Avoiding the theorem of Lerche and Shore

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Abstract
Supersymmetric $\sigma$-models obtained by constraining linear supersymmetric field theories are ill defined. Well defined subsectors parametrising Kahler manifolds exist but are not believed to arise directly from constrained linear ones. A counterexample is offered using improved understanding of membranes in superstring theories leading to crucial central terms modifying the algebra of supercharge densities.

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Massless Goldstone bosons arise from components of global symmetries which are spontaneously broken. There is no extra symmetry for Goldstone bosons in supersymmetry. Instead the supersymmetry forces complexification of scalars. This leads to an increased number of massless excitations in general, with complete doubling of the original number in some cases. The special cases when the coset space manifold, $G/H$, of the original Goldstone bosons is Kahler might be expected to be an exception in view of the seminal work of Zumino [1]. This does indeed confirm that a non-linear supersymmetry model can be established without any increase in the number of Goldstone bosons. However the theorem frequently attributed to Lerche and Shore, following early work by Ong [2], appears to prove that such models can never result from constraining linear supersymmetric ones. The formal proofs in Lerche [3], and in Kotcheff and Shore [4], reveal a striking similarity to the work of Witten [5], in which the impossibility of partially breaking extended global supersymmetries ($N > 1$) to lower values was proposed. Indeed it was this similarity which prompted the current work. Once Bagger and Wess [6], and subsequently Hughes, Liu and Polchinski [7] had produced (non–linear) counter examples to the Witten analysis it seemed likely that the Lerche and Shore proofs would also fail. The key contribution could be argued to be that of Hughes and Polchinski [8], which revealed that the original anticommutator algebra for supersymmetric charges had to be generalised to include a central term at the underlying current density level. They attributed this revision to the more modern viewpoint that supermembranes are just as fundamental as elementary particles in string theory.

This reinterpretation is the current starting point. The generalisation of

$$\{Q_{A\alpha}, Q_{B\beta}\} = 2(\sigma^\mu)_{\alpha\beta}\delta_{AB} \quad (1)$$

to local form is
\[
\partial_\mu T \left( j^\mu_{\alpha A}(x) j^\nu_{\beta B}(y) \right) = 2 (\sigma^\rho)_{\alpha\beta} T^\rho_\nu \delta^4(x-y) \delta_{AB} + 2 (\sigma^\nu)_{\alpha\beta} C_{AB} \delta^4(x-y),
\]

(2)

where Schwinger terms which are irrelevant to this analysis are ignored [8]. The key feature is provided by the central terms \( C_{AB} \) which give infinities of unclear covariance on integration over fixed volume. Possibly this was why this was previously overlooked. One might wonder if equation (2) could be restricted by the fact that the Hughes and Polchinski treatment was in two dimensions. But they appear to be taking advantage of the fact that \( T^{\mu\nu} \) is not the unique conserved symmetric tensor since \( T^{\mu\nu} + C^{\mu\nu} \) is also conserved. This does not depend on being in two or fewer dimensions. It seems that this is one of those situations where the symmetry of the Hamiltonian is larger than the symmetry of the \( S \)-matrix. At any rate equation (2) is clearly finite and Lorentz invariant. From it follow the usual consequences of degenerate multiplets for unbroken supersymmetries and Goldstone fermions for those that are broken. In momentum space, with \( C_{AB} \) diagonal and \( \langle T^{\mu\nu} \rangle = \Lambda \eta^{\mu\nu} \), we have

\[
q_\mu < J^\mu_{\alpha A}(q) j^\nu_{\alpha A}> = 2 (\sigma^\nu)_{\alpha\beta} (\Lambda + C_{AA}) + 0(q)
\]

(3)

where there is no sum over \( A \). For those \( A \) such that \( \Lambda + C_{AA} \neq 0 \), equation (3) implies a \( 1/\hat{q} \) singularity in the two current correlations, \( J^\mu_{\alpha A} \) couples the vacuum to a massless fermion with coupling strength \( [2(\Lambda + C_{AA})]^{1/2} \). It also follows that \( \Lambda + C_{AA} \geq 0 \). It should now be obvious how to evade the extra unwanted Goldstone bosons, in the case where the underlying coset manifold is indeed Kahler. The classical analyses of Coleman, Wess and Zumino [9], and Callan, Coleman, Wess and Zumino [10] were extended in the case of non-linearly realised supersymmetry by Volkov and Akulov [11]. This paper follows the elegant treatment of Itoh, Kugo and Kunitoma [12] based upon the very complete generalisations of the classical analyses by Bando, Kuramoto, Maskawa and Uehara [13], and the same authors in [14] and [15]. Finally,
we bring attention to the further clarifications made by Volkov [16] and so elegantly presented by Ogievetsky [17].

The crucial point of extending the underlying algebra of supercharge current densities by central terms, has to be combined not merely with a Kahler $G/H$, but that manifold has to be re-expressed as a quotient space of the complexified $G$ (usually called $G^c$) by a maximally extended complex extension of $H$ (usually called $\hat{H}$). In this treatment this will appear as an explicit mapping manifesting the homeomorphism between $G/H$ and $G^c/\hat{H}$.

A concrete example is offered in the form of the simplest possible case of $G/H = SU_2/U_1$, usually called the complex projective space $CP2$, although it is a straightforward task to extend to all similar (i.e. Kahler) but more complicated cases. The starting point is a recent, interesting but incomplete, attempt to generalise the ideas of chiral perturbation theory to the supersymmetric level by Barnes, Ross and Simmons [18]. It is instructive to see how the ambiguities arise in this chiral $SU_2 \times SU_2$ based model and we adapt the notation of the original only slightly. The original (unconstrained) supersymmetric action is constructed from four (complex) chiral superfields. In components, with

$$y^m = x^m + i\theta\sigma^m\bar{\theta},$$

these have the form

$$\Phi(x, \theta\bar{\theta}) = \phi(y) + \sqrt{2}\theta\lambda_\phi(y) + \theta^2 F_\phi(y),$$

$$\Sigma_3(x, \theta, \bar{\theta}) = \sigma_3(y) + \sqrt{2}\theta\lambda_3(y) + \theta^2 F_\sigma(y),$$

$$\Pi_A(x, \theta, \bar{\theta}) = \pi_A(y) + \sqrt{2}\lambda_A(y) + \theta^2 F_A(y),$$

where $\sigma^m = (-1, \tau^a)$, and the $\tau^a$ are the Pauli matrices ($a = 1, 2, 3$). The chiral superfields $\Pi_A(A = 1, 2)$, and $\Sigma_3$ transform as a triplet under $SU_2$, where the third direction is that of the intended spontaneous breaking, and $\Phi$ is a singlet. It has
previously been noted by Barnes, Generowicz and Grimshare [12] that the chiral $SU_2$ generated by the first two components of the axial generators together with the third component of the vector generators leads indeed to a Kahler manifold of the type $SU_2/U_1$. This is embedded in the chiral $SU_2 \times SU_2$ structure exactly so as to give the $\pi_A$ pseudoscalar nature in their real parts, and correspondingly $\phi$ and $\sigma_3$ scalar nature. The most general supersymmetric action is then written as

$$I = \int d^8z (\bar{\Phi} \Phi + \bar{\Sigma}_3 \Sigma_3 + \bar{\Pi}^A \Pi_A) + \int d^6s W + \int d^6\bar{s} \bar{W},$$

(8)

where the superpotential $W$ is a functional of chiral superfields only. Combining the $\Sigma_3$ and $\Pi_A$ fields into the matrix

$$M = \Sigma_3 \tau^3 + \Pi_A \tau^A,$$

(9)

where the chiral $\gamma_5$ factors are now suppressed, reveals that, under chiral $SU_2 \times SU_2$, $M$ transforms as

$$M \to LMR^\dagger,$$

(10)

and taking

$$W = k(\det M + f_\pi^2)\Phi,$$

(11)

where $k$ is a constant, ensures that the model reduces to the usual bosonic chiral model below the supersymmetry breaking scale provided that $f_\pi$ is required to be real.

Notice that $\sigma^2$ of reference [18] now appears in the guise of $(-\sigma_3)^2$. The advantage of this change of notation will become clear later. This starting action now yields the potential

$$V = F_\sigma \bar{F}_\sigma + F_A \bar{F}_A + F_\phi \bar{F}_\phi$$

$$= 4k^2 (\sigma_3 \bar{\sigma}_3 + \pi_A \bar{\pi}_A)$$

$$+ k^2 [f_\pi^2 - \sigma_3^2 - \pi_A \bar{\pi}_A] [f_\pi^2 - \bar{\sigma}_3^2 - \bar{\pi}_A \bar{\pi}_A].$$

(12)

The minimum of this potential is clearly $V = 0$ which may be achieved by giving the fields the following $SU_2 \times SU_2$ breaking vacuum expectation values (VEVs)

$$< \sigma_3 > = f_\pi$$

$$< \pi_A > = 0 = < \phi >.$$  

(13)
Importantly no auxiliary field acquires a VEV with these assignments so supersymmetry is manifestly not broken in this model. The formal limit $k \to \infty$ leaves the action

$$I = \int d^8z (\Sigma_3 \bar{\Sigma}_3 + \Pi_A \bar{\Pi}_A + \Phi \bar{\Phi}),$$  \hspace{1cm} (14)$$

with the superfields subject to the constraint

$$\Sigma_3^2 + \Pi_A \bar{\Pi}_A = f_\pi^2,$$  \hspace{1cm} (15)$$

with the consequence that the superfield $\Phi$ takes no part in the interactions and can be ignored as spectator field. Eliminating $\sigma_3$, $\lambda_3$ and $F_3$ by substituting the constraints into the kinetic part of the Lagrangian to obtain the leading term in the low momentum expansion (each fermion is considered to have associated with it a factor of the square root of the momentum scale) gives exactly the non-linear (Zumino) Lagrangian as reported in reference [18]. As stated there, the interaction terms involving pseudo-Goldstone bosons and, or, the fermionic superpartners are not uniquely specified. However, the structure of the non-linear Lagrangian describing the Goldston pions alone (when the scalar fields are taken to be real, and the fermions supressed), is quite independent of the structure in which it is is now embedded. This applies also to the Kahler subset of fields, but in the previous conventional wisdom this subsector alone was prohibited from arising in this manner by the theorem of Lerche and Shore. Finally, it is necessary to express the manifold in the $G^c/\hat{H}$ complex form. The key is to introduce the projection operator $\eta$ with the properties $\eta^2 = \eta$ and $\eta^\dagger = \eta$ which is made possible because of the change in the underlying algebra of supercharge densities with the central terms. This can be taken, in this notation, to be

$$\eta = \frac{1 + \tau_3}{2},$$  \hspace{1cm} (16)$$

and it is trivial to confirm that the property

$$\hat{h}\eta = \eta\hat{h}$$, \hspace{1cm} (17)$$
picks out $\tau^+$ and $\tau^3$ as the four members of the complex subgroup. (There is a two way alternative choice at this point, but this has become standard.) With this in mind rewriting $M$ as

$$M = \Sigma_3 \tau^3 + \frac{i \Delta \tau^3}{2} - \frac{i \Gamma \tau^-}{2},$$  \hspace{1cm} (18)$$

so that, now leaving out the $f_\pi^2$ terms,

$$I = \int d^8z (\Sigma_3 \bar{\Sigma}_3 + \frac{\Gamma \bar{\Gamma}}{4} + \frac{\Delta \bar{\Delta}}{4} + \Phi \bar{\Phi}),$$  \hspace{1cm} (19)$$

and

$$V = 4k^2 \phi \bar{\phi} \left( \sigma_3 \bar{\sigma}_3 + \gamma \gamma + \delta \bar{\delta} \right) + k^2 \left[ \sigma^2_3 + \frac{2 \delta}{4} \right] \left[ \bar{\sigma}^2_3 + \frac{2 \bar{\delta}}{4} \right].$$  \hspace{1cm} (20)$$

In the formal limit as $k \to \infty$, the action becomes

$$I = \int d^8z \frac{\Gamma \bar{\Gamma}}{4},$$  \hspace{1cm} (21)$$

as the constraints are satisfied by the superfield conditions

$$\Sigma_3 = 0 \quad \text{and} \quad \Delta = 0.$$  \hspace{1cm} (22)$$

The superfield $\Phi$ can again be ignored as a non-interacting spectator. Notice that the single complex superfield $\Gamma$ is all that remains in the action, and it is not constrained.

To describe the coset space of the chiral sphere [19], the real part of $\pi^A$ is written as $M^A$, and so

$$L = \exp \left\{ \frac{-i}{2} \theta(\phi) \frac{M_A \tau^A}{\phi} \right\},$$  \hspace{1cm} (23)$$

where $\theta(\phi)$ is any arbitrary function of

$$\phi = [M_A M^A]^{1/2},$$  \hspace{1cm} (24)$$

divided by the pion decay constant $f_\pi$, and where the chiral $\gamma_5$ dependence is again suppressed. The arbitrariness may be viewed as the freedom to change coordinate systems on the surface of the sphere, or to redefine the field variables describing the pions. Now this can alternatively be written in the form [12]

$$L = \exp \left( \frac{-i \gamma \tau^-}{2} \right) \exp \left( \frac{-i \delta \tau^+}{2} \right) \exp \left( -\frac{V \tau^3}{2} \right),$$  \hspace{1cm} (25)$$

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using the complex subgroup \( \hat{H} \). Moreover the expression

\[
\exp\left( -\frac{i\gamma \tau^-}{2} \right)
\]

gives the explicit mapping of the homeomorphism between \( G/H \) and \( G^c/\hat{H} \). In the general coordinate system

\[
\frac{\gamma \bar{\gamma}}{4} = \tan^2 \left( \frac{\theta}{2} \right),
\]

(26)

and it is known from reference [12] that the Kahler potential is given by

\[
K = \ln \det_{\eta} \left\{ \exp \left( \frac{i\bar{\gamma} \tau^+}{2} \right) \exp \left( -\frac{-i\gamma \tau^-}{2} \right) \right\},
\]

(27)

where the notation indicates that the determinant is to be taken in the top left hand corner of the matrix in this representation. This reveals at once that

\[
K = \ln \left[ 1 + \frac{\bar{\gamma} \gamma}{2} \right] = \ln \left[ \sec^2 \left( \frac{\theta}{2} \right) \right] = V
\]

(28)

which is the desired result. Note that, although the general coordinate notation is most convenient in this context, reliance is placed on the results of reference [12]. The Kahler nature of the potential is demonstrated by diagonalizing the metric in stereographic coordinates, \( z \) and \( \bar{z} \), and revealing the holomorphic nature of the transformations in the usual manner.

Although this demonstration that Kahler potentials can arise from constrained linear supersymmetric schemes has used only \( CP2 \), there seems no reason whatsoever why this can not be generalised directly to larger Kahler manifolds – in particular to \( CPN \).

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