A renormalizable theory of gravity, so-called the Hořava–Lifshitz gravity, has been attracting great interest since the advent of the seminal works of Hořava [1, 2]. This theory does not have the full diffeomorphism invariance, but the following anisotropic scaling with dynamical critical exponent $z$ larger than the spatial dimensions $d$ exists

$$x^i \to b x^i \quad (i = 1, \ldots, d), \quad t \to b^z t. \quad (1)$$

This theory has a remarkable property such that it describes the non-relativistic theory of gravity in the UV regime, but becomes the Einstein gravity in the IR region. It was extensively applied to a resolution of the cosmological problem including inflation and non-Gaussianity in [3–21], and new solutions were constructed in [22–28]. For the other recent progress, refer the reader to [29–53].

It becomes convenient in the following discussion to take the Wick rotation to the Euclidean space. The ADM decomposition of the $(d + 1)$-dimensional Euclidean space is

$$ds^2 = N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (2)$$

and the extrinsic curvature for the spacelike slice is given by

$$K_{ij} = \frac{1}{N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (3)$$
The action of the Hořava–Lifshitz gravity is defined as \[ S = \frac{2}{\kappa^2} \int_{t_0}^{t} dt \int_{\Sigma_t} d^d x \sqrt{g} N \left( K_{ij} G^{ijkl} K_{kl} + \frac{\kappa^4}{16} E^{ij} E_{ijkl} E^{kl} \right), \] (4)
where we set a future boundary at \( t = t_0 \) and \( G_{ijkl} \) is the inverse of the DeWitt metric \( G^{ijkl} \) of the space of metrics
\[ G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl}, \] (5)
\[ G_{ijkl} = \frac{1}{2} \left( g_{ik} g_{jl} + g_{il} g_{jk} \right) - \xi g_{ij} g_{kl}, \] (6)
The first and second terms in (4) are the kinetic and potential terms, respectively, and \( E^{ij} \) does not contain the time derivative of \( g_{ij} \). When we require \( E^{ij} \) to be a gradient of some function \( W[g] \) with respect to the metric \( g_{ij} \)
\[ E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W[g]}{\delta g_{ij}}, \] (7)
it is called a ‘detailed balance condition’ [1, 2]. In general, in the context of condensed matter physics, it is known that theories which satisfy the detailed balance condition have simpler quantum properties than a generic theory, and that the renormalization properties in \((d + 1)\) dimensions are often inherited from the simpler renormalization of the theory in \( d \) dimensions with the action \( W \). It still remains to be understood what the detailed balance condition means in the context of the Hořava–Lifshitz gravity, however. We would like to get a deep insight into the role of it.
In this paper, we derive the detailed balance condition as a solution to the Hamilton–Jacobi equation in the Hořava–Lifshitz gravity. This result leads us to propose the existence of the \( d \)-dimensional quantum field theory with the effective action \( W \) on the future boundary of the \((d + 1)\)-dimensional Hořava–Lifshitz gravity from the viewpoint of the holographic renormalization group [54]. This proposal reminds us of the dS/CFT correspondence [55], while the detailed balance condition forces the cosmological constant to be always negative\(^1\). In addition, we obtain a Ricci flow equation of the boundary theory as the holographic RG flow, which is the Hamilton equation in the bulk gravity.
Although this proposal is the most salient feature of our work, we should emphasize that we can derive the detailed balance condition and the Ricci flow equation based on the Hamiltonian formulation of the Hořava–Lifshitz gravity without the holography.
The Hamiltonian formulation in the bulk gravity affords us the renormalization group flow of the field theory on the future boundary \( t = t_0 \) along the time direction as the Hamilton equation. Such a flow is termed a holographic renormalization group flow initiated by [54] (for a comprehensive review see [56] and references therein).
The ADM decomposition (2) is suitable for the Hamiltonian formulation. The conjugate momentum associated with \( g_{ij} \) in the Hořava–Lifshitz action (4) is \((\dot{\pi}^{ij} \equiv \partial / \partial t)\)
\[ \dot{\pi}^{ij} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \dot{g}_{ij}} = G^{ijkl} K_{kl}, \] (8)
and the momenta conjugate to \( N \) and \( N_i \) are identically zero. The Hamiltonian is
\[ H = \int_{\Sigma_t} d^d x \sqrt{g} \left[ N H + N_i \dot{P}^i + 2 \nabla_j (\pi^{ij} N_i) \right], \] (9)
\(^1\) We are grateful to S Mukohyama for pointing out this feature.
The last term in the second line of (9) vanishes when $\Sigma_t$ is compact space; this is the case we focus on here.

In the Hamiltonian formulation, the conjugate momentum $\pi^{ij}$ becomes the independent variable instead of $\dot{g}^{ij}$. Using the momentum, we can take the action to the one in the first-order form

$$S[g_{ij}, \pi^{ij}, N^i, N_i] = \int dt \int_{\Sigma_t} d^d x \sqrt{g} \left[ \pi^{ij} \dot{g}^{ij} - N^i \dot{H} - N_i \dot{P}^i \right].$$

(12)

Varying this action, we obtain the Hamilton equation

$$\dot{g}^{ij} = 2N G^{ijkl} \pi^{kl} + \nabla_i N_j + \nabla_j N_i,$$

(13)

with the Hamiltonian and momentum constraints

$$H = P = 0.$$  

(14)

Substituting the classical solution $g_c$ into the action (12) and integrating it along the time direction, one can express $S[g]$ as a surface integral with respect to $g_c(x, t_0)$

$$S[g = g_c] = S_{bdy} [g_c(x, t_0)].$$

(15)

It follows from this relation that $S_{bdy}$ is the effective action of the $d$-dimensional quantum field theory on the future boundary $\Sigma_t$, using the bulk/boundary relation $Z_{\text{gravity}} = Z_{\text{QFT}}$ and $Z_{\text{gravity}} = \exp(-S)$ in a manner similar to the AdS/CFT correspondence [57–59].

In this case, the momentum is expressed in terms of the boundary action (see [56] for a careful derivation)

$$\pi^{ij}(x, t_0) = \frac{1}{\sqrt{g}} \frac{\delta S_{bdy}[g_{ij}]}{\delta g_{ij}}.$$  

(16)

We use the same notation $g_{ij}$ to denote $g_c(x, t_0)$ for simplicity hereafter. Inserting these relations into the constraints (14), we obtain the momentum constraint

$$\nabla_j \left[ \frac{1}{\sqrt{g}} \frac{\delta S_{bdy}}{\delta g_{ij}} \right] = 0,$$

(17)

which indicates the conservation law of the energy momentum tensor in the $d$-dimensional QFT, and the Hamilton–Jacobi equation from the Hamiltonian constraint

$$\left( \frac{1}{\sqrt{g}} \frac{\delta S_{bdy}}{\delta g_{ij}} \right) G_{ijkl} \left( \frac{1}{\sqrt{g}} \frac{\delta S_{bdy}}{\delta g_{kl}} \right) = \frac{k^4}{16} E^{ij} G_{ijkl} E^{kl}.$$  

(18)

This equation is easily solved

$$E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W[g]}{\delta g_{ij}},$$

(19)

We must mention that there might exist other solutions than the one (19) we obtained here. In fact, there is an ambiguity such that we can add a zero norm term to the right (or left)-hand side of (19), though we can remove this ambiguity by adding a zero norm term to $E^{ij}$ in the action from the beginning.
where \( W \) stands for the rescaled effective action of the \( d \)-dimensional QFT
\[
W[g] = \frac{4}{\kappa^2} S_{\text{bdy}}[g].
\] (20)
The solution (19) to the Hamiltonian constraint results in the detailed balance condition (7) and we find that \( W \) is the (rescaled) effective action of QFT on the future boundary of the Hořava–Lifshitz gravity.

It is worth mentioning that the Hamilton equation (13) gives us the holographic renormalization group flow after substituting (16) into it
\[
\dot{g}_{ij}|_{t=t_0} = \frac{\kappa^2}{2N} G_{ijkl} \frac{1}{\sqrt{g}} \frac{\delta W}{\delta g_{kl}} + \nabla_i N_j + \nabla_j N_i.
\] (21)
This is a simpler equation in first order as opposed to the original equation of motion which is second order in time derivatives. The Hamilton equation for \( \pi_{ij} \) is automatically satisfied and it is enough to solve (13). As was mentioned in [1, 2], we can easily find classical solutions in the Hořava–Lifshitz gravity by just solving this equation.

Moreover, this equation has a remarkable property as follows. When we take the \( d \)-dimensional effective action
\[
W = \frac{1}{\kappa_W^d} \int d^d x \sqrt{-g} \left( -R + \Lambda_W \right),
\] (22)
the \((d+1)\)-dimensional theory becomes the Hořava–Lifshitz gravity with dynamical critical exponent \( z = 2 \). The holographic RG flow (21) is given by [1]
\[
\dot{g}_{ij}|_{t=t_0} = -\frac{\kappa^2}{2\kappa_W^d} N \left( R_{ij} + \frac{1-2\lambda}{2(d\lambda - 1)} (R - 2\Lambda_W)g_{ij} \right) + \nabla_i N_j + \nabla_j N_i.
\] (23)
If we take \( \lambda = 1/2, \kappa_W = \kappa/2, N = 1 \) and \( N_i = 0 \), this becomes the Ricci flow equation. From this viewpoint one may say that the Ricci flow in \( d \) dimensions is the holographic RG flow to the \((d+1)\)-dimensional Hořava–Lifshitz gravity with \( z = 2 \) and \( \lambda = 1/2 \).

One simple but interesting application of our results is that static solutions in the Hořava–Lifshitz gravity are obtainable by solving
\[
\frac{\delta W}{\delta g_{ij}} = 0.
\] (24)
In the case of the four-dimensional Hořava–Lifshitz gravity with \( z = 3 \), \( W \) is the Einstein–Hilbert action with the gravitational Chern–Simons term in three dimensions (i.e. the topologically massive gravity). Equation (24) is just the Einstein equation in TMG and the interesting solutions were constructed in [60–62]. For example, we can construct the four-dimensional solitonic solution by the use of the Euclidean warped AdS3 black hole. It would be of wide interest to investigate such a solution in higher dimensions with different dynamical exponents \( z \).

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\[^4\] We are grateful to K Izumi for informing us of this point.

\[^5\] We can set \( N_i = 0 \) using the \( d \)-dimensional diffeomorphism, but we are not sure if we can take \( N = 1 \) by the reparametrization of the time \( t \rightarrow f(t) \).

\[^6\] One can find the same discussion in [1] under the assumption of (21).
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