EYM equations in the presence of q-stars

Athanasios Prikas

Physics Department, National Technical University, Zografou Campus, 157 80 Athens, Greece.¹

Abstract

We study Einstein-Yang-Mills equations in the presence of gravitating non-topological soliton field configurations, of q-ball type. We produce numerical solutions, stable with respect to gravitational collapse and to fission into free particles, and we study the effect of the field strength and the eigen-frequency to the soliton parameters. We also investigate the formation of such soliton stars when the spacetime is asymptotically anti de Sitter.

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¹e-mail: aprikas@central.ntua.gr
1 Introduction

Coupled Einstein-Yang-Mills (EYM) equations have been studied in various contexts, [1]. Bartnik and McKinnon found particle-like non-abelian solutions of the coupled EYM theory, [2, 3]. The coupling of the EYM system with a scalar field leads to several theories. We mention gravitating Skyrmions, [4], black hole solutions in dilaton, [5, 6], and massive dilaton and axion gravity, [7], and other field configurations in the EYM-Higgs theory with a Higgs doublet, [8] or with a Higgs triplet, [9, 10, 11].

Non-topological solitons appeared initially as an attempt to explain the structure of hadrons with various “bag” models, or independently, as mathematical objects, [20]. There are two main kinds of non-topological solitons and soliton-stars, q-solitons (q-balls and q-stars) and “large” non-topological solitons. “Large” non-topological solitons are characterized by a potential of the form \( m^2|\Phi|^2(1 - |\Phi|^2/|\Phi_0|^2)^2 \). Inside the soliton or the soliton-star, \( \phi \simeq \phi_0 \), [20, 21, 22, 23] and the kinetic terms arising from the temporal variation of the scalar field dominate. In q-balls the total energy is minimized at the minimum of the \( \sqrt{U/|\Phi|^2} \), [12, 13]. Their relativistic generalizations, may consist of one or two scalar fields, [15], in a Lagrangian with a global U(1) symmetry, or of a non-abelian scalar field in the adjoint representation of SU(3), [16], or with a scalar and a fermion field, in asymptotically flat or anti de Sitter spacetime, [18]. Q-solitons with local symmetries have also been investigated. There are charged q-balls, [14], and charged q-stars, [19].

The purpose of the present work is to find numerical solutions of the EYM equations in the presence of a Higgs doublet in the fundamental representation of SU(2), reducing the EYM-Higgs equations to a field configuration corresponding to a (charged) q-star. In the absence of the gauge field, the equations of motion give rise to a gravitating non-topological soliton, when using a special potential for which \( \omega_E \equiv \sqrt{U/|\Phi|^2} \), \( \min < m \) where \( m \) is the mass of the free particles and imposing an harmonic time-dependence on the scalar field with the frequency equal to \( \omega_E \). Our gravitating soliton is non-topological in the sense that \( \Phi, U \to 0 \) for \( \rho \to \infty \) according to [20]. It is a q-type non-topological soliton in the sense that in the absence of both gravitational and gauge fields one can find by simple calculations that this spherically symmetric Higgs field rotates within its symmetry space with a frequency \( \omega_E \) equal to the minimum of the \( \sqrt{U/|\Phi|^2} \) quantity. The difference between this soliton and the usual non-abelian q-balls is that the symmetry space in the case of non-abelian q-balls is the entire SU(3) space but in our case is an abelian U(1) subgroup of the SU(2), though both field configurations are non-abelian, and that non-abelian q-balls consist of fields in the adjoint representation of SU(3), when our Higgs are in the fundamental representation. We also study the solutions of the above soliton when the spacetime is asymptotically anti de Sitter. In any case, we find an analyti-
cal solution for the scalar field within the soliton, using the approximation known by the study of q-stars.

2 EYM equations in the framework of soliton stars

The action of a Yang-Mills gauge field coupled to a Higgs scalar in the fundamental representation and to gravity in $3+1$ dimensions is:

$$ S_{\text{HEYM}} = \int \left( -\frac{R - 2\Lambda}{16\pi G} + \frac{1}{4Kg^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi) \dagger (D^\mu \Phi) - U \right) \sqrt{-g} d^4x , $$

(1)

where $\Lambda$ stands for the cosmological constant, $U$ is the potential and:

$$ D_\mu \Phi = \partial_\mu \Phi - iA_\mu \Phi $$

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] . $$

(2)

One may use the $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \text{ form and eq.}$ 2 may be reproduced by redefining $A_\mu \rightarrow A_\mu / g$, where $g$ is the field strength, or coupling constant. The one-form gauge field $A$ is: $A \equiv A_\mu dx^\mu \equiv T_a A_a^\mu dx^\mu$, with $T_a = \frac{1}{2} \tau_a$ and $\tau_a$ the Pauli matrices. The factor $K$ appearing in the action is defined by the relation $\text{Tr}(T_a T_b) = K \delta_{ab}$, here we find $K = 1/2$.

We will choose a general spherically symmetric field configuration, defining:

$$ n^a \equiv (\sin \vartheta \cos \phi, \sin \vartheta \sin \phi, \cos \vartheta) \text{ and:} $$$$ T_\rho = n^a T_a , \quad T_\vartheta = \partial_\vartheta T_\rho \quad T_\varphi = \frac{1}{\sin \vartheta} \partial_\varphi T_\rho . $$

(3)

The gauge and scalar fields take respectively the forms:

$$ A = a T_\rho + \nu(1 - \text{Re} \omega)[T_\rho, dT_\rho] + \text{Im} \omega T_\rho = $$

$$ a T_\rho + [\text{Im} \omega T_\vartheta + (\text{Re} \omega - 1)T_\varphi] d\vartheta + [\text{Im} \omega T_\varphi + (1 - \text{Re} \omega)T_\vartheta] \sin \vartheta d\varphi , $$

(4)

$$ \Phi = \sigma \exp(\text{i} \xi T_\rho) |b\rangle $$

(5)

with $\sigma = \sigma(\rho, t)$, $\xi = \xi(\rho, t)$, $|b\rangle$ a constant unit vector of the internal $SU(2)$ space of the scalar (Higgs) field and $a = a_0 dt + a_\rho d\rho$.

Our purpose is to find a special spherically symmetric, static field configuration resulting to a non-topological soliton. For the sake of simplicity we choose $a_\rho = 0$ and $a_0 = a_0(\rho)$, $\sigma(\rho, t) = \sigma(\rho)$ and $\xi = \omega_E t$ in order to form a static configuration. The ansatz $\sigma(\rho, t) = \sigma(\rho)$ and $\xi = \omega_E t$ is the obvious generalization to the $\phi(\rho, t) = \sigma(\rho)e^{\omega t}$ ansatz, known from q-solitons. The spherically symmetric, static metric is written:

$$ ds^2 = -\frac{1}{B} dt^2 + \frac{1}{A} d\rho^2 + \rho^2 d\vartheta^2 + \rho^2 \sin^2 \vartheta d\varphi^2 . $$

(6)
Inserting eqs. 4-6 in eq. 1 we find for the matter action:

\[
S_{\text{matter}} = \int \rho^2 \sin \vartheta \frac{1}{\sqrt{AB}} \left[ -\frac{1}{2g^2} \left( a_0^2 AB + 2 \frac{\omega^2 a_0^2}{\rho^2} + \frac{(|\omega|^2 - 1)^2}{\rho^2} \right) + \sigma^2 A - \frac{1}{4} (\omega_E - a_0)^2 \sigma^2 B + \frac{\sigma^2}{2\rho^2} \left[ (\text{Re} \omega - \cos(\omega_E t))^2 + (\text{Im} \omega - \sin(\omega_E t))^2 \right] - U \right]
\]

(7)

In order the action to be time-independent we may choose \( \text{Re} \omega = \cos(\omega_E t) \) and \( \text{Im} \omega = \sin(\omega_E t) \), but this choice is not a solution of the equation of motion for \( \omega \), or \( \omega = 0 \), which is a solution of the equation of motion, so, our solution is embedded abelian.

In the absence of gauge fields, the Higgs field forms a soliton, which when coupled to gravity is a soliton star. This simplifies drastically the equations of motion and separate the total 3-dimensional space in three regions, the interior, the very thin surface and the exterior, where the matter scalar can be regarded as zero. Let \( U \) be equal to:

\[
U = m^2 \sigma^2 \left( 1 - \frac{4\sigma^2}{m^2} + \frac{16}{3m^4} \sigma^4 \right).
\]

(8)

The quantity \( \sqrt{U/\sigma_0^2} \) is smaller than the mass of the free particles, so for the “q-ball-type solution”, we write \( \sigma = \sigma_0 \), where \( \sigma_0 \) is the value that minimizes the above quantity, and \( \omega_E = \sqrt{U/\sigma_0^2} \) is the minimum energy frequency, [12, 13]. In the original papers referring to q-solitons, the non-abelian scalar field is in the adjoint representation of the \( SU(2) \). The results can be straightforward generalized for the fundamental representation. As one can see, \( \sigma_0 \) and \( \omega_E \) are of the same order of magnitude for q-type solitons (q-balls and q-stars), in contrast with the non-topological soliton (stars), investigated in [20], where \( \omega_E \ll |\phi|, m \). We define:

\[
W = \left( \frac{d \Phi}{dt} \right) \dagger \left( \frac{d \Phi}{dt} \right), \quad V = \left( \frac{d \Phi}{d \rho} \right) \dagger \left( \frac{d \Phi}{d \rho} \right).
\]

(9)

The “q-ball-type solution” (in the so-called “thick-wall” approximation) means that the scalar field within the soliton is approximately constant, \( W \) and \( U \) are of the same order of magnitude, when \( V \) is negligible and the energy density is \( \sim m^4 \). Gravity becomes important when \( R \sim 8\pi GM_R \) where \( M_R \) is the mass trapped within a sphere of radius \( R \). We find that \( R \sim 1/(8\pi G m^4)^{1/2} \).

We define:

\[
\epsilon \equiv \sqrt{8\pi G m^2}.
\]

(10)

If \( \sigma(0) \approx \sigma_0 \) and \( \sigma(\rho) \approx 0 \) for \( \rho > R \), then \( V \approx \epsilon^2 m^4 \). For \( m \sim GeV \) the \( O(\epsilon) \) quantities are negligible. The equation of motion for the Higgs field is:

\[
A \left[ \frac{d^2 \sigma}{d \rho^2} + \frac{1}{2} \left( \frac{4}{\rho} + \frac{1}{A} \frac{d A}{d \rho} - \frac{1}{B} \frac{d B}{d \rho} \right) \frac{d \sigma}{d \rho} \right] + \frac{B \sigma^2}{4} \frac{d U}{d \sigma^2} \sigma = 0,
\]

(11)
where we define a useful quantity:

$$\theta_0 = \omega_E - a_0.$$  \hfill (12)

We make the following rescalings:

$$\tilde{\rho} = 2m\rho, \quad \tilde{\omega}_E = \frac{\omega_E}{2m}, \quad \tilde{a}_0 = \frac{a_0}{2m}, \quad \tilde{\sigma} = \frac{\sigma}{m^2},$$

$$\tilde{r} = \epsilon\tilde{\rho}, \quad \tilde{g} = g\epsilon^{-1}, \quad \tilde{\Lambda} \equiv \frac{\Lambda}{8\pi G m^4}. \hfill (13)$$

With the above rescalings $\tilde{r} \sim 1$. In our solutions we find $\tilde{\Lambda} \sim 1$, so, the “true” cosmological constant, $\Lambda$, is of $O(8\pi G m^4)$ order. From now on the prime denotes the differentiation with respect to $\tilde{r}$. The potential takes a simple form, regarding $m = 1$, namely:

$$U = \frac{1}{12}(1 + \theta_0^2 B^{3/2}), \quad W = \theta_0^2 B(1 + \theta_0 B^{1/2}), \quad V \simeq 0. \hfill (14)$$

The above equation holds true within the soliton interior. The surface width is of $m^{-1}$. The scalar field within the surface changes rapidly from a $\sigma \simeq \sigma_0$ value at the inner edge of the surface to a $\sigma \simeq 0$ value at the outer one. Integrating the equation of motion within the surface we find:

$$V + W - U = 0. \hfill (15)$$

At the inner edge of the surface $\sigma' = 0$ so as to match the interior with the surface solution. Using eqs. 14 15 we find the eigenvalue equation for the new field $\theta_0$:

$$\theta_{0\text{sur}} = \frac{A_{\text{sur}}^{1/2}}{2} = \frac{B_{\text{sur}}^{-1/2}}{2}, \hfill (16)$$

where $\theta_{0\text{sur}}$ is the value of $\theta_0$ within the thin surface.

The Einstein equations are: $G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda \delta_{\mu\nu}$, where the energy-momentum tensor, $T_{\mu\nu}$, is defined by the relation:

$$T_{\mu\nu} = \frac{2}{g^2} \text{Tr} \left( g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) + (D_\mu \Phi)^\dagger (D_\nu \Phi) + (D_\mu \Phi)^T (D_\nu \Phi)^* - g_{\mu\nu} [g^{\alpha\beta} (D_\alpha \Phi)^\dagger (D_\beta \Phi)] - g_{\mu\nu} U. \hfill (17)$$

Dropping the $O(\epsilon)$ quantities, the independent Einstein equations take the simple form:

$$\frac{A - 1}{r^2} + \frac{A'}{r} = -U - W - \frac{\theta_0^2}{2g^2} AB - \Lambda, \hfill (18)$$
Figure 1: The value of $\sigma(0)$ as a function of the coupling constant $g^2$. Dashed lines correspond to $\Lambda = -0.02$ and solid lines to asymptotically flat space-time.

Figure 2: The radius of the soliton as a function of $g^2$.

\[
\frac{A - 1}{r^2} - \frac{AB'}{B}r = W - U - \frac{\theta_0^2}{2g^2}AB - \Lambda, \tag{19}
\]

and the equation of motion for the gauge field is:

\[
\theta''_0 + \left(\frac{2}{r} + \frac{A'}{2A} + \frac{B'}{2B}\right)\theta'_0 - \frac{g^2\theta_0(1 + \theta_0B^{1/2})}{2A} = 0. \tag{20}
\]

There are three Noether currents, related to the generators of the Lie algebra, which are given by the relation:

\[
j_{\alpha\alpha} = \left(\frac{\partial L}{\partial (\partial_\alpha \Phi)} \begin{array}{cc} \partial L \\ \partial (\partial_\alpha \Phi^*) \end{array}\right) \begin{pmatrix} iT_\alpha & 0 \\ 0 & -iT_\alpha \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi^* \end{pmatrix}. \tag{21}\]
Figure 3: The asymptotically anti de Sitter mass, $M_{\text{AADS}}$, as a function of $g^2$.

Figure 4: The particle number, $N$, of the soliton as a function of $g^2$.

Figure 5: The value of the gauge field, $\alpha$, at the center of the soliton as a function of $g^2$. 
We can find that:

\[
\begin{align*}
  j_{01} &= \frac{1}{2} \sigma^2 \theta_0 \sin \vartheta \cos \varphi \\
  j_{02} &= \frac{1}{2} \sigma^2 \theta_0 \sin \vartheta \sin \varphi \\
  j_{03} &= \frac{1}{2} \sigma^2 \theta_0 \cos \vartheta 
\end{align*}
\]

and

\[
  j_0 \equiv \sqrt{j_{01}^2 + j_{02}^2 + j_{03}^2} = \frac{1}{2} \sigma^2 \theta_0 .
\]

The particle number is:

\[
N = 2\pi \int \sigma^2 \theta_0 \sqrt{\frac{A}{B}} r^2 dr .
\]

The asymptotically anti de Sitter energy of the field configuration, \( M_{\text{AADS}} \), dropping the \( O(\epsilon) \) quantities, may be calculated by \(-T_{0}^{0}\), or by:

\[
M_{\text{AADS}} = 4\pi r \left( 1 - A(r) - \frac{1}{3} \Lambda r^2 + \frac{g^2 N^2}{32\pi^2 r^2} \right) , \quad r \to \infty ,
\]

resulting from the (unrescaled) relation:

\[
A(\rho) = 1 - \frac{2 GM_{\text{AADS}}}{\rho} - \frac{1}{3} \Lambda \rho^2 + \frac{Gg^2 N^2}{4\pi \rho^2} , \quad \rho \to \infty ,
\]

with the appropriate rescalings. We numerically solve the coupled system of eqs. [13][20] Eqs. [14][21] and [25] give some of the parameters of the soliton.

We will now discuss the sufficient conditions for the existence of q-solitons in theories with local symmetries, [24], and prove that our soliton meets them.
These conditions refer to solitons with zero overall non-abelian charge, which is the case for the embedded abelian solutions. In this case, let $\hat{B}$ be the generator of a global $U(1)$ symmetry (our theory possesses such a symmetry) and $\hat{T}^\alpha$ the generators of the non-abelian group. The energy can be written as, \[24\]:

$$E_{\lambda,\xi} = \int \sqrt{-g} d^3 x \left( p^\dagger p + |\partial_i \phi|^2 + U \right) - \lambda \left[ \int \sqrt{-g} d^3 x \hat{B} - Q \right] - \int \sqrt{-g} d^3 x \xi^\alpha(x) \hat{T}^\alpha , \tag{26}$$

with

$$p(x) = -i[\lambda \hat{B} + \xi^\alpha(x) \hat{T}^\alpha] \phi .$$

The equations for the Lagrange multipliers, $\lambda$ and $\xi^\alpha$ are:

$$
\lambda \phi^\dagger \hat{T}^\alpha \phi + \xi^\beta \phi^\dagger \{\hat{T}^\alpha, \hat{T}^\beta\} \phi = 0 , \quad a = 1, \ldots, \dim(G) ,
$$

$$
\int \sqrt{-g} d^3 x [\lambda \phi^\dagger \hat{B}^2 \phi + \xi^\beta(x) \phi^\dagger \hat{B} \hat{T}^\beta \phi] = Q . \tag{27}
$$

A q-soliton exists if the system of equations \[27\ 28\] has a solution. As one can easily see in the thin-wall limit we discuss in the present work, the \[27\ 28\] become a system of linear equations in $\lambda$ and $\xi$ and can easily be solved. There is also another condition for the existence of q-solitons: the total energy energy should be less than the energy of the free particles with the same charge. This condition can be handled only numerically: We solve the coupled system of eqs. \[18\ 20\] and include the solution in our figures only when the soliton is stable with respect to fission into free particles.

3 Conclusions

We studied a spherically symmetric doublet of scalar Higgs fields in the fundamental representation of the $SU(2)$ group, coupled to a gauge field. The EYM-Higgs equations reduced to an easy-manageable system of equations, corresponding to (charged) soliton-stars, by choosing a special potential for the Higgs field and imposing a certain time-dependence to it. The independent parameters are the coupling constant $g$, which represents the “strength” of the gauge field, the $\theta_{\text{sur}}$ eigenvalue, straightforward connected to the surface gravity through eq. \[16\] and equal to the eigen-frequency of the soliton star in the absence of gauge fields, and the cosmological constant. All the matter and metric field configurations correspond to stars and not to black holes (i.e.: no horizon and no anomalies at the center are formed), they are time-independent and stable with respect to gravitational collapse.
the field configurations discussed in the above section are stable with respect to decay into free particles, because their asymptotically anti de Sitter mass is smaller than the energy of the free particles. When the energy of the free fields with the same particle number, equal to $mN$ with $m$ unity for the sake of convenience, tends to become larger than the soliton energy, we interrupt calculations. The field configuration in the absence of gravitational and gauge fields corresponds to a soliton with $\omega_E = \sqrt{U/\sigma^2}|_{\min}$, $W = \omega_E^2 \sigma^2/4 = \omega_E^2 (1 + \omega_E)/4$, $U = (1 + \omega_E^3)/12$ and $V \simeq 0$ as one can find by some simple algebra, i.e.: corresponds to a q-ball type soliton. In the absence of gauge fields, but in the presence of gravity the field configuration corresponds to a soliton star with $\omega_E B = \sqrt{U/\sigma^2}|_{\min}$, $W = B \omega_E^2 (1 + \omega_E B^{1/2})/4$, $U = (1 + \omega_E^3 B^{3/2})/12$ and $V \simeq 0$ which corresponds to a q-type soliton star.

The numbers within the figures denote the value of $\theta_{0\text{sur}}$, which plays the same role as the frequency in solitons with global symmetries and the numbers within the figure denote $g^2$. All the field configurations depicted in our figures are stable with respect to fission into free particles.

The total mass increases with the increase of the coupling constant due to the electrostatic-type terms arising in the $-T_{00}$ component of the energy-momentum tensor. The value of $\sigma$ at the center of the soliton decreases due to the electrostatic repulsion between the different parts of the soliton. The influence of the cosmological constant to the energy, particle number and radius of the soliton is clear. The values of the above parameters are in general larger in an anti de Sitter spacetime.

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