Abstract

Experiments on the oscillatory motion of a suspended bar magnet throws light on the damping effects acting on the pendulum. The viscous drag offered by air was found to be the main contributor for slowing the pendulum down. The nature and magnitude of the damping effects were shown to be strongly dependent on the amplitude.
I. INTRODUCTION

The simple pendulum is pedagogically a very important experiment. As a result of its simplicity, it is introduced at high school level. The equation of motion (EOM) of a pendulum undergoing simple harmonic motion (SHM) is given as

\[
\frac{d^2y}{dt^2} = -\frac{g}{L}y = -\omega^2_0 y
\]

whose solution is easily derivable and can be taught in a class which has been introduced to calculus. The EOM can be modified to account for damping as seen in a real pendulum and yet the equation and it’s solution remains trivial as

\[
\frac{d^2y}{dt^2} + \left(\frac{b}{L}\right) \frac{dy}{dt} + \omega^2_0 y = 0
\]

\[
y(t) = e^{-\beta t}(A\cos\omega' t + B\sin\omega' t)
\]  

(1)

where \( \beta = \frac{b}{2mL} \) and \( \omega' = \sqrt{\omega^2_0 - \beta^2} \). However, this approach taken by textbooks oversimplifies the complex motion of the pendulum and implies that only the pendulum’s amplitude attenuates with time. On the contrary along with the amplitude even the oscillation’s time period varies\(^2\), a feature overlooked in classroom physics and carried forward for a long time by students. The difficulty in measuring these variations also does not encourage routine experimentation in high schools/undergraduate laboratories. However, with the advent of micro-computers such measurements can now be made easily. Most of the experiments reported have measured the change in amplitude\(^2,3,4\) while examples of measuring variation in time period is rare\(^5\). Since, both amplitude and the time period varies with successive oscillation, one can expect the pendulum’s velocity to vary with time at a given position. While Gregory\(^1\) used knowledge of the oscillation time period to extract information on the pendulum’s velocity, Avinash Singh et al\(^6\) used a novel method to estimate the pendulum’s velocity. A bar magnet was attached to a rigid semicircular aluminum frame of radius ’L’ which pivoted about the center of the circle such that the bar magnet oscillates through a coil kept at the mean position. As the magnet periodically passed through the coil, it generated a series of emf pulses. The arrangement with proper circuitry determined the peak emf. Avinash et al\(^6\) approximated the
peak emf ($\xi_{\text{max}}$) as

$$\xi_{\text{max}} \approx \left( \frac{d\phi}{dt} \right)_{\text{max}} \omega_{\text{max}}$$

where $\omega_{\text{max}}$ is the maximum velocity as the bar magnet passed through the mean position. This method has its advantage when one proposes to study the damping effects in a pendulum. Most of the works studying the variation in oscillation amplitude \cite{4,7,8} with time have the pendulum’s suspension connected to a variable resistance (potentiometer) which introduces a sliding friction in the pendulum’s motion. Complex mathematics with assumption that all damping contributors act independently is then used to filter out information of each contribution. Wang et al \cite{9} used a novel but costly method using Doppler effect to monitor the position of the pendulum to study its damping. Thus, Avinash et al \cite{6} provides an interesting yet cheap method to study the damped pendulum. While they rightly pointed out that several parameters of the experiment such as velocity and strength of the magnet and the number of turns in the coil can be varied, they did not explicitly discuss them theoretically or study these factors experimentally. Hence, in this manuscript, we have furthered the study made in ref 6 and have tried to address these issues.

II. EXPERIMENTAL SETUP

Our pendula was made by suspending a standard bar magnet by a cotton thread. The thread was fastened to a small hook drilled into one pole of the bar magnet. The length of the bar magnet (2$l$) was 7cm and the cotton thread ($L_s$) used was 53cm long. A coil of 1000 turns was kept near the pendula’s mean position at a distance ‘d’ from the magnet’s lower pole (see fig 1). The magnetic field at point ‘A’ is evaluated by

$$B = \frac{\mu_0 m}{4\pi} \left[ \frac{1}{BA^2} - \frac{1}{CA^2} \right]$$

(2)

where $m$ is the dipole moment. ‘AC’ and ‘BA’ can be written in terms of the pendulum’s position (angle $\Theta$) using the cosine law. That is,

$$BA^2 = OB^2 + OA^2 - OB.OAcos\Theta$$

$$AC^2 = OC^2 + OA^2 - OC.OAcos\Theta$$
where

\[ OC = L_s + 2l \]
\[ OB = L_s \]
\[ OA = L_s + 2l + d \]

hence,

\[
BA^2 = L_s^2 + (L_s + 2l + d)^2 - L_s(L_s + 2l + d) \cos \Theta \\
AC^2 = (L_s + 2l)^2 + (L_s + 2l + d)^2 - (L_s + 2l)(L_s + 2l + d) \cos \Theta
\]

Based on the assumption that \(2l\) and \(d\) are relatively small compared to \(L_s\), the higher powers of \(2l\) and \(d\) can be neglected. Hence, eqn(2) can be written as

\[
B \approx \mu_o M \left[ \frac{2l}{4\pi \left( \frac{L_s^2(L_s + 4l + d)(2 - \cos \Theta) \right)} \right]
\]

The induced emf is proportional to the rate of change in the number of magnetic lines cutting the coil.

\[
\frac{dB}{dt} \approx \mu_o M \left[ \frac{2l}{4\pi \left( \frac{L_s^2(L_s + 4l + d) \right)} \right] \left[ \frac{\sin \Theta}{(2 - \cos \Theta)^2} \right] \frac{d\Theta}{dt}
\]
Based on this, the respective induced \textit{emf} can be written as

$$\xi = -N \frac{dB}{dt} \approx \frac{\mu_0 MN}{4\pi} \left[ \frac{2l}{L_s^2(L_s + 4l + d)} \right] \left[ \frac{\sin \Theta}{(2 - \cos \Theta)^2} \right] \frac{d\Theta}{dt} \quad (3)$$

where \(N\) is the number of turnings in the coil. Eqn(3) can be written in a compact form

$$\xi = \xi_o \left[ \frac{\sin \Theta}{(2 - \cos \Theta^2)} \right] \left( \frac{d\Theta}{dt} \right) \quad (4)$$

where

$$\xi_o = \frac{\mu_0 MN}{4\pi} \left[ \frac{2l}{L_s^2(L_s + 4l + d)} \right] \quad (5)$$

Thus, as the distance between the magnet and detecting coil is increased, the induced \textit{emf} decreases. Infact the induced \textit{emf} is quite weak and is amplified by an op amp circuit. The high input impedance of the IC741 opamp ensures that a true measurement of the \textit{emf} is made. To digitalise this analog signal (see fig 1, ref 6) using an Analog to Digital Convertor ADC-0809 (see fig 2a), the amplified output is rectified and the peak value is held by charging a capacitor. The capacitor is discharged via a large resistance so that it retains the peak value till the next peak value arrives.

We require the ADC to start conversion once the peak value is attained by the capacitor. This implies a synchronization between the input \textit{emf} pulses and the ADC’s start of conversion (SOC) pulses. To achieve this synchronization it is best to generate the required SOC pulse by wave-shaping the input itself. The amplified input after rectification is fed to a comparator which compares to +1v. This is to avoid spurious/accidental triggering due to noise. The infinite gain results in pulses with sharp edges. The width of these pulses are approximately \(T_o/4\) (for our pendula \(\approx 390\text{ms}\)). This would be too large for serving as a SOC and hence is reduced to a 5\(\mu\)s pulse using a monostable timer made with IC555\textsuperscript{10}. The sequencing and synchronization can be understood from the various waveforms shown in fig 2b. The designed circuit digitalises the analog \textit{emf} and on completion sends an EOC to the computer or microprocessor kit (in case of a microprocessor this is done through a programmable I/O IC8155 chip, details of which can be found in the book by Goankan\textsuperscript{11}) which then reads the eight bit data and stores it for retrival. This project was done using an 8085 microprocessor kit. The programme and flowchart used is detailed in the Appendix.
FIG. 2: The (a) schematic diagram of the circuit used and the (b) important waveforms at the points marked in the circuit.

The reliability of our circuit can be tested by measuring the maximum emf induced in the coil for varying distances ‘d’. Eq(4) shows that the measured maximum emf would be directly proportional to $\xi_o$ which in turn is inversely proportional to ‘d’ (see eqn 5). Fig 3
FIG. 3: Variation in the maximum induced emf with increasing distance between the coil and the magnet.

shows the variation in the experimentally determined $\xi_{\text{max}}$ with 'd'. While the inverse nature is evident, the value of $(L_s + 4l)$ as returned by curve fitting eqn(5) on our data is substantially off mark from the actual lengths. This is expected since eqn(4) and (5) are very simplified approximations.

III. VARIATION OF INDUCED EMF WITH INITIAL DISPLACEMENT

A. While undergoing undamped oscillation

The velocity of an undamped pendulum undergoing SHM is given as

$$\left( \frac{d\Theta}{dt} \right) = \omega_o \sqrt{\Theta_m^2 - \Theta^2}$$

where $\omega_o = \sqrt{g/(L_s + 2l)}$ is the frequency of oscillation and $\Theta_m$ is the initial displacement given to the pendulum. Therefore, the emf induced by our pendulum undergoing undamped SHM
would be given as (using eq. 4)

\[
\xi \approx \omega_o \xi_o \left[ \frac{\sin\Theta \sqrt{\Theta_m^2 - \Theta^2}}{(2 - \cos\Theta)^2} \right]
\] (6)

The variation in induced emf with time of an undamped pendulum undergoing SHM is as calculated using eqn(6) is shown in fig(4). The maximum angular displacement used to generate the graph using eqn(6) was 5°. The emf pulse shown in fig 4 is only for half a cycle starting from one extreme position to the opposite extreme. As the magnet approaches the coil, the flux increases and as it crosses the mean position, the emf is negative since the magnet is receding from the coil. Eventhough the velocity \((d\Theta/dt)\) is maximum at the mean position, since the variation in flux \((d\phi/dt)\) is zero, the induced emf is zero as the pendulum passes the mean position.

![Graph showing induced emf with oscillating angle.](image)

FIG. 4: A measure of the induced emf with oscillating angle. The graph was generated using eqn(6) with \(\Theta_m = 5^\circ\).

The position of the pendulum when the maximum emf is generated (between \(0 < \Theta < \Theta_m\)) can be found as a problem of maxima and minima

\[
\frac{d\xi}{d\Theta} = \xi_o \left\{ \frac{(2 - \cos\Theta)^2 \left[ \sin\Theta \frac{d\Theta}{dt} \left( \frac{d\Theta}{dt} \right) + \cos\Theta \left( \frac{d\Theta}{dt} \right) \right] - (2 - \cos\Theta) \sin^2\Theta \left( \frac{d\Theta}{dt} \right)}{(2 - \cos\Theta)^3} \right\}
\]

\[
= \xi_o \left\{ \frac{(2 - \cos\Theta) \left[ \sin\Theta \frac{d\Theta}{dt} \left( \frac{d\Theta}{dt} \right) + \cos\Theta \left( \frac{d\Theta}{dt} \right) \right] - \sin^2\Theta \left( \frac{d\Theta}{dt} \right)}{(2 - \cos\Theta)^2} \right\}
\]
For cases of small angle oscillations eqn(7) reduces to

\[ \frac{d\xi}{d\Theta} = \xi_o \left[ \frac{d\Theta}{d\Theta} \left( \frac{d\Theta}{dt} \right) + (1 - \Theta^2) \left( \frac{d\Theta}{dt} \right) \right] \]  

\[ \frac{d\xi}{d\Theta} = \omega_o \xi_o \left[ \frac{-2\Theta^2}{2\sqrt{\Theta^2_m - \Theta^2}} + (1 - \Theta^2)\sqrt{\Theta^2_m - \Theta^2} \right] \]

\[ \Theta^2 = (1 - \Theta^2)(\Theta^2_m - \Theta^2) \]

Solving the quadratic equation

\[ \Theta^4 - (2 + \Theta^2_m)\Theta^2 + \Theta^2_m = 0 \]

we have

\[ \Theta_{\text{peak}} = \pm \frac{\Theta_m}{\sqrt{2}} \]  

Since eqn(7) was used to determine position of extrema, the above condition is only valid for undamped small angle oscillations. The maxima as per this condition for magnet oscillating through \( \Theta_m = 5^o \) occurs at \( \pm 0.0617 \) radians (or \( \pm 3.53^o \)). The maximum \textit{emf} that is induced, hence is (use eqn 6)

\[ \xi_{\text{max}} = \omega_o \xi_o \sqrt{\frac{\Theta_m}{2}} \times \frac{\sin \frac{\Theta_m}{\sqrt{2}}}{\left( 2 - \cos \frac{\Theta_m}{\sqrt{2}} \right)^2} \]

Since, these equations and conditions are essentially valid for small angles,

\[ \xi_{\text{max}} = \left( \frac{\omega_o \xi_o}{2} \right) \Theta_m^2 \]  

However, a physical pendulum is prone to damping and hence in the next section we investigate as to how the maximum induced emf varies with initial displacement for a damped pendulum.
FIG. 5: Variation of induced emf when oscillation is damped ($\beta = 0.45\text{s}^{-1}$) is compared with the case of no damping (i.e. $\beta = 0.0\text{s}^{-1}$).

FIG. 6: The variation of induced emf of fig 3 is plotted w.r.t. angular position of the pendulum (i.e. $\Theta$) for the cases (a) $\beta = 0.0\text{s}^{-1}$ and (b) $\beta = 0.45\text{s}^{-1}$.

B. While undergoing damped oscillation

We have already stated in our introduction that the damped motion described by eqn(1) exhibits how the pendulum’s oscillation amplitude decreases exponentially with time. The
EOM whose solution is given by eqn(1) describes a linear system. The solution can be further trivialized without losing any generality as

$$\Theta = \Theta_m e^{-\beta t} \sin(\omega't)$$  \hspace{1cm} (10)

from which the velocity can be calculated as

$$\frac{d\Theta}{dt} = \omega' \Theta_m e^{-\beta t} \cos(\omega't) - \beta \Theta_m e^{-\beta t} \sin(\omega't)
= \Theta_m e^{-\beta t} [\omega' \cos(\omega't) - \beta \sin(\omega't)]$$  \hspace{1cm} (11)

Substituting the above expression in eqn(4) we obtain the relation showing the variation of induced *emf* with time. This variation is shown in fig 5. It is also clear from the figure that the peaks in the induced *emf* occurs at $\omega t = (2n + 1)\pi/4$. Hence, the angles at which maxima occur in general is written as

$$\Theta_{\text{peak}} = \pm \frac{\Theta_m}{\sqrt{2}} e^{-\frac{(2n+1)\pi}{4 \tan \phi}}$$  \hspace{1cm} (12)

where $\tan \phi = \omega'/\beta$. Our circuit is designed only to measure peak *emfs* at $n=0,2,4,6,\ldots$, where only the positive solutions of eqn(12) would contribute. Using our condition on eqn(11) and eqn(1) we have

$$\xi_{\text{peak}} = (\omega' - \beta) \xi_o \left[ \frac{\sin \frac{\Theta_m}{\sqrt{2}} e^{-\frac{(2n+1)\pi}{4 \tan \phi}}}{(2 - \cos \frac{\Theta_m}{\sqrt{2}} e^{-\frac{(2n+1)\pi}{4 \tan \phi}})^2} \right] \frac{\Theta_m}{\sqrt{2}} e^{-\frac{(2n+1)\pi}{4 \tan \phi}}$$  \hspace{1cm} (13)

For small angle oscillations eqn(13) reduces to

$$\xi_{\text{peak}} = \frac{(\omega' - \beta) \xi_o}{2} \Theta_m^2 e^{-\frac{(2n+1)\pi}{2 \tan \phi}}$$  \hspace{1cm} (14)

The variation in *emf* (seen w.r.t time in fig 5) when viewed w.r.t oscillating angle $\Theta$ shows how the peak position decreases (eq(12)) as also the amplitude of the maximum induced *emf* decreases (eqn(14)) with each half cycle. Eqn(9) and eqn(14) shows that the maximum *emf* induced for damped pendulums undergoing SHM is directly proportional to the square of maximum angular displacement given to the pendulum. We have recorded the first maxima
FIG. 7: The variation of induced emf with the initial angular displacement of the pendulum. It shows the expected parabolic dependence (i.e. $\Theta_m^2$).

reading (i.e. n=0) of the induced emf for various angles upto 40°. The linear relation between $\xi_{\text{max}}$ and $\Theta_m^2$ is evident. Before commenting further, it must be recollected that eq(1) is valid for small angle oscillations, i.e. for $\Theta_m < 5^\circ$, yet a good linearity is obtained till $\Theta = 40^\circ$.

Experimental data for $\Theta_m \geq 45^\circ$ deviate markedly from this linear trend. Remember eqn(12) was obtained with the assumption that the pendula’s motion is described by eqn(10). This equation describes the motion of a pendulum oscillating in a viscous medium with small velocity. It would be shown in the next section that the pendulum’s velocity is quite appreciable for $\Theta_m \geq 45^\circ$ and hence it’s motion is not described as in eqn(10), explaining the departure for linearity.

IV. RESULTS AND OBSERVATIONS

Our preliminary measurements are in good correspondence with commonly known notions and hence we proceed to investigate further the nature of damping in our pendula. It should be noted that the amount of damping and it’s nature are strongly pendula dependent and all results reported here are specific to our experiment and can not be taken as general. We have
recorded the maxima in induced *emfs* for 80 oscillations for various initial displacements. Since for each oscillations, we get two positive maxima in induced *emf*, figure 8 shows the variation in maxima reading of induced *emf* for 160 peaks.

A general expression $ae^{-bn}$ was fitted to the data of $\xi_{\text{max}}$ w.r.t $n$ using a standard and freely available curve fitting software called ”Curxpt v3.0”. Good fits were obtained for oscillations set by initial displacements up to 40$^\circ$. The exponential fall in *emf* is indicative of the rate of loss of energy from the oscillating system. It indicates the loss to follow the relation

$$\frac{dE}{dt} \propto -E$$

or

$$\frac{dE}{dt} = -bE$$

This indicates that the velocity is low and hence the damping/resistive force acting on the pendulum is proportional to the velocity.\(^{12}\) Figure 9 shows the variation of the decay constant 'b' (of $ae^{-bn}$) with respect to the maximum displacement ($\Theta_m$) given to the pendulum. The graph indicates that as $\Theta_m$ increases, the velocity with which the pendulum moves increases with which the damping constant increases.
Fig. 9: The variation decay constant (b of general equation $ae^{-bn}$) of the exponentially decaying region. The continuous line is the parabolic fit for the data points $(0.00056\sqrt{\Theta_m})$.

Fig. 10: The variation decay in 'b', a measure of decay in the early oscillations when pendulum was set in motion with displacements $> 45^\circ$.

For displacement angles beyond $45^\circ$ ($\Theta_m \geq 45^\circ$), even though visual examination of the curves in fig 8 suggests an exponential damping, the data points do not fit to an exponential fall relation. A detailed examination suggests a more complex process is taking place with initial damping being sharper. In fact, the initial 25-30 data points fit to $a/n^b$. The data
beyond this fit to the exponential fall equation. The $a/n^b$ fit corresponds to the damping force being proportional to higher powers of velocity ($v^\gamma$, where $\gamma > 1$) and in turn the rate of energy loss also being proportional to higher terms of energy. That is, the rate of energy loss for our pendulum set into oscillations with a displacement angle $\Theta > 45^\circ$ is given as

$$\frac{dE}{dt} = -\alpha E^{\frac{1+b}{b}}$$

Figure 10 plots 'b' versus $\Theta_m$. The power term (b) is being treated as a measure of damping and is consistent with the results of fig 9, i.e. as initial displacement increases the damping becomes large with a proportionality to the pendulum’s velocity.

The resistive force being proportional to higher powers of velocity has been reported earlier also. A system is reported to have a constant friction ($\gamma = 0$) or a linear dependence of velocity ($\gamma = 1$) or a quadratic dependence of velocity ($\gamma = 2$). Corresponding to which the pendulum’s amplitude decays linearly, exponentially and inverse power decay respectively with time. It hence maybe concluded that for our pendulum sent into motion by initial displacements $\Theta \geq 45^\circ$, the damping force is proportional to $v^\gamma$ where $\gamma > 1$.

V. CONCLUSION

A simple experiment of setting a suspended bar magnet into oscillations, is a rich source of information. Not only does it give exposure to Faraday’s induction law and a basic understanding of induced emf’s dependence on angle of oscillation, it enables us to study the damping effects on the pendulum. This method is better than previously used methods since the measuring technique does not introduce additional contributions to damping. When the oscillation imparted to the pendulum is very large, the damping effect is also strong with the damping force being proportional to $v^\gamma$, where $\gamma > 1$ and ‘v’ is the pendulum’s velocity. This brings down the oscillation amplitude of the pendulum and it’s velocity. As the velocity becomes low, the resistive force acting on the pendulum changes it’s nature and becomes proportional to ‘v’. Considering the rich information obtained from the experiment and the simplicity of the experiment, it allows the method to be easily implemented as a routine experiment in undergraduate laboratories.
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Table 1. Program used to collect data.

| Address | Mnemonics | Hex Code | Address | Mnemonics | Hex Code |
|---------|-----------|----------|---------|-----------|----------|
| C400    | LXI SP    | 31       | C40A    | 01H       | 01       |
| C401    | 00H       | 00       | C40B    | JZ        | CA       |
| C402    | C3H       | C3       | C40C    | 07H       | 07       |
| C403    | MVI A     | 3E       | C40D    | C4H       | C4       |
| C404    | 00H       | 00       | C40E    | IN        | DB       |
| C405    | OUT       | D3       | C40F    | 09H       | 09       |
| C406    | 08H       | 08       | C410    | PUSH PSW  | F5       |
| C407    | IN        | DB       | C411    | JMP       | C3       |
| C408    | OBH       | OB       | C412    | 07H       | 07       |
| C409    | ANI       | E6       | C413    | C4H       | C4       |
Set Control word making Port A and Port C input ports

Initialise Stack Pointer

Read port C, (contains EOC)
AND content with 01H

Is EOC high

No

Read Data at Port A
Push PSW to save data

Unconditional Jump

Yes

FIG. 11: Flowchart.