Locally simple subalgebras of diagonal Lie algebras

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Abstract

We describe, up to isomorphism, all locally simple subalgebras of any diagonal locally simple Lie algebra.

Key words (2010 MSC): 17B05 and 17B65.

1 Introduction

A Lie algebra \( g \) is locally finite if any finite subset \( S \) of \( g \) is contained in a finite-dimensional Lie subalgebra \( g(S) \) of \( g \). If, for any \( S \), \( g(S) \) can be chosen simple (semisimple), \( g \) is called locally simple (semisimple). In 1998, A. Baranov introduced the class of diagonal locally finite Lie algebras and established their general properties, see [B1], [B2]. Moreover, an explicit description of the more special class of diagonal locally simple Lie algebras was obtained by A. Baranov and A. Zhilinskii in [BZ], where they classified diagonal direct limits of simple complex Lie algebras up to isomorphism. In the present paper we work with the latter class of Lie algebras, and throughout the paper a diagonal Lie algebra will be assumed locally simple. Particular examples of such algebras are the classical infinite-dimensional complex Lie algebras \( \text{sl}(\infty) \), \( \text{so}(\infty) \), and \( \text{sp}(\infty) \), which can be defined as the unions \( \bigcup_{i \in \mathbb{Z}_{\geq 1}} \text{sl}(i) \), \( \bigcup_{i \in \mathbb{Z}_{\geq 1}} \text{o}(i) \), and \( \bigcup_{i \in \mathbb{Z}_{\geq 1}} \text{sp}(2i) \), respectively, for any inclusions \( \text{sl}(i) \subset \text{sl}(i+1) \), \( \text{o}(i) \subset \text{o}(i+1) \), and \( \text{sp}(2i) \subset \text{sp}(2i+2) \), \( i \geq 1 \). Moreover, the latter Lie algebras are the only countable-dimensional finitary locally simple complex Lie algebras, see [B3], [B4], [BS].

The semisimple subalgebras of semisimple finite-dimensional complex Lie algebras were described by A. Malcev and E. Dynkin more than half a century ago [M], [D]. Recently, I. Dimitrov and I. Penkov characterized all locally semisimple subalgebras of \( \text{sl}(\infty) \), \( \text{o}(\infty) \), and \( \text{sp}(\infty) \) [DP]. The same problem is of interest for the more general class of diagonal Lie algebras. It makes sense to first restrict the problem to describing, up to isomorphism, all locally simple subalgebras of diagonal Lie algebras. The purpose of this paper is to present a solution of the latter problem.

2 Preliminaries

The base field is \( \mathbb{C} \). We assume that all Lie algebras considered are finite dimensional or countable dimensional. When considering classical simple Lie algebras, we consider the three types \( A, C, \) and \( O \), where \( O \) stands for both types \( B \) and \( D \).

A classical simple Lie subalgebra \( g_1 \) of a finite-dimensional classical simple Lie algebra \( g_2 \) is called diagonal if there is an isomorphism of \( g_1 \)-modules

\[
V_2 \downarrow g_1 \cong V_1 \oplus \cdots \oplus V_{i-1} \oplus V_i^* \oplus \cdots \oplus V_{i+1}^* \oplus T_1 \oplus \cdots \oplus T_{i+1},
\]

where \( V_i \) is the natural \( g_i \)-module \( (i = 1, 2) \), \( V_i^* \) is the dual of \( V_i \), and \( T_1 \) is the one-dimensional trivial \( g_1 \)-module. The triple \( (i, r, z) \) is called the signature of \( g_1 \) in \( g_2 \). An injective homomorphism \( \varepsilon : g_1 \rightarrow g_2 \) is diagonal if \( \varepsilon(g_1) \) is a diagonal subalgebra of \( g_2 \). The signature of \( \varepsilon \) is by definition the signature of \( \varepsilon(g_1) \) in \( g_2 \).

An exhaustion

\[
g_1 \subset g_2 \subset \cdots
\]

of a locally finite Lie algebra \( g \) is a direct system of finite-dimensional Lie subalgebras of \( g \) such that the direct limit Lie algebra \( \varinjlim g_i \) is isomorphic to \( g \). A locally simple Lie algebra \( s \) is diagonal if it admits an exhaustion by simple subalgebras \( s_i \) such that all inclusions \( s_i \subset s_{i+1} \) are diagonal.

The following result is due to A. Baranov.

**Proposition 2.1.** Any locally simple subalgebra of a diagonal Lie algebra is diagonal.
Proof. Let $\mathfrak{s}$ be a locally simple subalgebra of a diagonal Lie algebra $\mathfrak{s}'$. Corollary 5.11 in [B1] claims that a locally simple Lie algebra is diagonal if and only if it admits an injective homomorphism into a Lie algebra associated with some locally finite associative algebra. Hence $\mathfrak{s}'$ admits an injective homomorphism into a Lie algebra $\mathfrak{g}$ associated with some locally finite associative algebra. Then there is an injective homomorphism $\mathfrak{s} \rightarrow \mathfrak{s}' \rightarrow \mathfrak{g}$, so $\mathfrak{s}$ is diagonal.

This result reduces the study of locally simple subalgebras of diagonal Lie algebras to the study of diagonal subalgebras.

Next we introduce the notion of index of a simple subalgebra in a simple Lie algebra. This notion goes back to E. Dynkin [D]. For a simple finite-dimensional Lie algebra $\mathfrak{g}$ we denote by $\langle , \rangle_\mathfrak{g}$ the invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ normalized so that $\langle \alpha, \alpha' \rangle_\mathfrak{g} = 2$ for any long root $\alpha$ of $\mathfrak{g}$. If $\varphi: \mathfrak{s} \rightarrow \mathfrak{g}$ is an injective homomorphism of simple Lie algebras, then $\langle x, y \rangle_\varphi := \langle \varphi(x), \varphi(y) \rangle_\mathfrak{g}$ is an invariant non-degenerate symmetric bilinear form on $\mathfrak{s}$. Consequently,

$$\langle x, y \rangle_\varphi = I^s_\varphi(x, y)$$

for some scalar $I^s_\varphi(\varphi)$. By definition $I^s_\varphi(\varphi)$ is the index of $\mathfrak{s}$ in $\mathfrak{g}$. If $\varphi$ is clear from the context, we will simply write $I^s_\varphi$. If $U$ is any finite-dimensional $\mathfrak{s}$-module, then the index $I^s(U)$ of $U$ is defined as $I^s(U)$, where $\mathfrak{s}$ is mapped into $\mathfrak{sl}(U)$ through the module $U$. The following properties of the index are established in [D].

**Proposition 2.2.** (i) $I^s_\varphi \in \mathbb{Z}_{\geq 0}$.

(ii) $I^s_\Phi = I^s_{\Phi}'$.

(iii) $I^s(U_1 + \cdots + U_n) = I^s(U_1) + \cdots + I^s(U_n)$.

(iv) If $U$ is an $\mathfrak{s}$-module with highest weight $\lambda$ (with respect to some Borel subalgebra), then $I^s(U) = \frac{\dim(\lambda)}{\dim \mathfrak{s}}(\lambda, \lambda + 2\rho)_{\mathfrak{s}}$, where $2\rho$ is the sum of all the positive roots of $\mathfrak{s}$.

**Corollary 2.3.** Let $\mathfrak{s}$ and $\mathfrak{g}$ be finite-dimensional classical simple Lie algebras of the same type $(A, C, \text{ or } O)$. If $\mathfrak{s}$ is a diagonal subalgebra of $\mathfrak{g}$ of signature $(l, r, z)$, then $I^s(\mathfrak{g}) = l + r$.

**Proof.** Indeed, if $V$ is the natural $\mathfrak{s}$-module then clearly $I^s(V) = I^s(V^{\ast})$, and (iii) implies the result for type $A$ algebras. If $\mathfrak{s}$ and $\mathfrak{g}$ are of type $O$ or $C$ then the result follows from the observation in [DP] that $I^s_{\mathfrak{g}}(U) = I^s(U)$ and $I^s_{\mathfrak{g}}(U) = \frac{2}{\dim \mathfrak{s}}I^s(U)$ when $U$ admits a corresponding invariant form. This latter observation is also a corollary from [D].

Let us now recall several notions introduced by Baranov and Zhilinskii, and state the main result of [BZ], namely the classification of diagonal Lie algebras.

Let $p_1 = 2, p_2 = 3, \ldots$ be the increasing sequence of all prime numbers. A map from the set $\{p_1, p_2, \ldots\}$ to the set $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a *Steinitz number*. The Steinitz number which has value $\alpha_1$ at $p_1$, $\alpha_2$ at $p_2$, etc. will be denoted by $p_1^{\alpha_1} p_2^{\alpha_2} \cdots$. Let $\Pi = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ and $\Pi' = p_1^{\alpha'_1} p_2^{\alpha'_2} \cdots$ be two Steinitz numbers. We put $\Pi \Pi' = p_1^{\alpha_1 + \alpha'_1} p_2^{\alpha_2 + \alpha'_2} \cdots$, and we say that $\Pi$ divides $\Pi'$ (or $\Pi | \Pi'$) if and only if $\alpha_1 \leq \alpha'_1, \alpha_2 \leq \alpha'_2, \ldots$. In the latter case we write $\frac{\alpha}{2} = (\Pi, \Pi') = p_1^{\alpha - \alpha_1} p_2^{\alpha - \alpha_2} \cdots$, where by convention $p_i^{\infty} = 1$ for any $i$. We also define the greatest common divisor GCD($\Pi, \Pi'$) as $p_1^{\min(\alpha_1, \alpha'_1)} p_2^{\min(\alpha_2, \alpha'_2)} \cdots$.

Let $q \in \mathbb{Q}$. We write $\frac{\alpha}{2} = \frac{\alpha}{2} + \frac{q}{\frac{\alpha}{2}}$ if there exists $n \in \mathbb{N}$ such that $nq \in \mathbb{Z}$. If there exists $0 < q < 2$ such that $\frac{\alpha}{2} = \frac{\alpha}{2} + \frac{q}{\frac{\alpha}{2}}$, then we say that $\Pi$ and $\Pi'$ are $Q$-equivalent and denote this relation by $\Pi \sim \Pi'$. Suppose $q \in \frac{\alpha}{2}$ for some $0 \neq q \in \mathbb{Q}$. If $p^\infty$ divides $\Pi$, then $p^\infty$ also divides $\Pi'$ and so $\Pi = \frac{p^k}{\Pi}$ for all $k \in \mathbb{Z}$. Hence in this case $\{p^k\} \in \mathbb{Z}$ is a subset of $\frac{\alpha}{2}$ in our notation. On the other hand, if there is no prime $p$ with $p^\infty$ dividing $\Pi$, then the set $\frac{\alpha}{2}$ consists of the only element $q$. If $S = (s_1, s_2, \ldots)$ is a sequence of positive integers, $\text{Stz}(S)$ denotes the infinite product $\prod_{i=1}^{\infty} s_i$ considered as a Steinitz number.

Let $s$ be an infinite-dimensional diagonal Lie algebra, so there is an exhaustion $\mathfrak{s} = \cup_s \mathfrak{s}_i$, with all inclusions $\mathfrak{s}_i \subset \mathfrak{s}_{i+1}$ being diagonal. Without loss of generality we may assume that all $\mathfrak{s}_i$ are of the same type $\chi(X = A, C, \text{ or } O)$, and we say that $\mathfrak{s}$ is of type $\chi$. Note that a diagonal Lie algebra can be of more than one type. The triple $(c_i, r_i, z_i)$ denotes the signature of the homomorphism $\mathfrak{s}_i \rightarrow \mathfrak{s}_{i+1}$ and $n_i$ denotes the dimension of the natural $\mathfrak{s}_i$-module. We assume that $c_1 = 0$ if $X$ is not $A$ (for all classical Lie algebras of type other than $A$ the natural representation is isomorphic to its dual). We also assume that $c_i \geq r_i$ for all $i$ for type $A$ algebras. (This does not restrict generality as one can apply outer automorphisms to a suitable subexhaustion if necessary.) Finally, if not stated otherwise, we assume that $n_1 = 1, l_1 = n_2, r_1 = z_1 = 0$. Denote by $\mathcal{T}$ the sequence of all such triples $\{(i, r_i, z_i)\}_{i \in \mathbb{N}}$. We will write $\mathfrak{s} = X(\mathcal{T})$ which makes sense up to isomorphism.
Set $s_i = l_i + r_i$, $c_i = l_i - r_i (i \geq 1)$. $S = (s_i)_{i \in \mathbb{N}}$, $C = (c_i)_{i \in \mathbb{N}}$. Put $\delta_i = \frac{s_1 \cdots s_{n-i}}{n_i+1}$. Then $\delta_{i+1} = \frac{s_1 \cdots s_{n-i-1}}{n_i+2} \leq \delta_i$. The limit $\delta = \lim_{i \to \infty} \delta_i$ is called the density index of $T$ and is denoted by $\delta(T)$. Since $\delta_2 = s_1/n_2 = 1$, we have $0 \leq \delta \leq 1$. If $\delta = 0$ then the sequence of triples $T$ is called sparse. If there exists $i$ such that $\delta_j = \delta_i \neq 0$ for all $j > i$, the sequence is called pure. We say that $T$ is dense if $0 < \delta < \delta_i$ for all $i$.

If there exists $i$ such that $c_j = s_j$ for all $j \geq i$, then $T$ is called one-sided (in which case we can and will assume that $c_j = s_j$ for all $j \geq 1$). Otherwise it is called two-sided. If, for each $i$, there exists $j > i$ such that $c_j = 0$, then $T$ is called symmetric. Otherwise it is called non-symmetric. In the latter case we will assume that $c_j > 0$ for all $i \geq 1$. Set $\sigma_i = \frac{T_{i+1}}{T_i}$. The limit $\sigma = \lim_{i \to \infty} \sigma_i$ is called the symmetry index of $T$ and is denoted by $\sigma(T)$. Observe that $0 \leq \sigma \leq 1$. Two-sided non-symmetric sequences $T$ with $\sigma(T) = 0$ are called weakly non-symmetric, and those with $\sigma(T) \neq 0$ are called strongly non-symmetric.

The classification of the infinite-dimensional diagonal locally simple Lie algebras is given by the following two theorems.

**Theorem 2.4. [BZ]** Let $X = A$, $C$, or $O$. Let $T = \{(l_i, r_i, z_i)\}$ and $T' = \{(l'_i, r'_i, z'_i)\}$, where $r_i = r'_i = 0$ if $X \neq A$. Set $\delta = \delta(T)$, $\sigma = \sigma(T)$, $\delta' = \delta(T')$, $\sigma' = \sigma(T')$. Then $X(T) \cong X(T')$ if and only if the following conditions hold.

1. The sequences $T$ and $T'$ have the same density type.
2. $\frac{\delta}{\sigma} \in \frac{\text{Stz}(S)}{\text{Stz}(S')}$ for dense and pure sequences.
3. The sequences $T$ and $T'$ have the same symmetry type.
4. $\text{Stz}(C) \cong \text{Stz}(C')$ for two-sided non-symmetric sequences.
5. There exists $\alpha \in \frac{\text{Stz}(S)}{\text{Stz}(S')}$ such that $\alpha \frac{\delta}{\sigma} \in \frac{\text{Stz}(C)}{\text{Stz}(C')}$ for two-sided strongly non-symmetric sequences.

Moreover, $\alpha = \frac{\delta}{\sigma}$ if in addition the triple sequences are dense or pure.

**Theorem 2.5. [BZ]** Let $T = \{(l_i, r_i, z_i)\}$, $T' = \{(l'_i, r'_i, z'_i)\}$, and $T'' = \{(l''_i, 0, z''_i)\}$.

1. $A(T) \cong O(T')$ (resp., $A(T) \cong C(T')$) if and only if $T$ is two-sided symmetric, $2^\infty$ divides $\text{Stz}(S')$, and the conditions (A1), (A2), (A3) of Theorem 2.4 hold.
2. $O(T') \cong C(T'')$ if and only if $2^\infty$ divides both $\text{Stz}(S')$, and the conditions (A1), (A2), (A3) of Theorem 2.4 hold.

**Remark.** It is easy to see from Theorem 2.4 that a diagonal Lie algebra $X(T)$ is finitary (i.e., isomorphic to $\text{sl}(\infty)$, $\text{so}(\infty)$, or $\text{sp}(\infty)$) if and only if $\text{Stz}(S)$ is finite.

As we see from the above classification, the density type and the symmetry type are well-defined for a diagonal Lie algebra. We will call an algebra pure, dense, or sparse if its sequence of triples $T$ can be chosen pure, dense, or sparse, respectively. We will also call an algebra one-sided, two-sided symmetric, two-sided strongly non-symmetric, or two-sided weakly non-symmetric if its sequence of triples $T$ can be chosen with the respective property.

For an arbitrary sequence $S = (s_i)_{i \geq 1}$ by $\text{sl}(\text{Stz}(S))$ (respectively, $\text{so}(\text{Stz}(S))$, $\text{sp}(\text{Stz}(S))$) we will denote the pure Lie algebra $A(\{(s_i, 0, 0)\}_{i \geq 1})$ (resp., $O(\{(s_i, 0, 0)\}_{i \geq 1})$, $C(\{(s_i, 0, 0)\}_{i \geq 1})$).

We need two branching rules for Lie algebras of type $A$. Throughout this paper $F^\lambda_n$ denotes an irreducible $\text{sl}(n)$-module with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{Z}$. Note that the isomorphism class of $F^\lambda_n$ is determined by the differences $\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n$.

**Theorem 2.6. (Gelfand-Tsetlin rule [Z])** Consider a subalgebra $\text{sl}(n) \subset \text{sl}(n + 1)$ of signature $(1, 0, 1)$. Then, there is an isomorphism of $\text{sl}(n)$-modules

$$F^\lambda_{n+1} \downarrow \text{sl}(n) \cong \bigoplus_{\mu} F^\mu_n,$$

where the summation runs over all integral weights $\mu = (\mu_1, \ldots, \mu_n)$ satisfying $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_{n+1}$.

Consider the $\text{sl}(n) \oplus \text{sl}(n)$-module $F^\lambda_n \otimes F^\mu_n$. By Theorem 2.1.1 of [HTW] its restriction to $\text{sl}(n) := \{x \oplus x, x \in \text{sl}(n)\}$ decomposes as $\bigoplus_{\lambda} e^\lambda_{\mu} F^\lambda_n$, where $e^\lambda_{\mu}$ is the Littlewood-Richardson coefficient. One can iterate this branching rule to obtain the decomposition for higher tensor products. Let $e^\lambda_{\mu_1 \cdots \mu_k}$ denote the coefficient obtained in this manner, so,

$$F^\lambda_{n+1} \otimes \cdots \otimes F^\mu_{n+k} \downarrow \text{sl}(n) \cong \bigoplus_{\lambda} e^\lambda_{\mu_1 \cdots \mu_k} F^\lambda_n,$$
where the summation runs over all integral dominant weights \( \lambda \) with \( \lambda_i \geq 0 \). We will call the numbers \( c^\lambda_{\mu_1 \ldots \mu_k} \) generalized Littlewood-Richardson coefficients.

The following branching rule was communicated to us by J. Willenbring.

**Proposition 2.7.** Consider a diagonal subalgebra \( \mathfrak{sl}(n) \subset \mathfrak{sl}(kn) \) of signature \((k,0,0)\). Then, there is an isomorphism of \( \mathfrak{sl}(n) \)-modules

\[
F^\lambda_{kn} \downarrow \mathfrak{sl}(n) \cong \bigoplus_{\nu} \left( \sum_{\mu_1, \ldots, \mu_k} c^\lambda_{\mu_1 \ldots \mu_k} c^\nu_{\mu_1 \ldots \mu_k} \right) F^\nu_{n},
\]

where one summation runs over all integral dominant weights \( \nu \) with \( \nu_i \geq 0 \) for all \( i \) and the other summation runs over all sets of integral dominant weights \( \mu_1, \ldots, \mu_k \) with \( (\mu_j)_i \geq 0 \) for all \( i,j \).

**Proof.** Consider the block-diagonal subalgebra \( \mathfrak{sl}(l) \oplus \mathfrak{sl}(m) \subset \mathfrak{sl}(n) \) \((n = l + m)\). By Theorem 2.2.1 of [HTW] \( F^\lambda_{kn} \downarrow \mathfrak{sl}(l) \oplus \mathfrak{sl}(m) \) decomposes as \( \bigoplus_{\mu} c^\lambda_{\mu} F^\mu_{l} \otimes F^\nu_{m} \). Let now the direct sum of \( k \) copies of \( \mathfrak{sl}(n) \) be a subalgebra \( \mathfrak{sl}(kn) \) with block diagonal inclusion. By iteration of this branching rule we see that the decomposition of \( F^\lambda_{kn} \downarrow \mathfrak{sl}(n) \oplus \cdots \oplus \mathfrak{sl}(n) \) is determined by the generalized Littlewood-Richardson coefficients:

\[
F^\lambda_{kn} \downarrow \mathfrak{sl}(n) \oplus \cdots \oplus \mathfrak{sl}(n) \cong \bigoplus_{\mu_1, \ldots, \mu_k} \left( \sum \right) c^\lambda_{\mu_1 \ldots \mu_k} F^\mu_{n} \oplus \cdots \oplus F^\nu_{n},
\]

where \( \mathfrak{sl}(n) \oplus \cdots \oplus \mathfrak{sl}(n) \) is the block diagonal subalgebra of \( \mathfrak{sl}(kn) \), and the summation runs over all integral dominant weights \( \mu_1, \ldots, \mu_k \) with \( (\mu_j)_i \geq 0 \).

Consider now a subalgebra \( \mathfrak{sl}(n) \subset \mathfrak{sl}(kn) \) of signature \((k,0,0)\). One can obtain (3) as a combination of the two branching rules 2 and 4.

**Remark.** In Proposition 2.7 the sum is taken over all integral dominant weights \( \nu \) with \( \nu_i \in \mathbb{Z}_{\geq 0} \) for all \( i \). In order for \( F^\nu_{n} \) to have a non-zero coefficient in (3) both Littlewood-Richardson coefficients \( c^\lambda_{\mu_1 \ldots \mu_k} \) and \( c^\nu_{\mu_1 \ldots \mu_k} \) must be non-zero for some \( \mu_1, \ldots, \mu_k \). But for that we must have \( \sum_{i=1}^{kn} \lambda_i = \sum_{i=1}^{n} \nu_i \). Therefore the summation in (3) may be taken to run over only those weights \( \nu \) with fixed \( \sum_{i=1}^{n} \nu_i \). Hence all modules \( F^\nu_{n} \) which are present in (3) with non-zero coefficients are pairwise non-isomorphic. Indeed, if \( F^\nu_{n} \cong F^\nu_{n} \) both have non-zero coefficients in (3), then the weight \( \nu' \) can be obtained by shifting the weight \( \nu \) by an integer, so \( \sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \nu_i' \) implies \( \nu' = \nu \). This argument allows us to refer to a non-zero coefficient

\[
( \sum_{\mu_1, \ldots, \mu_k} c^\lambda_{\mu_1 \ldots \mu_k} c^\nu_{\mu_1 \ldots \mu_k} )
\]

as the multiplicity of \( F^\nu_{n} \) in (3).

**Corollary 2.8.** For a diagonal subalgebra \( \mathfrak{sl}(n) \subset \mathfrak{sl}(kn) \) of signature \((k,0,0)\) the restriction \( F^\lambda_{kn} \downarrow \mathfrak{sl}(n) \) has a submodule with highest weight

\[
(\nu_1, \ldots, \nu_n) = (\lambda_1 + \cdots + \lambda_k, \lambda_{k+1} + \cdots + \lambda_{2k}, \ldots, \lambda_{kn-k+1} + \cdots + \lambda_{kn}).
\]

**Proof.** Indeed, if we set \( \mu_i = (\lambda_1, \lambda_{k+i}, \ldots, \lambda_{kn-k+i}) \) for \( i \in \{1, \ldots, k\} \), then it easy to check that both coefficients \( c^\lambda_{\mu_1 \ldots \mu_k} \) and \( c^\nu_{\mu_1 \ldots \mu_k} \) are non-zero, and therefore the highest weight module \( F^\nu_{n} \) is present in (3) with non-zero multiplicity.

If \( \mathfrak{s} \) and \( \mathfrak{g} \) are two diagonal Lie algebras, then constructing a homomorphism \( \theta : \mathfrak{s} \rightarrow \mathfrak{g} \) is equivalent to constructing commutative diagram

\[
\begin{array}{cccc}
\mathfrak{s}_1 & \xrightarrow{\varphi_1} & \mathfrak{s}_2 & \rightarrow \cdots \\
\mathfrak{s}_1 & \xrightarrow{\psi_1} & \mathfrak{g}_1 & \xrightarrow{\varphi_2} & \mathfrak{s}_2 & \rightarrow \cdots \\
\mathfrak{s}_1 & \xrightarrow{\psi_2} & \mathfrak{g}_2 & \xrightarrow{\varphi_3} & \mathfrak{s}_3 & \rightarrow \cdots \\
\end{array}
\]

for some exhaustions \( \mathfrak{s}_1 \xrightarrow{\varphi_1} \mathfrak{s}_2 \xrightarrow{\varphi_2} \cdots \) and \( \mathfrak{g}_1 \xrightarrow{\psi_1} \mathfrak{g}_2 \xrightarrow{\psi_2} \cdots \) of \( \mathfrak{s} \) and \( \mathfrak{g} \) respectively. An injective homomorphism \( \theta \) is called diagonal if all \( \theta_i \) can be chosen diagonal for sufficiently large \( i \).

To deal with diagonal homomorphisms we will need the following result.

**Lemma 2.9.** Let \( \varepsilon_1 : \mathfrak{s}_1 \rightarrow \mathfrak{s}_2 \) and \( \varepsilon_2 : \mathfrak{s}_1 \rightarrow \mathfrak{g} \) be diagonal injective homomorphisms of finite-dimensional simple classical Lie algebras of signatures \((l,r,z)\) and \((p,q,u)\) respectively. Let a triple of non-negative integers \((p,q,u)\) satisfy the following conditions:

\[
p + q = (l + r)(p' + q'), \quad p - q = (l - r)(p' - q'), \quad n = n_1(p' + q') + u',
\]

where \( n_1 \) is any non-negative integer.

\[\text{[4]}\]
where \( n \) and \( n_2 \) are the dimensions of the natural \( g \)- and \( g_2 \)-modules respectively. Then, under the assumption that \( g_2 \) and \( g \) are of the same type \( X \), there exists a diagonal injective homomorphism \( \theta : g_2 \to g \) of signature \((p',q',u')\) such that \( \varphi_2 = \theta \circ \varphi_1 \). If \( g_2 \) and \( g \) are of different types \( X \) and \( Y \), the statement holds under the following additional conditions on the triple \((p',q',u')\):

\[
\begin{align*}
p' &= q' \text{ if } (X,Y) = (A,O) \text{ or } (X,Y) = (A,C); 
p' &\text{ is even if } (X,Y) = (O,C) \text{ or } (X,Y) = (C,O).
\end{align*}
\]

Proof. Lemma 2.6 in [BZ] states the same result in case all Lie algebras \( g_1, g_2, g \) are of the same type. The proof of Lemma 2.6 in [BZ] works also when the three algebras are not of the same type, but only if \( g_2 \) can be mapped into \( g \) by an injective homomorphism of signature \((p',q',u')\). It is easy to check that the additional conditions guarantee the existence of such a homomorphism.

Consider the diagram in (5) without the commutativity assumption. Lemma 2.9 implies that if all \( \theta_i \) are diagonal injective homomorphisms such that for all \( i \geq 1 \) the two diagonal injective homomorphisms \( \psi_i \circ \theta_i \) and \( \theta_i+1 \circ \varphi_i \) of \( g_i \) into \( g_{i+1} \) have the same signature, then there are diagonal injective homomorphisms \( \theta'_i \) with the same property making the diagram commutative. Later on in this paper when constructing diagrams as in (5) in concrete situations, we will check commutativity by showing only that the signatures of \( \psi_i \circ \theta_i \) and \( \theta_{i+1} \circ \varphi_i \) coincide for all \( i \geq 1 \). It will then be assumed that \( \theta_i \) are replaced by corresponding diagonal injective homomorphisms \( \theta'_i \) making the diagram commute.

The following result can be found in [BZ] (see also all references in there, for instance [B2]).

**Lemma 2.10.** Let \( h \subset g \subset s \) be finite-dimensional classical simple Lie algebras, \( \text{rk} \ h > 10 \). Assume that the inclusion \( h \subset s \) is diagonal. Then the inclusions \( h \subset g \) and \( g \subset s \) are also diagonal.

**Corollary 2.11.** Let \( h \subset g \subset s \) be infinite-dimensional diagonal Lie algebras. Assume that the inclusion \( h \subset s \) is diagonal. Then the inclusions \( h \subset g \) and \( g \subset s \) are also diagonal.

We conclude this section by introducing a notion of equivalence of infinite-dimensional Lie algebras. We say that \( g_1 \) is equivalent to \( g_2 \) \( (g_1 \sim g_2) \) if there exist injective homomorphisms \( g_1 \to g_2 \) and \( g_2 \to g_1 \). For finite-dimensional Lie algebras this equivalence relation is the same as isomorphism, but this is no longer the case for infinite-dimensional Lie algebras.

### 3 Classification of locally simple subalgebras of diagonal Lie algebras

In this section all diagonal Lie algebras considered are assumed to be infinite dimensional.

We start the classification by asking whether \( \mathfrak{sl}(\infty) \) admits an injective homomorphism into any non-finitary diagonal Lie algebra. As it turns out, the most basic example sufficed to answer this question, as we were able to construct an injective homomorphism of \( \mathfrak{sl}(\infty) \) into \( \mathfrak{sl}(2^\infty) \), so the answer is yes. The following construction was suggested to us by I. Dimitrov.

Let \( F_n \) be the natural representation of \( \mathfrak{sl}(n) \). Note that under the injective homomorphism \( \mathfrak{sl}(n) \to \mathfrak{sl}(n+1) \) of signature \((1,0,1)\), the exterior algebra \( \bigwedge (F_{n+1}) \) decomposes as two copies of \( \bigwedge (F_n) \) as an \( \mathfrak{sl}(n) \)-module. Fix a map \( \theta_n : \mathfrak{sl}(n) \to \mathfrak{sl}(2^n) \) such that the natural representation of \( \mathfrak{sl}(2^n) \) decomposes as \( \bigwedge (F_n) \) as an \( \mathfrak{sl}(n) \)-module. Then there exists a map \( \theta_{n+1} : \mathfrak{sl}(n+1) \to \mathfrak{sl}(2^{n+1}) \) such that the natural representation of \( \mathfrak{sl}(2^{n+1}) \) decomposes as \( \bigwedge (F_{n+1}) \) as an \( \mathfrak{sl}(n+1) \)-module making the following diagram commute:

\[
\begin{array}{ccccccc}
\mathfrak{sl}(2) & \to & \cdots & \to & \mathfrak{sl}(n) & \to & \mathfrak{sl}(n+1) & \to & \cdots \\
\downarrow \theta_2 & & & & \downarrow \theta_n & & \downarrow \theta_{n+1} & & \\
\mathfrak{sl}(2^2) & \to & \cdots & \to & \mathfrak{sl}(2^n) & \to & \mathfrak{sl}(2^{n+1}) & \to & \cdots
\end{array}
\]

where the lower row consists of injective homomorphisms of signature \((2,0,0)\). Therefore by induction, the diagram yields an injective homomorphism of \( \mathfrak{sl}(\infty) \) into \( \mathfrak{sl}(2^\infty) \).

We will prove now that similar injective homomorphisms exist in a more general setting. The following result will be used later to prove that in fact any finitary diagonal Lie algebra can be similarly mapped into any diagonal Lie algebra.

**Proposition 3.1.** \( \mathfrak{sl}(\infty) \) admits an injective homomorphism into any pure one-sided Lie algebra \( s \) of type \( A \).
Indeed, the diagram in (7) yields an injective homomorphism $\text{sl}(k) \rightarrow \text{sl}(k+1)$ for suitable $\{n_i\}$, where $\theta_i$ are injective homomorphisms and $n_1, n_2, \ldots$ are chosen so that $\prod_{i=1}^{\infty} n_i = \Pi$. Indeed, the diagram in (7) yields an injective homomorphism $\text{sl}(\infty) \rightarrow \text{sl}(n_1n_2 \cdots)$, and $\text{sl}(n_1n_2 \cdots)$ is isomorphic to $\mathfrak{s}$ by Theorem 2.4. We will choose the homomorphisms $\theta_k$ so that

$$
V_k \downarrow \text{sl}(k) \cong a_0^k \bigwedge (F_k) \oplus a_1^k \bigwedge (F_k) \oplus \cdots \oplus a_k^k \bigwedge (F_k)
$$

as $\text{sl}(k)$-modules. Here $V_k$ stands for the natural $\text{sl}(n_1 \cdots n_k)$-module, $F_k$ is the natural $\text{sl}(k)$-module and the coefficients $a_i^k, i = 0, \ldots, k$ are non-negative integers. The above injective homomorphism of $\text{sl}(\infty)$ into $\text{sl}(2^\infty)$ corresponds to the particular case $n_k = 2$ and $a^k_i = 1$ for all $k \geq 2, i = 0, \ldots, k$.

We see that if the numbers $\{a^k_i\}$ satisfy the conditions $a^k_i + a^k_{i+1} = n_k a^{k-1}_k$, $k \geq 3$, $i = 0, \ldots, k - 1$ and $a^0_0 + 2a^2_1 + a^2_2 = n_1n_2$, then the homomorphisms $\theta_k$ can be chosen so that the diagram in (7) commutes.

We will add numbers $a^0_0, a^1_1, a^2_2$ to the set of coefficients $\{a^k_i\}$ and will require $a^k_0 + a^k_1 = n_2a^k_0, a^2_1 + a^2_2 = n_2a^1_1, a^1_0 + a^1_1 = n_1$, and $a^0_0 = 1$. Then the numbers $\{a^k_i\}$ will form an infinite triangle

$$
a^0_0, a^1_1, a^2_2, \ldots
$$

such that

$$
a^k_0 + a^k_{i+1} = n_i a^{k-1}_i, \ k \geq 1 \text{ and } a^0_0 = 1.
$$

It is enough to prove that a triangle of non-negative integers satisfying (8) exists for a suitable choice of $n_i$. Set $b_k := \frac{a^k_{i-1}}{n_i - n_k}$ for $k \geq 1$. A simple calculation shows that $a^k_k = n_1 \cdots n_k (a^0_0 - b_1 - b_2 - \cdots - b_k)$. Notice that since $a^0_0 = 1$, the numbers $b_1, b_2, \ldots$ uniquely determine the entire triangle, as the $l^{th}$ “diagonal” $\{a^{k+l}_l\}_{k \geq 0}$ of the triangle is determined by the previous diagonal $\{a^{k+l-1}_l\}_{k \geq 0}$ and the sequence $n_1, n_2, \ldots$.

Now we will find conditions on $b_k$ under which all $a^k_i$ will be non-negative. Since $a^k_{k+1} = 0$, the numbers $b_k$ should be non-negative. In order for $a^k_k$ to be non-negative we should have

$$
\sum_{i=1}^{\infty} b_i < a^0_0
$$

for all $k$ (since $b_i$ are non-negative, we can rewrite these conditions as

$$
\sum_{i=1}^{\infty} b_i \leq 1
$$

). The entries of the diagonal $\{a^k_{k+2}\}_{k \geq 0}$ can be found from (8): $a^k_{k+2} = n_1 \cdots n_{k+2} (b_{k+1} - b_{k+2})$ for $k \geq 0$. This requires the sequence $\{b_k - b_{k+1}\}$ to be non-negative. If we set $b^{(1)}_k := b_k - b_{k+1}$ for $k \geq 1$, then in a similar way we obtain $a^k_{k+3} = n_1 \cdots n_{k+3} (b^{(1)}_{k+1} - b^{(1)}_{k+2})$. This requires the sequence $\{b^{(2)}_k := b^{(1)}_k - b^{(1)}_{k+1}\}$ to be non-negative. Continuing this procedure, we get $a^k_{k+l} = n_1 \cdots n_{k+l} b^{(l-1)}_{k+1}$ for all $l \geq 3$, where by definition $b^{(l+1)}_{k+1} = b^{(l)}_{k+1} - b^{(l)}_{k+1}$. Now we see that the non-negative integers $a^k_i$ satisfying (8) exist if there exists a non-negative sequence $\{b_k\}_{k \geq 1}$ with $b_k \in \frac{1}{n_k} \mathbb{Z}_{\geq 0}$ and $\sum_{k=1}^{\infty} b_k \leq 1$ such that

$$
\text{all iterated sequences of differences } (b^{(l)}_k)_{k \geq 1} \text{ are non-negative.}
$$

Note that the sequence $\{b_k = \frac{1}{q^k}\}, q > 1$ satisfies (9) as $b^{(l)}_k = \frac{1}{q^k} (1 - \frac{1}{q^k})^l > 0$ for all $k, l \geq 1$. (In the case $n_k = n$ for all $k$, taking $q = n$ yields an injective homomorphism $\text{sl}(\infty) \leftrightarrow \text{sl}(n^\infty)$.) We will find the desired sequence $\{b_k\}$ as a convergent infinite linear combination of geometric sequences.

Let us put $q = 4$ (the following construction would work for any $q \geq 4$) and let $\Pi = m_1m_2 \cdots$. Choose a strictly increasing sequence of integers $\{l_k\}_{k \geq 0}$ so that $l_0 = 0$ and $m_1m_2 \cdots m_{l_k} > \frac{(q-1)q^{l_k+1}}{q-2}$ for $k \geq 1$, which is possible as $\Pi$ is infinite. Take $n_k = m_{l_{k-1}+1} \cdots m_{l_k}$ for $k \geq 1$. Then clearly $n_1n_2 \cdots = \Pi$.

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Let us now construct the sequence \( \{b_k\} \) for the chosen \( n_1, n_2, \ldots \). For \( i \geq 1 \) we denote \( c_i = 1 + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{q^j(q^j - 1)(q^j - q)} \), where the numbers \( \varepsilon_j \), satisfying
\[
0 \leq \varepsilon_j < \frac{q - 2}{(q - 1)q^{i+1}},
\]
are to be chosen later, and put \( b_k = \sum_{i=1}^{\infty} c_i \left( \frac{1}{q^i} \right)^k \). We will show that for the numbers \( \varepsilon_j \), satisfying (10), the series for \( c_i \) converges to a positive number for \( i \geq 1 \), the series for \( b_k \) converges for \( k \geq 1 \), and \( \sum_{k=1}^{\infty} b_k \leq 1 \). Moreover, we will show that by varying \( \varepsilon_j \) inside corresponding intervals we can make each \( b_k \) to be of the form \( \frac{1}{n_1 \cdots n_k} Z \geq 0 \). We will have then \( b_k^{(i)} = \sum_{i=1}^{\infty} c_i \left( \frac{1}{q^i} \right)^k \left( 1 - \frac{1}{q^i} \right)^j \geq 0 \), so \( \{b_k^{(i)}\} \) will be a sequence of non-negative numbers for any \( l \). Hence the final condition in (9) will be satisfied.

As a matter of convenience we denote \( q_l = \frac{q_l}{q_l-1} \). Then let \( c_{ij} = \frac{1}{q^{q_l(q_l-1)(q_l-q-1)(q_l-q-2)(q_l-q-3)}(q_l-q-4)} \) for \( i \leq j \). We see that \( c_i = 1 + \sum_{j=i}^{\infty} c_{ij} \). Let us prove that this series converges absolutely. We have
\[
|c_i - 1| = \left| \sum_{j=i}^{\infty} \frac{\varepsilon_j}{1 - q^j} \right| = \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{j-1}}{q^j - 1} \leq \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{j}}{q^j - q^i} \leq \frac{\varepsilon_j q^{j}}{q^j - q^i} \leq \frac{\varepsilon_j q^{j}}{(q-1)} \leq \frac{\varepsilon_j q^{j}}{q^j - 1} \leq \frac{\varepsilon_j q^{j}}{q^j - q^i} \leq \frac{\varepsilon_j q^{j}}{q^j - q^i} = \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{j}}{q^j - 1} = \varepsilon_j q^{j} (q - 1).
\]

Then, using (10), we obtain
\[
|c_i - 1| \leq \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{j}}{q^j - 1} \leq \frac{1}{q} - \frac{1}{q^2} + \frac{1}{q^3} + \ldots < \frac{1}{q} + \frac{1}{q^2} + \ldots = \frac{1}{q - 1}.
\]

Thus, the series \( 1 + \sum_{j=i}^{\infty} c_{ij} \) converges absolutely and its sum \( c_i \) is a number from the interval \( \left( \frac{2}{q - 1}, \frac{1}{q - 1} \right) \) (in particular, \( c_i \) is positive) for all \( i \). Furthermore,
\[
\sum_{k=1}^{\infty} b_k = \sum_{i=1}^{\infty} \frac{c_i}{q^i} + \sum_{i=1}^{\infty} \frac{c_i}{(q^i)^2} + \cdots < \frac{q}{q - 1} \left( \sum_{i=1}^{\infty} \frac{1}{q^i} + \sum_{i=1}^{\infty} \frac{1}{(q^i)^2} + \cdots \right) = \frac{q}{q - 1} \left( \frac{1}{q - 1} + \frac{1}{q^2 - 1} + \frac{1}{q^3 - 1} + \cdots \right) = \frac{q}{q - 1} \cdot \frac{1}{q - 1} < 1 \because q \geq 4.
\]

Since every term in these expressions is non-negative, the convergence of each series \( b_k = \sum_{i=1}^{\infty} c_i \left( \frac{1}{q^i} \right)^k \) follows.

Finally, let us show that the numbers \( \varepsilon_j \), satisfying (10), can be chosen so that \( b_k \in Z \geq 0 \). We know that \( b_k = \sum_{i=1}^{\infty} c_i q_i^k = \sum_{i=1}^{\infty} q_i^k + \sum_{i=1}^{\infty} \sum_{j=1}^{i} c_{ij} q_i^k \). From what we proved it follows that the latter sum is absolutely convergent. Therefore we can rewrite it as \( b_k = \sum_{i=1}^{\infty} q_i^k + \sum_{j=1}^{\infty} \sum_{i=1}^{j} c_{ij} q_i^k \). Note that the numbers \( c_{ij} \) were defined as solutions of the equation
\[
\begin{pmatrix}
q_1 & \cdots & q_j \\
\vdots & \ddots & \vdots \\
q_{i-1} & \cdots & q_j \\
\end{pmatrix}
\begin{pmatrix}
c_{ij} \\
0 \end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0 \\
\varepsilon_j \\
\end{pmatrix}
\]
using the well-known formula for inverting a Vandermonde matrix. Thus, \( \sum_{i=1}^{\infty} q_i^k c_{ij} = 0 \) for \( k < j \) and \( \sum_{i=1}^{\infty} q_i^k c_{ij} = \varepsilon_j \).
Lemma 3.2. Homomorphisms of non-finitary diagonal Lie algebras.

Proof. Hence,

\[
b_k = \sum_{i=1}^{\infty} q_i^k + \sum_{j=1}^{k-1} \sum_{i=1}^{j} c_{ij} q_i^k + \varepsilon_k,
\]

so \(b_k - \varepsilon_k\) depends only on \(\varepsilon_1, \ldots, \varepsilon_{k-1}\). Let us introduce the notation

\[
f_k(\varepsilon_1, \ldots, \varepsilon_{k-1}) = \sum_{i=1}^{\infty} q_i^k + \sum_{j=1}^{k-1} \sum_{i=1}^{j} c_{ij} q_i^k \quad \text{for } k \geq 2 \quad \text{and} \quad f_1 = \sum_{i=1}^{\infty} q_i = \sum_{i=1}^{\infty} \frac{1}{q_i} = \frac{1}{q-1}.
\]

Now we define inductively the numbers \(\varepsilon_k\). We choose \(\varepsilon_1\) such that \(b_1 = f_1 - f_1 < \frac{1}{q_1^2} \leq \frac{2 (q-1)^2}{(q-2)^2}\) (because of the choice of \(n_1\)), so \(\varepsilon_1\) lies inside the corresponding interval in (10). For fixed \(\varepsilon_1, \ldots, \varepsilon_{k-1}\) we choose \(\varepsilon_k\) to make \(b_k\) the smallest number of the form \(\frac{n_1}{n_k} + r_{n_k} Z \geq 0\) which is not less than \(f_k(\varepsilon_1, \ldots, \varepsilon_{k-1})\). Then

\[
0 \leq \varepsilon_k = b_k - f_k(\varepsilon_1, \ldots, \varepsilon_{k-1}) - \frac{1}{n_1 - n_k} < \frac{2 (q-1)^2}{(q-2)^2} \quad \text{(again, because of the choice of } n_1, \ldots, n_k\), so \(\varepsilon_k\) satisfies (10). Therefore the sequence \(\{b_k\}\) satisfies all the required conditions, and the statement follows.

Remark. Since \(so(\infty)\) and \(sp(\infty)\) are subalgebras of \(sl(\infty)\), each of them admits also an injective homomorphism into any one-sided pure diagonal Lie algebra of type \(A\).

The following two lemmas show that certain conditions guarantee the existence of injective homomorphisms of non-finitary diagonal Lie algebras.

Lemma 3.2. Let \(s_1 = X(T_1)\) and \(s_2 = X(T_2)\) be diagonal Lie algebras of the same type \((X = A, C, \text{or } O)\), neither of them finitary. Set \(S_i = Stz(s_i), S = GCD(S_1, S_2), R_i = \delta(S_i, S), \delta_i = \delta(T_i), C_i = Stz(C_i), C = GCD(C_1, C_2), B_i = \sigma(T_i)\) for \(i = 1, 2\). We assume that \(R_1\) is finite.

(i) Assume that \(s_1\) and \(s_2\) are non-sparse of type \(A\), both \(R_1\) and \(R_2\) are finite, and \(S\) is not divisible by an infinite power of any prime number. If \(2 \frac{R_1}{R_1} < \frac{R_2}{R_2}\), then \(s_1\) admits an injective homomorphism into \(s_2\). If \(2 \frac{R_1}{R_1} = \frac{R_2}{R_2}\), \(s_1\) admits an injective homomorphism into \(s_2\) unless \(s_1\) is pure and \(s_2\) is dense.

(ii) Assume that \(s_1\) and \(s_2\) are non-sparse, both \(R_1\) and \(R_2\) are finite, and \(S\) is not divisible by an infinite power of any prime number. In addition, assume that one of the following is true:

- both \(s_1\) and \(s_2\) are one-sided;
- \(B_1\) is finite, either \(s_1\) is one-sided and \(s_2\) is two-sided non-symmetric or \(s_2\) is two-sided weakly non-symmetric and \(s_1\) is two-sided non-symmetric;
- \(B_1\) is finite, both \(s_1\) and \(s_2\) are two-sided strongly non-symmetric, either \(B_2\) is finite or \(C\) is divisible by an infinite power of some prime number;
- both \(B_1\) and \(B_2\) are finite, both \(s_1\) and \(s_2\) are two-sided strongly non-symmetric, \(C\) is not divisible by an infinite power of some prime number, and \(\frac{R_2}{R_1} > \frac{R_1}{R_2}\).

Then, if \(\frac{R_1}{R_1} < \frac{R_2}{R_2}\), \(s_1\) admits an injective homomorphism into \(s_2\). If \(\frac{R_1}{R_1} = \frac{R_2}{R_2}\), \(s_1\) admits an injective homomorphism into \(s_2\) unless \(s_1\) is pure and \(s_2\) is dense.

(iii) Assume that \(s_1\) and \(s_2\) are non-sparse. If \(R_2\) is infinite or \(S\) is divisible by an infinite power of some prime number, then \(s_1\) admits an injective homomorphism into \(s_2\).

(iv) If \(s_2\) is sparse, then \(s_1\) admits an injective homomorphism into \(s_2\).

Proof. The Steinitz numbers \(S_1, C_1\) and the indices \(\delta_1, \sigma_1\) are in general not well-defined for a Lie algebra \(s_1\); these values characterize a given exhaustion of \(s_1\). However, if \(s_1\) is non-sparse and \(S_1\) is not divisible by an infinite power of any prime number, then the number \(\frac{R_1}{R_1}\) does not depend on the exhaustion of \(s_1\) (because then by condition \(A_2\) of Theorem 2.4 \(\frac{Stz(S_1)}{Stz(S)}\) is a set containing exactly one element for \(S_i\) corresponding to any other exhaustion of \(s_1\), and therefore \(\frac{R_1}{R_1}\) is well-defined by condition \(A_3\)). Also, under the assumptions made in the last statement of (ii) the number \(\frac{2 \frac{R_1}{R_1}}{R_1}\) does not depend on the exhaustion of \(s_1\) (this follows from condition \(B_3\) of Theorem 2.4). The finiteness of \(R_1, R_2, B_1, B_2\) does not depend on the exhaustion either, so in the proofs of all the statements we can exhaust \(s_1\) in any convenient way. The same applies to \(s_2\).

We will assume that \(X = A\) and prove all four statements for type \(A\) Lie algebras. If \(s_1, s_2\) are of type \(O\) or \(C\), then both \(s_1\) and \(s_2\) are one-sided and the proof is analogous to the proof in the type \(A\) case when \(s_1\) and \(s_2\) are one-sided.

Let us now set up the notations for the proof of all four statements. Let \(s_1\) be exhausted as \(sl(n_0) \subset sl(n_1) \subset \cdots\), each inclusion \(sl(n_i) \rightarrow sl(n_{i+1})\) being of signature \((l_i, r_i, z_i), \ i \geq 0\). By possibly changing some first terms of the exhaustion, we can choose \(n_0\) to be divisible by \(R_1\). Similarly, let \(sl(n_{i+1}) \subset \cdots\) be the exhaustion of \(s_2\), each inclusion \(sl(n_i) \rightarrow sl(n_{i+1})\) being of signature \((l_i', r_i', z_i'), \ i \geq 0\). Set \(s_i = l_i + r_i, c_i = l_i - r_i, s_i' = l_i' + r_i', \text{ and } c_i' = l_i' - r_i'\) for \(i \geq 0\). Then \(s_1 = n_0 s_0 s_1, \cdots, C_1 = n_0 c_0 c_1, \cdots,\)
\[ S_2 = m_0 s'_0 s'_1 \cdots, C_2 = m_0 c'_0 c'_1 \cdots, \delta_1 = \lim_{i \to \infty} \frac{n_0 s_0 \cdots s_{i-1}}{n_i}, \delta_2 = \lim_{i \to \infty} \frac{m_0 s'_0 \cdots s'_{i-1}}{m_i}, \sigma_1 = \lim_{i \to \infty} \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_i} \]

and \( \sigma_2 = \lim_{i \to \infty} \frac{c'_0 \cdots c'_{i-1}}{s'_0 \cdots s'_{i}} \).

Consider a diagram

\[
\begin{array}{c}
\text{sl}(n_0) \rightarrow \text{sl}(n_1) \rightarrow \cdots \rightarrow \text{sl}(n_i) \rightarrow \text{sl}(n_{i+1}) \rightarrow \cdots \\
\downarrow \theta_i \downarrow \theta_i \downarrow \theta_{i+1} \\
\text{sl}(m_{k_0}) \rightarrow \text{sl}(m_{k_1}) \rightarrow \cdots \rightarrow \text{sl}(m_{k_i}) \rightarrow \text{sl}(m_{k_{i+1}}) \rightarrow \cdots
\end{array}
\]

where \( \theta_i \) is a diagonal homomorphism of signature \((x_i, y_i, m_{k_i} - (x_i + y_i)n_i), i \geq 0 \). Taking into consideration our remark at the end of section 2, we see that to make such a diagram well-defined and commutative it is enough to have

\[
s_i(x_{i+1} + y_{i+1}) = (x_i + y_i)s'_{k_i} \cdots s'_{k_{i+1} - 1},
\]

\[
c_i(x_{i+1} - y_{i+1}) = (x_i - y_i)c'_{k_i} \cdots c'_{k_{i+1} - 1},
\]

and

\[
m_{k_i} \geq (x_i + y_i)n_i
\]

for \( i \geq 0 \). Finally, we set \( p_0 = \frac{m_0}{n_0} \) and \( p_i = \frac{m_0 s_0 \cdots s_{i-1}}{n_i} \) for \( i \geq 1 \). We are now ready to prove that there exist numbers \( x_i, y_i, i \geq 0 \) satisfying (12) - (14) in all cases.

(i) The Steinitz number \( R_2 \) is finite in this case. Possibly by changing the exhaustion of \( \sigma_2 \) we can choose \( m_0 \) to be divisible by \( R_2 \). Choose also each \( k_i \) large enough so that \( m_0 s'_0 \cdots s'_{k_i - 1} \) is divisible by \( R_2 p_i \) (this is possible since \( p_i \) divides \( S \)) and put \( q_i = \frac{m_0 s'_0 \cdots s'_{k_i - 1}}{R_2 p_i} \) for \( i \geq 0 \). Put \( x_i = y_i = q_i \). Then it is easy to verify that (12) and (13) hold, and (14) is equivalent to

\[
\frac{m_0 s'_0 \cdots s'_{k_i - 1}}{R_2 m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{R_2 n_i}
\]

Suppose that \( \frac{m_0}{n_0} < \frac{R_2}{R_2} \). Pick \( \alpha \in \left( \frac{m_0}{n_0}, \frac{R_2}{R_2} \right) \). Since \( \delta_1 = \lim_{i \to \infty} \frac{n_0 s_0 \cdots s_{i-1}}{n_i} \) and \( \delta_2 = \lim_{i \to \infty} \frac{m_0 s'_0 \cdots s'_{i-1}}{m_i} \), we have \( \frac{m_0 s'_0 \cdots s'_{k_i - 1}}{R_2 m_{k_i}} \leq \alpha \leq \frac{n_0 s_0 \cdots s_{i-1}}{R_2 n_i} \) for \( i \geq t_0, k_i \geq j_0 \). Obviously we can choose each \( k_i \) greater than \( j_0 \). Also we can construct \( \theta_i \) only for \( i \geq t_0 \) and the diagram in (11) will still give us an injective homomorphism of \( S_1 \) into \( \sigma_2 \).

Let now \( \frac{m_0}{n_0} = \frac{R_2}{R_2} \). If \( \sigma_2 \) is pure then \( \frac{m_0 s'_0 \cdots s'_{k_i - 1}}{R_2 m_{k_i}} = \frac{\delta_2}{R_2} = \frac{\delta_1}{R_2} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2 R_2 n_i} \), where the latter inequality holds because the sequence \( \frac{n_0 s_0 \cdots s_{i-1}}{n_i} \) is decreasing. Finally, if both \( S_1 \) and \( \sigma_2 \) are dense, then for each \( i \) we have \( \frac{m_0 s'_0 \cdots s'_{k_i - 1}}{R_2 m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2 R_2 n_i} \), where \( k_i \) is chosen large enough to make \( R_2 p_i \) divide \( m_0 s'_0 \cdots s'_{k_i - 1} \).

(ii) Possibly by changing the exhaustions of \( S_1 \) and \( \sigma_2 \) we choose \( n_0 \) to be divisible by \( R_2^2 \) and \( m_0 \) to be divisible by \( R_2^2 \), where \( u \) is the maximal power of 2 dividing \( S \) (\( u \) is finite because \( 2^u \) does not divide \( S \)). We also choose \( m_0 \) large enough so that \( \frac{m_0}{n_0} \geq \frac{R_2}{R_1} \). Denote again \( q_i = \frac{m_0 s'_0 \cdots s'_{k_i - 1}}{R_2 p_i} \), \( i \geq 0 \) (\( k_i \) is chosen large enough to make \( R_2 p_i \) divide \( m_0 s'_0 \cdots s'_{k_i - 1} \)).

If both \( S_1 \) and \( \sigma_2 \) are one-sided, we put \( x_i = q_i, y_i = 0 \). In the other three cases \( B_1 \) is finite, so \( c_0 c_1 \cdots \) divides \( M c_0 c'_1 \cdots \) for some finite \( M \). By changing the exhaustion of \( S_1 \) we can make \( c_0 c_1 \cdots \) divide \( c'_0 c'_1 \cdots \). For that we replace the signature \((l_i, r_i, z_i)\) with \((l_i + r_i + 1)(2, l_i + r_i - 1)/2, z_i)\) for finitely many \( i \) \((l_i + r_i) \) is odd for all \( i \geq 0 \) because \( s_0, s_1, \ldots \) is not divisible by \( 2 \). Now we can choose each \( k_i \) large enough so that \( c_0 c_1 \cdots \) divides \( c'_0 c'_1 \cdots \). Then denote \( \delta_i = \frac{m_0 s'_0 \cdots s'_{k_i - 1}}{m_0 s_0 \cdots s_{i-1}} \) for \( i \geq 1 \) and \( \delta_0 = 1 \). Notice that for each \( i \geq 0 \) the numbers \( c_i \) and \( c'_i \) have the same parities as the numbers \( s_i \) and \( s'_i \) respectively. But all \( s_i \) and \( s'_i \) are odd, so \( c_i \) and \( c'_i \) are odd as well. Hence \( t_i \) and \( q_i \) are odd, and we put \( x_i = (q_i + t_i)/2 \) and \( y_i = (q_i - t_i)/2 \). Let us check that \( y_i \geq 0 \) \((q_i \geq t_i)\). This is obvious for \( i = 0 \). For \( i \geq 1 \) the inequality \( y_i \geq 0 \) is equivalent to \( \frac{R_2}{m_0} \cdot \frac{c'_0 \cdots c'_{i-1}}{s_0 \cdots s_{i-1}} \leq \frac{R_1}{n_0} \cdot \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_{i-1}} \) or

\[
\frac{R_2}{m_0} (\sigma_2)_{k_i} \leq \frac{R_1}{n_0} (\sigma_1)_{t_i},
\]

where \( (\sigma_1)_{t_i} = \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_{i-1}} \) is a decreasing sequence which tends to \( \sigma_1 \) and \( (\sigma_2)_{t_i} = \frac{c'_0 \cdots c'_{i-1}}{s'_0 \cdots s'_{i-1}} \) is a decreasing sequence which tends to \( \sigma_2 \). Let us verify the inequality in (15) case by case.

If \( S_1 \) is one-sided, then \( (\sigma_1)_{t_i} = 1 \) for \( i \geq 1 \) and our inequality is equivalent to \( (\sigma_2)_{t_i} \leq \frac{m_0 R_1}{n_0 R_2} \).

This holds in case \( \sigma_2 \) is two-sided non-symmetric because of the assumption \( \frac{m_0 R_1}{n_0 R_2} \leq \frac{m_0 R}{n_0 R_2} \) made at the
beginning of the proof. If \( g_2 \) is two-sided weakly non-symmetric, then \( \lim_{i \to \infty} (\sigma_2)_k = \sigma_2 = 0 \), and therefore \( (\sigma_2)_k_i \leq \frac{m_0 R_1}{m_0 R_2} (\sigma_1)_i \) for large enough \( k_i \) in case \( g_1 \) is two-sided non-symmetric.

Let now both \( g_1 \) and \( g_2 \) be two-sided strongly non-symmetric, \( B_2 \) be infinite or \( C \) be divisible by an infinite power of some prime number. In this case there exists an infinite Steinitz number \( C \) such that \( c_0 C \cdot \cdots \) divides \( d_0 C \cdot' \cdots \). Since \( \sigma_1 = \lim (\sigma_1)_i > 0 \) and the sequence \( (\sigma_1)_i \), decreases, to verify \( \lim_{i \to \infty} \sigma_2 = 0 \) it suffices to prove that \( (\sigma_2)_k_i \leq \frac{m_0 R_1}{m_0 R_2} (\sigma_1)_i \). We have \( \frac{m_0}{m_2} \geq \frac{m_0}{m_1} \) therefore it is enough to prove that \( (\sigma_2)_k_i \leq \sigma_1 \). This clearly holds for large enough \( k \), if \( \sigma_2 < \sigma_1 \). Otherwise we change the exhaustion of \( g_2 \) such that the new symmetry index \( \tilde{\sigma}_2 = 2 / N \) is less than \( \sigma_1 \) for a finite \( N \), we replace \( \sigma'_i \) by \( \sigma'_i + 1 / N \), \( \sigma'_i - 1 / N \) respectively, where \( \sigma'_i = \sigma_i + 1 / N \) and \( \sigma'_i = \sigma_i - 1 / N \) for finitely many \( i \) and repeat the same construction of \( x_i, y_i \). Then \( \sigma_1 \) stays the same and in the new construction the inequality \( (\sigma_2)_k_i \leq \sigma_1 \) holds for large enough \( k_i \).

Finally, let both \( B_1 \) and \( B_2 \) be finite, both \( g_1 \) and \( g_2 \) be two-sided strongly non-symmetric, \( C \) be not divisible by an infinite power of a prime number, and \( R_1 x_1 \geq R_2 x_2 \). Then \( c_0 C \cdot \cdots \cdot = c_0 c_1 \cdot \cdots \cdot \) for an odd number \( N \), and by possibly changing the exhaustion of \( g_2 \) we can make \( c_0 C \cdot \cdots \cdot = c_0 c_1 \cdot \cdots \cdot \) and repeat the same construction. Then \( \frac{m_0}{m_2} \geq \frac{m_0}{m_1} \) and therefore \( \frac{m_0 R_1}{m_0 R_2} \geq \frac{m_0}{m_1} \). Then \( \lim_{i \to \infty} (\sigma_2)_k_i = \sigma_2 \leq \frac{m_0 R_1}{m_0 R_2} (\sigma_1)_i \), for all \( i \), since \( (\sigma_1)_i \) is a decreasing sequence which does not stabilize. Now clearly \( \lim_{i \to \infty} \sigma_2 = 0 \).

So far we have proven that in all cases we can choose exhaustions of \( g_1 \) and \( g_2 \) such that \( x_i = \frac{1}{d_i} (q_i + t_i) \) and \( y_i = \frac{1}{d_i} (q_i - t_i) \) are non-negative integers (in the first case, where both \( g_1 \) and \( g_2 \) are one-sided, we just put \( t_i = q_i \), so \( x_i = q_i, y_i = 0 \)). Since we have \( x_i + y_i = q_i \) and \( x_i - y_i = t_i \), it is easy to check \( \sigma_1 \). The condition in \( \sigma_2 \) is equivalent to \( m_0 R_1 x_i \leq m_0 R_2 x_i \), and under the assumption \( \frac{m_0}{m_1} \geq \frac{m_0}{m_2} \) or \( \frac{m_0}{m_1} = \frac{m_0}{m_2} \) its proof is analogous to that in (i).

Let us fix an exhaustion of \( g_1 \) and choose \( m_0 \) in the exhaustion of \( g_2 \) such that \( R_1 x_1 \geq R_2 x_2 \) and \( \frac{m_0 R_1 x_1}{m_0 R_2 x_2} \cdot = \) divisible by \( S \) for some finite \( R_2 \). Moreover, we can choose \( R_2 \) to be arbitrary large (if \( R_2 \) is finite, then \( R_2 \) can be any divisor of \( R_2 \); if \( p^N \cdot S \), then \( R_2 \) can be \( p^N \) for any \( N \).)

Denote \( q_i = \frac{m_0 R_1 x_i}{m_0 R_2 x_2} \cdot \) and put \( x_i = y_i = q_i \). \( x_i = 2q_i, y_i = 0 \) for types \( O \) and \( C \). Similar to the proof of (i), the conditions in \( \sigma_2 \) are satisfied, and \( \sigma_2 \) is equivalent to the inequality \( m_0 R_1 x_i \leq m_0 R_2 x_i \). Since the exhaustion of \( g_1 \) is fixed, the right-hand side is bounded by \( \frac{m_0}{m_1} \) from below. But \( \frac{m_0 R_1 x_i}{m_0 R_2 x_2} \cdot \leq \frac{1}{m_1} \), and therefore it is enough to choose \( R_2 \) to be greater than \( \frac{m_0 R_1 x_i}{m_0 R_2 x_2} \). \( \sigma_2 \).

(ii) Choose each \( k_i \) large enough so that \( m_0 R_1 x_i \cdot \cdot = \) divisible by \( p_i \) and denote \( q_i = \frac{m_0 R_1 x_i}{m_0 R_2 x_2} \cdot \cdot \), \( i \geq 0 \). Then put \( x_i = y_i = q_i \), \( x_i = 2q_i, y_i = 0 \) for types \( O \) and \( C \). The conditions in \( \sigma_2 \) are again satisfied, and \( \sigma_2 \) is equivalent to the inequality \( m_0 R_1 x_i \cdot \cdot \leq m_0 R_2 x_i \cdot \cdot \cdot \). But \( g_2 \) is sparse, therefore \( \lim_{i \to \infty} \frac{m_0}{m_i} \cdot \cdot = 0 \), so the inequality holds for large enough \( k_i \).

\begin{lemma}
Let \( g_1 = X_1(T_1) \) and \( g_2 = X_2(T_2) \) be diagonal Lie algebras, neither of them finitary. Set \( S_i = \text{St}(S_i), \ S = \text{GCD}(S_1, S_2), \ R_i = \delta_i(S_i, S), \) and \( \delta_i = \delta(T_i) \) for \( i = 1, 2 \). We assume that \( R_1 \) is finite.

(i) Assume that \( g_1 \) and \( g_2 \) are non-sparse, both \( R_1 \) and \( R_2 \) are finite, and \( S \) is not divisible by an infinite power of any prime number. In addition, let \( (X_1, X_2) = (A, C) \), \( (A, O) \), \( (O, C) \), or \( (C, O) \). If \( \frac{m_0}{m_1} < \frac{m_0}{m_2} \), then \( g_1 \) admits an injective homomorphism into \( g_2 \). If \( \frac{m_0}{m_1} = \frac{m_0}{m_2} \), \( g_1 \) admits an injective homomorphism into \( g_2 \) unless \( g_1 \) is pure and \( g_2 \) is dense.

(ii) Assume that \( g_1 \) and \( g_2 \) are non-sparse, both \( R_1 \) and \( R_2 \) are finite, and \( S \) is not divisible by an infinite power of any prime number. In addition, assume that \( (X_1, X_2) = (C, A) \) or \( (O, A) \). If \( \frac{m_0}{m_1} < \frac{m_0}{m_2} \), then \( g_1 \) admits an injective homomorphism into \( g_2 \). If \( \frac{m_0}{m_1} = \frac{m_0}{m_2} \), \( g_1 \) admits an injective homomorphism into \( g_2 \) unless \( g_1 \) is pure and \( g_2 \) is dense.

(iii) Assume that \( g_1 \) and \( g_2 \) are non-sparse. If \( R_1 \) is infinite or \( S \) is divisible by an infinite power of some prime number, then \( g_1 \) admits an injective homomorphism into \( g_2 \).

(iv) If \( g_2 \) is sparse, then \( g_1 \) admits an injective homomorphism into \( g_2 \).
\end{lemma}

\begin{proof}
The proofs of all four statements in the lemma are analogous to the corresponding proofs of Lemma 3.2. We will point out only the essential differences.

(i) If \( (X_1, X_2) = (A, C) \) or \( (A, O) \), we put \( x_i = y_i = q_i \) as in the proof of Lemma 3.2 (i). If \( (X_1, X_2) = (O, C) \) or \( (C, O) \), we put \( x_i = 2q_i, y_i = 0 \). Since we are dealing with Lie algebras of different types we have to pay attention the additional conditions of Lemma 2.9 which are obviously
satisfied. The rest of the proof is the same and the diagram in (8) (with Lie algebras of corresponding types) yields an injective homomorphism of \( s_1 \) into \( s_2 \).

(ii) Since \( s_1 \) is of type \( O \) or \( C \), \( s_1 \) is one-sided. The Lie algebra \( s_2 \) is not two-sided symmetric because \( 2^\infty \) does not divide \( S_2 \). Thus \( s_2 \) is either one-sided or two-sided non-symmetric. Both cases were considered in Lemma 3.2(ii) for type \( A \) Lie algebras. The construction of an injective homomorphism of \( s_1 \) into \( s_2 \) is the same in the case we now consider.

(iii), (iv) If \( (X_1, X_2) = (A, C) \) or \( (A, O) \), we put \( x_i = y_i = q_i \), and if \( (X_1, X_2) = (C, A), (O, A), (O, C) \), or \( (C, O) \), we put \( x_i = 2q_i, y_i = 0 \). The proofs of (iii) and (iv) are completed in a similar way to the proofs of Lemma 3.2(iii) and (iv).

\[ \square \]

Corollary 3.4. The three finitary Lie algebras \( sl(\infty), so(\infty), \) and \( sp(\infty) \) admit an injective homomorphism into any diagonal Lie algebra.

\[ \text{Proof.} \] Let \( \mathfrak{s} \) be a diagonal Lie algebra. If \( \mathfrak{s} \) is finitary, then \( \mathfrak{s} \) is isomorphic to one of the three Lie algebras \( sl(\infty), so(\infty), sp(\infty) \). Hence \( sl(\infty), so(\infty), \) admit \( sp(\infty) \) admit an injective homomorphism into \( \mathfrak{s} \). If \( \mathfrak{s} \) is not finitary, then (by an easy corollary from Lemma 3.3(iii), (iv)) there exists a pure one-sided Lie algebra of type \( A \) \( \mathfrak{s}' \) which admits an injective homomorphism into \( \mathfrak{s} \). Then each of the Lie algebras \( sl(\infty), so(\infty), sp(\infty) \) can be mapped by an injective homomorphism into \( \mathfrak{s}' \) by Proposition 3.1 and the statement follows.

\[ \square \]

Proposition 3.5. Let \( s_1 = X_1(T_1) \) be a subalgebra of \( s_2 = X_2(T_2) \). Set \( S_1 = Stz(S_1), S_2 = Stz(S_2) \). Then \( S_1|S_2 N \) for some \( N \in \mathbb{Z}_{>0} \).

\[ \text{Proof.} \] We take \( \mathfrak{s} := s_1 \) and \( \mathfrak{g} := s_2 \), in order to use the notation \( S_i \) for an exhaustion of \( \mathfrak{s} \). Since \( \mathfrak{s} \) admits an injective homomorphism into \( \mathfrak{g} \) there is a commutative diagram

\[ \begin{array}{c}
\mathfrak{s}_1 \rightarrow \cdots \rightarrow \mathfrak{s}_i \rightarrow \cdots \\
\mathfrak{g}_1 \rightarrow \cdots \rightarrow \mathfrak{g}_i \\
\end{array} \]

Set \( M = I_{\mathfrak{s}_1}^{k_1}(\theta_1) \). Then, by Proposition 3.2(ii), we have \( I_{\mathfrak{s}_1}^{k_1} M = I_{\mathfrak{s}_i}^{k_i} I_{\mathfrak{s}_i}^{k_i}(\theta_1) \) for \( i \geq 1 \). Then

\[ \prod_{j=1}^{k_1-1} I_{\mathfrak{s}_j}^{k_j+1} M \prod_{j=1}^{k_i-1} I_{\mathfrak{s}_j}^{k_j+1} \]

for \( i \geq 1 \). Thus, \( S_1|S_2 M n_1 \), where \( n_1 \) is the dimension of the natural representation of \( \mathfrak{s}_1 \).

\[ \square \]

Proposition 3.6. Let \( \mathfrak{s} \) be a sparse one-sided Lie algebra of type \( A \) not isomorphic to \( sl(\infty) \). Then \( \mathfrak{s} \) admits no non-trivial homomorphism into a pure one-sided Lie algebra of type \( A \).

\[ \text{Proof.} \] Assume for the sake of a contradiction that there is an injective homomorphism of \( \mathfrak{s} \) into some pure one-sided Lie algebra of type \( A \). Let \( \mathfrak{s} \) be exhausted as \( \mathfrak{s}(n_1) \subset \mathfrak{s}(n_2) \subset \cdots \), each inclusion \( \mathfrak{s}(n_i) \rightarrow \mathfrak{s}(n_{i+1}) \) being of signature \((l_i, 0, z_i)\). Recall that by the definition of a sparse Lie algebra, \( \lim_{i \rightarrow \infty} \frac{n_1 l_1 \cdots l_{i-1}}{n_i} = 0 \). Then there is a commutative diagram

\[ \begin{array}{c}
\mathfrak{s}(n_1) \rightarrow \cdots \rightarrow \mathfrak{s}(n_i) \rightarrow \mathfrak{s}(n_{i+1}) \rightarrow \cdots \\
\mathfrak{s}(m_1) \rightarrow \cdots \rightarrow \mathfrak{s}(m_i \cdots m_{i+1}) \rightarrow \cdots \\
\end{array} \]

The lower row constitutes an exhaustion of the pure Lie algebra \( \mathfrak{s}(m_1 m_2 \cdots) \).

Denote by \( V_i \) the natural \( \mathfrak{s}(m_1 \cdots m_i) \)-module for \( i \geq 1 \). Note that \( \theta_i \) makes \( V_i \) into an \( \mathfrak{s}(n_i) \)-module. Let

\[ V_i \downarrow \mathfrak{s}(n_i) \cong \bigoplus_{\lambda \in \mathcal{H}_i} T_\lambda \otimes F_{n_i}^\lambda \]

be the decomposition into a direct sum of isotypic components. Here \( T_\lambda = \text{Hom}_{\mathfrak{s}(n_i)} (F_{n_i}^\lambda, V_i \downarrow \mathfrak{s}(n_i)) \) is a trivial \( \mathfrak{s}(n_i) \)-module, and \( \mathcal{H}_i \) is the set of all highest weights appearing in this decomposition. We can rewrite (17) (non-canonically) as

\[ V_i \downarrow \mathfrak{s}(n_i) \cong \bigoplus_{\lambda \in \mathcal{H}_i} F_{n_i}^\lambda \oplus \cdots \oplus F_{n_i}^\lambda. \]
where $t_\lambda = \dim T_\lambda$. Since all weights $\lambda \in H_i$ are dominant, for each $\lambda = (\lambda_1, \ldots, \lambda_{n_i})$, $\lambda_1 - \lambda_{n_i}$ is a non-negative integer. Let $d_i = \max(\lambda_1 - \lambda_{n_i})$. We define $H(\varphi)$ and $d(\varphi)$ in a similar way for an arbitrary injective homomorphism $\varphi$ of finite-dimensional classical simple Lie algebras of type $A$, so that $H(\theta_i) = H_i$ and $d(\theta_i) = d_i$.

Let us show that $d_i \geq d_{i+1}$ for $i \geq 1$. By $\varphi_i$ we denote the injective homomorphism $s l(m_1 \cdots m_i) \to s l(m_1 \cdots m_{i+1})$ as in (16). Notice first that $H(\varphi_i \circ \theta_i) = H(\theta_i)$ and $d(\varphi_i \circ \theta_i) = d(\theta_i)$ for all $\lambda \in H_i$. Furthermore, $d(\varphi_i \circ \theta_i) = d(\theta_i) = d_i$.

Let $\lambda \in H_{i+1}$ be a weight such that $\lambda_1 - \lambda_{n_{i+1}} = d_{i+1}$. Since $(1, 0, z_i)$ is the signature of the diagonal injective homomorphism $s l(n_i) \to s l(n_{i+1})$, there is a chain of inclusions $s l(n_i) \subset s l(n_{i+1}) \subset s l(n_i + 1) \subset \cdots \subset s l(n_{i+1} + z_i) = s l(n_{i+1})$ such that their composition is the original map in (16). Applying Gelfand-Tsetlin rule repeatedly we obtain that $F^\lambda_{n_{i+1}} \downarrow s l(n_i + z_i - j)$ has a submodule with highest weight $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n_{i+1} - j - 1}, \lambda_{n_{i+1}})$ for $j = 1, \ldots, n_i$. We then apply Corollary 2.3 to the submodule of $F^\lambda_{n_{i+1}} \downarrow s l(n_i)$ with highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_{n_{i+1} - j - 1}, \lambda_{n_{i+1}})$ and see $\lambda := (\lambda_1 + \cdots + \lambda_k, \lambda_{k+1} + \cdots + \lambda_{n_{i+1}}) \in H(\varphi_i \circ \theta_i)$, i.e. the $s l(n_i)$-module with highest weight $\lambda$ is a constituent of $F^\lambda_{n_{i+1}} \downarrow s l(n_i)$. Hence, $d(\varphi_i \circ \theta_i) \geq \lambda_1 - \lambda_{j - 1}, \lambda_{j} - \lambda_{j - 1}, \ldots, \lambda_{n_{i+1}}$ is a decreasing sequence of positive integers, it stabilizes, so there exists $\theta \in Z_{\geq 0}$ such that $d_i = d$ for all $i \geq \theta$. Pick $k$ such that $l_j \cdots l_{K-1} > d$ (this is possible since $\theta$ is not isomorphic to $s l(\infty)$, and therefore $\prod_{i=1}^{\infty} l_i$ is infinite). Consider now the following part of the diagram in (16):

\[
\begin{array}{ccc}
\theta(n_j) & \rightarrow & \cdots \\
\downarrow & & \downarrow \\
\theta_K(n_j) & \rightarrow & \cdots \\
\theta(n_1 \cdots m_j) & \rightarrow & \cdots \\
\end{array}
\]

The injective homomorphism $\theta(n_j) \to \theta(n_K)$ is diagonal of signature $(l, 0, z)$, where $l = l_j \cdots l_{K-1}$ and $z = n_K - l_{l_j}$. Using similar arguments as above we obtain that $\lambda = (\lambda_1 + \cdots + \lambda_k, \lambda_{k+1} + \cdots + \lambda_{n_K - j - 1} + \cdots + \lambda_{n_K}) \in H(\varphi_\theta \circ \theta_i)$ for any $\lambda \in H_K$. Then we have $\lambda_1 + \cdots + \lambda_k - (\lambda_{n_K - j - 1} + \cdots + \lambda_{n_K}) \leq \theta$. If $\lambda_{d+1} \neq \lambda_{n_K - d}$, then $\lambda_{d+1} \geq \lambda_{n_K - d} + 1$, in which case $\lambda_1 + \cdots + \lambda_k - (\lambda_{n_K - j - 1} + \cdots + \lambda_{n_K}) \geq \lambda_1 + \cdots + \lambda_{d+1} \geq \lambda_{n_K - d} + \cdots + \lambda_{n_K} \geq \lambda_{n_K - d}$. Hence, $\lambda_{d+1} = \lambda_{n_K - d}$, which yields $\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_{n_K - d}$. We thus conclude that for $i \geq K$ each integral dominant weight appearing in $H_i$ has the property that all its values apart from the first $d$ and the last $d$ must be equal.

Using Proposition 2.2(ii) and Corollary 2.3 we compute $I_{s l(n_1)}^{\lambda_{1} \cdots m_i} = I(\theta_1)m_2 \cdots m_i$ by following down $\theta_1$ and to the right; similarly we compute $I_{s l(n_1)}^{\lambda_{1} \cdots m_i} = l_1 \cdots l_{i-1}I(\theta_i)$ by going to the right and then down $\theta_i$. By Proposition 2.2(iii), (iv) we have

\[
I(\theta_i) = \sum_{\lambda \in H_i} t_\lambda J(\varphi_i)(\lambda) = \frac{1}{n_i! - 1} \sum_{\lambda \in H_i} t_\lambda \dim F_{n_i}^\lambda(\lambda, \lambda + 2\rho)_{s l(n_i)},
\]

where $2\rho$ is the sum of all the positive roots of $s l(n_i)$.

Note that $\langle \lambda, \lambda + 2\rho \rangle_{s l(n_i)} = (\hat{\lambda}, \hat{\lambda} + 2\rho)$, where $\hat{\lambda}_j = \lambda_j - \frac{1}{n_i} \sum_{k=1}^{n_i} k \lambda_k$ for $j = 1, \ldots, n_i, 2\rho = (n_i - 1, n_i - 3, \ldots, -(n_i - 1))$, and $(,)$ is the usual scalar product on $\mathbb{C}^{n_i}$.

Fix $i \geq K$, using the notation from above, so that $\lambda_1 - \lambda_{n_i} \leq d$ and $\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_{n_i - d}$. Set $\alpha = \lambda_{d+1}$, so that $|\lambda_j - \alpha| = 0$ for $j = d + 1, d + 2, \ldots, n_i - d$. Then $|\lambda_j - \alpha| = |\lambda_j - \lambda_{d+1}| \leq d$ for all $j$. 

\[
12
\]
Let there exists a pure one-sided Lie algebra of type \( k \) holds.

Since \( \lambda_1 + \cdots + \lambda_d = (n_1 - 2d) + \lambda_{n_1-d+1} + \cdots + \lambda_{n_1} = 0 \) (which implies \( |\alpha| \leq \frac{2d^2}{n_1 m_1} \)), we obtain the following inequality:

\[
|\langle \lambda, \lambda + 2\rho \rangle|_{sl(n_1)} \leq 2d^2 n_1 + 4d^2 \left( \frac{n_1}{n_1 - 2d} \right) + \frac{2d^2 n_1 (n_1 + 1)}{n_1 - 2d} = c_0 n_1
\]

for all \( i \geq K \), where \( c_0 \) is some positive constant. Then from \( \{19\} \) we have \( I(\theta_i) \leq \sum_{\lambda H_1} t_{\lambda} \dim F_{n_1}^{\lambda} = \frac{c_0 n_1}{n_1^2 - 1} m_1 \cdots m_1 \). Hence, \( I(\theta_i)_{m_2 \cdots m_1} = \frac{sl(m_1, \cdots m_1)}{sl(n_1)} = l_1 \cdots l_{i-1} I(\theta_i) \leq l_1 \cdots l_{i-1} \frac{c_0 n_1}{n_1^2 - 1} m_1 \cdots m_1 \). This implies \( \frac{c_0 n_1}{n_1^2 - 1} \frac{l_1 \cdots l_{i-1}}{n_i} \geq c_1 \) for some positive constant \( c_1 \). The last inequality contradicts the fact that \( \lim_{i \to \infty} \frac{n_1 l_1 \cdots l_{i-1}}{n_i} = 0 \), so the proposition follows.

**Corollary 3.7.** Let \( s_1, s_2 \) be non-finitary diagonal Lie algebras. Assume that \( s_1 \) is sparse and there is an injective homomorphism of \( s_1 \) into \( s_2 \). Then \( s_2 \) must be sparse as well.

**Proof.** Suppose, on the contrary, that \( s_2 \) is pure or dense. Lemma \( 3.3 \) (iv) implies that there exists a sparse one-sided Lie algebra of type \( A \) \( s'_1 \) which admits an injective homomorphism into \( s_1 \). By Lemma \( 3.3 \) (iii) there exists a pure one-sided Lie algebra of type \( A \) \( s'_2 \) such that \( s_2 \) admits an injective homomorphism into \( s'_2 \). If \( s_1 \) would admit an injective homomorphism into \( s_2 \), then \( s'_1 \) would admit an injective homomorphism into \( s'_2 \) through the chain \( s'_1 \subset s_1 \subset s_2 \subset s'_2 \), which would contradict Proposition \( 3.6 \). Hence the statement holds.

**Proposition 3.8.** Let \( s_1 = A(T_1) \) and \( s_2 = A(T_2) \) be pure one-sided Lie algebras, neither of them finitary. Set \( S_i = Stz(S_i) \) for \( i = 1, 2 \), and \( S = GCD(S_1, S_2) \). Assume that both Steinits numbers \( \hat{\nu}(S_i, S) \) and \( \hat{\nu}(S, S) \) are finite and \( S \) is not divisible by an infinite power of any prime number. An injective homomorphism of \( s_1 \) into \( s_2 \) is necessarily diagonal.

**Proof.** Let \( S = p_1^{\ell_1} p_2^{\ell_2} \cdots \) for the increasing sequence \( \{p_i\} \) of all prime numbers dividing \( S \). Denote \( n_i = \frac{S_2(p_i)}{S_1(p_i)} \cdots (p_{N+1})^{N+1} \) for \( i \geq 0 \), with integer \( N \) to be fixed later. Suppose that there is an injective homomorphism \( \theta : s_1 \to s_2 \). Then it is given by the following commutative diagram:

\[
\begin{array}{ccc}
\text{sl}(n_0) & \cdots & \text{sl}(n_i) & \cdots & \text{sl}(n_{i+1}) & \cdots \\
\theta_0 & & \theta_i & & \theta_{i+1} & \\
\text{sl}(m_0) & \cdots & \text{sl}(m_i) & \cdots & \text{sl}(m_{i+1}) & \cdots \\
\end{array}
\]

where \( m_i = \frac{S_2(p_i)}{S_1(p_i)} \cdots (p_{N+k_i})^{N+k_i} \) for \( i \geq 0 \) for some \( k_0, k_1, \ldots \). By possibly shifting the bottom row of the diagram we may assume that \( k_i \geq i + 1 \) for each \( i \geq 0 \).
Denote by $W_i$ the natural $\text{sl}(n_i)$-module. Let $H(\varphi)$ and $d(\varphi)$ be as in the proof of Proposition 3.5 for an arbitrary injective homomorphism $\varphi$ of finite-dimensional classical simple Lie algebras of type $A$. Set $H_i = H(\theta_i)$ and $d_i = d(\theta_i)$ for $i \geq 0$. Similarly to (18) we then have

$$W_i \downarrow \text{sl}(n_i) \cong \bigoplus_{\lambda \in \Pi_i} F^\lambda_{n_i} \oplus \cdots \oplus F^\lambda_{n_i},$$

where $\ell_{\lambda,i} = \dim \text{Hom}_{\text{sl}(n_i)}(F^\lambda_{n_i}, W_i \downarrow \text{sl}(n_i))$.

Similarly to the proof of Proposition 3.5, $\{d_i\}$ is a decreasing sequence, and therefore $d_i = d$ for $i \geq i_0$. By choosing $N$ large enough we make $d_i = d$ and $p_{N+i} > d + 1$ for all $i \geq 0$. Take now $0 \leq i < j \leq k_i$ and consider the following piece of the diagram in (20):

$$\begin{array}{c}
\text{sl}(n_i) \longrightarrow \cdots \longrightarrow \text{sl}(n_j) \\
\theta \downarrow \text{sl}(n_i) \longrightarrow \cdots \longrightarrow \text{sl}(n_j) \\
\text{sl}(n_j) \longrightarrow \cdots \longrightarrow \text{sl}(n_m).
\end{array}$$

Here the injective homomorphism $\text{sl}(n_i) \to \text{sl}(n_j)$ is of signature $(q, 0, 0)$, where $q = (p_{N+i+1})^{N+i+1} \cdots (p_{N+j})^{N+j}$.

Take an arbitrary non-trivial highest weight $\lambda$ in $H_j$, yielding the $\text{sl}(n_i)$-module $F^\lambda_{n_i}$. Since $n_j = qn_i$, by Proposition 2.4 we have

$$F^\lambda_{qn_i} \downarrow \text{sl}(n_i) \cong \bigoplus_{\nu} \left( \sum_{\mu_1, \ldots, \mu_q} c^\lambda_{\mu_1, \ldots, \mu_q} q^\nu_{\mu_1, \ldots, \mu_q} F^\nu_{n_i} \right).$$

Since the coefficients $c^\lambda_{\mu_1, \ldots, \mu_q}$ and $q^\nu_{\mu_1, \ldots, \mu_q}$ are independent of the order of $\mu_1, \ldots, \mu_q$, we can rewrite this as

$$F^\lambda_{qn_i} \downarrow \text{sl}(n_i) \cong \bigoplus_{\nu} \left( \sum_{[\mu_1, \ldots, \mu_q]} c^{q\nu_{\mu_1, \ldots, \mu_q}}_{\mu_1, \ldots, \mu_q} F^{q\nu_{\mu_1, \ldots, \mu_q}}_{n_i} \right).$$

Here $[\mu_1, \ldots, \mu_q]$ denotes the multiset with these elements, and $q_1, \ldots, q_r$ are the corresponding multiplicities, so that $q_1 + \cdots + q_r = q$.

Fix a highest weight $\nu$ such that $F^\nu_{n_i}$ has non-zero multiplicity in (22) and fix a multiset of integral dominant weights $[\mu_1, \ldots, \mu_q]$ making both generalized Littlewood-Richardson coefficients $c^{q\nu_{\mu_1, \ldots, \mu_q}}_{\mu_1, \ldots, \mu_q}$ and $q^{\nu_{\mu_1, \ldots, \mu_q}}$ non-zero. We will show that $\nu$ divides $q^{\nu_{\mu_1, \ldots, \mu_q}}$ (and hence the contribution from $[\mu_1, \ldots, \mu_q]$ to the multiplicity of $F^\nu_{n_i}$) if the module $F^\nu_{n_i}$ is non-trivial. Suppose that $p_i$ divides all $q_1, \ldots, q_r$ for some $N+i+1 \leq l \leq N+j$. Note that the $\text{sl}(n_i)$-module $F^\nu_{n_i}$ for $\nu' = \mu_1 + \cdots + \mu_q$ also has non-zero multiplicity in (22) because $c^{\nu_{\mu_1, \ldots, \mu_q}}_{\mu_1, \ldots, \mu_q} \neq 0$. Since all $q_1, \ldots, q_r$ are divisible by $p_i$, we have $\nu' = p_i \nu'$ for some integral dominant weight $\nu'$. Since $F^\nu_{n_i}$ has non-zero multiplicity in $W_j$ considered as an $\text{sl}(n_i)$-module using the path along $\theta_i$ in (21), and since $W_j \downarrow \text{sl}(n_i)$ is a direct sum of copies of $W_i$, it must be that $F^\nu_{n_i}$ has non-zero multiplicity in $W_j \downarrow \text{sl}(n_i)$, i.e. $\nu' \in H_i$. Since $d_i = d < p_i - 1$ we have $p_i > \nu' - \nu'_{\nu'} = p_i(\mu_1 - \mu'_{\nu'})$ which is possible only if $\mu_1 = \mu'_{\nu'}$ (equivalently, $\nu' = \nu'_{\nu'}$). Therefore $\nu'$ is a trivial highest weight, and hence all $\mu_1, \ldots, \mu_q$ are trivial as well. Then the coefficient $c^{\nu_{\mu_1, \ldots, \mu_q}}_{\mu_1, \ldots, \mu_q}$ is non-zero only if $\nu$ is trivial, so $F^\nu_{n_i}$ is the trivial module.

Suppose now that $p_i$ does not divide at least one of $q_1, \ldots, q_r$, for each $l$ such that $N+i \leq l \leq N+j$. A combinatorial argument shows that $C^{q^{\nu_{\mu_1, \ldots, \mu_q}}}_{q^{\nu_{\mu_1, \ldots, \mu_q}}} = q^{\nu_{\mu_1, \ldots, \mu_q}}$ is divisible by $q$ if each prime divisor of $q$ fails to divide at least one of $q_1, \ldots, q_r$. We thus conclude that each non-trivial $\text{sl}(n_i)$-module $F^\nu_{n_i}$ with non-zero multiplicity in (22), has multiplicity divisible by $q$. As a corollary, any non-trivial simple constituent of $W_j \downarrow \text{sl}(n_i)$ appears with multiplicity divisible by $q$.

By following the diagram in (21) down $\theta_i$ and then to the right, we get $W_j \downarrow \text{sl}(n_i) \cong \bigoplus_{\nu \in \Pi_i} t_{\nu,i} F^\nu_{n_i}$.

Since $q = (p_{N+i+1})^{N+i+1} \cdots (p_{N+j})^{N+j}$ is relatively prime to $m_{n_i} = (p_{N+i+k_i})^{N+k_i+1} \cdots (p_{N+j+k_i})^{N+k_i}$ (as $j \leq k_i$), the commutativity of the diagram in (21) implies that $t_{\nu,i}$ is divisible by $q$ for any non-trivial $\nu$ in $H_i$.

Let us introduce a new notation. For an arbitrary injective homomorphism $\varphi : g_1 \to g_2$ of finite-dimensional classical simple Lie algebras of type $A$ we denote by $N(\varphi)$ the number (counting multiplicities) of simple non-trivial constituents of the natural representation of $g_2$ considered as a $g_1$-module via $\varphi$. Then $N_i := N(\theta_i)$ is divisible by $q = (p_{N+i+1})^{N+i+1} \cdots (p_{N+j})^{N+j}$ by the above argument. Taking $j = k_i$, we obtain that $N_i$ is divisible by $(p_{N+i+k_i})^{N+k_i+1} \cdots (p_{N+j+k_i})^{N+k_i}$. Fix now $j = i+1$ in the diagram in (24), and let $\varphi : \text{sl}(n_1) \to \text{sl}(n_{i+1})$ denote the map produced by this diagram. As shown above, each non-trivial weight $\lambda \in H_{i+1}$ yields a non-trivial weight in $H(\varphi) = H_i$ with non-zero multiplicity divisible by $(p_{N+i+k_i})^{N+i+1}$, and hence at least $(p_{N+i+1})^{N+i+1}$. Therefore by following the diagram to the right and then down $\theta_{i+1}$, we obtain $N(\varphi) \geq (p_{N+i+1})^{N+i+1} N_{i+1}$. Note
also that equality holds here only if for each non-trivial \( \lambda \in H_{i+1} \) we have \( F_{\nu_1} \downarrow \text{sl}(n_i) \cong q F_{\nu'} \oplus T \) for a non-trivial \( \nu \in H_i \), where \( T \) is a trivial (possibly 0-dimensional) module. Meanwhile, by following the diagram down \( \theta_i \) and to the right we have \( N(\psi) = (p_{N-k_i+1})^{N+k_i+1} \cdots (p_{N-k_i+1})^{N+k_i+1} N_i \). As a result we obtain the inequality \( (p_{N-k_i+1})^{N+k_i+1} \cdots (p_{N-k_i+1})^{N+k_i+1} N_i \geq (p_{N+i+1})^{N+i+1} N_{i+1} \), i.e. \( \alpha_i \geq \alpha_{i+1} \), where \( \alpha_i := \frac{c_{N+k_i} \cdots c_{N}}{N_{i+1} N_i} \) are integers for \( i \geq 0 \). Since \( \{\alpha_i\} \) is a decreasing sequence of positive integers it stabilizes, and by choosing \( N \) sufficiently large we can assume that \( \alpha_0 = \alpha_1 = \alpha_2 = \cdots \).

Now take an arbitrary non-trivial \( \lambda \in H_{i+1} \). Since \( \alpha_i = \alpha_{i+1} \), the decomposition in (22) becomes \( F_{\nu_1} \downarrow \text{sl}(n_i) \cong q F_{\nu'} \oplus T \) for some non-trivial \( \nu \in H_i \), where \( T \) is some trivial (possibly 0-dimensional) module. Since the contribution from each multiset \( [\mu_1, \ldots, \mu_q] \) to the multiplicity of \( F_{\nu_i} \) in (22) is divisible by \( q \), there exists exactly one multiset \( [\mu_1, \ldots, \mu_q] \) making a non-zero contribution to the multiplicity of \( F_{\nu_i} \). Moreover, the fact that \( C^{\lambda}_{\nu_1} \cdots \nu_r c_{\mu_1} \cdots c_{\mu_1} \cdots c_{\mu_q} = q \) together with the fact that \( q \) divides \( C^{\lambda}_{\nu_1} \cdots \nu_r \) implies \( C^{\lambda}_{\nu_1} \cdots \nu_r = q \). It is easy to check that \( q_{\nu_1}^{\lambda} = q \) only if \( r = 2 \) and \( \{q_1, q_2\} = \{1, q-1\} \). Then we safely can assume that \( \mu_1 = \mu_2 = \cdots = \mu_{q-1} \). Since \( \nu' = \mu_1 + \cdots + \mu_q \) is a non-trivial weight satisfying \( c_{\mu_1} \cdots \mu_q \neq 0 \), the module \( F_{\nu_1} \) also has non-zero multiplicity in (22), and therefore \( \nu = \nu' \). Hence \( \nu = (q-1)\mu_1 + \mu_q \), and since \( \nu = \nu_1 = \frac{c_{N+k_i} \cdots c_{N}}{N_{i+1} N_i} \), we immediately get that \( \mu_1 = \mu_2 = \cdots = \mu_{q-1} \) is a trivial weight. Then the only multiset \( [\mu_1, \ldots, \mu_q] \) making \( c_{\mu_1} \cdots \mu_q \) non-zero has \( q-1 \) trivial weights. One can check this can only be possible if \( \alpha_1 \) is of the form \((c+1, c, \ldots, c, c)\) or \((c, c, \ldots, c, c+1)\). Thus, all non-trivial highest weights from \( H_{i+1} \) are either those of the natural or of the conatural representation. This means precisely that all homomorphisms \( \theta_i \) are diagonal.

**Corollary 3.9.** Let \( s_1 = X_1(T_1) \) and \( s_2 = X_2(T_2) \) be non-sparse Lie algebras, neither of them finitary. Set \( S_i = \text{Stz}(S_i), S = \text{GCD}(S_1, S_2) \), and \( R_i = \frac{\sigma_i}{\delta_i} (S_i, S) \) for \( i = 1, 2 \). Assume that \( S \) is not divisible by an infinite power of any prime number, and that both \( R_1 \) and \( R_2 \) are finite. An injective homomorphism of \( s_1 \) into \( s_2 \) is necessary diagonal.

**Proof.** Denote \( \delta_1 = \delta(T_1), \delta_2 = \delta(T_2) \), where \( R_1 > 2\delta_1 \) is some finite divisor of \( S_1 \), and \( R_2 = \text{sl}(S_2 R_2) \), where \( R_2 \) is finite and \( R_2 > R_1 \). Then, by Lemma 3.2 (i) and Lemma 3.3 (i), (ii), \( \delta_1 \) admits an injective homomorphism into \( s_1 \) and \( \delta_2 \) admits an injective homomorphism into \( s_2 \). Then there exists an injective homomorphism of \( s_1 \) into \( s_2 \) through the chain \( s_1' \subset s_2' \subset s_2' \) and this homomorphism is diagonal because the Lie algebras \( s_1' \) and \( s_2' \) satisfy the conditions of Proposition 3.8. Finally, it follows from Corollary 2.11 that the injective homomorphism of \( s_1 \) into \( s_2 \) has to be diagonal as well.

**Lemma 3.10.** Let \( s_1 = X_1(T_1) \) and \( s_2 = X_2(T_2) \) be non-sparse Lie algebras, neither of them finitary. Set \( S_i = \text{Stz}(S_i), S = \text{GCD}(S_1, S_2) \), \( R_i = \frac{\sigma_i}{\delta_i} (S_i, S) \), \( \delta_i = \delta(T_i), \delta_i = \text{Stz}(C_i) \), \( C_i = \text{GCD}(C_1, C_2) \), \( B_i = \frac{\sigma_i}{\delta_i} (C_i, C) \), and \( \sigma_1 = \sigma(T_1) \) for \( i = 1, 2 \). Assume that \( S \) is not divisible by an infinite power of any prime, and that both \( R_1 \) and \( R_2 \) are finite. If \( s_1 \) admits a diagonal injective homomorphism into \( s_2 \), then the following holds.

(i) \( \frac{R_1}{R_1} \leq \frac{R_2}{R_2} \). The inequality is strict if \( s_1 \) and \( s_2 \) are dense.

(ii) \( \frac{R_1}{R_1} \leq \frac{R_2}{R_2} \). When one of the following additional hypotheses holds:

- \( (X_1, X_2) = (A, C), (A, O), (O, C), \) or \( (C, O) \);
- \( (X_1, X_2) = (A, A), B_1 \) is infinite;
- \( (X_1, X_2) = (A, A), B_1 \) is finite, \( s_1 \) is two-sided weakly non-symmetric, \( s_2 \) is either one-sided or two-sided strongly non-symmetric;
- \( (X_1, X_2) = (A, A), B_1 \) and \( B_2 \) are finite, \( C \) is not divisible by an infinite power of a prime number, both \( s_1 \) and \( s_2 \) are two-sided strongly non-symmetric, and \( \frac{R_1 B_1}{R_1} < \frac{R_2 B_2}{R_2} \).

Again the inequality is strict if \( s_1 \) and \( s_2 \) are dense.

**Proof.** As it was explained in the proof of Lemma 3.2, we can choose suitable exhaustions of \( s_1 \) and \( s_2 \).

(i) Assume that \( (X_1, X_2) = (A, A) \) (the other cases are analogous). Let \( s_1 \) be exhausted as \( \text{sl}(m_0) \subset \text{sl}(m_1) \subset \cdots \), each inclusion \( \text{sl}(m_0) \to \text{sl}(m_{i+1}) \) being of signature \( (l_i, r_i, z_i) \), \( i \geq 0 \) and \( s_2 \) as \( \text{sl}(m_0) \subset \text{sl}(m_1) \subset \cdots \) with \( \text{sl}(m_0) \to \text{sl}(m_{i+1}) \) being of signature \( (l'_i, r'_i, z'_i) \), \( i \geq 0 \). Moreover, we choose \( m_0 \) to be divisible by \( R_1 \) and \( m_0 \) to be divisible by \( R_2 \).

There is a commutative diagram

\[
\begin{array}{c}
\text{sl}(m_0) \longrightarrow \text{sl}(m_1) \longrightarrow \cdots \\
\downarrow \theta_0 \quad \downarrow \theta_1 \quad \downarrow \theta_i \\
\text{sl}(m_k) \longrightarrow \cdots \\
\end{array}
\]
where each injective homomorphism \( \theta \) is diagonal of signature \((x_i, y_i, m_{k_i} - (x_i + y_i)n_i)\). Denote \( q_i = x_i + y_i \). Then, using Corollary 2.5 \([BZ]\), we get

\[
q_is_i \cdots s_ij = s_i \cdots sj-1q_j \quad \text{for all } j \geq i \geq 0.
\]

Hence \( s_is_{i+1} \cdots \) divides \( q_is_{i+1} \cdots \) for \( i \geq 0 \), so \( S_1m_0s_0 \cdots s_{k_i-1} \) divides \( q_iS_2m_0s_0 \cdots s_{k_i-1} \). Since \( S \) is not divisible by an infinite power of any prime number, the first Steinitz number will still divide the second one after cancellation of both of them by \( S \). Therefore \( \delta(q_iR_2m_0s_0 \cdots s_{k_i-1}, R_1m_0s_0 \cdots s_{k_i-1}) \) is a Steinitz number which is moreover finite, and thus it is a positive integer. So \( \frac{m_0s_0 \cdots s_{k_i-1}}{R_2m_{k_i+1}} \leq \frac{m_0s_0 \cdots s_{k_i}}{R_1m_{k_i}} \).

Taking the limit of both sides for \( i \to \infty \) we get \( \frac{\delta}{R_2} \leq \frac{\delta}{R_1} \). Moreover, if \( \delta_1 \) is pure and \( \delta_2 \) is dense, then \( \frac{m_0s_0 \cdots s_{k_i-1}}{R_2m_{k_i+1}} \leq \frac{\delta}{R_1} \) for large enough \( i \). But the decreasing sequence \( \frac{m_0s_0 \cdots s_{k_i-1}}{m_{k_i}} \) does not stabilize, so we obtain the strict inequality \( \frac{\delta}{R_2} < \frac{\delta}{R_1} \).

(ii) We keep the notations from (i). The injective homomorphism of \( \sigma_1 \) into \( \sigma_2 \) is given again by \([BZ]\).

If the pair \((X_1, X_2)\) is one of \((A, C), (A, O), (O, C), \) and \((C, O)\), then, by Proposition 2.3 \([BZ]\), for any diagonal injective homomorphism of a type \( X_1 \) algebra into a type \( X_2 \) algebra of signature \((l, r, z)\) the integer \( l + r \) is even. Therefore \( q_i \) is divisible by 2 for any \( j \) and it follows from \([BZ]\) that \( q_is_i \cdots s_{k_i-1} \) is divisible by \( 2, s_{i+1}, \ldots \). The rest of the proof is analogous to (i).

In the other three cases both \( \sigma_1 \) and \( \sigma_2 \) are of type \( A \). Notice that neither \( \sigma_1 \) nor \( \sigma_2 \) is two-sided symmetric (otherwise \( S \) would be divisible by \( 2^\infty \)). Thus we can assume that \( c_i > 0 \) and \( c_i' > 0 \) for all \( i \geq 0 \). Denote \( t_i = x_i - y_i \). It is enough to prove that \( t_i = 0 \) for infinitely many \( i \) (because then \( q_i \) is even for infinitely many \( i \) and the statement can be proven similarly to the first case). Assume the contrary, i.e. let \( t_i > 0 \) for \( i \geq 0 \). Without loss of generality we can assume that \( t_i > 0 \) for all \( i \geq 0 \). Let us show that this contradicts with the assumptions of the lemma in all three cases.

Let \( B_1 \) be infinite. By Corollary 2.5 in \([BZ]\),

\[
t_0c_0 \cdots c_{k_i-1} = c_0, \cdots c_{i-1}t_i \quad \text{for } i \geq 1.
\]

Then clearly \( c_0c_1 \cdots c_{k_i-1} \) divides \( t_0c_0 \cdots c_{k_i-1} \), and therefore \( B_1 \) divides \( n_0t_0 \). This contradicts \( B_1 \) being infinite.

For the next case, combining \([24]\) and \([24]\), we obtain \( \lim_{q_0 \to \infty} \frac{c_0 \cdots c_i}{s_0 \cdots s_i} = \frac{\delta}{R_2} \). By definition \( \sigma_1 = \lim_{i \to \infty} c_0 \cdots c_i \), and since \( \sigma_1 \) is two-sided weakly non-symmetric we have \( \lim_{i \to \infty} \frac{t_i c_0 \cdots c_i}{s_0 \cdots s_i} = 0 \). But

\[
\lim_{i \to \infty} \frac{\delta}{q_0} \frac{c_0 \cdots c_i}{s_0 \cdots s_i} = \frac{\delta}{q_0} \frac{c_0 \cdots c_i}{s_0 \cdots s_i} \geq 0.
\]

Finally, let both \( \sigma_1 \) and \( \sigma_2 \) be two-sided strongly non-symmetric. Since \( t_i \leq q_i \) for \( i \geq 0 \), we have \( \frac{c_0 \cdots c_i}{s_0 \cdots s_i} \leq \frac{c_0 \cdots c_i}{s_0 \cdots s_i} \). Taking the limit we obtain

\[
\frac{\delta}{q_0} \frac{c_0 \cdots c_i}{s_0 \cdots s_i} \leq \frac{\delta}{q_0} \frac{c_0 \cdots c_i}{s_0 \cdots s_i} \leq \sigma_1.
\]

Let us go back to \([24]\). We know that \( q_0s_0 \cdots s_{i-1} = s_0 \cdots s_{i-1}q_i \). If \( q_i \) is divisible by some prime number \( p \) for infinitely many \( i \), then by an argument similar to that in (i) one derives the inequality \( \frac{p^{\infty}}{2^{\infty}} \leq \frac{p^{\infty}}{2^{\infty}} \), from which the statement follows. So we can assume that every \( p \) divides at most finitely many \( q_i \). Then it is easy to see that the Steinitz numbers \( s_0s_1s_2 \cdots s_{k_0} \) have equal values at every prime \( p \), so they coincide. Hence,

\[
\frac{R_2}{R_1} = \frac{m_0s_0 \cdots s_{k_0}}{q_0n_0}.
\]

From \([25]\) \( c_0c_1 \cdots c_0 \cdots c_{k_i+1} \cdots \) and therefore \( \frac{R_2}{R_1} \geq \frac{m_0s_0 \cdots s_{k_0}}{q_0n_0} \). Combining the latter inequality with \([26]\) and \([27]\) we obtain \( \frac{\delta}{2^{\infty}} \geq \frac{R_2B_1}{R_1B_2} \), which contradicts an assumption in the statement of the lemma. □

We are now able to prove the main result of the paper.

**Theorem 3.11.**  
1) The three finitary Lie algebras \( sl(\infty) \), \( so(\infty) \), \( sp(\infty) \) admit an injective homomorphism into any infinite-dimensional diagonal Lie algebra. An infinite-dimensional non-finitary diagonal Lie algebra admits no injective homomorphism into \( sl(\infty) \), \( so(\infty) \), \( sp(\infty) \).
b) Let $\mathfrak{s}_1 = X_1(T_1)$, $\mathfrak{s}_2 = X_2(T_2)$ be infinite-dimensional non-finitary diagonal Lie algebras. Set $S_i = \text{Stz}(S_i)$, $S = \text{GCD}(S_1, S_2)$, $R_i = \sum(S_i, S)$, $\delta_i = \delta(T_i)$, $C_i = \text{Stz}(C_i)$, $C = \text{GCD}(C_1, C_2)$, $B_i = \sum(C_i, C)$, and $\sigma_i = \sigma(T_i)$ for $i = 1, 2$. Then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ if and only if the following conditions hold.

1) $R_1$ is finite.
2) $\mathfrak{s}_2$ is sparse if $\mathfrak{s}_1$ is sparse.
3) If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse, both $R_1$ and $R_2$ are finite, and $S$ is not divisible by an infinite power of any prime number, then $\epsilon_2 S_2 \leq \frac{R_2}{2} \leq \frac{R_1}{S_1}$ for $\epsilon$ as specified below. The inequality is strict if $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense. We have $\epsilon = 2$, except in the cases listed below, in which $\epsilon = 1$:

3.1) $(X_1, X_2) = (C, C)$, $(O, O)$, $(C, A)$, $(O, A)$, and $(X_1, X_2) = (A, A)$ with both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ being one-sided;

3.2) $(X_1, X_2) = (A, A)$, $B_1$ is finite, either $\mathfrak{s}_1$ is one-sided and $\mathfrak{s}_2$ is two-sided non-symmetric or $\mathfrak{s}_2$ is two-sided non-symmetric and $\mathfrak{s}_1$ is two-sided non-symmetric;

3.3) $(X_1, X_2) = (A, A)$, $B_1$ is finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, either $B_2$ is infinite or $C$ is divisible by an infinite power of any prime number;

3.4) $(X_1, X_2) = (A, A)$, both $B_1$ and $B_2$ are finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, $C$ is not divisible by an infinite power of any prime number, and $\frac{R_1}{B_1} \leq \frac{R_2}{B_2}$.

Proof. a) The statement follows directly from Corollary 3.4 and Proposition 3.5.

b) The sufficiency of the conditions follows directly from Lemma 3.2 and Lemma 3.6.

The necessity of conditions 1 and 2 follows from Proposition 3.5 and Corollary 3.7 respectively. Let us prove the necessity of condition 3. Note that the assumptions of this condition satisfy Corollary 3.9. Hence in this case an injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$, if it exists, has to be diagonal. Therefore we can apply Lemma 3.10 and this lemma implies the necessity of condition 3 (it is easy to check that under corresponding assumptions the cases which are not listed in 3.1–3.4 are exactly the cases listed in Lemma 3.10 (ii)).

The following corollary gives a description of equivalence classes of diagonal Lie algebras with respect to the equivalence relation introduced earlier in this paper.

Corollary 3.12.  

a) The three finitary Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{so}(\infty)$, and $\mathfrak{sp}(\infty)$ are pairwise equivalent. None of them is equivalent to any non-finitary diagonal Lie algebra.

b) Let $\mathfrak{s}_1 = X_1(T_1)$ and $\mathfrak{s}_2 = X_2(T_2)$ be infinite-dimensional non-finitary diagonal Lie algebras. Set $S_i = \text{Stz}(S_i)$, $S = \text{GCD}(S_1, S_2)$, $R_i = \sum(S_i, S)$, $\delta_i = \delta(T_i)$, $C_i = \text{Stz}(C_i)$, $C = \text{GCD}(C_1, C_2)$, $B_i = \sum(C_i, C)$, and $\sigma_i = \sigma(T_i)$ for $i = 1, 2$. Then $\mathfrak{s}_1$ is equivalent to $\mathfrak{s}_2$ if and only if the following conditions hold.

1) $S_1 \cong S_2$.
2) Both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are either sparse or non-sparse.
3) If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse and $S$ is not divisible by an infinite power of any prime number, then:

3.1) $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$;

3.2) $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have the same density type;

3.3) $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are of the same type ($X_1 = X_2$);

3.4) $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have the same symmetry type;

3.5) $C_1 \cong C_2$ if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided non-symmetric;

3.6) $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$ if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric and $C$ is not divisible by an infinite power of any prime number.

Proof. a) The statement follows directly from Theorem 3.11 a).

b) To prove sufficiency it is easy to check case by case that all the conditions of Theorem 3.11 b) are satisfied for both pairs $\mathfrak{s}_1 \subset \mathfrak{s}_2$ and $\mathfrak{s}_2 \subset \mathfrak{s}_1$.

Let us prove necessity. Assume that there exist injective homomorphisms $\mathfrak{s}_1 \rightarrow \mathfrak{s}_2$ and $\mathfrak{s}_2 \rightarrow \mathfrak{s}_1$. Conditions 1 and 2 are obviously satisfied. Suppose that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both non-sparse and $S$ is not divisible by an infinite power of any prime number. Then $\epsilon_1 \frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$ and $\epsilon_2 \frac{R_2}{\delta_2} \leq \frac{R_1}{\delta_1}$ by Theorem 3.11 b). Clearly, this is only possible if $\epsilon_1 = \epsilon_2 = 1$ and $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$. Then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have the same density type (otherwise one of the inequalities would be strict). Conditions 3.3–3.6 follow from conditions 3.1–3.4 of Theorem 3.11 b) for both pairs ($\mathfrak{s}_1, \mathfrak{s}_2$) and ($\mathfrak{s}_2, \mathfrak{s}_1$).
Remark. Isomorphic Lie algebras are clearly equivalent. If two Lie algebras satisfy Theorem 2.4 (or Theorem 2.5), then they satisfy also Corollary 3.12. One can check that conditions $A_3$ and $B_3$ of Theorem 2.4 correspond respectively to conditions 3.1 and 3.6 of Corollary 3.12.

Let $\mathbb{D}$ denote the set of equivalence classes of infinite-dimensional diagonal Lie algebras. If we write $s_1 \rightarrow s_2$ in case there exists an injective homomorphism from $s_1$ into $s_2$, then the relation $\rightarrow$ is a partial order on $\mathbb{D}$. It follows from Theorem 3.11 a) that $\mathbb{D}$ has the only minimal element (which also is the least element) with respect to the order $\rightarrow$: the equivalence class consisting of the three finitary Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{so}(\infty)$, $\mathfrak{sp}(\infty)$. The following statement shows that there exist precisely one maximal element of $\mathbb{D}$ (which also is the greatest element).

**Corollary 3.13.** Let $s = X(T)$ be a diagonal Lie algebra. Set $S = \text{Stz}(S)$. The following are equivalent.

1) Any diagonal Lie algebra admits an injective homomorphism into $s$.

2) $s$ is sparse and $S = p_1^\infty p_2^\infty \cdots$, where $p_1, p_2, \ldots$ is the increasing sequence of all prime numbers.

**Proof.** 1)$\Rightarrow$2): Consider a Lie algebra $s' = A(T')$, where $T'$ is sparse and $\text{Stz}(S') = p_1^\infty p_2^\infty \cdots$. Since $s'$ admits an injective homomorphism into $s$, the Steinitz number $\div(p_1^\infty p_2^\infty \cdots, S)$ is finite and $s$ is sparse by Theorem 3.11 b). Then clearly $S = p_1^\infty p_2^\infty \cdots$.

2)$\Rightarrow$1): It follows immediately from Theorem 3.11.

The equivalence class corresponding to the maximal element of $\mathbb{D}$ consists of infinitely many pairwise non-isomorphic Lie algebras. Indeed, by Theorem 2.4 there is only one, up to isomorphism, sparse one-sided Lie algebra of type $A$ satisfying property 2 of Corollary 3.13 but there are infinitely many sparse two-sided Lie algebras of type $A$ with this property. By Theorem 2.5 any Lie algebra of type other than $A$ satisfying property 2 of Corollary 3.13 is isomorphic to the sparse two-sided symmetric Lie algebra of type $A$ with $\text{Stz}(S) = p_1^\infty p_2^\infty \cdots$.

4 Acknowledgements

I want to express my gratitude to my thesis adviser Ivan Penkov for sharing his many ideas with me. This project would never have been without his guidance and support. I also thank Elizabeth Dan-Cohen and Ivan Penkov for their invaluable assistance in the writing of this text.

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