Splitting of two-component solitons from collisions with narrow potential barriers

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We consider the interaction of two component bright–bright solitons with a narrow potential barrier (splitter) in the framework of the two-component Gross–Pitaevskii (nonlinear-Schrödinger) system, with self-atraction within each component and cross-attraction between the components, motivated by the splitting of composite solitons facilitating the design of two-component soliton-interferometer schemes. We determine approximate analytic results by assuming a weak potential barrier and applying perturbation theory to the limit where the system becomes integrable. We do this for the case of negligible interspecies interactions, and also when the nonlinearities are strongly asymmetric, such that the wavefunction for the component with a smaller population is solved by neglecting its self-interaction term and the other component constitutes a bright soliton. We use numerical simulations to find the transmissions of both components in regions outside of these approximations and to compare with the approximate analytics, finding that there is an appreciable parameter range where one component is essentially entirely transmitted and the other reflected.

I. INTRODUCTION

Solitons are manifest in a broad range of physical settings \cite{1,2}, including in particular nonlinear matter waves in atomic Bose–Einstein condensates (BECs) \cite{3–6}. In the mean-field approximation, the commonly adopted dynamical model of a BEC is based on the Gross–Pitaevskii equation (GPE) for a single-component condensate, and a system of coupled GPEs for binary (two-component) mixtures \cite{7}. One of the potential applications of matter-wave solitons is in the design of interferometers, in which an incident soliton splits into two fragments upon hitting a narrow potential barrier, followed by recombination of the fragments after rebounding from the confining potential. An object to be probed by the interferometer is placed as an obstacle through which one fragment will pass, which will affect the outcome of the recombination \cite{8–10}. Soliton interferometers have been elaborated theoretically in various configurations \cite{11–20} (including the case when the splitter is inserted as a localized self-repulsive nonlinearity, or its combination with the usual potential barrier \cite{21}) and realized in experiment \cite{9}. Interactions of matter-wave solitons with local potentials have also been studied in other contexts, such as an analytical treatment of the collisions \cite{22}, rebound from potential wells \cite{23,24}, dynamics of solitons in a dipolar BEC \cite{12}, and probing effects of interparticle interactions on tunneling \cite{25,26}. However, the splitting of a fundamental soliton by a linear and/or nonlinear potential barrier implies, in a sense, the application of “brute force” to a soliton, as its intrinsic structure does not resonate with the action of the splitter. A more natural option, which was elaborated recently, is fission of a 2-soliton (breather) into its fundamental-soliton constituents \cite{27} (see also Ref. \cite{28}), with the amplitude ratio close to the natural value, $3:1$ \cite{29} (see also \cite{30} for a similarly motivated protocol involving a laser pulse in combination with control of the scattering length). These settings may also be realized in the context of optics, in which case the GPE is replaced by the nonlinear Schrödinger equation (NLSE) for the spatial-domain propagation of light in planar waveguides \cite{2}.

In this work, we aim to elaborate another natural scheme for the splitting, when an incident two-component soliton, governed by a pair of nonlinearly coupled GPEs, hits a narrow splitting barrier. The situation under consideration is one with equal atomic masses and equal negative scattering lengths within the two components, and attractive interaction between the components, while there is no linear coupling (interconversion) between them (any interconversion would make splitting of a composite soliton into single-component ones impossible). We note that replacing time in the coupled GPEs by the propagation distance, $z$, this model also applies to bimodal light propagation in a Kerr-nonlinear waveguide with transverse coordinate $x$, while $\psi_1$ and $\psi_2$ are amplitudes of two components of the electromagnetic wave, corresponding to different carrier wavelengths \cite{2}, and where the potential represents transverse modulation of the refractive index. However, in this case the strength of the cross-interaction can only take a single value ($g = 2$, as defined later in the paper) as there is no straightforward optical counterpart to the Feshbach-resonance technique. Alternatively, if $\psi_1$ and $\psi_2$ represent the amplitudes of two waves with mutually orthogonal linear polarizations, the relevant value is $g = 2/3$, provided that rapidly oscillating four-wave-mixing terms may be neglected \cite{2}.

As mentioned above, previous works have addressed collisions of single-component solitons with potential barriers, represented by an ideal 6-function or a narrow Gaussian potential barrier, aiming to identify outcomes of the collisions as functions of the velocity of the incoming soliton and the barrier’s height and width \cite{8,22}. Two-component soliton dynamics have been studied with regard to their intrinsic vibrations in free space \cite{31}, as well as collisions between two solitons on a narrow Gaussian barrier added to Manakov’s system...
also the scattering of dark-bright solitons by impurities [33]. The main objective of the present work is to identify a parameter region in which the collision of a composite soliton with the barrier effectively splits it into single-component constituents. The primary control parameters we consider are the relative norm of the components, defined by parameter $f$, the velocity of the incident soliton, the strength of the barrier $\varepsilon$ [see Eq. (4)], and the strength of the interspecies attraction $g$ [see Eqs. (5a) and (5b)]. We first report approximate analytical results, obtained for the system with a weak barrier, in Section II. We then summarize results of systematic numerical simulations of the collisions in Section III. We compare analytical results to their numerical counterparts in Section III, and conclude the paper with Section IV.

II. SYSTEM OVERVIEW

We consider a two-component BEC system, where the two components are provided by different internal states of the same atomic species, and collisions are dominated by s-wave scattering. We describe this system in terms of two coupled Gross–Pitaevskii equations, where the component atoms are radially confined by a far-off-resonant optical waveguide providing approximately harmonic trapping in the $y$ and $z$ directions, but are axially ($x$ direction) relatively weakly confined, if at all. In addition, we impose an off-resonant sheet of light propagating perpendicular to the axial direction with peak beam strength $E_B$ and an axial direction 1/e radius $x$, which provides a barrier potential for both components centered at $x = 0$ [10, 34]. We assume the off-resonant optical waveguide and barrier potentials to be insensitive to the atomic internal state, allowing the coupled Gross–Pitaevskii equations to take the form

$$i\frac{\partial}{\partial t} \Psi_1(r) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V(x) + ma_\perp^2 y^2 + \frac{z^2}{2} \right] \Psi_1(r) + \frac{4\pi\hbar^2 N}{m} \left[ a_{11} |\Psi_1(r)|^2 + a_{12} |\Psi_2(r)|^2 \right] \Psi_1(r),$$

$$i\frac{\partial}{\partial t} \Psi_2(r) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V(x) + ma_\perp^2 y^2 + \frac{z^2}{2} \right] \Psi_2(r) + \frac{4\pi\hbar^2 N}{m} \left[ a_{22} |\Psi_2(r)|^2 + a_{12} |\Psi_1(r)|^2 \right] \Psi_2(r),$$

(1a)

(1b)

where $m$ is the atomic mass, $a_{11}$, $a_{22}$, and $a_{12}$ are the intra- and inter-species s-wave scattering lengths, $V(x) = m\omega_r^2 x^2/2 + E_B e^{-2x^2/\xi^2}$ combines a weak harmonic axial trapping potential and the barrier potential, $\omega_T$ and $\omega_r$ are axial and radial harmonic frequencies, and $N$ is the total particle number. Note that we have chosen the normalisation convention $\int dr |\Psi_1(r)|^2 = f$, $\int dr |\Psi_2(r)|^2 = 1 - f$, such that

$$\int dr \left[ |\Psi_1(r)|^2 + |\Psi_2(r)|^2 \right] = 1,$$

(2)

and the numbers of particles in the two components are given by $N_1 = f N$ and $N_2 = (1 - f) N$.

Strong radial confinement then permits us to assume a Gaussian ansatz $\phi(y, z) = (m\omega_r/\pi)^{1/2} \exp(-m\omega_r[y^2 + z^2]/2\hbar)$ for the radial degrees of freedom of the condensate wavefunctions $\Psi_1(r)$, $\Psi_2(r)$. We integrate over $y$ and $z$, define $g_{11} \equiv 2\hbar \omega_d a_{11}$, and express the result in terms of a units system with unit position $\hbar$, unit time $\hbar/m g_{11} N^2$, and unit energy $m g_{11} N/\hbar^2$ [directly implying a unit velocity of $|g_{11}| N/\hbar$, and that after integration over $y$ and $z$ we effectively multiply the the condensate wavefunctions by $h/(m g_{11} N)^{1/2}$ to render them dimensionless].

This yields

$$i\frac{\partial}{\partial t} \psi_1(x) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{\omega_r^2 x^2}{2} + \eta(x, \sigma) \right] \psi_1(x) - \left[ \psi_1(x)^2 + g \psi_2(x)^2 \right] \psi_1(x),$$

(3a)

$$i\frac{\partial}{\partial t} \psi_2(x) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{\omega_r^2 x^2}{2} + \eta(x, \sigma) \right] \psi_2(x) - \left[ g \psi_2(x)^2 + g \psi_1(x)^2 \right] \psi_2(x),$$

(3b)

where $g = a_{12}/a_{11}$, $g' = a_{22}/a_{11}$, $\omega_r = \omega_T \hbar/m g_{11} N^2$, $\epsilon = E_B x_2 (\pi/2)^{1/2}/m g_{11} N^3$, $\sigma = x/2\hbar^2/m g_{11} N$, and

$$\eta(x, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-x^2/2\sigma^2),$$

(4)

such that $\lim_{\sigma \to 0} \eta(x, \sigma) = \delta(x)$, and we have assumed all the scattering lengths to be negative. Here $g \equiv a_{12}/a_{11}$ is the relative strength of the cross-attraction between the components, which can be effectively adjusted by means of the Feshbach-resonance technique [35, 36], and $\epsilon > 0$ is the strength of the splitting barrier. Direct control of the properties of binary BECs has been demonstrated in [37] for a heteronuclear BEC and in [38] for a BEC consisting of different hyperfine states of the same species where the interspecies interaction was varied to probe the miscibility–immiscibility threshold. One must bear in mind that although it has become a fairly standard technique to control the scattering length in BEC systems, conventionally using magnetic Feshbach resonances, there are limitations as to what can be achieved in multicomponent BEC systems. For instance, when exploiting a magnetic Feshbach resonance, in principle all of the scattering lengths are dependent on the value of the applied magnetic field and therefore cannot be varied independently. Hence, the Feshbach resonance technique must be utilised in such a way that the three scattering lengths in the two component system studied here are brought as close as possible to their desired values. For the numerical results presented in Section III we have fixed $g' = 1$ (implying that $a_{11} = a_{22}$) and varied $g$. We point out that in the particular case of $g = 0$, in the Gross–Pitaevskii treatment considered in this paper, the value of $g'$ has a role equivalent to that of $f$ in that it defines the relative self-interaction strength of the two condensate components. Furthermore, there may be regimes where for example
energy functional which also yields the total (conserved) system energy in the mean-field treatment.

Setting aside the barrier potential (i.e., setting \( \varepsilon = 0 \)), two exactly integrable cases present themselves. If \( g = 0 \), then Eq. (5) corresponds to two decoupled integrable NLSEs; and, if \( g = g' = 1 \), Eq. (5) then constitutes the integrable Manakov system [42]. More generally, the \( g = -1 \) case describing repulsive interspecies interactions is also exactly integrable, and it should also be noted that, even in the general attractive case where \( g = g' = 1 \) is not fulfilled, two-component bright soliton solutions, where both components have identical mode profiles proportional to \( \text{sech}(F[x - (x_0 + vt)]/2) \) [i.e., of the same form as in Eq. (8)], exist if \( f + (1 - f)g = (1 - f)g' + fg' \equiv F \). This equality condition is equivalent to requiring \( f = (g' - g)/(g' + 1 - 2g) \), or, in terms of the more immediately physical quantities of particle numbers and (assumed negative) scattering lengths:

\[
\frac{N_1}{N} = \frac{a_{12} - a_{22}}{2a_{12} - a_{11} - a_{22}}.
\]  

In summary, the existence of very well behaved regimes for certain parameter values provides scope for making use of perturbation theory in the regime of relatively small \( \varepsilon \).

**B. Limit of negligible interspecies interactions**

1. **Basic system properties**

We are interested in a solution of Eq. (5) with \( g = 0 \) (no interspecies interactions) consisting initially of a composite soliton of both species centered at the same point in space, i.e., such that the peak of both components of the composite soliton move according to \( x_0 + vt \). Assuming the composite soliton to be located sufficiently far from the potential barrier for its influence to be considered negligible, this is given by

\[
\psi_1(x,t) = \frac{f}{2} \exp \left( i \left[ vx - \mu_1 t \right] \right) \times \text{sech} \left( \frac{f}{2} [x - (x_0 + vt)] \right),
\]

\[
\psi_2(x,t) = \frac{(1 - f) \sqrt{g}}{2} \exp \left( i \left[ vx - \mu_2 t \right] \right) \times \text{sech} \left( \frac{[1 - f]g'}{2} [x - (x_0 + vt)] \right),
\]

with associated chemical potentials

\[
\mu_1 = f^2/8 + v^2/2,
\]

\[
\mu_2 = -(1 - f)^2 g^2/8 + v^2/2.
\]

The center-of-mass kinetic energies of each component (essentially half the total mass of the soliton multiplied by the square of the velocity with which it is moving) are given in terms of our unit energy \( m(g_{11}N)/h^2 \) by:

\[
(E_{\text{cm}})_1 = f^2/2,
\]

\[
(E_{\text{cm}})_2 = (1 - f)^2 v^2/2.
\]
and we determine the intraspecies interaction potential energies from terms on the second line of Eq. (6):

\begin{align}
(E_{\text{int}})_1 &= -f^3/12, \quad (11a) \\
(E_{\text{int}})_2 &= -(1 - f)^3g^2/12. \quad (11b)
\end{align}

The potential energy of each component associated with the weak potential barrier can be easily found in the framework of perturbation theory (with the parameter \(\varepsilon\) assumed to be sufficiently small), which neglects deformation of the soliton under the influence of the barrier potential \([43]\):

\begin{align}
U_1(t) &\equiv \varepsilon \int_{-\infty}^{\infty} dx \, \delta(x) |\psi_1(x, t)|^2 \\
&= \frac{1}{4} \varepsilon f^2 \text{sech}^2 \left( \frac{f}{2} |x_0 + vt| \right), \quad (12a) \\
U_2(t) &\equiv \varepsilon \int_{-\infty}^{\infty} dx \, \delta(x) |\psi_2(x, t)|^2 \\
&= \frac{1}{4} \varepsilon (1 - f)^2 g^2 \text{sech}^2 \left( \frac{11 - f \delta g'}{2} |x_0 + vt| \right). \quad (12b)
\end{align}

The perturbation theory we have used applies (to each component) provided that the magnitudes of the interaction potential energies [Eq. (11)] dominate the peak values of the barrier potential) provided that the magnitudes of the interaction potential energies, as determined by Eq. (12) when \(x_0 + vt = 0 \Rightarrow t = -x_0/v\). This gives the potential energy associated with the unmodified component solitons being located exactly on top of the barrier,

\begin{align}
(U_{\text{max}})_1 &= \varepsilon f^2/4, \quad (13a) \\
(U_{\text{max}})_2 &= \varepsilon (1 - f)^2 g'/4, \quad (13b)
\end{align}

which, compared with Eq. (11), reveals the condition for sufficiently small \(\varepsilon\) to be

\begin{equation}
\varepsilon \ll f/3. \quad (14)
\end{equation}

2. Splitting on a barrier

In the \(g = 0\) case we are considering, we can reasonably estimate that each component transmits through the barrier under the condition that the respective peak potential energy, as given by Eq. (13), is exceeded by that component’s center-of-mass kinetic energy. Combining the results of Eq. (10) and Eq. (13) therefore results in the conditions

\begin{align}
v^2 &> \varepsilon f^2/2, \quad (15a) \\
v^2 &> \varepsilon (1 - f) g'/2, \quad (15b)
\end{align}

for components 1 and 2, respectively, to be transmitted. We therefore predict an incident composite soliton to be split into a pair of transmitted component 1 and reflected component 2 solitons within the following interval of velocities:

\begin{equation}
\sqrt{\varepsilon f^2/2} < |v| < \sqrt{\varepsilon (1 - f) g'/2} \quad (16)
\end{equation}

[recall that we have previously assumed \(f \leq (1 - f)g'\) by definition]. We can reasonably expect the same prediction to be valid in the case of nonzero but weak interspecies attraction, \(g \ll 1\).

3. Extension to the nonlinear splitter

It is relatively straightforward to extend the theoretical treatment to the case of a nonlinear splitter, as described in Refs. [21] and [27]. In these works this takes the form of a localized self-repulsive nonlinearity (this can be created by a tightly focused laser beam which locally applies an optical Feshbach resonance\([44]\)). The equivalent modified system of Eq. (5) then becomes

\begin{align}
i \frac{\partial \psi_1}{\partial t} &= -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \varepsilon_{\text{nonlin}} \delta(x) |\psi_1|^2 - |\psi_1|^2 - g |\psi_2|^2 \psi_1, \quad (17a) \\
i \frac{\partial \psi_2}{\partial t} &= -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \varepsilon'_{\text{nonlin}} \delta(x) |\psi_2|^2 - g' |\psi_2|^2 - g |\psi_1|^2 \psi_2, \quad (17b)
\end{align}

where \(\varepsilon_{\text{nonlin}}\) and \(\varepsilon'_{\text{nonlin}}\) both \(> 0\) quantify the strengths of the nonlinear splitters, which in principle should be considered distinct for the two atomic species.

Similarly to the linear case, when \(g \ll 0\) one can determine velocity intervals in which we predict splitting of an incident composite soliton into a lighter transmitted soliton and a heavier reflected soliton:

\begin{equation}
\sqrt{\varepsilon_{\text{nonlin}} f^2/4} < |v| < \sqrt{\varepsilon'_{\text{nonlin}} (1 - f)^3/4}. \quad (18)
\end{equation}

Note that the form of this interval implicitly assumes \(f^{3/2} < g' (1 - f)^{3/2} (\varepsilon'_{\text{nonlin}}/\varepsilon_{\text{nonlin}})^{1/2}\), which is only automatically fulfilled by the previously assumed condition that \(f \leq (1 - f)g'\) when \(g' (\varepsilon'_{\text{nonlin}}/\varepsilon_{\text{nonlin}})^{1/2} \leq 1\). In the case of \(g' = 8\) and \(f = 13/15\) (with \(\varepsilon'_{\text{nonlin}} = \varepsilon_{\text{nonlin}}\)), for example, the upper and lower bounds of Eq. (18) would need to be transposed.

We can readily determine nonlinear equivalents to Eq. (13)

\begin{align}
(U_{\text{max nonlin}})_1 &= \varepsilon_{\text{nonlin}} f^4/32, \quad (19a) \\
(U_{\text{max nonlin}})_2 &= \varepsilon'_{\text{nonlin}} (1 - f)^4 g'^2/32, \quad (19b)
\end{align}

which, compared with Eq. (11), reveal the condition for sufficiently small \(\varepsilon_{\text{nonlin}}\) is that should be significantly less than \(8/3f\), where similarly \(\varepsilon'_{\text{nonlin}}\) should be significantly less than \(8/3(1 - f)\). If we consider \(\varepsilon_{\text{nonlin}}\) and \(\varepsilon'_{\text{nonlin}}\) to be broadly similar in magnitude, this may be simplified to setting the minimum of \(\varepsilon_{\text{nonlin}}, \varepsilon'_{\text{nonlin}}\) \(< 8/3\).

Comparing Eq. (18) with Eq. (16), we see that the nonlinear splitter manifests much stronger dependence on the norm-distribution parameter \(f = N_1/N\), as well as a stronger dependence on the relative magnitude of the intraspecies scattering lengths \(g' = a_{22}/a_{11}\). In the numerical analysis presented below, however, we address solely the case of the previously described linear splitter, which we believe would be experimentally more straightforward.
This simply describes a one-dimensional Schrödinger particle
which simplify to the passing lighter and bouncing heavier components close to the splitting-unsplitting boundary, at parameter values \( \sigma = 0.4, \varepsilon = 0.07, f = 0.3, g = 0.2, \) and \( v = 0.155. \)

C. Limit of a strongly asymmetric two-component soliton

1. Setup of the equations

One can also carry out a perturbative analysis, assuming sufficiently small \( \varepsilon, \) on the case when the intraspecies mean-field attraction of component 1 is much smaller than the interspecies mean-field attraction of component 2 on component 1, i.e., \( f \ll (1 - f)g, \) and also the intraspecies mean-field attraction of component 2 is much greater than the interspecies mean-field attraction of component 1 on component 2, i.e., \( fg \ll (1 - f)g'. \) These conditions can be summarised as

\[
\frac{f}{(1 - f)} \ll g \ll \frac{g'(1 - f)}{f},
\]

which simplify to \( f/(1 - f) \ll g \ll (1 - f)/f \) when \( g' = 1, \) and to \( 1 \ll g \ll g' \) when \( f = 1 - f = 1/2. \) In this case component 2 of the incident mode is essentially the usual NLSE bright soliton, as described by Eq. (8b) and Eq. (9b). We can then seek a solution describing component 1 having the form

\[
\psi_1(x, t) = \exp \left( i \left[ \frac{1}{2} \varepsilon + \frac{v^2}{2} t \right] \right) u_1(x - (x_0 + vt)),
\]

with, subsequent to carrying out a change of variables such that \( x - (x_0 + vt) \equiv X, \) \( u_1 \) determined by a stationary linear Schrödinger equation:

\[
\mu_1^{(0)} u_1(X) = -\frac{1}{2} \frac{d^2}{dX^2} u_1(X) - \frac{g(1 - f)g'}{4} \text{sech}^2 \left( \frac{1}{2} \frac{(1 - f)g'}{X} \right) u_1(X).
\]

This simply describes a one-dimensional Schrödinger particle in a Pöschl–Teller potential, which can be generally solved in terms of special functions.\(^2\)

\(^2\)This is made substantially simpler upon implementing the change of variable \( Y = (1 - f)g'X. \)

2. Ground state of component 1

The exact ground-state solution to Eq. (22) and its corresponding eigenvalue are given by

\[
u_1(0) = A_1 \left[ \text{sech} \left( \frac{(1 - f)g'}{2} \right) \right]^\alpha,
\]

\[
(\mu_1^{(0)})_{\text{ground}} = -\frac{\alpha(1 - f)g'}{8},
\]

where

\[
\alpha = \sqrt{1 + \frac{2g}{g'} - \frac{1}{2}}.
\]

The amplitude \( A_1 \) is [with respect to Eq. (22)] in principle arbitrary, however as we have already set the norm of component 1 to be \( f, \) these can be related to each other via

\[
f = \int dX |u_1(X)|^2 = A_1^2 \frac{2\sqrt{\pi} \Gamma(\alpha)}{(1 - f)g' \Gamma(\alpha + 1/2)},
\]

where \( \Gamma \) is the Gamma function.

3. Determining the splitting interval

The kinetic energy, \( (E_{\text{kin}})_1, \) of component 1 is given by Eq. (10a), while the height of the energy barrier generated by the splitter, in similar fashion to Eq. (12a), can be determined from Eq. (25) to be

\[
(U_{\text{max}})_1 = \mathcal{E} A_1^2 = \frac{\mathcal{E} f(1 - f)g'}{2 \sqrt{\pi} \Gamma(\alpha)}.
\]

Combining these expressions within the energy condition for component 1 to be transmitted through the barrier, \( (E_{\text{kin}})_1 > (U_{\text{max}})_1 \) [cf. Eq. (15a)], yields the following result:

\[
v^2 > \frac{\mathcal{E}(1 - f)g' \Gamma(\alpha + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}.
\]
On the other hand, the condition for the reflection of component \( 2 \) remains, in the first approximation, the same as that given by Eq. (15b). Hence, this condition becomes \( \nu^2 < \varepsilon g'(1 - f)/2 \), and there are intervals of velocities defined by

\[
\sqrt{\frac{\nu(1 - f)g'\Gamma(\alpha + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}} < |\nu| < \sqrt{\frac{\nu(1 - f)g'}{2}},
\]

(cf. Eq. (16)) in which collision of the incident composite soliton with the barrier leads to splitting, with component \( 1 \) transmitted and component \( 2 \) reflected. Note that when \( \alpha = 1 \) [revealed from Eq. (24) to be when \( g/g' = 1 \)] it follows that \( \sqrt{\pi} \Gamma(\alpha) = 2 \Gamma(\alpha + 1/2) \), and the interval shrinks to nil; this, in particular, applies for the Manakov system, when \( g = g' = 1 \). For \( g/g' > 1 \) the situation inverts, and there is instead a velocity interval, defined by transposing the upper and lower bounds in Eq. (28), in which component \( 1 \) is reflected and component \( 2 \) transmitted.

It is important to note that in the limiting case of \( g' = 1 \), this regime is accessed purely by control of the relative particle numbers in each component, with component \( 1 \) having a much smaller population. While this is entirely reasonable within the Gross–Pitaevskii description, in any atomic physics realisation this runs the risk of component \( 1 \) being so small that it is difficult to image, or even of the particle numbers being so low that a Gross–Pitaevskii description is no longer valid.

### 4. Interaction energy condition

Strictly speaking, there is an additional condition necessary for the complete collision-induced splitting in free space. The kinetic energy of the transmitted component must exceed its binding energy in the composite soliton, determined by the cross-attraction \( E_{\text{cross}} = -g \int_{-\infty}^{\infty} dx |\psi_1(x)|^2 |\psi_2(x)|^2 \) [in this analysis we assume the smallness of \( \varepsilon \) means consideration of the energy described by Eq. (26) can be neglected altogether], otherwise component \( 1 \) will not become a free soliton. Hence, substituting in Eq. (8b) and Eq. (23a),

\[
E_{\text{cross}} = -gA_1^2(1 - f)^2 g' \left[ \frac{\nu(1 - f)g'}{2} \right]^{\alpha+3/2} + \frac{2}{2\alpha + 1}
\]

where we have also made use of Eq. (25). Comparing the expressions of Eq. (10) and Eq. (29) via \( |E_{\text{kin}}| > |E_{\text{cross}}| \) yields,

\[
E_{\text{cross}} = -gA_1^2(1 - f)^2 g' \left( \frac{1}{2} \right)^{\alpha+1} + \frac{2}{2\alpha + 1}
\]
finally, the condition

$$|v| > \sqrt{\frac{a g (1 - f)^2 g'}{2 a + 1}}.$$  

(30)

IV. NUMERICAL RESULTS

A. Details of the numerical approach

In our numerical simulations, we typically consider a Gaussian rather than a $\delta$-function barrier, $V(x) = \epsilon \eta(x, \sigma)$, as defined in Eq. (4). The intention here is to model an experimentally relevant Gaussian-profile off-resonant sheet of light, however in a formal sense this could also be considered a regularized approximation to a “true” $\delta$-function with finite width $\sigma$. Except for Fig. 9, all of our numerical results in this paper when considering such a Gaussian barrier are for $\sigma = 0.4$. As argued in [8], this can generally be considered a reasonable low value when taking experimental practicalities into account.

We numerically integrated Eqs. (5a) and (5b), including the potential barrier described by Eq. (4), by means of the well-known Fourier-transform split-step method [45, 46]. We have displayed typical examples of collisions with the barrier leading to either the splitting of two-component solitons, or their mutual passage through the barrier, in Figs. 1 and 2, respectively. These two examples have slightly different collision velocities but otherwise identical parameter values, and correspond to situations close to the boundary between the presence and absence of splitting.

We quantify the transmission of the two components through the barrier by the coefficients

$$T_1 = f^{-1} \int_0^\infty dx |\psi_1(x, t = t_f)|^2,$$  

(31)

$$T_2 = (1 - f)^{-1} \int_0^\infty dx |\psi_2(x, t = t_f)|^2,$$  

(32)

which we compute at the “final time” $t_f$. This is given by $t_f \geq L/2v$ in the cases when we do not consider an axial trapping potential, where $v$ is the incoming velocity, and $L$ is the size of the numerical spatial domain. We set the value of $L$ to 160, which we chose such that at the starting location ($x = -L/4$) and approximate end location of any transmitted component ($x = L/4$), the soliton components are spatially well resolved from the splitting barrier. Note that interaction with the barrier can slow the trajectory of any transmitted components, meaning that $t_f$ must be increased accordingly.

We have performed systematic simulations to produce $T_{1,2}$ as functions of the four control parameters, viz., $v$, $f$, $\epsilon$ and $g$. The repulsive barrier cannot intrinsically trap any part of the wave functions, meaning that in the absence (or negligibility) of axial confinement (see [8], however) the reflection coefficients for the two components are simply $R_{1,2} = 1 - T_{1,2}$.

FIG. 4. Similar to Fig. 3, but in the $(v, f)$ parameter plane at different values of $g$ and $\epsilon$. (a) $\epsilon = 0.04$, (b) $\epsilon = 0.06$, and (c) $\epsilon = 0.08$. 

IV. NUMERICAL RESULTS

A. Details of the numerical approach

In our numerical simulations, we typically consider a Gaussian rather than a $\delta$-function barrier, $V(x) = \epsilon \eta(x, \sigma)$, as defined in Eq. (4). The intention here is to model an experimentally relevant Gaussian-profile off-resonant sheet of light, however in a formal sense this could also be considered a regularized approximation to a “true” $\delta$-function with finite width $\sigma$. Except for Fig. 9, all of our numerical results in this paper when considering such a Gaussian barrier are for $\sigma = 0.4$. As argued in [8], this can generally be considered a reasonable low value when taking experimental practicalities into account.

We numerically integrated Eqs. (5a) and (5b), including the potential barrier described by Eq. (4), by means of the well-known Fourier-transform split-step method [45, 46]. We have displayed typical examples of collisions with the barrier leading to either the splitting of two-component solitons, or their mutual passage through the barrier, in Figs. 1 and 2, respectively. These two examples have slightly different collision velocities but otherwise identical parameter values, and correspond to situations close to the boundary between the presence and absence of splitting.

We quantify the transmission of the two components through the barrier by the coefficients

$$T_1 = f^{-1} \int_0^\infty dx |\psi_1(x, t = t_f)|^2,$$  

(31)

$$T_2 = (1 - f)^{-1} \int_0^\infty dx |\psi_2(x, t = t_f)|^2,$$  

(32)

which we compute at the “final time” $t_f$. This is given by $t_f \geq L/2v$ in the cases when we do not consider an axial trapping potential, where $v$ is the incoming velocity, and $L$ is the size of the numerical spatial domain. We set the value of $L$ to 160, which we chose such that at the starting location ($x = -L/4$) and approximate end location of any transmitted component ($x = L/4$), the soliton components are spatially well resolved from the splitting barrier. Note that interaction with the barrier can slow the trajectory of any transmitted components, meaning that $t_f$ must be increased accordingly.

We have performed systematic simulations to produce $T_{1,2}$ as functions of the four control parameters, viz., $v$, $f$, $\epsilon$ and $g$. The repulsive barrier cannot intrinsically trap any part of the wave functions, meaning that in the absence (or negligibility) of axial confinement (see [8], however) the reflection coefficients for the two components are simply $R_{1,2} = 1 - T_{1,2}$.
B. Comparison of numerical results for transmission with the analytical predictions

1. Case of small interspecies interactions

In Figs. 3 and 4 we show the results of our comparison of the analytical prediction given by Eq. (16) with numerical simulations. In general the agreement is quite good, provided that $g$ is small (a significant region for the value of $T_1 - T_2 = 1$ is visible for up to about $g = 0.1$), and that $\varepsilon$ is also relatively small (up to the regime of about $\varepsilon \approx 0.1$, as expected for $f = 0.3$ from Eq. (14), as shown in Fig. 3).

In Figs. 3 and 4, we have mapped out the degree of splitting, as produced by the simulations, in detail by plotting the difference $T_1 - T_2$ as a function of all the control parameters, $\{\varepsilon, v, g, f\}$. The same figures display the boundaries (dashed lines) within which the analytical prediction, given by Eq. (16), predicts splitting to occur. To reiterate, the analytical result implies that $T_1 = 1$ and $T_2 = 0$ in the interval of velocities of the incident composite soliton given by Eq. (16), and, on the other hand, $T_1 = T_2$ outside the interval, where the incident soliton does not split. We can clearly see in Figs. 3 and 4 (as well as in Fig. 10, which is produced below with an effectively exact numerical implementation of a $\delta$-function barrier) that the prediction gives a good indication of where near-complete splitting occurs for $g \leq 1$, gradually deteriorating with increasing $g$. This is explained, in particular, by the fact that, at relatively large values of $g$, the attraction between the components naturally tends to suppress the collision-induced splitting. It is also generally the case that, as the barrier area $\varepsilon$ increases (which, for fixed width $\sigma$, effectively corresponds to increasing the height), each component is only partly transmitted and partly reflected, i.e., $T_1$ and $T_2$ take intermediate values between 0 and 1. The figures also corroborate the prediction of Eq. (16), that the splitting region shrinks markedly as $f$ approaches 1/2, i.e., the components of the incident composite solitons become nearly equal in population.

2. Case of asymmetric nonlinearities

We have collected numerical results for the case of strong asymmetry between the two components of the incident composite soliton, i.e., situations satisfying Eq. (20), where we have confined ourselves to considering $g' = 1$ only. These are collected in Fig. 5, which clearly demonstrates that the analytical prediction, elaborated for this case in the form of Eq. (28), is quite accurate, at least up to $f = 0.05$, in significantly broad intervals of values of $g$ and $\varepsilon$. As in the case of small interspecies interactions, we do not expect perfect splitting (i.e. $T_1 - T_2 \neq 1$) for larger values of $g$. This is codified in the case of strongly asymmetric nonlinearities in the additional condi-
FIG. 6. The transmission coefficients of the two components, produced by the simulations of Eqs. (5a) and (5b) for the composite soliton, with the relative norm of the first component \( f = 0.3 \) incident on the splitter with strength \( \varepsilon = 0.07 \), in the parameter plane of the collision velocity \( v \) and the interspecies attraction \( g \). Panels (a), (b), and (c) display, severally, the transmission coefficients of the first and second components, and their difference.

C. Continuous variation of the interspecies interaction strength with and without weak axial harmonic confinement

In addition to considering the situation where a soliton moves with a given velocity in free space, we consider a setting in which the soliton begins at rest from an initial position \( x_0 \) on one side of an external harmonic-oscillator potential,

\[ U = \omega_0^2 x^2 / 2, \tag{33} \]

accelerating to an equivalent incident velocity

\[ v = \omega_0 x_0, \tag{34} \]

when it meets a barrier centered at \( x = 0 \).

Figure 6 shows how the transmission in both components varies in the \( (v, g) \) parameter space for \( \varepsilon = 0.07 \) and \( f = 0.3 \) in the free space configuration. In this case, Eq. (16) predicts that both components of the composite soliton pass the barrier, without splitting, at \( v > \sqrt{\varepsilon (1-f) / 2} \approx 0.1565 \). The numerical findings collected in Fig. 6 generally support this prediction. We can compare this with the case of axial harmonic confinement [as given by Eq. (33)], displayed in Fig. 7, which shows the results over an equivalent range of parameters with the collision velocity given by Eq. (34).

We display another aspect of the results collected in Figs. 6 and 7 in Fig. 8 by means of boundaries between the parameter regions where the second component is effectively reflected or transmitted for different values of the barrier’s strength, \( \varepsilon \), while fixing the proportion of the total population in this component at \( 1 - f = 0.7 \). We define this boundary by the condition \( T_2 = 0.5 \).

Both sets of Figs. 6 and 8(a), which pertain to the soliton-barrier collision in free space, and Figs. 7 and 8(b), that display the numerical findings for the splitter embedded in the external trapping potential [Eq. (33)], demonstrate that the stronger the attraction between the two components, quantified by increasing cross-attraction strength \( g \), while keeping other parameter values fixed (relative population \( f \) and barrier area \( \varepsilon \)), leads to multiple transitions between the transmission and reflection of both components. This effect is more strongly pronounced in the presence of the trapping potential, which is explained by the fact that it produces an additional reflecting effect on the moving matter-wave pulses.

D. Effect of finite barrier width

In Fig. 9 we show boundaries corresponding to \( T_1 = 0.5 \), which separate the effective reflection and transmission of the first component in the parameter space made up by the collision velocity \( v \) and the barrier area \( \varepsilon \), for different fixed values of \( f \), \( g \), and the barrier width \( \sigma \) [see Eq. (4)]. Note that we obtained the results for \( \sigma = 0 \) by means of the numerical method outlined in Appendix A, in which we represent a \( \delta \)-function barrier in Fourier space, and incorporate it in the split-step simulation algorithm in the same step as the kinetic
FIG. 8. Boundaries in the \((v, g)\) parameter plane (the collision velocity and relative cross-attraction strength) between regions where the second component of the incident composite soliton, with a fixed relative share of the total norm, \(1 - f = 0.7\), effectively bounces (left of the boundary) or passes (right of the boundary), for varying values of the barrier’s strength, \(\varepsilon\). (a) The case of the incident solitons arriving with velocity \(v\); (b) for the soliton accelerated by the trapping potential \((33)\) as per Eq. \((34)\).

energy term [see Eq. \((A2)\)].

We choose the ranges of the parameters in Fig. 9 in such a way as to be representative of the values used in Figs. 1–8. From Fig. 9 one can see that the location on the \((\varepsilon, v)\) parameter plane where \(T_1 = 0.5\) is more sensitive to the width \(\sigma\) of the barrier when \(g\) is relatively large, and that for the (comparable) range of values of \(f\) and \(g\) considered, the effect of varying \(g\) appears more significant with regard to influencing the sensitivity of the dynamics to \(\sigma\). Increasing \(\sigma\) while keeping other parameter values constant can cause the value of \(v\) where \(T_1 = 0.5\) to become either lower or higher, depending on the other parameter values. However, for the value of \(\sigma = 0.4\), this value of \(v\) is consistently larger than that for the \(\delta\)-function (\(\sigma = 0\)) barrier case, although this difference is never greater than 0.01.

V. CONCLUSIONS

We have examined the transmission properties of two-component bright–bright solitary waves colliding with a narrow potential barrier, and considered in detail the effect of varying the barrier strength, incoming soliton velocity, populations and scattering lengths. We carried this out with the main objective of identifying parameter regions which split the incident composite soliton into its components such that one was reflected and the other transmitted, which is an effect of major importance to the design of matter-wave soliton interferometers. For small values of the barrier strength \(\varepsilon\), we developed a perturbation theory that effectively predicts the velocity interval in which the splitting takes place for relatively small interspecies interactions \((g \to 0)\), as well as for the case where the intraspecies interactions are significant for one species only (fulfilled by \(f \ll 1\) if the two intraspecies scattering lengths are comparable). In order to obtain analytical estimates of these intervals we considered a \(\delta\)-function barrier with the same area as a corresponding Gaussian. An additional parameter is therefore the width of the Gaussian barrier used, and we have demonstrated that the value of \(\sigma = 0.4\) taken in the majority of the simulations behaves similarly to the numerically exact \(\delta\)-function case. We have carried out systematic simulations to identify regimes in which the perturbation theory accurately predicts the transmissions of the two components, and then extended the parameter space beyond these regions by increasing the value of the interspecies interaction strength \(g\) for the estimates found assuming \(g = 0\), and the population in component 1, \(f\), for the estimates found assuming asymmetric nonlinearities. Possible extensions of this work include numerical treatments considering unequal intra-component scattering lengths, and different atomic masses in the two components, which would make it possible to address heteronuclear binary BECs. Finally, it may also be relevant to consider in detail the splitting by a localized nonlinear potential, as briefly introduced above in Section III B 3.

The data presented in this paper can be found in Ref. [47].

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FIG. 10. The same as in Fig. 3, but produced by the numerical algorithm which implements the δ-function in Fourier space, as per Eq. (A2), instead of a Gaussian barrier given by Eq. (4) in coordinate space.

Apple A: Numerical simulations with regularized δ-functions

The scheme for handling the δ-function barrier in the simulations is adapted from [48]. This incorporates the Fourier transform of a δ-function, δ(k), into the part of the split-step method which implements the kinetic-energy term. The relevant expression for the split-step algorithm in the Fourier space is then

\[ \mathcal{F}[T + \varepsilon \delta(x)] = \frac{1}{2} k^2 + \varepsilon \delta(k). \]  

(A1)

Due to the fact that one is conflating an analytical expression for the Fourier transform and its discrete computational counterpart, one must be careful while defining the periodic domain for the Fourier transform. To use the discrete Fourier transform, in the numerical computations we choose the domain as \(-L/2 \leq x < L/2\), placing the δ-function at the center. The corresponding operator for the kinetic energy, combined with the δ-function, is then written as

\[ (M_1)_{mn} = \mathcal{F}[T + \varepsilon \delta(x)]_{mn} = \frac{k^2}{2} \delta_{mn} + \frac{\varepsilon}{L} \cdot \exp \left( \frac{iL}{2} [k_m - k_n] \right), \]  

where \(k\) is defined as a discrete variable running between \(-\pi/L\) and \(+\pi/L\) with \(N\) entries, indexed by integers \((m, n)\), and \(\delta_{mn}\) is the Kronecker delta. When using standard FFT routines, in the current context they must be used in conjunction with two shifting protocols (which shift the location of zero frequency to the centre of the array) whenever they are applied in order to behave in a way which is consistent with the physically relevant boundary conditions.

Alternatively, one can use only one shifting protocol by accounting for a phase offset in the resulting expression for the δ-function in Fourier space. The expression for the sum of the kinetic energy with the δ-function barrier is then

\[ (M_2)_{mn} = \mathcal{F}[T + \varepsilon \delta(x)]_{mn} = \frac{k^2}{2} \delta_{mn} + \frac{\varepsilon}{L} \exp \left( \frac{iL}{2} [k_m - k_n] \right). \]  

(A3)

where the variables are as defined by Eq. A2. Note that if we were only considering the kinetic energy this would make no difference, and it is in fact common practice to only use one shifting protocol in this case.

In order to execute this step in the split-step algorithm, one must diagonalize the matrix \(M_1\) or \(M_2\) and combine the associated amount of shifts with the Fourier transforms, as mentioned above. Note that the diagonalization need only be done once as it is constant throughout the simulations (recalculation is required only if \(\varepsilon\) or \(L\) is altered). The need, on a grid with \(N\) spatial points, for \(N \times N\) dimensional matrix multiplications at each timestep in order to implement this method increases the computational time. As a result the resolution of parameter space as plotted in Fig. 10 is reduced relative to comparable plots presented in this paper when considering

the UK EPSRC.
Figure 10 shows a counterpart of Fig. 3, produced by the numerical scheme described in this appendix. Comparison of the plots suggest that when \( g = 0 \) the results are quite comparable, and that the analytical treatment gives a good indication of what to expect. Deviations of the numerical results from the approximate analytics therefore appear to be primarily due to the nonlinearity describing atom–atom interactions rather than not having an exact \( \delta \)-function barrier (so long as it is sufficiently narrow), as assumed by the analytical treatment.
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