ON THE DYNAMICS OF MINIMAL HOMEOMORPHISMS OF $\mathbb{T}^2$ WHICH ARE NOT PSEUDO-ROTATIONS

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Abstract. We prove that any minimal 2-torus homeomorphism which is isotopic to the identity and whose rotation set is not just a point exhibits uniformly bounded rotational deviations on the perpendicular direction to the rotation set. As a consequence of this, we show that any such homeomorphism is topologically mixing and we prove Franks-Misiurewicz conjecture under the assumption of minimality.

1. Introduction

The study of the dynamics of orientation preserving circle homeomorphisms has a long and well established history that started with the celebrated work of Poincaré [Poi80]. If $f : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denotes such a homeomorphism and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a lift of $f$ to the universal cover, he showed there exists a unique $\rho \in \mathbb{R}$, the so called rotation number of $\tilde{f}$, such that

$$\frac{\tilde{f}^n(z) - z}{n} \to \rho, \quad \text{as } n \to \infty, \; \forall z \in \mathbb{R},$$

where the convergence is uniform in $z$. Moreover, in this case a stronger (and very useful, indeed) condition holds: every orbit exhibits uniformly bounded rotational deviations, i.e.

$$|\tilde{f}^n(z) - z - n\rho| \leq 1, \quad \forall n \in \mathbb{Z}, \; \forall z \in \mathbb{R}.$$

In this setting, the homeomorphism $f$ has no periodic orbit if and only if the rotation number is irrational; and any minimal circle homeomorphism is topologically conjugate to a rigid irrational rotation.

However, in higher dimensions the situation dramatically changes. If $f : \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ is a homeomorphism homotopic to the identity and $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ is a lift of $f$, then one can define its rotation set by

$$\rho(\tilde{f}) := \left\{ \rho \in \mathbb{R}^d : \exists n_k \uparrow +\infty, \; z_k \in \mathbb{R}^d, \; \rho = \lim_{k \to +\infty} \frac{\tilde{f}^{n_k}(z_k) - z_k}{n_k} \right\}.$$

This set is always compact and connected, and as we mentioned above, it reduces to a point when $d = 1$. But for $d \geq 2$ some examples with larger rotation sets can be easily constructed.

In the two-dimensional case, which is the main scenario of this work, Misiurewicz and Ziemian showed in [MZ89] that the rotation set is not just connected but convex. So, when $d = 2$ all torus homeomorphisms of the identity isotopy class can be classified according to the geometry of their rotation sets: they can either have non-empty interior, or be a non-degenerate line segment, or be just a point. In the last case, such a homeomorphism is called a pseudo-rotation.

Regarding the boundedness of rotational deviations, this property has been shown to be very desirable in the study of the dynamics of pseudo-rotations (see for instance the works of Jäger and collaborators [J09, Jäg09, JT17]). However, it
has been proved in [KK09] and [KT14a] that, in general, pseudo-rotations does not exhibit bounded rotational deviations in any direction of \(\mathbb{R}^2\), i.e. it can hold
\[
\sup_{z \in \mathbb{R}^2, n \in \mathbb{Z}} \langle \bar{f}^n(z) - z - n\rho, v \rangle = +\infty, \quad \forall v \in \mathbb{S}^1.
\]

When \(\rho(\bar{f})\) is a (non-degenerate) line segment, of course there exist points with different rotation vectors, so we cannot expect to have any boundedness at all for rotational deviations on the plane. However, in such a case there exists a unit vector \(v \in \mathbb{S}^1\) and a real number \(\alpha\) such that \(\rho(\bar{f})\) is contained in the line \(\{z \in \mathbb{R}^2 : \langle z, v \rangle = \alpha\}\), so one can analyze the boundedness of rotational deviations, i.e. whether there exist constants \(M(z) \in \mathbb{R}\) such that
\[
|\langle \bar{f}^n(z) - z - n\rho, v \rangle| = |\langle \bar{f}^n(z) - z, v \rangle - n\alpha| \leq M(z), \quad \forall n \in \mathbb{Z},
\]
and any \(\rho \in \rho(\bar{f})\).

Unlike the case of pseudo-rotations, when \(\rho(\bar{f})\) is a non-degenerate line segment in general it is expected to have uniformly bounded rotational \(v\)-deviations, i.e. the constant \(M(z)\) can be taken independently of \(z\). This result has already proved by Dávalos [Dáv16] in the case where \(\rho(\bar{f})\) has rational slope and intersects \(\mathbb{Q}^2\), extending a previous result of Guelman, Koropecki and Tal [GKT14]. In those works periodic orbits of \(f\) play a key role.

However, the situation is considerably subtler when dealing with periodic point free homeomorphisms. So far there did not exist any \textit{a priori} boundedness of rotational deviations of torus homeomorphisms which are not pseudo-rotations and with no periodic points. In fact, it had been conjectured that any periodic point free homeomorphism should be a pseudo-rotation. More precisely, Franks and Mi- siurewicz had proposed in [FM90] the following

**Conjecture 1.1** (Franks-Misiurewicz Conjecture). Let \(f : T^2 \rightarrow T^2\) be a homeomorphism homotopic to the identity and \(\bar{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) be a lift of \(f\) such that \(\rho(\bar{f})\) is a non-degenerate line segment.

Then, the following dichotomy holds:

(i) either \(\rho(\bar{f})\) has irrational slope and one of its extreme points belongs to \(\mathbb{Q}^2\);

(ii) or \(\rho(\bar{f})\) has rational slope and contains infinitely many rational points.

Recently Avila has announced the existence of a minimal smooth diffeomorphism whose rotation set is an irrational slope segment containing no rational point, providing in this way a counter-example to the first case of Franks-Misiurewicz Conjecture. On the other hand, Le Calvez and Tal have proved in [LCT18] that if \(\rho(\bar{f})\) has irrational slope and contains a rational point, then this point is an extreme one.

The second case of Conjecture 1.1 remains open, i.e. whether there exists a homeomorphism \(f\) such that \(\rho(\bar{f})\) has rational slope and \(\rho(\bar{f}) \cap \mathbb{Q}^2 = \emptyset\), and in fact this is one of the main motivations of our work.

The main result of this paper is the following \textit{a priori boundedness} for rotational deviations of minimal homeomorphisms:

**Theorem A.** Let \(f : T^2 \rightarrow T^2\) be a minimal homeomorphism homotopic to the identity which is not a pseudo-rotation. Then there exists a unit vector \(v \in \mathbb{R}^2\) and a real number \(M > 0\) such that for any lift \(\bar{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2\), there is \(\alpha \in \mathbb{R}\) so that
\[
|\langle \bar{f}^n(z) - z, v \rangle - n\alpha| \leq M, \quad \forall z \in \mathbb{R}^2, \quad \forall n \in \mathbb{Z}.
\]

As a consequence of Theorem A and a recent result due to Koropecki, Passeggi and Sambarino [KPS16], we get a proof of the second case of Franks-Misiurewicz Conjecture (Conjecture 1.1) under minimality assumption. More precisely we get the following:
Theorem B. There is no minimal homeomorphism of $\mathbb{T}^2$ in the identity isotopy class such that its rotation set is a non-degenerate rational slope segment.

As a consequence of Theorem B, some results of [KPR18] and a recent generalization of a theorem of Kwapisz [Kwa02] due to Beguin, Crovisier and Le Roux [BCLR17], we have the following

Theorem C. If $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a minimal homeomorphism homotopic to the identity and is not a pseudo-rotation, then $f$ is topologically mixing.

Moreover, in such a case the rotation set of $f$ is a non-degenerate irrational slope line segment and its supporting line does not contain any point of $\mathbb{Q}^2$.

1.1. Strategy of the proof Theorem A. Theorem A is certainly the most important result of the paper and its proof is rather long and technical. So, for the sake of readability, here we summarize the main steps of the proof in a rather informal way.

We proceed by contradiction. First of all one can observe that there is no loss of generality assuming the rotation set $\rho(f)$ is transversal to the horizontal axes, i.e. it intersects the upper and lower horizontal semi-planes (see Propositions 2.5 and 2.16 for details). This means there exist points with positive asymptotic vertical mean speed and others with negative one.

Then we define the stable sets at infinity $\Lambda^+_h, \Lambda^-_h \subset \mathbb{R}^2$ as the unbounded connected components of the maximal $f$-invariant sets of the upper and lower semi-plane, respectively. More precisely,

$$\Lambda^+_h := \text{cc} \left( \{ z \in \mathbb{R}^2 : \text{pr}_2(f^n(z)) \geq 0, \forall n \in \mathbb{Z} \} , \infty \right)$$

and

$$\Lambda^-_h := \text{cc} \left( \{ z \in \mathbb{R}^2 : \text{pr}_2(f^n(z)) \leq 0, \forall n \in \mathbb{Z} \} , \infty \right),$$

where $\text{pr}_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes the projection on the second coordinate and $\text{cc}(\cdot, \infty)$ the union of the unbounded connected components of the corresponding set. In $\S$ 5, we study the geometry of these stable sets at infinity, showing in particular that they are non empty (Theorem 5.1), and they are in fact the union of “infinitely long hairs”. Then, assuming estimate (1) is false, we show in Theorem 5.5 that these “hairs” exhibit arbitrarily large oscillations in the horizontal direction.

Then, in $\S$ 6 we define the stable sets at infinity but this time with respect to the direction determined by the rotation set. At this point some new important technical problems appear. In fact, the number $\alpha$ in (1) represents the mean asymptotic speed of every point with respect to the perpendicular direction to the rotation set, and we know a posteriori, by Theorem C, that it is always irrational, and in particular, non-zero. That means if we just define theses sets analogously to what we did above for $\Lambda^+_h$ and $\Lambda^-_h$, we shall just get empty sets. On the other hand, if we modify the definition writing

$$\Lambda^+_v := \text{cc} \left( \{ z \in \mathbb{R}^2 : \langle f^n(z), v \rangle - n \alpha \geq 0, \forall n \in \mathbb{Z} \} , \infty \right)$$

and

$$\Lambda^-_v := \text{cc} \left( \{ z \in \mathbb{R}^2 : \langle f^n(z), v \rangle - n \alpha \leq 0, \forall n \in \mathbb{Z} \} , \infty \right),$$

we get non-empty sets, but they are not dynamically defined. So, this is the reason why we have to introduce the fiber-wise Hamiltonian skew-products in $\S$ 6.1 in order to get these sets $\Lambda^+_v$ and $\Lambda^-_v$ as dynamical ones (see $\S$ 6.2 for details).

Then, always assuming that (1) does not hold, we use these sets to show that the induced fiber-wise Hamiltonian skew-product exhibits certain form of topologically mixing behavior along the fibers (Theorem 6.7). Then we use this dynamical information to show the sets $\Lambda^+_h, \Lambda^-_h, \Lambda^+_v, \Lambda^-_v$ are pairwise disjoint (Proposition 7.1).

Finally, we finish the proof showing that this disjointedness is incompatible with
the large horizontal oscillation of the connected components of the sets $\Lambda^+_h$ and $\Lambda^-_h$ we proved in Theorem 5.5.

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2. Preliminaries and notations

2.1. Maps, topological spaces and groups. Given any map $f : X \ni \omega$, we write $\text{Fix}(f)$ for its set of fixed points and $\text{Per}(f) := \bigcup_{n \geq 1} \text{Fix}(f^n)$ for the set of periodic ones. If $A \subseteq X$ denotes an arbitrary subset, we define its positively maximal $f$-invariant subset by

$$\mathcal{I}^+(A) := \bigcap_{n \geq 0} f^{-n}(A).$$

When $f$ is bijective, we can also define its maximal $f$-invariant subset by

$$(2) \quad \mathcal{I}(A) := \mathcal{I}^+(A) \cap \mathcal{I}^-(A) = \bigcap_{n \in \mathbb{Z}} f^n(A).$$

When $X$ is a topological space and $A \subseteq X$ is any subset, we write $\text{int} A$ for the interior of $A$ and $\text{cl} A$ for its closure. When $A$ is connected, we write $\text{cc}(X, A)$ for the connected component of $X$ containing $A$. As usual, $\pi_0(X)$ denotes the set of connected components of $X$. When $X$ is connected and $A \subseteq X$, we say that $A$ disconnects $X$ when $X \setminus A$ is not connected. Given two connected sets $U, V \subseteq X$, we say that $A$ separates $U$ and $V$ when $\text{cc}(X \setminus A, U) \neq \text{cc}(X \setminus A, V)$.

The space $X$ is said to be a continuum when is compact, connected and non-trivial, i.e. it is neither empty nor a singleton.

A homeomorphism $f : X \ni \omega$ is said to be non-wandering when given any non-empty open set $U \subseteq X$, there exists a positive integer $n$ such that $f^n(U) \cap U \neq \emptyset$.

We say that $f$ is topologically mixing when for every pair of non-empty open sets $U, V \subseteq X$, there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for every $n \geq N$.

The homeomorphism $f$ is said to be minimal when it does not exhibit any proper $f$-invariant closed set, i.e. $X$ and $\emptyset$ are the only closed $f$-invariant sets. If $(X, d)$ is a metric space, the open ball of radius $r > 0$ and center at $x \in X$ will be denoted by $B_r(x)$. Given an arbitrary set $A \subseteq X$ and a point $x_0 \in X$, we write

$$d(x_0, A) := \inf_{y \in A} d(x_0, y).$$

For any $\varepsilon > 0$, the $\varepsilon$-neighborhood of $A$ is given by

$$(3) \quad A_\varepsilon := \{ x \in X : d(x, A) < \varepsilon \} = \bigcup_{x \in A} B_\varepsilon(x).$$

The diameter of $A \subseteq X$ is defined by $\text{diam} A := \sup_{x, y \in A} d(x, y)$ and we say $A$ is unbounded whenever $\text{diam} A = +\infty$. Making a slight abuse of notation, we shall write $\text{cc}(A, \infty)$ to denote the union of the unbounded connected components of $A$.

The space of (non-empty) compact subsets of $X$ will be denoted by

$$\mathcal{K}(X) := \{ K \subseteq X : K \text{ is compact}, K \neq \emptyset \}.$$ 

and we endow this space with its Hausdorff distance $d_H$ defined by

$$(d_H(K_1, K_2) := \max \left\{ \max_{x \in K_1} d(x, K_2), \max_{y \in K_2} d(y, K_1) \right\},$$

for every $K_1, K_2 \in \mathcal{K}(X)$.

Whenever $M_1, M_2, \ldots, M_n$ are $n$ arbitrary sets, we shall use the generic notation $\text{pr}_i : M_1 \times M_2 \times \ldots \times M_n \rightarrow M_i$ to denote the $i$-th-coordinate projection map.
Finally, when $X$ is a compact topological space, we shall always consider the vector space of continuous functions $C^0(X, \mathbb{R}^d)$ endowed with the uniform norm given by
\[
\|\phi\|_{C^0} := \max_{1 \leq i \leq d} \max_{x \in X} |\text{pr}_i(\phi(x))|, \quad \forall \phi \in C^0(X, \mathbb{R}^d).
\]

2.2. **Topological factors and extensions.** Let $(X, d_X)$ and $(Y, d_Y)$ be two compact metric spaces. We say that a homeomorphism $f: X \subset Y$ is a **topological extension** of a homeomorphism $g: Y \subset X$ when there exists a continuous surjective map $h: X \to Y$ such that $h \circ f = g \circ h$; and we say $g$ is a **topological factor** of $f$. In such a case, $h$ is called a **semi-conjugacy**.

As usual, when $h$ is a homeomorphism, $f$ and $g$ are said to be **topologically conjugate**, and $h$ is said to be a **conjugacy**.

2.3. **Euclidean spaces, tori and the annulus.** We consider $\mathbb{R}^d$ endowed with its usual Euclidean structure, which is denoted by $\langle \cdot, \cdot \rangle$. We write $\|v\| := \langle v, v \rangle^{1/2}$ for its induced norm and $d(v, w) := \|v - w\|$ for its induced distance function.

The unit $(d - 1)$-sphere is denoted by $S^{d-1} := \{v \in \mathbb{R}^d : |v| = 1\}$. For any $v \in \mathbb{R}^d \setminus \{0\}$ and any $r \in \mathbb{R}$ we define the (open) **half-space**
\[
\mathbb{H}^v_r := \{z \in \mathbb{R}^d : \langle z, v \rangle > r\}.
\]

Given any $a \in \mathbb{R}^d$, we write $T_a$ for the translation $T_a : z \mapsto z + a$ on $\mathbb{R}^d$.

The $d$-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ will be denoted by $\mathbb{T}^d$ and we write $\pi: \mathbb{R}^d \to \mathbb{T}^d$ for the canonical quotient projection. We will always consider $\mathbb{T}^d$ endowed with the distance
\[
d_{\mathbb{T}^d}(x, y) := \min \{d(\tilde{x}, \tilde{y}) : \tilde{x} \in \pi^{-1}(x), \ \tilde{y} \in \pi^{-1}(y)\}, \quad \forall x, y \in \mathbb{T}^d
\]

Given any $a \in \mathbb{T}^d$, we write $T_a$ for the torus translation $T_a : \mathbb{T}^d \ni z \mapsto z + a$. A point $a \in \mathbb{R}^d$ is said to be **totally irrational** when $T_{\pi(a)}$ is minimal on $\mathbb{T}^d$.

In several places along this paper the symbol $\pm^\alpha$ shall have the following meaning: given $v \in \mathbb{R}^d$, we write $\pm^\alpha v$ to denote either the vector $v$ or $-v$.

2.3.1. **The plane** $\mathbb{R}^2$. In the particular case of $d = 2$, given any $v = (a, b) \in \mathbb{R}^2$, we define $v^\perp := (-b, a)$. For any $\alpha \in \mathbb{R}$ and any $v \in \mathbb{S}^1$, we shall use the following notation for the straight line through the point $\alpha v$ and perpendicular to $v$:
\[
\ell_\alpha^v := \alpha v + \mathbb{R}v^\perp = \{\alpha v + tv^\perp : t \in \mathbb{R}\}.
\]

We say a vector $v \in \mathbb{R}^2 \setminus \{0\}$ has **rational slope** when there exists $\alpha > 0$ such that $\alpha v \in \mathbb{Z}^2$; and it is said to have **irrational slope** otherwise.

We will also need the following notation for strips on $\mathbb{R}^2$: given any $v \in \mathbb{S}^1$ and $r < s$, we define the (closed) strip
\[
A_{r,s}^v := \mathbb{H}^v_r \cap \mathbb{H}^v_s = \{z \in \mathbb{R}^2 : r \leq \langle z, v \rangle \leq s\}.
\]

A **Jordan curve** is any subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{S}^1$. A **Jordan domain** is any bounded open subset of $\mathbb{R}^2$ whose boundary is a Jordan curve.

We shall need the following theorem due to Janiszewski (see for instance, [Kur68, Chapter X, Theorem 2]):

**Theorem 2.1.** If $X_1, X_2 \subset \mathbb{S}^2$ are two continua such that $X_1 \cap X_2$ is not connected, then $X_1 \cup X_2$ is not connected.
2.3.2. The annulus. The open annulus is given by \( A := \mathbb{T} \times \mathbb{R} \). Its universal covering map will be denoted by \( P : \mathbb{R}^2 \to A \) and is defined by
\[
P(x, y) := (\pi(x), y) = (x + \mathbb{Z}, y), \quad \forall (x, y) \in \mathbb{R}^2.
\]
We will always considered the annulus endowed the distance
\[
d_A(x, y) := \min \{ ||\tilde{x} - \tilde{y}|| : \tilde{x} \in P^{-1}(x), \tilde{y} \in P^{-1}(y) \}, \quad \forall x, y \in A.
\]
We write \( \hat{A} := A \cup \{-\infty, +\infty\} \) for the two-end compactification of the annulus \( A \). Observe that \( \hat{A} \) is homeomorphic to the 2-sphere \( \mathbb{S}^2 \).

We will need the following elementary result about unbounded connected subsets of \( A \):

**Lemma 2.2.** Let \( C \subset A \) be a closed connected unbounded set. If \( P \) is the covering map given by (7), then every connected component of \( P^{-1}(C) \) is unbounded in \( \mathbb{R}^2 \).

**Proof.** Reasoning by contradiction, let us suppose there is a bounded connected component \( K \) of \( P^{-1}(C) \subset \mathbb{R}^2 \).

If we write \( \overline{C} \) for the closure of \( C \) in \( \hat{A} \), we get that \( \overline{C} \) is compact and connected as well, and contains at least one of the two ends. Without loss of generalization, we can assume the upper end \(+\infty\) belongs to \( \overline{C} \).

Then we consider a sequence of open connected subsets \( \{U_n\}_{n \geq 1} \) of \( \hat{A} \) satisfying the following properties: \( \overline{C} \subset U_1 \subset A \) and \( \overline{U_{n+1}} \subset U_n \), for every \( n \geq 1 \) and
\[
\overline{C} = \bigcap_{n \geq 1} U_n = \bigcap_{n \geq 1} \overline{U_n}.
\]

Such a sequence of nested open sets can be constructed as follows: one consider a distance function \( d_A \) on \( \hat{A} \) which is compatible with its topology and then defines \( U_n := \overline{C_{1/n}} \), for every \( n \geq 1 \), where \( \overline{C_{1/n}} \) denotes the \( 1/n \)-neighborhood of \( \overline{C} \) with respect to the distance \( d_A \) as defined by (3).

Now let \( z_0 \) be an arbitrary point of \( K \subset \mathbb{R}^2 \). So we have \( P(z_0) \in C \subset U_n \), for every \( n \geq 1 \). Since \( U_n \) is open and connected, there is a continuous curve \( \gamma_n : [0, 1] \to U_n \) such that \( \gamma_n(0) = P(z_0) \), \( \gamma_n(1) = +\infty \) and \( \gamma_n(t) \in A \), for each \( n \in \mathbb{N} \) and every \( t \in [0, 1] \).

Then observe that, since \( P \) is a covering map, there exists a unique continuous curve \( \tilde{\gamma}_n : [0, 1] \to \mathbb{R}^2 \) such that \( \tilde{\gamma}_n(0) = z_0 \) and \( P \circ \tilde{\gamma}_n = \gamma_n \).

If \( \mathbb{R}^2 := \mathbb{R}^2 \cup \{ \infty \} \) denotes the one-point compactification of \( \mathbb{R}^2 \), one sees that each \( \tilde{\gamma}_n \) has a unique continuous extension from \([0, 1]\) to \( \mathbb{R}^2 \) just defining \( \tilde{\gamma}_n(1) := \infty \). In this way, each \( \tilde{\gamma}_n([0, 1]) \) is a compact subset of \( \mathbb{R}^2 \), and by compactness of the Hausdorff space \( K(\mathbb{R}^2) \), there exists a subsequence \( n_j \to \infty \) and a non-empty compact subset \( L \subset \mathbb{R}^2 \) such that \( \tilde{\gamma}_{n_j}([0, 1]) \to L \), as \( n_j \to \infty \), where the convergence is respect to the the Hausdorff distance. One can easily verify that \( L \) is connected, both points \( z_0 \) and \( \infty \) belong to \( L \) and \( P(L\setminus\{\infty\}) \subset C \). In particular, \( L\setminus\{\infty\} \) is a closed, connected, unbounded subset of \( P^{-1}(C) \subset \mathbb{R}^2 \) and \( K \cap (L\setminus\{\infty\}) \neq \emptyset \), contradicting the fact that \( K \) were a bounded connected component of \( P^{-1}(C) \). \( \square \)

### 2.4. Ergodic theory and cocycles.

Given a topological space \( X \), we write \( \mathcal{B}_X \) to denote its Borel \( \sigma \)-algebra.

The Haar (probability) measure on \((\mathbb{T}^d, \mathcal{B}_{\mathbb{T}^d})\), also called Lebesgue measure, will be denoted by \( \text{Leb}_d \). By a slight abuse of notation, we will also write \( \text{Leb}_d \) for the Lebesgue measure on \( \mathbb{R}^d \), and for the sake of simplicity of notation, we shall just write \( \text{Leb} \) instead of \( \text{Leb}_1 \).

Given an arbitrary \( \sigma \)-finite measure space \((X, \mathcal{B}, \mu)\), a map \( f : (X, \mathcal{B}) \to \mathbb{R} \) is said to be non-singular (respect to \( \mu \)) when it is measurable and, for every \( B \in \mathcal{B} \), it holds \( \mu(f^{-1}(B)) = 0 \) if and only if \( \mu(B) = 0 \). A non-singular map \( f : (X, \mathcal{B}) \to \mathbb{R} \) is
is said to be conservative (with respect to \( \mu \)) when for every \( B \in \mathcal{B} \) such that \( \mu(B) > 0 \), there exists \( n \geq 1 \) satisfying \( \mu(B \cap f^{-n}(B)) > 0 \).

As usual, we say that a measurable map \( f : (X, \mathcal{B}) \to \mathbb{R} \) preserves \( \mu \) when \( f_\ast \mu = \mu \), where \( f_\ast \mu(B) := \mu(f^{-1}(B)) \), for every \( B \in \mathcal{B} \); and \( f \) is said to be an automorphism of \( (X, \mathcal{B}, \mu) \) when it is bijective and its inverse is measurable and preserves \( \mu \), too.

Given an invertible map \( f : X \to X \), a function \( \phi : X \to \mathbb{R} \) and any \( n \in \mathbb{Z} \), one defines the Birkhoff sum

\[
S_f^n(\phi) := \begin{cases} 
\sum_{j=0}^{n-1} \phi \circ f^j, & \text{if } n \geq 1; \\
0, & \text{if } n = 0; \\
-\sum_{j=1}^{-n} \phi \circ f^{-j}, & \text{if } n < 0.
\end{cases}
\]

Putting together two classical results of Atkinson [Atk76, Theorem] and Schmidt [Sch06, Proposition 6], we get the following

**Theorem 2.3.** Let \((X, \mathcal{B}, \mu)\) be a probability space, \( f : (X, \mathcal{B}, \mu) \to \mathbb{R} \) be an ergodic automorphism and \( \phi \in L^1(X, \mathcal{B}, \mu) \) be a real function such that \( \int_X \phi \, d\mu = 0 \). Then, the skew-product automorphism \( F : X \times \mathbb{R} \to \mathbb{R} \) given by

\[
F(x, t) := (f(x), t + \phi(x)), \quad \forall (x, t) \in X \times \mathbb{R},
\]

is conservative with respect to the \( F \)-invariant \( \sigma \)-finite measure \( \mu \otimes \text{Leb} \).

### 2.5. Groups of homeomorphisms.

From now on and until the end of this section, \( M \) will denote an arbitrary topological manifold. We write \( \text{Homeo}(M) \) for the group of homeomorphisms from \( M \) onto itself. The subgroup formed by those homeomorphisms which are homotopic to the identity \( \text{id}_M \) will be denoted by \( \text{Homeo}_0(M) \).

**2.5.1. Torus homeomorphisms and their lifts.** The group of lifts of torus homeomorphisms which are homotopic to the identity will be denoted by

\[
\text{Homeo}_0(\mathbb{T}^d) := \{ \hat{f} \in \text{Homeo}_0(\mathbb{R}^d) : \hat{f} - \text{id}_{\mathbb{R}^d} \in C^0(\mathbb{T}^d, \mathbb{R}^d) \}.
\]

Notice that in this definition, as it is usually done, we are identifying the elements of \( C^0(\mathbb{T}^d, \mathbb{R}^d) \) with those \( \mathbb{Z}^d \)-periodic continuous functions from \( \mathbb{R}^d \) to itself.

Making some abuse of notation, we also write \( \pi : \text{Homeo}_0(\mathbb{T}^d) \to \text{Homeo}_0(\mathbb{T}^d) \) for the map that associates to each \( \hat{f} \) the only torus homeomorphism \( \pi \hat{f} \) such that \( \hat{f} \) is a lift of \( \pi \hat{f} \). Notice that with our notations, it holds \( \pi T_\alpha = T_{\pi(\alpha)} \in \text{Homeo}_0(\mathbb{T}^d) \), for every \( \alpha \in \mathbb{R}^d \).

Given any \( \hat{f} \in \text{Homeo}_0(\mathbb{T}^d) \), we define its displacement function by

\[
\Delta \hat{f} := \hat{f} - \text{id}_{\mathbb{R}^d} \in C^0(\mathbb{T}^d, \mathbb{R}^d).
\]

Observe that this function can be naturally considered as a cocycle over \( f := \pi \hat{f} \) because

\[
\Delta f_n = \sum_{j=0}^{n-1} \Delta \hat{f} \circ f^j, \quad \forall n \geq 1.
\]

For the sake of readability, we shall use the usual notation for cocycles defining

\[
\Delta \hat{f}^{(n)} := \Delta \hat{f}_n, \quad \forall n \in \mathbb{Z}.
\]

The map \( \mathbb{R}^d \ni \alpha \mapsto T_\alpha \in \text{Homeo}_0(\mathbb{T}^d) \) defines an injective group homomorphism, and hence, \( \mathbb{R}^d \) naturally acts on \( \text{Homeo}_0(\mathbb{T}^d) \) by conjugacy. However, since every element of \( \text{Homeo}_0(\mathbb{T}^d) \) commutes with \( T_p \), for all \( p \in \mathbb{Z}^d \), we conclude \( \mathbb{T}^d \) itself acts...
on \( \text{Homeo}_0(\mathbb{T}^d) \) by conjugacy, i.e. the map \( \text{Ad}: \mathbb{T}^d \times \text{Homeo}_0(\mathbb{T}^d) \to \text{Homeo}_0(\mathbb{T}^d) \) given by
\[
\text{Ad}_t(f) := T_t^{-1} \circ f \circ T_t, \quad \forall (t, f) \in \mathbb{T}^d \times \text{Homeo}_0(\mathbb{T}^d), \quad \forall t \in \pi^{-1}(t),
\]
is well-defined.

2.5.2. Invariant measures. We write \( \mathfrak{M}(M) \) for the space of Borel probability measures on \( M \). A measure \( \mu \in \mathfrak{M}(M) \) is said to have total support when \( \mu(A) > 0 \) for every non-empty open set \( A \subset M \). We say \( \mu \) is a topological measure if it has total support and no atoms.

For every \( \mu \in \mathfrak{M}(M) \), we consider the group of homeomorphisms
\[
\text{Homeo}_\mu(M) := \{ f \in \text{Homeo}(M) : f_* \mu = \mu \}.
\]
Given \( f \in \text{Homeo}(M) \), we write \( \mathfrak{M}(f) := \{ \nu \in \mathfrak{M}(M) : f_* \nu = \nu \} \).

The following classical result is due to Oxtoby and Ulam [OU41]:

**Theorem 2.4.** Let \( M \) be a compact topological manifold and \( \mu, \nu \in \mathfrak{M}(M) \) two topological measures. Then, there exists \( h \in \text{Homeo}(M) \) such that \( h_* \mu = \nu \).

For the sake of simplicity of notation, on the two-dimensional torus we define the group of symplectomorphisms (also called area-preserving homeomorphisms) by
\[
\text{Symp}^+(\mathbb{T}^2) := \{ f \in \text{Homeo}(\mathbb{T}^2) : \text{Leb}_2 \in \mathfrak{M}(f) \}.
\]

It is well known that its connected component containing the identity, which will be denoted by \( \text{Symp}_0(\mathbb{T}^2) \), coincides with \( \text{Symp}^+(\mathbb{T}^2) \cap \text{Homeo}_0(\mathbb{T}^2) \). We write
\[
\text{Symp}_0(\mathbb{T}^2) := \pi^{-1}(\text{Symp}_0(\mathbb{T}^2)) \subset \text{Homeo}_0(\mathbb{T}^2).
\]

2.6. Rotation set and rotation vectors. Let \( f \in \text{Homeo}_0(\mathbb{T}^d) \) be an arbitrary homeomorphism and \( \tilde{f} \in \text{Homeo}_0(\mathbb{T}^d) \) be a lift of \( f \). The rotation set of \( \tilde{f} \) is given by
\[
\rho(\tilde{f}) := \bigcap_{m \geq 0} \bigcup_{n \geq m} \left\{ \frac{\Delta_{\tilde{f}}^{(n)}(z)}{n} : z \in \mathbb{R}^d \right\}.
\]

It can be easily shown that \( \rho(\tilde{f}) \) is non-empty, compact and connected.

When \( d = 1 \), by classical Poincaré theory of circle homeomorphisms [Poi80] we know that \( \rho(\tilde{f}) \) reduces to a point, but in general this does not hold in higher dimensions.

We summarized some elementary facts about rotation sets which are due to Misiurewicz and Ziemian [MZ89, Proposition 2.1]:

**Proposition 2.5.** Given any \( \tilde{f} \in \text{Homeo}_0(\mathbb{T}^d) \), the following properties hold:
\begin{enumerate}[(i)]
    \item \( \rho(\tilde{f}^n) = n \rho(\tilde{f}) := \{ np \in \mathbb{R}^d : \rho \in \rho(\tilde{f}) \} \), for any \( n \in \mathbb{Z} \);
    \item \( \rho(T_p \circ \tilde{f}) = T_p(\rho(\tilde{f})) \), for any \( p \in \mathbb{Z}^d \).
\end{enumerate}

As a consequence of (ii) of Proposition 2.5, we see that given any \( f \in \text{Homeo}_0(\mathbb{T}^d) \) and any lift \( \tilde{f} : \mathbb{R}^d \to \mathbb{T}^d \), we can define
\[
\rho(f) := \pi(\rho(\tilde{f})) \subset \mathbb{T}^d.
\]

We say that \( f \in \text{Homeo}_0(\mathbb{T}^d) \) is a pseudo-rotation when \( \rho(f) \) is a singleton.

By (11) and (13), we know the rotation set is formed by accumulation points of Birkhoff averages of the displacement function. So given any \( \mu \in \mathfrak{M}(f) \), one can define its rotation vector by
\[
\rho_\mu(\tilde{f}) := \int_{\mathbb{R}^d} \Delta_{\tilde{f}} \, d\mu.
\]
Thus, by Birkhoff ergodic theorem we get $\rho_\mu(f) = \rho(f)$, for every $f$-invariant ergodic probability measure $\mu$. Moreover, the following holds:

**Theorem 2.6** (Theorem 2.4 in [MZ89]). Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ and $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $f$. Then, for every extreme point $w \in \rho(f)$, there exists an ergodic measure $\mu \in \mathcal{M}(f)$ such that $\rho_\mu(f) = w$. Consequently, it holds

$$\text{Conv}\{(\rho(f)) : \nu \in \mathcal{M}(\pi f)\},$$

where Conv(·) denotes the convex hull operator.

However, in the two-dimensional case rotation sets are always convex:

**Theorem 2.7** (Theorem 3.4 in [MZ89]). For every $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$, we have

$$\rho(\tilde{f}) = \{\rho_\nu(\tilde{f}) : \nu \in \mathcal{M}(\pi \tilde{f})\}.$$

2.7. Hamiltonian homeomorphisms. In the symplectic setting, that is when $\tilde{f} \in \text{Symp}_0(\mathbb{T}^2)$, the rotation vector of Leb$_2$ is also called the flux of $\tilde{f}$ and is usually denoted by $\text{Flux}(\tilde{f}) := \rho_{\text{Leb}_2}(\tilde{f})$. In this case, it can be easily shown that the flux map $\text{Flux} : \text{Symp}_0(\mathbb{T}^2) \to \mathbb{R}^2$ is indeed a group homomorphism. Since $\text{Flux}(T_p \circ \tilde{f}) = T_p(\text{Flux}(\tilde{f}))$, $\forall p \in \mathbb{Z}^2$, $\forall \tilde{f} \in \text{Symp}_0(\mathbb{T}^2)$ this homomorphism clearly induces a map $\text{Symp}_0(\mathbb{T}^2) \to \mathbb{T}^2$ which, by some abuse of notation, will be denoted by $\text{Flux}$, too.

The kernel of this homomorphism $\text{Flux} : \text{Symp}_0(\mathbb{T}^2) \to \mathbb{T}^2$ is denoted by

$$\text{Ham}(\mathbb{T}^2) := \{f \in \text{Symp}_0(\mathbb{T}^2) : \text{Flux}(f) = 0\} \subset \text{Symp}_0(\mathbb{T}^2),$$

i.e. it is a normal subgroup of $\text{Symp}_0(\mathbb{T}^2)$. The elements of $\text{Ham}(\mathbb{T}^2)$ are called Hamiltonian homeomorphisms.

Analogously, the kernel of $\text{Flux} : \text{Symp}_0(\mathbb{T}^2) \to \mathbb{R}^2$ is denoted by

$$\widetilde{\text{Ham}}(\mathbb{T}^2) := \{\tilde{f} \in \text{Symp}_0(\mathbb{T}^2) : \text{Flux}(\tilde{f}) = 0\}.$$

**Remark 2.8.** Notice that $\text{Ham}(\mathbb{T}^2)$ and $\widetilde{\text{Ham}}(\mathbb{T}^2)$ can be naturally identified. In fact, the restriction $\pi_{\widetilde{\text{Ham}}(\mathbb{T}^2)} : \text{Ham}(\mathbb{T}^2) \to \widetilde{\text{Ham}}(\mathbb{T}^2)$ is a continuous group isomorphism.

Observe the following short exact sequence splits:

$$0 \to \text{Ham}(\mathbb{T}^2) \to \text{Symp}_0(\mathbb{T}^2) \xrightarrow{\text{Flux}} \mathbb{T}^2 \to 0.$$

In fact, the map $\mathbb{T}^2 \ni \alpha \mapsto T_\alpha$ is a section of $\text{Flux}$, and thus, the group $\text{Symp}_0(\mathbb{T}^2)$ can be decomposed as a semi-direct product $\text{Symp}_0(\mathbb{T}^2) = \mathbb{T}^2 \ltimes \text{Ham}(\mathbb{T}^2)$. In other words, given $\alpha, \beta \in \mathbb{T}^2$ and $h, g \in \text{Ham}(\mathbb{T}^2)$, we have

$$(T_\alpha \circ h) \circ (T_\beta \circ g) = T_{\alpha + \beta} \circ (\text{Ad}_\beta(h) \circ g),$$

where the $\mathbb{T}^2$-action $\text{Ad}$ is given by (12).

This elementary fact about the group structure of $\text{Symp}_0(\mathbb{T}^2)$ is our main inspiration for the construction of the fiber-wise Hamiltonian skew-product we will perform in §6.1.

2.8. Rotation set, periodic points and minimality. The following result due to Handel asserts that the rotation set of a periodic point free homeomorphism has empty interior:

**Theorem 2.9** (Handel [Han89]). Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be such that $\text{Per}(f) = \emptyset$ and $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ be a lift of $f$. Then, there exist $v \in S^1$ and $\alpha \in \mathbb{R}$ so that

$$\frac{\langle \tilde{f}^n(z), v \rangle}{n} \to \alpha, \quad \text{as } n \to \infty,$$
where the convergence in uniform in \( z \in \mathbb{R}^2 \). In other words, the rotation set \( \rho(\tilde{f}) \subseteq \ell^\alpha \), where the straight line \( \ell^\alpha \) is given by (5).

The following result due to Franks will play a fundamental role in our work:

**Theorem 2.10** (Franks [Fra95]). If \( \tilde{f} \in \text{Symp}_0(\mathbb{T}^2) \) and \( \text{Flux}(\tilde{f}) = (p_1/q, p_2/q) \in \mathbb{Q}^2 \), then there exists \( z \in \mathbb{R}^2 \) such that

\[
\tilde{f}^q(z) = z + (p_1, p_2).
\]

In particular, \( \pi(z) \in \text{Per}(\pi\tilde{f}) \).

Every probability measure which is invariant under a minimal homeomorphism is necessarily a topological measure. Hence, as a straightforward consequence of Theorems 2.4 and 2.10, we get the following

**Corollary 2.11.** If \( f \in \text{Homeo}_0(\mathbb{T}^2) \) is minimal and \( \tilde{f} \in \text{Homeo}_0(\mathbb{T}^2) \) is a lift of \( f \), then

\[
\rho(\tilde{f}) \cap \mathbb{Q}^2 = \emptyset.
\]

### 2.9. Classification of plane fixed points

Let \( V, V' \subseteq \mathbb{R}^2 \) be two non-empty open sets and let \( f : V \to V' \) be a homeomorphism. Following the terminology of Le Calvez [LC03], a fixed point \( z_0 \in \text{Fix}(f) \) is said to be:

- **isolated** when it is an isolated point of the set \( \text{Fix}(f) \);
- **accumulated** when every neighborhood of \( z_0 \) contains a periodic orbit of \( f \) different from \( z_0 \);
- **dissipative** when \( z_0 \) admits a local basis \( \{U_n\}_{n \geq 0} \) of neighborhoods such that \( f(U_n) \cap \partial U_n = \emptyset \), for every \( n \geq 0 \), i.e. each neighborhood is either attractive or repulsive;
- **indifferent** when there exists a neighborhood \( W \) of \( z_0 \) such that \( \partial W \subseteq V \) and for every Jordan domain \( U \subseteq W \) which is a neighborhood of \( z_0 \) it holds

\[
\text{cc}(\mathcal{I}_f(U), z_0) \cap \partial U \neq \emptyset,
\]

where \( \mathcal{I}_f(U) \) denotes the maximal \( f \)-invariant subset of \( U \) given by (2).

### 2.10. Fixed point indexes

If \( f : V \to V' \) is as in §2.9, and \( \gamma : S^1 \to V \) is a Jordan curve such that \( \gamma(S^1) \cap \text{Fix}(f) = \emptyset \), then one defines the **index of \( f \) along \( \gamma \)** as the integer

\[
i(f, \gamma) := \text{deg} \left( S^1 \ni t \mapsto \frac{f(\gamma(t)) - \gamma(t)}{|f(\gamma(t)) - \gamma(t)|} \in S^1 \right),
\]

where \( \text{deg}(\cdot) \) denotes the topological degree.

When \( z_0 \in \text{Fix}(f) \) is isolated, then one can define the **index of \( f \) at \( z_0 \)** as

\[
i(f, z_0) := i(f, \partial U),
\]

where \( U \) denotes any Jordan domain satisfying \( \overline{U} \subseteq V \) and \( \overline{U} \cap \text{Fix}(f) = \{z_0\} \).

Since this index does not depend on the choice of \( U \), this notion is well-defined.

We will need the following topological version of Leau-Fatou’s flower theorem due to Le Calvez [LC99], that has been lately improved by Le Roux [LR04]:

**Theorem 2.12.** Let us suppose \( f : V \to V' \) is an orientation-preserving homeomorphism and \( z_0 \in \text{Fix}(f) \) is an isolated fixed point such that \( i(f, z_0) \geq 2 \). Then there exist two open non-empty subsets \( W^+, W^- \subseteq V \setminus \{z_0\} \) such that

(i) \( f^n(W^+) \) is well-defined for every \( n \geq 0 \), \( f^m(W^+) \cap f^n(W^+) = \emptyset \) whenever \( m \) and \( n \) are different non-negative integers and \( \omega_f(z) = \{z_0\} \), for every \( z \in W^+ \);

(ii) \( f^{-n}(W^-) \) is well-defined for every \( n \geq 0 \), \( f^{-m}(W^-) \cap f^{-n}(W^-) = \emptyset \) whenever \( m \) and \( n \) are different non-negative integers and \( \alpha_f(z) = \{z_0\} \), for every \( z \in W^- \);
where $\alpha_f$ and $\omega_f$ denote the $\alpha$- and $\omega$-limit sets, respectively.

The following result about indexes of iterates of non-accumulated fixed points is due to Le Calvez and Yoccoz [LCY] but its proof has never been published (see for instance [LC03, Proposition 3.3]):

**Theorem 2.13.** If $f$ is an orientation-preserving homeomorphism and $z_0 \in \text{Fix}(f)$ is isolated, non accumulated, non indifferent and non dissipative, then there exist integers $q \geq 1$ and $r \geq 1$ such that
\[
\begin{cases}
i(f^k, z_0) = 1, & \text{if } k \notin q\mathbb{Z}; \\
i(f^k, z_0) = 1 - rq, & \text{if } k \in q\mathbb{Z}.
\end{cases}
\]

2.11. Minimal homeomorphisms. In this paragraph we recall some classical and elementary results about minimal homeomorphisms that we shall frequently use all along the paper.

We say that a subset $A \subset \mathbb{Z}$ has *bounded gaps* if there exists $N \in \mathbb{N}$ such that
\[
A \cap \{n, n + 1, \ldots, n + N\} \neq \emptyset, \quad \forall n \in \mathbb{Z}.
\]

The minimum natural number $N$ such that (14) holds shall be denoted by $\mathcal{G}(A)$.

The following three results are very well-known, but we decided to include them here just for the sake of reference:

**Proposition 2.14.** If $(X, d)$ is a compact metric space and $f: X \supset \subset$ is a minimal homeomorphism, then for every non-empty open set $U \subset X$ and any $x \in X$, the visiting time set
\[
\tau(x, U, f) := \{n \in \mathbb{Z} : f^n(x) \in U\}
\]
has bounded gaps.

As a consequence of this result, one can easily show the following:

**Corollary 2.15.** For every $\alpha \in \mathbb{T}^d$, any $x \in \mathbb{T}^d$ and any neighborhood $V \subset \mathbb{T}^d$ of $x$, the visiting time set $\tau(x, V, T_\alpha)$ has bounded gaps.

**Proposition 2.16.** If $(X, d)$ is a connected compact metric space and $f: X \supset \subset$ is a minimal homeomorphism, then $f^n$ is minimal for every $n \in \mathbb{Z}\setminus\{0\}$.

The last result we recall here is due to Gottschalk and Hedlund and deals with cohomological equations:

**Theorem 2.17** (Gottschalk, Hedlund [GH55]). Let $X$ be a compact metric space and $f: X \supset \subset$ be a minimal homeomorphism. Let $\phi: X \to \mathbb{R}$ be a continuous function and assume there exists $x_0 \in X$ such that
\[
\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n-1} \phi(f^j(x_0)) \right| < \infty.
\]

Then, there is a continuous function $u: X \to \mathbb{R}$ such that $u \circ f - u = \phi$. In particular,
\[
\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n-1} \phi(f^j(x)) \right| \leq 2 \|u\|_{C^0} < \infty, \quad \forall x \in X.
\]

3. An ergodic deviation result

This section is devoted to prove an abstract ergodic deviation theorem that will play a key role in §7. Even though this result might be already known to some experts, we were not able to find any reference in the literature and thus we have decided to include its proof here.
Theorem 3.1. Let \((X, \mathcal{B}, \mu)\) be a probability space, \(f: (X, \mathcal{B}, \mu) \to \) an ergodic automorphism and \(\phi \in L^1(X, \mathcal{B}, \mu)\) such that \(\int_X \phi \, d\mu = 0\). Let us suppose that
\[
\sup_{n \geq 0} S_f^m \phi(x) = +\infty, \quad \text{and} \quad \inf_{n \geq 0} S_f^m \phi(x) > -\infty,
\]
for \(\mu\)-a.e. \(x \in X\), where \(S_f^m \phi\) denotes the Birkhoff sum given by (8).

Then, it holds
\[
\sup_{n \geq 0} S_f^n \phi(x) = +\infty, \quad \text{and} \quad \inf_{n \geq 0} S_f^n \phi(x) > -\infty,
\]
for \(\mu\)-a.e. \(x \in X\).

Proof of Theorem 3.1. Of course we can assume \(|\phi(x)| < \infty\), for every \(x \in X\). By (15), there exist \(K > 0\) and \(\tilde{\mathbb{A}} \in \mathcal{B}\) with \(\mu(\tilde{\mathbb{A}}) > 0\) such that
\[
\mu \otimes \text{Leb}(\tilde{\mathbb{A}}) \cap F^{-k_j}(\tilde{\mathbb{A}}) > 0, \quad \forall j \in \mathbb{N}.
\]
This is equivalent to say that the set
\[
B_j := \left\{ x \in \tilde{\mathbb{A}} : f^{k_j}(x) \in \mathbb{A}, \quad \left| S_{f_j}^{k_j} \phi(x) \right| \leq \frac{1}{n} \right\}
\]
has positive \(\mu\)-measure, for every \(j\).

Now, choosing \(j\) large enough in order to have \(k_j > N\), and combining this with (18), we get
\[
S_{f_j}^{k_j-N}(f^N(x)) = S_{f_j}^{k_j}(x) - S_{f_j}^N(x) > -\frac{1}{n} + M + \frac{1}{n} = M,
\]
for \(\mu\)-a.e. \(x \in B_j\). Since \(\mu(B_j) > 0\), this contradicts (16). \(\square\)

Proof of Theorem 3.1. Of course we can assume \(|\phi(x)| < \infty\), for every \(x \in X\). By (15), there exist \(K > 0\) and \(\mathbb{A} \in \mathcal{B}\) with \(\mu(\mathbb{A}) > 0\) such that
\[
S_f^n \phi(x) > -K, \quad \forall n \geq 0,
\]
for \(\mu\)-a.e. \(x \in \mathbb{A}\).

Consider the functions \(\tau_+^A, \tau_-^A: \mathbb{R} \to \mathbb{N}_0 \cup \{\infty\}\) given by
\[
\tau_+^A(x) := \min\{n \geq 0 : f^n(x) \in \mathbb{A}\},
\]
\[
\tau_-^A(x) := \min\{n \geq 0 : f^{-n}(x) \in \mathbb{A}\}, \quad \forall x \in X.
\]
Since \( f \) is ergodic, \( \tau_A^+ \) and \( \tau_A^- \) are finite \( \mu \)-a.e. We also define the first return time to \( A \) by \( r_A := \tau_A^+ \circ f|_A^{-1} + 1 \).

Now, we consider the probability space \((A, \mathcal{B}_A, \mu_A)\) given by \( \mathcal{B}_A := \{ B \in \mathcal{B} : B \subset A \} \) and \( \mu_A := \mu(A)^{-1} \mu|_A \) and the ergodic automorphism \( f_A : (A, \mathcal{B}_A, \mu_A) \circlearrowright \) given by the first return map:

\[
f_A(x) := f^{\tau_A^+(f(x))}(f(x)) = f^{r_A(x)}(x),
\]

for \( \mu \)-a.e. \( x \in A \). We also define the function \( \phi_A(x) := S_{f_A}^{r_A(x)} \phi(x) \). Then it holds \( \phi_A \in L^1(A, \mathcal{B}_A, \mu_A) \) and \( \int_A \phi_A \, d\mu = 0 \). On the other hand, by (19) we know

\[
S_{f_A}^n \phi_A(x) > -K, \quad \forall n \geq 0,
\]

for \( \mu_A \)-a.e. \( x \in A \).

Then, applying Lemma 3.2 in this context, we conclude that for \( \mu_A \)-a.e. \( x \in A \) it holds

\[
S_{f_A}^n \phi_A(x) \leq K, \quad \forall n \geq 0.
\]

Now, let us consider the measurable functions \( M : A \to \mathbb{R} \) and \( N : A \to \mathbb{N}_0 \) given by

\[
M(x) := \sup_{1 \leq n \leq r_A(x)} S_{f_A}^{n-1} \phi(x),
\]

and

\[
N(x) := \inf \left\{ n \geq 0 : n < r_A(x), \quad S_{f_A}^n \phi(x) = M(x) \right\},
\]

and notice they are well define \( \mu \)-a.e. \( x \in A \).

For each pair \( (m, n) \in \mathbb{N} \times \mathbb{N}_0 \), let us consider the set

\[
A_{m}^n := \{ x \in A : m \leq M(x) < \infty, \quad N(x) = n \}.
\]

Putting together (15) and (21) it follows that

\[
\mu\left( \bigcup_{n \geq 0} A_{m}^n \right) > 0, \quad \forall m \in \mathbb{N}.
\]

By (24), for each \( m \in \mathbb{N} \) there exists \( n_m \in \mathbb{N} \) so that \( \mu(A_{m}^{n_m}) > 0 \). So, let us define the set

\[
B := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq 0} f^j(A_{m}^{n_m}).
\]

Since \( f \) is an ergodic automorphism, \( B \) has full \( \mu \)-measure.

Now, let us consider an arbitrary point \( x \in B \) and any \( m \in \mathbb{N} \). Since \( A_{m}^{n_m} \subset A \), it clearly holds \( \tau_A^-(x) < \infty \). By (25), there exists a natural number \( j = j(x, m) > n_m \) such that

\[
f^{-j} \left( f^{-\tau_A^-(x)}(x) \right) \in A_{m}^{n_m}.
\]

Since both points \( f^{-\tau_A^-(x)}(x) \) and \( f^{-j} \left( f^{-\tau_A^-(x)}(x) \right) \) belong to \( A \), there exists \( j_A \in \mathbb{N} \) such that

\[
f_A^{-j_A} \left( f^{-\tau_A^-(x)}(x) \right) = f^{-j} \left( f^{-\tau_A^-(x)}(x) \right).
\]

Now invoking (20), (22), (23) and (26), we get

\[
S_{f_A}^{-\tau_A^-(x) - j_A + n_m} \phi(x) = S_{f_A}^{-\tau_A^-(x)} \phi(x) + S_{f_A}^{-j_A + n_m} \phi \left( f^{-\tau_A^-(x)}(x) \right)
\]

\[
= S_{f_A}^{-\tau_A^-(x)} \phi(x) + S_{f_A}^{-j_A} \phi_A \left( f^{-\tau_A^-(x)}(x) \right) + S_{f_A}^{n_m} \phi \left( f_A^{-j_A} \left( f^{-\tau_A^-(x)}(x) \right) \right)
\]

\[
\geq S_{f_A}^{-\tau_A^-(x)} \phi(x) - K + m.
\]
Since \( m \) is arbitrary in (27), \( \mu(B) = 1 \) and \( j > n_m \), we have proved that
\[
\sup_{n \leq 0} S_f^n \phi(x) = +\infty, \quad \text{for } \mu\text{-a.e. } x \in X.
\]

On the other hand, let us consider the set
\[
C : = \bigcap_{i \geq 0} \bigcup_{j \geq i} f^j(A).
\]
Since, \( f \) is ergodic, it holds \( \mu(C) = 1 \).

Now, consider any \( x \in C \) and any \( n \in \mathbb{N} \). Thus, \( \tau_A^{\pm n}(x) \) and \( \tau_A^{-n}(f^{-n}(x)) \) are both finite, and both points \( f^{-\tau_A^{\pm n}(x)}(x) \) and \( f^{-n-\tau_A^{-n}(f^{-n}(x))}(x) \) belong to \( A \). So, there exists \( l_A = l_A(x,n) \in \mathbb{N} \) such that
\[
f_A^{l_A}(f^{-\tau_A^{\pm n}(f^{-n}(x))}(x)) = f^{-\tau_A^{-n}(f^{-n}(x))}(x).
\]
Then, we have
\[
S_f^{-n} \phi(x) = S_f^{-\tau_A^{-n}(f^{-n}(x))} \phi(x) + S_f^{\tau_A^{-n}(f^{-n}(x))} \phi(f^{-n-\tau_A^{-n}(f^{-n}(x))}(x))
\]
\[
> S_f^{-\tau_A^{-n}(f^{-n}(x))} \phi(x) - K
\]
\[
= S_f^{-\tau_A^{-n}(f^{-n}(x))} \phi(x) - S_f^{n+\tau_A^{-n}(f^{-n}(x))} \phi(f^{-n-\tau_A^{-n}(f^{-n}(x))}(x)) - K
\]
\[
= S_f^{-\tau_A^{-n}(f^{-n}(x))} \phi(x) - S_f^{l_A} \phi_A(f^{-n-\tau_A^{-n}(f^{-n}(x))}(x)) - K
\]
\[
> S_f^{-\tau_A^{-n}(f^{-n}(x))} \phi(x) - 2K,
\]
where the first inequality follows from (19) and the second one from (21).

From this last estimate, and since \( \mu(C) = 1 \), it follows that
\[
\inf_{n \leq 0} S_f^n \phi(x) > -\infty, \quad \text{for } \mu\text{-a.e. } x \in X.
\]

\[
4. \text{ Rotational deviations}
\]

In this section we enter into the core of this work: the study of rotational deviations for 2-torus homeomorphisms in the identity isotopy class.

Let us start recalling some definitions we introduced in [KPR18]. Let \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) be a homeomorphism homotopic to the identity and \( \tilde{f} \in \text{Homeo}_0(\mathbb{T}^2) \) be a lift of \( f \). Let us suppose there exist \( v \in S^1 \) and \( \alpha \in \mathbb{R} \) such that
\[
\rho(\tilde{f}) \subset \ell_{\alpha} = \{ \alpha v + tv^1 : t \in \mathbb{R} \}.
\]
Observe that, by Theorem 2.9, this hypothesis is always satisfied when \( f \) is periodic point free.

We say that a point \( z_0 \in \mathbb{T}^2 \) exhibits \textit{bounded} \( v \)-\textit{deviations} when there exists a real constant \( M = M(z_0, f) > 0 \) such that
\[
\langle \tilde{f}^n(z_0) - z_0 - n\rho, v \rangle = \langle \Delta_f^{(n)}(z_0), v \rangle - n\alpha \leq M, \quad \forall n \in \mathbb{Z},
\]
for any \( z_0 \in \pi^{-1}(z_0) \), any \( \rho \in \rho(\tilde{f}) \) and where \( \Delta_f \) denotes the displacement cocycle of \( \tilde{f} \) given by (10).

Moreover, we say that \( f \) exhibits \textit{uniformly bounded} \( v \)-\textit{deviation} when there exists \( M = M(f) > 0 \) such that
\[
\langle \Delta_f^{(n)}(z), v \rangle - n\alpha \leq M, \quad \forall z \in \mathbb{T}^2, \quad \forall n \in \mathbb{Z}.
\]

Even though the straight lines \( \ell_{\alpha} \) and \( \ell_{-\alpha} \) coincide as subsets of \( \mathbb{R}^2 \), there is no \textit{à priori} obvious relation between boundedness of \( v \)-deviation and \((-v)\)-deviation.
because in our definition of “bounded $v$-deviations” given by (29) we are just considering boundedness from above.

However, we got the following result that relates both $v$- and $\tilde{v}$-deviations:

**Theorem 4.1** (Corollary 3.2 in [KPR18]). If $f \in \text{Homeo}_0(T^2)$ and $\tilde{f} : \mathbb{R}^2 \circlearrowleft$ is a lift of $f$ such that condition (28) holds, then $f$ exhibits uniformly bounded $v$-deviations if and only if it exhibits uniformly bounded $\tilde{v}$-deviations.

As a particular case of our definition of boundedness rotational deviations, let us recall that a homeomorphism $f \in \text{Homeo}_0(T^2)$ is said to be annular (see for instance [KT14b, JT17]) when there exist a lift $\tilde{f} \in \text{Homeo}_0(T^2)$, $M > 0$ and $v \in \mathbb{S}^1$ with rational slope such that

\[ \left| \langle \Delta^{(n)}_f(z), v \rangle \right| \leq M, \quad \forall z \in T^2, \forall n \in \mathbb{Z}. \]

Observe that in such a case, the rotation set $\rho(\tilde{f})$ is contained in the line $\ell^0_v$, i.e. the straight line parallel to $v$ and passing through the origin.

On the other hand, a homeomorphisms $f \in \text{Homeo}_0(T^2)$ is said to be eventually annular when there exists $k \in \mathbb{N}$ such that $f^k$ is annular.

In [KPR18] we proved that boundedness of $v$-deviations is equivalent to the existence of certain invariant topological object called torus pseudo-foliation.

### 4.1. Pseudo-foliations

In this paragraph we recall the notions of **plane and torus pseudo-foliations** we introduced in [KPR18].

**4.1.1. Plane pseudo-foliations.** Let $\mathcal{F}$ be a partition of $\mathbb{R}^2$. We say that $\mathcal{F}$ is a **plane pseudo-foliation** when every atom of $\mathcal{F}$ is closed, connected, has empty interior and disconnects $\mathbb{R}^2$ in exactly two connected components.

Given any $z \in \mathbb{R}^2$, we write $\mathcal{F}_z$ for the atom of the partition $\mathcal{F}$ containing the point $z$. If $h : \mathbb{R}^2 \circlearrowleft$ is an arbitrary map, we say that $\mathcal{F}$ is $h$-invariant when

\[ h(\mathcal{F}_z) = \mathcal{F}_{h(z)}, \quad \forall z \in \mathbb{R}^2. \]

Let us recall the following result of [KPR18, Preposition 5.1]:

**Proposition 4.2.** If $\mathcal{F}$ is a plane pseudo-foliation, then both connected component of $\mathbb{R}^2 \setminus \mathcal{F}_z$ are unbounded, for every $z \in \mathbb{R}^2$.

**4.1.2. Torus pseudo-foliations.** A partition $\mathcal{F}$ of $T^2$ is said to be a **toral pseudo-foliation** whenever there exists a plane pseudo-foliation $\tilde{\mathcal{F}}$, called the **lift** of $\mathcal{F}$, satisfying

\[ \pi \left( \tilde{\mathcal{F}}_z \right) = \mathcal{F}_{\pi(z)}, \quad \forall z \in \mathbb{R}^2. \]

Notice that such a plane pseudo-foliation is $\mathbb{Z}^2$-invariant, i.e. $\tilde{\mathcal{F}}$ is $T_p$-invariant for every $p \in \mathbb{Z}^2$.

In [KPR18] we have gotten the following result that guaranties the existence of an asymptotic homological direction for torus pseudo-foliations:

**Theorem 4.3.** If $\tilde{\mathcal{F}}$ is the lift of torus pseudo-foliation, then there exists $v \in \mathbb{S}^1$ and $M > 0$ such that

\[ \left| \langle w - z, v \rangle \right| \leq M, \quad \forall z \in \mathbb{R}^2, \forall w \in \tilde{\mathcal{F}}_z. \]

The vector $v$ given by Theorem 4.3 is unique up to multiplication by $(-1)$. So, we will call it the **asymptotic direction** of either the torus pseudo-foliation $\mathcal{F}$ or its lift $\tilde{\mathcal{F}}$.

One of the main results of [KPR18] is the following:
Theorem 4.4 (Theorem 5.5 in [KPR18]). Let \( f \in \text{Homeo}_0(\mathbb{T}^2) \) be a periodic point free, area-preserving, non-wandering and non-eventually annular homeomorphism. If \( f \) exhibits uniformly bounded \( v \)-deviations, for some \( v \in \mathbb{S}^1 \), then there exists an \( f \)-invariant pseudo-foliation whose asymptotic direction is given by \( v^+ \).

4.2. Rotational deviations for minimal homeomorphisms. In this paragraph we present some simple results about rotational deviations of minimal homeomorphisms. So, from now on let us assume \( f \in \text{Homeo}_0(\mathbb{T}^2) \) is minimal and \( \tilde{f} : \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \) is a lift of \( f \). By Theorem 2.9 we know there exist \( v \) and \( \alpha \) such that the rotation set of \( f \) is contained in the line \( \ell_{\alpha}^v \), i.e. inclusion (28) holds.

The following result is an improvement of Theorem 4.1 under the minimality assumption:

Proposition 4.5. If \( f \) is minimal, \( \tilde{f} \) is a lift of \( f \) and \( v \) and \( \alpha \) are such that condition (28) holds, then the following properties are equivalent:

(i) \( f \) does not exhibit uniformly bounded \( v \)-deviations;
(ii) \( f \) does not exhibit uniformly bounded \((-v)\)-deviations;
(iii) for every \( z \in \mathbb{T}^2 \) it holds
\[
\sup_{n \geq 0} \left| \left\langle \Delta_f^{(n)}(z), v \right\rangle - na \right| = \sup_{n \leq 0} \left| \left\langle \Delta_f^{(n)}(z), v \right\rangle - na \right| = \infty.
\]

Proof. This is a straightforward consequence of Theorems 2.17 and 4.1. \( \square \)

For the proof of Theorem B we shall need the following

Proposition 4.6. If \( f \in \text{Homeo}_0(\mathbb{T}^2) \) is a minimal homeomorphism, then it is not eventually annular.

Proof. By Proposition 2.16, \( f^k \) is minimal for any \( k \in \mathbb{N} \). So, it is enough to show that \( f \) is not annular.

Reasoning by contradiction, let us suppose \( f \) is annular. Then, there exist a lift \( \tilde{f} : \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \) and \( v \in \mathbb{S}^1 \) with rational slope such that
\[
\sup_{n \in \mathbb{Z}, z \in \mathbb{T}^2} \left| \left\langle \Delta_{\tilde{f}}^{(n)}(z), v \right\rangle \right| < \infty.
\]

Since \( v \) has rational slope, by Proposition 2.5 there is no loss of generality assuming \( v = (1, 0) \). So, by Theorem 2.17, there exists \( u \in \text{C}^0(\mathbb{T}^2, \mathbb{R}) \) satisfying
\[
\text{pr}_1 \circ \Delta_{\tilde{f}} = u \circ f - u.
\]

Now, let us consider the continuous maps \( \tilde{g}, \tilde{h} : \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \) given by
\[
\tilde{g}(x, y) := (x, y + \text{pr}_2 \circ \Delta_f(x, y)),
\]
\[
\tilde{h}(x, y) := (x - u(x, y), y),
\]
for every \( (x, y) \in \mathbb{R}^2 \).

As consequence of (31), we know \( \tilde{h} \circ \tilde{f} = \tilde{g} \circ \tilde{h} \), and hence, \( h \circ f = g \circ h \), where \( g \) and \( h \) are the continuous torus maps whose lifts are \( \tilde{g} \) and \( \tilde{h} \), respectively. However, since \( h \) is homotopic to the identity, it is surjective and \( g \) is clearly not minimal, contradicting the minimality of \( f \).

So, \( f \) is not annular. \( \square \)

Even though our next result is rather simple, it may be useful in future works:

Theorem 4.7. Let \( f \in \text{Homeo}_0(\mathbb{T}^2) \) be a minimal homeomorphism, \( \tilde{f} : \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \) a lift of \( f \) and \( p \) any point in \( \rho(\tilde{f}) \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( n \in \mathbb{Z} \) satisfying \( d(np, \mathbb{Z}^2) < \delta \), there is \( z \in \mathbb{R}^2 \) such that
\[
\left\| \tilde{f}^n(z) - z - np \right\| < \varepsilon.
\]
Proof. By Theorem 2.7, there exists $\mu \in \mathcal{M}(f)$ such that $\rho_\mu(\tilde{f}) = \rho$. Since $f$ is minimal, $\mu$ is a topological measure (i.e. has total support and no atoms). So, by Theorem 2.4, there exists $h \in \text{Homeo}(T^2)$ such that $h_*\text{Leb}_2 = \mu$. Moreover, after pre-composing with a linear automorphism of $\mathbb{T}^2$ if necessary, we can assume that $h$ is isotopic to the identity. Then, if $\hat{h} \in \text{Homeo}_0(T^2)$ is a lift of $h$ and we write $\tilde{g} := \hat{h}^{-1} \circ \tilde{f} \circ h$, we have $\tilde{g} \in \text{Symp}_0(T^2)$ and Flux$(\tilde{g}) = \rho$.

Observing the displacement function $\Delta_{\hat{h}}$ is $\mathbb{Z}^2$-periodic, and hence, uniformly continuous, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Delta_{\hat{h}}(x) - \Delta_{\hat{h}}(y)\| < \varepsilon$ whenever $d_T(x, y) < \delta$.

On the other hand, we have

$$\text{Flux}(T_{\rho}^{-n} \circ \tilde{g}^n) = \text{Flux}(T_{\rho}^{-n}) + \text{Flux}(\tilde{g}) = -n\rho + n\rho = 0.$$ 

So, $T_{\rho}^{-n} \circ \tilde{g}^n \in \overline{\text{Ham}(T^2)}$ for every $n \in \mathbb{Z}$. By Theorem 2.10, for each $n$ there exists $z_n \in \mathbb{R}^2$ such that $T_{\rho}^{-n} \circ \tilde{g}^n(z_n) = z_n$. Then,

$$\tilde{f}^n(\tilde{h}(z_n)) = \tilde{h}(z_n + n\rho) = z_n + n\rho + \Delta_{\tilde{h}}(z_n + n\rho),$$

and consequently, defining $w_n := \tilde{h}(z_n)$, we get

$$\|\tilde{f}^n(w_n) - w_n - n\rho\| = \|\Delta_{\tilde{h}}(z_n + n\rho) - \Delta_{\tilde{h}}(z_n)\| < \varepsilon,$$

whenever $d(n\rho, \mathbb{Z}^2) < \delta$. □

5. Stable sets at infinity: transverse direction

All along this section $f \in \text{Homeo}_0(\mathbb{T}^2)$ will continue to denote a minimal homeomorphism and $\tilde{f} \in \overline{\text{Homeo}_0(\mathbb{T}^2)}$ a lift of $f$. Here we start the study of stable sets at infinity associated to (certain lifts of) $f$. We begin considering stable sets at infinity with respect to the horizontal direction assuming there exists a lift $\tilde{f}$ such that the rotation set $\rho(\tilde{f})$ intersects the horizontal axis.

We call this “a transverse direction” because, under the hypotheses of Theorem A, there is no loss of generality assuming $\rho(\tilde{f})$ intersects transversely the horizontal axis, modulo finite iterates, conjugacy and appropriate choice of the lift.

To simplify our notation, in this section we will write $u$ to denote either the vector $(0, 1)$ or $(0, -1)$.

Theorem 5.1. Let $f$ and $\tilde{f}$ be as above, and assume $\rho(\tilde{f})$ intersects the horizontal axis, i.e.

$$\rho(\tilde{f}) \cap \ell^{(0,1)} \neq \emptyset.$$

For each $r \in \mathbb{R}$ and $u \in \{(0, 1), (0, -1)\}$, consider the set

$$\Lambda^u_r := \text{cc} \left( \mathcal{F}(\tilde{f}^r, \mathbb{H}^u), \infty \right),$$

where $\mathcal{F}(\tilde{f}^r, \mathbb{H}^u)$ denotes the maximal invariant set given by (2).

Then, it holds:

(i) $\Lambda^u_r = T_{(1, 0)}(\Lambda^u_s)$, $T_{(0, 1)}(\Lambda^u_s) = \Lambda^u_{r+1}$, and $\Lambda^u_r \subset \Lambda^u_s$, for any $r \in \mathbb{R}$ and any $s < r$;

(ii) $\Lambda^u_r \cap \ell^u_s = \emptyset$, for every $r \in \mathbb{R}$;

(iii) $\mathcal{F}(\tilde{f}^r, \mathbb{H}^u) \cap \mathcal{F}(\tilde{f}^r, \mathbb{H}^{-n}) = \emptyset$, for every $r, r' \in \mathbb{R}$;

(iv) given any $r$ and any connected unbounded closed subset $\Gamma \subset \Lambda^u_r$, it holds

$$\Gamma \cap \ell^u_s \neq \emptyset, \quad \forall s > \inf\{|\text{pr}_2(z)| : z \in \Gamma\};$$

(see Figure 1);
Figure 1. \( \Gamma \subset \Lambda^u \) intersects \( \ell^u \) for \( u = (0,1) \) and \( s > \text{pr}_2(z) \).

(v) \[
\bigcup_{r \in \mathbb{R}} \Lambda^u_r = \mathbb{R}^2;
\]

(vi) For each \( r \in \mathbb{R} \), the set \( \mathcal{J}_f(\mathbb{H}^u_r) \) has empty interior and does not disconnect \( \mathbb{R}^2 \), i.e. \( \mathbb{R}^2 \setminus \mathcal{J}_f(\mathbb{H}^u_r) \) is connected. In particular, this implies \( \Lambda^u_r \) has empty interior and does not disconnect \( \mathbb{R}^2 \) as well.

Proof. Statement (i) easily follows from the fact that \( \mathbb{H}^u_r \subset \mathbb{H}^u \) whenever \( r > s \) and recalling that \( \tilde{f} \) commutes with every integer translation.

To show (ii), let \( \mathbb{A} \) be the open annulus and \( P: \mathbb{R}^2 \to \mathbb{A} \) the covering map as defined in §2.3.2. Since \( \tilde{f} \) commutes with all deck transformations of \( P \), it induces a homeomorphism on \( \mathbb{A} \); and if we write \( \hat{\mathbb{A}} := \mathbb{A} \cup \{-\infty, +\infty\} \) for the two-end compactification of the annulus \( \mathbb{A} \), this homeomorphism admits a unique extension to \( \hat{\mathbb{A}} \). More precisely, one can define \( \hat{f} \in \text{Homeo}(\hat{\mathbb{A}}) \) by

\[
\hat{f}(z) := \begin{cases} 
+\infty, & \text{if } z = +\infty; \\
-\infty, & \text{if } z = -\infty; \\
P(f(\bar{z})), & \text{if } z \in \mathbb{A}, \bar{z} \in P^{-1}(z). 
\end{cases}
\]

Now, we want to show both fixed points \(-\infty\) and \(+\infty\) are indifferent for \( \hat{f} \), according to the classification of fixed points given in §2.9.

Since \( f \) is minimal, it holds \( \text{Fix}(\hat{f}) = \text{Per}(\hat{f}) = \{-\infty, +\infty\} \). In particular, both fixed points are non-accumulated.

On the other hand, since \( \hat{\mathbb{A}} \) is homeomorphic to \( S^2 \) and \( \hat{f} \) is isotopic to the identity, by Lefschetz fixed point theorem one gets

\[
i(\hat{f}^n, -\infty) + i(\hat{f}^n, +\infty) = L(\hat{f}^n) = \chi(\hat{\mathbb{A}}) = 2, \quad \forall n \in \mathbb{Z} \setminus \{0\}.
\]

(34)

Then, we make the following

Claim 5.2. Fixed point indexes at \(+\infty\) and \(-\infty\) satisfy

\[
i(\hat{f}^n, \pm \infty) \leq 1, \quad \forall n \in \mathbb{Z} \setminus \{0\}.
\]

(35)
Since the statement is completely symmetric, we will just prove Claim 5.2 for the point $+\infty$. Let us proceed by contradiction. Suppose that $i(\hat{f}, +\infty) \geq 2$. Then, let $W^+$ and $W^- \subset \hat{A}$ denote the open sets given by Theorem 2.12.

Let $\nu \in \mathcal{M}(f)$ be any ergodic $f$-invariant measure. We will consider the three possible cases: $\text{pr}_2(\rho_\nu(\hat{f}))$ is either positive, negative or zero. Let us start assuming $\text{pr}_2(\rho_\nu(\hat{f})) > 0$. Then by Birkhoff ergodic theorem, for $\nu$-almost every $z \in \mathbb{T}^2$ and every $\hat{\nu} \in (id \times \pi)^{-1}(z)$, it holds

$$\hat{f}^{-n}(\hat{\nu}) \to -\infty \in \hat{A}, \quad \text{as } n \to +\infty,$$

where $id \times \pi: \hat{A} = \mathbb{T} \times \mathbb{R} \to \mathbb{T}^2$ denotes the natural covering map. Since $\hat{f}$ is minimal, $\nu$ is a topological measure and hence, taking into account $W^-$ is open, there exists a point $\hat{\nu} \in W^-$ satisfying (36). This contradicts the fact that $\alpha_f(\hat{\nu}) = \{+\infty\}$, for every $\hat{\nu} \in W^-$. Analogously, one gets a contradiction assuming $\text{pr}_2(\rho_\nu(\hat{f})) < 0$.

So, it just remains to consider the case $\text{pr}_2(\rho_\nu(\hat{f})) = 0$. In such a case, as consequence of by Theorem 2.3 we know $f: \hat{A} \subset \mathbb{T} \times \mathbb{R}$ is non-wandering, i.e. $\Omega(f) = \hat{A}$. In fact, let $\hat{V} \subset \hat{A}$ be a non-empty open set and suppose $\text{diam} \hat{V} < 1/4$; let us define $V := (id \times \pi)(\hat{V}) \subset \mathbb{T}^2$. Since $id \times \pi$ is a covering map, $V$ is open and thus, $\nu(V) > 0$. On the other hand, since we are assuming $\text{pr}_2(\rho_\nu(\hat{f})) = 0$, we have

$$\int_{\mathbb{T}^2} \text{pr}_2 \circ \Delta_f \, d\nu = 0.$$

Then, invoking Theorem 2.3 we know there exists $z \in V$ and $n \geq 1$ such that $f^n(z) \in V$ and $|S^n_f(\text{pr}_2 \circ \Delta_f)(z)| = |\text{pr}_2 \circ \Delta_f^n(z)| < 1/4$. This implies $\hat{V} \cap \hat{f}^{-n}(\hat{V}) = \emptyset$. So, $\hat{A} \subset \Omega(f)$, then clearly we have $\Omega(f) = \hat{A}$. But the both sets $W^+$ and $W^-$ given by Theorem 2.12 are wandering for $\hat{f}$, getting a contradiction.

So, $i(\hat{f}, +\infty) \leq 1$. By Proposition 2.16, one can easily adapt the previous reasoning for the general case, i.e. where $i(f^n, +\infty) \geq 2$, with $|n| \geq 2$; and Claim 5.2 is proven.

Now, putting together (34) and (35) we conclude that

$$i(\hat{f}^k, -\infty) = i(\hat{f}^k, +\infty) = 1, \quad \forall k \in \mathbb{Z}\setminus\{0\}.$$  

By an argument similar to that one we used to prove Claim 5.2, one can show both fixed points $-\infty$ and $+\infty$ are not dissipative (i.e. they are neither attractive nor repulsive). In fact, arguing by contradiction let us suppose, for instance, that there is a trapping neighborhood $V$ of $+\infty$, i.e. $V \subset \hat{A}$ is an open set such that $+\infty \in V$, $-\infty \notin V$ and $\hat{f}(V) \subset V$. Following the very same reasoning we used to prove Claim 5.2 one can conclude in this case that

$$\text{pr}_2(\rho_\nu(\hat{f})) > 0, \quad \forall \nu \in \mathcal{M}(f).$$

However this last estimate is incompatible with the convexity of $\rho(\hat{f})$ and our hypothesis (32). So, $-\infty$ and $+\infty$ are non-dissipative.

Now, combining this last assertion, Theorem 2.13 and (37) we show that $+\infty$ and $-\infty$ are indifferent (according to the classification of fixed points given in §2.9). In other words, if we define

$$V_r^{\pm(0,1)} := P\left(\mathbb{H}_r^{\pm(0,1)}\right) \cup \{\pm\infty\} \subset \hat{A},$$

it can be easily verified that $V_r^{\pm(0,1)}$ is a neighborhood of $\pm\infty$ and therefore,

$$\text{cc} \left(\mathcal{S}_f(V_r^{\pm(0,1)}), \pm\infty\right) \cap \partial V_r^{\pm(0,1)} \neq \emptyset, \quad \forall r \in \mathbb{R},$$
Figure 2. $\hat{\Lambda}_r^u \subset \Lambda_r^u$ where $\partial V_r^{(0,1)} = \mathbb{T} \times \{r\} \subset A$. In particular, the set
\begin{equation}
\hat{\Lambda}_r^{(0,1)} := \text{cc}(\mathcal{F}_f(V_r^u), +\infty) \subset \mathbb{R}^2
\end{equation}
intersects the horizontal line $\ell_r^{(0,1)}$, for every $r \in \mathbb{R}$. By Lemma 2.2, every connected component of $\hat{\Lambda}_r^{(0,1)}$ is unbounded, so it holds
\begin{equation}
\hat{\Lambda}_r^u \subset \Lambda_r^u, \quad \forall r \in \mathbb{R}.
\end{equation}
Hence, (ii) is proven (see Figure 2 for an illustrative representation of the construction we have performed).

Assertion (iii) easily follows from Proposition 4.5 and Proposition 4.6. In fact, let us assume there exists $z \in \mathcal{F}_f(H_r^{(0,1)}) \cap \mathcal{F}_f(H_r^{(-0,-1)})$, for some $r, r' \in \mathbb{R}$. Then, this implies
\begin{equation}
r \leq \text{pr}_2\left(\Delta_f^{(n)}(z)\right) = \sum_{j=0}^{n-1} \text{pr}_2 \circ \Delta_f(\check{f}^j(z)) \leq -r', \quad \forall n \in \mathbb{N}.
\end{equation}
So, by Proposition 4.5, $f$ should be annular and by Proposition 4.6, this is incompatible with minimality of $f$.

Then, let us prove (iv) reasoning by contradiction. Suppose there exists a connected closed unbounded set $\Gamma \subset \Lambda_r^u$ such that $\Gamma \cap \ell_r^u = \emptyset$, for some real number $s > \inf\{|\text{pr}_2(z)| : z \in \Gamma\}$. This means $\Gamma$ is contained in $A_{r,s}^u$, where the strip $A_{r,s}^u$ is given by (6).

By (i) we know that $\Lambda_r^u$ is $T_{1,0}$-invariant. So,
\begin{equation}
\Gamma' := \bigcup_{n \in \mathbb{Z}} T_{1,0}^n(\Gamma) \subset \Lambda_r^u,
\end{equation}
and $\Gamma'$ is contained in $A_{r,s}^u$ as well. Moreover, since $\mathcal{F}_f(H_r^u)$ is a closed $\check{f}$-invariant set, and $\Gamma' \subset \Lambda_r^u \subset F_f(H_r^u)$, we conclude that
\begin{equation}
\Gamma' \subset \mathcal{F}_f(H_r^u) \cap A_{r,s}^u.
\end{equation}
On the other hand, since $\Gamma$ is unbounded, one sees that $\Gamma'$ is contained in the interior of the strip $A_{r-1,s+1}$ and separates both connected components of its boundary.
By (40), this implies

\[ V \] where \( \text{sign} t \) denotes the covering map given by (7). Observe that, since \( (vi) \) is proved.

\[ s \] and condition (32) holds. Fixing a real number \( V \)

\[ \Lambda r, s \]

\[ \Gamma^u_z (s) \]

\[ \ell^u_r \]

\[ \ell^u_s \]

\[ \text{Figure 3. Definition of the set } \Gamma^u_z (s) \text{ sets for } u = (0, 1) \]

Then let us write \( \hat{\Gamma}' := P(\hat{\Gamma}) \) and \( \hat{\Lambda}^u_{T^{-1}, s+1} := P(\hat{\Lambda}^u_{T^{-1}, s+1}) \), where \( P: \mathbb{R}^2 \to \hat{\Lambda} \) denotes the covering map given by (7). Observe that, since \( \hat{\Gamma}' \) is \( T_{1,0} \)-invariant and \( T_{1,0} \) generates the group of deck transformations of \( P \), \( \hat{\Gamma}' \) is a compact subset of \( \hat{\Lambda} \).

So \( \hat{\Gamma}' \subset \hat{\Lambda}^u_{T^{-1}, s+1} \) and when \( \hat{\Gamma}' \) is considered as a compact subset of \( \hat{\Lambda} = \hat{\Lambda} \cup \{ -\infty, +\infty \} \), it separates the horizontal circle \( P(\ell^u_{s+1}) \) and the point \(-\text{sign}(u)\infty \in \hat{\Lambda} \), where \( \text{sign}(u) = 1 \), for \( u = (0, 1) \) and \( \text{sign}(u) = -1 \), for \( u = (0, -1) \).

In the proof of (ii) we have shown that the set \( \text{cc} \left( \mathcal{F}_f(V_{-s-1}^u), -\text{sign}(u)\infty \right) \)

intersects the boundary of \( V_{-s-1}^u \), the set \( V_{-s-1}^u \) is given by (38), and so we have

\[ \text{cc} \left( \mathcal{F}_f(V_{-s-1}^u), -\text{sign}(u)\infty \right) \cap \hat{\Gamma}' \neq \emptyset. \]

By (40), this implies

\[ \emptyset \neq \hat{\Lambda}^u_{-s-1} \cap \Gamma^u_z \subset \mathcal{F}_f(\hat{\Lambda}^u_{-s-1}) \cap \mathcal{F}_f(\hat{\Gamma}'_z), \]

contradicting (iii).

In order to prove (v), first notice that, as a consequence of (i), the set \( \bigcup_{p \in \mathbb{Z}^2} \Lambda^u_r \)

is \( \mathbb{Z}^2 \)-invariant, i.e. it is \( T_p \)-invariant, for every \( p \in \mathbb{Z}^2 \).

On the other hand, since the set \( \Lambda^u_r \) is defined as the union of unbounded connected components of an \( f \)-invariant closed set, it is \( f \)-invariant itself.

So, the set \( \bigcup_{p \in \mathbb{Z}^2} \Lambda^u_p \) is invariant under the abelian subgroup \( \langle f, (T_p)_{p \in \mathbb{Z}^2} \rangle < \text{Homeo}_+(\mathbb{R}^2) \) which acts minimally on \( \mathbb{R}^2 \). Then, \( \bigcup_{p \in \mathbb{Z}^2} \Lambda^u_p \) is dense in \( \mathbb{R}^2 \), as desired.

Last assertion (vi) is a rather straightforward consequence of (iii), (iv) and (v).

In fact, first observe that, combining (iii) and (v) one easily shows that \( \Lambda^u_r \) has empty interior.

On the other hand, if \( \mathbb{R}^2 \setminus \Lambda^u_r \) were not connected, then there should exist a connected component \( V \subset \pi_0(\mathbb{R}^2 \setminus \Lambda^u_r) \) such that \( V \subset \mathbb{H}^u_r \).

By (v), there exists \( r' \in \mathbb{R} \) such that \( \Lambda^u_{-r'} \cap V \neq \emptyset \). If \( z_0 \) is any point in \( \Lambda^u_{-r'} \cap V \), then by (iv) we know that \( \text{cc}(\Lambda^u_{-r'}, z_0) \) is not contained in the strip \( A^u_{r, -r'} \). Consequently, the connected set \( \text{cc}(\Lambda^u_{-r'}, z_0) \) is not contained in \( V \). So it intersects the boundary of \( V \) which is contained in \( \Lambda^u_r \). This contradicts (iii) and (vi) is proved.

Let us continue assuming \( \tilde{f} \in \text{Homeo}_0(\mathbb{T}^2) \) is a lift of a minimal homeomorphism \( f \) and condition (32) holds. Fixing a real number \( r \), for each \( z \in \Lambda^u_r \cap \ell^u_r \) and every \( s > r \) we define the set

\[ \Gamma^u_z(s) := \text{cc} \left( \Lambda^u_r \cap \Lambda^u_{r,s}, z \right), \]
where the strip $\Lambda_{r,s}^u$ is given by (6) (see Figure 3).

As consequence of Theorem 5.1, we get the following result about the geometry of the sets $\Gamma^u_z$:

**Corollary 5.3.** For every $r \in \mathbb{R}$ and $u \in \{(0,1); (0,-1)\}$ the following conditions are satisfied:

(i) for every $z \in \Lambda^u_r \cap \ell^u_r$ and any $s > r$,

\begin{equation}
T^n_{1,0} \left( \Gamma^u_z(s) \right) \cap \Gamma^u_z(s) = \emptyset, \quad \forall n \in \mathbb{Z}\setminus\{0\};
\end{equation}

(ii) for any $s > r$, there exists a real number $D = D(f,s,r) > 0$ such that

\begin{equation}
\diam \left( \pr_1 \left( \Gamma^u_z(s) \right) \right) \leq D, \quad \forall z \in \Lambda^u_r \cap \ell^u_r,
\end{equation}

and so, $\Gamma^u_z(s)$ is compact;

(iii) for every $U \in \pi_0 \left( \mathbb{H}^u_r \setminus \Lambda^u_r \right)$,

\begin{equation}
T^n_{1,0}(U) \cap U = \emptyset, \quad \forall n \in \mathbb{Z}\setminus\{0\}.
\end{equation}

See Figure 4 for a graphical representation of these properties.

**Proof.** Let us fix real numbers $s > r$ and let $z$ denote an arbitrary point in $\Lambda^u_r \cap \ell^u_r$. Reasoning by contradiction, let us start supposing $\diam \left( \pr_1 \left( \Gamma^u_z(s) \right) \right)$ is infinite. Then, the set

\[ \Gamma := \bigcup_{n \in \mathbb{Z}} T^n_{1,0} \left( \Gamma^u_z(s) \right) \]

disconnects $\mathbb{R}^2$ and since $\mathcal{F}_j \left( \mathbb{H}^u_r \right)$ is $T_{1,0}$-invariant, $\Gamma \subset \mathcal{F}_j \left( \mathbb{H}^u_r \right)$. So, $\mathcal{F}_j \left( \mathbb{H}^u_r \right)$ disconnects $\mathbb{R}^2$ contradicting (vi) of Theorem 5.1. Thus it holds

\begin{equation}
\diam \left( \pr_1 \left( \Gamma^u_z(s) \right) \right) < \infty, \quad \forall z \in \Lambda^u_r \cap \ell^u_r, \quad \forall s > r.
\end{equation}

Now suppose (43) is false, i.e. there exist $z$, $s$ and $n$ such that

\( T^n_{1,0} \left( \Gamma^u_z(s) \right) \cap \Gamma^u_z(s) \neq \emptyset \),

with $n \neq 0$. Then, since $\Lambda^u_r$ is $T_{1,0}$-invariant, the set

\[ \bigcup_{n \in \mathbb{Z}} T^n_{1,0} \left( \Gamma^u_z(s) \right) \subset \Lambda^u_r \]

is connected, contains $\Gamma^u_z(s)$ and so, it coincides with $\Gamma^u_z(s)$. Since $n \neq 0$, we conclude $\diam(\pr_1(\Gamma^u_z(s))) = \infty$, contradicting (46).

Property (44) is a straightforward consequence of (43) and (46). In fact, for fixed real numbers $s > r$ and any point $w \in \Lambda^u_r \cap \ell^u_r$, by (46) we know that $\diam(\pr_1(\Gamma^u_w(s)))$ is finite. Then, given any $z \in \Lambda^u_r \cap \ell^u_r$, there exists a unique
Claim 5.4. Every boundary at level

\[ B_n \]

given by

\[ \text{(47)} \]

and \( (44) \) is proved. Thus, it holds

\[ \text{diam} \left( \text{pr}_1 \left( \Gamma_u^n \right) \right) \leq \text{diam} \left( \text{pr}_1 \left( \Gamma_u^\ell(s) \right) \right) + 2, \quad \forall z \in \Lambda^u \cap \ell^u, \]

and \( (44) \) is proved.

To prove \( (45) \), we first need to introduce some definitions: for each connected component \( U \in \pi_0(\mathbb{H}^n \setminus \Lambda^u) \) we define its boundary at level \( r \) by

\[ \partial_r^U := (\partial U \cap \ell_r^u) \setminus \Lambda^u = (U \cap \ell_r^u) \setminus \Lambda^u, \]

where \( \partial U \) denotes the boundary of \( U \) in \( \mathbb{R}^2 \) (see Figure 5).

So, the boundary at level \( r \) operator satisfies the following property:

**Claim 5.4.** Every boundary at level \( r \) is connected and non-empty. In other words, the operator

\[ \partial_r^U : \pi_0(\mathbb{H}^n \setminus \Lambda^u) \to \pi_0(\ell_r^u \setminus \Lambda^u) \]

given by \( (47) \) is a well-defined bijection.

To prove our claim, let us consider an arbitrary point \( z \in \Lambda^u \cap \ell^u \) whose existence is guaranteed by (ii) of Theorem 6.1 and define \( \Gamma^u_z := \text{cc}(\Lambda^u, z) \). Then, notice that \( \Gamma^u_z \) disconnects the half-plane \( \mathbb{H}^n \). In fact, let us consider the one-point compactification of the plane \( \mathbb{R}^2 := \mathbb{R}^2 \cup \{ \infty \} \). Then, the closures \( \ell^u_2 \) and \( \Gamma^u_2 \) in \( \mathbb{R}^2 \) are compact and connected, and the points \( z \) and \( \infty \) belong to both continua. On the other hand, since \( \Lambda^u \) does not disconnect \( \mathbb{R}^2 \) and is \( T_{1,0} \)-invariant, then the intersection \( \ell^u_2 \cap \Gamma^u_2 \) cannot be connected. Thus, by Theorem 2.1, \( \ell^u_2 \cup \Gamma^u_2 \) disconnects \( \mathbb{R}^2 \) and consequently, \( \mathbb{H}^n \setminus \Gamma^u_z \) is not connected. This implies \( \partial_r^U \) is connected for every \( U \in \pi_0(\mathbb{H}^n \setminus \Lambda^u) \), and Claim 5.4 is proved.

On the other hand, since \( \mathbb{H}^n, \ell^u \) and \( \Lambda^u \) are \( T_{1,0} \)-invariant, we observe the translation \( T_{1,0} \) naturally acts on \( \pi_0(\mathbb{H}^n \setminus \Lambda^u) \) and, consequently, on \( \pi_0(\ell^u \setminus \Lambda^u) \), too. Moreover, it can be easily seen that the following diagram commutes

\[
\begin{array}{ccc}
\pi_0(\mathbb{H}^n \setminus \Lambda^u) & \xrightarrow{T_{1,0}} & \pi_0(\mathbb{H}^n \setminus \Lambda^u) \\
\downarrow{\partial^u_z} & & \downarrow{\ell^u_2} \\
\pi_0(\ell^u \setminus \Lambda^u) & \xrightarrow{T_{1,0}} & \pi_0(\ell^u \setminus \Lambda^u)
\end{array}
\]
being the boundary at level \( r \) operator \( \tilde{\sigma}^r_n \) bijective. Hence, the actions of \( T_{1,0} \) on both sets are conjugate and, clearly, there is no periodic orbit for \( T_{1,0} : \pi_0(\ell^u_r \setminus \Lambda^+_s) \subset \). So, there is no periodic orbit for \( T_{1,0} : \pi_0(\mathbb{H}^+_r \setminus \Lambda^+_s) \subset \) either, and (45) is proved. \( \square \)

The rest of this section is devoted to study the geometry of sets \( \Gamma^u_r(s) \) given by (42), assuming \( f \) is not a pseudo-rotation and does not exhibit uniformly bounded \( v \)-deviations. We will show that the connected sets \( \Gamma^u_r(s) \) exhibit unbounded oscillations in the \( v \) direction, as \( s \to +\infty \):

**Theorem 5.5.** Let us assume \( f \) is minimal and \( \tilde{f} \) is a lift of \( f \) such that its rotation set \( \rho(\tilde{f}) \) intersects the upper and lower open semi-planes, i.e., with our notation it holds

\[
\rho(\tilde{f}) \cap \mathbb{H}^{(0,1)}_0 \neq \emptyset \quad \text{and} \quad \rho(\tilde{f}) \cap \mathbb{H}^{(0,-1)}_0 \neq \emptyset.
\]

On the other hand, we know there exist \( \rho \) and \( \rho(\tilde{f}) \) holds

\[
\rho \text{ set relations in the } (42), \text{ assuming } \rho \text{ such that inclusion (28) holds.}
\]

If \( f \) does not exhibits uniformly bounded \( v \)-deviations, then for every \( r \in \mathbb{R} \), any \( z \in \Lambda^+_r \cap \ell^u_r \) and \( s > r \), it holds

\[
\lim_{s \to +\infty} \sup_{w \in \Gamma^+_r(s)} \langle w, v \rangle = +\infty,
\]

and

\[
\lim_{s \to +\infty} \inf_{w \in \Gamma^+_r(s)} \langle w, v \rangle = -\infty.
\]

The proof of Theorem 5.5 will follow combining Theorem 3.1 and the following

**Lemma 5.6.** Under hypotheses of Theorem 5.5, the following holds:

\[
\lim_{s \to +\infty} \sup_{w \in \Gamma^+_r(s)} |\langle w, v \rangle| = +\infty.
\]

**Proof of Lemma 5.6.** For the sake of simplicity of notation, all along this proof we shall just write \( \Lambda^+_r \) and \( \Lambda^-_r \) instead of \( \Lambda^+_r(0,1) \) and \( \Lambda^-_r(0,-1) \), and do the same for any object that depends on the vectors \((0,1)\) or \((0,-1)\). Let us fix an arbitrary real number \( r \). We will just prove (51) for \( \Lambda^+_r \). The other case is completely analogous.

By our hypothesis (48) and Theorem 2.6, there exist two ergodic measures \( \mu^+, \mu^- \in \mathcal{M}(f) \) such that \( \text{pr}_2(\nu^+_{\mu^+}(\tilde{f})) > 0 \) and \( \text{pr}_2(\nu^-_{\mu^-}(\tilde{f})) < 0 \).

By Birkhoff ergodic theorem, for \( \mu^+ \)-almost every \( x \in \mathbb{T}^2 \) and any \( \tilde{x} \in \pi^{-1}(x) \), it holds \( \text{pr}_2(\tilde{f}^n(\tilde{x})) \to +\infty \) and \( \text{pr}_2(\tilde{f}^{-n}(\tilde{x})) \to -\infty \), as \( n \to +\infty \). Analogously, for \( \mu^- \)-almost every \( x \in \mathbb{T}^2 \) and any \( \tilde{x} \in \pi^{-1}(x) \), it holds \( \text{pr}_2(\tilde{f}^n(\tilde{x})) \to -\infty \) and \( \text{pr}_2(\tilde{f}^{-n}(\tilde{x})) \to +\infty \), as \( n \to -\infty \). In particular, this implies

\[
\mu^+(\pi(\Lambda^+_r) \cup \pi(\Lambda^-_r)) = \mu^-(\pi(\Lambda^+_r) \cup \pi(\Lambda^-_r)) = 0.
\]

Then, given any \( \mu^+ \)-generic point \( x^+ \in \mathbb{T}^2, \pi(\Lambda^+_s) \), we can find a point \( z^+ \in \pi^{-1}(x^+) \) such that

\[
\text{pr}_2(\tilde{f}^n(z^+)) > r + 2\left\|\Delta f\right\|_{C^0}, \quad \forall n \geq 0.
\]

So we can define

\[
U_n := cc(\mathbb{H}^+_r \setminus \Lambda^+_s, \tilde{f}^n(z^+)) \in \pi_0(\mathbb{H}^+_r \setminus \Lambda^+_s), \quad \forall n \geq 0.
\]

We claim that the sequence of boundaries at level \( r \), i.e. \( \tilde{\sigma}^+_r(U_n) \) where \( \tilde{\sigma}^+_r \) is given by (47), exhibits bounded “rotational deviations”. More precisely, we make the following
Claim 5.7. There exists a constant $C > 0$ such that for any sequence of real numbers $(a_n)_{n \geq 0}$ satisfying

$$(a_n, r) \in \partial^+_v(U_n), \quad \forall n \geq 0,$$

it holds

$$\left| a_n - n \frac{\alpha}{\text{pr}_1(v)} \right| \leq C, \quad \forall n \geq 0.$$ 

To prove our claim, first observe that, since we are assuming condition (48), $v$ is not vertical, i.e. its first coordinate $\text{pr}_1(v)$ does not vanish, because the rotation set is not horizontal.

Since the measure $\mu^-$ has total support on $\mathbb{T}^2$ and the set $U_0 \subset \mathbb{R}^2$ is open, there is a point $w^- \in U_0$ such that $\pi(w^-)$ is $\mu^-$-generic and consequently, it holds $\text{pr}_2(\tilde{f}^n(w^-)) \to -\infty$, as $n \to +\infty$. Since $z^+, w^- \in U_0$ and $U_0$ is arc-wise connected, there is a continuous path $\gamma : [0, 1] \to U_0$ connecting $w^-$ and $z^+$. Then for every $n$ sufficiently large, $\tilde{f}^n(w^-)$ belongs to semi-plane $\mathbb{H}^-_1$ and, so, there exists $t_n \in [0, 1]$ such that $\tilde{f}^n(\gamma(t_n)) \in \partial^+_v(U_n)$. By inclusion (28) we know that

$$\frac{1}{n} \langle \tilde{f}^n(\gamma(t_n)) - \gamma(t_n), v \rangle \to \alpha, \quad \text{as } n \to +\infty.$$ 

By Claim 5.4 we know $\text{diam}(\partial^+_v(U_n)) \leq 1$ and since both points $\tilde{f}^n(\gamma(t_n))$ and $(a_n, r)$ belong to $\partial^+_v(U_n)$ for $n$ sufficiently big, we conclude that

$$\lim_{n \to +\infty} \frac{a_n}{n} = \frac{\alpha}{\text{pr}_1(v)}.$$ 

To finish the proof of our claim, we use a classical sub-additive argument: let us show there exists $C > 0$ such that

$$\left| a_{m+n} - a_m - a_n \right| \leq C, \quad \forall m, n \geq 0.$$ 

To prove this, let us define the following total order on $\pi_0(\mathbb{H}^+_1 \setminus \Lambda^+_1)$: given any pair of connected components $V, V' \in \pi_0(\mathbb{H}^+_1 \setminus \Lambda^+_1)$, we write

$$V < V' \iff \text{pr}_1(w) < \text{pr}_1(w'), \quad \forall w \in \partial^+_v V, \forall w' \in \partial^+_v V'.$$

For each $n \geq 0$, let us write $p_n := [a_n - a_0] \in \mathbb{Z}$, where $[\cdot]$ denotes the integer part operator. Then, observe that

$$T_{1,0}^{p_n-1}(U_0) < U_n < T_{1,0}^{p_n+1}(U_0), \quad \forall n \geq 0.$$ 

Since $\tilde{f}$ commutes with every integer translation, preserves orientation and the point $z^+$ has been chosen such that (52) holds, we have

$$T_{1,0}^{p_n-1}(U_m) < U_{m+n} < T_{1,0}^{p_n+1}(U_m), \quad \forall m, n \geq 0.$$ 

In particular, this implies that $a_m + p_n - 1 < a_{m+n} < a_m + p_n - 1$ and then,

$$\left| a_{m+n} - a_m - a_n \right| \leq \left| a_0 \right| + 1, \quad \forall m, n \geq 0.$$ 

Then, Claim 5.7 easily follows from (53), (54) and an elementary fact about sub-additive sequences (see for instance [Nav11, Lemma 2.2.1]).

Continuing with the notation we introduced in the proof of Claim 5.7 and since we are assuming $f$ exhibits unbounded $v$-deviations, by (iii) of Proposition 4.5 we know that for every $M > 0$ there exists $n = n(M) \geq 0$ such that

$$\left| \langle \tilde{f}^n(z^+), v \rangle - na \right| > M.$$ 

Hence,

$$\left| \langle \tilde{f}^n(z^+), v \rangle - (a_n, r), v \rangle \right| \geq \left| \langle \tilde{f}^n(z^+), v \rangle - a_n \text{pr}_1(v) \rangle - \left| \text{pr}_2(v) \right|$$

$$\geq \left| \langle \tilde{f}^n(z^+), v \rangle - na \right| - \left| \text{pr}_1(v)C \right| - \left| \text{pr}_2(v) \right|$$

$$\geq M - \left| \text{pr}_1(v)C \right| - \left| \text{pr}_2(v) \right|,$$
where $C$ is the constant given by Claim 5.7. Finally, estimate (51) easily follows from Corollary 5.3, (55) and the fact that $M$ is arbitrary.

Then, Theorem 5.5 will follow combining Theorem 3.1 and Lemma 5.6.

**Proof of Theorem 5.5.** By Lemma 5.6 we know that, for each $u \in \{(0,1), (0,-1)\}$, either (49) or (50) holds.

Reasoning by contradiction and for the sake of concreteness, let us suppose that for every $r \in \mathbb{R}$ and every $z \in \Lambda_r^-$ condition (50) does not hold. Notice here we continue using the notation we introduced in the proof of Lemma 5.6. Analyzing the argument we used in the proof of Lemma 5.6, this implies that

$$\sup_{n \geq 0} \langle \Delta_f^{(n)}(x), v \rangle - n\alpha = +\infty \quad \text{and} \quad \inf_{n \geq 0} \langle \Delta_f^{(n)}(x), v \rangle - n\alpha > -\infty,$$

for $\mu^+$-almost every $x \in \mathbb{T}^2$.

On the other hand, by (iii) of Theorem 5.1 and invoking Lemma 5.6 for $\Lambda_r^-$, we conclude that (50) holds and (49) does not, for any $z \in \Lambda_r^-$ and every $r \in \mathbb{R}$.

However, applying Theorem 3.1 to the ergodic system $(f, \mu^+)$ and the real function $\phi := \langle \Delta_f, v \rangle - \alpha$, and taking into account (56), we conclude that

$$\sup_{n \geq 0} \langle \Delta_f^{(-n)}(x), v \rangle - n\alpha = +\infty \quad \text{and} \quad \inf_{n \geq 0} \langle \Delta_f^{(-n)}(x), v \rangle - n\alpha > -\infty,$$

for $\mu^+$-a.e. $x \in \mathbb{T}^2$.

Now, taking into account that $\text{pr}_2(\Delta_f^{(-n)}(x)) \to -\infty$, as $n \to +\infty$ and $\mu^+$-a.e. $x \in \mathbb{T}^2$, we can repeat the argument we used to prove Lemma 5.6 for negative times and a $\mu^+$-generic point to show that (49) holds for the set $\Lambda_r^-$ as well. Then, we have gotten a contradiction and Theorem 5.5 is proved.

Combining Corollary 5.3 and Theorem 5.5 one can easily strengthen this last result getting the following

**Corollary 5.8.** If $f$, $\tilde{f}$, $v$ and $r$ are as in Theorem 5.5 and $\Gamma$ is a closed connected unbounded subset of $\Lambda_r^+$, then it holds

$$-\inf_{z \in \Gamma} \langle z, v \rangle = \sup_{z \in \Gamma} \langle z, v \rangle = +\infty.$$

6. Stable sets at infinity: parallel direction

Our next purpose consists in defining stable sets at infinity with respect to the same direction of a supporting line of the rotation set. More precisely, if $f \in \text{Homeo}_0(\mathbb{T}^2)$, $\tilde{f} : \mathbb{R}^2 \to \mathbb{S}^1$ is a lift of $f$ and we suppose there are $v \in \mathbb{S}^1$ and $\alpha$ such that the rotation set $\rho(\tilde{f})$ is contained in the line $\ell^\alpha_{v}$, we want to define stable sets at infinity with respect to $v$, i.e. associated to the families of semi-planes $H^+_{v_r}$ and $H^-_{v_r}$, with $r \in \mathbb{R}$.

In such a case it might happen that there is no lift $\tilde{f}$ of $f$ such that the supporting line of $\rho(\tilde{f})$ pass through the origin, and therefore, if we naively defined $\Lambda^+_r(\tilde{f}) = \mathcal{A}_f(H^+_{\tilde{f}})$, we would get $\Lambda^+_r(\tilde{f}) = \emptyset$ for every $\tilde{f}$ and every $r \in \mathbb{R}$.

To overpass this difficulty, we shall use the fiber-wise Hamiltonian skew-product to define such stable sets at infinity.
6.1. The fiber-wise Hamiltonian skew-product. Since we are assuming \( f \) is minimal, by Theorem 2.4 we do not lose any generality assuming \( f \) is area-preserving, i.e. \( f \in \text{Symp}_0(\mathbb{T}^2) \) and let \( \tilde{f} \in \text{Symp}_0(\mathbb{T}^2) \) denote an arbitrary lift of \( f \).

Then, we define the fiber-wise Hamiltonian skew-product associated to \( \tilde{f} \), which can be considered as a particular case of the construction performed in \([KPR18]\).

In very rough words the main idea of this construction consists in splitting our homeomorphism \( f \) into a “rotational” part and a “Hamiltonian” or “rotationless” one. Doing that, the “Hamiltonian” part is responsible by rotational deviations. The main technical advantage of dealing with such skew-products is that an arbitrary point exhibits bounded rotational deviations if and only if its orbit is bounded.

This novel object is certainly the main character of this work and will play a fundamental role in the rest of the paper.

For the sake of simplicity, we fix some notations we shall use until the end of the paper: we write \( \hat{\rho} := \text{Flux}(\tilde{f}) \in \mathbb{R}^2 \) and \( \dot{\rho} := \pi(\hat{\rho}) = \text{Flux}(f) \in \mathbb{T}^2 \).

Then we define the map \( H: \mathbb{T}^2 \to \text{Ham}(\mathbb{T}^2) \) by

\[
H_t := \text{Ad}_t (T_{\hat{\rho}}^{-1} \circ \tilde{f}) = T_{\hat{\rho}}^{-1} \circ \tilde{f} \circ T_t, \quad \forall t \in \mathbb{T}^2, \quad \forall \hat{\rho} \in \pi^{-1}(t),
\]

where \( \text{Ad} \) denotes \( \mathbb{T}^2 \)-action given by (12).

Considering \( H \) as a cocycle over the torus translation \( T_{\rho}: \mathbb{T}^2 \to \mathbb{T}^2 \), one defines the fiber-wise Hamiltonian skew-product associated to \( f \) as the map \( F: \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) given by

\[
F(t, z) := (T_\rho(t), H_t(z)), \quad \forall (t, z) \in \mathbb{T}^2 \times \mathbb{R}^2.
\]

Notice that \( F \) depends just on \( f \) and not on the chosen lift \( \tilde{f} \).

One can easily verify that

\[
F(t, z) = \left( t + \rho, z + \Delta_\tilde{f}(t + \pi(z)) - \hat{\rho} \right), \quad \forall (t, z) \in \mathbb{T}^2 \times \mathbb{R}^2,
\]

where \( \Delta_\tilde{f} \in C^0(\mathbb{T}^2, \mathbb{R}^2) \) is the displacement function given by (10). We will use the following usual notation for cocycles: given \( n \in \mathbb{Z} \) and \( t \in \mathbb{T}^2 \), we write

\[
H_t^{(n)} := \begin{cases} id_{\mathbb{T}^2}, & \text{if } n = 0; \\
H_{t+(n-1)\rho} \circ H_{t+(n-2)\rho} \circ \cdots \circ H_t, & \text{if } n > 0; \\
H_{t-n\rho}^{-1} \circ \cdots \circ H_{t-2\rho}^{-1} \circ H_{t-\rho}^{-1}, & \text{if } n < 0. 
\end{cases}
\]

Then we have

\[
(57) \quad F^n(t, z) = \left( T^n_\rho(t), H_t^{(n)}(z) \right) = \left( t + n\rho, \text{Ad}_t \left( T_{\tilde{\rho}}^{-n} \circ \tilde{f}^n \right)(z) \right),
\]

for every \( (t, z) \in \mathbb{T}^2 \times \mathbb{R}^2 \) and every \( n \in \mathbb{Z} \).

Hence, if \( \rho(\tilde{f}) \) has empty interior, there exist \( \alpha \in \mathbb{R} \) and \( v \in S^1 \) such that inclusion (28) holds, and from (57) it easily follows that a point \( z \in \mathbb{R}^2 \) exhibits bounded \( v \)-deviations if and only if

\[
\left\langle H_0^{(n)}(z) - z, v \right\rangle \leq M, \quad \forall n \in \mathbb{Z},
\]

where \( M = M(z, f) \) denotes the positive constant given by (29).

6.2. Fibered stable sets at infinity. Continuing with previous notation, let \( \alpha \in \mathbb{R} \) and \( v \in S^1 \) be such that property (28) holds. Notice that in such a case, \( \langle \tilde{\rho}, v \rangle = \alpha \).

Then, for each \( r \in \mathbb{R} \) and each \( t \in \mathbb{T}^2 \) we define the fibered \((r, v)\)-stable set at infinity by

\[
(58) \quad \Lambda_r^v(f, t) := \text{pr}_2 \left( \text{cc} \left( \{ t \} \times \mathbb{R}^2 \cap \mathcal{F}_r(\mathbb{T}^2 \times \mathbb{R}^2), \infty \right) \right) \subset \mathbb{R}^2,
\]
Proposition 6.3. Continuing with the same notation of Proposition 6.2, the map 
set such that 
Proposition 6.2
infinity:

Let us also define the \((r, v)\)-stable set at infinity as 

\[ \Lambda_r^v(t) := \bigcup_{t \in \mathbb{T}^2} \{ t \} \times \Lambda_r^v(f, t) \subset \mathbb{T}^2 \times \mathbb{R}^2. \]

For the sake of simplicity, if there is no risk of confusion we shall just write \(\Lambda_r^v(t)\) and \(\Lambda_r^v\) instead of \(\Lambda_r^v(f, t)\) and \(\Lambda_r^v(f)\), respectively.

Now we recall some results of [KPR18]:

\[\text{Theorem 6.1 (Theorem 3.4 in [KPR18]). Assuming inclusion } (28) \text{ holds, for every } r \in \mathbb{R} \text{ the set } \Lambda_r^v \text{ is non-empty, closed and } F\text{-invariant. Moreover, } \Lambda_r^v(t) \neq \emptyset, \text{ for every } t \in \mathbb{T}^2. \]

\[\text{Analogously, the same assertions hold for the } (r, -v)\text{-stable set at infinity.} \]

The following result describes some elementary properties of \((r, v)\)-stable sets at infinity:

**Proposition 6.2** (Proposition 3.6 in [KPR18]). For each \(t \in \mathbb{T}^2\) and any \(r \in \mathbb{R}\), the following properties hold:

1. \(\Lambda_r^v(t) \subset \Lambda_r^{v'}(t)\), for every \(v' < r\);
2. \(\Lambda_r^v(t) = \bigcap_{s < r} \Lambda_s^v(t)\);
3. \(T_{r+\ell,v}( t' - \pi(\ell)) = T_{\ell}(\Lambda_r^{v'}(t'))\), for all \(\ell \in \mathbb{R}^2\) and every \(t' \in \mathbb{T}^2\);
4. \(T_p(\Lambda_r^v(t)) = \Lambda_{r+\ell,p,v}^v(t)\), for every \(p \in \mathbb{Z}^2\).

We shall need the following additional regularity result:

**Proposition 6.3.** Continuing with the same notation of Proposition 6.2, the map 
\(t \to \Lambda_r^v(t)\) is compactly upper semi-continuous, i.e. if \(t_0 \in \mathbb{T}\) and \(U \subset \mathbb{R}^2\) is an open set such that \(\Lambda_r^v(t_0) \subset U\) and \(\mathbb{R}^2\backslash U\) is compact, then there is a neighborhood \(W(t_0)\) of \(t_0\) in \(\mathbb{T}^2\) such that 

\[\Lambda_r^v(t) \subset U, \quad \forall t \in W(t_0).\]

**Proof.** This is a straightforward consequence of the very definition of \((r, v)\)-stable sets at infinity given by (58).

In fact, arguing by contradiction, let us suppose there exists a sequence \(\{t_n\}_{n \geq 1}\) of points of \(\mathbb{T}\) such \(t_n \to t_0\) as \(n \to +\infty\) and 

\[\Lambda_r^v(t_n) \cap (\mathbb{R}^2\backslash U) \neq \emptyset, \quad \forall n \geq 1.\]

For each \(n \geq 1\), let us consider a point \(z_n \in \Lambda_r^v(t_n) \cap (\mathbb{R}^2\backslash U)\). Since the complement of \(U\) is compact, there exists a sub-sequence \(\{z_{n_j}\}_{j \geq 1}\) converging to a point \(z_\infty \in \mathbb{R}^2\backslash U\). However, the whole set \(\Lambda_r^v\) is closed in \(\mathbb{T}^2 \times \mathbb{R}^2\) and thus, \(z_\infty \in \Lambda_r^v(t_0)\) as well, contradicting the fact that \(\Lambda_r^v(t_0) \subset U\). \(\square\)

We also need the following

**Theorem 6.4** (Theorem 4.1 in [KPR18]). If \(f \in \text{Symp}_0(\mathbb{T}^2)\) is periodic point free, i.e. \(\text{Per}(f) = \emptyset\), then for every \(t \in \mathbb{T}^2\) the set 

\[\bigcup_{r > 0} \Lambda_r^v(t)\]

is dense in \(\mathbb{R}^2\).
As a rather straightforward consequence of Theorems 2.17 and 6.4 we get the following

**Corollary 6.5.** If \( f \in \text{Symp}_0(\mathbb{T}^2) \) is minimal and does not exhibit uniformly bounded \( v \)-deviations, then, for every \( r \in \mathbb{R} \) and any \( t \in \mathbb{T}^2 \), the following assertions hold:

(i) \( \Lambda^v_r(t) \cap \Lambda^{-v}_{r'}(t) = \emptyset \), for any \( r' \in \mathbb{R} \);
(ii) \( \Lambda^v_r(t) \) has empty interior;
(iii) \( \Lambda^v_r(t) \) does not disconnect \( \mathbb{R}^2 \), i.e. \( \mathbb{R}^2\backslash \Lambda^v_r(t) \) is connected.

**Proof.** To prove (i) let us start assuming there exists \( z \in \Lambda^v_r(t) \cap \Lambda^{-v}_{r'}(t) \). Thus, putting together (57) and (58) we get
\[
 r \leq \langle T^{-1}_t \circ T^{-n}_r \circ \hat{f}^n \circ T_t(z), v \rangle \leq -r', \quad \forall n \in \mathbb{Z}, \forall \hat{t} \in \pi^{-1}(t),
\]
and consequently, if we define \( \phi \in C^0(\mathbb{T}^2, \mathbb{R}) \) by \( \phi(x) := \langle \Delta f(x), v \rangle \), then it holds
\[
 r \leq S^f_{\hat{t}} \phi(t + \pi(z)) \leq -r', \quad \forall n \in \mathbb{Z}.
\]
Then, since \( f \) is minimal, by Theorem 2.17 we conclude that \( \phi \) is a continuous coboundary for \( f \), and hence, \( f \) exhibits uniformly bounded \( v \)-deviations, contradicting our assumption.

Property (ii) is a straightforward consequence of Theorem 6.4 and property (i).

Finally, in order to show (iii) let us suppose there exists \( r \in \mathbb{R} \) and \( t \in \mathbb{T} \) such that \( \Lambda^v_r(t) \) disconnects \( \mathbb{R}^2 \). So, there exists \( U \in \pi_0 \left( \mathbb{R}^2; \Lambda^v_r(t) \right) \) such that \( U \cap \mathbb{R}^2 \backslash \Lambda^v_r(t) = \emptyset \). Then one can easily check that the boundary of \( U \) is completely contained in \( \Lambda^v_r(t) \) and thus, \( \overline{U} \) is contained in \( \Lambda^v_r(t) \) as well, contradicting property (ii).

\[ \square \]

### 6.3. Rotational deviations and the spreading property.

From now on and until the end of this section, we shall assume \( f \in \text{Symp}_0(\mathbb{T}^2) \) is a minimal homeomorphism such that there is a lift \( \hat{f} \in \text{Symp}_0(\mathbb{T}^2) \) satisfying

\[
\hat{p} = \text{Flux}(\hat{f}) = (\hat{p}_1, 0) \in \mathbb{R}^2
\]

\[
\rho(\hat{f}) \cap \mathbb{R}^{(0,1)} \neq \emptyset \quad \text{and} \quad \rho(\hat{f}) \cap \mathbb{R}^{(-1,0)} \neq \emptyset.
\]

Notice that, since \( f \) is minimal, by Corollary 2.11 we know \( \hat{p}_1 \in \mathbb{R}\backslash \mathbb{Q} \).

Then, if \( F: \mathbb{T}^2 \times \mathbb{R}^2 \ract \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{R}^2 \) denotes the fiber-wise Hamiltonian skew-product induced by \( \hat{f} \), the closed set \( \mathbb{T} \times \{0\} \times \mathbb{R}^2 \subset \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{R}^2 \) is \( F \)-invariant. Making some abuse of notation and for the sake of simplicity, we shall write \( F \) to denote the restriction of the fiber-wise Hamiltonian skew-product to this set. More precisely, from now on we have \( F: \mathbb{T} \times \mathbb{R}^2 \ract \mathbb{T} \times \mathbb{R}^2 \) where

\[
F(t, z) = \left( t + \rho_1, z + \Delta f((t, 0) + \pi(z)) - (\hat{p}_1, 0) \right), \quad \forall (t, z) \in \mathbb{T} \times \mathbb{R}^2,
\]

and \( \rho_1 := \pi(\hat{p}_1) \).

In a joint work with Koropecki [KK09], we introduced the notion of topological spreading, which is stronger than topological mixing:

**Definition 6.6.** A homeomorphism \( h \in \text{Homeo}(\mathbb{T}^2) \) is said to be spreading when for any lift \( \hat{h} \in \text{Homeo}(\mathbb{T}^2) \), any \( R, \varepsilon > 0 \) and any non-empty open set \( U \subset \mathbb{R}^2 \), there exists \( N \in \mathbb{N} \) such that for every \( n \geq N \), there exists a point \( z_n \in \mathbb{R}^2 \) such that \( \hat{h}^n(U) \) is \( \varepsilon \)-dense in the ball \( B_R(z_n) \).

Motivated by this notion, we will prove the following theorem which the main result of this section:
Theorem 6.7. Let us suppose $f$ does not exhibit uniformly bounded $v$-deviations. Then, for every pair of non-empty open sets $U, V \subset \mathbb{R}^2$, there exists $N \in \mathbb{N}$ such that for every $t \in T$ it holds

$$F^n \{(t) \times U \} \cap T \times V \neq \emptyset, \quad \forall n \geq N.$$ 

We shall divide the proof of Theorem 6.7 in several lemmas. Notice that without loss of generality we can assume the open set $V$ in Theorem 6.7 is bounded.

Lemma 6.8. There exists $r \in \mathbb{R}$ such that

$$\Lambda_r^v(t) \cap V \neq \emptyset \quad \text{and} \quad \Lambda_{r}^{-v}(t) \cap V \neq \emptyset, \quad \forall t \in T.$$ 

Proof. This is a straightforward consequence of Theorem 6.4, properties (i) and (iii) of Proposition 6.2, and compactness of $T$. □

From now on we fix a real number $r \in \mathbb{R}$ such that the conclusion of Lemma 6.8 holds.

Since we are assuming $V$ is bounded and $f$ does not exhibit uniformly bounded $v$-deviations, by Theorem 2.1 we know that the set $\Lambda_r^v(t) \cup \Lambda_{r}^{-v}(t) \cup V$ disconnects $\mathbb{R}^2$, for every $t \in T$. For the sake of simplicity of notation, for each $t \in T$ let us write

$$\Gamma_t := \Lambda_r^v(t) \cup \Lambda_{r}^{-v}(t) \cup V.$$ 

Now for each $\varepsilon > 0$, we define the following set:

$$Z_2(v, \varepsilon) := \{ p \in \mathbb{Z}^2 : |\langle p, v \rangle| \leq \varepsilon \}.$$ 

Notice that by (60), the rotation set $\rho(f)$ is not a horizontal segment, so $v$ is not vertical, i.e. $\text{pr}_1(v) \neq 0$. By classical arguments about approximations by rational numbers one can easily get the following

Lemma 6.9. For each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{Z}$, there exist $p \in \{ n, n + 1, \ldots, n + N \}$ and $p \in \mathbb{Z}^2(v, \varepsilon)$ satisfying $\text{pr}_2(p) = p$.

Proof. This easily follows from Corollary 2.15 and the fact that $v$ is not horizontal. □

Lemma 6.10. For every $t \in T^2$ there exist two connected components $W_t^+, W_t^-$ in $\pi_0(\mathbb{R}^2 \setminus \Gamma_t)$ such that the following property holds: for every $z \in \mathbb{R}^2 \setminus \Gamma_t$, there exist $\varepsilon > 0$ and $M \in \mathbb{N}$ such that

$$T_p(z) \in W_t^+, \quad \text{and} \quad T_p^{-1}(z) \in W_t^-,$$

for every $p \in \mathbb{Z}^2(v, \varepsilon)$ satisfying $\text{pr}_2(p) > M$.

Proof. Let $z$ be any point in $\mathbb{R}^2 \setminus \Gamma_t$. By statement (ii) of Proposition 6.2, there exists $\varepsilon > 0$ such that $z \notin \Lambda_{r-2 \varepsilon}(t) \cup \Lambda_{r+2 \varepsilon}(t)$, and so we can consider the positive number

$$\delta := \frac{1}{2} d(z, \Lambda_{r-2 \varepsilon}(t) \cup \Lambda_{r+2 \varepsilon}(t)).$$

Hence, we have

$$T_p(B_\delta(z)) \cap (\Lambda_r^v(t) \cup \Lambda_{r}^{-v}(t)) = \emptyset, \quad \forall p \in \mathbb{Z}^2(v, \varepsilon).$$

Now, since by Corollary 6.5 the set $\Lambda_{r-2 \varepsilon}(t) \cup \Lambda_{r+2 \varepsilon}(t)$ has empty interior and does not disconnect $\mathbb{R}^2$, for each $m \in \mathbb{Z}^2$ we can find a continuous path $\gamma_m : [0, 1] \to \mathbb{R}^2$ such that $\gamma_m(0) = z$, $\gamma_m(1) = T_m(B_\delta(z))$ and

$$\gamma_m(s) \notin \Lambda_{r-2 \varepsilon}(t) \cup \Lambda_{r+2 \varepsilon}(t), \quad \forall s \in [0, 1].$$

Let $N$ denote the natural number given by Lemma 6.9 for $v$ and $\varepsilon$ as above and consider the set

$$A := \{ p \in \mathbb{Z}^2(v, 2 \varepsilon) : 0 \leq \text{pr}_2(p) \leq N \}.$$
Since $A$ is a non-empty finite set and $V$ is bounded, we can define the real number
\[(63) \quad M := 1 + \sup_{m \in A} \sup_{s \in [0,1]} |pr_2(\gamma_m(s))| + \sup_{w \in V} |pr_2(w)| < \infty.\]

Consider any two points $m, p \in \mathbb{Z}^2(v, \varepsilon)$ satisfying $0 \leq pr_2(p) - pr_2(m) \leq N$ and $pr_2(m) > M$. Thus, we have $T_m \circ \gamma_{p-m}$ is a continuous path connecting $T_m(z)$ and the ball $T_p(B_r(z))$; since $m \in \mathbb{Z}^2(v, \varepsilon)$ and (62) holds, the image of $T_m \circ \gamma_{p-m}$ does not intersect $\Lambda^v_r(t) \cup \Lambda^{-v}_r(t)$; and since $p - m \in A$ and $pr_2(m) > M$, invoking (63) we conclude that the image of $T_m \circ \gamma_{p-m}$ does not intersect $V$. So, by (61), $T_m(z)$ and $T_p(z)$ belong to the same connected component of $\mathbb{R}^2 \setminus \Gamma_1$.

Hence, choosing any $m \in \mathbb{Z}^2(v, \varepsilon)$ satisfying $pr_2(m) > M$, we can define
\[W^+_t := cc(\mathbb{R}^2, \Gamma_1, T_m(z)), \]
and combining the last argument with Lemma 6.9, one can shows that $T_p(z) \in W^+_t$, for any $p \in \mathbb{Z}^2(v, \varepsilon)$ such that $pr_2(p) > M$.

To prove the uniqueness of $W^+_t$, let $w$ be any other point in $\mathbb{R}^2 \setminus \Gamma_1$. Let $\varepsilon' \leq \varepsilon$ be any positive number such that $w \in \mathbb{R}^2(\Lambda^v_{r-2\varepsilon'}(t) \cup \Lambda^{-v}_{r-2\varepsilon'}(t))$. So, by Corollary 6.8, the set $\Lambda^v_{r-2\varepsilon'}(t) \cup \Lambda^{-v}_{r-2\varepsilon'}(t)$ does not disconnect $\mathbb{R}^2$, then there exists a continuous path $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(0) = w$, $\gamma(1) = z$ and
\[
\gamma(s) \notin \Lambda^v_{r-2\varepsilon'}(t) \cup \Lambda^{-v}_{r-2\varepsilon'}(t), \quad s \in [0,1].
\]
So, the image of $T_p \circ \gamma$ does not intersect $\Lambda^v_{r}(t) \cup \Lambda^{-v}_{r}(t)$, for any $p \in \mathbb{Z}^2(v, \varepsilon')$, and does not intersect $V$ either, provided $pr_2(p)$ is sufficiently large. Thus, $T_p(w) \in W^+_t$ for such a $p$, and uniqueness of $W^+_t$ is proven.

Finally, defining $W^{-}_t := cc(\mathbb{R}^2, \Gamma_1, T_m^{-1}(z))$ for $m$ as above, one can easily show that analogous properties hold.

In order to finish the proof of Theorem 6.7, we fix a non-empty open set $U \subset \mathbb{R}^2$. Without loss of generality we can assume that $U$ is bounded, connected and
\[(64) \quad \overline{U} \cap (\Lambda^v_{r}(0) \cup \Lambda^{-v}_{r}(-1)) = \emptyset,
\]
where $r$ is the real number we fixed after Lemma 6.8. Since $\overline{U}$ is compact and $\Lambda^v_{r} \cup \Lambda^{-v}_{r}$ is contained in $\mathbb{R}^2 \setminus \overline{U}$, by Proposition 6.3 we know the maps $t \mapsto \Lambda^v_{r}(t)$ and $t \mapsto \Lambda^{-v}_{r}(t)$ are both compactly upper semi-continuous. Thus, there is $\eta > 0$ such that
\[(65) \quad B_\eta(0) \times \overline{U} \cap (\Lambda^v_{r} \cup \Lambda^{-v}_{r}) = \emptyset,
\]
where $B_\eta(0)$ denotes the $\eta$-ball centered at $0 \in \mathbb{T}$ with respect to the distance $d_\mathbb{T}$.

Now, by minimality of $f$ and recalling that $\rho_1 = \pi(\tilde{\rho}_1)$ where $\tilde{\rho}_1 \in \mathbb{R} \setminus \mathbb{Q}$, there exists $k \geq 1$ such that
\[(66) \quad \bigcup_{i=0}^{k} f^i(\pi(U)) = \mathbb{T}^2, \quad \text{and} \quad \bigcup_{i=0}^{k} T^{\rho_1}_t(B_\eta(0)) = \mathbb{T}.
\]

Let us define
\[\mathcal{U} := \bigcup_{i=0}^{k} F^i(B_\eta(0) \times U) \subset \mathbb{T} \times \mathbb{R}^2,
\]
and for every $t \in \mathbb{T}$, let us write
\[\mathcal{U}(t) := pr_2(\mathcal{U} \cap \{t\} \times \mathbb{R}^2) \subset \mathbb{R}^2.
\]
Notice that, by (65), $\mathcal{U} \cap (\Lambda^v_{r} \cup \Lambda^{-v}_{r}) = \emptyset$. So, by (ii) of Proposition 6.2, there exists $\varepsilon > 0$ such that
\[\overline{\mathcal{U}(t) \varepsilon} \cap (\Lambda^v_{r-2\varepsilon}(t) \cup \Lambda^{-v}_{r-2\varepsilon}(t)) = \emptyset, \quad \forall t \in \mathbb{T},
\]
where $(\cdot)\varepsilon$ denotes the $\varepsilon$-neighborhood given by (3).
On the other hand, by our hypothesis (60) there exists \( \hat{\rho}^+ \in \rho(\hat{f}) \) such that \( \text{pr}_2((\hat{\rho}^+)) > 0 \). So, let \( \delta \) be a positive number given by Theorem 4.7 for \( f, \hat{\rho}^+ \) and \( \varepsilon/2 \). Without loss of generality we can assume \( \delta < \min\{\eta, \frac{\varepsilon}{2}\} \), where \( \eta \) was chosen in (65).

Now, consider the translation \( T := T_{\rho_1, \pi \hat{\rho}^+} : \mathbb{T} \times \mathbb{T}^2 = \mathbb{T}^3 \) \( \supseteq \) and the visiting time set \( \tau := \tau(0, B_\delta(0), T) \) defined in Corollary 2.15.

Then, by Theorem 4.7 and (66) we get that, for each \( n \in \tau \) there exist \( z_n \in U \), \( j_n \in \{0, 1, \ldots, k\} \), \( p_n \in \mathbb{Z}^2 \) and \( q_n \in \mathbb{Z} \) such that \( |p_n - n\hat{\rho}^+| < \delta, |q_n - n\hat{\rho}| < \delta \) and

\[
\| \hat{f}^n(z_n) \| - \hat{f}^n(z_n) - n\hat{\rho}^+ \| < \frac{\varepsilon}{2},
\]

or equivalently,

\[
F^n(F^0(z_n)) = \{(j_n + n)p_1 \times T_{\rho_1, \pi \hat{\rho}^-}^{-1} \circ T_{p_n}((\mathcal{U}(j_n, p_1), \varepsilon)) \cap \{(i, \tau) \times \mathcal{W}(i, \varepsilon, \hat{f}) \}\}
\]

Observe that

\[
\| \langle v, p_n - (q_n, 0) \rangle \| = \| \langle v, p_n - n\hat{\rho}^+ + n(\hat{\rho}_1, 0) - (q_n, 0) \rangle \|
\leq \| \langle v, n(\hat{\rho}_1, 0) \rangle \| - \| \langle v, n(\hat{\rho}_1, 0) - (q_n, 0) \rangle \| \leq 2\delta < \frac{\varepsilon}{2}.
\]

So, in particular, this implies \( p_n - (q_n, 0) \in \mathbb{Z}^2(v, \varepsilon) \).

Then observe that since we are assuming \( U \) is connected, \( \mathcal{U}(t, \varepsilon) \) has finitely many connected components for every \( t \in \mathbb{T} \), and then we can apply Lemma 6.10 to conclude there exists \( M > 0 \) such that

\[
F^n((t) \times T_{p}((\mathcal{U}(t), \varepsilon)) \) \subset \{(t + i\rho_1) \times W_{\rho_1}^+\}
\]

for every \( p \in \mathbb{Z}^2(v, \varepsilon) \) satisfying \( \text{pr}_2(p) > M \), any \( t \in \mathbb{T} \) and every \( 0 < i \leq \max\{k, \mathcal{U}(\tau)\} \), where \( \mathcal{U}(\tau) \) denotes the maximum length gap of \( \tau \), just defined after (14).

Putting together (67), (68) and (69), and observing \( \mathcal{U} \) is open, we conclude there is a positive number \( \eta_0^+ > 0 \) such that fixing any \( N_0^+ \in \tau \) verifying \( \text{pr}_2(p) > M, \) where \( p_{N_0}^+ \in \mathbb{Z}^2 \) is chosen as above, it holds

\[
F^m((t) \times U) \subset \{(t + m\rho_1) \times W_0^+ \}, \forall m \geq N_0^+, \forall t \in B_{\eta_0}(0).
\]

Analogously, one may prove a similar statement for some \( \hat{\rho}^- \in \rho(\hat{f}) \cap \mathbb{H}^-, \) showing that there exist \( \eta_0^- > 0 \) and \( N_0^- \in \mathbb{N} \) such that

\[
F^m((t) \times U) \subset \{(t + m\rho_1) \times W_0^- \}, \forall m \geq N_0^-, \forall t \in B_{\eta_0}(0).
\]

Putting together, (64), (70) and (71) we can conclude that

\[
F^m((t) \times U) \cap \{(t + m\rho_1) \times V \} \neq \emptyset, \forall m \geq N_0^+, \forall t \in B_{\eta_0}(0),
\]

where \( N_0 := \max\{N_0^+, N_0^-\} \) and \( \eta_0 := \min\{\eta_0^-, \eta_0^+\} \).

Then, invoking property (57) one can repeat above argument to show that property (72) in fact holds for any \( s \in \mathbb{T} \), i.e. given any \( s \in \mathbb{T} \), there exist \( \eta_s > 0 \) and \( N_s \in \mathbb{N} \) such that

\[
F^m((t) \times U) \cap \{(t + m\rho_1) \times V \} \neq \emptyset, \forall m \geq N_s, \forall t \in B_{\eta_s}(s).
\]

Finally, by compactness of \( \mathbb{T} \) there are points \( s_1, s_2, \ldots, s_r \in \mathbb{T} \) such that

\[
\bigcup_{j=1}^{r} B_{\eta_j}(s_j) = \mathbb{T}.
\]

Defining \( N := \max\{N_j : 1 \leq j \leq r\} \), one can easily verify that the conclusion of Theorem 6.7 holds for any \( n \geq N \).
7. Proof of Theorem A

In this section we finish the proof of Theorem A. To do this let us suppose \( f \) does not exhibit uniformly bounded \( v \)-deviations. By Proposition 2.5, Theorem 2.4 and Proposition 2.16 there is no loss of generality if we assume that \( f \) is a minimal symplectic homeomorphism and admits a lift \( \tilde{f} \in \text{Symp}_0(T^2) \) whose rotation set \( \rho(\tilde{f}) \) is transversal to the horizontal axis and they intersect at the rotation vector of Lebesgue, i.e. it holds

\[
\rho(\tilde{f}) \cap \mathbb{H}^{(0,1)}_0 \neq \emptyset, \quad \text{and} \quad \rho(\tilde{f}) \cap \mathbb{H}^{(0,-1)}_0 \neq \emptyset,
\]

and where \( \text{Flux}(\tilde{f}) = (\tilde{p}_1, 0) \), for some \( \tilde{p}_1 \in \mathbb{R} \). Notice that by Corollary 2.11, \( \tilde{p}_1 \) is irrational.

So, we can define the fiber-wise Hamiltonian skew-product \( F : T \times \mathbb{R}^2 \rightharpoonup \mathbb{R}^2 \) as in §6.3.

By analogy with (33), for each \( r \in \mathbb{R} \) and \( u \in \{(0,1), (0,-1)\} \) we define the stable set at infinity with respect to horizontal direction (we called it the transversal direction in [5]) by

\[
\Lambda^u_r(t) := \text{pr}_2 \left( \text{cc} \left( \{t\} \times \mathbb{R}^2 \cap \mathcal{F}(T^2 \times \mathbb{R}^2), \infty \right) \right) \subset \mathbb{R}^2.
\]

One can easily see that stable sets at infinity defined by (33) and (73) are very close related and, in fact,

\[
\Lambda^u_r(t) = T_{t,0}^{-1}(\Lambda^u_{r+1}(t)), \quad \forall t \in \pi^{-1}(t), \forall r \in \mathbb{R}.
\]

In particular, this implies that all topological and geometric results we proved in Theorems 5.1 and 5.5, and Corollary 5.3 for the sets \( \Lambda^u_r \) continue to hold \textit{mutatis mutandis} for the new ones \( \Lambda^u_r(t) \).

Then we have the following

**Proposition 7.1.** If \( f \) does not exhibit uniformly bounded \( v \)-deviations, then

\[
\Lambda^{(0,1)}_r(t) \cap \Lambda^u_r(t) = \emptyset,
\]

for every \( r, s \in \mathbb{R} \) and every \( t \in T \).

**Proof.** Arguing by contradiction, let us suppose there exist \( r, s \in \mathbb{R} \) and \( t \in T \) such that \( C := \Lambda^{(0,1)}_r(t) \cap \Lambda^u_r(t) \neq \emptyset \). We claim that, in such a case, every connected component of \( C \) is bounded in \( \mathbb{R}^2 \). In order to prove our claim, let us suppose there exists an unbounded closed connected component \( \Gamma \in \pi_0(C) \).

Since \( \Gamma \) is contained in \( \Lambda^{(0,1)}_r(t) \), invoking (iv) of Theorem 5.1 we know that \( \Gamma \) is “vertically unbounded”, i.e. it is not contained in any horizontal strip. On the other hand, since \( \Gamma \subset \Lambda^u_r(t) \subset \mathbb{H}^2_{\mathbb{C}} \), we get that

\[
\langle z, v \rangle \geq s, \quad \forall z \in \Gamma,
\]

which contradicts Corollary 5.8.

So, every connected component of \( C \) is bounded in \( \mathbb{R}^2 \). Invoking Theorem 2.1 and taking into account that \( \Lambda^{(0,1)}_r(t) \cup \Lambda^u_r(t) \) is unbounded, we conclude that \( \Lambda^{(0,1)}_r(t) \cup \Lambda^u_r(t) \) should disconnect \( \mathbb{R}^2 \). Now let us consider two different connected components \( V_1 \) and \( V_2 \) of \( \mathbb{R}^2 \backslash \left( \Lambda^{(0,1)}_r(t) \cup \Lambda^u_r(t) \right) \), and let \( U \subset \mathbb{R}^2 \) be a non-empty connected open set and \( \varepsilon > 0 \) such that

\[
U \cap \left( \Lambda^u_r(t') \cup \Lambda^u_r(t') \right) = \emptyset,
\]

for \( t' \in T \) satisfying \( d_T(0, t') < \varepsilon \).
Then, invoking Theorem 6.7 we know that there is a natural number $N$ such that
\[ F^n(B_x(0) \times U) \cap \mathbb{T} \times V_i \neq \emptyset, \]
for every $i = 1, 2$ and every $n \geq N$. In particular, there is some $n_0 \geq N$ and $t' \in B_x(0)$ such that $R^n(t') = t$ and $F^{n_0}(t') \times U$ intersects $\{t\} \times V_1$ and $\{t\} \times V_2$, and therefore, intersects $\{t\} \times (\Lambda_\alpha^{(0,1)}(t) \cup \Lambda_\alpha^s(t))$ as well, getting a contradiction. \(\Box\)

Now, by Proposition 7.1, given any $r \in \mathbb{R}$ and any $z \in \Lambda_\alpha^r(0)$, we can define the set
\[ U_s := \text{cc}(\mathbb{H}_s^{(0,1)}(\Lambda_\alpha^{(0,1)}(t), z), \forall s < \text{pr}_2(z). \]
Since $\Lambda_\alpha^r(t) \subset \mathbb{H}_r^+$ and is connected, combining Theorem 5.5 and Proposition 7.1 we conclude that
\[ \Lambda_\alpha^r(t) \cap \mathbb{H}_s^{(0,1)}(U_s) \neq \emptyset, \forall s < \text{pr}_2(z), \]
where $\mathbb{H}_s^{(0,1)}$ denotes the boundary operator as level $s$ given by (47).

However, Theorem 5.5 also implies that there is some $s_0 < \text{pr}_2(z)$ such that
\[ \mathbb{H}_s^{(0,1)}(U_{s_0}) \subset \mathbb{H}^{-v}. \]
Since $\Lambda_\alpha^r(t) \subset \mathbb{H}_r^+$, we see that (74) and (75) cannot simultaneously hold, and Theorem A is proved.

8. Proof of Theorem B

Let us suppose there exists a minimal homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$ such that its rotation set is a non-degenerate rational slope segment. So, if $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ is a lift of $f$, then there are $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that inclusion (28) holds. We know that $v$ has rational slope and, by Corollary 2.11, $\alpha$ is an irrational number.

Then, by Theorem A $f$ exhibits uniformly bounded $v$-deviations, i.e. estimate (1) holds. As a straightforward consequence of Theorem 2.17 one can show that $f$ is a topological extension of an irrational circle rotation (see [Jäg09, Proposition 2.1] for details). But this contradicts the following result due to Koropecki, Passaggi and Sambarino [KPS16, Theorem 1]:

**Theorem 8.1.** If $f \in \text{Homeo}_0(\mathbb{T}^2)$ is a topological extension of an irrational circle rotation, then $f$ is a pseudo-rotation.

9. Proof of Theorem C

Let $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that $\rho(\tilde{f}) \subset \ell_\alpha^v$. By Theorem A we know that $f$ exhibits uniformly bounded $v$-deviations. On the other hand, invoking Theorem B we conclude $v$ has irrational slope.

Then, the first step of proof consists in showing the existence of an $f$-invariant torus pseudo-foliation (see §4.1.1 for definitions). Since $f$ is minimal, by Theorem 2.4 there is no loss of generality assuming it is area-preserving, and by Proposition 4.6, $f$ is not eventually annular. So we can invoke Theorem 4.4 to conclude $f$ leaves invariant a torus pseudo-foliation $\mathcal{F}$. Let $\tilde{\mathcal{F}}$ denote its lift to $\mathbb{R}^2$.

In order to study some topological and geometric properties of $\tilde{\mathcal{F}}$, let us recall some simple steps of its construction from [KPR18]. Since $f$ exhibits uniformly bounded $v$-deviations, by [KPR18, Corollary 3.2], there exists a constant $C > 0$ such that every $(r,v)$-stable set at infinity given by (58) satisfies
\[ \mathbb{H}_r^v \subset \Lambda_\alpha^r(0), \forall r \in \mathbb{R}. \]
So, for each \( r \in \mathbb{R} \), we define the open set \( U_r := \text{cc} \left( \int (\Lambda^+_r(0)), \mathbb{H}^+_{r+C} \right) \); and then, we consider the function \( H: \mathbb{R}^2 \to \mathbb{R} \) given by
\[
H(z) := \sup \{ r \in \mathbb{R} : z \in U_r \}, \quad \forall z \in \mathbb{R}^2.
\]
(76)

In the proof of [KPR18, Theorem 5.5], we showed that
\[
H(\bar{f}(z)) = H(z) + \alpha, \quad \forall z \in \mathbb{R}^2,
\]
and then we defined the pseudo-leaves (i.e. the atoms of the partition \( \mathcal{F} \)) by
\[
\mathcal{F}_z := H^{-1}(H(z)), \quad \forall z \in \mathbb{R}^2.
\]

In general the function \( H \) is just semi-continuous, but under our minimality assumption, we will show it is indeed continuous. In fact, let \( \phi: \mathbb{T}^2 \to \mathbb{R} \) be given by
\[
\phi(z) := \left\langle \Delta f(z), v \right\rangle - \alpha, \quad \forall z \in \mathbb{T}^2.
\]
(78)

Since \( f \) exhibits uniformly bounded \( v \)-deviations, invoking Theorem 2.17 we know there is \( u \in C^0(\mathbb{T}^2, \mathbb{R}) \) satisfying
\[
\phi = u \circ f - u.
\]
(79)

However, putting together (58), (76) and (78) one can show that the function \( u': \mathbb{R}^2 \to \mathbb{R} \) given by
\[
u' \left( z,v \right) := \left\langle z, v \right\rangle - H(z), \quad \forall z \in \mathbb{R}^2,
\]
(80)
is semi-continuous, \( \mathbb{Z}^2 \)-periodic and satisfies \( \phi = u' \circ f - u' \), as well. By minimality and classical arguments on semi-continuous functions, we have that \( u - u' \) is a constant, and thus, \( u' \) is continuous. Consequently, function \( H \) given by (76) is continuous as well.

So, for simplicity from now on we can assume that functions \( H \) and \( u \) given by (76) and (79), respectively, satisfy
\[
H(z) = \left\langle z, v \right\rangle - u(z), \quad \forall z \in \mathbb{R}^2,
\]
(81)
and
\begin{equation}
\min_{t \in [0,1]} \text{pr}_2 \circ \gamma(t) < -M < M < \max_{t \in [0,1]} \text{pr}_2 \circ \gamma(t),
\end{equation}
there exists \( t_2 \in [0,1] \) such that \( \gamma(t_2) \in T_{\text{pr}_2}(\tilde{V}) \).

Now, let \( z \) be an arbitrary point of \( \tilde{U} \), and write \( r := H(z) \) and
\begin{equation}
\tilde{U}_z := \text{cc}(\tilde{U} \cap H^{-1}(r - \delta, r + \delta), z).
\end{equation}

Since we are assuming the rotation set \( \rho(\tilde{f}) \) is an irrational slope segment, there exists two \( f \)-invariant Borel probability measures \( \mu \) and \( \nu \) such that \( \text{pr}_2(\rho_\mu(\tilde{f})) \neq \text{pr}_2(\rho_\nu(\tilde{f})) \). By minimality of \( f \), \( \mu \) and \( \nu \) have total support, and this means there exist two points \( z_\mu, z_\nu \in \tilde{U}_z \) such that
\begin{equation}
\frac{\tilde{f}^n(z_\mu) - z_\mu}{n} \to \rho_\mu(\tilde{f}), \quad \text{and} \quad \frac{\tilde{f}^n(z_\nu) - z_\nu}{n} \to \rho_\nu(\tilde{f}),
\end{equation}
as \( n \to \infty \).

By (83), \( \tilde{U}_z \) is arc-wise connected. So, there is a continuous curve \( \eta \in [0,1] \to \tilde{U}_z \) such that \( \eta(0) = z_\mu \) and \( \eta(1) = z_\nu \). Then, by (84) we conclude there exists a natural number \( N_M \) such that
\begin{equation}
|\text{pr}_2 \circ \tilde{f}^n(\gamma(0)) - \text{pr}_2 \circ \tilde{f}^n(\gamma(1))| > 2M, \quad \forall n \geq N_M,
\end{equation}
where \( M \) is the positive real constant invoked in (82).

Observing that, by (77) and (83), one has
\[ \tilde{f}^n(\tilde{U}_z) = \text{cc}(\tilde{f}^n(\tilde{U}) \cap H^{-1}(r + na - \delta, r + na + \delta), \tilde{f}^n(z)), \quad \forall n \in \mathbb{Z}. \]
This implies that, putting together (81), (82) and (85) one can conclude that, for each \( n \geq N_M \) there exists \( q_n \in \mathbb{Z}^2 \) and \( i_n \in \{1, \ldots, k\} \) such that
\[ T_{q_n} \circ \tilde{f} \circ \gamma[0,1] \cap T_{\text{pr}_n}(\tilde{V}) \neq \emptyset, \quad \forall n \geq N_M, \]
and this proves \( f \) is topologically mixing, because the image is of \( \gamma \) is completely contained in \( \tilde{U} \).

In order to show that \( \ell_j^\circ \cap \mathbb{Q}^2 = \emptyset \), we invoke a recent result of Beguin, Crovisier and Le Roux [BCLR17] which extends a previous one due to Kwapisz [Kwa02]. In fact, if there is any rational point on \( \ell_j^\circ \), one can show that \( f \) is flow equivalent to a homeomorphism \( g \in \text{Homeo}_0(\mathbb{T}^2) \) such that \( \rho(\tilde{g}) \) is a vertical line segment, for any lift \( \tilde{g} : \mathbb{R}^2 \to \tilde{g} \) of \( g \) (see [Kwa02, §§2,3] and [BCLR17] for details). Since minimality is preserved under flow equivalence, we conclude that \( g \) is a minimal homeomorphism and its rotation set is a non-degenerate rational slope segment, contradicting Theorem B.

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