Abstract. We review the mechanism of sequential right-handed neutrino dominance proposed in the framework of the type I see-saw mechanism to account for bi-large neutrino mixing and a neutrino mass hierarchy in a natural way. We discuss how sequential dominance (SD) may also be applied to the right-handed charged leptons, which alternatively allows for bi-large lepton mixing from the charged lepton sector. We review how such SD models may be upgraded to include type II see-saw contributions, resulting in a partially degenerate neutrino mass spectrum with the bi-large lepton mixing arising from SD. We also summarize the model-building applications and the phenomenological implications of SD.
1. Introduction

Neutrino masses and mixing angles must now be regarded as unavoidable consequences of the firmly established atmospheric and solar neutrino oscillation experiments [1]. A profound consequence of this is that the minimal Standard Model must necessarily be incomplete, and must be extended in some way to account for neutrino masses and mixings. The simplest way to do this appears to be to add right-handed neutrinos to the Standard Model. Since the right-handed neutrinos are gauge singlets, electroweak symmetry does not prevent them from having large Majorana masses ranging from a few TeV up to the Planck scale. The right-handed neutrinos may also couple with left-handed leptons via the usual Higgs doublets. The combination of very large right-handed neutrino Majorana masses and weak scale Dirac masses from the Higgs couplings leads to suppressed left-handed Majorana neutrino masses, which may be identified with the physical neutrino masses responsible for atmospheric and solar neutrino oscillations. This scenario, proposed some time ago in [2], is known as the see-saw mechanism.

Given the simplicity of the see-saw mechanism, it has been widely applied to understanding the pattern of neutrino masses and mixings implied by the atmospheric and solar neutrino
oscillation data [1]. Although there are alternatives to the see-saw mechanism involving large extra dimensions [3] or R-parity-violating supersymmetry [4], we shall consider only the see-saw mechanism in the present paper, although we shall later consider a more complicated version of the see-saw mechanism, called the type II see-saw mechanism, which also involves heavy Higgs triplets (see e.g. [5]). We shall also only consider the case of three active neutrinos, which is the minimal case consistent with the confirmed atmospheric and solar neutrino oscillation data.

Within the framework as described above, the goal of the see-saw mechanism is to account for large atmospheric neutrino mixing ($\theta_{23} \sim 45^\circ$, close to maximal) and large solar neutrino mixing ($\theta_{12} \sim 30^\circ$, but not maximal) together with the observed atmospheric and solar neutrino mass-squared differences [1]. The solar data are consistent with the so-called large mixing angle (LMA) MSW solution [6]. The third remaining mixing angle associated with the three active neutrinos is so far unmeasured but must be quite small ($\theta_{13} \lesssim 13^\circ$) [7]. The neutrino oscillation data neither determines the absolute scale of neutrino masses nor fixes uniquely the ordering of neutrino masses; however in a normally ordered hierarchical scheme, the neutrino mass values would be roughly given by $m_3 \sim 0.05$ eV, $m_2 \sim 0.008$ eV, with $m_1 \ll m_2$. However $m_1$ could, in principle, be substantially larger, up to the cosmological limit of about 0.23 eV [8].

It has frequently been observed that the simultaneous appearance of hierarchical neutrino masses and two LMAs is not natural in the see-saw mechanism. An important exception to this is the sequential dominance (SD) mechanism [9]–[13] (see also [14]), which is the subject of this focus. SD is not in itself a model, but is a submechanism within the general framework of the see-saw mechanism, which may be applied to constructing different classes of models. The starting point of SD is to assume that one of the right-handed neutrinos contributes dominantly in the see-saw mechanism to the heaviest neutrino mass, with the atmospheric mixing angle being determined by a simple ratio of two Yukawa couplings [9, 10], which is sometimes referred to as single right-handed neutrino dominance (SRHND). SD corresponds to the further assumption that, together with SRHND, a second right-handed neutrino contributes dominantly to the second heaviest neutrino mass, with the large solar mixing angle interpreted as a ratio of Yukawa couplings [11, 12]. The third right-handed neutrino is effectively decoupled from the see-saw mechanism, and plays no part in determining the neutrino mass spectrum, although it may play a cosmological role. If the decoupled right-handed neutrino is also the heaviest then the SD is effectively equivalent to having two right-handed neutrinos [10, 11].

We also review how SD may be generalized to include the right-handed charged leptons [15], which allows bi-large charged lepton mixing consistent with a neutrino mass hierarchy. We then show how such SD models may be upgraded to include type II see-saw contributions [16], resulting in a partially degenerate neutrino mass spectrum with bi-large lepton mixing arising from SD.

In section 2, we recall the type I and II see-saw mechanisms. Section 3 shows how the type I see-saw mechanism can lead to a hierarchical pattern of neutrino masses, with bi-large neutrino mixing in a natural way using sequential right-handed neutrino dominance. Section 4 shows how the type I see-saw mechanism can lead to bi-large charged lepton mixing, with naturally small neutrino mixing and hierarchical neutrino masses, using SD in the right-handed lepton sector. Section 5 shows how a partially degenerate neutrino mass spectrum could originate from the type II see-saw mechanism, with the neutrino mass splittings and mixings controlled by SD. Section 6 briefly reviews some of the model-building applications, whereas section 7 discusses the phenomenological implications of SD. Section 8 concludes the review. Our conventions are stated in an appendix.
2. The see-saw mechanism

The see-saw mechanism provides a convincing explanation for the smallness of neutrino masses. In this section, we review its simplest form, the type I see-saw mechanism and its generalization to the type II see-saw mechanism.

2.1. Type I see-saw

Before discussing the see-saw mechanism it is worth first reviewing the different types of neutrino mass that are possible. So far, we have been assuming that neutrino masses are Majorana masses of the form

$$m_{\nu}^{LL} \nu_L \nu_C^L,$$

(1)

where $\nu_L$ is a left-handed neutrino field and $\nu_C^L$ is the CP conjugate of a left-handed neutrino field; in other words, a right-handed antineutrino field. Such Majorana masses are possible since both the neutrino and the antineutrino are electrically neutral and so Majorana masses are not forbidden by electric charge conservation. For this reason, a Majorana mass for the electron would be strictly forbidden. However, such Majorana neutrino masses violate lepton number conservation, and in the standard model, assuming only Higgs doublets are present, are forbidden at the renormalizable level by gauge invariance. The idea of the simplest version of the see-saw mechanism is to assume that such terms are zero to begin with, but are generated effectively, after right-handed neutrinos are introduced [2].

If we introduce right-handed neutrino fields then there are two sorts of additional neutrino mass terms that are possible. There are additional Majorana masses of the form

$$M_{RR} \nu_R \nu_C^R,$$

(2)

where $\nu_R$ is a right-handed neutrino field and $\nu_C^R$ is the CP conjugate of a right-handed neutrino field; in other words a left-handed antineutrino field. In addition, there are Dirac masses of the form

$$m_{\nu}^{LR} \nu_L \nu_R.$$

(3)

Such Dirac mass terms conserve lepton number, and are not forbidden by electric charge conservation even for the charged leptons and quarks.

Once this is done then the types of neutrino mass discussed in equations (2) and (3) (but not equation (1) since we do not assume direct mass terms, e.g. from Higgs triplets, at this stage) are permitted, and we have the mass matrix

$$\begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix} \begin{pmatrix} 0 & m_{\nu}^{\nu_T} \\ m_{\nu}^{LR} & M_{RR} \end{pmatrix} \begin{pmatrix} \nu_C^L \\ \nu_R \end{pmatrix}.$$  

(4)

Since the right-handed neutrinos are electroweak singlets, the Majorana masses of the right-handed neutrinos $M_{RR}$ may be orders of magnitude larger than the electroweak scale. In the approximation that $M_{RR} \gg m_{\nu}^{LR}$, the matrix in equation (4) may be diagonalized to yield effective Majorana masses of the type in equation (1),

$$m_{\nu}^{LL} = -m_{\nu}^{LR} M_{RR}^{-1} m_{\nu}^{\nu_T}. $$

(5)
The effective left-handed Majorana masses $m_{\nu LL}^\nu$ are naturally suppressed by the heavy scale $M_{RR}$. In a one family example, if we take $m_{\nu LR} = M_W$ and $M_{RR} = M_{\text{GUT}}$, then we find $m_{\nu LL}^\nu \sim 10^{-3} \text{eV}$ which looks good for solar neutrinos. Atmospheric neutrino masses would require a right-handed neutrino with a mass below the grand unified theory (GUT) scale.

With three left-handed neutrinos and three right-handed neutrinos the Dirac masses $m_{\nu LR}^\nu$ are a $3 \times 3$ (complex) matrix and the Majorana masses $M_{RR}^\nu$ form a separate $3 \times 3$ (complex symmetric) matrix. The light effective Majorana masses $m_{\nu LL}^\nu$ are also a $3 \times 3$ (complex symmetric) matrix and continue to be given from equation (5) which is now interpreted as a matrix product. From a model-building perspective the fundamental parameters which must be input into the see-saw mechanism are the Dirac mass matrix $m_{\nu LR}^\nu$ and the heavy right-handed neutrino Majorana mass matrix $M_{RR}^\nu$. The light-effective left-handed Majorana mass matrix $m_{\nu LL}^\nu$ arises as an output according to the see-saw formula in equation (5).

The version of the see-saw mechanism discussed so far is sometimes called the type I see-saw mechanism. It is the simplest version of the see-saw mechanism, and can be thought of as resulting from integrating the heavy right-handed neutrinos to produce the effective dimension 5 neutrino mass operator

$$\frac{1}{4} (H_u \cdot L^T) \kappa (H_u \cdot L),$$

where the dot indicates the SU(2)$_L$-invariant product and

$$\kappa = 2Y_vM_{RR}^{-1}Y_v^T$$

with $Y_v$ being the neutrino Yukawa couplings and $m_{\nu LR} = Y_v v_u$ with $v_u = \langle H_u \rangle$. The type I see-saw mechanism is illustrated diagramatically in figure 1.

2.2. Type II see-saw

In models with a left–right symmetric particle content such as minimal left–right symmetric models, Pati-Salam models or GUTs based on SO(10), the type I see-saw mechanism is often generalized to a type II see-saw (see e.g. [5]), where an additional direct mass term $m_{\nu LL}^{\nu II}$ for the light neutrinos is present.

With such an additional direct mass term, the general neutrino mass matrix is given by

$$\begin{pmatrix} \nu_L^C & \nu_R \\ \nu_L & \nu_R \end{pmatrix} \begin{pmatrix} m_{\nu LL}^{\nu II} & m_{\nu LR}^\nu \\ m_{\nu LR}^\nu & M_{RR} \end{pmatrix} \begin{pmatrix} \nu_L^C \\ \nu_R \end{pmatrix}. \tag{8}$$
Figure 2. Diagram leading to a type II contribution $m_{II}^{LL}$ to the neutrino mass matrix via an induced vev of the neutral component of a triplet Higgs $\Delta$.

Under the assumption that the mass eigenvalues $M_{Ri}$ of $M_{RR}$ are very large compared with the components of $m_{II}^{LL}$ and $m_{LR}$, the mass matrix can approximately be diagonalized yielding effective Majorana masses

$$m_{LL}^\nu \approx m_{II}^{LL} + m_{LL}^I$$

with

$$m_{LL}^I \approx -m_{LR}^\nu M_{RR}^{-1} m_{LR}^{\nu T}$$

for the light neutrinos. The direct mass term $m_{II}^{LL}$ can also provide a naturally small contribution to the light neutrino masses if it stems, e.g. from a see-saw suppressed induced vev. We will refer to the general case, where both possibilities are allowed, as the II see-saw mechanism. Realizing the type II contribution by generating the dimension 5 operator in equation (6) via the exchange of heavy Higgs triplets of SU(2)$_L$ is illustrated diagrammatically in figure 2.

3. Sequential right-handed neutrino dominance in the type I see-saw mechanism

In this section we discuss an elegant and natural way of accounting for a neutrino mass hierarchy and two LMAs, called SD. The idea of SD is that one of the right-handed neutrinos contributes dominantly to the see-saw mechanism and determines the atmospheric neutrino mass and mixing. A second right-handed neutrino contributes subdominantly and determines the solar neutrino mass and mixing. The third right-handed neutrino is effectively decoupled from the see-saw mechanism.

3.1. Single right-handed neutrino dominance

Consider the case of full neutrino mass hierarchy $m_3 \gg m_2 \gg m_1 \approx 0$. From the appendix we see that in the diagonal charged lepton basis, ignoring phases, the neutrino mass matrix is given by

$$m_{LL}^\nu \approx \begin{pmatrix}
\frac{1}{\sqrt{2}}(m_2 s_{12} c_{12} + m_3 \theta_{13}) & -\frac{1}{\sqrt{2}}(m_2 s_{12} c_{12} - m_3 \theta_{13}) \\
\frac{1}{2} (m_3 + m_2 c_{12}^2) & \frac{1}{2} (m_3 - m_2 c_{12}^2) \\
\frac{1}{2} (m_3 + m_2 c_{12}^2) & \frac{1}{2} (m_3 - m_2 c_{12}^2)
\end{pmatrix},$$

(11)
neglecting terms like $m^2_\theta$ and setting $\theta_{23} \approx \pi/4$. Clearly, this expression reduces to

$$m^\nu_{LL} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \frac{m^2}{2},$$

(12)

with $m = m^3$ in the approximation that $m^2$ and $\theta_{13}$ are neglected. However, the more exact expression in equation (11) shows that the required form of $m^\nu_{LL}$ should have a very definite detailed structure. The requirement $m^2 \ll m^3$ implies that the sub-determinant of the mass matrix $m^\nu_{LL}$ is small:

$$\det \begin{pmatrix} m^{22} & m^{23} \\ m^{23} & m^{33} \end{pmatrix} \ll m^2_3.$$  

(13)

This requirement in equation (13) is satisfied by equation (11), as may be readily seen, and this condition must be reproduced in a natural way (without fine-tuning) by any successful theory.

The goal of see-saw model building for hierarchical neutrino masses is therefore to choose input see-saw matrices $m^\nu_{LR}$ and $M_{RR}$ that will give rise to the form in equation (11). We now show how the input see-saw matrices can be simply chosen to give this form, with the property of a naturally small sub-determinant in equation (13) using a mechanism first suggested in [9] (see also [14]). The idea was developed in [10] where it was called SRHND. SRHND was first successfully applied to the LMA MSW solution in [11].

To understand the basic idea of dominance, it is instructive to begin by discussing a simple $2 \times 2$ example, where we have in mind applying this to the atmospheric mixing in the 23 sector:

$$M_{RR} = \begin{pmatrix} Y & 0 \\ 0 & X \end{pmatrix}, \quad m^\nu_{LR} = \begin{pmatrix} e & b \\ f & c \end{pmatrix}.$$  

(14)

The see-saw formula in equation (5), $m^\nu_{LL} = -m^\nu_{LR} M^{-1}_{RR} m^{\nu T}_{LR}$, gives

$$-m^\nu_{LL} = \begin{pmatrix} e^2 + f^2 \sum \frac{e^2 f^2}{X} + \frac{b c}{X} \\ \frac{e^2 f^2}{Y} + \frac{b c}{X} \end{pmatrix} \approx \begin{pmatrix} e^2 \frac{Y}{Y} \\ f^2 \frac{Y}{Y} \end{pmatrix},$$  

(15)

where the approximation in equation (15) assumes that the right-handed neutrino of mass $Y$ is sufficiently light that it dominates in the see-saw mechanism:

$$\frac{e^2, f^2, e f}{Y} \gg \frac{b^2, c^2, b c}{X}.$$  

(16)

The neutrino mass spectrum from equation (15) then consists of one neutrino with mass $m^2 \approx (e^2 + f^2)/Y$ and one naturally light neutrino $m^2 \ll m^3$, since the determinant of equation (15) is clearly approximately vanishing, due to the dominance assumption [9]. The atmospheric angle from equation (15) is $\tan \theta_{23} \approx e f/\sum$, which can be large or maximal providing $e \approx f$, even in the case $e, f, b \ll c$ that the neutrino Dirac mixing angles arising from equation (14) are small. Thus, two crucial features, namely a neutrino mass hierarchy $m^2_2 \gg m^2_3$ and a large neutrino mixing angle $\tan \theta_{23} \approx 1$, can arise naturally from the see-saw mechanism assuming the dominance of a single right-handed neutrino. It was also realized that small perturbations from the sub-dominant right-handed neutrinos can then lead to a small solar neutrino mass splitting [9], as we now discuss.
3.2. Sequential right-handed neutrino dominance

To account for the solar and other mixing angles, we must generalize the above discussion to the $3 \times 3$ case. The SRHND mechanism is most simply described assuming three right-handed neutrinos in the basis where the right-handed neutrino mass matrix is diagonal although it can also be developed in other bases [10, 11]. In this basis we write the input see-saw matrices as

$$M_{RR} = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix}, \quad (17)$$

$$m_{LR}^{1} = \begin{pmatrix} a & d & p \\ b & e & q \\ c & f & r \end{pmatrix}. \quad (18)$$

Each right-handed neutrino in the basis of equation (17) couples with a particular column of $m_{LR}^{1}$ in equation (18). There is no mass ordering of $X, Y, Z$ implied in equation (17). In [9] it was suggested that one of the right-handed neutrinos may dominate the contribution to $m_{LL}^{1}$ if it is lighter than the other right-handed neutrinos. The dominance condition was subsequently generalized to include other cases where the right-handed neutrino may be heavier than the other right-handed neutrinos but dominates due to its larger Dirac mass couplings [10]. In any case the dominant right-handed neutrino may be taken to be the one with mass $Y$ without loss of generality.

It was subsequently shown how to account for the LMA MSW solution with a large solar angle [11] by careful consideration of the sub-dominant contributions. SD occurs when the right-handed neutrinos dominate sequentially [11],

$$\frac{|e^2|, |f^2|, |ef|}{Y} \gg \frac{|xy|}{X} \gg \frac{|x'y'|}{Z}, \quad (19)$$

which is the straightforward generalization of equation (16) where $x, y \in a, b, c$ and $x', y' \in p, q, r$. Assuming SRHND with sequential sub-dominance as in equation (19), then equations (5), (17) and (18) give

$$-m_{LL}^{1} \approx \begin{pmatrix} \frac{a^2}{X} + \frac{d^2}{Y} & \frac{ae}{X} + \frac{df}{Y} & \frac{ad}{X} \\ \frac{be}{X} + \frac{ef}{Y} & \frac{b^2}{X} + \frac{c^2}{Y} & \frac{bc}{X} \\ \frac{ce}{X} + \frac{f^2}{Y} & \frac{cf}{X} & \frac{c^2}{X} + \frac{f^2}{Y} \end{pmatrix}, \quad (20)$$

where the contribution from the right-handed neutrino of mass $Z$ may be neglected according to equation (19). If the couplings satisfy the SD condition in equation (19) then the matrix in equation (20) resembles the Type IA matrix and, furthermore, has a naturally small sub-determinant as in equation (13). This leads to a full neutrino mass hierarchy

$$m_{3}^{2} \gg m_{2}^{2} \gg m_{1}^{2}, \quad (21)$$

and, ignoring phases, the solar angle only depends on the sub-dominant couplings and is given by $\tan \theta_{12} \approx a/(c_{23}b - s_{23}c)$ [11]. The simple requirement for large solar angle is then $a \sim b - c$ [11].

New Journal of Physics 6 (2004) 110 (http://www.njp.org/)
Including phases the neutrino masses are given to leading order in \(m_2/m_3\) by diagonalizing the mass matrix in equation (20) using the analytic procedure described in appendix D of [12].

In the case that \(d = 0\), corresponding to a 11 texture zero in equation (18), we have [12, 13]

\[
m_1 \sim O\left(\frac{x'y'}{Z}\right), \tag{22}
\]

\[
m_2 \approx \frac{|a|^2}{Xs_{12}^2}, \tag{23}
\]

\[
m_3 \approx \frac{|e|^2 + |f|^2}{Y}, \tag{24}
\]

where \(s_{12} = \sin \theta_{12}\) is given below. Note that with SD each neutrino mass is generated by a separate right-handed neutrino, and the SD condition naturally results in a neutrino mass hierarchy \(m_1 \ll m_2 \ll m_3\). The neutrino mixing angles are given to leading order in \(m_2/m_3\) by [12, 13]

\[
\tan \theta_{23} \approx \frac{|e|}{|f|}, \tag{25}
\]

\[
\tan \theta_{12} \approx \frac{|a|}{c_{23}|b| \cos(\phi_b) - s_{23}|c| \cos(\phi_c)}, \tag{26}
\]

\[
\theta_{13} \approx e^{i(\phi_{b} + \phi_{c} - \phi_{a})} \left|a\right| \left(e^{*}b + f^{*}c\right) Y \frac{\left[|e|^2 + |f|^2\right]^{1/2}}{X}, \tag{27}
\]

where we have written some (but not all) complex Yukawa couplings as \(x = |x|e^{i\phi}\). The phase \(\delta\) is fixed to give a real angle \(\theta_{12}\) by

\[
c_{23}|b| \sin(\phi_b) \approx s_{23}|c| \sin(\phi_c), \tag{28}
\]

where

\[
\tilde{\phi}_b \equiv \phi_b - \phi_a - \tilde{\phi} + \delta, \quad \tilde{\phi}_c \equiv \phi_c - \phi_a + \phi_f - \tilde{\phi} + \delta. \tag{29}
\]

The phase \(\tilde{\phi}\) is fixed to give a real angle \(\theta_{13}\) by

\[
\tilde{\phi} \approx \phi_c - \phi_a - \arg(e^{*}b + f^{*}c). \tag{30}
\]

Physically, these results show that in SD the atmospheric neutrino mass \(m_3\) and mixing \(\theta_{23}\) is determined by the couplings of the dominant right-handed neutrino of mass \(Y\). The solar neutrino mass \(m_2\) and mixing \(\theta_{12}\) is determined by the couplings of the sub-dominant right-handed neutrino of mass \(X\). The third right-handed neutrino of mass \(Z\) is effectively decoupled from the see-saw mechanism and leads to the vanishingly small mass \(m_1 \approx 0\).

### 3.3. Types of sequential right-handed neutrino dominance

Assuming SD, there is still an ambiguity regarding the mass ordering of the heavy Majorana right-handed neutrinos. So far we have assumed that the dominant right-handed neutrino of
Table 1. Types of SD, classified according to the mass ordering of the right-handed neutrinos. LSD corresponds to the dominant right-handed neutrino of mass $Y$ being the lightest, ISD to the dominant right-handed neutrino of mass $Y$ being the intermediate one, and HSD to the dominant right-handed neutrino of mass $Y$ being the heaviest. The fourth column shows the leading-order form for $Y_v$ under the assumption of a large 33-element in the Yukawa matrix.

| Type of SD | $M_{RR}$ | $m_{LR}^{1R} = Y_v v_u$ | Leading $Y_v$ |
|-----------|----------|------------------------|--------------|
| LSDa $Y < X < Z$ | $\begin{pmatrix} Y & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Z \end{pmatrix}$ | $\begin{pmatrix} d & a & p \\ e & b & q \\ f & c & r \end{pmatrix}$ | $0 \ 0 \ 0$ |
| LSDb $Y < Z < X$ | $\begin{pmatrix} Y & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & X \end{pmatrix}$ | $\begin{pmatrix} d & p & a \\ e & q & b \\ f & r & c \end{pmatrix}$ | $0 \ 0 \ 1$ |
| ISDa $X < Y < Z$ | $\begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix}$ | $\begin{pmatrix} a & d & p \\ b & e & q \\ c & f & r \end{pmatrix}$ | $0 \ 0 \ 0$ |
| ISDb $Z < Y < X$ | $\begin{pmatrix} Z & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X \end{pmatrix}$ | $\begin{pmatrix} p & d & a \\ q & e & b \\ r & f & c \end{pmatrix}$ | $0 \ 0 \ 1$ |
| HSDa $Z < X < Y$ | $\begin{pmatrix} Z & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Y \end{pmatrix}$ | $\begin{pmatrix} p & a & d \\ q & b & e \\ r & c & f \end{pmatrix}$ | $0 \ 0 \ 0$ |
| HSDb $X < Z < Y$ | $\begin{pmatrix} X & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & Y \end{pmatrix}$ | $\begin{pmatrix} a & p & d \\ b & q & e \\ c & r & f \end{pmatrix}$ | $0 \ 0 \ 1$ |

mass $Y$ is dominant because it is the lightest one. We emphasize that this need not be the case. The neutrino of mass $Y$ could be dominant even if it is the heaviest right-handed neutrino, providing its Yukawa couplings are strong enough to overcome its heaviness and satisfy the condition in equation (19). In hierarchical mass matrix models, it is natural to order the right-handed neutrinos so that the heaviest right-handed neutrino is the third one, the intermediate right-handed neutrino is the second one and the lightest right-handed neutrino is the first one. It is also natural to assume that the 33 Yukawa coupling is of order unity, due to the large top quark mass. It is therefore possible that the dominant right-handed neutrino is the heaviest (called heavy sequential dominance or HSD), the lightest (called light sequential dominance or LSD) or the intermediate one (called intermediate sequential dominance or ISD). This leads to the six possible types of SD corresponding to the six possible mass orderings of the right-handed neutrinos as shown in table 1. In each case the dominant right-handed neutrino is the one with mass $Y$, and the leading sub-dominant right-handed neutrino is the one with mass $X$. The resulting see-saw matrix $m_{LL}^{\nu}$ is invariant under re-orderings of the right-handed neutrino columns, but the leading order form of the neutrino Yukawa matrix $Y_v$ is not.

It is worth emphasizing that, since all the forms above give the same light effective see-saw neutrino matrix $m_{LL}^{\nu}$ in equation (20), under the SD assumption in equation (19), this implies
that the analytic results for neutrino masses and mixing angles applies to all of these forms. They are distinguished theoretically by different preferred leading-order forms of the neutrino Yukawa matrix $Y_\nu$ shown in the table. These leading-order forms follow from the the LMA requirements $e \sim f$ and $a \sim b - c$.\footnote{Note that the leading-order $Y_\nu$ in table 1 only gives the independent order unity entries in the matrix, so that, for example, in LSDb we would expect $b - c \sim 1$ in general, and not zero.} Thus we see that LSDa and ISDa are consistent with a form of Yukawa matrix with small Dirac mixing angles, while HSDa and HSDb correspond to the so-called ‘lop-sided’ forms.

4. Sequential right-handed lepton dominance in the type I see-saw mechanism

In this section we show how bi-large mixing could originate from the charged lepton sector using a generalization of sequential right-handed neutrino dominance \cite{11, 12} to all right-handed leptons \cite{15}. We write the mass matrices for the charged leptons $m_E$ as

$$m_E = \begin{pmatrix} p' & d' & a' \\ q' & e' & b' \\ r' & f' & c' \end{pmatrix}.$$  \hspace{1cm} (31)

In our notation, each right-handed charged lepton couples with a column in $m_E$. For the charged leptons, the SD conditions are \cite{15}

$$|a'|, |b'|, |c'| \gg |d'|, |e'|, |f'| \gg |p'|, |q'|, |r'|.$$ \hspace{1cm} (32)

They imply the desired hierarchy for the charged lepton masses $m_\tau \gg m_\mu \gg m_e$ and small right-handed mixing of $U_{eR}$. We assume zero mixing from the neutrino sector which corresponds to the MNS matrix being given by $U_{MNS} = U_e \text{ diag}(1, e^{i\beta_2}, e^{i\beta_3})$ in the conventions in the appendix. A natural possibility for obtaining a small $\theta_{13}$ is \cite{15}

$$|d'|, |e'| \ll |f'|.$$ \hspace{1cm} (33)

In leading order in $|d'|/|f'|$ and $|e'|/|f'|$, for the mixing angles $\theta_{12}, \theta_{23}$ and $\theta_{13}$, we obtain

$$\tan(\theta_{12}) \approx \frac{|a'|}{|b'|},$$ \hspace{1cm} (34a)

$$\tan(\theta_{23}) \approx \frac{s_{12} |d'| + c_{12} |b'|}{|c'|},$$ \hspace{1cm} (34b)

$$\tan(\theta_{13}) \approx \frac{s_{12} |e'| e^{i(\phi_e - \phi_\nu + \phi_e + \delta)} - c_{12} |d'| e^{i(\phi_e + \delta)} |f'| e^{i(\phi_e - \phi_\nu + \phi_f + \delta)}}{|f'| e^{i(\phi_e - \phi_\nu + \phi_f + \delta)}},$$ \hspace{1cm} (34c)

where the Dirac CP phase $\delta$ is determined such that $\theta_{13}$ is real, which requires

$$\tan(\delta) \approx \frac{c_{12} |d'| \sin(\phi_e - \phi_\nu + \phi_f) - s_{12} |e'| \sin(\phi_\nu - \phi_e - \phi_\nu + \phi_f)}{c_{12} |d'| \cos(\phi_e - \phi_\nu + \phi_f) - s_{12} |e'| \cos(\phi_\nu - \phi_e - \phi_\nu + \phi_f)}.$$ \hspace{1cm} (35)
Given \( \tan(\delta) \), \( \delta \) has to be chosen such that \( \tan(\theta_{13}) \geq 0 \) in order to match with the usual convention \( \theta_{13} \geq 0 \). The phases \( \beta_2^e \) and \( \beta_3^e \) from the charge lepton sector are given by

\[
\beta_2^e \approx \phi_{u'} - \phi_{u} + \pi, \\
\beta_3^e \approx \phi_{u'} - \phi_{e}.
\]

(36a)

(36b)

Note that in the case that the neutrino sector induces Majorana phases, the total Majorana phases \( \beta_2 \) and \( \beta_3 \) of the MNS matrix are given by

\[
\beta_2 \approx \beta_2^e + \beta_2^\nu, \\
\beta_3 \approx \beta_3^e + \beta_3^\nu.
\]

(37a)

(37b)

\( \theta_{13} \) only depends on \( d'/f' \) and \( e'/f' \) from the Yukawa couplings to the sub-dominant right-handed muon and on \( \theta_{12} \). We find that in the limit \( |d'|, |e'| \ll |f'| \), the two LMAs \( \theta_{12} \) and \( \theta_{23} \) approximately depend only on \( a'/c' \) and \( b'/c' \) from the right-handed tau Yukawa couplings. Both mixing angles are large if \( a', b' \) and \( c' \) are of the same order.

In addition to achieving bi-large mixing from the charged lepton sector, we also require now small mixing from the neutrino sector. Usually, sequential RHND [11, 12] is viewed as a framework for generating large solar mixing \( \theta_{12} \) and large atmospheric mixing \( \theta_{23} \) in the neutrino mass matrix. However, given SD in the neutrino sector in equation (19) which guarantees a neutrino mass hierarchy, one can easily find the conditions for small mixing from the neutrinos as well from equations (25) and (26). Using the notation of section 3.2, we need \( d, e \ll f \) and \( a \ll b, c \). Small mixing from the neutrino sector thus requires three small entries in \( m_{\nu}^{LR} \).

In type II see-saw models, it seems to be difficult to obtain a partially degenerate or quasi-degenerate neutrino mass spectrum in a natural way, whereas hierarchical masses seem to be natural. The direct mass term in type II models, on the other hand, has the potential to provide a natural way for generating neutrino masses with a partial degeneracy. In this section, we show that it is possible to obtain a partially degenerate neutrino mass spectrum by essentially adding a type II direct neutrino mass contribution proportional to the unit matrix. In this case, the neutrino mass scale is controlled by the type II direct mass term, while the neutrino mass splittings (which are generally now much smaller) and mixings continue to be determined by the type I see-saw matrix using SD as described earlier.

Thus we shall consider a type II upgrade [16], where the mass matrix of the light neutrinos in equation (9) has the particular form

\[
m_{\nu}^{LL} \approx m^{II} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} (m_{\nu}^{LL})_{11} & (m_{\nu}^{LL})_{12} & (m_{\nu}^{LL})_{13} \\ (m_{\nu}^{LL})_{21} & (m_{\nu}^{LL})_{22} & (m_{\nu}^{LL})_{23} \\ (m_{\nu}^{LL})_{31} & (m_{\nu}^{LL})_{32} & (m_{\nu}^{LL})_{33} \end{pmatrix},
\]

(38)

in the basis where the mass matrix \( M_{RR} \) of the heavy right-handed neutrinos is diagonal.
To understand the effect of the type II contribution $m_{1LL}^{\text{II}}$, we consider the diagonalization of $m_{1LL}^{\text{I}}$ by a unitary transformation $(m_{1LL}^{\text{I}})_{\text{diag}} = V m_{1LL}^{\text{I}} V^T$. If we assume for the moment that the type I see-saw mass matrix $m_{1LL}^{\text{I}}$ is real, which implies that $V$ is an orthogonal matrix, we obtain

$$(m_{1LL}^{\nu})_{\text{diag}} = m_{1LL}^{\text{II}} V V^T + (m_{1LL}^{\text{I}})_{\text{diag}} = m_{1LL}^{\text{II}} + (m_{1LL}^{\text{I}})_{\text{diag}}.$$  

(39)

The additional direct mass term leaves the predictions for the mixings from the type I see-saw contribution unchanged in this case. This allows us to transform many type I see-saw models for hierarchical neutrino masses into type II see-saw models for partially degenerate or quasi-degenerate neutrino masses while maintaining the predictions for the mixing angles. Obviously, in the general complex case, it is no longer that simple since for a unitary matrix $VV^T \neq 1$ and the phases will have impact on the predictions for the mixings. However, as we will see below, with sequential right-handed neutrino dominance [11, 12] for the type I contribution to the neutrino mass matrix, and a particular phase structure, the known techniques and mechanisms for explaining the bi-large lepton mixings can be directly applied also in the presence of CP phases.

5.1. Type II upgrade of a ISD model

As an example of a type II model where the bi-large lepton mixing stems from the neutrino mass matrix, we now consider explicitly the model A1 of table 4 in [16] with sequential right-handed neutrino dominance [11, 12] for the type I part $m_{1\nu}^\nu$ of the neutrino mass matrix. The leading-order Dirac mass matrices are

$$m_{1\nu}^{\nu} = \begin{pmatrix} a e^{i \delta_1} & 0 & 0 \\ b e^{i \delta_1} & e e^{i \delta_2} & 0 \\ c e^{i \delta_1} & f e^{i \delta_2} & r e^{i \delta_3} \end{pmatrix}, \quad m_{E} = \begin{pmatrix} a' e^{i \delta'_1} & 0 & 0 \\ b' e^{i \delta'_1} & e' e^{i \delta'_2} & 0 \\ c' e^{i \delta'_1} & f' e^{i \delta'_2} & r' e^{i \delta'_3} \end{pmatrix},$$  

(40)

where here $a, b, c, e, f, r$ and $a', b', c', e', f', r'$ are real. $M_{RR}$ and the type II contribution $m_{1LL}^{\text{II}}$ are given by

$$M_{RR} = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix}, \quad m_{1LL}^{\text{II}} = m_{\nu}^{\text{II}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  

(41)

denoting the mass of the dominant right-handed neutrino by $Y$ and the mass of the sub-dominant one by $X$. The sequential RHND condition we impose is then

$$\left| \frac{e^2}{Y} \right| \gg \left| \frac{a^2, b^2, c^2}{X} \right| \gg \left| \frac{r^2}{Z} \right|.$$  

(42)

The leading-order type II neutrino mass matrix is given by the type II see-saw formula of equation (9).

The masses of the charged leptons are given by $m_\tau = r'$, $m_\mu = e'$ and $m_e = a'$. In addition we note that the mixings $\theta_{12}^e$, $\theta_{13}^e$ and $\theta_{23}^e$, which stem from $U_{eR}$ and could contribute to the MNS matrix, are very small. Furthermore, in leading order each column of $M_\nu$ has a common complex phase, which can be absorbed by $U_{eR}$. Therefore, the charged leptons do not influence the leptonic CP phases in this approximation.
Using the analytical methods for diagonalizing neutrino mass matrices with small $\theta_{13}$ derived in [12], from $m_{\nu}^{\text{LL}} = m_{\nu}^{\text{II LL}} + m_{\nu}^{\text{I LL}}$, we find for the mixing angles

\[
\tan(\theta_{23}) \approx \frac{|e|}{|f|}, \tag{43a}
\]

\[
\tan(2\theta_{13}) \approx \frac{2|a|}{X} \frac{\sin(\theta_{23})|b| + \cos(\theta_{23})|c| \text{sign}(bcef)}{|2m^{\text{II}} \sin(\delta) + m_{3}^{1} e^{i(2\delta_{1} + 3\pi/2 - \delta)}|}, \tag{43b}
\]

\[
\tan(\theta_{12}) \approx \frac{|a|}{|\cos(\theta_{23})| |b| - \sin(\theta_{23})|c| \text{sign}(bcef)|} \tag{43c}
\]

where $m_{i}^{1}$ ($i \in \{1, 2, 3\}$) are the mass eigenvalues of the hierarchical $m_{\nu}^{\text{LL}}$ given by

\[
m_{1}^{1} = \mathcal{O} \left( \frac{r^{2}}{Z} \right) \approx 0, \tag{44a}
\]

\[
m_{2}^{1} \approx \frac{(|a|^{2} + |\cos(\theta_{23})| |b| - \sin(\theta_{23})|c| \text{sign}(bcef)|^{2})}{X} \approx \frac{|a|^{2}}{\sin^{2}(\theta_{12}) X}, \tag{44b}
\]

\[
m_{3}^{1} \approx \frac{(|e| \sin(\theta_{23}) + |f| |\cos(\theta_{23})|^{2})}{Y}, \tag{44c}
\]

and with $\tilde{\delta}$ defined by

\[
\tan(\tilde{\delta}) := \frac{m_{1}^{1} \sin(2\delta_{2} - 2\delta_{1})}{m_{3}^{1} \cos(2\delta_{2} - 2\delta_{1}) - 2m^{\text{II}} \cos(2\delta_{1})}. \tag{45}
\]

Given $\tan(\tilde{\delta})$, $\tilde{\delta}$ has to be chosen such that

\[
\frac{\sin(\theta_{23})|b| + \cos(\theta_{23})|c| \text{sign}(bcef)}{\text{sign}(ab)[2m^{\text{II}} e^{-i(2\delta_{1} + 3\pi/2)} \sin(\tilde{\delta}) + m_{3}^{1} e^{-i(2\delta_{1} - 2\delta_{2} + \delta)}]} \geq 0. \tag{46}
\]

This does not affect $\theta_{13}$, which we have defined to be $\geq 0$; however it is relevant for extracting the Dirac CP phase $\delta$, given by

\[
\delta \approx \begin{cases} 
\tilde{\delta} & \text{for } P \geq 0, \\
\tilde{\delta} + \pi & \text{for } P < 0,
\end{cases} \tag{47}
\]

with $P$ being defined by

\[
P := \frac{\cos(\theta_{23})|b| - \sin(\theta_{23})|c| \text{sign}(bcef)}{\text{sign}(ab)[(\cos(\theta_{23})|b| - \sin(\theta_{23})|c| \text{sign}(bcef))^{2} - |a|^{2}]} \tag{48}
\]
The mass eigenvalues of the complete type II neutrino mass matrix are given by

\[ m_1 \approx |m^{\text{II}}| , \]  

\[ m_2 \approx |m^{\text{II}} - m^1 e^{i \delta_1}| , \]  

\[ m_3 \approx |m^{\text{II}} - m^3 e^{i \delta_2}| , \]

and, for \( m^{\text{II}} \neq 0 \), the Majorana phases \( \beta_2 \) and \( \beta_3 \) can be extracted by

\[ \beta_2 \approx \frac{1}{2} \arg (m^{\text{II}} - m^1 e^{i \delta_1}) , \]  

\[ \beta_3 \approx \frac{1}{2} \arg (m^{\text{II}} - m^3 e^{i \delta_2}) . \]

In the classes of type II see-saw models with sequential right-handed neutrino dominance for the type I contribution to the neutrino mass matrix and real vacuum alignment leading to the phase structure of the Yukawa matrices as in equation (40), the solar and the atmospheric neutrino mixings \( \theta_{12} \) and \( \theta_{23} \) are independent of the type II mass scale \( m^{\text{II}} \) and of the complex phases of the neutrino Yukawa matrix. We can thus upgrade these types of models continuously from hierarchical neutrino mass spectra to partially degenerate ones, while maintaining the predictions for the two large lepton mixings.

6. Model-building applications of SD

We have seen that SD is not a model, but is a general sub-mechanism within the see-saw mechanism. SD may be used to obtain hierarchical type I neutrino masses, together with bi-large lepton mixing, in a completely natural way, overcoming the usual naturalness objection to the see-saw mechanism in this case. We have also seen that SD may be extended to the case where the lepton mixing arises from the charged lepton sector. Furthermore, we have seen that the SD mechanism is also useful within the framework of the type II see-saw mechanism in the case that the additional type II mass contributions are proportional to the unit matrix. Despite the successes of SD, the conditions on which it is based have just been stated without any explanation. It also remains to be seen how the mechanism of SD can be used to construct realistic unified models of flavour.

In this section, we discuss some of the model-building applications of SD. We shall see that the use of SD is ideally suited to GUTs and family symmetry models, has already been used in quite a number of works of this nature. SD also makes contact with studies based on two right-handed neutrinos. Finally, there have been some interesting cosmological applications that have recently been proposed. Given the simplicity and naturalness of SD, it is reasonable to expect that it will continue to be exploited increasingly in the future.

6.1. Effective two right-handed neutrino models

In SD we have seen that one of the right-handed neutrinos effectively decouples from the see-saw mechanism. Without loss of generality, we have denoted the mass of this decoupled right-handed neutrino as \( Z \). From table 1 we see that the decoupled right-handed neutrino of mass \( Z \) may be
the lightest, the heaviest of the intermediate mass right-handed neutrino. If it is the lightest or
the second lightest then it could, in principle, play an important part in leptogenesis or inflation
and so have cosmological relevance even though it is decoupled from the see-saw mechanism. However if it is the heaviest right-handed neutrino, as in LSDa or ISDa in table 1, then it would be expected to play no part in phenomenology. In these cases, the heaviest neutrino of mass \( Z \) is completely decoupled from physics, and SD reduces to effectively two right-handed neutrino models, as pointed out in [11, 13]. Recently, there have been several studies based on the ‘minimal see-saw’ involving two right-handed neutrinos [17], and it is worth bearing in mind that such models could naturally arise as the limiting case of SD.

6.2. GUT and family symmetry models

There are many models in the literature based on SRHND or SD. For example, explicit realizations of the small determinant condition of Altarelli and Feruglio implicitly involve SRHND, or SD, together with U(1) family symmetry and SU(5) GUTs [18]. An example of SD of the HSD type in Pati-Salam models with U(1) family symmetry was considered in [19]. Single right-handed neutrino dominance has also been applied to SO(10) GUT models involving a U(2) family symmetry [20]. SD of the LSD type with SU(3) family symmetry and SO(10) GUTs has been considered in [21]. Type II upgradable models based on SD of the ISD type with SO(3) family symmetry have been considered in [15, 16]. This list is not exhaustive, but represents a subset of models based on single or sequential right-handed neutrino dominance. The main point is that SD can readily be included in a wide range GUT and family symmetry models, and it enhances the naturalness of such models.

6.3. Sneutrino inflation models

SD has recently also been applied to sneutrino inflation [22]. Requiring a low reheat temperature after inflation, in order to solve the gravitino problem, forces the sneutrino inflation to couple very weakly with ordinary matter and its superpartner almost to decouple from the see-saw mechanism. This decoupling of a right-handed neutrino from the see-saw mechanism is a characteristic of SD.

7. Phenomenological implications of SD

We now review phenomenological consequences of type I see-saw models with SD and their type II upgrades for the low-energy neutrino parameters and high-energy mechanisms as leptogenesis and minimal lepton flavour violation (LFV). In order to compare the predictions of see-saw models based on SD with the experimental data obtained at low energy, the renormalization group (RG) running of the effective neutrino mass matrix has to be taken into account.

7.1. RG corrections

For type I models with SD, the running of the mixing angles is generically small [23] since the mass scheme is strongly hierarchical. When the neutrino mass scale is lifted, e.g. via a type II upgrade, a careful treatment of the RG running of the neutrino parameters, including the energy ranges between and above the see-saw scale [23, 24], is required. For convenient estimates of
the running below the see-saw scales, the approximate analytical formulae for the running of the parameters [25] can be used. Dependent on $\tan \beta$ in the MSSM, on the size of the neutrino Yukawa couplings and on the neutrino mass scale, the RG effects can be sizable or cause only small corrections.2

7.2. Dirac and Majorana CP phases and neutrinoless double beta decay

At present, the CP phases in the lepton sector are unconstraint by experiment. In type I see-saw models based on SD, there is no restriction on them from a theoretical point of view. The type II upgrade scenario however predicts that all observable CP phases, i.e. the Dirac CP phase $\delta$ relevant for neutrino oscillations and the Majorana CP phases $\beta_2$ and $\beta_3$, become small as the neutrino mass scale increases.

The key process for measuring the neutrino mass scale could be neutrinoless double beta decay. The decay rates depend on an effective Majorana mass defined by $\langle m_\nu \rangle = |\sum_i (U_{\text{MNS}})^2_{1i} m_i|$. Future experiments which are under consideration at present might increase the sensitivity to $\langle m_\nu \rangle$ by more than an order of magnitude. For type I models with SD, which have a hierarchical mass scheme, $\langle m_\nu \rangle$ can be very small, below the accessible sensitivity.

For models where the neutrino mass scale is lifted via a type II upgrade [16], there is a close relation between the neutrino mass scale, i.e. the mass of the lightest neutrino and $\langle m_\nu \rangle$. Since the CP phases are small, there can be no significant cancellations in $\langle m_\nu \rangle$. This implies that the effective mass for neutrinoless double beta decay is approximately equal to the neutrino mass scale $\langle m_\nu \rangle \approx m_\nu^{\text{II}}$ and, therefore, neutrinoless double beta decay will be observable in the next round of experiments if the neutrino mass spectrum is partially degenerate.

7.3. Theoretical expectations for the mixing angles

In order to discriminate between models, precision measurements of the neutrino mixing angles have the potential to play an important role.

One important parameter is the value of the mixing angle $\theta_{13}$, which is at present only bounded from above to be smaller than approximately $13^\circ$. In the type I SD case, the mixing angle $\theta_{13}$ is typically of the order $O(m_2^1/m_3^1)$. In the type II upgrade scenario this ratio decreases with increasing neutrino mass scale and is smaller than $\approx 5^\circ$ for partially degenerate neutrinos even if it was quite large in the type I limit. Sizable RG corrections, which are usually expected for partially degenerate neutrinos, are suppressed in the type II upgrade scenario due to small CP phases $\beta_2$, $\beta_3$ and $\delta$ [25].

Another important parameter is $\theta_{23}$. Its present best-fit value is close to $45^\circ$; however comparably large deviations are experimentally allowed as well. With SD, we expect minimal deviations of $\theta_{23}$ from $45^\circ$ of the order $O(m_2^1/m_3^1)$, which could be observed by future long-baseline experiments in the type I see-saw case.3 In the type II upgraded version, the corrections can be significantly smaller since the ratio $m_2^1/m_3^1$ decreases with increasing neutrino mass scale [16]. For large $\tan \beta$ in the MSSM, the major source for the corrections can be RG effects [25], which are unsuppressed for small CP phases.

2 A complete list of references for the $\beta$-functions of the neutrino mass operator can be found in [25].

3 For sensitivities of future long-baseline experiments for measuring deviation from $\theta_{23} = 45^\circ$ and their potential for discriminating between models, see [26].
7.4. Minimal LFV

At leading order in a mass insertion approximation the branching fractions of LFV processes are given by\(^4\)

\[
\text{BR}(l_i \rightarrow l_j \gamma) \approx \frac{\alpha^3}{G_F} f(M_2, \mu, m_{\tilde{\nu}})|m_{\tilde{L}_{ij}}|^2 \tan^2 \beta, \tag{51}
\]

where \(l_i = e, l_2 = \mu, l_3 = \tau\), and where the off-diagonal slepton doublet mass squared is given in the leading log approximation (LLA) by

\[
m^2_{\tilde{L}_{ij}}^{(\text{LLA})} \approx -\frac{(3m_0^2 + A_0^2)}{8 \pi^2} C_{ij}. \tag{52}
\]

With SD, using the notation of equations (17) and (18), the leading log coefficients relevant for \(\mu \rightarrow e\gamma\) and \(\tau \rightarrow \mu\gamma\) are given approximately as

\[
C_{21} = ab \ln \frac{M_U}{X} + de \ln \frac{M_U}{Y}, \quad C_{32} = bc \ln \frac{M_U}{X} + ef \ln \frac{M_U}{Y}. \tag{53}
\]

From table 1 and equation (53) it can be seen which types of SD will lead to large rates for \(\mu \rightarrow e\gamma\) and \(\tau \rightarrow \mu\gamma\). For example, the results for HSD show a large rate for \(\tau \rightarrow \mu\gamma\) which is the characteristic expectation of lop-sided models in general [27] and HSD in particular. A global analysis of LFV has been performed in the constrained minimal supersymmetric standard model (CMSSM) for the case of SD, focusing on the two cases of HSDa and LSDa [28]. The results in [28] are based on an exact calculation, and the error incurred compared with the LLA study [29] can be as much as 100%. For LSDa, \(\tau \rightarrow \mu\gamma\) is well below observable values. Therefore \(\tau \rightarrow \mu\gamma\) provides a good discriminator between the HSDa and LSDa types of dominance. In [28] it is shown that the rate for \(\mu \rightarrow e\gamma\) may determine the order of the sub-dominant neutrino Yukawa couplings in the flavour basis.

7.5. Leptogenesis

Leptogenesis and LFV are important indicators that can help to resolve the ambiguity of right-handed neutrino masses in table 1. In the LSD and HSD cases of SD, leptogenesis has been studied with some interesting results [30]. In general, successful leptogenesis for such models requires the mass of the lightest right-handed neutrino to be quite high, and generally to exceed the gravitino constraints if supersymmetry is assumed. However, putting this to one side for the moment, interesting links between the phase relevant for leptogenesis and the phase \(\delta\) measurable in neutrino oscillation experiments have been made. The precise link depends on how many ‘texture’ zeros are assumed to be present in the neutrino Dirac mass matrix. For example, if two texture zeros are assumed then there is a direct link between \(\delta\) and the leptogenesis phase, with the sign of \(\delta\) being predicted from the fact that we are made of matter rather than antimatter. On the other hand, if only the physically motivated texture zero in the 11 entry of the Dirac mass matrix is assumed, then the link is more indirect [13].

\(^4\) The mass insertion approximation given in equation (51) is for illustrative purposes only. The conclusions quoted below from [28] do not rely on this approximation.

*New Journal of Physics* 6 (2004) 110 (http://www.njp.org/)
More generally, in three right-handed neutrino models with SD, if the dominant right-handed neutrino is the lightest one (LSD) then the washout parameter $\tilde{m}_1 \sim O(m_3)$, which is rather too large compared with the optimal value of around $10^{-3}$ eV, while if the dominant right-handed neutrino is either the intermediate one or the heaviest one then one finds $\tilde{m}_1 \sim O(m_2)$ or arbitrary $\tilde{m}_1$, which can be closer to the desired value [30].

8. Discussion and conclusions

Neutrino masses and mixings are now established experimental phenomena which must be included in some extended version of the Standard Model. The simplest mechanism for describing small neutrino masses is the see-saw mechanism; however the simultaneous appearance of hierarchical neutrino masses and two LMAs is not natural in the see-saw mechanism. The simplest solution to this difficulty is to assume SD which has been the subject of this review.

We have reviewed the mechanism of sequential right-handed neutrino dominance which was proposed in the framework of the type I see-saw mechanism to account for bi-large neutrino mixing and a neutrino mass hierarchy in a natural way. We have discussed how SD may also be applied to the right-handed charged leptons, which alternatively allows bi-large lepton mixing in the charged lepton sector. We reviewed how such SD models may be upgraded to include type II see-saw contributions, resulting in a partially degenerate neutrino mass spectrum with bi-large lepton mixing arising from SD. We also saw that the use of SD is ideally suited to GUTs and family symmetry models, and mentioned some examples of such models. We also pointed out the interesting case where SD reduces effectively to the case of two right-handed neutrinos, and mentioned some interesting cosmological applications that have recently been proposed such as sneutrino inflation.

We also reviewed some phenomenological consequences of type I see-saw models with SD and their type II upgrades for the low-energy neutrino parameters and high-energy mechanisms as leptogenesis and minimal LFV, both of which can be probes of different types of SD. While RG effects are expected to be quite small for type I SD, they become increasingly important for the type II upgrade SD as the neutrino mass scale increases. We noted that neutrinoless double beta decay is practically unobservable in type I SD, but may well be observed in the next round of experiments in the type II upgrade sequential models if the neutrino masses are partially degenerate. We have seen that both $\theta_{13}$ and the correction to $\theta_{23}$ are controlled by the ratio $m_2^I/m_3^I$ which decreases with increasing neutrino mass scale, with interesting consequences.

Given the simplicity and naturalness of SD, we expect it to continue to be used and exploited ubiquitously in the future.

Acknowledgments

We acknowledge the support from the PPARC grant PPA/G/O/2002/00468.

Appendix. Our conventions

For the mass matrix of the charged leptons $m_E = Y_e v_d$, where $v_d = (H_d)$, defined by $\mathcal{L}_e = -m_E \bar{e}_L e_R^c + h.c.$ and for the neutrino mass matrix $m^c_{\nu L L}$, the change from flavour basis to mass
The eigenbasis can be performed with the unitary diagonalization matrices $U_{eL}$, $U_{eR}$ and $U_{\nu L}$ by

\[
U_{eL}m_{e}U_{eR}^{\dagger} = \begin{pmatrix} m_{e} & 0 & 0 \\ 0 & m_{\mu} & 0 \\ 0 & 0 & m_{\tau} \end{pmatrix}, \quad U_{\nu L}m_{\nu L}^{\dagger}U_{\nu L}^{\dagger} = \begin{pmatrix} m_{1} & 0 & 0 \\ 0 & m_{2} & 0 \\ 0 & 0 & m_{3} \end{pmatrix}.
\]

The MNS matrix is then given by

\[
U_{\text{MNS}} = U_{eL}U_{\nu L}^{\dagger}.
\]

We use the parametrization $U_{\text{MNS}} = R_{23}U_{13}P_{0}$ with $R_{23}$, $U_{13}$, $R_{12}$ and $P_{0}$ being defined as

\[
R_{12} := \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{13} := \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix},
\]

\[
R_{23} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad P_{0} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta_{2}} & 0 \\ 0 & 0 & e^{i\beta_{3}} \end{pmatrix}
\]

and where $s_{ij}$ and $c_{ij}$ stand for $\sin(\theta_{ij})$ and $\cos(\theta_{ij})$, respectively. The matrix $P_{0}$ contains the possible Majorana phases $\beta_{2}$ and $\beta_{3}$. $\delta$ is the Dirac CP phase relevant for neutrino oscillations.

References

[1] For a review see e.g. King S F 2004 Rep. Prog. Phys. 67 107 (Preprint hep-ph/0310204)
[2] Yanagida T 1979 Proc. Workshop on Unified Theory and Baryon Number of the Universe, KEK, Japan
  Glashow S L 1979 Cargese Lectures
  Gell-Mann M, Ramond P and Slansky R 1979 Sanibel Talk, CALT-68-709, February
  Gell-Mann M, Ramond P and Slansky R 1979 Supergravity (Amsterdam: North-Holland)
  Mohapatra R N and Senjanovic G 1980 Phys. Rev. Lett. 44 912
[3] Arkani-Hamed N, Dimopoulos S, Dvali G R and March-Russell J 2002 Phys. Rev. D 65 024032 (Preprint hep-ph/9811448)
[4] Hirsch M and Valle J W F 2004 Preprint hep-ph/0405015
[5] Lazarides G, Shafi Q and Wetterich C 1981 Nucl. Phys. B 181 287
  Mohapatra R N and Senjanovic G 1981 Phys. Rev. D 23 165
  Wetterich C 1981 Nucl. Phys. B 187 343
[6] Wolfenstein L 1978 Phys. Rev. D 17 2369
  Mikheyev S and Smirnov A Yu 1985 Sov. J. Nucl. Phys. 42 913
[7] Apollonio M et al [CHOOZ Collaboration] 1999 Phys. Lett. B 466 415 (Preprint hep-ex/9907037)
[8] Elgaroy O et al 2002 Preprint astro-ph/0204152
[9] King S F 1998 Phys. Lett. B 439 350 (Preprint hep-ph/9806440)
[10] King S F 1999 Nucl. Phys. B 562 57 (Preprint hep-ph/9904210)
[11] King S F 2000 Nucl. Phys. B 576 85 (Preprint hep-ph/9912492)
[12] King S F 2002 J. High Energy Phys. JHEP09 (2002) 011 (Preprint hep-ph/0204360)
[13] King S F 2003 Phys. Rev. D 67 113010 (Preprint hep-ph/0211228)
[14] Davidson S and King S F 1998 Phys. Lett. B 445 191 (Preprint hep-ph/9808296)
[15] Antusch S and King S F 2004 Preprint hep-ph/0403053
[16] Antusch S and King S F 2004 Preprint hep-ph/0402121

New Journal of Physics 6 (2004) 110 (http://www.njp.org/)
[17] Frampton P H, Glashow S L and Yanagida T 2002 Phys. Lett. B 548 119 (Preprint hep-ph/0208157)
    Raidal M and Strumia A 2003 Phys. Lett. B 553 72 (Preprint hep-ph/0210021)
    King S F 2003 Phys. Rev. D 67 113010 (Preprint hep-ph/0211228)
    Ibarra A and Ross G G 2003 Preprint hep-ph/0312138
[18] Altarelli G and Feruglio F 1998 Phys. Lett. B 439 112 (Preprint hep-ph/9807353)
    Altarelli G and Feruglio F 1998 J. High Energy Phys. JHEP11 (1998) 021 (Preprint hep-ph/9809596)
    Altarelli G and Feruglio F 1999 Phys. Lett. B 451 388 (Preprint hep-ph/9812475)
    Altarelli G, Feruglio F and Masina I 2000 Phys. Lett. B 472 382 (Preprint hep-ph/9907532)
[19] King S F and Oliveira M 2001 Phys. Rev. D 63 095004 (Preprint hep-ph/0009287)
    Blazek T, King S F and Parry J K 2003 J. High Energy Phys. JHEP05 (2003) 016 (Preprint hep-ph/0303192)
[20] Barbieri R, Creminelli P and Romanino A 1999 Nucl. Phys. B 559 17 (Preprint hep-ph/9903460)
    Raby S 2003 Phys. Lett. B 561 119 (Preprint hep-ph/0302027)
[21] King S F and Ross G G 2001 Phys. Lett. B 520 243 (Preprint hep-ph/0108112)
    Ross G G and Velasco-Sevilla L 2003 Nucl. Phys. B 653 3 (Preprint hep-ph/0208218)
    King S F and Ross G G 2003 Preprint hep-ph/0307190
[22] Ellis J R, Raidal M and Yanagida T 2004 Phys. Lett. B 581 9 (Preprint hep-ph/0303242)
    Chankowski P H, Ellis J R, Pokorski S, Raidal M and Turzynski K 2004 Preprint hep-ph/0403180
[23] King S F and Singh N N 2000 Nucl. Phys. B 591 3 (Preprint hep-ph/0006229)
[24] Antusch S, Kersten J, Lindner M and Ratz M 2002 Phys. Lett. B 538 87 (Preprint hep-ph/0203233)
[25] Antusch S, Kersten J, Lindner M and Ratz M 2003 Nucl. Phys. B 674 401 (Preprint hep-ph/0305273)
[26] Antusch S, Huber P, Kersten J, Schwetz T and Winter W 2004 Preprint hep-ph/0404268
[27] Blazek T and King S F 2001 Phys. Lett. B 518 109 (Preprint hep-ph/0105005)
[28] Blazek T and King S F 2003 Nucl. Phys. B 662 359 (Preprint hep-ph/0211368)
[29] Lavignac S, Masina I and Savoy C A 2002 Preprint hep-ph/0202086
[30] Hirsch M and King S F 2001 Phys. Rev. D 64 113005 (Preprint hep-ph/0107014)