Numerical integration of stochastic contact Hamiltonian systems via stochastic Herglotz variational principle

Qingyi Zhan, Jinqiao Duan, Xiaofan Li and Yuhong Li

1 College of Computer and Information Science, Fujian Agriculture and Forestry University, Fuzhou, Fujian, 350002, People’s Republic of China
2 Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL, 60616, United States of America
3 Department of Mathematics and Department of Physics, Great Bay University, Dongguan, Guangdong, 523000, People’s Republic of China
4 Center for Mathematical Sciences, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, People’s Republic of China
5 School of Civil and Hydraulic Engineering, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, People’s Republic of China

Abstract

In this work, we establish a stochastic contact variational integrator and its discrete version via stochastic Herglotz variational principle for stochastic contact Hamiltonian systems. A general structure-preserving stochastic contact method is provided to seek the stochastic contact variational integrators. Numerical experiments are performed to verify the validity of this approach.

1. Introduction

Contact geometry was introduced in Sophus Lie’s study of differential equations. It has been the subject of intense research, especially related to low-dimensional topology [1–4]. In the last few years, stochastic contact Hamiltonian systems become interesting subject of research [5]. These stochastic differential equations (SDEs) are important in modelling natural phenomena, such as gravity, thermodynamics and dissipative systems [6]. Stochastic contact Hamiltonian systems constitute a rather important class of SDEs, whose contact structures are similar to symplectic structures in the odd dimension. Therefore, there is a demand for the investigation of numerical integrators to preserve the contact structures.

As the counterpart of contact case, much attention have been paid to stochastic symplectic methods [7–12]. As we know, structure-preserving algorithm has been widely applied in many aspects. The deterministic contact Hamiltonian systems have been studied recently. For example, [3] exploited the symplectification and the corresponding generating functions. However, so far [3] has not attracted much attention most likely due to its missing of a variational approach, which would make it hard to construct a numerical contact method [13–15] investigated variational integrators for stochastic dissipative Hamiltonian systems, which is the Hamiltonian systems in even dimension. That is, they are one class of dissipative Hamiltonian systems.

Fortunately, contact flows possess geometric integrators, which precisely parallel their symplectic counterparts. Contact variational integrators have been an important approach to creating contact methods [16]. They have relationship with stochastic Herglotz variational principle and its discrete version.

For stochastic contact Hamiltonian systems, the main difficulty in constructing stochastic contact variational integrators is the formulation of the stochastic Herglotz variational principle [17, 18].

In this paper, the generalized Herglotz variational principle is utilized to investigate the numerical contact integrators for stochastic contact Hamiltonian systems. It is motivated by two factors. First, as we know, the variational principle has been widely used to deal with fractal and fractional differential equations [19]. An associated variational principle for contact Hamiltonian systems and its contact methods are of great interest [16, 20]. Then it is natural to expect to expand it to the stochastic contact Hamiltonian systems. Second, many contributions have been made to the numerical analysis of SDEs [12, 21, 22]. The readers can find more...
information on numerical topic in these references. These are the foundations of our present work. However, to
the best of our knowledge, systematic construction of contact scheme of stochastic contact Hamiltonian systems
is still an open problem.

This work contributes to the efforts in improving the structure-preserved method for stochastic contact Hamiltonian systems. On one hand, there is a theoretical gap in this research field. For example, no literature on
the construction and numerical analysis of the structure-preserved method for this systems is presented up to
now. It is needed to fill in this gap. On the other hand, there is a practical demand to the structure-preserved
methods for those stochastic dissipative systems, which usually are needed to simulate the dynamical behaviour
in a long time interval.

Our results furnished a contact scheme for stochastic contact Hamiltonian systems in which the contact
structure is preserved a.s. In the numerical experiments, we compare the numerical dynamical behaviors of
contact scheme with non-contact scheme in several aspects, such as the orbits in a long time interval, the
preservations of contact structure and the conformal factor. For our purpose the numerical experiments are
realizable by programming.

The contents are arranged as follows. Section 2 deals with some preliminaries. In section 3 the theoretical
results on stochastic Herglotz variational principle are summarized. The discrete version of stochastic Herglotz
variational principle is proved. Illustrative numerical experiments are included in section 4. Finally, the last
section is addressed to summarize the conclusions of the paper.

2. Preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), and let \(W(t) = (W^1(t), W^2(t), \ldots, W^m(t))\) be
m-dimensional standard Wiener processes on this probability space. Here \(\Omega\) is a set, \(\omega \in \Omega\) is a sample function,
\(\mathcal{F}\) is a \(\sigma\)-algebra of subsets of \(\Omega\), \(\mathbb{P}\) is a probability measure, and \(\{\mathcal{F}_t\}_{t \geq 0}\) is the \(\sigma\)-algebra generated by the
random variables \(\{W^k(\tau)\}_{1 \leq k \leq m, 0 \leq \tau \leq t}\). We consider the following stochastic Hamiltonian systems in the sense of
Stratonovitch on a smooth \(d = (2n + 1)\)-dimensional contact manifold \(\mathbb{M}\),

\[
\begin{align*}
    dX(t) &= f(t, X(t))dt + \sum_{k=1}^m g_k(t, X(t))dW^k(t), \\
    X(t_0) &= x \in \mathbb{M},
\end{align*}
\]

where \(X, f(t, x^1, \ldots, x^d), g(t, x^1, \ldots, x^d)\) are \(d\)-dimensional column-vectors with the compo-
ents \(X^i, f^i, g^i, i = 1, \ldots, d\).

In Darboux coordinates \((q, p, s) = (q^1, q^2, \ldots, q^n, p^1, p^2, \ldots, p^n, s)\), a canonical stochastic contact Hamiltonian
system can be rewritten as

\[
\begin{align*}
    dq &= \frac{\partial H_0}{\partial p}dt + \sum_{k=1}^m \frac{\partial H_k}{\partial q}dW^k(t), \\
    dp &= -\left(\frac{\partial H_0}{\partial q} + p\frac{\partial H_0}{\partial s}\right)dt + \sum_{k=1}^m \left(\frac{\partial H_k}{\partial q} + p\frac{\partial H_k}{\partial s}\right)dW^k(t), \\
    ds &= \left(p\frac{\partial H_0}{\partial p} - H_0\right)dt + \sum_{k=1}^m \left(p\frac{\partial H_k}{\partial p} - H_k\right)dW^k(t).
\end{align*}
\]

with initial condition \((q(t_0), p(t_0), s(t_0)) = (q_0, p_0, s_0), t_0 \geq 0\), where \(H_0 = H_0(q, p, s)\) is a smooth Hamiltonian
function on \(\mathbb{M}\), and \(\{H_k\}_{k=1}^m = \{H_k(q, p, s)\}_{k=1}^m\) are a family of smooth functions on \(\mathbb{M}\). In fact, equations (2)
are the generalization of Hamilton's equation to a contact manifold. In particular, if \(\{H_k\}_{k=0}^m\) do not depend on \(s\),
equations (2) give a stochastic Hamiltonian equations in the symplectic phase space. Therefore, equations (2)
generate the equations of motion for the positions, the momenta and Hamilton's principal function of the
standard Hamilton's theory, and can include a large class of models, such as dissipative systems.

We introduce the following notations.

Let \(L^2(\Omega, \mathbb{P})\) be the space of all bounded square-integrable random variables \(X: \Omega \to \mathbb{R}^d\). For random
vector \(X = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d\), the norm of \(X\) is defined in the form of

\[
\|X\|_2 = \left[\int_{\Omega} [x^1(\omega)^2 + |x^2(\omega)|^2 + \ldots + |x^d(\omega)|^2]d\mathbb{P}\right]^{1/2} < \infty.
\]

We define the norm of random matrices as follows [23]

\[
\|G\|_2(\Omega, \mathbb{P}) = \left[\mathbb{E}(G^2)\right]^{1/2},
\]

where \(G\) is a random matrix and \(\| \cdot \|\) is the operator norm.

For simplicity, the norms \(\| \cdot \|_2\) and \(\| \cdot \|_2(\Omega, \mathbb{P})\) are usually written as \(\| \cdot \|\).
3. Theoretical results on stochastic Herglotz variational principle

3.1. Stochastic Herglotz variational principle

Variational principle is a powerful tool to study the dynamics of contact Hamiltonian systems, which is similar to symplectic Hamiltonian systems. This was originally created by Herglotz in 1930. Then there are many related works, such as [16]. It follows from [24] that the Lagrange equation can be written in the expanded variable set $(q, p, \dot{q}, \dot{p}, t)$ [11]. Therefore, we introduce the following definition.

**Definition 3.1.** (Deterministic Herglotz variational principle [16, 17, 25]) Let $\mathcal{Q}$ be an n-dimensional manifold with local coordinates $q^i$, $p^i$, and $T\mathcal{Q}$ be the tangent space of $\mathcal{Q}$. Consider a continuous Lagrangian $L: \mathbb{R} \times T\mathcal{Q} \times \mathbb{R} \times T\mathcal{Q} \times \mathbb{R} \to \mathbb{R}$. For any given curves $q, p: [0, T] \to \mathcal{Q}$, which connect $q(0) = q_0$, $p(0) = p_0$ and $q(T) = q_N$, $p(T) = p_N$, with $\delta q(0) = \delta q(T) = \delta p(0) = \delta p(T) = 0$, satisfying the following deterministic contact Hamiltonian system,

$$
\begin{align*}
\frac{dq}{dt} &= \frac{\partial H_0}{\partial p}, \\
\frac{dp}{dt} &= -\left(\frac{\partial H_0}{\partial q} + p \frac{\partial H_0}{\partial s}\right), \\
\frac{ds}{dt} &= \left(p \frac{\partial H_0}{\partial p} - H_0\right),
\end{align*}
$$

where the contact Hamiltonian $H_0$ is given by Legendre transformation [25]

$$
H_0(q, p, s) = p\dot{q} - L(t, q(t), \dot{q}(t), p(t), \dot{p}(t), s).
$$

Then the initial value problem is given as follows

$$
\dot{s} = L(t, q(t), \dot{q}(t), p(t), \dot{p}(t), s).
$$

Therefore, the value $s(T)$ is called the action functional of the curves $q(t)$ and $p(t)$. The curves $q(t)$ and $p(t)$ are called critical if and only if $s(T)$ is invariant under infinitesimal variations of $q$ and $p$ that vanish at the boundary of $[0, T]$, where the notation $s$ is the same as in the contact Hamiltonian system, that is, the scalar quantity $s$ in the contact Hamiltonian system can be regarded as the action functional.

It is natural to generalize it to the stochastic case [2, 11, 17].

**Definition 3.2.** (Stochastic Herglotz variational principle) For any given curves $q, p: [0, T] \to \mathcal{Q}$, with $q(0) = q_0$, $p(0) = p_0$ and $p(T) = p_N$, $q(T) = q_N$, satisfying the stochastic contact Hamiltonian system (2), the initial value problem is given in the form of

$$
\dot{s} = L(t, q(t), \dot{q}(t), p(t), \dot{p}(t), s) - \sum_{k=1}^{m} H_k \circ W^k, \quad s(0) = s_0,
$$

where $\dot{W}^k dt = dW^k$. Therefore, the curves $q(t)$ and $p(t)$ are called critical if and only if $s(T)$ is invariant under infinitesimal variations of $q$ and $p$ that vanish at the boundary of $[0, T]$.

**Theorem 3.3.** If the curves $q(t)$ and $p(t)$ are solutions to the following stochastic Euler–Lagrange equations,

$$
\begin{align*}
\frac{\partial L}{\partial q} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial q} \circ W^k - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \left(\frac{\partial L}{\partial s} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial s} \circ W^k\right) \frac{\partial L}{\partial \dot{s}} = 0, \\
\frac{\partial L}{\partial p} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial p} \circ W^k - \frac{d}{dt} \frac{\partial L}{\partial \dot{p}} + \left(\frac{\partial L}{\partial s} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial s} \circ W^k\right) \frac{\partial L}{\partial \dot{s}} = 0,
\end{align*}
$$

then the action functional $s: [0, T] \to \mathbb{R}$ with an initial condition $s(0) = s_0$ can be minimized by this curves $q(t)$ and $p(t)$.

**Proof.** According to (5), we obtain the variation of $\dot{s}$ in the form of

$$
\delta \dot{s} = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial p} \delta \dot{p} + \frac{\partial L}{\partial \dot{s}} \delta \dot{s} - \sum_{k=1}^{m} \left(\frac{\partial H_k}{\partial q} \delta q + \frac{\partial H_k}{\partial p} \delta \dot{p} + \frac{\partial H_k}{\partial s} \delta \dot{s}\right) \circ W^k.
$$

Here we set

$$
A(t) = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial q} \delta q \circ W^k,
$$

with $\delta q, \delta \dot{q}, \delta \dot{p}, \delta \dot{s}$ being the infinitesimal variations of $q, \dot{q}, \dot{p}, \dot{s}$, respectively.
\[ B(t) = \frac{\partial L}{\partial p} \dot{\delta p} + \frac{\partial L}{\partial \dot{p}} \delta \dot{p} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial \dot{p}} \delta \omega W^k, \]

and

\[ C(t) = \int_{0}^{t} \left( \frac{\partial L}{\partial \dot{q}} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial \dot{q}} \delta \omega W^k \right) dt. \]

Then we obtain the differential equation as follows

\[ \dot{\delta s}(t) = A(t) + B(t) + \frac{dC(t)}{dt} \delta s, \]

whose solution is

\[ \delta s(t) = \exp(C(t)) \left[ \int_{0}^{t} (A(\tau) + B(\tau)) \exp(-C(\tau)) d\tau + \delta s(0) \right]. \]

Utilizing the integration by parts and the expression of \( A(t), B(t) \) and \( C(t) \), we have

\[
\begin{align*}
\delta s(T) &= \exp(C(t)) \cdot \left[ \int_{0}^{T} \left( \frac{\partial L}{\partial q} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial q} \delta \omega W^k - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}} \delta \dot{p} \right) \right) \exp(-C(\tau)) \delta q d\tau \\
&\quad + \frac{\partial L}{\partial \dot{q}} (T) \exp(-C(T)) \delta q(T) - \frac{\partial L}{\partial \dot{q}} (0) \delta q(0) \\
&\quad + \int_{0}^{T} \left( \frac{\partial L}{\partial p} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial p} \delta \omega W^k - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}} \delta \dot{p} \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \right) \exp(-C(\tau)) \delta p d\tau \\
&\quad + \frac{\partial L}{\partial \dot{p}} (T) \exp(-C(T)) \delta p(T) - \frac{\partial L}{\partial \dot{p}} (0) \delta p(0) + \delta s(0) \right],
\end{align*}
\]

where

\[ \frac{dC}{dt} = \frac{\partial L}{\partial s} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial s} \delta \omega W^k. \]

Due to the boundary conditions \( \delta q(T) = \delta q(0) = \delta \dot{p}(T) = \delta \dot{p}(0) = 0 \), we obtain that the action \( s(T) \) is critical if and only if equation (6) holds.

The proof of theorem 3.3 is completed. \( \square \)

**Remark 3.4.** [11, 25–27] Here we present the relationship between the stochastic Herglotz variational principle and the Hamiltonian formulation of stochastic contact Hamiltonian systems. Let \( L \) be the Lagrangian function with respect to the deterministic part of the stochastic contact systems (2), and it is connected with the deterministic Hamiltonian function \( H_0 \) through the Legendre transformation

\[ L = p^T \dot{q} - H_0. \]

**Theorem 3.5.** The stochastic Lagrangian and Hamiltonian formalisms are equivalent. That is, the expression of \( L \) and stochastic Euler-Lagrangian equations (6) can be utilized to seek equations (2).

**Proof.** According to the explicit expression of \( L \), we have

\[ \frac{\partial L}{\partial q} = \frac{\partial H_0}{\partial \dot{q}}, \quad \frac{\partial L}{\partial \dot{q}} = \dot{p}, \quad \frac{\partial L}{\partial s} = - \frac{\partial H_0}{\partial \dot{s}}. \]

Then substituting these into the first equation of equations (6), we have

\[ -\frac{\partial H_0}{\partial q} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial q} \delta \omega W^k - \dot{\delta p} + \left( -\frac{\partial H_0}{\partial \dot{s}} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial \dot{s}} \delta \omega W^k \right) \delta \dot{s} = 0, \]

This is equivalent to the second equation of equations (2).

Similarly, we have

\[ \frac{\partial L}{\partial p} = \dot{q} - \frac{\partial H_0}{\partial \dot{p}}, \quad \frac{\partial L}{\partial \dot{p}} = 0. \]
Then applying these to the second equation of equations (6), we obtain
\[ \dot{q} - \frac{\partial H_0}{\partial p} - \sum_{k=1}^{m} \frac{\partial H_k}{\partial p} \circ W^k = 0, \]
which is equivalent to the first equation of equations (2).

Lastly, by equation (5), we have
\[ \dot{s} = p \dot{q} - H_0 - \sum_{k=1}^{m} H_k \circ W^k. \]

Applying the result of \( \dot{q} \), we get
\[ \begin{bmatrix} \dot{s} \\ \dot{p} \end{bmatrix} = -\begin{bmatrix} \frac{\partial H_0}{\partial p} + \sum_{k=1}^{m} \frac{\partial H_k}{\partial p} \circ W^k \end{bmatrix} - H_0 - \sum_{k=1}^{m} H_k \circ W^k. \]

It is easy to check that this is equivalent to the third equation of equations (2).

We have proved theorem 3.5.

3.2. Discrete stochastic Herglotz variational principle

In view of the stochastic Herglotz variational principle and stochastic Euler–Lagrange equations (6), we will investigate the construction of the contact methods for the stochastic contact Hamiltonian systems. We first present the following definition, which is the discrete version of stochastic Herglotz variational principle.

**Definition 3.6.** Let \( \mathcal{Q} \) be an n-dimensional manifold with local coordinates \( q^i, p^i \), and let \( L : \mathcal{Q}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) with \( h > 0 \). For any given discrete curves \( q := (q_0, q_1, \ldots, q_N) \in \mathcal{Q}^{N+1} \) and \( p := (p_0, p_1, \ldots, p_N) \in \mathcal{Q}^{N+1} \), we define the curve \( s := (s_0, s_1, \ldots, s_N) \in \mathbb{R}^{N+1} \) with \( s_0 = 0 \) and
\[ s_{j+1} - s_j = hL_j - \sum_{k=1}^{m} H_k^{(j)} \circ \Delta W^k_j, \text{ almost surely}, \]
where
\[ L_j := L(q_j, q_{j+1}, p_j, p_{j+1}, s_j, s_{j+1}), \quad H_k^{(j)} := H_k(q_j, q_{j+1}, p_j, p_{j+1}, s_j, s_{j+1}), \]
and
\[ \Delta W^k_j = W^k(t_{j+1}) - W^k(t_j). \]

Then the curve \( s_N \) is a functional of the discrete curves \( q \) and \( p \). The discrete curves \( q \) and \( p \) are the solutions to the discrete stochastic Herglotz variational principle if and only if
\[ \frac{\partial s_j}{\partial q_j} = 0, \quad \forall j \in \{1, 2, \ldots, N - 1\}, \text{ almost surely}. \]

**Definition 3.7.** If the numerical solutions \( \{(q_j, p_j, s_j)_{j=0}^{N}\} \) of the stochastic contact Hamiltonian system (2) are obtained by a discrete scheme and satisfy the following equality,
\[ d_{j+1} - p_{j+1} d_{j+1} = \lambda_j (d_j - p_j d), \quad j = 0, 1, 2, \ldots, N - 1, \text{ almost surely}, \]
this discrete scheme is called contact scheme, where \( \lambda_j \) is the conformal factor with \( \lambda_0 = 1 \).

Now we will show the fact that the discrete scheme obtained by theorem 3.8 is contact one. That is, this scheme preserves contact structure in the case of discrete time.

**Theorem 3.8.** Let
\[ p^j = \frac{D^{j-1}}{1 - E^{j-1}}, \quad p^j = -\frac{D^j}{1 + E^j}, \]
where
\[ \begin{cases} D^{j-1} = h \frac{\partial L_{j-1}}{\partial q_j} - \sum_{k=1}^{m} \frac{\partial H_k^{j-1}}{\partial q_j} \circ \Delta W^k_{j-1}, \\ E^{j-1} = h \frac{\partial L_{j-1}}{\partial s_j} - \sum_{k=1}^{m} \frac{\partial H_k^{j-1}}{\partial s_j} \circ \Delta W^k_{j-1}, \end{cases} \]
and

\[
D^j = h \frac{\partial L_j}{\partial q_j} - \sum_{k=1}^m \frac{\partial H^j_k}{\partial q_j} \circ \Delta W^k_j,
\]

\[
E^j = h \frac{\partial L_j}{\partial s_j} - \sum_{k=1}^m \frac{\partial H^j_k}{\partial s_j} \circ \Delta W^k_j.
\]

Then the solutions to the discrete generalized Herglotz variational principle are obtained by

\[
P_j = P_j^+.
\]

And the map \((q_j, P_j, s_j) \rightarrow (q_{j+1}, P_{j+1}, s_{j+1})\) induced by a critical discrete curve preserves the contact structure, i.e.,

\[
ds_{j+1} - P_{j+1} dq_{j+1} = \lambda_j (ds_j - P_j dq_j), j = 0, 1, \ldots, N - 1, \text{ almost surely},
\]

where the conformal factor \(\lambda_j\) is

\[
\lambda_j = \frac{1 + h \frac{\partial L_j}{\partial q_j} - \sum_{k=1}^m \frac{\partial H^j_k}{\partial q_j} \circ \Delta W^k_j}{1 - h \frac{\partial L_j}{\partial s_j} - \sum_{k=1}^m \frac{\partial H^j_k}{\partial s_j} \circ \Delta W^k_j}.
\]

**Proof.** First, we will prove the discrete stochastic Herglotz variational principle. It follows from equation (7) that

\[
\frac{\partial s_{j+1}}{\partial q_j} = \frac{\partial s_j}{\partial q_j} + h \frac{\partial L_j}{\partial q_j} \frac{\partial q_j}{\partial s_j} + h \frac{\partial L_j}{\partial s_j} \frac{\partial s_j}{\partial q_j} + \sum_{k=1}^m \frac{\partial H^j_k}{\partial q_j} \circ \Delta W^k_j \frac{\partial s_j}{\partial q_j} + \sum_{k=1}^m \frac{\partial H^j_k}{\partial s_j} \circ \Delta W^k_j \frac{\partial s_j}{\partial q_j}
\]

Then we have

\[
\left[ 1 - h \frac{\partial L_j}{\partial s_j} - \sum_{k=1}^m \frac{\partial H^j_k}{\partial s_j} \circ \Delta W^k_j \right] \frac{\partial s_j}{\partial q_j} = h \frac{\partial L_j}{\partial q_j} - \sum_{k=1}^m \frac{\partial H^j_k}{\partial q_j} \circ \Delta W^k_j.
\]

It also follows from (7) that we have

\[
s_j - s_{j-1} = h L_{j-1} - \sum_{k=1}^m H_{j-1}^k \circ \Delta W^k_{j-1},
\]

where

\[
L_{j-1} = L(q_{j-1}, q_j, P_{j-1}, P_j, s_{j-1}, s_j), H_{j-1}^k = H_k(q_{j-1}, q_j, P_{j-1}, P_j, s_{j-1}, s_j),
\]

and

\[
\Delta W^k_{j-1} = W^k(t_j) - W^k(t_{j-1}).
\]

So according to (13) we get

\[
\frac{\partial s_j}{\partial q_j} = h \frac{\partial L_{j-1}}{\partial q_j} + h \frac{\partial L_{j-1}}{\partial s_j} \frac{\partial s_j}{\partial q_j} - \sum_{k=1}^m \frac{\partial H_{j-1}^k}{\partial q_j} \circ \Delta W^k_{j-1} - \sum_{k=1}^m \frac{\partial H_{j-1}^k}{\partial s_j} \circ \Delta W^k_{j-1} \frac{\partial s_j}{\partial q_j}.
\]

Then it follows from (14) that

\[
\frac{\partial s_j}{\partial q_j} = h \frac{\partial L_{j-1}}{\partial q_j} - \sum_{k=1}^m \frac{\partial H_{j-1}^k}{\partial q_j} \circ \Delta W^k_{j-1}.
\]
Substituting (15) into (12) gives

\[
1 - \frac{\partial L_j}{\partial s_{j+1}} - m \frac{\partial H_j^i}{\partial s_{j+1}} \frac{\partial \Delta W_j^k}{\partial s_{j+1}} \right] ds_j + 1 = h \frac{\partial L_j}{\partial q_j} + m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j}
\]

which is the same as

\[
1 + h \frac{\partial L_j}{\partial s_j} - m \frac{\partial H_j^i}{\partial s_j} \frac{\partial \Delta W_j^k}{\partial s_j}
\]

Due to (9) and (10), it is obvious that solutions to the discrete generalized Herglotz variational principle are obtained by

\[ p_j = p_j^- = p_j^+ \]

if and only if the following equality holds

\[ \frac{\partial s_{j+1}}{\partial q_j} = 0. \]

Second, we will prove the preserve of the contact structure. Consider the relation of (7), we obtain

\[
ds_{j+1} - ds_j = h \frac{\partial L_j}{\partial q_j} dq_j + h \frac{\partial L_j}{\partial q_{j+1}} dq_{j+1} + h \frac{\partial L_j}{\partial s_j} ds_j + h \frac{\partial L_j}{\partial s_{j+1}} ds_{j+1}
\]

\[- m \sum_{k=1}^{m} \left( \frac{\partial H_j^i}{\partial q_j} dq_j + \frac{\partial H_j^i}{\partial q_{j+1}} dq_{j+1} + \frac{\partial H_j^i}{\partial s_j} ds_j + \frac{\partial H_j^i}{\partial s_{j+1}} ds_{j+1} \right) \frac{\partial \Delta W_j^k}{\partial q_j} ds_j.
\]

Motivated by (9), (10) and (10), we have

\[
1 - h \frac{\partial L_j}{\partial s_{j+1}} - m \frac{\partial H_j^i}{\partial s_{j+1}} \frac{\partial \Delta W_j^k}{\partial s_{j+1}} \right] ds_j + 1 = h \frac{\partial L_j}{\partial q_j} + m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j}
\]

\[
1 + h \frac{\partial L_j}{\partial s_j} - m \frac{\partial H_j^i}{\partial s_j} \frac{\partial \Delta W_j^k}{\partial s_j}
\]

\[
1 - h \frac{\partial L_j}{\partial s_{j+1}} - m \frac{\partial H_j^i}{\partial s_{j+1}} \frac{\partial \Delta W_j^k}{\partial s_{j+1}} \right] ds_j + 1 - h \frac{\partial L_j}{\partial q_j} + m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j} ds_j.
\]

Therefore, we obtain that

\[
ds_{j+1} = \frac{h \frac{\partial L_j}{\partial q_j} + m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j} \right] dq_j + 1 = h \frac{\partial L_j}{\partial s_{j+1}} - m \frac{\partial H_j^i}{\partial s_{j+1}} \frac{\partial \Delta W_j^k}{\partial s_{j+1}} \right] dq_j + 1 - h \frac{\partial L_j}{\partial q_j} - m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j} ds_j.
\]

Together with (9)–(10), equation (10) implies

\[ ds_{j+1} - p_j^+ dq_j^+ = \lambda_j (ds_j - p_j dq_j).
\]

Here the conformal factor \( \lambda_j \) is

\[ \lambda_j = \frac{1 + h \frac{\partial L_j}{\partial q_j} - m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j}}{1 - h \frac{\partial L_j}{\partial q_j} - m \frac{\partial H_j^i}{\partial q_j} \frac{\partial \Delta W_j^k}{\partial q_j}} \]

which is the same as (10). The proof of theorem 3.8 is finished.

\[ \square \]
Remark 3.9. By Taylor expansion, it is clear that

$$\lambda_j = 1 + \left[ \frac{\partial L}{\partial s_j} - m \frac{\partial H^j_k}{\partial s_j} \Delta W^j_k \right] + o(h^2)$$

Therefore, the conformal factor \( \lambda_j \) is consistent with \( \lambda(t_j) \) in the continuous case [5]

$$\lambda(t_j) = \exp \left[ \int_{t_j}^{t_{j+1}} \frac{\partial L}{\partial s} \, dt - \sum_{k=1}^{m} \int_{t_j}^{t_{j+1}} \frac{\partial H^k_j}{\partial s} \, dW^k_j \right].$$

4. Numerical experiments

We apply theorem 3.8 to construct contact schemes via stochastic Herglotz variational principle for several stochastic contact Hamiltonian systems. These resulted schemes illustrate the validity of stochastic Herglotz variational principle and the effectiveness of the numerical experiments.

4.1. Example 1

We consider the following stochastic damped mechanical system driven by additive Gaussian noise,

$$\begin{aligned}
 dq &= \rho dt, \\
 dp &= (\alpha p + V(q)) dt, \\
 ds &= (p^2 - H_0) dt - \varepsilon dW(t),
\end{aligned}$$

where the coordinates \( q, p \) and \( s \) are one dimension. It is easy to generalize to the high-dimensional case. The constant \( \alpha \) is positive, \( V(q) \) is a potential function, and the Hamiltonians are of the form

$$H_0 = \frac{1}{2} p^2 + V(q) + \alpha s, \quad H_1 = \varepsilon.$$

Obviously, it is a special linear stochastic contact Hamiltonian systems with additive Gaussian noise.

Motivated by [25–27], we can obtain the Lagrangian function \( L \) with respect to the deterministic part of (10) is

$$L = pq - H_0 = \frac{1}{2} q^2 - V(q) - \alpha s,$$

where \( \dot{q} = \frac{dq}{dt} = p \).

One of the discretizations of the Lagrangian \( L \) is

$$L_j = L(q_j, q_{j+1}; p_j, p_{j+1}; s_j, s_{j+1})$$

$$= \frac{1}{2} \left( \frac{q_{j+1} - q_j}{h} \right)^2 - \frac{V(q_j) + V(q_{j+1})}{2} - \alpha s_j,$$

where the midpoint quadrature is applied to approximate the differential, \( \dot{q} \approx \frac{q_{j+1} - q_j}{h} \), with \( \Delta W_j = W(t_{j+1}) - W(t_j) \), and \( \Delta t_j = t_{j+1} - t_j, j = 0, 1, \ldots, N \).

Substituting the expression of (10) into the relations (10) and (10), we get

$$p_j = \frac{\partial L_{j+1}}{\partial q_j} - \frac{\partial H^{j+1}_j}{\partial q_j} \Delta W_{j+1} = \frac{q_j - q_{j+1}}{h} - \frac{V'(q_j)}{2} h,$$

and

$$p_{j-1} = \frac{\partial L_{j-1}}{\partial q_{j-1}} - \frac{\partial H^{j-1}_{j-1}}{\partial q_{j-1}} \Delta W_{j-1} = \frac{q_j - q_{j-1}}{h} + \frac{V'(q_{j-1})}{2} h.$$
Therefore, the contact scheme of (10) can be obtained explicitly,
\[
\begin{align*}
q_j &= q_{j-1} + h(1 - h\alpha)p_{j-1} - \frac{h^2}{2}V'(q_{j-1}), \\
p_j &= (1 - h\alpha)p_{j-1} - \frac{h}{2}[V'(q_{j-1}) + V'(q_j)], \\
s_j &= s_{j-1} + \frac{1}{2h}(q_j - q_{j-1})^2 - \frac{h}{2}[V(q_j) + V(q_{j-1})] - \alpha hs_{j-1} - \varepsilon\Delta W_{j-1}.
\end{align*}
\] (20)

The next part is devoted to two aspects. One is the preservation of contact structure of the scheme (10), whose conformal factor is \(\lambda_j = \exp(-\alpha\Delta t_j)\) in the continuous time case. The other is the comparison with Euler-Maruyama scheme, i.e., non-contact scheme. The Euler-Maruyama scheme of (10) is
\[
\begin{align*}
q_j &= q_{j-1} + hp_{j-1} \\
p_j &= p_{j-1} - h[V'(q_{j-1}) + \alpha p_{j-1}], \\
s_j &= s_{j-1} + \frac{1}{2}h(p_{j-1})^2 - hV(q_{j-1}) - \alpha hs_{j-1} - \varepsilon\Delta W_{j-1}.
\end{align*}
\] (21)

To investigate the difference between the contact and non-contact scheme, we can compare long time behaviors of the numerical solutions obtained by the schemes (10) and (10). In order to improve the accuracy of the comparison, let \(V(q) = \frac{1}{2}q^2\). The initial conditions are selected as follows, \(h = 0.1, T = 200.0, \alpha = 0.1, \varepsilon = 0.02, N=2000.0\) and the initial value \((q(0), p(0), s(0)) = (0.75, -0.25, 0.08)\).

As shown in figure 1, the contact scheme has better performance than the non-contact scheme. That is, the dissipation phenomenon of SDE (10) is better simulated by contact scheme (10) than non-contact scheme (10). Figure 2 illustrates that the contact structure of (10) is almost surely preserved in the time interval \([0, 200]\). As we can see from figure 2, although there are some small size perturbations in the trajectory of contact structure in the time interval \([0, 40]\), it is stable in the time interval \([40, 200]\). That is to say, it is flat a.s. in the time interval...
Figure 3 shows the fact that the conformal factor is preserved well by the contact scheme. Due to continuous inputting of the Gaussian noise, the curve has some fluctuations in some uncertain time moments, but it almost lies near the straight line of the continuous case, whose value is $\lambda = \exp(-\omega h)$. Therefore, these phenomena verify the results of theorem 3.8.

4.2. Example 2

Now we illustrate the method of stochastic Herglotz variational principle in constructing contact schemes by another example. For a more general case, we consider the following stochastic damped mechanical systems driven by multiplicative Gaussian noise,
To illustrate the performances of contact scheme and Substituting the expression of According to section 18. The Hamiltonians are of the form

\[ H_0 = \frac{1}{2}p^2 + V(q) + \frac{1}{2}\alpha s^2, \quad H_1 = \sin q. \]

According to [25–27], the Lagrangian function \( L \) with respect to the deterministic part of (10) is

\[ L = p\dot{q} - H_0 = \frac{1}{2}q^2 - V(q) - \frac{1}{2}\alpha s^2. \]

We consider a discrete form of the Lagrangian as follows

\[ L_j = \frac{1}{2}\left(\frac{q_{j+1} - q_j}{h}\right)^2 - \frac{V(q_j) + V(q_{j+1})}{2} - \frac{1}{4}\alpha s_j^2 - \frac{1}{4}\alpha s_{j+1}^2. \]

Substituting the expression of (10) into the relations (10) and (10), we have

\[ p_j = \frac{q_j - q_{j-1}}{h} - \frac{V'(q_j)}{2h}, \]

and

\[ p_{j-1} = \frac{q_{j-1} - q_{j-2}}{h} + \frac{V'(q_{j-1})}{2h} + \cos(q_{j-2})\Delta W_{j-1}. \]

Therefore, an implicit contact scheme of (22) can be obtained,

\[
\begin{align*}
q_j &= q_{j-1} + h\left(1 - \frac{1}{2}h\alpha s_{j-1}\right)p_{j-1} - h\cos(q_{j-1})\cdot\Delta W_{j-1} - \frac{V'(q_{j-1})}{2}h^2, \\
p_j &= \frac{1}{2}\left(1 - \frac{1}{2}h\alpha s_{j-1}\right)p_{j-1} - \cos(q_{j-1})\cdot\Delta W_{j-1} \\\n&\quad - \frac{1}{2}h(V'(q_j) + V'(q_{j-1}))\left(1 + \frac{1}{2}h\alpha s_j\right), \\
s_j &= s_{j-1} + \frac{1}{2h}(q_j - q_{j-1})^2 - \frac{h}{2}[V(q_j) + V(q_{j-1})] \\
&\quad - \frac{1}{4}\alpha hs_{j-1} - \frac{1}{4}\alpha s_j^2 - \sin(q_{j-1})\cdot\Delta W_{j-1}. 
\end{align*}
\]

Now we devote to two aspects. One is the preservation of contact structure of the scheme (10), whose conformal factor is \( \lambda_j = \exp(-\alpha \Delta t_j) \) in the continuous case. The other is the comparison with Euler-Maruyama scheme. The Euler-Maruyama scheme of (10) is shown in the form of

\[
\begin{align*}
q_j &= q_{j-1} + hp_{j-1}, \\
p_j &= p_{j-1} - h\left(V'(q_{j-1}) + \alpha s_{j-1}\frac{q_j - q_{j-1}}{h}\right) - \cos(q_{j-1})\cdot\Delta W_{j-1}, \\
s_j &= s_{j-1} + h\left(\frac{q_j - q_{j-1}}{h}\right)^2 - V(q_{j-1}) - \frac{1}{2}\alpha s_{j-1}^2 - \sin(q_{j-1})\cdot\Delta W_{j-1}. 
\end{align*}
\]

To illustrate the performances of contact scheme (10), except for the parameters \( T \) and \( N \), the initial conditions are the same as those in section 4.1. In order to present the figures clearly and show more details, we only consider the case in the time interval \([0, 600]\).

As we can see from figure 4, the contact scheme has better performance than the non-contact scheme in the simulation of dissipation. Figure 5 shows that the trajectory of contact structure is stable in the time interval \([0, 600]\). Although it has some random oscillations, it most fluctuates between \(-2\) and \(2\). This is due to the
multiplicative noise pumped into this system constantly. The contact structure is preserved almost surely. In figure 6, as we can see, although there are some jumps in the trajectory of the conformal factor with the influence of noise, the contact structure is preserved as well as the continuous case a.s., too. Therefore, these phenomena verify the results of theorem 3.8. That is, the preservation of the contact structure is the distinguished feature.

Figure 4. Comparison of sample trajectories of (10) and (10) in the coordinates $q$, $p$ and $s$, respectively.

Figure 5. Preservation of contact structure (10).
4.3. Example 3

Now we consider the following stochastic Kepler problem equations \[28\],

\[
\begin{align*}
    dq &= pdt, \\
    dp &= -\left[\frac{1}{q^2} + \beta p\right] dt - \gamma q \circ dW(t), \\
    ds &= \left[\frac{p^2}{2} + \frac{1}{|q|} - \beta s\right] dt - \gamma q p \circ dW(t),
\end{align*}
\]

(26)

where \(q, p, s \in \mathbb{R}\). The Hamiltonians are

\[
H_0 = \frac{|p|^2}{2} - \frac{1}{|q|} + \beta s, \quad H_1 = \gamma q.
\]

The Lagrangian function \(L\) with respect to the deterministic part of \((10)\) is

\[
L = pq - H_0 = \frac{q^2}{2} + \frac{1}{|q|} - \beta s.
\]

We consider one discrete form of the Lagrangian

\[
L_j = \frac{1}{2} \left(\frac{q_{j+1} - q_j}{h}\right)^2 + 2 \frac{2h}{|q_j + q_{j+1}|} - \frac{1}{2} \beta (s_j + s_{j+1}).
\]

(27)

Substituting the expression of \((10)\) into the relations \((10)\) and \((10)\), we have

\[
P_j = \frac{q_j - q_{j-1}}{h} \frac{2h}{(q_j + q_{j-1})^2},
\]

and

\[
P_{j-1} = \frac{q_{j-1} - q_j}{h} \frac{2h}{(q_{j-1} + q_j)^2} + \gamma q \circ \Delta W_{j-1}.
\]
Therefore, an implicit contact scheme of (10) is
\[
\begin{align*}
q_j &= q_{j-1} + \frac{h}{2} \left( 1 + \frac{1}{2} \beta h \right) p_j + \left( 1 - \frac{1}{2} \beta h \right) p_{j-1} - \gamma \sigma \Delta W_{j-1}, \\
p_j &= \left[ \frac{q_j - q_{j-1}}{h} - \frac{2h}{h} \left( q_j + q_{j-1} \right)^2 \right] \left( 1 + \frac{1}{2} \beta h \right), \\
s_j &= s_{j-1} + \frac{1}{2h} (q_j - q_{j-1})^2 - \frac{2h}{2h} \left( q_j + q_{j-1} \right) - \frac{\beta h}{2} (s_j + s_{j-1}) - \gamma q_{j-1} \sigma \Delta W_{j-1}.
\end{align*}
\]  
(28)

Here we devote oneself to two things. One is the preservation of contact structure of the scheme (10), whose conformal factor is \( \lambda_j = \exp(-\beta \Delta t) \) in the continuous case. The other is the comparison with Euler-Maruyama scheme. The Euler-Maruyama scheme of (10) is
\[
\begin{align*}
q_j &= q_{j-1} + \beta h p_{j-1}, \\
p_j &= p_{j-1} - \gamma \sigma \Delta W_{j-1}, \\
s_j &= s_{j-1} + \frac{1}{2} \beta h^2 - \gamma \sigma \Delta W_{j-1}.
\end{align*}
\]  
(29)

To examine the performances of contact scheme (10), the initial conditions are chosen as follows, \( (q_0, p_0, s_0) = (0.75, -0.25, 0.08), \beta = 0.01, \gamma = 0.1, h = 0.1, N = 20000. \)

As shown in figure 7, the trajectory of \( s \) is stable almost surely in the time interval \([0, 2000]\), and the trajectories of \( q \) and \( p \) are stable, too. Figure 7 also shows that the contact scheme has better performance than the non-contact scheme in the simulation of dissipation. Figure 8 shows that the trajectory of contact structure is stable in the time interval \([0, 2000]\). Although it increases in the time interval \([0, 500]\), it is stable in the time interval \([500, 2000]\).
interval [500, 2000]. And the trend of the trajectory is flat almost surely in the time interval [500, 2000]. Therefore, the contact structure is preserved almost surely in [0, 2000]. Figure 9 shows that the trajectories of the conformal factor of the contact scheme and the continuous case does overlap each other almost surely with some jumps. It means that the conformal factor is preserved well a.s., too. Therefore, these results verify the results of theorem 3.8.

Remark 4.1. This paper mainly devotes to the construction of contact scheme for the contact Hamiltonian stochastic systems, which is one of frontier problems in nonlinear dynamics. Therefore, some further research topics, such as convergence analysis and stability, will be future work. Compared with the past works[13, 14], several new low-stage stochastic symplectic methods are presented and tested numerically to demonstrate their

Figure 8. Preservation of contact structure (10).

Figure 9. Comparison of the conformal factor of contact scheme (28) and the continuous case of (26).
better long-time numerical stability and energy behavior than in References [13, 14]. They motivate us to check the long-time dynamical behavior in our numerical experiments, considering the time interval [0, 2000].

**Remark 4.2.** The construction of the Lagrangian of stochastic contact Hamiltonian systems is still an open problem. Here we only present some examples, whose Lagrangian can be explicitly expressed via the deterministic part of the systems. The method to investigate the contact structure via the power invariance or energy balance equation will be our next focus.

5. Conclusion

This paper focuses on the construction and proof of the stochastic contact variational integrator via stochastic Herglotz variational principle. We investigate the dynamics of stochastic contact Hamiltonian systems and the validation through performing three numerical experiments. These numerical experiments are conducted to demonstrate the effectiveness and superiority of the proposed method by the simulations of its orbits, contact structure and conformal factor over a long time interval. The results show that the method is effective. This is due to the fact that the numerical experiments match the results of theoretical analysis. Construction of various methods via the stochastic contact variational integrator theory will be our further work.

**Acknowledgments**

This work is supported by NSFC (No. 61841302, and 12141107), The Fund of Fujian Agriculture and Forestry University, No. 111422138.X. Li was partially supported by the grant DOE DE-SC00222766.

**Data availability statement**

The data that support the findings of this study are openly available at the following URL/DOI: https://github.com/zhaniiit2020/-Numerical-integration-of-stochastic-contact-Hamiltonian-systems.git.

**Statements**

All data in this manuscript is available. And all programs will be available on the WEB GitHub [29].

**ORCID iDs**

Qingyi Zhan @ https://orcid.org/0000-0003-0110-6029

Jinqiao Duan @ https://orcid.org/0000-0002-2077-990X

**References**

[1] Arnold V I 2013 *Mathematical Methods of Classical Mechanics* vol 60 (Springer Science and Business Media)
[2] Bravetti A, Cruz H and Tapias D 2017 Contact Hamiltonian mechanics *Ann. Phys.* 376 17–39
[3] Feng K 1998 Contact algorithms for contact dynamical systems *J. Computational Mathematics* 16 1–14
[4] Geiges H 2008 *An Introduction to Contact Topology* vol 109 (Cambridge University Press)
[5] Wei P and Wang Z 2021 Formulation of stochastic contact Hamiltonian systems *Chaos* 31 041101
[6] Duan J 2015 *An Introduction to Stochastic Dynamics* (Cambridge University Press)
[7] Hairer E, Lubich C and Wanner G 2002 *Geometric Numerical Integration* (Springer–Verlag)
[8] Milstein G, Repin Y and Tretyakov M 2002 Numerical methods for stochastic systems preserving symplectic structure *SIAM J. Numer. Anal.* 40 1583–604
[9] Milstein G, Repin Y and Tretyakov M 2002 Symplectic integration of Hamiltonian systems with additive noise *SIAM J. Numer. Anal.* 39 2066–88
[10] Misawa T 2010 Symplectic integrators to stochastic Hamiltonian dynamical systems derived from composition methods *Mathematical Problems in Engineering* 2010 384937
[11] Wang L, Hong J, Scherer R and Bai F 2009 Dynamics and variational integrators of stochastic Hamiltonian systems *International J. of Numerical Analysis and Modeling* 6 586–602
[12] Zhan Q, Duan J and Li X 2020 Symplectic Euler scheme for Hamiltonian stochastic differential equations driven by Lévy noise arXiv:2006.15500
[13] Bou-Rabee N and Owhadi H 2009 Stochastic variational integrators *IMA J. Numer. Anal.* 29 421–43
[14] Holm D D and Tyrronowski T M 2018 Stochastic discrete Hamiltonian variational integrators *BIT Numerical Mathematics* 58 1009–48
[15] Kraus M and Tyrronowski T M 2020 Variational integrators for stochastic dissipative Hamiltonian systems *IMA J. Numer. Anal.* 00 1–50
[16] Georgieva B, Guenther R and Bodurov T 2003 Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem *J. Math. Physics* 44 3911–27
[17] Bravetti A, Seri M, Vermeeren M and Zadra F 2020 Numerical integration in Celestial Mechanics: a case for contact geometry Celest. Mech. Dyn. Astron. 132 1–29
[18] de Leon M and Valcazar M I 2019 Contact Hamiltonian systems J. Math. Phys. 60 102902
[19] Wang K 2022 Exact traveling wave solutions for the local fractional Kadomtsov-Petviashvili-Benjamin-Bona-Mahony model by variational perspective Fractals 30 2250101
[20] Liu Q, Torres P J and Wang C 2018 Contact Hamiltonian dynamics: variational principles, invariants, completeness and periodic behavior Ann. Phys. 395 26–44
[21] Milstein G 1995 Numerical Integration of Stochastic Differential Equations (Kluwer Academic Publishers)
[22] Wang X, Duan J, Li X and Luan Y 2015 Numerical methods for the mean exit time and escape probability of two-dimensional stochastic dynamical systems with non-Gaussian noises Appl. Math. Comput. 258 282–95
[23] Golub G and Van Loan C 2013 Matrix Computations IV edn (The Johns Hopkins University Press)
[24] Tveter F T 1998 Deriving the Hamilton equations of motion for a nonconservative system using a variational principle J. Math. Physics 39 1495–500
[25] Vermeeren M, Bravetti A and Seri M 2019 Contact variational integrators J. Phys. A: Math. Theor. 445206
[26] Cieslinski J I and Nikiciuk T 2010 A direct approach to the construction of standard and nonstandard Lagrangians for dissipative-like dynamical systems with variable coefficients J. Phys. A: Math. Theor. 43 175205
[27] Musielak Z 2008 Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients J. Phys. A: Math. Theor. 41 055205
[28] Margheri A, Ortega R and Rebelo C 2014 Dynamics of Kepler problem with linear drag Celest. Mech. Dyn. Astron. 120 19–38
[29] Zhan Q 2022 GitHub