SPECTRUM OF WEIGHTED COMPOSITION OPERATORS
PART VII
ESSENTIAL SPECTRA OF WEIGHTED COMPOSITION OPERATORS ON $C(K)$. THE CASE OF NON-INVERTIBLE HOMEOMORPHISMS.

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ABSTRACT. We provide a complete description of the spectrum and the essential spectra of weighted composition operators $T = wT_\varphi$ on $C(K)$ in the case when the map $\varphi$ is a non-invertible homeomorphism of $K$ into itself.

1. Introduction

Let $K$ be a compact Hausdorff space and $C(K)$ be the space of all complex-valued continuous functions on $K$. A weighted composition operator $T$ on $C(K)$ is an operator of the form

$$(Tf)(k) = w(k)f(\varphi(k)), k \in K, f \in C(K),$$

where $\varphi$ is a continuous map of $K$ into itself and $w \in C(K)$.

The spectrum of arbitrary weighted composition operators on $C(K)$ was investigated by the first named author in [3, Theorems 3.10, 3.12, and 3.23]. On the other hand, the full description of essential spectra (in particular Fredholm and semi-Fredholm spectra) of such operators is, as far as we are informed, still not known. In a special case, when the map $\varphi$ is a homeomorphism of $K$ onto itself, such a description was obtained in [4, Theorems 2.7 and 2.11]. In this paper we provide a description of the spectrum (Theorem 3.1) and the essential spectra (Theorems 5.1, 5.2, and 5.6) of a weighted composition operator $T = wT_\varphi$ in the case when $\varphi$ is a non-surjective homeomorphism of $K$ into itself.

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2. Preliminaries

In the sequel we use the following standard notations.
\( \mathbb{N} \) is the semigroup of all natural numbers.
\( \mathbb{Z} \) is the ring of all integers.
\( \mathbb{R} \) is the field of all real numbers.
\( \mathbb{C} \) is the field of all complex numbers.
\( \mathbb{T} \) is the unit circle. We use the same notation for the unit circle considered as a subset of the complex plane and as the group of all complex numbers of modulus 1.
\( \mathbb{U} \) is the open unit disc.
\( \mathbb{D} \) is the closed unit disc.
All the linear spaces are considered over the field \( \mathbb{C} \) of complex numbers.
The algebra of all bounded linear operators on a Banach space \( X \) is denoted by \( L(X) \).

Let \( E \) be a set and \( \varphi : E \to E \) be a map of \( E \) into itself. Then \( \varphi^n, n \in \mathbb{N} \), is the \( n \)th iteration of \( \varphi \),
\( \varphi^0(e) = e, e \in E \),
If \( F \subseteq E \) then \( \varphi^{(-n)}(F) \) means the full \( n \)th preimage of \( F \), i.e. \( \varphi^{(-n)}(F) = \{ e \in E : \varphi^n(e) \in F \} \).
If the map \( \varphi \) is injective then \( \varphi^{-n}, n \in \mathbb{N} \), is the \( n \)th iteration of the inverse map \( \varphi^{-1} \). In this case we will write \( \varphi^{-n}(F) \) instead of \( \varphi^{(-n)}(F) \).

Let \( w \) be a complex-valued function on \( E \). Then \( w_0 = 1 \) and \( w_n = w(w \circ \varphi) \ldots (w \circ \varphi^{n-1}), n \in \mathbb{N} \).

Recall that an operator \( T \in L(X) \) is called semi-Fredholm if its range \( R(T) \) is closed in \( X \) and either \( \dim \ker T < \infty \) or \( \operatorname{codim} R(T) < \infty \).

The index of a semi-Fredholm operator \( T \) is defined as
\[
\operatorname{ind} T = \dim \ker T - \operatorname{codim} R(T).
\]
The subset of \( L(X) \) consisting of all semi-Fredholm operators is denoted by \( \Phi \).
\[
\Phi_+ = \{ T \in \Phi : \operatorname{null}(T) = \dim \ker T < \infty \}
\]
is the set of all upper semi-Fredholm operators in \( L(X) \).
\[
\Phi_- = \{ T \in \Phi : \operatorname{def}(T) = \operatorname{codim} R(T) < \infty \}
\]
is the set of all lower semi-Fredholm operators in \( L(X) \).
\[
\mathcal{F} = \Phi_+ \cap \Phi_-
\]
is the set of all Fredholm operators in \( L(X) \).
\[
\mathcal{W} = \{ T \in \mathcal{F} : \operatorname{ind} T = 0 \}
\]
is the set of all Weyl operators in \( L(X) \).

Let \( T \) be a bounded linear operator on a Banach space \( X \). As usual, we denote the spectrum of \( T \) by \( \sigma(T) \) and its spectral radius by \( \rho(T) \).
We will consider the following subsets of \( \sigma(T) \).
\[
\sigma_p(T) = \{ \lambda \in \mathbb{C} : \exists x \in X \setminus \{0\}, Tx = \lambda x \}.
\]
\[
\sigma_{a.p.}(T) = \{ \lambda \in \mathbb{C} : \exists x_n \in X, \|x_n\| = 1, Tx_n - \lambda x_n \to 0 \}.
\]
\[ \sigma_r(T) = \sigma(T) \setminus \sigma_{a.p.}(T) = \left\{ \lambda \in \sigma(T) : \text{the operator } \lambda I - T \text{ has the left inverse} \right\}. \]

**Remark 2.1.** It is clear that \( \sigma_{a.p.}(T) \) is the union of the point spectrum \( \sigma_p(T) \) and the approximate point spectrum \( \sigma_a(T) \) of \( T \), while \( \sigma_r(T) \) is the residual spectrum of \( T \). We have to notice that the definition of the residual spectrum varies in the literature.

**Remark 2.2.** If needed to avoid an ambiguity, we will use notations \( \sigma(T, X), \rho(T, X) \), et cetera.

Following [2] we consider the following essential spectra of \( T \).

- \( \sigma_1(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \not\in \Phi \} \) is the semi-Fredholm spectrum of \( T \).
- \( \sigma_2(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \not\in \Phi_+ \} \) is the upper semi-Fredholm spectrum of \( T \).
- \( \sigma_2(T') = \{ \lambda \in \mathbb{C} : \lambda I - T \not\in \Phi_- \} \) is the lower semi-Fredholm spectrum of \( T \).
- \( \sigma_3(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \not\in F \} \) is the Fredholm spectrum of \( T \).
- \( \sigma_4(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \not\in W \} \) is the Weyl spectrum of \( T \).
- \( \sigma_5(T) = \sigma(T) \setminus \{ \zeta \in \mathbb{C} : \text{there is a component } C \text{ of the set } \mathbb{C} \setminus \sigma_1(T) \text{ such that } \zeta \in C \text{ and the intersection of } C \text{ with the resolvent set of } T \text{ is not empty} \} \) is the Browder spectrum of \( T \).

The Browder spectrum was introduced in [1] as follows: \( \lambda \in \sigma(T) \setminus \sigma_5(T) \) if and only if \( \lambda \) is a pole of the resolvent \( R(\lambda, T) \). It is not difficult to see ([2, p. 40]) that the definition of \( \sigma_5(T) \) cited above is equivalent to the original definition of Browder.

It is well known (see e.g. [2]) that the sets \( \sigma_i(T), i \in \{1, \ldots, 5\} \) are nonempty closed subsets of \( \sigma(T) \) and that

\[ \sigma_i(T) \subseteq \sigma_j(T), 1 \leq i < j \leq 5, \]

where all the inclusions can be proper. Nevertheless all the spectral radii \( \rho_i(T), i = 1, \ldots, 5 \) are equal to the same number, \( \rho_e(T) \), (see [2, Theorem I.4.10]) which is called the essential spectral radius of \( T \). It is also known (see [2]) that the spectra \( \sigma_i(T), i = 1, \ldots, 4 \) are invariant under compact perturbations, but \( \sigma_5(T) \) in general is not.

It is immediate to see that \( \sigma_1(T) = \sigma_2(T) \cap \sigma_2(T') \) and that \( \sigma_3(T) = \sigma_2(T) \cup \sigma_2(T') \).

Let us recall that a sequence \( x_n \) of elements of a Banach space \( X \) is called *singular* if it does not contain any norm convergent subsequence. We will use the following well known characterization of \( \sigma_2(T) \) (see e.g. [2]). The following statements are equivalent

(a) \( \lambda \in \sigma_2(T) \).
(b) There is a singular sequence \( x_n \) such that \( \|x_n\| = 1 \) and \( \lambda x_n - T x_n \to 0 \).
3. The Spectrum of $T = wT_\varphi$

Let $K$ be a compact Hausdorff space, $\varphi$ be a homeomorphism of $K$ into itself, and $w \in C(K)$. We consider the weighted composition operator $T = wT_\varphi$ on $C(K)$ defined as

$$(Tf)(k) = w(k)f(\varphi(k)), \ f \in C(K), \ k \in K.$$  \hfill (1)

By the reasons outlined in the introduction we will always assume that $\varphi(K) \not\subseteq K$. \hfill (2)

We have to introduce some additional notations.

$L = \bigcap_{n=0}^{\infty} \varphi^n(K), \ M = K \setminus \text{Int}_KL, \ N = L \setminus \text{Int}_KL.$ \hfill (3)

Obviously, $\varphi$ is a homeomorphism of $L$ and $N$ onto themselves and (1) defines the action of $T$ on the spaces $C(L)$, $C(M)$, and $C(N)$.

**Theorem 3.1.** Let $K$ be a compact Hausdorff space, $\varphi$ be a homeomorphism of $K$ into itself, and $w \in C(K)$. Let $T$ be the operator on $C(K)$ defined by (1). Assume (2) and notations in (3). Then

(I) $\sigma(T, C(M))$ is either the disk $\rho(T, C(M))U$ or the singleton $\{0\}.$

(II) $\sigma(T) = \sigma(T, C(M)) \cup \sigma(T, C(L)).$

**Proof.** (I) follows from (2) and Theorems 3.10 and 3.12 in [3]. The proof of (II) will be divided into several steps. Step 1. We will prove the inclusion $\sigma(T, C(M)) \subseteq \sigma(T)$. Assume to the contrary that there is a $\lambda \in \mathbb{C}$, $\lambda \in \sigma(T, C(M)) \setminus \sigma(T)$. Because $0 \in \sigma(T)$, we can assume without loss of generality that $\lambda = 1$. Then $(I - T)C(K) = C(K)$ and because $\varphi(\text{Int}_KL) = \text{Int}_KL = \varphi^{-1}(\text{Int}_KL)$ we also have $(I - T)C(M) = C(M)$. Because $1 \in \sigma(T, C(M))$ there is an $f \in C(M)$ such that $f \neq 0$ and $Tf = f$. Then it follows from Lemma 3.6 in [3] that there is a point $k \in M$ such that

$$|w_n(k)| \geq 1, \ |w_n(\varphi^{-n})| \leq 1, \ n \in \mathbb{N}.$$ \hfill (4)

The point $k$ is either not $\varphi$-periodic or, in virtue of (2), a limit point of the set of all non $\varphi$-periodic points in $K$. It follows from the proof of Theorem 3.7 in [3] that $T \subset \sigma(T)$, in contradiction with our assumption.

Step 2. On this step we prove the inclusion $\sigma(T, C(L)) \subseteq \sigma(T)$. Let $\lambda \in \sigma(T, C(L)) \setminus \sigma(T)$. We can assume that $\lambda = 1$, and like on the previous step $(I - T)C(K) = C(K)$ implies that $(I - T)C(L) = C(L)$. Therefore there is an $f \in C(L)$, $f \neq 0$, such that $Tf = f$. Consider two possibilities.
(a) \( f \not\equiv 0 \) on \( L \setminus \text{Int}_K L \). Let \( k \in L \setminus \text{Int}_K L \) be such that \( |f(k)| = \max_{L \setminus \text{Int}_K L} |f| \). Then like on step 1 we see that \( T \subseteq \sigma(T) \).

(b) \( f \equiv 0 \) on \( L \setminus \text{Int}_K L \). We will define the function \( \tilde{f} \in C(K) \) as

\[
\tilde{f}(k) = \begin{cases} 
  f(k) & \text{if } k \in L \\
  0 & \text{if } k \in K \setminus L.
\end{cases}
\]

Then \( T \tilde{f} = \tilde{f} \), and \( 1 \in \sigma(T) \) contrary to our assumption.

Combining steps 1 and 2 we see that \( \sigma(T, \text{C}(M)) \cup \sigma(T, \text{C}(L)) \subseteq \sigma(T) \).

Step 3. We prove the inclusion \( \sigma(T) \subseteq \sigma(T, \text{C}(M)) \cup \sigma(T, \text{C}(L)) \). Let \( \lambda \in \sigma(T) \). If \( \lambda = 0 \) then \( \lambda \in D \) and therefore without loss of generality we can assume that \( \lambda = 1 \).

Consider first the case when \( 1 \in \sigma_{\text{ap}}(T) \). Then there is a sequence \( f_n \in C(K) \), \( \|f_n\| = 1 \) and \( f_n \to 0 \). But then clearly either \( \|f_n\|_{\text{C}(L)} \not\to 0 \) or \( \|f_n\|_{\text{C}(M)} \not\to 0 \), and therefore

\[
1 \in \sigma_{\text{ap}}(T, \text{C}(L) \cup \sigma_{\text{ap}}(T, \text{C}(M)) \subseteq D \cup \sigma(T, \text{C}(L)).
\]

If on the other hand \( 1 \in \sigma_r(T, \text{C}(K)) \) then there is a regular nonzero Borel measure \( \mu \) on \( K \), \( \mu \in C(K)' \), such that \( T'\mu = \mu \). It is easy to see that \( \text{supp}(\mu) \subseteq L \) whence \( 1 \in \sigma(T, \text{C}(L)) \). \( \square \)

4. Some axillary results

To obtain a description of the essential spectra of \( T \) we will need a series of lemmas. In the statements of all of the lemmas we will assume, without mentioning it explicitly, that \( T \) is an operator on \( C(K) \) defined by \( (\text{I}) \), that \( \varphi \) is a homeomorphism of \( K \) into itself, and that \( (2) \) holds. We will also assume notations from \( (3) \).

**Lemma 4.1.** Assume that \( T \) is invertible on \( \text{C}(L) \) and that \( 0 < |\lambda| < 1/\rho(T^{-1}, \text{C}(L)) \). Then \( (\lambda I - T)C(K) = C(K) \).

**Proof.** It is enough to prove that the operator \( \lambda I - T' \) is bounded from below, where \( T' \) is the Banach dual of \( T \). Assume to the contrary that there is a sequence \( \mu_n \in (C(K))' \) such that \( \|\mu_n\| = 1 \) and \( T'\mu_n - \lambda\mu_n \to 0 \). Because the operator \( T' \) preserves disjointness (see e.g. \( (4) \) Lemma 5.13) we have \( |T'||\mu_n| - |\lambda||\mu_n| \to 0 \). Let \( \mu \in C(K)' \) be a limit point of the set \( \{|\mu_n|\} \) in the weak* topology. Then \( \mu \) is a probability measure on \( K \). Because the operator \( |T'| = |T|' \) is weak* continuous we have \( |T'|\mu = |\lambda|\mu \). But then \( \text{supp}(\mu) \subseteq L \) whence \( |\lambda| \in \sigma(|T'|, \text{C}(L)) \). The last statement involves a contradiction because the operator \( |T| \) is invertible on \( \text{C}(L) \) and \( \rho(|T|^{-1}, \text{C}(L)) = \rho(T^{-1}, \text{C}(L)) \). \( \square \)
Lemma 4.2. (1) Let $\lambda \in \sigma_{ap}(T, C(N))$. Then $\lambda T \subseteq \sigma_2(T)$.

(2) Let $\lambda \in \sigma_{ap}(T', C'(N))$. Then $\lambda T \subseteq \sigma_2(T')$.

Proof. We divide the proof into four steps.

(I) Let $\lambda = 0 \in \sigma_{ap}(T, C(N))$. Then the weight $w$ takes value 0 on $N$. It follows from the definition of $N$ that there are pairwise distinct points $k_n \in K$ such that $|w(k_n)| \leq 1/n$. Let $u_n$ be the characteristic function of the singleton $\{k_n\}$. Then $u_n \in C''(K), \|u_n\| = 1$, the sequence $u_n$ is singular, and $T''u_n \to 0$. Thus $0 \in \sigma_2(T'') = \sigma_2(T)$.

(II) Let $0 \in \sigma_{ap}(T', C'(N))$. Because $T' = (T, \varphi)^\delta w'$ and $(T, \varphi)^\delta$ is a homeomorphism of $C'(N)$ the weight $w$ takes value 0 on $N$. Let $k_n$ be as in part (I) of the proof and $\delta_n$ be the Dirac measure corresponding to the point $k_n$. Then the sequence $\delta_n$ is singular and $T'\delta_n \to 0$.

(III) Let $\lambda \in \sigma_{ap}(T, C(N))$ and $\lambda \neq 0$. Without loss of generality we can assume that $|\lambda| = 1$. Recall that the restriction of $\varphi$ on $N$ is a homeomorphism of $N$ onto itself. Therefore by [3, Lemma 3.6] there is a point $k \in N$ such that $|w_n(k)| \geq 1$ and $|w_n(\varphi^{-n}(k))| \leq 1$, $n \in \mathbb{N}$. Let us fix an $m \in \mathbb{N}$. From the definition of the set $N$ follows that there is a net $\{k_n\}$ of points in $K \backslash L$ convergent to $\varphi^{-m}(k)$. From this trivial observation and from the fact that $K \backslash L$ does not contain $\varphi$-periodic points easily follows the existence of points $k_n \in K \backslash L, n \in \mathbb{N}$ with the properties.

(a) The points $\varphi^i(k_n)$, $-n - 1 \leq i \leq n + 1$ are pairwise distinct.

(b) The sets $A_n = \{\varphi^i(k_n), -n - 1 \leq i \leq n + 1\}$ are pairwise disjoint.

(c) For any $n \in \mathbb{N}$ the following inequalities hold

$$|w_i(k_n)| \geq 1/2 \text{ and } |w_i(\varphi^{-i}(k_n))| \leq 2. \quad (5)$$

Let $u_n$ be the characteristic function of the singleton $\{\varphi^n(k_n)\}$. Then $u_n \in C''(K)$. Let us fix $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$. Consider $F_n \in C''(K)$,

$$F_n = \sum_{i=0}^{2n} (1 - \frac{1}{\sqrt{n}})^{|i-n|} \alpha^{-i}(T'')^iu_n. \quad (6)$$

It follows from [5] and [6] by the means of a simple estimate (see also [3, Proof of Theorem 3.7]) that

$$\|T''F_n - \alpha F_n\| = o(\|F_n\|), \ n \to \infty. \quad (7)$$

Condition (b) guarantees that the sequence $F_n$ is singular and therefore (7) implies that $\alpha \in \sigma_2(T'') = \sigma_2(T)$.

(IV) Let $\lambda \in \sigma_{ap}(T', C'(N))$. It was proved in [3] that there is a point $k \in N$ such that $|w_n(k)| \leq 1$ and $|w_n(\varphi^{-n}(k))| \geq 1$. Then we
can find points $k_n \in K \setminus L$ satisfying conditions (a) and (b) above and also the following condition
\[ |w_i(k_n)| \leq 2 \text{ and } |w_i(\varphi^{-i}(k_n))| \geq 1/2, \; n \in \mathbb{N}. \tag{8} \]
Let $\nu_n$ be the Dirac measure $\delta_{\varphi^{-i}(k_n)}, \alpha \in T,$ and
\[ \mu_n = \sum_{i=0}^{2n} \left( 1 - \frac{1}{\sqrt{n}} \right)^{|i-n|} \alpha^{-i(T')}^i \nu_n, \tag{9} \]
It follows from (8) and (9) that
\[ \|T'\mu_n - \alpha \mu_n\| = o(\|\mu_n\|), \; n \to \infty \tag{10} \]
Condition (b) guarantees that the sequence $\mu_n$ is singular, and therefore (10) implies that $\alpha \in \sigma_2(T').$

**Lemma 4.3.** $\sigma_2(T, C(L)) \subseteq \sigma_2(T) \text{ and } \sigma_2(T', C'(L)) \subseteq \sigma_2(T').$

**Proof.** Let $\lambda \in \sigma_2(T, C(L)).$ Then there is a singular sequence $f_n \in C(L)$ such that $\|f_n\| = 1$ and $T f_n - \lambda f_n \to 0.$ We have to consider two possibilities.
\(1\) $\|f_n\|_{C(N)} \neq 0.$ Then $\lambda \in \sigma_2(T)$ by Lemma 4.2 (1).
\(2\) $\|f_n\|_{C(N)} \to 0.$ Then we can find $g_n \in C(L)$ such that $f_n - g_n \to 0$ and $g_n \equiv 0$ on $N.$ Clearly, the sequence $g_n$ is singular in $C(L).$ We define the function $h_n \in C(K)$ as follows
\[ h_n(k) = \begin{cases} g_n(k) & \text{if } k \in L \\ 0 & \text{if } k \in K \setminus L. \end{cases} \]
The sequence $h_n$ is singular in $C(K)$ and $Th_n - \lambda h_n \to 0.$ Therefore $\lambda \in \sigma_2(T)$.

The second inclusion is trivial. \qed

**Lemma 4.4.** Let $|\lambda| > \rho(T, C(N))$ and $\lambda \notin \sigma_2(T, C(L)).$ Then $\lambda \notin \sigma_2(T).$

**Proof.** Assume to the contrary that there is a singular sequence $f_n \in C(K)$ such that $\|f_n\| = 1$ and $T f_n - \lambda f_n \to 0.$ Because $|\lambda| > \rho(T, C(N))$ and $\rho(T, C(M)) = \rho(T, C(N))$ (see e.g. [3] Theorem 3.23), we have $\|f_n\|_{C(M)} \to 0.$ Therefore, if $g_n$ is the restriction of $f_n$ on $L$ then the sequence $g_n$ is singular in $C(L), ||g_n|| \to 1,$ and $T g_n - \lambda g_n \to 0.$ Thus, $\lambda \in \sigma_2(T, C(L)),$ a contradiction. \qed

**Lemma 4.5.** Let $T$ be invertible on $C(N)$ and $|\lambda| < 1/\rho(T^{-1}, C(N)).$ Assume also that $\lambda \notin \sigma_2(T, C(L)).$ Then the following statements are equivalent.
\(1\) $\lambda \in \sigma_2(T)$.
\(2\) $\text{card}(K \setminus \varphi(K)) = \infty$. 
Proof. By Theorem 3.1 we have \( \lambda \in \sigma(T) \) and by Lemma 4.1 \( (\lambda I - T)C(K) = C(K) \). Therefore \( \lambda \notin \sigma(T) \) if and only if \( \dim \ker (\lambda I - T) = \dim \ker (\lambda I - T\|) < \infty \).

Assume that \( \text{card}(K \setminus \varphi(K)) < \infty \). This condition combined with \( \lambda \notin \sigma(T, C(N)) \) provides that \( \dim \ker ((\lambda I - T), C(M)) < \infty \). Combining it with the condition \( \lambda \notin \sigma_2(T, C(L)) \) we see that \( \dim \ker (\lambda I - T) < \infty \). Thus, \( (1) \Rightarrow (2) \).

Assume next that \( \text{card}(K \setminus \varphi(K)) = \infty \). Then clearly \( \dim \ker (\lambda I - T) = \infty \) and therefore \( \lambda \in \sigma_2(T) \). \( \square \)

Lemma 4.6. The set \( \sigma_2(T, C(M)) \) is rotation invariant and for a \( \lambda \in \sigma(T, C(M)), \lambda \neq 0 \), the following conditions are equivalent.

1. \( \lambda T \cap \sigma_2(T, C(M)) = \emptyset \).
2. \( M \) is the union of two clopen (in \( M \)) subsets \( M_1 \) and \( M_2 \) such that
   a. \( M_2 \neq \emptyset \),
   b. \( \varphi(M_i) \subseteq M_i, i = 1, 2 \),
   c. If \( M_1 \neq \emptyset \) then \( \rho(T, C(M_1)) < |\lambda| \),
   d. \( T \) is invertible on \( C(N_2) \) and \( |\lambda| < 1/\rho(T^{-1}, C(N_2)) \) where \( N_2 = \bigcap_{n=0}^{\infty} \varphi^n(M_2) \),
   e. \( \text{card}(M_2 \setminus \varphi(M_2)) < \infty \).

Proof. The implication \( (2) \Rightarrow (1) \) follows from Lemmas 4.4 and 4.5.

To prove that \( (1) \Rightarrow (2) \) notice that if \( \lambda \in \sigma(T, C(M)) \setminus \sigma_2(T, C(M)) \) then by Lemma 4.2 we have \( \lambda T \cap \sigma_{ap}(T, C(N)) = \emptyset \). We have to consider two possibilities.

(I) \( \lambda T \cap \sigma(T, C(N)) = \emptyset \). Then (see [3]) \( N \) is the union of two clopen (in \( N \)) subsets \( N_1 \) and \( N_2 \) (one of them might be empty), such that

\[
\varphi(N_i) = N_i, i = 1, 2,
\]

\[
\rho(T, C(N_i)) < |\lambda|,
\]

\( T \) is invertible on \( C(N_2) \) and \( |\lambda| < 1/\rho(T^{-1}, C(N_2)) \).

It follows from the definition of \( N \) that \( M \) is the union of two clopen (in \( M \)) subsets \( M_1 \) and \( M_2 \) such that \( N_i = \bigcap_{n=0}^{\infty} \varphi^n(M_i), i = 1, 2 \). It remains to apply Lemmas 4.4 and 4.5.

(II) \( \lambda T \subseteq \sigma_r(T, C(N)) \). Then (see [3] Theorem 3.29]) \( N \) is the union of three pairwise disjoint nonempty subsets \( N_1, N_2, \) and \( O \) such that

a. \( N_i, i = 1, 2 \) are closed subsets of \( N \),

b. \( \varphi(N_i) = N_i, i = 1, 2 \),

c. \( \rho(T, C(N_i)) < |\lambda| \),

d. The operator \( T \) is invertible on \( C(N_2) \) and \( |\lambda| < 1/\rho(T, C(N_2)) \),
(ε) If $V_1$ and $V_2$ are open neighborhoods in $N$ of $N_1$ and $N_2$, respectively, then there is an $n \in \mathbb{N}$, such that for any $m \geq n$ we have $\varphi^m(N \setminus V_2) \subseteq V_1$.

We need to consider two subcases.

(IIa) For any open (in $M$) neighborhood $V$ of $N_2$ there is an infinite subset $E$ of $M \setminus \varphi(M)$ such that

$$\forall k \in E \exists n = n(k) \in \mathbb{N} \text{ such that } \varphi^n(k) \in V.$$  

It follows from (δ) that there are a positive number $\varepsilon$ and open (in $M$) neighborhoods $V_n, n \in \mathbb{N}$ of $N_2$ such that

$$|w_n(t)| \geq (|\lambda| + \varepsilon)^n, t \in V_n. \quad (11)$$

By our assumption there are pairwise distinct points $k_n, n \in \mathbb{N}$ and positive integers $m_n$ such that $k_n \in M \setminus \varphi(M)$ and $u_n = \varphi^{m_n}(k_n) \in V_n$.

We define $f_n \in C''(M)$ as follows.

$$f_n(u_n) = 1,$$

$$f_n(\varphi^{-l}(u_n)) = \frac{w_l(\varphi^{-l}(u_n))}{\lambda^l}, l = 1, \ldots, m_n,$$

$$f_n(\varphi^l(u_n)) = \frac{\lambda^l}{w_l(u_n)}, l = 1, \ldots, n,$$

$$f_n(k) = 0 \text{ otherwise.}$$

It follows from the definition of $f_n$ and (II) that $\|f_n\| \geq 1$ and $T'' f_n - \lambda f_n \to 0$. Because the sequence $f_n$ is singular we get $\lambda \in \sigma_2(T'', C''(M)) = \sigma_2(T, C(M))$, a contradiction.

(IIb) There is an open (in $M$) neighborhood $V$ of $N_2$ such that the set

$$F = \{k \in M \setminus \varphi(M) : \exists n \in \mathbb{N} \text{ such that } \varphi^n(k) \in V\}$$

is at most finite. It follows from the definition of $N$ that $F$ cannot be empty. Clearly $F$ consists of points isolated in $M$. We will bring the assumption that $F$ is finite to a contradiction. It is not difficult to see from (ε) that there is a $k \in F$ such that the intersection of $\text{cl}\{\varphi^n(k) : n \in \mathbb{N}\}$ with each of the sets $N_1$, $N_2$, and $O$ is not empty. Therefore we can assume without loss of generality that $M = \text{cl}\{\varphi^n(k) : n \in \mathbb{N}\}$.

Let $W$ be an open neighborhood of $N_1$ in $M$ such that $\text{cl}W \cap N_2 = \emptyset$.

It follows from (ε) that there is an $m \in \mathbb{N}$ such that $\varphi^m(W) \subseteq W$. Considering, if necessary, the operator $T^m$ instead of $T$ we can assume that $m = 1$. There is a $p \in \mathbb{N}$ such that $\varphi^p(k) \in W$. Then $\varphi^n(k) \in W$ for any $n \geq p$, a contradiction. \hfill \Box

**Lemma 4.7.**

$$\sigma_2(T', C'(M)) \cup \{0\} = \sigma_2(T', C'(N)) \cap \{0\}. $$
Proof. The inclusion $\sigma_2(T', C'(N)) T \cup \{0\} \subseteq \sigma_2(T', C''(M)) \cup \{0\}$ follows from Lemma 4.2.

To prove the converse inclusion consider $\lambda \in \sigma_{ap}(T', C'(N)) \setminus \{0\}$. The proof of Lemma 4.1 shows that $|\lambda| \in \sigma_{ap}([T'], C'(N))$. But then (see [4]) $\lambda \in \sigma_{ap}(T', C'(N)) T$. \hfill \qed

5. Description of essential spectra of $T = wT_{\varphi}$

Finally we can provide a complete description of essential spectra of weighted composition operators on $C(K)$ induced by non-surjective homeomorphisms. The statements of Theorems 5.1 and 5.2 below follow from the previous lemmas.

Theorem 5.1. Let $K$ be a compact Hausdorff space, $\varphi$ be a homeomorphism of $K$ into itself, and $w \in C(K)$. Let $T$ be the operator on $C(K)$ defined by (7). Assume (2) and notations in (3). Let $\lambda \in \sigma(T) \setminus \{0\}$. The operator $\lambda I - T$ is upper semi-Fredholm if and only if the following conditions are satisfied

(a) The operator $\lambda I - T$ is upper semi-Fredholm on $C(L)$.

(b) The set $M$ is the union of two $\varphi$-invariant disjoint closed subsets $M_1$ and $M_2$ such that

(c) if $M_1 \neq \emptyset$ then $\rho(T, C(M_1)) < |\lambda|$, 

(d) if $M_2 \neq \emptyset$ then $T$ is invertible on $C(N_2)$, where $N_2 = \bigcap_{n=0}^{\infty} \varphi^n(M_2)$, 

$|\lambda| < 1/\rho(T^{-1}, C(N_2))$, and the set $M_2 \setminus \varphi(M_2)$ is finite.

Moreover,

$\dim \ker (\lambda I - T) = \dim \ker (\lambda I - T, C(L)) + \text{card}(M_2 \setminus \varphi(M_2))$.

Theorem 5.2. Let $K$ be a compact Hausdorff space, $\varphi$ be a homeomorphism of $K$ into itself, and $w \in C(K)$. Let $T$ be the operator on $C(K)$ defined by (7). Assume (2) and notations in (3). Let $\lambda \in \sigma(T) \setminus \{0\}$. The operator $\lambda I - T$ is lower semi-Fredholm if and only if the following conditions are satisfied

(a) The operator $\lambda I - T$ is lower semi-Fredholm on $C(L)$.

(b) $\lambda T \subseteq \sigma_{r}(T', C''(N))$.

Moreover, $\text{def}(\lambda I - T) = \text{def}(\lambda I - T, C(L))$.

Corollary 5.3. Assume conditions of Theorem 5.1. Let $\lambda \in \sigma(T) \setminus \{0\}$. The operator $\lambda I - T$ is Fredholm if and only if it is Fredholm on $C(L)$ and conditions (b) - (d) from the statement of Theorem 5.1 are satisfied.

Moreover $\text{ind}(\lambda I - T) = \text{ind}(\lambda I - T, C(L)) + \text{card}(M_2 \setminus \varphi(M_2))$.

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1 In particular, if $\lambda \notin \sigma(T, C(L))$. 
Corollary 5.4. Assume conditions of Theorem 5.1. Assume additionally that the set of all $\varphi$-periodic points is of first category in $K$. Then the spectrum $\sigma(T)$ and the essential spectra $\sigma_i(T), i = 1, \cdots, 5$ are rotation invariant.

Corollary 5.5. Assume conditions of Theorem 3.1.

(1) If the set of all isolated $\varphi$-periodic points is empty, then $\sigma_5(T) = \sigma(T)$.

(2) If $K$ has no isolated points (in particular, if $\text{Int}_K L = \emptyset$), then $\sigma_3(T) = \sigma(T)$

Proof. The proof follows from Theorems 3.1 and 5.1, and from [4, Theorems 2.7 and 2.11].

To finish our description of essential spectra of $T$ it remains to look at the case $\lambda = 0$;

Theorem 5.6. Assume conditions of Theorem 3.1. Then

(1) The operator $T$ is upper semi-Fredholm if and only if the following two conditions are satisfied

(a) The set $Z(w) = \{k \in K : w(k) = 0\}$ is either empty or all of its points are isolated in $K$,

(b) the set $K \setminus \varphi(K)$ is finite.

Moreover, $\dim \ker T = \text{card}(\{K \setminus \varphi(K)\} \cup Z(w))$.

(2) The operator $T$ is lower semi-Fredholm if and only if the set $Z(w) = \{k \in K : w(k) = 0\}$ is either empty or all of its points are isolated in $K$.

Moreover, $\text{def} T = \text{card} Z(w)$.

(3) The operator $T$ is Fredholm if and only if it is semi-Fredholm and $\dim \ker T < \infty$.

(4) The operator $T$ is Fredholm and $\text{ind} T = 0$ if and only if $T$ is Fredholm and $w \equiv 0$ on $K \setminus \varphi(K)$.

(5) $0 \in \sigma_5(T)$.

Proof. (1) Assume that $T$ is semi-Fredholm and that $\dim \ker T < \infty$. Then the same is true for $T''$. If $k \in K \setminus \varphi(K)$ then $T'' \chi_k = 0$ where $\chi_k \in C(K)''$ is the characteristic function of the singleton $\{k\}$. Therefore $\text{card}(K \setminus \varphi(K)) < \infty$.

Similarly, if $k \in \varphi(K)$ and $w(k) = 0$ then $T'' \chi_{\varphi(k)} = 0$ whence $Z(w)$ is finite or empty. Assume now that $w(k) = 0$ but $k$ is not isolated in $K$. Then there is a sequence of pairwise distinct points $k_n \in K$ such that $|w(k_n)| \leq 1/n$. The sequence $\chi_{\varphi(k_n)}$ is singular in $C(K)''$ and $T'' \chi_{\varphi(k_n)} \to 0$ whence $0 \in \sigma_2(T)$.
Conversely, assume conditions (a) and (b). Assume also, contrary to the statement of the theorem that there is a singular sequence \( f_n \in C(K) \) such that \( \|f_n\| = 1 \) and \( Tf_n \rightarrow 0 \). It is immediate to see that \( f_n \rightarrow 0 \) uniformly on \( E = \varphi(K) \setminus \varphi(Z(w)) \). Because the set \( K \setminus E \) is finite the sequence \( f_n \) contains a convergent subsequence and thus cannot be singular.

Finally, if \( Tf = 0 \) then \( \text{supp } f \subseteq K \setminus \varphi(K) \cup Z(w) \) whence \( \text{dim ker } T = \text{card}(K \setminus \varphi(K) \cup Z(w)) \).

(2) Assume that \( T \) is semi-Fredholm and that \( \text{def } T < \infty \). If \( k \in Z(w) \) then \( T'\delta_k = 0 \) whence \( Z(w) \) is either finite or empty.

Conversely, if \( \text{card } Z(w) < \infty \), \( \|\mu_n\| = 1 \), and \( T'\mu_n \rightarrow 0 \) then (because \( T' \) preserves disjointness) \( |T'\mu_n| = |T'\mu_n| \rightarrow 0 \). Let \( \nu_{2n} \) be the restrictions of the measure \( |\mu_n| \) on \( Z(w) \) and \( K \setminus Z(w) \), respectively. Because \( Z(w) \) is finite there is a positive constant \( c \) such that \( |w| > c \) on \( K \setminus Z(w) \). Therefore \( \nu_{2n} \rightarrow 0 \) and we can find a norm convergent subsequence of the sequence \( \mu_n \). Therefore, \( 0 \not\in \sigma_2(T') \).

It is immediate to see that if \( T'\mu = 0 \) then \( \text{supp } \mu \subseteq Z(w) \) whence \( \text{def } T = \text{card } Z(w) \).

(3) and (4) follow immediately from (1) and (2).

(5) If \( \sigma(T, C(M)) \) is a disk of positive radius then it follows directly from the definition of \( \sigma_5(T) \) that \( 0 \in \sigma_5(T) \). On the other hand, if \( \rho(T, C(M)) = 0 \) then there is a point \( k \in N \) such that \( w(k) = 0 \). Because \( k \) is not an isolated point of \( K \) we see that \( 0 \in \sigma_2(T) \cap \sigma_2(T') = \sigma_1(T) \subseteq \sigma_5(T) \). □

**Example 5.7.** Let \( Tf(x) = f(x/2), f \in C[0, 1], x \in [0, 1] \). Then

1. \( \sigma_5(T) = \sigma_4(T) = \sigma_3(T) = \sigma_2(T) = \sigma_1(T) = \sigma(T) = \mathbb{D} \).
2. \( \sigma_2(T') = \sigma_1(T) = T \).

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