Analyticity of parametric elliptic eigenvalue problems and applications to quasi-Monte Carlo methods

Van Kien Nguyen

Department of Mathematical Analysis, University of Transport and Communications, Hanoi, Vietnam

ABSTRACT
In the present paper, we study the analyticity of the leftmost eigenvalue of the linear elliptic partial differential operators with random coefficient and analyse the convergence rate of the quasi-Monte Carlo method for approximation of the expectation of this quantity. The random coefficient is assumed to be represented by an affine expansion $a_0(x) + \sum_{j \in \mathbb{N}} y_j a_j(x)$, where elements of the parameter vector $y = (y_j)_{j \in \mathbb{N}} \in \mathbb{U}^{\infty}$ are independent and identically uniformly distributed on $\mathbb{U} = [-\frac{1}{2}, \frac{1}{2}]$. Under the assumption $\| \sum_{j \in \mathbb{N}} \rho_j |a_j| \|_{L_\infty(D)} < \infty$ with some positive sequence $(\rho_j)_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ for $p \in (0, 1]$ we show that for any $y \in \mathbb{U}^\infty$, the elliptic partial differential operator has a countably infinite number of eigenvalues $(\lambda_j(y))_{j \in \mathbb{N}}$ which can be ordered non-decreasingly. Moreover, the spectral gap $\lambda_2(y) - \lambda_1(y)$ is uniformly positive in $\mathbb{U}^\infty$. From this, we prove the holomorphic extension property of $\lambda_1(y)$ to a complex domain in $\mathbb{C}^\infty$ and estimate partial derivatives of $\lambda_1(y)$ with respect to the parameter $y$ by using Cauchy’s formula for analytic functions. Based on these bounds we prove the dimension-independent convergence rate of the quasi-Monte Carlo method to approximate the expectation of $\lambda_1(y)$.

1. Introduction
In the last two decades there has been a tremendous growth of interest in uncertainty quantification for physical, biological, or geological models such as groundwater flow, heat transfer, or risk management in financial mathematics. Normally, these models are described by partial differential equations where the input data may be a random variable or a random field. This induces that the derived quantity of interest will in general also be a random variable or a random field. The computational goal is usually to find the expected value or high-order moments of these derived quantities in which the calculation of high-dimensional (or even infinite) integrals is required. Due to its immunity to the dimension of integration, recently, there is a huge interest in treating uncertainty quantification problems by quasi-Monte Carlo method (QMC) such as G. Dick et al. [1–3], Gantner et al. [4], Gilber et al. [5–8], Graham et al. [9–11], Herrmann and Schwab [12–14], Kazashi [15], Kuo
Let $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded Lipschitz domain. In this paper, we consider a family of real parametric eigenvalue problems (EVP) of the form
\begin{equation}
- \text{div}(a(y)(x) \nabla \omega(y)(x)) = \lambda(y) \omega(y)(x), \quad x \in D,
\end{equation}
with the homogeneous Dirichlet boundary condition, i.e. $\omega(y)(x)|_{\partial D} = 0$. We assume that the coefficient $a(y)(x)$ has an expansion of the form
\begin{equation}
a(y)(x) = a_0(x) + \sum_{j \in \mathbb{N}} y_j a_j(x),
\end{equation}
where $a_0$ and $(a_j)_{j \in \mathbb{N}}$ belong to $L_\infty(D)$ and elements of the parameter $y = (y_j)_{j \in \mathbb{N}} \in U_\infty$, $U := [-\frac{1}{2}, \frac{1}{2}]$, are independent and identically uniformly distributed on $U$. Hence, the distribution of $y$ is given by the product measure $dy = \bigotimes_{j \in \mathbb{N}} dy_j$ on $U_\infty$.

We denote by $\langle \cdot, \cdot \rangle$ the inner product in $L_2(D)$ and by $V := H^1_0(D)$ the Sobolev space of real-valued functions vanishing on the boundary in the sense of trace. The norm of the function $v \in V$ is defined by
\[ \|v\|_V := \|\nabla v\|_{L_2(D)}. \]
The dual space $H^{-1}(D)$ of $V$ is denoted by $V^*$ and $\langle \cdot, \cdot \rangle_{V \times V^*}$ is the duality pairing on $V$ and $V^*$. For $y \in U_\infty$ we define the symmetric bilinear forms $B(y, \cdot, \cdot) : V \times V \to \mathbb{R}$ by
\begin{equation}
B(y, u, v) := \int_D a(y)(x) \nabla u(x) \cdot \nabla v(x) \, dx.
\end{equation}

The variational formulation of the parametric EVP (1) reads as follows. For any given $y \in U_\infty$, find $(\lambda(y), \omega(y)) \in \mathbb{R} \times V$, with $\omega(y) \neq 0$ such that
\begin{align*}
B(y, \omega(y), v) &= \lambda(y) \langle \omega(y), v \rangle, \quad \text{for all } v \in V \\
\|\omega(y)\|_{L_2(D)} &= 1.
\end{align*}
Under some assumptions on the systems $(a_j)_{j \in \mathbb{N}}$ it will be proven (see Section 2) that for any $y \in U_\infty$ the problem (4) has a countably infinite number of eigenvalues denoted by $(\lambda_k(y))_{k \in \mathbb{N}}$. Moreover, they can be ordered in a non-decreasing sequence (associated with $y$). In the present paper, we are interested in studying the analyticity of the leftmost eigenvalue $\lambda_1(y)$ and analysing the convergence rate of approximating the expectation
\begin{equation}
\mathbb{E}_y[\lambda_1] := \int_{U_\infty} \lambda_1(y) \, dy
\end{equation}
by randomized QMC rules.

The eigenvalue problems of parametric or stochastic elliptic differential operators have been of interest for the past fifty years, see [5–8,26–34] and references therein. These problems appear in many areas of engineering and physics, for example, in nuclear reactor physics; photonics; quantum physics; acoustic; or in electromagnetic. In applications, the leftmost eigenvalue $\lambda_1$ and its corresponding eigenfunction $\omega_1$ usually have an important physical meaning, see [32,35–37]. For example, in the model of nuclear reactor the
eigenvalue \( \lambda_1 \) characterizes the physical state of the core reactor (critical, supercritical, or subcritical) while the eigenfunction \( \omega_1 \) models the associated neutron flux. Therefore, the calculation of the smallest eigenvalue \( \lambda_1 \) and its eigenfunction \( \omega_1 \) is one of the primary objectives of nuclear reactor analysis.

The analytic dependence of the eigenvalue \( \lambda_1(y) \) on the parameter \( y \) of the parametric EVP (4) has been considered in Ref. [26]. The analysis of the QMC method for \( \lambda_1(y) \) was studied in Refs [5,7,8]. However, in these mentioned papers, the authors have not taken into account the amount of overlap between the supports of the functions \((a_j)_{j \in \mathbb{N}}\). It has been observed in many situations that, for example in elliptic PDEs with stochastic diffusion coefficients [38,39], disjoint supports or finite overlap of the functions \((a_j)_{j \in \mathbb{N}}\) (such as wavelet-type representations) generally leads to simpler analysis and a lower computational cost compared to global supports representations. Motivated by this fact, the present paper aims at proving the holomorphic extensions of the QMC convergence theory of [5] accounting for possible locality of the supports of the functions \((a_j)_{j \in \mathbb{N}}\) in the representation (2). For the analysis relevant, it requires that the spectral gap \( \lambda_2(y) - \lambda_1(y) \) is bounded away from zero uniformly in \( U^\infty \). In this paper, we give a simple proof for the uniform positivity of this spectral gap under a weaker assumption compared to [5,6]. To do this we point out that the set \( \mathcal{K} = \{a(y)(x) : y \in U^\infty\} \) is compact in \( L^\infty(D) \) and the mapping \( \mathcal{K} \ni a \mapsto \lambda_2(a) - \lambda_1(a) \in \mathbb{R} \) is Lipschitz continuous.

The main tool in analysing the error of approximating the integral (5) by QMC formula is the bound on the partial derivative \( |\partial^\nu \lambda_1(y)| \) with respect to the parameter \( y \), where \( \nu = (\nu_j)_{j \in \mathbb{N}} \) is a multi-index with finitely many non-zero entries. One of such a bound was obtained in [5] where the authors have proved that

\[
|\partial^\nu \lambda_1(y)| \leq C(|\nu|!)(C_\varepsilon \beta)^\varepsilon, \quad \forall \ y \in U^\infty \tag{6}
\]

with \( \beta = (||a_j||_{L^\infty(D)})_{j \in \mathbb{N}} \) and \( \varepsilon \) arbitrarily close to zero. This estimate highly depends on \( \varepsilon \), in particular, the constant \( C_\varepsilon \) tends to infinity when \( \varepsilon \) approaches zero, see [5, Lemma 3.3.]. The assumptions for the QMC convergence in Ref. [5] relied on the \( p \)-summability of the sequence \( \beta \), in detail, it was assumed that \( \sum_{j \in \mathbb{N}} ||a_j||_{L^\infty(D)}^p < \infty \) for some \( p \in (0, 1) \). One disadvantage of the estimate (6) is that it does not allow the authors in Ref. [5] to study the convergence of the QMC quadrature in the case \( p = 1 \).

In this paper, by using Cauchy’s formula for analytic functions, we give a new bound for partial derivatives of \( \lambda_1(y) \) with respect to \( y \). Under the assumption that \( \rho = (\rho_j)_{j \in \mathbb{N}} \) is a sequence of positive numbers satisfying

\[
\left\| \sum_{j \in \mathbb{N}} \rho_j |a_j| \right\|_{L^\infty(D)} < \infty
\]

we prove that

\[
|\partial^\nu \lambda_1(y)| \leq K \frac{\nu!}{(\eta \rho)^\nu}, \quad \forall \ y \in U^\infty \tag{7}
\]

for some positive constants \( K \) and \( \eta \). Hence our analysis in this paper improves the result in Ref. [5] to the case of the sequence of functions \((a_j)_{j \in \mathbb{N}}\) having disjoint supports or finite overlap. Moreover, the estimate (7) also allows us to consider the QMC method in the case \( p = 1 \) which was left open in Ref. [5], see Remark 3.3 and Theorem 4.3.
The outline of this paper is as follows. In Section 2, we prove the well-posedness of the EVP (4) and recapitulate some basic properties of eigenpairs of this problem. In particular, in this section, we give a simple proof showing that the spectral gap of the EVP (4) is uniformly positive in $U^\infty$. Section 3 is devoted to prove the analytic dependence on the parameter $y$ of $\lambda_1(y)$ and the corresponding eigenfunction $\omega_1(y)$ under a weaker assumption where the locality in the supports of the system $(a_j)_{j \in \mathbb{N}}$ is considered. This analytic property is then employed to bound the partial derivatives of the eigenvalue $\lambda_1(y)$ and eigenfunction $\omega_1(y)$ with respect to $y \in U^\infty$. In Section 4, we apply the result in Section 3 to study the convergence of the QMC method for the expectations of $\lambda_1$ and $G(\omega_1)$, where $G$ is an element in $V^*$.

Notation: We use standard notations. We denote by $\mathbb{N}_0^\infty$ the set of all sequence $\nu = (\nu_j)_{j \in \mathbb{N}}$ with $\nu_j \in \mathbb{N}_0$. Similarly, we define $\mathbb{C}^\infty$ and $U^\infty$. Denote by $\mathbb{F}$ the set of all $\nu \in \mathbb{N}_0^\infty$ such that $\supp(\nu) = \{j \in \mathbb{N} : \nu_j \neq 0\}$ is finite. If $\nu \in \mathbb{F}$, we define

$$\nu! := \prod_{j \in \mathbb{N}} \nu_j,$$

$$|\nu| := \sum_{j \in \mathbb{N}} \nu_j,$$

and

$$\rho^\nu := \prod_{j \in \mathbb{N}} \rho_j^{\nu_j}$$

for a sequence $\rho = (\rho_j)_{j \in \mathbb{N}}$ of positive numbers.

2. Well-posedness of the parametric eigenvalue problems

The main purpose of this section is to give a condition on which the EVP (4) is well-posed and to show the uniform positivity of spectral gap of this problem.

Let $\chi_1, \chi_2, \ldots$ be eigenvalues of the negative Laplacian on $D$ with the homogeneous boundary condition. They are strictly positive and the eigenvalue $\chi_1$ is isolated and non-degenerate. By the min-max principle for the variational characterization of eigenvalues of self-adjoint operators, see, e.g. [40], we have

$$\chi_k = \min\limits_{S_k \subset V, \dim(S_k) = k} \max\limits_{0 \neq u \in S_k} \frac{\|u\|^2_V}{\|u\|^2_{L_2(D)}}. \quad (8)$$

When $k = 1$ we have the Poincaré inequality

$$\|v\|_{L_2(D)} \leq \chi_1^{-1/2} \|v\|_V, \quad \text{for } v \in V.$$ 

Throughout this paper, we use the following assumption.

Assumption 2.1: (1) The functions $a_0, (a_j)_{j \in \mathbb{N}}$ belong to $L_\infty(D)$ and there exist $\alpha_{\min}, \alpha_{\max} \in \mathbb{R}$ such that

$$0 < \alpha_{\min} \leq a_0(x) \leq \alpha_{\max} < \infty, \quad \text{for all } x \in D.$$ 

(2) There exists a positive sequence $\rho = (\rho_j)_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} \rho_j^{-1} = 0$ and $\sum_{j \in \mathbb{N}} \rho_j |a_j|$ belongs to $L_\infty(D)$. We put

$$\Lambda_1 := \left\| \sum_{j \in \mathbb{N}} \rho_j |a_j| \right\|_{L_\infty(D)} \quad \text{and} \quad \Lambda_0 := \left\| \sum_{j \in \mathbb{N}} \frac{1}{2} |a_j| \right\|_{L_\infty(D)} \quad (9)$$

and assume that $\Lambda_0 < \alpha_{\min}$. 
From Assumption 2.1 and the representation (2), we immediately obtain
\[ \alpha_{\text{min}} - \Lambda_0 \leq a(y)(x) \leq \alpha_{\text{max}} + \Lambda_0, \quad \forall \, x \in D. \]

Therefore, we have the following result.

**Lemma 2.1:** Let Assumption 2.1 hold. Then the bilinear form \( B(y, \cdot, \cdot) \) defined in (3) is coercive and bounded, uniformly in \( y \), i.e.
\[ B(y, v, v) \geq (\alpha_{\text{min}} - \Lambda_0) \| v \|_V^2, \quad \text{for all } v \in V \]

and
\[ B(y, u, v) \leq (\alpha_{\text{max}} + \Lambda_0) \| u \| \| v \|_V, \quad \text{for all } u, v \in V. \]

Let \( y \in U^\infty \). For any function \( f \in L_2(D) \), we consider the operator
\[ T(y) : L_2(D) \ni f \mapsto T(y)f \in V \subset L_2(D) \]
defined by
\[ B(y, T(y)f, v) = \langle f, v \rangle, \quad \text{for all } v \in V. \quad (10) \]

Under Assumption 2.1, it has been proved that the operator \( T(y) \) is self-adjoint, compact, and positive from \( L_2(D) \) to \( L_2(D) \), see [41, Section 1.2.2]. Then, there exist a real positive sequence \( \mu_k(y) \) converging to zero and a sequence of functions \( \omega_k(y) \) with \( \| \omega_k(y) \|_{L_2(D)} = 1 \) such that \( T(y)\omega_k(y) = \mu_k(y)\omega_k(y) \). Putting \( \lambda_k(y) = \frac{1}{\mu_k(y)} \) we obtain
\[ B(y, \omega_k(y), v) = \lambda_k(y) \langle \omega_k(y), v \rangle, \quad \text{for all } v \in V. \quad (11) \]

The pair \( (\lambda_k(y), \omega_k(y)) \in \mathbb{R} \times V \) is called the eigenpair of the bilinear form \( B(y, \cdot, \cdot) \). We have the following estimates.

**Lemma 2.2:** Under Assumption 2.1, for any \( k \in \mathbb{N} \) and any \( y \in U^\infty \) we have
\[ (\alpha_{\text{min}} - \Lambda_0) \chi_k \leq \lambda_k(y) \leq (\alpha_{\text{max}} + \Lambda_0) \chi_k \]

and
\[ \| \omega_k(y) \|_V \leq \left( \frac{(\alpha_{\text{max}} + \Lambda_0) \chi_k}{\alpha_{\text{min}} - \Lambda_0} \right)^{1/2}. \]

**Proof:** Using the min-max principle, see [42, Exercise D2, page 1543] Lemma 2.1, and (8) we obtain
\[
\lambda_k(y) = \min_{\dim(S_k) = k} \max_{\substack{S_k \subset V \\text{dim}(S_k) = k \\\text{max}}} \frac{B(y, u, u)}{\langle u, u \rangle} \\
\leq \min_{\dim(S_k) = k} \max_{\substack{S_k \subset V \\\text{max}}} \frac{(\alpha_{\text{max}} + \Lambda_0) \langle \nabla u, \nabla u \rangle}{\langle u, u \rangle} = (\alpha_{\text{max}} + \Lambda_0) \chi_k.
\]
Similarly, we have
\[
\lambda_k(y) = \min_{\dim(S_k) = k} \max_{\psi \in \mathcal{V}, \dim(S_k) = k} \frac{\mathcal{B}(y, u, u)}{\langle u, u \rangle} \geq \min_{\dim(S_k) = k} \max_{\psi \in \mathcal{V}, \dim(S_k) = k} \frac{(\alpha_{\text{min}} - \Lambda_0) \langle \nabla u, \nabla u \rangle}{\langle u, u \rangle} = (\alpha_{\text{min}} - \Lambda_0) \chi_k.
\]

Furthermore, taking \( \nu = \omega_k(y) \) as a test function in (11), we obtain \( \mathcal{B}(y, \omega_k(y), \omega_k(y)) = \lambda_k(y) \). This and Lemma 2.1 lead to
\[
(\alpha_{\text{min}} - \Lambda_0) \|\omega_k(y)\|^2 \leq \lambda_k(y)
\]
or
\[
\|\omega_k(y)\|^2 \leq \frac{\lambda_k(y)}{\alpha_{\text{min}} - \Lambda_0} \leq \frac{(\alpha_{\text{max}} + \Lambda_0) \chi_k}{\alpha_{\text{min}} - \Lambda_0}
\]
which is the needed claim.

To prove the uniformly positivity of spectral gap \( \lambda_2(y) - \lambda_1(y) \) in \( U^\infty \) we need an auxiliary lemma.

**Lemma 2.3:** Let \( \psi_j \in L_\infty(D) \) for all \( j \in \mathbb{N}_0 \). If there exists a positive sequence \( q = (q_j)_{j \in \mathbb{N}} \) such that \( \sum_{j \in \mathbb{N}} q_j \|\psi_j\| = L_\infty(D) \) and \( \lim_{j \to \infty} q_j^{-1} = 0 \), then the set
\[
\psi(U^\infty) := \left\{ \psi(y) := \psi_0 + \sum_{j \in \mathbb{N}} y_j \psi_j : y = (y_j)_{j \in \mathbb{N}} \in U^\infty \right\}
\]
is compact in \( L_\infty(D) \).

**Proof:** One can follow the argument in [43, Lemma 2.7] by showing that for every sequence in \( \psi(U^\infty) \) we can extract a subsequence whose limit belongs to \( \psi(U^\infty) \). In the following we show that the Kolmogorov \( n \)-width of the set \( \psi(U^\infty) \) in \( L_\infty(D) \) tends to zero as \( n \) tends to \( \infty \). For a Banach space \( X \) and a subset \( F \) in \( X \), the Kolmogorov \( n \)-width of \( F \) is defined by
\[
d_n(F)_X = \inf_{L_n} \sup_{f \in F} \inf_{g \in L_n} \|f - g\|_X,
\]
where the left-most infimum is taken over all \( n \)-dimensional subspaces \( L_n \) of \( X \). It has been shown that the set \( F \) is compact in \( X \) if and only if \( d_n(F)_X \to 0 \) as \( n \to \infty \) and \( F \) is bounded, see, e.g. [44, Proposition 1.2].

By the definition of the Kolmogorov \( n \)-width we have
\[
d_n(\psi(U^\infty))_{L_\infty(D)} = \inf_{L_n} \sup_{\psi(y) \in \psi(U^\infty)} \inf_{g \in L_n} \|\psi(y) - g\|_{L_\infty(D)}
\]
with the left-most infimum being taken over all \( n \)-dimensional subspaces \( L_n \) of \( L_\infty(D) \). It follows immediately that
\[
d_n(\psi(U^\infty))_{L_\infty(D)} \leq \sup_{\psi(y) \in \psi(U^\infty)} \left\|\psi(y) - \psi_0 - \sum_{j=1}^{n-1} y_j \psi_j\right\|_{L_\infty(D)}
\]
\[
\sup_{y \in U^\infty} \left\| \sum_{j=\infty}^{\infty} y_j \psi_j \right\|_{L_\infty(D)} 
\leq \frac{1}{2} \left( \sup_{j \geq n} \varrho_j^{-1} \right) \left\| \sum_{j=n}^{\infty} \varrho_j |\psi_j| \right\|_{L_\infty(D)} 
\leq C \sup_{j \geq n} \varrho_j^{-1},
\]

where in the last estimate we have used the assumption \( \sum_{j \in \mathbb{N}} \varrho_j |\psi_j| \in L_\infty(D) \) and \( C := \frac{1}{2} \left\| \sum_{j \in \mathbb{N}} \varrho_j \psi_j \right\|_{L_\infty(D)} \). Due to \( \lim_{j \to \infty} \varrho_j^{-1} = 0 \) we have \( \lim_{n \to \infty} (\sup_{j \geq n} \varrho_j^{-1}) = 0 \) which implies \( d_n(\psi(U^\infty))_{L_\infty(D)} \to 0 \) as \( n \to \infty \) and therefore the set \( \psi(U^\infty) \) is compact in \( L_\infty(D) \).

We define

\[ \mathcal{K} := \{ a(y) \in L_\infty(D), y \in U^\infty \}. \]

In the following proposition, we will show that the map

\[ \lambda_k : \mathcal{K} \ni a \mapsto \lambda_k(a) \in \mathbb{R} \]

is Lipschitz continuous with respect to \( a \). As a consequence we conclude that the spectral gap of the EVP (4) is uniformly positive in \( U^\infty \).

**Proposition 2.4:** Under Assumption 2.1, for any \( y \in U^\infty \) the EVP (4) has the following properties:

1. There are countably-many eigenvalues \( (\lambda_k(y))_{k \in \mathbb{N}} \) which are all positive, have finite multiplicity and accumulate at infinity. Counting multiplicities we can write

\[ 0 < \lambda_1(y) < \lambda_2(y) \leq \ldots. \]

Additionally, \( \lambda_1(y) \) is isolated and non-degenerate.

2. There exist four positive constants \( \gamma_{\min}, \gamma_{\max}, \delta_{\min}, \delta_{\max} \) independent of \( y \) such that

\[ 0 < \gamma_{\min} \leq \gamma(y) := \lambda_2(y) - \lambda_1(y) \leq \gamma_{\max} < \infty, \tag{12} \]

and

\[ 0 < \delta_{\min} \leq \delta(y) := \frac{\lambda_2(y) - \lambda_1(y)}{\lambda_1(y)} \leq \delta_{\max} < \infty. \tag{13} \]

3. For any \( k \in \mathbb{N} \) the eigenvalue \( \lambda_k(y) \) is Lipschitz continuous in \( y \) (with \( \ell_\infty(\mathbb{N}) \)-norm).

**Proof:** Because the bilinear form \( \mathcal{B}(y, \cdot, \cdot) \) is coercive and bounded, the first claim has been proved in [26, Section 2], see also [5, Section 2.1] (by using the Krein–Rutman theorem).
Since $T(y)$ are self-adjoint, compact, and positive operators, from [41, Theorem 2.3.1] for $y, \tilde{y} \in U^\infty$ we have

$$|\mu_k(y) - \mu_k(\tilde{y})| \leq \|T(y) - T(\tilde{y})\|_{L_2(D) \to L_2(D)}$$

which is equivalent to

$$|\lambda_k(y) - \lambda_k(\tilde{y})| \leq \lambda_k(y)\lambda_k(\tilde{y})\|T(y) - T(\tilde{y})\|_{L_2(D) \to L_2(D)}. \quad (14)$$

In the following, we will show that

$$\|T(y) - T(\tilde{y})\|_{L_2(D) \to L_2(D)} \leq \frac{1}{\sqrt{\lambda_1}} \frac{\alpha_{\max} + \Lambda_0}{(\alpha_{\min} - \Lambda_0)^2} \|a(y) - a(\tilde{y})\|_{L_\infty(D)} \quad (15)$$

by using the same argument as in the proof of [5, Proposition 2.3]. From (10) we have

$$B(y, T(y)f, v) = B(\tilde{y}, T(\tilde{y})f, v)$$

for $f \in L_2(D)$ and $v \in V$. This implies

$$B(y, T(y)f - T(\tilde{y})f, v) = B(\tilde{y}, T(\tilde{y})f, v) - B(y, T(\tilde{y})f, v)$$

$$= \int_D [a(\tilde{y})(x) - a(y)(x)] \nabla T(\tilde{y})f(x) \cdot \nabla v(x) dx.$$

Replacing $v$ by $T(y)f - T(\tilde{y})f$ we get

$$B(y, T(y)f - T(\tilde{y})f, T(y)f - T(\tilde{y})f)$$

$$= \int_D [a(\tilde{y})(x) - a(y)(x)] \nabla T(\tilde{y})f(x) \cdot \nabla [T(y)f - T(\tilde{y})f](x) dx. \quad (16)$$

From Lemma 2.1 we have

$$(\alpha_{\min} - \Lambda_0)\|T(y)f - T(\tilde{y})f\|_V^2 \leq B(y, T(y)f - T(\tilde{y})f, T(y)f - T(\tilde{y})f).$$

By the Cauchy-Schwarz inequality we can bound the right term in (16) as

$$\left| \int_D [a(\tilde{y})(x) - a(y)(x)] \nabla T(\tilde{y})f(x) \cdot \nabla [T(y)f - T(\tilde{y})f](x) dx \right|$$

$$\leq \|a(y) - a(\tilde{y})\|_{L_\infty(D)} \|T(\tilde{y})f\|_V \|T(y)f - T(\tilde{y})f\|_V.$$
Therefore we obtain
\[(\alpha_{\text{min}} - \Lambda)\|T(y)f - T(\tilde{y})f\|_V^2 \leq \|a(y) - a(\tilde{y})\|_{L_\infty(D)} \|T(\tilde{y})f\|_V \|T(y)f - T(\tilde{y})f\|_V\]
which leads to
\[\|T(y)f - T(\tilde{y})f\|_V \leq \frac{1}{\alpha_{\text{min}} - \Lambda_0} \|a(y) - a(\tilde{y})\|_{L_\infty(D)} \|T(\tilde{y})f\|_V. \tag{17}\]

Using Lax-Milgram Theorem for (10) and Poincaré inequality we find
\[\|T(\tilde{y})f\|_V \leq \frac{\alpha_{\text{max}} + \Lambda_0}{\alpha_{\text{min}} - \Lambda_0} \|f\|_V \leq \frac{1}{\sqrt{\chi_1}} \frac{\alpha_{\text{max}} + \Lambda_0}{\alpha_{\text{min}} - \Lambda_0} \|f\|_{L_2(D)}.
\]
Inserting this into (17) we find
\[\|T(y)f - T(\tilde{y})f\|_V \leq \frac{1}{\alpha_{\text{min}} - \Lambda_0} \|a(y) - a(\tilde{y})\|_{L_\infty(D)} \frac{1}{\sqrt{\chi_1}} \frac{\alpha_{\text{max}} + \Lambda_0}{\alpha_{\text{min}} - \Lambda_0} \|f\|_{L_2(D)}\]
which implies (15). Combining Lemma 2.2 with (14) we infer the existence of a constant 
\[C_k > 0\] such that \[\lambda_k(y) - \lambda_k(\tilde{y}) \leq C_k \|a(y) - a(\tilde{y})\|_{L_\infty(D)} . \tag{18}\]

Consequently, we obtain
\[\left|\lambda_2(y) - \lambda_1(y)\right| - \left|\lambda_2(\tilde{y}) - \lambda_1(\tilde{y})\right| \leq (C_1 + C_2) \|a(y) - a(\tilde{y})\|_{L_\infty(D)}\]
or with \(a = a(y)\) and \(\tilde{a} = a(\tilde{y})\) we can write
\[\left|\lambda_2(a) - \lambda_1(a)\right| - \left|\lambda_2(\tilde{a}) - \lambda_1(\tilde{a})\right| \leq (C_1 + C_2) \|a - \tilde{a}\|_{L_\infty(D)} .\]

This implies that the map \(a \mapsto \lambda_2(a) - \lambda_1(a)\) is continuous in the set \(\mathcal{K} \subset L_\infty(D)\). We know from Lemma 2.3 that \(\mathcal{K}\) is compact in \(L_\infty(D)\). Moreover, since \(0 < \lambda_2(y) - \lambda_1(y) < \infty\) for all \(y \in U_\infty\), we conclude that there exits two constants \(\gamma_{\text{min}}\) and \(\gamma_{\text{max}}\) such that \[0 < \gamma_{\text{min}} \leq \gamma(y) = \lambda_2(y) - \lambda_1(y) \leq \gamma_{\text{max}} < \infty.\]

The estimate (13) then follows from Lemma 2.2. This is the second claim.

Finally, we have
\[\|a(y) - a(\tilde{y})\|_{L_\infty(D)} = \left\| \sum_{j \in \mathbb{N}} (y_j - \tilde{y}_j) a_j \right\|_{L_\infty(D)} \leq 2 \|y - \tilde{y}\|_{\ell_\infty(\mathbb{N})} \left\| \sum_{j \in \mathbb{N}} \frac{1}{2} |a_j| \right\|_{L_\infty(D)} = 2 \Lambda_0 \|y - \tilde{y}\|_{\ell_\infty(\mathbb{N})}.\]

Inserting this into (18) we obtain the last claim. The proof is finished. \(\blacksquare\)
3. Parametric analyticity and bound of mixed derivatives

The analytic dependence of the eigenpair \((\lambda_1(y), \omega_1(y))\) on the parameter \(y\) of the parametric EVP (4) has been proved in Ref. [26] under the assumption that \(\sum_{j \in \mathbb{N}} \|a_j\|_{L^p(D)}^p < \infty, \quad p \in (0, 1)\). In this section, we will extend this result to a weaker assumption where the locality in the supports of the system \((a_j)_{j \in \mathbb{N}}\) is considered. Afterward, we use Cauchy’s formula to bound the partial derivatives of the eigenvalue \(\lambda_1(y)\) and eigenfunction \(\omega_1(y)\) with respect to \(y \in U^\infty\). We assume in this section that \(L^2(D)\) and \(V\) are complex-valued function spaces. We consider the coefficients of the form

\[
a(z)(x) = a_0(x) + \sum_{j \in \mathbb{N}} z_j a_j(x),
\]

where \(z = (z_j)_{j \in \mathbb{N}} \in \mathbb{C}^\infty\). We define the associated sesquilinear forms \(B(z, \cdot, \cdot)\) and \(B_j(\cdot, \cdot)\) from \(V \times V\) to \(\mathbb{C}\) by

\[
B(z, u, v) := \int_D a(z)(x) \nabla u(x) \cdot \nabla \overline{v(x)} \, dx
\]

and

\[
B_j(u, v) := \int_D a_j(x) \nabla u(x) \cdot \nabla \overline{v(x)} \, dx.
\]

Let \(\mathcal{L}(V, V^*)\) denote the set of all continuous linear mappings from \(V\) to \(V^*\). We define \(A(z) \in \mathcal{L}(V, V^*)\) for \(j \in \mathbb{N}_0\) the operators corresponding to \(B(z, \cdot, \cdot)\) and \(B_j(\cdot, \cdot)\) by identifications

\[
B(z, u, v) = \langle u, A(z)v \rangle_{V \times V^*}, \quad B_j(u, v) = \langle u, A_jv \rangle_{V \times V^*}, \quad \text{for } u, v \in V.
\]

The complex version of the EVP (4) reads as follows. Find \((\lambda(z), \omega(z)) \in \mathbb{C} \times V\), with \(\omega(z) \neq 0\) such that

\[
B(z, \omega(z), v) = \lambda(z) \langle \omega(z), v \rangle, \quad \text{for all } v \in V
\]

\[
\|\omega(z)\|_{L^2(D)} = 1. \tag{19}
\]

This problem is well-posed as long as there exist positive constants \(C\) and \(\gamma\) (might depend on \(z\)) such that

\[
|B(z, u, v)| \leq C \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V
\]

\[
\inf_{0 \neq u \in V} \sup_{0 \neq v \in V} \frac{|B(z, u, v)|}{\|u\|_V \|v\|_V} \geq \gamma,
\]

and

\[
\sup_{u \in V} |B(z, u, v)| > 0, \quad \text{for all } 0 \neq v \in V.
\]

see [26, Section 2].

Before formulating our main results in this section we recall the notion of separate holomorphy of countable product spaces over \(\mathbb{C}\). Let \((Z_j)_{j \in \mathbb{N}}\) be a family of Banach spaces over
and \( Y \) also a Banach space over \( \mathbb{C} \). Let \( S \subset \bigotimes_{j \in \mathbb{N}} Z_j \) be an open set and \( z = (z_j)_{j \in \mathbb{N}} \in S \). For a finite set \( J \subset \mathbb{N} \) we denote

\[
S_J(z) := \{(y_j)_{j \in J} : \exists (\nu_j)_{j \in \mathbb{N}} \in S \text{ with } \nu_j = y_j, \ j \in J \text{ and } \nu_j = z_j, \ j \notin J \}.
\]

We say that the map \( u : S \to Y \) is separately holomorphic if for every finite set \( J \subset \mathbb{N} \) and \( z \in S \) the map \( u \) is holomorphic as a function of variables in \( S_J(z) \). We have the following result.

**Theorem 3.1:** Let Assumption 2.1 hold and \( \gamma_{\text{max}}, \delta_{\text{min}} \) be given in Proposition 2.4. For \( \varepsilon \in (0, 1) \) we put \( \kappa := \frac{1 - \varepsilon}{2(1 + \delta_{\text{min}}^{-1})} < 1 \) and define the sequence \( \tau = (\tau_j)_{j \in \mathbb{N}} \) where

\[
\tau_j := \eta_{\varepsilon} \rho_j \quad \text{with} \quad \eta_{\varepsilon} := (1 - \varepsilon) \frac{\alpha_{\text{min}} - \Lambda_0}{2\Lambda_1(1 + \delta_{\text{min}}^{-1})}
\]

and

\[
\mathcal{E}(\tau) := \bigotimes_{j \in \mathbb{N}} \mathcal{E}_j(\tau) \quad \text{with} \quad \mathcal{E}_j(\tau) := \left\{ z_j \in \mathbb{C} : \text{dist} \left( z_j, \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) < \tau_j \right\}.
\]

Then the eigenpair \( (\lambda_1(y), \omega_1(y)) \) of the EVP (4), as functions of \( y \), can be extended to separate holomorphic functions on \( \mathcal{E}(\tau) \). Moreover, we have

\[
\sup_{z \in \mathcal{E}(\tau)} |\lambda_1(z)| \leq \frac{\gamma_{\text{max}}}{2} + (\alpha_{\text{max}} + \Lambda_0) \chi_1 =: K_\lambda,
\]

and

\[
\sup_{z \in \mathcal{E}(\tau)} \|\omega_1(z)\|_V \leq \left( \frac{\gamma_{\text{max}} + 2(\alpha_{\text{max}} + \Lambda_0) \chi_1}{2(1 - \kappa)(\alpha_{\text{min}} - \Lambda_0)} \right)^{1/2} =: K_\omega.
\]

**Proof:** We follow the proof of [26, Theorem 2.13]. For \( z = (z_j)_{j \in \mathbb{N}} \in \mathcal{E}(\tau) \) we take \( y = (y_j)_{j \in \mathbb{N}} \in U^\infty \) such that \( |z_j - y_j| < \tau_j \) and denote \( \xi := z - y = (\xi_j)_{j \in \mathbb{N}} \). Hence, \( A(z) \) can be written as \( A(z) = A(y) + B(\xi) \), where \( B(\xi) := \sum_{j \in \mathbb{N}} \xi_j A_j \). Define the complex-analytic operator-valued function

\[
t \mapsto A(y) + tB(\xi).
\]

From Lemma 2.1 we know that

\[
\langle v, A(y)v \rangle_{V^* \times V} \geq (\alpha_{\text{min}} - \Lambda_0) \|v\|_V^2, \quad \text{for all } v \in V.
\]

We now estimate the norm \( \|B(\xi)\|_{\mathcal{L}(V, V^*)} \). Since \( |\xi_j| < \tau_j \) for all \( j \in \mathbb{N} \), under Assumption 2.1 we obtain

\[
\langle u, B(\xi)v \rangle_{V^* \times V} \leq \int_D \left| \sum_{j \in \mathbb{N}} \tau_j a_j(x) \right| |\nabla u(x) \cdot \nabla v(x)| \, dx \leq \eta_{\varepsilon} \Lambda_1 \int_D |\nabla u(x) \cdot \nabla v(x)| \, dx
\]

\[
\leq \frac{(1 - \varepsilon)(\alpha_{\text{min}} - \Lambda_0)}{2(1 + \delta_{\text{min}}^{-1})} \|u\|_V \|v\|_V = \kappa (\alpha_{\text{min}} - \Lambda_0) \|u\|_V \|v\|_V.
\]
This implies
\[ \|B(\xi)\|_{\mathcal{L}(V, V^*)} \leq \kappa (\alpha_{\min} - \Lambda_0). \]

Therefore, according to [26, Theorem 2.6 and Corollary 2.8] there is a complex-valued function \( t \rightarrow \tilde{\lambda}_1(y + t\xi) \) depending holomorphically on the parameter \( t \) in the disk
\[
\left\{ t \in \mathbb{C}, |t| < \frac{1}{1 - \varepsilon} \right\} = \left\{ t \in \mathbb{C}, |t| < \frac{1}{2\kappa(1 + \delta_{\min}^{-1})} \right\}.
\]

Moreover, we have \( \tilde{\lambda}_1(y + t\xi) \) is an isolated and non-degenerated eigenvalue of \( A(y) + tB(\xi) \).

Thus \( \tilde{\lambda}(y + \xi) \) is a candidate for the holomorphic extension of \( \lambda_1(z) \) of the parametric eigenvalue. It has been shown in [26, Theorem 2.13] that \( \tilde{\lambda}(y + \xi) \) is independent of the choice \( y \in U^\infty \) and \( \xi \) satisfying \( z = y + \xi \). Therefore, \( \lambda_1(z) := \tilde{\lambda}(y + \xi) \) is well-defined. Using the same argument at the end of the proof of [26, Theorem 2.13] we obtain the holomorphic extension of \( \lambda_1 \) on \( \mathcal{E}(\tau) \).

Similar considerations apply for eigenfunction \( \omega_1 \). Checking the proofs of [26, Theorem 2.6] and [45, Theorems XII.8 and XII.11] we find that \( \lambda_1(z) \) satisfies
\[ |\lambda_1(z) - \lambda_1(y)| \leq \frac{1}{2} \gamma(y) \leq \frac{\gamma_{\max}}{2}, \]
where \( \gamma_{\max} \) is given in (12). Consequently, we obtain from Lemma 2.2
\[ |\lambda_1(z)| \leq \frac{\gamma_{\max}}{2} + \lambda_1(y) \leq \frac{\gamma_{\max}}{2} + (\alpha_{\max} + \Lambda_0) \chi_1. \]  
\[ \text{(25)} \]

To show uniformly boundedness of \( \|\omega_1(z)\|_V \) in \( \mathcal{E}(\tau) \) we use \( v = \omega_1(z) \) as a test function in (19) to get
\[ |B(z, \omega_1(z), \omega_1(z))| = |\lambda_1(z)|. \]  
\[ \text{(26)} \]

Since \( A(z) = A(y) + B(\xi) \), we have
\[ B(z, \omega_1(z), \omega_1(z)) = \langle \omega_1(z), A(z)\omega_1(z) \rangle_{V \times V^*} \]
\[ = \langle \omega_1(z), A(y)\omega_1(z) \rangle_{V \times V^*} + \langle \omega_1(z), B(\xi)\omega_1(z) \rangle_{V \times V^*} \]

From (23) and (24) we get
\[ |B(z, \omega_1(z), \omega_1(z))| \geq (\alpha_{\min} - \Lambda_0)(1 - \kappa)\|\omega_1(z)\|_V^2 \]
which together with (26) and (25) leads to
\[ \|\omega_1(z)\|_V^2 \leq \frac{|B(z, \omega_1(z), \omega_1(z))|}{(1 - \kappa)(\alpha_{\min} - \Lambda_0)} = \frac{|\lambda_1(z)|}{(1 - \kappa)(\alpha_{\min} - \Lambda_0)} \leq \frac{\gamma_{\max}}{2} + (\alpha_{\max} + \Lambda_0) \chi_1. \]

The proof is completed.  

The analyticity of the eigenpair \((\lambda_1(y), \omega_1(y))\) leads to the following.
**Theorem 3.2:** Let Assumption 2.1 hold. Then for any \( y \in U^\infty \) and any \( v \in \mathbb{F} \) the partial derivative of eigenvalue \( \lambda_1(y) \) and eigenfunction \( \omega_1(y) \) of the EVP (4) can be estimated as

\[
|\partial^p \lambda_1(y)| \leq K_{\lambda} \frac{v!}{(\eta \rho)^{\nu}} \quad \text{and} \quad \|\partial^p \omega_1(y)\|_V \leq K_{\omega} \frac{v!}{(\eta \rho)^{\nu}},
\]

where \( \eta := \frac{\alpha_{\min} - \Lambda_0}{2\Lambda_1(1 + \delta_{\min})} \) and \( K_{\lambda} \) and \( K_{\omega} \) are given in (21) and (22).

**Proof:** Let \( \varepsilon \in (0, 1) \) and \( \eta \varepsilon \) be given in (20). From Theorem 3.1 we know that eigenpair \((\lambda_1(y), \omega_1(y))\) of the EVP (4) can be extended to separately complex-analytic functions on \( \mathcal{E}(\tilde{\tau}) \) with \( \tilde{\tau} = \eta \varepsilon / 2 \rho \). Hence, for any \( y \in U^\infty \) and \( v \in \mathbb{F} \) with \( u := \text{supp}(v) \) applying Cauchy's formula gives

\[
\partial^p \lambda_1(y) = \frac{v!}{(2\pi i)^{|v|}} \int_{C_u(y, \tau)} \frac{\lambda_1(z_u)}{\prod_{j \in u}(z_j - y_j)^{\nu_j} + 1} \prod_{j \in u} dz_j,
\]

where

\[
C_u(y, \tau) := \bigotimes_{j \in u} C_j(y, \tau) \quad \text{with} \quad C_j(y, \tau) := \{z_j \in \mathbb{C} : |z_j - y_j| = \tau_j\},
\]

and

\[
z_u \in C_u(y, \tau) := \{(z_j)_{j \in \mathbb{N}} \in \mathbb{C}^\infty : z_j \in C_j(y, \tau) \text{ if } j \in u \text{ and } z_j = y_j \text{ if } j \notin u\}.
\]

Using (21) we obtain

\[
|\partial^p \lambda_1(y)| \leq \frac{v!}{(2\pi i)^{|v|}} \sup_{z_u \in C_u(y, \tau)} |\lambda_1(z_u)| \int_{C_u(y, \tau)} \prod_{j \in u} dz_j \prod_{j \in u} |z_j - y_j|^{\nu_j} + 1
\]

\[
\leq K_{\lambda} \frac{v!}{\tau^v} = K_{\lambda} \frac{v!}{(\eta \varepsilon \rho)^v}.
\]

Similar considerations give an estimate for \( \|\partial^p \omega_1(y)\|_V \). Since these bounds hold for any \( \varepsilon \in (0, 1) \) we obtain the desired results. The proof is completed. \( \blacksquare \)

**Remark 3.3:** We give a comment when the system \((a_j)_{j \in \mathbb{N}}\) has arbitrary supports. Let \( \beta_j = \|a_j\|_{L^\infty(D)} \) and \( \beta = (\beta_j)_{j \in \mathbb{N}} \). Assume that the second condition in (9) is replaced by \( \Lambda_0 := \sum_{j \in \mathbb{N}} \frac{1}{2} \beta_j < \alpha_{\min} \). This assumption guarantees that the set \( \mathcal{K} := \{a(y) \in L^\infty(D), \ y \in U^\infty\} \) is compact in \( L^\infty(D) \), see [43, Lemma 2.7]. As a consequence, Proposition 2.4 holds. Now for each \( v \in \mathbb{F} \) fixed we define the sequence \( \rho_v = (\rho_j)_{j \in \mathbb{N}} \) with \( \rho_j = \frac{v_j}{|v| \beta_j} \) if \( j \in \text{supp}(v) \) and \( \rho_j = 0 \) otherwise. From this we have \( \Lambda_1 := \|\sum_{j \in \mathbb{N}} \rho_j |a_j|\|_{L^\infty(D)} = 1 \).
Next, following argument in the proof of Theorems 3.1 and 3.2 we can show that
\[ |\partial^\nu \lambda_1(y)| \leq K_\lambda \left| \frac{\nu!}{(\eta \rho \nu)^\nu} \right|, \]
with \( \eta := \frac{\alpha_{\min} - \Lambda_0}{2(1 + \delta_{\min})} \). Employing the estimate \( |v| \leq e^{v} \), see, e.g. [43, Page 61] we get
\[ |\partial^\nu \lambda_1(y)| \leq K_\lambda |\nu|! \left( \frac{e^\beta}{\eta} \right)^\nu. \]

A similar argument applies for \( \omega_1(y) \). This estimate improves (6) and can be used to consider the QMC error in the case \( (\beta_j)_{j \in \mathbb{N}} \in \ell_1(\mathbb{N}) \) which is excluded in Ref. [5].

4. Analysis of the quasi-Monte Carlo method

In this section we apply the estimate of partial derivatives of \( \lambda_1(y) \) and \( \omega_1(y) \) with respect to the parameter \( y \) to analyse the convergence rate of QMC method for \( \mathbb{E}_y(\lambda_1) \) and \( \mathbb{E}_y(\mathcal{G}(\omega_1)) \) where \( \mathcal{G} \in V^* \).

We review some basic results about the QMC quadratures for approximating the \( s \)-dimensional integrals, following [16]. For a measurable function \( F : U^s \to \mathbb{R} \) we seek to approximate the integral of the form
\[ I_s(F) = \int_{U^s} F(\xi) d\xi. \]
To approximate \( I_s(F) \) we use the randomly shifted lattice rule which is given by the QMC quadrature
\[ Q_{s,N}^\Delta(F) = \frac{1}{N} \sum_{i=1}^{N} F\left( \left\{ \frac{iz}{N} + \Delta \right\} - \frac{1}{2} \right), \quad (27) \]
where \( z \in \mathbb{N}^s \) is the generating vector and \( \Delta \) is a random shift which is uniformly distributed over the cube \((0, 1)^s\). The braces in (27) indicate that we take the fractional parts of each component in a vector. We want to evaluate the root-mean-square error given by
\[ \sqrt{\mathbb{E}^\Delta(|I_s(F) - Q_{s,N}^\Delta(F)|^2)} \]
where \( \mathbb{E}^\Delta \) is the expectation with respect to the random shift \( \Delta \).

It is well-known that good randomly shifted lattice rules can be constructed to achieve the optimal rate of convergence close to \( \mathcal{O}(n^{-1}) \) provided that the integrand belongs to a certain weighted Sobolev space. Denote \( [s] = \{1, \ldots, s\} \) and let \( y = (\gamma_u)_{u \subseteq [s]} \) be a sequence of positive weights. We define the weighted Sobolev space of mixed first order derivatives \( \mathcal{W}_{\mathcal{V}}(U^s) \) as the collection of all functions \( F : U^s \to \mathbb{R} \) such that
\[ \|F\|_{\mathcal{W}_{\mathcal{V}}(U^s)}^2 = \sum_{u \subseteq [s]} \frac{1}{\gamma_u} \left( \int_{U^{|u|}} \left| \frac{\partial^{|u|} F}{\partial \xi_u} (\xi) \right| d\xi_u \right)^2 < \infty. \]
Here \( \bar{u} := [s] \setminus u \) and \( \frac{\partial^{|u|} F}{\partial \xi_u} \) denotes the mixed first partial derivatives of \( F \) with respect to the variable \( \xi_u = (\xi_j)_{j \in u} \). The weight sequence \( (\gamma_u)_{u \subseteq \bar{u}} \) is associated with each subset of variables to moderate the relative importance between the different sets of variables. With an
appropriate choice of weight, we can get the error bound independent of the dimension \( s \). Moreover, we need some structure of the weight for the Component-by-Component (CBC) construction cost to be feasible. Different types of weights have been considered depending on the problem and the estimation of \( \frac{\partial |F|}{\partial \nu} \). In the case of product weights, the cost of the fast CBC algorithm for constructing a randomly shifted lattice rule with \( N \) points is \( \mathcal{O}(sN \log N) \) while \( \mathcal{O}(sN \log N + s^2N) \) operations needed in the case of product and order-dependent weight, see [16, Section 5].

We have the following result on the error of the QMC quadrature (27), see, e.g. [16, Theorem 5.1].

**Proposition 4.1:** Let \( s \in \mathbb{N} \) and \( (\gamma_j)_{j=1}^s \) be a positive sequence. We define the product weight by \( \mathbf{y} = (\gamma_j)_{u \subseteq [s]} \) where \( \gamma_u = \prod_{j \in u} \gamma_j \). Then a randomly shifted lattice rule with \( N \) points can be constructed in \( \mathcal{O}(sN \log N) \) operations using the fast CBC algorithm such that for every \( F \in \mathcal{W}_\mathbf{y}(U^s) \) and for every \( \lambda \in (\frac{1}{2}, 1] \) there holds the error bound

\[
\sqrt{\mathbb{E}^\Delta (|I_\lambda(F) - Q_{sN}^\Delta(F)|^2)} \leq \left( \sum_{u \subseteq [s]} \gamma_u^\lambda \left( \frac{2 \zeta(2\lambda)}{(2\pi^2)^\lambda} \right)^{|u|} \right)^{\frac{1}{2\lambda}} \varphi(N)^{-\frac{1}{2\lambda}} \| F \|_{\mathcal{W}_\mathbf{y}(U^s)},
\]

where \( \varphi(N) \) denotes Euler's totient function and \( \zeta(x) := \sum_{k \in \mathbb{N}} k^{-x} \) denotes the Riemann zeta function.

**Remark 4.2:** It is known that for any fixed \( \delta \in (0, 1) \), we have \( \frac{\varphi(N)}{N^\delta} \to \infty \) when \( N \to \infty \). If \( N \) is a prime then we have \( \varphi(N) = N - 1 \). We can verify that \( \varphi(N) > \frac{N}{2} \) for \( N \leq 10^{30} \). Hence, in practice one can replace \( \varphi(N) \) by \( N \) multiplying with an appropriate constant.

For \( \mathbf{y} = (y_j)_{j \in \mathbb{N}} \in U^\infty \) we denote \( \mathbf{y}_s = (y_1, \ldots, y_s, 0, \ldots) \) and \( \mathbf{y}_s = (0, \ldots, 0, y_{s+1}, y_{s+2}, \ldots) \), \( \lambda_{1,s}(\mathbf{y}) := \lambda_1(\mathbf{y}_s) \) and \( \omega_{1,s} := \omega_1(\mathbf{y}_s) \). Our result in this section reads as follows.

**Theorem 4.3:** Let \( s \in \mathbb{N} \) and \( N \in \mathbb{N} \) be prime, \( \mathcal{G} \in V^* \). Let Assumption 2.1 hold with a non-increasing sequence \( (\rho_j^{-1})_{j \in \mathbb{N}} \) and \( (\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N}) \) for \( p \in (0, 1] \). Then a randomly shifted lattice rule with \( N \) points can be constructed in \( \mathcal{O}(sN \log N) \) operations using the fast CBC algorithm such that

\[
\sqrt{\mathbb{E}^\Delta (|E_{\mathcal{G}}[\lambda_{1}] - Q_{sN}^\Delta(\lambda_{1,s})|^2)} \leq C_\lambda \left( \min \left\{ \rho_{s+1}^{-1}, s^{-2(\frac{1}{p} - 1)} \right\} + N^{-\alpha} \right)
\]

and

\[
\sqrt{\mathbb{E}^\Delta (|E_{\mathcal{G}}[\mathcal{G}(\omega_1)] - Q_{sN}^\Delta(\mathcal{G}(\omega_{1,s}))|^2)} \leq C_\omega \begin{cases} s^{-2(\frac{1}{p} - 1)} + N^{-\alpha} & \text{if } p < 1 \\ \left( \sum_{j=s+1}^\infty \rho_j^{-1} \right)^2 + N^{-\frac{1}{2}} & \text{if } p = 1 \end{cases}
\]

where

\[
\alpha = \begin{cases} 1 - \delta, \text{ for arbitrary } \delta \in (0, \frac{1}{2}) & \text{if } p \in (0, \frac{2}{3}] \\ \frac{1}{p} - \frac{1}{2} & \text{if } p \in (\frac{2}{3}, 1] \end{cases}
\]

and the positive constants \( C_\lambda \) and \( C_\omega \) are independent of \( s \) and \( N \).
Before proving the above theorem, we need a truncation estimation. Using the analyticity of $\lambda_1(y)$ and $\omega_1(y)$ we can prove the following.

**Lemma 4.4:** Let $s \in \mathbb{N}$ and Assumption 2.1 hold. Let $G \in V^\ast$. Assume that $(\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ with $p \in (0, 1]$ and $(\rho_j^{-1})_{j \in \mathbb{N}}$ is non-increasing. If $p \in (0, 1)$ then we have

$$|E_y[\lambda_1 - \lambda_{1,s}]| \leq C_0^2 \frac{K_\lambda}{4\eta^2} s^{-2(\frac{1}{p} - 1)}$$

and

$$|E_y[G(\omega_1) - G(\omega_{1,s})]| \leq C_0^2 \frac{\|G\|_{V^\ast} K_\omega}{4\eta^2} s^{-2(\frac{1}{p} - 1)},$$

(30)

where $C_0 = \min(\frac{p}{1-p}, 1)\|\rho_j^{-1}\|_{\ell_p(\mathbb{N})}$ and $K_\lambda, K_\omega$ are given in Theorem 3.1 and $\eta$ in Theorem 3.2. When $p = 1$ it holds

$$|E_y[\lambda_1 - \lambda_{1,s}]| \leq \frac{K_\lambda}{4\eta^2} \left( \sum_{j=s+1}^{\infty} \rho_j^{-1} \right)^2$$

and

$$|E_y[G(\omega_1) - G(\omega_{1,s})]| \leq \frac{\|G\|_{V^\ast} K_\omega}{4\eta^2} \left( \sum_{j=s+1}^{\infty} \rho_j^{-1} \right)^2.$$

**Proof:** We prove (30). The other bounds are carried out similarly. First, we recall Stechkin’s estimate

$$\sum_{j=s+1}^{\infty} \rho_j^{-1} \leq \min\left( \frac{p}{1-p}, 1 \right) \|\rho_j^{-1}\|_{\ell_p(\mathbb{N})} s^{-\left(\frac{1}{p} - 1\right)},$$

(31)

see [19, Theorem 5.1]. Since $\omega_1(y)$ is analytic, see Theorem 3.1, by Taylor’s theorem with integral form of the remainder we obtain

$$\omega_1(y) = \omega_1(y_s) + \sum_{j=s+1}^{\infty} y_j \int_0^1 \partial^j \omega_1(y_s + ty_s) dt,$$

where $e_j = (\delta_{j,\ell})_{\ell \in \mathbb{N}}$ and $\delta_{j,\ell}$ denotes the Kronecker delta. Employing Theorem 3.2 with the fact that $|y_j| \leq \frac{1}{2}$ yields

$$\|\omega_1(y) - \omega_{1,s}(y)\|_V \leq \frac{1}{2} \sum_{j=s+1}^{\infty} \int_0^1 \|\partial^j \omega_1(y_s + ty_s)\|_V dt$$

$$\leq \frac{1}{2} \sum_{j=s+1}^{\infty} \frac{K_\omega}{\eta} \rho_j^{-1} \leq \frac{K_\omega}{2\eta} C_0 s^{-(\frac{1}{p} - 1)},$$
where in the last inequality we used (31). It has been proved in [5, Theorem 4.1] that
\[
\mathbb{E}_y\{G(\omega_1) - G(\omega_{1,s})\} = \sum_{\ell,j=s+1}^{\infty} \mathbb{E}_y \left[ G \left( \frac{2y_\ell y_j}{(e_\ell + e_j)!} \int_0^1 (1-t) \partial^{e_\ell+e_j} \omega_1(y_s + ty_\ell) dt \right) \right].
\]

By the linearity of $G$, from Theorem 3.2 we obtain
\[
\left| \mathbb{E}_y\{G(\omega_1) - G(\omega_{1,s})\} \right| \leq \frac{1}{4} \|G\|_{V^*} \frac{K_\omega}{\eta^2} \rho^{-1} \sum_{j=s+1}^{\infty} \rho_j^{-1}
\]
\[
= \frac{\|G\|_{V^*} K_\omega}{4\eta^2} \left( \sum_{j=s+1}^{\infty} \rho_j^{-1} \right)^2 \leq \frac{\|G\|_{V^*} K_\omega}{4\eta^2} C_0 s^{-2(\frac{1}{2}-1)}.
\]

This leads to
\[
\left| \mathbb{E}_y\{G(\omega_1) - G(\omega_{1,s})\} \right| \leq \frac{1}{4} \|G\|_{V^*} K_\omega \rho^{-1} \sum_{j=s+1}^{\infty} \rho_j^{-1}
\]
\[
= \frac{\|G\|_{V^*} K_\omega}{4\eta^2} \left( \sum_{j=s+1}^{\infty} \rho_j^{-1} \right)^2 \leq \frac{\|G\|_{V^*} K_\omega}{4\eta^2} C_0 s^{-2(\frac{1}{2}-1)}.
\]

The proof is finished.

We also have the following.

**Lemma 4.5:** Let $s \in \mathbb{N}$. Under Assumption 2.1, we have
\[
\left| \mathbb{E}_y(\lambda_{1} - \lambda_{1,s}) \right| \leq C_1 \frac{A_1}{2} \sup_{j \geq s+1} \rho_j^{-1},
\]
where $C_1$ is given in (18).

**Proof:** From (18) we have
\[
\left| \lambda_{1}(y) - \lambda_{1,s}(y) \right| \leq C_1 \|a(y) - a(y_s)\|_{L_\infty(D)} = C_1 \left\| \sum_{j=s+1}^{\infty} y_j a_j \right\|_{L_\infty(D)}
\]
\[
\leq \frac{C_1}{2} \left( \sup_{j \geq s+1} \rho_j^{-1} \right) \left\| \sum_{j=s+1}^{\infty} |\rho_j a_j| \right\|_{L_\infty(D)} \leq C_1 \frac{A_1}{2} \sup_{j \geq s+1} \rho_j^{-1}.
\]

This leads to the desired result.

**Remark 4.6:** By using estimates in Remark 3.3 we obtain a similar result as [5, Theorem 4.1] for $p \in (0, 1]$. Note that the constant $C_{trunc}$ in [5, Theorem 4.1] depends on the other constant $C_e$ and $C_{trunc}$ blows up when $e$ goes to zero.

We are now in the position to prove Theorem 4.3.
\textbf{Proof:} We proof the error bound (28). The error bound (29) is carried out similarly. Using triangle inequality we get
\[
\sqrt{\mathbb{E}^A(|E_y[\lambda_1] - Q_{s,N}^A(\lambda_{1,s})|^2)} \\
\leq C \left( \sqrt{\mathbb{E}^A(|E_y[\lambda_1] - E_y[\lambda_{1,s}]|^2)} + \sqrt{\mathbb{E}^A(E_y[\lambda_1,s] - Q_{s,N}^A(\lambda_{1,s})|^2)} \right).
\]

Since the first term on the right-hand side is independent of the random shift and \((\rho_j^{-1})_{j \in \mathbb{N}}\) is non-increasing, by Lemmas 4.5 and 4.4 we get
\[
\sqrt{\mathbb{E}^A(|E_y[\lambda_1] - E_y[\lambda_{1,s}]|^2)} = \sqrt{|E_y[\lambda_1] - E_y[\lambda_{1,s}]|^2} \leq \tilde{C} \min \left\{ \rho_{s+1}^{-1}, \rho^{-2(1/p-1)} \right\}
\]
for some positive constant \(\tilde{C}\). For \(u \subset [s]\) we put \(v = (\nu_j)_{j \in \mathbb{N}} \in \mathbb{F}\) with \(\nu_j \leq 1\) and \(\text{supp}(v) = u\). From Theorem 3.2 we get
\[
\left| \frac{\partial |u| \lambda_{1,s}}{\partial \xi_u} (\xi) \right| \leq K_1 \frac{\nu^!}{(\eta \rho)^\nu} = K_1 \prod_{j \in u} \frac{1}{\eta \rho_j}, \quad \xi = (\xi_1, \ldots, \xi_s)
\]
which implies
\[
\|\lambda_{1,s}\|_{\mathcal{V}_y(U^n)}^2 = \sum_{u \subset [s]} \frac{1}{\gamma_u} \int_{U^{|u|}} \left( \int_{U^{|u|}} K_1 \prod_{j \in u} \frac{1}{\eta \rho_j} d\xi_u \right)^2 d\xi_u \leq \sum_{u \subset [s]} \frac{1}{\gamma_u} \left( K_1 \prod_{j \in u} \frac{1}{\eta \rho_j} \right)^2.
\]
By Proposition 4.1 we find
\[
\sqrt{\mathbb{E}^A(E_y[\lambda_{1,s}] - Q_{s,N}^A(\lambda_{1,s})|^2)} \leq K_\lambda \left( \sum_{u \subset [s]} \gamma_u^2 \left( \frac{2\zeta(2\lambda)}{(2\pi^2)^\lambda} \right)^{|u|} \right)^{\frac{1}{2\lambda}} \left( \sum_{u \subset [s]} \frac{1}{\gamma_u} \prod_{j \in u} \frac{1}{(\eta \rho_j)^2} \right)^{\frac{1}{2}} N^{-\frac{1}{2\lambda}}.
\]
Choosing
\[
\gamma_j = \left( \frac{1}{(\eta \rho_j)^2} : \frac{2\zeta(2\lambda)}{(2\pi^2)^\lambda} \right)^{\frac{1}{1+\frac{1}{\lambda}}}, \quad j \in u
\]
we get
\[
\sqrt{\mathbb{E}^A(E_y[\lambda_{1,s}] - Q_{s,N}^A(\lambda_{1,s})|^2)} \leq K_\lambda \left( \sum_{u \subset [s]} \prod_{j \in u} (\eta \rho_j)^{-\frac{2\lambda}{1+\lambda}} \left( \frac{2\zeta(2\lambda)}{(2\pi^2)^\lambda} \right)^{\frac{1}{1+\lambda}} \right) \left( \sum_{j \in \mathbb{N}} (\eta \rho_j)^{-\frac{2\lambda}{1+\lambda}} \right) N^{-\frac{1}{2\lambda}}
\]
where we have applied [19, Lemma 6.3] in the second inequality. We consider two cases. If \(p \in (0, \frac{2}{3}]\), for \(d \in (0, \frac{1}{2})\) we choose \(\lambda = \frac{1}{2(1-d)}\). By this choice it is easily seen that \(\frac{2\lambda}{1+\lambda} \geq \frac{2}{3}\). Therefore \(\sum_{j \in \mathbb{N}} (\eta \rho_j)^{-\frac{2\lambda}{1+\lambda}} < \infty\) since \((\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})\). If \(p \in (\frac{2}{3}, 1]\) we choose \(\lambda\) such that \(\frac{2\lambda}{1+\lambda} = p\) which implies \(\frac{1}{2\lambda} = \frac{1}{p} - \frac{1}{2}\). The proof is completed. \(\blacksquare\)
Acknowledgments

A part of this work was done when the author was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for providing a fruitful research environment and working condition.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This research is funded by Vietnam Ministry of Education and Training [grant number B2023-CTT-08.]

References

[1] Dick J, Kuo FY, Le Gia QT, et al. Higher order QMC Galerkin discretization for parametric operator equations. SIAM J Numer Anal. 2014;52:2676–2702.
[2] Dick J, Kuo FY, Le Gia QT, et al. Multi-level higher order QMC Galerkin discretization for affine parametric operator equations. SIAM J Numer Anal. 2016;54:2541–2568.
[3] Dick J, Kuo FY, Sloan IH. High-dimensional integration: the quasi-Monte Carlo way. Acta Numer. 2013;22:133–288.
[4] Gantner RN, Herrmann L, Schwab C. Quasi-Monte Carlo integration for affine-parametric, elliptic PDEs: local supports and product weights. SIAM J Numer Anal. 2018;56:111–135.
[5] Gilbert AD, Graham IG, Kuo FY, et al. Analysis of quasi-Monte Carlo methods for elliptic eigenvalue problems with stochastic coefficients. Numer Math. 2019;142:863–915.
[6] Gilbert AD, Graham IG, Scheichl R, et al. Bounding the spectral gap for an elliptic eigenvalue problem with uniformly bounded stochastic coefficients. In: Wood D, de Gier J, Praeger C, Tao T editors. MATRIX annals. Cham: Springer; 2018. p. 29–43 (MATRIX book series; 3)
[7] Gilbert AD, Scheichl R. Multilevel quasi-Monte Carlo for random elliptic eigenvalue problems I: Regularity and error analysis. arXiv:2010.01044. 2020.
[8] Gilbert AD, Scheichl R. Multilevel quasi-Monte Carlo for random elliptic eigenvalue problems II: Efficient Algorithms and Numerical results. arXiv:2103.03407. 2021.
[9] Graham IG, Kuo FY, Nichols JA, et al. Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients. Numer Math. 2015;131:329–368.
[10] Graham IG, Kuo FY, Nuyens D, et al. Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications. J Comput Phys. 2011;230:3668–3694.
[11] Graham IG, Kuo FY, Nuyens D, et al. Circulant embedding with QMC: analysis for elliptic PDE with lognormal coefficients. Numer Math. 2018;140:479–511.
[12] Herrmann L, Schwab C. QMC algorithms with product weights for lognormal-parametric, elliptic PDEs. In: Monte Carlo and quasi-Monte Carlo methods. Cham: Springer; 2018. p. 313–330. (Springer proceedings in mathematics & statistics; 241).
[13] Herrmann L, Schwab C. Multilevel quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients. ESAIM: Math Model Numer Anal. 2019;53:1507–1552.
[14] Herrmann L, Schwab C. QMC integration for lognormal-parametric, elliptic PDEs: local supports and product weights. Numer Math. 2019;141:63–102.
[15] Kazashi Y. Quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients. IMA J Numer Anal. 2019;39:1563–1593.
[16] Kuo FY, Nuyens D. Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients – a survey of analysis and implementation. Found Comput Math. 2016;16:1631–1696.
[17] Kuo FY, Nuyens D. Hot new directions for quasi-Monte Carlo research in step with applications. In: Monte Carlo and quasi-Monte Carlo methods. Cham: Springer; 2018. p. 123–144. (Springer proceedings in mathematics and statistics; 241).

[18] Kuo FY, Scheichl R, Schwab C, et al. Multilevel quasi-Monte Carlo methods for lognormal diffusion problems. Math Comp. 2017;86:2827–2860.

[19] Kuo FY, Schwab C, Sloan IH. Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. SIAM J Numer Anal. 2012;50:3351–3374.

[20] Kuo FY, Schwab C, Sloan IH. Multi-level quasi-Monte Carlo finite element methods for a class of elliptic PDEs with random coefficients. Found Comput Math. 2015;15:411–449.

[21] Lemieux C. Monte Carlo and quasi-Monte Carlo integration and applications. Springer; 2014.

[22] Nguyen DTP, Nuyens D. MDFEM: multivariate decomposition finite element method for elliptic PDEs with uniform random diffusion coefficients using higher-order QMC and FEM. Numer Math. 2021;148:633–669.

[23] Nguyen DTP, Nuyens D. MDFEM: multivariate decomposition finite element method for elliptic PDEs with lognormal diffusion coefficients using higher-order QMC and FEM. ESAIM: Math Model Numer Anal. 2021;55:1461–1505.

[24] Nichols JA, Kuo FY. Fast CBC construction of randomly shifted lattice rules achieving $O(n^{-1+\delta})$ convergence for unbounded integrands over $\mathbb{R}^d$ in weighted spaces with POD weights. J Complex. 2014;30:444–468.

[25] Andreev R, Schwab C. Sparse tensor approximation of parametric eigenvalue problems. In: Graham IG et al. editors. Numerical Analysis of Multiscale Problems, Lecture Notes in Computational Science and Engineering. Berlin Heidelberg: Springer-Verlag; 2012. p. 203–241.

[26] Elman HC, Su T. Low-rank solution methods for stochastic eigenvalue problems. SIAM J Sci Comp. 2019;41:A2657–A2680.

[27] Ghanem R, Ghosh D. Efficient characterization of the random eigenvalue problem in a polynomial chaos decomposition. Int J Numer Methods Eng. 2007;72:486–504.

[28] Hakula H, Kaarnioja V, Laaksonen M. Approximate methods for stochastic eigenvalue problems. Appl Math Comput. 2015;267:664–681.

[29] Hakula H, Laaksonen M. Asymptotic convergence of spectral inverse iterations for stochastic eigenvalue problems. Numer Math. 2019;142:577–609.

[30] Shinozuka M, Astill CJ. Random eigenvalue problems in structural analysis. R, AIAA Journal. 1972;10:456–462.

[31] Wachspress EL. Iterative solution of elliptic systems and applications to the neutron diffusion equations of reactor physics. Englewood-Cliffs, NJ, USA: Prentice-Hall, Inc.; 1966.

[32] Williams MMR. A method for solving stochastic eigenvalue problems. Appl Math Comput. 2010;215:3906–3928.

[33] Williams MMR. A method for solving stochastic eigenvalue problems II. Appl Math Comput. 2013;219:4729–4744.

[34] Dobson DC. An efficient method for band structure calculations in 2D photonic crystals. J Comput Phys. 1999;149:363–376.

[35] Duderstadt JJ, Hamilton LJ. Nuclear reactor analysis. New York-London-Sydney-Toronto: John Wiley and Sons, Inc; 1976.

[36] Giani S, Graham IG. Adaptive finite element methods for computing band gaps in photonic crystals. Numer Math. 2012;121:31–64.

[37] Bachmayr M, Cohen A, DeVore R, et al. Sparse polynomial approximation of parametric elliptic Pdes. Part II: lognormal coefficients. ESAIM Math Model Numer Anal. 2017;51:341–363.

[38] Bachmayr M, Cohen A, Migliorati G. Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients. ESAIM Math Model Numer Anal. 2017;51:321–339.

[39] Babuška I, Osborn J. Finite element-Galerkin approximation of eigenvalues and eigenvectors of self-adjoint problems. Math Comp. 1988;52:275–297.
[41] Henrot A. Extremum problems for eigenvalues of elliptic operators. Basel, Switzerland: Birkhäuser Verlag; 2006.

[42] Dunford N, Schwartz JT. Linear operators. Part 2: spectral theory. Self adjoint operators in Hilbert space. New York-London-Sydney: Interscience Publishers, A Division of John Wiley & Sons; 1963.

[43] Cohen A, DeVore R. Approximation of high-dimensional parametric PDEs. Acta Numer. 2015;24:1–159.

[44] Pinkus A. $n$-widths in approximation theory. Berlin: Springer; 1985.

[45] Reed M, Simon B. Methods of modern mathematical physics. IV. Analysis of operators. New York: Academic Press [Harcourt Brace Jovanovich Publishers]; 1978.