Actions of the map algebras on random hypergraphs
and random simplicial complexes of
Erdös-Rényi-Types

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Abstract

In recent years, random hypergraphs and random simplicial complexes have been extensively studied. Let $H$ be a hypergraph with vertex set $V$. Consider the complement $\gamma H$ of $H$ in the complete complex $\Delta[V]$, the minimal complex $\Delta H$ that $H$ can be embedded in, and the maximal complex $\delta H$ that can be embedded in $H$. The map algebra is the semi-group generated by $\Delta$, $\delta$ and $\gamma$. Let $p : V \to [0, 1]$. In this paper, by considering the minimal independence hypergraph $\overline{\Delta} H$ that $H$ can be embedded in and the maximal independence hypergraph $\overline{\delta} H$ that can be embedded in $H$, we give the explicit expression of the action of the map algebra on the Erdös-Rényi-type model $\overline{P}_p$ of random hypergraphs. As a by-product, we give the explicit expression of the action of the map algebra on the Erdös-Rényi-type model $P_p$ of random simplicial complexes.

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1 Introduction

1.1 Models of random graphs and random simplicial complexes

Let $V$ be a vertex set of cardinality $N$ with a total order $\prec$. Let $0 \leq p \leq 1$. Let $0 \leq n \leq N - 1$ be an integer. We write an $n$-hyperedge $\sigma^{(n)} = \{v_0, v_1, \ldots, v_n\}$ on $V$ such that $v_0 \prec v_1 \prec \cdots \prec v_n$. Let $\Delta[V] = 2^V \setminus \{\emptyset\}$, where $2^V$ is the power-set of $V$. We call an element of $\Delta[V]$ a hyperedge or a simplex on $V$. We also denote $\Delta[V]$ as $\Delta_N$. Let $sk_r(\Delta_N)$ be the $r$-skeleton of $\Delta_N$. Consider the probability space $\Omega_N^r$ consisting of all sub-complexes of $sk_r(\Delta_N)$ and the probability function $P_{r,N,p} : \Omega_N^r \to \mathbb{R}$ given by

$$P_{r,N,p}(\mathcal{K}) = \prod_{\sigma \in \mathcal{K}} p(\sigma) \prod_{\sigma \in E(\mathcal{K})} (1 - p(\sigma)),$$
where \( p = (p_0, p_1, \ldots, p_r), 0 \leq p_0, p_1, \ldots, p_r \leq 1, E(K) \) is the set of all the external faces of \( K \) in \( \text{sk}_r(\Delta_N) \), and \( p(\sigma) = p_i \) if \( \sigma \) is an \( i \)-simplex. This model \( P_{r,N,p} \) was given by A. Costa and M. Farber in [14, 15, 18]. The connectivity, the fundamental group, the dimension, and the Betti number of the random simplicial complex given by \( P_{r,N,p} \) have been studied by A. Costa and M. Farber in [14, 15, 16, 17, 18]. The followings are special cases of \( P_{r,N,p} \):

- \( P_{1,N,p} \) with \( p = (1, p, 0, \ldots, 0) \) is the Erdös-Rényi model \( G(N, p) \).
  P. Erdös and A. Rényi [19] and E.N. Gilbert [21] constructed the Erdös-Rényi model \( G(N, p) \) by choosing each pair of vertices in \( V \) as an edge independently at random with probability \( p \). Thresholds for connectivity of \( G(N, p) \) were proved by P. Erdös and A. Rényi in [20].

- \( P_{2,N,p} \) with \( \mathbf{p} = (1, 1, p, 0, \ldots, 0) \) is the Linial-Meshulam model \( Y_2(N, p) \).
  N. Linial and R. Meshulam [31] constructed a random 2-complex \( Y_2(N, p) \) by taking the complete graph on \( V \) as the 1-skeleton and then choosing each 2-simplex independently at random with probability \( p \). The fundamental group, the homology groups, and the asphericity as well as the hyperbolicity of \( Y_2(N, p) \) were respectively studied in [5], [9, 10], and [11, 12].

- \( P_{d,N,p} \) with \( \mathbf{p} = (1, \ldots, 1, p, 0, \ldots, 0) \) is the Meshulam-Wallach model \( Y_d(N, p) \).
  R. Meshulam and N. Wallach [33] constructed a random \( d \)-complex \( Y_d(N, p) \) by taking the \((d-1)\)-skeleton of the complete complex on \( V \) and then choosing each \( d \)-simplex independently at random with probability \( p \). The (co)homology groups, the phase transition of the homology groups, the eigenvalues of the Laplacian, the collapsibility, and the topological minor were respectively studied in [4, 25, 29, 30], [32], [22], [3, 4] and [23].

- \( P_{N-1,N,p} \) with \( \mathbf{p} = (1, p, 1, \ldots, 1) \) is the random flag complex (random clique complex) \( X(N, p) \) of \( G(N, p) \).
  The random flag complex (random clique complex) \( X(N, p) \) of \( G(N, p) \) were studied by M. Kahle in [26, 28] and A. Costa, M. Farber and D. Horak in [13]. Sharp vanishing thresholds for the cohomology of \( X(N, p) \) were proved by M. Kahle in [28].

### 1.2 Map algebras on hypergraphs and simplicial complexes

The map algebra on hypergraphs was initially studied by S. Ren, C. Wu and J. Wu in [37]. A hypergraph \( \mathcal{H} \) on \( V \) is a subset of \( \Delta[V] \). The associated simplicial complex \( \Delta \mathcal{H} \) of \( \mathcal{H} \) is the smallest simplicial complex containing \( \mathcal{H} \) (cf. firstly studied in [35] and later studied
in [8]). The lower-associated simplicial complex \( \delta H \) of \( H \) is the largest simplicial complex contained in \( H \). The complement \( \gamma H \) is the hypergraph \( \Delta[V] \setminus H \). We call \( \gamma K \), where \( K \) is any sub-complex of \( \Delta[V] \), an independence hypergraph. Let \( H(V) \) be the category whose objects are hypergraphs on \( V \) and whose morphisms are morphisms of hypergraphs. The map algebra \( G \), which acts on \( \text{Obj}(H(V)) \), is the semi-group generated by the operators \( \Delta, \delta \) and \( \gamma \) whose multiplication is the composition and whose unit is the identity.

Let \( K(V) \) be the category whose objects are simplicial complexes on \( V \) and whose morphisms are simplicial maps. The map algebra \( G'_K \), which acts on \( \text{Obj}(K(V)) \), is the semi-group generated by \( \alpha = \Delta \gamma \) and \( \beta = \delta \gamma \).

Write \( L \) for an independence hypergraph on \( V \). An co-external face of \( L \) is a hyperedge \( \sigma \in \Delta[V] \) such that \( \sigma \notin L \) and \( \tau \in L \) for any proper superset \( \tau \supseteq \sigma \), where \( \tau \in \Delta[V] \). Let \( \bar{E}(L) \) be the collection of all the co-external faces of \( L \). Let \( L(V) \) be the category whose objects are independence hypergraphs on \( V \) and whose morphisms are morphisms of hypergraphs.

1.3 Random hypergraphs, random simplicial complexes and the actions of the map algebras

Random hypergraphs are useful in data sciences and complex networks, for examples [2, 7]. A random hypergraph on \( V \) is a probability function on \( \text{Obj}(H(V)) \). Let \( D(H(V)) \) be the functional space of all the probability functions on \( \text{Obj}(H(V)) \). Let \( p : \Delta[V] \to [0, 1] \). Consider the Erdös-Rényi-type model \( P_p \) of random hypergraphs given by

\[
\bar{P}_p(H) = \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1 - p(\sigma))
\]

for any \( H \in \text{Obj}(H(V)) \). We choose each element \( \sigma \in \Delta[V] \) to be a hyperedge of \( H \) independently at random with probability \( p(\sigma) \). This randomly generated hypergraph \( H \) satisfies the probability function \( \bar{P}_p \), denoted as \( \bar{H} \sim \bar{P}_p \). The map algebra \( G \) acts on \( D(H(V)) \) by

\[
(Dw)(H') = \text{Prob}[wH = H' \mid \bar{H} \sim \bar{P}_p] = \sum_{wH=H'} \bar{P}_p(H),
\]

where \( H' \in \text{Obj}(H(V)) \) and \( w \in G \).

A random simplicial complex on \( V \) is a probability function on \( \text{Obj}(K(V)) \). Let \( D(K(V)) \) be the functional space of all the probability functions on \( \text{Obj}(K(V)) \). Consider the Erdös-Rényi-type model \( P_p \) of random simplicial complexes given by

\[
P_p(K) = \prod_{\sigma \in K} p(\sigma) \prod_{\sigma \in E(K)} (1 - p(\sigma)),
\]

where \( K \in \text{Obj}(K(V)) \). In particular, if there exist constants \( 0 \leq p_0, p_1, \ldots, p_{|V|-1} \leq 1 \) such that \( p(\sigma) = p_{\dim \sigma} \) for each \( \sigma \in K \), then letting \( p = (p_0, p_1, \ldots, p_{|V|-1}) \), we have that
\( p \) is \( P_{N-1,N,p} \). The map algebra \( G'_K \) acts on \( D(K(V)) \) by

\[
(Dw)(K') = \text{Prob}\left[wK = K' \mid K \sim P_p\right] = \sum_{wK = K'} P_p(K),
\]

where \( K' \in \text{Obj}(K(V)) \) and \( w \in G'_K \).

A random independence hypergraph on \( V \) is a probability function on \( \text{Obj}(L(V)) \). Let \( D(L(V)) \) be the functional space of all the probability functions on \( \text{Obj}(L(V)) \). Consider the Erdös-Rényi-type model \( Q_p \) of random independence hypergraphs given by

\[
Q_p(L) = \prod_{\sigma \in L} p(\sigma) \prod_{\sigma \in E(L)} (1 - p(\sigma)),
\]

where \( L \in \text{Obj}(L(V)) \).

In [37, Theorem 1.5 (1) - (3)], it is proved that

\[
\begin{align*}
((D\gamma)(\bar{P}_p))(\mathcal{H}) &= P_{1-p}(\mathcal{H}), \quad (1.1) \\
((D\Delta)(\bar{P}_p))(\mathcal{K}) &= \prod_{\tau \in \text{max}(\mathcal{K})} p(\tau) \prod_{\tau \not\in \mathcal{K}} (1 - p(\tau)), \quad (1.2) \\
((D\delta)(\bar{P}_p))(\mathcal{K}) &= \sum_{\delta \mathcal{H} = \mathcal{K}} \prod_{\sigma \in \mathcal{K}} p(\sigma) \prod_{\sigma \in E(\mathcal{K})} (1 - p(\sigma)). \quad (1.3)
\end{align*}
\]

Note that the RHS of (1.1) and (1.2) are explicit in terms of \( \mathcal{K} \) while the RHS of (1.3) is not explicit.

### 1.4 Results of this paper

In this paper, we define the associated independence hypergraph \( \tilde{\Delta} \mathcal{H} \) of \( \mathcal{H} \) to be the smallest independence hypergraph containing \( \mathcal{H} \). We define the lower-associated independence hypergraph \( \delta \mathcal{H} \) of \( \mathcal{H} \) to be the largest independence hypergraph contained in \( \mathcal{H} \). We prove the next theorem for the map algebra \( G \). Let \( L(V) \) be the category whose objects are independence hypergraphs on \( V \) and whose morphisms are morphisms of independence hypergraphs.

**Theorem 1.1.** Each of the triple

\[ \{ \gamma, \Delta, \delta \} \]

\[ \{ \gamma, \Delta, \bar{\delta} \} \]

\[ \{ \gamma, \bar{\Delta}, \bar{\delta} \} \]

\[ \{ \gamma, \delta, \bar{\delta} \} \]

multiplicatively generates \( G \).

Let \( \sigma \in \mathcal{H} \). We call \( \sigma \) a maximal hyperedge if there does not exist any \( \tau \in \mathcal{H} \) such that \( \sigma \subseteq \tau \). We denote the collection of all the maximal hyperedges of \( \mathcal{H} \) as \( \text{max}(\mathcal{H}) \). We call \( \sigma \) a minimal hyperedge if there does not exist any \( \tau \in \mathcal{H} \) such that \( \sigma \supseteq \tau \). We denote the collection of all the minimal hyperedges of \( \mathcal{H} \) as \( \text{min}(\mathcal{H}) \). With the help of Theorem 1.1, we calculate the explicit action of \( G \) on \( \bar{P}_p \) in the next theorem.
Theorem 1.2 (Main Result). The action of $G$ on $\bar{P}_p$ is given by

(i). $((D\gamma)(\bar{P}_p))(\mathcal{H}) = \bar{P}_{1-p}(\mathcal{H})$,

(ii). $((D\Delta)(\bar{P}_p))(\mathcal{K}) = Q_{1-p}(\gamma\mathcal{K})$,

(iii). $((D\Delta)(\bar{P}_p))(\mathcal{L}) = P_{1-p}(\gamma\mathcal{L})$,

(iv). $((D\delta)(\bar{P}_p))(\mathcal{K}) = P_p(\mathcal{K})$,

(v). $((D\delta)(\bar{P}_p))(\mathcal{L}) = Q_p(\mathcal{L})$,

where $\mathcal{H} \in \text{Obj}(\mathcal{H}(V))$, $\mathcal{K} \in \text{Obj}(\mathcal{K}(V))$ and $\mathcal{L} \in \text{Obj}(\mathcal{L}(V))$.

As a by-product, we calculate the explicit action of $G'_K$ on $P_p$ in the next theorem.

Theorem 1.3. The action of $G'_K$ on $P_p$ is given by

(i). $(D\alpha)(P_p)(\mathcal{K}) = \begin{cases} \prod_{\sigma \in \Delta[V]} p(\sigma) & \text{if } \mathcal{K} = \emptyset, \\ \prod_{v \in V} (1 - p(\{v\})) & \text{if } \mathcal{K} = \Delta[V], \\ 0 & \text{if } \mathcal{K} \neq \emptyset, \Delta[V] \end{cases}$

(ii). $(D\beta)(P_p)(\mathcal{K}) = \begin{cases} \prod_{\{v\} \in \gamma\text{St}(\mathcal{K})} p(\{v\}) \prod_{\{v\} \notin \gamma\text{St}(\mathcal{K})} (1 - p(\{v\})) & \text{if } \mathcal{K} = \gamma\text{St}(V \setminus K_0), \\ 0 & \text{otherwise,} \end{cases}$

where $\mathcal{K} \in \text{Obj}(\mathcal{K}(V))$.

2 Preliminaries on hypergraphs and simplicial complexes

2.1 Simplicial complexes and their complements

An (abstract) simplicial complex $\mathcal{K}$ on $V$ is a collection of hyperedges on $V$, which are called simplices, such that for any $\sigma \in \mathcal{K}$ and any nonempty subset $\tau \subseteq \sigma$, $\tau$ is in $\mathcal{K}$ (cf. [24 p. 107]).

Let $\mathcal{K}$ and $\mathcal{K}'$ be two simplicial complexes. Let $V$ be the vertex set of $\mathcal{K}$ and let $V'$ be the vertex set of $\mathcal{K}'$. A simplicial map $f : \mathcal{K} \rightarrow \mathcal{K}'$ is a map $f : V \rightarrow V'$ such that for any simplex $\sigma = \{v_0, v_1, \ldots, v_n\}$ in $\mathcal{K}$, its image $f(\sigma)$ is a simplex in $\mathcal{K}'$ spanned by the (not necessarily distinct) vertices $f(v_0), f(v_1), \ldots, f(v_n)$.

An independence hypergraph $\mathcal{L}$ on $V$ is a collection of hyperedges on $V$ such that for any $\sigma \in \mathcal{L}$ and any finite superset $\tau \supseteq \sigma$, where $\tau \subseteq V$, $\tau$ is in $\mathcal{L}$. A hypergraph $\mathcal{L}$ on $V$ is an independence hypergraph iff there exists a simplicial complex $\mathcal{K}$ on $V$ (we allow $\mathcal{K}$ to be the emptyset) such that $\mathcal{L} = \Delta[V] \setminus K$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be two independence hypergraphs. Let $V$ be the vertex set of $\mathcal{L}$ and let $V'$ be the vertex set of $\mathcal{L}'$. A morphism of independence hypergraphs $f : \mathcal{L} \rightarrow \mathcal{L}'$ is a
map $f : V \rightarrow V'$ such that for any hyperedge $\sigma = \{v_0, v_1, \ldots, v_n\}$ in $\mathcal{L}$, its image $f(\sigma)$ is a hyperedge in $\mathcal{L}'$ spanned by the (not necessarily distinct) vertices $f(v_0), f(v_1), \ldots, f(v_n)$.

### 2.2 Hypergraphs, associated simplicial complexes and associated independence hypergraphs

A hypergraph $\mathcal{H}$ on $V$ is just a collection of hyperedges on $V$. Let $\mathcal{H}$ be a hypergraph on $V$. The associated simplicial complex of $\mathcal{H}$ is the simplicial complex (cf. [35], later [1] and [8])

$$\Delta \mathcal{H} = \{ \tau \in \Delta[V] \mid \tau \subseteq \sigma \text{ for some } \sigma \in \mathcal{H} \}.$$ 

The lower-associated simplicial complex of $\mathcal{H}$ is the simplicial complex (cf. [38, 37])

$$\delta \mathcal{H} = \{ \sigma \in \mathcal{H} \mid \tau \in \mathcal{H} \text{ for any } \tau \subseteq \sigma \text{ and } \tau \neq \emptyset \},$$

We define the associated independence hypergraph of $\mathcal{H}$ to be the independence hypergraph

$$\bar{\Delta} \mathcal{H} = \{ \tau \in \Delta[V] \mid \tau \supseteq \sigma \text{ for some } \sigma \in \mathcal{H} \}$$

and define the lower-associated independence hypergraph of $\mathcal{H}$ to be the independence hypergraph

$$\bar{\delta} \mathcal{H} = \{ \sigma \in \mathcal{H} \mid \tau \in \mathcal{H} \text{ for any } \tau \supseteq \sigma \text{ and } \tau \subseteq V \}.$$

We have a diagram

$$\begin{array}{ccc}
\delta \mathcal{H} & \xrightarrow{i_\delta} & \Delta \mathcal{H} \\
\downarrow{i_\Delta} & & \downarrow{i_\Delta} \\
\bar{\delta} \mathcal{H} & \xrightarrow{i_\Delta} & \bar{\Delta} \mathcal{H}
\end{array}$$

such that each arrow is the canonical inclusion of hypergraphs. We observe the followings

(i). $\mathcal{H}$ is a simplicial complex $\iff \delta \mathcal{H} = \mathcal{H} \iff \Delta \mathcal{H} = \mathcal{H} \iff \delta \mathcal{H} = \Delta \mathcal{H}$,

(ii). $\mathcal{H}$ is an independence hypergraph $\iff \bar{\delta} \mathcal{H} = \mathcal{H} \iff \bar{\Delta} \mathcal{H} = \mathcal{H} \iff \bar{\delta} \mathcal{H} = \bar{\Delta} \mathcal{H}$,

(iii). $\mathcal{H}$ is a simplicial complex and also an independence hypergraph $\iff \mathcal{H} = \Delta[V]$.

**Example 2.1.** Let $n \geq 4$ be an integer. Let $V = \{v_0, v_1, \ldots, v_n\}$ where $v_0 < v_1 < \cdots < v_n$. Fix an integer $2 \leq k \leq n - 2$. 

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(i). Consider the hypergraph
\[
\mathcal{H} = \{\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\} \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n\}.
\]  
(2.2)

We have \(\delta \mathcal{H} = \bar{\delta} \mathcal{H} = \emptyset\) and
\[
\Delta \mathcal{H} = \{\{v_{i_0}, v_{i_1}, \ldots, v_{i_l}\} \mid 0 \leq i_0 < i_1 < \cdots < i_l \leq n, 0 \leq l \leq k\},
\]
\[
\bar{\Delta} \mathcal{H} = \{\{v_{i_0}, v_{i_1}, \ldots, v_{i_l}\} \mid 0 \leq i_0 < i_1 < \cdots < i_l \leq n, k \leq l \leq n\}.
\]

(ii). Consider the hypergraph
\[
\mathcal{H}' = \mathcal{H} \cup \{\{v_0\}, \{v_1\}, \ldots, \{v_n\}\}.
\]  
(2.3)

We have \(\bar{\delta} \mathcal{H}' = \emptyset\), \(\Delta \mathcal{H}' = \Delta \mathcal{H}, \bar{\Delta} \mathcal{H}' = \Delta[V]\), and
\[
\delta \mathcal{H}' = \{\{v_0\}, \{v_1\}, \ldots, \{v_n\}\}.
\]

(iii). Consider the hypergraph
\[
\mathcal{H}'' = \mathcal{H} \cup \{\{v_0, v_1, \ldots, v_n\}\}.
\]  
(2.4)

We have \(\delta \mathcal{H}'' = \emptyset\), \(\Delta \mathcal{H}'' = \Delta[V], \bar{\Delta} \mathcal{H}'' = \Delta \mathcal{H}\), and
\[
\bar{\delta} \mathcal{H}'' = \{\{v_0, v_1, \ldots, v_n\}\}.
\]

(iv). Consider the hypergraph
\[
\mathcal{H}''' = \mathcal{H} \cup \{\{v_0\}, \{v_1\}, \ldots, \{v_n\}\} \cup \{\{v_0, v_1, \ldots, v_n\}\}.
\]  
(2.5)

We have \(\Delta \mathcal{H}''' = \bar{\Delta} \mathcal{H}''' = \Delta[V], \delta \mathcal{H}''' = \delta \mathcal{H}', \bar{\delta} \mathcal{H}''' = \bar{\delta} \mathcal{H}''\).

Let \(\mathcal{H}\) and \(\mathcal{H}'\) be two hypergraphs. Let \(V\) be the vertex set of \(\mathcal{H}\) and let \(V'\) be the vertex set of \(\mathcal{H}'\). A morphism of hypergraphs \(f : \mathcal{H} \to \mathcal{H}'\) is a map \(f : V \to V'\) such that for any hyperedge \(\sigma = \{v_0, v_1, \ldots, v_n\}\) in \(\mathcal{H}\), its image \(f(\sigma)\) is a hyperedge in \(\mathcal{H}'\) spanned by the (not necessarily distinct) vertices \(f(v_0), f(v_1), \ldots, f(v_n)\). Let \(f : \mathcal{H} \to \mathcal{H}'\) be a morphism of hypergraphs. Then \(f\) induces two simplicial maps (cf. [38])
\[
\Delta f : \quad \Delta \mathcal{H} \to \Delta \mathcal{H}',
\]
\[
\delta f : \quad \delta \mathcal{H} \to \delta \mathcal{H}'
\]

sending a simplex \(\{v_0, v_1, \ldots, v_n\}\) to the simplex spanned by the (not necessarily distinct) vertices \(f(v_0), f(v_1), \ldots, f(v_n)\). We say that \(f\) is vertex-bijective if the map \(f : V \to V'\) is bijective.
Lemma 2.2. Let $f : \mathcal{H} \rightarrow \mathcal{H}'$ be a morphism of hypergraphs. Then $f$ induces a morphism of independence hypergraphs

$$\bar{\Delta}f : \bar{\Delta}\mathcal{H} \rightarrow \bar{\Delta}\mathcal{H}'. \quad (2.6)$$

Suppose in addition that $f$ is vertex-bijective. Then $f$ induces a morphism of independence hypergraphs

$$\bar{\delta}f : \bar{\delta}\mathcal{H} \rightarrow \bar{\delta}\mathcal{H}'. \quad (2.7)$$

Proof. Let $f : \mathcal{H} \rightarrow \mathcal{H}'$ be a morphism of hypergraphs. Let $\sigma \in \bar{\Delta}\mathcal{H}$. Then there exists $\tau \in \Delta[V]$ such that $\sigma \subseteq \tau$ and $\tau \in \mathcal{H}$. Thus $f(\tau) \in \Delta[V']$ satisfies $f(\sigma) \subseteq f(\tau)$ and $f(\tau) \in \mathcal{H}'$. It follows that $f(\sigma) \in \bar{\Delta}\mathcal{H}'$. We obtain (2.6).

Suppose in addition that $f$ is vertex-bijective. Let $\sigma \in \bar{\delta}\mathcal{H}$. We want to prove that $f(\sigma) \in \bar{\delta}\mathcal{H}'$. Let $\tau' \in \Delta[V']$ such that $\tau' \supseteq f(\sigma)$. Then the pre-image $f^{-1}(\tau')$ is in $\Delta[V]$ and $f^{-1}(\tau') \supseteq \sigma$. It follows by the definition of $\bar{\delta}\mathcal{H}$ that $f^{-1}(\tau') \in \bar{\delta}\mathcal{H}$. Since $f$ is a morphism of hypergraphs, we have $f(f^{-1}(\tau')) \in \mathcal{H}'$. Since $f$ is vertex-bijective, we have $\tau' = f(f^{-1}(\tau'))$. Therefore, $\tau' \in \mathcal{H}'$, which implies $f(\sigma) \in \bar{\delta}\mathcal{H}'$. We obtain (2.7). \qed

The morphisms $i_\Delta$, $i_\delta$ and $i_{\bar{\Delta}}$ in diagram (2.1) are functorial with respect to morphisms of hypergraphs; and the morphism $i_{\bar{\delta}}$ in diagram (2.1) is functorial with respect to vertex-bijective morphisms of hypergraphs. The following diagrams commute

$$
\begin{array}{ccc}
\bar{\delta}\mathcal{H} & \xrightarrow{\delta f} & \delta\mathcal{H}' \\
\downarrow & & \downarrow \\
\bar{\Delta}\mathcal{H} & \xrightarrow{\Delta f} & \Delta\mathcal{H}' \\
\downarrow & & \downarrow \\
\bar{\Delta}\mathcal{H}' & \xrightarrow{\Delta f} & \Delta\mathcal{H'},
\end{array}
$$

where $f$ is any morphism of hypergraphs and $f^{vb}$ is any vertex-bijective morphism of hypergraphs.

3 Map algebras on hypergraphs and simplicial complexes

3.1 Some categories, functors, and maps

(a). Categories of hypergraphs and simplicial complexes. Let $\mathcal{X}$ be a fixed hypergraph on $V$. We call a sub-hypergraph $\mathcal{H}$ of $\mathcal{X}$ an $\mathcal{X}$-relative simplicial complex if for any $\sigma \in \mathcal{H}$ and any $\tau \in \mathcal{X}$ such that $\tau \subseteq \sigma$, it always holds $\tau \in \mathcal{H}$. We call a sub-hypergraph $\mathcal{H}$ of $\mathcal{X}$ an $\mathcal{X}$-relative independence hypergraph if for any $\sigma \in \mathcal{H}$ and any $\tau \in \mathcal{X}$ such that $\tau \supseteq \sigma$, it always holds $\tau \in \mathcal{H}$. Consider the categories
(i). $H(\mathcal{X})$ (resp. $H^{vb}(\mathcal{X})$), whose objects are sub-hypergraphs of $\mathcal{X}$ and whose morphisms are morphisms (resp. vertex-bijective morphisms) of hypergraphs;

(ii). $K(\mathcal{X})$ (resp. $K^{vb}(\mathcal{X})$), whose objects are $\mathcal{X}$-relative simplicial complexes and whose morphisms are morphisms (resp. vertex-bijective morphisms) of $\mathcal{X}$-relative simplicial complexes;

(iii). $L(\mathcal{X})$ (resp. $L^{vb}(\mathcal{X})$), whose objects are $\mathcal{X}$-relative independence hypergraphs and whose morphisms are morphisms (resp. vertex-bijective morphisms) of $\mathcal{X}$-relative independence hypergraphs.

Let $\mathcal{Y}$ be a fixed simplicial complex on $V$. By (ii), $K(\mathcal{Y})$ (resp. $K^{vb}(\mathcal{Y})$) is the category whose objects are simplicial sub-complexes of $\mathcal{Y}$ and whose morphisms are (resp. vertex-bijective) simplicial maps.

Let $\mathcal{Z}$ be a fixed independence hypergraph on $V$. By (iii), $L(\mathcal{Z})$ (resp. $L^{vb}(\mathcal{Z})$) is the category whose objects are independence sub-hypergraphs of $\mathcal{Z}$ and whose morphisms are morphisms (resp. vertex-bijective morphisms) of independence hypergraphs.

(b). Some functors among the categories. By Section 2(c), we have the following functors

$$
\begin{align*}
\Delta : & \quad H(\mathcal{X}) \to K(\Delta \mathcal{X}), \quad H^{vb}(\mathcal{X}) \to K^{vb}(\Delta \mathcal{X}), \\
\delta : & \quad H(\mathcal{X}) \to K(\delta \mathcal{X}), \quad H^{vb}(\mathcal{X}) \to K^{vb}(\delta \mathcal{X}), \\
\bar{\delta} : & \quad H^{vb}(\mathcal{X}) \to L^{vb}(\bar{\delta} \mathcal{X}).
\end{align*}
$$

The following diagrams commute

(c). The complement hypergraphs. For any sub-hypergraph $\mathcal{H}$ of $\mathcal{X}$, let $\gamma_\mathcal{X} \mathcal{H} = \{ \sigma \in \mathcal{X} \mid \sigma \notin \mathcal{H} \}$. We have a map

$$
\gamma_\mathcal{X} : \quad \text{Obj}(H(X)) \to \text{Obj}(H(X)).
$$
The following diagram commutes

\[
\begin{array}{ccc}
\text{Obj}(K(X)) & \xrightarrow{\gamma_X} & \text{Obj}(L(X)) \\
\downarrow & & \uparrow \\
\text{Obj}(H(X)) & \xrightarrow{\gamma_X} & \text{Obj}(H(X)).
\end{array}
\]

(d). The special case \( \mathcal{X} = \Delta[V] \). Let \( \mathcal{X} \) be \( \Delta[V] \) throughout the above of this subsection. We denote the categories \( H(\Delta[V]) \) (resp. \( K(\Delta[V]) \) and \( L(\Delta[V]) \)) as \( H(V) \) (resp. \( K(V) \) and \( L(V) \)) and denote \( H^{vb}(\Delta[V]) \) (resp. \( K^{vb}(\Delta[V]) \) and \( L^{vb}(\Delta[V]) \)) as \( H^{vb}(V) \) (resp. \( K^{vb}(V) \) and \( L^{vb}(V) \)). We have the following functors

\[
\begin{align*}
\Delta, \delta : & \quad H(V) \longrightarrow K(V), \quad H^{vb}(V) \longrightarrow K^{vb}(V), \\
\bar{\Delta} : & \quad H(V) \longrightarrow L(V), \quad H^{vb}(V) \longrightarrow L^{vb}(V), \\
\bar{\delta} : & \quad H^{vb}(V) \longrightarrow L^{vb}(V).
\end{align*}
\]

The following diagrams commute

For any hypergraph \( \mathcal{H} \) on \( V \), let \( \gamma \mathcal{H} = \{ \sigma \in \Delta[V] \mid \sigma \notin \mathcal{H} \} \). We have a map

\[
\gamma : \quad \text{Obj}(H(V)) \longrightarrow \text{Obj}(H(V)).
\]

The following diagrams commute

\[
\begin{array}{ccc}
\text{Obj}(K(V)) & \xrightarrow{\gamma} & \text{Obj}(L(V)) \\
\downarrow & & \uparrow \\
\text{Obj}(H(V)) & \xrightarrow{\gamma} & \text{Obj}(H(V)).
\end{array}
\]
3.2 The map algebra on hypergraphs

(a). The map algebra of sub-hypergraphs of a simplicial complex. Consider
the sub-hypergraphs of a fixed simplicial complex $Y$ on $V$. The map algebra on these
sub-hypergraphs is initially studied in [37, Subsection 2.1]. The following relations among
$\Delta$, $\delta$ and $\gamma_Y$ are satisfied:

1. as functors, $\Delta\delta = \delta$ and $\delta\Delta = \Delta$;
2. as functors, $\Delta^2 = \Delta$ and $\delta^2 = \delta$;
3. as self-maps on $\text{Obj}(H(Y))$, $(\gamma_Y)^2 = \text{id}$; 
4. as self-maps on $\text{Obj}(H(Y))$, $(\Delta\gamma_Y\Delta)\gamma_Y)^2 = \Delta\gamma_Y\Delta\gamma_Y$ and $(\delta\gamma_Y\delta\gamma_Y)^2 = \delta\gamma_Y\delta\gamma_Y$.

Let $G_Y$ be the semi-group of maps generated by $\Delta$, $\delta$, $\gamma_Y$ modulo the relations (1) -
(4). The multiplication of $G_Y$ is the composition of maps and the unit of $G_Y$ is id. For
any $w_1, w_2 \in G_Y$, define

$$w_1 + w_2, w_1 \wedge w_2 : \text{Obj}(H(Y)) \times \text{Obj}(H(Y)) \rightarrow \text{Obj}(H(Y))$$

by

(I). $(w_1 + w_2)(H_1, H_2) = w_1(H_1) \cup w_2(H_2)$,

(II). $(w_1 \wedge w_2)(H_1, H_2) = w_1(H_1) \cap w_2(H_2)$

for any sub-hypergraphs $H_1$ and $H_2$ of $Y$. We call the triple $(G_Y, +, \wedge)$ the map algebra
on $\text{Obj}(H(Y))$ (cf. [37 Subsection 2.1]).

Lemma 3.1. For any sub-hypergraph $H$ of $Y$, define $\bar{\Delta}_Y(H) = Y \cap \bar{\Delta}(H \cup \gamma_Y)$ and
$\bar{\delta}_Y(H) = Y \cap \bar{\delta}(H \cup \gamma_Y)$. Then $\delta(H) = \gamma_Y\bar{\Delta}_Y\gamma_Y(H)$ and $\bar{\delta}_Y(H) = \gamma_Y\Delta\gamma_Y(H)$.

Proof. Let $H$ be a sub-hypergraph of $Y$. We have

$$\gamma_Y\bar{\Delta}_Y\gamma_YH = \{ \sigma \in Y \mid \sigma \notin \bar{\Delta}_Y\gamma_YH \}$$
$$= \{ \sigma \in Y \mid \sigma \notin \bar{\Delta}((\gamma_YH \cup \gamma_Y)) \}$$
$$= \{ \sigma \in Y \mid \sigma \not\supset \tau \text{ for any } \tau \in (\gamma_YH \cup \gamma_Y) \}$$
$$= \{ \sigma \in Y \mid \sigma \not\supset \tau \text{ for any } \tau \in \gamma_YH \text{ and any } \tau \notin Y \}$$
$$= \{ \sigma \in Y \mid \sigma \not\supset \tau \text{ for any } \tau \not\in H, \text{ where } \tau \in \Delta[V] \}$$
$$= \{ \sigma \in Y \mid \tau \in H \text{ for any } \tau \subseteq \sigma, \text{ where } \tau \in \Delta[V] \}$$
$$= \delta H$$
\[ \gamma_{\mathcal{Y}} \Delta \gamma_{\mathcal{H}} = \{ \sigma \in \mathcal{Y} \mid \sigma \notin \Delta \gamma_{\mathcal{Y}} \mathcal{H} \} \]
\[ = \{ \sigma \in \mathcal{Y} \mid \sigma \not\subseteq \tau \text{ for any } \tau \in \gamma_{\mathcal{Y}} \mathcal{H} \} \]
\[ = \{ \sigma \in \mathcal{Y} \mid \sigma \not\subseteq \tau \text{ for any } \tau \notin \mathcal{H} \text{ and } \tau \in \mathcal{Y} \} \]
\[ = \{ \sigma \in \mathcal{Y} \mid \tau \in \mathcal{H} \text{ for any } \tau \supseteq \sigma \text{ and } \tau \in \mathcal{Y} \} \]
\[ = \{ \sigma \in \mathcal{Y} \mid \tau \in \mathcal{H} \cup \gamma_{\mathcal{Y}} \text{ for any } \tau \supseteq \sigma \} \]
\[ = \{ \sigma \in \mathcal{Y} \mid \tau \in \delta(\mathcal{H} \cup \gamma_{\mathcal{Y}}) \} \]
\[ = \delta_{\mathcal{Y}}(\mathcal{H}). \]

(b). The map algebra of sub-hypergraphs of an independence hypergraph.

Consider the sub-hypergraphs of a fixed independence hypergraph \( \mathcal{Z} \) on \( V \). Similar to (1) - (4), we have the following relations among \( \bar{\Delta}, \bar{\delta} \) and \( \gamma_{\mathcal{Z}} \):

(1)' as functors, \( \bar{\Delta} \bar{\delta} = \bar{\delta} \) and \( \bar{\delta} \bar{\Delta} = \bar{\Delta} \);

(2)' as functors, \( \bar{\Delta}^2 = \bar{\Delta} \) and \( \bar{\delta}^2 = \bar{\delta} \);

(3)' as self-maps on \( \text{Obj}(\mathcal{H}(\mathcal{Z})) \), \( (\gamma_{\mathcal{Z}})^2 = \text{id} \);

(4)' as self-maps on \( \text{Obj}(\mathcal{H}(\mathcal{Z})) \), \( (\bar{\Delta} \gamma_{\mathcal{Z}} \bar{\Delta} \gamma_{\mathcal{Z}})^2 = \bar{\Delta} \gamma_{\mathcal{Z}} \bar{\Delta} \gamma_{\mathcal{Z}} \) and \( (\bar{\delta} \gamma_{\mathcal{Z}} \bar{\delta} \gamma_{\mathcal{Z}})^2 = \bar{\delta} \gamma_{\mathcal{Z}} \bar{\delta} \gamma_{\mathcal{Z}} \).

The proofs of (1)' - (3)' are direct. We prove (4)'.

Proof of (4)'. Let \( \mathcal{H} \) be a sub-hypergraph of the independence hypergraph \( \mathcal{Z} \). Then

\[ \bar{\Delta} \gamma_{\mathcal{Z}} \bar{\Delta} \gamma_{\mathcal{Z}} \mathcal{H} = \bar{\Delta} \gamma_{\mathcal{Z}} \bar{\Delta} \{ \sigma \in \mathcal{Z} \mid \sigma \notin \mathcal{H} \} \]
\[ = \bar{\Delta} \gamma_{\mathcal{Z}} \{ \sigma \in \mathcal{Z} \mid \text{there exists } \tau \subseteq \sigma \text{ with } \tau \in \mathcal{Z}, \text{ such that } \tau \notin \mathcal{H} \} \]
\[ = \bar{\Delta} \{ \sigma \in \mathcal{Z} \mid \text{for any } \tau \subseteq \sigma \text{ with } \tau \in \mathcal{Z}, \text{ it holds } \tau \in \mathcal{H} \} \]
\[ = \bar{\Delta}(\min(\mathcal{Z}) \cap \mathcal{H}). \tag{3.1} \]

Thus

\[ (\bar{\Delta} \gamma_{\mathcal{Z}} \bar{\Delta} \gamma_{\mathcal{Z}})^2 \mathcal{H} = \bar{\Delta}(\min(\mathcal{Z}) \cap \bar{\Delta}(\min(\mathcal{Z}) \cap \mathcal{H})) \]
\[ = \bar{\Delta}(\min(\mathcal{Z}) \cap \mathcal{H}) \]
\[ = \bar{\Delta} \gamma_{\mathcal{Z}} \bar{\Delta} \gamma_{\mathcal{Z}} \mathcal{H}. \tag{3.2} \]
On the other hand,
\[
\bar{\delta} \gamma_2 \bar{\delta} \gamma_2 \mathcal{H} = \bar{\delta} \gamma_2 \bar{\delta} \{ \sigma \in \mathcal{Z} \mid \sigma \notin \mathcal{H} \} \\
= \bar{\delta} \gamma_2 \{ \sigma \in \mathcal{Z} \mid \text{for any } \tau \supseteq \sigma \text{ with } \tau \in \mathcal{Z}, \text{it holds } \tau \notin \mathcal{H} \} \\
= \bar{\delta} \{ \sigma \in \mathcal{Z} \mid \text{there exists } \tau \supseteq \sigma \text{ such that } \tau \in \mathcal{H} \} \\
= \bar{\delta} (\Delta \mathcal{H} \cap \mathcal{Z}) \\
= \begin{cases} \\
\bar{\delta} \mathcal{Z} & \text{if } \Delta \mathcal{H} = \Delta [V], \\
\emptyset & \text{otherwise.} \\
\end{cases} \tag{3.3}
\]

Thus
\[
(\bar{\delta} \gamma_2 \bar{\delta} \gamma_2)^2 \mathcal{H} = \begin{cases} \\
\bar{\delta} \bar{\delta} \mathcal{Z} = \bar{\delta} \mathcal{Z} & \text{if } \Delta \mathcal{H} = \Delta [V], \\
\emptyset & \text{otherwise,} \\
\end{cases}
\]

which implies
\[
(\bar{\delta} \gamma_2 \bar{\delta} \gamma_2)^2 \mathcal{H} = \bar{\delta} \gamma_2 \bar{\delta} \gamma_2 \mathcal{H}. \tag{3.4}
\]

Let \( \mathcal{G} \) be the semi-group of maps generated by \( \bar{\Delta}, \bar{\delta}, \gamma_2 \) modulo the relations (1)' - (4)'. The multiplication of \( \mathcal{G} \) is the composition of maps and the unit of \( \mathcal{G} \) is \( \text{id} \). For any \( w_1, w_2 \in \mathcal{G} \), define
\[
w_1 + w_2, w_1 \wedge w_2 : \text{Obj}(\mathcal{H}(\mathcal{Z})) \times \text{Obj}(\mathcal{H}(\mathcal{Z})) \longrightarrow \text{Obj}(\mathcal{H}(\mathcal{Z}))
\]
by (I) and (II) for any sub-hypergraphs \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) of \( \mathcal{Z} \).

**Lemma 3.2.** For any sub-hypergraph \( \mathcal{H} \) of \( \mathcal{Z} \), define \( \Delta_2(\mathcal{H}) = \mathcal{Z} \cap \Delta (\mathcal{H} \cup \gamma_2 \mathcal{Z}) \) and \( \delta_2(\mathcal{H}) = \mathcal{Z} \cap \delta (\mathcal{H} \cup \gamma_2 \mathcal{Z}) \). Then \( \bar{\delta}(\mathcal{H}) = \gamma_2 \Delta_2 \gamma_2(\mathcal{H}) \) and \( \delta_2(\mathcal{H}) = \gamma_2 \Delta \gamma_2(\mathcal{H}) \).

**Proof.** Let \( \mathcal{H} \) be a sub-hypergraph of \( \mathcal{Z} \). We have
\[
\gamma_2 \Delta_2 \gamma_2 \mathcal{H} = \{ \sigma \in \mathcal{Z} \mid \sigma \notin \Delta_2 \gamma_2 \mathcal{H} \} \\
= \{ \sigma \in \mathcal{Z} \mid \sigma \nsubseteq \tau \text{ for any } \tau \in (\gamma_2 \mathcal{H} \cup \gamma_2 \mathcal{Z}) \} \\
= \{ \sigma \in \mathcal{Z} \mid \sigma \nsubseteq \tau \text{ for any } \tau \in \gamma_2 \mathcal{H} \text{ and any } \tau \notin \mathcal{Z} \} \\
= \{ \sigma \in \mathcal{Z} \mid \sigma \nsubseteq \tau \text{ for any } \tau \notin \mathcal{H}, \text{ where } \tau \in \Delta [V] \} \\
= \{ \sigma \in \mathcal{Z} \mid \tau \in \mathcal{H} \text{ for any } \tau \supseteq \sigma, \text{ where } \tau \in \Delta [V] \} \\
= \bar{\delta} \mathcal{H}
\]
and
\[ \gamma_Z \Delta \gamma_Z \mathcal{H} = \{ \sigma \in \mathcal{Z} \mid \sigma \notin \Delta \gamma_Z \mathcal{H} \} \]
\[ = \{ \sigma \in \mathcal{Z} \mid \sigma \not\supset \tau \text{ for any } \tau \in \gamma_Z \mathcal{H} \} \]
\[ = \{ \sigma \in \mathcal{Z} \mid \sigma \not\supset \tau \text{ for any } \tau \notin \mathcal{H} \text{ and } \tau \in \mathcal{Z} \} \]
\[ = \{ \sigma \in \mathcal{Z} \mid \tau \in \mathcal{H} \cup \gamma Z \text{ for any } \tau \subseteq \sigma \text{ and } \tau \in \mathcal{Z} \} \]
\[ = \{ \sigma \in \mathcal{Z} \mid \tau \not\subseteq \sigma \text{ for any } \tau \in \mathcal{H} \text{ and } \tau \in \mathcal{Z} \} \]
\[ = \{ \sigma \in \mathcal{Z} \mid \tau \in \mathcal{H} \cup \gamma Z \text{ for any } \tau \subseteq \sigma, \text{ where } \tau \in \Delta[V] \} \]
\[ = \{ \sigma \in \mathcal{Z} \mid \sigma \in \delta(\mathcal{H} \cup \gamma Z) \} \]
\[ = \delta_Z \mathcal{H}. \]

\[ \Box \]

(c). The map algebra of hypergraphs on \( V \). Consider all the hypergraphs on \( V \), that is, all the sub-hypergraphs of \( \Delta[V] \). Consider the functors \( \Delta, \delta, \bar{\Delta}, \bar{\delta} \) and the map \( \gamma \). Let \( Y = \Delta[V] \) in (1) - (4), Subsection 3.2 (a). Let \( Z = \Delta[V] \) in (1)' - (4)', Subsection 3.2 (b). We obtain

(1)". as functors, \( \Delta \delta = \delta, \delta \Delta = \Delta, \bar{\Delta} \bar{\delta} = \bar{\delta} \) and \( \bar{\delta} \Delta = \bar{\Delta} \);

(2)". as functors, \( \Delta^2 = \Delta, \delta^2 = \delta, \bar{\Delta}^2 = \bar{\Delta} \) and \( \bar{\delta}^2 = \bar{\delta} \);

(3)". as self-maps on \( \text{Obj}(\mathcal{H}(V)) \), \( \gamma^2 = \text{id} \);

(4)". as self-maps on \( \text{Obj}(\mathcal{H}(V)) \), \( \Delta \gamma \Delta \gamma = \Delta \gamma \Delta \gamma, (\delta \gamma \delta \gamma)^2 = \delta \gamma \delta \gamma, (\bar{\Delta} \gamma \Delta \gamma)^2 = \bar{\Delta} \gamma \Delta \gamma, (\bar{\delta} \gamma \bar{\delta} \gamma)^2 = \bar{\delta} \gamma \bar{\delta} \gamma \).

Let \( Y \) be \( \Delta[V] \) in Lemma 3.1 or alternatively let \( Z \) be \( \Delta[V] \) in Lemma 3.2. We obtain

(5)". as self-maps on \( \text{Obj}(\mathcal{H}(V)) \), \( \gamma \Delta \gamma = \bar{\delta} \) and \( \gamma \delta \gamma = \bar{\Delta} \).

Consider the following functors:

(i). The \( \chi \) Functor. Let \( \chi \) be the functor from \( \mathcal{H}^v (V) \) to \( \mathcal{H}^v (V) \) such that for any hypergraph \( \mathcal{H} \) on \( V \),

\[ \chi(\mathcal{H}) = \begin{cases} \Delta[V] & \text{if } V \in \mathcal{H}, \\ \emptyset & \text{if } V \notin \mathcal{H} \end{cases} \]

and for any vertex-bijection morphism \( f : \mathcal{H} \rightarrow \mathcal{H}' \),

\[ \chi(f) = \begin{cases} \Delta f : \Delta[V] \rightarrow \Delta[V] & \text{if } V \in \mathcal{H} \cap \mathcal{H}', \\ * & \text{otherwise}. \end{cases} \]

Here we use * to denote the trivial map whose domain or target is the emptyset.
(ii). The 0-Functor. For any hypergraph $\mathcal{H}$ on $V$, let $\mathcal{H}_0$ be the sub-hypergraph consisting of all the 0-hyperedges of $\mathcal{H}$. For any morphism of hypergraphs $f : \mathcal{H} \rightarrow \mathcal{H}'$, there is an induced map $f_0 : \mathcal{H}_0 \rightarrow \mathcal{H}_0'$. Let $\mathbf{H}_0(V)$ be the category whose objects are hypergraphs on $V$ with only 0-hyperedges and whose morphisms are maps of sets. Consider the functor $\mathbf{0} : \mathbf{H}(V) \rightarrow \mathbf{H}_0(V)$ such that $\mathbf{0}(\mathcal{H}) = \mathcal{H}_0$ for any $\mathcal{H} \in \text{Obj}(\mathbf{H}(V))$ and $\mathbf{0}(f) = f_0$ for any $f \in \text{Mor}(\mathcal{H}, \mathcal{H}')$, where $\mathcal{H}, \mathcal{H}' \in \text{Obj}(\mathbf{H}(V))$.

(iii). The $2^0$-Functor. For any hypergraph $\mathcal{H}$ on $V$, let $\Delta[\mathcal{H}_0]$ be the hypergraph consisting of all the hyperedges $\{v_0, v_1, \ldots, v_k\}$, where $\{\{v_0\}, \{v_1\}, \ldots, \{v_k\}\}$ is any subset of $\mathcal{H}_0$. For any morphism of hypergraphs $f : \mathcal{H} \rightarrow \mathcal{H}'$, there is an induced morphism $2^{f_0} : \Delta[\mathcal{H}_0] \rightarrow \Delta[\mathcal{H}_0']$ induced by the map $f_0$. Consider the functor $2^0$ from $\mathbf{H}(V)$ to itself sending a hypergraph $\mathcal{H}$ to $\Delta[\mathcal{H}_0]$ and sending a morphism $f$ to $2^{f_0}$.

We have

$$\Delta \bar{\delta} = \chi, \ \delta \bar{\Delta} = 2^0, \ \bar{\Delta} \delta = \bar{\Delta}0 \text{ and } \bar{\delta} \Delta = \chi.$$

Proof of (6)". Let $\mathcal{H}$ be a hypergraph on $V$. By (5)" and [37, Subsection 2.1, proof of (vi)],

$$\Delta \bar{\delta} \mathcal{H} = \Delta \gamma \Delta \gamma \mathcal{H} \quad = \Delta(\max(\Delta[V]) \cap \mathcal{H}) \quad = \chi(\mathcal{H}).$$

By (5)" and [37, Subsection 2.1, proof of (vii)],

$$\delta \bar{\Delta} \mathcal{H} = \delta \gamma \delta \gamma \mathcal{H} \quad = \{\sigma \in \Delta[V] \mid \{v\} \in \mathcal{H} \text{ for any } v \in \sigma\} \quad = 2^0(\mathcal{H}).$$

By (5)" and (3.4),

$$\bar{\Delta} \delta \mathcal{H} = \bar{\Delta} \gamma \bar{\Delta} \gamma \mathcal{H} \quad = \bar{\Delta}(\min(\Delta[V]) \cap \mathcal{H}) \quad = \bar{\Delta}0(\mathcal{H}).$$

By (5)" and (3.3),

$$\bar{\delta} \Delta \mathcal{H} = \bar{\delta} \gamma \bar{\delta} \gamma \mathcal{H} \quad = \begin{cases} \bar{\delta} \Delta[V] & \text{if } \Delta \mathcal{H} = \Delta[V], \\ \emptyset & \text{otherwise} \end{cases} \quad = \chi(\mathcal{H}).$$
For any morphism $f : \mathcal{H} \rightarrow \mathcal{H}'$, where $\mathcal{H}$ and $\mathcal{H}'$ are hypergraphs on $V$, we have $\delta \Delta(f) = 2^0(f)$ and $\Delta \delta(f) = \Delta 0(f)$. In addition, if $f$ is vertex-bijective, then we have $\Delta \delta(f) = \tilde{\delta} \Delta(f) = \chi(f)$. We obtain (6)". \hfill \Box

Let $G$ be the semi-group of maps generated by $\Delta$, $\delta$, $\gamma$ modulo the relations (1)" - (6)". For any $w_1, w_2 \in G$, define

$$w_1 + w_2, w_1 \wedge w_2 : \text{Obj}(H(V)) \times \text{Obj}(H(V)) \rightarrow \text{Obj}(H(V))$$

by (I) and (II) for any hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ on $V$. We call the triple $(G, +, \wedge)$ the map algebra on $\text{Obj}(H(V))$. 

Proof of Theorem Let By (3)" and (5)" , $\Delta = \gamma \bar{\Delta} \gamma$ and $\delta = \gamma \bar{\Delta} \gamma$. Hence each of the triple $\{\gamma, \Delta, \delta\}, \{\gamma, \Delta, \bar{\delta}\}, \{\gamma, \Delta, \bar{\Delta}\}$ and $\{\gamma, \delta, \bar{\delta}\}$ could multiplicatively generate $\gamma$, $\Delta$, $\delta$, $\bar{\Delta}$ and $\bar{\delta}$. Therefore, each of the triple could multiplicatively generate $G$. \hfill \Box

3.3 The map algebra on simplicial complexes

The map algebra on simplicial complexes is initially studied in [37, Subsection 2.2]. Let $\mathcal{Y}$ be a simplicial complex on $V$. Consider the semi-group $G_{\mathcal{Y}}$ generated by $\Delta$, $\delta$, $\gamma_{\mathcal{Y}}$ modulo the relations (1) - (4) in Subsection 3.2(a). Let $w \in G_{\mathcal{Y}}$. We call a subset $S$ of $\text{Obj}(H(\mathcal{Y}))$ $w$-invariant if $w(\mathcal{H}) \in S$ for any $\mathcal{H} \in S$. The collection of all the $w \in G_{\mathcal{Y}}$ such that $\text{Obj}(K(\mathcal{Y}))$ is a $w$-invariant subset of $\text{Obj}(H(\mathcal{Y}))$ forms a sub-semi-group of $G_{\mathcal{Y}}$, denoted by $G_{K, \mathcal{Y}}$. Since both $\Delta$ and $\delta$ act on $\text{Obj}(K(\mathcal{Y}))$ identically, we take an equivalent relation identifying both $\Delta$ and $\delta$ as the unit element of $G_{K, \mathcal{Y}}$. We denote the quotient semi-group as $G'_{K, \mathcal{Y}}$. Precisely, $G'_{K, \mathcal{Y}}$ is the semi-group generated by $\alpha = \Delta \gamma_{\mathcal{Y}}$ and $\beta = \delta \gamma_{\mathcal{Y}}$ modulo the relations (i). $\alpha^4 = \alpha^2$, (ii). $\beta^4 = \beta^2$. An element in $G'_{K, \mathcal{Y}}$ is of the form $\alpha^{m_1} \beta^{n_1} \alpha^{m_2} \beta^{n_2} \cdots \alpha^{m_k} \beta^{n_k}$ where $m_1, n_k = 0, 1, 2, 3$ and $m_2, m_3, \ldots, m_k, n_1, n_2, \ldots, n_{k-1} = 1, 2, 3$. By (I) and (II) in Subsection 3.2(a), the operations $+$ and $\wedge$ on $G'$ induce operations $+$ and $\wedge$ on $G'_{K, \mathcal{Y}}$. We call the triple $(G'_{K, \mathcal{Y}}, +, \wedge)$ the map algebra on $\text{Obj}(K(\mathcal{Y}))$. 

Proposition 3.3. For any simplicial sub-complex $K$ of $\mathcal{Y}$,

$$\alpha(K) = \Delta(\mathcal{Y} \cap \gamma K) = \mathcal{Y} \cap \gamma \bar{\Delta}(K \cup \gamma \mathcal{Y}),$$
$$\beta(K) = \delta(\mathcal{Y} \cap \gamma K) = \mathcal{Y} \cap \gamma \bar{\Delta}(K \cup \gamma \mathcal{Y}).$$

In particular, let $\mathcal{Y} = \Delta[V]$. Then for any simplicial complex $K$ on $V$,

$$\alpha(K) = \begin{cases} \Delta[V] & \text{if } K \neq \Delta[V], \\ \emptyset & \text{if } K = \Delta[V], \end{cases}$$
$$\beta(K) = \gamma \text{St}(K),$$

where $\text{St}(K) = \cup_{\sigma \in K} \text{St}(\sigma)$, $\text{St}(\sigma) = \{\tau \subseteq V \mid \tau \cap \sigma \neq \emptyset\}$, is the open star of $K$ in $\Delta[V]$. 

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Proof. Let $\mathcal{K}$ be a simplicial sub-complex of $\mathcal{Y}$. With the help of Lemma 3.1 we have

$$
\alpha(\mathcal{K}) = \Delta_{\mathcal{Y}}(\mathcal{K}) = \Delta(\mathcal{Y} \cap \gamma \mathcal{K}) = \gamma_{\mathcal{Y}}(\mathcal{Y} \cap \gamma(\mathcal{K} \cup \gamma \mathcal{Y})) = \mathcal{Y} \cap \gamma_{\mathcal{Y}}(\mathcal{K} \cup \gamma \mathcal{Y}),
$$

$$
\beta(\mathcal{K}) = \delta_{\mathcal{Y}}(\mathcal{K}) = \delta(\mathcal{Y} \cap \gamma \mathcal{K}) = \gamma_{\mathcal{Y}}(\mathcal{Y} \cap \Delta(\mathcal{K} \cup \gamma \mathcal{Y})) = \mathcal{Y} \cap \gamma_{\mathcal{Y}}(\mathcal{K} \cup \gamma \mathcal{Y}).
$$

Let $\mathcal{Y} = \Delta[\mathcal{V}]$. Then $\alpha \mathcal{K} = \Delta \gamma \mathcal{K}$. Since $\mathcal{K}$ is a simplicial complex, $\gamma \mathcal{K}$ is an independence hypergraph. Thus $\mathcal{V} \in \gamma \mathcal{K}$ if $\mathcal{K} \neq \Delta[\mathcal{V}]$ and $\mathcal{V} \notin \gamma \mathcal{K}$ otherwise. Therefore, $\alpha \mathcal{K} = \Delta[\mathcal{V}]$ if $\mathcal{K} \neq \Delta[\mathcal{V}]$ and $\alpha \mathcal{K} = \emptyset$ otherwise. Moreover, $\beta \mathcal{K} = \gamma \Delta \mathcal{K}$. Since $\mathcal{K}$ is a simplicial complex, $\Delta \mathcal{K} = \Delta(\mathcal{K}_0) = \text{St}(\mathcal{K}_0) = \text{St}(\mathcal{K})$. Therefore, $\beta(\mathcal{K}) = \gamma \text{St}(\mathcal{K})$. \hfill \qed

Remark 1: The open star of $\mathcal{K}$ in $\Delta[\mathcal{V}]$ satisfies $(\text{St}(\mathcal{K}))_0 = \mathcal{K}_0$ and $(\gamma \text{St}(\mathcal{K}))_0 = \mathcal{V} \setminus \mathcal{K}_0$.

In particular, if $\mathcal{Y} = \Delta[\mathcal{V}]$, then we denote $\mathcal{G}'_{\mathcal{K}, \mathcal{Y}}$ as $\mathcal{G}'_{\mathcal{K}}$. The map algebra on $\text{Obj}(\mathcal{K}(\mathcal{V}))$ is the triple $(\mathcal{G}'_{\mathcal{K}}, +, \wedge)$.

### 3.4 The map algebra on independence hypergraphs

Let $\mathcal{Z}$ be an independence hypergraph on $\mathcal{V}$. Consider the semi-group $\mathcal{G}^\mathcal{Z}$ generated by $\bar{\Delta}$, $\bar{\delta}$, $\gamma_{\mathcal{Z}}$ modulo the relations (1)’ - (4)’ in Subsection 3.2(b). Similar with Subsection 3.3, the collection of all the $w \in \mathcal{G}^\mathcal{Z}$ such that $\text{Obj}(\mathcal{L}(\mathcal{Z}))$ is a $w$-invariant subset of $\text{Obj}(\mathcal{H}(\mathcal{Z}))$ forms a sub-semi-group of $\mathcal{G}$, denoted by $\mathcal{G}^\mathcal{Z}_\mathcal{L}$. Since both $\bar{\Delta}$ and $\bar{\delta}$ act on $\text{Obj}(\mathcal{L}(\mathcal{Z}))$ identically, we take an equivalent relation identifying both $\bar{\Delta}$ and $\bar{\delta}$ as the unit element of $\mathcal{G}^\mathcal{Z}_\mathcal{L}$. We denote the quotient semi-group as $\mathcal{G}^\mathcal{Z}_\mathcal{L}$. Precisely, $\mathcal{G}^\mathcal{Z}_\mathcal{L}$ is the semi-group generated by $\bar{\alpha} = \bar{\Delta} \gamma_{\mathcal{Z}}$ and $\bar{\beta} = \bar{\delta} \gamma_{\mathcal{Z}}$ modulo the relations (i)’. $\bar{\alpha}^4 = \bar{\alpha}^2$, (ii)’. $\bar{\beta}^4 = \bar{\beta}^2$.

An element in $\mathcal{G}^\mathcal{Z}_\mathcal{L}$ is of the form $\bar{\alpha}^{m_1} \bar{\beta}^{m_2} \gamma \bar{\alpha}^{m_3} \bar{\beta}^{m_4} \ldots \bar{\alpha}^{m_k} \bar{\beta}^{m_s}$ where $m_1, n_k = 0, 1, 2, 3$ and $m_2, m_3, \ldots, m_k, n_1, n_2, \ldots, n_{k-1} = 1, 2, 3$. The operations $+$ and $\wedge$ on $\mathcal{G}$ induce operations $+$ and $\wedge$ on $\mathcal{G}^\mathcal{Z}_\mathcal{L}$. We call the triple $(\mathcal{G}^\mathcal{Z}_\mathcal{L}, +, \wedge)$ the map algebra on $\text{Obj}(\mathcal{L}(\mathcal{Z}))$.

**Proposition 3.4.** For any independence sub-hypergraph $\mathcal{L}$ of $\mathcal{Z}$,

$$
\bar{\alpha}(\mathcal{L}) = \bar{\Delta}(\mathcal{Z} \cap \gamma \mathcal{L}) = \mathcal{Z} \cap \bar{\delta}(\mathcal{L} \cup \gamma \mathcal{Z}),
$$

$$
\bar{\beta}(\mathcal{L}) = \bar{\delta}(\mathcal{Z} \cap \gamma \mathcal{L}) = \mathcal{Z} \cap \gamma \bar{\Delta}(\mathcal{L} \cup \gamma \mathcal{Z}).
$$

In particular, let $\mathcal{Z} = \Delta[\mathcal{V}]$. Then for any independence hypergraph $\mathcal{L}$ on $\mathcal{V}$,

$$
\bar{\alpha}(\mathcal{L}) = \text{St}(\gamma \mathcal{L}),
$$

$$
\bar{\beta}(\mathcal{L}) = \begin{cases} 
\Delta[\mathcal{V}] & \text{if } \mathcal{L} = \emptyset, \\
\emptyset & \text{if } \mathcal{L} \neq \emptyset,
\end{cases}
$$

where $\text{St}(\gamma \mathcal{L}) = \bigcup_{\sigma \in \gamma \mathcal{L}} \text{St}(\sigma)$ is the open star of $\gamma \mathcal{L}$ in $\Delta[\mathcal{V}]$. 

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\textbf{Proof.} Let \( L \) be an independence sub-hypergraph of \( Z \). With the help of Lemma 3.2 we have
\[
\bar{\alpha}(L) = \bar{\Delta} \gamma(Z) = \bar{\Delta} (Z \cap \gamma L) \\
= \gamma Z \delta(Z) = \gamma(Z \cap \delta(L \cup \gamma Z)) = Z \cap \gamma \delta(L \cup \gamma Z),
\]
\[
\bar{\beta}(L) = \bar{\delta} \gamma(Z) = \bar{\delta}(Z \cap \gamma L) \\
= \gamma Z \Delta(Z) = \gamma(Z \cap \Delta(L \cup \gamma Z)) = Z \cap \gamma \Delta(L \cup \gamma Z).
\]

Let \( Z = \Delta[V] \). Then \( \bar{\alpha} L = \bar{\Delta} \gamma L \). Since \( L \) is an independence hypergraph, \( \gamma L \) is a simplicial complex. By the proof of Proposition 3.3, \( \bar{\Delta} \gamma L = \text{St}(\gamma L) \). Therefore, \( \bar{\alpha} L = \text{St}(\gamma L) \). Moreover, \( \bar{\beta} L = \gamma \Delta L \). Note that if \( L \neq \emptyset \), then \( V \subseteq L \) and consequently \( \Delta L = \Delta[V] \). Therefore, \( \bar{\beta} L = \emptyset \) if \( L \neq \emptyset \) and \( \bar{\beta} L = \Delta[V] \) otherwise.

In particular, if \( Z = \Delta[V] \), then we denote \( G'_L Z \) as \( G'_L \). The map algebra on \( \text{Obj}(L(V)) \) is the triple \((G'_L, +, \wedge)\).

\section{Random hypergraphs and random simplicial complexes}

Let \( \mathcal{X} \) be a fixed hypergraph on \( V \). Let \( \mathcal{Y} \) be a fixed simplicial complex on \( V \). Let \( Z \) be a fixed independence hypergraph on \( V \).

\subsection{Random hypergraphs and actions of the map algebra}

\textbf{(a). Random hypergraphs.} A \textit{random sub-hypergraph} of \( \mathcal{X} \) is a probability function on \( \text{Obj}(\mathcal{H}(\mathcal{X})) \). Let \( D(\mathcal{H}(\mathcal{X})) \) be the functional space of all the probability functions on \( \text{Obj}(\mathcal{H}(\mathcal{X})) \). A self-map \( F \) on \( \text{Obj}(\mathcal{H}(\mathcal{X})) \) induces a self-map \( DF \) on \( D(\mathcal{H}(\mathcal{X})) \) by
\[
(DF)(\varphi)(\mathcal{H}) = \sum_{\mathcal{H}' \in \text{Obj}(\mathcal{H}(\mathcal{X}))} \varphi(\mathcal{H}'),
\]
for any \( \varphi \in D(\mathcal{H}(\mathcal{X})) \) and any \( \mathcal{H} \in \mathcal{H}(\mathcal{X}) \). Any two self-maps \( F_1 \) and \( F_2 \) on \( \text{Obj}(\mathcal{H}(\mathcal{X})) \) induce three self-maps \( D(F_1 F_2), D(F_1 + F_2) \) and \( D(F_1 \wedge F_2) \) on \( D(\mathcal{H}(\mathcal{X})) \) by
\[
D(F_1 F_2)(\varphi)(\mathcal{H}) = D(F_1)D(F_2)(\varphi)(\mathcal{H}),
\]
\[
(D(F_1 + F_2))(\varphi)(\mathcal{H}) = \sum_{\mathcal{H}_1, \mathcal{H}_2 \in \text{Obj}(\mathcal{H}(\mathcal{X}))} \varphi(\mathcal{H}_1)\varphi(\mathcal{H}_2),
\]
\[
(D(F_1 \wedge F_2))(\varphi)(\mathcal{H}) = \sum_{\mathcal{H}_1, \mathcal{H}_2 \in \text{Obj}(\mathcal{H}(\mathcal{X}))} \varphi(\mathcal{H}_1)\varphi(\mathcal{H}_2).
\]

Consider the particular cases:
• $\mathcal{X} = \mathcal{Y}$.

Then we have induced maps $D\Delta$, $D\delta$ and $D\gamma$. The map algebra $(G_{\mathcal{Y}}, +, \wedge)$ acts on $D(\mathbf{H}(\mathcal{Y}))$ by letting $\mathcal{X} = \mathcal{Y}$ in (4.1) - (4.4) with the multiplicative generators $D\Delta$, $D\delta$ and $D\gamma$.

• $\mathcal{X} = Z$.

Then we have induced maps $D\bar{\Delta}$, $D\bar{\delta}$ and $D\gamma$. The map algebra $(G_{\mathcal{Z}}, +, \wedge)$ acts on $D(\mathbf{H}(\mathcal{Z}))$ by letting $\mathcal{X} = \mathcal{Z}$ in (4.1) - (4.4) with the multiplicative generators $D\bar{\Delta}$, $D\bar{\delta}$ and $D\gamma$.

• $\mathcal{X} = \Delta[V]$.

Then we have induced maps $D\Delta$, $D\delta$, $D\bar{\Delta}$, $D\bar{\delta}$ and $D\gamma$. The map algebra $(G, +, \wedge)$ acts on $D(\mathbf{H}(V))$ by letting $\mathcal{X} = \Delta[V]$ in (4.1) - (4.4).

By Theorem [11] we have the next corollary.

**Corollary 4.1.** Each of the triple

\[ \{D\gamma, D\Delta, D\delta\}, \quad \{D\gamma, D\bar{\Delta}, D\bar{\delta}\}, \quad \{D\gamma, D\Delta, D\Delta\}, \quad \{D\gamma, D\delta, D\bar{\delta}\} \]

multiplicatively generates the induced action of $(G, +, \wedge)$ on $D(\mathbf{H}(V))$.

(b). Random simplicial complexes. A random simplicial sub-complex of $\mathcal{Y}$ is a probability function on Obj$(\mathbf{K}(\mathcal{Y}))$. Let $D(\mathbf{K}(\mathcal{Y}))$ be the functional space of all the probability functions on Obj$(\mathbf{K}(\mathcal{Y}))$. A self-map $F$ on Obj$(\mathbf{K}(\mathcal{Y}))$ induces a self-map $DF$ on $D(\mathbf{K}(\mathcal{Y}))$ by

\[ (DF)(\varphi)(\mathcal{K}) = \sum_{\mathcal{K}' \in \text{Obj}(\mathbf{K}(\mathcal{Y})) \atop F(\mathcal{K}') = \mathcal{K}} \varphi(\mathcal{K}') \tag{4.5} \]

for any $\varphi \in D(\mathbf{K}(\mathcal{Y}))$ and any $\mathcal{K} \in \text{Obj}(\mathbf{K}(\mathcal{Y}))$. Any two self-maps $F_1$ and $F_2$ on $\mathbf{K}^{vb}(\mathcal{Y})$ induce three self-maps $D(F_1F_2)$, $D(F_1 + F_2)$ and $D(F_1 \wedge F_2)$ on $D(\mathbf{K}(\mathcal{Y}))$ by

\[ D(F_1F_2)(\varphi)(\mathcal{K}) = D(F_1)D(F_2)(\varphi)(\mathcal{K}), \tag{4.6} \]
\[ (D(F_1 + F_2))(\varphi)(\mathcal{K}) = \sum_{\mathcal{K}_1, \mathcal{K}_2 \in \text{Obj}(\mathbf{K}(\mathcal{Y})) \atop F_1(\mathcal{K}_1) = F_2(\mathcal{K}_2) = \mathcal{K}} \varphi(\mathcal{K}_1)\varphi(\mathcal{K}_2), \tag{4.7} \]
\[ (D(F_1 \wedge F_2))(\varphi)(\mathcal{K}) = \sum_{\mathcal{K}_1, \mathcal{K}_2 \in \text{Obj}(\mathbf{K}(\mathcal{Y})) \atop F_1(\mathcal{K}_1) \cap F_2(\mathcal{K}_2) = \mathcal{K}} \varphi(\mathcal{K}_1)\varphi(\mathcal{K}_2). \tag{4.8} \]

The map algebra $(G'_{\mathbf{K}}, +, \wedge)$ acts on $D(\mathbf{K}(\mathcal{Y}))$ by (4.5) - (4.8) with the multiplicative generators $D\alpha$ and $D\beta$.

(c). Random independence hypergraphs. A random independence sub-hypergraph of $\mathcal{Z}$ is a probability function on Obj$(\mathbf{L}(\mathcal{Z}))$. Let $D(\mathbf{L}(\mathcal{Z}))$ be the functional space of all
the probability functions on \( \text{Obj}(L(Z)) \). A self-map \( F \) on \( \text{Obj}(L(Z)) \) induces a self-map \( DF \) on \( D(L(Z)) \) by

\[
(DF)(\varphi)(L) = \sum_{L' \in \text{Obj}(L(Z)), F(L') = L} \varphi(L'),
\]

(4.9)

for any \( \varphi \in D(L(Z)) \) and any \( L \in L(Z) \). Any two self-maps \( F_1 \) and \( F_2 \) on \( \text{Obj}(L(Z)) \) induce three self-maps \( D(F_1 F_2), D(F_1 + F_2) \) and \( D(F_1 \wedge F_2) \) on \( D(L(Z)) \) by

\[
D(F_1 F_2)(\varphi)(L) = D(F_1)D(F_2)(\varphi)(L),
\]

(4.10)

\[
(D(F_1 + F_2))(\varphi)(L) = \sum_{L_1, L_2 \in \text{Obj}(L(Z))} \varphi(L_1)\varphi(L_2),
\]

(4.11)

\[
(D(F_1 \wedge F_2))(\varphi)(L) = \sum_{L_1, L_2 \in \text{Obj}(L(Z))} \varphi(L_1)\varphi(L_2).
\]

(4.12)

The map algebra \( (G^L_{Z}, +, \wedge) \) acts on \( D(L_{vb}(Z)) \) by \((4.5) - (4.8)\) with the multiplicative generators \( D\bar{\alpha} \) and \( D\bar{\beta} \).

(d). Some commutative diagrams. With the help of Subsection 3.1 (b), we have the following commutative diagrams

\[
\begin{align*}
D(K(X)) & \quad D(K(\delta X)) \xleftarrow{D\delta} D(H(X)) \xrightarrow{D\Delta} D(K(\Delta X)), \\
D(L(X)) & \quad D(L(\delta X)) \xleftarrow{D\delta} D(H(X)) \xrightarrow{D\Delta} D(L(\Delta X)), \\
D(K(X)) & \quad D((L(X)) \xrightarrow{D\gamma_X} D((H(X)) \xrightarrow{D\gamma_X} D((H(X)).
\end{align*}
\]

Let \( X = \Delta[V] \). We have the following commutative diagrams

\[
\begin{align*}
D(K(V)) & \quad D(K(V)) \xleftarrow{D\delta} D(H(V)) \xrightarrow{D\Delta} D(K(V)),
\end{align*}
\]
4.2 The map algebra on the Erdős-Rényi-type model of random hypergraphs

Consider a function \( p : \Delta[V] \rightarrow [0,1] \) assigning a number \( 0 \leq p(\sigma) \leq 1 \) for each \( \sigma \in \Delta[V] \).

Consider the probability function \( \bar{P}_p \in D(H(\mathcal{X})) \) given by (cf. [37, Definition 1.2])

\[
\bar{P}_p(\mathcal{H}) = \prod_{\sigma \in \mathcal{H}} p(\sigma) \prod_{\tau \in \mathcal{X} \setminus \mathcal{H}} (1 - p(\sigma)), \quad \mathcal{H} \in \text{Obj}(H(\mathcal{X})).
\]

The probability function \( \bar{P}_p \) is of Erdős-Rényi-type and can be generated as follows:

Algorithm (H). We choose each hyperedge \( \sigma \) of \( \mathcal{X} \) independently at random with probability \( p(\sigma) \). By taking all these independent trials, the final randomly generated sub-hypergraph of \( \mathcal{X} \) has the probability distribution \( \bar{P}_p \).

Let \( \mathcal{H} \) be a random sub-hypergraph of \( \mathcal{X} \) with the probability function \( \bar{P}_p \). By [37, Theorem 1.5 (1)],

\[
(D(\Delta)(\bar{P}_p))(\mathcal{H}) = \bar{P}_{1-p}(\mathcal{H})
\]

for any sub-hypergraph \( \mathcal{H} \) of \( \mathcal{X} \).

Lemma 4.2. [37, Theorem 1.5 (2)] Consider the random sub-hypergraph of a fixed simplicial complex \( \mathcal{Y} \) with the probability function \( \bar{P}_p \). Then

\[
((D_{\gamma,\mathcal{X}})(\bar{P}_p))(\mathcal{K}) = \prod_{\tau \in \max(\mathcal{K})} p(\tau) \prod_{\tau \in \mathcal{Y} \setminus \mathcal{K}} (1 - p(\tau))
\]

for any simplicial sub-complex \( \mathcal{K} \) of \( \mathcal{Y} \).

Lemma 4.3. Consider the random sub-hypergraph of a fixed independence hypergraph \( \mathcal{Z} \) with the probability function \( \bar{P}_p \). Then

\[
((D_{\bar{\Delta}})(\bar{P}_p))(\mathcal{L}) = \prod_{\tau \in \min(\mathcal{L})} p(\tau) \prod_{\tau \in \mathcal{Z} \setminus \mathcal{L}} (1 - p(\tau))
\]

for any independence sub-hypergraph \( \mathcal{L} \) of \( \mathcal{Z} \).
Proof. Let $L$ be an independence sub-hypergraph of $Z$. Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be any set (allowed to be the emptyset) of distinct hyperedges in $L$ such that for each $\sigma_i$, $i = 1, 2, \ldots, s$, there exists $\tau \in \min(L)$ such that $\sigma_i \supseteq \tau$. Suppose $S$ runs over all such sets of hyperedges in $L$. Then $H = \min(L) \cup S$ runs over all the sub-hypergraphs of $Z$ such that $\bar{\Delta}H = L$. Consequently, with the help of (4.1),

$$
((D\Delta)(\bar{P}_p))(L) = \sum_{\Delta H = L} \bar{P}_p(H)
= \sum_{\Delta H = L} \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1 - p(\sigma))
= \sum_{H = \min(L) \cup S} \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1 - p(\sigma))
= \prod_{\sigma \in \min(L)} p(\sigma) \prod_{\sigma \notin L} (1 - p(\sigma)) \left( \sum_{S \subseteq L \setminus \min(L)} \prod_{\sigma \in S} p(\sigma) \prod_{\sigma \in L \setminus \min(L)} (1 - p(\sigma)) \right)
= \prod_{\tau \in \min(L)} p(\tau) \prod_{\tau \notin L} (1 - p(\tau)).
$$

We obtain (4.15).

Let both $\mathcal{Y}$ in Lemma 4.2 and $Z$ in Lemma 4.3 be $\Delta[V]$. The next theorem follows.

**Theorem 4.4.** The action of $G$ on $\bar{P}_p$ is given by

(i). $((D\gamma)(\bar{P}_p))(H) = P_{1-p}(H)$,

(ii). $((D\Delta)(\bar{P}_p))(\mathcal{K}) = \prod_{\tau \in \max(\mathcal{K})} p(\tau) \prod_{\tau \notin \mathcal{K}} (1 - p(\tau))$,

(iii). $((D\Delta)(\bar{P}_p))(L) = \prod_{\tau \in \min(L)} p(\tau) \prod_{\tau \notin L} (1 - p(\tau))$,

(iv). $((D\delta)(\bar{P}_p))(\mathcal{K}) = \prod_{\tau \in \min(\gamma \mathcal{K})} (1 - p(\tau)) \prod_{\tau \in \mathcal{K}} p(\tau)$,

(v). $((D\delta)(\bar{P}_p))(L) = \prod_{\tau \in \min(\gamma L)} (1 - p(\tau)) \prod_{\tau \in L} p(\tau)$,

where $H \in \text{Obj}(H(V))$, $\mathcal{K} \in \text{Obj}(K(V))$ and $L \in \text{Obj}(L(V))$.

**Proof.** Let $\mathcal{X}$ be $\Delta[V]$ in (4.13). We obtain (i). Let $\mathcal{Y}$ be $\Delta[V]$ in (4.14). We obtain (ii). Let $Z$ be $\Delta[V]$ in (4.15). We obtain (iii). It follows from (5)" (4.12), (i), (iii) and (4.16).

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that
\[(Dδ)(\bar{P}_p))(K) = \sum_{\gamma \in \mathcal{K}} (D\Delta \circ D\gamma)(\bar{P}_p)(\mathcal{L}) \]
\[= \sum_{\gamma \in \mathcal{K}} \sum_{\Delta \mathcal{H} = \mathcal{L}} (D\gamma)(\bar{P}_p)(\mathcal{H}) \]
\[= \sum_{\gamma \in \mathcal{K}} \sum_{\Delta \mathcal{H} = \mathcal{L}} \sum_{\gamma \mathcal{H}' = \mathcal{H}} \bar{P}_p(\mathcal{H}') \]
\[= \sum_{\Delta \mathcal{H} = \gamma \mathcal{K}} \bar{P}_p(\gamma \mathcal{H}) \]
\[= \prod_{\tau \in \min(\gamma \mathcal{K})} (1 - p(\tau)) \prod_{\tau \in \mathcal{K}} p(\tau). \]

The last equality follows by a analogous calculation of \[4.1\]. We obtain (iv). Similarly,
\[(D\delta)(\bar{P}_p))(\mathcal{L}) = \sum_{\gamma \in \mathcal{L}} (D\Delta \circ D\gamma)(\bar{P}_p)(\mathcal{K}) \]
\[= \sum_{\gamma \in \mathcal{L}} \sum_{\Delta \mathcal{H} = \mathcal{K}} (D\gamma)(\bar{P}_p)(\mathcal{H}) \]
\[= \sum_{\gamma \in \mathcal{L}} \sum_{\Delta \mathcal{H} = \mathcal{K}} \sum_{\gamma \mathcal{H}' = \mathcal{H}} \bar{P}_p(\mathcal{H}') \]
\[= \sum_{\Delta \mathcal{H} = \gamma \mathcal{L}} \bar{P}_p(\gamma \mathcal{H}) \]
\[= \prod_{\tau \in \max(\gamma \mathcal{L})} (1 - p(\tau)) \prod_{\tau \in \mathcal{L}} p(\tau). \]

The last equality follows by a analogous calculation of \[3.7\] Lemma 4.2. We obtain (v).

4.3 The map algebra on the Erdös-Rényi-type model of random simplicial complexes

For any simplicial sub-complex \(\mathcal{K}\) of \(\mathcal{Y}\), an external face of \(\mathcal{K}\) in \(\mathcal{Y}\) is a hyperedge \(\sigma \in \mathcal{Y}\) such that \(\sigma \notin \mathcal{K}\) and \(\tau \in \mathcal{K}\) for any nonempty proper subset \(\tau \subsetneq \sigma\) (cf. \[15\] Definition 2.1]). Denote by \(E_{\mathcal{Y}}(\mathcal{K})\) the set of all the external faces of \(\mathcal{K}\) in \(\mathcal{Y}\). In particular, if \(\mathcal{Y} = \Delta[V]\), then we write \(E_{\mathcal{Y}}(\mathcal{K})\) as \(E(\mathcal{K})\). We have

Observation 1. for any simplicial complex \(\mathcal{K}\) on \(V\) it holds \(E(\mathcal{K}) = \min(\gamma \mathcal{K})\).

Consider the probability function \(P_p \in D(\mathcal{K}(\mathcal{Y}))\) given by (cf. \[3.7\] Definition 1.1])
\[P_p(\mathcal{K}) = \prod_{\sigma \in \mathcal{K}} p(\sigma) \prod_{\sigma \in E_{\mathcal{Y}}(\mathcal{K})} (1 - p(\sigma)), \quad \mathcal{K} \in \text{Obj}(\mathcal{K}(\mathcal{Y})). \]
The probability function \( P_p \) is of Erdös-Rényi-type and can be generated as follows:

**Algorithm (K).** We generate the 0-skeleton by choosing each 0-hyperedge \( \{v\} \) of \( \mathcal{Y} \) independently at random with probability \( p(\{v\}) \). By induction, suppose the \( k \)-skeleton is already randomly generated. We generate the \((k + 1)\)-skeleton by choosing each external \((k + 1)\)-face \( \sigma \) of the \( k \)-skeleton in \( \mathcal{Y} \) independently at random with probability \( p(\sigma) \). By taking all these independent trials, the final randomly generated simplicial complex on \( V \) has the probability distribution \( P_p \).

**Lemma 4.5.** Let \( U \) be a subset of \( V \). Then

\[
\sum_{(\Delta[U])_0 \subseteq K \subseteq \Delta[U]} P_p(K) = \prod_{v \in U} p(\{v\}) \prod_{v \in V \setminus U} (1 - p(\{v\})).
\]

**Proof.** Let \( K \) be the random generated simplicial complex by Algorithm (K). Then \( K \sim P_p \). Thus

\[
\sum_{(\Delta[U])_0 \subseteq K \subseteq \Delta[U]} P_p(K) = \sum_{(\Delta[U])_0 \subseteq K \subseteq \Delta[U]} \text{Prob}[\text{Algorithm (K) generates } K]
\]

\[
= \text{Prob}\bigg[ \bigcup_{(\Delta[U])_0 \subseteq K \subseteq \Delta[U]} \{\text{Algorithm (K) generates } K\} \bigg]
\]

\[
= \text{Prob}\bigg[ \text{each } v \in U \text{ is chosen and each } v \in V \setminus U \text{ is not chosen} \bigg]
\]

\[
= \prod_{v \in U} \text{Prob}[v \text{ is chosen}] \prod_{v \in V \setminus U} \text{Prob}[v \text{ is not chosen}]
\]

\[
= \prod_{v \in U} p(\{v\}) \prod_{v \in V \setminus U} (1 - p(\{v\})).
\]

\[\square\]

**Lemma 4.6.** Let \( K \) and \( K' \) be simplicial complexes on \( V \).

(i). If \( \gamma\text{St}(K') = K \), then \( 0\gamma\text{St}(K') \subseteq K' \subseteq 2^0\gamma\text{St}(K) \);

(ii). If \( 0\gamma\text{St}(K) \subseteq K' \subseteq 2^0\gamma\text{St}(K) \), then \( \gamma\text{St}(K') = \gamma\text{St}(V \setminus K_0) \).

**Proof.** (i). Suppose \( \gamma\text{St}(K') = K \). We prove (i) by two steps.

**Step 1.** \( 0\gamma\text{St}(K') \subseteq K' \).

Let \( \{v\} \in 0\gamma\text{St}(K) \). Then for any \( \sigma \in K \), we have \( \{v\} \cap \sigma = \emptyset \), i.e. \( v \notin \sigma \). To prove \( \text{Step 1} \), it suffices to prove that \( \{v\} \in K' \). Suppose to the contrary, \( \{v\} \notin K' \). Then \( v \notin \sigma' \) for any \( \sigma' \in K' \). Thus \( \{v\} \in \gamma\text{St}(K') \), i.e. \( \{v\} \in K \) (cf. Remark \[\square\]). We get a contradiction. Therefore, there exists \( \sigma' \in K' \) such that \( v \in \sigma' \). We have proved \( \text{Step 1} \).

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Lemma 4.7. Let $K$ be a simplicial complex on $V$. Then there exists $K'$ such that $\gamma \text{St}(K') = K$ iff $K = \gamma \text{St}(V \setminus K_0)$. In this case, all the simplicial complexes $K'$ on $V$ satisfying $\gamma \text{St}(K') = K$ are given by $0 \gamma \text{St}(K) \subseteq K' \subseteq 2^0 \gamma \text{St}(K)$.

**Proof.** (\(\implies\)): Suppose there exists $K'$ such that $\gamma \text{St}(K') = K$. Then by Lemma 4.6 (i), $0 \gamma \text{St}(K) \subseteq K' \subseteq 2^0 \gamma \text{St}(K)$. By Lemma 4.6 (ii), $\gamma \text{St}(K') = \gamma \text{St}(V \setminus K_0)$. Thus $K = \gamma \text{St}(V \setminus K_0)$.

(\(\impliedby\)): Suppose $K = \gamma \text{St}(V \setminus K_0)$. By Lemma 4.6 (ii), for any simplicial complex $K'$ on $V$ satisfying $0 \gamma \text{St}(K) \subseteq K' \subseteq 2^0 \gamma \text{St}(K)$, it holds $\gamma \text{St}(K') = \gamma \text{St}(V \setminus K_0)$. Thus there exists $K'$ such that $\gamma \text{St}(K') = K$.

Summarizing Lemma 4.6 (i) and (ii), we have that if $K = \gamma \text{St}(V \setminus K_0)$, then all the simplicial complexes $K'$ on $V$ satisfying $\gamma \text{St}(K') = K$ are given by $0 \gamma \text{St}(K) \subseteq K' \subseteq 2^0 \gamma \text{St}(K)$.

**Proof of Theorem 1.3.** Let $K$ be a simplicial complex on $V$. By Proposition 3.3 and (4.5),

$$
(D\alpha)(P_p)(K) = \sum_{aK'=K} P_p(K')
$$

$$
= \begin{cases} 
P_p(\Delta[V]) & \text{if } K = \emptyset, \\
P_p(\emptyset) & \text{if } K = \Delta[V], \\
0 & \text{if } K \neq \emptyset, \Delta[V]. 
\end{cases}
$$

$$
= \begin{cases} 
\prod_{\sigma \in \Delta[V]} p(\sigma) & \text{if } K = \emptyset, \\
\prod_{v \in V} (1 - p(\{v\})) & \text{if } K = \Delta[V], \\
0 & \text{if } K \neq \emptyset, \Delta[V]. 
\end{cases}
$$
By Proposition 3.3, (4.5), Lemma 4.5, and Lemma 4.7

\[(D\beta)(P_p)(\mathcal{K}) = \sum_{\beta\mathcal{K}' = \mathcal{K}} P_p(\mathcal{K}')\]

\[= \sum_{\gamma \text{St}(\mathcal{K}') = \mathcal{K}} P_p(\mathcal{K}')\]

\[= \begin{cases} 
\sum_{0 \gamma \text{St}(\mathcal{K}) \subseteq \mathcal{K}' \subseteq 2^0 \gamma \text{St}(\mathcal{K})} P_p(\mathcal{K}') & \text{if } \mathcal{K} = \gamma \text{St}(V \setminus \mathcal{K}_0), \\
0 & \text{otherwise.}
\end{cases}\]

\[= \begin{cases} 
\prod_{\{v\} \in \gamma \text{St}(\mathcal{K})} p(\{v\}) \prod_{\{v\} \notin \gamma \text{St}(\mathcal{K})} (1 - p(\{v\})) & \text{if } \mathcal{K} = \gamma \text{St}(V \setminus \mathcal{K}_0), \\
0 & \text{otherwise.}
\end{cases}\]

4.4 The Erdős-Rényi-type model of random independence hypergraphs

For any independence sub-hypergraph \(\mathcal{L}\) of \(\mathcal{Z}\), we define a co-external face of \(\mathcal{L}\) to be a hyperedge \(\sigma \in \mathcal{Z}\) such that \(\sigma \notin \mathcal{L}\) and \(\tau \in \mathcal{L}\) for any proper superset \(\tau \supseteq \sigma\), where \(\tau \in \Delta[V]\). Denote by \(\tilde{E}^\mathcal{Z}(\mathcal{L})\) the set of all the co-external faces of \(\mathcal{L}\). In particular, if \(\mathcal{Z} = \Delta[V]\), then we write \(\tilde{E}^\mathcal{Z}(\mathcal{L})\) as \(\tilde{E}(\mathcal{L})\). We have

Observation 2. for any independence hypergraph \(\mathcal{L}\) on \(V\) it holds \(\tilde{E}(\mathcal{L}) = \max(\gamma \mathcal{L})\).

Consider the probability function \(Q_p \in D(L(\mathcal{Z}))\) given by

\[Q_p(\mathcal{L}) = \prod_{\sigma \in \mathcal{L}} p(\sigma) \prod_{\sigma \in \tilde{E}^\mathcal{Z}(\mathcal{L})} (1 - p(\sigma)), \quad \mathcal{L} \in \text{Obj}(L(\mathcal{Z})).\]

Algorithm (L). We choose the (|\(V\)| - 1)-hyperedge \(\{V\}\) of \(\mathcal{Z}\) at random with probability \(p(\{V\})\). By induction, suppose all the \(i\)-hyperedges, \(i = k + 1, k + 2, \ldots, |\(V\)| - 1, are already randomly generated. We generate the \(k\)-hyperedges by choosing each co-external face \(\sigma\) of the collection of the \((k+1)\)-hyperedges independently at random with probability \(p(\sigma)\). By taking all these independent trials, the final randomly generated independence hypergraph on \(V\) has the probability distribution \(R_p\).

4.5 Proof of Theorem 1.2

Proof of Theorem 1.2. Substitute Observation 1 and Observation 2 as well as the expressions of \(P_p\) and \(Q_p\) into Theorem 4.4. Then we obtain Theorem 1.2. \(\square\)
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