A short proof that $\chi$ can be bounded $\epsilon$ away from $\Delta + 1$ towards $\omega$

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Abstract

In 1998 the second author proved that there is an $\epsilon > 0$ such that every graph satisfies $\chi \leq \lceil (1 - \epsilon)(\Delta + 1) + \epsilon \omega \rceil$. The first author recently proved that any graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$ contains a stable set intersecting every maximum clique. In this note we exploit the latter result to give a much shorter, simpler proof of the former. We include, as a certificate of simplicity, an appendix that proves all intermediate results with the exception of Hall’s Theorem, Brooks’ Theorem, the Lovász Local Lemma, and Talagrand’s Inequality.

1 Introduction

Much work has been done towards bounding the chromatic number $\chi$ of a graph in terms of the clique number $\omega$ and the maximum size of a closed neighbourhood $\Delta + 1$, which are trivial lower and upper bounds on the chromatic number, respectively. Recently, much of this work has been done in pursuit of a conjecture of Reed, who proposed that the average of the two should be an upper bound for $\chi$, modulo a round-up:

Conjecture 1 ([15]). Every graph satisfies $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.

This conjecture has been proven for some restricted classes of graphs [2 7 9 10 13], sometimes in the form of a stronger local conjecture posed by King [3 7]; both forms are known to hold in the fractional relaxation [7 12].

For general graphs, we only know that we can bound the chromatic number by some nontrivial convex combination of $\omega$ and $\Delta + 1$:

Theorem 2 ([15]). There exists an $\epsilon > 0$ such that every graph satisfies

$$\chi \leq \lceil (1 - \epsilon)(\Delta + 1) + \epsilon \omega \rceil.$$  

The original proof of this theorem is quite long and complicated. In this note we give a much shorter, simpler proof that exploits the following new existence condition for a stable set hitting every maximum clique, the proof of which from first principles is itself short and fairly simple:

Theorem 3 ([8]). Every graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$ contains a stable set hitting every maximum clique.

This result is a strengthening of a result of Rabern [14], which could be used to similar effect.
2 A proof sketch

We sketch the proof here, prove the necessary lemmas in the following two sections, then finally prove the theorem more formally.

Suppose \( G \) is a minimum counterexample to Theorem 2 for some fixed \( \epsilon \). Applying Theorem 3 and Brooks’ Theorem tells us that \( G \) satisfies \( \omega \leq \frac{2}{3}(\Delta + 1) \) and \( \Delta > \frac{1}{\epsilon} \). Our proof then considers two cases: If every neighbourhood contains much fewer than \( \left( \frac{\Delta}{2} \right) \) edges, we can apply a simple probabilistic argument. Otherwise we have a vertex \( v \) whose neighbourhood contains almost \( \left( \frac{\Delta}{2} \right) \) edges. The fact that \( \omega \leq \frac{2}{3}(\Delta + 1) \) tells us that there is a large antimatching in \( N(v) \), and since there are few edges between \( N(v) \) and \( G - v \), we can take an optimal colouring of \( G - N(v) - v \) and extend it to a colouring of \( G \) in which many pairs of the antimatching are monochromatic, which is enough to contradict the minimality of \( G \).

3 Dealing with sparse neighbourhoods

Theorem 10.5 in \cite{12}, which is a straightforward application of Talagrand’s Inequality, gives us a bound on the chromatic number when no neighbourhood contains almost \( \left( \frac{\Delta}{2} \right) \) edges:

**Theorem 4.** There is a \( \Delta_0 \) such that for any graph with maximum degree \( \Delta > \Delta_0 \) and for any \( B > \Delta(\log \Delta)^3 \), if no \( N(v) \) contains more than \( \left( \frac{\Delta}{2} \right) - B \) edges then \( \chi(G) \leq (\Delta + 1) - \frac{B}{e^\Delta} \).

We restate this theorem as follows:

**Corollary 5.** There is a \( \Delta_0 \) such that for any graph with maximum degree at most \( \Delta > \Delta_0 \) and for any \( \alpha > 2(\log \Delta)^3/(\Delta - 1) \), if no \( N(v) \) contains more than \( (1 - \alpha)\left( \frac{\Delta}{2} \right) \) edges then

\[
\chi(G) \leq (\Delta + 1) - \frac{\alpha(\Delta - 1)}{2e^6} \leq \left( 1 - \frac{\alpha}{2e^6} \right)(\Delta + 1).
\]

This is all we need for the case in which no neighbourhood contains almost \( \left( \frac{\Delta}{2} \right) \) edges.

4 Dealing with dense neighbourhoods

We need the following theorem to extend a colouring when we have a dense neighbourhood.

**Theorem 6.** Let \( \alpha \) be any positive constant and let \( \epsilon \) be any constant satisfying \( 0 < \epsilon < \frac{1}{6} - 2\sqrt{\alpha} \). Let \( G \) be a graph with \( \omega \leq \frac{2}{3}(\Delta + 1) \) and let \( v \) be a vertex whose neighbourhood contains more than \( (1 - \epsilon)(\frac{\Delta}{2}) \) edges. Then

\[
\chi(G) \leq \max\{\chi(G - v), (1 - \epsilon)(\Delta + 1)\}.
\]

This immediately implies:

**Corollary 7.** Let \( \rho \) be a positive constant satisfying \( \rho \leq \frac{1}{100} \), let \( G \) be a graph with maximum degree at most \( \Delta \), \( \omega \leq \frac{2}{3}(\Delta + 1) \) and let \( v \) be a vertex whose neighbourhood contains at least \( (1 - \rho)(\frac{\Delta}{2}) \) edges. Then

\[
\chi(G) \leq \max\{\chi(G - v), (1 - \rho)(\Delta + 1)\}.
\]

Before we prove Theorem 6 we need to lay out one more simple fact:
Lemma 8. Every graph G contains an antimatching of size $\left\lceil \frac{1}{2}(n - \omega(G)) \right\rceil$.

Proof. Let M be a maximum antimatching; there are $n - 2|M|$ vertices outside M, and these vertices must form a clique. Thus $\omega(G) \geq n - 2|M|$; the result follows. \qed

Proof of Theorem We may assume that $d(v) = \Delta$ since if this is not the case we can hang pendant vertices from $v$, and we may assume $\alpha < \frac{1}{14\pi}$, otherwise no valid value of $\epsilon$ exists. Our approach is as follows. We first partition $\tilde{N}(v)$ into sets $D_1$, $D_2$, and $D_3$ such that $D_1$ and $D_2$ are small, each $u \in D_2$ has few neighbours outside $D_2 \cup D_3$, and each $u \in D_3$ has very few neighbours outside $D_3$. In particular, $v \in D_3$. Then, using at most $\max\{\chi(G - v), (1 - \epsilon)(\Delta + 1)\}$ colours, we first colour $G - (D_2 \cup D_3)$, then greedily extend the colouring to $D_2$. Finally, we exploit the existence of a large antimatching in $G|D_3$ and extend the colouring to $D_3$ using an elementary result on list colourings.

It is straightforward to confirm that there are at most $\alpha (\Delta^2 - \Delta)$ edges between $G - \tilde{N}(v)$ and $\tilde{N}(v)$. We set $c_1 = \frac{1}{2}$ and $c_2 = \sqrt{\alpha}$. We partition $\tilde{N}(v)$ into $D_1$, $D_2$, and $D_3$ as follows:

$\begin{align*}
D_1 &= \{ u \in N(v) \mid u \text{ has more than } c_1(\Delta + 1) \text{ neighbours outside } \tilde{N}(v) \} \\
D_2 &= \{ u \in N(v) \setminus D_1 \mid u \text{ has more than } c_2(\Delta + 1) \text{ neighbours outside } \tilde{N}(v) \setminus D_1 \} \\
D_3 &= \tilde{N}(v) \setminus (D_1 \cup D_2)
\end{align*}$

Let $\beta_1$ denote $|D_1|/(\Delta + 1)$ and let $\beta_2$ denote $|D_2|/(\Delta + 1)$. Thus $|D_3| = (1 - \beta_1 - \beta_2)(\Delta + 1)$. Since there fewer than $\alpha \Delta^2$ edges between $\tilde{N}(v)$ and $G - \tilde{N}(v)$, we can see that $|D_1| < 2\alpha \Delta < \frac{1}{2}\sqrt{\alpha}(\Delta + 1)$. Note that every vertex in $D_1$ has more neighbours outside $\tilde{N}(v)$ than in $\tilde{N}(v)$, so there are fewer than $\alpha \Delta^2$ edges between $D_2 \cup D_3$ and $G - (D_2 \cup D_3)$. Thus $|D_2| < \sqrt{\alpha}(\Delta + 1)$. Therefore $\beta_1 < 2\alpha < \frac{1}{6}\sqrt{\alpha}$ and $\beta_2 < c_2 = \sqrt{\alpha}$. By the first of these two facts, we can see that $v$ is in $D_3$.

Now let $k$ denote $|(1 - \epsilon)(\Delta + 1)|$, let $k'$ denote $\max\{k, \chi(G - v)\}$, and take a $k'$-colouring of $G - (D_2 \cup D_3)$. We greedily extend this to a $k'$-colouring of $G - D_3$. To see that this is possible, note that while extending, every vertex in $D_2$ has at most $|D_1| + |D_2| + c_1(\Delta + 1) - 1 = (\beta_1 + \beta_2 + c_1)(\Delta + 1) - 1$ available colours, so each vertex at least $k - (\beta_1 + \beta_2 + c_1)(\Delta + 1) + 1 > (\frac{1}{2} - \epsilon - \frac{7}{6}\sqrt{\alpha})(\Delta + 1) > 0$ available colours, so we can indeed extend to all vertices of $D_2$ greedily.

Extending the partial colouring to $D_3$ takes a little more finesse. By assumption, $\omega(G|D_3) \leq \frac{1}{2}(\Delta + 1)$. Let $M$ be a maximum antimatching in $G|D_3$. We now define the graph $G_3$ as a clique of size $|D_3|$ minus $|M|$ vertex-disjoint edges. Note that $G|D_3$ is a subgraph of $G_3$. Lemma along with a classical result of Erdős, Rubin, and Taylor on list colourings of complete multipartite graphs with parts of size $\leq 2$ tells us that $\chi_t(G|D_3) \leq \chi_t(G_3) = \chi(G_3) = |D_3| - |M| \leq \frac{5}{6}(\Delta + 1)$ (this can be proven easily using induction and Hall’s Theorem). It follows that if we give each vertex of $D_3$ a list of at least $\frac{5}{6}(\Delta + 1)$ colours, we can find a colouring of $G|D_3$ such that every vertex gets a colour from its list.

We extend the partial colouring of $G - D_3$ to a colouring of $G$ by assigning each vertex $u$ in $D_3$ a list $\ell_u$ consisting of all colours from 1 to $k$ not appearing in $N(u) \setminus D_3$. Each list has size at least $k - (\beta_2 + c_2)(\Delta + 1) > (1 - \epsilon - \beta_2 - c_2)(\Delta + 1) - 1 > (1 - \epsilon - 2\sqrt{\alpha})(\Delta + 1) - 1 > \frac{5}{6}(\Delta + 1) - 1$. Since the list sizes are integers, each list has size at least $|D_3| - |M|$. Therefore we can extend the $k'$-colouring of $G - D_3$ to a $k'$-colouring of $G$. This completes the proof. \qed

5 Putting it together

We can now prove Theorem
Proof of Theorem 2. Take $\Delta_0$ from the statement of Corollary 5 and set $\epsilon$ as $\min\{1/\Delta_0, 1/320e^6\}$.

Let $G$ be a counterexample on a minimum number of vertices and denote its maximum degree and clique number by $\Delta$ and $\omega$ respectively. If $\Delta \leq \Delta_0$ then the result is implied by Brooks’ Theorem, so we can assume $\Delta > \Delta_0$. If $\omega < \frac{2}{3}(\Delta + 1)$, then Theorem 3 guarantees that we have a maximal stable set $S$ such that $\Delta(G - S) < \Delta$ and $\omega(G - S) < \omega$. By the minimality of $G$ we have a proper colouring of $G - S$ using

$$[(1 - \epsilon)(\Delta(G - S) + 1) + \epsilon\omega(G - S)] < [(1 - \epsilon)(\Delta + 1) + \epsilon\omega]$$

colours, to which we can add $S$ as a colour class, giving the desired colouring of $G$. So $G$ satisfies $\omega \leq \frac{2}{3}(\Delta + 1)$.

Now $G$ must be vertex-critical, must satisfy $\omega \leq \frac{2}{3}(\Delta + 1)$ and $\Delta > \Delta_0$, and must have chromatic number $> (1 - \frac{1}{320e^6})(\Delta + 1)$. Thus by Corollary 7 there is no vertex $v$ such that the neighbourhood of $v$ contains more than $\left(1 - \frac{1}{160}\right)\Delta$ edges. The theorem now follows immediately from Corollary 5.

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A Proving the intermediate results

To support the claim that our new proof is short, we offer proofs of the results that we have used, namely the choosability result of Erdős, Rubin and Taylor, Theorem 3, and Theorem 4. We omit proofs of Hall’s Theorem, Brooks’ Theorem, the Lovász Local Lemma, and Talagrand’s Inequality. We begin with choosability.

A.1 Chromatic choosability in the complement of a matching

Theorem 9 ([4]). If $G$ is a graph obtained from $K_n$ by removing a matching of size $\ell$, then $G$ is $(n - \ell)$-chooseable.

Proof. We proceed by induction on $n$; the basis $n = 1$ clearly holds. Let $G$ be a graph obtained from $K_n$ by removing a matching of size $\ell$, in which every vertex $v$ is assigned a list $L(v)$ of at least $n - \ell$ colours.

If $\ell < n/2$ we may take a universal vertex $v$ in $G$, assign it any colour from its list, and delete this colour from all other lists, proceeding by induction in the obvious way. Thus we may assume $\ell = n/2$. Call the vertices of $G$ $u_1, \ldots, u_\ell$ and $v_1, \ldots, v_\ell$ such that for $1 \leq i \leq \ell$, $u_i$ is nonadjacent to $v_i$. By the same argument we used for a universal vertex, we can see that $G$ is $n - \ell$-chooseable if some $u_i$ and $v_i$ have non-disjoint lists, so we may assume that for all $i$, the lists of $u_i$ and $v_i$ are disjoint.

We now construct an auxiliary bipartite graph $H$ with parts $V$ and $V'$ in which $V = V(G)$, $V'$ is the set of colours in some list, and $v \in V$ is adjacent to $v' \in V'$ precisely if $v' \in L(v)$. It suffices to prove that there is a $V$-saturating matching in $H$. Observe that for a set $W \subseteq V$, $|\bigcup L(v) \mid v \in W\} \geq n/2$ if $|W| \geq 1$, and that $|\bigcup L(v) \mid v \in W\} \geq n$ if $W$ intersects both $\{u_i\}_{i=1}^\ell$ and $\{v_i\}_{i=1}^\ell$, which is always the case if $|W| > n/2$. Therefore the result follows immediately from Hall’s Theorem.

A.2 Proof of Theorem 3

This subsection is essentially a terse version of [8] with proofs of two lemmas added. The main intermediate result is the following extension of Haxell’s Theorem [6], the proof of which we postpone until we have proved Theorem 3.

Theorem 10. For a positive integer $k$, let $G$ be a graph with vertices partitioned into cliques $V_1, \ldots, V_r$. If for every $i$ and every $v \in V_i$, $v$ has at most $\min\{k, |V_i| - k\}$ neighbours outside $V_i$, then $G$ contains a stable set of size $r$.

To prove Theorem 3 we must investigate intersections of maximum cliques. Given a graph $G$ and the set $\mathcal{C}$ of maximum cliques in $G$, we define the clique graph $G(\mathcal{C})$ as follows. The vertices
of \( G(\mathcal{C}) \) are the cliques of \( \mathcal{C} \), and two vertices of \( G(\mathcal{C}) \) are adjacent if their corresponding cliques in \( G \) intersect. For a connected component \( G(\mathcal{C}_i) \) of \( G(\mathcal{C}) \), let \( D_i \subseteq V(G) \) and \( F_i \subseteq V(G) \) denote the union and the mutual intersection of the cliques of \( \mathcal{C}_i \) respectively, i.e. \( D_i = \bigcup_{C \in \mathcal{C}_i} C \) and \( F_i = \cap_{C \in \mathcal{C}_i} C \).

The proof uses three intermediate results. The first, due to Hajnal \([5]\) (also see \([14]\)), tells us that for each component of \( G(\mathcal{C}) \), \(|D_i| + |F_i|\) is large:

**Lemma 11** (Hajnal). Let \( G \) be a graph and \( \mathcal{C} = \{C_1, \ldots, C_r\} \) be a nonempty collection of maximum cliques in \( G \). Then

\[
|\cap \mathcal{C}| + |\cup \mathcal{C}| \geq 2\omega(G).
\]

**Proof.** Fix \( G \) and proceed by induction on \( r \), with the base case \( r = 1 \) being clear. Fix \( r > 1 \) and let \( \mathcal{C}' \) denote \( \{C_1, \ldots, C_{r-1}\} \).

Let \( A \) denote \( \cap \mathcal{C}' \setminus \cap \mathcal{C} = \cap \mathcal{C}' \setminus C_r \), and let \( B \) denote \( \cup \mathcal{C} \setminus \cup \mathcal{C}' = \cup \mathcal{C}' \cup C_r \). The result holds by induction if \( |A| \leq |B| \) since \(|\cap \mathcal{C}| + |\cup \mathcal{C}| = |\cap \mathcal{C}'| + |\cup \mathcal{C}'| + (B - A)\), so assume \(|A| > |B|\). But observe that \((C_r \setminus B) \cup A\) is a clique that is larger than \( C_r \), a contradiction. The lemma follows. \( \square \)

The second is due to Kostochka \([11]\). It tells us that if \( \omega(G) \) is sufficiently close to \( \Delta(G) + 1 \), then \(|F_i|\) is large:

**Lemma 12** (Kostochka). Let \( G \) be a graph with \( \omega(G) > \frac{2}{3}(\Delta(G) + 1) \) and let \( \mathcal{C} \) be the set of maximum cliques in \( G \). Then for each connected component \( G(\mathcal{C}_i) \) of \( G(\mathcal{C}) \),

\[
|\cap \mathcal{C}| \geq 2\omega(G) - (\Delta(G) + 1).
\]

**Proof.** We proceed by induction on \(|\mathcal{C}|\), with the base case \(|\mathcal{C}| = 1\) being a consequence of Lemma \([11]\). We may consider the components of \( G(\mathcal{C}_i) \) individually, so assume \( G(\mathcal{C}_i) \) is connected. Let the set \( \mathcal{C} \) of maximum cliques be \( \{C_1, \ldots, C_r\} \), and let \( \mathcal{C}' \) denote \( \{C_1, \ldots, C_{r-1}\} \). Observe that it suffices to deal with the case in which every vertex of \( G \) is in \( \mathcal{C} \).

Since \( \omega(G) > \frac{2}{3}(\Delta(G) + 1) \) any two intersecting maximum cliques intersect in more that \( \omega/2 \) vertices, so the relation of two maximum cliques intersecting is transitive and therefore an equivalence relation. Furthermore, since \(|\cap \mathcal{C}'| > \omega/2\), \( \cap \mathcal{C} \) is nonempty and therefore \(|V(G)| = |\cup \mathcal{C}| = \Delta + 1\). Thus the result follows from Lemma \([11]\). \( \square \)

The third intermediate result is Theorem \([10]\). Combining them to prove Theorem \([2]\) is a simple matter.

**Proof of Theorem \([2]\)** Let \( \mathcal{C} \) be the set of maximum cliques of \( G \), and let the connected components of \( G(\mathcal{C}) \) be \( G(\mathcal{C}_1), \ldots, G(\mathcal{C}_r) \). It suffices to prove the existence of a stable set \( S \) in \( G \) intersecting each clique \( F_i \).

Lemma \([12]\) tells us that \(|F_i| > \frac{1}{3}(\Delta(G) + 1)\). Consider a vertex \( v \in F_i \), noting that \( v \) is universal in \( G[D_i] \). By Lemma \([11]\) we know that \(|F_i| + |D_i| > \frac{2}{3}(\Delta(G) + 1)\). Therefore \( \Delta(G) + 1 - |D_i| < |F_i| - \frac{1}{3}(\Delta(G) + 1) \), so \( v \) has fewer than \(|F_i| - \frac{1}{3}(\Delta(G) + 1)\) neighbours in \( \cup_{j \neq i} F_j \). Furthermore \( v \) certainly has fewer than \( \frac{1}{3}(\Delta(G) + 1)\) neighbours in \( \cup_{j \neq i} F_j \).

Now let \( H \) be the subgraph of \( G \) induced on \( \cup_i F_i \), and let \( k = \frac{1}{3}(\Delta(G) + 1) \). Clearly the cliques \( F_1, \ldots, F_r \) partition \( V(H) \). A vertex \( v \in F_i \) has at most \( \min\{k, |F_i| - k\} \) neighbours outside \( F_i \). Therefore by Theorem \([10]\) \( H \) contains a stable set \( S \) of size \( r \). This set \( S \) intersects each \( F_i \), and consequently it intersects every clique in \( \mathcal{C} \), proving the theorem. \( \square \)

It remains to prove Theorem \([10]\).
A.2.1 Independent transversals with lopsided sets

Suppose we are given a finite graph whose vertices are partitioned into stable sets $V_1, \ldots, V_r$. An independent system of representatives or ISR of $(V_1, \ldots, V_r)$ is a stable set of size $r$ in $G$ intersecting each $V_i$ exactly once. A partial ISR, then, is simply a stable set in $G$ intersecting no $V_i$ more than once.

A totally dominating set $D$ is a set of vertices such that every vertex of $G$ has a neighbour in $D$, including the vertices of $D$. Given $J \subseteq [m]$, we use $V_J$ to denote $(V_i \mid i \in J)$. Given $X \subseteq V(G)$, we use $I(X)$ to denote the set of partitions intersected by $X$, i.e., $I(X) = \{i \in [r] \mid V_i \cap X \neq \emptyset\}$.

For an induced subgraph $H$ of $G$, we implicitly consider $H$ to inherit the partitioning of $G$.

To prove our lopsided existence condition for ISRs, we use a slight strengthening of a lemma of Aharoni, Berger, and Ziv [1].

**Lemma 13.** Let $x_1$ be a vertex in $V_r$, and suppose $G[V_{[r-1]}]$ has an ISR. Suppose there is no $J \subseteq [r-1]$ and $D \subseteq V_J \cup \{x_1\}$ totally dominating $V_J \cup \{x_1\}$ with the following properties:

1. $D$ is the union of disjoint stable sets $X$ and $Y$.
2. $Y$ is a (not necessarily proper) partial ISR for $V_J$. Thus $|Y| \leq |J|$.
3. Every vertex in $Y$ has exactly one neighbour in $X$. Thus $|X| \leq |Y|$.
4. $X$ contains $x_1$.

Then $G$ has an ISR containing $x_1$.

**Proof.** Let $G$ be a minimum counterexample; we can assume $G = G[V_{[r-1]}] \cup \{x_1\}$. Furthermore, $r > 1$ otherwise the lemma is trivial. Let $R_1$ be an ISR of $G[V_{[r-1]}]$ chosen such that the set $Y_1' = Y_1 = R_1 \cap N(x_1)$ has minimum size. We know that $R_1$ exists because $G[V_{[r-1]}]$ has at least one ISR, and we know that $Y_1'$ is nonempty because $G$ does not have an ISR. Now let $X_1 = \{x_1\}$ and let $D_1 = X_1 \cup Y_1$.

We now construct an infinite sequence of partial ISRs $Y_1 \subset Y_2 \subset \ldots$, which contradicts the fact that $G$ is finite. Let $i > 1$, and suppose we have sets $\{R_j, Y_j, X_j \mid 1 \leq j < i\}$ such that:

- $X_j$ is a stable set consisting of distinct vertices $\{x_1, \ldots, x_j\}$. For $j > 1$, $x_j$ is a vertex in $G[V_{[Y_{j-1}]}]$ with no neighbour in $X_{j-1} \cup Y_{j-1}$.
- $R_j$ is an ISR of $G[V_{[r-1]}]$ such that for every $1 \leq \ell < j$, $R_j \cap N(X_{\ell}) = Y_{\ell}$. Subject to that, $R_j$ is chosen so that $Y_{j}' = R_j \cap N(x_j)$ is minimum. For $1 \leq j < i$, $Y_{j}'$ is nonempty.
- $Y_j = \cup_{i=1}^{j} Y_{j}'$.

To find $x_i$, $Y_{i}'$, and $R_i$, we proceed as follows.

1. Let $x_i$ be any vertex in $G[V_{[Y_{i-1}]}]$ with no neighbour in $X_{i-1} \cup Y_{i-1}$. We know that $x_i$ exists, otherwise the set $D_{i-1} = X_{i-1} \cup Y_{i-1}$ would be a total dominating set for $G[V_{[Y_{i-1}]}] \cup \{x_1\}$, contradicting the fact that $G$ is a counterexample.

2. Let $R_i$ be an ISR of $G[V_{[r-1]}]$ chosen so that for all $1 \leq j < i$, $R_i \cap N(x_j) = R_j \cap N(x_j) = Y_{j}'$. Subject to that, choose $R_i$ so that $Y_{i}' = R_i \cap N(x_i)$ is minimum. We know that $R_i$ exists because $R_{i-1}$ is a possible candidate for the ISR.
3. It remains to show that \( Y'_i \) is nonempty, i.e. that \( Y_i \neq Y_{i-1} \). Suppose \( Y'_i = \emptyset \). We will show that this contradicts our choice of \( R_i \) for the unique \( j < i \) such that \( x_i \in V_i(Y'_j) \). Let \( y \) be the unique vertex in \( R_i \cap V_i(x_i) \). Construct \( R'_i \) from \( R_i \) by removing \( y \) and inserting \( x_i \). Now for every \( \ell \) such that \( 1 \leq \ell < j \), \( R'_i \cap N(x_\ell) = Y'_i = R_j \cap N(x_\ell) \). For \( j \), \( R'_j \cap N(x_j) = (R_j \cap N(x_j)) \setminus \{y\} \), a contradiction. Thus \( Y'_i \) is nonempty.

4. Set \( X_i = X_{i-1} \cup \{x_i\} \) and \( Y_i = Y_{i-1} \cup Y'_i \).

This choice of \( X_i \), \( R_i \), and \( Y_i \) sets up the conditions so that we can repeat our argument indefinitely for increasing \( i \), a contradiction since \( G \) is finite. \( \square \)

Theorem 10 follows immediately from the following consequence of the previous lemma:

**Theorem 14.** Let \( k \) be a positive integer and let \( G \) be a graph partitioned into stable sets \( \{V_1, \ldots, V_r\} \). If for each \( i \in [r] \), each vertex in \( V_i \) has degree at most \( \min\{k, |V_i| - k\} \), then for any vertex \( v \), \( G \) has an ISR containing \( v \).

**Proof.** Suppose \( G \) is a minimum counterexample for a given value of \( k \). Clearly we can assume each \( V_i \) has size greater than \( k \), and that \( G[V_j] \) has an ISR for all \( J \subseteq [r] \). Take \( v \) such that \( G \) does not have an ISR containing \( v \); we can assume \( v \in V_r \). By Lemma 13 there is some \( J \subseteq [r-1] \) and a set \( D \subseteq V_J \cup \{v\} \) totally dominating \( V_J \cup \{v\} \) such that (i) \( D \) is the union of disjoint stable sets \( X \) and \( Y \), (ii) \( Y \) is a partial ISR of \( V_J \), (iii) \(|X| \leq |Y| \leq |J| \), and (iv) \( v \in X \).

Since \( D \) totally dominates \( V_J \cup \{v\} \), the sum of degrees of vertices in \( D \) must be greater than the number of vertices in \( V_J \). That is, \( \sum_{v \in D} d(v) > \sum_{i \in J} |V_i| \). Clearly \( \sum_{v \in X} d(v) \leq k \cdot |J| \) and \( \sum_{v \in Y} d(v) \leq \sum_{i \in J} (|V_i| - k) \), so \( \sum_{v \in X} d(v) \leq \sum_{i \in J} |V_i| \). Thus \( D \) cannot totally dominate \( V_J \cup \{v\} \), giving us the contradiction that proves the theorem. \( \square \)

### A.3 Proof of Theorem 4

Here we prove Theorem 10.5 from [12], which is a straightforward application of Talagrand’s Inequality and the Lovász Local Lemma.

**Lemma 15** (Lovász Local Lemma). Let \( A \) be a set of events in a probability space and take \( p \in \mathbb{R} \) and \( d \in \mathbb{Z} \) such that for every \( A \in A \),

- \( \Pr(A) \leq p \) and
- \( A \) is independent of all but at most \( d \) other events in \( A \).

Then if \( 4pd \leq 1 \), with nonzero probability no event in \( A \) occurs.

**Theorem 16** (Talagrand’s Inequality). Let \( X \) be a non-negative random variable, not identically 0, which is determined by \( n \) independent trials \( T_1, \ldots, T_n \), and satisfying the following for some \( c, r > 0 \):

- changing the outcome of any one trial changes the value of \( X \) by at most \( c \), and
- for any \( s \), if \( X \geq s \) then there is a set of at most \( rs \) trials whose outcomes certify that \( X \geq s \),

then for any \( 0 \leq t \leq E(X) \),

\[
\Pr(|X - E(X)| > t + 60c\sqrt{rE(X)}) \leq 4e^{-\frac{t^2}{8c^2rE(X)}}.
\]
Proof of Theorem \[ \text{A} \]. A simple embedding argument allows us to assume that \( G \) is \( \Delta \)-regular (take two copies of \( G \), add an edge between the two copies of each vertex of minimum degree, and repeat as necessary). Set \( C = \lfloor \Delta / 2 \rfloor \) and assign every vertex a colour in \( \{1, \ldots, C\} \) uniformly at random. If a vertex \( w \) is assigned a colour that appears on some neighbour, we uncolour \( w \) and all its neighbours of the same colour. Otherwise we say \( w \) retains its colour.

We wish to lower-bound the number of colours retained by at least two non-adjacent neighbours of a given vertex. To do so we will underestimate this number with the more manageable variable \( X_v \), which we define as the number of colours assigned to at least two non-adjacent neighbours of \( v \) and retained by all neighbours of the same colour. Otherwise we say \( w \) retains its colour.

We consider two closely related variables for each vertex, which we may as well introduce now. Let the variable \( AT_v \) (assigned twice) count the number of colours assigned to at least two non-adjacent neighbours of \( v \), and let the variable \( Del_v \) (deleted) count the number of colours assigned to at least two non-adjacent neighbours of \( v \) but removed (i.e. uncoloured) from at least one neighbour of \( v \). Note that \( X_v = AT_v - Del_v \).

For all \( v \in V(G) \) let \( A_v \) be the event that \( X_v < \frac{B}{e^2 \Delta} \). To prove the theorem it suffices to prove that with nonzero probability, \( A_v \) holds for no vertex \( v \). To see this, note that if we have a colouring using \( C \) colours in which every vertex has at least \( \frac{B}{e^2 \Delta} \) repeated colours in its neighbourhood, we can complete a \( \Delta + 1 - \frac{B}{e^2 \Delta} \) colouring of \( G \) as follows: First we extend each of the \( C \) colour classes such that if a vertex \( v \) is uncoloured, all \( C \) colours appear on its neighbourhood; we can do this greedily one colour at a time. We then delete these \( C \) colour classes, giving us a graph of maximum degree at most \( \Delta - C - \frac{B}{e^2 \Delta} \), which we can then colour greedily using \( \Delta + 1 - C - \frac{B}{e^2 \Delta} \) colours.

We will prove in three separate lemmas that:

- \( \mathbb{E}(X_v) \geq \frac{1.99 B}{e^2 \Delta} \) (Lemma 17).
- \( \mathbb{E}(X_v) \leq \mathbb{E}(AT_v) \leq \frac{3B}{\Delta} \leq \frac{6}{\Delta} \mathbb{E}(X_v) \) (Lemma 18).
- \( \mathbb{P}(|X_v - \mathbb{E}(X_v)| > \log \Delta \sqrt{\mathbb{E}(X_v)}) < \frac{1}{\Delta^3} \) (Lemma 19).

Now for \( B \geq \Delta (\log \Delta)^3 \) with sufficiently large \( \Delta \), we have

\[
\frac{1.99 B}{e^2 \Delta} - \log \Delta \sqrt{\mathbb{E}(X_v)} \geq \frac{1.99 B}{e^2 \Delta} - \log \Delta \sqrt{\frac{3B}{\Delta}} > \frac{B}{e^2 \Delta}.
\]

Therefore for any \( v \) we have \( \mathbb{P}(A_v) < 1/(4\Delta^5) \). Since \( A_v \) only depends on the colours assigned to vertices at distance at most two from \( v \), an event \( A_u \) is independent from \( A_v \) unless \( u \) is at distance at most four from \( v \); there are at most \( \Delta^4 \) such events. Thus setting \( p = 1/(4\Delta^5) \) and \( d = \Delta^4 \), the result follows from the Local Lemma.

Lemma 17. \( \mathbb{E}(X_v) \geq \frac{1.99 B}{e^2 \Delta} \).

Proof. For every vertex \( v \) we define \( X'_v \) to be the number of colours assigned to exactly two non-adjacent neighbours of \( v \) and retained by both. Note that \( X_v \geq X'_v \).

Two vertices \( u, w \in N(v) \) will both retain the colour \( \alpha \) if \( \alpha \) is assigned to both \( u \) and \( v \) but no vertex in \( S = N(u) \cup N(w) \cup N(v) - u - w \). Because \( |S| \leq 3\Delta - 3 \leq 6C \), for any colour \( \alpha \) the probability that this occurs is at least \( \left( \frac{1}{C} \right)^2 (1 - \frac{1}{C})^{6C} \). There are \( C \) choices for \( \alpha \) and at least \( B \) choices for \( \{u, v\} \). Therefore by Linearity of Expectation for sufficiently large \( C = \lfloor \Delta / 2 \rfloor \),
we have
\[ \mathbb{E}(X_v) \geq CB \left( \frac{1}{C} \right)^2 \left( 1 - \frac{1}{C} \right)^{6C} \geq \frac{B}{C} \left( 1 - \frac{1}{C} \right)^{6C} \geq \frac{2B}{\Delta} \left( 1 - \frac{1}{C} \right)^{6C} \geq 1.99B e^6 \Delta. \]

**Lemma 18.** \( \mathbb{E}(X_v) \leq \mathbb{E}(AT_v) \leq \frac{2B}{\Delta} \leq \frac{e^6}{3} \mathbb{E}(X_v). \)

**Proof.** The probability of a colour \( \alpha \) being assigned to at least two nonadjacent neighbours of \( v \) is at most \( \frac{B}{6} \), therefore \( \mathbb{E}(AT_v) \leq \frac{B}{C} \leq \frac{2B}{\Delta} \leq \frac{e^6}{3} \mathbb{E}(X_v) \), the last inequality coming from Lemma [17].

**Lemma 19.** \( \Pr \left( |X_v - \mathbb{E}(X_v)| > \Delta \log \mathbb{E}(X_v) \right) < \frac{1}{e^\Delta}. \)

**Proof.** To prove the lemma it suffices to prove that the following concentration bounds hold for \( t > \sqrt{\Delta \log \Delta} \):

- **Claim 1:** \( \Pr(|AT_v - \mathbb{E}(AT_v)| > t) < 4e^{-\frac{t^2}{100\mathbb{E}(AT_v)}}. \)
- **Claim 2:** \( \Pr(|Del_v - \mathbb{E}(Del_v)| > t) < 4e^{-\frac{t^2}{100e^6\mathbb{E}(AT_v)}}. \)

To see that these claims imply the lemma, we first observe that \( \mathbb{E}(X_v) = \mathbb{E}(AT_v) - \mathbb{E}(Del_v). \) Therefore if \( |X_v - \mathbb{E}(X_v)| > \log \Delta \sqrt{\mathbb{E}(X_v)} \), setting \( t = \frac{1}{4} \log \Delta \sqrt{\mathbb{E}(X_v)} > \sqrt{\Delta \log \Delta} \), either \( |AT_v - \mathbb{E}(AT_v)| > t \) or \( |Del_v - \mathbb{E}(Del_v)| > t \). Thus by the claims, and noting that

\[ t^2 \geq \frac{1}{4}(\log \Delta)^2 \left( \frac{3}{e^6} \mathbb{E}(AT_v) \right) > \frac{1}{2e^6} (\log \Delta)^2 \mathbb{E}(AT_v), \]

the probability of this happening is at most

\[ 8e^{-\frac{t^2}{100\mathbb{E}(AT_v)}} < 8e^{-\frac{(\log \Delta)^2 \mathbb{E}(AT_v)}{200e^6 \mathbb{E}(AT_v) / 200e^6}} = 8e^{-\frac{(\log \Delta)^2}{200e^6}} < \frac{1}{4 \Delta^\Delta}. \]

We now prove Claim 1. The value of \( AT_v \) only depends on the colours assigned to \( N(v) \), and changing any of these assignments can affect \( AT_v \) by at most 2. If \( AT_v \geq s \) then there is a set of at most 2s assignments that certify this. Therefore Talagrand’s Inequality with \( c = 2 \) and \( r = 2 \) gives us:

\[ \Pr(|AT_v - \mathbb{E}(AT_v)| > t) < 4e^{-\frac{(t-120\sqrt{2E(AT_v)})^2}{64E(AT_v)}} < 4e^{-\frac{t^2}{100\mathbb{E}(AT_v)}}, \]

the latter inequality following from the fact that \( t \geq \sqrt{\Delta \log \Delta} \geq \sqrt{\mathbb{E}(AT_v) \log \Delta}. \)

We now prove Claim 2 in the same way. The value of \( Del_v \) depends on at most \( \Delta^2 + 1 \) colour assignments. As with \( AT_v \), changing a colour assignment changes \( Del_v \) by at most 2. If \( Del_v \geq s \), this can be certified by a set of at most 3s assignments (the two nonadjacent vertices in \( N(v) \) and a third vertex adjacent to one of the first two). Therefore we can apply Talagrand’s Inequality with \( c = 2 \) and \( r = 3 \), which gives us:

\[ \Pr(|Del_v - \mathbb{E}(Del_v)| > t) < 4e^{-\frac{(t-120\sqrt{2E(Del_v)})^2}{96E(Del_v)}} < 4e^{-\frac{t^2}{100\mathbb{E}(AT_v)}}, \]

since \( \mathbb{E}(AT_v) \geq \mathbb{E}(Del_v) \) and \( t \geq \sqrt{\Delta \log \Delta} \geq \sqrt{\mathbb{E}(AT_v) \log \Delta}. \)