Phase modulated solitary waves controlled by bottom boundary condition

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A forced KdV equation is derived to describe weakly nonlinear, shallow water surface wave propagation over non trivial bottom boundary condition. We show that different functional forms of bottom boundary conditions self-consistently produce different forced KdV equations as the evolution equations for the free surface. Solitary wave solutions have been analytically obtained where phase gets modulated controlled by bottom boundary condition whereas amplitude remains constant.

1. Introduction

The dynamics of the shallow water nonlinear, unidirectional, dispersive, gravity induced surface waves is described by the celebrated KdV equation that admits solitary wave solutions. The derivation of such an equation assumes that the fluid is incompressible and inviscid, bounded below by a rigid, impermeable bottom and above by a free surface. The generalization of the KdV equation to higher order nonlinearities lead to a multitude of nonlinear equations that found potential applications in various physical situations. The classical KdV equation was also extended to admit arbitrary forcing functions leading to the forced KdV equation, subject to asymptotic initial conditions depending on the forcing disturbances and also including the effects of surface pressure and topography. In this work, we also encounter a series of forced KdV equations as the evolution equations of shallow water, nonlinear dispersive waves over non-trivial bottom boundary conditions. The different functional nature of this fixed bottom condition self-consistently generates different types of forced KdV equations.

In most of the realistic situations, water waves propagate over a porous bed so that one needs to consider the transformation of the waves brought about by the permeability of the bottom bed. Several important problems exist concerning the effects of soil dynamics on waves. Many kinds of structures such as ships, buoys, breakwaters, submersibles supporting oil drillings rigs are directly or indirectly supported by the bottom bed that can be made up of a variety of different soils ranging from solid rock, to sand to fine clay. Persistent attack by the waves causes the varying wave pressure to induce considerable stress and strain on the bottom bed. This in turn effects the dynamic stability of the bed causing fatigue of the structures supported by the bed. A description of the porous bed needs different types of constitutive equations depending on the soil type and strain magnitude, while the fluid medium is itself described by Darcy’s law. Mei has developed several theoretical concepts needed to pursue the problems of wave induced stresses in a porous media using the boundary layer approximation that facilitates even the nonlinear modelling of seabeds.

The dynamics of the linear water waves in a channel of permeable bottom has been one of the interesting research problems in water wave theory undertaken from the early times. Rigorous development of mathematical models for nonlinear, diffusive, weakly dispersive water waves interacting with a porous bottom has begun only in the last decade, with the description based on the Boussinesq approximation. In shallow water, Boussinesq equation gives wave solutions propagating in both positive and negative directions. However, for unidirectional wave propagation in shallow water, the KdV equation appears as a reasonable dynamical equation when the vertical fluid velocity at the bottom is assumed to be zero. In this work, we consider the non zero vertical fluid velocity at the bottom that leads to a series of forced KdV equations self-consistently where the functional forms of the leakage velocity appears as forcing function. One of the boundary condition to the problem arises, as is the usual practice, by considering the upper boundary of the fluid to be a free surface. For the other fixed boundary condition, a nonzero value of the vertical fluid velocity at the bed of the channel is considered that physically describes the existence of a leakage velocity at the interface of the fluid and the bed. This fixed bottom boundary condition is the key feature in the present problem. Situations describing different types of interactions of the fluid and porous bed can be analyzed by considering different mathematical forms of this leakage velocity as functions of time, space or both. The basic features of analysis of this work run parallel to the derivation of KdV equation in a hard bottom channel. The novelty of the present work is that exact solitary wave solutions of the forced KdV equations have been obtained analytically for different leakage conditions. For example, constant, time dependent, space dependent or both space-time dependent forms of leakage velocity have been considered that control the phase modulation of the obtained solitary wave solutions leading to different types of dynamical behaviour of such waves.

The arrangement of the paper is as follows. In section 2, the foundation of the problem has been established for a nonlinear shallow water wave propagation using a general space time dependent bottom boundary condition. For better understanding of the problem, different types of functional dependence of the vertical fluid velocity at the
bottom have been considered. In section 2.1, it is assumed to be constant leading to a constantly decelerating solitary wave. In section 2.2, only temporal dependence is considered, which also leads to decelerating solitary wave with the phase modulation explicitly depending on the nature of time dependence. Both of these two cases have been analytically solved to obtain exact solutions. In the next subsection, spatial dependence has been considered, leading to inhomogeneous KdV, where the inhomogeneity is in the form of a kdv like equation. Using the interesting nature of the inhomogeneity, exact and approximate solutions are obtained. In the last subsection, the more general space-time dependent bottom boundary condition is assumed. In order to obtain exact analytic solitary wave solution, bilinearization technique is applied where also an arbitrary function dependent on time modulates the phase of the solution. In all the cases considered, the amplitude of the solitary wave remains constant. Appropriate three dimensional plots of the solution have been presented for each case.

2. Derivation of the free surface evolution equation in presence of water leakage at the bottom

A one dimensional unidirectional wave motion propagating through a shallow water channel of permeable bottom is considered. The channel is of uniform cross section and of constant depth \( h \). The fluid is assumed to be incompressible with the wavelength, amplitude and velocity of the wave represented by \( l, a \) and \( v \) respectively (as shown in Figure 1). The surface tension and viscosity have been neglected throughout this calculation. At an arbitrary \((x, t)\) the free surface displacement is denoted by \( \eta(x, t) \). Two natural small parameters \( \epsilon = a/h \) and \( \delta = h/l \) are introduced, both of which are \( \approx 1 \), and further \( \epsilon \approx \delta^2 \).

\[ \text{FIG. 1: Shallow water solitary wave in a water channel of permeable bottom} \]

The fluid motion can be described by the velocity vector \( \vec{V} = V_h \vec{i} + V_v \vec{j} \) where the subscripts \( h, v \) denote horizontal and vertical components of the fluid velocity respectively. From the condition of irrotational flow of the fluid we can introduce a velocity potential \( \phi(x, y, t) \) such that \( \vec{V} = \nabla \phi \).

Since the fluid is incompressible, the mass conservation equation leads to

\begin{equation}
\nabla^2 \phi = 0
\end{equation}

Again from the momentum conservation equation, i.e, Euler equation, we get,

\begin{equation}
\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + \frac{p}{\rho} + g y = 0,
\end{equation}

where \( \rho, p, g \) are density, pressure of the fluid and acceleration due to gravity respectively. Eq. \( (2) \) is well known as the Lagrange equation. When \( \phi \) is independent of time, then the above equation is called Bernoulli’s condition. Equations \( (1), (2) \) are the two main equations of the problem which must be supplemented by appropriate boundary conditions.

The fluid is bounded by two surfaces, one is the fixed bottom and other is the free boundary. Since at the upper free surface, \( p = 0 \), hence taking derivative of eq. \( (2) \) along the direction of propagation, we obtain

\begin{equation}
\frac{\partial V_h}{\partial t} + V_h \frac{\partial V_h}{\partial x} + V_v \frac{\partial V_v}{\partial x} + \frac{g}{\rho} \frac{\partial \eta}{\partial x} = 0
\end{equation}

Again, at the free surface

\begin{equation}
y(x, t) = h + \eta(x, t)
\end{equation}
Taking time derivative of eq. (4), we get

\[ V_v = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} V_h \]  

(5)

These two equations (3) and (5) are defined at the free surface of the wave. Since the upper free surface is movable, hence these equations are called variable boundary conditions.

Since some amount of water is considered to be leaking through the fixed bottom of the channel, hence the downward vertical fluid velocity at the bottom is nonzero. This constitutes the fixed boundary condition defined by the equation

\[ V_v(x, 0, t) = \frac{\partial \phi(x, 0, t)}{\partial y} = C(x, t) \]  

(6)

where \( C(x, t) \) is the vertical fluid velocity at the bottom of the channel. This equation is called the penetration condition.

Thus ultimately we get two equations (1), (2) that are valid in the bulk of the fluid. Taking the derivative of eq. (2) and eq. (4) at the free boundary, we get two nonlinear boundary conditions (3), (5) respectively and the penetration condition given by (6).

The velocity potential \( \phi \) is expanded in Taylor series as follows

\[ \phi(x, y, t) = \sum_{n=0}^{\infty} y^n \phi_n(x, t) \]  

(7)

Where \( \phi_n(x, t) = \frac{\partial^n \phi}{\partial y^n} \) at \( y = 0 \). Substituting this in the Laplace’s equation (1) the following recurrence relation is obtained

\[ \frac{\partial^2 \phi_n}{\partial x^2} = -(n + 2)(n + 1)\phi_{n+2} \]  

(8)

Using penetration condition (6) we can arrive at \( \phi_1(x, t) = C(x, t) \) and using this in the recurrence relation (8) the expression for \( \phi(x, y, t) \) is obtained as

\[ \phi(x, y, t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} F_{2m} y^{2m} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)!} C_{2m} y^{2m+1} \]  

(9)

where \( F = \phi_0(x, t) \) and the subscript -(2m) in C and F denotes 2m-th order derivative w.r.to x. The horizontal and vertical components of the fluid velocity at the free surface are determined as,

\[ V_h = y C_x - \frac{1}{3!} C_{xxx} y^3 + f - \frac{1}{2!} y^2 f_{xx} + h.o.t \]  

(10)

\[ V_v = C - \frac{1}{2!} C_{xx} y^2 - y f_x + \frac{1}{3!} y^3 f_{xxx} + h.o.t \]  

(11)

where \( x \) in the subscript denotes partial derivative with respect to \( x \), \( f = \frac{\partial F}{\partial x} \) and h.o.t denotes higher order terms in \( y \). The different dimensional variables that have made their appearance in the problem will be made dimensionless by incorporating the following scaling of variables so that small parameters \( \epsilon, \delta \) creep into the equations and smaller terms can be neglected in comparison to them.

2.1 Scaling of variables

All the dependent and independent variables occurring in the above equations are scaled in the following way by taking account of the smallness parameters \( \epsilon, \delta \)

\[ x \Rightarrow lx', \eta \Rightarrow a \eta', t \Rightarrow \frac{l}{\sqrt{gh}} t', V_h \Rightarrow \epsilon \sqrt{gh} V'_h, V_v \Rightarrow \epsilon \delta \sqrt{gh} V'_v, \]
\( f \Rightarrow \epsilon \sqrt{gh}f', y \Rightarrow h(1 + \epsilon y'), C \Rightarrow \epsilon \delta \sqrt{gh}C', \)

where the variables in prime are dimensionless and henceforth all terms \( \leq \epsilon \delta^2 \) will be neglected by considering them to be small compared to terms of the order of \( \epsilon \) or \( \delta^2 \). Using this scaling in equations (10), (11) dimensionless velocity components are obtained as:

\[
V'_h = \delta^2 C'_x + f' - \frac{1}{2} \delta^2 f'_{xx}
\]

\[
V'_v = C' - \frac{1}{2} \delta^2 C'_x x - (1 + \epsilon \eta' f' + \frac{1}{6} \delta^2 f'_{xx})
\]

Hence from the two nonlinear boundary conditions (5), (3) we get

\[
\eta'_t + f'_x + \epsilon \eta' \eta_x + \epsilon \eta f'_{x} - \frac{1}{6} \delta^2 f'_{xxx} = C' - \frac{1}{2} \delta^2 C'_x x
\]

\[
\eta'_x + f'_t + \epsilon f' f_x - \frac{1}{2} \delta^2 f'_{xt} = -\delta^2 C'_x x
\]

For notational convenience the prime symbol will be omitted in all the variables in the subsequent analysis, remembering however that all variables correspond to rescaled quantities. These are the equations related to the displacement of the free surface wave \( \eta \), function related to velocity potential \( f \) and the leakage velocity \( C \).

In order to formulate the problem in a more general way \( C(x,t) \) is considered to have different forms. In section(2.2), \( C \) is considered to be a constant i.e., leakage velocity of water at the bottom is constant throughout its motion.

### 2.2 \( C \) is pure constant

Considering \( C \) to be constant, equations (14), (15) can be written as

\[
\eta_t + f_x + \epsilon f \eta_x + \epsilon \eta f_x - \frac{1}{6} \delta^2 f_{xxx} = C
\]

\[
\eta_x + f_t + \epsilon f f_x - \frac{1}{2} \delta^2 f_{xt} = 0
\]

Expanding \( f \) in a series of small parameters as

\[
f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + h.o.t
\]

and neglecting higher order terms in \( \epsilon \) or \( \delta^2 \), equations (16), (17) converge to

\[
\eta_t + f^{(0)}_x + \epsilon(f^{(1)}_f + \eta f^{(0)}_x + \eta_x f^{(0)}_f) + \delta^2(f^{(2)}_f - \frac{1}{6} f^{(0)}_{xxx}) = C
\]

\[
\eta_x + f^{(0)}_t + \epsilon(f^{(1)}_f + f^{(0)}_f f^{(0)}_x) + \delta^2(f^{(2)}_f - \frac{1}{2} f^{(0)}_{xt}) = 0
\]

In order that equations (19), (20) are self-consistent as evolution equations for a one-dimensional wave propagating along the positive x-axis, the following choice is made:

\[
f^{(0)} = \eta - Ct + O(\epsilon \delta^2)
\]

where \( O(\epsilon \delta^2) \) denote terms \( \sim \epsilon \delta^2 \). Thus from equations (19), (20), we get

\[
\ddot{\eta}_t + \ddot{\eta}_x + \epsilon(f^{(1)}_f + 2\ddot{\eta} f + \ddot{\eta}_x C + \eta f^{(2)} - \frac{1}{6} \dddot{\eta}_{xxx}) = 0
\]

where \( \ddot{\eta}_t \) and \( \dddot{\eta}_{xxx} \) denote terms \( \sim \epsilon \delta^2 \). Thus from equations (19), (20), we get
\[ \tilde{\eta}_t + \tilde{\eta}_x + \epsilon(f^{(1)}_t + \tilde{\eta}\tilde{\eta}_x) + \delta^2(f^{(2)}_t - \frac{1}{2}\tilde{\eta}_xxx) = 0 \]  

where \( \tilde{\eta} = \eta - Ct \).

Let \( f^{(1)}, f^{(2)} \) be functions of \( \tilde{\eta} \) and its spatial derivatives. This leads to \( f^{(1)}_t = -\tilde{\eta}_x f^{(1)}_{\tilde{\eta}} + f^{(1)}_\tilde{\eta} O(\epsilon, \delta^2) \). where \( O(\epsilon, \delta^2) \) is the term proportional to \( \epsilon \) or \( \delta^2 \). Since terms of the order of \( \epsilon\delta^2 \) are being neglected in the present work, the following relations are obtained

\[
\begin{align*}
    f^{(1)}_t &\approx -\tilde{\eta}_x f^{(1)}_{\tilde{\eta}} = -f^{(1)}_x \\
    f^{(2)}_t &\approx -\tilde{\eta}_x f^{(2)}_{\tilde{\eta}} = -f^{(2)}_x \\
    \tilde{\eta}_xxx &\approx -\tilde{\eta}_xxx
\end{align*}
\]

Using these results in equations (22) and (23), the condition for compatibility of these two equations leads to

\[
\begin{align*}
    f^{(1)}_x &= -\frac{1}{2} \tilde{\eta}_x (\tilde{\eta} + Ct) \\
    f^{(2)}_x &= \frac{1}{3} \tilde{\eta}_xxx
\end{align*}
\]

These results when substituted into any of the equations (22), (23), the following single evolution equation is obtained as

\[
\tilde{\eta}_t + \tilde{\eta}_x + \epsilon\left(\frac{3}{2} \tilde{\eta}_x + \frac{1}{2} \tilde{\eta}_x Ct\right) + \delta^2\left(\frac{1}{6} \tilde{\eta}_{xxx}\right) = 0
\]

Equation 26 can be converted to a forced KdV equation with a constant forcing term by redefining the dependent variable as \( \tilde{\eta} = 3\eta + Ct \).

Equation 26 is similar to KdV equation except for the 4-th term which comes from the leakage. A suitable transformation into a moving frame can remove this term so that the standard form of KdV equation is recovered. We use,

\[
X = x - t - bt^2, T = t
\]

where \( b \) is a constant denoting the acceleration of the frame. In the new frame \( (X, T) \), equation (26) will look like

\[
\tilde{\eta}_T + \epsilon\left(\frac{3}{2} \tilde{\eta}_X + \frac{1}{2} \tilde{\eta}_X CT\right) + \delta^2\left(\frac{1}{6} \tilde{\eta}_{XXX}\right) - 2bT \tilde{\eta}_X = 0
\]

Choosing \( b = \frac{\epsilon C}{4} \) and defining new variables \( u = (3\epsilon/2\delta^2)\tilde{\eta}, T' = (\delta^2/6)T \) the following standard form of KdV equation is obtained which is in the accelerated frame.

\[
u_{T'} + 6uu_X + u_{XXX} = 0
\]

### 2.2.1 Solitary wave solution

The well known one-Soliton solution of equation (28) is given by

\[
u(X, T') = \frac{\beta}{2} Sech^2\left[\frac{\sqrt{\beta}}{2}(X - \beta T')\right]
\]

where \( \beta \) is a constant. Back boosting the solution to the rest frame we get

\[
u(x, t) = \frac{\beta}{2} Sech^2\left[\frac{\sqrt{\beta}}{2}(x - t\{1 + \frac{\beta\delta^2}{6} - \frac{\epsilon |C|}{4}\})\right]
\]

where \( |C| \) is the absolute value of \( C \). The argument of \( u(x, t) \) contains linear and quadratic terms in \( t \) and the term inside the second bracket behaves like the velocity of the wave.
When $t$ starts increasing from a very small value, the term inside the second parentheses decreases i.e. the wave retards. At a critical time given by

$$t = t_c = \frac{4(1 + \beta \delta^2)}{\epsilon |C|},$$

the second bracketed term vanishes and the wave stops. After $t_c$, the wave propagates in the negative $x$-axis with increasing speed. Since the wave moving in positive $x$ axis is only of concern to us, this oppositely moving wave can be neglected in a practical situation. Since $\epsilon$ is a small parameter, hence $t_c$ is large for small values of fluid leakage. For large leakage velocity, $t_c$ become small causing reflection of wave at earlier time. A $x$-$t$ plot of the solution (30) is shown in Figure (2) indicating that the wave gets reflected at time at $t_c$ and then moves in opposite direction.

### 2.3 $C$ is function of $t$ only

In the subsection (2.2) solitary wave solution has been obtained by considering a constant leakage velocity. The problem is now generalized by assuming $C$ to be function of $t$ only. From equations (14), (15)

$$\eta_t + f_x + \epsilon f_{xx} + \eta f_{x} - \frac{1}{6} \delta^2 f_{xxx} = C(t) \quad (31)$$

$$\eta_x + f_t + \epsilon f_{xx} - \frac{1}{2} \delta^2 f_{xxt} = 0 \quad (32)$$

Carrying out series expansion for $f$ as in equation (18) and neglecting higher order terms in $\epsilon$ or $\delta^2$ the following equations are obtained from equations (31) and (32)

$$\eta_t + f_x^{(0)} + \epsilon f_{x}^{(1)} + \eta f_{x}^{(0)} + \eta_x f_x^{(0)} + \delta^2 (f_x^{(2)} - \frac{1}{6} f_x^{(0)}) = C(t) \quad (33)$$

$$\eta_x + f_t^{(0)} + \epsilon (f_{t}^{(1)} + f_x^{(0)} f_x^{(0)}) + \delta^2 (f_t^{(2)} - \frac{1}{2} f_{xxt}^{(0)}) = 0 \quad (34)$$

In order to make equations (33) and (34) self-consistent, the following choice is made

$$f^{(0)} = \eta - \int C(t) dt + O(\epsilon \delta^2) \quad (35)$$

Thus from equation (33), (34) the following equations are obtained -

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon (f_x^{(1)} + 2 \tilde{\eta}_x + N(b) \tilde{\eta}(t)) + \delta^2 (f_x^{(2)} - \frac{1}{6} \tilde{\eta}_{xxx}) = 0 \quad (36)$$
\[ \eta_t + \eta_x + \epsilon (f_t^{(1)} + \eta \eta_x) + \delta^2 (f_t^{(2)} - \frac{1}{2} \eta_{xxx}) = 0 \]  

(37)

where \( \eta = \eta - B(t) \) where \( B(t) = \int C(t) \, dt \).

Considering \( f^{(1)} \), \( f^{(2)} \) to be functions of \( \eta \) and its spatial derivatives, \( f_t^{(1)} = -\eta_x f_{\eta}^{(1)} + f_{\eta}^{(1)} O(\epsilon, \delta^2) \). Since terms of the order of \( \epsilon \delta^2 \) are neglected, the following relations are obtained

\[ f_t^{(1)} \approx -\eta_x f_{\eta}^{(1)} = -f_x^{(1)} \]
\[ f_t^{(2)} \approx -\eta_x f_{\eta}^{(2)} = -f_x^{(2)} \]

\[ \eta_{xxx} \approx -\eta_{xx} \]

Using these results in any of the equations (36) and (37) together with the compatibility condition leads to the following single equation

\[ \eta_t + \eta_x + \epsilon \left( \frac{3}{2} \eta \eta_x + \frac{1}{2} \eta_x B(t) \right) + \delta^2 (\frac{1}{6} \eta_{xxx}) = 0 \]

(38)

Equation (38) can be cast in the form of a forced KdV equation by redefining a new variable as given in the previous subsection with a time dependent forcing term. An analytical treatment of the influence of the time dependent random external noise on the propagation of nonlinear waves has been carried out by Orlowski\[7\] by considering a forced KdV equation. Considering the Gaussian character of the noise, the nature of deformation of the stationary solution of KdV-Burgers equation was studied\[8\]-\[9\]. during its propagation in randomly excited media.

In order to arrive at a standard form of the KdV equation from equation (38) a transformation to a moving frame given by

\[ X = x - t - a(t), \quad T = t \]

is carried out, where \( a(t) \) is a function dependent on \( t \). This leads to

\[ \eta_T + \frac{3 \epsilon}{2} \eta_X + \delta^2 \eta_{XX} + \left( \frac{\epsilon B(t)}{2} - \frac{\partial a}{\partial t} \right) \eta_X = 0 \]

(40)

With the choice \( \partial a/\partial t = \epsilon B(t)/2 \), and defining \( u = (3\epsilon/2\delta^2) \eta, \quad T' = (\delta^2/6)T \), we get the standard form of KdV equation in the moving frame

\[ u_{T'} + 6uu_X + u_{XXX} = 0 \]

(41)

### 2.3.1 Nature of the Solution

One-Soliton solution of equation (41) is of the standard form

\[ u(X, T') = \frac{\alpha}{2} \text{Sech}^2 \left[ \frac{\sqrt{\alpha}}{2} (X - \alpha T') \right] \]

(42)

where \( \alpha \) is a constant. Transforming this solution to the rest frame

\[ u(x, t) = \frac{\alpha}{2} \text{Sech}^2 \left[ \frac{\sqrt{\alpha}}{2} (x - (1 + \frac{\alpha \delta^2}{6}) t - \frac{\epsilon}{2} \int B(t) \, dt) \right] \]

(43)

It should be noted that when \( C \) is constant then the integral term inside the argument of \( u(x, t) \) is \( \frac{C_{abs} t^2}{4} \) which is consistent with the solution of the previous case.

The functional form of \( B(t) \) controls the motion of the solution (43). For different choices of the function \( B(t) \) the 3D x-t plot of the solution will have different shapes. If the leakage velocity is dependent on the fluid velocity such that when a large upsurge of water arrives in a region, large leakage occurs and when \( t \rightarrow \pm \infty \) the leakage goes to zero i.e. leakage is localized in time. As an example we can choose the functional form of \( C \) as, \( C(t) = b_0/(b_1 + b_2 t^2) \), where \( b_0, b_1, b_2 \) are constants, which is localized in time. The corresponding 3D plot of the solution is given as Figure (3). The solution gets curved due to the presence of nonlinear function of \( t \).

#### 2.4 C is a function of x only
FIG. 3: $x-t$ plot of the solution \([43]\) with $\alpha = 1$, $\delta = 0.01$, $\epsilon = 0.0001$, $b_0 = -20000$, $b_1 = 1$, $b_2 = 1$. $x$ and $t$ are plotted in the 2 horizontal axes and $u(x,t)$ is plotted in the vertical axis.

The next case of interest deals with the situation where the leakage velocity is dependent on spatial coordinate only. This has physical relevance in practical situations where the presence of porous medium in different parts of the bottom of the channel makes the leakage velocity also to be dependent on $x$.

From equations \([14]\), \([15]\) we obtain

$$\eta_t + f_x + \epsilon f \eta_x + \epsilon \eta f_x = \frac{1}{6} \delta^2 f_{xxx} = C - \frac{1}{2} \delta^2 C_{xx} \quad (44)$$
$$\eta_x + f_t + \epsilon f f_x - \frac{1}{2} \delta^2 f_{xxt} = 0 \quad (45)$$

The function $f$ is expanded in a perturbation series as in earlier sections and the following choice is made for $f^{(0)}$,

$$f^{(0)}(x,t) = \eta(x,t) + B(x) + O(\epsilon \delta^2) \quad (46)$$

where $B(x) = \int C(x) dx$.

The compatibility of the 2 equations \([44]\) and \([45]\) leads to,

$$f^{(1)}_x = -\frac{1}{2} (\eta_x \eta - BC) \quad (47)$$
$$f^{(2)}_x = \frac{1}{3} (\eta_{xxx}) - \frac{1}{6} C_{xx} \quad (48)$$

Substituting these values of $f^{(1)}_x$, $f^{(2)}_x$ in any of the equations \([44]\) and \([45]\) leads to a single evolution equation

$$\ddot{\eta}_t + \ddot{\eta}_x + \frac{3}{2} \dddot{\eta}_x + \frac{\delta^2}{6} \dddot{\eta}_{xxx} = \frac{2}{3} B_x + \frac{\epsilon}{6} B B_x - \frac{\delta^2}{18} B_{xxx} \quad (49)$$

where $\ddot{\eta} = \eta + \frac{\epsilon}{3} B$ and the equation \([49]\) has the form of a forced KdV equation.

2.4.1 Nature of the Solution

For the case when the vertical fluid velocity at the bottom is a function of spatial coordinate only, the nonlinear equation for water wave propagation given by equation \([49]\) is obtained as an inhomogeneous KdV equation. The right hand side of eq. \([49]\) has a space KdV like form with the time coordinate replaced by the spatial coordinate (here $x$). The mathematical elegance of this equation enables one to obtain its solitary wave solutions in a simple manner considering the following two different cases:

Case (a)

A very simple situation occurs if the right hand side of the equation \([49]\) is taken to be zero. Using the scaling \((X = x/\delta, B' = \epsilon B)\), the inhomogeneous part of equation \([49]\) reduces to the following:

$$12B'_X + 3B' B'_X - B'_{XXX} = 0 \quad (50)$$
A one-soliton like solution of this space KdV equation is obtained as

\[ B'(X) = (-12) sech^2[\sqrt{3}(X - X_0)] \]  

and the corresponding leakage velocity is given by

\[ C'(X) = \frac{\partial B'}{\partial X} = (24\sqrt{3}) sech^2[\sqrt{3}(X - X_0)] tanh[\sqrt{3}(X - X_0)] \]  

This leads to the conclusion that if \( B' \) has the functional form that satisfies (50), then the evolution equation for the waves would be the standard KdV equation. Hence its solution will also be given by the standard KdV solitary wave solution moving with constant velocity. Thus for those functional forms of \( B' \) and the corresponding leakage velocity, the wave solution will be practically unaffected by the leakage. Since the solution of this case is the solution of standard KdV, it is not being shown explicitly. The leakage velocity given in equation (52) is plotted as Figure 4.

![FIG. 4: Leakage velocity profile for \( X_0 = 0 \) when \( B' \) satisfies (50)](image)

**Case (b):**

An interesting analytic solution of the equation (49) arises if the leakage velocity function has a preassigned form. Considering the leakage velocity to be localized in space, (that is often consistent with certain physical condition), the following relevant form of \( B(x) \) is chosen

\[ B(x) = m \tanh(nx), \]  

where \( m \) and \( n \) are two external small parameters dependent on the leakage profile.

Since \( m, n \) are small parameters, in the following calculation terms up to ~ \( mn \) will be retained and terms beyond this order are neglected. Hence second and third terms in r.h.s of the equation (49) will produce higher order terms in \( m, n \) and hence these are neglected.

Transforming to a new variable \( u = \bar{\eta} - 2B_xt/3 \), the following equation is obtained from equation (49)

\[ u_t + u_x + \frac{3}{2}uu_x + \frac{\delta^2}{6}u_{xxx} + \epsilon B_xu_xt = 0 \]  

Further, a transformation to a moving frame given by

\[ X = x - \frac{\epsilon B_x}{2}t^2 - t, T = t \]

is carried out to obtain

\[ u_T + (\frac{3}{2})uu_X + (\frac{\delta^2}{6})u_{XXX} = 0 \]  

The above equation is finally cast in the standard form of a KdV equation by defining the following variables \( U = (3\epsilon/2\delta^2)u, T' = (\frac{\delta^2}{6})T \)

\[ U_{T'} + 6UU_X + U_{XXX} = 0 \]
The corresponding solitary wave solution in the rest frame is given by

\[
U(x, t) = \frac{\gamma}{2} \text{Sech}^2\left[\frac{\sqrt{\gamma}}{2} (x - \frac{\epsilon B_x t^2}{2} - t(1 + \frac{\gamma \delta^2}{6}))\right]
\]

\[
= \frac{\gamma}{2} \text{Sech}^2\left[\frac{\sqrt{\gamma}}{2} (x - \frac{\epsilon m n \text{Sech}^2(nx)}{2} t^2 - t(1 + \frac{\gamma \delta^2}{6}))\right],
\]

where \(\gamma\) is a constant. The critical time \(t_c\) at which the velocity of the wave becomes zero is also a function of \(x\). The 3D plot of the solution is shown in Figure (5).

![3D plot of the solution](image)

**FIG. 5:** \(x - t\) plot of the solution (57) with \(m = -0.1, n = 0.1, \gamma = 1\)

### 2.5 When \(C\) is function of both \(x\) and \(t\)

Finally the problem will be treated in the most general manner by treating the leakage velocity \(C\) as a function of both \(x\) and \(t\). The mathematical analysis is carried out in the following way. As in all the earlier cases \(f\) is expanded as

\[
f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + \text{h.o.t,}
\]

where \(f^{(i)}\)'s are functions of \(\eta\) and its spatial derivatives. In the present case, the leakage velocity \(C(x, t)\) is also expanded in a series of small parameters in the following way

\[
C(x, t) = \epsilon C_1(x, t) + \delta^2 C_2(x, t) + \text{h.o.t}
\]

Carrying out the same kind of mathematical analysis as in the earlier cases the following two equations are obtained from equations (14) and (15)

\[
\eta_t + f_x^{(0)} + \epsilon (f_x^{(1)} + \eta f_x^{(0)} + \eta_x f^{(0)} - C_1) + \delta^2 (f_x^{(2)} - \frac{1}{6} f^{(0)}_{xxx} - C_2) = 0
\]

\[
\eta_x + f_t^{(0)} + \epsilon (f_t^{(1)} + f_x^{(0)} f^{(0)} + \delta^2 (f_t^{(2)} - \frac{1}{2} f_{xx}^{(0)}) = 0
\]

In order to make equations (60), (61) self-consistent as evolution equation for a 1d wave propagating to the right, the following transformation is carried out

\[
f^{(0)}(x, t) = \eta(x, t) + \text{h.o.t}
\]

Since terms of the order of \(\epsilon \delta^2\) are neglected, the following relations are obtained

\[
f_t^{(1)} \approx -\eta_x f_{\eta}^{(1)} = -f_x^{(1)}
\]

\[
f_t^{(2)} \approx -\eta_x f_{\eta}^{(2)} = -f_x^{(2)}
\]
\[ \eta_{xx} t \approx -\eta_{xxx} \]

From the compatibility of the two equations (60) and (61) we get,

\[ f^{(1)}_x = \frac{1}{2} (C_1 - \eta_x \eta) \]
\[ f^{(2)}_x = \frac{1}{3} (\eta_{xxx}) + \frac{1}{2} C_2 \] (63)

Finally, a single evolution equation is obtained utilizing these functional forms of \( f^{(1)}_x, f^{(2)}_x \) in any of the equations (60), (61) as:

\[ \eta_t + \eta_x + \left( \frac{3}{2} \right) \eta \eta_x + \left( \frac{\delta^2}{6} \right) \eta_{xxx} = \frac{1}{2} (\epsilon C_1 + \delta^2 C_2) \] (64)

For the sake of mathematical simplicity the condition \( C_2(x,t) = 0 \) is assumed. Since \( \epsilon \approx \delta^2 \), it does not break the generality of the problem. Using Galilean transformation \( \xi = x - t \) and \( \tau = \left( \frac{\delta^2}{6} \right) t \) and defining the variable \( u(\xi, \tau) = \left( \frac{3}{2} \right) \eta(\xi, \tau) \), the final form of the evolution equation is obtained as,

\[ u_\tau + 6u u_\xi + u_{\xi\xi\xi} = \frac{9}{2} C_1(\xi, \tau) \] (65)

The above equation has the form of a forced kdv equation.

### 2.5.1 Nature of Solutions

For the sake of notational simplicity equation (65) is expressed in variables \((x, t)\) as

\[ u_t + 6u u_x + u_{xxx} = \frac{9}{2} C_1(x, t) \] (66)

In order to obtain a solitary wave solution of the forced KdV equation, the bilinearization technique[16] is used. Assuming

\[ \frac{9}{2} C_1(x, t) = \frac{\partial D(x, t)}{\partial x} \] (67)

and using the bilinear transformation

\[ u = 2 \frac{\partial^2}{\partial x^2} [\ln(F)], \quad D = \frac{G}{F} \]

equation(66) transforms to the following bilinear equation

\[ FF_{xt} - F_x F_t + FF_{xxxx} + 3F_x^2 - 4F_x F_{xxx} = GF \] (68)

In the previous cases, we have attempted to obtain the final solutions in the form of solitary waves where phase gets modulated and amplitude remains constant. In order to obtain similar solitary wave solutions in this case also the following choice for the functions \( G \) and \( F \) have to be made,

\[ G = h(t) Sech[x - vt - p(t)] \] (69)

\[ F = h_1(t)(\exp[x - vt - q(t)] + \exp[-x + vt + q(t)]) \] (70)

where \( h(t), h_1(t), p(t) \) and \( q(t) \) are all arbitrary functions of time. Substituting the expressions for \( F \) and \( G \) in equation (68) leads to

\[ p(t) = q(t), \quad h_1(t) = -\frac{h(t)}{2(-4 + v + \frac{dp}{dt})} \] (71)
Thus, the analytic solution of the forced KdV equation \( \text{(66)} \) is obtained as
\[
 u = 2(\text{Sech}[x - vt - p(t)])^2
\]  
(72)
with the forcing term \( D \) given by
\[
 D(x, t) = -(4 + v + \frac{\partial q}{\partial t})(\text{Sech}[x - vt - p(t)])^2
\]  
(73)
Hence the leakage velocity \( C_1(x, t) \) is obtained in the form
\[
 C_1(x, t) = -2(-4 + v + \frac{\partial p}{\partial t})(\text{Sech}[x - vt - p(t)])^2 \tanh[x - vt - p(t)]
\]  
(74)
Since there is an arbitrary function \( p(t) \) in the leakage velocity, as well as in the solution, one can observe different types of waves excited by different forcing sources, i.e. different functional forms of \( p(t) \). Here also the amplitude of the wave solution remains constant.

![FIG. 6: x – t plot of the solution \( \text{(72)} \) for \( q(t) = \text{Sech}(10t) \), v = 1](image)

**Conclusive remarks**

The problem of shallow water, unidirectional, weakly nonlinear, surface wave propagation in a water channel is treated (in a way distinct from existing literature) by incorporating all the information related to the interactions of water and the bed in the form of a non-trivial penetration condition. When the vertical fluid velocity at the bottom is constant then it is shown that the evolution equation is given by a forced KdV equation with a constant forcing term. The solitary wave solutions of such equation will contain a constant retardation term in the argument of the function, while the amplitude will remain constant. When the leakage is a function of time, a time dependent retardation appears in the argument of the solution while the amplitude still remains constant. When the leakage velocity is only space dependent, two different kinds of solitary wave solutions are obtained analytically. For those functional forms of \( B(x) \), satisfying the stated space KdV like equation, the solution will remain unaffected by the leakage and becomes identical to the standard KdV soliton with constant velocity. When the leakage velocity is a slowly varying function localized in space a soliton solution with constant amplitude is obtained under certain approximation related to the slowness of variation of the leakage velocity profile. The technique yields a solitary wave solution with a constant amplitude and an arbitrary function of time appearing in the argument of the solution as well as in the argument of the leakage velocity profile. The nature of the solution can be modulated by choosing different forms of this arbitrary function.

The present work can be extended to address certain interesting problems. Retaining the higher powers of the expansion parameters \( \epsilon, \delta^2 \), other evolution equations will be obtained that are applicable to the other portions of the channel. Also, in the present work, the channel depth is assumed to be constant. If the variation of bathymetry is considered, then solving the problem analytically in presence of leakage velocity, is a challenging task. If transverse directions are involved then instead of KdV equation, multidimensional type equations will evolve which are nearer to the actual real physical situations.
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[1] D.J. Korteweg and G. de Vries, Philos. Mag., 39, 422(1895).
[2] G.I. Burde, Phys. Rev. E, 84, 026615 (2011).
[3] E. Infeld, A. Senatorski, and A. A. Skorupski, Phys. Rev. E, 51, 3183 (1995).
[4] M.J. Ablowitz and D.E. Baldwin, Phys. Rev. E, 86, 036305 (2012).
[5] T.Y. Wu, J. Fluid Mech. 184, 75 (1987).
[6] W.K. Melville and K.R. Helfrich, J. Fluid Mech., 178, 31 (1987).
[7] A. Orlowski, Phys. Rev. E, 49, 2465 (1994).
[8] M. Wadati, J. Phys. Soc. Jpn. 52, 2642 (1983).
[9] N. Zahibo, E. Pelinovsky and A. Sergeeva, Chaos, Solitons and Fractals, 39, 1645 (2009).
[10] R. Hirota, Direct Method in Soliton Theory, Springer, Berlin (1980).
[11] Z. Jun-Xiao and G. Bo-Ling Commun. Theor. Phys. (Beijing, China) 52 (2009) 279.