Bang-Xian Han

Sharp \( p \)-Poincaré inequalities under measure contraction property

Received: 11 June 2019 / Revised: 2 July 2019 / Accepted: 13 July 2019 / Published online: 19 July 2019

Abstract. We obtain sharp estimate on \( p \)-spectral gaps, or equivalently optimal constant in \( p \)-Poincaré inequalities, for metric measure spaces satisfying measure contraction property. We also prove the rigidity for the sharp \( p \)-spectral gap.

Contents

1. Introduction .................................... 457
2. Prerequisites .................................... 459
3. One dimensional models .............................. 460
   3.1. One dimensional MCP densities ....................... 460
   3.2. One dimensional \( p \)-Poincaré inequalities ................... 462
4. \( p \)-spectral gap ................................... 468
   4.1. Sharp \( p \)-spectral gap estimates ........................ 468
   4.2. Rigidity for \( p \)-spectral gap .......................... 469
References ....................................... 470

1. Introduction

Sharp estimates on spectral gap for \( p \)-Laplacian, or equivalently, the optimal constant in \( p \)-Poincaré inequalities is a classical problem in comparison geometry. It addresses the following basic problem. Given a family \( \mathcal{F} := \{ (X_\alpha, d_\alpha, m_\alpha) : \alpha \in \mathcal{A} \} \) of metric measure spaces, the corresponding optimal constant \( \lambda_{\mathcal{F}} \) in \( p \)-Poincaré inequalities is defined by

\[
\lambda_{\mathcal{F}} := \inf_{\alpha \in \mathcal{A}} \inf \left\{ \frac{\int_{X_\alpha} |\nabla_{d_\alpha} f|^p \, d m_\alpha}{\int_{X_\alpha} |f|^p \, d m_\alpha} : f \in \text{Lip} \cap L^p, \int_{X_\alpha} |f|^p \, d m_\alpha = 0, f \neq 0 \right\},
\]

(1.1)

where the local Lipschitz constant \( |\nabla_{d_\alpha} f| : X_\alpha \mapsto \mathbb{R} \) is defined by

\[
|\nabla_{d_\alpha} f|(x) := \lim_{y \to x} \frac{|f(y) - f(x)|}{d_\alpha(y, x)}.
\]

B.-X. Han (✉): Department of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel. e-mail: hanbangxian@gmail.com

Mathematics Subject Classification: 35P15 · 58J50 · 53C23 · 51F99

https://doi.org/10.1007/s00229-019-01138-5
One of the most studied families of metric measure spaces is Riemannian manifolds with lower Ricci curvature bound $K \in \mathbb{R}$, upper dimension bound $N > 0$ and diameter bound $D > 0$. In this case, $\lambda_1$ is the minimum of all first positive eigenvalues of the $p$-Laplacian (assuming Neumann boundary conditions if the boundary is not empty). Based on a refined gradient comparison technique and a careful analysis of the underlying model spaces, sharp estimate on the first eigenvalue of the $p$-Laplacian was finally obtained by Valtorta and Naber [22,26].

Another important family is weighted Riemannian manifolds (called smooth metric measure spaces) satisfying BE($K, N$) curvature-dimension condition à la Bakry–Émery [5,6]. More generally, thanks to the development of optimal transport theory, it was realized that Bakry–Émery’s condition in smooth setting can be equivalently characterized by convexity of an entropy functional along $L^2$-Wasserstein geodesics (c.f. [14] and [27]). In this direction, metric measure spaces satisfying CD($K, N$) condition was introduced by Lott–Villani [20] and Sturm [24,25]. This class of metric measure spaces with synthetic lower Ricci curvature bound and upper dimension bound includes the previous smooth examples, and is closed in the measured Gromov–Hausdorff topology. Recently, using measure decomposition technique on Riemannian manifolds developed by Klartag [19] (and by Cavalletti–Mondino [10] on metric measure spaces), sharp $p$-Poincaré inequalities under the BE($K, N$) condition and the CD($K, N$) condition have been obtained by E. Calderon in his Ph.D thesis [9].

In addition, Measure Contraction Property MCP($K, N$) was introduced independently by Ohta [23] and Sturm [25] as a weaker variant of CD($K, N$) condition. The family MCP($K, N$) is strictly larger than CD($K, N$). It was discovered by Juillet [18] that the $n$-th Heisenberg group equipped with the left-invariant measure, which is the simplest sub-Riemannian space, does not satisfy any CD($K, N$) condition but do satisfy MCP($0, N$) for $N \geq 2n + 3$. More recently, interpolation inequalities à la Cordero–Erausquin–McCann–Schmuckenshläger [14] were obtained, under suitable modifications, by Barilari and Rizzi [8] in the ideal sub-Riemannian setting, Badreddine and Rifford [4] for Lipschitz Carnot group, and by Balogh, Kristály and Sipos [7] for the Heisenberg group. As a consequence, more and more examples of spaces verifying MCP but not CD have been found, e.g. the generalized H-type groups and the Grushin plane (for more details, see [8]).

In [17], the author and Milman proved a sharp Poincaré inequality for subsets of (essentially non-branching) MCP($K, N$) metric measure spaces, whose diameter is bounded from above by $D$. The current paper is a subsequent work of [17]. We will study the general $p$-Poincaré inequality within the class of spaces verifying measure contraction property. Thanks to measure decomposition theorem (c.f. Theorem 3.5 [12]), it suffices to study the corresponding eigenvalue problems on one-dimensional model spaces introduced by Milman [21]. In particular, we identify a family of one-dimensional MCP($K, N$)-densities with diameter $D$, not verifying CD($K, N$), achieving the optimal constant $\lambda_{K,N,D}^p$.

As a basic problem in metric geometry, the rigidity theorem helps us to understand more about the spaces under study. For the family of metric measure spaces satisfying RCD($K, N$) condition with $K > 0$, a space that reaches the equality
in (1.1) must have maximal diameter $\pi \sqrt{\frac{N-1}{K}}$. By maximal diameter theorem this space is isomorphic to a spherical suspension (see [11] and references therein for details). For MCP($K, N$) spaces, the situation is very different. For $K > 0$, due to lack of monotonicity, we do not know whether a space that reaches the minimal spectrum has maximal diameter. For $K \leq 0$, by monotonicity (Proposition 3.6) and one-dimensional rigidity (Theorem 3.13) we can prove the rigidity Theorem 4.2.

2. Prerequisites

Let $(X, d)$ be a complete metric space and $m$ be a locally finite Borel measure with full support. Denote by Geo($X, d$) the space of geodesics. We say that a set $\Gamma \subset$ Geo($X, d$) is non-branching if for any $\gamma_1, \gamma_2 \in \Gamma$, it holds:

$$\exists t \in (0, 1) \text{ s.t. } \gamma_s^1 = \gamma_s^2, \forall s \in [0, t] \Rightarrow \gamma_s^1 = \gamma_s^2, \forall s \in [0, 1].$$

Let $(\mu, t)$ be a $L^2$-Wasserstein geodesic. Denote by OptGeo($\mu_0, \mu_1$) the space of all probability measures $\Pi \in \mathcal{P}$ (Geo($X, d$)) such that $(e_t)_\sharp \Pi = \mu$, (c.f. Theorem 2.10 [1]) where $e_t$ denotes the evaluation map $e_t(\gamma) := \gamma_t$. We say that $(X, d, m)$ is essentially non-branching if for any $\mu_0, \mu_1 \ll m$, any $\Pi \in$ OptGeo($\mu_0, \mu_1$) is concentrated on a set of non-branching geodesics.

It is clear that if $(X, d)$ is a smooth Riemannian manifold then any subset $\Gamma \subset$ Geo($X, d$) is a set of non-branching geodesics, in particular any smooth Riemannian manifold is essentially non-branching. In addition, many sub-Riemannian spaces are also essentially non-branching, which follows from the existence and uniqueness of the optimal transport map on some ideal sub-Riemannian manifolds (c.f. [15]).

Given $K, N \in \mathbb{R}$, with $N > 1$, we set for $(t, \theta) \in [0, 1] \times \mathbb{R}^+$,

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K \theta^2 \geq (N-1)\pi^2, \\ \sin(t \sqrt{K/(N-1)}) \sin(\theta \sqrt{K/(N-1)}), & \text{if } 0 < K \theta^2 < (N-1)\pi^2, \\ t, & \text{if } K \theta^2 = 0, \\ \sinh(t \sqrt{K/(N-1)}) \sinh(\theta \sqrt{K/(N-1)}), & \text{if } K \theta^2 < 0. \end{cases}$$

and

$$\tau_{K,N}^{(t)} := \frac{1}{\Gamma} \left( \sigma_{K,N-1}^{(t)} \right)^{1-\frac{1}{N}}.$$

Definition 2.1. (Measure Contraction Property MCP($K, N$)) We say that an essentially non-branching metric measure space $(X, d, m)$ satisfies measure contraction property MCP($K, N$) if for any point $o \in X$ and Borel set $A \subset X$ with $0 < m(A) < \infty$ (and with $A \subset B(o, \sqrt{(N-1)/K}$ if $K > 0$), there is $\Pi \in$ OptGeo($\frac{1}{m(A)}m|_A, \delta_o$) such that the following inequality holds for all $t \in [0, 1]$

$$\frac{1}{m(A)}m \geq (e_t)_\sharp \left[ \tau_{K,N}^{(1-t)} d(\gamma_0, \gamma_1) \right]^N \Pi(d\gamma). \quad (2.1)$$
Theorem 2.2. (Localization for MCP($K, N$) spaces, Theorem 3.5 [12]) Let $(X, d, m)$ be an essentially non-branching metric measure space satisfying MCP($K, N$) condition for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Then for any 1-Lipschitz function $u$ on $X$, the non-branching transport set $T_u$ associated with $u$ (roughly speaking, $T_u$ coincides with $\{|\nabla u| = 1\}$ up to $m$-measure zero set) admits a disjoint family of unparameterized geodesics $(X_q)_{q \in \Omega}$ such that

$$m(T_u \setminus \cup X_q) = 0,$$

and

$$m|_{T_u} = \int_{\Omega} m_q \, dq(q), \quad q(\Omega) = 1 \text{ and } m_q(X_q) = 1 \text{ q-a.e. } q \in \Omega.$$ 

Furthermore, for q-a.e. $q \in \Omega$, $m_q$ is a Radon measure with $m_q \ll H^1|_{X_q}$ and $(X_q, d, m_q)$ satisfies MCP($K, N$).

3. One dimensional models

3.1. One dimensional MCP densities

Let $h \in L^1(\mathbb{R}^+, \mathcal{L}^1)$ be a non-negative Borel function. It is known (see e.g. Lemma 4.1 [17]) that $(\text{supp } h, |\cdot|, h\mathcal{L}^1)$ satisfies MCP($K, N$) condition if and only if $h$ is a MCP($K, N$) density in the following sense

$$h(tx_1 + (1-t)x_0) \geq \sigma_{K, N-1}(|x_1 - x_0|)N^{-1}h(x_0)$$

for all $x_0, x_1 \in \text{supp } h$ and $t \in [0, 1]$.

Definition 3.1. Given $K \in \mathbb{R}$, $N > 1$. Denote by $D_{K, N}$ the Bonnet–Meyers diameter upper-bound:

$$D_{K, N} := \begin{cases} \frac{\pi}{\sqrt{K/(N-1)}} & \text{if } K > 0 \\ +\infty & \text{otherwise} \end{cases}.$$ 

For any $D > 0$, we define $\mathcal{F}_{K, N, D}$ as the collection of MCP($K, N$) densities $h \in L^1(\mathbb{R}^+, \mathcal{L}^1)$ with supp $h = [0, D \wedge D_{K, N}]$.

For $\kappa \in \mathbb{R}$, we define the function $s_\kappa : [0, +\infty) \mapsto \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$)

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta), & \text{if } \kappa > 0, \\ \theta, & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta), & \text{if } \kappa < 0. \end{cases}$$

It can be seen that (3.1) is equivalent to

$$\left(\frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}\right)^{N-1}$$

for all $[x_0, x_1] \subset [a, b] \subset \text{supp } h$.

Furthermore, we have the following characterization.
Lemma 3.2. Given $D \leq D_{K,N}$, a density $h$ is in $\mathcal{F}_{K,N,D}$ if and only if
\[
\left( \frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)} \right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left( \frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)} \right)^{N-1} \quad \forall \ 0 \leq x_0 \leq x_1 \leq D.
\]

Furthermore, $h \in \mathcal{F}_{K,N,D}$ if and only if $\ln h$ is $L^1$-a.e. differentiable and
\[-h(x) \cot_{K,N,D}(D-x) \leq h'(x) \leq h(x) \cot_{K,N,D}(x), \quad L^1 \text{ - a.e. } x \in [0,D]\]
where the function $\cot_{K,N,D} : [0,D] \mapsto \mathbb{R}$ is defined by
\[
\cot_{K,N,D}(x) := \begin{cases} \sqrt{K(N-1)} \cot \left( \frac{K}{N-1} x \right), & \text{if } K > 0, \\ (N-1)/x, & \text{if } K = 0, \\ \sqrt{-K(N-1)} \coth \left( \frac{-K}{N-1} x \right), & \text{if } K < 0. \end{cases}
\]

Proof. It can be checked that the function
\[a \mapsto \frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}\]
is non-decreasing on $[0,x_0]$, and the function
\[b \mapsto \frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}\]
is non-decreasing on $[x_1,D]$. Thus, (3.4) follows from (3.3).

Furthermore, for any $h \in \mathcal{F}_{K,N,D}$, it can be seen that (3.4) holds if and only if
\[x \mapsto \frac{\left( s_{K/(N-1)}(D-x) \right)^{N-1}}{h(x)} \text{ is non-increasing,} \quad (3.5)\]
and
\[x \mapsto \frac{\left( s_{K/(N-1)}(x) \right)^{N-1}}{h(x)} \text{ is non-decreasing.} \quad (3.6)\]

From (3.4) we can see that $\ln h$ is locally Lipschitz, so $\ln h$ is differentiable almost everywhere. So, by (3.5) and (3.6) we know (3.4) is equivalent to
\[
\left( \ln s_{K/(N-1)}^{N-1}(D-\cdot) \right)' \leq (\ln h)' = \frac{h'}{h} \leq \left( \ln s_{K/(N-1)}^{N-1} \right)' \quad L^1 \text{ - a.e. on } [0,D]\]
which is the thesis. \(\square\)

Notice that the function
\[[0,D] \ni x \mapsto \frac{s_{K/(N-1)}(D-x)}{s_{K/(N-1)}(x)}\]
is decreasing. By Lemma 3.2 (or (3.5) and (3.6)) we immediately obtain the following rigidity result.
Lemma 3.3. (One dimensional rigidity) Denote \( h_{K,N,D}^1 = \left( \frac{s_K}{(N-1)}(x) \right)^{N-1} |_{[0,D]} \) and \( h_{K,N,D}^2 = \left( \frac{s_K}{(N-1)}(D - x) \right)^{N-1} |_{[0,D]} \). Then we have \( h_{K,N,D}^1, h_{K,N,D}^2 \in \mathcal{F}_{K,N,D} \). Furthermore, \( h_{K,N,D}^1 \) is the unique \( \mathcal{F}_{K,N,D} \) density (up to multiplicative constants) satisfying

\[
h'(x) = h(x) \cot K_{N,D}(x)
\]

and \( h_{K,N,D}^2 \) is the unique \( \mathcal{F}_{K,N,D} \) density satisfying

\[
h'(x) = -h(x) \cot K_{N,D}(D - x).
\]

3.2. One dimensional \( p \)-Poincaré inequalities

Definition 3.4. For \( p \in (1, \infty) \) and \( h \in \mathcal{F}_{K,N,D} \), the \( p \)-spectral gap associated with \( h \) is defined by

\[
\lambda^p_{h} := \inf \left\{ \int |u'|^p h \, dx : u \in \text{Lip} \cap L^p, \int u |u|^{p-2} h \, dx = 0, u \neq 0 \right\}.
\]

Definition 3.5. Let \( K \in \mathbb{R}, D > 0 \) and \( N > 1 \). The optimal constant \( \lambda^p_{K,N,D} \) is defined as the infimum of all \( p \)-spectral gaps associated with admissible densities, i.e., \( \lambda^p_{K,N,D} \) is given by

\[
\lambda^p_{K,N,D} := \inf_{h \in \bigcup \mathcal{D} \leq D} \lambda^p_{h}.
\]

Proposition 3.6. Let \( K \in \mathbb{R}, D > 0 \) and \( N > 1 \). The function \( D \mapsto \lambda^p_{K,N,D} \) is non-increasing, and

\[
\lambda^p_{K,N,D} = \inf_{h \in \bigcup \mathcal{D} \leq D} \lambda^p_{h}.
\]

If \( K \leq 0 \), the map \( D \mapsto \lambda^p_{K,N,D} \) is strictly decreasing, and

\[
\lambda^p_{K,N,D} = \inf_{h \in \mathcal{F}_{K,N,D} \cap C^\infty} \lambda^p_{h}.
\]

Proof. By Lemma 3.2 we know MCP densities are locally Lipschitz. Thus, using a standard mollifier we can approximate \( h \) uniformly by smooth MCP densities. Then by a simple approximation argument (see e.g. Proposition 4.8 [17]) we can prove

\[
\lambda^p_{K,N,D} = \inf_{h \in \bigcup \mathcal{D} \leq D} \lambda^p_{h}.
\]

Let \( h \in \mathcal{F}_{K,N,D} \) be a MCP density for some \( D' > 0 \), and \( u \) be an admissible function in (3.7). Then \( \tilde{h}(x) := h(D'/D \cdot x) \in \mathcal{F}_{K',N,D} \) with \( K' = \left( \frac{D'}{D} \right)^2 K', \)
and \( \tilde{u}(x) := u(D'x) \) is also an admissible function. By computation, we have
\[
\frac{\int |\tilde{u}'|^p \tilde{h} \, dx}{\int |\tilde{u}'|^p h \, dx} = \left( \frac{D'}{D} \right)^p \frac{\int |u'|^p h \, dx}{\int |u'|^p h \, dx}. 
\]
Therefore, if \( K \leq 0 \) and \( D' < D \), we have
\[
\inf_{h \in \mathcal{F}_{K,N,D}} \lambda^{p,h} \leq \inf_{h \in \mathcal{F}_{K',N,D'}} \lambda^{p,h} \leq \left( \frac{D'}{D} \right)^p \left( \inf_{h \in \mathcal{F}_{K,N,D'}} \lambda^{p,h} \right) < \inf_{h \in \mathcal{F}_{K,N,D'}} \lambda^{p,h} \quad \text{and so}
\]
\[
\lambda^{p}_{K,N,D} < \lambda^{p}_{K,N,D'}. 
\]
Then we obtain (3.9). \( \square \)

**Remark 3.7.** The difference between the cases \( K \leq 0 \) and \( K > 0 \) was already observed in [13] in the isoperimetric context and in [17] in the 2-Poincaré context. It is known that the monotonicity property (3.9) is false when \( K > 0 \).

In order to study the equation (3.18) in Theorem 3.10, we recall some basic facts about generalized trigonometric functions \( \sin_p \) and \( \cos_p \).

**Definition 3.8.** For \( p \in (1, +\infty) \), define \( \pi_p \) by
\[
\pi_p := \int_{-1}^{1} \frac{dt}{1 - |t|^p} = \frac{2\pi}{p \sin(\pi/p)} > 0. 
\]
The periodic \( C^1 \) function \( \sin_p : \mathbb{R} \mapsto [-1, 1] \) is defined on \([-\pi/p, 2\pi/p] \) by:
\[
\sin_p(t) = \sin_p(\pi_p - t) \quad \text{if} \quad t \in \left[ -\frac{\pi_p}{2}, \frac{\pi_p}{2} \right], \\
\sin_p(t) = \frac{\sin \left( \int_{0}^{t} \sin_p(t) \, dx \right)}{(1-|s|^p)^{1/2}} \quad \text{if} \quad t \in \left[ \frac{\pi_p}{2}, \frac{3\pi_p}{2} \right]. 
\]
It can be seen that \( \sin_p(0) = 0 \) and \( \sin_p \) is strictly increasing on \([-\frac{\pi_p}{2}, \frac{\pi_p}{2}] \). Define \( \cos_p(t) = \frac{d}{dt} \sin_p(t) \), then we have the following generalized trigonometric identity
\[
|\sin_p(t)|^p + |\cos_p(t)|^p = 1. 
\]

**Definition 3.9.** Let \( h^i_{K,N,D}, i = 1, 2 \) be MCP(\( K, N \)) densities defined in Lemma 3.3. Define \( h_{K,N,D} \) by
\[
h_{K,N,D}(x) := \begin{cases} 
    h^1_{K,N,D}(x) & \text{if } x \in \left[ \frac{D}{2}, D \right], \\
    h^2_{K,N,D}(x) & \text{if } x \in \left[ 0, \frac{D}{2} \right].
\end{cases}
\]
Define \( T_{K,N,D} \) by
\[
T_{K,N,D} := (\ln h_{K,N,D})' = \begin{cases} 
    \cot_{K,N,D}(x) & \text{if } x \in \left[ \frac{D}{2}, D \right], \\
    -\cot_{K,N,D}(D - x) & \text{if } x \in \left[ 0, \frac{D}{2} \right].
\end{cases}
\]
By Lemma 3.2 we know \( h_{K,N,D} \) is a MCP(\( K, N \)) density. It can be seen that (c.f. Lemma 3.4 [13]) \( h_{K,N,D} \) does not satisfy any forms of CD condition.
Theorem 3.10. (One dimensional $p$-spectral gap) Let $K \in \mathbb{R}$, $N > 1$, $D > 0$. Denote by $\hat{\lambda}^p_{K,N,D}$ the minimal $\lambda$ such that the following initial value problem has a solution:

\[
\begin{align*}
\varphi' &= \left( \frac{\lambda}{p-1} \right)^{\frac{1}{p}} + \frac{1}{p-1} T_{K,N,D} \cos_p^{-1}(\varphi) \sin_p(\varphi), \\
\varphi(0) &= -\frac{\pi_p}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(D) = \frac{\pi_p}{2}.
\end{align*}
\] (3.11)

Then $\lambda^{p,h} \geq \hat{\lambda}^p_{K,N,D}$ for any $h \in \mathcal{F}_{K,N,D}$.

Proof. Step 1. Firstly we will show the existence of $\hat{\lambda}^p_{K,N,D}$.

By Lemma 3.2 we know $T_{K,N,D} \in C^\infty((0, \frac{D}{2}) \cup (\frac{D}{2}, D))$ and $-\cot_{K,N,D}(D - \cdot) \leq T'_{K,N,D} \leq \cot_{K,N,D}$. Denote $T = T_{K,N,D}$, and denote by $u = u^{T,\lambda}$ the (unique) solution of the following equation:

\[
\begin{align*}
\left(u'|u'|^{p-2}\right)' + Tu'|u'|^{p-2} + \lambda u|u|^{p-2} &= 0, \\
u\left(\frac{D}{2}\right) &= 0.
\end{align*}
\] (3.12)

Next we will study the equation (3.12) using a version of the so-called Pfüfer transformation. Define the functions $e = e^{T,\lambda}$ and $\varphi = \varphi^{T,\lambda}$ by:

\[
\alpha := \left( \frac{\lambda}{p-1} \right)^{\frac{1}{p}}, \quad \alpha u = e \sin_p(\varphi), \quad u' = e \cos_p(\varphi).
\]

By Lemma 3.11 we know that $\varphi, e$ solve the following equation:

\[
\begin{align*}
\varphi' &= \alpha + \frac{1}{p-1} T \cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\frac{d}{dt} \ln e &= \frac{\varphi}{e} = -\frac{1}{p-1} T \cos_p(\varphi)|^{p-1}.
\end{align*}
\] (3.13)

Consider the following initial valued problem on $(0, \frac{D}{2}) \cup (\frac{D}{2}, D)$.

\[
\begin{align*}
\varphi' &= \alpha + \frac{1}{p-1} T \cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\varphi\left(\frac{D}{2}\right) &= 0.
\end{align*}
\] (3.14)

By Cauchy’s theorem we have the existence, uniqueness and continuous dependence on the parameters. Fix an $\epsilon \in (0, \frac{D}{2})$. We can find $\alpha = \alpha(\epsilon) > 0$, such that $\varphi'(x) > \frac{\pi_p}{D - 2\epsilon} > 0$ for all $x \in (\epsilon, \frac{D}{2})$. So there exists $a_\alpha \in (0, \frac{D}{2})$ such that $\varphi(a_\alpha) = -\frac{\pi_p}{2}$. Similarly, there is $b_\alpha \in (\frac{D}{2}, D]$ such that $\varphi(b_\alpha) = \frac{\pi_p}{2}$. Conversely, assume there is $\alpha > 0$ such that the following problem has a solution $\varphi$ for some $a_\alpha \in (0, \frac{D}{2})$ and $b_\alpha \in (\frac{D}{2}, D)$:

\[
\begin{align*}
\varphi' &= \alpha + \frac{1}{p-1} T \cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\varphi(a_\alpha) &= -\frac{\pi_p}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(b_\alpha) = \frac{\pi_p}{2}.
\end{align*}
\] (3.15)

Then for any $\alpha' > \alpha$, the following problem also has a solution for some $a_{\alpha'} \in (a_\alpha, \frac{D}{2})$ and $b_{\alpha'} \in (\frac{D}{2}, b_\alpha)$:

\[
\begin{align*}
\varphi' &= \alpha' + \frac{1}{p-1} T \cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\varphi(a_{\alpha'}) &= -\frac{\pi_p}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(b_{\alpha'}) = \frac{\pi_p}{2}.
\end{align*}
\] (3.16)
Therefore, by connectedness, there is a minimal $\tilde{\lambda} \geq 0$ such that for any $\lambda > \tilde{\lambda}$, there exist $\varphi = \varphi^{T, \lambda}$, $0 \leq a^\lambda < \frac{D}{2}$ and $\frac{D}{2} < b^\lambda \leq D$ such that

$$
\begin{cases}
\varphi' = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} T \cos_p^{p-1}(\varphi) \sin_p(\varphi), \\
\varphi(a^\lambda) = -\frac{\pi_p}{2}, \varphi\left(\frac{D}{2}\right) = 0, \varphi(b^\lambda) = \frac{\pi_p}{2}.
\end{cases}
$$

(3.17)

By continuous dependence on the parameter $\lambda$, we know (3.17) has a solution $\varphi_\infty$ for $\gamma = \tilde{\gamma}$, some $a^{\lambda} \in [0, \frac{D}{2})$ and $b^{\lambda} \in (\frac{D}{2}, D]$. In particular, $\tilde{\lambda} > 0$.

Step 2. Given $h \in F_{K, N, D} \cap C^\infty$, we will show that $\hat{\lambda}_{K, N, D}^p \leq \lambda_{p, h}^p$.

First of all, by a standard variational argument we can see that $\hat{\lambda}_{p, h}^p$ is the smallest positive real number such that there exists a non-zero $u \in W^{1, p}([0, D], hL^1)$ solving the following equation (in weak sense):

$$\Delta_p^h u = -\lambda u |u|^{p-2}$$

with Neumann boundary condition, where $\Delta_p^h u$ is the weighted $p$-Laplacian on $([0, D], |\cdot|, hL^1)$:

$$\Delta_p^h u = \Delta_p u + u' |u'|^{p-2} (\log h)' = \left(u' |u'|^{p-2}\right)' + u' |u'|^{p-2} \frac{h'}{h}.$$  

By regularity theory we know $u \in C^{1, \alpha} \cap W^{1, p}$ for some $\alpha > 0$, and $u \in C^{2, \alpha}$ if $u' \neq 0$. Conversely, for any $u$ solving the Neumann problem (3.18), we have $\int u u'|u'|^{p-2} h \, dx = 0$ and $\int |u'|^p h \, dx = \lambda \int |u|^p h \, dx$.

Assume by contradiction that $\lambda_{p, h} < \hat{\lambda}_{K, N, D}^p$. From the monotonicity argument in Step 1, we can see that there is $\lambda < \hat{\lambda}_{K, N, D}^p$ such that the following equation has a (monotone) solution $\varphi = \varphi^{h', \lambda}_{\bar{T}, D}$:

$$
\begin{cases}
\varphi' = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} h' \cos_p^{p-1}(\varphi) \sin_p(\varphi), \\
\varphi(0) = -\frac{\pi_p}{2}, \varphi(D) = \frac{\pi_p}{2}.
\end{cases}
$$

(3.19)

Without loss of generality (or by symmetry), we may assume there is $a' \in [\frac{D}{2}, D]$ such that $\varphi^{h', \lambda}_{\bar{T}, D}(a') = 0$. Suppose there is a point $x_0 \in (a', D)$ such that $\varphi^{h', \lambda}_{\bar{T}, D}(x_0) = \varphi^{T, \lambda}_{K, N, D}(x_0)$. From Lemma 3.2 we know that $h' \leq T_{K, N, D}$. So we know

$$(\varphi^{h', \lambda}_{\bar{T}, D})' (x_0) < (\varphi^{T, \lambda}_{K, N, D})' (x_0).$$

Therefore,

$$\varphi^{h', \lambda}_{\bar{T}, D} (x) < \varphi^{T, \lambda}_{K, N, D} (x)$$

for all $x \in (a', D)$, which contradicts to the fact that $\varphi^{h', \lambda}_{\bar{T}, D}(\frac{D}{2}) = \varphi^{T, \lambda}_{K, N, D}(\frac{D}{2}) = \frac{\pi_p}{2}$. \qed
The following formulas have been used in [22, 26]. We give a proof for completeness.

**Lemma 3.11.** Let $e, \varphi, T$ be functions defined in the proof of Theorem 3.10. Then we have

\[
\begin{align*}
\varphi' &= \alpha + \frac{1}{p-1} T \cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\
\frac{d}{dt} \ln e &= \frac{\varphi'}{e} = -\frac{1}{p-1} T |\cos_p(\varphi)|^p.
\end{align*}
\]

(3.20)

**Proof.** Firstly, we have

\[
(u' | u |^{p-2})' = (e \cos_p(\varphi) | e \cos_p(\varphi) |^{p-2})'.
\]

\[
= |e \cos_p(\varphi)|^{p-2} (e' \cos_p(\varphi) + e \sin'_p(\varphi) \varphi')
\]

\[+ e \cos_p(\varphi) (p-2) e \cos_p(\varphi) | e \cos_p(\varphi) |^{p-4} \]

\[\times (e' \cos_p(\varphi) + e \sin'_p(\varphi) \varphi')
\]

\[= |e \cos_p(\varphi)|^{p-2} (p-1) (e' \cos_p(\varphi) + e \sin'_p(\varphi) \varphi').
\]

Combining with (3.12) we obtain

\[
|e \cos_p(\varphi)|^{p-2} (e' \cos_p(\varphi) + e \sin'_p(\varphi) \varphi') \sin_p(\varphi)
\]

\[+ \frac{1}{p-1} T e \cos_p(\varphi) \sin_p(\varphi) | e \cos_p(\varphi) |^{p-2} + \frac{\lambda}{p-1} \alpha^{1-p} e^{p-1} | \sin_p(\varphi) |^p = 0.
\]

Differentiating the equation $\alpha u = e \sin_p(\varphi)$ and substituting $u'$ by $e \cos_p(\varphi)$, we get

\[
\alpha e \cos_p(\varphi) = e' \sin_p(\varphi) + e \cos_p(\varphi) \varphi'.
\]

Differentiating the identity $| \sin_p(t) |^p + | \cos_p(t) |^p = 1$ we also have

\[
| \sin_p(t) |^{p-2} \sin_p(t) \cos_p(t) + | \cos_p(t) |^{p-2} \cos_p(t) \sin'_p(t) = 0.
\]

Therefore,

\[
|e \cos_p(\varphi)|^{p-2} (e' \cos_p(\varphi) + e \sin'_p(\varphi) \varphi') \sin_p(\varphi)
\]

\[= |e \cos_p(\varphi)|^{p-2} (\alpha e \cos^2_p(\varphi) - e \cos_p^2(\varphi) \varphi' + e \sin'_p(\varphi) \sin_p(\varphi) \varphi')
\]

\[= \alpha e^{p-1}(\varphi) | \cos_p(\varphi) |^p - e^{p-1}(\varphi) \left( | \cos_p(\varphi) |^p + | \sin_p(\varphi) |^p \right) \varphi'
\]

\[= \alpha e^{p-1}(\varphi) | \cos_p(\varphi) |^p - e^{p-1} \varphi'.
\]

Combining the results above, we prove the lemma. \[\square\]

Combining Proposition 3.6 and Theorem 3.10, we get the following corollary immediately.

**Corollary 3.12.** We have the following sharp $p$-spectral gap estimates for one dimensional models:
\[ \lambda^p_{K,N,D} = \begin{cases} \hat{\lambda}^p_{K,N,D} & \text{if } K \leq 0 \\ \inf_{D' \in \{0, \min(D,D_{K,N})\}} \hat{\lambda}^p_{K,N,D'} & \text{if } K > 0 \end{cases} \]

**Theorem 3.13.** (One dimensional rigidity) Given \( K \leq 0 \), \( N > 1 \) and \( D > 0 \). If \( \lambda^{p,h} = \hat{\lambda}^p_{K,N,D} \) for some \( h \in \mathcal{F}_{K,N,D} \). Then \( h = h_{K,N,D} \) up to a multiplicative constant.

**Proof.** Assume \( \lambda^{p,h} = \hat{\lambda}^p_{K,N,D} \) for some \( h \in \mathcal{F}_{K,N,D} \). Then there is \( h_n \in \mathcal{F}_{K,N,D} \cap C^\infty \) with \( h_n \to h \) uniformly, and a decreasing sequence \( (\lambda^{p,h_n}) \) with \( \lambda^{p,h_n} \to \hat{\lambda}^p_{K,N,D} \), such that \( \varphi_n = \varphi_{h_n}^{-1} \lambda^{p,h_n} \) solves the following equation:

\[
\begin{align*}
\varphi_n'(x) &= \left( \frac{\lambda^{p,h_n}}{p-1} \right)^{\frac{1}{p}} + \frac{1}{p-1} \frac{h_n'}{h_n} \cos^p(\varphi_n) \sin_p(\varphi_n), \\
\varphi_n(0) &= -\frac{\pi_p}{2}, \quad \varphi_n(D) = \frac{\pi_p}{2}.
\end{align*}
\] (3.21)

From Lemma 2.1 we know that \( \{\varphi'_n\}_n \) and \( \{\varphi_n\}_n \) are uniformly bounded. By Arzelà–Ascoli theorem we may assume \( \varphi_n \to \varphi_\infty \) uniformly for some Lipschitz function \( \varphi_\infty \).

By minimality of \( \hat{\lambda}^p_{K,N,D} \) and symmetry, we can see that \( \lim_{n \to \infty} \varphi_n^{-1}(t) \) exists for any \( t \in [-\frac{\pi_p}{2}, \frac{\pi_p}{2}] \) and

\[
\lim_{n \to \infty} \varphi_n^{-1}(t) = (\varphi_{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})^{-1}(t).
\]

In fact, assume by contradiction that \( \lim_{n \to \infty} \varphi_n^{-1}(t) \neq (\varphi_{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})^{-1}(t) \) for some \( t \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}) \). By symmetry we may assume there are \( N_1 \in \mathbb{N} \) and \( \delta > 0 \), such that \( \delta_n := (\varphi_{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})^{-1}(t) - \varphi_n^{-1}(t) \geq \delta \) for all \( n \geq N_1 \). Define a MCP(\( K, N \)) density \( \tilde{h}_n \) by

\[
\tilde{h}_n(x) := \begin{cases} 
\frac{h_n(x)}{h_n(\varphi_n^{-1}(t))} h_{K,N,D}(x + \delta_n) & \text{if } x \in [0, \varphi_n^{-1}(t)], \\
\frac{h_n(\varphi_n^{-1}(t) + \delta_n)}{h_{K,N,D}(\varphi_n^{-1}(t) + \delta_n)} h_{K,N,D}(x + \delta_n) & \text{if } x \in [\varphi_n^{-1}(t), D - \delta_n].
\end{cases}
\]

Then \( \bar{\varphi}_n = \varphi_{\delta_n}^{h_n} \lambda^{p,h_n} \) satisfies \( (\bar{\varphi}_n)^{-1}(\frac{\pi_p}{2}) < D - \frac{\delta}{2} \) for \( n \) large enough, which contradicts to Proposition 3.6 and the minimality of \( \hat{\lambda}^p_{K,N,D} \).

In conclusion, \( \varphi_\infty = \varphi_{T_{K,N,D},\hat{\lambda}^p_{K,N,D}} \) and we have \( \varphi_n \to \varphi_{T_{K,N,D},\hat{\lambda}^p_{K,N,D}} \) uniformly.
Then we get
\[
\frac{\pi_p}{2} = \varphi_n(\varphi_n^{-1}(0)) - \varphi_n(0)
\]
\[
= \lim_{n \to \infty} \int_0^{\varphi_n^{-1}(0)} \frac{D}{2} \left( \frac{\lambda_p h_n}{p - 1} \right)^{\frac{1}{p}} + \frac{1}{p - 1} h_n' \cos_p^{-1}(\varphi_n) \sin_p(\varphi_n) \, dx
\]
\[
\leq \lim_{n \to \infty} \int_0^{\varphi_n^{-1}(0)} \frac{D}{2} \left( \frac{\lambda_p h_n}{p - 1} \right)^{\frac{1}{p}} + \frac{1}{p - 1} T_{K,N,D} \cos_p^{-1}(\varphi_n) \sin_p(\varphi_n) \, dx
\]
\[
= \int_0^{D} \left( \frac{\hat{\lambda}_{K,N,D}}{p - 1} \right)^{\frac{1}{p}} + \frac{1}{p - 1} T_{K,N,D} \cos_p^{-1}(\varphi_n) \sin_p(\varphi_n) \, dx
\]
\[
= \frac{\pi_p}{2}.
\]

Therefore,
\[
(\ln h_n)' = \frac{h_n'}{h_n} \to T_{K,N,D} = \frac{h_n'}{h_{K,N,D}}
\]
in $L^1([0, D \frac{D}{2}])$, $\cos_p^{-1}(\varphi_{T_{K,N,D},\hat{\lambda}_{K,N,D}}) \sin_p(\varphi_{T_{K,N,D},\hat{\lambda}_{K,N,D}}) \mathcal{L}^1$). By symmetry, we can see that $(\ln h_n)' \to (\ln h_{K,N,D})'$ in $L^1([0, D], \mathcal{L}^1)$. Hence $h = h_{K,N,D}$ up to a multiplicative constant. \hfill \Box

4. $p$-spectral gap

4.1. Sharp $p$-spectral gap estimates

Using standard localization argument (c.f. Theorem 1.1 [17], Theorem 4.4 [11]), we can prove the sharp $p$-Poincaré inequality with one dimensional results.

**Theorem 4.1.** (The sharp $p$-spectral gap under MCP($K$, $N$)) Let $(X, \text{d}, \text{m})$ be an essentially non-branching metric measure space satisfying MCP($K$, $N$) for some $K \in \mathbb{R}$, $N \in (1, \infty)$ and $\text{diam}(X) \leq D$. For any $p > 1$, define $\lambda_p(X,\text{d},\text{m})$ as the optimal constant in $p$-Poincaré inequality on $(X, \text{d}, \text{m})$:

$$\lambda_p(X,\text{d},\text{m}) := \inf \left\{ \frac{\int |\nabla f|^p \, dm}{\int |f|^p \, dm} : f \in \text{Lip} \cap L^p, \int |f|^p |f|^{-2} \, dm = 0, f \neq 0 \right\}. $$

Then we have the following sharp estimate

$$\lambda_p(X,\text{d},\text{m}) \geq \lambda_p(K,N,D) = \begin{cases} \frac{\hat{\lambda}_{K,N,D}}{p} & \text{if } K \leq 0 \\ \inf_{D' \in (0, \min(D,D,K,N))} \frac{\hat{\lambda}_{K,N,D'}}{p} & \text{if } K > 0. \end{cases}$$
Proof. Let \( \bar{f} = f |f|^{p-2} \) be a Lipschitz function with \( \int \bar{f} = 0 \). Let \( \bar{f}^\pm \) denote the positive and the negative parts of \( \bar{f} \) respectively. Then we have \( \int \bar{f}^+ = -\int \bar{f}^- \). Consider the \( L^1 \)-optimal transport problem from \( \mu_0 := \bar{f}^+ m \) to \( \mu_1 := -\bar{f}^- m \). By Theorem 2.2, there exists a family of disjoint unparameterized geodesics \( \{X_q\}_{q \in \Omega} \) of length at most \( D \), such that

\[
m(X \setminus \bigcup X_q) = 0, \quad m = \int_{\Omega} m_q \, dq(q)
\]

where \( m_q = h_q \mathcal{H}^1|_{X_q} \) for some \( h_q \in \mathcal{F}_{K,N,D} \) with \( D'_q \leq D \), \( m_q(X_q) = m(X) \) and

\[
\int \bar{f} h_q \, d\mathcal{H}^1|_{X_q} = 0
\]

for \( q \)-a.e. \( q \in \Omega \).

Denote \( f_q = f |_{X_q} \). By definition we obtain

\[
\int |f'_q|^p h_q \, d\mathcal{H}^1|_{X_q} \geq \lambda^{p,h_q} \int |f_q|^p h_q \, d\mathcal{H}^1|_{X_q} \geq \lambda^{p}_{K,N,D} \int |f_q|^p h_q \, d\mathcal{H}^1|_{X_q}.
\]

Notice that \( |f'_q| \leq |\nabla f| \). Thus, we have

\[
\lambda^{p}_{K,N,D} \int |f|^p \, dm = \lambda^{p}_{K,N,D} \int_{\Omega} \int_{X_q} |f_q|^p m_q \, dq(q)
\]

\[
\leq \int_{\Omega} \int_{X_q} |f'_q|^p m_q \, dq(q)
\]

\[
= \int |\nabla f|^p \, dm.
\]

Combining with Corollary 3.12 we prove the theorem. \( \square \)

4.2. Rigidity for \( p \)-spectral gap

In this part, we will study the rigidity for \( p \)-spectral gap under the measure contraction property. We adopt the notation \( |Df| \) to denote the weak upper gradient of a Sobolev function \( f \). We refer the readers to [2] and [16] for details about Sobolev space theory and calculus on metric measure spaces.

**Theorem 4.2.** (Rigidity for \( p \)-spectral gap) Let \((X, d, m)\) be an essentially non-branching metric measure space satisfying \( \text{MCP}(K, N) \) for some \( K \leq 0, N \in (1, \infty) \) and \( \text{diam}(X) \leq D \). Assume there is a non-zero Sobolev function \( f \in W^{1,p}(X, d, m) \) with \( \int f |f|^{p-2} \, dm = 0 \) such that

\[
\int |Df|^p \, dm - \hat{\lambda}^p_{K,N,D} \int |f|^p \, dm = 0.
\]
Then \( \text{diam}(X) = D \) and there are disjoint unparameterized geodesics \( \{X_q\}_{q \in \Omega} \) of length \( D \) such that \( \mathbf{m}(X \setminus \bigcup X_q) = 0 \). Moreover, \( \mathbf{m} \) has the following representation

\[
\mathbf{m} = \int_{\Omega} h_q \, d\mathcal{H}^1|_{X_q} \, dq(q),
\]

where \( \frac{h_q'}{h_q} = T_{K,N,D}^p \) for \( q \)-a.e. \( q \in \Omega \).

**Proof.** Similar to the proof of Theorem 4.1, we can find a measure decomposition associated with \( \tilde{f} := f | f |^{p-2} \), such that

\[
\mathbf{m}(X \setminus \bigcup X_q) = 0, \quad \mathbf{m} = \int_{\Omega} m_q \, dq(q)
\]

where \( m_q = h_q \mathcal{H}^1|_{X_q} \) for some \( h_q \in \mathcal{F}_{K,N,D'} \) with \( D_q' \leq D \), \( m_q(X_q) = m(X) \) and

\[
\int \tilde{f} h_q \, d\mathcal{H}^1|_{X_q} = 0
\]

for \( q \)-a.e. \( q \in \Omega \).

By Theorem 7.3 [3] we know \( f_q := f |_{X_q} \in W^{1,q}(X_q) \) and \( |D f_q| \leq |D f| \).

Then from the proof of Theorem 4.1 we can see that \( \lambda^{p,h_q} = \tilde{\lambda}_{K,N,D}^p \) for \( q \)-a.e. \( q \in \Omega \). By Proposition 3.6 we know that the function \( D \mapsto \tilde{\lambda}_{K,N,D}^p \) is strictly decreasing, so \( D_q' = D \) and \( \text{diam}(X) = D \). Finally, by Theorem 3.13 we know that \( \frac{h_q'}{h_q} = T_{K,N,D}^p \). \( \square \)

**Acknowledgements.** This research is part of a project which has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant Agreement No. 637851). The author thanks Emanuel Milman for helpful discussions and comments.

**References**

[1] Ambrosio, L., Gigli, N.: A user’s guide to optimal transport. In: Modelling and Optimisation of Flows on Networks, Lecture Notes in Mathematics, vol. 2062. Springer, Berlin, Heidelberg (2011)

[2] Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. 195, 289–391 (2014)

[3] Ambrosio, L., Gigli, N., Savaré, G.: Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. Rev. Mat. Iberoam. 29, 969–996 (2013)

[4] Badreddine, Z., Rifford, L.: Measure contraction properties for two-step analytic sub-Riemannian structures and Lipschitz Carnot groups. Preprint arXiv:1712.09900 (2017)

[5] Bakry, D.: L’hypercontractivité et son utilisation en théorie des semigroupes. In: Lectures on Probability Theory (Saint-Flour, 1992), vol. 1581 of Lecture Notes in Mathematics, pp. 1–114. Springer, Berlin (1994)
Sharp $p$-Poincaré inequalities under measure contraction property

[6] Bakry, D., Émery, M.: Diffusions Hypercontractives. In: Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Mathematics, pp. 177–206. Springer, Berlin (1985)

[7] Balogh, Z.M., Kristály, A., Sipos, K.: Geometric Inequalities on Heisenberg Groups. Calc. Var. Partial Differ. Equ. 57, 61 (2018)

[8] Barilari, D., Rizzi, L.: Sub-Riemannian interpolation inequalities. Invent. Math. (2018). https://doi.org/10.1007/s00222-018-0840-y

[9] Calderon, E.: Functional inequalities on weighted Riemannian manifolds subject to curvature-dimension conditions. Ph.D Thesis, Technion - I.I.T. arXiv:1905.08866 (2019)

[10] Cavalletti, F., Mondino, A.: Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. 208, 803–849 (2017)

[11] Cavalletti, F., Mondino, A.: Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds. Geom. Topol. 21, 603–645 (2017)

[12] Cavalletti, F., Mondino, A.: New formulas for the Laplacian of distance functions and applications. Preprint arXiv:1803.09687 (2018)

[13] Cavalletti, F., Santarcangelo, F.: Isoperimetric inequality under measure-contraction property. Preprint arXiv:1810.11289 (2018)

[14] Cordero-Erausquin, D., McCann, R.J., Schmuckenschläger, M.: A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. Invent. Math. 146, 219–257 (2001)

[15] Figalli, A., Rifford, L.: Mass transportation on sub-Riemannian manifolds. Geom. Funct. Anal. 20, 124–159 (2010)

[16] Gigli, N.: On the differential structure of metric spaces and applications. Mem. Am. Math. Soc. 236, vi+91 (2015)

[17] Han, B.-X., Milman, E.: Sharp poincaré inequalities under measure contraction property. Preprint arXiv:1905.05465 (2019)

[18] Juillet, N.: Geometric inequalities and generalized Ricci bounds in the Heisenberg group. Int. Math. Res. Not. IMRN 2009, 2347–2373 (2009)

[19] Klartag, B.: Needle decompositions in Riemannian geometry. Mem. Am. Math. Soc. 249, vi +77 (2017)

[20] Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. Math. (2) 169, 903–991 (2009)

[21] Milman, E.: Sharp isoperimetric inequalities and model spaces for the curvature-dimension-diameter condition. J. Eur. Math. Soc. (JEMS) 17, 1041–1078 (2015)

[22] Naber, A., Valtorta, D.: Sharp estimates on the first eigenvalue of the $p$-Laplacian with negative Ricci lower bound. Math. Z. 277, 867–891 (2014)

[23] Ohta, S.-I.: On the measure contraction property of metric measure spaces. Comment. Math. Helv. 82, 805–828 (2007)

[24] Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. 196, 65–131 (2006)

[25] Sturm, K.-T.: On the geometry of metric measure spaces. II. Acta Math. 196, 133–177 (2006)

[26] Valtorta, D.: Sharp estimate on the first eigenvalue of the $p$-Laplacian. Nonlinear Anal. 75, 4974–4994 (2012)

[27] von Renesse, M.-K., Sturm, K.-T.: Transport inequalities, gradient estimates, entropy, and Ricci curvature. Commun. Pure Appl. Math. 58, 923–940 (2005)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.