On the questions $P \overset{?}{=} NP \cap \text{co-NP}$ and $NP \overset{?}{=} \text{co-NP}$ for infinite time Turing machines

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Schindler recently addressed two versions of the question $P \overset{?}{=} NP$ for Turing machines running in transfinite ordinal time. These versions differ in their definition of input length. The corresponding complexity classes are labelled $P$, NP and $P^+, NP^+$. Schindler showed that $P \neq NP$ and $P^+ \neq NP^+$. We show that $P = NP \cap \text{co-NP}$ and $NP \neq \text{co-NP}$, whereas $P^+ \subset NP \cap \text{co-NP}$ and $NP^+ \neq \text{co-NP}^+$.

Key Words: Complexity theory, Descriptive set theory.

1. INTRODUCTION

The fundamental open problems in complexity theory for Turing machines running in finite time are whether $P \overset{?}{=} NP$, whether $P \overset{?}{=} NP \cap \text{co-NP}$ and whether $NP \overset{?}{=} \text{co-NP}$.

After Hamkins and Lewis [1] formalized the notion of a Turing machine running in transfinite ordinal time, the corresponding questions about infinite time Turing machines were posed. Schindler recently defined two
versions of the classes P and NP in a Turing machine running in transfinite ordinal time. He labelled them P, NP and P+, NP+ and showed that P ≠ NP and P+ ≠ NP+. The proofs used a classical theorem from descriptive set theory stating that not all analytic sets are Borel. Analogies between the classes P and NP, and the classes of Borel and analytic sets respectively, had earlier been drawn by Sipser and others (cf. [3] and the references therein).

We address the problems of whether P \supseteq NP \cap \text{co-NP} (resp. P+ \supseteq NP+ \cap \text{co-NP}+) and whether NP \supseteq \text{co-NP} (resp. NP+ \supseteq \text{co-NP}+). We use a classical result by Suslin which characterizes analytic sets that are Borel as those whose complements are also analytic. Using Suslin’s theorem, we prove that P = NP \cap \text{co-NP}. We then show that P+ is a strict subset of NP+ \cap \text{co-NP}+ using properties of projections of hyperarithmetical sets.

We also observe that NP ≠ \text{co-NP} and NP+ ≠ \text{co-NP}+.

2. PRELIMINARIES

This section will provide some notions from descriptive set theory needed in order to state our results. For details, the reader is referred to [4, 5].

We first fix our notation. The first infinite ordinal is denoted by \omega, the first uncountable ordinal by \omega_1, the first non-recursive ordinal (i.e. the Church-Kleene ordinal) by \omega^\text{CK}_1, and the ordinal of Gödel’s constructible universe \text{L} by \omega^\text{L}_1. A Polish space, i.e. a space that is separable and completely metrizable, is denoted by \text{X}. In this paper, we are particularly interested in Cantor space \omega^2 and Baire space \omega^\omega.
The class of Borel sets in $X$ are denoted by $\mathcal{B}(X)$. This class ramifies in an infinite Borel Hierarchy whose classes are denoted by $\Sigma^0_\xi$ and $\Pi^0_\xi$. The projective sets obtained from Borel sets by the operations of projection and complementation ramify into a projective hierarchy of length $\omega$ whose classes are denoted by $\Sigma^1_n$ and $\Pi^1_n$. The ambiguous classes are denoted by $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$. Members of the class $\Sigma^1_1$ are called analytic, members of $\Pi^1_1$ are called co-analytic and members of $\Delta^1_1$ are called bi-analytic. The effective analogues of the Borel classes are denoted by lightface $\Sigma^0_\xi$ and $\Pi^0_\xi$.

Similarly the effective analogues of the projective classes are denoted by $\Sigma^1_n$, $\Pi^1_n$ and $\Delta^1_n$.

We need the following two central results in descriptive set theory.

**Theorem 2.1** (Lusin’s Separation Theorem). If $X$ is a standard Borel space, and $A, B \subseteq X$ are two disjoint analytic sets, then there is a Borel set $C \subseteq X$ such that $A \subseteq C$ and $C \cap B = \emptyset$.

**Theorem 2.2** (Suslin’s Theorem). For a Polish space $X$, $\mathcal{B}(X) = \Delta^1_1(X)$.

**Proof.** The Effros Borel space of the set of closed subsets of a Polish space is standard so the Lusin separation theorem can be applied. The proof of Suslin’s theorem is immediate by letting $B$ be the complement of $A$ in the Lusin separation theorem. 

**Definition 2.1.** Let $\alpha$ be a countable ordinal. We say that $A \subseteq \omega^2$ is $\Delta^1_1(\alpha)$ in $\alpha$ if it is uniformly $\Delta^1_1$ in any real $x$ coding a well-order of order type $\alpha$. 

Lemma 2.1. Let $A$ be $\Delta^1_1(\alpha)$ in a countable ordinal $\alpha$ as defined above. Let $B$ be its projection. If $B$ is a Borel set, then there exists a countable ordinal $\beta$ such that $B$ is $\Delta^1_1(\beta)$.

Proof. Let $WO$ be the set of codes of countable ordinals, and $WO(\alpha)$ be the set of codes of the ordinal $\alpha$. Let $B^c$ be the complement of $B$. Since $B$ is $\Sigma^1_1$ in $\alpha$, it follows that there is a $\Pi^1_1$ set $D$ such that

$$x \in B^c \iff (x, w) \in D, \text{ for any real } w \text{ coding } \alpha.$$ 

Let $f$ be a recursive function such that

$$(x, w) \in P \iff f(x, w) \in WO.$$ 

Then $B^c \times WO(\alpha)$ is Borel and contained in $D$, therefore its image under $f$ is analytic and contained in $WO$. By the Boundedness theorem, there is an ordinal $\delta$, such that if $|w|$ denotes the ordinal coded by $w$, we have

$$x \in B^c \iff |f(x, w)| < |v|$$

for any code $w$ of $\alpha$ and any code $v$ of $\delta$. It immediately follows that $B$ is $\Delta^1_1(\beta)$, for some $\beta$. $\blacksquare$

The countable ordinal $\beta$ is possibly much larger than $\alpha$. We have the following result from [6, Theorems 1.4,2.3].

Lemma 2.2. There exist $\Delta^1_1$ sets with projections that are $\Delta^1_1(\omega^L_1)$, but not $\Delta^1_1(\alpha)$ for any $\alpha < \omega^L_1$.

Corollary 2.1. There exist hyperarithmetical sets whose projections are Borel but not hyperarithmetical.
Proof. Immediate from the Lemma 2.2 by letting $\alpha$ be recursive and observing that $\omega^L_1 > \omega^C_1$. □

3. INFINITE TIME TURING MACHINES

Infinite time Turing machines were introduced in [1]. There has been growing research interest in these machines especially after Schindler [2] showed that $\text{P} \neq \text{NP}$ in this transfinite setting. We recall the following definitions from [2].

**Definition 3.1.** Let $A \subseteq \omega^2$ and let $\alpha \leq \omega_1 + 1$. Then $A$ is in $P_\alpha$ if there exists a Turing machine $T$ and some $\beta < \alpha$ such that
(a) $T$ decides $A$
(b) $T$ halts on all inputs after $< \beta$ many steps.

**Definition 3.2.** Let $A \subseteq \omega^2$ and let $\alpha \leq \omega_1 + 1$. Then $A$ is in $\text{NP}_\alpha$ if there exists a Turing machine $T$ and some $\beta < \alpha$ such that
(a) $x \in A$ if and only if $(\exists y$ such that $T$ accepts $x \oplus y)$
(b) $T$ halts on all inputs after $< \beta$ many steps.

3.1. The classes $P$ and $\text{NP}$ in infinite time Turing machines

If we let all inputs $x \in \omega^2$ as having the same length $\omega$, then we have the following characterization of the classes $P$ and $\text{NP}$.

**Definition 3.3.** $P = P_{\omega^2}$ and $\text{NP} = \text{NP}_{\omega^2}$.

The following description of the class $P$ is given in [2, Lemmas 2.5, 2.6].
Lemma 3.1. Let $A \subset \omega^2$. Then $A \in P_{\omega_1^{CK}}$ if and only if $A$ is a hyperarithmetic set. Furthermore, $A \in P_{\omega_1}$ if and only if $A$ is $\Delta^1_1(\alpha)$ in a countable ordinal $\alpha$.

In particular, for a Borel set to be outside of $P_{\omega_1}$, it must be $\Delta^1_1$ in a real $x$ that does not code any countable well-order.

Now we state our main theorem.

Theorem 3.1. $P = NP \cap co-NP$.

Proof. Since $P \subset NP$ and $P = co-P$, it follows that $P \subset co-NP$ and therefore $P \subseteq NP \cap co-NP$. Let $B \subset \omega^2$ satisfy $B \in NP \cap co-NP$. Since $B \in NP$, it is the projection of a set in $P$. However, $P \subset \Delta^1_1$, which implies that $B \in \Sigma^1_1$. Let $B^c$ be the complement of $B$. Since $B \in co-NP$, it follows that $B^c \in NP$. Thus $B^c \in \Sigma^1_1$, which implies that $B \in \Pi^1_1$. Therefore $B \in \Sigma^1_1 \cap \Pi^1_1 = \Delta^1_1$. From Suslin’s theorem, it follows that $B$ must be Borel.

We cannot immediately infer that $B$ is in $P$ because not all Borel sets are in $P$. Those in $P$ are precisely those that are $\Delta^1_1(\alpha)$ in a countable ordinal $\alpha$ (note the use of lightface font as opposed to the boldface in the previous paragraph). We need to show that $B$ is actually a Borel set in a countable ordinal. Now $B$ is a projection of a set in $P$, which means it is the projection of a set that is $\Delta^1_1(\alpha)$ in a countable ordinal $\alpha$. Further, we have shown that $B$ is Borel. Thus $B$ satisfies the hypothesis of Lemma 2.1. It follows that $B$ itself is $\Delta^1_1(\beta)$ in a countable ordinal $\beta$. In other words, $B$ is in $P$. Since this argument holds for any element of $NP \cap co-NP$, this completes the proof.
Theorem 3.2. \(\text{NP} \neq \text{co-NP}\).

Proof. This follows immediately from \(\text{P} \neq \text{NP}\) and \(\text{P} = \text{NP} \cap \text{co-NP}\). An alternative proof is obtained by observing that in [2, Section 1] and using the notation there, the lightface analytic set \(\Delta\) which is the projection of a lightface \(G_\delta\) set \(G\) is in \(\text{NP}\) but not in \(\text{co-NP}\).

Remark 3.1. There is an interesting interplay between the answers to the questions \(\text{P} \equiv \text{NP}\), \(\text{P} \equiv \text{NP} \cap \text{co-NP}\), and \(\text{NP} \equiv \text{co-NP}\). Without the result \(\text{P} = \text{NP} \cap \text{co-NP}\), \(\text{NP} \neq \text{co-NP}\) is a stronger statement than \(\text{P} \neq \text{NP}\) because it implies \(\text{P} \neq \text{NP}\), however the converse is not true. On the other hand, if \(\text{P} = \text{NP} \cap \text{co-NP}\), then \(\text{NP} \neq \text{co-NP}\) if and only if \(\text{P} \neq \text{NP}\).

3.2. The classes \(P^+\) and \(NP^+\) in infinite time Turing machines

If we regard an input \(x \in \omega^2\) as having length \(\omega^x\), which is the least \(x\)-admissible ordinal greater than \(\omega\), then the versions of the classes \(P\) and \(NP\) obtained are labelled \(P^+\) and \(NP^+\). We recall the following definitions from [2]

Definition 3.4. Let \(A \subseteq \omega^2\). We say that \(A\) is in \(P^+\) if there exists a Turing machine \(T\) such that

(a) \(x \in A\) if and only if \(T\) accepts \(x\)
(b) \(T\) halts on all inputs after \(\omega^x\) many steps.

Definition 3.5. Let \(A \subseteq \omega^2\). We say that \(A\) is in \(NP^+\) if there exists a Turing machine \(T\) such that

(a) \(x \in A\) if and only if \((\exists y\) such that \(T\) accepts \(x \oplus y\))
(b) \(T\) halts on all inputs \(x \oplus y\) after \(\omega^x\) many steps.
Theorem 3.3. \([2, \text{Theorem } 2.13]\) \(P^+ = P_{\omega_1^{CK}} = \Delta_1^1\)

Corollary 3.1. \([2, \text{Corollary } 2.14]\) \(P^+ \neq NP^+\).

We now show that the class \(P^+\) is a strict subset of \(NP^+ \cap \text{co–NP}^+\).

Theorem 3.4. \(P^+ \subset NP^+ \cap \text{co–NP}^+\).

Proof. Follows from Corollary 2.1. \(\blacksquare\)

Theorem 3.5. \(NP^+ \neq \text{co–NP}^+\)

Proof. Follows immediately from the proof of \([2, \text{Theorem } 2.13]\). \(\blacksquare\)

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