On well-posedness of a velocity-vorticity formulation of the Navier–Stokes equations with no-slip boundary conditions

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Abstract

We study well-posedness of a velocity-vorticity formulation of the Navier–Stokes equations, supplemented with no-slip velocity boundary conditions, a no-penetration vorticity boundary condition, along with a natural vorticity boundary condition depending on a pressure functional. In the stationary case we prove existence and uniqueness of a suitable weak solution to the system under a small data condition. The topic of the paper is driven by recent developments of vorticity based numerical methods for the Navier–Stokes equations.

1 Introduction

The evolution of an incompressible, viscous Newtonian fluid is governed by the Navier–Stokes equations (NSE), which for a given bounded, connected domain $\Omega \subset \mathbb{R}^3$ with a piecewise, smooth, Lipschitz boundary $\partial \Omega$, an end time $T$, an initial condition $u_0 : \Omega \rightarrow \mathbb{R}^3$ and force field $f : (0,T] \times \Omega \rightarrow \mathbb{R}^3$ read: find a velocity field $u : [0,T] \times \Omega \rightarrow \mathbb{R}^3$ and a pressure field $p : (0,T] \times \Omega \rightarrow \mathbb{R}$ such that

$$\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= f \quad \text{in } (0,T] \times \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } (0,T] \times \Omega, \\
u u|_{t=0} &= u_0 \quad \text{in } \Omega,
\end{align*}$$

(1.1)

where $\nu > 0$ is the kinematic viscosity. The system must also be equipped with appropriate boundary conditions.

For suitably smooth solutions, equations (1.1) can be re-formulated in other equivalent forms, in particular through the introduction of new variables such as vorticity, streamfunction, rate of strain tensor, Bernoulli pressure, etc., see, e.g., [13, 17, 22]. While different formulations often provide useful insights into both physical and mathematical properties of NSE solutions, it is believed that in general they do not include more information than that already contained in (1.1). The situation changes, however, if one is interested in numerical solutions. It is well known that by applying a discretization method to different equivalent formulations of the equations, one commonly obtains non-equivalent discrete problems with strikingly different numerical properties and, for particular problems, appropriate discretizations of certain formulations outperform others. For this reason, the literature on computational methods for (1.1) often considers formulations different from the primitive variables “convective” formulation in (1.1). Among these, we can mention the conservative form, skew-symmetric form, vorticity-streamfunction formulation, streamfunction formulation,
rotational form, and EMAC form. We refer the reader to [5, 14, 19, 27] for a description of these formulations and their discretizations.

For numerical analysis of discretization methods, such as a finite element method, one typically needs discrete counterparts of fundamental a priori bounds and well-posedness results. However due to the reasons outlined above, for discrete solutions such results often cannot be obtained by resorting to the primitive variables that describe the equations in (1.1). This is the main motivation of this work where, through direct arguments, we show well-posedness of a particular formulation of NSE based on vorticity dynamics. Our analysis overcomes two major (technical) difficulties:

(i) The implicit enforcement of the relation $w = \text{rot } u$ between two independent variables, vorticity and velocity of the fluid. In fact, our arguments extend to the more general case when this identity is no longer true.

(ii) Imposing the practically important no-slip conditions on solid walls on the velocity $u$ leads to natural boundary conditions for the vorticity $w$, which involve boundary values of the pressure variable.

The two observations above have two immediate consequences: first, the energy estimate for $u$ does not necessarily imply any bound for $w$; and, second, one has to look for more regular weak solutions to give sense to the vorticity boundary conditions on solid walls. A key new technical result in this paper is the well-posedness and regularity of a Stokes-type problem with non-standard boundary conditions. The problem was introduced in [12] but, to the best of our knowledge, it has not been further explored in the literature.

Velocity-vorticity formulations have been widely used in the numerical literature and have been found to provide greater accuracy in simulations (compared to more common formulations), especially for flows where boundary effects are critical [28, 21, 23, 10, 16, 29, 18]. Such formulations directly build on the equation of the vorticity dynamics,

$$
\frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla) w - (w \cdot \nabla) u = \text{rot } f. 
$$

(1.2)

There are several ways to close the system of equations for $w$ and $u$. One common way is to supplement (1.2) with the vector Poisson problem for the velocity, $\Delta u = \text{rot } w$. In this paper, we consider the formulation that couples (1.2) with the rotational form of the momentum equation,

$$
\frac{\partial u}{\partial t} - \nu \Delta u + w \times u + \nabla P = f. 
$$

(1.3)

We are motivated to follow this path by the recent development of numerical methods for this coupling [23, 20, 3, 26, 18]. As these references show, these schemes provide superior long-time numerical stability and more accurate fulfillment of discrete conservation laws (in addition to other well-known benefits of vorticity based numerical formulations).

The system of velocity-vorticity equations should be supplemented with suitable boundary conditions for both $u$ and $w$. The most practically important are no-slip, no-penetration conditions on solid boundaries of the fluid domain, i.e., $u = 0$. The corresponding vorticity boundary conditions are not straightforward to obtain and were actually the subject of a long and controversial discussion in the literature; see, e.g., [24] for a brief overview. The normal condition $u \cdot n = 0$ on $\partial \Omega$ is the immediate consequence of $u = 0$ and $w = \text{rot } u$ for $u$ smooth up to the boundary. Here and further $n$ is the unit normal vector on $\partial \Omega$. To define tangential vorticity boundary conditions, we follow [24], where the following physically consistent boundary condition on solid walls was derived,

$$
\nu (\text{rot } w) \times n = (f - \nabla P) \times n. 
$$

(1.4)

This boundary condition can be obtained by taking the tangential component of (1.3) on $\partial \Omega$, and recalling that $\Delta u = \text{rot } w$ and $u = 0$ on $\partial \Omega$. It can be shown, cf. [24], that for $f = 0$, condition
characterizes the production of the streamwise and spanwise vorticity on no-slip boundaries due to the tangential pressure gradient and vorticity intensification, depending on the shape of $\partial \Omega$.

The vorticity boundary condition (1.4) was used in [24, 18] to produce numerical simulations with excellent results. We shall see that the condition in (1.4) is a natural boundary condition, which enters the weak formulation as a functional.

Based on the numerical success of the velocity-vorticity formulation (1.2)–(1.3) with boundary conditions $u = 0$, $w \cdot n = 0$ on $\partial \Omega$, and (1.4), a theoretical study of the underlying PDE system is warranted. Hence, the purpose of this work is to initiate the study of the velocity-vorticity system equipped with no-slip velocity boundary conditions, and the corresponding vorticity boundary conditions. We shall focus on a stationary version of the problem:

$$\begin{align*}
\alpha u - \nu \Delta u + w \times u + \nabla P &= f, \\
\text{div } w &= \text{div } u = 0, \\
\alpha w - \nu \Delta w + (u \cdot \nabla)w - (w \cdot \nabla)u + \nabla \eta &= \text{rot } f, \\
u w \cdot n\big|_{\partial \Omega} &= 0, \\
w \cdot n\big|_{\partial \Omega} &= 0, \\
(\nu \text{ rot } w \times n - (f - \nabla P) \times n)\big|_{\partial \Omega} &= 0,
\end{align*}$$

(1.5)

with $\alpha \geq 0$. For this system we will show existence and uniqueness of solutions, provided that the problem data is sufficiently small, or $\alpha$ is sufficiently large. We introduced the non-negative parameter $\alpha$ in the system, with the aim of covering the case when an additional zero order term results from the approximation of time derivative by a finite difference. One, of course, would be also interested in treating the true semi-discrete problem and passing to the limit of time step turning to zero. However, this seems beyond our reach, and so we cannot provide an existence result for the full time dependent problem. This, in particular, is due to the presence of the pressure in the vorticity boundary conditions (1.4). Nevertheless, the partial analysis that we provide here comes with its own set of unique features. As mentioned above, to overcome them, we need to analyze a class of Stokes-type problems with nonstandard boundary conditions, and Helmholtz-Weyl decompositions of the space $L^r(\Omega)$, $r \neq 2$.

**Remark 1.6 (solenoidal vorticity).** We note that (1.5) explicitly enforces the divergence free condition for vorticity with the help of the Lagrange multiplier $\eta$ in the vorticity equation. There are several reasons for doing this. First, we find that this helps us in the analysis. Second, this is motivated by numerical experience, since a common finite element method for (1.2) does not yield divergence free discrete solutions and enforcing it as a constraint benefits the accuracy of the recovered vorticity field. Finally, all the arguments in this paper extend now in a straightforward way to a more generic system of equations, where rot $f$ on the right-hand side of the vorticity equation in (1.5) is replaced by a sufficiently regular source term $f_w$, not necessary divergence free, which can be interpreted as modeling external sources of vorticity production.

We also note the alternative way of enforcing div $w = 0$ by re-writing the vorticity equation (1.2) in the form

$$\frac{\partial w}{\partial t} - \nu \Delta w + 2D(w)u - \nabla \eta = \text{rot } f.$$  

(1.7)

In this equation, $\eta$ has the physical meaning of the *helical density* (rather than an auxiliary variable) and $D(w)u = u \cdot \nabla w - \frac{1}{2}(\text{rot } w) \times u$. When (1.7) is used as the vorticity equation in (1.5), the system is called the velocity-vorticity-helicity (VVH) formulation of the NSE [25]. It is unique in the fact that it naturally enforces the mathematical constraint that the vorticity be divergence-free, and also that it solves for helicity directly (via helical density). The analysis of the paper can be extended to the system (1.5), where the vorticity equation is written in the vorticity–helical density form (1.7), but we shall not elaborate further on this.
Our presentation is organized as follows. In section 2 we provide some preliminary results along with some new results concerning the Stokes problem with nonstandard boundary conditions. In section 3 we show that system (1.5) is well posed. The analysis follows carefully the dependence of all estimates on problem parameters, since such dependence is of interest for understanding the properties of numerical solutions. In particular, we are interested in the dependence on α, since it comes from the time-discretization and so is “user-controlled”. Conclusions and potential directions for future work are provided in section 4.

2 Preliminaries

In this work we assume that the fluid occupies a bounded, simply connected domain Ω ⊂ ℝ³, with ∂Ω ∈ C³. We will follow standard notation concerning function spaces. In particular, we will denote the L²(Ω) inner product and norm by ⟨·, ·⟩ and ||·||, respectively. All other norms will be clearly labeled.

The natural function spaces for velocity, vorticity and pressure in the presence of solid walls are

\[ \mathbf{X} = \mathbf{H}^1_0(Ω), \quad \mathbf{W} = \{ \mathbf{v} ∈ \mathbf{H}^1(Ω) : \mathbf{v} \cdot \mathbf{n}_{|∂Ω} = 0 \}, \quad Q = L^2_0(Ω) \cap H^1(Ω), \]

where L²₀(Ω) denotes the subspace of L²(Ω) consisting of functions with zero mean. The Nečas inequality implies that there exists β₁ > 0 such that

\[ β₁ ||q|| \leq ||\nabla q||_{X'} \quad \forall q ∈ L^2_0(Ω). \quad (2.1) \]

2.1 The operators rot and div

Let us here recall some well-known results about the operators rot and div, and explore some of their consequences that shall be useful in the sequel. The assumptions on Ω yield that \( \mathbf{W} = \mathbf{H}_0(\text{div}) \cap \mathbf{H}(\text{rot}) \) both algebraically and topologically and, as shown in [13, section 3.5], we additionally have that

\[ ||\mathbf{v}|| ≤ C₀(||\text{rot} \mathbf{v}|| + ||\text{div} \mathbf{v}||) \quad ∀ \mathbf{v} ∈ \mathbf{W}, \]

\[ ||\nabla \mathbf{v}||² ≤ C₁(||\text{rot} \mathbf{v}||² + ||\text{div} \mathbf{v}||²) \quad ∀ \mathbf{v} ∈ \mathbf{W}, \]

\[ ||\mathbf{v}||_{\mathbf{H}^2} ≤ C₂(||\mathbf{v}|| + ||\text{rot} \mathbf{v}||_{\mathbf{H}^1} + ||\text{div} \mathbf{v}||_{\mathbf{H}^1}) \quad ∀ \mathbf{v} ∈ \mathbf{W} \cap \mathbf{H}²(Ω). \]

(2.4)

If Ω is convex, then C₁ = 1 in (2.3).

Since Ω is assumed to be bounded, simply connected, and have a smooth boundary, we have the following result about the solvability of the equation rot \( \mathbf{v} = \mathbf{u} \), see [13, Theorems 3.5 and 3.6].

**Lemma 2.5** (right inverse of rot). Let \( \mathbf{w} ∈ \mathbf{H}_0(\text{div}) \) be divergence free, i.e. \( \text{div} \mathbf{w} = 0 \) a.e. in Ω. Then there is a unique vector potential \( \mathbf{u} ∈ \mathbf{H}^1(Ω) \) characterized by

\[ \text{rot} \mathbf{u} = \mathbf{w}, \quad \text{div} \mathbf{u} = 0 \quad \text{in} \ Ω, \quad \mathbf{u} \times \mathbf{n} = 0 \quad \text{on} \ ∂Ω, \quad ||\mathbf{u}||_{\mathbf{H}^1} \leq c||\mathbf{w}||. \]

Notice that if \( \mathbf{v} ∈ \mathbf{H}(\text{rot}) \), then \( \text{rot} \mathbf{v} ∈ \mathbf{H}(\text{div}) \), and so the condition \( \mathbf{n} \cdot \text{rot} \mathbf{v} = 0 \) on ∂Ω is well-defined. We therefore define the space

\[ \widehat{\mathbf{H}}(\text{rot}) := \{ \mathbf{v} ∈ \mathbf{H}(\text{rot}) : \mathbf{n} \cdot \text{rot} \mathbf{v} = 0 \quad \text{on} \ ∂Ω \}. \]

Owing to Lemma 2.5 we can characterize this space.

**Lemma 2.6** (characterization of \( \widehat{\mathbf{H}}(\text{rot}) \)). A function \( \mathbf{v} ∈ \widehat{\mathbf{H}}(\text{rot}) \) iff the following decomposition holds

\[ \mathbf{v} = \mathbf{u} + \nabla q, \quad q ∈ \mathbf{H}^1(Ω), \quad \mathbf{u} ∈ \mathbf{H}^1(Ω), \quad \text{s.t.} \quad \text{div} \mathbf{u} = 0 \quad \text{in} \ Ω, \quad \mathbf{u} \times \mathbf{n} = 0 \quad \text{on} \ ∂Ω. \]

(2.7)
The result of the Lemma is given in [12, Remark 2.1], but we prove it here for completeness. For a given \( \mathbf{v} \in \mathbf{H}(\text{rot}) \), let \( \mathbf{w} = \text{rot} \mathbf{v} \) in Lemma 2.3 and note that \( \text{rot}(\mathbf{u} - \mathbf{v}) = 0 \) implies that \( \mathbf{u} - \mathbf{v} = \nabla q \) for some \( q \in H^1(\Omega) \). The reverse implication, i.e. \( \mathbf{u} + \nabla q \in \mathbf{H}(\text{rot}) \) for all \( q \in H^1(\Omega) \), \( \mathbf{u} \in \mathbf{H}^1(\Omega) \) satisfying the conditions in (2.7), is also straightforward since \( \mathbf{u} \times \mathbf{n} = 0 \) on \( \partial \Omega \) implies \( \mathbf{n} \cdot \text{rot} \mathbf{u} = 0 \) on \( \partial \Omega \).

We also define the subspace of \( \mathbf{H}(\text{rot}) \) consisting of divergence free functions with zero normal component on \( \partial \Omega \) as

\[
\mathbf{H}^0(\text{rot}) := \{ \mathbf{v} \in \mathbf{H}(\text{rot}) : (\mathbf{v}, \nabla q) = 0 \quad \forall q \in H^1(\Omega) \}.
\]

Notice that if \( \mathbf{v} \in \mathbf{H}^0(\text{rot}) \), we have in particular that \( (\mathbf{v}, \nabla q) = 0 \quad \forall q \in \dot{C}(\Omega) \), where \( \dot{C}(\Omega) \) denotes the set of \( C^\infty(\Omega) \) functions with compact support. Hence \( \text{div} \mathbf{v} = 0 \) in \( \dot{C}(\Omega) \) and so, owing to a density argument, \( \text{div} \mathbf{v} = 0 \) in \( L^2(\Omega) \). Therefore, \( \mathbf{v} \in \mathbf{H}^0(\text{rot}) \) implies \( \mathbf{v} \in \mathbf{H}(\text{div}) \) and the condition \( \mathbf{n} \cdot \mathbf{v} = 0 \) holds in \( H^{-\frac{1}{2}}(\partial \Omega) \), and so we conclude \( \mathbf{n} \cdot \mathbf{v} = 0 \) a.e. on \( \partial \Omega \). Therefore, for \( \mathbf{v} \in \mathbf{H}^0(\text{rot}) \), we get due to the regularity of \( \partial \Omega \) that \( \mathbf{v} \in \mathbf{H}^1(\Omega) \), and because of the boundary condition we have \( \mathbf{v} \in \mathbf{W} \). Summarizing, we have shown that

\[
\mathbf{H}^0(\text{rot}) \subset \{ \mathbf{v} \in \mathbf{W} : \text{div} \mathbf{v} = 0 \}.
\]

Owing to the fact that we are assuming \( \partial \Omega \subset C^3 \) we have the following Helmholtz-Weyl decomposition of \( \mathbf{u} \in \mathbf{L}^r(\Omega) \), \( r \in (1, \infty) \), see [9] Theorems III.1.2 and III.2.3:

\[
\mathbf{u} = \nabla q + \psi,
\]

where \( q \in W^{1,r}(\Omega)/\mathbb{R} \) is unique,

\[
\psi \in \mathbf{H}^0(\text{div}) := \{ \psi \in \mathbf{L}^r(\Omega) : \text{div} \psi = 0, \quad \psi \cdot \mathbf{n}|_{\partial \Omega} = 0 \},
\]

and \( \|\psi\|_{\mathbf{L}^r} + \|\nabla q\|_{\mathbf{L}^r} \leq c_0 \|\mathbf{u}\|_{\mathbf{L}^r} \). The existence of the Helmholtz-Weyl decomposition [2.9] implies the well-posedness of the following problem [9]: Given \( g \in \mathbf{L}^r(\Omega) \) with \( r \in (1, \infty) \), find \( \psi \in W^{1,r}(\Omega)/\mathbb{R} \) such that

\[
(\nabla \psi, \nabla \phi) = (g, \nabla \phi) \quad \forall \phi \in W^{1,r}(\Omega), \quad \frac{1}{r} + \frac{1}{t} = 1.
\]

The space \( \mathbf{H}^0(\text{div}) \) from the Helmholtz-Weyl decomposition admits the following characterization for simply connected domains \( \Omega \) for which \( \partial \Omega \subset C^2 \), see [11, Corollary 2.3].

**Lemma 2.11** (characterization of \( \mathbf{H}^0(\text{div}) \)). Let \( \Omega \) be bounded, simply connected with \( \partial \Omega \subset C^2 \) and connected. If \( \mathbf{u} \in \mathbf{H}^0(\text{div}) \) with \( r \in (1, \infty) \), then there is a unique \( \mathbf{v} \in \mathbf{W}^{1,r}(\Omega) \) such that

\[
\mathbf{u} = \text{rot} \mathbf{v}, \quad \Delta \text{div} \mathbf{v} = 0 \quad \text{in} \quad \Omega,
\]

with the estimate \( \|\mathbf{v}\|_{\mathbf{W}^{1,r}} \leq c \|\mathbf{u}\|_{\mathbf{L}^r} \).

This characterization yields the following result.

**Corollary 2.13** (gradient estimate). In the setting of Lemma 2.11, if \( \mathbf{u} \in \mathbf{H}^0(\text{div}) \) is such that \( \text{rot} \mathbf{u} \in \mathbf{H}^0(\text{div}) \), then \( \mathbf{u} \in \mathbf{W}^{1,r}(\Omega) \) and \( \|\nabla \mathbf{u}\|_{\mathbf{L}^r} \leq c \|\text{rot} \mathbf{u}\|_{\mathbf{L}^r} \).

**Proof.** Note that \( \text{rot} \mathbf{u} \in \mathbf{H}^0(\text{div}) \) implies \( \text{rot} \mathbf{u} = \text{rot} \mathbf{v} \) with \( \mathbf{v} \) as in (2.12), satisfying \( \|\nabla \mathbf{v}\|_{\mathbf{L}^r} \leq c \|\text{rot} \mathbf{u}\|_{\mathbf{L}^r} \).

Define \( \mathbf{e} = \mathbf{u} - \mathbf{v} \), and notice that we have \( \text{div} \mathbf{e} = \text{div} \mathbf{v} \), \( \mathbf{e} \cdot \mathbf{n} = 0 \) on \( \partial \Omega \), and \( \text{rot} \mathbf{e} = 0 \). This, in particular, implies that \( \mathbf{e} = \nabla \psi \). Therefore, the function \( \psi \) solves the Neumann problem \( \Delta \psi = -\text{div} \mathbf{v} \) in \( \Omega \) with \( \mathbf{n} \cdot \nabla \psi = 0 \) on \( \partial \Omega \). Using a regularity result for this problem, see [13, Theorem 2.4.2.7], we get

\[
\|\nabla \mathbf{u}\|_{\mathbf{L}^r} \leq \|\nabla \mathbf{v}\|_{\mathbf{L}^r} + \|\nabla^2 \psi\|_{\mathbf{L}^r} \leq c(\|\text{rot} \mathbf{u}\|_{\mathbf{L}^r} + \|\text{div} \mathbf{v}\|_{\mathbf{L}^r}) \leq c \|\text{rot} \mathbf{u}\|_{\mathbf{L}^r},
\]

where we used that \( \|\nabla \mathbf{v}\|_{\mathbf{L}^r} \leq c \|\text{rot} \mathbf{u}\|_{\mathbf{L}^r} \).
2.2 The Stokes problem with nonstandard boundary conditions

Consider first the classical Stokes problem, supplemented with no-slip boundary conditions:

\[-\nu \Delta u + \nabla p = F \quad \text{in } \Omega,\]
\[\text{div } u = 0 \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega.\]

(2.14)

Recall the following regularity result, [9, Lemma IV.6.1], if \(\partial \Omega \in C^2\) and \(F \in L^r(\Omega)\) with \(r \in (1, \infty)\), then the solution to (2.14) satisfies \(u \in W^{2,r}(\Omega), p \in W^{1,r}(\Omega)/\mathbb{R},\) and

\[\nu \|u\|_{W^{2,r}} + \|p\|_{W^{1,r}/\mathbb{R}} \leq c\|F\|_{L^r}.\]

(2.15)

Our analysis in section 3 will also require regularity results for the following non-standard Stokes-type problem,

\[\alpha u - \nu \Delta u + a \times \text{rot } u + \nabla p = g \quad \text{in } \Omega,\]
\[\text{div } u = 0 \quad \text{in } \Omega,\]
\[n \cdot \text{rot } u = n \cdot \text{rot } u = u \cdot n = 0 \quad \text{on } \partial \Omega,\]

(2.16)

with \(a \in X\) and \(\text{div } a = 0\). The problem with \(a = 0, \alpha = 0\) was discussed in [12].

We will now establish the necessary well-posedness and regularity for (2.16). A weak formulation reads: For \(g \in L^2(\Omega)\) find \((u, p) \in \tilde{H}(\text{rot}) \cap L^3(\Omega) \times W^{1,\frac{4}{3}}(\Omega)/\mathbb{R}\) satisfying

\[\alpha(u, \psi) + \nu(\text{rot } u, \text{rot } \psi) + (a \times \text{rot } u, \psi) + (\nabla p, \psi) = (g, \psi),\]
\[(\nabla q, u) = 0,\]

(2.17)

for all \((\psi, q) \in \tilde{H}(\text{rot}) \cap L^3(\Omega) \times W^{1,\frac{4}{3}}(\Omega)\).

In light of the decomposition given in Lemma 2.6 the variables \(u\) and \(p\) can be decoupled and we can consider, instead, the following weak formulation: find \((u, p) \in \tilde{H}^0(\text{rot}) \times W^{1,\frac{4}{3}}(\Omega)/\mathbb{R}\) that solve

\[\alpha(u, \psi) + \nu(\text{rot } u, \text{rot } \psi) + (a \times \text{rot } u, \psi) = (g, \psi), \quad \forall \psi \in \tilde{H}^0(\text{rot}),\]
\[(\nabla p, \nabla q) = (g - a \times \text{rot } u - \alpha u, \nabla q), \quad \forall q \in W^{1,3}(\Omega).\]

(2.18)

To see this it suffices to set, in (2.17), \(\psi = v \in \tilde{H}^0(\text{rot})\) and \(\psi = \nabla q\) with \(q \in W^{1,3}(\Omega)\) and recall that (2.8) implies that \(\tilde{H}^0(\text{rot}) \subset L^3(\Omega)\).

We will now prove the well-posedness of (2.17) and (2.18). This is done as an auxiliary step towards showing that (1.5) is well-posed for \(\alpha \geq 0\). For this reason, we need to make sure that certain constants in our estimates are independent of \(\alpha \geq 0\). Since this includes two extreme cases: \(\alpha = 0\) and \(\alpha \to +\infty\), it is helpful to introduce the following parameter,

\[\alpha_+ = \max\{\alpha, C_P^{-2}\nu\},\]

(2.19)

where \(C_P\) is the optimal constant in the Poincaré inequality \(\|v\| \leq C_P\|\nabla v\|, v \in \tilde{W}\). We introduce it in the definition of \(\alpha_+\) so that it has the proper (physical) scaling. The definition of \(\alpha_+\) allows us to obtain the following simple bound:

\[|(g, v)| \leq \|g\| \min\{\|v\|, C_P\|\nabla v\|\} \leq \sqrt{2}\alpha_+^{-\frac{1}{2}}\|g\|\|v\|, \quad \forall g \in L^2, v \in \tilde{W},\]

(2.20)

with

\[\|v\| = (\alpha\|v\|^2 + \nu\|\nabla v\|^2)^{\frac{1}{2}}.\]
To check the last inequality in (2.20) it is helpful to note the trivial bound \( \min\{a,b\} \max\{c,d\} \leq ac+bd \leq \sqrt{2}(ac^2+(bd)^2)^{\frac{1}{2}} \), for non-negative reals \( a, b, c, d \). Notice, in addition, that by considering two cases \( \alpha \leq C_P^{-2} \nu \) and \( \alpha > C_P^{-2} \nu \) and also using Poincare’s or Young’s inequality, we obtain that

\[
\|v\|\|\nabla v\| \leq \alpha_{+}^{-\frac{1}{2}} \nu^{-\frac{1}{2}} \|v\|_*^2 \quad \forall \ v \in W. \tag{2.22}
\]

These two inequalities will help us to obtain conditions to guarantee that problems (2.17) and (2.18) are well posed.

**Lemma 2.23** (conditional well-posedness). There is a constant \( C^* \) which depends only on \( \Omega \) such that, if

\[
\|\nabla a\| \leq C^* \nu^{3/4} \alpha_{+}^{1/4} \tag{2.24}
\]

then problems (2.17) and (2.18) are well-posed and have the same unique solution.

**Proof.** We begin the proof by noting that all bilinear forms in (2.17) and (2.18) are well-defined and continuous on the corresponding spaces. In particular, we have

\[
|(a \times \text{rot } u, \psi)| \leq \|a\|_{L^6} \text{rot } u\|\psi\|_{L^3} \leq C_3 \|\nabla a\|\text{rot } u\|\psi\|_{L^3}. \tag{2.25}
\]

The Gagliardo–Nirenberg interpolation inequality [1] Theorem 4.17] provides

\[
\|u\|_{L^3} \leq \tilde{C}_5 \|u\|^{1/2}\|\nabla u\|^{1/2} \leq C_5 \|u\|^{1/2}\|\text{rot } u\|^{1/2}, \quad u \in \tilde{H}^0(\text{rot}), \tag{2.26}
\]

where in the last step we applied (2.3). Now (2.25)–(2.26) yields the bound for \( u \in \tilde{H}^0(\text{rot}) \),

\[
|(a \times \text{rot } u, u)| \leq C\|\nabla a\|\|\text{rot } u\|^{3/2}\|u\|^{1/2} \leq \frac{C}{\alpha^{1/3}}\|\nabla a\|^{4/3}\|\text{rot } u\|^2 + \frac{\alpha}{2}\|u\|^2, \tag{2.27}
\]

where the constant \( C \) depends only on \( \Omega \). In view of estimate (2.25) for \( 0 \leq \alpha < \alpha_{+} \) and (2.27) for \( \alpha = \alpha_{+} \), respectively, we see that there is a constant \( C^* = C^*(\Omega) \) such that, whenever (2.24) holds, we have, for every \( u \in \tilde{H}^0(\text{rot}) \), that

\[
\alpha(u, u) + \nu(\text{rot } u, \text{rot } u) + (a \times \text{rot } u, u) \geq \frac{\alpha}{2}\|u\|^2 + \frac{\nu}{2}\|\text{rot } u\|^2 \geq \frac{\alpha}{2}\|u\|^2 + c\nu(\|\text{rot } u\|^2 + \|u\|^2_{L^3}), \tag{2.28}
\]

where the constant \( c > 0 \) depends only on \( \Omega \). With these estimates at hand, we can now show the well posedness of each problem.

We begin with (2.18) since it is simpler. Notice that, in light of (2.25) and the Banach–Nečas–Babuška theorem [7] Theorem 2.6], the \( u \)-problem is well posed. Now since \( u \in \tilde{H}^0(\text{rot}) \) is uniquely defined, estimate (2.25) together with the well posedness of (2.10) show that the \( p \)-problem is well-defined as well. In addition, estimates (2.20) and (2.28) yield

\[
c\|u\|^2 \leq \|(g, u)\| \leq \sqrt{2}\alpha_{+}^{\frac{1}{2}}\|g\|^* \Rightarrow \alpha\|u\|^2 + \frac{1}{\alpha_{+}^{\frac{1}{2}}}\nu^{\frac{1}{2}}\|\nabla u\| \leq C\|g\|. \tag{2.29}
\]

From the second equation in (2.18), (2.25) and (2.29) we conclude that, if (2.24) holds, we have

\[
\|\nabla p\|_{L^2} \leq \|g\|_{L^2} + \alpha\|u\|_{L^2} + \|a \times \text{rot } u\|_{L^2} \leq c\|g\| + \|\nabla a\|\nu^{\frac{1}{2}}\|\nabla u\| \leq C\|g\|. \tag{2.30}
\]

where, for the last inequality, we used that \( \nu^{-\frac{1}{2}}\alpha_{+}^{-\frac{1}{2}} \leq c\nu^{-\frac{1}{4}}\alpha_{+}^{-\frac{1}{4}} \).
We now proceed with (2.17). For that we begin by noticing that both spaces $\tilde H(\text{rot}) \cap L^3(\Omega)$ and $W^{1,3}(\Omega)/\mathbb{R}$ are Banach and reflexive. Hence the well-posedness of (2.17) follows from the theory of saddle point problems as detailed, for instance, in [7, Theorem 2.34]. Indeed, inequality (2.28) gives the inf-sup property and nondegeneracy of the $u$-form over the kernel of the $p$-form, which happens to coincide with $\tilde H^0(\text{rot}) \cap L^3(\Omega)$. On the other hand, the decomposition (2.7) and the Helmholtz-Weyl decomposition of $L^3(\Omega)$ yield the inf-sup property of the $p$-form:

$$\sup_{\psi \in \tilde H(\text{rot}) \cap L^3(\Omega)} \frac{\langle \nabla p, \psi \rangle}{\| \text{rot} \psi \| + \| \psi \|_{L^3}} \geq \sup_{q \in W^{1,3}(\Omega)} \frac{\langle \nabla p, \nabla q \rangle}{\| \nabla q \|_{L^3}} \geq c \sup_{v \in L^3(\Omega)} \frac{\langle \nabla p, v \rangle}{\| v \|_{L^3}} = c\| \nabla p \|_{L^2}$$

$$\geq c\| p \|_{W^{1,3}} \quad \forall \ p \in W^{1,\frac{3}{2}}(\Omega)/\mathbb{R}.$$

It remains to show that the solutions coincide, but this is immediate upon choosing appropriate test functions.

**Remark 2.31** (large $\alpha$). Notice that, for any given viscosity $\nu$ and vector $a \in X$ with $\text{div} \ a = 0$, there exists $\alpha$ large enough so that condition (2.21) is satisfied.

We now establish a regularity result for the weak solution of (2.17) and (2.18). Although relatively straightforward, such a result is not found in [12] nor seemingly anywhere else in the literature.

**Lemma 2.32** (regularity). Assume that $\Omega$ is simply connected and $\partial \Omega \in C^3$. If (2.21) holds, then the solution to (2.17) satisfies $(u, p) \in H^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ and $\nu \| u \|_{H^2} + \| \nabla p \| \leq C \| g \|$, with $C = C(\Omega)$ independent of $\alpha$ and $\nu$.

**Proof.** By definition, the solution to (2.18) satisfies $(u, p) \in \tilde H^0(\text{rot}) \times W^{1,\frac{3}{2}}(\Omega)$. Thanks to the embedding (2.8), we have that $\text{div} \ u = 0$ in $\Omega$ and $u \cdot n = 0$ on $\partial \Omega$. We want to apply the regularity result in (2.14) and so it remains to show rot $u \in H^1(\Omega)$ together with a suitable bound on $\| \text{rot} u \|_{H^1}$.

First, we note that rot $u \in L^2(\Omega)$ and $a \in L^6(\Omega)$ imply $g - \nabla p - a \times \text{rot} \ u - \omega u \in L^3(\Omega)$. Thus, from (2.17), we have that rot rot $u \in L^2(\Omega)$ and $n \cdot \text{rot} \ u = 0$ on $\partial \Omega$. Indeed, the integrability follows by taking $\psi \in \hat C(\Omega)$ in the first equation of (2.17) and noting that

$$\nu(\text{rot} \text{rot} \ u, \psi)_{\hat C \times \hat C} = \nu(\text{rot} \ u, \text{rot} \ \psi) = \langle \tilde g, \psi \rangle \leq c \| \psi \|_{L^3}, \quad \text{with} \ \tilde g = g - \nabla p - a \times \text{rot} \ u - \omega u.$$

Since $\hat C(\Omega)$ is dense in $L^3(\Omega)$, rot rot $u$ defines a bounded linear functional on $L^3(\Omega)$, and hence rot rot $u \in (L^3(\Omega))' = L^3(\Omega)$. By a similar argument, but now setting $\psi = \nabla \phi$ with $\phi \in C^\infty(\Omega)$ in (2.17) we show $n \cdot \text{rot} \ u = 0$ a.e. on $\partial \Omega$. Therefore, rot $u$ satisfies, for $r = \frac{3}{2}$, the assumptions of Corollary 2.13 and we conclude rot $u \in W^{1,\frac{3}{2}}(\Omega)$.

Having obtained this, let us focus now on obtaining a bound on $\| \tilde g \|_{L^{3/2}}$. In particular, Hölder’s inequality and the embedding $H^1 \hookrightarrow L^6$ yield

$$\| a \times \text{rot} \ u \|_{L^{3/2}} \leq \| a \|_{L^6} \| \text{rot} \ u \| \leq C \| \nabla a \| \| \nabla u \|,$$

for a constant $C$ that depends only on $\Omega$. Using now condition (2.24) and estimate (2.29) we obtain

$$\| a \times \text{rot} \ u \|_{L^{3/2}} \leq C \nu^{3/4}\alpha^1\| \nabla u \| \leq C \nu^{1/4}\alpha^{-1/4} \| g \|. \quad (2.33)$$

Using, again, estimate (2.29) in conjunction with (2.30) and (2.33) finally yields

$$\| \text{rot} u \|_{L^3} \leq \| \text{rot} u \|_{W^{1,3/2}} \leq c \nu^{-1} \| \tilde g \|_{L^3} \leq C \nu^{-3/4}\alpha^{-1/4} \| g \|, \quad (2.34)$$

for a constant $C$ that only depends on $\Omega$. We can now bootstrap this estimate to conclude that $a \times \text{rot} u \in L^2$ with the estimate

$$\| a \times \text{rot} u \| \leq \| a \|_{L^6} \| \text{rot} u \|_{L^3} \leq C \nu^{3/4}\alpha^{1/4}\nu^{-3/4}\alpha^{-1/4} \| g \| \leq C \| g \|, \quad (2.35)$$

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where we also used (2.23), and the constant depends only on $\Omega$.

The estimates above and the second equation in (2.18) imply the claimed regularity for the pressure: $\nabla p \in L^2(\Omega)$. Moreover, (2.29) and (2.35) yield the estimate
\[
\|\nabla p\| \leq \|g\| + \alpha\|u\| + \|a \times \text{rot } u\| \leq C\|g\|,
\]
with a constant that depends only on $\Omega$.

It remains to show the regularity of $u$. Since $\nabla p, g, a \times \text{rot } u, \alpha u \in L^2(\Omega)$ we have $\text{rot } u \in L^2(\Omega)$. Recalling that $u \cdot \text{rot } u = 0$ on $\partial\Omega$ we have $u \in H(\text{rot}) \cap H_0(\text{div})$ and so, owing to the regularity of $\partial\Omega$, $\text{rot } u \in H^1(\Omega)$, as we intended to show. Moreover, we have the estimate
\[
\nu\|u\|_{H^2} \leq C\|\text{rot } u\| \leq C\|\nabla p\| + \|g\| + \alpha\|u\| + \|a \times \text{rot } u\| \leq C\|g\|,
\]
where $C$ only depends on $\Omega$.

This completes the proof. \qed

For given $\alpha \geq 0$ and $a \in X$, with $a$ solenoidal and satisfying (2.23) we denote by $\widehat{\Delta}_a^{-1} : L^2(\Omega) \to H^2(\Omega) \cap W$ the (velocity-)solution operator to the Stokes problem (2.16). Owing to Lemma 2.32 $\widehat{\Delta}_a^{-1}$ is well-defined. We will denote by $\widehat{\Delta}_0^{-1}$ the solution operator with $a = 0$, $\alpha = 0$. This operator has the following properties:
\[
\nu\|\widehat{\Delta}_a^{-1} w\|_{H^2} \leq C(\Omega)\|w\|, \quad \text{div } \widehat{\Delta}_a^{-1} w = 0, \quad u \cdot \text{rot } \widehat{\Delta}_a^{-1} w = 0 \text{ on } \partial\Omega,
\]
\[
\alpha(w, \widehat{\Delta}_a^{-1} w) + \nu(\text{rot } w, \text{rot } \widehat{\Delta}_a^{-1} w) + (a \cdot \nabla w - w \cdot \nabla a, \widehat{\Delta}_a^{-1} w) = \|w\|^2 \quad \forall \, w \in W, \quad \text{s.t. } \text{div } w = 0,
\]
\[
(w, \widehat{\Delta}_a^{-1} w) = \nu\|\text{rot } \widehat{\Delta}_a^{-1} w\|^2 + \alpha\|\widehat{\Delta}_a^{-1} w\|^2 + (a \times \text{rot } \widehat{\Delta}_a^{-1} w, \widehat{\Delta}_a^{-1} w) \quad \forall \, w \in W, \quad \text{s.t. } \text{div } w = 0.
\]

The first three are obvious. For the fourth one we note that $w \in W$ is well-defined on $\partial\Omega$ by $\hat{\Delta}_0^{-1} w = 0$ yields $w \in H^1(\partial\Omega)$. Now the identity follows by taking $g = w$ and $\psi = w$ in (2.18). The last one follows by taking $g = w$ and $\psi = \hat{\Delta}_a^{-1} w$ in (2.18).

### 2.3 Additional forms and their associated bounds

As a last preliminary step, we define some forms that will be needed for the analysis of the velocity-vorticity formulation (1.15). We first define the bilinear form $f_{bc} : Q \times W \to \mathbb{R}$ by
\[
f_{bc}(P, \chi) := \int_{\partial\Omega} (\nabla P \times n) \cdot \chi \, ds.
\]
The form $f_{bc}$ is well-defined on $Q \times W$ and continuous, since for any smooth $P$ and $\chi$ it holds that
\[
\int_{\partial\Omega} (\nabla P \times n) \cdot \chi \, ds = -\int_{\partial\Omega} (\nabla \chi) \cdot n \, ds = \int_{\Omega} \text{div}(\nabla P \times \chi) \, ds
\]
\[
= \int_{\Omega} \nabla P \cdot \text{rot } \chi \, dx \leq \|\nabla P\| \|\text{rot } \chi\|.
\]
We also note the identities
\[
f_{bc}(P, \chi) = \int_{\Omega} \nabla P \cdot \text{rot } \chi \, dx = \int_{\partial\Omega} P \cdot \text{rot } \chi \cdot n \, ds,
\]
which implies that, for any $a \in X$ with $\text{div } a = 0$ such that $\hat{\Delta}_a^{-1}$ is well defined, we have
\[
f_{bc}(P, \hat{\Delta}_a^{-1} w) = 0 \quad \forall \, w \in W.
\]
Finally, define the trilinear form $b : W \times W \times W \to \mathbb{R}$ by
\[
b(u, v, w) := (u \cdot \nabla v, w),
\]
and note that, whenever $\text{div } u = 0$, this form is skew-symmetric, i.e. $b(u, v, w) = 0$.

We will utilize the following bounds for the nonlinear terms that arise in our analysis.
Lemma 2.39 (bounds on b). There exists a constant $M = M(\Omega)$ such that, for every $(u, v, w) \in W^3$ we have

$$\| (w \times u, v) \| \leq M \| w \| \| \nabla u \|^{\frac{1}{2}} \| u \| \| \nabla v \|, \quad (2.40)$$

$$|b(u, w, \bar{\Delta}^{-1}_0 v) + |b(w, u, \bar{\Delta}^{-1}_0 v)| \leq M \nu^{-1} \| \nabla u \|^{\frac{1}{2}} \| u \| \| w \| \| v \|, \quad \text{if } \text{div } u = \text{div } w = 0, \quad (2.41)$$

$$|(w \times u, \text{rot } \bar{\Delta}^{-1}_0 w)| \leq M \nu^{-1} \| w \|^{2} \| \nabla u \|^{\frac{1}{2}} \| u \|^{\frac{3}{2}}. \quad (2.42)$$

**Proof.** The constant $M$ will depend only on $\Omega$. Each of the bounds in this lemma will hold with a potentially different constant that depends only on $\Omega$, and we take $M$ to be the maximum of these constants. Estimate (2.40) follows from Hölder’s inequality, Sobolev inequalities and (2.26).

For (2.41), to bound the first term, we use the fact that $\text{div } u = 0$, Hölder’s inequality and the embedding $H^1 \hookrightarrow L^6$ to obtain

$$|b(u, w, \bar{\Delta}^{-1}_0 v)| = \left| (u \cdot \nabla \bar{\Delta}^{-1}_0 v, w) \right| \leq \| u \|_{L^3} \| \nabla \bar{\Delta}^{-1}_0 v \|_{L^6} \| w \| \leq C \| u \|_{L^3} \| \bar{\Delta}^{-1}_0 v \|_{H^2} \| w \|,$$

where in the last step we used (2.36) and (2.26). Similarly, for the second term, we obtain

$$|b(w, u, \bar{\Delta}^{-1}_0 v)| = \left| (w \cdot \nabla \bar{\Delta}^{-1}_0 v, u) \right| \leq C \| u \|_{L^3} \| \nabla \bar{\Delta}^{-1}_0 v \|_{L^6} \leq C \nu^{-1} \| \nabla u \|^{\frac{1}{2}} \| u \|^{\frac{3}{2}} \| v \| \| w \|.$$
### 3.1 A priori estimates and uniqueness

We begin the analysis of (3.1)–(3.4) with a priori bounds on solutions.

**Lemma 3.5** (a priori bounds). Assume that system (3.1)–(3.4) has solutions. Then they satisfy

\[ \|u\|_* \leq \sqrt{2} \alpha_{+}^{-1/2} \|f\|, \]  

(3.6)

where the norm \( \| \cdot \|_* \) was defined in (2.21) and \( \alpha_{+} \) in (2.19). If, in addition, the forcing term \( f \) satisfies

\[ \|f\| \leq C^* \sqrt{2} 5/4 \alpha_{+}^{3/4}, \]  

(3.7)

where the constant \( C^* \) is the same as in (2.24) of Lemma 2.23, then they verify

\[ \|w\| \leq K_1 := \nu^{-1}(\nu \Omega) \|f\|_{H^{-1}}, \quad \nu \|u\|_{H^2} + \|\nabla P\| \leq K_2, \quad \|w\|_* + \|\eta\| \leq K_3, \]  

(3.8)

with \( K_2 = K_2(\nu, \alpha_{+}, f, \Omega) \) and \( K_3 = K_3(\nu, \alpha_{+}, f, \Omega) \). The constants \( K_2 \) and \( K_3 \), however, remain uniformly bounded as \( \alpha \to \infty \), while all the other data of the problem remain fixed.

**Proof.** We begin with the velocity bound. Setting \( v = u \) in (3.1) and using (2.20) yields estimate (3.6).

Notice now that, if (3.7) holds, then (3.6) implies that \( a = u \) satisfies condition (2.21) of Lemma 2.23 and, consequently, the operator \( \hat{\Delta}^{-1}_u \) is well defined. We set \( \chi = \hat{\Delta}^{-1}_u w \) in (3.1) and observe that, due to (2.35) and (2.36), the \( \eta \)-term and boundary functional vanish, thus (3.1) reduces to

\[ \alpha(w, \hat{\Delta}^{-1}_u w) + \nu(\text{rot } w, \text{rot } \hat{\Delta}^{-1}_u w) + b(u, w, \hat{\Delta}^{-1}_u w) - b(w, u, \hat{\Delta}^{-1}_u w) = (f, \text{rot } \hat{\Delta}^{-1}_u w), \]

which when compared with (2.36) yields

\[ \|w\|^2 = (f, \text{rot } \hat{\Delta}^{-1}_u w) \leq \|f\|_{H^{-1}} \|\text{rot } \hat{\Delta}^{-1}_u w\|_{H^1} \leq C\|f\|_{H^{-1}} \|\hat{\Delta}^{-1}_u w\|_{H^2} \leq \nu^{-1}(\nu \Omega) \|f\|_{H^{-1}} \|w\|, \]

which is the first estimate in (3.8).

To obtain the velocity–pressure part of estimate (3.8) we employ a bootstrapping argument. Hölder’s inequality, the embedding \( H^1 \hookrightarrow L^6 \), the bound (3.6) on the velocity, and the \( L^2 \)-bound on the vorticity yield the existence of a constant \( C(\Omega) \) that depends only on the domain \( \Omega \), for which

\[ \|w \times u\|_{L^3/2} \leq \|w\| \|u\|_{L^6} \leq C(\Omega) \nu^{-1/2} \alpha_{+}^{-1/2} K_1 \|f\| =: C_0(\nu, \alpha_{+}, f, \Omega), \]

and, consequently \( w \times u \in L^3(\Omega) \). We now apply the regularity result for the Stokes problem (2.15) using \( F = f - w \times u - \alpha u \in L^3(\Omega) \) with the bound

\[ \|F\|_{L^3} \leq C(\|f\| + \alpha \|u\|) + \|w \times u\|_{L^3} \leq C\|f\| + C_0(\nu, \alpha_{+}, f, \Omega) =: C_1(\nu, \alpha_{+}, f, \Omega), \]

where we used (3.6) and the fact that \( \alpha \leq \alpha_{+} \). The estimate given above shows that the velocity part of the solution to (3.1)–(3.4) satisfies

\[ \|u\|_{W^{2,2}} \leq \nu^{-1} C_1(\nu, \alpha_{+}, f, \Omega). \]  

(3.9)

We now invoke that, for every \( r < \infty \), we have the embedding \( W^{2,2} \hookrightarrow L^r \) to obtain

\[ \|w \times u\|_{L^r} \leq \|w\| \|u\|_{L^r} \leq C\|w\| \|u\|_{W^{2,2}} \leq C(\Omega) \nu^{-1} K_1 C_1(\nu, \alpha_{+}, f, \Omega), \]
where, in the last step, we used (3.9) and the $L^2$-estimate on the vorticity. This shows that $F \in L^2(\Omega)$ with the bound

\[
\|F\|_{L^2} \leq C(\|f\| + \alpha \|u\|) + \|w \times u\|_{L^2} \leq C\|f\| + C(\Omega)\nu^{-1}K_1C_1(\nu, \alpha_+, f, \Omega) =: C_2(\nu, \alpha_+, f, \Omega).
\]

Using, once again, (2.15) yields that $u \in W^{2,7}(\Omega)$ with the estimate

\[
\|u\|_{W^{2,7}} \leq \nu^{-1}C_2(\nu, \alpha_+, f, \Omega).
\]

Finally, we use the embedding $W^{2,7} \hookrightarrow L^\infty$ to assert that

\[
\|w \times u\| \leq \|w\|\|u\|_{L^\infty} \leq C(\Omega)\nu^{-1}K_1C_2(\nu, \alpha_+, f, \Omega).
\]

This gives us that $F \in L^2(\Omega)$ with the estimate

\[
\|F\| \leq \|f\| + \alpha \|u\| + \|w \times u\| \leq C\|f\| + C\nu^{-1}K_1C_2(\nu, \alpha_+, f, \Omega),
\]

so that invoking, one last time, (2.15) we obtain

\[
\nu\|u\|_{H^2} + \|\nabla P\| \leq C\|f\| + C\nu^{-1}K_1C_2(\nu, \alpha_+, f, \Omega) =: K_2(\nu, \alpha_+, f, \Omega). \tag{3.10}
\]

Let us now bound the $H^1$-norm of the vorticity and the $L^2$-norm of $\eta$. Setting $\chi = w$ in (3.4) yields

\[
\alpha\|w\|^2 + \nu\|\text{rot} \, w\|^2 + \nu\|\text{div} \, w\|^2 = b(w, u, w) + (f, \text{rot} \, w) - f_{bc}(P, w).
\]

Applying (2.3) and then using the continuity of the functionals on the right hand side we obtain

\[
\min\{1, C^{-1}_1\} \|w\|^2_s \leq C(\|\nabla P\| + \|f\|)\|\nabla w\| + b(w, u, w).
\]

To control the trilinear term, we use (3.8) and the embedding $H^2 \hookrightarrow W^{1,6}$ to obtain

\[
|b(w, u, w)| \leq \|w\|\|
abla u\|_{L^6}\|w\|_{L^3} \leq C(\Omega)\nu^{-1}K_1K_2\|\nabla w\|,
\]

and, as a consequence,

\[
\min\{1, C^{-1}_1\} \|w\|^2_s \leq C \left(\|f\| + K_2(\nu, \alpha_+, f, \Omega) + \nu^{-1}\|f\|K_2(\nu, \alpha_+, f, \Omega)\right)\|\nabla w\|,
\]

from which the bound

\[
\|w\|^s_s \leq C\nu^{-1/2} \left(\|f\| + K_2 + \nu^{-1}K_1K_2\right) =: K_3(\nu, \alpha_+, f, \Omega)
\]

follows. An application of (2.11) yields the desired bound for $\eta$.

It remains then to show that $K_2$ and $K_3$ are bounded as $\alpha$ grows large. To see this, we first observe that for $\alpha$ sufficiently large we have $\alpha_+ = \alpha$, and so we need to study the dependence on $\alpha_+$. In the course of the proof of estimates (3.8), we obtained that

\[
K_3 = C\nu^{-1/2} \left(\|f\| + K_2 + \nu^{-1}K_1K_2\right).
\]

Since $K_1$ does not depend on $\alpha_+$, it then is sufficient to show that $K_2$ remains bounded as $\alpha \to \infty$. From (3.10) we have

\[
K_2 = C \left(\|f\| + \nu^{-1}K_1C_2\right) = C \left[\|f\| + \nu^{-1}K_1 \left(\|f\| + \nu^{-1}K_1C_1\right)\right]
\]

\[
= C \left\{\|f\| + \nu^{-1}K_1 \left[\|f\| + \nu^{-1}K_1 \left(\|f\| + \nu^{-1/2}a_+^{-1/2}\right)\right]\right\},
\]

where we successively applied the definitions of $C_j$, $j = 0, 1, 2$. The only power of $\alpha_+$ that appears is negative and so we are able to conclude. $\square$
Lemma 3.5 will all hold if 

Theorem 3.12 somewhat more restrictive conditions on the data.

and this identity using (2.41), (2.22) and (3.6) and obtain the bound where we used (2.36). We now estimate each one of the trilinear terms on the right hand side of nonlinear terms vanish. Inequality (2.40) then gives

Remark 3.11 (large α). If all the problem data besides α is kept fixed, the a priori bounds of Lemma 3.5 will all hold if α is taken sufficiently large.

With the a priori bounds on the solution of Lemma 3.5 we are able to prove uniqueness, under somewhat more restrictive conditions on the data.

Theorem 3.12 (uniqueness). Assume that, in addition to (3.7), the problem data satisfies

\[ \alpha_1 := 1 - 2\sqrt{2}M^{\nu - \frac{3}{2}}\alpha_+^{-\frac{3}{2}}\|f\| > 0, \]

and

\[ 2\sqrt{2}M^{2\nu - 2}\alpha_+^{-1}\|f\|K_1 < 1 \]

where \( K_1 \) is the data dependent constant from (3.8) for which, every vorticity solution to (3.1)–(3.4) verifies \( \|w\| \leq K_1 \). Then solutions to (3.1)–(3.4) are unique.

Proof. Suppose there are two solutions to (3.1)–(3.4), \((u_1, P_1, w_1, \eta_1)\), \((u_2, P_2, w_2, \eta_2)\) \(\in X \times Q \times W \times L^2_0(\Omega)\), and set

\[ e_u := u_1 - u_2, \ e_p := P_1 - P_2, \ e_w := w_1 - w_2, \ e_\eta := \eta_1 - \eta_2. \]

Subtracting (3.1)–(3.4) for each one of these two solutions gives, for all \((v, q, \chi, \lambda) \in X \times Q \times W \times L^2_0(\Omega)\),

\[ \alpha(e_u, v) + \nu(\nabla e_u, \nabla v) - (e_p, \text{div } v) + (w_1 \times e_u, v) + (e_w \times u_2, v) = 0, \]

\[ (\text{div } e_u, \lambda) = (\text{div } e_u, q) = 0, \]

\[ \alpha(e_w, \chi) + \nu(\text{rot } e_w, \text{rot } \chi) + \nu(\text{div } e_w, \text{div } \chi) + b(e_u, w_2, \chi) - b(e_u, u_2, \chi) + b(u_1, e_w, \chi) - b(w_1, e_u, \chi) - (e_\eta, \text{div } \chi) = -f(e_p, \chi). \]

Set now \( \chi = \hat{\Delta}^{-1}_0 e_w \), which makes the pressure boundary term vanish, to obtain

\[ \|e_w\|^2 = -b(e_u, w_2, \hat{\Delta}^{-1}_0 e_w) + b(e_w, u_2, \hat{\Delta}^{-1}_0 e_w) - b(u_1, e_u, \hat{\Delta}^{-1}_0 e_w) + b(w_1, e_u, \hat{\Delta}^{-1}_0 e_w), \]

where we used (2.36). We now estimate each one of the trilinear terms on the right hand side of this identity using (2.41), (2.22) and (3.6) and obtain the bound

\[ \|e_w\|^2 \leq M^{\nu - 1}\left(\|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{3}{2}}\|w_2\|\|e_w\| + \|e_w\|^2\|\nabla u_2\|^{\frac{1}{2}}\|u_2\|^{\frac{5}{2}} + \|\nabla u_1\|^{\frac{1}{2}}\|u_1\|^{\frac{5}{2}}\|e_w\|^2 + \|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{5}{2}}\|w_1\|\|e_w\|\right) \]

\[ \leq M^{\nu - 1}\left(\|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{5}{2}}\|w_2\| + \|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{5}{2}}\|w_1\|\right)\|e_w\| + 2\sqrt{2}M^{\nu - \frac{3}{4}}\alpha_+^{-\frac{3}{4}}\|f\|\|e_w\|^2, \]

which using the definition of \( \alpha_1 \), the bound given in (3.8), and (2.22) yields

\[ \alpha_1\|e_w\| \leq M^{\nu - 1}\left(\|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{5}{2}}\|w_2\| + \|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{5}{2}}\|w_1\|\right) \leq 2M^{\nu - \frac{3}{4}}\alpha_+^{-\frac{1}{4}}K_1\|e_u\|. \] (3.13)

Next, set \( v = e_u \) in the velocity error equation. This makes the pressure term and one of the nonlinear terms vanish. Inequality (2.40) then gives

\[ \|e_u\|^2 \leq M\|e_w\|\|\nabla u_2\|\|\nabla e_u\|^{\frac{1}{2}}\|e_u\|^{\frac{5}{2}} \leq M^{\nu - \frac{3}{4}}\alpha_+^{-\frac{3}{4}}\|e_w\|\|\nabla u_2\|\|e_u\|, \] (3.14)

\[ \leq \sqrt{2}M^{\nu - \frac{3}{4}}\|f\|\|e_w\|\|e_u\|, \]
where, in the last step, we used the a priori bounds of Lemma 3.5 and (2.22). Now using (3.13) in (3.14), we obtain
\[ \|e_u\|_r^2 \leq 2\sqrt{2}M^2\nu^{-2}\alpha_1^{-1}\|f\|_r^2\|e_u\|_r^2, \]
which, by the second smallness assumption on the data yields \( \|e_u\|_r = 0 \). From (3.13) we immediately get that \( \|e_u\| = 0 \) and from Poincaré-Friedrichs’ inequality \( \|e_u\| = 0 \). Now that we have established \( e_u = 0 \) and \( e_w = 0 \) and \( e_\eta = 0 \) follow from (2.1).

**Remark 3.15** (large \( \alpha \)). As in Lemma 3.5 if all the data besides \( \alpha \) is fixed, Theorem 3.12 implies that uniqueness can be obtained by taking \( \alpha \) sufficiently large.

### 3.2 Existence

To prove the existence of solutions, we will utilize the following fixed point theorem, referred to as Shaefer’s fixed point theorem in [8] and as Leray-Schauder’s fixed point theorem in [11].

**Lemma 3.16** (fixed point). Let \( Y \) be a real Banach space and \( F : Y \to Y \) a compact map. Assume that the set of solutions to the family of fixed point problems:

\[
\text{find } y_\lambda \in Y \text{ satisfying } y_\lambda = \lambda F(y_\lambda), \quad 0 \leq \lambda \leq 1,
\]

are uniformly bounded. Then the problem \( y^* = F(y^*) \) has a solution \( y^* \in Y \).

We will proceed by constructing a compact map whose fixed points are solutions of (3.1)–(3.4), then consider the family of fixed point problems, and finally apply Lemma 3.16.

Define \( T : L^2(\Omega) \times W' \to X \times Q \times W \times L^2_0(\Omega) \) to be the solution operator of the following problem: Find \((u, P, w, \eta) = T(g, l) \in X \times Q \times W \times L^2_0(\Omega)\) satisfying for all \((v, \pi, \chi, \lambda) \in X \times Q \times W \times L^2_0(\Omega)\)

\[
\begin{align*}
\alpha(u, v) + \nu(\nabla u, \nabla v) - (P, \operatorname{div} v) &= (g, v), \\
(\operatorname{div} w, \lambda) &= (\operatorname{div} u, \pi) = 0, \\
\alpha(w, \chi) + \nu(\operatorname{rot} w, \operatorname{rot} \chi) + \nu(\operatorname{div} w, \operatorname{div} \chi) - (\eta, \operatorname{div} \chi) &= l(\chi) - f_{bc}(P, \chi).
\end{align*}
\]

**Lemma 3.20** (\( T \) is well-defined). Let \((g, l) \in L^2(\Omega) \times W'\). Then, problem (3.17)–(3.19) is well-posed and, as a consequence, \( T \) is well-defined and continuous. Moreover, \( T(g, l) = (u, P, w, \eta) \) satisfies the following bounds:

\[
\begin{align*}
\|\nabla u\| &\leq \nu^{-1}\|g\|_{H^{-1}}, \\
\nu\|u\|_{H^2} + \|\nabla P\| &\leq C_S\|g\|, \\
\|w\|_r + \|\eta\| &\leq C(\|l\|_{W'} + \|g\|),
\end{align*}
\]

where \( C \) and \( C_S \) are constants depending only on \( \Omega \).

**Proof.** The bounds follow standard arguments. The first bound repeats the proof of (3.6), while the second is (2.15) for \( r = 2 \). Once this is established, we invoke (2.3) to conclude the last one. □

Next, for \( f \in L^2(\Omega) \) we define the nonlinear operator \( N : X \times Q \times W \times L^2_0(\Omega) \to L^2(\Omega) \times W' \) as follows:

\[
N(u, P, w, \eta)_1 := f - w \times u \in L^2(\Omega),
\]

\[
\langle N(u, P, w, \eta)_2, \chi \rangle_{W', W} := (f, \operatorname{rot} \chi) + b(w, u, \chi) - b(u, w, \chi)
\]
where we have that the first component $N(u, P, w, \eta)_1$ belongs to $L^2(\Omega)$ because we have that $u, w \in W \hookrightarrow L^4(\Omega)$. Moreover, the estimates of Lemma 2.39 guarantee that $N(u, P, w, \eta)_2 \in W'$. Note also that, for $(u, w) \in X \times W$ we have

$$\int_\Omega |\nabla (w \cdot u)|^\frac{3}{2} \, dx \leq C \left( \int_\Omega |\nabla w|^\frac{3}{2} |u|^\frac{3}{2} \, dx + \int_\Omega |\nabla u|^\frac{3}{2} |w|^\frac{3}{2} \, dx \right) \leq C \left( \|\nabla w\|_{L^6}^\frac{3}{2} \|u\|_{L^6}^\frac{3}{2} + \|\nabla u\|_{L^6}^\frac{3}{2} \|w\|_{L^6}^\frac{3}{2} \right) \leq C \|\nabla u\|_{L^6}^\frac{3}{2} \|w\|_{L^6}^\frac{3}{2}.$$

Hence $w \cdot u \in W^{1,\frac{3}{2}}(\Omega)$, which is compactly embedded in $L^2(\Omega)$. By the same arguments we also have that $(w \cdot \nabla)u - (u \cdot \nabla)w \in L^2(\Omega)$ for $u, w \in X \times W$. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have the compact embedding $L^2(\Omega) = (L^2(\Omega))' \hookrightarrow (H^1(\Omega))' \subset W'$; see [6, Theorem 5.11.2]. Therefore, we conclude that the operator $N$ is well defined and compact.

Define now $F : X \times Q \times W \times L^2_0(\Omega) \to X \times Q \times W \times L^2_0(\Omega)$ as

$$F(u, P, w, \eta) = T(N(u, P, w, \eta)).$$

By a superposition of continuous and compact operators, this defines a compact operator. We have now established that the operator defined by (3.21) is a compact map from $X \times Q \times W \times L^2_0(\Omega)$ into itself. Further, note that solutions of the fixed point problem

$$(u, P, w, \eta) = F(u, P, w, \eta),$$

are solutions of (3.1)–(3.4).

We are now ready to show the existence of solutions.

**Theorem 3.22** (existence). Assume that $f \in L^2(\Omega)$ is such that (3.7) holds, then problem (3.1)–(3.4) has at least one solution. Thus, for large enough $\alpha$ the solution exists and is unique.

**Proof.** Consider the family of fixed point problems, with $\lambda \in [0, 1]$:

$$(u_\lambda, P_\lambda, w_\lambda, \eta_\lambda) = \lambda F(u_\lambda, P_\lambda, w_\lambda, \eta_\lambda).$$

Decomposing $F$ and noting that $\lambda T(g) = T(\lambda g)$, we have that

$$(u_\lambda, P_\lambda, w_\lambda, \eta_\lambda) = \lambda T(N(u_\lambda, P_\lambda, w_\lambda, \eta_\lambda)) = T(\lambda N(u_\lambda, P_\lambda, w_\lambda, \eta_\lambda)),$$

and thus for a given $\lambda \in [0, 1]$, solutions to the associated fixed point problem satisfy

$$\alpha(u_\lambda, v) + \nu(\nabla u_\lambda, \nabla v) + \lambda(w_\lambda \times u_\lambda, v) - (P_\lambda, \text{div } v) = (\lambda f, v),$$

$$\text{(div } w_\lambda, q) = (\text{div } u_\lambda, \pi) = 0,$$

$$\alpha(w_\lambda, \chi) + \nu(\text{rot } w_\lambda, \text{rot } \chi) + \nu(\text{div } w_\lambda, \text{div } \chi) + \nu(b(u_\lambda, w_\lambda, \chi)

- \lambda b(w_\lambda, u_\lambda, \chi) = (\lambda f, \text{rot } \chi) - f_{bc}(P_\lambda, \chi),$$

for every $(v, \pi, \chi, q) \in X \times Q \times W \times L^2_0(\Omega)$. To obtain a priori bounds on the solutions, we note that this system is identical to (3.1)–(3.4), except that the right hand side $f$ is scaled by $\lambda$, as are the nonlinear terms in the velocity and vorticity equations. Since $0 \leq \lambda \leq 1$, the same proof for a priori bounds as is done for (3.1)–(3.4) in Lemma 3.5 can be repeated for this system, and the only difference in the bounds is the dependence of the constants on $\lambda^k$, with $k \geq 0$. But since $\lambda \leq 1$, the following bounds hold uniformly in $\lambda$:

$$\|\nabla u_\lambda\| + \|\nabla P_\lambda\| + \|\nabla w_\lambda\| + \|\eta_\lambda\| \leq C,$$

where the constant $C$ depends only on the problem data. Thus by Lemma 3.16 there exists a fixed point for $F$ and thus a solution for (3.1)–(3.4). □
4 Conclusions and Outlook

We have proven well-posedness of a steady velocity-vorticity system with no-slip velocity boundary conditions, no penetration vorticity boundary conditions, and a natural boundary condition for vorticity involving a pressure functional. The well-posedness result provides a mathematical foundation for numerical methods, which exploit vorticity equations with these boundary conditions, and suggests a possible framework for their numerical analysis.

Significant technical difficulties arose due to the use of vorticity, vorticity boundary conditions, and the boundary conditions containing a pressure functional, and were overcome by a long, technical analysis that combined and extended analyses from [12, 13, 4]. These results are important for at least two reasons in addition to that stated above: First, there are very few analytical results known for fluid problems involving vorticity boundary conditions. Second, several results in section 2 are new, and can aid in future works for related PDEs which use non-standard boundary conditions.

Moving forward, these results will allow for improved analysis of numerical methods for NSE in velocity-vorticity formulations, and thus likely also to improved algorithms. To date, analysis of numerical schemes for these systems has been limited to 2D or the 3D steady case, and without physically derived boundary conditions, e.g. [2, 20, 18].

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