Microlocal Versal Deformations of Plane Curves
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Abstract
We introduce the notion of microlocal versal deformation of a plane curve. We construct equisingular versal deformations of Legendrian curves that are the conormal of a semi-quasi-homogeneous branch.

1 Contact Geometry
Schlessinger, Tyurina and Grauert initiated the study of versal deformations of analytic spaces (cf. [2]). We show in remark 2 that the obvious definition of deformation of a Legendrian curve is not very interesting. There would be too many rigid Legendrian curves. We consider an alternative approach, recovering Lie’s original point of view: to look at contact transformations as maps that take plane curves into plane curves.

We follow the terminology of [2] regarding deformations of analytic spaces. When we refer to a curve or a deformation, we are identifying it with its germ at a convenient point. If $Y \rightarrow T$ is a deformation of a plane curve $Y$ over a complex manifold $T$, we assume that $Y$ is reduced. We say that a deformation of a plane curve is equisingular if all of its fibers have the same topological type. Given a germ of an analytic set $Y$ we will denote by $\text{mult} Y$ the multiplicity of $Y$.

We set $C^A = \{(t_a)_{a \in A} : t_a \in \mathbb{C}\}, t_A = (t_a)_{a \in A}$, for each finite set $A$. In general, we denote $\partial f/\partial t$ by $\partial_t f$.

Let $(X, \mathcal{O}_X)$ be a complex manifold. A differential form of degree 1 is called a contact form if $\omega \wedge d\omega$ never vanishes. If $\omega$ is a contact form there is a system of local coordinates $(x, y, p)$ such that $\omega = dy - pdx$. A locally free $\mathcal{O}_X$-module $\Omega$ of rank 1 is called a contact structure if $\Omega$ is locally generated by a contact form. The pair $(X, \Omega)$ is called a contact threefold. Let $\xi dx + \eta dy$ denote the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. We will identify the open set $\{\eta \neq 0\}$ of $\mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1$ with the vector space $\mathbb{C}^3$, endowed with the coordinates $(x, y, p)$, where $p = -\xi/\eta$. The contact structure $\Omega$ of $\mathbb{C}^3$ generated by the differential form $\omega = dy - pdx$ is the restriction to $\mathbb{C}^3$ of the canonical contact structure of $\mathbb{P}^*\mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the canonical projection. Let $C(Z)$ be the tangent cone of a germ of an analytic

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set \( Z \). A Legendrian curve \( L \subset (\mathbb{P}^*\mathbb{C}^2, \sigma) \) is in **strong generic position** if 
\[ C(L) \cap (D\pi(\sigma))^{-1}(0) = \{0\}. \]

**Remark 1** (i) Let \( Y_1, Y_2 \) be germs of irreducible plane curves at \( o \). Let 
\( (L_i, \sigma_i) \) be the conormal of \( Y_i, \ i = 1, 2 \). Then \( \sigma_1 = \sigma_2 \) if and only if \( Y_1 \) and \( Y_2 \) have the same tangent cone. Hence the plane curve \( \pi(L) \) has irreducible tangent cone for each Legendrian curve \( L \subset \mathbb{P}^*\mathbb{C}^2 \). We will assume that all plane curves have irreducible tangent cone.

(ii) If \( Y \hookrightarrow \mathcal{Y} \to T \) is an equisingular deformation of a plane curve \( Y \), 
\( Y \hookrightarrow \mathcal{Y} \to T \) is isomorphic to a deformation \( Y \hookrightarrow Z \to T \) such that 
the tangent cone of \( Z \) does not depend on \( t \). We will assume that 
all the fibers of an equisingular deformation of a plane curve have the 
same tangent cone.

(iii) Let \( Y \) be a germ of an irreducible plane curve with tangent cone \( \{ y = 0 \} \). Let 
\( y = \sum_{\alpha \in \mathbb{Q}} a_\alpha x^\alpha \) be the Puiseux expansion of \( Y \). Set \( \delta = \inf\{ \alpha : a_\alpha \neq 0 \} \). The tangent cone of the conormal of \( Y \) equals 
\( \{ x = y = 0 \} \) if \( \delta < 2 \), \( \{ y = p - 2a_2x = 0 \} \) if \( \delta = 2 \) and 
\( \{ y = p = 0 \} \) if \( \delta > 2 \).

Let \( (X, \Omega_X) \) be a contact threefold. Let \( T \) be a complex manifold. Let 
\( q_X : X \times T \to X, r_X : X \times T \to T \) be the canonical projections. The pair 
\( (X \times T, q_X^*\Omega_X) \) is called a **relative contact threefold**. Let \( \mathcal{L} \) be an analytic 
set of dimension \( \dim T + 1 \) of \( X \times T \). We say that \( \mathcal{L} \) is a relative Legendrian 
curve if \( q_X^*\omega \) vanishes on the regular part of \( \mathcal{L} \), for each section \( \omega \) of \( \Omega_X \).
The intersection of \( \mathcal{L} \) with each fiber of \( r_X \) is a Legendrian curve. Given 
two relative contact threefolds \( X \times T \) and \( Y \times T \), a biholomorphic map 
\( \Phi : X \times T \to Y \times T \) is called a relative contact transformation if 
\( r_Y \circ \Phi = r_X \) and \( \Phi^*q_Y^*\Omega_Y = q_X^*\Omega_X \). Given \( t \in T \), let \( \Phi_t \) be the induced map from \( X \times \{ t \} \) 
into \( Y \times \{ t \} \). Two relative Legendrian curves \( \mathcal{L}_1 \subset X \times T, \mathcal{L}_2 \subset Y \times T \) are **isomorphic** if there is a relative contact transformation \( \Phi : X \times T \to Y \times T \) 
such that \( \Phi(\mathcal{L}_1) = \mathcal{L}_2 \). Let \( Y \hookrightarrow \mathcal{Y} \to T \) be a deformation of a plane curve. 
Let \( \pi_T : \mathbb{P}^*\mathbb{C}^2 \times T \to \mathbb{C}^2 \times T \) be the canonical projection. The **conormal** of \( \mathcal{Y} \) 
the smallest relative Legendrian curve \( \mathcal{L} \subset \mathbb{P}^*\mathbb{C}^2 \times T \) such that \( \pi_T(\mathcal{L}) = \mathcal{Y} \).
Let \( \mathcal{H} \) be the group of contact transformations of the type \( (x, y, p) \mapsto (\lambda x, \mu y, \mu^{-1} p), \lambda, \mu \in \mathbb{C}^* \). Set \( \ell = \{ y = p = 0 \} \). Let \( \mathcal{G} \) be the group of germs 
of contact transformations \( \varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) such that \( D\varphi(0)(\ell) = \ell \).
Let \( \mathcal{J} \) be the group of germs of contact transformations 
\( (x, y, p) \mapsto (x + \alpha, y + \beta, p + \gamma) \), where \( \alpha, \beta, \gamma, \partial_x \alpha, \partial_y \beta, \partial_p \gamma \in (x, y, p) \). (1)
Theorem 1.1 (cf. [1]) The group \( \mathcal{J} \) is an invariant subgroup of \( \mathcal{G} \). Moreover, \( \mathcal{G}/\mathcal{J} \) is isomorphic to \( \mathcal{H} \).

Theorem 1.2 (cf. [1]) Set \( t = (t_1, \ldots, t_m) \). Let \( \alpha \in \mathbb{C}\{x, y, p, t\} \) and \( \beta_0 \in \mathbb{C}\{x, y, t\} \) be power series such that \( \alpha, \partial_x \alpha, \beta_0, \partial_y \beta_0 \in (x, y, p) \). There are \( \beta, \gamma \in \mathbb{C}\{x, y, p, t\} \) such that \( \beta - \beta_0 \in (p) \), \( \gamma \in (x, y, p) \) and \( \mathcal{J} \) is a relative contact transformation. Moreover, \( \beta \) is the solution of the Cauchy problem

\[
\begin{pmatrix} 1 + \frac{\partial \alpha}{\partial x} + p \frac{\partial \alpha}{\partial y} \end{pmatrix} \frac{\partial \beta}{\partial p} - p \frac{\partial \alpha \partial \beta}{\partial p \partial y} - \frac{\partial \alpha \partial \beta}{\partial p \partial x} = p \frac{\partial \alpha}{\partial p}, \quad \beta - \beta_0 \in (p).
\]

We set \( \alpha = \sum_{i \geq 0} \alpha_ip^i \) and \( \beta = \sum_{i \geq 0} \beta_ip^i \), where \( \alpha_i, \beta_i \in \mathbb{C}\{x, y, t\} \) for all \( i \).

Theorem 1.3 Let \( L_1, L_2 \) be two germs of Legendrian curves of \( \mathbb{P}^1 \mathbb{C}^2 \) in strong generic position. Set \( Y_i = \pi(L_i), i = 1, 2 \). If there is a contact transformation \( \varphi \) such that \( \varphi(L_1) = L_2 \), there is a local homeomorphism \( \psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) such that \( \psi(Y_1) = Y_2 \).

Proof. Assume that \( L_1 \) is irreducible. Assume that \( (n_i, k_i) = 1, 1 \leq i \leq l \), and \( n_1/k_1 < \cdots < n_l/k_l \). Set \( k_0 = 1, k = k_l \). The curve \( Y_1 \) has Puiseux pairs \( n_i/k_i, 1 \leq i \leq l \), if and only if the following conditions hold for \( 1 \leq i \leq l \):

(i) \( a_{n_i/k_i} \neq 0 \); (ii) if \( a_j \neq 0 \) and \( k/k_{i-1} \) does not divide \( j, j \geq n_i/k_i \). Let \( i \in \{1, 2\} \). Following the notations of remark 1(iii), we can assume that \( C(Y_i) = \{y = 0\}, \delta > 2 \) and the tangent cone of \( L_i \) equals \( \{y = p = 0\} \). By theorem 1.1 we can assume that \( \varphi \) is of the type \( 1 \). Composing \( \varphi \) with the contact transformation induced by \( (x, y) \mapsto (x, y - \beta_0) \), we can assume that \( \beta_0 = 0 \). Hence \( \partial_x \beta, \partial_y \beta \in (p) \). Setting \( p = 0 \) in \( 2 \), we conclude that \( \beta_1 = 0 \). Let

\[
x = s^k, \quad y = s^n + \sum_{r \geq n+1} a_rs^r, \quad p = \frac{n}{k}s^{n-k} + \sum_{r \geq n+1} \frac{r}{k}a_rs^{r-k}.
\]

be a parametrization of \( L_1 \). The curve \( Y_2 \) admits a parametrization of the type \( x = s^k + \alpha(s), y = s^n + \sum_{r \geq n+1} a_rs^r + \sum_{l \geq 2} \beta_l(s)p(s)^l = s^k + \sum_{r \geq n+1} b_rs^r \), where \( \alpha \in (s^{k+1}) \). Since \( 2(n_i - k_i) > n_i \), the coefficients of \( \beta_l(s)p(s)^l \), \( l \geq 2 \), verify condition (ii). Hence the coefficients \( b_r \) verify conditions (i),(ii). Replacing the parameter \( s \) by a parameter \( t \) such that \( t^k = s^k + \alpha(s) \) one obtains a parametrization of \( Y_2 \) that still verifies conditions (i) and (ii) (cf. Lemma 3.5.4 of [4]).

If \( L_1' \) and \( L_2'' \) are irreducible Legendrian curves, a similar argument shows that the contact order (cf. Section 4.1 of [5]) of \( \pi(\varphi(L_1')) \) and \( \pi(\varphi(L_2'')) \) equals the contact order of \( \pi(L_1') \) and \( \pi(L_2'') \). □
Definition 1.4 Let $\mathcal{L}$ be a relative Legendrian curve of a relative contact threefold $X \times (T, o)$. We call plane projection of $\mathcal{L}$ to an analytic set $\pi_T(\Phi(\mathcal{L}))$, where $\Phi : X \times T \to \mathbb{P}^* \mathbb{C}^2 \times T$ is a relative contact transformation. The plane projection $\pi_T(\Phi(\mathcal{L}))$ is called generic if $\Phi_o(\mathcal{L}_o)$ is in strong generic position. Two Legendrian curves are equisingular if their generic plane projections have the same topological type.

Lemma 1.5 If $Y$ is a plane projection of a Legendrian curve $L$, $\text{mult} \ Y \geq \text{mult} \ L$. Moreover, $\text{mult} \ Y = \text{mult} \ L$ if and only if $Y$ is a generic plane projection of $L$. Hence the multiplicity is an equisingularity invariant of a Legendrian curve.

Proof. It's enough to prove the result when $L$ is irreducible. Let $\varphi : X \to \mathbb{P}^* \mathbb{C}^2$ be a contact transformation. Let $(3)$ be a parametrization of $\varphi(L)$. Then $\text{mult} \ Y = k$. Moreover, $\text{mult} \ L = k$ if $n \geq 2k$ and $\text{mult} \ L = n - k$ if $n < 2k$. The result follows from remark 1(iii). □

2 Microlocal Versal Deformations

Let $L$ be a Legendrian curve. A relative Legendrian curve $\mathcal{L} \subset X \times (T, o)$ is called a deformation of $L$ if the sets $\mathcal{L}_o$ and $L$ are equal. The deformation $\mathcal{L}$ is called equisingular if its fibers are equisingular. We do not demand the flatness of the morphism $\mathcal{L} \hookrightarrow X \times T \to T$. Let $Y$ be a plane curve with conormal $L$. A deformation $Y$ of $L$ is called a microlocal deformation of $Y$ if the conormal of $Y$ is a deformation of $L$.

Let $J'_T$ be the group of relative contact transformations $\Phi : (\mathbb{C}^3, 0) \times (T, o) \to (\mathbb{C}^3, 0) \times (T, o)$ such that $\Phi_o = \text{id}_{C^3}$. Let $m$ be the maximal ideal of $\mathcal{O}_{T, o}$. Given $\Phi \in J'_T$, there are $\alpha, \beta, \gamma \in m$ such that $\Phi$ equals $\mathbf{1}$. Two deformations $\mathcal{L}_1, \mathcal{L}_2$ of $L$ over $(T, o)$ are isomorphic if there is $\Phi \in J'_T$ such that $\Phi(\mathcal{L}_1) = \mathcal{L}_2$. Two microlocal deformations of a plane curve are microlocally equivalent if their conormals are isomorphic.

Remark 2 Let $\mathcal{L}$ be a flat equisingular deformation of the conormal of $\{y^k = x^n\}$ along $(T, o)$, where $n > k > 1$ and $(k, n) = 1$. Since $\mathcal{L}$ is flat and $ny - kxp$ vanishes on $\mathcal{L}_o$, there is an holomorphic function $f$ on $X \times T$ such that $f$ vanishes on $\mathcal{L}$ and $f(x, y, p, o) = ny - kxp$. Hence, if $t \in T$ is close enough to $o$, $\mathcal{L}_t$ is contained in a smooth hypersurface. By theorem 8.3 of $\mathbf{4}$, there are integers $l, m$ and a system of local coordinates $(x', y', p')$ on $X$ such that $m > l > 1$, $(l, m) = 1$ and $\mathcal{L}_t$ is the conormal of $\{y'^l = x'^m\}$. Since the deformation is equisingular, $l = k$ and $m = n$. Hence $\mathcal{L}_t$ is isomorphic
to $\mathcal{L}_o$ when $t$ is close enough to $o$. Modifying the proof of the theorem referred above one can show that de deformation $\mathcal{L}$ is trivial even without the assumption that $\mathcal{L}$ is equisingular.

**Theorem 2.1** Let $\mathcal{L}$ be an equisingular deformation of a Legendrian curve $L$ over $(T,o)$. A generic plane projection of $\mathcal{L}$ is an equisingular deformation of $\pi(\mathcal{L})$.

**Proof.** Let $\Phi : X \times T \to \mathbb{P}^*\mathbb{C}^2 \times T$ be relative contact transformation such that $\Phi_o(\mathcal{L}_o)$ is in strong generic position. It follows from lemma 1.5 and the upper-semicontinuity of the map $t \mapsto \text{mult} \pi(\Phi_t(\mathcal{L}_t))$ that $\Phi_t(\mathcal{L}_t)$ is in strong generic position for each $t$ in a neighbourhood $W$ of $o$. Hence the topological type of $\pi(\Phi_t(\mathcal{L}_t))$ equals the topological type of $\pi(\Phi_o(\mathcal{L}_o))$ for $t \in W$. Moreover, the multiplicity of the reduced curve $\pi(\Phi_t(\mathcal{L}_t))$ is constant near $o$. Therefore $\pi_T(\Phi(\mathcal{L}))_o$ is reduced. □

**Definition 2.2** Let $Y$ be a plane curve with an irreducible tangent cone. Let $L$ be the conormal of $Y$. We say that an [equisingular] microlocal deformation $\mathcal{Y}$ of $Y$ over $T$ is an [equisingular] microlocal versal deformation of $Y$ if (i) for each [equisingular] microlocal deformation $Y \hookrightarrow Z \to (R,o)$, there is $\psi : R \to T$ such that $\psi^*Y$ is microlocally equivalent to $Z$; (ii) the derivative of $\psi$ at $o$ only depends on $Z$.

**Definition 2.3** An [equisingular] deformation $\mathcal{L}$ of a Legendrian curve $L$ over $T$ is an [equisingular] versal deformation of $L$ if (i) for each [equisingular] deformation $K$ of $L$ over $(R,o)$ there is $\psi : R \to T$ such that $\psi^*L$ is isomorphic to $K$; (ii) The derivative of $\psi$ at $o$ only depends on $K$.

Let $k,n$ be integers such that $n > k > 1$ and $(k,n) = 1$. Set $Y = \{y^k - x^n = 0\}$. Let $c$ be the conductor of the semi-group of $Y$. Let $\mathcal{L}$ be an equisingular deformation of the conormal $L$ of $Y$.

**Remark 3** An equisingular deformation $\mathcal{Y}$ of $Y$ admits a parametrization of the type $x = s^k$, $y = s^n + \sum_{r \geq n+1} a_r s^r$, where $a_r \in \mathcal{O}_{T,o}$ and $a_r(o) = 0$. Hence the conormal $K$ of $\mathcal{Y}$ admits a parametrization $\sigma : (\mathbb{C},0) \times (T,o) \to K$ of type [3]. For each $t$, $s \mapsto \sigma(s,t)$ defines a parametrization of $K_t$, hence $K_t$ is the conormal of $\mathcal{Y}_t$. Therefore $\mathcal{Y}$ is a microlocal deformation of $Y$. The parametrization $\sigma$ defines a valuation $w$ of the ring $\mathbb{C}[[x,y,p,t]]$, the one that associates to $f \in \mathbb{C}[[x,y,p,t]]$ the multiplicity of the zero of $f \circ \sigma$ has an element of $\mathbb{C}[[t]][t^{-1}][[s]]$. 5
Lemma 2.4 Let \( i, j, l \) be non negative integers such that \( ki + nj + (n-k)l > kn \). There are non negative integers \( a, b \) such that \( w(x^iy^jp^l) = w(x^ay^b) \), \( x^iy^jp^l \equiv (n/k)^l x^ay^b \mod (t) + I_\mathcal{C} \) and \( w(x^iy^jp^l - (n/k)^l x^ay^b) > w(x^ay^b) \). Moreover, \( p^l \equiv (n/k)^k x^{n-k} \mod (t) + I_\mathcal{C} \).

\[
\text{Proof.} \quad \text{By remark 3, } x^p \equiv (n/k)y, x^{i+1}y^{j+1}p^l \equiv (n/k)x^iy^{j+1}p^l \text{ and } p^l \equiv (n/k)^k_x x^{n-k} \mod (t) + I_\mathcal{C}. \text{ Hence we can assume that } l < k. \text{ If } i = 0, \text{ then } j + l > k \text{ and } y^jp^l \equiv (n/k)^l x^{n-l}y^{i+l-k} \mod (t) + I_\mathcal{C}. \text{ Given an analytic set } Z \text{ of } (T, o), \text{ let } I_Z[I_Z] \text{ be the ideal of elements of } \mathcal{O}_{T,o} \text{ that vanish on } Z. \]

Lemma 2.5 Given \( u \in \mathbb{C}\{x, y, p, t\} \) such that \( w(u) \geq c \), there is \( v \in \mathbb{C}\{x, y, p, t\} \) such that \( u - v \in I_\mathcal{C} \).

\[
\text{Proof.} \quad \text{There are non negative integers } a, b \text{ and } \xi \in \mathbb{C}\{t\} \text{ such that } w(u) = w(x^aw^b) \text{ and } w(u - \xi x^aw^b) > w(u). \text{ We iterate the procedure, producing in this way } v \in \mathbb{C}\{x, y, t\} \text{ such that } w(u - v) = \infty. \]

Lemma 2.6 (cf. [1]) Given non negative integers \( a, b \text{ and } \lambda \in \mathbb{C}, \text{ there is a contact transformation } \phi \text{ of type } (1) \text{ and } \varepsilon \in \mathbb{C}\{x, y, p\} \text{ such that } \alpha = \lambda y^ap^b, \beta = \lambda y^b \phi^b/(b+1) + \varepsilon \text{ and } w(\varepsilon) \geq w(y^{2a-1}p^{2b+2}). \]

Theorem 2.7 The function \( F(x, y, t) = y^k - x^n + \sum_{(i,j) \in B} t_{i,j} x^iy^j \) defines an equisingular versal deformation \( \mathcal{F} \) of \( \{y^k - x^n = 0\} \over \mathbb{C}^B \), where \( B = \{(i,j) : ki + nj > kn, i \leq n-2, j \leq k-2\} \).

\[
\text{Proof.} \quad \text{It is a corollary of theorem II.1.16 of [2].} \]

Theorem 2.8 If \( n > 2k \), the function \( G(x, y, t) = y^k - x^n + \sum_{(i,j) \in C} t_{i,j} x^iy^j \) defines an equisingular microlocal versal deformation \( \mathcal{G} \) of \( \{y^k - x^n = 0\} \over \mathbb{C}^C \), where \( C = \{(i,j) \in B : i + j \leq n-2\} \).

\[
\text{Proof.} \quad \text{We can assume that } R = (\mathbb{C}^a, 0) \text{ and } \mathcal{O}_{\mathbb{C}^a,0} = \mathbb{C}\{r\}. \text{ Let } \mathcal{H} \in \mathbb{C}\{x, y, r\} \text{ be a generator of the ideal } I_Z. \text{ By theorem 2.7 we can assume that there are } \xi_{i,j} \in \mathbb{C}\{r\}, (i, j) \in B, \text{ such that } \xi_{i,j} \in (r) \text{ and } H(x, y, r) = y^k - x^n + \sum_{(i,j) \in B} \xi_{i,j} x^iy^j. \text{ Moreover, the functions } \xi_{i,j} \text{ are uniquely determined mod } (t^2). \text{ Let } \mathcal{K} \text{ be the conormal of } Z. \text{ Assume that there are } \psi : \mathbb{C}^a \to \mathbb{C}^C, \psi = (\psi_{i,j}), \text{ and } \Phi \in \mathcal{J}_{\mathbb{C}^a} \text{ such that } \Phi^{-1}(\mathcal{K}) \text{ equals the conormal of } \psi^* \mathcal{G}. \text{ There are } \alpha, \beta, \gamma \in \mathbb{C}\{x, y, p, r\} \text{ such that } \Phi \text{ equals } (1). \text{ Hence there is } \varepsilon \in (r). \]
such that $H(x+\alpha,y+\beta,r) = (1+\varepsilon(x,y,r))G(x,y,\psi(r)) \mod I_{\Phi^{-1}(\mathcal{K})}$. Notice that

$$H(x+\alpha,y+\beta,r) \equiv H(x,y,r) + ky^{k-1}\beta - nx^{n-1}\alpha \mod (r)^2.$$  

By lemma 2.4, $x^{n-1}\alpha, y^{k-1}\beta$ are congruent modulo $(r)^2 + \tilde{I}_{\Phi^{-1}(\mathcal{K})}$ with elements of $(x,y)^{n-1}\mathbb{C}[[x,y,r]]$. Hence $\psi^{i,j} \equiv \xi_{i,j} \mod (r)^2$ for each $(i, j) \in C$.

By theorem 2.7, it is enough to show that there is $\psi : \mathbb{C}^B \to \mathbb{C}^C$ such that $\psi^*\mathcal{G}$ is microlocally equivalent to $\mathcal{F}$. Set $N = \#(B \setminus C)$. Let us order the pairs $(i, j) \in B \setminus C$ by the value of $ki + nj$. Let $B_l$ be the set $B$ minus the set of the $l$ smaller ordered pairs of $B \setminus C$. Set $H_l(x,y,t_{B_l}) = y^k - x^n + \sum_{(i,j) \in B_l} t_{i,j} x^i y^j$. Let $\mathcal{L}_l$ be the conormal of $H_l = \{H_l = 0\}$. Let $0 \leq l \leq N - 1$. It is enough to show that there is $\psi_l : \mathbb{C}^{B_l} \to \mathbb{C}^{B_{l+1}}$ such that $\psi_l^* H_{l+1}$ is microlocally equivalent to $H_l$. Let $(a, b)$ be the smallest element of $B_l \setminus C$. Let $v = w_l(x^ay^b)$. Let $\Phi_l$ be the contact transformation of the type (10) associated by theorem 2.2 to $\alpha = \lambda y^{a+b-(n-1)p^{n-1}}$ and $\beta = 0$. Let $\psi_l$ be the valuation associated to $\oplus^{-1}_l(\mathcal{L}_l)$. By lemma 2.6, $\beta = \lambda(n-1-a)y^{a+b-(n-1)p^{n-1}}/(n-a) + \mu$, with $w_l(\mu) > w_l((y^{a+b-(n-1)p^{n-1}}).$

There is $\delta \in \mathbb{C}\{x,y,p,t_{B_l}\}$ such that $w_l(\delta) > v$ and

$$H_l(x+\alpha,y+\beta,t_{B_l}) \equiv H_{l+1}(x,y,t_{B_{l+1}}) + t_{a,b} x^a y^b + ky^{k-1}\beta - nx^{n-1}\alpha + \delta \mod I_{\Phi_l^{-1}(\mathcal{L}_l)}.$$  

Set $\lambda = t_{a,b}(n-a)(n/k)^{n-1-a}/n$. By lemma 2.3, there is $\tilde{\delta} \in \mathbb{C}\{x,y,t_{B_l}\}$ such that $w_l(\tilde{\delta}) > v$ and $H_l(x+\alpha,y+\beta,t_{B_l}) \equiv H_{l+1}(x,y,t_{B_{l+1}}) + \tilde{\delta} \mod I_{\Phi_l^{-1}(\mathcal{L}_l)}$. Since $\pi_{B_l}(\Phi_l^{-1}(\mathcal{L}_l))$ is a convergent hypersurface, we can assume that $\tilde{\delta} \in \mathbb{C}\{x,y,t_{B_l}\}$. By the proof of theorem 2.7, there are $\alpha, \beta, \varepsilon \in \mathbb{C}\{x,y\}[[t_{B_l}]]$ and $\psi^{i,j} \in \mathbb{C}[t_{B_l}]$, such that $\alpha, \beta, \varepsilon \in (t_{B_l})$ and

$$(H_{l+1} + \tilde{\delta})(x+\alpha,y+\beta,t_{B_l}) = (1+\varepsilon)F(x,y,\psi_l(t_{B_l})),  

where $F_l = (\psi^{i,j}).$ Since $y^{k-1} - x^n$ is quasi-homogeneous, we can assume that $\varepsilon$ vanishes. Set $\alpha = \sum_\alpha^q \alpha_q$, $\beta = \sum_\beta q$, $\psi^{i,j} = \sum_\psi^{i,j} q^{i,j}$, where $\alpha_q, \beta_q, \psi^{i,j}_q$ are homogeneous functions of degree $q$ on the variables $t_{i,j}, (i,j) \in B_l$. For each $q$ there is an homogeneous part $g_q$ of degree $q$ of an element of the ideal $(\partial_x H_{l+1}, \partial_y H_{l+1})$ such that $\alpha_q nx^{n-1} + \beta_q ky^{k-1} = g_q$. Moreover, each $g_q$ depends on the choices of $\alpha_p, \beta_p$, $p < q$. Notice that $\alpha_0, \beta_0, g_0 = 0$, $g_1$ is the linear part of $\tilde{\delta}$, $\psi^{i,j}_1 = t_{i,j}$ if $(i,j) \in B_{l+1}$ and $\psi^{i,j}_1 = 0$ otherwise. We choose $\alpha_1, \beta_1$ such that $\beta_1 ky^{k-1}$ is the rest of the division of $g_1$ by $nx^{n-1}$. Since $w_l(g_1) > v$, then $w_l(\alpha_1 x^{n-1}), w_l(\beta_1 y^{k-1}), w_l(g_2) > v$ and $\psi^{i,j}_1 = 0$ if
\( ki + nj \leq v \). Moreover, we can iterate the procedure. We show in this way that \( F(x, y, \psi(t_B)) = H_{i+1}(x, y, \psi(t_B)) \). Since our choices of \( \alpha_q, \beta_q \) are the ones of lemma 1 of [3] and of theorem II.D.2 of [4], the functions \( \alpha, \beta, \psi^{i,j}, (i, j) \in B \), converge. \( \square \)

**Corollary 2.9** Each irreducible Legendrian curve \( L \) contained in a smooth surface admits an equisingular versal deformation. This deformation is trivial if and only if \( L \) is isomorphic to the conormal of a curve defined by one of the functions \( y^2 - x^{2n+1}, n \geq 1, y^3 - x^7, y^3 - x^8 \).

**Proof.** By theorem 8.3 of [5], there are integers \( k, n \) such that \( n > 2k > 1, (k, n) = 1 \) and the Legendrian curve is isomorphic to the conormal of \( \{y^k = x^n\} \). The corollary follows from results [2.1] and [2.8] \( \square \)

By the canonical properties of versal deformations (cf. theorem II.1.15 of [2]), the previous result still holds if \( L \) is the conormal of a semi-quasihomogeneous branch.

**Example 1** Set \( f_0(x, y) = y^4 - x^{11}, f_1(x, y) = y^4 - x^{11} + x^6 y^2, f_2(x, y) = y^4 - x^{11} + x^7 y^2 \). Let \( L_i \) be the conormal of \( \{f_i = 0\}, 0 \leq i \leq 2 \). A Legendrian curve equisingular to \( L_0 \) is isomorphic to one and only one of the curves \( L_0, L_1, L_2 \). The equisingular microlocal versal deformation of \( y^4 - x^{11} \) equals \( G(x, y, t_2, t_6) = y^4 - x^{11} + t_2 x^6 y^2 + t_6 x^7 y^2 \). Moreover,

\[
G(\lambda^4 x, \lambda^{11} y, t_2, t_6) = \lambda^{44} G(x, y, \lambda^2 t_2, \lambda^6 t_6), \quad \text{for each } \lambda \in \mathbb{C}^*. \quad (4)
\]

If \( t_2 \neq 0 \), it follows from [4] that we can assume \( t_2 = 1 \). Moreover, there are \( \varepsilon, \delta \in (x, y) \) such that \( G(x - 2t_6 x^2, y - 11t_6 x y / 2, 1, t_6) = (1 - 22 t_6 x + \varepsilon)(y^4 - x^{11} + x^6 y^2 + \delta) \). By theorem 2.8 we can assume that \( \delta = 0 \). If \( t_2 = 0 \) it follows from [1] that we can assume \( t_6 = 0 \) or \( t_6 = 1 \).

Since \( L_0 \) is contained in the surface \( 11y - 4xp = 0 \) and \( L_1, L_2 \) are not contained in a smooth surface, \( L_0 \) cannot be isomorphic to \( L_1 \) or \( L_2 \). Let \( \varphi \in \mathcal{J} \). There is \( \varepsilon \in \mathbb{C}\{x, y\} \) such that \( f_1(x + \alpha, y + \beta) \equiv f_1(x, y) + \varepsilon(x, y) \mod I_{\varphi(L_1)} \). If \( f_1(x, y) + \varepsilon(x, y) \equiv f_2, w(\varepsilon) \geq 46 \). Hence \( w(\varepsilon) \geq 47 \). Therefore \( f_1 + \varepsilon \neq f_2 \). By theorem 1.1, \( \varphi(L_1) \neq L_2 \) for each \( \varphi \in \mathcal{G} \).

**References**

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