Thermodynamics of scalar–tensor gravity

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Previously, the Einstein equation has been described as an equation of state, general relativity as the equilibrium state of gravity, and \( f(R) \) gravity as a non-equilibrium one. We apply Eckart’s first order thermodynamics to the effective dissipative fluid describing scalar-tensor gravity. Surprisingly, we obtain simple expressions for the effective heat flux, “temperature of gravity”, shear and bulk viscosity, and entropy density, plus a generalized Fourier law in a consistent Eckart thermodynamical picture. Well-defined notions of temperature and approach to equilibrium, missing in the current thermodynamics of spacetime scenarios, naturally emerge.

I. INTRODUCTION

The idea that there is a deep connection between gravity and thermodynamics, originating in black hole thermodynamics, took a new meaning with Jacobson’s seminal work \[1\] in which the Einstein field equation of general relativity (GR) was derived as an equation of state using purely thermodynamic considerations. This fact has deep implications for viewing gravity as an emergent, rather than fundamental, phenomenon and for quantum gravity as well. In this picture, quantizing the Einstein equation would make no more sense than quantizing the macroscopic ideal gas equation of state, which cannot produce fundamental results such as the energy spectrum and eigenfunctions of the hydrogen atom. In quantum gravity, the “atoms of spacetime” (if they exist) would have to be found with a radically different approach.

A second, equally important, idea was proposed in Ref. \[2\], which derived the field equation of fourth order metric \( f(R) \) gravity using thermodynamics. This modification of GR would correspond to dissipative, non-equilibrium “thermodynamics of gravitational theories” in which a “bulk viscosity of spacetime” is introduced to explain dissipation, while GR corresponds to equilibrium thermodynamics instead. These works have generated a very large literature. In particular, Ref. \[3\] stressed the essential role of shear viscosity while eliminating bulk viscosity from this picture. In spite of the large literature, the equations ruling how modified gravity approaches the equilibrium state of gravity, and \( f(R) \) gravity using thermodynamics. This approach is described as an effective relativistic dissipative fluid \[4, 5\]. We then apply Eckart’s first order thermodynamics \[10\] to this effective fluid and extract explicit expressions for the relevant effective thermodynamic quantities, including the heat current density, “temperature of modified gravity”, viscosity coefficients, and entropy density.

To summarize our results, the temperature is positive-definite and vanishes at the GR equilibrium state; the bulk viscosity is absent, and the shear viscosity coefficient is negative, which allows for the possibility that the entropy density decreases, in agreement with the fact that the system (the \( \phi \)-fluid) is not isolated. What is more, we provide an equation describing explicitly the approach of scalar-tensor gravity to the GR equilibrium state, which is Eckart’s generalization of the Fourier law modelling diffusion \[10\].

Begin with the scalar-tensor action \[1\]

\[
S_{ST} = \int d^4x \sqrt{-g} \left[ \phi \nabla^2 \phi - \frac{\omega(\phi)}{\phi} \nabla^a \phi \nabla_a \phi - V(\phi) \right] + S^{(m)},
\]

where \( R \) is the Ricci scalar, the Brans-Dicke scalar \( \phi > 0 \) is approximately the inverse of the effective gravitational coupling, \( \omega(\phi) \) is the “Brans-Dicke coupling”, \( V(\phi) \) is a potential, and \( S^{(m)} = \int d^4x \sqrt{-g} L^{(m)} \) is the matter action. The corresponding field equations \[\delta(1), \delta(2)\] are written as the effective Einstein equations

\[
\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = -\frac{8\pi}{\phi} \mathcal{T}^{(m)}_{ab} + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi \right) - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi,
\]

\[
\square \phi = \frac{1}{2\omega + 3} \left( \frac{8\pi}{\phi} \mathcal{T}^{(m)} + \phi \frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi} \nabla^a \phi \nabla_a \phi \right),
\]

We follow the notation of Ref. \[6\] and we use units in which Newton’s constant \( G \) and the speed of light \( c \) are unity.

\vspace{1cm}

\[\delta(1)\]
where $R_{ab}$ is the Ricci tensor and $T^{(m)}_{ab}$ is the trace of the matter stress-energy tensor $T^{(m)}_{ab}$. The terms containing $\phi$ and its derivatives form an effective $\phi$-fluid with stress-energy tensor

$$8\pi T^{(\phi)}_{ab} = \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla_c \phi \right) + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi} g_{ab} ,$$

(4)

II. EFFECTIVE SCALAR FIELD FLUID

When the gradient $\nabla^a \phi$ is timelike, it is used to construct the normalized effective fluid 4-velocity

$$u^a = \frac{\nabla^a \phi}{\sqrt{-\nabla^a \phi \nabla_a \phi}} .$$

(5)

The 3+1 splitting of spacetime into the time direction $u^c$ and the 3-dimensional space of the observers comoving with the fluid follows. Their 3-space is endowed with the Riemannian metric $h_{ab} = g_{ab} + u_a u_b$ and $h^a_b$ is the projection operator on this 3-space. The effective fluid 4-acceleration $u^a \equiv u^b \nabla_b u^a$ is orthogonal to the 4-velocity. The spatial projection of the velocity gradient

$$V_{ab} \equiv h^c_a h^d_b \nabla_c u_d = \sigma_{ab} + \frac{\theta}{3} h_{ab} ,$$

(6)

coincides with the symmetric expansion tensor $\sigma_{ab}$ since its antisymmetric part (the vorticity $\omega_{ab}$) vanishes, as $u^a$ derives from a gradient. $u^a$ is irrotational and hypersurface-orthogonal \[8, 9\]. Here $\theta \equiv \theta^c_c = \nabla_c u^c$, while $\sigma_{ab} = \theta_{ab} - \theta h_{ab}/3$ is the symmetric, trace-free shear tensor. The velocity gradient splits as \[9\]

$$\nabla_b u_a = \sigma_{ab} + \frac{\theta}{3} h_{ab} - \dot{u}_a u_b .$$

(7)

When these general definitions \[8, 9\] are specialized to the effective $\phi$-fluid, one obtains \[9\]

$$\dot{u}_a = \left(- \nabla^c \phi \nabla_c \phi \right)^{-2} \nabla^b \phi \left[ - \left(- \nabla^c \phi \nabla_c \phi \right) \nabla_a \nabla_b \phi + \nabla^c \phi \nabla_b \nabla_c \phi \nabla_a \phi \right] .$$

(8)

$$\theta = \frac{\Box \phi}{\left(- \nabla^c \phi \nabla_c \phi \right)^{1/2}} + \frac{\nabla_a \nabla_b \phi \nabla^a \phi \nabla^b \phi}{\left(- \nabla^c \phi \nabla_c \phi \right)^{3/2}} ,$$

(9)

$$\sigma_{ab} = \left(- \nabla^c \phi \nabla_c \phi \right)^{-3/2} \left[ - \left(- \nabla^c \phi \nabla_c \phi \right) \nabla_a \nabla_b \phi + \nabla^c \phi \nabla_b \nabla_c \phi \nabla_a \phi - \frac{1}{3} \left( \nabla_a \phi \nabla_b \phi - g_{ab} \nabla^c \phi \nabla_c \phi \right) \right] ,$$

(10)

The effective stress-energy tensor \[11\] of the Brans-Dicke-like field takes the imperfect fluid form

$$T_{ab} = \rho u_a u_b + g_{ab} u^c u_c + \Pi_{ab} , \quad \Pi_{ab} = P h_{ab} + \pi_{ab} ,$$

(11)

with effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stresses \[1, 5\]

$$8\pi \rho^{(\phi)} = \frac{-\omega}{2\phi^2} \nabla^c \phi \nabla_c \phi + \frac{V}{2\phi} + \frac{1}{\phi} \left( \Box \phi - \frac{\nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi}{\nabla^c \phi \nabla_c \phi} \right) ,$$

(12)

$$8\pi q_a^{(\phi)} = \frac{\nabla^c \phi \nabla^d \phi}{\phi \left(- \nabla^c \phi \nabla_c \phi \right)^{3/2}} \left( \nabla_d \phi \nabla_c \phi - \nabla_a \phi \nabla_c \nabla_d \phi \right) ,$$

(13)

$$8\pi \Pi_{ab}^{(\phi)} = \left(- \frac{\omega}{2\phi^2} \nabla^c \phi \nabla_c \phi - \frac{\Box \phi}{\phi} - \frac{V}{2\phi} \right) h_{ab} + \frac{1}{\phi} h^c_a h^d_b \nabla_c \nabla_d \phi ,$$

(14)

$$8\pi P^{(\phi)} = \frac{-\omega}{2\phi^2} \nabla^c \phi \nabla_c \phi - \frac{V}{2\phi} - \frac{1}{3\phi} \left( 2 \Box \phi + \frac{\nabla^a \phi \nabla^b \phi \nabla_a \phi \nabla_b \phi}{\nabla^c \phi \nabla_c \phi} \right) ,$$

(15)

$$8\pi \pi_{ab}^{(\phi)} = \frac{1}{\phi \nabla^c \phi \nabla_c \phi} \left[ \frac{1}{3} \left( \nabla_a \phi \nabla_b \phi - g_{ab} \nabla^c \phi \nabla_c \phi \right) \left( \Box \phi - \frac{\nabla^c \phi \nabla^d \phi \nabla_d \nabla_c \phi}{\nabla^c \phi \nabla_c \phi} \right) + \nabla^d \phi \left( \nabla_d \phi \nabla_a \nabla_b \phi - \nabla_b \phi \nabla_a \nabla_d \phi - \nabla_a \phi \nabla_b \nabla_d \phi + \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi \right) \right] .$$

(16)

III. ECKART'S THERMODYNAMICS FOR SCALAR-TENSOR GRAVITY

Since it is a fact that the field equations of scalar-tensor gravity assume the form of effective Einstein equations
as source, it makes sense to take this property further and examine the thermodynamical context of this imperfect fluid, which is Eckart’s first order thermodynamics. The non-causal spacelike heat flow finds its place in Eckart’s theory which, albeit non-causal, is widely used as a first approach to relativistic thermodynamics.

In Eckart’s first order thermodynamics \([10, \text{see also } 11]\), the dissipative quantities (viscous pressure \(P_{\text{vis}}\), heat current density \(q^a\), and anisotropic stresses \(\pi_{ab}\)) are related to the expansion \(\theta\), temperature \(T\), and shear tensor \(\sigma_{ab}\) by the constitutive equations \([10]\)

\[
P_{\text{vis}} = -\zeta \theta, \tag{17}
\]
\[
q_a = -K \left( h_{ab} \nabla^b T + \mathcal{T}_a \right), \tag{18}
\]
\[
\pi_{ab} = -2\eta \sigma_{ab}, \tag{19}
\]
where \(\zeta\) is the bulk viscosity, \(K\) is the thermal conductivity, and \(\eta\) is the shear viscosity. The comparison of Eqs. \([10] \) and \([8]\) yields \([2]\)

\[
q_a^{(\phi)} = -\frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8\pi \phi} \mathcal{U}_a. \tag{20}
\]

In the comoving frame, the spatial temperature gradient vanishes identically and the heat flow arises solely from the inertia of energy. The Eckart temperature of the \(\phi\)-fluid, which can be called the “temperature of scalar-tensor gravity”, is then \([3]\)

\[
\mathcal{T} = \frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8\pi K \phi}; \tag{21}
\]

it is positive definite and vanishes when \(\phi = \text{const}\), which corresponds to GR.

The structure of \(T_{ab}^{(\phi)}\) does not allow for bulk viscosity, hence \(\zeta = 0\). Comparing Eqs. \([10] \) and \([10] \) for \(\pi_{ab}^{(\phi)}\) and \(\sigma_{ab}^{(\phi)}\) and using Eq. \([19]\) yields

\[
\eta = -\frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{16\pi \phi} = -\frac{KT}{2} < 0. \tag{22}
\]

Negative viscosities are common in fluid mechanics (including, \(e.g.,\) atmospheric physics, ocean currents) in the presence of turbulence and appear in non-isolated systems into which energy is fed from the outside (\(e.g.,\) \([12]\)). Indeed, the \(\phi\)-fluid is not isolated. In the action \([11]\), \(\phi\) couples explicitly to gravity through the term \(\phi R\).

In the different context of spacetime thermodynamics, Ref. \([3]\) stressed the importance of shear viscosity and the absence of bulk viscosity in \(f(R)\) gravity, contrary to the previous interpretation of \([2]\). These results are echoed in our approach.

In Eckart’s formalism, the entropy current due to the heat flux is \(R^0 = q^0 / T \ [10, 11]\), with components \((0, q/T)\) in the comoving frame. The entropy current density in a fluid with particle density \(\rho\) and entropy density \(s\) is \(s^a = \rho u^0 + R^0\), where \(R^0\) (here equal to \(-K \dot{u}^0\)) describes entropy generation due to dissipative processes. While, in a non-dissipative fluid, entropy is conserved (\(\nabla_c s^c = 0\)), with dissipation in an isolated system it is \(\nabla_c s^c > 0\) due to \(R^0\).

Using Eqs. \([12]\), \([15]\), and \([21]\), the entropy density obtained from the first law of thermodynamics is

\[
s \equiv \frac{dS}{dV} = \frac{\rho + P}{T} = \frac{K}{\sqrt{-\nabla^c \phi \nabla_c \phi}} \times \left[ -\frac{\omega}{\phi} \nabla^c \phi \nabla_c \phi + \frac{\Box \phi}{3} - 4 \frac{\nabla^a \phi \nabla_b \phi \nabla_a \nabla_b \phi}{\nabla^c \phi \nabla_c \phi} \right], \tag{23}
\]
assuming a closed (yet, not isolated) system. Furthermore, in a fluid in which the particle number is conserved, \(\nabla_a n^a = 0\) (where \(n^a = n u^a\) is the particle current density), one has \([10, 11]\)

\[
\nabla_c s^c = \frac{P_{\text{vis}}}{\zeta T} + \frac{q^c q^c}{K T^2} + \frac{\pi_{ab} \pi^{ab}}{2\eta T}, \tag{24}
\]

where the bulk viscosity term is absent for the effective \(\phi\)-fluid. The comparison of the components of the effective stress-energy tensor of the \(\phi\)-fluid based on Eckart’s constitutive laws imply Eqs. \([19], [20], [21]\) and a vanishing contribution of the bulk viscosity term, then Eq. \([21]\) reduces to

\[
\nabla_c s^c = K \left( \dot{u}^a \dot{u}_a + \frac{K T \sigma^2}{\eta} \right) = K \left( \dot{u}^a \dot{u}_a - \sigma_{ab} \sigma^{ab} \right), \tag{25}
\]
where \(\sigma^2 = \sigma_{ab} \sigma^{ab} / 2\). Since the second term in round brackets is negative, the entropy does not always increase.\(^2\) Indeed, if energy is injected into the \(\phi\)-fluid coupled to gravity, \(s\) can decrease.

\section{The Approach to the GR Equilibrium State}

An effective heat equation for the \(\phi\)-fluid, governing the approach to equilibrium, can easily be obtained by differentiating the quantity \(K T\) in Eq. \([21]\), which yields

\[
\frac{d \left( K T \right)}{d\tau} = 8\pi (K T)^2 - \theta K T + \frac{\Box \phi}{\sqrt{-\nabla^c \phi \nabla_c \phi}}, \tag{26}
\]
which differs from the standard result of Eckart’s first-order thermodynamics \([10, 11]\).

\(^2\) If they are possible, situations in which the \(\phi\)-fluid is geodesic, \(\dot{u}^a = 0\), correspond to decreasing entropy density, consistent with the fact that then \(R^0 = 0\) and shear contributes to decreasing \(s\) due to the negative \(\eta\), as described by Eq. \([25]\).
The physical interpretation of Eq. (26) is rather tricky, however one can gain some insight into the approach to equilibrium of the system by considering simplified scenarios. First, let us consider electrovacuum, \( \omega = \text{const.} \), and \( V(\phi) = 0 \); the field equations then imply that \( \Box \phi = 0 \). Therefore, if \( \theta < 0 \), then
\[
\frac{d(K T)}{d\tau} > 8\pi(K T)^2 ,
\]
meaning that \( K T \) diverges from the GR equilibrium state extremely fast. Thus, one can easily observe that near spacetime singularities, where the worldlines of the \( \phi \) field converge, the deviations of scalar-tensor gravity from GR will be extreme. Such a scenario is therefore worthy of further investigation in relation with analytic solutions of scalar-tensor theories involving naked singularities (see e.g. \[14\]). As a second scenario, let us consider again the same electrovacuum with \( \theta > 0 \). Given these assumptions one has that, in principle, the second term in the right-hand side of Eq. (26) can dominate over the first one. This implies that the solution \( K T \) approaches zero or, in other words, the theory approaches the GR equilibrium. One can understand this in a rather evocative way observing that the expansion seems to cool down gravity. Nonetheless, if \( K T \) is large, the positive term will dominate the right-hand side and drive the solution away from GR. Therefore, the approach to the GR equilibrium state is not necessarily granted. Examples of analytic solutions supporting these simple arguments will be presented elsewhere.

V. CONCLUSIONS AND OUTLOOKS

Contrary to Jacobson’s thermodynamics of spacetime \[1,2\], our approach has minimal assumptions. Rewriting the scalar-tensor field equations in the form of Einstein equations with an imperfect fluid does not entail any extra assumption. The limitations of our approach are those intrinsic to Eckart’s first order thermodynamics: it is not causal and suffers from instabilities, but is nevertheless the most widely used model of relativistic thermodynamics due to its relative simplicity. Causality violation is also present in the imperfect fluid model \[11\] which is also the most common model of dissipative fluid. This fact leads us to associate, rather naturally, Eckart’s thermodynamics with the effective fluid description of scalar-tensor gravity. While, \textit{a priori}, Eckart’s thermodynamics applied to an effective fluid could have led nowhere, it surprisingly makes sense for the \( \phi \)-fluid and one can read expressions of the heat current density, effective temperature (the sought-for order parameter), shear viscosity, (vanishing) bulk viscosity, entropy density, and the approach to the GR equilibrium state is described by Eckart’s generalization \[13\] of the Fourier law.

Two unexpected results are that \( i) \) the heat flux is purely inertial in the frame comoving with the \( \phi \)-fluid, in which the spatial gradient of \( T \) vanishes; \( ii) \) the shear viscosity is negative, leading to the possibility of decreasing entropy. This fact is not disconcerting because the system is not isolated and negative (turbulent) viscosities are common in the fluid dynamics of non-isolated systems. An important consequence is that gravity does not necessarily relax to the GR state but can depart from it. This fact opens the possibility that other states of equilibrium, different from GR, exist.

We stress that the effective fluid formalism is not an analogy, but a definite approach to the problem of gravity reaching the GR state of equilibrium through dissipation. Separate publications will widen the approach of this Letter\[4\] and will attempt to generalize it to causal (second order) thermodynamics to overcome the limitations intrinsic to Eckart’s formalism.

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\[\text{An alternative approach used, thus far, in braided kinetic gravity \[13\] would trade temperature with chemical potential and assign zero temperature and entropy to the effective fluid.}\]
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