Additional Explanatory Notes on the Analytic Proof of the Finite Generation of the Canonical Ring

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INTRODUCTION. This set of notes is put together to provide some additional explanatory material on the analytic proof of the finite generation of the canonical ring for use by the participants in the Workshop on Minimal and Canonical Models in Algebraic Geometry in April 16-20, 2007 at the Mathematical Sciences Research Institute at Berkeley, California.

In late March 2007 before the Workshop Mihai Paun of Strasbourg came to Harvard for two weeks to pose to me some questions which he and other people have about the analytic proof of the finite generation of the canonical ring which I posted in October 2006 on the arXiv.org server [Siu 2006]. Besides orally answering his questions I also wrote up some notes for him to give him more precise details. This set of notes is compiled by putting together the notes which I wrote up to answer his questions. This set of notes consists of two parts.

Part I is about how to apply the general nonvanishing theorem to prove the precise achievement of stable vanishing order in codimension one. Part II gives the argument for the precise achievement of the stable vanishing order for higher codimension. For both Part I and Part II the new powerful tool in the analytic proof is the use of the second case of the dichotomy of the curvature current of the metric of minimum singularity for the canonical line bundle.

Let $V = V_k$ be an irreducible subvariety of codimension $k$ in the compact complex algebraic manifold $X$ of finite type whose canonical ring is to be proved to be finitely generated such that $V$ is an embedded branch of the stable base point set. A modified restriction $\Theta_V$ to $V$ of the curvature current of the canonical line bundle is given as follows. For some $\eta_{\nu} \geq 0$ and $V_\nu$ irreducible of codimension 1 in $V_{\nu-1}$ ($k < \nu < 0$) with $V_0 = X$ and $\eta_0 = 0$, let $\Theta_0$ be the curvature current $\Theta_{K_X}$ on $X$ of the metric of minimum singularity for the canonical line bundle $K_X$ and inductively

$$\Theta_\nu = (\Theta_{\nu-1} - \eta_{\nu-1}V_{\nu-1})|_{V_\nu}.$$  

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Then $\Theta_k$ is the modified restriction $\Theta_V$ to $V$ of the curvature current of $K_X$ for the sequence $V_{k-1} \subset \cdots \subset V_1 \subset X$ of nested subvarieties. For the canonical decomposition

$$\Theta_V = \sum_{j=1}^{J} \gamma_j [Y_j] + R$$

of $\Theta_V$ on $V$ there are two cases. The first case is either $J = \infty$ or $R \neq 0$ and the second case is $J < \infty$ and $R = 0$, where $Y_j$ is a subvariety of codimension one in $V$ and $R$ is a closed positive current on $V$ whose Lelong number is zero outside a countable union of subvarieties of codimension at least two in $V$. For Part I on how to apply the general nonvanishing theorem to prove the precise achievement of stable vanishing order in codimension one, the result of the use of this new powerful tool is that the second case of the dichotomy must always eventually occur at some positive dimensional $V$, because if the first case occurs all the time down to dimension zero then there is some improvement in the stable vanishing at some point of $V$ in the multi-directions defined by the sequence of nested subvarieties. The second case of the dichotomy gives a explicitly constructed section on $V$ (unique up to a nonzero constant factor) which belongs to the multiplier ideal sheaf corresponding to $\Theta_V$.

In Part I of these additional explanatory notes, in order to make the arguments of the general nonvanishing theorem more transparent, I treat separately the two cases of the dichotomy. We use the process of using the techniques for the Fujita conjecture by constructing singular metrics successively to cut down on the dimension of (the projection of) the zero-set of the multiplier ideal sheaf until we end up with the inevitable second case of the dichotomy and in the second case of the dichotomy we use the sections explicitly constructed from the special form of the canonical decomposition of the modified restriction of the curvature current. In extending the explicitly constructed sections all the way back to the ambient manifold $X$ we introduce the technique of constrained minimum center of log canonical singularity.

In Part II of these additional explanatory notes about the argument for the precise achievement of the stable vanishing order for higher codimension, we give its illustration in the low dimensional cases of complex surfaces and complex threefolds. The illustration in the low dimensional cases avoids the encumberment of complicated notations in the case of general dimension and at the same time contains the essence of the argument of the general case.
For the case of a threefold $X$ in Part II, the key case of higher codimension is that of an irreducible curve $C$ which is an embedded curve in the stable base point set. The important ingredient of using the second case of the dichotomy is the following. For simplicity assume that there is no codimension-one base point set.

Suppose $s_1, \ldots, s_k$ are pluricanonical sections on $X$ such that $C$ is cut out by the family $S_\sigma$ of surfaces defined by pluricanonical sections $s_\sigma = \sum_{j=1}^{k} \sigma_j s_j$ parametrized by $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{C}^k - \{0\}$. Suppose $s$ is a pluricanonical section whose restriction to some $S_\sigma$ achieves the restriction of the stable vanishing order $\alpha_\sigma$ on $S_\sigma$ at some point $P_\sigma$ of $C$. Then, after taking away the restriction of the stable vanishing order $\alpha_\sigma$ of $s|_{S_\sigma}$ across $C$, the resulting section $\left(\frac{s|_{S_\sigma}}{s_{C,\sigma}}\right)_\sigma$ on $S_\sigma$ (where $s_{C,\sigma}$ is the canonical section of $C$ in $S_\sigma$), when restricted to $C$, is the section (up to a nonzero constant factor) explicitly constructed from the second case of the dichotomy for $C$ in $S_\sigma$. This means that at every point of $C$ outside the finite zero-set $Z_\sigma$ of the resulting section $\left(\frac{s|_{S_\sigma}}{s_{C,\sigma}}\right)_\sigma$ on $C$, the section $\left(\frac{s|_{S_\sigma}}{s_{C,\sigma}}\right)_\sigma$ achieves the restriction of the stable vanishing order $\alpha_\sigma$ on $S_\sigma$.

Suppose for every parameter $\sigma$ there is a parameter $\tau$ and some point $P_{\sigma,\tau}$ in $C$ such that the restriction of the pluricanonical section $s_\tau$ to $S_\sigma$ achieves the restriction of the stable vanishing order $\alpha_\sigma$ on $S_\sigma$ at the point $P_{\sigma,\tau}$ of $C$. Let $Z$ be the finite subset of $C$ such that as a point $P$ varies in $C$ outside $Z$ the Artinian subscheme defined by $s_1, \ldots, s_k$ in the normal direction of $C$ at $P$ does not change. In other words, every point $Q \in C - Z$ admits an open neighborhood $U$ in $C - Z$ and a holomorphic family of local nonsingular complex surfaces $M_P$ in $X$ parametrized by $P \in U$ such that each $M_P$ intersecting $C$ normally at $P$ and

$$\dim_C \mathcal{O}_{M_P} / \sum_{j=1}^{k} \mathcal{O}_{M_P} (s_j|_{M_P})$$

is independent of $P \in U$. From the above discussion about $s$ being the explicitly constructed section and about the achievement of the restriction to $S_\sigma$ of stable vanishing order at points of $C$ outside $Z_\sigma$ it follows that the stable vanishing order is achieved at points of $C - Z$.

This use of the sections, explicitly constructed from the curvature current in the second case of the dichotomy, in order to come up with a finite subset $Z$ of $C$ can be regarded as the higher-codimensional way of constructing
sections of vector bundles explicitly from the second case of the dichotomy of the modified restriction of the curvature current. Note that the codimension one case for a surface is invoked for us to get the rationality of the restriction of the stable vanishing order to a surface and its achievement on the surface at some point of $C$. This part of the hard work already done for the codimension one case is used here. This vector bundle section on $C$ explicitly constructed from the second case of the dichotomy of the modified restriction of the curvature current is obtained, so to speak, by piecing together the line bundle sections on $C$ explicitly constructed by using the surface slices $S$ from the second case of the dichotomy of the modified restriction to the surface slice $S$ of the curvature current. This is the reason why the usual worry about interminable blowups to transform into hypersurfaces newly appearing higher-codimensional base point sets is not a problem here.

Before presenting Part I and Part II, we would like to make two remarks. The first remark is about the finite generation of the canonical ring without the assumption of finite type. In analysis, when $K_X$ is not big, adding a line bundle $E$ to $K_X$ to have a big bundle $K_X + E$ and working with $K_X + E$ instead of $K_X$ would only mean a standard modification of the argument for the finite generation of the canonical ring for the case of general type. (This is actually used in the form of $K_X - \gamma Y$, with $\gamma$ being the stable vanishing order across the hypersurface $Y$, in the proof of the finite generation of the canonical ring for the case of general type.) However, getting rid of $E$ afterwards needs the justification of a rather involved limiting process. Such a limiting process is of the same nature as the limiting process which is used to extend the proof of the deformational invariance of the plurigenera for the case of general type [Siu 1998] to the general algebraic case [Siu 2002].

For the finite generation of the canonical ring, when $K_X$ is big, we introduce the curvature $\Theta_{K_X}$ of the metric of minimum singularity constructed from pluricanonical sections. When $K_X$ is not big, we use an ample line bundle $A$ on $X$ and the limit curvature current

$$\tilde{\Theta}_{K_X} := \lim_{m \to \infty} \Theta_{K_X + \frac{1}{m} A}$$

to replace $\Theta_{K_X}$. All the diophantine approximation arguments deal with the modified restriction of the canonical decomposition of $\tilde{\Theta}_{K_X}$. This makes it possible to go ahead with the limiting process.
In the case of the deformational invariance of the plurigenera the limiting process is handled by abandoning the metric of minimum singularity and replacing it by another metric with maximum allowable singularity which is still good enough for the finiteness of the $L^2$ norm of the section to be extended, because there is a need to bound the dimension of the space of $L^2$ holomorphic $m$-canonical sections with the bound independent of $m$ or at least growing more slowly than $m$. It is not yet clear what the analogous procedure is for the problem of the finite generation of the canonical ring.

The second remark is to answer the question why one does not simply take a pluricanonical section and get the finite generation of the canonical ring of its divisor and then use extension of pluricanonical sections from that divisor to the entire ambient space. This procedure is explained in §8 of the posted notes of my analytic proof [Siu 2006]. The extension technique for pluricanonical sections in the problem of deformational invariance of the plurigenera was developed with that particular application in mind [Siu 1998, Siu 2006]. Though the pluricanonical extension technique for deformational invariance of the plurigenera is for extension from the fiber at a point to the open unit disk, in analysis it does not matter whether one deals with a compact manifold or a Stein manifold if the Stein manifold is a Zariski open subset and one keeps track of the $L^2$ bounds.

However, the situation of the singularities in the pluricanonical divisor poses technical problems in analysis. In my article in the proceedings of a 2001 conference in Hanoi [Siu 2004] I addressed the problem of pluricanonical extension from a singular divisor. There are technical problems concerning finite $L^2$ bounds. The difficulty in handling such technical problems concerning finite $L^2$ bounds is the same as getting the comparability of the metric of minimum singularity for $K_X$ and that from an appropriate truncated finite sum in the application of Skoda’s theorem of ideal generation [Skoda 1972]. Moreover, the Ohsawa-Takegoshi-type extension theorem used in the pluricanonical extension is closely related to Skoda’s theorem of ideal generation (see, for example, Ohsawa’s article in the Festschrift for Grauert’s 70th birthday [Ohsawa 2002]) and the constant in Skoda’s theorem is more precise. That is why we use the approach by Skoda’s theorem instead of the essentially equivalent approach of pluricanonical extension.

It seems that at this point any approach to the problem of finite generation of the canonical ring has to rely, as starting point, on the technique of pluricanonical extension. Any difference in the different approaches are
more a matter of technical handling of the singularities of the pluricanonical divisor.

In the definition of a stable metric in [Siu 2006, (6.1)] there are some typographical errors and inaccuracies. The correct definition is as follows. A metric $e^{-\kappa}$ of a line bundle $L$ on a compact Kähler manifold $M$ is said to be stable if $\kappa$ is locally plurisubharmonic and there exists some $\varepsilon > 0$ with the following property. If $U$ is an open neighborhood of a point $P \in M$, and $\varphi$ and $\psi$ are plurisubharmonic functions on $U$ such that the total mass, with respect to the Kähler form of $M$, of the sum of the two closed positive $(1,1)$-currents $\Theta\varphi$ and $\Theta\psi$ is less than $\varepsilon$ and if $\kappa - \varphi$ is plurisubharmonic, then there exists an open neighborhood $U'$ of $P$ in $M$ such that the multiplier ideal sheaf $\mathcal{I}_{\kappa+\psi}$ of the metric $e^{-\kappa-\psi}$ on $U'$ is equal to the multiplier ideal sheaf $\mathcal{I}_{\kappa-\varphi}$ of the metric $e^{-\kappa+\varphi}$ on $U'$. 
PART I
How to Apply Nonvanishing Theorem to Precisely Achieve Stable Vanishing Order in Codimension One

The proof of the precise achievement of stable vanishing order in codimension one has the following ingredients.

(a) The techniques for the Fujita conjecture which consists of

   (i) constructing singular metrics with curvature current of strict positive lower bound whose multiplier ideal sheaf has high vanishing order at a prescribed point,

   (ii) blowing up the zero-set of the multiplier ideal sheaf of the new singular metric,

   (iii) repeating the procedure so that the zero-set of the multiplier ideal sheaf of the new singular metric on a blow-up space is projected down to lower and lower dimension in the original manifold, until one gets to the case of a singular point in the original manifold, and

   (iv) extending a section defined on the single point to over all of the original manifold by the vanishing theorem of Kawamata-Viehweg-Nadel.

Because of the need of adding one canonical line bundle in the application of the vanishing theorem of Kawamata-Viehweg-Nadel, the singular metric is for a multiple of the line bundle in question minus the canonical line bundle.

(b) Introducing a dichotomy depending on the canonical decomposition of some curvature current so that

   (i) in the first case of the dichotomy the technique of the Fujita conjecture for the construction of singular metric can be carried out, and

   (ii) in the second case of the dichotomy a section can be explicitly constructed which can be extended to all of the original manifold.
Once we get to the second case of the dichotomy there is no need to use the technique of the Fujita conjecture to construct any more singular metric. The process is stopped and complete. For the precise achievement of stable vanishing order, the second case of the dichotomy must arise, otherwise the stable vanishing order can be improved, which contradicts the definition of a stable vanishing order. The explicitly constructed section is rather rigid in the sense that there is no choice and the section comes from the curvature current in a rather unique way. This uniqueness and rigidity of the explicitly constructed section will be a key ingredient in the proof of the precise achievement of the stable vanishing order in higher codimension.

(c) The new technique of constrained minimum center of log canonical singularity, whose motivation and precise application will be explained below. This new technique is needed, because of the undesirable additional vanishing order in the process of constructing singular metrics, which will be explained in detail below.

(d) Use of diophantine approximation to handle irrational coefficients. Since the use of diophantine approximation has already been described in detail in the posted notes of my proof of the finite generation of the canonical ring [Siu 2006], we will not discuss diophantine approximation in these additional explanatory notes and we just assume that the relevant coefficients are known to be rational numbers.

As the first step we will consider the first case of the dichotomy and describe how to construct an appropriate singular metric. Then we will give the motivation for constrained minimum center of log canonical singularity by reviewing the goal and the strategy of the standard technique for the Fujita conjecture. Finally we consider the second case of the dichotomy where a rather rigid section is explicitly constructed.

**PROPOSITION** (*Construction of Metric with Multiplier Ideal Sheaf Vanishing to High Order at a Prescribed Point for the First Case of the Dichotomy of the Curvature Current*). Let $M$ be a compact complex projective algebraic manifold of complex dimension $n$. Let $L$ be a holomorphic line bundle on $M$ with a (possibly singular) metric $e^{-\varphi}$ along its fibers whose curvature current
\( \Theta \) is a closed positive \((1, 1)\)-current. Let 

\[
\Theta = \sum_{j=1}^{J} \tau_j [V_j] + R
\]

be the canonical decomposition of \( \Theta \), where \( J \in \mathbb{N} \cup \{0, \infty\} \) and the Lelong number of \( R \) is zero outside a countable union \( Z \) of subvarieties of codimension at least two in \( M \) and \( V_j \) is an irreducible hypersurface in \( M \) and \( \tau_j > 0 \). Assume that either \( J = \infty \) or \( R \neq 0 \) (that is, one is in the first case of the dichotomy for the curvature current \( \Theta \)). Assume that for some positive integer \( p_0 \) there is a (possibly singular) metric \( e^{-\chi} \) along the fibers of \( p_0 L - K_M \) which is stable and whose curvature current \( \Theta_\chi \) is a closed positive \((1, 1)\)-current which dominates some strictly positive smooth \((1, 1)\)-form on \( M \). Let \( P_0 \) be a point of \( M \) such that the Lelong number of \( \Theta \) is zero at \( P_0 \). Let \( q \in \mathbb{N} \). Then for some sufficiently divisible \( m \in \mathbb{N} \) the line bundle \((m + p_0) L - K_M \) admits a metric \( e^{-\tilde{\chi}} \) whose curvature current dominates some strictly positive smooth \((1, 1)\)-form on \( M \) such that its multiplier ideal sheaf \( I_{\tilde{\chi}} \) is contained in the maximum ideal of \( M \) at \( P_0 \) raised to the \( q \)-th power. Moreover, if \( M \) is a hypersurface in some compact complex algebraic manifold \( X \) of general type so that \( L \) is the restriction of some line bundle \( \tilde{L} \) on \( X \) and \( K_M = K_X + M \) and the metric \( e^{-\chi} \) is defined by a convergent infinite sum of multi-valued holomorphic sections of \( p_0 \tilde{L} - K_X - M \) over \( X \), then the metric \( e^{-\tilde{\chi}} \) can be chosen to be defined also by a convergent infinite sum of multi-valued holomorphic sections of \((p + p_0) \tilde{L} - K_X - M \) over \( X \).

**Proof.** The idea of the proof is to use the techniques for Fujita’s conjecture (see, for example, [Angehrn-Siu 1995]).

**Slicing by an Ample Divisor.** Let \( A \) be a very ample line bundle over \( M \) such that \( A - K_M \) is ample. Let \( h_A \) be a smooth metric of \( A \) whose curvature form \( \omega_A \) is positive on \( M \). We assume that \( A \) is chosen to be sufficiently ample so that for each point \( P \in M \) the proper transform of \( A \) in the manifold obtained from \( M \) by blowing up \( P \) is still very ample. This technical assumption will enable us to choose a generic element of \( \Gamma(M, A) \) vanishing at \( P_0 \) which is not a zero-divisor of a prescribed coherent ideal sheaf.

Let \( p \) and \( k \) be positive integers and we will impose more conditions on \( p \) and \( k \) later. Let \( s_1 \) be a generic element of \( \Gamma(M, A) \) vanishing at \( P_0 \) so that
the short exact sequence

\[
0 \to \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + kA) \xrightarrow{\theta_{s_1}} \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + (k + 1) A) \\
\to (\mathcal{I}_{p_\varphi + \chi} / s_1 \mathcal{I}_{p_\varphi + \chi}) ((p + p_0) L - K_M + (k + 1) A) \to 0
\]

is exact, where \( \theta_{s_1} \) is defined by multiplication by \( s_1 \). Let \( M_1 \) be the zero-set of \( s_1 \) and

\[
\mathcal{O}_{M_1} = (\mathcal{O}_M / s_1 \mathcal{O}_M) |_{M_1},
\]

which we can assume to be regular with ideal sheaf equal to \( s_1 \mathcal{O}_M \) because \( s_1 \) is generic element of \( \Gamma (M, A) \) vanishing at \( P_0 \). By choosing \( s_1 \) generically we can also assume that \( \mathcal{I}_{(p_\varphi + \chi)|_{M_1}} = \mathcal{I}_{p_\varphi + \chi} / s_1 \mathcal{I}_{p_\varphi + \chi} \) and \( \mathcal{I}_{(p_\varphi)|_{M_1}} = \mathcal{I}_{p_\varphi} / s_1 \mathcal{I}_{p_\varphi} \) and \( \mathcal{I}_{\chi|_{M_1}} = \mathcal{I}_{\chi} / s_1 \mathcal{I}_{\chi} \). We use \( \chi (\cdot, \cdot) \) to denote the arithmetic genus which means

\[
\chi (\cdot, \cdot) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \dim_{\mathbb{C}} H^{\nu} (\cdot, \cdot).
\]

From the long cohomology exact sequence of the above short exact sequence we obtain

\[
\chi (M, \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + (k + 1) A)) = \\
\chi (M, \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + kA)) + \chi \left( M_1, \mathcal{I}_{(p_\varphi + \chi)|_{M_1}} ((p + p_0) L - K_M + (k + 1) A) |_{M_1} \right).
\]

Since \( A - K_M \) is ample and \( 2A - K_M \) is also ample, when we assume \( k \geq 1 \), by the theorem of Kawamata-Viehweg-Nadel

\[
H^{\nu} (M, \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + kA)) = 0 \quad \text{for } \nu \geq 1,
\]

\[
H^{\nu} \left( M_1, \mathcal{I}_{(p_\varphi + \chi)|_{M_1}} (((p + p_0) L - K_M + (k + 1) A) |_{M_1}) \right) = 0 \quad \text{for } \nu \geq 1
\]

so that

\[
\dim_{\mathbb{C}} \Gamma (M, \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + (k + 1) A)) \\
= \dim_{\mathbb{C}} \Gamma (M, \mathcal{I}_{p_\varphi + \chi} ((p + p_0) L - K_M + kA)) \\
+ \dim_{\mathbb{C}} \Gamma \left( M_1, \mathcal{I}_{(p_\varphi + \chi)|_{M_1}} (((p + p_0) L - K_M + (k + 1) A) |_{M_1}) \right)
\]

\[
\geq \dim_{\mathbb{C}} \Gamma \left( M_1, \mathcal{I}_{(p_\varphi + \chi)|_{M_1}} (((p + p_0) L - K_M + (k + 1) A) |_{M_1}) \right).\]

*Slicing by Ample Divisors Down to a Curve.* Instead of one single element \( s \in \Gamma (M, A) \), we can choose generically

\[
s_1, \ldots, s_{n-1} \in \Gamma (M, A)
\]
all vanishing at \( P_0 \) so that inductively for \( 1 \leq \nu \leq n-1 \) the common zero-set \( M_\nu \) of \( s_1, \cdots, s_\nu \) with the structure sheaf

\[
\mathcal{O}_{M_\nu} := \left( \mathcal{O}_M / \sum_{j=1}^\nu s_j \mathcal{O}_M \right) \bigg|_{M_\nu}
\]

is regular and we end up with the inequality

\[
\dim \Gamma (M, \mathcal{I}_{(p\varphi + \chi) (p + p_0) L - K_M + (k + n - 1) A}) \\
\geq \dim \Gamma (M, \mathcal{I}_{\mathcal{I}_{(p\varphi + \chi)}|_{M_{n-1}}} ((p + p_0) L - K_M + (k + n - 1) A)|_{M_{n-1}})).
\]

Since \( M_{n-1} \) is a curve, all coherent ideal sheaves on it are principal and are locally free and they come from holomorphic line bundles.

We would like to remark also that this particular step of slicing by \( n - 1 \) ample divisors to get down to a curve roughly corresponds to the step in Shokurov’s proof of his non-vanishing theorem [Shokurov 1985] where he takes the product of his numerically effective divisor in his \( n \)-dimensional manifold with the \( (n-1) \)-th power of a numerically effective big line bundle.

**Application of the Theorem of Riemann-Roch to a Curve and Comparing Contributions from the Curvature Current and the Multiplier Ideal Sheaves.**

Let \( b \) be the Chern class of the line bundle on \( M_{n-1} \) defined by the multiplier ideal sheaf \( \mathcal{I}_{(p\varphi + \chi)}|_{M_{n-1}} \) of the restriction to \( M_{n-1} \) of the metric \( e^{-\chi} \). Let \( c \) be the nonnegative number

\[
\int_{M_{n-1}} R = \int_M R \wedge (\omega_A)^{n-1}.
\]

Then

\[
\dim \Gamma (M, \mathcal{I}_{p\varphi + \chi} ((p + p_0) L - K_M + (k + n - 1) A)) \\
\geq \dim \Gamma (M, \mathcal{I}_{(p\varphi + \chi)|M_{n-1}} ((p + p_0) L - K_M + (k + n - 1) A)|_{M_{n-1}})) \\
\geq 1 - \text{genus} (M_{n-1}) + b + (k + n - 1) A^{n-1} M_{n-1} \\
+ \sum_{j=1}^J (p \tau_j - [p \tau_j]) V_j \cdot A^{n-1} + p \int_{M_{n-1}} R,
\]

where the last identity is from the theorem of Riemann-Roch applied to the regular curve \( M_{n-1} \) and the locally free sheaf

\[
\mathcal{I}_{(p\varphi + \chi)|M_{n-1}} ((p + p_0) L - K_M + (k + n - 1) A)|_{M_{n-1}})
\]
on $\mathbb{P}_{k-1}$. From the assumption that $J = \infty$ or $R \neq 0$, we conclude that the right-hand side of (†) goes to $\infty$ as $p$ goes to $\infty$ through an appropriate sequence, where for the case of $J = \infty$ and $R = 0$ a diophantine argument has to be used whereas for the case $R \neq 0$ we simply need to use $c > 0$.

Construction of Sections with Extra Vanishing Order from Dimension Counting and Construction of Metrics by Canceling Contributions from Ample Divisors by Using the General Type Property. For any $\ell \in \mathbb{N}$ the number of terms in a polynomial of degree $\ell$ in $d$ variables is $(d + \ell + 1)$. Take a positive integer $N$ and we will impose more condition on $N$ later. By the behavior of the right-hand side of (†) as $p \to \infty$, there exists $p \in \mathbb{Z}$ such that

$$\dim \Gamma (M, I_{p\varphi + \chi} ((p + p_0) L - K_M + (k + n - 1) A)) \geq 1 + \left( n + N (k + n - 1) q \right)$$

and we can find some non identically zero element $s$ of

$$\Gamma (M, I_{p\varphi + \chi} ((p + p_0) L - K_M + (k + n - 1) A))$$

which vanishes to order at least $N (k + n - 1) q$ at $P_0$ so that $s^{\frac{1}{N(k+n-1)}}$ is a multi-valued holomorphic section of the $\mathbb{Q}$-line-bundle $\frac{p}{N(k+n-1)} L + \frac{p}{N} A$ over $M$ which vanishes to order at least $q$ at $P_0$. We assume that $N$ is chosen so large that the curvature current $\Theta_{\varphi}$ dominates $\frac{2}{N} \omega_A$. Let $\hat{p}$ to be the round-up of $\frac{p}{N(k+n-1)}$ and $\delta_p = \hat{p} - \frac{p}{N(k+n-1)}$. We introduce the metric

$$e^{-\hat{\chi}} := \frac{e^{-\chi - \delta_p \varphi}}{(h_A)^{\frac{1}{N}} |s|^\frac{2}{N(k+n-1)}}$$

of $(p + p_0) L - K_M$ so that the multiplier ideal of $I_{\hat{\chi}}$ at $P_0$ is contained in $(\mathfrak{m}_{M,P_0})^{\left(\frac{2}{N}\right)}$. Q.E.D.

**Remark on Application of the Proposition on Construction of Metric for the First Case of the Dichotomy.** The above application is applied in the following manner. Let $X$ be the compact complex algebraic manifold of finite type whose canonical ring is to be proved to be finitely generated. Let $Y$ be a hypersurface in $X$ across which the stable vanishing order is $\gamma > 0$. Let $P_0$ be a generic point of $Y$. We start out with $M = Y$ and $L = K_X - \gamma Y$. After we get the new metric $e^{-\hat{\chi}}$, we use an interpolation between $e^{-\chi}$ and $e^{-p\varphi - \chi}$ and a slight modification to get to a minimum center of log canonical
singularity which, after blow-up, projects down to a proper subvariety of $M$ containing $P_0$. Then we replace $X$ by its blow-up and replace $M$ by the new minimum center of log canonical singularity and replace $L$ by its pullback to the blowup of $X$. We continue doing this until we inevitably come to the second case of dichotomy eventually as explained in the Introduction.

**Remark on the Second Case of the Dichotomy.** Suppose

$$\Theta_\varphi = \sum_{j=1}^{J} \tau_j [V_j]$$

with $J < \infty$. Then we can explicitly construct a section $s_0$ of $pL$ over $M$. The reason why a minimal center of log canonical singularity is used in the techniques for the Fujita conjecture is to make sure that when we take the subspace defined by the multiplier ideal sheaf, the subspace has a reduced structure. In our case we have to introduce the notion of constrained minimal center of log canonical singularity so that the center is not completely contained in the zero-set of $s_0$. For that we have to pay the price that the subspace defined by the multiplier ideal sheaf may have an unreduced structure, but the set where nonzero nilpotent elements of its structure sheaf occurs is contained in the zero-set of $s_0$. By raising $s_0$ to a sufficiently high power, we can handle the unreduced structure and get the extension of a sufficiently high power of $s_0$ to the ambient manifold $X$ by using the vanishing theorem from the metric of $pL - K_M$. We are going to elaborate on this by reviewing the goal of the techniques for the Fujita conjecture and the use of minimal center of log canonical singularity and also how we are naturally and by necessity led to the concept of constrained minimal center of log canonical singularity.

**Main Idea of the Techniques for the Fujita Conjecture.** For the discussion about the main idea of the technique for the Fujita conjecture, we forget the above meaning of $X$ and $L$ and for the time being use the symbols $X$ and $L$ in another context. We will so indicate when we later return to the above meaning of $X$ and $L$. The goal of the technique for the Fujita conjecture is to find global sections to generate some positive power $mL$ of a line bundle $L$ over a compact complex algebraic manifold $X$.

For a more general setting, the goal is to find global sections to globally generate $\mathcal{J}(mL)$ over $X$ for some given coherent ideal sheaf $\mathcal{J}$ at points
outside some given subvariety $Z$ of $X$. The problem of proving the finite generation of the canonical ring by verifying the precise achievement of stable vanishing orders actually involves this more general setting. There the even more complicated situation of supremum norm is used instead of just the $L^2$ norm. However, for the sake of simplicity in our discussion of the main idea of the techniques of the Fujita conjecture, we stick with the simpler goal of find global sections to generate some positive power $mL$ of a line bundle $L$ over a compact complex algebraic manifold $X$.

The main idea of the technique is to focus on the subvariety where the global generation fails. We take a basis of $s_1, \cdots, s_k \in \Gamma (X, mL)$ and let $Y$ be their common zero-set so that global generation precisely fails at points of $Y$. The main idea of the technique is simply to focus on $Y$ if $Y$ is nonempty. We seek elements of $\Gamma (Y, mL|_Y)$ which are not identically zero and then extend them to elements of $\Gamma (X, mL)$, which would then contradict the definition of $Y$. Hopefully the extension of elements of $\Gamma (Y, mL|_Y)$ to elements of $\Gamma (X, mL)$ could be done by the vanishing of some appropriate first cohomology group. Usually this first cohomology comes from the vanishing theorem of Kawamata-Viehweg-Nadel. We seek a metric $e^{-\phi}$ of $mL - K_X$ so that

(i) the zero-set of its multiplier ideal sheaf $I_\phi$ is $Y$, and

(ii) the curvature current of $e^{-\phi}$ dominates some strictly positive smooth $(1, 1)$-form on $X$.

In such a case we have

$$H^1 (X, I_\phi (mL)) = 0$$

from the vanishing theorem of Kawamata-Viehweg-Nadel and the map

$$\Gamma (X, mL) \rightarrow \Gamma (Y, (\mathcal{O}_X / I_\phi) (mL))$$

is surjective. The next step is to come up with some element of

$$\Gamma (Y, (\mathcal{O}_X / I_\phi) (mL))$$

which induces a non identically zero element of $\Gamma (Y, mL|_Y)$.

It is at this point that the question of a possibly *unreduced* complex structure $\mathcal{O}_X / I_\phi$ arises. It means that the structure sheaf $\mathcal{O}_X / I_\phi$ for $Y$
may have nonzero nilpotent elements. This is the case when \( I \) is a proper subsheaf of the full ideal sheaf \( I_Y \) of \( Y \) and is not equal to \( I_Y \). An element of \( I_Y \) which is not in \( I_\varphi \) would yield a nonzero nilpotent element of the structure sheaf \( \mathcal{O}_X / I_\varphi \) for \( Y \). When we have an unreduced structure \( \mathcal{O}_X / I_\varphi \) for \( Y \), it is more difficult to produce some element of

\[
\Gamma (Y, (\mathcal{O}_X / I_\varphi) (mL))
\]

which induces a non identically zero element of \( \Gamma (Y, mL|_Y) \). To handle the problem of unreduced structure sheaf, the technique is to use minimum centers of log canonical singularity whose role we are going to explain.

**Minimum Center of Log Canonical Singularity.** The idea is to seek a metric \( e^{-\psi} \) for \( mL - K \) which is less singular than \( e^{-\varphi} \) so that the multiplier ideal sheaf \( I_\psi \) of \( e^{-\psi} \) contains the multiplier ideal sheaf \( I_\varphi \) of \( e^{-\varphi} \). This procedure usually involves the interpolation of two metrics and a slight modification of the result of the interpolation. We seek to make the metric of \( e^{-\psi} \) for \( mL - K \) to be as least singular as possible, with just enough singularity to make the multiplier ideal sheaf \( I_\psi \) of \( e^{-\psi} \) not equal to \( \mathcal{O}_X \). Let \( Y' \) be the support of \( \mathcal{O}_X / I_\psi \). This kind of least or minimum singularity for the choice of \( e^{-\psi} \) gives us a reduced complex structure \( \mathcal{O}_X / I_\psi \) for \( Y' \). The reduced complex subspace \((Y', \mathcal{O}_X / I_\psi)\) is called a minimum center of log canonical singularity. (Usually for this technique of minimum center of log canonical singularity one requires, in addition, that the proper transform of \( Y' \) in some appropriate blow-up \( \tilde{X} \) of \( X \) to be a nonsingular hypersurface in \( \tilde{X} \).) Now one replaces \( Y \) by \( Y' \) and uses the vanishing of

\[
H^1 (X, I_\psi (mL)) = 0
\]

and the surjectivity of the map

\[
\Gamma (X, mL) \to \Gamma (Y', (\mathcal{O}_X / I_\psi) (mL))
\]

to reduce the problem to the construction of nonzero element of

\[
\Gamma (Y', (\mathcal{O}_X / I_\psi) (mL)) = \Gamma (Y', mL|_{Y'}). \]

**Constrained Minimum Center of Log Canonical Singularity.** For our case at hand for the finite generation of the canonical ring one modification has to be added in the application of the technique of minimum center of log
canonical singularity. This modification necessitates the introduction of a new concept which we give the name constrained minimum center of log canonical singularity just to make it easier to refer to. Let us now describe the situation. In the second case of the dichotomy of the curvature current, there is some non identically zero element \( s_0 \) of \( \Gamma (Y, mL|_Y) \) which is explicitly constructed from the canonical decomposition on \( Y \) of a modified restriction of the curvature current.

The section \( s_0 \) may have a nonempty zero-set \( W \) in \( Y \). If we just use the technique of minimum center of log canonical singularity without any modification, we may end up with a minimum center of log canonical singularity \( Y' \) which is completely contained inside the zero-set \( W \) of \( s_0 \). In such a case the extension of \( s_0|_{Y'} \) is useless, because the restriction \( s_0|_{Y'} \) of \( s_0 \) to \( Y' \) is identically zero on \( Y' \). So we need to introduce a modification to the technique of minimum center of log canonical singularity. In the procedure of using a metric \( e^{-\psi} \) with least singularity to get the minimum center of log canonical singularity, we introduce the additional condition that the support of \( \mathcal{O}_{X/\mathcal{I}_\psi} \) is not contained entirely in the zero-set \( W \) of \( s_0 \). With this additional condition we can no longer require that the structure sheaf \( \mathcal{O}_{X/\mathcal{I}_\psi} \) of \( Y' \) is reduced, but we can require that the set \( E \) of points where the structure sheaf \( \mathcal{O}_{X/\mathcal{I}_\psi} \) of \( Y' \) fails to be reduced is entirely contained in \( W \). We call \( Y' \), which is obtained from this procedure of the additional condition, a constrained minimum center of log canonical singularity.

The key point about the use of a constrained minimum center of log canonical singularity is the following. Though the restriction \( s_0|_{Y'} \) to \( Y' \) is only holomorphic on the reduced structure of \( Y' \), yet since \( s_0 \) vanishes on \( E \) we can take a sufficiently high power \( s^N \) of \( s \) so that \( s^N|_{Y'} \) is holomorphic on the unreduced structure \( \mathcal{O}_{X/\mathcal{I}_\psi} \) of \( Y' \). We now extend \( s^N|_{Y'} \) to \( X \). Of course, we have to replace \( m \) by \( Nm \).

**Proposition (Global Generation of the Pluricanonical Bundle at Points of Zero Stable Vanishing Order).** Let \( X \) be a compact complex algebraic manifold of complex dimension \( n \). Let \( e^{-\varphi} \) be the metric of \( K_X \) of minimum singularity and let \( \Theta_\varphi \) be its curvature current. Then there exist a positive integer \( m_0 \) such that the common zero-set \( W \) of a \( \mathbb{C} \)-basis of \( \Gamma (X, m_0K_X) \) is precisely the set of points where the Lelong number of \( \Theta_\varphi \) is positive.

**Proof.** We use the technique for the Fujita conjecture and constrained minimum centers of log canonical singularity. We use \( W \) as the set of constraint
for the constrained minimum center of log canonical singularity. We will not go into further details here, because a similar but harder situation will be handled in the proof of precisely achieving stable vanishing order $\gamma > 0$ for codimension one in the case of a hypersurface $Y$ whose coefficient in $\Theta_\phi$ is $\gamma$. The only difference is that here the number $\gamma$ is replaced by 0 and we do the argument in the ambient space $X$ instead of in the hypersurface $Y$ taking the place of $X$.

We now finish the use of the temporary meaning of $X$ and $L$ and return to the earlier meaning of $X$ and $L$.

**Proposition (Extension of Explicitly Constructed Section to Ambient Manifold by Constrained Minimum Center of Log Canonical Singularity).** Let $X$ be a compact complex algebraic manifold of complex dimension $n$. Let $e^{-\varphi}$ be the metric of $K_X$ of minimum singularity and let $\Theta_\varphi$ be its curvature current. Let $M$ be a nonsingular hypersurface in $X$ such that the stable vanishing order $\eta$ for $M$ is a positive rational number. Let $(\Theta_\varphi - \eta M)|_M = \sum_{j=1}^J \gamma_j Y_j$ be the canonical decomposition of the closed positive $(1,1)$-current $(\Theta_\varphi - \eta M)|_M$ on $M$ with $J < \infty$ and each $\gamma_j$ being rational. Then the stable vanishing order $\eta$ for $M$ is precisely achieved.

**Proof.** By the previous Proposition we find a positive integer $m_0$ such that the common zero-set of a $C^\infty$-basis $s_1, \ldots, s_k$ of $\Gamma(X, m_0 K_X)$ is precisely the set of points where the Lelong number of $\Theta_\varphi$ is positive. If any of the elements of $\Gamma(X, m_0 K_X)$ precisely achieves the stable vanishing order $\eta$ for $M$, we are already done. By replacing $m_0$ by another sufficiently large integer, we can make the vanishing order across $M$ of $\sum_{j=1}^k |s_j|_{m_0}$ to be as close to $\eta$ as prescribed (though still $> \eta$). On the other hand, by raising $\sum_{j=1}^k |s_j|_{m_0}$ to a positive integral power afterwards and using interpolation, we can adjust the stable vanishing order across $M$ to $\eta$ plus any positive prescribed number.

Let $L = K_X - \eta M$. We can thus use $s_1, \ldots, s_k$ and interpolation of metrics and their slight modifications to construct a metric $e^{-\chi}$ of $mL - K_X = (m-1)L - (\eta + 1)M$ of strictly positive curvature current so that

(i) the zero-set of its multiplier ideal sheaf $\mathcal{I}_\chi$ is contained in the set of points where the Lelong number of $\Theta_\varphi$ is positive, and
(ii) the generic vanishing order of the its multiplier ideal sheaf $\mathcal{I}_\chi$ across $M$ is precisely 1.

We are able to fulfill Condition(i), because we can construct the metric $e^{-\chi}$ by using the $(m - 1)$-th power $\sum_{j=1}^k |s_j|^{m_0}$ and since the common zero-set of a $\mathbb{C}$-basis $s_1, \cdots, s_k$ of $\Gamma (X, m_0 K_X)$ is precisely the set of points where the Lelong number of $\Theta_{\phi}$ is positive. We are able to fulfill Condition(ii), because we have the additional order $\eta + 1$ across $M$ to spare when we use the $(m - 1)$-th power $\sum_{j=1}^k |s_j|^{m_0}$ and require only the higher generic vanishing order of $m\eta + 1 = (m - 1)\eta + (\eta + 1)$ across $M$ instead of the order $(m - 1)\eta$ from the stable vanishing order $\eta$ across $M$.

The vanishing theorem of Kawamata-Viehweg-Nadel gives

$$H^1 (X, \mathcal{I}_\chi (mL)) = 0.$$ 

To get elements of $\Gamma (X, mL)$, we need to use elements of $\Gamma (X, (\mathcal{O}_X / \mathcal{I}_\chi) (mL))$. At points of $Y$ the additional vanishing order of the multiplier ideal sheaf $\mathcal{I}_\chi$ of $e^{-\chi}$ occurs only at points where the Lelong number of $\Theta_{\phi}$ is positive. When we construct a constrained minimum center $M'$ of log canonical singularity in $M$ by interpolation (with the subvariety $\bigcup_{j=1}^l Y_j$ as the set of constraint), the complex structure of $M'$ is reduced outside $\bigcup_{j=1}^l Y_j$. Now we can explicitly construct the section

$$s_0 = \prod_{j=1}^k (s_{Y_j})^{\gamma_j N}$$

of the line bundle $NL|_M$ on the reduced structure of $M$ for some appropriately chosen positive integer $N$, where $s_{Y_j}$ is the canonical section of $Y_j$ on $M$. Since $s_0$ vanishes on $s_{Y_j}$, by replacing $N$ by a large positive integral multiple, we can assume that the restriction of $s_0$ to $M'$ can be extended to a holomorphic section over any unreduced structure of $M'$ which is reduced outside $\bigcup_{j=1}^l Y_j$ and, moreover, can be extended to an element of $\Gamma (X, (\mathcal{O}_X / \mathcal{I}_\chi) (NL))$. Note that the support of $\mathcal{O}_X / \mathcal{I}_\chi$ may be more than just $Y$, but its intersection with $Y$ is contained in the zero-set $\bigcup_{j=1}^l Y_j$ of $s_0$ so that the last extension to an element of $\Gamma (X, (\mathcal{O}_X / \mathcal{I}_\chi) (NL))$ is possible for a sufficiently large $N$. Since $s_0$ is nonzero at some point $P_0$ of $M'$, the stable vanishing order across $M$ is precisely achieved by

$$(\tilde{s}_0 (s_M)^{Nq})^q \in \Gamma (X, Nq K_X)$$

at the point $P_0$ of $M$. Q.E.D.
PART II
Illustration in Low Dimension of the Argument of Precise Achievement of Stable Vanishing Order for Higher Codimension

We now illustrate the argument of the precise achievement of the stable vanishing order for higher codimension by using the low dimensional cases of complex surfaces and complex threefolds. First we consider the case of surfaces. For surfaces codimension two means isolated points. For isolated points for any dimension there is a simple direct argument, which is given in the following proposition. After we present the case of threefolds, we will remark on how this simple direct argument can be interpreted in the context of the argument for any dimension.

PROPOSITION (Precise Achievement of Stable Vanishing Order at a Finite Set). The stable vanishing order is automatically precisely achieved everywhere when it is precisely achieved outside a finite set of a compact complex algebraic manifold $X$.

PROOF. Let $X$ be a compact complex algebraic manifold of general type. Let $e^{-\varphi}$ be the metric of minimum singularity for the canonical line bundle $K_X$ of $X$. Suppose it has been proved that the stable vanishing order is precisely achieved outside a finite number of points $P_1, \ldots, P_k$ of $X$ by using

$$\sigma_1, \ldots, \sigma_\ell \in \Gamma (X, m_0 K_X).$$

We are going to show that this finite number of points must be the empty set, otherwise there is a contradiction. Let

$$e^{-\psi} = \frac{1}{\sum_{j=1}^{\ell} |\sigma_j|^2}.$$

We take $p \in \mathbb{N}$ sufficiently large to magnify the discrepancy of the vanishing orders of $e^{-pm_0\varphi}$ and $e^{-p\psi}$ so that the support of the quotient

$$\mathcal{I}_{pm_0\varphi} / \mathcal{I}_{p\psi}$$

of the multiplier ideal sheaves $\mathcal{I}_{pm_0\varphi}$ and $\mathcal{I}_{p\psi}$ of $e^{-pm_0\varphi}$ and $e^{-p\psi}$ respectively is the finite set $\{P_1, \ldots, P_k\}$. We now apply a slight modification to $\mathcal{I}_{p\psi}$ to get a metric with strictly positive curvature current. Let $e^{-\theta}$ be a metric
of $K_X$ with strictly positive curvature current. Since $e^{-\varphi}$ is the metric of minimum singularity for $K_X$, it follows that when $\varepsilon > 0$ is sufficiently small (which we assume to be the case) the multiplier ideal sheaf $\mathcal{I}_{(p-\varepsilon)\psi+\varepsilon m_0\theta}$ of the metric $e^{-(p-\varepsilon)\psi+\varepsilon m_0\theta}$ agrees with the multiplier ideal sheaves $\mathcal{I}_{p\varphi}$ of $e^{-p\varphi}$ on $X \setminus \{P_1, \ldots, P_k\}$ and the support of the quotient

$$\mathcal{I}_{p\varphi} / \mathcal{I}_{(p-\varepsilon)\psi+\varepsilon m_0\theta}$$

of the multiplier ideal sheaves $\mathcal{I}_{p\varphi}$ and $\mathcal{I}_{p\psi}$ of $e^{-p\varphi}$ and $e^{-(p-\varepsilon)\psi+\varepsilon m_0\theta}$ respectively is the finite set $\{P_1, \ldots, P_k\}$. By the vanishing theorem of Kawamata-Viehweg-Nadel, we have

$$H^1 (X, \mathcal{I}_{p\psi} ((pm_0 + 1) K_X)) = 0.$$ 

Then the map

$$\Gamma (X, \mathcal{I}_{p\varphi} ((pm_0 + 1) K_X)) \to \Gamma (X, (\mathcal{I}_{p\varphi} / \mathcal{I}_{(p-\varepsilon)\psi+\varepsilon m_0\theta}) ((pm_0 + 1) K_X))$$

is surjective. Note that for this we do not need the vanishing of the cohomology group

$$H^1 (X, \mathcal{I}_{p\varphi} ((pm_0 + 1) K_X)).$$

Since

$$\Gamma (X, (\mathcal{I}_{p\varphi} / \mathcal{I}_{(p-\varepsilon)\psi+\varepsilon m_0\theta}) ((pm_0 + 1) K_X)) \approx \bigoplus_{j=1}^k (\mathcal{I}_{p\varphi} / \mathcal{I}_{(p-\varepsilon)\psi+\varepsilon m_0\theta})_{P_j},$$

it follows from Nakayama’s lemma and the surjectivity of

$$\Gamma (X, \mathcal{I}_{p\varphi} ((pm_0 + 1) K_X)) \to \bigoplus_{j=1}^k (\mathcal{I}_{p\varphi} / \mathcal{I}_{(p-\varepsilon)\psi+\varepsilon m_0\theta})_{P_j},$$

that the map

$$\Gamma (X, \mathcal{I}_{p\varphi} ((pm_0 + 1) K_X)) \to \bigoplus_{j=1}^k (\mathcal{I}_{p\varphi})_{P_j}$$

is surjective. This actually gives a contradiction, because the stable vanishing order of $(pm_0 + 1) K_X$ should be given by $e^{-(pm_0 + 1)\varphi}$ instead of by $e^{-pm_0\varphi}$. Q.E.D.
Remarks. (a) With this proposition, to get the finite generation of the canonical ring for the case of a compact complex algebraic surface of general type it suffices to show that the stable vanishing order is precisely achieved at codimension one.

(b) For the analytic proof of the finite generation of the canonical ring, when we get down to the point of having already verified the precise achievement of the stable vanishing order outside a finite set of points, we do not need to blow up the points to reduce the argument to the case of a hypersurface in the new blown-up manifold. Therefore the difficulty does not exist, in the case of a surface, of blowing up a point $P_0$ to get a curve $C_1$ and then locating some bad point $P_1$ (where the precise achievement of stable vanishing order fails) in the curve $C_1$ and then blowing up $P_1$ to get a curve $C_2$ and locating some bad point $P_2$ (where the precise achievement of stable vanishing order fails) in the curve $C_2$ and then blowing up $P_2$ and possibly finally ending up with an unending infinite sequence of bad points, each in a tower of successively blown-up surfaces.

Higher Codimension Argument for Threefold Case. Let $X$ be a complex complex algebraic threefold of general type. Let $e^{-\varphi}$ be the metric of minimum singularity for the canonical line bundle $K_X$ of $X$. Suppose it has been proved that the stable vanishing order is precisely achieved outside a curve $C = \bigcup_j C_j$ (where each $C_j$ is irreducible) by

$$s_1, \cdots, s_k \in \Gamma (X, m_0K_X).$$

Note that here we have used the Proposition given above to rule out the possibility that, besides at the curve $C$, the stable vanishing order may fail to be precisely achieved at a finite set of points in $X - C$.

On $X$ we have the canonical decomposition of

$$\Theta_{K_X} = \sum_{j=1}^{J} \gamma_j Y_j + R$$

of the curvature current of the metric of minimum singularity for $K_X$. We are skipping the diophantine argument which is explained in detail in [Siu 2006] and consider as verified the rationality of each $\gamma_j$. By replacing $m_0$ by an appropriate integral multiple, we can assume without loss of generality that each $m_0\gamma_j$ is a positive integer. Let $s^* = \prod_{j=1}^{J} (s_{Y_j})^{m_0\gamma_j}$, where $s_{Y_j}$ is
the canonical section of $Y_j$. Let $L = K_X - \sum_{j=1}^{J} \gamma_j Y_j$. Since we can replace $\sigma_1, \cdots, \sigma_\ell$ by

$$s_1 \frac{s_1}{s^*}, \cdots, \frac{s_k}{s^*} \in \Gamma(X, m_0 L)$$

and the essence of the rest of the argument that is to follows does not change, for notational simplicity we are going to assume that all $\gamma_j = 0$ so that $L = K_X$. Also, for notational simplicity we are going to assume that the curve $C$ is irreducible.

For $\sigma = (\sigma_1, \cdots, \sigma_k) \in \mathbb{C}^k - \{0\}$ let $s_\sigma = \sum_{j=1}^{k} \sigma_j s_j$ and let $S_\sigma$ be the surface in $X$ defined by $s_\sigma$. We consider those $\sigma$ for which $S_\sigma$ is irreducible across which $s_\sigma$ vanishes to order 1. By considering the blow-up of $X$ by the ideal generated by $s_1, \cdots, s_k$ and the precise achievement of stable vanishing order for codimension one, after replacing $m_0$ by a positive integral multiple if necessary, we have the following situation.

(a) For $\sigma = (\sigma_1, \cdots, \sigma_k) \in \mathbb{C}^k - \{0\}$ for which $S_\sigma$ is irreducible where $s_\sigma$ vanishes to order 1, there exist

(i) some $\tau = (\tau_1, \cdots, \tau_k) \in \mathbb{C}^k - \{0\}$ and

(ii) some finite subset $Z_{\tau, \sigma}$ of $C$

such that the section $s_\tau$ is not identically zero on $S_\sigma$ and the multi-valued section

$$\frac{s_\tau|_{S_\sigma}}{(s_{C, \sigma})^{\eta_\sigma}}$$

on a neighborhood of $C - Z_{\tau, \sigma}$ in $S_\sigma$ is nonzero at points of $C - Z_{\tau, \sigma}$, where $\eta_\sigma$ is the stable vanishing order at $C$ on the surface $S_\sigma$ and $s_{C, \sigma}$ is the canonical section of $C$ in $S_\sigma$.

(b) One has the second case of the dichotomy for the curvature current

$$(\Theta_{K_X} | s_\sigma - \eta_\sigma [C])|_C$$

on $C$.

(c) The multi-valued section

$$\frac{s_\tau|_{S_\sigma}}{(s_{C, \sigma})^{\eta_\sigma}}$$

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on $C$ is constructed from the second case of the dichotomy for the curvature current
\[(\Theta_{K_X}|s_\sigma - \eta_\sigma [C])|_C\]
on $C$.

(c) The stable vanishing order $\eta_\sigma$ for $C$ on $S_\sigma$ is achieved by $s_\tau$ at points of $C - Z_{\tau,\sigma}$.

Note that, because of Condition(b) and Condition(c) we can choose $Z_{\tau,\sigma}$ independent of $\tau$ so that $Z_{\tau,\sigma} = Z_\sigma$ for some finite subset $Z_\sigma$ of $C$ depending only on $\sigma$. To finish the proof of the precise achievement of the stable vanishing order for the threefold $X$, it suffices to show that there is a finite subset $Z$ of $C$ such that every $Z_\sigma$ is contained in $Z$.

At a regular point $P_0$ of $C$, there is some open neighborhood $U$ of $P_0$ in $X$ such that

(i) the pair $(U,C)$ is biholomorphic to the pair $(\Delta^3, \{(0,0)\} \times \Delta)$, where $\Delta$ is the open unit disk in $\mathbb{C}$, and

(ii) $s_j|U$ is represented by a holomorphic function $f_j(z_1, z_2, t)$ on $\Delta^3$ for $1 \leq j \leq k$.

The existence of the finite subset $Z$ of $C$ with the property that $Z_{\tau,\sigma} \subset Z$ follows from Lemma 2 given below.

Lemma 1. Let $f_0, \ldots, f_k$ be holomorphic function germs on $\mathbb{C}^2$ at the origin so that the origin is the common zero-set of any two of the holomorphic function germs $f_0, \ldots, f_k$. Let $C_0$ be the complex curve germ at the origin defined by $f_0 = 0$, which is assumed to be irreducible and across which $f_0$ vanishes to order 1. Then the following numbers are the same.

(i) The multiplicity of the ideal $\sum_{j=0}^k \mathcal{O}_{\mathbb{C}^2,0} f_j$ at the origin.

(ii) The dimension over $\mathbb{C}$ of
\[\mathcal{O}_{\mathbb{C}^2,0} \left/ \sum_{j=0}^k \mathcal{O}_{\mathbb{C}^2,0} f_j \right..\]
(iii) The Lelong number \( \lambda \) of

\[
\hat{\Theta} := \left( \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \sum_{j=1}^{k} |f_j|^2 \right) \bigg|_{C_0}
\]

on \( C_0 \) at the origin. Here the Lelong number \( \lambda \) means the Lelong number of the pullback of the closed positive \((1,1)\)-current \( \hat{\Theta} \) to the normalization of \( C_0 \).

(iv) The number \( \eta \) such that

\[
\frac{\sum_{j=1}^{k} |f_j|^2}{\left( \sum_{j=1}^{2} |z_j|^2 \right)^\eta}
\]

is bounded between two positive numbers near the origin on \( C_0 \).

**Lemma 2.** Let \( k \geq 2 \) and let \( f_j(z_1, z_2, t) \), for \( 1 \leq j \leq k \), be holomorphic functions on the tri-disk \( \Delta^3 \) with coordinates \( z_1, z_2, t \). Assume that the common zero-set of any two of \( f_1, \cdots, f_k \) is \( \{(0,0)\} \times \Delta = \{z_1 = z_2 = 0\} \). For any \( k \)-tuple \((a_1, \cdots, a_k)\) of complex numbers not all zero let \( S_{a_1, \cdots, a_k} \) be the zero-set of \( \sum_{j=1}^{k} a_j f_j \). Then there exists a discrete subset \( Z \) of \( \{(0,0)\} \times \Delta \) with the following property. For any \( k \)-tuple \((a_1, \cdots, a_k)\) of complex numbers not all zero there exists some nonnegative number \( \gamma_{a_1, \cdots, a_k} \) such that

\[
\frac{\sum_{j=1}^{k} |f_j|^2}{\left( \sum_{j=1}^{2} |z_j|^2 \right)^{\gamma_{a_1, \cdots, a_k}}}
\]

is continuous nonzero on some neighborhood of \((S_{a_1, \cdots, a_k} \cap \{(0,0)\} \times \Delta) - Z \in S_{a_1, \cdots, a_k} \) if \( S_{a_1, \cdots, a_k} \) is irreducible and \( \sum_{j=1}^{k} a_j f_j \) vanishes to order 1 across \( S_{a_1, \cdots, a_k} \).

**Proof.** Observe that for fixed \( t \) if \( a_1 \neq 0 \) then each of \( \sum_{j=1}^{k} |f_j|^2 \) and \( |a_1 f_1 + \cdots + a_k f_k|^2 + \sum_{j=2}^{k} |f_j|^2 \) is bounded by a positive constant times the other on some neighborhood of the origin in \( \mathbb{C}^2 \). Use Lemma 1. Q.E.D.

**Remarks.** (a) We would like to highlight the intuitive geometric reason for the existence of a discrete set \( Z \) in \( C \) such that the “bad set” \( Z_\sigma \) in \( C \) for the restriction of the closed positive \((1,1)\)-current \( \Theta_{K_X} \) restricted to \( S_\sigma \) is
contained in $Z$. The key point is that the surface $S_{\sigma}$ is sliced out by a $\C$-linear combination of the pluricanonical sections $s_1, \cdots, s_k$ and that these pluricanonical sections $s_1, \cdots, s_k$ have the property that the “bad set” $Z_{\sigma}$ of $C$ can be described by the extra vanishing of $\sum_{j=1}^{k} |s_j|^2$ beyond their generic vanishing order at points of $C$.

(b) We need to restrict $\Theta_{K_X}$ to $S_{\sigma}$, because we have to take away the vanishing order of $\Theta_{K_X}$ at $C$ and it is only for codimension one we can take away the vanishing order. The vanishing order on different $S_{\sigma}$ may be different.

(c) The “bad set” $Z_{\sigma}$ in $C$ describes the points where the relative position of the pair of two Artinian subschemes in the normal direction of $C$ jumps. One Artinian subscheme comes from the restriction of $s_1, \cdots, s_k$ to a local surface $T$ normal to $C$ and the other one comes from $s_1, \cdots, s_k$ plus a $\C$-linear combination $\sigma_1 s_1 + \cdots + \sigma_k s_k$ after restricting them to $T$. The main point is that there is a finite subset $Z$ of $C$ such that $Z_{\sigma} \subset Z$ with $Z$ independent of the choice of $\sigma$.

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