Spherically Symmetric Black Holes and Wormholes in Hybrid Metric-Palatini Gravity# 

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Abstract—The so-called hybrid metric-Palatini theory of gravity (HMPG), proposed in 2012 by T. Harko et al., is known to successfully describe both local (solar-system) and cosmological observations. This paper gives a complete description of static, spherically symmetric vacuum solutions of HMPG in the simplest case where its scalar-tensor representation has a zero scalar field potential \( V(\phi) \), and both Riemannian \( (R) \) and Palatini \( (\mathcal{R}) \) Ricci scalars are zero. Such a scalar-tensor theory coincides with general relativity with a phantom conformally coupled scalar field as a source of gravity. Generic asymptotically flat solutions either contain naked central singularities or describe traversable wormholes, and there is a special two-parameter family of globally regular black hole solutions with extremal horizons. In addition, there is a one-parameter family of solutions with an infinite number of extremal horizons between static regions and a spherical radius monotonically changing from region to region. It is argued that the obtained black hole and wormhole solutions are unstable under monopole perturbations. As a by-product, it is shown that a scalar-tensor theory with \( V(\phi) = 0 \), in which there is at least one nontrivial \( (\phi \neq \text{const}) \) vacuum solution with \( R \equiv 0 \), necessarily reduces to a theory with a conformal scalar field (the latter may be usual or phantom).

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1. INTRODUCTION

The century-old general relativity (GR) theory is still successfully passing all local observational tests but faces well-known difficulties in describing large-scale phenomena, such as rotation curves of galaxies and clusters of galaxies and the cosmological evolution (these difficulties are frequently designated as the Dark Matter and Dark Energy problems). One of the ways of addressing these problems is to assume the existence of still unobserved kinds of matter in the framework of GR, for example, weakly interacting massive particles (WIMPs) as dark matter, a cosmological constant or a scalar “quintessence” field as dark energy, etc. [1]. Another way is to generalize or modify GR, invoking more general gravitational Lagrangians than in GR (as, for instance, in \( f(R) \) theories), additional degrees of freedom (such as fundamental scalar fields in scalar-tensor theories), extra dimensions or new geometric quantities like torsion and nonmetricity, etc. [2, 3].

The recently proposed [4] hybrid metric-Palatini gravity (HMPG) theory belongs to the second approach, sometimes called “dark gravity” to distinguish it from the “dark energy” approach that remains in the framework of GR. In HMPG, one separately introduces the Riemannian metric \( g_{\mu\nu} \) and the independent connection \( \hat{\Gamma}_{\mu\nu}^\alpha \) considering the total action [4]

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}[R + F(\mathcal{R})] + S_m, \tag{1}
\]

where \( R = R[g] \) is the Ricci scalar corresponding to \( g_{\mu\nu} \), \( F(\mathcal{R}) \) is a function of the scalar \( \mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \), with the Ricci tensor \( \mathcal{R}_{\mu\nu} \) built in the standard way from the connection \( \hat{\Gamma}_{\mu\nu}^\alpha \); as usual, \( g = \det(g_{\mu\nu}) \), \( \kappa^2 \) is the gravitational constant, and \( S_m \) is the action of nongravitational matter.

Since the action (1) is varied in both \( g^{\mu\nu} \) and \( \hat{\Gamma}_{\mu\nu}^\alpha \), this theory combines the metric and Palatini approaches to gravity and thus extends the \( f(R) \) theories. It has been shown that this theory does not contradict the local tests of gravity in the Solar system [6], fairly well describes the dynamics of galaxies and galaxy clusters thus addressing the dark matter...
problem[7], and is able to explain the modern cosmological observations that tell us about the properties of an accelerating Universe. A detailed exposition of HMPG and its achievements may be found in the reviews [10, 11]. An even more general theory, containing an arbitrary function of two variables, \( R \) and \( \mathcal{R} \), is presented in [12], see a recent further work on this theory in [13].

This paper is devoted to a study of exact static, spherically symmetric solutions of HMPG in the simplest case where \( F(R) \propto \mathcal{R} \). This version of HMPG is equivalent to a scalar–tensor theory (STT) of gravity [4, 10] with zero scalar field potential \( V(\phi) \), and this theory coincides with GR with a source in the form of a phantom conformally invariant scalar field. The solutions are obtained in the standard way, by reducing the STT to the Einstein conformal frame, in which they are well known (the so-called anti–Fisher family of solutions). Many of the resulting HMPG solutions possess naked singularities, and, due to a phantom nature of the effective scalar field \( \phi \), there are generic wormhole solutions. In addition, under some special condition between the integration constants, there are nonsingular black hole solutions and a solution with infinitely many Killing horizons, each having its own radius.

The paper is organized as follows. The next section discusses the STT representation of HMPG [4, 10] and its basic features. It is shown, in particular, that if the STT with \( V \equiv 0 \) contains at least one nontrivial (that is, with \( \phi \neq \text{const} \)) vacuum solution where \( R \equiv 0 \), this theory describes a conformally invariant scalar field. Section 3 describes static, spherically symmetric solutions in this massless case \((V(\phi) = 0, F(R) \propto \mathcal{R})\) with special attention to globally regular solutions containing Killing horizons. These are a two-parameter family of asymptotically flat black holes and a one-parameter family of geometries with an infinite number of horizons. There is also a discussion of possible black hole solutions with nonzero potentials, and a brief consideration of the stability of the presently obtained solutions. Section 4 contains some concluding remarks.

2. SOME BASIC FEATURES OF HMPG

Varying the action separately in \( g^{\mu \nu} \) and \( \Gamma^\alpha_{\mu \nu} \), one obtains the field equations according to which [5, 10] \( \Gamma^\alpha_{\mu \nu} \) is a Riemannian connection corresponding to the metric \( h_{\mu \nu} = \phi g_{\mu \nu} \), conformal to \( g_{\mu \nu} \), and, as a result, there is only one dynamic degree of freedom in addition to \( g_{\mu \nu} \), expressed in the scalar field \( \phi = F_{\mathcal{R}} \equiv df/d\mathcal{R} \). The whole theory can then be reformulated in terms of \( g_{\mu \nu} \) and \( \phi \) as a scalar–tensor theory with the gravitational part of the action [4, 10]

\[
S_g = \int d^4x \sqrt{-g} \left[ (1 + \phi)R - \frac{3}{2\phi} (\partial \phi)^2 - V(\phi) \right]
\]

where the scalar field potential is related to \( f(\mathcal{R}) \) by

\[
V(\phi) = \mathcal{R} F_{\mathcal{R}} - F(\mathcal{R}).
\]

The action (2) corresponds to the original formulation of the HMPG in [4]. In the review paper [10] the authors introduce one more constant \( \Omega_A \) into the action, so that in (2) one has \((\Omega_A + \phi)\) before \( R \) instead of \((1 + \phi)\). They further remark that in the limit \( \Omega_A \to 0 \) the theory (1) turns into pure Palatini \( F(R) \) gravity in which the scalar field has no dynamics (precisely as in the Brans–Dicke theory with \( \omega = -3/2 \)), to which (2) is converted if one replaces \((1 + \phi \mapsto \phi)\), while in the limit \( \Omega_A \to \infty \) the theory (1) turns into metric \( F(R) \) gravity. In what follows we adhere to the original STT formulation (2).

It is easy to notice that Eq. (2) represents a special case of the Bergmann–Wagoner–Nordtvedt STT [14–16] whose gravitational action reads

\[
S_g = \int d^4x \sqrt{-g} \left[ f(\phi)R + h(\phi)(\partial \phi)^2 - V(\phi) \right],
\]

such that

\[
f(\phi) = 1 + \phi, \quad h(\phi) = -\frac{3}{2\phi}
\]

In the general theory (4), there is a standard transformation [15] from Eq. (4) (describing what is called the Jordan conformal frame of the STT) to the Einstein conformal frame which is free from a nonminimal coupling between the scalar field and the space–time curvature:

\[
\bar{g}_{\mu \nu} = f(\phi) g_{\mu \nu}, \quad \frac{df}{d\phi} = f(\phi)|D(\phi)|^{-1/2},
\]

\[
D(\phi) = f(\phi) h(\phi) + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2.
\]

1 Unlike [4, 5, 10] and many other papers, we are using here the metric signature \((+, −, −, −)\); therefore, the sign before \((\partial \phi)^2 = g^{\mu \nu} \partial \phi \mu \partial \phi \nu \) is different from that in the cited papers (the minus here corresponds to a phantom), while the meaning of (2) remains the same as there. We will also safely omit the factor \(1/(2x^2)\) before the gravitational part of the action integral because we will only deal with vacuum configurations where \( S_m = 0 \).
resulting in a simpler form of the action:

$$S_g = \int d^4x \sqrt{-g} \left\{ \bar{R} + n\bar{g}^{\mu\nu} \phi,_{\mu} \phi,_{\nu} - \frac{V(\phi)}{f^2(\phi)} \right\},$$  \hspace{1cm} (7)

where bars mark quantities obtained using the transformed metric $\bar{g}_{\mu\nu}$, and $n = \text{sign}D(\phi)$. In the theory (1) under consideration, we have

$$D = -\frac{3}{2\phi}, \quad n = -1, \quad \phi = \tan^2 \frac{\varphi}{\sqrt{6}}. \quad \hspace{1cm} (8)$$

We have to put $n = -1$ since $\phi > 0$ by construction. It is a very important point, indicating a phantom nature of the scalar field $\phi$ and hence $\phi$ which, in particular, favors the existence of wormholes.

The transition (6) from (4) to (7) is a map from the Jordan-frame manifold $\mathbb{M}_J$ with the metric $g_{\mu\nu}$ to the Einstein-frame manifold $\mathbb{M}_E$ with the metric $\bar{g}_{\mu\nu}$. This map is completely reversible if the conformal factor $f(\phi)$ is regular and turns neither to zero nor to infinity in the whole range of $\phi$ relevant to the problem under study; in this case, one may assert that there is a single manifold equipped with two different Riemannian metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. If $f(\phi)$ is zero or infinity at some value of $\phi$, it may happen that a singularity in $\mathbb{M}_E$ maps to a regular surface in $\mathbb{M}_J$ (or vice versa), and $\mathbb{M}_J$ then has a continuation beyond this surface. This phenomenon, called conformal continuation [17], occurs in many known scalar–vacuum and scalar–electrovacuum solutions including black holes and wormholes with conformally coupled scalar fields [18, 19] as well as the so-called cold black holes in Brans–Dicke theory [22, 23].

Meanwhile, the transition (6) is also a well-known efficient method of finding exact or approximate solutions to the equations of the theory (4) due to the comparative simplicity of the action (7). I can hardly agree with the statement from [5] “A crucial mathematical requirement for transformations (28) [(6) in this paper] to be valid is that they must be nonsingular for the considered range of the scalar field variable.” In fact, the transformation (6) is, from the viewpoint of solving differential equations, just a substitution which can in principle work in only a part of the relevant range of the variables involved. It can then happen that the solutions obtained with such a substitution will be incomplete, covering only a part of the range of independent variables or parameters, and the full range may be then obtained using analytical extensions. Conformal continuations are just an example of such extensions. But in any case a solution obtained using this substitution is a valid solution of the original set of equations. Therefore the above-mentioned regularity is not a “crucial mathematical requirement”: its possible violation only requires due attention and study.

Even more than that: as is clear from [19, 23], conformal continuations emerge only in special cases, for example, black holes and wormholes with a conformally coupled scalar field [19, 24] are found at special values of the integration constants, whereas in more general cases singularities in $\mathbb{M}_E$ are mapped into singularities in $\mathbb{M}_J$ (maybe of another nature).

As to the action in the form (2), we notice that, as follows from Eq. (8), the whole range of the field $\phi$ is covered by a single suitable segment of $\phi$, for example, $\phi \in (0, \sqrt{6}\pi/2)$, and the conformal factor $f(\phi) = 1 + \phi = 1/\cos^2(\varphi/\sqrt{6})$ is regular in the same range. Therefore, solutions to the field equations must be equivalently described in terms of $\phi$ and $\varphi$, up to possible special cases with conformal continuations. We shall clearly see that in the next section.

One more important observation is in order: after the scalar field reparametrization in (2),

$$\phi \mapsto \chi, \quad \phi = \chi^2/6,$$  \hspace{1cm} (9)

the gravitational field action reads

$$S_g = \int d^4x \sqrt{-g} \left\{ (1 + \chi^2/6)R - (\partial\chi)^2 - W(\chi) \right\}, \quad \hspace{1cm} (10)$$

which is nothing else but the action of GR with a source of gravity in the form of a conformally coupled scalar field (as mentioned in [10]) of phantom nature, with the potential $W(\chi) = V(\phi)$. Curiously, this version of STT was as early as in 1970 considered by Zaitsev and Kolesnikov [25] as a viable alternative to GR in cosmological and astrophysical applications.

In the case of a massless field $\phi$, $V(\phi) = 0$, it follows from the field equations due to (2) or (10) that any vacuum solution ($S_m = 0$) has a zero Ricci scalar, $R = 0$ (see [5]). However, one can prove an inverse general result:

*If there is a vacuum solution with $R \equiv 0$ and a non-constant scalar field $\phi$ in a theory (4), with $V \equiv 0$, then this STT reduces either to GR with a conformally coupled scalar field or to pure conformal field theory.*

Indeed, consider a general STT with $V(\phi) = 0$ in the parametrization of the kind (10), that is,

$$S_g = \int d^4x \sqrt{-g} \left\{ f(\phi)R + \varepsilon(\partial\phi)^2 \right\},$$  \hspace{1cm} (11)

where $\varepsilon = \pm 1$, so that $\varepsilon = +1$ corresponds to a canonical scalar and $\varepsilon = -1$ to a phantom one. The vacuum field equations read

$$-\frac{1}{2} \delta^\nu_\mu f R + (R^\nu_\mu + \nabla^\nu \nabla_\mu - \delta^\nu_\mu \square) f$$
\[ + \varepsilon \phi_{,\mu} \phi^{,\mu} - \frac{1}{2} \varepsilon \delta_{\mu}^{\nu} (\partial \phi)^2 = 0, \tag{12} \]
\[ 2 \Box \phi = \varepsilon f_{,\phi} R, \tag{13} \]
where \( \Box = g^{\mu \nu} \nabla_\mu \nabla_\nu \), and \( f_{,\phi} = df / d\phi \). The trace of (12) is
\[ \varepsilon (\partial \phi)^2 + f R + 3 \Box f = 0, \tag{14} \]
Expressing \( \Box f \) in terms of the derivatives of \( \phi \) and using (13) for \( \Box \phi \), we obtain
\[ (\varepsilon + 3f_{,\phi})(\partial \phi)^2 + \left( f + \frac{3}{2} \varepsilon f_{,\phi}^2 \right) R = 0. \tag{15} \]
Now, assuming a nontrivial \( \phi(x) \), such that \((\partial \phi)^2 \neq 0\), let us look under which conditions this equation is compatible with \( R = 0 \): evidently, we should require
\[ \varepsilon + 3f_{,\phi} = 0, \tag{16} \]
so that
\[ f_{,\phi} = -\frac{1}{3} \varepsilon (\phi + C_1), \]
\[ f = -\frac{\varepsilon}{6} (\phi + C_1)^2 + C_2, \tag{17} \]
with \( C_{1,2} = \text{const} \), or, denoting \( \tilde{\phi} = \phi + C_1 \),
\[ f(\phi) = C_2 - \frac{1}{6} \varepsilon \tilde{\phi}^2, \tag{18} \]
which means that \( f(\phi) \) corresponds to conformal coupling of the scalar \( \phi \) to the metric, while \( \phi \) is allowed to be canonical or phantom. If \( C_2 = 0 \), we are dealing with pure conformal field theory, and \( C_2 \neq 0 \) describes the presence of the Einstein-Hilbert term in the action.

3. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

Consider static, spherically symmetric vacuum configurations in the theory (2). Evidently, instead of solving the Jordan-frame field equations that directly follow from variation of (2) in \( g^{\mu \nu} \) and \( \phi \), that is,
\[ (G_{\mu}^{\nu} + \nabla_\mu \nabla_\nu - \delta_{\mu}^{\nu} \Box)(1 + \phi) \]
\[ = \frac{3}{2} \varepsilon \delta_{\mu}^{\nu} (\partial \phi)^2 + \frac{1}{2} V \delta_{\mu}^{\nu} = 0, \tag{19} \]
\[ - \frac{3}{2} \phi \Box \phi - \frac{3}{2} \delta_{\mu}^{\nu} (\partial \phi)^2 + R - \frac{dV}{d\phi} = 0 \tag{20} \]
\( (G_{\mu}^{\nu} = R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} R \) is the Einstein tensor), it is easier to solve the Einstein equations that follow from (7),
\[ G_{\mu}^{\nu} - \phi_{,\mu} \phi_{,\nu} + \frac{1}{2} \delta_{\mu}^{\nu} (\partial \phi)^2 + \frac{1}{2} \delta_{\mu}^{\nu} W = 0, \tag{21} \]
\[ 2 \Box \phi - \frac{dW}{d\phi} = 0, \quad W(\phi) := \frac{V(\phi)}{(1 + \phi)^2}, \tag{22} \]
and transform the solutions back to \( g_{\mu \nu} \) and \( \phi \) according to (6). As is stressed in [5], the HMPG theory is naturally formulated in terms of the Jordan-frame metric \( g_{\mu \nu} \) and the scalar field \( \phi \).

3.1. Solutions for \( V(\phi) = 0 \)

In the case of a massless field \((V(\phi) = 0)\), the Einstein-frame solution is well known: it is the phantom solution with a canonical massless minimally coupled scalar field [28]. This solution with a phantom scalar (often called the “anti-Fisher” solution) splits into three branches, which can be presented in a unified way using the notations of [19]. Namely, consider the general static, spherically symmetric metric in \( \mathcal{M}_E \)
\[ ds^2_e = e^{2\gamma} dt^2 - e^{2a} du^2 - e^{2\beta} d\Omega^2, \tag{23} \]
where \( \alpha, \beta, \gamma \) are functions of an arbitrarily chosen radial coordinate \( u \), and \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \) is the line element on a unit sphere. Then, using the harmonic coordinate condition
\[ \alpha(u) = 2\beta(u) + \gamma(u), \tag{24} \]
one can write the anti-Fisher solution as follows [19]:
\[ \bar{\phi} = \bar{C} u + \bar{\phi}_0, \quad \gamma(u) = -hu, \]
\[ e^{-\beta(u) - \gamma(u)} = s(k, u) := \begin{cases} k - 1 \sinh k u, & k > 0 \\ u, & k = 0 \\ k - 1 \sin k u, & k < 0, \end{cases} \tag{25} \]
where \( u > 0 \) while the integration constants \( h, \bar{C}, \bar{\phi}_0 = \text{const} \), and \( k \) are related by the equality
\[ 2k^2 \text{sign} k = 2h^2 - \bar{C}^2. \tag{26} \]
Detailed descriptions of its properties may be found, e.g., in [29, 30].

Accordingly, the Jordan-frame metric and the scalar field \( \phi \) in the theory (2) with \( V(\phi) = 0 \) may be presented as
\[ ds^2_J = \cos^2(Cu + \psi_0) \left\{ e^{-2hu} dt^2 - \frac{e^{2hu}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right] \right\}, \tag{27} \]

\( \text{2 The (anti-)Fisher metric is given by the expression in curly brackets in (27) and corresponds to a minimally coupled scalar \( \tilde{\phi} \) or \( \phi \). The only difference between the solutions for a canonical scalar (Fisher’s) and that for a phantom scalar (discussed here) is that in the canonical case there is a plus instead of a minus before \( C^2 \) in (26) and (29); it leads to \( k > 0 \). and so there is only branch A with \( a < 1 \) according to Eq. (31) below. In the phantom case, \( k \) has any sign, which leads to three branches.} \]
\( \phi(u) = \tan^2 \psi, \quad \psi := \phi/\sqrt{6} = C u + \psi_0, \)

where the notation \( \psi = \phi/\sqrt{6} \) has been introduced for convenience, \( C = \bar{C}/\sqrt{6} \), and \( \psi_0 = \bar{\phi}/\sqrt{6} \). The integration constants \( k, h \) and \( C \) are related by

\[
k^2 \text{sign} k = h^2 - 3C^2. \tag{29}
\]

Without loss of generality we assume \( |\psi_0| < \pi/2 \).

A direct inspection shows that these \( g_{\mu \nu} \) and \( \phi \) do satisfy the field equations (19) and (20) with \( V(\phi) \equiv 0 \). It is also directly confirmed that the scalar curvature is zero for \( g_{\mu \nu} \) given by (27), as should be the case for \( V \equiv 0 \). (According to (3), \( V = 0 \) implies \( \mathcal{R} = \text{const} \cdot R \), therefore in this case \( \mathcal{R} \) is also zero [5].)

Let us briefly describe the properties of this solution. To begin with, in all cases the metric (27) is asymptotically flat at \( u = 0 \) which corresponds to the spherical radius \( r \equiv \sqrt{-g_{00}} \) tending to infinity, so that \( r \propto 1/u \) as \( u \to 0 \).\(^3\) A comparison with the Schwarzschild metric at small \( u \) using a transition to the coordinate \( r \) leads to the following expression for the Schwarzschild mass:

\[
m = h \cos \psi_0 + C \sin \psi_0. \tag{30}
\]

Other properties of the solution are different for different values of \( k \), comprising three branches according to the definition of \( s(k, u) \) in (25).

**Branch A:** \( k > 0 \). It is helpful to put

\[
e^{-2ku} = P(x) := 1 - \frac{2k}{x},
\]

\[
e^{-2hu} = P(x)^a, \quad a = \frac{h}{k} = \pm \sqrt{1 + \frac{3C^2}{k^2}}. \tag{31}
\]

The metric acquires the form

\[
ds_J^2 = \cos^2 \psi \left[ P^a dt^2 - P^{-a} dx^2 - x^2 P^{-a} d\Omega^2 \right],
\]

\[
\psi = \psi_0 - \frac{C}{2k} \ln P. \tag{32}
\]

With the new coordinate \( x \), flat spatial infinity corresponds to \( x \to \infty \). As \( x \) decreases from infinity, \( P(x) \) decreases beginning from unity, ultimately reaching the value where \( \cos \psi = 0 \) and, according to Eq. (8), \( \phi \to \infty \). This happens where \( \ln P(x) = -(2k/C)(\pi/2 - \psi_0) \) if \( C > 0 \) and where \( \ln P(x) = (2k/C)(\pi/2 + \psi_0) \) if \( C < 0 \). It evidently corresponds to finite \( P \), that is, \( x > 2k \): it is a naked central (i.e., where the radius \( r = 0 \)) singularity which is attractive for test particles since \( g_{tt} \) is nonzero.

**Branch B:** \( k = 0 \). In this case, it is helpful to substitute \( u = 1/x \), so that \( x = \infty \) corresponds to flat spatial infinity. The solution has the form

\[
ds_J^2 = \cos^2 \psi \left[ e^{-2hu} dt^2 - e^{-2hu} (dx^2 + x^2 d\Omega^2) \right],
\]

\[
\psi = \psi_0 + C/x, \quad h^2 = 3C^2. \tag{33}
\]

The coordinate \( x \) ranges from \( x_s \) to infinity, where \( x_s \) is the value of \( x \) where \( \cos \psi = 0 \) and \( \phi = \infty \). We have \( x_s = C/(\pi/2 - \psi_0) \) if \( C > 0 \) and \( x_s = -C/(\pi/2 + \psi_0) \) if \( C < 0 \). In both cases, as in branch A, it is a central attractive singularity.

**Branch C:** \( k < 0 \). In this case, it makes sense to use the original coordinate \( u \) (which is harmonic in the Einstein frame), and the solution reads

\[
ds_J^2 = \cos^2 \psi \left[ e^{-2hu} dt^2 - \frac{k^2 e^{2hu}}{\sin^2 ku} \left( \frac{k^2 du^2}{\sin^2 ku} + d\Omega^2 \right) \right],
\]

\[
\psi = \psi_0 + C u. \quad h^2 = 3C^2 - k^2. \tag{34}
\]

As already mentioned, \( u = 0 \) is flat spatial infinity with the Schwarzschild mass (30). However, on the whole, the nature of the solution crucially depends on an interplay between the constants \( k, C, \) and \( \psi_0 \), depending on which of the functions \( \sin |k|u \) or \( \cos \psi \) is the first to vanish at growing \( u \) beginning from zero. Adhering to asymptotically flat solutions, we require that at \( u = 0 \) the conformal factor \( \cos^2 \psi \) should be nonzero, therefore without loss of generality we require \( |\psi_0| < \pi/2 \).

Three possible behaviors should be singled out (we assume, for certainty, \( C > 0 \)), see Fig. 1.

**C1:** \( (\pi/2 - \psi_0)/C < |k| \). The coordinate \( u \) ranges from zero to \( u_s = (\pi/2 - \psi_0)/C \), at which

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\(^3\) The conformal factor \( \cos^2 \psi \) is not normalized to unity at \( u = 0 \) if \( \psi_0 \neq 0 \), which, however, does not affect the further description.
\( \cos \psi = 0 \), and \( u = u_s \) is a central naked singularity quite similar to the one in branches A and B.

**C2:** \((\pi/2 - \psi_0)/C > \pi/|k|\). The coordinate \( u \) ranges from zero to \( u_s = \pi/|k| \) where \( \sin ku = 0 \), and there occurs the second spatial infinity: as \( u \to \pi/|k| \), the radius \( r \to \infty \) while \( g_{tt} \) and \( \phi \) remain finite. This second infinity is also flat, and the Schwarzschild mass\(^4\) is there equal to

\[
m_\ast = -e^{h_\ast}(h \cos \psi_\ast + C \sin \psi_\ast), \tag{35}\]

where \( \psi_\ast = \psi_0 + C\pi/|k| < \pi/2 \) is the value of \( \psi \) at \( u = u_\ast \). So, it is a wormhole configuration, only quantitatively different from its anti-Fisher and Brans-Dicke counterparts, see, e.g., [19, 20].

**C3:** \((\pi/2 - \psi_0)/C = \pi/|k|\). In this special case, at \( u = u_1 = \pi/|k| \) both \( \sin |k|u \) and \( \cos \psi \) turn to zero, while the spherical radius \( r = \sqrt{-g_{tt}} \) is finite but \( \phi = \infty \). Near \( u = u_1 \), the metric looks like

\[
ds^2 = C^2 \left[ e^{-2h_{u_1}} \Delta u^2 dt^2 - e^{2h_{u_1}} \frac{du^2}{\Delta u^2} - e^{2h_{u_1}} dr^2 \right], \tag{36}\]

where \( \Delta u = u_1 - u \). It shows that \( u = u_1 \) is a double (extremal) horizon, and it is the only special case of the solution under study in which it describes a black hole.

Quite similar three cases are observed if \( C < 0 \), hence \( \psi \) decreases at increasing \( u \), and \( \cos \psi = 0 \) corresponds to \( \psi = -\pi/2 \).

### 3.2. Black Holes with \( V = 0 \)

The black hole solutions deserve a more detailed discussion.

The condition \((\pi/2 - \psi_0)/C = \pi/|k|\) leads to \( \psi_0 = \pi(1/2 - C/|k|) \). The requirement \( \psi_0 > -\pi/2 \) then implies \( C < |k| \). This inequality, obtained formally, has an evident meaning: since \( \cos \psi \neq 0 \) at \( u = 0 \), the plot of \( \cos \psi \) must be wider than that of \( \sin ku \) in order that their first positive zeros coincide (\( u = u_1 \)). It is then clear that, as the coordinate \( u \) further increases (that is, when we are looking what happens beyond the horizon), the next zero of \( \sin ku \), namely, \( u_2 = 2\pi/|k| \) is reached earlier than a zero of \( \cos \psi \).

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\(^4\) As can be directly verified, for a general asymptotically flat static, spherically symmetric metric in the form (23), with an arbitrarily chosen radial coordinate \( u \), the Schwarzschild mass at a value \( u = u_\ast \) corresponding to flat infinity is determined as

\[
m_\ast = -\lim_{u \to u_\ast} \left( e^{\delta_1} \psi_\prime /\beta_1 \right),
\]

where the prime stands for \( d/du \).

---

This in turn completely reveals the global structure of our black hole configuration. Indeed, the value \( u = u_2 \) with \( \sin |k|u = 0 \) corresponds to a second spatial infinity, quite similar to that in a wormhole solution. Like the wormholes described above, this space-time is globally regular, but now it is not two-sided traversable because of the horizon. Its Carter-Penrose diagram is shown in Fig. 2.

It should be noted that the present black hole solution has much in common with the well-known solution for a black hole supported by a conformally coupled scalar field [18, 19, 21] with the metric

\[
ds^2 = \left( 1 - \frac{m}{r} \right)^2 dt^2 - \left( 1 - \frac{m}{r} \right)^{-2} dr^2 - r^2 d\Omega^2 \tag{37}\]

and \( \phi = \sqrt{6m/(r - m)} \), \( m = \text{const} \) being the mass. In both cases,

(i) the black holes are described by special solutions to the Einstein-scalar equations;

(ii) all of them are asymptotically flat and extremal and therefore have zero Hawking temperature;

(iii) the supporting scalar fields are infinite on the horizon, but the effective stress-energy tensor components \( T^\mu_\nu \) are finite there (as follows from finite values of the Einstein tensor components \( G^\mu_\nu \));

(iv) lastly, in both cases the scalar curvature is zero in the whole space.

However, there are important differences:

(i) the solution (37) corresponds to a canonical scalar field in the Einstein frame while for (34) such a field is phantom;

(ii) the solution (37) has a singular center \( r = 0 \) (it has the same geometry as the extreme Reissner-Nordström space-time) while the present black hole space-time has no center and is globally regular;
(iii) the solution (37) has only one free parameter \( m \) while the present one contains two independent parameters, for example, \( k \) and \( C \);

(iv) the static region of the solution (37) can be obtained from Fisher’s solution [28] in the Einstein frame only with the help of a conformal continuation [17, 19], whereas for the black hole case of (34) the Einstein-frame image of the horizon is the second spatial infinity of an anti-Fisher wormhole. A conformal continuation is only required for the extension beyond the horizon.

One can notice that none of the static, spherically symmetric solutions of the theory (2) with \( V \equiv 0 \) possess simple horizons with finite temperature, contrary to the results announced in [5].

3.3. A Geometry with Infinitely Many Horizons

A one-parameter family of geometries of interest is obtained if we abandon the asymptotic flatness requirement and put

\[
C = |k|, \quad \psi_0 = -\pi/2,
\]

so that \( \cos^2 \psi = \sin^2 ku \), and the Jordan-frame metric then reads

\[
ds_J^2 = \sin^2 ku e^{-2hu} dt^2 - k^2 e^{2hu} \left( \frac{k^2 du^2}{\sin^2 ku} + d\Omega^2 \right),
\]

where \( h = \pm \sqrt{2k^2} \) according to (34). This space-time, with \( u \in \mathbb{R} \), is a union of an infinite number of static regions, each being described by a single half-wave of the function \( \sin ku \), and double horizons between them at each \( u = \pi n/|k| \), with any integer \( n \). A symbolic picture of this geometry is presented in Fig. 3. Though, the space-time \( M_J \) with the metric (39) is, in a clear sense, “much larger” than depicted since the spatial distance from any regular point in any static region to each of its two nearest horizons is infinite due to divergence of the integral

\[
\int \sqrt{|g_{uu}|} du
\]

which is a common feature of all double horizons.

Thus the Jordan-frame manifold \( M_J \) maps to a countable number of Einstein-frame manifolds \( M_{E} \), each of the latter being an anti-Fisher wormhole whose both flat infinities map into horizons in \( M_J \). One more example of a construction with an infinite number of conformal continuations was built in [17]: there, it is presented by a solution for a conformally coupled scalar field \( \phi \) with the normal sign of kinetic energy and a nonzero potential \( U(\phi) \). In that example, the continuation took place through ordinary surfaces \( S_{trans} \) of finite radius, and the whole \( M_J \) had no horizons and was either completely static or completely cosmological. In the first case, its shape was that of an infinitely long tube with a periodically changing diameter. In the second case, \( M_J \) represented a \((2 + 1)\)-dimensional cosmology with a periodically (and isotropically) changing scale factor. Unlike that, in our case, all transition surfaces \( S_{trans} \) are double horizons between static regions, and the spherical radius monotonically changes from one region to another.

The global causal structure of the manifold \( M_J \) with the metric (39) is described by the Carter-Penrose diagram that occupies the whole plane. Horizons with \( u = -\pi/|k|, 0, \pi/|k| \) are shown by the dashed, thick and double broken lines, respectively.

\[
\frac{u = -\pi}{|k|} \quad \frac{u = 0}{|k|} \quad \frac{u = \pi}{|k|}
\]

Fig. 3. A symbolic representation of the geometry (39) as a horn with cross-sections whose radius grows with growing \( u \). For certainty, we have put \( |k| = 1 \). The selected sections at \( u = \pi \pi \) represent the horizon spheres.

Fig. 4. Carter-Penrose diagram of the manifold \( M_J \) with the metric (39). The diagram occupies the whole plane. Horizons with \( u = -\pi/|k|, 0, \pi/|k| \) are shown by the dashed, thick and double broken lines, respectively.
branching since any point of the diagram in Fig. 4 (including the knots) corresponds to regular spheres, and there is no spatial infinity.

Another example of a manifold with an infinite number of horizons was obtained among solutions for phantom dilaton-Einstein-Maxwell black holes in [35].

3.4. On Solutions with Nonzero Potentials $V(\phi)$

If $V(\phi) \neq 0$, it is not so easy to solve the field equations, and there is a modest number of examples of static, spherically symmetric solutions with some special potentials even in the case of minimally coupled scalar fields. For the latter, there exist some general theorems allowing for judging on possible solution behaviors without having these solutions. These are, above all, the no-hair theorems (see, e.g., [32] for a recent review) indicating the conditions that exclude horizons, and the global structure theorems [33] which work when the no-hair theorems can be violated and tell us about the maximum possible number of horizons: this number is two, and it is only one for asymptotically flat configurations.

What can be said on Jordan-frame metrics, which are conformal to Einstein-frame ones with minimally coupled scalars? If the conformal factor $f(\phi)$ in the mapping (6) is regular and finite in its whole range, most of the theorems actually extend to $M_J$ since (while mapping to either side) a flat infinity maps to a flat infinity, a horizon maps to a horizon, and the scalar field potential preserves its sign. However, if $f(\phi)$ is somewhere singular, there emerges the possibility of conformal continuations, and how might they be, is demonstrated by the above example of an infinite number of horizons in $M_J$. An important restriction is that horizons coinciding with transition surfaces at continuations are double (extreme); however, one cannot exclude that $M_E$ contains a simple horizon, but as a result of a continuation the new region of $M_J$ maps to the same $M_E$, and it will then contain two simple horizons. We can conclude that $M_J$ may be much wider than $M_E$ and even contain more horizons; however, it should be kept in mind that conformal continuations emerge only at special values of integration constants.

Simple horizons in $M_E$ with a nontrivial scalar field can certainly be obtained if the relevant no-hair theorems do not work. For example, an important no-hair theorem [32, 34] claims that (in GR) the domain of outer communication of an asymptotically flat black hole cannot contain a non-constant canonical minimally coupled scalar field $\psi$ with a potential $V(\psi) \geq 0$. However, examples of such black holes where $V < 0$ in at least a part of the range of $\psi$ do exist, see, e.g., [36]. Meanwhile, for HMPG, relevant results valid for phantom scalars, for which there is no such restriction, and black holes can in principle appear even with $V \geq 0$; it is known that phantom fields can make black holes globally regular [35, 37, 38]. Numerically obtained examples of black holes in HMPG with a Higgs-like potential have been considered in [5], and further work in this direction should also be of interest.

3.5. The Stability Problem

The stability properties of static configurations in STT under small perturbations may be studied, as well as their geometries themselves, with the aid of field equations transformed to the Einstein frame, in which there are quite numerous results concerning the fate of perturbations of various scalar-vacuum space-times, see, e.g., [30, 40–47]. It is clear that perturbations in Jordan’s frame are governed by the same equations as in Einstein’s, though being expressed in other variables after the substitution (6). What can significantly change after this transformation are the boundaries and the boundary conditions that should now be imposed by physical requirements formulated in $M_J$.

Let us briefly outline the expected results on the stability properties of the solutions considered here with respect to radial perturbations (a more thorough analysis is postponed for the future).

To begin with, in the case of spherically symmetric (monopole) perturbations of static scalar-vacuum space-times, the only dynamic degree of freedom is related to scalar field perturbations $\delta \phi(u, t)$, or those of the Einstein-frame field, $\delta \psi(u, t)$, because the tensor degrees of freedom only begin with the quadrupole. Accordingly, the perturbations are governed by a single linear equation for $\delta \psi$ while all the accompanying perturbations $\delta \alpha, \delta \beta, \delta \gamma$ of the metric functions (in the notations of the metric (23)) can be excluded from this “master equation” with the help of the Einstein equations. The resulting single equation can be written for a spectral component $\delta \psi = \Psi(u) e^{i \omega t}$ as the Schrödinger-like equation

$$\frac{d^2 Y}{dz^2} + \left( \omega^2 - W(z) \right) Y = 0. \quad (40)$$

Here, $z$ is the so-called tortoise coordinate related to an arbitrary coordinate $u$ (in the notations of (23)) by $du/\sqrt{e^{\psi}}$, the unknown function is $Y(z) = \Psi(u) e^{\beta}$, and the effective potential $W(z)$ is given by [30, 45, 46]

$$W(z) = e^{2\gamma} \left[ \frac{3^{\gamma/2} \beta^2}{\beta^2} (2e^{2\beta} - U) + \frac{\psi'}{\beta} U_{\psi} - \frac{1}{12} U_{\psi \psi} \right] + e^{2\gamma - 2\alpha} \beta'' \left( \beta' + \gamma' - \alpha' \right), \quad (41)$$
where the index $\psi$ stands for $d/d\psi$, the prime for $d/du$ ($u$ is an arbitrary radial coordinate in the background metric (23)), and $U = U(\psi) = V(\phi)/(1 + \phi)^2$, $V(\phi)$ being the scalar field potential in the original theory (2).

In the solutions under study, Eqs. (27)–(29), all branches A–C contain throats ($\beta' = 0$) in their Einstein-frame manifolds $M_E$ (even though not all of them correspond to wormholes), hence the potential $W(z)$ contains a singularity due to $\beta'$ in the denominator. This singularity admits regularization by replacing $W(z)$ with a new potential $W_{\text{reg}}(z)$ which is finite in the whole range of $u$ (or $z$) and suitable for considering the relevant boundary value problems, see detailed descriptions in [30, 42, 44, 47].

The particular form of $W_{\text{reg}}(z)$ is different for different branches of the solution (27)–(29) and will not be presented here, see [30]. It is important that all branches of the anti-Fisher solution turn out to be unstable [30, 42], the instability being connected with a potential well in $W_{\text{reg}}(z)$ (see examples in Fig. 5). Let us discuss, which of these instability conclusions can be extended to the present HMPG solutions in the Jordan frame, for which $W_{\text{reg}}(z)$ is the same but the range of $z$ and the boundary conditions may be different.

For branches A and B ($k \geq 0$), in $M_E$ we have $u > 0$ corresponding to $z \in \mathbb{R}$, and an unstable mode is found [30] under the boundary conditions $\delta \psi \to 0$ as $z \to \pm \infty$. In $M_J$, due to the factor $\cos^2 \psi$ in the metric, the range of $u$ extends from zero to the singular point $u_s$ such that $\psi = C u_s + \psi_0 = \pi/2$, and the range of $z$ is also diminished: $z \in (z(u_s), \infty)$. Therefore, the instability conclusion cannot be directly extended to $M_J$, and a separate study is necessary. Its results must depend on the solution parameters (including $\psi_0$) and on the adopted boundary condition at the singularity.

The same argument applies to branch C1, also with a singularity caused by $\cos \psi = 0$. On the contrary, in branches C2 (a black hole) and C3 (a wormhole), the whole wormhole space-time $M_E$ maps in $M_J$ either to a wormhole or to a static region of a black hole from the horizon to infinity. Therefore, if we adopt for the horizons the same boundary condition $\delta \psi = 0$ as is used at infinity (which looks reasonable), then the instability result for spherically symmetric perturbations also extends to $M_J$. The same is true for each static region between horizons in the solution described in Sec. 3.3.

This brief analysis argues that all regular solutions considered here are unstable.

4. CONCLUDING REMARKS

Considering exact vacuum solutions of HMPG with zero potential $V$, we have found that many of them describe space-times with naked singularities, other generic solutions correspond to traversable wormholes, while only solutions of a special two-parameter family contain extremal Killing horizons and describe globally regular black holes with zero Hawking temperature. These results disagree with some of those obtained in [5], where the same set of equations was solved numerically directly in Jordan’s frame, and the existence of black holes with finite temperature was stated. The reason for such a discrepancy is yet to be understood. An independent analysis of solutions with nonzero potentials can probably shed light on this question.

The black hole solutions obtained here are also of interest as phantom counterparts of those with a usual conformal scalar [18, 21]. However, a tentative analysis indicates that these new solutions are unstable under monopole perturbations. One may recall that it was first concluded [48] that the black hole solutions of [18, 21] are also unstable but a later and more thorough analysis [49] revealed their stability.

The wormhole configurations described here are quite a natural consequence of the phantom nature of the action (2). It is therefore not surprising that examples of wormhole solutions with matter respecting the standard energy conditions were previously obtained in [50, 51] since a violation of the Null Energy Condition, which is necessary for their existence, was actually provided by the scalar $\phi$, and, as we saw, wormholes with two flat infinities are obtained even without matter. Though, precisely as their counterparts in GR (anti–Fisher wormholes), they turn out to be unstable since their stability analysis directly extends in this case to the Jordan frame.

In the massless case ($V = 0$) it is quite straightforward to include into consideration electromagnetic fields $F_{\mu\nu}$ in the HMPG framework, in full analogy
with studies in other scalar-tensor theories [19, 39]. Solutions with $F_{\mu\nu}$ and nonzero potentials can also be studied both analytically and numerically, for example, by analogy with [31, 45].

Lastly, it is of interest to extend the present study to the so-called extended HMPG with functions $f(R, \mathcal{R})$ of two curvatures [12, 13], this work is in progress.

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