DELOCALIZED ETA INVARIANTS, ALGEBRAICITY, AND K-THEORY OF GROUP C*-ALGEBRAS

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Abstract. In this paper, we establish a precise connection between higher rho invariants and delocalized eta invariants. Given an element in a discrete group, if its conjugacy class has polynomial growth, then there is a natural trace map on the $K_0$-group of its group $C^*$-algebra. For each such trace map, we construct a determinant map on secondary higher invariants. We show that, under the evaluation of this determinant map, the image of a higher rho invariant is precisely the corresponding delocalized eta invariant of Lott. As a consequence, we show that if the Baum-Connes conjecture holds for a group, then Lott’s delocalized eta invariants take values in algebraic numbers. We also generalize Lott’s delocalized eta invariant to the case where the corresponding conjugacy class does not have polynomial growth, provided that the strong Novikov conjecture holds for the group.

1. Introduction

Let $X$ be a complete manifold of dimension $n$ with a discrete group $\Gamma$ acting on it properly and cocompactly through isometries. Each $\Gamma$-equivariant elliptic differential operator $D$ on $X$ gives rise to a higher index class $\text{Ind}_\Gamma(D) \in K_n(C^*_\Gamma(\Gamma))$. This higher index class is an obstruction to the invertibility of $D$. It is a far-reaching generalization of the classical Fredholm index and plays a fundamental role in the studies of many problems in geometry and topology such as the Novikov conjecture, the Baum-Connes conjecture and the Gromov-Lawson-Rosenberg conjecture. Higher index classes are often referred to as primary invariants. When the higher index class of an operator vanishes, a secondary index theoretic invariant naturally arises. One such example is the associated Dirac operator $\tilde{D}$ on the universal covering $\tilde{M}$ of a closed spin manifold $M$, which is equipped with a positive scalar curvature metric $g$. In this case, it follows from the Lichnerowicz formula that the higher index of the Dirac operator vanishes. And there is a natural secondary higher invariant – introduced by Higson and Roe [12, 13, 14, 30] – called the higher rho invariant of $\tilde{D}$ (with respect to the metric $g$), cf. Section 2.1 and Section 4 below for details. This higher rho invariant is an obstruction to the inverse of the Dirac operator being local, and has important applications to geometry and topology.

On the other hand, for the same Dirac operator $\tilde{D}$ above, Lott introduced the following delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ [23]:

$$\eta_{\langle h \rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_h(\tilde{D}e^{-t^2\tilde{D}^2}) dt,$$  \hspace{1cm} (1)

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under the condition that the conjugacy class $\langle h \rangle$ of $h \in \pi_1 M$ has polynomial growth. Here $\pi_1 M$ is the fundamental group of $M$, and $\text{tr}_h$ is the following trace map (see Section 4 for more details):

$$\text{tr}_h(A) = \sum_{g \in \langle h \rangle} \int_{\mathcal{F}} A(x, gx) dx$$

on $\Gamma$-equivariant Schwartz kernels $A \in C^\infty(\tilde{M} \times \tilde{M})$, where $\mathcal{F}$ is a fundamental domain of $\tilde{M}$ under the action of $\Gamma$.

In this paper, we shall devise a conceptual $K$-theoretic approach to establish a precise connection between Higson-Roe’s $K$-theoretic higher rho invariants and Lott’s delocalized eta invariants. More precisely, we have the following theorem.

**Theorem 1.1.** Let $M$ be a closed odd-dimensional spin manifold equipped with a positive scalar curvature metric $g$. Suppose $\tilde{M}$ is the universal cover of $M$, $\tilde{g}$ is the Riemannian metric on $\tilde{M}$ lifted from $g$, and $\tilde{D}$ is the associated Dirac operator on $\tilde{M}$. Suppose the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \pi_1 M$ has polynomial growth, then we have

$$\tau_h(\rho(\tilde{D}, \tilde{g})) = \frac{1}{2} \eta_{\langle h \rangle}(\tilde{D}),$$

where $\rho(\tilde{D}, \tilde{g})$ is the $K$-theoretic higher rho invariant of $\tilde{D}$ with respect to the metric $\tilde{g}$, and $\tau_h$ is a canonical determinant map associated to $\langle h \rangle$.

While the definition of Lott’s delocalized eta invariant requires certain growth conditions on $\pi_1 M$ (e.g. polynomial growth on a conjugacy class), the $K$-theoretic higher rho invariant can be defined in complete generality, without any growth conditions on $\pi_1 M$. We shall show how to generalize Lott’s delocalized eta invariant without imposing any growth conditions on $\pi_1 M$, provided that the strong Novikov conjecture holds for $\pi_1 M$. This is achieved by using the Novikov rho invariant introduced in [36, Section 7].

As an application of Theorem 1.1 above, we have the following algebraicity result concerning the values of delocalized eta invariants.

**Theorem 1.2.** With the same notation as above, if the (rational) Baum-Connes conjecture holds for $\Gamma$, and the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, then the delocalized eta invariant $\eta_{\langle h \rangle}(\tilde{D})$ is an algebraic number. Moreover, if in addition $h$ has infinite order, then $\eta_{\langle h \rangle}(\tilde{D})$ vanishes.

This theorem follows from the construction of the determinant map $\tau_h$ and a $L^2$-Lefschetz fixed point theorem of B.-L. Wang and H. Wang [33, Theorem 5.10]. When $\Gamma$ is torsion-free and satisfies the Baum-Connes conjecture, and the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, Piazza and Schick have proved the vanishing of $\eta_{\langle h \rangle}(\tilde{D})$ by a different method [25, Theorem 13.7].

In light of this algebraicity result, we propose the following question.

**Question.** What values can delocalized eta invariants take in general? Are they always algebraic numbers?
In particular, if a delocalized eta invariant is transcendental, then it will lead to a counterexample to the Baum-Connes conjecture [4, 5, 9]. Note that the above question is a reminiscent of Atiyah’s question concerning rationality of $\ell^2$-Betti numbers [1]. Atiyah’s question was answered in negative by Austin, who showed that $\ell^2$-Betti numbers can be transcendental [2].

As another application of Theorem 1.1, we give a $K$-theoretic proof of a version of delocalized Atiyah-Patodi-Singer index theorem. See Theorem 5.3 below.

Some of the main results in this paper are inspired by previous work of Lott, Leichtnam, Piazza and Schick [22, 23][25][21]. A key new ingredient of our approach is the construction of an explicit determinant map $\tau_h$ on $K_1(C^*_{\rho,0}(\tilde{M}^{\pi_1 M}))$ for each non-identity conjugacy class $\langle h \rangle$ with polynomial growth. The definition of $K_1(C^*_{\rho,0}(\tilde{M}^{\pi_1 M}))$ is reviewed in Section 2.1 below. Each such determinant map is induced by the corresponding trace map $\text{tr}_h$ on $K_0(C^*_{\rho}(\pi_1 M))$, and our construction is inspired by the work of de la Harpe and Skandalis [11] and Keswani [17]. In fact, combined with finite propagation speeds of wave operators, our $K$-theoretic approach above can also be used to give a uniform treatment of various vanishing results and homotopy invariance results for delocalized eta variants in [34, 17, 18, 25, 15, 7]. See a brief discussion in Remark 3.11 below. We will present the details in a separate paper [32].

The paper is organized as follows. In Section 2, we recall some basic definitions of certain geometric $C^*$-algebras. Given a discrete group, for each conjugacy class with polynomial growth, we review how to extend a trace on the group algebra to the Connes-Moscovici smooth dense subalgebra of the corresponding reduced group $C^*$-algebra. In Section 3, we then use this extended trace map to define an explicit determinant map on secondary higher invariants. In Section 4, we establish a precise connection between Higson-Roe’s $K$-theoretic higher rho invariants and Lott’s delocalized eta invariants. We then apply it in Section 5 to prove an algebraicity result concerning the values of delocalized eta invariants and a version of delocalized Atiyah-Patodi-Singer index theorem.

2. Conjugacy classes with polynomial growth and extension of traces

Let $\Gamma$ be a discrete group and $\mathbb{C}\Gamma$ the corresponding group algebra. For each $h \in \Gamma$, there is a natural trace map on $\mathbb{C}\Gamma$ defined as follows:

$$\text{tr}_h(a) = \sum_{g \in \langle h \rangle} a_g$$

where $\langle h \rangle$ is the conjugacy class of $h$ and $a = \sum_{g \in \Gamma} a_g g \in \mathbb{C}\Gamma$. In this section, we give a brief construction on how to extend this trace to a smooth dense subalgebra of the reduced group $C^*$-algebra $C^*_{\tau}(\Gamma)$, provided that $\langle h \rangle$ has polynomial growth. In particular, such a trace map induces a map on $K_0(C^*_{\tau}(\Gamma))$. We shall use this induced map on $K_0(C^*_{\tau}(\Gamma))$ to define a determinant map on secondary higher invariants in the next section.

2.1. Roe algebras and localization algebras. In this subsection, we briefly recall some standard definitions of certain geometric $C^*$-algebras. We refer the reader to [29, 39] for more details. Let $X$ be a proper metric space. That is, every closed ball in $X$ is compact. An $X$-module is a separable Hilbert space equipped with a
*-representation of $C_0(X)$, the algebra of all continuous functions on $X$ which vanish at infinity. An $X$-module is called nondegenerate if the *-representation of $C_0(X)$ is nondegenerate. An $X$-module is said to be standard if no nonzero function in $C_0(X)$ acts as a compact operator.

**Definition 2.1.** Let $H_X$ be a $X$-module and $T$ a bounded linear operator acting on $H_X$.

(i) The propagation of $T$ is defined to be $\sup\{d(x, y) \mid (x, y) \in \text{supp}(T)\}$, where $\text{supp}(T)$ is the complement (in $X \times X$) of the set of points $(x, y) \in X \times X$ for which there exist $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x) \neq 0$, $g(y) \neq 0$;

(ii) $T$ is said to be locally compact if $fT$ and $Tf$ are compact for all $f \in C_0(X)$;

(iii) $T$ is said to be pseudo-local if $[T, f]$ is compact for all $f \in C_0(X)$.

**Definition 2.2.** Let $H_X$ be a standard nondegenerate $X$-module and $B(H_X)$ the set of all bounded linear operators on $H_X$.

(i) The Roe algebra of $X$, denoted by $C^*(X)$, is the $C^*$-algebra generated by all locally compact operators with finite propagations in $B(H_X)$.

(ii) $D^*(X)$ is the $C^*$-algebra generated by all pseudo-local operators with finite propagations in $B(H_X)$. In particular, $D^*(X)$ is a subalgebra of the multiplier algebra of $C^*(X)$.

(iii) $C^*_L(X)$ (resp. $D^*_L(X)$) is the $C^*$-algebra generated by all bounded and uniformly norm-continuous functions $f : [0, \infty) \to C^*(X)$ (resp. $f : [0, \infty) \to D^*(X)$) such that

\[
\text{propagation of } f(t) \to 0, \text{ as } t \to \infty.
\]

Again $D^*_L(X)$ is a subalgebra of the multiplier algebra of $C^*_L(X)$.

(iv) $C^*_L,0(X)$ is the kernel of the evaluation map

\[
ev : C^*_L(X) \to C^*(X), \quad ev(f) = f(0).
\]

In particular, $C^*_L,0(X)$ is an ideal of $C^*_L(X)$. Similarly, we define $D^*_L,0(X)$ as the kernel of the evaluation map from $D^*_L(X)$ to $D^*(X)$.

Now in addition we assume that a discrete group $\Gamma$ acts properly and cocompactly on $X$ by isometries. Let $H_X$ be a $X$-module equipped with a covariant unitary representation of $\Gamma$. If we denote the representation of $C_0(X)$ by $\varphi$ and the representation of $\Gamma$ by $\pi$, this means

\[
\pi(\gamma)(\varphi(f)v) = \varphi(f(\gamma))(\pi(\gamma)v),
\]

where $f \in C_0(X)$, $\gamma \in \Gamma$, $v \in H_X$ and $f^\gamma(x) = f(\gamma^{-1}x)$. In this case, we call $(H_X, \Gamma, \varphi)$ a covariant system.

**Definition 2.3** ([41]). A covariant system $(H_X, \Gamma, \varphi)$ is called admissible if

1. the $\Gamma$-action on $X$ is proper and cocompact;
2. $H_X$ is a nondegenerate standard $X$-module;
3. for each $x \in X$, the stabilizer group $\Gamma_x$ acts on $H_X$ regularly in the sense that the action is isomorphic to the action of $\Gamma_x$ on $l^2(\Gamma_x) \otimes H$ for some infinite dimensional Hilbert space $H$. Here $\Gamma_x$ acts on $l^2(\Gamma_x)$ by translations and acts on $H$ trivially.

We remark that for each locally compact metric space $X$ with a proper and cocompact isometric action of $\Gamma$, there exists an admissible covariant system $(H_X, \Gamma, \varphi)$. Also,
we point out that the condition (3) above is automatically satisfied if $\Gamma$ acts freely on $X$. If no confusion arises, we will denote an admissible covariant system $(H_X, \Gamma, \varphi)$ by $H_X$ and call it an admissible $(X, \Gamma)$-module.

**Definition 2.4.** Let $X$ be a locally compact metric space $X$ with a proper and cocompact isometric action of $\Gamma$. If $H_X$ is an admissible $(X, \Gamma)$-module, we denote by $\mathbb{C}[X]^\Gamma$ the $\ast$-algebra of all $\Gamma$-invariant locally compact operators with finite propagations in $\mathcal{B}(H_X)$. We define $C^*(X)^\Gamma$ to be the completion of $\mathbb{C}[X]^\Gamma$ in $\mathcal{B}(H_X)$.

Since the action of $\Gamma$ on $X$ is cocompact, we have $C^*(X)^\Gamma \cong C^*_r(\Gamma) \otimes \mathcal{K}$, where $C^*_r(\Gamma)$ is the reduced group $C^*$-algebra of $\Gamma$ and $\mathcal{K}$ is the algebra of all compact operators, cf. [30, Lemma 5.14].

Similarly, we can also define $D^*(X)^\Gamma$, $C^*_L(X)^\Gamma$, $D^*_L(X)^\Gamma$, $C^*_{L,0}(X)^\Gamma$, $D^*_{L,0}(X)^\Gamma$, $C^*_L(Y; X)^\Gamma$ and $C^*_{L,0}(Y; X)^\Gamma$.

**Remark 2.5.** Up to isomorphism, $C^*(X) = C^*(X, H_X)$ does not depend on the choice of the standard nondegenerate $X$-module $H_X$. The same holds for $D^*(X)$, $C^*_L(X)$, $D^*_L(X)$, $C^*_{L,0}(X)$, $D^*_{L,0}(X)$, $C^*_L(Y; X)$, $C^*_{L,0}(Y; X)$ and their $\Gamma$-equivariant versions.

**Remark 2.6.** Note that we can also define maximal versions of all the $C^*$-algebras above. For example, we define the maximal $\Gamma$-invariant Roe algebra $C^\text{max}_r(X)^\Gamma$ to be the completion of $\mathbb{C}[X]^\Gamma$ under the maximal norm:

$$\|a\|_{\text{max}} = \sup_{\phi} \left\{ \|\phi(a)\| \mid \phi : \mathbb{C}[X]^\Gamma \to \mathcal{B}(H^\prime) \text{ a } \ast\text{-representation} \right\}.$$

### 2.2. Extensions of traces to smooth dense subalgebras

In this subsection, we give a brief construction on how to extend the trace $\text{tr}_h : \mathbb{C}\Gamma \to \mathbb{C}$ to a trace map on a smooth dense subalgebra of the reduced group $C^*$-algebra $C^*_r(\Gamma)$, provided that $\langle h \rangle$ has polynomial growth. Also see [31] for an alternative approach and some interesting applications.

Let $M$ be a closed oriented Riemannian manifold. Let $\mathcal{R}$ be the algebra of smoothing operators on $M$. Fix a basis of $L^2(M)$, then $\mathcal{R}$ can be identified with the algebra of matrices $(a_{ij})_{i,j \in \mathbb{N}}$ such that

$$\sup_{i,j} i^k j^l |a_{ij}| < \infty \text{ for all } k, l \in \mathbb{N}.$$

Let us recall the following smooth dense subalgebra of $C^*_r(\Gamma) \otimes \mathcal{K}$, due to Connes and Moscovici [10]. Let $\Delta$, resp. $D$, be the unbounded operator in $\ell^2(\mathbb{N})$, resp. $\ell^2(\Gamma)$, defined by

$$\Delta(\delta_j) = j \delta_j \text{ for } j \in \mathbb{N}, \text{ resp. } Dg = |g| \cdot g \text{ for } g \in \Gamma.$$

Consider the unbounded derivations $\partial = [D, \cdot]$ of $\mathcal{B}(\ell^2(\Gamma))$ and $\widetilde{\partial} = [D \otimes I, \cdot]$ of $\mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\mathbb{N}))$, and set:

$$\mathcal{R}(\widetilde{M})^\Gamma = \{ A \in C^*_r(\Gamma) \otimes \mathcal{K} \mid \widetilde{\partial}^k(A) \circ (I \otimes \Delta)^2 \text{ is bounded } \forall k \in \mathbb{N} \}.$$

It follows from [10, Lemma 6.4](and its proof) that $\mathcal{R}(\widetilde{M})^\Gamma$ contains $\mathbb{C}\Gamma \otimes \mathcal{R}$ and is closed under holomorphic functional calculus. Denote the operator norm of the element

\[1\]To be precise, the algebra $\mathcal{R}(\widetilde{M})^\Gamma$ defined here is slightly different from Connes-Moscovici’s algebra $\mathcal{R}$ in [10, Lemma 6.4]. Both of them are smooth dense subalgebras of $C^*_r(\Gamma) \otimes \mathcal{K}$. In this paper, the algebra $\mathcal{R}(\widetilde{M})^\Gamma$ works better for our purposes.
\( \partial^k(A) \circ (I \otimes \Delta)^2 \) by \( \|A\|_k \). Then \( \mathcal{B}(\widetilde{M})^\Gamma \) is a Fréchet algebra under the sequence of seminorms \( \{\| \cdot \|_k : k \in \mathbb{N}\} \).

We associate to each element \( h \in \Gamma \) the following trace on \( \mathbb{C}\Gamma \otimes \mathcal{R} \):

\[
\text{tr}_h(\gamma \otimes \omega) = \begin{cases} 
\text{trace}(\omega) & \text{if } \gamma \in \langle h \rangle, \\
0 & \text{if } \gamma \notin \langle h \rangle, 
\end{cases}
\]

for \( \gamma \in \Gamma \) and \( \omega \in \mathcal{R} \). Now suppose \( h \in \Gamma \) such that its conjugacy class \( \langle h \rangle \) has polynomial growth, that is, there exists \( C \) and \( d \) such that

\[
\sharp \{g \in \langle h \rangle : \|g\| \leq n \} \leq C \cdot n^d.
\]

The following lemma shows that the trace \( \text{tr}_h \) extends to a continuous trace on \( \mathcal{B}(\widetilde{M})^\Gamma \), provided that \( \langle h \rangle \) has polynomial growth.

**Lemma 2.7.** If \( \langle h \rangle \) has polynomial growth, then \( \text{tr}_h \) extends to a continuous trace on \( \mathcal{B}(\widetilde{M})^\Gamma \).

**Proof.** If \( A \in \mathcal{B}(\widetilde{M})^\Gamma, A = (a_{ij})_{i,j \in \mathbb{N}} \) with \( a_{ij} \in C^*_\tau(\Gamma) \), we define

\[
\text{tr}_h(A) = \sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} a_{jj}(g). \tag{2}
\]

We need to verify that the summation on the right side converges. Consider the following inequality:

\[
\sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} |a_{jj}(g)| \leq \left( \sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} j^2(1 + |g|)^{2k} |a_{jj}(g)|^2 \right) \left( \sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} j^{-2}(1 + |g|)^{-2k} \right)
\]

Since \( \langle h \rangle \) has polynomial growth, the term \( \sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} j^{-2}(1 + |g|)^{-2k} \) converges by choosing a sufficiently large \( k \), for example, \( k > (d+1)/2 \). Moreover, it follows from the proof of [10, Lemma 6.4] that \( \sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} j^2(1 + |g|)^{2k} |a_{jj}(g)|^2 \) is finite for all \( k \in \mathbb{N} \). In fact, there exists a fixed constant \( C_k > 0 \) such that

\[
\left( \sum_{j \in \mathbb{N}} \sum_{g \in \langle h \rangle} j^2(1 + |g|)^{2k} |a_{jj}(g)|^2 \right) \leq C_k \cdot \|A\|_k.
\]

Therefore, \( \text{tr}_h \) extends to a continuous linear map on \( \mathcal{B}(\widetilde{M})^\Gamma \).

Now let us verify that \( \text{tr}_h \) is a trace on \( \mathcal{B}(\widetilde{M})^\Gamma \), that is, \( \text{tr}_h(AB) = \text{tr}_h(BA) \) for \( A, B \in \mathcal{B}(\widetilde{M})^\Gamma \). First, assume that \( A, B \in \mathbb{C}\Gamma \otimes \mathcal{R} \). In this case, a straightforward calculation shows that \( \text{tr}_h(AB) = \text{tr}_h(BA) \). Now the general case follows, since \( \text{tr}_h \) is continuous and \( \mathbb{C}\Gamma \otimes \mathcal{R} \) is dense in \( \mathcal{B}(\widetilde{M})^\Gamma \). This finishes the proof. \( \square \)

Since \( \mathcal{B}(\widetilde{M})^\Gamma \) is a dense subalgebra of \( C^*_\tau(\Gamma) \otimes \mathcal{K} \) and is closed under holomorphic functional calculus, we see that the trace \( \text{tr}_h \) induces a homomorphism:

\[
\text{tr}_h : K_0(\mathcal{B}(\widetilde{M})^\Gamma) = K_0(C^*_\tau(\Gamma) \otimes \mathcal{K}) \to \mathbb{C}.
\]
3. Secondary higher invariants and determinant maps

In this section, we will use the trace maps from the previous section to construct certain determinant maps on secondary higher invariants. More precisely, for each map
\[ \text{tr}_h: K_0(\mathcal{B}(\tilde{M})^\Gamma) = K_0(C_\ast^r(\Gamma) \otimes K) \to \mathbb{C}, \]
we will construct a linear map \( \tau_h: K_1(C_\ast^r_{L,0}(\tilde{M})^\Gamma) \to \mathbb{C} \). Let us first introduce some notation.

**Definition 3.1.** We define \( \mathcal{B}_L(\tilde{M})^\Gamma \) to be the dense subalgebra of \( C_\ast^r(\tilde{M})^\Gamma \) consisting of elements \( f \in C_\ast^r(\tilde{M})^\Gamma \) such that \( f \) is piecewise smooth and \( f(t) \in \mathcal{B}(\tilde{M})^\Gamma \) for all \( t \in [0, \infty) \).

\( \mathcal{B}_L(\tilde{M})^\Gamma \) is a dense subalgebra of \( C_\ast^r(\tilde{M})^\Gamma \) and is closed under holomorphic functional calculus. Similarly, we define \( \mathcal{B}_{L,0}(\tilde{M})^\Gamma \) to be the kernel of the evaluation map
\[ \text{ev}: \mathcal{B}_L(\tilde{M})^\Gamma \to \mathcal{B}(\tilde{M})^\Gamma \] defined by \( f \mapsto f(0) \).

The following lemma is an immediate consequence of the above definitions.

**Lemma 3.2.** The inclusion maps \( \mathcal{B}_L(\tilde{M})^\Gamma \hookrightarrow C_\ast^r(\tilde{M})^\Gamma \) and \( \mathcal{B}_{L,0}(\tilde{M})^\Gamma \hookrightarrow C_\ast^r_{L,0}(\tilde{M})^\Gamma \) induce natural isomorphisms:

\[ K_j(\mathcal{B}_L(\tilde{M})^\Gamma) \cong K_j(C_\ast^r(\tilde{M})^\Gamma) \text{ and } K_j(\mathcal{B}_{L,0}(\tilde{M})^\Gamma) \cong K_j(C_\ast^r_{L,0}(\tilde{M})^\Gamma). \]

Now for each non-identity element \( h \in \Gamma \) such that the conjugacy class \( \langle h \rangle \) has polynomial growth, we shall construct a determinant map
\[ \tau_h: K_1(\mathcal{B}_{L,0}(\tilde{M})^\Gamma) \to \mathbb{C}, \]
which can be equivalently viewed as a map \( \tau_h: K_1(C_\ast^r_{L,0}(\tilde{M})^\Gamma) \to \mathbb{C} \). Roughly speaking, the explicit formula for \( \tau_h \) is given by
\[ \tau_h(u) := \frac{1}{2\pi i} \int_0^\infty \text{tr}_h(\dot{u}(t)u^{-1}(t))dt. \]

for each \( [u] \in K_1(\mathcal{B}_{L,0}(\tilde{M})^\Gamma) \), where \( \dot{u} \) is the derivative of \( u \). In order to justify the validity of this integral, we need the following technical results.

**Definition 3.3.** Let \( A \) be a \( C^* \)-algebra and \( \delta \) a sufficiently small positive number. An element \( p \in A \) is called a \( \delta \)-quasi-projection, if \( p^* = p \) and \( \|p - p^2\| < \delta \).

In what follows, the explicit value of \( \delta \) is not important, as long as it is sufficiently small (e.g. \( \leq \frac{1}{100} \)). So we will suppress \( \delta \), and simply call \( p \) a quasi-projection in this case.

**Definition 3.4.** Let \( (\mathcal{B}(\tilde{M})^\Gamma)^+ \) be the unitization of \( \mathcal{B}(\tilde{M})^\Gamma \). A piecewise smooth loop \( \varphi: S^1 = [0, 1]/\{0, 1\} \to (\mathcal{B}(\tilde{M})^\Gamma)^+ \) of invertible elements such that \( \varphi(1) = 1 \) is called local if for all \( \varepsilon > 0 \), there exists a piecewise smooth loop \( \psi: S^1 \to (\mathcal{B}(\tilde{M})^\Gamma)^+ \) of invertible elements such that \( \psi(1) = 1 \), the propagation of \( \psi(\theta) \) is \( \leq \varepsilon \) for all \( \theta \in S^1 \), and \( \varphi \) is homotopic to \( \psi \) through a piecewise smooth family of loops of invertible elements.

Local loops of invertible elements have the following characterization.
Lemma 3.5. If $\varphi : S^1 \to (\mathcal{B}(\widetilde{M})^F)^+$ is a local loop of invertible elements, then for all $\varepsilon > 0$, there exists a quasi-projection $p \in (\mathcal{B}(\widetilde{M})^F)^+$ such that the propagation of $p$ is $\leq \varepsilon$ and $\varphi$ is homotopic to the loop $u(\theta) = e^{2\pi i\theta}p + (1 - p)$ through a piecewise smooth family of loops of invertible elements.

Proof. Let $SC^*(\widetilde{M})^F$ be the suspension of $C^*(\widetilde{M})^F$. Recall that for any finite-dimensional simplicial complex $X$ endowed with the simplicial metric, there exists $\varepsilon_0 > 0$ (which depends only on the dimension of $X$) such that the following holds: if an invertible element $u \in SC^*(\widetilde{X})^F$ satisfies that the propagation of $u(\theta)$ is $\leq \varepsilon_0$ for all $\theta \in S^1$, then $u$ represents a $K$-homology class of $X$ [40, Proposition 7.5]. In this case, $u$ is homotopic to the image of an invertible element $U \in SC^*_L(\widetilde{X})^F$ under the evaluation map. Since $\varphi$ is a local loop, it follows from the Bott periodicity map

$$\beta : K_0(C^*_L(\widetilde{M})^F) \xrightarrow{\cong} K_1(SC^*_L(\widetilde{M})^F), \quad P(t) \mapsto e^{2\pi i\theta}P(t) + (1 - P(t)),$$

that for every $\varepsilon > 0$, there exists a quasi-projection $p \in C^*(\widetilde{M})^F$ with propagation $\leq \varepsilon$ such that $\varphi$ is homotopic to the loop $e^{2\pi i\theta}p + (1 - p)$ through a piecewise smooth family of loops of invertible elements. Now the statement follows from Lemma 3.2. \qed

Now we prove that every element of $K_1(\mathcal{B}_{L,0}(\widetilde{M})^F)$ has a nice representative with certain regularities. This allows us to rigorously define the determinant map

$$\tau_h : K_1(\mathcal{B}_{L,0}(\widetilde{M})^F) \to \mathbb{C}$$

by using these nice representatives.

Proposition 3.6. Every element $[u] \in K_1(\mathcal{B}_{L,0}(\widetilde{M})^F)$ has a representative $w : [0, \infty) \to (\mathcal{B}(\widetilde{M})^F)^+$ such that

$$w(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1, \\ h(t) & \text{if } 1 \leq t \leq 2, \\ e^{2\pi i\frac{F(t-1)+1}{2}} & \text{if } t \geq 2, \end{cases}$$

where $h$ is a piecewise smooth path of invertible elements connecting $u(1)$ and $e^{2\pi i\frac{F(1)+1}{2}}$, and $F$ is a piecewise smooth map $F : [1, \infty) \to D^*(\widetilde{M})^F$ satisfying

1. $F(t)^2 - 1 \in \mathcal{B}(\widetilde{M})^F$ and $F^*(t) = F(t)$,
2. its derivative $F'(t) \in \mathcal{B}(\widetilde{M})^F$,
3. and propagation of $F(t)$ goes to 0, as $t \to \infty$.

Moreover, if $v$ is another such representative, then there exists a piecewise smooth family of invertibles $u_s \in \mathcal{B}_{L,0}(\widetilde{M})^F$ and a piecewise smooth family of maps $F_s : [1, \infty) \to D^*(\widetilde{M})^F$ satisfying conditions (1), (2) and (3) above for each $s \in [0, 1]$, such that

1. $u_0 = w$,
2. $u_s(t) = e^{2\pi i\frac{F_s(t-1)+1}{2}}$ for all $t \geq 2$;
3. $u_1(t) = v(t)$ for all $t \notin (1, 2)$ and $u_1v^{-1} : [1, 2] \to \mathcal{B}(\widetilde{M})^F$ is a local loop of invertible elements.

An element $p \in (\mathcal{B}(\widetilde{M})^F)^+$ is called quasi-projection, if $p$ is a quasi-projection when viewed as an element of $C^*(\widetilde{M})^F)^+$.

Note that $u(\theta)$ is invertible, since $p$ is a quasi-projection.
Remark 3.7. For simplicity, we shall call a representative as in the proposition a regularized representative.

Proof. View the invertible element \( u \in \mathcal{B}_{L,0}(\widetilde{M})^\Gamma \) as an invertible element in \( \mathcal{B}_{L}(\widetilde{M})^\Gamma \). Consider the element\(^4 \hat{u} = u: [1, \infty) \to (\mathcal{B}(\widetilde{M})^\Gamma)^+ \) in \( K_1(\mathcal{B}_{L}(\widetilde{M})^\Gamma) \). Since the \( K \)-theory of \( \mathcal{B}_{L}(\widetilde{M})^\Gamma \) is the \( K \)-homology of \( M \), it follows from the Baum-Douglas geometric description of \( K \)-homology [6, Section 11] that \( \hat{u} \) can be represented by a twisted Dirac operator over a \( \text{spin}^c \) manifold. It follows that there exists a piecewise smooth map \( F: [1, \infty) \to D^*(\widetilde{M})^\Gamma \) satisfying

1. \( F(t)^2 - 1 \in \mathcal{B}(\widetilde{M})^\Gamma \) and \( F^*(t) = F(t) \),
2. its derivative \( F'(t) \in \mathcal{B}(\widetilde{M})^\Gamma \),
3. propagation of \( F(t) \) goes to 0, as \( t \to \infty \);

and \( \hat{u}(t) \) is homotopic to the path \( e^{2\pi i \frac{F(t)+1}{2}} \) with \( t \in [1, \infty) \). In particular, there is a path of invertible elements, denoted by \( h \), connecting \( u(1) \) and \( e^{2\pi i \frac{F(1)+1}{2}} \). Then \( u \) is homotopic to the invertible element \( w \) defined by

\[
w(t) = \begin{cases} 
  u(t) & \text{if } 0 \leq t \leq 1, \\
  h(t) & \text{if } 1 \leq t \leq 2, \\
  e^{2\pi i \frac{F(t-1)+1}{2}} & \text{if } t \geq 2,
\end{cases}
\]

cf. Figure 1 below.

![Figure 1. homotopy between u and w.](image)

Now suppose \( v \) is another representative of \( [u] \) such that

\[
v(t) = \begin{cases} 
  u(t) & \text{if } 0 \leq t \leq 1, \\
  g(t) & \text{if } 1 \leq t \leq 2, \\
  e^{2\pi i \frac{G(t)+1}{2}} & \text{if } t \geq 2,
\end{cases}
\]

where \( g \) is a path of invertible elements connecting \( u(1) \) and \( e^{2\pi i \frac{G(1)+1}{2}} \), and \( G \) is a piecewise smooth map \( G: [1, \infty) \to D^*(\widetilde{M})^\Gamma \) satisfying that \( G(t)^2 - 1 \in \mathcal{B}(\widetilde{M})^\Gamma \) and \( G^*(t) = G(t) \); its derivative \( G'(t) \in \mathcal{B}(\widetilde{M})^\Gamma \); and propagation of \( G(t) \) goes to 0, as \( t \to \infty \).

\(^4\) Note that \( \hat{u} \) starts at \( t = 1 \) instead of \( t = 0 \).
By [16, Theorem 3.8], there exists a piecewise smooth family $F_s: [1, \infty) \to D^*(\tilde{M})^\Gamma$ with $s \in [0, 1]$ such that $F_0 = F$ and $F_1 = G; F_s(t)^2 - 1 \in \mathcal{B}(\tilde{M})^\Gamma$ and $F_s^*(t) = F_s(t)$; its derivative $\frac{\partial}{\partial t} F_s(t) \in \mathcal{B}(\tilde{M})^\Gamma$; and propagation of $F_s(t)$ goes to 0, as $t \to \infty$.

Let $\varpi: [0, \infty) \to (\mathcal{B}(\tilde{M})^\Gamma)^+$ be the path of invertibles defined as

$$
\varpi(t) = \begin{cases} 
  u(t) & \text{if } 0 \leq t \leq 1, \\
  h(t) & \text{if } 1 \leq t \leq 2, \\
  e^{2\pi i F_s(1) + 1} & \text{if } 2 \leq t = s + 2 \leq 3, \\
  e^{2\pi i G(t-2) + 1} & \text{if } t \geq 3,
\end{cases}
$$

Clearly, $w$ is homotopic to $\varpi$. On the other hand, after a re-parametrization, it is not difficult to see that $\varpi$ differs from $v$ by the loop $f: [0, 1] \to (\mathcal{B}(\tilde{M})^\Gamma)^+$ with

$$
f(t) = \begin{cases} 
  (g(t)^{-1}) h(2t) & \text{if } 0 \leq t \leq 1/2 \\
  g(t)^{-1} e^{2\pi i F_s(t) + 1} & \text{if } 1/2 \leq t \leq 1.
\end{cases}
$$

Moreover, $f$ is a local loop in the sense of Definition 3.4 (cf. Figure 2). This finishes the proof.

![Figure 2](image)

**Figure 2.** $\varpi$ and $v$ differs by a local loop. The picture should be viewed as 3-dimensional, where the circular sector on the left is seen to be homotopic to the circular sector on the right through the cylindrical surface between them.

As before, suppose that $h$ is a non-identity element in $\Gamma$ such that its conjugacy class $\langle h \rangle$ has polynomial growth.

**Definition 3.8.** For each $[u] \in K_1(\mathcal{B}_{L,0}(\tilde{M})^\Gamma)$, let $w$ be a regularized representative of $u$ as in Proposition 3.6. We define

$$
\tau_h(u) := \frac{1}{2\pi i} \int_0^\infty \text{tr}_h(\dot{w}(t)w^{-1}(t))dt. \tag{3}
$$

where $\dot{w}$ is the derivative of $w$. 


Proposition 3.9. If the conjugacy class \( \langle h \rangle \) of a non-identity element \( h \in \Gamma \) has polynomial growth, then the map \( \tau_h : K_1(C^*_L(\hat{M})^\Gamma) \to \mathbb{C} \) is well-defined.

Proof. Let \([u] \in K_1(C^*_L(\hat{M})^\Gamma) \cong K_1(\mathfrak{B}_{L,0}(\hat{M})^\Gamma)\). If \( w \) is a regularized representative of \([u]\) as in Proposition 3.6, then

\[
\frac{1}{2\pi i} \int_0^\infty \tr_h(\dot{w}(t)w^{-1}(t))dt = \frac{1}{2\pi i} \int_0^2 \tr_h(\dot{w}(t)w^{-1}(t))dt + \frac{1}{2\pi i} \int_2^\infty \tr_h(\dot{F}(t))dt
\]

The first integral on the right is clearly well-defined. Observe that there exists \( \varepsilon > 0 \) (depending only on \( \hat{M} \)) such that \( \tr_h(\dot{F}(t)) = 0 \) as long as the propagation of \( \dot{F}(t) \) is less than \( \varepsilon \). Since the propagation of \( \dot{F}(t) \) goes to 0 as \( t \) goes to \( \infty \), it follows that the second integral on the right is well-defined.

Now let us show that \( \tau_h([u]) \) is independent of the choice of a regularized representative. Suppose \( \tilde{v} \) is another regularized representative of \([u]\). By Proposition 3.6, there exists a piecewise smooth family of invertibles \( u_s \in \mathfrak{B}_{L,0}(\hat{M})^\Gamma \) with the stated properties as in Proposition 3.6. In particular, we have

\[
\frac{\partial}{\partial s} \tau_h(u_s) = \int_0^\infty \partial_s \tr_h((\partial_s u)u^{-1})dt
\]

where \( f \) stands for \( \frac{1}{2\pi i} \int \). It follows that \( \tau_h(w) = \tau_h(u_0) = \tau_h(u_1) \). On the other hand, \( v \) and \( u_1 \) differ by a local loop \( \varphi \). By Lemma 3.5, a local loop \( \varphi : S^1 \to (\mathfrak{B}(\hat{M})^\Gamma)^+ \) is homotopic to a loop \( e^{2\pi i} p + (1 - p) \) with the propagation of \( p \) being sufficiently small. It follows that

\[
\frac{1}{2\pi i} \int_0^1 \tr_h(\dot{\varphi}(\theta)\varphi^{-1}(\theta))d\theta = \int_0^1 \tr_h(p)d\theta = 0,
\]

since the propagation of \( p \) is sufficiently small. Therefore, we have \( \tau_h(v) = \tau_h(u_1) = \tau_h(w) \). This finishes the proof. \( \square \)

The determinant map \( \tau_h : K_1(C^*_L,0(\hat{M})^\Gamma) \to \mathbb{C} \) is related to the trace map

\[
\tr_h : K_0(C^*_r(\Gamma)) \to \mathbb{C}
\]
as follows.
Lemma 3.10. With the same notation as above, if the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, then the following diagram commutes:

\[
\begin{array}{ccc}
K_0(C^*_r(\Gamma)) & \xrightarrow{\partial} & K_1(C^*_L(\tilde{M})^\Gamma) \\
\downarrow \text{tr}_h & & \downarrow \tau_h \\
\mathbb{C} & \xrightarrow{\partial} & \mathbb{C}
\end{array}
\]

where $\partial: K_0(C^*_r(\Gamma)) \to K_1(C^*_L(\tilde{M})^\Gamma)$ is the connecting map in the six-term $K$-theory long exact sequence for the short exact sequence:

$0 \to C^*_L(\tilde{M})^\Gamma \to C^*_L(\tilde{M})^\Gamma \to C^*(\tilde{M})^\Gamma \to 0.$

Proof. For each $[p] \in K_0(C^*_r(\Gamma))$, recall that $\partial[p]$ is defined as follows: let $\{a(t)\}_{t \in [0, \infty)}$ be a lift of $p$ in $C^*_L(\tilde{M})^\Gamma$, in particular, $a(0) = p$, then

$\partial p := u$ with $u(t) = e^{2\pi i a(t)}$ for $t \in [0, \infty)$. It follows that

$\tau_h(\partial p) = \frac{1}{2\pi i} \int_0^\infty \text{tr}_h(\dot{u}(t)u^{-1}(t))dt = \int_0^\infty \text{tr}_h(\dot{u}(t))dt = -\text{tr}_h(p)$.

$\square$

Remark 3.11. The same method in this section can be also applied to the following (relative) traces defined on $C^*_r(\Gamma)$ or $C^*_\max(\Gamma)$. Throughout this remark, we do not assume any growth conditions on the group $\Gamma$.

(1) Let $\sigma_1: \Gamma \to U(n)$ and $\sigma_2: \Gamma \to U(n)$ be two unitary representations of $\Gamma$ of the same dimension. They induce traces $\text{tr}_{\sigma_i}$ on $C^*_\max(\Gamma)$ by

$\gamma \mapsto \text{tr}(\sigma_i(\gamma))$.

By using regularized representatives as in Proposition 3.6, the relative trace $\text{tr}_{\sigma_1} - \text{tr}_{\sigma_2}$ induces a homomorphism $\tau_{\sigma_1,\sigma_2}: K_1(C^*_L(\tilde{M})^\Gamma_{\max}) \to \mathbb{C}$ by

$\tau_{\sigma_1,\sigma_2}(u) = \frac{1}{2\pi i} \int_0^\infty (\text{tr}_{\sigma_1} - \text{tr}_{\sigma_2})(\dot{u}(t)u^{-1}(t))dt.$

A key observation here is again that there exists $\varepsilon > 0$ such that

$(\text{tr}_{\sigma_1} - \text{tr}_{\sigma_2})(a) = 0$

if the propagation of $a \in C^*_\max(\tilde{M})^\Gamma$ is less than $\varepsilon$. Combined with finite propagation speeds of wave operators, this provides a conceptual approach to some results of Keswani [17], Piazza and Schick [25] and Higson and Roe [15]. We shall present the details in a separate paper [32].

5To be precise, since $C^*_\max(\tilde{M})^\Gamma \cong C^*_\max(\Gamma) \otimes \mathcal{K}$, one needs to pass to an appropriate smooth dense subalgebra of $C^*_\max(\Gamma) \otimes \mathcal{K}$ on which the traces $\text{tr}_{\sigma_1}$ and $\text{tr}_{\sigma_2}$ are defined. Such smooth dense subalgebras always exist.
Let $\nu$ be the $L^2$-trace on the group von Neumann algebra $\mathcal{N}_\Gamma$ of $\Gamma$. It induces a trace, still denoted by $\nu$, on $C^*_\mathrm{max}(\Gamma)$ by the natural map $C^*_\mathrm{max}(\Gamma) \to C^*_\rho(\Gamma) \to \mathcal{N}_\Gamma$. Now suppose $\lambda : C^*_\rho(\Gamma) \to \mathbb{C}$ is the trivial representation. Then the formula
\[
\rho_{(2)}(u) := \frac{1}{2\pi i} \int_0^\infty (\nu - \lambda)(\bar{u}(t)u^{-1}(t)) \, dt
\]
defines a homomorphism $\rho_{(2)} : K_1(C^*_\rho(\widetilde{M})^{\Gamma}_{\mathrm{max}}) \to \mathbb{C}$, which is precisely the $L^2$-$\rho$-invariant of Cheeger and Gromov \cite{CheegerGromov}. This provides a more conceptual approach to some results of Keswani \cite{Keswani} and Benamour and Roy \cite{BenamourRoy}. Again, the details will be given in \cite{notesK}.}

\section{Higher rho invariants and delocalized eta invariants}

In this section, we shall establish a precise connection between higher rho invariants and delocalized eta invariants. More precisely, let $M$ be an odd dimensional\footnote{The even dimensional case is completely parallel. For simplicity, we will only discuss the odd dimensional case here.} closed spin manifold equipped with a positive scalar curvature metric. Denote its fundamental group $\pi_1 M$ by $\Gamma$. Suppose $\widetilde{M}$ is the universal cover of $M$. Let $\tilde{g}$ be the Riemannian metric on $\widetilde{M}$ lifted from $M$. The Dirac operator $\widetilde{D}$ on $\widetilde{M}$ with respect to $\tilde{g}$ naturally defines a higher rho invariant $\rho(\widetilde{D}, \tilde{g}) \in K_1(C^*_\rho(\widetilde{M})^{\Gamma})$. We shall show that if the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, then $\tau_h(\rho(\widetilde{D}, \tilde{g}))$ is equal to the delocalized eta invariant of Lott.

Let us briefly review the construction of $\rho(\widetilde{D}, \tilde{g}) \in K_1(C^*_\rho(\widetilde{M})^{\Gamma})$. Recall that
\[
\widetilde{D}^2 = \nabla^* \nabla + \frac{\kappa}{4},
\]
where $\nabla : C^{\infty}(\widetilde{M}, S) \to C^{\infty}(\widetilde{M}, T^*\widetilde{M} \otimes S)$ is the connection on the spinor bundle $S$ over $\widetilde{M}$, $\nabla^*$ is the adjoint of $\nabla$, and $\kappa$ is the scalar curvature of the metric $\tilde{g}$. By assumption, $\kappa > \varepsilon$ for some $\varepsilon > 0$, it follows immediately that $\widetilde{D}$ is invertible in this case. We define
\[
F = \widetilde{D}|\widetilde{D}|^{-1}.
\]
Now let $\{U_{n,j}\}$ be a $\Gamma$-invariant locally finite open cover\footnote{If $n = 0$, we choose the open cover to be $\{\widetilde{M}\}$ consisting of a single open set $\widetilde{M}$ itself.} of $\widetilde{M}$ with diameter($U_{n,j}$) < $1/n$ and $\{\phi_{n,j}\}$ a $\Gamma$-invariant partition of unity subordinate to $\{U_{n,j}\}$. We define
\[
F(t) = \sum_j (1 - (t - n))\phi_{n,j}^{1/2} F \phi_{n,j}^{1/2} + (t - n)\phi_{n+1,j}^{1/2} F \phi_{n+1,j}^{1/2}
\]
for $t \in [n, n + 1]$. Form the path of unitaries
\[
u(t) = e^{2\pi i \frac{F(t)+1}{2}}, \quad 0 \leq t < \infty.
\]
Note that $F^{1/2}$ is a genuine projection, hence $\nu(0) = 1$. So the path $\nu(t), 0 \leq t < \infty$, defines a class in $K_1(C^*_\rho(\widetilde{M})^{\Gamma})$.

**Definition 4.1.** The higher rho invariant $\rho(\widetilde{D}, \tilde{g})$ is defined to be the $K$-theory class
\[
[u] \in K_1(C^*_\rho(\widetilde{M})^{\Gamma}).
\]
Now let us also recall the definition of delocalized eta invariants due to Lott.

**Definition 4.2** ([23, Definition 7]). With the above notation, the delocalized eta invariant of $\tilde{D}$ at $\langle h \rangle$ is defined to be

$$\eta_{\langle h \rangle}(\tilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_h(\tilde{D} e^{-t^2 \tilde{D}^2}) dt. \quad (5)$$

Here the convergence of the integral does not hold in general, and relies on the growth rate of the conjugacy class $\langle h \rangle$, cf. [26, Section 3] for a more thorough discussion. A sufficient condition for the convergence of the integral is that the conjugacy class $\langle h \rangle$ has polynomial growth.

We have the following main result of this section.

**Theorem 4.3.** Let $M$ be a closed odd-dimensional spin manifold equipped with a positive scalar curvature metric $g$. Suppose $\tilde{M}$ is the universal cover of $M$, $\tilde{g}$ is the Riemannian metric on $\tilde{M}$ lifted from $g$, and $\tilde{D}$ is the associated Dirac operator. Suppose the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \pi_1(M)$ has polynomial growth. Then we have

$$\tau_h(\rho(\tilde{D}, \tilde{g})) = \frac{1}{2} \eta_{\langle h \rangle}(\tilde{D}).$$

Before we prove the theorem, let us point out that the definition of higher rho invariant $\rho(\tilde{D}, \tilde{g})$ does not require any growth condition on $\langle h \rangle$ or $\pi_1 M$. In fact, if the strong Novikov conjecture holds for $\Gamma = \pi_1(M)$, then we can generalize Lott’s delocalized eta invariant without any growth conditions of the conjugacy class of $h$. This can be achieved by using the Novikov rho invariant introduced in [36, Section 7]. Let us briefly recall the construction below, and refer the reader to [36, Section 7] for more details. Consider the following commutative diagram:

$$
\begin{array}{cccc}
K_1^\Gamma(ET, \tilde{M}) & \longrightarrow & K_0^\Gamma(\tilde{M}) & \longrightarrow & K_0^\Gamma(ET) & \longrightarrow & K_0^\Gamma(ET, \tilde{M}) \\
\Lambda & & \cong & & \mu_* & & \Lambda \\
K_0(C^*_L(\tilde{M})^\Gamma) & \longrightarrow & K_0(C^*_L(\tilde{M})^\Gamma) & \longrightarrow & K_0(C^*_r(\Gamma)) & \longrightarrow & K_1(C^*_L(\tilde{M})^\Gamma) \\
\end{array}
$$

(6)

where $ET$ is the universal space for proper $\Gamma$-actions and $K_1^\Gamma(ET, \tilde{M})$ is the $\Gamma$-equivariant relative $K$-homology group for the pair of $\Gamma$-spaces $(ET, \tilde{M})$. Let us assume that $\mu_* : K_1^\Gamma(ET) \to K_1(C^*_r(\Gamma))$ is a split injection\(^8\). In this case, a simple diagram chasing shows that there is a natural surjective morphism

$$\beta : K_1(C^*_L(\tilde{M})^\Gamma) \rightarrow K_0^\Gamma(ET, \tilde{M})$$

such that $\beta \circ \Lambda = \text{Id}$. By composing $\beta$ with the natural morphism

$$K_0^\Gamma(ET, \tilde{M}) \rightarrow K_0^\Gamma(ET, ET)$$

induced by the inclusion $(ET, \tilde{M}) \hookrightarrow (ET, ET)$, where $ET$ is the universal space for free and proper $\Gamma$-actions, and the Chern character map

$$K_0^\Gamma(ET, ET) \rightarrow \bigoplus_{k \in \mathbb{Z}} H_{2k}^\Gamma(ET, ET) \otimes \mathbb{C},$$

\(^8\)So far, in all known cases where the strong Novikov conjecture holds, the split injectivity of the Baum-Connes assembly map is known to be true as well.
we get a morphism
\[ \Theta: K_1(C^{*}_{L,0}(\tilde{M})^\Gamma) \to \bigoplus_{k \in \mathbb{Z}} H_{2k}^\Gamma(E\Gamma, E\Gamma) \otimes \mathbb{C}. \]

Recall that (cf. \cite{3})
\[ \bigoplus_{k \in \mathbb{Z}} H_{2k}(E\Gamma, E\Gamma) \otimes \mathbb{C} \cong \bigoplus_{\gamma \text{ finite order and } \gamma \neq e} H_{2k}(\mathbb{Z}_\gamma; \mathbb{C}), \]
where \( e \) is the identity element of \( \Gamma \), \( \langle \gamma \rangle \) runs through all conjugacy classes of finite order elements \( \gamma \) with \( \gamma \neq e \), and \( \mathbb{Z}_\gamma \) is the centralizer group of \( \gamma \) in \( \Gamma \).

**Definition 4.4.** If the Baum-Connes assembly map for \( \pi_1(M) \) is a split injection\(^9\), then we define the generalized delocalized eta invariant \( \tilde{D} \) at \( \langle h \rangle \) to be the complex number in the \( H_0(\mathbb{Z}_h; \mathbb{C}) \)-component of \( \rho(\tilde{D}, \tilde{g}) \) under the map \( \Theta \).

Note that if the conjugacy class \( \langle h \rangle \) has polynomial growth and in addition the Baum-Connes assembly map is an isomorphism, then the above generalized delocalized eta invariant coincides with the delocalized eta invariant of Lott.

Now let us proceed to prove Theorem 4.3. In order to make the exposition more transparent, we shall work with the following alternative smooth dense subalgebra of \( C^{*}(\tilde{M})^\Gamma = C^{*}(\Gamma) \otimes \mathcal{K} \).

Let \( \mathcal{A}(\tilde{M})^\Gamma \) be the convolution algebra of all elements \( A \in C^\infty(\tilde{M} \times \tilde{M}) \) satisfying
1. \( A \) is \( \Gamma \)-invariant, that is, \( A(gx, gy) = A(x, y) \) for all \( g \in \Gamma \),
2. \( A \) has finite propagation, that is, there exists \( R > 0 \) such that \( A(x, y) = 0 \) for all \( x, y \in \tilde{M} \) with \( d(x, y) \geq R \).

The algebra \( \mathcal{A}(\tilde{M})^\Gamma \) acts on \( L^2(\tilde{M}) \) by
\[ (Af)(x) = \int_{\tilde{M}} A(x, y) f(y) dy, \]
for \( A \in \mathcal{A}(\tilde{M})^\Gamma \) and \( f \in L^2(\tilde{M}) \).

Fix a point \( x_0 \in \tilde{M} \) and let \( \sigma: \tilde{M} \to \mathbb{R} \) be the distance function \( \sigma(x) = d(x, x_0) \) on \( \tilde{M} \).

In fact, we shall modify \( \sigma \) near \( x_0 \) to make it smooth on \( \tilde{M} \). For notational simplicity, we shall continue to denote this modified distance function by \( \sigma \). Multiplication by the function \( \sigma \) acts as an unbounded operator on \( L^2(\tilde{M}) \). Taking commutator with \( \sigma \) defines a derivation on \( \mathcal{A}(\tilde{M})^\Gamma \):
\[ \tilde{\partial} = [\sigma, \cdot]: \mathcal{A}(\tilde{M})^\Gamma \to \mathcal{A}(\til{M})^\Gamma. \]

Now let \( \Delta \) be the Laplace operator on \( \tilde{M} \) and \( r \) an integer \( > \dim M \). We define
\[ \mathcal{A}(\tilde{M})^\Gamma = \{ A \in C^{*}(\tilde{M})^\Gamma \mid \tilde{\partial}^k(A) \circ (\Delta + 1)^r \text{ is bounded for } \forall k \in \mathbb{N} \}. \]

The same proof from \cite{10, Lemma 6.4} shows that \( \mathcal{A}(\tilde{M})^\Gamma \) contains \( \mathcal{A}(\til{M})^\Gamma \) and is closed under holomorphic functional calculus.

\(^9\)We remark that this definition of generalized delocalized eta invariants depends on the choice of the split injection in general.
We associate to each element \( h \in \Gamma \) the following trace on \( \mathcal{S}(\widetilde{M})^\Gamma \):
\[
\text{tr}_h(A) = \sum_{g \in \langle h \rangle} \int_{\mathcal{F}} A(x, gx) dx
\]
where \( \mathcal{F} \) is a fundamental domain of \( \widetilde{M} \) under the action of \( \Gamma \). Here we have identified \( L^2(\widetilde{M}) \) with \( L^2(\mathcal{F}) \otimes \ell^2(\Gamma) \) through the mapping \( f \to \hat{f} \) by the formula \( \hat{f}(x, \alpha) = f(\alpha x) \) for \( x \in \mathcal{F} \) and \( \alpha \in \Gamma \). In particular, each element \( A \in \mathcal{S}(\widetilde{M})^\Gamma \) becomes a finite sum \( \sum_{g \in \Gamma} A_g R_g \), where \( A_g(x, y) = A(x, gy) \) for \( x, y \in \mathcal{F} \) and \( R \) denotes the right regular representation of \( \Gamma \).

The following lemma and its proof are essentially the same as Lemma 2.7. We shall be brief.

**Lemma 4.5.** If \( \langle h \rangle \) has polynomial growth, then \( \text{tr}_h \) extends to a continuous trace on \( \mathcal{S}(\widetilde{M})^\Gamma \).

**Proof.** Let \( A \) be an element in \( \mathcal{S}(\widetilde{M})^\Gamma \). By assumption, \( \partial^k \circ (\Delta + 1)^r \) is bounded for all \( k \in \mathbb{N} \). It follows from the Sobolev embedding theorem that the Schwartz kernel \( \partial^k(A)(x, y) \) of \( \hat{A} \) is a uniformly bounded continuous function on \( \widetilde{M} \times \widetilde{M} \) for each \( k \in \mathbb{N} \). We define
\[
\text{tr}_h(A) = \sum_{g \in \langle h \rangle} \int_{\mathcal{F}} A(x, gx) dx. \tag{8}
\]
We need to verify that the summation on the right side converges. Observe that
\[
\partial^{2k}(A)(x, gx) = (\sigma(x) - \sigma(gx))^{2k} A(x, gx),
\]
and furthermore \( |\sigma(x) - \sigma(gx)| \geq |g| - \text{diam}(\mathcal{F}) \) for all \( x \in \mathcal{F} \), where \( \text{diam}(\mathcal{F}) \) is the diameter of \( \mathcal{F} \). It follows that there exists a fixed constant \( C_k > 0 \) such that
\[
(1 + |g|)^{4k} \int_{\mathcal{F}} |A(x, gx)|^2 dx \leq C_k \int_{\mathcal{F}} |\partial^{2k}(A)(x, gx)|^2 dx. \tag{9}
\]
Now for all \( k \in \mathbb{N} \), we have
\[
|\text{tr}_h(A)| \leq \sum_{g \in \langle h \rangle} \int_{\mathcal{F}} |A(x, gx)| dx
\]
\[
\leq \left( \sum_{g \in \langle h \rangle} (1 + |g|)^{2k} \int_{\mathcal{F}} |A(x, gx)|^2 dx \right)^{1/2} \left( \sum_{g \in \langle h \rangle} (1 + |g|)^{-2k} \right)^{1/2}
\]
\[
\leq C_k^{1/2} \left( \sum_{g \in \langle h \rangle} (1 + |g|)^{-2k} \int_{\mathcal{F}} |\partial^{2k}(A)(x, gx)|^2 dx \right)^{1/2} \left( \sum_{g \in \langle h \rangle} (1 + |g|)^{-2k} \right)^{1/2},
\]
which is finite for \( k \) sufficiently large, since \( \langle h \rangle \) has polynomial growth and \( \partial^{2k} A(x, gx) \) is uniformly bounded for all \( g \). Now a similar argument\(^\text{10}\) as in Lemma 2.7 shows that \( \text{tr}_h \) is a continuous map and \( \text{tr}_h(AB) = \text{tr}_h(BA) \) for \( A, B \in \mathcal{S}(\widetilde{M})^\Gamma \). \( \square \)

\(^\text{10}\)Note that there exists a fixed constant \( \Lambda_j \) such that the supremum norm of the continuous function \( \partial^j(A)(x, gy) \) is less than or equal to \( \Lambda_j \cdot \| \partial^j(A) \circ (\Delta + 1)^r \| \) for all \( A \in \mathcal{S}(\widetilde{M})^\Gamma \).
Let $S$ be the associated spinor bundle on $\tilde{M}$ and $S^*$ its dual bundle. Consider the bundle $\text{End}(S) = p_1^*(S) \otimes p_2^*(S^*)$ on $\tilde{M} \times \tilde{M}$, where $p_i: \tilde{M} \times \tilde{M} \rightarrow \tilde{M}$ is the projection onto the first and second component respectively. There is a natural diagonal action of $\Gamma$ on $\text{End}(S)$. Define $C^\infty(\tilde{M} \times \tilde{M}, \text{End}(S))$ to be the set of all smooth sections of the bundle $\text{End}(S)$ over $\tilde{M} \times \tilde{M}$. Let $\mathcal{S}(\tilde{M}, S)^\Gamma$ be the convolution algebra of all $\Gamma$-invariant finite propagation elements in $C^\infty(\tilde{M} \times \tilde{M}, \text{End}(S))$. The algebra $\mathcal{S}(\tilde{M}, S)^\Gamma$ acts on $L^2(\tilde{M}, S)$ by
\[(Af)(x) = \int_{\tilde{M}} A(x, y) f(y) dy,\]
for $A \in \mathcal{S}(\tilde{M}, S)^\Gamma$ and $f \in L^2(\tilde{M}, S)$, where $L^2(\tilde{M}, S)$ is the space of $L^2$-sections of $S$ over $\tilde{M}$.

Recall that the $\Gamma$-equivariant Roe algebra $C^*(\tilde{M})^\Gamma$ is (up to isomorphism) independent of the choice of an admissible $(\tilde{M}, \Gamma)$-module. We shall still denote the $\Gamma$-equivariant Roe algebra obtained from the $(\tilde{M}, \Gamma)$-module $L^2(\tilde{M}, S)$ by $C^*(\tilde{M})^\Gamma$. Similar to line (7), we define
\[\mathcal{A}(\tilde{M}, S)^\Gamma = \{ A \in C^*(\tilde{M})^\Gamma \mid \tilde{D}^k(A) \circ (\tilde{D}^{2n} + 1) \text{ is bounded for } \forall k \in \mathbb{N} \}.
\]
where $\tilde{D}$ is the Dirac operator on $\tilde{M}$ and $n$ is a fixed integer $> \dim M$. As in Definition 3.1, we can similarly define the algebras $\mathcal{A}_L(\tilde{M}, S)^\Gamma$ and $\mathcal{A}_{L,0}(\tilde{M}, S)^\Gamma$.

Recall that a smooth function $\varphi$ on $\mathbb{R}$ is called a Schwartz function if the function $x^k \varphi^{(j)}(x)$ is bounded on $\mathbb{R}$ for each $k, j \in \mathbb{N}$, where $\varphi^{(j)}$ is the $j$-th derivative of $\varphi$.

**Proposition 4.6.** Suppose $\varphi$ is a Schwartz function, then $\varphi(\tilde{D})$ is an element in $\mathcal{A}(\tilde{M}, S)^\Gamma$.

**Proof.** Recall that
\[\varphi(\tilde{D}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(s) e^{is\tilde{D}} ds,
\]
where $\hat{\varphi}$ is the Fourier transform of $\varphi$, which is also a Schwartz function, since $\varphi$ is. Let us first verify that $[\varphi(\tilde{D}), \sigma]$ is a bounded operator. Consider
\[f(s) = e^{is\tilde{D}} A e^{-is\tilde{D}} - A.
\]
Then
\[f'(s) = e^{is\tilde{D}} i\tilde{D} A e^{-is\tilde{D}} - e^{is\tilde{D}} A(i\tilde{D}) e^{-is\tilde{D}} = ie^{is\tilde{D}} [\tilde{D}, A] e^{-is\tilde{D}}
\]
and $f(0) = 0$, which implies that
\[f(s) = e^{is\tilde{D}} A e^{-is\tilde{D}} - A = i \int_0^s e^{it\tilde{D}} [\tilde{D}, A] e^{-it\tilde{D}} dt,
\]
or equivalently,
\[[e^{is\tilde{D}}, A] = i \int_0^s e^{it\tilde{D}} [\tilde{D}, A] e^{i(s-t)\tilde{D}} dt.
\]
It follows that
\[[\varphi(\tilde{D}), \sigma] = \frac{i}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(s) \int_0^s e^{it\tilde{D}} [\tilde{D}, \sigma] e^{i(s-t)\tilde{D}} dt ds.
\]
Note that \( \|e^{it\tilde{D}}[\tilde{D},\sigma]e^{i(s-t)\tilde{D}}\| \leq \|[\tilde{D},\sigma]\| \) for all \( s,t \in \mathbb{R} \), thus
\[
\left\| \int_0^s e^{it\tilde{D}}[\tilde{D},\sigma]e^{i(s-t)\tilde{D}}\,dt \right\| \leq s \cdot \|[\tilde{D},\sigma]\|.
\]
This implies that \( [\varphi(\tilde{D}),\sigma] \) has finite operator norm, since the Fourier transform \( \hat{\varphi} \) is a Schwartz function. Now a straightforward inductive argument shows that \( \tilde{D}^k(\varphi(\tilde{D})) \) is a bounded operator for each \( k \in \mathbb{N} \).

Now let us show that \( \tilde{D}^k(\varphi(\tilde{D})) \circ \tilde{D}^{2n} \) is bounded. Note that
\[
[\sigma,\varphi(\tilde{D})]\tilde{D}^{2n} = [\sigma,\varphi(\tilde{D})\tilde{D}^{2n}] - \varphi(\tilde{D})[\sigma,\tilde{D}^{2n}].
\]
By the same argument as above, the operator \( [\sigma,\varphi(\tilde{D})\tilde{D}^{2n}] \) is bounded, since \( \varphi(\tilde{D})\tilde{D}^{2n} = \psi(\tilde{D}) \), where \( \psi(x) = \varphi(x)x^{2n} \) is a Schwartz function. Also, observe that
\[
\varphi(\tilde{D})[\sigma,\tilde{D}^{2n}] = \sum_{k=0}^{n-1} \varphi(\tilde{D})\tilde{D}^k[\sigma,\tilde{D}]\tilde{D}^{n-k-1}
\]
\[
= \sum_{k=0}^{n-1} \left( \varphi(\tilde{D})\tilde{D}^k(\tilde{D}^{2n} + 1) \right) (\tilde{D}^{2n} + 1)^{-1}[\sigma,\tilde{D}]\tilde{D}^{2n-k-1}.
\]
Note that \( (\tilde{D}^{2n} + 1)^{-1}[\sigma,\tilde{D}]\tilde{D}^{2n-k-1} \) is bounded for all \( 0 \leq k \leq 2n - 1 \). It follows that \( \varphi(\tilde{D})[\sigma,\tilde{D}^{2n}] \) is bounded. This finishes the proof.

Now let us prove Theorem 4.3.

**Proof of Theorem 4.3.** Consider the normalizing function
\[
\varphi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.
\]
Define \( F_t = \varphi(t^{-1}\tilde{D}) \) for \( t \in (0,\infty) \). Since the scalar curvature of \( \tilde{g} \) is uniformly bounded below by a positive number, it follows that \( \tilde{D} \) is invertible. In particular, there is a spectral gap near 0 in the spectrum of \( \tilde{D} \). This implies that \( F_t \) converges to \( \text{sign}(\tilde{D}) = \tilde{D}|\tilde{D}|^{-1} \) in operator norm, as \( t \to 0 \). Define \( F_0 = \text{sign}(\tilde{D}) \). The path
\[
u_t = e^{2\pi i \frac{\bar{\epsilon}(t)}{2}} \text{ with } t \in [0,\infty)
\]
defines an element in \( K_1(C^*_L(\tilde{M})) \). By construction, the class \([\nu]\) is the higher rho invariant \( \rho(\tilde{D},\tilde{g}) \).

Moreover, it is not difficult to see that \( 1 - \exp(2\pi i \frac{\bar{\epsilon}(t)}{2}) \) is a Schwartz function. By Proposition 4.6, it follows that \( \nu \in (\mathcal{A}_{L,0}(\tilde{M},\mathcal{S}))^+ \). Now a direct calculation shows that
\[
\tau_h(\rho(\tilde{D},\tilde{g})) = \frac{1}{2\pi i} \int_0^\infty \text{tr}_h(\tilde{u}_t\nu_t^{-1})\,dt = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr}_h(\tilde{D}e^{-t^2\tilde{D}^2})\,dt = \frac{1}{2}\eta_{(h)}(\tilde{D}).
\]
\( \square \)

**Remark 4.7.** One can use the same techniques developed in this paper to show that the analogues of Theorem 4.3 hold for hyperbolic groups, if one uses Puschnigg’s smooth dense subalgebras [28]. In fact, from the viewpoint of cyclic cohomology, the various traces considered in this paper are degree zero cyclic cocycles in the cyclic cohomology...
of the corresponding group algebra. For example, we can restate Theorem 4.3 by saying that Lott’s delocalized eta invariants equal the pairings between Higson-Roe’s higher rho invariants and degree zero cyclic cocycles. Theorem 4.3 and various other results of this paper have natural analogues for higher degree cyclic cocycles. We shall apply the techniques from this paper to investigate the pairings between higher rho invariants and cyclic cocycles of higher degrees in a sequel paper, where in particular we will show that analogues of the main results of this paper hold for higher degree cyclic cocycles, if $\pi_1 M$ has polynomial growth or $\pi_1 M$ is hyperbolic.

5. Baum-Connes conjecture and algebraicity of delocalized eta invariants

In this section, we prove an algebraicity result concerning the values of delocalized eta invariants. We also give a $K$-theoretic proof of a version of delocalized Atiyah-Patodi-Singer index theorem.

**Definition 5.1.** Given a discrete group $\Gamma$, let $Q_{\Gamma}$ be the field extension of $Q$ by the following set of roots of unity:

$$\{e^{2\pi i/n} \mid \text{there exists } \alpha \in \Gamma \text{ such that the order of } \alpha \text{ is } n \}.$$  

We have the following algebraicity result concerning the values of delocalized eta invariants.

**Theorem 5.2.** Assume the same notation as in Theorem 4.3. If the (rational) Baum-Connes conjecture holds for $\Gamma$, that is, the assembly map $\mu: K^*_1(E\Gamma) \to K^*_1(\Gamma)$ is a (rational) isomorphism, and the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, then $\eta_{\langle h \rangle}(D)$ is an element in $Q_{\Gamma}$. If in addition $h$ has infinite order, then $\eta_{\langle h \rangle}(D) = 0$.

**Proof.** Consider the following long exact sequence:

$$\begin{align*}
K_0(C^*_L(ET)^F) \otimes Q &\longrightarrow K_0(C^*_L(ET)^F) \otimes Q \\
&\longrightarrow K_0(C^*_r(\Gamma)) \otimes Q \\
K_1(C^*_r(\Gamma)) \otimes Q &\longleftarrow K_1(C^*_L(ET)^F) \otimes Q \\
&\longleftarrow K_1(C^*_L,0(ET)^F) \otimes Q
\end{align*}$$

Recall that the morphism $K_1(C^*_L(ET)^F) \to K_1(C^*_L(E\Gamma)^F)$ induced by the natural inclusion from $ET$ to $E\Gamma$ is rationally injective. It follows that, if the rational Baum-Connes conjecture holds for $\Gamma$, that is, the assembly map $\mu: K^*_1(E\Gamma^F) \to K^*_1(\Gamma)$ is rationally isomorphic, then the map $\mu$ is injective and the map $\partial$ is surjective. In particular, rationally every element $u \in K_1(C^*_L,0(ET)^F)$ is the image of some $p \in K_0(C^*_r(\Gamma))$ under the map $\partial$. By Lemma 3.10, we have

$$\tau_h(u) = -\tau_h(p).$$

Moreover, the map $\tau_h: K_1(C^*_L,0(M)^F) \to C$ factors through $K_1(C^*_L,0(ET)^F)$. Therefore, the image of the map $\tau_h: K_1(C^*_L,0(M)^F) \to C$ is (up to rationals) equal to the image of the map $\tau_h: K_0(C^*_r(\Gamma)) \to C$.

By Theorem 4.3, it suffices to show that image of the map $\tau_h: K_0(C^*_r(\Gamma)) \to C$ is contained in $Q_{\Gamma}$. By using the Baum-Douglas model of $K$-homology, $K_0(C^*_L(E\Gamma)^F)$ is
generated by \((M, E, \varphi)\), where \(M\) is a complete spin\(^c\) manifold equipped with a proper and cocompact \(\Gamma\)-action, \(E\) is a \(\Gamma\)-equivariant bundle over \(M\), and \(\varphi : M \to \overline{E}\Gamma\) is a \(\Gamma\)-equivariant map. In this case, the Baum-Connes assembly map takes \((M, \overline{E}, \varphi)\) to its higher index \(\text{Ind}_\Gamma(D_E)\), where \(D_E\) is the associated Dirac operator on \(M\) twisted by \(E\). Now by a modified version of the \(L^2\)-Lefschetz fixed point theorem of B.-L. Wang and H. Wang [33] (see Appendix below for details), we see that \(\text{tr}_h(\text{Ind}_\Gamma(D_E))\) is an algebraic number in \(\mathbb{Q}_\Gamma\). Moreover, if \(h\) has infinite order, then there are no points fixed by \(h\), thus \(\text{tr}_h(\text{Ind}_\Gamma(D_E)) = 0\) in this case. This finishes the proof.

\[\square\]

In light of the above theorem, we propose the following question.

**Question 1.** What values can delocalized eta invariants take in general? Are they always algebraic numbers?

In particular, if a delocalized eta invariant is transcendental, then it would lead to a counterexample to the Baum-Connes conjecture. Note that the above question is a reminiscent of Atiyah’s question concerning rationality of \(\ell^2\)-Betti numbers [1]. Atiyah’s question was answered in negative by Austin, who showed that \(\ell^2\)-Betti numbers can be transcendental [2].

Now let us turn to a delocalized Atiyah-Patodi-Singer index theorem. Let \(W\) be a compact \(n\)-dimensional spin manifold with boundary \(\partial W\). Suppose \(W\) is equipped with a Riemannian metric \(g_W\) which has product structure near \(\partial W\) and in addition has positive scalar curvature on \(\partial W\). Let \(\widetilde{W}\) be the universal covering of \(W\) and \(g_{\widetilde{W}}\) the Riemannian metric on \(\widetilde{W}\) lifted from \(g_W\). Denote \(\pi_1(W)\) by \(\Gamma\). With respect to the metric \(g_{\widetilde{W}}\), the associated Dirac operator \(\widetilde{D}\) on \(\widetilde{W}\) naturally defines a higher index, denoted by \(\text{Ind}_\Gamma(\widetilde{D}, g_{\widetilde{W}})\), in \(K_n(C^*(\widetilde{W})^\Gamma) = K_n(C^*_\gamma(\Gamma))\), cf. [37, Section 3]. Let \(\tilde{g}_\partial\) be the restriction of \(g_{\widetilde{W}}\) on \(\partial \widetilde{W}\). As we have seen above, with respect to the metric \(\tilde{g}_\partial\), the associated Dirac operator \(\tilde{D}_\partial\) on \(\partial \widetilde{W}\) naturally defines a higher rho invariant \(\rho(\tilde{D}_\partial, \tilde{g}_\partial)\) in \(K_{n-1}(C^*_\partial, \partial \widetilde{W})^\Gamma\). If no confusion is likely to arise, the image of \(\rho(\tilde{D}_\partial, \tilde{g}_\partial)\) in \(K_{n-1}(C^*_\partial, \partial \widetilde{W})^\Gamma\) under the natural morphism \(K_{n-1}(C^*_\partial, \partial \widetilde{W})^\Gamma \to K_{n-1}(C^*_\partial, \partial \widetilde{W})^\Gamma)\) will still be denoted by \(\rho(\tilde{D}_\partial, \tilde{g}_\partial)\).

We denote by \(\partial : K_n(C^*(\widetilde{W})^\Gamma) \to K_{n-1}(C^*_\partial, \partial \widetilde{W})^\Gamma\) the connecting map in the K-theory long exact sequence induced by the short exact sequence of \(C^*\)-algebras:

\[0 \to C^*_{\partial, 0}(\widetilde{W})^\Gamma \to C^*_{\partial}(\widetilde{W})^\Gamma \to C^*(\widetilde{W})^\Gamma \to 0.\]

Then we have

\[
\partial(\text{Ind}_\Gamma(\tilde{D}, g_{\widetilde{W}})) = \rho(\tilde{D}_\partial, \tilde{g}_\partial) \text{ in } K_{n-1}(C^*_\partial(\widetilde{W})^\Gamma),
\]

cf. [27, Theorem 1.14][37, Theorem A].

We have the following version of delocalized Atiyah-Patodi-Singer index theorem.

**Theorem 5.3.** Let \(W\) be a compact even-dimensional spin manifold with boundary \(\partial W\). Suppose \(W\) is equipped with a Riemannian metric \(g_W\) which has product structure near \(\partial W\) and in addition has positive scalar curvature on \(\partial W\). If the conjugacy class \(\langle h \rangle\) of a non-identity element \(h \in \Gamma = \pi_1(W)\) has polynomial growth, then

\[
\text{tr}_h(\text{Ind}_\Gamma(\tilde{D}, g_{\widetilde{W}})) = \frac{-\eta(h)(\tilde{D}_\partial)}{2}.
\]
Proof. Since
\[ \partial(\text{Ind}_{\Gamma}(\tilde{D}, g_\gamma)) = \rho(\tilde{D}_\theta, \tilde{g}_\theta) \text{ in } K_1(C^*_r(\tilde{W})\Gamma), \]
the theorem follows immediately from Theorem 4.3 and Lemma 3.10. \Box

Remark 5.4. When \( \Gamma \) is virtually nilpotent, a similar result has been proved at the level of noncommutative de Rham homology by Leichtnam and Piazza [20, Theorem 14.1].

Appendix A. \( L^2 \)-Lefschetz fixed point theorem

In this appendix, we shall briefly review a modified version of the \( L^2 \)-Lefschetz fixed point theorem of B.-L. Wang and H. Wang for proper actions of discrete groups on complete manifolds [33, Theorem 5.10].

Let \( X \) be a complete \( \text{spin}^c \) manifold of dimension \( n \), and \( E \) a Hermitian vector bundle over \( X \). Suppose a discrete group \( \Gamma \) acts properly and cocompactly on \( X \) through isometries. Moreover, assume this action lifts to actions on the associated \( \text{spin}^c \) bundle \( S \) and the bundle \( E \) over \( X \) through isometric bundle morphisms.

For each \( h \in \Gamma \), let \( X^h = \{ x \in X \mid h \cdot x = x \} \) the fixed point set of \( h \). Denote the normal bundle of \( X^h \) by \( N \). The bundle \( N \) admits a decomposition
\[ N = N(\pi) \oplus \bigoplus_{0 < \theta < \pi} N(\theta) \]
where the differential \( d_h \) of the map \( h \) acts on \( N(\pi) \) by multiplication by \(-1\), and for each \( 0 < \theta < \pi \), \( N(\theta) \) is a complex bundle in which \( d_h \) acts by multiplication by \( e^{i\theta} \).

Since \( h \) is orientation preserving, the bundle \( N(\pi) \) is an oriented even-dimensional real bundle.

The \( L^2 \)-Lefschetz fixed point theorem will be expressed in terms of characteristic classes as follows. We refer to [19, Chapter III, Section 14] for more details. If \( V \) is a complex vector bundle with formal splitting \( V = \ell_1 \oplus \cdots \oplus \ell_k \) into line bundles with the corresponding first Chern class denoted by \( c_1(\ell_j) = x_j \), then for each \( 0 < \theta < \pi \), we define
\[ \hat{A}_\theta(V) = 2^{-k} \prod_{j=1}^{k} \frac{1}{\sinh \left( \frac{1}{2}x_j + i\theta \right)} = \prod_{j=1}^{k} \frac{e^{\frac{i}{2}(x_j + i\theta)}}{e^{(x_j + i\theta)} - 1} \]
When \( \theta = \pi \), we define a characteristic class \( \hat{A}_\pi(V) \) for any oriented real \( 2k \)-dimensional bundle as follows. Let \( V = V_1 \oplus \cdots \oplus V_k \) be a formal splitting into oriented 2-plane bundles, and set \( x_j = \chi(V_j) \) the Euler class of \( V_j \). We define
\[ \hat{A}_\pi(V) = 2^{-k} \prod_{j=1}^{k} \frac{1}{\sinh \left( \frac{1}{2}x_j + i\pi \right)} = (2i)^{-k} \prod_{j=1}^{k} \frac{1}{\cosh(x_j/2)}. \]
Moreover, suppose \( \ell \) is the associated line bundle for the \( \text{spin}^c \)-structure of \( X \), and \( c_1 = c_1(\ell) \) its first Chern class. Suppose \( d_h \) acts on \( \ell \) by multiplication by \( e^{i\beta} \). We define an associated characteristic class to be \( e^{\frac{c_1 + i\beta}{2}} \).

Let \( D_E \) be the associated Dirac operator twisted by \( E \) on \( X \) and denote its higher index by \( \text{Ind}_{\Gamma}(D_E) \in K_n(C^*_r(\Gamma)) \). We have the following modified version of \( L^2 \)-Lefschetz fixed point theorem of B.-L. Wang and H. Wang [33, Theorem 5.10 & Theorem 6.1].
Theorem A.1. With the same notation as above, if the conjugacy class \( \langle h \rangle \) of \( h \in \Gamma \) has polynomial growth, then
\[
\text{tr}_h(\text{Ind}_\Gamma(D_E)) = \int_{\mathcal{F}} \left( \prod_{0 < \theta \leq \pi} \hat{A}_\theta(N(\theta)) \cdot \hat{A}(X^h) \cdot e^{-\frac{c_1 + i \beta}{2}} \cdot \text{ch}(E) \right).
\] (10)

Here \( \text{tr}_h(\text{Ind}_\Gamma(D_E)) \) stands for the evaluation of the linear map \( \text{tr}_h : K_0(C^*_r(\Gamma)) \to \mathbb{C} \) on the higher index class \( \text{Ind}_\Gamma(D_E) \), and \( \mathcal{F} \) is a fundamental domain of \( X^h \) under the action of \( Z_h \), where \( Z_h \) is the centralizer group of \( h \) in \( \Gamma \).

Although B.-L. Wang and H. Wang made the assumption that \( \text{tr}_h \) extends to a trace map on \( C^*_r(\Gamma) \) in [33, Theorem 5.10], we point out that it suffices to have the less restrictive assumption that the conjugacy class \( \langle h \rangle \) of \( h \in \Gamma \) has polynomial growth. Indeed, the higher index class \( \text{Ind}_\Gamma(D_E) \) can be represented by elements in terms of the heat kernel operator \( e^{-(\tilde{D}_E)^2} \) (see for example [10, Page 356]). Such a representative lies in \( \mathcal{A}(\tilde{X}, S \otimes E)^\Gamma \) (see Proposition 4.6 above). Now the same proof as in [33, Theorem 6.1] gives the above theorem.

Remark A.2. The assumption that the conjugacy class \( \langle h \rangle \) has polynomial growth is only used to guarantee that the trace map \( \text{tr}_h : \mathbb{C} \Gamma \to \mathbb{C} \) induces a linear map \( \text{tr}_h : K_0(C^*_r(\Gamma)) \to \mathbb{C} \). On the other hand, the \( L^2 \)-Lefschetz fixed point theorem continues to hold in complete generality, without any growth conditions on \( \langle h \rangle \). Indeed, observe that \( \text{Ind}_\Gamma(D_E) \) can be represented by an element with finite propagation. Now the local index formula can be calculated by using finite propagation speed methods such as those employed in [24, Theorem 3.4].

As an immediate consequence of the theorem above, we have the following corollary.

Definition A.3. Let \( Q_\Gamma \) be the field extension of \( \mathbb{Q} \) by the following set of roots of unity:
\[
\{ e^{2\pi i/n} \mid \text{there exists } \alpha \in \Gamma \text{ such that the order of } \alpha \text{ is } n \}.
\]

Corollary A.4. With the same notation as above, \( \text{tr}_h(\text{Ind}_\Gamma(D_E)) \) is an algebraic number in \( Q_\Gamma \). If in addition \( h \) has infinite order, then \( \text{tr}_h(\text{Ind}_\Gamma(D_E)) = 0 \).

Proof. Suppose \( h \) has finite order \( n \). Since the action of \( h \) on \( X \) lifts to an action on the spin\(^c\)-bundle, it follows that the only possible values for \( \theta/2 \) and \( \beta/2 \) that appear in the formula (10) are \((2k\pi/n)\) for \( 0 \leq k < [n/2] \). This proves the first part. Now if \( h \) has infinite order, then the fixed point set of \( h \) is empty, since the action of \( \Gamma \) on \( X \) is proper. It follows that \( \text{tr}_h(\text{Ind}_\Gamma(D_E)) = 0 \). \( \square \)

References

[1] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In Colloque “Analyse et Topologie” en l’Honneur de Henri Cartan (Orsay, 1974), pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris, 1976.
[2] Tim Austin. Rational group ring elements with kernels having irrational dimension. Proc. Lond. Math. Soc. (3), 107(6):1424–1448, 2013.
[3] Paul Baum and Alain Connes. Chern character for discrete groups. In A fête of topology, pages 163–232. Academic Press, Boston, MA, 1988.
[4] Paul Baum and Alain Connes. K-theory for discrete groups. In Operator algebras and applications, Vol. I, volume 135 of London Math. Soc. Lecture Note Ser., pages 1–20. Cambridge Univ. Press, Cambridge, 1988.
[5] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and $K$-theory of group $C^*$-algebras. In $C^*$-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of Contemp. Math., pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[6] Paul Baum and Ronald G. Douglas. $K$ homology and index theory. In Operator algebras and applications, Part I (Kingston, Ont., 1980), volume 38 of Proc. Sympos. Pure Math., pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.

[7] Moulay-Tahar Benameur and Indrava Roy. The Higson-Roe exact sequence and $\ell^2$ eta invariants. J. Funct. Anal., 268(4):974–1031, 2015.

[8] Jeff Cheeger and Mikhael Gromov. Bounds on the von Neumann dimension of $L^2$-cohomology and the Gauss-Bonnet theorem for open manifolds. J. Differential Geom., 21(1):1–34, 1985.

[9] Alain Connes. Noncommutative geometry. Academic Press Inc., San Diego, CA, 1994.

[10] Alain Connes and Henri Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. Topology, 29(3):345–388, 1990.

[11] P. de la Harpe and G. Skandalis. Déterminant associé à une trace sur une algèbre de Banach. Ann. Inst. Fourier (Grenoble), 34(1):241–260, 1984.

[12] Nigel Higson and John Roe. Mapping surgery to analysis. I. Analytic signatures. K-Theory, 33(4):277–299, 2005.

[13] Nigel Higson and John Roe. Mapping surgery to analysis. II. Geometric signatures. K-Theory, 33(4):301–324, 2005.

[14] Nigel Higson and John Roe. Mapping surgery to analysis. III. Exact sequences. K-Theory, 33(4):325–346, 2005.

[15] Nigel Higson and John Roe. K-homology, assembly and rigidity theorems for relative eta invariants. Pure Appl. Math. Q., 6(2, Special Issue: In honor of Michael Atiyah and Isadore Singer):555–601, 2010.

[16] Navin Keswani. Geometric $K$-homology and controlled paths. New York J. Math., 5:53–81 (electronic), 1999.

[17] Navin Keswani. Relative eta-invariants and $C^*$-algebra $K$-theory. Topology, 39(5):957–983, 2000.

[18] Navin Keswani. Von Neumann eta-invariants and $C^*$-algebra $K$-theory. J. London Math. Soc. (2), 62(3):771–783, 2000.

[19] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. Spin geometry, volume 38 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1989.

[20] Eric Leichtnam and Paolo Piazza. The $b$-pseudodifferential calculus on Galois coverings and a higher Atiyah-Patodi-Singer index theorem. Mém. Soc. Math. Fr. (N.S.), (68):iv+121, 1997.

[21] Eric Leichtnam and Paolo Piazza. On higher eta-invariants and metrics of positive scalar curvature. K-Theory, 24(4):341–359, 2001.

[22] John Lott. Higher eta-invariants. K-Theory, 6(3):191–233, 1992.

[23] John Lott. Delocalized $L^2$-invariants. J. Funct. Anal., 169(1):1–31, 1999.

[24] H. Moscovici and F.-B. Wu. Localization of topological Pontryagin classes via finite propagation speed. Geom. Funct. Anal., 4(1):52–92, 1994.

[25] Paolo Piazza and Thomas Schick. Bordism, rho-invariants and the Baum-Connes conjecture. J. Noncommut. Geom., 1(1):27–111, 2007.

[26] Paolo Piazza and Thomas Schick. Groups with torsion, bordism and rho invariants. Pacific J. Math., 232(2):355–378, 2007.

[27] Paolo Piazza and Thomas Schick. Rho-classes, index theory and Stolz’ positive scalar curvature sequence. J. Topol., 7(4):965–1004, 2014.

[28] Michael Puschnigg. New holomorphically closed subalgebras of $C^*$-algebras of hyperbolic groups. Geom. Funct. Anal., 20(1):243–259, 2010.

[29] John Roe. Coarse cohomology and index theory on complete Riemannian manifolds. Mem. Amer. Math. Soc., 104(497):x+90, 1993.

[30] John Roe. Index theory, coarse geometry, and topology of manifolds, volume 90 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.

[31] Sıleynan Kağan Samurkaş. Bounds for the rank of the finite part of operator. J. Noncommut. Geom., 2017. arXiv:1705.07378v2.
[32] Xiang Tang, Yi-Jun Yao, Zhizhang Xie, and Guoliang Yu. Higher rho invariants, wave operator and rigidity theorems of rho invariants. under preparation, 2018.
[33] Bai-Ling Wang and Hang Wang. Localized index and $L^2$-Lefschetz fixed-point formula for orbifolds. *J. Differential Geom.*, 102(2):285–349, 2016.
[34] Shmuel Weinberger. Homotopy invariance of $\eta$-invariants. *Proc. Nat. Acad. Sci. U.S.A.*, 85(15):5362–5363, 1988.
[35] Shmuel Weinberger. Higher $\rho$-invariants. In *Tel Aviv Topology Conference: Rothenberg Festschrift (1998)*, volume 231 of *Contemp. Math.*, pages 315–320. Amer. Math. Soc., Providence, RI, 1999.
[36] Shmuel Weinberger, Zhizhang Xie, and Guoliang Yu. Additivity of higher rho invariants and nonrigidity of topological manifolds. submitted, 2016.
[37] Zhizhang Xie and Guoliang Yu. Positive scalar curvature, higher rho invariants and localization algebras. *Adv. Math.*, 262:823–866, 2014.
[38] Zhizhang Xie and Guoliang Yu. Higher rho invariants and the moduli space of positive scalar curvature metrics. *Adv. Math.*, 307:1046–1069, 2017.
[39] Guoliang Yu. Localization algebras and the coarse Baum-Connes conjecture. *K-Theory*, 11(4):307–318, 1997.
[40] Guoliang Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)*, 147(2):325–355, 1998.
[41] Guoliang Yu. A characterization of the image of the Baum-Connes map. In *Quanta of Maths*, volume 11 of *Clay Math. Proc.*, pages 649–657. Amer. Math. Soc., Providence, RI, 2010.

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