Multipliers and Closures of Besov-Type Spaces in the Bloch Space

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Abstract. Let \( p > 1 \) and let \( \rho \) be a non-negative function defined in \( \mathbb{R}^+ \). A function \( f \in H(\mathbb{D}) \) belongs to the space \( B_p(\rho) \) (see [4]) if
\[
\|f\|_{B_p(\rho)} = |f(0)|^p + \int_{\mathbb{D}} \left| (1 - |z|^2) f'(z) \right|^p \frac{\rho(1 - |z|^2)}{(1 - |z|^2)^2} \, dA(z) < \infty.
\]

In this paper, motivated by the works of Békollé and Bao and Göğüş, under some conditions on the weight function \( \rho \), we investigate the closures \( C_{B_p(\rho)} \) of the spaces \( B \cap \mathcal{B}_{p}(\rho) \) in the Bloch space. Moreover we prove that interpolating Blaschke products in \( C_{B_p(\rho)} \) are multipliers of \( B_{p}(\rho) \cap \text{BMOA} \).

1. Introduction

We denote the unit disk \( \{z \in \mathbb{C} : |z| < 1\} \) by \( \mathbb{D} \) and its boundary \( \{z \in \mathbb{C} : |z| = 1\} \) by \( \partial \mathbb{D} \). Let \( H(\mathbb{D}) \) be the space of all analytic functions in \( \mathbb{D} \).

Let \( H^p \) (see [11]) denote the space of those analytic functions \( f \in H(\mathbb{D}) \) such that
\[
\|f\|_{H^p} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.
\]

Let \( \text{BMOA} \) denote the space of those analytic functions \( f \) in the Hardy space \( H^p \) whose boundary functions have bounded mean oscillation on \( \partial \mathbb{D} \). \( \text{BMOA} \) ([17, 19]) is a Banach space under the following norm:
\[
\|f\|_{\text{BMOA}} = |f(0)|^p + \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^p}^p,
\]
where \( \varphi_a(z) = \frac{z - a}{1 - \overline{a}z}, a, z \in \mathbb{D} \) and \( p \geq 1 \).

Recall that the Bloch space ([2, 34]) is the class of functions \( f \in H(\mathbb{D}) \) satisfying
\[
\|f\|_B = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.
\]

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Let $p > 1$ and let $\rho$ be a non-negative function defined in $\mathbb{R}^+$. A function $f \in H(\mathbb{D})$ belongs to the space $B_p(\rho)$ if
\[ |f|^p_{B_p(\rho)} = |f(0)|^p + \int_{\mathbb{D}} (1 - |z|^2)^p |f'(z)|^p \rho \left( \frac{1 - |z|^2}{1 - |z|^2} \right) dA(z) < \infty, \]
where $dA(z)$ is the usual normalized Lebesgue measure on $\mathbb{D}$. This space is introduced by Arcozzi, Rochberg and Sawyer in [4]. They considered Carleson measures for $B_p(\rho)$ spaces under the condition that the weight function $\rho$ is $p$-admissible, or admissible, that is, $\rho$ satisfies the following conditions:

(i) $\rho(z)$ is regular, i.e., there exist $\epsilon > 0, C > 0$ such that $\rho(z) \leq C \rho(w)$ when $z$ and $w$ are within hyperbolic distance $\epsilon$. Equivalently, there are $\delta < 1, C' > 0$ so that $\rho(z) \leq C' \rho(w)$ whenever
\[ \frac{|z - w|}{1 - \overline{w}z} \leq \delta < 1. \]

(ii) The weight $\rho_1(z) = (1 - |z|^2)^{-\frac{p}{2}} \rho(z)$ satisfies the Békollé-Bonami $B_p$ condition([7, 8]): There is a $C(\rho, p)$ so that for all $a \in \mathbb{D}$ we have
\[ \left( \int_{S(a)}\rho_1(z)dA(z) \right) \left( \int_{S(a)}\rho(z)^{-\frac{1}{q}}dA(z) \right)^{\frac{1}{q-1}} \leq C(\rho, p) \left( \int_{S(a)}dA(z) \right)^{\frac{1}{q}}. \]

Where $1/p + 1/q = 1$, and
\[ S(a) = \{ z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, \left| \frac{\arg(az)}{2\pi} \right| \leq \frac{1 - |a|}{2}, a \in \mathbb{D}. \]

In the case $\rho(t) = t^s, 0 \leq s < \infty$, the space $B_p(\rho)$ gives the usual Besov type space $B_p(s)$. In particular, if $s = 0$, this gives the classical Besov space $B_p$. We refer to [5], [9], [10] and [12] for $B_p(s)$ spaces and [30], [31] and [32] for $B_2(s) = D_s$ spaces. The space $B_p(\rho)$ has been extensively studied. For example, under some conditions on $\rho$, N. Arcozzi, R. Kerman and E. Sawyer [4] give many results on $B_p(\rho)$ spaces using Carleson measures. When $p = 2$, $B_2(\rho) = D_\rho$, weighted Dirichlet spaces. Using maximal operators, R. Kerman and E. Sawyer [21] characterized the Carleson measures and multipliers of $D_\rho$ spaces. For more informations on $D_\rho$ spaces, we refer to [1] and the paper referinhere.

Let us recall that a weight $\rho$ is of upper (resp.lower) type $\gamma$ ($0 \leq \gamma < \infty$) ([20]), if
\[ \rho(s) \leq Cs^\gamma \rho(t), \ s \geq 1 \ \text{(resp.} s \leq 1) \ \text{and} \ 0 < t < \infty. \]

We say that $\rho$ is of upper type less than $\gamma$ if it is of upper type $\delta$ for some $\delta < \gamma$ and similarly for lower type greater than $\delta$. From [20], we see that an increasing function $\rho$ is of finite upper type if and only if $\rho(2t) \leq C\rho(t)$ for some positive constant $C$ and all $t$. It is not hard to verify that $\rho$ satisfies (i) and (ii), if $\rho$ is of upper type less than $1$ and lower type greater than $0$.

In [2], Anderson, Clunie and Pommerenke raised the question of determining the closure of $H^\infty$ in the Bloch norm. Until now, the problem is still unsolved. Jones [3, Theorem 9] gave an unpublished characterization of the closure of BMOA in Bloch space. Zhao [33] studied the closures of some Möbius invariant spaces in the Bloch space. Lou and Chen [22] generalize [33] later. For $1 < p < \infty$, Monreal Galán and Nicolau in [24] characterized the closure in the Bloch norm of the space $H^p \cap B$. Galanopoulos, Monreal Galán and Pau [16] generalize [24] to $0 < p < \infty$ later. Recently, Bao and Göğüs [6] and Galanopoulos and Girela [15] have investigated the closures in $B$ of $B \cap D^\rho$ for certain spaces of Dirichlet Type $D^\rho$. For more results on closures of analytic function spaces in the Bloch space, we refer to [28] and [29]. In this paper, we study the closures of the $B_p(\rho)$ spaces, generalizing the main results in [6] and [15]. Meanwhile, interpolating Blaschke products in $C_p(B \cap B_p(\rho))$ as multipliers of $B_p(\rho) \cap BMOA$ are also investigated.

Throughout this paper, let $\rho : [0, \infty) \to [0, \infty]$ be a right continuous and nondecreasing function with $\rho(0) = 0$ and $\rho(t) > 0$ if $t > 0$. The symbol $A \sim B$ means that $A \leq B \leq A$. We say that $A \leq B$ if there exists a constant $C$ such that $A \leq CB$. 
Remark 1. Using [20, Lemma 4], the fact that $\rho$ is increasing, and the above mentioned fact that $\rho$ is of finite upper if and only if $\rho(2t) \leq C \rho(t)$ $(t \geq 0)$, we deduce the following:

If $\rho$ is of finite upper type, then $\rho$ is of upper type less than $\rho$ if and only if
\[
\frac{\rho(t)}{t^p} \approx \int_t^\infty \rho(s) \frac{ds}{s^{1+p}}.
\]

Remark 2. Let $0 \leq a < 1$ and $p > 1$. If $\rho$ is of finite upper type $a$, we can deduce that $B_p(\rho) \subseteq H^p$. Indeed, take $b$ with $a < b < 1$, using Remark 1, we deduce that

Thus, $B_p(\rho) \subseteq B_p(b)$. Then the inclusion $B_p(\rho) \subseteq H^p$ follows from the well known fact that $B_p(b) \subseteq H^p$ because $0 < b < 1$.

2. Equivalent Characterizations of closures of $B_p(\rho)$ spaces in Bloch space

Theorem 1. Let $\rho$ be of finite lower type greater than 0 and upper type less than 1. Suppose that $1 < p \leq \infty$. Then the following conditions are equivalent.

(1) $f \in \mathcal{C}_0(B_p(\rho) \cap \mathcal{B})$.

(2) For any $\epsilon > 0$,
\[
\int_{\Omega_\epsilon(f)} \frac{\rho(1 - |z|^2)}{(1 - |z|^2)^2} dA(z) < \epsilon,
\]
where
\[
\Omega_\epsilon(f) = \{z \in \mathbb{D} : (1 - |z|^2)|f'(z)| \geq \epsilon\}.
\]

(3) For $\epsilon > 0$ and $s > 1$,
\[
\int_{\Gamma_{p\epsilon}(f)} \frac{\rho(1 - |z|^2)}{(1 - |z|^2)^s} dA(z) < \epsilon,
\]
where
\[
\Gamma_{p\epsilon}(f) = \left\{z \in \mathbb{D} : \int_{\mathbb{D}} |f'(w)|^p(1 - |w|^2)^{p-2}(1 - |\rho_z(w)|^2)^s dA(w) \geq \epsilon\right\}.
\]

Proof. (2) $\Rightarrow$ (1). Following [33], without loss of generality, we may assume that $f(0) = 0$. By Proposition 4.27 in [34], we have that
\[
f(z) = \frac{1}{(a + 1)} \int_D \frac{f'(w)(1 - |w|^2)^{1+a}}{b(1 - zw)^{2+a}} dA(w), \ z \in \mathbb{D},
\]
where $\alpha > 0$. Set $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{\alpha + 1} \int_{\Omega_1(f)} \frac{f'(w)(1 - |w|^2)^{\alpha + 1}}{|w|^{3 + \alpha}} dA(w)$$

and

$$f_2(z) = \frac{1}{\alpha + 1} \int_{\Omega_2(f)} \frac{f'(w)(1 - |w|^2)^{\alpha + 1}}{|w|^{3 + \alpha}} dA(w).$$

Clearly,

$$|f_1'(z)| \leq \int_{\Omega_1(f)} \frac{|f'(w)(1 - |w|^2)^{\alpha + 1}}{|w|^{3 + \alpha}} dA(w)$$

and

$$|f_2'(z)| \leq \int_{\Omega_2(f)} \frac{|f'(w)(1 - |w|^2)^{\alpha + 1}}{|w|^{3 + \alpha}} dA(w).$$

Let $F = f_1 - f_1(0)$. Then $F(0) = 0$ and

$$\|f - F\|_{B_\rho} = \sup_{z \in D} (1 - |z|^2)|f_2'(z)|$$

$$\leq \sup_{z \in D} (1 - |z|^2) \int_{\Omega_2(f)} \frac{|f'(w)(1 - |w|^2)^{\alpha + 1}}{|w|^{3 + \alpha}} dA(w)$$

$$\leq \epsilon \sup_{z \in D} (1 - |z|^2) \int_{D} \frac{(1 - |w|^2)^{\alpha}}{|1 - z\bar{w}|^{3 + \alpha}} dA(w).$$

Using [34, Lemma 3.10] with $t = \alpha$ and $c = 1$, we see that $\|f - F\|_{B_\rho} \leq \epsilon$. It remains to prove that $F \in B_p(\rho)$. Using Fubini’s theorem, we have

$$\int_{\Omega_1(f)} |f'(w)(1 - |w|^2)^{\alpha - 2}\rho(1 - |w|^2) dA(w)$$

$$\leq \int_{\Omega_1(f)} |f_1'(w)(1 - |w|^2)^{-2}\rho(1 - |w|^2) dA(w)$$

$$\leq \frac{1}{\alpha + 1} \int_{\Omega_1(f)} \frac{|f'(w)(1 - |w|^2)^{\alpha + 1}}{|w|^{3 + \alpha}} dA(w)$$

$$\leq \epsilon \sup_{z \in D} (1 - |z|^2) \int_{D} \frac{(1 - |w|^2)^{\alpha}}{|1 - z\bar{w}|^{3 + \alpha}} dA(w).$$

Since $\rho$ is of finite lower type greater than 0 and upper type less than 1, there exist $\gamma$ and $\delta$ with $0 < \gamma < \delta < 1$, such that

$$\rho(st) \leq s^\gamma \rho(t), \ s \leq 1$$

(A)

and

$$\rho(st) \leq s^\delta \rho(t), \ s \geq 1,$$

(B)
where $0 < t < \infty$. Using this and [34, Lemma 3.10], we obtain
\[
\int_{D} \frac{\rho(1-|z|^2)}{1-\overline{z}^2(1-|z|^2)} dA(z)
\]
\[
= \rho(1-|w|^2) \int_{D} \frac{\rho(1-|z|^2)}{\rho(1-|w|^2)} dA(z)
\]
\[
\leq \rho(1-|w|^2) \int_{D} \left( \frac{(1-|z|^2)}{(1-|w|^2)} \right)^{\gamma} + \left( \frac{(1-|z|^2)}{(1-|w|^2)} \right)^{\delta} dA(z)
\]
\[
\leq \rho(1-|w|^2) \frac{(1-|z|^2)^{\gamma+\delta}}{(1-|w|^2)^{\gamma+\delta}}.
\]
Combining this with (2), we have
\[
\int_{D} |f''(z)|^p (1-|z|^2)^{\gamma+2} \rho(1-|z|^2) dA(z) \leq \int_{D} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) < \infty.
\]
Hence, $F \in B_p(\rho)$. This finishes the proof.

(1) $\Rightarrow$ (3). It is well known that $||f||_{B_2}$ is equivalent to
\[
||f||_{B_2} = |f(0)| + \left( \sup_{z \in D} \int_{D} |f''(w)|^p (1-|w|^2)^{\gamma} (1-|\varphi_z(z)|^2)^{\nu} dA(w) \right)^{1/p},
\]
where $p > 1$ and $s > 1$. Let $f \in C_0(B_p(\rho) \cap \mathcal{B})$. Then for any $\epsilon > 0$, there exists $g \in B_p(\rho) \cap \mathcal{B}$ such that $||f-g||_{B_2} \leq \frac{\epsilon}{2^p}$.

For any $z \in D$, we have
\[
\int_{D} |f''(w)|^p (1-|w|^2)^{\gamma-2} (1-|\varphi_z(w)|^2)^{\nu} dA(w)
\]
\[
\leq C \int_{D} |f''(w) - g''(w)|^p (1-|w|^2)^{\gamma-2} (1-|\varphi_z(w)|^2)^{\nu} dA(w) + C \int_{D} |g''(w)|^p (1-|w|^2)^{\gamma-2} (1-|\varphi_z(w)|^2)^{\nu} dA(w).
\]
Thus, $\Gamma_{p,\mu}(f) \subseteq \Gamma_{p,\mu}(g)$. Note that
\[
1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}.
\]
Using Fubini's theorem, we have
\[
\int_{\Gamma_{p,\mu}(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z)
\]
\[
\leq 2^p C \int_{\Gamma_{p,\mu}(g)} \left( 1-|z|^2 \right)^{\gamma-2} \rho(1-|z|^2) \left[ \int_{D} |g''(w)|^p (1-|w|^2)^{\gamma} \frac{(1-|z|^2)^{\nu}}{|1-\overline{z}w|^2} dA(w) \right] dA(z)
\]
\[
\leq \int_{D} |g''(w)|^p (1-|w|^2)^{\gamma-2} \left[ \int_{D} \left( 1-|z|^2 \right)^{\gamma-2} \rho(1-|z|^2) \frac{(1-|z|^2)^{\nu}}{|1-\overline{z}w|^2} dA(z) \right] dA(w).
\]
Combining (A) with (B), we deduce that
\[
\int_{D} \left( 1-|z|^2 \right)^{\gamma-2} \rho(1-|z|^2) |1-\overline{z}w|^2 dA(z) \leq \rho(1-|w|^2) \frac{(1-|w|^2)^{\gamma}}{(1-|w|^2)^{\gamma+2}}, s > 1.
\]
Thus,
\[
\int_{\Gamma_{p,\mu}(f)} \frac{\rho(1-|z|^2)}{(1-|z|^2)^2} dA(z) \leq \int_{D} |g''(w)|^p (1-|w|^2)^{\gamma-2} \rho(1-|w|^2) dA(w) < \infty.
\]
(3) ⇒ (2). Let \( E(z, 1/2) = \{ w \in \mathbb{D} : |\varphi_z(w)| < 1/2 \} \) be a pseudo-hyperbolic disk of center \( z \in \mathbb{D} \) and radius 1/2. Recall that
\[
1 - |w| \approx |1 - \overline{z}w| \approx 1 - |z|, \quad w \in E(z, 1/2)
\]
and \( |E(z, 1/2)| \approx (1 - |z|)^2 \) (see [34, Page 69]). Using the subharmonicity of \( |f''| \), we obtain
\[
\int_{\mathbb{D}} |f''(w)|^p (1 - |w|^2)^{p-2} (1 - |\varphi_z(w)|^2)^\rho \, dA(w) 
\geq \int_{E(z, 1/2)} |f''(w)|^p (1 - |w|^2)^{p-2} (1 - |\varphi_z(w)|^2)^\rho \, dA(w) 
\geq (1 - |z|)^p |f''(z)|^p.
\]
Therefore, \( \Omega_z(f) \subseteq \Gamma_{\rho,p}(f) \). The proof is complete.

3. Interpolating Blaschke product in \( C_H(B_\rho(\rho) \cap \mathcal{B}) \) as multipliers

An analytic function in the unit disc \( \mathbb{D} \) is called an inner function if it is bounded and has radial limits of absolute value 1 at almost every point of the boundary \( \partial \mathbb{D} \). It is well known that every inner function has a factorization \( e^{iy} B(z) S(z) \), where \( y \in \mathbb{R} \), \( B(z) \) is a Blaschke product and \( S(z) \) is a singular inner function. A Blaschke product \( B \) with sequence of zeros \( \{a_k\}_{k=1}^\infty \) is called interpolating if there exists a positive constant \( \delta \) such that
\[
\prod_{j \neq k} |\varphi_{a_j}(a_k)| \geq \frac{1}{2}, \quad k = 1, 2, \ldots
\]
We also say that \( \{a_k\}_{k=1}^\infty \) is an interpolating sequence or a uniformly separated sequence. The following notions were introduced by Dyakonov [14]:

Suppose \( X \) and \( Y \) are two classes of analytic functions on \( \mathbb{D} \), and \( X \subseteq Y \). Let \( \theta \) be an inner function, \( \theta \) is said to be \((X,Y)\)-improving, if every function \( f \in X \) satisfying \( f \theta \in Y \) must actually satisfy \( f \theta \in X \). For more information related to improving multipliers, we refer to [27]. Motivated by the works of Dyakonov and Peláez, we have the following result.

**Theorem 2.** Let \( \rho \) be of finite lower type greater than 0 and upper type less than 1. Suppose that \( 1 < p < \infty \) and \( B(z) \) is an interpolating Blaschke product with zeros \( \{a_k\}_{k=1}^\infty \). Then following are equivalent:

1. \( B \in C_H(B_\rho(\rho) \cap \mathcal{B}) \).
2. \( \sum_{k=1}^\infty \rho (1 - |a_k|^2) < \infty \).
3. \( B \) is \((B_\rho(\rho) \cap \text{BMOA}), \text{BMOA})\)-improving.
4. \( B \) is \((B_\rho(\rho) \cap \text{BMOA}), \mathcal{B})\)-improving.

Before we get into the proof, we need some lemmas.

**Lemma 1.** ([25, Lemma 2.5]) Let \( s > -1, r, t > 0, \) and \( t < s + 2 < r \). Then
\[
\int_{\mathbb{D}} \frac{(1 - |w|^2)^s}{|1 - \overline{w}z||1 - \overline{w}z|^t} \, dA(w) \leq \frac{(1 - |z|^2)^{2s+r-1}}{|1 - \overline{z}w|^t}, \quad z, \zeta \in \mathbb{D}.
\]

**Lemma 2.** Let \( \rho \) be of finite lower type greater than 0 and upper type less than 1. Suppose that \( f \in H(\mathbb{D}) \) and \( a \in \mathbb{D} \), then
\[
\int_{\mathbb{D}} |f(z) - f(0)|^p \rho \left( \frac{1 - |a|^2}{1 - |z|^2} \right) \, dA(z) 
\leq \int_{\mathbb{D}} |f''(z)|^2 (1 - |z|^2)^{p-1} \rho (1 - |a|^2)^2 \, dA(z).
\]
Proof. Let $\epsilon > 0$ be sufficiently small. From the proof of Lemma 2.1 of [9], we see that

\[
|f(z) - f(0)|^p \leq \left( \int_{D} |f'(w)|^p \frac{(1 - |w|^2)^{(1+\epsilon)(p-\epsilon)}}{|1 - wz|^{2\sigma}} dA(w) \right) (1 - |z|^2)^{-\varrho(p-1)},
\]

where $\sigma - \epsilon > -1$. Using Fubini’s theorem, we have

\[
\int_{D} |f(z) - f(0)|^p \frac{\rho(1 - \varphi_{\sigma}(z))}{1 - |z|^2} dA(z)
\]

\[
\leq \int_{D} \left( \int_{D} |f'(w)|^p \frac{(1 - |w|^2)^{(1+\epsilon)(p-\epsilon)}}{|1 - wz|^{2\sigma}} dA(w) \right) \rho(1 - \varphi_{\sigma}(z)) (1 - |z|^2)^{-\varrho(p-1)} dA(z)
\]

\[
= \int_{D} |f'(w)|^p (1 - |w|^2)^{p+\epsilon(p-1)} \left( \int_{D} \frac{\rho(1 - \varphi_{\sigma}(z))}{(1 - |w|^2)^{1+\epsilon(p-1)}|1 - wz|^{2\sigma}} dA(w) \right) dA(w).
\]

Using conditions (A) and (B), combining (C) with Lemma 1, we deduce

\[
\int_{D} \frac{\rho(1 - \varphi_{\sigma}(z))}{(1 - |z|^2)^{1+\epsilon(p-1)}|1 - wz|^{2\sigma}} dA(z)
\]

\[
= \rho(1 - \varphi_{\sigma}(w)) \int_{D} \frac{\rho(1 - \varphi_{\sigma}(w))}{(1 - |z|^2)^{1+\epsilon(p-1)}|1 - wz|^{2\sigma}} dA(z)
\]

\[
\leq \rho(1 - \varphi_{\sigma}(w)) \left( \int_{D} \frac{(1 - \varphi_{\sigma}(z))^\gamma}{(1 - \varphi_{\sigma}(w))^\gamma} + \frac{(1 - \varphi_{\sigma}(z))^\delta}{(1 - \varphi_{\sigma}(w))^\delta} \right) dA(z)
\]

\[
\leq \rho(1 - \varphi_{\sigma}(w))^\gamma (1 - |w|^2)^{1-\sigma(p-1)},
\]

where $\gamma + \epsilon(p-1) < \delta + \epsilon(p-1) < 1$. Thus,

\[
\int_{D} |f(z) - f(0)|^p \frac{\rho(1 - \varphi_{\sigma}(z))^\gamma}{1 - |z|^2} dA(z)
\]

\[
\leq \int_{D} |f'(w)|^p (1 - |w|^2)^{p-1} \rho(1 - \varphi_{\sigma}(w))^\gamma dA(w).
\]

The proof is complete. \qed

Lemma 3. ([23]) Let $\{a_k\}_{k=1}^\infty$ be a sequence in $D$. Then the measure $d\mu_n = \sum_{k=1}^\infty (1 - |a_k|^2)\delta_{a_k}$ is a Carleson measure, i.e.

\[
\sup_{\omega \in D} \sum_{k=1}^\infty (1 - |\varphi_{\omega}(a_k)|^2) < \infty,
\]

if and only if $\{a_k\}_{k=1}^\infty$ is a finite union of interpolating sequences.

Lemma 4. Let $\rho$ be of finite lower type greater than 0 and upper type less than 1. Suppose that $1 < p < \infty$, $B(\zeta)$ is an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^\infty$ and $f \in B_p(\rho)$. If $\sum_{k=1}^\infty |f(a_k)|^p \rho(1 - |a_k|^2) < \infty$, then $fB \in B_p(\rho)$. 

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Proof. Suppose that $f \in B_p(\rho)$ and $B(z)$ is an interpolating Blaschke product with zeros $\{a_k\}_{k=1}^\infty$. Since

$$
\int_D |fB'(z)|^p(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z)
$$

and

$$
\int_D |f'(z)|^pB(z)^p(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z)
$$

we have

$$
\int_D |f(z)|^p|B'(z)|^p(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z)
\leq \int_D |f(z)|^p|B'(z)|^{p}(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z) + \int_D |f(z)|^p|B'(z)|^{p}(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z)
$$

It is enough to prove

$$
\int_D |f(z)|^p|B'(z)|^p(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z) < \infty.
$$

Notice the fact that

$$(1 - |z|^2)|B'(z)| \leq 1$$

and

$$|B'(z)| \leq \sum_{k=1}^\infty \frac{1 - |a_k|^2}{1 - |a_k|^2},$$

we have

$$
\int_D |f(z)|^p|B'(z)|^p(1 - |z|^2)^{p-2}\rho(1 - |z|^2)dA(z)
\leq \int_D |f(z)|^p|B'(z)|(1 - |z|^2)^{-1}\rho(1 - |z|^2)dA(z)
$$

$$
\leq \sum_{k=1}^\infty (1 - |a_k|^2) \int_D \frac{|f(a_k)|^p}{1 - |a_k|^2}|1 - |z|^2|^p\rho(1 - |z|^2)dA(z)
$$

$$+ \sum_{k=1}^\infty (1 - |a_k|^2) \int_D \frac{|f(z) - f(a_k)|^p}{1 - |a_k|^2}(1 - |z|^2)^p\rho(1 - |z|^2)dA(z)
$$

$$= M + N.
$$

Since

$$
\int_D \frac{\rho(1 - |z|^2)}{1 - a_k |z|^2(1 - |z|^2)}dA(z) \leq \frac{\rho(1 - |a_k|^2)}{(1 - |a_k|^2)},
$$

we deduce that

$$M := \sum_{k=1}^\infty (1 - |a_k|^2) \int_D \frac{|f(a_k)|^p}{1 - |a_k|^2}|1 - |z|^2|^p\rho(1 - |z|^2)dA(z)
$$

$$\leq \sum_{k=1}^\infty |f(a_k)|^p\rho(1 - |a_k|^2) < \infty.
$$

Making the change of variables $z = \varphi_{a_k}(w)$, we obtain

$$N := \sum_{k=1}^\infty (1 - |a_k|^2) \int_D \frac{|f(z) - f(a_k)|^p}{1 - |a_k|^2}(1 - |z|^2)^p\rho(1 - |z|^2)dA(z)
$$

$$= \sum_{k=1}^\infty \int_D |f \circ \varphi_{a_k}(w) - f \circ \varphi_{a_k}(0)|^p\rho(1 - |\varphi_{a_k}(w)|^2)(1 - |w|^2)dA(w).
$$
Using Fubini’s theorem and Lemma 2, we have
\[
N = \sum_{k=1}^{\infty} \int_{\mathbb{D}} |f \circ q_{a_k}(w) - f \circ q_{a_k}(0)|^p \frac{\rho(1 - |q_{a_k}(w)|^2)}{(1 - |w|^2)^2} dA(w)
\]
\[
\leq \sum_{k=1}^{\infty} \int_{\mathbb{D}} |(f \circ q_{a_k})'(w)|^p \rho(1 - |q_{a_k}(w)|^2)(1 - |w|^2)^{-p} dA(w)
\]
\[
= \sum_{k=1}^{\infty} \int_{\mathbb{D}} |f'(w)|^p(1 - |w|^2)^{-p} \rho(1 - |q_{a_k}(w)|^2)dA(w).
\]

Since \(\{a_k\}_{k=1}^{\infty}\) is an interpolating sequences, using Lemma 3, we have \(N \leq \|f\|_{B_{0,1}^p}\), that is,
\[
\int_{\mathbb{D}} |f(z)|^p|B'(z)|^p(1 - |z|^2)^{-p} \rho(1 - |z|^2)dA(z)
\]
\[
\leq \sum_{k=1}^{\infty} |f(a_k)|^p(1 - |a_k|^2) + \|f\|_{B_{0,1}^p}^p.
\]

The proof is complete. \(\Box\)

We also need the following lemma.

**Lemma 5.** ([13, Theorem 1]) If \(f \in BMOA\) and \(\theta\) is an inner function, then the following conditions are equivalent:

1. \(f \theta \in BMOA;\)
2. \(\sup_{z \in \mathbb{D}} |f(z)|^p(1 - |\theta(z)|^2) < \infty;\)
3. \(\sup_{z \in \mathbb{D}(\theta, \epsilon)} |f(z)| < \infty, \text{for every } \epsilon, 0 < \epsilon < 1;\)
4. \(\sup_{z \in \mathbb{D}(\theta, \epsilon)} |f(z)| < \infty, \text{for some } \epsilon, 0 < \epsilon < 1.\)

**Proof of Theorem 2.**

Proof. (1) \(\Rightarrow\) (2). Let \(B\) be an interpolating Blaschke product with zeros \(\{a_k\}_{k=1}^{\infty}\) and \(B \in C_B(B \cap B_p(\rho)).\) From [18, Page 681], we know that there exist a \(\delta > 0,\) such that
\[
(1 - |z|^2)|B'(z)| \geq \frac{\delta(1 - \delta)}{8}, \quad z \in E(a_k, \frac{\delta}{4}).
\]

Thus,
\[
\bigcup_{k=1}^{\infty} E(a_k, \frac{\delta}{4}) \subseteq \left\{ z \in \mathbb{D} : (1 - |z|^2)|B'(z)| \geq \frac{\delta(1 - \delta)}{8} \right\}.
\]

Since \(\left\{E(a_k, \frac{\delta}{4})\right\}_{k=1}^{\infty}\) are pairwise disjoint, using the fact that
\[
|E(a_k, \frac{\delta}{4})| \approx (1 - |z|^2)^2, \quad z \in E(a_k, \frac{\delta}{4}),
\]
we obtain
\[
\sum_{k=1}^{\infty} \rho(1 - |a_k|^2) \leq \sum_{k=1}^{\infty} \int_{E(a_k, \frac{\delta}{4})} \rho \left( 1 - |z|^2 \right) \frac{dA(z)}{(1 - |z|^2)^2}
\]
\[
\leq \int_{\mathbb{D}(1 - |\theta|)|B'(z)| \leq \frac{\delta(1 - \delta)}{8}} \rho \left( 1 - |z|^2 \right) \frac{dA(z)}{(1 - |z|^2)^2} < \infty.
\]
(2) ⇒ (3). Suppose that \( f \in B_p(\rho) \cap BMOA \) and \( fB \in BMOA \). We only need to prove that \( fB \in B_p(\rho) \).

Using Lemma 5, we obtain

\[
\sum_{k=1}^{\infty} |f(a_k)|^p \rho(1 - |a_k|^2) \leq \sup_{a_k} |f(a_k)|^p \sum_{k=1}^{\infty} \rho(1 - |a_k|^2) < \infty.
\]

By Lemma 4, we have \( fB \in B_p(\rho) \).

(3) ⇒ (4). Let \( f \in B_p(\rho) \cap BMOA \subseteq BMOA \) and \( fB \in B \). From [27, Corollary 1], we see that every interpolating Blaschke product \( B \) is \((BMOA, B)\)-improving. Hence, we have \( fB \in BMOA \). Notice that \( B \) is \((B_p(\rho) \cap BMOA, BMOA)\)-improving, we have \( fB \in B_p(\rho) \cap BMOA \). Thus, \( B \) is \((B_p(\rho) \cap BMOA, B)\)-improving.

(4) ⇒ (1). Suppose that \( B \) is \((B_p(\rho) \cap BMOA, B)\)-improving. Note that \( 1 \in B_p(\rho) \cap BMOA \) and \( B \in H^\infty \subseteq B \). Thus, \( B \in B_p(\rho) \cap BMOA \subseteq B_p(\rho) \cap B \cap C_B(B_p(\rho) \cap B) \). The proof is complete.

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