THE EFFECT OF MAGNETIC TOPOLOGY ON THERMALLY DRIVEN WIND: TOWARD A GENERAL FORMULATION OF THE BRAKING LAW

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ABSTRACT

Stellar wind is thought to be the main process responsible for the spin down of main-sequence stars. The extraction of angular momentum by a magnetized wind has been studied for decades, leading to several formulations for the resulting torque. However, previous studies generally consider simple dipole or split monopole stellar magnetic topologies. Here we consider, in addition to a dipolar stellar magnetic field, both quadrupolar and octupolar configurations, while also varying the rotation rate and the magnetic field strength. Sixty simulations made with a 2.5D cylindrical and axisymmetric set-up, and computed with the PLUTO code, were used to find torque formulations for each topology. We further succeed to give a unique law that fits the data for every topology by formulating the torque in terms of the amount of open magnetic flux in the wind. We also show that our formulation can be applied to even more realistic magnetic topologies, with examples of the Sun in its minimum and maximum phases as observed at the Wilcox Solar Observatory, and of a young K-star (TYC-0486-4943-1) whose topology has been obtained by Zeeman–Doppler Imaging.

Key words: magnetohydrodynamics (MHD) – stars: low-mass – stars: magnetic field – stars: rotation – stars: winds, outflows

1. INTRODUCTION

The evolution of the angular momentum of a solar-like star is the result of interactions with its environment: a disk when it is an accreting young star or its expanding atmosphere—the stellar wind—during the main sequence (MS). Despite a broad distribution of rotation rates in very young star clusters (Irwin & Bouvier 2009), MS stars spin down and are observed to approximately follow the empirical Skumanich’s law: $\Omega(t) \propto t^{-1/2}$ (Skumanich 1972). This law has been deduced from observations of the Pleiades, Ursa Major, and the Hyades. Since then, observations of low-mass stars’ open clusters have confirmed this trend. The braking is the consequence of the magnetized wind that carries angular momentum away from the star (Schatzman 1959, 1962), and close planets could play a role as well (Strugarek et al. 2014b). Also, the magnetic activity of the star is stronger with higher rotation rate (Noyes et al. 1984; Brandenburg & Saar 2000; Brun et al. 2004; García et al. 2014) so that more rapidly rotating stars brake more than slower ones. Thus, this convergence can be understood, and models have been developed (Reiners & Mohanty 2012; Gallet & Bouvier 2013; Brown 2014) with the goal of being able to infer age from the stellar rotation period (gyrochronology, see Barnes 2003) or magnetic activity (magnetochronology, see Vidotto et al. 2014b).

For sun-like stars, the wind accelerates mostly due to the pressure gradient (Parker 1958) (for a more precise description, see Cranmer et al. 2007). In the case of rapid rotators, both magnetic pressure and centrifugal force add to acceleration (Weber & Davis 1967). The wind reaches the Alfvén speed at the Alfvén surface. Weber & Davis (1967) demonstrated that for a one-dimensional magnetized wind, the loss of angular momentum is proportional to the Alfvén radius squared, which acts as a lever arm. Hence, models that have been proposed to explain rotation rate evolution need to tie the Alfvén radius to the parameters of the problem. Kawaler (1988), following the formulation of Mestel (1984), introduced a power-law dependence of the Alfvén radius on the strength of the magnetic field over the mass-loss rate. This power-law formulation has been investigated further by Matt & Pudritz (2008) and Matt et al. (2012) who included the influence of the rotation rate.

In those torque formulations, the mass-loss rate is assumed to be known, and analytical techniques have been proposed to compute it from stellar parameters (Cranmer & Saar 2011), but it can also be observed (Wood 2004). Indeed, the wind is eventually stopped by the interstellar medium pressure and becomes subsonic again at the termination shock. Beyond this point, a contact surface between the stellar wind and the interstellar plasma, the astropause, contains heated hydrogen that produces H I Lyman-α absorption, which is detectable in the UV. Those data can be used to infer the mass-loss rate of sun-like stars (Wood et al. 2002) and thus braking models can be constrained by observations and provide a solid base for gyrochronology.

However, to date, studies have mainly considered simple magnetic topologies such as split monopoles (Weber & Davis 1967; Kawaler 1988) and dipoles (Mestel 1968; Washimi & Shibata 1993; Matt & Pudritz 2008; Matt et al. 2012; Cohen & Drake 2014). The real topology of the stellar magnetic field can be much more complex. For instance, during the 22 year solar cycle, the sun oscillates between dipolar and quadrupolar dominant topology (DeRosa et al. 2012). It is now generally agreed that magnetic activity in solar-like stars owes its origin to a nonlinear dynamo process operating in and at the base of the stars’ convective envelopes (Moffatt 1978; Brun et al. 2004, 2013; Charbonneau 2010). High performance numerical simulations are now able to reproduce key characteristics of stellar magnetic activity, such as global scale organization of the magnetic field, regular cycles, and flux concentration and emergence (Ghizaru et al. 2010; Brown et al. 2011; Racine et al. 2014b).
These simulations inform us on the organization of the large-scale magnetic topology in solar-like stars and hence on its impact on their coronal field.

The large-scale coronal magnetic fields influence the wind driving and may be responsible for the changes in velocities and the mass-loss rate over the solar cycle (Pinto et al. 2011). Thus, models are needed to quantify this effect on the wind driving and the associated extraction of angular momentum. Three-dimensional simulations of stellar winds evaluating the mass and angular momentum loss rates have already been made, for instance in the work of Cohen & Drake (2014), who used the BATS-R-US code with a dipolar topology and an axisymmetric set-up. Vidotto et al. (2014b) introduced realistic topologies of six M-dwarfs, where they provide physical based relations between output and parameters of the simulations. However, for the sample of stars, they used a wide range of stellar parameters whose influence could not be clearly isolated. We chose to work in 2.5D, which means that we have two spatial dimensions, assumed axisymmetry, and vectors have all three components, to be able to perform more than 60 wind simulations in a systematic parametric study. We derive braking laws from our simulation results, which accurately include the influence of magnetic topologies that are more complex than a dipole, in order to improve rotation evolution models.

In our study, we focused on thermally driven winds that are thought to exist in every cool star with an outer convective zone. To include this in our work, we used the maximum of cycle 22 using Wilcox magnetograms. These simulations inform us on the organization of the large-scale magnetic topology in solar-like stars and hence on its impact on their coronal field.

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3. NUMERICAL SETUP

In this work, we use the compressible magnetohydrodynamic (MHD) code PLUTO (Mignone et al. 2007). All simulations are performed in 2.5D (two spatial dimensions, three vector components), assuming axisymmetry and using cylindrical coordinates (hereafter $(R, \phi, Z)$). Since PLUTO is a multi-physics, multi-solver code, we chose a finite-volume method using an approximate Riemann Solver, here the HLL solver (Einfeldt 1988).

PLUTO uses a reconstruct–solve–average approach using a set of primitive variables $(\rho, \mathbf{v}, p, \mathbf{B})$ to solve the Riemann problem corresponding to the following conservative ideal MHD equations (expressed here with the set of conservative variables $(\rho, \mathbf{m}, E, \mathbf{B})$):

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial}{\partial t} \mathbf{m} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \rho \mathbf{a}$$
$$\frac{\partial}{\partial t} E + \nabla \cdot ((E + p) \mathbf{v}) = \mathbf{m} \cdot \mathbf{a}$$
$$\frac{\partial}{\partial t} \mathbf{B} + \nabla \cdot (\mathbf{v} \times \mathbf{B}) = 0,$$

where $\rho$ is the density, $p$ and $\mathbf{v}$ are the pressure and the velocity field, $\mathbf{m} = \rho \mathbf{v}$ is the momentum density, $E = \rho e + \frac{1}{2} \mathbf{m}^2/(2\rho)$ + $\mathbf{B}^2/2$ is the energy density using the ideal equation of state: $\rho e = p/(\gamma - 1)$ ($\gamma$ being the adiabatic exponent, and $\epsilon$ the internal energy per mass), $\mathbf{B}$ is the magnetic field, and $\mathbf{a}$ is a source term (gravitational acceleration in our case).

Our domain is $[R, Z] \in [0, 100R_\odot] \times [-100R_\odot, 100R_\odot]$ with 768 $\times$ 1536 grid points. We use a mixed grid (uniform-stretched) so that 256 $\times$ 512 grid points uniformly mesh the domain $[0, 2.5R_\odot] \times [-2.5R_\odot, 2.5R_\odot]$, which surrounds the star. The grid spacing then grows geometrically with the distance to the star.

The stellar wind solutions are obtained by setting boundary conditions at the border and inside the computational domain. Those boundary conditions are described in Appendix B and summed-up in Figure 10. PLUTO solves normalized equations. Three normalization values set all the others: length, speed, and density. Thus, in our set-up, the radius of the star is $R_\star/R_\odot = 1$, the density and the keplerian speed at the surface of the star are $\rho_\star/\rho_\odot = 1$ and $v_{\text{kep}}/V_\odot = \sqrt{GM_\odot/R_\odot}/V_\odot = 1$.

By choosing the physical values of those normalizations, for example $R_\odot = 6.96 \times 10^{10}$ cm (the radius of the Sun) one can deduce all of the other values output by the code. The magnetic field normalization is, for instance, given by $B_\odot = \sqrt{4\pi \rho_\odot V_\odot}$. Our simulations are then controlled with five parameters: $\gamma$ the adiabatic exponent (ratio of specific heats), the initial magnetic field topology, which can be a dipole, a quadrupole, or an octupole, and three speeds normalized by the escape speed, taken at the equator and at the surface of the star: $v_{\text{rot}}/v_{\text{esc}}$ the rotation speed, $v_{A}/v_{\text{esc}}$ the surface Alfvén speed at the equator (directly related to the strength of the magnetic field), and $c_s/v_{\text{esc}}$ the speed of sound (giving the pressure over density ratio since $c_s = \sqrt{\gamma p/\rho}$).

We then let the code evolve the set of Equations (8)–(11) until it reaches a steady-state solution. We check the quality of this steady state with various criteria. For instance, by looking at the mass flux versus time, one can be sure that a constant value is reached. Another method first introduced in Keppens & Goedbloed (1999) is to look at several quantities that should be conserved along the field lines in ideal MHD. We used this technique with several quantities, especially the effective rotation rate. More details are given in Appendix B.

Our purpose here is to investigate the effect of the magnetic topology on the magnetic braking of sun-like stars. We use for this the same method as the one developed in Matt & Balick (2004), Matt & Pudritz (2008), and Matt et al. (2012), i.e., 2.5D axisymmetric ideal MHD simulations. However, a different code is used to compute the wind solutions: PLUTO (Mignone et al. 2007). We kept fixed $\gamma$ and $c_s/v_{\text{esc}}$, at the fiducial values for sun-like stars (Matt et al. 2012; Washimi & Shibata 1993). The parameter $c_s/v_{\text{esc}} = 0.222$ corresponds to a $\sim 10^6$ K hot corona for solar parameters and $\gamma = 1.05$. This choice of $\gamma$ is dictated by the need to maintain an almost constant temperature as the wind expands, which is observed in the solar wind. Hence, choosing $\gamma \neq 5/3$ is a simplified way of taking into account heating that is not modeled here. For combined values of $v_{A}/v_{\text{esc}}$ and $\gamma$ we chose 20 cases from Matt et al. (2012). For each of these cases we run three different magnetic topologies for the star. The value of the parameters are summed-up in Table 1. For a solar mass and radius, the values for the rotation period range from 1167 days till 0.3 days approximately (from $10^{-4}$ to $0.4$ in terms of break-up ratio), while the strength of the magnetic field at the equator (controlled by $v_{A}/v_{\text{esc}}$) goes from 0.9 to 35 G for a base coronal density of $2.9 \times 10^{15}$ g cm$^{-3}$. However, changing normalizations naturally changes the physical parameter range (see Section 7).

Figure 1 shows these three topologies in the initial state (dashed field lines) and once a steady-state has been reached (continuous field lines). We now discuss the results of our 60 simulations.

4. PARAMETRIC DEPENDENCE OF THE MAGNETIC TORQUE

Magnetic field topology has a strong influence on the outflow. Figure 1 illustrates how the thermal and dynamical pressure of

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**Table 1**

| Case | $v_A/v_{esc}$ | $f$ | $\langle R_A \rangle$ Dip. | $\langle R_A \rangle$ Quad. | $\langle R_A \rangle$ Oct. |
|------|---------------|------|---------------------------|---------------------------|---------------------------|
| 1    | 0.0753        | 9.95e-5 | 6.3                     | 3.6                       | 3.0                       |
| 2    | 0.301         | 9.95e-5 | 12.5                    | 5.3                       | 4.0                       |
| 3    | 1.51          | 9.95e-5 | 32.3                    | 9.3                       | 5.9                       |
| 4    | 2.00          | 9.95e-5 | 36.4                    | 9.9                       | 6.3                       |
| 5    | 0.0753        | 9.95e-4 | 6.3                     | 3.6                       | 3.0                       |
| 6    | 0.301         | 9.94e-4 | 12.5                    | 5.3                       | 4.0                       |
| 7    | 1.51          | 9.94e-4 | 32.3                    | 9.3                       | 5.9                       |
| 8    | 0.0753        | 3.93e-3 | 6.3                     | 3.6                       | 3.0                       |
| 9    | 0.301         | 3.93e-3 | 12.6                    | 5.3                       | 4.0                       |
| 10   | 1.51          | 3.93e-3 | 32.3                    | 9.3                       | 5.9                       |
| 11   | 0.0753        | 4.03e-2 | 5.9                     | 3.5                       | 3.1                       |
| 12   | 0.301         | 4.03e-2 | 11.7                    | 5.2                       | 4.1                       |
| 13   | 1.51          | 4.03e-2 | 30.3                    | 9.2                       | 5.8                       |
| 14   | 0.301         | 5.94e-2 | 10.6                    | 4.9                       | 4.0                       |
| 15   | 0.301         | 9.86e-2 | 8.7                     | 4.4                       | 3.6                       |
| 16   | 0.301         | 1.97e-1 | 5.5                     | 3.4                       | 2.9                       |
| 17   | 1.51          | 1.97e-1 | 13.4                    | 5.7                       | 4.4                       |
| 18   | 0.753         | 4.03e-1 | 4.9                     | 3.1                       | 2.6                       |
| 19   | 1.51          | 4.03e-1 | 7.1                     | 3.7                       | 3.2                       |
| 20   | 3.01          | 4.03e-1 | 11.4                    | 5.0                       | 4.3                       |

Notes. Parameters of our 60 simulations (20 for each topology) with $\gamma = 1.05$ and $c_s/v_{\text{esc}} = 0.222$ in Columns 2 and 3. The number the cases refer to Matt et al. (2012). The resulting average Alfvén radii are given in the last three columns. They decrease with higher order topology, and with rotation starting at $f = 0.04$ (cases 23, 24, 25).
Generally speaking, we can extract trends for each independent parameter variation.

4.1. Effect of Magnetic Field Strength

The magnetic field strength has quite a straightforward influence on the average Alfvén radius: a more intense magnetic field results in a larger torque (see Table 1). This is because the Alfvén speed is higher, and thus it takes longer for the wind to accelerate and reach the Alfvén surface. The magnetic field strength also has a weak influence on the mass-loss rate (compared to the rotation rate, see Section 7). The latter decrease with stronger magnetic fields since magnetic forces are able to confine more plasma in the dead zones. However, the Alfvén radius does not grow linearly with the magnetic field strength. For instance, between cases 2 and 3, the magnetic field strength is multiplied by 5, whereas the average Alfvén radius is multiplied by 2.6 (see Table 1).

Matt & Pudritz (2008) precisely described this effect, and for weakly rotating cases, we are able to fit the average Alfvén radius with the same formulation (Equation (A5)). More details on the fit will be given in Section 4.2.

4.2. Magnetocentrifugal Regime and Force Budget (Effect of Rotation)

In Figure 2, several solutions illustrate the influence of rotation and topology on the wind. For the three different topologies, the rotation rate is increased while the magnetic field strength is held fixed (the top panel is case 2, the bottom panel is case 31). We plot the surfaces corresponding to each mode of ideal MHD along with the sonic surface. Fast and slow magnetosonic surfaces are often merged with the sonic and Alfvén surfaces, and switch when those two cross so that the fast magnetosonic surface is always further from the star than the slow one. The poloidal speed (in color background) is strongly affected by the change in rotation rate in the bottom panels.

However, the influence of the rotation rate appears through two different phenomena depending on latitude. The most relevant aspect for angular momentum loss is that the Alfvén surface comes closer to the star at low and mid latitudes with a higher rotation rate (see Figure 2 and Table 1). This is a simple consequence of the magnetocentrifugal effect described by Weber & Davis (1967), Sakurai (1985), and Washimi & Shibata (1993). The magnetized wind is rotating with the star and is thus accelerated by a centrifugal effect. A magnetic pressure gradient is also responsible for an additional acceleration. All forces projected along a magnetic field line can be expressed as follows (Ustyugova et al. 1999):

\[
\begin{align*}
    f_p &= -\frac{1}{\rho} \frac{\partial p}{\partial s} \\
    f_c &= -\frac{\partial \Phi}{\partial s} \\
    f_m &= -\frac{1}{8\pi \rho R^2} \frac{\partial (RB \phi)^2}{\partial s} \\
    f_g &= -\frac{v^2}{s} \cdot \hat{s} \cdot \hat{R},
\end{align*}
\]

where the subscripts \(p\), \(g\), \(m\), and \(c\) refer, respectively, to the pressure gradient and the gravitational, magnetic, and centrifugal forces, while \(s\) is the curvi-linear abscissa directed
Figure 2. Steady-state solutions for a wind with dipolar, quadrupolar, and octupolar magnetic fields from left to right and for two different rotation rates: case 2 ($f = 9.95 \times 10^{-5}$) on top and 31 ($f = 5.94 \times 10^{-2}$) below. The simulations are initialized with the same coronal temperature and the same magnetic field strength. The thick white line with a black core is the Alfvén surface and the thick gray line is the sonic surface. The slow and the fast magneto-sonic surfaces, respectively, are the dot–dashed and dashed lines. When rotation increases, the Alfvén surface gets closer to the star at the equator and further at the pole. For higher order multipoles the Alfvén radius is also generally closer to the star. The poloidal speed normalized to the keplerian speed (437 km/s for the Sun) is represented by the color background with the same scale on all the panels.

along the field line and $\Phi$ the gravitational potential. Vectors with a hat are unit vectors. The force budget in slow and fast rotators is given in Figure 3. One can see that at high rotation (20% of the break-up speed) the magnetic and centrifugal forces become comparable to or even higher than the pressure gradient. A toroidal component of the magnetic field is created, which contributes to the driving (Equation (14)). This additional acceleration can double the poloidal speed around the equatorial streamer (for the highest values of $f$, see Figure 2) and thus allow the outflow to reach the Alfvén speed much closer to
Figure 3. On the two panels are the force budget along a field line for the dipolar cases 2, 45, and 47. The pressure gradient and gravity only remain close between the different cases, while the magnetic and centrifugal forces go from negligible (top panel for case 2) to comparable (colored lines for case 45 (red) and 47 (blue) in the bottom panel) to the pressure gradient. The field lines have their seeds at 75°, 55°, 60° of latitude for cases 2, 45, and 47, respectively, so that they remain approximately radial. With higher magnetic field strength, the amplitudes of the centrifugal and magnetic forces increase.

Figure 4. Three fits made for each topology (parameters are in Table 2). The crosses are for the dipole, the diamonds for the quadrupole, and the triangles for the octupole. Colors go from red to blue when rotating faster. With the growing complexity of the magnetic field, the braking efficiency decreases as shown by the values of the fits.

20 R_∗ (the window of Figure 2 is too small to see those effects). However, this effect does not seem to have a strong influence on the average global value (K_A) since angular momentum flux is relatively weak near the rotation axis (see Table 1).

In our parameter study (see Table 1), the effect of rotation on the average Alfvén radius starts to be noticeable with cases 23, 24, and 25 with a break-up ratio f = 0.04 for all topologies. There, magnetic and centrifugal forces reach a few percent of the pressure gradient. Up to f ≈ 0.1, this effect is still weak and the simple power law (Equation A5) provides a good fit. Beyond this value (f > 0.1), the decrease of the Alfvén radius becomes stronger, justifying a three parameter regression, as in Matt et al. (2012). Formulation 5 accurately describes our results and Figure 4 illustrates this fit made independently for the three topologies (three black lines). All of the results and outputs of our simulations are given in Table 4 while the results of the fits are in table 2.

4.3. Effect of Topology

Qualitatively, in all three topologies, we observe streamers at latitudes where magnetic forces are high enough to counterbalance the thermal and the ram pressure. Around the dead zones, the plasma is moving slower than in open field line regions (see Figure 2). The magnetic field reaches a minimum (in ideal MHD it should be zero and the Alfvén surface should touch the last closed magnetic field loop) on top of the dead zones, while on both sides of this minimum the Alfvén surface is slightly extended due to the slow wind. At those latitudes, beyond dead zones, thin current sheets are created, one for the dipole, two for the quadrupole, and three for the octupole.

Interestingly, the value of Υ is usually comparable (same order of magnitude) for the three topologies. Colored points of Figure 4 are almost vertically aligned, which means that the mass-loss rate does depend weakly on the magnetic topology (see Equation (6)). However, a trend for the effect of topology on the mass loss can be detected with the strongest magnetic fields (see Table 4), where lower-order topologies have higher Υ (see Appendix C).

The key point of our parameter study is that with an increasing complexity of the topology (higher-order multipole), the
Table 2
Fit Parameters for the Three Topologies

| Topology          | $K_1$  | $K_2$  | $m$   | $m_{\text{fit}}(q = 0.7)$ | $m_{\text{fit}}(q = -1/2)$ |
|-------------------|--------|--------|-------|---------------------------|-----------------------------|
| Dipole ($l = 1$)  | 2.0 ± 0.1 | 0.2 ± 0.1 | 0.235 ± 0.007 | 0.21                       | 0.29                        |
| Quadrupole ($l = 2$) | 1.7 ± 0.3 | 0.2 ± 0.1 | 0.15 ± 0.02   | 0.15                       | 0.18                        |
| Octupole ($l = 3$) | 1.7 ± 0.3 | 0.2 ± 0.1 | 0.11 ± 0.02   | 0.11                       | 0.13                        |
| Radial/open field ($l = 0$) |         |         | 0.37 ± 0.01 | 0.37                       | 0.66                        |

| $K_3$  | $K_4$  | $m$   |
|--------|--------|-------|
| Topology independent | 1.4 ± 0.1 | 0.06 ± 0.01 | 0.31 ± 0.02 |

Notes. Parameters of the fit to Equation (5) made independently for each topology. The values $K_1$ and $m$ correspond to three black lines of Figure 4 for the dipolar, quadrupolar, and octupolar configurations. The parameters $K_3$ and $K_4$ for the topology independent formulation 18 (see Section 5) are given as well. The expected values for $m_{\text{fit}}$ through analytical models are given for different values of the parameter $q$ for comparison with the fitted value (see Section 5.2 and Appendix A).

In our 60 simulations, we chose to use the parameter $v_A/v_{\text{esc}}$ to control magnetic field strength. This parameter is proportional to $B_\star$ which is the magnetic field amplitude taken at the surface of the star, on the equator. Thus, for different topologies, the magnetic surface flux varies slightly at $r/r_*=1$ as it can be seen in Figure 5. The variation is smaller than 29% when comparing the various topologies used in our study, independently of the value of $B_\star$. Hence, a parametric study that varies the surface magnetic flux would give similar braking laws. On the other hand, the value of the open flux is a complex consequence of the dynamics of the wind, and topology has a strong influence on it (see Figure 5). We will show in the next section how the open flux can be used to derive a topology-independent braking law.

5. TOWARD A GENERAL BRAKING LAW

5.1. A Topology-independent Formulation for the Magnetic Torque

Topologies of cool stars’ magnetic fields include combinations of dipolar, quadrupolar, and octupolar components as well as higher order multipoles. This configuration changes over magnetic cycles that are likely to occur in most solar-like stars (see Pinto et al. (2011) for the effect of the 11 yr cycle variability of the Sun on the wind topology). We attempt to find a topology-independent formulation in order to take into account this complexity for stellar evolution models. This section gives a single law fit for our 60 simulations and all three topologies. The key idea is to consider the dependency of the Alfvén radius on the open magnetic flux instead of the surface magnetic field strength. In all of our cases, the open flux is the constant value of the integrated magnetic flux beyond the last magnetic loops (see previous section, Figure 5).

Thus, introducing the open flux into a new parameter $\gamma_{\text{open}}$ defined as follows:

$$\gamma_{\text{open}} \equiv \frac{\Phi_{\text{open}}^2}{R_\star^2 M_\star v_{\text{esc}}} \, ;$$

and using a similar formulation to the previous section,

$$\langle R_A \rangle = K_3 \left[ \frac{\gamma_{\text{open}}}{(1 + f^2/K_4^2)^{1/2}} \right]^m \, ,$$

we are able to fit all of our 60 simulations into one single law, as shown in Figure 6. The value for the parameters of this law are in Table 2.

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The flux at the surface of the star is $\Phi(R_\star) = a \pi B_\star R_\star^2$ where $a = 4, 16/(3\sqrt{3}) \approx 3.1$ and $52/15 \approx 3.5$ for the dipole, the quadrupole, and the octupole, respectively.

5.1. A Topology-independent Formulation for the Magnetic Torque

Topologies of cool stars’ magnetic fields include combinations of dipolar, quadrupolar, and octupolar components as well as higher order multipoles. This configuration changes over magnetic cycles that are likely to occur in most solar-like stars (see Pinto et al. (2011) for the effect of the 11 yr cycle variability of the Sun on the wind topology). We attempt to find a topology-independent formulation in order to take into account this complexity for stellar evolution models. This section gives a single law fit for our 60 simulations and all three topologies. The key idea is to consider the dependency of the Alfvén radius on the open magnetic flux instead of the surface magnetic field strength. In all of our cases, the open flux is the constant value of the integrated magnetic flux beyond the last magnetic loops (see previous section, Figure 5).

Thus, introducing the open flux into a new parameter $\gamma_{\text{open}}$ defined as follows:

$$\gamma_{\text{open}} \equiv \frac{\Phi_{\text{open}}^2}{R_\star^2 M_\star v_{\text{esc}}} \, ;$$

and using a similar formulation to the previous section,

$$\langle R_A \rangle = K_3 \left[ \frac{\gamma_{\text{open}}}{(1 + f^2/K_4^2)^{1/2}} \right]^m \, ,$$

we are able to fit all of our 60 simulations into one single law, as shown in Figure 6. The value for the parameters of this law are in Table 2.

The flux at the surface of the star is $\Phi(R_\star) = a \pi B_\star R_\star^2$ where $a = 4, 16/(3\sqrt{3}) \approx 3.1$ and $52/15 \approx 3.5$ for the dipole, the quadrupole, and the octupole, respectively.

Figure 5. Magnetic flux for all three topologies in case 3. The black lines are the magnetic flux in the steady-state winds while the dashed lines are the fluxes of the initial potential field as a function of the distance from the star.

magnetic braking is less efficient for a given Alfvén speed at the base of the corona (see Figure 2 for relative positions of the Alfvén surfaces and Table 1 for the average value). Dipolar cases are always above quadrupolar cases, which are always above octupolar cases, in Figure 4 for a given $\Upsilon$. The slope of the fits also decreases with a higher order of multipoles (see Table 2). This is a consequence of the faster decay with distance to the star of the magnetic field for more complex topologies. This decay is shown in Figure 5, where the absolute magnitude of the magnetic flux is integrated over concentric spheres:

$$\Phi(r) = \int S_r |B \cdot dS| \, ,$$

(16)

As shown in Figure 5, the magnetic flux shows two regimes. Close to the star, the magnetic flux decreases as $1/r^l$ where $l$ is 1 for the dipole, 2 for the quadruple, and 3 for the octupole. In other words, the flux follows the radial dependence expected for single-mode topology. However, the wind opens the field lines and beyond some stellar radii, the field becomes completely open, and thus the flux becomes constant. This constant is the value we define as the open flux, which is numerically computed from our simulations (see Section 5). The decay of the three modes is such that even if the field lines open closer to the star in higher order multipole, the amount of magnetic flux is higher for lower order multipoles. Hence the Alfvén speed is logically reached closer for more complex topologies since the amplitude of the magnetic field is still lower.

![Figure 5](image-url)

Figure 5. Magnetic flux for all three topologies in case 3. The black lines are the magnetic flux in the steady-state winds while the dashed lines are the fluxes of the initial potential field as a function of the distance from the star.
In the end, our parameter study gave us three topology associated braking laws, using the same formulation as Matt et al. (2012), but with different coefficients and a topology independent formulation. For the dipolar cases, comparing our results to Matt et al. (2012), it is interesting to note that despite very different values of $\gamma$, the exponents $m$ for the dipolar case, which are $0.235$ in our case and $0.22$ in Matt et al. (2012), are almost statistically indistinguishable. This shows that even if the prediction of $\dot{M}_w$ is highly sensitive to numerical and thermodynamical properties of the simulations, $\gamma$ is the relevant control parameter for stellar wind braking.

Interestingly, the values of $K_1$ and $K_2$ do not vary much with the topology. The value of $K_1$ is somewhat constrained by a similar behavior of the Alfvén surface when $\gamma \sim 1$, i.e., for very weak magnetic fields. The coefficient $K_2$ is logically similar since the same acceleration process occurs in the open field lines with the three topologies. However, the topology has a strong influence on the braking through the $m$ coefficient and, to account for the complex magnetic field of solar-like stars in a single equation, the new formulation that we propose is needed.

Our dipolar value for $K_1$ is a bit smaller than the one obtained in Matt et al. (2012). We think that this value is very sensitive to our change of boundary conditions. It is the most likely to change with the thermal driving, i.e., the value of $\gamma$ and $c_s/v_{\text{esc}}$ that remained fixed in our parameter study (see Matt & Pudritz 2008 for some cases with different $c_s/v_{\text{esc}}$ and $\gamma$, and Section 7.1).

For the three topologies, our value for the $K_2$ constant is larger ($\sim 0.2$ rather than $0.07$) from the one found in Matt et al. (2012). However, as we see in Table 1, rotation starts to influence the average Alfvén radius around $f = 0.04$, but the errors are significant in those parameters and we have less coverage of the rotation rate than in Matt et al. (2012). The difference might also be due to the slight changes of boundary conditions in comparison with this work.

The parameter $K_4$ that appears in the open flux formulation seems to be the most relevant to consider the influence of the rotation rate. The value of $0.06$ is coherent with our forces analysis. It is also very close to the one obtained in Matt et al. (2012; $K_2 = 0.0716$), and in Washimi & Shibata (1993) where it is also found that above $f = 0.079$ the centrifugal effect begins to be prominent.

For the exponent $m$ of the power law, analytical calculations give $m_{\text{th}} = 1/(2l + 2 + q)$, where $1/l^{1+2}$ is the radial dependency of the magnetic field and $q$ is the exponent of the power law that describe the Alfvén velocity dependency on the Alfvén radius (see Appendix A). Kawaler (1988) and Reiners & Mohanty (2012) used a value of $-1/2$ in their studies. This value should be positive since the wind speed matches the Alfvén speed at the Alfvén radius and that our winds accelerate with distance to the star. In order to propose a value for $q$, we computed an average value of the Alfvén speed along the Alfvén surface, for slowly rotating cases ($f \lesssim 10^{-2}$), since the rotational influence is already included through the parameters $K_2$ and $K_4$. We also focused our sample on the range where most of our Alfvén radii are, namely $R_A/R_*= [2, 13]$, which represents 26 out of our 60 simulations. Then, as shown in Figure 7, the variation of the Alfvén speed (or wind speed) with the Alfvén radius can be approximated with the following equation:

\[
v(R_A) = v_{\text{esc}}(a(R_A/R_*)^b + b),
\]

with $q = 0.7$, $a = 0.063$ and $b = -0.084$.

The offset is due to the fact that the wind starts to accelerate at the surface of the star, i.e., at $R/R_* = 1$. Indeed, when magnetic

---

Over the 60 simulations, the average deviation to the fit is around $4\%$ while six cases reach between $15\%$ and $20\%$. This formulation shows how relevant the open flux in this situation is. Indeed, the mass and angular momentum losses occur through open field lines. Also, $\gamma_{\text{open}}$ contains the relevant information about the open field line region and the size and the number of dead zones, since these determine how much open flux there is compared to the surface magnetic flux. As a consequence, the open flux and the mass-loss rate are coupled (see Vidotto et al. 2012). Higher mass loss corresponds to higher open flux for a given $B_\ast$, that is to say rotation increases both. Thus, the effect of rotation can be seen earlier in the open flux formulation (the fitted value of $K_4$ is $0.06$ rather than $K_2 = 0.2$ in Section 4).

We show in Figure 13 (in Appendix C) how important it is to introduce a three parameter fit, taking into account the rotation rate. Comparing Figure 13 and 6, we see that the denominator $(1 + f^2/K_4^2)^{0.5}$ in formulation 18 collapses all the points on the single power law.

5.2. Comments

In our work, we have been through 20 cases of Matt et al. (2012) parameters. Comparing both works, we can see that our values for $\gamma$ are in all cases larger, sometimes by an order of magnitude. Different computed mass-loss rates are responsible for this mismatch. Our trend for higher $\gamma$ and thus a lower mass-loss rate can be understood. We implemented a somewhat different boundary condition that sets the pressure and density gradients inside the star using a polytropic (rather than isothermal) hydrodynamical wind solution (Keppens & Goedbloed 2000). This makes our simulation more stable numerically since an inherent irregularity of the speed solution at the surface of the star is removed thanks to this method. Indeed, in Matt et al. (2012), the wind is isothermal inside the star and polytropic outside, and for $\gamma = 1.05$ the speed profile is below the isothermal solution at a given radius. Even though the outside domain is ruled by a polytropic equation of state in both works, the isothermal boundary condition seems to maintain the steady-state solution with a higher $M_w$ than the fully polytropic simulation.
field tends toward zero, the Alfvén surface is located just outside the surface where the wind speed starts to increase. Interestingly, the value of \( m_{th} \) for the quadrupolar and the octupolar topologies (see Table 2) are very close to the fitted value obtained with our simulations. Moreover, in the case of a purely radial field (or a split monopole), we find that the value of \( m_{th} \) is close to the value obtained with our topology independent formulation using the open flux, i.e., the part of the flux created by the open/radial field lines. However, even if the value \( q = 0.7 \) yields good fits for small Alfvén radii, it is too high to fit our Alfvén radii around \( 30 R_\odot \). Indeed, as shown in Parker (1958), the speed profile behaves asymptotically as the square root of a logarithm in the hydrodynamical case and \( q \) diminishes with the distance to the star. This is why the \( m_{th} \) value for the dipolar cases is less satisfactory and would need a lower value of \( q \). Nevertheless, any estimation should be positive and thus different than the one from Kawaler (1988) and Reiners & Mohanty (2012).

6. THE CASE OF REALISTIC MIXED TOPOLOGIES OF THE SUN AND YOUNG STARS

The formulation given in the previous section seems to work well on single-mode topologies. However, what happens if we mix all three components in order to make more realistic simulations? The topology of the Sun changes during its cycle, from mostly quadrupolar during activity maximum to a mostly dipolar global topology at its minimum. This topology change has been measured by DeRosa et al. (2012) with data of the Wilcox Solar Observatory and the MDI instrument on board Solar and Heliospheric Observatory. This work gives the coefficients of each component of the spherical harmonics decomposition of the magnetic field normal to the solar photosphere. We used the classical formalism of this decomposition (Donati et al. 2006; DeRosa et al. 2012), but using only the three axisymmetric components \( l = 1, 2, 3 \):

\[
B_\alpha(\theta, \phi, t) = \sum_{l=1}^{3} B_l^\alpha(t) Y_l^\alpha(\theta, \phi),
\]

\[
B_l^\alpha(t) = \left( B_l^\alpha \right)_0(1 + \dot{f})
\]

\[
\dot{f} = \frac{\dot{\rho}}{\rho}
\]

\[
Y_l^\alpha(\theta, \phi) = \frac{2l + 1}{4\pi} P_l^\alpha(\cos(\theta)),
\]

where

\[
C_l^0 \approx \sqrt{\frac{2l + 1}{4\pi}}
\]

and \( P_l^\alpha \) are the Legendre polynomials. We performed two simulations with the coefficients given in Table 3; the results for the Sun are given there as well. Adding those three components together creates a complex topology (see Figure 8). In the case of the Solar maximum, the topology is close to quadrupolar, while at the minimum, a strong dipole dominates.

In order to test our stellar wind model and topology-independent formulation on another star than the Sun, we used output from ZDI maps (Donati et al. 2006) of young stars (observed with NARVAL at Télescope Bernard Lyot) with a strong non-dipolar magnetic field. As young stars are generally more magnetic than the Sun, we expect a larger Alfvén radius, although this depends strongly on the unknown mass-loss rate. The case of the 70 million year old star TYC-0486-4943-1 is interesting because it has a dominant quadrupolar component and a strong magnetic activity. The radius and the mass of this star are \( R_{TYC} = 0.68 R_\odot \) and \( M_{TYC} = 0.69 M_\odot \), so the speed normalization can remain the same since the Keplerian speed \( v_{kep} = \sqrt{GM_*/R_*} \) is very similar to the Sun. Its rotation period is 3.75 days and thus falls within the range of our parameter study (\( f = 0.03 \)). The coronal temperature \( (T_c) \) may be higher (Preibisch 1997) in this star, since it is more active magnetically, but without more information we chose to keep the same \( c_1/v_{esc} \) for all simulations. Changing the value of the coronal temperature is likely to change the values of the constants \( K_1 \) and \( K_2 \) that would diminish with higher \( T_c \) (Matt & Pudritz 2008; Matt et al. 2012; Ud-Doula et al. 2009), and the values of \( K_1 \) and \( K_2 \). We expect the exponent \( m \) to be robust to this change but a more systematic study is needed, which is beyond the scope of this paper.

The winds created by all those configurations are shown in Figure 8, and the results are listed in Table 3. We see in Figure 6 that the torque of those wind are well described by our formulation. We tested other configurations of topologies and all fall onto our law (not shown here). Interestingly, despite a much higher magnetic energy density for the young K-star TYC-0486 (\( \propto \sum (B_m^l)^2 \)), the average Alfvén radius is only slightly
larger than for the Solar minimum. This result demonstrates how relevant the topology parameter for the calculation of the magnetic torque is. It is also interesting to note that during one cycle the magnetic braking (which is proportional to $R_m^2$) vary by a factor of four (Pinto et al. 2011 found a factor of 16 over the solar cycle; see also Vidotto et al. 2012). This could have an effect on the long time rotational evolution of the Sun.

The value for the Alfvén radius in both solar cases is low compared to the expected value of 10–12$R_\odot$ (Pizzo et al. 1983). However, depending on models the range can vary between 2.5$R_\odot$ and 60$R_\odot$. Isothermal models, from which we are close tend to give a lower limit for $R_A$ (Pneuman & Kopp 1971), whereas conductive models give the highest estimates (Durney & Pneuman 1975). The mass-loss rates are also lower than the usual value for the Sun (2–4×10$^{-14}$ $M_\odot$ yr$^{-1}$), and since we do not know the mass-loss rate of TYC-0486-4943-1 we kept the same density normalization $\rho_0 = 2.9 \times 10^{-15}$ g cm$^{-3}$ for all of the values of mass-loss rates given in Table 3. We will come back to this point in Section 7.1.

7. DISCUSSION

7.1. Mass-loss Rate

In Section 6, we give values for the solar mass-loss rate. Here normalization plays an important role. We chose the density normalization to be $\rho_0 = 2.9 \times 10^{-15}$ g cm$^{-3}$. The mass-loss rate of the two cases is then around 0.8 × 10$^{-14}$ $M_\odot$ yr$^{-1}$. We could simply change the density normalization to get solar mass-loss rates, but we would then have to lower values of the Alfvén radii in our Solar cases, given that the magnetic field normalization is imposed by the density normalization.

However, the mass-loss rate is also very sensitive to the parameter we kept fixed: $c_s/v_{\text{esc}}$ and $\gamma$. For instance, increasing temperature by 12% to get $c_s/v_{\text{esc}} = 0.235$ multiplies the mass-loss rate by five. Thus, there is some freedom that can be used to get closer to solar values, changing temperature and normalizations.

Also the mass-loss rate depends on physics not included in our simulations. For instance, our simulations include no heating, and the driving could be better physically modeled including Alfvén waves. This is why we give our braking laws as a function of the mass-loss rate so that the rotation evolution model can take into account much more physics than we do here, for instance, using the method proposed by Cranmer & Saar (2011).

Nevertheless, we performed a simple fit of the mass-loss rate in the case of our coronal temperature, as a function of the parameters we varied: $v_A/v_{\text{esc}}$ and $v_{\text{rot}}/v_{\text{esc}}$. As $M_w$ is an increasing function of $f$ and a decreasing function of $v_A/v_{\text{esc}}$, we propose the following formulation:

$$M_w = A_1(v_A/v_{\text{esc}})^{-p_1}(1 + f^2 A_2^{p_2})$$

$$= A_1 \left( B_\odot \frac{R_\odot}{8\pi\rho_0 GM_\odot} \right)^{-p_1} \left( 1 + f^2 A_2^{p_2} \right).$$

Fitting this formula with our set of simulations, we find: $A_1 = 0.28$, $p_1 = 0.19$, $A_2 = 0.087$, and $p_2 = 1.6$. It means that $M_w$ is strongly increased at high rotation while the dependence on the magnetic field strength is rather weak (there is a factor of eight between $p_1$ and $p_2$). This fit is shown in Figure 9.
Cohen & Drake (2014) found $\dot{M}_w \propto B_s$, which seems contradictory with our results. However, their wind driving is related to an energy source term that changes accordingly with the magnetic field. As we see with the value of our fit, the dependence of $\dot{M}_w$ on the magnetic field strength is weak, whereas the coronal temperature (which we held fixed) has a very strong influence on $\rho v$. Hence, a trend such as that found in Cohen & Drake (2014) is expected because the coronal heating is a consequence of the magnetic activity. However, in our parameter study with a fixed temperature polytropic wind, a higher strength of the magnetic field reduces the mass-loss rate since more magnetic loops are formed and confined the plasma.

7.2. Higher Order Components of the Magnetic Field and Non-axisymmetry

We have limited our study to three spherical harmonics modes. ZDI maps offer many more non-zero components of stars’ magnetic fields and our formulation could be further generalized. For axisymmetric modes, our formulation should be robust. Indeed, as we go to higher orders, the radial decay of the magnetic field increases and we can already see that the octupole has very little influence on the large-scale topology of the steady-state solution of the wind when mixed with a dipolar or a quadrupolar field (if the surface amplitudes are comparable). In any case, the open flux captures the effect of higher order fields by decreasing if more plasma is kept in magnetic loops.

However, because all of our study is axisymmetric, a full three-dimensional study of the torque created by complex topologies must be investigated in order to fully test our formulation.

8. CONCLUSION

In this work, we give the first quantitative results on the systematic influence of the topology of stellar magnetic fields on the Alfvén radius and the torque applied by a magnetized wind on its star. Our formulation in the simplest dipolar case is very similar to Matt et al. (2012). We found that the more complex the field is the smaller is the torque. Qualitatively, this has been expected from simple scaling arguments (see quadrupolar simulations in Matt & Pudritz 2008) and we are now able to give braking laws in three ideal axisymmetric cases: a dipole, a quadrupole, and an octupole. Furthermore, we derived a unified topology-independent formulation from our set of 60 simulations. In the case of our Sun today, the first two topologies are dominant during the activity cycle. We have performed simulations of the two cases of the solar minimum and maximum, whose torque is well predicted by our formulation. In the cases of more active stars, such as the young K-star TYC-0486-4943-1, even higher order magnetic multipoles can be significant during an activity cycle, and our topology-independent formulation is a first step to understanding the influence of a fully realistic topology on the angular momentum loss. With this formulation, the equivalent torque is given as follows:

$$\tau_w = \dot{M}_w \Omega \, R^2 \kappa^2 \left( \frac{\gamma_{\text{open}}}{1 + f^2 / \kappa^2} \right)^{2m}$$

$$= \dot{M}_w \Omega \, R^2 \kappa^2 \left( \frac{\Phi_{\text{open}}^2}{v_{\text{esc}} \left(1 + f^2 / \kappa^2 \right)^{1/2}} \right)^{2m}.$$  

However, considering the present day knowledge of the mass-loss rate and the wind velocities at 1 AU, our model might need to include more physics or at least an exploration of the other parameters, i.e., $\gamma$ and $c_s / v_{\text{esc}}$. Also, even if the open flux is more difficult to get than a star’s magnetic field strength at the equator from observations, our study shows that it might be more meaningful for torque calculations. Further works could tackle the prediction of open flux from knowledge of the surface field from ZDI maps (as in Vidotto et al. 2014b) and provide torque for distant stars.

A major step to further confirm this formulation is to test it on non-axisymmetric realistic topologies, and this is left for further works.

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APPENDIX A

DERIVATION OF TORQUE FORMULATIONS

In order to have a semi-analytic expression for the Alfvén radius and thus a torque formulation, a one-dimensional approach has been introduced by Kawaler (1988). Assuming that the magnetic field strength decreases as a single power law $B = B_0 (R_s / R)^{\nu + 2}$, at the Alfvén radius, we have the following equality:

$$v(R_A)^2 = \frac{B^2}{4 \pi \rho R_A} = \frac{B_0^2 R_0^{2+2 \nu}}{R_A^{2+4} 4 \pi \rho R_A}.$$  

and thus,

$$\left( \frac{R_A}{R_s} \right)^{2+2 \nu} = \frac{B_0^2 R_0^2}{4 \pi R_0^2 \rho R_s \rho (\nu v) (R_A)^{\nu v}}.$$  

Interpreting $4 \pi \rho R_s \nu (R_A) R_A^4$ as the mass-loss rate at this radius and considering a simple model for the dependence of the Alfvén speed at the Alfvén radius

$$v(R_A) \propto v_{\text{esc}} (R_A / R_s)^{\nu},$$

Figure 9. Fit of $\dot{M}_w$ with the stellar parameters $f$ and $v_A / v_{\text{esc}}$ (see Equation (24)).
we are left with
\[
\left( \frac{R_A}{R_*} \right)^{2 + 2q} \propto \frac{B_*^2 R_*^2}{M_* v_{\text{esc}}} \equiv \Upsilon. \tag{A4}
\]

The parameter \( q \) is likely positive (the wind accelerates) even though in the literature it has often been set to \(-1/2\) (Kawaler 1988; Reiners & Mohanty 2012). Hence, we have a semi-analytical formulation for the Alfvén radius
\[
\frac{R_A}{R_*} = K \Upsilon^m, \tag{A5}
\]
where
\[
m = 1/(2l + 2 + q). \tag{A6}
\]

Although in two dimensions the Alfvén surface is not spherically symmetric and the magnetic field does not follow a single power law, Matt & Pudritz (2008) demonstrated that the formulation (A5) fits the simulations results precisely when all the constituting parameters of \( \Upsilon (B_*, v_{\text{esc}}, M_*) \) vary.

In Matt et al. (2012), 50 simulations were performed allowing the magnetic field strength and the rotation rate to vary. Considering that the rotation rate mainly appears in the magnetocentrifugal effect, one can write a modified speed:
\[
v_{\text{mod}}^2 = v_{\text{esc}}^2 + \frac{2 \Omega_* R_*^2}{K_2^2} = v_{\text{esc}}^2 \left( 1 + f^2/K_2^2 \right), \tag{A7}
\]
where \( f \) is the fraction of break-up rate defined in Section 2.

The formulation for the Alfvén radius as a function of \( (\Omega_*, \, \Upsilon) \) is given by replacing \( v_{\text{esc}} \) in Equations (A3) and (A4) by this modified speed \( v_{\text{mod}} \):
\[
\frac{R_A}{R_*} = K_1 \left[ \frac{\Upsilon}{(1 + f^2/K_2^2)^{1/2}} \right]^m. \tag{A8}
\]

This formulation precisely captures the influence of the rotation rate of the star on the magnetic torque in the 50 simulations of Matt et al. (2012). The values found for the three models parameters are
\[
K_1 = 2.5, \quad K_2 = 0.07, \quad m = 0.22. \tag{A9}
\]

The parameter \( K_2 \) is the value from which the rotation rate starts to have an influence on the magnetic braking. Indeed, the magnetocentrifugal effect adds acceleration to the wind and thus the Alfvén radius gets closer to the star. Rapid rotation can also increase the mass-loss rate (for a given \( v_A/v_{\text{esc}} \)), but this effect is already taken care of in the parameter \( \Upsilon \).

APPENDIX B
BOUNDARY CONDITIONS AND CONSERVATION PROPERTIES

In this work, the boundary conditions at the external limits of the domain are outflow conditions except for the axisymmetric axis.7 The star is then modeled through a three-layer boundary condition described in Figure 10 (Strugarek et al. 2014a). Each layer is at least one grid cell thick. The interior of the star is not modeled so that quantities into the gray area on Figure 10 are set to constant values. In the red layer, the poloidal field is set to be the same as in the initial state (dipolar, quadrupolar, octupolar, or a mix of the three) and since we use a background field splitting, we set the fluctuating field to zero. The poloidal speed is set to zero, while the star is in solid body rotation. The density and pressure profile are fixed to the polytropic solution of Parker’s equations for a one-dimensional hydrodynamical wind with \( \gamma = 1.05 \) (see polytropic solutions of Keppens & Goedbloed 2000; Lamers & Cassinelli 1999). A sophisticated condition is used on the toroidal field, which is set to maintain the poloidal current as close to zero as possible. This is an empirical method first used in Matt & Balick (2004) that allows good conservation properties of the effective rotation, which is key for torque calculations.

However, this condition is only necessary where the field lines are open and we set the toroidal field to zero on closed field line regions to avoid artificial creation of \( B_\phi \) in the dead zones. We dynamically discriminate open/closed field line regions using an empirical criterion on the Alfvén speed associated with the azimuthal magnetic field in the upper layers of the boundary conditions.

In the green layer, the same conditions are applied to the pressure, density, rotation rate, and the poloidal speed is set again to zero, but we let the magnetic field evolve. In the blue layer, we keep the same conditions for \( \rho \) and \( p \) while the poloidal speed and magnetic field are forced to be parallel.

We give the analytic expressions of the magnetic field used to initialize our simulations, for the dipole, the quadrupole, or the octupole in cylindrical coordinates, where \( B_\phi \) is the amplitude at the equator:

Figure 10. Three-layer boundary condition is used to ensure the conservation of quantities such as \( \Omega_{\text{eff}} \), which measure the effective rotation of the magnetic field lines with the star.
Using a variational principle with the Lagrangian of ideal MHD equations in the axisymmetric case, it can be demonstrated that some scalar quantities must be conserved along the magnetic field lines. Among them is the energy (the Bernoulli function), the entropy, and the effective rotation rate of the magnetic field lines, which is the derivative of the electric field potential (Keppens & Goedbloed 2000; Zanni & Ferreira 2009), defined as follows:

\[
\Omega_{\text{eff}}(\psi) \equiv \frac{1}{R} \left( \frac{v_\psi - v_\rho B_\phi}{B_\rho} \right),
\]

where the subscript \(\rho\) stands for the poloidal component of the field, and the subscript \(\phi\) for the toroidal component, thus we have \(B_\rho = \sqrt{B_R^2 + B_Z^2}\). \(\Omega_{\text{eff}}\) is expressed as a function of \(\psi\), the stream function of the magnetic field, which can be computed as \(\psi = RA\), where \(R\) is the cylindrical radius and \(A\) the scalar potential of the magnetic field (\(B = \nabla \times (Ae_\phi)\)). As a consequence, since each value of \(\psi\) can be associated with a magnetic field line, the conservation of this quantity is visualized on the full two-dimensional grid through little vertical spread in the \(\Omega_{\text{eff}}/\Omega\) versus \(\psi\) plots in Figure 11.

In steady-state ideal MHD, \(\Omega_{\text{eff}}\) should be the same on all of the magnetic field lines and equal to the rotation rate of the star in order to behave as field lines anchored into the rotating stellar surface. Thus, good conservation properties of \(\Omega_{\text{eff}}\) are necessary to compute an accurate value of the angular momentum flux. The conservation is affected by boundary conditions. For instance, if our deepest layer boundary condition on \(B_\phi\) is not applied, the value of \(\Omega_{\text{eff}}\) is below one by more than 20% on the left arm on Figure 11 for the dipolar case, which corresponds to open field lines (see Strugarek et al. 2014a).
Figure 12. Parameter space explored for the three different topologies: crosses are for the dipole, diamonds are for the quadrupole, and triangles are for the octupole. As in Figures 4, 13, 6, and 9, colors are associated with rotation.

Table 4
Results of Our 60 Simulations

| Case | \(\langle R_A \rangle\) | \(\Upsilon\) | \(\Upsilon\) open |
|------|----------------|-------------|----------------|
|      | Dipole | Quadrupole | Octupole | Dipole | Quadrupole | Octupole | Dipole | Quadrupole | Octupole |
| 1    | 6.3    | 3.6        | 3.0      | 115    | 98         | 113      | 123    | 16.5       | 8        |
| 2    | 12.5   | 5.3        | 4.0      | 2360   | 1830       | 1880     | 886    | 80         | 32       |
| 3    | 32.3   | 9.3        | 5.9      | 115000 | 70000      | 59600    | 21400  | 652        | 151      |
| 3+   | 36.4   | 9.9        | 6.3      | 239000 | 151000     | 114000   | 50600  | 1140       | 199      |
| 5    | 6.3    | 3.6        | 3.0      | 115    | 98         | 113      | 123    | 16.5       | 8        |
| 6    | 12.5   | 5.3        | 4.0      | 2360   | 1830       | 1880     | 886    | 80         | 32       |
| 7    | 32.3   | 9.3        | 5.9      | 115000 | 70000      | 59600    | 20300  | 646        | 151      |
| 8    | 6.3    | 3.6        | 3.0      | 115    | 98         | 113      | 122    | 16.5       | 8        |
| 10   | 12.6   | 5.3        | 4.0      | 2340   | 1830       | 1875     | 890    | 81         | 32       |
| 13   | 32.3   | 9.3        | 5.9      | 101000 | 70600      | 59900    | 20300  | 646        | 145      |
| 23   | 5.9    | 3.5        | 3.1      | 83     | 99.5       | 95       | 127    | 20         | 9        |
| 24   | 11.7   | 5.2        | 4.1      | 1450   | 1380       | 1500     | 998    | 85         | 33       |
| 25   | 30.3   | 9.2        | 5.8      | 52000  | 49000      | 46200    | 15000  | 680        | 151      |
| 31   | 10.6   | 4.9        | 4.0      | 1030   | 1045       | 1180     | 937    | 85         | 31       |
| 37   | 8.7    | 4.4        | 3.6      | 492    | 560        | 635      | 758    | 77.5       | 30       |
| 45   | 5.5    | 3.4        | 2.9      | 102    | 143        | 135      | 407    | 60         | 20       |
| 47   | 13.4   | 5.7        | 4.4      | 3430   | 4085       | 4510     | 5170   | 438        | 126      |
| 48   | 4.9    | 3.1        | 2.6      | 81     | 104        | 94       | 545    | 79         | 37       |
| 49   | 7.1    | 3.7        | 3.2      | 431    | 447        | 486      | 1920   | 193        | 87       |
| 50   | 11.4   | 5.0        | 4.3      | 2280   | 2390       | 2710     | 7100   | 524        | 252      |

Notes. We give here all of the computed values of \(R_A\), \(\Upsilon\), and \(\Upsilon\) open that were used to perform the fits of this paper. The parameters used for each case are listed in Table 1.

For other topologies, the value of \(\Omega_{\text{eff}} / \Omega_\ast\) for open field lines remain close to unity even if non-ideal features (current sheets) have more influence than in the dipolar case. Those non-ideal features are due to a current sheet around the streamers, created by numerical diffusion. However, using our resolution, they have negligible influence on our average value of \(R_A\) (this has been checked by increasing resolution, while those features diminish, our integrated values remain the same within 1%).

Appendix C

Exhaustive Results

Table 4 gives all of the results that we used to fit our formulations. Figure 12 shows the parameter space explored in terms of \(\Upsilon\) and \(\Upsilon\) open and also visualizes the dependence of these quantities on the rotation rate. For instance, we can see that \(\Upsilon\) depends weakly on the topology (the different symbols...
are merged), this is not the case of $\Upsilon_{\text{open}}$, for which all three topologies are well separated for a given case. This is necessary to get a topology-independent formulation.

Eventually, we would like to show how important it is to take into account the rotation rate in our formulation. For instance, considering a simple formulation such as

$$\frac{R_A}{R_*} = K_1 \Upsilon_{\text{open}}^{m}$$  \hspace{1cm} (C1)$$

gives a general trend (see Figure 13). For a given rotation rate, points of all topologies follow the same power law. However, for rotation rates beyond $f = 0.05$ (green, light blue, and blue points), the power law is shifted downward.

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Figure 13. Alfvén radius vs. $\Upsilon_{\text{open}}$. Colors and symbols are the same as in Figure 4. A constant slope (represented by the black line) is observed between simulations at the same rotation rate, but higher rotators (green, light blue, and blue) are shifted to smaller $R_A/R_*$. 

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