Abstract. Hofer’s norm (metric) is an important and interesting topic in symplectic geometry. In the present paper, we define fragmented Hofer’s norms which are Hofer’s norms controlled by fragmentation norms and give some observations on fragmented Hofer’s norms.

1. Introduction

Let \((M, \omega)\) be a symplectic manifold. For a Hamiltonian function \(F: M \to \mathbb{R}\) with compact support, we define the Hamiltonian vector field \(X_F\) associated with \(F\) by
\[
\omega(X_F, V) = -dF(V) \quad \text{for any } V \in \mathfrak{X}(M),
\]
where \(\mathfrak{X}(M)\) is the set of smooth vector fields on \(M\).

Let \(a, b\) be real numbers with \(a < b\). For a (time-dependent) Hamiltonian function \(F: M \times [a, b] \to \mathbb{R}\) with compact support and \(t \in [a, b]\), let \(F_\cdot: M \to \mathbb{R}\) denote the time-independent function defined by \(F_\cdot(x) = F(x, t)\). Let \(X_F^t\) denote the Hamiltonian vector field associated with \(F_\cdot\) and \(\{\phi_F^t\}_{t \in [a, b]}\) denote the isotopy generated by \(X_F^t\) with \(\phi_F^a = \text{id}\). The time-b map \(\phi_F^b\) of \(\{\phi_F^t\}\) is called the Hamiltonian diffeomorphism generated by the Hamiltonian function \(F\) and denoted by \(\phi_F\). For a symplectic manifold \((M, \omega)\), let \(\text{Ham}(M, \omega)\) denote the group of Hamiltonian diffeomorphisms of \((M, \omega)\).

We define the Hofer’s length of a Hamiltonian function \(F: M \times [a, b] \to \mathbb{R}\) as follows:
\[
||F|| = \int_a^b (\max_M F_t - \min_M F_t) dt.
\]
We review the definition of the original Hofer’s norm \(||\cdot||_{\text{Ho}}\).

**Definition 1.1.** Let \((M, \omega)\) be a symplectic manifold. For a Hamiltonian diffeomorphism \(\phi\), we define Hofer’s norm of \(\phi\) by
\[
||\phi|| = \inf\{||F||\},
\]
where the infimum is taken over Hamiltonian functions \(F: M \times [0, 1] \to \mathbb{R}\) such that \(\phi = \phi_F\).
The original Hofer’s norm $|| \cdot ||$ is known to be non-degenerate i.e. $||\phi||$ is positive for any Hamiltonian diffeomorphism $\phi$ which is not the identity map $1([LM])$.

In [Ka], the author studied the commutator length controlled by a fragmentation norm (see also [Ki]). In the present paper, we study the Hofer’s norm controlled by a fragmentation norm which we call a fragmented Hofer’s norm. For a time-dependent Hamiltonian function $H : M \times [a, b] \to \mathbb{R}$, we define the support $\text{Supp}(H)$ of $H$ by $\text{Supp}(H) = \bigcup_{t \in [a, b]} \text{Supp}(H_t)$.

**Definition 1.2.** Let $(M, \omega)$ be a symplectic manifold and $U$ be a non-empty open subset of $M$. We define *Hofer’s norm of $\phi$ fragmented by $U$* by

$$||\phi||_U = \inf \{ ||F^1|| + \cdots + ||F^k|| \},$$

where the infimum is taken over Hamiltonian functions $F^1, \ldots, F^k : M \times [0, 1] \to \mathbb{R}$ and Hamiltonian diffeomorphisms $h_1, \ldots, h_k$ such that $\text{Supp}(F^i) \subset U$ for any $i$ and $\phi = h_1 \phi_{F^1} h_1^{-1} \cdots h_k \phi_{F^k} h_k^{-1}$.

Banyaga’s fragmentation lemma ([B]) states that the above decomposition exists and thus $|| \cdot ||_U$ is well-defined i.e. $||\phi||_U < \infty$ for any Hamiltonian diffeomorphism $\phi$. By the definition, we see that

- the original Hofer’s norm is equal to Hofer’s norm fragmented by the whole symplectic manifold $M$ i.e. $||\phi|| = ||\phi||_M$ holds for any Hamiltonian diffeomorphism $\phi$,
- For open subsets $U, V$ of $M$ with $U \subset V$, $||\phi||_V \leq ||\phi||_U$ holds for any Hamiltonian diffeomorphism $\phi$.

In particular, $||\phi|| \leq ||\phi||_U$ holds for any Hamiltonian diffeomorphism $\phi$ and any open subset $U$ of $M$.

We also easily verify that the following proposition holds.

**Proposition 1.3.** Let $(M, \omega)$ be a symplectic manifold and $U$ an open subset of $M$. Then $|| \cdot ||_U$ is a conjugation-invariant norm in the sense of Burago, Ivanov and Polterovich ([BIP]) i.e. $|| \cdot ||_U$ satisfies the following conditions.

1. $||1||_U = 0$,
2. $||f||_U = ||f^{-1}||_U$ for any Hamiltonian diffeomorphism $f$,
3. $||fg||_U \leq ||f||_U + ||g||_U$ for any Hamiltonian diffeomorphisms $f, g$,
4. $||f||_U = ||gfg^{-1}||_U$ for any Hamiltonian diffeomorphisms $f, g$,
5. $||f||_U > 0$ for any Hamiltonian diffeomorphism $f$ with $f \neq 1$. 
Conjugation-invariant norms \( ||·||_1 \) and \( ||·||_2 \) on \( G \) are equivalent if there exists positive numbers \( a \) and \( b \) such that \( \frac{1}{a}||φ||_1 - b \leq ||φ||_2 \leq a||φ||_1 + b \) for any element \( φ \) of \( G \).

We state that any fragmented Hofer’s norm is equivalent to the original Hofer’s norm on a compact symplectic manifold.

**Theorem 1.4.** Let \((\hat{M}, ω)\) be a compact symplectic manifold which can have a smooth boundary and \( M \) the interior of \( \hat{M} \). For any open subset \( U \) of \( M \), there is a positive number \( C_U \) such that \( ||φ||_U \leq C_U ||φ|| \) for any Hamiltonian diffeomorphism \( φ \) on \((M, ω)\).

We prove Theorem 1.4 in Section 2.

However, some fragmented Hofer’s norms are not equivalent to the original Hofer’s norm on \( \text{Ham}(\mathbb{R}^{2n}, ω_0) \) where \( ω_0 = dx_1 ∧ dy_1 + · · · + dx_n ∧ dy_n \) is the standard symplectic form on the Euclidean space \( \mathbb{R}^{2n} \) with coordinates \((x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)\).

To prove it, we introduce the Calabi homomorphism. A symplectic manifold \((M, ω)\) is called exact if the symplectic form \( ω \) is exact.

**Definition 1.5 ([B]).** Let \((M, ω)\) be a \( 2n \)-dimensional exact symplectic manifold. The Calabi homomorphism \( \text{Cal}: \text{Ham}(M, ω) → \mathbb{R} \) is defined by

\[
\text{Cal}(h) = \int_0^1 \int_M H_t ω^n dt
\]

for a Hamiltonian diffeomorphism \( h \),

where \( H: M × [0, 1] → \mathbb{R} \) is a Hamiltonian function which generates \( h \). \( \text{Cal}(h) \) does not depend on the choice of generating Hamiltonian function \( H \) ([B] and [Hu]). Thus the functional \( \text{Cal} \) is a well-defined homomorphism.

By using the Calabi homomorphism, we obtain the following lower bound of fragmented Hofer’s norms.

**Proposition 1.6.** Let \( U \) be an open subset of a \( 2n \)-dimensional exact symplectic manifold \((M, ω)\). If \( \text{Vol}(U, ω) < ∞ \), then for any Hamiltonian diffeomorphism \( φ \),

\[
||φ||_U \geq \text{Vol}(U, ω)^{-1} |\text{Cal}(φ)|,
\]

where \( \text{Vol}(U, ω) \) is the volume \( \int_U ω^n \) of \( U \).

It is known \( ||φ_H|| \geq \text{Vol}(M, ω)^{-1} |\text{Cal}(φ_H)| \) on the original Hofer’s norm. We prove Proposition 1.6 in Section 2.

Sikorav [Si] (see also subsection 5.6 of [HZ]) proved that the original Hofer’s norm \( ||·|| \) on the Euclidean space is stably bounded i.e. \( \lim_{k → ∞} \frac{||φ^k||}{k} = \frac{1}{1/ω^n} \).
0 for any Hamiltonian diffeomorphism $\phi$ on $\mathbb{R}^{2n}$. Thus Proposition 1.6 implies the following corollary.

**Corollary 1.7.** Let $U$ be an open subset of $\mathbb{R}^{2n}$ with $\text{Vol}(U, \omega_0) < \infty$. Then the fragmented Hofer’s norm $\| \cdot \|_U : \text{Ham}(\mathbb{R}^{2n}, \omega_0) \to \mathbb{R}$ is not equivalent to the original Hofer’s norm $\| \cdot \|$. The author does not know whether the fragmented Hofer’s norm $\| \cdot \|_U$ is equivalent to the original Hofer’s norm $\| \cdot \|$ on $\text{Ham}(\mathbb{R}^{2n}, \omega_0)$ when $U = \{ |x_1|^2 + |y_1|^2 < 1 \}$ and $n \geq 2$.

**Definition 1.8.** Let $(M, \omega)$ be a symplectic manifold and $\mathcal{U} = \{ U_i \}_{i=1,2 \ldots}$ a sequence of open subsets in $M$. $\mathcal{U}$ has uniformly bounded fragmented Hofer’s geometry (UBFH) if there are a non-trivial Hamiltonian diffeomorphism $\phi$ and a positive number $C$ such that $\| \phi \|_U < C$ for any $i$.

If $(M, \omega)$ is an exact symplectic manifold or a closed symplectic manifold, UBFH naturally induces a stronger property.

**Proposition 1.9.** Let $(M, \omega)$ be a symplectic manifold and $\mathcal{U} = \{ U_i \}_{i=1,2 \ldots}$ a sequence of open subsets in $M$ with $\lim_{i \to \infty} \text{Vol}(U_i, \omega) = 0$ and UBFH.

- Let $(M, \omega)$ be an exact one. Then, for any Hamiltonian diffeomorphism $\phi$, there exists a positive number $C_\phi$ such that $\| \phi \|_U < C_\phi$ for any $i$ if and only if $\text{Cal}(\phi) = 0$.
- Let $(M, \omega)$ be a closed one. Then, for any Hamiltonian diffeomorphism $\phi$, there exists a positive number $C_\phi$ such that $\| \phi \|_U < C_\phi$ for any $i$.

**Proof.** Define a subset $N_\mathcal{U}$ of $\text{Ham}(M, \omega)$ by

$$N_\mathcal{U} = \{ \phi \in \text{Ham}(M, \omega) ; \exists C_\phi > 0 \text{ such that } \forall i, \| \phi \|_U < C_\phi \}.$$  

Since $N_\mathcal{U}$ is a conjugation-invariant norm, $N_\mathcal{U}$ is a normal subgroup of the group $\text{Ham}(M, \omega)$. Since $\mathcal{U}$ has UBFH, $N_\mathcal{U}$ is non-trivial.

Let $(M, \omega)$ be an exact symplectic manifold. Then, by Proposition 1.6 and $\lim_{i \to \infty} \text{Vol}(U_i, \omega) = 0$, $N_\mathcal{U} \subset \text{Ker(Cal)}$. Thus, since $\text{Ker(Cal)}$ is a simple group ([B]), $N_\mathcal{U} = \text{Ker(Cal)}$.

Let $(M, \omega)$ be a closed symplectic manifold. Then, since $\text{Ham}(M, \omega)$ is a simple group ([B]), $N_\mathcal{U} = \text{Ham}(M, \omega)$.

Any symplectic manifold $(M, \omega)$ admits a sequence $\mathcal{U} = \{ U_i \}_{i=1,2 \ldots}$ of open subsets in $M$ with $\lim_{i \to \infty} \text{Vol}(U_i, \omega) = 0$ and UBFH. To construct such a sequence, we introduce some notions.
For a positive number \( r \), let \( Q_r \) be a cube defined by
\[
Q_r = \{(x, y) \in \mathbb{R}^{2n}; |x_1|^2 + \cdots + |x_n|^2 + |y_1|^2 + \cdots + |y_n|^2 < r^2\}.
\]

For a positive integer \( l \) and a subset \( X \) of \( \mathbb{R}^{2n} \), let \( \partial_l X \) denote the \( 1/l \)-neighborhood of the boundary \( \partial X \) of \( X \) in \( \mathbb{R}^{2n} \). Then we obtain the following theorem.

**Theorem 1.10.** Let \((M, \omega)\) be a symplectic manifold and \( \iota: Q_{5r} \to M \) a symplectic embedding. Let \( W_l \) denote the open subset \( \iota(\partial_l Q_r) \) of \( M \) and \( || \cdot ||_{W_l} \) denote the norm \( || \cdot ||_{W_l} \). There exists a positive constant \( C \) such that
\[
||[\phi_F, \phi_G]||_{l} < C
\]
for any Hamiltonian functions \( F, G: M \times [0, 1] \to \mathbb{R} \) whose support is in \( \iota(Q_r) \) and any positive integer \( l \). In particular, \( \{W_l\}_{l=1,2,\ldots} \) has UBFH.

We prove Theorem 1.10 in Section 3. The author does not know whether the sequence \( \{\iota(Q_{r/l})\}_{l=1,2,\ldots} \) has UBFH or not.

Many researchers have studied functionals \( c: C^\infty_c(M \times [0, 1]) \to \mathbb{R} \) which are called spectral invariants or action selectors ([V], [HZ], [Sc], [FS], [O], [EnP] and [FOOO]). We obtain a lower bound of fragmented Hofer’s norm by spectral invariants.

**Proposition 1.11.** Let \((M, \omega)\) be an exact symplectic manifold. Assume that a functional \( c: C^\infty_c(M \times [0, 1]) \to \mathbb{R} \) satisfies the following conditions.

1. **invariance:** Assume that Hamiltonian functions \( F \) and \( G: M \times [0, 1] \to \mathbb{R} \) satisfy \( \phi_F^1 = \phi_G^1 \). Then \( c(a, F) = c(a, G) \).
2. **triangle inequality:** \( c(F \ast G) \leq c(F) + c(G) \) for any Hamiltonian functions \( F, G: M \times [0, 1] \to \mathbb{R} \). Here \( F \ast G \) is the Hamiltonian function defined by \( (F \ast G)(x, t) = F(x, t) + G((\phi_F^1)^{-1}(x), t) \) whose Hamiltonian isotopy is \( \{\phi_F^t \phi_G^t\} \).
3. **stability:** \( -\int_0^1 \max_M (F_t - G_t) dt \leq c(F) - c(G) \leq -\int_0^1 \min_M (F_t - G_t) dt \) for any Hamiltonian functions \( F, G: M \times [0, 1] \to \mathbb{R} \).
4. **conjugation invariance:** \( c(H \circ \phi) = c(H) \) for any Hamiltonian function \( H: M \times [0, 1] \to \mathbb{R} \) and any Hamiltonian diffeomorphism \( \phi \).
5. \( c(0) = 0 \).

Then, for any Hamiltonian function \( H: M \times [0, 1] \to \mathbb{R} \),
\[
c(H) + \text{Vol}(U, \omega)^{-1} \text{Cal}(\phi_H) \leq ||\phi_H||_U.
\]

It is known that
\[
c(H) + \text{Vol}(M, \omega)^{-1} \text{Cal}(\phi_H) \leq ||\phi_H||,
\]
on the original Hofer’s norm. We prove Proposition 1.11 in Section 4.
Example 1.12. Frauenfelder and Schlenk ([FS]) proved that there exists a spectral invariant $c: C_c^\infty(M \times [0, 1]) \to \mathbb{R}$ which satisfies the conditions of Proposition 1.11 if $(M, \omega)$ is an exact compact convex symplectic manifold.

The most general construction of spectral invariants is Oh’s one ([O]) and its bulk-deformation ([FOOO]). However, Oh’s spectral invariant does not satisfy the above condition (1) in general. For conditions to satisfy the above condition (1), refer [Sc], [EnP], [M] and [Sc].

Remark 1.13. There are other conventions of spectral invariants. For instance, under the convention of [EnP], spectral invariants satisfy the following condition instead of the above condition (4).

\[(4)\': \int_0^1 \min_M (F_t - G_t) dt \leq c(F) - c(G) \leq \int_0^1 \max_M (F_t - G_t) dt \quad \text{for any Hamiltonian functions } F, G: M \times [0, 1] \to \mathbb{R}.\]

If a functional $c: C_c^\infty(M \times [0, 1]) \to \mathbb{R}$ satisfies the conditions (1), (2), (3), (4)', and (5), the following inequality holds instead of Proposition 1.11

\[c(H) - \text{Vol}(U, \omega)^{-1} \text{Cal}(\phi_H) \leq ||\phi_H||_U.\]

For the proof of this inequality, see Remark 4.4.

Remark 1.14. In many cases, asymptotic spectral invariants satisfy the conditions of Proposition 1.11. For instance, Monzner-Vichery-Zapolsky’s spectral invariant ([MVZ]) does not satisfy the condition (4), but the asymptotization of their spectral invariant satisfies the condition (4).

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2. Proof of Theorem 1.4 and Proposition 1.6

To prove Theorem 1.4, we use the following lemma.

Lemma 2.1. Let $(\hat{M}, \omega)$ be a compact symplectic manifold which can have a smooth boundary and $M$ the interior of $\hat{M}$. For any open subset $U$ of $M$, there is a positive number $C_U$ such that $||\phi_H||_U \leq C_U ||H||$ for any $C^1$-small Hamiltonian function $H: M \times [0, 1] \to \mathbb{R}$. 
For a subset $X$ of $\hat{M}$, the topological closure of $X$ is denoted by $\bar{X}$.

**Proof.** Since $\hat{M}$ is compact, we can take finite open coverings $\mathcal{V} = \{V_i\}_{i=1 \ldots l}$ and $\hat{\mathcal{V}} = \{\hat{V}_i\}_{i=1 \ldots l}$ of $\hat{M}$ such that

- $\hat{V}_i \subset \hat{V}_i$,
- for any compact subset $K$ of $M$ and $i$, there exists a Hamiltonian diffeomorphism $\phi_{i,K}$ with compact support in $M$ such that $\phi_{i,K}(\hat{V}_i \cap K) \subset U$.

For $j = 0, \ldots, l$, let $W_j$ and $\hat{W}_j$ denote $\bigcup_{i=1}^{j} V_i$ and $\bigcup_{i=1}^{j} \hat{V}_i$, respectively. Now, we define $W_0$ and $\hat{W}_0$ as the empty set $\emptyset$.

Since $(\hat{W}_{j+1} \setminus \hat{V}_{j+1}) \cap (V_{j+1} \setminus W_j) = \emptyset$, we can take smooth functions $\rho_j: M \to [0, 1]$ $(j = 0, \ldots, l - 1)$ such that

- $\rho_j = 1$ on some open neighborhood of $\overline{(W_{j+1} \setminus \hat{V}_{j+1})}$,
- $\rho_j = 0$ on some open neighborhood of $(V_{j+1} \setminus \hat{W}_{j})$.

Let $\chi_l$ be a constant function 1 on $M$ and define functions $\chi_j: M \to [0, 1]$ $(j = l-1, \ldots, 0)$ inductively by $\chi_j = \rho_j \cdot \chi_{j+1}$. Then we define Hamiltonian functions $K_j$ $(j = 0, \ldots, l)$ and $L_j$ $(j = 0, \ldots, l)$ by

- $K^j(x, t) = \chi_j(x) \cdot H(x, t)$,
- $L^{j+1}(x, t) = -K^j(\phi_{K,j}^t(x), t) + K^{j+1}(\phi_{K,j}^t(x), t)$.

Note that $L^{j+1}$ generates the Hamiltonian diffeomorphism $\phi_{K,j}^{-1} \phi_{K,j+1}$ and thus $\phi_{K,j+1} = \phi_{K,j} \phi_{L,j+1}$. Since $K^l(x, t) = H(x, t)$ and $K^0(x, t) = 0$, $\phi_H = \phi_{K^l} = \phi_{K^l} \phi_{L^l} = \cdots = \phi_{K^0} \phi_{L^1} \cdots \phi_{L^j}$.

Now, we claim $\text{Supp}(L_j) \subset \overline{V_j}$. Fix a point $x$ in $M$ with $x \notin \overline{V_j}$. Note that $M \setminus \overline{V_j} \subset (\hat{W}_j \setminus \hat{V}_j) \cup (M \setminus (V_j \setminus \hat{W}_{j-1}))$ since $M \setminus \hat{W}_j \subset M \setminus (V_j \setminus \hat{W}_{j-1})$.

- Assume $x \in \hat{W}_j \setminus \hat{V}_j \subset \hat{W}_{j-1}$. Since $H$ is a $C^1$-small Hamiltonian function, $K^{j-1}$ is also $C^1$-small. Thus $\phi_{K,j-1}^t(\hat{W}_j \setminus \hat{V}_j) \subset \text{Supp}(1-\rho_{j-1})$ for any $t$ and therefore $\chi_{j-1}(\phi_{K,j-1}^t(x)) = \chi_j(\phi_{K,j-1}^t(x))$ for any $t$. Hence $L^j(x, t) = 0$ for any $t \in [0, 1]$.
- Assume $x \notin V_j \setminus \hat{W}_{j-1}$. Since $K^{j-1}$ is also $C^1$-small, $\phi_{K,j-1}^t(V_j \setminus \hat{W}_{j-1}) \subset \text{Supp}(\rho_{j-1})$. Thus $\chi_{j-1}(\phi_{K,j-1}^t(x)) = \chi_j(\phi_{K,j-1}^t(x)) = 0$ and therefore $L^j(x, t) = 0$ for any $t \in [0, 1]$.

Hence $L^j(x, t) = 0$ for any $x \notin \overline{V_j}$ and any $t \in [0, 1]$ and we complete the proof of $\text{Supp}(L_j) \subset \overline{V_j}$.

By $\text{Supp}(L_j) \subset \overline{V_j}$ and the first condition of $\hat{V}$, there exists a Hamiltonian diffeomorphism $h_j$ such that $h_j(\text{Supp}(L_j)) \subset U$. Therefore $||\phi_{L^j}||_U =}$
\[ \|h_j \phi_Lg_j^{-1}\| \leq \|Lj\| \leq \|H\| \text{ and} \]
\[ \|\phi_H\| U \leq \|\phi_L\| U + \cdots + \|\phi_L\| U \leq l \cdot \|H\|. \]

\[ \|\phi_H\| U \leq \|\phi_H\| U + \cdots + \|\phi_H\| U \leq C \cdot \|H\|. \]

Proof of Theorem 1.4. Let \( H : M \times [0, 1] \rightarrow \mathbb{R} \) be a Hamiltonian function. For positive integers \( N \) and \( n \) with \( n \leq N \), we define a Hamiltonian function \( H^{n,N} : M \times [0, 1] \rightarrow \mathbb{R} \) by
\[ H^{n,N}(x, t) = \frac{1}{N} H(x, \frac{n - 1 + t}{N}). \]
Then \( \phi_{H^{n,N}} = \phi_{H^{n,N}|_{M \times \{(n-1)/N,n/N\}}} \) holds. If \( N \) is sufficiently large, then \( H^{n,N} \) is sufficiently \( C^1 \)-small to satisfy the assumption of Lemma 2.1 for any \( n = 1, \ldots, N \). Thus Lemma 2.1 implies
\[ \|\phi_H\| U \leq \|\phi_{H^{1,N}}\| U + \cdots + \|\phi_{H^{n,N}}\| U \leq C \cdot \|H\|. \]

Proof of Proposition 1.6. Let \( F^1, \ldots, F^k \) and \( h_1, \ldots, h_k \) be Hamiltonian functions and Hamiltonian diffeomorphisms such that \( \text{Supp}(F^i) \subset U \) for any \( i \) and \( \phi = h_1 \phi_{F^1}h_1^{-1} \cdots h_k \phi_{F^k}h_k^{-1} \), respectively. By the definition of the Calabi homomorphism, \( |\text{Cal}(\phi_{F^i})| \leq \text{Vol}(U, \omega)||F^i|| \) for any \( i \). Since the Calabi homomorphism is a homomorphism,
\[ |\text{Cal}(\phi)| = |\text{Cal}(\phi_{F^1})| + \cdots + |\text{Cal}(\phi_{F^k})| \leq \text{Vol}(U, \omega)||F^1|| + \cdots + \text{Vol}(U, \omega)||F^k|| = \text{Vol}(U, \omega)(||F^1|| + \cdots + ||F^k||). \]

By taking the infimum, we complete the proof.

3. Uniformly bounded fragmented Hofer’s geometry

To prove Theorem 1.10, we use the beautiful argument by Eliashberg and Polterovich ([EP], see also Section 2 of [P]).

Proof of Theorem 1.10. For any \( 0 \leq t \leq 1 \), define a subset \( Q^t_r \) of \( Q_3r \) by
\[ Q^t_r = \{(x, y) \in \mathbb{R}^{2n}; (x_1 + t, x_2, \ldots, x_n, y) \in Q_r\}. \]
For a sufficiently large integer $l$, we can take a Hamiltonian function $H: Q_{5r} \times [0, 1] \to \mathbb{R}$ such that

- $H(x, y, t) = 3ry_1$ if $(x, y) \in \partial_3 Q_r^{3r}$,
- $\text{Supp}H_t \subset \partial_2 Q_r^{3r}$,
- $-4r^2 \leq H(x, y, t) \leq 4r^2$.

Since $\text{Supp}H_t \subset \partial_2 Q_r^{3r}$ for any $t$, $\bigcup_{i=0}^{[\frac{3r}{2l}] - 1} \text{Supp}H_t \subset \partial_2 Q_r^{\frac{3r(2i+1)}{2l}}$ holds for any $i = 0, \ldots, l - 1$. Let $H^i$ be the restriction of $H: M \times [0, 1] \to \mathbb{R}$ to $M \times \left[\frac{2i}{2l}, \frac{2i+2}{2l}\right]$ for $i = 0, \ldots, l - 1$. Then $\phi_H = \phi_{H^{l-1}} \cdots \phi_{H^0}$. Since $-4r^2 \leq H(x, y, t) \leq 4r^2$, $||H^i|| \leq \frac{8r^2}{l}$. Since $\partial_t Q_r$ and $\partial_2 Q_r$ are conjugate by a Hamiltonian diffeomorphism on $Q_{5r}$,

$$||\phi_H||_{\partial_2 Q_r} \leq ||H^0|| + \cdots + ||H^{l-1}|| \leq 8r^2.$$  

Since $H(x, y, t) = 3ry_1$ for any $(x, y) \in \partial_3 Q_r^{3r}$, $\phi^t_H(\partial Q_r) = \partial Q_r^{3r}$. In particular, $\phi_H(\partial Q_r) \cap \partial Q_r = \emptyset$. Thus $\phi_H(Q_r) \cap Q_r = \emptyset$. Now, we regard $H$ as a Hamiltonian function on $M$ through $\iota$. Then, for any two Hamiltonian functions $F, G: \iota(Q_r) \times [0, 1] \to \mathbb{R}$ on $\iota(Q_r)$, $[\phi_F, \phi_G] = [\phi_F, [\phi_G, \phi_H]]$ holds. Since $||\cdot||_l$ is a conjugation-invariant norm,

$$||[\phi_F, [\phi_G, \phi_H]]_l \leq ||\phi_F[\phi_G, \phi_H](\phi_F)^{-1}||_l + ||[\phi_G, \phi_H]^{-1}||_l$$

$$= 2||[\phi_G, \phi_H]||_l \leq 2(||\phi_G\phi_H\phi_F^{-1}||_l + ||(\phi_H)^{-1}||_l)$$

$$= 4||\phi_H||_l \leq 4||\phi_H||_{\partial_2 Q_r} \leq 32r^2.$$  

Thus $||[\phi_F, \phi_G]||_l = ||[\phi_F, [\phi_G, \phi_H]]||_l \leq 32r^2$.  

4. Lower bound by spectral invariants

Let $(M, \omega)$ be an exact symplectic manifold, $c: C_0^\infty(M \times [0, 1]) \to \mathbb{R}$ a functional satisfying the conditions of Proposition 1.11 and $U$ an open subset of $M$. We define functionals $c^U: C_0^\infty(U \times [0, 1]) \to \mathbb{R}$ and $c_U: \text{Ham}(M, \omega) \to \mathbb{R} \cup \{-\infty\}$ by

$$c^U(H) = c(H) + \text{Vol}(U, \omega)^{-1} \cdot \text{Cal}(\phi_H),$$

$$c_U(\phi) = \inf\{c^U(F^1) + \cdots + c^U(F^k)\},$$

where the infimum is taken over Hamiltonian functions $F^1, \ldots, F^k: M \times [0, 1] \to \mathbb{R}$ and Hamiltonian diffeomorphisms $h_1, \ldots, h_k$ such that $\text{Supp}(F^i) \subset$
U for any i and \( \phi = h_1 \phi_{F_1} h_1^{-1} \cdots h_k \phi_{F_k} h_k^{-1} \). To prove Proposition 1.11, we use the following lemmas and proposition.

**Lemma 4.1.** For any Hamiltonian function \( H : M \times [0, 1] \to \mathbb{R} \),

\[
    c^U(H) \leq ||H||.
\]

**Proof.** Since \( \int_U (H_t - \text{Vol}(U, \omega))^{-1} \cdot \int_U H_t \omega^n \omega^n = 0 \) for any \( t \),

\[
    \min_U H_t - \text{Vol}(U, \omega)^{-1} \cdot \int_U H_t \omega^n \leq 0 \leq \max_U H_t - \text{Vol}(U, \omega)^{-1} \cdot \int_U H_t \omega^n.
\]

By the conditions (3) and (5) of Proposition 1.11,

\[
    c^U(H) = c(H) + \text{Vol}(U, \omega)^{-1} \cdot \text{Cal}(\phi_H)
\]

\[
    \leq - \int_0^1 \min_U H_t dt + c(0) + \text{Vol}(U, \omega)^{-1} \cdot \text{Cal}(\phi_H)
\]

\[
    = - \int_0^1 (\min_U H_t - \text{Vol}(U, \omega)^{-1} \cdot \int_U H_t \omega^n) dt + c(0)
\]

\[
    \leq - \int_0^1 (\min_U H_t - \text{Vol}(U, \omega)^{-1} \cdot \int_U H_t \omega^n) dt
\]

\[
    + \int_0^1 (\max_U H_t - \text{Vol}(U, \omega)^{-1} \cdot \int_U H_t \omega^n) dt + c(0)
\]

\[
    = \int_0^1 (\max_U H_t - \min H_t) dt.
\]

\( \square \)

**Lemma 4.2.** For any Hamiltonian diffeomorphism \( \phi \),

\[
    c_U(\phi) \leq ||\phi||_U.
\]

**Proof.** Let \( F^1, \ldots, F^k \) and \( h_1, \ldots, h_k \) be Hamiltonian functions and Hamiltonian diffeomorphisms such that \( \phi = h_1 \phi_{F_1} h_1^{-1} \cdots h_k \phi_{F_k} h_k^{-1} \) and \( \text{Supp}(F^i) \subset U \) for any \( i \), respectively. By Lemma 4.1,

\[
    c^U(F^1) + \cdots + c^U(F^k) \leq ||F^1|| + \cdots + ||F^k||.
\]

By taking the infimum, we prove \( c_U(\phi) \leq ||\phi||_U \).

\( \square \)

**Proposition 4.3.** For any Hamiltonian function \( H : M \times [0, 1] \to \mathbb{R} \),

\[
    c(H) + \text{Vol}(M, \omega)^{-1} \text{Cal}(\phi_H) \leq c_U(\phi_H).
\]
Proof. Let $F^1, \ldots, F^k$ and $h_1, \ldots, h_k$ be Hamiltonian functions and Hamiltonian diffeomorphisms such that $\text{Supp}(F^i) \subset U$ for any $i$ and $\phi_H = h_1 \phi_{F^1} h_1^{-1} \cdots h_k \phi_{F^k} h_k^{-1}$, respectively. Define a Hamiltonian function $F: M \times [0, 1] \to \mathbb{R}$ by $F(x, t) = F^1(h_1^{-1} x, t) \# \cdots \# F^k(h_k^{-1} x, t)$. Then $\phi_F = \phi_H$. By the conditions (2) and (4) of Proposition 1.11,

$$c(F) \leq c(F^1 \circ h^{-1}) + \cdots + c(F^k \circ h^{-1})$$

$$= c(F^1) + \cdots + c(F^k).$$

Since $\phi_F = \phi_H$, by the condition (1) of Proposition 1.11 and well-definedness of the Calabi homomorphism, $c(F) = c(H)$ and $\text{Cal}(\phi_F) = \text{Cal}(\phi_H)$. Thus, since the Calabi homomorphism is a homomorphism,

$$c(H) + \text{Vol}(U, \omega)^{-1} \text{Cal}(\phi_H)$$

$$\leq c(F^1) + \cdots + c(F^k) + \text{Vol}(U, \omega)^{-1} (\text{Cal}(\phi_{F^1}) + \cdots + \text{Cal}(\phi_{F^k}))$$

$$= (c(F^1) + \text{Vol}(U, \omega)^{-1} \text{Cal}(\phi_{F^1})) + \cdots + (c(F^k) + \text{Vol}(U, \omega)^{-1} \text{Cal}(\phi_{F^k}))$$

$$= c^U(F^1) + \cdots + c^U(F^k).$$

By taking the infimum, we complete the proof. \qed

Proposition 1.11 immediately follows from Lemma 4.2 and Proposition 4.3.

**Remark 4.4.** If we use the convention in Remark 1.13, we should define $c^U$ by

$$c^U(H) = c(H) - \text{Vol}(U, \omega)^{-1} \cdot \text{Cal}(\phi_H).$$

Then our argument goes well similarly and we prove the inequality in Remark 1.13.

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