Fermionic functional renormalization group approach to superfluid phase transition

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A fermionic functional renormalization group (FRG) is applied to describe the superfluid phase transition of the two-component fermionic system with attractive contact interaction. The connection between the fermionic FRG approach and the conventional Bardeen–Cooper–Schrieffer (BCS) theory with Gorkov and Melik-Barkhudarov (GMB) correction are clarified in detail in the weak coupling region by using the renormalization group flow of the fermionic four-point vertex with particle–particle and particle–hole scattering contributions. To go beyond the BCS+GMB theory, coupled FRG flow equations of the fermion self-energy and the four-point vertex are studied under an Ansatz concerning their frequency/momentum dependence. We found that the fermion self-energy turns out to be substantial even in the weak coupling region, and the frequency dependence of the four-point vertex is essential to obtain the correct asymptotic-ultraviolet behavior of the flow for the self-energy. The superfluid transition temperature and the associated chemical potential are calculated in the region of negative scattering lengths.

Subject Index A40, A63, B32, I22

1. Introduction

Superfluidity in many-fermion systems is one of the central problems in condensed matter, atomic, nuclear, and particle physics. Examples include liquid superfluid \(^3\)He, electron superconductivity, cold atoms, nucleon superfluidity, color superconductivity [1], and so on. Among others, ultracold atomic gases play an important role in revealing the nature of superfluidity from weak to strong couplings in a single physical system: inter-atomic interactions in dilute systems are rather simple and the strength of the interaction is tunable via the Feshbach resonance technique [2]. The experimental discovery of the Bardeen–Cooper–Schrieffer (BCS) to Bose–Einstein condensate (BEC) crossover in two-component fermionic atoms [3–6] is a characteristic example of such a high level of control.

Two seemingly different kinds of superfluidity, BCS superfluid of weakly coupled fermions and BEC of Bose gas, turned out to be the same phenomenon connected via a smooth crossover for two-component fermionic systems. On the BCS side, weak attraction causes pairing instability against the Fermi surface leading to the formation of Cooper pairs [7], while on the BEC side, pairs of fermions form tightly bound composite bosons and superfluidity occurs due to their condensation at low temperature. Furthermore, the crossover between those two forms of superfluidity can be studied...
by combining the number equation and the assumption of the ground state being a BCS pairing wave function [8–10].

From a theoretical point of view, quantitative description of fermionic superfluidity in the weak coupling limit requires not only the BCS theory but also its Gorkov and Melik-Barkhudarov (GMB) correction [11,12]. However, the BCS+GMB theory still ignores higher-order many-body correlations which become important when the scattering length between fermions ($a_s$) becomes large in the so-called unitary regime. This is the reason why various non-perturbative techniques such as Monte Carlo simulations [13–17], the $\epsilon$-expansion [18], the functional renormalization group (FRG) method with auxiliary bosonic field [19–23], and the $t$-matrix approach [24–27] have been developed to attack the problems at and around unitarity.

The purpose of this paper is to develop a fermionic FRG (f-FRG) method without introducing the auxiliary bosonic field, not only to make a firm connection between the non-perturbative FRG approach and the conventional BCS+GMB theory but also to go beyond the BCS+GMB theory on solid ground. First, we will show how BCS+GMB theory is obtained from the renormalization group (RG) flow of the fermionic four-point vertex with particle–particle and particle–hole interactions, and then we explore the role of the RG flow of the fermion self-energy to go beyond BCS+GMB theory. We note that such analyses can be best achieved by fermionic FRG without introducing bosonic auxiliary fields. Compared with the auxiliary field method, which contains ambiguities in how to introduce the auxiliary field and usually requires a priori knowledge of the ground state property of the system, fermionic FRG can provide systematic and unbiased study of interacting fermions [28–31].

Throughout this paper, the main focus will be on the weak and intermediate coupling regime, where $a_s$ is negative. We will clarify the physical meaning of each approximation of f-FRG, and give physical interpretations of our results, based on the analysis of the flow equations in detail. Although our formalism itself is not limited to this regime, the case of $a_s > 0$ is not accessible at the level of approximations in the present paper, as will be discussed later. Nevertheless, we will extrapolate our results for the critical temperature and associated chemical potential to the unitary regime to see their qualitative behavior.

The contents of this paper are as follows. In Sec. 2, we introduce fermionic FRG formalism and explain how the critical temperature of the superfluid phase transition and the number density of fermions is calculated. In Sec. 3, we construct approximate flow equations of the effective coupling, which reproduce the results of the critical temperature of particle–particle random phase approximation (i.e. BCS theory) and the GMB correction. In Sec. 4, we concentrate on the flow equation of the self-energy correction, describing its problems and their possible resolution. In Sec. 5, we solve the coupled flow equations numerically and derive the self-energy correction, the critical temperature, and the associated chemical potential. Section 6 is devoted to a summary and concluding remarks. The reader can find some useful analytic formulas in the appendices.

### 2. Fermionic FRG formalism

We consider non-relativistic two-component fermions $\psi = \left( \begin{array}{c} \psi_\uparrow \\ \psi_\downarrow \end{array} \right)$ with a contact interaction:

$$
S[\overline{\psi}, \psi] = \int_0^\beta d\tau \int d^3x \left[ \overline{\psi} \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi + g \overline{\psi}_\uparrow \overline{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right],
$$

(1)
where $\beta (= 1/T)$, $\mu$, $m$, and $g$ are the inverse temperature, the chemical potential, the mass, and the bare coupling constant, respectively. The action $S$ can be written in momentum space as

$$
S[\bar{\psi}, \psi] = \int p (T) \bar{\psi}(p) G^{-1}(p) \psi(p) + g \int_p (T) e^{-i p^0 + \int_{q, q'} (T) \bar{\psi}_+(p + q) \bar{\psi}_+(p/2 - q) \psi_+(p/2 - q') \psi_+(p/2 + q').
$$

where $G^{-1}(p) = i p^0 + \frac{p^2}{2m} - \mu$ is the inverse propagator with $p = (p^0, p)$. Also, we adopt an abbreviated notation, $\int p (T) \equiv \int \frac{dp}{(2\pi)^3} T \sum p^0$. The factor $\exp(-i p^0)$ originates from the normal ordering of fermionic operators in the interaction term. The classical action (2) is symmetric under global $U(1)$ phase rotation, $SU(2)$ spin rotation, and space-time translation.

Following the idea of the FRG method [32–34], we define a scale-dependent generating functional $W_k[\eta, \bar{\eta}]$ by

$$
\exp(W_k[\eta, \bar{\eta}]) = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \left[ -S[\bar{\psi}, \psi] + \int_p (T) \bar{\psi}(p) R_k(p) \psi(p) \right]
$$

where $\eta$ and $\bar{\eta}$ are fermionic Grassmannian sources and $R_k$ is an infrared (IR) regulator which suppresses modes with momentum smaller than scale $k$. We require this function to satisfy conditions $R_k \to 0$ ($+\infty$) as $k \to 0$ ($+\infty$). Then the scale-dependent one-particle-irreducible (1PI) effective action $\Gamma_k[\bar{\psi}, \psi]$ is defined via Legendre transformation:

$$
\Gamma_k[\bar{\psi}, \psi] = \int_p (T) \left( \eta(p) \bar{\psi}(p) + \bar{\eta}(p) \psi(p) \right) - W_k[\eta, \bar{\eta}] - \int_p (T) \bar{\psi}(p) R_k(p) \psi(p).
$$

where $\eta, \bar{\eta}$ in (4) are determined by inverting the relations $\delta_L W_k / \delta \eta = \bar{\psi}$ and $\delta_L W_k / \delta \bar{\eta} = \psi$. Here, we distinguish left and right derivatives via subscripts, $\delta_L$ and $\delta_R$, respectively. Due to the property of $R_k$, $\Gamma_k \to \infty$ reduces to the classical action $S$ (up to a constant), while $\Gamma_{k=0}$ is the full 1PI quantum effective action $\Gamma$. $\Gamma_k[\bar{\psi}, \psi]$ obeys the flow equation [32–34]

$$
\partial_k \Gamma_k[\bar{\psi}, \psi] = -\frac{1}{2} \text{Tr} \left[ \frac{1}{\Gamma_k^{(2)}[\bar{\psi}, \psi] + R_k} \partial_k R_k \right],
$$

where the negative sign on the right-hand side can be interpreted as the result of a closed fermionic loop, and $\text{Tr}$ denotes the trace operation in both matrix and functional spaces with

$$
\left( \Gamma_k^{(2)}[\bar{\psi}, \psi] + R_k \right)_{p, q} = \begin{pmatrix}
\frac{\delta L}{\delta \eta} \frac{\delta \Gamma_k[\bar{\psi}, \psi]}{\delta \psi(p)} & \frac{\delta L}{\delta R} \frac{\delta \Gamma_k[\bar{\psi}, \psi]}{\delta \psi(p)} \\
\frac{\delta \psi(p)}{\delta \psi(q)} & \frac{\delta \psi(p)}{\delta \psi(q)}
\end{pmatrix} + \begin{pmatrix}
0 & R_k(p) \delta_{p, q} \\
-R_k(p) \delta_{p, q} & 0
\end{pmatrix}.
$$

Equation (5) implies that the change of the 1PI effective action in terms of $k$ is given by one-loop Feynman diagrams with dressed Green functions, 1PI interaction vertices, and a single insertion of a two-point vertex $\partial_k R_k$. Integrating (5) starting from a large enough but otherwise arbitrary scale (denoted by $\Lambda_{UV}$) down to $k = 0$, we can obtain $\Gamma = \Gamma_{k=0}$. Since we can obtain the dressed fermion
propagator and 1PI vertex functions, all information on thermodynamics is available using fermionic FRG formalism.

To solve the flow equation (5), we consider the following vertex expansion of $\Gamma_k$:

$$
\Gamma_k[\overline{\psi}, \psi] = \int_p (\overline{\psi}(p)G^{-1} - \Sigma_k)(p)\psi(p)
+ \int_{p,q,q'} \Gamma_k^{(4)}(p; q, q') \overline{\psi}^\dagger \left( \frac{p}{2} + q \right) \overline{\psi} \left( \frac{p}{2} - q \right) \psi \left( \frac{p}{2} + q' \right)
+ O((\overline{\psi}\psi)^2),
$$

(7)

where $\Sigma_k$ and $\Gamma_k^{(4)}$ are the self-energy and the four-point vertex, respectively. In the following, we truncate the expansion up to fourth order and study the flow equations of $\Sigma_k$ and $\Gamma_k^{(4)}$ without the introduction of auxiliary bosonic fields. (We call such an approach a fermionic FRG method or f-FRG method.) The critical temperature $T_c$ of the superfluid transition will always be obtained from the high-temperature side, which means that we will work in the normal phase. To study the superfluid phase below $T_c$, one needs to introduce an explicit symmetry-breaking term in the classical action. Otherwise, a second-order phase transition would occur in terms of $k$ at a nonzero value (since the flow always crosses the critical surface) leading to the divergence of the four-point function and the breakdown of the flow equation [30,35].

Let us now discuss the actual choice of the IR regulator $R_k$, which needs to suppress low-energy excitations of the system properly. In the weak coupling regime, where fermions form a Fermi sphere, $R_k$ must suppress both particle and hole excitations around the Fermi level defined through $p^2/2m = \mu + \sigma_0$; here, $\sigma_0$ is a constant part of the self-energy $\Sigma_0$ near the Fermi sphere. A suitable regulator for this purpose is Litim’s optimized regulator [36] with a self-energy term [22];

$$
\int_p (\overline{\psi}(p)R_k(p)\psi(p) \equiv \int_p (\overline{\psi}(p)\left[\frac{k^2}{2m}\text{sgn}(\xi(p)) - \xi(p)\right]\theta\left(\frac{k^2}{2m} - |\xi(p)|\right)\psi(p),
$$

(8)

where $\xi(p) \equiv \frac{p^2}{2m} - \mu - \sigma_0$ denotes the energy relative to the Fermi level. For a second-order phase transition, the critical point $T = T_c$ can be determined from the Thouless criterion [37] by looking at the divergence of the fermion–fermion scattering matrix at the total momentum $p = 0$. In our case, this reads

$$
\left[\Gamma_k^{(4)}(p = 0)\right]^{-1} = 0 \text{ at } T = T_c.
$$

(9)

Our primary goal is to calculate the ratios $T_c/\varepsilon_F$ and $\mu/\varepsilon_F$ as a function of the dimensionless constant $1/(k_F a_s)$. Here, $a_s$ is the scattering length between fermions, $k_F \equiv (3\pi^2 n)^{1/3}$ with $n$ being the fermion number density, and $\varepsilon_F \equiv k_F^2/2m = (3\pi^2 n)^{2/3}/2m$. Note that the number density $n$ is related to $T$ and $\mu$ through the number equation:

$$
n = \langle \overline{\psi}\psi \rangle = 2 \int_p (\overline{\psi}(p)G^{-1}(p) - \Sigma_0(p)).
$$

(10)

3. BCS+GMB theory from fermionic FRG method

Taking the vertex expansion (7) of the scale-dependent 1PI effective action $\Gamma_k[\overline{\psi}, \psi]$ up to $n = 2$ and applying the flow equation (5), a closed set of equations for the self-energy and four-point vertex
find

\[ \Gamma_k^{(4)}(p; q, l) \equiv \Gamma_k^{(4,PP)}(p; q, l, q') \]

where \( p \) is the center-of-mass momentum, and \( q, q' \) are relative momenta. On the right-hand side of (11a) and (11b), the differential operator \( \tilde{\partial}_k \) is defined to act only on the regulator \( R_k \).

A diagrammatic representation of (11) is given in Fig. 1. The first term on the right-hand side of (11b) describes particle–particle (PP) correlations, and it corresponds to the first diagram of Fig. 1(b), while the second term describes particle–hole (PH) correlations, and it corresponds to the second diagram of Fig. 1(b). Note that the PP diagram is the essential contribution forming Cooper pairs; PH contributions will only correct the value of the transition temperature. In the f-FRG approach, Eqs. (11) start to entangle with each other throughout the FRG flow \( k = \Lambda_{UV} \to k = 0 \).

3.1. BCS theory from f-FRG method

Let us first briefly review how the BCS theory is reproduced from the FRG method in the weak coupling limit [28]. Let us consider the flow of the four-point vertex ignoring the self-energy correction \( \Sigma_k \). Also, let us take into account only the PP correlation on the right-hand side of (11b). Then we find

\[ \Gamma_k^{(4,PP)}(p; q, l, q') = \tilde{\partial}_k \int l^{(T)} \frac{\Gamma_k^{(4,PP)}(p; q, l) \Gamma_k^{(4,PP)}(p; l, q')}{[G^{-1} + R_k]([G^{-1} + R_k](q + q' + l))}, \]

where we use the label “PP” to identify the approximation in question. A diagrammatic representation of this equation is shown in Fig. 2(a). Since the bare coupling in our model is a constant (see (1)), \( \Gamma_k^{(4)} \) does not depend on any momenta. Since the flow equation (12) does not have any explicit relative momentum dependence, it cannot be produced for \( \Gamma_k^{(4,PP)} \) either. Therefore, at any \( k \) it is only a function of the center-of-mass momentum \( p \), and we can replace \( \Gamma_k^{(4,PP)}(p; q, l) \) by \( \Gamma_k^{(4,PP)}(p) \).
Now we consider the particle–hole correlation (the second term on the right-hand side of Fig. 1) in
3.2. GMB correction from f-FRG method
Equation (12) can be rewritten as
\[
\frac{1}{\Gamma_k^{(4,PP)}(p)} = \frac{1}{\Gamma_k^{(4,PP)}(\mu)} + \int_{|l|<\Lambda_{UV}}^{(T)} \frac{1}{[G^{-1} + R_k](\frac{p}{2} + l)[G^{-1} + R_k](\frac{p}{2} - l) - \frac{m\Lambda_{UV}}{6\pi^2}}. 
\]
Taking \( k = 0 \) and defining \( \frac{1}{\tilde{\Gamma}} \equiv \frac{1}{\Gamma_k^{(4,PP)}} - m\frac{\Lambda_{UV}}{6\pi^2} \), we end up with the standard BCS result above \( T_c \), which corresponds to Fig. 2(b). Note that, the UV cutoff \( \Lambda_{UV} \) in (14) can be removed by the standard renormalization procedure \( \Gamma_k^{(4,PP)}(0)|_{\mu=0, T=0} = \frac{4\pi a_s}{m} \), which requires
\[
(\Gamma_{\Lambda_{UV}}^{(4,PP)})^{-1} = \frac{m}{4\pi a_s} - \left( \int_0^{\Lambda_{UV}} \frac{d^3l}{(2\pi)^3} \frac{1}{2l^2 / 2m} - \frac{m\Lambda_{UV}}{6\pi^2} \right) = \frac{m}{4\pi a_s} - \frac{m\Lambda_{UV}}{3\pi^2}. 
\]
After renormalization, the Thouless criterion \( |\Gamma_k^{(4)}(p = 0)|^{-1} |_{T=T_c} = 0 \), together with (14), leads to the standard form
\[
\frac{\pi}{2|a_s|} = \frac{1}{2} \int_0^{\infty} \sqrt{2m\epsilon d\epsilon} \left[ \frac{\tanh((\epsilon - \mu)/2T_c)}{\epsilon - \mu} - \frac{1}{\epsilon} \right]. 
\]
If we further take the weak coupling limit \( (k_F a_s)^{-1} \to -\infty \) with \( \mu = \epsilon_F \), (16) provides the standard BCS result
\[
\frac{T_{c}^{BCS}}{\epsilon_F} \equiv \frac{8}{\pi} e^{\gamma} e^{-\frac{\pi}{2k_F|a_s|}}. 
\]
3.2. GMB correction from f-FRG method
Now we consider the particle–hole correlation (the second term on the right-hand side of Fig. 1) in
the flow equation of \( \Gamma_k^{(4)} \). As reported in Ref. [20], this leads to the Gorkov and Melik-Barkhudarov (GMB) correction [11]. The main aim of this subsection is to clarify the origin of the GMB correction in detail and to analyze the approximations to be employed in a precise way. Such an analysis is particularly useful for going beyond the BCS+GMB theory in later sections.
In general, the four-point vertex function \( \Gamma_k^{(4)}(p; q, q') \) depends not only on the center-of-mass momentum \( p \) but also on relative momenta \( q \) and \( q' \). If we restrict ourselves to particle–particle correlations, the dependence on \( q \) and \( q' \) disappears as we have shown in the previous subsection. However, once we have particle–hole correlations, such a simplification does not occur anymore. Since we are interested in s-wave superfluidity, instead of solving the full momentum dependence,
we can consider the case $|q| = |q'| = k_F$ and take the s-wave projection averaged over the directions of $q$ and $q'$: $\Gamma^{(4)}_k(p; q^0, q'^0) = \int d^2\hat{q} \hat{d}^2\hat{q}' \Gamma^{(4)}_k(p; q, q')\bigg|_{|q|=|q'|=k_F}$. This is motivated by the fact that, at low energy, single-particle excitations are given by fermionic quasi-particles in the vicinity of the Fermi surface, and $q$ and $q'$ will be of the order of the Fermi momentum $k_F \approx \sqrt{2m\mu}$, with the loop momentum $l$ being also restricted to the region $|l| \sim k_F$ due to the presence of the regulator $R_k$. Finally, we define $\Gamma^{(4)}_k(p) \equiv \Gamma^{(4)}_k(p; 0, 0)$, so that $\Gamma^{(4)}_k(p = 0)$ can be regarded as the effective coupling constant at scale $k$. The corresponding flow equation is given by

$$\begin{align*}
\partial_k \left( \frac{1}{\Gamma^{(4, PP+PH)}_k(0)} \right) &= \partial_k \int_{(1)}^{(T)} \frac{1}{[G^{-1} + R_k(l)][G^{-1} + R_k(-l)]} \\
+ \partial_k \int_{|q|=k_F} \frac{d^2\hat{q} d^2\hat{q}'}{4\pi 4\pi} \int_{(1)}^{(T)} \frac{1}{[G^{-1} + R_k(l)][G^{-1} + R_k(q - q' + l)]},
\end{align*}$$

(18)

where we have used the label “PP + PH” to identify the approximation taken here. Note that the angular integration of $\hat{q}$ and $\hat{q}'$ still remain in the particle–hole contribution through the regularized fermion propagator. Further discussions on the approximation adopted here are given at the end of this subsection on the basis of the detailed flow pattern as a function of $k$ in the weak coupling regime.

Since both sides of (18) became total derivatives, it can be solved as

$$\frac{1}{\Gamma^{(4, PP+PH)}_k(0)} = \frac{1}{\Gamma^{(4, PP)}_k(0)} + \frac{8m\mu}{8m\mu} \int_0^{8m\mu} d\Omega^2 \int_0^{8m\mu} d\Omega^2 \int (2\pi)^3 \frac{n_F \left( (l+Q)^2 - \mu + R_k(l + Q) \right) - n_F \left( (l+Q)^2 - \mu + R_k(l) \right)}{(2\pi)^3 \left( (l+Q)^2 - \mu + R_k(l + Q) \right) - \left( (l+Q)^2 - \mu + R_k(l) \right)},$$

(19)

where $Q \equiv q - q'$ and $n_F$ is the Fermi–Dirac distribution function. The first term in the right-hand side of (19) has already been evaluated in (14). Also, considering the fact that the critical temperature $T_c$ is much smaller than the Fermi energy $\varepsilon_F$ in the weak coupling regime, the second term in the right-hand side of (19) can be evaluated at $T = 0$ as

$$\frac{8m\mu}{8m\mu} \int_0^{8m\mu} d\Omega^2 \int_0^{8m\mu} d\Omega^2 \int (2\pi)^3 \frac{\theta \left( (l+Q)^2 - \mu \right) - \theta \left( (l+Q)^2 - \mu \right)}{(2\pi)^3 \left( (l+Q)^2 - \mu \right) - \left( (l+Q)^2 - \mu \right)} = -\frac{1}{3} \ln 4 \frac{mk_F}{2\pi^2}.$$

(20)

Then we find that, at the end of the flow ($k \to 0$), the particle–hole contribution simply shifts the value of the inverse scattering length from $a_s$ to $a_s^{\text{eff}}$:

$$\frac{1}{a_s^{\text{eff}}} = \frac{1}{a_s} - \frac{2k_F}{\pi} \frac{1 + \ln 4}{3}.$$

(21)

Therefore, taking into account the correction coming from PH correlations, the BCS critical temperature in the weak coupling regime is corrected as

$$\frac{T_c^{\text{GMB}}}{\varepsilon_F} = \frac{1}{(4e)^{1/3}} \cdot \frac{T_c^{\text{BCS}}}{\varepsilon_F}.$$

(22)

This is exactly the Gorkov and Melik-Barkhudarov result [11,12], showing the reduction of $T_c$ due to the effect of order parameter fluctuations, and is in agreement with previous FRG studies as well [20].

Let us now discuss the structure of the flow equation (18) in more detail to gain a deeper understanding of why exactly the GMB result was reproduced from f-FRG within the above approximation.
Table 1. Contributions from the particle–particle (PP) correlation and particle–hole (PH) correlation to $\partial_k (\Gamma_k^{(4)}(0))^{-1}$ in the right-hand side of Eq. (18). This behavior is consistent with that of the prediction from Shanker’s RG analysis [28].

| $k^2/2m$ | $\pi T_c$ | $k^2/2m \ll k_F^2/2m$ |
|----------|-----------|-------------------------|
| PP       | $O(k_F^3/T_c^3)$ | $O(k_F/k)$ |
| PH       | $O(k^2/k_F T_c^2)$ | $O(k/k_F)$ |

Fig. 3. Derivative of $1/\Gamma_k^{(4)}(0)$ at $T_c/\mu = 0.06$ (which corresponds to $(|k_F a_s|)^{-1} \sim O(1)$) with and without the PH loop included. The figure shows that only when $k \sim k_F \approx \sqrt{2m\mu}$, particle–hole fluctuations change the behavior of the flow.

The parametric $k$-dependence of the first term (particle–particle correlation) and the second term (particle–hole correlation) of the right-hand side of (18) are summarized in Table 1 (the corresponding explicit formulas are given in Appendix B). We mention here that the PH contribution has the opposite sign to the PP contribution. From Table 1, we observe that the PH contribution is much smaller than the PP contribution in the high-energy region ($k \gg k_F$) and the low-energy region ($k \ll k_F$). Only for $k \approx k_F$ do the PH and PP contributions become comparable. Such a behavior can be seen explicitly in Fig. 3, where numerical solutions of $\partial_k (\Gamma_k^{(4)}(0))^{-1}$ with and without the PH correlation are plotted. The figure shows that the screening of the effective coupling from the PH correlation takes place only around $k \sim k_F$. This implies that we can neglect the momentum dependence of $\Gamma_k^{(4)}$ in the PH contribution, at least for weak couplings, where we have the hierarchy $1/a_s \gg k_F \gg T_c/k_F$. In such a regime, $1/\Gamma_k^{(4)}$ is $O(1/a_s)$ around $k \sim k_F$, so that the momentum dependence of $\Gamma_k^{(4)}$ in the PH contribution is negligible. Although such a justification is questionable beyond the weak coupling limit, we will later adopt this approximation as a working hypothesis to make a qualitative study towards the unitary regime.

For later purposes, we show here the structure of $\Gamma_k^{(4,PP+PH)}$ as a function of the center-of-mass momentum $p$ for large values of the flow parameter $k$. In such an asymptotic regime, PP correlation

\footnote{The behavior of these terms in the f-FRG method is qualitatively consistent with Shankar’s RG analysis of interacting fermions [28].}
is the dominant contribution, as discussed above. Therefore, (14) leads to

$$\frac{1}{\Gamma_k^{(4,PP+PH)}(p)} \xrightarrow{k \to \infty} - \frac{2m}{6 \pi^2} \left[ k - \frac{3 \pi}{4 a_s} + \frac{2m}{k} \left( i p^0 + \frac{p^2}{4m} - \frac{3}{2} \mu \right) \right]. \quad (23)$$

4. f-FRG with fermion self-energy

So far, we have not taken into account the effect of the self-energy correction $\Sigma_k$ in the flow equations. Since $\sigma_k$ (the constant part of $\Sigma_k$ near the Fermi sphere) gives a shift of the chemical potential $\mu \to \mu_k = \mu + \sigma_k$, its importance grows as the system approaches the unitary regime. In the following, we focus only on the lowest Matsubara frequency part of $\Sigma_k$ and define $\sigma_k$ as its real part: $\sigma_k \equiv \Re \Sigma_k(\pm \pi T, \theta)$. We note that even with the present (constant) self-energy, non-trivial resummation of the original perturbative series occurs via the self-consistent nature of the coupled flow equations, and therefore it is a good starting point for analyzing the role of the self-energy going beyond BCS+GMB theory. In this section we will first study the behavior of $\sigma_k$ for asymptotically large $k$, and formulate coupled flow equations of $\sigma_k$ and $\Gamma_k^{(4)}$, and then proceed to obtain a numerical solution of them.

4.1. Asymptotic behavior of the self-energy at large $k$

Within the momentum-independent vertex discussed in the previous section, using (11a), the constant part of the self-energy near the Fermi sphere (i.e. $\sigma_k$) satisfies

$$\partial_k \sigma_k = \tilde{g}_k \left\{^{(T)} \int_l \frac{\Gamma_k^{(4)}(0)}{G^{-1}(l) - \sigma_k + R_k(l)} \right\}, \quad (24)$$

which can be rewritten using an effective Fermi level $\mu_k = \mu + \sigma_k$ as

$$\partial_k \mu_k = - \frac{(2m)^{1/2} k \Gamma_k^{(4)}(0)}{3 \pi^2} \left[ \left( (\mu_0 + k^2/2m)^{3/2} - (\mu_0)^{3/2} \right) n_F(\omega_+) \right. \left. - \left( (\mu_0)^{3/2} - \Re (\mu_0 - k^2/2m)^{3/2} \right) n'_F(\omega_-) \right], \quad (25)$$

where $\omega_\pm = \pm k^2/2m + \mu_0 - \mu_k$, and $n_F$ is the Fermi–Dirac distribution function. By taking $k \to \infty$ (note that $ka_s$ can take any negative values) and using the asymptotic form of $\Gamma_k^{(4)}(0)$ given in (23), one finds

$$\sigma_k = e^{-k^2/2m} \left( \frac{k^2/2m}{1 - 3 \pi/4ka_s} + O(1) \right). \quad (26)$$

This asymptotic behavior is, however, not correct, as can be seen from the following argument. First of all, the present theory is asymptotically free in the sense that $\Gamma_k^{(4)}(0) \simeq -(3 \pi^2)/(m(k - 3 \pi/4a_s))$ for large $k$ as obtained from (23). Then, since perturbative analysis is valid for large $k$, we can evaluate $\sigma_k$ for large $k$ as

$$\sigma_k \simeq \Re \left\{^{(T)} \int_l \frac{\Gamma_k^{(4)}(p + l)e^{-ip^0l_0}}{ip^0l_0 + \frac{P^2}{2m} - \mu + R_k(l)} \right\}, \quad (27)$$
where $p^0 = \pm \pi T$. Taking into account the correct leading-order frequency dependence as $\Gamma_k^{(4)}(l) \simeq -\frac{(3\pi^2 k/2m^2)}{(i l^0 + k^2/2m - 3\pi k/8ma_s)}$ and carrying out the integral, one finds

$$\sigma_k = \frac{\gamma l}{\mathcal{F}} \int l T \left[ \frac{\Gamma_k^{(4)}(\pm \pi T + l^0, l)}{G^{-1}(l) - \sigma_k + R_k(l)} \right]$$

$$\simeq \frac{\sqrt{2m(\mu + \sigma_0)^{3/2}}}{2k(1 - 3\pi/8ka_s)},$$

(28)

where we omitted the spatial-momentum dependence of $\Gamma_k(p + l)$ giving only subleading contributions. Thus we observe that $\sigma_k$ must decrease as $1/k$, unlike (26). Another way to see the problem of (26) is that the number density $n_k = \langle \bar{\psi} \psi \rangle$ for large $k$ simply vanishes contrary to the true behavior expected from the lowest-order perturbation at large $k$; $n_k \rightarrow (2m\mu_0)^{3/2}/(3\pi^2)$.

4.2. Coupled flow equations for self-energy and four-point vertex

From the discussion of the previous subsection, we see that the momentum dependence of $\Gamma_k^{(4)}$, especially its frequency dependence, is essential to determine $\sigma_k$ in the f-FRG approach. To study such a frequency dependence by avoiding the quite demanding numerical calculation of solving the coupled flow-equation of $\Sigma_k$ and $\Gamma_k$ with full momentum dependence, we adopt the hybrid approach described below as a first step, in order to explore the effect of the self-energy.

Let us start with the following coupled equations:

$$\partial_l \sigma_k = \gamma l \partial_k \int l T \frac{\Gamma_k^{(4)}(\pm \pi T + l^0, l)}{G^{-1}(l) - \sigma_k + R_k(l)},$$

(29)

$$\partial_l \Gamma_k^{(4)-1}(0) = \partial_k \int l T \frac{1}{[G^{-1}(l) - \sigma_k + R_k(l)][G^{-1}(-l) - \sigma_k + R_k(-l)]}$$

$$+ \partial_k \int_0^{8m\mu} \frac{d|Q|^2}{8m\mu} \int l T \frac{1}{[G^{-1} - \sigma_k + R_k(l)][G^{-1} - \sigma_k + R_k(l)](Q + l)},$$

(30)

with $Q = (0, Q)$. In (29), the frequency of the fermion self-energy is restricted to the lowest values $\pm \pi T$. After performing the Matsubara sum, $\sigma_k$ and $\sigma_0$ appear as a combination $\sigma_0 - \sigma_k (= \mu_0 - \mu_k)$. Since the approximate particle–hole symmetry would make this combination small for $k < k_F$, where the PH contribution is already not significant, we take $\sigma_k = \sigma_0$ in the actual calculation of the PH contribution.

To take into account the momentum dependence in a minimal, but sufficient way we make the following expansion and keep the first few terms:

$$\Gamma_k^{(4)-1}(p^0 + l^0, l) \approx -Z_k^{-1} \left[ i(l^0 + p^0) + S_k^{(1)} \cdot |l| + S_k^{(2)} \cdot l^2 + |\mu_k^B| \right],$$

(31)

where we have introduced the notations

$$Z_k^{-1} = i\partial_{p^0} \Gamma_k^{(4)-1}(0), \quad |\mu_k^B| = -Z_k \Gamma_k^{(4)-1}(0),$$

$$S_k^{(1)} = -Z_k \partial_{|l|} \Gamma_k^{(4)-1}(0), \quad S_k^{(2)} = -Z_k \partial_{l^2} \Gamma_k^{(4)-1}(0).$$

(32)

Now we adopt a hybrid approach in which $\Gamma_k^{(4)-1}(0)$ is calculated from the flow equation, while its derivatives with respect to frequency and momentum are estimated by the RPA analysis in the
Now we are ready to solve (35a) and (35b) together with the number equation, (10). In order to get

\[ \partial_k \sigma_k = \sqrt{2m \mu_0 + k^2} \int d\omega Z_k \Re \left[ \frac{\partial \omega n_F(\omega_{k^+})}{\omega_{k^+} - \omega_k^B(I) + i\pi T} - \frac{n_B(\omega_k^B(I)) + n_F(\omega_{k^+})}{(\omega_{k^+} - \omega_k^B(I) + i\pi T)^2} \right] \]

and

\[ \partial_k \Gamma_k^{(4)} = \frac{(2m)^{1/2} k}{3\pi^2} \left( -1 + \frac{n_F(\omega_{k^+})}{\omega_{k^+}} - \frac{n'_F(\omega_{k^+})}{\omega_{k^+}^2} \right) \left[ (\mu_0 + k^2/2m)^{3/2} - (\mu_0)^{3/2} \right] \]

\[ \quad - \frac{(2m)^{1/2} k}{3\pi^2} \left( -1 + \frac{n_F(\omega_{k^-})}{\omega_{k^-}} - \frac{n'_F(\omega_{k^-})}{\omega_{k^-}^2} \right) \left[ (\mu_0)^{3/2} - \Re(\mu_0 - k^2/2m)^{3/2} \right] \]

\[ \quad + \tilde{\delta}_k \int_0^\infty \frac{dQ^2}{8m \mu} \int \frac{d^3 l}{(2\pi)^3} \frac{n_F\left( \frac{l^2}{2m} + R_k(I) - \mu_k \right) - n_F\left( \frac{l^2}{2m} + R_k(I + Q) - \mu_k \right)}{l^2/2m + R_k(I) - (l + Q)^2/2m - R_k(I)} \]

where \( \omega_{k^\pm} = \pm k^2/2m + \mu_0 - \mu_k. \) The effective Fermi level \( \mu_k \) starts to flow from the chemical potential at \( k = \infty, \mu_{k=\infty} = \mu, \) and converges to \( \mu_0 = \mu + \sigma_0. \) Because of the appearance of bosonic (Cooper pair) contributions to the flow equation for \( \mu_k = \mu + \sigma_k, \) Eq. (35a) indeed shows the correct asymptotic behavior at large \( k \) consistent with the discussion in the previous subsection:

\[ \partial_k \mu_k = -\frac{\sqrt{2m \mu_0}^{3/2}}{2(k - 3\pi/8d_s)^2} + \mathcal{O}(1/k^4) \Rightarrow \sigma_k = \frac{\sqrt{2m} (\mu + \sigma_0)^{3/2}}{2k(1 - 3\pi/8d_s)} + \mathcal{O}(1/k^3) \]

5. Numerical results

Now we are ready to solve (35a) and (35b) together with the number equation, (10). In order to get

the transition temperature \( T_c, \) one may start from a large \( k \) at the UV cutoff scale, and let the flow

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Fig. 4. Flow of the inverse of the four-point function at unitarity in the PP + PH + SE approximation. The plot demonstrates that the singularity only appears at \( k = 0 \).

equations run towards \( k = 0 \). Then \( T_c \) is obtained through the Thouless criterion, \( \Gamma_k^{(4)^{-1}}(0) = 0 \), while the chemical potential \( \mu \) and the number density \( n \) are related via the number equation (10).

However, the actual numerical calculation is much more efficient by starting from \( k = 0 \) with \( \Gamma_k^{(4)^{-1}}(0) = 0 \) and letting the flow move towards large \( k \). Keeping the temperature at a certain value and choosing \( \mu = 1 \), the flow of \( \sigma_k \) and \( \Gamma_k^{(4)^{-1}} \) are numerically obtained on a grid with a step size of \( \Delta k = 10^{-4} \). Then we fit these quantities by their known asymptotic behavior for large \( k \) to determine \( \sigma_{k=\infty} \) and \( \Gamma_k^{(4)^{-1}} \). We choose the highest scale to be \( k_{UV} = 50.0 \) and fit the functions in the interval \([49.0, 50.0]\). After subtracting the “divergent” \(- mk/3\pi^2\) piece in (23), \( \Gamma_k^{(4)^{-1}} \) should be equal to \( m/4\pi a_s \), from which we obtain the function \( a_k = a_s(T_c) \). Since the self-energy always has to approach zero at large \( k \), a unique solution is obtained by adjusting \( \sigma_{k=0} \) this way.

Even after taking into account the self-energy correction, we found numerically that the flow of the four-point vertex \( \Gamma_k^{(4)} \) shows a qualitatively similar behavior to the PP + PH flow shown in Fig. 3. Also, the scalings with respect to flow parameter \( k \) are the same as those given in Table 1. Therefore, we can safely use the Thouless criterion (9) to determine the critical temperature of the superfluid transition without singular behavior of the flow at \( k > 0 \). We show a typical flow of \( [\Gamma_k^{(4)}(0)]^{-1} \) in Fig. 4, which indicates that \( [\Gamma_k^{(4)}(0)]^{-1} \) increases monotonically as \( k \) decreases.

Flows of the self-energy are shown in Fig. 5 for inverse scattering lengths \((k_Fa_s)^{-1} = -2, -1, 0\) at the corresponding critical temperatures. Figures 6 and 7 show our numerical results for the critical temperature \((T_c/\epsilon_F)\) and the chemical potential \((\mu/\epsilon_F)\) as a function of the inverse scattering length \((k_Fa_s)^{-1} < 0\), respectively.

For large \( k \) \((\gg k_F)\), the flow of the self-energy in Fig. 5 is well described by its asymptotic behavior given in (28). On the other hand, the flow of the self-energy almost stops for small \( k \) \((\ll k_F)\). This is due to the fact that approximate particle–hole symmetry for small \( k \) protects the shift of the Fermi level. Figure 5 shows that the magnitude of the self-energy \( \sigma_0 \) becomes larger as the coupling becomes strong, but the saturation of \( \sigma_k \) at \( k \sim k_F \) is irrespective of the coupling strength.
Fig. 5. Self-energy $\sigma_k = \mu_k - \mu$ as a function of $k$ for $(k_F a_s)^{-1} = -2, -1, 0$ at $T = T_c$.

Fig. 6. $T_c/\varepsilon_F$ as a function of the dimensionless scattering strength, $(k_F a_s)^{-1}$, at different levels of approximation: PP (particle–particle correlation only), PP + PH (particle–particle and particle–hole correlation without self-energy), PP + PH + SE (particle–particle and particle–hole correlation with self-energy).

As can be seen from Fig. 6, PH correlation is the dominant source of the reduction of $T_c/\varepsilon_F$, and its physical origin is the screening of the coupling strength.\(^2\) We note that this does not imply that the self-energy correction is negligible in the weak coupling region. Since the flow of $\sigma_k$ stops for $k < k_F$, most of its effect is absorbed by shifting the Fermi level as $\mu \mapsto \mu_0 = \mu + \sigma_0$. The self-energy (SE) correction still leads to further reduction of $T_c/\varepsilon_F$ since it makes $\varepsilon_F$ or the number density $n$ bigger.

\(^2\) Note that our numerical solutions for $T_c/\varepsilon_F$ with the PP correlation and PP + PH correlation are smaller than the results of the BCS formula, Eq. (17), and the GMB formula, Eq. (22), respectively, valid only in the weak coupling limit.
Fig. 7. $\mu(T_c)/\varepsilon_F$ as a function of the dimensionless scattering strength, $(k_F a_s)^{-1}$. The meaning of each line is as in Fig. 6.

Table 2. Numerical values of $T_c/\varepsilon_F$ and $\mu/\varepsilon_F$ at $(k_F a_s)^{-1} = -2, -1, 0$ at different levels of approximation. The meaning of PP, PP + PH, and PP + PH + SE are the same as those in Fig. 6.

| $(k_F a_s)^{-1}$ | $T_c/\varepsilon_F$ | $\mu/\varepsilon_F$ |
|------------------|---------------------|---------------------|
|                  | $-2$    | $-1$    | $0$    | $-2$    | $-1$    | $0$    |
| PP               | 0.027   | 0.126   | 0.496  | 1       | 0.987   | 0.747  |
| PP + PH          | 0.012   | 0.058   | 0.276  | 1       | 0.997   | 0.929  |
| PP + PH + SE     | 0.012   | 0.053   | 0.237  | 0.828   | 0.713   | 0.367  |

as the coupling strength becomes stronger towards the unitary regime, but the effect is rather limited because $\sigma_0 - \sigma_k$ for $k \sim k_F$ cannot be large due to the particle–hole symmetry, as discussed above.

Figure 7 shows that the effect of the PH correlation on $\mu/\varepsilon_F$ is the opposite of what we find for $T_c/\varepsilon_F$. This is because the critical temperature $T_c$ of PP is higher than that of PP + PH for the same chemical potential $\mu$ due to the GMB correction. Therefore, the number of fermions in PP is larger than that of PP + PH. When the self-energy correction is taken into account in PP + PH + SE, the number density $n$ and hence the Fermi energy $\varepsilon_F$ increase for given $\mu$, so that a considerable decrease of $\mu/\varepsilon_F$ arises. It is interesting to see that even in the weak coupling limit, $\varepsilon_F$ is comparable with $\mu$, even though the self-energy has no significant effect on $T_c/\varepsilon_F$ in this regime.

In Table 2 we pick up some values of $T_c/\varepsilon_F$ and $\mu/\varepsilon_F$ obtained from our numerical calculations at $(k_F a_s)^{-1} = -2$ (weak coupling), $(k_F a_s)^{-1} = -1$ (intermediate coupling), and at $(k_F a_s)^{-1} = 0$ (unitarity). Although it is beyond our scope to predict a quantitatively correct $T_c$ in the unitary regime, it is still instructive to compare our result at unitarity, $(T_c/\varepsilon_F, \mu/\varepsilon_F) = (0.237, 0.367)$, with the previous FRG results using an auxiliary bosonic field, $(T_c/\varepsilon_F, \mu/\varepsilon_F) \approx (0.264, 0.68)$ [20], $\approx (0.276, 0.63)$ [21], $\approx (0.248, 0.51)$ [22,23], and with the results of the t-matrix approach, $T_c/\varepsilon_F \approx 0.16$ [26], and $T_c/\varepsilon_F \approx 0.217$ [27,38]. Note that quantum Monte Carlo simulations [13–17] for $T_c/\varepsilon_F$ vary in the range 0.15–0.3.
6. Summary and concluding remarks

We developed a fermionic FRG (f-FRG) method with a modified Litim’s regulator to describe the superfluid phase transition for two-component fermions with a contact interaction. By making vertex expansion of the 1PI effective action $\Gamma_k[\bar{\psi}, \psi]$ up to the four-point vertex and solving the RG flow equation, we determined the critical temperature $T_c$ in the regime of negative scattering lengths using the Thouless criterion $[\Gamma_k^{(4)}(p = 0)]^{-1} = 0$.

In order to clarify the relation between the FRG approach and conventional many-body theory such as the BCS theory + GMB correction, we have taken into account the particle–particle correlation, the particle–hole correlation and the self-energy correction step by step in the flow equations. In agreement with the literature, we saw that the flow equation with a single PP correlation reduces to the BCS theory, and we have also found that, equipped with the PH correlation in the flow equation, a momentum-independent vertex approximation precisely reproduces the GMB correction. The major part of the PH contribution originated from the region $k \sim k_F$.

To go beyond the BCS + GMB theory in our framework, we considered the flow of the constant part of the self-energy $\sigma_k$, together with the flow of the four-point vertex $\Gamma_k^{(4)}(p = 0)$. We found that the self-energy decreases slowly as $\sigma_k \sim 1/(k - 3\pi/(8a_s))$ for $k \to \infty$, while it saturates as $\sigma_k \sim \sigma_0$ for $k (\lesssim k_F)$ due to approximate particle–hole symmetry. We also carried out a numerical calculation with a hybrid approach, in which $\Gamma_k^{(4)}(p = 0)$ is determined from the flow equation, while the $p$-dependent part of $\Gamma_k^{(4)}(p)$ necessary to reproduce correct asymptotic behavior of $\sigma_k$ is evaluated using PP-RPA. The resulting value of $T_c/\varepsilon_F$ does not receive a large correction from the self-energy except for the unitary regime $(1/(k_F a_s) \to 0)$. On the other hand, $\mu/\varepsilon_F$ shows a large reduction by the self-energy correction due to the increase of the Fermi energy given $\mu$, even for the relatively weak coupling region $(k_F a_s)^{-1} \lesssim -1$. Extrapolation to unitarity gives $T_c/\varepsilon_F = 0.237$ and $\mu/\varepsilon_F = 0.367$.

We note that the approximations employed in the present paper, particularly the assumption of constant self-energy, is valid only in the regime $1/(k_F a_s) < 0$. To enter into the BEC side $(1/(k_F a_s) > 0)$, momentum-dependent self-energy is crucial in f-FRG method, since it provides the information for composite bosons in the number equation. Therefore, solving the flow equations with momentum-dependent self-energy is one of the important problems to be studied in the future. More sophisticated treatments could also be employed, e.g. the use of the two-particle-irreducible formalism [39–41] combined with the f-FRG method is expected to be an efficient way to describe the superfluid phase transition, with its power of resumming an even wider class of Feynman diagrams. Another direction from which to study the superfluid phase transition is to consider f-FRG with an IR regulator in the fermion vertex [42], which enables the treatment of the non-trivial momentum dependence of the fermion self-energy. These methods may open new ways for unbiased studies of the BCS–BEC crossover in the future by extending the f-FRG approach given in the present paper.

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Appendix A. RPA results for the vertex function at finite $k$

In this appendix, we show the results of the four-point vertex function in particle–particle RPA (PP-RPA). According to (14), the basic equation is given by

$$
\frac{1}{\Gamma_k^{(4,\text{PP})}(p)} = \frac{1}{g} + \int_{|l|<\Lambda_{\text{UV}}} \frac{d^3l}{(2\pi)^3} \frac{1 - \sum_{\pm} n_F \{ (\frac{l}{2} \pm I)^2 - \mu + R_k (\frac{l}{2} \pm I) \}}{1p^0 + \sum_{\pm} \{ (\frac{l}{2} \pm I)^2 - \mu + R_k (\frac{l}{2} \pm I) \}},
$$

where we take $2m = 1$ for simplicity, and $g$ is the bare coupling constant, as we indicated after (14).

We consider the expansion given in (31) with the parameterization denoted as

$$
\Gamma_k^{(4,\text{PP})}(p) = -\frac{Z_k}{ip^0 + S_k^{(1)}|p| + S_k^{(2)}p^2 + |\mu_B|^2},
$$

where parameters $Z_k$, $S_k^{(1)}$, $S_k^{(2)}$, and $|\mu_B|^2$ are positive.

Let us first consider $Z_k$, which may be regarded as the wave function renormalization constant of Cooper pairs. According to (A.2), we obtain,

$$
Z_k^{-1} = -\frac{\partial \Gamma_k^{(4,\text{PP})}}{\partial (ip^0)}(0) = \int \frac{d^3l}{(2\pi)^3} \frac{1 - 2n_F(l^2 - \mu + R_k(I))}{[2(l^2 - \mu + R_k(I))]^2}.\]

(A.3)

Using the notation $k^2_\pm = \pm k^2 + \mu$, we find that

$$
Z_k^{-1} = \frac{1}{8\pi^2} \left[ \frac{1}{2} \left( \frac{k^2_+}{k^2_+ - \mu} + \frac{1}{\sqrt{\mu}} \tanh^{-1} \frac{1}{\sqrt{\mu}} \right) + \frac{k^3_+ - \sqrt{\mu}^3}{3} \frac{1 - 2n_F(k^2_+ - \mu)}{(k^2_+ - \mu)^2} \\
- \frac{1}{2} \left( \frac{k^2_-}{k^2_- - \mu} + \frac{1}{\sqrt{\mu}} \tanh^{-1} \frac{1}{\sqrt{\mu}} \right) + \frac{\sqrt{\mu}^3 - 3k^3_-}{3} \frac{1 - 2n_F(k^2_- - \mu)}{(k^2_- - \mu)^2} \\
- 2 \left( \int_0^{\infty} \int_{k^+_+}^{\infty} \frac{d(l^2 - \mu)}{(l^2 - \mu)^2} \right). \]

(A.4)

According to the expression (A.4), the asymptotic behavior of $Z_k$ in the large $k$ limit is given by

$$
Z_k^{-1} = \frac{1}{6\pi^2 k^2_+} \left( 1 + \frac{\mu}{k^2_+} \right) + O(1/k^4).
$$

(A.5)

In order to get the spatial momentum dependence of the four-point function, we must perform the integration (A.1) with great care concerning the singularities associated with the IR regulator. The coefficient of the $|p|$-linear term is given by

$$
S_k^{(1)} = \frac{\mu Z_k}{8\pi^2} \left( \frac{\tanh \frac{\beta}{2} k^2}{k^2} - \frac{\beta/2}{\cosh^2 \frac{\beta}{2} k^2} \right).
$$

(A.6)
This term originates from the discontinuity of the regulator $R_k(I)$ at $|l| = \sqrt{\mu}$, and it vanishes at $k = 0$. As for the quadratic term in $p$, we find that

$$S_k^{(2)} = \frac{Z_k}{16\pi^2} \left[ \frac{1}{2} \left( \frac{k_+}{k_+^2 - \mu} + \frac{1}{\sqrt{\mu}} \right) \tanh^{-1} \frac{\sqrt{\mu}}{k_+} \right] + \frac{k_+^3}{3} \left( \frac{1}{(k_+^2 - \mu)^2} - \frac{1}{(k_+^2 - \mu)^2} \right) + \frac{2}{3} \frac{k_+^3 n_F(k_+^2 - \mu)}{k_+^2 - \mu} - \frac{1}{2} \left( \frac{\Re k_-}{k_-^2 - \mu} + \frac{1}{\sqrt{\mu}} \tanh^{-1} \frac{\Re k_-}{\sqrt{\mu}} \right)$$

$$+ 2 \int_0^{\Re k_-} + \left[ \int_{k_-}^{\infty} l^2 dl \left( \frac{n_F(l^2 - \mu)}{(l^2 - \mu)^2} + \frac{n'_F(l^2 - \mu)}{(l^2 - \mu)^2} + \frac{3}{2} l^2 n''_F(l^2 - \mu) \right) \right]$$

$$= \frac{\Re k_-}{3} \left( \frac{2n'_F(k_+^2 - \mu)}{k_+^2 - \mu} + \frac{1}{(k_+^2 - \mu)^2} \right) \left( k_+^3 \frac{1 - 2n_F(k_+^2 - \mu)}{(k_+^2 - \mu)^2} \right) \right]. \tag{A.7}$$

The large-$k$ behavior of this quantity reads as

$$S_k^{(2)} = \frac{1}{2} \left( \frac{\mu^3}{4k^3} \right) + O(1/k^5), \tag{A.8}$$

which implies that the effective mass of the two-particle resonance is nothing but twice the fermion mass, if $k$ is sufficiently large.

Although we are not using the PP-RPA estimate of $|\mu_k^B|$ in the text, we show it here for the sake of completeness:

$$|\mu_k^B| = \frac{1}{8\pi a_s} + \int \frac{d^3 l}{(2\pi)^3} \left( \frac{1 - 2n_F(l^2 - \mu) + R_k(I)}{2(l^2 - \mu) + R_k(I)} - \frac{1}{2l^2} \right), \tag{A.9}$$

whose large-$k$ asymptotic behavior is given by

$$\mu_k^B = k^2 - \mu - \frac{3\pi}{4a_s} \sqrt{k^2 + \mu} + O(1/k). \tag{A.10}$$

### Appendix B. Asymptotic behavior of the PP and PH contributions

Here we list the analytic formulas for the asymptotic $k$ behavior of the right-hand side of (18). Considering the $k$-derivative of the inverse coupling, $\partial_k(1/\Gamma_k^{(4)})$, the first term in the right-hand side of (18), the PP contribution, reads

$$\partial_k \int_{-l}^{(T)} \frac{1}{[G^{-1} + R_k(I)][G^{-1} + R_k(-l)]}$$

$$\approx \begin{cases} \left( \frac{m}{3\pi^2} + \frac{m k_F^2}{2\pi k^2} \right), & (k^2/2m \ll k_F^2/2m, \pi T), \\ \left( -\frac{m k_F}{\pi^2 k} \right), & (\pi T \ll k^2/2m \ll k_F^2/2km), \\ \left( -\frac{3k_F}{96\pi^2 m^2 T^3 k^5} \right), & (k^2/2m \ll k_F^2/2m, \pi T), \end{cases} \tag{B.1}$$
with $k_F^2 = 2m\mu$. The second term in the right-hand side of (18), the PH contribution, reads

$$
\partial_k \int \frac{d^2q}{4\pi} \frac{d^2q'}{4\pi} \int_l \left( \frac{1}{[G^{-1} + R_k(l)][G^{-1} + R_k(q - q' + l)]} \right) \left[ \begin{array}{l}
\frac{8m}{15\pi^2} k_F^3, \\
\frac{3m}{4\pi^2} k_F^2, \\
f(T/\mu) \frac{k^5}{2\pi^5 m k_F T^2},
\end{array} \right] \left( k^2/2m \gg k_F^2/2m, \pi T \right), \tag{B.2}
$$

where $f(x)$ is given by

$$
f(x) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left[ \int_0^2 \! dQ \left( \pi - \tan^{-1} \left( \frac{(2n-1)\pi x}{1 - (Q - 1)^2} \right) - \tan^{-1} \left( \frac{(2n-1)\pi x}{(Q + 1)^2 - 1} \right) \right) + \frac{(2n-1)\pi x}{16} \ln \left( 1 + \left( \frac{8}{(2n-1)\pi x} \right)^2 \right) \right]. \tag{B.3}
$$

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