Non-termination using Regular Languages
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Abstract
We describe a method for proving non-termination of term rewriting systems that do not admit looping reductions. As certificates of non-termination, we employ regular (tree) automata.

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1 Introduction
We describe a method for proving non-termination of term rewriting systems that do not admit looping reductions, that is, reductions from a term $t$ to a term $C[t\sigma]$ containing a substitution instance of $t$. For this purpose, we employ tree automata as certificates of non-termination. For proving non-termination of a term rewriting system $R$, we search a tree automaton $A$ whose language $L(A)$ is not empty, weakly closed under rewriting and every term of the language contains a redex occurrence. We have fully automated the search for these certificates employing SAT-solvers.

All automata that we use as example in this paper have been found automatically; this concerns in particular fully automated proofs of non-termination for the following two rewrite systems.

► Example 1. We consider the following string rewriting system:
\[
\begin{align*}
zL & \rightarrow Lz \\
Rz & \rightarrow zR \\
bL & \rightarrow bR \\
Rb & \rightarrow Lzb
\end{align*}
\]
This rewrite system admits no reductions of the form $s \rightarrow^* \ell sr$.

► Example 2. We consider the $S$-rule from combinatory logic:
\[
ap(ap(ap(S, x), y), z) \rightarrow ap(ap(x, z), ap(y, z))
\]
For the $S$-rule it is known that there are no reductions $t \rightarrow^* C[t]$ for ground terms $t$, see \cite{15}. For open terms $t$ the existence of reductions $t \rightarrow^* C[t\sigma]$ is open.

It turns out that the method can be fruitfully applied to obtain non-termination proofs of several string rewriting systems that have remained unsolved in the last full run of the termination competition.
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Related Work

The paper [11] investigates necessary conditions for the existence of loops. The work [17] employs SAT solvers to find loops. [15] uses forward closures to find loops efficiently, and the work [16] introduces ‘compressed loops’ to find certain forms of (possibly very long) loops.

Non-termination beyond loops has been investigated in [14] and [2]; we note that Example 2 cannot be handled by these techniques.

Here we prove non-looping non-termination on regular languages. The converse, local termination on regular languages, has been investigated in [3]. Regular (tree) automata have been fruitfully applied to a wide range of properties of term rewriting systems: for proving termination [10, 8, 12], for infinitary normalization [4], for proving liveness [13], and for analysing reachability and deciding the existence of common reducts [9, 5].

2 Non-termination and Weakly Closed Languages

Definition 3. Let $L \subseteq T(\Sigma, \emptyset)$ a language and $R$ a TRS over $\Sigma$. Then $L$ is called:

- closed under rewriting if for every $t \in L$ and $s$ such that $t \rightarrow s$, one has $s \in L$, and
- weakly closed under rewriting if for every $t \in L$ that is not in normal form, there exists $s \in L$ such that $t \rightarrow_R s$.

The following theorem describes the basic idea that we employ for proving non-termination.

Theorem 4. A term rewriting system $R$ over $\Sigma$ is non-terminating if and only if there exists a non-empty language $L \subseteq T(\Sigma, \emptyset)$ such that

(i) every $t \in L$ contains a redex (that is, $t \rightarrow s$ for some term $s$), and
(ii) $L$ is weakly closed under rewriting.

A language fulfilling the properties of Theorem 4 is also called a recurrence set, see [1].

To automate this method, we need to restrict to a certain family of languages. In this paper, we consider regular tree languages. To guarantee that the language of a tree automaton is weakly closed under rewriting, we check that the language is not empty and that the automaton is a quasi-model (see Definition 13) for the rewrite system. The latter condition is actually too strict; it implies that the languages is not only weakly closed, but also closed under rewriting. In future, we plan to relieve this restriction.

3 Tree Automata

Definition 5. A (nondeterministic finite) tree automaton $A$ over a signature $\Sigma$ is a tuple $A = \langle Q, \Sigma, F, \delta \rangle$ where

(i) $Q$ is a finite set of states,
(ii) $F \subseteq Q$ is a set of accepting states, and
(iii) $\{\delta_f\}_{f \in \Sigma}$ is a family of transition relations such that for every $f \in \Sigma$:

$$\delta_f \subseteq Q^n \times Q$$

where $n$ is the arity of $f$.

In examples, we often write the transition relation $\delta_f$ as $\rightarrow_f$. 
The transition relation for \( \varepsilon \) can be thought of as defining the initial states (here 0) of a word automaton.

**Example 7.** The following is a tree automaton for Example 2. Let \( A_S = (Q, \Sigma, F, \to) \) where \( Q = \{0, 1, 2, 3, 4\} \), \( \Sigma = \{\text{ap}, S\} \), \( F = \{4\} \) and

\[
\begin{align*}
\to_S 0 & \quad (0, 0) \to_{\text{ap}} 1 & \quad (1, 0) \to_{\text{ap}} 2 & \quad (2, 2) \to_{\text{ap}} 3 & \quad (3, 3) \to_{\text{ap}} 3 \\
(0, 2) & \quad \to_{\text{ap}} 2 & \quad (2, 3) \to_{\text{ap}} 3 & \quad (3, 3) \to_{\text{ap}} 4 \\
(0, 3) & \quad \to_{\text{ap}} 2 \\
\end{align*}
\]

In Example 12 we show that this automaton accepts the term \( SSS(SSS)(SSS(SSS)) \).

**Example 9.** We use the automaton \( A_S \) from Example 2. Let \( \alpha(x) = \{2\} \), then we have:

\[
\begin{align*}
[S, \alpha] & = \{0\} & [\text{ap}(S, S), \alpha] & = \{1\} & [\text{ap}(\text{ap}(S, S), S), \alpha] & = \{2\} \\
[\text{ap}(x, x), \alpha] & = \{3\} & [\text{ap}(\text{ap}(x, x), \text{ap}(x, x)), \alpha] & = \{3, 4\} \\
\end{align*}
\]

**Definition 10.** Let \( A = (Q, \Sigma, F, \delta) \) be a tree automaton over \( \Sigma \). The language \( \mathcal{L}(A) \) accepted by \( A \) is the set \( \mathcal{L}(A) = \{t | t \in T(\Sigma, \emptyset), [t]_A \cap F \neq \emptyset \} \) of ground terms.

**Example 10.** The automaton in Example 6 accepts all words of the form \( b \varepsilon^* (L/R) \varepsilon^* b \), that is, all words that start with \( b \), end with \( b \), contain one \( L \) or \( R \) and otherwise only \( z \).

**Example 12.** We continue Example 9. Let \( \alpha(x) = \{2\} \), then we have:

\[
\begin{align*}
[\text{ap}(\text{ap}(S, S), S)] & = \{2\} & [\text{ap}(\text{ap}(\text{ap}(S, S), S), \text{ap}(\text{ap}(S, S), S))] & = \{3\} \\
[\text{ap}(\text{ap}(\text{ap}(S, S), S), \text{ap}(\text{ap}(S, S), S))] & = \{3, 4\} \\
\end{align*}
\]

Thus \( F \cap [SSS(SSS)(SSS(SSS))] = \{4\} \neq \emptyset \) and hence the term is accepted by the automaton.

**4 Closure under Rewriting**

**Definition 13.** A tree automaton \( A = (Q, \Sigma, F, \delta) \) is a quasi-model for a term rewriting system \( R \) over \( \Sigma \) if \( [\ell, \alpha]_A \subseteq [r, \alpha]_A \) for every \( \ell \to r \in R \) and \( \alpha : \mathcal{X} \to \mathcal{P}(Q) \).
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Actually, it suffices to check the property \([\ell, \alpha]_A \subseteq [r, \alpha]_A\) for assignments \(\alpha : X \rightarrow \mathcal{P}(Q)\) that map variables to singleton sets.

\textbf{Lemma 14.} A tree automaton \(A = \langle Q, \Sigma, F, \delta \rangle\) is a quasi-model for a term rewriting system \(R\) over \(\Sigma\) if \(\forall \ell \rightarrow r \in R\) and \(\alpha : X \rightarrow \{\{q\} \mid q \in Q\}\).

\textbf{Example 15.} It is not difficult to check that the automaton \(A_{LR}\) from Example 6 is a quasi-model for rewrite system in Example 4.

\textbf{Example 16.} We consider the automaton \(A_S\) from Example 7. We write \((a, b, c) \rightarrow d\) if \(d \in [\ell, \alpha]\) when \(\alpha(x) = \{a\}, \alpha(y) = \{b\}, \alpha(z) = \{c\}\). Then for \([\ell, \alpha]\) we have:

\[
\begin{align*}
(0, 0, 2) \rightarrow 1 & \quad (2, 2, 3) \rightarrow 3 & \quad (2, 3, 3) \rightarrow 3 & \quad (3, 2, 3) \rightarrow 3 & \quad (3, 3, 3) \rightarrow 3 \\
(0, 0, 3) \rightarrow 1 & \quad (2, 2, 3) \rightarrow 4 & \quad (2, 3, 3) \rightarrow 4 & \quad (3, 2, 3) \rightarrow 4 & \quad (3, 3, 3) \rightarrow 4
\end{align*}
\]

The interpretation \([r, \alpha]\) has all the above and additionally:

\[
\begin{align*}
(0, 2, 2) \rightarrow 3 & \quad (1, 1, 0) \rightarrow 3 & \quad (2, 2, 2) \rightarrow 3 \\
(0, 2, 3) \rightarrow 3 & \quad (2, 2, 2) \rightarrow 4 \\
(0, 3, 3) \rightarrow 3 &
\end{align*}
\]

As a consequence \(A_S\) is a quasi-model for the \(S\)-rule.

The following theorem is immediate:

\textbf{Theorem 17.} Let \(A = \langle Q, \Sigma, F, \delta \rangle\) be a tree automaton and \(R\) a term rewriting system over \(\Sigma\). If \(A\) is a quasi-model for \(R\) then the language of \(A\) is closed under rewriting.

## 5 Ensuring Redex Occurrences

Next, we want to guarantee that every term in the language \(\mathcal{L}(A)\) of an automaton \(A\) contains a redex with respect to the term rewriting system \(R\). For left-linear systems \(R\), this problem can be reduced to deciding the inclusion of regular languages.

Let \(R\) be a left-linear term rewriting system. Then the set of ground terms containing a redex is a regular tree language. A deterministic automaton \(B\) for this language can be constructed using the overlap-closure of subterms of left-hand sides, see further [6, 7].

\textbf{Example 18.} The following tree automaton \(C = \langle Q, \Sigma, F, \rightarrow \rangle\) accepts the language of ground terms that contain a redex occurrence with respect to the \(S\)-rule. Here \(Q = \{0, 1, 2, 3\}, \Sigma = \{ap, S\}, F = \{3\}\) and

\[
\begin{align*}
\rightarrow_S & 0 \quad (0, q) \rightarrow_{ap} 1 & \quad (1, q) \rightarrow_{ap} 2 & \quad (2, q) \rightarrow_{ap} 3 & \quad (3, q) \rightarrow_{ap} 3 & \quad (q, 3) \rightarrow_{ap} 3
\end{align*}
\]

for all \(q \in \{0, 1, 2\}\).

As a consequence the problem of checking whether every term in \(\mathcal{L}(A)\) contains a redex boils down to checking that \(\mathcal{L}(A) \subseteq \mathcal{L}(B)\). For non-deterministic \(A\) and deterministic \(B\), this property can be decided by constructing the product automaton and considering the reachable states.

\textbf{Definition 19.} The product \(A \cdot B\) of tree automata \(A = \langle Q, \Sigma, F, \delta \rangle\) and \(B = \langle Q', \Sigma, F', \delta' \rangle\) is the automaton \(C = \langle Q \times Q', \Sigma, \emptyset, \gamma \rangle\) where for every \(f \in \Sigma\) of arity \(n\), we define the transition relation \(\gamma_f \subseteq (Q \times Q')^n \times (Q \times Q')\) by

\[
\langle (q_1, p_1), \ldots, (q_n, p_n) \rangle \gamma (q', p') \iff \langle q_1, \ldots, q_n \rangle \delta_f q' \land \langle p_1, \ldots, p_n \rangle \delta' f p'
\]


Definition 20. The set of reachable states of a tree automaton $A = \langle Q, \Sigma, F, \delta \rangle$ is the smallest set $S \subseteq Q$ such that $q \in S$ whenever $\langle q_1, \ldots, q_n \rangle \delta_f q$ for some $q_1, \ldots, q_n \in S$ and $f \in \Sigma$ with arity $n$.

The following theorem gives a method for checking $L(A) \subseteq L(B)$ without the need for determinising $A$ (only $B$ needs to be deterministic).

Theorem 21. Let $A = \langle Q, \Sigma, F, \delta \rangle$ and $B = \langle Q', \Sigma, F', \delta' \rangle$ be tree automata such that $B$ is deterministic. Let $S$ be the set of reachable states of the product automaton $A \cdot B$. Then $L(A) \subseteq L(B)$ if and only if for all $(q, p) \in S$ it holds that $q \in F \implies p \in F'$.

Example 22. The reachable states of product automaton $A_S \cdot C$ of the automata $A_S$ from Example 7 and $C$ from Example 18 are $(0, 0), (1, 1), (2, 2), (2, 1), (3, 3), (3, 2), (2, 3), (4, 3)$. The only state $(q, q')$ such that $q$ is accepting in $A_S$ is $(4, 3)$ and 3 is an accepting state of $C$. Thus the conditions of Theorem 21 are fulfilled and hence $L(A_S) \subseteq L(C)$. Thus every term accepted by $A_S$ contains a redex.

6 Future Work

We plan to investigate whether the method described in this paper can be fruitfully extended from regular automata to pushdown automata, that is, context-free languages. For this purpose, it is important that it is decidable whether a context-free language is a subset of a regular language (the language of terms containing left-linear redex occurrences). However, it remains to be investigated whether context-free certificates can be found efficiently.
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