Yau, we also have \( h^{n,0} = 1 \) and \( h^{p,0} = 0 \) for any \( p \neq 0, n \) (cf. [21], Prop. 6.2.6). In particular we have \( b_2(M) = 2 h^{2,0} + h^{1,1}, b_3(M) = 2 h^{3,0} + 2 h^{1,2}, b_4(M) = 2 h^{4,0} + 2 h^{1,3} + h^{2,2} \).

Since the canonical bundle is trivial the spinor bundle of a Calabi-Yau manifold is associated to the \( \text{SU}(n) \)-representation \( \Sigma_{1/2} = \oplus_{p=0}^{n} \Lambda^0, p \). For \( p = 0 \) and \( n = 1 \) one has two trivial summands corresponding to the two parallel spinors of a Calabi-Yau manifold. Then the bundle \( S_{3/2} \) is associated to the representation

\[
\Sigma_{3/2} = \oplus_{p=0}^{n} \Lambda^0, p \oplus (\Lambda^1,0 \oplus \Lambda^{0,1}) \oplus \oplus_{p=0}^{n} \Lambda^0, p = \oplus_{p=0}^{n} (\Lambda^1,0 \oplus \Lambda^{n-p},1) \oplus \oplus_{p=0}^{n} \Lambda^0, p .
\]
We see that all summands of $\Sigma_{3/2}$ appear in the form representation. Thus the standard Laplace operator $\Delta_{S_{3/2}}$ coincides with the Hodge-Laplace operator and the dimension of the kernel of $Q$ is given by a sum of Hodge numbers of the form components of $\Sigma_{3/2}$. Since we have to subtract the two parallel forms in $\Lambda^{0,0}$ and $\Lambda^{0,n}$, and since $h^{1,p} = h^{p,1}$ we obtain
\[ \dim \ker Q = -2 + 2 \sum_{p=1}^{n-1} h^{1,p}. \]

Note that in the decomposition (13) of $\Sigma_{3/2}$ the two trivial summands of $\Sigma_{1/2}$ cancel with the trivial summands in $\Lambda^{1,p}$ for $p = 1$ and in $\Lambda^{n-p,1}$ for $p = n-1$. Indeed $\Lambda^{1,1}$ contains a trivial summand corresponding the Kähler form. We see that there are no trivial summands in $\Sigma_{3/2}$ and hence there are no parallel Rarita-Schwinger fields on a Calabi-Yau manifold.

The half-spinor representations $\Sigma_{1/2}^\pm$ are given by the sum of the spaces $\Lambda^{0,p}$ with all $p$ even for $\Sigma_{1/2}^+$ and all $p$ odd for $\Sigma_{1/2}^-$. Then the decomposition of $\Sigma_{3/2}^\pm$ easily follows together with a formula for $\dim \ker Q^\pm$. Summarizing we have the following

**Proposition 4.6.** Let $(M^{2n}, g, J)$ be a compact Calabi-Yau manifold, then

\[ \dim \ker Q = -2 + 2 \sum_{p=1}^{n-1} h^{1,p}. \]

If $n$ is odd the index of $Q$ vanishes, whereas for even $n$ it holds that

\[ \text{ind } Q = 2 + 2 \sum_{p=1}^{n-1} (-1)^p h^{1,p}. \]

**Example:** (1) $n = 2$ : then $M$ is a K3 surface and it follows from Proposition 4.6 that $\dim \ker Q = 2h^{1,1} - 2 = 38$. In this case the space of Rarita-Schwinger fields is isomorphic to two copies of the space of harmonic primitive $(1,1)$-forms. (2) $n = 3$ : then we have $\dim \ker Q = 2(h^{1,1} + h^{1,2}) - 2 = 2b_2(M) + b_2(M) - 4$, e.g. the Fermat surface $X_3(5)$ has $h^{1,1} = 1$ and $h^{1,2} = 101$, thus $\dim \ker Q = 202$. (3) $n = 4$ then $\dim \ker Q = 2b_2(M) + b_3(M) + 2h^{1,3} - 2$, e.g. the Fermat surface $X_4(6)$ has $h^{1,1} = 1, h^{1,2} = 0$ and $h^{1,3} = 426$, thus $\dim \ker Q = 852$. For the computation of Hodge numbers of complete intersections we refer to [15], Section 22. In particular it holds that $h^{p,q} = \delta_{p,q}$ for all $p, q$ with $p + q \neq n$.

**Remark:** By a different method M.Y. Wang computed in [11] the dimension of the kernel of the twisted Dirac operator $D_{TM}$ on Calabi-Yau manifolds of complex dimension $n = 2, 3$ and 4. Taking into account that the two parallel spinors are in the kernel of $D_{TM}$ and that for $n = 4$ we have the formula $b_4^-(M) = b_2(M) + 2h^{1,3} - 1$ the numbers agree.

4.7. **Rarita-Schwinger fields on hyperkähler manifolds.** Let $(M^{4n}, g)$ be a compact hyperkähler manifold with $\text{Hol}(M, g) = \text{Sp}(n)$. As for Calabi-Yau manifolds, hyperkähler manifolds are spin, Ricci-flat and $Q^2$ coincides with the standard Laplace operator $\Delta_{S_{3/2}}$. Again we will see that the representations appearing in $\Sigma_{3/2}$ are all form representations, thus the Rarita-Schwinger fields are given by certain harmonic forms.
Let $E = \mathbb{C}^{2n}$ be the standard representation of $\text{Sp}(n)$. Then $E \cong \bar{E}$ and the representation $\Lambda^k E$ decomposes into irreducible summands as $\Lambda^k E = \Lambda^k_0 E \oplus \Lambda^k_{-2} E \oplus \ldots$. If $k$ is even the sum ends in the trivial representation $\mathbb{C}$, for $k$ odd the last summand is $E$. Here the primitive part $\Lambda^k_0 E$ is defined as the kernel of the contraction with the symplectic form of $E$.

The complexified tangent bundle is associated to the representation $E \oplus E = 2E$ and the space of $(p, q)$-forms is associated to $\Lambda^{p,q} := \Lambda^p E \otimes \Lambda^q E$. We need the following decomposition into irreducible summands: $\Lambda^k_0 E \otimes E = \Lambda^{k+1}_0 E \oplus \Lambda^{k-1}_0 E \oplus \Lambda^{k,1}_0 E$, where $\Lambda^{k,1}_0 E$ is the Cartan summand in $\Lambda^k_0 E \otimes E$ corresponding the sum of highest weights of $\Lambda^k_0 E$ and $E$.

The Hodge numbers $h^{p,q}$ of a compact hyperkähler manifold satisfy the additional symmetry $h^{p,q} = h^{2n-p,q}$. Moreover we have $h^{2q+1,0} = 0$ for all $q$ and $h^{2q,0} = 1$ for $0 \leq q \leq n$ (cf. [21], Prop. 7.4.9). In particular, it follows that $b_2(M) = 2 + h^{1,1}$, $b_3(M) = 2h^{1,2}$ and $b_4(M) = 2 + 2h^{1,3} + h^{2,2}$.

We will use the following notation: let $V$ be any $\text{Sp}(n)$-representation then we define $h(V) := \dim \ker \Delta_V$. If $V$ is a summand of the form representation then $h(V)$ is the dimension of the space of harmonic forms in the bundle associated to $V$ and in this sense a refined Betti or Hodge number. In particular, the Hodge numbers of $M$ are given as $h^{p,q} = h(\Lambda^{p,q})$.

**Lemma 4.7.** The refined Hodge numbers $h(\Lambda^{k,0}_0 E)$ vanish for all $k \geq 1$ and $h(\Lambda^{0,1}_0 E) = 1$. Moreover, it holds for all $k \geq 0$ that $h(\Lambda^{k,1}_0 E) = h^{k,1} - h^{k-2,1}$, where $h^{-2,1} = 0$ and $h^{-1,1} = 1$.

**Proof.** If $k$ is odd then there are no harmonic $(k,0)$-forms, hence there are also no harmonic forms corresponding to $\Lambda^k_0 E \subset \Lambda^k E = \Lambda^{k,0}_0$. If $k$ is even then the space of harmonic $(k,0)$-forms is one-dimensional, corresponding to the one-dimensional trivial representation in the decomposition of $\Lambda^k E$ given above. Hence, there are again no harmonic forms in the primitive part $\Lambda^k_0 E$. The case $k = 0$ is trivial since by definition $\Lambda^0_0 E = \mathbb{C}$. In order to prove the second statement we consider the decomposition

$$\Lambda^{k,1} = \Lambda^k E \otimes E = (\Lambda^k_0 E \oplus \Lambda^k_{-2} E) \otimes E = (\Lambda^{k+1}_0 E \oplus \Lambda^{k-1}_0 E \oplus \Lambda^{k,1}_0 E) \oplus \Lambda^{k-2,1}$$

As shown in the first part, there are no harmonic forms corresponding to $\Lambda^k_0 E$. Hence, we conclude $h^{k,1} = h(\Lambda^{k,1}_0 E) + h^{k-2,1}$. This proves the formula for $k \geq 2$. The cases $k = 0$ and $k = 1$ are trivial. Recall that $E \otimes E = \Lambda^2_0 E \oplus \mathbb{C} \oplus \Lambda^{1,1}_0 E$ and $\Lambda^2_0 E = E$. \hfill $\Box$

Hyperkähler manifolds can be considered as a special case of quaternion Kähler manifolds with trivial bundle $H$. Then $\text{Sym}^{n-k} H$ is a trivial representation of complex dimension $n - k + 1$ and it follows that the spinor bundle is associated to the $\text{Sp}(n)$ representation $\Sigma_{1/2} = \oplus_{k=0}^n (n - k + 1) \Lambda^k_0 E$. Note, that $\Sigma_{1/2}$ contains a $(n+1)$-dimensional trivial representation, inducing a $(n+1)$-dimensional space of parallel spinors. The bundle $S_{3/2}$ is then associated to the $\text{Sp}(n)$ representation:

$$\Sigma_{3/2} = (\Sigma_{1/2} \otimes 2E) \oplus \Sigma_{1/2} = \oplus_{k=0}^n 2(n - k + 1) \Lambda^k_0 E \otimes E \oplus \oplus_{k=0}^n (n - k + 1) \Lambda^k_0 E$$

$$= \oplus_{k=0}^n 2(n - k + 1) \Lambda^{k+1}_0 E \oplus \Lambda^{k-1}_0 E \oplus \Lambda^{k,1}_0 E \oplus \oplus_{k=0}^n (n - k + 1) \Lambda^k_0 E.$$
Note, that the first sum contains a $2n$-dimensional trivial representation: the summand $\Lambda^k E$ for $k = 1$, and the second sum contains a $(n + 1)$-dimensional trivial representation: the summand for $k = 0$, which has to be subtracted. All together the representation $\Sigma_{3/2}$ contains a $(n - 1)$-dimensional trivial summand corresponding to a space of parallel Rarita-Schwinger fields of the same dimension.

Again all summands in $\Sigma_{3/2}$ are components of the form representation. Hence the kernel of $Q$ is realized by harmonic forms. Using Lemma 4.7 we conclude

$$\dim \ker Q = h(\Sigma_{3/2}) = (n - 1) + \sum_{k=0}^n 2(n - k + 1)h(\Lambda^k E)$$

$$= (n - 1) + \sum_{k=0}^n 2(n - k + 1)(h^{k,1} - h^{k-2,1}) = -(n + 1) + 2h^{n,1} + 4 \sum_{k=1}^{n-1} h^{k,1}.$$

The decomposition of the half-spinor representations $\Sigma_{1/2}^\pm$ is similar to the Calabi-Yau case and it is then not difficult to give the decomposition of $\Sigma_{3/2}$ and to compute the dimension of $\ker Q^\pm$ in order to obtain a formula for the index of $Q$. Summarizing we have

**Proposition 4.8.** Let $(M^n, g)$ be a compact Hyperkähler manifold, then

$$\dim \ker Q = -(n + 1) + 2h^{n,1} + 4 \sum_{k=1}^{n-1} h^{k,1}.$$

The index of the Rarita-Schwinger operator is given by

$$\text{ind } Q = (n + 1) + (-1)^n 2h^{n,1} + 4 \sum_{k=1}^{n-1} (-1)^k h^{k,1}.$$

In particular, any compact hyperkähler manifold admits an $(n - 1)$-dimensional space of parallel Rarita-Schwinger fields.

**Examples:** (1) If $n = 2$ then Proposition 4.8 gives: $\dim \ker Q = -3 + 2h^{2,1} + 4h^{1,1} = 4b_2(M) + b_3(M) - 11$. (2) For $n = 3$ we have $\dim \ker Q = -4 + 4(h^{1,1} + h^{2,1}) + 2h^{3,1} = -12 + 4b_2(M) + 2b_3(M) + 2h^{3,1}$.

**Remark:** The dimension of $\ker Q$ in the case $n = 2$ also follows from results in [11] using the formula $b_7(M) = 3b_2(M) - 9$ and subtracting 3 for three parallel spinors.

4.8. Rarita-Schwinger fields on Spin(7)-manifolds. Let $(M^8, g)$ be a compact Spin(7)-manifold, i.e. the holonomy group of $g$ is the group Spin(7) $\subset$ SO(8). Then $M$ is spin and the metric $g$ is Ricci-flat. Hence, the kernel of the Rarita-Schwinger operator $Q$ coincides with the kernel of $\Delta_{S_{3/2}}$.

First, we recall the well-known decompositions into irreducible summands of the Spin(7)-representations on forms $\Lambda^k T$, $k = 2, 3, 4$ and the spinor representation $\Sigma_{1/2} = \Sigma_{1/2}^+ \oplus \Sigma_{1/2}^-$. 
Here \( T \) denotes the (complexified) holonomy representation of \( \text{Spin}(7) \subset \text{SO}(8) \) on \( \mathbb{C}^8 \). Then we have \( \Lambda^2 T = \Lambda^2_7 \oplus \Lambda^2_{21} \), \( \Lambda^3 T = \Lambda^3_8 \oplus \Lambda^3_{48} \), \( \Lambda^4 T = \Lambda^4_1 T \oplus \Lambda^4_7 T \) with \( \Lambda^4_1 T = \mathbb{C} \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \) and \( \Lambda^4 T = \Lambda^4_{35} \) and \( \Sigma_{1/2} = \Sigma^+_{1/2} \oplus \Sigma^-_{1/2} \) with \( \Sigma^+_{1/2} = \mathbb{C} \oplus \Lambda^2_7 \) and \( \Sigma^-_{1/2} = T \), as usual the index denotes the dimension of the irreducible summands, which in the given cases uniquely defines the corresponding \( \text{Spin}(7) \)-representation (cf. \[21\], Proposition 10.5.4). Note, that the trivial representation in \( \Sigma^+_{1/2} \) corresponds to the parallel spinor of the \( \text{Spin}(7) \)-manifold \( M \). We still need the following two tensor product decompositions

\[
\Lambda^2_7 \otimes T = \Lambda^3_{35} \oplus T \quad \text{and} \quad T \otimes T = \Lambda^2_{35} \oplus \Lambda^2_{21} \oplus \Lambda^2_7 \oplus \mathbb{C}.
\]

Comparing the decompositions of \( \Sigma_{1/2} \) and \( \Sigma_{1/2} \otimes T \) we find for \( \Sigma_{3/2} \) considered as \( \text{Spin}(7) \)-representation the decomposition \( \Sigma_{3/2} = \Sigma^+_{3/2} \oplus \Sigma^-_{3/2} = (T \oplus \Lambda^3_{48}) \oplus (\Lambda^3_{35} \oplus \Lambda^3_{21}) \). It follows that all parallel subbundles of \( S_{3/2} \) are isomorphic to subbundles of the bundle of differential forms and since the restriction of the standard Laplace operator \( \Delta_{S_{3/2}} \) to these subbundles coincides with the Hodge-Laplace operator \( \Delta \) we conclude that the kernel of the Rarita-Schwinger operator is realized by harmonic forms. In particular, we can express the dimension of \( \ker Q \) using the refined Betti numbers (cf. \[21\], Def. 10.6.3). On a compact manifold with holonomy equal to \( \text{Spin}(7) \) there are no non-trivial harmonic 1-form and no non-trivial harmonic 2-forms corresponding to \( \Lambda^2_7 \) (\[21\], Prop. 10.6.5). Hence, the only non-vanishing refined Betti numbers are \( b^2_{21} = b_2(M) \), \( b^3_{48} = b_3(M) \), \( b^4_{27} = b_4^+(M) - 1 \) and \( b^4_{35} = b_4^-(M) \). Checking the representations appearing in \( \Sigma_{3/2} \) we obtain

**Proposition 4.9.** Let \((M^8, g)\) be a compact Riemannian manifold with holonomy \( \text{Spin}(7) \). Then the dimension of the kernel of the Rarita-Schwinger operator \( Q \) is given in terms of refined Betti numbers as \( \dim \ker Q = b^2_{21} + b^3_{48} + b^4_{35} = b_2(M) + b_3(M) + b_4(M) \).

Note, that on all compact \( \text{Spin}(7) \)-manifolds described in \[21\], cf. Tables 14.1 - 14.3 and 15.1, the Rarita-Schwinger operator has a non-trivial kernel and because of Proposition 4.11 all these examples are in the kernel of \( P^* \), i.e. are Rarita-Schwinger fields. It is also clear that they are non-parallel. It would be interesting to know whether there are \( \text{Spin}(7) \)-manifolds with \( \dim \ker Q = 0 \). These example would have a rather simple cohomology.

The arguments above can also be used to calculate the index of \( Q \). Since \( \Sigma^+_{3/2} = T \oplus \Lambda^3_{48} \) and \( \Sigma^-_{3/2} = \Lambda^3_{35} \oplus \Lambda^3_{21} \), we obtain the equation \( \text{ind } Q = b^3_{48} - b^4_{35} - b^2_{21} \). Expressing the Euler characteristic and the signature in terms of refined Betti numbers (cf. \[21\]) we again obtain the formulas \( \text{ind } Q = 25\hat{A}(M) - \sigma(M) = 9\hat{A}(M) - \frac{1}{8}\chi(M) \) of Proposition 3.1 and Remark 3.2. Recall, that with our choice of orientation \( \hat{A}(M) = 1 \) for compact \( \text{Spin}(7) \)-manifolds.

**Remark:** The formula for \( \dim \ker Q \) on compact \( \text{Spin}(7) \)-manifolds was also proved by M.Y. Wang (cf. \[41\], Th. 3.8.) using a different and more complicated approach. Note, that in \[41\] the opposite orientation was used.

**4.9. Rarita-Schwinger fields on \( G_2 \)-manifolds.** Let \((M^7, g)\) be a compact Riemannian manifold with holonomy group \( G_2 \subset \text{SO}(7) \). Then the manifold is spin and Ricci-flat and
again the kernel of the Rarita-Schwinger operator \( Q \) coincides with the kernel of the standard Laplace operator \( \Delta_{S_{3/2}} \). The decompositions into irreducible summands of the \( G_2 \)-representation on forms, spinors and of the tensor product \( T \otimes T \) are: \( \Lambda^2 T = T \oplus \Lambda^2_{14} \), \( \Lambda^3 T = \mathbb{C} \oplus T \oplus \Lambda^3_{27} \), \( \Sigma_{1/2} = \mathbb{C} \oplus T \) and \( T \otimes T = \mathbb{C} \oplus T \oplus \Lambda^3_{27} \oplus \Lambda^2_{14} \). Hence, \( \Sigma_{3/2} = T \oplus \Lambda^3_{27} \oplus \Lambda^2_{14} \).

Again, all parallel subbundles of \( S_{3/2} \) are also subbundles of the form bundle. Note, that \( \Sigma_{3/2} \) contains no trivial representation, i.e. on a compact \( G_2 \)-manifolds there are no parallel Rarita-Schwinger fields. Since there are no harmonic 1-forms on compact \( G_2 \)-manifolds the dimension of the kernel of the Rarita-Schwinger operator can be expressed with refined Betti numbers as

**Theorem 4.10.** Let \((M^7, g)\) be a compact \( G_2 \)-manifold. Then the dimension of the kernel of the Rarita-Schwinger operator \( Q \) is given as \( \dim \ker Q = b^3_{27} + b^3_{14} = b^2(M) + b^3(M) - 1 \).

We note that all examples of compact \( G_2 \)-manifolds described in [21], Tables 12.1 - 12.7, have a Rarita-Schwinger operator with a non-trivial kernel. There seems to be no example of a \( G_2 \)-manifold where the Rarita-Schwinger operator has a trivial kernel. In fact such a manifold would have in some sense a minimal cohomology. Indeed \( \dim \ker Q = 0 \) implies \( b^2(M) = 0, b^3(M) = 1 \), i.e. the only non-trivial harmonic forms would be the 3-form defining the \( G_2 \)-structure and its Hodge dual.

**Remark:** As in the Spin(7)-case the formula for \( \dim \ker Q \) on compact \( G_2 \)-manifolds was first proved by M.Y. Wang (cf. [41], Th. 3.7) using different and more complicated methods.

4.10. **Parallel Rarita-Schwinger fields.** Let \((M^n, g)\) be a compact irreducible Riemannian spin manifold. Then \( M \) is either a symmetric space or its holonomy group belongs to the Berger list. Any parallel section in a vector bundle associated to a representation of the holonomy group corresponds to a trivial summand in this representation.

The case of the spinor bundle, i.e. the study of parallel spinors, was already done by M.Y. Wang in [40]. In this section we want to consider parallel Rarita-Schwinger fields. In fact we only have to give a summary since we already determined the space of parallel Rarita-Schwinger fields for irreducible symmetric spaces and all holonomy groups of the Berger list except the generic case \( SO(n) \) and the Kähler case \( U(n) \). However, in the first case the representations \( \Sigma_{3/2} \) and \( \Sigma_{3/2}^\perp \) are irreducible and in the second case it is easy to check that these representations do not contain a trivial summand. Thus we have proved the following

**Proposition 4.11.** Let \((M^n, g)\) be an irreducible compact spin manifold admitting a parallel Rarita-Schwinger field. Then \( M \) is either one of the symmetric spaces listed in Theorem 4.5 or a hyperkähler manifold.

Also in the case of product manifolds \( M^m \times N^n \) it is possible to say something about parallel Rarita-Schwinger fields. First, one has to decompose the spinor bundle of a product manifold (cf. [23]), e.g. if the dimensions \( m \) and \( n \) are even, then the spinor bundle of \( M \times N \) is just the tensor product of the spinor bundles of \( M \) and \( N \), i.e. if \( S_M^{3/2} \) denotes
the spinor bundle of a spin manifold $M$ we have $S^{3/2}_{M \times N} = S^{1/2}_{M} \otimes S^{1/2}_{N}$. It follows that the bundle $S^{3/2}_{M \times N}$, i.e. the kernel of the Clifford multiplication on $M \times N$, has a decomposition into three parallel subbundles isomorphic to $S^{1/2}_{M} \otimes S^{1/2}_{N}$, $S^{3/2}_{M} \otimes S^{1/2}_{N}$ and $S^{1/2}_{M} \otimes S^{3/2}_{N}$. Moreover, one easily can give the explicit embeddings. This shows that a parallel spinor on $M$ and parallel Rarita-Schwinger field on $N$ (or vice versa), or two parallel spinors on $M$ and $N$, respectively, give rise to a parallel Rarita-Schwinger field on $M \times N$. Conversely, given a parallel Rarita-Schwinger field we can project it onto the three parallel subbundle. At least one projection is non-zero and we get either a parallel Rarita-Schwinger field on one factor and a parallel spinor on the other factor, or one parallel spinor on each of the factors.

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