Envariance, or environment-assisted invariance, is a recently identified symmetry for maximally entangled states in quantum theory with important ramifications for quantum measurement, specifically for understanding Born’s rule [3]. We benchmark the degree to which nature respects this symmetry by using entangled photon pairs. Our results show quantum states can be (99.66 ± 0.04)% envariant as measured using the quantum fidelity [4], and (99.963 ± 0.005)% as measured using a modified Bhattacharya Coefficient [5], as compared with a perfectly envariant system which would be 100% in either measure. The deviations can be understood by the less-than-maximal entanglement in our photon pairs.

Symmetries play a central role in physics with wide-reaching implications in fields as diverse as spectroscopy and particle physics. It is therefore of fundamental importance to identify and understand new symmetries of nature. One of these more recently identified symmetries in quantum mechanics has been named environment-assisted invariance, or envariance [2]. It applies in certain cases where a composite quantum object consists of a system part, labelled $S$, and an environment part, labelled $E$. If some action is applied to the system part only, described by some unitary transformation, $U_S$, then the state is said to be envariant under $U_S$ if another unitary applied to the environment, $U_E$, can restore the initial state. This can be expressed,

$$U_S|\psi_{SE}\rangle = (U_S \otimes \mathbb{1}_E)|\psi_{SE}\rangle = |\eta_{SE}\rangle$$ (1)

$$U_E|\eta_{SE}\rangle = (\mathbb{1}_S \otimes U_E)|\eta_{SE}\rangle = |\psi_{SE}\rangle.$$ (2)

Envariance is an example of an assisted symmetry [1] where once the system is transformed under some unitary $U_S$, it can be restored to its original state by another operation on a physically distinct system: the environment.

Envariance is a uniquely quantum symmetry in the following sense. A pure quantum state represents complete knowledge of the quantum system. In an entangled quantum state, however, complete knowledge of the whole system does not imply complete knowledge of its parts. It is therefore possible that an operation on one part of a quantum state can alter the global state, but its local effects are masked by incomplete knowledge of that part; the effect on the global state can then be undone by an action on a different part. In contrast, complete knowledge of a composite classical system implies complete knowledge of each of its parts. Thus transforming one part of a classical system cannot be masked by incomplete knowledge and cannot be undone by a change on another part.

Envariance plays a prominent role in work related to fundamental issues of decoherence and quantum measurement [11,13]. Decoherence converts amplitudes in coherent superposition states to probabilities in mixtures and is central to the emergence of the classical world from quantum mechanics [6,7]. Mathematically the mixture appears in the reduced density operator of the system which is extracted from the global wavefunction by a partial trace $\rho_S$. This partial trace limits the approach for deriving, as opposed to separately postulating, the connection between the wavefunction and measurement probabilities known as Born’s rule [10], since the partial trace assumes Born’s rule is valid [2,11]. Envariance was employed in a derivation of Born’s rule which sought to avoid circularity inherent to approaches which rely on partial trace [2]. For comments on this derivation, see for example [11,13].

In the present work, we subject envariance to experimental test in an optical system. We use the polarization of a single photon to encode the system, $S$, and the polarization of a second single photon to encode the environment, $E$. We subject the system photon to a wide range of polarization rotations with the goal of benchmarking the degree to which we can restore the initial state by applying a second transformation on the environment photon.

Our test requires a source of high-quality two-photon polarization entanglement, an optical set-up to perform unitary operations on zero, one, or both of the photons, and polarization analyzers to characterize the final state of the light. Our experimental setup is shown in Fig. [1]

We produce pairs of polarization-entangled photons using spontaneous parametric down-conversion (SPDC) in a Sagnac interferometer [14,16]. In the ideal case, this source produces pairs of photons in the singlet state,

$$|\psi_{SE}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_S|V\rangle_E - |V\rangle_S|H\rangle_E),$$ (3)

where $|H\rangle$ ($|V\rangle$) represents horizontal (vertical) polarization, and $S$ and $E$ label the photons. This state is envariant under all unitary transformations and has the convenient symmetry that $u_S = u_E$ for all $u_S$. We pump a 10 mm periodically-poled KTP crystal (PPKTP), phase-matched to produce photon pairs at 809.8 nm and 809.3 nm from type-II down-conversion using 6 mW from...
and 5 for 5 s. We typically measured total coincidence rates of
coincidence logic with a 1 ns coincidence window, counting
(Perkin-Elmer SPCM-AQ4C) and analyzed using coin-
ination transformations. Photons from both ports of each
into the beam paths to implement controlled polariza-
analyzers are two sets of wave plates—a QWP, a HWP,
beams, where polarization controllers correct unwanted
fibres, where polarization controllers correct unwanted
settings used to implement polarization
ations. Each analyzer consists of a half-wave
rotation by an angle \( \theta \) about the \( \hat{x} \), \( \hat{y} \), and \( \hat{z} \) axes are
in the first QWP, the HWP and the second QWP respectively.
The angles \( \alpha, \beta, \gamma \) are the wave plate angles for
the first QWP, the HWP and the second QWP respectively.

\[
\begin{array}{ccc}
\text{Rotation Axis} & \alpha(\theta) & \beta(\theta) & \gamma(\theta) \\
\hat{x} & \pi/2 & -\theta/4 & \pi/2 \\
\hat{y} & \pi/2 + \theta/2 & 0/4 & \pi/2 \\
\hat{z} & \pi/4 & -\pi/4 - \theta/4 & \pi/4 \\
\end{array}
\]

TABLE I: Wave plate settings used to implement polarization
ations. The angles \( \alpha, \beta, \gamma \) are the wave plate angles for
the specified axis on the Bloch sphere.

For our experiment, we implemented rotations about the
standard \( \hat{x}, \hat{y}, \hat{z} \) axes of the Bloch sphere; in
addition we implemented rotations about an axis \( \hat{m} = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3} \). The wave plate angles used to implement
ations by an angle \( \theta \) about the \( \hat{x}, \hat{y}, \hat{z} \) axes are
shown in Table I; the angles to implement rotations about
\( \hat{m} \) were determined numerically using MATHEMATICA.

Our experiment proceeds in three stages as depicted in
Fig. 2: first characterizing the initial state (I), then
characterizing the state after a transformation is applied
to the system photon (II), and finally characterizing the state after that same transformation is applied to both system and environment (III).

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Provides a good measure of the source stability. Specifically, the same (no additional waveplates inserted) and thus produced fluctuations in the state produced by the source itself by comparing the state produced with expectation. We considered the effects of Poissonian noise and waveplate calibration on our results and found that these effects were too small to explain the deviation between the expected state and the reconstructed state. This state has a fidelity very high, close to 1, in all cases and we see reasonable agreement with the expected state calculated by transforming the state from stage I with the reconstructed state from stage II. The open circles show the theoretical expectation for the fidelity between the measured state at stage I and the expected state calculated by acting the unitaries on the measured state from stage I. The fidelities are very high, close to 1, in all cases and we see reasonable agreement with expectation.

We considered the effects of Poissonian noise and waveplate calibration on our results and found that these effects were too small to explain the deviation between the expected state and the reconstructed state. To account for this, we characterized the fluctuations in the state produced by the source itself by comparing the state produced in subsequent stage I states in the data collection; recall that stage I for each choice of unitary is always the same (no additional waveplates inserted) and thus provides a good measure of the source stability. Specifically, we calculated the standard deviation in the fidelity of the state produce at a stage I in the i-th round of the experiment to that produced in the next, (i + 1)th, stage. We found that the standard deviation in these fidelities calculated from the data taken within each set of rotation axes are shown as representative error bars on the plots in Figs. 4a)–d). The standard deviation of this quantity over all the experiments was 0.0008. We characterize the difference between the measured and expected fidelities by calculating the standard deviation in the quantity, $F(\rho_{\text{expt}}^{I}, \rho_{\text{th}}^{I} - F(\rho_{\text{expt}}^{I+1}, \rho_{\text{th}}^{I+1})$, for each experiment. (This is the difference between the coloured and open data points in Figs. 4a)–d).) Over all experiments to be 0.002. This value is comparable to the error in the fidelity due to source fluctuations. Refer to the appendix to see the comparison between stage I and stage II, which would not fit on the scale of Fig. 4. From our data, we extract the average fidelity $F(\rho_{\text{expt}}^{I}, \rho_{\text{expt}}^{II})$ for the set of measurements made for each unitary axis and show the results in Table II. As measured by the average fidelity, our experiment benchmarks envariance to 0.9966 ± 0.0004, (99.66 ± 0.04)% of the ideal) averaged over all rotations.

Fidelity has conceptual problems as a measure for testing quantum mechanics, since the density matrix we used to compute the fidelity is reconstructed using state tomography, which is under the assumption of Born’s rule. The Bhattacharyya Coefficient (BC) is a measure of the overlap between two discrete distributions $P$ and $Q$, where $p_i$ and $q_j$ are the probabilities of the $i$th element for $P$ and $Q$ respectively. The BC is defined as:

$$BC = \sum_i \sqrt{p_i q_i}.$$ (5)

If we normalize the measured tomographic data by dividing by the sum of the counts, we can treat this as a probability distribution. The BC then can be calculated using the distribution of measurements at each stage in the experiment, directly analogous to the approach used with fidelity. It should be noted that the BC has some limitations when applied in this case. If two quantum states produce identical measurement outcomes, its value is 1. Unlike fidelity though, it is not the case that the BC goes to 0 for orthogonal quantum states. For example, the BC for two orthogonal Bell states measured with

| Rotation Axis | Average Fidelity | Average BC |
|---------------|------------------|------------|
| $\hat{x}$     | 0.9977 ± 0.0001  | 0.9997 ± 0.0001 |
| $\hat{y}$     | 0.9973 ± 0.0007  | 0.9966 ± 0.0008 |
| $\hat{z}$     | 0.9984 ± 0.0006  | 0.9975 ± 0.0007 |
| $\hat{m}$     | 0.9941 ± 0.0007  | 0.9994 ± 0.0001 |
| Overall average: | 0.9966 ± 0.0004  | 0.9963 ± 0.0005 |

TABLE II: Summary of the results for comparing stages I and III using fidelity and Bhattacharyya Coefficient (BC) analysis and averaging over each unitary rotation. The overall average is representative of the overall envariance of our state.
FIG. 4: Analysis of the experimental results. Panels a)–d) show the fidelity analysis results for unitary rotations about \( \hat{x} \), \( \hat{y} \), \( \hat{z} \), and \( \hat{m} \) axes as functions of rotation angle. The coloured data points are the comparison between stage I and stage III (comparing the source state and the state after the unitary has been applied to both qubits). The open circles show a theoretical comparison. Panels e)–h) show the quantum Bhattacharyya results comparing stage I and stage III in the coloured data points for each of the four axes, with the open circles being the theoretical comparison. For plots which include a comparison of stage I and II (applying the unitary to one qubit only) and theoretical comparisons, see the appendix. The error bar for each graph is the standard deviation of comparisons of source state measurements during the experiment.

The Bhattacharyya Coefficients from our measured data are shown in Fig. 4a)–h). We normalize the measured counts from stages I and III to give us probability distributions \( p_{\text{expt}}^{I} \) and \( p_{\text{expt}}^{III} \). The coloured data points in Figs. 4a)–h) show the BC between these distributions, \( BC(p_{\text{expt}}^{I}, p_{\text{expt}}^{III}) \). The open circles are a theoretical expectation of the BC given the tomographic measurements from stage I; for these theoretical values we used state tomography, and thus assumed quantum mechanics, to obtain the expected distribution \( p_{\text{th}}^{III} \) and calculate the expected BC, \( BC(p_{\text{expt}}^{I}, p_{\text{th}}^{III}) \).

Using an analogous procedure to that employed with the fidelity, we estimate the uncertainty in the BC by comparing subsequent measured distributions in stage I throughout the experiment, i.e., \( BC(p_{\text{expt}}^{I}, p_{\text{expt}}^{I+1}) \). A representative error bar calculated from the data for a set of unitaries around the same axis are shown in Fig. 4a)–h). The standard deviation in this quantity over all the data is 0.00005. As before we characterize the difference between the measured and expected BCs as the standard deviation of the quantity \( BC(p_{\text{expt}}^{I}, p_{\text{expt}}^{III}) - BC(p_{\text{expt}}^{I}, p_{\text{th}}^{III}) \) which is 0.00009 over all experiments. As before, this value is comparable to the error due to source fluctuations. Data showing the BC between stage I and II are shown in the appendix along with analogous theoretical comparison. A summary of the BC analysis results are in Table 1. The average measured BC is 0.99963 ± 0.00005 ((99.963 ± 0.005)% of the ideal) across all tested unitaries.

Our deviation from perfect invariance can be understood from our imperfect state fidelity. However, we also consider the magnitude of the violation of Born’s rule if one instead assumes all of the deviation stems from such a violation. One recently proposed extension of Born’s rule
determines probabilities by raising the wavefunction to the power of \( n \) rather than Born’s rule which raises the wavefunction to the power of 2. In this theory, the correlation between measurement outcomes as a function of measurement setting on a singlet state depends on the power of \( n \), thus we can test this theory using our experimental data. Fitting our experimental data to this model, we find \( n = 2.01 \pm 0.02 \) in good agreement with Born’s rule. More details are included in the supplementary materials.

We have experimentally tested the property of envariance on an entangled two-qubit quantum state. Over a wide range of unitary transformations, we experimentally showed envariance at \( (99.66 \pm 0.04)\% \) when measured using the fidelity and \( (99.963 \pm 0.005)\% \) using the Bhattacharyya Coefficient. Deviations from perfect envariance are in good agreement with theory and can be explained by our initial state fidelity and fluctuations in the properties of our state. Fitting our results to a recently published model which does not explicitly assume Born’s rule yields nevertheless good agreement with it. Our results serve as a benchmark for the property of envariance, as improving the envariance of the state significantly would require substantive improvements in source delity and stability. It would be interesting to extend tests of envariance to higher dimensional quantum state and to other physical implementations.

Acknowledgements- We thank D. Hamel, and K. Fisher for valuable discussions. We are grateful for financial support from NSERC, QuantumWorks, MRI ERA, Ontario Centres of Excellence, Industry Canada, Canada Research Chairs, CFI and CIFAR.

I. SUPPLEMENTARY INFORMATION

A. Additional Experimental Data

Our experiment procedure included three stages, I measurements of the source, II measurements after we apply the unitary to only one qubit, and III measurements after applying the same unitary to both qubits. The fidelities and Bhattacharyya Coefficients between stages I and II, and stages I and III as a function of the rotation angle are shown in Fig. 5 for rotation axes, \( \hat{x} \), \( \hat{y} \), \( \hat{z} \), and \( \hat{m} \). Panels a)–d) show the fidelity, and panels e)–h) show the Bhattacharyya Coefficient \( (BC) \). The open circles show the theoretical expectation for various unitaries. For the fidelity comparison the theoretical model applies perfect unitaries to the imperfect measured state. For the \( BC \) comparison the theoretical model applies perfect unitaries to the reconstructed state from stage I. We observe very good agreement between the measured and predicted results.

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B. Fitting Son’s theory to experimental data as a test of Born’s rule

In our experiment, we place a bound on the degree of envariance. It has been shown that envariance can be used to derive Born’s rule \[2, 10\]. However, the derivation does not relate bounds on Born’s rule to bound on envariance. In order to do so, we explore a recently proposed extension of quantum mechanics by Son \[18\]. Son’s theory generalizes Born’s rule, replacing the familiar power of 2 which relates wavefunctions to probabilities with a power of \(n\). In this section, we summarize Son’s theory and use it to put a bound on \(n\) using our experimental data.

We first consider measurements on a pair of qubits in the maximally entangled singlet state using standard quantum mechanics. We define measurement observables \(\hat{a} = \vec{\alpha} \cdot \sigma_1^x\) and \(\hat{b} = \vec{\beta} \cdot \sigma_2^z\) where \(\vec{\alpha}, \vec{\beta}\) are unit vectors and \(\sigma_1^x, \sigma_2^z\) are the Pauli matrices for the two qubits. The result of measurements of \(a\) and \(b\) for qubits 1 and 2 respectively can take on the values \(\pm 1\). The correlation function is defined by

\[ E = \langle ab \rangle = P_a=b - P_{a\neq b}, \quad (6) \]

where \(P_a=b\) and \(P_{a\neq b}\) are probabilities that \(a = b\) and \(a \neq b\) respectively. The correlation function only depends on the angle \(2\theta\) between \(\vec{\alpha}\) and \(\vec{\beta}\) for the singlet state. From Born’s rule, we have the probability amplitudes \(\psi_{a=b}\) and \(\psi_{a\neq b}\) satisfy \(P_{a=b} = |\psi_{a=b}|^2\) and \(P_{a\neq b} = |\psi_{a\neq b}|^2\). Therefore, the correlation function in standard quantum mechanics is given by

\[ E_{QM}(\theta) = |\psi_{a=b}|^2 - |\psi_{a\neq b}|^2 = -\cos 2\theta. \quad (7) \]

We now consider Son’s theory, where Born’s rule is generalized to be \(P_{a=b} = |\psi_{a=b}|^n\) and \(P_{a\neq b} = |\psi_{a\neq b}|^n\), and the correlation function is thus,

\[ E(\theta, n) = |\psi_{a=b}|^n - |\psi_{a\neq b}|^n, \quad (8) \]

where standard quantum mechanics is the special case \(E(\theta, 2) = E_{QM}(\theta)\). As in standard quantum mechanics, Son assumed that the correlation function depends only on the angle between measurement settings. Son showed
that the constraints $|\partial_{\theta}|^2 + |\partial_{\phi}|^2 \propto 1$ and $|\psi_{a=b}|^n + |\psi_{a\neq b}|^n = 1$ and the boundary condition $E(0, n) = -1$ and $E(\pi/2, n) = 1$ are sufficient to solve for $E(\theta, n)$. See [18] for further details on the deviation. Figure 6 shows the results of fitting the correlation function $E(\theta, n)$ for different value $n$.

In the experiment, we rotated one qubit while leaving the other qubit unchanged during the stage II (See Figure 2). If we use the same measurement basis on both qubits for that rotated state, we are effectively measuring the singlet state input with two measurement basis. For a realistic state, the correlation function will not necessarily depend only on $\theta$. In his derivation, Son additionally assumed $E(0, n) = -1$ and $E(\pi/2, n) = 1$, i.e., perfect correlations, which are not experimentally achievable. To relax these assumptions, we consider the difference between two correlation functions measured for a general state $\rho$ and the ideal state $|\psi^\text{\textdagger}$, $E(\phi, n, \rho)$ and $E(\phi, n, |\psi^\text{\textdagger})$ where $\phi$ is the rotation angle of one of the settings. For $n \approx 2$, we make the assumption that $E(\phi, n, \rho) - E(\phi, n, |\psi^\text{\textdagger}) \approx E(\phi, 2, \rho) - E(\phi, 2, |\psi^\text{\textdagger})$. Thus for states close to the ideal singlet state and for $n$ close to 2, we have the relation:

$$E(\phi, n, \rho) \approx E(\phi, n, |\psi^\text{\textdagger}) + E(\phi, 2, \rho) - E(\phi, 2, |\psi^\text{\textdagger})$$

(9)

We calculated $E(\phi, 2, \rho)$ and $E(\phi, 2, |\psi^\text{\textdagger})$ from standard quantum mechanics, and use Son’s theory to calculate $E(\phi, n, |\psi^\text{\textdagger})$. For a given set of data $E_{\text{exp}}(\phi_i)$, we find $\rho$ and $n$ to minimize the objective function $L = \sum_i [E(\phi_i, n, \rho) - E_{\text{exp}}(\phi_i)]^2 / [\delta E_{\text{exp}}(\phi_i)]^2$, where $\delta E_{\text{exp}}(\phi_i)$ is the standard deviation of correlation function $E_{\text{exp}}(\phi_i)$ predicted assuming Poissonian count statistics. Figure 7 shows the results of fitting the correlation functions for 6 sets of data. From this, we extracted $n = 2.04, 2.01, 2.00, 2.01, 2.01, 2.00$; averaging these results and using their standard deviation to estimate the uncertainty yields $n = 2.01 \pm 0.02$ in good agreement with Born’s rule where $n = 2$. 

FIG. 6: Generalized correlations for the singlet state as a function of $n$ using Son’s theory [18]. The correlation as a function of $\theta$ is shown for $n = 1$ (blue line), $n = 2$ (purple line), $n = 5$ (brown line) and $n = 10$ (green line). The $n = 2$ case corresponds to standard quantum mechanics.

FIG. 7: Correlation functions versus the rotation angle $\phi$. The experimental correlations are extracted from our data for the case where the rotation axis and the measurement basis are given by {}[\{Z,(D,A)\}, [Z,(R,L)], [Y,(D,A)], [Y,(H,V)], [X,(R,L)], [X,(H,V)]} shown as {red squares, blue circles, green up triangles, yellow down triangles, black empty squares, pink diamonds} as a function of the rotation angle $\phi$. The best fit using Eq. 9 for each correlation is shown as a line whose colour matches the corresponding data points. These fits yield estimates for the value of $n$ of \{2.04, 2.01, 2.00, 2.01, 2.01, 2.00\}, respectively.