Localization effects due to a random magnetic field on heat transport in a harmonic chain

Gaëtan Cane\textsuperscript{1}, Junaid Majeed Bhat\textsuperscript{2}, Abhishek Dhar\textsuperscript{2} and Cédric Bernardin\textsuperscript{1,3,}\large{*}

\textsuperscript{1} Université Côte d’Azur, CNRS, LJAD, Parc Valrose, 06108 NICE Cedex 02, France
\textsuperscript{2} International Centre for Theoretical Sciences, Bengaluru, 560089, India
\textsuperscript{3} Interdisciplinary Scientific Center Poncelet (CNRS IRL 2615), 119002 Moscow, Russia
E-mail: Gaetan.cane@univ-cotedazur.fr, junaid.bhat@icts.res.in, abhishek.dhar@icts.res.in and cbernard@unice.fr

Received 9 September 2021
Accepted for publication 21 October 2021
Published 30 November 2021

Abstract. We consider a harmonic chain of \(N\) oscillators in the presence of a disordered magnetic field. The ends of the chain are connected to heat baths and we study the effects of the magnetic field randomness on heat transport. The disorder, in general, causes localization of the normal modes, due to which a system becomes insulating. However, for this system, the localization length diverges as the normal mode frequency approaches zero. Therefore, the low frequency modes contribute to the transmission, \(T_N(\omega)\), and the heat current goes down as a power law with the system size, \(N\). This power law is determined by the small frequency behaviour of some Lyapunov exponents, \(\lambda(\omega)\), and the transmission in the thermodynamic limit, \(T_\infty(\omega)\). While it is known that in the presence of a constant magnetic field \(T_\infty(\omega) \sim \omega^{3/2}, \omega^{1/2}\) depending on the boundary conditions, we find that the Lyapunov exponent for the system behaves as \(\lambda(\omega) \sim \omega\) for \(\langle B \rangle \neq 0\) and \(\lambda(\omega) \sim \omega^{2/3}\) for \(\langle B \rangle = 0\). Therefore, we obtain different power laws for current vs \(N\) depending on \(\langle B \rangle\) and the boundary conditions.

\[\textit{CrossMark}\]

© 2021 IOP Publishing Ltd and SISSA Medialab srl
Localization effects due to a random magnetic field on heat transport in a harmonic chain

Keywords: heat conduction, transport processes/heat transfer, Anderson model, random/ordered microstructure

Contents

1. Introduction .................................................................................. 2
2. The model and heat current by NEGF .......................................... 3
   2.1. The model ............................................................................. 3
   2.2. Heat current ......................................................................... 4
   2.3. Effect of localization due to random magnetic field on the net transmission .... 5
3. Analysis of the Lyapunov exponents ............................................ 6
   3.1. Theoretical results for Lyapunov exponents .............................. 7
   3.2. Numerical results for Lyapunov exponents .............................. 8
4. Size dependence of the current .................................................. 10
5. Conclusion .................................................................................. 12
Acknowledgments ............................................................................. 13
Appendix A. Lyapunov exponent for a harmonic oscillator with parametric noise ................................................. 13
References ...................................................................................... 17

1. Introduction

In his seminal paper [1], Anderson studied the conductance of electrons and explained how the presence of impurities in metal could drastically reduce the diffusive motion of the electrons up to a complete halt, thus giving place to an insulator. This phenomenon strongly depends on the dimension and the metal insulator transition is not instantaneous with respect to the disorder strength only in dimensions greater or equal to three. Nowadays, Anderson localization is seen as a generic phenomenon present in disordered media, whereby the addition of random defects in the medium has the tendency to localize in space the normal modes of the system. As a consequence, it drastically reduces the transport coefficient. Originally, Anderson’s work took place in a quantum context but the phenomenon he explained also appears in a classical one. In the 70s, Lebowitz and others [2–5] started to investigate the effect of impurities (random masses) on the transport properties of a one dimensional harmonic chain, arguing in particular that the conductivity $\kappa(N)$ of the chain (which is proportional to the system size $N$ for a purely harmonic chain), loses some order of magnitude because of disorder: $\kappa(N) \sim N^a$, $a < 1$ (see [6] for a mathematical proof). But at the difference with respect to original Anderson localization, the conductivity does not become exponentially small in the

https://doi.org/10.1088/1742-5468/ac32b8

2
system size and, depending on the physical boundary conditions and thermostats, it can vanish \((a < 0)\), diverge \((0 < a < 1)\) or even converge \((a = 0)\) \([7–9]\). The reason for this is roughly due to the fact that for disordered harmonic chains, normal modes with frequency \(\omega\) become localized but with a length of localization \(\ell(\omega) \sim \omega^{-2}\). The role of thermostats and boundary conditions is more difficult to explain without going into computational details. Thus, we note that in disordered harmonic chains the low frequency modes still have the possibility to transport energy \([10]\). The case where the disorder is in the interparticle springs instead of the masses has recently been addressed in \([11, 12]\). In higher dimensions, the situation is less understood \([13, 14]\). More recently, there has been a renewed interest for these questions with respect to the effect of nonlinearities \([15, 16]\) or of an energy conserving noise \([17–20]\).

In this paper, we consider an ordered (constant masses) one-dimensional chain of two-dimensional charged oscillators subject to a random transverse magnetic field on every lattice site (or equivalently a chain of disorderly charged oscillators subjected to a constant magnetic field on the lattice). In \([21]\), by using the non-equilibrium Green’s function (NEGF) formalism, we obtained an explicit expression of the heat current in the steady state and then investigated the transport properties of such a system when all of the charges are the same. We established that transport is ballistic-like for ordered harmonic chains. The aim of this paper is therefore to describe the effect of the charge impurities on the behaviour of the conductivity of the system. This will require us to investigate the frequency dependence of the localization lengths of normal modes. We show that due to charge disorder the current shows different scaling with the system size, which depends on the boundary conditions, as well as on the expectation value of the magnetic field.

This paper is structured as follows: in section 2, we introduce the model and state the results for heat current using the NEGF formalism. We also present numerical results for the transmission function and discuss the effects of localization due to the random magnetic field on the transmission function. In section 3, we use the Green’s function expression as a product of random matrices to determine the Lyapunov exponents. We also present numerical results for Lyapunov exponents, which are consistent with our theoretical results. Using the results for the Lyapunov exponents, we finally determine the size dependence of the mean of the heat current in section 4 and compare with direct numerical calculations of the current. We conclude in section 5.

2. The model and heat current by NEGF

2.1. The model

We consider a chain of \(N\) harmonic oscillators each having two transverse degrees of freedom so that every oscillator is free to move in a plane perpendicular to the length of the chain. We choose the plane of motion to be the \(x-y\) plane and denote the positions and momenta of the \(n\)th oscillator by \((x_n, y_n)\) and \((p_{x,n}, p_{y,n})\), respectively, with \(n = 1, 2, \ldots, N\). The oscillators are assumed to have unit masses and each carry a positive unit charge.

We consider a magnetic field \(\vec{B}_n = B_n \vec{e}_z\) perpendicular to the plane of motion, which can be obtained from a vector potential \(\vec{A}_n = (-B_n y_n, B_n x_n, 0)\) at each lattice site. In
Localization effects due to a random magnetic field on heat transport in a harmonic chain

this paper, we assume that \((B_n)_n\) form a sequence of independent identically distributed random variables with average \(\langle B \rangle\) and variance \(\sigma^2\). The Hamiltonian of the chain is given by:

\[
H = \sum_{n=1}^{N} \left( \frac{(p_n^2 + B_n y_n)^2}{2} + \frac{(p_n^2 - B_n x_n)^2}{2} \right) + \sum_{n=0}^{N} \left( (x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2 \right),
\]

where the inter particle spring constant has been fixed to 1. We will consider the two different boundary conditions: (i) fixed boundaries with \(x_0 = x_{N+1} = 0\) and (ii) free boundaries with \(x_0 = x_1, x_N = x_{N+1}\). In order to study heat current through this system, we consider the 1st and the \(N\)th oscillators to be connected to heat reservoirs at temperatures \(T_L\) and \(T_R\), respectively. The heat reservoirs are modelled using dissipative and noise terms leading to the following Langevin equations of motion:

\[
\begin{align*}
\dot{x}_n &= (x_{n+1} + x_{n-1} - c_n x_n) + B_n \dot{y}_n + \eta^x_n(t) \delta_{n,1} + \eta^x_n(t) \delta_{n,N} - (\gamma \delta_{n,1} + \gamma \delta_{n,N}) \dot{x}_n, \\
\dot{y}_n &= (y_{n+1} + y_{n-1} - c_n y_n) - B_n \dot{x}_n + \eta^y_n(t) \delta_{n,1} + \eta^y_n(t) \delta_{n,N} - (\gamma \delta_{n,1} + \gamma \delta_{n,N}) \dot{y}_n,
\end{align*}
\]

for \(n = 1, 2, \ldots, N\). Here, \(\eta^x_n(t) := (\eta^x_n(t), \eta^x_n(t))\) and \(\eta^y_n(t) := (\eta^y_n(t), \eta^y_n(t))\) are Gaussian white noise terms acting on the 1st and \(N\)th oscillators, respectively. These follow the regular white noise correlations, \(\langle \eta^x(t) \eta^y(t') \rangle = \sqrt{2 \gamma T_L/R} \delta(t - t')\) (Boltzmann’s constant is fixed to one to simplify), where \(\gamma\) is the dissipation strength at the reservoirs. The coefficients \(c_n\) fix the boundary conditions of the problem. For fixed boundaries \(c_n = 2\) for all \(n\), while for free boundary conditions \(c_n = 2 - \delta_{n,1} - \delta_{n,N}\).

\[.2. Heat current\]

In [21], by using the NEGF formalism, we obtained an exact expression for the heat current, \(J_N\), in the steady state of the chain. More exactly, let us define the processes \((f_n^\pm)_{n \geq 0}\) and \((g_n^\pm)_{n \geq 0}\) as

\[
\begin{align*}
f_{n+1}^\pm &= (c_{n+1} - \omega^2 \pm \omega B_{n+1}) f_n^\pm - f_{n-1}^\pm, \quad f_0^\pm = 1, \quad f_1^\pm = c_1 - \omega^2 \pm \omega B_1, \\
g_{n+1}^\pm &= (c_{n+1} - \omega^2 \pm \omega B_{n+1}) g_n^\pm - g_{n-1}^\pm, \quad g_0^\pm = 0, \quad g_1^\pm = 1.
\end{align*}
\]

Then we introduce

\[
F_N^\pm := F_N^\pm(\omega) = f_N^\pm + i \gamma \omega (g_N^\pm + f_{N-1}^\pm) - \gamma^2 \omega^2 g_{N-1}^\pm,
\]

and the heat current is equal to

\[
J_N = (T_L - T_R) \int_{-\infty}^{\infty} d\omega \mathcal{T}_N(\omega) = 2(T_L - T_R) \int_{0}^{\infty} d\omega \mathcal{T}_N(\omega)
\]

with the net transmission function \(\mathcal{T}_N\) defined for any frequency \(\omega\) by

\[
\mathcal{T}_N(\omega) := \frac{\gamma^2}{\pi} \omega^2 \left[ \frac{1}{|F_N^+(\omega)|^2} + \frac{1}{|F_N^-(\omega)|^2} \right].
\]

We denote by \(\langle J_N \rangle\) the expectation of the heat current with respect to the magnetic field distribution \(\langle \cdot \rangle\) and our goal is to understand its scaling behaviour in \(N\).

https://doi.org/10.1088/1742-5468/ac32b8 4
Localization effects due to a random magnetic field on heat transport in a harmonic chain

Observe that the stochastic processes \((f_n^-)_{n \geq 0}\) and \((g_n^-)_{n \geq 0}\) are defined in terms of the two dimensional discrete time Markov chain \((U_n)_{n \geq 0}\) given by

\[ U_{n+1} = \begin{pmatrix} 2 - \omega^2 - \omega B_{n+1} & -1 \\ 1 & 0 \end{pmatrix} U_n, \quad \text{where } U_n := \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \]

(7)

by choosing suitable initial conditions. By replacing the \(B_n\)'s by \(-B_n\)'s in the last display, we see that \((f_n^+)_{n \geq 0}\) and \((g_n^+)_{n \geq 0}\) can also be expressed in terms of \((U_n)_{n \geq 0}\). The state of the Markov chain is nothing but the result of a product of \(2 \times 2\) product of independent and identically distributed random matrices. Roughly, the behaviour of \(F_X^\pm\) is related to the growth of \(\|U_N(\omega)\|\), which will be in the form \(e^{2\lambda(\omega)N}\), where

\[ \lambda(\omega) = \lim_{n \to \infty} \frac{1}{2n} \langle \log \|U_n(\omega)\| \rangle = \lim_{n \to \infty} \frac{1}{n} \langle \log |u_n(\omega)| \rangle > 0, \]

(8)

with \(\langle \ldots \rangle\) denoting a disorder average, half the Lyapunov exponent associated to the Markov chain \((U_n)_{n \geq 0}\), or equivalently of the corresponding product of random matrices. The limit exists by Furstenberg’s theorem [22], is non-negative, independent of the initial condition \(U_0\) and the limit also holds in fact for any realisation and not only by averaging over the magnetic field distribution.

For now, we quickly discuss the effect of localization due to the random magnetic field on the heat transport and the need for calculating the Lyapunov exponent \(\lambda(\omega)\) for small frequencies \(\omega\).

2.3. Effect of localization due to random magnetic field on the net transmission

Using equation (6), we can calculate the net transmission \(T_N(\omega)\) for any spatial configuration of the magnetic field using a computer programme. In figures 1(a) and (b), we plot the net transmission function with \(\omega\) for a uniform magnetic field and for a random magnetic field for different system sizes, respectively. In the comparison of the two plots, we can see that the randomness causes suppression of the net transmission and also the net transmission for the random magnetic field case goes down with system size, while the system size has nearly no effect on the transmission for the uniform magnetic field. The suppression in case of random magnetic field is due to localization of the normal modes of the system. The normal modes of frequency \(\omega\) get exponentially localized due to randomness with a localization length given by \(1/\lambda(\omega)\), where \(\lambda(\omega)\) is the Lyapunov exponent defined in equation (8). As a result of this they \(a \text{ priori}\) do not contribute to the transmission. However, note that the transmission for random magnetic field is higher near \(\omega = 0\) and goes down as we move away, which means that the normal modes with energies closer to \(\omega = 0\) have a larger localization length, i.e. \(\lambda(\omega) \to 0\) as \(\omega \to 0\). Since we are eventually interested in the size dependence of the current, for large \(N\), which is the integral of the transmission over all \(\omega\), we can reduce the integration limit to values of \(\omega\) for which the localization length is greater than the system size. For the remaining \(\omega\) values for which the localization length is less than the system size, the transmission would be negligible. Hence, we cut off the integral limit to \(\omega = \omega_{\max}\) where
Figure 1. Variation of the net transmission, in units of $k_B = 1$, with $\omega$ for uniform magnetic field (a), and random magnetic field (b). The axes are in log scale and $\gamma = 0.2$. The magnetic field in (a) is set to be 1 on all oscillators and in (b) it was uniformly chosen from the interval $(0, 2)$. As can be seen clearly from the plots, the localization effects cause suppression of the transmission.

$$\frac{1}{\lambda(\omega_{\text{max}}^N)} = N$$ and the current is then given by

$$\langle J_N \rangle \approx 2(T_L - T_R) \int_0^{\omega_{\text{max}}^N} \omega \lim_{N \rightarrow \infty} \langle T_N(\omega) \rangle = 2(T_L - T_R) \int_0^{\omega_{\text{max}}^N} \omega \langle T_\infty(\omega) \rangle. \quad (9)$$

Note that the frequency $\omega_{\text{max}}^N$ would be very small for large $N$, and for such small frequencies we expect $T_\infty(\omega)$ to have a weak dependence on disorder (since in the recursion equation (7), the randomness is multiplied by $\omega$)—hence in the above equation $T_\infty(\omega)$ is written without a disorder average and can in fact be determined by considering the chain in a constant magnetic field of strength $\langle B \rangle$. In [21], we proved that for constant magnetic field $\langle B \rangle \neq 0$, $T_\infty(\omega) \sim \omega^{3/2}$ and $\sim \omega^{1/2}$ for fixed and free boundaries, respectively, while for $\langle B \rangle = 0$ it goes as $\omega^2$ and $\omega^0$ for the two boundary conditions, respectively. To determine the size dependence of the current in addition to the small $\omega$ behaviour of $T_\infty(\omega)$ we also need the small $\omega$ behaviour of $\lambda(\omega)$. We now proceed to the next section where we discuss the Lyapunov exponents of this equation.

3. Analysis of the Lyapunov exponents

In this section, we present theoretical and numerical results on the asymptotics of Lyapunov exponents for small $\omega$ for the Markov processes defined by equation (7). The Lyapunov exponents are independent of the boundary conditions—so for this section we only work with fixed boundary conditions by setting $c_n = 2$ for all $n$—and of the initial condition of the process—i.e. it is the same for $f_n^\pm$ and $g_n^\pm$. We show that equation (7) has three different behaviours for the Lyapunov exponent depending on the expected
Localization effects due to a random magnetic field on heat transport in a harmonic chain

value \langle B \rangle of the random magnetic field. For \langle B \rangle > 0 the Lyapunov exponent satisfies \lambda(\omega) \sim \omega and for \langle B \rangle < 0, \lambda(\omega) \sim \omega^{1/2}. However, for \langle B \rangle = 0, \lambda(\omega) \sim \omega^{2/3}. Similar Lyapunov exponent behaviours are found for a harmonic oscillator with parametric noise [23], and we will see that equation (7) could be written exactly in this form in the continuum limit.

3.1. Theoretical results for Lyapunov exponents

Let \((z_t)_{t \geq 0} \in \mathbb{R}^2\) be the solution of the following stochastic differential equation (with arbitrary initial condition)

\[ \dot{z}_t = A_0 z_t + \varepsilon \sigma \xi_t A_1 z_t, \tag{10} \]

where \(\varepsilon\) is a small positive parameter, \(\sigma > 0\) a constant, \(\xi_t\) a one-dimensional standard white noise and \(A_0\) and \(A_1\) are \(2 \times 2\) matrices, such that

\[ A_0 = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \text{ with } c \in \mathbb{R}. \]

The Lyapunov exponent \(\lambda_z(\varepsilon)\) of the process \((z_t)_{t \geq 0}\) is defined by

\[ \lambda_z(\varepsilon) = \lim_{t \to \infty} \frac{1}{t} \langle \log \| z_t \| \rangle, \tag{11} \]

where \(\langle \cdot \rangle\) denotes the expectation with respect to the white noise. It is proved in appendix A that if we denote \(z_t = (u_t, v_t)^\top\) then the Lyapunov exponent for \((u_t)_{t \geq 0}\) is the same as for \((z_t)_{t \geq 0}\):

\[ \lambda_z(\varepsilon) = \lim_{t \to \infty} \frac{1}{t} \langle \log | u_t | \rangle. \tag{12} \]

The following result, proved in [24], gives the behaviour of the Lyapunov exponent \(\lambda_z(\varepsilon)\) for small noise

(a) If \(c = 0\) then \(\lambda_z(\varepsilon) = \hat{\lambda}(\sigma) \varepsilon^{2/3}\), where \(\hat{\lambda}(\sigma)\) is defined in equation (A.9).

(b) If \(c > 0\) then \(\lambda_z(\varepsilon) \sim \frac{\sigma^2}{8c} \varepsilon^2\).

(c) If \(c < 0\) then \(\lambda_z(\varepsilon) \sim \sqrt{|c|}\).

A sketch of the proof of this result is given in appendix A.

Now consider equation (7) defining the discrete time Markov chain \(U_n = (u_n, u_{n-1})^\top\) and rewrite it in the following form, for small \(\omega\),

\[ u_{n+1} + u_{n-1} - 2u_n = -\omega \langle B \rangle u_n - \omega (B_{n+1} - \langle B \rangle) u_n + O(\omega^2). \]

In the continuum limit, the discrete time process \((u_n)_{n \geq 0}\) then becomes the continuous time process \((u_t)_{t \geq 0}\) solution of

\[ \ddot{u}_t = -\omega \langle B \rangle u_t - \omega \sigma \xi_t u_t \tag{13} \]

https://doi.org/10.1088/1742-5468/ac32b8
Localization effects due to a random magnetic field on heat transport in a harmonic chain

where \((\xi_t)_{t \geq 0}\) is a standard white noise and \(\sigma^2\) the variance of the \((B_n)_n\). Defining \(w_t = (u_t, \dot{u}_t)^\top\), we see that the previous equation reads

\[
\dot{w}_t = \begin{pmatrix} 0 & 1 \\ -\omega \langle B \rangle & 0 \end{pmatrix} w_t + \sigma \omega \xi_t \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} w_t.
\]

We are interested in the Lyapunov exponent of the process \((u_t)_{t \geq 0}\) (or equivalently of the process \((w_t)_{t \geq 0}\) as said before):

\[
\lambda_w(\omega) = \lim_{t \to \infty} \frac{1}{t} \langle \log \|w_t\| \rangle = \lim_{t \to \infty} \frac{1}{t} \langle \log |u_t| \rangle.
\]

Equation (14) looks similar to equation (10) but to fit perfectly with equation (10), we perform the time scaling

\[
\tilde{u}_t = u_t / \sqrt{\omega}
\]

in equation (13), which gives by scaling invariance of white noise

\[
\ddot{\tilde{u}}_t = -\langle B \rangle \tilde{u}_t - \omega^{1/4} \sigma \xi_t \tilde{u}_t
\]

or equivalently for \(\tilde{z}_t = (\tilde{u}_t, \dot{\tilde{u}}_t)^\top\) the equation

\[
\dot{\tilde{z}}_t = \begin{pmatrix} 0 & 1 \\ -\langle B \rangle & 0 \end{pmatrix} \tilde{z}_t + \sigma \omega^{1/4} \xi_t \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tilde{z}_t.
\]

With the previous notation, we have hence

\[
\lambda_w(\omega) = \sqrt{\omega} \lambda_z(\omega^{1/4}).
\]

Equation (17) perfectly fits equation (10) with \(c = \langle B \rangle\) and \(\varepsilon = \omega^{1/4}\). Then using point (a), (b) and (c) of equations (10) and (18), we get

(a) If \(\langle B \rangle = 0\), \(\lambda_w(\omega) = \hat{\lambda}(\sigma) \omega^{2/3}\), where \(\hat{\lambda}(\sigma)\) is defined in equation (A.9).

(b) If \(\langle B \rangle > 0\), \(\lambda_w(\omega) \sim \frac{\sigma^2}{\langle B \rangle^{1/2}} \omega^{1/2}\).

(c) If \(\langle B \rangle < 0\), \(\lambda_w(\omega) \sim \sqrt{\langle B \rangle} |\omega|^{1/2}\).

It makes sense to believe that \(\lambda(\omega)\) defined by equation (8) and \(\lambda_w(\omega)\) defined by equation (15) have roughly the same behaviour as \(\omega \to 0\) but a strong theoretical argument supporting this belief is missing. However, in the case \(\langle B \rangle > 0\), we can directly obtain the behaviour of \(\lambda(\omega)\) by following the approach of [25] and we then observe a good agreement at first order between \(\lambda(\omega)\) and \(\lambda_w(\omega)\), not only at the level of the exponent in \(\omega\) but also at the level of the prefactor; see Table 1. Unfortunately, we were not able to carry this approach for \(\langle B \rangle < 0\) or \(\langle B \rangle = 0\) and we hence decided to not pursue this approach. However, numerical results presented in the next section strongly support the claim that \(\lambda(\omega) \sim \lambda_w(\omega)\) for \(\omega \to 0\).

3.2. Numerical results for Lyapunov exponents

We numerically calculate the Lyapunov exponents by using equation (3) to generate \(u_N\) for 100 realizations of the random magnetic field. The Lyapunov exponent would then

https://doi.org/10.1088/1742-5468/ac32b8
Localization effects due to a random magnetic field on heat transport in a harmonic chain

Table 1. Comparison of analytical prefactor for the three cases with the numerical prefactor. For this table, \( N = 10^7 \).

| Case          | Range of \( B_n \) | \( s : \lambda(\omega) \sim C \omega^s \) | \( C \)   | \( C_{\text{theoretical}} \) |
|---------------|--------------------|---------------------------------|--------|----------------|
| \( \langle B \rangle > 0 \) | \((0, 0.25)\) | 0.986 0.0045 0.0052 | \( \langle B \rangle > 0 \) | \((0.25, 0.75)\) | 1.0005 0.0156 0.0156 | 0.986 0.0045 0.0052 | \( \langle B \rangle > 0 \) |
| \( \lambda(\omega) \sim \sigma^2 \omega \) | \((0.5)\) | 0.999 0.0102 0.0104 | \( \langle B \rangle > 0 \) | \((0.75)\) | 1.0005 0.0156 0.0156 | 0.999 0.0102 0.0104 | \( \langle B \rangle > 0 \) |
| \( \langle B \rangle < 0 \) | \((-0.25, 0.25)\) | 0.492 0.315 0.353 | \( \langle B \rangle < 0 \) | \((-0.75, 0.75)\) | 0.491 0.532 0.612 | 0.492 0.315 0.353 | \( \langle B \rangle < 0 \) |
| \( \lambda(\omega) \sim \sqrt{|\langle B \rangle|} \omega^{1/2} \) | \((-0.5, 0)\) | 0.492 0.444 0.5 | \( \langle B \rangle < 0 \) | \((-0.75, 0.75)\) | 0.649 0.136 0.167 | \(-0.5, 0\) | \( \langle B \rangle < 0 \) |
| \( \langle B \rangle = 0 \) | \((-0.25, 0.25)\) | 0.658 0.073 0.079 | \( \langle B \rangle = 0 \) | \((-0.75, 0.75)\) | 0.491 0.532 0.612 | 0.658 0.073 0.079 | \( \langle B \rangle = 0 \) |
| \( \lambda(\omega) = \hat{\lambda}(\sigma) \omega^{2/3} \) | \((-0.5, 0.5)\) | 0.658 0.115 0.127 | \( \langle B \rangle = 0 \) | \((-0.75, 0.75)\) | 0.649 0.136 0.167 | \(-0.5, 0.5\) | \( \langle B \rangle = 0 \) |

Figure 2. Variation of numerically calculated Lyapunov exponent, \( \lambda = \frac{1}{N} \langle \log |u_N| \rangle \), with \( \omega \). \( \langle \log |u_N| \rangle \) denotes the average of \( \log |u_N| \) over 100 realizations of the random magnetic field. (a)–(c) The magnetic fields were chosen randomly from the intervals \((0, 1), (-1, 1)\) and \((-1, 0)\), respectively. The solid line is the data from the simulation, while the dashed line is a power law fit, \( C \omega^s \), to the data with \( C \) and \( s \) as fitting parameters. The obtained values of the fitting parameters agree appreciably with the theoretical values.

be given by \( \lambda = \frac{1}{N} \langle \log |u_N| \rangle \), where \( N \) is the number of oscillators. We plot in figure 2 the numerical data thus obtained for different \( \omega \) and the power law fit, \( C \omega^s \), for the data with \( C \) and \( s \) as fitting parameters. We see that the values of \( s \) obtained for the three cases, \( \langle B \rangle > 0 \), \( \langle B \rangle < 0 \) and \( \langle B \rangle = 0 \), agree reasonably well with the theoretically expected values. The prefactor, \( C \), obtained for the three cases also seems to agree with the expected values from theory; see table 1.

We now have the behaviour of the Lyapunov exponents at small \( \omega \) for equation (3) and we found this to be different depending on the expectation value of the random magnetic field. The transmission is determined by \( f_N^+ \), as well as \( f_N^- \) and these two have different Lyapunov exponents for \( \langle B \rangle \neq 0 \), therefore the larger of the two exponents

https://doi.org/10.1088/1742-5468/ac32b8
Localization effects due to a random magnetic field on heat transport in a harmonic chain

Table 2. Power law for the current for different boundary conditions and average magnetic fields.

| Boundary conditions | Average magnetic field | $T_\infty(\omega)$ | $\lambda(\omega)$ | Power law for the current $\langle J_N \rangle$ |
|---------------------|------------------------|-------------------|------------------|-----------------------------------------------|
| Fixed               | $\langle B \rangle \neq 0$ | $\sim \omega^{3/2}$ | $\sim \omega$ | $\sim 1/N^{5/2}$ |
| Fixed               | $\langle B \rangle = 0$   | $\sim \omega^2$    | $\sim \omega^{3/2}$ | $\sim 1/N^{9/2}$ |
| Free                | $\langle B \rangle \neq 0$ | $\sim \omega^{1/2}$ | $\sim \omega$ | $\sim 1/N^{3/2}$ |
| Free                | $\langle B \rangle = 0$   | $\sim \omega^0$    | $\sim \omega^{2/3}$ | $\sim 1/N^{3/2}$ |

will dominate in the transmission. This is the Lyapunov exponent for $f_{N+1}$ for $\langle B \rangle > 0$, while for $\langle B \rangle = 0$, $f_N$ and $f_{N+1}$ have the same Lyapunov exponent. In the next section, we determine the size dependence of the current using these results for the Lyapunov exponents.

4. Size dependence of the current

We now have the small $\omega$ behaviour of $\lambda(\omega)$ for the transmission. We found this to be different for $\langle B \rangle \neq 0$ and $\langle B \rangle = 0$, so we expect different power laws for the current for the two cases. The boundary conditions will also play a role in the power law via the small $\omega$ behaviour of $T_\infty(\omega)$. We therefore take the cases $\langle B \rangle \neq 0$ and $\langle B \rangle = 0$ separately for the two boundary conditions (table 2).

Fixed boundary conditions:

(a) For $\langle B \rangle \neq 0$, $T_\infty(\omega) \sim \omega^{3/2}$ and $\lambda(\omega) \sim \omega$. Therefore, using these in equation (9), we have $\langle J_N \rangle \sim 1/N^{5/2}$.

(b) For $\langle B \rangle = 0$, $T_\infty(\omega) \sim \omega^2$ and $\lambda(\omega) \sim \omega^{2/3}$, which gives $\langle J_N \rangle \sim 1/N^{9/2}$.

Free boundary conditions:

(a) For $\langle B \rangle \neq 0$, $T_\infty(\omega) \sim \omega^{1/2}$ and $\lambda(\omega) \sim \omega$, which gives $\langle J_N \rangle \sim 1/N^{3/2}$.

(b) For $\langle B \rangle = 0$, $T_\infty(\omega) \sim \omega^0$ and $\lambda(\omega) \sim \omega^{2/3}$, which gives $\langle J_N \rangle \sim 1/N^{3/2}$.

The results are summarized in table 2. Figure 3 shows the numerically obtained power laws for $\langle B \rangle \neq 0$ and $\langle B \rangle = 0$. Numerically, the power laws are obtained by calculating $T_N(\omega)$ for different $\omega$ and then performing the integration numerically. We expect to see the power law behaviour at some large enough $N$. We see a reasonable agreement with the theoretically expected power laws, except for the case with $\langle B \rangle = 0$ and free BC, where we get $\langle J_N \rangle \sim 1/N^2$ instead of the expected $\langle J_N \rangle \sim 1/N^{3/2}$.

The case with $\langle B \rangle = 0$ seems to be quite subtle because of the following reasons:

- The assumption that $T_\infty(\omega)$ may be replaced by the transmission for the uniform case for small $\omega$ does not hold well for the $\langle B \rangle = 0$ case. This can be clearly seen from figure 4, where we show a comparison of the transmission for small $\omega$ for $\langle B \rangle \neq 0$ and $\langle B \rangle = 0$ with their respective uniform cases. While $\langle B \rangle \neq 0$ shows a clear agreement

https://doi.org/10.1088/1742-5468/ac32b8
Localization effects due to a random magnetic field on heat transport in a harmonic chain

Figure 3. Numerically obtained power laws for the average current, averaged over 100 realizations of the disorder, with fixed and free boundary conditions. For $\langle B \rangle > 0$, $B_n$ is chosen from $(1, 3)$ while for $\langle B \rangle = 0$, $B_n$ is chosen from $(-2, 2)$.

Figure 4. Comparison for the transmission for disordered and uniform cases for the two boundary conditions. For $\langle B \rangle \neq 0$, $B_n$ is chosen from $(1, 3)$, while for $\langle B \rangle = 0$, $B_n$ is chosen from $(-1, 1)$. These are compared with the transmission for the uniform cases with $B_n = \langle B \rangle$, respectively.

with the corresponding uniform case, the $\langle B \rangle = 0$ case shows a clear disagreement. It is not clear how to estimate $T_\infty(\omega)$ for this case.

- Interestingly we note that the transmission coefficient has peaks at much lower frequencies than the ordered case. These peaks correspond to the normal modes of the...
isolated chain and it is then of interest to study the system size dependence of the lowest allowed normal mode frequency, $\omega_s^N$, for the disordered chains with $\langle B \rangle \neq 0$ and $\langle B \rangle = 0$, and the ordered case with $B = 0$. In figure 5, we show the scaling of $\omega_s^N$ with $N$. We see that, for $\langle B \rangle \neq 0$, $\omega_{\text{max}}^N \sim 1/N$ while $\omega_s^N \sim 1/N^2$. Thus, for any finite but large $N$, we have $\omega_{\text{max}}^N > \omega_s^N$ and there are a sufficient number of conducting modes. On the other hand, for $\langle B \rangle = 0$, both $\omega_{\text{max}}^N$ and $\omega_s^N$ scale as $1/N^{3/2}$ and this could be the reason why our heuristic approach for current scaling fails for this case.

5. Conclusion

We considered a harmonic chain of charged particles in the presence of random magnetic fields and derived power laws for the current with respect to the system size. The power laws were found to be sensitive to boundary conditions and the expectation value of the magnetic field. This was understood as arising from the different behaviours of the Lyapunov exponent $\lambda(\omega)$ and $T_{\infty}(\omega)$ for small frequency $\omega$.

Arguing that the small $\omega$ behaviour of $T_{\infty}(\omega)$ was the same as that for the ordered chain, we used the results obtained in our previous paper [21] using the NEGF approach. It was found there that this behaviour strongly depends on the presence (or absence) of the magnetic field, but also on the boundary conditions imposed. To estimate the Lyapunov exponent, we mapped the discrete time process, which determines Green’s functions to the motion of a harmonic oscillator with parametric noise. This not only
Localization effects due to a random magnetic field on heat transport in a harmonic chain revealed an interesting connection between the Lyapunov exponents of the two systems but also showed that the Lyapunov exponent have different behaviours for different expectation values of the magnetic field. For $\langle B \rangle > 0$, $\langle B \rangle = 0$ and $\langle B \rangle < 0$, we find that the Lyapunov exponents were of order $\omega, \omega^{2/3}$ and $\omega^{1/2}$, respectively. These behaviours of the Lyapunov exponent were also verified numerically.

Using the results for the $T_\infty(\omega)$ and $\lambda(\omega)$, we make analytic predictions of different system-size dependences of the current, depending on the expectation value of the magnetic field and the boundary conditions. For free boundary conditions, the current decreases as $1/N^{3/2}$ irrespective of the expectation value of the magnetic field. However, for fixed boundary conditions the current decreases as $1/N^{3/2}$ and $1/N^{9/2}$ for $\langle B \rangle \neq 0$ and $\langle B \rangle = 0$, respectively. Our direct numerical estimates show disagreement for the case $\langle B \rangle = 0$, and this is especially clear for the case with free boundary conditions. We discussed the possible reasons for the disagreement, amongst which is the intriguing numerical observation of the $1/N^{3/2}$ system-size dependence of the lowest normal mode frequency for the $\langle B \rangle = 0$ case. The resolution of this issue remains an interesting outstanding problem.

Acknowledgments

A D and J M B acknowledge support of the Department of Atomic Energy, Government of India, under Project No. RTI4001. The work of C B and G C has been supported by the projects LSD ANR-15-CE40-0020-01 of the French National Research Agency (ANR), by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovative programme (Grant Agreement No. 715734) and the French-Indian UCA project ‘Large deviations and scaling limits theory for non-equilibrium systems’.

Appendix A. Lyapunov exponent for a harmonic oscillator with parametric noise

In order to obtain an expansion of $\lambda_\varepsilon(\varepsilon)$, we follow the strategy developed by Pardoux and Wihstutz in [26] and by Wihstutz in [27]. The first step of the proof is to use the ergodic theorem to obtain an explicit formula (see equation (A.6)) for $\lambda_\varepsilon(\varepsilon)$ instead of equation (11). In the second step, we perform a perturbation analysis in $\varepsilon$ with this new expression.

First, we express the solution of the two-dimensional SDE $(z_t)_{t \geq 0}$ in terms of a one-dimensional SDE. Define $(\theta_t)_{t \geq 0}$ to be the solution of

$$\dot{\theta}_t = h_0(\theta_t) + \frac{1}{2}\varepsilon^2 \partial_\theta h_1(\theta_t) h_1(\theta_t) + \varepsilon h_1(\theta_t) \xi_t, \quad (A.1)$$

with

$$h_0(\theta) = \sin^2(\theta)(c - 1) - c \quad \text{and} \quad h_1(\theta) = -\sigma \cos^2(\theta). \quad (A.2)$$

https://doi.org/10.1088/1742-5468/ac32b8
One can check that
\[ z_t = R_t(\cos(\theta_t), \sin(\theta_t))^T \]
where
\[ R_t = ||z_0|| \exp \left( \int_0^t [g_0(\theta_r) + \varepsilon^2 r(\theta_r)] \, d\tau + \varepsilon \int_0^t q_1(\theta_r) \xi_r \, d\tau \right), \tag{A.3} \]
with
\[ q_0(\theta) = (1 - c) \cos(\theta) \sin(\theta), \quad q_1(\theta) = \sigma^2 \cos(\theta) \sin(\theta), \tag{A.4} \]
\[ r(\theta) = \frac{\sigma^2 \cos^2(\theta)}{2} [2 \cos^2(\theta) - 1]. \tag{A.5} \]
Observe that \( ||z_|| = R_0 \). Moreover, since in equation (A.1) the noise vanishes exactly at the points \( \theta_k = (2k + 1)\pi/2, \) \( k \in \mathbb{Z} \), and the drift in equation (A.1) at \( \theta_k \) is equal to \(-1\), we see that starting from \( \theta_0 \in [\theta_k, \theta_{k+1}] \) the process \( \hat{(\theta_t)}_{t \geq 0} \) will pass successively in the intervals \( \hat{\theta}_0 \in [\theta_{k+1}, \theta_{k+2}] \) for \( k \leq k - 1 \) without coming back to an interval previously visited. This defines a sequence of random times \( t_{k} = \inf \{ t \geq 0 \; ; \; \hat{\theta}_0 \in [\theta_{k}, \theta_{k+1}] \} \) for \( k \geq 0 \) with \( t_0 = 0 \). The process is thus clearly not ergodic. A simple way to restore this ergodicity (that will be needed later) is to consider the process \( \hat{(\theta_t)}_{t \geq 0} \), living in \([\pi/2, \pi/2]\), and defined by \( \hat{\theta}_t = \theta_t + (k - \ell)\pi \) for \( t \in [t_{k}, t_{k+1}] \). The process \( \hat{\theta}_t \) satisfies the same stochastic differential equation as \( (\theta_t)_{t \geq 0} \) but when it reaches \(-\pi/2\) it is immediately reset to \( \pi/2 \). Equivalently, \( \hat{(\theta_t)}_{t \geq 0} \) is the solution of equation (A.1) but seen as an SDE on the torus \([-\pi/2, \pi/2]\), where the two end points of the interval have been identified. The process \( \hat{(\theta_t)}_{t \geq 0} \) now has the nice property to be ergodic. We denote by \( \rho_\varepsilon(\theta)d\theta \) its invariant measure, which is computed below. Moreover, observe that equation (A.3) still holds by replacing \( \theta \) by \( \hat{\theta} \) because the functions \( q_0, q_1, r \) are \( \pi \)-periodic. In order to keep notation simple, we denote in the sequel the process \( \hat{\theta} \) by \( \theta \).

By definition (11) of Lyapunov exponent and equation (A.3), we get that
\[ \lambda_\varepsilon(\varepsilon) = \lim_{t \to \infty} \frac{1}{t} \left\langle \int_0^t [g_0(\theta_r) + \varepsilon^2 r(\theta_r)] \, d\tau + \varepsilon \int_0^t q_1(\theta_r) \xi_r \, d\tau \right\rangle = \lim_{t \to \infty} \frac{1}{t} \left\langle \int_0^t [g_0(\theta_r) + \varepsilon^2 r(\theta_r)] \, d\tau \right\rangle, \]
since \( \left\langle \int_0^t q_1(\theta_r) \xi_r \, d\tau \right\rangle = 0 \). Then by using the ergodic theorem, we obtain
\[ \lambda_\varepsilon(\varepsilon) = \int_{-\pi/2}^{\pi/2} [g_0(\theta) + \varepsilon^2 r(\theta)] \rho_\varepsilon(\theta)d\theta. \tag{A.6} \]
The expansion in \( \varepsilon \) for \( \lambda_\varepsilon(\varepsilon) \) can then be obtained from the expansion of \( \rho_\varepsilon \).

Before doing this we prove equation (12), i.e. that the process \( (z_t)_{t \geq 0} = ((u_t, v_t)^T)_{t \geq 0} \) and the process \( (u_t)_{t \geq 0} \) have the same Lyapunov exponent. By definition, we have
\[ \lim_{t \to \infty} \frac{1}{t} \langle \log |u_t| \rangle = \lim_{t \to \infty} \frac{1}{t} \langle \log ||z_t|| \rangle + \lim_{t \to \infty} \frac{1}{t} \langle \log |\cos(\theta_t)| \rangle. \tag{A.7} \]
Since \((\theta_t)\) is an ergodic process, we obtain that
\[
\lim_{t \to \infty} \frac{1}{t} \langle \log |\cos(\theta_t)| \rangle = \lim_{t \to \infty} \frac{1}{t} \int_{-\pi/2}^{\pi/2} \rho_\varepsilon(\theta) \log (|\cos(\theta)|) \, d\theta = 0.
\]
This proves the claim.

Let us now compute \(\rho_\varepsilon\) which is the solution of the stationary Fokker–Planck equation
\[
\partial_\theta \left[ \frac{\varepsilon^2}{2} \partial_\theta (h_1^2 \rho_\varepsilon) - (h_0 + \frac{\varepsilon^2}{2} h_1 \partial_b h_1) \rho_\varepsilon \right] = 0. \tag{A.8}
\]
If we look for a solution such that \(\frac{\varepsilon^2}{2} \partial_\theta (h_1^2 \rho_\varepsilon) - (h_0 + \frac{\varepsilon^2}{2} h_1 \partial_b h_1) \rho_\varepsilon = 0\), we get \(\rho_\varepsilon(\theta) \propto \cos^{-2}(\theta) e^{-\frac{2\varepsilon^2}{3} \tan^3(\theta) - \frac{2\varepsilon^2}{3} \tan(\theta)}\), which is not normalisable. Hence, we have to look for a normalisable solution, such that \(\frac{\varepsilon^2}{2} \partial_\theta (h_1^2 \rho_\varepsilon) - (h_0 + \frac{\varepsilon^2}{2} h_1 \partial_b h_1) \rho_\varepsilon = A\) for some constant \(A\). We then get that
\[
\rho_\varepsilon(\theta) = Z_\varepsilon^{-1} v_\varepsilon(\theta) \cos^{-2}(\theta) \int_{-\infty}^{\tan(\theta)} \exp \left( \frac{2\varepsilon^{-2}}{3\sigma^2} u^3 + \frac{2c\varepsilon^{-2}}{\sigma^4} \tan(\theta) \right) \, du
\]
with
\[
v_\varepsilon(\theta) = \exp \left\{ -\frac{2\varepsilon^{-2}}{3\sigma^4} \tan^3(\theta) - \frac{2c\varepsilon^{-2}}{\sigma^4} \tan(\theta) \right\}
\]
and \(Z_\varepsilon\) the partition function making \(\rho_\varepsilon\) a probability. Injecting this in equation (A.6), we may derive the results claimed by a careful saddle point analysis. We instead prefer to rely on a more heuristic analysis to bypass boring computations.

It is natural to expect that as \(\varepsilon \to 0\) the stationary measure \(\rho_\varepsilon(\theta) d\theta\) will converge to the one of \(\hat{\theta}_t = h_0(\theta_t)\) (i.e. equation (A.1) with \(\varepsilon = 0\)). However, as we will see, this deterministic dynamical system has different behaviours depending on the value of \(c\) and in some cases we also have to compute the next order corrections.

If \(c > 0\), the deterministic dynamical system has a unique invariant state \(\rho_0(\theta) d\theta\) with \(\rho_0(\theta) = -\frac{\varepsilon}{2} h_0^{-1}(\theta)\) because \(h_0\) never vanishes on \([-\pi/2, \pi/2]\). Hence, \(\rho_\varepsilon \to \rho_0\) as \(\varepsilon \to 0\). However, since \(\int_0^\pi g_0(\theta) \rho_0(\theta) \, d\theta = 0\), we have to expand \(\rho_\varepsilon\) at order \(\varepsilon^2\) to obtain the behaviour of \(\lambda_\varepsilon\) in equation (A.6). Let us assume that \(\rho_\varepsilon = \rho_0 + \varepsilon^2 \delta \rho_0 + o(\varepsilon^2)\), inject this in equation (A.8) and identify the powers in \(\varepsilon\). We obtain that
\[
\partial_\theta [h_0 (\delta \rho_0)] = \frac{1}{2} \partial_\theta \left[ \partial_\theta (h_1^2 \rho_0) - (h_1 \partial_b h_1) \rho_0 \right]
\]
which implies, since \(\int_{-\pi/2}^{\pi/2} (\delta \rho_0)(\theta) \, d\theta = 0\), that
\[
\delta \rho_0 = \frac{A}{h_0} + \frac{1}{2h_0} \left[ \partial_\theta (h_1^2 \rho_0) - (h_1 \partial_b h_1) \rho_0 \right].
\]
We deduce that
\[
\delta \rho_0 = \frac{A}{h_0} + \frac{\varepsilon^2 \sqrt{c}}{2\pi} \left( \frac{\sin(\theta) \cos^3(\theta)}{h_0^2} + (c - 1) \left( \frac{\cos^5(\theta) \sin(\theta)}{h_0^3(\theta)} \right) \right).
\]
Localization effects due to a random magnetic field on heat transport in a harmonic chain

Since \( \int_{-\pi/2}^{\pi/2} (\delta \rho_0)(\theta) d\theta = 0 \), we obtain \( A = 0 \) and

\[
\delta \rho_0 = \frac{\sigma^2 \sqrt{c}}{\pi} \left( \frac{\sin(\theta) \cos^3(\theta)}{h_0^2} + (c - 1) \frac{\cos^5(\theta) \sin(\theta)}{h_0^3} \right).
\]

Hence, we get that

\[
\lambda_\varepsilon(\varepsilon) = \varepsilon^2 \int_{-\pi/2}^{\pi/2} (r(\theta)\rho_0(\theta) + q_0(\theta)\delta \rho_0(\theta)) d\theta + o(\varepsilon^2).
\]

By the change of variable \( x = \tan(\theta) \), we get

\[
\int_{-\pi/2}^{\pi/2} r(\theta)\rho_0(\theta) d\theta = \frac{\sigma^2 \sqrt{c}}{2\pi} \int_{-\infty}^{\infty} \frac{x^2 - 1}{(1 + x^2)(x^2 + c)} dx = \frac{\sigma^2}{2(\sqrt{c} + 1)^2}.
\]

\[
\int_{-\pi/2}^{\pi/2} q_0(\theta)\delta \rho_0(\theta) d\theta = \sigma^2 \left( \frac{(4\sqrt{c} + 1)(c - 1)^2}{8(\sqrt{c} + 1)^4 c} + \frac{1 - c}{2(\sqrt{c} + 1)^3} \right).
\]

Hence, we finally get

\[
\lambda_\varepsilon(\varepsilon) = \varepsilon^2 \frac{\sigma^2}{8c} + o(\varepsilon^2).
\]

This proves case (b).

If \( c < 0 \), then \( \frac{c}{c-1} \in (0, 1) \) and the function \( h_0 \) vanishes on \([-\pi/2, \pi/2]\) if and only if \( \theta \in [-\pi/2, \pi/2] \) is solution of

\[
\sin^2(\theta) = \frac{c}{c-1}.
\]

There are two solutions \( \theta^* > 0 \) and \( -\theta^* < 0 \). The deterministic dynamical has two extremal invariant probability measures given by \( \delta_{\theta^*} \). Since \( h_0''(\theta^*) < 0 < h_0''(-\theta^*) \), \( \delta_{-\theta^*} \) is unstable while \( \delta_{\theta^*} \) is stable. By introducing noise in this dynamical system the stable stationary state is selected when the intensity of the noise is sent to zero afterwards, i.e. \( \rho_\varepsilon(\theta) d\theta \rightarrow \delta_{\theta^*} \). We conclude that

\[
\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon(\varepsilon) = q_0(\theta^*) = \sqrt{|c|}.
\]

This proves case (c).

The case \( c = 0 \) is more delicate. Since \( h_0(\cdot) = -\sin^2(\cdot) \), the unique invariant measure for the deterministic dynamical system is \( \delta_0 \) (stable) and we expect that \( \rho_\varepsilon(\theta) d\theta \rightarrow \delta_0 \) as \( \varepsilon \rightarrow 0 \). However, observe that \( q_0(0) = 0 \) so that we have to find the first correction to the approximation of \( \rho_\varepsilon \) to \( \delta_0 \). Due to the singularity of the Dirac mass, we cannot perform an expansion analysis in \( \varepsilon \). Hence, we will use another argument to get item (a). Consider the following linear transformation \( T_\varepsilon = \begin{pmatrix} \varepsilon^{2/3} & 0 \\ 0 & 1 \end{pmatrix} \), which is such that \( \varepsilon^{2/3} \|z\| \leq \|T_\varepsilon z\| \leq \|z\| \) for any \( z \in \mathbb{R}^2 \) and \( \varepsilon \leq 1 \). This implies that \( (z_t)_{t \geq 0} \) and \( (T_\varepsilon z_t)_{t \geq 0} \) have the same Lyapunov exponent. Expressing as we did before

\[
\hat{z}_t = T_\varepsilon z_t = \|\hat{z}_t\| (\cos \hat{\theta}_t, \sin \hat{\theta}_t)^T
\]

https://doi.org/10.1088/1742-5468/ac32b8
we notice that
\[ \dot{\hat{\theta}_t} = \varepsilon^{2/3} \left( -\sin^2(\hat{\theta}_t) - \sigma^2 \sin(\hat{\theta}_t) \cos^3(\hat{\theta}_t) \right) - \varepsilon^{1/3} \sigma \cos^2(\hat{\theta}_t) \xi_t, \]
which implies by scaling invariance of the white noise that
\[ \hat{\theta}_t = \alpha_t \varepsilon^{2/3} \]
where
\[ \dot{\alpha}_t = -\sin^2(\alpha_t) - \sigma^2 \sin(\alpha_t) \cos^3(\alpha_t) - \sigma \cos^2(\alpha_t) \xi_t. \]

If \( \hat{\rho}(\alpha) d\alpha \) is the unique invariant measure for \( (\alpha_t)_{t \geq 0} \), we have by a scaling argument that the Lyapunov exponent satisfies
\[ \lambda_{\varepsilon}(\varepsilon) = \varepsilon^{2/3} \hat{\lambda}(\sigma) \]
with
\[ \hat{\lambda}(\sigma) = \int_{-\pi/2}^{\pi/2} (q_0(\alpha) + r(\alpha)) \hat{\rho}(\alpha) d\alpha, \quad (A.9) \]
where \( q_0 \) and \( r \) are defined respectively in equations (A.4) and (A.5) with \( c = 0 \). To obtain the value of \( \hat{\lambda}(\sigma) \), it is sufficient to find \( \hat{\rho} \), which is the unique normalisable function of the Fokker–Planck equation associated to the process \( (\alpha_t)_{t \geq 0} \), i.e.
\[ \hat{\rho}(\alpha) = \hat{Z}^{-1} \cos^{-2}(\alpha) e^{-\frac{1}{3\sigma^2} \tan^3(\alpha)} \int_{-\infty}^{\tan(\alpha)} \exp \left( \frac{2u^3}{3\sigma^2} \right) du, \]
where \( \hat{Z} \) is the normalisation constant making \( \hat{\rho} \) a probability measure.

References

[1] Anderson P W 1958 Absence of diffusion in certain random lattices Phys. Rev. 109 1492–505
[2] Casher A and Lebowitz J L 1971 Heat flow in regular and disordered harmonic chains J. Math. Phys. 12 1701–11
[3] Rubin R J and Greer W L 1971 Abnormal lattice thermal conductivity of a one-dimensional, harmonic, isotopically disordered crystal J. Math. Phys. 12 1686–701
[4] O’Connor A J and Lebowitz J L 1974 Heat conduction and sound transmission in isotopically disordered harmonic crystals J. Math. Phys. 15 692–703
[5] Verheggen T 1979 Transmission coefficient and heat conduction of a harmonic chain with random masses: asymptotic estimates on products of random matrices Commun. Math. Phys. 68 69–82
[6] Ajanki O and Huveneers F 2011 Rigorous scaling law for the heat current in disordered harmonic chain Commun. Math. Phys. 301 841–83
[7] Dhar A 2001 Heat conduction in the disordered harmonic chain revisited Phys. Rev. Lett. 86 5882–5
[8] Roy D and Dhar A 2008 Role of pinning potentials in heat transport through disordered harmonic chains Phys. Rev. E 78 051112
[9] De Roeck W, Dhar A, Huveneers F and Schütz M 2017 Step density profiles in localized chains J. Stat. Phys. 167 1143–63
[10] Bernardin C, Huveneers F and Olla S 2019 Hydrodynamic limit for a disordered harmonic chain Commun. Math. Phys. 365 215–37
[11] Amir A, Oreg Y and Imry Y 2018 Thermal conductivity in 1D: disorder-induced transition from anomalous to normal scaling Europhys. Lett. 124 16001
[12] Ash B, Amir A, Bar-Sinai Y, Oreg Y and Imry Y 2020 Thermal conductance of one-dimensional disordered harmonic chains Phys. Rev. B 101 121403
[13] Lee L W and Dhar A 2005 Heat conduction in a two-dimensional harmonic crystal with disorder Phys. Rev. Lett. 95 094302

https://doi.org/10.1088/1742-5468/ac32b8
Localization effects due to a random magnetic field on heat transport in a harmonic chain

[14] Chaudhuri A, Kundu A, Roy D, Dhar A, Lebowitz J L and Spohn H 2010 Heat transport and phonon localization in mass-disordered harmonic crystals Phys. Rev. B 81 064301
[15] Dhar A and Lebowitz J L 2008 Effect of phonon-phonon interactions on localization Phys. Rev. Lett. 100 134301
[16] Dhar A and Saito K 2008 Heat conduction in the disordered Fermi–Pasta–Ulam chain Phys. Rev. E 78 061136
[17] Bernardin C 2008 Thermal conductivity for a noisy disordered harmonic chain J. Stat. Phys. 133 417–33
[18] Dhar A, Venkateshan K and Lebowitz J L 2011 Heat conduction in disordered harmonic lattices with energy-conserving noise Phys. Rev. E 83 021108
[19] Bernardin C and Huveneers F 2013 Small perturbation of a disordered harmonic chain by a noise and an anharmonic potential Probab. Theor. Relat. Field 157 301–31
[20] Bernardin C, Huveneers F, Lebowitz J L, Liverani C and Olla S 2015 Green-Kubo formula for weakly coupled systems with noise Commun. Math. Phys. 334 1377–412
[21] Bhat J, Cane G, Bernardin C and Dhar A 2021 Heat transport in an ordered harmonic chain in presence of a uniform magnetic field J. Stat. Phys. (arXiv:2106.12069)
[22] Furstenberg H 1963 Noncommuting random products Trans. Am. Math. Soc. 108 377
[23] Crauel H, Imkeller P and Steinkamp M 1999 Bifurcations of One-Dimensional Stochastic Differential Equations (New York: Springer)
[24] Wihstutz V 1999 Perturbation Methods for Lyapunov Exponents (New York: Springer) pp 209–39
[25] Matsuda H and Ishii K 1970 Localization of normal modes and energy transport in the disordered harmonic chain Prog. Theor. Phys. Suppl. 45 56–86
[26] Pardoux E and Wihstutz V 1988 Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion SIAM J. Appl. Math. 48 442–57
[27] Wihstutz V 1981 Ergodic Theory of Linear Parameter-Excited Systems (Berlin: Springer) pp 205–18

https://doi.org/10.1088/1742-5468/ac32b8