Derivation of Non-Local Macroscopic Traffic Equations and Consistent Traffic Pressures from Microscopic Car-Following Models

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Abstract. This contribution compares several different approaches allowing one to derive macroscopic traffic equation directly from microscopic car-following models. While it is shown that some conventional approaches lead to theoretical problems, it is proposed to use a smooth particle hydrodynamic approach and to avoid gradient expansions. The derivation circumvents approximations and, therefore, demonstrates the large range of validity of macroscopic traffic equations, without the need of averaging over many vehicles. It also gives an expression for the “traffic pressure”, which generalizes previously used formulas. Furthermore, the method avoids theoretical inconsistencies of macroscopic traffic models, which have been criticized in the past by Daganzo and others.

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1 Introduction

In order to describe the dynamics of traffic flows, a large number of mathematical models has been developed. The analysis of the spatio-temporal features and statistics of traffic patterns has often been done with methods from statistical physics and non-linear dynamics. An overview of modeling approaches and methods is, for example, given in Refs. [1, 2, 3, 4], among them cellular automata, “microscopic” car-following models, “mesoscopic” gas-kinetic, and macroscopic traffic models.

Cellular automata can often be interpreted as discretized versions of car-following models, while gas-kinetic models have frequently been used to derive macroscopic from microscopic models. Such derivations were driven by the desire to improve phenomenological specifications of macroscopic traffic models [5, 10, 12, 13], which were criticized to have unrealistic properties [11]. However, the derivation of gas-kinetic models from car-following models usually simplifies the interactions among vehicles by a collisional approach assuming immediate braking maneuvers. Moreover, the derivation of macroscopic traffic models from gas-kinetic ones terminates an infinite and poorly converging series expansion, which replaces dynamical equations for higher moments of the velocity distribution by simplified equilibrium relationships [11].

Although this leads to macroscopic equations which work well in most theoretical and practical aspects [12], the implications of the approximations are hardly known. Moreover, the approach seems to require an averaging over at least 100 vehicles for each speed class and spatial location. While this constitutes no problem for gases with $10^{23}$ particles within a small volume, for traffic flows this would require an averaging over spatial intervals much greater than the scale on which traffic flow changes. Hence, it is not well understood, whether or why macroscopic traffic equations can be used at all.

In this paper, we will therefore focus on attempts to derive macroscopic traffic equations directly from microscopic ones. Doing so, we will compare three different approaches: First, we study the gradient expansion approach in Sec. 2. Second, we turn to the linear interpolation approach in Sec. 3. Third, we discuss the smooth particle hydrodynamics approach in Sec. 4 and compare the results with macroscopic traffic models such as the Payne model, the Aw-Rascle model, and a non-local traffic model. In the Conclusions, we summarize and discuss our results, in particular with regard to the mathematical form of the traffic pressure and the theoretical consistency of macroscopic traffic models.

2 The Gradient Expansion Approach

Already in the 1970’s, Payne [10, 11] used a gradient expansion approach to derive a macroscopic velocity equation complementing the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} [\rho(x, t)V(x, t)] = 0. \quad (1)$$
It relates the vehicle density $\rho(x, t)$ at location $x$ and time $t$ with the average velocity $V(x, t)$ or the vehicle flow
\[ Q(x, t) = \rho(x, t)V(x, t) , \tag{2} \]
respectively, and describes the conservation of the number of vehicles $[12]$.

Payne derived his model from Newell’s car-following model $[13]$
\[ v_i(t + \tau) = v_o(d_i(t)) , \tag{3} \]
which assumes that the speed $v_i(t)$ of vehicle $i$ at time $t$ will be adjusted with a delay of $\tau$ to some optimal speed $v_o$, which depends on the distance $d_i(t) = x_{i-1}(t) - x_i(t)$ between the location of the leading vehicle $x_{i-1}(t)$ and the location $x_i(t)$ of the following car.

Payne identified microscopic and macroscopic velocities as follows:
\[ v_i(t + \tau) = V(x + V\tau, t + \tau) \approx V(x, t) + V\tau \frac{\partial V(x, t)}{\partial x} + \tau \frac{\partial V(x, t)}{\partial t} . \tag{4} \]

Then, Taylor approximations (gradient expansions) were used in several places. For example, Payne substituted the inverse of the distance $d_i$ to the leading vehicle by the density $\rho$ at the place $x + d_i/2$ in the middle between the leading and the following vehicle. In this way, he obtained
\[
\frac{1}{d_i(t)} = \rho \left( x + \frac{d_i(t)}{2}, t \right) = \rho \left( x + \frac{1}{2\rho}, t \right) \\
\approx \rho(x, t) + \frac{1}{2\rho} \frac{\partial \rho(x, t)}{\partial x} . \tag{5} 
\]

When defining the so-called equilibrium velocity $V_e(\rho)$ through
\[ V_e(\rho) = v_o \left( \frac{1}{\rho} \right) \quad \text{or} \quad V_e \left( \frac{1}{d_i} \right) = v_o(d_i) , \tag{6} \]
a first order Taylor approximation and Eq. (5) imply
\[
v_o(d_i(t)) = V_e \left( \frac{1}{d_i(t)} \right) = V_e(\rho(x, t)) + \frac{1}{2\rho(x, t)} \frac{dV_e(\rho)}{d\rho} \frac{\partial \rho(x, t)}{\partial x} . \tag{7} \]

Starting from the previous equations, one finally arrives at Payne’s macroscopic velocity equation
\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{1}{\tau} \left[ V_e(\rho) - \frac{D(\rho)}{\rho} \frac{\partial \rho}{\partial x} - V(x, t) \right] , \tag{8} 
\]
where we have introduced the density-dependent diffusion
\[ D(\rho) = -\frac{1}{\rho} \frac{dV_e(\rho)}{d\rho} \geq 0 . \tag{9} \]

The single terms of Eq. (8) have the following interpretation: The term $\partial V/\partial x$ is called the transport term and describes a motion of the velocity profile with the vehicles. The term $-D(\rho)/(\rho \Delta t) \partial \rho/\partial x$ is called anticipation term, as it reflects the reaction of drivers to the traffic situation in front of them. The relaxation term $[V_e(\rho) - V]/\Delta t$ delineates the adaptation of the average velocity $V(x, t)$ to the density-dependent equilibrium velocity $V_e(\rho)$ with a delay $\tau$.

Other authors have applied similar gradient expansions to the optimal velocity model defined by
\[
\frac{dv_i(t)}{dt} = \frac{1}{\tau} \left[ v_o(d_i(t)) - v_i(t) \right] \tag{10} \]
with $dd_i/dt = v_{i-1}(t) - v_i(t)$, see e.g. Refs. [15, 16]. Equation (10) results from the Newell model (3) by a first-order Taylor approximation $v_i(t + \tau) \approx v_i(t) + \tau dv_i/dt$. Regarding the derivation of macroscopic traffic equations from the optimal velocity model, it is also worth reading Refs. [15, 16].

One weakness of the gradient expansion approach is that it implicitly assumes small gradients in order to be mathematically valid. It is well-known, however, that many microscopic and macroscopic traffic equations give rise to emergent traffic jams, which are related with steep gradients. That would require the consideration of higher-order terms and lead to macroscopic traffic equations that are not anymore simple and well tractable (even numerically). Let us, therefore, study other approaches to determine macroscopic from microscopic equations.

### 3 The Linear Interpolation Approach

The optimal velocity model may be also written in the form
\[
\frac{dv_i(t)}{dt} = a_i(t) = v^0 - \frac{v_i(t)}{\tau} + f(d_i(t)) , \tag{11} \]
where $a_i(t)$ denotes the acceleration, $v^0$ the “desired velocity” or “free speed”, and
\[ f(d_i) = \frac{v_0(d_i) - v^0}{\tau} \leq 0 \tag{12} \]
the repulsive interaction among the leading vehicle $i - 1$ and its follower $i$.

In Ref. [17], it has been suggested to establish a micro-macro link between microscopic and macroscopic traffic variables by the definitions
\[
\rho(x, t) = \frac{1}{x_i(t) - x_{i+1}(t)} \left[ x_{i-1}(t) - x \right] \tag{13} \]
\[ \frac{1}{x_i(t) - x_{i-1}(t)} \left[ x - x_i(t) \right] \]
\[ \frac{1}{x_{i-1}(t) - x} \]
\[ \frac{1}{x_i(t) - x_{i-1}(t)} \]
These definitions assume that the macroscopic variables in
the vehicle locations \( x = x_i(t) \) would be given by the micro-
scopic ones, while in locations \( x \) between two vehicles, they
would be defined by linear interpolation.

Let us consider the consequences of such an approach.

For this, we determine the partial derivative of
\[
G(x, t) = \frac{g_i(t)[x_{i-1}(t) - x] + g_{i-1}(t)[x - x_i(t)]}{x_{i-1}(t) - x_i(t)}
\]
with respect to \( x \), which gives
\[
\frac{\partial G(x, t)}{\partial x} = -\frac{g_i(t) + g_{i-1}(t)}{x_{i-1}(t) - x_i(t)}
\]
for any specification of \( g_i(t) \), for example, \( g_i(t) = v_i(t) \).

The partial derivative with respect to time is
\[
\frac{\partial G(x, t)}{\partial t} = \frac{dg_i(t)[x_{i-1}(t) - x] + g_{i-1}(t)dx_i(t)}{x_{i-1}(t) - x_i(t)}
\]
for which we find the exact relationship
\[
\frac{dg_i(t)}{dt} = \frac{\partial G(x, t)}{\partial x} \frac{dx_i(t)}{dt}.
\]

4 The Smooth Particle Hydrodynamics Approach

4.1 Derivation of the Continuity Equation

In this section, we will derive macroscopic traffic equations
directly from microscopic ones. We will start with the
derivation of the continuity equation from the equation
of motion \( dx_i/dt = v_i \), using a “trick” that I learned
from Isaac Goldhirsch. For this, we represent the location
\( x_i(t) \) of an element \( i \) in space by a delta function
\( \delta(x - x_i(t)) \), which may be treated here like a very narrow
Gaussian distribution. Moreover, we introduce a symmetri-
cal smoothing function
\[
s(x' - x) = s(|x' - x|) = s(x - x'),
\]
for example, a Gaussian distribution with a finite variance or a differentiable approximation of a triangular function or a rectangular one. The smoothing function shall be nor-
malized by demanding
\[
\int_{-\infty}^{\infty} dx' \ s(x' - x) = 1
\]
for any value of \( x \). With this, we define the local density
\[
\rho(x, t) = \int_{-\infty}^{\infty} dx' \ s(x' - x) \sum_i \delta(x' - x_i(t)) = \sum_i s(x_i(t) - x).
\]

Herein, we sum up over all particles \( i \). Note that the re-
placement of the conventional formula \( \sum_i \delta(x_i(t) - x) \) for
the particle density by the formula \( \sum_i s(x_i(t) - x) \) cor-
sponds to a substitution of point-like particles by “fuzzy”
particles, which is the idea behind smooth particle hydro-
dynamics.

Now, we define the average velocity \( V(x, t) \) as usual
via a weighted average with the weight function \( \delta(x' - x_i(t))s(x' - x) \):
\[
V(x, t) = \frac{\int_{-\infty}^{\infty} dx' \sum_i v_i(t)\delta(x' - x_i(t))s(x' - x)}{\int_{-\infty}^{\infty} dx' \sum_i \delta(x' - x_i(t))s(x' - x)} = \frac{\int_{-\infty}^{\infty} dx' \sum_i v_i(t)s(x' - x_i(t))s(x' - x)}{\int_{-\infty}^{\infty} dx' \sum_i s(x' - x_i(t))s(x' - x)} = \frac{\sum_i v_i(t)s(x_i(t) - x)}{\rho(x, t)}.
\]
This implies the well-known fluid-dynamic flow relationship

\[ Q(x, t) = \rho(x, t)V(x, t). \]  

(27)

Differentiation of Eq. \((24)\) with respect to time and application of the chain rule gives

\[
\frac{\partial \rho(x, t)}{\partial t} = \int_{-\infty}^{\infty} dx' \sum_i \left[ \frac{dx_i}{dt} \right] \rho(x', x_i(t)) s(x' - x) + \sum_i v_i(t) \frac{\partial}{\partial x_i} \left[ s(x_i(t) - x) \right] \frac{dx_i(t)}{dt}
\]

(32)

Introducing \(\delta v_i(x, t) = v_i(t) - V(x, t)\) and defining the velocity variance

\[
\theta(x, t) = \frac{\int_{-\infty}^{x} dx' \sum_i [v_i(t) - V(x, t)]^2 \delta(x' - x_i(t)) s(x' - x)}{\int_{-\infty}^{\infty} dx' \sum_i \delta(x' - x_i(t)) s(x' - x)}
\]

(33)

Similarly to the average velocity \((26)\), we can make the decomposition

\[
\sum_i [v_i(t)]^2 s(x_i(t) - x)
\]

\[
= \sum_i [V(x, t) + \delta v_i(x, t)]^2 s(x_i(t) - x)
\]

\[
= \sum_i \left\{ [V(x, t)]^2 + 2V(x, t) \delta v_i(x, t) + [\delta v_i(x, t)]^2 \right\} s(x_i(t) - x)
\]

\[
= \rho(x, t) [V(x, t)]^2 + 2 \rho(x, t)V(x, t) [V(x, t) - V(x, t)] + \rho(x, t) \theta(x, t),
\]

(34)

where we have considered

\[
\sum_i \delta v_i(x_i, t) s(x_i(t) - x)
\]

\[
= \sum_i \left[ v_i(t) - V(x, t) \right] s(x_i(t) - x)
\]

\[
= Q(x, t) - \rho(x, t)V(x, t) = 0,
\]

(35)

see Eqs. \((26)\) and \((27)\). Altogether, we get

\[
\frac{\partial}{\partial t} \left[ \rho(x, t)V(x, t) \right] = -\frac{\partial}{\partial x} \left\{ \rho(x, t) [V(x, t)]^2 + \theta(x, t) \right\}
\]

\[
+ \sum_i a_i(t) s(x_i(t) - x).
\]

(36)

Now, we carry out the partial differentiation applying the product rule of Calculus. Taking into account

\[
\rho(x, t) \frac{\partial V(x, t)}{\partial t} = -V(x, t) \frac{\partial \rho(x, t)}{\partial t} + \frac{\partial}{\partial t} \left[ \rho(x, t)V(x, t) \right]
\]

(37)

4.2 Derivation of the Macroscopic Velocity Equation

In order to derive the equation for the average velocity, we start by deriving the formula

\[
\rho(x, t)V(x, t) = \sum_i v_i(t) s(x_i(t) - x)
\]

(31)

for the vehicle flow with respect to time. This gives

\[
\frac{\partial}{\partial t} \left[ \rho(x, t)V(x, t) \right] = \sum_i \frac{dv_i(t)}{dt} s(x_i(t) - x)
\]
and
\[
\frac{\partial}{\partial x} \left\{ \left[ \rho(x, t)V(x, t) \right] V(x, t) \right\} \\
= \rho(x, t)V(x, t) \frac{\partial V}{\partial x} + V(x, t) \frac{\partial}{\partial x} \left[ \rho(x, t)V(x, t) \right],
\]
we obtain with Eq. (36)
\[
\rho(x, t) \frac{\partial V(x, t)}{\partial t} \\
= -V(x, t) \frac{\partial \rho(x, t)}{\partial t} - V(x, t) \frac{\partial}{\partial x} \left[ \rho(x, t)V(x, t) \right] \\
= \rho(x, t)V(x, t) \frac{\partial V(x, t)}{\partial x} - \frac{\partial}{\partial x} \left[ \rho(x, t)\theta(x, t) \right] \\
+ \sum_i a_i(t) s(x_i(t) - x).
\]
Inserting the continuity equation for \(\partial \rho/\partial t\) and dividing the above equation by \(\rho(x, t)\) finally gives the velocity equation
\[
\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} \\
= -\frac{1}{\rho(x, t)} \frac{\partial}{\partial x} \left[ \rho(x, t)\theta(x, t) \right] \\
+ \frac{1}{\rho(x, t)} \sum_i a_i(t) s(x_i(t) - x).
\]
Inserting Eq. (11) for \(a_i(t)\), we find
\[
\sum_i a_i(t) s(x_i(t) - x) \\
= \sum_i \left[ \frac{v_0 - v_i}{\tau} + \sum_i f(d_i(t)) \right] s(x_i(t) - x) \\
= \frac{v_0 - V(x, t)}{\tau} + \sum_i f(d_i(t)) s(x_i(t) - x).
\]
For further simplification, let us now specify the smoothing function by the rectangular function
\[
s(x_i - x) = \frac{\varrho}{2} \begin{cases} 1 & \text{if } |x_i - x| \leq 1/\varrho \\ 0 & \text{otherwise}, \end{cases}
\]
with a large enough smoothing window of length \(\Delta x = 2/\varrho\). Then, the number of vehicles \(i\) within the smoothing interval \([x - 1/\varrho, x + 1/\varrho]\) is expected to be \(\rho \Delta x = 2\rho/\varrho\), where \(\rho\) represents the average vehicle density in this interval. Therefore,\[
\rho(x, t) = \sum_i s(x_i(t) - x) = \frac{2\rho \varrho}{\varrho 2} = \rho,
\]
which shows the consistency of this approach.

If the smoothing parameter \(\varrho\) is specified via the in\-verse vehicle distance
\[
\varrho = \varrho_k = \frac{1}{d_k} = \frac{1}{x_{k-1} - x_k} = \rho(x, t) \quad \text{for } x_k < x \leq x_{k-1},
\]
the smoothing window of length \(\Delta x = 2/\varrho\) will usually contain only two vehicles \(k - 1\) and \(k\) with \(x_k \leq x \leq x_{k-1}\). With this, the sum over \(i\) reduces to two terms with \(i = k\) and \(i = k - 1\) only. This finally yields
\[
V(x, t) = \sum_i v_i(t)s(x_i(t) - x) \\
= v_k(t)s(x_k(t) - x) + v_{k-1}(t)s(x_{k-1}(t) - x) \\
= \frac{\varrho}{2} [v_{k-1}(t) + v_k(t)] \\
= \rho(x, t) \frac{v_{k-1}(t) + v_k(t)}{2}
\]
and, considering Eq. (44),
\[
\sum_i s(x_i(t) - x) f(d_i(t)) = \frac{\varrho}{2} f(d_k) + \frac{\varrho}{2} f(d_{k-1}) \\
= \frac{\varrho}{2} f \left( \frac{1}{\varrho_k} \right) + \frac{\varrho}{2} f \left( \frac{1}{\varrho_{k-1}} \right) \\
= \frac{\rho(x, t)}{2} f \left( \frac{1}{\rho(x, t)} \right) \\
+ \frac{\rho(x, t)}{2} f \left( \frac{1}{\rho(x + 1/\varrho, t)} \right).
\]
In summary, the macroscopic velocity equation corresponding to the optimal velocity model corresponds to
\[
\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} \\
= -\frac{1}{\rho(x, t)} \frac{\partial}{\partial x} \left[ \rho(x, t)\theta(x, t) \right] + \varrho V(x, t) \frac{\partial}{\partial x} \left[ \rho(x, t)\theta(x, t) \right] \\
+ \frac{1}{2} \frac{1}{\rho(x, t)} f \left( \frac{1}{\rho(x, t)} \right) + \frac{1}{2} \frac{1}{\rho(x + 1/\varrho, t)} f \left( \frac{1}{\rho(x + 1/\varrho, t)} \right).
\]
It should be noted that this equation is non-local due to the dependence on \(x + 1/\rho(x, t)\). This reflects the anticipatory behavior of drivers, who react to the traffic situation ahead of them. From the point of view of traffic simulation, the non-locality does not constitute a problem. Non-local traffic models such as the gas-kinetic based traffic model summarized in Appendix B can be even numerically more efficient than local ones with diffusion terms, that would result from a gradient expansion. In fact, the reason for the numerical inefficiency of explicit solvers for partial differential equations is the diffusion instability, which must be avoided by small time discretizations \cite{15}. As pointed
\footnote{If another smoothing function is applied, the last term of Eq. (47) is replaced by a similar weighted mean value, as Eq. (11) reveals, but the essence stays the same. That is, the way of looking at the microscopic equations (i.e. the way of defining the density and velocity moments) potentially has some influence on the dynamics, but it is expected to be small.}
out by Daganzo [19], a diffusion term also implies theoretical inconsistencies such as the occurrence of negative velocities at the end of jam fronts. Therefore, it should be underlined that numerical inefficiencies and theoretical inconsistencies can be avoided by working with the non-local velocity equation rather than with the gradient expansion of it, which will be looked at in the next section.

4.3 Comparison with Other Macroscopic Traffic Models

According to the discussion above, a gradient expansion is acceptable in case of small gradients, e.g., when a linear stability analysis is performed. It is also useful to compare different macroscopic traffic models. For this purpose, let us carry out a Taylor approximation of first order. It gives

\[
f \left( \frac{1}{\rho(x+1/\rho, t)} \right) \\
\approx f \left( \frac{1}{\rho(x,t)} + \frac{\partial \rho(x,t)}{\partial x} \frac{1}{\rho(x,t)^2} \right) \\
\approx f \left( \frac{1}{\rho(x,t)} \right) + \frac{df(d)}{dd} \left( \frac{1}{\rho(x,t)} \rho(x,t)^2 \right),
\]

where we have applied the geometric series expansion \( 1/(1-z) \approx 1 + z + \ldots \). Note that the relation \( \rho = 1/d \) and

\[ V_c(\rho) = V_c \left( \frac{1}{d} \right) = v_0(d) = v^0 + \tau f(d) = v^0 + \tau f \left( \frac{1}{\rho} \right) \]

imply

\[
\frac{df(d)}{dd} = \left( \frac{d V_c(\rho) - v^0}{\tau} \right) \frac{d \rho}{dd} = \frac{d V_c(\rho)}{d \rho} \left( \frac{1}{\rho^2} \right) = -\frac{\rho^2}{\tau} \frac{d V_c(\rho)}{d \rho}.
\]

Therefore, using Eq. (10), we finally obtain:

\[
\sum_i s(x_i(t) - x)f(t) \approx \rho(x,t) f \left( \frac{1}{\rho(x,t)} \right) + \frac{1}{2\tau} \frac{d V_c(\rho)}{d \rho} \frac{\partial \rho(x,t)}{\partial x}.
\]

Considering \( V_c(\rho) = v^0 + \tau f(\rho) \) and defining the “traffic pressure” as

\[ P(x,t) = \rho(x,t) \theta(x,t) + \frac{v^0 - V_c(\rho)}{2\tau}, \]

the corresponding macroscopic velocity equation becomes

\[
\frac{\partial V(x,t)}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} = \frac{1}{\rho(x,t)} \frac{\partial P(x,t)}{\partial x} + \frac{V_c(\rho) - V(x,t)}{\tau}.
\]

If the velocity variance \( \theta \) is zero, this model corresponds exactly to Payne’s macroscopic traffic model with the pressure term \( P(\rho) = \frac{V^0 - V_c(\rho)}{2\tau} \).

It should be noted that the gradient \( \partial P/\partial x = [dP(\rho)/d\rho] \cdot \partial \rho/\partial x \) of this pressure becomes zero, whenever the density becomes zero or maximum, as the derivative of \( dV_c(\rho)/d\rho \) vanishes in these situations.

4.3.1 The Macroscopic Traffic Model by Aw and Rascle

Note that Daganzo has seriously criticized macroscopic traffic equations of the type [53] [20]. For example, he considered the case of a vehicle queue of maximum density \( \rho = \rho_{jam} \) and speed \( V = V_c(\rho_{jam}) = 0 \), the end of which was assumed to be at some location \( x = x_0 \). Then, for the last vehicle in the queue, Eq. (53) predicts \( V = 0 \) and \( dV/dt = \partial V/\partial t + \partial \rho V/\partial x < 0 \), i.e., the occurrence of negative velocities, if pressure relations such as \( P = \rho \theta_0 - \eta_0 \partial V/\partial x \) with non-negative parameters \( \theta_0 \) and \( \eta_0 \) are assumed [7], as it was common at the time when Ref. [19] was published.

In order to overcome Daganzo’s criticism, Aw and Rascle [20] have proposed the macroscopic velocity equation

\[
\frac{\partial}{\partial t} [V + p(\rho)] + V \frac{\partial}{\partial x} [V + p(\rho)] = 0 \quad (55)
\]

with \( p(\rho) = \rho' \). Let us study, how this model relates to the previous macroscopic models. For this purpose, let us apply the chain rule of Calculus to obtain

\[
\frac{\partial V(x,t)}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} = -\frac{d p(\rho)}{d \rho} \frac{\partial \rho(x,t)}{\partial t} - V(x,t) \frac{d p(\rho)}{d \rho} \frac{\partial \rho(x,t)}{\partial x}.
\]

Inserting the continuity equation (30) for \( \partial \rho/\partial t \) on the right-hand side, we get

\[
\frac{\partial V(x,t)}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} = \frac{d p(\rho)}{d \rho} \frac{\partial \rho(x,t)}{\partial x} \frac{\partial V(x,t)}{\partial x}.
\]

By comparison with the macroscopic velocity equation (53) we see that the model by Aw and Rascle does not have a relaxation term \( [V_c(\rho) - V(x,t)]/\tau \), which would correspond to the limit \( \tau \to \infty \). Moreover, we find

\[
-\frac{1}{\rho(x,t)} \frac{\partial P(x,t)}{\partial x} = \rho(x,t) \frac{d p(\rho)}{d \rho} \frac{\partial V(x,t)}{\partial x}.
\]

Therefore, the traffic pressure according to the model of Aw and Rascle is a function of the velocity gradient rather
than the density gradient, in contrast to Payne’s pressure term [34]. Consequently, Aw’s and Raschle’s pressure term must result in a different way than Payne’s one. In order to demonstrate this, let us now discuss a generalization of the optimal velocity model and its macroscopic counterpart.

4.3.2 Non-Local Macroscopic Traffic Models

It is well-known that the optimal velocity model may produce accidents, if the initial condition, the optimal velocity function \( v_0(d) \), and the parameter \( \tau \) are not carefully chosen. In order to have both, the emergence of traffic jams and the avoidance of accidents, we need to assume that the repulsive interaction force among vehicles does not only depend on the vehicle distance \( d_i(t) = x_{i-1}(t) - x_i(t) \), but also on the vehicle velocity \( v_i(t) \) (to reflect the dependence of the safe distance on the vehicle speed) or on the relative velocity

\[
\Delta v_i(t) = v_i(t) - v_{i-1}(t) = -\frac{dd_i}{dt}.
\]

The corresponding generalization of the acceleration equation (11) reads

\[
\frac{dv_i}{dt} = a_i(t) = \frac{v^0 - v_i(t)}{\tau} + f\left(d_i(t), v_i(t), \Delta v_i(t)\right).
\]

This also changes the associated macroscopic traffic equation. Namely, equation (17) has to be replaced by

\[
\frac{\partial V(x,t)}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} = -\frac{1}{\rho(x,t)} \frac{\partial}{\partial x} \left[ \rho(x,t) \theta(x,t) \right] + \frac{v^0 - V(x,t)}{\tau} + \frac{1}{2} f \left( \frac{1}{\rho(x,t)} V(x,t), \Delta V(x,t) \right)
\]

\[
+ \frac{1}{2} f \left( \frac{1}{\rho(x + 1/\rho, t)} V(x + 1/\rho, t), \Delta V(x + 1/\rho, t) \right).
\]

(61)

For the sake of comparison with other macroscopic traffic models and linear stability analyses, let us perform a Taylor approximation of this. First, we may write

\[
V(x + 1/\rho, t) - V(x, t) \approx \frac{\partial V}{\partial x} \frac{1}{\rho}.
\]

Furthermore, considering \( \Delta v_i(t) = -dd_i/\rho, \rho(x,t) = 1/d_i(t) \), and the continuity equation \( d\rho/dt = \partial\rho/\partial t + V\partial\rho/\partial x = -\rho \partial V/\partial x \), we get

\[
\Delta V(x,t) = -\frac{d}{dt} \left( \frac{1}{\rho(x,t)} \right) = \frac{1}{\rho(x,t)} \frac{d\rho(x,t)}{dt} = -\frac{\partial V(x,t)}{\partial x} \frac{1}{\rho(x,t)} \approx V(x,t) - V(x + 1/\rho, t)
\]

and

\[
\Delta V(x + 1/\rho, t) - \Delta V(x, t) \approx \frac{\partial}{\partial \rho} \left( \frac{1}{\rho^2} \frac{\partial V}{\partial \rho} \right) + \frac{\partial \rho}{\partial V} \frac{\partial V}{\partial x} + \frac{\partial}{\partial \Delta V} \left( \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \Delta V \partial x^2} \right).
\]

(66)

As a linearization drops products of gradient terms such as \( \rho \partial \rho/\partial x \partial V/\partial x \) (which are assumed to be smaller than the linear terms). Altogether, with \( dd/d\rho = -1/\rho^2 \) we can write

\[
\frac{\partial V(x,t)}{\partial \rho} + V(x,t) \frac{\partial V(x,t)}{\partial x} = -\frac{1}{\rho} \frac{\partial}{\partial x} \left[ \rho(x,t) \theta(x,t) \right] + \frac{v^0 - V(x,t)}{\tau} + \frac{1}{2} f \left( \frac{1}{\rho(x + 1/\rho, t)} V(x + 1/\rho, t), \Delta V(x + 1/\rho, t) \right)
\]

\[
\approx \frac{v^0 - V(x,t)}{\tau} + \frac{1}{2} f \left( \frac{1}{\rho(x + 1/\rho, t)} V(x + 1/\rho, t), \Delta V(x + 1/\rho, t) \right) \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{\partial \rho}{\partial V} \frac{\partial V}{\partial x} + \frac{\partial}{\partial \Delta V} \left( \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \Delta V \partial x^2} \right).
\]

(67)

With the definition

\[
V_o(\rho, V, \Delta V) = v^0 + \tau f \left( \rho_0 V, \Delta V \right),
\]

we may finally write

\[
\frac{\partial V(x,t)}{\partial \rho} + V(x,t) \frac{\partial V(x,t)}{\partial x} = -\frac{1}{\rho} \frac{\partial}{\partial x} \left[ \rho(x,t) \theta(x,t) \right] + \frac{V_o(\rho, V, \Delta V) - V(x,t)}{\tau} + \frac{1}{2} f \left( \frac{1}{\rho(x + 1/\rho, t)} V(x + 1/\rho, t), \Delta V(x + 1/\rho, t) \right) \frac{1}{\rho} \frac{\partial V}{\partial \rho}
\]

\[
\quad + \frac{\partial \rho}{\partial V} \frac{\partial V}{\partial x} + \frac{\partial}{\partial \Delta V} \left( \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \Delta V \partial x^2} \right).
\]

(69)

Furthermore, let us assume that the variance can be approximated as a function of the density and the average velocity:

\[
\theta(x,t) = \theta_e(\rho(x,t), V(x,t)).
\]

(70)

With the definitions

\[
\frac{\partial P_1}{\partial \rho} = \theta_e(\rho, V) + \rho \frac{\partial \theta_e(\rho, V)}{\partial \rho} + \frac{1}{2} \frac{\partial}{\partial d} \left( \frac{1}{\rho} V, \Delta V \right),
\]

(71)

\[
\frac{\partial P_2}{\partial V} = \rho \frac{\partial \theta_e(\rho, V)}{\partial V} - \frac{\partial}{\partial \Delta V} \left( \frac{1}{\rho} V, \Delta V \right),
\]

(72)

\[
\eta = -\frac{1}{2} \frac{\partial}{\partial \Delta V} \left( \frac{1}{\rho} V, \Delta V \right).
\]

(73)
(where \(\eta\) should be greater than zero), we may also write the linearized macroscopic traffic equations as

\[
\frac{\partial V(x,t)}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} = - \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{1}{\rho} \frac{\partial P}{\partial x} \frac{\partial V}{\partial x} + \frac{\eta}{\tau} \frac{\partial^2 V}{\partial x^2} + \frac{V_c(\rho, \Delta V, V) - V(x,t)}{\tau}.
\]

(74)

The term \(\eta\frac{\partial^2 V}{\partial x^2}\) can be interpreted as viscosity term and has some smoothing effect. Further viscosity (and diffusion) terms may be derived by second-order Taylor expansions.

Note that the pressure term \(P_2\) looks similar to Eq. (55). Therefore, it should be possible to derive a pressure term corresponding Aw’s and Rascle’s model from a suitable microscopic traffic model, but one would expect additional terms such as (52) to occur as well.

5 Summary, Discussion, and Conclusions

In this paper, we have discussed several approaches to derive macroscopic traffic equations from microscopic car-following models. It has been pointed out that a Taylor approximation should be used only for linear stability analyses, as the gradients may otherwise be too large for the approximation to work. Further undesirable consequence of a gradient expansion are the possible occurrence of negative velocities, diffusion instabilities, and inefficient numerical solution methods.

The linear interpolation approach often works well in practice, but it is theoretically inconsistent as it violates the continuity equation which is required for the conservation of the vehicle number. In contrast, the smooth particle hydrodynamics approach was suited in all respects. It led to a non-local macroscopic traffic model, as did the gas-kinetic based traffic model. In order to have a realistic traffic dynamics (in particular accident avoidance if a vehicle goes to zero, but it will be positive otherwise. It should be noted that the variance prefactor \(A\) is higher in congested traffic than in free traffic.
The “effective cross section” is, finally, specified via

\[ [1 - p(\rho)]\chi(\rho) = \frac{v_0^2 \rho T^2}{\tau A(\rho_{jam})(1 - \rho/\rho_{jam})^2}, \tag{82} \]

where \( T \) is the safe time headway and \( \rho_{jam} \) the maximum vehicle density. This formula makes also sense in the low-density limit \( \rho \to 0 \), where \( \chi \to 1 \) and \( p \to 1 \).

A linear stability analysis of the non-local traffic model can be done via a gradient expansion. It results in equations of the kind \( \{74\} \) and further viscosity and diffusion terms \( \{22\} \).

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