The geometry of learning

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We establish a correspondence between classical conditioning processes and fractals. The association strength at a given training trial corresponds to a point in a disconnected set at a given iteration level. In this way, one can represent a training process as a hopping on a fractal set, instead of the traditional learning curve as a function of the trial. The main advantage of this novel perspective is to provide an elegant classification of associative theories in terms of the geometric features of fractal sets. In particular, the dimension of fractals is a parameter that can both measure the efficiency of a given conditioning model (in terms of the characteristics of the stimuli) and compare the efficiency of different models. We illustrate the correspondence with the examples of the Hull, Rescorla–Wagner, and Mackintosh models and show that they are equivalent to a Cantor set. In doing so, we approximate the Mackintosh model with a new formulation in terms of a nonlinear recursive equation for the strength of association.

Keywords: Pavlovian conditioning; associative theories; fractal geometry

Introduction

Making psychology quantitative has been a difficult but feasible challenge since the first laboratory of psychophysiology established by Wundt. Making it a mathematical theory under analytic control has been, and possibly will always be, a utopia. Nevertheless, there are a plethora of analytic models which are able to fit and explain data coming from the observation of subjects in specific experiments. For instance, in the context of behavioral theories of classical conditioning, one can study the interplay between a conditioned stimulus (CS) and the subsequent occurrence of an unconditioned stimulus (US) of typically high relevance for the subject, such as food or an electric discharge. Despite their limited range of applicability, associative conditioning models are useful for several reasons. First, they express in a compact and economic way concepts that took time and their limited range of applicability, associative conditioning models are useful for several reasons. First, they express in a compact and economic way concepts that took time and

The classic 1950s theoretical approaches to simple cases of conditioning are cast in the language of probability theory [see, e.g., the works by Estes (Estes, 1950, Estes and Burke, 1953), Bush and Mosteller (1951a; 1951b), and the reviews by Mosteller (1958) and Bower (1994)]. In these models, one considers the probability of a given conditioned response (CR) as a function of the trial number \( n \). The increment \( \Delta p_{n} \) at each trial is linear in \( p_{n} \); by evaluating \( p_{n} \) iteratively, one obtains a learning curve. Alternatively, the probability \( p \) can be replaced by the strength of association \( V \). This change of variable is useful for phenomenological applications because \( V \) can directly be measured by several performance indicators. For instance, the quality of surprise in the US as a function of the appearance of the CS was first suggested by Kamin in relation with cue competition (Kamin, 1968; Kamin, 1969). For a single CS, the evolution of the novelty (or “surprisiness”) of the US along the learning curve had been made quantitative already by Hull in his linear model of classical conditioning (Hull, 1943). Recast in modern terminology by Rescorla and Wagner (Rescorla and Wagner, 1972, Wagner and Rescorla, 1972), this model states that the change \( \Delta V_{n} \) in the strength of the association at the \( n \)-th trial is

\[
\Delta V_{n} = \alpha \beta (\lambda - V_{n-1}) \quad \text{for } n = 1, 2, 3, \ldots \tag{1}
\]

where \( 0 \leq \alpha \leq 1 \) is the salience of the US, \( 0 \leq \beta \leq 1 \) is the salience of the US, and \( 0 \leq \lambda \leq 1 \) is the magnitude or probability of occurrence of the US (i.e., the asymptote of learning). The sign of \( V \) depends on whether the conditioning is excitatory (\( V > 0 \)) or inhibitory (\( V < 0 \)). The term \( \lambda - V_{n-1} \) indicates the novelty of the US, which decreases as the associative strength increases. The association strength

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The solution of this equation is
\[ V_n = \lambda [1 - (1 - \alpha \beta)^{n-1}] . \]

By definition, at the beginning of the first trial \( V_1 = 0 \) (no association has been made yet), which fixes the unphysical constant \( \lambda \neq 1 \) being the case where the US always occurs after the CS. Maximum learning is achieved when \( \lambda = 0 \). If \( \lambda = 0 \), the US does not show up after the CS and the conditioning is inhibitory. Rescorla and Wagner extended the linear model to the case of the presentation of multiple CSs (Rescorla and Wagner, 1972; Wagner and Rescorla, 1972), as we will discuss later.

The main contribution of this paper is to propose a geometric interpretation of learning processes which is useful at the time of assessing their efficiency quantitatively, both within a given model (how efficiency is affected by the salience of the stimuli for the subject) and when comparing different models. Specifically, we obtain the following results. (i) First of all, we recognize Eq. (2) as one of the contraction maps defining Cantor sets, which are an example of peculiar, totally disconnected sets known as deterministic fractals. (ii) Second, we calculate the Hausdorff dimension of the set and show that it depends on the parameters \( \alpha \), \( \beta \), and \( \lambda \) in such a way that the smaller the dimension, the more efficient the conditioning. (iii) Third, we conjecture that this picture can be generalized to any other conditioning described by iterative equations. As an example, (iv) we approximate Mackintosh theory with a new model where the recursive equation describes slow learning during the first trials; the dimension of this conditioning process is calculated and shown to be greater than in the Hull model, in agreement with (ii).

The presentation is rather mathematical since it borrows a few concepts from fractal geometry (Falconer, 2003). The reader interested only in the practical advantage of our construction to psychology can jump to Eq. (8) and skip other parts that will be indicated in due course.

**Fractals and the Cantor set**

Let
\[ S_1(x) = a_1 x + b_1 , \quad S_2(x) = a_2 x + b_2 , \]
be two similarity maps, where \( a_{1,2} \) (called similarity ratios) and \( b_{1,2} \) (called shift parameters) are real constants and \( x \in I \) is a point in the unit interval \( I = [0, 1] \). The image \( S_i(A) \) of a subset \( A \subset I \) is the set of all points \( S_i(x) \) where \( x \in A \). A Cantor set or Cantor dust \( C \) is given by the union of the image of itself under the two similarity maps \( \mathbb{C} = S_1(C) \cup S_2(C) \). For instance, the ternary (or middle-third) Cantor set \( C_3 \) (Cantor, 1883) has \( a_1 = 1/3 = a_2, b_1 = 2/3, and b_2 = 0: \)
\[ S_1(x) = \frac{1}{3} x + \frac{2}{3}, \quad S_2(x) = \frac{1}{3} x . \]

This is rescaled in Fig. 1. At the first iteration, the interval \( [0, 1] \) is rescaled by \( 1/3 \) and duplicated in two copies: one copy (corresponding to the image of \( S_1 \)) at the leftmost side of the unit interval and the other one (corresponding to \( S_2 \)) at the rightmost side. In other words, one removes the middle third of the interval \( I \). In the second iteration, each small copy of \( I \) is again contracted by \( 1/3 \) and duplicated, i.e., one removes the middle third of each copy thus producing four copies 9 times smaller than the original; and so on. Iterating infinitely many times, one obtains \( C_3 \), a dust of points sprinkling the line. The set is self-similar inasmuch as, if we zoom in by a multiple of 3, we will observe exactly the same structure.

It is easy to determine the dimensionality of the Cantor set \( C \). Since this dust does not cover the whole line, it has less than one dimension. Naively, one might expect that the dimension of \( C \) is zero, since it is the collection of disconnected points (which are zero-dimensional). However, there are “too many” points of \( C \) on \( I \) and, as it turns out, the dimension of

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1The efficiency of an excitatory conditioning can be roughly defined as the inverse of the number of trials necessary to increase the associative strength from 0 to, say, 0.9.1.
the set is a real number between 0 and 1. In particular, given
\( N \) similarity maps all with ratio \( a \), the similarity dimension
or capacity of the set is
\[
d_c(C) := -\frac{\ln N}{\ln a}.
\] (6)

This formula is valid for an exactly self-similar set made of
\( N \) copies of itself, each of size \( a \). Note that \( a = N^{-1/d_c} \)
the smaller the size \( a \), the smaller the copies at each iteration
and the smaller the dimensionality of the set. In the case
of the middle-third Cantor set, \( N = 2 \) and \( a = 1/3 \), so that
\( d_c = \ln 2/\ln 3 \approx 0.63 \). Sets with noninteger dimension-
ality are called fractals, a term coined by Mandelbrot (1967).
For the class of fractals we will consider here, other important
guiding indicators such as the box-counting dimension and the
Hausdorff dimension \( d_\infty \) (perhaps the most popular
among the fractal dimensions) are equal to the capacity.

**Geometric interpretation of the Hull and Rescorla–Wagner models**

Let us now apply these results, which are quite standard in
fractal geometry, to simple cases of the Hull and Rescorla–
Wagner models: (a) excitation, (b) extinction, and (c) inhibition.
We consider two conditioning experiments, one where
the CS is an excitatory stimulus (\( \lambda \neq 0 \), the US follows
the EC) and another where the same stimulus is inhibitory
(\( \lambda = 0 \), the US never follows the EC). For instance, the EC
can be a sound or a light and the US food or a discharge.
Interpreting the association strength \( \tilde{V} \) as the point in the
\( n \)th iteration of a set and comparing Eqs. 4 and 5, we see
that the similarity ratio and shift parameter in the Hull model
(one CS) is
\[
\begin{align*}
a_1 &= a_2 = a = 1 - a\beta, \\
b_1 &= a\beta \lambda, \quad b_2 = 0.
\end{align*}
\] (7)

The images \( S_1(I) \) and \( S_2(I) \) correspond to the set of association
strengths measured in, respectively, excitatory and in-
hibitory conditioning. We can pair the Hull and Rescorla–
Wagner models with a Cantor set \( C = S_1(C) \cup S_2(C) \) with
parameters (7). In the example of Fig. 4, \( 1 - a\beta = 1/3 \), which
obtained with \( \beta = 1 \) (maximum salience of the US)
and \( a = 2/3 \approx 0.66 \) (a highly salient CS).

The actual excitatory conditioning process (a) is shown
by the red learning curve in Fig. 1[1] touching upon the \( S_1(I) \)
branch with initial condition \( V_1 = 0 \), while the extinction
curve is shown in blue [touching the \( S_2(I) \) branch with initial
condition \( V_1 = 1 \)]. In the case of extinction (b), we can
consider a two-phase experiment. The first phase is excitat-
ory with a given CS, corresponding to the red curve. Then,
the \( n = 1 \) point of the blue curve corresponds to the first trial
in a second phase of training where the association strength
\( V^{CS} \) relates the absence of the US (\( \lambda = 0 \)) with the same
CS used in phase 1, so that \( V^{CS} \) decreases from 1 to zero.

In the case of an inhibitory conditioning process (c), the
red curve of phase 1 describes the association strength \( V^{CS1} \)
when pairing a given conditioned stimulus CS1 to the US. In
phase 2, we pair a second stimulus CS2 to CS1 and the blue curve
runs over trials where the associative strength \( V^{CS1+CS2} \)
of the combined conditioned stimulus \( \text{Rescorla and Wagner, 1972, Wagner and Rescorla, 1972} \) decreases to zero. This is
standard inhibitory conditioning. Alternatively, one can con-
sider a backward pairing between the US and a previously
untrained stimulus CS-. Then, we identify the coordinate
\( \tilde{V} =: \tilde{V} + 1 \) in the figure with the association strength \( \tilde{V} \)
shifted by 1, of CS-. The blue curve then runs from \( \tilde{V} = 0 \)
to \( \tilde{V} = -1 \) (perfect inhibition). Trace conditioning follows a
similar rule.

The rest of the set (in black) represents experiments where
trials with excitatory pairing (single CS followed by the US)
are mixed with trials where the US does not follow the CS.
The learning curve will change according to the session pat-
tern (positive or negative contingency of the US) and one
will have infinitely many possible experiments or natural sit-
uations with nooptimal learning.

By interpreting a model of classical conditioning as a col-
lection of processes taking place on a fractal set, we gain
a number in insights that can be described in a very mini-
imalistic but effective fashion. For instance, the notion that
excitation and inhibition are the two extremes, with oppo-
site sign, of the same process [Rescorla, 1967] translates
into a precise mathematical statement. For the Hull and
Rescorla–Wagner models, excitation and extinction corre-
spond to processes living on, respectively, the two comple-
mentary branches \( S_1(I) \) and \( S_2(I) \) of the Cantor set with ini-
tial condition \( V_1^{\text{excit}} = 0 \) and \( V_1^{\text{inhib}} = 0 \).

Also, the Hausdorff dimension of \( C \) is equal to the capac-
ity, which is found from Eqs. (6) and (7):
\[
d_a(C) = d_c(C) = -\frac{\ln 2}{\ln(1 - a\beta)}.
\] (8)

In the example of Fig. 1 \( d_a = \ln 2/\ln 3 \approx 0.63 \). The profile
(8) is shown in Fig. 2.

Note that \( d_a \) is independent of the magnitude \( \lambda \) of the
US but depends on the learning-rate parameters \( a \) and \( \beta \),
the salience of the CS and of the US. We can identify two
regimes:
- For \( 0 < a\beta < 0.5 \), the fractal is degenerate, in the sense
that it fills the whole line. A lower salience of the CS in-
creases the similarity ratio \( a = 1 - a\beta \) and produces longer
segments, leading to a Cantor set with a larger dimension.
Strictly speaking, although \( d_a > 1 \), the dimension of the set
is just equal to 1, since the continuous line fixes, so to speak,
the maximum number of points and it cannot be overflown.
However, conditioning makes sense in this region and \( d_a \) is a
good learning index even in this range of values. The latter
is consistent both with a CS with very low salience \( \alpha \) (resulting in difficult or no conditioning) and with the expectation that, at the beginning of the training, a very salient CS would actually compete with the US (very small \( \beta \) versus relatively large \( \alpha \)) rather than being a neutral stimulus. For \( \beta = 1 \) and the typical CS salience \( \alpha = 0.5 \), one has \( d_n = 1 \).

- For \( \alpha \beta > 0.5 \), we have \( 0 < d_n < 1 \). The closer the value of \( \alpha \beta \) to 1, the larger the denominator of Eq. (8): as \( \alpha \beta \rightarrow 1 \) and \( 1 - \alpha \beta \rightarrow 0 \), we move towards the limit \( d_n \rightarrow 0 \). The interpretation of this feature in terms of classical conditioning is straightforward. The larger the salience of the US, the more efficient the conditioning and the fewer the trials needed to achieve complete training. Perfect training corresponds to infinitely many iterations, but efficient training requires less iterations to reach optimal learning. But the fewer the trials, the “fewer the points” in \( \mathcal{C} \), which means the smaller the dimension \( d_n \).

We have thus established that the capacity or Hausdorff dimension of the fractal associated with the conditioning model decreases with the increase in the efficiency of the training. We conjecture that this conclusion may be valid for any other conditioning that can be defined by an iterative equation. Of course, different models will correspond to different fractals, not necessarily Cantor dusts.

Interestingly, our result (8) does not depend on the magnitude \( \lambda \) of the US, which is another factor affecting the efficiency of a conditioning. We must conclude that the capacity dimension is insufficient to fully characterize the efficiency. However, it is a first step towards a geometric classification of conditioning methods.

**Generalizations**

So far, we have considered treatments with a single conditioned stimulus. Actually, the Rescorla–Wagner model was proposed to describe the case of compound stimuli, in which case the iterative evolution is more complicated. For two cues \( A \) and \( B \) with salience \( \alpha_A \) and \( \alpha_B \), one has two iterative processes \( V^{(A)}_n \) and \( V^{(B)}_n \), with Eq. (1) replaced by \( \Delta V^{(A)}_n = \alpha_A \beta \lambda - (V^{(A)}_{n-1} + V^{(B)}_{n-1}) \) and \( \Delta V^{(B)}_n = \alpha_B \beta \lambda - (V^{(A)}_{n-1} + V^{(B)}_{n-1}) \). Clearly, this extension of the single-CS case is nontrivial only if \( A \) and \( B \) are not presented together at all sessions (otherwise, all said above would hold for the composite strength \( V_n = V^{(A)}_n + V^{(B)}_n \)). Also, at different phases one might want to couple different CSs with different USs. The above pair of equations would then be augmented by another identical pair with a different US with salience \( \beta \) and asymptote \( \lambda \), and possibly a different CS compound \( AB \). Systems of recurrence equations with \( N \) variables \( V^{(i)}_n \), \( i = 1, \ldots, N \), can be much more difficult to solve analytically than stand-alone expressions such as (2), depending on how such variables are mutually entangled. Conceptually, there should be no problem in extending our geometric interpretation and one may still be able to construct fractal sets by joining excitatory and inhibitory branches. However, the techniques needed to do that go beyond the scope of the present investigation. In fact, they involve either the product of \( N \) one-dimensional fractals or the study of \( N \)-dimensional nondecomposable fractals, both of which cases require a machinery more sophisticated than the one developed for one-dimensional fractals [Falconer, 2003].

A somewhat easier but still nontrivial possibility is to analyze learning processes which are not described by linear recurrence equations. Mathematically, this is the most natural way to generalize the Cantor construction to generic one-dimensional deterministic fractals \( \mathcal{F} \). Just like the Cantor set, deterministic fractals can be defined as the union of the image of several maps, \( \mathcal{F} = S_1(\mathcal{F}) \cup S_2(\mathcal{F}) \cup \cdots \cup S_K(\mathcal{F}) \), but now the functions \( S_i(x) \), \( k = 1, \ldots, K \), are contractions nonlinear in \( x \). A map \( \mathcal{S} \) is called a contraction if there exists a constant \( 0 < a < 1 \) such that \( |\mathcal{S}(y) - \mathcal{S}(x)| \leq a|y - x| \). When the inequality is saturated, we have a similarity, of which the formulae in (4) are an example [in Eq. (5), \( a = 1/3 \)]. The following neat example, presented after recalling some alternatives to the Rescorla–Wagner model, can help the reader to appreciate that such abstract constructions can have direct applications to psychology.

**Mackintosh and Pearce–Hall models**

It is well-known that the Rescorla–Wagner model suffers from several limitations [Miller et al. 1995]. Among them, we recall that it predicts the extinction of conditioned inhibition (which does not occur actually) and it regards extinction as an unlearning process (i.e., the blue and red curves...
in Fig. 1 are perfectly specular). Thus, it cannot explain either spontaneous recovery (when a CR that had been extinguished reappears) or other effects such as preconditioning exposure to the CS [i.e., latent inhibition, which may occur also in conjunction with a reinforcer (Hall and Pierce, 1979), augmentation (or counter-blocking), first-trial unblocking (Mackintosh, 1975a), or unblocking by the surprising omission of part of a compound US (Dickinson et al., 1976). Moreover, the blocking effect on a CS2 (by a CS1 associated with the US in a preliminary training phase) is explained as the absence of novelty in the US after its pairing with the CS1, but this interpretation has been ruled out in an experiment by Mackintosh and Turner (1971). Latent inhibition was accounted for by Wagner (1978), while first-trial unblocking and unblocking by omission were explained by Mackintosh model of attention (Mackintosh, 1975b), according to which blocking occurs because predictive but redundant stimuli such as CS2 are ignored. Finally, Pearce and Hall (1980) proposed a model that could encompass all these cases and explain latent inhibition as well as various phenomena of unblocking.

These and other elemental theories of associative learning are reviewed by Le Pelley (2004) and Wagner and Vogel (2005); some have been developed more recently (Esber and Haselgrove, 2011).

Most of these proposals require a quantitative modification of the Rescorla–Wagner model by replacing all constant α’s with trial-dependent parameters αn that change with the subject’s experience. In the case of Mackintosh model, the rate of change is assumed to follow a linear law. For instance, given two cues A and B, one has

\[ \Delta \alpha_n^{(A)} = \gamma_A \left( \lambda - V_n^{(B)} - |\lambda - V_n^{(A)}| \right) \]

analogous expression for \( \Delta \alpha_n^{(B)} \), where \( \gamma_A \) is a constant. The recursive law governing the evolution of \( V_n^{(A)} \) is \( \Delta V_n^{(A)} = \alpha_n^{(A)} \beta (|\lambda - V_n^{(A)}|) \) and the influence of other stimuli is encoded exclusively in the way \( \alpha_n^{(A)} \) varies, via Eq. (9). This is in contrast with the Rescorla–Wagner prescription, according to which the size of the associative change depends on the strength of all the stimuli, \( \Delta V_n^{(A)} = \alpha_n \beta (|\lambda - \sum V_n^{(i)}|) \).

For a single conditioned stimulus, we can write

\[ \alpha_n = \alpha_{n-1} + \Delta \alpha_n = \alpha_{n-1} + \gamma (\lambda - V_{n-1}) \]

which should be coupled with Eq. (2). We were unable to find explicit solutions \( V_n \) and \( \alpha_n \) but it is easy to see that, compared with the Hull learning curve [3], this model predicts a systematically slower learning during the first trials (Fig. 3).

Also the Pearce–Hall model assumes that the effectiveness α of a CS changes with its predictive strength V but, contrary to Mackintosh theory, the magnitude of the US now varies with the experience (Pearce and Hall, 1980): \( \alpha_n = |\lambda_{n-1} - V_{n-1}| \), where the US magnitude is bound to lie in the range \( 0 < \lambda_n < 1 \). In a variant of the model which fixes some issues of the original proposal, this expression is replaced by \( \alpha_n = \gamma (\lambda_{n-1} - V_{n-1}) + (1 - \gamma) \alpha_{n-1} \), where \( 0 < \gamma < 1 \) (Pearce, Kave, and Hall, 1982). This reflects the idea that stimuli always have the possibility to gain access to the subject’s processing, but less surprising stimuli will have limited access. The recursive law for \( \alpha_n \) is combined with \( \Delta V_n = \beta \lambda \alpha_n \). Without any specific iteration rule for \( \lambda_n \) [absent in the original paper by Pearce and Hall (1980)], we cannot solve the system analytically.

**Nonlinear model**

In this section, we present (to the best of our knowledge, for the first time) a simplified model of variable salience which approximates Mackintosh model. Equation (10) is replaced by

\[ \alpha_n = \alpha_{\text{min}} + (\alpha_{\text{max}} - \alpha_{\text{min}}) \frac{V_n}{\lambda}, \quad 0 < \alpha_{\text{min}} < \alpha_{\text{max}} \leq 1. \]

Using Eq. (1), it is easy to see that (11) is an approximation of (10) when \( \alpha_n \) does not vary much during conditioning (\( \alpha_{\text{max}} \approx \alpha_{\text{min}} \)), corresponding to a small parameter \( \gamma \approx (\alpha_{\text{max}} - \alpha_{\text{min}}) a \beta / \lambda \ll 1 \). In excitatory conditioning, \( \lambda = \lambda \). When \( n = 1 \) (first trial), \( V_1 = 0 \) and the salience of the CS has some default value \( 0 < \alpha_1 = \alpha_{\text{min}} \ll 1 \) which depends on the nature of the CS and on its salience for the subject. As the excitatory training proceeds, the salience of the CS approaches the asymptotic value \( \alpha_{\text{max}} > \alpha_1 \). In the case of extinction, the association strength decreases from \( V_1 = \lambda \) to zero and the salience of the CS drops from its maximal value

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**Figure 3.** Learning progression for excitatory conditioning with \( V_1 = 0, \beta = 1 \) and \( \lambda = 1 \). Light-gray circles: Hull’s model (3) for \( \alpha \approx 0.664 \approx 2 / 3 \). Black squares: Mackintosh model (2) and (10) for \( \alpha_1 = 0.5 \) and \( \gamma = 0.09 \) (the asymptote is \( \alpha_n \rightarrow 0.664 \)). Gray diamonds: the nonlinear model (11) with \( \lambda = \lambda, \alpha_{\text{min}} = 0.5 \), and \( \alpha_{\text{max}} \approx 0.664 \).
\( \alpha_1 = \alpha_{\text{max}} \) down to \( \alpha_{\text{min}} \). Plugging Eq. (11) into (2), we obtain a nonlinear law:

\[
V_n = (1 - \alpha_n - \beta)V_{n-1} + \alpha_{n-1}\beta \lambda
\]

\[
\frac{\alpha_n\beta \lambda}{1 - \beta \alpha_{\text{min}}} V_{n-1} - \frac{(\alpha_{\text{max}} - \alpha_{\text{min}})\beta^2}{1 - \beta \alpha_{\text{min}}} V_{n-1}.
\]

(12)

For small \( \gamma \), this is a good approximation of Mackintosh model (Fig. 3). Notice that the conditioning described by Eq. (12) differs from other nonlinear models described in the past [Brandon et al., 2003; Le Pelley, 2004]. In particular, Le Pelley’s hybrid model [Le Pelley, 2004] is an extension, rather than an approximation, of Mackintosh theory.

The excitation branch \( \mathcal{S}_1(I) \) of the associated fractal \( \mathcal{C} \) (in this case, a nonlinear Cantor set) is given by the contraction (12) with \( \lambda = \lambda \), while the extinction branch \( \mathcal{S}_1(I) \) is given by setting first \( \lambda = 0 \) and then \( \lambda = 1 \) (i.e., in the final expression this is the magnitude of the previously trained US):

\[
\mathcal{S}_1(x) = (1 - \beta \alpha_{\text{min}})x - (\alpha_{\text{max}} - \alpha_{\text{min}})\frac{\beta}{\lambda} x^2,
\]

(13a)

\[
\mathcal{S}_2(x) = \alpha_{\text{min}}\beta \lambda + [1 + \beta (\alpha_{\text{max}} - 2\alpha_{\text{min}})]x
\]

\[-(\alpha_{\text{max}} - \alpha_{\text{min}})\frac{\beta}{\lambda} x^2.
\]

(13b)

Notice that, in general, the appearance of two different conditioning laws for excitatory and inhibitory processes is nothing new and it was already employed in Pearce and Hall (1980). The novelty here, apart from the specific form of Eq. (13), is the geometric interpretation of conditioning in terms of branches of a fractal. As an application, we now show that the capacity of this set is larger than the one of Hull’s model, thus giving a quantitative estimate of the “uphill learning” depicted in Fig. (3). Although it is not always possible to find the exact value of the dimension of a fractal, there are some powerful theorems that make use of the properties of contractions. The reader uninterested in technicalities can skip this part and go directly to Eq. (15).

The first step consists in checking whether the maps \( \mathcal{S}_2 \) are bi-Lipschitz, meaning that there exist two positive and finite constants \( a_b \) and \( b_t \) such that \( b_t |y - x| \leq |\mathcal{S}_2(y) - \mathcal{S}_2(x)| \leq a_b |y - x| \). In that case, \( a_b = \sup |\mathcal{S}_2'(x)| \) and \( b_t = \inf |\mathcal{S}_2'(x)| \), where \( \mathcal{S}_2 \) denotes the derivative with respect to \( x \). For the system (13), it is easy to find these constants. Since \( \mathcal{S}'_2(x) = 1 - \beta \alpha_{\text{min}} - 2(\alpha_{\text{max}} - \alpha_{\text{min}})(\beta/\lambda) x, \mathcal{S}'_2(x) = 1 + \beta (\alpha_{\text{max}} - 2\alpha_{\text{min}}) + 2(\alpha_{\text{max}} - \alpha_{\text{min}})(\beta/\lambda) x, \alpha_{\text{max}} - \alpha_{\text{min}} > 1, \) and \( 0 \leq x < 1, \) one has that the highest and lowest (respectively sup and inf) value of \( |\mathcal{S}'_2(x)| \) is attained at, respectively, \( x = 0 \) and \( x = 1 \). Then, we find \( a_1 = 1 - \beta \alpha_{\text{min}}, a_2 = 1 + \beta (\alpha_{\text{max}} - 2\alpha_{\text{min}}), b_1 = 1 - \beta \alpha_{\text{min}} - 2\beta (\alpha_{\text{max}} - \alpha_{\text{min}})/\lambda, \) and \( b_2 = 1 + \beta (\alpha_{\text{max}} - 2\alpha_{\text{min}}) - 2\beta (\alpha_{\text{max}} - \alpha_{\text{min}})/\lambda \). For \( \lambda = 1 \) and assuming that there is not much difference between the

Initial and final value of \( \alpha_n \), these expressions reduce to

\[
a_1 = 1 - \beta \alpha_{\text{min}}, \quad a_2 = 1 - \beta (2\alpha_{\text{min}} - \alpha_{\text{max}}),
\]

(14a)

\[
b_1 = 1 - \beta (2\alpha_{\text{max}} - \alpha_{\text{min}}), \quad b_2 = 1 - \beta \alpha_{\text{max}}.
\]

(14b)

In particular, for Hull’s model \( \alpha_{\text{min}} = \alpha_{\text{max}} = \alpha \) and all the coefficients collapse to 1 - \( \beta \alpha \). Next, one recalls that the Hausdorff dimension of a fractal \( F = \mathcal{S}_1(F) \cup \mathcal{S}_2(F) \) is bounded from above and from below by \( s_b \leq d_H(F) \leq s_a \), where \( s_b \) and \( s_a \) are two constants determined implicitly by the relations \( b_1^n + b_2^n = 1 \) and \( a_1^n + a_2^n = 1 \) [Falconer, 2003]. For instance, consider \( \beta = 1, \lambda = 1, \alpha_{\text{min}} = 0.5 \) and \( \alpha_{\text{max}} = 2/3 \) in Eq. (14). Then, \( 0.55 < d_H(C) < 1.10 \), which can be further refined to \( 0.59 < d_H(C) < 1.01 \) by taking an extra iteration and the maps \( \{\mathcal{S}_1, \mathcal{S}_2 \} : k, l, q = 1, 2 \). Taking a large number of iterations, the lower bound coincides with the Hausdorff dimension of the ternary Cantor set, while the upper bound tends to 1 (the dimension of the unit line). For general values of the saliences, one has

\[
d_H(C) > d_H(C)
\]

(15)

and, therefore, the nonlinear model describes less efficient learning than Hull’s. This conclusion is obvious by looking at Fig. (3) but we have just made it quantitative in a precise sense, since the Hausdorff dimension of the nonlinear model can be computed numerically and compared with the dimension of the linear model.

Conclusions

It is easy to generalize the above construction to other experiments. For instance, an asymmetric Cantor set is obtained not only in the nonlinear model proposed above, but also in the linear case if we choose different saliences \( \alpha_1 \neq \alpha_2 \) and \( \beta_1 = \beta_2 \) in Eq. (3). Also, if we let any of the parameters \( \lambda, \alpha, \) and \( \beta \) vary randomly in the interval (0, 1) at each iteration, we would be in a situation where the strength and appearance rate of the unconditioned stimulus is governed by a random generator at each conditioning trial. In the language of fractal geometry, this would be a *random fractal*. We expect the conditioning-to-fractal correspondence to hold only by considering both the excitatory and inhibitory branches at the same time (otherwise, the iterative process would collapse the initial set to a point). While the ideal excitatory conditioning in a controlled environment is only a portion of the sequence of iterations generating the fractal [in

3For inhibitory conditioning, either one makes the change of variables \( V = \hat{V} + 1 \) explained below Eq. (7) or one considers the compound case. [4]

The reason behind this name and the details of the approximation linking Mackintosh’s model and Eq. (5) will be discussed in a separate publication.
the example of Fig. 11 from the interval (2/3, 1) to the point \( V = 1 \), we can interpret the whole fractal as a description of the most varied pairings one can find in Nature or in the laboratory between two given stimuli. Due to an incomplete definition of the Pearce–Hall model, we have not managed to analyze it with our formalism, but we do not expect to find any conceptual difficulty in that direction.

Eventually, single nonlinear equations cannot account for the variety of conditioning processes we are aware of. Systems of coupled iterative laws with multiple entangled variables are typical in modern approaches such as the SOP model of memory processing [Brandon et al., 2003; Wagner, 1981], where elements of different nodes in a neural graph interact nontrivially. Sensitivity to context further complicates the way different stimuli interact, as reflected by later elemental theories (Brandon and Wagner, 1998; Wagner, 2003; Wagner and Brandon, 2001) [see Wagner (2003) and the crystal-clear reviews by Wagner (2008) and Wagner and Vogel (2009), also for an account on configural theories]. Nevertheless, simple models such as Rescorla–Wagner and Pearce–Hall have not exhausted their usefulness. For instance, they are still topical in as hot a field of research as neuroscience and they may actually coexist in decision and some implications.\cite{Estes59}

In Mosteller (2006), pp. 235–250.

References

- \textit{Essentials of Psychology}, 1st ed., McGraw-Hill, New York.
- \textit{Selected Papers of Frederick Mosteller}, S.E. Fienberg & D.C. Hoaglin (Eds.), Appleton-Century-Crofts.
- \textit{Punishment and Adversive Behavior} (pp. 299–296).

New York, NY: Appleton-Century-Crofts.

Le Pelley, M.E. (2004). The role of associative history in models of associative learning: a selective review and a hybrid model. \textit{Q. J. Exp. Psychol.} \textbf{57}, 193.

Mackintosh, N.J. (1975). Blocking of conditioned suppression: Role of the first compound trial. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{1}, 335.

Mackintosh, N.J. (1975). A theory of attention: Variations in the associability of stimuli with reinforcement. \textit{Psychol. Rev.} \textbf{82}, 276.

Mackintosh, N.J., & Turner, C. (1971). Blocking as a function of novelty of CS and predictability of UCS. \textit{Q. J. Exp. Psychol.} \textbf{23}, 359.

Mandelbrot, B. (1967). How long is the coast of Britain? Statistical self-similarity and fractional dimension. \textit{Science} \textbf{156}, 636.

Miller, R.R., Barnet, R.C., & Grahame, N.J. (1995). Assessment of the Rescorla–Wagner model. \textit{Psychol. Bull.} \textbf{117}, 363.

Mosteller, F. (2006). S.E. Fienberg & D.C. Hoaglin (Eds.), \textit{Selected Papers of Frederick Mosteller}. New York, NY: Springer.

Mosteller, F. (1958). Stochastic models for the learning process. \textit{Proc. Amer. Phil. Soc.} \textbf{102}, 53 (1958); \textit{reprinted in} Mosteller (2006), pp. 295–307.

Pearce, J.M., & Hall, G. (1980). A model for Pavlovian

Dewey, J. (1896). The reflex arc concept in psychology. \textit{Psychol. Rev.} \textbf{3}, 357.

Dickinson, A., Hall, G., & Mackintosh, N.J. (1976). Surprise and the attenuation of blocking. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{2}, 313.

Esber, G.R., & Haselgrove, M. (2011). Reconciling the influence of predictiveness and uncertainty on stimulus salience: a model of attention in associative learning. \textit{Proc. Roy. Soc. B} \textbf{278}, 2553.

Estes, W.K. (1950) Toward a statistical theory of learning. \textit{Psychol. Rev.} \textbf{57}, 94.

Estes, W.K., & Burke, C.J. (1953). A theory of stimulus variability in learning. \textit{Psychol. Rev.} \textbf{60}, 276.

Falconer, K. (2003). \textit{Fractal Geometry}. New York, NY: Wiley.

Hall G., & Pierce, J.M. (1979). Latent inhibition of a CS during CS-US pairings. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{5}, 31.

Hull, C.L. (1943). \textit{Principles of Behavior}. New York, NY: Apple-Century-Crofts.

Kamin, L.J. (1968). “Attention-like” processes in classical conditioning. In M.R. Jones (Ed.), \textit{Miami Symposium on the Prediction of Behavior: Aversive Stimulation} (pp. 9–31). Miami, FL: University of Miami Press.

Kamin, L.J. (1969). Predictability, surprise, attention, and conditioning. In B.A. Campbell & R.M. Church (Eds.), \textit{Punishment and Adversive Behavior} (pp. 279–296). New York, NY: Appleton-Century-Crofts.

Dehaene, S., & Changeux, J.P. (1993). Retrieval models of the Rescorla–Wagner model. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{19}, 395.

Dickinson, A., Hall, G., & Mackintosh, N.J. (1976). Surprise and the attenuation of blocking. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{2}, 313.

Esber, G.R., & Haselgrove, M. (2011). Reconciling the influence of predictiveness and uncertainty on stimulus salience: a model of attention in associative learning. \textit{Proc. Roy. Soc. B} \textbf{278}, 2553.

Estes, W.K. (1950) Toward a statistical theory of learning. \textit{Psychol. Rev.} \textbf{57}, 94.

Estes, W.K., & Burke, C.J. (1953). A theory of stimulus variability in learning. \textit{Psychol. Rev.} \textbf{60}, 276.

Falconer, K. (2003). \textit{Fractal Geometry}. New York, NY: Wiley.

Hall G., & Pierce, J.M. (1979). Latent inhibition of a CS during CS-US pairings. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{5}, 31.

Hull, C.L. (1943). \textit{Principles of Behavior}. New York, NY: Apple-Century-Crofts.

Kamin, L.J. (1968). “Attention-like” processes in classical conditioning. In M.R. Jones (Ed.), \textit{Miami Symposium on the Prediction of Behavior: Aversive Stimulation} (pp. 9–31). Miami, FL: University of Miami Press.

Kamin, L.J. (1969). Predictability, surprise, attention, and conditioning. In B.A. Campbell & R.M. Church (Eds.), \textit{Punishment and Adversive Behavior} (pp. 279–296). New York, NY: Appleton-Century-Crofts.

Le Pelley, M.E. (2004). The role of associative history in models of associative learning: a selective review and a hybrid model. \textit{Q. J. Exp. Psychol.} \textbf{57}, 193.

Mackintosh, N.J. (1975). Blocking of conditioned suppression: Role of the first compound trial. \textit{J. Exp. Psychol.: Animal Behav. Proc.} \textbf{1}, 335.

Mackintosh, N.J. (1975). A theory of attention: Variations in the associability of stimuli with reinforcement. \textit{Psychol. Rev.} \textbf{82}, 276.

Mackintosh, N.J., & Turner, C. (1971). Blocking as a function of novelty of CS and predictability of UCS. \textit{Q. J. Exp. Psychol.} \textbf{23}, 359.

Mandelbrot, B. (1967). How long is the coast of Britain? Statistical self-similarity and fractional dimension. \textit{Science} \textbf{156}, 636.

Miller, R.R., Barnet, R.C., & Grahame, N.J. (1995). Assessment of the Rescorla–Wagner model. \textit{Psychol. Bull.} \textbf{117}, 363.

Mosteller, F. (2006). S.E. Fienberg & D.C. Hoaglin (Eds.), \textit{Selected Papers of Frederick Mosteller}. New York, NY: Springer.

Mosteller, F. (1958). Stochastic models for the learning process. \textit{Proc. Amer. Phil. Soc.} \textbf{102}, 53 (1958); \textit{reprinted in} Mosteller (2006), pp. 295–307.

Pearce, J.M., & Hall, G. (1980). A model for Pavlovian
learning: variations in the effectiveness of conditioned but not of unconditioned stimuli. *Psychol. Rev.* **87**, 532.

Pearce, J.M., Kaye, H., & Hall, G. (1982). Predictive accuracy and stimulus associability: development of a model for Pavlovian learning. In M.L. Commons, R.J. Herrnstein & A.R. Wagner (Eds.), *Quantitative Analyses of Behavior* (pp. 241–255). Cambridge, MA: Ballinger.

Rescorla, R.A. (1967). Inhibition of delay in Pavlovian fear conditioning. *J. Comp. Physiol. Psychol.* **64**, 114.

Rescorla, R.A., & Wagner, A.R. (1972). A theory of Pavlovian conditioning: variations in the effectiveness of reinforcement and nonreinforcement. In A.H. Black & W.F. Prokasy (Eds.), *Classical Conditioning II* (pp. 64–99). New York, NY: Appleton-Century-Crofts.

Roesch, M.R., Esber, G.R., Li, J., Daw, N.D., & Schoenbaum G. (2012). Surprise! Neural correlates of Pearce–Hall and Rescorla–Wagner coexist within the brain. *Eur. J. Neurosci.* **35**, 1190.

Wagner, A.R. (1978). Expectancies and the priming of STM. In S.H. Hulse, H. Fowler, & W.K. Honig (Eds.), *Cognitive Processes in Animal Behavior* (pp. 177–209). Hillsdale, NJ: Erlbaum.

Wagner, A.R. (1981). SOP: a model of automatic memory processing in animal behavior. In N.E. Spear & R.R. Miller (Eds.), *Information Processing in Animals: Memory Mechanisms* (pp. 5–47). Hillsdale, NJ: Erlbaum.

Wagner, A.R. (2003). Context-sensitive elemental theory. *Q. J. Exp. Psychol. B* **56**, 7.

Wagner, A.R. (2008). Evolution of an elemental theory of Pavlovian conditioning. *Learn. Behav.* **36**, 253.

Wagner, A.R., & Brandon, S.E. (2001). A componential theory of Pavlovian conditioning. In R.R. Mower & S.B. Klein (Eds.), *Handbook of Contemporary Learning Theories* (pp. 23–64). Mahwah, NJ: Erlbaum.

Wagner, A.R., & Rescorla, R.A. (1972). Inhibition in Pavlovian conditioning: applications of a theory. In M.S. Halliday & R.A. Boakes (Eds.), *Inhibition and Learning* (pp. 301–336). London, U.K.: Academic Press.

Wagner, A.R., & Vogel, E.H. (2009). Conditioning: theories. *Encyclopedia of Neuroscience* **3**, 49.