SPECIAL GEOMETRY AND PERTURBATIVE ANALYSIS OF $N = 2$ HETEROTIC VACUA†

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ABSTRACT

The requirement of target-space duality and the use of nonrenormalization theorems lead to strong constraints on the perturbative prepotential that encodes the low-energy effective action of $N = 2$ heterotic superstring vacua. The analysis is done in the context of special geometry, which governs the couplings of the vector multiplets. The presentation is kept at an introductory level.

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1 Special geometry

Special geometry refers to the target-space geometry of $N = 2$ supersymmetric vector multiplets, possibly coupled to supergravity. The physical states of a vector multiplet are described by gauge fields $W^{IJ}_\mu$, doublets of Majorana spinors $\Omega^I_I$ and complex scalars $X^I$. The kinetic term for the scalars is a nonlinear sigma model which defines the metric of the target space, the space parametrized by the scalar fields. Special geometry has the following characteristic features. The Lagrangian is encoded in a holomorphic prepotential $F(X)$. In rigid supersymmetry the fields $X^I$ can be regarded as independent coordinates $(I = A = 1, \ldots, n)$. In the local case there is one extra vector multiplet labeled by $I = 0$, which provides the graviphoton, but the $n + 1$ fields $X^I$ are parametrized in terms of $n$ holomorphic coordinates $z^A$. Often one chooses special coordinates defined by $z^A = X^A/X^0$. The target space is Kählerian and the Kähler potential is given by (the subscripts on $F$ denote differentiation)

$$K(X, \bar{X}) = -i \bar{X}^A F_A(X) + i X^A \bar{F}_A(\bar{X}),$$
$$K(z, \bar{z}) = -\log(-i \bar{X}^I F_I + i X^I \bar{F}_I),$$

(1)

for rigid and for local supersymmetry, respectively. In the latter case, the $2n + 2$ quantities $(X^I, F_I)$ are parametrized by $n$ complex coordinates $z^A$. In more mathematical terms they define holomorphic symplectic sections. The ensuing metric satisfies the following curvature relations

$$R^A_{BCD} = -W_{BCDE} \bar{W}^{EAD},$$
$$R^A_{BCD} = 2\epsilon_{BCD} \delta^A_E - \epsilon_{BCDE} W_{BCDE} \bar{W}^{EAD},$$

(2)

respectively, for the two cases. Here the tensor $W$ is related to the third derivative of $F(X)$. Special geometry is also the geometry of the moduli of Calabi-Yau spaces. This intriguing connection can be understood in the context of type-II superstrings, whose compactification on Calabi-Yau manifolds leads to four-dimensional low-energy field theories with local $N = 2$ supersymmetry.

The bosonic kinetic terms read

$$\mathcal{L} = \pm \left( D_\mu F_I D^\mu \bar{X}^I - D_\mu X^I D^\mu \bar{F}_I \right) - \frac{i}{16\pi} \left( \hat{N}_{IJ} \bar{F}_{\mu}^{+I} F^{+\mu I} - \hat{N}_{IJ} F_{\mu}^{-I} F^{-\mu J} \right),$$

(3)

where $F_{\mu}^{\pm I}$ denote the selfdual and anti-selfdual field-strength components, and

$$\mathcal{N}_{IJ} = \bar{F}_{I,J} + 2i \frac{\text{Im}(F_{IK}) \text{Im}(F_{IL}) X^K X^L}{\text{Im}(F_{KL}) X^K X^L}. \quad \text{(4)}$$

2 Symplectic reparametrizations

From the Lagrangian (3) one defines the tensors

$$G_{\mu\nu I} = \hat{N}_{IJ} F_{\mu}^{+J}, \quad G_{\mu\nu I} = \hat{N}_{IJ} F_{\mu}^{-J}, \quad \text{so that the Bianchi identities and equations of motion for the Abelian gauge fields can be written as}$$

$$\partial^\mu (F_{\mu}^{+I} - F_{\mu}^{-I}) = 0, \quad \partial^\mu (G_{\mu\nu I}^{+} - G_{\mu\nu I}^{-}) = 0. \quad \text{(6)}$$

These are invariant under the transformation

$$\left( \begin{array}{c} F_{\mu}^{+I} \\ G_{\mu\nu I}^{+} \end{array} \right) \rightarrow \left( \begin{array}{cc} U & Z \\ W & V \end{array} \right) \left( \begin{array}{c} F_{\mu}^{+I} \\ G_{\mu\nu I}^{+} \end{array} \right) \quad \text{(7)}$$

where $U_I, V_I, W_{IJ}$ and $Z^{IJ}$ are constant real $(n + 1) \times (n + 1)$ submatrices. The transformations for the anti-selfdual tensors follow by complex conjugation. From (4) one derives that $\mathcal{N}$ transforms as

$$\mathcal{N}_{IJ} \rightarrow (V_I^K \mathcal{N}_{KL} + W_{IL}) \left[ (U + Z\mathcal{N})^{-1} \right]_L^J. \quad \text{(8)}$$

To ensure that $\mathcal{N}$ remains a symmetric tensor, at least in the generic case, the transformation (9) must be an element of $Sp(2n + 2, \mathbb{R})$ (disregarding a uniform scale transformation). The required change of $\mathcal{N}$ is induced by a change of the scalar fields, implied by

$$\left( \begin{array}{c} X^I \\ F_I \end{array} \right) \rightarrow \left( \begin{array}{c} \bar{X}^I \\ \bar{F}_I \end{array} \right) = \left( \begin{array}{cc} U & Z \\ W & V \end{array} \right) \left( \begin{array}{c} X^I \\ F_I \end{array} \right). \quad \text{(9)}$$

In this transformation we include a change of $F_I$. Because the transformation belongs to $Sp(2n + 2, \mathbb{R})$, one can show that the new quantities $\tilde{F}_I$ can be written as the derivatives of a new function $\tilde{F}(\bar{X})$. The new but equivalent set of equations of motion one obtains by means of the symplectic transformation (properly extended to other fields), follows from the Lagrangian based on $\tilde{F}$. In special cases $F$ remains unchanged, $\tilde{F}(X) = F(X)$, so that the theory is invariant under the corresponding transformations.

The symplectic transformations (9) cause electric fields to transform into magnetic fields and vice versa. The interchange of electric and magnetic fields is known as electric-magnetic duality. Under the transformation with $U = V = 0$ and $W = -Z = 1$, $F_{\mu}^{+I}$ and $G_{\mu\nu I}^{+}$ are simply interchanged, while $\mathcal{N}$ transforms into $-\mathcal{N}^{-1}$.

In the rigid case, $\mathcal{N}$ consists of only the first term and the $I = 0$ component is suppressed. In general, $\mathcal{N}$ is complex. Its imaginary part is related to the gauge coupling constant, its real part to a generalization of the $\theta$ angle.
Since the coupling constants are thus replaced by their inverses, electric-magnetic duality relates the strong- and weak-coupling description of the theory. Electric-magnetic duality is a special case of so-called $S$ duality. The coupling constant inversion is then part of an $Sl(2, \mathbb{Z})$ group. This situation is known in the context of string theory and lattice gauge theories. Other symplectic transformations (with $Z=0$) can be discussed at the perturbative level and may involve a shift of the generalized $\theta$ angles. In nonabelian gauge theories $\theta$ is periodic, so that $\mathcal{N}$ is defined up to the addition of certain discrete real constants.

3 Semiclassical theory of monopoles and dyons

To elucidate some important features of the symplectic reparametrizations, let us discuss the effective action of abelian gauge fields, possibly obtained from a nonabelian reparametrizations, let us discuss the effective action of nonabelian gauge fields, possibly obtained from a nonabelian matrix theory by integrating out certain fields. We write the matrix $\mathcal{N}$ in terms of generalized coupling constants and $\theta$ angles, according to

$$\mathcal{N}_{lj} = \frac{\theta_{lj}}{2\pi} - i\frac{4\pi}{g_{lj}}. \quad (10)$$

This matrix can be compared to a generalization of the permeability and permittivity that is conventionally used in the treatment of electromagnetic fields in the presence of a medium. The fields $G_{\mu\nu}$ are thus generalizations of the displacement and magnetic fields, while $F_{\mu\nu}$ corresponds to the electric fields and magnetic inductions. So far we have considered an abelian theory without charges. It is straightforward to introduce electric charges by introducing an electric current in the Lagrangian. However, to consider duality transformations one must also introduce magnetic currents into the field equations, so that when electric fields transform into magnetic fields and vice versa, the electric and magnetic currents transform accordingly. The magnetic currents occur as sources in the Bianchi identity and describe magnetic monopoles.

Electric and magnetic charges are conveniently defined in terms of flux integrals over closed spatial surfaces that surround the charged objects,

$$\oint_{\partial V}(F^{+} + F^{-}) = 2\pi q_{l}^{e},$$

$$\oint_{\partial V}(G^{+} + G^{-}) = -2\pi q_{l}^{m}. \quad (11)$$

With these definitions a static point charge at the origin exhibits magnetic inductions and electric fields equal to $\vec{r}/(4\pi r^{3})$ times $2\pi q_{l}^{e}$ and $\frac{1}{2}\, g^{2}(q_{l}^{e} + q_{m}^{e}\theta_{lj}/2\pi)$, respectively. Note that $q_{e}$ does not coincide with the electric charge. From (11) it follows that the charges must transform under symplectic rotations according to

$$\begin{pmatrix} q_{m}^{e} \\ -q_{l}^{e} \end{pmatrix} \to \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} q_{m}^{e} \\ -q_{l}^{e} \end{pmatrix}. \quad (12)$$

As is well known, the charges are subject to a generalized Dirac quantization condition, due to Schwinger and Zwanziger. To derive this condition, consider a test particle with charges $q_{e}$ and $q_{m}$ in the field of a heavy dyon with charges $q_{e}$ and $q_{m}$ (for simplicity we restrict ourselves to a single gauge field). The equation of motion of the test particle is assumed to be invariant under duality transformations. There is only one symplectic invariant that one can construct from the test particle charges and the dyon fields, namely $q_{m}^{e}G_{\mu\nu} + q_{e}^{e}F_{\mu\nu}$, where $F_{\mu\nu}$ and $G_{\mu\nu}$ represent the fields induced by (11). Inserting this combination into the field equation of the test particle yields a generalization of the Lorentz force,

$$m \frac{d^{2}x^{\mu}}{dt^{2}} = \left\{ \frac{1}{2} \left( q_{e}^{e} + \frac{\theta}{2\pi q_{m}^{e}} \right) F_{\mu\nu}^{\nu} + \frac{1}{2} q_{m}^{m} 4\pi q_{m}^{e} G_{\mu\nu} F^{\mu\nu} \right\} \frac{dx^{\nu}}{dt}. \quad (13)$$

The angular momentum $\vec{L} = m\vec{r} \times \dot{\vec{r}}$ of the test particle is not invariant in the dyon field (taken at the origin) and one must include the contribution of the electromagnetic fields. The total angular momentum vector,

$$\vec{j} = \vec{L} + \frac{\vec{r} \times \vec{F}}{r} \frac{q_{e}^{e} q_{m}^{m} - q_{m}^{e} q_{e}^{m}}{4},$$

is indeed a constant of the motion. Quantum-mechanically the component of this vector along $\vec{r}$ must be an integer times $\hbar/2$, so that one obtains a quantization condition for $q_{e}^{e} q_{m}^{m} - q_{m}^{e} q_{e}^{m}$. It implies that the allowed electric and magnetic charges comprise a lattice such that surface elements spanned by the lattice vectors are equal to a multiple of the Dirac unit $2\hbar$, as shown in fig. 1. In addition, the lattice should be consistent with the periodicity of the $\theta$ angle; $\theta \to \theta + 2\pi$ corresponding to $\mathcal{N} \to \mathcal{N} + 1$. This shift is associated with a symplectic transformation with $U = V = W = 1$ and $Z = 0$, so that the charges transform according to $q_{e} \to q_{e} + q_{m}$, while $q_{m}$ remains invariant. This transformation is contained in the discrete subgroup $Sp(2n + 2, \mathbb{Z})$ that leaves the charge lattice invariant.

\footnote{The normalization of the $\theta$ angle is fixed by the assumption that instantons yield an integer value for the Pontryagin index $(32\pi^{2})^{-1} \int d^{4}x \, \ast FF$ in a nonabelian extension of the theory.}
4 Wilsonian action and nonrenormalization

We now elucidate some consequences of special geometry in the example of $SU(2)$ $N = 2$ supersymmetric gauge theory. The classical theory is described by $F_{\text{class}}(\vec{X}) = \frac{1}{2} S \vec{X}^2$, where $\vec{X}$ is an $SU(2)$ vector and $i S = \theta/2\pi + i 4\pi/\gamma^2$. In the context of heterotic string theory $S$ is the dilaton field, which is the scalar component of a vector multiplet. We momentarily restrict this multiplet to be constant, which preserves supersymmetry. The nonabelian theory has a potential with flat valleys whenever the real and imaginary parts of $\vec{X}$ are parallel, so that $SU(2)$ is broken down to $U(1)$. In the Wilsonian action we integrate out the fields of momenta higher than a certain scale $\Lambda$. Restricting ourselves to the vector multiplet associated with the unbroken $U(1)$, the Wilsonian effective action is then, by $N = 2$ supersymmetry, encoded in a holomorphic function $F_W(X, S)$; because $S$ can be regarded as the scalar component of a vector multiplet, holomorphicity applies also to $S$. However, at the same time $S$ is a loop-counting parameter. The one-loop result is therefore $S$-independent and explicit calculation shows that it is equal to $(i/2\pi)X^2 \ln(X^2/\Lambda^2)$. The coefficient in front of this expression is directly related to the one-loop beta function and the chiral anomaly: the latter is related to the fact that $X^{-2} F_W(X, S)$ changes under a phase transformation $X \to e^{i\alpha} X$ by an additive term $-\alpha/\pi$.

Now we invoke a nonrenormalization argument which hinges on the fact that, perturbatively, the result should be independent of the $\theta$ angle, as this parameter multiplies only the total divergence *FF* in the Lagrangian. However, the requirement that the Wilsonian action should be independent of $\theta$, while the corresponding prepotential should depend holomorphically on $S$, excludes all perturbative corrections beyond the one-loop level. Therefore we may write

$$F_W(X, S) = i \frac{X^2}{2\pi} \left\{ \ln \frac{X^2}{\Lambda e^{-\pi S}} + f^{n-p} \left( \frac{\Lambda^4 e^{-2\pi S}}{X^4} \right) \right\},$$

where the last term denotes the nonperturbative contributions. These take a restricted form. First of all, nonperturbatively, the action is invariant under discrete shifts of $\theta$ equal to multiples of $2\pi$. Secondly, corrections from instantons break the invariance under phase transformations $X \to e^{i\alpha} X$ to $\mathbb{Z}_4$. This is tied to the 8 independent fermionic zero-modes that exist in the instanton background (the number of zero-modes is related to the one-loop axial anomaly coefficient through the Atiyah-Singer index theorem). So we write the nonperturbative corrections as a function of the dimensionless $X_4$ invariant $X^4/\Lambda^4$. Thirdly, by assigning an extra transformation to $S$, namely $S \to S - 2i\alpha/\pi$, the chiral anomaly cancels at one loop. This can be generalized beyond one loop by properly adjusting the $S$-dependent subtractions, so that the combined transformation of $X$ and $S$ will constitute an exact invariance. The explicit form of the function $f^{n-p}$ is of course subtraction dependent. The parametrization is entirely in agreement with explicit instanton calculations (note that the real part of $2\pi S$ equals the one-instanton action $8\pi^2/\gamma^2$) and exhibits the cut-off dependence characteristic for supersymmetric gauge theories.

We draw attention to the fact that the function is not single valued. Because of the (perturbative) logarithmic correction, $F_W$ is determined up to a quadratic function $X^2$ with an integer coefficient. The logarithmic singularity is due to the fact that the mass of the charged particles that we integrated out, tends to zero when $|X|$ vanishes. Going around the branch-cut by $X \to e^{i\pi} X$ is equivalent to a symplectic transformation with $U = V = -1$, $W = 2$ and $Z = 0$. These monodromies play a central role in understanding the ground-state structure of $N = 2$ supersymmetric Yang-Mills theories, as demonstrated by Seiberg and Witten. By exploiting symplectic reparametrizations they describe similar singularities in the nonperturbative domain. Again these singularities can be understood as the result of certain electrically charged states becoming massless, where ‘electric’ refers to the new basis obtained after applying the symplectic reparametrization. In the original (semiclassical) basis these states then correspond to dyons with nonzero magnetic charge.

5 $N = 2$ Heterotic vacua

Finally we consider the prepotential of vector multiplets relevant for $N = 2$ heterotic vacua. In such vacua there are at least two abelian gauge fields, one associated with the graviphoton and one with the dilaton field. (This can be deduced from supergravity alone, provided the dilaton is contained in a so-called vector-tensor multiplet.) In toroidal compactifications there are two extra vector multiplets associated with the complex toroidal moduli $T$ and $U$. The classical prepotential is uniquely determined (up to symplectic reparametrizations) by the requirement that the dilaton couples universally at the string tree level and the effective action does not depend on the $\theta$ angle,

$$F_{\text{class}}(X) = -\frac{X^4}{X_0} \left[ X^2 X^3 - \sum_{I \geq 4} (X^I)^2 \right].$$

Including the one-loop correction the gauge coupling $N$ becomes equal to $N_{\text{eff}} = \frac{\theta/2\pi - i \pi/2}{2\gamma^2} - (i/\pi)(\ln(X^2/\Lambda^2) + 3)$ and satisfies $\Lambda^2 \partial/\partial X^2 N_{\text{eff}} = -(i/\pi)[\frac{11}{12} + \frac{2}{\pi} C_2]$, where the separate terms refer to the beta-function contribution from vectors, scalars and spinors, respectively; $C_2$ is the second-order Casimir invariant, which equals 2 for $SU(2)$. Nonperturbatively $N$ also receives real $A$-dependent corrections, which lead to a renormalization of the $\theta$ angle. See ref. 2 and references quoted therein.
This function corresponds to the product manifold $[SU(1,1)/U(1)] \times [SO(2,n-1)/SO(2) \times SO(n-1)]$. The $SU(1,1)/U(1)$ coordinate is the dilaton field $iS = X^1/X^0$, whose real part correlates the string coupling constant; other moduli are given by $iT = X^2/X^0$, $iU = X^3/X^0$, etc. The objective is to consider the perturbative corrections which, as above, originate entirely from one-loop effects and cause an $S$-independent addition to $[1]$. An immediate problem is that the gauge couplings do not uniformly vanish in the large dilaton limit. To set up string perturbation theory consistently we must therefore change our basis by means of a symplectic reparametrization. As it turns out, this reparametrization is such that the prepotential $F$ no longer exists. Fortunately the latter is merely a technical problem. In the new basis the classical Lagrangian is $SO(2,n-1)$ invariant.

The one-loop correction should be invariant under target-space dualities, which are perturbative and expected to leave the dilaton invariant. Classically they coincide with $SO(2,n-1)$, but for finite string coupling we expect only a discrete subgroup to be relevant. Not surprisingly, in view of the high symmetry of the classical result, there are no modifications of $[1]$ that preserve the invariance under the full $SO(2,n-1)$. Furthermore, we expect the corrections to exhibit a similar lack of single-valuedness as noted in the previous section. The corresponding monodromy, whose identification depends on the proper choice for the new symplectic basis, is induced by a symplectic transformation that interferes with the $SU(1,1)$ invariance of the dilaton.

It turns out that the one-loop contribution to the function $F_W \equiv i(X^0)^{-2}F_W$ must therefore be invariant under target-space duality transformations, up to a restricted polynomial of the moduli with discrete real coefficients. For example, in toroidal compactifications of six-dimensional $N = 1$ string vacua, where we have only $T$ and $U$, the transformation $T \to (aT - ib)/(icT + d)$ with integer parameters satisfying $ad - bc = 1$ induces the following result on the one-loop correction:

$$F^{(1)}(T,U) \to (icT + d)^{-2}[F^{(1)}(T,U) + \Xi(T,U)]; \quad (17)$$

where $\Xi$ is a quadratic polynomial in the variables $(1, iT, iU, TU)$. Hence $\partial^2_{T,U} \Xi = \partial^2_{T,U} \Xi = 0$. The appearance of $\Xi$ complicates the symmetry properties of the one-loop moduli prepotential, which would otherwise be a modular function of weight $-2$. It encodes the monodromies at singular points in the moduli space (for instance, at $T \approx U$) where one has an enhancement of the gauge symmetry. Knowledge of these singularities allows one to determine certain derivatives of $F^{(1)}$ in terms of standard modular functions. For a detailed discussion of the monodromies in the toroidal case, we refer to refs. 8 and 9.

In the presence of the one-loop correction the dilaton field is no longer invariant under target-space duality. This can be understood from the fact that the dilaton belongs originally to a vector-tensor multiplet and is only on-shell equivalent to a vector multiplet. One can always redefine $S$ such that it becomes invariant, but then it can no longer be interpreted as the scalar component of a vector multiplet. Interestingly enough, these perturbative results have been confirmed by explicit calculations based on ‘string duality’. For this we refer to A. Klemm’s contribution to these proceedings.

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