Moduli-space structure of knots with intersections

Norbert Grot *, Carlo Rovelli †
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pa 15260, U.S.A.
(October 24, 2018)

It is well known that knots are countable in ordinary knot theory. Recently, knots with intersections have raised a certain interest, and have been found to have physical applications. We point out that such knots —equivalence classes of loops in $R^3$ under diffeomorphisms— are not countable; rather, they exhibit a moduli-space structure. We characterize these spaces of moduli and study their dimension. We derive a lower bound (which we conjecture being actually attained) on the dimension of the (non-degenerate components) of the moduli spaces, as a function of the valence of the intersection.

I. INTRODUCTION

At the end of his delicious booklet on catastrophe theory [1], Arnold notices the following. Consider a set of $n$ lines through the origin in the plane. Call two such sets equivalent if they can be mapped into each other by a linear transformation of the plane. The equivalence classes are discrete for $n = 1, 2, 3$; but for $n = 4$, a moment of reflection shows that the equivalence classes are parametrized by a continuous parameter. Precisely this phenomenon is at the root of the emergence of a rich moduli space structure in the spaces of knots with intersections.

Knots play an increasingly important role in various areas of mathematics and physics [2–4]. Classical knot theory [3] deals with knots without intersections, but recent applications of knot theory require knots with intersections to be considered as well [4]. For instance, quantum states of the gravitational field are labeled by knots with intersections in the loop representation approach to quantum gravity [5]. Knots can be defined in two ways: as equivalence classes of loops in $R^3$ under continuous deformations (ambient isotopy) of the image of the loop —“c-knots”; or as equivalence classes (of unparametrized loops) under invertible smooth transformations (diffeomorphisms) of $R^3$ —“d-knots”. For the non-self intersecting loops, the two definitions are equivalent and there is no distinction between c-knots and d-knots. But the two definitions cease to be equivalent in the case with intersections. Intersecting d-knots are different than intersecting c-knots. The case of intersecting d-knots is of particular interest in physics [7]; these knots display a remarkable novel phenomenon, which, to our knowledge, has been rarely noticed (the only mention to this phenomenon we could find in the literature is in Ref. [5]): unlike ordinary knot spaces, the space $K_d$ of the intersecting d-knots is not countable.

The continuous dimensions of the space $K_d$ come from the differential structure of the underlying manifold. The differential structure gives rise to a tangent space $T_p$ at intersection points, loops define lines in $T_p$, and diffeomorphisms act linearly on $T_p$. Equivalence under diffeomorphisms imply equivalence under linear transformations of $T_p$. We are therefore precisely in the situation of Arnold’s example —one dimension up. For a large enough number of lines, linear transformations of $T_p$ fail to be able to align all the lines, and a moduli space structure emerge. Let us illustrate more in detail how this comes about by means of an example. Consider a smooth loop $\alpha$ in $R^3$, with a self-intersection point $p \in R^3$, and assume that $\alpha$ goes through $p$ five times, so that it has five tangents $\vec{v}_1, \ldots, \vec{v}_5$ at $p$ (assume any three of the five are linearly independent). Let us denote by $K_c[\alpha]$ the c-knot to which $\alpha$ belongs. Consider a loop $\beta$ in the same c-knot $K_c[\alpha]$. The loop $\beta$ will have an intersection point as well, say $q$, and five tangents $\vec{w}_1, \ldots, \vec{w}_5$ at $q$. In order for $\alpha$ and $\beta$ to be in the same d-knot, there must be a diffeomorphism $f : R^3 \rightarrow R^3$ sending $\alpha$ into $\beta$. In particular, $f$ maps $p$ to $q$. The tangent map $f^*$ maps the tangent space at $p$, $T_p$, to the tangent space at $q$, $T_q$, and it should align the tangents $\vec{v}_i$ ($i = 1, \ldots, 5$) to the tangents $\vec{w}_i$. But $f^*$ is a linear map between three-dimensional spaces, given by the Jacobian matrix of $f$ at $p$; it is a $GL(3)$ transformation depending on 9 parameter. Since the directions of five vectors $\vec{v}_i$ depend on 10 parameters, it is clear that generically no linear transformation exists that aligns five given vectors $\vec{v}_i$ to five given vectors $\vec{w}_i$. Generically $\alpha$ and $\beta$ will not belong to the same d-knot. There will be at least one continuous parameter $\lambda$ —function of the angles between the five tangents— which is invariant under diffeomorphisms and distinguishes $\alpha$ from $\beta$. Actually, as we shall see, in this example there are two such parameters, $\lambda_1$ and $\lambda_2$; we will give them explicitly below. d-knots are distinguished by such continuous parameters, and therefore fail to be countable. The space of all d-knots in $K_c[\alpha]$ is a finite dimensional space obtained by quotienting the infinite.
dimensional space $K_c[\alpha]$ by the infinite dimensional group $Diff_R^3$. Namely, it is a moduli space, coordinatized by the two moduli $\lambda_1$ and $\lambda_2$.

The same phenomenon repeats in the higher jets – namely for derivatives of the loops higher than the tangents – in a more intricate manner. Derivatives of order $n$ transform under diffeomorphisms according to (non-trivial) transformation formulas that depend only on the derivatives of order $n-1$ (or lower) of the Jacobian matrix. Since the last have a finite number of components, a sufficiently high number of segments crossing one intersection will always give rise to new moduli. Thus, d-knots are not countable and exhibit a very rich moduli space structure, coming from the jets of all orders.

In this paper, we make the above observation precise, we define the moduli spaces of intersecting d-knots, and study their general structure and their dimension. We derive some general results on these dimensions. In particular, our main result is a formula for the dimension of the (generic components) of these spaces. We show that this formula gives indeed the correct dimension. Our original motivations came from quantum gravity, and we expect, in particular, that our results could be of interest for that field.

II. STRUCTURE OF THE INTERSECTING D-KNOT SPACE

By loop, we indicate here a smooth map $\alpha : S_1 \to M$ from the circle $S_1$ to a three-dimensional manifold $M$, which we assume for simplicity having the topology of $R^3$. We indicate loops by Greek letters $\alpha, \beta, ...$, and denote the space of the loops in $R^3$ as $\mathcal{L}$. We consider two equivalence relations in $\mathcal{L}$. We say that $\alpha$ and $\beta$ are c-equivalent, and write $\alpha \sim_c \beta$ if there exist a smooth one-parameter family $c_t, t \in [0,1]$ of smooth, invertible maps from the image of $\alpha$ to $R^3$ such that $c_0\alpha = \alpha$ and $c_1\alpha = \beta$. Namely, if the image of $\alpha$ can be smoothly deformed to the image of $\beta$. This is clearly an equivalence relation; we call the corresponding equivalence classes in $\mathcal{L}$ c-knots, and denote them as $K_c$. We denote the equivalence class to which $\alpha$ belongs as $K_c[\alpha]$ and the space of c-knots as $K_c$. Next we say that $\alpha$ and $\beta$ are d-equivalent, and write $\alpha \sim_d \beta$, if there exist a diffeomorphism $t : S_1 \to M$ connected to the identity – such that $\alpha = f \circ \beta \circ t$. This too is an equivalence relation. We call the corresponding d-equivalence classes in $\mathcal{L}$ d-knots, and denote them as $K_d$. We denote the equivalence class to which $\alpha$ belongs as $K_d[\alpha]$, and the space of d-knots as $K_d$. Thus

$$K_c = \mathcal{L} \sim_c, \quad K_d = \mathcal{L} \sim_d.$$  \hspace{1cm} (1)

Our aim is to study the structure of $K_d$. In particular we want to investigate its continuous dimensions. d-knots can be labeled by a set of discrete parameters $k_j$ and continuous parameters $\lambda_j$. We will use a Dirac-like notation $K_d = (k_j, \lambda_j)$, suggested by the fact that d-knots label quantum state of spacetime in loop quantum gravity. We are interested in studying the appearance and the number of continuous parameters $\lambda_j$, namely the dimensions of the d-knot moduli spaces.

The space of the c-knots $K_c$ is countable. Since every diffeomorphism in the connected component of the identity induces a smooth deformation of the image of the loop, $\alpha \sim_d \beta$ implies $\alpha \sim_c \beta$, and therefore every d-knot is contained inside a single c-knot. Thus we have a well defined map $i : K_d \to K_c$ sending $K_d[\alpha]$ to $K_c[\alpha]$. As the example of the introduction shows, the map $i$ is not injective: a c-knot is formed, in general, by many d-knots. We call $K_d^{(K_c)}$ the inverse image of $K_c$ under $i$, namely the set of the d-knots that correspond to the c-knot $K_c$. The space $K_d$ is thus the union of a countable number of components $K_d^{(K_c)}$, one for every c-knot $K_c$

$$K_d = \bigcup_{K_c \in K_c} K_d^{(K_c)}.  \hspace{1cm} (2)$$

In other words, the first discrete parameter that characterizes a d-knot $K_d$ is the c-knot $K_c$ to which it belongs, and we can write $K_d = |K_c, \text{ other parameters}|$.

Let us consider one of the components $K_d^{(K_c)}$. A continuous map cannot change the number of intersections $I$ of a loop. Therefore this number is well defined for a c-knot $K_c$. Each intersection $i$ is further characterized by the number $N_i$ of times the loop crosses it, which we call the valence of the intersection, following the literature. Thus a set of integers $N_i, i = 1...I$ – the valence of its intersections – is associated with every c-knot. Imagine now that three segments cross at the intersection $i$, namely $N_i = 3$. Imagine that the corresponding three tangents at the intersection are linearly dependent. A continuous transformation can alter this linear dependence, but a diffeomorphism cannot. Thus, the presence of linear dependency between tangents distinguishes d-knot. We denote an intersection of valence three with linearly dependent tangents as a degenerate intersection. Similarly, we denote an intersection of higher valence
degenerate if at least one triple of its tangents is linearly dependent. As we shall better illustrate below, degeneracy of this kind – a relation between derivatives of the loop that cannot be removed by a diffeomorphism – may happen for higher than first derivatives of the loops as well. The information about the presence of degeneracy is discrete, and we represent it collectively by a discrete parameter $k_i$ for every intersection $i$. We write $K_d = \{K_c, k_i \mathrm{other~parameters}\}$ and denote the set of d-knots in the same c-knot and with the same degeneracies as $K_d^{(K_c, k_i)}$. We shall write $k_i = 0$, or just omit the $k_i$ to indicate that the $i$ intersection has no degeneracies.

This exhausts the discrete parameters that characterize d-knots. The remaining parameters distinguishing d-knots are continuous moduli parameters. Thus, the space of intersecting d-knots $\mathcal{K}_d$ can be written as the union of a denumerable set of components $K_d^{(K_c, k_i)}$ as

$$\mathcal{K}_d = \bigcup_{K_c \in \mathcal{K}_c} \bigcup_{k_i} K_d^{(K_c, k_i)},$$

where the spaces $K_d^{(K_c, k_i)}$ are finite dimensional moduli spaces, whose dimensions we are now going to study.

Let us consider one of these moduli spaces $K_d^{(K_c, k_i)}$. A moment of reflection shows that each modulus is attached to one of the intersections, and that there is no relation between moduli of different intersections. As we will show below, the number of moduli that characterize a d-knot at one intersection depends on the valence $N_i$ of the intersection, and the possible presence of degeneracies described by $k_i$. Let $d(N_i, k_i)$ be the number of moduli that characterize an intersection $i$. Then, there will be $d(N_i, k_i)$ continuous parameters $\lambda_j^{(i)}, j = 1 \ldots d(N_i, k_i)$ characterizing each intersection $i$. The d-knot is then fully characterized by all these parameter for each of its intersections. Namely

$$K_d = \{K_c, k_i, \lambda_j^{(i)}\},$$

where $i = 1 \ldots I$ and $j_i = 1 \ldots d(N_i, k_i)$. In other words, the moduli space $K_d^{(K_c, k_i)}$ is the cartesian product of one moduli-space per each intersection. We denote the moduli space of an intersection of valence $N$ and (possible) degeneracy $k_i$ by $K_{N, k_i}$. We thus have

$$\mathcal{K}_d = \bigcup_{K_c \in \mathcal{K}_c} \bigcup_{k_i} \bigotimes_{i \in K_c} K_{N_i, k_i}.$$  

It follows that it is sufficient to study intersections (of any valence $N$ and with any degeneracy $k_i$) in order to fully determine the general structure of $\mathcal{K}_d$. Below, we will discuss the moduli space $K_N = K_{N, 0}$ of the intersections of arbitrary valence, but with no degeneracy. The case with degeneracy $k \neq 0$ can be treated along similar lines.

### III. THE MODULI SPACE $K_N$

Let $p$ be a non-degenerate intersection point of valence $N$ (we drop the suffix $i$ since we deal here with a single intersection) in a loop $\alpha$. We denote by $s$ (or $t, u, ...$) a coordinate on the circle $S_1$ and use coordinates $x^a$ with $a = 1, 2, 3$ from an atlas of $M$. Thus $p^a$ will be the coordinates of $p$ and we write $\alpha : s \mapsto \alpha^a(s)$. There are $N$ segments of $\alpha$ crossing $p$ (intersection between $\alpha$ and a sufficiently small $M$-neighborhood of $p$); we denote them by $\alpha^a_i(s)$, where $i = 1 \ldots N$, and we call $s_i$ the $N$ points in $S_1$ defined by $\alpha(s_i) = p$. Similarly, we consider a second loop $\beta$ in the same moduli space, namely in the same c-knot and with the same degeneracies as $\alpha$. Let $q$ be its intersection point (corresponding to $p$) and $\beta^a_i(s)$ the coordinates of the segments crossing in $q$. The two loops are in the same d-knot if there is a diffeomorphism of the three manifold $f : x^a \mapsto f^a(x)$ and a diffeomorphism of the circle $t : s \mapsto t(s)$ such that

$$f^a(\alpha(t(s))) = \beta^a(s).$$

If we Taylor expand this condition around the intersection point, for each of the $N$ segments, we obtain

$$\beta^a_i(s_i) + \frac{d}{ds} \beta^a_i(s) \bigg|_{s_i} (s - s_i) + \frac{1}{n!} \frac{d^n}{ds^n} \beta^a_i(s) \bigg|_{s_i} (s - s_i)^n + ... = f^a(\alpha(t(s_i)))$$

$$+ \frac{d}{ds} f^a(\alpha(t(s))) \bigg|_{s_i} (s - s_i) + \frac{1}{n!} \frac{d^n}{ds^n} f^a(\alpha(t(s))) \bigg|_{s_i} (s - s_i)^n + ...$$

From here on, the following notation will be used (for the sake of tradition and brevity):

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3
\[ \frac{d}{ds} \alpha^a(s) \bigg|_{s_i} , \]
\[ \frac{d^n}{ds^n} \alpha^a(s) \bigg|_{s_i} , \]
\[ f^a_{b_1...b_n} = \frac{\partial^n f^a}{\partial x^{b_1}...\partial x^{b_n}} \bigg|_{t} , \]
\[ t_i^{(n)} = \frac{d^n}{ds^n} f(s) \bigg|_{s_i} . \]

(8)

We now consider each term of the expansion (8) separately. To zero order, we have

\[ f^a(p) = q^a. \]

(9)

To first order

\[ f^a_{b} \alpha_i^{b(1)} = \beta_i^{a} . \]

(10)

Indices are summed if repeated on different levels. To second order we have

\[ f^a_{b_1} \alpha_i^{b(1)} \alpha_i^{b(1)} + f^a_{b_2} \alpha_i^{b(2)} = - f^a_{b_1} \alpha_i^{b(1)} \alpha_i^{b(2)} + \beta_i^{(2)a} . \]

(11)

And for any order \( n \geq 2 \)

\[ f^a_{b_1...b_n} \alpha_i^{b_1...b_n} (t_i^{(1)})^n + f^a_{b_2} \alpha_i^{b_2} (t_i^{(1)})^n + f^a_{b_3} \alpha_i^{b_3} (t_i^{(1)})^n + F_i^{a} = \beta_i^{(n)a} , \]

or

\[ f^a_{b_1...b_n} \alpha_i^{b_1...b_n} (t_i^{(1)})^n + f^a_{b_2} \alpha_i^{b_2} (t_i^{(1)})^n = \beta_i^{(n)a} - f^a_{b_3} \alpha_i^{b_3} (t_i^{(1)})^n - F_i^{a} . \]

(12)

(13)

where \( F_i^{a} \) is a function of the derivatives of \( f \), \( \alpha_i^{a} \), and \( t \) of orders lower then \( n \) (namely of \( f^a_{b_1...b_n} \); \( \alpha_i^{(k)a} \); \( t_i^{(k)} \); ... with \( k = 1, ..., n - 1 \)). Equation (8) is equivalent to the infinite system (1-13).

Now, the two loops \( \alpha \) and \( \beta \) are d-equivalent if this system can be solved for the functions \( f \) and \( t \), namely for the infinite tower of variables \( f^a_{b_1...b_n} \), \( t_i^{(n)} \). Therefore, we may regard (1-13) as a system of equations for the unknowns \( f^a_{b_1...b_n} \) and \( t_i^{(n)} \). If the system can be solved for every \( \alpha \) and \( \beta \), then all such loops are in the same d-knot and there are no moduli. Namely the moduli space has zero dimension. This is the case, for instance, if \( N = 2 \) (the lowest valence intersection, formed by a single crossing). In fact, one can check in this case that for each order \( n \) the number of unknowns is larger than the number of equations, and the system can be solved. For higher valence intersections, however, the system cannot be solved for arbitrary \( \alpha \) and \( \beta \). This means that there are loops that are not in the same d-knot, and we have a moduli space structure.

A moment of reflection shows that the dimension of the moduli space is equal to the number of (independent) equations that overdetermine the system. To clarify this point, imagine that the system is solvable for general \( \alpha \) and \( \beta \) only if we leave, say, \( d \) (independent) equations out. By inserting \( f \) and \( t \) that solve the rest of the system into these equations we obtain \( d \) equations relating \( \alpha \) and \( \beta \). If we imagine that \( \beta \) is fixed, we obtain then \( d \) conditions on \( \alpha \), determining the set of \( \alpha \)'s d-equivalent to \( \beta \). Thus, this set has codimension \( d \) in the space of the \( \alpha \)'s. This means that a d-knot has codimension \( d \) in the space of the loops in \( \mathcal{K}_N \), and therefore that there is a \( d \) parameters space of d-knots in \( \mathcal{K}_N \). Namely \( \mathcal{K}_N \) is \( d \)-dimensional.

Our task is therefore to find –for every given \( N \)– the number \( d \) of independent equations by which the system (1-13) is overdetermined. This may seem a hard task, given that the system has an infinite number of equations, but there is a key observation that simplifies the matter. First observe that the system has a rather simple structure. As we increase the order \( n \), at each new order there are only a finite number of new unknowns that appear. Indeed the unknowns \( f^a_{b_1...b_n} \) and \( t_i^{(n)} \) appear only at order \( n \) or higher. We denote them as unknowns of order \( n \). For instance, at order zero, the only unknowns are the three \( f^a \). At order one, we have the new unknowns \( f^a_{b_1} \) (nine of them) and \( t_i^{(1)} \) (nine of them), and so on. Now, at each order \( n \), we have the same number \( 3N \) of equations in the system. But the number of unknowns increases rapidly, because the number of components of \( f^a_{b_1...b_n} \) increases with \( n \). Indeed, \( f^a_{b_1...b_n} \) has \( 3 \times I_n \) independent entries, where

\[ I_n = \frac{(n+1)(n+2)}{2} \]

(14)
is the number of independent components in a completely symmetrized $3 \times 3 \times \ldots \times 3$ $n$-dimensional matrix. It is then easy to see that (for fixed $N$) the equations of sufficiently high order can always be solved. More precisely, for every $N$, there is a number $m$, which we determine below, such that all equations of order higher than $m$ can always be solved, and we can safely forget them. This fact essentially reduces the system to a finite dimensional system, making the problem treatable.

A. A gauge

One is now tempted to immediately proceed to determine the number of equations by which the system is overdetermined by naively counting equations and unknowns order by order, and subtracting. Unfortunately, there is a complication. At every order $n$, the actual number of unknowns is less than what a simple count would suggest, because of the particular structure of our equations. Consider first equation (10). The unknowns are the 9 components of $f^a_b$ and the $N$ quantities $t_i^{(1)}$. However, if $f^a_b, t_i^{(1)}$ solve (10), so do

$$\tilde{f}^a_b = T f^a_b, \quad \tilde{t}_i^{(1)} = T^{-1} t_i^{(1)}$$

for every non vanishing $T$. Therefore, the overall scale $T$ can never be determined by equation (10). In other words, equation (10) depends on only $(9 + N - 1)$ functions of the $(9 + N)$ quantities $f^a_b, t_i^{(1)}$. The remaining one cannot be determined by this equation.

The same happens at higher orders. It is easy to verify that if $f^a_{b_1 \ldots b_n}, t_i^{(n)}$ solve the equation of order $n$, so do

$$\tilde{f}^a_{b_1 \ldots b_n} = f^a_{b_1 \ldots b_n} + f^a_{(b_1} T_{b_2 \ldots b_n)}, \quad \tilde{t}_i^{(n)} = t_i^{(n)} - T a_{a_1 \ldots a_{n-1}} \tilde{t}^{(1)}_{a_1} \ldots \tilde{t}^{(1)}_{a_{n-1}} (t_i^{(1)})^2. $$

for every symmetric tensor $T a_{a_1 \ldots a_{n-1}}$. This tensor has $I_{n-1}$ components. We call this transformation the $n$-order gauge of the system. Because of the gauge, if we cut off the system at order $n$, we have indeed $I_{n-1}$ less unknowns entering the system than what a naive counting would suggest. Are there other degeneracies beside the gauge we have just described? We suspect there aren’t, but we have not been able to prove this in general. Because of this incompleteness, we cannot claim that the number we compute below is in fact the dimension of the moduli space, but only that it is the dimension’s lower bound.

B. Size of the space of solutions

Consider the order $n = 0$, equation (10). We have three unknowns ($f^a$) and three equations there. The system is linear and can obviously always be solved. Consider next the order $n = 1$, equation (10). There are $9 + N$ unknowns, and $3N$ equations. But, because of the gauge described above, only $9 + N - 1$ unknowns can be determined by the equations. Generically, the system can be solved if the the number of equations is less than or equal to the number of unknowns, namely if

$$3N \leq 9 + N - 1. $$

In this case, if $N \leq 4$. A simple inspection of the equation confirms that for $N \leq 4$ the 3 equation can indeed be solved, and thus they do not give rise to any continuous dimension. (Since we assumed absence of degeneracies at the beginning of our analysis.) What happens if $N = 5$? In this case we have 15 equations and 13 (independent) unknowns. Which means that the system is overdetermined by two equations. Again, inspection shows that this is indeed the case. Correspondingly, we expect to have at least a two-dimensional moduli space for $N = 5$.

Let us study this $N = 5$ case. At order 2 we have 3N = 15 equations and $3I_2 + N - I_{2-1} + I_{1-1} = 21$ unknowns, where the term $I_{2-1}$ represents the number of irrelevant variables (the ones that cannot be solved for) because of the gauge of order 2, and the term $I_{1-1}$ represents the gauge unknown of order 1 that becomes relevant (can be solved for) at order 2. We have more equations than unknowns and so we expect the system to be solvable. Indeed, it is solvable. The same happens for higher orders, and thus we can conclude that the only equations for which the system is overdetermined are the two of order 1. Thus, an intersection of valence 5 has a two dimensional moduli space. In the next section, we will study this example in detail for illustration. Here let us continue the general analysis.

At any given order $n$, we have $3N$ equations and $3I_n + N - I_{n-1} + I_{n-2}$ new unknowns where, again the term $I_{n-1}$ is for the gauge terms of order $n$ (that are not being solved for at order $n$) and the term $I_{n-2}$ is for the gauge terms
of order $n - 1$ (that can be solved for in order $n$ as opposed to order $n - 1$). Generically, the system can be solved if the number of equations is less than the number of unknowns:

$$3N \leq 3I_n + N - I_{n-1} + I_{n-2},$$

which yields

$$N \leq \frac{3n^2 + 7n + 6}{4}.$$  

Solving this for $n$, we find that the system is solvable at any order $n > m(N)$ where

$$m(N) =: \text{Int}^{-1} \left( \frac{\sqrt{48N - 23} - 7}{6} \right),$$

where $\text{Int}^{-1}(x)$ is the largest integer smaller than $x$. Thus we can forget all equations of order larger than $m(N)$ in the system (19)-(23). The remaining system is formed by the $3N \times m$ equations of order $n \leq m(N)$. (We do not count the 3 equations of order zero and the 3 unknowns $f^a$, which can be always found.) At each order $n$ (less or equal to $m$), the number $d_n$ of overdetermined equations is

$$d_n = \#\text{equations} - \#\text{unknowns} = 3N - (3I_n + N - I_{n-1} + I_{n-2}),$$

(where $I_0 = I_{-1} = 0$), and the total number of equations by which the system is overdetermined is

$$d = \sum_{n=1,m} d_n = \sum_{n=1,m} 2N - 3I_n + I_{n-1} - I_{n-2} = 2mN + I_{n-1} - 3 \sum_{n=1,m} I_n.$$  

Since we are under the assumption that the intersection is nondegenerate, there are no additional degeneracies in the linear system, and the $d$ equations by which the system is overdetermined are independent. Thus $d$ gives the lower bound on the dimension of the moduli space that we are searching. Performing the sum, we get

$$d(N) = \left(2N - 5\right)m - \frac{5}{2} m^2 - \frac{1}{2} m^3,$$

where $m$ is a function of $N$, given in Eq. (21). Equation (23) is our main result. It gives (a lower bound on) the dimension of the moduli space of a single nondegenerate intersection of order $N$.

**IV. AN EXAMPLE: $N = 5$**

Let’s consider again the simplest non-degenerate case in which a moduli-space appear, which is $N = 5$. Thus, we have an intersection point $p$ crossed by the loop $\alpha$ five times. From Eq. (21) we have $m = 1$ and from Eq. (23) $d = 2$, as anticipated. It is instructive to identify the two continuous degrees of freedom of the knot space. We will do this in two different ways. First we give a geometrical construction of these two degrees of freedom, and then we give an explicit algebraic expression for the two moduli.

Let us fix an arbitrary coordinate chart in the neighborhood of $p$, and let $\hat{\alpha}_i^a$ for $i = 1, 2, 3, 4, 5$ be the components of the five tangents of $\alpha$ at $p$. Let us (arbitrarily) pick three of these five tangents, say $\hat{\alpha}_k^a$ for $k = 1, 2, 3$. The three vectors $\hat{\alpha}_k^a$ define a basis in the tangent space at $p$. Clearly the components of the other two tangents, $\hat{\alpha}_4^a$ and $\hat{\alpha}_5^a$ on this basis are quantities that do not depend on the coordinate chosen, and are consequently invariant under diffeomorphisms. If we indicate by $(\hat{\alpha}^{-1})_a^k$ the 3x3 matrix inverse to the 3x3 matrix $\hat{\alpha}_k^a$, such components are given by

$$\beta_4^a = (\hat{\alpha}^{-1})_a^k \hat{\alpha}_4^k,$$

$$\beta_5^a = (\hat{\alpha}^{-1})_a^k \hat{\alpha}_5^k.$$  

The quantities $\beta_4^a$ and $\beta_5^a$ are invariant under diffeomorphisms. They transform under a reparametrization of the loop as $\beta_4^a \mapsto t_i t_k^{-1} \beta_4^a$, where $t_i$ are the 5 derivative of the reparametrization in the intersection point. Assuming, for instance, that the components of $\beta_4^a$ are positive, we can always choose these derivatives in such a way that, say, $\beta_4^a = (1, 1, 1)$. The length of the last vector, $\beta_5^a$, can be arbitrarily rescaled, by fixing $t_5$, but its direction is uniquely determined. This direction gives the two dimensions of the moduli space.
Notice that the sign of the components of $\beta^4_k$ determine eight disconnected sectors of the moduli space, at the boundary of which are degenerate intersections. This is a general feature: the moduli spaces in general have disconnected components, separated by the degenerate cases.

Given the above discussion, it is not too hard to write the two moduli explicitly. This can be done, for instance, in the following manner:

$$\lambda_1 = \frac{(\hat{\alpha}^{-1})^1_{a} \hat{\alpha}^a_4 (\hat{\alpha}^{-1})^2_{b} \hat{\alpha}^b_5}{(\hat{\alpha}^{-1})^3_{c} \hat{\alpha}^c_4 (\hat{\alpha}^{-1})^1_{d} \hat{\alpha}^d_5}, \quad \lambda_2 = \frac{(\hat{\alpha}^{-1})^1_{a} \hat{\alpha}^a_4 (\hat{\alpha}^{-1})^3_{b} \hat{\alpha}^b_5}{(\hat{\alpha}^{-1})^2_{c} \hat{\alpha}^c_4 (\hat{\alpha}^{-1})^1_{d} \hat{\alpha}^d_5}. \quad (25)$$

It is easy to see that these two quantities are independent and are invariant under diffeomorphisms of the manifold and reparametrization of the loops.

We thank Ted Newman for some suggestions. We are particularly indebted to Lee Smolin: the problem considered here emerged in a discussion with him. This work was partially supported by NFS Grant 5-39634.

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