On Dirac structure of infinite-dimensional stochastic port-Hamiltonian systems

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Abstract

Stochastic infinite-dimensional port-Hamiltonian systems (SPHSs) with multiplicative Gaussian white noise are considered. In this article we extend the notion of Dirac structure for deterministic distributed parameter port-Hamiltonian systems to a stochastic ones by adding some additional stochastic ports. Using the Stratonovich formalism of the stochastic integral, the proposed extended interconnection of ports for SPHSs is proved to still form a Dirac structure. This constitutes our main contribution. We then deduce that the interconnection between (stochastic) Dirac structures is again a (stochastic) Dirac structure under some assumptions. These interconnection results are applied on a system composed of a stochastic vibrating string actuated at the boundary by a mass-spring system with external input and output. This work is motivated by the problem of boundary control of SPHSs and will serve as a foundation to the development of stabilizing methods.

Keywords: Infinite-dimensional systems – Stochastic partial differential equations – Dirac structures – Boundary control

1. Introduction

Linear distributed port-Hamiltonian systems constitute a powerful class of systems for the modelling, the analysis, and the control of distributed parameter systems. It enables us to model many physical systems such as beam equations, transport equations or wave equations, see for instance Jacob and Zwart (2012). A comprehensive overview of the literature on this class of systems can be found in Rashad Hashem et al. (2020). In order to cover an even larger set of systems that admits a port-Hamiltonian representation, some authors have defined the notion of dissipative or irreversible port-Hamiltonian systems, see e.g. Mora et al. (2021a,b), Caballeria et al. (2021) and Ramirez et al. (2022). Port-Hamiltonian systems (PHSs) are characterized by a Dirac structure and an Hamiltonian. Dirac structures consist of the power-preserving interconnection of different ports elements and were first introduced in the context of port-Hamiltonian systems in van der Schaft and Maschke (2002). It was then extended for higher-order PHSs in Le Gorrec et al. (2005) and Villegas (2007). A fundamental property of Dirac structures is that the composition of Dirac structures still forms a Dirac structure, provided some assumptions. This induces the main aspect of the port-Hamiltonian modelling, which is that the power-conserving interconnection of PHSs is still a port-Hamiltonian system.

Stochastic models are powerful to take into account neglected random effects that may occur when working with real plants. Especially, random forcing, parameter uncertainty or even boundary noise can impact the behavior of dynamical systems. In particular, PHSs interact with their environment through external ports, which can be a cause of randomness in many different ways as explained in Lamoline (2021). The stochastic extension of port-Hamiltonian systems was first proposed in Lázaro-Camí and Ortega (2008) for Poisson manifolds. On finite-dimensional spaces the class of nonlinear time-varying stochastic port-Hamiltonian systems (SPHSs) was presented in Satoh and Fujimoto (2013). A stochastic extension of distributed port-Hamiltonian systems was first developed in Lamoline (2019) and in Lamoline and Winkin (2020). In addition the passivity property of SPHSs was investigated in Lamoline and Winkin (2017) and Lamoline (2021). However, in these works only a state space representation described by a stochastic differential equation (SDE) is given. As far as known, few efforts have been done for describing the underlying geometric structure of SPHSs. One can cite Cordoni et al. (2019), where finite-dimensional stochastic port-Hamiltonian systems are modelled using the Stratonovich and Ito formalisms.

In this paper the notion of Dirac structure with stochastic port-variables is explored. Our central idea consists in extending the original Dirac structure of deterministic first-order PHSs by adding further noise ports to the port-based structure. These specific noise ports are devoted to represent the interaction of the dynamical system with...
its random environment. In order to preserve the power-preserving interconnection we consider the Stratonovich formulation of the stochastic integral, see for instance Duan and Wang (2014). Interested readers may also be referred to Ruth F. Curtain (1978) and Da Prato and Zabczyk (2014) for further details on infinite-dimensional SDEs.

The content of this article is as follows. In Section 2 we introduce the basic concepts on Dirac structures together with the class of deterministic port-Hamiltonian systems. In Section 3 a port-based representation for SPHs is presented and it is shown to form a Dirac structure, which is the main contribution of the paper. Section 4 is dedicated to the illustration of our central result, by showing that some interconnection between the newly defined Dirac structure and another arbitrary Dirac structure that shares common ports is still a Dirac structure. A stochastic damped vibrating string actuated by a mass-spring system at the boundary is then presented as an example. We conclude and discuss some future works in 5.

2. Background on Dirac structure

In this section we introduce some notions on distributed port-Hamiltonian systems, Tellegen structures and Dirac structures. Let us first recall the definitions of Tellegen and Dirac structures for linear distributed PHSs, see e.g. van der Schaft and Maschke (2002), Le Gorrec et al. (2005) and Kurula et al. (2010). Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert spaces endowed with the inner products $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{F}}$, respectively. The spaces $\mathcal{E}$ and $\mathcal{F}$ denote the effort and the flow spaces, respectively. We define the bond space $\mathcal{B} := \mathcal{F} \times \mathcal{E}$ equipped with the following inner product

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix} , \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_{\mathcal{B}} := \langle f_1 , f_2 \rangle_{\mathcal{F}} + \langle e_1 , e_2 \rangle_{\mathcal{E}}$$

(1)

for all $(f_1, e_1), (f_2, e_2) \in \mathcal{B}$. To define Tellegen or Dirac structures, the bond space is endowed with the bilinear symmetric pairing given by

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix} , \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_{+} := \langle f_1, j^{-1} f_2 \rangle_{\mathcal{E}} + \langle e_1, j f_2 \rangle_{\mathcal{E}},$$

(2)

with $j : \mathcal{F} \to \mathcal{E}$ being an invertible linear mapping. The bilinear pairing $(\cdot , \cdot)_+$ represents the power. Let $\mathcal{V}$ be a linear subspace of $\mathcal{B}$. The orthogonal subspace of $\mathcal{V}$ with respect to the bilinear pairing $(\cdot , \cdot)_+$ is defined as

$$\mathcal{V}^\perp := \{ b \in \mathcal{B} : \langle b, v \rangle_+ = 0, \text{ for all } v \in \mathcal{V} \}. $$

(3)

These tools enable us to define Tellegen and Dirac structures, see Kurula et al. (2010) Definition 2.1).

Definition 2.1. A linear subspace $\mathcal{D}$ of the bond space $\mathcal{B} := \mathcal{F} \times \mathcal{E}$ is called a Tellegen structure if $\mathcal{D} \subset \mathcal{D}^\perp$, where the orthogonal complement is understood with respect to the bilinear pairing $(\cdot , \cdot)_+$, see (2).

Definition 2.2. A linear subspace $\mathcal{D}$ of the bond space $\mathcal{B}$ is said to be a Dirac structure if

$$\mathcal{D}^\perp = \mathcal{D}. $$

(4)

Note that the condition (4) implies that the power of any element of the Dirac structure is equal to zero, i.e.,

$$\left\langle \begin{pmatrix} f \\ e \end{pmatrix} , \begin{pmatrix} f \\ e \end{pmatrix} \right\rangle_{+} = 2 \langle f, j^{-1} e \rangle_{\mathcal{F}} = 0,$$

for any $(f, e) \in \mathcal{D}$, where the relation $\langle f, j^{-1} e \rangle_{\mathcal{F}} = \langle f, e \rangle_{\mathcal{E}}$ has been used. The underlying structure of port-Hamiltonian systems forms a Dirac structure, which links the port-variables in a way that the total power is equal to zero. A distributed port-Hamiltonian system is described by the following partial differential equation

$$\frac{\partial}{\partial \epsilon} \langle \varphi, \chi \rangle_{\mathcal{H}} = \langle \mathcal{P} \varphi, \chi \rangle_{\mathcal{H}} + \langle \mathcal{P}_0 \varphi, \chi \rangle_{\mathcal{H}},$$

(5)

where $\varphi(\zeta, \epsilon) \in \mathbb{R}^n$ for $\zeta \in [a, b]$ and $t \geq 0$. In addition, $\mathcal{P}_1 = \mathcal{P}_1^T \in \mathbb{R}^{n \times n}$ is invertible, $\mathcal{P}_0 = -\mathcal{P}_0^T \in \mathbb{R}^{n \times n}$ and $\mathcal{H} \in L^\infty([a, b]; \mathbb{R}^{n \times n})$ is symmetric and satisfies $m I \leq \mathcal{H}(\zeta)$ for all $\zeta \in [a, b]$ and some constant $m > 0$. The state space $\mathcal{X} := L^2([a, b]; \mathbb{R}^n)$ is endowed with the energy inner product $\langle \varphi_1, \varphi_2 \rangle_{\mathcal{X}} = \langle \varphi_1, \mathcal{H} \varphi_2 \rangle_{\mathcal{L}^2} = \int_a^b \varphi_1^T(\mathcal{H}(\zeta) \varphi_2) d\zeta$, for all $\varphi_1, \varphi_2 \in \mathcal{X}$. The energy associated to (5) is given by $E(\epsilon) = \frac{1}{2} \| \varphi(\epsilon) \|^2_{\mathcal{H}}$.

The boundary ports denoted by $f_\partial$ and $e_\partial$ are given by

$$\left\langle \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \right\rangle_{\mathcal{E}} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_0 \end{pmatrix}^{-1} \right) \left( \begin{pmatrix} \mathcal{H}(\epsilon(t)) & 0 \\ 0 & \mathcal{H}(\epsilon(t)) \end{pmatrix} \right) \left( \begin{pmatrix} \mathcal{H}(\epsilon(t)) & 0 \\ 0 & \mathcal{H}(\epsilon(t)) \end{pmatrix} \right)^{-1} =: \mathcal{R}_0 \left( \begin{pmatrix} \mathcal{H}(\epsilon(t)) & 0 \\ 0 & \mathcal{H}(\epsilon(t)) \end{pmatrix} \right)^{-1}$$

(6)

and represent a linear combination of the restriction at the boundary variables. Note that the notation $(\mathcal{H}(\epsilon(t))(a) := \mathcal{H}(a) \epsilon(a, t)$ has been used. We complete the PDE (5) with the following homogeneous boundary conditions

$$0 = W_B \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}_{\mathcal{E}}.$$

(7)

where $W_B \in \mathbb{R}^{n \times 2n}$. It can be easily seen that the PDE (5) together with the boundary conditions (7) admits the abstract representation $\varphi(\epsilon) = \mathcal{A} \varphi(\epsilon), \varphi(0) = e_\partial \in \mathcal{X}$ where the linear (unbounded) operator $\mathcal{A}$ is defined by

$$\mathcal{A} \varphi := \mathcal{P}_1 \frac{d}{d \epsilon}(\mathcal{H} \varphi) + \mathcal{P}_0 \mathcal{H} \varphi$$

(8)

for $\varphi$ on the domain

$$D(\mathcal{A}) = \left\{ \varphi \in \mathcal{X} : \mathcal{H} \varphi \in H^1([a, b]; \mathbb{R}^n), W_B \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = 0 \right\}.$$

(9)

Before introducing the concept of Dirac structure for (5) and (7), let us recall the following two results from Jacob and Zwart (2012).

Lemma 2.1. Consider the operator $\mathcal{A}$ defined by (8) with domain $\mathcal{D}$. Then the following result holds:

$$\langle \mathcal{A} \varphi, \varphi \rangle_{\mathcal{X}} + \langle \varphi, \mathcal{A} \varphi \rangle_{\mathcal{X}} = 2 f_\partial^T e_\partial.$$

(10)
Theorem 2.1. Let $W_B$ be a $n \times 2n$ real matrix. Then the operator $\mathcal{A}$ defined in (3) on the domain (4) generates a contraction $C_0$-semigroup of bounded linear operators if and only if $W_B$ is full rank and satisfies $W_B \Sigma W_B^T \geq 0$, with $\Sigma := (0 \ 1)$. Furthermore, the energy balance equation
\[
\frac{dE(t)}{dt} = f^T(t)e_\partial(t) \tag{11}
\]
holds.

**Proof.** For the fact that the operator $\mathcal{A}$ generates a contraction $C_0$-semigroup, we refer to (Jacob and Zwart 2012 Theorem 7.2.4). Now observe that
\[
\langle \cdot, \cdot \rangle \quad \text{(15)}
\]
and only if $\mathcal{A}$ is full rank and satisfies $B \in \mathcal{L}(Z, \mathcal{X})$. The SPDE
\[
\frac{\partial E}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}(H(\zeta)\varepsilon(\zeta, t)) + P_0 H(\zeta)\varepsilon(\zeta, t) + H(\varepsilon(\zeta, t))(\dot{\varepsilon}(t))(\zeta) \tag{16}
\]
with the boundary conditions (1). Let $Z$ be a Hilbert space. The noise process $(\dot{\varepsilon}(t))_{t \geq 0}$ is a $\mathbb{Z}$-valued Gaussian white noise process with intensity operator $H \in \mathcal{L}(\mathcal{X}, \mathcal{L}(Z, \mathcal{X}))$. The SPDE (16) can be rewritten as a SDE on $\mathcal{X}$ of the form
\[
\delta \varepsilon(t) = \mathcal{A}e(t)\delta t + H(\varepsilon(t))\delta w(t), \varepsilon(0) = \varepsilon_0, \tag{17}
\]
where the operator $\mathcal{A}$ is given in (8) with domain (9). Note that $(\varepsilon(t))$ is a $\mathbb{Z}$-valued Wiener process with covariance $Q$. We further assume that $Q$ is nonnegative, self-adjoint and of trace class, i.e. $\text{Tr}[Q] < \infty$. The differential term $H(\varepsilon(t))\delta w(t)$ has to be understood under the Stratonovich definition of a stochastic integral. For further details on the Stratonovich stochastic integral on Hilbert spaces, we refer to (Duan and Wang 2014) among others. It is worth pointing out that the Stratonovich integral satisfies the standard rules of the chain calculus. The well-posedness of the SDE (17) in terms of existence and uniqueness of a mild solution is extensively studied in Lamoline and Winkin (2020) and Lamoline (2021). By a mild solution of (17), we mean a $\mathcal{F}$-adapted and mean-square continuous solution of the integral form of (17), i.e. a solution $\varepsilon(t)$ which satisfies
\[
\varepsilon(t) = T(t)\varepsilon_0 + \int_0^t T(t-s)H(\varepsilon(s))\delta w(s),
\]
where $(T(t))_{t \geq 0}$ is the $C_0$-semigroup \footnote{We assume being in the conditions of Theorem 2.1} whose operator $\mathcal{A}$ is the infinitesimal generator and $\varepsilon_0 \in \mathcal{X}$ denotes the initial condition. The power-balance equation associated to (16) can be expressed as follows
\[
\delta E(\varepsilon(t)) = \delta f^T_\partial(t)e_\partial(t) + \langle H^*(\varepsilon(t))\mathcal{H}(\varepsilon(t), \delta w(t)) \rangle_Z,
\]
which is equivalent to
\[
0 = -\langle \delta f_{\varepsilon}(t), e_{\varepsilon}(t) \rangle_{L^2} + \delta f^T_\partial(t)e_\partial(t) + \langle \delta w(t), e_w(t) \rangle_Z, \tag{18}
\]
where the ports $\delta f_e = \delta \varepsilon$ and $\varepsilon_e = \mathcal{H} \varepsilon$ while the new ports (noise ports) due to the stochastic nature of $\delta f_\beta$ are defined as $\delta f_\omega := \delta w_t(t)$ and $\varepsilon_\omega(t) = H^* \varepsilon_e$. Note that the boundary ports $\delta f_\beta$ and $\varepsilon_\beta$ are now expressed as $\delta f_\beta = \frac{1}{\sqrt{2}}(P_1(\varepsilon_e)(b) - P_1(\varepsilon_e)(a))\delta t$ and $\varepsilon_\beta = \frac{1}{\sqrt{2}}(\varepsilon_e(b) - \varepsilon_e(a))$, respectively. In that way the pairing (13) is extended as follows:

$$
\langle (\delta f^1, \delta f_\omega^1, \varepsilon^1, \varepsilon_\omega^1), (\delta f^2_\omega, \delta f^2_\omega, \varepsilon_\omega^2, \varepsilon_\omega^2) \rangle +
= (\varepsilon_e^1, \delta f^1_\omega)_{L^2} + (\varepsilon_e^2, \delta f^2_\omega)_{L^2} - (\varepsilon_e^1, \delta f^2_\omega)_{L^2} - (\varepsilon_e^2, \delta f^1_\omega)_{L^2}.
$$

Observe that $\varepsilon_\omega(t)$ represents the power-conjugated effort coupled to the stochastic input $w(t)$. It will be shown in Theorem 3.1 it it to preserve Dirac structure of port-Hamiltonian systems when subject to stochastic disturbances, see [Lamontage 2021].

**Remark 3.1.** We stress that the notations $\delta f_e, \delta f_\beta$ and $\delta f_\omega$ are used for the flows as they are defined from infinitesimal variations resulting from $\delta t$ and $\delta w(t)$.

In order to write SPHSs as Dirac structures, we complete the flow and the efforts spaces in the following way:

$$
\mathcal{F} = \mathcal{E} := \mathcal{X} \times \mathcal{Z} \times \mathbb{R}^n.
$$

Once more, the comparison can be made with (2) where the invertible linear map $j : \mathcal{F} \to \mathcal{E}$ is expressed as $j = \begin{pmatrix} I_{2n} & 0 & 0 \\ 0 & -I_4 & 0 \\ 0 & 0 & -I_4 \end{pmatrix}$. As a result of the pairing (19), let us consider the following structure for SPHSs described by (16)

$$
\mathcal{D} = \left\{ \begin{array}{ll}
\delta f_e &= \mathcal{J} \varepsilon_e \delta t + H \delta f_\omega, \\
\delta f_\omega &= \delta f_\beta, \\
\varepsilon_e &= \varepsilon_\beta, \\
\varepsilon_\omega &= \varepsilon_\omega
\end{array} \right\} \in \mathcal{F} \times \mathcal{E} | e_e \in H^1([a, b]; \mathbb{R}^n),
$$

$$
\delta f_e = \mathcal{J} \varepsilon_e \delta t + H \delta f_\omega, \delta f_\beta = \frac{1}{\sqrt{2}}(P_1(e_e)(b) - P_1(e_e)(a))\delta t, \\
\varepsilon_\omega = \frac{1}{\sqrt{2}}(e_e(b) - e_e(a)), e_\beta = H^* e_e
$$

Note that the boundary flow and effort variables, $\delta f_\beta$ and $\varepsilon_\beta$ respectively, can be written in a more compact form as

$$
\begin{pmatrix} \delta f_\beta \\ \varepsilon_\beta \delta t \end{pmatrix} = R_0 \begin{pmatrix} (e_e(b)) \\ (e_e(a)) \end{pmatrix} \delta t.
$$

We can now prove the main result, which states that $\mathcal{D}$ given by (21) forms a Dirac structure as defined in Definition 2.2.

**Theorem 3.1.** The subspace $\mathcal{D}$ of $\mathcal{B}$ given by (21) is a Dirac structure.

**Proof.** We first prove that $\mathcal{D} \subset \mathcal{D}^\perp$. This is equivalent to the canonical product $(b, b)_+$ being set to zero for any $b \in \mathcal{D}$. Let us consider $(\delta f_e, \delta f_\omega, \delta f_\beta, e_e, e_\omega, e_\beta) \in \mathcal{F} \times \mathcal{E}$. From (19), we get that

$$
\langle (\delta f_e, \delta f_\omega, \delta f_\beta, e_e, e_\omega, e_\beta) \rangle = (\delta f_e, e_\omega)_L^2 - (\delta f_\omega, e_\omega)_R^2
$$

Moreover, since $\delta f_e(t) = \mathcal{J} \varepsilon_e(t) \delta t + H(\varepsilon(t)) \delta f_\omega(t)$, we obtain that

$$
\langle (\delta f_\beta, e_\omega) \rangle = (\mathcal{J} \varepsilon_e(t) \delta t + H(\varepsilon(t)) \delta f_\omega(t), e_\omega)_L^2 - (\delta f_\omega(t), H^*(\varepsilon(t)) e_\omega)_R^2
$$

By using (21), there holds

$$
\langle (\delta f_e, \delta f_\omega, \delta f_\beta, e_e, e_\omega, e_\beta) \rangle = (\mathcal{J} \varepsilon_e(t) \delta t + H(\varepsilon(t)) \delta f_\omega(t), e_\omega)_L^2 \delta t - 2 (\delta f_\omega(t), e_\omega)_R^2 = 0.
$$

In order to prove that $\mathcal{D}^\perp \subset \mathcal{D}$, let us pick any $(\delta f_e, \delta f_\omega, \delta f_\beta, e_e, e_\omega, e_\beta) \in \mathcal{D}^\perp$ and $(\delta f_\beta, \delta f_\omega, \delta f_\omega, e_e, e_\omega, e_\beta) \in \mathcal{D}$. By orthogonality, we have that

$$
0 = \langle (\delta f_e, \delta f_\omega, \delta f_\beta, e_e, e_\omega, e_\beta) \rangle = (\mathcal{J} \varepsilon_e(t) \delta t + H(\varepsilon(t)) \delta f_\omega(t), e_\omega)_L^2 + (\delta f_\omega(t), e_\omega)_R^2,
$$

Note that the time dependency of the variables has been willingly omitted for the sake of readability.

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3The time dependency of the variables has been willingly omitted for the sake of readability.
Therefore, we obtain $\delta f_c(t) = J\bar{c}_c(t)dt + H(\varepsilon(t))\delta f_w(t)$.

Step 3. Let us take now $\delta f_w = \tilde{e}_w = 0$ such that $(J\bar{c}_c,\delta t, 0, \delta f_\partial, \bar{e}_\partial, 0, \bar{e}_\partial) \in \mathcal{D}$. From (23), we have

$$0 = (\delta f_c, \bar{e}_c)_L + (\tilde{f}_c, \bar{e}_c)_L = (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n} - (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n}$$

$$= (J\bar{c}_c, \bar{e}_c)_L \delta t + (J\bar{e}_c, \bar{e}_c)_L \delta t = -\langle \delta f_\partial, \bar{e}_\partial \rangle_{\mathbb{R}^n} = -\langle \tilde{f}_\partial, \bar{e}_\partial \rangle_{\mathbb{R}^n}$$

$$= \left[ e^T \tilde{P}_1 e \right] \delta t - (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n} = (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n}$$

$$= \left[ e^T \tilde{P}_1 e \right] \delta t - (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n} = (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n}$$

$$\sum\delta f_\partial = \left[ e^T \tilde{P}_1 e \right] \delta t - (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n} = (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n}$$

$$= \left[ e^T \tilde{P}_1 e \right] \delta t - (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n} = (\delta f_\partial, \bar{e}_\partial)_{\mathbb{R}^n}$$

Since the above equality has to hold for all $\bar{e}_\partial$ and $\tilde{f}_\partial$, we deduce that

$$\delta f_\partial = R_0 \left( \left( \begin{array}{c} e^T \tilde{P}_1 e \end{array} \right) \delta t, \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \right)$$

This proves that $\mathcal{D}^+ \subset \mathcal{D}$, which completes the proof. \hfill \Box

**Remark 3.2.** Dissipative elements have not been considered to focus on proof arguments regarding the noise elements. Note that dissipative elements could be added independently to the Dirac structure. Theorem 3.1 then readily extends, see [27], Theorem 6.5.

4. Illustration: boundary control as interconnection of stochastic Dirac structures.

In this section, we illustrate a central feature of Dirac structures, their ability to be interconnected between each other, under appropriate assumptions. We investigate the interconnection of a stochastic Dirac structure of the form [21] with an arbitrary Dirac structure that shares the same space of interconnection. Therefore, the notions of split Tellegen and split Dirac structures are introduced, see [21], Definitions 3.1.

**Definition 4.1.** Let us suppose that the flow and the effort spaces may be decomposed as $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$, respectively. Furthermore, assume that $j_i : \mathcal{F}_i \to \mathcal{E}_i, i = 1, 2$ are unitary linear and invertible mappings. A linear subspace $\mathcal{D} \subset \mathcal{B}$ is called a split Tellegen (split Dirac) structure if it is a Tellegen (Dirac) structure in the sense of Definition 2.2 with $j = \left( \begin{array}{c} j_1 \\ j_2 \end{array} \right)$.

We are now in position to explicit what we mean by the interconnection of two (stochastic) Dirac structures. Therefore, let us consider $\mathcal{D}^A \subset \mathcal{B} = (\mathcal{F}_1 \times \mathcal{F}_2) \times (\mathcal{E}_1 \times \mathcal{E}_2)$ being a split Dirac structure of the form [21] and $\mathcal{D}^B \subset \mathcal{B} = (\mathcal{F}_3 \times \mathcal{F}_2) \times (\mathcal{E}_3 \times \mathcal{E}_2)$ being an arbitrary split Dirac structure where $\mathcal{F}_3$ and $\mathcal{E}_3$ are Hilbert spaces. As $\mathcal{D}^A$ is of the form [21], we define the spaces $\mathcal{F}_1, \mathcal{E}_1$ and $\mathcal{F}_2, \mathcal{E}_2$ as $\mathcal{F}_1 = \mathbb{R} \times \mathbb{R}^{n-p} = \mathcal{E}_1$ and $\mathcal{F}_2 = \mathbb{R}^p = \mathcal{E}_2$, $1 \leq p \leq n$, respectively. With the split Dirac structures $\mathcal{D}^A$ and $\mathcal{D}^B$, we define the unitary operators $j_1 : \mathcal{F}_1 \to \mathcal{E}_1, j_2 : \mathcal{F}_2 \to \mathcal{E}_2$ and $j_3 : \mathcal{F}_3 \to \mathcal{E}_2$ where $j_1$ and $j_2$ are given by

$$j_1 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{array} \right), \quad j_2 = -I_{\mathbb{R}^p}.$$ 

The proposed interconnection between $\mathcal{D}^A$ and $\mathcal{D}^B$ is inspired by [21], Definition 3.2 in which first-order PHSs are considered. It is performed via some of the boundary ports through the space $\mathcal{F}_2 \times \mathcal{E}_2$. The dimension of $\mathcal{F}_2$ gives the number of boundary ports used for the interconnection.

**Definition 4.2.** The composition of $\mathcal{D}^A$ and $\mathcal{D}^B$ is denoted $\mathcal{D}^A \circ \mathcal{D}^B \subset (\mathcal{F}_1 \times \mathcal{F}_3) \times (\mathcal{E}_1 \times \mathcal{E}_3)$ and is defined as

$$\mathcal{D}^A \circ \mathcal{D}^B := \left\{ \left( \begin{array}{c} \delta f^A \\ \delta f^B \\ \delta f_{\partial,n-p}^A \end{array} \right) \in (\mathcal{F}_1 \times \mathcal{F}_3) \times (\mathcal{E}_1 \times \mathcal{E}_3) \right\}$$

with $j = \left( \begin{array}{c} j_1 \\ j_2 \end{array} \right)$, where $\delta f^A = (\delta f^A_1, \delta f^A_2, \delta f_{\partial,n-p}^A), e^A = (e^A_1, e^A_2, e^A_{\partial,n-p})$.

An illustration of the proposed interconnection is given in Figure 1. In that way, it is easy to see that the pairing whose $\mathcal{D}^A \circ \mathcal{D}^B$ is equipped with, denoted $\langle \cdot, \cdot \rangle_t$, is expressed as

$$\langle \delta f^A_1, e^A_1 \rangle_t + \langle \delta f^A_2, e^A_2 \rangle_t + \langle \delta f_{\partial,n-p}^A, e^A_{\partial,n-p} \rangle_{\mathbb{R}^p} = \langle \delta f^B_1, e^B_1 \rangle_t + \langle \delta f^B_2, e^B_2 \rangle_t + \langle \delta f_{\partial,n-p}^B, e^B_{\partial,n-p} \rangle_{\mathbb{R}^p}$$

$$+ \langle \delta f_{\partial,n-p}^A, e^A_{\partial,n-p} \rangle_{\mathbb{R}^p} + \langle \delta f_{\partial,n-p}^B, e^B_{\partial,n-p} \rangle_{\mathbb{R}^p} + \langle e^A_1, \delta f^B_1 \rangle_{\mathbb{R}^p} + \langle e^A_2, \delta f^B_2 \rangle_{\mathbb{R}^p} + \langle e^A_{\partial,n-p}, \delta f^B_{\partial,n-p} \rangle_{\mathbb{R}^p}.$$ 

We shall now focus on the nature of the structure introduced in [21]. First let us consider the following lemma.

**Lemma 4.1.** The structure $\mathcal{D}^A \circ \mathcal{D}^B$ defined in (24) has zero power, i.e. $\langle \cdot, \cdot \rangle_t = 0$ for any $\delta \in \mathcal{D}^A \circ \mathcal{D}^B$. 

\footnote{From this definition of $j_1$ and $j_2$, there holds $j_1 = j_1^{-1}$ and $j_2 = j_2^{-1}$.}

\footnote{This definition of $j$ entails that $j_1^{-1} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right)$.

\footnote{The notation $\delta f_{\partial,n-p}$ is used to emphasize the fact that the vector $\delta f_{\partial,n-p}$ is of size $n-p$.}
Figure 1: Interconnection of the Dirac structures $\mathcal{D}^A$ and $\mathcal{D}^B$. The resulting structure is denoted by $\mathcal{D}^A \circ \mathcal{D}^B$. The interconnection is performed at the boundary thanks to the boundary ports $f_p$ and $e_p$.

**Proof.** Let us consider $\vartheta = (f) \in \mathcal{D}^A \circ \mathcal{D}^B$, where $f = (\delta f_A^e, \delta f_B^e, \delta f_{\partial,n-p}^A, f^B)$, $e = (e_A^e, e_B^e, e_{\partial,n-p}^A, e^B)$. Showing that $\langle \vartheta, \vartheta \rangle = 0$ is equivalent in showing that

$$
\langle \delta f_A^e, e_A^e \rangle - \langle \delta f_B^e, e_B^e \rangle - \langle \delta f_{\partial,n-p}^A, e_{\partial,n-p}^A \rangle - \langle f^B, j_3^{-1} e^B \rangle + \langle e_A^e, \delta f_A^e \rangle - \langle e_B^e, \delta f_B^e \rangle - \langle e_{\partial,n-p}^A, \delta f_{\partial,n-p}^A \rangle - \langle e^B, j_3 f^B \rangle = 0,
$$

see (26). According to the definition of $\mathcal{D}^A \circ \mathcal{D}^B$, there exist $f_p, e_p \in \mathbb{R}^p$ such that $(\delta f_A^e, \delta f_B^e, \delta f_{\partial,n-p}^A, f_p, e_A^e, e_B^e, e_{\partial,n-p}^A, e_p) \in \mathcal{D}^A$ and $(f^B, j_3, e^B, e_p) \in \mathcal{D}^B$. As $\mathcal{D}^A$ and $\mathcal{D}^B$ are split Dirac structures, they have zero power. In particular, there holds

$$
0 = \langle \delta f_A^e, e_A^e \rangle - \langle \delta f_B^e, e_B^e \rangle - \langle \delta f_{\partial,n-p}^A, e_{\partial,n-p}^A \rangle - \langle f_p, e_p \rangle + \langle e_A^e, \delta f_A^e \rangle - \langle e_B^e, \delta f_B^e \rangle - \langle e_{\partial,n-p}^A, \delta f_{\partial,n-p}^A \rangle - \langle e^B, j_3 f^B \rangle - \langle e_p, f_p \rangle,
$$

and

$$
0 = \langle f^B, j_3^{-1} e^B \rangle + \langle f_p, e_p \rangle + \langle e^B, j_3 f^B \rangle + \langle e_p, f_p \rangle.
$$

Making the sum of (27) and (28) implies (26).

This results is also known as the fact that the interconnection of split Tellegen structures remains a Tellegen structure, when the interconnection is expressed like it is in Definition 1.2 see [Kurula et al., 2010] (Corollary 3.9). This means that $\mathcal{D}^A \circ \mathcal{D}^B \subset (\mathcal{D}^A \circ \mathcal{D}^B)^\perp$ where the orthogonal complement $\perp$ has to be understood with respect to the new pairing $\langle \cdot, \cdot \rangle$. However, showing the inclusion $(\mathcal{D}^A \circ \mathcal{D}^B)^\perp \subset \mathcal{D}^A \circ \mathcal{D}^B$ poses delicate problems and comes with conditions since it depends on the structures $\mathcal{D}^A, \mathcal{D}^B$ and on the nature of the interconnection. In our setting, the following proposition holds.

**Proposition 4.1.** The structure $\mathcal{D}^A \circ \mathcal{D}^B$ defined in (1.2) is a split Dirac structure.

**Proof.** The inclusion $\mathcal{D}^A \circ \mathcal{D}^B \subset (\mathcal{D}^A \circ \mathcal{D}^B)^\perp$ holds from Lemma 1.1. The other inclusion follows by the fact that the space $\mathcal{F}_2 \times \mathcal{E}_2 = \mathbb{R}^p \times \mathbb{R}^p$ through which the interconnection takes place is finite-dimensional, see [Kurula et al., 2010] (Corollary 3.9).

**Remark 4.1.** 1. As it is highlighted in [Kurula et al., 2010] (Corollary 3.9), the dimensionality of the interconnection plays an important role in determining whether $\mathcal{D}^A \circ \mathcal{D}^B$ is a split Dirac structure or not. No conclusion could have been possible without a finite-dimensional space of interconnection $\mathcal{F}_2 \times \mathcal{E}_2$. In a more general case, one should consider the scattering operators describing each of the split Dirac structures $\mathcal{D}^A$ and $\mathcal{D}^B$, see [Kurula et al., 2010, Corollary 2.8]. Then, conditions on these scattering operators are proposed in [Kurula et al., 2010, Theorem 3.8] to ensure that $\mathcal{D}^A \circ \mathcal{D}^B$ is a split Dirac structure. This result should be of interest in the case where the interconnection is performed via Hilbert spaces-valued ports.

2. The definition of the structure $\mathcal{D}^B$ does not exclude stochastic ports. This could be envisaged as well through the spaces $\mathcal{F}_3$ and $\mathcal{E}_3$.

3. Without any loss of generality the problem of interconnections of multiple Dirac structures can be reduced to the problem of interconnection of two Dirac structures.

In terms of control practice, one usage of the Dirac structure consists in taking advantage of their nice geometric properties to design control laws for achieving certain goals via the interconnection of subsystems. Most of the current methods developed for the stabilization of infinite-dimensional port-Hamiltonian systems deal with boundary controllers, see [Rashad Hashem et al., 2020]. Generalization to the stochastic setting leads to even more difficulties.
as the noise of the plant cannot be controlled. In [Hadj- 
dad et al. 2018] noise was assumed to be vanishing at the 
equilibrium, which in practice would be quite restrictive in 
terms of configurations. Recently, theses restrictions were 
left in [Cordoni et al. 2022] for the generalization of en-
ergy shaping techniques using weaker concepts of Casimir 
function and passivity. The development of adapted con-
trol methods for infinite-dimensional SPHSs remains an 
uncultivated field. The authors believe that Proposition 
4.1 would open the way to the development of stabiliza-
tion method for infinite-dimensional SPHSs via Casimir 
generation or energy shaping approaches.

To illustrate Proposition 4.1, we study the example of a 
boundary controlled stochastic vibrating string de-
scribed by coupled SPDE-ODE. The string is assumed to 
be fixed at one extremity, free at the other, and subject to 
some stochastic damping. Moreover, we assume that 
some boundary conditions are dynamic. More particu-
larly, those are actuated by a mass-spring system. The 
dynamics of such a stochastic adaptive control system are 
written as

\[
\rho(z) \frac{\partial^2 z}{\partial t^2}(z, t) = \frac{\partial}{\partial z} \left( T(z) \frac{\partial z}{\partial z}(z, t) \right) - (R_\ell + \bar{w}(t)) \frac{\partial z}{\partial t},
\]

(29)

\[
\frac{\partial z}{\partial t}(a, t) = 0, \quad T(b) \frac{\partial z}{\partial z}(b, t) = 0,
\]

(30)

\[
\frac{1}{\sqrt{2}} \frac{\partial^2 z}{\partial t^2}(b, t) = p(t) m - \frac{1}{\sqrt{2}} T(a) \frac{\partial z}{\partial z}(a, t) = kq(t)
\]

(31)

where the control variables \( p \) and \( q \) are updated adaptively as

\[
\dot{p}(t) = -kq(t) + u(t),
\]

\[
\dot{q}(t) = \frac{1}{m} p(t),
\]

\[
y(t) = \frac{1}{m} \dot{p}(t),
\]

(32)

with \( k \) and \( m \) being positive parameters. Here, \( \rho(z) \) and 
\( T(z) \) are the mass density and the Young modulus of the 
string at position \( z \in [a, b] \). The variable \( t \geq 0 \) denotes 
the time and \( z(t) \) is the displacement of the string at 
\( z \in [a, b] \) and \( t \geq 0 \). The positive frictional damping pa-
rameter \( R_\ell \) is perturbated by a real-valued white noise \( \dot{w}(t) \)

\[
\text{covariance } \sigma^2. \text{ By considering } \epsilon_1(z, t) := \rho(z) \frac{\partial}{\partial z}(z, t) \text{ and } \epsilon_2 := \frac{\partial}{\partial z}(z, t) \text{ as the momentum and the strain,}
\]

respectively, the SDE (29) with the homogeneous boundary 
conditions (30) admits the following port-Hamiltonian for-
mulation

\[
\begin{align*}
\frac{\partial}{\partial t} = W_B f_0(t) & = P_1 \frac{\partial}{\partial t}(H_0(t) - C_0 H(t) + H(\dot{x}(t))(\dot{w}(t))), \\
\end{align*}
\]

(33)

with 
\[
W_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}
\]

\[
f_0(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} T(b) \frac{\partial}{\partial z}(b, t) - T(a) \frac{\partial}{\partial z}(a, t) \\
T(b) \frac{\partial}{\partial z}(b, t) - T(a) \frac{\partial}{\partial z}(a, t) \\
T(b) \frac{\partial}{\partial z}(b, t) + T(a) \frac{\partial}{\partial z}(a, t) \end{bmatrix},
\]

(34)

\[
e_0(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial}{\partial z}(b, t) + \frac{\partial}{\partial z}(a, t) \\
\frac{\partial}{\partial z}(b, t) + \frac{\partial}{\partial z}(a, t) \\
\frac{\partial}{\partial z}(b, t) - \frac{\partial}{\partial z}(a, t) \end{bmatrix}.
\]

The Hamiltonian operator \( \mathcal{H} \), the matrix \( P_1 \) and the 
matrix \( G_0 \) are respectively given by

\[
\mathcal{H}(z) = \begin{bmatrix} \frac{\partial}{\partial z} & 0 \\ 0 & T(z) \end{bmatrix}, P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad G_0 = \begin{bmatrix} R_\ell & 0 \\ 0 & 0 \end{bmatrix}.
\]

As it is made in [Lamoline 2021], we set \( Z = \mathbb{R} \). In that 
way, \( H \in \mathcal{L}(X, \mathcal{L}(Z, X)) \) with \( H(\dot{x}) = [-\epsilon_1, 0]^T \) for all 
\( \dot{x} \in X \). Here we consider \( X = \mathcal{L}^2([a, b]; \mathbb{R}^2) \) as en-
ergy space. It is then easy to see that (33) defines a Dirac 
structure thanks to Theorem 3.1. Let us now focus on (32) 
with the boundary conditions (31). The system (32) may 
be regarded as a mass-spring system (harmonic oscillator) 
with \( q \) and \( p \) being the deviation from the zero position 
and the momentum, respectively. The first equation of 
(32) is due to the force and the second equation is for the 
velocity. By defining the associated potential and kinetic 
energies as

\[
H_p = \frac{1}{2} kq^2, \quad H_c = \frac{1}{2m}p^2,
\]

(35)

system (32) admits a port-Hamiltonian formulation in 
terms of the following Dirac structure

\[
\mathcal{D}_c := \left\{ \begin{bmatrix} f_1 & u & f_2 & e_1 & y \\ f_2 & -y & -e_1 & -u \end{bmatrix} \right\} \in \mathbb{R}^3 \times \mathbb{R}^3, \quad f_2 = -e_1 = -y,
\]

(36)

where the variables \( f_1 = -\dot{p}, f_2 = -\dot{q} \) while the variables 
\( e_1 \) and \( e_2 \) are the derivatives of the Hamiltonian \( H_p + H_c \) 
with respect to \( p \) and \( q \), respectively. The variables \( u \) 
and \( y \) are external input and output whose objective could 
be the stability of the closed-loop system (29)–(32) for 
instance. As a particularity of Dirac structure, note that 
the power associated to \( \mathcal{D}_c \) is zero, i.e. \( f_1 e_1 + f_2 e_2 + 
u y = 0 \). Now remark that (29)–(32) may be regarded as the 
interconnection of the homogeneous stochastic port-
Hamiltonian system (33) with the Dirac structure \( \mathcal{D}_c \) 
in the following way

\[
\begin{bmatrix} f_{0,2}(t) = -f_2(t), \\
e_{0,2}(t) = e_2(t), \end{bmatrix}
\]

(37)

where \( f_{0,2} \) and \( e_{0,2} \) are the second components of the vari-
ables defined in (34). The interconnection (37) is the same 
as the one performed in (12), which, thanks to Proposi-
tion 1.1, implies that the controlled and observed system 
(29)–(32) may be written as a Dirac structure. In par-
cular, as an interesting feature, the power of the total 
system (29)–(32) is zero.

5. Conclusion & perspectives

In this work we introduced and studied the notion of 
Dirac structure for stochastic port-Hamiltonian systems 
with multiplicative Gaussian white noise. Taking advan-
tage of the nice geometrical properties of the Stratonovich 
formalism, the Dirac structure for deterministic infinite-
dimensional PHSSs as studied in [Kurula et al. 2010] was
extended to a stochastic setting. We showed that a newly defined subset of the Cartesian product between extended effort and flow spaces related to a class of SPHSs forms a Dirac structure. As an illustration, we showed that the system composed of a stochastic vibrating string and a mass-spring damper forms a Dirac structure, when interconnected in a power-conserving way. These results should be considered as a first step towards the development of boundary controllers of SPHSs.

This work opens the way to further research questions and investigations. It would be of great interest to generalize the Dirac structure proposed here for SPHSs by considering various sources of noise entering such as boundary and interconnection noises. Moreover, higher-order stochastic port-Hamiltonian systems will also be considered by the authors in future works.

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