Differential forms on locally convex spaces and
the Stokes formula

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Abstract

We prove a version of the Stokes formula for differential forms on
locally convex spaces announced in [10]. The main tool used for prov-
ing this formula is the surface layer theorem proved in the paper [6] by
the author. Moreover, for differential forms of a Sobolev-type class rel-
relative to a differentiable measure [1], we compute the operator adjoint
to the exterior differential in terms of standard operations of calculus
of differential forms and the logarithmic derivative. Previously, this
connection was established under essentially stronger assumptions on
the space [8], the measure [7], or smoothness of differential forms [5].
See also [4].

1. Calculus on a Sobolev-type class of differential forms
on a locally convex space

Let $H$ be a Hilbert space with the scalar product $(\cdot, \cdot)$, $\{e_n\}_{n=1}^{\infty}$ be an or-
thonormal basis of $H$. Let $\Gamma(n)$ denote the set of increasing sequences of
natural numbers of length $n \in \mathbb{N}$, $\Gamma(0) = \{0\}$. If $\gamma_1 \in \Gamma(n)$, $\gamma_2 \in \Gamma(m)$,
$n \geq m$, then we consider the sequences $\gamma_1 \cup \gamma_2$ and $\gamma_1 \setminus \gamma_2$ as elements of
$\Gamma(m + n)$ and $\Gamma(n - m)$ respectively, putting them in the increasing order,
if necessarily. For every $n \in \mathbb{N}$ and $\gamma = (i_1, \ldots, i_n) \in \Gamma(n)$, the symbol $e_\gamma$
denotes $e_{i_1} \wedge \ldots \wedge e_{i_n}$, $e_0 = 1$, where the vectors $e_{i_1}, \ldots, e_{i_n}$ are considered as
linear continuous functionals on $H$ (the operation $\wedge$ is defined, for example,
in [2]). By $L_n(H)$ we denote the space of antisymmetric $n$-linear Hilbert-
Schmidt functionals. Note that $L_n(H)$ is a Hilbert space with the scalar
product \((g_1, g_2)_n = \sum_{i_1 < \cdots < i_n} g_1(e_{i_1}, \ldots, e_{i_n}) g_2(e_{i_1}, \ldots, e_{i_n})\), and \(\{e_\gamma\}_{\gamma \in \Gamma(n)}\) is the orthonormal basis in \(L_n(H)\). Let \(\| \cdot \|_n\) denote the norm which corresponds to \((\cdot, \cdot)_n\). Let us show that if \(f \in L_n(H)\), \(g \in L_m(H)\), then \(f \land g \in L_{m+n}(H)\). Indeed, let \(e_\gamma = e_{i_1} \land \ldots \land e_{i_n}\), and let \(g(e_\gamma)\) denote \((g, e_\gamma)_n = g(e_{i_1}, \ldots, e_{i_n})\). We obtain:

\[
\sum_{\gamma \in \Gamma(m+n)} (f \land g)^2(e_\gamma) = \sum_{\gamma \in \Gamma(m+n)} \left( \sum_{\gamma_1 \in \Gamma(n)} \varepsilon(\sigma) f(e_{\gamma_1}) g(e_{\gamma_\gamma_1}) \right)^2 \\
\leq C_{m+n}^m \sum_{\gamma \in \Gamma(m+n)} \sum_{\gamma_1 \in \Gamma(n)} f^2(e_{\gamma_1}) g^2(e_{\gamma_\gamma_1}) = C_{m+n}^m \sum_{\gamma_1 \in \Gamma(n)} f^2(e_{\gamma_1}) g^2(e_{\gamma_2}) \\
\leq C_{m+n}^m \sum_{\gamma_1 \in \Gamma(n)} f^2(e_{\gamma_1}) \sum_{\gamma_2 \in \Gamma(m)} g^2(e_{\gamma_2}) < \infty.
\]

Analogously to the finite dimensional case [11], for elements \(f \in L_m(H)\) and \(g \in L_n(H)\), \(m > n\), one can define the element \(g \downarrow f \in L_{m-n}(H)\) by the formula \((g \downarrow f, h)_{m-n} = (f, g \land h)_n\), which holds for all \(h \in L_{m-n}(H)\). Let us show that the operation \(\downarrow\) is well defined for \(f \in L_m(H)\) and \(g \in L_n(H)\). Specifically, we have to show that

\[
\sum_{\gamma \in \Gamma(m-n)} (g \downarrow f, e_\gamma)_{m-n}^2 = \sum_{\gamma \in \Gamma(m-n)} (f, g \land e_\gamma)_{m}^2 < \infty.
\]

Note that

\[
(g \land e_\gamma)(e_\gamma') = \sum_{\substack{\gamma_1 \in \Gamma(n), \\ \gamma_2 \in \Gamma(m-n): \\ \gamma_1 \cup \gamma_2 = \gamma}} \varepsilon(\sigma) g(e_{\gamma_1}) e_{\gamma_2} = \begin{cases} 
0, & \gamma \not\subset \gamma' \\
\varepsilon(\sigma) g(e_{\gamma \\setminus \gamma'}), & \gamma \subset \gamma'
\end{cases}
\]

where \(\varepsilon(\sigma)\) is the permutation parity of \(\sigma\). Also, we used here the definition of the exterior multiplication [2]. Taking into account the latter relation, we obtain:

\[
\sum_{\gamma \in \Gamma(m-n)} (f, g \land e_\gamma)_{m}^2 = \sum_{\gamma \in \Gamma(m-n)} \left( \sum_{\gamma' \in \Gamma(m)} f(e_{\gamma'}) \left( g \land e_\gamma)(e_{\gamma'}) \right) \right)^2 =
\]
\[
\sum_{\gamma \in \Gamma(m-n)} \left( \sum_{\gamma' \in \Gamma(m) : \gamma \subset \gamma'} \varepsilon(\sigma) f(e_{\gamma'}) g(e_{\gamma' \setminus \gamma}) \right)^2 \\
\leq \sum_{\gamma \in \Gamma(m-n)} \left( \sum_{\gamma_1 \in \Gamma(n) : \gamma \cap \gamma_1 = \emptyset} f^2(e_{\gamma \cup \gamma_1}) \sum_{\gamma_1 \in \Gamma(n) : \gamma \cap \gamma_1 = \emptyset} g^2(e_{\gamma_1}) \right) \\
\leq \sum_{\gamma_1 \in \Gamma(n)} g^2(e_{\gamma_1}) \sum_{\gamma \in \Gamma(m-n), \gamma \cap \gamma_1 = \emptyset} f^2(e_{\gamma \cup \gamma_1}) \leq C_m \sum_{\gamma_1 \in \Gamma(n)} g^2(e_{\gamma_1}) \sum_{\gamma_2 \in \Gamma(m)} f^2(e_{\gamma_2}) < \infty.
\]

Now let \( X \) be a locally convex space, and the Hilbert space \( H \) be a vector subspace of \( X \).

**Definition 1.** A mapping \( f : X \to L_n(H) \) is called a differential form of degree \( n \) (or differential \( n \)-form) on \( X \).

Note that every differential form \( f \) can be presented as: \( f = \sum_{\gamma \in \Gamma(n)} f_\gamma e_\gamma \), where \( f_\gamma \) are real-valued functions. The operations of exterior and interior multiplications are defined for differential forms pointwise.

Let \( f \) be a differential form of degree \( n \).

**Definition 2.** We say that \( f \) possesses a differential if its coefficients \( f_\gamma \) are differentiable in each direction \( e_p \) for \( p \notin \gamma \), and for all \( x \in X \),

\[
\sum_{\gamma \in \Gamma(n), p \notin \gamma} d_{e_p} f_\gamma(x) e_p \wedge e_\gamma \in L_{n+1}(H).
\]

The differential \((n+1)\)-form \( df = \sum_{\gamma \in \Gamma(n), p \notin \gamma} d_{e_p} f_\gamma e_p \wedge e_\gamma \) is called the differential of \( f \).

**Definition 3.** We say that \( f \) possesses a codifferential if its coefficients \( f_\gamma \) are differentiable in each direction \( e_p \) for \( p \in \gamma \), and for all \( x \in X \),

\[
\sum_{p \in \gamma \in \Gamma(n)} d_{e_p} f_\gamma(x) e_p \perp e_\gamma \in L_{n-1}(H).
\]

The differential \((n-1)\)-form \( \delta f = \sum_{p \in \gamma \in \Gamma(n)} d_{e_p} f_\gamma e_p \perp e_\gamma \) is called the codifferential of \( f \).
Let $\mathcal{B}_X$ be the $\sigma$-algebra of Borel subsets of the space $X$. A measure on $X$ means a $\sigma$-additive Hilbert space valued function on $\mathcal{B}_X$.

**Definition 4.** A $\sigma$-additive $L_n(H)$-valued measure on $X$ is called a differential form of codegree $n$.

Every differential form $\omega$ of codegree $n$ can be decomposed as: $\omega = \sum_{\gamma \in \Gamma(n)} \omega_{\gamma} e_{\gamma}$ where $\omega_{\gamma}$ are real-valued $\sigma$-additive measures.

**Definition 5.** Let $g$ be a bounded differential form of degree $m$, $\omega$ be a differential form of codegree $n \geq m$ which is a measure of bounded variation. The differential form $g \wedge \omega$ of codegree $n - m$ defined as

$$(g \wedge \omega)(A) = \int_A g(x) \lhd \omega(dx)$$

is called the exterior product of $g$ and $\omega$.

The differential form $g \wedge \omega$ is well defined. Indeed, the differential form $\omega$ can be presented in the form $\omega = f \cdot |\omega|$ (see [3]), where $|\omega|$ denotes the variation of $\omega$, and $f$ is a differential form of degree $n$ such that $\|f(x)\|_n = 1$ for $|\omega|$-almost all $x$. We have

$$(g \wedge \omega)(A) = \int_A (g(x) \lhd f(x)) |\omega|(dx).$$

Further,

$$\sum_{\gamma \in \Gamma(n-m)} ((g \wedge \omega)(A), e_{\gamma})_{n-m}^2 = \sum_{\gamma \in \Gamma(n-m)} \left(\int_A (g(x) \lhd f(x), e_{\gamma})_{n-m} |\omega|(dx)\right)^2$$

$$\leq |\omega|(A) \sum_{\gamma \in \Gamma(n-m)} \int_A (g(x) \lhd f(x), e_{\gamma})_{n-m}^2 |\omega|(dx)$$

$$= |\omega|(A) \int_A \|g(x) \lhd f(x)\|_{n-m}^2 |\omega|(dx) \leq C_n^m (|\omega|(A))^2 \sup_x \|g(x)\|_m < \infty.$$ 

**Definition 6.** We say that the differential form $\omega$ of codegree $n$ possesses a differential if its coefficients $\omega_{\gamma}$ are differentiable in all directions $e_{p}$ for $p \in \gamma$. 

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and \( \sum_{p \in \gamma \in \Gamma(n)} d_{e_p} \omega_\gamma(A) e_p \wedge e_\gamma \in L_{n-1}(H) \) for all \( A \in \mathcal{B}_X \). The differential form \( d\omega \) of codegree \( n-1 \) defined by

\[
d\omega = (-1)^{n-1} \sum_{p \in \gamma \in \Gamma(n)} d_{e_p} \omega_\gamma e_p \wedge e_\gamma,
\]

is called the differential of \( \omega \).

**Lemma 1.** Let \( g \) and \( \omega \) be differential forms of degree \( m \) and codegree \( n+1 > m \) respectively, both possess differentials. Further let \( g \) and \( \omega \) be such that \( g \) and \( dg \) are bounded, \( \omega \) and \( d\omega \) are of bounded variation. Then the differential form \( g \wedge \omega \) possesses a differential, and

\[
d(g \wedge \omega) = g \wedge d\omega + (-1)^n dg \wedge \omega. \tag{1}
\]

The equality (1) can be easily obtained. Indeed, one should use the definitions of differentials for \( g \) and \( \omega \), the definition of the operation \( \wedge \), and compare the coefficients at each \( e_\gamma \).

**2. The operator adjoint to the differential**

Now we compute the operator adjoint to the operator \( d \) for differential forms of a Sobolev-type class relative to a real- or complex-valued \( \sigma \)-additive measure on \( X \). Let \( \mu \) be such a measure. We assume that \( \mu \) is differentiable in each direction \( e_p \) [1], and for all \( x \), \( \| \beta_\mu(x) \|_H < \infty \), where \( \beta_\mu(x) = \sum_p \beta_{e_p}(x) e_p \), and \( \beta_{e_p} \) is the logarithmic derivative of the measure \( \mu \) in the direction \( e_p \) [1, 9].

Further let the numbers \( p > 1 \) and \( q > 1 \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). By \( \Omega^n_p \), we denote the vector space of differential \( n \)-forms \( f \) satisfying the condition \( \int_X \| f(x) \|_p \mu(dx) < \infty \). Define a norm on \( \Omega^n_p \) by

\[
\| f \|_{n,p} = \left( \int_X \| f(x) \|_p^n \mu(dx) \right)^{1/p}.
\]

For elements \( f = \sum_{\gamma \in \Gamma(n)} f_\gamma(x) e_\gamma \in \Omega^n_p \) and \( \omega = \sum_{\gamma \in \Gamma(n)} \omega_\gamma(x) e_\gamma \in \Omega^n_q \), we define the bilinear operation

\[
\langle \omega, f \rangle_n = \int_X (\omega(x), f(x))_n \mu(dx). \tag{2}
\]
The integral on the right-hand side exists by Hölder’s inequality and by
the definition of $\Omega_p^n$. By the definition of the scalar product $(\cdot,\cdot)_n$ and by
Lebesgue’s theorem, we rewrite (2):

$$\langle \omega, f \rangle_n = \sum_{\gamma \in \Gamma(n)} \int_X f_\gamma(x) \omega_\gamma(x) \mu(dx).$$

Let $A_p^n$ be the vector subspace of $\Omega_p^n$, consisting of differential forms $f$
possessing the codifferential $\delta f \in \Omega_p^{n-1}$, satisfying the inequality

$$\int_X \|\beta^n(x)\|_p \|f(x)\|_p \mu(dx) < \infty, \quad (3)$$

and such that the following condition (i) is fulfilled. Condition (i): for every
$\gamma \in \Gamma(n)$, there exists a $\delta > 0$ and non-negative functions $g_\gamma(x), g_{1\gamma}(x),$ and
$g_{2\gamma}(x)$, such that $g_\gamma(x)$ is $d_\gamma \mu$-summable for every $p \notin \gamma, g_{1\gamma}(x)$ and $g_{2\gamma}(x)$ are
$\mu$-summable, and for all $p \notin \gamma$, for $|t| < \delta, |f_\gamma(x + te_p)| < \min\{g_\gamma(x), g_{1\gamma}(x)\}$
and $|d_\gamma f(x + te_p)| < g_{2\gamma}(x)$. On $A_p^n$ we define the norm

$$\|f\|_{A_p^n} = \|f\|_{n,p} + \|\delta f\|_{n-1,p} + \left(\int_X \|\beta^n(x)\|_p \|f(x)\|_p \mu(dx)\right)^{1/p}.$$  

Further let $B_q^n$ be the vector subspace of $\Omega_q^n$ consisting of differential forms $\omega$
possessing the differential $d\omega \in \Omega_q^{n+1}$ and satisfying the following condition
(ii). Condition (ii): for every $\gamma \in \Gamma(n)$ there exists a $\delta > 0$ and non-negative functions $g_\gamma(x), g_{1\gamma}(x), g_{2\gamma}(x)$, such that $g_{2\gamma}(x)$ $d_\gamma \mu$-summable for all $q \in \gamma, g_{2\gamma}(x)$ and $g_{2\gamma}(x)$ are $\mu$-summable, and for all $q \in \gamma$, for $|t| < \delta, |\omega_\gamma(x + te_q)| < \min\{g_\gamma(x), g_{1\gamma}(x)\}$ and $|d_\gamma \omega_\gamma(x + te_q)| < g_{2\gamma}(x)$. On $B_q^n$ we define the norm

$$\|\omega\|_{B_q^n} = \|\omega\|_{n,q} + \|d\omega\|_{n+1,q}.$$  

It is clear that $d : B_q^n \rightarrow \Omega_q^{n+1}$ is a linear continuous operator. Conditions (i)
and (ii) are necessary to satisfy the assumptions of the integration by parts
formula proved in [1] which we apply to compute the operator $d^*.$

**Theorem 1.** For every pair of elements $f \in A_p^{n+1}$ and $\omega \in B_q^n, 1/p + 1/q = 1,$
the element $-\beta^\mu \triangledown f - \delta f$ belongs to $\Omega_p^n,$ and

$$\langle d\omega, f \rangle_{n+1} = \langle \omega, -\beta^\mu \triangledown f - \delta f \rangle_n,$$

i.e. the adjoint operator $d^* : A_p^{n+1} \rightarrow \Omega_p^n$ is represented by the formula:

$$d^* = -(\beta^\mu \triangledown + \delta).$$
Proof. Let $\omega = \sum_{\gamma \in \Gamma(n)} \omega_\gamma e_\gamma$, $f = \sum_{\gamma_1 \in \Gamma(n+1)} f_{\gamma_1} e_{\gamma_1}$. We have

$$
\parallel \beta^\mu(x) \parallel_n \parallel f(x) \parallel_n
$$

$$
= \parallel \sum_{p \in \Gamma(n+1)} \beta^\mu_{ep}(x) f_{\gamma_1}(x) e_p \parallel_n
$$

$$
= \parallel \sum_{\gamma \in \Gamma(n)} \gamma \notin \Gamma(n+1)} \beta^\mu_{ep}(x) f_{\gamma_1}(x) e_p \parallel_n
$$

$$
= \sqrt{\sum_{\gamma \in \Gamma(n)} \sum_{p \in \Gamma(n+1)}} \beta^\mu_{ep}(x) f_{\gamma_1}(x) e_p \parallel_n
$$

$$
\leq \sqrt{\sum_{\gamma \in \Gamma(n)} \sum_{p \in \Gamma(n+1)}} \beta^\mu_{ep}(x) f_{\gamma_1}(x) e_p \parallel_n
$$

$$
= \sqrt{n+1} \parallel \beta^\mu(x) \parallel_n \parallel f(x) \parallel_n + 1. \tag{4}
$$

From this and (3) it follows that $\parallel \beta^\mu \parallel f \parallel_{n,p} < \infty$. By (3), $\parallel \delta f \parallel_{n,p} < \infty$, and hence, $\parallel \beta^\mu \parallel f + \delta f \parallel_{n,p} < \infty$, i.e. $-\beta^\mu \parallel f - \delta f \in \Omega_p^n$. Next,

$$
d\omega = \sum_{\gamma \in \Gamma(n), p \in \Gamma(n+1)} d_{ep} \omega_\gamma e_p \wedge e_\gamma = \sum_{\gamma \in \Gamma(n), p \in \Gamma(n+1)} (-1)^{k_{p-1}} d_{ep} \omega_\gamma e_{\gamma_1}$$

where $k_p$ is the number of $p$ in the sequence $\gamma_1$. Applying the integration by parts formula [1] (one can easily verify the conditions under which this formula holds), we obtain:

$$
\langle d\omega, f \rangle_n = \sum_{\gamma_j \in \Gamma(n+1)} \int_{\chi} f_{\gamma_j}(x) \sum_{\gamma \in \Gamma(n+1)} (-1)^{k_{p-1}} d_{ep} \omega_\gamma(x) \mu(dx)
$$

$$
= \sum_{\gamma_j \in \Gamma(n+1), \gamma \in \Gamma(n+1), \gamma \in \Gamma(n) \setminus \gamma} (-1)^{k_{p-1}} \int_{\chi} f_{\gamma_j}(x) d_{ep} \omega_\gamma(x) \mu(dx)
$$

$$
= \sum_{\gamma_j \in \Gamma(n+1), \gamma \in \Gamma(n+1), \gamma \in \Gamma(n) \setminus \gamma} (-1)^{k_{p-1}} \int_{\chi} \omega_\gamma(x) (d_{ep} f_{\gamma_j}(x) + f_{\gamma_j}(x) \beta^\mu_{ep}(x)) \mu(dx) =
$$
\[= - \int_X \sum_{\gamma \in \Gamma(n), \gamma \neq \gamma} (-1)^{k_p-1} \omega_\gamma(x) \left( d_{e_p} f_{\gamma \cup p}(x) + f_{\gamma \cup p}(x) \beta_{e_p}^\mu(x) \right) \mu(dx). \tag{5} \]

We changed the order of summation and integration when passing to the latter expression in (5), and applied Lebesgue’s theorem. Clearly, the sequence of partial sums under the last integral sign in (5) is majorized by an integrable function. This follows from the definition of the spaces \( A_{p+1}^n \) and \( B_q^n \), from Cauchy-Bunyakovsky-Schwarz’s inequality, and from the inequality

\[
\sqrt{\sum_{\gamma \in \Gamma(n)} \left( \sum_{p \not\in \gamma} \beta_{e_p}^\mu(x) f_{\gamma \cup p}(x) \right)^2} \leq \sqrt{n+1} \| \beta^\mu(x) \|_H \| f(x) \|_{n+1},
\]

which was proved, in fact, together with the estimate (4). By the same argument, all the series in (5) converge absolutely, and hence the order of summation of these series can be chosen arbitrary. We rewrite the expressions for \( \delta f \) and \( \beta^\mu \uplus f \):

\[
\delta f(x) = \sum_{p \in \gamma_1 \in \Gamma(n+1)} d_{e_p} f_{\gamma_1}(x) e_p \uplus e_{\gamma_1} = \sum_{\gamma \in \Gamma(n)} \left( \sum_{p \not\in \gamma} (-1)^{k_p-1} d_{e_p} f_{\gamma \cup p}(x) \right) e_\gamma,
\]

\[
\beta^\mu(x) \uplus f(x) = \sum_{p \in \gamma_1 \in \Gamma(n+1)} f_{\gamma}(x) \beta_{e_p}^\mu(x) e_p \uplus e_{\gamma_1} = \sum_{\gamma \in \Gamma(n), p \not\in \gamma} (-1)^{k_p-1} f_{\gamma \cup p}(x) \beta_{e_p}^\mu(x) e_\gamma.
\]

This and (5) imply the statement of the theorem. \( \square \)

3. The Stokes formula

3.1 Assumptions and notation

As before, let \( X \) be a locally convex space, \( H \) be its vector subspace which is a Hilbert space relative to the scalar product \( \langle \cdot, \cdot \rangle \), and let \( \{ e_n \}_{n=1}^\infty \) be an orthonormal basis of \( H \). We assume that \( H \) is dense in \( X \) and the identical embedding of \( H \) into \( X \) is continuous. Let \( \Xi_n, n \in N \), denote the vector space of bounded differential forms of degree \( n \) differentiable along \( H \) and possessing bounded differentials. Let \( S_n \) denote the space of differential forms of codegree \( n \) which are Radon measures differentiable along \( H \). Also, we
assume that the differential forms from \( S_n \) and their differentials are measures of bounded variation. Let \( \bar{S}_n \) and \( \bar{\Xi}_n \), \( n \in \mathbb{N} \), denote pseudo-topological vector spaces of linear continuous functionals on \( \Xi_n \) and \( S_n \), respectively. We assume that \( \bar{S}_n \) and \( \bar{\Xi}_n \) contain \( S_n \) and \( \Xi_n \) as dense subsets.

In addition to that, we assume that the mapping \( d : \Xi_0 \to \Xi_1 \) can be extended to a continuous mapping \( \bar{\Xi}_0 \to \bar{\Xi}_1 \), and the mapping \( d : S_1 \to S_0 \), to a continuous mapping \( \bar{S}_1 \to \bar{S}_0 \). Further we assume that for every measure \( \nu \in S_0 \), the mapping \( \Xi_1 \to S_1 \), \( f \mapsto f
\nu \) can be extended to a continuous mapping \( \bar{\Xi}_1 \to \bar{\Xi}_1 \), and the mapping \( \Xi_1 \times S_1 \to S_0 \), \( (f, \omega) \mapsto f \wedge \omega \), to a continuous mapping \( \bar{\Xi}_1 \times \bar{\Xi}_1 \to \bar{\Xi}_0 \). We denote the extended mappings by the same symbols. When the functional \( f \in \bar{S}_1 \) acts on the element \( g \in \Xi_1 \), we write \( \langle f, g \rangle \). Further let us assume that every sequence of elements from \( \Xi_n \), \( n = 1, 2 \), converging pointwise to an element from \( \bar{\Xi}_n \), converges to this element also with respect to the \( \bar{\Xi}_n \)-topology.

Let \( V \) be a domain in \( X \) such that its boundary \( \partial V \) can be covered with a finite number of surfaces \( U_i \) of codimension 1. Everywhere below, a surface of codimension 1 is the object defined in [6], p. 552. We assume that the indicator \( \mathbb{I}_V \) of the domain \( V \) is an element of the space \( \bar{\Xi}_0 \). Let the sets \( U_i \) covering \( \partial V \) and their intersections possess the property (\( \ast \)) formulated in [6], p. 559. In what follows, we will use the notations introduced in [6]. Here we briefly repeat their meaning:

\[ n \partial V : \partial V \to H \] is the normal vector (with respect to the \( H \)-topology) to \( \partial V \); let \( B \subset \partial V \) be a Borel subset, then \( B^\varepsilon = \{ y \in X : y = x + tn^{\partial \nu}(x), x \in B, |t| < \varepsilon \} \) is the \( \varepsilon \)-layer of \( B \); \( \varepsilon_b \) is the maximal number for which \( \varepsilon_b \)-layers are well defined; \( \nu^\varepsilon \) is a measure on \( \partial V \), \( \nu^\varepsilon(B) = \frac{\nu(B^\varepsilon)}{2\varepsilon} \); \( P^\varepsilon : (\partial V)^{\varepsilon_b} \to \partial V : x + tn^{\partial \nu}(x) \mapsto x \) means the projector of \( \varepsilon \)-layers to the surface; \( \nu^{\partial \nu} \) is the surface measure generated by the measure \( \nu \) ([12], [6]). Rigorous definitions of these objects as well as lemmas proving their existence are given in [6]. Note that by Theorem 2 of [6], \( \lim_{\varepsilon \to 0} \nu^\varepsilon(\partial V) = \nu^{\partial \nu}(\partial V) \).

3.2 A connection between a measure and the generated surface measure in terms of differential forms

**Theorem 2.** Let \( \nu \in S_0 \). Then \( d\mathbb{I}_V \cdot \nu \) is an \( H \)-valued Radon measure on \( X \) concentrated on \( \partial V \). Moreover, \( n^{\partial \nu} \cdot \nu^{\partial \nu} \in \bar{S}_1 \), and the measures \( \nu \) and \( \nu^{\partial \nu} \) are related through the identity:

\[ d\mathbb{I}_V \cdot \nu = -n^{\partial \nu} \cdot \nu^{\partial \nu}. \tag{6} \]
Note that by assumption, \( dI_N \in \bar{\Xi}_1 \) and \( dI_N \cdot \nu \in \bar{S}_1 \).

**Proof.** Let \( h^\varepsilon : (-\varepsilon_b, \varepsilon_b) \rightarrow [0, 1], \varepsilon < \varepsilon_b, \)

\[
h^\varepsilon(\tau) = \begin{cases} 
-\frac{\tau}{2\varepsilon} + \frac{1}{2}, & \text{if } \tau \in (-\varepsilon - \varepsilon^2, \varepsilon - \varepsilon^2), \\
1, & \text{if } \tau \in (-\varepsilon_b, -\varepsilon), \\
0, & \text{if } \tau \in (\varepsilon, \varepsilon_b), 
\end{cases}
\]

be \( C^\infty \)-smooth functions such that on the intervals \((-\varepsilon, -(\varepsilon - \varepsilon^2))\) and \((\varepsilon - \varepsilon^2, \varepsilon)\), the absolute values of their derivatives change monotonically from 0 to \(\frac{1}{2\varepsilon}\), and from \(\frac{1}{2\varepsilon}\) to 0, respectively. For \(\varepsilon < \varepsilon_b\), we define the functions \(f^\varepsilon : X \rightarrow \mathbb{R}, \)

\[
f^\varepsilon(x) = \begin{cases} 
h^\varepsilon(\tau), & \text{if } x = y + \tau n^{ov}(y), y \in \partial V, \tau \in (-\varepsilon_b, \varepsilon_b), \\
1, & \text{if } x \in V \setminus (\partial V)^{\varepsilon_b}, \\
0, & \text{if } x \notin V \cup (\partial V)^{\varepsilon_b}.
\end{cases}
\]

Let us calculate

\[
\frac{d}{dt} f^\varepsilon(x + \tau n^{ov}(x)) = \left. \frac{d}{dt} f^\varepsilon(x + \tau n^{ov}(x) + t e_p) \right|_{t=0}
\]

for \(\tau \in (-\varepsilon - \varepsilon^2, \varepsilon - \varepsilon^2)\) and \(x \in \partial V\). If \(t\) is sufficiently small, then there exist \(x_t\) and \(\tau_t\), such that

\[
x + \tau n^{ov}(x) + t e_p = x_t + \tau_t n^{ov}(x_t).
\]

Let \(n^{ov}_p\) be coordinates of the vector \(n^{ov}\) in the basis \(\{e_p\}_{p=1}^\infty\). Subtracting \(x_t\) from the both sides of (8), and multiplying by \(n^{ov}(x)\), we obtain:

\[
(x - x_t, n^{ov}(x)) + \tau + t n^{ov}_p(x) = \tau_t (n^{ov}(x), n^{ov}(x_t)).
\]

Hence,

\[
\tau_t = \frac{tn^{ov}_p(x)}{(n^{ov}(x), n^{ov}(x_t))} + \frac{\tau + (x - x_t, n^{ov}(x))}{(n^{ov}(x), n^{ov}(x_t))}.
\]

Note that \(x_t = P_{ov}(x + \tau n^{ov}(x) + t e_p)\). We can prove that the derivative \(\frac{d}{dt} x_t|_{t=0}\) exists with respect to the \(H\)-topology similarly to how it was done in the proof of Lemma 6 of \cite{6}. From the results of \cite{6} (Lemmas 1 and 2), it follows that the derivative \(\frac{d}{dt} n^{ov}(x_t)|_{t=0}\) exists with respect to the \(H\)-topology.
as well. Taking into account this, we show that \( \frac{d}{dt}(n^{\partial V}(x), n^{\partial V}(x_t)) \bigg|_{t=0} = 0 \) and \( \frac{d}{dt}(x - x_t, n^{\partial V}(x)) \bigg|_{t=0} = 0 \). The latter identity is obvious since \( \frac{d}{dt} x_t \bigg|_{t=0} \in H_x \), and \( n^{\partial V}(x) \) is orthogonal to \( H_x \), where \( H_x \) is the intersection of the tangent space at \( x \in \partial V \) with \( H \) (see [6]). Further we have:

\[
0 = \frac{d}{dt} \| n^{\partial V}(x_t) \|^2 \bigg|_{t=0} = 2 \left( n^{\partial V}(x), \frac{d}{dt} n^{\partial V}(x_t) \bigg|_{t=0} \right) = 2 \frac{d}{dt}(n^{\partial V}(x), n^{\partial V}(x_t)) \bigg|_{t=0}.
\]

From this and from (9), it follows that \( \frac{d}{dt} \tau_t \bigg|_{t=0} = n^{\partial V}_p(x) \). Taking into account that \( f^\varepsilon(x + \tau n^{\partial V}(x + t e_p)) = -\frac{n^{\partial V}_p(x)}{2 \varepsilon} + \frac{1}{2} \), we obtain that \( d_{e_p} f^\varepsilon(x + \tau n^{\partial V}(x)) = -\frac{n^{\partial V}_p(x)}{2 \varepsilon} \), and hence,

\[
df^\varepsilon(x + \tau n^{\partial V}(x)) = -\frac{n^{\partial V}_p(x)}{2 \varepsilon}.
\]

(10)

For \( \tau \) which belongs to one of the intervals \(( -\varepsilon, - (\varepsilon - \varepsilon^2)) \) or \(( \varepsilon - \varepsilon^2, \varepsilon) \), the differential \( df^\varepsilon(x + \tau n^{\partial V}(x)) \) can be calculated in the same way. Indeed, \( f^\varepsilon(x + \tau n^{\partial V}(x + t e_p)) = h^\varepsilon(\tau_t) \), and

\[
df^\varepsilon(x + \tau n^{\partial V}(x)) = (h^\varepsilon)'(\tau) n^{\partial V}(x).
\]

(11)

This implies that

\[
\| df^\varepsilon(x) \| < \frac{1}{2 \varepsilon} \quad \text{for all } x \in X.
\]

Note that as \( \varepsilon \to 0 \), \( f^\varepsilon \to I_V \) pointwise, and hence with respect to the \( \Xi_0 \)-topology. By assumption, \( df^\varepsilon \to dI_V \) in the \( \Xi_1 \)-topology. Let \( g \in \Xi_1 \). We have:

\[
\langle dI_V \cdot \nu, g \rangle = \lim_{\varepsilon \to 0} \langle df^\varepsilon \cdot \nu, g \rangle = \lim_{\varepsilon \to 0} \int_X (df^\varepsilon(x), g(x)) \nu(dx)
\]

\[
= \lim_{\varepsilon \to 0} \int_{(\partial V)^\varepsilon} (df^\varepsilon(x), g(x)) \nu(dx).
\]

Let \( x \in \partial V \). The function \([0, \varepsilon_b) \to \mathbb{R}, t \mapsto (n^{\partial V}(x), g(x + t n^{\partial V}(x))) \) is differentiable. By assumption, \( g \) has a bounded derivative, say by a constant
$M$, along $H$. For all $x \in \partial V$, $t \in (-\varepsilon, \varepsilon)$, we obtain:

$$\left| (n^{\partial V}(x), g(x + tn^{\partial V}(x))) - (n^{\partial V}(x), g(x)) \right|$$

$$\leq \left| \left( n^{\partial V}(x), \frac{d}{dt}g(x + tn^{\partial V}(x)) \right) \right| \cdot t \leq \|g'(x + t_0 n^{\partial V}(x)) n^{\partial V}(x)\| \cdot t$$

$$\leq \|g'(x + t_0 n^{\partial V}(x))\|_2 \cdot t < M \varepsilon,$$

where $t_0 < t$. Define a function $\tilde{g} : (\partial V)^{\varepsilon_b} \to H$ in the following way: for $x \in \partial V$, $t \in (-\varepsilon_b, \varepsilon_b)$, we set $\tilde{g}(x + tn^{\partial V}(x)) = g(x)$. Then, taking into account the above sequence of inequalities, formulas (10), (11), and the definition of $h^{\varepsilon}$, for all $x \in (\partial V)^{\varepsilon}$ we obtain that

$$\left| (df^{\varepsilon}(x), g(x)) - (df^{\varepsilon}(x), \tilde{g}(x)) \right| < \frac{M}{2}.$$

This implies:

$$\langle (d\mathbb{I}_V) \cdot \nu, g \rangle = \lim_{\varepsilon \to 0} \int_{(\partial V)^{\varepsilon}} (df^{\varepsilon}(x), \tilde{g}(x)) \nu(dx)$$

$$= -\lim_{\varepsilon \to 0} \int_{(\partial V)^{\varepsilon}} \frac{1}{2\varepsilon} (n^{\partial V}(P_{\partial V}x), g(P_{\partial V}x)) \nu(dx)$$

$$+ \lim_{\varepsilon \to 0} \int_{(\partial V)^{\varepsilon \setminus (\partial V)^{\varepsilon - \varepsilon^2}}} (df^{\varepsilon}(x) - \frac{1}{2\varepsilon} n^{\partial V}(P_{\partial V}x), \tilde{g}(x)) \nu(dx)$$

$$= -\lim_{\varepsilon \to 0} \int_{\partial V} (n^{\partial V}(x), g(x)) \nu^{\varepsilon}(dx).$$

(12)

Indeed,

$$\int_{(\partial V)^{\varepsilon}} \frac{1}{2\varepsilon} (n^{\partial V}(P_{\partial V}x), g(P_{\partial V}x)) \nu(dx) = \int_{\partial V} (n^{\partial V}(x), g(x)) \nu^{\varepsilon}(dx),$$

and as $\varepsilon \to 0$,

$$\left| \int_{(\partial V)^{\varepsilon \setminus (\partial V)^{\varepsilon - \varepsilon^2}}} (df^{\varepsilon}(x) - \frac{1}{2\varepsilon} n^{\partial V}(P_{\partial V}x), \tilde{g}(x)) \nu(dx) \right|$$

$$\leq M \nu((\partial V)^{\varepsilon \setminus (\partial V)^{\varepsilon - \varepsilon^2}}) \varepsilon \to 0.$$
Further, we fix an arbitrary \( \sigma > 0 \), and let \( \sigma' = \frac{\sigma}{2(M + \nu^\partial V(\partial V))} \). Since \( \nu^{\partial V} \) is a Radon measure (see [12], [14]), then there exists a compact \( K_\sigma \subset \partial V \), such that \( \nu^{\partial V}(\partial V \setminus K_\sigma) < \sigma' \). For each point \( x_0 \in K_\sigma \) we fix a neighborhood \( U_{x_0} \), which is contained in one of \( U_i \), possesses the property (*) formulated in [6], and such that for the function \( \varphi(x) = (n^{\partial V}(x), g(x)) \), the inequality \( |\varphi(x) - \varphi(x_0)| < \sigma' \) holds for all \( x \in U_{x_0} \). We choose a finite number of neighborhoods \( U_x, x \in K_\sigma \), covering \( K_\sigma \) (let them be neighborhoods \( U_i \) of points \( x_i \), and denote their union by \( O_\sigma \). It is clear that \( \nu^{\partial V}(\partial V \setminus O_\sigma) < \sigma' \), and by the construction of \( O_\sigma \), there exists the limit \( \lim_{\varepsilon \to 0} \nu^\varepsilon(O_\sigma) = \nu^{\partial V}(O_\sigma) \). Hence the limit \( \lim_{\varepsilon \to 0} \nu^\varepsilon(\partial V \setminus O_\sigma) = \nu^{\partial V}(\partial V \setminus O_\sigma) \) exists too. Further let \( B_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j \), and \( \varphi_\sigma : O_\sigma \to \mathbb{R} \) be such that \( \varphi_\sigma = \sum_i \varphi(x_i) \mathbb{I}_{B_i} \). It is clear that on \( O_\sigma \), \( |\varphi(x) - \varphi_\sigma(x)| < \sigma' \). We have:

\[
\lim_{\varepsilon \to 0} \int_{\partial V} \varphi(x) \nu^\varepsilon(dx) = \lim_{\varepsilon \to 0} \int_{O_\sigma} \varphi_\sigma(x) \nu^\varepsilon(dx) \\
+ \lim_{\varepsilon \to 0} \left( \int_{O_\sigma} (\varphi(x) - \varphi_\sigma(x)) \nu^\varepsilon(dx) + \int_{\partial V \setminus O_\sigma} \varphi(x) \nu^\varepsilon(dx) \right). \tag{13}
\]

By the definition of \( \varphi_\sigma \), \( \int_{O_\sigma} \varphi_\sigma(x) \nu^\varepsilon(dx) = \sum_i \varphi(x_i) \nu^\varepsilon(B_i) \), where the sum contains a finite number of terms. We observe that for every set \( B_i \), \( \lim_{\varepsilon \to 0} \nu^\varepsilon(B_i) = \nu^{\partial V}(B_i) \) by the construction of \( B_i \) and by Theorem 2 of [6]. Hence \( \lim_{\varepsilon \to 0} \int_{O_\sigma} \varphi_\sigma(x) \nu^\varepsilon(dx) = \int_{O_\sigma} \varphi_\sigma(x) \nu^{\partial V}(dx) \). The limit of the second term in (13) exists by the existence of the two other limits. We continue (13):

\[
\lim_{\varepsilon \to 0} \int_{\partial V} \varphi(x) \nu^\varepsilon(dx) = \int_{\partial V} \varphi(x) \nu^{\partial V}(dx) \\
- \int_{\partial V \setminus O_\sigma} \varphi(x) \nu^{\partial V}(dx) - \int_{O_\sigma} (\varphi(x) - \varphi_\sigma(x)) \nu^{\partial V}(dx) \\
+ \lim_{\varepsilon \to 0} \left( \int_{O_\sigma} (\varphi(x) - \varphi_\sigma(x)) \nu^\varepsilon(dx) + \int_{\partial V \setminus O_\sigma} \varphi(x) \nu^\varepsilon(dx) \right).
\]

Let us estimate the last three terms. We have:

\[
\left| \int_{O_\sigma} (\varphi(x) - \varphi_\sigma(x)) \nu^\varepsilon(dx) + \int_{\partial V \setminus O_\sigma} \varphi(x) \nu^\varepsilon(dx) \right| \leq \sigma' \nu^\varepsilon(O_\sigma) + M \nu^\varepsilon(\partial V \setminus O_\sigma).
\]

Passing to the limit as \( \varepsilon \to 0 \) in the both sides of this inequality, and taking
into account that the limit on the left-hand side exists, we obtain:

\[
\left| \lim_{\varepsilon \to 0} \left( \int_{O_{\sigma}} (\varphi(x) - \varphi_{\sigma}(x)) \nu^\varepsilon(dx) + \int_{\partial V \setminus O_{\sigma}} \varphi(x) \nu^\varepsilon(dx) \right) \right| \leq \sigma' \nu^\sigma(\partial V) + M \sigma'.
\]

Analogously, we prove the two other estimates:

\[
\left| \int_{\partial V \setminus O_{\sigma}} \varphi(x) \nu^{\sigma}(dx) \right| < M \sigma', \quad \left| \int_{O_{\sigma}} (\varphi(x) - \varphi_{\sigma}(x)) \nu^{\sigma}(dx) \right| < \sigma' \nu^{\sigma}(\partial V).
\]

From this it follows that

\[
\left| \lim_{\varepsilon \to 0} \int_{\partial V} \varphi(x) \nu^\varepsilon(dx) - \int_{\partial V} \varphi(x) \nu^{\sigma}(dx) \right| < \sigma.
\]

Since \( \sigma > 0 \) was chosen arbitrary, we conclude that

\[
\lim_{\varepsilon \to 0} \int_{\partial V} (n^{\sigma}(x), g(x)) \nu^\varepsilon(dx) = \int_{\partial V} (n^{\sigma}(x), g(x)) \nu^{\sigma}(dx).
\]

Together with (12) this implies that

\[
\langle (dII_V) \cdot \nu, g \rangle = - \int_{\partial V} (n^{\sigma}(x), g(x)) \nu^{\sigma}(dx)
\]

which is equivalent to (6). The theorem is proved.

**Corollary 1.** Let \( \omega = \sum_{p=1}^{\infty} \omega_p e_p \in S_1 \), and \( \omega^{\sigma} = \sum_{p=1}^{\infty} \omega_p^{\sigma} e_p \in \bar{S}_1 \), where \( \omega_p^{\sigma} \) are the surface measures generated by the measures \( \omega_p \). Then \( n^{\sigma} \in \Xi_1 \), and the measures \( \omega \) and \( \omega^{\sigma} \) are related through the identity:

\[
dII_V \wedge \omega = - n^{\sigma} \wedge \omega^{\sigma}.
\]

**Proof.** Let us prove that \( n^{\sigma} \in \Xi_1 \). Indeed, \( n^{\sigma} \) originally defined on \( \partial V \) can be extended to \( X \) by setting \( n^{\sigma}(x) = 0 \) for \( x \notin \partial V \). Let us consider the functions \( f_\varepsilon \) defined by (7). By (10), \( -2\varepsilon df_\varepsilon \) converges to \( n^{\sigma} \) pointwise, and hence with respect to the \( \Xi_1 \)-topology by assumption. Note that by assumption, the operation \( \wedge \) can be extended from \( \Xi_1 \times S_1 \) to \( \Xi_1 \times \bar{S}_1 \) so that \( (f, \omega) \mapsto f \wedge \omega \) is a continuous mapping \( \Xi_1 \times \bar{S}_1 \to S_0 \). Applying Theorem 2 to each pair of real-valued measures \( \omega_p \) and \( \omega_p^{\sigma} \) we obtain:

\[
dII_V \wedge \omega = \sum_{p=1}^{\infty} dII_V \cdot \omega_p \wedge e_p = - \sum_{p=1}^{\infty} (n^{\sigma} \cdot \omega_p^{\sigma}) \wedge e_p = - n^{\sigma} \wedge \omega^{\sigma}.
\]
3.3 Derivation of the Stokes formula

**Definition 7.** Let \( \omega \in S_1 \) and \( \omega^{\partial V} \in \bar{S}_1 \). We define the integral of \( \omega \) over the surface \( \partial V \) by the identity:

\[
\int_{\partial V} \omega = \int_{\partial V} (n^{\partial V}(x), \omega^{\partial V}(dx)).
\]

**Theorem 3 (The Stokes formula).** Let \( \omega \in S_1 \) and \( \omega^{\partial V} \in \bar{S}_1 \), then

\[
\int_{\partial V} \omega = \int_V d\omega.
\]

**Proof.** Corollary 1 and Definition 7 imply:

\[
\int_{\partial V} \omega = \int_{\partial V} (n^{\partial V}(x), \omega^{\partial V}(dx)) = -\langle d\mathbb{I}_V \wedge \omega, 1 \rangle.
\]

Let us consider again the functions \( f_\varepsilon \) defined by (7). We proved that \( f_\varepsilon \to \mathbb{I}_V \) pointwise and in the \( \mathcal{E}_0 \)-topology, and that \( df_\varepsilon \to d\mathbb{I}_V \) in the \( \mathcal{E}_1 \)-topology. By Lemma 1,

\[
0 = d(f_\varepsilon \wedge \omega)(X) = (df_\varepsilon \wedge \omega)(X) + (f_\varepsilon \wedge d\omega)(X).
\]

Hence,

\[
\int_V d\omega = \lim_{\varepsilon \to 0} (f_\varepsilon \wedge d\omega)(X) = -\lim_{\varepsilon \to 0} (df_\varepsilon \wedge \omega)(X) = -\langle d\mathbb{I}_V \wedge \omega, 1 \rangle = \int_{\partial V} \omega.
\]

The theorem is proved. \( \square \)

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