On Spectral Radius of Biased Random Walks on Infinite Graphs

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Abstract

We consider a class of biased random walks on infinite graphs and present several general results on the spectral radius of biased random walk.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a locally finite, connected infinite graph, where \( V(G) \) is the set of its vertices and \( E(G) \) is the set of its edges. Fix a vertex \( o \) of \( G \) as the root. For any reversible Markov chain on \( G \), there is a stationary measure \( \pi(\cdot) \) such that for any two adjacent vertices \( x \) and \( y \), \( \pi(x)p(x, y) = \pi(y)p(y, x) \), where \( p(x, y) \) is the transition probability of the Markov chain. For the edge joining vertices \( x \) and \( y \), we assign a weight

\[
c(x, y) = \pi(x)p(x, y),
\]

and call by conductances the weights of the edges. We study the biased random walks on the rooted graph \((G, o)\) defined as follows:

For any vertex \( x \) of \( G \) let \( |x| \) denote the graph distance between \( x \) and \( o \). Let \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). For any \( n \in \mathbb{Z}_+ \):

\[
B_G(n) = \{ x \in V(G) : |x| \leq n \}, \quad \partial B_G(n) = \{ x \in V(G) : |x| = n \}.
\]

Let \( \lambda \in [0, \infty) \). If an edge \( e = \{x, y\} \) is at distance \( n \) from \( o \), i.e., \( \min(|x|, |y|) = n \), its conductance is defined as \( \lambda^{-n} \). Denote by \( \text{RW}_\lambda \) the nearest-neighbour random walk \( (X_n)_{n=0}^{\infty} \) among such conductances and call it the \( \lambda \)-biased random walk. In other words, \( \text{RW}_\lambda \) has the following transition probabilities: for \( v \sim u \) (i.e., if \( u \) and \( v \) are adjacent on \( G \)),

\[
p(v, u) := p^G_\lambda(v, u) = \begin{cases} 
\frac{\lambda}{d_v} & \text{if } v = o, \\
\frac{\lambda}{d_v + (\lambda - 1)d_u} & \text{if } u \in \partial B_G(|v| - 1) \text{ and } v \neq o, \\
\frac{1}{d_v + (\lambda - 1)d_u} & \text{otherwise}. 
\end{cases}
\]  

(1.1)

Here, \( d_v \) is the degree of vertex \( v \), and \( d_v^-, d_v^0 \) and \( d_v^+ \) are the numbers of edges connecting \( v \) to \( \partial B_G(|v| - 1) \), \( \partial B_G(|v|) \) and \( \partial B_G(|v| + 1) \) respectively. Note that

\[
d_v^+ + d_v^0 + d_v^- = d_v, \quad d_v^- \geq 1, \quad v \neq o, \quad d_o^- = d_v^0 = 0,
\]

and that \( \text{RW}_{\lambda=1} \) is the simple random walk (SRW) on \( G \).

By Rayleigh’s monotonicity principle (see [22], p. 35), there is a critical value \( \lambda_c(G) \in [0, \infty] \) such that \( \text{RW}_\lambda \) is transient for \( \lambda < \lambda_c(G) \) and is recurrent for \( \lambda > \lambda_c(G) \). Let \( M_n = \#(\partial B_G(n)) \) be the cardinality of \( \partial B_G(n) \) for any \( n \in \mathbb{Z}_+ \). Define the volume growth rate of \( G \) as

\[
gr(G) = \liminf_{n \to \infty} M_n^{1/n}.
\]

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When \( G \) is a tree, \( \lambda_c(G) \) is exactly the exponential of the Hausdorff dimension of the tree boundary, namely the branching number of the tree ([11], [17], [22]). When \( G \) is a transitive graph, \( \lambda_c(G) = \text{gr}(G) \) (see [19] and [22]). Let

\[
\text{gr}_+(G) = \liminf_{n \to \infty} \left( \sum_{x \in \partial B_G(n-1)} d_x^n \right)^{1/n}.
\]

Clearly \( \text{gr}_+(G) \geq \text{gr}(G) \). If \( G \) either is a tree or satisfies

\[
\limsup_{n \to \infty} \left( \max_{x \in [n]} d_x^n \right)^{1/n} = 1,
\]

then \( \text{gr}_+(G) = \text{gr}(G) \).

From the Nash-Williams criterion ([22] Section 2.5), it follows that for any \( G \) with \( \text{gr}_+(G) < \infty \), \( \text{RW}_\lambda \) is recurrent for \( \lambda > \text{gr}_+(G) \) and thus \( \lambda_c(G) \leq \text{gr}_+(G) \). If \( G \) is spherically symmetric then \( \lambda_c(G) = \text{gr}_+(G) \) ([22] Section 3.4, Exercise 3.11).

An original motivation for introducing \( \text{RW}_\lambda \) by Berretti and Sokal [7] was to design a Monte-Carlo algorithm for self-avoiding walks. See [15, 25, 23] for refinements of this idea. Since the 1980s biased random walks and biased diffusions in disordered media have attracted much attention in mathematical and physics communities due to their interesting phenomenology and similarities to concrete physical systems ([2, 9, 10, 12]). In the 1990s, Lyons ([17, 18, 19]), and Lyons, Pemantle and Peres ([20, 21]) made a fundamental advance in the study of \( \text{RW}_\lambda \)'s. \( \text{RW}_\lambda \) has also received attention recently, see [5, 1, 4, 13] and the references therein. For a survey on biased random walks on random graphs see Ben Arous and Fribergh [3].

This paper focuses on a specific properties of spectral radius of \( \text{RW}_\lambda \)'s on non-random infinite graphs. The uniform spanning forests of the network associated with \( \text{RW}_\lambda \) on the Euclidean lattices are studied in a companion paper [24].

Let us introduce some basic notation. Write

\[
p^{(n)}(x, y) := p^{(n)}_{\lambda}(x, y) = \mathbb{P}_x(X_n = y),
\]

where \( \mathbb{P}_x := \mathbb{P}^{G}_x \) is the law of \( \text{RW}_\lambda \) starting at \( x \). The Green function is given by

\[
G(x, y \mid z) := G_\lambda(x, y \mid z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \quad x, y \in V(G), \quad z \in \mathbb{C}, \quad |z| < R_G,
\]

where \( R_G = R_G(\lambda) = R_G(\lambda, x, y) \) is its convergence radius. Note that

\[
R_G = R_G(\lambda) = \limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, y)}
\]

is independent of \( x, y \) when \( \text{RW}_\lambda \) is irreducible, i.e., \( \lambda > 0 \). When \( \lambda = 0 \), \( R_G(0) = \infty \). Call

\[
\rho_\lambda = \rho(\lambda) = \frac{1}{R_G} = \limsup_{n \to \infty} p^{(n)}(x, x)^{1/n} = \limsup_{n \to \infty} p^{(n)}(o, o)^{1/n}
\]

the spectral radius of \( \text{RW}_\lambda \).

We are ready to state our first main result.

**Theorem 1.1.** Let \( G \) be a locally finite, connected infinite graph.

(i) The spectral radius \( \rho_\lambda \) is continuous in \( \lambda \in (0, \infty) \), and \( \rho(\lambda_c) = 1 \).

(ii) If \( \rho_\lambda \) is continuous at 0, then there are no adjacent vertices in \( \partial B_G(n) \) for any \( n \in \mathbb{N} \), and \( d_v - d_v' \geq 1 \) for any vertex \( v \).

Conversely, on any infinite graph \( G \), if for any \( n \in \mathbb{N} \) there are no adjacent vertices in \( \partial B_G(n) \), and if there exists \( \delta > 0 \) such that \( d_v - d_v' \geq \delta d_v \) for any vertex \( v \), then \( \rho_\lambda \) is continuous at 0.
Let $d \in \mathbb{N}$, $d \geq 2$, and $\mathcal{G}_d$ denotes the set of all $d$-regular infinite connected graphs.

**Theorem 1.2.** Let $G \in \mathcal{G}_d$, and $\lambda \in (0, \lambda_c(\mathbb{T}_d) = d - 1)$.

(i) We have

$$\rho_G(\lambda) \geq \rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}.$$ 

(ii) Assume $G$ is transitive. Then

$$\rho_G(\lambda) = \rho_{\mathbb{T}_d}(\lambda) \text{ if and only if } G \text{ is isomorphic to } \mathbb{T}_d.$$ 

In the case $\lambda = 1$, Theorem 1.2 follows from Kesten [14, Theorem 2] (see [28, p. 122 Corollary 11.7] and [22, Theorem 6.11]). For this case ($\lambda = 1$) our proof of Theorem 1.2 differs from the proofs in [14], [28] and [22].

The rest of the paper is organized as follows. We prove Theorem 1.1 and 1.2 in Section 2. In Section 3, we focus on the spectral radius and the speed for RW's on free product of graphs.

When emphasizing that a function $g(\cdot)$ depends on the underlying graph $G$, we will use $g_G(\cdot)$ or $g^G(\cdot)$ to replace $g(\cdot)$.

## 2 Proofs of Theorem 1.1 and Theorem 1.2

For any vertex set $A$, let

$$\tau_A = \inf\{n \geq 0 \mid X_n \in A\}, \quad \tau_A^+ = \inf\{n \geq 1 \mid X_n \in A\}.$$ 

When $A = \{y\}$, write $\tau_y = \tau_{\{y\}}$, $\tau_y^+ = \tau_{\{y\}}$. Put

$$f^{(n)}(x, y) := f^{(n)}_\lambda(x, y) = P_x(\tau_y^+ = n), \quad U(x, y \mid z) := U(x, y \mid z) = \sum_{n=1}^{\infty} f^{(n)}(x, y)z^n, \quad x, y \in V(G), \quad z \in \mathbb{C}, \; |z| < R_U, \quad (2.1)$$

where $R_U = R_U(\lambda) = R_U(\lambda, x, y)$ is the convergence radius of $U$, which is also independent of $x, y$ for $\lambda > 0$. When $\lambda = 0$, $R_U(0) = \infty$.

### 2.1 Proof of Theorem 1.1 part (i)

**Proof.** It suffices to verify that the convergence radius $R_G(\lambda)$ is continuous in $\lambda \in (0, \infty)$. This is done in tree steps.

**Step 1.** For any sequence $\{\lambda_k\}_{k \geq 1} \subset (0, \lambda_c(G)]$ converging to a limit $\lambda_0 \in (0, \lambda_c(G)]$, we claim that

$$\limsup_{k \to \infty} R_G(\lambda_k) \leq R_G(\lambda_0).$$

Assume

$$\limsup_{k \to \infty} R_G(\lambda_k) > R_G(\lambda_0) = z_\ast.$$ 

then there exists a subsequence $\{\lambda_{n_k}\}_{k \geq 1}$, such that $\alpha = \lim_{k \to \infty} R_G(\lambda_{n_k}) > z_\ast$. For any $z_0 \in (z_\ast, \alpha)$, when $k$ is sufficiently large,

$$U_{\lambda_{n_k}}(\alpha, \alpha \mid z_0) = \sum_{n=1}^{\infty} f^{(n)}_{\lambda_{n_k}}(\alpha, \alpha)z_0^n < 1,$$

because $G_{\lambda_{n_k}}(\alpha, \alpha \mid z_0) = \sum_{n=0}^{\infty} \{U_{\lambda_{n_k}}(\alpha, \alpha \mid z_0)\}^n < \infty$. By Fatou's lemma,

$$1 \geq \liminf_{k \to \infty} \sum_{n=1}^{\infty} f^{(n)}_{\lambda_{n_k}}(\alpha, \alpha)z_0^n \geq \sum_{n=1}^{\infty} \liminf_{k \to \infty} f^{(n)}_{\lambda_{n_k}}(\alpha, \alpha)z_0^n = \sum_{n=1}^{\infty} f^{(n)}_{\lambda_0}(\alpha, \alpha)z_0^n = U_{\lambda_0}(\alpha, \alpha \mid z_0). \quad (2.3)$$
We now distinguish two possible cases. First case: $R_G(\lambda_0) = R_U(\lambda_0)$. Since $z_0 > z_\ast = R_G(\lambda_0)$, we would have $U_{\lambda_0}(o, o | z_0) = \infty$, leading to a contradiction. Second case: $R_G(\lambda_0) < R_U(\lambda_0)$. For any $z > z_\ast$, $\infty = G_{\lambda_0}(o, o | z) = \sum_{n=0}^{\infty} U_{\lambda_0}(o, o | z)^n$, so $U_{\lambda_0}(o, o | z) \geq 1$. Since $U_{\lambda_0}(o, o | z)$ is strictly increasing in $z \in [0, R_U(\lambda_0))$, this would again contradict (2.3).

**Step 2.** We prove in this step that $\liminf_{k \to \infty} R_G(\lambda_k) \geq R_G(\lambda_0)$ for any sequence $\{\lambda_k\}_{k \geq 1}$ converging to a limit $\lambda_0 \in (0, \infty)$.

For any $n \in \mathbb{Z}_+$, let

\[ \Pi_n = \{ \text{paths } \gamma \text{ in } G \text{ staring and ending at } o \text{ with length } n \}, \]

\[ \mathbb{P}(\gamma, \lambda) = \prod_{i=0}^{n-1} p_\lambda(w_i, w_{i+1}), \gamma = w_0 w_1 \cdots w_n \in \Pi_n. \]

Note that for $0 < \lambda_1 \leq \lambda_2 < \infty$ and $v \sim u$ we have

\[ \frac{\lambda_1}{\lambda_2} \leq \frac{p_\lambda(v, u)}{p_\lambda(v, u)} \leq \frac{\lambda_2}{\lambda_1}. \]  \hspace{1cm} (2.4)

Thus, for any $\delta > 0$, there is a constant $\varepsilon > 0$ such that $p_\lambda(v, u) \leq (1 + \delta)p_{\lambda_0}(v, u)$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Consequently, we have $\mathbb{P}(\gamma, \lambda) \leq (1 + \delta)^n \mathbb{P}(\gamma, \lambda_0)$ for $\gamma \in \Pi_n$ and

\[ p_\lambda^{(n)}(o, o) = \sum_{\gamma \in \Pi_n} \mathbb{P}(\gamma, \lambda) \leq \sum_{\gamma \in \Pi_n} (1 + \delta)^n \mathbb{P}(\gamma, \lambda_0) = (1 + \delta)^n p_{\lambda_0}^{(n)}(o, o). \]

Therefore we have for $k$ large enough,

\[ G_{\lambda_k}(o, o | z) = \sum_{n=0}^{\infty} p_{\lambda_k}^{(n)}(o, o) z^n \leq \sum_{n=0}^{\infty} p_{\lambda_0}^{(n)}(o, o) ((1 + \delta)z)^n < \infty, \]

provided $(1 + \delta)z < R_G(\lambda_0)$. Since $\delta$ is arbitrary, we have that $\liminf_{k \to \infty} R_G(\lambda_k) \geq R_G(\lambda_0)$.

**Step 3.** It remains to prove $R_G(\lambda_c) = 1$. Suppose $R_G(\lambda_c) > 1$, then for $\lambda > \lambda_c$ and $z > 1$ with $1 < \frac{\lambda}{\lambda_c} < R_G(\lambda_c)$, we would have from (2.4) that

\[ \sum_{n=0}^{\infty} p_\lambda^{(n)}(o, o) z^n \leq \sum_{n=0}^{\infty} p_{\lambda_c}^{(n)}(o, o) \left( \frac{\lambda z}{\lambda_c} \right)^n < \infty. \]

Then $R_G(\lambda) > 1$. This contradicts to the fact that $\text{RW}_\lambda$ is recurrent for $\lambda > \lambda_c$.

### 2.2 Proof of Theorem 1.1 part (ii)

We split the proof of (ii) into three steps.

**Step 1.** For any given locally finite, connected infinite graph $G$, such that $\partial B_G(n_0)$ contains adjacent vertices for some $n_0$ we prove that $\rho_\lambda$ is not continuous at 0.

Let $u$ and $v$ be adjacent vertices in $\partial B_G(n_0)$. Let $e = \{u, v\}$ and $x_0 = o$. For $\text{RW}_\lambda$ (with $\lambda > 0$) to return to $o$, it suffices to walk first along a path $\gamma = x_0 x_1 \cdots x_{n_0}$ of length $n_0$ to a vertex $u \in \partial B_G(n_0)$ in $n_0$ steps, then walk $2n$ steps between $u$ and $v$, and finally returns to $o$ from $u$ along $\tilde{\gamma} = x_{n_0} x_{n_0-1} \cdots x_1 x_0$. Accordingly,

\[ p_\lambda^{(2n+2n_0)}(o, o) \geq \mathbb{P}(\gamma, \lambda) \mathbb{P}(\tilde{\gamma}, \lambda) \left( \frac{1}{d_u + (\lambda - 1)d_v} \right)^n \left( \frac{1}{d_v + (\lambda - 1)d_u} \right)^n, \]  \hspace{1cm} (2.5)

where for any $\lambda > 0$,

\[ \mathbb{P}(\gamma, \lambda) = \prod_{i=0}^{n_0-1} p_\lambda(x_i, x_{i+1}) > 0, \quad \mathbb{P}(\tilde{\gamma}, \lambda) = \prod_{i=0}^{n_0-1} p_\lambda(x_{n_0-i}, x_{n_0-i-1}) > 0. \]
So for any $\lambda > 0$,

$$\rho_\lambda \geq \limsup_{n \to \infty} \left\{ p_\lambda^{(2n+2n_0)}(o, o) \right\}^{\frac{1}{2n+2n_0}} \geq \frac{1}{\left[ (d_u + (\lambda - 1)d_u^-) (d_v + (\lambda - 1)d_v^-) \right]^{1/2}} > 0.$$  

Letting $0 < \lambda \to 0$, we immediately get

$$\liminf_{\lambda \to 0^+} \rho_\lambda \geq \frac{1}{\left[ (d_u - d_u^-) (d_v - d_v^-) \right]^{1/2}} > 0 = \rho_0.$$  

**Step 2.** Assume that there is a vertex $v$ such that $d_v - d_v^- = 0$. Let $u$ be a vertex adjacent to $v$. Let $\gamma$ be a path from $o$ to $u$ of length $n_0$, and denote by $\tilde{\gamma}$ the reverse path. Similar to the arguments in the previous step, we have for any $n$,

$$p_\lambda^{(2n_0+2n)}(o, o) \geq P(\gamma, \lambda)P(\tilde{\gamma}, \lambda) \left( \frac{1}{d_u + (\lambda - 1)d_u^-} \right)^n \left( \frac{1}{d_v} \right)^n.$$  

Then for any $\lambda > 0$,

$$\rho_\lambda \geq \left( \frac{1}{d_v(d_u + (\lambda - 1)d_u^-)} \right)^{1/2} > 0.$$  

Hence $\rho_\lambda$ is not continuous at 0.

**Step 3.** Assume that there are no adjacent vertices in $\partial B_G(n)$ for any $n \in \mathbb{N}$, and there exists $\delta > 0$ such that $d_v - d_v^- \geq \delta d_v$ for any vertex $v$. Then for any $\lambda > 0$ and the RW $X_n$ of steps to return to $o$, among these $2m$ steps, $m$ steps are upward and the other $m$ steps are downward.

When $v \sim u$ and $|u| = |v| - 1$, we have

$$p_\lambda(v, u) = \frac{\lambda}{d_v + (\lambda - 1)d_v^-} \leq \frac{\lambda}{d_v - d_v^-} \leq \lambda d_v^- \delta^{-1}, \quad \lambda > 0.$$  

When $v \sim u$ and $|u| = |v| + 1$, we have $p_\lambda(v, u) \leq d_v^- \delta^{-1}$. Hence for any path $\gamma = w_0w_1 \cdots w_{2n} \in \Pi_{2n}$,

$$P(\gamma, \lambda) = \prod_{i=0}^{2n-1} p_\lambda(w_i, w_{i+1}) \leq \lambda^n \delta^{-2n} P(\gamma, 1), \quad \lambda > 0,$$

which implies that for any $\lambda > 0$,

$$p_\lambda^{(2n)}(o, o) = \sum_{\gamma \in \Pi_{2n}} P(\gamma, \lambda) \leq \lambda^n \delta^{-2n} \sum_{\gamma \in \Pi_{2n}} P(\gamma, 1) \leq \lambda^n \delta^{-2n} p_1^{(2n)}(o, o).$$  

Hence

$$\rho_\lambda = \limsup_{n \to \infty} \left\{ p_\lambda^{(2n)}(o, o) \right\}^{\frac{1}{2n}} \leq \delta^{-1} \rho_1^{1/2},$$

proving that $\lim_{\lambda \to 0^+} \rho_\lambda = 0 = \rho_0$.

### 2.3 Proof of Theorem 1.2

We start with the lemma, which will be used in the proof of Theorem 1.2. For readers’ convenience we provide the proof in Appendix A.
Lemma 2.1. For the $d$-regular tree $\mathbb{T}_d$, the following holds:

$$\theta_{\mathbb{T}_d}(\lambda) = \frac{\lambda}{d-1}, \quad \rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1 + \lambda}, \quad \lambda \in [0, \lambda_c(\mathbb{T}_d)] = [0, d-1],$$

and for $\lambda \in (0, \infty)$ and $n \to \infty$,

$$f^{(2n)}(\lambda, o, o) = \frac{1}{\sqrt{\pi}} \left( \frac{2\sqrt{(d-1)\lambda}}{d-1 + \lambda} \right)^n n^{-3/2}. \quad (2.7)$$

Moreover,

$$P^{(2n)}_\lambda(o, o) \sim \begin{cases} 
\frac{(d-1-\lambda)^2}{16\pi^2(d-1)^2} \rho_{\mathbb{T}_d}(\lambda)^2 n^{-3/2} & \text{if } \lambda \in (0, d-1), \\
\frac{1}{\sqrt{\pi}} & \text{if } \lambda = d-1.
\end{cases} \quad (2.8)$$

Now we are ready to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** (i) Fix $\lambda \in (0, \lambda_c(\mathbb{T}_d)]$. Define $g = g_\lambda : \mathbb{Z}_+ \to \mathbb{R}$ by

$$g(n) = g_\lambda(n) := \left( 1 + \frac{d-1-\lambda}{d-1+\lambda} \right) ((d-1)/\lambda)^{-n/2},$$

and $f = f_\lambda : G \to \mathbb{R}$ by

$$f(x) := f_\lambda(x) = g(|x|), \quad \forall x \in V(G). \quad (2.9)$$

Clearly, $g$ is non-increasing on $\mathbb{Z}_+$. Recall $p(x, y)$ from (1.1). For any $h : G \to \mathbb{R}$, let

$$P(h) := \sum_{y \sim x} p(x, y)h(y), \quad x \in V(G). \quad (2.10)$$

Then $P(o) = \rho_{\mathbb{T}_d}(\lambda)f(o)$, and for $x \neq o$,

$$Pf(x) = \frac{d^+_x g(|x|+1) + d^0_x g(|x|) + \lambda d^-_x g(|x|-1)}{d^+_x + d^0_x + \lambda d^-_x} \geq \frac{(d^+_x + d^0_x) g(|x|+1) + \lambda d^-_x g(|x|-1)}{d^+_x + d^0_x + \lambda d^-_x}. \quad (2.11)$$

Since $g(|x|-1) \geq g(|x|+1)$ and $d^+_x \geq 1$ (so $d^+_x + d^0_x \leq d-1$), this leads to:

$$Pf(x) \geq \frac{(d-1)g(|x|+1) + \lambda g(|x|-1)}{d-1 + \lambda} = \rho_{\mathbb{T}_d}(\lambda) f(x), \quad x \neq o. \quad (2.11)$$

For further use, we notice that for $x \neq o$, if $Pf(x) = \rho_{\mathbb{T}_d}(\lambda)f(x)$, then $d^+_x = 1$, $d^0_x = 0$ and $d^-_x = d-1$. For any $n \in \mathbb{N}$, put $f_n := f I_{B_G(n)}$. For $x \in B_G(n)$,

$$Pf_n(x) = Pf(x) - \frac{d^+_x g(n+1)}{d^+_x + d^0_x + \lambda d^-_x} I_{\{x = n\}}.$$

Define $\mu$ as follows: $\mu(o) = d_o$ and $\mu(x) = (d^+_x + d^0_x + \lambda d^-_x) \lambda^{-|x|}$ for $x \neq o$. Let $M_n := |\partial B_G(n)|$ as before. Denote by $(\cdot, \cdot)$ the inner product of $L^2(G, \mu)$. Then

$$(Pf_n, f_n) = \sum_{x \in B_G(n)} Pf(x)f(x)\mu(x) - \sum_{x \in \partial B_G(n)} \frac{d^+_x g(n+1)}{d^+_x + d^0_x + \lambda d^-_x} f(x)\mu(x).$$
For the sum $\sum_{x \in B_G(n)}$ on the right-hand side, we observe that by (2.11), for $x \in B_G(n)$, $Pf(x) \leq \rho_{\tau_d}(\lambda)f(x) = \rho_{\tau_d}(\lambda)f_n(x)$. For the sum $\sum_{x \in \partial B_G(n)}$, we note that for $x \in \partial B_G(n)$, since $d_x^+ \leq d - 1$ and $f(x) = g(n)$, we have $\frac{n(x)}{d_x^+ + d_x^- + \lambda d_x^-} = \lambda^{-n}$. Accordingly,

$$(P f_n, f_n) \geq \rho_{\tau_d}(\lambda)(f_n, f_n) - (d - 1)M_n g(n)g(n + 1) \lambda^{-n} \geq \rho_{\tau_d}(\lambda)(f_n, f_n) - (d - 1)M_n g(n)^2 \lambda^{-n},$$

which implies that

$$\rho_G(\lambda) = \sup_{h \in L^1(G, \rho)} \frac{(Ph, h)}{(h, h)} \geq \frac{(P f_n, f_n)}{(f_n, f_n)} \geq \rho_{\tau_d}(\lambda) - \frac{(d - 1)M_n g(n)^2 \lambda^{-n}}{(f_n, f_n)}.$$

Observe that

$$(f_n, f_n) = \sum_{k=0}^{n} \sum_{x \in B_G(k)} g(k)^2 \mu(x) = \sum_{k=0}^{n} \sum_{x \in \partial B_G(k)} g(k)^2 (d_x^+ + d_x^- + \lambda d_x^-) \lambda^{-|x|} \geq (\lambda + 1)^d \sum_{k=0}^{n} M_k g(k)^2 \lambda^{-k}.$$

Hence

$$\rho_G(\lambda) \geq \rho_{\tau_d}(\lambda) - \frac{d - 1}{d(\lambda + 1)} \sum_{k=0}^{n} M_k g(k)^2 \lambda^{-k}. $$

It remains to prove that

$$\lim_{n \to \infty} \frac{M_n g(n)^2 \lambda^{-n}}{\sum_{k=0}^{n} M_k g(k)^2 \lambda^{-k}} = 0.$$ 

For $k \leq n$,

$$M_n g(n)^2 \lambda^{-n} \leq M_k (d - 1)^{n-k} g(n)^2 \lambda^{-n} = M_k g(k)^2 \lambda^{-k} \left(\frac{(d - 1 - \lambda)n + d - 1 + \lambda}{(d - 1 - \lambda)k + d - 1 + \lambda}\right)^2,$$

which implies that

$$\frac{\sum_{k=0}^{n} M_k g(k)^2 \lambda^{-k}}{M_n g(n)^2 \lambda^{-n}} \geq \sum_{k=0}^{n} \left(\frac{(d - 1 - \lambda)k + d - 1 + \lambda}{(d - 1 - \lambda)n + d - 1 + \lambda}\right)^2.$$ 

Since $\lambda \leq d - 1$, the sum on the right-hand side goes to infinity as $n \to \infty$.

(ii) For $d = 2$, $G_d = \{T_2\}$, the result holds trivially. So we assume $d \geq 3$. It suffices to prove that for any transitive $G \in G_d$ with the minimal cycle length $\ell \geq 3$,

$$\rho_G(\lambda) > \rho_{\tau_d}(\lambda), \quad \forall \lambda \in (0, \lambda_c(T_d)).$$

(2.12)

**Step 1.** $\lambda_c(G) < \lambda_c(T_d) = d - 1$.

Let $\Gamma_\ell := \langle a_1, \ldots, a_{d-2}, b | a_1^2 = 1, b^\ell = 1 \rangle$ be a finitely-presented group with generating set $S = \{a_1, \ldots, a_{d-2}, b, b^{-1}\}$, and $X_{d, \ell} := (\mathbb{Z}_2 * \cdots * \mathbb{Z}_2)(d - 2 \text{ folds})*\mathbb{Z}_d$ the corresponding Cayley graph; then the transitive graph $G$ is covered by $X_{d, \ell}$ (see Theorem 11.6 of [28]). From this result, we obtain

$$\lambda_c(G) = \text{gr}(G) \leq \text{gr}(X_{d, \ell}).$$

For $z \geq 0$, define

$$k_\ell(z) = \begin{cases} 
2z + 2z^2 + \cdots + 2z^{\frac{\ell-1}{2}}, & \text{if } \ell \text{ is odd,} \\
2z + 2z^2 + \cdots + 2z^{\frac{\ell-2}{2}} + z^{\ell-1}, & \text{if } \ell \text{ is even};
\end{cases}$$

$$h_\ell(z) = \frac{(d - 2)z}{1 + z} + \frac{k_\ell(z)}{1 + k_\ell(z)}.$$
Then \( \text{gr}(X_{d,\ell}) = \frac{1}{z} \) where \( z \) is the unique positive number satisfying \( h_\ell(z) = 1 \) (see [8] p. 28; it will also be recalled in more details in (3.1) below). Since \( j_\ell := \frac{k_\ell(z_\ell)}{1+k_\ell(z_\ell)} \) is strictly increasing in \( \ell \), and \( \lim_{\ell \to \infty} j_\ell = \frac{2}{d} \), we have \( j_\ell < \frac{2}{d} \), which implies \( h_\ell(\frac{1}{d-1}) < 1 \). Notice that \( h_\ell(z) \) is strictly increasing in \( z \geq 0 \). So \( z_\ell > \frac{1}{d-1} \) and \( \text{gr}(X_{d,\ell}) = \frac{1}{z_\ell} < d - 1 \), which implies \( \lambda_c(G) < d - 1 \).

**Step 2.** Fix \( \lambda \in (0, d - 1) \). Let as before \( \mu(o) := d_o \) and \( \mu(x) := (d^+_x + d^-_x + \lambda d^-_x)\lambda^{-|x|} \) if \( x \neq o \). Let \( f : G \to \mathbb{R} \) be the function defined in (2.9). Then \( f \in L^2(G, \mu) \).

Since \( G \) is transitive, \( \lambda_c(G) = \text{gr}(G) = \lim_{n \to \infty} M_n^{1/n} \). By Step 1, for any \( \varepsilon \in (0, d - 1 - \lambda_c(G)) \), there is a constant \( c_\varepsilon > 0 \) such that

\[
M_n \leq c_\varepsilon (\lambda_c(G) + \varepsilon)^n, \quad \forall n \geq 0.
\]

Thus

\[
\sum_{x \in V(G)} f^2(x) \mu(x) = \sum_{x \in V(G)} \left( \frac{1}{1 + d - 1 - \lambda |x|} \right)^2 \left( \frac{\lambda}{d - 1} \right)^{|x|} \left( d^+_x + d^-_x + \lambda d^-_x \right) \lambda^{-|x|}
\]

\[
\leq (\lambda \vee 1)d \sum_{n=0}^{\infty} M_n \left( \frac{1}{1 + d - 1 - \lambda |n|} \right)^2 \left( \frac{1}{d - 1} \right)^n
\]

\[
\leq (\lambda \vee 1)d c_\varepsilon \sum_{n=0}^{\infty} \left( \frac{\lambda_c(G) + \varepsilon}{d - 1} \right)^n \left( 1 + \frac{d - 1 - \lambda |n|}{d - 1} \right)^2
\]

\[
< \infty.
\]

**Step 3.** (2.12) is true.

Let \( \lambda \in (0, \lambda_c(T_d)) \). We have noticed in the proof of (i) that \( Pf(o) = \rho_{T_d}(\lambda)f(o) \) and that for \( x \neq o \),

\[
Pf(x) \geq \rho_{T_d}(\lambda)f(x), \quad \text{and} \quad = \text{ implies } d^-_x = 1, d^+_x = 0, d^-_x = d - 1.
\]

Since the transitive \( G \) has the minimal cycle length \( \ell \geq 3 \), we cannot have \( d^-_x = 1, d^+_x = 0, d^-_x = d - 1 \) for any \( x \in V(G) \setminus \{o\} \). Note that \( f(\cdot) \) and \( \mu(\cdot) \) are strictly positive on \( G \). Hence

\[
(Pf, f) = \sum_{x \in V(G)} Pf(x) f(x) \mu(x) > \sum_{x \in V(G)} \rho_{T_d}(\lambda) f^2(x) \mu(x) = \rho_{T_d}(\lambda)(f, f).
\]

By Step 2, \( f \in L^2(G, \mu) \), which implies that

\[
\rho_G(\lambda) = \sup_{h \in L^2(G, \mu) \setminus \{0\}} \langle Ph, h \rangle \geq \frac{( Pf, f )}{( f, f )} > \rho_{T_d}(\lambda),
\]

proving (2.12).

Since for some \( G \in \mathcal{G}_d \) that are not trees, one may have \( \text{gr}(G) = d - 1 \), in general it is not true that \( f \in L^2(G, \mu) \) for \( \lambda \in (0, d - 1) \). However, for any transitive graph \( G \in \mathcal{G}_d \) that is not isomorphic to \( T_d \), we have \( \text{gr}(G) < d - 1 \), which ensures \( f \in L^2(G, \mu) \) in the proof of Theorem 1.2 (ii).

### 3 Biased random walks on free product of graphs

The study of random processes on free products of graphs goes back at least to Teh and Gan [27], Znoiko [29] and Lyndon and Schupp [16]. The recursive structure of such graphs often makes it possible to do explicit computations, leading to close-form analytical formulas. For simple random walks on free products of graphs, the spectral radius (see, for example, Woess [28] p. 101-110) and the critical percolation probability (Špakulová [26]) are known. When \( \lambda \neq 1 \), the biased random walks are not transitive any more, making computations more delicate. In this section, we determine the spectral radius and the speed of the biased random walk on the free product of two complete graphs.
Let $r \in \mathbb{N}$, $r \geq 2$. Write $\mathcal{I} = \{1, \ldots, r\}$. Let $\{G_i = (V_i, E_i, o_i)\}_{i \in \mathcal{I}}$ be a family of connected finite rooted graphs with vertex sets $V_i$, edge sets $E_i$ and roots $o_i$. Call a copy of $G_i$ an $i$-cell. Assume that each $|V_i| \geq 2$ for all $i$, and that all $V_i$’s are disjoint. Put

$$V_i^x := V_i \setminus \{o_i\}, \quad \text{and} \quad \langle x \rangle := i, \quad \text{if } x \in V_i^x, \ i \in \mathcal{I}.$$  

Define

$$V := V_1 \ast \cdots \ast V_r = \left\{ x_1 x_2 \cdots x_n \, | \, n \in \mathbb{N}, x_i \in \bigcup_{j \in \mathcal{I}} V_j^x, \langle x_i \rangle \neq \langle x_{i+1} \rangle \right\} \cup \{o\}.$$ 

We can also view $V$ as the set of words over the alphabet $\bigcup_{i \in \mathcal{I}} V_i^x$ without two consecutive letters from the same $V_j^x$, with $o$ denoting the empty word in $V$. Let

$$\langle x_1 \cdots x_n \rangle := \langle x_n \rangle, \ \forall x_1 \cdots x_n \in V; \quad \langle o \rangle := 0.$$ 

For any pair of words $x = x_1 \cdots x_m$ and $y = y_1 \cdots y_n \in V$ with $\langle x_m \rangle \neq \langle y_1 \rangle$, the concatenation $xy$ of $x$ and $y$ is an element of $V$. In particular, $x o = o x = x$. When $\langle x \rangle \neq i \in \mathcal{I}$, we set $x o_i = o_i x = x$.

Define the set $E$ of edges on $V$ as follows: If $x, y \in V_i$ with $i \in \mathcal{I}$ and $x \sim y$, then

$$wx \sim wy \text{ for any } w \in V \text{ with } \langle w \rangle \neq i.$$ 

Then $G = (V, E, o)$ is the free product of the graphs $G_1, \ldots, G_r$, denoted by

$$G = G_1 \ast G_2 \ast \cdots \ast G_r.$$ 

By [28, Theorem 10.10], $G$ is nonamenable if $r \geq 3$ or if $\max_{i \in \mathcal{I}} |V_i| \geq 3$.

Let

$$\partial B_{G_i}(n) := \{ x \in V_i : |x| = n \}, \quad \psi_i(z) := \sum_{n \geq 1} |\partial B_{G_i}(n)| z^n, \quad z \geq 0.$$ 

From [8, Lemma 4.15], we have

$$\text{gr}(G) = \frac{1}{z_*}, \quad \text{where } z_* \text{ is the unique positive number satisfying } \sum_{i=1}^r \frac{\psi_i(z_*)}{1 + \psi_i(z_*)} = 1. \quad (3.1)$$ 

Let $m_1$ and $m_2$ be positive integers such that $m_1 m_2 \geq 2$, and $K_{m_i+1}$ the complete graph on $m_i + 1$ vertices (for $i = 1$ and $2$). We observe that by (3.1), $\lambda_c(G) = \sqrt{m_1 m_2}$ when $G = K_{m_1+1} \ast K_{m_2+1}$.

**Theorem 3.1.** Let $G := K_{m_1+1} \ast K_{m_2+1}$ and $\lambda \in (0, \lambda_c(G))$. Let $m = m_1 + m_2$. For RW$_\lambda$ on $G$, the following hold:

(i) The speed exists and equals

$$S(\lambda) = \frac{2(m_1 m_2 - \lambda^2)}{(2 \lambda + m) (\lambda + m - 1)}.$$ 

In particular, $S(\lambda) > 0$ is smooth and strictly decreasing on $(0, \lambda_c(G))$.

(ii) RW$_\lambda$ has the non-Liouville property, namely, RW$_\lambda$ has a non-constant bounded harmonic function.

(iii) The spectral radius

$$\rho(\lambda) = \frac{m - 2 + [(m_1 - m_2)^2 + 4 \lambda (\sqrt{m_1} + \sqrt{m_2})]^2]^{1/2}}{2 (m + \lambda - 1)}.$$ 

In particular, $\lambda \mapsto \rho(\lambda) \in (0, 1)$ is strictly increasing on $(0, \lambda_c(G))$. Moreover, for some constant $c > 0$,

$$p^{(n)}_{\lambda}(o, o) \sim c \rho(\lambda)^n n^{-3/2} \text{ as } n \to \infty.$$ 

The proof of Theorem 3.1 is presented in Section 3.2.
3.1 Spectral radius for free product of complete graphs

Let \( r \geq 2 \) and \( m_i \geq 1, \ 1 \leq i \leq r \). Let \( G \) be the free product of the complete graphs \( K_{m_i+1} \) with \( m_i+1 \) vertices. Let \( z_* \) denote the unique positive number satisfying

\[
\sum_{i=1}^{r} \frac{m_i z_*}{1 + m_i z_*} = 1. \tag{3.2}
\]

By (3.1),

\[
\lambda_\ast(G) = \text{gr}(G) = \frac{1}{z_*}. \tag{3.3}
\]

Write \( m := \sum_{i=1}^{r} m_i \). The transition probability of \( RW_\lambda \) from \( v \) to an adjacent vertex \( u \) is

\[
p(v, u) = \begin{cases} \frac{1}{m} & \text{if } v = o, \\ \frac{m - \lambda}{m + \lambda - 1} & \text{if } u \in \partial B(|v| - 1) \text{ and } v \neq o, \\ \frac{m - \lambda}{m + \lambda - 1} & \text{otherwise}. \end{cases}
\]

**Theorem 3.2.** For \( \lambda \in [0, \lambda_\ast(G)] \), we have \( \rho(\lambda) < 1 \). Moreover,

\[
\rho(0+) = \frac{\max_{1 \leq i \leq r} (m_i - 1)}{m - 1}. \tag{3.4}
\]

**Proof.** **Step 1.** Recall \( U(o, o | z) \) and \( R_U \) from (2.2). For \( z \in (-R_U, R_U) \),

\[
U(o, o | z) = \sum_{i=1}^{r} \frac{-(\phi_i(z) - mU(o, o | z))}{2m} + \sum_{i=1}^{r} \frac{[(\phi_i(z) - mU(o, o | z))^2 + 4\lambda m_i z^2]^{1/2}}{2m}, \tag{3.5}
\]

where \( \phi_i(z) := m - 1 + \lambda - (m_i - 1)z \).

To this end, let \( \tau_0^+ := \inf\{n \geq 1 | X_n = o\} \) as before, and for \( i = 1, 2, \ldots, r \), let \( f_i^{(n)}(o, o) := \mathbb{P}_o(\tau_0^+ = n, (X_1) = i) \). Define

\[
U_i(o, o | z) := \sum_{n=1}^{\infty} f_i^{(n)}(x, y) z^n, \quad z \geq 0.
\]

Then

\[
U(o, o | z) = \sum_{i=1}^{r} U_i(o, o | z), \quad z \geq 0.
\]

Note the tree-like structure of \( G \). When the event \( \{\tau_0^+ = n, (X_1) = i\} \) occurs, \( RW_\lambda \) must visit an edge in \( i \)-cell attached at \( o \) at step 1 and return to \( o \) the first time by an edge in the same \( i \)-cell at step \( n \). Each vertex of the \( i \)-cell is attached to a certain \( j \)-cell (with \( j \neq i \)). From the spherical symmetry of each \( K_{m_i+1} \), we obtain

\[
U_i(o, o | z) = \frac{m_i z^\lambda}{m + \lambda - 1} \sum_{n=0}^{\infty} \left(M_1(z) + M_2(z) + \cdots + \tilde{M}_i(z) + M_{i+1}(z) + \cdots + M_r(z)\right)^n,
\]

where, for \( j \neq i \),

\[
M_j(z) := \sum_{n=1}^{\infty} \mathbb{P}_x[\tau_x^+ = n, (X_1) = j] z^n, \quad x \in V(G), (x) = i, |x| = 1,
\]

which does not depend on \( (x) = i \), and

\[
\tilde{M}_i(z) := \mathbb{P}_x[(X_1) = i] z = \frac{m_i - 1}{m + \lambda - 1} z, \quad x \in V(G), (x) = i, |x| = 1.
\]
By the similarity structure of $G$, 
\[ M_j(z) = \left( \frac{m_j}{m + \lambda - 1} \right) U_j(o, o \mid z) = \frac{m}{m + \lambda - 1} U_j(o, o \mid z). \]

So when $|z| < R_U$ (where $R_U$ denotes as before the convergence radius of $U$), 
\[ U_i(o, o \mid z) = \frac{\lambda m_i}{m(m + \lambda - 1)} z^2 \left( 1 - \frac{m}{m + \lambda - 1} \sum_{j=1}^{r} U_j(o, o \mid z) - \frac{m}{m + \lambda - 1} U_i(o, o \mid z) + M_i(z) \right) \]
\[ = \frac{\lambda m_i}{m(m + \lambda - 1)} z^2 \left( 1 - \frac{m}{m + \lambda - 1} U(o, o \mid z) - \frac{m}{m + \lambda - 1} U_i(o, o \mid z) + M_i(z) \right). \]

Since $\tilde{M}_i(z) = \frac{m}{m + \lambda - 1} U_j(o, o \mid z)$, this yields, with the notation $\phi_i(z) := m - 1 + \lambda - (m_i - 1)z$, 
\[ U_i(o, o \mid z) = \frac{-(\phi_i(z) - mU(o, o \mid z))}{2m} + \left[ (\phi_i(z) - mU(o, o \mid z))^2 + 4\lambda m_i z^2 \right]^{1/2}, \]

which implies (3.5). 

**Step 2.** For any $0 < \lambda < \lambda_c(G)$, $G(o, o \mid R_G) < \infty$, $U(o, o \mid R_G) < 1$, and $R_G = R_U$. 

Note that $R_G < R_U$, and that for $|z| < R_G$, $|U(o, o \mid z)| < 1$, $G(o, o \mid z) = \frac{1}{1 - U(o, o \mid z)}$. So $U(o, o \mid R_G) = \lim_{r \to R_G} U(o, o \mid z) < 1$. 

Recall Pringsheim’s Theorem: For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$, its convergence radius is the smallest positive singularity point of $f(z)$. As such, the smallest positive singularity point $R_G$ of $G(o, o \mid z)$ is either the smallest positive number $z_1$ with $U(o, o \mid z_1) = 1$ if exists, or the convergence radius $R_U$ for $U(o, o \mid z)$. Since $U(o, o \mid z)$ is strictly increasing in $z \geq 0$, and $z_1$ is the unique positive number satisfying $U(o, o \mid z_1) = 1$ if exists, it remains to prove that $U(o, o \mid R_G) < 1$ (which implies $G(o, o \mid R_G < \infty$ and $R_G = R_U$). 

Assume this were not true; so $U(o, o \mid R_G) = 1$. We exclude the trivial case where $m_i = 1$ for $1 \leq i \leq r$ (in which case the result holds trivially; see the proof of Lemma 2.1). Note that $R_G \geq 1$. If $R_G = 1$, then $U(o, o \mid R_G) = U(o, o \mid 1) < 1$ due to transience. So we assume $R_G > 1$. 

By (3.5), 
\[ 1 = \sum_{i=1}^{r} \frac{-(\lambda - 1) - (m_i - 1)R_G + [(\lambda - 1) - (m_i - 1)R_G]^2 + 4\lambda m_i R_G^2]^{1/2}}{2m} \]
\[ = \sum_{i=1}^{r} \frac{(1 - \lambda) + (m_i - 1)R_G + [(1 - \lambda) + (m_i - 1)R_G]^2 + 4\lambda m_i R_G^2]^{1/2}}{2m}. \] 

(3.6) 

We deduce a contradiction by distinguishing two possible cases. 

**Case 1.** $0 < \lambda \leq 1$. For any $1 \leq i \leq r$, 
\[ (1 - \lambda) + (m_i - 1)R_G \geq (1 - \lambda) + (m_i - 1) \geq 0, \]
and the inequality is strict for at least one $i$. Thus by (3.6), 
\[ 1 > \sum_{i=1}^{r} \frac{(1 - \lambda) + (m_i - 1) + [(1 - \lambda) + (m_i - 1)]^2 + 4\lambda m_i R_G^2]^{1/2}}{2m} \]
\[ = \sum_{i=1}^{r} \frac{(m_i - \lambda) + (m_i + \lambda)}{2m} = 1, \]
which leads to a contradiction. Consequently, in this case $U(o, o \mid R_G) < 1$. 

**Case 2.** $1 < \lambda < \lambda_c(G)$. Write 
\[ [\lambda - 1 - (m_i - 1)R_G]^2 + 4\lambda m_i R_G^2 = [\lambda - 1 + (m_i + 1)R_G]^2 + 4\lambda m_i R_G^2 - 4m_i R_G^2 - 4(\lambda - 1)m_i R_G. \]
Since \( \lambda > 1 \) and \( R_G > 1 \),

\[
4\lambda m_i R_G^2 - 4m_i R_G^2 - 4(\lambda - 1)m_i R_G = 4R_G(\lambda - 1)m_i(R_G - 1) > 0.
\]

So \([\lambda - 1 - (m_i - 1)R_G]^2 + 4\lambda m_i R_G^2 > [\lambda - 1 + (m_i + 1)R_G]^2\). By (3.6),

\[
1 > \sum_{i=1}^r \frac{(1 - \lambda) + (m_i - 1)R_G + \lambda - 1 + (m_i + 1)R_G}{2m} = \sum_{i=1}^r \frac{m_i R_G}{m} = R_G,
\]

contradicting the assumption \( R_G > 1 \). Hence \( U(0, o|R_G) < 1 \) in this case as well.

**Step 3.** Let \( \phi_i(z) := m - 1 + \lambda - (m_i - 1)z \) for \( 1 \leq i \leq r \), and

\[
F(z, U) := \frac{1}{2m} \sum_{i=1}^r \left\{ -\phi_i(z) - mU + (\phi_i(z) - mU)^2 + 4\lambda m_i z^2 \right\}^{1/2}.
\]

Then \( U(z) := U(o, o|z) \) solves the equation \( U = F(z, U), \) \( |z| < R_U \), and \( \rho(\lambda)^{-1} \) is the smallest positive number \( z \) such that \( \frac{\partial F}{\partial U}(z, U(z)) = 1 \). Therefore, to obtain \( \rho(\lambda) < 1 \), it suffices to prove that

\[
\left| \frac{\partial F}{\partial U}(1, U(1)) \right| < 1.
\]

To prove this, we observe that

\[
F(1, 0) = \frac{1}{2m} \sum_{i=1}^r \left\{ -(m - m_i + \lambda) + (m_i - \lambda)^2 + 4\lambda m_i \right\} > 0,
\]

\[
F(1, 1) = \frac{1}{2m} \sum_{i=1}^r \left\{ m_i - \lambda + (m_i - \lambda)^2 + 4\lambda m_i \right\} = \frac{1}{2m} \sum_{i=1}^r \left\{ m_i - \lambda + m_i + \lambda \right\} = 1.
\]

Moreover,

\[
\frac{\partial F}{\partial U}(1, U) = \frac{r}{2} - \frac{1}{2} \sum_{i=1}^r \frac{m - m_i + \lambda - mU}{\left\{ (m - m_i + \lambda - mU)^2 + 4\lambda m_i \right\}^{1/2}} > 0,
\]

\[
\frac{\partial^2 F}{\partial U^2}(1, U) = \frac{m}{2} \sum_{i=1}^r \frac{4\lambda m_i}{\left\{ (m - m_i + \lambda - mU)^2 + 4\lambda m_i \right\}^{3/2}} > 0.
\]

Hence \( F(1, U) \) is strictly increasing and convex in \( U \in \mathbb{R} \). By (3.3), for any \( \lambda \in (0, \lambda_c(G)) \),

\[
\frac{\partial F}{\partial U}(1, 1) = \frac{r}{2} - \frac{1}{2} \sum_{i=1}^r \frac{-m_i + \lambda}{\left\{ (m_i - \lambda)^2 + 4\lambda m_i \right\}^{1/2}} = \sum_{i=1}^r \frac{m_i}{m_i + \lambda} > 1.
\]

As a consequence, \( U(1) \) is the smallest positive solution to \( U = F(1, U) \) and \( 0 < \frac{\partial F}{\partial U}(1, U(1)) < 1 \). Therefore we have proved that \( \rho(\lambda) < 1 \).

**Step 4.** Now we prove (3.4).

If \( m_i = 1 \) for all \( i \), then \( G \) is the \( r \)-regular tree, and by Theorem 1.1(ii), \( \rho(0+) = \rho(0) = 0 \), so (3.4) holds.

Assume that

\[
\max_{1 \leq i \leq r} m_i = m_{i_*} > 1 \text{ for some } i_* \in \{1, \ldots, r\}.
\]

For any \( 1 \leq i \leq r \) and \( n \geq 3 \), let

\[
A_i(n) := \{ X_0 = o, \langle X_1 \rangle = \langle X_2 \rangle = \cdots = \langle X_{n-1} \rangle = i, \ X_n = o \}.
\]

Then

\[
p^{(n)}(o, o) \geq P^\infty(A_{i_*}(n)) = \frac{m_{i_*}}{m} \left( \frac{m_{i_*} - 1}{m - 1 + \lambda} \right)^{n-2} \frac{\lambda}{m - 1 + \lambda},
\]

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which implies that
\[ \rho(\lambda) \geq \lim_{n \to \infty} \{ \mathbb{P}_o[A_{i*}(n)] \}^{1/n} = \frac{m_{i*} - 1}{m - 1 + \lambda}. \]

Consequently,
\[ \lim_{\lambda \downarrow 0} \inf \rho(\lambda) \geq \frac{m_{i*} - 1}{m - 1}. \]

It remains to prove that \( \limsup_{\lambda \downarrow 0} \rho(\lambda) \leq \frac{m_{i*} - 1}{m - 1} \). Let us make a few simple observations concerning the transition probability of \( \{ |X_n| \}_{n \geq 0} \). Let \( \ell \in \mathbb{N} \) and let \( k \in \mathbb{N} \).

For any \( j \in \{1, \ldots, r\} \) such that \( \mathbb{P}_o(|X_k| = j, |X_k| = \ell) > 0 \), we have
\[ \mathbb{P}_o[|X_{k+1}| = \ell \mid |X_k| = j, |X_k| = \ell, |X_{k-1}|, \ldots, |X_0|] = \frac{m_j - 1}{m - 1 + \lambda} = \frac{m_j - 1}{m - 1 + \lambda}, \]
so that
\[ \mathbb{P}_o[|X_{k+1}| = \ell \mid |X_k| = \ell, |X_{k-1}|, \ldots, |X_0|] \leq \frac{m_{i*} - 1}{m - 1 + \lambda}. \]

On the other hand,
\[ \mathbb{P}_o[|X_{k+1}| = \ell - 1 \mid |X_k| = \ell, |X_{k-1}|, \ldots, |X_0|] = \frac{\lambda}{m - 1 + \lambda}, \]
and trivially,
\[ \mathbb{P}_o[|X_{k+1}| = \ell + 1 \mid |X_k| = \ell, |X_{k-1}|, \ldots, |X_0|] \leq 1. \]

For any \( n \geq 3 \), let \( S_n \) denote the set of all vectors \( \vec{s} = \{ s_k \}_{1 \leq k \leq n} \) such that
\[ s_1 = 1, \ s_n = -1, \ s_k \in \{ -1, 0, +1 \}, \sum_{j=1}^{k} s_j \geq 0, 1 \leq k \leq n - 1, \sum_{j=1}^{n} s_j = 0. \]

For \( \vec{s} \in S_n \), let
\[ a_{+}(\vec{s}) := \# \{ k \leq n : s_k = +1 \}, \ a_{-}(\vec{s}) := \# \{ k \leq n : s_k = -1 \}, \ a_0(\vec{s}) := \# \{ k \leq n : s_k = 0 \}. \]

Clearly, \( a_{+}(\vec{s}) = a_{-}(\vec{s}) \), \( 2a_{-}(\vec{s}) + a_0(\vec{s}) = n \). Moreover, if \( |X_n| = 0 \), then \( \{ |X_k| - |X_{k-1}| \}_{1 \leq k \leq n} \in S_n \).

By our discussions on transition probabilities of \( \{ |X_n| \}_{n \geq 0} \), it is seen that for \( 3 \leq n \) and \( \vec{s} \in S_n \),
\[ \mathbb{P}_o[|X_n| = 0 \mid \{ |X_k| - |X_{k-1}| \}_{1 \leq k \leq n} = \vec{s} \in S_n] \leq 1^{a_{+}(\vec{s})} \left( \frac{\lambda}{m - 1 + \lambda} \right)^{a_{-}(\vec{s})} \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^{a_0(\vec{s})}. \]

For sufficiently small \( \lambda > 0 \), we have \( \frac{\lambda}{m - 1 + \lambda} \leq \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^2 \), so that
\[ \left( \frac{\lambda}{m - 1 + \lambda} \right)^{a_{-}(\vec{s})} \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^{a_0(\vec{s})} \leq \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^{2a_{-}(\vec{s})} \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^{a_0(\vec{s})} = \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^n. \]

Consequently,
\[ \mathbb{P}_o[|X_n| = 0] \leq \left( \frac{m_{i*} - 1}{m - 1 + \lambda} \right)^n. \]

Hence,
\[ \limsup_{\lambda \downarrow 0} \rho(\lambda) \leq \limsup_{\lambda \downarrow 0} \frac{m_{i*} - 1}{m - 1 + \lambda} = \frac{m_{i*} - 1}{m - 1} = \frac{\max_{1 \leq i \leq r} m_i - 1}{m - 1}, \]
completing the proof of (3.4).
3.2 Proof of Theorem 3.1

Recall that $G$ is the free product of two complete graphs $K_{m_1+1}$ and $K_{m_2+1}$ and that $(X_n)_{n=0}^\infty$ is the $\lambda$-biased random walk on $G$. Recall that $\lambda_c(G) = \sqrt{m_1 m_2}$.

Define

$$f(x) = \begin{cases} 
\frac{m_2 - \lambda}{m - 1 + \lambda} & \text{if } x = o, \\
\frac{m_1 - \lambda}{m - 1 + \lambda} & \text{if } \langle x \rangle = 1, \\
\frac{m_2 + \lambda}{m - 1 + \lambda} & \text{if } x = 2. 
\end{cases}$$

Then $\{|X_n| - |X_{n-1}| - f(X_{n-1})\}_{n=1}^\infty$ is a martingale-difference sequence. It follows from the strong law of large numbers for uncorrelated random variables ([22, Theorem 13.1]) that

$$\lim_{n \to \infty} \frac{1}{n} \left| X_n - \sum_{k=0}^{n-1} f(X_k) \right| = 0 \quad \text{a.s.}$$

Note that $\sum_{k=0}^{n-1} f(X_k) = \sum_{k=0}^{n-1} f(o) I_{\{X_k = o\}} + \sum_{i=1}^{\lambda} \sum_{k=0}^{n-1} f(i) I_{\{X_k = i\}}$. Since the walk is transient

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = 0 \quad \text{a.s. for } 0 \leq \lambda < \lambda_c(G).$$

Consequently, we have

$$\lim_{n \to \infty} \frac{1}{n} \left| X_n - \frac{m_2 - \lambda}{m - 1 + \lambda} \sum_{k=0}^{n-1} I_{\{X_k = 1\}} - \frac{m_1 - \lambda}{m - 1 + \lambda} \sum_{k=0}^{n-1} I_{\{X_k = 2\}} \right| = 0 \quad \text{a.s.} \quad (3.8)$$

For any $\lambda \in [0, \infty)$, let

$$F(\lambda) := \frac{m_2 - \lambda}{\lambda + m - 1} \frac{m_1 + \lambda}{2\lambda + m} + \frac{m_1 - \lambda}{\lambda + m - 1} \frac{m_2 + \lambda}{2\lambda + m} = \frac{2m_1 m_2 - 2\lambda^2}{(\lambda + m - 1)(2\lambda + m)}.$$ 

Note that $\lambda \mapsto F(\lambda)$ is strictly decreasing on $[0, \infty)$. 

**Lemma 3.3.** For any $0 \leq \lambda < \lambda_c(G)$, the speed $S(\lambda) := \lim_{n \to \infty} \frac{|X_n|}{n}$ exists almost surely, is deterministic and equals $F(\lambda)$. In particular,

$$S(\lambda) > 0 \text{ is smooth and strictly decreasing in } \lambda \in [0, \lambda_c(G)], \text{ and } \lim_{\lambda \uparrow \lambda_c(G)} S(\lambda) = 0.$$ 

**Proof.** **Step 1.** Consider the process $(|X_n|, \langle X_n \rangle)_{n=0}^\infty$. For any type 1 (resp. type 2) vertex $x$, all its $m_2$ (resp. $m_1$) neighbours in $\partial B_G(|x| + 1)$ are of type 2 (resp. 1), and its unique neighbour $x_-$ in $\partial B_G(|x| - 1)$ is of type 2 (resp. type 1) if $|x| \geq 2$, and is $o$ if $|x| = 1$. The vertex $o$ has exactly $m_1$ type 1 neighbours and $m_2$ type 2 neighbours in $\partial B_G(1)$. The process $(|X_n|, \langle X_n \rangle)_{n=0}^\infty$ is a Markov chain on state space $(\mathbb{N} \times \{1, 2\}) \cup \{(0, 0)\}$ with transition probability function $q(\cdot, \cdot)$ given by

$$q((0, 0), (1, 1)) = \frac{m_1}{m}, \quad q((0, 0), (1, 2)) = \frac{m_2}{m},$$

$$q((1, 1), (0, 0)) = \frac{\lambda}{m - 1 + \lambda}, \quad q((1, 2), (0, 0)) = \frac{\lambda}{m - 1 + \lambda},$$

$$q((1, 1), (1, 1)) = \frac{m_1 - 1}{m - 1 + \lambda}, \quad q((1, 1), (2, 2)) = \frac{m_2}{m - 1 + \lambda},$$

$$q((1, 2), (1, 2)) = \frac{m_2 - 1}{m - 1 + \lambda}, \quad q((1, 2), (2, 1)) = \frac{m_1}{m - 1 + \lambda};$$

and for any $k \geq 2$,

$$q((k, 1), (k, 1)) = \frac{m_1 - 1}{m - 1 + \lambda}, \quad q((k, 2), (k, 2)) = \frac{m_2 - 1}{m - 1 + \lambda},$$

$$q((k, 1), (k - 1, 2)) = \frac{\lambda}{m - 1 + \lambda}, \quad q((k, 1), (k + 1, 2)) = \frac{m_2}{m - 1 + \lambda},$$

$$q((k, 2), (k - 1, 1)) = \frac{\lambda}{m - 1 + \lambda}, \quad q((k, 2), (k + 1, 1)) = \frac{m_1}{m - 1 + \lambda}.$$
Step 2. Define, for \( i \in \{1, 2\} \),
\[
\sigma^i_1 := \inf\{ n \geq 0 : \langle X_n \rangle = i \}, \quad \tau^i_1 := \inf\{ n > \sigma^i_1 : \langle X_n \rangle \neq i \},
\]
and recursively for any \( k \in \mathbb{N} \),
\[
\sigma^i_{k+1} := \inf\{ n > \tau^i_k : \langle X_n \rangle = i \}, \quad \tau^i_{k+1} := \inf\{ n > \sigma^i_{k+1} : \langle X_n \rangle \neq i \}.
\]
Set
\[
p_1 := \frac{m_1 - 1}{m - 1 + \lambda}, \quad p_2 := \frac{m_2 - 1}{m - 1 + \lambda}.
\]

By Step 1 and the strong Markov property, all stopping times \( \tau^i_k \) and \( \sigma^i_k \) are finite, and \( \{\tau^i_k - \sigma^i_k - 1\}_{k \geq 1} \) is an i.i.d. sequence with \( \mathbb{P}(\tau^i_k - \sigma^i_k - 1 = j) = p^i_j (1 - p_i) \) for \( j \geq 0 \). In particular, \( \mathbb{E}(\tau^i_k - \sigma^i_k - 1) = \frac{p_i}{1 - p_i} \).

Notice that for any \( n \geq 1 + \sigma^i_1 \), there exists a unique random integer \( k^i_n \) such that \( \sigma_{k^i_n} \leq n - 1 < \sigma_{k^i_n + 1} \). Therefore, for any \( n \geq 1 + \sigma^i_1 \vee \sigma^2_1 \) and \( i \in \{1, 2\} \),
\[
\frac{1}{n} \sum_{j=1}^{k^i_n - 1} (\tau^i_j - \sigma^i_j) \leq \frac{1}{n} \sum_{k=0}^{n-1} I_{\{\langle X_k \rangle = i\}} \leq \frac{1}{n} \sum_{j=1}^{k^i_n} (\tau^i_j - \sigma^i_j).
\]

Since \( \{\tau^i_k - \sigma^i_k - 1\}_{k \geq 1} \) is i.i.d. with \( \mathbb{E}(\tau^i_1 - \sigma^i_1) < \infty \), we have \( \frac{1}{n} (\tau^i_{k^i_n} - \sigma^i_{k^i_n}) \to 0 \) a.s. Consequently, for \( i \in \{1, 2\} \),
\[
\frac{1}{n} \sum_{j=1}^{k^i_n} (\tau^i_j - \sigma^i_j) - \frac{1}{n} \sum_{k=0}^{n-1} I_{\{\langle X_k \rangle = i\}} \to 0 \quad \text{a.s.}
\]

Step 3. Almost surely,
\[
\lim_{n \to \infty} \frac{k^1_n}{n} = \lim_{n \to \infty} \frac{k^2_n}{n} = \frac{(m_1 + \lambda)(m_2 + \lambda)}{(m + 2\lambda)(m - 1 + \lambda)}.
\]

Indeed, \( \frac{1}{n} \sum_{k=0}^{n-1} I_{\{\langle X_k \rangle = \sigma^i_1\}} \to 0 \) a.s., thus
\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} I_{\{\langle X_k \rangle = 1\}} + \frac{1}{n} \sum_{k=0}^{n-1} I_{\{\langle X_k \rangle = 2\}} \right) = 1 \quad \text{a.s.}
\]
By Step 2, this implies that
\[
\lim_{n \to \infty} \frac{1}{n} \left( \sum_{j=1}^{k^1_n} (\tau^1_j - \sigma^1_j) + \sum_{j=1}^{k^2_n} (\tau^2_j - \sigma^2_j) \right) = 1 \quad \text{a.s.} \quad (3.9)
\]
On the other hand, each \( \{\tau^i_k - \sigma^i_k - 1\}_{k \geq 1} \) is an i.i.d. sequence with \( \mathbb{E}(\tau^i_k - \sigma^i_k - 1) = \frac{p_i}{1 - p_i} \), thus by the strong law of large numbers, for \( i \in \{1, 2\} \),
\[
\lim_{n \to \infty} \frac{1}{k^i_n} \sum_{j=1}^{k^i_n} (\tau^i_j - \sigma^i_j) = 1 + \frac{p_i}{1 - p_i} = \frac{1}{1 - p_i} \quad \text{a.s.}
\]
In view of (3.9), we obtain:
\[
\lim_{n \to \infty} \left( \frac{k^1_n}{n} \frac{1}{1 - p_1} + \frac{k^2_n}{n} \frac{1}{1 - p_2} \right) = 1 \quad \text{a.s.} \quad (3.10)
\]

Observe that \( \langle X_{\tau^i_k} \rangle \) is either \( j \) for \( j \in \{1, 2\} \setminus \{i\} \) or \( X_{\tau^i_k} = a \), and that when \( X_{\tau^i_k} = a \), \( X_{\tau^i_{k+1}} \) must be of type 1 or 2. Since
\[
\frac{1}{n} \sum_{k=0}^{n-1} I_{\{\langle X_k \rangle = a\}} \to 0 \quad \text{a.s.}
\]
that, for \( n \to \infty \), \( k_n^1 \) (the number of jumps of RW\(_{λ}\) from \( o \) or type 2 vertex to type 1 vertex up to time \( n - 1 \)) differs by \( o(n) \) from \( k_n^2 \) (the number of jumps of RW\(_{λ}\) from \( o \) or type 1 vertex to type 2 vertex up to time \( n - 1 \)). In other words, \( \frac{k_n^1 - k_n^2}{n} \to 0 \) a.s. In view of (3.10), we get

\[
\lim_{n \to \infty} \frac{k_n^1}{n} = \lim_{n \to \infty} \frac{k_n^2}{n} = \frac{(m_1 + \lambda)(m_2 + \lambda)}{(m + 2\lambda)(m - 1 + \lambda)} \quad \text{a.s.}
\]

**Step 4.** By Steps 2 and 3, for \( i \in \{1, 2\} \),

\[
\frac{1}{n} \sum_{k=0}^{n-1} I_{\{X_k = i\}} \to \frac{m_i + \lambda}{m + 2\lambda} \quad \text{a.s.}
\]

This and (3.8) complete the proof of this lemma.

The next lemma concerns the non-Liouville property of RW\(_{λ}\) with \( 0 \leq \lambda < λ_c(G) \).

**Lemma 3.4.** For any \( 0 \leq \lambda < λ_c(G) \), RW\(_{λ}\) has a non-constant bounded harmonic function.

**Proof.** Take \( y \in ∂B_G(1) \) with \( \langle y \rangle = 2 \). Let \( A \) be the induced subgraph consisting of \( y \) and all words (vertices) of forms \( yw \). Let \( (X_n)_{n=0}^{∞} \) be RW\(_{λ}\) on \( G \), and let \( P_z \) denote the law of \( (X_n)_{n=0}^{∞} \) starting at \( z \). Notice that every vertex \( z \in G \) is a cutpoint in the sense that \( G \setminus \{z\} \) has two disjoint connected components. By the transience of RW\(_{λ}\), \( \lim_{n \to ∞} I_{\{X_n \in A\}} \) exists \( P_z \)-a.s.

For any vertex \( z \) of \( G \), let

\[
f(z) := P_z[(X_n)_{n=0}^{∞} \text{ ends up in } A].
\]

Then \( f \) is a bounded harmonic function. Let \( x \in ∂B_G(1) \) with \( \langle y \rangle = 1 \). Let

\[
a := P_x[(X_n)_{n=0}^{∞} \text{ never hits } o].
\]

Since the walk is transient, we have \( a \in (0, 1) \), and \( f(x) = (1 - a)f(o) \). Note that \((G, o)\) is quasi-spherically symmetric, so the transience of the walk implies \( f(o) > 0 \). Hence \( f \) is a non-constant harmonic function.

**Proof of Theorem 3.1.** By Lemmas 3.3-3.4, we obtain Theorem 3.1(i)-(ii). It remains to prove Theorem 3.1(iii).

**Step 1. Computation of \( ρ(λ) \).** Recall from Step 3 in the proof of Theorem 3.2 (Section 3.1) that \( U(z) := U(o, o| z) \) solves the equation \( U = F(z, U), \ |z| < R_G \), and \( z_0 := ρ(λ)^{-1} \) is the smallest positive number \( z \) such that \( \frac{∂F}{∂U}(z, U(z)) = 1 \), where the function \( F(z, U) \) is defined by (3.7) with \( r = 2 \):

\[
F(z, U) := \frac{1}{2m} \sum_{i=1}^{2} \{-(φ_i(z) - mU) + [(φ_i(z) - mU)^2 + 4λm_i z^2]^{1/2}\}.
\]

Since

\[
\frac{∂F}{∂U}(z, U) = 1 + \sum_{i=1}^{2} \frac{- (φ_i(z) - mU)}{[(φ_i(z) - mU)^2 + 4λm_i z^2]^{1/2}},
\]

we have

\[

\frac{φ_1(z_0) - mU(z_0)}{[(φ_1(z_0) - mU(z_0))^2 + 4λm_1 z_0^2]^{1/2}} = \frac{mU(z_0) - φ_2(z_0)}{[(φ_2(z_0) - mU(z_0))^2 + 4λm_2 z_0^2]^{1/2}},
\]

which implies

\[

\frac{φ_1(z_0) - mU(z_0)}{\sqrt{4λm_1 z_0^2}} = \frac{mU(z_0) - φ_2(z_0)}{\sqrt{4λm_2 z_0^2}}.
\]

Recall that \( φ_i(z) = m + λ - 1 - (m_i - 1)z \). This yields

\[
mU(z_0) = m + λ - 1 - (\sqrt{m_1 m_2} - 1)z_0;
\]

(3.12)
hence \( \phi(z_0) - mU(z_0) = (\sqrt{m_1m_2} - m)z_0 \). Consequently,

\[
F(z_0, U(z_0)) = \frac{1}{2m} \sum_{i=1}^{2} \{(\sqrt{m_1m_2} - m_i)z_0 + [(\sqrt{m_1m_2} - m_i)^2 z_0^2 + 4\lambda m_i z_0^2]^{1/2}\}
\]

\[
= -\left(\frac{\sqrt{m_1m_2}}{m} - \frac{1}{2}\right)z_0 + \frac{1}{2m} \sum_{i=1}^{2} m_i^{1/2} [(\sqrt{m_1} - \sqrt{m_2})^2 + 4\lambda]^{1/2} z_0
\]

\[
= -\left(\frac{\sqrt{m_1m_2}}{m} - \frac{1}{2}\right)z_0 + \frac{1}{2m} [(m_1 - m_2)^2 + 4\lambda(\sqrt{m_1} + \sqrt{m_2})^2]^{1/2} z_0. \quad (3.13)
\]

On the other hand, \( F(z_0, U(z_0)) = U(z_0) \), which is \( m + \lambda - 1 = \sqrt{m_1m_2} - 1 \) \( z_0 \) (by (3.12)). Combining this with (3.13) yields

\[
\rho(\lambda)^{-1} = z_0 = \frac{2(m + \lambda - 1)}{m - 2 + [(m_1 - m_2)^2 + 4\lambda(\sqrt{m_1} + \sqrt{m_2})^2]^{1/2}}. \quad (3.14)
\]

Taking limit \( \lambda \to 0^+ \), we have

\[
\lim_{\lambda \to 0^+} \rho(\lambda) = \frac{(m_1 \lor m_2) - 1}{m - 1}.
\]

**Step 2. Strictly increasing property for \( \rho(\lambda) \).** By a change of variables

\[
x = m - 2 + [(m_1 - m_2)^2 + 4\lambda(\sqrt{m_1} + \sqrt{m_2})^2]^{1/2},
\]

\( \lambda = \frac{(x-m+2)^2-(m_1-m_2)^2}{4\sqrt{m_1+m_2}^2} \), we see that

\[
z_0 = \frac{1}{2(\sqrt{m_1} + \sqrt{m_2})^2} \left[ x + \frac{4(m - 1 + \sqrt{m_1m_2})^2}{x} - 2(m - 2) \right],
\]

which is strictly decreasing in \( x < 2(m - 1 + \sqrt{m_1m_2}) \), i.e., \( \lambda < \sqrt{m_1m_2} = \lambda_G(G) \). Thus \( \rho(\lambda) \) is strictly increasing in \( \lambda \in (0, \sqrt{m_1m_2}) \).

**Step 3. Asymptotics for \( p_\lambda^{(n)}(o, o) \).** Write for simplicity \( G(z) := G(o, o \mid z) \). Note that \( U(z) = \frac{G(z)}{G(z)-1}, |z| < R_G \). We have from \( U(z) = F(z, U(z)) \) that

\[
2(m + \lambda - 1) - (m - 2)z = \sum_{i=1}^{2} \left[ \left( \phi_i(z) - \frac{mG(z)}{G(z) - 1} \right)^2 + 4\lambda m_iz_0^2 \right]^{1/2} \quad (3.15)
\]

Set

\[
\Psi(u, v) := 2(m + 1 - \lambda) - (m - 2)u - \sum_{i=1}^{2} \left[ \left( \phi_i(u) - \frac{mv}{v - 1} \right)^2 + 4\lambda m_iu^2 \right]^{1/2}.
\]

Notice that \( \Psi(z, G(z)) = 0 \). By (3.11), there exists \( \theta_0 \in (0, \pi) \) such that

\[
\cos \theta_0 = \frac{\phi_1(z_0) - mU(z_0)}{[(\phi_1(z_0) - mU(z_0))^2 + 4\lambda m_1 z_0^2]^{1/2}} = \frac{mU(z_0) - \phi_2(z_0)}{[(\phi_2(z_0) - mU(z_0))^2 + 4\lambda m_2 z_0^2]^{1/2}}.
\]

By direct computations, we have

\[
\frac{\partial \Psi}{\partial v}(z_0, G(z_0)) = 0,
\]

\[
\frac{\partial^2 \Psi}{\partial v^2}(z_0, G(z_0)) = -m^2 \sum_{i=1}^{2} \frac{(G(z_0) - 1)^{-4} \sin^2 \theta_0}{[(\phi_i(z_0) - mU(z_0))^2 + 4\lambda m_i z_0^2]^{1/2}} \neq 0
\]

\[
\frac{\partial \Psi}{\partial u}(z_0, G(z_0)) = -(m - 2) + (m_1 - m_2) \cos \theta_0 - 2\sqrt{\lambda}(\sqrt{m_1} + \sqrt{m_2}) \sin \theta_0 \neq 0.
\]

Applying the method of Darboux (see [6] Theorem 5) as in the proof of Lemma 2.1, we obtain the desired asymptotics for \( p_\lambda^{(n)}(o, o) \).
A Proof of Lemma 2.1

Proof of Lemma 2.1. The lemma holds trivially for $\lambda = 0$. So assume $\lambda > 0$. Notice that $\text{RW}_\lambda (X_n)_{n=0}^{\infty}$ must return to $o$ in even steps, and that $\{|X_n|\}_{n=0}^{\infty}$ with $|X_0| = 0$ is a Markov chain on $\mathbb{Z}_+$ with transition probabilities given by

$$p(x, y) = \begin{cases} 
1 & \text{if } x = 0, \ y = 1 \\
\frac{\lambda}{d-1+\lambda} & \text{if } y = x - 1 \text{ and } x \neq 0,
\end{cases}$$

otherwise.

Recall for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$,

$$f^{(2n)}(o, o) = \mathbb{P}_o \left( \tau_o^+ = 2n \right), \quad f^{(2n-1)}(o, o) = 0, \quad \lambda \in (0, \infty),$$

and the $k$th Catalan number given by $c_k = \frac{1}{k+1} \binom{2k}{k}$, with the associated related generating function

$$C(x) := \sum_{k=0}^{\infty} c_k x^k = \frac{1 - \sqrt{1 - 4x}}{2x}, \quad x \in \left[ -\frac{1}{4}, \frac{1}{4} \right].$$

(A.1)

Note the number of all $2n$-length nearest-neighbour paths $\gamma = w_0w_1 \cdots w_{2n}$ on $\mathbb{Z}_+$ such that

$$w_0 = w_{2n} = 0, \quad w_j \geq 1, \quad 1 \leq j \leq 2n - 1$$

is precisely $c_{n-1}$. Hence for any $\lambda > 0$,

$$f^{(2n)}(o, o) = c_{n-1} \left( \frac{d-1}{d-1+\lambda} \right)^{n-1} \left( \frac{\lambda}{d-1+\lambda} \right)^{\lambda z} \lambda^2 C \left( \frac{\lambda}{d-1+\lambda} \right),$$

which readily yields (2.7) by means of Stirling’s formula.

By definition, for $\lambda > 0$,

$$U_\lambda(o, o \mid z) = \sum_{n=1}^{\infty} f^{(2n)}(o, o) z^{2n} = \sum_{n=1}^{\infty} c_{n-1} \left( \frac{d-1}{d-1+\lambda} \right)^{n-1} \left( \frac{\lambda}{d-1+\lambda} \right)^{\lambda z} \frac{\lambda z^2}{(d-1+\lambda)^2},$$

which, in view of (A.1), implies that for $|z| \leq \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$,

$$U_\lambda(o, o \mid z) = \frac{(d-1+\lambda) - \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}{2(d-1)}.$$  \quad (A.2)

Taking $z = 1$ gives that

$$\theta_{\lambda, d}(\lambda) = U_\lambda(o, o \mid 1) = \frac{\lambda \wedge (d-1)}{d-1}.$$

Notice from (A.2) that when $0 < \lambda \leq d - 1$,

$$U_\lambda \left( o, o \mid \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}} \right) = \frac{d-1+\lambda}{2(d-1)} \leq 1.$$  

Hence, for $|z| < \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$ and $0 < \lambda \leq d - 1$,

$$G_\lambda(o, o \mid z) = \frac{1}{1 - U_\lambda(o, o \mid z)}$$

$$= \frac{2(d-1)}{2(d-1) - (d-1+\lambda) + \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}.$$

(A.3)
This implies that the convergence radius for $\mathbb{G}_\lambda(o, o \mid z)$ is $\frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$. In other words,

$$\rho(\lambda) := \rho_{\mathbb{G}}(\lambda) = \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda}, \quad 0 < \lambda \leq d-1.$$ 

It remains to show (2.8) for $\lambda \in (0, d-1)$. Write $a(\lambda) = \frac{2(d-1)}{d-1+\lambda}$ and $b(\lambda) = \frac{d-1-\lambda}{d-1+\lambda}$. Then for any $|z| \leq R_{\mathbb{G}}(\lambda) = \frac{1}{\rho(\lambda)}$,

$$G_{\lambda}(o, o \mid z) = \frac{2(d-1)}{d-1+\lambda} \frac{1}{d-1} + \frac{1}{\rho(\lambda)^2 z^2} = \frac{a(\lambda)}{b(\lambda) + \frac{1}{\rho(\lambda)^2 z^2}}.$$ 

Let

$$\Phi(t) := \Phi_{\lambda}(t) = -\frac{a(\lambda)b(\lambda) + \sqrt{a(\lambda)^2 + \rho(\lambda)^2(1-b(\lambda))^2}}{1-b(\lambda)^2}, \quad t \in \mathbb{R}.$$ 

Then for any $|z| \leq R_{\mathbb{G}}(\lambda)$,

$$G_{\lambda}(o, o \mid z) = \Phi(z \mathbb{G}_\lambda(o, o \mid z)).$$

Define

$$\Psi(u, v) := \Phi(\lambda v) - v, \quad u, v \in \mathbb{R}.$$ 

Then

$$\frac{\partial \Psi(u, v)}{\partial v} \bigg|_{(u,v)=(\frac{2\pi\lambda}{a(\lambda)}, \mathbb{G}_\lambda(o, o \mid \frac{2\pi\lambda}{a(\lambda)}))} = 0,$$

$$c_1(\lambda) := \frac{\partial^2 \Psi(u, v)}{\partial v^2} \bigg|_{(u,v)=(\frac{2\pi\lambda}{a(\lambda)}, \mathbb{G}_\lambda(o, o \mid \frac{2\pi\lambda}{a(\lambda)}))} = \frac{(d-1-\lambda)^3}{2(d-1)(d-1+\lambda)^2} \neq 0,$$

$$c_2(\lambda) := \frac{\partial \Psi(u, v)}{\partial u} \bigg|_{(u,v)=(\frac{2\pi\lambda}{a(\lambda)}, \mathbb{G}_\lambda(o, o \mid \frac{2\pi\lambda}{a(\lambda)}))} = \frac{2\rho(\lambda)(d-1)}{d-1-\lambda} \neq 0.$$ 

Applying the method of Darboux (see [6] Theorem 5), we obtain that

$$p^{(2n)}_\lambda(o, o) \sim \left(\frac{c_1(\lambda)}{2\pi \rho(\lambda)c_2(\lambda)}\right)^{1/2} \rho(\lambda)^{2n}(2n)^{-3/2} = \frac{(d-1-\lambda)^2}{16(\pi\lambda)^{1/2}(d-1)^{3/2}} \rho(\lambda)^{2n-3/2}.$$ 

The idea of using the method of Darboux to establish the asymptotics for $p^{(2n)}_\lambda(o, o)$ is not new. For example, in Woess [28] Chapter III Section 17 pp. 181–189, examples of random walk on groups are given such that $p^{(n)}(o, o) \sim c \rho^{n-3/2}$ for some constant $c > 0$. The exact value of $c$ is not known in general.

For $z \in (-1, 1)$, $G_{d-1}(o, o \mid z) = \frac{1}{1-z^2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!} z^{2n}$. Thus

$$p^{(2n)}_{d-1}(o, o) = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{1}{\sqrt{\pi n}}.$$ 

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