Mixed pooling of seasonality in time series pallet forecasting

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Abstract

Multiple seasonal patterns play a key role in time series forecasting, especially for business time series where seasonal effects are often dramatic. Previous approaches including Fourier decomposition, exponential smoothing, and seasonal autoregressive integrated moving average (SARIMA) models do not reflect the distinct characteristics of each period in seasonal patterns, such as the unique behavior of specific days of the week in business data. We propose a multi-dimensional hierarchical model. Intermediate parameters for each seasonal period are first estimated, and a mixture of intermediate parameters is then taken, resulting in a model that successfully reflects the interactions between multiple seasonal patterns. Although this process reduces the data available for each parameter, a robust estimation can be obtained through a hierarchical Bayesian model implemented in Stan. Through this model, it becomes possible to consider both the characteristics of each seasonal period and the interactions among characteristics from multiple seasonal periods. Our new model achieved considerable improvements in prediction accuracy compared to previous models, including Fourier decomposition, which Prophet uses to model seasonality patterns. A comparison was performed on a real-world dataset of pallet transport from a national-scale logistic network.

Keywords: Forecasting, Logistics, Multiple Seasonality, Bayesian Hierarchical Model

1. Introduction

Seasonality is the main component of time series, and the consideration of seasonality has become more important with the increasing frequency of time series produced in industry. For example, business time series data are collected and recorded on a much shorter interval. The number of web page users, the amount of electricity used, and the amount withdrawn from cash dispensers are representative examples of business time series data that had been previously collected on a quarterly or monthly basis, but is now aggregated weekly, daily, or even hourly (Hyndman, 2018). The emergence of high-frequency data, with time series containing closely spaced time intervals, necessitates a model capable of accurately modeling seasonal patterns of time series.

Two characteristics of seasonality are the main motivation for this paper: periodic scale jumps and interactions between seasonal patterns. Business time series often contain periodic scale jumps, such as
drops in trade volume over the weekend or extreme spikes due to stock adjustments immediately before the end-of-month evaluation. Previous seasonality models that fit entire seasonal periods with a single set of parameters can hardly capture these extreme patterns. Model structures enforce trend and seasonality parameter to be shared among different seasonal periods which make them inflexible. Moreover, business time series generally exhibit multiple seasonal patterns. As each seasonal pattern has distinct periods and effects, it is not trivial to design a model that can capture multiple seasonal patterns at once.

It has become essential to consider these two factors as the frequency of data collection has increased, as scale jumps and interactions occur compositely among multiple seasonalities. Despite the increasing need to model seasonality intricately, few models directly address these seasonal traits. Previous approaches to multiple seasonal patterns include Fourier decomposition, exponential smoothing, and seasonal autoregressive integrated moving average (SARIMA) models. These approaches often fail to reflect the distinct characteristics of each period, such as the unique behavior of specific days of the week in business data.

Apart from the two specific seasonal characteristics, the increased variety of recent data also motivates our research. Previously, many time series models that “adaptively” or “automatically” learn data have been suggested, with illustrative examples including an auto-ARIMA model that searches through a range of different parameters and selects the best model and the Prophet forecasting model, which utilizes a Bayesian based curve fitting method to learn trend and seasonality parameters. In addition, many applied automatic frameworks for specific types of time series have been introduced (Gooijer and Hyndesman, 2006). Most of these methods, although they greatly alleviate the burden of forecasting, have failed to adaptively detect multiple seasonal effects. Therefore, there is a need for a model that could automatically capture various seasonal patterns from the data.

We suggest a new time series forecasting framework, namely the mixed hierarchical seasonality (MHS) model. This approach makes use of the hierarchical structure of years, quarters, months, weeks, days, and hours inherent to time. Both the accuracy and interpretability of predictions are improved through this model, as discussed in greater depth below. A Bayesian hierarchical model is the main theoretical basis of our model. According to Gelman et al. (2013), modeling data generated in a hierarchical structure as a non-hierarchical model may result in overfitting. Through the hierarchical structure, the aforementioned two problems of seasonality can be improved. We solve the problem of scale jumps by separating parameters for each seasonal component, such as day of the week or day of the month. Each parameter is fit using the hierarchical model to achieve partial pooling. As for multiple seasonal patterns, we construct a mixed pooling model that estimates the final parameter distributions by taking a mixture of intermediate parameter distributions for each seasonal pattern.

The proposed model could bring great benefits to business management, as the degree to which seasonality patterns are reflected has a major influence on prediction accuracy, especially in demand time series. This is because demand time series are mainly generated by the direct influence of
customers’ cyclical activities, and the multiple seasonalties are relatively clear (Taylor & Letham, 2018). Demand forecasting is often performed as the basis for optimizing subsequent decisions. Therefore, if a predictive model yields analytical results in addition to the predicted results, it is possible to make more effective decisions in the next step. Periodic components among time series are important for decision-makers (Stevenson, 2015). For example, in the field of supply chain management, one purpose of demand forecasting is to prepare an appropriate supply. As demand forecasts become more accurate, even though many demand forecasting tasks are now automated, it is not easy to automate the entire process of supply chain management decision-making. Therefore, decision-makers often make final decisions based on analyses and forecasts provided to them. In this context, MHS which enhance the interpretability of the forecast, especially in seasonal respect, contribute to efficient management; as it can help decision-makers, who are usually suppliers, to better understand the system, control sales to buyers, and properly manage inventories (Stevenson, 2015).

The following paper consists of five further sections. Section 2 introduces the key concepts upon which the MHS model is based, and section 3 explains the partial pooling model and the mixed partial pooling model. In section 4, real-world pallet transport data is introduced and experimental models are described. Section 5 contains an analysis of the experimental models, and lastly, conclusions are presented in section 6.

2. Literature Review

2.1. Past approaches to multiseasonal time series

Multiple seasonal patterns in demand time series vary widely across industries. For example, in the clothing industry (e.g., swimwear), the quarterly periodicity is more pronounced than the daily periodicity, and the weekly periodicity in the food industry (e.g., bakeries) is more influential than the monthly periodicity. However, current time series models are not able to find seasonally adaptive data. In order to reflect periodicity, it is first necessary to decompose components corresponding to the periodicity. However, the X11 and seasonal extraction in ARIMA time series (SEATS) methods for time series decomposition have the disadvantage of only reflecting monthly and quarterly periodicity (Dagum and Bianconcini, 2016). In contrast, the seasonal and trend decomposition using Loess (STL) method can reflect various and time-varying periodic components. In addition, the decision-maker can adjust the trend-cycle smoothness of the trend cycle, and this method is robust to outliers (Cleveland et al., 1990). However, the STL method cannot automatically reflect the volatility of the calendar, and it has the disadvantage of only being applicable when a time series is in a decomposable form (Hyndman and Athanasopoulos, 2018).

Fourier regression is a popular method for modeling single or multiple seasonality. Regression of a time series with Fourier terms $X_t$ and regression parameter $\beta$ is modeled as below:

$$y(t) \sim N(k_t + X_t \cdot \beta + m_t, \sigma)$$
The Fourier term $X_t$ can be constructed as:

$$X_t = \left[ \frac{\cos (2\pi t)}{P}, \frac{\sin (2\pi t)}{P}, \ldots, \frac{\cos (2\pi nt)}{P}, \frac{\sin (2\pi nt)}{P} \right]$$

where $P$ is the regular period of the seasonality and $n$ is the order of the Fourier series.

Taylor and Letham (2018) compared the performance of a model based on the Fourier seasonality with \textit{tbats}, \textit{auto.arima}, and \textit{ets}. They suggested that \textit{tbats} and \textit{auto.arima} are not effective for detecting multiple time series, and that Fourier regression can detect multiple seasonality better than the \textit{ets} method.

2.2. Cluster-then-predict models

Cluster then predict forecasting is a method of creating independent prediction models for each cluster after clustering the population. Bertsimas et al. (2008) explained that since the health patterns of patients leading to heart failure vary widely, it is more accurate to group the patients’ data first and then create a model based on each cluster after collecting similar patients. Venkatesh et al. (2013) clustered the automatic withdrawal period of regions using a time series pattern and then made independent predictions for each community. The clustering criterion used by Venkatesh was daily cash withdrawal, grouped by the sequence-alignment method (SAM) with a sequence of length 7, which discretized the day-of-the-week effect. This method yielded more accurate predictions than the previous method of creating predictive models, without clustering preprocessing. Based on the clustering results, operators can make cash replenishment plans more efficiently.

The cluster-then-predict model provides a background for the mixed hierarchical seasonal model, as discrete models should be constructed for data with different traits. However, clustering could be too inflexible in some cases, because it does not allow for ‘partial’ divisions. This inflexibility could be improved through the concept of partial pooling in a hierarchical model.

2.3. Hierarchical model

Gelman et al. (2005) explained that hierarchical models are highly predictive because of partial pooling. When updating the model parameters, such as prior parameters, the relationship between the part of the data being used and the whole population should always be considered. If the focus is limited to the part, over-fitting occurs, whereas when the focus is only on the whole, under-fitting occurs. In a Bayesian hierarchy, the balance of fit can be learned by using hyperpriors. By properly setting the hyperprior structure, we can find a reasonable balance between over-fitting and under-fitting, as hyperpriors are known to serve as a regularizing factor. Many examples of applying hierarchical structure in cross-sectional data exist, including the eight-schools and rat tumor experiments. The structure of cross-sectional data where the whole population is divided into multiple and nested subcategories provides an excellent environment for a hierarchical model. A specific example is furnished by the structure of the eight-schools experiment, where one class belongs to one school and one school belongs to one city.
In the time series domain, the term “hierarchical time series” is not necessarily related to partial pooling. Hyndman and Athanasopoulos (2018) describe a “hierarchical time series” as time series data that can be decomposed and combined according to a category. The decomposition is often caused primarily by geographical differences, and each category may be part of a larger category, so that a bundle of time series data has a hierarchical group structure. Examples include decomposing the number of Australian tourists by city, and selling bicycles according to product type, such as mountain bicycles, general bicycles, and infant bicycles. Moreover, the concept of grouped time series was developed to explain that the hierarchical structure of time series is not unique. Taieb et al. (2017) presented various methods of combining these structures harmoniously to reach a final forecast. Hyndman et al. (2011) introduced optimal decomposition and aggregation in a time series hierarchy.

2.4. Model evaluation measures

Information criteria can be used to measure the fit of a model. To estimate pointwise out-of-sample prediction accuracy in a Bayesian model, widely applicable information criterion (WAIC) and the leave-one-out cross-validation (LOOCV) are preferred to Akaike information criterion (AIC) and deviance information criterion (DIC). Due to computational problems, approximate LOOCV methods exist, including Pareto smoothed importance sampling (PSIS-LOO) which are implemented in R package called loo (Vehtari et al. 2017). However, k-fold cross-validation could instead be used if diagnostics hint that importance weights are not good or if data points have certain dependencies (Vehtari and Lampinen, 2002). The diagnostics loo package provides is Pareto tail shape parameter, k.

Time series cross-validation and k-fold cross-validation, along with the expanding forecast method, can be used to measure forecast accuracy in time series (Hyndman and Athanasopoulos, 2018). Several sets of training and test data are created in a walk-forward mode, and forecast accuracy is computed by averaging over the test sets. Various measures of forecast error exist, including the mean absolute, root mean squared, and mean absolute percentage error. When a large difference of scale exists in the data, using a scaled error measure is recommended. The mean absolute scaled error is recommended for comparing forecast accuracy across multiple time series (Hyndman & Koehler, 2006).

3. Mixed Hierarchical Seasonality Model and Data Analysis

Previous hierarchical models were mainly attempted on cross-sectional data, and no studies have focused on hierarchical structure in time series data with multiple seasonality. However, seasonal (or recurring) characteristics of days of the week and month can be regarded as parallels to subcategories in cross-sectional data, and a considerable increase in accuracy is expected when the concept of a hierarchy is applied to model multiple seasonal patterns in time series data. The same assertion made in many previous studies regarding the effectiveness of cross-sectional hierarchical models can therefore be applied to the time series domain. Therefore, in this paper we focus on modeling the seasonal component of time series using a hierarchical model. Two previously suggested problems of original seasonal models (shared trend and seasonal parameters for each seasonal component and
failure to model interactions between the week and month) can be improved by using the partial pooling framework.

To introduce the concept of mixed hierarchical model, three models of seasonality are compared. These models focus on how to manage seasonal parameters in terms of pooling, as follows: complete, partial, and mixed. For the Stan code (Carpenter et al. 2017) of the entire model, see Appendix.

The results are compared with those obtained using a Fourier decomposition model, which is a popular method for modeling single or multiple seasonalities that is used by Prophet. The classical time series decomposition of data into trend, seasonal, and irregular components is used. Our base model assumes a time series with a linear trend. It models the value $y(t)$ at time $t$ as follows:

$$y(t) = kt + m + \epsilon_t$$  \hspace{1cm} (1)

where $k$ is the growth rate, $m$ is the offset parameter, and $\epsilon_t$ is the error term.

3.1. Complete pooling model

In the complete pooling model, all data points share a single set of parameters $k$ and $m$. It can be represented as:

$$y(t) \sim N(kt + m, \sigma)$$  \hspace{1cm} (2)

Entire seasonal periods are fit with a single set of parameters in complete pooling. Extreme seasonal patterns, such as periodic scale jumps, are hardly captured. This observation leads to the suggestion of a partial pooling model.

3.2. Partial pooling model

In a partial pooling model, we group data points by periods in seasonality, such as day of the week or day of the month. Each period $i$ in seasonality has its own set of parameters $k_i$ and $m_i$, and those parameters share single set of priors $N(\mu_k, \sigma_k)$ and $N(\mu_m, \sigma_m)$. This forms a hierarchical model where parameters share information through their priors. This structure can be represented as:

$$k_i \sim N(\mu_k, \sigma_k)$$  \hspace{1cm} (3a)

$$m_i \sim N(\mu_m, \sigma_m)$$  \hspace{1cm} (3b)

$$y(t) \sim N(k_i t + m_i, \sigma)$$  \hspace{1cm} (3b)

Just as different hierarchical structures exist in cross-sectional data, time series data could also be represented with more than one structure. For daily data, pooling could be done between parameters
reflecting days of the week, days of the month, or even both. For pooling based on days of the week, the total data is divided based on the day of the week (from Monday to Sunday) into 7 submodels, while 31 submodels are created for pooling based on the day of the month. Note that considering the months with 28, 29 or 30 days, only the available data is used for days of the month models. For example, only the 31th day of the month is used for the 31th submodel.

However, in reality, certain days reflect the result of mixed seasonal effects of both days of the week and days of the month; the effect of multiple seasonalities is not the simple sum of each seasonal effect in this case. Interactions exist, leading to the last model – the mixed pooling model.

3.3. Mixed pooling model

The partial pooling model provides a distinct set of parameters for each seasonal period. However it becomes unclear how to propose a likelihood for an observed value \( y \) when we model a time series with more than one seasonal pattern. In the mixed pooling model, we take the linear mixture of each parameter to obtain a single set of parameters. The seasonal contribution factor, or mixture weight \( \theta \) represents the contribution of each seasonal pattern to the time series. It models \( k_{ij} \) \( m_{ij}, y(t) \) at time \( t \) as follows:

\[
\begin{align*}
    k_{ij} &\sim N(\mu_k, \sigma_k) \\
    m_{ij} &\sim N(\mu_m, \sigma_m) \\
    y(t) &\sim N((k_i * \theta)t + (m_i * \theta), \sigma)
\end{align*}
\]

Parameter \( \theta \) can be regarded as a factor reflecting the contribution of each seasonal pattern. If the data displays stronger weekly seasonality than monthly seasonality, the corresponding component of \( \theta \) is bigger. It should be noted that \( \theta \) is modeled as a simplex vector, the components of which sum to 1. We named this model, mixed hierarchical seasonality model.
4. Data and Experiments

To evaluate this model, we use a real-world dataset from a national-scale logistic network. Each daily time series represents the number of logistic units transported in certain logistic flows: delivery, restocking, and shipment. The specific form of the logistic unit is a pallet, a product container. We chose data from logistics because transport data displays strong and multiple seasonalities. For each set of data, the results obtained using a partial pooling models are compared with a baseline seasonal model, Fourier regression.

![Graphs showing values and autocorrelations of delivery, restocking, and shipment.](image)

Fig. 1. Values and autocorrelations of delivery, restocking, and shipment.

Fig. 1 shows a plot of the values and autocorrelations of the three datasets. Deliveries show a strong weekly periodicity, with a sharp fall observed every Sunday. Restocking shows a clear monthly periodicity, with extreme spikes observed at the beginning and end of every month. However, a graph of the autocorrelation obscures the periodicity because the length of each month differs. Shipments show a strong weekly periodicity, as well as a subtle monthly periodicity.

Based on the seasonal patterns observed, we constructed complete and partial pooling models for the delivery and restocking data and compared them with the Fourier decomposition model. Since the delivery data shows weekly seasonality and the restocking data shows monthly seasonality, the partial pooling models for the two datasets differ in that pooling was performed on the scale of weeks for the former, but on the scale of months for the latter. In other words, pooling was performed by day of the week for delivery data and by day of the month for restocking data. For the shipment data, which displays both weekly and monthly seasonality and therefore necessitates the mixed hierarchical seasonal model, all four models were fitted and compared. For the Fourier decomposition model, the period \( P \) and Fourier order \( n \) were set as 7 (days) and 3 for weekly seasonality and 30.4375 (days) and 5 for monthly seasonality. Both weekly and monthly seasonality were considered in the Fourier model for the shipment data.
The pooling models use the Newton algorithm for robust convergence of optimization. The Pareto tail shape parameter $k$ exceeded 0.7, indicating that PSIS-LOO would not be reliable. Therefore, expanding-window cross-validation was used. Although we had nine years of data from 2010 to 2018, immense change of trend was observed in 2015. Therefore, to clearly model the seasonality, we used recent two years of data, 2017-01-01 to 2018-12-31, which was enough fit the model. 12 folds of validation, with each fold corresponding to a month in 2018, were performed. The model was fitted using all past data of each validation set and accuracy was averaged. Since our dataset contains values with various scales, relative errors would be overly sensitive to data points at a tiny scale. Therefore, mean absolute percentage error was used.

5. Results and Discussion

5.1. Accuracy

The forecast results of the three datasets are shown in Table 1 and Fig. 2, 3, and 4.

Table 1. Forecast accuracy results of the pooling models and Fourier decomposition model

|                  | Complete pooling | Partial pooling | Mixed pooling | Fourier decomposition |
|------------------|------------------|-----------------|---------------|----------------------|
| **Delivery**     | 0.4099           | 0.1049          |               | 0.1084               |
| (week pool)      |                  | (week pool)     |               |                      |
| **Restocking**   | 1.6892           | 0.5872          |               | 1.3635               |
| (month pool)     |                  | (month pool)    |               |                      |
| **Shipment**     | 0.3828           | 0.2800          | 0.2126        | 0.2672               |
| (week pool)      |                  | (week pool)     |               |                      |
From Fig. 2, it can be seen that the complete pooling model does not consider weekly seasonality at all, because every data point shares a single set of parameters. The Fourier decomposition model and partial pooling model behave almost identically and a negligible difference in performance is observed.

The partial polling models shows a significantly higher performance than the other models. Fig. 3 shows that the Fourier decomposition model with a given Fourier order does not fit the extreme spikes correctly.
5.2. Parameters

5.2.1. Parameter divergence in partial pooling

In the delivery dataset, there is a sharp fall in the time series every Sunday (Fig. 1). The parameter plots from the partial pooling model in Fig. 5 shows that the probability distribution of the parameters $m$ and $k$ of the sixth day of the week is differentiated from the other parameters. Similarly, in the restocking dataset, the probability distributions of the parameters $m$ and $k$ at the beginning and end of the month are differentiated.
5.2.2. Seasonality contribution factor

The $\theta$ parameter in equation (4c) can be used to interpret the effect of each seasonal pattern, as well as to account for the interaction of seasonal patterns. In the left of Fig. 6, posterior distribution of theta for the delivery dataset where monthly seasonality is hardly observed, the weight for monthly seasonality is distributed near zero. A seasonal pattern with higher mixture weight has a greater effect on the time series. In Fig. 6, the left plot shows probability distribution of $\theta$ of the weekly seasonality parameters and monthly seasonality parameters in the shipment dataset is shown. The maximum a posteriori estimates of each weight are about 0.6 and 0.4, so it can be inferred that the weekly seasonality has an linear effect of about twice the monthly seasonality.

![Fig. 6. Posterior distribution of seasonal factors for delivery(left) and shipment(right)](image)

6. Conclusions

We have proposed MHS, a hierarchical model for time series data with multiple seasonal patterns. We demonstrated the applicability of the model using a real-world dataset of pallet movement from a logistic network in comparison to simpler models and models using previous methods. Through these comparisons, we confirmed that the prediction performance of our novel model in the given dataset was greatly improved. Moreover, we have shown that the fitted parameters can reasonably explain the contribution of each seasonal pattern to the time series.

Although only the seasonal periods of week and month are introduced and modeled in this paper, other periods ranging from the minute, hour, and day to the quarter or year could be modeled using MHS; it is possible to choose different seasonal periods and to choose the number of seasonal periods to be considered compositely, meaning that the resulting model would be a true multidimensional hierarchical model.

Two improvements could be noted for further studies. First is trend change detection. Current model could be improved so that immense change of the trend could be automatically detected and reflected in the trend parameter. This is important as the time series is a combined result of trend and seasonality. Accurate trend model as well as clear distinction between the trend and seasonal effects is needed for better seasonality model. Second is using various evaluation measures. PSIS-LOO could not be used in this paper as Pareto
tail shape parameter \( k \) exceeded 0.7. However, comparing models on diverse measures lead to reliable results. As it is noted in the previous literature that large \( k \) could indicate the need to make the model more robust, different information criteria methods could be used by improving the robustness of the model; this can be achieved by applying different priors or reparameterization.

MHS utilized the strengths of the hierarchical Bayesian approach and was able to achieve both enhanced estimation robustness and model interpretability. The new model which addresses seasonality in a nuanced manner, could bring great benefits to business management, where seasonal patterns are salient. It could efficiently support the data-driven decision making process.

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**Appendix: Implementation in Stan**

1. Complete pooling model

```stan
data {
  int<lower=1> T;
  vector[T] t;
  vector[T] y;
}
parameters {
  real k;
  real m;
  real<lower=0> sigma_obs;
}
transformed parameters {
  vector[T] yhat;
  yhat = k * t + m;
}
model {
  k ~ normal(0, 5);
  m ~ normal(0, 5);

  y ~ normal(yhat, sigma_obs);
}
```

2. Partial pooling model

```stan
data {
  int<lower=1> T;
  int<lower=1> P;
  vector[T] t;
  vector[T] y;
}
parameters {
  real<lower=0> sigma_t;
}
transformed parameters {
  real<lower=0> sigma_y;
  sigma_y = sigma_t / P;
}
model {
  for (t in 1:T) {
    y ~ normal(k * t + m, sigma_y);
  }
  for (p in 1:P) {
    k[p] ~ normal(0, 5);
    m[p] ~ normal(0, 5);
  }
  sigma_t ~ exponential(1);
}
```

int pool[T];
}
parameters {
  real k_mu;
  real<lower=0> k_sigma;
  real m_mu;
  real<lower=0> m_sigma;
  real k[P];
  real m[P];
  real<lower=0> sigma_obs;
}
transformed parameters {
  vector[T] p_k;
  vector[T] p_m;
  vector[T] yhat;
  for (i in 1:T) {
    p_k[i] = k[pool[i]];
    p_m[i] = m[pool[i]];
  }
  yhat = p_k .* t + p_m;
}
model {
  k_mu ~ normal(0, 5);
  k_sigma ~ exponential(1);
  m_mu ~ normal(0, 5);
  m_sigma ~ exponential(1);
  k ~ normal(k_mu, k_sigma);
  m ~ normal(m_mu, m_sigma);
  sigma_obs ~ normal(0, 0.5);
  y ~ normal(yhat, sigma_obs);
}

3. Mixed model

data {
  int<lower=1> T;
  int<lower=1> D;
  int<lower=1> P;
  vector[T] t;
  vector[T] y;
  int pool[T, D];
}
parameters {
  real k_mu;
  real<lower=0> k_sigma;
  real m_mu;
  real<lower=0> m_sigma;
  real k[D, P];
  real m[D, P];
  simplex[D] theta;
  real<lower=0> sigma_obs;
}
transformed parameters {
    matrix[T, D] p_k;
    matrix[T, D] p_m;
    vector[T] yhat;
    for (i in 1:T) {
        for (d in 1:D) {
            p_k[i, d] = k[d, pool[i, d]];
            p_m[i, d] = m[d, pool[i, d]];
        }
    }
    yhat = (p_k * theta) .* t + (p_m * theta);
}

model {
    k_mu ~ normal(0, 5);
    k_sigma ~ exponential(1);
    m_mu ~ normal(0, 5);
    m_sigma ~ exponential(1);

    for (d in 1:D) {
        k[d] ~ normal(k_mu, k_sigma);
        m[d] ~ normal(m_mu, m_sigma);
    }
    sigma_obs ~ normal(0, 0.5);

    y ~ normal(yhat, sigma_obs);
}