ON REPRESENTATIONS OF $U'_q\mathfrak{so}_n$

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Abstract. We study representations of the non-standard quantum deformation $U'_q\mathfrak{so}_n$ of $U\mathfrak{so}_n$ via a Verma module approach. This is used to recover the classification of finite-dimensional modules for $q$ not a root of unity, given by classical and non-classical series. We obtain new results at roots of unity, in particular for self-adjoint representations on Hilbert spaces.

It is well-known that there exists another $q$-deformation $U'_q\mathfrak{so}_n$ of the universal enveloping algebra of $\mathfrak{so}_n$ which differs from the Drinfeld-Jimbo quantum group, see [5], [12], [15]. Unlike the latter one, it can be embedded as a co-ideal subalgebra into the Drinfeld-Jimbo quantum group $U_q\mathfrak{sl}_n$. Besides its occurrences in connection with quantum symmetric spaces and in mathematical physics, it has more recently also appeared as a centralizer algebra for tensor products of spinor representations of $U_q\mathfrak{so}_N$ ([20]) and for $q$-Howe duality for orthogonal quantum groups ([17]), and in connection with von Neumann subfactors and in quantum computing, ([21], [16]). In the last two areas mentioned, one has to deal with representations of $U'_q\mathfrak{so}_n$ at roots of unity. A detailed classification of simple finite-dimensional representations $U'_q\mathfrak{so}_n$ has been obtained in the work of Klimyk and his collaborators if $q$ is not a root of unity (see [9] and references therein). Some representations have also been obtained for $q$ a root of unity in [10]. Unfortunately, this does not help for the representations in connection with subfactors and quantum computing. This lead to the approach in this paper, which differs from the ones by other authors in several substantial ways. It is also hoped that at least parts of this approach may be useful for the study of representations of more general coideal algebras, see e.g. [3], [17], [13] and the discussion at the end of this paper.

The first step in our approach is to define and construct Verma modules. Unfortunately, the algebras $U'_q\mathfrak{so}_n$ do not have canonical raising and lowering operators, which makes the construction more difficult (see Section 7 for details). Nevertheless, one can construct an analog of a Verma module $V_m$ also for the algebra $U'_q\mathfrak{so}_n$, for arbitrary weight $m$. As a first application, we obtain the classification of finite-dimensional simple modules for $q$ not a root of unity. Unlike for Drinfeld-Jimbo quantum groups, there also exist finite dimensional simple representations of $U'_q\mathfrak{so}_n$ which are not deformations of representations of $U\mathfrak{so}_n$; following the notation in [9], we call them non-classical representations. Moreover, we construct canonical finite-dimensional quotients in these cases which are well-defined for all values of $q \neq 0$. Their dimensions are given by Weyl’s character formula. If $q$ is not a root of unity, they are irreducible for the classical modules, but reducible for the non-classical modules. In the
latter case, the quotient decomposes into the direct sum of $2^{[(n-1)/2]}$ mutually non-equivalent irreducible modules, all of which have the same character with respect to our chosen Cartan subalgebra. With this approach, one can now also extend our classification to certain classes of representations at roots of unity. This includes representations on Hilbert spaces on which the generators act as self-adjoint operators. These representations are different from the ones constructed in [10].

Here is the contents of our paper in more detail. After basic definitions, we first give a detailed discussion of known results about representations of $U_q^\prime \mathfrak{so}_n$ in Section 1.2. In the second section we define the notion of a Verma module for $U_q^\prime \mathfrak{so}_n$, and we determine a certain canonical spanning set. In the third section, we prove all the necessary results for Verma modules for $U_q^\prime \mathfrak{so}_3$ via elementary methods which more or less have been known before. In particular, we recuperate the classification of all finite-dimensional $U_q^\prime \mathfrak{so}_3$-modules. Using this and known results about representations of $U_q^\prime \mathfrak{so}_n$, as reviewed in the first section, we prove that the spanning set in the second section is indeed a basis for any Verma module. Section 5 starts out with an elementary study of representations of $U_q^\prime \mathfrak{so}_4$. This is then used to classify all weights for which the corresponding Verma module allows a finite-dimensional quotient, and to construct such a quotient. These modules can be viewed as analogs of Weyl modules and their dimensions are given by Weyl’s character formula. As already mentioned above, they are reducible in the case of non-classical representations. In Section 6, we apply our results for representations at roots of unity. In particular, we show that for representations on Hilbert spaces for which the generators of $U_q^\prime \mathfrak{so}_n$ act as self-adjoint operators, they are again classified by their highest weights. The final section contains a brief discussion comparing the approach in this paper with other approaches and possible applications.

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1. Definitions and known representations

1.1. Definitions. We are primarily interested in representations over the complex numbers, with $q$ also a complex number. Occasionally, it will be convenient to view $q$ as a variable over the complex numbers. Many of the results hold over far more general rings (e.g. the results in Section 2 hold over any domain in which $[2] = q + q^{-1}$ is invertible). The algebra $U_q^\prime \mathfrak{so}_n$, often referred to as non-standard deformation of the universal enveloping algebra $U \mathfrak{so}_n$ of the orthogonal Lie algebra $\mathfrak{so}_n$ (see [5], [12], [15]), is defined via generators $B_i$, $1 \leq i \leq n - 1$ and relations $B_i B_j = B_j B_i$ for $|i - j| > 1$ and

$$B_i^2 B_{i \pm 1} - [2] B_i B_{i \pm 1} B_i + B_{i \pm 1} B_i^2 = B_{i \pm 1},$$

where the same sign is chosen in each summand. We identify roots and weights of $U_q^\prime \mathfrak{so}_n$ with vectors in $\mathbb{R}^k$, where $k = n/2$ or $(n - 1)/2$ depending on the parity of $n$, as usual. We shall
use the notation \( \epsilon_i \) for the \( i \)-th standard basis vector of \( \mathbb{R}^k \). As usual, we denote the simple roots \( \alpha_i \) for \( \mathfrak{so}_n \) by \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( 1 \leq i \leq (n-2)/2 \), and by \( \alpha_k = \epsilon_{k-1} + \epsilon_k \) for \( n = 2k \) even, and \( \alpha_k = \epsilon_k \) for \( n = 2k + 1 \) odd.

The analog of the Cartan subalgebra in \( U'_q\mathfrak{so}_n \) is the algebra generated by \( B_1, B_3, \ldots B_{2k-1} \) for \( n = 2k \) or \( n = 2k + 1 \). A vector \( v \) in a \( U'_q\mathfrak{so}_n \)-module is said to have weight \( m \) if \( B_{2i-1}v = m_i v \) for all \( B_{2i-1} \in \mathfrak{h} \). If \( \lambda \in \mathbb{R}^k \) we shall often use the notation \([\lambda]\) for the weight whose coordinates are \([\lambda_i]\); here, as usual, \([n] = (q^n - q^{-n})/(q - q^{-1})\). Another important type of weights are denoted by \([\lambda]_+\), where the \( i \)-th coordinate is given by \([\lambda_i]_+\), where now \([n]_+ = \sqrt{-1}(q^n + q^{-n})/(q - q^{-1})\).

1.2. Known representations. We review some of the known representations of \( U'_q\mathfrak{so}_n \). At this point, it mainly serves as a motivation for this paper. The results of this subsection will only be used later when we prove the linear independence of the spanning set of a Verma module, as established in the next section.

1. A classification of finite dimensional representations of \( U'_q\mathfrak{so}_n \) for \( q \) not a root of unity has been given by Klimyk and his coauthors in a series of papers, see [9] or [11] and the literature quoted there. They derived explicit matrix representations for analogs of Gelfand-Tsetlin bases. In particular, they found, besides the expected deformations of representations \( U\mathfrak{so}_n \) (called classical representations), additional non-classical representations which are not well-defined in the classical limit \( q = 1 \). Not surprisingly, the classical representations are classified by the dominant integral weights of \( \mathfrak{so}_n \). The non-classical representations are classified by half-integer dominant weights, up to possible sign changes for the generators, see Theorem 5.9 for a precise statement. We shall reprove their results (or results equivalent to theirs) in the special cases of \( U'_q\mathfrak{so}_3 \) and \( U'_q\mathfrak{so}_4 \), essentially in the same form. The results in the general case are of similar nature; however, the matrix coefficients in these explicit representations are considerably more complicated in the higher rank cases. Moreover, these representations are not well-defined for \( q \) a root of unity if the highest weight \( \lambda \) is an integral dominant weight. We will define representations for \( n > 4 \) in a different way for both the classical and the non-classical representations.

2. Quite different representations have been found in [20]. Let \( S = S_+ \oplus S_- \) be the direct sum of the two spinor representation of the Drinfeld-Jimbo quantum group \( U'_q\mathfrak{so}_N \) for \( N \) even. Then a matrix \( C \in \text{End}(S^\otimes 2) \) was found in [20] such that the map

\[
B_i \mapsto 1_{i-1} \otimes C \otimes 1_{n-1-i} \in \text{End}(S^\otimes n), \quad 1 \leq i \leq n-1
\]

defines a representation of \( U'_q\mathfrak{so}_n \). In particular, any irreducible classical representation of \( U'_q\mathfrak{so}_n \) does appear in these representations, see e.g. [16], Theorem 2.2. Unfortunately, this approach only gives us the classical representations. On the other hand, the relations are much easier to check as for the representations discussed before, and the representations are also well-defined for roots of unity. In particular, it follows from [19] and [20] that for special roots of unity (usually of the form \( q = e^{\pm 2\pi i/\ell} \)) one can obtain representations of \( U'_q\mathfrak{so}_n \) on Hilbert spaces on which the generators \( B_i \) act via self-adjoint operators. We may sometimes
call such representations of $U'_q\mathfrak{so}_n$ unitary, as they are (at least for these specific examples) closely related to unitary representations of braid groups.

3. Representations of $U'_q\mathfrak{so}_n$ for $q$ a primitive $\ell$-th root of unity also appear in [16], Section 4. We define them for $n$ odd, with the even case obtained by restricting to $U'_q\mathfrak{so}_{n-1}$. Let $V$ be a $\ell^{(n-1)/2}$-dimensional vector space with basis $v(i)$, where $i \in \{0,1,\ldots,\ell-1\}^{(n-1)/2}$. The action of $u_{2s-1}$ on $V$ is defined by $u_{2s-1}v(i) = q^{i_s}v(i)$. The action of $u_{2s}$ is given by the rule (indices modulo $\ell$):

$$u_{2s}(v(i_1,\ldots,i_s,i_{s+1},\ldots,i_{n-1})) = v(i_1,\ldots,i_s+1,i_{s+1}-1,i_{s+2},\ldots,i_{n-1});$$

in other words, the even indexed generators $u_{2s}$ permute the vectors $v(i_1,\ldots,i_{n-1})$ by shifting the $s$-th index up by 1 and the $(s+1)$-st index down by 1, except for $s = (n-1)/2$ where there is no index left for shifting down. It is easy to check that these operators satisfy the relations $u_iu_{i+1} = qu_{i+1}u_i$ and $u_iu_j = u_ju_i$ for $|i-j| > 1$. It was shown in [16], Lemma 4.3 that we obtain two representations of $U'_q\mathfrak{so}_n$ on $V$ given by the maps

\begin{align*}
B_j &\mapsto b_j = \pm \frac{\sqrt{-1}}{q-q^{-1}}(u_j + u_j^{-1}), & 1 \leq j \leq n-1, \\
B_j &\mapsto b_j = \frac{u_j - u_j^{-1}}{q-q^{-1}}, & 1 \leq j \leq n-1.
\end{align*}

Remark 1.1. The main motivation for this paper came from the paper [16]. Both in the context of Examples 2 and 3 we have unitary braid representations which are of interest in quantum computing. It was conjectured that the braid representations from Example 3 correspond to certain cases within the framework of Example 2. The proof of showing this correspondence was reduced to showing that the corresponding representations of $U'_q\mathfrak{so}_n$ are isomorphic. However, as we could not find a general representation theory which would also involve roots of unity, this was finally done via a somewhat indirect procedure. One of the aims of this paper is to close this gap by extending existing results also to roots of unity, at least to the extent that it would cover the just mentioned examples.

2. Spanning set for Verma modules

2.1. Basic exchange relations. It has already been observed in [5] that a PBW type theorem holds for the algebra $U'_q\mathfrak{so}_n$, using its embedding into the quantum group $U_q\mathfrak{sl}_n$. We will give a direct proof here of a weaker version of this statement, by exhibiting an explicit spanning set. This will then be used to construct an explicit spanning set for an analog of a Verma module.

Lemma 2.1. We define $B_{n,k} = B_{n-1}B_{n-2} \ldots B_k$. Then we have the following equalities, where $w_1$, $w_2$ and $w_3$ are linear combinations of products of generators with coefficients being Laurent polynomials in $q$:
as follows: If two monomials \( w_1 \) and \( w_2 \) have different length, i.e. the number of factors is different, then the word of shorter length is the smaller one. For words of same length, it will be convenient to define inverse alphabetic order, reading from the right. E.g. we get for \( B_1, B_2, B_3 \):

\[
[B_1, B_2] = B_1 B_2 - B_2 B_1 = [2] B_1 B_2 - B_2 B_1 = [2] (B_2 B_1 - B_1 B_2) = 1
\]

A similar calculation also shows that

\[
[B_{n-2}, B_{n,k}] = [B_{n-2}, B_{n-2,k} B_{n-2,n} - 1].
\]

The general case for statement (b) can be reduced to these calculations by observing that \( B_r B_n, k = B_{n,r+2}(B_r B_{r+2,k}) \) and \( B_n, k B_r = (B_{n,r-1} B_r B_{r-2,k}) \). Statement (c) can now be deduced from (a) and (b).

2.2. Ordering. We will need an ordering on monomials in the generators which is defined as follows: If two monomials \( w_1 \) and \( w_2 \) have different length, i.e. the number of factors is different, then the word of shorter length is the smaller one. For words of same length, it will be convenient to define inverse alphabetic order, reading from the right. E.g. we get for words of length 3 in \( U'_\mathfrak{so}_n \) the ordering

\[
B_3^2 < B_1 B_2 < B_2 B_1 < B_1^2 B_2 < B_2 B_1^2 < B_1 B_2 B_1 < B_2 B_1^2 < B_1^3
\]

Lemma 2.2. Let \( k > l \). Then \( B_n, k B_n, l = [2] B_n, l B_n, k + \text{lower terms} \).

Proof. The proof goes by downward induction on \( k \). If \( k = n - 1 \), then \( B_n, n-1 = B_{n-1} \), and the claim follows from Lemma 2.1, (a). For the induction step, we use Lemma 2.1, (b) for the second line below, and the induction assumption for the third line.

\[
B_{n, k} B_{n, l} = B_{n, k+1} B_{k, l} B_{n, l} = B_{n, k+1} B_{k, l} B_{n, l} + B_{n, k+1} (B_{k, l} B_{k, l} B_{n, l}) = ([2] B_{n, l} B_{n, k+1} + \text{lower terms}) B_{k, l} + B_{n, l} B_{n, k+1} - B_{k, l} B_{k, l} B_{n, l}.
\]

It follows directly from the definitions for words \( w_1 \) and \( w_2 \) in the generators \( B_j \) that if \( w_1 < w_2 \), then also \( w_1 B_k < w_2 B_k \). The claim follows from this.
2.3. A spanning set for a Verma module. Let \( m, \tilde{n} \) be weights. Then we define the left ideal \( I_{m, \tilde{n}} \) by

\[
I_{m, \tilde{n}} = U'_q \mathfrak{so}_n \langle (B_{2i-1} - m_i 1), (B_{2i-1} B_{2i} - \tilde{n}_i B_{2i}) \rangle
\]

for all values of \( i \) for which the indices \( 2i-1 \) and \( 2i \) are between (including) \( 1 \) and \( (n-1) \). It can be easily shown, see Lemma 4.1 that for given \( m_i \) we obtain a nonzero vector \( B_{2i-1} B_{2i} \) mod \( I_{m, \tilde{n}} \) only for two special values of \( \tilde{n}_i \). In the classical case these two cases would correspond to \( 1 \) being a highest resp a lowest weight vector mod \( I_{m, \tilde{n}} \). This will be made more precise in the following sections.

**Proposition 2.3.** (a) The algebra \( U'_q \mathfrak{so}_n \) is spanned by monomials of the form

\[
B_{e(2,1)}^1 B_{e(3,1)}^3 B_{e(3,2)}^3 B_{e(4,1)}^4 \ldots B_{e(n,n-2)}^e(n,n-2) B_{e(n,n-1)}^e(n,n-1)
\]

where the \( e(i,j) \) are nonnegative integers.

(b) The quotient \( V_{m, \tilde{n}} = U'_q \mathfrak{so}_n / I_{m, \tilde{n}} \) is spanned by the subset \( \mathcal{B} \) of the set in (a) which only contains factors \( B_{k,r} \) for which \( r \) is even.

**Proof.** For (a), we first show that any element in \( U'_q \mathfrak{so}_n \) is a linear combination of elements of the form \( \prod_i B_{n_i,k_i} \) with \( n_i \leq n_j \) for \( i < j \); here we assume an ordered product of the form \( B_{n_1,k_1} B_{n_2,k_2} \ldots \) with \( n_i > k_i \) for all \( i \). Indeed, if this was not the case, we would have within the word an expression of the form \( B_{n,k} B_r \), with \( k \leq r < n - 1 \). By Lemma 2.1, (a) and (b), we can replace this by a linear combination of shorter terms. We can continue this until we get a linear combination of products as stated. The fact that we can also assume \( k_i \geq k_j \) for \( i < j \) if \( n_i = n_j \) follows from Lemma 2.2, using the same strategy as before.

To prove part (b), we will show that whenever a product \( \prod_i B_{n_i,k_i} \) contains an odd \( k_i \), we can replace it by a linear combination of products of generators modulo \( I_{m, \tilde{n}} \) where each product has fewer factors than the original element. For the purpose of the induction, we will prove such a statement by induction on \( s \) for any element of the form \( B_{k} w \) or \( B_{k} B_{k+1} w \), with \( k \) odd and \( w = \prod_{j=1}^s B_{n_j,k_j} \) with all \( k_j \) even. If \( s = 0 \), this follows directly from the definition of \( I_{m, \tilde{n}} \). For the induction step, we only need to observe that we can replace \( B_{k} B_{n_1,k_1} \) resp \( B_{k} B_{k+1} n_{1,k_1} \) by suitable linear combinations of elements ending with an element \( C \), where \( C \) is equal to \( B_k, B_k B_{k+1} \) or \( B_{k+2} \), by Lemma 2.1. Now, by induction assumption, we can replace \( C \prod_{j=2}^s B_{n_j,k_j} \) by a linear combination of shorter elements.

Let \( s = (s_i) \), with \( s_i \) non-negative integers for \( 1 \leq i \leq \lfloor (n - 1)/2 \rfloor \). We define the subspace \( V_{m, \tilde{n}}[[s]] \) to be the span of all monomials in \( \mathcal{B} \) which contain the generator \( B_{2i} \) at most \( s_i \) times. The proof of the following corollary is essentially just a special case of the proof of Prop. 2.3.(b).

**Corollary 2.4.** The subspace \( V_{m, \tilde{n}}[[s]] \) is a module of the Cartan algebra, i.e. the subalgebra generated by \( B_{2i-1}, i \leq n/2 \).
3. Representations of $U'_q\mathfrak{so}_3$

As in the classical case, the representation theory for the simplest nontrivial case, $U'_q\mathfrak{so}_3$ is both elementary and important. Most or probably all of the results in this section have been obtained before, see e.g. [9] and references there, or [6]. We reprove the results here, as it will fix our notations and the applied methods will be useful for the general case.

3.1. General weights and Verma modules. As usual, we identify weights of $U'_q\mathfrak{so}_3$ with eigenvalues of $B_1$. So the ideal $I_{m,n}$ defined in the last section would depend on two numbers $m$ and $n$. We also use the notation $v_0$ for the image of 1 in the quotient $V_{m,n} = U'_q\mathfrak{so}_3/I_{m,n}$, and the notation

$$V_{m,n}[k] = \text{span}\{B_j^jv_0, \ 0 \leq j \leq k \}.$$  

In order to describe the weights of $V_{m,n}$, we define, for given weight $m$, a sequence $(m_j)$ by $m_0 = m$,

$$(3.2) \quad m_1 = \frac{1}{2}(2m \pm \sqrt{(q-q^{-1})^2m^2 + 4}), \quad \text{and} \quad m_{j+1} = [2]m_j - m_{j-1} \quad \text{for} \ j > 0.$$  

This sequence is uniquely determined by $m$ and $m_1$, the choice of the root of the polynomial $x^2 - [2]mx + m^2 - 1$. Such sequences will only appear in this section. So no confusion with coordinates $m_i$ of a weight $m$ for higher rank cases should occur. The following examples of sequences $(m_j)$ will be particularly important in our paper:

(a) If $m = [\lambda] = \frac{q^\lambda - q^{-\lambda}}{q - q^{-1}}$, then $m_j = [\lambda \pm j]$, with the sign fixed by the choice of sign for $m_1$.

(b) If $m = [\lambda]_+ = i\frac{q^\lambda + q^{-\lambda}}{q - q^{-1}}$, then $m_j = [\lambda \pm j]_+$, with the sign fixed by the choice of sign for $m_1$.

Lemma 3.1. (a) The numbers $m_{j+1}$ are the zeros of $x^2 - [2]m_jx + m_j^2 - 1$ for all $j$.

(b) We have $m_{j+k} = m_j$ for some $k > 0$ only if $q^k = 1$ or $m_j = [k/2]_+$.

Proof. Part (a) is shown by induction on $j$, with $j = 0$ being trivially true. It follows from the induction assumption for $j$ that $m_{j+1}^2 - [2]m_jm_{j+1} + m_j^2 = 0$; it follows that $m_j$ is a root of the polynomial $x^2 - [2]m_jx + m_j^2 - 1$. But then the second root is equal to $[2]m_{j+1} - m_j = m_{j+2}$. To prove (b), let us assume $j = 0$ for ease of notation. It is easy to prove by induction that

$$(3.3) \quad m_k = [k]m_1 - [k-1]m_0 \quad \text{and} \quad m_{-1} = [k+2]m_k - [k+1]m_{k+1}.$$  

Solving for $m_1$ in the first formula, we obtain from $m_k = m_0$ that

$$(2 + [2]k - [2][k])m_0 = \pm [k]\sqrt{(q-q^{-1})^2m_0^2 + 4}.$$  

This can be transformed to

$$2 - q^k - q^{-k})m_0^2 = |k|^2.$$  

Hence, if $0 \neq -(q^{k/2} - q^{-k/2})^2 = -q^{-k}(1 - q^k)^2$, then $m_0 = \pm |k/2|_+.$
Lemma 3.2. (a) If \( v \) is a vector in the \( U_q^* \mathfrak{so}_3 \) module \( V \) of weight \( \mu \), then \( (B_1^2 - [2] \mu B_1 + \mu^2 - 1)B_2 v = 0 \).

(b) The quotient \( U_q^* \mathfrak{so}_3 / I_{m,n} \) can have dimension \( > 1 \) only if \( n \) is a root of the polynomial \( x^2 - [2]m x + m^2 - 1 \).

(c) \( B_1 B_2^k v_0 = m_k B_2^k v_0 + v_k' \), where \( v_k' \in V_{m,n}[k - 1] \), as defined in 3.1.

(d) We have \( \prod_{j=0}^k (B_1 - m_j) = 0 \) on \( V_{m,n}[k] \).

Proof. Claim (a) follows from \( B_1^2 B_2 v = ([2] B_1 B_2 B_1 - B_2 B_1^2 + B_2) v \), using \( B_1 v = \mu v \). As \( B_2 \mod I_{m,n} \) would be an eigenvector of \( B_1 \) with eigenvalue \( n \), it can only be nonzero mod \( I_{m,n} \) if \( n \) is a root of the polynomial as stated, by (a). If \( n \) is not a root of the polynomial, it follows that \( B_2^j = 0 \mod I_{m,n} \) for all \( j \geq 1 \). Hence the quotient has at most dimension 1. Claim (c) is obviously true for \( k = 0, 1 \). For \( k > 1 \), it follows by induction, using

\[
B_1 B_2^k = [2] B_2 B_1 B_2^{k-1} - B_2^2 B_1 B_2^{k-2} + B_1 B_2^{k-2}.
\]

Assuming that the vectors \( B_2^j \), \( 0 \leq j \leq k \) are linearly independent, it follows from part (c) that \( B_1 \) acts on \( V_{m,n}[k] \) via a triangular matrix with diagonal entries \( m_j \), \( 0 \leq j \leq k \). If \( B_2^{k+1} v_0 \subset V_{m,n}[k] \), then so are all higher powers of \( B_2 \), and the same argument can be used for the smallest \( k \) for which the \( B_2^j v_0 \) are linearly independent, \( 0 \leq j \leq k \).

3.2. Finite-dimensional modules. The statement of the following proposition is not true for \( q \neq 1 \) a root of unity.

Proposition 3.3. If \( q \) is not a root of unity, then any finite dimensional simple \( U_q^* \mathfrak{so}_3 \)-module has to be a quotient of a Verma module.

Proof. Let \( V \) be a finite dimensional \( U_q^* \mathfrak{so}_3 \)-module, and let \( v \in V \) be an eigenvector of \( B_1 \) with eigenvalue \( m_0 \). If we can find such an eigenvector \( v \) for which also \( B_2 v \) is an eigenvector, with eigenvalue \( n \), then the map \( u \in U_q^* \mathfrak{so}_3 \rightarrow uv \) has a kernel containing the ideal \( I_{m_0,n} \). As \( V \) is simple, the claim would follow.

By Lemma 3.2 (a), the vector \( B_2 v \) lies in an at most 2-dimensional \( B_1 \)-invariant subspace. If we can not find a \( v \) as in the previous paragraph, we can inductively construct a sequence of eigenvectors \((v_j), j \in \mathbb{Z} \) with eigenvalues \( m_j \), by Lemma 3.2(a) and Lemma 3.1(a). As \( V \) is finite dimensional, this sequence can only take finitely many values and we have \( m_j = m_{j+k} \) for some \( j \) and some \( k > 0 \). If \( q \) is not a root of unity, then \( m_0 = \pm [k/2]_+ \), by Lemma 3.1(b). But then \( m_k = [k/2 - j]_+ \), and the sequence \((m_j) \) would take infinitely many distinct values. The claim now follows from Lemma 3.1(b).

3.3. Basis for Verma module. We have seen in the previous section that, for given \( m \), we only obtain nontrivial quotients for two values of \( n \). As a consequence of this, we will simplify the notation by writing \( I_m \) for \( I_{m,n} \), and \( V_m = U_q^* \mathfrak{so}_3 / I_m, V_m[k] \) etc, with the understanding that one of the two possible choices \( m_1 \) for \( n \) has been fixed.
Proposition 3.4. (a) Assume $m$ is such that $m_j \neq \pm \frac{2i}{q-1}$ for all $j > 0$, and that $q$ is not a root of unity. Then the Verma module $V_m$ has a basis of weight vectors $v_j \in V_m[j]$ with weight $m_j$, $j \geq 0$. We can choose them such that $B_2v_j = v_{j+1} + \alpha_{j-1,j}v_{j-1}$, where

$$\alpha_{j-1,j} = \frac{m_0m_{-1} - m_{j-1}m_j}{(m_j - m_{j+1})(m_{j-2} - m_j)}.$$ 

(b) We are not in case (a) if and only if $m = \pm \lfloor \lambda \rfloor_+$ for a positive integer $\lambda$ and $m_1 = \lfloor \lambda - 1 \rfloor_+$. In this case, there still exist weight vectors $v_j$, $0 \leq j \leq \lambda$ and $v_{2\lambda+1}$, with $m_j = [\lambda - j]_+$ and $m_{2\lambda+1} = [\lambda + 1]_+ = [-\lambda - 1]_+$.

Proof. Let us first assume that the vectors $B_2^j v_0$, $j = 0, 1, 2, \ldots$ are linearly independent. Then it follows from Lemma 3.2(c) and (d) that $B_1$ acts via a triangular matrix with respect to this basis, with diagonal entries $m_j$. It follows from Lemma 3.2(a) by induction on $j$ that $B_1$ can be diagonalized provided $m_{j+1} \neq m_{j-1}$ for all $j > 0$. Again, by Lemma 3.2(a), we have that $m_{j+1}$ are the roots of the polynomial $x^2 - 2m_j x + m_j^2 - 1$ (this is easy to check for the special case $m_j = [\lambda - j]$, from which the general case follows via a Zariski density argument). We deduce from this that

$$(m_{j+1} - m_{j-1})^2 = (2m_j)^2 - 4(m_j^2 - 1) = (q - q^{-1})^2 m_j^2 + 4.$$ 

Hence $m_{j+1} = m_{j-1}$ if and only if $m_j = \pm 2i/(q-q^{-1})$, as stated. We normalize the eigenvector $v_k$ with eigenvalue $m_k$ such that $v_k = B_2^j v + \sum_{j=0}^{k-1} \beta_j B_2^j v$. From this follows the expression for $B_2 v_j$, up to determining the scalar $\alpha_{j-1,j}$. To do this, first observe that

$$B_2^j v_j = v_{j+2} + (\alpha_{j,j+1} + \alpha_{j-1,j})v_j + \alpha_{j-2,j-1} \alpha_{j-1,j} v_{j-2}. \tag{3.4}$$

We now calculate $B_1 B_2^j v_j$ in two different ways. First by applying $B_1$ to Eq 3.4 directly, using $B_1 v_i = m_i v_i$. Secondly, using the relations, we also obtain

$$B_1 B_2^j v_j = [2] B_2 B_1 B_2 v - B_2^2 B_1 v_j + B_1 v_j,$$

after which we again expand it into a linear combination of $v_{j+2}$, $v_j$ and $v_{j-2}$ using the formula for $B_2 v_j$ in statement (a). Comparing the coefficients of $v_j$ in these two expressions we obtain

$$m_j (\alpha_{j,j+1} + \alpha_{j-1,j}) = [2] (m_{j+1} \alpha_{j,j+1} + m_{j-1} \alpha_{j-1,j}) - m_j (\alpha_{j,j+1} + \alpha_{j-1,j}) + m_j. \tag{3.5}$$

Using $m_{j+2} = [2] m_{j+1} - m_j$, we can simplify the expression above to

$$(m_{j+2} - m_j) \alpha_{j,j+1} = (m_j - m_{j-2}) \alpha_{j-1,j} - m_j.$$ 

Hence we can express $\alpha_{j,j+1}$ in terms of $\alpha_{j-1,j}$. To get the induction going, we calculate $B_1 B_2^j v_0$ in two different ways as before. One checks directly that

$$\alpha_{0,1} = \frac{m_0}{m_0 - m_2} = \frac{m_0 (m_{-1} - m_1)}{(m_0 - m_2)(m_{-1} - m_1)}.$$ 

The general form for $\alpha_{j,j+1}$ now follows from the two previous formulas by induction on $j$.

To prove the linear independence of the vectors $B_2^j v_0$, we consider a vector space $V$ with basis $v'_j$, on which we define the action of $B_1$ and $B_2$ by $B_2 v'_j = v'_{j+1} + \alpha_{j-1,j} v'_{j-1}$ and by
$B_1 v_j^r = m_j v_j^r$. It follows essentially from the same calculations as before that this does define a representation of $U_q^s\mathfrak{so}_3$. It is now straightforward to check that the kernel of the map $u \in U_q^s\mathfrak{so}_3 \mapsto uv_0^r$ contains the ideal $I_m$, and that the vectors $B_2 v_0^r$ are linearly independent.

Finally, to check (b), one observes that $2i/(q - q^{-1}) = [0]_+$. One deduces from this that $m_{j-r} = [r]_+ = [-r]_+$, as $[n]_+ = [-n]_+$. For the last claim, one observes that the matrix representing $B_1$ on $V_m[2j+1]$ for $m = [j]_+$ has the eigenvalue $[-j-1]_+ = [j+1]_+$ with multiplicity 1, which does not exist for its restriction to $V_m[2j]$, by Lemma 3.2.(d). Hence the projection onto the eigenspace of $B_1$ corresponding to eigenvalue $[\lambda - 2r - 1]_+$ is well-defined for $\lambda = r$, and $v_{2r+1}$ is its unique eigenvector with coefficient 1 for $B_2^{2r+1}v_0$ in its expansion with respect to the basis $(B_2^jv_0, 0 \leq j \leq 2r + 1)$.

**Corollary 3.5.** (a) If $m = [\lambda]$ and $m_1 = [\lambda - 1]$, then
\[
\alpha_{j-1,j}^r = \frac{[2\lambda+1-j]_+}{(q^\lambda-j+q^{-\lambda})(q^{\lambda-j+1}+q^{j-1}-\lambda)}.
\]
(b) If $m = [\lambda]_+$ and $m_1 = [\lambda - 1]_+$, then
\[
\alpha_{j-1,j}^r = \frac{[2\lambda+1-j]_+}{(q^\lambda-j+q^{-\lambda})(q^{\lambda-j+1}+q^{j-1}-\lambda)}.
\]

**Corollary 3.6.** Define polynomials $P_n$ inductively by $P_0 = 1$, $P_1(x) = x$ and $P_{n+1}(x) = xP_n(x) - \alpha_{n-1,n}P_{n-1}(x)$. Then $P_n(B_1)v_0 = v_n$.

### 3.4. Highest weight vectors

The following lemma will be useful for determining non-trivial highest weight vectors in $V_m$.

**Lemma 3.7.** Assume $\alpha_{j,j+1} = 0$.

(a) If $j = 2k$ is even, then $m_k = 0$ or $m_k = \pm [0]_+ = \pm 2i/(q - q^{-1})$.

(b) If $j = 2k + 1$ is odd, then $m_k = \pm [1/2]_+$ or $m_k = \pm [1/2]_+$.

**Proof.** We will need the following identities:

(A) $m_{-1}m_0 = [j+1]m_0m_{j-1} - [j]m_0m_j$,

(B) $m_jm_{k+1} = [k+1]m_jm_1 - [k]m_jm_0$,

(C) $m_{1}m_{j} - m_{0}m_{j-1} = m_{k+1}m_{j-k} - m_{k}m_{j-1-k}$.

Claim (A) and (B) are easily proved by induction on $j$ (for (A)) and on $k$ (for (B)), using Eq 3.3. Claim (C) is shown by induction on $k$, applying the two formulas in Eq 3.3 to $m_{k+1}$ and to $m_{j-1-k}$. Applying (A) and (B) for $j = k$, we obtain
\[
m_{-1}m_0 - m_jm_{j+1} = [j+1](m_0m_{j-1} - m_jm_1).
\]

Hence if $j = 2k$, we obtain from (C) that
\[
m_0m_{j-1} - m_jm_1 = m_k m_{k-1} - m_{k+1}m_k = m_k(m_{k+1} - m_{k-1}).
\]

Hence $\alpha_{j,j+1} = 0$ implies that either $m_k = 0$ or $m_{k+1} - m_{k-1} = 0$, which implies $m_k = [0]_+$. Similarly, if $j = 2k + 1$, we have
\[
m_0m_{j-1} - m_jm_1 = m_k^2 - m_{k+1}^2 = (m_{k+1} + m_k)(m_{k+1} - m_k).
\]
Observing that \( m_{k+1} = \frac{1}{2}([2]m_k \pm \sqrt{(q-1)^2 m_k^2 + 4}) \), we deduce from \( m_{k+1} = \pm m_k \) that

\[
(2 \pm [2])m_k = \pm \sqrt{(q-1)^2 m_k^2 + 4}.
\]

This can be transformed to

\[
m_k^2 = \frac{1}{2} \pm (q + q^{-1}) = \left[ \frac{q^{1/2} \pm q^{-1/2}}{q - q^{-1}} \right]^2,
\]

from which one deduces \( m_k = \pm [1/2]_+ \) or \( m_k = [1/2]_0 \), depending on the choice of sign and the choice of square root of \( q \).

3.5. Finite-dimensional modules for generic \( q \).

**Theorem 3.8.** We assume \( q \) not to be a root of unity except possibly for \( q = \pm 1 \). The Verma module \( V_m \) of \( U_q \mathfrak{so}_3 \) has a finite-dimensional quotient if

(a) \( m = [\lambda] \), \( m_1 = [\lambda - 1] \) or \( m = [-\lambda] \), \( m_1 = [1 - \lambda] \), for \( \lambda \in \frac{1}{2} \mathbb{Z} \), \( \lambda > 0 \), or

(b) \( m = \pm [\lambda]_+ \), \( m_1 = \pm [\lambda - 1]_+ \) for \( \lambda \in \frac{1}{2} + \mathbb{Z} \), \( \lambda > 0 \), with matching signs.

In both cases, the largest finite-dimensional quotient has dimension \( 2\lambda + 1 \). It is simple in case (a), and the direct sum \( L_+ \oplus L_- \) of two simple submodules of dimension \( \lambda + 1/2 \) each in case (b). These submodules correspond to the eigenspaces of \( B_2 \) with eigenvalues \( [\lambda - j]_+ \) resp with eigenvalues \( -[\lambda - j]_+ \), \( j = 0, 1, \ldots \lambda - 1/2 \).

**Proof.** By Proposition 3.4, the module \( V_m \) has a basis of weight vectors \( v_j \) in the cases listed here. It follows from a standard argument that then also any submodule of \( V_m \) would have a basis of weight vectors. If \( v_j \) is the vector of highest weight of such a submodule, it follows that \( \alpha_{j-1,j} = 0 \). One checks from Corollary 3.5 that this happens for \( j = 2\lambda + 1 \) in both cases (a) and (b), and that \( \alpha_{j-1,j} \neq 0 \) for all \( j \neq 2\lambda + 1 \). This shows that we have finite dimensional quotients of dimension \( 2\lambda + 1 \), and any other finite-dimensional quotient would have to be smaller. If \( m = [\lambda] \), all eigenvalues of \( B_1 \) are distinct in that quotient \( L \). As \( \alpha_{j,j+1} \neq 0 \) for \( 0 \leq j < 2\lambda \), one checks easily that \( L \) is simple.

If \( m = [\lambda]_+ \), the vectors \( v_j \) and \( v_{2\lambda - j} \) have the same eigenvalue for \( B_1 \), for \( 0 \leq j \leq 2\lambda \). We leave it to the reader to check that after a suitable rescaling of these basis vectors, \( B_2 \) is represented by a symmetric matrix \( A \) such that \( a_{j-1,j} = a_{j,j-1} = \sqrt{\alpha_{j-1,j}} \), and \( a_{i,j} = 0 \) if \( |i - j| \neq 1 \). Hence \( V_m/M \) is equal to the direct sum \( L_+ \oplus L_- \), where the modules \( L_\pm \) have bases \( v_j \pm v_{2\lambda - j} \), \( 0 \leq j \leq \lambda - 1/2 \). Now again the eigenvalues of \( B_1 \) are mutually distinct on each of these submodules, and one shows irreducibility as for case (a). We will prove the last statement in the proof of Theorem 3.11.

3.6. The case \( m = [\lambda]_+ \), with \( \lambda \in \mathbb{Z} \). As noted in Proposition 3.4, we do not have a basis of weight vectors for the Verma module in this case. In the following let \( r \) be a positive integer.
We then define, for generic highest weight $m$, the vectors $v'_{r+j}$, $1 \leq j \leq r$ by

$$v'_{r+j} = v_{r+j} + \prod_{i=1}^{j} \alpha_{r-i, r-i+1} v_{r-j}, \quad 1 \leq j \leq r.$$ 

**Lemma 3.9.** The vectors $v'_{r+j}$, $1 \leq j \leq r$ as well as the vector $v_{2r+1}$ are well-defined also if $m = [r]_+$.

**Proof.** We prove the claim by induction on $j$. Observe that $v'_{r+1} = B_2 v_r$, by Prop. 3.4. Using Eq 3.4, one obtains

$$B_2 v'_{r+1} = B_2^2 v_r = v'_{r+2} + (\alpha_{r, r+1} + \alpha_{r-1, r}) v_r,$$

where one checks that the coefficient of $v_r$ is well-defined also for $m = [r]_+$. Using the definitions, we similarly obtain, for $j > 1$,

$$B_2 v'_{r+j} = v'_{r+j+1} + \alpha_{r+j-1, r+j} v'_{r+j-1} + (\alpha_{r-j, r-j+1} - \alpha_{r+j-1, r+j}) \prod_{i=1}^{j-1} \alpha_{r-i-1, r-i} v_{r-j+1}.$$

It can then be shown by a direct calculation that the scalar of $v_{r-j+1}$ is also well-defined for $m = [r]_+$; the pole at $\alpha_{r-1, r}$ cancels with the zero at $\alpha_{r-j, r-j+1} - \alpha_{r+j-1, r+j}$ for $\lambda = r$. Hence we can express $v'_{r+j-1}$ by an expression of vectors which are also well-defined at $\lambda = r$, using the last formula. The existence of $v_{2r+1}$ follows from Proposition 3.4(b).

**Lemma 3.10.** The Verma module $V_m$ has no nontrivial submodule for $m = [r]_+$, with $r$ a positive integer.

**Proof.** By the previous lemma, we have a basis for $V_m$ consisting of the vectors $v_j$, with $0 \leq j \leq r$ or $j \geq 2r + 1$, and the vectors $v'_{r+j}$, $1 \leq j \leq r$. Now observe that

$$B_2 v_{2r+1} = v_{2r+2} + \alpha_{2r, 2r+1} v_{2r}$$

$$= v_{2r+2} + \alpha_{2r, 2r+1} v'_{2r} - \alpha_{2r, 2r+1} \prod_{i=1}^{r-1} \alpha_{r-i-1, r-i} v_0$$

$$= v_{2r+2} - \beta v_0 \quad \text{for } m = [r]_+,$$

(3.6)

where one checks as in the proof of Lemma 3.9 that for $m = [r]_+$ the scalar $\beta$ is well-defined and non-zero. Assume now $M$ is a submodule of $V_m$, and $v \in M$ a nonzero vector. Writing it as a sum of generalized eigenvectors of $B_1$, we can also assume that each of the generalized eigenvectors is in $M$. Applying $B_1$ to a generalized eigenvector, if necessary, we can also assume that the corresponding eigenvector is contained in $M$, i.e. a vector $v_j$ with $j \leq r$ or $j \geq 2r$. In both cases, we can show that then also $v_{2r+1}$ is in $M$, and hence also the generating vector $v_0$, by the calculation above.
3.7. Irreducible $U'_q\mathfrak{so}_3$-modules. We can now classify irreducible $U'_q\mathfrak{so}_3$-modules for $q$ not a root of unity. These results have been obtained before, see [9] and references there, or [6].

**Theorem 3.11.** Let $q$ not be a root of unity, except possibly for $q = \pm 1$. Then the following gives a complete list of finite dimensional simple modules $L$ of $U'_q\mathfrak{so}_3$, up to isomorphism:

(a) The module $L$ has highest weight $m = [\lambda]$ with $m_1 = [\lambda - 1]$ and $\lambda \in \frac{1}{2}\mathbb{Z}$, $\lambda \geq 0$. In this case, $L$ has dimension $2\lambda + 1$, and it is uniquely determined by its highest weight.

(b) The module $L$ has highest weight $m = \pm[\lambda]_+$ with $m_1 = \pm[\lambda - 1]_+$ (matching signs) and $\lambda \in \frac{1}{2} + \mathbb{Z}$, $\lambda \geq 0$. In each of these cases, $L$ has dimension $\lambda + \frac{1}{2}$, and there are exactly two non-equivalent modules with these properties. They are related by the outer automorphism $B_2 \mapsto -B_2$, $B_1 \mapsto B_1$.

(c) For any positive integer $k$, there are exactly 5 non-equivalent simple $U'_q\mathfrak{so}_3$ modules of dimension $k$, one in case (a) and four in case (b). Those four cases are related by possible sign changes $B_j \mapsto -B_j$, $j = 1, 2$.

(d) In every simple finite dimensional representation $L$ of $U'_q\mathfrak{so}_3$, $B_1$ is conjugate to $B_2$ or to $-B_2$.

*Proof.* We have shown in Lemma 3.10 that the Verma module $V_m$ does not have a finite-dimensional quotient if $m = [\lambda]_+$ for $\lambda$ a positive integer. In all other cases, the Verma module $V_m$ has a basis of weight vectors, by Prop. 3.4. We can have a nontrivial submodule in $V_m$ only if $\alpha_{j-1,j} = 0$ for some $j$, see e.g. the proof of Theorem 3.8. It follows from Lemma 3.7 that this is possible only if the weights are of the form $[\mu]$ with all $\mu \in \mathbb{Z}$ or all $\mu \in \frac{1}{2} + \mathbb{Z}$, or all weights are of the form $[\mu]_+$, with $\mu \in \frac{1}{2} + \mathbb{Z}$ (see also Lemma 3.10). Excluding the cases with $v_0$ being a lowest-weight vector in the usual sense, we are left with the cases already listed in Theorem 3.8.

Now observe that for $m = [\lambda]_+$, $B_1$ has trace $\sum_{j=0}^{\lambda-1/2}[j]_+ \neq 0$ on $L_\pm$, and $B_2$ has only one nonzero diagonal element, which is equal to $\pm a_{r-1,r}$, where $\lambda = r + \frac{1}{2}$. This shows that $L_\pm$ are two non-equivalent $U'_q\mathfrak{so}_3$ modules. On the other hand, if $m = [\lambda]$, both $B_1$ and $B_2$ are represented by matrices with trace equal to 0. This proves (c), as we can also replace $B_1$ by $-B_1$ in case (b); changing the sign of the generators in (a) does not change the weights, and hence must produce an equivalent module. Part (d) now follows from the fact that constructing modules using the weight spaces of $B_2$ also only gives us 5 non-equivalent modules. By the trace argument above, $B_1$ being in case (a) is equivalent to $B_2$ being in case (a). Hence, as there is only one irreducible module in that case, $B_1$ and $B_2$ must have the same eigenvalues. The same argument also works for case (b), up to possible sign changes. Finally, statements (d) and (c) and the arguments in this paragraph also imply the last statement of Theorem 3.8.

3.8. Examples. We now want to apply our results so far for the representations of $U'_q\mathfrak{so}_3$ on the $\ell$-dimensional module $V$ defined in Eq 1.3 and 1.4 for $q$ a primitive $\ell$-th root of unity. For $\ell$ even, this has already more or less appeared in [16], Lemma 4.3.
Lemma 3.12. (a) If $\ell$ is even, the module $V$ is a direct sum of two irreducible modules of dimensions $\ell/2 \pm 1$ with highest weights $[\ell/4]$ and $[\ell/4 - 1]$ for Eq 1.3.

(b) If $\ell$ is odd, the module $V$ is the direct sum of two modules of $U'_q\mathfrak{so}_3$ of dimensions $(\ell \pm 1)/2$ and highest weights $-\lfloor \ell/2 \rfloor_+$ and $-\lfloor \ell/2 - 1 \rfloor_+$ for Eq 1.3.

(c) We only have highest weight modules for representations given by Eq 1.4 if $\ell$ is divisible by 4. In this case they are equivalent to representations in (a).

Proof. Let us do case (b) in some detail. Let $q^{1/2}$ be the square root of $q$ for which $q^{\ell/2} = -1$. Using this and Eq 1.3, the eigenvalue of $B_1$ for the $j$-th basis vector is

$$\frac{q^j + q^{-j}}{q - q^{-1}} = -\frac{q^{\ell/2-j} + q^{-\ell/2}}{q - q^{-1}}.$$

This shows that the character of $V$ is the sum of the characters of the two highest weight modules as claimed. Indeed, $v_0$ and $v_1 - v_{-1}$ are highest weight vectors for the given weights.

It follows from Cor. 3.5 that the coefficients $\alpha_{i-1,j}$ in Prop. 3.4 are well-defined and non-zero for the highest weights $[\ell/2]_+$ and $[\ell/2 - 1]_+$ also for our choice of $q$. Hence these modules are irreducible. Part (a) is shown similarly and was already done in [16]. Part (c) follows from the observation that we can not find a weight vector $v \in V$ such that also $B_2v$ is a weight vector unless we can find a weight $\pm 2i/(q - q^{-1})$.

4. Basic results for Verma modules of $U'_q\mathfrak{so}_n$

4.1. Weights. The following notation will be convenient: If $\mu = (\mu_i)_i \in \mathbb{R}^k$, we use the notations $[\mu]$ for the vector $([\mu_i])$, and $[\mu]^+$ for the vector $([\mu_i]^+)$, where, as usual $[k] = (q^k - q^{-k})/(q - q^{-1})$ and $[k]^+ = (q^k + q^{-k})/(q - q^{-1})$. If $k = r/s$ is rational, we assume that a choice for an $s$-th root of $q$ has been made. The following lemma clarifies how generators which are not in the Cartan algebra act on weight vectors. It will be convenient to use the notation $\alpha_i^+ = \epsilon_i + \epsilon_{i+1}$; here $\epsilon_i$ is as in Section 1.1. Observe that $\alpha_i^+$ is a positive root.

Lemma 4.1. Let $v$ be a vector in a $U'_q\mathfrak{so}_n$ module with weight $[\mu]$. Then

(a) $(B_{2i-1} - [\mu_i + 1])(B_{2i-1} - [\mu_i - 1])B_{2i}v = 0$.

(b) The vector $B_{2i}v$ can be written as a linear combination of at most four weight vectors, with weights $[\mu \pm \alpha_i]$ and $[\mu \pm \alpha_i^+]$.

(c) The analogous statements also hold if every weight $[\nu]$ in (a) and (b) is replaced by $[\nu]^+$.

Proof. These are straightforward calculations. E.g. for (a) we have

$$B_{2i-1}^2B_{2i}v = ([2][\mu_i]B_{2i-1}B_{2i} - B_{2i}B_{2i-1} + B_{2i})v$$

$$(4.1) = ([2][\mu_i]B_{2i-1} - ([\mu_i]^2 - 1))B_{2i}v.$$

We now get the claimed factorization in (a) using the identities $[2][\mu_i] = [\mu_i + 1] + [\mu_i - 1]$ and $[\mu_i]^2 - 1 = [\mu_i + 1][\mu_i - 1]$. For part (b) observe that a similar calculation also holds with $2i - 1$ replaced by $2i + 1$. The claim follows from this. Part (c) can be proved in the same way as parts (a) and (b).
4.2. Applications to general $U'_q\mathfrak{so}_n$ modules.

**Corollary 4.2.** Let $V$ be a finite-dimensional simple $U'_q\mathfrak{so}_n$ module, for $q$ not a root of unity except for $q = \pm 1$. Then we have, using the notations of Section 1.1 that

(a) all its weights are of the form $[\mu]$ with all $\mu_i \in \mathbb{Z}$ or all $\mu_i \in \frac{1}{2} + \mathbb{Z}$, or

(b) all its weights are of the form $(\pm [\mu_i]_+)$ with all $\mu_i \in \frac{1}{2} + \mathbb{Z}$ (where $q \neq \pm 1$).

**Proof.** This follows for $\mu_1$ from our results for $U'_q\mathfrak{so}_3$. Let $U'_q\mathfrak{so}_3(i)$ be the algebra generated by $B_i$ and $B_{i+1}$. Then we can show by induction on $i$, using Theorem 3.11, that also the eigenvalues of $B_{i+1}$ have to be the same as the ones of $B_i$, up to a possible sign change.

**Definition 4.3.** We say that a weight $m$ of $U'_q\mathfrak{so}_n$ is regularly dominant if $m = [\lambda]$ or $m = [\lambda]_+$ with $\lambda$ a dominant integral weight of $\mathfrak{so}_n$ with all $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ if $m = [\lambda]_+$

**Proposition 4.4.** Let $q$ not be a root of unity. Then any simple finite dimensional $U'_q\mathfrak{so}_n$-module $V$ is a quotient of a Verma module with highest weight $m$ being regularly dominant, after possible sign changes $B_{2i-1} \mapsto \pm B_{2i-1}$, for $1 \leq 2i - 1 \leq n$.

**Proof.** Corollary 4.2 already associates the weights of $V$ to weights in the classical case. If we are in case (b), we can assume that our module has at least one weight of the form $[\mu]_+$ with all $\mu_i > 0$, after possible sign changes $B_{2i-1} \mapsto \pm B_{2i-1}$. As our module is finite dimensional, it must have a highest weight. The rest of the proof now follows from usual standard arguments. If we are in case (a) of Corollary 4.2, the same arguments apply, without having to worry about possible sign changes.

**Definition 4.5.** We call $V_m = U'_q\mathfrak{so}_n/I_m$ a standard Verma module if $I_m = I_{m,n}$ is as before Prop. 2.3 with $m$ being a regularly dominant weight, and with $n_i = [\lambda_i - 1]_+$ resp $n_i = [\lambda_i - 1]_+$ if $m_i = [\lambda_i]_+$ resp $m_i = [\lambda_i]_+$.

**Remark 4.6.** If $q = 1$ and $m = [\lambda] = \lambda$, a standard Verma module will be the usual Verma module with highest weight $\lambda$. As we do not have explicit raising and lowering operators in the algebra $U'_q\mathfrak{so}_n$ any vector whose weight only differs by sign changes in some of its coordinates from the highest weight could also be made into a highest weight vector of $U'_q\mathfrak{so}_n$ according to the definition before Prop. 2.3. E.g. the lowest weight vector of a simple finite dimensional $U'_q\mathfrak{so}_3$-module would also be a highest weight vector. We can also carry out the theory of Verma modules for such weights. As this would only lead to more tedious notations, without any new results, we will restrict ourselves to standard Verma modules in the following.

4.3. Classical case $q = 1$. In order to show linear independence of the spanning sets in Prop. 2.3 we will appeal to the classical case at $q = 1$. Using matrix units $E_{ij}$, we can identify $B_j$ with $\sqrt{-1}(E_{j,j+1} - E_{j+1,j})$. It can be easily checked by explicit matrix calculations that we get root vectors of $\mathfrak{so}_n$ as follows, where, for simplicity, we write $ad_i$ for $ad_{B_i}$, the adjoint operation by $B_i$:

1. If $\alpha = \pm \epsilon_i \pm \epsilon_j$, we define $X_{\alpha} = (1 \pm ad_{2i-1})(1 \pm ad_{2j-1})ad_{2i+1} \cdots ad_{2j-3}B_{2j-2}$.

2. If $\alpha = \pm \epsilon_i$ is a short root for $\mathfrak{so}_n$, with $n$ odd, we have the root vector $X_{\alpha} = (1 \pm ad_{2i-1})ad_{2i+1} \cdots ad_{n-2}(B_{n-1})$. 

It will be convenient to use the following elements for describing a basis for the Verma module:

\[ Y_{-\epsilon_j} = X_{\epsilon_j} + X_{-\epsilon_j} = ad_2 ad_{2i+1} \ldots ad_{n-2}(B_{n-1}), \]
\[ Y_{-\epsilon_i+\epsilon_j} = X_{\epsilon_i+\epsilon_j} + X_{\epsilon_i-\epsilon_j} + X_{-\epsilon_i+\epsilon_j} + X_{-\epsilon_i-\epsilon_j} = ad_{2i-1} ad_{2i+1} \ldots ad_{2j-3}(B_{2j-2}), \]
\[ Y_{-\epsilon_i-\epsilon_j} = X_{\epsilon_i+\epsilon_j} - X_{\epsilon_i-\epsilon_j} - X_{-\epsilon_i+\epsilon_j} + X_{-\epsilon_i-\epsilon_j} = ad_{2i} ad_{2i+1} \ldots ad_{2j-3}(B_{2j-2}). \]

**Lemma 4.7.** Let \( V_m \) be a standard Verma module, see Def. 4.5, with \( m = [\lambda] \) for an integral dominant weight \( \lambda \). Then the spanning set in Prop. 2.3.(b) is linearly independent for \( q = 1 \).

**Proof.** We will use the following simple observation. Assume \((Y_i)_i \) is a set of elements in a semisimple Lie algebra \( g \) whose residues mod \( b_+ \) form a basis of \( g/b_+ \); here \( b_+ \) is the Borel algebra spanned by the Cartan algebra and the root vectors corresponding to positive roots. Hence we get a triangular transformation matrix between the spanning set of Prop. 2.3,(b) and the basis in Eq 4.2.

\[
\prod_i Y_i^{m_i} v_{\lambda}, \quad m_i \geq 0,
\]

where \( v_{\lambda} \) is the highest weight vector. Indeed, this is well-known if the \( Y_i = X_i \in n_- \) form a basis of the nilpotent part spanned by weight vectors corresponding to negative roots. In the general case, we can write \( Y_i = X_i + Z_i \), with \( X_i \in n_- \) and \( Z_i \in b_+ \). It is well-known that the Verma module \( V_{\lambda} \) has a filtration whose \( m \)-th subspace \( V_{\lambda}(m) \) is spanned by all products of \( \leq m \) generators \( X_i \in n_- \) modulo \( I_{\lambda} \), and that \( ZV_{\lambda}(m) \subset V_{\lambda}(m) \) for all \( Z \in b_+ \). It now follows easily that we get the same filtration also in terms of the generators \( Y_i \).

We now use the same argument for the ordering as defined before Lemma 2.2. First observe that with respect to this ordering we have

\[ ad_i ad_{i+1} \ldots ad_{j-1}(B_j) = (-1)^j i B_j B_{j-1} \ldots B_i B_i + \text{lower terms}. \]

Hence, mapping \( Y_\alpha \), with \( \alpha \) a negative root, to its highest term in its expansion of products of \( B_m \)'s, we obtain a bijection

\[ \Phi : \{ Y_\alpha, \alpha < 0 \} \leftrightarrow \{ B_{j,i}, 1 \leq i < j \leq n, i \text{ even} \}. \]

As the ordering is preserved by multiplication, we obtain in general that

\[ \Phi(\prod Y_\alpha^{m(\alpha)}) = \prod \Phi(Y_\alpha)^{m(\alpha)} + \text{lower terms}. \]

Hence we get a triangular transformation matrix between the spanning set of Prop. 2.3,(b) and the basis in Eq 4.2.

**4.4. Linear independence.** We will prove linear independence of the set in Prop 2.3 (b) for standard Verma modules by appealing to the already developed representation theory of \( U_q\mathfrak{so}_n \), see Section 1.2. We give the standard arguments in the proof of the following theorem for the reader’s convenience. We will use the following well-known result, which follows from a more precise result by Harish-Chandra (see e.g. [18], Theorem 4.7.3).
Proposition 4.8. Let \( q = 1 \) and let \( \lambda \) be an integral dominant weight, with \( V_\lambda \) and \( L_\lambda \) the Verma module and the irreducible module with highest weight \( \lambda \) respectively. Let \( \mu = \lambda - \sum r_i \alpha_i \) for some non-negative integers \( r_i \), and let \( \mathbf{r} = (r_i) \). We denote by \( V_\lambda[\mathbf{r}] \) and \( L_\lambda[\mathbf{r}] \) the span of weight vectors in the respective modules with weights of the form \( \lambda - \sum \tilde{r}_i \alpha_i \), with \( 0 \leq \tilde{r}_i \leq r_i \). Then, for given \( \mathbf{r} \), the restriction of the canonical map \( \Phi : V_\lambda \to V_\lambda[\mathbf{r}] \) is injective for all but finitely many dominant integral weights \( \lambda \).

Proof. We give the short proof for the reader’s convenience. By Harish-Chandra’s theorem (see e.g. [18], Theorem 4.7.3), the kernel of \( \Phi \) is spanned by submodules of \( V_\lambda \) with highest weights \( s_i \lambda = \lambda - ((\lambda, \alpha_i) + 1) \alpha_i \). As all the weights \( \mu \) of such a submodule satisfy \( \mu \leq s_i \lambda \), it follows that no weight vector of \( V_\lambda \) with weight \( \lambda - \sum \tilde{r}_i \alpha_i \) for which \( \tilde{r}_i \leq (\lambda, \alpha_i) \) for all \( i \) can be in the kernel of \( \Phi \). Hence, if \( r_i = (\lambda, \alpha_i) \), none of the weights of \( V_\lambda[\mathbf{r}] \) can be in the kernel of \( \Phi \).

Theorem 4.9. The set \( \mathcal{B} \) in Prop. 2.3 (b) is a basis for the Verma module \( V_{m,n} \), provided that \( m_i^2 - 2m_i n_i + n_i^2 = 1 \) for \( 1 \leq i \leq \lfloor n/2 \rfloor \). This includes, in particular, standard Verma modules, see Def 4.5.

Proof. Let \( \mathcal{B} \) be as in Prop. 2.3,(b), and let \( S = \{ M_j, 1 \leq j \leq m \} \) be a finite subset of \( \mathcal{B} \). By Lemma 4.7, their image in the Verma module \( V_\lambda \) is linearly independent for any dominant weight \( \lambda \) for \( q = 1 \). By Lemma 4.1,(b), there exists a vector \( \mathbf{r} \) with non-negative integer entries such that the image of \( S \) is contained in \( V_\lambda[\mathbf{r}] \). By Prop. 4.8, linear independence even holds for the image of \( S \) in \( L_\lambda \), for all but finitely many dominant integral weights. As discussed in Section 1, the representations \( L_\lambda \) are also well-defined for generic \( q \). In particular, the image of \( S \) is also linearly independent in \( L_\lambda \) for all but finitely many dominant integral weights, and for generic \( q \).

We now want to prove linear independence for general \( q \) for Verma modules \( V_{m,n} \) with \( m_i = [\lambda_i] \) and \( n_i = [\lambda_i - 1] \); for simplicity of notation, we will write \( V_\lambda \) for \( V_{m,n} \). We define a linear action of the generators \( B_i \) on a vector space whose basis elements are labeled by the monomials in \( \mathcal{B} \), using the relations in the proof of Prop. 2.3 and its preceding lemmas. Fix a monomial \( M \in \mathcal{B} \) and generators \( B_i \) and \( B_{i+1} \). Apply each monomial which appears in relation 1.1 to \( M \) and expand this as a linear combination of elements in \( \mathcal{B} \). Let \( S \) be the finite subset of \( \mathcal{B} \) consisting of all those elements appearing in these expansions. It follows from the discussion in the previous paragraph that the image of \( S \) is linearly independent in \( L_\lambda \) for all but finitely many dominant integral weights \( \lambda \). As \( L_\lambda \) is a \( U'_q \mathfrak{so}_n \)-module, it follows that the relation 1.1 holds, if applied to \( M \), for all but finitely many \( \lambda \)’s. Observe that the matrix coefficients for \( B_i \) and \( B_{i+1} \) are Laurent polynomials in the variables \( q \) and \( q^{\lambda_i} \), and the elements \( (q - q^{-1})^{-1} \) and \( (q + q^{-1})^{-1} \) (see Prop. 2.3 and the lemmas used for it). Hence the relation holds for a Zariski-dense set of values \( m_i = [\lambda_i] \) and \( n_i = [\lambda_i - 1] \), for highest weight \( m = [\lambda] \); this implies it holds for such \( m \) and \( n \) for any choice of values for \( \lambda \). It follows that we have constructed a highest weight representation of \( U'_q \mathfrak{so}_n \) with a basis labeled by the elements of \( \mathcal{B} \). Hence \( \mathcal{B} \) itself must be linearly independent for these choices of parameters.
To conclude the proof, we also need to show the claim for all possible choices of \( n_i \), for given \( m \). We have already seen that if \( m_i = [\lambda_i] \), the two possible solutions for \( n_i \) are \([\lambda_i \pm 1]\) (see the examples before Lemma 3.1). We can mimic the proof above by using integral weights of the form \((\lambda_i)\), where we choose negative entries for those coordinates \( i \) for which we want \( n_i = [\lambda_i + 1] \). By defining the Weyl chamber of this \( \lambda \) to be the dominant Weyl chamber, we again obtain a finite dimensional quotient; it is isomorphic to \( L_\lambda \), where \( |\lambda| \) is the weight in the usual dominant Weyl chamber which is conjugate to \( \lambda \). We can now show as in the previous paragraph that \( \mathcal{B} \) is linearly independent for any \( \lambda \), with \( m_i = [\lambda_i] \) and \( n_i = [\lambda_i \pm 1] \) for our given choice of signs. This concludes the proof of linear independence of \( \mathcal{B} \) for any \( m \) and \( n \) as given in the statement.

**Corollary 4.10.** If \( m = [\lambda] \) and \( V_m \) is a standard Verma module (see Def. 4.5), we have the ‘same’ weight multiplicities for \( V_m \) as in the classical case \( q = 1 \), i.e. the weight \([\mu]\) has the same multiplicity in \( V_m \) as the weight \( \mu \) has in \( V_\lambda \) for \( q = 1 \).

**Proof.** If \( V_m \) is a standard Verma module with \( m = [\lambda] \), then its weights are of the form \([\lambda - \omega]\), with \( \omega \) in the root lattice, by Lemma 4.1. This is also true for its finite-dimensional subspaces \( V_m[[s]] \), as defined before Cor. 2.4. The multiplicities of the (generalized) weight spaces are obtained from the characteristic polynomials of the \( B_i \)'s acting on \( V_m[[s]] \). It follows that the statement of the corollary is true for these subspaces. For any given weight \( \mu \), we can find a vector \( s \) with sufficiently large coordinates such that the multiplicity of \( \mu \) in \( V_m[[s]] \) coincides with the one of \( V_m \). The claim follows from this.

**Remark 4.11.** If \( m = [\lambda]_+ \) a slight subtlety occurs stemming from the fact that \([r]_+ = [-r]_+ \) for any rational number \( r \). However, if the \( \lambda \)'s are not integers, we can inductively decide for each eigenvector of \( B_{2i-1} \) whether it has formal eigenvalue \([\nu_i]_+ \) or \([-\nu_i]_+ \), by Lemma 4.1.(a). Indeed, if \( v \) is a weight vector with weight \([\mu]_+ \), and if we write \( B_{2i}v \) as a linear combination of weight vectors, we would only have difficulties in determining their formal weights if \( \mu_i = 0 \) or \( \mu_{i+1} = 0 \). Hence we can and will define distinct weight spaces \( V_m[\mu] \) and \( V_m[\tilde{\mu}] \) for weights \( \mu \neq \tilde{\mu} \) for which \(|\mu_i| = |\tilde{\mu}_i| \) for all \( i \), even though the corresponding weights \([\mu]_+ \) and \([\tilde{\mu}]_+ \) coincide.

5. **Representations of \( U'_q so_n \), \( n \geq 4 \)**

We assume for this section that \( q \) is not a root of unity, except for \( q = \pm 1 \).

### 5.1. Preliminaries

We first need some explicit results for \( U'_q so_4 \). It follows from Proposition 4.4 that any finite-dimensional \( U'_q so_4 \) module is a quotient of a standard Verma module \( V_m \). Moreover, the weights of \( V_m \) are of the form \([\mu] = [\lambda - r_1 \alpha_1 - r_2 \alpha_2] \) for \( m = [\lambda] \) or of the form \([\mu]_+ = [\lambda - r_1 \alpha_1 - r_2 \alpha_2]_+ \) for \( m = [\lambda]_+ \), where \( r_1, r_2 \geq 0 \) and \( \alpha_1 = \epsilon_1 - \epsilon_2 \) and \( \alpha_2 = \epsilon_1 + \epsilon_2 \). Using the well-known isomorphism \( so_4 \cong sl_2 \oplus sl_2 \), we see each weight in the Verma module has multiplicity 1 in the classical case. So we have a basis \((v_\mu)\), with \( \mu \) running through the set as described above for any standard Verma module, uniquely determined up to scalar multiples. In view of Theorem 4.9, we can use the same notation also for a basis of weight...
vectors for a standard Verma module $V_m$ of $U_q\mathfrak{so}_4$. In the following, it will be convenient to use the notation and easily proved identity

\begin{equation}
\{ k \} = q^k + q^{-k} = \lfloor k + 1 \rfloor - \lfloor k - 1 \rfloor.
\end{equation}

**Lemma 5.1.** Let $\mu = \lambda - r_1\alpha_1 - r_2\alpha_2$. The following statements are given for $m = \lfloor \lambda \rfloor$. They similarly hold for $m = \lfloor \lambda \rfloor_+$ after replacing any numbers $[a]$ by $[a]_+$. Let $\lambda \in \mathfrak{b}$. Then

(a) There exists a unique weight vector $v_\mu$ of weight $\mu$ of the form

\[ v_\mu = ((B_3B_2)^{r_1+r_2} + \text{lower terms})v_\lambda. \]

(b) If $\mu = \lambda - r_1\alpha_i$, $v_\mu$ can be defined inductively by $v_\mu = (B_3 - [\lambda_2 \pm (r - 2)]B_2)v_{\lambda - (r_1, -1)\alpha_i}$, where we have a plus sign for $i = 1$ and a minus sign for $i = 2$ respectively. In particular, there exists a polynomial $P_{\lambda, r, i}$ in two non-commuting variables such that $v_{\lambda - r_1\alpha_i} = P_{\lambda, r, i}(B_2, B_3)v_\lambda$.

**Proof.** We do the proof here for $m = \lfloor \lambda \rfloor$. It goes by induction on $r_1 + r_2$, which is trivially true for $r_1 + r_2 = 0$. It follows from Lemma 4.1 that $B_2v_\mu$ is a linear combination of weight vectors of weights $(\mu_1 \pm 1, \mu_2 \pm 1)$. Hence $(B_3 - [\mu_2 - 1])B_2v_\mu$ is a linear combination of two weight vectors of weights $(\mu_1 \pm 1, \mu_2 + 1)$. Subtracting a suitable multiple of $v_{(\mu_1 - 1, \mu_2 + 1)}$, we obtain a vector of weight $(\mu + 1, \mu_2 + 1)$. The statement about the leading term follows by induction assumption. Part (b) is a special case of part (a). Here it is easier to write down an explicit formula due to the fact that $\lambda - r_1\alpha_1 + \alpha_2$ and $\lambda - r_2\alpha_2 + \alpha_1$ are not weights for $V_m$.

**Lemma 5.2.** (see [7], Section VI) Let $q$ not be a root of unity, let $m = \lfloor \lambda \rfloor$ and let $(v_\mu)$ be the basis of weight vectors of $V_m$ as in Lemma 5.1. The action of the generators on $v_\mu$, with $\mu = \lambda - r_1\alpha_1 - r_2\alpha_2$, is given by $B_1v_\mu = [\mu_1]v_\mu$, $B_3v_\mu = [\mu_2]v_\mu$ and

\begin{equation}
B_2v_\mu = -\frac{\lambda_1 - r_1 + 1}{\mu_1}\frac{\lambda_2 + r_1}{\mu_2}\frac{[\lambda_1 - \lambda_2 - r_1 + 1]}{\{\mu_1\}\{\mu_1 + 1\}\{\mu_2\}}v_{\mu + \alpha_1}
\end{equation}

\begin{equation}
-\frac{\lambda_1 - r_2 + 1}{\mu_1}\frac{\lambda_2 - r_2}{\mu_2}\frac{[\lambda_1 + \lambda_2 - r_2 + 1]}{\{\mu_1\}\{\mu_1 + 1\}\{\mu_2\}}v_{\mu + \alpha_2}
\end{equation}

\begin{equation}
+ \frac{1}{\mu_2}[v_{\mu - \alpha_1} - v_{\mu - \alpha_2}].
\end{equation}

**Proof.** Up to a renormalization of the basis vectors, this is essentially just an extension of the results in e.g. [7], Section VI from the finite dimensional modules to the full Verma module. We give a brief outline here how this could be proved similar to our approach for $U_q\mathfrak{g}$ modules, see Prop. 3.4. It follows from Lemma 4.1 that we can write $B_2v_\mu$ as

\begin{equation}
B_2v_\mu = a_{\mu, r_1}v_{\mu + \alpha_1} + a_{\mu, r_2}v_{\mu + \alpha_2} + \frac{1}{\mu_2}[v_{\mu - \alpha_1} - v_{\mu - \alpha_2}],
\end{equation}

for suitable scalars $a_{\mu, r_i}$, $i = 1, 2$; here the scalars for $v_{\mu - \alpha_1}$ are determined by the normalization of our weight vectors as in Lemma 5.1. The remaining scalars are calculated by induction on $r_1 + r_2$ by explicitly checking the relation $B_2^2B_1 - [2]B_2B_1 + B_1B_2^2 = B_1$ at the vector
Corollary 5.3. Comparing the coefficients of the vectors $v_{\mu-\alpha_1+\alpha_2}$ and $v_{\mu+\alpha_1-\alpha_2}$, we obtain the recursion relations

$$\{\mu_1 - 1\}\{\mu_2 + 1\}a_{\mu-\alpha_1,\alpha_2} = \{\mu_1 + 1\}\{\mu_2\}a_{\mu,\alpha_2},$$

$$\{\mu_1 - 1\}\{\mu_2 - 1\}a_{\mu-\alpha_2,\alpha_1} = \{\mu_1 + 1\}\{\mu_2\}a_{\mu,\alpha_1}.$$ 

Moreover, comparing the coefficients of $v_\mu$ in the same relation, one obtains

$$\{\mu_1 - 1\}\{\mu_2\}(a_{\mu-\alpha_2,\alpha_2} - a_{\mu-\alpha_1,\alpha_1}) = -[\mu_1] + (\mu_1 + 1)(\frac{1}{\{\mu_2 + 1\}}a_{\mu,\alpha_2} - \frac{1}{\{\mu_2 - 1\}}a_{\mu,\alpha_1}).$$

Finally, comparing the coefficients of $v_\mu$ after applying the relation $B_2^2B_3 - [2]B_2B_3B_2 + B_3B_2^2 = B_3$ to that vector, we also obtain

$$\frac{1}{\{\mu_2\}}(\{\mu_2 - 1\}a_{\mu-\alpha_2,\alpha_2} + (\mu_2 + 1)a_{\mu-\alpha_1,\alpha_1}) = -[\mu_2] + a_{\mu,\alpha_1} + a_{\mu,\alpha_2}.$$ 

Using the last two recursion relations, one can calculate the coefficients $a_{\lambda-r_1\alpha_1,\alpha_1}$ and $a_{\lambda-r_2\alpha_2,\alpha_2}$ by induction on $r_1$ and $r_2$ respectively. Using the first two relations then allows us to calculate the coefficients in general.

**Corollary 5.3.** If $q$ is not a root of unity and $m = [\lambda]_+$, with $\lambda_i$ not a positive integer for $i = 1, 2$, the Verma module $V_m$ has a basis of weight vectors and we obtain representations as in the previous lemma by $B_1v_\mu = [\mu_1]_+v_\mu$ and $B_3v_\mu = [\mu_2]_+v_\mu$ and by replacing every occurrence of a factor $\{m\}$ by $i(q^m - q^{-m}) = [m + 1]_+ - [m - 1]_+$ in the definition of the action of $B_2$.

**Proof.** The proof goes completely analogous to the one of Lemma 5.2.

**5.2. Quotients of Verma modules for $U'_q\mathfrak{so}_4$.** As usual, we define the dot action of the Weyl group $W$ on weights by $w.\mu = w(\mu + \rho) - \rho$. If $m = [\lambda]$ or $m = [\lambda]_+$, we define $w.m$ to be equal to $[w.\lambda]$ and to $[w.\lambda]_+$ respectively. If $g = \mathfrak{so}_4$ and $s_1(\mu_1, \mu_2) = (\mu_2, \mu_1)$ and $s_2(\mu_1, \mu_2) = (-\mu_2, -\mu_1)$, we obtain $s_1.\lambda = \lambda - (r_1 + 1)\alpha_1$, $s_2.\lambda = \lambda - (r_2 + 1)\alpha_2$ and $s_1s_2.\lambda = \lambda - (r_1 + 1)\alpha_1 - (r_2 + 1)\alpha_2$, where $r_1 = \lambda_1 - \lambda_2$ and $r_2 = \lambda_1 + \lambda_2$.

**Proposition 5.4.** Let $V_m$ be a standard Verma module (see Def. 4.5) of $U'_q\mathfrak{so}_4$, with $m = [\lambda]$ or $m = [\lambda]_+$. We will write $w.\lambda$ for $w.m$ for simplicity of notation.

(a) The module $V_m$ has highest weight vectors $v_{s_i.\lambda} = P_{\lambda,(\lambda,\alpha_i)+1,i}(B_2, B_3)v_\lambda$ with weights $s_i.m$, $i = 1, 2$, where $P_{\lambda,(\lambda,\alpha_i)+1,i}$ is as in Lemma 5.1. Moreover, each of these vectors generates a submodule of $V_m$ which is isomorphic to the Verma module with the same weight.

(b) Let $M$ be a $U'_q\mathfrak{so}_4$ module with highest weight $\lambda$. Assume there exists a constant $K$ such that $|\mu_1 - \mu_2| \leq K$ for each weight $\mu$ of $M$. Then the weight spaces $M[\mu]$ and $M[s_1(\mu)]$ have the same dimensions for each weight $\mu$ of $M$.
Lemma 5.5. Then $V_s$.

Let $s$ be a simple reflection of $V$. Fix a simple reflection $s$. Let $M$ be the quotient module (see e.g. [18], Theorem 4.7.3 and preceding lemmas/theorems).

But as $s$, it is an element of it. One deduces from this that the vector $\pm .\lambda = \frac{1}{2} + Z, i = 1, 2$ go similarly.

By abuse of notation, we will denote the submodules of $V_m$ generated by the vector $v_w, \lambda$ by $V_m, \lambda$. By assumption, we have a surjective map $\Phi$ from $V_m$ onto $M$. It follows again from Lemma 5.2 that the only nontrivial submodule of $V_{s, \lambda}$ is $V_{s, s_2, \lambda}$. We do know all the weights of $V_{s, \lambda}$ and $V_{s, \lambda}/V_{s, s_2, \lambda}$, from which we see that these two modules do not satisfy the condition on the weights as in the statement. Hence the kernel of $\Phi$ must contain $V_{s, \lambda}$. We conclude that $M$ must be isomorphic to $V_{\lambda}/V_{s, \lambda}$ or to $V_{\lambda}/(V_{s, \lambda} + V_{s_2, \lambda})$, both of which satisfy the claim.

5.3. Quotients of Verma modules for $U'_q\mathfrak{so}_n$.

Lemma 5.5. Let $V_m$ be a standard Verma module for $U'_q\mathfrak{so}_n$. If $s_i$ is a simple reflection of the Weyl group $W$ of $\mathfrak{so}_n$, then $V_m$ has a submodule with highest weight $s_i m$.

Proof. Let $U'_q\mathfrak{so}_4(i)$ be the subalgebra of $U'_q\mathfrak{so}_n$ which is generated by $B_{2i}$ and $B_{2i+1}$. It is isomorphic to $U'_q\mathfrak{so}_4$. We define $v_{s_i, \lambda} = m_{\lambda, \alpha_i} + 1, 1, 2, 2, \ldots, 2, 1, 1, 2, 2, \ldots, (B_n-2, B_n-1)v_{1, M}$, and if $n$ is odd and $i = (n-1)/2$, we define $v_{s_i, \lambda} = P_2\alpha_{n-1/2} + 1, 1, 2, 2, \ldots, 2, 1, 1, 2, 2, \ldots, (B_n-2, B_n-1)v_{1, M}$, with the polynomial as in Cor. 3.6. We claim that each $v_{s_i, \lambda}$ is a highest weight vector. If $j$ is odd, then the generator $B_j$ either commutes with $U'_q\mathfrak{so}_4(i)$, or it is an element of it. One deduces from this that the vector $v_{s_i, \lambda}$ is a weight vector. By the same argument, one easily checks the highest weight property for all $2j$, except for $2j+2$. By Lemma 4.1, $B_{2i+2}v_{s_i, \lambda}$ is a linear combination of vectors of weights $s_i, \lambda + \alpha_{i+1}$ or $s_i, \lambda + \alpha_{i-1}$. But as $s_i, \lambda = \lambda - r\alpha_i$ for a suitable multiple $r$, the weights above can only be weights of the Verma module if we have a minus sign at $\pm$. Hence $B_{2i+1}B_{2i+2}v_{s_i, \lambda}$ is a multiple of $B_{2i+2}v_{s_i, \lambda}$, i.e. the highest weight property is satisfied.

Theorem 5.6. Let $V_m$ be a standard Verma module, (see Def. 4.5) with $m = [\lambda]$ or $m = [\lambda]_+$. Let $I(\lambda) = \sum V_{s_i, \lambda}$, where $V_{s_i, \lambda}$ is the highest weight module generated by $v_{s_i, \lambda}$, as defined in Lemma 5.5. Then $V_m/I(\lambda)$ is a finite dimensional module whose character and, in particular, its dimension coincides with the one of the irreducible $\mathfrak{so}_n$ module with highest weight $\lambda$.

Proof. We mimic the classical proof by showing that the Weyl group permutes the weight spaces of the quotient module (see e.g. [18], Theorem 4.7.3 and preceding lemmas/theorems).

Fix a simple reflection $s_i$, where we assume $i \leq (n-2)/2$ for the moment. Let $U$ be the subspace of $V_m/I(\lambda)$ consisting of vectors $u$ which generate a $U'_q\mathfrak{so}_4(i)$ module $M(u)$ which is symmetric under $s_i$; by this we mean that the weight spaces $M(u)[\mu]$ and $M(u)[s_i(\mu)]$ have the same dimensions, for all weights $\mu$ of $M(u)$. It follows from Prop. 5.4 that $v_{\lambda} \in U$. 


Next we claim that if \( u \in U \), then so is \( B_j u \), for \( 1 \leq j < n \). This is easy to see if \( B_j \) commutes with \( U'_q\mathfrak{so}_4(i) \), and obvious if \( B_j \in U'_q\mathfrak{so}_4(i) \). Hence we only need to worry about \( B_{2i\pm 2} \). Now observe that

\[
U'_q\mathfrak{so}_4(i)B_{2i+2}M(u) \subset M(u) + \sum_{j=2i-1}^{2i+2} B_j B_{j+1} \ldots B_{2i+2} M(u);
\]

to see this, it suffices to show that the right hand side is a \( U'_q\mathfrak{so}_4(i) \)-module which contains \( B_{2i+2} M(u) \). This is a consequence of the relations proved in the lemmas before Prop. 2.3. As \( M(u) \) is symmetric under \( s_i \), it follows that \( |\mu_i - \mu_{i+1}| \leq K \) for a fixed constant, and any weight \( \mu \) of \( M(u) \). But then it follows from the last inclusion that there also exists a finite constant for the weights of \( U'_q\mathfrak{so}_4(i)B_{2i+2}M(u) \). The proof for \( B_{2i-2} \) goes similarly. Hence this module is symmetric under \( s_i \), by Prop. 5.4. It follows that \( U = V_m/I(\lambda) \). The proof for \( i = n/2 \) for \( n \) even goes similarly, while the proof for \( i = (n-1)/2 \) is easier, only involving the corresponding argument in the setting of \( \mathfrak{so}_3 \).

It follows from the last paragraph that \( V_m/I(\lambda) \) is symmetric with respect to every simple reflection, and hence with respect to its Weyl group. Hence the dimension of the weight space for \( \mu \) is equal to the dimension of the weight space belonging to its conjugate in the dominant Weyl chamber. As our ideal does not contain any weights in that region, by definition, this dimension coincides with the dimension of the weight space in the Verma module. Hence the character of our quotient module coincides with the character of the simple \( \mathfrak{so}_n \) module with highest weight \( \lambda \).

**Corollary 5.7.** If \( m = [\lambda]_+ \) with \( \lambda_i \in \frac{1}{2} + \mathbb{Z} \), then \( V_m/I(\lambda) \) decomposes into the direct sum of \( 2^{[(n-1)/2]} \) representations of equal dimension. Any two of these can be made equivalent after suitable sign changes \( B_{2i} \rightarrow \pm B_{2i} \). Its weights are given by all weights \( \mu \) of \( V_m/I(\lambda) \) with \( \mu_i > 0 \) for all \( i \), with the same multiplicity as in \( V_m/I(\lambda) \), except possibly if \( n \) is even, and \( \lambda_{n/2} < 0 \). In this case, the character is again the same as for \( \lambda \) which coincides with \( \lambda \) except for the last coordinate, where \( \lambda_{n/2} = -\lambda_{n/2} \).

**Proof.** According to Theorem 3.11, (c) the simple \( U'_q\mathfrak{so}_3 \) module \( L_m, m = [\lambda]_+ \) with \( \lambda \in \frac{1}{2} + \mathbb{Z} \) decomposes into a direct sum \( L_{m+} \oplus L_{m-} \), where \( L_{m\pm} \) are the direct sum of eigenspaces of \( B_2 \) corresponding to the eigenvalues \( \pm [\lambda - j]_+ \); in the following, we also refer to these spaces \( L_{m\pm} \) as eigenspaces of \( B_2 \). Using the same argument for the subalgebra of \( U'_q\mathfrak{so}_4 \) generated by \( B_2 \) and \( B_3 \), we similarly conclude that also \( B_3 \) leaves invariant the eigenspaces \( L_{m\pm} \) of \( B_2 \). We conclude that also \( V_m/I([\lambda]_+) \) decomposes into the direct sum of two submodules for any admissible dominant weight \( \lambda \) of \( U'_q\mathfrak{so}_4 \).

For the general case, we just construct analogous subspaces for each \( B_{2i} \). The statement about the character comes from the fact that \( [\mu_i]_+ = [-\mu_i]_++ \).

**Remark 5.8.** We have not shown here that the quotients constructed in Theorem 5.6 for \( m = [\lambda] \) respectively each of the \( 2^{[(n-1)/2]} \) components for \( m = [\lambda]_+ \) in Corollary 5.7 are irreducible. This was shown in [9] in general, and for \( m = [\lambda] \) also in [20]. It should be possible
to prove this result directly in the usual way. Casimir elements have been constructed in [4] and [14], and the scalar $c_\lambda$ via which it acts on an irreducible module with highest weight $\lambda$ has been calculated in [4]. So it would suffice to check that $c_\mu < c_\lambda$ whenever $\mu < \lambda$, i.e. $\lambda - \mu$ is a sum of positive roots.

**Theorem 5.9.** (see [11], [9]) Let $q$ not be a root of unity except for $q = \pm 1$, and let $V$ be a simple finite-dimensional $U'_q\mathfrak{so}_n$-module. Then $V$ is either a classical simple module with highest weight $[\lambda]$, where $\lambda$ is a dominant integral weight for $\mathfrak{so}_n$, or it is one of $2^{n-1}$ representations belonging to the highest weight $[\lambda]_+$, where now $\lambda$ is a weight of $\mathfrak{so}_n$ whose coordinates are all in $\mathbb{Z} + \frac{1}{2}$ and positive. In the second case, the dimension is $1/2^{(n-1)/2}$ times the dimension of the $\mathfrak{so}_n$ module with highest weight $\lambda$. The different modules can be obtained from each other by multiplying generators with even indices by $-1$.

**Proof.** The restriction on the highest weights follows from Prop. 4.4. The existence of modules for such highest weights modules follows from Theorem 5.6 and Corollary 5.7. The dimensions of the finite-dimensional simple modules for these highest weights was shown in [9] in general, and for $m = [\lambda]$ also in [20]. See also the discussion in Remark 5.8.

**Remark 5.10.** The approach in [9] and its predecessors consisted of constructing explicit matrix representations. Our explicit representations for $U'_q\mathfrak{so}_3$ and $U'_q\mathfrak{so}_4$ can be considered as slight generalizations of special cases of their results. Unfortunately their formulas become quite involved for larger $n$, and they are not well-defined for $q$ a root of unity.

6. **Roots of Unity**

Experience with the small quantum group at roots of unity (see e.g. [1], [2]) as well as results by Iorgov and Klimyk [10] suggest that the general representation theory of $U'_q\mathfrak{so}_n$ will be quite complicated for $q$ a root of unity. Fortunately, the motivation for this paper came from studying unitary representations, where the situation is less complicated.

6.1. **Generic modules at roots of unity.** In the following we assume the representations to be defined over the complex numbers $\mathbb{C}$, with $q$ being a primitive $\ell$-th root of unity. Observe that in this case we have

$$[\lambda]_+ = [\lambda + \ell/4].$$

So, for $\ell$ even, we would not have to distinguish between classical and nonclassical representations. The following is a special case of the main result of [10]. These modules can be considered as analogs of what is denoted as baby-Verma modules for small quantum groups, see [1].

**Theorem 6.1.** Let $\lambda = (\lambda_i)$ be a weight such that $\lambda_i \notin \frac{1}{4}\mathbb{Z}$, $1 \leq i \leq n/2$, and let $m = [\lambda]$ or $m = [\lambda]_+$. Then there exists a module $M_m$ of dimension $\ell^d$, where $d$ is the number of positive roots in $\mathfrak{so}_n$ and with highest weight $\lambda$. 
Observe that all the coefficients are well-defined in the explicit representations of $\mathfrak{so}_3$ and $\mathfrak{so}_4$ in this and the previous sections. One directly reads off these representations that $v_{\lambda-\ell \alpha_i}$ is a well-defined highest weight vector in the Verma module $V_m$ for $\mathfrak{so}_4$ and $\mathfrak{so}_3$, a result analogous to the one of Proposition 5.4. For $n > 4$, we can now define a submodule $V_i$ with highest weight $\lambda - \ell \alpha_i$ as in Lemma 5.5. We now define $M_m = V_m/\sum V_i$.

To determine the dimension and character of $M_m$, we map the weights of $V_\lambda$ to the weights of $V_{(\ell-1)\rho}$, via the map $\mu \mapsto \mu - \lambda + (\ell-1)\rho$. Using this map, we pull back the dot-action of the Weyl group on the weight lattice to the lattice $\mu + P$, where $P$ is the weight lattice. Then one shows as in the proof of Theorem 5.6 that the weight spaces of $M_m$ are symmetric under this action of the Weyl group. In particular, we obtain that

$$\dim M_m = \dim M_{|(\ell-1)\rho|} = \prod_{\alpha \not= 0} (\langle \ell \rho, \alpha \rangle) = \ell^d.$$ 

**Remark 6.2.** The same proof also works if we define $V_i(a_i)$ to be the Verma module generated by $a_i v_\lambda + v_{\lambda-\ell \alpha_i}$, where $v_{\lambda-\ell \alpha_i}$ is the highest weight vector of the module $V_i$ in the proof of Theorem 6.1 and $a_i \in \mathbb{C}$. Hence we obtain a multiparameter family of modules $V/\sum V_i(a_i)$ with highest weight $\lambda$. It is easy to see that these modules do not have a lowest weight vector if $a_i \neq 0$ for some $i$. There are even more non-isomorphic modules with the same weight structure, see [10], which do not have highest weight vectors.

6.2. **Weyl modules.** If the highest weight $m$ is equal to $[\lambda]$ or $[\lambda]_+$ as in Theorem 5.6, we obtain additional finite-dimensional representations besides the ones in Theorem 6.1.

**Theorem 6.3.** Let $m = [\lambda]$ or $m = [\lambda]_+$ as in Theorem 5.6 resp Cor. 5.7. Then we obtain a finite-dimensional module with highest weight $m$ and the same dimension as the module in Theorem 5.6 resp Cor. 5.7.

**Proof.** We first observe that the statement is true for $U_q^\prime \mathfrak{so}_3$ and $U_q^\prime \mathfrak{so}_4$. Indeed, it follows from the explicit formulas for the highest weight vectors $v_{s,\lambda}$ in Cor 3.6 and in Lemma 5.1 that they are also well-defined at a root of unity. The basis in Prop. 2.3 is well-defined also for a root of unity. Hence the highest weight property also holds for $q$ a root of unity, by continuity (or Zariski density). In particular, also the representation on the quotient module is well-defined over the ring $R$ in Prop. 2.3, and hence also at a root of unity. We can now prove the general case as in Theorem 5.6, using the same arguments.

**Corollary 6.4.** Let $m = [\lambda]$ or $m = [\lambda]_+$ as in Theorem 5.6 with $\ell/4 \geq \lambda_1 \geq \ldots \geq |\lambda_{[n/2]}|$. Then the quotient in Theorem 6.3 has a unique maximum ideal.

**Proof.** Let us consider the case $m = [\lambda]$ first. As $|\mu_i| \leq \ell/4$, it follows that $[\mu_i] = [\nu_i]$ also implies $\mu_i = \nu_i$ for any coordinates of weights $\mu$ and $\nu$. In particular, the highest weight $\lambda$ has multiplicity 1 in the finite-dimensional module with highest weight $[\lambda]$ in Theorem 6.3. Assume that we have two different maximum submodules, $N_1$ and $N_2$. None of them can have the highest weight $[\lambda]$. But then also their sum $N_1 + N_2$ can not contain the highest weight, so also $N_1 + N_2$ is a maximum submodule. This is only possible if $N_1 = N_2 = N_1 + N_2$. 




















































































































































































































































































































































The same argument also works for each of the $2^{[(n-1)/2]}$ summands for the module constructed in Theorem 5.6 with highest weight $m = [\lambda]_+$. 

6.3. **Unitary representations.** One of the motivations for this paper came from the need to identify certain representations of $U'_q\mathfrak{so}_n$ on a Hilbert space. See Example 2 in Section 1.2 for the choice of name in the following definition. We call a representation of $U'_q\mathfrak{so}_n$ on a Hilbert space $V$ with inner product $(\,,\,)$ a **unitary** representation if

$$(B_i v, w) = (v, B_i w) \quad \text{for } v, w, \in V, \quad 1 \leq i \leq n - 1 \quad \text{for all } v, w \in V.$$ 

We are going to show that unitary representations with highest weights will have to factor over the quotient in the previous section. The idea is to prove that the image of the highest weight vectors $v_{\alpha,\lambda}$ in a unitary representation will have to have length 0, using the explicit representations for $U'_q\mathfrak{so}_3$ and $U'_q\mathfrak{so}_4$. There are some minor complications as the weight vectors there are not always well-defined at a root of unity.

**Lemma 6.5.** Let $(\,,\,)$ be a bilinear or sesquilinear form on the Verma module $V_m$ of $U'_q\mathfrak{so}_3$ with respect to which the generators $B_1$ and $B_2$ are self-adjoint.

(a) If the weight vectors $v_i$ are well-defined and mutually orthogonal for $j-1 \leq i \leq j+2$, then $\|v_{j+1}\|^2 = \alpha_{j,j+1}\|v_j\|^2$.

(b) The weight $m = [\pm \ell/4] = [\pm 0]_+$ can appear in a unitary highest weight representation $V$ of $U'_q\mathfrak{so}_3$ only for highest/lowest weight vectors. In particular, if $w_j \in V$ is a weight vector with weight $[\pm 0]_+$, then $B_2w_j$ is a weight vector with weight $[\pm 1]_+$ and with $\|B_2w_j\|^2 = -2/(q - q^{-1})^2\|w_j\|^2$.

(c) Let $m = [\lambda]$, $\lambda \in \frac{1}{2}\mathbb{Z}$ or $m = [\lambda]_+$, $\lambda \in \frac{1}{2} + \mathbb{Z}$, with $\lambda > 0$ in both cases. If $V$ is a unitary representation of $U'_q\mathfrak{so}_3$ with highest weight $m$ and $\Phi : V_m \rightarrow V$ the canonical epimorphism, then $\Phi(v_{2\lambda+1}) = 0$.

**Proof.** Using the assumptions and the definition of the action of $B_2$, see Prop. 3.4, we obtain

$$(v_{j+1}, v_{j+1}) = (B_2v_j - \alpha_{j-1,j}v_{j-1}, v_{j+1}) = (v_j, B_2v_{j+1}) = \alpha_{j,j+1}(v_j, v_j).$$

To prove (b), assume to the contrary that a weight $[\pm 0]_+$ does occur in a module with highest weight $[\lambda]_+$ for the weight vector $v_j$, $0 < j$ such that $B_2v_j$ is not a multiple of $v_{j-1}$. Observe that $[\lambda - j]_+ = [0]_+$ implies $j = \lambda$. We proceed as in Lemma 3.9 by defining the vector $v'_{j+1} = B_2v_j$. One calculates from the definitions that

$$(B_1v'_{j+1}) = [\lambda - j - 1]_+v'_{j+1} + (q - q^{-1})[\lambda - j]\alpha_{j-1,j}v_{j-1} = [\lambda - j - 1]_+v'_{j+1} + \frac{[\lambda + 1][\lambda]}{q - q^{-1}}v_{j-1},$$

where we only used $\lambda = j$ for the second equality. Observe that $[\lambda - j - 1]_+ = [\lambda - j + 1]_+$. If $[\lambda + 1][\lambda] \neq 0$, $B_1$ would act as a Jordan block on span $\{v'_{j+1}, v_{j-1}\}$, a contradiction to it being self-adjoint. If $m = [\ell/2 - 1]_+ = [\ell/4 - 1]$, one checks, using the formulas of $\alpha_{j-1,j}$ with $m = [\ell/4 - 1]$, that the vectors $v_j$ are well-defined for $\ell/2 - 3 \leq j \leq \ell/2$. One deduces
from this that \( \|v_{\ell/2-1}\| = 0 \), as \( \alpha_{\ell/2-2,\ell/2-1} = 0 \). Hence a unitary module with highest weight \([\ell/4 - 1]\) does not contain the weight \( \pm[0]_+ = [\pm\ell/4] \).

If \( m = [\ell/2]_+ = [\ell/4]_+ \), one checks, using the formula for \( \alpha_{\ell/2-2,\ell/2} \) with \( m = [\ell/4]_+ \) that \( v_{\ell/2+1} = B_2v_{\ell/2} - \frac{(-1)^{m}}{[q^{-1}q^{1/2}]} v_{\ell/2-1} \) is a well-defined weight vector. Using the recursion formula 3.5 for \( j = \ell/2 \) and \( m_j = \pm[0]_+ \), we see that \( \alpha_{\ell/2,\ell/2+1} = 0 \) also for our choice of \( q \). This implies that \( v_{\ell/2+1} \) is a highest weight vector in the Verma module \( V_m \). If its image \( w_{\ell/2+1} \) in \( V \) was non-zero, it would generate a simple module \( U \subset V \) which would not contain the weight \( \pm[0]_+ \), by the results of the previous paragraph. In particular, the highest weight vector would be contained in the orthogonal complement of \( U \), which is itself a \( U'_q\text{so}_3 \)-module. This contradicts the fact that \( V \) is generated by a highest weight vector with weight \( \pm[0]_+ \). Hence \( w_{\ell/2+1} = 0 \) and \( w_{\ell/2} \) is a highest (or lowest) weight vector. The norm of the weight vector \( B_2w_{\ell/2} \) can now be calculated as in (a).

For part (c) we can calculate the norms of the weight vectors \( v_j \) by (a), as long as the weights \( \pm[0]_+ \) do not appear. By (b), this is the case for all unitary highest weight representations except when the highest weight is equal to \( \pm[0]_+ = [\pm\ell/4] \). If \( m = [\lambda] \neq [\pm\ell/4] \), it follows as in the classical case that \( \|v_{2\lambda+1}\|^2 = a_{2\lambda,2\lambda+1}\|v_{2\lambda}\|^2 = 0 \). If \( m = [\pm\ell/4] \), \( \Phi(v_{\ell/2+1}) = w_{\ell/2+1} = 0 \) by the previous paragraph.

6.4. Characterization by highest weight. We can now give a similar characterization of certain unitary representations for \( q \) a root of unity via their highest weights as in the generic case. We remind the reader that we need to specify, besides the highest weight \( m \), also the weights of the vectors \( B_2v_m \). We will assume the situation of a standard Verma module, see Def. 4.5.

**Theorem 6.6.** Let \( \ell \) be even and let \( q \) be a primitive \( \ell^{th} \) root of unity. Let \( m = [\lambda] \) or \( m = [\lambda]_+ \), with \( \ell/4 \geq \lambda_1 \geq \ldots \geq |\lambda_{\ell/2}| \), and with \( \lambda_i \in \frac{1}{2} + \mathbb{Z} \) for \( m = [\lambda]_+ \). Then there exists at most one unitary \( U'_q\text{so}_n \) module with highest weight \( m \) which would be a quotient of the standard Verma module \( V_m \).

**Proof.** Let \( \Phi \) be the surjective map from the Verma module \( V_m \) onto the unitary representation \( V \), and let (, ) also denote the pull-back of the inner product on \( V \) to \( V_m \). The claim follows if we can prove that \( \|v_{\lambda_1,m}\|^2 = 0 \) for the vectors \( v_{\lambda_1,m} \) in Theorem 6.3, by Corollary 6.4. This has already been shown for \( U'_q\text{so}_3 \) in Lemma 6.5. In the case of \( U'_q\text{so}_4 \), let \( m = [\lambda]_+ \), let \( \mu = \lambda - r\alpha_1 \), and let \( v_\mu \) be the weight vector as defined in Lemma 5.1,(b). Observe that these vectors are well-defined also for \( q \) a root of unity. We are going to use Eq 5.4 for the action of \( B_2 \) on \( v_\mu \). Observe that even if \( \{\mu_2\} = 0 \), the expression

\[
\frac{1}{\{\mu_2\}}(v_{\mu-\alpha_1} - v_{\mu-\alpha_2}) = B_2v_\mu - \frac{[r_1][(\lambda,\alpha_1) + 1 - r]}{\{\mu_1\}} v_{\mu+\alpha_1}
\]

is well-defined, provided \( \{\mu_1\} = [\mu_1 + 1] - [\mu_1 - 1] \neq 0 \). In particular, in this case this expression is an eigenvector of \( B_1 \) with eigenvalue \([\mu_1 - 1] \), and hence orthogonal to \( v_{\mu+\alpha_1} \).
Using this for the last equality below, as well as Eq. 5.4, we obtain
\[\|v_{\mu - \alpha_1}\|^2 = ((B_3 - [\lambda_2 + r - 1])B_2v_\mu, v_{\mu - \alpha_1}) = (v_\mu, B_2(B_3 - [\lambda_2 + r - 1])v_{\mu - \alpha_1}) = \{\lambda_2 + r\}v_\mu, B_2v_{\mu - \alpha_1}\]
\[= \frac{\{\lambda_2 + r\}}{\{\lambda_1 - r - 1\}}[r + 1][\lambda_1 - \lambda_2 - r] \|v_\mu\|^2. \tag{6.4}\]

Now observe that if \(r = (\lambda, \alpha_1) = \lambda_1 - \lambda_2\) and \(\{\lambda_1 - r - 1\} = \{\lambda_2 - 1\} \neq 0\), we have \(s_1, \lambda = \mu - \alpha_1\) and hence \(\|v_{s_1, \lambda}\|^2 = 0\). So except if \(\lambda_2 = 1 - \ell/4\), we have shown that \(v_{s_1, \lambda}\) is in the kernel of \(\Phi\). One shows by the same approach that also \(\|v_{s_2, \lambda}\|^2 = 0\) as long as \(\lambda_2 \neq \ell/4 - 1\). This applies, in particular, to the remaining cases with \(\lambda_2 = 1 - \ell/4\). We leave it to the reader to check that the quotient \(V_\lambda/V_{s_2, \lambda}\) is isomorphic to \(V_{\ell/4 - 1}\) as a \(U_q^4\mathfrak{so}_2\)-module for \(\lambda = \ell/4 - 1\), and it is isomorphic to \(V_{\ell/4} + V_{\ell/4 - 1}\) as a \(U_q^4\mathfrak{so}_2\)-module for \(\lambda = \ell/4\). One can now check for \(\lambda = (\ell/4, \ell/4 - 1)\) that the vector \(v_\lambda = [(\lambda, \alpha_2)]v_{\lambda - \alpha_1} + [(\lambda, \alpha_1)]v_{\lambda - \alpha_2}\) does generate the \(U_q^4\mathfrak{so}_2\) Verma module with highest weight \([\lambda_1 - 1]\), and, using Lemma 6.5, that the vector \(v_{\ell/2 - 1}\) in this Verma module is a nonzero multiple of \(v_{s_1, \lambda}\) mod \(V_{s_2, \lambda}\). Lemma 6.5 then implies that \(\|v_{s_1, \lambda}\| = 0\). The case with \(\lambda = (\ell/4 - 4, 1 - \ell/4)\) is similar and easier. This proves the claim for \(U_q^4\mathfrak{so}_4\) with \(m = [\lambda]\). The proof for \(m = [\lambda]_+\) is similar and easier, as we need not worry about eigenvalues \(\pm 2i/(q - q^{-1})\) for \(B_1\) or \(B_3\). If it occurs in a highest weight vector, it would already be covered in the previous case, using \([0]_+ = [\ell/4]\).

Let now \(V\) be a simple unitary representation of \(U_q^4\mathfrak{so}_n\) with highest weight \(m = [\lambda]\), for \(n \geq 4\). Restricting the representation to the subalgebra \(U_q^4\mathfrak{so}_4(i)\), see the proof of Theorem 5.6 for definitions, we see from the previous paragraph that also the submodule \(V_{s_1, \lambda} \subset V_m\) is in the kernel of the map from \(V_m\) onto \(V\). This shows that our unitary representation factors over the ideal in Theorem 6.3. The uniqueness statement now follows from Cor. 6.4.

**Remark 6.7.**
1. Theorem 6.6 does not make any statements about existence of unitary modules. Unitary modules for \(m = [\lambda]\) and \(q = e^{\pm 2\pi i/\ell}\) appeared in [16] and in [19], see the discussion in the first section. There are at least some non-classical unitary representations for \(m = [\lambda]_+\) with \(q = -e^{\pm 2\pi i/\ell}\). It should be possible to determine, at least in some special cases, for which \(q\) our quotients are unitary by using the formulas for the norms \(\|v_\mu\|^2\) of weight vectors \(v_\mu\) in this section.
2. Theorem 6.6 can be used to reprove the main technical result of [16]. It was shown there that the unitary representations of \(U_q\mathfrak{so}_n\) of Section 1.2, Example 3 had the same characters, up to multiplicities, as certain representations listed under Example 2. Hence their images are isomorphic. The proof in [16] used a Verma module approach only up to \(U_q\mathfrak{so}_5\). It then took advantage of the special nature of the representations in that particular situation to proceed by induction for \(n > 5\).
3. For a root of unity, the representations of Theorem 5.6 are usually not simple. Assuming that the duality between the actions of \(U_q\mathfrak{so}_n\) and \(U_q^4\mathfrak{so}_n\) on \(S^\otimes n\) also holds for \(q\) a root of unity, where \(S\) is the spin representation of \(U_q\mathfrak{so}_N\) (see Example 2 and [20] for details), the
multiplicities of simple modules in a filtration should be related to certain parabolic Kazhdan-
Lusztig polynomials. The latter ones are used to describe characters of tilting modules of
Drinfeld-Jimbo quantum groups.

7. Discussion of our results

7.1. Technicalities in this approach. The main results in this paper concern the construc-
tion and classification of highest weight modules of $U'_q so_n$ via a Verma module approach. One
of the difficulties in this approach is the lack of generic raising and lowering operators: while
one can define, for a given weight vector $v$ of weight $\mu$ and a given simple root $\alpha$, an expres-
sion $E$ in terms of our generators such that $Ev$ has weight $\mu - \alpha$ (if it exists), this expression
depends on the given weight $\mu$; the same expression applied to another weight vector usually
does not even give a weight vector anymore. This makes it difficult, perhaps impossible to
define analogs of Borel algebras in this setting. So we decided to define Verma modules by a
more involved quotient construction. Due to this more complicated setting, we have decided
to appeal to already known representations for proving linear independence of our spanning
set in general. It would be interesting to see whether this could be avoided without too much
additional work. On the other hand, finding sufficiently many representations of general
coidal subalgebras of quantum groups should not be terribly hard in view of the established
representation theory of quantum groups; we only needed the classical representations for
proving linear independence. A short-coming in our approach so far would be the fact that
we did not independently prove irreducibility of the finite-dimensional modules coming from
our quotient construction. This should not be overly hard to fix using Casimir elements for
$U'_q so_n$, see Remark 5.8 for details.

7.2. Comparison with the approach by Klimyk et al. Recall our brief discussion in
Section 1 of the classification of representations of $U'_q so_n$ in [9]. We think the approach in this
paper has two advantages. It also works for representations at roots of unity with integral
dominant weight, allowing us to identify at least certain representations at roots of unity via
their highest weight. These include the ones studied in [16]. Secondly, the approach in this
paper is less computational. While this may just be a matter of personal taste, we hope that
the approach here may be more suitable for generalizations to other co-ideal algebras (see also
next subsection). On the other hand, one can criticize our approach for the time being for
not proving irreducibility of the finite-dimensional modules. But see the remark at the end of
the previous subsection and Remark 5.8.

7.3. Generalizations to other co-ideal algebras. More general co-ideal subalgebras of
quantum groups appear in several contexts where it would be useful to know their representa-
tion theory such as e.g. in $q$-Howe duality and categorification of representations of quantum
groups (see e.g. [3], [17]). So it is a natural problem to classify the representations of general
coidal subalgebras. As a first general result, Letzter has determined analogs of Cartan
subalgebras for all co-ideal subalgebras in [13]. As already mentioned, it may be difficult,
or perhaps not even possible, to define analogs of Borel subalgebras. But it might still be
possible to generalize the direct quotient construction for Verma modules in this paper to more general coideal algebras.

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