Relativistic diffusion of massless particles

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Abstract

We obtain a limit when mass tends to zero of the relativistic diffusion of Schay and Dudley. The diffusion process has the log-normal distribution. We discuss Langevin stochastic differential equations leading to an equilibrium distribution. We show that for the Jüttner equilibrium distribution the relativistic diffusion is a linear approximation to the Kompaneetz equation describing a photon diffusion in an electron gas. The stochastic equation corresponding to the Jüttner distribution is explicitly soluble. We relate the relativistic diffusion to imaginary time quantum mechanics. Some astrophysical applications (including the Sunyaev-Zeldovich effect) are briefly discussed.

1 Introduction

An extension of the idea of diffusion to relativistic theories poses some problems of conceptual as well as physical character (for a discussion of these problems and different approaches to solve them see ([1][2][3][4][5][6]; for a review and further references see [7][8]). In [9] we developed the diffusion theory of Schay [1] and Dudley [2] of massive relativistic particles discussing friction terms leading to an equilibrium. In refs.[1][2] the relativistic Brownian motion is uniquely defined by the requirement that this is the diffusion whose four momentum stays on the mass-shell. The formulas become singular as \( m \to 0 \).

The photon is a massless particle which could experience a diffusive behavior. We could think of a multiple scattering of the photon in a charged gas as resembling the Brownian motion. The number of such encounters must be large if the approximation is to be feasible. This happens in astrophysical applications when the photon travels over large distances in intergalactic space filled by a ionized gas. The diffusive approximation could also apply to the early stages of the Big Bang and to the photon motion inside the plasma shell of the
compact stellar objects (stars, black holes) The relativistic Boltzmann equation is the standard tool in a study of the evolution of the photon in a gas of charged particles [10][11]. The diffusion approximation to the Boltzmann equation has been applied in many models of the transport phenomena [12]. In the context of the photon propagation such a diffusion approximation is known as the (non-linear) Kompaneetz equation [13][14][10]. The Kompaneetz equation arises as an approximation to the Boltzmann equation describing an evolution of the photon in a gas of electrons. In such a case the evolution of a beam of photons is disturbed by Compton scattering and Bremsstrahlung. These are the processes widely studied in astrophysics [10][15][16].

In this paper we investigate a diffusion of massless particles without spin in the framework of classical relativistic diffusion theory. We show that there is a massless counterpart of Schay [1] and Dudley [2] relativistic diffusion (the formula for a diffusion of a massless particle has been also derived in [17] as a continuum limit of discrete dynamics of space-time, see also [18]). We derive the diffusion as a classical limit of quantum mechanics (the Boltzmann equation usually also incorporates quantum scattering processes). Such a diffusion process has no limit as time grows to infinity. We discuss (sec.3) drags which lead to an equilibration of the process. Subsequently, we show in sec.4 that if the Kompaneetz equation is expanded around its equilibrium solution (Jüttner or Bose-Einstein) and non-linear correction terms are neglected then the resulting diffusion equation is the same as our relativistic diffusion with the friction leading to the same equilibrium. In secs.5 and 6 we discuss the dynamics in an affine time parameter. It is shown that the dynamics (in particular, the approach to the equilibrium) can be well controlled by methods of Langevin theory (sec.5) or quantum mechanics (sec.6). In the final sec.7 we discuss the problem of time in the relativistic diffusion. We show that it can be approached in a similar way as in the deterministic relativistic dynamics. The affine parameter on the trajectory can be replaced by the (random) laboratory time. As a result from a solution of the diffusion equation in the affine time we can obtain a solution of the transport equation involving only the physical variables (in particular, the laboratory time).

2 Quantum origin of the relativistic diffusion equation

In our earlier paper [9], following Schay[1] and Dudley[2], we have defined the generator of the relativistic diffusion as the second order differential operator on the mass-shell $H_m$

$$p^2 = p_0(\tau)^2 - p_1(\tau)^2 - p_2(\tau)^2 - p_3(\tau)^2 = m^2c^2$$  \hspace{1cm} (1)

The generator is singular at $m = 0$. For this reason in order to define a diffusion of massless particles we propose another method. A diffusion should be con-
sidered as an approximation of a complex multi-particle dynamics. In our case
the relativistic dynamics. It is not clear how to describe many particle systems
in a relativistic way. Quantum field theory leads to a relativistic description of
scattering processes. However, its reduction to particle dynamics has not been
explored. The photon as a massless particle is a typically relativistic object. At
the same time this is a quantum system with an internal angular momentum
(spin). We should describe it by a relativistic wave function. In this paper we
neglect the spin. In [19] we discuss the diffusion of particles with a spin (the
helicity in the massless case). It is shown in [19] that the dissipative part of the
evolution does not depend on the helicity if \( m = 0 \). Hence, the neglect of spin
in this paper is justified.

The explicitly Lorentz invariant description of dynamics must in fact be
static. It has to be described in terms of space-time trajectories. The laboratory
time \( x^0 \) is one of the coordinates. Hence, the evolution as a function of this
coordinate should be inferred at the later stage from a study of the set of paths.
In the massive case the position of a particle on the space-time path is most
conveniently described by its own time : the proper time. If a particle is massless
then the notion of the proper time as a time of an observer moving with the
particle does not make sense. Nevertheless, it is still convenient to introduce an
affine parameter \( \tau \) describing the fictitious motion along the particle trajectory
(we shall still call it a proper time although no observer can move together with
the particle).

The proper time as an additional parameter in a description of a relativistic
quantum particle has been introduced in [20][21] (in quantum theory the relation
to the time of an observer moving with the particle is lost anyhow). It can be
treated as a convenient tool in a formulation of relativistic quantum mechanics
and quantum field theory. The equation for a massless free field is

\[
\imath \partial_\tau \phi = (\partial_0^2 - \Delta)\phi
\]

where \( \partial_0^2 - \Delta \) is the wave operator.

We may consider a perturbation of (2) by some other fields (an environment)
so that eq.(2) remains invariant under the Lorentz group. When we average over
the environment then in the Markovian approximation the system will be de-
scribed by the master equation for the density matrix \( \rho \). A preservation of the
trace and positivity of the density matrix requires that the master equation
should have the Lindblad form [22]. Then, the Lorentz invariance and an as-
sumption that the Lindblad generators are built from the Lorentz generators
leads in the simplest (linear) case to the equation

\[
i\partial_\tau \rho = \{\partial_0^2 - \Delta, \rho\} + \frac{1}{4} \gamma^2 [M_{\mu\nu}, [M^{\mu\nu}, \rho]]
\]  

where \( M_{\mu\nu} \) are the generators of the algebra of the Lorentz group. Eq.(3) can
be rewritten as an equation for the Wigner function [23] (proper time equations
for the Wigner function in relativistic quantum mechanics appear in [24] and in quantum field theory in [25][26])

$$\frac{\partial \tau}{\partial t} W = p^\mu \frac{\partial}{\partial p^\mu} W + \frac{1}{4} \gamma^2 L_{\mu\nu} L^{\mu\nu} W \quad (4)$$

where

$$L_{\mu\nu} = -i(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}) \quad (5)$$

is a realization of the algebra of the Lorentz group in the momentum space.

It can be checked that the diffusion generator of [9] can be expressed as

$$\triangle H = \frac{1}{2} L_{\mu\nu} L^{\mu\nu} \quad (6)$$

The derivation of the relativistic diffusion equation from quantum mechanics is similar to a derivation of the Boltzmann equation from quantum mechanics in [27]. Eq.(4) makes sense in the massless case. Choosing $p$ as coordinates on $H_0$ we obtain

$$\triangle H = p_j p_k \partial^j \partial^k + 3 p^k \partial_k \quad (7)$$

where $k = 1, 2, 3$ and $\partial^j = \frac{\partial}{\partial p_j}$. We note that the generator (7) is the limit $m \to 0$ of $m^2 c^2 \triangle H$ of ref.[9].

The relativistic diffusion equation in the momentum space (a relativistic analog of the Brownian motion) reads

$$\frac{\partial \tau}{\partial t} \phi_\tau = \frac{1}{2} \gamma^2 \triangle H \phi_\tau \quad (8)$$

$\gamma^2$ is a diffusion constant which has the dimension of $\tau^{-1}$ (our choice in the next section: momentum divided by length).

### 3 An approach to the equilibrium

We add a drag term

$$X = Y_j \frac{\partial}{\partial p_j} + \gamma^2 R_j \frac{\partial}{\partial p_j} + p_\mu \frac{\partial}{\partial x_\mu} \quad (9)$$

to the diffusion (8). The vector field (9) defines a perturbation of the relativistic dynamical system (at $\gamma = 0$)

$$\frac{dp_j}{d\tau} = Y_j \quad (10)$$

$$\frac{dx^\mu}{d\tau} = p^\mu \quad (11)$$
where from eq.(1) (for \(m = 0\)) we have \(p^0 = |p|\). The zero component of eq.(11) determines a relation between the parameter \(\tau\) on the trajectory and the laboratory time \(x^0\). Let

\[
G = \frac{1}{2} \gamma^2 \Delta H + X
\]  

(12)

A solution of the diffusion equation

\[
\partial_\tau \phi_\tau = G \phi_\tau
\]

(13)

with the initial condition \(\phi(x, p)\) defines a distribution of random paths (see sec.5) starting from the point \((x, p)\) of the phase space. At \(\gamma = 0\) we have a distribution of deterministic trajectories. The dynamics in the laboratory time \(x^0\) must be derived from the static picture of space-time paths (see the discussion of relativistic statistical dynamics in [3]).

We are interested in the behavior of the relativistic dynamics at large values of \(\tau\). In particular, whether the diffusion generated by \(G\) can have an equilibrium limit. We say that the probability distribution \(dxdp \Phi\) is the invariant measure for the diffusion process (see [28]) if

\[
\int dxdp \phi_\tau(p, x) \phi_\tau(p, x) = \text{const}
\]

(14)

where

\[
\partial_\tau \Phi_\tau = G^* \Phi_\tau
\]

(15)

and \(G^*\) is the adjoint of \(G\) in \(L^2(dpdx)\) (the Lebesgue measure \(dp\) is not Lorentz invariant, the factor \(|p|^{-1}\) necessary for the Lorentz invariance is contained in \(\Phi\), see [9]). Hence, the invariant measure \(dxdp \Phi_R\) is the solution of the transport equation \(G^* \Phi_R = 0\). Explicitly,

\[
\frac{1}{2} \gamma^2 \partial^i p_j p_k \Phi_R - \frac{3}{2} \gamma^2 \partial^j p_j \Phi_R - \gamma^2 \partial^j R_j \Phi_R = p_\mu \partial_\mu \Phi_R
\]

(16)

(derivatives over the space-time coordinates will have an index \(x\)). We denote by \(\Phi_{ER}\) the \(x\)-independent solution of eq.(16). Then, \(R\) can be expressed in terms of \(\Phi_{ER}\)

\[
R_j = \frac{1}{2} p_j + \frac{1}{2} p_j p_k \partial^k \ln \Phi_{ER}
\]

(17)

We assume that \(\Phi_{ER}\) is a function of the energy \(p_0 c\) multiplied by a constant \(\beta\) of the dimension inverse to the dimension of the energy \(\beta = \frac{1}{\mu} \) in the conventional notation. In such a case from eq.(17)

\[
R_j = \frac{1}{2} p_j + \frac{1}{2} \beta c p_j |p| (\ln \Phi_{ER})'(c\beta |p|)
\]

(18)
Now, eq.(13) reads
\[ \partial_\tau \phi_\tau = \frac{1}{2} \gamma^2 \triangle_\mathcal{H} \phi_\tau + \frac{1}{2} \gamma^2 p_j (1 + \beta c \text{p}) (\ln \Phi_{ER})' \partial^j \phi_\tau \] (19)
and eq.(16)
\[ \frac{1}{2} \gamma^2 \partial^j \partial^k p_j p_k \Phi_R - 2 \gamma^2 \partial^j p_j \Phi_R - \frac{1}{2} \gamma^2 \beta c \partial^j p_j (\ln \Phi_{ER})' \Phi_R = \mu \partial_\mu x \Phi_R \] (20)

Let us consider an initial probability distribution of the form \( \Phi = \Phi_{ER} \Psi \). Then, from eq.(15) we obtain its evolution
\[ \Phi_\tau = \Phi_{ER} \Psi_\tau \] (21)
as a solution of the equation (we do not make any assumptions on the form of \( \Phi_{ER} \) here)
\[ \partial_\tau \Psi_\tau = \frac{1}{2} \gamma^2 p_j p_k \partial^j \partial^k \Psi_\tau + 2 \gamma^2 p_j \partial^j \Psi_\tau + \frac{1}{2} \gamma^2 p_j p_k (\partial^j \ln \Phi_{ER}) \partial^k \Psi_\tau - \mu \partial_\mu \Psi_\tau \] (22)

4 Spherical coordinates and the Kompaneetz equation

In this section we restrict ourselves to the diffusion in momentum space. The diffusion generator(7) is degenerate. This is easy to see if we express it in the spherical coordinates on \( \mathcal{H}_0 \)
\[ p_0 = r \]
\[ p_1 = r \cos \phi \sin \theta, \quad p_2 = r \sin \phi \sin \theta, \quad p_3 = r \cos \theta. \]
In these coordinates
\[ \triangle_\mathcal{H} = r^2 \frac{\partial^2}{\partial r^2} + 3r \frac{\partial}{\partial r} = \frac{\partial^2}{\partial u^2} + 2 \frac{\partial}{\partial u} \] (24)
where we introduced an exponential parametrization of \( r \)
\[ r = \exp u \] (25)
Here, \( u \) varies over the whole real axis. It follows that the three-dimensional diffusion of massive particles becomes one-dimensional in the limit \( m \to 0 \).

In the coordinates (23) it is sufficient if we restrict ourselves to the drags
\[ X = \gamma^2 R \frac{\partial}{\partial u} \] (26)
The diffusion (12) is generated by
\[ \mathcal{G} = \frac{\gamma^2}{2} (\partial_u^2 + 2 \partial_u) + \gamma^2 R \partial_u \] (27)
We may write for integrals of spherically symmetric functions

\[ dp = 4\pi du \exp(3u) \]  

(28)

Then

\[ G^* = e^{-3u}G^+e^{3u} \]  

(29)

where

\[ G^+ = \frac{2}{\gamma^2}(\partial_u^2 - 2\partial_u) - \gamma^2\partial_u R \]  

(30)

is the adjoint of \( G \) in \( L^2(du dx) \). The probability distribution evolves according to eq.(15). The invariant measure \( dx du \Phi_R \equiv dx du \Phi_I \) solves an analog of eq.(16)

\[ G^* \Phi_R = G^+ \Phi_I = 0 \]  

(31)

where

\[ \Phi_I = \exp(3u)\Phi_R \]  

(32)

We can express eq.(31) as an evolution equation in the laboratory time \( x^0 \)

\[
\frac{\partial}{\partial x^0} \Phi_I = \exp(-u)p \nabla_x \Phi_I + \frac{\gamma^2}{2} \exp(-u)(\partial_u^2 - 2\partial_u)\Phi_I - \gamma^2 \exp(-u)\partial_u(R\Phi_I)
\]  

(33)

where \( n = \exp(-u)p \) is a unit vector in the direction of motion (equations of this type describe a diffusive perturbation of the propagation of a massless particle with the velocity of light [29]).

The \( x^0 \) independent solution of eq.(33) is denoted \( \Phi_{EI} \). The drift \( R \) is related to \( \Phi_{EI} \)

\[ R = \frac{1}{2}(\partial_u \ln \Phi_{EI} - 2) \]  

(34)

If the diffusion (15) starts from an initial distribution

\[ \Phi_I = \Phi_{EI} \Psi_I \]  

(35)

then from eq.(15), using the equilibrium relation (34), we obtain the evolution of \( \Psi \)

\[ \partial_t \Psi_I = \frac{1}{2}\gamma^2 \partial_u^2 \Psi_I + \frac{1}{2}\gamma^2(\partial_u \ln \Phi_{EI})\partial_u \Psi_I \]  

(36)

We consider Jüttner equilibrium distribution [30] in \( L^2(du) \)

\[ \Phi_{EI}^J = r^3 \exp(-\beta cr) \equiv r^3 n_J \]  

(37)

Then, from eq.(34)

\[ R = -\frac{1}{2}(-1 + \beta c \exp(u)) \]  

(38)

The diffusion generator corresponding to the Jüttner distribution reads

\[ G = \frac{1}{2}\gamma^2 \left( \partial_u^2 + 3\partial_u - \beta c \exp(u)\partial_u \right) \]

\[ = \frac{1}{2}\gamma^2 \left( r^2 \partial_r^2 + 4r\partial_r - \beta cr^2 \partial_r \right) \]  

(39)
For the Bose-Einstein distribution (in Compton scattering the photon number is preserved; if an equilibrium is achieved, then the chemical potential $\mu \neq 0$, see [14])

$$\Phi^B_{EI} = r^3 \left( \exp(\beta(\mu + cr)) - 1 \right)^{-1} \equiv r^3 n_E(\mu) \quad (40)$$

we have

$$R = -\frac{1}{2} \gamma^2 \left( -1 + \beta c \exp(u) \left( 1 - \exp(-\beta \mu - \beta c \exp u) \right)^{-1} \right) \quad (41)$$

Eq.(36) for an evolution of $\Psi^I$ (with the drift (41) leading to the Bose-Einstein equilibrium distribution) in the $r$-coordinates reads

$$\partial_\tau \Psi^I = \frac{1}{2} \gamma^2 r^{-2} \left( \partial_r r^4 \partial_r \Psi^I + r^4 (\partial_r \ln n_E) \partial_r \Psi^I \right) \quad (42)$$

where

$$\partial_r \ln n_E = -\beta c (1 - \exp(-\beta \mu - \beta cr))^{-1} \quad (43)$$

In the low temperature limit (or for the Jüttner distribution with the generator (39)) we have in the diffusion equation (42)

$$\partial_\tau \ln n_E \to -\beta c$$

We compare the relativistic diffusion equation with the (non-linear) Kompaneetz equation usually written in the form [13]

$$\partial_\tau n = \kappa^2 x^{-2} \partial_x \left( x^4 (\partial_x n + n + n^2) \right) \quad (44)$$

Eqs.(42) and (44) coincide if the non-gradient and friction terms are neglected. Note that the form of the differential operators on the rhs of eqs.(42) and (44) follows from the conservation of probability for eq.(42) and conservation of the number of photons for the Kompaneetz equation, i.e.,

$$\partial_\tau \int dr r^2 \Phi^I = \partial_\tau \int dx x^2 n = 0$$

In order to explore the appearance of friction in eq.(44) we expand the photon distribution of the Kompaneetz equation around its equilibrium value in the same way as we expanded the distribution of the diffusion process (15) around the equilibrium in eq.(36). If we neglect $n^2$ on the rhs of eq.(44) then the rhs disappears (because $\partial_x n_J = -n_J$, with $x = \beta cr$) for the Jüttner distribution. Let

$$n = \exp(-x) \Psi$$

Then, neglecting the $n^2$ term in the Kompaneetz equation (44) we obtain (after an elementary rescaling $r \to \beta cr = x$, $\kappa^2 = \frac{1}{2} \gamma^2$) the diffusion equation (42) with $n_E \to n_J$. 

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In general, the Kompaneetz distribution $n$ equilibrates to $n_{E}$ (as $\partial_x n_{E} + n_{E} + n_{E}^2 = 0$ on the rhs of eq.(44)). Let us write the initial distribution in the form

$$n = n_{E} \Gamma$$

then the evolution of $\Gamma$ is determined by the equation

$$\partial_\tau \Gamma = \kappa^2 x^{-2} \left( \partial_x x^4 \partial_x \Gamma + x^4 (\partial_x \ln n_{E}) \partial_x \Gamma \right) + \kappa^2 x^{-2} n_{E}^{-1} \partial_x \left( x^4 n_{E}^2 (\Gamma^2 - \Gamma) \right)$$

(45)

The last term in eq.(45) is absent in eq.(42). It describes the effect of the quantum statistics which promotes bosons to condense. As long as the density of photons is low this term will be negligible.

It remains to consider the Lorentz transformations of the equilibrating diffusion equation. The diffusion equation (8) (without friction) is explicitly Lorentz invariant as follows from eq.(6). However, the equilibrium measure cannot be Lorentz invariant if it is to be normalizable (in a finite volume) and if the mass shell condition (1) is to be satisfied (there is only one invariant measure on the mass-shell, this is $\delta p \delta p_0^{-1}$). The equilibration to the Jüttner or Bose-Einstein equilibrium takes place in a preferred Lorentz frame [31]. In a general frame described by the four-velocity $w^\nu$ we should write the equilibrium distribution in the form

$$n_{E}(w, \mu) = \left( \exp(\beta w^\nu p_\nu + \beta w^\nu \mu_\nu) - 1 \right)^{-1}$$

(46)

where (by convention) in the rest frame $w = (1, 0, 0, 0)$ and $\mu_0 = \mu$.

We can write the Kompaneetz equation (44) in an arbitrary frame

$$\partial_\tau n(w) = \frac{\kappa^2}{2} \left( L_{\nu \rho} L^{\nu \rho} n(w) + 2 i w_\nu L^{\nu \rho} p_\rho n(w) (n(w) + 1) + 2 w^\nu p_\nu n(w) (n(w) + 1) \right)$$

(47)

In the rest frame eq.(47) coincides with eq.(44) as can be checked using eq.(5) (with $L_{0j} = -i |p| \partial_j$ on the mass shell). In an arbitrary frame $n_\tau(w)$ has the Bose-Einstein distribution $n_{E}(w, \mu)$ as an equilibrium measure.

The Kompaneetz equation finds applications in astrophysics [10][32][33][34][35][36]. The CMBR photons described by $n_{E}(\mu = 0)$ are scattered when passing clusters of galaxies and other reservoir of hot plasma. Eq.(45) could be solved with $n_{E}(\mu = 0)$ and $\Gamma = 1$ as an initial condition (see the end of the next section). Then, it describes the photon density evolution as it passes through the electron gas. The resulting distortion of the black-body spectrum (Sunyaev-Zeldovich effect, see [34][37] for reviews) can be compared with observations (as discussed in [19] taking into account the photon helicity does not change the diffusion equations (8) and (13), the spin dependent part disappears in the limit $m \to 0$). The Kompaneetz equation has been derived from the Boltzmann equation [13][10] under an assumption that the electron velocities are non-relativistic. Relativistic corrections to the Boltzmann equation
describing the Compton scattering in a gas of electrons and its diffusion limit have been calculated in [16][35][38]. These computations involve relativistic corrections to the photon-electron cross section in the laboratory rest frame. They are unrelated to the generalization (47) of the diffusion equation to an arbitrary frame.

5 Langevin stochastic equations

We wish to express the solution of the diffusion equation (8) by a stochastic process

$$\phi_\tau(p, x) = E[\phi(p(\tau), x(\tau))]$$

Here,

$$x'^\mu(\tau) = x^\mu + \int_0^\tau p'^\mu(s)ds$$

The diffusion process $p(\tau)$ as well as its generator $\mathcal{G}$ can be expressed in various coordinates. In the coordinates (7) we have to find a frame $e_{ja}$ ($j, a = 1, 2, 3$) such that

$$p_j p_k = e_{ja}e_{ka}$$

A solution of eq.(50) is

$$e_{ja} = p_j e_a$$

where $e_a e_a = 1$. Then, the stochastic process solving eqs.(8) and (48) reads [28][39]

$$dp_j = \frac{3}{2}\gamma^2 p_j d\tau + \gamma p_j e_a db^a = \gamma^2 p_j d\tau + \gamma p_j e_a \circ db^a$$

where $b^a$ are independent Brownian motions defined as the Gaussian process with the covariance

$$E[b^a(\tau)b^c(\tau')] = \delta^{ac} \min(\tau, \tau')$$

(we denote the expectation values by $E[...]$). The first of eqs.(52) is the Ito equation whereas the circle in the second equation denotes the Stratonovitch differential (the notation is the same as in [28]). Eq.(52) is equivalent to the simpler equation for $r = |p|

$$dr = \gamma^2 r d\tau + \gamma r \circ db$$

where $b$ is the one-dimensional Brownian motion. The solution of eq.(52) is

$$p_j(\tau) = p_j \exp(\gamma^2 \tau + \gamma e_a b^a(\tau))$$

whereas the one of eq.(54)

$$r_\tau = r \exp(\gamma^2 \tau + \gamma b(\tau))$$
We could derive eq.(56) from eq.(55) setting \( b = e_a b^a \). Eq.(56) means that (in the coordinates (25))

\[
u_\tau = u + \gamma^2 \tau + \gamma b(\tau)
\] (57)

When we know the probability distribution of \( u_\tau \), then we can calculate the probability distribution of the momenta. We have (in the coordinates (23) the angles do not change in time)

\[
E[f(p_\tau)] = E[f(r_\tau, \phi, \theta)] = (2\pi \gamma^2)^{-\frac{3}{2}} \int_0^\infty \frac{dy}{y} f(y, \phi, \theta) \exp \left( -\frac{1}{2\pi \gamma^2} (\ln \frac{y}{|p|} - \gamma^2 \tau)^2 \right)
\]

Hence, \(|p|\) has the log-normal distribution. Note that if we wish to express the probability distribution \( \Psi (\mu = 0) \) in terms of a stochastic process \( \tilde{u}_\tau \), then \( \tilde{u}_\tau \) is the solution of the stochastic equation

\[
d\tilde{u}_\tau = \frac{3}{2} \gamma^2 d\tau + \gamma db_\tau
\]

In such a case

\[
\Psi_\tau(r) = E[\exp(\tilde{u}_\tau)] = (2\pi \gamma^2)^{-\frac{3}{2}} \int_0^\infty \frac{dy}{y} \Psi(y) \exp \left( -\frac{1}{2\pi \gamma^2} (\ln \frac{y}{r} - \frac{3}{2} \gamma^2 \tau)^2 \right)
\]

is the solution of the equation

\[
\partial_\tau \Psi_\tau = \frac{1}{2} \gamma^2 r^{-2} \partial_r r^4 \partial_\tau \Psi_\tau \equiv G^* \Psi_\tau
\]

with the initial condition \( \Psi \). The solution (58) is discussed in [33][36]. For small \( \tau \) and the initial condition \( \Psi = n_E(\mu = 0) \) the solution can be approximated by

\[
\Psi_\tau = n_E(\mu = 0) + \tau G^* n_E(\mu = 0)
\]

This approximate solution of the diffusion equation without friction is usually applied in a discussion of Sunyaev-Zeldovich CMBR spectrum distortion [33][34][35][36][37].

The stochastic equation for the diffusion with the Jüttner friction (38) generated by (39) reads

\[
du = \gamma^2 d\tau - \frac{1}{2} \gamma^2 \beta c \exp(u) d\tau + \gamma db_\tau
\] (59)

It has the solution

\[
u_\tau = u + \gamma^2 \tau + \gamma b_\tau - \ln \left( 1 + \frac{1}{2} \gamma^2 \beta c \exp(u) \int_0^\tau \exp(\gamma b_s + \gamma^2 s) ds \right)
\] (60)

It is not easy to calculate the correlation functions of \( u_\tau \) or \( r_\tau \) in a closed form (we could do it as an expansion in \( \beta \)). However, correlation functions of \( \frac{1}{\tau_\tau} = \exp(-u_\tau) \) can be calculated explicitly.
6 A relation to the imaginary time quantum mechanics

We can express solutions of the diffusion equations by an imaginary time evolution generated by a quantum mechanical Hamiltonian. The diffusion generator is of the form

\[ G = \frac{1}{2} \gamma^2 \partial_u^2 - \omega \partial_u \]  

(61)

Let

\[ \Omega(u) = \int_u^\omega \]  

(62)

Then

\[ \exp(-\Omega) G \exp(\Omega) \equiv -H = \frac{1}{2} \gamma^2 \partial_u^2 - V \]  

(63)

where

\[ V = \frac{1}{2} \omega^2 - \frac{1}{2} \partial_u \omega \]  

(64)

We have the standard Feynman-Kac formula [28][39] for \( \exp(-\tau H) \)

\[ (\exp(-\tau H) \psi)(u) = E \left[ \exp \left( - \int_0^\tau V(u + \gamma b_s) ds \right) \psi(u + \gamma b_\tau) \right] \]  

(65)

Let

\[ \psi = \exp(-\Omega) \phi \]  

(66)

Then, from eq.(63) it follows

\[ \phi_\tau(u) = (\exp(\tau G) \phi)(u) = E[\phi(u_\tau(u))] = \exp(\Omega(u))(\exp(-\tau H) \psi)(u) \]  

(67)

where \( u_\tau(u) \) (for the Jüttner distribution) is the solution (60) of the stochastic equation (59) with the initial condition \( u \). Inserting \( \phi = 1 \) in eq.(67) we obtain \( \exp(-\tau H) \exp(-\Omega) = \exp(-\Omega) \). Hence,

\[ H \exp(-\Omega) = 0 \]  

(68)

We can relate statistical expectation values of the process \( u_\tau \) to some expressions from an operator theory. Applying eq.(67) we obtain

\[ \int du \exp(-\Omega(u))(\exp(-\tau H) \exp(-\Omega) \phi)(u) = \int du \exp(-2\Omega(u)) E[\phi(u_\tau(u))] \]  

(69)

In the Jüttner model

\[ V = \frac{9}{8} \gamma^4 + \frac{1}{8} \gamma^4 \beta^2 c^2 \exp(2u) - \frac{1}{4} (1 + 3\gamma^2) \gamma^2 \beta c \exp(u) \]  

(70)

The eigenstate of \( H \) with the zero eigenvalue is

\[ \exp(-\Omega) = \exp \left( \frac{3}{2} u - \frac{1}{2} \beta c \exp(u) \right) \]  

(71)
The statistical weight in eq.(69)
\[ \exp(-2\Omega(u)) = r^3 \exp(-\beta c r) = \Phi_E^J \] (72)
is just the Jüttner distribution. It can be seen from eq.(69) that in general
\[ \exp(-2\Omega) = \Phi_E \]
for any equilibrium distribution \( \Phi_E \).

The potential \( V \) has a minimum. There is a well which supports some bound states of \( H \). Eq.(72) defines one of these (normalizable) bound states. All the eigenstates of the potential (70) are known (this is the soluble Morse potential [40]). On the set of functions
\[ \phi(u) = \sum_n c_n \exp(-un) \]
(\( n \) a natural number) the expectation value (67) could be calculated explicitly using the solution (60). The formula
\[ \exp(\tau G)\phi = \sum_k a_k \exp(-\epsilon_k \tau)\phi_k \] (73)
allows to calculate the eigenvalues \( \epsilon_k \) and eigenfunctions \( \phi_k \) from eqs.(60) and (67). We obtain in this way the well-known eigenfunctions and eigenvalues of the Morse potential [40]. However, for an arbitrary friction we would apply the relation (63) the other way round exploiting quantum mechanical methods for estimates of the diffusion equilibration.

7 Discussion: The problem of time

In general, we have two candidates for time: \( \tau \) and \( x^0 \). In the relativistic dynamics, from the zero component of eq.(11), we can express \( \tau \) by \( x^0 \). Hence, we can consider the evolution as a function of \( \tau \) or as a function of \( x^0 \). Let us discuss this problem for the diffusion equation (22) assuming (for simplicity of the argument) that \( \Phi_E \) depends only on the momenta. First, we find the solution \( (x(\tau, x, p), p(\tau, p)) \) of the stochastic equations
\[ dp_j = 2\gamma^2 p_j d\tau + \frac{1}{2}\gamma^2 p_k \partial^j \ln \Phi_{ER} d\tau + \gamma p_j e dB(\tau) \] (74)
\[ dx^j = p_j d\tau \] (75)
\[ dx^0 = -|p| d\tau \] (76)
with the initial condition \( (x, p) \). Then,
\[ \Psi_\tau(x, p) = E[\Psi(x(\tau, x, p), p(\tau, p))] \] (77)
is the solution of eq.(22).

We can obtain $\Phi_R$ from $\Phi_\tau$. For this purpose integrate both sides of eq.(15) in the interval $[0,T]$. We have

$$\lim_{T \to \infty} T^{-1} \int_0^T d\tau \Phi_\tau = \Phi_R \quad (78)$$

because the lhs of eq.(78) satisfies the equation $G^* \Phi_R = 0$. Hence, if we denote

$$\Phi_\tau = \Phi_{ER} \Psi_\tau \quad (79)$$

Then, the $\tau$-average of $\Psi_\tau$ satisfies the transport equation (16)

$$|p| \partial_0 \Psi_R = \frac{1}{2} \gamma^2 p_j p_k \partial^j \partial^k \Psi_R + 2 \gamma^2 p_j \partial^j \Psi_R + \frac{1}{2} \gamma^2 p_j p_k (\partial^j \ln \Phi_{ER}) \partial^k \Psi_R + p_j \partial^j \Psi_R \quad (80)$$

We can obtain a solution of eq.(80) from eq.(22) by a random change of time.

Let us treat the formula (49) for $x_0(\tau)$ as a definition of $\tau$

$$\tau = \int_0^{x^0} |p_s|^{-1} ds \quad (81)$$

(here $p_s$ is the solution of eq.(74); $x^0(\tau)$ can be defined implicitly by (81)). We can see from eq.(81) that $\tau$ depends only on events earlier than $x^0$. As a consequence $p_j(x^0) = p_j(x^0(\tau))$ is again a Markov process. Then, differentiating the momenta and coordinates according to the rules of the Ito calculus [28] we obtain the following Langevin equations (for mathematical details of a random change of time see [39][28])

$$dp_j(x^0) = 2 \gamma^2 |p(x^0)|^{-1} p_j dx^0 + \frac{1}{2} \gamma^2 |p(x^0)|^{-1} p_j p_k \partial^j \ln \Phi_{ER} dx^0 + \gamma p_j |p(x^0)|^{-1} dx^0 \quad (82)$$

$$dx^j = p_j |p(x^0)|^{-1} dx^0 \quad (83)$$

Let $\Psi(x,p)$ be an arbitrary function of $x$ and $p$ and $(x(x^0,x,p), p(x^0,p))$ the solution of eqs.(82)-(83) with the initial condition $(x,p)$ then

$$\Psi_R(x^0,x,p) = E \left[ \Psi \left(x(x^0,x,p), p(x^0,x,p)\right) \right] \quad (84)$$

is the solution of the transport equation (80) with the initial condition $\Psi(x,p)$. We can prove that eq.(80) is satisfied by differentiation of eq.(84) applying the rules of the Ito calculus (or the well-known relation between Langevin equation and the diffusion equation). Hence, in principle, having a solution $E[\Psi(x_\tau,p_\tau)]$ of the diffusion equation (22) as a function of $\tau$ we can obtain a solution $E[\Psi(x(x^0), p(x^0))]$ of the transport equation (80) as a function of $x^0$. This is like in the deterministic case. The difference is in the averaging over events expressed by the expectation value (84)( a choice of the time parameter
and a random change of time in a relativistic diffusion is also discussed in [41]). Surprisingly, the new diffusion obtained in this way is the same as the one derived by the averaging over \( \tau \). If the limit \( \tau \to \infty \) of \( \Psi_\tau \) exists then this limit is equal to the one resulting from the averaging (78) (i.e., \( \Psi_\infty = \Psi_R \)). We can show this property using the equivalence to quantum mechanics of sec.6 . We apply the expansion (73) of \( \Psi_\tau \) and perform the \( \tau \) averaging term by term. We can see that

\[
\lim_{T \to \infty} T^{-1} \int_0^T d\tau \exp(-\epsilon_k \tau)
\]

exists and is different from zero if and only if \( \epsilon_k = 0 \). Hence, the averaging over \( \tau \) gives the same result as the limit \( \tau \to \infty \) of \( \Psi_\tau \). In this sense for large \( \tau \) the probability distribution \( \Psi_\tau \) depends only on \( x^0 \).

If the initial distribution \( \Phi \) depends on \( x^0 \) then we must perform the limit \( \tau \to \infty \) (or averaging) in order to get rid off the additional time in \( \Phi_\tau \). However, if \( \Phi \) does not depend on \( x^0 \) (what is a reasonable choice of physical initial phase space distribution) then apart from the procedure discussed above which leads from \( \Phi_\tau(x, p) \) to \( \Phi_R(x^0, x, p) \) there is still the question whether \( \Phi_\tau(x, p) \) could possibly have a physical meaning (the solution of the Kompaneetz equation is of this type). It satisfies eq.(22) instead of the transport equation (80). The difference in the \( \tau \) and \( x^0 \) evolutions is in the \( p_0^{-1} \) factor in the collision term of the Boltzmann equation. In the massive case \( p_0 \approx mc \) in the non-relativistic approximation. Hence, the difference between \( \tau \) and \( x^0 \) may be insignificant. However, in the massless case \( p_0 = |p| \) can never be approximated by a constant. For a small time \( \tau \approx |p|^{-1} x^0 \). We can make this replacement in all solutions of proper time equations (in particular in the Kompaneetz equation). However, in general, we should make the random time change discussed in this section. Nevertheless, for an estimation of the Sunyaev-Zeldovich CMBR spectrum distortion [33][36][34] this problem is not relevant because the time \( \tau \) is small and chosen as a physical parameter (plasma penetration depth). In such a case the spectral deformation is \( n_E + \tau \partial^\tau n_E \) (as discussed at the end of sec.5).

It can be seen from our discussion that only a small part of the theory of relativistic Brownian motion is applied in astrophysics. The behavior of the diffusion at large time (apart from its final effect: the equilibrium) seems to have no observational consequences. However, with the increasing sophistication of the astrophysical observations the large time effects of Compton scattering may give important information about the sources of the CMBR distortions. In this paper we have completely neglected the effect of the gravity. The relativistic diffusion of massive particles on a general gravitational background is discussed in [42][43][44][45]. We elaborate relativistic diffusion with a friction leading to equilibration in a gravitational field for massive as well as massless particles in a forthcoming paper [46]. In such a framework the role of strong gravity (resulting from compact stars or dark matter) and the effect of the expansion of the Universe upon the photon diffusion can be investigated.
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