GROUP GRADED PI-ALGEBRAS AND THEIR CODIMENSION GROWTH

ELI ALJADEFF

ABSTRACT. Let $W$ be an associative PI-algebra over a field $F$ of characteristic zero. Suppose $W$ is $G$-graded where $G$ is a finite group. Let $\text{exp}(W)$ and $\text{exp}(W_e)$ denote the codimension growth of $W$ and of the identity component $W_e$, respectively. The following inequality had been conjectured by Bahturin and Zaicev: $\text{exp}(W) \leq |G|^2 \text{exp}(W_e)$. The inequality is known in case the algebra $W$ is affine (i.e. finitely generated). Here we prove the conjecture in general.

1. INTRODUCTION

Let $W$ be any PI-algebra over a field of characteristic zero. Suppose $W$ is $G$-graded where $G$ is any finite group. Let $\text{exp}(W)$ and $\text{exp}(W_e)$ be the exponents of the algebra $W$ and of its $e$-component (determined by the $G$-grading). The main objective of this paper is to prove the following conjecture (see [8]):

**Conjecture 1.1.**

$$\text{exp}(W) \leq |G|^2 \text{exp}(W_e).$$

The case where $W$ is affine was proved in [1]. Here we show that the conjecture holds also when $W$ is non affine.

**Theorem 1.2.** Let $W$ be as above. Then $\text{exp}(W) \leq |G|^2 \text{exp}(W_e)$.

The proof is based on three ingredients. The first is the representability theorem for $G$-graded PI-algebras which allows us to "pass" from arbitrary algebras to finite dimensional algebras. The second is a precise description of the structure of a $G$-graded simple algebra (as determined by Bahturin, Sehgal and Zaicev) and the third is the relation (established by Giambruno and Zaicev) between the $\text{exp}(W)$ and the structure of $W$ when $W$ is a finite dimensional algebra. In fact these ingredients were also used in the proof of the conjecture in the affine case. The point here is that when passing to finite dimensional algebras (with the representability theorem for non-affine $G$-graded algebras), we are "forced" to consider some statements which are more general than Bahturin-Zaicev conjecture. In section 1 (after some preliminaries) we formulate these statements and in particular translate the problem to finite dimensional algebras. In section 2 we solve the problem for finite dimensional algebras in sufficient generality to imply Bahturin-Zaicev conjecture for non affine algebras.

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2. Preliminaries and Translation of the Problem to Finite Dimensional Algebras

We start this section by recalling some definitions. Let \( X \) be a countable set of indeterminates and \( F(X) \) the corresponding free algebra over \( F \). Let \( \text{id}(W) \) be the \( T \)-ideal of identities of \( W \) in \( F(X) \) and let \( W = F(X)/\text{id}(W) \) be the relatively free algebra of \( W \). We denote by \( c_n(W) \) the dimension of the subspace spanned by multilinear elements in \( n \) free generators of \( W \). We refer to \( c_n(W) \) as the \( n \)-th term of the codimension sequence of the algebra \( W \). Our interest is in the (exponential) asymptotic behavior of the sequence of codimensions, namely in

\[
\exp(W) = \lim_{n \to \infty} \sqrt[n]{c_n(W)}.
\]

It is known that \( \exp(W) \) exists and moreover it assumes only integer values (see \([14, 15, 20]\)).

If the algebra \( W \) is \( G \)-graded, where \( G \) is any group, we may consider \( G \)-graded polynomial identities and the corresponding \( G \)-graded exponent. Since our main theorem will be formulated in these terms, we recall them here as well. Let \( X_G = \bigcup_{g \in G} X_g \) where \( X_g = \{x_1.g, x_2.g, \ldots \} \) is a countable set of indeterminates and let \( F(X_G) \) be the free \( G \)-graded algebra on the set \( X_G \) (here, a monomial \( x_{i_1.g_1}x_{i_2.g_2} \cdots x_{i_r.g_r} \) is homogeneous of degree \( g = g_1g_2 \cdots g_r \)). In \( F(X_G) \) we consider \( \text{id}_G(W) \), the ideal of \( G \)-graded identities of \( W \), and denote by \( W_G = F(X_G)/\text{id}_G(W) \) the corresponding relatively free \( G \)-graded algebra. For \( n = 1, 2, \ldots \), we denote by \( c_n^G(W) \) the dimension of the subspace of \( W_G \) spanned by multilinear elements in \( n \) (\( G \)-graded) free generators. We call \( \{c_n^G(W)\}_{n=1}^\infty \) the sequence of \( G \)-codimensions of \( W \). The \( G \)-graded exponent of \( W \) is given (when it exists) by

\[
\exp_G(W) = \lim_{n \to \infty} \sqrt[n]{c_n^G(W)}.
\]

**Remark 2.1.** The limit \( \lim_{n \to \infty} \sqrt[n]{c_n^G(W)} \) is known to exist when \( G \) is abelian (see \([2, 4]\) in case \( W \) is affine and \([11]\) in case \( W \) is non affine). Moreover in that case it assumes only integer values. In the general case (i.e. when \( G \) is not necessarily abelian) it is known only that the sequence of codimensions \( c_n^G(A) \) is exponentially bounded.

As mentioned in the introduction, we wish to translate the problem to the “world” of finite dimensional algebras. Before doing it let us recall two results, the first of which is Kemer’s representability theorem for arbitrary PI-algebras (see \([17, 18, 19]\)).

**Theorem 2.2 (PI-equivalence).** Let \( W \) be a PI-algebra over \( F \). Then there exists a field extension \( L \) of \( F \) and a finite dimensional \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra \( A \) over \( L \) such that \( \text{id}(W) = \text{id}(A^*) \) where \( A^* \) is the Grassmann envelope of \( A \).

Next we recall a result of Giambruno and Zaicev (see \([15]\)) which describes the (exponential) codimension growth of an algebra \( W \) in terms of the structure of the \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra \( A \) appearing in Kemer’s result.

**Theorem 2.3.** Let \( W \) be an arbitrary PI-algebra over a field of characteristic zero \( F \). Let \( A \) be a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra as in Kemer’s theorem. Then \( \exp(W) = \exp_{\mathbb{Z}/2\mathbb{Z}}(A) \).
Remark 2.4. In fact, Theorem 2.3 was stated in [15] in a different way. The result there says that \( \exp(W) \) is equal to the dimension of a certain semisimple subalgebra of \( \overline{A} \), where \( \overline{A} = A \otimes_L \overline{L} \) and \( \overline{L} \) is the algebraic closure of \( L \). Then one finds that this dimension is actually equal to \( \exp_{\mathbb{Z}/2\mathbb{Z}}(A) \) (see [2]).

The other “player” in Conjecture 1.1 is of course \( \exp(W) \). We could describe it in similar terms as we did for \( \exp(W) \), i.e. using a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra \( A' \) say, but by doing it we won’t be able to compare \( \exp_{\mathbb{Z}/2\mathbb{Z}}(A) \) and \( \exp_{\mathbb{Z}/2\mathbb{Z}}(A') \). We need therefore a \( G \)-graded representability theorem which will produce the \( \mathbb{Z}/2\mathbb{Z} \)-graded algebras \( A \) and \( A' \) simultaneously. This is achieved by invoking the \( G \)-graded representability theorem for arbitrary \( G \)-graded PI-algebras (see [3]).

Theorem 2.5 (\( G \)-graded PI-equivalence). Let \( W \) be a PI and \( G \)-graded algebra over \( F \) where \( G \) is any finite group. Then there exists a field extension \( L \) of \( F \) and a finite dimensional \( \mathbb{Z}/2\mathbb{Z} \times G \)-graded algebra \( A \) over \( L \) such that \( \text{id}_G(W) = \text{id}_G(A') \) where \( A' \) is the Grassmann envelope of \( A \).

Let us draw some consequences from Theorem 2.5 (\( W \) and \( A' \) are as in the theorem).

1. \( \text{id}(W) = \text{id}(A') \) (as ungraded algebras). Indeed, by linearity, any ordinary polynomial identity is equivalent to a set of \( G \)-graded identities.
2. \( \text{id}(W_e) = \text{id}(A'^e) \).
3. Note that \( A_{\mathbb{Z}/2\mathbb{Z} \times e} \), the \( \mathbb{Z}/2\mathbb{Z} \times e \)-component of \( A \), is a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra and we may consider its Grassmann envelope \( (A_{\mathbb{Z}/2\mathbb{Z} \times e})' \). It is easily seen that \( (A_{\mathbb{Z}/2\mathbb{Z} \times e})' = (A')^e \).

In view of Theorem 2.5 and its consequences we conclude that Conjecture 1.1 will be proved if we prove the following statement.

Theorem 2.6. Let \( A \) be a finite dimensional \( \mathbb{Z}/2\mathbb{Z} \times G \)-graded algebra. Let \( A' \) be its Grassmann envelope (with respect to the \( \mathbb{Z}/2\mathbb{Z} \)-grading) and let \( (A')^e \) be its \( e \)-component. Then we have

\[
\exp(A^e) \leq |G|^2 \exp((A_{\mathbb{Z}/2\mathbb{Z} \times e})').
\]

Invoking Theorem 2.3 we see that Theorem 2.6 can be replaced by the following theorem.

Theorem 2.7. With the above notation:

\[
\exp_{\mathbb{Z}/2\mathbb{Z}}(A) \leq |G|^2 \exp_{\mathbb{Z}/2\mathbb{Z}}(A_{\mathbb{Z}/2\mathbb{Z} \times e}).
\]

As mentioned in Remark 2.4 if \( F \) is algebraically closed, then the \( \mathbb{Z}/2\mathbb{Z} \)-exponent of a finite dimensional \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra is determined in terms of the dimension of a certain semisimple subalgebra of \( A \). More generally, this is known to be true for finite dimensional \( G \)-graded algebras where \( G \) is a finite abelian group. In [11] it was conjectured that this is the case when \( G \) is any finite group. For the reader convenience we recall here the precise construction in the general case, namely where \( G \) is any finite group.

Let \( A \) be a finite dimensional algebra over an algebraically closed field \( F \) of characteristic zero. Suppose \( A \) is \( G \)-graded where \( G \) is a finite group. It is well known (see [10]) that the radical \( J \) of \( A \) is \( G \)-graded as well. Moreover there exists a \( G \)-graded semisimple subalgebra \( S_G \) of \( A \) which supplements \( J \) (in \( A \)) as
vector spaces (see [21]). Let \( S_G = (S_G)_1 \oplus \cdots \oplus (S_G)_q \) be the decomposition of \( S_G \) into its \( G \)-simple components and consider nonzero products in \( A \) of the form \((S_G)_{i_1} J(S_G)_{i_2} \cdots J(S_G)_{i_k}\). For any product of that form we consider the sum of the dimensions (as \( F \)-vector spaces) of the different \( G \)-simple components that appear in the product. It was conjectured in [1] that the exponent of the algebra \( A \) as a \( G \)-graded algebra, is equal to the maximum of these sums. Let us denote this maximum by \( \exp_{G}^{\text{Conj}}(A) \).

The following lemma is clear.

**Lemma 2.8.** Let \( G \) be a finite group and \( H \) a normal subgroup. Let \( A \) be a finite dimensional algebra over an algebraically closed field \( F \) of characteristic zero graded by \( G \). Then \( A \) is graded by \( G/H \) in a natural way and we have \( \exp_{G}^{\text{Conj}}(A) \geq \exp_{G/H}^{\text{Conj}}(A) \).

Let us pause for a moment and explain what we have so far. Bahturin-Zaicev conjecture (Conjecture 1.1) for an algebra \( W \) over a field \( F \) is given in terms of the growth of the codimensions of the ideals of identities of \( W \). Applying representability for \( G \)-graded algebras, namely Theorem 2.7, the conjecture is translated with the same terms (namely growth of codimensions) into finite dimensional algebras. Giambruno-Zaicev result translates the problem in terms of the structure of the finite dimensional algebras but for this the field \( F \) is required to be algebraically closed. In order to fill this “gap” we need to pass to the algebraic closure \( \bar{F} \) of \( F \). Indeed, it is well known that the ideal of identities (defined over \( F \)) of \( W \) and \( W \otimes_F L \) coincide where \( L \) is any field extension of \( F \) and in particular \( \bar{F} \).

Of course, the algebra \( W \otimes_F L \) is an \( L \)-algebra and we may consider its identities over \( L \). The point is that since the field is infinite, these are linear combinations with coefficient in \( L \) of \( F \)-identities and hence by changing the ground field from \( F \) to \( L \) we do not change the codimensions.

It follows from Lemma 2.8 and the last paragraph that Theorem 2.7 (and hence Conjecture 1.1) will be proved if we show the following theorem.

**Theorem 2.9.** Let \( A \) be a \( \mathbb{Z}/2\mathbb{Z} \times G \)-graded finite dimensional over an algebraically closed field \( F \) of characteristic zero. Then

\[
\exp_{\mathbb{Z}/2\mathbb{Z} \times G}^{\text{Conj}}(A) \leq |G|^2 \exp_{\mathbb{Z}/2\mathbb{Z}}^{\text{Conj}}(A_{\mathbb{Z}/2\mathbb{Z}})\\(A_{\mathbb{Z}/2\mathbb{Z}})\]

Note that since the group \( \mathbb{Z}/2\mathbb{Z} \) is abelian, by the paragraph following Theorem 2.7, the word “Conj” could be omitted from the right hand expression.

Our aim is to prove a substantially generalization of the last statement.

Let \( A \) be a finite dimensional algebra over an algebraically closed field \( F \) of characteristic zero. Let \( G \) be a finite group and assume \( A \) is \( G \)-graded. Let \( K \) be any subgroup of \( G \) and let \( A_K \) be the subalgebra of \( A \) which corresponds to \( K \) (viewed as a \( K \)-graded algebra). The following theorem is the main result of this paper.

**Theorem 2.10.** Let \( A, G \) and \( K \) as above. Then

\[
\exp_{G}^{\text{Conj}}(A) \leq [G : K]^2 \exp_{K}^{\text{Conj}}(A_K)\]

Note that the case where \( K = \{e\} \) was proved in [1].
3. Proof of Theorem 3.1

As in [1], the key here is a result of Bahturin, Sehgal and Zaicev in which they fully describe the structure of finite dimensional $G$-graded simple algebras over an algebraically closed field $F$ of characteristic zero.

Let us recall their theorem.

**Theorem 3.1 ([6]).** Let $B$ be a $G$-simple algebra. Then there exists a subgroup $H$ of $G$, a 2-cocycle $f : H \times H \rightarrow F^*$ where the action of $H$ on $F$ is trivial, an integer $r$ and an $r$-tuple $(g_1, \ldots, g_r) \in G^r$ such that $B$ is $G$-graded isomorphic to $C = F^f H \otimes M_2(F)$, where $C_g = \text{span}_F \{ u_{i,j} \otimes e_{i,j} : g = g_{s_i}^{-1} h g_{s_j} \}$. Here $F^f H = \sum_{h \in H} fh$ is the twisted group algebra of $H$ over $F$ with the 2-cocycle $f$ and $e_{i,j} \in M_2(F)$ is the $(i,j)$-elementary matrix.

In particular the idempotents $1 \otimes e_{i,i}$ as well as the identity of $B$ are homogeneous of degree $e \in G$.

**Remark 3.2.** Note that the grading above induces $G$-gradings on the subalgebras $F^f H \otimes F$ and $F \otimes M_2(F)$ which are fine and elementary respectively (see [3], [5], [4] for the precise definitions).

Let us analyze the $K$ component of a $G$-graded simple algebra $B$ using the terminology of Theorem 3.1.

**Lemma 3.3.** Let $B$ as in Theorem 3.1. Then the following hold.

1. A basis element $u_h \otimes e_{i,j}$ is in the $K$-homogeneous component of $B$ if and only if $g_{s_i}^{-1} h g_{s_j} \in K$.
2. Let $(i,j)$ be any pair of indices $1 \leq i, j \leq r$ (in the elementary grading of $M_2(F)$). Then there exists a basis element $u_h \otimes e_{i,j}$ in the $K$-homogeneous component of $B$ if and only if $g_{s_i}$ and $g_{s_j}$ determine the same $H - K$ double coset in $G$.
3. The relation “$i \sim j$ if and only if $g_{s_i}$ and $g_{s_j}$ determine the same $H - K$ double coset in $G”$ is an equivalence relation on the set of indices $\{1, \ldots, r\}$.
4. Fix $i$, $1 \leq i \leq r$, and let $[i]$ be the equivalence class it represents. Then the set of basis elements $u_h \otimes e_{i,k}$ which belong to the $K$-homogeneous component $B_{K,i}$ of $B$ such that $[j] = [k] = [i]$ span a $K$-graded simple algebra. Furthermore, if we denote this $K$-graded simple algebra by $B_{K,i}$, then $B_{K,i} \cong F^{g_{s_i}^{-1}}(g_{s_i}^{-1} H g_{s_i} \cap K) \otimes M_2(F)$ where $\pi_i$ is the number of indices in $\{1, \ldots, r\}$ which are equivalent to $i$ and $g_{s_i}(f)$ is the action of $g_{s_i}$ on the 2-cocycle $f$ given by $g_{s_i}(f)(g_{s_i}^{-1} h g_{s_i}, g_{s_i}^{-1} \tilde{h} g_{s_i}) = f(h, \tilde{h})$ for any $h, \tilde{h}$ in $H$.

In particular, the dimension of $B_{K,i}$ is $|g_{s_i}^{-1} H g_{s_i} \cap K| \pi_i^2$.

5. Let $\Theta$ be a set of indices in $\{1, \ldots, r\}$ which represent the different equivalence classes. Then there is an isomorphism of $K$-graded algebras $B_{K} \cong \bigoplus_{i \in \Theta} B_{K,i}$.

**Proof.** We prove (4) (1-3 and 5 are clear). Let $I$ be a nonzero $K$-graded 2-sided ideal in $B_{K,i}$. We need to show $I = B_{K,i}$. Observe that the idempotents $1 \otimes e_{j,j}$, $[j] = [i]$, belong to the $e$-component of $B$ and hence are in $B_{K}$. Now, if $z = \sum_{s,j,k} \alpha_{s,j,k} u_h \otimes e_{j,k}$ is a non-zero element in $I$, multiplying from left and right by suitable idempotents $1 \otimes e_{j,j}$ we may assume that $z = z_{j,k} = (\sum \alpha_{s,k} u_{h_s}) \otimes e_{j,k}$. But for different $h_s$ in $H$, the basis elements $u_{h_s} \otimes e_{j,k}$ belong to different homogeneous components and so, since $I$ is $K$-graded, we may assume $z = u_h \otimes e_{j,k}$. Take now
any basis element $u_{h,i} \otimes e_{u,v}$ in $B_{K,i}$. We need to show it is in $I$. Indeed, since $[u] = [j] = [k] = [v]$, there exist $h_1$ and $h_2$ in $H$ such that $u_{h_1} \otimes e_{u,j}$ and $u_{h_2} \otimes e_{k,v}$ are in $B_{K,i}$. Multiplying $z$ from left and right by these elements we obtain that $I$ contains a basis element $u_{\hat{h}} \otimes e_{u,v}$ for some $\hat{h} \in H$. Finally we multiply $u_{h'} \otimes e_{u,v}$ from the right by $(u_{h'})^{-1}u_{h'} \otimes e_{v,v} \in B_{K,i}$ and the first part of (4) is proved. To see the second part of (4) we fix $h_t \in H$ such that $g_{s_i}^{-1}h_t g_{s_i} \in K$ for every $t \sim i$ (if $t = i$ we may take $h_t = e$). Then, any basis element $u_k \otimes e_{t,k} \in B_{K,i}$ can be written uniquely as $\lambda u_{\hat{h}} u_{h_k} \otimes e_{t,k}$ where $\lambda \in F^*$ and $g_{s_i}^{-1}h_k g_{s_i} \in g_{s_i}^{-1}H g_{s_i} \cap K$. It follows that if we equip $M_{n_1}(F)$ with the elementary $K$-grading given by the $\pi_{r}$-tuple

$$(e = g_{s_i}^{-1}e_{g_{s_i}}, g_{s_i}^{-1}h_{j_1}h_{j_2}g_{s_{i1}}, g_{s_i}^{-1}h_{j_2}g_{s_{i2}}, \ldots, g_{s_i}^{-1}h_{j_r}g_{s_{ir}})$$

where $(i = j_1, j_2, j_3, \ldots, j_r)$ runs over all indices in $(1, \ldots, r)$ which are equivalent to $i$, then the map

$$u_{h_i}^{-1}u_{k} u_{h_k} \otimes e_{t,k} \mapsto v_{g_{s_i}^{-1}h_{j_i}} \otimes e_{\mu,\nu}, \quad j_i = t, j_{i'} = k$$

is an isomorphism of $K$-graded algebras of $B_{K,i}$ with $F^{n_1}(g_{s_i}^{-1}H g_{s_i} \cap K) \otimes M_{n_1}(F)$. Details are omitted. □

Consider now the decomposition $A \cong S_G \oplus J$ where $S_G$ is a $G$-graded semisimple algebra and $J$ is the Jacobson radical. Let $S_G = (S_G)_1 \oplus \ldots \oplus (S_G)_q$ be the decomposition of $S_G$ into the direct sum of its $G$-simple components. In view of the above decomposition we will consider homogeneous elements that belong to $J$ (radical elements) and basis elements of the form $u_{k} \otimes e_{i,j} \in F^{j} H \otimes M_{r}(F)$ (semisimple elements) where $F^{j} H \otimes M_{r}(F)$ is a $G$-simple component of $S_G$.

We can now proceed to prove Theorem 2.10.

Let

$$\Lambda = z_1 v_1 z_2 \cdots z_n v_n z_{n+1}$$

be a (nonzero) monomial of elements of $A$ that realizes the value of $\exp_{\text{conj}}^{G_{\text{conj}}} (A)$.

By linearity we may assume that the $z_i$’s are semisimple (homogeneous) elements and the $v_i$’s are radical (homogeneous). Note also that we may assume that the semisimple elements belong to different $G$-simple components. Indeed, those that repeat may be “swallowed” by the radical elements.

Our goal (as in [1]) is to replace each one of the semisimple elements $z_i$ by a certain monomial $E_i$ (products of homogeneous elements) which roughly speaking, is sufficiently “rich”, namely it “visits” different basis elements in $(S_G)_i$ in a suitable way.

We describe the construction in four steps. Before we start, let us recall a lemma which is well known.

**Lemma 3.4.** Consider the $r \times r$ elementary matrices $\{e_{i,j}\}_{i,j=1}^r$. Then there is a nonzero product of the $r^2$ different elementary matrices of the form $e_{1,1} e_{1,2} e_{2,1} \cdots e_{i,1}$. Clearly the total value is $e_{1,1}$.

Furthermore by decomposing this product into two parts and switching them we obtain a nonzero monomial of that kind which starts with $e_{i,1}$ for any $i = 1, \ldots, r$.

**Step 1 (construction of $\Lambda_1$)**

Multiply each one of the $z_i$’s in $\Lambda$ by a primitive idempotent $1 \otimes e_{i,i}$ of the corresponding $G$-simple component, say on the right, so that the product remains...
non zero (clearly such idempotent exists by linearity). We denote these idempotents by \( x_t, t = 1, \ldots, n + 1 \).

Step 2 (construction of \( \Lambda_2 \))
Replace each one of the idempotents \( x_t = 1 \otimes e_{i,i} \) by a nonzero monomial \( X_t \) of the form \( 1 \otimes e_{i,i} \times \cdots \times 1 \otimes e_{j,i} \) (see Lemma 3.4). Here, if the idempotent \( x_t = 1 \otimes e_{i,i} \) belongs to the \( G \)-simple component \( (S_G)_i \) then the cardinality of \( X_t \) is \( r^2 \).

Step 3 (construction of \( \Lambda_3 \))
Replace each one of the idempotents \( \alpha_t,k = 1 \otimes e_{k,k} \) in \( X_t \), \( t = 1, \ldots, n + 1 \), by a monomial \( Y_{t,k} \) of the form

\[
\alpha_t,k = (u_h \otimes e_{k,k} \times a_k \times u_{h_2} \otimes e_{k,k} \times \cdots \times u_{h_p} \otimes e_{k,k} \times a_k)
\]

where \( p = ord(H) \) and the set \( \{h_1, h_1 h_2, \ldots, h_1 h_2 \cdots h_p\} \) runs over all elements of \( H \).

We denote by \( \hat{X}_t \) the monomial obtained from \( X_t \) and by \( \Lambda_3 \) the monomial obtained from \( \Lambda_2 \).

Step 4 (construction of \( \Lambda = \Lambda_4 \))
The objective of this step is nothing but to “clean” the monomial

\[
\Lambda_4 = z_1 \hat{X}_1 v_1 \hat{X}_2 \cdots \hat{X}_n v_n \hat{X}_{n+1}
\]
Indeed, we throw \( z_1 \) from \( \Lambda_3 \) and replace \( v_t \hat{X}_{t+1} \) by a new radical homogeneous element which we denote again by \( v_t \).

Remark 3.5. Clearly (by construction) the monomial

\[
\hat{\Lambda} = \hat{X}_1 v_1 \hat{X}_2 \cdots \hat{X}_n v_n \hat{X}_{n+1}
\]
is non zero and realizes \( \exp_{G}^{\text{Conj}}(A) \).

Our next task, roughly speaking, is to construct a certain subset of \( G \), denoted by \( \Omega_0 \), which will “determine” different decompositions of the monomial \( \hat{\Lambda} \). The set \( \Omega_0 \) will play a decisive role in the proof of Theorem 2.10.

Write \( \hat{\Lambda} = b_1 b_2 b_3 \cdots b_d \) where \( b_t \) is either a basis element of the form \( u_{h_t} \otimes e_{i,j} \) which appears in \( \hat{X}_t \), \( t = 1, \ldots, n + 1 \) or a radical element \( v_t \) for some \( t = 1, \ldots, n \).

Consider the subwords of \( \hat{\Lambda} \) of the form

\[
b_1, b_1 b_2, b_1 b_2 b_3, \ldots, b_1 b_2 \cdots b_d.
\]

For each subword we consider its homogeneous degree in \( G \) and we let \( \Omega \) be the set of elements in \( G \) obtained in that way. Clearly, any element in \( g \in \Omega \) may be the homogeneous degree of more than one subword of the form \( b_1 b_2 \cdots b_j \) in \( \Lambda \) and so we let \( \mu(g) \in \{1, \ldots, d\} \) be the minimal length \( j \) it appears. We refer to \( \mu(g) \) as the length of \( g \in \Omega \) in \( \Lambda \). Next we consider the set \( \Pi \) of left \( K \)-cosets of \( G \) that are represented by elements of \( \Omega \) and let \( \Omega_0 \) be the set of representatives for the left \( K \)-cosets (in \( \Pi \)) of minimal length in \( \Lambda \).

Definition 3.6. Let \( b_j b_{j+1} \cdots b_{j+k} \) be a \( G \)-graded monomial and \( S \) a nonempty subset of \( G \).

(1) We say that \( b_j b_{j+1} \cdots b_{j+k} \) is an \( S \)-monomial, if its homogeneous degree belongs to \( S \).
(2) We say that $b_{j+1} \cdots b_{j+k}$ has no proper $S$-submonomial if either $k = 0$ or else, there is no $S$-monomial of the form $b_{j+1} \cdots b_{j+p}$ where $0 \leq p < k$. Note that still there may exist an $S$-monomial of the form $b_{j+u} \cdots b_{j+q}$ where $1 < u \leq q \leq k$.

For any $g \in \Omega_0$ we consider the decomposition of

$$\widehat{\Lambda} = X_g \Sigma_{K,1} \Sigma_{K,2} \cdots \Sigma_{K,d} Y(g)$$

where

1. $X_g$ is a $g$-monomial and has no $g$-submonomial.
2. Each $\Sigma_{K,j}$ is a $K$-monomial and has no $K$-submonomial.
3. $Y(g)$ is an homogeneous monomial (which depends on $g \in G$) and has no $K$-submonomial.

**Remark 3.7.** Note that by the definition of the set $\Omega_0$, the above decomposition of $\widehat{\Lambda}$ exists and is unique for any $g \in \Omega_0$.

Consider the decomposition of $\widehat{\Lambda}$ determined by $g \in \Omega_0$. Clearly, the $g$-monomial $X_g$ and the $K$-monomials $\Sigma_{K,j}$ may be multiplied from the right (and give a nonzero product) by a unique primitive idempotent of the form $1 \otimes e_{i,j}$ which belongs to one of the $G$-simple components $(S_G)_t$ of $S$. We refer to these idempotents as the $K$-stops determined by $g \in \Omega_0$ in $\Lambda$.

**Lemma 3.8.** With the above notation we have:

1. Each $g \in \Omega_0$ determines an ordered set $\{\alpha_{g,1}, \ldots, \alpha_{g,n_g}\}$ of $K$-stops in $\widehat{\Lambda}$.
2. Each $K$-stop $\alpha_{g,i}$ determines uniquely a $G$-simple component $(S_G)_{\alpha_{g,i}}$. Moreover it determines uniquely a $K$-simple component $(S_G)_{K,\alpha_{g,i}}$. We refer to these $K$-simple components as the $K$-simple components determined by $g \in \Omega_0$.
3. If two different $K$-stops $\alpha_{g,i}$ and $\alpha_{g,j}$ (same $g$) determine the same $G$-simple component then they determine also the same $K$-simple component.
4. Each $K$-simple component (in any $G$-simple component) is determined by a $K$-stop $\alpha_{g,i}$ for some $g \in \Omega_0$.

**Proof.** Parts (1) and (2) follow directly from the construction. Part (3) follows easily from the fact that between two $K$-stops in the same $G$-simple component we have only semisimple $K$-monomials. Hence having $K$-stops which belong to different $K$-simple components would yield a zero product.

For the proof of (4) we recall first that any $K$-simple component in $F^r H \otimes M_r(F)$ is determined by an equivalence class of indices $\{1, \ldots, r\}$ (see Lemma 3.3). Fix a $K$-simple component $[i]$ in a $G$-simple component, say $(S_G)_t \cong F^r H \otimes M_r(F)$. This says that there is a primitive idempotent $1 \otimes e_{j,i}$ in $(S_G)_t$, with $j \in [i]$, that can be inserted on the right of one of the $b$'s, say $b_m$ in $\widehat{\Lambda} = b_1 b_2 b_3 \cdots b_d$ and yield a non zero product. Let $g \in G$ be the homogeneous degree of the monomial $b_1 b_2 b_3 \cdots b_m$. Clearly $g \in \Omega$ and hence $gK$, the left $K$-coset represented by $g$, intersects $\Omega$ non-trivially. Observe that by the definition of $\Omega$, any element $g' \in gK \cap \Omega$ determines a $K$-stop in $\widehat{\Lambda} = b_1 b_2 b_3 \cdots b_d$ and hence if $\mu(g) \leq \mu(g')$ (the lengths in $\widehat{\Lambda}$) we have that any $K$-stop in $\widehat{\Lambda} = b_1 b_2 b_3 \cdots b_d$ determined by $g'$ is determined also by $g$. We conclude that if $g$ is the representative $gK \cap \Omega$ of minimal length in $\widehat{\Lambda}$ (hence in $\Omega_0$) it determines the $K$-simple component represented by $[i]$. □
The next lemma is key. It computes the number of elements in \( \Omega_0 \) which determine the different \( K \)-simple components.

Fix \( g \) in \( \Omega_0 \) and let

\[
\hat{\Lambda} = X_g \Sigma_{K,1} \Sigma_{K,2} \cdots \Sigma_{K,d} Y(g)
\]

be the corresponding decomposition of \( \hat{\Lambda} \). Consider the \( K \)-stop \( \alpha_{g,j} \) determined by \( X_g \Sigma_{K,1} \Sigma_{K,2} \cdots \Sigma_{K,j} \).

By construction, \( \alpha_{g,j} \) is a primitive idempotent of the form \( 1 \otimes e_{i,i} \), \( 1 \leq i \leq r \), in the \( t \)-th \( G \)-simple component \( (S_G)_t \cong F^{H} \otimes M_{r}(F) \). Let \( k_1 \in K \) be the homogeneous degree of the monomial \( \Sigma_{K,1} \Sigma_{K,2} \cdots \Sigma_{K,j} \). In the next lemma we use the same notation as above (in particular \((g_{s_1}, \ldots, g_{s_r})\) is the \( r \)-tuple in \( G^{(r)} \) which determines the elementary \( G \)-grading on \( M_{r}(F) \)).

**Lemma 3.9.** The following hold.

1. Any element in \( g_{k_1} g_{s_i}^{-1} H g_{s_j} K \) which determines a \( K \)-simple component in the \( t \)-th \( G \)-simple component \((S_G)_t\), determines the same \( K \)-simple component as \( g \) does.
2. Any \( K \)-left coset which is contained in \( g_{k_1} g_{s_i}^{-1} H g_{s_j} K \), is represented in \( \Omega \) and hence is represented in \( \Omega_0 \).
3. Any \( \hat{g} \in G \) that determines the same \( K \)-simple component in the \( t \)-th \( G \)-simple component as \( g \) does, is in \( g_{k_1} g_{s_i}^{-1} H g_{s_j} K \).

**Proof.** Recall first that the homogeneous degree of a semisimple element \( b_t = u_h \otimes e_{i,j} \) is \( g_{s_i}^{-1} h g_{s_j} \). Suppose \( \hat{g} = g_{k_1} g_{s_i}^{-1} h_1 g_{s_j} k_2 \) is an element in \( g_{k_1} g_{s_i}^{-1} H g_{s_j} K \) which determines a \( K \)-stop in the \( t \)-th \( G \) simple component \((S_G)_t\) with primitive idempotent \( 1 \otimes e_{j,j} \). We need to show that \( j \sim i \). Indeed, by construction there is \( k_3 \in K \) and \( h_2 \in H \) such that \( \hat{g} k_3 = g_{k_1} g_{s_i}^{-1} h_2 g_{s_j} \) that is

\[
g_{k_1} g_{s_i}^{-1} h_1 g_{s_j} k_2 k_3 = g_{k_1} g_{s_i}^{-1} h_2 g_{s_j}.
\]

This shows that \( h_1 g_{s_j} k_2 k_3 = h_2 g_{s_j} \) and \( j \sim i \) as desired.

Let us prove (2). We are assuming that \( g \) determines \( 1 \otimes e_{i,j} \) in the \( t \)-th \( G \)-simple component. Consider the set \( g_{k_1} g_{s_i}^{-1} H g_{s_j} \). Clearly it contains a set of representatives for the left \( K \)-cosets which are contained in \( g_{k_1} g_{s_i}^{-1} H g_{s_j} K \). On the other hand, by the construction of the monomial \( \hat{\Lambda} \) (step 3), each element in \( g_{k_1} g_{s_i}^{-1} H g_{s_j} \) determines a \( K \)-stop \( 1 \otimes e_{i,i} \) and the result follows.

To prove (3) suppose \( \hat{g} \in G \) determines the same \( K \)-simple component as \( g \) does in the \( t \)-th \( G \)-simple component. This means that \( \hat{g} \) determines a \( K \)-stop \( 1 \otimes e_{j,j} \) in the \( t \)-th simple component and \( j \sim i \). It follows that there exist elements \( k_1 \) and \( k_2 \) in \( K \) and \( h \) in \( H \) such that

\[
\hat{g} k_2 = g_{k_1} g_{s_i}^{-1} h g_{s_j}.
\]

But from the condition on relation of \( j \) and \( i \) we know that \( g_{s_j} \) and \( g_{s_i} \) represent the same \( H - K \) double coset of \( G \) and hence there exist \( k_3 \) in \( K \) and \( h_2 \) in \( H \) such that \( g_{s_j} = h_2 g_{s_j} k_3 \). This implies

\[
\hat{g} k_2 = g_{k_1} g_{s_i}^{-1} h h_2 g_{s_j} k_3
\]

and hence \( \hat{g} \in g_{k_1} g_{s_i}^{-1} H g_{s_j} K \) as desired. This completes the proof of the lemma. \( \square \)
Remark 3.10. We emphasize that parts (1) and (3) of the lemma are not exactly opposite to each other for it may be the case that an element in \(g k_1 g_s^{-1} H g_s, K\) does not determine any \(K\)-simple component in \((S_G)_t\). This is the reason that we couldn’t get through with the set \(\Omega\) and we needed a finer set, namely a set of left \(K\)-cosets representatives \(\Omega_0\).

From our construction we see that any \(g \in \Omega_0\) determines a unique \(K\)-simple component in some of the \(G\)-simple components \((S_G)_1, \ldots, (S_G)_{n+1}\). We denote the \(K\)-simple components by \(B_{g,K,1}, \ldots, B_{g,K,n+1}\) where the parameter \(g\) says that the \(K\)-components depend on \(g\). We put \(B_{g,K,j} = 0\) for those \(G\)-simple component which are not visited in the decomposition of \(\hat{\Lambda}\) determined by \(g\).

Corollary 3.11. Consider all decompositions of the form

\[
\hat{\Lambda} = X_g \Sigma_{K,1} \Sigma_{K,2} \cdots \Sigma_{K,j} \cdots \Sigma_{K,d} V(g)
\]

where \(g \in \Omega_0\).

Take any \(K\)-simple component, say the \(K\)-simple component \([i]\) in the \(t\)-th \(G\)-simple components \((S_G)_t\). Let \(N_{s_G}([i])\) be the number of such decompositions in which the \(K\)-simple component \([i]\) is “visited” (that is the decompositions that determine a \(K\)-stop \(1 \otimes e_{j,j}\) in \((S_G)_t\) and \(j \sim i\)). Then

\[
N_{s_G}([i]) = |H g_s, K|/|K|
\]

Proof. By Lemma 3.8 part (4), we know that the \(K\)-simple component \([i]\) is “visited” in the decomposition of \(\hat{\Lambda}\) determined by some element \(g \in \Omega_0\). Then by Lemma 3.9 we conclude that the number is

\[
N_{s_G}([i]) = |g k_1 g_s^{-1} H g_s, K|/|K| = |H g_s, K|/|K|.
\]

We are now ready to complete the proof of Theorem 2.10. Let us assume

\[
\exp_G^{\text{Conj}}(A) > [G : K]^2 \exp_K^{\text{Conj}}(A_K).
\]

and get a contradiction.

Recall we are assuming that the monomial

\[
\hat{\Lambda} = X_1 v_1 X_2 \cdots X_n v_n X_{n+1}
\]

realizes \(\exp_G^{\text{Conj}}(A)\). Consider the different decompositions

\[
\hat{\Lambda} = X_g \Sigma_{K,1} \Sigma_{K,2} \cdots \Sigma_{K,j} \cdots \Sigma_{K,d} V(g)
\]

where \(g \in \Omega_0\). By the definition of \(\exp_K^{\text{Conj}}(A_K)\) we have that

\[
\exp_K^{\text{Conj}}(A_K) \geq \dim_F(B_{g,K,1}) + \cdots + \dim_F(B_{g,K,n+1})
\]

for every \(g \in \Omega_0\). Consequently for every \(g \in \Omega_0\)

\[
\exp_G^{\text{Conj}}(A) > [G : K]^2 \sum_{j=1}^{n+1} \dim_F(B_{g,K,j}).
\]

Summing over all elements of \(\Omega_0\) we have
\[
\sum_{g \in \Omega_0} \exp_{G}^{Conj}(A) > [G : K]^2 \sum_{g \in \Omega_0} \sum_{j=1}^{n+1} \dim_{F}(B_{g,K,j}).
\]

Let us show that this is not the case, that is
\[
\sum_{g \in \Omega_0} \exp_{G}^{Conj}(A) \leq [G : K]^2 \sum_{g \in \Omega_0} \sum_{j=1}^{n+1} \dim_{F}(B_{g,K,j}).
\]

Now by construction,
\[
\exp_{G}^{Conj}(A) = (\pi_1 + \cdots + \pi_m)^2 \leq [G : K] \sum_{j=1}^{m} (|H_{g_s,K}|/|K|) |g^{-1}_{s_i}H_{g_s,K} \cap K| \pi_j^2.
\]

But one knows that
\[
(|H_{g_s,K}|/|K|) |g^{-1}_{s_i}H_{g_s,K} \cap K| = |H| \]

and hence we need to show that
\[
|\pi_1 + \cdots + \pi_m|^2 \leq [G : K] \sum_{i=1}^{m} \pi_i^2.
\]

As \(m\) is the number of double \(H - K\)-cosets in \(G\), we have that \(m \leq [G : K]\), and hence the result will follow if we prove
\[
(\pi_1 + \cdots + \pi_m)^2 \leq m \sum_{i=1}^{m} \pi_i^2.
\]
But this is equivalent to $\sum_{i<j} (\pi_j - \pi_i)^2 \geq 0$ and Theorem 2.10 is proved.

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Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: aljadeff@tx.technion.ac.il