A CLASSIFICATION OF CONFORMALLY FLAT GENERALIZED RICCI RECURRENT PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. Conformally flat pseudo-Riemannian manifolds with generalized Ricci recurrent, $(GR)_n$ structure are completely classified in this short report. A conformally flat generalized Ricci recurrent pseudo-Riemannian manifold is shown to be either a de Sitter space or an anti-de Sitter space. In particular, a conformally flat generalized Ricci recurrent spacetime must be either a de Sitter spacetime or an anti-de Sitter spacetime.

1. Introduction

The importance of the Generalized Ricci recurrent structure and its interaction with several gravity theories, like the standard theory of gravity [5], the modified $f(R)$-theory [1], the modified Gauss-Bonnet $f(R,G)$-theory [2] are well established. And of course, mathematically the structure raised curiosity among researchers in several occasions, details can be seen in [5] and the references therein.

De et al. [3] introduced the notion of $(GR)_n$ as an $n$-dimensional non-flat pseudo-Riemannian manifold whose Ricci tensor $R_{ij}$ satisfies the following:

$$\nabla_i R_{jl} = A_i R_{jl} + B_i g_{jl},$$  

where $A_i$ and $B_i$ are two non-zero 1-forms. The structure is considered to be a generalization of Patterson introduced Ricci recurrent manifolds $R_n$ [7] in which the Ricci tensor satisfies $\nabla_i R_{jl} = A_i R_{jl}$, with a non-zero 1-form $A_i$. Obviously, if the one-form $B_i$ vanishes, it reduces to a $R_n$.

A $(GR)_4$ spacetime is a generalized Robertson Walker spacetime with Einstein fibre for a Codazzi type $R_{ij}$ [1]. In [5], a conformally flat $(GR)_4$ was shown to be a perfect fluid. Its interaction with general relativistic cosmology was discussed thoroughly in the same paper. Continuing to this study, in [1], the authors studied a conformally flat $(GR)_4$ with constant $R$ as a solution of $f(R)$-gravity theory. The presently accepted homogeneous and isotropic model of our universe, the Robertson-Walker spacetime is $(GR)_4$ if and only if it is Ricci symmetric. The equation of state (EoS) was shown to have $\omega = -1$. Several energy conditions were also analyzed and validated by current observational dataset. Very recently, the impact of $(GR)_4$ structure is investigated in modified Gauss-Bonnet, $f(R,G)$ theory of gravity [2]. The obtained results were examined for two particular $f(R,G)$-models and for them both, the weak, null and dominant energy conditions were validated while the strong energy condition was violated, which is a good agreement with

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the recent observational studies revealing that the current universe is in accelerating phase. Despite having such recognised physical significance, a complete classification of conformally flat Lorentzian manifolds with generalized Ricci recurrent structure is still due. This motivated us to do a careful scrutiny through all the possible cases. We shall give a complete classification of this structure in a general setting as below:

**Theorem 1.1.** A conformally flat generalized Ricci recurrent pseudo-Riemannian manifold is an Einstein manifold, in particular, it is either a de Sitter space or an anti-de Sitter space.

We can then deduce a classification of conformally flat generalized Ricci recurrent spacetime.

**Corollary 1.1.** A conformally flat generalized Ricci recurrent spacetime is either a de Sitter spacetime or an anti-de Sitter spacetime.

2. **The Proof**

Let $M$ be an $n$-dimensional pseudo-Riemannian manifold, $n \geq 3$, with index $q$ ($0 \leq q \leq n$). Suppose $M$ is a conformally flat $(GR)_n$, that is, the Ricci tensor satisfies (1) and the Riemannian curvature tensor satisfies $[6, \text{pp.302}]

\begin{equation}
R^l_{ijk} = \frac{R_{ki}\delta^l_j - R_{ji}\delta^l_k + g_{ki}R^l_j - g_{ji}R^l_k}{n-2} - \frac{g_{ki}\delta^l_j - g_{ji}\delta^l_k}{n-1} R
\end{equation}

(2)

The divergence of the above equation gives

\begin{equation}
\nabla_j R_{ik} - \nabla_k R_{ij} = \frac{1}{2} \nabla_j R g_{ki} - \nabla_k R g_{ji}.
\end{equation}

(3)

Contracting $j$ and $l$ in (1) we obtain

\begin{equation}
\nabla_i R = RA_i + nB_i.
\end{equation}

(4)

Contracting $i$ and $l$ in (1) we obtain

\begin{equation}
\frac{1}{2} \nabla_j R = A^l R_{lj} + B_j.
\end{equation}

(5)

It follows from (4)–(5) that

\begin{equation}
2A^l R_{lj} = RA_j + (n-2)B_j.
\end{equation}

(6)

By using (1), (3)–(4) we calculate

\begin{equation}
2(n-1)\{A_i R_{kl} - A_k R_{il}\} = RA_i g_{kl} - RA_k g_{il} - (n-2)B_i g_{kl} + (n-2)B_k g_{il}.
\end{equation}

(7)

Next we split into two cases: $A^i$ is not a lightlike vector and $A^i$ is a lightlike vector.

(I) $A^i$ is either a timelike or spacelike vector. Write

\begin{equation}
A_i = \varepsilon \alpha U_i; \quad U_l U^l = \varepsilon = \pm 1; \quad \alpha \neq 0.
\end{equation}
Then (6)–(7) become
\[ 2\varepsilon\alpha U^l R_{ij} = \varepsilon\alpha R U_j + (n - 2) B_j, \]
\[ 2(n - 1)\varepsilon\alpha \{U_i R_{kl} - U_k R_{il}\} = \varepsilon\alpha \{RU_i g_{kl} - RU_k g_{il}\} - (n - 2) B_i g_{kl} + (n - 2) B_k g_{il}. \]
Transvecting (9) with $U^i$ we obtain
\[ 2(n - 1)\alpha R_{kl} = \{R - (n - 2)\beta\} g_{kl} + (n - 2)\{\varepsilon\alpha R U_k U_l + U_k B_i + (n - 1) U_k B_l\}, \]
where $\beta = B_i U^i$. The skew-symmetry part gives
\[ U_i B_k + (n - 1) U_B = U_k B_i + (n - 1) U_i B_k, \]
which implies that
\[ B_i = \varepsilon\beta U_i, \quad (\beta = U^i B_i). \]
Armed with this significant result, using (4), (8) and (10) we can conclude that the Ricci curvature tensor and the Ricci scalar satisfy
\[ 2(n - 1)\varepsilon\alpha R_{kl} = \{\varepsilon\alpha R - (n - 2)\lambda\} g_{kl} + (n - 2)\{R + n\lambda\} \varepsilon U_k U_l, \]
\[ \nabla_i R = \{R + n\lambda\} \alpha \varepsilon U_i, \]
\[ 2U^l R_{ij} = \{R + (n - 2)\lambda\} U_j, \]
where
\[ \lambda = \frac{\beta}{\alpha}. \]
Covariantly differentiating (12) with respect to $i$ we obtain
\[ 2(n - 1)\nabla_i R_{kl} = \{\nabla_i R - (n - 2)\nabla_i \lambda\} g_{kl} + (n - 2)\{\nabla_i R + n\nabla_i \lambda\} \varepsilon U_k U_l + (n - 2) (R + n\lambda) \varepsilon \nabla_i (U_k U_l) \]
\[ + (n - 2) \{ - \nabla_i \lambda g_{kl} + n \nabla_i \lambda \varepsilon U_k U_l + (R + n\lambda) \varepsilon \nabla_i (U_k U_l) \} \]
\[ + \nabla_i R \{g_{kl} + (n - 2) \varepsilon U_k U_l\}. \]
On the other hand, (11), (11) ̅(12) give
\[ 2(n - 1)\nabla_i R_{kl} = 2(n - 1) \alpha \varepsilon U_i R_{kl} + 2(n - 1) \lambda \alpha \varepsilon U_i g_{kl} \]
\[ = \{R - (n - 2) \lambda\} \alpha \varepsilon U_i g_{kl} + (n - 2)\{R + n\lambda\} \alpha \varepsilon U_i U_k U_l \]
\[ + 2(n - 1) \lambda \alpha \varepsilon U_i g_{kl} \]
\[ = (R + n\lambda) \alpha \varepsilon U_i \{g_{kl} + (n - 2) \varepsilon U_k U_l\} \]
\[ = \nabla_i R \{g_{kl} + (n - 2) \varepsilon U_k U_l\}. \]
These two equations give
\[ - \nabla_i \lambda g_{kl} + n \nabla_i \lambda \varepsilon U_k U_l + (R + n\lambda) \varepsilon \nabla_i (U_k U_l) = 0. \]
Transvecting with $U^k$ and $U^l$ we obtain
\[ \nabla_i \lambda = 0. \]
It follows from these two equations that
\[(R + n\lambda)\nabla_i(U_kU_l) = 0.\]

We consider two subcases: \( R + n\lambda \neq 0 \) and \( R + n\lambda = 0 \).

\textbf{a:} If \( R + n\lambda \neq 0 \), we have \( \nabla_i U_l = 0 \) and so \( U_l R^l_i = 0 \). It follows from (14) that
\[ R + (n - 2)\lambda = 0. \]

It follows from this equation and (15) that
\[ \nabla_i R = -(n - 2)\nabla_i \lambda = 0. \]

After applying this to (13) gives \( R + n\lambda = 0 \). Hence we have \( R = \lambda = 0 \) and so \( R_{lk} = 0 \); violating the hypothesis of being \((GR)\). Hence this case is impossible.

\textbf{b:} Suppose that \( R + n\lambda = 0 \).

Then, using (12) we obtain
\[ R_{kl} = -\lambda g_{kl}, \]
meaning that it is an Einstein space.

\textbf{(II) \( A^i \) is a lightlike vector.} Take another lightlike vector \( K^i \) such that
\[ A_l A^i = K_l K^i = 0; \quad A_l K^i = -1. \]

Transvecting (7) with \( A^i \), with the help of (6), we obtain
\[ -A^j B_j g_{kl} + RA_k A_l + A_l B_k + (n - 1)A_k B_l = 0. \tag{16} \]

The skew-symmetry part gives
\[ A_l B_k = A_k B_l, \]
which implies that \( A^k B_k = 0 \) and
\[ B_i = -\lambda A_i, \quad (\lambda = K^l B_l). \tag{17} \]

Substituting these into (14) and (16) respectively give
\[ \nabla_i R = R - n\lambda = 0, \tag{18} \]
and so
\[ \nabla_i \lambda = 0. \tag{19} \]

By using (17)–(18), we can simply (7) as
\[ A_l R_{kl} - A_k R_{il} = \lambda A_l g_{kl} - \lambda A_k g_{il}. \tag{20} \]

Transvecting with \( K^i \) and \( K^l \) gives
\[ K^l R_{lk} = -\tau A_k + \lambda K_k; \quad (\tau = K_i^l K^j R_{ij}). \]

Transvecting (20) with \( K^i \) and applying the above equation gives
\[ R_{lk} = \lambda g_{lk} + \tau A_l A_k. \tag{21} \]
Suppose \( \tau \neq 0 \). Applying (18) and (21), we can simplify (2) as
\[
R_{ijk} = \frac{\lambda}{n-1} \{g_{ki} \delta^l_j - g_{ji} \delta^l_k\} + \frac{\tau}{n-2} \{A_k A_i \delta^l_j - A_j A_i \delta^l_k + g_{ki} A^l A_j - g_{ji} A^l A_k\}. \tag{23}
\]
On the other hand, by substituting (17) and (21) into (1) give
\[
\nabla_i R_{jk} = \tau A_i A_j A_k. \tag{24}
\]
It follows from (2), (18)–(19) and (21) that
\[
\nabla_i R_{hj} = \nabla_h R_{ij} = \nabla_h (\tau A_i A_j).
\]
Covariantly differentiating (24) with respect to \( h \) gives
\[
\nabla_h \nabla_i R_{jk} = \nabla_h (\tau A_i A_j A_k) + \tau A_i A_j \nabla_h A_k = \nabla_h R_{ij} A_k + \tau A_i A_j \nabla_h A_k,
\]
and so
\[
\tau A_i A_j \nabla_h A_k - \tau A_j A_k \nabla_i A_k = (\nabla_h \nabla_i - \nabla_i \nabla_h) R_{jk} = -R^l_{jhi} R_{lk} - R^l_{khi} R_{lj}.
\]
Next, by applying (21), the properties \( R_{kjh} + R_{jkh} = 0 \) and (22) we have
\[
A_j A_i \nabla_h A_k - A_j A_h \nabla_i A_k = -R^l_{jhi} A_l A_k + R^l_{khi} A_l A_j.
\]
Furthermore, by using (23) we obtain
\[
A_j A_i \left\{ \nabla_h A_k - \frac{\lambda}{n-1} g_{hk} \right\} - A_j A_h \left\{ \nabla_i A_k - \frac{\lambda}{n-1} g_{ik} \right\} = \frac{\lambda}{n-1} \{ -g_{ij} A_h + g_{hj} A_i \} A_k. \tag{25}
\]
Transvecting with \( K^i \) and \( K^j \) we obtain
\[
\nabla_h A_k - \frac{\lambda}{n-1} g_{hk} = -A_h \Omega_k - \frac{\lambda}{n-1} K_h A_k; \quad (\Omega_k = K^i \nabla_i A_k - \frac{\lambda}{n-1} K_k).
\]
Substituting this into (25) gives
\[
\frac{\lambda}{n-1} \{ -A_j A_i K^h + A_j A_h K_i \} A_k = \frac{\lambda}{n-1} \{ -g_{ij} A_h + g_{hj} A_i \} A_k.
\]
Since \( \lambda \neq 0 \), it is descended to
\[
-A_j A_i K^h + A_j A_h K_i = -g_{ij} A_h + g_{hj} A_i.
\]
Transvecting with \( K^h \) and \( g^{ij} \) we obtain \( 1 = n - 1 \). This is impossible. Hence (22) is false or we must have \( \tau = 0 \). It follows from (21) that it is also an Einstein space in this case.

Hence we conclude that \( M \) is an Einstein space in both cases. As a conformally flat Einstein space is of constant sectional curvature and the manifold is non-flat, We conclude that a conformally flat generalized Ricci recurrent spacetime is either a de Sitter space or an anti-de Sitter space.
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