Matrix Models, Argyres-Douglas singularities and double scaling limits

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Abstract: We construct an $\mathcal{N} = 1$ theory with gauge group $U(nN)$ and degree $n + 1$ tree level superpotential whose matrix model spectral curve develops an $A_{n-1}$ Argyres-Douglas singularity. We evaluate the coupling constants of the low-energy $U(1)^n$ theory and show that the large $N$ expansion is singular at the Argyres-Douglas points. Nevertheless, it is possible to define appropriate double scaling limits which are conjectured to yield four dimensional non-critical string theories as proposed by Ferrari. In the Argyres-Douglas limit the $n$-cut spectral curve degenerates into a solution with $n^2$ cuts for even $n$ and $\frac{n+1}{2}$ cuts for odd $n$.

Keywords: matrix models, Argyres-Douglas points, double scaling limits, non-critical strings.

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1. Introduction

In [1, 2, 3], Dijkgraaf and Vafa conjectured that the exact superpotential and gauge couplings of a class of \( \mathcal{N} = 1 \) super Yang-Mills theories can be calculated by doing perturbative computations in an auxiliary matrix model. They considered theories
with a polynomial superpotential $W(\Phi)$ for the chiral adjoint field $\Phi$ and proposed that $W$ is actually the potential in the related matrix model. Furthermore, only planar diagrams in the matrix model contribute to the effective superpotential. This striking result was later proved with perturbative field theory arguments in [4] and by the analysis of the generalized Konishi anomaly in [5]. The solution of the matrix model in the planar limit is captured by the so-called spectral curve, which is given by

$$y^2 = W_n'(x)^2 + f_{n-1}(x), \quad (1.1)$$

where $n$ is the degree of $W'(x)$ and $f_{n-1}(x)$ is a polynomial of degree $n - 1$. The above curve is a hyperelliptic Riemann surface of genus $n - 1$, a double cover of the $x$ plane with $2n$ branch points. The values of the glueball superfields $S_k$ and the expression of the effective superpotential are related to integrals of the meromorphic one-form $y dx$ over the curve (1.1).

In [6], Ferrari studied an $\mathcal{N} = 1 U(N)$ gauge theory with cubic superpotential and using the results of [7] discovered that, in the phase where the gauge group is unbroken, there are critical values of the superpotential couplings where the effective superpotential is non-analytic and the standard large $N$ expansion is singular, namely its coefficients are divergent. In fact, the tension of supersymmetric domain walls scales as a fractional power of $N$ at the critical points. This breakdown of the $1/N$ expansion can actually be compensated by taking the limit $N \to \infty$ and approaching the critical points in a correlated way. Furthermore, these double scaling limits are conjectured to define a four dimensional non-critical string theory. This relies on a proposal made by Ferrari on how to generalize the old matrix model approach to non-critical strings [8] to the four dimensional case [8, 14, 15, 16, 17].

It was also shown in [6] that, from the matrix model point of view, the singularity corresponds to a transition from a two-cut solution to a one-cut solution. A cycle of the genus 1 spectral curve that describes the two-cut solution shrinks to zero size. In field theory language, this is a contact point between two different patterns of gauge symmetry breaking. Ferrari also raised the question of the structure of higher order critical points à la Argyres-Douglas [14]. A matrix model spectral curve undergoes such a degeneration when two or more cycles with non-vanishing intersection shrink to zero size simultaneously. These singularities were first investigated in the context of $\mathcal{N} = 2$ super Yang-Mills theories, whose low-energy physics is encoded by Seiberg-Witten hyperelliptic curves [15, 16, 17]. Their importance lies in the fact that, since the vanishing cycles have non-trivial intersection, the low-energy theory contains both electric and magnetic charges [14]. Furthermore, they are non-trivial interacting $\mathcal{N} = 2$ conformal field theories [14, 18, 19, 20] and they provided the first quantitative check of the scenario advocated by Ferrari [11].

In this paper, higher critical points à la Argyres-Douglas in $\mathcal{N} = 1$ theories are constructed and studied. In particular, a $U(nN)$ gauge theory breaking to $U(N)^n$ in
the presence of the one-parameter superpotential

\[ W(\Phi) = g_n \left( \frac{1}{n+1} \Phi^{n+1} - u \Phi \right), \quad n \geq 3, \quad (1.2) \]

is analysed in detail. There are two values of the parameter \( u \) where the spectral curve develops an \( A_{n-1} \) Argyres-Douglas singularity.

The plan of the paper is as follows. In section 2, Argyres-Douglas singularities are briefly reviewed. In section 3, the strong coupling approach to the study of softly broken \( \mathcal{N} = 1 \) theories \([21]\) is also reviewed. This is the essential instrument to engineer models whose spectral curve develops an Argyres-Douglas singularity on-shell. The choice of the particular model (1.2) and the reason why it is expected to develop an Argyres-Douglas singularity are thus explained.

In section 4, the values of the glueball superfields \( S_k, k = 1, \ldots n \), are calculated exactly by first showing that they are solutions of a linear second-order differential equation in \( u \) and then evaluating them in the semiclassical limit. They are non-analytic at the Argyres-Douglas points. The differential equation is the Picard-Fuchs equation for the periods of the meromorphic one-form \( ydx \) on the spectral curve.

In section 5, the multiplication map by \( N \) \([21]\) is used to map the original \( U(n) \) theory breaking to \( U(1)^n \) to a \( U(nN) \) theory breaking to \( U(N)^n \) with the same superpotential (1.2), thereby enabling us to take the large \( N \) limit. The single vacuum of the \( U(n) \) theory corresponding to the above symmetry breaking is mapped to \( N \) vacua of the \( U(nN) \) theory.

In section 6, it is shown that the effective superpotential for all the vacua is vanishing. In section 7, it is proved that the ansatz for the spectral curve satisfies the matrix model equations of motion consistent with the above symmetry breaking pattern, \( U(nN) \rightarrow U(N)^n \). In particular, the results found via the multiplication map are reproduced. This paves the way for the evaluation of the coupling constants of the low-energy \( U(1)^n \) theory in section 8. They are non-analytic at the critical points. Furthermore, in the Argyres-Douglas limit the \( n \)-cut spectral curve degenerates into a solution with \( \frac{n}{2} \) cuts for even \( n \) and \( \frac{n+1}{2} \) cuts for odd \( n \). Finally, in section 9, it is shown that the large \( N \) expansion is singular at the Argyres-Douglas points, with the coupling constants scaling in general as a fractional power of \( N \). However, as in \([6]\), there exists a well-defined double scaling limit

\[ x \rightarrow 1, \quad N \rightarrow \infty, \quad N(1-x) = \text{cnst} = \frac{1}{\kappa}, \quad x = \frac{4\Lambda^{2n}}{u^2}, \quad (1.3) \]

which is conjectured to yield a non-perturbative definition of a four dimensional non-critical string theory.

### 2. Argyres-Douglas singularities

Argyres-Douglas singularities were originally investigated in the context of \( \mathcal{N} = 2 \)
super Yang-Mills theories, whose low-energy physics is encoded by Seiberg-Witten hyperelliptic curves \[15, 16, 17\]. The fact that the vanishing cycles have non-trivial intersection implies that the low-energy $\mathcal{N} = 2$ theory has massless solitons with mutually non-local charges \[14\]. Namely, these solitons are both electrically and magnetically charged under the same $U(1)$ factor. The theories at these points are actually superconformal \[18\]. As an illustration, let us consider the $\mathcal{N} = 2$ Seiberg-Witten curve for $SU(3)$ \[15, 16, 17\]

$$y^2 = (x^3 - ux - v)^2 - 4\Lambda^6.$$ 

The above genus 2 hyperelliptic curve is singular whenever the polynomial on the r.h.s. has at least a double root, which is equivalent to the vanishing of its discriminant

$$\Delta = 2^{12}\Lambda^{18} \left(4u^3 - 27(v + 2\Lambda^3)^2\right) \left(4u^3 - 27(v - 2\Lambda^3)^2\right).$$

For instance for $v = 0$, $u = 3 e^{2\pi ik/3} \Lambda^2$, $k = 0, 1, 2$, the curve reduces to

$$y^2 = (x - e^{\pi ik/3} \Lambda)^2(x + e^{\pi ik/3} \Lambda)^2(x^2 - 4e^{2\pi ik/3} \Lambda^2),$$

which has two double roots. This is the limit where two mutually local dyons become massless. Argyres-Douglas points in moduli space, however, correspond to higher order singularities. An example is given by $u = 0$ and $v = \pm 2\Lambda^3$, where the curve becomes \[14\]

$$y^2 = x^3(x^3 - 2v), \quad (2.1)$$

and correspondingly two cycles with non-vanishing intersection shrink to zero size.

In the following, a simple generalization of the above singularity will be considered. In particular, the on shell spectral curve of the $\mathcal{N} = 1$ system studied in the paper is going to be

$$y^2 = (x^n - u)^2 - 4\Lambda^{2n}, \quad n \geq 3.$$ 

It is easy to recognize that for $u = \pm 2\Lambda^n$, $n$ out of the $2n$ branch points coalesce leading to an $A_{n-1}$ Argyres-Douglas singularity

$$y^2 \sim x^n. \quad (2.2)$$

Before introducing the specific model which is object of study, it is necessary to review the strong coupling approach to the study of $\mathcal{N} = 1$ gauge theories with polynomial superpotentials $\mathcal{W}(\Phi) \[21, 22\].

3. The strong coupling approach

The dynamics of $\mathcal{N} = 1$ $U(N)$ gauge theories with polynomial superpotentials can be studied by treating $\mathcal{W}(\Phi)$ as a perturbation of the underlying strongly coupled
gauge theory with $W = 0$. The latter system has $\mathcal{N} = 2$ supersymmetry and a Coulomb moduli space of vacua described by a Seiberg-Witten curve \cite{15, 16, 17}

$$y^2 = P_N^2(x) - 4\Lambda^{2N},$$

where the coefficients of $N$-th order polynomial $P_N(x)$ depend on the $N$ moduli $\langle tr\Phi^r \rangle$, $r = 1, \ldots, N$. In this strong coupling approach, which was developed in \cite{21} using the methods of \cite{22}, $W$ is regarded as an effective superpotential on the moduli space. The generic low energy group on the Coulomb moduli space is $U(1)^N$. Vacua in which the low energy group of the $\mathcal{N} = 1$ theory is $U(1)^n$, for $n < N$, can be found by extremizing the superpotential on submanifolds of the Coulomb branch where $N - n$ monopoles of the $\mathcal{N} = 2$ theory are massless. The superpotential lifts all of the moduli space except for a finite set of vacua. At points where $N - n$ mutually local monopoles become massless, the Seiberg-Witten curve has the following factorization

$$y^2 = P_N(x)^2 - 4\Lambda^{2N} = F_{2n}(x)H_{N-n}^2(x), \quad (3.1)$$

where the polynomials on the r.h.s. have simple roots. This factorization is satisfied on an $n$-dimensional submanifold of the Coulomb moduli space on which the superpotential should be extremized in order to find the $\mathcal{N} = 1$ vacua. In \cite{21}, Cachazo, Intriligator and Vafa showed that this yields an on shell relation between the tree level superpotential and the polynomial $F_{2n}(x)$. In particular, when the degree of $W'(x)$ is equal to $n$, the highest $n + 1$ coefficients of $F_{2n}(x)$ are given in terms of $W'(x)$ as follows

$$F_{2n}(x) = \frac{1}{g_n^2}W'(x)^2 + O(x^{n-1}). \quad (3.2)$$

Given $W$, the above relation determines $F_{2n}(x)$ in terms of $n$ unknown coefficients that are fixed by requiring the existence of a polynomial $H_{N-n}(x)$ such that the factorization (3.1) holds. This determines $P_N(x)$ or equivalently the $\mathcal{N} = 2$ vacuum.

Conversely, given a polynomial $P_N(x)$ with the above factorization, one may look for a superpotential consistent with this vacuum. This inverse technique was used in \cite{23} to rederive the $\mathcal{N} = 2$ solution using the geometric engineering approach of \cite{21}. It was also used in \cite{24} to study the various phases of such $\mathcal{N} = 1$ gauge theories and the structure of their parameter space. The same authors also provided a generalization of the above approach to include the cases $\text{deg} W(\Phi) > n + 1$ and $\text{deg} W(\Phi) > N$.

Therefore, in order to construct examples of matrix model spectral curves which develop Argyres-Douglas singularities, one can start from a $p$-parameter family of $\mathcal{N} = 2$ hyperelliptic curves that displays such a degeneration in some appropriate limit. Then, using the inverse technique, it is possible to determine a superpotential consistent with these curves. The procedure yields the corresponding on shell family of $\mathcal{N} = 1$ spectral curves and once this is given one can study the gauge theory along
the lines of \[21, \] 1, 2, 3. In the following, a one-parameter family of genus \( n - 1 \) hyperelliptic curves that can develop Argyres-Douglas singularities is introduced. Then, a consistent order \((n + 1)\) superpotential \( W(\Phi) \) is determined. Finally, the effective superpotential, glueball superfields \( S_k \) and coupling constants of the low-energy abelian gauge theory will be evaluated.

### 3.1 The model

Consider a \( U(n) \) Seiberg-Witten curve of the following form

\[
y^2 = P_n(x)^2 - 4\Lambda^{2n} = (x^n - u)^2 - 4\Lambda^{2n}.
\]  
(3.3)

The above curve has genus \( n - 1 \) and is singular whenever the discriminant \( \Delta \) of the polynomial on the right hand side of (3.3) vanishes

\[
\Delta = (2n)^{2n}(-4\Lambda^{2n})^n(u^2 - 4\Lambda^{2n})^{n-1}.
\]  
(3.4)

In particular, in the limit \( u \to \pm 2\Lambda^n \), \( n \) branch points collide and the curve reduces to

\[
y^2 = x^n (x^n \mp 4\Lambda^n).
\]

For \( n \geq 3 \), these are Argyres-Douglas singularities. Note that the curve (3.3) depends on one parameter only, \( u \), and that it has a \( \mathbb{Z}_n \) symmetry generated by

\[
(x, y) \to (e^{2\pi i/n}x, y).
\]  
(3.5)

In this case the factorization of the Seiberg-Witten curve is trivial, namely for a generic value of \( u \) there are no double roots

\[
y^2 = P_n^2(x) - 4\Lambda^{2n} = F_{2n}(x).
\]  
(3.6)

Then, the low energy group of the above theory is \( U(1)^n \) and the degree of a polynomial superpotential \( W(\Phi) \) consistent with the above Seiberg-Witten curve has to be at least \( n + 1 \). When the degree of \( W(\Phi) \) is equal to \( n + 1 \), the matrix model spectral curve actually coincides with the Seiberg-Witten curve 23

\[
y^2 = P_n^2(x) - 4\Lambda^{2n} = F_{2n}(x) = \frac{1}{g_n^2} \left( \mathcal{W}_n'(x)^2 + f_{n-1}(x) \right),
\]  
(3.7)

where

\[
\mathcal{W}_n'(x) = g_n P_n(x), \quad f_{n-1}(x) = -4 g_n^2 \Lambda^{2n}.
\]  
(3.8)

Then, modulo the addition of a constant, the tree level superpotential is

\[
\mathcal{W}(\Phi) = \frac{g_n}{n + 1} \Phi^{n+1} - g_n u \Phi.
\]  
(3.9)

When \( u = 0 \), the field \( \Phi \) becomes critical and there is a singularity in the classical space of parameters. Note also that, since \( f_{n-1}(x) \) is constant, the total glueball
superfield $S = \sum_{k=1}^{n} S_k$ vanishes identically \[23, 5\]. Finally, by the *multiplication map* introduced in \[21\], the above $U(n)$ theory can be mapped to a $U(nN)$ theory with the same superpotential. In the remainder of the paper we are going to set $g_n = 1$.

![Figure 1](image_url): The $A$ and $B$ cycles for $n = 3$, $(\text{Im } u = 0)$. The path $\gamma$ enters in the rigorous definition of the matrix model conjecture recently studied by Lazaroiu \[27\] involving holomorphic matrix models. By definition, $\gamma$ threads the branch cuts and fixes the basis of $A$ and $B$ cycles on the spectral curve.

### 4. The glueball superfields

The glueball superfields $S_k$ are given by the period integrals of the meromorphic one-form $y dx$ along the closed loops $A_k$ surrounding the $k$-th branch cut \[21, 1, 2, 3\]

\[
S_k = \frac{1}{2\pi i} \oint_{A_k} y \, dx = \frac{1}{2\pi i} \oint_{A_k} \sqrt{(x^n - u)^2 - 4\Lambda^2 n} \, dx. \tag{4.1}
\]

We are going to find the exact expression of the above periods by first deriving a second-order linear differential equation in $u$ satisfied by them, a so-called Picard-Fuchs equation. The evaluation of the semiclassical limit of $S_k$ will then allow to
fix the linear combination of the two independent solutions. A similar analysis was
carried out in the context of $\mathcal{N} = 2$ Seiberg-Witten theories in [25]. First of all
\[ \partial_u S_k = \frac{1}{2\pi i} \oint_{A_k} \partial_u (y \, dx) = -\frac{1}{2\pi i} \oint_{A_k} \frac{(x^n - u)}{y} \, dx. \]
Taking another derivative we find
\[ \partial_u^2 S_k = \frac{1}{2\pi i} \oint_{A_k} \partial_u^2 (y \, dx) = -\frac{1}{2\pi i} \oint_{A_k} \frac{4\Lambda^{2n}}{y^3} \, dx. \]
By the following identity
\[ \partial_u^2 (y \, dx) = \left( n - \frac{1}{n} \right) \frac{4\Lambda^{2n}}{4\Lambda^{2n} - u^2} \frac{dx}{y} + \frac{(x^n - u)u}{n(4\Lambda^{2n} - u^2)} \, dx + \partial_x \left( \frac{x(u^2 - ux^n + 4\Lambda^{2n})}{n(4\Lambda^{2n} - u^2)y} \right) \, dx, \]
we conclude that
\[ \partial_u^2 S_k = \left( n - \frac{1}{n} \right) \frac{1}{u^2 - 4\Lambda^{2n}} \Lambda \partial_u S_k + \frac{1}{n} \frac{u}{(u^2 - 4\Lambda^{2n})} \partial_u S_k, \]
(4.2)
since the integral of an exact differential along a closed cycle is zero.
By (4.1), we can also see that $S_k$ is a homogeneous function of degree $n + 1$, which yields
\[ (\Lambda \partial_u + nu \partial_u) S_k = (n + 1) S_k. \]
(4.3)
Finally, by (4.2) and (4.3), we find
\[ \left[ \partial_u^2 + \left( n - \frac{2}{n} \right) \frac{u}{u^2 - 4\Lambda^{2n}} \partial_u - \left( \frac{n^2 - 1}{n^2} \right) \frac{1}{u^2 - 4\Lambda^{2n}} \right] S_k = 0. \]
(4.4)
Note that every period of $y \, dx$ is a solution of the above Picard-Fuchs equation as long as the cycle is closed. In particular, the difference of any two derivatives of the prepotential $\mathcal{F}$ with respect to the glueball superfields is also a solution of (4.4).

4.1 Solution of the Picard-Fuchs equation

By the following change of variables
\[ z = \frac{u^2}{4\Lambda^{2n}} \]
Eq.(4.4) becomes a hypergeometric equation
\[ \left[ z(1-z) \partial_z^2 + \left( \frac{1}{2} - \frac{n-1}{n} z \right) \partial_z + \frac{n^2 - 1}{4n^2} \right] S_k = 0, \]
(4.5)
which is solved by \[26\]

\[
C_{1,k} F \left( \frac{1}{2} - \frac{1}{2n}, -\frac{1}{2} - \frac{1}{2n}, \frac{1}{2}; \frac{u^2}{4\Lambda^{2n}} \right) + C_{2,k} u F \left( 1 - \frac{1}{2n}, -\frac{1}{2} - \frac{3}{2n}; \frac{u^2}{4\Lambda^{2n}} \right). \tag{4.6}
\]

In order to fix the coefficients \(C_{1,k}\) and \(C_{2,k}\), one can evaluate \(S_k\) in the semiclassical limit \(\Lambda \to 0\), where the glueball superfield vanishes. Since \(\lim_{\Lambda \to 0} z = \infty\), one needs to perform the analytic continuation of the above hypergeometric functions, which are defined as power series in the disc \(|z| \leq 1\). Alternatively, one can rewrite the Picard-Fuchs equation in terms of a new variable that vanishes as \(\Lambda \to 0\).

In the limit \(u \to \infty\), which is dual to \(\Lambda \to 0\), the solutions of Eq.(4.4) are asymptotic to

\[u^{\alpha_{\pm}} f_{\pm}(u),\]

where \(\alpha_{\pm} = \frac{1}{n} \pm 1\), \(\lim_{u \to \infty} f_{\pm}(u) \neq 0\).

Setting \(S_k = u^{\alpha-} f_k(u)\) and changing variables to \(z = \frac{4\Lambda^{2n}}{u^2}\), Eq.(4.4) is equivalent to

\[
\left[ z(1 - z) \partial_z^2 + \left( 2 - \left( \frac{5}{2} - \frac{1}{n} \right) z \right) \partial_z - \frac{2n^2 - 3n + 1}{4n^2} \right] f_k = 0, \tag{4.7}
\]

which is again a hypergeometric equation. The condition that \(S_k\) vanishes in the limit \(z = \frac{4\Lambda^{2n}}{u^2} \to 0\) implies that

\[S_k = C_{3,k} u^{-1 + \frac{1}{n}} F \left( \frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}; 2, \frac{4\Lambda^{2n}}{u^2} \right). \tag{4.8}\]

Note that the above expression is a power series in \(\Lambda^{2n}\), namely \(S_k\) is given by an instanton sum. The value of \(C_{3,k}\) can be found by evaluating the semiclassical limit of \(S_k\) more carefully

\[
S_k = \frac{1}{2\pi i} \oint_{\gamma_k} \sqrt{P_n^2 - 4\Lambda^{2n}} \, dx = -\frac{2\Lambda^{2n}}{2\pi i} \oint_{\gamma_k} \frac{1}{P_n(x)} \, dx + O(\Lambda^{4n}), \tag{4.9}
\]

where \(\gamma_k\) is a counterclockwise loop around the \(k\)-th root of \(P_n(x) = x^n - u\),

\[x_k = e^{2\pi i k/n} u^{1/n}.
\]

Then

\[-\frac{2\Lambda^{2n}}{2\pi i} \oint_{\gamma_k} \frac{1}{P_n(x)} \, dx = -2\Lambda^{2n} \frac{1}{P'(x_k)} = -\frac{2\Lambda^{2n}}{n} e^{2\pi i k/n} u^{-1 + \frac{1}{n}}.
\]

Finally, by (4.8), we find

\[C_{3,k} = -\frac{2\Lambda^{2n}}{n} e^{2\pi i k/n}, \tag{4.10}\]
and
\[ S_k = -\frac{2\Lambda^{2n}}{n} e^{2\pi i/k} u^{-1+1/n} F \left( \frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}, 2, \frac{4\Lambda^{2n}}{u^2} \right). \]

(4.11)

Note that this is consistent with the fact that
\[ \sum_{k=1}^{n} S_k = 0, \]

since \( \sum_{k=1}^{n} e^{2\pi i/k} = 0 \). Actually, \( S_{k+1} = e^{2\pi i/k} S_k \) as a direct consequence of the symmetry (3.5). In fact
\[ S_k = \frac{1}{2\pi i} \oint_{A_k} y dx = \frac{2}{2\pi i} \int_{x_{k,1},-}^{x_{k,1,+}} y dx, \]

where
\[ x_{k,\pm} = e^{2\pi i/k} (u \pm 2\Lambda^n)^{1/n}, \]

are the branch points of the spectral curve (3.3) and by the change of variable \( x = e^{2\pi i/n} \tilde{x} \), we find
\[ S_k = \frac{2}{2\pi i} e^{2\pi i/n} \int_{x_{k,1,-}}^{x_{k,1,+}} y d\tilde{x} = e^{2\pi i/n} S_{k-1}. \]

(4.12)

Similarly, we can show that \( \frac{\partial F}{\partial S_k} = e^{2\pi i(k-1)/n} \frac{\partial F}{\partial S_1} \). Performing the analytic continuation of (4.6), which defines an absolutely convergent series for \(|u| \leq 2\Lambda^n\), to the region \(|u| > 2\Lambda^n\), where (4.8) is valid, we are going to find relations among the various \( C_k \)'s. The details are given in the Appendix. The results are
\[ C_{2,k} = -\frac{e^{i\pi/2}}{2\Lambda^n} \frac{\Gamma(1-1/2n)\Gamma(3/2+1/2n)}{\Gamma(1/2-1/2n)\Gamma(1+1/2n)} C_{1,k}. \]

(4.13)

and
\[ C_{3,k} = e^{i\pi/2+i\pi/2n} \frac{2\pi}{\sin(\pi/n)} \left( \frac{\sqrt{\pi} 2^{n-1}}{2n\Gamma(1+1/2n)\Gamma(-1/2-1/2n)} \right) \Lambda^{n-1} C_{1,k}. \]

(4.14)

4.2 Non-analytic behaviour close to the Argyres-Douglas points

By analytic continuation of (4.8) (4.11), we find [26]
\[ S_k = C_{3,k} u^{-1+1/n} \left( A_1 F \left( \frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}, 2, \frac{1}{u^2} - 4\Lambda^{2n} \right) \right) \]
\[ + A_2 (1 - 4\Lambda^{2n}/u^2) \frac{n+2}{2n} F \left( \frac{3}{2} + \frac{1}{2n}, \frac{1}{2n} + \frac{3}{2}, 1 - 4\Lambda^{2n}/u^2 \right), \]

(4.15)

which implies that close to the singularity at \( u^2 = 4\Lambda^{2n} \)
\[ S_k \approx A_1 + A_2 \left( \frac{u^2 - 4\Lambda^{2n}}{4\Lambda^{2n}} \right)^{n+2/2n} + \mathcal{O}(u^2 - 4\Lambda^{2n}), \]

(4.16)
where
\[ A_1 = \frac{\Gamma(1/2 + 1/n)}{\Gamma(3/2 + 1/2n)\Gamma(1 + 1/2n)}, \quad A_2 = \frac{\Gamma(-1/2 - 1/n)}{\Gamma(1/2 - 1/2n)\Gamma(1 - 1/2n)}. \] (4.17)

Thus, we see that \( S_k \) is non-vanishing in the limit \( u^2 \to 4\Lambda^{2n} \) and that it is non-analytic due to the fractional exponent \( \frac{n+2}{2n} \), \( n \geq 3 \).

5. The multiplication map

By the so-called multiplication map by \( N \) introduced in [21], the above \( U(n) \) theory with superpotential (3.9) can be mapped to a \( U(nN) \) theory with the same tree level superpotential. In particular, the vacuum considered up to now, which is a Coulomb vacuum with unbroken \( U(1)^n \) gauge group, is associated to \( N \) different vacua with unbroken \( U(N)^n \). In fact, given a set of polynomials \( P_n(x) \), \( F_{2m}(x) \) and \( H_{n-m}(x) \), all with the highest coefficient equal to 1, that satisfy the following relations
\[ P_n^2(x) - 4\Lambda^{2n} = F_{2m}(x)H_{n-m}^2(x), \quad F_{2m}(x) = \mathcal{W}'(x)^2 + f_{m-1}(x), \] (5.1)

it is possible to show that \( Q_{nN}(x) \equiv 2\Lambda_{0}^{nN}\eta^{N}T_{N}(\frac{P_{n}(x)}{2\eta\Lambda_{0}^{n}}) \), where \( T_{N} \) is the \( N \)-th Chebishev polynomial of the first kind and \( \eta \) is a \( 2N \)-th root of unity, is a polynomial of degree \( nN \) with highest coefficient equal to 1 and that it satisfies
\[ Q_{nN}^2(x) - 4\Lambda_{0}^{2nN} = 4\Lambda_{0}^{2nN}\left( T_{N}^2\left(\frac{P_{n}(x)}{2\eta\Lambda_{0}^{n}}\right) - 1 \right) \]
\[ = 4\Lambda_{0}^{2nN}\left[ \left( \frac{P_{n}(x)}{2\eta\Lambda_{0}^{n}} \right)^2 - 1 \right] U_{N-1}^2\left( \frac{P_{n}(x)}{2\eta\Lambda_{0}^{n}} \right) \]
\[ = \left( P_{n}^2(x) - 4\eta^2\Lambda_{0}^{2n} \right) \left( \eta^{-1}\Lambda_0^{n(N-1)}U_{N-1}\left( \frac{P_{n}(x)}{2\eta\Lambda_{0}^{n}} \right) \right)^2, \]

where \( U_{N-1} \) is the \( (N-1) \)-th Chebishev polynomial of the second kind and we used the following relation
\[ T_{N}^2(x) - 1 = (x^2 - 1)U_{N-1}^2(x). \] (5.2)

Therefore, if one sets
\[ \Lambda^{2n} = \eta^2\Lambda_0^{2n} = e^{2\pi ip/N}\Lambda_0^{2n}, \quad p = 0, \ldots, N - 1, \] (5.3)

one finds that \( Q_{nN}(x) \) satisfies the following identity
\[ Q_{nN}^2(x) - 4\Lambda_{0}^{2nN} = \tilde{F}_{2m}(x)\tilde{H}_{n-m}^2(x) \left( \eta^{-1}\Lambda_0^{n(N-1)}U_{N-1}\left( \frac{P_{n}(x)}{2\eta\Lambda_{0}^{n}} \right) \right)^2 \]
\[ = \tilde{F}_{2m}(x)\tilde{H}_{nN-m}^2(x), \] (5.4)
where
\[ \tilde{F}_{2m}(x) = F_{2m}(x), \quad \tilde{H}_{n-m}(x) = H_{n-m}(x) \eta^{N-1} \Lambda_0^{n(N-1)} U_{N-1} \left( \frac{P_n(x)}{2n \Lambda_0^n} \right). \] (5.5)

As was explained in section 3, if one wants to find vacua of a \( U(nN) \) theory with low energy group \( U(1)^m \), one has to solve the following factorization problem
\[ P_{nN}^2(x) - 4 \Lambda_0^{2nN} = \tilde{F}_{2m}(x) \tilde{H}_{n-m}^2(x). \]

Then, Eq.(5.4) implies that \( P_{nN}(x) = Q_{nN}(x) \) is a solution. Furthermore, since \( \tilde{F}_{2m}(x) = F_{2m}(x) \), the vacua of the \( U(nN) \) theory have the same superpotential as the vacua of the \( U(n) \) theory. It was also shown in [21], that a classical limit with unbroken \( \prod_{j=1}^{k} U(n_j) \) is mapped to a classical limit with unbroken \( \prod_{j=1}^{k} U(Nn_j) \). Finally, Eq.(5.3) implies that for each vacuum of the \( U(n) \) theory one has \( N \) vacua of the \( U(nN) \) theory.

By Eqs.(5.7), (5.3) and (5.5), the spectral curve relative to one of the \( N \) vacua of the \( U(nN) \) theory breaking to \( U(N)^n \) is given by
\[ y^2 = (x^n - u)^2 - 4 e^{2\pi ip/N} \Lambda_0^{2n}, \quad p = 0, \ldots, N - 1. \] (5.6)

This implies that the glueball superfields \( S_k, k = 1, \ldots, n \) of each \( U(N) \) factor are given by
\[ S_k = -2 e^{2\pi ip/N} \Lambda_0^{2n} \frac{1}{n} e^{2\pi ik/n} u^{-1+1/n} F \left( \frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}, 2, \frac{4 e^{2\pi ip/N} \Lambda_0^{2n}}{u^2} \right). \] (5.7)

6. The effective superpotential

Since the glueball superfield \( S = \sum_{k=1}^{n} S_k \) vanishes exactly, the effective superpotential is given by the classical expression. In fact
\[ \Lambda^{2nN} \frac{\partial}{\partial \Lambda^{2nN}} W_{eff} = \langle S \rangle = 0, \]
which implies that there are no quantum corrections to \( W_{eff} \). The superpotential (1.2) has \( n \) classical ground states where
\[ \langle \Phi \rangle = e^{2\pi ik/n} u^{1/n}, \quad k = 1, \ldots, n. \]

Then, for the \( U(nN) \) theory breaking to \( U(N)^n \) we find that
\[ W_{eff} = W_{cl} = g_n \left( \frac{1}{n+1} \langle \Phi^{n+1} \rangle - u \langle \Phi \rangle \right) = g_n \left( \frac{1}{n+1} \sum_{k=1}^{n} e^{2\pi ik/n} u^{(n+1)/n} - u \sum_{k=1}^{n} e^{2\pi ik/n} u^{1/n} \right) = 0. \]
7. The Matrix Model analysis

In this section, we are going to show that the ansatz (3.7),(3.8),(3.9) for the spectral curve indeed satisfies the matrix model equations of motion consistent with the symmetry breaking pattern \( U(nN) \rightarrow U(N)^n \). In particular, we are going to find \( N \) different vacua, characterized by a different value of \( \Lambda^{2n} \) as in Eq.(5.3). This will pave the way for the evaluation of the coupling constants of the low-energy \( U(1)^n \) theory in section 8. As is shown in [6], the extremization of the superpotential

\[
\mathcal{W} = -\sum_{k=1}^{C} N_k \partial S_k \mathcal{F}, \tag{7.1}
\]

where the prepotential \( \mathcal{F} \) is given by the planar approximation to a holomorphic integral over complex \( n \times n \) matrices [1, 2, 3]

\[
\exp \left( \frac{n^2 \mathcal{F}}{S^2} \right) = \int_{\text{planar}} d^n \phi \exp \left[ -\frac{n}{S} \text{tr} \mathcal{W}_{\text{tree}}(\phi) \right], \tag{7.2}
\]

is equivalent to solving the following equations

\[
\sum_{l=1}^{C} N_l t_{lk} = p_k \mod N_k, \quad k = 1, \ldots, C, \tag{7.3}
\]

where

\[
t_{lk} = -\frac{1}{2\pi i} \partial_{S_l} \partial_{S_k} \mathcal{F}, \tag{7.4}
\]

and \( C \) is the number of cuts. In particular, \( t_{lk} \) is the generalized period matrix

\[
t_{lk} = \lim_{\ell_0 \to \infty} \left( \frac{1}{2\pi i} \int_{B_l} \psi_k dx + \frac{i}{\pi} \log \ell_0 \right),
\]

associated to the curve

\[
z^2 = \prod_{k=1}^{C} (x - a_k)(x - b_k),
\]

which is the non-trivial part of the spectral curve

\[
y^2 = \mathcal{W}_{\text{tree}}'(x)^2 + f_{n-1}(x) = M(x)^2 \prod_{k=1}^{C} (x - a_k)(x - b_k).
\]

The differentials \( \psi_k dx \) form a basis of log-normalizable holomorphic one-forms dual to the contours \( A_l \) surrounding the \( C \) cuts, namely they satisfy the following normalization condition [6]

\[
\frac{1}{2\pi i} \oint_{A_l} \psi_k dx = \delta_{kl}. \tag{7.5}
\]
It is possible to show that
\[ \psi_k(x) = \frac{N_k(x)}{z}, \]
where
\[ N_k(x) = x^{C-1} + \ldots, \]
is a polynomial of degree \( C - 1 \) whose coefficients are determined by the normalization condition (7.5). Furthermore, the coupling constants of the \( U(1)^C \) low-energy theory are given by
\[ \tau_{lk} = -\frac{1}{2\pi i} \left( \frac{\partial^2 \mathcal{F}}{\partial S_l \partial S_k} - \delta_{lk} \frac{1}{N_l} \sum_{m=1}^{C} N_m \frac{\partial^2 \mathcal{F}}{\partial S_l \partial S_m} \right) = t_{lk} - \delta_{lk} \frac{1}{N_l} \sum_{m=1}^{C} t_{lm}, \quad (7.6) \]
and satisfy
\[ \sum_{k=1}^{C} \tau_{lk} N_k = 0, \]
which signals the decoupling of the diagonal \( U(1) \).

### 7.1 The canonical basis

In our case \( M(x) \equiv 1 \) and we need to find a basis of one-forms such that
\[ \psi_k(x) = \frac{N_k(x)}{y}, \]
where
\[ N_k(x) = x^{n-1} + \ldots, \]
is a polynomial of degree \( n - 1 \) whose coefficients are determined by the normalization condition (7.3). In particular, the \( \psi_k \)'s will be a linear combination of the one-forms \( \omega_m = \frac{x^{m-1}}{y} dx, m = 1, \ldots, n \)
\[ \psi_k dx = \sum_{m=1}^{n} \alpha_{km} \omega_m. \]

In order to find the appropriate combination, we need to compute integrals of the form
\[ \mathcal{I}_{ml} \equiv \frac{1}{2\pi i} \oint_{A_l} \omega_m = \frac{1}{2\pi i} \oint_{A_l} \frac{x^{m-1}}{y} dx, \quad l, m = 1, \ldots, n. \quad (7.7) \]
Then
\[ \frac{1}{2\pi i} \oint_{A_l} \psi_k dx = \frac{1}{2\pi i} \oint_{A_l} \alpha_{km} \omega_m = \alpha_{km} \mathcal{I}_{ml} = \delta_{kl}, \]
which means that the matrix \( \alpha \) is the inverse of \( \mathcal{I} \). It is convenient to consider integrals of the \( \omega_m \)'s since, by virtue of \( Z_n \) the symmetry (3.3)
\[ \mathcal{I}_{ml} = e^{2\pi i m l} \mathcal{I}_{ml-1} = \eta^m \mathcal{I}_{ml-1}. \quad (7.8) \]
This identity will drastically simplify the analysis. Define
\[ \hat{I}_{kj} = \eta^{-(k-1)} \hat{I}_{1j}, \quad \hat{I}_{1j} = \frac{1}{\hat{I}_{11}}. \]
Then
\[ \hat{I}_{kj} \hat{I}_{jl} = \sum_{j=1}^{n} \eta^{-(k-1)} \hat{I}_{kj} \eta^{l-1} \hat{I}_{1j} = \sum_{j=1}^{n} \eta^{l-1} \hat{I}_{1j} = n \delta_{lk}, \]
which implies that
\[ \alpha = I^{-1} = \frac{1}{n}. \]
The value of \( I_{n1} = I_{nj} \) can be evaluated using (7.8) and a simple residue calculation and it is given by
\[ n I_{n1} = \frac{1}{2\pi i} \oint A \frac{x^{n-1}}{y} \, dx = \frac{1}{2\pi i} \oint A_{\infty} \frac{x^{n-1}}{y} \, dx = 1 \Rightarrow I_{n1} = \frac{1}{n}. \]
Thus, we can verify that the basis one-forms \( \psi_k \) have the correct asymptotic behaviour
\[ \psi_k \, dx \sim \alpha_{kn} \frac{x^{n-1}}{y} \, dx = \frac{1}{n} \hat{I}_{kn} \frac{x^{n-1}}{y} \, dx = \frac{1}{n} \hat{I}_{1n} \frac{x^{n-1}}{y} \, dx = \frac{1}{n} \frac{x^{n-1}}{y} \, dx. \]
In summary
\[ \psi_k \, dx = \frac{1}{n} \sum_{m=1}^{n} \eta^{-(k-1)m} \frac{\omega_m}{I_{m1}}. \]  
\[ (7.9) \]

### 7.2 The period matrix

The final goal is to evaluate the generalized period matrix
\[ t_{lk} = t_{kl} \equiv \frac{1}{2\pi i} \int_{B_l} \psi_k \, dx = \frac{1}{2\pi i} \int_{B_l} \alpha_{km} \omega_m = \frac{1}{n} \sum_{m=1}^{n} \hat{I}_{km} L_{ml}, \]
where
\[ L_{ml} \equiv \frac{1}{2\pi i} \int_{B_l} \omega_m. \]  
\[ (7.10) \]
As before we find
\[ L_{ml} = e^{2\pi i m} L_{ml-1} = \eta^m L_{ml-1}. \]  
\[ (7.11) \]
Then
\[ t_{lk} = \frac{1}{n} \sum_{m=1}^{n} \hat{I}_{km} L_{ml} = \frac{1}{n} \left( \sum_{m=1}^{n} \eta^{-m(k-1)} \hat{I}_{1m} \eta^{m(l-1)} L_{ml} \right) \]
\[ = \frac{1}{n} \left( \sum_{m=1}^{n} \eta^{m(l-k)} \hat{I}_{1m} L_{ml} \right) \equiv \frac{1}{n} \left( \sum_{m=1}^{n-1} \eta^{m(l-k)} c_m + c_n \right). \]  
\[ (7.12) \]
Note that the above matrix is symmetric if and only if \( c_m = c_{n-m}, m = 1, \ldots, n - 1 \).

In fact
\[
 t_{lk} = \frac{1}{n} \left( \sum_{m=1}^{n-1} \eta^{m(l-k)} c_m + c_n \right) = t_{kl} = \frac{1}{n} \left( \sum_{m=1}^{n-1} \eta^{-m(l-k)} c_m + c_n \right)
\]

\[
 = \frac{1}{n} \left( \sum_{m=1}^{n-1} \eta^{(n-m)(l-k)} c_m + c_n \right) = \frac{1}{n} \left( \sum_{m=1}^{n-1} \eta^{m(l-k)} c_{n-m} + c_n \right)
\]

\[\iff c_m = c_{n-m}, \quad m = 1, \ldots, n - 1.\]

This identity will be verified in section (8.1).

### 7.3 The equations of motion

The equations of motion (7.3) for \( N_l = N \) reduce to

\[
 \sum_{l=1}^{n} N_l t_{lk} = \sum_{l=1}^{n} N \frac{1}{n} \left( \sum_{m=1}^{n-1} \eta^{m(l-k)} \hat{L}_{1m} L_{m1} \right)
\]

\[
 = N \hat{L}_{1n} L_{n1} = N n L_{n1} = p_k \mod N = p \mod N.
\]

Since \( \omega_n \) is actually a logarithmic derivative
\[
 \omega_n = \frac{x^{n-1}}{y} \, dx = \frac{1}{n} \frac{d}{dx} \log (x^n - u + y), \tag{7.13}
\]

it follows that

\[
 L_{nl} = \frac{1}{2\pi i} \int_{B_l} \omega_n = \lim_{\ell_0 \to \infty} \frac{1}{\pi i} \int_{b_l} \omega_n - \frac{1}{\pi i} \log \ell_0 = -\frac{1}{n\pi i} \log(-\Lambda^n).
\]

where \( b_l = e^{2\pi i/n}(u - 2\Lambda^n)^{1/n} \).

Therefore the equations of motion are equivalent to

\[
 \sum_{l=1}^{n} N_l t_{lk} = -\frac{N}{\pi i} \log(-\Lambda^n) = p \mod N = -p' \mod N,
\]

which yields
\[
 \Lambda^{2n} = e^{2\pi ip'/N}, \quad p' = 0, 1, \ldots, N - 1. \tag{7.14}
\]

This reproduces the results of the strong coupling analysis (5.3).
8. The $U(1)^n$ coupling constants

The coupling constant matrix $\tau_{lk}$ of the low-energy $U(1)^n$ theory is given by

$$\tau_{lk} = t_{lk} - \delta_{lk} \frac{1}{N_l} \sum_{m=1}^{n} t_{lm},$$

(8.1)

and satisfies $\sum_{k=1}^{n} \tau_{lk} N_k = 0$, which signals the decoupling of the diagonal $U(1)$.

In Appendix B, we show that the periods of $\omega_m$ satisfy the following Picard-Fuchs equation

$$\left( \frac{\partial^2}{\partial u^2} + \frac{\alpha(m) u}{(u^2 - 4\Lambda^2 n)} \frac{\partial}{\partial u} + \frac{\beta(m)}{(u^2 - 4\Lambda^2 n^2)} \right) \oint \omega_m = 0,$$

(8.2)

where

$$\alpha(m) = \frac{3n - 2m}{n}, \quad \beta(m) = \left(\frac{\alpha(m) - 1}{2}\right)^2 = \left(\frac{n - m}{n}\right)^2.$$

(8.3)

In terms of the variable $z = \frac{4\Lambda^2 n}{u^2}$, Eq. (8.2) is equivalent to

$$\left( \frac{\partial^2}{\partial z^2} + \frac{3(1 - z) - \alpha(m) z}{2z(1 - z)} \frac{\partial}{\partial z} + \frac{\beta(m)}{4z^2(1 - z)} \right) \phi(z) = 0.$$

(8.4)

The indicial equation at $z = 0$ has a double root equal to $(\alpha(m) - 1)/4$. This matches the behaviour of the integrals of $\omega_m$ around the cuts in the classical limit $z \to 0$

$$\int_{A_k} \frac{x^{m-1}}{y} \, dx \sim \oint_{x_k} \frac{x^{m-1}}{P_n(x)} \sim u^{\frac{m-n}{n}} = u^{(1 - \alpha(m))/2} \sim z^{(\alpha(m) - 1)/4},$$

where $x_k$ is the $k$-th root of $P_n(x)$.

Setting $\phi(z) = z^{(\alpha - 1)/4} \psi(z)$, we find that Eq. (8.4) is equivalent to the following hypergeometric equation for $\psi(z)$

$$\left( z(1 - z) \frac{\partial^2}{\partial z^2} + \left(1 - \frac{\alpha(m) + 2}{2} z\right) \frac{\partial}{\partial z} - \frac{\alpha^2(m) - 1}{16} \right) \psi(z) = 0,$$

(8.5)

with coefficients

$$a = \frac{\alpha(m) - 1}{4} = \frac{n - m}{2n}, \quad b = \frac{\alpha(m) + 1}{4} = \frac{2n - m}{2n}, \quad c = 1.$$

In terms of $w = 1 - z = 1 - \frac{4\Lambda^2 n}{u^2}$, which is the appropriate variable in the neighbourhood of the Argyres-Douglas point, Eq. (8.5) becomes

$$\left( w(1 - w) \frac{\partial^2}{\partial w^2} + \left(\frac{\alpha(m)}{2} - \frac{\alpha(m) + 2}{2} w\right) \frac{\partial}{\partial w} - \frac{\alpha^2(m) - 1}{16} \right) \psi(1 - w) = 0,$$

(8.6)

which is a hypergeometric equation with coefficients

$$a' = a, \quad b' = b, \quad c' = \frac{\alpha(m)}{2} = \frac{3n - 2m}{2n}.$$
Thus, Eq. (8.5) has two linearly independent solutions, namely
\[ \psi_1(z) = F \left( \frac{n-m}{2n}, \frac{2n-m}{2n}, 1, z \right), \]  
(8.7)\[ \psi_2(z) = F \left( \frac{n-m}{2n}, \frac{2n-m}{2n}, \frac{3n-2m}{2n}, 1 - z \right). \]  
(8.8)

We need to find the appropriate linear combination of \( \psi_1(z) \) and \( \psi_2(z) \) corresponding to each of the periods of \( \omega_m \). Let us first consider the integral of \( \omega_m \) around the \( k \)-th branch cut
\[ \oint_{A_k} \omega_m = \oint_{A_k} \frac{x^{m-1}}{y}, m = 1, \ldots, n. \]

By evaluating the above integral in the semiclassical limit \( \frac{A^2}{u^2} \to 0 \), we find
\[ \frac{1}{2\pi i} \oint_{\mathcal{A}_k} \frac{x^{m-1}}{y} = \frac{1}{n} \eta^{(k-1)m} u^{\frac{m-n}{n}} \psi_1 \left( \frac{4A^2}{u^2} \right), \quad m = 1, \ldots, n. \]  
(8.9)

Eq. (7.12) reduces the calculation the period matrix \( t_{lk} \) to the evaluation of the integrals \( I_1 m \) and \( L_1 m \)
\[ I_{1m} \equiv \frac{1}{2\pi i} \oint_{\mathcal{A}_1} \omega_m, \quad L_{1m} \equiv \frac{1}{2\pi i} \oint_{\mathcal{B}_1} \omega_m. \]

By virtue of (7.11), the integral of \( \omega_m \) along the non-closed cycle \( \mathcal{B}_1 \) can be related to a period integral along a closed one. In fact
\[ L_{m2} - L_{m1} = (\eta^m - 1) L_{m1} = \frac{1}{2\pi i} \oint_{\mathcal{B}_2 - \mathcal{B}_1} \omega_m. \]

Thus \( L_{1m} \) will also be a solution of the Picard-Fuchs equations derived above. In the semiclassical limit, we find
\[ \frac{1}{2\pi i} \oint_{\mathcal{B}_2 - \mathcal{B}_1} \frac{x^{m-1}}{y} dx = -\frac{2}{2\pi i} \int_{\mathcal{B}_2 - \mathcal{B}_1} \frac{\eta^{(u+2A^2)^{1/n}}}{y} x^{m-1} \frac{x^{m-1}}{y} dx \]
\[ = -\frac{2}{2\pi i} u^{\frac{m-n}{n}} \frac{1}{n} \left( \eta^m - 1 \right) \left( \gamma + \psi(m/n) + \frac{1}{2} \log \left( \frac{4A^2}{u^2} \right) + i\pi \eta^m \right) + \mathcal{O} \left( \frac{4A^2}{u^2} \right). \]

This fixes the integral to be
\[ L_{m1} = -\frac{2}{2\pi i} u^{\frac{m-n}{n}} \frac{1}{n} \left[ \log 2 \psi_1 - \frac{n}{2m-n} \frac{1}{A_2(m)} \left( \frac{\psi_1}{A_1(m)} - \psi_2 \right) + i\pi \frac{\eta^m}{\eta^m - 1} \psi_1 \right], \]
(8.10)

where
\[ A_1(m) = \frac{\Gamma \left( \frac{(2m-n)/2n}{2} \right)}{\Gamma \left( \frac{(m/2n)}{2} \right) \Gamma \left( \frac{(n+m)/2n}{2} \right)}, \quad A_2(m) = \frac{\Gamma \left( \frac{(n-2m)/2n}{2} \right)}{\Gamma \left( \frac{(n-m)/2n}{2} \right) \Gamma \left( \frac{(2n-m)/2n}{2} \right)}. \]  
(8.11)
Thus
\[
c_m \equiv \mathcal{L}_{m,1} = \oint_{b_2=b_1} \frac{x^{m-1}}{\eta^m - 1} \, dx / \oint_A \frac{x^{m-1}}{y} \, dx
\]
\[= \frac{-2}{2\pi i} \left[ \log 2 - \frac{n}{2m-n} \frac{1}{A_1(m)A_2(m)} + \frac{n}{(2m-n)A_2(m)} \psi_2 + \frac{i\pi \eta^m}{\eta^m - 1} \right]. \tag{8.12}
\]

8.1 The map \( m \to n - m \)

As was remarked above, the period matrix (7.12) is symmetric if and only if \( c_{n-m} = c_m, m = 1, \ldots, n - 1 \). In Appendix C, it is shown that
\[
\psi_1(n - m) = \left( 1 - \frac{4\Lambda^2}{u^2} \right) \frac{n^{2m}}{\eta^{2m}} \psi_1(m), \tag{8.13}
\]
\[
\psi_2(n - m) = \left( 1 - \frac{4\Lambda^2}{u^2} \right) \frac{n^{2m}}{\eta^{2m}} \left( \frac{\psi_1(m)}{A_2(m)} - \frac{A_1(m)}{A_2(m)} \psi_2(m) \right). \tag{8.14}
\]

Then
\[
c_{n-m} = \frac{-2}{2\pi i} \left( \log 2 - \frac{n}{(n-2m)A_2(m)} \psi_2(m) + i\pi \frac{1}{1 - \eta^m} \right),
\]
and
\[
c_{n-m} - c_m = \frac{-2}{2\pi i} \left( \frac{n}{2m-n} \frac{1}{A_1(m)A_2(m)} + i\pi \frac{1}{1 - \eta^m} - i\pi \frac{\eta^m}{\eta^m - 1} \right)
\]
\[= \frac{-2}{2\pi i} \left( \pi \cot \left( \frac{\pi m}{n} \right) + i\pi \frac{1 + \eta^m}{1 - \eta^m} \right) = 0, \tag{8.15}
\]
where we used the fact that
\[A_1(m)A_2(m) = \frac{n \tan(\pi m/n)}{\pi(2m-n)}.\]

8.2 The diagonalized coupling constant matrix

By Eq.(7.12) we can immediately see that the following
\[(v_{\pm p})_k = \eta^{\pm pk}, \quad k = 1, \ldots, n, \; p = 1, \ldots, n,\]
are eigenvectors of \( t \) with eigenvalues \( c_p = c_{n-p} \). In fact
\[
\sum_{k=1}^{n} t_{ik} (v_{\pm p})_k = \frac{1}{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \eta^{m(l-k)} c_m \eta^{\pm pk} = \frac{1}{n} \sum_{m=1}^{n} \sum_{k=1}^{n} \eta^{(\pm p-m)k} \eta^{ml} c_m
\]
\[= c_p \eta^{\pm pl} = c_p (v_{\pm p})_l, \tag{8.16}
\]
where we used the fact that
\[
\sum_{k=1}^{n} \eta^{(p-m)k} = n \delta_{p,m}, \quad \sum_{k=1}^{n} \eta^{(-p-m)k} = n \delta_{n-p,m},
\]
and $c_{n-p} = c_p$. Taking linear combinations of $v_{\pm p}$ we obtain two linearly independent real eigenvectors. Thus we can immediately conclude that the period matrix $t$ has $c_m, m = 1, \ldots, n$ as its eigenvalues. Due to the symmetry $c_{n-m} = c_m, m = 1, \ldots, n-1$ they come in pairs. Finally, the matrix $\tau_{ik}$ of $U(1)$ couplings (8.1) becomes

$$\tau_{ik} = t_{ik} - \delta_{ik} \left( \frac{1}{N_k} \sum_{m=1}^{n} N_m t_{mk} \right) = t_{ik} - \delta_{ik} c_n.$$ (8.17)

Therefore, the eigenvalues of $\tau$ are given by 0 and $\tau_m = c_m - c_n, m = 1, \ldots, n-1$.

### 8.3 Non-analytic behaviour close to the Argyres-Douglas points

The hypergeometric function $\psi_1$ (8.7) has the following analytical continuation

$$\psi_1 = A_1(m) F\left( \frac{n-m}{2n}, \frac{2n-m}{2n}, \frac{3n-2m}{2n}, 1 - \frac{4 \Lambda^{2n}}{u^2} \right)$$

$$+ A_2(m) \left( 1 - \frac{4 \Lambda^{2n}}{u^2} \right)^{\frac{2m-n}{4n}} F\left( \frac{n+m}{2n}, \frac{m}{2n}, \frac{n+2m}{2n}, 1 - \frac{4 \Lambda^{2n}}{u^2} \right), \quad m = 1, \ldots, n-1,$$ (8.18)

except for the case $m = \frac{n}{2}$ when

$$\psi_1 = \frac{1}{\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{(\frac{1}{4})_p (\frac{3}{4})_p}{p!p!} \left[ k_p - \log \left( 1 - \frac{4 \Lambda^{2n}}{u^2} \right) \right] \left( 1 - \frac{4 \Lambda^{2n}}{u^2} \right)^p,$$ (8.19)

where $k_p = 2 \psi(p+1) - \psi\left( \frac{1}{4} + p \right) - \psi\left( \frac{3}{4} + p \right)$. Then, by (8.12), the eigenvalues of the period matrix are non-analytic at the Argyres-Douglas points. In fact

$$c_m = c_{n-m} \approx \left( 1 - \frac{4 \Lambda^{2n}}{u^2} \right)^{\frac{n-2m}{2n}}, \quad m < \frac{n}{2}, \quad c_{\frac{n}{2}} \approx \frac{1}{\log \left( 1 - \frac{4 \Lambda^{2n}}{u^2} \right)}.$$ (8.20)

### 8.4 Transition to a solution with a lower number of cuts

By (8.9),(8.18) and (8.19)

$$\lim_{u^2 \to 4 \Lambda^{2n}} I_{m,l} \equiv \frac{1}{2\pi i} \oint_{\gamma_l} \frac{x^{m-1}}{y} dx \to \infty, \quad m \leq \frac{n}{2}.$$  

Then

$$\lim_{u^2 \to 4 \Lambda^{2n}} \psi_k = \sum_{m>n/2} \frac{1}{I_{mk}} \omega_m.$$ (8.21)

Note that in this limit

$$\lim_{u \to \pm 2 \Lambda^n} \omega_m = \frac{x^{m-1}}{\sqrt{x^m (x^m \mp 4 \Lambda^{2n})}} dx.$$
Therefore, for \( n = 2p \) and \( m > \frac{n}{2} = p \) we find
\[
\lim_{u \to \pm 2\Lambda^n} \omega_m = \frac{x^{m-1-p}}{\sqrt{(x^{2p} \mp 4\Lambda^{2n})}} \ dx = \frac{x^{m'-1}}{y} \ dx \equiv \tilde{\omega}_{m'}, \quad m' = 1, \ldots, p. \tag{8.22}
\]
Likewise, for \( n = 2p + 1, m > p \)
\[
\lim_{u \to \pm 2\Lambda^n} \omega_m = \frac{x^{m-1-p}}{\sqrt{x(x^{2p+1} \mp 4\Lambda^{2n})}} \ dx = \frac{x^{m'-1}}{y} \ dx \equiv \tilde{\omega}_{m'}, \quad m' = 1, \ldots, p + 1. \tag{8.23}
\]
Hence, by (8.21), (8.22) and (8.23), we can conclude that in the Argyres-Douglas limit the \( n \)-cut solution degenerates into one with \( \frac{n}{2} \) cuts for \( n \) even and \( \frac{n+1}{2} \) cuts for \( n \) odd, in short an \( \left[ \frac{n+1}{2} \right] \)-cut solution. The relevant curves are given respectively by
\[
\tilde{y}^2 = x^n \mp 4\Lambda^{2n},
\]
and
\[
\tilde{y}^2 = x(x^n \mp 4\Lambda^{2n}).
\]
The above generalizes a result of [6], where the singularity corresponded to a transition from a two-cut solution to a one-cut solution.

9. The large \( N \) limit

Following [3, 9], we expect a non-trivial behaviour of the large \( N \) limit at the Argyres-Douglas critical points. Using Eqs. (8.12) and (8.17), we can analyze the behaviour of the coupling constants of the low-energy \( U(1)^n \) theory in the \( p \)-th vacuum, \( p = 1, \ldots, N \). Let us denote by \( \tau_m, m = 1, \ldots, n-1 \) the non-trivial eigenvalues of the coupling constant matrix \( \tau_{lk} \). We find that
\[
\tau_m = c_m(e^{2\pi ip/N} x) - c_n = \frac{p}{N} + c_m(e^{2\pi ip/N} x)
\]
\[
= \frac{p}{N} + c_m(x) + \frac{2\pi ip}{N} x \ c_m'(x) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{9.1}
\]
where we used the fact that by Eq. (7.14)
\[
\frac{4\Lambda^{2n}}{u^2} = \frac{4e^{2\pi ip/N}}{u^2} \equiv e^{2\pi ip/N} x, \quad p = 1, \ldots, N.
\]
However, the expansion (9.1) is singular in the vicinity of the Argyres-Douglas points, \( x = x_c = 1 \), because by (8.20), \( c_m'(x) \) is not defined for \( x = x_c \). Actually, at the Argyres-Douglas points, \( \tau_m \) becomes
\[
\tau_m \approx \left(\frac{p}{N}\right)^{\frac{n-2m}{2n}}, \quad m < \frac{n}{2}, \quad \tau_{\pm} \approx \frac{1}{\log(\frac{x}{x_c})}. \tag{9.2}
\]
This is clearly a signal of the breakdown of the large $N$ expansion. In fact, the $1/N$ corrections to (9.1) for $x \neq x_c$ read

$$
\tau_m \approx (1 - x)^{\frac{n - 2m}{2n}} \left[ -2\pi i \left( \frac{n - 2m}{2n} \right) \frac{p}{N(1 - x)} + \mathcal{O} \left( \frac{1}{(N(1 - x))^2} \right) \right], m < \frac{n}{2},
$$

(9.3)

$$
\frac{1}{\tau_n^2} \approx \log(1 - x) - 2\pi i \frac{p}{N(1 - x)} + \mathcal{O} \left( \frac{1}{(N(1 - x))^2} \right)
$$

(9.4)

which make the singularity manifest.

9.1 The double scaling limit

Eqs. (9.3) and (9.4) suggest that the divergences at $x = 1$ can be compensated by taking the limits $N \to \infty$ and $x \to x_c = 1$ in a correlated way as follows

$$
x \to 1, \quad N \to \infty, \quad N(1 - x) = \text{cnst} = \frac{1}{\kappa}.
$$

(9.5)

In particular, the rescaled couplings

$$
\tau_m^{scaled} = (1 - x)^{\frac{2m - n}{2n}} \tau_m,
$$

(9.6)

have a finite universal limit given by

$$
\tau_m^{scaled} \sim (1 - 2\pi i p\kappa)^{\frac{n - 2m}{2n}}.
$$

(9.7)

Similarly, for $1/\tau_n/2$, after subtracting a term proportional to $\log(1 - x)$, one obtains

$$
\frac{1}{\tau_n^{scaled}} \sim \log (1 - 2\pi i p\kappa).
$$

(9.8)

In a series of papers [9, 10, 11, 12], Ferrari made a proposal to generalize the matrix model approach to non-critical strings [8] to the four dimensional case. The basic idea is to replace matrix integrals with four dimensional gauge theory path integrals with $N \times N$ adjoint Higgs fields.

It was shown in [10] that the large $N$ expansion of pure $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory becomes singular at special points on the moduli space due to IR divergences. However, these divergences can be compensated by taking the limit $N \to \infty$ and approaching the critical points in a correlated manner. These double scaling limits were then conjectured to define four dimensional string theories [11]. In this paper, the Seiberg-Witten period integrals of the $A_{n-1}$ Argyres-Douglas singularities were analyzed in detail. It is crucial to note there are non-trivial contributions in powers of $1/N$, which signals the presence of open strings. These terms are generated by fractional instantons [10].

In [1], the analysis was extended to the $\mathcal{N} = 1$ case. In particular, a $U(N)$ gauge theory with cubic superpotential was studied and it was shown that there
are critical values of the superpotential couplings where glueballs are massless, there are tensionless domain walls and confinement without a mass gap. At these critical points, the large $N$ expansion is singular and the tension of domain walls scales as a fractional power of $N$. Nevertheless, double scaling limits analogous to (9.5) exist and are again conjectured to define a four dimensional non-critical string theory.

The double scaling limits (9.5) fit into the above scenario and are consistent with the $\mathcal{N} = 2$ analysis of [11]. The conjecture is that they define a four dimensional non-critical string theory.

10. Conclusion

Using the techniques of [21], we constructed an $\mathcal{N} = 1$ theory with gauge group $U(nN)$ and degree $n+1$ tree level superpotential whose matrix model spectral curve develops an $A_{n-1}$ Atyreas-Douglas singularity. This theory is closely related to an underlying $\mathcal{N} = 2 U(n)$ model. In fact, the one-dimensional parameter space of the $U(nN)$ theory is actually isomorphic to a slice of the $\mathcal{N} = 2$ Coulomb moduli space of the $U(n)$ theory: $n - 1$ parameters of the $U(n)$ Seiberg-Witten curve are set to zero and the remaining one parametrizes the most relevant deformation away from the singularity. In particular, only a finite, $N$-independent, number of parameters is adjusted. The expression of the coupling constants of the $U(1)^n$ low-energy theory shows that the $1/N$ expansion is singular at the Atyreas-Douglas points. Nevertheless, it is possible to define appropriate double scaling limits (9.5) which are conjectured to define four dimensional non-critical string theories as proposed by Ferrari in [3, 10, 11, 12, 13]. At the Atyreas-Douglas points, the $n$-cut matrix model spectral curve degenerates into a curve with $\frac{n}{2}$ cuts for $n$ even and $\frac{n+1}{2}$ cuts for $n$ odd.

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Appendix A

Performing the analytic continuation of (4.6) we find

\[ S_k = C_{1,k} F\left( -\frac{1}{2} - \frac{1}{2n} - \frac{1}{2} + \frac{1}{2n} \right) + C_{2,k} u F\left( u^2 \right) \]

\[ = \frac{C_{1,k} \Gamma(1/2)}{\Gamma(1/2 - 1/2n)} \left[ \frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1 + 1/2n)} \sum_{k=0}^{\infty} \frac{(-1/2 - 1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!} \left( \frac{4\Lambda^{2n}}{u^2} \right)^k \right] \times \left( \log(-u^2/4\Lambda^{2n}) + h_{1,k} \right) \]

\[ + \frac{C_{2,k} \Gamma(3/2)}{\Gamma(1 - 1/2n)} \left[ \frac{(-u^2/4\Lambda^{2n})^{-1+1/2n}}{\Gamma(3/2 + 1/2n)} \sum_{k=0}^{\infty} \frac{(-1/2 - 1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!} \left( \frac{4\Lambda^{2n}}{u^2} \right)^k \right] \times \left( \log(-u^2/4\Lambda^{2n}) + h_{2,k} \right) \]

In the above expression there is a term proportional to \( u^{1+1/n} \) that we need to set to zero. This determines \( C_{2,k} \) as a function of \( C_{1,k} \)

\[ C_{2,k} = -\frac{e^{i\pi/2} \Gamma(1 - 1/2n) \Gamma(3/2 + 1/2n)}{2\Lambda^n \Gamma(1/2 - 1/2n) \Gamma(1 + 1/2n)} C_{1,k} . \quad (10.1) \]

We are left with

\[ S_k = \frac{C_{1,k} \Gamma(1/2)}{\Gamma(1/2 - 1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1 + 1/2n)} \sum_{k=0}^{\infty} \frac{(-1/2 - 1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!} \left( \frac{4\Lambda^{2n}}{u^2} \right)^k \times \left( h_{1,k} - h_{2,k} \right) \]

where

\[ h_{1,k} - h_{2,k} = -\psi(1/2 - 1/2n + k) - \psi(1/2n - k) + \psi(-1/2n + 1 + k) + \psi(1/2 + 1/2n - k) \]

\[ = \left[ \psi(1/2 + 1/2n - k) - \psi(1/2 - 1/2n + k) \right] - \left[ \psi(1/2n - k) - \psi(-1/2n + 1 + k) \right] \]

\[ = [\pi \tan \pi(1/2n - k)] - [-\pi \cot \pi(1/2n - k)] = \frac{2\pi}{\sin(\pi/n)} . \]

Finally

\[ S_k = \frac{2\pi}{\sin(\pi/n)} \frac{C_{1,k} \Gamma(1/2)}{\Gamma(1/2 - 1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1 + 1/2n)} \times \]

\[ \times \sum_{k=0}^{\infty} \frac{(-1/2 - 1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!} \left( \frac{4\Lambda^{2n}}{u^2} \right)^k . \]
Comparing with (4.8), we find

\[
(1/2 - 1/2n)_{k+1} = \frac{\Gamma(-1/2 - 1/2n + k + 1)}{\Gamma(-1/2 - 1/2n)} = \frac{\Gamma(1/2 - 1/2n + k)}{\Gamma(1/2 - 1/2n)} \frac{\Gamma(1/2 - 1/2n)}{\Gamma(-1/2 - 1/2n)}
\]

\[
= (1/2 - 1/2n)^k \frac{\Gamma(1/2 - 1/2n)}{\Gamma(-1/2 - 1/2n)},
\]

\[
(1/2n)_{k+1} = \frac{\Gamma(-1/2n + k + 1)}{\Gamma(-1/2n)} = \frac{\Gamma(1 - 1/2n + k)}{\Gamma(1 - 1/2n)} \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)}
\]

\[
= (1 - 1/2n)^k \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)},
\]

and

\[
\Gamma(2)(2)_k = (2)_k = \Gamma(2 + k) = (k + 1)!
\]

the expression is equivalent to

\[
S_k = \frac{2\pi}{\sin(\pi/n)} \frac{C_{1,k} \Gamma(1/2)}{\Gamma(1/2 - 1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2} + 1/2n}{\Gamma(1 + 1/2n)} \frac{\Gamma(1/2 - 1/2n)}{\Gamma(-1/2 - 1/2n)} \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)}
\]

\[
= \frac{2\pi}{\sin(\pi/n)} \frac{\sqrt{\pi} C_{1,k}}{\Gamma(1 + 1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2} + 1/2n}{\Gamma(-1/2 - 1/2n)} \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)}
\]

\[
F \left( \frac{1/2 - 1/2n}{1/2 - 1/2n}, \frac{1 - 1/2n}{1 - 1/2n}, 2, \frac{4\Lambda^{2n}}{u^2} \right)
\]

Comparing with (4.8), we find

\[
C_{3,k} = e^{i\pi/2 + i\pi/2n} \frac{2\pi}{\sin(\pi/n)} \left( \frac{\sqrt{\pi} 2^{n-1}}{2n \Gamma(1 + 1/2n) \Gamma(-1/2 - 1/2n)} \right) \Lambda^{n-1} C_{1,k}.
\]  

(10.2)

Appendix B: The Picard-Fuchs equations for the coupling constants

In order to determine the coupling constants explicitly, we will derive the Picard-Fuchs equations satisfied by the periods of \( \omega_m \). The first derivatives of a period w.r.t. \( u \) and \( \Lambda^{2n} \) are given by

\[
\frac{\partial}{\partial u} \int \omega_m = \frac{\partial}{\partial u} \int \frac{x^{m-1}}{y} \, dx = \int \frac{x^{n+m-1} - ux^{m-1}}{y^3} \, dx,
\]  

(10.3)

and

\[
\frac{\partial}{\partial \Lambda^{2n}} \int \frac{x^{m-1}}{y} \, dx = 2 \int \frac{x^{m-1}}{y^3} \, dx.
\]  

(10.4)
Furthermore, since a period of $\omega_m$ is a homogeneous function of degree $m - n$, we find
\[
\left(2n\Lambda^{2n} \frac{\partial}{\partial \Lambda^{2n}} + nu \frac{\partial}{\partial u} - (m - n)\right) \int \frac{x^{m-1}}{y} \, dx = 0. \tag{10.5}
\]
The second derivative w.r.t. $u$ reads
\[
\frac{\partial^2}{\partial u^2} \int \frac{x^{m-1}}{y} \, dx = 12n\Lambda^{2n} \int \frac{x^{m-1}}{y^5} \, dx + 2 \int \frac{x^{m-1}}{y^3} \, dx.
\]
Then
\[
x^{m-1} \frac{1}{y^5} \, dx = -\left(\frac{4\Lambda^{2n}(m - 3n) + mu^2}{12n\Lambda^{2n}(u^2 - 4\Lambda^{2n})}\right) x^{m-1} \frac{1}{y^3} \, dx - \left(\frac{(m - 2n)u}{12n\Lambda^{2n}(u^2 - 4\Lambda^{2n})}\right) x^{m+n-1} \frac{1}{y^3} \, dx,
\]
up to a total derivative, which yields
\[
\frac{\partial^2}{\partial u^2} \int \frac{x^{m-1}}{y} \, dx = \frac{(m - 2n)u}{n(u^2 - 4\Lambda^{2n})} \int \frac{x^{m+n-1}}{y^3} \, dx
\]
\[-\frac{(4\Lambda^{2n}(m - n) + (m - 2n)u^2)}{n(u^2 - 4\Lambda^{2n})} \int \frac{x^{m-1}}{y^3} \, dx. \tag{10.6}\]
Note that all the above equations involve periods of $\frac{x^{m-1}}{y^5} \, dx$ and $\frac{x^{m+n-1}}{y^3} \, dx$ only.
Inverting these relations yields the Picard-Fuchs equation for the periods of $\omega_m$
\[
\left(\frac{\partial^2}{\partial u^2} + \frac{\alpha(m)u}{(u^2 - 4\Lambda^{2n})} \frac{\partial}{\partial u} + \frac{\beta(m)}{u^2 - 4\Lambda^{2n}}\right) \int \omega_m = 0, \tag{10.7}
\]
\[
\alpha(m) = \frac{3n - 2m}{n}, \quad \beta(m) = \left(\frac{\alpha(m) - 1}{2}\right)^2 = \left(\frac{n - m}{n}\right)^2. \tag{10.8}
\]

**Appendix C**

First of all
\[
\psi_1(n - m) = F\left(a(n - m), b(n - m), c(n - m), \frac{4\Lambda^{2n}}{u^2}\right)
\]
\[
= F\left(c(m) - b(m), c(m) - a(m), c(m), \frac{4\Lambda^{2n}}{u^2}\right)
\]
\[
= \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{a(m)+b(m)-c(m)} \psi_1(m) = \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{\frac{u^{2m}}{2m}} \psi_1(m),
\]
where the following Kummer’s relation was used
\[
F(a, b, c, z) = (1 - z)^{c-a-b}F(c - a, c - b, c, z).
\]
Another Kummer’s relation implies that
\[
\psi_2(n-m) = F\left(a(n-m), b(n-m), c(n-m), 1 - \frac{4\Lambda^{2n}}{u^2}\right)
\]
\[
= F\left(1 - a(m), 1 - b(m), 2 - c(m), 1 - \frac{4\Lambda^{2n}}{u^2}\right)
\]
\[
= \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{(m) - 1} \left(\frac{4\Lambda^{2n}}{u^2}\right)^{a(m) + b(m) - c(m)} U_5 \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)
\]

\[= \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{(m) - 1} U_5 \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)
\]

Likewise
\[
U_6(1-z) = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c+1-a-b)\Gamma(c-1)} U_6(1-z) - \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(1-c)}{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)} U_1(1-z)
\]
\[
= \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c+1-a-b)\Gamma(c-1)} (z)^{c-a-b} F(c-a, c-b, c+1-a-b, z)
\]
\[\quad \times \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(1-c)}{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)} F(a, b, c, 1-z),
\]

implies that
\[
\psi_2(n-m) = \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{(m) - 1} U_5 \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)
\]
\[
= \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{(m) - 1} \left(\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c-1)} \psi_1(m) - \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(1-c)}{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)} \psi_2(m)\right)
\]
\[
= \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{(m) - 1} \left(\frac{\Gamma((n-m)/2n)\Gamma((2n-m)/2n)}{\Gamma((n-2m)/2n)} \psi_1(m)
\]
\[\quad - \frac{\Gamma((n-m)/2n)\Gamma((2n-m)/2n)\Gamma((2m-n)/2n)}{\Gamma(m/2n)\Gamma((n+m)/2n)\Gamma((n-2m)/2n)} \psi_2(m)\right)
\]
\[= \left(1 - \frac{4\Lambda^{2n}}{u^2}\right)^{(m) - 1} \left(\frac{\psi_1(m)}{A_2(m)} - \frac{A_1(m)}{A_2(m)} \psi_2(m)\right).
\]

**Appendix D**

In order to derive the various Picard-Fuchs equations in the paper, we make use of the following identity
\[
a(x)p(x) + b(x)p'(x) = u^2 - 4\Lambda^{2n}, \quad (10.9)
\]
where
\[
p(x) = y^2 = (x^n - u)^2 - 4\Lambda^{2n},
\]
and
\[
a(x) = 1 - \frac{u}{4\Lambda^2}x^n, \quad b(x) = \frac{x}{2n} \left(-1 + \frac{u}{4\Lambda^2}(x^n - u)\right).
\]
By (10.9), we see that
\[
\frac{\phi(x)}{y^n} = \frac{1}{u^2 - 4\Lambda^2} \left( \frac{a(x)\phi(x)}{y^{n-2}} + \frac{2}{n-2} \left( \frac{b(x)\phi(x)}{y^{n-2}} \right)' \right),
\]
up to a total derivative.

References

[1] R. Dijkgraaf and C. Vafa, “Matrix Models, Topological Strings, and Supersymmetric Gauge Theories”, Nucl. Phys. B644 (2002) 3-20, [arXiv:hep-th/0206255].

[2] R. Dijkgraaf and C. Vafa, “On Geometry and Matrix Models”, Nucl. Phys. B644 (2002) 21-39, [arXiv:hep-th/0207106].

[3] R. Dijkgraaf and C. Vafa, “A Perturbative Window into Non-Perturbative Physics”, [arXiv:hep-th/0208048].

[4] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, “Perturbative Computation of Glueball Superpotentials”, [arXiv:hep-th/0211017].

[5] F. Cachazo, M. Douglas, N. Seiberg and E. Witten, “Chiral Rings and Anomalies in Supersymmetric Gauge Theories”, J. High Energy Phys. 0212 (2002) 071, [arXiv:hep-th/0211170].

[6] F. Ferrari, “Quantum parameter space and double scaling limits in N=1 super Yang-Mills theory”, [arXiv:hep-th/0211069];

[7] F. Ferrari, “On exact superpotentials in confining vacua”, Nucl. Phys. B648 (2003) 161-173, [arXiv:hep-th/0210135].

[8] É. Brézin and V.A. Kazakov, Phys. Lett. B236 (1990) 144; M.R. Douglas and S. Shenker, Nucl. Phys. B355 (1990) 635; D.J. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 127.

[9] F. Ferrari, Phys. Lett. B496 (2000) 212, [arXiv:hep-th/0003142]; J. High Energy Phys. 0106 (2001) 057 [arXiv:hep-th/0102041].

[10] F. Ferrari, Nucl. Phys. B612 (2001) 151, [arXiv:hep-th/0106192].

[11] F. Ferrari, Nucl. Phys. B617 (2001) 348, [arXiv:hep-th/0107096].

[12] F. Ferrari, J. High Energy Phys. 0205 (2002) 044, [arXiv:hep-th/0202002]; Int. J. Mod. Phys. A18 (2003) 577, [arXiv:hep-th/0202205];
[13] F. Ferrari, “Four dimensional non-critical strings”, Les Houches summer school 2001, Session LXXVI, l’ Unité de la Physique fondamentale: Gravité, Théorie de Jauge et Cordes”, A. Bilal, F. David, M.R Douglas and N. Nekrasov editors, [arXiv:hep-th/0205171].

[14] P.C. Argyres and M. Douglas, “New Phenomena in SU(3) Supersymmetric Gauge Theory”, Nucl. Phys. B448 (1995) 93-126 [arXiv:hep-th/9505062].

[15] N. Seiberg and E. Witten, “Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory”, Nucl. Phys. B426, (1994) 19, [Erratum-ibid. B430, (1994) 485] [arXiv:hep-th/9407087].

[16] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, “Simple singularities and N=2 supersymmetric Yang-Mills theory”, Phys. Lett. B344, (1995) 169, [arXiv:hep-th/9411048].

[17] P.C. Argyres and A.E. Faraggi, “The vacuum structure and spectrum of N=2 supersymmetric SU(n) gauge theory”, Phys. Rev. Lett. 74, (1995) 3931, [arXiv:hep-th/9411057].

[18] P.C. Argyres, R.N. Plesser, N. Seiberg and E. Witten, “New N=2 Superconformal Field Theories in Four Dimensions”, Nucl. Phys. B461 (1996) 71-84, [arXiv:hep-th/9511154].

[19] T. Eguchi, K. Hori, K. Ito and S. Yang, “Study of N = 2 Superconformal Field Theories in 4 Dimensions”, Nucl. Phys. B471 (1996) 430-444, [arXiv:hep-th/9603002].

[20] T. Eguchi, K. Hori, “N = 2 Superconformal Field Theories in 4 Dimensions and A-D-E classification”, Talk given by T. Eguchi at “Mathematical Beauty of Physics in Memory of Claude Itzykson”, Saclay, June 5-7, 1996, [arXiv:hep-th/9607125].

[21] F. Cachazo, K. Intriligator and C. Vafa, “A Large N Duality via a Geometric Transition”, Nucl. Phys. B603 (2001) 3-41, [arXiv:hep-th/0103067].

[22] J. de Boer and Y. Oz, “Monopole Condensation and confining phase of N=1 gauge theories via M-theory fivebrane”, Nucl. Phys. B511 (1998) 155, [arXiv:hep-th/9708044].

[23] F. Cachazo and C. Vafa, “N = 1 and N = 2 Geometries from Fluxes”, [arXiv:hep-th/0206017].

[24] F. Cachazo, N. Seiberg and E. Witten, “Phases of N=1 Supersymmetric Gauge Theories and Matrices”, J. High Energy Phys. 0302 (2003) 042, [arXiv:hep-th/0301006].

[25] A. Klemm, W. Lerche and S. Theisen, “Nonperturbative Effective Actions of N=2 Supersymmetric Gauge Theories”, Int.J.Mod.Phys. A11 (1996) 1929-1974, [arXiv:hep-th/9505150].

[26] Bateman Manuscript Project, “Higher transcendental functions”, Vol. I, New York, McGraw-Hill, 1953-55, Arthur Erdélyi editor.
[27] C. I. Lazaroiu, “Holomorphic Matrix Models”, [arXiv:hep-th/0303008].