THE NODAL BASIS OF $C^m$-$P_k^{(3)}$ AND $C^m$-$P_k^{(4)}$ FINITE ELEMENTS ON TETRAHEDRAL AND 4D SIMPLICIAL GRIDS

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Abstract. We construct the nodal basis of $C^m$-$P_k^{(3)}$ ($k \geq 2^4 m + 1$) and $C^m$-$P_k^{(4)}$ ($k \geq 2^4 m + 1$) finite elements on 3D tetrahedral and 4D simplicial grids, respectively. $C^m$-$P_k^{(n)}$ stands for the space of globally $C^m$ ($m \geq 1$) and locally piecewise $n$-dimensional polynomials of degree $k$ on $n$-dimensional simplicial grids. We prove the uni-solvency and the $C^m$ continuity of the constructed $C^m$-$P_k^{(3)}$ and $C^m$-$P_k^{(4)}$ finite element spaces. A computer code is provided which generates the index set for the nodal basis of $C^m$-$P_k^{(n)}$ finite elements on $n$-dimensional simplicial grids.

1. Introduction

The Argyris finite element is one of the first finite elements, cf. [4]. It is a $C^1$-$P_5^{(2)}$ finite element on triangular grids. Here $C^1$-$P_5^{(2)}$ denotes the space of globally $C^1$ and locally piecewise polynomials of degree 5 on 2 dimensional triangular grids. In general $C^m$-$P_k^{(n)}$ stands for the space of globally $C^m$ ($m \geq 1$) and locally piecewise $n$-dimensional polynomials of degree $k$ on $n$-dimensional simplicial grids. It is straightforward to extend the Argyris finite element to $C^1$-$P_k^{(2)}$ ($k > 5$) finite elements, as follows. We introduce $(k-5)$ function values at $(k-5)$ internal points on each edge; we introduce (additional) first-order normal derivatives at $(k-4)$ internal points each edge; and we introduce additional function values at $\dim P_{k-6}$ internal points on the triangle.

In 1970, Bramble and Zl´amal [2] and Ženíšek [6] extended the above $C^1$-$P_k^{(2)}$ finite element to $C^m$-$P_k^{(4m+1)}$ finite elements for all $m \geq 1$. In fact, Ženíšek [6] defined all $C^m$-$P_k^{(2)}$ finite elements for $k \geq 4m+1$. We have a perfect partition of index in two space-dimensions. That is,

• we require $2m$th order continuity at each vertex of the triangulation, and the degrees of freedom are exactly the function value, the two first derivatives, and up to the $2m + 1$ $2m$th derivatives at each vertex;
• we require $m$th order continuity on each edge of the triangulation, and the degrees of freedom are exactly the function value at $(k-4m-1)$ internal points, the first normal derivatives at $(k-4m)$ internal points, and up to the $m$th normal derivatives at $(k-3m-1)$ internal points inside each edge;

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• the degrees of freedom are exactly the function value at $\dim P_{k-3}^{2}$ internal points inside each triangle.

For $C^{m}-P_{4m+2}^{2}$ finite elements, the above sets of index form exactly seven triangles, three at three vertices, three at three edges, and one at the center of the triangle. Such a nice partition of index does not exist in three and higher dimensions.

The first $C^{1}$ element in 3D was constructed by Ženěšek in 1973 [7], a $C^{1}-P_{9}^{3}$ finite element. By avoiding high-order derivatives in the degrees of freedom of Ženěšek [7], the author extended this finite element to all $C^{1}-P_{k}^{3}$ ($k \geq 9$) finite elements [9] in 2009. Also Ženěšek extended the $C^{1}-P_{9}^{3}$ finite element to $C^{m}-P_{8m+1}^{3}$ in 1974 [8]. Again high-order derivatives (above the continuity order) were used as degrees of freedom in [8]. The author could not extend the 3D $C^{m}$ Ženěšek finite element to all $C^{m}-P_{k}^{3}$ ($k \geq 8m+1$) finite elements, but constructed a family of $C^{2}-P_{k}^{3}$ ($k \geq 17$) finite elements using the continuity-equal order derivatives in [10], in 2016. Also the author defined a family of $C^{1}-P_{k}^{4}$ ($k \geq 17$) finite elements [10].

For $C^{m}-P_{k}^{n}$ ($k \geq 2^{n}m+1$) finite elements on $n$ ($n \geq 3$) space-dimensional simplicial grids, by the author’s limited knowledge, Alfeld, Schumaker and Sirvent are the first to introduce the distance concept to low-dimensional simplex and to define recursively the index for the nodal basis (degrees of freedom), cf. Equation (36) in [1]. As claimed by [1], it is very difficult to find explicit definitions of these index sets for general or given $m$, $n$ and $k$.

Similar to [1], [5] and [3] studied the index partition recently. But no closed formula is obtained for the index sets for general $C^{m}-P_{k}^{n}$ finite elements. For example, [5] obtained the index sets (degrees of freedom) for the $C^{4}-P_{33}^{3}$ finite elements on tetrahedral grids.

Mathematically it is a challenge to find explicit definitions of basis functions for general $C^{m}-P_{k}^{n}$ finite elements. For so many years we could not even complete the work in 3D, except for $m = 1$ ([9]) and $m = 2$ ([10]). In this work, we give explicitly the index set of the nodal basis for all $C^{m}-P_{k}^{3}$ ($k \geq 2^{4}m+1$) and $C^{m}-P_{k}^{4}$ ($k \geq 2^{4}m+1$) finite elements, on tetrahedral grids and four-dimensional simplicial grids, respectively. We did not use any formula of [1], [5], or [3], in giving the closed formula on the index of the nodal bases of $C^{m}-P_{k}^{3}$ and $C^{m}-P_{k}^{4}$ finite elements. We prove the uni-solvency and the $C^{m}$ continuity of the constructed $C^{m}-P_{k}^{3}$ and $C^{m}-P_{k}^{4}$ finite element spaces.

We also provide a computer code which produces the index sets for any given $n$, $m$ and $k$. The computer code is listed in Section A. The computer code follows the continuity requirements and goes exhaustively and non-overlappingly from nodal indices on lower dimensional simplicial faces to those on higher ones. The computer code does not solve this mathematical problem, but does give explicit indices for practical computation need. With the computer output, we study the patterns of overlapping index so that we can give explicit nodal basis definitions in 3D and 4D.
The computer code verifies the index sets, for some low \( n \)'s, \( m \)'s and \( k \)'s, constructed in this manuscript. In particular, it verifies the indices of the \( C^4-P_{33}^{(3)} \) finite element in [5].

2. THE 3D \( C^m-P_k^{(3)} \) FINITE ELEMENTS

Let \( T_h = \{ K \} \) be a 3D tetrahedral grid where a tetrahedron \( K \) has 4 vertices \( \{ x_i = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}), \ i = 0, \ldots , 3 \} \), the intersection of of two \( K \)'s is either a common face-triangle, or a common edge, or a common vertex, or an empty set, and the maximal diameter of \( \{ K \} \) is \( h \).

For 3D \( C^m-P_k \) (\( k \geq 2^3m+1 \)) finite elements, we have 4 types of nodal basis functions.

1: At 4 vertices, the function value, the three first derivatives and up to the \( \dim P_{4m}^{(2)} \) 4\( m \)th order derivatives, are selected. There are

\[
\sum_{i=0}^{4m} \dim P_i^{(2)} = \dim P_{4m}^{(3)} = \frac{(4m+1)(4m+2)(4m+3)}{6}
\]

degrees of freedom at each vertex.

2: At 6 edges, the function values at \( k_1 \) internal points (\( k_1 = k - 2^3m - 1 \)) and the 2 first derivatives at \( k_1 + 1 \) internal points, and up to \( 2m+1 \) 2\( m \)th order normal derivatives (normal to the edge) at \( (k_1 + 2m) \) internal points, are selected. On one edge there are

\[
\sum_{i=0}^{2m} (1+i) \dim P_{k_1+i-1}^{(1)} = k_1 \frac{(2m+1)^2 + (2m+1)}{2} + \frac{(2m+1)^3 - (2m+1)}{3}
\]

degrees of freedom.

3: On 4 face-triangles, the function values at \( \dim P_{k-6m-3}^{(2)} \) internal points, and the first normal derivatives at \( \dim P_{k-6m-1}^{(2)} \) internal points, and up to order \( m \) normal derivatives at \( \dim P_{k-4m-3}^{(2)} \) internal points, except near each of 3 corners of a triangle \( \dim P_{m-2}^{(3)} \) order 2 to order \( m \) normal derivatives, are selected. That is, we drop one order 2 normal derivative at \( \dim P_0^{(2)} \) internal corner point(s), the order 3 normal derivative at \( \dim P_1^{(2)} \) internal corner points, and up to the order \( m \) normal derivative at \( \dim P_{m-2}^{(3)} \) internal corner points. On one face-triangle, there are

\[
\sum_{i=0}^{m} \dim P_{k-6m-3+2i}^{(2)} - 3 \dim P_{m-2}^{(3)} = \sum_{i=0}^{m} \frac{(2m-1+k_1+2i)(2m+k_1+2i)}{2} - \frac{(m-1)m(m+1)}{2}
\]

\[
= (m+1) \frac{3k_1^2 + 3k_1(6m-1) + 25m^2 - 4m}{6}
\]
degrees of freedom.

4: Inside the tetrahedron, the \( \dim P_{k-4m-4}^{(3)} \) function values, except near each of four corners \( \dim P_{m-3}^{(3)} \) internal function values, are selected. There are

\[
\begin{align*}
\dim P_{k-4m-4}^{(3)} - 4 \dim P_{m-3}^{(3)} &= \frac{(4m + k_1 - 2)(4m + k_1 - 1)(4m + k_1)}{6} - \frac{4(m - 2)(m - 1)m}{6} \\
\end{align*}
\]

degrees of freedom.

Adding all degrees of freedom in (2.1)–(2.4), we obtain

\[
dof_3 = 4 \cdot \frac{(4m + 1)(4m + 2)(4m + 3)}{6} \\
+ 6 \cdot (k_1 \frac{(2m + 1)^2 + (2m + 1)}{2} + \frac{(2m + 1)^3 - 2m - 1}{3}) \\
+ 4 \cdot (m + 1) \frac{3k_1^2 + 3k_1(6m - 1) + 25m^2 - 4m}{6} \\
+ \frac{(4m + k_1 - 2)(4m + k_1 - 1)(4m + k_1)}{6} \\
- \frac{4(m - 2)(m - 1)m}{6} \\
= \frac{(8m + k_1 + 2)(8m + k_1 + 3)(8m + k_1 + 4)}{6}.
\]

(2.5) \[ \dim P_k^{(3)} = \frac{(8m + k_1 + 2)(8m + k_1 + 3)(8m + k_1 + 4)}{6} = \text{dof}_3. \]

The two numbers match.

We next limit ourselves to the unit right tetrahedron

\[
K = \{(x_1, x_2, x_3) : 0 \leq x_1, x_2, x_3, x_1 + x_2 + x_3 \leq 1 \}. \]

For a general tetrahedron \( K = (\lambda_1, \lambda_2, \lambda_3) \), the only change is the coordinate system, using the barycentric coordinates instead of \((x_1, x_2, x_3)\). The restriction of \( K \) by \( x_3 = 0 \) is the 2D unit right triangle

\[
T = \{(x_1, x_2) : 0 \leq x_1, x_2, x_1 + x_2 \leq 1 \}. \]

Lemma 2.1. The subset of linear functionals in (2.1)–(2.3), when restricted on the face-triangle \( T \), uniquely determines a 2D polynomial \( p_k \in P_k^{(2)}(T) \).

Proof. The restricted sub-subset of linear functionals is also described in the Introduction of this manuscript, for the \( C^{2m}P_k^{(2)} \) (not \( C^m \)) finite element. The lemma is proved by [2]. \( \square \)

Theorem 2.1. The set of linear functionals in (2.1)–(2.4) uniquely determines a 3D polynomial \( p_k \in P_k^{(3)}(K) \).

Proof. By (2.5), we have a square linear system of finite equations so that we only need to prove the uniqueness of the solution. Let \( p_k \in P_k^{(3)}(K) \) having all dofs in (2.1)–(2.4) zero.
By Lemma 2.1 and (2.1)–(2.3),

\[ p_k = p_{k-1}x_3, \text{ for some } p_{k-1} \in P_{k-1}^{(3)}(K). \]

By (2.2), \( p_{k-1} \) vanishes at all three vertices of \( T \):

\[ p_{k-1}(v_i) = 0, \; v_i \in T, i = 0, \ldots, 2. \]  

(2.6)

Using the first normal derivative and all directional derivatives \( \partial_{m_1,m_2} p_{k-1} \) in (2.1)–(2.3), all degrees of freedom of \( C_{m-1}^2 P_{k-1}^{(2)} \) of \( p_{k-1} \) vanish. By Lemma 2.1, we get

\[ p_{k-1}|_T = 0 \text{ and } p_{k-1} = p_{k-2}x_3, \text{ for some } p_{k-2} \in P_{k-2}^{(3)}(K). \]

Doing this \( m-1 \) times more, and on the other three face triangles, we get

\[ p_k = p_{k-4m-4}B, \text{ for some } p_{k-4m-4} \in P_{k-4m-4}^{(3)}(K), \]

where \( B \in P_{4m+4} \) has its zeroth to \( m \)-th normal derivatives vanished on the 4 face-triangles of \( K \).

By (2.1)–(2.3),

\[ \partial_{m_1,m_2,m_3} p_{k-4m-4}(v_i) = 0, \; 0 \leq m_1 + m_2 + m_3 \leq (m - 3), \; i = 0, \ldots, 3, \]

(2.8)

where \( \{v_i\} \) are 4 vertices of \( K \). We note that, in (2.8), the high order derivatives on \( p_k \) from (2.1)–(2.3) become low order derivatives on \( p_{k-4m-4} \). By (2.4) and (2.8),

\[ p_{k-4m-4} = 0. \]

The proof is complete. \( \Box \)

**Theorem 2.2.** The finite element space

\[ V_h = \left\{ v \in L^2(\Omega) : v|_K = \sum_{i=1}^{dof_3} f_i(v)\phi_i, \quad K \in T_h \right\} \subset C^m(\Omega), \]

where linear functionals \( \{f_i\} \) are defined in (2.1)–(2.4), and \( \{\phi_i\} \) is the dual basis of \( \{f_i\} \) on \( K \).

**Proof.** Let \( F \) be a common face-triangle of \( K_1 \) and \( K_2 \). Let \( \lambda_1 = 0 \) be a linear equation for the plane on which \( F \) is. By (2.7),

\[ v_1 - v_2 = \lambda_1^{m+1}p_{k-m-1}, \text{ for some } p_{k-m-1} \in P_{k-m-1}^{(3)}(\Omega), \]

where \( v_1 \) and \( v_2 \) are global polynomials whose restrictions are \( v|_{K_1} \) and \( v|_{K_2} \), respectively. Therefore the function value and all the normal derivatives up to \( m \)-th order, on \( F \), are zero. Thus \( v \) is \( C^m \) on face \( F \). The continuity on edges and at vertices are direct corollaries of face-continuity. The proof is complete. \( \Box \)
3. The 4D $C^m$-$P_k^{(4)}$ Finite Elements

Let $T_h = \{S\}$ be a 4D simplicial grid where $S$ has 5 vertices $\{x_i = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}), i = 0, \ldots, 4\}$, the intersection of of two $S$’s is either a common face-tetrahedron, or a common face-triangle, or a common edge, or a common vertex, or an empty set, and the maximal diameter of $\{S\}$ is $h$.

For 4D $C^m$-$P_k^{(4)}$ $(k \geq 2^4m + 1)$ finite elements, we have 5 types of nodal basis functions.

1: At 5 vertices, the function value and the derivatives up to order $2^3m$ are selected. There are

\[
dof_{4,0} = \dim P_{2^3m}^{(4)} = \frac{(8m + 1)(8m + 2)(8m + 3)(8m + 4)}{24}
\]

degrees of freedom at each vertex.

2: At 10 edges, the $k_1$ function values, $(k_1 = k - 2^4m - 1)$ and the $\dim P_{1}^{(2)}$ first derivatives at $k_1 + 1$ points, and up to order $4m \ dim P_{4m}^{(2)}$ normal derivatives at $k_1 + 4m$ points, are selected. Inside one edge there are

\[
dof_{4,1} = \sum_{i=0}^{4m} (k_1 + i) \dim P_{1}^{(2)}
\]

\[
= k_1 m_1(m_1 + 1)(m_1 + 2) \frac{6}{6} + (m_1 - 1)m_1(m_1 + 1)(m_1 + 2) \frac{8}{8}
\]

degrees of freedom, where $m_1 = 4m + 1$.

3: On each of 10 face-triangles, with two normal vectors on each triangle, we are supposed to select the function values at $\dim P_{k-12m-3}^{(2)}$ points, and the $\dim P_{1}^{(1)}$ first derivatives at $\dim P_{2^3m-1}^{(2)}$ points, and up to order-$2m \ \dim P_{2m}^{(1)}$ normal derivatives at $\dim P_{8m-3}^{(2)}$ points. However due to corner overlaps, since the second derivative, we drop $\dim P_{2}^{(1)}$ third normal derivatives at $\dim P_{0}^{(2)}$ points at each of three corners of the triangle, $\dim P_{3}^{(1)}$ fourth normal derivatives at $\dim P_{1}^{(2)}$ points at each of three corners of the triangle, and up to $\dim P_{2m}^{(2)}$ $2m$-th normal derivatives at $\dim P_{2m-2}^{(2)}$ points at each of three corners of the triangle. On one face-triangle, there are

\[
dof_{4,2} = \sum_{i=0}^{2m} \left( \dim P_{i}^{(1)} \ \dim P_{k-12m-3+2i}^{(2)} - 3 \dim P_{i}^{(1)} \ \dim P_{i-2}^{(2)} \right)
\]

\[
= (m + 1)(2m + 1) \frac{3k_1^2 + 40k_1m + 118m^2 - 3k_1 - 7m}{6}
\]

degrees of freedom inside each triangle.

4: On each of 5 face-tetrahedra of the 4D simplex, we are supposed to select the function value at $\dim P_{k-8m-4}^{(3)}$ internal points, the first normal derivative at $\dim P_{k-8m-1}^{(3)}$ internal points, and up to the $m$-th normal derivative at $\dim P_{k-5m-3}^{(3)}$ internal points. For the first derivative and higher normal derivatives, we have vertex overlapping and we drop near each of 4 corners of a tetrahedron, the $i$-th norm derivative $(0 \leq i \leq m)$ at $\dim P_{m+2i-2}^{(3)}$...
internal points. The overlap with edge degrees of freedom is complicated. The overlapped indices form a wedge with two end-triangles non-parallel. That is, along each edge, we drop second normal derivatives at $4m$ internal (tetrahedron) points, third normal derivatives at $2(4m) + (4m - 1)$ points, fourth normal derivatives at $3(4m) + 2(4m - 1) + (4m - 2)$ points, and so on until $m$-th normal derivatives. There are

$$
dof_{4,3} = \sum_{i=0}^{m} \left( \dim P_{k-8m-4+3i}^{(3)} - 4 \dim P_{i-1}^{(3)} \right) - 6 \sum_{i=2}^{m} \sum_{j=2}^{i} (i - j + 1)(4m + 2 - j) = (m + 1) \left( m(2945m^2 - 491m + 6) \right)^{24} + \frac{(546m^2 - 105m + 4)k_1}{12} + \frac{(19m - 2)k_1^2}{4} + \frac{k_1^3}{6} \right)
$$

degrees of freedom inside each tetrahedron.

5: Inside the 4D simplex, the function values at $\dim P_{k-5m-5}^{(4)}$ internal points are supposedly selected, but near each of five corners we drop $\dim P_{4m-4}^{(4)}$ internal function values, and for $m \geq 3$ near each of ten edges we drop the function value at internal points which form 4-dimensional wedges. The barycentric coordinates of these 4-dimensional wedge points, near the edge $\mathbf{x}_4 \mathbf{x}_5$, have the form $(c_1, c_2, c_3, c_4, c_5)$ with $\max\{c_i, c_5\} \leq 2^3 m + k_1$, $c_4 + c_5 = (2^2 + 2^3)m + k_1 + (i - 3)$ where $i = 3, \ldots, m$, and $c_1 + c_2 + c_3 \leq 4m - (i - 3)$. There are, inside the 4D simplex,

$$
dof_{4,4} = \dim P_{k-5m-5}^{(4)} - 5 \dim P_{4m-4}^{(4)} - 10 \sum_{i=3}^{m} \dim_{m-i}^{(2)}(4m + k_1 - i + 3) = \frac{(11m - 3 + k_1)(11m - 2 + k_1)(11m - 1 + k_1)(11m + k_1)}{24} - \frac{5(4m - 3)(4m - 2)(4m - 1)(4m)}{24} - \frac{10(m - 2)(m - 1)m(4k_1 + 15m + 3)}{24}
$$

degrees of freedom.

Adding all degrees of freedom in (3.1)–(3.5), we obtain the total degrees of freedom on one 4D simplex,

$$
dof_t = 5 \dof_{4,0} + 10 \dof_{4,1} + 10 \dof_{4,2} + 5 \dof_{4,3} + \dof_{4,4} = \frac{(16m + k_1 + 2)(16m + k_1 + 3)(16m + k_1 + 4)(16m + k_1 + 5)}{24} = \frac{(k + 1)(k + 2)(k + 3)(k + 4)}{24}.
$$
By \((3.1)–(3.4)\), we have

\[
\dim P_k^{(4)} = \frac{(k + 1)(k + 2)(k + 3)(k + 4)}{24} = \text{dof}_4.
\]

**Theorem 3.1.** The set of linear functionals in \(3.11\)–\(3.5\) uniquely determines a 4D polynomial \(p_k \in P_k^{(4)}(S)\).

**Proof.** By \((3.6)\), we have a square linear system of finite equations. The uniqueness implies the existence of solution. Let \(p_k \in P_k^{(4)}(S)\) having all dof’s in \(3.11–3.5\) zero. By Theorem 2.1 and \(3.1–3.4\), as all dof’s of \(C^{2m}P_k^{(3)}(K)\) of \(p_k|_K\) vanish,

\[
p_k = p_{k-1}x_4, \text{ for some } p_{k-1} \in P_{k-1}^{(4)}(S),
\]

where we assume

\[
S = \{(x_1, x_2, x_3, x_4) : 0 \leq x_1, x_2, x_3, x_4, x_1 + x_2 + x_3 + x_4 \leq 1\}.
\]

By \(3.11\) and \(3.3\), \(p_{k-1}\) vanishes at all four vertices of a face-tetrahedron \(K = \{x_4 = 0\}\) of \(S\), because all first order derivatives of \(p_k\) vanish at those points,

\[
p_{k-1}(v_i) = 0, \ v_i \in K, i = 0, \ldots, 3.
\]

Using the first normal derivative and all directional derivatives \(\partial_{1,2,3,4}^{m_1,m_2,m_3,4} p_k\) in \(3.1–3.4\), by Theorem 2.1 and \(3.1\), we have

\[
p_{k-1} = p_{k-2}x_4, \text{ for some } p_{k-2} \in P_{k-2}^{(4)}(S).
\]

By \(3.11\) and \(3.3\), \(p_{k-2}\) and all \(4(3\) on face tetrahedron) first order derivatives vanish at all four vertices of a face-tetrahedron \(K = \{x_4 = 0\}\) of \(S\),

\[
\partial_{1,2,3,4}^{m_1,m_2,m_3,0} p_{k-2} = 0, \ 0 \leq m_1 + m_2 + m_3 \leq 1, \ i = 0, \ldots, 3.
\]

Additionally, for \(m \geq 2\), by \(3.1\) and \(3.3\), \(p_{k-2}\) vanishes at \(4m\) internal points on the 6 edges of \(K\),

\[
p_{k-2}(m_{i,j}) = 0, \ 0 \leq i, j \leq 6, \ j = 1, \ldots, 4m.
\]

By \(3.8\)–\(3.9\) and \(3.2\), \(p_{k-2}\) on \(K\) is zero and

\[
p_k = p_{k-3}x_4^3, \text{ for some } p_{k-3} \in P_{k-3}^{(4)}(S).
\]

Repeating this \(m - 2\) times and also on the other 4 face-tetrahedra of \(S\), we get

\[
p_k = p_{k-5m-5}B, \text{ for some } p_{k-5m-5} \in P_{k-5m-5}^{(4)}(S),
\]

where \(B \in P_{5m+5}(S)\) is a bubble polynomial having its zeroth to \(m\)-th normal derivatives vanishing on the 5 face-tetrahedra of \(S\).

By \(3.1\)–\(3.4\), we have

\[
\partial_{1,2,3,4}^{m_1,m_2,m_3,4} p_{k-5m-5}(v_i) = 0, \ 0 \leq \sum_{i=1}^4 m_i \leq m - 1, \ i = 0, \ldots, 4,
\]

where \(\{v_i\}\) are the 5 vertices of \(S\). For \(m \geq 3\), by \(3.1\)–\(3.4\), we have

\[
p_{k-5m-5}(m_{i,j}) = 0, \ j = 1, \ldots, 12m, \ i = 1, \ldots, 10,
\]
where \( \{ m_{i,j} \} \) are 1D Lagrange points inside the 10 edges of \( S \). Further, for \( m \geq 4 \), by (3.1)–(3.4), we have
\[
\partial^{m_1,m_2,m_3}_{n_{i,1,n_{i,2},n_{i,3}}} p_{k-5m-5}(m_{i,j}) = 0, \quad m_1 + m_2 + m_3 = 1,
\]
\[ j = 1, \ldots, k - 12m - 1, \quad i = 1, \ldots, 10, \]
where \( \{ n_{i,j} \} \) are three unit normal vector on an edge \( E_i \) of \( S \), and \( \{ m_{i,j} \} \) are 1D Lagrange points inside the 10 edges of \( S \). We note that each such first order derivative can replace a function value, in (3.5), at a node internal to a face-triangle (of \( S \)) which has this edge as one of its three edges. The Lagrange points in (3.12) and (3.13) are different as they belong to different degree polynomials. Combining (3.12), (3.13) and equations for higher order derivatives, we have
\[
\partial^{m_1,m_2,m_3}_{n_{i,1,n_{i,2},n_{i,3}}} p_{k-5m-5}(m_{i,j}) = 0, \quad m_0 = \sum_{i=l}^{m_1} m_i = 0, \ldots, m - 3,
\]
\[ j = 1, \ldots, k - 12m - m_0, \quad i = 1, \ldots, 10. \]

By (3.11), (3.14) and (3.5), we conclude with \( p_{k-5m-5} = 0 \). The proof is complete.

\[ \square \]

**Theorem 3.2.** The finite element space

\[
V_h = \left\{ v \in L^2(\Omega) : \left. v \right|_S = \sum_{i=1}^{\text{def}_S} F_i(v) \Phi_i, \quad S \in T_h \right\} \subset C^m(\Omega),
\]
where linear functionals \( \{ F_i \} \) are defined in (3.1)–(3.5), and \( \{ \Phi_i \} \) is the dual basis of \( \{ F_i \} \) on \( S \).

**Proof.** Using (3.10) instead of (2.7), the proof is identical to that of Theorem 2.2.

\[ \square \]

4. A code computing degrees of freedom of \( C^m-P_k^{(n)} \)

A computer code in Fortran is listed in Section A.

(1) The code uses \( n = 3, \ m = 3 \) and \( k_1 = 2 \) to compute the nodal basis of the \( C^m-P_2^{(n)} \) finite element. One can change c0 for computing other cases.

(2) The output in Section A lists the barycentric indices of the first function value, the first first derivative, the first second derivative and the first third derivative at a face triangle in top four lines, respectively. This is for studying the index. It can be commented out by c6.

(3) Also by changing c6 (and the if statement above it) we can output other index or all \( \text{dim} P_k^{(n)} \) indices.

(4) The output in Section A lists the degrees of freedom at each one sub-simplex (0=vertex, 1=edge, and so on.) At the end of each level of simplex, the number of sub-simplex and the subtotal of degrees of freedom are listed.
At the end, the dimension of $P_k^{(n)}$ and the number of degrees of freedom are listed. They match each other.

(5) By changing $c1$, we can list each index as soon as it is assigned into an index set.

(6) By changing $c2$, we can list an index on high-dimension sub-simplex if it is assigned to a low-dimensional sub-simplex. This would help us to find overlapping structure. This change requires corresponding change on $c7$.

(7) By changing $c3$, $c4$ and $c5$, we can output all indices of one particular group.

**Appendix A. The code computing dof of $C^m-P_k(n)$ and its output**

A Fortran computer code computes the index set of nodal basis functions of $C^m-P_k(n)$ finite elements, for any space dimension $n$.

```fortran
subroutine nDCmPk(n,m,k1)
integer ix(4000000,10),ii(12,120,30,10),iz(8,120,10)
k=m*2**n+1+k1
idim=1
do i=1,n
    idim=idim*(k+i)/i
endo
if(idim.gt.4000000) stop 'inc ix dim'
call izindex(n,iz)
call baryc(n,k,idim,ix)
do i=2,7
    iz(1,1,1)=0
endo
do i0=0,n
    is=0
    do j=0,i0
        is=max(is,iz(8,1,i0+1))
    enddo
if(is.gt.30) stop 'inc ii in nDCmPk'
m1=m*2*(n-i0)+1
if(i0.eq.n) m1=1
do j2=1,is
    do j1=1,m1
        ii(1,j1,j2,1+i0)=0
    enddo
endo
endo
```

do il=0,n
    call subs(n,m,k,idim,il,ix,ii,iz)
enddo
itl=0
do i0=0,n
    ic=0
    do il=0,m*2**(n-i0-1)
        is=0
        do i2=1,1
            do i=1,idim
                if(((ix(i,n+2).eq.i0+1).and.(ix(i,n+3).eq.i1+1))
                    .and. (ix(i,n+4).eq.i2) ) then
                    is=is+1
            endif
        enddo
        ic=ic+is
    enddo
    write(6,33) i0,i1, is,ic
    c1     if(i0.eq.3) print*, ' overlap ', (iz(j,i1,10),j=1,i0)
    endif
    ic=ic+is
    write(6,33) i0,i1, is,ic
    c2     if(i0.eq.3) print*, ' overlap ', (iz(j,i1,10),j=1,i0)
    endif
    write(6,43) i0,iz(8,1,i0+1), ic,ic*iz(8,1,i0+1)
    itl=itl+ic*iz(8,1,i0+1)
enddo
write(6,53) n,m,k1,k, idim, itl
end

subroutine indexing(n,k,ix,i1,i2,i3,i4,i5,i6,j,i)
    integer ix(4000000,10),id(10)
    j=j+1
    if(i.eq.j) then
        id(1)=i1
        id(2)=i2
        id(3)=i3
        id(4)=i4
        id(5)=i5
        id(6)=i6
        ix(i,1+n)=0
        do l=1,n
            ix(i,l)=id(l)
        enddo
        ix(i,1+n)=k-ix(i,1+n)
    endif
end
ix(i,2+n)=0
endif
end

subroutine baryc(n,k,idim,ix,iz)
integer ix(4000000,10)
do i=1,idim
  j=0
do i1=0,k
do i2=0,k-i1
  if(n.eq.2)then
    call indexing(n,k,ix,i1,i2,i3,i4,i5,i6,j,i)
  else
    do i3=0,k-i1-i2
      if(n.eq.3)then
        call indexing(n,k,ix,i1,i2,i3,i4,i5,i6,j,i)
      else
        do i4=0,k-i1-i2-i3
          if(n.eq.4)then
            call indexing(n,k,ix,i1,i2,i3,i4,i5,i6,j,i)
          else
            do i5=0,k-i1-i2-i3-i4
              if(n.eq.5)then
                call indexing(n,k,ix,i1,i2,i3,i4,i5,i6,j,i)
              else
                do i6=0,k-i1-i2-i3-i4-i5
                  call indexing(n,k,ix,i1,i2,i3,i4,i5,i6,j,i)
                enddo
              enddo
            enddo
          enddo
        enddo
      enddo
    enddo
  endif
enddo
enddo
enddo
enddo
end

subroutine dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
integer ix(4000000,10),ii(12,120,30,10),iz(8,120,10)
integer id(10)
id(1)=i1+1
id(2)=i2+1
id(3)=i3+1
id(4)=i4+1
id(5)=i5+1
id(6)=i6+1
do kd=0,m*2**(n-il-1)
doi=1,idim
  if((ix(i,n+2).eq.0).or. (il.eq.3) )
    if (ix(i,n+2).eq.0)
      call ixy(n,m,kd,ix,ii,iz,i,id)
    enddo
  enddo
enddo
end

subroutine subs(n,m,k,idim,il,ix,ii,iz)
integer ix(4000000,10),ii(12,120,30,10),iz(8,120,10)
do i1=0,n
  if(il.eq.0)then
    call dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
  else
    do i2=i1+1,n
      if(il.eq.1)then
        call dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
      else
        do i3=i2+1,n
          if(il.eq.2)then
            call dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
          else
            do i4=i3+1,n
              if(il.eq.3)then
                call dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
              else
                do i5=i4+1,n
                  if(il.eq.4)then
                    call dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
                  else
                    do i6=i5+1,n
                      call dof(n,m,k,il,idim,ix,ii,iz,i1,i2,i3,i4,i5,i6)
                    enddo
                    endif
                  enddo
                endif
              enddo
            endif
          enddo
        endif
      endif
    enddo
  endif
enddo
endo
endo
endo
endo
end

subroutine izes(j,j0,iz,i1,i2,i3,i4,i5,i6)
integer iz(8,120,10)
j=j+1
if(j.gt.120) stop ' inc j in izs '
iz(1,j,j0)=i1+1
iz(2,j,j0)=i2+1
iz(3,j,j0)=i3+1
iz(4,j,j0)=i4+1
iz(5,j,j0)=i5+1
iz(6,j,j0)=i6+1
end

subroutine izindex(n,iz)
integer iz(8,120,10)
do j0=1,6
  j=0
do i1=0,n
  if(j0.eq.1) then
    call izs(j,j0,iz,i1,i2,i3,i4,i5,i6)
  else
    do i2=i1+1,n
      if(j0.eq.2) then
        call izs(j,j0,iz,i1,i2,i3,i4,i5,i6)
      else
        do i3=i2+1,n
          if(j0.eq.3) then
            call izs(j,j0,iz,i1,i2,i3,i4,i5,i6)
          else
            do i4=i3+1,n
              if(j0.eq.4) then
                call izs(j,j0,iz,i1,i2,i3,i4,i5,i6)
              else
                do i5=i4+1,n
                  if(j0.eq.5) then
                    call izs(j,j0,iz,i1,i2,i3,i4,i5,i6)
                  else
                    do i6=i5+1,n
                      call izs(j,j0,iz,i1,i2,i3,i4,i5,i6)
                    enddo
                  endif
                enddo
              endif
            enddo
          endif
        enddo
      endif
    enddo
  endif
enddo
end if
enddo
enddo
FINITE ELEMENTS

iz(8,1,j0)=j
enddo
end

subroutine ixy(n,m,k,il,kd,ix,ii,iz,i,id)
integer ix(4000000,10),ii(12,120,30,10),iz(8,120,10)
integer id(10)
is=0
do l=1,il+1
  is=is+ ix(i,id(l))
  enddo
if(is.eq.k-kd) then
  do j=1,iz(8,1,il+1)
    is=0
    do l=1,il+1
      is=is+abs(iz(l,j,il+1)-id(l))
    enddo
    if(is.eq.0) goto 2
  enddo
  print*, ' not found ', il, (id(l),l=1,il+1)
  stop ' n-found'
continue
2  if(ix(i,n+2).eq.3) then
    if(((il.eq.3).and.(kd.eq.3)).and.(j.eq.1)) then
      c3 iz(ix(i,n+2),kd+1,10)=iz(ix(i,n+2), kd+1,10)+1
      c4 write(6,4) i,(ix(i,l),l=1,n+4), il+1,1+kd,j,
      c5 > iz(ix(i,n+2), kd+1,10)
      4 format(i8, 5i3, ' used', 3i4, ' try', 3i3, ' #', i5)
    endif
  else
    ii(1,j,1+kd,il+1)=ii(1,j,1+kd,il+1)+1
    ix(i,n+2)=il+1
    ix(i,n+3)=1+kd
    ix(i,n+4)=j
    ix(i,n+5)=ii(1,j,1+kd,il+1)
    if(((il.eq.2).and.(j.eq.1)).and.(kd.ge.0))then
      iz(kd+2,1,1)=iz(kd+2,1,1)+1
      if(iz(kd+2,1,1).lt.2) then
        c6 write(6,13) iz(kd+1,1,1), ii(1,j,1+kd,il+1), j, kd,il
        write(6,12) (ix(i,l),l=1,n+4), (iz(l,j,il+1),l=1,il+1)
        endif
    endif
  13 format(2i8, ' ',5i4, ' ',10i4)
  12 format('check: ',5i4, ' ',10i4)
endif
endif
endif
end

The output for $C^3-P_3^{(3)}$:
check: 7 7 13 0 3 1 1 1 2 3
check: 6 6 14 1 3 2 1 1 2 3
check: 5 6 14 2 3 3 1 1 2 3
check: 4 6 14 3 3 4 1 1 2 3

simplex 0 derivative 0 dof 1 sum= 1
simplex 0 derivative 1 dof 3 sum= 4
simplex 0 derivative 2 dof 6 sum= 10
simplex 0 derivative 3 dof 10 sum= 20
simplex 0 derivative 4 dof 15 sum= 35
simplex 0 derivative 5 dof 21 sum= 56
simplex 0 derivative 6 dof 28 sum= 84
simplex 0 derivative 7 dof 36 sum= 120
simplex 0 derivative 8 dof 45 sum= 165
simplex 0 derivative 9 dof 55 sum= 220
simplex 0 derivative 10 dof 66 sum= 286
simplex 0 derivative 11 dof 78 sum= 366
simplex 0 derivative 12 dof 91 sum= 455
level 0 #simplex 4 dofs 455 total 1820

simplex 1 derivative 0 dof 2 sum= 2
simplex 1 derivative 1 dof 6 sum= 8
simplex 1 derivative 2 dof 12 sum= 20
simplex 1 derivative 3 dof 20 sum= 40
simplex 1 derivative 4 dof 30 sum= 70
simplex 1 derivative 5 dof 42 sum= 112
simplex 1 derivative 6 dof 56 sum= 168
level 1 #simplex 6 dofs 168 total 1008

simplex 2 derivative 0 dof 28 sum= 28
simplex 2 derivative 1 dof 45 sum= 73
simplex 2 derivative 2 dof 63 sum= 136
simplex 2 derivative 3 dof 82 sum= 218
level 2 #simplex 4 dofs 218 total 872

simplex 3 derivative 0 dof 360 sum= 360
level 3 #simplex 1 dofs 360 total 360

(n m k_1)= 3 3 2, dim P_{27}= 4060 C^m-P_k^n= 4060

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