Locality and applications to subsumption testing and interpolation in $\mathcal{EL}$ and some of its extensions

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Abstract In this paper we show that subsumption problems in lightweight description logics (such as $\mathcal{EL}$ and $\mathcal{EL}^+$) can be expressed as uniform word problems in classes of semilattices with monotone operators. We use possibilities of efficient local reasoning in such classes of algebras, to obtain uniform PTIME decision procedures for CBox subsumption in $\mathcal{EL}$, $\mathcal{EL}^+$ and extensions thereof. These locality considerations allow us to present a new family of (possibly many-sorted) logics which extend $\mathcal{EL}$ and $\mathcal{EL}^+$ with $n$-ary roles and/or numerical domains. As a by-product, this allows us to show that the algebraic models of $\mathcal{EL}$ and $\mathcal{EL}^+$ have ground interpolation and thus that $\mathcal{EL}$, $\mathcal{EL}^+$, and their extensions studied in this paper have interpolation. We also show how these ideas can be used for the description logic $\mathcal{EL}^{++}$.

1 Introduction

Description logics are logics for knowledge representation used in databases and ontologies. They provide a logical basis for modeling and reasoning about objects, classes of objects (concepts), and relationships between them (roles). Recently, tractable description logics such as $\mathcal{EL}$ [2] have attracted much interest. Although they have restricted expressivity, this expressivity is sufficient for formalizing the type of knowledge used in widely used ontologies such as the medical ontology SNOMED [28,29]. Several papers were dedicated to studying the properties of $\mathcal{EL}$ and its extensions $\mathcal{EL}^+$ [4,6] and $\mathcal{EL}^{++}$ [5], and to understanding the limits of tractability in extensions of $\mathcal{EL}$. Undecidability results for extensions of $\mathcal{EL}$ are obtained in [1] using a reduction to the word problem for semi-Thue systems.

In this paper we show that the subsumption problem in $\mathcal{EL}$ and $\mathcal{EL}^+$ can be expressed as a uniform word problem in certain varieties of semilattices with monotone operators. We identify a large class of such algebras for which the uniform word problem is decidable in PTIME. For this, we use results on so-called local theory extensions which we introduced in [21] and further developed in [22,23,27]. In [15,14,26] we proved that local theory extensions occur in a natural way in verification (especially in program verification, and in the verification of parametric systems) and in mathematics. The purpose of this paper is to
show that the concept of local theory extension turns out to be useful also for identifying and studying tractable extensions of $\mathcal{EL}$. General results on local theories allow us to:

- uniformly present extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$ with $n$-ary roles (and concrete domains);
- provide uniform complexity analysis for $\mathcal{EL}$ and $\mathcal{EL}^+$ and their extensions;
- analyze interpolation in the corresponding algebraic models and its consequences.

The concept constructors, role constructors and role inclusions we can consider are summarized in Figure 1. The main contributions of the paper are:

- We show that the subsumption problem in $\mathcal{EL}$ (resp. $\mathcal{EL}^+$) can be expressed as a uniform word problem in classes of semilattices with monotone operators (possibly satisfying certain composition laws).
- We show that the corresponding classes of semilattices with operators have local presentations and we use methods for efficient reasoning in local theories or in local theory extensions in order to obtain PTIME decision procedures for $\mathcal{EL}$ and $\mathcal{EL}^+$.
- These locality considerations allow us to present new families of PTIME logics with $n$-ary roles (and possibly also concrete domains) which extend $\mathcal{EL}$ and $\mathcal{EL}^+$.
- In particular, we identify a PTIME extension of $\mathcal{EL}$ with two sorts, concept and num, where the concepts of sort num are interpreted as elements in the ORD-Horn, convex fragment of Allen’s interval algebra.
- We notice that the axioms which correspond, at an algebraic level, to the role inclusions in $\mathcal{EL}^+$ are exactly of the type studied in the context of hierarchical interpolation in \[22\]. As a by-product, we thus show that the algebraic models of $\mathcal{EL}$ and $\mathcal{EL}^+$ have the ground interpolation property and infer that $\mathcal{EL}$, $\mathcal{EL}^+$, and their extensions studied in this paper have interpolation.
- We end the paper with some considerations on possibilities of handling $\mathcal{EL}^{++}$ constructors and ABoxes.

Some of the results of this paper were reported – in preliminary form – in \[24,25\]. At that time we could only prove a weak locality property in the presence of role inclusions. In
Fig. 2 A L C constructors and their semantics

| Constructor name | Syntax  | Semantics             |
|------------------|---------|-----------------------|
| bottom           | ⊥       | D                     |
| top              | ⊤       | D                     |
| negation         | ¬C      | D\(\lor C\)           |
| conjunction      | C\(\land C_2\) | C_I\(\land C_I_2\) |
| disjunction      | C\(\lor C_2\) | C_I\(\lor C_I_2\) |
| existential restriction | \(\exists r.\{x\mid \exists y((x,y)\in r\land y\in C_I}\}\) | |
| universal restriction | \(\forall r.\{x\mid \forall y((x,y)\in r\rightarrow y\in C_I)}\)| |

In this paper we considerably improve the results presented in [24,25] by showing that \(\mathcal{EL}\) and \(\mathcal{EL}^+\) as well as some of their extensions enjoy the same type of locality property, which allows to reduce, ultimately, CBox subsumption checking to checking the satisfiability of ground clauses in the theory of partially-ordered sets. We thus obtain a cubic time decision procedures for CBox subsumption in a class of extensions of \(\mathcal{EL}\). New contributions of this paper are also (i) the applications of our results on interpolation in local theory extensions [22,23] to interpolation in \(\mathcal{EL}^+\) and (ii) the presentation of PTIME results in \(\mathcal{EL}^{++}\) in the framework of locality.

**Structure of the paper.** In Sect. 2 we present generalities on description logic and introduce the description logics \(\mathcal{EL}\) and \(\mathcal{EL}^+\). In Sect. 3 we provide the notions from algebra and correspondence theory needed in the paper. In Sect. 4 we show that for many extensions of \(\mathcal{EL}\) CBox subsumption can be expressed as a uniform word problem in the class of semilattices with monotone operators satisfying certain composition axioms. In Sect. 5 we present general definitions and results on local theory extensions and in Sect. 6 we show that the algebraic models of \(\mathcal{EL}\) and \(\mathcal{EL}^+\) have local presentations, thus providing an alternative proof of the fact that CBox subsumption in \(\mathcal{EL}\) and \(\mathcal{EL}^+\) is decidable in PTIME. locality results for more general classes of semilattice with operators are used in Sect. 6.4 for defining extensions of \(\mathcal{EL}\) and \(\mathcal{EL}^+\) with a subsumption problem decidable in PTIME. In Sect. 7 we use these results for obtaining interpolation results for \(\mathcal{EL}\) and its extensions. The results in Sect. 8 show that also PTIME decidability of CBox subsumption in \(\mathcal{EL}^{++}\) can be explained within the framework of locality.

**2 Description logics: generalities**

The central notions in description logics are concepts and roles. In any description logic a set \(N_C\) of concept names and a set \(N_R\) of roles is assumed to be given. Complex concepts are defined starting with the concept names in \(N_C\), with the help of a set of concept constructors. The available constructors determine the expressive power of a description logic. The semantics of description logics is defined in terms of interpretations \(\mathcal{I} = (D^\mathcal{I},r^\mathcal{I})\), where \(D^\mathcal{I}\) is a non-empty set, and the function \(\cdot^\mathcal{I}\) maps each concept name \(C \in N_C\) to a set \(C^\mathcal{I} \subseteq D^\mathcal{I}\) and each role name \(r \in N_R\) to a binary relation \(r^\mathcal{I} \subseteq D^\mathcal{I} \times D^\mathcal{I}\). Fig. 2 shows the constructor names used in the description logic \(\mathcal{L}\) and their semantics. The extension of \(\cdot^\mathcal{I}\) to concept descriptions is inductively defined using the semantics of the constructors.

**Definition 1 (Terminology)** A terminology (or TBox, for short) is a finite set consisting of primitive concept definitions of the form \(C \equiv D\), where \(C\) is a concept name and \(D\) a concept
description; and general concept inclusions (GCI) of the form $C \sqsubseteq D$, where $C$ and $D$ are concept descriptions.

**Definition 2 (Interpretation)** An interpretation $\mathcal{I}$ is a model of a TBox $\mathcal{T}$ if it satisfies:

- all concept definitions in $\mathcal{T}$, i.e. $C^\mathcal{I} \vdash D^\mathcal{I}$ for all definitions $C \equiv D \in \mathcal{T}$;
- all general concept inclusions in $\mathcal{T}$, i.e. $C^\mathcal{I} \sqsubseteq D^\mathcal{I}$ for every $C \sqsubseteq D \in \mathcal{T}$.

Since definitions can be expressed as double inclusions, in what follows we will only refer to TBoxes consisting of general concept inclusions (GCI) only.

**Definition 3 (TBox subsumption)** Let $\mathcal{T}$ be a TBox, and $C_1, C_2$ two concept descriptions. $C_1$ is subsumed by $C_2$ w.r.t. $\mathcal{T}$ (for short, $C_1 \sqsubseteq_C C_2$) if and only if $C_1^\mathcal{I} \subseteq C_2^\mathcal{I}$ for every model $\mathcal{I}$ of $\mathcal{T}$.

### 2.1 The description logics $\mathcal{EL}$, $\mathcal{EL}^+$ and some extensions

By restricting the type of allowed concept constructors less expressive but tractable description logics can be defined. If we only allow intersection and existential restriction as concept constructors, we obtain the description logic $\mathcal{EL}$ [2], a logic used in terminological reasoning in medicine [28,29]. In [4,6], the extension $\mathcal{EL}^+$ of $\mathcal{EL}$ with role inclusion axioms is studied. Relationships between concepts and roles are described using CBoxes.

**Definition 4 (Constraint box)** A CBox consists of a terminology $\mathcal{T}$ and a set $RI$ of role inclusions of the form $r_1 \circ \ldots \circ r_n \subseteq s$. Since terminologies can be expressed as sets of general concept inclusions, we will view CBoxes as unions $GCI \cup RI$ of a set $GCI$ of general concept inclusions and a set $RI$ of role inclusions of the form $r_1 \circ \ldots \circ r_n \subseteq s$, with $n \geq 1$.

**Definition 5 (Models of CBoxes)** An interpretation $\mathcal{I}$ is a model of the CBox $\mathcal{C} = GCI \cup RI$ if it is a model of GCI and satisfies all role inclusions in $\mathcal{C}$, i.e. $r_1^\mathcal{I} \circ \ldots \circ r_n^\mathcal{I} \subseteq s^\mathcal{I}$ for all $r_1 \circ \ldots \circ r_n \subseteq s \in RI$.

**Definition 6 (CBox subsumption)** If $\mathcal{C}$ is a CBox, and $C_1, C_2$ are concept descriptions then $C_1 \sqsubseteq_{\mathcal{C}} C_2$ if and only if $C_1^\mathcal{I} \subseteq C_2^\mathcal{I}$ for every model $\mathcal{I}$ of $\mathcal{C}$.

In [4] it was shown that subsumption w.r.t. CBoxes in $\mathcal{EL}^+$ can be reduced in linear time to subsumption w.r.t. normalized CBoxes, in which all GCIs have one of the forms: $C \sqsubseteq D, C_1 \sqcap C_2 \sqsubseteq D, C \sqsubseteq \exists r_1 D, \exists r_2 \sqcap C \subseteq D$, where $C, C_1, C_2, D$ are concept names, and all role inclusions are of the form $r \subseteq s$ or $r_1 \circ r_2 \subseteq r$. Therefore, in what follows, we consider w.l.o.g. that CBoxes only contain role inclusions of the form $r \subseteq s$ and $r_1 \circ r_2 \subseteq r$.

In [5], the extension $\mathcal{EL}^{++}$ of $\mathcal{EL}^+$ is introduced. In addition to the constructions in $\mathcal{EL}^+$, $\mathcal{EL}^{++}$ can be parameterized by one or more concrete domains $\mathcal{D}_1, \ldots, \mathcal{D}_m$, which correspond to standard data types and permit reference to concrete data objects such as strings and integers. Formally, a concrete domain is a pair $\mathcal{D} = (D^\mathcal{D}, P^\mathcal{D})$, where $D^\mathcal{D}$ is a set and $P^\mathcal{D}$ is a family of predicate names with given (strictly positive) arity, and given interpretations as relations on $D^\mathcal{D}$. The link between the description logic and the concrete domains is established by means of a set of feature names $\mathcal{H}_F$, interpreted as maps $f : D \rightarrow D_i$, where $D$ is the universe of the interpretation $\mathcal{I}$ of the description logic and $D_i$ is the...
Locality and applications to subsumption testing

universe of a concrete domain \( \mathcal{D} \). TBoxes can contain constraints referring to features and concrete domains, of the form

| Name         | Syntax                                      | Semantics                                                                 |
|--------------|---------------------------------------------|---------------------------------------------------------------------------|
| concrete domains | \( p(f_1, \ldots, f_n) \) | \( \{ x \in D^\mathcal{D} \mid \exists y_1, \ldots, y_k \in D^\mathcal{D} : f_i^\mathcal{D}(x) = y_i \text{ for } 1 \leq i \leq k \} \text{ and } p^\mathcal{D}\{y_1, \ldots, y_k\} \) |

In this paper we show that CBox subsumption for \( \mathcal{E}L \) and \( \mathcal{E}L^+ \) can be expressed as a uniform word problem for classes of semilattices with monotone operators. We then analyze various other types of axioms leading to extensions of \( \mathcal{E}L \) and \( \mathcal{E}L^+ \), including a variant of \( \mathcal{E}L^+ \) without ABoxes.

We start by presenting the necessary notions from algebra.

3 Algebra: preliminaries

We assume known notions such as partially-ordered set and order filter/ideal in a partially-ordered set. For further information cf. [18]. In what follows we will use one-sorted as well as many-sorted algebraic structures.

Let \( \Sigma \) be a (one-sorted) signature consisting of a set of function symbols, together with an arity function \( a : \Sigma \to \mathbb{N} \) which associates with every function symbol its arity. An algebraic structure (over \( \Sigma \)) is a tuple \( \mathcal{A} = (A, \{f_\alpha\}_{\alpha \in \Sigma}) \), where \( A \) is a non-empty set (the universe of \( \mathcal{A} \)) and for every \( f \in \Sigma \), if \( a(f) = n \) then \( f_A : A^n \to A \).

Let \( (S, \Sigma) \) be a many-sorted signature consisting of a set \( S \) of sorts and a set \( \Sigma \) of function symbols, together with an arity function \( a : \Sigma \to (S^* \to S) \) which associates with every function symbol \( f \) its arity \( a(f) = s_1, \ldots, s_n \to s \) (which specifies the sorts of the \( n \) arguments of \( f \) and the sort of the output). A (many-sorted) algebraic structure (over \( (S, \Sigma) \)) is a tuple \( \mathcal{A} = (\{A_s\}_{s \in S}, \{f_\alpha\}_{\alpha \in \Sigma}) \), where for every \( s \in S, A_s \) is a non-empty set (the universe of \( \mathcal{A} \) of sort \( s \)) and for every \( f \in \Sigma \), if \( a(f) = s_1 \ldots s_n \to s \) then \( f_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s \).

3.1 Semilattices, (distributive) lattices, Boolean algebras

An algebraic structure \( (L, \wedge) \) consisting of a non-empty set \( L \) together with a binary operation \( \wedge \) is called semilattice if \( \wedge \) is associative, commutative and idempotent. An algebraic structure \( (L, \lor, \wedge) \) consisting of a non-empty set \( L \) together with two binary operations \( \lor \) and \( \land \) on \( L \) is called lattice if \( \lor \) and \( \land \) are associative, commutative and idempotent and satisfy the absorption laws. A distributive lattice is a lattice that satisfies either of the distributive laws \( (D_\land) \) or \( (D_\lor) \), which are equivalent in a lattice.

\[
(D_\land) \quad \forall x, y, z \quad x \land (y \lor z) = (x \land y) \lor (x \land z)
\]

\[
(D_\lor) \quad \forall x, y, z \quad x \lor (y \land z) = (x \lor y) \land (x \lor z)
\]

In any semilattice \( (L, \wedge) \) or lattice \( (L, \lor, \wedge) \) an order can be defined in a canonic way by

\[ x \leq y \text{ if and only if } x \land y = x. \]

An element 0 which is smaller than all other elements w.r.t. \( \leq \) is called first element; an element 1 which is larger than all other elements w.r.t. \( \leq \) is called last element. A lattice having both a first and a last element is called bounded. A Boolean algebra is a structure
(B, ∨, ∧, ¬, 0, 1), such that (B, ∨, ∧, 0, 1) is a bounded distributive lattice and ¬ is a unary operation that satisfies:

\[(\text{Complement}) \quad \forall x \quad \neg x \lor x = 1 \quad \forall x \quad \neg x \land x = 0\]

Let \( \mathcal{Y} \) be a class of algebras. The *universal Horn theory* of \( \mathcal{Y} \) is the collection of those closed formulae valid in \( \mathcal{Y} \) which are of the form

\[
\forall x_1 \ldots \forall x_n \left( \bigwedge_{i=1}^{\mu} s_{i1} = s_{i2} \rightarrow t_1 = t_2 \right)
\]

(1)

The formula (1) above is valid in \( \mathcal{Y} \) if for each algebra \( \mathcal{A} \in \mathcal{Y} \) with universe \( A \) and for each assignment \( v \) of values in \( A \) to the variables, if \( v(s_{i1}) = v(s_{i2}) \) for all \( i \in \{1, \ldots, n\} \) then \( v(t_1) = v(t_2) \). The problem of deciding the validity of universal Horn sentences in a class \( \mathcal{Y} \) of algebras is also called the *uniform word problem* for \( \mathcal{Y} \). It is known that the uniform word problem is decidable for the following classes of algebras: The class SL of semilattices (in PTIME), the class DL of distributive lattices (coNP-complete), and the class Bool of Boolean algebras (NP-complete).

### 3.2 Boolean algebras with operators

In what follows we will consider the following class of Boolean algebras with operators:

**Definition 7** Let \( \text{BAO}(\Sigma) \) be the class of Boolean algebras with operators in \( \Sigma \), of the form

\[
(B, \lor, \land, \neg, 0, 1, \{f_{\Sigma}\}_{f \in \Sigma})
\]

such that for every \( f \in \Sigma \) of arity \( n = a(f) \), \( f_B : B^n \rightarrow B \) is a join-hemimorphism, i.e.

\[
\forall x_1, \ldots, x_i, \ldots, x_n \quad f(x_1, \ldots, x_i \lor x'_i, \ldots, x_n) = f(x_1, \ldots, x_i, \ldots, x_n) \lor f(x_1, \ldots, x'_i, \ldots, x_n)
\]

\[
\forall x_1, \ldots, x_n \quad f(x_1, \ldots, 0, \ldots, x_n) = 0.
\]

With every join-hemimorphism on a Boolean algebra \( B \), \( f_B : B^n \rightarrow B \) we can associate a map \( g_B : B^n \rightarrow B \) defined for every \( (x_1, \ldots, x_n) \in B^n \) by \( g_B(x_1, \ldots, x_n) = \neg \land (\neg x_1, \ldots, \neg x_n) \). The map \( g_B \) is a meet-hemimorphism in every argument, i.e. it satisfies, for every \( 1 \leq i < n \):

\[
\forall x_1, \ldots, x_i, x'_i, \ldots, x_n \quad g(x_1, \ldots, x_i \land x'_i, \ldots, x_n) = g(x_1, \ldots, x_i, \ldots, x_n) \land g(x_1, \ldots, x'_i, \ldots, x_n)
\]

\[
\forall x_1, \ldots, x_n \quad g(x_1, \ldots, x_1, \ldots, x_n) = 1.
\]

In relationship with \( \mathcal{E}_L \) and \( \mathcal{E}_L^+ \) we will also use the following types of algebras:

- \( \text{DL}(\Sigma) \) the class of bounded distributive lattices with operators \( (L, \lor, \land, 0, 1, \{f_L\}_{f \in \Sigma}) \), such that \( f_L : L^n \rightarrow L \) is a join-hemimorphism of arity \( n = a(f) \);
- \( \text{SLO}(\Sigma) \) the class of all \( \land \)-semilattices with operators \( (S, \land, 0, 1, \{f_S\}_{f \in \Sigma}) \), such that \( f_S \) is monotone and \( f_S(0) = 0 \).

In what follows we will denote join-hemimorphisms by \( f_\Sigma \) and the associated meet-hemimorphisms by \( f_\Sigma \). The reason for this notation will become clear in Section 6.3 and especially in Section 4.
3.3 Correspondence theory

We now present some links between axioms satisfied in Boolean algebras with operators and properties of relational spaces.[2]

Definition 8 (Duals of Boolean algebras with operators) Let $B = (B, \land, \lor, \neg, 0, 1, \{f_\alpha\}_{\alpha \in \Sigma})$ be a Boolean algebra with operators having the property that for every $f \in \Sigma$, $f_\alpha : B^{\alpha} \rightarrow B$ is a join-hemimorphism in every argument, and let $f_\alpha : B^{\alpha} \rightarrow B$ be defined by $f_\alpha(x_1, \ldots, x_n) = \neg f_\alpha(-x_1, \ldots, -x_n)$ for every $x_i \in B^{\alpha_i}$ (a meet-hemimorphism in every argument).

The *Stone dual* of $B$ is the topological relational space $D(B) = (\mathcal{P}(B), \{f_\alpha\}_{\alpha \in \Sigma})$ having as support the set $\mathcal{P}(B)$ of all prime filters of $B$ with the Stone topology, and relations associated with the operators of $B$ in a canonical way by:

$$r_\alpha (F, F_1, \ldots, F_n) \text{ iff } f_\alpha (F_1, \ldots, F_n) \subseteq F.$$ 

Definition 9 (Canonical extension of a Boolean algebra with operators) The canonical extension of $B$ is the Boolean algebra of subsets of the Stone dual $D(B)$ of $B$, $\mathcal{P}(D(B)) = (\mathcal{P}(\mathcal{P}(B)), \cup, \cap, 0, \mathcal{P}(B), \{f_\alpha\}_{\alpha \in \Sigma})$, where

$$f_\alpha (U_1, \ldots, U_n) = \{F \mid \exists F_1, \ldots, F_n \in \mathcal{P}(B), r_\alpha (F, F_1, \ldots, F_n)\}.$$

3.3.1 From algebras to relational spaces

We now analyze the link between properties of Boolean algebras with operators and properties of their duals. We focus on the properties related to the role inclusions considered in the study of $\mathcal{L}^\ast$.

We consider slightly more general guarded role inclusions of the form:

$$\forall x \ (x \in C \land r(x, y) \rightarrow s(x, y))$$

$$\forall x, y \ (x \in C \land r_1 \circ s(x, y) \rightarrow r_2(x, y))$$

$$\forall x, y \ (x \in C \land r_1 \circ s(x, y) \rightarrow x = y)$$

Theorem 1 Let $B \in BAO(\Sigma)$, let $f, g, h \in \Sigma$ be unary join-hemimorphisms on $B$; and let $c$ be a constant and $C$ be the predicate associated in a canonical way with $c$ in $D(B)$ by

$$C(F) \text{ iff } c \in F.$$

(1) If $B \models \forall x (x \leq c \rightarrow g(x) \leq h(x))$ then $D(B) \models \forall x, y (y \in C \land r_2(x, y) \rightarrow r_3(x, y))$.

(2) If $B \models \forall x (x \leq c \rightarrow f(g(x)) \leq h(x))$ then $D(B) \models \forall x, y (y \in C \land r_2 \circ r_3(x, y) \rightarrow r_4(x, y))$.

(3) If $B \models \forall x (x \leq c \rightarrow f(g(x)) \leq x)$ then $D(B) \models \forall x, y (x \in C \land r_2 \circ r_4(x, y) \rightarrow x = y)$.

Proof: (1) Assume that $B \models \forall x (x \leq c \rightarrow g(x) \leq h(x))$. Let $F, G \in \mathcal{P}(B)$. Assume that $G \in C$ and $r_3(F, G)$. Then $c \in G$ and $g(G) \subseteq F$. We show that $h(G) \subseteq F$. Let $x \in G$. Then $c \land x \in G$. As $c \land x \leq x$, $g(c \land x) \leq h(c \land x) \leq h(x)$. Hence, $h(G) \subseteq F$. Hence, $(F, G) \in r_3$.

(2) Let $F, G \in \mathcal{P}(B)$. Assume that $G \in C$ (i.e. $c \in G$) and $(F, G) \in r_2 \circ r_3$. Then there exists $H \in \mathcal{P}(B)$ such that $(F, H) \in r_2$ and $(H, G) \in r_3$, i.e. such that $f(H) \subseteq F$ and $g(G) \subseteq H$. Then $f(g(G)) \subseteq f(H) \subseteq F$. Let $x \in G$. Then $c \land x \in G$. Hence, $f(g(c \land x)) \in F$. As $c \land x \leq x$, for every $x \in G$, $f(g(c \land x)) \leq h(c \land x) \leq h(x)$, so $h(G) \subseteq F$. This shows that $h(G) \subseteq F$, i.e. $(F, G) \in r_3$. The proof of (3) is analogous to that of (2).

In the particular case when $c = 1$ we obtain the following correspondence result:

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[2] Most calculations in the results presented here are simple; the correspondence results presented here could also be obtained as a consequence of a general result in algebraic logic, namely Sahlqvist’s theorem.
Corollary 1 Let $B \in BAO(\Sigma)$, and let $f, g, h \in \Sigma$ be unary join-hemimorphisms on $B$.

(1) If $B \models g(x) \leq h(x)$ then in $D(B)$, $r_f \subseteq r_h$.
(2) If $B \models f(g(x)) \leq h(x)$ then in $D(B)$, $r_f \circ r_g \subseteq r_h$.
(3) If $B \models f(g(x)) \leq x$ then in $D(B)$, $r_f \circ r_g \subseteq \text{id}$, where $\text{id} = \{(x,x) \mid x \in \mathcal{F}_p(B)\}$.

Analogons of Theorem 1 and Corollary 1 can also be proved for operators with higher arity:

Theorem 2 Let $B \in BAO(\Sigma)$, and let $f, g, g_1, \ldots, g_n, h \in \Sigma$ such that $f, g$ are $n$-ary, $g_i$ are $n_i$-ary, and $h$ is an $m$-ary join-hemimorphism on $B$, and let $c_i$ (resp. $c_i'$) be constants and $C_i$ (resp. $C_i'$) the predicates associated in a canonical way with $c_i$ in $D(B)$ as explained above. Then:

(1) If $n = m$ and $B \models \forall x_1, \ldots, x_n (\bigwedge_i x_i \leq c_i \rightarrow g(x_1, \ldots, x_n) \leq h(x_1, \ldots, x_n))$ then
$$D(B) \models \forall x_1, \ldots, x_n (x_1 \in C_1 \land \cdots \land x_n \in C_n \land r_f(x_1, \ldots, x_n) \rightarrow r_h(x_1, \ldots, x_n)).$$

(2) If $B \models \forall x_1, \ldots, x_n (\bigwedge_i x_i \leq c_i \rightarrow f(g_1(x_1), \ldots, g_n(x_n)) \leq h(x_1, \ldots, x_n))$ (where $\sum_{i=1}^n n_i = m$) then
$$D(B) \models \forall x_1, \ldots, x_n (\bigwedge_{i=1}^n x_i \in C_i' \land \cdots \land x_n \in C_n' \land r_f(x, r_{g_1}(x_1), \ldots, r_{g_n}(x_n)) \rightarrow r_h(x_1, \ldots, x_n)).$$

(3) If $g_i$ are unary and $B \models \forall x \leq c \rightarrow f(g_1(x), \ldots, g_n(x)) \leq x)$ then
$$D(B) \models \forall y (y \in C \land r_f(x, r_{g_1}(y), \ldots, r_{g_n}(y)) \rightarrow x = y).$$

Proof: The proof of (1) is similar to the proof of item (1) in Theorem 1. Let $F \in \mathcal{F}_p(B)$ and $F_1, \ldots, F_n$ be tuples of prime filters such that $F_i$’s length corresponds to the arity of $g_i$. Assume that $F_i' \in C_i'$ (i.e. $c_i' \in F_i'$) and that $(F, F_1', \ldots, F_n') \in r_f \circ (r_{g_1}, \ldots, r_{g_n})$. Then there exist $F_1, \ldots, F_n \in \mathcal{F}_p(B)$ such that $(F, F_1, \ldots, F_n) \in r_f$ and $(F_i, F_i') \in r_{g_i}$. Then $f(F_1, \ldots, F_n) \subseteq F$ and $g_i(F_i) \subseteq F_i$. It follows that $f(F, g_1(F_1), \ldots, g_n(F_n)) \subseteq F$. As in the proof of (2) in Theorem 1 we can then conclude that $(F, F_1', \ldots, F_n') \in r_h$. The proof of (3) is similar. □

Corollary 2 Let $B \in BAO(\Sigma)$, and let $f, g, g_1, \ldots, g_n, h \in \Sigma$ be such that $f, g$ are $n$-ary, $g_i$ are $n_i$-ary, and $h$ is an $m$-ary join-hemimorphism on $B$. Then:

(1) If $n = m$ and $B \models g(x) \leq h(x)$ then in $D(B)$, $r_g \subseteq r_h$.
(2) If $B \models f(g_1(x_1), \ldots, g_n(x_n)) \leq h(x_1, \ldots, x_l)$ (where $\sum_{i=1}^n n_i = m$) then in $D(B)$,
$$r_f \circ (r_{g_1}, \ldots, r_{g_n}) \subseteq r_h.$$

(3) If $g_i$ are unary and $B \models f(g_1(x), \ldots, g_n(x)) \leq x$ then in $D(B)$, $r_f \circ (r_{g_1}, \ldots, r_{g_n}) \subseteq \text{id}$, where $\text{id} = \{(x,x) \mid x \in \mathcal{F}_p(B)\}$ is the identity relation.
3.3.2 From relational spaces to algebras

We now consider relational spaces, i.e., structures of the form \( D = (D, \{ r_p \}_{p \in \Sigma}) \), where \( D \) is a set and for every \( r \in \Sigma, r_p \) is a relation on \( D \). The dual of a Boolean algebra (if we ignore the topology) is a relational space. The canonical extension associated with a Boolean algebra \( B \) is the Boolean algebra
\[
(\mathcal{P}(\mathcal{F}_p(B)), \cup, \cap, \neg, \emptyset, \mathcal{F}_p \{ f_{sr, f} \}_{f \in \Sigma})
\]
of subsets of \( \mathcal{F}_p(B) \), with operators \( f_{sr, f} \) defined from the relations \( r_f \) by:
\[
\begin{align*}
\ f_{sr, f}(U_1, \ldots, U_n) &= \{ x \mid \exists y_1, \ldots, y_n (x, y_1, \ldots, y_n) \in r_f \text{ and } y_i \in U_i \text{ for } 1 \leq i \leq n \} \\
\ f_{sr}(U_1, \ldots, U_n) &= \{ x \mid \forall y_1, \ldots, y_n ((x, y_1, \ldots, y_n) \in r_f \Rightarrow y_i \in U_i \text{ for } 1 \leq i \leq n \}.
\end{align*}
\]

With every relational space one can associate a Boolean algebra, with the universe consisting of all subsets of \( D \).

**Theorem 3** Let \( D = (D, \{ r_p \}_{p \in \Sigma}) \) be a relational space, let \( C \subseteq D \) and let \( \mathcal{D} \) be the Boolean algebra with operators \( (\mathcal{P}(D), \cup, \cap, \emptyset, D, \{ f_{sr} \}_{r \in \Sigma}) \), where \( f_{sr} \) is as in definition 3.3.2 above, and \( c \) be a constant symbol with interpretation \( C \). Then the following hold:

1. If \( D \models \forall x, y (x \in C \land r_1(x, y) \rightarrow r_2(x, y)) \) then \( \mathcal{D} \models \forall x \ (x \leq c \rightarrow f_{sr_1}(x) \leq f_{sr_2}(x)). \)
2. If \( D \models \forall x, y (y \in C \land r_1(x, y) \rightarrow r_2(x, y)) \) then \( \mathcal{D} \models \forall x \ (x \leq c \rightarrow f_{sr_1}(x) \leq f_{sr_2}(x)). \)
3. If \( D \models \forall x, y (y \in C \land r_1 \circ s(x, y) \rightarrow x = y) \) then \( \mathcal{D} \models \forall x (x \leq c \rightarrow f_{sr_1}(x) \leq x). \)

**Proof:** Clearly, \( \mathcal{D} \models BAO(\Sigma) \). Let \( r_1, r_2, r \in N_r \) and \( U \in \mathcal{P}(D) \) with \( U \subseteq C \).

1. Assume that \( D \models \forall x, y (y \in C \land r_1(x, y) \rightarrow r_2(x, y)) \). Let \( x \in f_{sr_1}(U) \). Then there exists \( y \in U \) such that \( r_1(x, y) \). As \( U \subseteq C \), \( y \in C \) so \( r_2(x, y) \).

2. Assume that \( D \models \forall x, y (y \in C \land r_1(x, y) \rightarrow r_2(x, y)) \). Let \( x \in f_{sr_1}(U) \). Then there exists \( y \in U \) such that \( (x, y) \in r_1 \circ s \). As before, \( y \in C \) so \( r_2(x, y) \). The proof of (3) is similar.

**Corollary 3** Let \( D = (D, \{ r_p \}_{p \in \Sigma}) \) be a relational space and let \( \mathcal{D} \) be the Boolean algebra with operators \( (\mathcal{P}(D), \cup, \cap, \emptyset, D, \{ f_{sr} \}_{r \in \Sigma}) \), where \( f_{sr} \) is as in definition 3.3.2 above. The following hold:

1. If \( D \models r_1 \subseteq r_2 \) then \( \mathcal{D} \models \forall x \ f_{sr_1}(x) \leq f_{sr_2}(x). \)
2. If \( D \models r_1 \circ s \subseteq \pi \) then \( \mathcal{D} \models \forall x \ f_{sr_1}(f_{sr_2}(x)) \leq f_{sr_2}(x). \)
3. If \( D \models r_1 \circ s \subseteq id \) then \( \mathcal{D} \models \forall x \ f_{sr_1}(f_{sr_2}(x)) \leq x. \)

Similar results hold also for \( n \)-ary relations.

**Theorem 4** Let \( D = (D, \{ r_p \}_{p \in \Sigma}) \) be a relational space and let \( \mathcal{D} \) be the Boolean algebra with operators \( (\mathcal{P}(D), \cup, \cap, \emptyset, D, \{ f_{sr} \}_{r \in \Sigma}) \in BAO(\Sigma) \), where for every \( r \in \Sigma, f_{sr} \) is defined as in formula (3.3.2) above. Let \( r_1, s_1, \ldots, s_n, r_2 \in \Sigma \) such that \( r_1 \) is an \( n + 1 \)-ary, \( s_i \) are \( n_i + 1 \)-ary, and \( r_2 \) an \( m + 1 \)-ary relations. Let \( C_1, C_2 \subseteq D \) and let \( c_i, c_k \) be constant symbols which are interpreted as \( C_i, C_k \). The following hold:

1. If \( n = m \) and \( D \models \forall x, \exists y (\exists_i y_i \in C_i \land r_1(x, y_i) \rightarrow r_2(x, y)) \) then \( \mathcal{D} \models \forall x_1, \ldots, x_n \left( \bigwedge_i x_i \leq c_i \rightarrow f_{sr_1}(x_1, \ldots, x_n) \leq f_{sr_2}(x_1, \ldots, x_n) \right). \)
If \( \mathbf{D} \models \forall x, y_1, \ldots, y_n (N_{a_i} \wedge \bigvee_{i,j} \neg \delta_i \leftarrow r_1 \circ (s_1, \ldots, s_n)(x, y_1, \ldots, y_n) \rightarrow r_2(x, y_1, \ldots, y_n)) \) then
\[
\mathcal{P}(\mathbf{D}) \models \forall x, y \in X \wedge \bigvee_{i,j} \neg \delta_i \leftarrow f_{\exists r_1}(f_{\exists a_1}(\overline{x}^1), \ldots, f_{\exists a_n}(\overline{x}^n)) \leq f_{\exists r_2}(\overline{x}^1, \ldots, \overline{x}^n).
\]
(3) If \( s_i \) are binary and \( \mathbf{D} \models \forall x, y \in C \wedge r_1 \circ (s_1, \ldots, s_n)(x, y) \leq x = y \) then
\[
\mathcal{P}(\mathbf{D}) \models \forall x \leq c \rightarrow f_{\exists r_1}(f_{\exists a_1}(x), \ldots, f_{\exists a_n}(x)) \leq x.
\]
Proof: Analogous to the proof of Theorem \[5\]

4 Algebraic semantics for description logics

A translation of concept descriptions into terms in a signature naturally associated with the set of constructors can be defined as follows. For every role name \( r \), we introduce unary function symbols, \( f_{\exists r} \) and \( f_{\forall r} \). The renaming is inductively defined by:
- \( \overline{C} \equiv C \) for every concept name \( C \);
- \( \overline{\neg C} = \neg \overline{C} \), \( \overline{C_1 \cap C_2} = \overline{C}_1 \cap \overline{C}_2 \), \( \overline{C_1 \cup C_2} = \overline{C}_1 \lor \overline{C}_2 \);
- \( \overline{\exists r C} = f_{\exists r}(\overline{C}) \), \( \forall r C = f_{\forall r}(\overline{C}) \).

There exists a one-to-one correspondence between interpretations \( \mathcal{I} = (D, r^\mathcal{I}) \) and Boolean algebras of sets with additional operators, \( (\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\exists r}, f_{\forall r}\}) \subseteq \mathcal{N}_K \), together with valuations \( v : N_C \rightarrow \mathcal{P}(D) \), where \( f_{\exists r}, f_{\forall r} \) are defined, for every \( U \subseteq D \), by:
\[
\begin{align*}
f_{\exists r}(U) &= \{ x \mid \exists y ((x, y) \in r^\mathcal{I} \text{ and } y \in U) \} \\
f_{\forall r}(U) &= \{ x \mid \forall y ((x, y) \in r^\mathcal{I} \Rightarrow y \in U) \}.
\end{align*}
\]

It is easy to see that, with these definitions:
- \( f_{\exists r} \) is a join-homomorphism, i.e. \( f_{\exists r}(x \lor y) = f_{\exists r}(x) \lor f_{\exists r}(y), f_{\exists r}(0) = \emptyset \);
- \( f_{\forall r} \) is a meet-homomorphism, i.e. \( f_{\forall r}(x \land y) = f_{\forall r}(x) \land f_{\forall r}(y), f_{\forall r}(1) = D \);
- \( f_{\forall r}(x) = \neg f_{\exists r}(\neg x) \) for every \( x \in B \).

Let \( v : N_C \rightarrow \mathcal{P}(D) \) with \( v(A) = A^\mathcal{I} \) for all \( A \in N_C \), and let \( v^r \) be the (unique) homomorphic extension of \( v \) to terms. Let \( C \) be a concept description and \( \overline{C} \) be its associated term. Then \( C^\mathcal{I} = \overline{v^r(C)} \) (denoted by \( C^\mathcal{I} \)).

The TBox subsumption problem for the description logic \( \mathcal{ALC} \) (which was defined in Section \[2\]) can be expressed as uniform word problem for Boolean algebras with suitable operators.

**Theorem 5** If \( \mathcal{I} \) is an \( \mathcal{ALC} \) TBox consisting of general concept inclusions between concept terms formed from concept names \( N_C = \{C_1, \ldots, C_n\} \), and \( D_1, D_2 \) are concept descriptions, the following are equivalent:

1. \( D_1 \subseteq \mathcal{I} D_2 \);
2. \( \mathcal{P}(\mathbf{D}) \models \forall C_1 \ldots C_n \left( \bigwedge_{C \in \mathcal{I} \subseteq \mathcal{P}(\overline{D})} \overline{C} \subseteq \overline{D} \right) \rightarrow \overline{D_1} \subseteq \overline{D_2}, \) for all interpretations \( \mathcal{I} = (D, r^\mathcal{I}) \), where \( \mathcal{P}(\mathbf{D}) = (\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\exists r}, f_{\forall r}\}) \subseteq \mathcal{N}_K \).
3. \( \text{BAO}_N \models \forall C_1 \ldots C_n \left( \bigwedge_{C \in \mathcal{I} \subseteq \mathcal{P}(\overline{D})} \overline{C} \subseteq \overline{D} \right) \rightarrow \overline{D_1} \subseteq \overline{D_2} \).
Proof: The equivalence of (1) and (2) follows from the definition of $D_1 \subseteq \mathcal{F} D_2$. (3) follows from the fact that every algebra in $\text{BAO}_N$ homomorphically embeds into a Boolean algebra of sets, its canonical extension.

An analogon of Theorem 5 can be used for more general description logics in which in addition to the TBoxes also properties of roles need to be taken into account. We consider properties $R$ of roles which can be expressed by sets $R_a$ of clauses at an algebraic level. The main restriction we impose is that the sets of clauses $R_a$ are preserved when taking canonical extensions of Boolean algebras. We denote by $\text{BAO}_N(R_a)$ the family of all algebras in $\text{BAO}_N$ which satisfy the axioms in $R_a$.

Theorem 6 Let $\mathcal{F}$ be an $\mathcal{ALC}$ TBox consisting of general concept inclusions between concept terms formed from concept names $N_C = \{C_1, \ldots, C_2\}$, and let $R$ be a family of general (e.g. guarded) role inclusions with the additional property that there exists a set $R_a$ of clauses in the signature of $\text{BAO}_N$ such that:

(i) For each interpretation $\mathcal{F} = (D, \cdot, \cdot)$, which satisfies the constraints on roles in $R$, we have that $\mathcal{P}(D) \models R_a$, where $\mathcal{P}(D)$ stands for $(\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\leq}, f_{\geq}\})_{r \in N_R}$.

(ii) Every $B \in \text{BAO}_N(R)$ embeds into an algebra of sets of the form $\mathcal{P}(D)$ (defined as above), where $(D, \{r\}_{r \in N_R})$ satisfies $R$.

Then for any concept descriptions $D_1, D_2$ the following are equivalent:

(1) $D_1 \subseteq \mathcal{F} D_2$,

(2) $\mathcal{P}(D) \models \forall C_1 \ldots C_n \left( (\bigwedge_{C \subseteq D} C \subseteq D \Rightarrow D_1 \subseteq D_2) \right)$ for all interpretations $\mathcal{F} = (D, \cdot, \cdot)$ which are models of $R$.

(3) $\text{BAO}_N \left( R_a \right) \models \forall C_1 \ldots C_n \left( (\bigwedge_{C \subseteq D} C \subseteq D \Rightarrow D_1 \subseteq D_2) \right)$.

Proof: (1) $\Leftrightarrow$ (2) Let $\mathcal{F} = (D, \cdot, \cdot)$ be an interpretation which is a model of $R$. Let $\nu : N_C \to \mathcal{P}(D)$ be a valuation with the property that $\nu(C) \subseteq \nu(D)$ for all $C \subseteq D \in \mathcal{F}$. Since $D_1 \subseteq \mathcal{F} D_2$, it follows that $\nu(D_1) \subseteq \nu(D_2)$. (3) $\Rightarrow$ (2) follows from the fact that, by assumption (i), $\mathcal{P}(D) = (\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\leq}, f_{\geq}\})_{r \in N_R} \in \text{BAO}_N(R_a)$. (2) $\Rightarrow$ (3) follows from the fact that, by Assumption (ii), for every Boolean algebra $B$ with operators there exists a relational space $D$ which satisfies $R$, such that $B$ homomorphically embeds into a Boolean algebra of sets of the form $\mathcal{P}(D)$ which satisfies the conditions in (2). Hence, $\mathcal{P}(D) \models \forall C_1 \ldots C_n \left( (\bigwedge_{C \subseteq D} C \subseteq D \Rightarrow D_1 \subseteq D_2) \right)$. Since $B$ is isomorphic to a subalgebra of $\mathcal{P}(D)$, it follows that $B \models \forall C_1 \ldots C_n \left( (\bigwedge_{C \subseteq D} C \subseteq D \Rightarrow D_1 \subseteq D_2) \right)$.

Example 1 Assume that $R$ consists of concept inclusions of the form $r \subseteq s$ and $r_1 \circ s \subseteq r_2$ and $r_1 \circ s \subseteq id$ and $R_a$ consists of the corresponding axioms $\forall x (f_{=} (x) \leq f_{=} (x))$, $\forall x(f_{=} (f_{=} (x)) \leq f_{=} (x))$, and $\forall x(f_{=} (f_{=} (x)) \leq x)$. Then, by Corollaries 3 and 4, premises (i) and (ii) of Theorem 5 hold, hence the CBox subsumption problem can be expressed as a uniform word problem in $\text{BAO}_N(R_a)$.

4.1 Algebraic semantics for $\mathcal{EL}$, $\mathcal{EL}^+$ and extensions thereof

In [20] we studied the link between TBox subsumption in $\mathcal{EL}$ and uniform word problems in the corresponding classes of semilattices with monotone functions. We now show that these results naturally extend to the description logic $\mathcal{EL}^+$. We will consider the following classes of algebras:
- \( \text{BAO}_{\mathcal{N}_0}^3 \): the class of boolean algebras with operators \((B, \vee, \wedge, \neg, 0, 1, \{f_B\}_{r \in \mathcal{N}_0})\), such that \( f_B \) is a unary join-hemimorphism;
- \( \text{DLO}_{\mathcal{N}_0}^3 \): the class of bounded distributive lattices with operators \((L, \vee, \wedge, 0, 1, \{f_L\}_{r \in \mathcal{N}_0})\), such that \( f_L \) is a unary join-hemimorphism;
- \( \text{SLO}_{\mathcal{N}_0}^3 \): the class of all \( \wedge \)-semilattices with operators \((S, \wedge, 0, 1, \{f_S\}_{r \in \mathcal{N}_0})\), such that \( f_S \) is a monotone unary function and \( f_S(0) = 0 \).

4.2 Algebraic semantics for \( \mathcal{L}^+ \)

In \( \mathcal{L}^+ \) the following types of role inclusions are considered:

\[
    r \sqsubseteq s \quad \text{and} \quad r_1 \circ s \sqsubseteq r_2.
\]

In \cite{7} it is proved that subsumption w.r.t. GCT’s in the extension \( \mathcal{E} \mathcal{L} \mathcal{F} \) of \( \mathcal{L} \mathcal{F} \) with inverse roles is ExpTime complete. It is also proved that subsumption w.r.t. general TBoxes in the extension \( \mathcal{E} \mathcal{L} \mathcal{F} \mathcal{S} \) of \( \mathcal{L} \mathcal{F} \) with symmetric roles is ExpTime complete. We will now start by considering also CBoxes containing role inclusion axioms which describe weaker, left- and right-inverse properties of roles, of the form: \( r \circ s \sqsubseteq id \).

Let \( RI \) be a set of axioms of the form \( r \sqsubseteq s \), \( r_1 \circ r_2 \sqsubseteq r \), and \( r_1 \circ r_2 \sqsubseteq id \) with \( r_1, r_2, r \in \mathcal{N}_R \). We associate with \( RI \) the following set \( RL \) of axioms:

\[
    RL = \{ \forall x \ f_{\mathcal{N}_0}(x) \leq f_{\mathcal{N}_0}(x) \mid r \sqsubseteq s \in RI \} \cup \ \{ \forall x \ (f_{\mathcal{N}_0} \circ f_{\mathcal{N}_0})(x) \leq f_{\mathcal{N}_0}(x) \mid r_1 \circ r_2 \subseteq r \in RI \} \cup \ \{ \forall x \ (f_{\mathcal{N}_0} \circ f_{\mathcal{N}_0})(x) \leq x \mid r_1 \circ r_2 \subseteq id \in RI \}.
\]

Let \( \text{BAO}_{\mathcal{N}_0}^3 (RI) \) (resp. \( \text{DLO}_{\mathcal{N}_0}^3 (RI) \), \( \text{SLO}_{\mathcal{N}_0}^3 (RI) \)) be the subclass of \( \text{BAO}_{\mathcal{N}_0}^3 \) (resp. \( \text{DLO}_{\mathcal{N}_0}^3 \), \( \text{SLO}_{\mathcal{N}_0}^3 \)) consisting of those algebras which satisfy \( RL \).

**Lemma 1** Let \( \mathcal{F} = (D, \cdot, \mathcal{F}) \) be a model of an \( \mathcal{E} \mathcal{L}^+ \) CBox \( \mathcal{C} = \text{GCI} \cup RI \). Then the algebra \( \mathcal{P}(D), \cap, \emptyset, D, \{f_B\}_{r \in \mathcal{N}_0} \) is a semilattice with operators in \( \text{SLO}_{\mathcal{N}_0}^3 (RI) \).

**Proof:** Clearly, \( \mathcal{P}(D), \cap, \emptyset, D, \{f_B\}_{r \in \mathcal{N}_0} \in \text{SLO}_{\mathcal{N}_0}^3 \). The proof of the second part uses exactly the same arguments as the proof of Theorem \cite{3} and Corollary \cite{3}

We will now show that every algebra in \( \text{SLO}_{\mathcal{N}_0}^3 (RI) \) embeds into (the bounded semilattice reduct of) an algebra in \( \text{BAO}_{\mathcal{N}_0}^3 (RI) \). We start with a more general lemma, which will be important also for proving the locality results in Section \cite{5}.

**Lemma 2** For every structure \( \mathcal{F} = (S, \wedge, 0, 1, \{f_S\}_{f \in \Sigma}) \) in which \( f_S \) are partial functions, if properties (i), (ii) and (iii) below hold, then \( \mathcal{F} \) embeds into a semilattice with operators in \( \text{SLO}_{\mathcal{N}_0}^3 (RI) \).

(i) \( (S, \wedge, 0, 1) \) is a bounded semilattice; \( \leq \) the partial order on \( S \) defined by \( x \leq y \iff x \wedge y = y \).

(ii) For every \( f \in \Sigma \) with arity \( n \), \( f_S \) is a partial \( n \)-ary function on \( S \) which satisfies the monotonicity axiom \( \text{Mon}(f) \) whenever all terms are defined.

\[
    \text{Mon}(f) \quad \forall x, y (x \leq y \implies f(x) \leq f(y))
\]

\footnote{For the sake of simplicity, in this paper we assume that the description logics \( \mathcal{E} \mathcal{L} \) and \( \mathcal{E} \mathcal{L}^+ \) contain the additional constructors \( \bot, \top \), which will be interpreted as 0 and 1. Similar considerations can be used to show that the algebraic semantics for variants of \( \mathcal{E} \mathcal{L} \) and \( \mathcal{E} \mathcal{L}^+ \) having only \( \top \) (or \( \bot \)) is given by semilattices with 1 (resp. 0).}
(iii) There exists a set $R^{\text{flat}}$ of axioms of the form\footnote{These axioms are logically equivalent with those discussed before; the reason for preferring the flat version will become apparent in Section\ref{sec:applications}.}:

$$\forall x \quad g(x) \leq h(x)$$

$$\forall x, y \quad x \leq g(y) \rightarrow f(x) \leq h(y)$$

$$\forall x, y \quad x \leq g(y) \rightarrow f(x) \leq y$$

such that:

- if $g, h$ appear in a rule as above and $g_S(s)$ is defined then also $h_S(s)$ is defined;
- for every $\beta : \{x, y\} \rightarrow S$, and every axiom $D \in R^{\text{flat}}$ if all terms in $\overline{D}(D)$ are defined, then $\overline{D}(D)$ is true in $S$ (where $\overline{D}$ is the canonical extension of $\beta$ to formulae).

Proof: Let $S = (S, \wedge, 0, 1, \{f_S \}_{f \in \Sigma})$ be a 0,1 semilattice, and let $f_S, g_S : S \rightarrow S$ be partially defined functions which satisfy the conditions above. Consider the lattice of all order-ideals of $S$, $\mathcal{O}(S) = (\mathcal{O}(S), \cap, \cup, \{0\}, S, \{f_S \}_{f \in \Sigma})$, where join is set union, meet is set intersection, and the additional operators in $\Sigma$ are defined, for every order ideal $U$ of $S$, by

$$\overline{f_S}(U) = \{ f_S(u) \mid u \in U, f_S(u) \text{ defined} \}.$$ 

Note that $\overline{f_S}((0)) = \{0\}$ and $\overline{f_S}(U_1 \cup U_2) = \overline{\overline{f_S}(U_1) \cup \overline{f_S}(U_2)} = \overline{f_S(U_1) \cup f_S(U_2)} = \overline{f_S(U_1)} \cup \overline{f_S(U_2)}$. Thus, $\mathcal{O}(S) \in \text{DLLO}_S$. Moreover, $\eta : S \rightarrow \mathcal{O}(S)$ defined by $\eta(x) := \{ x \}$ is an injective homomorphism w.r.t. the bounded semilattice operations and $\eta(f_S(x)) = f_S(\overline{\eta(x)}) = \overline{f_S}(|x|)$. We prove that $\overline{f_S}, \overline{g_S}, \overline{h_S}$ satisfy the axioms in $R^{\text{flat}}$. Consider first the axiom:

$$\forall x, y \quad (y \leq g(x) \rightarrow f(y) \leq x)$$

(2)

Let $U, V \in S$ be such that $U \subseteq \overline{V}(V)$. Let $x \in \overline{f_S}(U)$. Then there exists $u \in U$ such that $f_S(u)$ is defined and $x \leq f_S(u)$. Since $U \subseteq \overline{g}(V)$, we know that there exists $v \in V$ with $g_S(v)$ defined and $u \leq g_S(v)$. Thus, $x \leq f_S(u) \leq v \in V$, so $x \in V$. This shows that for all $U, V \in S$:

$$U \subseteq \overline{V}(V) \rightarrow \overline{f_S}(U) \subseteq V.$$ 

We now check preservation of the axioms of the form:

$$\forall x \quad g(x) \leq h(x)$$

(3)

$$\forall x, y \quad (y \leq g(x) \rightarrow f(y) \leq h(x))$$

(4)

We assume that $S$ has the property that $h_S(a)$ is defined whenever $g_S(a)$ is defined. We have to show that if $f_S, g_S, h_S$ are monotone whenever defined and satisfy one of the axioms above (say $\text{Axiom } 3$) is similar) whenever defined then $\overline{f_S}, \overline{g_S}$ and $\overline{h_S}$ satisfy $\text{Axiom } 3$.

Let $U, V \in S$ be such that $U \subseteq \overline{V}(V)$. Let $x \in \overline{f_S}(U)$. Then there exists $u \in U$ such that $f_S(u)$ is defined and $x \leq f_S(u)$. Since $U \subseteq \overline{V}(V)$, we know that there exists $v \in V$ with $g_S(v)$ defined and $u \leq g_S(v)$. Due to the first condition in (iii), $h_S(v)$ must be defined as well. Since $S$ satisfies Axiom $\text{Axiom } 3$ and $g_S(v), f_S(u), h_S(v)$ are defined and $u \leq g_S(v)$ it follows that $f_S(u) \leq h_S(v)$. Thus, there exists $v \in V$ such that $x \leq f_S(u) \leq h_S(v)$, so $x \in \overline{V}(V)$. This shows that for all $U, V \in S$:

$$U \subseteq \overline{V}(V) \rightarrow \overline{f_S}(U) \subseteq \overline{h_S}(V).$$

\[\square\]
Lemma 3 Every $S \in \text{SLO}_{N_k}(RI)$ embeds into (the bounded semilattice reduct of) a lattice in $\text{DLO}_{N_k}(RI)$. Every lattice in $\text{DLO}_{N_k}(RI)$ embeds into (the bounded lattice reduct of) an algebra in $\text{BAO}_{N_k}(RI)$.

Proof: The first part follows from Lemma 2. The second statement is a consequence of Priestley duality for distributive lattices. Let $L \in \text{DLO}_{N_k}(RI)$. Let $\mathcal{F}$ be the set of prime filters of $L$, and $B(L) = \langle \mathcal{D}(\mathcal{F}), \cup, \cap, \{\mathcal{P}\} \rangle$, for where $r \in R$, $\mathcal{P}$ is defined by

\[ \mathcal{P}(U) = \{ F \in \mathcal{F} \mid \exists G \in U : f_2(G) \subseteq F \} \]

Let $i : L \rightarrow B(L)$ be defined by $i(x) = \{ F \in \mathcal{F} \mid x \in F \}$. Obviously, $i$ is a lattice homomorphism. We show that $i(f_2(x)) = \mathcal{P}(i(x))$.

To prove the converse inclusion, let $F \in i(f_2(x))$. Then $F \in \mathcal{F}$ and $f_2(x) \in F$. Let $G = f_2^{-1}(F)$. As $F$ is a prime filter, and $f_2$ is a join-homomorphism, $G$ is a prime filter with $x \in G$ and $f_2(G) \subseteq F$, so $F \in \mathcal{P}(i(x))$. Finally, we show that $B(L)$ satisfies the axioms in $RI_y$. Let $U \in B(L)$. By definition,

\[ \mathcal{P}(U) = \{ F \in \mathcal{F} \mid \exists G \in U : f_2(G) \subseteq F \} \]

Assume that $r_1 \subseteq r \in RI$. We know that $L \models \forall x, f_2(x) \subseteq f_2(x)$. Let $F \in \mathcal{P}(U)$. Then $f_2(G_1) \subseteq F$ for some $G_1 \in U$, so also $f_2(G_1) \subseteq F$. Hence, $\mathcal{P}(U) \subseteq \mathcal{P}(U)$. Similarly we can prove that if $r_1 \circ r_2 \subseteq r \in RI$ then $\mathcal{P}(\mathcal{P}(U)) \subseteq \mathcal{P}(U)$ and that if $r_1 \circ r_2 \subseteq r \in RI$ then $\mathcal{P}(\mathcal{P}(U)) \subseteq U$.

Theorem 7 If the only constructor concepts are intersection and existential restriction, then for all concept descriptions $D_1$, $D_2$, and every $\mathcal{E}^\mathcal{C}$ CBox $\mathcal{C}=\mathcal{GCl}\cup\mathcal{RI}$, with concept names $N_C = \{C_1, \ldots, C_n\}$ the following are equivalent:

(1) $D_1 \subseteq_k D_2$.
(2) $\text{SLO}_{N_k}(RI) \models \forall C_1 \ldots C_n (\langle \wedge_{C \subseteq D \in \mathcal{C}} C \leq \mathcal{D} \rangle \rightarrow \mathcal{D}_1 \leq \mathcal{D}_2)$.

Proof: We know that $C_1 \subseteq_k C_2$ if and only for every model $\mathcal{I}$ of the CBox $\mathcal{C}$. Assume first that (2) holds. Let $\mathcal{I} = \langle D, C \rangle$ be an interpretation that satisfies $\mathcal{C}$. Then $\mathcal{I} = \langle \mathcal{I}(D), \cap, \emptyset, D, \{f_2\} \rangle \in \text{SLO}_{N_k}(RI)$, hence $\mathcal{I} = \langle \wedge_{C \subseteq D \in \mathcal{C}} C \leq \mathcal{D} \rangle \rightarrow \mathcal{D}_1 \leq \mathcal{D}_2$. As $\mathcal{I}$ is a model of $\mathcal{GCl}, C \subseteq \mathcal{D}$ for all $C \subseteq D \in \mathcal{C}$, so $D_1 \subseteq \mathcal{D}_1 \subseteq D_2 \subseteq D_2$. To prove (1) \rightarrow (2) note first that in this case the premises of Thm. 6 are fulfilled. By Thm. 6 if $D_1 \subseteq_k D_2$ then $\text{BAO}_{N_k}(RI) \models \langle \bigwedge_{C \subseteq D \in \mathcal{C}} C \leq \mathcal{D} \rangle \rightarrow \mathcal{D}_1 \leq \mathcal{D}_2$. Let $S \in \text{SLO}_{N_k}(RI)$. By Lemma 3 $S$ embeds into an algebra in $\text{BAO}_{N_k}(RI)$ which satisfies $RI_y$. Therefore, $S \models \langle \bigwedge_{C \subseteq D \in \mathcal{C}} C \leq \mathcal{D} \rangle \rightarrow \mathcal{D}_1 \leq \mathcal{D}_2$. \qed
We will show that the word problem for the class of algebras $SLO^3_{\exists R}(RI)$ is decidable in PTIME. For this we will prove that $SLO^3_{\exists R}(RI)$ has a “local” presentation. The general locality definitions, as well as methods for recognizing local presentations are given in Sect.5. The application to the class of models for $EL$ and $EL^+$ are given in Sect.6. Before doing this, we present some additional types of constraints on the roles which can be handled similarly. This will allow us to obtain a new tractable extension of $EL^+$.

4.3 Guarded role inclusions

In applications it may be interesting to consider role inclusions guarded by membership to a certain concept, i.e. role inclusions of the form:

\[\forall x, y \ (y \in C \land r(x, y) \rightarrow r'(x, y))\]  
\[\forall x, y \ (y \in C \land r \circ s(x, y) \rightarrow r'(x, y))\]  
\[\forall x, y \ (y \in C \land r \circ s(x, y) \rightarrow x = y)\]  

The corresponding axioms at the algebra level we consider are:

\[\forall x \ (x \leq C \rightarrow f_r(x) \leq f_{r'}(x))\]  
\[\forall x \ (x \leq C \rightarrow f_{r \circ s}(x) \leq f_{r' \circ s}(x))\]  
\[\forall x \ (x \leq C \rightarrow f_{r \circ s}(x) \leq f_{r'(x)})\]  

**Theorem 8** Assume that the only concept constructors are intersection and existential restriction. Let $\mathcal{C} = \mathcal{GCI} \cup \mathcal{RI} \cup \mathcal{GRI}$ be a CBox containing a set $\mathcal{GCI}$ of general concept inclusions, a set $\mathcal{RI}$ of role inclusions of the type considered in Sect.4.2 and a set $\mathcal{GRI}$ of guarded role inclusions of the form (5)–(7), with concept names $\mathcal{N}_C = \{C_1, \ldots, C_n\}$. Then for all concept descriptions $D_1, D_2$ the following are equivalent:

1. $D_1 \sqsubseteq_C D_2$;
2. $\mathcal{GRI}(C_1, \ldots, C_n) \land (\bigwedge_{C \in \mathcal{GCI}} C \leq D) \land D_1 \not\leq D_2$ is unsatisfiable w.r.t. $BAO^3_{\exists R}(RI)$.
3. $\mathcal{GRI}(C_1, \ldots, C_n) \land (\bigwedge_{C \in \mathcal{GCI}} C \leq D) \land D_1 \not\leq D_2$ is unsatisfiable w.r.t. $SLO^3_{\exists R}(RI)$.

**Proof**: The proof is analogous to that of Theorem7 and uses the results in Theorems1 and 3, as well as an analogon of Theorem4.

4.4 Extensions of $EL^+$ with $n$-ary roles and concrete domains

We now present a possibility of extending $EL^+$ with concrete domains, which is a natural generalization of the extension in Section4.1. This extension is different from the extensions with concrete domains and those with $n$-ary quantifiers studied in the description logic literature (cf. e.g. [5,3]).

Later, in Section8 we will present another extension (the one used in $EL^{++}$).

We consider $n$-ary roles because in relational databases, relations of higher arity are often used. This is especially important when we need to express dependencies between several (not only two) individuals.

**Example 2** We would like to express, for instance, information about all the routes from cities in a set $C_1$ to cities in a set $C_2$ passing through cities in a set $C_3$. This could be done using ternary roles interpreted as ternary relations.
4.4.1 An extension of $\mathcal{EL}^+$ with n-ary roles

An extension of the description logic $\mathcal{EL}$, containing n-ary roles instead of binary roles (interpreted as n-ary relations) can easily be defined. The definition of TBox subsumption can be extended naturally to the n-ary case. In this paper we will restrict to $\mathcal{EL}$ (cf. Figure 4.4), i.e. consider only existential restrictions, which are in this case n-ary – of the form $\exists r(C_1, \ldots, C_n)$ – and are interpreted in any interpretation $\mathcal{F} = (D, \cdot^\mathcal{F})$ as:

$$\exists r(C_1, \ldots, C_n)^\mathcal{F} = \{ x \mid \exists y_1, \ldots, y_n (y_1 \in C_1 \land \cdots \land y_n \in C_n \land r^\mathcal{F}(x, y_1, \ldots, y_n)) \}.$$

A translation of concept descriptions into terms can be defined in a natural way also in this case, with the difference that for every role name $r$ with arity $n+1$, we introduce an n-ary function symbol $f_{\exists r}$. The renaming is inductively defined as in the binary case, with the difference that:

$$\exists r(C_1, \ldots, C_n)^\mathcal{F} = f_{\exists r}(\overline{C_1}, \ldots, \overline{C_n}).$$

Also in the n-ary case we denote by $\text{BAO}_n^\mathcal{F}$ the class of Boolean algebras with operators $(\text{B}, \lor, \land, \lnot, 0, 1, \{ f_{\exists r} \}_{r \in N_n})$, such that for every $r \in N_n$ with arity $n+1$, $f_{\exists r}$ is a join-homomorphism with arity $n$; $\text{DLO}_n^\mathcal{F}$ and $\text{SLO}_n^\mathcal{F}$ are defined similarly. An extension of $\mathcal{EL}^+$ with n-ary roles can be obtained by allowing role inclusions of type:

$$r_1 \sqsubseteq r_2 \quad (11)$$
$$r_1 \circ (s_1, \ldots, s_n) \sqsubseteq r_2 \quad (12)$$
$$r_1 \circ (s_1, \ldots, s_n) \sqsubseteq \text{id} \quad \text{for binary relations } s_i \quad (13)$$

An interpretation $\mathcal{F} = (D, \cdot^\mathcal{F})$ satisfies a role inclusion type (12) if it satisfies the formula:

$$\forall x, x', y' \left( (r_1(x, x_1, \ldots, x_n) \land \bigwedge_{k=1}^n \exists y_k(x_k, y_k^1, \ldots, y_{m_k}^k)) \rightarrow r_2(x, y_1^1, \ldots, y_{m_1}^n, \ldots, y_1^n, \ldots, y_{m_n}^n) \right).$$

The truth of role inclusions of type (11) resp. (13) is defined in a similar way. As in the case of $\mathcal{EL}^+$ we can also prove that TBox subsumption can be expressed as a uniform word problem w.r.t. the class of semilattices with monotone operators associated with the roles, satisfying axioms corresponding in a natural way to the role inclusion laws above:

$$\forall x_1, \ldots, x_n \quad f_{\exists r_1}(x_1, \ldots, x_n) \leq f_{\exists r_2}(x_1, \ldots, x_n)$$
$$\forall y_1^1, \ldots, y_{m_1}^n \quad f_{\exists r_1}(y_1^1, \ldots, y_{m_1}^n, \ldots, y_1^n, \ldots, y_{m_n}^n) \leq f_{\exists r_2}(y_1^1, \ldots, y_{m_1}^n, \ldots, y_1^n, \ldots, y_{m_n}^n)$$
$$\forall x \quad f_{\exists r_1}(f_{\exists r_1}(x), \ldots, f_{\exists r_1}(x)) \leq x$$

This type of inequalities are exactly of the form studied in Section 3.3 A straightforward generalization of Theorem[7] using the corresponding corrolaries of Theorem[2]and[4] yields:
Theorem 9 If the only concept constructors are intersection and existential restriction, then for all concept descriptions \(D_1, D_2\) and every \(\mathcal{L}^+\) CBox \(\mathcal{C} = GCI \cup RI\) – where \(RI\) consists of role inclusions of type (11)–(13) – with concept names \(N_C = \{C_1, \ldots, C_n\}\) the following are equivalent:

1. \(D_1 \subseteq \mathcal{E} D_2\).
2. \(\text{BAO}^\mathcal{I}_{\mathcal{N}_A} (RI) \models \forall C_1 \ldots C_n \left( (\bigwedge_{C \in GCI} C \leq D) \rightarrow \mathcal{D}_1 \leq \mathcal{D}_2 \right)\).
3. \(\text{SLO}^\mathcal{I}_{\mathcal{N}_A} (RI) \models \forall C_1 \ldots C_n \left( (\bigwedge_{C \in GCI} C \leq D) \rightarrow \mathcal{D}_1 \leq \mathcal{D}_2 \right)\).

A similar result is obtained if we also consider guarded role inclusions.

Theorem 10 Assume that the only concept constructors are intersection and existential restriction. Let \(\mathcal{C} = GCI \cup RI \cup \text{GRI}\) be a CBox containing a set GCI of general concept inclusions, a set RI of role inclusions and a set GRI of guarded role inclusions of the form discussed above, with concept names \(N_C = \{C_1, \ldots, C_n\}\). Then for all concept descriptions \(D_1, D_2\) the following are equivalent:

1. \(D_1 \subseteq \mathcal{E} D_2\).
2. \(\text{GRI}(C_1, \ldots, C_n) \wedge (\bigwedge_{C \in GCI} C \leq D) \wedge \mathcal{D}_1 \leq \mathcal{D}_2\) is unsatisfiable w.r.t. \(\text{BAO}^\mathcal{I}_{\mathcal{N}_A} (RI)\).
3. \(\text{GRI}(C_1, \ldots, C_n) \wedge (\bigwedge_{C \in GCI} C \leq D) \wedge \mathcal{D}_1 \leq \mathcal{D}_2\) is unsatisfiable w.r.t. \(\text{SLO}^\mathcal{I}_{\mathcal{N}_A} (RI)\).

### 4.4.2 \(\mathcal{L}^+\) with \(n\)-ary roles and concrete domains

A further extension is obtained by allowing for certain concrete sorts – having the same support in all interpretations; or additionally assuming that there exist specific concrete concepts which have a fixed semantics (or additional fixed properties) in all interpretations.

Example 3 Consider a description logic having a usual (concept) sort and a 'concrete' sort \(\text{num}\) with fixed domain \(\mathbb{R}\). We may be interested in general concrete concepts of sort \(\text{num}\) (interpreted as subsets of \(\mathbb{R}\)) or in special concepts of sort \(\text{num}\) such as \(\uparrow n\), \(\downarrow n\), or \(\left[ n, m \right]\) for \(m, n \in \mathbb{R}\). For any interpretation \(\mathcal{I}\), \(\uparrow n^\mathcal{I} = \{x \in \mathbb{R} | x \geq n\}\), \(\downarrow n^\mathcal{I} = \{x \in \mathbb{R} | x \leq n\}\), and \(\left[ n, m \right]^\mathcal{I} = \{x \in \mathbb{R} | n \leq x \leq m\}\). We will denote the arities of roles using a many-sorted framework. Let \((D_1, D_2, \ldots)^\mathcal{I}\) be an interpretation with two sorts concept and num. A role with arity \((s_1, \ldots, s_n)\) is interpreted as a subset of \(D_{s_1} \times \cdots \times D_{s_n}\), where \(D_{\text{concept}} = D\) and \(D_{\text{num}} = \mathbb{R}\).

1. Let price be a binary role or arity (concept, num), which associates with every element of sort concept its possible prices. The concept
   \[
   \exists \text{price}. \uparrow n = \{x \mid \exists k \geq n : \text{price}(x, k)\}
   \]
   represents the class of all individuals with some price greater than or equal to \(n\).
2. Let has-weight-price be a role of arity (concept, num, num). The concept
   \[
   \exists \text{has-weight-price}. \{\{y, p\} | \exists y' \geq y, \exists p' \leq p \text{ and has-weight-price}(x, y', p')\}
   \]
   denotes the family of individuals for which a weight above \(y\) and a price below \(p\) exist.

The example below can be generalized by allowing a set of concrete sorts. We discuss the algebraic semantics of this type of extensions of \(\mathcal{L}^+\).

Let \(\text{SLO}^\mathcal{I}_{\mathcal{N}_A}\) denote the class of all structures \((S, \mathcal{P}(A_1), \ldots, \mathcal{P}(A_n), \{f_r \mid r \in \mathcal{N}_R\})\), where \(S\) is a semilattice, \(A_1, \ldots, A_n\) are concrete domains, and \(\{f_r \mid r \in \mathcal{N}_R\}\) are \(n\)-ary monotone operators. We may allow constants of concrete sort, interpreted as sets in \(\mathcal{P}(A_i)\).
Theorem 11. If the only concept constructors are intersection and existential restriction, then for all concept descriptions \( D_1, D_2 \), and every CBox \( \mathcal{C} = \text{GCI} \cup \text{RI} \) consisting of general concept inclusions GCI with concrete domains as defined above, and role inclusions RI of the type considered in Sect. 4.2 or Sect. 4.4.1 the following are equivalent:

1. \( D_1 \subseteq \mathcal{C} D_2 \).
2. \( \text{SLO}_{N_{\geq 3}}(\text{RI}) \models \forall C_1, \ldots, C_n \left( \left( \bigwedge_{C \subseteq \mathcal{C} \geq \text{GCI}} \mathcal{C} \leq \mathcal{D} \right) \Rightarrow \mathcal{D}_1 \leq \mathcal{D}_2 \right) \).

Proof: Analogous to the proof of Theorem 7.

We will also consider relationships of the form

\[ \exists \{ \text{Assume that } s \} \]

Lemma 4. Assume that \( s = \exists r, (i, C) \). Then

\[ f_{\exists r, (i, C)}(U_1, \ldots, U_i-1, U_{i+1}, \ldots, U_n) = \{ x \mid \exists x_i \in U_i, i \in \{ 1, \ldots, n \} \cup \{ i \} : s(x, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \} \]

The axioms which corresponds to role restrictions are of the type:

\[ f_{\exists r, (i, C)}(x_1, \ldots, x_i, \ldots, x_n) = f_{\exists r}(x_1, \ldots, x_i, C, x_{i+1}, \ldots, x_n). \] (14)

All results established for \( \mathcal{L}^+ \) hold also if this kind of role constructions are considered.

4.5 Existential restrictions for roles

We will also consider relationships of the form

\[ \exists \{ \text{Assume that } s \} \]

Example 4. Consider a database where we can express relationships of the form:

\[ r_{\text{int}}(x, y, z) \quad (\text{there exists a route from } x \text{ to } y \text{ passing through } z) \]

\[ r(x, y) \quad (\text{there exists a route from } x \text{ to } y). \]

We will also want to express relationships of the form “For all \( x_1, x_2 \), if there exists a route from \( x_1 \) to \( x_2 \) passing through some city in \( C_3 \), then there exists a route from \( x_1 \) to \( x_2 \).” We need therefore to express a new relation \( r' \) where \( r'(x_1, x_2) \) stands for there exists a route from \( x_1 \) to \( x_2 \) passing through some city in \( C_3 \). For this we will need constructors of the type \( \exists r, (i, C) \). They help to formulate the property above as \( \exists r_{\text{int}}(3, C_3) \subseteq r \), interpreted as:

\[ \{ (x, x_2) \mid \exists x_3 \in C_3 : r_{\text{int}}(x_1, x_2, x_3) \} \subseteq r, \text{ i.e. } \forall x_1, x_2 \exists r_{\text{int}}(3, C_3)(x_1, x_2) \Rightarrow r(x_1, x_2). \]
Theorem 12 Assume the only concept constructors are intersection and existential restriction. Let \( C = \text{GCI} \cup \text{RI} \cup \text{GRI} \cup \text{ER} \) be a CBox containing general concept inclusions (GCI), (guarded) role inclusions (RI, resp. GRI) and a set \( \text{ER} \) of definitions of roles by existential restrictions with concept names \( N_C = \{ C_1, \ldots, C_n \} \). Then for all concept descriptions \( D_1, D_2 \) the following are equivalent:

1. \( D_1 \subseteq \subseteq D_2; \)
2. \( \text{GRI}(C_1, \ldots, C_n) \wedge \left( \bigwedge_{C \in \text{GCI}} C \subseteq D_1 \right) \wedge D_1 \not\subseteq D_2 \) is unsatisfiable w.r.t. \( \text{SLO}_{\text{Nk}}^3(\text{RI} \cup \text{ER}) \).

In addition, we may also need to express numerical information.

Example 5 Consider a variant of Example 4 in which we use a role with arity 4, \( r_{\text{il}} \), where \( r_{\text{il}}(x_1, x_2, x_3, n) \) expresses the fact that there exists a route from \( x_1 \) to \( x_2 \) passing through \( x_3 \) of length \( n \). Also in this situation we would like to talk about all routes from \( x_1 \) to \( x_2 \) passing through \( x_3 \) which are shorter than a certain length \( l \). This can also be expressed using projections as the relation \( \exists r_{\text{il}}(4, l, l) \), where:

\[
\exists r_{\text{il}}(4, l, l) = \{ (x_1, x_2, x_3) \mid \exists x_4 (x_4 \leq l \wedge r_{\text{il}}(x_1, x_2, x_3, x_4)) \}.
\]

We will show that the axioms describing the algebraic models for the extensions of \( \mathcal{E} \mathcal{L}^+ \) we considered here are “local”, a property which ensures that the uniform word problem (resp. the problem of checking the validity of a set of ground unit clauses) is decidable in \( \text{PTIME} \). We start by presenting a few important results on local theories and local theory extensions.

5 Local theories; local theory extensions

First-order theories are sets of formulae (closed under logical consequence), typically the set of all consequences of a set of axioms. Alternatively, we may consider the set of all models of a theory. In this paper we consider theories specified by their sets of axioms. (At places, however, – usually when talking about local extensions of a theory – we will refer to a theory, and mean the set of all its models.)

Before defining the notion of local theory and local theory extension we will introduce some preliminary notions on partial models of a theory.

Definition 10 (Partial and total models) Let \( \Pi = (S, \Sigma, \text{Pred}) \) be a many-sorted signature with set of sorts \( S \), set of function symbol \( \Sigma \) and set of predicates \( \text{Pred} \). A partial \( \Pi \)-structure is a structure \( \{ (A_i)_{i \in S}, \{ f_A \}_{f \in \Sigma}, \{ P_A \}_{P \in \text{Pred}} \} \) in which for some function symbols \( f \in \Sigma \), \( f_A \) may be partial.

Definition 11 A weak \( \Pi \)-embedding between the partial structures \( A = \{ (A_i)_{i \in S}, \{ f_A \}_{f \in \Sigma}, \{ P_A \}_{P \in \text{Pred}} \} \) and \( B = \{ (B_i)_{i \in S}, \{ f_B \}_{f \in \Sigma}, \{ P_B \}_{P \in \text{Pred}} \} \) is a (many-sorted) family \( i = (i_s)_{s \in S} \) of total maps \( i_s : A_s \rightarrow B_s \) such that

(i) if \( f_A(a_1, \ldots, a_n) \) is defined (in \( A \)) then also \( f_B(i_{s_1}(a_1), \ldots, i_{s_n}(a_n)) \) is defined (in \( B \)) and \( i_s(f_A(a_1, \ldots, a_n)) = f_B(i_{s_1}(a_1), \ldots, i_{s_n}(a_n)) \);

(ii) for each sort \( s \), \( i_s \) is injective and an embedding w.r.t. \( \text{Pred} \), i.e. for every \( P \in \text{Pred} \) with arity \( s_{1} \ldots s_{n} \) and every \( a_{1}, \ldots, a_{n} \) where \( a_{i} \in A_{s_{i}} \), \( P_A(a_{1}, \ldots, a_{n}) \) if and only if \( P_B(i_{s_1}(a_1), \ldots, i_{s_n}(a_n)) \).

In this case we say that \( A \) weakly embeds into \( B \).
In [11], Ganzinger established a link between proof theoretic and semantic concepts for function symbols. We will also consider extensions of theories, in which the signature is extended by new function symbols. In what follows, when we refer to sets $G$ of ground clauses we assume that they are in the signature $\Pi' = (S, \Sigma \cup \Sigma_c, \text{Pred})$, where $\Sigma_c$ is a set of new constants.

We will focus on the following type of locality of a theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$, where $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ with $\mathcal{K}$ a set of (universally quantified) clauses:

(Loc) For every finite set $G$ of ground clauses $\mathcal{T}_0 \cup G \models \bot$ iff $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \bot$ has no weak partial model with all terms in $st(\mathcal{K}, G)$ defined.
Here, \( \text{st}(\mathcal{X}, G) \) is the set of all ground terms occurring in \( \mathcal{X} \) or \( G \).

We say that an extension \( \mathcal{T}_0 \subseteq \mathcal{T}_1 \) is local if it satisfies condition \((\text{Loc})\). (Note that a local equational theory \[11\] is a local extension of the pure theory of equality (with no function symbols).) A more general notion, namely \( \Psi \)-locality of an extension theory (in which the instances to be considered are described by a closure operation \( \Psi \)) is introduced in \[13\]. Let \( \mathcal{X} \) be a set of clauses. Let \( \Psi' \) be a function associating with any set \( T \) of ground terms a set \( \Psi'_\mathcal{T}(T) \) of ground terms such that

(i) all ground subterms in \( \mathcal{X} \) and \( T \) are in \( \Psi'\mathcal{T}(T) \);
(ii) for all sets of ground terms \( T, T' \) if \( T \subseteq T' \) then \( \Psi'_\mathcal{T}(T) \subseteq \Psi'_\mathcal{T}(T') \);
(iii) for all sets of ground terms \( T, \Psi'_\mathcal{T}(\Psi'_\mathcal{T}(T)) \subseteq \Psi'_\mathcal{T}(T) \);
(iv) \( \Psi \) is compatible with any map \( h \) between constants, i.e. for any map \( h: C \rightarrow C, \Psi'_\mathcal{T}(h(T)) = h(\Psi'_\mathcal{T}(T)) \), where \( h \) is the unique extension of \( h \) to terms.

Let \( \mathcal{X}[\Psi'_\mathcal{X}(G)] \) be the set of instances of \( \mathcal{X} \) where the variables are instantiated with terms in \( \Psi'_\mathcal{X}(\text{st}(\mathcal{X}, G)) \) (set denoted in what follows by \( \Psi'_\mathcal{X}(G) \)), where \( \text{st}(\mathcal{X}, G) \) is the set of all ground terms occurring in \( \mathcal{X} \) or \( G \). We say that \( \mathcal{X} \) is \( \Psi \)-stably local if it satisfies:

\((\text{Loc}'(\Psi))\) for every finite set \( G \) of ground clauses, \( \mathcal{X} \cup G \) has a model which is a model of \( \mathcal{T}_0 \) iff \( \mathcal{X}[\Psi'_\mathcal{X}(G)] \cup G \) has a partial model which is a total model of \( \mathcal{T}_0 \) and in which all terms in \( \Psi'_\mathcal{X}(G) \) are defined.

If \( \Psi'_\mathcal{X}(G) = \text{st}(\mathcal{X}, G) \) we recover the definition of local theory extension.

In \( \Psi \)-local theories and theory extensions hierarchical reasoning is possible. We present the ideas for the case of local theories.

### 5.1.2 Hierarchical reasoning

Consider a \( \Psi \)-local theory extension \( \mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{X} \). The locality conditions defined above require that, for every set \( G \) of ground clauses, \( \mathcal{T}_1 \cup G \) is satisfiable if and only if \( \mathcal{T}_0 \cup \mathcal{X}[\Psi'_\mathcal{X}(G)] \cup G \) has a weak partial model with additional properties. All clauses in the set \( \mathcal{X}[\Psi'_\mathcal{X}(G)] \cup G \) have the property that the function symbols in \( \Sigma_1 \) have as arguments only ground terms. Therefore, \( \mathcal{X}[\Psi'_\mathcal{X}(G)] \cup G \) can be flattened and purified (i.e. the function symbols in \( \Sigma_1 \) are separated from the other symbols) by introducing, in a bottom-up manner, new constants \( c_i \) for subterms \( t = f(g_1, \ldots, g_n) \) with \( f \in \Sigma_1, g_i \) ground \( \Sigma_0 \cup \Sigma_1 \)-terms (where \( \Sigma_1 \) is a set of constants which contains the constants introduced by flattening, resp. purification), together with corresponding definitions \( c_i = t \). The set of clauses thus obtained has the form \( \mathcal{X}_0 \cup G_0 \cup \text{Def} \), where \( \text{Def} \) is a set of ground unit clauses of the form \( f(g_1, \ldots, g_n) = c_i \), where \( f \in \Sigma_1, c \) is a constant, \( g_1, \ldots, g_n \) are ground terms without function symbols in \( \Sigma_1 \), and \( G_0 \) and \( G_0 \) are clauses without function symbols in \( \Sigma_1 \). Flattening and purification preserve both satisfiability and unsatisfiability w.r.t. total algebras, and also w.r.t. partial algebras in which all ground subterms which are flattened are defined \[21\].

For the sake of simplicity in what follows we will always flatten and then purify \( \mathcal{X}[\Psi'_\mathcal{X}(G)] \cup G \). Thus we ensure that \( \text{Def} \) consists of ground unit clauses of the form \( f(c_1, \ldots, c_n) = c \), where \( f \in \Sigma_1 \), and \( c_1, \ldots, c_n, c \) are constants.

**Theorem 13** \[2114\] Let \( \mathcal{X} \) be a set of clauses. Assume that \( \mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{X} \) is a \( \Psi \)-local theory extension, and that for every finite set \( T \) of terms \( \Psi'_\mathcal{T}(T) \) is finite. For any set \( G \) of ground clauses, let \( \mathcal{X}_0 \cup G_0 \cup \text{Def} \) be obtained from \( \mathcal{X}[\Psi'_\mathcal{X}(G)] \cup G \) by flattening and purification, as explained above. Then the following are equivalent:
(1) $G$ is satisfiable w.r.t. $\mathcal{T}_1$.

(2) $\mathcal{T}_0 \cup \mathcal{K} \cup G$ has a partial model with all terms in $\text{st}(\mathcal{K}, G)$ defined.

(3) $\mathcal{T}_0 \cup \mathcal{K} \cup G \cup \text{Def}$ has a partial model with all terms in $\text{st}(\mathcal{K}, G)$ defined.

(4) $\mathcal{T}_0 \cup \mathcal{K} \cup G \cup \text{Con}[G]$ has a (total) model, where

$$\text{Con}[G] = \{ \bigwedge_{i=1}^{n} c_i = d_i \rightarrow c = d \mid f(c_1, \ldots, c_n) = c, f(d_1, \ldots, d_n) = d \in \text{Def} \}.$$

5.1.3 Parameterized decidability and complexity

Theorem 13 allows us to show that:

- decidability of checking satisfiability in a $\Psi$-local extension of a theory $\mathcal{T}_0$ is a consequence of the decidability of the problem of checking the satisfiability of ground clauses in $\mathcal{T}_0$, and
- the complexity of the task of checking the satisfiability of sets of ground clauses w.r.t. a $\Psi$-local extension of a base theory $\mathcal{T}_0$ can be expressed as a function of the complexity of checking the satisfiability of sets of ground clauses in $\mathcal{T}_0$.

Theorem 14 (21) Assume that the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies condition $(\text{Loc})$. If all variables in the clauses in $\mathcal{K}$ occur below some function symbol from $\Sigma_1$ and if testing satisfiability of ground clauses in $\mathcal{T}_0$ is decidable, then testing satisfiability of ground clauses in $\mathcal{T}_1$ is decidable.

Assume in addition that the complexity of testing the satisfiability of a set of ground clauses of size $m$ w.r.t. $\mathcal{T}_0$ can be described by a function $g(m)$. Let $G$ be a set of $\mathcal{T}_1$-clauses of size $n$. Then the complexity of checking the satisfiability of $G$ w.r.t. $\mathcal{T}_1$ is of order $g(n^k)$, where $k$ is the maximum number of free variables in a clause in $\mathcal{K}$, at least 2.

Proof: This follows from the fact that:

- the number of clauses in $\mathcal{K}_0$ is polynomial in the size of $\Psi_{\mathcal{X}}(G)$, where the degree $d$ of the polynomial is at most the maximum number of free variables in a clause in $\mathcal{K}$;
- the number of clauses in $G_0$ is linear in the size of $G$;
- the number of clauses in $\text{Con}[G]_0$ is quadratic in the size of $G$. □

5.1.4 Recognizing local theory extensions

The locality of an extension can be recognized by proving embeddability of partial models into total models [21,27,14]. We will use the following notation:

$$\text{PMod}^w_{\Psi}(\Sigma_1, \mathcal{T}_1)$$

is the class of all weak partial models $A$ of $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{X}$ in which the $\Sigma_1$-functions are partial, the $\Sigma_0$-functions are total, and the set of terms $\{ f(a_1, \ldots, a_n) \mid f(a_1, \ldots, a_n) \text{ defined} \}$ is closed under $\Psi_{\mathcal{X}}$.

For extensions $\mathcal{R}_0 \subseteq \mathcal{R}_1 = \mathcal{R}_0 \cup \mathcal{X}$, where $\mathcal{X}$ is a set of clauses, we consider the condition:

(Emb$^w_{\Psi}$) Every $A \in \text{PMod}^w_{\Psi}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

In what follows we say that a non-ground clause is $\Sigma_1$-flat if function symbols (including constants) do not occur as arguments of function symbols in $\Sigma_1$. A $\Sigma_1$-flat non-ground clause

\footnote{This requirement ensures that all variables are instantiated in $\mathcal{X}(G)$, and that therefore the satisfiability problem can be reduced without problems to testing the satisfiability of a set of ground clauses.}
Analyzing the proof of Theorem 15 we notice that the $\Sigma_1$-linearity restriction can be relaxed. We can allow a variable $x$ to occur below two unary function symbols $g$ and $h$ in a clause $C$ if $\Psi_x$ has the property that for every constant $c$, if $g(c) \in \Psi_x(G)$ then $h(c) \in \Psi_x(G)$ or vice versa. (In terms of partial models this means that we consider models $A$ with the property that if $g_A(a)$ is defined then $h_A(a)$ is defined or vice versa.)

The linearity condition can be similarly relaxed in the presence of $n$-ary functions, namely for groups of function symbols $(g_1, \ldots, g_n, h)$ — which occur in axioms containing

---

Note: The text contains mathematical symbols and logical notation which might not render clearly in this format. For a complete understanding, it is recommended to view the original document or reference the source material.
In [20] we proved that the algebraic counterpart of the description logic – i.e. an axiomatization with the property that for every set \( G \) of ground clauses

\[
\mathcal{S} \cup \text{Mon}(\Sigma) \cup G \models \bot \quad \text{if and only if} \quad (\mathcal{S} \cup \text{Mon}(\Sigma))[G] \cup G \models \bot.
\]

We denoted by \( \text{Mon}(\Sigma) \) the set \( \{ \text{Mon}(f) \mid f \in \Sigma \} \), where

\[
\text{Mon}(f) = \forall x, y (x \leq y \rightarrow f(x) \leq f(y)).
\]

In [21] we showed that the extension \( \text{SL} \cup \text{Mon}(\Sigma) \) of the theory \( \text{SL} \) of bounded semilattices with a family of monotone functions is local.

**Theorem 16** ([21,27]) Let \( G \) be a set of ground clauses. The following are equivalent:

1. \( \text{SL} \cup \text{Mon}(\Sigma) \cup G \models \bot \).
2. \( \text{SL} \cup \text{Mon}(\Sigma)[G] \cup G \) has no partial model \( A \) such that its \( \{\land, 0, 1\} \)-reduct is a (total) bounded semilattice, the functions in \( \Sigma \) are partial and all \( \Sigma \)-subterms of \( G \) are defined.

Let \( \text{Mon}(\Sigma)[G]_0 \cup G_0 \cup \text{Def} \) be obtained from \( \text{Mon}(\Sigma)[G] \cup G \) by purification, i.e. by replacing, in a bottom-up manner, all subterms \( f(g) \) with \( f \in \Sigma \), with newly introduced constants \( c_{f(g)} \) and adding the definitions \( f(g) = c_f \) to the set \( \text{Def} \).

**Theorem 17** The following are equivalent (and equivalent to (1) and (2) above):

3. \( \text{Mon}(\Sigma)[G]_0 \cup G_0 \cup \text{Def} \) has no partial model \( A \) such that its \( \{\land, 0, 1\} \)-reduct is a (total) bounded semilattice, the functions in \( \Sigma \) are partial and all \( \Sigma \)-subterm of \( G \) are defined.
4. \( \text{Mon}(\Sigma)[G]_0 \cup G_0 \) is unsatisfiable in \( \text{SL} \).

(Note that in the presence of \( \text{Mon}(\Sigma) \) the instances \( \text{Con}[G]_0 \) of the congruence axioms for the functions in \( \Sigma \) are not necessary.)

\[
\text{Con}[G]_0 = \{ g = g' \rightarrow c_{f(g')} = c_{f(g')} \mid f(g) = c_{f(g)}, f(g') = c_{f(g')} \in \text{Def} \}.
\]

This equivalence allows us to hierarchically reduce, in polynomial time, proof tasks in \( \text{SL} \cup \text{Mon}(\Sigma) \) to proof tasks in \( \text{SL} \) (cf. e.g. [21]) which can then be solved in polynomial time.
Example 6: We illustrate the method on an example first considered in [2]. Consider the $\mathcal{E \mathcal{L}^+}$ TBox $\mathcal{T}$ consisting of the following definitions:

\[
\begin{align*}
A_1 &= P_1 \sqcap A_2 \sqcap \exists r_1. \exists r_2. A_3 \\
A_2 &= P_2 \sqcap A_3 \sqcap \exists r_2. \exists r_1. A_1 \\
A_3 &= P_3 \sqcap A_2 \sqcap \exists r_1. (P_1 \sqcap P_2)
\end{align*}
\]

We want to prove that $P_1 \sqcap A_2 \sqcap \exists r_1. (A_1 \sqcap A_2) \subseteq A_3$. We translate this subsumption problem to the following satisfiability problem:

\[
\text{SL} \cup \text{Mon}(f_1, f_2) \cup \{ a_1 = (p_1 \land a_2 \land f_1(f_2(a_3))), \\
a_2 = (p_2 \land a_3 \land f_2(f_1(a_1))), \\
a_3 = (p_3 \land a_2 \land f_1(p_1 \land p_2)), \\
\neg(p_3 \land a_2 \land f_1(a_1 \land a_2) \leq a_3) \} \models \bot.
\]

We proceed as follows: We flatten and purify the set $G$ of ground clauses by introducing new names for the terms starting with the function symbols $f_1$ or $f_2$. Let Def be the corresponding set of definitions. We then take into account only those instances of the monotonicity and congruence axioms for $f_1$ and $f_2$ which correspond to the instances in Def, and purify them as well, by replacing the terms themselves with the constants which denote them. We obtain the following separated set of formulae:

| Def  | $G_0 \cup \text{Mon}(f_1, f_2)[G]_0 \cup \text{Con}[G]_0$ |
|------|----------------------------------------------------------|
| $f_2(a_3) = c_1$ | $(a_1 = p_1 \land a_2 \land c_2)$, $a_2 \leq c_2$, $R \in \{\leq, \geq, =\}$ |
| $f_1(c_1) = c_2$ | $(a_2 = p_2 \land a_3 \land c_4)$, $a_3 \leq c_4$, $R \in \{\leq, \geq, =\}$ |
| $f_1(a_1) = c_3$ | $(a_3 = p_3 \land a_2 \land d_1)$, $a_1 \leq c_3$, $R \in \{\leq, \geq, =\}$ |
| $f_2(c_3) = c_4$ | $(p_3 \land a_2 \land d_2 \leq a_3)$, $a_2 \leq c_4$, $R \in \{\leq, \geq, =\}$ |
| $f_1(e_1) = d_1$ | $(p_1 \land p_2 = e_1)$, $e_1 \leq d_1$, $R \in \{\leq, \geq, =\}$ |
| $f_1(e_2) = d_2$ | $a_1 \land a_2 = e_2$ |

The subsumption is true iff $G_0 \cup (\text{Mon}(f_1, f_2)[G]_0 \cup \text{Con}[G]_0$ is unsatisfiable in the theory of semilattices. We can see this as follows: note that $a_1 \land a_2 \leq p_1 \land p_2$, i.e. $e_2 \leq e_1$. Then (using an instance of monotonicity) $d_2 \leq d_1$, so $p_3 \land a_2 \land d_1 \leq p_3 \land a_2 \land d_1 = a_3$.

This can also be checked automatically in PTIME either by using the fact that there exists a local presentation of SL (cf. also Sect. 6.3) or using the fact that $\text{SL} = \text{ISP}(S_2)$ (i.e. every semilattice is isomorphic with a sublattice of a power of $S_2$), where $S_2$ is the semilattice with two elements, hence SL and $S_2$ satisfy the same Horn clauses. Since the theory of semilattices is convex, satisfiability of ground clauses w.r.t. SL can be reduced to SAT solving.

6.2 Locality and $\mathcal{E \mathcal{L}^+}$

We prove that similar results hold for the class $\text{SLO}_{g}(RI)$ of semilattices with monotone operators in a set $\Sigma$ satisfying a family $RI$ axioms of the form:

\[
\forall x \quad g(x) \leq h(x) \\
\forall x \quad f(g(x)) \leq x \\
\forall x \quad f(g(x)) \leq h(x)
\]
Since the characterization of locality in Theorem 15 refers to sets of flat clauses, instead of RI we consider the flat versions $R_{\text{flat}}$ of this family of axioms:

\[ \forall x \ g(x) \leq h(x) \]
\[ \forall x, y \ (y \leq g(x) \rightarrow f(y) \leq x) \]
\[ \forall x, y \ (y \leq g(x) \rightarrow f(y) \leq h(x)) \]

**Theorem 18**  The extension $SL \cup \text{Mon}(\Sigma) \cup R_{\text{flat}}(2)$ of the theory of semilattices with monotone functions $f, g$ satisfying axioms of the second type in $R_{\text{flat}}$ above is local.

**Proof:** We have to prove that every weak partial model of $SL \cup \text{Mon}(\Sigma) \cup R_{\text{flat}}(2)$ weakly embeds into a total model of $SL \cup \text{Mon}(\Sigma) \cup R_{\text{flat}}(2)$. This follows from Lemma 2.

**Theorem 19**  The extension $SL \cup \text{Mon}(\Sigma) \cup R_{\text{flat}}(1, 3)$ of the theory of lattices with monotone functions satisfying axioms of the first or third type in $R_{\text{flat}}$ above is $\Psi$-local, where $\Psi(T) = \bigcup_{i \geq 1} \Psi_i(T)$, with $\Psi_0(T) = T$, and

\[ \Psi_{i+1}(T) = \{ h(c) \mid \forall x(g(x) \rightarrow h(x)) \in R_{\text{flat}} \text{ and } g(c) \in T \} \cup \{ h(c) \mid \forall x(y \leq g(x) \rightarrow f(y) \leq h(x)) \in R_{\text{flat}} \text{ and } g(c) \in T \} . \]

**Proof:** Note first that the clauses we consider (see below) are flat, but not linear.

\[ \forall x \ g(x) \leq h(x) \]
\[ \forall x, y \ (y \leq g(x) \rightarrow f(y) \leq h(x)) \]

As mentioned before, a small change in the proof of Theorem 15 allows us to relax the linearity condition on the sets of clauses. By Theorem 15, an extension of $SL$ with monotonicity axioms and clauses of the type above is $\Psi$-local provided that every partial model $S$ of $SL \cup \text{Mon}(\Sigma) \cup RI(1, 3)$ with a total bounded semilattice reduct and with the property that if $g_S(a)$ is defined then $h_S(a)$ is defined (for all $g$ and $h$ occurring at the positions they have in the axioms above) weakly embeds into a total model of $SL \cup \text{Mon}(\Sigma) \cup RI(1, 3)$. The proof of the fact that this embeddability result holds is a consequence of Lemma 2.

**Theorem 20**  Any extension of the theory $SL$ of semilattices with a set of monotone functions satisfying axioms of type $RI$ is $\Psi$-local, where $\Psi$ is defined as above.

**Proof:** This is a consequence of Theorems 18 and 19 and of the fact that the same completion was used in all cases.

**Theorem 21**  Any theory of the form $SL \cup \text{Mon}(\Sigma) \cup RI \cup GRI(c_1, \ldots, c_n)$ – where GRI are guarded forms of axioms corresponding to role inclusions, as discussed in Section 7.4 – is $\Psi$-local, where $\Psi(T)$ is as defined above.

**Proof:** The proof is analogous to the proof of Theorems 18 and 19. We illustrate, as an example, the completion process for the case of axioms of the type

\[ \forall x (x \leq c \land y \leq g(x) \rightarrow f(y) \leq h(x)) . \]

Let $S$ be a bounded semilattice with partial operators satisfying the axioms in $\text{Mon}(\Sigma) \cup RI \cup GRI(c_1, \ldots, c_n)$. We extend the functions to $\mathfrak{S} = \mathfrak{S}(S)$ as explained in Lemma 2.
Let $\eta : S \rightarrow 2^S$ defined by $\eta(x) = \downarrow x$. Then $\iota(c) = \downarrow c$. Let now $U, V \in S$ be such that $V \subseteq \downarrow c$ and $U \subseteq \Omega(V)$. Let $x \in \Omega(U)$, so there exist $u \in U$ for which $f(u)$ is defined, and $v \in V$ with $g(v)$ defined such that $v < c$, $x \leq f(u)$ and $u \leq g(v)$. By the $\Psi$-closure condition, $h(v)$ is defined as well. Thus, $x \leq f(u) \leq h(v)$, i.e. $x \in \Omega(V)$. The other guarded cases can be handled similarly.

Example 7 We illustrate the ideas on an example presented in [4] (here slightly simplified). Consider the CBox $\mathcal{C}$ consisting of the following GCI:

- Endocard $\sqsubseteq$ Tissue $\sqcap \exists$ cont-in.HeartWall $\sqcap \exists$ cont-in.HeartValve
- HeartWall $\sqsubseteq$ part-of Heart
- HeartValve $\sqsubseteq$ part-of Heart
- Endocarditis $\sqsubseteq$ Inflammation $\sqcap \exists$ has-loc. Endocard
- Inflammation $\sqsubseteq$ Disease
- Heartdisease $=$ Disease $\sqcap \exists$ has-loc. Heart

and the following role inclusions $RI$:

- part-of $\circ$ part-of $\sqsubseteq$ part-of
- part-of $\circ$ cont-in
- has-loc $\circ$ cont-in $\sqsubseteq$ has-loc

We want to check whether Endocarditis $\sqsubseteq$ Heartdisease. This is the case iff (with some abbreviations – e.g. $f_{ci}$ stands for $f_{\exists \text{cont-in}}$ and $f_{po}$ for $f_{\exists \text{part-of}}$, $h_u$ and $h_v$ for HeartWall resp. HeartValve, $e$ for Endocard, $h$ for Heart, etc.):

$$\begin{align*}
SL \cup \text{Mon}(f_{ci}, h_u, f_{po}) \cup \{ & \forall x y \leq f_{ci}(x) \rightarrow f_{ci}(y) \leq f_{ci}(x), \\
& \forall x f_{po}(x) \leq f_{ci}(x), \\
& \forall x y \leq f_{ci}(x) \rightarrow f_{po}(y) \leq f_{po}(x) \}\ \\
\cup \{ & e \leq t \uparrow f_{ci}(h_u) \uparrow f_{ci}(h_v), h_u \leq f_{po}(h), h_v \leq f_{po}(h), \\
& \text{Endocarditis} \leq t \uparrow f_{po}(e), \ i \leq t, \ \text{Heartdisease} = d \uparrow f_{po}(h), \\
& \text{Endocarditis} \leq \text{Heartdisease} \} \models \bot.
\end{align*}$$

Then $\text{st}(\mathcal{C}, G) = \{f_{ci}(h_u), f_{ci}(h_v), f_{po}(h), f_{po}(e), f_{po}(h)\}$. It follows that $\Psi_{\mathcal{C}}(G)$ consists of the following terms: $\{f_{ci}(h_u), f_{ci}(h_v), f_{ci}(e), f_{po}(h), f_{po}(e), f_{po}(h), f_{po}(h), f_{po}(h)\}$. After computing $(\text{RL}_{\Delta} \cup \text{Mon}(f_{ci}, h_u, f_{po}) \cup \text{Con})[\Psi(G)]$ we obtain:

| $G$ | $(\text{RL}_{\Delta} \cup \text{Mon} \cup \text{Con})[\Psi(G)]$ |
|-----|--------------------------------------------------|
| $e \leq t \uparrow f_{ci}(h_u) \uparrow f_{ci}(h_v)$ | $y \leq f_{ci}(x) \rightarrow f_{ci}(y) \leq f_{ci}(x)$ for $x, y \in \{h_u, h_v\}, x \neq y$ |
| $h_u \leq f_{po}(h)$ | $f_{po}(h) \leq f_{ci}(h)$ |
| $h_v \leq f_{po}(h)$ | $y \leq f_{ci}(x) \rightarrow f_{po}(y) \leq f_{po}(x)$ for $x \in \{h_v, h_u, h_v\}, y \in \{e, h, h_u, h_v\}, x \neq y$ |
| Endocarditis $\leq i \uparrow f_{po}(e)$ | $xRy \rightarrow f_{ci}(x)Rf_{ci}(y)$ for $x, y \in \{h_u, h_v\}, x \neq y$ |
| $i \leq d$ | $xRy \rightarrow f_{po}(x)Rf_{po}(y)$ for $x, y \in \{e, h, h_u, h_v\}$ |
| Heartdisease $= d \uparrow f_{po}(h)$ | $R \in \{\leq, \geq\}$ |
| Endocarditis $\leq$ Heartdisease | |

We can simplify the problem even further by replacing the ground terms in $\Psi(G)$ with new constants, and taking into account the corresponding definitions $c_i = t$. Let $(\text{RL}_{\Delta} \cup \text{Mon} \cup$
\[
\phi \models \chi_{\text{Endocarditis}} \sqsubseteq \chi_{\text{Heartdisease}} \quad \text{iff} \quad G_0 \cup (RL_d \cup \text{Mon} \cup \text{Con})|\Psi(G)|_0 \models \frac{\chi_{\text{Endocarditis}}}{\chi_{\text{Heartdisease}}} \quad \text{(i.e. it is unsatisfiable w.r.t. the theory of semilattices with 0 and 1)}.
\]

The satisfiability of \( \phi \) can therefore be checked automatically in polynomial time in the size of \( \phi \) which in its turn is polynomial in the size of \( \Psi_x(G) \). Hence, in this case, the size of \( \phi \) is polynomial in the size of \( G \).

Unsatisfiability can also be proved directly: \( G \) entails the inequalities:

1. \( \text{Endocarditis} \leq (d \land f_{hl}(e)) \);
2. \( e \leq (f_c(h_w) \land f_c(h_t)) \);
3. \( h_w \leq f_{po}(h) \);
4. \( h_t \leq f_{po}(h) \).

Hence \( G \land (RL_d \land \text{Mon} \land \text{Con})|\Psi(G)| \models e \leq f_c(f_{po}(h)) \leq f_c(f_c(h)) \leq f_c(h) \). Thus, \( G \land (RL_d \land \text{Mon} \land \text{Con})|\Psi(G)| \models (d \land f_{hl}(h)) \), which together with \( d \land f_{hl}(h) = \text{Heartdisease} \) and \( \text{Endocarditis} \nleq \text{Heartdisease} \) leads to a contradiction.

6.3 Complexity

We now analyze the complexity of the problem of checking CBox subsumption in the extensions of \( \mathcal{E} \cup \mathcal{L}^+ \) considered in this paper. Note that by Theorems 16 and 20 in all cases considered in Section 6.1 and 6.2 we can reduce CBox subsumption to the task of checking the satisfiability of a set of constraints of the form

\[
R'\{\Psi(G)\}_0 \cup \text{Mon}(\Sigma)|\Psi(G)|_0 \cup G_0
\]

w.r.t. the theory of bounded semilattices.

**Lemma 5** For the specific closure operator \( \Psi \) we consider, the following hold:

- The size of \( \Psi(G) \) is linear in the size of \( |\text{st}(G)| \), where \( |\text{st}(G)| \) is the number of subterms of \( G \) that start with a function symbol in \( \Sigma \).
- The size of \( \text{Mon}(\Sigma)|\Psi(G)| \) (and hence also the size of \( \text{Mon}(\Sigma)|\Psi(G)| \)) is \( 2|\Psi(G)|^2 \), hence it is quadratic in the size of \( |\text{st}(G)| \).
- The size of \( R'|\Psi(G)| \) (hence also the size of \( R'|\Psi(G)|_0 \)) is quadratic in the size of \( \Psi(G) \), hence also in the size of \( |\text{st}(G)| \).
We reduced the initial problem to the problem of checking satisfiability w.r.t. the theory of bounded semilattices of a conjunction between a set $G_0$ of ground unit clauses of the form

$$c_1 \land c_2 \leq c, \quad c_1 \leq c_2 \quad d_1 \not\leq d_2$$

of size linear in $|st(G)|$ and a set of Horn clauses of length at most $n + 1$, where $n$ is the maximal arity of a function symbol in $\Sigma$ of the form

$$c_1 \leq c'_1 \land \cdots \land c_n \leq c' \rightarrow c \leq c'.$$

It is easy to see (cf. also [22,23]) that one can give a polynomial decision procedure for checking the satisfiability of such sets of clauses, by noticing that if the set of clauses is unsatisfiable then there exists an instance of monotonicity with all premises entailed by the unit clauses from $G_0$. We can add the conclusion to $G_0$ and recursively repeat the argument.

In order to obtain an even more efficient method for checking TBox subsumption we use a reduction to reachability in the theory of posets. It is known that the theory of semilattices allows a local Horn axiomatization (cf. e.g. [19,19]), by means of the following axioms:

\[
\begin{align*}
(S1) & \quad \forall x, y, z \quad (x \leq y \land y \leq z \rightarrow z \leq z) \\
(S2) & \quad \forall x \quad (0 \leq x \land x \leq 1) \\
(S3) & \quad \forall x, y \quad (x \land y \leq x \land x \land y \leq y) \\
(S4) & \quad \forall x, y, z \quad (z \leq x \land z \leq y \rightarrow z \leq x \land y)
\end{align*}
\]

We denote by $\mathcal{L}$ this set of axioms for the theory of bounded semilattices.

**Theorem 22** The set of Horn clauses $\mathcal{L}$ define a local extension of the pure theory of bounded partial orders, i.e. for every set $G$ of ground clauses in the signature of bounded semilattices, $\mathcal{L} \cup G \models \bot$ if $\mathcal{L}[G] \cup G \models \bot$.

**Proof:** Let $(P, \leq, \land, 0, 1)$ be a weak partial model of $\mathcal{L}$. Then $(P, \leq, 0, 1)$ is a poset with first and last element. Let $\mathcal{O}(P) = (\mathcal{O}(P), \leq, \cap, \{0\}, P)$ be the semilattice of all order ideals of $P$. We show that the map $i : P \rightarrow \mathcal{O}(P)$ defined by $i(x) = \downarrow x$ is a weak embedding: $i$ is obviously injective and an order embedding. Clearly, $i(0) = \downarrow 0 = \{0\}$ and $i(1) = P$. Assume that $x \land y$ is defined in $P$. Then $i(x \land y) = \downarrow(\downarrow x \land \downarrow y)$. If $x \land y$ is defined in $P$, since $P$ weakly satisfies (S3), $x \land y \leq x$ and $x \land y \leq y$, so $x \land y \in \downarrow x \land \downarrow y$. Hence, $\downarrow(\downarrow x \land \downarrow y) \subseteq \downarrow x \land \downarrow y$. Conversely, let $z \in \downarrow x \land \downarrow y$. Then $z \leq x$ and $z \leq y$ and as $x \land y$ is defined and $P$ weakly satisfies (S4), $z \leq x \land y$. It follows that $i(x \land y) = \downarrow(\downarrow x \land \downarrow y) = \downarrow x \land \downarrow y = i(x) \land i(y)$.

**Corollary 4** The following are equivalent:

1. $SL \cup \text{Mon}(\Sigma) \cup RI \models \forall x, y, z \quad (x \leq y \land y \leq z \rightarrow z \leq z)$.
2. $SL \cup \text{Mon}(\Sigma) \cup RI \cup G \models \bot$, where $G = \bigcup_{i=0}^{n_s} s_i(\Sigma) \land s(\Sigma) \not\leq s'_i(\Sigma)$.
3. $SL \cup \text{Mon}(\Sigma) \cup RI \cup \Psi(G) \models \bot$, where $G = \bigcup_{i=0}^{n_s} s_i(\Sigma) \land s(\Sigma) \not\leq s'_i(\Sigma)$, and $\Psi$ is defined as in Theorem [19]
4. $\mathcal{L} \cup \text{Mon}(\Sigma) \cup RI |\Psi(G)| \models \bot$, for the purified semilattice part of the problem.
5. $\mathcal{L}[G] \cup G' \models \bot$, where $G' = (\text{Mon}(\Sigma) \cup RI) |\Psi(G)| \cup G_0$.  
6. $\mathcal{L}[G]_0 \cup G_0 \cup \text{Con}(\Sigma)[G] \models \bot$.

**Theorem 23** CBox subsumption can be checked in cubic time in the size of the original CBox for all CBoxes in the language of the extension of $\mathcal{L}^+$ considered in this paper.
Proof: We analyze the complexity of the problem in item (6) of Corollary 4 as a function of the size of the input CBox, i.e. as a function of the size of \( \text{RI} \) and \( G \). We first estimate the size of \( G' \). Note that \( \Psi(G) \) can have at most \(|\text{st}(G)| \cdot |N_R| \) elements. Thus, its size is linear in the size of \( G \) if \( N_R \) is fixed. The number of clauses in \((\text{Mon}(\Sigma) \cup \text{RI})[\Psi(G)]\) is quadratic in \(|\Psi(G)|\). By purification, the size grows linearly. Thus:

- The size of \( G' \) is quadratic in the number of subterms of \( G \).
- \( G' \) contains a set of ground unit clauses (of size linear in the size of \( G \)) and a set of ground Horn clauses (of size quadratic in the size of \( G \)).
- The number of subterms in \( G' \) is linear in the number of subterms of \( G \).

If we consider the form of the clauses in \( \mathcal{L} \) we note that the number of clauses in \( \mathcal{L}[G'] \cup \text{Con}(\land)[G'] \) is at most cubic in the number of subterms in \( G' \), i.e. cubic in the number of subterms of \( G \). The conclusion of the theorem now follows easily if we note that

- \( \mathcal{L}[G'] \cup \text{Con}(\land)[G'] \) is a set of ground Horn clauses, and
- in order to check the satisfiability of any set of \( N \) ground clauses w.r.t. the theory of posets we only need to take into account those instances of the poset axioms in which the variables are instantiated with the (ground) terms occurring in \( N \).

We can thus reduce the verification problem to the problem of checking the satisfiability of a set of Horn clauses of size at most cubic in the number of subterms of \( G \). Since the satisfiability of Horn clauses can be tested in linear time [10], this shows that the uniform word problem for the class \( \text{SLO}_\mathcal{L}(\text{RI}) \) (and thus for \( \text{SLO}_\mathcal{L}(\exists)\)) is decidable in cubic time. \( \square \)

6.4 Extensions of \( \mathcal{L} \) with \( n \)-ary roles and concrete domains

The previous results can easily be generalized to semilattices with \( n \)-ary monotone functions satisfying composition axioms.

6.4.1 Extensions of \( \mathcal{L} \) with \( n \)-ary roles

We now consider the extensions of \( \mathcal{L} \) with \( n \)-ary roles introduced in Section 4.4.1. The semantics is defined in terms of interpretations \( \mathcal{J} = (D^\mathcal{J}, \cdot^\mathcal{J}) \), where \( D^\mathcal{J} \) is a non-empty set, concepts are interpreted as usual, and each \( n \)-ary role \( R \in N_R \) is interpreted as an \( n \)-ary relation \( R^\mathcal{J} \subseteq (D^\mathcal{J})^n \). All results in the previous section extend in a natural way to this case, because, independently of the arities of the functions, the extension of the theory of bounded semilattices with monotone functions is local and the number of instances of the monotonicity axioms in \( \text{Mon}[\Psi(G)] \) is quadratic in the size of \( \Psi(G) \).

6.4.2 Extensions of \( \mathcal{L}^+ \) with \( n \)-ary roles

In this case we need to take into account role inclusions of type:

\[
\begin{align*}
    r_1 & \subseteq r_2 \\
    r_1 \circ (s_1, \ldots, s_n) & \subseteq r_2 \\
    r_1 \circ (s_1, \ldots, s_n) & \subseteq \text{id} \quad \text{for binary relations } s_i
\end{align*}
\]

We proved that TBox subsumption can be expressed as a uniform word problem w.r.t. the class of semilattices with monotone operators associated with the roles, satisfying axioms
RL₀ corresponding to subsumption testing

In the presence of existential role restrictions we can prove the following result.

6.4.3 Extensions with existential role restrictions

In the presence of existential role restrictions we can prove the following result.

Theorem 24 Any extension of the theory SL of lattices with a set of monotone functions satisfying any combination of axioms containing axioms of type $R\text{flat}^l$ is $Ψ$-local, where $Ψ(T) = \bigcup_{j \geq 1} Ψ_j(T)$, with $Ψ_0(T) = T$, and

$$Ψ_j+1(T) = \{ h(\mathcal{r}) | \exists \forall \mathcal{r} \bigwedge g_i(\mathcal{r}) \rightarrow h(\mathcal{r}) \} \in R\text{flat}^l \text{ and } g(\mathcal{r}) \in T \}\cup$$

$$\{ h(c) | \exists \forall x, y \bigwedge y \leq g(x) \rightarrow f(y) \leq h(x) \} \in R\text{flat}^l \text{ and } \forall g_i(c) \in T \}\cup$$

$$\{ h(\mathcal{r}_1, \ldots, \mathcal{r}_n) | \exists \forall \mathcal{r}_i \bigwedge y_i \leq g_i(\mathcal{r}_i) \rightarrow f(y_1, \ldots, y_n) \leq h(\mathcal{r}_1, \ldots, \mathcal{r}_n) \} \in R\text{flat}^l$$

Proof: The proof is analogous to the proof of Theorem 20. We illustrate as an example the flatness restriction of the extensions of $GRL_0$ axioms, $G\text{flat}^r_0$ which are defined analogously to $R\text{flat}^l$.

6.4.3 Extensions with existential role restrictions

In the presence of existential role restrictions we can prove the following result.

Theorem 25 Any extension of the theory SL of lattices with a set of monotone functions satisfying any combination of axioms containing axioms of type $R\text{flat}^l$, $G\text{flat}^r_0$ and existential role restrictions $ER$ of the form:

$$\forall x_1, x_2, \ldots, n \ g(x_1, x_2, \ldots, x_n) = h(x_1, \ldots, x_n)$$

is $Ψ$-local, where $Ψ(T) = \bigcup_{j \geq 1} Ψ_j(T)$, with $Ψ_0(T) = T$, and

$$Ψ_j+1(T) = \{ h(\mathcal{r}) | \exists \forall \mathcal{r} \bigwedge g_i(\mathcal{r}) \rightarrow h(\mathcal{r}) \} \in G\text{flat}^l \text{ and } g(\mathcal{r}) \in T \}\cup$$

$$\{ h(c) | \exists \forall x, y \bigwedge y \leq g(x) \rightarrow f(y) \leq h(x) \} \in G\text{flat}^l$$

and $g_i(c) \in T$ for all $i$.

$$\{ h(\mathcal{r}_1, \ldots, \mathcal{r}_n) | \exists \forall \mathcal{r}_i \bigwedge y_i \leq g_i(\mathcal{r}_i) \rightarrow f(y_1, \ldots, y_n) \leq h(\mathcal{r}_1, \ldots, \mathcal{r}_n) \} \in G\text{flat}^l$$

where $g$ are either true or a suitable conjuction of guards of the form $x_i \leq d_i$. 
Proof: The only issue to be clarified is the locality of the extension with axioms in $ER$. The axioms in $ER$ are extensions by definitions like the ones considered in [27]. Due to arity reasons, they are acyclic. Thus, we have the following chain of extensions: $SLO_2 \subseteq SLO_2(ER) \subseteq SLO_2(RI \cup ER)$. \hfill $\square$

6.4.4 Extensions with n-ary roles and concrete domains

We now consider the extension with concrete domains studied in Section 4.2.2. We showed that an algebraic semantics can be given in terms of the class $SL_2$ of all structures $\mathcal{A} = (A, \mathcal{P}(A_1), \ldots, \mathcal{P}(A_n))$, with signature $\Pi = \{S, \{\wedge\} \cup \Sigma, \mathcal{Pred}\}$ with $S = \{\text{concept}, s_1, \ldots, s_n\}$, $\text{Pred} = \{\leq \} \cup \{\leq_i | 1 \leq i \leq n\}$, where $A \in SL$, the support of sort concept $\mathcal{A}$ is $A$, and for all $i$ the support sort $s_i$ of $\mathcal{A}$ is $\mathcal{P}(A_i)$.

**Theorem 26** ([27]) Every structure $(A, \mathcal{P}(A_1), \ldots, \mathcal{P}(A_n), \{f_i\}_{i \in \Sigma})$, where

(i) $(A, \mathcal{P}(A_1), \ldots, \mathcal{P}(A_n)) \in SL_2$, and
(ii) for every $f \in \Sigma$ of arity $s'_1 \ldots s'_{n} \rightarrow s$, with $s'_1, \ldots, s'_n, s \in S$, $f_A$ is a partial function from $\prod_{i=1}^{n} U_i \rightarrow U_i$ which is monotone on its domain of definition (here $U_{\text{concept}} = A$ and $U_{s_i} = \mathcal{P}(A_i)$ are the universes of the many-sorted structure in (i)).

weakly embeds into a total model of $SL_2 \cup \text{Mon} (\Sigma)$.

**Corollary 5** Let $G = \bigwedge_{i=1}^{n} s_i (\bar{v}) \leq s'_i (\bar{v}) \wedge s (\bar{v}) \leq s'_i (\bar{v})$ be a set of ground unit clauses in the extension $\Pi^*$ of $\Pi$ with new constants $\Sigma$. The following are equivalent:

(1) $SL_2 \cup \text{Mon} (\Sigma) \cup G = \bot$.
(2) $SL_2 \cup \text{Mon} (\Sigma) \cup G$ has no partial model with a total $\{\wedge \}$-reduct in which all terms in $G$ are defined.

A hierarchical reduction to the problem of checking satisfiability of constraints in the disjoint combinations of the theory of semilattices and the theories $\mathcal{P}(A_i)$ follows immediately from this locality result. Let $\bigcup_{i=0}^{n} \text{Mon}(\Sigma)[G_i] \cup G_i \cup \text{Def}$ be obtained from $\text{Mon}(\Sigma)[G_i] \cup G_i$ by purification, i.e. by replacing, in a bottom-up manner, all subterms $f(g)$ of sort $s$ with $f \in \Sigma$, with newly introduced constants $c_{f(g)}$ of sort $s$ and adding the definitions $f(g) = c_i$ to the set $\text{Def}$. We thus separate $\text{Mon}(\Sigma)[G_i] \cup G_i$ into a conjunction of constraints $\Gamma_i = \text{Mon}(\Sigma)[G_i] \cup G_i$, where $\Gamma_0$ is a constraint of sort semilattice and for $1 \leq i \leq n$, $\Gamma_i$ is a set of constraints over sort $i$ (i being the concrete sort with fixed support $\mathcal{P}(A_i)$).

**Corollary 6** The following are equivalent (and are also equivalent to (1) and (2)):

(3) $\bigcup_{i=0}^{n} \text{Mon}(\Sigma)[G_i] \cup G_i \cup \text{Def}$ has no partial model with a total $\{\wedge, 0, 1\}$-reduct in which all terms in $\text{Def}$ are defined.
(4) $\bigcup_{i=0}^{n} \text{Mon}(\Sigma)[G_i] \cup G_i$ is unsatisfiable in the many-sorted disjoint combination of $SL$ and the concrete theories of $\mathcal{P}(A_i)$, $1 \leq i \leq n$.

The complexity of the uniform word problem of $SL_2 \cup \text{Mon}(\Sigma)$ depends on the complexity of the problem of testing the satisfiability — in the many-sorted disjoint combination of $SL$ with the concrete theories of $\mathcal{P}(A_i)$, $1 \leq i \leq n$ — of sets of clauses $\text{C}_{\text{concept}} \bigcup \bigcup_{i=1}^{n} C_i \bigcup \text{Mon}$, where $\text{C}_{\text{concept}}$ and $C_i$ are unit clauses of sort concept resp. $s_i$, and $\text{Mon}$ consists of possibly mixed ground Horn clauses.

Specific extensions of the logic $\mathcal{E} \mathcal{L}$ can be obtained by imposing additional restrictions on the interpretation of the “concrete”-type concepts within $\mathcal{P}(A_i)$. For instance, we can require that numerical concepts are always interpreted as intervals, as in Example 3.
Theorem 27 Consider the extension of $\mathcal{EL}$ with two sorts, concept and num, where the semantics of classical concepts is the usual one, and the concepts of sort num are interpreted as elements in the ORD-Horn, convex fragment of Allen’s interval arithmetic [17], where any CBox can contain many-sorted GCI’s over concepts, as well as constraints over the numerical data expressible in the ORD-Horn fragment.

In this extension, CBox subsumption is decidable in PTIME.

Proof: The assumption on the semantics of the extension of $\mathcal{EL}$ we made ensures that all algebraic models are two-sorted structures of the form $\mathcal{A} = (\{A, \land\}, \text{Int}(\mathbb{R}, O), \{f_{\mathcal{A}}\}_{f \in \Sigma})$, with sorts $\{\text{concept}, \text{num}\}$, such that $(\{A, \land\})$ is a semilattice, $\text{Int}(\mathbb{R}, O)$ is an interval algebra in the Ord-Horn fragment of Allen’s interval arithmetic [17], and for all $f \in \Sigma, f_\mathcal{A}$ is a monotone (many-sorted) function. We will denote the class of all these structures by $\mathcal{SL}_{\text{ORD-Horn}}$.

Note that the Ord-Horn fragment of Allen’s interval arithmetic has the property that all operations and relations between intervals can be represented by Ord-Horn clauses, i.e. clauses over atoms $x \leq y, x = y$, containing at most one positive literal (or $x \leq y$ or $x = y$) and arbitrarily many negative literals (of the form $x \neq y$). Nebel and Bückert [17] proved that a finite set of Ord-Horn clauses is satisfiable over the real numbers iff it is satisfiable over posets. As the theory of partial orders is convex, this means that although the theory of reals is not convex w.r.t. $\leq$, we can always assume that the theory of Ord-Horn clauses is convex. The main result in Corollary 5 can be adapted without problems to show that if $G = \bigwedge_{i=1}^{n} s(\tau) \leq s'(\tau) \land s(\tau) \leq s'(\tau)$ is a set of ground unit clauses in the extension $\Pi'$ of $\Pi$ with new constants $\Sigma_i$, and if $\text{Mon}(\Sigma)[G] \cup \text{Mon}(\Sigma)[\text{num}] \cup G_c \cup G_{\text{num}} \cup \text{Def}$ are obtained from $\text{Mon}(\Sigma)[G] \cup G$ by purification, the following are equivalent:

1. $\mathcal{SL}_{\text{OrdHorn}} \cup \text{Mon}(\Sigma) \cup G \models \bot$;
2. $\text{Mon}(\Sigma)[G]_0 \cup G_0 \cup \text{Con}[\text{Def}_0]$ is unsatisfiable in the combination of $\mathcal{SL}$ and the Ord-Horn fragment of Allen’s interval arithmetic.

In order to test the unsatisfiability of the latter problem we proceed as follows. We first note that, due to the convexity of the theories involved and to the fact that all constraints in $G_0 \cup \text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0]$ are separated (in the sense that there are no mixed atoms) if

1. $G_0 \cup \text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0] \models \bot$;
2. there exists a clause $\text{C} = (\bigwedge_{i=1}^n c_i = d_i \to c = d)$ in $\text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0]$ such that $G_0 \models \bigwedge_{i=1}^n c_i = d_i$ and $G_0 \cup \{c = d\} \cup (\text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0]) \models \bot$.

In order to prove this, let $\mathcal{D}$ be the set of all atoms $c_i R d_i$ occurring in premises of clauses in $\text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0]$. As every model of $G_0 \land \bigwedge_{(c R d) \in \mathcal{D}} \neg(c R d)$ is also a model of $G_0 \cup \text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0]$, and the last formula is by (1) unsatisfiable, it follows that $G_0 \land \bigwedge_{(c R d) \in \mathcal{D}} \neg(c R d) \models \bot$ in the combination of the Ord-Horn fragment over posets with the theory of semilattices. Let $G_{cRd}^G$ be the conjunction of all atoms in $G_0$, and $G_{cRd}$ be the set of all negative literals in $G_0$. Then $G_{cRd}^G \models \bigvee_{(c R d) \in \mathcal{D}} (c R d) \lor \bigvee_{-e \in G_0} \bot$. Since the constraints are sort-separated and both theories involved are convex, it follows that either $G_0 \models \bot$ or else $G_0 \models c R d$ for some $(c R d) \in \mathcal{D}$. We can repeat the process until all the premises of some clause in $\text{Mon}(\Sigma)[G]_0 \cup \text{Con}[\text{Def}_0]$ are proved to be entailed by $G_0$. Thus, (2) holds.

By iterating the argument above we can always – if (1) holds – successively entail sufficiently many premises of monotonicity and congruence axioms in order to ensure that, in the end,
(3) there exists a set \(\{C_1, \ldots, C_n\}\) of clauses in \(\text{Mon}(\Sigma)[G_0] \cup \text{Con}[\text{Def}]_0\) with \(C_j = \bigwedge c^j_1 = d^j_1 \rightarrow c^j \equiv d^j\), such that for all \(k \in \{0, \ldots, n-1\}\),

\[
G_0 \land \bigwedge_{j=1}^{k}(c^j = d^j) \models \bigwedge_{j=k+1}^{n} c^{j+1} = d^{j+1} \text{ and } G_0 \land \bigwedge_{j=1}^{n}(c^j = d^j) \models \bot.
\]

Note that (3) implies (1), since the conditions in (3) imply that \(G_0 \land \bigwedge_{j=1}^{n}(c^j = d^j)\) is logically equivalent with \(G_0 \land C_1 \land \ldots \land C_n\), which (as set of clauses) is contained in the set of clauses \(G_0 \cup \text{Mon}(\Sigma)[G_0] \cup \text{Con}[\text{Def}]_0\).

This means that in order to test satisfiability of \(G_0 \cup \text{Mon}(\Sigma)[G_0] \cup \text{Con}[\text{Def}]_0\) we need to test entailment of the premises of \(\text{Mon}(\Sigma)[G_0] \cup \text{Con}[\text{Def}]_0\) from \(G_0\); when all premises of some clause are provably true we delete the clause and add its conclusion to \(G_0\). The PTIME assumptions for concept subsumption and for the Ord-Horn fragment ensure that this process terminates in PTIME.

\[\Box\]

**Example 8** Consider the special case described in Example 8. Assume that the concepts of sort \(\text{num}\) used in any TBox are of the form \(\forall n, \exists m\) and \([n,m]\). Consider the TBox \(\mathcal{T}\) consisting of the following GCI’s:

\[
\{\exists \text{price}(n) \sqsubseteq \text{affordable}, \exists \text{weight}(m) \sqcap \text{car} \sqsubseteq \text{truck}, \\
\text{has-weight-price}(n, m) \sqsubseteq \exists \text{price}(n) \sqcap \exists \text{weight}(m), \\
\forall n \sqsubseteq \downarrow n, \forall m \sqsubseteq \downarrow m, C \sqsubseteq \text{car}, C \sqsubseteq \exists \text{has-weight-price}(m, n) \}
\]

In order to prove that \(C \sqsubseteq \mathcal{T} \text{ affordable} \sqcap \text{truck}\) we proceed as follows. We refute \(\bigwedge_{D \in \mathcal{T}} D \leq \text{affordable} \sqcap \text{truck}\). We purify the problem introducing definitions for the terms starting with existential restrictions, and express the interval constraints using constraints over \(Q\) and obtain the following set of constraints:

| Def       | F-num | C-concept                  | Mon                      |
|-----------|-------|---------------------------|--------------------------|
| \(f_{\text{price}}(\downarrow n) = c\) | \(n \leq n_1\) | \(c_1 \sqsubseteq \text{affordable}\) | \(n_1 \leq n \rightarrow c_1 \leq c\) |
| \(f_{\text{price}}(\downarrow m) = c\) | \(m \geq m_1\) | \(c_1 \sqcap \text{car} \sqsubseteq \text{truck}\) | \(m_1 \geq m \rightarrow c_1 \geq c\) |
| \(f_{\text{weight}}(\downarrow m) = d\) | \(m \geq m_1\) | \(c \leq c \sqcap d\) | \(m_1 \geq m \rightarrow d_1 \leq d\) |
| \(f_{\text{weight}}(\downarrow m) = d\) | \(m \geq m_1\) | \(c \sqsubseteq \text{car}\) | \(m_1 \geq m \rightarrow d_1 \leq d\) |
| \(f_{\text{weight}}(\downarrow m, \downarrow n) = e\) | \(c \leq e\) | \(C \leq e\) | \(m_1 \leq m \rightarrow d_1 \leq d\) |
| \(f_{\text{weight}}(\downarrow m, \downarrow n) = e\) | \(c \leq e\) | \(C \sqsubseteq \text{affordable} \sqcap \text{truck}\) | \(m_1 \leq m \rightarrow d_1 \leq d\) |

The task of proving \(C \sqsubseteq \mathcal{T} \text{ affordable} \sqcap \text{truck}\) can therefore be reduced to checking whether \(C_{\text{num}} \land C_{\text{concept}} \land \text{Mon}\) is satisfiable w.r.t. the combination of \(\text{SL}\) (sort concept) with \(L(I(\mathbb{Q}))\) (sort num). For this, we note that \(C_{\text{num}}\) entails the premises of the first, second, and fourth monotonicity rules. Thus, we can add \(c \leq c_1\) and \(d \leq d_1\) to \(C_{\text{concept}}\). Thus, we deduce that \(C \leq e \sqcap \text{car} \leq (c \sqcap d) \sqcap \text{car} \leq c_1 \land (d_1 \sqcap \text{car}) \leq \text{affordable} \sqcap \text{truck}\), which contradicts the last clause in \(C_{\text{concept}}\).

A similar procedure can be used in general for testing (in PTIME) the satisfiability of mixed constraints in the many-sorted combination of \(\text{SL}\) with concrete domains of sort \(\text{num}\), assuming that all concepts of sort \(\text{num}\) are interpreted as intervals and the constraints \(C_{\text{num}}\) are expressible in a PTIME, convex fragment of Allen’s interval algebra.

These results lift in a natural way to \(n\)-ary roles satisfying (guarded) role inclusion axioms.
7 Interpolation in semilattices with operators and applications

Interpolation theorems are important in the study of distributed or evolving ontologies.

A theory \( \mathcal{T} \) has interpolation if, for all formulae \( \phi \) and \( \psi \) in the signature of \( \mathcal{T} \), if \( \phi \models \mathcal{T} \psi \) then there exists a formula \( I \) containing only symbols which occur in both \( \phi \) and \( \psi \) such that \( \phi \models \mathcal{T} I \) and \( I \models \mathcal{T} \psi \). First order logic has interpolation but – for an arbitrary theory \( \mathcal{T} \) – even if \( \phi \) and \( \psi \) are e.g. conjunctions of ground literals, \( I \) may still be an arbitrary formula, containing alternations of quantifiers. It is often important to identify situations in which ground clauses have ground interpolants. In recent literature, when defining ground interpolation, instead of considering formulae \( \phi \) and \( \psi \) such that \( \phi \models \mathcal{T} \psi \), formulae \( A \) and \( B \) are considered such that \( A \land B \models \mathcal{T} \bot \). The two formulations are clearly equivalent. In what follows we will use the second one.

**Definition 14 (Ground interpolation)** We say that a theory \( \mathcal{T} \) has the ground interpolation property (or, shorter, that \( \mathcal{T} \) has ground interpolation) if for all ground clauses \( A(\overline{a}, \overline{d}) \) and \( B(\overline{c}, \overline{e}) \), if \( A(\overline{a}, \overline{d}) \land B(\overline{c}, \overline{e}) \models \mathcal{T} \bot \) then there exists a ground formula \( I(\overline{d}) \), containing only the constants \( \overline{d} \) occurring both in \( A \) and \( B \) (and, ideally, only function symbols shared by \( A \) and \( B \)), such that \( A(\overline{a}, \overline{d}) \models \mathcal{T} I(\overline{d}) \) and \( B(\overline{c}, \overline{e}) \models \mathcal{T} \bot \).

**Definition 15 (Equational interpolation property)** An equational theory \( \mathcal{T} \) (in signature \( \Pi = (\Sigma, \text{Pred}) \) where \( \text{Pred} = \{ \approx \} \)) has the equational interpolation property if whenever

\[
\bigwedge_i A_i(\overline{a}, \overline{c}) \land \bigwedge_j B_j(\overline{c}, \overline{b}) \land \neg B(\overline{c}, \overline{b}) \models \mathcal{T} \bot,
\]

where \( A_i, B_j \) and \( B \) are ground atoms, there exists a conjunction \( I(\overline{c}) \) of ground atoms containing only the constants \( \overline{c} \) occurring both in \( \bigwedge_i A_i(\overline{a}, \overline{c}) \) and \( \bigwedge_j B_j(\overline{c}, \overline{b}) \land \neg B(\overline{c}, \overline{b}) \), such that \( \bigwedge_i A_i(\overline{a}, \overline{c}) \models \mathcal{T} I(\overline{c}) \) and \( I(\overline{c}) \land \bigwedge_j B_j \models \mathcal{T} B \).

There exist results which relate ground interpolation to amalgamation or the injection transfer property \([16][18][30]\) and thus allow us to recognize many theories with ground interpolation. However, just knowing that ground interpolants exist is usually not sufficient: we would like to construct the interpolants fast. In \([22][23]\) a class of theory extensions was identified which have ground interpolation, and for which hierarchical methods for computing the interpolants exist. We present the results below. The theories we consider are theory extensions \( \mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K} \) which satisfy the following assumptions:

\( \mathcal{T}_0 \) is a theory with the following properties:

**Assumption 1:** \( \mathcal{T}_0 \) is convex w.r.t. the set \( \text{Pred} \) (including equality \( \approx \)), i.e., for all conjunctions \( \Gamma \) of ground atoms, relations \( R_1, \ldots, R_m \in \text{Pred} \) and ground tuples of corresponding arity \( \overline{t}_1, \ldots, \overline{t}_m \), if \( \Gamma \models \mathcal{T}_0 \bigwedge_{i=1}^m R_i(\overline{t}_i) \) then there exists \( j \in \{1, \ldots, m\} \) such that \( \Gamma \models \mathcal{T}_0 R_j(\overline{t}_j) \).

**Assumption 2:** \( \mathcal{T}_0 \) is \( P \)-interpolating w.r.t. a subset \( P \subseteq \text{Pred} \) and the separating terms \( t \) can be effectively computed, i.e. for all conjunctions \( A \) and \( B \) of ground literals, all binary predicates \( R \in P \) and all constants \( a \) and \( b \) such that \( a \) occurs in \( A \) and \( b \) occurs in \( B \) (or vice versa), if \( A \land B \models \mathcal{T}_0 aRb \) then there exists a term \( t \) containing only constants common to \( A \) and \( B \) with \( A \land B \models \mathcal{T}_0 aRt \land tRb \). (If we can always find a term \( t \) containing only constants common to \( A \) and \( B \) with \( A \models \mathcal{T}_0 aRt \) and \( B \models \mathcal{T}_0 tRb \) we say that \( \mathcal{T}_0 \) is strongly \( P \)-interpolating.)

**Assumption 3:** \( \mathcal{T}_0 \) has ground interpolation.
The extension \( \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K} \) of \( \mathcal{T}_0 \) has the following properties:

Assumption 4: \( \mathcal{T}_1 \) is a local extension of \( \mathcal{T}_0 \); and

Assumption 5: \( \mathcal{K} \) consists of the following type of combinations of clauses:

\[
\begin{align*}
\{ x_1 R_1 x_1 \land \cdots \land x_n R_n x_n & \rightarrow f(x_1, \ldots, x_n) Rg(y_1, \ldots, y_n) \\
 x_1 R_1 y_1 \land \cdots \land x_n R_n y_n & \rightarrow f(x_1, \ldots, x_n) Rf(y_1, \ldots, y_n)
\}\end{align*}
\]

where \( n \geq 1 \), \( x_1, \ldots, x_n \) are variables, \( R_1, \ldots, R_n \) are binary relations, \( R_1, \ldots, R_n \in P \), \( R \) is transitive, and each \( s_i \) is either a variable among the arguments of \( g \), or a term of the form \( f_i(z_1, \ldots, z_k) \), where \( f_i \in \Sigma_1 \) and all the arguments of \( f_i \) are variables occurring among the arguments of \( g \).

Because of the presence of several function symbols in the axioms in \( \mathcal{K} \) we need to define a more general notion of “shared function symbols”.

**Definition 16 (Shared function symbols)** We define a relation \( \sim \) between extension functions, where \( f \sim g \) if \( f \) and \( g \) occur in the same clause in \( \mathcal{K} \). We henceforth consider that a function \( f \in \Sigma_1 \) is common to \( A \) and \( B \) if there exist \( g, h \in \Sigma_1 \) such that \( f \sim g, f \sim h \), \( g \) occurs in \( A \) and \( h \) occurs in \( B \).

**Theorem 28** Assume that the theories \( \mathcal{T}_0 \) and \( \mathcal{T}_0 \cup \mathcal{K} \) satisfy Assumptions 1–5.

For every conjunction \( A \land B \) of ground unit clauses in the signature \( \Pi^* \) of \( \mathcal{T}_1 \) (possibly containing additional constants) with \( A \land B = \bot \), a ground interpolant \( I \) for \( A \land B \) exists.

In \[22,23\] a procedure for hierarchically computing interpolants is given.

If in addition \( \mathcal{T}_0 \) is strongly \( P \)-interpolating and the interpolants for conjunctions of ground literals are again conjunctions of ground literals, the same is true in the extension.

The theory \( \mathcal{T}_0 \) of bounded semilattices has the following properties (cf. \[22,23\]):

- it is convex w.r.t. \( \leq \); and \( \preceq \);
- it is strongly \( P \)-interpolating w.r.t. \( \leq \) and separating terms can be effectively computed;
- it has ground interpolation (in fact, the equational interpolation property (cf. \[23\])).

Thus, Assumptions 1, 2 and 3 above are fulfilled. The class \( \text{SLO}_2(\Sigma) \) of all semilattices with monotone operators which satisfy a set \( RI \) of axioms satisfies also Assumptions 4 and 5 provided that \( RI \) contains (flat) axioms of the following types:

\[
\begin{align*}
\forall x & \quad f(x) \leq g(x) \\
\forall x, y & \quad x \leq g(y) \rightarrow f(x) \leq h(y) \\
\forall x, y & \quad x \leq g(y) \rightarrow f(x) \leq y
\end{align*}
\]

as well as of the more general type:

\[
\begin{align*}
\forall x_1, \ldots, x_n & \quad f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n) \\
\forall y_1, \ldots, x_n, y_1, \ldots, y_n & \quad \bigwedge_k x_k \leq g_k(y_1, \ldots, y_n) \rightarrow f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n) \\
\forall y_1, \ldots, x_n, y_1, \ldots, y_n & \quad \bigwedge_k x_k \leq g_k(y) \rightarrow f(x_1, \ldots, x_n) \leq y
\end{align*}
\]

**Corollary 7** The class \( \text{SLO}_2(\Sigma) \) has ground interpolation (in fact the equational interpolation property) and interpolants can be computed in a hierarchical manner.
Example 9 (cf. also [23]) Let $\mathcal{T}_1 = \text{SL} \cup \text{SGC}(f, g) \cup \text{Mon}(f, g)$ be the extension of the theory of semilattices with two monotone functions $f, g$ satisfying the semi-Galois condition

$$\text{SGC}(f, g) \quad \forall x, y \quad x \leq g(y) \rightarrow f(x) \leq y.$$  

Consider the following ground formulae $A, B$ in the signature of $\mathcal{T}_1$:

$$A : \quad d \leq g(a) \land a \leq c \quad B : \quad b \leq d \land f(b) \leq c.$$  

where $c$ and $d$ are shared constants. We proved that $\mathcal{T}_1$ is a local extension of the theory of (bounded) semilattices. To prove that $A \land B \models \mathcal{T}_1$ we proceed as follows:

**Step 1:** Use locality. By the locality condition, $A \land B$ is unsatisfiable w.r.t. $\text{SL} \land \text{SGC}(f, g) \land \text{Mon}(f, g)$ iff $\text{SL} \land \text{SGC}(f, g)[A \land B] \land \text{Mon}(f, g)[A \land B] \land A \land B$ has no weak partial model in which all terms in $A$ and $B$ are defined. The extension terms occurring in $A \land B$ are $f(b)$ and $g(a)$, hence:

$$\text{Mon}(f, g)[A \land B] = \{ a \leq a \rightarrow g(a) \leq g(a), \quad b \leq b \rightarrow f(b) \leq f(b) \}$$

$$\text{SGC}(f, g)[A \land B] = \{ b \leq g(a) \rightarrow f(b) \leq a \}$$

**Step 2:** Flattening and purification. We purify and flatten the formula $\text{SGC}(f, g) \land \text{Mon}(f, g)$ by replacing the ground terms starting with $f$ and $g$ with new constants. The clauses are separated into a part containing definitions for terms starting with extension functions, $D_A \land D_B$, and a conjunction of formulae in the base signature, $A_0 \land B_0 \land \text{SGC}_0 \land \text{Mon}_0$.

**Step 3:** Reduction to testing satisfiability in $\mathcal{T}_0$. As the extension $\text{SL} \subseteq \mathcal{T}_1$ is local, we have:

$$A \land B \models \mathcal{T}_1 \quad \text{iff} \quad A_0 \land B_0 \land \text{SGC}_0 \land \text{Mon}_0 \land \text{Con}_0 \text{ is unsatisfiable w.r.t. } \text{SL},$$

where $\text{Con}_0 = \text{Con}[A \land B]_0$ consists of the flattened form of those instances of the congruence axioms containing only $f$- and $g$-terms which occur in $D_A$ or $D_B$, and $\text{SGC}_0 \land \text{Mon}_0$ consists of those instances of axioms in $\text{SGC}(f, g) \land \text{Mon}(f, g)$ containing only $f$- and $g$-terms which occur in $D_A$ or $D_B$.

| Extension $D_A \land D_B$ | Base $A_0 \land B_0 \land \text{SGC}_0 \land \text{Mon}_0 \land \text{Con}_0$ |
|--------------------------|-------------------------------------------------|
| $a_1 \approx g(a)$       | $A_0 = d \leq a_1 \land a_1 \leq c$             |
| $b_1 \approx f(b)$       | $B_0 = b \leq d \land b_1 \not\leq c$          |
|                           | $\text{SGC}_0 = b \leq a_1 \rightarrow b_1 \leq a$ |
|                           | $\text{Con}_A \land \text{Mon}_A = a \leq a \rightarrow a_1 \leq a_1, \quad \forall \in \{\approx, \leq\}$ |
|                           | $\text{Con}_B \land \text{Mon}_B = b \leq b \rightarrow b_1 \leq b_1, \quad \forall \in \{\approx, \leq\}$ |

It is easy to see that $A_0 \land B_0 \land \text{SGC}_0 \land \text{Mon}_0 \land \text{Con}_0$ is unsatisfiable w.r.t. $\mathcal{T}_0$: $A_0 \land B_0$ entails $b \leq a_1$; together with $\text{SGC}_0$ this yields $b_1 \leq a$, which together with $a \leq c$ and $b_1 \not\leq c$ leads to a contradiction.

In order to compute an interpolant we proceed as follows: Consider the conjunction $A_0 \land D_A \land B_0 \land D_B \land \text{Con}[D_A \land D_B]_0 \land \text{Mon}_0 \land \text{SGC}_0$. The $A$ and $B$-part share the constants $c$ and $d$, and no function symbols. However, as $f$ and $g$ occur together in $\text{SGC}, f \sim g$, so they are considered to be all shared. (Thus, the interpolant is allowed to contain both $f$ and $g$.)

We obtain a separation for the clause $b \leq a_1 \rightarrow b_1 \leq a$ of $\text{SGC}_0$ as follows:

(i) We note that $A_0 \land B_0 \models b \leq a_1$. 

(ii) We can find an interpolant \( C \) such that \( A_0 \land B_0 \models b \leq t \land t \leq a_1 \). (Indeed, such a term is \( t = d \).)

(iii) We show that, instead of the axiom \( b \leq g(a) \rightarrow f(b) \leq a \), whose flattened form is in \( SGc \), we can use, without loss of unsatisfiability:

1. an instance of the monotonicity axiom for \( f \): \( b \leq d \rightarrow f(b) \leq f(d) \),
2. another instance of \( SGc \), namely: \( d \leq g(a) \rightarrow f(d) \leq a \).

For this, we introduce a new constant \( c_{f(d)} \) for \( f(d) \) (its definition, \( c_{f(d)} \approx f(d) \), is stored in a set \( D_f \)), and the corresponding instances \( H_{\text{sep}} = H_{\text{sep}}^A \land H_{\text{sep}}^B \) of the congruence, monotonicity and \( SGc \)-axioms, which are now separated into an \( A \)-part \( (H_{\text{sep}}^A : d \leq a_1 \rightarrow c_{f(d)} \leq a) \) and a \( B \)-part \( (H_{\text{sep}}^B : b \leq d \rightarrow b_1 \leq c_{f(d)}) \). We thus obtain a separated conjunction \( C_0 \land B_0 \) (where \( A_0 = H_{\text{sep}}^A \land A_0 \) and \( B_0 = H_{\text{sep}}^B \land B_0 \)), which can be proved to be unsatisfiable in \( \mathcal{F}_1 = \mathcal{E}L \).

(iv) To compute an interpolant in \( SL \) for \( A_0 \land B_0 \) note that \( C_0 \) is logically equivalent to the conjunction of unit literals \( d \leq a_1 \land a \leq c \land c_{f(d)} \leq a \) and \( B_0 \) is logically equivalent to \( b \leq d \land b_1 \leq c \land b_1 \leq c_{f(d)}. \) An interpolant is \( I_0 = c_{f(d)} \leq c. \)

(v) By replacing the new constants with the terms they denote we obtain the interpolant \( I = f(d) \leq c \) for \( A \land B \).

An immediate consequence of Corollary 7 is interpolation in \( \mathcal{E}L^+ \) and their extensions considered in this paper. A variant of the result for the case of \( \mathcal{E}L \) occurs in [31].

Theorem 29. \( \mathcal{E}L^+ \) has the interpolation property, i.e. if \( \mathcal{F} \cup RI \models C \subseteq D \) then there exists a finite set \( \mathcal{T} \) of general concept inclusions containing only concept names and role names common* to \( \mathcal{F} \) and \( C \subseteq D \) such that \( \mathcal{F} \cup RI \models \mathcal{T} \) and \( \mathcal{T} \cup RI \models C \subseteq D \). The same holds also for the generalization of \( \mathcal{E}L^+ \) with \( n \)-ary roles.

Proof: Assume that \( \mathcal{F} \cup RI \models C \subseteq D \). Then \( SLO^+_R(RI) \land A \land B \models \bot \), where \( A = \bigwedge_{C \subseteq C_1} C_1 \leq C_2 \) and \( B = \bigvee_{C \subseteq C_1} C_1 \leq C_2 \). By Corollary 7 there exists a formula \( I \) containing only concept names and role names common to \( A \) and \( B \) such that \( SLO^+_R(RI) \land A \models I \) and \( SLO^+_R(RI) \land I \land B \models \bot \). We actually showed that \( SLO^+_C(RI) \) has the equational interpolation property, so we can find an interpolant \( I \) which is a conjunction of (positive) literals. Then \( \mathcal{T} \) is this interpolant.

8. \( \mathcal{E}L^{++} \) constructors

In the definition of \( \mathcal{E}L^{++} \) the following concept constructors are considered:

\[
\text{ConcDom } p(f_1, \ldots, f_n) = \{ x \mid \exists y_1, \ldots, y_n : f_1(x) = y_1 \text{ and } p(y_1, \ldots, y_n) \}.
\]

Here, we show how to approach this type of problems, as well as the related concept constructions of the following type\(^*\) (where \( D_1, \ldots, D_n \) are concepts terms in the concrete domains):

\[
\text{ConcDom } p(f_1, \ldots, f_n)(D_1, \ldots, D_n) = \{ x \mid \exists y_1 \in D_1, \ldots, y_n \in D_n : f_1(x) = y_1 \text{ and } p(y_1, \ldots, y_n) \}
\]

within the framework of locality. Note that the following transfer of locality results holds:

\(^*\) In the case of roles, by “common” we mean common or “shared” according to Definition 15.

\(^*\) These constructors are allowed if we allow concept construction also on the concrete domains.
Theorem 30 Let $\mathcal{T}_0$ be a theory and let $\mathcal{T}'_0$ be another theory, in the same signature $(\Sigma_0, \text{Pred})$, with the property that every model of $\mathcal{T}'_0$ is a model of $\mathcal{T}_0$. Let $\Sigma_1$ be an additional set of function symbols, not contained in the signature of $\mathcal{T}_0$, and let $\mathcal{K}$ be a set of clauses over the signature $(\Sigma_0 \cup \Sigma_1, \text{Pred})$. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ has the property that every model in $\text{PMod}_\Sigma(\Sigma_1, \mathcal{T}_0 \cup \mathcal{K})$ weakly embeds into a total model of $\mathcal{T}_0 \cup \mathcal{K}$ then every model in $\text{PMod}_\Sigma(\Sigma_1, \mathcal{T}'_0 \cup \mathcal{K})$ weakly embeds into a total model of $\mathcal{T}'_0 \cup \mathcal{K}$.

Theorem 31 Assume that the only concept constructors are intersection, existential restriction, and $\text{ConcDom}$. Let $\mathcal{G} = \text{GCI} \cup \text{CD} \cup \mathcal{RI}$ be a CBox containing a set $\mathcal{GCI}$ of general concept inclusions, a set $\mathcal{CD}$ of definitions of domains $\{c_1, \ldots, c_k\}$ using rules in $\text{ConcDom}$:

$$C_k = p_k(f_1^1, \ldots, f_m^k)$$

and a set $\mathcal{RI}$ of (guarded) role inclusions. Assume that the only concepts names that appear are $N_C = \{c_1, \ldots, c_k\}$. Then for all concept descriptions $D_1, D_2$ the following are equivalent:

1. $D_1 \subseteq \mathcal{G} D_2$.
2. $\bigwedge_{C \subseteq D \in \mathcal{GCI}} (C \leq D) \land D_1 \not\leq D_2$ is unsatisfiable w.r.t. the class $\text{SetBAO}(c_1, \ldots, c_n)(\mathcal{RI})$ of all Boolean algebras of sets with monotone operators satisfying $\mathcal{RI}$ (of the form $\mathcal{P}(D) = (\mathcal{P}(D), \cap, \cup, \lor, \emptyset, D, \{f_r\}_{r \in \mathcal{RI}}, c_1, \ldots, c_k)$).

3. $\bigwedge_{C \subseteq D \in \mathcal{GCI}} (C \leq D) \land D_1 \not\leq D_2$ is unsatisfiable w.r.t. the class $\text{SetSL}(c_1, \ldots, c_n)(\mathcal{RI})$ of all semilattices of sets with monotone operators (i.e. semilattices of the form $\mathcal{P}(D) = (\mathcal{P}(D), \cap, \emptyset, D, \{f_r\}_{r \in \mathcal{RI}}, c_1, \ldots, c_k)$) which satisfy $\mathcal{RI}$.

Proof: (2) $\Rightarrow$ (1) follows from the definition of $D_1 \subseteq \mathcal{G} D_2$, and (3) $\Rightarrow$ (2) is immediate. To prove that (1) $\Rightarrow$ (3), assume that (1) holds and (3) does not. Then there would exist a model $\mathcal{P}(D) = (\mathcal{P}(D), \cap, \cup, \emptyset, D, \{f_r\}_{r \in \mathcal{RI}}, c_1, \ldots, c_k)$ in $\text{SetSL}(c_1, \ldots, c_n)(\mathcal{RI})$ of

$$G = \left( \bigwedge_{C \subseteq D \in \mathcal{GCI}} (C \leq D) \right) \land D_1 \not\leq D_2.$$ 

Then $\mathcal{P}(D) = (\mathcal{P}(D), \cap, \cup, \emptyset, D, \{f_r\}_{r \in \mathcal{RI}}, c_1, \ldots, c_k) \in \text{SetSL}(c_1, \ldots, c_n)(\mathcal{RI})$ is a model of $G$. As the set of maximal filters of $\mathcal{P}(D)$ is in bijective correspondence with $D$, the canonical definition of relations associated with the monotone functions $f_r$ on the Stone dual of $\mathcal{P}(D)$ induces a model $\mathcal{F} = (D, \cdot)$ which satisfies $G, \mathcal{RI}$ and also $\mathcal{CD}$. This contradicts (1). \hfill \Box

We now show that $\text{SetSL}(c_1, \ldots, c_n)(\mathcal{RI})$ is a local extension of $\text{SetSL}(c_1, \ldots, c_n)$. We use the criterion in Theorem 30.

Lemma 6 Let $S = (\text{S}, \lor, \land, 0, 1, \{f_r\}_{r \in \Sigma})$ be a bounded semilattice with partial unary functions $f_r$ satisfying the monotonicity axioms and the RI axioms. Then $S$ weakly embeds into a total semilattice reduct of sets with monotone operators satisfying the axioms $\mathcal{RI}$.

Proof: By the proof of Theorem 30, $S$ weakly embeds into the total semilattice reduct (in $\text{SLO}_2$) of the distributive lattice $L = \mathcal{F}(S) \in \text{DLO}_2^\circ(\mathcal{RI})$. We can now use the proof of the last part in Lemma 5 to show that if $\mathcal{F}_P$ is the set of prime filters of $L$ then the Boolean algebra of sets $B(L) = (\mathcal{P}(\mathcal{F}_P), \cap, \cup, \emptyset, \mathcal{F}_P, \{f_r\}_{r \in \mathcal{RI}})$ (defined in Lemma 5) is a Boolean algebra in $\text{BAO}_2^\circ(\mathcal{RI})$. \hfill \Box

We therefore can hierarchically reduce the problem of checking if $D_1 \subseteq \mathcal{G} D_2$ as follows:
Corollary 8 Assume that the only concept constructors are intersection, existential restriction, and ConcDom. Let $C = GCI \cup CD \cup RI$ be a CBox containing a set $GCI$ of general concept inclusions, a set $CD$ of definitions of domains $\{c_1, \ldots, c_n\}$ using rules in ConcDom, as:

$$c_k = p_k(f_1^k, \ldots, f_n^k)$$

and sets $RI$, $GRI$ of (guarded) role inclusions. Assume that the concepts names that appear are $N_C = \{c_1, \ldots, c_n\}$. Then for all concept descriptions $D_1, D_2$ the following are equivalent:

1. $D_1 \subseteq_D D_2$.
2. $CD \cup G$ — where $G = (\wedge_{CD \cup GCI} \exists \leq D) \wedge \exists \leq D_2$ — is unsatisfiable w.r.t. the class $SetSL(c_1, \ldots, c_n)(RI)$ of all semilattices of sets with monotone operators satisfying $RI_a$, of the form $\mathcal{P}(D) = (\mathcal{P}(D), \cap, \emptyset, D, \{f_r\}_{r \in \mathbb{N}}, c_1, \ldots, c_k)$.
3. $CD \cup G_0 \cap RI[G_{0} \cap Con_0 \cup Def]$ is unsatisfiable w.r.t. the class $SetSL(c_1, \ldots, c_n)(RI)$ of all semilattices of sets with monotone operators satisfying $RI_a$, of the form $\mathcal{P}(D) = (\mathcal{P}(D), \cap, \emptyset, D, \{f_r\}_{r \in \mathbb{N}}, c_1, \ldots, c_k)$.
4. $CD_0 \cup G_0 \cap RI[G_0 \cup Mon_0]$ is unsatisfiable w.r.t. the extensions with free function symbols $\{f_1, \ldots, f_n\}$ of the many-sorted disjoint combination (SetSL, Dom) of the theory $SetSL$ of sets with intersection and the theory Dom of the concrete domains.

Proof: (1) and (2) are equivalent by Theorem 31. It is obvious that (3) implies (2). We show that (2) implies (3). Assume that $CD \cup G_0 \cap RI[G_{0} \cap Con_0 \cup Def]$ has a (partial) model $S = (\mathcal{P}(D), \cap, \emptyset, D, \{f_r\}_{r \in \mathbb{N}}, c_1, \ldots, c_k)$. By Theorem 6, $S$ weakly embeds into a semilattice with operators $S' = (\mathcal{P}(D'), \cap, \emptyset, D, \{f_r\}_{r \in \mathbb{N}})$ which satisfies $RI \cup GRI$ (the interpretation of the constants is translated too). Then $S'$ is also a model of $G_0 \cup CD_0$ and $Def$, hence of $G \cup CD$. Contradiction. The equivalence of (3) and (4) follows as a special case of Theorem 13.

9 Conclusions

In this paper we have shown that subsumption problems in $\mathcal{EL}$ can be expressed as uniform word problems in classes of semilattices with monotone operators, and that subsumption problems in $\mathcal{EL}^+$ can be expressed as uniform word problems in classes of semilattices with monotone operators satisfying certain composition laws. This allowed us to obtain, in a uniform way, PTIME decision procedures for $\mathcal{EL}$, $\mathcal{EL}^+$, and extensions thereof. The use of the notion of local theory extensions allowed us to present a new family of PTIME (many-sorted) logics which extend $\mathcal{EL}$ with $n$-ary roles, (guarded) role inclusions, existential role restrictions and/or with numerical domains. These extensions are different from other types of extensions studied in the description logic literature such as extensions with $n$-ary existential quantifiers (cf. e.g. $\mathcal{ALC}$) or with concrete domains $\mathcal{EL}^c$, but are, in our opinion, very natural and very likely to occur in ontologies. Moreover, we showed that the results in this paper can also be used for the extension $\mathcal{EL}^+$ introduced in [3] (it seems that the results on $\mathcal{EL}^+$ can be extended to tackle also ABoxes). In the future we would like to also analyze generalizations of existential concept restrictions in $\mathcal{EL}$ to existential relation restrictions of the form $\exists r, r_1 \text{interpreted as}$

$$\{x \mid \exists x_1, x_2 : r(x, x_1, x_2) \land r_1(x_1, x_2)\},$$

implications of the form:

$$r_1(x, y) \land r_2(x, y) \rightarrow r_3(x, y)$$
and guarded role inclusions of the form:

\[ r(x_1, x_2) \land r_1(x, x_1, x_2) \rightarrow r_2(x, x_1, x_2). \]

We also showed that the results in [22] can be used to prove that the class of semilattices with monotone operations satisfying the types of axioms considered here allows ground (equational) interpolation. We used this for proving interpolation properties in extensions of \( \mathcal{EL} \) and \( \mathcal{EL}^+ \). We would like to further explore the area of applications of such results for efficient (modular) reasoning in combinations of ontologies based on extensions of \( \mathcal{EL} \) and \( \mathcal{EL}^+ \).

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