Exponential Convergence for Semilinear SDEs
Driven by Lévy Processes on Hilbert Spaces

Yulin Song
School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, P.R.China

Tiange Xu
School of Mathematics, University of CAS, Beijing, 100049, P.R.China

Abstract
In this paper, an integration by parts formula was derived for jump processes on Hilbert spaces. Using this formula, we investigated derivative formula and exponential convergence for semilinear SDEs driven by purely jump processes on Hilbert spaces.

Keywords: jump processes; integration by parts formula; derivative formula; exponential convergence.

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1 Introduction
In recent years, integration by parts formula, derivative formula and exponential convergence have been extensively studied for stochastic differential equations with jumps. The topics on the two above formulas for jump processes on \( \mathbb{R}^d \), we refer to [1], [2], [5], [9], [10], [17], [18], [19], [22] and so on. In the finite-dimensional case, the authors investigated the coupling property for linear SDEs in [3], [13], [15], [14] and references within. For nonlinear SDEs driven by jump processes, coupling property was derived in [16]. In most of references mentioned above the shift-invariance of the Lebesgue measure plays an essential role. But in infinite-dimensional setting, there is no Lebesgue measure available. The authors in [20] investigated the strong Feller and coupling properties for transition semigroups of linear SDEs driven by Lévy processes on a Banach space equipped with a nice reference measure, which has quasi-invariance property. Exponential convergence for SDEs driven by Lévy processes was studied in [8], [10], [21] and references therein for finite-dimensional case. Ergodicity and exponential mixing properties of SPDEs driven by cylindrical stable processes were derived in [10], [11] and [12]. In this paper we aim to investigate integration by parts formula, derivative formula and

\footnote{Corresponding author. Email: songy10@amss.ac.cn.}

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exponential convergence for semilinear SDEs driven by non-cylindrical purely jump processes on a separable Hilbert space.

Let \((\mathbb{H}, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space and \(\mu\) be a Gaussian measure on \(\mathbb{H}\) with covariance operator \(Q\), which is nonnegative, symmetric and has finite trace. Its square root, denoted by \(Q^{\frac{1}{2}}\), is a nonnegative and symmetric Hilbert-Schmidt operator. Let \(\text{Im}Q^{\frac{1}{2}}\) be the image space of \(Q^{\frac{1}{2}}\), i.e., \(\text{Im}Q^{\frac{1}{2}} = \{ Q^{\frac{1}{2}}x | x \in \mathbb{H} \}\). As is known, \(\text{Im}Q^{\frac{1}{2}}\) is a Hilbert space with the induced inner product
\[
\langle x, y \rangle_0 := \langle Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y \rangle, \quad x, y \in \text{Im}Q^{\frac{1}{2}},
\]
where \(Q^{-\frac{1}{2}}\) is the pseudo inverse of \(Q\) in the case that it is not one-to-one, that is, for \(h \in \text{Im}Q^{\frac{1}{2}}\),
\[
Q^{-\frac{1}{2}}h = x, \text{ if } Q^{\frac{1}{2}}x = h \text{ and } \|x\| = \inf\{\|y\| : Q^{\frac{1}{2}}y = h\}.
\]
Equivalently, we also have
\[
\langle x, y \rangle_0 = \sum_{k=1}^{\infty} \frac{\langle x, e_k \rangle \langle y, e_k \rangle}{\lambda_k} I_{[\lambda_k > 0]},
\]
where \(\{e_k\}_{k \in \mathbb{N}}\), the eigenvectors of \(Q\) with eigenvalues \(\{\lambda_k\}_{k \in \mathbb{N}}\), consists of an orthonormal basis of \(\mathbb{H}\). The space \((\text{Im}Q^{\frac{1}{2}}, \langle \cdot, \cdot \rangle_0)\) is called the Reproducing Kernel Hilbert Space of \(\mathbb{H}\). As is known, the Gaussian measure \(\mu\) has quasi-invariant property under the shift \(z \mapsto z + h\) for any \(h \in \text{Im}Q^{\frac{1}{2}}\), i.e., \(\mu(\cdot + h)\) and \(\mu\) are mutually absolutely continuous. The Randon-Nikodym derivative of \(\mu(\cdot + h)\) with respect to \(\mu\) is
\[
\varphi(z, h) := \frac{\mu(dz + h)}{\mu(dz)} = \exp\{\langle h, z \rangle_0 - \frac{1}{2} \langle h, h \rangle_0\}, \quad \mu - a.s.
\]
For \(i = 1, 2\), let \(W_i\) be the space of all càdlàg functions from \([0, \infty)\) to \(\mathbb{H}\) vanishing at 0, which is endowed with the Skorohod topology and the probability measure \(\mathbb{P}_i\) such that the coordinate process \(L^i_t(w_i) = w_i(t)\) is a Lévy process. Furthermore, we assume that \(L^1 := \{L^1_t\}_{t \geq 0}\) is a purely jump process with Lévy measure \(\rho(z)\mu(dz)\), where \(\rho : \mathbb{H} \to (0, \infty)\) is a differentiable function satisfying \(\int_{\mathbb{H}} (|z|^2 \wedge 1) \rho(z)\mu(dz) < \infty\).

Consider the following product probability space
\[
(\Omega, \mathcal{F}, \mathbb{P}) := (W_1 \times W_2, \mathcal{B}(W_1) \times \mathcal{B}(W_2), \mathbb{P} \times \mathbb{P}^2)
\]
and define
\[
L_t = L^1_t + L^2_t.
\]
That is, for \(w = (w_1, w_2) \in \Omega\),
\[
L_t(w) = w_1(t) + w_2(t).
\]
Then \( \{L_t\}_{t \geq 0} \) is a Lévy process on \((\Omega, \mathcal{F}, \mathbb{P})\) with two independent parts and its Lévy measure denoted by \( \nu \) satisfies \( \nu(dz) \geq z \mu(dz) \). Denote by \( \{\mathcal{F}_t\}_{t \geq 0} \) the smallest filtration generated by \( \{L_t\}_{t \geq 0} \). We use \( N^1 \) and \( N^2 \) to be the jump measure and martingale measure of \( \{L_t\}_{t \geq 0} \). Let \( \mathbb{E} \) and \( \mathbb{E}^1 \) be the associated expectations of \( P \) and \( P^1 \) respectively.

In this paper, we consider the following stochastic equation on \( \mathbb{H} \)

\[
\begin{align*}
    dX_t &= AX_t dt + F(X_t) dt + dL_t, \\
    X_0 &= x,
\end{align*}
\]

where \( A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H} \) is an adjoint, unbounded and linear operator generating a \( C_0 \)-semigroup \( \{S_t\}_{t \geq 0} \) on \( \mathbb{H} \) and \( F : \mathbb{H} \to \mathbb{H} \) is measurable and bounded. The mild solution of Eq.(1.1), if it exists, can be formulated as

\[
X_t = S(t)x + \int_0^t S(t-s)F(X_s)ds + \int_0^t S(t-s)dL_s, \quad t \geq 0.
\]

We gather here all hypotheses which will be made on Eq.(1.1).

(H1) There exists a differentiable function \( \rho : \mathbb{H} \to (0, \infty) \) with bounded derivative such that

\[
\lambda := \int_{\mathbb{H}} \rho(z) \mu(dz) < \infty \quad \text{and} \quad \int_{\mathbb{H}} |z|^2 \rho(z) \mu(dz) < \infty.
\]

(H2) \( \text{Im} S(t) \subset \text{Im} Q \) holds for any \( t > 0 \).

(H3) \( A \) is a dissipative operator defined by

\[
A = \sum_{k \geq 1} (-\gamma_k) e_k \otimes e_k, \quad (1.2)
\]

for \( 0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k \leq \cdots \) and \( \gamma_k \to \infty \) as \( k \to \infty \).

(H4) \( F : \mathbb{H} \to \mathbb{H} \) is a bounded and Lipschitz continuous function with the smallest Lipschitz constant \( \|F\|_{\text{Lip}} \).

Let \( \mathcal{B}_b(\mathbb{H}) \) be the class of all bounded measurable function on \( \mathbb{H} \). We use \( C_b^2(\mathbb{H}) \) to denote the family of \( C^2 \) real-valued functions \( f \) such that \( f \) and its derivatives of order up to 2 are bounded. \( \| \cdot \|, \| \cdot \|_\infty \) and \( \| \cdot \|_{\text{Var}} \) stand for the operator norm, uniform norm and total variation norm respectively. Denote the associated jump measure of \( L^1 \) by \( N^1 \) and the counting process by \( N^1_t \), that is, \( N^1_t := N^1([0,t] \times \mathbb{H}) \). We aim to derive the Bismut type formula for semigroups \( \{P_t^1\}_{t \geq 0} \) and the exponential convergence for \( \{P_t\}_{t \geq 0} \) defined as

\[
P_t^1 f(x) := \mathbb{E}\{f(X^x_t)I_{\{N^1_t \leq 1\}}\}, \quad P_t f(x) := \mathbb{E}f(X^x_t), \quad x \in \mathbb{H}, \ t \geq 0, \ f \in \mathcal{B}_b(\mathbb{H}). \quad (1.3)
\]

In this paper, it is difficult to investigate a derivative formula for \( P_t \). Fortunately, associated formula for \( P_t^1 \) is derived, which is enough for our further work.

Let \( J_t \xi \) be the derivative of \( X^x_t \) w.r.t. the initial value \( x \). Denote

\[
C_b^1(\mathbb{H} \to \mathbb{H}) = \{ G : \mathbb{H} \to \mathbb{H} | G \text{ is bounded, differential, with bounded and continuous derivatives up to order 1} \}.
\]

We have the following main results.
Theorem 1.1. Assume \((H1) - (H2)\) hold. Let \(F \in C_b^1(\mathbb{H} \to \mathbb{H})\) and \(\nabla F\) be Lipschitz continuous. If \(\int_0^t \|Q^{-1}S(s)\|^2 ds < \infty\), then for \(f \in C_b^2(\mathbb{H})\) and \(\xi \in \mathbb{H}\)

\[
\nabla_x P_t^1 f(x) = -\mathbb{E} \left\{ f(X_t^x) \frac{I_{[N_t^1 \geq 1]}}{N_t^1} \int_0^t \int_{\mathbb{H}} \left( \langle z, Q^{-1}J_s \xi \rangle + \langle \nabla \log \rho(z), J_s \xi \rangle \right) \tilde{N}^1(dz, ds) \right\}.
\]

As a direct application of the formula, exponential convergence of the transition semigroup is derived.

Theorem 1.2. Assume \((H1) - (H4)\) hold. If \(\gamma_1 > \|F\|_{Lip}\) and \(\lim_{t \to \infty} \int_0^t \|Q^{-1}S(s)\|^2 ds < \infty\), then there exists a constant \(C > 0\) such that

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{Var} \leq C(1 + |x - y|) \exp \left\{ -\frac{\lambda(\gamma_1 - \|F\|_{Lip})t}{\lambda + \gamma_1 - \|F\|_{Lip}} \right\}.
\]

Remark 1.1. Under the conditions of Theorem 1.1, if the solution admits invariant probability measures and each invariant measure is integrable, then exponential ergodicity follows from (1.4).

An example is given to show the conditions of Theorem 1.1 and Theorem 1.2 on \(Q\) and \(S(t)\) can be satisfied.

Example: Let \((H3)\) hold. For \(0 < \delta < \frac{1}{2}\), the fractional power \((-A)^\delta\) of \(-A\) is defined by

\[
(-A)^\delta = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{-\delta} S(t) dt,
\]

where \(\Gamma\) is the Euler function. It can be proved that \(S(t)\mathbb{H} \subset D((-A)^\delta)\) and for any \(t > 0\),

\[
\|(-A)^\delta S(t)\| \leq C_\delta t^{-\delta}
\]

for a suitable positive constant \(C_\delta\). Take \(Q = ((-A)^\delta)^{-1}\), then we have \(S(t)\mathbb{H} \subset \text{Im}Q\). Moreover,

\[
\lim_{t \to \infty} \int_0^t \frac{\|Q^{-1}S(s)\|^2 ds}{t} \leq \lim_{t \to \infty} \frac{C_\delta^2}{t} \int_0^t s^{-2\delta} ds = \lim_{t \to \infty} \frac{C_\delta^2 t^{1-2\delta}}{t} = 0.
\]

The rest of this paper is organized as follows: in section 2, we shall investigate an integration by parts formula for jump processes valued on \(\mathbb{H}\); the proofs of the main results will be presented in section 3.

2 Integration by Parts Formula

An integration by parts formula can enable one to derive the derivative formula, and it is a powerful tool in stochastic analysis. The associated formula for jump processes on \(\mathbb{R}^d\) can be found in [2], [5], [9],[17] and so on. But so far, there are few references studying the formula for jump processes in infinite-dimensional case.
For fixed $T > 0$, denote
\[
L^1(\Omega \times [0, T]) = \left\{ V : \Omega \times [0, T] \to \mathbb{H} \left| \int_0^T \mathbb{E}|V(s)|ds < \infty \right. \right\}.
\]

Define a perturbed random measure $N_{1, \epsilon}$ by
\[
N_{1, \epsilon}(\Gamma \times [0, t]) = \int_0^t \int_{\mathbb{H}} I_{\Gamma}(z + \epsilon V(s))N^1(dz, ds), \quad \Gamma \in \mathcal{B}(\mathbb{H}),
\]
for $V \in L^1(\Omega \times [0, T])$. Let $\{L_{t, \epsilon}\}_{0 \leq t \leq T}$ be the associated Lévy process perturbed by $\epsilon V$, that is,
\[
L_{t, \epsilon} = L_t + \epsilon \int_0^t \int_{\mathbb{H}} V(s)N^1(dz, ds).
\]

Recall the following notion of $L^1$-derivative, which was first introduced in [2] and can also be found in [5] and [16].

**Definition 2.1.** A functional $G_t = G(\{L_s\}_{s \leq t})$ is called to have an $L^1$-derivative in the direction $V$, if there exists an integrable random variable denoted by $D_{V}G_t$, such that
\[
\lim_{\epsilon \to 0} \mathbb{E} \left| \frac{G(\{L_{s, \epsilon}\}_{s \leq t}) - G(\{L_s\}_{s \leq t})}{\epsilon} - D_{V}G_t \right| = 0.
\]

In order to obtain the integration by parts formula, we would like to construct a weighted probability measure such that the distribution of the perturbed process under this weighted probability equals the one of original processes under the reference measure $P^1$. For the sake of convenience, denote
\[
\mathcal{Y}_1 = \left\{ V : \Omega \times [0, T] \to \text{ Im}Q \big| V \text{ is predictable and } \sup_{s \leq T} |Q^{-1}V(s)| < \infty \right\},
\]
and
\[
\mathcal{Y}_2 = \left\{ V : \Omega \times [0, T] \to \text{ Im}Q \big| V \text{ is predictable and } \int_0^T \mathbb{E}|Q^{-1}V(s)|ds < \infty \right\}.
\]

For $\epsilon > 0$, $V \in \mathcal{Y}_1$ and $w_2 \in \Omega_2$, we set
\[
\lambda_{\epsilon,w_2}(s, z) = \varphi(z, \epsilon V_{w_2}(s)) \frac{\rho(z + \epsilon V_{w_2}(s))}{\rho(z)}.
\]

Take
\[
Z_{t, \epsilon,w_2} = \exp \left\{ \int_0^t \int_{\mathbb{H}} \log \lambda_{\epsilon,w_2}(s, z)N^1(dz, ds) - \int_0^t \int_{\mathbb{H}} (\lambda_{\epsilon,w_2}(s, z) - 1)\rho(z)\mu(dz)ds \right\},
\]
then $\{Z_{t, \epsilon,w_2}\}_{t \leq T}$ satisfies the following equation
\[
\begin{cases}
\frac{dZ_{t, \epsilon,w_2}}{Z_{t-}, \epsilon,w_2} = Z_{t-}(\lambda_{\epsilon,w_2}(t, z) - 1)\tilde{N}^1(dz, dt), \\
Z_{0, \epsilon,w_2} = 1,
\end{cases}
\]
and can be formulated as
\[
Z_t^{\epsilon,w_2} = 1 + \int_0^t \int_{\mathbb{R}} Z_{s-}^{\epsilon,w_2}(\lambda^{\epsilon,w_2}(s, z) - 1)N_1(dz, ds).
\]
(2.2)

So \(\{Z_t^{\epsilon,w_2}\}_{0 \leq t \leq T}\) is a martingale and \(Z_0^{\epsilon,w_2} = 1\). Define a probability measure \(\mathbb{P}^{1,\epsilon,w_2}\) as
\[
d\frac{d\mathbb{P}^{1,\epsilon,w_2}}{d\mathbb{P}^1}|_{\mathcal{F}_t} = Z_t^{\epsilon,w_2}, \quad t \leq T.
\]

Lemma 2.1. For any \(V \in \mathcal{Y}_1\), the law of \(L_t^{1,\epsilon}\) under \(\mathbb{P}^{1,\epsilon,w_2}\) is equal to the one of \(L_t^{1}\) under \(\mathbb{P}^1\).

Proof. For any test function \(\phi \leq 0\) and fixed \(w_2 \in \Omega_2\), denote
\[
Y_t^{\epsilon,w_2} = \exp \left\{ \int_0^t \int_{\mathbb{R}} \phi(s, z)N^{1,\epsilon,w_2}(dz, ds) \right\}, \quad G_t^{\epsilon,w_2} = Y_t^{\epsilon,w_2}Z_t^{\epsilon,w_2}.
\]

We just check that \(\mathbb{E}^{1,\epsilon,w_2}G_t^{\epsilon,w_2}\) does not depend on \(\epsilon\), where \(\mathbb{E}^{1,\epsilon,w_2}\) denotes the associated expectation of conditional probability \(\mathbb{P}^{1,\epsilon,w_2}\). Actually, note that
\[
Y_t^{\epsilon,w_2} = \exp \left\{ \int_0^t \int_{\mathbb{R}} \phi(s, z)N^{1,\epsilon,w_2}(dz, ds) \right\}
\]
\[= \exp \left\{ \int_0^t \int_{\mathbb{R}} \phi(s, z + \epsilon V^{w_2}(s))N^1(dz, ds) \right\}
\]
\[= 1 + \int_0^t \int_{\mathbb{R}} Y_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)N^1(dz, ds),
\]
and
\[
[Y_t^{\epsilon,w_2}, Z_t^{\epsilon,w_2}]_t = \int_0^t \int_{\mathbb{R}} Y_{s-}^{\epsilon,w_2}Z_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)N^1(dz, ds).
\]
Applying Itô formula, it yields
\[
G_t^{\epsilon,w_2} = 1 + \int_0^t \int_{\mathbb{R}} Y_{s-}^{\epsilon,w_2}dZ_{s-}^{\epsilon,w_2} + \int_0^t \int_{\mathbb{R}} Z_{s-}^{\epsilon,w_2}dY_{s-}^{\epsilon,w_2} + [Y_t^{\epsilon,w_2}, Z_t^{\epsilon,w_2}]_t
\]
\[= 1 + \int_0^t \int_{\mathbb{R}} Y_{s-}^{\epsilon,w_2}dZ_{s-}^{\epsilon,w_2} + \int_0^t \int_{\mathbb{R}} Z_{s-}^{\epsilon,w_2}Y_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)N^1(dz, ds)
\]
\[+ \int_0^t \int_{\mathbb{R}} Y_{s-}^{\epsilon,w_2}Z_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)N^1(dz, ds)
\]
\[= 1 + \int_0^t \int_{\mathbb{R}} Y_{s-}^{\epsilon,w_2}dZ_{s-}^{\epsilon,w_2} + \int_0^t \int_{\mathbb{R}} G_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)N^1(dz, ds).
\]
Furthermore,
\[
\mathbb{E}^1G_t^{\epsilon,w_2} = 1 + \mathbb{E}^1 \int_0^t \int_{\mathbb{R}} G_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)\phi(z, \epsilon V^{w_2}(s))\frac{\rho(z + \epsilon V^{w_2}(s))}{\rho(z)}\rho(z)\mu(dz)ds
\]
\[= 1 + \mathbb{E}^1 \int_0^t \int_{\mathbb{R}} G_{s-}^{\epsilon,w_2}(e^{\phi(s, z) + \epsilon V^{w_2}(s)}) - 1)\phi(h, \epsilon V^{w_2}(s))\rho(z + \epsilon V^{w_2}(s))\mu(dz)ds
\]
\[ = 1 + \mathbb{E}^1 \int_0^t \int_H G_s^{\epsilon, w_2} (e^{\phi(s, z + \epsilon V(w_2(s))} - 1) \rho(z + \epsilon V(w_2(s))) \mu(dz + \epsilon V(w_2(s))) ds \]

\[ = 1 + \int_0^t \mathbb{E}^1 G_s^{\epsilon, w_2} \int_H (e^{\phi(s, z)} - 1) \rho(z) \mu(dz) ds. \]

Therefore,

\[ \mathbb{E}^1 G_t^{\epsilon, w_2} = \exp \left\{ \int_0^t \int_H (e^{\phi(s, z)} - 1) \rho(z) \mu(dz) ds \right\}. \]

\[ \square \]

**Lemma 2.2.** Assume (H1) holds. If there exists a constant \( \delta > 0 \) such that \( \rho(z) \geq \delta \) for \( \forall z \in H \), then for any fixed \( t \in [0, T] \) and \( V \in \mathcal{V} \),

\[ \sup_{\epsilon \leq 1} \mathbb{E}^1 \left\{ \left| \frac{Z_t^{\epsilon, w_2} - 1}{\epsilon} \right|^2 \right\} < \infty. \]

for any fixed \( w_2 \in \Omega_2 \).

**Proof.** For fixed \( \epsilon \in (0, 1) \), it follows (2.2) that

\[ Z_t^{\epsilon, w_2} - 1 = \int_0^t \int_H Z_s^{\epsilon, w_2}(\lambda^{\epsilon, w_2}(s, z) - 1) \tilde{N}(dz, ds). \]

Triangle inequality and B-D-G inequality yield

\[
\begin{aligned}
\mathbb{E}^1 \left\{ \sup_{s \leq t} \left| \frac{Z_s^{\epsilon, w_2} - 1}{\epsilon} \right|^2 \right\} &= \mathbb{E}^1 \left\{ \sup_{s \leq t} \int_0^s \int_H \left| \frac{Z_{r-}^{\epsilon, w_2}(\lambda^{\epsilon, w_2}(r, z) - 1) - 1}{\epsilon} \tilde{N}(dz, dr) \right|^2 \right\} \\
&\leq 2 \mathbb{E}^1 \left\{ \sup_{s \leq t} \int_0^s \int_H \left| \frac{Z_{r-}^{\epsilon, w_2} - 1}{\epsilon} (\lambda^{\epsilon, w_2}(r, z) - 1) \tilde{N}(dz, dr) \right|^2 \right\} \\
&+ 2 \mathbb{E}^1 \left\{ \sup_{s \leq t} \int_0^s \int_H \left| \frac{\lambda^{\epsilon, w_2}(r, z) - 1}{\epsilon} \tilde{N}(dz, dr) \right|^2 \right\} \\
&\leq C \mathbb{E}^1 \left\{ \int_0^t \left( \sup_{r \leq s} \left| \frac{Z_{r-}^{\epsilon, w_2} - 1}{\epsilon} \right| \right) \left| \lambda^{\epsilon, w_2}(s, z) - 1 \right|^2 \rho(z) \mu(dz) ds \right\} \\
&+ C \mathbb{E}^1 \left\{ \int_0^t \int_H \left| \frac{\lambda^{\epsilon, w_2}(s, z) - 1}{\epsilon} \right|^2 \rho(z) \mu(dz) ds \right\},
\end{aligned}
\]

where \( C \) is a constant, which may change value from line to line and is independent of \( \epsilon \). Note that

\[
\begin{aligned}
\left| \frac{\lambda^{\epsilon}(s, z) - 1}{\epsilon} \right|^2 &= \left| \varphi(z, \epsilon V(s)) \frac{\rho(z + \epsilon V(s))}{\rho(z)} - 1 \right|^2 \\
&\leq 2 \varphi^2(z, \epsilon V(s)) \left| \frac{\rho(z + \epsilon V(s)) - \rho(z)}{\rho^2(z)} \epsilon \right|^2 \\
&+ 2 \left| \frac{\varphi(z, \epsilon V(s)) - 1}{\epsilon} \right|^2,
\end{aligned}
\]

(2.4)
\[
\varphi^2(z, \epsilon V^{w_2}(s)) = \exp \left\{ \langle z, 2\epsilon Q^{-1}V^{w_2}(s) \rangle - \epsilon^2 \langle V^{w_2}(s), Q^{-1}V^{w_2}(s) \rangle \right\}
\]

\[
= \varphi(z, 2\epsilon V^{w_2}(s)) \exp \left\{ \epsilon^2 \langle V^{w_2}(s), Q^{-1}V^{w_2}(s) \rangle \right\}.
\] (2.5)

Therefore, by mean value theorem and (2.5), there exists constant \( C_1 > 0 \) independent of \( \epsilon \) such that

\[
I_1 := \int_{\mathbb{H}} \frac{\varphi^2(z, \epsilon V^{w_2}(s)) - \varphi(z, \epsilon V^{w_2}(s))}{\epsilon} \frac{\rho(z)}{\mu(dz)} \leq \frac{1}{\delta} \| \nabla \rho \|_\infty^2 \langle V^{w_2}(s) \rangle \mu(dz)
\]

\[
= \frac{1}{\delta} \| \nabla \rho \|_\infty^2 \langle V^{w_2}(s) \rangle \mu(dz)
\]

\[
\leq C_1,
\] (2.6)

and

\[
I_2 := \int_{\mathbb{H}} \varphi^2(z, \epsilon V^{w_2}(s)) - 1 \frac{\rho(z)}{\mu(dz)} - 2 \langle z - \epsilon V^{w_2}(s), Q^{-1}V^{w_2}(s) \rangle \rho(z) \mu(dz)
\]

\[
= \int_{\mathbb{H}} \varphi^2(z, \epsilon V^{w_2}(s)) - 1 \frac{\rho(z)}{\mu(dz)} \mu(dz)
\]

\[
\leq 2\langle Q^{-1}V^{w_2}(s) \rangle \int_{\mathbb{H}} \varphi^2(z, \epsilon V^{w_2}(s)) \mu(dz)
\]

\[
+ 2 \langle Q^{-1}V^{w_2}(s) \rangle \int_{\mathbb{H}} \varphi^2(z, \epsilon V^{w_2}(s)) \rho(z) \mu(dz)
\]

\[
:= I_{21} + I_{22},
\] (2.7)

where \( \epsilon_2 \in (0, \epsilon) \) is a proper constant. Furthermore, there exist constants \( \epsilon_3 \in (0, \epsilon_2) \), \( C_2 \) and \( C_3 \) independent of \( \epsilon \) such that

\[
I_{21} \leq 2\langle Q^{-1}V^{w_2}(s) \rangle \int_{\mathbb{H}} |z|^2 \rho(z) \mu(dz) + 2\epsilon_2 V^{w_2}(s)
\]

\[
\leq 4\langle Q^{-1}V^{w_2}(s) \rangle ^2 \int_{\mathbb{H}} |z|^2 \rho(z) \mu(dz) + 2\epsilon_2 V^{w_2}(s)
\]

\[
\times \int_{\mathbb{H}} \left( |z + \epsilon_2 V^{w_2}(s)|^2 + |\epsilon_2 V^{w_2}(s)|^2 \right)
\]

\[
\times \left( |\rho(z + \epsilon_2 V^{w_2}(s)) - \rho(z)| + |\rho(z + \epsilon_2 V^{w_2}(s))| \right) \mu(dz) + \epsilon_2 V^{w_2}(s)
\]

\[
\leq 4\langle Q^{-1}V^{w_2}(s) \rangle ^2 \int_{\mathbb{H}} |z + \epsilon_2 V^{w_2}(s)|^2 \mu(dz)
\]

\[
\times \left( |\nabla \rho(z + \epsilon_3 V^{w_2}(s)) \rangle, \epsilon_2 V^{w_2}(s) \rangle \right) \mu(dz) + \epsilon_2 V^{w_2}(s)
\]

\[
+ \int_{\mathbb{H}} \left( |z + \epsilon_2 V^{w_2}(s)|^2 \rho(z + \epsilon_2 V^{w_2}(s)) \mu(dz) + \epsilon_2 V^{w_2}(s)
\]

\[
+ |V^{w_2}(s)|^2 \int_{\mathbb{H}} |\langle \nabla \rho(z + \epsilon_3 V^{w_2}(s)) \rangle, \epsilon_2 V^{w_2}(s) \rangle \mu(dz) + \epsilon_2 V^{w_2}(s)
\]
\[
\begin{align*}
+ |V^{w_2}(s)|^2 & \int_{\mathbb{H}} \rho(z + \epsilon_2 V^{w_2}(s)) \mu(\rho(z + \epsilon_2 V^{w_2}(s))) \\
\leq & 4|Q^{-1}V^{w_2}(s)|^2 \exp \left\{ (V^{w_2}(s), Q^{-1}V^{w_2}(s)) \right\} \\
\times & \left\{ |V^{w_2}(s)||\nabla \rho||_{\infty} \int_{\mathbb{H}} |z|^2 \mu(dz) + \int_{\mathbb{H}} |z|^2 \rho(z) \mu(dz) + |V^{w_2}(s)|^3 ||\nabla \rho||_{\infty} + \lambda |V^{w_2}(s)|^2 \right\} \\
\leq & C_2,
\end{align*}
\] (2.8)

and
\[
I_{22} = 2|Q^{-1}V^{w_2}(s)||V^{w_2}(s)|^2 \int_{\mathbb{H}} \varphi^2(z, \epsilon_2 V^{w_2}(s)) \rho(z) \mu(dz) \\
\leq & 2|Q^{-1}V^{w_2}(s)||V^{w_2}(s)|^2 \exp \left\{ \epsilon^2(V^{w_2}(s), Q^{-1}V^{w_2}(s)) \right\} \\
\times & \int_{\mathbb{H}} \left| \rho(z + \epsilon_2 V^{w_2}(s)) - \rho(z) + \rho(z + \epsilon_2 V^{w_2}(s)) \right| \mu(dz + 2\epsilon_2 V^{w_2}(s)) \\
\leq & 2|Q^{-1}V^{w_2}(s)||V^{w_2}(s)|^2 \exp \left\{ \epsilon^2(V^{w_2}(s), Q^{-1}V^{w_2}(s)) \right\} \left\{ ||\nabla \rho||_{\infty} |V^{w_2}(s)| + \lambda \right\} \\
\leq & C_3.
\] (2.9)

Combining (2.6), (2.7), (2.8), (2.9) with (2.11), we arrive at
\[
\int_{\mathbb{H}} \left| \frac{\lambda^{\epsilon,w_2}(s, z) - 1}{\epsilon} \right|^2 \rho(z) \mu(dz) \leq 2(C_1 + C_2 + C_3).
\] (2.10)

It follows from (2.3) and (2.11) that
\[
\mathbb{E}^1 \left\{ \sup_{s \leq t} \left| \frac{Z_{\epsilon,w_2}^s - 1}{\epsilon} \right|^2 \right\} \leq C \mathbb{E}^1 \left\{ \int_0^t \left( \sup_{r \leq s} \left| \frac{Z_{\epsilon,w_2}^r - 1}{\epsilon} \right|^2 \right) ds \right\} + C.
\]

Applying Gronwall’s inequality, we deduce
\[
\mathbb{E}^1 \left\{ \sup_{s \leq t} \left| \frac{Z_{\epsilon,w_2}^s - 1}{\epsilon} \right|^2 \right\} < C(t),
\]
where \(C(t)\) is a constant independent of \(\epsilon\). Consequently, the claim is proved. \(\Box\)

With the help of above two lemmas, we are ready to derive the following integration by parts formula.

**Theorem 2.2.** Suppose \((H_1)\) holds. For \(V \in \mathcal{Y}_2\) and \(f \in C_b^2(\mathbb{H})\),
\[
\mathbb{E} \left\{ D_V f(L_t) \right\} = -\mathbb{E} \left\{ f(L_t) M_t \right\}, \quad t \leq T;
\] (2.11)

where \(M_t = \int_0^t \int_{\mathbb{H}} \left( \langle z, Q^{-1}V(s) \rangle + \langle \nabla \log \rho(z), V(s) \rangle \right) \tilde{N}(dz, ds).\)
Proof. We give the proof in three steps.

Step 1. Assume $V \in \mathcal{V}_1$ and $\rho \geq \delta$ for some $\delta > 0$. By virtue of Lemma 2.1, for any fixed $w_2 \in \Omega_2$ and $\epsilon \in (0, 1)$, we have

$$\mathbb{E}^1 f(L^w_t) = \mathbb{E}^1 \left\{ f(L^{\epsilon, w_2}_t)Z^{\epsilon, w_2}_t \right\}.$$ 

Therefore,

$$\mathbb{E}^1 \frac{f(L^{\epsilon, w_2}_t)Z^{\epsilon, w_2}_t - f(L^w_t)}{\epsilon} = 0.$$ 

Furthermore,

$$\mathbb{E}^1 \frac{f(L^{\epsilon, w_2}_t) - f(L^w_t)}{\epsilon} + \mathbb{E}^1 \frac{f(L^{\epsilon, w_2}_t)(Z^{\epsilon, w_2}_t - 1 - \epsilon M^{w_2}_t)}{\epsilon} + \mathbb{E}^1 f(L^{\epsilon, w_2}_t)M^{w_2}_t = 0, \quad (2.12)$$

where

$$M^{w_2}_t = \int_0^t \int_{\mathbb{H}} \left( \langle z, Q^{-1}V^{w_2}(s) \rangle + \langle \nabla \log \rho(z), V^{w_2}(s) \rangle \right) \tilde{N}^1(dz, ds).$$

Taking expectation w.r.t. $\mathbb{P}^2$ in both sides of (2.12), we have

$$\mathbb{E}^1 \frac{f(L^{\epsilon, w_2}_t) - f(L^w_t)}{\epsilon} + \mathbb{E}^1 \frac{f(L^{\epsilon, w_2}_t)(Z^{\epsilon, w_2}_t - 1 - \epsilon M^{w_2}_t)}{\epsilon} \mathbb{P}^2(dw_2) + \mathbb{E} f(L^\epsilon_t)M^w_t = 0. \quad (2.13)$$

The Definition 2.1 implies

$$\lim_{\epsilon \to 0} \mathbb{E}^1 \frac{f(L^\epsilon_t) - f(L^w_t)}{\epsilon} = \mathbb{E} D_V f(L^w_t). \quad (2.14)$$

Moreover,

$$\lim_{\epsilon \to 0} \mathbb{E} \{ f(L^\epsilon_t)M^w_t \} = \mathbb{E} \{ f(L^w_t)M^w_t \}. \quad (2.15)$$

Therefore, it is sufficient to prove

$$\lim_{\epsilon \to 0} \mathbb{E}^1 \frac{f(L^{\epsilon, w_2}_t)(Z^{\epsilon, w_2}_t - 1 - \epsilon M^{w_2}_t)}{\epsilon} = 0.$$ 

By (2.12) and the fact that $\lambda < \infty$, one has

$$\lim_{\epsilon \to 0} \frac{Z^{\epsilon, w_2}_t - 1}{\epsilon} = \lim_{\epsilon \to 0} \int_0^t \int_{\mathbb{H}} \frac{Z^{\epsilon, w_2}_s(\lambda^{\epsilon, w_2}(s, z) - 1)}{\epsilon} \tilde{N}^1(dz, ds)$$

$$= \int_0^t \int_{\mathbb{H}} \lim_{\epsilon \to 0} \frac{Z^{\epsilon, w_2}_s(\lambda^{\epsilon, w_2}(s, z) - 1)}{\epsilon} \tilde{N}^1(dz, ds)$$

$$= \int_0^t \int_{\mathbb{H}} \frac{d}{d\epsilon} \Big|_{\epsilon = 0} \left( \varphi(z, \epsilon V^{w_2}(s)) \frac{\rho(z + \epsilon V^{w_2}(s))}{\rho(z)} \right) \tilde{N}^1(dz, ds)$$

$$= \int_0^t \int_{\mathbb{H}} \left( \langle z, Q^{-1}V^{w_2}(s) \rangle + \langle \nabla \log \rho(z), V^{w_2}(s) \rangle \right) \tilde{N}^1(dz, ds).$$

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Combining this with Lemma 2.2, we derive
\[
\lim_{\epsilon \to 0} \mathbb{E}^{1} \frac{|Z_{t}^\epsilon - 1 - \epsilon M_t^\epsilon|}{\epsilon} = 0. \tag{2.16}
\]
By the dominated convergence theorem and (2.13)-(2.16), we obtain (2.11).

Step 2. Assume \( V \in \mathcal{V}_1 \). For each \( n \in \mathbb{N} \), let \( P^{1,n} \) be a probability measure on path space \( W_{1,n} \) such that the coordinate process \( L_{t}^{1,n} = w_{1,n}(t) \) is a purely jump Lévy process with characteristic measure \( \frac{1}{n} \mu(dz) \). The associated jump measure is denoted by \( N^{1,n} \). Define
\[
\hat{\Omega} = \prod_{n=1}^{\infty} W_{1,n} \times \Omega, \quad \hat{P} = \prod_{n=1}^{\infty} P^{1,n} \times \mathbb{P},
\]
and
\[
L^n_t = L^{1,n}_t + L_t.
\]
Then the jump measure and characteristic measure of \( L^1_t + L^{1,n}_t \) are \( N_n(dz, ds) := N^1(dz, ds) + N^{1,n}(dz, ds) \) and \( (\rho(z) + \frac{1}{n}) \mu(dz) \) respectively. By Step 1, for \( V \in \mathcal{V}_1 \)
\[
\hat{\mathbb{E}} D_V f(L^n_t) = \mathbb{E} \{ f(L^n_t)M^n_t \}, \tag{2.17}
\]
where
\[
M^n_t = \int_0^t \int_{\mathbb{H}} \left( \langle z, Q^{-1}V(s) \rangle + \langle \nabla \rho(z) + \frac{1}{n}, V(s) \rangle \right) \tilde{N}_n(dz, ds).
\]
Note that,
\[
|\hat{\mathbb{E}} D_V f(L^n_t) - \mathbb{E} D_V f(L_t)| = |\hat{\mathbb{E}} D_V f(L^n_t) - \hat{\mathbb{E}} D_V f(L_t)|
\]
\[
= |\hat{\mathbb{E}} \langle \nabla f(L^n_t), \int_0^t \int_{\mathbb{H}} V(s)N_n(dz, ds) \rangle - \hat{\mathbb{E}} \langle \nabla f(L_t), \int_0^t \int_{\mathbb{H}} V(s)N^1(dz, ds) \rangle |
\]
\[
\leq |\hat{\mathbb{E}} \langle \nabla f(L^n_t), \int_0^t \int_{\mathbb{H}} V(s)N_n(dz, ds) \rangle - \hat{\mathbb{E}} \langle \nabla f(L^n_t), \int_0^t \int_{\mathbb{H}} V(s)N^n(dz, ds) \rangle |
\]
\[
+ |\hat{\mathbb{E}} \langle \nabla f(L^n_t), \int_0^t \int_{\mathbb{H}} V(s)N^n(dz, ds) \rangle - \hat{\mathbb{E}} \langle \nabla f(L_t), \int_0^t \int_{\mathbb{H}} V(s)N^1(dz, ds) \rangle |
\]
\[
\leq \| \nabla f \|_{\infty} \hat{\mathbb{E}} \int_0^t \int_{\mathbb{H}} |V(s)|N^{1,n}(dz, ds)
\]
\[
+ \| \nabla^2 f \|_{\infty} \mathbb{E} \left\{ \int_0^t \int_{\mathbb{H}} |z|N^{1,n}(dz, ds) \int_0^t \int_{\mathbb{H}} |V(s)|N^1(dz, ds) \right\}
\]
\[
\leq \frac{1}{n} \left\{ \| \nabla f \|_{\infty} \| V \|_{\infty} t + \| \nabla^2 f \|_{\infty} \| V \|_{\infty} \lambda \int_{\mathbb{H}} |z|\mu(dz) t^2 \right\} \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty,
\]
and
\[
|\hat{\mathbb{E}} \{ f(L^n_t)M^n_t \} - \mathbb{E} \{ f(L_t)M_t \}| = |\hat{\mathbb{E}} \{ f(L^n_t)M^n_t \} - \hat{\mathbb{E}} \{ f(L_t)M_t \}|
\]
\[ \leq |\mathbb{E}\{f(L^n_t)M^n_t\} - \mathbb{E}\{f(L^n_t)M_t\}| + |\mathbb{E}\{f(L^n_t)M_t\} - \mathbb{E}\{f(L_t)M_t\}| := I_1 + I_2. \]

As for \( I_1 \), we have
\[
I_1 \leq \|f\|_{\infty} \mathbb{E}|M^n_t - M_t| \leq \|f\|_{\infty} \mathbb{E} \left| \int_0^t \int_\mathbb{H} \left( \langle z, Q^{-1}V(s) + (\nabla \rho(z) + \frac{\nabla \rho(z)}{\rho(z)}, V(s) \rangle \right) \tilde{N}_n(dz, ds) \right| \\
- \int_0^t \int_\mathbb{H} \left( \langle z, Q^{-1}V(s) + (\nabla \rho(z) + \frac{\nabla \rho(z)}{\rho(z)}, V(s) \rangle \right) \tilde{N}^1(dz, ds) \right| \\
\leq \|f\|_{\infty} \left\{ \mathbb{E} \left| \int_0^t \int_\mathbb{H} \langle z, Q^{-1}V(s) \rangle \tilde{N}^1_n(dz, ds) \right| \\
+ \mathbb{E} \left| \int_0^t \int_\mathbb{H} \left( \frac{\nabla \rho(z)}{\rho(z)} - \frac{\nabla \rho(z)}{\rho(z)}, V(s) \rangle \tilde{N}^1_1(dz, ds) \right| \right\} \\
\leq 2\|f\|_{\infty} \left\{ \frac{t}{n} \|Q^{-1}V\|_{\infty} \int_\mathbb{H} |z|\mu(dz) + t\|V\|_{\infty} \int_\mathbb{H} \left| \frac{\rho(z)}{\rho(z)} + \frac{1}{n} \right| |\nabla \rho(z)| \mu(dz) \right\} \to 0, \text{ as } n \to \infty. \]

Meanwhile,
\[
I_2 = |\mathbb{E}\{f(L^n_t)M_t\} - \mathbb{E}\{f(L_t)M_t\}| \leq \|\nabla f\|_{\infty} \mathbb{E}|L^n_t, M_t| \leq \|\nabla f\|_{\infty} \mathbb{E}|L^n_t, \mathbb{E}|M_t| \leq \|\nabla f\|_{\infty} \frac{t}{n} \int_\mathbb{H} |z|\mu(dz) \mathbb{E}|M_t| \to 0, \text{ as } n \to \infty. \]

Considering above estimates and letting \( n \to \infty \) in (2.17), we get (2.11) for \( V \in \mathcal{Y}_1 \).

Step 3. Assume \( V \in \mathcal{Y}_2 \). For \( n \in \mathbb{N} \), define
\[
V_n(t) = V(t)I_{[0,\infty]}(|Q^{-1}V(t)|), \quad t \geq 0.
\]

By Step 2, one arrives at
\[
\mathbb{E}D_{V_n}f(L_t) = -\mathbb{E}\{f(L_t)M^n_t\}, \quad (2.18)
\]
for \( M^n_t = \int_0^t \int_\mathbb{H} \left( \langle z, Q^{-1}V_n(s) + (\nabla \log \rho(z), V_n(s)) \rangle \tilde{N}_1(dz, ds) \right. \). It is easy to check that
\[
\mathbb{E}|D_{V_n}f(L_t) - D_V f(L_t)| \to 0 \text{ and } \mathbb{E}|M^n_t - M_t| \to 0, \quad n \to \infty,
\]
Therefore, let \( n \to \infty \) in (2.18) and we finish the proof. \( \square \)
3 Proofs of Main Results.

In this section, we would like to give the proofs of main results. Denote

\[ C^2_0(\mathbb{H} \to \mathbb{H}) = \left\{ G : \mathbb{H} \to \mathbb{H} | G \text{ is bounded, differential, with bounded} \right. \\
\left. \quad \text{and continuous derivatives up to order 2.} \right\} \]

Before we move on, it is necessary for us to prove the existence of \( L^1 \)-derivative of \( [1.1] \).

**Proposition 3.1.** Assume \( A \) generates a \( C_0 \)-semigroup \( \{ S(t) \}_{t \geq 0} \) and \( F \in C^2_0(\mathbb{H} \to \mathbb{H}) \). If a predictable process \( V \) satisfies \( \mathbb{E} \int_0^T |V(s)|^2 ds < \infty \), then \( X_t \) has an \( L^1 \)-derivative in direction \( V \). Moreover, the \( L^1 \)-derivative satisfies

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
    dD_V X_t = AD_V X_t dt + \nabla F(X_t) D_V X_t dt + \int_\mathbb{H} V(s) N^1(dz, dt) \\
    D_V X_0 = 0.
\end{array}
\right.
\end{aligned}
\]

(3.1)

**Proof.** By classical results of SPDEs, the solution of Eq. (3.1) admits a unique solution

\[ D_V X_t = \int_0^t S(t-s)\nabla F(X_s) D_V X_s ds + \int_0^t \int_\mathbb{H} V(s) N^1(dz, ds). \]

(3.2)

Now we aim to prove \( D_V X_t \) is the \( L^1 \)-derivative of \( X_t \). It is easy to check the integrability of \( D_V X_t \). We shall prove

\[ E\left\{ \sup_{s \leq t} \left| \frac{X_t^\epsilon - X_s}{\epsilon} - D_V X_s \right| \right\} \to 0, \quad \epsilon \to 0. \]

(3.3)

In fact,

\[ X_t^\epsilon = x + \int_0^t S(t-s)F(X_s^\epsilon) ds + \int_0^t S(t-s) dL_s^\epsilon \]

\[ = x + \int_0^t S(t-s)F(X_s^\epsilon) ds + \int_0^t S(t-s) dL_s + \epsilon \int_0^t \int_\mathbb{H} S(t-s) V(s) N^1(dz, ds), \]

then

\[ X_t^\epsilon - X_t = \int_0^t S(t-s)(F(X_s^\epsilon) - F(X_s)) ds + \epsilon \int_0^t \int_\mathbb{H} S(t-s) V(s) N^1(dz, ds). \]

(3.4)

Therefore,

\[ |X_t^\epsilon - X_t| \leq \int_0^t |S(t-s)(F(X_s^\epsilon) - F(X_s))| ds + \epsilon \int_0^t \int_\mathbb{H} |S(t-s) V(s)| N^1(dz, ds) \]

\[ \leq C \| \nabla F \| \infty e^{\delta t} \int_0^t e^{-\delta s} |X_s^\epsilon - X_s| ds + \epsilon Ce^{\delta t} \int_0^t \int_\mathbb{H} e^{-\delta s} |V(s)| N^1(dz, ds). \]

Furthermore,

\[ \sup_{s \leq t} \left\{ e^{-\delta s} |X_s^\epsilon - X_s| \right\} \]
\[
\leq C\| \nabla F \|_\infty \int_0^t \sup_{r \leq s} \left\{ e^{-\delta r} |X^\epsilon_r - X_r| \right\} ds + \epsilon C \int_0^t \int_H e^{-\delta s} |V(s)| N^1(dz, ds).
\]

Using Gronwall’s inequality, one obtains

\[
\sup_{s \leq t} \left\{ e^{-\delta s} |X^\epsilon_s - X_s| \right\} \leq \epsilon C \exp \{ C\| \nabla F \|_\infty t \} \int_0^t \int_H e^{-\delta s} |V(s)| N^1(dz, ds),
\]

which yields

\[
\sup_{s \leq t} |X^\epsilon_s - X_s| \leq \epsilon C \exp \{ \delta t + C\| \nabla F \|_\infty t \} \int_0^t \int_H e^{-\delta s} |V(s)| N^1(dz, ds). \tag{3.5}
\]

From (3.2), (3.4) and Taylor’s formula, it follows that

\[
\left\| \frac{X^\epsilon_t - X_t}{\epsilon} - D_V X_t \right\| \leq C \int_0^t e^{\delta(t-s)} \left\| \nabla F \right\|_\infty \left\{ \frac{X^\epsilon_s - X_s}{\epsilon} - D_V X_s \right\} ds + \| \nabla^2 F \|_\infty \frac{|X^\epsilon_s - X_s|^2}{\epsilon} ds.
\]

By the similar argument above, we have

\[
\sup_{s \leq t} \left\| \frac{X^\epsilon_s - X_s}{\epsilon} - D_V X_s \right\| \leq C\| \nabla^2 F \|_\infty \exp \{ C\| \nabla F \|_\infty t + \delta t \} \sup_{s \leq t} \frac{|X^\epsilon_s - X_s|^2}{\epsilon}.
\]

Combining it with (3.5), one arrives at

\[
\mathbb{E} \left\{ \sup_{s \leq t} \left\| \frac{X^\epsilon_s - X_s}{\epsilon} - D_V X_s \right\| \right\} \leq \hat{C}_1 \epsilon \mathbb{E} \left\{ \int_0^t \int_H |V(s)| N^1(dz, ds) \right\}^2 \leq \hat{C}_2 \epsilon.
\]

where \( \hat{C}_1 \) and \( \hat{C}_2 \) are constants independent of \( \epsilon \). \( \square \)

Let \( J_t = \nabla_x X^\epsilon_t \). For \( s \leq t \), let \( J_{st} \) be the solution of the following equation:

\[
\begin{aligned}
&dJ_{st} = AJ_{st} dt + \nabla F(X_t) J_{st} dt, \\
&J_{ss} = I.
\end{aligned} \tag{3.6}
\]

Then it can be proved that

\[ J_t = J_0 t = J_{st} J_s \]

By (3.2) and (3.6), we can deduce

\[
D_V X_t = \int_0^t \int_H J_{st} V(s) N(dz, ds). \tag{3.7}
\]
**Proof of Theorem 1.1.** We show the proof in two steps.

**Step 1:** Assume \( F \in C^2_b(\mathbb{H} \to \mathbb{H}) \). By (3.6), one has

\[
J_t = S(t) + \int_0^t S(t-s) \nabla F(X_s) J_s ds.
\]

Then

\[
\|J_t\| \leq e^{-\gamma_1 t} + \|\nabla F\|_{\infty} \int_0^t \|J_s\| e^{-\gamma_1 (t-s)} ds.
\]

Gronwall’s inequality implies

\[
\|J_t\| \leq \exp\{(\gamma_1 - \|\nabla F\|_{\infty}) t\}. \tag{3.8}
\]

Observe that

\[
Q^{-1}J_t = Q^{-1}S(t) + Q^{-1} \int_0^t S(t-s) \nabla F(X_s) J_s ds
\]

\[
= Q^{-1}S(t) + \lim_{\epsilon \to 0} \int_0^t (Q + \epsilon I)^{-1} S(t-s) \nabla F(X_s) J_s ds
\]

\[
= Q^{-1}S(t) + \int_0^t \lim_{\epsilon \to 0} (Q + \epsilon I)^{-1} S(t-s) \nabla F(X_s) J_s ds
\]

\[
= Q^{-1}S(t) + \int_0^t Q^{-1}S(t-s) \nabla F(X_s) J_s ds, \tag{3.9}
\]

where in the fourth equality we use the dominated convergence theorem. By (3.8) and (3.9), we get

\[
\int_0^t \|Q^{-1}J_s\| ds \leq \int_0^t \|Q^{-1}S(s)\| ds + \int_0^t \int_0^s \|Q^{-1}S(s-r)\| \|\nabla F(X_r)\| \|J_r\| dr ds
\]

\[
\leq (1 + t \|\nabla F\|_{\infty}) \int_0^t \|Q^{-1}S(s)\| ds < \infty.
\]

Taking \( V(s) = J_s \xi \) in (3.7), we obtain \( D_V X_t = N_t^1 J_t \xi \), and

\[
\frac{I_{[N_t^1 \geq 1]}}{N_t^1} D_V X_t = J_t I_{[N_t^1 \geq 1]} \xi. \tag{3.10}
\]

Since \( D_V N_t^1 = 0 \), then

\[
D_V \left\{ \frac{I_{[N_t^1 \geq 1]}}{N_t^1} \right\} = 0. \tag{3.11}
\]

It follows from (3.10), (3.11) and (2.11) that

\[
\nabla \xi P_t^1 f(x)
\]
\[
\begin{align*}
&= \nabla \xi \mathbb{E}\left\{ f(X^\xi_t) I_{\{N^1_t \geq 1\}} \right\} \\
&= \mathbb{E} \left\langle \nabla f(X^\xi_t), J_t \xi I_{\{N^1_t \geq 1\}} \right\rangle \\
&= \mathbb{E} \left\langle \nabla f(X^\xi_t), \frac{I_{\{N^1_t \geq 1\}}}{N^1_t} D_V X^\xi_t \right\rangle \\
&= \mathbb{E} \left\{ D_V f(X^\xi_t) \frac{I_{\{N^1_t \geq 1\}}}{N^1_t} \right\} \\
&= - \mathbb{E} \left\{ f(X^\xi_T) \frac{I_{\{N^1_T \geq 1\}}}{N^1_T} \right\} \\
&= - \mathbb{E} \left\{ f(X^\xi_T) \frac{I_{\{N^1_T \geq 1\}}}{N^1_T} \right\} \\
&= \int_0^T \int_{\mathbb{H}} \left( \langle z, Q^{-1} J_s \xi \rangle + \langle \nabla \log \rho(z), J_s \xi \rangle \right) \tilde{N}^1(dz, ds) \right\}. \quad (3.12)
\end{align*}
\]

Step 2: Assume \( F \in C^1_b(\mathbb{H} \to \mathbb{H}) \) and \( \nabla F \) is Lipschitz continuous. We aim to construct approximation sequence \( \{F_k\}_{k \geq 1} \subset C^2_b(\mathbb{H} \to \mathbb{H}) \) such that \( F_k \to F \) and \( \nabla F_k \to \nabla F \) in pointwise sense as \( k \to \infty \). For \( k \geq 1 \), we take a sequence of non-negative, twice differential function \( \{g_k\}_{k \geq 1} \) such that

\[
\text{Supp}\{g_k\} \subset \{ y \in \mathbb{R}^k : |y|_{\mathbb{R}^k} \leq \frac{1}{k} \}
\]

and

\[
\int_{\mathbb{R}^k} g_k(y) dy = 1.
\]

Identifying \( \mathbb{R}^k \) with span\( \{e_1, \ldots, e_k\} \), we define

\[
F_k(x) = \int_{\mathbb{R}^k} g_k(y - \Pi_k x) F \left( \sum_{i=1}^k y_i e_i \right) dy,
\]

then \( F_k \) is a twice differentiable function with bounded and continuous derivatives. Moreover,

\[
\nabla F_k(x) = \int_{\mathbb{R}^k} g_k(y) \nabla F \left( \sum_{i=1}^k y_i e_i + \Pi_k x \right) \Pi_k dy.
\]

For any \( x, \tilde{x} \in \mathbb{H} \),

\[
\| \nabla F_k(x) - \nabla F_k(\tilde{x}) \| \\
= \| \int_{\mathbb{R}^k} g_k(y) \left[ \nabla F \left( \sum_{i=1}^k y_i e_i + \Pi_k x \right) - \nabla F \left( \sum_{i=1}^k y_i e_i + \Pi_k \tilde{x} \right) \right] \Pi_k dy \|
\leq \int_{\mathbb{R}^k} g_k(y) \| \nabla F \|_{Lip} |x - \tilde{x}| dy
\leq \| \nabla F \|_{Lip} |x - \tilde{x}|,
\]

where \( \| \nabla F \|_{Lip} \) denotes the smallest Lipschitz constant. This implies

\[
\sup_{k \geq 1} \| \nabla^2 F_k \|_\infty \leq \| \nabla F \|_{Lip}.
\]

(3.15)
Consider the following equation for any $k \geq 1$,
\[
\begin{cases}
    dX^k_t = AX^k_t dt + F_k(X^k_t) dt + dL_t, \\
    X^k_0 = x.
\end{cases} \quad (3.16)
\]

Let $\{X^k_t\}_{t \geq 0}$ be the solution of Eq. (3.16) and $\{J^k_t\}_{t \geq 0}$ be its derivative w.r.t. the initial value. Then $\{J^k_t\}_{t \geq 0}$ satisfies
\[
\begin{cases}
    dJ^k_t = A J^k_t dt + \nabla F_k(X^k_t) J^k_t dt, \\
    J^k_0 = I.
\end{cases} \quad (3.17)
\]

Since
\[
|X^k_t - X_t| = \left| \int_0^t S(t-s) (F_k(X^k_s) - F(X_s)) ds \right|
\leq \left| \int_0^t S(t-s) (F_k(X^k_s) - F_k(X_s)) ds \right| + \left| \int_0^t S(t-s) (F_k(X_s) - F(X_s)) ds \right|
\leq \int_0^t e^{-\gamma_1(t-s)} \sup_{k \geq 1} \|\nabla F_k\|_{\infty} |X^k_s - X_s| ds + \int_0^t e^{-\gamma_1(t-s)} |F_k(X_s) - F(X_s)| ds,
\]
then with the help of Gronwall's inequality and dominated convergence theorem, we obtain
\[
\lim_{k \to \infty} |X^k_t - X_t|
\leq \exp \left\{ - (\gamma_1 + \sup_{k \geq 1} \|\nabla F_k\|_{\infty}) t \right\} \lim_{k \to \infty} \int_0^t e^{-\gamma_1(t-s)} |F_k(X_s) - F(X_s)| ds = 0.
\]

Based on the above estimates, by (3.8), we deduce
\[
\begin{align*}
\|J^k_t - J_t\| &\leq \int_0^t \|S(t-s)\|_{\infty} \|\nabla F_k(X^k_s) J^k_s - \nabla F(X_s) J_s\| ds \\
&\leq \int_0^t \|S(t-s)\|_{\infty} \|\nabla F_k(X^k_s) J^k_s - \nabla F_k(X^k_s) J_s\| ds \\
&\quad + \int_0^t \|S(t-s)\|_{\infty} \|\nabla F_k(X^k_s) J_s - \nabla F_k(X^k_s) J_s\| ds \\
&\quad + \int_0^t \|S(t-s)\|_{\infty} \|\nabla F_k(X_s) J_s - \nabla F(X_s) J_s\| ds \\
&\leq \sup_{k \geq 1} \|\nabla F_k\|_{\infty} \int_0^t e^{-\gamma_1(t-s)} \|J^k_s - J_s\| ds \\
&\quad + \sup_{k \geq 1} \|\nabla^2 F_k\|_{\infty} \int_0^t e^{-\gamma_1(t-s) + \gamma_1 s} \|\nabla F\|_{\infty} \|X^k_s - X_s\| ds \\
&\quad + \int_0^t e^{-\gamma_1(t+s)} \|\nabla F(X_s) - \nabla F(X_s)\| ds \\
&\quad := I_1(k) + I_2(k) + I_3(k),
\end{align*}
\]
which implies
\[
\|J^k_t - J_t\| \leq \exp \{ - \gamma_1 t + \sup_{k \geq 1} \|\nabla F_k\|_{\infty} t \} \left( I_2(k) + I_3(k) \right) \to 0, \quad k \to \infty.
\]
Define

\[ P^k_t f(x) = \mathbb{E}\{ f(X^k_t) I_{|N^1_t| \geq 1} \} . \]

By (3.12), we have

\[
\nabla P^k_t f(x)
= -\mathbb{E}\left\{ f(X^k_t) \frac{I_{|N^1_t| \geq 1}}{N^1_t} \int_0^t \int_H \left( \langle z, Q^{-1} J_s^k \xi \rangle + \langle \nabla \log \rho(z), J_s^k \xi \rangle \right) \tilde{N}^1(dz, ds) \right\}.
\]

(3.19)

Note that as \( k \to \infty \),

\[ |\nabla P^k_t f(x) - \nabla P^1_t f(x)| \leq \| \nabla f \|_\infty \| J^k_t - J_t \| \to 0, \]

and

\[ \mathbb{E}\left| \int_0^t \int_H \left( \langle z, Q^{-1} (J^k_s - J_s) \xi \rangle + \langle \nabla \log \rho(z), (J^k_s - J_s) \xi \rangle \right) \tilde{N}^1(dz, ds) \right| \to 0. \]

We finish the proof by letting \( k \to \infty \) in (3.19).

**Proof of Theorem 1.2.**

**Step 1:** Assume \( F \in C^2_b(\mathbb{H} \to \mathbb{H}) \). By Theorem 1.1, triangle inequality and H"older inequality, one arrives at

\[
|\nabla \xi P^1_t f(x)|
= \left| -\mathbb{E}\left\{ f(X^1_t) \frac{I_{|N^1_t| \geq 1}}{N^1_t} \int_0^t \int_H \left( \langle z, Q^{-1} J_s \xi \rangle + \langle \nabla \log \rho(z), J_s \xi \rangle \right) \tilde{N}^1(dz, ds) \right\} \right|
\leq \| f \|_\infty \| \xi \| \left\{ \frac{\int_H |z|^2 \rho(z) \mu(dz) \mathbb{E}\left\{ \frac{I_{|N^1_t| \geq 1}}{(N^1_t)^2} \right\} \int_0^t \mathbb{E}\| Q^{-1} J_s \|^2 ds \}}{2} \right\}^{\frac{1}{2}}
\leq 2\mathbb{E}\int_0^t \int_H |\nabla \log \rho(z)||J_s\|\rho(z)\mu(dz)ds \right\}.
\]

(3.20)

Note that

\[
\mathbb{E}\left\{ \frac{I_{|N^1_t| \geq 1}}{(N^1_t)^2} \right\} = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n^n n!}
= \frac{e^{-\lambda t}}{(\lambda t)^2} \sum_{n=1}^{\infty} \frac{(n + 2)(n + 1)}{n^2} \frac{(\lambda t)^{n+2}}{(n + 2)!}
\leq 6 \frac{e^{-\lambda t}}{(\lambda t)^2} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n+2}}{(n + 2)!}
\leq 6 \frac{6}{(\lambda t)^2},
\]

(3.21)

and by (3.8), (3.9) and H"older inequality, we have

\[ \int_0^t \mathbb{E}\| Q^{-1} J_s \|^2 ds \]
= \int_0^t \mathbb{E} \left\| Q^{-1} S(s) + \int_0^s Q^{-1} S(s - r) \nabla F(X_r) J_r dr \right\|^2 ds \\
\leq 2 \int_0^t \left\| Q^{-1} S(s) \right\|^2 ds + 2 \int_0^t \mathbb{E} \left\| \int_0^s Q^{-1} S(s - r) \nabla F(X_r) J_r dr \right\|^2 ds \\
\leq 2 \|
abla F\|_\infty^2 \int_0^t \left\{ s \int_0^s \left\| Q^{-1} S(s - r) \right\|^2 \exp \{-2(\gamma_1 - \|
abla F\|_\infty) r \} dr \right\} ds \\
+ 2 \int_0^t \left\| Q^{-1} S(s) \right\|^2 ds \\
= 2 \|
abla F\|_\infty^2 \Gamma_t + 2 \int_0^t \left\| Q^{-1} S(s) \right\|^2 ds. \tag{3.22}

where

\[ \Gamma_t = \int_0^t \left\{ s \int_0^s \left\| Q^{-1} S(s - r) \right\|^2 \exp \{-2(\gamma_1 - \|
abla F\|_\infty) r \} dr \right\} ds. \]

In addition,

\[ \mathbb{E} \int_0^t \int_{\mathbb{H}} |\nabla \log \rho(z)| \| J_s \rho(z) \mu(dz) \] \[ \leq \int_{\mathbb{H}} |\nabla \rho(z)| \mu(dz) \int_0^t \exp \{-2(\gamma_1 - \|
abla F\|_\infty) s \} ds \\
\leq \frac{\int_{\mathbb{H}} |\nabla \rho(z)| \mu(dz)}{\gamma_1 - \|
abla F\|_\infty}. \tag{3.23} \]

With the help of (3.20)-(3.23), we can obtain

\[ |\nabla \xi P_t^1 f(x)| \leq 2 \|f\|_\infty |\xi| \left\{ \sqrt{6} \left( \frac{\int_{\mathbb{H}} |z|^2 \rho(z) \mu(dz) \|
abla F\|_\infty^2 \Gamma_t + \int_0^t \left\| Q^{-1} S(s) \right\|^2 ds}{\lambda^2 \tau^2} \right)^{1/2} \\
+ \frac{\int_{\mathbb{H}} |\nabla \rho(z)| \mu(dz)}{\gamma_1 - \|
abla F\|_\infty} \right\}. \]

Furthermore,

\[ \|\nabla P_t^1 f\|_\infty := \sup_{|\xi| \leq 1} \sup_{x \in \mathbb{H}} |\nabla \xi P_t^1 f(x)| \leq 2 \|f\|_\infty \left\{ \sqrt{6} \left( \frac{\int_{\mathbb{H}} |z|^2 \rho(z) \mu(dz) \|
abla F\|_\infty^2 \Gamma_t + \int_0^t \left\| Q^{-1} S(s) \right\|^2 ds}{\lambda^2 \tau^2} \right)^{1/2} \\
+ \frac{\int_{\mathbb{H}} |\nabla \rho(z)| \mu(dz)}{\gamma_1 - \|
abla F\|_\infty} \right\}. \tag{3.24} \]

Since \( \lim_{t \to \infty} \int_0^t \left\| Q^{-1} S(s) \right\|^2 ds < \infty \), then \( \lim_{t \to \infty} \left\| Q^{-1} S(t) \right\|^2 < \infty \). Moreover,

\[ \lim_{t \to \infty} \frac{\Gamma_t}{(\lambda t)^2} = \lim_{t \to \infty} \frac{\int_0^t \left\{ s \int_0^s \left\| Q^{-1} S(s - r) \right\|^2 \exp \{-2(\gamma_1 - \|
abla F\|_\infty) r \} dr \right\} ds}{t^2} \]
such that \( F(3) \).

Step 2: Assume \((H4)\).

Therefore,

\[
|P_t f(x) - P_t f(y)| = |P_t^1 f(x) - P_t^1 f(y)| + |\mathbb{E}\{f(X_t^x)I_{[N_t=0]}\} - \mathbb{E}\{f(X_t^y)I_{[N_t=0]}\}| \\
\leq \|\nabla P_t f\|_{\infty} |x - y| + 2\|f\|_{\infty} \mathbb{P}(N_t^1 = 0) \\
\leq 2\|f\|_{\infty} \left\{ \left\{ \sqrt{6C_1\left(\|\nabla F\|_{\infty}^2 + 1\right)} \right\} \right\} |x - y| \\
+ e^{-\lambda t}. \tag{3.26}
\]

Step 2: Assume \((H4)\).

Making use of \((3.13)\), we can construct \(\{F_n\}_{n \geq 1} \subset C^2_b(\mathbb{H} \to \mathbb{H})\) such that \(F_n \to F\) as \(n \to \infty\) in pointwise sense and \(\sup_{n \geq 1} \|\nabla F_n\|_{\infty} \leq \|F\|_{\text{Lip}}\). It follows from \((3.26)\) that

\[
|P_t^n f(x) - P_t^n f(y)| \\
\leq 2\|f\|_{\infty} \left\{ \left\{ \sqrt{6C_1\left(\|\nabla F_n\|_{\infty}^2 + 1\right)} \right\} \right\} |x - y| \\
+ e^{-\lambda t} \tag{3.27}
\]
where \( \{P^n_t\}_{t\geq 0} \) denotes the transition semigroup of \( \{X^n_t\}_{t\geq 0} \). Letting \( n \to \infty \) in (3.27), we get
\[
|P_t f(x) - P_t f(y)| \leq 2\|f\|_\infty \left\{ C_2|x - y| + e^{-\lambda t} \right\},
\] (3.28)
for some constant \( C_2 > 0 \). Since for \( x, y \in \mathbb{H} \),
\[
\mathbb{E}|X^x_t - X^y_t| \leq |S(t)(x - y)| + \mathbb{E}\int_0^t |S(t-s)(F(X^x_s) - F(X^y_s))|ds
\]
\[
\leq e^{-\gamma_1 t}|x - y| + \int_0^t e^{-\gamma_1(t-s)}\|F\|_{\text{Lip}}\mathbb{E}|X^x_s - X^y_s|ds,
\]
then one obtain
\[
\mathbb{E}|X^x_t - X^y_t| \leq \exp\left\{-(\gamma_1 + \|F\|_{\text{Lip}})t\} |x - y|.
\]
Combining this with (3.28) and using the Markov property, we have for \( t > s \geq 1 \)
\[
|P_t f(x) - P_t f(y)|
\leq E|P_s f(X^x_{t-s}) - P_s f(X^y_{t-s})|
\leq 2\|f\|_\infty \left\{ C_2\mathbb{E}|X^x_{t-s} - X^y_{t-s}| + e^{-\lambda s} \right\}
\leq 2\|f\|_\infty \left\{ C_2 \exp\left\{-\gamma_1 + \|F\|_{\text{Lip}})(t-s)\} |x - y| + e^{-\lambda s} \right\}.\] (3.29)

Let \( t > \frac{\gamma_1 - \|F\|_{\text{Lip}} + \lambda}{\gamma_1 - \|F\|_{\text{Lip}}} \) and take \( s = \frac{\gamma_1 - \|F\|_{\text{Lip}}}{\gamma_1 - \|F\|_{\text{Lip}} + \lambda} \) in (3.29), then there exists a constant \( C > 0 \) such that
\[
|P_t f(x) - P_t f(y)| \leq C\|f\|_\infty (1 + |x - y|) \exp\left\{-\frac{\lambda(\gamma_1 - \|F\|_{\text{Lip}})t}{\lambda + \gamma_1 - \|F\|_{\text{Lip}}} \right\},
\]
which implies
\[
\|P_t (x, \cdot) - P_t (y, \cdot)\|_{\text{Var}} \leq C(1 + |x - y|) \exp\left\{-\frac{\lambda(\gamma_1 - \|F\|_{\text{Lip}})t}{\lambda + \gamma_1 - \|F\|_{\text{Lip}}} \right\}.
\]
The proof is completed by noting that the inequality trivially holds with a suitable constant \( C > 0 \) for \( t \leq \frac{\gamma_1 - \|F\|_{\text{Lip}} + \lambda}{\gamma_1 - \|F\|_{\text{Lip}}} \). \( \square \)

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