Galilean Conformal Algebra in Semi-Infinite Space

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Abstract

In the present work we considered Galilean conformal algebras (GCA), which arises as a contraction relativistic conformal algebras \((x_i \rightarrow \epsilon x_i, \quad t \rightarrow t, \quad \epsilon \rightarrow 0)\). We can use the Galilean conformal symmetry to constrain two-point and three-point functions. Correlation functions in space-time without boundary condition were found in \cite{19}. In real situations there are boundary conditions in space-time, so we have calculated correlation functions for Galilean conformal invariant fields in semi-infinite space with boundary condition in \(r = 0\). We have calculated two-point and three-point functions with boundary condition in fixed time.
1 Introduction

Recently, there has been some interest in extending the AdS/CFT correspondence to non-relativistic field theories [1, 2]. The Kaluza-Klein type framework for non-relativistic symmetries, used in Refs. [1, 2, 3], is basically identical to the one introduced in [4] (see also [5]). The study of a different non-relativistic limit was initiated in [6], where the non-relativistic conformal symmetry was obtained by a parametric contraction of the relativistic conformal group. Galilean conformal algebra (GCA) arises as a contraction relativistic conformal algebras [6, 7, 8], where in \( d = 4 \) the Galilean conformal group is a fifteen parameter group which contains the ten parameter Galilean subgroup. Beside Galilean conformal algebra, there is another Galilean algebra, the twelve parameter schrödinger algebra [1, 2]. The dilatation generator in the schrödinger group scales space and time differently, \( x_i \rightarrow \lambda x_i, \quad t \rightarrow \lambda^2 t \), but in contrast the corresponding generator in GCA scales space and time in the same way, \( x_i \rightarrow \lambda x_i, \quad t \rightarrow \lambda t \). Infinite dimensional Galilean conformal group has been reported in [7], the generators of this group are:

\[
L^n = -(n+1)t^n x_i \partial_i - t^{n+1} \partial_t, \quad M^n_i = t^{n+1} \partial_i \quad \text{and} \quad J^n_{ij} = -t^n (x_i \partial_j - x_j \partial_i)
\]

for an arbitrary integer \( n \), where \( i \) and \( j \) are specified by the spatial directions. There is a finite dimensional subgroup of the infinite dimensional Galilean conformal group which is generated by \( (J^0_{ij}, L^\pm, L^0, M^\pm_i, M^0_i) \). These generators are obtained by contraction ( \( t \rightarrow t, \quad x_i \rightarrow \epsilon x_i, \quad \epsilon \rightarrow 0, \quad v_i \sim \epsilon \) ) of the relativistic conformal generators. Several conformal extensions of the Galilean Lie algebra have been obtained recently [9].

The representation for finite GCA was found in [10] and [11]. According to the theorem [12] "Every representation \( D(G) \) of a finite group on an inner product space is equivalent to a unitary representation," the representations of finite Galilean conformal group are unitary and we can use this group for physical applications. For example, non-relativistic limit of conformal hydrodynamic describes the small fluctuations from thermal equilibrium. Whenever the conformal invariant dissipative hydrodynamics are considered in nonrelativistic limit, the relations \( \partial_\nu T^{\mu\nu} = 0 \) and \( T^{\mu}_\mu = 0 \) (The stress-energy tensor of the CFT obeys these relations) are converted to incompressible Navier-Stokes (NS) equation [13]. From reference [14] it can be inferred that, the Navier-stokes equation for incompressible flow (\( \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} \) is velocity of fluid) is covariant under the infinite GCA. The NS equation for ideal hydrodynamics (without viscosity) changes to Euler equation. The finite Galilean conformal group is symmetry of the Euler equation [7] (see also [15, 16]). The gravity dual of finite GCA was considered in [7, 8, 17] and the metric with finite 2d GCA isometry was obtained in
We can use the Galilean conformal symmetry to constrain correlation function \[19\]. Correlation functions of Galilean conformal invariant fields without any boundary condition were found in \[19\]. The presence of free surfaces or walls in macroscopic systems which are at the critical point, lead to the large variety of physical effects. Since, using boundary condition effects is shown to be very helpful in various branch in physics, the systems with boundary conditions have been considered by both theorists \[20\] and experimentalists \[21\]. The situation with walls or free surfaces opens a new area in condensed matter physics \[22\]. In reference \[23\], the research on semi-infinite systems which exhibits a non-equilibrium bulk phase transitions was initiated and the effects of boundary condition on direct percolation were considered.

Correlation functions near the boundary are different from other places, and in real situation there are boundary conditions in space coordinates. By using some methods in non-relativistic conformal field theory, we have obtained an analytic expression for correlation functions with boundary condition. The situation when a system is in a predefined initial state and relaxes toward its critical equilibrium considered as a situation with a boundary condition at fixed time \[23\], for these reasons, in this paper we calculated two-point and three-point functions in space-time with boundary condition. The discussion will be exclusively in two dimensions, but the extension to arbitrary dimension is immediate. A system with boundary condition at a surface \(r = 0\) was kept invariant under the transformations were generated by \([L_{-1}, L_0, L_1]\), but space translations, Galilean Boost transformation and space special Galilean conformal transformation no longer leave the system invariant. A system with boundary condition at a fixed time was kept invariant by the subalgebra \([M_{-1}, M_0, M_1, L_0]\). Correlation functions with these boundary conditions are different with the corresponding results found from Galilean conformal invariance without boundary condition \[19\]. If the domain of space-time coordinates are infinite in extent (without boundary condition) and the scalar fields are invariant under the Galilean conformal (GC) group, the two-point and three-point functions are completely determined \[19\]. For example, these results apply to the calculation of time-delayed correlation functions of non-relativistic (small viscosity) systems at equilibrium and at static critical point. If the space geometry is semi-infinite, a form of correlation function will be derived, see Eqs. \[12\] and \[27\]. These results may be relevant to critical dynamics close to a surface, see \[24\] for an example. when a system is in a predefined initial state, it can be seen that critical relaxation towards equilibrium displays scaling at intermediate time \[23\]. (This situation is discribed as a system with boundary condition
in time coordinate.) We calculate the form of correlation functions with this condition, see Eq. (20).

The paper organized as follows. Section 2 is a brief review of GCA. In section 3, we calculate the form of correlation functions with a boundary condition at surface \( r = 0 \). Then in section 4, we calculate the form of two-point function in space-time with a boundary condition at a fixed time. Then in section 5, we extend these calculation to the three-point correlation function. Finally, in section 5, we close by some concluding remarks.

2 Representations of Galilean conformal group

Galilean conformal algebras (GCA) was obtained via a direct contraction of conformal generators. Physically, this comes from taking \( t \rightarrow t \), \( x_i \rightarrow \epsilon x_i \) where \( \epsilon \rightarrow 0 \). The generators of conformal group are

\[
P_\mu = -i\partial_\mu, \quad D = -ix^\mu \partial_\mu, \quad J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)
\]

(1)

(1)

\( \mu, \nu = 0, 1, \ldots, d \).

From the above scaling we obtain the Galilean conformal vector field generators

\[
L_{-1} = \partial_r, \quad M_{-1} = \partial_t, \quad L_0 = (t \partial_t + r \partial_r), \quad J_{ij} = -(x_i \partial_j - x_j \partial_i),
\]

\[
L_1 = (2tr \partial_r + t^2 \partial_t), \quad M_1 = -t^2 \partial_r, \quad M_0 = -t \partial_r,
\]

(2)

\( L_{-1}, M_{-1}, L_0, J_{ij}, M_0, L_1, M_1 \) are spatial translations, time translation, dilatation, rotations, boosts, time component and space components of special conformal transformation respectively. In following, the discussion will be exclusively in two dimensions, but the extension to arbitrary dimension is immediate. The above generators obey the following commutation relations in two dimensions, where define the Galilean conformal algebras.

\[
[L_m, L_n] = (m - n)L_{m+n}
\]

\[
[L_m, M_n] = (m - n)M_{m+n}
\]

\[
[M_n, M_m] = 0
\]

\[
[\delta r, \delta t] = -\partial_\delta r + \partial_\delta t
\]
The above symmetries use to constrain two-point function (see section (3, 4)). The representations of this group was obtained in [10]. According to the theorem [12] ”every representation $D(G)$ of a finite group on an inner product space is equivalent to a unitary representation”, the representations of $2d$ finite Galilean conformal group are unitary and it can be used for physical applications. The representations of Galilean conformal group are built by Hausdorff formula [19].

\[
\begin{align*}
[L_{-1}, \phi] &= \partial_t \phi \\
[L_0, \phi] &= (t \partial_t + r \partial_r + \Delta)\phi \\
[L_1, \phi] &= (t^2 \partial_t + 2tr\partial_r + 2t\Delta - r\xi)\phi
\end{align*}
\]

and

\[
\begin{align*}
[M_{-1}, \phi] &= \partial_r \phi \\
[M_0, \phi] &= (-t\partial_r + \xi)\phi \\
[M_1, \phi] &= (-t^2 \partial_r + 2t\xi)\phi
\end{align*}
\]

where $\Delta$ is scaling dimension and $\xi$ is rapidity.

3 Two-point function in semi-infinite space

Galilean conformal symmetry is related to massless non-relativistic systems. Correlations of Galilean-quasiprimary fields were studied without any boundary condition [19], but in real systems with boundary conditions, we are interested in correlation function near the boundary, so we consider the effect of surface at $r = 0$. It is kept invariant under the transformations were generated by $[L_{-1}, L_0, L_1]$, but space translations, Galilean Boost transformation and space special Galilean conformal transformation no longer leave the system invariant. Nevertheless, it is known that Galilean conformal invariance can be used in analogous situations to constrain the two-point function [19]. For Galilean-quasiprimary fields, we require covariance only under the subalgebra $[L_{-1}, L_0, L_1]$.

Consider the two-point function of Galilean-quasiprimary fields

\[
G = G(r_a, r_b, t_a, t_b) = \langle \phi_a(r_a, t_a)\phi_b(r_b, t_b) \rangle
\]

and we require space points to be in the right half-plane, i.e. $r_a, r_b \geq 0$. Time translation invariance gives $G = G(r_a, r_b, \tau)$, with $\tau = t_a - t_b$. From scale invariance we obtain
\[ \langle 0 \mid [L_0, \phi_a \phi_b] \mid 0 \rangle = 0 \quad (7) \]
\[ \Rightarrow \sum_{i=a}^{i=b} (t_i \partial_i + r_i \partial_i + \Delta_i)G = 0 \]
\[ = (\tau \partial_{\tau} + r_a \partial_{r_a} + r_b \partial_{r_b} + \Delta)G = 0 \]

where \( \Delta = \Delta_a + \Delta_b \). On the other hand, from the invariance under the time special Galilean conformal transformation we have

\[ \langle 0 \mid [L_1, \phi_a \phi_b] \mid 0 \rangle = 0 \quad (8) \]
\[ \Rightarrow \sum_{i=a}^{i=b} (t_i^2 \partial_i + 2t_i r_i \partial_i + 2t_i \Delta_i - r_i \xi_i)G = 0 \]
\[ = ((t_a^2 - t_b^2) \partial_{\tau} + 2(t_a r_a \partial_{r_a} + t_b r_b \partial_{r_b}) + 2(t_a \Delta_a + t_b \Delta_b - r_a \xi_a + r_b \xi_b))G \]
\[ = (\tau^2 \partial_{\tau} + 2t_b(\tau \partial_{\tau} + r_a \partial_{r_a} + r_b \partial_{r_b}) + 2(t_a \Delta_a + t_b \Delta_b))G \]
\[ = (\tau^2 \partial_{\tau} + 2(r_a \xi_a - r_b \xi_b) + 22r_a \partial_{r_a} + 2\tau \Delta_a)G = 0 \]

where in the last equation the scale invariance of \( G \) was used.

Now, we make the ansatz

\[ G(r_a, r_b, \tau) = \tau^{-2\Delta_a} G'(u, v), \quad u = \tau \frac{r_a}{\tau}, \quad v = \frac{r_b}{\tau} \quad (9) \]

which solves for scale invariance, while Eq. (8) gives

\[ (u \partial_u - v \partial_v - 2(u \xi_a + v \xi_b))G'(u, v) = 0, \quad (10) \]

Note that, the change of variables in Eq. (9) are not singular, because \( \tau = t_a - t_b \neq 0 \). The general solution of Eq. (10) is found by the method of characteristics (20 and 25).

\[ G'(u, v) = \chi(uv) \exp(2(u \xi_a - v \xi_b)) \quad (11) \]

where \( \chi \) is an arbitrary function. The final result is

\[ G(r_a, r_b, \tau) = \delta_{\Delta_a, \Delta_b} \tau^{-2\Delta_a} \chi \left( \frac{r_a r_b}{\tau} \right) \exp \left( \frac{2}{\tau} (r_a \xi_a - r_b \xi_b) \right) \quad (12) \]
It is clear that, two-point function near the boundary is different from other places. We note that analogously to the conformal result [19], the scaling dimension have to agree, while in this case we do not have a constrain on the rapidity \( \xi_a \), since the system is not space special Galilean conformal invariant. One can use the relation (12) for nonrelativistic conformal hydrodynamics with small viscosity near the boundary. Two-point function in the bulk (out of boundary) for nonrelativistic conformal hydrodynamics was found in [19].

4 Two-point function for a non-stationary state

We now consider a situation with a boundary condition at a fixed time. Boundary conditions of this type are kept invariant by the subalgebra \([M_{-1}, M_0, M_1, L_0]\). For example, this may correspond to the situation when a system is in a predefined initial state and relaxes toward its critical equilibrium state [23]. Consider two-point function of quasiprimary fields

\[ G = G(r_a, r_b, t_a, t_b) = \langle \phi_a(r_a, t_a) \phi_b(r_b, t_b) \rangle \]

Invariance under space translation implies \( G = G(r, t_a, t_b) \) with \( r = r_a - r_b \). We next demand invariance under Galilean Boost transformation.

\[
\begin{align*}
< 0 | [M_0, \phi_a \phi_b] | 0 > & = 0 \\
\Rightarrow \sum_{i=a}^{b} (-t_i \partial_{r_i} + \xi_i)G &= (-t_a \partial_{r_a} + \xi_a - t_b \partial_{r_b} + \xi_b)G \\
&= (-t_a - t_b) \partial_r + (\xi_a + \xi_b)G = 0
\end{align*}
\]

Space special Galilean conformal invariance demands that

\[
\begin{align*}
< 0 | [M_1, \phi_a \phi_b] | 0 > & = 0 \\
\Rightarrow \sum_{i=a}^{b} (-t_i^2 \partial_{r_i} + 2t_i \xi_i)G &= 0
\end{align*}
\]

which gives respectively

\[ \xi_a = \xi_b \]
So, the two-point function reads:

\[ G = G'(t_a, t_b) \exp\left(\frac{2\xi_a r}{t_a - t_b}\right) \]  

(17)

In analogy to what was done before, we demand invariance under scale invariance

\[ < 0 | [L_0, \phi_a \phi_b] | 0 > = 0 \]  

(18)

\[ \Rightarrow \sum_{i=a}^{b}(t_i \partial_t + r_i \partial_i + \Delta_i)G \]

\[ = (t_a \partial_{t_a} + t_b \partial_{t_b} + \Delta_a + t_b \partial_{t_b} + \Delta_b)G \]

\[ = (t_a \partial_{t_a} + t_b \partial_{t_b} + r \partial_r + \Delta_a + \Delta_b)G = 0 \]

we find

\[ G'(t_a, t_b) = t_a^{-\Delta_a} \Phi\left(\frac{t_a}{t_b}\right) \]  

(19)

where \( \Phi \) is an arbitrary function. The final result is

\[ G(t_a, t_b) = t_a^{-(\Delta_a + \Delta_b)} \Phi\left(\frac{t_a}{t_b}\right) \exp\left(\frac{\xi r}{t_a - t_b}\right) \]  

(20)

Note that here we have no condition on the exponents because the system is not invariant under the time special Galilean conformal transformation. We are able to use the above relation for nonrelativistic conformal hydrodynamics with small viscosity in critical equilibrium [23].

Our study in sections 3, 4 was an extension of the work of Henkel in [20]. This paper has devoted to the study of the schrödinger symmetry algebra, the maximal symmetry of free schrödinger equations, and one of the things has done there was considering two-point functions invariant under a subalgebra of the schrödinger algebra. Here we applied the construction to the Galilean Conformal Algebra. Our results in Eq. (12) and Eq. (20) are correspond to the Eq. (3.35) and Eq. (3.44) respectively in [20]. The difference between these results is the form of the exponential piece (\( r/t \) in the case of the GCA as opposed to \( r^2/t \) in the schrödinger algebra) which is due to the difference in the form of the dilatation generator.

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Three-point function in space-time with boundary condition

Three-point function for Galilean conformal invariant fields in the bulk without boundary condition was calculated in [19]. In this section, we calculated three-point function near the boundary for Galilean conformal invariant fields. Three-point function in real situation with a boundary condition at fixed time was calculated. Along lines similar to section (3), we wish to construct three-point function of three GC-invariant fields \( \phi_i(\Delta_i, \xi_i) \) with \( i = a, b, c \). As we see in section (3), the correlation functions near the boundary are kept invariant under the transformations were generated by \([L_{-1}, L_0, L_1]\). Consider the three-point function of GC-invariant fields

\[
G(r_a, r_b, r_c, t_a, t_b, t_c) = \langle \phi_a(r_a, t_a) \phi_b(r_b, t_b) \phi_c(r_c, t_c) \rangle = 0
\]  

(21)

G is invariant under the time translation which is generated by \( L_0 \), so \( G = G(r_a, r_b, r_c, \tau, \sigma) \) where \( \tau = t_a - t_c \) and \( \sigma = t_b - t_c \). From scale invariance one can obtain

\[
\langle 0 | [L_0, \phi_a \phi_b \phi_c] | 0 \rangle = 0
\]  

(22)

\[
\Rightarrow \sum_{i=a} \sum_{i=r} \left( t_i \partial_{t_i} + r_i \partial_{r_i} + \Delta_i \right) G = 0
\]

From the invariance under the temporal non-relativistic special conformal transformation we have

\[
\langle 0 | [L_1, \phi_a \phi_b \phi_c] | 0 \rangle = 0
\]  

(23)

\[
\Rightarrow \sum_{i=a} \sum_{i=r} \left( t_i^2 \partial_{t_i} + 2t_i r_i \partial_{r_i} + 2t_i \Delta_i - r_i \xi_i \right) G = 0
\]

\[
= \left( (t_a^2 - t_c^2) \partial_{\tau} + (t_b^2 - t_c^2) \partial_{\sigma} + 2(t_a r_a \partial_{r_a} + t_b r_b \partial_{r_b} + t_c r_c \partial_{r_c}) + 2(t_a \Delta_a + t_b \Delta_b + t_c \Delta_c) - r_a \xi_a - r_b \xi_b - r_c \xi_c \right) G
\]

\[
= \left( \tau^2 \partial_{\tau} + \sigma^2 \partial_{\sigma} + 2t_c(\tau \partial_{\tau} + \sigma \partial_{\sigma} + r_a \partial_{r_a} + r_b \partial_{r_b} + r_c \partial_{r_c}) - r_a \xi_a - r_b \xi_b - r_c \xi_c + 2\tau r_a \partial_{r_a} + 2\sigma r_b \partial_{r_b} + 2(t_a \Delta_a + t_b \Delta_b + t_c \Delta_c)\right) G
\]

\[
= \left( (\tau^2 \partial_{\tau} - r_a \xi_a - r_b \xi_b - r_c \xi_c + 2\tau r_a \partial_{r_a} + 2\sigma r_b \partial_{r_b} + 2(t_a \Delta_a + t_b \Delta_b + t_c \Delta_c)\right) G
\]

\[
+ \sigma^2 \partial_{\sigma} + 2\sigma r_b \partial_{r_b} + 2\tau \Delta_a + 2\sigma \Delta_b) G = 0
\]

where in last equation the scale invariance (22) of three-point function \( G \) was used. The above equations have this solution

\[
G = \delta_{\Delta_a + \Delta_b, \Delta_c} \tau^{-2\Delta_a} \sigma^{-2\Delta_b} G'
\]  

(24)
If we introduce $G' = G'_1(r_a, r_c, \tau)G'_2(r_b, \sigma)$ or $G' = G'_1(r_b, r_c, \sigma)G'_2(r_a, \tau)$, the above equations can be simplified as

\[
\begin{align*}
\tau^2 \partial_{\tau} + 2\tau r_a \partial_{r_a} - r_a \xi_a - r_c \xi_c & = 0 \\
\sigma^2 + 2\sigma r_b \partial_{r_b} - r_b \xi_b & = 0 \\
\sigma \partial_{\sigma} + r_b \partial_{r_b} & = 0
\end{align*}
\]

or

\[
\begin{align*}
\sigma^2 + 2\sigma r_b \partial_{r_b} - r_b \xi_b - r_c \xi_c & = 0 \\
\sigma \partial_{\sigma} + r_b \partial_{r_b} & = 0 \\
\tau^2 + 2\tau r_a \partial_{r_a} - r_a \xi_a & = 0 \\
\tau \partial_{\tau} + r_a \partial_{r_a} & = 0
\end{align*}
\]

The general solution of these equations are found by using the method of characteristic \[12\].

\[
G = \delta_{\Delta_a+\Delta_b,\Delta_c}(t_a - t_c)^{-2\Delta_a}(t_b - t_c)^{-2\Delta_b} \exp\left(\frac{r_a \xi_a}{t_a - t_c} + \frac{r_b \xi_b}{t_b - t_c}\right)
\]

\[
(\chi_1 \frac{r_a r_c}{(t_a - t_c)^2} \exp\left(-\frac{r_c \xi_c}{t_a - t_c}\right) + \chi_2 \frac{r_b r_c}{(t_b - t_c)^2} \exp\left(-\frac{r_c \xi_c}{t_b - t_c}\right))
\]

+ $\Sigma$ (exchanging $b \leftrightarrow c$ or $a \leftrightarrow c$)

where $\chi_1$ and $\chi_2$ are arbitrary functions. Three-point function near the boundary is different from other places. In the following, we calculate three-point function for a situation with boundary condition at fixed time. Similar to section (4) three-point function \[21\] is kept invariant by the subalgebra $[M_{-1}, M_0, M_1, L_0]$. Invariance under the spatial translation which is generated by $M_{-1}$ implies $G = G(r, s, t_a, t_b, t_c)$ with $r = r_a - r_c$ and $s = r_b - r_c$. Invariance under non-relativistic Boost transformation is demanded

\[
< 0 | [M_0, \phi_a \phi_b \phi_c] | 0 >= 0
\]

\[
\sum_{i=a}^{i=c} (-t_i \partial_{t_i} + \xi_i) G = (-t_a \partial_{t_a} + \xi_a - t_b \partial_{t_b} + \xi_b - t_c \partial_{t_c} + \xi_c) G
\]

\[
= (-t_a - t_c) \partial_{t_r} - (t_b - t_c) \partial_{t_s} + \xi_a + \xi_b + \xi_c) G = 0
\]

so, three-point function reads

\[
G = G'(t_a, t_b, t_c) \exp\left(\frac{r_c \xi_c}{t_a - t_c} + \frac{s \xi_a}{t_b - t_c}\right)
\]
Space special Galilean conformal invariance demands that
\[ <0 | [M_1, \phi_a \phi_b \phi_c] | 0 > = 0 \quad (30) \]
\[ \Rightarrow \sum_{i=a}^c (-t_i^2 \partial_{t_i} + 2t_i \xi_i) G = 0 \]
which gives
\[ \xi_c = 2\xi_b \quad \xi_a = \xi_b \quad (31) \]
Finally we demand invariance under non-relativistic dilatation
\[ <0 | [L_0, \phi_a \phi_b \phi_c] | 0 > = 0 \quad (32) \]
\[ \Rightarrow \sum_{i=a}^b (t_i \partial_{t_i} + r_i \partial_i + \Delta_i) G \]
\[ = (t_a \partial_{t_a} + 2r_a \partial_a + \Delta_a + t_b \partial_{t_b} + r_b \partial_b + \Delta_b + t_c \partial_{t_c} + r_c \partial_c + \Delta_c) G \]
\[ = (t_a \partial_{t_a} + t_b \partial_{t_b} + t_c \partial_{t_c} + r \partial_r + s \partial_s + \Delta_a + \Delta_b + \Delta_c) G = 0 \]
\[ \Rightarrow (t_a \partial_{t_a} + t_b \partial_{t_b} + t_c \partial_{t_c} + \Delta_a + \Delta_b + \Delta_c) G' = 0 \]
we find
\[ G' = t_a^{-(\Delta_a+\Delta_b+\Delta_c)} \Phi_1(t_b t_c) + t_b^{-(\Delta_a+\Delta_b+\Delta_c)} \Phi_2(t_a t_c) + t_c^{-(\Delta_a+\Delta_b+\Delta_c)} \Phi_3(t_a t_c) \quad (33) \]
where \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) are arbitrary functions. The general solution is
\[ G(r_a, r_b, r_c, t_a, t_b, t_c) = \delta_{\xi_a, \xi_b} \delta_{\xi_c, \xi_a + \xi_b} \exp\left[ \frac{(r_a - r_c)\xi_c}{t_a - t_c} + \frac{(r_b - r_c)(\xi_a + \xi_b)}{t_b - t_c} \right] \quad (34) \]
\[ \left[ t_a^{-(\Delta_a+\Delta_b+\Delta_c)} \Phi_1(t_b t_c) + t_b^{-(\Delta_a+\Delta_b+\Delta_c)} \Phi_1(t_a t_c) + t_c^{-(\Delta_a+\Delta_b+\Delta_c)} \Phi_1(t_a t_c) \right] \]
\[ + \Sigma (\text{exchanging } b \leftrightarrow c \text{ or } a \leftrightarrow c) \]
The above result is different from three-point function of the GCA without boundary condition.

6 Conclusion

We can use finite Galilean conformal group to constrain two-point and three-point functions. Correlation functions of Galilean conformal invariant fields in the bulk out of boundary were found in [19]. Correlation functions near
the boundary are different from other places, and in real situation there are boundary conditions in space coordinates. When a system is in an initial state and relaxes toward its critical equilibrium considered as a situation with a boundary condition at fixed time. In this paper we considered two real situations:

1. A system with boundary in space coordinate \( r = 0 \) was considered in sections (3), (5).

2. A system with boundary condition at fixed time was considered in sections (4), (5).

The main results of this paper are the explicit expressions for Galilean conformal invariant correlation functions in a semi-infinite geometry as given in Eqs. (12), (20), (27) and (34). We calculated two-point function with boundary conditions at fixed time and surface \( r = 0 \), the form of two-point functions (12), (20) obviously are different with the corresponding results found from Galilean conformal invariance without boundary condition [19]. The form of three-point functions are obviously different with corresponding results found from GC-invariant without boundary condition [19].

Since Galilean conformal symmetry is related to massless (small viscosity) non-relativistic systems, one of the applications of GCA is considering nonrelativistic conformal hydrodynamics with small viscosity. One can use the Eqs. (12), (27) and (20), (34) for nonrelativistic conformal hydrodynamics with boundary condition in surface \( r = 0 \) and fixed time respectively. We know that in 2-dimensional space, there is a so-called exotic central extension of the GCA [26]. Martelli and Tachikawa [27] looked into the two-point functions in the bulk, but what happens close to a surface no-one has studied yet, as far as we know. We keep this interesting study for our future work in this topic.

7 Acknowledgments

We thank Prof. Malte Henkel for reading the paper and helpful comments and suggestions.

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