BRATTELI DIAGRAMS WHERE ALMOST ALL ORDERS ARE IMPERFECT

J. JANSSEN AND R. YASSAWI

Abstract. We show that for a large family of infinite rank Bratteli diagrams $B$ with the equal path number property, a random order on $B$ does not admit a continuous Vershik map.

1. Introduction

Consider the following random process. For each natural $n$, we have a collection of finitely many individuals. Each individual in the $n+1$-st collection randomly picks a parent from the $n$-th collection, and this is done for all $n$. If we know how many individuals there are at each stage, the question “How many infinite ancestral lines are there?” almost always has a common answer $j$: what is it? We can also make this game more general, by for each individual, changing the odds that he choose a certain parent, and ask the same question.

The information that we are given will come as a Bratteli diagram $B$ (Definition 2.1), where each “individual” at stage $n$ is represented by a vertex in the $n$-th vertex set $V_n$, and the chances that an individual $v \in V_{n+1}$ chooses $v' \in V_n$ as a parent is the ratio of the number of edges incoming to $v$ with source $v'$ to the total number of edges incoming to $v$.

We consider the space $O_B$ of orders on $B$ (Definition 2.4) as a measure space equipped with the uniform product measure $\mu$. A result in [BKY14] (stated as Theorem 3.1 here) tells us that there is some $j$, either a positive integer or infinite, such that a $\mu$-random order $\omega$ possesses $j$ maximal paths.

In this article we compute $j$ for a large family of infinite rank Bratteli diagrams (Definition 2.3). Namely, in Theorem 4.2 we show that $j = \infty$ for the situation where any individual at stage $n$ is equally likely to be chosen as a parent by any individual at stage $n+1$, provided that the sets of individuals grow super-quadratically. In Theorem 4.9 we generalise our result to a family of Bratteli diagrams which have the equal path number property, (Definition 4.4, also known as the equal row sum property). We can draw the following conclusions from these results. First we show in Corollary 4.3 that $j$ is not an invariant of $B$’s dimension group [EFS1]. Second, an order $\omega$ is called perfect if it admits a Bratteli-Vershik map (Definition 2.4). For “most” Bratteli diagrams (including the ones we identify in Theorems 4.2 and 4.9), if $j > 1$, then a $\mu$-random order is not perfect (Theorem 5.3). This is in contrast to the case for finite rank diagrams, where almost any order put on “almost any” finite rank Bratteli diagram is perfect (Section 5, [BKY14]). Indeed, one wonders whether for a “reasonable” infinite rank diagram, it is always the case that $j = \infty$. Here the word reasonable needs to

1991 Mathematics Subject Classification. Primary 37B10, Secondary 37A20.

Key words and phrases. Bratteli diagrams, Vershik maps.

The first author is partially supported by an NSERC Discovery Grant.
be defined: we suspect that if the growth rate of the vertex sets in the diagrams of Theorem 4.2 is subquadratic, then \( j = 1 \).

2. Bratteli diagrams and Vershik maps

In this section, we collect the notation and basic definitions that are used throughout the paper.

2.1. Bratteli diagrams.

**Definition 2.1.** A Bratteli diagram is an infinite graph \( B = (V, E) \) such that the vertex set \( V = \bigcup_{i \geq 0} V_i \) and the edge set \( E = \bigcup_{i \geq 1} E_i \) are partitioned into disjoint subsets \( V_i \) and \( E_i \) where

(i) \( V_0 = \{v_0\} \) is a single point;

(ii) \( V_i \) and \( E_i \) are finite sets;

(iii) there exists a range map \( r \) and a source map \( s \), both from \( E \) to \( V \), such that \( r(E_i) = V_i, \ s(E_i) = V_{i-1} \), and \( s^{-1}(v) \neq \emptyset, r^{-1}(v') \neq \emptyset \) for all \( v \in V \) and \( v' \in V \setminus V_0 \).

The pair \((V_i, E_i)\) or just \( V_i \) is called the \( i \)-th level of the diagram \( B \). A finite or infinite sequence of edges \((e_i : e_i \in E_i)\) such that \( r(e_i) = s(e_{i+1}) \) is called a finite or infinite path, respectively.

For \( m < n \), \( v \in V_m \) and \( w \in V_n \), let \( E(v, w) \) denote the set of all paths \( \overline{e} = (e_1, \ldots, e_p) \) with \( s(e_1) = v \) and \( r(e_p) = w \). For a Bratteli diagram \( B \), let \( X_B \) be the set of infinite paths starting at the top vertex \( v_0 \). We endow \( X_B \) with the topology generated by cylinder sets \( \{U(e_j, \ldots, e_n) : j, n \in \mathbb{N}, \ (e_j, \ldots, e_n) \in E(v, w), v \in V_{j-1}, w \in V_n \} \), where \( U(e_j, \ldots, e_n) := \{x \in X_B : x_i = e_i, i = j, \ldots, n\} \). With this topology, \( X_B \) is a 0-dimensional compact metric space.

**Definition 2.2.** Given a Bratteli diagram \( B \), the \( n \)-th incidence matrix \( F_n = (f^{(n)}_{v, w}), n \geq 0 \), is a \(|V_{n+1}| \times |V_n|\) matrix whose entries \( f^{(n)}_{v, w} \) are equal to the number of edges between the vertices \( v \in V_{n+1} \) and \( w \in V_n \), i.e.

\[
f^{(n)}_{v, w} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.
\]

Next we define some popular families of Bratteli diagrams that we work with in this article.

**Definition 2.3.** Let \( B \) be a Bratteli diagram.

(1) We say \( B \) has finite rank if for some \( k \), \( |V_n| \leq k \) for all \( n \geq 1 \).

(2) We say that \( B \) is simple if for any level \( n \) there is \( m > n \) such that \( E(v, w) \neq \emptyset \) for all \( v \in V_m \) and \( w \in V_m \).

In this article we work only with simple Bratteli diagrams.

2.2. Orderings on a Bratteli diagram.

**Definition 2.4.** A Bratteli diagram \( B = (V, E) \) is called ordered if a linear order ‘\( > \)’ is defined on every set \( r^{-1}(v), v \in \bigcup_{n \geq 1} V_n \). We use \( \omega \) to denote the corresponding partial order on \( E \) and write \((B, \omega)\) when we consider \( B \) with the ordering \( \omega \). Denote by \( \mathcal{O}_B \) the set of all orderings on \( B \).
Every \( \omega \in \mathcal{O}_B \) defines the lexicographic ordering on the set of finite paths between vertices of levels \( V_k \) and \( V_l \): \((e_1, \ldots, e_l) > (f_1, \ldots, f_l)\) if and only if there is \( i \) with \( k + 1 \leq i \leq l \), \( e_j = f_j \) for \( i < j \leq l \) and \( e_i > f_i \). It follows that, given \( \omega \in \mathcal{O}_B \), any two paths from \( E(v_0, v) \) are comparable with respect to the lexicographic ordering generated by \( \omega \). If two infinite paths are tail equivalent, i.e. agree from some vertex \( v \) onwards, then we can compare them by comparing their initial segments in \( E(v_0, v) \). Thus \( \omega \) defines a partial order on \( X_B \), where two infinite paths are comparable if and only if they are tail equivalent.

**Definition 2.5.** We call a finite or infinite path \( e = (e_i) \) maximal (minimal) if every \( e_i \) is maximal (minimal) amongst the edges from \( r^{-1}(r(e_i)) \).

Notice that, for \( v \in V_i \), \( i \geq 1 \), the minimal and maximal (finite) paths in \( E(v_0, v) \) are unique. Denote by \( X_{\max}(\omega) \) and \( X_{\min}(\omega) \) the sets of all maximal and minimal infinite paths in \( X_B \), respectively. It is not hard to show that \( X_{\max}(\omega) \) and \( X_{\min}(\omega) \) are non-empty closed subsets of \( X_B \); in general, \( X_{\max}(\omega) \) and \( X_{\min}(\omega) \) may have interior points. For a finite rank Bratteli diagram \( B \), the sets \( X_{\max}(\omega) \) and \( X_{\min}(\omega) \) are always finite for any \( \omega \), and if \( B \) has rank \( d \), then each of them have at most \( d \) elements (Proposition 6.2 in [BKM09]).

**Definition 2.6.** A Bratteli diagram \( B \) is called regular if for any ordering \( \omega \in \mathcal{O}_B \) the sets \( X_{\max}(\omega) \) and \( X_{\min}(\omega) \) have empty interior.

In particular, finite rank diagrams are always regular. We shall see that all Bratteli diagrams we study here are regular.

Given a Bratteli diagram \( B \), we can describe the set of all orderings \( \mathcal{O}_B \) in the following way. Given a vertex \( v \in V \setminus V_0 \), let \( P_v \) denote the set of all orders on \( r^{-1}(v) \); an element in \( P_v \) is denoted by \( \omega_v \). Then \( \mathcal{O}_B \) can be represented as

\[
\mathcal{O}_B = \prod_{v \in V \setminus V_0} P_v.
\]

The set of all orderings \( \mathcal{O}_B \) on a Bratteli diagram \( B \) can be considered also as a measure space whose Borel structure is generated by cylinder sets. On the set \( \mathcal{O}_B \) we take the product measure \( \mu = \prod_{v \in V \setminus V_0} \mu_v \) where \( \mu_v \) is a measure on the set \( P_v \). The case where each \( \mu_v \) is the uniformly distributed measure on \( P_v \) is of particular interest: \( \mu_v(\{i\}) = (|r^{-1}(v)!|)^{-1} \) for every \( i \in P_v \) and \( v \in V \setminus V_0 \).

**Definition 2.7.** Let \( B \) be a Bratteli diagram, and \( n_0 = 0 < n_1 < n_2 < \ldots \) be a strictly increasing sequence of integers. The telescoping of \( B \) to \( (n_k) \) is the Bratteli diagram \( B' \), whose \( k \)-level vertex set \( V'_k = V_{n_k} \) and whose incidence matrices \( (F'_k) \) are defined by

\[
F'_k = F_{nk+1-1} \circ \ldots \circ F_{nk},
\]

where \( (F_n) \) are the incidence matrices for \( B \).

Note that unless \( |V_{n_k}| = 1 \) for almost all \( n \), if \( B' \) is a telescoping of \( B \), then the lexicographical injection of \( \mathcal{O}_B \) in \( \mathcal{O}_{B'} \) is a set of zero measure.

2.3. Vershik maps.

**Definition 2.8.** Let \((B, \omega)\) be an ordered Bratteli diagram. We say that \( \varphi = \varphi_\omega : X_B \to X_B \) is a (continuous) Vershik map if it satisfies the following conditions:

(i) \( \varphi \) is a homeomorphism of the Cantor set \( X_B \);
(ii) $\varphi(X_{\text{max}}(\omega)) = X_{\text{min}}(\omega)$;

(iii) if an infinite path $x = (x_1, x_2, \ldots)$ is not in $X_{\text{max}}(\omega)$, then $\varphi(x_1, x_2, \ldots) = (x_0^0, \ldots, x_k^0, x_k^1, x_k^2, \ldots)$, where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\pi_k^e$ is the successor of $x_k$ in $r^{-1}(r(x_k))$, and $(x_0^0, \ldots, x_k^0)$ is the minimal path in $E(v_0, s(\pi_k^e))$.

If $\omega$ is an ordering on $B$, then one can always define the map $\varphi_0$ that maps $X_B \setminus X_{\text{max}}(\omega)$ onto $X_B \setminus X_{\text{min}}(\omega)$ according to (iii) of Definition 2.9. The question about the existence of the Vershik map is equivalent to that of an extension of $\varphi_0 : X_B \setminus X_{\text{max}}(\omega) \to X_B \setminus X_{\text{min}}(\omega)$ to a homeomorphism of the entire set $X_B$. If $\omega$ has a unique minimal and a unique maximal path, then $\varphi_\omega$ is a homeomorphism. For a finite rank Bratteli diagram $B$, the situation is simpler than for a general Bratteli diagram because the sets $X_{\text{max}}(\omega)$ and $X_{\text{min}}(\omega)$ are finite. Note that if the diagram is regular, and there is an extension of the Vershik map to the whole space, then this extension is unique.

**Definition 2.9.** Let $B$ be a Bratteli diagram $B$. We say that an ordering $\omega \in \mathcal{O}_B$ is perfect if $\omega$ admits a Vershik map $\varphi_\omega$ on $X_B$. If $\omega$ is not perfect, we call it imperfect.

Let $\mathcal{P}_B \subset \mathcal{O}_B$ denote the set of perfect orders on $B$.

### 3. The size of certain sets in $\mathcal{O}_B$.

The following result was shown for finite rank Bratteli diagrams in [BKY14]; the proof for non-finite rank diagrams is very similar.

**Theorem 3.1.** Let $B$ be an aperiodic Bratteli diagram. Then there exists $j \in \mathbb{N} \cup \{\infty\}$ such that $\mu$-almost all orderings have $j$ maximal and $j$ minimal paths.

**Example 3.2.** It is not difficult, though contrived, to find a simple finite rank Bratteli diagram $B$ where almost all orderings are not perfect. Let $V_n = V = \{v_1, v_2\}$ for $n \geq 1$, and define $m_{v_i,v}^{(n)} := \frac{f_{v_i,v}^{(n)}}{\sum_w f_{v_i,w}^{(n)}}$; i.e., $m_{v_i,v}^{(n)}$ is the proportion of edges with range $v \in V_{n+1}$ that have source $w \in V_n$. Suppose that $\sum_{n=1}^{\infty} m_{v_i,v_j}^{(n)} < \infty$ for $i \neq j$. Then for $\mu$-almost all orderings, there is some $K$ such that for $k > K$, the sources of the two maximal/minimal edges at level $n$ are distinct, i.e. $j = 2$.

The following result is proved for finite rank diagrams in Theorem 5.4 of [BKY14]; here we give a general proof that works for more general diagrams.

**Theorem 3.3.** Suppose that $B$ is a regular simple Bratteli diagram such that $\mu$-almost all orderings have $j$ maximal and minimal elements, with $j > 1$. Then $\mu$-almost all orderings are imperfect.

**Proof.** Note that if $|V_n| = 1$ for infinitely many $n$, then any order on $B$ has exactly one maximal and one minimal path. So we shall assume that $|V_n| \geq 2$ for all large $n$.

Let $C_{n,N}$ be the set of orders $\omega$ such that if the non-maximal paths $x$ and $y$ agree to level $N$, then $\varphi_\omega(x)$ and $\varphi_\omega(y)$ agree to level $n$. Note that $\mathcal{P}_B = \cap_{n=1}^{\infty} \liminf N C_{n,N}$.

Fix an order $\omega$, and for the time being let us suppose that we know everything about the extremal edges for $\omega$. In other words, suppose that for each $v \in V \setminus V_0$, we know the minimal incoming edge and the maximal incoming edge, and let $\overline{\mathcal{M}} = \{M_i : i \in J\}$ and $\overline{\mathcal{M}} = \{m_i : i \in J\}$ denote the resulting sets of maximal and minimal paths respectively, where $J = |J|$.
We set up some notation that we shall need. Given a path \( x \) in \( X_B \), let \( v_n(x) \) denote the vertex at level \( n \) through which \( x \) passes. If \( n < N \) and \( v \in V_N \), let \( \overline{e}(V_n, v) \) and \( \overline{s}(V_n, v) \) denote the maximal and minimal paths respectively from level \( n \) to level \( N \) with range \( v \). Given a minimal path \( m \) in \( X_B \), define
\[
[v_n(m), N] := \{ v \in V_N : s(\overline{e}(V_n, v)) = v_n(m) \},
\]
i.e \([v_n(m), N]\) is the set of all vertices in \( V_N \) such that their minimal incoming path agrees with \( m \) from the top of the diagram to level \( n \). For a maximal path \( M \) we define \([v_n(M), N]\) in an analogous manner.

We claim that for each \( j \) there exists an \( K \) such that for all \( k \geq K \),
\[
\bigcup_{M \in \tilde{M}} [v_j(M), k] = \bigcup_{m \in \tilde{M}} [v_j(m), k] = V_k.
\]

For, let \( \tilde{V}_j \) be the set of vertices in \( V_j \) through which a maximal path flows. The compactness of \( X_B \) implies that the set of finite maximal paths with source in \( V_j \) is finite. Choose \( K \) large enough such that the range of these (finitely many) maximal paths lies in a level \( V_m \) with \( m < K \). Similarly for the analogous minimal statement.

Fix a large \( n \). If \( \omega \) is to be perfect, then there is a \( K \) such that \( \omega \in C_{n,k} \) for all \( k \geq K \). We use \( e_v(\alpha) \) to denote the incoming edge to \( v \) that is labelled \( \alpha \) by \( \omega \). Choose \( L \) so that
\[
(3.1) \quad \bigcup_{M \in \tilde{M}} [v_K(M), L] = \bigcup_{m \in \tilde{M}} [v_n(m), L] = V_L.
\]

We can rephrase continuity of \( \varphi_\omega \) as follows: If \( v \in V_{L+1} \) and \( s(e_v(\alpha)) \in [v_K(M), L] \) with \( \alpha \) non-maximal, then \( s(e_v(\alpha + 1)) \in [v_n(\varphi_\omega(M)), L] \). This is true for any \( v \in V_{L+1} \) and any \( \alpha \) such that \( e_v(\alpha) \) is non maximal. In other words, if an order is to be perfect, then (if \( L \) is large) for each maximal path \( M \) there is an appropriate choice of a minimal path \( m \) such that any non-maximal path with range in \( V_L \) which has a non maximal edge in \( E_{L+1} \) and which passes through \( v_K(M) \) must be succeeded by a path which passes through \( v_n(m) \).

We identify a combinatorial fact which we shall use. Suppose that \( A, B \subseteq C \) where \( |A| = |B|, |C| = n, |A \cap B| = \ell \) and \( |A \setminus B| = |B \setminus A| = k \). Then there are \((n-k+\ell)!(k+\ell-1)!k^n\) linear orderings of elements in \( C \) in each element of \( A \) is succeeded by an element of \( B \) and, for such an ordering, each element of \( A \setminus B \) must be followed by a sequence of elements of \( A \cap B \), followed by an element of \( B \setminus A \). There are \((n-k+1)!\) ways to order \( C \setminus B \), and \((k+\ell)!\) ways to divide the elements of \( B \) into an ordered list of \( k \) sequences of this kind. These sequences are then inserted after the elements of \( A \setminus B \) in the initial ordering. This quantity \((n-k+\ell)!(k+\ell-1)!k^n\) is at most \((n-k+\ell)!(k+\ell)!\), so \(\frac{(n-k+\ell)!(k+\ell-1)!k^n}{n!} \leq \frac{1}{(k+\ell)!} \). If \( 1 \leq k + \ell \leq n - 1 \), this quantity is at most \( \frac{1}{n!} \).

We apply this as follows. Fix \( M \); if \( \omega \) were perfect, let \( m = \varphi_\omega(M) \). In fact we only need to know information about \( m \)'s location at the first \( n \) levels. Let \( C := E_{L+1} \), and define
\[
A^* = \bigcup_{v \in V_{L+1}} \bigcup_{\tilde{M} : v_n(\varphi_\omega(M)) = v_n(m)} r^{-1}(v) \cap s^{-1}([v_K(\tilde{M}), L])
\]
and let \( A \) be all elements of \( A^* \), except all maximal edges are removed. Let
\[
B^* = \bigcup_{v \in V_{L+1}} r^{-1}(v) \cap s^{-1}([v_n(m), L])
\]
\[1\]The authors thank Bing Zhou for this observation.
and let $B$ be all elements of $B^*$ except all minimal edges are removed. Define $\tilde{f}_{vw}^{(L)} = f_{vw}^{(L)} - 1$ if $\tilde{e}_v$ has source $w$; $f_{vw}^{(L)} = f_{vw}^{(L)}$ otherwise. Then both $A$ and $B$ each have size $\sum_{v \in E_{L+1}} \sum \tilde{\omega}_n(\tilde{\omega}(\tilde{M})) = \omega_n(m) \sum_{w \in r^{-1}(v) \cap s^{-1}((v_K, L))] \tilde{f}_{vw}^{(L)}$. Since $j > 1$, we can assume that $n$ is large enough so that there are at least two elements in $\tilde{V}_n$ and also $\nabla_n$ (the set of vertices at level $n$ through which a minimal path flows). So each of the sets $[v_K(M), L]$ and $[v_n(m), L]$ have cardinality at most $|V_L| - 1$, and $|A|$ is at most $(|E_{L+1}| - 1)!$. Thus the probability that an order satisfies all these constraints at level $L + 1$ is at most $\frac{1}{|E_{L+1}|} \leq \frac{1}{2}$, since $B$ is simple and there are at least two vertices at each level. Note also that this last bound does not depend on how $\omega_n$ was extended to $\tilde{M}$, or what maximal edge structure $\omega$ possesses, though the value of $L$ depends on the extremal edge structure of $\omega$, and the argument works for this value of $n$ if there are at least two elements in $\tilde{V}_n$ and also $\nabla_n$.

Also, this constraint works for all levels larger than level $L$.

Choose an $n$ large enough so that for a set of orders of size at least $1 - \frac{1}{2}$, there are at least two elements in each of $\tilde{V}_n$ and $\nabla_n$. Fix $K > n$, and choose an $L$ large enough so that for a set of orders of size at least $1 - \frac{1}{2}$, Identity 3.1 holds. For $\ell \geq L$, Let $C_{n,K,\ell}$ denote the orders such that an element in $A$ is succeeded by an element of $B$ in the edge set $E_{\ell+1}$. Then $\mu(C_{n,K,\ell}) \geq \frac{1}{2} - \epsilon$. By the Borel Cantelli Lemma, $C_{n,K,\ell}$ occurs for infinitely many $\ell$ on a set of size at least $1 - \epsilon$. This means that for a set of $\omega$’s of measure at least $1 - \frac{1}{2}$, $\omega \in C_{n,K}$. Now let $K$ increase and repeat the argument. But $P_B \subset \lim \inf_K C_{n,K} \approx \mu \cap_{k \geq K} C_{n,k} \subset C_{n,K}$.

4. Diagrams whose Orders are Almost Always Imperfect

**Notation:** Given $v \in V \setminus V_0$ and an order $\omega \in \mathcal{O}_B$, we use $\tilde{e}_v = \tilde{e}_v(\omega)$ to denote the maximal edge with range $v$. Let $J$ denote a matrix (size determined by the context) all of whose entries are $1$.

**Definition 4.1.** Let $B$ be a Bratteli diagram. We say that $B$ is superquadratic if there exists $\delta > 0$ so that $|V_n| \geq n^{2+\delta}$ for all large $n$.

**Theorem 4.2.** Let $B$ be a superquadratic Bratteli diagram whose incidence matrices are $F_0 = J$ for each $n$. Then $\mu$-almost all orders on $B$ have infinitely many maximal paths, and $\mu$-almost orders are imperfect.

**Proof.** The idea behind the proof is as follows. For large enough $N$, we split the $N$-th vertex set $V_N$ into two equal pieces, and call one piece $A_N$. Then, for a random order, we look at the set of vertices in $V_{N+1}$ whose maximal edge has source in $A_N$, and call this set $A_{N+1}$. Since the maximal edge is determined independently for each vertex in $V_{N+1}$, the size of $A_{N+1}$ is the sum of independent indicator variables with expectation $\frac{1}{2}$. Thus we argue that for most orders $A_{N+1}$ is a set whose size, relative to that of $V_{N+1}$, stays within a small amount $\epsilon_1$ of $\frac{1}{2}$. We repeat this procedure, working with $A_{N+1}$, and then recursively with $A_{N+k}$, at each stage making sure that the size of our sets do not drift too far from $\frac{1}{2}$. In this way, we see that for a random order, the Bratteli diagram is partitioned into two sub diagrams, each of which has at least one maximal path running through it. A similar argument shows that for any $k$, for a random order, the Bratteli diagram can be partitioned into $k$ sub diagrams, each of which contains a maximal path.
Formally, choose a sequence of small non-negative numbers \((\epsilon_j)_{j=1}^{\infty}\) such that

\[\sum_{j=1}^{\infty} \epsilon_j < \infty\]  

(4.1)

and

\[|V_j|^2 \geq j^{\gamma}\] for some \(\gamma > 0\) and large enough \(j\).

(4.2)

Since \(B\) is superquadratic, such numbers must exist. Fix \(N\) so that (4.2) holds for all \(j \geq N\), and let \(N\) be large enough so that \(\sum_{j=N}^{\infty} \epsilon_j < \frac{1}{2}\). Choose \(A_N \subset V_N\) such that \(|A_N| = \frac{|V_N|}{2}\); we shall assume, without loss of generality, that \(\frac{|A_N|}{|V_N|} = \frac{1}{2}\).

For all integers \(k > 0\) and all \(v \in V_{N+k}\), define the random variables \(X_v : O_B \to \{0,1\} : v \in V_{N+k}\), the random sets \(\{A_{N+k} : O_B \to 2^{V_{N+k}} : k \geq 1\}\), and random variables \(\{Y_{N+k} : O_B \to [0,1] : k \geq 1\}\) recursively as follows.

(4.3)

\[X_v(\omega) = \begin{cases} 1 & \text{if } s(\tilde{c}_v) \in A_{N+k-1} \\ 0 & \text{otherwise} \end{cases}\]

and define the random set \(A_{N+k} = A_{N+k}(\omega) = \{v \in V_{N+k} : X_v(\omega) = 1\}\). Thus, the set \(A_{N+k}\) is the set of all vertices \(v \in V_{N+k}\) such that the maximal path from level \(N\) to \(v\) has source in \(A_N\). Finally, define the random variable

(4.4)

\[Y_{N+k}(\omega) := \frac{1}{|V_{N+k}|} \sum_{v \in V_{N+k}} X_v = \frac{|A_{N+k}|}{|V_{N+k}|}.

For \(k \geq 1\), let \(E_{N+k}\) be the event

\[E_{N+k} := \{\omega : |Y_{N+k} - \frac{1}{2}| \leq \sum_{j=1}^{k} \epsilon_{N+j}\}.

We will prove the following inequality for all \(k \geq 1\).

(4.5)

\[\mu(E_{N+k}^{c} \mid E_{N+1} \cap \ldots \cap E_{N+k-1}) \leq 2e^{-2|V_{N+k}|\epsilon_{N+k}^2}.

(Note that for \(k = 1\), the probability above is not conditioned on any event.)

We first consider the case where \(k = 1\). By definition, for any \(v \in V_{N+1}\), \(r^{-1}(v)\) contains exactly one edge originating from any vertex in \(V_N\), and the maximal edge is chosen uniformly from this edge set. Thus, \(s(\tilde{c}_v) \in A_N\) exactly when one of the edges originating from vertices in \(A_N\) is chosen as maximal edge. Therefore \(\mu(X_v = 1) = \frac{|A_N|}{|V_N|} = \frac{1}{2}\). Thus, \(\{X_v : v \in V_{N+1}\}\) are independent and identically distributed Bernoulli random variables with mean \(\frac{1}{2}\). Then Hoeffding’s inequality [Hoe63] tells us that

(4.6)

\[\mu(E_{N+1}^{c}) = \mu(\{|Y_{N+1} - \frac{1}{2}| \geq \epsilon_{N+1}\}) \leq 2e^{-2|V_{N+1}|\epsilon_{N+1}^2}.

In summary, starting with a fixed set \(A_N \subset V_N\), the above procedure defines, for each order \(\omega\), a set \(A_{N+1} = A_{N+1}(\omega) \subset V_{N+1}\); Hoeffding’s inequality tells us that if \(|V_{N+1}|\) is sufficiently large, then for a set of \(\omega\)'s of large \(\mu\)-measure, the ratio \(\frac{|A_{N+1}|}{|V_{N+1}|}\) remains close to \(\frac{1}{2}\). This completes the proof of (4.3) for \(k = 1\).
Next we assume that our claim has been shown for some \( k \geq 1 \), and work at level \( N + k + 1 \). Define the conditional random variables \( \{X_y^v : v \in V_{N+k+1}, y \in [0,1]\} \) where

\[
X_y^v = X_v(Y_{N+k} = y) .
\]

Here the only values of \( y \) that \( Y_{N+k} \) can take are \( y \in \{0, \frac{1}{|V_{N+k}|}, \ldots, \frac{|V_{N+k}|}{|V_{N+k}|}\} \). By a similar argument as for the previous case, we see that for each \( y \) and \( v \), \( X_y^v \) is Bernoulli with mean \( y \), and \( \{X_y^v : v \in V_{N+k+1}\} \) are independent. Define also the random variable

\[
Y_{N+k+1}^y = Y_{N+k+1}|Y_{N+k} = y .
\]

Note that \( Y_{N+k+1}, \ldots, Y_{N+k-1} \), conditioned on knowledge of \( Y_{N+k} \), is independent of \( Y_{N+1}, \ldots, Y_{N+k-1} \). For convenience, define \( \alpha = \frac{1}{2} - \sum_{j=1}^{k} \epsilon_{N+j} \). By the condition on \( N, \alpha > 0 \). Define the set of integers \( S = \{i : i \in \mathbb{N}, \alpha \leq \frac{i}{|V_{N+k}|} \leq 1 - \alpha\} \). Thus, \( S \) is the set of values \( |A_{N+k}| \) can take if \( E_{N+k} \) holds.

\[
\mu(E_{N+k+1}^c \mid E_{N+1} \cap \ldots \cap E_{N+k}) = \mu(\{|Y_{N+k+1} = \frac{i}{|V_{N+k}|} - 1| \geq \sum_{l=1}^{k+1} \epsilon_{N+l} | E_{N+k})
= \frac{1}{\mu(E_{N+k})} \sum_{i \in S} \mu(\{|Y_{N+k+1} = \frac{i}{|V_{N+k}|} - 1| \geq \sum_{l=1}^{k+1} \epsilon_{N+l} \}) \mu(\{|Y_{N+k+1} = \frac{i}{|V_{N+k}|} \})
\leq \frac{1}{\mu(E_{N+k})} \sum_{i \in S} 2e^{-2|V_{N+k+1}|\alpha^2 |N+k+1|} \mu(\{|Y_{N+k+1} = \frac{i}{|V_{N+k}|} \})
= 2e^{-2|V_{N+k+1}|\alpha^2 |N+k+1|} .
\]

The last inequality again uses Hoeffding’s inequality.

This proves Equation \( \text{[13]} \). In other words, we have shown that sequences \( (A_n)_{n \geq N} \) with the asserted properties exist.

Finally we show that our work implies that a random order has at least two maximal paths. Let \( \alpha = \frac{1}{2} - \sum_{j=N}^{\infty} \epsilon_j \). By the condition on \( N, \alpha > 0 \). We have that

\[
\mu(\{\omega : |X_{\max}(\omega)| \geq 2\}) \geq \mu(\bigcap_{k=1}^{\infty} \{\omega : \alpha \leq Y_{N+k} \leq 1 - \alpha, \})
\geq \mu(\bigcap_{k=1}^{\infty} E_{N+k}) = \lim_{n \to \infty} \mu(E_{N+1}) \prod_{k=1}^{n} \mu(E_{N+k+1}|E_{N+k})
\geq \lim_{n \to \infty} \mu(E_{N+1}) \prod_{k=1}^{n} (1 - 2e^{-2|V_{N+k+1}|\alpha^2 |N+k+1|}) ,
\]

and Condition \( \text{[12]} \) ensures that \( |V_j|\alpha^2 \geq j^\delta \) for large \( j \), so this last term converges to a non-zero value. By Theorem \( \text{[11]} \), \( \mu(\{\omega : |X_{\max}(\omega)| \geq 2\}) = 1 \). This proof can be repeated, to show that for any natural \( k, \mu(\{\omega : |X_{\max}(\omega)| \geq k\}) = 1 \). Our claim that \( \mu \)-almost all orders have infinitely many maximal paths follows.
To show that almost any order is not perfect, we need only verify that $B$ is a regular diagram so that we can apply Theorem 3.3. Suppose that for some order $\omega$, the interior of $X_{\text{max}}(\omega)$ is non-empty. Then for some maximal path $x = (x_n)_{n \geq 1}$, there exists an $N$ such that $U(x_1, \ldots, x_N) \subset X_{\text{max}}(\omega)$. But this means that all finite paths having $r(x_N)$ as source must be maximal, and the incidence matrices of $B$ make this impossible.

**Corollary 4.3.** The number of maximal paths that a random order on $B$ possesses is not invariant under telescoping of $B$.

**Proof.** Consider the Bratteli diagram $B$ which has one vertex at odd levels and $n^3$ vertices at level $2n$. Let the incidence matrices of $B$ all be $F_n = J$ for each $n$. Any order on $B$ has one maximal path. Let $B'$ be the diagram with $n^3$ vertices at level $n$, and let the incidence matrices of $B'$ all be $F_n = J$ for each $n$. By Theorem 4.2, a random order on $B'$ has infinitely many maximal paths. On the other hand, $B$ can be telescoped to $B'$.

**4.1. Other Brattelli diagrams whose orders support many maximal paths.** Next we generalize Theorem 4.2 to the family of Bratteli diagrams that satisfy the following properties.

**Definition 4.4.** Let $B$ be a Bratteli diagram.

- We say that $B$ has the *equal path number property* if for each $n$, each row of $F_n$ has the same sum. In other words, for each $n$ there is an $s_n$ such that $\sum_{j=1}^{\vert V_n \vert} f_{ij}^{(n)} = s_n$ for each $i \in \{1, \ldots, \vert V_{n+1} \vert \}$.
- Let $B$ be superquadratic. We say that $B$ is *exponentially bounded* if for all $\delta > 0$, $\sum_{n=1}^{\infty} \vert V_{n+1} \vert e^{-\frac{\delta}{3}}$ converges.

We remark that the condition that $B$ is exponentially bounded is very mild.

In Theorem 4.9 below we show that Bratteli diagrams satisfying these conditions have infinitely many maximal paths. The proof of Theorem 4.9 is similar to that of Theorem 4.2 though we will have to demand more of the set $A_N$ which starts off our procedure, which in turn will impose some restrictions on the diagram $B$. Given $v \in V_{n+1}$, define $V_{n+1}^{v,i} := \{w \in V_n : f_{v,w}^{(n)} = i\}$, so that if the incidence matrix entries for $B$ are all positive and bounded above by $k$, then $V_n = \bigcup_{i=1}^{k} V_{n+1}^{v,i}$ for each $v \in V_{n+1}$.

**Definition 4.5.** Let $B$ be a Bratteli diagram with positive incidence matrices. We say that $B$ is *impartial* if there exists an integer $k$ so that all of $B$’s incidence matrix entries are bounded above by $k$, and if there exists some $\alpha \in (0, 1)$ such that for any $n$, any $i \in \{1, \ldots, k\}$ and any $v \in V_{n+1}$, $|V_{n+1}^{v,i}| \geq \alpha |V_n|$.

In other words, $B$ is impartial if for any row of any incidence matrix, no entry is disproportionately small or large with respect to the others. Note that our diagrams in Theorem 4.2 have the equal row sum property and are impartial. However the vertex sets can grow as fast as we want, so the diagrams are not necessarily exponentially bounded.
Definition 4.6. Suppose that $B$ is a Bratteli diagram all of whose incidence matrix entries are bounded above by a fixed integer $k$. We say that $A \subset V_n$ is $\epsilon$-equitable for $B$ if for each $v \in V_{n+1}$ and for each $i = 1, \ldots, k$,

\[
\frac{|V_n^v \cap A|}{|V_n^v|} - \frac{1}{2} \leq \epsilon.
\]

Lemma 4.7. Suppose that $B$ is impartial, superquadratic and exponentially bounded. Then for any $\epsilon$ small there exist $n$ and $A \subset V_n$ such that $A$ is $\epsilon$-equitable.

Proof. Let $r$ be so that all incidence matrix entries are bounded above by $r$. Fix $\epsilon > 0$. We will show that for large $n$, a randomly chosen subset of $V_n$ is with large probability $\epsilon$-equitable.

Precisely, for each vertex $v \in V_n$, let $Z_v$ be a Bernoulli random variable taking the values 1 or 0 with equal probability. For each $w \in V_{n+1}$ and $1 \leq i \leq r$, let $Y_{w,i} = \frac{1}{|V_n|} \sum_{v \in V_n^{w,i}} Z_v$. Since $B$ is impartial, there exists $\alpha \in (0, 1)$ such that for any $n$, any $i \in \{1, \ldots, r\}$ and any $v \in V_{n+1}$, $|V_n^{v,i}| \geq \alpha |V_n|$.

Hoeffding’s inequality tells us

\[
\mu(\{|Y_{w,i} - \frac{1}{2}| \geq \epsilon\}) \leq 2e^{-2\epsilon^2 |V_n^{v,i}|} \leq 2e^{-2\epsilon^2 \alpha |V_n|}.
\]

Then

\[
\mu(\bigcup_{w \in V_{n+1}} \bigcup_{i=1}^r \{|Y_{w,i} - \frac{1}{2}| \geq \epsilon\}) \leq 2r |V_{n+1}| e^{-2\epsilon^2 \alpha |V_n|},
\]

and if $n$ is large, the assumption that $B$ is superquadratic and exponentially bounded exponentially bounded implies that the above probability is small. Thus for such large $n$, if $A = \{v \in V_n : Z_v = 1\}$, with positive probability, $A$ is $\epsilon$-equitable. □

Lemma 4.8. Suppose that $B$ has the equal path number property and $B$ is impartial. Let $A \subset V_n$ be $\epsilon$-equitable, and $v \in V_{n+1}$. Let the random variable $X_v$ be defined as

\[
X_v(\omega) = \begin{cases} 
1 & \text{if } s(\bar{e}_v) \in A, \\
0 & \text{otherwise}.
\end{cases}
\]

Then $\frac{1}{2} - \epsilon \leq E(X_v) \leq \frac{1}{2} + \epsilon$.

Proof. Suppose that the sum of any row in $F_n$ is $s_n$, and all incidence matrices are bounded above by $r$. Then

\[
(4.7) \quad \sum_{j=1}^r j |V_n^{v,j}| = s_n \text{ for each } v \in V_{n+1}.
\]

Also

\[
E(X_v) = \frac{1}{s_n} \sum_{j=1}^k j |A \cap V_n^{v,j}| \leq \frac{1}{s_n} \sum_{j=1}^k j |V_n^{v,j}| \left(\frac{1}{2} + \epsilon\right) = \frac{1}{2} + \epsilon,
\]

the last inequality following since $A$ is $\epsilon$-equitable. Similarly, $E(X_v) \geq \frac{1}{2} - \epsilon$. □

Theorem 4.9. Suppose that $B$ is a Bratteli diagram with the equal path number property. Suppose also that $B$ is impartial, superquadratic and exponentially bounded. Then $\mu$-almost all orders on $B$ have infinitely many maximal paths.
Proof. As $B$ is superquadratic, we find a sequence $(\epsilon_j)$ satisfying Conditions (4.1) and (4.2) as described in Theorem 4.2. Fix $N$ so that (4.2) holds for all $j \geq N$, and let $N$ be large enough so that $\sum_{j=N}^{\infty} \epsilon_j < \frac{1}{2}$. Moreover, we can also choose our sequence $(\epsilon_j)$ and our $N$ large enough so that there exists a set $A_N \subset V_N$ which is $\epsilon_N$-equitable: by Lemma 4.7 this can be done. For all $k \geq 0$, define also

$$\delta_{N+k} = \sum_{i=0}^{k} \epsilon_{N+i}.$$ 

Finally, let $r$ be so that all entries of all $F_n$ are bounded above by $r$.

We continue to use the notation set up in Theorem 4.2. In particular for all integers $k > 0$ and all $v \in V_{N+k}$, let the random variables $\{X_v : \mathcal{O}_B \to \{0,1\} : v \in V_{N+k}\}$, be defined as in (4.3), and let the random sets $\{A_{N+k} : \mathcal{O}_B \to 2^{V_{N+k}} : k \geq 1\}$ be defined as $A_{N+k} = \{v \in V_{N+k} : X_v = 1\}$.

We shall show that for a large set of $\omega$, each set $A_{N+k}$ is $\delta_{N+k}$-equitable. This implies that the size of $A_{N+k}$ is not far from $\frac{1}{2}|V_{N+k}|$. For, if $k \geq 1$, define the event

$$E_{N+k} := \{\omega : \left|\frac{|A_{N+k}|}{|V_{N+k}|} - \frac{1}{2}\right| \leq \delta_{N+k}\},$$

and the event

$$D_{N+k} := \{\omega : A_{N+k} \text{ is } \delta_{N+k} \text{-equitable}\}.$$ 

Then $D_{N+k} \subset E_{N+k}$. For, suppose that for some $n$ and $\epsilon > 0$, a set $A \subseteq V_n$ is $\epsilon$-equitable. Let $w \in V_{n+1}$. Recall we have assumed that $f_{n,w}^{(n)} \geq 1$ for all $v \in V_n$. Thus the sets $V_n^{w,i}$, $1 \leq i \leq r$ form a partition of $V_n$, and

$$|A| = \sum_{i=1}^{r} |A \cap V_n^{w,i}| \leq \sum_{i=1}^{r} \left(\frac{1}{2} + \epsilon\right) |V_n^{w,i}| = \left(\frac{1}{2} + \epsilon\right) |V_n|.$$ 

Similarly, we can show that $|A| \geq \left(\frac{1}{2} - \epsilon\right) |V_n|$.

Let $B$ be impartial with “impartiality constant” $\alpha$. We will prove the following inequality for all $k \geq 1$:

\begin{equation}
\mu(D_{N+k}^c \mid D_{N+1} \cap \ldots \cap D_{N+k-1}) \leq 2r|V_{N+k+1}|e^{-2\alpha |V_{N+k+1}|^3}. \tag{4.8}
\end{equation}

We first prove the statement for $k = 1$. As before, $\{X_v : v \in V_{N+1}\}$ are independent Bernoulli random variables. From Lemma 4.3, their mean satisfies $\frac{1}{2} - \epsilon_N \leq E(X_v) \leq \frac{1}{2} + \epsilon_N$.

For $u \in V_{N+2}$ and $1 \leq i \leq r$, define

\begin{equation}
Y_{u,i} := \frac{1}{|V_{N+1}^{u,i}|} \sum_{v \in V_{N+1}^{u,i}} X_v = \frac{|\{v \in V_{N+1}^{u,i} : s(\overline{v}) \in A_N\}|}{|V_{N+1}^{u,i}|} = \frac{|A_{N+1} \cap V_{N+1}^{u,i}|}{|V_{N+1}^{u,i}|}. \tag{4.9}
\end{equation}

Using Hoeffding’s inequality, since $\frac{1}{2} - \epsilon_N \leq E(Y_{u,i}) \leq \frac{1}{2} + \epsilon_N$ we have that

$$\mu(\{|Y_{u,i} - \frac{1}{2}| \geq (\epsilon_N + \epsilon_{N+1})\}) \leq \mu(\{|Y_{u,i} - E(Y_{u,i})| \geq (\epsilon_{N+1})\}) \leq 2e^{-2|V_{N+1}^{u,i}|^3} \leq 2e^{-2|V_{N+1}^{u,i}|^3}.$$ 

This implies that

\begin{equation}
\mu(\bigcup_{i=1}^{r} \bigcup_{u \in V_{N+2}} \{|Y_{u,i} - \frac{1}{2}| \geq \delta_{N+1}\}) \leq 2r|V_{N+2}|e^{-2|V_{N+1}^{u,i}|^3}. \tag{4.10}
\end{equation}
which proves (4.8) for $k = 1$.

As in the proof of Theorem 4.2, we now assume that our Claim (4.8) has been shown for some $k \geq 1$, and work at level $N + k + 1$. First, we generalize Definition (4.9). Define the random variables $\{Y_{v',i,N+k} : \mathcal{O}_B \to [0,1] : v' \in V_{N+k+1}, 1 \leq i \leq r, k \geq 1\}$ as follows:

$$Y_{v',i,N+k} := \frac{1}{|V_{N+k}|} \sum_{u \in V_{N+k}} X_u = \frac{|A_{N+k} \cap V_{N+k}^v|}{|V_{N+k}|}.$$ 

Now define the conditional random variables $\{X^y_v : v \in V_{N+k+1}, y \in [0,1]^{V_{N+k+1} \times [0,1]^r}\}$ where

$$X^y_v = X_v \bigg| \bigcap_{v' \in V_{N+k+1}} \bigcap_{i=1}^r Y_{v',i,N+k} = y_{v',i}.$$ 

Thus, $y$ is a vector indexed by pairs $(v', i)$ where $v' \in V_{N+k+1}$ and $1 \leq i \leq r$, which gives the values of the size of $V_{N+k}^v \cap A_{N+k}$.

Define a vector $y \in [0,1]^{V_{N+k+1} \times [0,1]^r}$ to be $\epsilon$-equitable if there exists an $\epsilon$-equitable set $A \subseteq V_{N+k}$ so that for all $v' \in V_{N+k+1}$ and $1 \leq i \leq r$, $|A \cap V_{N+k}^v| = y_{v',i}$. By a similar argument as for the previous case, for each $v$ and $y$, $X^y_v$ is Bernoulli, and $\{X^y_v : v \in V_{N+k+1}\}$ are independent. Moreover, if $y$ is $\epsilon$-equitable, then by Lemma 4.8 each $X^y_v$ has mean between $\frac{1}{2} - \epsilon$ and $\frac{1}{2} + \epsilon$.

For $u \in V_{N+k+2}, i \in \{1, \ldots, r\}$, and $y \in [0,1]^{V_{N+k+1} \times [0,1]^r}$, define also the conditional random variable

$$Y^y_{u,i} := \frac{1}{|V_{N+k+1}|} \sum_{v \in V_{N+k+1}} X^y_v = Y_{u,i,N+k+1} \bigg| \bigcap_{v' \in V_{N+k+1}} \bigcap_{i=1}^r Y_{v',i,N+k} = y_{v',i}.$$ 

Now define the set $S \subseteq [0,1]^{V_{N+k+1} \times [0,1]^r}$ to be the set of all $\epsilon_{N+k}$-equitable vectors $y$. Thus, elements of $S$ represent the values which $\frac{|A_{N+k} \cap V_{N+k}^v|}{|V_{N+k}^v|}$ can take if $D_{N+k}$ holds. Specifically, for $y \in S$ and $v \in V_{N+k+1}$, we have that $|E(X^y_v) - \frac{1}{2}| \leq \delta_{N+k}$. Therefore, for all $u \in V_{N+k+2}$ and $i \in \{1, \ldots, r\}$, and $y \in S$,

$$|E(Y^y_{u,i}) - \frac{1}{2}| \leq \delta_{N+k},$$ 

and, by Hoeffding’s inequality,

$$\mu(\{|Y^y_{u,i} - E(Y^y_{u,i})| \geq \epsilon_{N+k+1}\}) \leq 2e^{-2|V_{N+k+1}|^2 \epsilon_{N+k+1}^2} \leq 2e^{-2|V_{N+k+1}|^2 \epsilon_{N+k+1}^2}.$$
We can now finish the argument. In the following, let $\epsilon = \epsilon_{N+k+1}$.

\[
\mu(\{D_{N+k+1}^c \mid D_{N+1} \cap \ldots \cap D_{N+k}\}) \\
= \mu(\{D_{N+k+1}^c \mid D_{N+k}\}) \\
= \mu(\bigcup_{u \in V_{N+k+2}} \bigcup_{i=1}^{r} \{ |Y_{u,i,N+k+1} - \frac{1}{2}| \geq \delta_{N+k+1} \mid D_{N+k}\}) \\
\leq \mu(\bigcup_{u \in V_{N+k+2}} \bigcup_{i=1}^{r} \{ |Y_{u,i,N+k} - E(Y_{u,i,N+k+1})| \geq \epsilon \mid D_{N+k}\}) \\
\leq \sum_{u \in V_{N+k+2}} \sum_{i=1}^{r} \mu(\{ |Y_{u,i,N+k} - E(Y_{u,i,N+k+1})| \geq \epsilon \mid D_{N+k}\}) \\
= \frac{1}{\mu(\{D_{N+k}\})} \sum_{u \in V_{N+k+2}} \sum_{i=1}^{r} \left( \sum_{y \in S} \mu(\{ |Y_{u,i} - E(Y_{u,i})| \geq \epsilon \}) \mu(\bigcap_{v \in V_{N+k+1}} \bigcap_{i=1}^{r} \{ Y_{v,i,N+k} = y_{v,i} \}) \right) \\
\leq 2r |V_{N+k+2}| e^{-2|V_{N+k+1}| \alpha^2} \mu(\bigcap_{v \in V_{N+k+1}} \bigcap_{i=1}^{r} \{ Y_{v,i,N+k} = y_{v,i} \}) \\
= 2r |V_{N+k+2}| e^{-2|V_{N+k+1}| \alpha^2} \\
\]

This completes the proof of (4.8) for $k+1$.

Finally we show that our work implies that a random order has at least two maximal paths. Let $\gamma = \frac{1}{2} - \sum_{j=N}^{\infty} \epsilon_j$. By our choice of $N$ and $\gamma > 0$ we have that

\[
\mu(\{ \omega : |X_{max}(\omega)| \geq 2 \}) \geq \mu(\bigcap_{k=1}^{\infty} \{ \omega : \gamma \leq \frac{|A_{N+k}|}{|V_{N+k}|} \leq 1 - \gamma, \}) \\
\geq \mu(\bigcap_{k=1}^{\infty} E_{N+k}) \\
\geq \mu(\bigcap_{k=1}^{\infty} D_{N+k}) \\
= \lim_{n \to \infty} \mu(D_{N+1}) \prod_{k=1}^{n} \mu(D_{N+k+1} \mid D_{N+k}) \\
\geq \lim_{n \to \infty} \mu(D_{N+1}) \prod_{k=1}^{n} (1 - 2r |V_{N+k+2}| e^{-2|V_{N+k+1}| \alpha^2_{N+k+1}}),
\]

and the condition that $B$ is superquadratic and exponentially bounded ensures that this last term converges to a non-zero value. The argument that $B$ is regular is the same as that for the diagrams satisfying the conditions of Theorem 4.2. By Theorem 3.1, $\mu(\{ \omega : |X_{max}(\omega)| \geq 2 \}) = 1$.

$\square$

**References**

[BKM09] S. Bezuglyi, J. Kwiatkowski, and K. Medynets. Aperiodic substitution systems and their Bratteli diagrams. *Ergodic Theory Dynam. Systems*, 29(1):37–72, 2009.
[BKY14] Sergey Bezuglyi, Jan Kwiatkowski, and Reem Yassawi. Perfect orderings on finite rank Bratteli diagrams. *Canad. J. Math.*, 66(1):57–101, 2014.

[Eff81] Edward G. Effros. *Dimensions and C*-algebras*, volume 46 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1981.

[Hoe63] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

DEPARTMENT OF MATHEMATICS AND STATISTICS, DALHOUSIE UNIVERSITY, CANADA, AND DEPARTMENT OF MATHEMATICS, TRENT UNIVERSITY, PETERBOROUGH, CANADA

E-mail address: Jeannette.Janssen@dal.ca, ryassawi@trentu.ca