Integrating the Jacobian equation

A. S. de Medeiros\textsuperscript{a}, R. R. Silva\textsuperscript{b,*}

\textsuperscript{a} Instituto de Matemática, Universidade Federal do Rio de Janeiro, Ilha do Fundão, CEP: 21941-909, Rio de Janeiro, RJ, Brazil

\textsuperscript{b} Departamento de Matemática, Universidade de Brasília, Campus Universitário Darcy Ribeiro, Asa Norte, CEP: 70910-900, Brasília, Brazil

Abstract

We show essentially that the differential equation \( \frac{\partial (P, Q)}{\partial (x, y)} = c \in \mathbb{C} \), for \( P, Q \in \mathbb{C}[x, y] \), may be ”integrated”, in the sense that it is equivalent to an algebraic system of equations involving the homogeneous components of \( P \) and \( Q \). Furthermore, the first equations in this system give explicitly the homogeneous components of \( Q \) in terms of those of \( P \). The remaining equations involve only the homogeneous components of \( P \).

Keywords: Jacobian equation, Jacobian conjecture, algebraic dependence

1. Introduction

The starting point of this article was a very naive attempt to introduce some geometry, via singularities of differential 1-forms, in the

**Jacobian problem.** Let \( F = (P, Q) : \mathbb{C}^2 \leftarrow \mathbb{C}^2 \) be a polynomial map such that \( \frac{\partial (P, Q)}{\partial (x, y)} \neq 0 \) on \( \mathbb{C}^2 \). Then, \( F \) is an injective map. (See, e.g., [1].)

The relation with differential 1-forms is attained by associating to \( F \) the differential form \( \omega = PdQ - QdP \).

Given \( z \in \text{Sing}(\omega) \) we have \( \omega(z) \wedge dP(z) = \omega(z) \wedge dQ(z) = 0 \) and, since \( dP \wedge dQ = \frac{\partial (P, Q)}{\partial (x, y)} dx \wedge dy \neq 0 \), we conclude that \( P(z) = Q(z) = 0 \), i.e., \( \text{Sing}(\omega) \subset Z(P, Q) \).

\textsuperscript{*}Corresponding author. Tel. +55 61 3107-6454

Email addresses: airtonsoh@yahoo.com.br (A. S. de Medeiros), rrsilva73@gmail.com (R. R. Silva)

Preprint submitted to Elsevier  September 25, 2014
On the other hand, since $Z(P, Q) \subset Sing(\omega)$ trivially holds, we have that $Sing(\omega) = Z(P, Q)$.

This leads at once to the following alternative statement of the Jacobian problem,

$$\omega = PdQ - QdP,$$

where $P, Q$ are polynomials on $\mathbb{C}^2$. If $d\omega$ has no singular points then, $\omega$ has at most one singular point.

Maybe that has led us naturally to make use of differential 1-forms in order to study the

**Jacobian equation.** $\frac{\partial (P,Q)}{\partial (x,y)} = c \in \mathbb{C}$, for $P, Q \in \mathbb{C}[x, y]$.

Which, in fact, has shown to be very efficient in establishing Theorem 4.1, where a system of algebraic equations involving the homogeneous components of $P$ and $Q$ is shown to be equivalent to the Jacobian equation.

### 2. Preliminaries

Henceforward we shall concentrate in investigating the solutions of the Jacobian equation, where $P$ and $Q$ have fixed degrees $k$ and $l$ respectively. In addition, by obvious reasons, $P$ and $Q$ are supposed to satisfy:

(i) $P(0) = Q(0) = 0$.

(ii) $P, Q \neq 0$.

(iii) $P$ and $Q$ are not both linear.

Let us now consider the decomposition of $P$ and $Q$ into their respective homogenous components,

$$P = P_1 + \ldots + P_k$$

$$Q = Q_1 + \ldots + Q_l$$

If $(dP \wedge dQ)_\mu$ denotes the homogeneous component of $dP \wedge dQ$ of degree $\mu$, the condition $\frac{\partial (P,Q)}{\partial (x,y)} \in \mathbb{C}$ is equivalent to,

$$(dP \wedge dQ)_\mu = 0, \ \mu = (k + l) - 2, \ldots, 1.$$
Which is, by its turn, equivalent to the following system of \( k+l-2 \) partial differential equations,

\[
\begin{align*}
\dot{d}P_k \wedge dQ_l &= 0 \\
\dot{d}P_k \wedge dQ_{l-1} + \dot{d}P_{k-1} \wedge dQ_l &= 0 \\
&\vdots \\
\dot{d}P_2 \wedge dQ_1 + \dot{d}P_1 \wedge dQ_2 &= 0
\end{align*}
\]

**Remark 2.1.** The above system may be written more conveniently as,

\((S) \quad \dot{d}P_k \wedge dQ_l - j + \dot{d}P_k - (j-1) \wedge dQ_l - j = 0, \quad j = 0, \ldots, k+l-3.\)

Where it is agreed that \( P_i = Q_i = 0 \), whenever \( i < 0 \).

Notice that the \( j \)-th equation of \((S)\) is,

\[(j) \quad \sum_{j'=0}^{j} \dot{d}P_{k-(j-j')} \wedge dQ_{l-j'} = 0.\]

Before we proceed to the investigation of the solutions of \((S)\), we present below,

**2.1. Some basic elementary results**

In what follows, \( \mathbb{C}(z) = \mathbb{C}(z_1, \ldots, z_n) \) denotes the field of rational functions on \( \mathbb{C}^n \). We shall agree that the zero polynomial is homogeneous of any degree.

(1) Given a non constant \( R \in \mathbb{C}(z) \) we shall denote by \( s(R) = \max\{m \in \mathbb{N} \mid R = X^m \text{ for some } X \in \mathbb{C}(z)\} \). The notation \( G = \sqrt{s} \) means that \( s = s(R) \) and that \( G^s = R \). Note that necessarily \( s(G) = 1 \), which is equivalent to saying that \( G \) is not the power of another rational function. Such a \( G \) will be referred to as being **simple**.

(2) Let \( H \) be a holomorphic homogeneous function of degree \( k \in \mathbb{Z} \) (defined in some region of \( \mathbb{C}^n \)). Then, \( i(\mathcal{R})dH = kH \), where \( \mathcal{R} \) denotes the radial vector field on \( \mathbb{C}^n \), i.e., \( \mathcal{R}(z) = z, \quad z \in \mathbb{C}^n \), and \( i(\mathcal{R})dH \) is the interior product (see, e.g., [2], p. 25) of the vector field \( \mathcal{R} \) and the differential 1-form \( dH \).

This is just a restatement of the classical Euler’s Formula for Homogeneous Functions, in the context of vector fields and differential forms.
(3) Let \( H, J \) be homogeneous holomorphic functions of integer degrees \( k, l \) respectively (defined in some region of \( \mathbb{C}^n \)), such that \( H \neq 0 \). Then, \( dH \wedge dJ = 0 \) if and only if there exists \( \lambda \in \mathbb{C} \) such that \( J^k = \lambda H^l \).

The necessity is an immediate consequence of (2) above. Indeed, from the equation \( dH \wedge dJ = 0 \) we have,

\[
0 = i(\mathcal{R})0 = i(\mathcal{R})(dH \wedge dJ) = (i(\mathcal{R})dH)dJ - dH(i(\mathcal{R})dJ) = kHdJ - lJdH.
\]

Now, let \( M = J^kH^l \) then \( dM = \frac{1}{H^l} (H^l dJ^k - J^k dH^l) = \frac{H^{l-1} J^{k-1}}{H^l} (kHdJ - lJdH) = 0. \)

Hence, there exists \( \lambda \in \mathbb{C} \) such that \( M = \lambda \), i.e., \( J^k = \lambda H^l \).

The converse is obvious.

(4) Let \( H, J \in \mathbb{C}(z), H \neq 0 \), be quotients of homogeneous polynomials. Then, \( dH \wedge dJ = 0 \) if and only if there exist \( \lambda \in \mathbb{C} \) and \( t \in \mathbb{Z} \) such that \( J = \lambda G^t \), where \( G = \sqrt[n]{H} \).

In fact, from \( G^s = H \), we conclude that \( G \) is, as well, a quotient of homogeneous polynomials.

If \( J = 0 \) there is nothing to prove. Otherwise, from \( 0 = dH \wedge dJ = sG^{s-1}dG \wedge dJ \) we deduce that \( dG \wedge dJ = 0 \). By (3) we have \( J^g = cG^l \), where \( c \in \mathbb{C} \) and \( g, l \) are the degrees of \( G \) and \( J \), respectively.

Now, the result follows by considering the factorizations of the rational functions \( G \) and \( J \), into irreducible factors, and by noting that \( G \) is simple, exactly when the \( \gcd \) of the exponents of the factors in its decomposition is equal to 1.

The converse is evident.

3. Definitions and notation

We shall denote by \( \Gamma \) the set of all sequences \( \alpha = (\alpha_1, \alpha_2, \ldots) \) of non-negative integers, having a finite number of nonzero terms.

Unless otherwise explicitly stated, any sequence appearing in the sequel lies in \( \Gamma \).
For $\alpha \in \Gamma$ we define,
\[ |\alpha| = \sum_{i \in \mathbb{N}} \alpha_i, \]
\[ \sigma(\alpha) = \sum_{i \in \mathbb{N}} i\alpha_i, \]
\[ D_\alpha = \{ i \in \mathbb{N} \mid \alpha_i \neq 0 \}. \]

The functions $|\alpha|$ and $\sigma(\alpha)$ will be referred to, respectively, as the modulus and the size of $\alpha$.

For each $j \in \mathbb{N}$, we denote by $e_j$ the sequence whose $j$-th term is 1 and all the others are zero.

Given a nonzero sequence $\alpha$, for each $i \in D_\alpha$, we denote by $\alpha(i)$ the sequence $\alpha - e_i$. The function $i \in D_\alpha \mapsto \alpha(i) \in \Gamma$ will be referred to as the function $\alpha(i)$.

If $t \in \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, 0, 0, \ldots) \in \Gamma$, we shall denote by $\binom{t}{\alpha + e_i}$ the usual multinomial coefficient $\binom{t}{\alpha_1, \ldots, \alpha_k} = \frac{(t)_{|\alpha|}}{\alpha!}$, where $\alpha! = \prod_{i \in \mathbb{N}} \alpha_i!$ and $(t)_{|\alpha|}$ is the Pochhammer symbol for the falling factorial $t(t-1)\ldots(t-|\alpha|+1)$. Recall that when $\alpha = 0$, $(t)_0 = 1$ by definition.

We point out, for further reference, the following elementary,

**Identity 3.1.** $\binom{t}{\alpha + e_i} = \frac{(t-|\alpha|)}{\alpha_i+1} \binom{t}{\alpha}. \]

Finally, let $X = (X_1, X_2, \ldots)$, where $X_1, X_2, \ldots$ are indeterminates. Given $\alpha \in \Gamma$ we shall adopt the usual notation $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \ldots$.

**4. Statement and proof of the result**

**Theorem 4.1.** Let $P, Q \in \mathbb{C}[x, y]$, of degrees $k$ and $l$ respectively, be such that $P(0) = Q(0) = 0$, and $kl > 1$. Then, $\frac{\partial^2(PQ)}{\partial(x,y)} \in \mathbb{C}$ if and only if there exist unique $\lambda_r \in \mathbb{C}$, $r = 0, \ldots, k+l-3$, such that,
\[ Q_{i-j} = \sum_{r=0}^{j} \sum_{\sigma(\alpha) = j-r} \lambda_r \binom{s_r/s}{\alpha} G^{s_r-s_{|\alpha|}} P_\alpha, \quad 0 \leq j \leq k+l-3, \]

where $G = \sqrt{F_k}$; $P_\alpha = (P_{k-1}, P_{k-2}, \ldots)$; $s_r = s\frac{(l-r)}{k}$, if $\lambda_r \neq 0$, and $s_r = 0$, if $\lambda_r = 0$. Furthermore, $s_r$ turns out to be an integer, whenever $\lambda_r \neq 0$. 

5
Proof.

We shall omit, along the proof, details that turn out to be mere elementary algebraic manipulations.

Henceforth, in order to simplify the typing, and the reading, we set $t_r = s_r/s$.

We shall see that the above expressions of $Q_{l-j}$ are obtained by solving recursively all equations of the system (S) for the $Q_{l-j}$.

In fact, we will show, by recurrence on $j$, the following assertion:

(Aj) Given $0 \leq j \leq k + l - 3$, the first equations of the system (S) up to the $j$-th, hold iff there exist unique $\lambda_r \in \mathbb{C}$, $r = 0, \ldots, j$, such that,

$$Q_{l-j} = \sum_{r=0}^{j'} \sum_{\sigma(\alpha) = j-r} \lambda_r \left( \frac{t_r}{\alpha} \right) G^{s_{r-s}|\alpha|} P_{-\alpha}, \ jt = 0, \ldots, j.$$  

For $j = 0$, the assertion is a straightforward consequence of (4) in subsection [2.1]

In order to complete the recurrence procedure let us prove that (Aj) implies (Aj-1), for $0 < j \leq k + l - 3$.

Indeed, by the recurrence hypothesis, (Aj) is equivalent to:

The $j$-th equation of the system holds iff there exists a unique $\lambda_j \in \mathbb{C}$ such that,

$$Q_{l-j} = \sum_{r=0}^{j} \sum_{\sigma(\alpha) = j-r} \lambda_r \left( \frac{t_r}{\alpha} \right) G^{s_{r-s}|\alpha|} P_{-\alpha},$$

where $s_j = s \left( \frac{l-j}{k} \right)$, if $\lambda_j \neq 0$, and $s_j = 0$, if $\lambda_j = 0$. Moreover, $s_j \in \mathbb{Z}$, if $\lambda_j \neq 0$.

By Remark 2.1, the $j$-th equation of (S) is,

$$(j) \quad dP_k \wedge dQ_{l-j} + \sum_{j=-1}^{j-1} dP_{k-(j-j)} \wedge dQ_{l-j} = 0.$$  

Since $P_k = G^s$, we have that $dP_k \wedge dQ_{l-j} = sG^{s-1}dG \wedge dQ_{l-j}$.

Now, we shall compute $dQ_{l-j}$ and $dP_{k-(j-j)} \wedge dQ_{l-j}$.
By the recurrence hypothesis we have,
\[ dQ_{l-j'} = d \left[ \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) G^{s_r-s|\alpha|} P_{-\alpha} \right] = \]
\[ = \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) [(s_r - s|\alpha|) G^{s_r-s|\alpha|} - 1 P_{-\alpha} dG + G^{s_r-s|\alpha|} dP_{-\alpha}] . \]

By taking the exterior product of \( dP_{k-(j-j')} \) and the expression above, we obtain,
\[ dP_{k-(j-j')} \land dQ_{l-j'} = \]
\[ \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) [(s_r - s|\alpha|) G^{s_r-s|\alpha|} - 1 P_{-\alpha} dP_{k-(j-j')} \land dG + G^{s_r-s|\alpha|} dP_{k-(j-j')} \land dP_{-\alpha}] , \]
\[ 0 \leq j' \leq j - 1. \]

Taking into account the above expressions, equation (j) is now,
\[ sG^{s-1} dG \land dQ_{l-j} + \]
\[ \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) (t_r - |\alpha|) G^{s_r-s(|\alpha|+1)} P_{-\alpha} dP_{k-(j-j')} \land dG + \]
\[ \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) G^{s_r-s|\alpha|} dP_{k-(j-j')} \land dP_{-\alpha} = 0. \]

By factoring out \( sG^{s-1} dG \), we obtain,
\[ sG^{s-1} dG \land [dQ_{l-j} - \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) (t_r - |\alpha|) G^{s_r-s(|\alpha|+1)} P_{-\alpha} dP_{k-(j-j')} ] + \]
\[ \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) G^{s_r-s|\alpha|} dP_{k-(j-j')} \land dP_{-\alpha} = 0. \]

Now, the recurrence procedure follows easily from the two statements below,

**Statement 4.1.**
\[ \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) G^{s_r-s|\alpha|} dP_{k-(j-j')} \land dP_{-\alpha} = 0, \quad 0 < j \leq k + l - 3. \]

**Statement 4.2.**
\[ \sum_{j'=0}^{j-1} \sum_{r=0}^{j'} \sum_{\sigma(\alpha)=j'-r} \lambda_r(\alpha) (t_r - |\alpha|) G^{s_r-s(|\alpha|+1)} P_{-\alpha} dP_{k-(j-j')} \land dG = \sum_{r=0}^{j-1} \sum_{\sigma(\alpha)=j-r} \lambda_r(\alpha) G^{s_r-s|\alpha|} dP_{-\alpha} , \]
\[ 0 < j \leq k + l - 3. \]
As a matter of fact, by taking for granted the two statements above, the equation \((j)\) becomes,

\[
0 = dG \wedge [dQ_{l-j} - \sum_{j=0}^{j-1} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} dP_\alpha] = \\
dG \wedge [Q_{l-j} - \sum_{j=0}^{j-1} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} P_\alpha], \text{ once } dG \wedge dG = 0.
\]

We notice that \(\deg(G^{s_r-s|\alpha} P_\alpha) = l - j, \) if \(\sigma(\alpha) = j - r \) and \(\lambda_r \neq 0.\) Thus, \(Q_{l-j} - \sum_{j=0}^{j-1} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} P_\alpha \) is a quotient of homogeneous polynomials.

Now, since \(G\) is simple, it follows from (4) of subsection 2.1, that the equation \((j)\) holds iff,

\[
Q_{l-j} = \sum_{r=0}^{j-1} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} P_\alpha = \lambda_j G^{s_j}, \text{ for some } \lambda_j \in \mathbb{C} \text{ and } s_j \in \mathbb{Z}.
\]

Clearly, the constant \(\lambda_j\) is uniquely determined by the above equation and, if \(\lambda_j \neq 0,\) this equation implies that \(l - j = s_j \deg(G) = s_j \frac{k}{n},\) i.e., \(s_j = \frac{k}{n}(l - j).\)

On the other hand, if \(\lambda_j = 0,\) we may clearly choose \(s_j = 0.\)

In other words, we have just shown that, under the recurrence hypothesis, the identity,

\[
Q_{l-j} = \sum_{r=0}^{j-1} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} P_\alpha + \lambda_j G^{s_j} = \sum_{r=0}^{j} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} P_\alpha,
\]

with \(\lambda_j\) and \(s_j\) as described above, is in fact equivalent to equation \((j).\)

Now, we will provide the proof of the two statements.

4.1. Proof of Statement 4.1

Before we proceed to the proof we set,

\[
A = \sum_{j=0}^{j-1} \sum_{r=0}^{j} \sum_{\sigma(\alpha) = j-r} \lambda_r(t^r_\alpha) G^{s_r-s|\alpha} dP_{k-\alpha} \wedge dP_\alpha.
\]

We are supposed to prove that \(A = 0.\)
First we remark that \( dP_0^\alpha = 0 \) if \( \alpha = 0 \) and, otherwise, \( dP_0^\alpha = \sum_{i \in D_\alpha} \alpha_i P_0^\alpha(i) \, dP_{k-i} \).

Consequently, we have that \( dP_{k-(j-j')}^\alpha \wedge dP_0^\alpha = 0 \) if \( \alpha = 0 \) and, otherwise, \( dP_{k-(j-j')}^\alpha \wedge dP_{k-i} = \sum_{i \in D_\alpha} \alpha_i P_{k-(j-j')}^\alpha \wedge dP_{k-i} \), if \( \alpha \neq 0 \).

Hence, in the expression of \( A \), we may restrict ourselves to those summands where \( \alpha \neq 0 \), if any exist. If not, \( A \) trivially vanishes.

The condition \( \alpha \neq 0 \) may be more appropriately expressed in terms of the indexes range, by observing that, \( \alpha \neq 0 \iff \sigma(\alpha) \neq 0 \iff j' - r \neq 0 \iff 0 \leq r \leq j' - 1 \leq j - 2 \), which is equivalent to \( 1 \leq j' \leq j - 1 \), \( 0 \leq r \leq j' - 1 \) and \( j \geq 2 \).

The above discussion may be summarized as follows:

\[ A = 0, \text{ if } j < 2, \text{ and for } j \geq 2 \text{ we have,} \]

\[ (*) \quad A = \sum_{j=1}^{j-1} \sum_{r=0}^{j'-1} \sum_{s=\sigma(\alpha)=j-r} \sum_{i \in D_\alpha} \lambda_r(\alpha_j) G^s_{\sigma(s)} \alpha_i P_{k-(j-j')}^\alpha \wedge dP_{k-i}. \]

Henceforward we shall presume \( j \geq 2 \).

Let us denote by \( \mathbb{A} \) the set of all \( 4 \)-tuples \( a = \left( \begin{array}{c} j \\ r \\ \alpha \\ i \end{array} \right) \), whose coordinates are subjected to the same constraints specified in \( (*) \) above.

Clearly, \( A = \sum_{a \in \mathbb{A}} \Phi(a), \) where \( \Phi(a) = \lambda_r(\alpha_j) G^s_{\sigma(s)} \alpha_i P_{k-(j-j')}^\alpha \wedge dP_{k-i}. \)

Now we set, for \( a \in \mathbb{A}, \) \( \tau(a) = \left( \begin{array}{c} j - i \\ r \\ \alpha(i) + e_{j-j'} \\ j - j' \end{array} \right). \) It is immediate to check that this defines, in fact, a bijective function \( \tau : \mathbb{A} \rightarrow \mathbb{A}. \) Such function satisfies:

\[ \Phi(\tau(a)) = -\Phi(a). \]
As a matter of fact, by the very definitions of $\Phi$ and $\tau$ we have, $\Phi(\tau(a)) = \lambda_r(t_r^{\tau}) \Phi_\alpha(s_{|\alpha|+e_j-j})G^{s_{|\alpha|+e_j-j}}P_\alpha dP_{k-i}dP_{k-j}$, and then, the fact that $\Phi(\tau(a)) = -\Phi(a)$ turns out to be an immediate consequence of Identity 3.1.

Hence we conclude that

$$A = \sum_{a \in A} \Phi(a) = \sum_{a \in A} \Phi(\tau(a)) = -A \ , \ i.e. \ A = 0 .$$

\[ \square \]

4.2. Proof of Statement 4.2

Let us set,

$$B = \sum_{j=0}^{j-1} \sum_{r=0}^{j} \lambda_r(t_r^{\tau}) \Phi_\alpha(s_{|\alpha|+e_j-j})G^{s_{|\alpha|+e_j-j}}P_\alpha dP_{k-j},$$

$$B' = \sum_{j=0}^{j-1} \sum_{r=0}^{j} \lambda_r(t_r^{\tau}) \Phi_\alpha(s_{|\alpha|+e_j-j})G^{s_{|\alpha|+e_j-j}}P_\alpha dP_{k-j},$$

this last equality is due to the fact that $\alpha \neq 0$, once $\sigma(\alpha) = j - r \neq 0$.

Recall we want to prove that $B = B'$.

The proof consists basically in showing that the summands in the expressions of $B$ and $B'$ are exactly the same. To this end, we shall express both, $B$ and $B'$, into the more suitable form:

$$B = \sum_{b \in B} \Psi(b), \text{ where } B = \left\{ \begin{pmatrix} j \alpha \\ i \end{pmatrix} | 0 \leq j \leq j-1, 0 \leq r \leq j, \sigma(\alpha) = j - r \right\},$$

and $\Psi(b) = \lambda_r(t_r^{\tau}) \Phi_\alpha(s_{|\alpha|+e_j-j})P_\alpha dP_{k-j}$.

$$B' = \sum_{b' \in B'} \Psi'(b'), \text{ where } B' = \left\{ \begin{pmatrix} r \alpha \\ i \end{pmatrix} | 0 \leq r \leq j - 1, \sigma(\alpha) = j - r, i \in D_\alpha \right\},$$

and $\Psi'(b') = \lambda_r(t_r^{\tau}) G^{s_{|\alpha|+e_j-j}}P_\alpha dP_{k-j}$.

Now, for $b \in B$, we set $g(b) = \begin{pmatrix} \alpha + e_{j-j} \\ j-j \end{pmatrix}$. It can be easily verified that this defines a bijective function $g : B \rightarrow B'$. 

10
Obviously, in order to conclude the proof, it suffices to show that $\Psi(b) = \Psi'(g(b))$. Indeed,

$$B' = \sum_{b' \in B'} \Psi'(b') = \sum_{b' \in g(B)} \Psi'(b') = \sum_{b \in B} \Psi'(g(b)) = \sum_{b \in B} \Psi(b) = B.$$ 

Finally, by a direct computation we find that,

$$\Psi'(g(b)) = \lambda r \left( \frac{t_r}{(\alpha + e_{j-j'})} \right) G_{s-\alpha-e_{j-j'}}(\alpha + e_{j-j'}) P_{(\alpha+e_{j-j})(j-j')} dP_{k-(j-j')}$$

And then, the fact that $\Psi'(g(b)) = \Psi(b)$ follows at once from Identity 3.1.

\[ \square \]

5. Final comments

It is worth mentioning that, when $k > 1$, the number of equations, provided by Theorem 4.1, is $k + l - 2 \geq l$. Hence, the first $l$ equations are explicit expressions of the homogeneous components of $Q$ in terms of those of $P$, whereas the remaining equations involve only the homogeneous components of $P$.

In particular, this holds when $k \geq l$, and, as far as our purpose is concerned, we could have restricted ourselves to this case, by simply reordering the pair $(P, Q)$, if necessary.

We have made our choice for the current statement of the theorem, mostly because of its "symmetric" character:

When $P$ and $Q$ are not linear we can, indistinctly, express the homogeneous components of $Q$ in terms of those of $P$, and conversely.

We point out that the theorem clearly holds in dimension $n \geq 2$, by replacing $\frac{\partial (P, Q)}{\partial (x, y)} \in \mathbb{C}$ by, $dP \wedge dQ$ is constant. The particular case $dP \wedge dQ = 0$ corresponds to the problem of algebraic dependence of the polynomials $P, Q$ (see, e.g., [3], Lemma 1, and [4], Ch. III).

Finally, it is evident that, mutatis mutandis, a real version of Theorem 4.1 is promptly available. For $c \neq 0$, it is related to the real Jacobian conjecture (see, e.g., [1], Part II, 10.1, and [5]).
List of notations

\( \mathbb{C} \), the field of complex numbers.
\( \mathbb{N} = \{1, 2, \ldots \} \), the set of natural numbers.
\( \mathbb{Z} \), the set of integer numbers.
\( \mathbb{C}[x, y] \), the ring of polynomials on \( \mathbb{C}^2 \).
\( \text{Sing}(\omega) = \{ z \in \mathbb{C}^2 \mid \omega(z) = 0 \} \), the set of singularities of the differential form \( \omega \).
\( \mathcal{Z}(P, Q) = \{ z \in \mathbb{C}^2 \mid P(z) = Q(z) = 0 \} \), the set of zeros of the mapping \( (P, Q) \).
\( \deg(H) \), the degree of the homogeneous function \( H \).

References

[1] A. van den Essen, Polynomial automorphisms and the Jacobian Conjecture, Progress in Mathematics, vol. 190, Birkhäuser, 2000.
[2] C. Godbillon, Géométrie Différentielle et Mécanique Analytique, Collection Méthodes Hermann, Paris, 1969.
[3] A.P. Petravchuk, O.G. Iena, On closed rational functions in several variables, Algebra Discrete Math. 2 (2007) 115–124.
[4] W.V.D. Hodge, D. Pedoe, Methods of algebraic geometry, vol. I. Reprint of the 1947 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1994.
[5] S. Pinchuk, A counterexample to the strong real Jacobian conjecture, Math. Z. 217 (1) (1994) 1–4.