A REDUCED-ORDER SHIFTED BOUNDARY METHOD FOR PARAMETRIZED INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We investigate a projection-based reduced order model of the steady incompressible Navier-Stokes equations for moderate Reynolds numbers. In particular, we construct an “embedded” reduced basis space, by applying proper orthogonal decomposition to the Shifted Boundary Method, a high-fidelity embedded method recently developed. We focus on the geometrical parametrization through level-set geometries, using a fixed Cartesian background geometry and the associated mesh. This approach avoids both remeshing and the development of a reference domain formulation, as typically done in fitted mesh finite element formulations. Two-dimensional computational examples for one and three parameter dimensions are presented to validate the convergence and the efficacy of the proposed approach.

1. Introduction

We consider a nonlinear system arising from computational fluid dynamics problems. In particular, a stationary Navier–Stokes system is examined and solved by the Shifted Boundary Method (SBM), an embedded boundary finite element method (EBM) that was recently proposed. The geometrical parametrization is described by means of level sets defined over a fixed (undeformed) background mesh. Moreover, a reduced order method (ROM), based on a proper orthogonal decomposition approach (POD-Galerkin) is tested, with the purpose of creating an appropriate embedded reduced order basis and decreasing the computational cost in the numerical solution procedure.

Aiming at reproducing and extending the early work of Peskin [1], the scientific community has shown great interest on embedded and immersed methods. Some recent developments are represented by ghost-cell finite difference methods [2], Cut-Cell Finite Volume Methods [3, 4], Immersed Interface Methods [5, 6], the Ghost Fluid Method [7, 8], the Volume Penalty Method [9, 10, 11], Cut-FEMs [12] and the Shifted Boundary Method (SBM) [13, 14, 15]. The interested reader can find additional details in [16, 17, 18, 19, 20, 21] and references therein. Very recently, in the context of Navier–Stokes equations, Cut-FEMs [22, 23] and unfitted discontinuous Galerkin method [24] were employed. In [13] the recently introduced Shifted Boundary Method was successfully applied to the Poisson and Stokes flow problems, with optimal convergence and robustness properties. The SBM approach was then extended in [14] to the scalar advection diffusion equation and to the Navier–Stokes equations for a wide range of Reynolds’ numbers.

Although immersed and embedded methods show better features than fitted mesh methods in the case of geometrical design changes, there are several partial differential systems where their approximate solutions become computationally unaffordable, e.g. in real time problems, uncertainty or parametrization of the geometry etc. In these cases reduced order methods appear beneficial, [25, 26, 27, 28].

The main goal of this paper is to show how the SBM can solve geometrically parametrized nonlinear partial differential systems within the ROM framework. For this purpose, recently developed POD techniques [29, 30, 31, 32] will be applied. The key feature of our approach is the avoidance of a remeshing stage and/or morphing (i.e., a mapping of all the deformed geometries to a reference domain, see e.g. [25, 33, 34, 35, 36, 37, 28] for the use of this strategy in traditional body-fitted mesh finite element methods). This contribution is organized as follows:

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We can now introduce a Nitsche's variational formulation of the Navier-Stokes equations: \( \chi \) of continuous, piecewise-linear, vector- and scalar-valued functions. Namely:

\[
\begin{align*}
\rho \nabla \cdot (u(\mu) \otimes u(\mu)) + \nabla p(\mu) - \nabla \cdot (2\nu \varepsilon (u(\mu))) - \rho g(\mu) &= 0, \quad \text{in } D(\mu), \\
\nabla \cdot u(\mu) &= 0, \quad \text{in } D(\mu), \\
\mu = g_D(\mu), & \quad \text{on } \Gamma_D(\mu), \\
-\rho(u(\mu) \otimes u(\mu)) \chi_{\Gamma_N(\mu)} + p(\mu) I - 2\nu \varepsilon (u(\mu))) n &= \mu_N(\mu), \quad \text{on } \Gamma_N(\mu),
\end{align*}
\]

where the variable \( \mu \) introduces a geometrical parameterization. The viscosity corresponds to \( \nu \), the density to \( \rho \), the velocity to \( u(\mu) \), the Dirichlet boundary velocity to \( g_D(\mu) \), the Neumann to \( g_N \) (normal stress), the pressure to \( p(\mu) \), the body force to \( g(\mu) \), and the strain tensor to \( \varepsilon(u(\mu)) = 1/2 \left( \nabla u(\mu) + \nabla u(\mu)^T \right) \).

We will use for inflow and outflow boundaries the notation \( \Gamma_N^{-}(\mu) = \{ x \in \Gamma_D(\mu) | g_D(\mu) \cdot n < 0 \} \), \( \Gamma_N^{+}(\mu) = \{ x \in \Gamma_N(\mu) | u(\mu) \cdot n < 0 \} \), and \( \Gamma_D^{+}(\mu) = \Gamma_D(\mu)/\Gamma_D^{+}(\mu) \) and \( \Gamma_N^{+}(\mu) = \Gamma_D(\mu)/\Gamma_N^{+}(\mu) \), where \( \overline{\text{Re}[\nu]} = \frac{2U}{\nu} \) is the Reynolds number, \( U \) and \( L \) are the characteristic speed and length of the problem, \( h \) is the characteristic mesh size, and \( \gamma_{N1} \) and \( \gamma_{N2} \) are penalty parameters, with \( \chi_{\Gamma_D(\mu)} \) and \( \chi_{\Gamma_N(\mu)} \) the characteristic functions of the boundaries \( \Gamma_D^{+}(\mu) \) and \( \Gamma_N^{+}(\mu) \), respectively.

### 2. The mathematical model and the high fidelity approximation

#### 2.1. Strong formulation of the steady Navier-Stokes problem.

The stationary Navier-Stokes systems for viscous incompressible flow describe how the velocity and pressure of a fluid are related in a medium where the convective forces prevail over viscous forces. In an open domain denoted by \( D \subset \mathbb{R}^d \), with dimension \( d = 2 \) or \( 3 \) we introduce a Lipschitz boundary \( \Gamma \), decomposed into Dirichlet and Neumann boundaries, \( \Gamma_D, \Gamma_N \), respectively. Let a parameter space \( \mathcal{P} \subset \mathbb{R}^k \) (k-dimensional) and \( \mu \in \mathcal{P} \) a parameter (vector).

The steady Navier-Stokes equations are given by:

\[
\begin{align*}
(1) \quad \rho \nabla \cdot (u(\mu) \otimes u(\mu)) + \nabla p(\mu) - \nabla \cdot (2\nu \varepsilon (u(\mu))) - \rho g(\mu) &= 0, \quad \text{in } D(\mu), \\
(2) \quad \nabla \cdot u(\mu) &= 0, \quad \text{in } D(\mu), \\
(3) \quad u(\mu) &= g_D(\mu), \quad \text{on } \Gamma_D(\mu), \\
(4) \quad -\rho(u(\mu) \otimes u(\mu)) \chi_{\Gamma_N(\mu)} + p(\mu) I - 2\nu \varepsilon (u(\mu))) n &= \mu_N(\mu), \quad \text{on } \Gamma_N(\mu),
\end{align*}
\]

where the variable \( \mu \) introduces a geometrical parameterization. The viscosity corresponds to \( \nu \), the density to \( \rho \), the velocity to \( u(\mu) \), the Dirichlet boundary velocity to \( g_D(\mu) \), the Neumann to \( g_N \) (normal stress), the pressure to \( p(\mu) \), the body force to \( g(\mu) \), and the strain tensor to \( \varepsilon(u(\mu)) = 1/2 \left( \nabla u(\mu) + \nabla u(\mu)^T \right) \).

We will use for inflow and outflow boundaries the notation \( \Gamma_N^{-}(\mu) = \{ x \in \Gamma_D(\mu) | g_D(\mu) \cdot n < 0 \} \), \( \Gamma_N^{+}(\mu) = \{ x \in \Gamma_N(\mu) | u(\mu) \cdot n < 0 \} \), and \( \Gamma_D^{+}(\mu) = \Gamma_D(\mu)/\Gamma_D^{+}(\mu) \) and \( \Gamma_N^{+}(\mu) = \Gamma_D(\mu)/\Gamma_N^{+}(\mu) \), where \( \overline{\text{Re}[\nu]} = \frac{2U}{\nu} \) is the Reynolds number, \( U \) and \( L \) are the characteristic speed and length of the problem, \( h \) is the characteristic mesh size, and \( \gamma_{N1} \) and \( \gamma_{N2} \) are penalty parameters, with \( \chi_{\Gamma_D(\mu)} \) and \( \chi_{\Gamma_N(\mu)} \) the characteristic functions of the boundaries \( \Gamma_D^{+}(\mu) \) and \( \Gamma_N^{+}(\mu) \), respectively.

#### 2.2. The conformal Nitsche's weak formulation.

For the sake of simplicity, in this subsection, we will omit the parameter dependency with respect to \( \mu \). Let \( V_h(\mathcal{D}(\mu)) \) and \( Q_h(\mathcal{D}(\mu)) \) be the spaces of continuous, piecewise-linear, vector- and scalar-valued functions. Namely:

\[
\begin{align*}
V_h(\mathcal{D}(\mu)) &= \{ v \in C^0(\mathcal{D}(\mu)) : v|_K \in (P^1(K))^d, \forall K \in \mathcal{T}(\mu) \}, \\
Q_h(\mathcal{D}(\mu)) &= \{ v \in C^0(\mathcal{D}(\mu)) : v|_K \in P^1(K), \forall K \in \mathcal{T}(\mu) \}.
\end{align*}
\]

We can now introduce a Nitsche's variational formulation of the Navier-Stokes equations:

**find** \( u \in V_h(\mathcal{D}(\mu)) \), \( p \in Q_h(\mathcal{D}(\mu)) \) such that, for all \( w \in V_h(\mathcal{D}(\mu)) \) and for all \( q \in Q_h(\mathcal{D}(\mu)) \),

\[
\begin{align*}
0 &= \langle w, \rho (\mu \cdot \nabla u - g) \rangle_{\mathcal{D}(\mu)} - \langle w, \rho g_D \cdot n (u - g_D) \rangle_{\Gamma_D^{-}(\mu)} - \langle w, \mu_N(\mu) \rangle_{\Gamma_N^{-}(\mu)} - \langle w, \mu(\mu) \rangle_{\Gamma_N^{+}(\mu)} \\
&\quad - \langle \nabla \cdot w, p \rangle_{\mathcal{D}(\mu)} - \langle q, \nabla \cdot u \rangle_{\mathcal{D}(\mu)} + \langle q, \mu_D \rangle_{\mathcal{D}(\mu)} \\
&\quad + \langle \varepsilon(w), 2 \nu \varepsilon(u) \rangle_{\mathcal{D}(\mu)} - \langle w \otimes \n, 2 \nu \varepsilon(u) - p I \rangle_{\mathcal{D}(\mu)} - \langle 2 \nu \varepsilon(w), (u - g_D) \otimes n \rangle_{\mathcal{D}(\mu)} \\
&\quad + \langle w, \nu / h (\gamma_{N1} I + \gamma_{N2} \overline{\text{Re}}[\nu] n \otimes n) (u - g_D) \rangle_{\mathcal{D}(\mu)}.
\end{align*}
\]

Here we use the standard notation \( \langle \cdot, \cdot \rangle_{\mathcal{D}}, \langle \cdot, \cdot \rangle_{\Gamma_D}, \langle \cdot, \cdot \rangle_{\Gamma_N} \) for the \( L^2(\mathcal{D}) \), \( L^2(\Gamma_D) \) and \( L^2(\Gamma_N) \) inner products over \( \mathcal{D} \), \( \Gamma_D \) and \( \Gamma_N \), respectively, and we denote the diameter of element \( K \in \mathcal{T} \) as \( h_K \). The overall size of the mesh is denoted by \( h = \max_{K \in \mathcal{T}} h_K \).
2.3. Shifted boundary variational formulation. In what follows, we outline the Shifted Boundary Method for the Navier-Stokes equations. A surrogate domain, $\tilde{D}$, and a surrogate boundary, $\tilde{\Gamma}$, that substitutes the original domain, $D$, and its boundary, $\Gamma$, is defined as visualized in Figure 1. The surrogate boundary $\tilde{\Gamma}$ is consisted of the closest to the boundary $\Gamma$ edges identified by the closest point projection rule.

The surrogate boundary $\tilde{\Gamma}$ encloses the surrogate domain $\tilde{D}$ (subset of $D$). Moreover, $\tilde{n}$ denotes the unit outward pointing normal to the $\tilde{\Gamma}$, and $n$ the outward pointing normal to $\Gamma$. The closest point projection is a piecewise Lipschitz continuous mapping $M$ from $\tilde{\Gamma}$ to $\Gamma$. Namely, $M: \tilde{x} \in \tilde{\Gamma} \rightarrow x \in \Gamma$.

The corresponding distance mapping vector $M$ is defined as

$$d = d_M(x) = x - \tilde{x} = [M - I](\tilde{x}),$$

and the $d = ||d||n$ (distance vector) has the same direction as $n$ (due to the closest point projection procedure), see Figure 1. The latter allows the original surface to be smooth (between edges and corners), assuming that $M$ is continuous and Lipschitz. Furthermore, the mapping $M$ allows to extend the unit-normal vector $n$ and the unit-tangential vectors $\tau_i$, $1 < i < d - 1$ to the boundary $\tilde{\Gamma}$ as $\tilde{n}(\tilde{x}) \equiv n(M(\tilde{x}))$ and $\tilde{\tau}_i(\tilde{x}) \equiv \tau_i(M(\tilde{x}))$. In what follows, $n(\tilde{x})$ is the $\tilde{n}(\tilde{x})$ at $\tilde{x} \in \tilde{\Gamma}$, and $\tilde{\tau}_i(\tilde{x})$ is the $\tilde{\tau}_i(\tilde{x})$. The mapping $g$ to the $\tilde{\tau}_i$ direction at $\tilde{x} \in \tilde{\Gamma}$ allows the derivative $\nabla_{\tilde{\tau}_i} g = \nabla g(\tilde{x}) \cdot \tilde{\tau}_i(\tilde{x})$.

In this way we extend the $\Gamma$ boundary conditions to the surrogate $\tilde{\Gamma}$ boundary.

We assume $n \cdot \tilde{n} \geq 0$, and this allows to a minimal-grid choice, see [13, 14]. Next we address the shifted boundary method weak form. We discretize the continuous system with the mesh $\tilde{D}_T$ composed of simplexes $K$ which construct a triangulation $\mathcal{T}$. Using the modified continuous piecewise-linear (discrete) spaces for the velocity/pressure,

$$V_h(\tilde{D}(\mu)) = \left\{ v \in (C^0(\tilde{D}(\mu)))^2 : v|_K \in (P^1(K))^d, \forall K \in \tilde{D}_T(\mu) \right\},$$

$$Q_h(\tilde{D}(\mu)) = \left\{ v \in C^0(\tilde{D}(\mu)) : v|_K \in P^1(K), \forall K \in \tilde{D}_T(\mu) \right\},$$

we can now introduce the Shifted boundary variational formulation:
find \( \mathbf{u} \in \mathbf{V}_h(\mathcal{D}(\mu)) \), \( p \in Q_h(\mathcal{D}(\mu)) \) such that, for all \( \mathbf{w} \in \mathbf{V}_h(\mathcal{D}(\mu)) \) and for all \( q \in Q_h(\mathcal{D}(\mu)) \),
\[
0 = \Pi_\mathbf{NS}[\nu](\mathbf{u}; p; w, q) + \Pi_\mathbf{STAB}[\nu](\mathbf{u}; p; w, q),
\]
where \( \Pi_\mathbf{NS} \) and \( \Pi_\mathbf{STAB} \) denote the SUPG/PSPG stabilization operators.

2.4. Variational multiscale stabilized finite element formulation. Because the proposed variational statement is not numerically stable, we introduce SUPG and PSPG stabilizing operators according to [41, 42], to which the reader can refer for more details. Since this is not the main focus of this paper, we refer the reader to [14, 43] for their implementation. The abstract variational form would then read:
\[
0 = \Pi_\mathbf{NS}[\nu](\mathbf{u}; p; w, q) + \Pi_\mathbf{STAB}[\nu](\mathbf{u}; p; w, q),
\]
where \( \Pi_\mathbf{STAB}[\nu] \) denotes the SUPG/PSPG stabilization operators.

In what follows, it is useful to define the Navier-Stokes operator
\[
\frac{d \mathbf{u}}{d t} + \nabla \cdot (\mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot \mathbf{f} + \mathbf{f}_N,
\]
and the right hand side
\[
\mathbf{f}(\mu) = \begin{bmatrix} \mathbf{f}_T(\mu) \\ \mathbf{f}_N(\mu) \end{bmatrix},
\]
consisting of forcing and boundary data related to stabilization and Nitsche weak enforcement boundary terms. Using these definitions we can express the following residual of the algebraic system of equations:
\[
R(U(\mu)) = G(U(\mu))U(\mu) - F(U(\mu)).
\]
Furthermore, the Jacobian of \( G(U(\mu))U(\mu) \) reads
\[
\nabla G(U^{n-1}(\mu)) = \begin{bmatrix} \partial_\mathbf{u}(\mu) + \mathbf{C}(\mathbf{u}(\mu)) \\ B + C \end{bmatrix} B^T_{\mathbf{u}(\mu)} + B + C_{p(\mu)}
\]
and yields the following iterative algebraic system of equations for the increment \( \delta U(\mu) = U^n(\mu) - U^{n-1}(\mu) \):
\[
\nabla G(U^{n-1}(\mu)) \delta U(\mu) = -R(U^{n-1}(\mu)).
\]
In the latter formulation we underline some features that are critical later in the ROM strategy described in § 3: additionally to the parameter dependent quantities \( \mathbf{A}, \mathbf{C}, \mathbf{B}, \bar{B} \) (saddle point structure) the stabilization operator \( \mathbf{C} \) has been employed. The latter high fidelity “tool” drives the system to fulfill the inf-sup property and allows the (normally unstable) \( \mathbf{P}_1/\mathbf{P}_1 \) finite elements pair, property which, as we will see later, propagates to the ROM.

The above formulation, solves the high fidelity system during the offline procedure, producing snapshots appropriate to derive a proper ROM as introduced in § 3.

3. ROM AND POD-GALERKIN METHOD

In the following, a POD-Galerkin framework will be analyzed as in [25, 44]. We will simplify the high fidelity model system to a reduced order one, which preserves the physics of the problem with reduced computational cost in a way adapted to embedded-immersed boundary finite element methods. This approach allows flexibility with geometrical changes and to effectively and efficiently overcome several related issues that appear when using traditional FEM (see for instance [45, 29, 30, 31]).

We focus on the geometrically parametrized nonlinear system of Navier-Stokes systems and on the benefits of the Shifted Boundary Method. We construct the reduced order basis in a classical way
According to the POD procedure, we collect a specific number of velocity vector $u(\phi)$ and pressure snapshots matrices: $S$. Following [62], one can transform (7) to the system:

$$\sum_{j=1}^{N_s} a_{ij}^k \phi_j \left| z(x) \right|^2 = 0,$$

where $(\phi_i, \phi_j)_D = \delta_{ij}$, for all $i, j = 1, \ldots, N_s$.

Following [62], one can transform (7) to the system:

$$\mathbf{X}^u Q^u = Q^u \mathbf{X}^u,$$

for $\mathbf{X}^u_{ij} = (u_i, u_j)_D$, $i, j = 1, \ldots, N_s$, the $\mathbf{X}^u$ corresponds to the square correlation matrix of eigenvectors derived from the parameter dependent realizations $\mathbf{S}_u$ and $Q^u$. We denote by $\mathbf{X}^u$ the (diagonal) matrix which contains the eigenvalues.

Thereafter, we can derive the basis components using the formula:

$$\phi_i = \frac{1}{N_s \lambda_{ii}^{1/2}} \sum_{j=1}^{N_s} u_j Q^u_{ij},$$

A similar mechanism is followed considering the pressure and we construct the snapshots matrix from the realizations $p_1, p_2, \ldots, p_{N_p}$, and the correlation matrix $C^p$ solving a similar eigenvalue problem.
\( C^p Q^p = Q^p \lambda^p \). Then the POD basis functions \( \chi_i \) is derived from the formula

\[
\chi_i = \frac{1}{N_s \lambda_i^{1/2}} \sum_{j=1}^{N_s} p_j Q^p_{ij}.
\]

Finally, we can take the velocity/pressure POD spaces:

\[
L_u = [\varphi_1, \ldots, \varphi_{N_u}] \in \mathbb{R}^{N_u \times N^e_u}, \quad L_p = [\chi_1, \ldots, \chi_{N_p}] \in \mathbb{R}^{N_p \times N^e_p}.
\]

We remark that the \( N_u^e \) and \( N_p^e \) are small numbers, (and much smaller than \( N_s \)) appropriately selected after examining the eigenvalue decay of \( \lambda^u \) and \( \lambda^p \), [37, 28].

Remark 3.1. Following the ideas of [29], in the out-of-interest area, i.e., the “ghost” region, we prefer to choose the extended solution as it is computed using the SBM (see the mapping \( \mathcal{M} \)). This approach employs a regular extension in the exterior of the boundary, onto the boundary neighboring simplices, while its values gradually are decreased to zero, see for example Figure 3. This strategy and mechanism permits a regular solution in the (background) mesh and domain, and consequently an effective reduced basis.
3.2. The ROM projection. Employing the aforementioned POD reduced basis, we can approximate the reduced order velocity, pressure with:
\[
\mathbf{u}^r \approx \sum_{i=1}^{N_r} a_i(\mu) \varphi_i(x) = L_a \mathbf{a}(\mu), \quad p^r \approx \sum_{i=1}^{N_r} b_i(\mu) \chi_i(x) = L_p \mathbf{b}(\mu).
\]

We emphasize that the reduced basis components \( \varphi_i, \chi_i \) are parameter independent while the coefficients \( \mathbf{a} \in \mathbb{R}^{N_r \times 1}, \mathbf{b} \in \mathbb{R}^{N_r \times 1} \) rely on the parameters. We remark that the \( \mathbf{L} = \begin{bmatrix} L_u & 0 \\ 0 & L_p \end{bmatrix} \)

\[
\mathbf{L}^T = \begin{bmatrix} L_u^T & 0 \\ 0 & L_p^T \end{bmatrix},
\]

the (unknown) coefficients \( \mathbf{V} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \) are derived using a Galerkin projection of the high fidelity solution onto the POD reduced basis spaces. The subsequent solution of a reduced iterative algebraic system for the increment \( \delta \mathbf{V}(\mu) \) then can be found using the formula
\[
\mathbf{L} \nabla G(U^{n-1}(\mu)) \mathbf{L}^T \delta \mathbf{V}(\mu) = -\mathbf{L}^T R(U^{n-1}(\mu)),
\]

which derives the subsequent reduced system of equations:
\[
\nabla G^*(V^{n-1}(\mu)) \delta \mathbf{V}(\mu) = -\mathbf{R}^*(V^{n-1}(\mu)).
\]

Concluding, we highlight in order to find the reduced order solution that we need to assemble the high fidelity matrices. Although, this expensive operation could be avoided, for example, using hyper reduction techniques [63, 64, 65, 66]. Moreover, during the online stage, also the stabilization term \( \mathbf{C} \) (as well as the nonlinear term \( \mathbf{D} \)) is projected onto the reduced basis space, which allows the inf-sup stabilization condition to propagate in an efficient way into the reduced model, as we will see below in the numerical examples.

Remark 3.2. We remark here that the reduced basis spaces have been generated using the iterative solution snapshots and not only the final solutions. This is justified by the fact that also at the reduced order level an iterative procedure is used to solve the non-linear problem and to obtain the reduced basis solutions.

4. Numerical experiments

We consider two different test cases, based on the setup shown in Figure 2. The first one consists of a geometrical parametrization using a one-dimensional parameter space where both parameters \( \mu_0, \mu_1 \) are fixed. The second one consists of a geometrical parametrization with a three-dimensional parameter space where all the parameters \( \mu_0, \mu_1, \mu_2 \) are left free. The problem domain is the rectangle \([-2, 2] \times [-1, 1]\), in which an embedded rectangular is immersed. The viscosity \( \nu \) is set to 1 and the Reynolds number is set to 100. A constant velocity in the \( x \) direction, \( u_{in} = 1 \) is applied at the left side of the domain, and an open boundary condition with \( \nabla \mathbf{u} \cdot \mathbf{n} = p_{out} = 0 \) on the right. In addition, a slip (no penetration) boundary condition is applied on the top and bottom edges while on the boundary of the embedded rectangle a no slip boundary condition is applied. The results for the test problems have been obtained with a mesh size of \( h = 0.0350 \) for the background mesh, using 150224 triangles for the discretization and \( P1/P1 \) polynomials.

4.1. Geometrical parametrization with one-dimensional parameter space. In this first experiment, the embedded domain consists of a rectangle of size \( \mu_1 \times \frac{\mu_2}{2} = 2 \times \frac{1}{2} \) where its aspect ratio inside the domain is parametrized with a geometrical parameter describing the size of the rectangle embedded domain with respect to its \( y \)-length with aspect ratio \( = \frac{\mu_1}{\mu_2} \) as in Figure 2. The horizontal \( x \)-length of the size of the box is not parametrized and the box’s center is located on the left bottom corner of the domain \([-2, -1]\). The ROM has been trained with 600 samples and tested onto 10 samples, for a parameter range \( \mu_2 \in [1, 2] \), chosen randomly inside the parameter space. To test the accuracy of the ROM we compared its results on 10 additional samples that were not used to create the ROM and were selected randomly within the same range. Under these considerations we record in Figure 4 the first six modes for the velocity magnitude and pressure, while in Table 1 the respective relative errors \( \| \mathbf{u}_{FOM} - \mathbf{u}_{ROM} \|_{L^2(D)} \), \( \| \mathbf{p}_{FOM} - \mathbf{p}_{ROM} \|_{L^2(D)} \) are summarized and they are depicted in Figure 8. In Figure 6 we depict the FOM and ROM solutions together with the relative error for both velocity magnitude and pressure, as well as the FOM and ROM streamlines for the velocity vector, for four distinct values of the input parameter.
Figure 4. The first 6 basis components of the velocity (first row) and pressure (second row) for one-dimensional geometrical parameter space case: $\mu \in [1, 2]$.

Figure 5. The first 6 basis components of the velocity (first row) and pressure (second row) for three-dimensional geometrical parameter space case: $\mu \in [-0.65, -0.45] \times [0.25, 0.45] \times [1.25, 1.45]$. 
Figure 6. Results for one-dimensional geometrical parameter space with $\mu_1 \in [0.5,1.1]$. In rows 1–3 we report the full-order solution, the reduced order solution and the absolute error plots for the velocity field, in rows 4–6 we report the same quantities for the pressure field while in rows 7–8 the FOM and ROM velocity streamlines. The different columns are for four distinct values of the parameter $\mu_1 = [0.6263, 1.0967, 1.0169, 0.98941569]$.

4.2. A geometrical parametrization study with a three-dimensional parameter space. The second case considers a geometrical parametrization with a three-dimensional parameter space in the range $(\mu_0, \mu_1, \mu_2) \in [-0.6, -0.5] \times [0.3, 0.4] \times [1.3, 1.4]$. We perform this test to examine the performances of the methodology on a more complex scenario where the box of the previous numerical
Figure 7. Results for three-dimensional geometrical parameter space with \((\mu_0, \mu_1, \mu_3) \in [-0.6, -0.5] \times [0.3, 0.4] \times [1.3, 1.4]\). In rows 1 – 3 we report the high fidelity solution, the reduced order solution and the absolute error plots for the velocity field while in rows 4 – 6 we report the same quantities for the pressure field. The different columns are for four distinct values of the parameter \(\mu = \{(-0.5365, 0.3452, 1.3214), (-0.5313, 0.3164, 1.3822), (-0.5927, 0.3496, 1.3305), (-0.5303, 0.3941, 1.3071)\}\). Finally in rows 7 – 8 the streamlines for the velocity in the full and in the reduced level are visualized.
Snapshots: 600

| Modes N  | relative error |   |   |
|----------|----------------|---|---|
|          | u             | p |   |
| 20       | 0.0752959     | 19.590245 |   |
| 40       | 0.0193090     | 1.0666177 |   |
| 60       | 0.0128775     | 0.8244898 |   |
| 80       | 0.0064458     | 0.5823617 |   |
| 100      | 0.0039197     | 0.3994657 |   |
| 120      | 0.0026156     | 0.0731744 |   |
| 140      | 0.0024058     | 0.0545712 |   |

Table 1. The relative errors (between the high fidelity solution and the reduced solution) are presented for both velocity and pressure in the case of one-dimensional geometrical parameter space. Results are reported for different dimensions of the reduced basis spaces.

Figure 8. Visualization of the relative errors for velocity and pressure for the case of one-dimensional geometrical parameter space and for varied number of modes.

4.3. Some Comments. Additional testing using more snapshots and with and without supremizers yielded very similar results, in which supremizers delivered slightly worse velocity results and slightly improved pressure results. We found that the supremizers did not substantially improved the errors. In our opinion, this is a consequence of the incremental iterative formulation in the offline solver, which preserves the effects of the SUPG and PSPG stabilization in the reduced model, which allows an inf-sup stable reduced basis. Some basics related to these stability issues, and numerical results can be found in Appendix A. Both the full-order and the ROM simulation were run in serial on an Intel® Core™ i7-4770HQ 3.70GHz CPU.

5. Conclusions and future developments

We have introduced a geometrically parametrised ROM model emulator of the two-dimensional Navier–Stokes equations, of much reduced computational cost. The ROM model evaluation through numerical tests shows good convergence properties and low errors. Comparing with high-fidelity solutions, numerical ROM errors improve with the increase of the size of training data and of the number
| Modes | (I) Relative Error | (II) Execution Time |
|-------|-------------------|---------------------|
|       | \(N\) | \(u\) | \(p\) | (sec) |
| 20    | 0.0794779 | 2.4790682 | 8.5617990 |
| 40    | 0.0388933 | 0.5546928 | 8.9083982 |
| 60    | 0.0294700 | 0.4594517 | 8.8653870 |
| 80    | 0.0200465 | 0.3642104 | 8.823756 |
| 100   | 0.0085595 | 0.0558062 | 8.7412506 |
| 120   | 0.0076601 | 0.0428119 | 8.753559 |
| 140   | 0.0068780 | 0.0182534 | 8.8129689 |

Table 2. (I) The relative errors (between the high fidelity solution and the reduced solution) for velocity and pressure for 900 snapshots in the case of three-dimensional geometrical parameter space. (II) Execution time at the reduced order level. The computation time includes the assembling of the full-order matrices, their projection and the resolution of reduced problem. Times are for the resolution of one random value of the input parameter. The time execution at full-order level is equal to \(\approx 50.70\) sec.

![Figure 9](image)

Figure 9. Visualization of the results for three-dimensional geometrical parameter space. On the left plot are depicted the relative errors for velocity and pressure. On the right plot we visualize the execution times of the reduced order problem for one random parameter value. In both plots, results are visualized for various number of modes.

of basis components. As future perspectives, we indicate applications to the time dependent Navier-Stokes systems, fluid structure interaction problems, and shallow water flows. Additionally, from the model reduction point of view, we will pursue further developments in hyper-reduction techniques [64, 65, 66, 63] and transportation methodologies [29].

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Table 3. Supremizers “inf-sup” condition enrichment: The relative errors (between the high fidelity solution and the reduced solution) are presented for both velocity and pressure and for three-dimensional geometrical parameter space. The results are visualized for various numbers of the reduced basis components.

| Modes N | relative error u | relative error p |
|---------|-----------------|-----------------|
| 20      | 5.2407257       | 46.628631       |
| 40      | 0.0699833       | 0.3109475       |
| 60      | 0.0537041       | 0.2613186       |
| 80      | 0.0374247       | 0.2116895       |
| 100     | 0.0107190       | 0.0365011       |
| 120     | 0.0093516       | 0.0237161       |
| 140     | 0.0062227       | 0.0132555       |

Appendix A

In this appendix we explore the experiment as described in Section 4.2. We will justify why in all previous experiments we did not use any additional RB stabilization, verifying numerically that the SUPG and PSPG stabilization which is applied on the high fidelity solver is strongly propagating through the reduced basis construction procedure to the reduced level. In contrast, we refer the interested reader to the classical works of [35, 33, 57, 58] where the reduced solution stabilization is deemed necessary. Subsequently, we introduce the basics related to the stability issues that could appear in the offline and online stage. In the full order method stage it is well known that the spaces have to satisfy the, also parametric in our case, Ladyzhenskaya-Brezzi-Babuska “inf-sup” condition see e.g. [67]. In particular, it is required that there should exist a constant \( \beta > 0 \), independent to the discretization parameter \( h \), such that

\[
\inf_{0 \neq q \in Q_h} \sup_{0 \neq v \in V_h} \frac{\langle \nabla \cdot v, q \rangle}{\|\nabla v\| \|q\|} \geq \beta > 0.
\]

In the present work this condition is fulfilled for the high fidelity solution through the SUPG and PSPG stabilization. Even if, the offline stage and the snapshots are realized and computed by a stabilized numerical method, it is not guaranteed that this stability is preserved onto the reduced basis spaces [33, 51, 35]. Next we briefly introduce the “inf-sup” condition enforcement in the reduced level using supremizers, we illustrate the relative errors results and we are comparing them with the case without supremizers enrichment. Within this approach, the velocity supremizer basis functions \( L_{\text{sup}} = [\eta_1, \ldots, \eta_{N_{\text{sup}}}^v] \in \mathbb{R}^{N_h^v \times N_{\text{sup}}} \) are constructed and added to the reduced velocity space (see Section 3.1) which is finally transformed into

\[
\tilde{L}_u = [\phi_1, \ldots, \phi_{N_u^v}, \eta_1, \ldots, \eta_{N_{\text{sup}}}^v] \in \mathbb{R}^{N_h^v \times (N_u^v + N_{\text{sup}})}.
\]

To obtain the latter enrichment for each pressure basis function \( \chi_i \) the auxiliary “supremizer” problem:

\[
\Delta s_i = -\nabla \chi_i \text{ in } \mathcal{D}(\mu^i), \quad s_i = 0 \text{ on } \Gamma(\mu^i)
\]

is solved with an SBM Poisson solver starting from the parameter value \( \mu^i \). For each pressure basis function the corresponding supremizer element can be found and the solution \( s_i \) permits the realization of the “inf-sup” condition.

We emphasize that the above supremizer basis functions do not depend on the particular pressure basis functions and on the geometrical parameters, they are computed during the offline phase, and their calculation is based on the pressure snapshots. Obviously, if someone compares the Table 3 with Table 2 and examines their visualization in Figure 10, supremizers drove the reduced solution to slightly worse velocity results and slightly improved pressure results. So, the supremizers did not substantially improved the errors and this is the reason that we avoided their application. In our opinion, this phenomenon is a consequence of the incremental iterative formulation in the offline solver, which preserves the effects of the SUPG and PSPG stabilization in the reduced model and allows an inf-sup stable reduced basis.
Figure 10. Supremizers “inf-sup” condition enrichment: Relative errors (between the high fidelity solution and the reduced solution) for velocity and pressure for the case of three-dimensional geometrical parameter space and for varied number of modes are presented.

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