Recurrent Lorentzian Weyl spaces and Riccati equation

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Abstract. We describe the local form of all non-closed Lorentzian Weyl structures $(M, c, \nabla)$ with recurrent curvature tensor. If the dimension of the manifold is bigger than 3, then the conformal structure is flat, and recurrent structures are in one-to-one correspondence with the solutions to the Riccati equation. The recurrent curvature tensor turns out to be a weighted parallel tensor.

Keywords: Weyl connection; recurrent curvature; holonomy group; Riccati equation.

1. Introduction and Main results

Let $(M, c, \nabla)$ be a Weyl manifold of Lorentzian signature, i.e., $c$ is a conformal class of Lorentzian metrics on a smooth manifold $M$, and $\nabla$ is a torsion-free affine connection on $M$ such that, for each $g \in c$, there exists a 1-form $\omega$ with

$$\nabla g = 2\omega \otimes g.$$  \hfill (1)

A metric $g \in c$ and the corresponding 1-form $\omega$ determine the connection $\nabla$, it holds

$$\nabla = \nabla^g + K, \quad g(K_X(Y), Z) = g(Y, Z)\omega(X) + g(X, Z)\omega(Y) - g(X, Y)\omega(Z),$$  \hfill (2)

where $\nabla^g$ is the Levi-Civita connection of the metric $g$, and $X, Y, Z$ are vector fields on $M$. The attention to Lorentzian Wayl structures is payed by many reasons in various recent works, e.g., [1, 4, 8, 10].

Let $R$ denote the curvature tensor of the affine connection $\nabla$. The aim of the paper is to give a description of the Lorentzian Weyl structures with recurrent curvature tensors $R$, i.e., satisfying

$$\nabla R = \theta \otimes R$$  \hfill (3)

for a 1-form $\theta$. A Weyl structure is called closed if $d\omega = 0$ for a $g \in c$ (equivalently, for all $g \in c$). If the structure is closed, then $\nabla$ is locally the Levi-Civita connection for some metric from the conformal class. The structure of Lorentzian recurrent spaces is known [12].

In what follows the dot over the function denotes the partial derivative in the direction of the coordinate $u$.

Theorem 1. Let $(M, c, \nabla)$ be a connected non-closed Lorentzian Weyl structure of dimension $n + 2 \geq 4$ with recurrent curvature tensor. Then the conformal class $c$ is flat; around each point of $M$ there exist coordinates $v, x^1, \ldots, x^n, u$ such that the class $c$ is represented by the metric

$$g = 2dvdu + \sum_{i=1}^{n-1} (dx^i)^2 + e^{-2F}(dx^n)^2 + a(u) \sum_{i=1}^{n-1} (x^i)^2(du)^2,$$

where

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the corresponding 1-form \( \omega \) is given by 
\[
\omega = \dot{F}du, 
\]
where \( F = F(x^n, u) \) is a function such that \( \dot{F} \) satisfies the Riccati 
\[
\dot{F} - \dot{F}^2 = -a(u) 
\]
and such that \( \partial_u \dot{F} \) is non-vanishing.

Let \( G = G(u, c) \) be the general solution of the Riccati equation \( \dot{G} - G^2 = -a(u) \), where \( c \) is the constant of integration, then the function \( F \) may be found form the condition \( \dot{F}(x^n, u) = G(u, c(x^n)) \), where \( c(x^n) \) is a function such that \( c'(x^n)\partial_x\dot{G} \) is non-vanishing. It is remarkable that the Riccati equation is also related to the Einstein-Weyl equation in dimension 2 \[2\] \[3\]. Similarly, the Einstein-Weyl equation in Lorentzian signature and dimension 3 is equivalent to the dKP equation, and that has led to many interesting results \[1\] \[6\] \[7\] \[8\].

**Theorem 2.** Let \((M, c, \nabla)\) be a connected non-closed Lorentzian Weyl structure of dimension 3 with recurrent curvature tensor. Then one of the following holds:

- around each point of \( M \) there exist coordinates \( v, x, u \) such that \( c = [g] \),
  \[
g = 2dvdu + e^{-2F}(dx)^2, \quad \omega = \dot{F}du, 
\]
  where \( F = F(x, u) \) is an arbitrary function with non-vanishing \( \partial_x \dot{F} \);
- around each point of \( M \) there exist coordinates \( v, x, u \) such that \( c = [g] \),
  \[
g = 2dvdu + (dx)^2 + H(du)^2, \quad \omega = a(u)xdu, 
\]
  \[
H = H(v, x, u) = a(u)vx + \frac{1}{12}a^2(u)x^4 - \frac{1}{3}a'(u)x^3 + c(u)x, 
\]
  where the function \( a(u) \) is non-vanishing.

In the settings of Theorem \[1\] the curvature tensor \( R \) satisfies
\[
\nabla R = \theta \otimes R, \quad \theta = \partial_u(F + \ln |\partial_u \dot{F}|)dx^n. 
\]

It holds \( d\theta = -3d\omega \), and
\[
\theta = -3\omega + d\varphi, \quad \varphi = F + \ln |\partial_u \dot{F}|. 
\]

Consider the metric
\[
\tilde{g} = e^{-\frac{1}{3}\varphi}g.
\]
Then it holds
\[
\nabla \tilde{g} = \tilde{\omega} \otimes \tilde{g}, \quad \nabla R = -3\tilde{\omega} \otimes R, \quad \tilde{\omega} = \omega - \frac{1}{3}d\varphi = -\frac{1}{3}\theta.
\]

Consider the density bundle with weight \( w \),
\[
\mathcal{L}^w = P_{CO} \times_{|\det \Omega|^\frac{1}{|\det \Omega|}} \mathbb{R},
\]
where \( P_{CO} \) is the conformal frame bundle. Each metric \( g \in c \) defines the section
\[
l_g = |\text{vol} g|^{-1} \tilde{\omega} \in \mathcal{L}^1
\]
satisfying
\[
\nabla l_g = \omega \otimes l_g.
\]
We conclude that
\[
\nabla (R \otimes l_g^3) = 0,
\]
i.e., the tensor field \( R \) is weighted parallel with the weight 3.

The curvature tensor of the first Weyl structure from Theorem \[2\] satisfies
\[
\nabla R = \theta R, \quad \theta = \partial_x(F + \ln |\partial_x \dot{F}|)dx + \partial_u(-2F + \ln |\partial_x \dot{F}|)du.
\]
Again we obtain \( \nabla (R \otimes l_g^3) = 0 \) for \( \tilde{g} = e^{-\frac{1}{3}\varphi}g, \ \varphi = F + \ln |\partial_x \dot{F}|. \)
For the second structure from Theorem 2 it holds\[
\nabla R = \theta \otimes R, \quad \theta = \left( \partial_u \ln |a(u)| - \frac{5}{2}a(u)x \right) du.
\]
In this case, \(\nabla (R \otimes \tilde{g}) = 0\) for \(\tilde{g} = e^{-\frac{5}{2}\varphi}g, \varphi = \ln |a(u)|\).

We conclude that

**Corollary 1.** If the curvature tensor of a non-closed Weyl Lorentzian structures is recurrent, then it is weighted parallel.

**Corollary 2.** Let \((M, c, \nabla)\) be a Lorentzian Weyl structure with \(\nabla R = 0\), then the structure is closed.

Non-closed Weyl structures with weighted parallel curvature tensors provide generalization of pseudo-Riemannian symmetric spaces, this is probably related to the work [9].

Finally note that the obtained spaces have another remarkable property: they admit a rather big number of weighted parallel spinors [5].

**Corollary 3.** Let \((M, c, \nabla)\) be a non-closed recurrent Lorentzian Weyl spin structure on a simply connected manifold \(M\) of dimension \(n + 2 \geq 4\), then it admits the space of weighted parallel spinors of complex dimension \(2\sqrt{2}\).

A similar result is know in Lorentzian geometry [11]: each simply connected indecomposable Cahen-Wallach spin manifold of dimension \(n + 2\) admits the space of parallel spinors of complex dimension \(2\sqrt{2}\) (Cahen-Wallach spaces are symmetric Lorentzian spaces represented by a special class of pp-waves).

2. The holonomy algebra of recurrent Lorentzian Weyl spaces

Let us first recall the classification of the holonomy algebras of Lorentzian Weyl structures [4]. Let \((M, c, \nabla)\) be a Weyl manifold of Lorentzian signature \((1, n + 1), n \geq 1\). Then its holonomy algebra is contained in the conformal Lorentzian algebra

\[\mathfrak{co}(1, n + 1) = \mathbb{R}
d_{1,n+1} \oplus \mathfrak{so}(1, n + 1).\]

If the Weyl structure is closed, then the holonomy algebra is contained in \(\mathfrak{so}(1, n + 1)\) and it is well-studied [11]. By that reason we suppose that the Weyl structure is non-closed, and the holonomy algebra is not contained in \(\mathfrak{so}(1, n + 1)\).

Fix a Witt basis \(p, e_1, \ldots, e_n, q\) of the Minkowski space \(\mathbb{R}^{1,n+1}\). With respect to that basis the subalgebra of \(\mathfrak{so}(1, n + 1)\) preserving the null line \(\mathbb{R}p\) has the following matrix form:

\[
\mathfrak{so}(1, n + 1)_{\mathbb{R}p} = \left\{ \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} : \begin{array}{c} a \in \mathbb{R} \\
A \in \mathfrak{so}(n) \\
X \in \mathbb{R}^n \end{array} \right\}.
\]

We identify the Lie algebra \(\mathfrak{so}(1, n + 1)\) with the space of bivectors \(\wedge^2 \mathbb{R}^{1,n+1}\) in such a way that

\[(X \wedge Y)Z = (X, Z)Y - (Y, Z)X.\]

Under this identification the above element of \(\mathfrak{so}(1, n + 1)_{\mathbb{R}p}\) corresponds to

\[-ap \wedge q + A - p \wedge X.\]

We get the decomposition

\[\mathfrak{so}(1, n + 1)_{\mathbb{R}p} = (\mathbb{R}p \wedge q \oplus \mathfrak{so}(n)) \ltimes p \wedge \mathbb{R}^n.\]

The holonomy algebra \(\mathfrak{g} \subset \mathfrak{co}(1, n + 1)\) of a non-closed Lorentzian Weyl structure of dimension \(n + 2, n \geq 1\), satisfies one of the following 3 conditions:

1. \(\mathfrak{g} \subset \mathfrak{co}(1, n + 1)\) is irreducible. In this case \(\mathfrak{g} = \mathfrak{co}(1, n + 1)\).
2. \( g \subseteq \mathfrak{co}(1, n + 1) \) preserves an orthogonal decomposition
\[
\mathbb{R}^{1,n+1} = \mathbb{R}^{1,k+1} \oplus \mathbb{R}^{n-k}, \quad -1 \leq k \leq n - 1.
\]
In this case \( g \) is one of the following:
- \( \mathbb{R} \operatorname{id}_{\mathbb{R}^{1,n+1}} \oplus \mathfrak{so}(1, k + 1) \oplus \mathfrak{so}(n - k), \quad -1 \leq k \leq n - 1; \)
- \( \mathbb{R} (\operatorname{id}_{\mathbb{R}^{1,n+1}} + p \wedge q) \oplus \mathfrak{t} \oplus \mathfrak{so}(n - k) \ltimes p \wedge \mathbb{R}^{k} \subseteq \mathfrak{co}(1, n + 1)_{\mathbb{R}^{p}}, \quad 0 \leq k \leq n - 1; \)
- \( \mathbb{R} \operatorname{id}_{\mathbb{R}^{1,n+1}} \oplus \mathfrak{t} \oplus \mathfrak{so}(n - k) \ltimes p \wedge \mathbb{R}^{k} \subseteq \mathfrak{co}(1, n + 1)_{\mathbb{R}^{p}}, \quad 1 \leq k \leq n - 1. \)
Here \( \mathfrak{t} \subseteq \mathfrak{so}(k) \) is the holonomy algebra of a Riemannian manifold.

3. \( g \) is contained in \( \mathfrak{co}(1, n + 1)_{\mathbb{R}^{p}}, \) and it does not preserves any proper non-degenerate subspace of \( \mathbb{R}^{1,n+1}. \) Such algebras may be divided into 6 types. For the results of the current paper it is enough to know that each such \( g \) contains the ideal \( p \wedge \mathbb{R}^{n}. \)

**Theorem 3.** Let \( (M, c, \nabla) \) be a recurrent non-closed Lorentzian Weyl structure of dimension \( n + 2 \geq 3, \) then one of the following conditions holds:

- \( n = 1 \) and the holonomy algebra of \( \nabla \) is one of the following
  \[
  g = \mathbb{R} \operatorname{id}_{\mathbb{R}^{1,2}} + p \wedge q, \quad \mathbb{R} (2 \operatorname{id}_{\mathbb{R}^{1,2}} + p \wedge q) \ltimes p \wedge \mathbb{R} e_{1};
  \]
- \( n \geq 2 \) and the holonomy algebra of \( \nabla \) is
  \[
  \mathbb{R} (\operatorname{id}_{\mathbb{R}^{1,n+1}} + p \wedge q) \ltimes p \wedge \mathbb{R}^{n-1} \subseteq \mathfrak{co}(1, n + 1)_{\mathbb{R}^{p}}.
  \]

**Proof of Theorem** Let \( (M, c, \nabla) \) be a recurrent non-closed Lorentzian Weyl structure and let \( g \subseteq \mathfrak{co}(1, n + 1) \) be its holonomy algebra at a point \( x \in M. \) Let us denote the tangent space \( T_{x}M \) by \( \mathbb{R}^{1,n+1}. \) Suppose that \( R_{x} \neq 0. \) Since \( R \) is recurrent, from the Ambrose-Singer Theorem it follows that
\[
(4) \quad g = \text{span}\{R_{x}(X, Y) | X, Y \in \mathbb{R}^{1,n+1}\}.
\]
Consider the space of algebraic tensors of type \( g \):
\[
\mathcal{R}(g) = \{ R \in \text{Hom}(\wedge^{2}\mathbb{R}^{1,n+1}, g) | R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \quad \forall X, Y, Z \in \mathbb{R}^{1,n+1}\}.
\]
From the Bianchi identity it follows that \( R_{x} \in \mathcal{R}(g). \) The holonomy algebra \( g \) acts on the space \( \mathcal{R}(g) \) in the following natural way
\[
(5) \quad \xi : R \mapsto \xi \cdot R, \quad (\xi \cdot R)(X, Y) = [\xi, R(X, Y)] - R(\xi X, Y) - R(X, \xi Y).
\]
Since the curvature tensor \( R \) is recurrent, by the holonomy principle, \( g \) preserves the line \( \mathbb{R}R_{x}, \) i.e., there exists a 1-form \( \rho \) on \( g \) such that
\[
(6) \quad \xi \cdot R_{x} = \rho(\xi)R_{x} \quad \forall \xi \in g.
\]
Now we are going to describe the holonomy algebras \( g \) of non-closed Weyl connections and elements \( R \in \mathcal{R}(g) \) satisfying the properties (4) and (5). Consider the 3 possibilities for \( g \) as above.

**Case 1.** Suppose that \( g = \mathfrak{co}(1, n + 1)_{\mathbb{R}^{p}} \) does not preserves any proper non-degenerate subspace of \( \mathbb{R}^{1,n+1}. \) There exists an isomorphism of \( \mathfrak{so}(1,n+1) \)-modules
\[
\mathcal{R}(\mathfrak{co}(1, n + 1)) \cong \mathcal{R} (\mathfrak{so}(1, n + 1)) \oplus \wedge^{2}\mathbb{R}^{1,n+1},
\]
where an element \( A \in \wedge^{2}\mathbb{R}^{1,n+1} \) defines an algebraic curvature tensor \( R_{A} \) by the equality
\[
(7) \quad R_{A}(X, Y) = AX \wedge Y + X \wedge AY + 2(A X, Y) id_{\mathbb{R}^{1,n+1}}.
\]
Let \( R \in \mathcal{R}(\mathfrak{co}(1, n + 1)). \) Property (6) implies that \( R \in \mathcal{R}(\mathfrak{so}(1, n + 1)). \) Such \( R \) does not satisfy property (4).

**Case 3.** Suppose that \( g \subseteq \mathfrak{co}(1, n + 1)_{\mathbb{R}^{p}}. \) Then \( g \) contains the ideal \( p \wedge \mathbb{R}^{n}. \) First suppose that \( n \geq 2. \) Let \( \xi = p \wedge Z, Z \in \mathbb{R}^{n}. \) Let \( R = R_{0} + R_{A}, \) where \( R_{0} \in \mathcal{R}(\mathfrak{so}(1, n + 1)) \) and an element \( A \in \wedge^{2}\mathbb{R}^{1,n+1} \) defines \( R_{A} \) by (7). The element \( A \) may be rewritten in the form
\[
A = ap \wedge q + p \wedge X + q \wedge Y + B, \quad a \in \mathbb{R}, \quad X, Y \in \mathbb{R}^{n}, \quad B \in \mathfrak{so}(n).
\]
The equality (6) implies
\[
[\xi, A] = \rho(\xi) A,
\]
or, equivalently,
\[ aZ - BZ = \rho(\xi)X, \quad Z \wedge Y = \rho(\xi)B, \quad \rho(\xi)a = 0, \quad \rho(\xi)b = 0. \]
If \( \rho(\xi) \neq 0 \) for some \( \xi \in p \wedge \mathbb{R}^{n} \), then \( A = 0 \), and \( R \) does not satisfy property \([4]\). This implies that \( \rho(\xi) = 0 \) for all \( \xi \in p \wedge \mathbb{R}^{n} \). Consequently, \( a = 0, B = 0, Y = 0 \), i.e., \( A = p \wedge X \). Combining this with the description of the element \( R \in \mathfrak{co}(1, n + 1)_{\mathbb{R}p} \) given in \([4]\), we conclude that \( R \) is determined by the equalities
\[
R(p, q) = -\lambda p \wedge q - p \wedge X_0, \quad R(p, V) = 0, \\
R(U, V) = -p \wedge (P(V)U - P(U)V - 2g(V, X)U + 2g(U, X)V), \\
R(U, q) = \gamma(U) \id - g(U, 2X + X_0)p \wedge q + P(U) - p \wedge K(U),
\]
where
\[
\lambda \in \mathbb{R}, \quad X_0, \in \mathbb{R}^{n}, \quad P \in \text{Hom}(\mathbb{R}^{n}, \mathfrak{so}(n)), \quad K \in \odot^{2}\mathbb{R}^{n}, \quad S \in \mathcal{R}(\mathfrak{so}(n))
\]
are fixed, and \( U, V \in \mathbb{R}^{n} \) are arbitrary. The condition \( \rho \wedge \mathbb{R}^{n} \cdot R = 0 \) implies \( \lambda = 0, S = 0, \) and \( 2P(V)U - P(U)V + 2g(U, X)V + g(V, X_0)U + (U, V)X_0 = 0 \) \( \forall U, V \in \mathbb{R}^{n} \).

Using this equality and the similar equality, where \( U \) is interchanged with \( V \), we get
\[
3P(V)U + g(U, 4X + X_0)V + 2g(V, X + X_0)V + 3g(U, V)X_0 = 0 \quad \forall U, V \in \mathbb{R}^{n}.
\]
The condition \( P(V) \in \mathfrak{so}(n) \) implies \( g(P(V)V, W) = -g(P(V)W, V) \) for all \( V, W \in \mathbb{R}^{n} \). Using this, we get \( X = X_0 \) and \( P(V) = -V \wedge X_0 \). Thus it holds
\[
R(p, q) = -p \wedge X_0, \quad R(p, V) = 0, \\
R(U, V) = -p \wedge (3(U \wedge X)0), \\
R(U, q) = -2g(U, X_0)\id - 3g(U, X_0)p \wedge q - U \wedge X_0 - p \wedge K(U).
\]
The condition \([4]\) implies \( X_0 \neq 0 \). We may suppose that \( X_0 = e_1 \). Since
\[
R(p, q) = -p \wedge e_1, \quad R(e_1 e_i) = 3p \wedge e_i, \quad R(e_i e_i) = -e_i \wedge e_i - p \wedge K(e_i), \quad i > 1,
\]
it holds \( -e_i \wedge e_i \in \mathfrak{g} \). From \([5]\) and \([6]\) for \( \xi = -e_i \wedge e_1, X = p, Y = q \) it follows that
\[
(\xi \cdot R)(p, q) = -p \wedge e_i = -\rho(\xi)p \wedge e_i,
\]
which gives a contradiction, i.e., it is impossible that \( \mathfrak{g} \) is contained in \( \mathfrak{co}(1, n + 1)_{\mathbb{R}p} \) and \( n \geq 2 \).

Now suppose that \( n = 1 \). Results of \([4]\) show that \( \mathfrak{g} \) is one of the following:
\[
a. \quad \mathbb{R} \id_{\mathbb{R}^{1} \wedge 2} \oplus \mathbb{R} p \wedge q \ltimes \mathbb{R} p \wedge e_1; \\
b. \quad \mathbb{R} \id_{\mathbb{R}^{1} \wedge 2} \ltimes \mathbb{R} p \wedge e_1; \\
c. \quad \mathbb{R}(\alpha \id_{\mathbb{R}^{1} \wedge 2} + p \wedge q) \ltimes \mathbb{R} p \wedge e_1, \quad \alpha \in \mathbb{R}.
\]
Let \( \xi = p \wedge e_1 \). If \( \rho(p \wedge e_1) \neq 0 \), then it is easy to show that the Weyl structure is closed. If \( \rho(p \wedge e_1) = 0 \), then as above we obtain the following equalities for \( R \):
\[
R(p, q) = -\beta p \wedge e_1, \quad R(p, e_1) = 0, \quad R(e_1, q) = \beta(2 \id_{\mathbb{R}^{1} \wedge 2} + p \wedge q) - kp \wedge e_1, \quad \beta, k \in \mathbb{R}.
\]
This and the property \([4]\) imply that \( \mathfrak{g} \) is the algebra \( c \). Moreover, it holds \( \alpha = 2, \rho(2 \id + p \wedge q) = -5, \) and \( k = 0 \).

**Case 2.** First consider the holonomy algebra \( \mathfrak{g} = \mathbb{R} \id_{\mathbb{R}^{1} \wedge n + 1} \oplus \mathfrak{so}(1, k + 1) \oplus \mathfrak{so}(n - k), \) \( -1 \leq k \leq n - 1 \). In \([4]\), it is shown that
\[
\mathcal{R}(\mathfrak{g}) \cong \mathcal{R}(\mathfrak{so}(1, k + 1)) \oplus \mathcal{R}(\mathfrak{so}(n - k)) \oplus \mathbb{R}^{1, k+1} \otimes \mathbb{R}^{n-k}.
\]
There exists an invariant line in the \( \mathfrak{so}(1, k + 1) \oplus \mathfrak{so}(n - k) \)-module \( \mathbb{R}^{1, k+1} \otimes \mathbb{R}^{n-k} \) if and only if \( 1 \leq k + 2 \leq 2 \) and \( n - k = 1 \). Hence, for \( n \geq 2 \) there is no \( R \in \mathcal{R}(\mathfrak{g}) \) satisfying \([4]\) and \([5]\). The only possible situation is \( k = 0, n = 1 \) and \( \mathfrak{g} = \mathbb{R} \id_{\mathbb{R}^{1} \wedge 2} \oplus \mathfrak{so}(1, 1) \oplus \mathfrak{so}(1) \cong \mathbb{R} \id_{\mathbb{R}^{1} \wedge 2} \oplus \mathbb{R} p \wedge q \subset \mathfrak{co}(1, 2)_{\mathbb{R}p} \).

According to \([4]\), \( R \) is uniquely determined by the following equalities:
\[
R(p, q) = -\lambda p \wedge q, \quad R(p, e_1) = 0, \quad R(e_1, q) = \gamma(\id_{\mathbb{R}^{1} \wedge 2} + p \wedge q), \quad \lambda, \gamma \in \mathbb{R}.
\]
From (6) for $\xi = a \text{id}_{\mathbb{R}^{1+2}}$, $a \in \mathbb{R}$ it follows that $\rho(a \text{id}_{\mathbb{R}^{1+2}}) = -2a$. Next, we use (5) and (6) for $\xi = bp \wedge q$, $b \in \mathbb{R}$. Substituting $X = e_1$, $Y = q$, we obtain $\rho(bp \wedge q) = -b$. Also, for $X = p$, $Y = q$ we have $\rho(bp \wedge q)R(p,q) = 0$ for all $b \in \mathbb{R}$; hence $\lambda = 0$ and $R$ does not satisfy property (4) (g does not contain $\mathbb{R}p \wedge q$).

Suppose that $g = \mathbb{R}(\text{id}_{\mathbb{R}^{1+n+1}} + p \wedge q) \oplus \mathfrak{t} \oplus \mathfrak{so}(n-k) \ltimes p \wedge \mathbb{R}^k \subset \mathfrak{co}(1,n+1)_{\mathbb{R}^p}$, $0 \leq k \leq n-1$.

According to (4), each $R \in \mathcal{R}(g)$ may be written in the form $R = R_1 + R_2 + R_3$, where $R_1 \in \mathcal{R}(\mathfrak{t} \ltimes p \wedge \mathbb{R}^k)$, $R_2 \in \mathcal{R}(\mathfrak{so}(n-k))$, and after a proper choice of the basis it holds

$$
R_3(e_i, e_{k+1}) = ap_i, \quad 1 \leq i \leq k,
R_3(e_{k+1}, q) = -a(id_{\mathbb{R}^{1+n+1}} + p \wedge q),
R_3(e_j, q) = -ae_{k+1} \wedge e_j, \quad k + 2 \leq j \leq n
$$

for some $a \in \mathbb{R}$. If $n - k \geq 2$ and $j > k + 1$ then $e_{k+1} \wedge e_j \in g$ and from (4) we get

$$
((e_{k+1} \wedge e_j) \cdot R_3)(e_j, q) = -a(e_{k+1} \wedge e_j, e_{k+1} \wedge e_j) + R_3(e_{k+1}, e_j) = -R_3(e_{k+1}, q) = \rho(e_{k+1} \wedge e_j)R_3(e_j, q).
$$

From the last equation it follows that $a = 0$, $R_3 = 0$ and $R$ does not satisfy property (4). We conclude that $n - k = 1$ and

$$
g = \mathbb{R}(\text{id}_{\mathbb{R}^{1+n+1}} + p \wedge q) \oplus \mathfrak{t} \ltimes p \wedge \mathbb{R}^{n-1}.
$$

Using arguments as in Case 1, it is not hard to show that $\mathfrak{t} = 0$, and it holds $R = R_3$, where the non-zero values of $R_3$ are the following:

$$
R_3(U, e_n) = ap \wedge U, \quad R_3(e_n, q) = -a(id_{\mathbb{R}^{1+n+1}} + p \wedge q)
$$

for all $U \in \mathbb{R}^{n-1}$.

Finally applying the arguments we used just above, it is easy to check that the holonomy algebra

$$
g = \mathbb{R}(\text{id}_{\mathbb{R}^{1+n+1}} \oplus \mathbb{R}p \wedge q \oplus \mathfrak{t} \oplus \mathfrak{so}(n-k) \ltimes p \wedge \mathbb{R}^k \subset \mathfrak{co}(1,n+1)_{\mathbb{R}^p}, \quad 1 \leq k \leq n-1
$$

does not satisfy the required properties. This proves the theorem.

3. Proof of the Main results

**Proof of Theorem 1.** Let $(M, c, \nabla)$ be a non-closed Lorentzian Weyl structure of dimension $n + 2 \geq 4$ with recurrent curvature tensor. By Theorem 3 its holonomy algebra coincides with

$$
\mathbb{R}(\text{id}_{\mathbb{R}^{1+n}} + p \wedge q) \ltimes p \wedge \mathbb{R}^{n-1} \subset \mathfrak{co}(1,n+1)_{\mathbb{R}^p}.
$$

Using the results from $\mathfrak{t}$, we obtain that around each point of $M$ there exists a coordinate neighborhood $U$ with coordinates $v, x^1, \ldots, x^n, u$ and a metric $g \in c$ such that

$$
g = 2dvdu + h + H(du)^2,
$$

$$
h = \sum_{i,j=1}^{n-1} \delta_{ij} dx^i dx^j + e^{-2F} (dx^n)^2,
$$

$$
\omega = f du, \quad f = \dot{F},
$$

where $H = H(x^1, \ldots, x^{n-1}, u)$ and $F = F(x^n, u)$ are functions. The Christoffel symbols of the connection $\nabla$ are the following:

$$
\Gamma_v = 0,
$$

$$
\Gamma_i = \begin{pmatrix} 0 & -(\delta_{ik}\dot{F})_{k=1}^{n} & \frac{1}{2}\partial_i H \\ 0 & 0 & ((\delta_{ik}\dot{F})_{k=1}^{n})^t \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n - 1
$$

$$
\Gamma_n = (-\delta_{bn}\delta_{kn}\partial_u F)_{b,c=1,\ldots,n,u},
$$

$$
\Gamma_{ii} = \frac{\partial_i \sqrt{G}}{\sqrt{G}} + \frac{\partial_i H}{2H} + \frac{\partial_i \varphi}{\varphi}.
$$
\[
\Gamma_u = \begin{pmatrix}
0 & \left(\frac{1}{2} \partial_k H\right)_{k=1}^{n-1} & 0 & 0 \\
0 & 0 & 0 & -((\frac{1}{2} \partial_k H)_{k=1}^{n-1})^t \\
0 & 0 & 0 & 2\hat{F} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The components of the curvature tensor are as it follows:

\[
R(\partial_v, \partial_u) = 0, \quad R(\partial_v, \partial_i) = 0, \quad R(\partial_v, \partial_n) = 0, \quad R(\partial_i, \partial_j) = 0, \quad i, j = 1, \ldots, n - 1,
\]

\[
R(\partial_i, \partial_n) = \partial_n \hat{F} \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -((\delta_{ik})_{k=1}^{n-1})^t \\
0 & 0 & 0
\end{pmatrix},
\]

\[
R(\partial_i, \partial_n) = \partial_n \hat{F} \begin{pmatrix} 0 & 0 \\
0 & 1 \\
0 & 0 & 2
\end{pmatrix}.
\]

Consider the field of frames

\[
p = \partial_v, \quad e_i = \partial_i, \quad q = \partial_u - \frac{1}{2} H \partial_v.
\]

In notation of the proof of Theorem 3 it holds \( R = R_3 \), consequently \( R(\partial_i, \partial_n) \) must be equal to zero, in other words,

\[
\frac{1}{2} \partial_n \partial_k H + \delta_{ik} (\hat{F} - \hat{F}^2) = 0, \quad \text{for all} \quad i, k = 1, \ldots, n - 1.
\]

From this we find out that

\[
H(x^1, \ldots, x^{n-1}, u) = a(u) \sum_{i=1}^{n-1} (x^i)^2 + \sum_{i=1}^{n-1} b_i(u)x^i + c(u),
\]

where \( a(u), b_i(u), c(u) \) are functions and

\[
(12) \quad \hat{F} - \hat{F}^2 = -a(u).
\]

It is easy to see that the coordinates \( v, x^1, \ldots, x^{n-1} \) may be change in such a way that \( c(u) = 0 \) and all \( b_i(u) = 0 \). Now it is easy to check that the only non-zero component of \( \nabla R \) is the following:

\[
\nabla_n R(\partial_i, \partial_n) = (\partial_n^2 \hat{F} + \partial_n \hat{F} \cdot \partial_n \hat{F}) R(\partial_i, \partial_n).
\]

Let \( \theta = \theta_v dv + \sum_{i=1}^n \theta_i dx^i + \theta_u du \). We get the equality

\[
\theta_n \partial_n \hat{F} = \partial_n^2 \hat{F} + \partial_n \hat{F} \cdot \partial_n \hat{F}.
\]

Let \( U_0 \subset U \) be the open subset, where \( \partial_n \hat{F} \) is non-vanishing. Suppose that \( U_0 \neq \emptyset \). Then on \( U_0 \) it holds

\[
\theta = \theta_n dx^n, \quad \theta_n = \partial_n F + \partial_n \ln |\partial_n \hat{F}|.
\]

Since \( \theta_n \) is smooth function on \( U, U_0 = U \). Note that \( d\omega = \partial_n \hat{F} dx^n \wedge du \). We conclude that for each coordinate neighborhood \( U \), either \( d\omega|_U = 0 \) or \( d\omega|_U \) is non-vanishing on \( U \). Since \( M \) is connected, we conclude that \( d\omega \) is non-vanishing on \( M \).

**Proof of Theorem 2.** Let \( (M, c, \nabla) \) be a non-closed Lorentzian Weyl structure of dimension \( n + 2 = 3 \) with recurrent curvature tensor. Then by Theorem 3 its holonomy algebra is one of the following:

\[
g = \mathbb{R}(id_{\mathbb{R}^2} + p \wedge q), \quad g = \mathbb{R}(2id_{\mathbb{R}^2} + p \wedge q) \cong \mathbb{R}p \wedge e_1.
\]

Consider the first case. Using the results from [7] we obtain that around each point of \( M \) there exist coordinates \( v, x, u \) and a metric \( g \in c \) such that

\[
g = 2dv du + e^{-2F}(dx)^2 + H(du)^2, \quad \omega = \hat{F} du,
\]
where $H = H(u)$ and $F = F(x, u)$ are functions. Applying a simple coordinate transformation, we may assume that $H = 0$. The only non-zero Christoffel symbols of the connection $\nabla$ are the following:

$$\Gamma_{xx}^x = -\partial_x F, \quad \Gamma_{ux}^u = 2\dot{F}.$$ 

The components of the curvature tensor are as follows

$$R(\partial_v, \partial_x) = 0, \quad R(\partial_v, \partial_u) = 0, \quad R(\partial_x, \partial_u) = \partial_x \dot{F} A, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For $\nabla R$ we obtain

$$\nabla R(\partial_v, \partial_x) = \nabla R(\partial_v, \partial_u) = 0, \quad \nabla_v R(\partial_x, \partial_u) = 0$$

$$\nabla_x R(\partial_x, \partial_u) = \left((\partial_x F)(\partial_x \dot{F}) + \partial^2_x \dot{F}\right) A,$$

$$\nabla_u R(\partial_x, \partial_u) = \left(\partial_x \dot{F} - 2(\partial_x F) \dot{F}\right) A.$$

The condition $\nabla R = \theta \otimes \mathbb{R}$ is equivalent to the following system of equations:

$$\theta_v \partial_v \dot{F} = 0, \quad \partial^2_x \dot{F} + (\partial_x F)(\partial_x \dot{F}) = \theta_x \partial_x \dot{F}, \quad \partial_x \dot{F} - 2(\partial_x F) \dot{F} = \theta_u \partial_u \dot{F}$$

where $\theta_\alpha = \theta(\partial_\alpha), \alpha = v, x, u$. Let $U_0$ be the set of non-zero points of the function $\partial_x \dot{F} = 0$. Then on $U_0$ it holds

$$\theta_v = 0, \quad \theta_x = \partial_x (F + \ln |\partial_x \dot{F}|), \quad \theta_u = \partial_u (-2\dot{F} + \ln |\partial_x \dot{F}|).$$

Hence, if $U_0 \neq \emptyset$, then $U_0$ is the entire coordinate neighbourhood.

Consider the second case. From the results of [5] we obtain that around each point of $M$ there exist coordinates $v, x, u$ and a metric $g \in \mathcal{C}$ such that

$$g = 2dvdu + (dx)^2 + H(du)^2,$$

where $H$ is a function and the corresponding 1-form $\omega$ satisfies

$$\omega = fdu, \quad \partial_v H = f.$$

The Christoffel symbols for the connection $\nabla$ are as follows

$$\Gamma_v = \begin{pmatrix} 0 & 0 & \frac{1}{2}f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_x = \begin{pmatrix} 0 & -f & \frac{1}{2}\partial_x H \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_u = \begin{pmatrix} \frac{1}{2}f & \frac{1}{2}\partial_x H & -\frac{1}{2}fH + \frac{1}{2}\dot{H} \\ 0 & f & -\frac{1}{2}\partial_x H \\ 0 & 0 & \frac{1}{2}f \end{pmatrix}.$$ 

Consider the field of frames

$$p = \partial_v, \quad e_1 = \partial_x, \quad q = \partial_u - \frac{1}{2}H \partial_v.$$

It holds $R^v_{uxx} = \partial_v f$. On the other hand, from [6] it follows that $R(\partial_v, \partial_x) = R(p, e_1) = 0$. Hence, $\partial_v f = 0$. Using this we find the components of the curvature tensor

$$R(\partial_v, \partial_x) = 0, \quad R(\partial_v, \partial_u) = \frac{1}{2} \partial_x f \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R(\partial_x, \partial_u) = \begin{pmatrix} \frac{1}{2} \partial_x f & -\frac{1}{2}f^2 + \frac{1}{2} \partial^2_x H & -\frac{1}{2}(\partial_x f)H \\ 0 & \partial_x f & \frac{1}{2}f^2 - \frac{1}{2} \partial^2_x H \\ 0 & 0 & \frac{3}{2} \partial_x f \end{pmatrix}.$$ 

From [6] it follows that

$$\frac{1}{2}f^2 - \dot{f} - \frac{1}{2} \partial^2_x H = 0 \quad \text{and} \quad R(\partial_x, \partial_u) = \frac{1}{2} \partial_x f \begin{pmatrix} 1 & 0 & -H \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$
The covariant derivatives of $R$ are as follows
\[ \nabla_a R(\partial_v, \partial_x) = 0, \quad a = v, x, u, \]
\[ \nabla_v R(\partial_v, \partial_u) = 0, \quad \nabla_x R(\partial_v, \partial_u) = \frac{1}{2} \partial_x^2 f \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nabla_u R(\partial_v, \partial_u) = \frac{1}{2} \partial_x^2 f \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \]
\[ \nabla_x R(\partial_x, \partial_u) = \frac{1}{2} \partial_x^2 f \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nabla_x R(\partial_x, \partial_u) = \frac{1}{2} \partial_x^2 f \begin{pmatrix} 1 & 0 & -H \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \]
\[ \nabla_u R(\partial_x, \partial_u) = \left( \frac{1}{2} \partial_x^2 f - \frac{5}{4} f \partial_x f \right) \begin{pmatrix} 1 & 0 & -H \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \]

The condition $\nabla R = \theta \otimes R$ is equivalent to the following system of equations:
\[ \partial_x^2 f = 0, \quad \theta_v \partial_x f = 0, \quad \partial_x f = \theta_x \partial_x f, \quad \partial_x \dot{f} - \frac{5}{2} f \partial_x f = \theta_u \partial_x f, \]
where $\theta_\alpha = \theta(\partial_\alpha)$, $\alpha = v, x, u$. We see that $f = f(x, u) = a(u)x + b(u)$. Applying a conformal rescaling of the metric and a simple coordinate transformation we may assume that $b(u) = 0$. Let $U_0$ be the set of points where $a(u)$ is non-vanishing. Then on $U_0$,
\[ \theta_u = \partial_u \ln |a(u)| - \frac{5}{2} f. \]

Since $\theta_u$ is smooth, $U_0$ is the entire coordinate neighborhood. This proves the theorem. \[\square\]

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