ON INEQUALITY $|z^n - 1| \geq |z - 1|$

RADOŠ BAKIĆ

Abstract. We prove that $|z^n - 1| \geq |z - 1|$ for all complex $z$ satisfying $|z - 1/2| \leq 1/2$ and all real $n \geq 1$.

1. Introduction

R. Spira proved that $|w^{n+1} - 1| \geq |w^n| |w - 1|$ for all Gaussian integers $w \in \mathbb{C}$ such that $\Re w \geq 1$ and all positive integers $n$, see [1]. He posed a question if this is true for all complex $w$ such that $\Re w \geq 1$, $n$ is again a positive integer, see [2]. The answer is affirmative, see [3], page 140. If we set $w = 1/z$, then the inequality is transformed into the following form: $|z^n - 1| \geq |z - 1|$ for $|z - 1/2| \leq 1/2$ and $n \geq 1$ is an integer.

In this note we prove that this inequality is valid for all real $n \geq 1$.

2. The Main Result

As noted in the Introduction, we prove the following result.

Theorem 1. For any real $n \geq 1$ we have

$$|z^n - 1| \geq |z - 1|, \quad |z - 1/2| \leq 1/2.$$  

If $n > 1$ and $z \neq 0, 1$, then the inequality is strict.

Our proof of the Theorem is based on the following lemma which is also of independent interest.

Lemma 1. If $n > 3$ then

$$\cos^n x < 1 - \sin x, \quad \frac{2\pi}{n + 1} \leq x < \frac{\pi}{2}.$$  

Proof of Lemma. Taking logarithms we transform our inequality into equivalent form

$$\frac{n}{2} \ln(1 - \sin^2 x) < \ln(1 - \sin x), \quad \frac{2\pi}{n + 1} \leq x < \frac{\pi}{2}.$$  

Next, using power series expansion of $\ln(1 - t)$ we obtain an equivalent inequality

$$\frac{n}{2} \sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} > \sum_{k=1}^{\infty} \frac{\sin^k x}{k}, \quad \frac{2\pi}{n + 1} \leq x < \frac{\pi}{2},$$  

or, by rearranging terms,

$$\sum_{k=1}^{\infty} \left( \frac{(n - 1) \sin^{2k} x}{2k} - \frac{\sin^{2k-1} x}{2k - 1} \right) > 0, \quad \frac{2\pi}{n + 1} \leq x < \frac{\pi}{2}.$$  

1 Mathematics Subject Classification 2010 Primary 30A10. Key words and Phrases: Complex powers, inequalities.
Clearly, it suffices to prove that each term in the sum is strictly positive, and since
\[ \sin x > 0 \]
for the allowed range of values of \( x \) this is equivalent to the following
inequality
\[ \sin x > \frac{2k}{(n-1)(2k-1)}, \quad k \geq 1, \quad \frac{2\pi}{n+1} < x < \frac{\pi}{2}, \quad n > 3. \]
However, the maximum of the right hand side over \( k \) is attained for \( k = 1 \) and the
minimum of the left hand side over \( x \) is attained for \( x = 2\pi/(n + 1) \), so it suffices
to verify the inequality
\[ \sin \frac{2\pi}{n+1} > \frac{2}{n-1}, \quad n > 3. \]
Since \( \sin x \) is strictly concave for \( 0 \leq x \leq \pi/2 \) we have \( \sin x > 2x/\pi \) for \( 0 < x < \pi/2 \),
setting \( x = 2\pi/(n + 1) \) this gives
\[ \sin \frac{2\pi}{n+1} > \frac{2\pi}{\pi n + 1} > \frac{2}{n-1}, \]
the last inequality relies on the assumption \( n > 3 \). This proves our Lemma.
\( \square \)

Proof of Theorem. Since
\[ f(z) = \frac{z-1}{z^n-1} \]
is analytic in a neighborhood of \( K = \{ z \in \mathbb{C} : |z - 1/2| \leq 1/2 \} \) it suffices, by
the Maximum Modulus Principle, to prove our inequality for \( z \in \partial K = C \). Let
\( z = re^{i\phi} \in C \). Since the inequality is obvious for \( z = 0 \) and for \( z = 1 \) we can assume
\( 0 < r < 1 \). Since both sides of inequality are invariant under complex conjugation
we can also assume \( 0 \leq \phi \leq \pi/2 \).

If \( n\phi \leq 2\pi - \phi \), then the inequality holds for elementary geometric reasons.
Indeed, in that case the point \( z^n \) lies on the circle \( \{ w \in \mathbb{C} : |w| = r^n \} \) and outside
the angle \( S_\phi = \{ w = re^{i\theta} : \rho \geq 0, -\phi < \theta < \phi \} \). Since \( r^n < r \) it is easily seen that
\( |z^n - 1| < |z - 1| \).

Therefore, we can assume that \( 2\pi/(n + 1) < \phi < \pi/2 \). Note that this implies
\( n > 3 \). Clearly \( |z - 1| = \sin \phi \), therefore we have to prove that \( z^n \) lies outside the
circle \( |z - 1| = \sin \phi \). In fact, we are going to prove a stronger assertion: \( z^n \) lies
to the left of the line \( l \) given by \( \Re z = 1 - \sin \phi \), which is tangent to the mentioned
circle. This can be expressed analytically as \( \Re z^n < 1 - \sin \phi \). Since \( r = \cos \phi \) this
can be written as \( \cos^n \phi \cos n\phi < 1 - \sin \phi \). However, in view of Lemma this is
immediate: \( \cos^n \phi \cos n\phi \leq \cos^n \phi < 1 - \sin \phi \) and the proof is finished. \( \square \)

References

[1] R. Spira, The complex sum of divisors, The American Mathematical Monthly, Vol. 68, No. 2
(Feb. 1961), 120-124.
[2] R. Spira, Problem 4975, The American Mathematical Monthly, Vol. 68, No. 6, 1961, p. 577.
[3] D. S. Mitrinović, Elementary inequalities, P. Noordhoff, Groningen, 1964.
[4] R. Breusch, Problem 5394, Amer. Math. Monthly 75,85 (1968).

Učiteljski Fakultet, Beograd, Serbia
E-mail address: bakicr@gmail.com