Stability Analysis of Thin-Shell Wormholes from Charged Black String

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Abstract

In this paper, we construct thin-shell wormholes from charged black string through cut and paste procedure and investigate its stability. We assume modified generalized Chaplygin gas as a dark energy fluid (exotic matter) present in the thin layer of matter-shell. The stability of these constructed thin-shell wormholes is investigated in the scenario of linear perturbations. We conclude that static stable as well as unstable configurations are possible for cylindrical thin-shell wormholes.

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1 Introduction

Wormholes are the hypothetical objects having a peculiar property of containing exotic matter (which violates null energy condition). The first ever

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wormhole model was known to be the Einstein-Rosen bridge [1], which was obtained as a part of the maximally extended Schwarzschild solution. The main problem with this wormhole is the existence of event horizon which prevents observers to move freely from one universe to the other. Later, Morris and Thorne [2] presented the first traversable Lorentzian wormhole as solution of the Einstein field equations. The key feature of this wormhole is that it does not contain event horizon and an observer may freely move in both universes through a handel (tunnel) known as wormhole throat.

Traversable wormholes have some issues such as their mechanical stability, the unavoidable amount of exotic matter present at the wormhole throat, etc. The violation of energy conditions due to the presence of exotic matter in these configurations is a debatable issue in general relativity which is the main hurdle in its observational evidence. To minimize the violation of energy conditions, Visser [3, 4] used the cut and paste technique on a black hole to build a thin-shell wormhole. He used the Darmois-Israel formalism [5, 6] to study the dynamical behavior of thin-shell wormhole made of two identical geometries.

Many authors studied the stability of thin-shell wormholes against linear perturbations through a standard potential approach. Poisson and Visser [7] explored the stability of the Schwarzschild thin-shell wormhole. Eiroa and Romero [8] generalized this analysis for the Reissner-Nordström thin-shell wormholes, while Lobo and Crawford [9] included the cosmological constant for the same analysis. For the sake of stable thin-shell wormhole configurations, people have also studied wormhole solutions in modified theories of gravity. For instance, Thibeault et al. [10] found stable thin-shell wormhole in Einstein-Maxwell theory with a Gauss-Bonnet term. Rahaman et al. [11, 12] explored thin-shell wormhole solutions in heterotic string theory and in the Randall-Sundrum scenario. Mazharimousavi et al. [13, 14] found viable thin-shell wormhole solutions in the Einstein-Hoffmann Born-Infeld theory and Einstein-Yang-Mills Dilaton gravity. In recent papers [15], we have investigated the stability of cylindrical and spherical geometries in Newtonian and post-Newtonian approximations and also spherically symmetric thin-shell wormholes.

Thorne [16] emphasized that models with cylindrical symmetry are the ideal one. These objects have widely been used to study cosmic strings [17] which play a vital role in different physical phenomena like gravitational lensing, galaxy formation, thin-shell wormholes, etc. Some literature [18–21] indicates keen interest in the study of cylindrical thin-shell wormholes.
In a sequence of papers [22]-[25], the stability of cylindrical thin-shell wormholes associated with local and global cosmic strings have been studied. It was found that the wormhole throat would expand or collapse according to the velocity sign and stable cylindrical thin-shell wormhole configurations could not be possible. Recently, we have explored stable thin-shell wormhole configurations supported by Chaplygin equation of state [26].

In this paper, we construct thin-shell wormholes from charged black string using cut and paste procedure and investigate its stability through linear perturbations. We consider the Darmois-Israel formalism for the dynamical analysis of the system with modified generalized Chaplygin gas (MGCG) to matter shell. The paper is organized as follows. Section 2 deals with the general formalism for the construction of thin-shell wormholes. In section 3, we discuss the linearized stability analysis and apply to charged black string thin-shell wormholes. In the last section, we summarize our results.

2 Thin-Shell Wormholes: General Formalism

The charged static cylindrically symmetric spacetime is given by [27]

\[ ds^2 = -\Phi(r)dt^2 + \Phi^{-1}(r)dr^2 + h(r)(d\varphi^2 + \alpha^2 dz^2), \]  

(1)

where

\[ \Phi(r) = \left( \alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} \right), \quad h(r) = r^2, \]

with the following constraints on the coordinates

\[-\infty < t < \infty, \quad 0 < r < \infty, \quad -\infty < z < \infty, \quad 0 \leq \varphi \leq 2\pi.\]

Here, the parameters \(M, Q\) are the ADM mass and charge density, respectively and \(\alpha = -\frac{1}{3} > 0\), \(\Lambda\) is the cosmological constant. The inner and outer event horizons of the charged black string are given as

\[ r_{\pm} = \frac{(4M)^{\frac{3}{2}}}{2\alpha} \left[ \sqrt{s} \pm \sqrt{2s^2 - Q^2 \left( \frac{2}{M} \right)^{\frac{3}{2}} - s} \right], \]  

(2)

provided that the inequality \(Q^2 \leq \frac{3}{4}M^\frac{3}{2}\) holds, where \(s\) is given by

\[ s = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{64Q^6}{27M^4}} \right)^{\frac{1}{4}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{64Q^6}{27M^4}} \right)^{\frac{1}{4}}. \]  

(3)
For $Q^2 > \frac{3}{4}M^4$, the given metric has no event horizon and represents a naked singularity. If $Q^2 = \frac{3}{4}M^4$, the inner and outer horizons merge, which corresponds to the extremal black strings.

We follow the Darmois-Israel formulation [5, 6] for the dynamical analysis of mathematically constructed thin-shell wormholes. For this purpose, we assume radius “a” greater than event horizon $r_h$ to avoid singularities and horizons in wormhole configuration. We take two identical copies $W^\pm$ with $r \geq a$ of the cylindrical vacuum solution defined as

$$W^\pm = \{ x^\mu = (t, r, \phi, z) / r \geq a \}. \tag{4}$$

We join these geometries at the timelike hypersurface $\Sigma = \Sigma^\pm = \{ r - a = 0 \}$ to get a geodesically complete manifold, i.e., $W = W^+ \cup W^-$ satisfying the radial flare-out condition, i.e., $h'(a) = 2a > 0 \ [21]$. The two regions are connected at the surface $\Sigma$ (surface of minimal area) with the throat radius $a$. The induced metric at the throat $\Sigma$ with coordinates $\eta^i = (\tau, \phi, z)$ is defined as

$$ds^2 = -d\tau^2 + a^2(\tau)(d\phi^2 + \alpha^2dz^2). \tag{5}$$

We take throat radius $a$ as a function of $\tau$ to understand the dynamics of the thin-shell wormhole. The presence of thin layer of matter at the shell leads to discontinuity in the extrinsic curvatures across a junction surface, where $K^+_ij - K^-ij = \kappa_{ij}$, and the extrinsic curvature $K^\pm_{ij}$ is defined on $\Sigma$

$$K^\pm_{ij} = -n^\pm_i \left( \frac{\partial^2 x^\pm_j}{\partial \eta^i \partial \eta^j} + \Gamma^\pm_{\gamma\mu\nu} \frac{\partial x^\pm_\mu}{\partial \eta^i} \frac{\partial x^\pm_\nu}{\partial \eta^j} \right), \quad (i, j = 0, 2, 3). \tag{6}$$

The 4-vector unit normals $n^\pm_\gamma$ to $W^\pm$ are

$$n^\pm_\gamma = \pm \left| g^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial f}{\partial x^\nu} \right| = \left( -\dot{a}, \sqrt{\Phi(r) + \dot{a}^2} \frac{\Phi(r)}{\Phi(r)}, 0, 0 \right), \tag{7}$$

satisfying the relation $n^\gamma n_\gamma = 1$. Using Eqs. (1) and (6), the non-trivial components of the extrinsic curvature are

$$K^\pm_{\tau\tau} = \pm \frac{\Phi'(a) + 2\ddot{a}}{2\sqrt{\Phi(a) + \dot{a}^2}}, \quad K^\pm_{\phi\phi} = \pm \frac{1}{a} \frac{\Phi(a) + \dot{a}^2}{\sqrt{\Phi(a) + \dot{a}^2}}, \quad K^\pm_{zz} = \alpha^2 K^\pm_{\phi\phi}, \tag{8}$$

where dot and prime mean derivative with respect to $\tau$ and $r$, respectively.
Now using the relations between the extrinsic curvatures
\[
[K_{ij}] = K^+_{ij} - K^-_{ij}, \quad K = tr[K_{ij}] = [K^i_i],
\]
the Einstein equations (called Lanczos equations) are defined on the shell as
\[
S_{ij} = \frac{1}{8\pi} \{ g_{ij} K - [K_{ij}] \}, \quad (9)
\]
where \( S_{ij} = diag(\sigma, p_\phi, p_z) \) is the surface energy-momentum tensor, \( \sigma \) and \( p_\phi, p_z \) are the surface energy density and surface pressures, respectively. With Eqs. (8) and (9), we obtain
\[
\sigma = -\frac{1}{2\pi a} \sqrt{\Phi(a) + \dot{a}^2}, \quad (10)
\]
\[
p = p_\phi = p_z = \frac{1}{8\pi a} \frac{2a\ddot{a} + 2\dot{a}^2 + 2\Phi(a) + a\Phi'(a)}{\sqrt{\Phi(a) + \dot{a}^2}}. \quad (11)
\]
The negative surface energy density (10) supports the presence of exotic matter at the throat.

For the dynamical characterization of the shell, we consider the MGCG as exotic matter on the shell. The equation of state for MGCG is defined as
\[
p = A\sigma - \frac{B}{\sigma^\beta}, \quad (12)
\]
where \( A, B \) are positive constants and \( 0 < \beta \leq 1 \). This equation combines various equations of state and reduces to the following classes for different values of the parameters \( A, B \) and \( \beta \), such as
- for \( A = 0, \beta = 1 \), it corresponds to the usual Chaplygin gas.
- for \( A = 0 \), it corresponds to pure generalized Chaplygin gas (GCG).
- for \( \beta = 1 \), it is another form of modified Chaplygin gas (MCG).

Ujjal [28] have generalized MGCG to variable MGCG by assuming \( B \) as a function of the scale factor \( a \), i.e., \( B = B(a) = B_0 a^{-m} \), where \( B_0, m \) are the positive constants. In this work, we have assumed \( B \) as a positive constant. Inserting Eqs. (10) and (11) in (12), we obtain a second order differential equation describing the evolution of the wormhole throat
\[
\left\{ [2\ddot{a} + \Phi'(a)] a^2 + \left( (\Phi(a) + \dot{a}^2) (1 + 2A) \right] 2a \right\} [2a]^\beta
- 2B(4\pi a^2)^{1+\beta} \left[ \Phi(a) + \dot{a}^2 \right]^{\frac{1-\beta}{2}} = 0.
\]
3 Linearized Stability Analysis: A Standard Approach

In this section, we analyze the stability of static solutions of thin-shell wormhole under the standard potential approach \[7, 8\]. For this purpose, the static configuration of surface energy density, surface pressure and dynamical equation for the thin-shell wormhole yields

\[
\sigma_0 = -\frac{\sqrt{\Phi(a_0)}}{2\pi a_0}, \quad p_0 = \frac{2\Phi(a_0) + a_0\Phi'(a_0)}{8\pi a_0 \sqrt{\Phi(a_0)}},
\]

\[14\]

\[
\{a_0^2\Phi'(a_0) + 2a_0(1 + 2A)\Phi(a_0)\} [2a_0]^{\beta} - 2B(4\pi a_0^2)^{1+\beta} [\Phi(a_0)]^{\frac{1-\beta}{2}} = 0. \quad 15
\]

The surface energy density and pressure satisfy the conservation equation

\[
\frac{d}{d\tau} (\sigma \Omega) + p \frac{d\Omega}{d\tau} = 0,
\]

\[16\]

where \(\Omega = 4\pi a^2\) is the area of the wormhole throat. This equation describes the change in internal energy of the throat plus the work done by the throat’s internal forces. We can write this equation as follows

\[
\dot{\sigma} = -2(\sigma + p) \frac{\dot{a}}{a}.
\]

\[17\]

Defining \(\sigma' = \frac{\dot{a}}{a}\), this equation takes the form

\[
aa\sigma' = -2(\sigma + p).
\]

\[18\]

For the stability of static configuration under the radial perturbations around \(a = a_0\), we rearrange Eq.(10) to obtain the thin-shell equation of motion

\[
\dot{a}^2 + V(a) = 0.
\]

\[19\]

This completely determines the dynamics of the thin-shell wormhole, where \(V(a)\) is known as potential function given by

\[
V(a) = \Phi(a) - [2\pi a\sigma(a)]^2.
\]

\[20\]
The stability of static solutions requires $V''(a_0) > 0$, $V(a_0) = 0 = V'(a_0)$. For this purpose, we apply the Taylor series expansion to $V(a)$ up to second order around $a_0$

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + O[(a - a_0)^3]. \quad (21)$$

Taking the first derivative of Eq. (20) and using (18), we obtain

$$V'(a) = \Phi'(a) + 8\pi^2a\sigma(a)[\sigma(a) + p(a)]. \quad (22)$$

We have another useful relation from the equation of state

$$p'(a) = \sigma'(a) \left[ (1 + \beta)A - \frac{\beta p(a)}{\sigma(a)} \right], \quad (23)$$

which may then be written as

$$\sigma'(a) + 2p'(a) = \sigma'(a) \left[ 1 + 2\{(1 + \beta)A - \frac{\beta p(a)}{\sigma(a)} \} \right]. \quad (24)$$

The second derivative of potential function along with the above equation leads to

$$V''(a) = \Phi''(a) - 8\pi^2 \left\{ [\sigma(a) + 2p(a)]^2 + 2\sigma(a)[\sigma(a) + p(a)] \right\} \times \left[ 1 + 2\left( (1 + \beta)A - \frac{\beta p(a)}{\sigma(a)} \right) \right]. \quad (25)$$

Using Eq.(14) both $V(a)$ and $V'(a)$ vanish at $a = a_0$, while $V''(a_0)$ becomes

$$V''(a_0) = \Phi''(a_0) + \frac{(\beta - 1)\Phi'(a_0)^2}{2\Phi(a_0)} + \frac{\Phi'(a_0)}{a_0} [1 + 2(1 + \beta)A] - \frac{2\Phi(a_0)(1 + \beta)}{a_0} (1 + A). \quad (26)$$

### 3.1 Charged Black String Thin-Shell Wormholes

In this section, we formulate the charged black string thin-shell wormholes and discuss the stability of their static solutions. The surface energy density and pressure for the charged black string wormhole with Eq.(14) becomes

$$\sigma_0 = -\frac{\sqrt{\alpha^4a_0^4 - 4M\alpha a_0 + 4Q^2}}{2\pi\alpha a_0}, \quad p_0 = \frac{\alpha^3a_0^3 - 4M}{2\pi a_0 \sqrt{\alpha^4a_0^4 - 4M\alpha a_0 + 4Q^2}}. \quad (27)$$
Using these values in Eqs. (12) and (25), the dynamical equation and the second derivative of potential for the thin-shell wormhole satisfied by the throat radius becomes

\[ \alpha^4 a_0^4 - M \alpha a_0 + A (\alpha^4 a_0^4 - 4M \alpha a_0 + 4Q^2) - B (2\pi \alpha a_0^2)^{1+\beta} \times (\alpha^4 a_0^4 - 4M \alpha a_0 + 4Q^2)^{\frac{1-\beta}{2}} = 0, \tag{28} \]

and

\[ V''(a_0) = \frac{4}{\alpha^2 a_0^4 (\alpha^4 a_0^4 - 4\alpha a_0 M + 4Q^2)} \times \left\{ (1 + \beta) \left[ -6\alpha^2 a_0^2 M^2 + (1 + A) - 32AQ^2 + (1 + A)\alpha^5 a_0^5 M + 8\alpha a_0 M Q^2 (1 + 7A) - 8(1 + A)\alpha^4 a_0^4 Q^2 + 16\alpha^4 a_0^4 Q^2 - 9\alpha^5 a_0^5 M \right] \right\}. \tag{29} \]

Now we explore the nature of static solutions whether they are stable or unstable. The existence of static solution is constrained by the condition \( a_0 > r_h \), i.e., throat radius must be greater than the horizon radius. On the other hand, for \( a_0 \leq r_h \), the static solutions do not exist which correspond to the non-physical zone (grey zone) as shown in Figures 1-6. Due to the complicated nature of Eq. (28), we solve this equation numerically and find \( a_0 \) for \( \beta = 0.2, 0.6, 1 \) and then replace the solution in Eq. (29). If \( V''(a_0) > 0 \), we have stable static solution which is represented by the solid curve, whereas for \( V''(a_0) < 0 \), the solution is unstable represented by the dotted curve. The behavior of static solutions depends upon the critical value of charge, \( Q_c = 0.866025 \). We can discuss the solutions given in Figures 1-2 for \( \beta = 0.2, 0.6 \) as follows:

- For \(|Q| = 0\), there exist both stable and unstable solutions for the black string thin-shell wormhole. The unstable solution approaches to the horizon radius for large values of \( B\alpha^{-(1+\beta)} \).

- For \(|Q| = 0.7Q_c\), this gives similar behavior as for the case \(|Q| = 0\).

- For \(|Q| = 0.999Q_c\), i.e., \(|Q|\) is nearly equal to the value of the critical charge. When \( \beta = 0.2 \), both stable and unstable solutions exist, while for \( \beta = 0.6 \), stable solution exists only for small values of \( B\alpha^{-(1+\beta)} \). Moreover, the horizon radius decreases with the increase of charge.
Figure 1: Charged black string thin-shell wormholes for $\alpha = 0.6$, $A = M = 1$ and $\beta = 0.2$ for different values of charge. The solid and dotted curves correspond to the stable and unstable static solutions under linear perturbations. The non-physical regions are represented by the shaded regions (grey zones), where $a_0 \leq r_h$. 
Figure 2: Charged black string thin-shell wormholes for $\alpha = 0.6$, $A = M = 1$ and $\beta = 0.6$. 
Figure 3: Charged black string thin-shell wormholes correspond to the parameters $\alpha = 0.6$, $A = M = 1$ and $\beta = 1$.

- When $|Q| = 1.1Q_c$, i.e., $|Q|$ has greater value than the critical value of the charge. In this case, stable and unstable solutions exist in each case for the increasing value of $B\alpha^{-(1+\beta)}$ and the horizon radius gradually disappears for $|Q| > Q_c$. We also explore the stability of static solutions corresponding to $\beta = 1$, which corresponds to the MCG as shown in Figure 3. Notice that for $|Q| < Q_c$, there always exists a stable static solution for small values of $B\alpha^{-(1+\beta)}$ and vanishes for large values of $B\alpha^{-(1+\beta)}$. When $|Q| > Q_c$, we again have two solutions stable for small values of $B\alpha^{-(1+\beta)}$ and unstable for large values of $B\alpha^{-(1+\beta)}$. Also, similar to the above cases, the horizon radius decreases and eventually disappears for the increasing value of $|Q|$. 

Figure 4: Charged black string thin-shell wormholes for $\alpha = 0.6$, $A = 0$, $M = 1$ and $\beta = 0.2$.

Now we analyze the stability of static solutions which correspond to GCG and usual Chaplygin gas. For this purpose, we take $A = 0$ and $\beta = 0.2$, 0.6, 1 in Eq. (12) and results are shown in Figures 4-6. When $\beta = 0.2$, we have only unstable solution for $|Q| = 0$, $0.7Q_c$, while for $|Q| = 0.999Q_c$ there exist three solutions two are unstable and one is stable. Finally, for $|Q| > Q_c$, the horizon radius disappears and both stable and unstable solutions exist similar to the above cases. For $\beta = 0.6$, the behavior of solutions presented in Figure 5 is similar to the case $\beta = 0.6$ for MGCG shown in Figure 2 for $|Q| = 0$, $0.7Q_c$, $0.999Q_c$, while for $|Q| > Q_c$, only stable solution exists in this case. When $\beta = 1$, we have only stable solutions for all values of $|Q|$ as shown in Figure 6.
Figure 5: Charged black string thin-shell wormholes for $\alpha = 0.6$, $A = 0$, $M = 1$ and $\beta = 0.6$. 
Figure 6: Charged black string thin-shell wormholes for $\alpha = 0.6$, $A = 0$, $M = 1$ and $\beta = 1$. 
4 Summary

The study of thin-shell wormholes has been the subject of interest due to the presence of exotic matter, which violates the null energy condition. The aim of this study is to construct cylindrical thin-shell wormholes and investigate the stability of these configurations. We have developed cylindrical thin-shell wormholes by joining two identical copies of the cylindrical manifold using cut and paste method. In order to explore the dynamics of thin-shell wormhole, we have applied the Darmois-Israel junction conditions along with MGCG for the description of exotic matter. We have explored the stability of static solutions numerically satisfying the condition \( a_0 > r_h \), under linear perturbations.

The stability of thin-shell wormhole solutions is examined for different values of parameter \( \beta = 0.2, 0.6, 1 \). It is found that for \( \beta = 0.2, 0.6 \), there always exist stable and unstable solutions except for \( \beta = 0.6 \) and \( |Q| = 0.999Q_c \), for which only stable static solution exists. Moreover, in both cases the horizon radius decreases with the increase of \( |Q| \). When \( \beta = 1 \), we have stable configurations for small values of \( B\alpha^{-1+\beta} \) with \( |Q| < Q_c \) and approaches to the horizon of the manifold, whereas for \( |Q| > Q_c \), we have obtained stable as well as unstable solutions like the other cases for \( \beta = 0.2, 0.6 \).

Further, we examine the stability of solutions for the GCG. It is found that for \( \beta = 0.2 \), there exists one unstable solution with \( |Q| = 0, 0.7Q_c \) and two unstable and one stable solutions for \( |Q| = 0.999Q_c \). Also, when \( |Q| > Q_c \), there exist both stable and unstable solutions. For \( \beta = 0.6 \), the solutions are similar to the case of MGCG with \( |Q| = 0, 0.7Q_c, 0.999Q_c \), while only stable solution exists for \( |Q| > Q_c \).

Recently, we have found stable configurations for the uncharged and charged black string wormholes supported by Chaplygin gas [26]. We would like to mention here that the results of this work reduce to [26] for \( \beta = 1 \) and \( A = 0 \) as shown in Figure 6. It is worth mentioning here that the literature [22]-[25] indicates only unstable solutions for cylindrical thin-shell wormhole. The apparent discrepancy of our results with those presented in the literature comes from the Einstein field equations and the choice of the equation of state for the exotic matter. We have concluded that there is a possibility of stable configuration for the cylindrical thin-shell wormholes.
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