Experimental comparison of tomography and self-testing in certifying entanglement

Koon Tong Goh,1,2 Chithrabhanu Perumangatt,1 Zhi Xian Lee,3 Alexander Ling,1,3 and Valerio Scarani1,3

1Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543
2Department of Electrical & Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117583
3Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542

We assess the quality of a source of allegedly pure two-qubit states using both standard tomography and methods inspired by device-independent self-testing. Even when the detection and locality loopholes are open, the latter methods can dispense with modelling of the system and the measurements. However, due to finite sample fluctuations, the estimated probability distribution usually does not satisfy the no-signaling conditions exactly. We implement data analysis that is robust against these fluctuations. We demonstrate a high ratio of self-testing fidelity to that from full tomography, proving high performance of self-testing methods.

I. INTRODUCTION

Like any physical device, quantum devices (for example, sources, transformations, and measurements) must be calibrated or certified. In this paper, we focus on certifying the properties of the state produced by a source. The obvious techniques are quantum state tomography (i.e. the reconstruction of the density matrix, whole or partial), or entanglement witnesses if one is interested only in certifying entanglement. The very idea of tomography is unavoidably based on modelling, and so are most entanglement witnesses: they assume the dimension of the Hilbert space, and often the perfect calibration of the measurement settings as well. Such modelling is superfluous if entanglement is certified by the violation of Bell inequalities. A loophole-free Bell test can be used for device-independent certification.

The requirement of performing a loophole-free Bell test is daunting for most laboratories: few can claim the label “device-independent”. However, diagnostics based on Bell nonlocality are of interest even if a loophole-free Bell test is not performed. Consider the analogy of tomography, called robust self-testing. It is a black-box certification of how close a state is to a target state, up to local isometries. Even if the detection and locality loopholes are not closed, self-testing presents some advantage over standard tomography (in which, incidentally, fair sampling and no-signaling are routinely assumed). First, it avoids the modelling assumptions on dimensions and measurements. Second, it requires estimating fewer average values than tomography: for bipartite systems, three measurements on Alice and four on Bob are sufficient to assess the closeness to any pure bipartite state, of any dimension. Self-testing tools could then replace or complement standard tomography, provided these new tools perform well: if self-testing were to yield a fidelity of 70% with the target state, where normal tomography yields 99%, there would be little incentive for experimentalists to relax assumptions.

In this paper, we implement tools for the assessment of the self-testing fidelity on finite samples. We then apply them to experimental measurements, to characterise a source that allegedly produces pure two-qubit entangled states. We find that the self-testing fidelity can match the tomography fidelity. Thus, self-testing certification can replace tomography for practical applications.

II. THEORY

A. Framework

A conceptual scheme of the setup is shown in Fig. 1. The source is designed to produce, ideally, two-qubit pure entangled states. In other words, we aim at certifying how close the actual state is to one of the states described as

\[ |\psi(\theta)\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle, \quad 0 < \theta \leq \frac{\pi}{4}, \tag{1} \]

up to local isometries. Each of the measurement devices, called Alice and Bob as usual, has a classical input (denoted respectively \( x \) and \( y \)) and a classical output (denoted respectively \( a \) and \( b \)). Ideally, the input determines which measurement is performed, and the output is the outcomes of the measurement; we emphasize that our treatment makes no assumption on how inputs are treated or how outputs are produced. In this work, we consider binary inputs and outputs, and denote them by \( x, y \in \{0, 1\} \) and \( a, b \in \{-1, +1\} \).

After performing several rounds of the experiment, we compute the frequencies \( f(a,b,x,y) \) of each of the sixteen 4-tuples \( (a,b,x,y) \); whence we estimate the conditional probabilities \( P(a,b|x,y) \) through

\[ P(a,b|x,y) \approx \frac{f(a,b,x,y)}{\sum_{a',b'} f(a',b',x,y)}. \tag{2} \]

It is in this estimate that we leave aside the possibility of device-independent certification. First, the probabilities are reconstructed only from events in which one detector fired on each side (thus, we assume fair sampling).
I. INTRODUCTION

The first examples of self-testing proved that the Clauser-Horne-Shimony-Holt (CHSH) inequality [17] can only be achieved by the state (1) with $\alpha = \sqrt{2\sin(2\theta)}$, measured according to

$$A_\alpha = \sigma_z, \quad B_\alpha = \cos \mu \sigma_z + \sin \mu \sigma_x,$$

where $\mu = \arctan(\sin(2\theta))$.

Since our source may not be ideal, we shall need a version of self-testing that is robust against experimental imperfections. The criterion itself can be made robust [19], but one can do better than simply checking the value of $I_\alpha$, since an estimate of all the $P(a, b|x, y)$ is available from the observed values. In this paper, we will adopt the SWAP method [21, 22], based on the Navascués-Pironio-Acín (NPA) [23] hierarchies of relaxation of the set of quantum correlations.

C. Finite Sample Size Effects

In the first report of the application of self-testing to experimental observations [15], the bound on the fidelity with the target state was estimated by plugging the observed frequencies into the expressions of suitable Bell-type inequalities. Here, we implement a previous data processing, addressing concerns that arise due to statistical fluctuations.

The awareness of the importance of statistical fluctuations due to the finite size of the samples is rather recent even in normal tomography [24, 25]. Notably, we mention two such concerns. The first is rather obvious: if the source is of high quality, $I_\alpha$ will be close to the quantum maximum. An estimate over few rounds may exceed that maximum, making it impossible to draw any conclusion from the point estimator. The second concern arises from the fact the probabilities inferred from the frequencies generally do not obey the no-signaling condition exactly. This is an issue because many tools in the theory of Bell nonlocality, including Bell inequalities themselves [26], cannot be properly used only in the no-signaling set. Using our measured results, we show an illustration of both these concerns in Fig. 2.

We address these issues following the proposal of Lin and co-workers [27]. Based on the work in [26], they devised a method to obtain a point estimator of correlation that is compatible with quantum theory from the raw observations. Since the quantum set cannot be efficiently parametrized, the point estimator is chosen as the one most compatible with the raw observations within the NPA relaxation of the quantum set of a given hierarchy level. In particular, the nearest quantum approximation (NQA$_2$) method, which uses 2-norm as a measure between correlations, can be computed efficiently using any semi-definite programming solver. We shall use this point estimator as the input for the SWAP method.

III. EXPERIMENTAL SETUP

The experimental setup is sketched in Fig 3. Polarization entangled photon pairs are generated using spontaneous parametric down-conversion (SPDC) where a pump photon undergoes frequency conversion within a $\chi^2$ non-linear crystal to generate photon pairs of lower frequency. We have used a type-1 critically phase-matched SPDC source that produced collinear non-degenerate photon pairs from two $\beta$- barium borate crystals whose axes are parallel [28]. A pump laser of wavelength 405nm is focused to two BBO crystals with a special wave
FIG. 2. (Color Online) Observed violation of the tilted-CHSH inequalities for various values of \( \theta \). The experimental results used for this plot consist of 500 trials per setting \((x, y)\). The error bars are obtained by assuming a Poisson distribution of photon counting. For the same experimental results, we compute \( \langle A_0 \rangle_y := \sum_{a,b} a P(a,b|x = 0, y) \) for both \( y = 0, 1 \). This plot is an example that illustrates two concerns that call for a proper finite sample analysis. First, for \( \theta = 32.5^\circ \) the point estimator clearly violate the no-signaling condition, as \( \langle A_0 \rangle_{y=0} \neq \langle A_0 \rangle_{y=1} \). Second, the plot in the inset are obtained from experimental results consist of 100 trials per setting \((x, y)\). For \( \theta = 42.5^\circ, 45^\circ \), the sample mean exceeds the theoretical quantum maximum.

The photons are split using a dichroic beam splitter. The vertically polarized pump generates photon pairs with horizontal polarization (with state \(|HH\rangle\)) in both the crystals. The waveplate in between the crystals rotates the polarization of the pairs produced in the first crystal \(|HH\rangle \rightarrow |VV\rangle\) without affecting the polarization of the pump. The wavelength dependent phase between \(|HH\rangle\) (produced in the second crystal) and \(|VV\rangle\) (generated at the first crystal) is compensated using a single a-cut yttrium orthovanadate crystal (YVO\(_4\)) with the correct orientation and length.

The experiments are performed for the quantum state verification. 1) 405nm laser 2) BBO1 on a linear translational stage to change the value of \( \theta \). 3) special waveplate 4) BBO2 5) dichroic mirror for removing the pump laser 6) temporal compensator (YVO\(_4\)) 7) dichroic beam splitter for splitting the signal and idler photons 8) quater wave plate (only for tomography) 9) polarizer for the projective measurements 10) single mode fiber connected single photon detectors 11) coincidence unit for calculating \( P(a,b|x, y) \).

FIG. 3. (Color Online) Experimental setup for the quantum state verification. 1) 405nm pump laser 2) BBO1 on a linear translational stage to change the value of \( \theta \). 3) special waveplate 4) BBO2 5) dichroic mirror for removing the pump laser 6) temporal compensator (YVO\(_4\)) 7) dichroic beam splitter for splitting the signal and idler photons 8) quater wave plate (only for tomography) 9) polarizer for the projective measurements 10) single mode fiber connected single photon detectors 11) coincidence unit for calculating \( P(a,b|x, y) \).
ever, in self-testing one can always obtain a lower bound for the fidelity even if the measurement angles are off from the ideal ones. Note that with non-ideal measurement angles, the fidelity obtained by self-testing will only be an underestimation of quantum correlation present in the source.

After the measurements are made, we apply the NQA₂ method on the frequencies of coincidences \( f(a, b, x, y) \) to obtain the nearest set of marginals and correlators that resides in the NPA relaxation to the set of quantum correlations. We used a \( 37 \times 37 \) NPA moment matrix detailed in Appendix A. Next, we apply the SWAP method on these marginals and correlators. Here, we used the same \( 37 \times 37 \) NPA moment matrix with additional two \( 16 \times 16 \) localising matrices. The resulting lower bound on the fidelity with the target state is also shown in Fig. 4.

![FIG. 4. (Color Online) The plot of the fidelity between the measured state and the ideal state, \(|\psi(\theta)\rangle\), against \( \theta \). The blue upright triangular points indicate the fidelity \( f_s \) obtained from quantum state tomography performed on the quantum state produced by the source in the experiment. The red inverted triangular points indicate the lower bound \( f_s \) obtained by self-testing (with the measurements that maximise the tilted-CHSH inequality). These plots are obtained using a \( 37 \times 37 \) NPA moment matrix and two \( 16 \times 16 \) localising matrices. The values of \( f_t \) and \( f_s \) are listed in Table 1.](image)

| \( \theta/\degree \) | 30 | 35 | 37.5 | 40 | 42.5 | 45 |
|----------------|----|----|------|----|------|----|
| \( f_t \)     | 0.990 | 0.996 | 0.984 | 0.992 | 0.992 | 0.988 | 0.991 |
| \( f_s \)     | 0.943 | 0.933 | 0.963 | 0.979 | 0.978 | 0.985 | 0.982 |
| \( f_s/f_t \) | 0.953 | 0.937 | 0.979 | 0.987 | 0.986 | 0.997 | 0.991 |

TABLE 1. Fidelities obtained via tomography (\( f_t \)) and self-testing (\( f_s \)) from the experimental results.

The fidelity obtained by tomography, denoted by \( f_t \), is always higher than that obtained by self-testing, denoted by \( f_s \) (see Table 1). Though expected, this is not trivial: it can be taken as a validation of the modelling assumptions made for tomography. That being said, the fidelities computed from self-testing are almost identical for \( \theta \geq 35\degree \). There is nothing fundamental in this number: the range of agreement could be improved by taking larger moment matrices in the NPA hierarchy. The average ratio \( f_s/f_t \) is 0.976, but for \( \theta = 30.0\degree, 32.5\degree \), the values of \( f_s/f_t \) are visibly smaller than other data points. These anomalies can be explained by the tilted-CHSH violation falling short from their maximal quantum value at these points. In Fig. 2, we can see that the probable regions for these points excludes the value of quantum maximal violation. If one considers only the data points where the probable region of the Bell violation includes the quantum maximal violation \( i.e. \theta = 35.0\degree, 37.5\degree, 40.0\degree, 42.5\degree, 45.0\degree \), the average ratio \( f_s/f_t \) is given by 0.988.

Our results demonstrate that even in the regime of near-maximal violation of the CHSH inequalities, self-testing (in a non-adversarial scenario) can provide a physically plausible point estimator of the bipartite entangled qubit state. In recent work [15], similar self-testing analysis has been done for pure bipartite entangled qubits states that maximally violates the Collins–Gisin-Linden-Massar-Popescu (CGLMP) [30, 31] and the Salavrakos–Augusiak–Tura–Wittek–Acín–Pironio (SATWAP) [32] inequalities for Hilbert space dimensions up to 8. In their work, violation of CGLMP and SATWAP inequalities are obtained and used to estimate the quality of their source. However, the violations observed were far their quantum maximal value and the analysis did not encounter the problems associated with near-maximal Bell violation using a finite sample size.

Similar can be said to any fully device-independent self-testing that was performed or will be performed in the near future. In another recent work [16], a fully device-independent certification of the singlet state was performed and yielded a fidelity of 0.5554 with a 99% confidence. In fact, the projected near-term achievable CHSH violation by a loophole-free Bell test is given by 2.47 [33] which gives a singlet fidelity of 0.752 using the method from [34]. For experimentalists who wish to check the serviceability of their entanglement source, such bounds are too pessimistic.

V. CONCLUSION AND OUTLOOK

In conclusion, we have shown that using existing quantum devices, we could provide a point estimator on the performance of a source of quantum states without assuming the characterisation of the measurements. Furthermore, this estimation is robust against false positive and requires less measurement settings as compared with quantum state tomography. This can be of great interest in practical deployment of ground or space based quantum communication systems since we can estimate the lower bound for the fidelity of the state even if the mea-
s

There is one final missing ingredient for the full solution to the problem: we could not propagate the error bars on the Bell violation and/or conditional probabilities to the error bars on the fidelity between the measured and ideal quantum states. We hope that this experimental demonstration in this paper would motivate researchers to find the full solution to the problem proposed.

ACKNOWLEDGEMENTS

The authors would like to thank Yeong-Cherng Liang, Jean-Daniel Bancal and Yanbao Zhang for the useful discussions. This research is supported by the National Research Foundation, Prime Minister’s Office, Singapore and the Ministry of Education, Singapore under the Research Centres of Excellence programme

[1] J. S. Bell, Physics 1, 195 (1964).

[2] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Nature 526, 682 (2015).

[3] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J.-A. Larsson, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, J. Beyer, T. Gerrits, A. E. Lita, L. K. Shalm, S. W. Nam, T. Scheidl, R. Ursin, B. Wittmann, and A. Zeilinger, Phys. Rev. Lett. 115, 250401 (2015).

[4] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, T. Gerrits, S. Glancy, D. R. Hamel, M. S. Allman, K. J. Coakley, S. D. Dyer, C. Hodge, A. E. Lita, V. B. Verma, C. Lambrocco, E. Tortorici, A. L. Migdall, Y. Zhang, D. R. Kumor, W. H. Farr, F. Marsili, M. D. Shaw, J. A. Stern, C. Abellán, W. Amaya, V. Pruneri, T. Jennewein, M. W. Mitchell, P. G. Kwiat, J. C. Bienfang, R. P. Mirin, E. Knill, and S. W. Nam, Phys. Rev. Lett. 115, 250402 (2015).

[5] W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, Phys. Rev. Lett. 119, 010402 (2017).

[6] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Physical Review Letters 98, 230501 (2007).

[7] J. Barrett, L. Hardy, and A. Kent, Physical review letters 95, 010503 (2005).

[8] A. K. Ekert, Physical review letters 67, 661 (1991).

[9] D. Mayers and A. Yao, Quantum Inf. Comput. 4, 273 (2004).

[10] B. Tóth, Hadronic Journal Supplement 8, 329 (1993).

[11] S. Popescu and D. Rohrlich, Phys. Lett. A 169, 411 (1992).

[12] S. J. Summers and R. Werner, J. Math. Phys. 28, 2440 (1987).

[13] F. Magniez, D. Mayers, M. Mosca, and H. Ollivier, in International Colloquium on Automata, Languages, and Programming (Springer, 2006) pp. 72–83.

[14] A. Coladangelo, K. T. Goh, and V. Scarani, Nature Communications 8, 15485 (2017).

[15] J. Wang, S. Paesani, Y. Ding, R. Santagati, P. Skrzypczyk, A. Salavarakos, J. Tura, R. Augusiak, L. Mančinska, D. Bacco, D. Boneau, J. W. Silverstone, Q. Gong, A. Acín, K. Rottwitt, L. K. Oxenløwe, J. L. O’Brien, A. Laing, and M. G. Thompson, Science 360, 285 (2018).

[16] J.-D. Bancal, K. Redeker, P. Sekatski, W. Rosenfeld, and N. Sangouard, arXiv preprint arXiv:1812.09117 (2018).

[17] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).

[18] T. H. Yang and M. Navascués, Phys. Rev. A 87, 050102 (2013).

[19] C. Bamps and S. Pironio, Phys. Rev. A 91, 052111 (2015).

[20] A. Acín, S. Massar, and S. Pironio, Phys. Rev. Lett. 108, 100402 (2012).

[21] J.-D. Bancal, M. Navascués, V. Scarani, T. Vértesi, and T. H. Yang, Physical Review A 91, 022115 (2015).

[22] T. H. Yang, T. Vértesi, J.-D. Bancal, V. Scarani, and M. Navascués, Physical review letters 113, 040401 (2014).

[23] M. Navascués, S. Pironio, and A. Acín, New Journal of Physics 10, 073013 (2008).

[24] X. Li, J. Shang, H. K. Ng, and B.-G. Englert, Physical Review A 94, 062112 (2016).

[25] J. Wang, V. B. Scholz, and R. Renner, arXiv preprint arXiv:1808.09988 (2018).

[26] Bell inequalities are hyperplanes bounding the set of local variables (a.k.a. local polytope), which is obviously a subset of the no-signaling set. If we consider the whole space of probabilities, there are infinitely many hyperplanes, whose intersections with the no-signaling set define the same Bell inequality. For a given signaling probability point, one can find hyperplanes, for which the point lies on the side of the local set; and others, for which the point lies on the other side. In other words, if one takes a signaling point and plugs it into the expression of a Bell inequality, the conclusion (violation or not) may be an artefact of the choice of the Bell expression.

[27] P.-S. Lin, D. Rosset, Y. Zhang, J.-D. Bancal, and Y.-C. Liang, Physical Review A 97, 032309 (2018).

[28] M. O. Renou, D. Rosset, A. Martin, and N. Gisin, Journal of Physics A: Mathematical and Theoretical 50, 255301 (2017).
Appendix A: Methods for robust self-testing

The techniques of robust self-testing that are employed in this paper will be documented in this appendix. In order to prove self-testing, it is sufficient to show the existence of a local isometry \( \Phi(\cdot) \) such that \( \Phi(\langle \psi \rangle) = \langle \text{junk} \rangle \otimes \langle \psi_{\text{target}} \rangle \) where \( \langle \psi \rangle \) is the measured quantum state, \( \langle \psi_{\text{target}} \rangle \) is the target quantum state and \( \langle \text{junk} \rangle \) can be any arbitrary quantum state.

Without loss of generality, one can pick the local isometry \( \Phi(\cdot) \) to be the quantum circuit given by Figure 5. After going through the computation of the circuit, we arrive at the following state:

\[
\Phi(\langle \psi \rangle_{AB}) = \frac{1}{4} (\langle \psi \rangle_{AB} | 0 \rangle_{A'}^0 | 0 \rangle_{B'}^0 + \langle \psi \rangle_{AB} | 0 \rangle_{A'}^1 | 1 \rangle_{B'}^1 + \langle \psi \rangle_{AB} | 1 \rangle_{A'}^0 | 1 \rangle_{B'}^0 + \langle \psi \rangle_{AB} | 1 \rangle_{A'}^1 | 0 \rangle_{B'}^0).
\]

The remaining task is to show that given the observed statistics or Bell violation, the bipartite qubits state in the Hilbert space \( A'B' \), denoted by \( \rho_{A'B'} \), is indeed \( \langle \psi_{\text{target}} \rangle \). In the case of this paper, the target states are the pure bipartite entangled states given by

\[
| \psi(\theta) \rangle = \cos \theta | 00 \rangle + \sin \theta | 11 \rangle.
\]

However, experimental results can never achieve the criteria for self-testing due to noise and error. Nonetheless, one can obtain a lower bound for the fidelity between a measured quantum state and the target quantum state, denoted by \( F \), given a certain amount of deviation from the ideal statistics. Since the target states are pure, we can define the fidelity as:

\[
F := \langle \psi(\theta) | \rho_{A'B'} | \psi(\theta) \rangle.
\]
Next, in order for the unitaries $Z_{A/B}$ and $X_{A/B}$ in the local isometry to simulate the effect of $\sigma$ and $\sigma_x$ operators respectively, we set the operators $Z_A, X_A, Z_B$ and $X_B$ as:

$$Z_A := A_0,$$  \hspace{1cm} (A3)

$$X_A := A_1,$$ \hspace{1cm} (A4)

$$Z_B := \frac{B_0 + B_1}{2\cos \mu},$$ \hspace{1cm} (A5)

$$X_B := \frac{B_0 - B_1}{2\sin \mu},$$ \hspace{1cm} (A6)

where $\tan \mu = \sin 2\theta$. Notice that we define the operators $\tilde{Z}_B$ and $\tilde{X}_B$ with tildes as we anticipate that they are not unitary in general. Hence, inserting $\tilde{Z}_B$ and $\tilde{X}_B$ in the quantum circuit will not result in a valid local isometry.

In order to circumvent this problem, we employ a method [22], which exploits a result from polar decomposition. For any operator $B$, there exist a decomposition such that $B = UP$, where $U$ is a unitary operator and $P$ is a positive semi-definite operator. Moreover, $P$ is unique and if $B$ is unitary, which implies $U = B$. Since, $\tilde{Z}_B$ and $\tilde{X}_B$ are Hermitian, one can show that the unitaries of their polar decomposition can be Hermitian. Hence, there exist some $\pm 1$ operators, $B_2$ and $B_3$, such that:

$$B_2(B_0 + B_1) \geq 0,$$  \hspace{1cm} (A7)

$$B_3(B_0 - B_1) \geq 0,$$ \hspace{1cm} (A8)

and we define

$$Z_A = A_0 , X_A = A_1,$$ \hspace{1cm} (A9)

$$Z_B = B_2 , X_B = B_3.$$ \hspace{1cm} (A10)

Using these relations, the optimisation to lower bound the fidelity, $F$, is given by:

$$\min F$$ \hspace{1cm} (A11)

s.t. \hspace{1cm} $\Gamma \geq 0$

$$B_2(B_0 + B_1) \geq 0,$$

$$B_3(B_0 - B_1) \geq 0,$$

where $\Gamma$ is the moment matrix associated with the Navascués-Pironio-Acin (NPA) [23] relaxation that is compatible with the observed statistics $P(a, b|x, y)$. The moment matrix $\Gamma_{ij} := \langle \psi|O_i^aO_j^b|\psi \rangle$ used in this paper employs the following set of operators $\{O_i\}: I, A_0, A_1, B_0, B_1, B_2, B_3, A_0A_1, A_1A_0, B_0B_1, B_1B_0, B_0B_2, B_2B_0, B_0B_3, B_3B_0, B_1B_2, B_2B_1, B_1B_3, B_3B_1, B_2B_3, B_3B_2, A_0A_0, A_0B_0, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0, A_0A_1, A_1B_0, A_0A_1, A_1B_0, B_0B_2, B_2B_0$. Hence, the $\Gamma$ we use in this paper is a $37 \times 37$ matrix.

Using the definition of $F$ in equation (A2) and the isometry given by Fig. 6, we can compute $F$ to be given by:

$$F = \frac{1}{4}(1 + \langle A_0B_2 \rangle + \cos 2\theta(\langle A_0 \rangle + \langle B_2 \rangle)$$ \hspace{1cm} (A12)

$$+ \frac{1}{2}(\cos \theta + \sin \theta)(\langle A_1B_3 \rangle + \langle A_1A_0B_3B_2 \rangle$$

$$+ \langle A_0A_1B_2B_3 \rangle + \langle A_0A_1A_0B_2B_3 \rangle - \langle A_0A_1B_2B_3 \rangle$$

$$- \langle A_0A_1B_2B_3 \rangle - \langle A_1A_0B_2B_3 \rangle)$$

However, the last two constraints of optimisation (A11) cannot be imposed in a numerical program. In order to impose the conditions (A7) to (A8) we use the method of matrix localization to provide a relaxation of the problem as it is a necessary condition that the localising matrix $\Gamma(B)_{ij} := \langle \psi|O_i^aO_j^b|\psi \rangle$ to be positive semi-definite if $B$ is positive semi-definite. Hence, we will perform the following optimisation:

$$\min F$$ \hspace{1cm} (A13)

s.t. \hspace{1cm} $\Gamma \geq 0$

$$\Gamma(B_2(B_0 + B_1)) \geq 0$$

$$\Gamma(B_3(B_0 - B_1)) \geq 0.$$
FIG. 6. (Color Online) The plot of lower bounds on the fidelity between the measured state and the ideal state, $|\psi(\theta)\rangle$, against the deviation from maximal violation of the tilted-CHSH inequality denoted by $\epsilon$ (see equation A14) over various values of $\theta$. These plots are obtained using a $37 \times 37$ NPA moment matrix and two $16 \times 16$ localising matrices.

Experimental results. The NQA$_2$ method can be phrased as the following semi-definite programming problem:

$$\tilde{P} := \arg\min_P s$$

s.t. $$\Gamma \geq 0,$$

$$\left( P - \bar{f} \right) \geq 0,$$

where $s$ is a real number, $\bar{f}$ is a vector with elements $\frac{f(a,b,x,y)}{\sum_{a',b',x',y'} f(a',b',x',y')}$, $P$ is a vector with elements $P(a,b|x,y)$ such that it is within some NPA relaxation of the set of quantum correlations i.e. $\Gamma \geq 0$ and $\tilde{P}$ is a vector with its elements consisting of the regularised conditional probabilities.