Accurate analytical solution of the circular sector oscillation by the modified harmonic balance method

Nadia M Farea¹, Mohra Zayed² and Gamal M Ismail³,⁴

Abstract
This paper aims to solve the nonlinear differential equation of the circular sector oscillator analytically via the modified harmonic balance method (MHBM). To assess the reliability and the precision of the present method, we have compared the obtained results with the global residue harmonic balance method, Akbari–Ganji’s method, and numerical Runge–Kutta method which reveals that the MHBM is more reliable than others methods.

Keywords
modified harmonic balance method, circular sector oscillator, nonlinear equation, analytical solutions

Introduction
Most oscillation systems used in engineering, biochemical, physical, and mechanical problems are generally revealed mathematically by nonlinear differential equations. Nonlinear differential equations are very important modern mathematics and are the basis for solving complex problems in many branches of sciences. In general, studying nonlinear oscillation differential equations that obtain exact solutions faces many difficulties. A few nonlinear systems of differential equations can be solved explicitly, and numerical methods, especially the Runge–Kutta method of the fourth order, are frequently used to calculate approximate solutions. Perturbation methods¹–⁴ were the first analytical and approximate methods to achieve approximate analytical solutions for nonlinear differential equations (NDEs). Recently, several methods have been introduced and developed to obtain approximate solutions for (NDEs) due to their complexity and the difficulty of solving them through traditional perturbation techniques. For example, variational iteration method,⁵ homotopy perturbation method,⁶ max-min approach,⁷–⁹ global residue harmonic balance method (GRHBM) for obtaining higher-order approximate solutions,¹⁰–¹² modified homotopy perturbation method,¹³–¹⁵ energy balance method,¹⁶,¹⁷ Hamiltonian approach,¹⁸–²⁰ iteration perturbation technique,²¹ coupled homotopy-variational approach,²²–²⁴ frequency-amplitude formulation,²⁵,²⁶ multiple scales technique,²⁷ parameter expansion method,²⁸ averaging method,²⁹ iteration method,³⁰ and Laplace variational iteration method.³¹

The harmonic balance method (HBM) is one of the main techniques for obtaining approximate analytical solutions to NDEs describing oscillatory systems.¹⁴,³²–³⁴ In recent decades, some researchers have studied the behavior of the circular sector oscillator generally modeled using NDEs. For example, Shaban et al.,¹³ investigated the numerical behavior of the nonlinear system using the modified homotopy perturbation method (HBM). To obtain the approximated solution with high accuracy, Hadi et al.,³⁵ considered Akbari–Ganji’s method (AGM) for solving this nonlinear oscillator. Moreover, Lu et al.,¹¹ used the GRHBM to obtain higher-order approximate solutions and compared it with the MHPM, AGM, and Runge–Kutta method while Pakar et al.,³⁶ used variational approach (VA).

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This paper extended the modified HBM up to second order to obtain the approximate analytical solutions for strongly nonlinear conservative systems. Comparing the approximate frequencies obtained with its numerical frequencies reveals that this method is effective and convenient for solving these analytical problems. Finally, an illustrative example demonstrates the validity and applicability of the method, which is further discussed in detail. The MHBM is suggested as a useful approach that can be easily extended to other strongly nonlinear oscillators.

**Basic concept of the proposed method**

We consider the governing differential equation in the following form

\[ \ddot{\theta} + \theta + \varepsilon f \left( \theta, \dot{\theta} \right) = 0 \]  

with the initial conditions

\[ \theta(0) = A, \quad \dot{\theta}(0) = 0 \]  

where \( \theta \) is the angular displacement, \( f(\theta, \dot{\theta}) \) is an odd nonlinear function, and \( \varepsilon \) is a constant parameter.

We also consider the approximate analytical solution to equation (1), which is in the following form

\[ \theta(t) = A(\rho \cos(\omega t) + \nu \cos(3\omega t) + \ldots), \]  

where \( A, \rho, \nu, \ldots \) are constants, \( \omega = \frac{2\pi}{T} \) is the frequency of the nonlinear oscillator, and \( T \) is the period. If \( \rho = \frac{1}{\omega^2} \) and the initial phase \( (\omega_0 t) = 0 \), the solution of equation (3) readily satisfies the given in equation (1).

Inserting equation (3) into equation (1) and using the Fourier series to expand the function \( f(\theta, \dot{\theta}) \), we finally can obtain the following

\[ A(\rho(1 - \omega^2)\cos(\omega_0 t) + \nu(1 - 9\omega^2)\cos(3\omega_0 t) + \ldots) \]

Comparing the coefficients of equation (4), we can obtain the following equations

\[ \rho(1 - \omega^2) = -\varepsilon F_1, \quad \nu(1 - 9\omega^2) = -\varepsilon F_3, \ldots \]  

With the first equation, \( \omega \) is eliminated from equation (5). Therefore, equation (5) takes the following form

\[ \rho \omega^2 = \rho + F_1, \quad 8\nu \rho = \rho F_3 - 9\nu F_1 \ldots \]  

By substituting \( \rho = (1 - \nu - \ldots) \), and simplifying, the second equation of equation (6) takes the following form

\[ \nu = G(\omega, \varepsilon, A, \ldots) \]  

where \( G \) exclude respectively the linear terms of \( \nu \)

Finally, we can obtain the value of \( \nu, \ldots \) and the approximate angular frequency \( \omega \) by solving equation (7) using a numerical technique.

**Application of the modified harmonic balance method**

In current work, we will consider the following nonlinear differential equation of the circular sector oscillator,\(^{13} \) which are widely used in many physical and engineering applications such as car spaces, the base of structures, and many other swing systems.

\[ \left( \frac{3}{2} R^2 - \frac{4}{3} \sin(\alpha) \right) \ddot{\theta} + R \left( \frac{2R \sin(\alpha)}{3\alpha} \sin(\theta) \right) \dot{\theta}^2 + \left( \frac{2 \sin(\alpha)}{3\alpha} g \right) \sin(\theta) = 0, \]

\[ \theta(0) = A, \quad \dot{\theta}(0) = 0 \]  

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\(^{1447}\) Farea et al.
Replacing the relatively accurate approximations: \( \cos \theta \approx \left( 1 - \frac{\theta^2}{2} \right) \), \( \sin \theta \approx \theta - \frac{\theta^3}{6} \), and \( k = 2 \frac{\sin \alpha}{3\alpha} \), into equation (8), the governing equation becomes

\[
\left( \frac{3}{2} R^2 - 2kR \left( 1 - \frac{\theta^2}{2} \right) \right) \ddot{\theta} + kR^2 \left( \theta - \frac{\theta^3}{6} \right) \dot{\theta}^2 + \frac{gk}{3} \left( \theta - \frac{\theta^3}{6} \right) = 0
\]

(9)

where \( \theta(t) \) denotes the angular displacement, \( R \) represents the semicircular radius, \( \alpha \) indicates the semicircular angle, \( g \) gravity acceleration, \( \dot{\theta} \) and \( \ddot{\theta} \) are the first and second differentiation with respect to \( t \), respectively. As illustrated in Figure 1 (see Ref. 13). The modified HBM was employed to obtain the approximate solution of equation (8).

We substitute \( \tau = \omega t \) with the unknown frequency \( \omega \) in the nonlinear differential equation (9), one can obtain

\[
\left\{ \begin{array}{l}
\frac{3}{2} R^2 - 2kR \left( 1 - \frac{\theta^2}{2} \right) \omega^2 \dddot{\theta} + kR^2 \left( \theta - \frac{\theta^3}{6} \right) \omega^2 \dot{\theta}^2 + \frac{gk}{3} \left( \theta - \frac{\theta^3}{6} \right) = 0,
\theta(0) = A, \quad \dot{\theta}(0) = 0.
\end{array} \right.
\]

(10)

**First-order modified harmonic balance method approximation**

From equation (4), the first-order analytical approximate solution can be expressed as follows

\[
\theta = A \cos (\omega_1 t)
\]

(11)

where \( \omega_1 \) is the angular frequency to be determined. By inserting equation (11) into equation (9) and considering the coefficient of \( \cos (\omega_1 t) \), we have

\[
- \frac{1}{48} A^5 k^2 R^2 \omega_1^2 - \frac{1}{8} A^3 gk + \frac{1}{4} A^3 kR^2 \omega_1^2 - \frac{3}{4} A^2 kR \omega_1^2 + Agk + 2AkR \omega_1^2 - \frac{3}{2} AR^2 \omega_1^2 = 0
\]

(12)

Finally, from equation (12), the first-order approximate angular frequency is

\[
\omega_1 = \frac{\sqrt{6} \sqrt{8gk - A^2 gk}}{\sqrt{4A^4 k^2 R^2 - 12A^2 k^2 R^2 + 36A^2 kR - 96kR + 72R^2}}
\]

(13)

Consequently, the first-order approximation for the approximate solution of equation (9) is given by equation (11) where \( \omega_1 \) is given by equation (13).

**Second-order modified harmonic balance method approximation**

The second-order approximation solution is assumed in the following form to improve the accuracy of the solution

\[
\theta = A(\rho \cos (\omega_2 t) + \nu \cos (3\omega_2 t))
\]

(14)

where \( \rho = 1 - \nu \). By inserting equation (14) into equation (9), and equating the coefficients of \( \cos (\omega_2 t) \) and \( \cos (3\omega_2 t) \), we have

![Figure 1. Geometric parameters of the homogeneous solid circular sector oscillator.](image-url)
Table 1. Comparison analytical solution, Akbari–Ganji’s method, and the numerical solution for $R = 15$, $A = \pi/8$, and $\alpha = 2\pi/3$.

| $t$  | Akbari–Ganji’s method$^{35}$ | Present method (modified harmonic balance method) | First solution | Second solution | Numerical solution |
|-----|-------------------------------|---------------------------------------------------|----------------|----------------|-------------------|
| 0   | 0.39269908                    | 0.39269908                                        | 0.39269908     | 0.39269908     | 0.39269908        |
| 10  | 0.24501045                    | 0.24407319                                        | 0.24408274     | 0.24512253     |                   |
| 20  | -0.09371990                   | -0.08930279                                       | -0.08927904    | -0.08993096    |                   |
| 30  | -0.36692738                   | -0.35508143                                       | -0.35506580    | -0.3555274     |                   |
| 40  | -0.35162485                   | -0.35208280                                       | -0.35210440    | -0.35254525    |                   |
| 50  | -0.06065639                   | -0.08257671                                       | -0.08263630    | -0.08307460    |                   |
| 60  | 0.26978783                    | 0.24943545                                        | 0.24937895     | 0.25054818     |                   |
| 70  | 0.39135302                    | 0.39263858                                        | 0.39264007     | 0.39263866     |                   |
| 80  | 0.21847002                    | 0.23871319                                        | 0.23961867     | 0.25054818     |                   |
| 90  | -0.12620233                   | -0.09600135                                       | -0.09589494    | -0.09675804    |                   |
| 100 | -0.37955984                   | -0.35790765                                       | -0.35792050    | -0.35844921    |                   |

Table 2. Comparison analytical solution, Akbari–Ganji’s method, and the numerical Solution for $R = 15$, $A = \pi/8$, and $\alpha = \pi/2$.

| $t$  | Akbari–Ganji’s method$^{35}$ | Present method (modiﬁed harmonic balance method) | First solution | Second solution | Numerical solution |
|-----|-------------------------------|---------------------------------------------------|----------------|----------------|-------------------|
| 0   | 0.39269908                    | 0.39269908                                        | 0.39269908     | 0.39269908     | 0.39269908        |
| 10  | 0.16964486                    | 0.16885445                                        | 0.16887735     | 0.16914305     |                   |
| 20  | -0.25944763                   | -0.24748960                                       | -0.2475017     | -0.24887845    |                   |
| 30  | -0.38749464                   | -0.38168768                                       | -0.38084918    | -0.38187454    |                   |
| 40  | -0.05286683                   | -0.08074978                                       | -0.08084918    | -0.08143699    |                   |
| 50  | 0.33172291                    | 0.31224542                                        | 0.31216834     | 0.31332380     |                   |
| 60  | 0.33408668                    | 0.34927099                                        | 0.34934062     | 0.34991364     |                   |
| 70  | -0.04827707                   | -0.01188337                                       | -0.01170568    | -0.01205181    |                   |
| 80  | -0.38611530                   | -0.35940032                                       | -0.35940850    | -0.36004875    |                   |
| 90  | -0.26304319                   | -0.29746998                                       | -0.29741627    | -0.29838583    |                   |
| 100 | 0.16555209                    | 0.10385008                                        | 0.10360515     | 0.10482796     |                   |

Table 3. Comparison analytical solution, Akbari–Ganji’s method, and the numerical Solution for $R = 15$, $A = \pi/8$, and $\alpha = \pi/3$.

| $t$  | Akbari–Ganji’s method$^{35}$ | Present method (harmonic balance method) | First solution | Second solution | Numerical solution |
|-----|-------------------------------|------------------------------------------|----------------|----------------|-------------------|
| 0   | 0.39269908                    | 0.39269908                               | 0.39269908     | 0.39269908     | 0.39269908        |
| 10  | 0.10762438                    | 0.10763527                               | 0.10765466     | 0.10875275     |                   |
| 20  | -0.35104698                   | -0.33369537                              | -0.33365131    | -0.33471506    |                   |
| 30  | -0.27870234                   | -0.29064536                              | -0.29065311    | -0.29201803    |                   |
| 40  | 0.20864948                    | 0.17441506                               | 0.17426529     | 0.17597552     |                   |
| 50  | 0.37789019                    | 0.38617222                               | 0.38621011     | 0.38631310     |                   |
| 60  | -0.00330218                   | -0.03727757                              | -0.03725716    | 0.03769972     |                   |
| 70  | -0.39470691                   | -0.36573733                              | -0.36563071    | -0.36626688    |                   |
| 80  | -0.18425738                   | -0.23776816                              | -0.23803414    | -0.23945080    |                   |
| 90  | 0.29236833                    | 0.23539712                               | 0.23505998     | 0.23706674     |                   |
| 100 | 0.33439609                    | 0.36680861                               | 0.36695763     | 0.36732664     |                   |
\[ \frac{1}{24}A^4 k R^2 \omega_2^2 \left( \frac{59}{4} v^5 - 25 v^4 + 17 v^3 - 5 v^2 - \frac{5}{4} v - \frac{1}{2} \right) + \frac{1}{4} A^2 g k \left( \frac{3}{2} v^2 + v - \frac{1}{2} \right) + 2 A k R \omega_2^2 \]

\[ (-v + 1) + \frac{1}{2} A^4 k R^2 \omega_2^2 \left( -7 v^3 + \frac{11}{2} v^2 + v + \frac{1}{2} \right) + \frac{3}{2} A R^2 \nu_0^2 (v - 1) + \frac{1}{2} A^3 k R \omega_2^2 \]

\[ (15v^3 - \frac{25}{2} v^2 - v - \frac{3}{2}) + A g k (v + 1) = 0 \]

\[ \frac{1}{6} A^2 k R^2 \omega_2^2 \left( \frac{67}{16} v^3 + \frac{95}{16} v^4 - \frac{13}{4} v^3 + v^2 - \frac{11}{16} v + \frac{1}{16} \right) + A^2 g k \left( -\frac{1}{3} v^3 + \frac{3}{8} v^2 - \frac{1}{8} v^2 - \frac{1}{24} \right) - \frac{27}{2} A R^2 \]

\[ \nu_0^2 + A^2 k R^2 \omega_2^2 \left( \frac{3}{4} v^3 - \frac{7}{2} v^2 + \frac{5}{2} v - \frac{1}{4} \right) + 18 A k R \nu_0^2 + A^2 k R \omega_2^2 \left( -12 v^3 + \frac{41}{4} v^2 - \frac{19}{4} v - \frac{1}{4} \right) + A g k v = 0 \]

After simplification, equation (15) can be appeared into another form as

![Figure 2. Comparison of the second-order analytical solution of equation (8) (—), second-order global residue harmonic balance method (— —) with the numerical solution (…).](image-url)
where

\[ \omega_2 = \frac{2\sqrt{3} \sqrt{gk(-2A^2v^3 + 3A^2v^2 - 2A^2v + A^2 + 8v - 8)}}{\sqrt{A^4kR^2\Gamma_1 + A^2kR^2\Gamma_2 + A^2kR\Gamma_3 + 192kR(-v + 1) + 144R^2(v - 1)}} \]  

(17)

By eliminating \( \omega_2 \) from equations (15) and (16), with the help of equation (17) and simplifying the result, we find the value of unknown constant \( v \).

Finally, the second-order analytical approximate solution of equation (8) is \( \theta = A(\rho \cos(\omega_2t) + v \cos(3\omega_2t)) \), where \( \rho \), \( v \), and \( \omega_2 \) are given respectively by equations (16)–(17).

**Results and discussion**

The modified harmonic balance method has been proposed and extended to obtain higher-order approximate analytical solutions and corresponding frequencies for the nonlinear circular sector oscillator differential equation. The results investigate the effects when comparing with Akbari–Ganj’s method, Global residue harmonic balance method, and the numerical solution for different values of \( R \) and \( \alpha \) as listed in Tables 1–3 and Figure 2. Tables 1–3 and Figure 2 show that approximate frequencies are better for present method than those obtained by GRHBHBM11 method and AGM. Also, to show the accuracy of the present method, the analytical solutions are compared with the numerical ones. The obtained results calculated by the MHBM were compared with the results produces by AGM, GRHBHBM, and numerical solutions, which let on that the MHBM has better accuracy with the numerical solutions than others methods as shown in Figure 2.
Conclusion

The modified harmonic balance method was presented for determining higher-order approximate solution to the nonlinear differential equation of the circular sector oscillator. The accuracy of the approximate analytical solution was verified by comparing the present results with the exact numerical solution and other analytical methods. It is obvious that the exact numerical results agree very well with the approximate periodic solutions. Tables 1–3 and Figure 2 indicate that the modified HBM has acceptable accuracy in comparison with the Runge–Kutta numerical solution than other methods. We can say that the MHBM is very strong analytical method to solve nonlinear differential equations.

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