A Poincaré lemma for Whitney–de Rham complex

CHEN, HOU-YI (*)

Abstract – Let $M$ be a real analytic manifold, $Z$ a closed subanalytic subset of $M$. We show that the Whitney–de Rham complex over $Z$ is quasi-isomorphic to the constant sheaf $C_Z$.

Mathematics Subject Classification (2010). 32B20, 32C38.

Keywords. Subanalytic sets, $\mathcal{D}$-modules.

1. Introduction

In [5], Kashiwara–Schapira introduced the Whitney functor (real case) and formal cohomology functor (complex case), then they introduced the notion of ind-sheaves and they also defined Grothendieck six operations in this framework in [6]. As applications, they defined the Whitney $C^\infty$ functions and Whitney holomorphic functions on the subanalytic site as examples of ind-sheaves. A more elementary study for sheaves on the subanalytic site is performed in [7] and [8].

Let $M$ be a real analytic manifold, by Poincaré lemma, it is well known that the de Rham complex over $M$ is isomorphic to $C_M$. The aim of this paper is to show that a theorem of [1] follows easily from a deep result of Kashiwara on regular holonomic $\mathcal{D}$-module in [3]. More precisely, we show that

Main theorem (Theorem 3.3). Let $M$ be a real analytic manifold of dimension $n$ and $Z$ a closed subanalytic subset of $M$. Then we have

$$
C_Z \xrightarrow{\sim} (0 \longrightarrow \mathcal{W}^{\infty}_{M,Z} \xrightarrow{d} \mathcal{W}^{(\infty,1)}_{M,Z} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}^{(\infty,n)}_{M,Z} \longrightarrow 0),
$$

(*) Indirizzo dell’A.: Hou-Yi Chen, Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
E-mail: d93221002@ntu.edu.tw
where $\mathcal{W}_{M,Z}^\infty$ denotes the sheaf of Whitney functions on $Z$ and $\mathcal{W}_{M,Z}^{(\infty,i)}$ denotes the sheaf of differential forms of degree $i$ with coefficients in $\mathcal{W}_{M,Z}^\infty$ for each $i$, i.e., the Whitney–de Rham complex is isomorphic to $\mathcal{C}_Z$.

2. Review on Whitney and formal cohomology functors

In this section, we review some results on Whitney and formal cohomology functors. References are made to \cite{five}, \cite{six}, \cite{seven}, \cite{eight}, and \cite{nine}.

Let $M$ be a real analytic manifold, we denote by $A_M$ the sheaf of complex-valued real analytic functions and $C^1_M$ the sheaf of differential forms of degree $i$ with coefficients in $A_M$ for each $i$, i.e., the Whitney–de Rham complex is isomorphic to $\mathcal{C}_Z$.

We denote by $\text{odd}_M$ the sheaf of rings on $M$ of finite-order differential operators with coefficients in $A_M$.

We denote by $\text{Mod}_{R-c}(C_M)$ the abelian category of $R$-constructible sheaves on $M$ and $\text{Mod}(\mathcal{D}_M)$ the abelian category of left $\mathcal{D}_M$-modules. We also denote by $\text{D}^{b}_{R-c}(C_M)$ the bounded derived category consisting of objects whose cohomology groups belong to $\text{Mod}_{R-c}(C_M)$ and $\text{D}^{b}(\mathcal{D}_M)$ the derived category of $\text{Mod}(\mathcal{D}_M)$ with bounded cohomologies.

**Definition 2.1.** Let $Z$ be a closed subset of $M$. We denote by $\mathcal{I}_{M,Z}$ the sheaf of $C^1_M$ functions on $M$ vanishing up to finite order on $Z$.

**Definition 2.2.** A Whitney function on a closed subset $Z$ of $M$ is an indexed family

$$F = (F^k)_{k \in \mathbb{N}^n}$$

consisting of continuous functions on $Z$ such that for all $m \in \mathbb{N}$ and $k \in \mathbb{N}^n$ with $|k| \leq m$, and all $x \in Z$ and $\varepsilon > 0$ there exists a neighborhood $U$ of $x$ such that

$$\left| F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leq \varepsilon d(y,z)^{m-|k|}, \quad \text{for all} \ y, z \in U \cap Z.$$ 

We denote by $W_{M,Z}^\infty$ the space of Whitney $C^\infty$ functions on $Z$. We denote by $W_{M,Z}^{\infty}_U$ the sheaf $U \mapsto W_{U, U \cap Z}^\infty$.

In \cite{five}, the authors defined the Whitney tensor product functor

$$\otimes_{C_M^\infty} : \text{Mod}_{R-c}(C_M) \longrightarrow \text{Mod}(\mathcal{D}_M)$$

in the following way. Let $U$ be an open subanalytic subset of $M$ and $Z = M \setminus U$. Then $C_U \otimes_{C_M^\infty} = \mathcal{W}_{M,Z}^\infty$ and $C_Z \otimes_{C_M^\infty} = W_{M,Z}^{\infty}$. This functor is exact and extends as a functor in the derived category, from $\text{D}^{b}_{R-c}(C_M)$ to $\text{D}^{b}(\mathcal{D}_M)$. Moreover, the sheaf $F \otimes_{C_M^\infty}$ is soft for any $R$-constructible sheaf $F$. 
Now let $X$ be a complex manifold and we denote by $\mathcal{D}_X$ the sheaf of rings on $X$ of finite-order differential operators. We still denote by $X$ the real underlying manifold and we denote by $\bar{X}$ the complex manifold conjugate to $X$. One defines the functor of formal cohomology as follows:

Let $F \in \mathcal{D}^b_{R-c}(\mathcal{C}_X)$, we set

$$F \otimes O_X = R\mathcal{H}om_{\mathcal{D}_X} (O_{\bar{X}}, F \otimes C^\infty_{\bar{X}}),$$

where $\mathcal{D}_{\bar{X}}$ denotes the sheaf of rings on $\bar{X}$ of finite-order differential operators.

Let $M$ be a real analytic manifold, $X$ a complexification of $M$, $i: M \hookrightarrow X$ the embedding. We recall the following result.

**Theorem 2.3 ([5], Theorem 5.10).** Let $F \in \mathcal{D}^b_{R-c}(\mathcal{C}_M)$. Then we have

$$i_* F \otimes O_X \simeq i_* (F \otimes C^\infty_M).$$

In particular,

$$C^\infty_M \otimes O_X \simeq C^\infty_M.$$

The following proposition is the key point of this paper which follows from a deep result of [3].

**Proposition 2.4 ([5], Corollary 6.2).** Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module, and let $F$ be an object of $\mathcal{D}^b_{R-c}(\mathcal{C}_X)$. Then, the natural morphism:

$$(2.1) \quad R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \otimes O_X) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \otimes O_X).$$

is an isomorphism.

**3. Main result**

Let $X$ be a complex manifold of dimension $n$. We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order differential operators and $\Theta_X$ the sheaf of vector fields on $X$.

First we recall the following basic result in $\mathcal{D}_X$-module theory.

**Proposition 3.1 ([4], Proposition 1.6).** The complex

$$0 \longrightarrow \mathcal{D}_X \otimes O_X \bigwedge^n \Theta_X \longrightarrow \cdots \longrightarrow \mathcal{D}_X \otimes O_X \bigwedge^2 \Theta_X$$

$$\longrightarrow \mathcal{D}_X \otimes O_X \Theta_X \longrightarrow \mathcal{D}_X \longrightarrow O_X \longrightarrow 0$$

is exact.
Lemma 3.2. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. Then we have

$$R\text{Hom}_{\mathcal{D}_X} (\mathcal{O}_X, \mathcal{M}) \simeq \left[ \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \bigwedge^n \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \right].$$

Proof. By Proposition 3.1, we have

$$R\text{Hom}_{\mathcal{D}_X} (\mathcal{O}_X, \mathcal{M})$$

$$\simeq \left[ \mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{D}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X, \mathcal{M}) \longrightarrow \cdots$$

$$\longrightarrow \mathcal{H}om_{\mathcal{D}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \mathcal{O}_X, \mathcal{M}) \right]$$

$$\simeq \left[ \mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{O}_X} (\mathcal{O}_X, \mathcal{M}) \longrightarrow \cdots \longrightarrow \mathcal{H}om_{\mathcal{O}_X} (\bigwedge^n \mathcal{O}_X, \mathcal{M}) \right]$$

$$\simeq \left[ \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \bigwedge^n \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \right]$$

where

$$\Omega^1_X := \mathcal{H}om_{\mathcal{O}_X} (\mathcal{O}_X, \mathcal{O}_X).$$

\qed

Let $\mathcal{M}$ be a real analytic manifold, $X$ a complexification of $\mathcal{M}$ and $Z$ a closed subanalytic subset of $\mathcal{M}$. We denote by $\Omega^1_X$ the sheaf of differential one-form on $X$ and

$$A^{(i)}_Z := \bigwedge^i \Omega^1_X \otimes_{\mathcal{O}_X} \left( \mathcal{C}_Z \otimes \mathcal{O}_X \right),$$

(3.1)

$$\mathcal{W}_{M,Z}^{(\infty,i)} := \bigwedge^i \Omega^1_X \otimes_{\mathcal{O}_X} \left( \mathcal{C}_Z \otimes \mathcal{O}_X \right) \simeq \bigwedge^i \Omega^1_X \otimes_{\mathcal{O}_X} \left( \mathcal{C}_Z \otimes \mathcal{C}_M^{\infty} \right).$$

(3.2)

Now we are ready to prove the main theorem of this paper below.

Theorem 3.3. Let $\mathcal{M}$ be a real analytic manifold of dimension $n$ and $Z$ a closed subanalytic subset of $\mathcal{M}$. Then we have

$$\mathcal{C}_Z \xrightarrow{\sim} (0 \longrightarrow \mathcal{W}_{M,Z}^{\infty} \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{W}_{M,Z}^{(\infty,n)} \longrightarrow 0),$$

where $\mathcal{W}_{M,Z}^{\infty}$ denotes the sheaf of Whitney functions on $Z$ and $\mathcal{W}_{M,Z}^{(\infty,i)}$ denotes the sheaf of differential forms of degree $i$ with coefficients in $\mathcal{W}_{M,Z}^{\infty}$ for each $i$ which are defined in (3.2), i.e., the Whitney–de Rham complex is isomorphic to $\mathcal{C}_Z$. 
A Poincaré lemma for Whitney–de Rham complex

Proof. Take $\Omega = \mathcal{O}_X$ and $F = C_Z$ in Proposition 2.4.

On the one hand, we show that the left hand side of (2.1) is $C_Z$. By Theorem 2.3 and Lemma 3.2, we get the following complex

$$0 \longrightarrow C_M \longrightarrow \mathcal{A}_M^{(0)} \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \mathcal{A}_M^{(n)} \longrightarrow 0$$

which is exact by Poincaré lemma where $\mathcal{A}_M^{(i)}$’s are defined in (3.1) by taking $Z = M$. Tensoring $C_Z$, we obtain the following exact sequence

$$0 \longrightarrow C_Z \longrightarrow \mathcal{A}_Z^{(0)} \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \mathcal{A}_Z^{(n)} \longrightarrow 0.$$  

Therefore,

$$C_Z \sim (0 \longrightarrow \mathcal{A}_Z^{(0)} \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \mathcal{A}_Z^{(n)} \longrightarrow 0).$$

On the other hand, the right hand side of (2.1) is the Whitney–de Rham complex

$$0 \longrightarrow \mathcal{W}^{\infty}_{M,Z} \overset{d}{\longrightarrow} \mathcal{W}^{(\infty,1)}_{M,Z} \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \mathcal{W}^{(\infty,n)}_{M,Z} \longrightarrow 0.$$  

Now the result follows from the isomorphism of (2.1).  

Remark 3.4. This theorem is in some sense dual to a theorem of Grothendieck in [2] which asserts that if $U$ is subanalytic then the de Rham cohomology may be calculated with holomorphic functions which are meromorphic on the complementary of $U$ ($U$ is the complementary of a closed hypersurface in the complex analytic space), that is, holomorphic functions with temperate growth.

Acknowledgements. I would like to thank Pierre Schapira for suggesting this problem to me and for many useful conversations.

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Manoscritto pervenuto in redazione il 17 aprile 2014.