Monopole type equations on compact symplectic 6-manifolds

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Abstract

In this article, we consider a gauge-theoretic equation on compact symplectic 6-manifolds, which forms an elliptic system after gauge fixing. This can be thought of as a higher-dimensional analogue of the Seiberg–Witten equation. By using the virtual neighbourhood method by Ruan [R], we define an integer-valued invariant, a 6-dimensional Seiberg–Witten invariant, from the moduli space of solutions to the equations, provided that the moduli space is compact; and it has no reducible solutions. We prove that the moduli space is compact if the underlying manifold is a compact Kähler threefold. We then compute the integers in some cases.

1 Introduction

Let $X$ be a compact symplectic 6-manifold with symplectic form $\omega$. We take an almost complex structure $J$ compatible with the symplectic form $\omega$. We fix a $\text{Spin}^c$-structure $c$ on $X$. We denote the characteristic line bundle for $c$ by $\xi$. Then there exists a line bundle $L$ such that $\xi = L^2 \otimes K_X^{-1}$, where $K_X^{-1}$ is the anti-canonical bundle of $X$. Let $A'$ be a connection on $\xi = L^2 \otimes K_X^{-1}$. We write $A' = A_c + 2A$, where $A_c$ is the canonical connection on $K_X^{-1}$, which is fixed, and $A$ is a connection of a line bundle $L$. We then consider the following equations on compact symplectic 6-manifolds (see Section 2 for more detail), seeking for a connection $A$ of $L$, $u \in \Omega^{0,3}(X)$, $\alpha \in C^\infty(L)$ and $\beta \in \Omega^{0,2}(L)$.

\[ \bar{\partial}_A \alpha + \bar{\partial}^* \alpha \beta = 0, \quad \bar{\partial}_A \beta = -\frac{1}{2} \alpha u, \]

\[ F_{A'}^{0,2} + \bar{\partial}^* u = \frac{1}{4} \alpha \beta, \quad \Lambda F_{A'}^{1,1} = -\frac{i}{8} (|u|^2 + |\beta|^2 - |\alpha|^2). \]

where $\Lambda = (\omega \wedge)^*$. 

Remark 1.1. Richard P. W. Thomas once considered similar equations in [T]. Our equations partially emerged out of discussion with Dominic Joyce around the end of 2010 together with the computation in the proof of Proposition 4.1.

These equations from an elliptic system with gauge fixing condition. We expect they enjoy nice properties similar to the original Seiberg–Witten equations such as the compactness of the moduli space.

In this article, we define an integer $n_X(c)$ for a $\text{Spin}^c$-structure $c$, a 6-dimensional Seiberg-Witten invariant, by using Ruan’s virtual neighbourhood method [R], provided that the moduli space is compact; and there are no reducible solutions. Examples are given in the Kähler case. We prove that the moduli spaces are compact if the underlying manifolds are compact Kähler threefolds. We then compute the integers in some cases as follows. These are analogies of those for the Seiberg–Witten invariants in 4 dimensions. Firstly, we have the following.

**Theorem 1.2** *(Corollary 4.10)*. Let $X$ be a compact Kähler threefold with $K_X < 0$, and let $c$ be a $\text{Spin}^c$-structure on $X$ with $\deg \xi < 0$, where $\xi$ is the characteristic line bundle of the $\text{Spin}^c$-structure. Then $n_X(c) = 0$.

For the case where $K_X > 0$, we get the following.

**Theorem 1.3** *(Theorem 4.11)*. Let $X$ be a compact Kähler threefold with $K_X > 0$. Let $s_c$ be the $\text{Spin}^c$-structure coming from the complex structure. We also assume that $c_2(X) = 0$. Then $n_X(s_c) = 1$.

The organisation of this article is as follows. In Section 2, we briefly describe $\text{Spin}^c$-structures and the Dirac operators on compact symplectic manifolds, and recall the Seiberg–Witten equation on compact symplectic 4-manifolds. Then we introduce our equation in six dimensions and describe its linearisation. In Section 3, we introduce an integer-valued invariant, which can be thought of as a 6-dimensional Seiberg–Witten invariant, from the moduli space of solutions to the equation by using Ruan’s virtual neighbourhood method. We then consider the Kähler case in Section 4 and compute the integers in some cases.

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2 Monopole type invariants for compact symplectic 6-manifolds

2.1 Spin\(^c\)-structure and the Dirac operator on compact symplectic manifolds

A general reference for Spin\(^c\)-structures and its Dirac operator is Lawson–Michelsohn [LM].

Spin\(^c\)-structure on compact symplectic manifolds. Let \(X\) be a compact symplectic manifold with symplectic form \(\omega\). We fix an almost complex structure \(J\) compatible with \(\omega\). Any almost complex manifold has the canonical Spin\(^c\) structure associated with the almost complex structure, whose spinor bundle is given by \(\bigoplus_i \Omega^{0,i}(X)\). The characteristic line bundle is given by \(K_X^{-1}\) if \(S\) is a spinor bundle on \(X\), then we can write

\[
S = \bigoplus_i \Omega^{0,i}(X) \otimes L, \quad S^+ = \bigoplus \Omega^{0,\text{even}}(X) \otimes L, \quad S^- = \bigoplus \Omega^{0,\text{odd}}(X) \otimes L,
\]

where \(L\) is some line bundle on \(X\). In particular, if underlying manifolds are symplectic 6-manifolds, we have \(S^+ = L \oplus (L \otimes \Omega^{0,2}(X))\).

The Dirac operator on symplectic manifolds. The Dirac operator \(D_{A'}\) associated to a connection \(A'\) on the characteristic line bundle \(\xi\) is given by the following composition.

\[
\Gamma(S) \xrightarrow{\nabla_{A'}} \Gamma(T^*X \otimes S) \xrightarrow{\text{metric}} \Gamma(TX \otimes S) \xrightarrow{\rho} \Gamma(S),
\]

where \(\rho\) is the Clifford multiplication. In almost complex case, it is written as \(D_A = \sqrt{2}(\bar\partial_A + \bar\partial_A^\ast)\), where \(A\) is a connection on \(L\).

2.2 The Seiberg–Witten equations on symplectic 4-manifolds

We recall the Seiberg–Witten equations in the original form first, which was introduced by Witten [W]. Let \(M\) be a compact, oriented, smooth 4-manifold. We fix a Riemannian metric and a Spin\(^c\) structure on \(M\). We denote by \(S^+\) the half spinor bundle over \(M\) associated to the Spin\(^c\)-structure, and by \(\xi\) the characteristic line bundle of the Spin\(^c\)-structure.

The Seiberg–Witten equations on \(M\) are equations seeking for a connection \(A'\) of \(\xi\) and a section of \(S^+\) satisfying the following.

\[
D_{A'}\psi = 0, \quad F_{A'}^+ = \frac{1}{4} \tau(\psi \otimes \psi^\ast),
\]
where \( D_{A'} \) is the Dirac operator associated to the connection \( A' \), \( F^+_{A'} \) is the self-dual part of the curvature \( F_{A'} \) of the connection \( A' \), \( \tau \) is a map \( \tau : \text{End} (S^+) \to \Lambda^+ \otimes \mathbb{C} \) defined by the Clifford multiplication on \( S^+ \).

We next consider these equations on a compact symplectic 4-manifold with symplectic structure \( \omega \) (see e.g. [HT] or [K] for more detail). We fix an almost complex structure compatible with \( \omega \). Then we have the following decomposition of the self-dual part of the curvature.

\[
F^+_{A'} = F_{A'}^{2,0} + F_{A'}^{0,2},
\]

where \( F_{A'}^{0,2} \) is the \( \omega \)-component of the curvature \( F_{A'} \). In addition, we can consider the canonical Spin\(^c\) structure whose characteristic line bundle is \( K^{-1}_M = \Lambda^0(T^*M \otimes \mathbb{C}) \). Using this canonical Spin\(^c\) structure, we can write the half-spinor bundle \( S^+ \) for any Spin\(^c\) structure as \( S^+ = L \oplus (L \otimes K^{-1}_M) \), where \( L \) is some complex line bundle on \( M \). We also have the canonical Spin\(^c\) connection \( A_c \) on \( K^{-1}_M \), and each connection \( A' \) of the characteristic line bundle is written by \( A' = A_c + 2A \), where \( A \) is a connection of \( L \).

We write a spinor as \( \psi = \varphi_0 u_0 + \varphi_2 \), where \( u_0 \) is a section which satisfies \( D_{A_c} u_0 = 0 \), and \( \varphi_0 \in \Gamma(L) \), \( \varphi_2 \in \Gamma(L \otimes K^{-1}) \). Then the Seiberg–Witten equations becomes as follows.

\[
\bar{\partial}_{A'} \varphi_0 + \bar{\partial}^*_{A'} \varphi_2 = 0,
\]

\[
F_{A'}^{0,2} = \frac{\varphi_0 \varphi_2}{2}, \quad \Lambda F_{A'}^{1,1} = -\frac{i}{4} (|\varphi_2|^2 - |\varphi_0|^2),
\]

where \( \Lambda := (\wedge \omega)^* \).

### 2.3 Equations in six dimensions

Let \( X \) be a compact symplectic 6-manifold with symplectic form \( \omega \). We fix an almost complex structure \( J \) compatible with \( \omega \). We take a Spin\(^c\)-structure \( s \) on \( X \), and denote by \( \xi \) the associated complex line bundle over \( X \).

There is a Spin\(^c\)-structure canonically determined by \( J \), which we denote by \( s_c \). The corresponding line bundle for \( s_c \) is given by the anti-canonical bundle \( K_X^{-1} \). For a Spin\(^c\)-structure \( s \), there is a complex line bundle \( L \), and the corresponding line bundle for \( s \), which we denote by \( \xi \), can be written as \( \xi = K_X^{-1} \otimes L^2 \), and a connection \( A' \) of \( \xi \) can be written as \( A' = A_c + 2A \), where \( A_c \) is a connection of \( K_X^{-1} \) and \( A \) is a connection of \( L \).

We consider the following equations for \( (A,u,(\alpha,\beta)) \), where \( A \) is a con-
connection of $L$, $u \in \Omega^{0,3}(X)$, $\alpha \in \Omega^0(X, L)$ and $\beta \in \Omega^{0,2}(X, L)$.

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0, \quad \bar{\partial}_A \beta = \frac{1}{2} \alpha u,$$

(2.1)

$$F_{\alpha'}^{0,2} + \bar{\partial}^* u = \frac{1}{4} \alpha \beta, \quad \Lambda F^{1,1}_{\alpha'} = - \frac{i}{8} (|u|^2 + |\beta|^2 - |\alpha|^2).$$

(2.2)

We call $\mathcal{G} := \Gamma(X, U(1))$ a gauge group. This is a set of all smooth $U(1)$-valued functions. This group acts on solutions to (2.1) and (2.2) by

$$A' \mapsto A' - \sigma^{-1} dg, \quad u \mapsto u, \quad \alpha \mapsto g \alpha, \quad \beta \mapsto g \beta,$$

where $g \in \mathcal{G}$. The equations (2.1) and (2.2) are equivariant under this action, namely, if $(A', u, (\alpha, \beta))$ is a solution to the equations (2.1) and (2.2), then so is $g(A', u, (\alpha, \beta))$ for any $g \in \mathcal{G}$. We say two solutions $(A'_1, u_1, (\alpha_1, \beta_1)), (A'_2, u_2, (\alpha_2, \beta_2))$ are gauge equivalent if there exists a gauge transformation $g \in \mathcal{G}$ such that $(A'_1, u_1, (\alpha_1, \beta_1)) = g(A'_2, u_2, (\alpha_2, \beta_2))$.

As in the Seiberg–Witten case, the stabilizer in $\mathcal{G}$ of $(A, u, (\alpha, \beta)) \in \mathcal{C}$ is trivial unless $\alpha = \beta = 0$. We then define the following.

**Definition 2.1.** $(A', u, (\alpha, \beta))$ is said to be reducible if $(\alpha, \beta) \equiv 0$. It is called irreducible otherwise.

Note that the stabilizer group in the case of reducibles is the group of constant maps from $X$ to $S^1$, namely, it is $S^1$.

### 2.4 Linearisation

The linearisation of the equation (2.2) fits into the following Atiyah–Singer–Hitchin type complex.

$$0 \to \Omega^0(X, i\mathbb{R}) \to \Omega^1(X, i\mathbb{R}) \oplus \Omega^{0,3}(X) \to \Omega^+(X, i\mathbb{R}) \to 0,$$

where $\Omega^+(X, i\mathbb{R}) := \Omega^0(X, i\mathbb{R})\omega \oplus \Omega^2(X, i\mathbb{R}) \cap (\Omega^{2,0} \oplus \Omega^{0,2})$. This is an elliptic complex, and the index of this can be computed by the following Dolbeault complex.

$$0 \to \Omega^{0,0}(X) \to \Omega^{0,1}(X) \to \Omega^{0,2}(X) \to \Omega^{0,3}(X) \to 0.$$  

(2.3)

Here we identified $\Omega^0(X) \oplus \Omega^0\omega$ with $\Omega^{0,0}(X)$ and $\Omega^1(X)$ with $\Omega^{0,1}(X)$.

On the other hand, the linearisation of (2.1) is given by

$$0 \to \Omega^{0,0}(X, L) \oplus \Omega^{0,2}(X, L) \to \Omega^{0,1}(X, L) \oplus \Omega^{0,3}(X, L) \to 0.$$  

(2.4)

Hence we obtain the following.
Proposition 2.2. The virtual dimension of the moduli space $\mathcal{M}$ is given by

$$-\frac{1}{12}c_1(X)c_2(X) - \frac{1}{24}c_1(L)\left(2c_1(X)^2 + 2c_2(X) + 6c_1(L)c_1(X) + 4c_1(L)^2\right).$$

(2.5)

**proof.** The virtual dimension is the sum of the indices of (2.3) and (2.4) with the opposite signs. By the index formula, it is

$$-\int_X \text{td}(X) - \int_X \text{ch}(L) \cdot \text{td}(X).$$

Here, $\text{td}(X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}c_1(X)^2 + \frac{1}{24}c_1(L)c_1(X) + \frac{1}{6}c_1(L)^3$, $\text{ch}(L) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \frac{1}{6}c_1(L)^3$. Thus, we get (2.5). $\square$

3 Invariant

Let $X$ be a compact symplectic 6-manifold with symplectic form $\omega$. We take an almost complex structure compatible with $\omega$, and a $\text{Spin}^c$-structure $c$ on $X$ with the characteristic line bundle $\xi$ being $K_X^{-1} \otimes L^2$, where $L$ is a line bundle on $X$.

**Moduli space.** We consider the following Sobolev completion of the configuration space.

$$\mathcal{C} := \mathcal{A}_{L^2_2} \times L^2_2(\Lambda^{0,3}) \times L^2_2((\Lambda^{0,0} \oplus \Lambda^{0,2}) \otimes L),$$

where $\mathcal{A}_{L^2_2}$ is the space of $L^2_2$-connections on $\xi$, and $L^2_2$-completion of the space of gauge group $\mathcal{G} := \text{Map}(X, U(1))$ to get smooth action on the configuration space $\mathcal{C}$. We then take the quotient

$$\mathcal{M} := \{(A, u, (\alpha, \beta)) \in \mathcal{C} : (A, u, (\alpha, \beta))\text{satisfies (2.1) and (2.2)}\}/\mathcal{G},$$

and call it the *moduli space* of solutions to the equations (2.1) and (2.2).

**Virtual Neighbourhood.** We consider the following case.

(A1) The moduli space is compact; and

(A2) there are no reducible solutions.
Assuming the above (A1) and (A2), one can invoke the virtual neighbourhood method by Ruan [R] to define an integer-valued invariant from the moduli space $\mathcal{M}$, since a triple $\left( \mathcal{B} := \mathcal{C}/\mathcal{G}, \mathcal{F}, F \right)$, where $\mathcal{F} := L^2_1(\Lambda^{0,1} \otimes \Lambda^{0,3}) \otimes L$, and $F : \mathcal{B} \to \mathcal{F}$ defined by the equations (2.1) and (2.2), forms a compact-smooth triple of $\mathcal{R}$ Def. 2.1.

From a compact-smooth triple, one can construct a virtual neighbourhood $\left( U, R^k, S \right)$ of $\mathcal{M}$ with $U$ being a smooth neighbourhood of dimension $-\text{ind}(L) + k$, where $L$ is the linearised operator of the equations (2.1) and (2.2), and $k \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{M} \times \{ 0 \} \subset U \subset \mathcal{B} \times \mathbb{R}^k$, and $S : U \to \mathbb{R}^k$ defined by the equations (2.1) and (2.2), forms a compact-smooth triple of $\mathcal{R}$ [Def. 2.1].

Invariants. We denote by $\mathcal{C}^*$ the open subset of $\mathcal{C}$ consisting of irreducible equivalence classes. We consider the subgroup $\mathcal{G}_0$ of $\mathcal{G}$ consisting of all gauge transformations which are trivial on the fibre over a fixed point $x \in X$. This is the kernel of the morphism $\mathcal{G} \to S^1$ defined by evaluating on the fibre over $x$. We then consider the quotient $\mathcal{B}^0 := \mathcal{C}^*/\mathcal{G}_0$. This is the total space of a principal $S^1$-bundle, we denote it by $\ell$, over $\mathcal{B}^*$. Then we define an integer $n_X(c)$ by (i) $\mu_F$ if $\text{ind}(L) = 0$; (ii) $\mu_F(c_1(\ell)^{-\text{ind}(L)/2})$ if $-\text{ind}(L) > 0$; and (iii) 0 if $-\text{ind}(L) < 0$.

Examples are given in the next section.

4 Invariants for compact Kähler threefolds

We describe the equation on compact Kähler threefolds in Section 4.1. We then prove the compactness of the moduli spaces in the Kähler case in Section 4.2. In Section 4.3 we compute the integers $n_X(c)$ defined in Section 3 in some cases.

4.1 The equations on compact Kähler threefolds

Firstly, we have the following.
Proposition 4.1. Let $X$ be a compact Kähler threefold. Then the equations (2.1) and (2.2) reduce to the following.

\begin{align}
\bar{\partial}_A\alpha &= \bar{\partial}_A \beta = \bar{\partial}_A^* \beta = \alpha u = 0, \\
F^{0,2}_{A'} &= \bar{\partial}^* u = \bar{\alpha} \beta = 0, \quad i\Lambda F^{1,1}_{A'} = \frac{1}{8} (|u|^2 + |\beta|^2 - |\alpha|^2) \tag{4.2}
\end{align}

**proof.** Using the second equation in (2.1), we get

\begin{align}
||\bar{\partial}_A\beta||^2_{L^2} &= \langle \beta, \bar{\partial}_A^* \bar{\partial}_A \beta \rangle_{L^2} \\
&= \frac{1}{2} \langle \beta, -\bar{\partial}_A^*(\alpha u) \rangle_{L^2} \\
&= -\frac{1}{2} \langle \beta \wedge \bar{\partial}_A \bar{\alpha}, u \rangle_{L^2} - \frac{1}{2} \langle \beta, (\bar{\partial}_A^* u) \alpha \rangle_{L^2}. \tag{4.3}
\end{align}

The second term in the last line of the above (4.3) can be computed as follows.

\begin{align}
\langle \beta, (\bar{\partial}_A^* u) \alpha \rangle_{L^2} &= \langle \beta, -F^{0,2}_{A'} \alpha \rangle_{L^2} + \frac{1}{4} \langle ||\alpha|| \beta \rangle^2_{L^2} \\
&= \langle \beta, -\bar{\partial}_A \bar{\partial}_A \alpha \rangle_{L^2} + \frac{1}{4} \langle ||\alpha|| \beta \rangle^2_{L^2} \\
&= \langle \beta, \bar{\partial}_A \bar{\partial}_A^* \beta \rangle_{L^2} + \frac{1}{4} \langle ||\alpha|| \beta \rangle^2_{L^2} \\
&= \langle \bar{\partial}_A^* \beta \rangle^2_{L^2} + \frac{1}{4} \langle ||\alpha|| \beta \rangle^2_{L^2}. \tag{4.4}
\end{align}

On the other hand, from the first equation in (2.2) and the identity $\bar{\partial}_A F^{0,2}_{A'} = 0$ which holds for an integrable complex structure, we get

$$\bar{\partial} \bar{\partial}^* u = \frac{1}{4} (\bar{\partial}_A \bar{\alpha}) \beta + \frac{1}{4} \bar{\alpha} \bar{\partial}_A \beta.$$ 

From this we obtain

\begin{align}
||\bar{\partial}^* u||^2_{L^2} &= \langle u, \bar{\partial} \bar{\partial}^* u \rangle_{L^2} \\
&= \frac{1}{4} \langle u, (\bar{\partial}_A \bar{\alpha}) \wedge \beta \rangle_{L^2} + \frac{1}{4} \langle u, \bar{\alpha} \bar{\partial}_A \beta \rangle_{L^2} \\
&= \frac{1}{4} \langle u, (\bar{\partial}_A \bar{\alpha}) \wedge \beta \rangle_{L^2} - \frac{1}{8} \langle ||\alpha||u \rangle^2_{L^2}. \tag{4.5}
\end{align}

Hence, from (4.3), (4.4) and (4.5), we get

$$||\bar{\partial}_A \beta||^2_{L^2} + 2||\bar{\partial}^* u||^2_{L^2} + \frac{1}{4} \langle ||\alpha||u \rangle^2_{L^2} + \frac{1}{2} \langle ||\bar{\partial}_A^* \beta \rangle^2_{L^2} + \frac{1}{8} \langle ||\alpha|| \beta \rangle^2_{L^2} = 0.$$ 

Thus, the assertion holds. \qed
We define the degree of $\xi$ by
\[
\deg \xi := c_1(\xi) \cdot [\omega^2] = \frac{i}{2\pi} \int_X F_{A'} \wedge \omega^2.
\]

**Proposition 4.2.** Let $X$ be a compact Kähler threefold, and let $\xi$ be the characteristic line bundle of a Spin$^c$-structure on $X$. Let $(A, u, (\alpha, \beta))$ be a solution to the equations (2.1) and (2.2). Then the following holds.

(i) If $\deg \xi < 0$, then $\beta \equiv 0$ and $u \equiv 0$.

(ii) If $\deg \xi > 0$, then $\alpha \equiv 0$.

(iii) If $\deg \xi = 0$, then $\alpha \equiv 0, \beta \equiv 0, u \equiv 0$.

**proof.** From $\bar{\alpha} \beta = 0$, either $\alpha$ or $\beta$ is zero on some open subset of $X$, thus either $\alpha$ or $\beta$ is zero on the whole of $X$ by unique continuation as $\bar{\partial}_A \alpha = 0$ and $\bar{\partial}_A \beta = \bar{\partial}_A^* \beta = 0$. Similarly, from $\alpha u = 0$, either $\alpha$ or $u$ is zero on an open set in $X$, so either $\alpha$ or $u$ is zero on $X$ again by unique continuation as $\bar{\partial}_A \alpha = 0$ and $\bar{\partial}^* u = 0$. On the other hand, from the second equation in (4.2), we have
\[
\deg \xi = \frac{i}{2\pi} \int_X F_{A'} \wedge \omega^2 = \frac{1}{16\pi} \int_X (|u|^2 + |\beta|^2 - |\alpha|^2) \text{ vol}. \tag{4.6}
\]

Firstly, we consider the case $\deg \xi < 0$. In this case, because of (4.6), $\alpha \equiv 0$ contradicts $\deg \xi < 0$, thus we have $\alpha \not\equiv 0$. Then, from the above reasoning in the top of this proof, we get $\beta \equiv 0, u \equiv 0$.

Secondly, we consider the case $\deg \xi > 0$. In this case, if $\beta \equiv 0$ and $u \equiv 0$, we get a contradiction again from (4.6). Thus, $\beta \not\equiv 0$ or $u \not\equiv 0$, and therefore $\alpha \equiv 0$.

Finally, we consider the case $\deg \xi = 0$. If $\alpha \not\equiv 0$, then we get $\beta \equiv 0, u \equiv 0$; and this results in $\deg \xi < 0$. Hence $\alpha \equiv 0$. Then, as $\alpha \equiv 0$, again from (4.6), we get $\beta \equiv 0$ and $u \equiv 0$. 

From Proposition 4.2 (iii) above, we immediately get the following.

**Proposition 4.3.** Let $X$ be a compact Kähler threefold, and let $\xi$ be the characteristic line bundle of a Spin$^c$-structure on $X$. Let $(A, u, (\alpha, \beta))$ be a solution to the equations (2.1) and (2.2). Then, if $\deg \xi = 0$, the moduli space is isomorphic to $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$. 
proof. From Proposition 4.2 (iii), we get $\alpha \equiv \beta \equiv u \equiv 0$. Then, from the Hitchin–Kobayashi correspondence of the Hermitian–Einstein connection for line bundles, the moduli space $M$ is identified with the moduli space of holomorphic structures on $\xi$. Since $\xi$ is topologically trivial, it is then isomorphic to $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$.

\section{Compactness of the moduli space}

In this subsection, we prove that the moduli space $M$ of solutions to the equations is compact if the underlying manifold is a compact Kähler threefold.

Firstly, we consider the case $\deg \xi < 0$. In this case, from Proposition 4.2, we have $\beta \equiv 0$ and $u \equiv 0$. We also have the following point-wise estimate on $\alpha$.

\begin{lemma}
Let $X$ be a compact Kähler threefold, and let $\xi$ be the characteristic line bundle of a Spin$^c$-structure on $X$. Let $(A, u, (\alpha, \beta))$ be a solution to the equations (2.1) and (2.2). Then, if $\deg \xi < 0$ then $|\alpha| \leq C$, where $C$ is a positive constant which depends upon the curvature of the canonical connection on $K_X^{-1}$.
\end{lemma}

\begin{proof}
For $\alpha \in \Omega^{0,0}(X, L)$, we have the following from the Kähler identity (see e.g. [K, § 2.2] or [DK, § 6.1.3]).

$$\bar{\partial}_A^* \bar{\partial}_A \alpha = \frac{1}{2} \nabla_A^* \nabla_A \alpha - \frac{1}{2} i(\Lambda F_A) \alpha. \quad (4.7)$$

Then, using the equations (4.1) and (4.2), we obtain

$$0 = \frac{1}{2} \nabla_A^* \nabla_A \alpha - \frac{1}{2} \left( -\frac{1}{2} F_{K_X^{-1}} - \frac{1}{8} |\alpha|^2 \right) \alpha.$$

On the other hand, we have $\frac{1}{8} \Delta |\alpha|^2 = \langle \nabla_A^* \nabla_A \alpha, \alpha \rangle - |\nabla_A \alpha|^2$, where $\Delta$ is the Laplacian on functions. Thus, we get

$$\frac{1}{2} \Delta |\alpha|^2 + |\nabla_A \alpha|^2 = -\frac{1}{2} F_{K_X^{-1}} |\alpha|^2 - \frac{1}{8} |\alpha|^4.$$

From the maximum principle, if $x_0$ is a local maximum of $|\alpha|^2(x)$, we obtain

$$-\frac{1}{2} F_{K_X^{-1}}(x_0) |\alpha|^2(x_0) - \frac{1}{8} |\alpha|^4(x_0) \geq 0.$$

Thus, we get either $|\alpha|^2(x_0) = 0$ or $|\alpha|^2(x_0) \leq -4 F_{K_X^{-1}}(x_0)$. Hence the assertion holds.
\end{proof}
If $\deg \xi > 0$, from Proposition 4.2, we have $\alpha \equiv 0$. In this case, similar to Lemma 4.4, we have the following point-wise estimate on $\beta$.

**Lemma 4.5.** Let $X$ be a compact Kähler threefold, and let $\xi$ be the characteristic line bundle of a Spin$^c$-structure on $X$. Let $(A, u, (\alpha, \beta))$ be a solution to the equations (2.1) and (2.2). Then, if $\deg \xi > 0$ then $|\beta| \leq C$, where $C$ is a positive constant which depends upon the curvature of the canonical connection on $K_X^{-1}$ and of a connection on $\Lambda^{0,2}$.

**proof.** For $\beta \in \Omega^{0,2}(X, L)$, we have the following again from the Kähler identity:

$$\partial_A \bar{\partial}_A \beta + \bar{\partial}_A \partial_A \beta = \frac{1}{2} \bar{\partial}_A \bar{\partial}_A \beta + \frac{1}{2} i(\Lambda(F' + F_A))\beta,$$

(4.8)

where $\bar{\partial}_A$ is the unitary connection of $\Lambda^{0,2} \otimes L$, and $F' + F_A$ is the curvature of $\bar{\partial}_A$. Then, using the equations (4.1) and (4.2), we obtain

$$0 = \frac{1}{2} \bar{\partial}_A \bar{\partial}_A \beta + \frac{1}{2} \left( i(\Lambda F') - \frac{1}{2} F_{K_X}^{-1} + \frac{1}{8} (|u|^2 + |\beta|^2) \right) \beta,$$

Thus, again using the maximum principle, if $x_1$ is a local maximum of $|\beta|^2(x)$, then we get

$$-\frac{1}{2} \left( i(\Lambda F') - \frac{1}{2} F_{K_X}^{-1} \right) |\beta|^2(x_1) - \frac{1}{8} (|u|^2(x_1) + |\beta|^2(x_1)) |\beta|^2(x_1) \geq 0.$$

Thus, either $|\beta|^2(x_1) = 0$ or $|\beta|^2(x_1) + |u|^2(x_1) \leq -i \Lambda F' + \frac{1}{2} F_{K_X}^{-1}$. Hence the assertion holds.

With these above in mind we prove the following.

**Proposition 4.6.** Let $X$ be a compact Kähler threefold. Then, the moduli space is compact.

**proof.** From Lemmas 4.4 and 4.5, we have $L^\infty$-bounds for $\alpha$ or $\beta$. In addition, from (4.1) and the Weitzenböck formulas (4.7), (4.8), we obtain bounds on $||\nabla_A \alpha||_{L^2}$ and $||\bar{\nabla}_A \beta||_{L^2}$. By the equations, we also get an $L^2$-bound on the self-dual part of the curvature $F_A$. Then we obtain a bound also on $||F_A^+||_{L^2}$ as we have $c_1(\xi) = \frac{1}{4\pi} \left( ||F_A^+||_{L^2}^2 - ||F_A^-||_{L^2}^2 \right)$.

On the other hand, by using the equations again, we get an $L^2$-bounds on $dF_A^0$ as $||\nabla_A \alpha||_{L^2}$ and $||\bar{\nabla}_A \beta||_{L^2}$ are bounded. Thus, we get a bound on $||F_A^0||_{L^4}$.

With all these above and the Uhlenbeck gauge fixing theorem, one can invoke the standard bootstrapping for an elliptic system to obtain $C^\infty$-bound on $A, u, \alpha$ and $\beta$. 




4.3 The invariant in some cases

In Section 4.2, we proved that the moduli spaces are compact in the Kähler case. Hence the assumption (A1) in Section 3 is satisfied in this case. Regarding the assumption (A2) in Section 3 on reducibles, firstly, we have the following. This holds for compact symplectic 6-manifolds.

**Proposition 4.7.** Let $X$ be a compact symplectic 6-manifold, and threefold, and let $\xi$ be the characteristic line bundle of a Spin$^c$-structure on $X$. If $\deg \xi < 0$, then there are no reducible solutions to the equations.

**proof.** If $\alpha = \beta = 0$, then it contradicts $\deg \xi < 0$ as $\deg \xi = \frac{1}{16\pi} \int_X (|u|^2 + |\beta|^2 - |\alpha|^2) \text{vol}$. Thus, the assertion holds.

The following holds if the underlying manifolds are Kähler threefolds.

**Proposition 4.8.** Let $X$ be a compact Kähler threefold, and threefold, and let $\xi$ be the characteristic line bundle of a Spin$^c$-structure on $X$. If $\deg \xi > 0$ and $K_X < 0$, then there are no reducible solutions to the equations.

**proof.** If $\deg \xi > 0$, then $\alpha \equiv 0$ from Proposition 4.2. Since we assume that $K_X < 0$, we have $u \equiv 0$. Suppose now for a contradiction that $\beta \equiv 0$. Then we get $\deg \xi = 0$ again from (4.6). This is a contradiction. Hence $\beta \not\equiv 0$.

Therefore, in these cases, one can define the integers $n_X(c)$ in Section 3 from the moduli space $\mathcal{M}$. We compute some of them below. These are analogies of those for the Seiberg–Witten invariants, presented as Proposition 7.3.1 in [M].

Firstly, we have the following.

**Proposition 4.9.** Let $X$ be a compact Kähler threefold with $K_X < 0$, and let $c$ be a Spin$^c$-structure on $X$ with $\deg \xi < 0$, where $\xi$ is the characteristic line bundle of the Spin$^c$-structure. Then there are no solutions to the equation (2.1) and (2.2). Namely, the moduli space is empty in this case.

**proof.** Suppose for a contradiction that there is a solution $(A, u, (\alpha, \beta))$ to the equation (2.1) and (2.2). Since $\deg \xi < 0$, we get $\beta \equiv 0$ from Proposition 4.2. As $\alpha$ is a holomorphic section of $L$, we get $\deg L > 0$. However, this contradicts the fact that $L^2 = K_X \otimes \xi$ has negative degree. Thus, the assertion holds.

Hence, we get the following.
Corollary 4.10. Let $X$ be a compact Kähler threefold with $K_X < 0$, and let $c$ be a Spin$^c$-structure on $X$ with $\deg \xi < 0$, where $\xi$ is the characteristic line bundle of the Spin$^c$-structure. Then $n_X(c) = 0$.

For the case $K_X > 0$, we have the following.

Theorem 4.11. Let $X$ be a compact Kähler threefold with $c_2(X) = 0$. Let $s_c$ be the Spin$^c$-structure coming from the complex structure. Assume that $K_X > 0$. Then $n_X(s_c) = 1$.

proof. Since $s_c$ is the Spin$^c$-structure coming from the complex structure, the corresponding line bundle $\xi$ is $K_X^{-1}$. As we assume that $K_X > 0$, thus $\xi < 0$; and $\beta \equiv 0$ and $u \equiv 0$. Because $L$ is trivial in this case, we only have a solution $(A_0, \alpha_0)$, where $A_0$ is the canonical connection of $\xi$, and $\alpha_0$ is a non-zero, constant section of $L$. Thus the moduli space $M$ contains only a single point.

We then prove that the moduli space $M$ is smooth, and its dimension is zero. Firstly, the index $(2.5)$ vanishes, since we assume that $c_2(X) = 0$, and $c_1(L) = 0$ as $L$ is trivial. Thus, the dimension of the moduli space is zero, if it is smooth.

We next prove that the moduli space is actually smooth. The proof goes in a similar way of that presented in [M, pp. 119–121] for the Seiberg–Witten case except that we have the extra terms coming from $u$ in the equations.

Firstly, recall that the following elliptic complex of the Atiyah–Hitchin–Singer type associated to a solution to the equations $(2.1)$ and $(2.2)$.

\[
i \Omega^0(\mathbb{R}) \xrightarrow{L_1} i \Omega^1 \oplus \Omega^{0,3}(X) \oplus (\Omega^{0,0}(L) \oplus \Omega^{0,2}(L)) \xrightarrow{L_2} \left(i \Omega^2 \cap (\Omega^{0,2} \oplus \Omega^{2,0}) \oplus i \Omega^0 \omega \right) \oplus (\Omega^{0,1}(L) \oplus \Omega^{0,3}(L))\]

We denote by $H^0, H^1, H^2$ the cohomology of the above complex. We now consider the above $L_1$ and $L_2$ at $(A, 0, (\alpha, 0))$. Then they become as follows.

\[
L_1(ig) = (2idg, 0, -iag, 0) \\
L_2(h, v, (a, b)) = (P^+ d(ih) - \frac{1}{8} i \text{Re}(a\bar{\alpha}) \omega + \bar{\partial}^* v + \frac{1}{4}(a\bar{b} - \bar{a}b), \partial a + \partial^* b + \pi^0.1(ih) \alpha/2, \bar{\partial}b + \alpha v/2).
\]

Firstly, since $\alpha$ is a non-zero constant section, the kernel of $L_1$ is trivial. Thus, $H^0 = 0$.

We next assume that $L_2(h, v, (a, b)) = 0$. Then, from the second component of $\bar{\partial}L_2(h, v, (a, b)) = 0$, we get $\bar{\partial}\bar{\partial} b + \frac{1}{2} \bar{\partial}(\pi^0.1(ih) \alpha) = 0$, where we used $\bar{\partial}\partial = 0$ and $\partial \alpha = 0$. 
On the other hand, we have $\bar{\partial}(\pi^{0,1}(ih)) = (d(ih))^{0,2} = -\bar{\partial}^*v + \bar{\alpha}b$. In addition, from the third component of $\bar{\partial}^*L_2(h,v,(a,b))$, we obtain $\bar{\partial}^*\bar{\partial}b + \frac{1}{2}\alpha\bar{\partial}^*v = 0$. Thus, we get

$$\bar{\partial}\bar{\partial}^*b + \bar{\partial}^*\bar{\partial}b + \frac{1}{4}|\alpha|^2b = 0.$$ 

By taking $L^2$-inner product of this with $b$, we get $||\bar{\partial}^*b||^2_{L^2} + ||\bar{\partial}b||^2_{L^2} + \frac{1}{4}||\bar{\alpha}b||^2_{L^2} = 0$. Thus, $\bar{\partial}^*b = \bar{\partial}b = 0, \bar{\alpha}b = 0$. As $\alpha \neq 0$, we get $b \equiv 0$ by the unique continuation.

We next write $ih = \bar{\chi} - \chi$ by some $\bar{\chi} \in \Omega^{0,1}(X,\mathbb{C})$. Then $L_2(ih,v,(a,b)) = 0$ becomes

$$P^+(\partial\bar{\chi} - \bar{\partial}\chi) - \frac{i}{2}\text{Re}(a\bar{\alpha})\omega + \bar{\partial}^*v = 0,$$ 

$$\bar{\partial}a + \frac{1}{2}\bar{\chi}\alpha = 0, \quad \alpha v = 0.$$ 

We now write $a = (p + iq)\alpha$, where $p$ and $q$ are real-valued functions. Then by adding $L_a(iq)$ to $(ih,v,(a,b))$, we can make $a = p\alpha$. By this, the first equation of (4.9) is now

$$\bar{\partial}(p\alpha) + \frac{1}{2}\bar{\chi}\alpha = 0.$$ 

From this, $\bar{\chi} = -2\bar{\partial}(p\alpha)$. Putting this into (4.10), we get $4\Delta p + p = 0$. Thus, we get $p = 0$. Hence $ih = 0$ and $v = 0$. Thus, the kernel of $L_2$ is in the image of $L_1$. Therefore $H^1 = 0$.

As we mentioned in the second paragraph of this proof, the index of the elliptic complex is zero; and since $H^0 = H^1 = 0$ as computed above, we conclude that $H^2 = 0$. Thus, $n_X(s_c) = \pm 1$.

One can easily check the sign to be positive following [M]. We omit the detail.

References

[DK] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, 1990.

[HT] M. Hutchings and C.H. Taubes, *An introduction to the Seiberg-Witten equations on symplectic four-manifolds*, Symplectic geometry and topology (Park City, UT, 1997), 103–142, Amer. Math. Soc., 1999.
[K] D. Kotschick, *The Seiberg–Witten invariants of symplectic four-manifolds (after C. H. Taubes)*, Séminaire Bourbaki, Vol 1995/96. Astérisque 241 (1997), 195–220.

[LM] H. B. Lawson, Jr. and M-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton NJ, 1989.

[M] J. W. Morgan, *The Seiberg–Witten equations and applications to the topology of smooth of smooth four-manifolds*, Mathematical Notes, 44, Princeton University Press NJ, 1996.

[R] Y. Ruan, *Virtual neighborhoods and monopole equations* in “Topics in Symplectic 4-manifolds”, Int. Press, Cambridge, MA, 1998.

[T] R. P. W. Thomas, *The Seiberg–Witten equations on complex 3-folds*, First Year dissertation, University of Oxford, 1995.

[W] E. Witten, *Monopoles and 4-manifolds*, Math. Res. Letters 1 (1994), 764–796.

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