SOME CHAIN MAPS ON KHOVANOV COMPLEXES AND REIDEMEISTER MOVES

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Abstract. We introduce some chain maps between Khovanov complexes. Each of the chain maps commutes with a chain homotopy map and a retraction maps which obtain a Reidemeister invariance of Khovanov homology.

1. Introduction.

Let \( C_3 = C \left( \begin{array}{cc} c & b \\ a & \end{array} \right) \) be a Khovanov complex of a link diagram and \( C_2 = C \left( \begin{array}{cc} b & \\ a & \end{array} \right) \), \( C_1 = C \left( \begin{array}{cc} b & \\ a & \end{array} \right) \), and \( C_{1'} = C \left( \begin{array}{cc} a & \\ b & \end{array} \right) \) be its subcomplexes. There exist chain homotopy maps \( h_J \) (\( J = 1, 1', 2, 3 \)) relating the identity maps \( C_J \rightarrow C_J \) with compositions in \( \circ \rho_J \) of an inclusion maps in and a retraction \( \rho_J \) for Reidemeister moves 1:

(Section 4, Appendix A, B). This paper will be show that the natural chain maps \( \pi_J : C_J \rightarrow C_{J-1} \) (\( J = 2, 3 \)) satisfy the relations \( h_{J-1} \circ \pi_J = \pi_J \circ h_J \) (Theorem 1) and similar relations for \( \rho_J \) (Theorem 2).

In section 1 chain maps \( \pi_J \) are defined. In section 2 relations of \( h_J, \rho_J, \) and \( \pi_J \) are given. In section 3 a map \( \tilde{\pi}_2 \) similar to \( \pi_2 \) is introduced. In section 4 we obtain the proof of the right twisted first Reidemeister invariance of Khovanov homology for a general differential. Appendix contains the definitions \( h_J \) and \( \rho_J \) provided by \( 1 \) \( 2 \). All notations in this paper and the definition of the differential \( \delta_s, t \) follows \( 2 \).

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2. The chain map $\pi_J$.

$\pi_3 : C_3 \to C_2$ is defined by

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array}
\mapsto \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array},
\]

(1)

$\pi_2 : C_2 \to C_1$ is defined by

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\mapsto \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array},
\]

(2)

Theorem 1. The chain maps $\pi_3$ and $\pi_2$ satisfy the following.

(3) $h_2 \circ \pi_3 = \pi_3 \circ h_3$,

(4) $h_1 \circ \pi_2 = \pi_2 \circ h_2$.

Proof. $\delta_{s,t} \circ \pi_J = \pi_J \circ \delta_{s,t}$ and $h_{J-1} \circ \pi_J = \pi_J \circ h_J$ $(J = 2, 3)$ are proved by direct computation for every generator of $C_J$.

Let $C_3' = C \left( \begin{array}{c}
\begin{array}{c}
\text{r} \\
\text{p} \\
\text{q}
\end{array}
\end{array} \otimes [xa] + \begin{array}{c}
\begin{array}{c}
\text{r} \\
\text{p}\colon q
\end{array}
\end{array} \otimes [xb], \begin{array}{c}
\begin{array}{c}
\text{q}\colon p
\end{array}
\end{array} \otimes [x] \right)$ and

\[
\begin{array}{c}
\begin{array}{c}
\text{p} \\
\text{q}
\end{array}
\end{array} \otimes [xa] + \begin{array}{c}
\begin{array}{c}
\text{p}\colon q
\end{array}
\end{array} \otimes [xb]
\]

$C_2'$ is a subcomplex of $C_J$ $(J = 2, 3)$.

We define $\pi_3' : C_3' \to C_2$ by

Theorem 2.

(5) $\rho_2 \circ \pi_3 = \pi_3' \circ \rho_3$. 

Proof. $\delta_{s,t} \circ \pi_3^t = \pi_3^t \circ \delta_{s,t}$ and $\rho_2 \circ \pi_3 = \pi_3^t \circ \rho_3$ are proved by direct computation for every generator of $\mathcal{C}_3$. □

3. A SIMILAR MAP $\tilde{\pi}_2$ TO $\pi_2$.

In this section we introduce a map $\tilde{\pi}_2$. It is not chain maps, but it has similar property $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$ (Theorem 3).

The map $\tilde{\pi}_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_{1'}$ is defined by

\[
(6) \quad b \rightarrow a \\
(7) \quad a \rightarrow 0.
\]

Theorem 3.

(8) $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$.

Proof. $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$ is proved by direct computation for every generator of $\mathcal{C}_2$. □

4. RIGHT TWISTED FIRST REIDEMEISTER INVARIANCE FOR THE GENERAL DIFFERENTIAL $\delta_{s,t}$.

In this section we will show that the right twisted first Reidemeister invariance of Khovanov homology because the proof of this case is missing in [2].

The right twisted first Reidemeister move is $D' = \overset{\sim}{\overset{\sim}{\overset{\sim}{a}}}$, we consider the composition

(9) $\mathcal{C}(D') = \mathcal{C} \oplus \mathcal{C}_{\text{contr}} \xrightarrow{\rho_1} \mathcal{C} \xrightarrow{\text{isom}} \mathcal{C}(D)$

where $a$ is a crossing and $\mathcal{C}, \mathcal{C}_{\text{contr}}, \rho_1$ and the isomorphism are defined in the following formulas (10)–(13).

First,

(10) $\mathcal{C} := \mathcal{C}
\left.
\begin{array}{c}
p \\
\otimes [x]
\end{array}
\right\},$

(11) $\mathcal{C}_{\text{contr}} := \mathcal{C}
\left.
\begin{array}{c}
p : p \\
\otimes [x], \overset{p}{\rightarrow} \overset{p}{\rightarrow} \otimes [x]\end{array}\right\}.$

Second, the retraction $\rho_1 : \mathcal{C}
\left.
\begin{array}{c}
\overset{\sim}{\overset{\sim}{\overset{\sim}{a}}}
\end{array}\right\} \rightarrow \mathcal{C}
\left.
\begin{array}{c}
p \\
\otimes [x]
\end{array}\right\}$ is defined by the formulas
\[ p \bigotimes [x] \mapsto p \bigotimes [x], \]

\[ p \bigoplus [x] \mapsto p \bigoplus [x] - p : p : p \bigotimes [x], \]

\[ \bigotimes [xa] \mapsto 0. \]

We can verify that \( \delta_{s,t} \circ \rho_1 = \rho_1 \circ \delta_{s,t} \). Then \( \rho_1 \) is a chain map.

Third, the isomorphism

\[ C \left( p \bigotimes [x] \right) \to C \left( \bigotimes [x] \right) \]

is defined by the formulas

\[ p \bigotimes [x] \mapsto \bigotimes [x]. \]

The homotopy connecting in \( \circ \rho_1 \) to the identity : \( C \left( \bigotimes \right) \to C \left( \bigotimes \right) \) such that \( \delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \text{id} - \circ \rho_1 \), is defined by the formulas:

\[ p \bigotimes [xa] \mapsto \bigotimes [x], \text{ otherwise } \mapsto 0. \]

Remark 1. The explicit formula (14) of the homotopy map \( h_1 \) in the case \( s = t = 0 \) of the original Khovanov homology is given by Oleg Viro [3, Subsection 5.5].

We can verify \( \delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \text{id} - \circ \rho_1 \) by a direct computation as follows.

\[ (h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( \bigotimes [x] \right) = h_1 \left( p : p \bigotimes [xa] \right) \]

\[ = \bigotimes [x] \]

\[ = (\text{id} - \rho_1) \left( \bigotimes [x] \right). \]

Similarly,

\[ (h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( \bigoplus [x] \right) = p : p \bigoplus [x] \]

\[ = (\text{id} - \rho_1) \left( \bigoplus [x] \right), \]

\[ (h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( \bigotimes [x] \right) = 0 = (\text{id} - \rho_1) \left( \bigotimes [x] \right). \]
Appendix A. Chain homotopy maps.

The homotopy connecting in $\circ \rho_1'$ to the identity $h_1' : \mathcal{C} \left( \begin{array}{c} a \\ \circ \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} a \\ \circ \end{array} \right)$ such that $\delta_{s,t} \circ h_1' + h_1' \circ \delta_{s,t} = \text{id} - \circ \rho_1'$, is defined by the formulas:

(18) \[ p \otimes [xa] \mapsto \begin{cases} p \otimes [x], & \text{otherwise} \rightarrow 0. \end{cases} \]

The homotopy connecting in $\circ \rho_2$ to the identity $h_2 : \mathcal{C} \left( \begin{array}{c} \ast \\ \circ \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} \ast \\ \circ \end{array} \right)$ such that $\delta_{s,t} \circ h_2 + h_2 \circ \delta_{s,t} = \text{id} - \circ \rho_2$, is defined by the formulas:

(19) \[ \begin{array}{lcl} p \otimes [xab] & \mapsto & \begin{cases} p \otimes [xb], & \text{otherwise} \rightarrow 0. \end{cases} \\
p \otimes [xb] & \mapsto & p \otimes [x], \end{array} \]

The homotopy connecting in $\circ \rho_3$ to the identity, that is, a map $h_3 : \mathcal{C} \left( \begin{array}{c} \ast \\ \circ \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} \ast \\ \circ \end{array} \right)$ such that $\delta_{s,t} \circ h_3 + h_3 \circ \delta_{s,t} = \text{id} - \circ \rho_3$, is defined by the formulas:

(20) \[ \begin{array}{lcl} r \otimes [xab] & \mapsto & \begin{cases} r' \otimes [xb], & \text{otherwise} \rightarrow 0. \end{cases} \\
r \otimes [xb] & \mapsto & r \otimes [x], \end{array} \]

Appendix B. Retractions.

The retraction $\rho_1' : \mathcal{C} \left( \begin{array}{c} + \\ \circ \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} p \oplus [x] - m(p : +) \end{array} \right) \otimes [x]$ is defined by the formulas
\begin{equation}
\begin{aligned}
p \uparrow \otimes [x] &\mapsto p \uparrow \otimes [x] - m(p : +) \otimes [x], \\
p \downarrow \otimes [x], &\mapsto 0.
\end{aligned}
\end{equation}

The retraction \( \rho_2 : C \left( \begin{array}{c} \scriptstyle b \\ \scriptstyle a \end{array} \right) \rightarrow C \left( \begin{array}{c} \scriptstyle q \otimes [xa] + \otimes [xb] \end{array} \right) \) is defined by the formulas

\begin{equation}
\begin{aligned}
\begin{array}{c}
p \\ q
\end{array} \otimes [xa] &\mapsto \begin{array}{c}
p \\ q
\end{array} \otimes [xa] + \begin{array}{c}
p : q \\ q : p
\end{array} \otimes [xb], \\
\begin{array}{c}
p \\ q \uparrow \otimes [xb] \mapsto - &\left( \begin{array}{c}
p : q \\ q : p
\end{array} \otimes [xa] + \otimes [xb] \right) ,
\end{array}
\end{aligned}
\end{equation}

otherwise \( \mapsto 0. \)

The retraction \( \rho_3 : C \left( \begin{array}{c} \scriptstyle b \\ \scriptstyle a \end{array} \right) \rightarrow C \left( \begin{array}{c} \scriptstyle p \otimes [xa] + \otimes [xb], \\
\end{array} \right) \) is defined by the formulas

\begin{equation}
\begin{aligned}
\begin{array}{c}
r \\ \scriptstyle p
\end{array} \otimes [xa] &\mapsto \begin{array}{c}
r \\ \scriptstyle p
\end{array} \otimes [xa] + \begin{array}{c}
r \rightarrow \scriptstyle p \\ \scriptstyle p : q \\ q : p
\end{array} \otimes [xb], \\
\begin{array}{c}
r \\ \scriptstyle p
\end{array} \otimes [x] &\mapsto \begin{array}{c}
r \\ \scriptstyle p
\end{array} \otimes [x],
\end{aligned}
\end{equation}
otherwise $\mapsto 0$.

**References**

[1] N. Ito, *On Reidemeister invariance of the Khovanov homology group of the Jones polynomial*, math.GT/0901.3952.

[2] N. Ito, *Chain homotopy maps and a universal differential for Khovanov-type homology*, math.GT/0907.2104.

[3] O. Viro, *Khovanov homology, its definitions and ramifications*, Fund. Math. 184 (2004), 317–342.

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