A Splitting Result for Real Submanifolds of a Kähler Manifold

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Abstract

Let \((Z, \omega)\) be a connected Kähler manifold with an holomorphic action of the complex reductive Lie group \(U^C\), where \(U\) is a compact connected Lie group acting in a hamiltonian fashion. Let \(G\) be a closed compatible Lie group of \(U^C\) and let \(M\) be a \(G\)-invariant connected submanifold of \(Z\). Let \(x \in M\). If \(G\) is a real form of \(U^C\), we investigate conditions such that \(G \cdot x\) compact implies \(U^C \cdot x\) is compact as well. The vice-versa is also investigated. We also characterize \(G\)-invariant real submanifolds such that the norm-square of the gradient map is constant. As an application, we prove a splitting result for real connected submanifolds of \((Z, \omega)\) generalizing a result proved in Gori and Podestà (Ann Global Anal Geom 26: 315–318, 2004), see also Bedulli and Gori (Results Math 47: 194–198, 2005), Biliotti (Bull Belg Math Soc Simon Stevin 16: 107–116 2009).

Keywords Gradient map · Real reductive Lie groups · Cartan decomposition

Mathematics Subject Classification 22E45 · 53D20

1 Introduction

Let \((Z, \omega)\) be a Kähler manifold. Assume that \(U^C\) acts holomorphically on \(Z\), that \(U\) preserves \(\omega\) and that there is a momentum map for the \(U\) action on \(Z\). This means there is a map \(\mu : Z \rightarrow u^*\), where \(u\) is the Lie algebra of \(U\) and \(u^*\) is its dual, which is \(U\)-equivariant with respect to the given action of \(U\) on \(Z\) and the coadjoint action \(\text{Ad}^*\) of \(U\) on \(u^*\) and satisfying the following condition. Let \(\xi \in u\). We denote

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by $\xi_Z$ the induced vector field on $Z$, i.e., $\xi_Z(p) = \frac{d}{dt}|_{t=0} \exp(t\xi)\cdot p$. Let $\mu^\xi$ be the function $\mu^\xi(z) := \mu(z)(\xi)$, i.e., the contraction of the momentum map along $\xi$. Then $d\mu^\xi := i_\xi \omega$.

Let $G$ be a closed connected subgroup of $U^C$ compatible with respect to the Cartan decomposition of $U^C$, i.e., $G = K \exp(p)$, for $K = U \cap G$ and $p = g \cap i u$ (Helgason 2001; Knapp 2002). The inclusion $ip \hookrightarrow u$ induces by restriction a $K$-equivariant map $\mu_{ip} : Z \rightarrow (ip)^*$ (Heinzner et al. 2008; Heinzner and Stötzel 2006).

Let $\langle \cdot, \cdot \rangle$ be a $U$-invariant scalar product on $u$. Let $\langle \cdot, \cdot \rangle$ denote also the inner product on $i u$ such that multiplication by $i$ be an isometry of $u$ into $i u$. Hence we may identify $u^*$ and $u$ by means of $\langle \cdot, \cdot \rangle$ and so we view $\mu$ as a map $\mu : Z \rightarrow u$. Therefore, we may view $\mu_{ip}$ as a map $\mu_p : Z \rightarrow p$ as follows:

$$\langle \mu_p(x), \beta \rangle = -\langle \mu(x), i\beta \rangle, \ \forall \beta \in p.$$ 

We call $\mu_p$ the $G$-gradient map associated with $\mu$. We also set $\mu^\beta_p := \langle \mu_p, \beta \rangle$. By definition, it follows that $\text{grad}\mu^\beta_p = \beta Z$. If $M$ is a $G$-stable locally closed real submanifold of $Z$, we may consider $\mu_p$ as a mapping $\mu_p : M \rightarrow p$ such that $\text{grad}\mu_p = \beta_M$, where the gradient is computed with respect to the induced Riemannian metric on $M$. Since $M$ is $G$-stable it follows $\beta_M(p) = \beta_M(p)$ for any $p \in M$.

Assume that $G$ is a real form of $U^C$. If $U^C \cdot x$ is compact, then it is well-known that $G$ has a closed orbit contained in $U^C \cdot x$ (Heinzner et al. 2008). On the other hand, if $G \cdot x$ is closed then it is not in general true that $U^C \cdot x$ is closed as well (Guivarc’h et al. 1998). In Sect. 2, we investigate conditions such that $G \cdot x$ compact implies $U^C \cdot x$ is compact. If $G \cdot x$ is compact then we give a necessary condition for $U^C \cdot x$ to be compact. If $M$ is Lagrangian and $x \in M$, then $U^C \cdot x$ being compact implies $G \cdot x$ is a Lagrangian submanifold of $U^C \cdot x$. Finally, we study the case when $Z$ is $U^C$-semistable, $M$ is $G$-semistable and is contained in the zero level set of the gradient map of $K^C$. As an application we obtain a well-known result of Birkes (Birkes 1971).

A strategy for analyzing the $G$-action on $M$ is to view the function $v_p : M \rightarrow \mathbb{R}$,

$$v_p(x) = \| \mu_p(x) \|^2$$

as a Morse-like function. The function $v_p$ is called the norm-square of the gradient map. If $M$ is compact or $\mu_p$ is proper, then associated to the critical points of $v_p$ we have $G$-stable submanifold of $M$ that they are strata of a Morse type stratification of $M$ (Heinzner et al. 2008; Kirwan 1984). In Sect. 3, we investigate under which condition $v_p$ is constant. The following result has some interest by itself.

**Proposition 1** Let $M$ be a $G$-stable connected submanifold of $Z$ and let $\mu_p : M \rightarrow p$ be the restricted gradient map. Then the norm-square of the gradient map $v_p : M \rightarrow \mathbb{R}$ is constant if and only if any $G$-orbit is compact.

By the stratification Theorem (Heinzner et al. 2008), it follows that $M$ coincides with a maximal pre-stratum and $\mu_p(M) = K \cdot \beta$. Moreover, $M = K \times K^\beta \mu_p^{-1}(\beta)$, where $K^\beta = \{ k \in K : \text{Ad}(k)(\beta) = \beta \}$. Let $x \in \mu_p^{-1}(\beta)$. By the $K$-equivariance of $\mu_p$, 

\[ \text{Ad}(k)(\beta) = \beta \].
it follows that the stabilizer $K_x \subseteq K^\beta$. Although $G \cdot x$ is compact, it is not true in general $K_x = K^\beta$. Indeed, let $U$ be a connected, compact semisimple Lie group and let $\rho : U \rightarrow \text{SL}(W)$ be a complex representation. Let $G$ be a noncompact connected semisimple real form of $U^C$. It is well known that $U^C$ has a compact orbit in $\mathbb{P}(W)$, which is a complex $U$-orbit (Guillemin and Sternberg 1990). Let $\mathcal{O}$ denote a compact orbit of $U^C$. If $x \in \mathcal{O}$ realizes the maximum of the norm-square of the $G$-gradient map restricted to $\mathcal{O}$, then $G \cdot x$ is closed and it is a $K$-orbit (Heinzner et al. 2008). Now, $K_x = K \cap U^\mu(x)$ and $U^\mu(x) = U_x$ since $U \cdot x$ is complex (Guillemin and Sternberg 1990). However, $\mu(x) \notin \mathfrak{p}$ and so $K_x$ does not coincide in general with $K^\mu_p(x)$. If $M$ is a $U$-invariant compact connected complex submanifold of $(Z, \omega)$, then $\nu_{\text{opt}}$ constant is equivalent to $U$ being semisimple and $M = U/U^\beta \times \mu^{-1}(\beta)$. The above splitting is Riemannian (Gori and Podestà 2004) (see also Bedulli and Gori 2005; Biliotti 2009 for the same result under the assumption that $M$ is symplectic). In this paper we prove this splitting result without any assumption on $M$.

**Theorem 2** Let $M$ be a $U^C$-stable connected submanifold of a Kähler manifold $Z$ and let $\mu : M \rightarrow \mathfrak{z}$ be the restricted momentum map. Then the norm-square of the momentum map $\| \mu \|^2$ is constant if and only if $U$ is semisimple and $M$ is $U$-equivariantly isometric to the product of a flag manifold and an embedded, closed submanifold which is acted on trivially by $U$.

Assume that $G$ is a real form of $U$. The momentum map of $U$ on $Z$ induces a gradient map $\mu_{i\mathfrak{g}}$ of $K^C$ in $Z$. We say that $M$ is $G$-semistable if $M = \{ p \in M : G \cdot p \cap \mu_{i\mathfrak{g}}^{-1}(0) \neq \emptyset \}$.

**Theorem 3** Assume that $Z$ is compact and $U^C$-semistable and $M$ is a $G$-semistable real connected submanifold of $Z$. Assume also $M$ is contained in the zero fiber of $\mu_{i\mathfrak{g}}$. Then the norm-square of the $G$-gradient map $\| \mu_p \|^2$ is constant if and only if $G$ is semisimple and $M$ is $K$-equivariantly isometric to the product of a real flag and an embedded closed submanifold which is acted on trivially by $K$.

### 2 Closed Orbits and Gradient Map

Let $(Z, \omega)$ be a Kähler manifold. Assume that $U^C$ acts holomorphically on $Z$, that $U$ preserves $\omega$ and that there is a momentum map for the $U$ action on $Z$. Let $G \subset U^C$ be a closed compatible subgroup and let $M$ be a $G$-invariant submanifold of $(Z, \omega)$ and let $\mu_p : M \rightarrow \mathfrak{p}$ be the associated $G$-gradient map.

**Lemma 4** Let $x \in M$. Then:

- if $x$ realizes a local maximum of $v_p$, then $G \cdot x = K \cdot x$ and so it is compact;
- if $G \cdot x$ is compact, then $G \cdot x = K \cdot x$ and $x$ is a critical point of $v_p$.

**Proof** If $x$ realizes a local maximum for $v_p$, then $v_p : G \cdot x \rightarrow \mathbb{R}$ has a local maximum at $x$. By Corollary 6.12, $p.21$ in Heinzner et al. (2008), it follows $G \cdot x = K \cdot x$.

Assume $G \cdot x$ is compact. Then $v_p : G \cdot x \rightarrow \mathbb{R}$ has a local maximum. Applying, again, Corollary 6.12 $p.21$ in Heinzner et al. (2008), we get $G \cdot x = K \cdot x$. We compute
the differential of \( \nu_p \) at \( x \). It is easy to check

\[
d\nu_p(v) = 2\langle (d\mu_p)_x(v), \mu_p(x) \rangle.
\]

Therefore, keeping in mind that \( \text{Ker}(d\mu_p)_x = (p \cdot x)^\perp \), see Heinzner and Schwarz (2007), it follows \( (d\nu_p)_x = 0 \) on \((p \cdot x)^\perp \). Since \( G \cdot x = K \cdot x \), it follows \( p \cdot x \subseteq \mathfrak{k} \cdot x \) and so, keeping in mind that \( \nu_p \) is \( K \)-invariant, \( (d\nu_p)_x = 0 \) on \( p \cdot x \) as well, proving \( x \) is a critical point of \( \nu_p \).

\[\square\]

**Lemma 5** Let \( x \in M \) be such that \( G \cdot x \) is compact. Let \( \beta = \mu_p(x) \). Then

\[\mathfrak{k} \cdot x = p \cdot x ^\perp \oplus \mathfrak{t}^\beta \cdot x.\]

Therefore \( \mathfrak{k} \cdot x = p \cdot x \) if and only if \( \dim K \cdot x = \dim K \cdot \beta \).

**Proof** Since \( G \cdot x \) is compact, by the above Lemma \( G \cdot x = K \cdot x \). By the \( K \)-equivariance of \( \mu_p \), it follows that \( \mu_p : K \cdot x \longrightarrow K \cdot \beta \) is a smooth fibration. Therefore, keeping in mind that \( \text{Ker}(d\mu_p)_x = (p \cdot x)^\perp \), we have

\[(p \cdot x)^\perp \cap \mathfrak{k} \cdot x = \mathfrak{t}^\beta \cdot x.\]

Since \( G \cdot x = K \cdot x \), we get

\[\mathfrak{k} \cdot x = p \cdot x ^\perp \oplus ((p \cdot x)^\perp \cap \mathfrak{k} \cdot x) = p \cdot x ^\perp \oplus \mathfrak{t}^\beta \cdot x.\]

This also implies \( \mathfrak{k} \cdot x = p \cdot x \) if and only if \( \dim K \cdot x = \dim K \cdot \beta \), concluding the proof.

\[\square\]

Assume that \( G \) is a real form of \( U^C \). If \( G \cdot x \) is compact then it is not in general true that \( U^C \cdot x \) is compact. Indeed, let \( V \) be a complex vector space and let \( \tau : G \longrightarrow \text{PGL}(V) \) be an irreducible faithful projective representation. Since the center of \( G \) acts trivially, we may assume that \( G \) is semisimple. The representation \( \tau \) extends to an irreducible projective representation of \( U^C \). It is well-known that \( U^C \) has a unique closed orbit (Guillemin and Sternberg 1990). It is the orbit through a highest vector. On the other hand \( G \) could have more than one closed orbit in \( \text{P}(V) \) Guivarc’h et al. (1998, Proposition 4.28, p. 58). The following result tells us that there exists a unique compact \( G \)-orbit contained in the unique compact orbit of \( U^C \).

**Proposition 6** Let \( M = U^C \cdot x \) be a compact orbit. If \( G \) is a real form of \( U \), then there exists exactly one closed \( G \)-orbit in \( M \).

**Proof** \( U^C \cdot x = U \cdot x \) and it is a flag manifold (Heinzner et al. 2008; Guillemin and Sternberg 1990). Applying a beautiful old Theorem of Wolf (1969), it follows that \( G \) has a unique closed orbit in \( M \). The \( G \) orbit is given by the orbit throughout the maximum of the norm square of the gradient map (Heinzner et al. 2008).

\[\square\]

The following result arises from Lemma 5.
Corollary 7 Let $x \in M$ be such that $G \cdot x$ is compact. If $\dim K \cdot x = \dim K \cdot \mu_p(x)$, then $U^C \cdot x$ is compact.

Proof Since $u = \ell \oplus ip$, it follows $u \cdot x = \ell \cdot x + ip \cdot x$. By Lemma 5, $\ell \cdot x = p \cdot x$ and so $u^C \cdot x = u \cdot x$. This implies $U \cdot x$ is open and closed in $U^C \cdot x$. Therefore $U^C \cdot x = U \cdot x$, concluding the proof.

The following result gives a necessary and sufficient condition so that $U^C \cdot x$ is compact whenever $G \cdot x$ is.

Proposition 8 Let $x \in M$ be such that $G \cdot x$ is compact. If $G$ is a real form of $U^C$, then $U^C \cdot x$ is compact if and only if $i\ell u_{p(x)} \cdot x \subseteq u \cdot x \cap i(\mathfrak{p} \cdot x)^{\perp}$. If $M$ is Lagrangian, then $U^C \cdot x$ is compact if and only if $\mu_p : K \cdot x \to K \cdot \mu_p(x)$ is a covering map. Moreover, $G \cdot x$ is a Lagrangian submanifold of $U^C \cdot x$.

Proof Set $\beta = \mu_p(x)$. By Lemma 5, $\ell \cdot x = \mathfrak{p} \cdot x \perp \ell^\beta \cdot x$. Therefore, keeping in mind $u = \ell \oplus ip$, we have

$$u \cdot x = \mathfrak{p} \cdot x \perp \ell^\beta \cdot x + ip \cdot x.$$ 

Since $i\ell \ell^\beta \cdot x$ is orthogonal to $ip \cdot x$, it follows that $u \cdot x = u^C \cdot x$, if and only if $i\ell \ell^\beta \cdot x \subseteq u \cdot x \cap i(\mathfrak{p} \cdot x)^{\perp}$. If $M$ is Lagrangian, then $T_x Z = T_x M \perp J(T_x M)$. Therefore

$$u \cdot x = \mathfrak{p} \cdot x \perp \ell^\beta \cdot x \perp ip \cdot x.$$ 

This implies $u \cdot x = u^C \cdot x$ if and only if $i\ell \ell^\beta \cdot x \subseteq ip \cdot x$. By the first part of the proof we get $U^C \cdot x$ is compact if and only if $i\ell \ell^\beta \cdot x = \{0\}$ and so if and only if $\dim K \cdot x = \dim K \cdot \beta$. In particular $\mathfrak{p} \cdot x = \ell \cdot x$. This implies $\dim_{\mathbb{R}} G \cdot x = \dim_{\mathbb{C}} U^C \cdot x$ and so $G \cdot x$ is a compact Lagrangian submanifold of $U^C \cdot x$.

Proposition 9 Let $M$ be a $G$-invariant Lagrangian submanifold of $(Z, \omega)$. Let $x \in M$. Then $U^C \cdot x$ is compact if and only if $\ell \cdot x = \mathfrak{p} \cdot x$. In particular $G \cdot x$ is compact and it is a Lagrangian submanifold of $U^C \cdot x$.

Proof Since $M$ is Lagrangian, we have

$$u \cdot x = \ell \cdot x \perp ip \cdot x.$$ 

Therefore $u \cdot x = u^C \cdot x$ if and only if $i\ell \cdot x \subseteq ip \cdot x$ and $\mathfrak{p} \cdot x \subseteq \ell \cdot x$ hence if and only if $\ell \cdot x = \mathfrak{p} \cdot x$. This also implies $G \cdot x$ is compact, $\dim_{\mathbb{R}} G \cdot x = \dim_{\mathbb{C}} U^C \cdot x$ and so $G \cdot x$ is a compact Lagrangian submanifold of $U^C \cdot x$.

Proposition 10 Let $x \in Z$. Assume that both $G \cdot x$ and $U^C \cdot x$ are compact. Then $\dim_{\mathbb{R}} U^C \cdot x \leq 2 \dim G \cdot x$. If the equality holds then $G \cdot x$ is totally real.

Proof By Lemma 4 $U^C \cdot x = U \cdot x$ and $G \cdot x = K \cdot x$. Since $u \cdot x = \ell \cdot x + ip \cdot x$ and $\mathfrak{p} \cdot x \subseteq \ell \cdot x$, it follows that

$$\dim_{\mathbb{R}} U^C \cdot x \leq 2 \dim G \cdot x.$$
Note also that $t^C \cdot x = u^C \cdot x$. This implies $K^C \cdot x$ is open in $U^C \cdot x$. This remark is not new, see Heinzner et al. (2008), Heinzner and Stötzel (2006), and it arises from the Matsuki duality (Matsuki 1982). Finally, $2 \dim G \cdot x = \dim_{\mathbb{R}} U^C \cdot x$ if and only if $t \cdot x = p \cdot x$ and $u \cdot x = t \cdot x \oplus i p \cdot x$. In particular $G \cdot x$ is totally real in $U^C \cdot x$. □

The momentum map of $U$ on $Z$ induces a gradient map $\mu_{i \mathfrak{k}}$ of $K^C$ in $Z$. Assume that $M$ is contained in the zero fiber of $\mu_{i \mathfrak{k}}$.

**Proposition 11** Let $x \in M$. If $U^C \cdot x$ is compact, then $G \cdot x$ is compact.

**Proof** Let $y \in U^C \cdot x$. Since $\mu = \mu_{i \mathfrak{k}} + \mu_p$, $M \subseteq \mu_{i \mathfrak{k}}^{-1}(0)$ and keeping in mind that $U^C \cdot x = U \cdot x$, it follows that

$$\| \mu_p(y) \|^2 \leq \| \mu(y) \|^2 = \| \mu(x) \|^2 = \| \mu_p(x) \|^2.$$

Hence $v_p : U^C \cdot x \rightarrow \mathbb{R}$ achieves its maximum in $x$. By Lemma 4, $G \cdot x$ is compact.

We say that $M$ is $G$-semistable if $M = \{ p \in M : \overline{U^C \cdot p \cap \mu_p^{-1}(0)} \neq \emptyset \}$. In the papers (Heinzner and Schwarz 2007; Heinzner et al. 2008), the authors proved if $M$ is $G$-semistable then $G \cdot x$ is closed if and only if $G \cdot x \cap \mu_p^{-1}(0) \neq \emptyset$. As an application we get the following result.

**Proposition 12** Assume that $(Z, \omega)$ is $U^C$-semistable and $M$ is $G$-semistable, it is contained in the zero fiber of $\mu_{i \mathfrak{k}}$. Let $x \in M$. If $G \cdot x$ is closed, then $U^C \cdot x$ is closed as well.

**Proof** If $G \cdot x$ is closed then $G \cdot x \cap \mu_p^{-1}(0) \neq \emptyset$. Since $\mu_p^{-1}(0) \cap M = \mu^{-1}(0) \cap M$, it follows that $U^C \cdot x \cap \mu^{-1}(0) \neq \emptyset$ as well and so $U^C \cdot x$ is closed. □

A corollary we prove a well-known result of Birkes (1971), see also Heinzner and Stötzel (1962).

**Corollary 13** Let $G$ be a real form of $U$. Let $V$ be complex vector space and $W$ be real subspace of $V$ such that $V = W^C$. Assume that $G$ acts on $W$. Let $w \in W$. Then $G \cdot w$ is closed if and only if $U^C \cdot w$ is closed.

**Proof** It is well-known that $V$, respectively $W$, is $U^C$-semistable, respectively $G$-semistable (Richardson and Slodowoy 1990), see also Biliotti (2021). Since $W$ is a Lagrangian subspace of $V$, applying the above Proposition it follows that $G \cdot x$ closed implies $U^C \cdot x$ is closed as well. Vice-versa, assume that $U^C \cdot x$ is closed. The $G$-action on $V$ is holomorphic and $W$ is a real form of $V$. Since $G$ is a real form of $U^C$ the dimension of any $G$-orbit in $U^C \cdot x \cap W$ coincides with the real dimension of $U^C \cdot x$ (Borel and Harish-Chandra 1962, Proposition 2.3). Then all these orbits, and so $G \cdot x$, are closed. □
3 Norm Square of the Gradient Map

We investigate splitting results for $G$-invariant real submanifolds of $(Z, \omega)$.

**Proposition 14** Let $M$ be a $G$-stable connected submanifold of $Z$ and let $\mu_p : M \rightarrow p$ be the restricted gradient map. Then the norm-square of the gradient map $\nu_p : M \rightarrow \mathbb{R}$ is constant if and only if any $G$-orbit is compact.

**Proof** Assume $\nu_p$ is constant. Let $x \in M$. Then $\nu_p : G \cdot x \rightarrow \mathbb{R}$ is constant and so $\nu_p$ has a maximum on $x$. By Lemma 4 $G \cdot x = K \cdot x$ and so it is compact. Vice-versa, assume that any $G$-orbit is compact. By Lemma 4 $(d\nu_p)_x = 0$ for any $x \in M$. Since $M$ is connected it follows $\nu_p$ is constant. $\square$

The following result is proved in Heinzner et al. (2008). For the sake of completeness we give a proof.

**Proposition 15** Let $M$ be a $G$-stable connected submanifold of $Z$ and let $\mu_p : M \rightarrow p$ be the restricted gradient map. If $\nu_p$ is constant, then $\mu_p(M) = K \cdot \beta$, $\mu^{-1}_p(\beta)$ is a submanifold and the following splitting

$$M = K \times_{K^\beta} \mu^{-1}_p(\beta),$$

holds.

**Proof** Since $\nu_p$ is constant, it follows that $M = S_\beta$, where $S_\beta$ is the maximal strata, and $\mu_p(S_\beta) = \mu_p(M) = K \cdot \beta$ (Heinzner et al. (2008, p. 21). In particular $M = K \mu^{-1}_p(\beta)$ and we may think $\mu_p : M \rightarrow K \cdot \beta$. Therefore $\beta$ is a regular value and so $\mu^{-1}_p(\beta)$ is a $K^\beta$-invariant submanifold of $M$.

Let $x \in \mu^{-1}_p(\beta)$. By the $K$-equivariance of $\mu_p$, it is easy to check $K \cdot x \cap \mu^{-1}_p(\beta) = K^\beta \cdot x$. We claim that the same holds infinitesimally, i.e., $T_x \mu^{-1}_p(\beta) \cap \mathfrak{k} \cdot x = \mathfrak{k}^\beta \cdot x$. Indeed, let $v \in T_x \mu^{-1}_p(\beta) \cap \mathfrak{k} \cdot x$. Let $\xi \in \mathfrak{k}$ such that $v = \xi_M(x)$. Since $T_x \mu^{-1}_p(\beta) = \ker (d\mu_p)_x$, we get

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mu_p(\exp(t\xi)x) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp(t\xi))\beta,$$

and so $v \in \mathfrak{k}^\beta \cdot x$.

We define the map

$$\Psi : K \times_{K^\beta} \mu^{-1}_p(\beta) \rightarrow M \quad [k, x] \mapsto kx.$$ 

It is easy to check that $\Psi$ is $K$-equivariant and smooth. Since $\mu_p(M) = K \cdot \beta$ it follows $M = K \cdot \mu^{-1}_p(\beta)$ and so $\Psi$ is surjective. It is also injective since $kx = k'x$ if and only if $k'^{-1}k \in K^\beta$, proving it is bijective. Now, we proof that $\Psi$ is a local diffeomorphism. This implies that $\Psi$ is a diffeomorphism concluding the proof. Note that it is enough to prove $d\Psi|_{[e, x]}$ is a diffeomorphism by the $K$-equivariance. Now,

$$T_x M = (p \cdot x) \oplus (p \cdot x)^\perp = (p \cdot x) \oplus T_x \mu^{-1}_p(\beta).$$
By Proposition 14 any $G$ orbit is a $K$ orbit. This implies $p \cdot x \subset \xi \cdot x$. Since $\xi^\beta \cdot x \subset (p \cdot x)^\perp$, it follows that the map

$$p \cdot x \leftrightarrow \xi \cdot x \longrightarrow \xi \cdot x/\xi^\beta \cdot x,$$

is injective. Therefore $d\Psi_{[\epsilon, x]}$ is surjective. Since $\Psi$ is bijective it follows that $d\Psi_{[\epsilon, x]}$ must be bijective. \hfill \Box

We are ready to prove the splitting results.

**Proof of Theorem 2** Since $v$ is constant, applying Proposition 14 it follows that any $U^\mathbb{C}$-orbit is compact and it is a complex $U$ orbit. Then for any $x \in M$, we have $U_x = U_{\mu(x)}$ (Guillemin and Sternberg 1990). Since $U_{\mu(x)}$ is a centralizer of a torus, then the center of $U$ does not act on $M$ and so $U$ is semisimple. By the above proposition $M = U/U^\beta \times \mu^{-1}(\beta)$ and for every $x \in \mu^{-1}(\beta)$, $U_x = U^\beta$ and so $U_x$ acts trivially on $\mu^{-1}(\beta)$. If $x \in \mu^{-1}(\beta)$, then

$$T_x M = (iu \cdot x) \oplus T_x \mu^{-1}(\beta) = T_x U \cdot x \oplus T_x \mu^{-1}(\beta).$$

This implies that the $U$-action on $M$ is polar with section $\mu^{-1}(\beta)$ (Dadok 1985) and so $\mu^{-1}(\beta)$ is totally geodesic. We claim that the above splitting is Riemannian.

Let $\xi \in u$ and let $\xi_M$ the induced vector field. It is enough to prove that the function $g(\xi_M, \xi_M)$ is constant when restricted to $\mu^{-1}(\beta)$.

Let $x \in \mu^{-1}(p)$ and $v \in T_x \mu^{-1}(p)$. We may extend $v$ to a vector field on a neighborhood of $p$, that we denote by $X$, such that $g(X, \xi_M) = 0$ for any $z \in W$ and for any $\xi \in u$. Indeed, let $\xi_1, \ldots, \xi_k \in u$ such that $(\xi_1)_M(x), \ldots, (\xi_k)_M(x)$ is a basis of $T_x U \cdot x$. Since the $U$ action on $M$ has only one type of orbit, it follows that there exists a neighborhood $W$ of $x$ such that $(\xi_1)_M(y), \ldots, (\xi_k)_M(y)$ is a basis of $T_y U \cdot y$ for any $y \in W$. Applying a Gram–Schmidt process we get an orthonormal basis $\{Y_1, \ldots, Y_k\}$ of $T_y U \cdot y$ for any $y \in W$. Let $\tilde{X}$ any local extension of $v$. Then

$$X = \tilde{X} - g(Y_1, \tilde{X})Y_1 - \cdots - g(Y_k, \tilde{X})Y_k,$$

satisfies the above conditions. Moreover, for any $z \in \mu^{-1}_p(\beta) \cap W$, the vector field $X$ lies in $T_z \mu^{-1}_p(\beta)$ due to the orthogonal splitting $T_z M = T_z U \cdot z \oplus T_z \mu^{-1}_p(\beta)$.

Let $v_M = -J(\xi_M)$ Then $J(v_M) = \xi_M$. Since $M = U/U^\beta \times \mu^{-1}(p)$, it follows $[X, \xi_M] = [X, v_M] = 0$ along $\mu^{-1}_p(\beta)$. By the closedness of $\omega$, we have

$$d\omega(v, v_M(x), \xi_M(x)) = 0.$$

On the other hand, by the expression of the differential (Kobayashi and Nomizu 1996), we have

$$d\omega(v, v_M(x), \xi_M(x)) = X \omega(v_M, \xi_M) + v_M \omega(\xi_M, X) + \xi_M \omega(X, v_M) - \omega([X, v_M], \xi_M) - \omega([v_M, \xi_M], X) - \omega([\xi_M, X], Y).$$

\[ Springer \]
Now, $\omega([X, v_M], \xi_M) = \omega([\xi_M, X], Y) = 0$ due to the fact that $[X, v_M](x) = [\xi_M, X](x) = 0$. The term $\omega([v_M, \xi_N], X) = 0$, since
$$\omega([v_M, \xi_N], X) = g(J([v_M, \xi_N], X) = 0$$
due to the facts that any $U$-orbit is complex and the splitting $T_x M = T_x \mu^{-1}(\beta) \oplus T_x U \cdot x$ holds. Finally, $v_M \omega(\xi_M, X) = 0$, respectively $\xi_M \omega(X, v_M) = 0$, due to the fact that
$$\omega(\xi_M, X) = g(J\xi_M, X) = 0,$$
respectively,
$$\omega(X, v_M) = g(JX, v_M) = -g(X, Jv_M) = 0,$$
along $U \cdot x$. Therefore
$$0 = d\omega(v, v_M(x), \xi_M(x)) = X\omega(v_M, \xi_M) = Xg(J(v_M), \xi_M) = Xg(\xi_M, \xi_M),$$
and so $g(\xi_M, \xi_M)$ is constant along $\mu^{-1}_p(\beta)$ and the result is proved.

**Proof of Theorem 3** By Proposition 15 $M = K \times K^\beta \mu^{-1}_p(\beta)$. By Proposition 14 it follows $G \cdot x$ is compact for any $x \in \mu^{-1}_p(\beta)$. Let $x \in \mu^{-1}_p(\beta)$. By Proposition 12, $U^C \cdot x$ is compact as well. Since $M$ is contained in the zero fiber of $\mu_t$, it follows that $\mu_p(x) = \mu(x) = \beta$. This implies $K_x = K \cap U_x = K \cap U^\beta = K^\beta$ for any $x \in \mu^{-1}_p(\beta)$ and so $M = K/K^\beta \times \mu^{-1}_p(\beta)$. The Lie algebra of the center of $G$ is contained in the Lie algebra of the center of $U^C$. On the other hand, the Lie algebra of the center of $U^C$ is the complexification of the Lie algebra of the center of $U$ which acts trivially on $M$. This implies $G$ is semisimple. Finally, keeping in mind that $\omega$ is closed and $U^C \cdot x$ is compact for any $x \in \mu^{-1}_p(\beta)$, applying the same idea of the above proof we get the splitting $M = K/K^\beta \times \mu^{-1}_p(\beta)$ is Riemannian.

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**References**

Bedulli, L., Gori, A.: A splitting result for compact symplectic manifolds. Results Math. 47, 194–198 (2005)
Biliotti, L.: A note on moment map on symplectic manifolds. Bull. Belg. Math. Soc. Simon Stevin 16, 107–116 (2009)
Biliotti, L.: The Kempf–Ness Theorem and invariant theory for real reductive representations. Sao Paulo J. Math. Sci. 15(1), 54–74 (2021)
Birkes, D.: Orbits of linear algebraic groups. Ann. Math. 93(2), 459–475 (1971)
Borel, A., Harish-Chandra: Arithmetic subgroups of algebraic groups. Ann. Math. 75(2), 485–535 (1962)
Dadok, J.: Polar coordinates induced by actions of compact Lie groups. Trans. Am. Math. Soc. 288(1), 125–137 (1985)
Gori, A., Podestà, F.: A note on the moment map on compact Kähler manifold. Ann. Global. Anal. Geom. 26, 315–318 (2004)
Guillemin, V., Sternberg, S.: Symplectic Techniques in Physics, 2nd edn. Cambridge University Press, Cambridge (1990)
Guivarc’h, Y., Ji, L., Taylor, J.C.: Compactifications of symmetric spaces. Progress in Mathematics, vol. 156. Birkhäuser Boston Inc., Boston (1998)
Heinzner, P., Schwarz, G.W.: Cartan decomposition of the moment map. Math. Ann. 337(1), 197–232 (2007)
Heinzner, P., Stötzel, H.: Critical Points of the Square of the Momentum Map, Global Aspects of Complex Geometry, pp. 211–226. Springer, Berlin (2006)
Heinzner, P., Schwarz, G.W., Stötzel, H.: Stratifications with respect to actions of real reductive groups. Compos. Math. 144(1), 163–185 (2008)
Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces, Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI (2001)
Kirwan, F.C.: Cohomology of Quotients in Symplectic and Algebraic Geometry. Mathematical Notes, vol. 31. Princeton University Press, Princeton (1984)
Knapp, A.W.: Lie Groups Beyond an Introduction. Progress in Mathematics, vol. 140, 2nd edn. Birkhäuser Boston Inc., Boston (2002)
Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. I. Wiley, New York (1996)
Matsuki, T.: Orbits on affine symmetric spaces under the action of parabolic subgroups. Hiroshima Math. J. 12, 307–320 (1982)
Richardson, R.W., Slodowy, P.J.: Minumum vectors for real reductive algebraic groups. J. Lond. Math. Soc. 42(2), 409–429 (1990)
Wolf, J.: The action of a real semisimple group on a complex flag manifold I. Orbit structure and holomorphic arc. Bull. Am. Math. Soc. 75, 1121–1237 (1969)

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