On the equivalence of internal and external habit formation models with finite memory

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Abstract

In this paper we use a dynamic programming approach to analytically solve an endogenous growth model with internal habits where the key parameters describing their formation, namely the intensity, persistence and lag structure (or memory), are kept generic. Then we show that external and internal habits lead to the same closed loop policy function and then to the same (Pareto) optimal equilibrium path of the aggregate variables when the utility function is subtractive nonseparable. The paper uses new theoretical results from those previously developed by the dynamic programming literature applied to optimal control problems with delay and it extends the existing results on the equivalence between models with internal and external habits to the case of finite memory.

JEL Classification C6; E1; E2.

Keywords Habit Formation; Optimal Control with Delay; Dynamic Programming.

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1 Introduction

Motivation and Results – In this paper we consider the simplest endogenous growth model with linear technology, as in Rebelo [34], and we assume that the representative household’s utility function depends also on internal habits, whose formation is based on the history (up to a given fixed lag $\tau$) of past consumption. The resulting instantaneous utility is

$$u(c(t), h(t))$$

while the habit formation is described by the following exponentially smoothed index of the past consumption rates

$$h(t) = \varepsilon \int_{t-\tau}^{t} c(s)e^{\eta(s-t)}du \quad \forall t \geq 0$$  \hspace{1cm} (1)

with $\varepsilon \geq 0$, $\eta \geq 0$, and $\tau \geq 0$ indicating respectively the intensity, persistence and lags structure (or memory) of the habits. The habit formation equation (1) is general in its assumptions on the intensity, persistence and lag structure and it embeds all the main specifications used in the literature; the role of $\tau$ is indeed critical in pinning down the different forms of the habits (e.g. $\tau = 1$ is the continuous time version of the case studied by Boldrin et al. [12] among others).\footnote{Equation (1) does not include the case with deep habits studied by Raven et al. [33] since we focus on a single consumption good economy.}

A generic and finite choice of the memory parameter $\tau$ is also consistent with recent empirical evidences.\footnote{Among them, Crawford [21] uses a revealed preference approach to characterize the internal habits. He finds no sharp result on the lag structure: increasing the number of period lags in the consumption of the good increases the “agreement between theory and data. However it (...) has a large negative effect on the power of the test compared with the one-lag version”. In his contribution, he looks at the cases $\tau = 1, 2$ and 3.}

Our objective is to solve analytically this problem with internal habits using a dynamic programming approach and to prove that, independently on the choice of the habits formation’s parameters, the solution of this problem coincides with the solution of the same problem but with external habits when the instantaneous utility function has the nonseparable subtractive form:\footnote{This instantaneous utility function is, together with the multiplicative nonseparable, one of the two most common specifications used in the habit formation literature.}

$$u(c(t), h(t)) = \frac{(c(t) - h(t))^{1-\gamma}}{1-\gamma} \quad \gamma > 0, \; \gamma \neq 1.$$  \hspace{1cm} (2)
To arrive to this result we prove that, keeping all the else equal, the problem with internal habits leads to the same solution path of its counterpart with external habits; in the latter the instantaneous utility function has the same functional form but the habits are now formed over the past average economy-wide consumption, $\bar{c}(\cdot)$:

$$h(t) = \varepsilon \int_{t-\tau}^{t} \bar{c}(u)e^{\eta(u-t)} du \quad \forall t \geq 0;$$

Our contribution is relevant for the following two main reasons. Firstly, it provides a full analytical characterization of an endogenous growth model with internal habits and finite memory. To achieve this result we have extended the dynamic programming approach to optimal control of Delay Differential Equations (DDE) first developed in Fabbri and Gozzi [23] to a different framework. In fact, the presence of the habit formation with a potentially finite lag parameter, $\tau$, implies a substantial analytical deviation from other problems studied in the literature since here the delay is contained in the objective function. Therefore, an extension of the previous results on dynamic programming approach to optimal control of DDE’s is necessary to find explicitly the policy function of our problem: in this extent, our paper represents a new contribution to the dynamic programming literature in infinite dimension, as it will be extensively explained later in this introduction and in Section 3. Secondly, it extends the existing results on consumption externalities not leading to economic distortions. More precisely, we prove that the equivalence holds when the three key parameters $\varepsilon$, $\eta$, and $\tau$ in the habit formation equation are kept generic, while in previous contributions were assumed either $\varepsilon = \eta$ and $\tau = 1$ (e.g. Alonso-Carrera et al. [3]) or $\varepsilon = \eta$ and $\tau = \infty$ (e.g. Gomez [31]).

**Methodology** – To prove our results we use a dynamic programming approach to find the solution of the model with internal habits, then we compare it with the solution in the case with external habits and finally we show that the two resulting closed loop policy functions are identical when the utility function has the nonseparable subtractive form.

The model with external habits was solved in Augeraud-Veron and Bambi [4] using a modified version of the Pontryagin Maximum Principle (PMP) (see e.g. [1]); the closed loop policy function was also found from the explicit computation of capital, consumption and habits. Such PMP approach has been recently used by several authors to solve vintage capital models (e.g. Barucci and Gozzi [8], [9], Boucekkine et al. [15], [16], [17], Bréchet et al. [18] Feichtinger et al [27], [28], Saglam and Veliov [35] Veliov [36]) and time to build models (e.g. Bambi [5] and Bambi and Gori [6]).

The same strategy can be applied to the case with internal habits but it won’t lead to an explicit formulation of the optimal policy because of the mixed type equation resulting from the PMP in presence of retarded control. So we proceed to solve the problem with internal habits and we find the closed loop policy function through the dynamic programming method; this approach successfully leads to identify the explicit form of the closed loop policy function as soon as its associated Hamilton-Jacobi-Bellman equation (HJB) can be solved explicitly. It must be noted that the delayed structure of the problem pins down an HJB equation which is a partial differential equation in infinite dimension without explicit solutions unless specific assumptions on the production and utility function are introduced. Luckily enough, the linear production function and the nonseparable subtractive form of the utility function let us develop an ad hoc approach in order to calculate explicitly the solutions of the HJB equation and then the closed loop policy functions which, as explained before, are crucial to prove the equivalence between the internal and external habit formation model.

The dynamic programming approach to optimal control problems with delay has found very
few applications in the economic literature. As far as we know the first to apply this method were Fabbri and Gozzi [23] in a vintage capital framework, and later Boucekkine et al. [13] and Bambi et al. [7], the latter in a time-to-build model (see also [6], [24] [25] and [26] for application of the same technique to models with age structure). More recently Boucekkine et al. [14] used it to investigate the compatibility of the optimal population size concepts produced by different social welfare functions and egalitarianism. We must note that, from a technical point of view, the problem we face in this paper is quite different from those in the papers just quoted because the delayed control appears both in the objective functional and in the constraints: then we have to extend the theoretical results, already used in the previously cited papers, to a different context, see on this Remark 19.

Plan of the paper – The paper is organized as follows. Section 2 presents the general model with habit formation where subsection 2.1 is devoted to further explain the case with internal habits. Section 3 explains how the problem can be rewritten in infinite dimension and how to arrive to the solution path using the Hamilton-Jacobi-Bellman equation. Section 4 states and proves the equivalence result. Finally Section 5 concludes the paper.

2 The model

Consider a standard neoclassical growth model, where the economy consists of a continuum of identical infinitely lived atomistic households, and firms. The households’ objective is to maximize over time the discounted instantaneous utility (here \( c(t) \) and \( h(t) \) are, respectively, the consumption and the habit at time \( t \)):

\[
u(c(t), h(t)) = \frac{(c(t) - h(t))^{1-\gamma}}{1-\gamma}, \tag{3}\]

for \( c(t) \geq h(t) \) and \( \gamma > 0 \). If \( c(t) < h(t) \) the utility function is not always well defined in the real field and it is never concave. For this reason, it is generally assumed that \( u(c(t), h(t)) = -\infty \) as soon as \( c(t) < h(t) \). The instantaneous utility function (3) clearly implies addiction in the habits since current consumption is forced to remain higher than the habits over time. The habits are formed according to the rule

\[
h(t) = \varepsilon \int_{t-\tau}^{t} \hat{c}(u) e^{\eta(u-t)} du \quad \forall t \geq 0 \tag{4}\]

where \( \hat{c}(t) \) indicates the customary consumption level, which is equal to \( c(t) \) in the case of internal habits or to the economy-wide average consumption, \( \bar{c}(t) \), when we consider external habits. Moreover \( \eta > 0 \) measures the persistence of habits, while \( \varepsilon > 0 \) the intensity of habits, i.e. the importance of the economy average consumption relative to current consumption. Finally the habits’ past history, \( h(t) \) with \( t \in [-\tau, 0] \) is given; also, in the case of external habits, the path of \( \bar{c}(t) \) is taken as given since no individual decision have an appreciable effect on the average consumption of the economy.

Differentiating (4) (this is possible e.g. in all continuity points of \( \hat{c}(\cdot) \)) we have

\[
\dot{h}(t) = \varepsilon (\hat{c}(t) - \bar{c}(t - \tau)e^{-\eta \tau}) - \eta h(t) \quad \forall t \geq 0 \tag{5}\]

Assuming a linear technology \( y(t) = Ak(t) \) and a depreciation factor \( \delta > 0 \) the optimal

\[\text{equation}^{4}\text{The case } \gamma = 1 \text{ can be treated exactly as the other ones. We do not do it here to make the analytical part less cumbersome.}\]
The social planner problem consists in finding the strategy \( c(\cdot) \) which maximizes the objective functional
\[
\max_{0}^{\infty} \frac{\left( c(t) - \varepsilon \int_{t-\tau}^{t} \hat{c}(u)e^{\eta(u-t)} du \right)^{1-\gamma}}{1 - \gamma} e^{-\rho t} dt \tag{6}
\]
under the state equation (which can be seen as an equality constraint)
\[
\dot{k}(t) = (A - \delta) k(t) - c(t), \quad t \geq 0, \tag{7}
\]
the positivity constraints
\[
k(t) \geq 0, \quad c(t) \geq 0, \tag{8}
\]
the constraint
\[
c(t) \geq \varepsilon \int_{t-\tau}^{t} \hat{c}(u)e^{\eta(u-t)} du, \tag{9}
\]
and with initial data \( k(0) = k_{0} > 0, c(s) = c_{0}(s) \) given for \( s \in [-\tau, 0) \), where we assume \( c_{0}(\cdot) \in L^{1}([-\tau, 0]; \mathbb{R}^{+}) \).\(^{5}\) Observe that we have already substituted in (6) and (9) the equation describing the internal habit formations:
\[
h(t) = \varepsilon \int_{t-\tau}^{t} c(u)e^{\eta(u-t)} du \quad \forall t \geq 0. \tag{10}
\]
whose initial value is known and is given by
\[
h_{0} := \varepsilon \int_{-\tau}^{0} c_{0}(u)e^{\eta(u-t)} du. \tag{11}
\]
In the following two sections we solve the social planner problem by using the dynamic programming approach. Then, in Section 4, we prove our equivalence theorem showing that the closed loop policy formula found in Section 3.2 is the same found in [4] for the market equilibrium problem.

\(^{5}\)\( L^{1}([-\tau, 0]; \mathbb{R}^{+}) \) indicates the space of functions from \([-\tau, 0)\) to \( \mathbb{R}^{+} \) which are Lebesgue measurable and integrable.
3 Solution of the internal habit problem

3.1 Preliminary results

We first introduce a notation useful to rewrite more formally equation (10) and the objective functional. We call \( c_0(\cdot) : [-\tau, 0) \to \mathbb{R}_+ \) the initial datum, \( c(\cdot) : [0, \infty) \to \mathbb{R}_+ \) the control strategy and \( \tilde{c} : [-\tau, \infty) \to \mathbb{R}_+ \) the function (sometimes called the concatenation of the two above)

\[
\tilde{c}(s) = \begin{cases} 
  c_0(s) & \text{for } s \in [-\tau, 0) \\
  c(s) & \text{for } s \in [0, \infty)
\end{cases}
\]

With this notation equation (10) is rewritten more precisely as

\[
h(t) = \varepsilon \int_{t-\tau}^{t} \tilde{c}(u)e^{\rho(u-t)}du \quad \forall t \geq 0.
\]

(12)

The state equation (7) is a standard linear Ordinary Differential Equation (ODE) and so it can be easily seen that, for every locally integrable control strategy \( c(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+^0 \), there exists a unique absolutely continuous solution of it, which will be denoted as \( k_{k_0,c(\cdot)}(\cdot) \) and which is given by

\[
k(t) = k_0e^{(A-\delta)t} - \int_{0}^{t} e^{(A-\delta)(t-u)}c(u)du.
\]

(13)

The objective functional to maximize is

\[
J(k_0, c_0(\cdot); c(\cdot)) := \int_{0}^{\infty} \frac{(c(t) - \varepsilon \int_{0}^{\infty} \tilde{c}(u)e^{\rho(u-t)}du)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt,
\]

over the set

\[
\mathcal{C}(k_0, c_0(\cdot)) = \left\{ c(\cdot) \in L^1_{\text{loc}}([0, \infty); \mathbb{R}_+) : k_{k_0,c(\cdot)}(\cdot) \geq 0 \right\}
\]

and \( c(t) \geq \varepsilon \int_{t-\tau}^{t} \tilde{c}(u)e^{\rho(u-t)}du \geq 0 \) for almost every \( t \in \mathbb{R}_+ \)

We call from now on (P) the problem of finding an optimal control strategy i.e. a strategy \( c^*(\cdot) \in \mathcal{C}(k_0, c_0(\cdot)) \) such that

\[
-\infty < J(k_0, c_0(\cdot); c^*(\cdot)) < +\infty
\]

and

\[
J(k_0, c_0(\cdot); c^*(\cdot)) = \sup_{c(\cdot) \in \mathcal{C}(k_0, c_0(\cdot))} \int_{0}^{\infty} \frac{(c(t) - \varepsilon \int_{0}^{\infty} \tilde{c}(u+t)e^{\rho u}du)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt
\]

As usual we call value function the map

\[
V(k_0, c_0(\cdot)) := \sup_{c(\cdot) \in \mathcal{C}(k_0, c_0(\cdot))} \int_{0}^{\infty} \frac{(c(t) - \varepsilon \int_{0}^{\infty} \tilde{c}(u+t)e^{\rho u}du)^{1-\gamma}}{1-\gamma} e^{-\rho t} dt
\]

We now give a preliminary study of the problem concerning the behavior of admissible control strategies and state trajectories.

First of all we give an estimate for the admissible control strategies and state trajectories.

\[\text{The space of such functions will be denoted from now on by } L^1_{\text{loc}}([0, \infty); \mathbb{R}_+).\]
Proposition 1 (Lower bound for admissible strategies) We consider any initial datum $(k_0, c_0(\cdot)) \in \mathbb{R}^+ \times L^1([-\tau, 0], \mathbb{R}^+)$ and any control strategy $c(\cdot) \geq 0$ satisfying (9). Then we have, for every $t \geq 0$,
\[
c(t) \geq c^m(t)
\]  
where $c^m(\cdot) \in L^1_{loc}([0, +\infty]; \mathbb{R}^+)$ is the unique solution of the equation
\[
c^m(t) = \varepsilon \int_{t-\tau}^{t} \tilde{c}^m(u) e^{\eta(u-t)} du
\]  \tag{14}
Moreover the state trajectory $k(\cdot)$ associated to $c(\cdot)$ is dominated at any time $t \geq 0$ by the solution $k^M(\cdot)$ obtained taking the same initial datum $k_0$ and the control $c^m(\cdot)$
\[
k(t) \leq k^M(t) = e^{(A-\delta)t} \left[ k_0 - \int_0^t c^m(u) e^{-(A-\delta)u} du \right].
\]  \tag{16}
Proof. First we observe that, thanks to standard existence theorems for DDE’s (see e.g. [32], Section 2.2) the equation (15) has a unique solution for every $c_0(\cdot) \in L^1([-\tau, 0]; \mathbb{R}^+)$. Now take a control strategy $c(\cdot) \in C(k_0, c_0(\cdot))$. The constraint (9) together with (15) implies that
\[
c(t) - c^m(t) \geq \varepsilon \int_{t-\tau}^{t} [\tilde{c}(u) - \tilde{c}^m(u)] e^{\eta(u-t)} du, \quad t \geq 0
\]
Clearly, since both functions $c(\cdot)$ and $c^m(\cdot)$ have the same past $c_0(\cdot)$, it must be $\tilde{c}(t) - \tilde{c}^m(t) = 0$ for $t \in [-\tau, 0)$. So, calling $c_1(t) := c(t) - c^m(t)$ we get, for $t \in [0, \tau]$,
\[
c_1(t) \geq \int_0^t c_1(u) e^{\eta(u-t)} du.
\]
This implies, by a simple application of Gronwall inequality (see e.g. [32],[p.15, Lemma 3.1], that $c_1(t) \geq 0$ for $t \in [0, \tau]$. Take now $t \in (\tau, 2\tau]$. As above we have, for any such $t$,
\[
c_1(t) \geq \int_{t-\tau}^{t} c_1(u) e^{\eta(u-t)} du + \int_{\tau}^{t} c_1(u) e^{\eta(u-t)} du.
\]
Since the function $t \rightarrow \int_{\tau}^{t} c_1(u) e^{\eta(u-t)} du$ is nonnegative for every $t \in (\tau, 2\tau]$, then applying again the Gronwall inequality we get that $c_1(t) \geq 0$ for $t \in (\tau, 2\tau]$. The claim (14) for every $t \geq 0$ then easily follows by induction. Finally the claim (16) follows by (14) and by the formula (13).

The characteristic equation associated to the delay equation (15) writes
\[
1 = \varepsilon \int_{-\tau}^{0} e^{(A+\eta)u} du
\]  \tag{17}
Proposition 2 (Properties of characteristic roots) The characteristic equation (17) admits a unique real root, $\lambda_0$. We have $\lambda_0 < \varepsilon - \eta$ and all complex roots have a real part smaller than $\lambda_0$. Moreover,
- if $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du < 0$ then $\lambda_0$ is the only root with positive real part;
- if $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du > 0$ all the roots have negative real part.
- if $1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du = 0$ then $\lambda_0 = 0$ and the other roots have negative real part.
Proof. We first study real roots. Consider the function

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi(\lambda) = 1 - \varepsilon \int_{-\tau}^{0} e^{(\lambda+\eta)u} du.$$  

Since $$\varphi'(\lambda) = -\varepsilon \int_{-\tau}^{0} ue^{(\lambda+\eta)u} du > 0,$$ then $$\varphi$$ is a strictly increasing function of $$\lambda$$. Moreover

$$\lim_{\lambda \to -\infty} \varphi(\lambda) = -\infty, \quad \lim_{\lambda \to +\infty} \varphi(\lambda) = 1$$  

and

$$\varphi(0) = 1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du \geq 1 - \varepsilon \tau, \quad \varphi(\varepsilon - \eta) = e^{-\varepsilon \tau} > 0.$$  

The above equation implies that there exists a unique real root of the equation $$\varphi(\lambda) = 0$$. Such root belongs to $$(-\infty, \varepsilon - \eta)$$. Moreover, all complex roots $$\lambda = p + iq$$ satisfy the system

$$1 - \varepsilon \int_{-\tau}^{0} e^{(p+\eta)u} \cos(qu) du = 0$$
$$\varepsilon \int_{-\tau}^{0} e^{(p+\eta)u} \sin(qu) du = 0$$

From the first equation we get that

$$1 < \varepsilon \int_{-\tau}^{0} e^{(p+\eta)u} du = 1 - \varphi(p).$$

This implies that $$\varphi(p) < 0$$ which implies $$p < \lambda_0$$.

Finally, let $$1 - \varepsilon \int_{-\tau}^{0} e^{\eta u} du < 0$$ and consider the function

$$a(\lambda) := (\lambda + \eta) \varphi(\lambda).$$

It can be easily seen that $$a(\lambda)$$ rewrites

$$a(\lambda) = \lambda + \eta - \varepsilon \left(1 - e^{-(\lambda+\eta)\tau}\right)$$

and that all complex roots of the characteristic equation $$\varphi(\lambda) = 0$$ are also zeros of $$a(\cdot)$$. Let us assume that there exists a complex root, $$\lambda = p + iq$$ with $$p \in (0, \lambda_0)$$. Then, we have

$$\text{Re}(a(\lambda)) = p + \eta - \varepsilon + \varepsilon e^{-(p+\eta)\tau} \cos(q\tau) < p + \eta - \varepsilon + \varepsilon e^{-(p+\eta)\tau} = a(p) < 0$$

which contradict the fact that $$\text{Re}(a(\lambda)) > 0$$. The rest of the claim is immediate. □

Now we have, as a consequence, the following result.

**Proposition 3 (Existence of admissible paths)**

(i) Fix an initial datum $$(k_0, c_0(\cdot)) \in (\mathbb{R}^+ \times L^1([-\tau,0];\mathbb{R}^+))$$. The set $$\mathcal{C}(k_0, c_0(\cdot))$$ is nonempty if and only if the control $$c^m(\cdot)$$ introduced in (15) is admissible, i.e. such that $$k^M(t) \geq 0$$ for every $$t \geq 0$$.

(ii) In particular, if $$\lambda_0 \geq A - \delta$$ then for any $$c_0(\cdot) \in L^1([-\tau,0];\mathbb{R}^+),$$ such that $$c_0(t) > 0$$ on a set of positive Lebesgue measure we have $$\mathcal{C}(k_0, c_0(\cdot)) = \emptyset.$$
Proof. The first statement is an immediate corollary of Proposition 1. Concerning the second statement we observe first that the solution of the equation (15) can be written with a series expansion (see e.g. Corollary 6.4, p.168 of [22]) as follows

\[ \tilde{c}_m(t) = \sum_{r=0}^{\infty} p_r(t)e^{\lambda_r t} \]  

where \( \{\lambda_r\}_{r \in \mathbb{R}} \) is the sequence of the roots of the characteristic equation (17) and the \( p_r(t) \) are polynomials of degree less or equal to \( m(r) - 1 \) where \( m(r) \) is the multiplicity of \( \lambda_r \).

Now, using e.g. [10], Section 6.7 (in particular Theorem 6.5) we can explicitly compute the coefficients of such solutions by using the Laplace transform.

In particular, since \( \lambda_0 \) is a simple root, we have

\[ p_0 = \frac{\psi(\lambda_0)}{\varphi'(\lambda_0)} \]

where

\[ \varphi(\lambda) = 1 - \varepsilon \int_{-\tau}^{0} e^{(\lambda + \eta)u} du \]

and

\[ \psi(\lambda) = (1 - \varphi(\lambda)) \int_{-\tau}^{0} c_0(u)e^{-\lambda u} du \]

Clearly, if \( c_0(\cdot) > 0 \) on a set of positive Lebesgue measure we have that \( p_0 > 0 \) and so the leading term of the series (18) is \( p_0 e^{\lambda_0 t} \) and all the others are complex exponentials with negative real part. So the corresponding state trajectory \( k_M(\cdot) \) is

\[ k_M(t) = e^{(A-\delta)t} \left[ k_0 - \int_{0}^{t} p_0 e^{(\lambda_0 - (A-\delta))u} du + \xi(t) \right] \]

where \( \xi(\cdot) : [0, +\infty) \to \mathbb{R} \) is a bounded function coming from the lower order term of the series (18). When \( \lambda_0 \neq A - \delta \) it follows

\[ k_M(t) = e^{(A-\delta)t} \left[ k_0 + \frac{p_0}{\lambda_0 - (A-\delta)} + \xi(t) \right] - \frac{p_0}{\lambda_0 - (A-\delta)} e^{\lambda_0 t} \]

Clearly, when \( \lambda_0 > A - \delta \) the limit of the above expression is \( -\infty \), so the claim follows. When \( \lambda_0 \neq A - \delta \) we have

\[ k_M(t) = e^{(A-\delta)t} [k_0 - p_0 t + \xi(t)] \]

and again the limit of the above expression is \( -\infty \), so the claim follows.

Due to the above proposition it makes sense to study the social planner problem when

\[ \lambda_0 < A - \delta. \]  

and the initial datum \( c_0(\cdot) \) is small enough so to guarantee that the corresponding \( k_M(\cdot) \) is always strictly positive. It is clear that \( \lambda_0 \) is the lowest possible growth rate of the habit: this growth rate has to be lower than the real interest rate of the economy \( r = A - \delta \), which coincides with the maximum growth rate of capital obtainable from the capital accumulation equation when consumption is set to zero. In fact an economy cannot sustain over time a growth rate which exceeds the real interest rate because capital does not accumulate sufficiently fast to sustain the higher and higher consumption.
Note in particular that (19) is surely true if
\[ \varepsilon - \eta \leq A - \delta. \] (20)
or, since \( \frac{\varepsilon}{\eta}(1 - e^{-\eta \tau}) < 1 \Leftrightarrow \lambda_0 < 0 \), if
\[ \frac{\varepsilon}{\eta}(1 - e^{-\eta \tau}) < 1 \quad \text{and} \quad A - \delta > 0. \] (21)

From now on, we will focus on the case
\[ \lambda_0 < \varepsilon - \eta \leq 0 < A - \delta. \] (22)

The condition \( \varepsilon - \eta \leq 0 \) is usually assumed in the economic literature (e.g. Constantinides [20]) because it prevents the economy to asymptotically converge to the corner solution \( c(t) = h(t) \).

We finally observe that strict positivity of \( k_M(\cdot) \) is guaranteed by assuming that, beyond (19)
\[ k_0 > \int_0^{+\infty} e^{-s(A-\delta)}c^m(s)ds \] (23)
where \( c^m(\cdot) \) is the unique solution of (15). The economic intuition behind this restriction on the initial condition of capital will be explained in Section 4.

Therefore conditions (19) and (23) are necessary to guarantee that the value function \( V \) is not always \(-\infty\) at a given point. Here we give a sufficient condition for the finiteness of \( V \).

**Proposition 4 (Finiteness of the value function \( V \))** Let us consider an initial datum \((k_0, c_0(\cdot)) \in (\mathbb{R}_+ \times L^1([-\tau, 0]; \mathbb{R}_+))\). Assume that (19) and (23) hold true, so \( \mathcal{C}(k_0, c_0(\cdot)) \neq \emptyset \). If
\[ \rho > (A - \delta)(1 - \gamma), \] (24)
then the value function is always finite.

**Proof.** To prove the claim it is enough to prove the following:

(i) If \( \gamma \in (0, 1) \) then there exists \( M_+ > 0 \) such that, for all \((k_0, c_0(\cdot)) \) in the space \((\mathbb{R}_+ \times L^1([-\tau, 0]; \mathbb{R}_+))\),
\[ 0 \leq V(k_0, c_0) \leq M_+ k_0^{1-\gamma}. \]

(ii) If \( \gamma \in (1, +\infty) \) and (21) holds, then there exists \( M_- < 0 \) such that, for all \((k_0, c_0(\cdot)) \) in the space \((\mathbb{R}_+ \times L^1([-\tau, 0]; \mathbb{R}_+))\),
\[ M_- k_0^{1-\gamma} \leq V(k_0, c_0) \leq 0. \]

We prove first (i). The first inequality is obvious since for \( \gamma \in (0, 1) \) we always have \( J(k_0, c_0(\cdot); c(\cdot)) \geq 0 \).

Concerning the other inequality setting (Fleming and Soner [29], p.30-32, Freni et al. [30])
\[ \zeta(s) = \int_0^s c(u)^{1-\gamma} du \]
and applying Hölder’s inequality to \( \zeta (s) = \int_0^s s^{1-\gamma} \left( \frac{c(u)}{s} \right)^{1-\gamma} du \) yields to

\[
\zeta (s) \leq \left( \int_0^s s^{1-\gamma} du \right)^{\gamma} \left( \int_0^s \left( \frac{c(u)}{s} \right)^{1-\gamma} du \right)^{1-\gamma} \\
\leq s^\gamma \left( \int_0^s c(u) du \right)^{1-\gamma}
\]
as \( c(u) = (A - \delta) k(u) - \dot{k}(u) \)

\[
\int_0^s c(u) du = \int_0^s (A - \delta) k(u) du - k(s) + k(0)
\]

Now, according to equation (13), \( k(s) \leq k(0) e^{(A-\delta)s} \). Thus using also that \( k(s) \geq 0 \) for \( s \geq 0 \) we get

\[
\int_0^s c(u) du \leq k_0 e^{(A-\delta)s}
\]

and so

\[
\zeta (s) \leq s^\gamma k_0^{1-\gamma} e^{(1-\gamma)(A-\delta)s}
\]

Now we have

\[
J (k_0, c_0(\cdot); c(\cdot)) \leq \int_0^{+\infty} \frac{c(s)^{1-\gamma}}{1-\gamma} e^{-\rho s} ds
\]

and, integrating by parts and using (25),

\[
J (k_0, c_0(\cdot); c(\cdot)) \leq \left( \frac{1-\gamma}{1-\gamma} \int_0^{+\infty} s^{\gamma} e^{((1-\gamma)(A-\delta)-\rho)s} ds \right)
\]

which gives the claim.

Now we prove (ii). The second inequality is obvious since for \( \gamma \in (1, +\infty) \) we always have \( J (k_0, c_0(\cdot); c(\cdot)) \leq 0 \).

Concerning the other inequality we observe that, calling \( c_m(\cdot) \) the unique solution of (15) we have, thanks to (23) and (21) that, for \( \alpha > 0 \) small enough, the control strategy defined as \( c_1(t) = c_m(t) + \alpha \) (\( t \geq 0 \)) is admissible.

Indeed, calling \( k_1(\cdot) \) the associated state trajectory we have

\[
k_1(t) = e^{(A-\delta)t} \left[ k_0 - \int_0^t e^{-s(A-\delta)u} (c_m(u) + \alpha) du \right] =
\]

\[
e^{(A-\delta)t} \left[ k_0 - \int_0^t e^{-s(A-\delta)u} c_m(u) du - \frac{\alpha}{A-\delta} \right]
\]

This remain always positive if

\[
\frac{\alpha}{A-\delta} \leq k_0 - \int_0^{+\infty} e^{-s(A-\delta)u} c_m(u) du
\]

which is possible by (23). Moreover the control \( c_1(\cdot) \) satisfy the constraint (9) since, substituting it into (15) we get

\[
\alpha \geq \alpha \frac{\varepsilon(1-e^{-\eta \tau})}{\eta}
\]
which is always true for positive $\alpha$ thanks to (21).

Since $c_1(\cdot)$ is admissible we have

$$V(k_0, c_0(\cdot)) \geq J(k_0, c_0(\cdot); c_1(\cdot)) = \frac{\alpha^{1-\gamma}}{\rho(1-\gamma)}$$

Now it is clear from what said above that it must be $\alpha \leq (A - \delta)k_0$, so the claim follows taking $M = \frac{(A - \delta)^{1-\gamma}}{\rho^{1-\gamma}}$. ■

Observe that condition (24) is the same condition which guarantees bounded utility in a standard AK model. Therefore habits formation does not affect this condition. Condition (24) will be assumed from now on without repeating it.

### 3.2 The equivalent infinite dimensional problem

We now rewrite our problem as an optimal control problem for ODE’s in an infinite dimensional space. Note that, differently from what has been done in the previous literature (see e.g. [23]) here the state equation (7) is not a DDE so the past of the control does not appear there. The past of the control strategy appears in the objective functional (6) and in the constraint (9). For this reason the way we choose to rewrite our problem is different from the one given in the previous literature.

We work in Hilbert space $M^2 = \mathbb{R} \times L^2([-\tau, 0]; \mathbb{R})$, with the scalar product defined by

$$\langle (x_0, x_1 (\cdot)), (y_0, y_1 (\cdot)) \rangle_{M^2} = x_0 y_0 + \int_{-\tau}^0 x_1(s) y_1(s) \, ds$$

for every $x = (x_0, x_1 (\cdot))$ and $y = (y_0, y_1 (\cdot))$ in $M^2$.

We first define, following e.g. [37], the structural state of the infinite dimensional system we want to study.

**Definition 5 (Structural state)** Given an initial datum $(k_0, c_0(\cdot)) \in \mathbb{R} \times L^1([-\tau, 0]; \mathbb{R})$, and a control strategy $c(\cdot) \in L^1_{\text{loc}}([0, +\infty); \mathbb{R})$ we define the structural state of our controlled dynamical system at time $t \geq 0$ as the element of $M^2$:

$$X(k_0, c_0(\cdot), c(\cdot)) (t) = \left( k_{k_0, c_0(\cdot)} (t), s \mapsto \int_{-\tau}^s \tilde{c}(t + u - s) e^{\eta u} \, du \right)$$

In the following we would write $X(t)$ for $X(k_0, c_0(\cdot), c(\cdot)) (t)$ when no confusion is possible. The second component of $X(t)$ is a function of $s \in [-\tau, 0)$ and we will usually write $X_1(t) [s]$ when we mean its value at time $t$ for given $s \in [-\tau, 0)$.

We now define the unbounded operator $A$ on $M^2$ by

$$D(A) = \{ (x_0, x_1(\cdot)) \in M^2, \ x_1 (\cdot) \in W^{1,2}([-\tau, 0]; \mathbb{R}), \ x_1(-\tau) = 0 \}$$

$$Ax = ((A - \delta) x_0, -x_1'(\cdot))$$

Moreover we define the operators

$$B : \mathbb{R} \to M^2, \quad Bc = c (-1, s \mapsto \varepsilon e^{\eta u})$$

and

$$D : \mathbb{R} \times C([-\tau, 0]; \mathbb{R}) \subset M^2 \to \mathbb{R}, \quad Dx = x_1(0).$$

Now we show that the structural state above satisfy a suitable ODE in the space $M^2$. 


Lemma 7 The adjoint of $\mathcal{A}$ and the claim follows.

The constraint $c \in \mathcal{C}_{ad}$ equivalent to our problem (or simply $X$). We now derive the adjoints of the operators $\mathcal{A}$ and $\mathcal{B}$. The proof easily follows. So the set of admissible control strategies for a given initial datum in $M^2$ is given by $\mathcal{C}_{ad}(x) = \{c(\cdot) \in L^1_{loc}([0,\infty);\mathbb{R})$, such that $X_0(t) \geq 0, c(t) \geq 0, c(t) \geq X_1(t)[0] \text{ for all } t\}$

The functional to be maximized becomes

$$J_0(x; c(\cdot)) := \int_0^{\infty} \frac{(c(t) - DX(t))^{1-\gamma}}{1-\gamma} e^{-\rho t} dt$$

The value function is defined as

$$V_0(x) := \max_{c(\cdot) \in \mathcal{C}_{ad}(x)} J_0(x; c(\cdot))$$

where we set $V_0(x) = -\infty$ if $\mathcal{C}_{ad}(x)$ is empty.

Now we derive the adjoints of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$.

Lemma 7 The adjoint of $\mathcal{A}$ in $M^2$ is the operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset M^2 \to M^2$ defined as

$$D(\mathcal{A}^*) = \{(y_0, y_1(\cdot)) \in M^2 : y_1(\cdot) \in W^{1,2}([-\tau,0];\mathbb{R}) \text{ and } y_1(0) = 0\}$$

$$\mathcal{A}^*(y_0, y_1(\cdot)) = (A - \delta) y_0, s \mapsto \frac{dy_1(s)}{ds}$$

Proof. Take $x \in D(\mathcal{A})$ and $y \in M^2$. We have

$$\langle \mathcal{A} x, y \rangle_{M^2} = (A - \delta) x_0 y_0 - \int_{-\tau}^{0} x_1'(s) y_1(s) ds$$

$$= (A - \delta) x_0 y_0 - x_1(0) y_1(0) + x_1(\tau) y_1(-\tau) + \int_{-\tau}^{0} x_1(s) y_1'(s) ds$$

It follows that the set of all $y \in M^2$ such that $x \to \langle \mathcal{A} x, y \rangle_{M^2}$ can be extended to a linear continuous functional on $M^2$ is given exactly by $D(\mathcal{A}^*)$. Then we have, for $x \in D(\mathcal{A})$ and $y \in D(\mathcal{A}^*)$,

$$\langle \mathcal{A} x, y \rangle_{M^2} = (A - \delta) x_0 y_0 + \int_{-\tau}^{0} x_1(s) y_1'(s) ds$$

and the claim follows. ■
Lemma 8 The adjoint of $\mathcal{B}$ is

$$\mathcal{B}^* : M^2 \to \mathbb{R}, \quad \mathcal{B}^*(y_0, y_1 (\cdot)) = -y_0 + \varepsilon \int_{-\tau}^{0} e^{\eta s} y_1 (s) \, ds.$$ Moreover the adjoint of $\mathcal{D}$ is

$$\mathcal{D}^* : \mathbb{R} \to \mathbb{R} \times [C([-\tau, 0]; \mathbb{R})]^*, \quad \mathcal{D}^* c = c(0, \delta_0)$$

where $\delta_0$ is the Dirac’s $\delta$ at the point $t = 0$.

Proof. We have

$$\langle \mathcal{B} c, (y_0, y_1 (\cdot)) \rangle_{M^2} = c \left( -y_0 + \varepsilon \int_{-\tau}^{0} e^{\eta s} y_1 (s) \, ds \right).$$

Moreover

$$\langle \mathcal{D} x, c \rangle_{\mathbb{R}} = cx_1 (0) = c(0 \cdot x_0 + \delta_0 x_1)$$

and the claim follows. \[ \blacksquare \]

3.3 The HJB equation and its explicit solution

The Current Value Hamiltonian $H_{CV}$ of our problem is a real valued function defined on the set

$$E \subset M^2 \times M^2 \times \mathbb{R}, \quad E = \{(x, p, c) \in D(A) \times M^2 \times \mathbb{R}\}$$

and is given by

$$H_{CV} (x, p; c) = \frac{(c - \mathcal{D} x)^{1-\gamma}}{1-\gamma} + \langle A x, p \rangle_{M^2} + \langle \mathcal{B}^* p, c \rangle_{\mathbb{R}} \quad (27)$$

When $\gamma > 1$, $H_{CV} (x, p; c)$ is not defined in the points such that $c = x_1 (0)$. In such points, since the utility is $-\infty$, we set $H_{CV} (x, p; c) = -\infty$. The maximum value of the Hamiltonian is defined by $H(x, p) = \sup_{c \geq x_1 (0)} H_{CV} (x, p; c)$. The HJB equation of the problem is then

$$\rho v (x) - H (x, Dv (x)) = 0 \quad (28)$$

where the unknown $v$ “should” be the value function $V_0$. We will use the following definition of solution. Note that it is different from the one used in the papers [7, 23].

Definition 9 We say that a function $v$ is a classical solution of the HJB equation (28) in an open set $\mathcal{Y} \subseteq M^2$ if it is differentiable at every $x \in \mathcal{Y}$ and if satisfies (28) in every point of $\mathcal{Y} \cap D(A)$.

Now we find a solution of the HJB equation. First we compute the maximum value Hamiltonian in the following lemma, whose proof is immediate.

Lemma 10 Given any $p \in M^2$ such that $\mathcal{B}^* p < 0$ and any $x \in D(A)$, the function

$$H_{CV} (x, p ; \cdot) : [x_1 (0), \infty) \to \mathbb{R}$$

admits a unique maximum point

$$c^{\max} = D x + (-\mathcal{B}^* p)^{-1/\gamma}.$$
So, in this case
\[ H(x,p) = \langle Ax, p \rangle_{M^2} + \frac{\gamma}{1 - \gamma} (-B^*p)^{\frac{1}{\gamma - 1}} + \langle Dx, B^*p \rangle_{\mathbb{R}}. \]

If, on the other hand, \( B^*p \geq 0 \), then
\[
\sup_{c \geq x_1(0)} H_{CV}(x,p;c) = +\infty.
\]

We now define, for \( x \in M^2 \),
\[
G(x) = \left( 1 - \varepsilon \int_{-\tau}^{0} e^{(A - \delta + \eta)s} ds \right) x_0 - \int_{-\tau}^{0} e^{(A - \delta)s} x_1(s) ds = \langle x, \kappa \rangle
\]
where \( \kappa = \left( 1 - \varepsilon \int_{-\tau}^{0} e^{(A - \delta + \eta)s} ds, s \mapsto -e^{(A - \delta)s} \right). \)

It is worth noting that \( \kappa_0 > 0 \) when we assume (22) i.e. that \( A - \delta > 0 \geq \varepsilon - \eta \). In fact looking at \( \kappa_0 \) as function of \( \tau \) we see that its derivative with respects to \( \tau \) is always negative. Since it converges to \( \frac{A - \delta + \eta - \varepsilon}{A - \delta + \eta} > 0 \) when \( \tau \to +\infty \), it must be always positive. We call \( \mathcal{X} \) the open subset of \( M^2 \) defined by
\[
\mathcal{X} = \{ x = (x_0, x_1(\cdot)) \in M^2, G(x) > 0 \}.
\]

**Proposition 11** The function \( v(x) = \nu(G(x))^{1-\gamma} \) with
\[
\nu = \frac{1}{1 - \gamma} \left( \rho - (A - \delta)(1 - \gamma) \right)^{-\gamma}
\]
is differentiable at all \( x \in \mathcal{X} \) and is a solution of the HJB equation in \( \mathcal{X} \).

**Proof.** Let \( v(x) = \nu(G(x))^{1-\gamma} \) for every \( x \in M^2 \).

Then
\[
Dv(x) = (1 - \gamma) \nu G(x)^{-\gamma} \kappa
\]

Since \( B^* \kappa = -1 \) we have
\[
B^* Dv(x) = (1 - \gamma) \nu G(x)^{-\gamma} B^* \kappa = -(1 - \gamma) \nu G(x)^{-\gamma}
\]
\[
\langle Dx, B^* Dv(x) \rangle_{\mathbb{R}} = -x_1(0)(1 - \gamma) \nu G(x)^{-\gamma}
\]

Now for \( x \in D(A) \) we have
\[
\langle Ax, Dv(x) \rangle_{M^2} = (1 - \gamma) \nu G(x)^{-\gamma} \langle Ax, \kappa \rangle_{M^2}.
\]

Moreover by definition of \( A \) and \( \kappa \) we have (integrating by parts and using that \( x(-\tau) = 0 \) since \( x \in D(A) \))
\[
\langle Ax, \kappa \rangle_{M^2} = (A - \delta) \left( 1 - \varepsilon \int_{-\tau}^{0} e^{(A - \delta + \eta)s} ds \right) x_0 + \int_{-\tau}^{0} x_1'(s) e^{(A - \delta)s} ds
\]
\[
= (A - \delta) \left( 1 - \varepsilon \int_{-\tau}^{0} e^{(A - \delta + \eta)s} ds \right) x_0 + x_1(0) - (A - \delta) \int_{-\tau}^{0} x_1(s) e^{(A - \delta)s} ds
\]
\[
= (A - \delta) \langle x, \kappa \rangle_{M^2} + x_1(0).
\]
It follows that
\[
H(x, Dv(x)) = (1 - \gamma) \nu G(x)^{1-\gamma} [(Ax, \kappa)]_{M^2} - x_1(0)] + \gamma \frac{1}{1 - \gamma} [(1 - \gamma) \nu] \nu G(x)^{1-\gamma} =
\]
\[
= (A - \delta)(1 - \gamma) \nu G(x)^{1-\gamma} + \gamma \frac{1}{1 - \gamma} [(1 - \gamma) \nu] \nu G(x)^{1-\gamma} =
\]
\[
= \nu G(x)^{1-\gamma} [(A - \delta)(1 - \gamma) + \gamma (1 - \gamma) \nu^{-\frac{1}{\gamma}}]
\]
We can now substitute all the above in the HJB equation getting
\[
\rho v(x) - H(x, Dv(x)) =
\]
\[
= \nu G(x)^{1-\gamma} \left[ \rho - (A - \delta)(1 - \gamma) - \gamma (1 - \gamma) \nu^{-\frac{1}{\gamma}} \right]
\]
and the claim follows by the definition of \( \nu \).

The optimal feedback policy associated to the above solution of the HJB equation (28) is easily found by Lemma 10 and is
\[
\varphi(x) = x_1(0) + \alpha G(x), \text{ for } x \in A
\]
where \( \alpha = \frac{\rho - (A - \delta)(1 - \gamma)}{\gamma} \). Observe that \( \alpha > 0 \) thanks to assumption (24).

3.4 Closed loop policy

We need to determine a set of admissible initial data included in \( A \) such that the candidate optimal feedback \( \varphi \) given in (29) is really optimal. For any \( x \) in this set we will have that \( v(x) = V_0(x) \).

We call \( C(M^2) \) the set of continuous functions from \( M^2 \) to \( \mathbb{R} \). As in Bambi et al. [7], we give definitions concerning feedback strategies.

**Definition 12** Given an initial condition \( q \in M^2 \), we call \( \psi \in C(M^2) \) a feedback strategy related to \( q \) if the equation
\[
\begin{cases}
\frac{dX(t)}{dt} = AX(t) + B(\psi(X(t))) \\
X(0) = q
\end{cases}
\]
has a unique solution \( X_\psi(t) \) in \( \Pi = \left\{ f \in C([0, \infty), M^2), \frac{df}{dt} \in L^2_{\text{loc}}([0, \infty), D(A')) \right\} \). The set of feedback strategies related to \( q \) is denoted \( FS_q \).

**Definition 13** Given an initial condition \( q \in M^2 \), and \( \psi \in FS_q \), we say that \( \psi \) is an admissible strategy if the unique solution \( X_\psi(t) \) of (30) satisfies \( \psi(X_\psi(\cdot)) \in C_{ad}(q) \). We denote \( AFS_q \) the set of admissible feedback strategies related to \( q \).

**Definition 14** We say that \( \psi \) is an optimal feedback strategy related to \( q \) if
\[
V(q) = \int_0^\infty \frac{(\psi(X_\psi(t)) - DX_\psi(t))^{1-\gamma}}{1 - \gamma} e^{-\rho t} dt
\]
We denote \( OFS_q \) the set of optimal feedback strategies related to \( q \).

We first prove that our candidate is always in \( FS_q \).
Lemma 15 For every $q \in M^2$, the map

$$\varphi : M^2 \to \mathbb{R}, \quad \varphi(x) = x_1(0) + \alpha G(x),$$

is in $FS_q$.

Proof. We have to prove that

$$\varphi(x) = x_1(0) + \alpha G(x).$$

Proof. It is enough to compute $\frac{d}{dt} \varphi(x(t))$.

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is in $FS_q$.

Proof. We have to prove that

$$\varphi(x) = x_1(0) + \alpha G(x).$$

Proof. It is enough to compute $\frac{d}{dt} \varphi(x(t))$.
Now we cannot, as done in other papers (see e.g. [7, 23], write
\[
\langle AX_\varphi (t), \kappa \rangle = \langle X_\varphi (t), A^* \kappa \rangle
\]
So we have to compute such term directly. Since
\[
A(x_0, x_1 (\cdot)) = (A - \delta) x_0, s \mapsto -\frac{dx_1 (s)}{ds}
\]
and
\[
\kappa = \left(1 - \varepsilon \int_{-\tau}^{0} e^{(A-\delta+\eta)s} ds, s \mapsto -e^{(A-\delta)s}\right),
\]
then, integrating by parts as in the proof of Proposition 11,
\[
\langle AX_\varphi (t), \kappa \rangle = (A - \delta) \langle X_\varphi (t), \kappa \rangle = X_{\varphi,1}(t)[0] + (A - \delta) G(X_\varphi (t))
\]
Moreover, since
\[
\varphi(X_\varphi(t)) = X_{\varphi,1}(t)[0] + \alpha G(X_\varphi (t))
\]
then
\[
\langle B\varphi(X_\varphi(t)), \kappa \rangle = \langle B(\varphi(X_{\varphi,1}(t)[0] + \alpha G(X_\varphi (t))), \kappa \rangle =
\]
\[
= (X_{\varphi,1}(t)[0] + \alpha G(X_\varphi (t))) \left(-1 + \varepsilon \int_{-\tau}^{0} e^{(A-\delta+\eta)s} ds - \varepsilon \int_{-\tau}^{0} e^{(A-\delta+\eta)s} ds\right)
\]
\[
= -X_{\varphi,1}(t)[0] - \alpha G(X_\varphi (t))
\]
Hence, summing up, we get
\[
\frac{d}{dt} G(X_\varphi (t)) = (A - \delta - \alpha) G(X_\varphi (t)).
\]
Using that
\[
A - \delta - \alpha = \Gamma
\]
the claim follows. ■

Now we define the set \( I \) and state a key invariance property of it.

**Proposition 17** The set \( I \) defined as
\[
I = X \cap \left\{ q = (x_0, x_1) \in \mathbb{R} \times W^{1,2}([-\tau, 0]; \mathbb{R}) \subset M^2, \begin{array}{l}
x_1(s) > 0 \text{ for all } s \in [-\tau, 0], \\
x_1(s) > 0 \text{ for all } s \in [-\tau, 0]
\end{array} \right\}
\]
is invariant for the flow of the autonomous ODE
\[
\frac{dX (t)}{dt} = AX (t) + B(\varphi (X (t))).
\]
Hence, if (22) holds, then for any \( q \in I \) we have \( \varphi \in AFS_q \).
Proof. Let \( q = (x_0, x_1(\cdot)) \in I \). We show that the associated solution \( X_\varphi(t) \) of (31) still belongs to \( I \) for every \( t > 0 \). Since we already know, by Proposition 16, that we always have \( G(X_\varphi(t)) > 0 \), it is enough to prove that, for every \( t > 0 \), \( X_\varphi,1(t)[0] > 0 \) and \( X_\varphi,1(t)[s] > 0 \) for almost all \( s \in [-\tau,0) \).

Let now \( t_0 \geq 0 \) be the supremum of all times \( t \) such that the above remain true. We show the \( t_0 = +\infty \). First of all observe that, by using the definition of structural state, we have, for \( t \geq 0 \) and \( s \in [-\tau,0] \),

\[
X_{\varphi,1} (t) [s] = \begin{cases} 
  & x_1(s-t) + \varepsilon \epsilon \eta(s-t) \int_0^t \epsilon (u) \epsilon (u) du, & \text{if } t - s - \tau < 0, \\
  & \epsilon (u) \epsilon (u) \int_0^t \epsilon (u) \epsilon (u) du, & \text{if } t - s - \tau \geq 0,
\end{cases}
\]

where

\[
\epsilon (u) = X_{\varphi,1}(u) [0] + \alpha G(X_\varphi(u)) \quad \text{for} \quad 0 \leq u < t.
\]

Since \( G(X_\varphi(t)) > 0 \) for every \( t \geq 0 \), from the above is clear that, for small \( t > 0 \) and for every \( s \in [-\tau,0] \) it must be \( X_{\varphi,1}(u) [s] > 0 \). So it must be \( t_0 > 0 \).

Now assume by contradiction the \( t_0 \) is finite. The we have

\[
\epsilon (u) = X_{\varphi,1}(u) [0] + \alpha G(X_\varphi(u)) > 0 \quad \text{for} \quad 0 \leq u < t_0.
\]

So according to (32) it must be

\[
X_{\varphi,1} (t_0) [s] > 0
\]

for every \( s \in [-\tau,0] \). This gives the contradiction showing the invariance of \( I \) since clearly the \( W^{1,2} \) regularity in \( s \) preserves due to (32).

Finally we observe that, if (22) holds, then \( X_{\varphi,0}(t) > 0 \) for all \( t \geq 0 \). Indeed, since \( G(X_\varphi(t)) > 0 \) we must have

\[
1 - \int_{-\tau}^0 e^{(A-\xi+\eta)s} ds < 0.
\]

Recalling that assumption (22) implies that \( 1 - \int_{-\tau}^0 e^{(A-\xi+\eta)s} ds > 0 \) (see the discussion before Proposition 11) we immediately get \( X_{\varphi,0}(t_0) > 0 \). Thus \( \varphi \in \text{OFS}_p \).

It now remains to prove that \( \varphi \in \text{OFS}_q \).

Proposition 18 If \( q \in I \) and (22) holds, then \( \varphi \in \text{OFS}_q \). The associated state-control couple is the unique optimal couple of the problem.

Proof. Let us consider the solution of the HJB equation \( v(x) = \nu(G(x))^{1-\gamma} \) and the function

\[
\bar{v}(t,x) : \mathbb{R} \times M^2 \to \mathbb{R},
\]

\[
\bar{v}(t,x) = e^{-\rho t} v(x)
\]

Now take \( q \in I \), take any admissible control \( \epsilon(\cdot) \in C_{ad}(q) \) and call \( X(\cdot) \) the associated state trajectory starting at \( q \). Then we have

\[
\frac{d\bar{v}(t,X(t))}{dt} = -\rho e^{-\rho t} v(X(t)) + e^{-\rho t} < Dv(X(t)) , AX(t) + Bc(t) >
\]

Observe that the above make sense since, by construction (see e.g. (32)) it must be \( X(t) \in D(A) \) when \( q \in I \).
Integrating on $[0, \tau]$ yields

\[ e^{-\rho \tau} v(X(\tau)) - v(X(0)) = \int_0^\tau e^{-\rho t} [-\rho v(X(t)) + <Dv(X(t)), AX(t)> + <B^*Dv(X(t)), c(t)>] \, dt \] (33)

Now

\[ G(X(t)) = \left(1 - \varepsilon \int_{-\tau}^0 e^{(A-\delta+\eta)s} \, ds \right) X_0(t) - \int_{-\tau}^0 e^{(A-\delta)s} X_1(t)[s] \, ds. \]

Since, as noted before Proposition 11, we have $1 - \varepsilon \int_{-\tau}^0 e^{(A-\delta+\eta)s} \, ds > 0$, then it must be $G(X(t)) \leq \left(1 - \varepsilon \int_{-\tau}^0 e^{(A-\delta+\eta)s} \, ds \right) X_0(t)$, thus

\[ e^{-\rho t} G(X(t))^{1-\gamma} \leq \left(1 - \varepsilon \int_{-\tau}^0 e^{(A-\delta+\eta)s} \, ds \right)^{1-\gamma} e^{-(\rho-(1-\gamma)(A-\delta))\tau} \left( X_0(t) \right)^{1-\gamma}. \]

According to Proposition 1 (since clearly $X_0(t) \leq kM(t)$) we thus have that

\[ \lim_{\tau \to \infty} e^{-\rho \tau} v(X(\tau)) = 0. \]

Hence, using that $q = X(0)$ and taking the limit as $\tau$ tends to infinite in (33), we obtain

\[ -v(q) = \int_0^\infty e^{-\rho t} [-\rho v(X(t)) + <Dv(X(t)), AX(t)> + <B^*Dv(X(t)), c(t)>] \, dt \] (34)

so using the definition (27) of current value Hamiltonian

\[ v(q) - J_0(q;c(\cdot)) = \int_0^\infty e^{-\rho t} \left( \rho v(X(t)) - H_{CV} (X(t), Dv(X(t)), c(t)) \right) \, dt \]

As the value function solves $\rho v(x) - H(x, Dv(x)) = 0$, the above implies that

\[ v(q) - J_0(q;c(\cdot)) = \int_0^\infty e^{-\rho t} \left[ H(X(t), Dv(X(t))) - H_{CV} (X(t), Dv(X(t)), c(t)) \right] \, dt \] (35)

According to the definition of $H$, for every admissible control the integrand of the above right hand side is always positive. This implies, according to the definition of $V_0$, that

\[ v(q) \geq V_0(q) \]

and this must be true for every $q \in I$. Moreover, choosing $c(t) = \varphi(X_\varphi(t))$ (which is admissible thanks to Proposition 17) clearly the right hand side becomes zero and so such control strategy is optimal. This implies that $v(q) = V_0(q)$ for every $q \in I$.

Finally, if $c^1(\cdot)$ is another optimal strategy (with associated state trajectory $X^1(\cdot)$) it must satisfy (35) (where now $v = V_0$ since they are equal on $I$). So it must be necessarily, for a.e $t \geq 0$,

\[ H(X^1(t), Dv(X^1(t))) - H_{CV} (X^1(t), Dv(X^1(t)), c^1(t)) = 0 \]

which implies that, t a.e., $c^1(t) = \varphi(X^1(t))$. By the uniqueness of the solutions of the closed loop equation (31) proved in Lemma 15, we then get that, t a.e., $c_1(t) = c(t)$. \(\blacksquare\)
Remark 19 To get the explicit solution \( v \) of the HJB equation and to find the optimal closed loop policy \( \varphi \), we cannot apply directly the approach used in [23] or in [7] due to the presence of the delay in the constraint and in the objective functional instead than in the state equation. This changes the structure of the problem. In particular, differently from the case treated in the previous papers, the gradient \( Dv \) of the solution of the HJB equation does not belong to \( D(A^*) \) and so the concept of solution of such equation must be changed (compare Definition 9 with the analogous one of such papers). This fact induces a change in the arguments of the main proofs: the fact that \( v \) solves the HJB equation and the fact that the feedback strategy \( \varphi \) is admissible and optimal.

It is worth noting that the optimality of \( \varphi \) depends on the initial datum \( q \) to belong to the set \( I \); this implicitly implies a restriction on the initial value of capital which we may choose. This restriction will be made explicit in the next Section after Proposition 20 and its economic meaning will be also explained.

4 Equivalence

The result of the previous section gives the optimal feedback map in the infinite dimensional setting. We now use this result to write the closed loop policy formula in the delay differential equation setting and we use it to prove the equivalence between the cases of internal and external habits.

Proposition 20 Given any initial datum \((k_0, c_0(\cdot))\) the problem (P) above has a unique optimal state-control couple \((k^*(\cdot), c^*(\cdot))\). Such couple is the only one that satisfies the closed-loop formula:

\[
\frac{c(t) - h(t)}{A - \delta - \Gamma} = \left(1 - \varepsilon \int^{t}_{t-\tau} e^{(A-\delta+\eta)s} ds\right) k(t) - \left[\frac{h(t)}{A - \delta + \eta} - \varepsilon e^{-(A-\delta+\eta)\tau} \int^{t}_{t-\tau} e^{(A-\delta)(t-s)} \tilde{c}(s) \ ds\right]
\]

where \( h(t) \) is given by (12).

Proof. We have, by the definition of the optimal feedback map \( \varphi \), that, on the optimal path,

\[
c(t) - h(t) = \alpha G(X(t))
\]

\[
= (A - \delta - \Gamma) \left[\left(1 - \varepsilon \int^{0}_{-\tau} e^{(A-\delta+\eta)s} ds\right) X_0(t) - \int^{0}_{-\tau} e^{(A-\delta)s} X_1(t)[s] ds\right]
\]

Now we know that \( X_0(t) = k(t) \) while \( X_1(t)[s] = \varepsilon \int^{s}_{-\tau} \tilde{c}(t + u - s) e^{\eta u} du \) so, substituting, we have,

\[
\frac{c(t) - h(t)}{A - \delta - \Gamma} = \left(1 - \varepsilon \int^{0}_{-\tau} e^{(A-\delta+\eta)s} ds\right) k(t) - \varepsilon \int^{0}_{-\tau} e^{(A-\delta)s} \int^{s}_{-\tau} \tilde{c}(t + u - s) e^{\eta u} du ds
\]

Now we integrate by parts obtaining, with straightforward computations,

\[
\int^{0}_{-\tau} e^{(A-\delta)s} \int^{s}_{-\tau} \tilde{c}(t + u - s) e^{\eta u} du ds = \frac{1}{A - \delta + \eta} \int^{0}_{-\tau} c(t + v) e^{\eta v} dv - e^{-(A-\delta+\eta)\tau} \int^{t}_{t-\tau} e^{(A-\delta)(t-s)} \frac{1}{A - \delta + \eta} c(s) \ ds
\]
which gives the claim. 

Using the above result it is not difficult to prove, by straightforward computations, that given any initial data \((k_0, c_0(\cdot))\), there exists a \(\Lambda\) such that, along an optimal trajectory, the optimal control \(c(\cdot)^*\) satisfies

\[
c(t) - h(t) = \Lambda e^{\Gamma t}
\]

with

\[
\Lambda = (A - \delta - \Gamma) \cdot \left(1 - \varepsilon \int_{-\tau}^{0} e^{(A-\delta+\eta)s} ds\right) \cdot \left(\frac{h(0)}{A - \delta + \eta} - \varepsilon e^{-(A-\delta+\eta)\tau} \int_{-\tau}^{0} e^{-(A-\delta+\eta)s} c(s) ds\right).
\]

(36)

It is worth noting that the constraint, \(c(t) \geq h(t)\), is respected if \(\Lambda > 0\) or equivalently, in term of the initial capital stock, if

\[
k(0) \geq \frac{h(0)}{A - \delta + \eta - \varepsilon + \varepsilon e^{-(A-\delta+\eta)\tau} \int_{-\tau}^{0} e^{-(A-\delta+\eta)u} c(u) du}.
\]

(37)

In the specific case, \(\tau = \infty\) and \(\varepsilon = \eta\) this condition becomes \(rk(0) > h(0)\) meaning that capital income (which in our context coincides with the initial wealth) has to be higher than the initial habits otherwise an initial consumption higher than \(h(0)\) will pin down a consumption path not sustainable over time since financed with the resources coming from disinvestments.

In the case with a finite \(\tau\) this condition becomes less restrictive as the first term in the right hand side of the inequality becomes smaller and the second negative term appears. The reason is that the stock of habits is now formed over a finite consumption history and therefore less resources are needed at the beginning because the past consumption affecting the habit formation will be completely “depreciated” after a period of length \(\tau\).

At this point we have all the information for proving the main result of the paper.

**Theorem 21 (Equivalence Theorem)** Consider an economy with subtractive nonseparable C.E.S. utility function and linear technology. Then internal and external habits lead to the same unique equilibrium path. This path is Pareto optimal.

**Proof.** The closed loop policy formula for the external case was found in Augeraud-Veron and Bambi [4] (see proof of Proposition 4) using a modified version of the Pontryagin Maximum Principle. Such function writes:

\[
\frac{c(t) - h(t)}{A - \delta - \Gamma} = k(t) - \frac{1}{A - \delta + \eta} \left[ h(t) + \varepsilon \left(1 - e^{-(A-\delta+\eta)\tau}\right) k(t) \right. \left. -\varepsilon e^{-\eta \tau} e^{-(A-\delta)\tau} \int_{t-\tau}^{t} e^{-(A-\delta)(u-t)} c(u) du\right]
\]

(37)

Therefore the equivalence between external and internal habits emerges immediately comparing it with the result of Proposition 20. It is worth noting that the initial value of the costate variable found in [4] is exactly equal to the constant \(\Lambda\) given in (36), as expected.

Therefore the market equilibrium path in the case of external habits has been proved to be Pareto optimal since it coincides with the optimal path derived by solving the problem with internal habits. Interestingly enough this result is robust to any selections of the key parameters in the economy and even more importantly to any specification of the parameter \(\tau\) capturing the consumption history relevant in the formation of the habits.
5 Concluding remarks

In this paper, we have shown that internal and external habits may lead to the same closed loop policy function and then to the same Pareto optimal equilibrium path. Interestingly enough this anomaly emerges when the utility function is subtractive nonseparable but not for the multiplicative nonseparable case as emerged from Carroll et al. [19]. Therefore models with habits formation and subtractive utility function needs a higher degree of heterogeneity, as for example different initial endowments across the households, to guarantee a clear distinction between the external and internal specification. Alternatively, the multiplicative nonseparable formulation should be preferred and used.

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