Quantum Inequalities from Operator Product Expansions

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Dedicated to the memory of Bernd Kuckert.

Abstract
Quantum inequalities are lower bounds for local averages of quantum observables that have positive classical counterparts, such as the energy density or the Wick square. We establish such inequalities in general (possibly interacting) quantum field theories on Minkowski space, using nonperturbative techniques. Our main tool is a rigorous version of the operator product expansion.

1 Introduction
The principal qualitative difference between classical and quantum physics lies in the fundamentally unsharp nature of the latter, quantitatively expressed by the uncertainty principle. This distinction becomes particularly acute when one seeks analogues in quantum theory of quantities that are classically positive. In quantum mechanics, for example, one replaces a probability distribution over classical phase space by the Wigner function, which is pointwise positive only for Gaussian states \cite{Hud74}. Consequently, Weyl quantization of classically positive observables does not generally yield positive operators. Similarly, a positive (local) quadratic form in a classical field and its derivatives, such as the energy density of a free minimally coupled scalar field, would not be expected to have a positive analogue in quantum field theory, owing to the subtractions necessary to renormalize products of fields at a point.

Nonetheless, positivity is not completely destroyed in quantization. The sharp Gårding inequalities \cite{FP81} show that classically positive symbols have Weyl quantizations that are positive modulo corrections of lower order; that is, operators corresponding to a lower rate of growth in momentum. The aim of this paper is to establish analogous results for quantum field theory in a model independent and nonperturbative setting. The key to our approach is a recently-developed microscopic phase space condition \cite{Bos05b} that controls the degrees of freedom available to the theory at small scales and bounded energy, and guarantees the existence of a rigorous operator product expansion (OPE) \cite{Bos05a}. In any theory obeying

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this condition (along with other standard criteria set out in Sec. [3]) we identify a class of ‘classically positive’ operator products and show how this classical positivity is reflected in estimates on suitable smearings of the composite fields appearing in the corresponding OPEs. If there is a distinguished normal product associated with the underlying classically positive expression the picture is closely analogous to that emerging from the Gårding inequalities: suitable smearings of the normal product are positive modulo corrections of a lower order. As we will describe, our results significantly generalize the quantum (energy) inequalities, developed over recent years, that provide lower bounds on smearings of quadratic normal ordered quantities in free field theories.

In the following subsections, we will describe the background and motivation for our study.

1.1 Quantum inequalities

It has been known for many years that expectation values of quantities such as the Wick square or energy density of a free scalar field may assume negative values and are pointwise unbounded from below as the quantum state is varied. Indeed, no local observable (other than the zero operator) can be both positive and have a vanishing vacuum expectation value [EGJ65]. Thirty years ago, Ford made the key observation that, as unrestricted negative energy densities or fluxes could produce macroscopic violations of the second law of thermodynamics, it was to be expected that QFT itself places strict limits on such departures from positivity [For78]. Subsequently, Ford and Roman were able to derive lower bounds, called quantum inequalities (QIs), on averaged energy densities for scalar fields in Minkowski space [For91, FR95, FR97]; these results were generalized to static curved spacetimes by Pfenning and Ford [PF98].

In the results just mentioned, the averaging is performed along a timelike geodesic with respect to a Lorentzian weight. With Eveson, one of us (CJF) obtained similar results for general weight functions [FE98]. As an example, the renormalized energy density :ρ: of the field of mass m in four-dimensional Minkowski space obeys the inequality

$$\int dt \omega(\cdot; (t,0))|g(t)|^2 \geq -Q[g] := \frac{1}{16\pi^3} \int_m^\infty du u^4 |\tilde{g}(u)|^2$$

(1.1)

for any g ∈ D(R) and all Hadamard states ω (this is slightly weaker than the bound of [FE98]). Here $\tilde{g}$ denotes the Fourier transform. Similar bounds are obeyed by any classically positive field of form $\sum_i : (P_i \phi)^2 :$, where the $P_i$ are partial differential operators with smooth real coefficients. We will understand the term ‘quantum inequality’ to apply to any bounds of this type, and not just those relating to the energy density (for which the more specific term ‘quantum energy inequality’ (QEI) is also used).

The basic technique of [FE98] generalizes straightforwardly to static spacetimes [FT99] and the electromagnetic field [Pfe02]. It also underlies the general and rigorous results of [Few00], which give QIs for averaging with arbitrary weights along arbitrary timelike curves in arbitrary globally hyperbolic spacetimes, valid for all Hadamard states. (The bound in [Few00] is expressed using a reference state; see [FS08] for analogous results with a purely local geometric bound.) Similar results hold for spin-1 fields [FP03]. We note that averaging in timelike directions is essential for establishing inequalities; while averaging over spacetime volumes also yields lower bounds (see, e.g., [FS08]), purely spatial [FHR02] or lightlike [FR03] averaging is known not to be sufficient for quantum inequalities.
An important feature of the lower bound in (1.1) is that it is independent of the state $\omega$, and can be rewritten as an operator inequality $\rho : (|g|^2) \geq -Q(g)1$. One cannot expect bounds of this type for general interacting theories [OG03] (although they do hold for conformal field theories in two dimensions [FH05]; see also [Pla97, Vol00] for precursors). Indeed, the nonminimally coupled scalar field provides an example of a free field theory in which averaged energy densities are unbounded from below [FO08]. The best that can be expected, in general, is an inequality of the form $\rho : (|g|^2) \geq -Q(g)$, where $Q(g)$ is now permitted to be an operator. As noted in [Few07], this would be a rather empty notion without some constraints on $Q(g)$ [for example, $Q(g) = -\rho : (|g|^2)$ gives a trivial inequality of this type]. To qualify as a nontrivial inequality, $Q$ should be of ‘lower order’ than $\rho$ in a defined sense. For example, the nonminimally coupled scalar field obeys bounds of the form

$$\rho : (|g|^2) \geq -Q_1(g)1 + 2\xi \phi^2 : (\dot{g}^2)$$

(1.2)

in four-dimensional Minkowski space for coupling $\xi \in [0, 1/4]$ [FO08]. Crucially, the right-hand side is bounded relative to $(1 + H)^p$ for any $p > 2$, while the left-hand side is not bounded relative to any $(1 + H)^q$ with $q < 3$, where $H$ is the Hamiltonian.

In the present paper we will weaken the criterion of nontriviality slightly owing to the approximate nature of OPEs. As we explain in outline in Sec. 2 and in detail in Sec. 6, we permit bounds containing a remainder term that is of higher order in energetic terms than the field of interest, but which is vanishing in the small distance limit.

All the results mentioned so far rely on positivity of an underlying classical expression, namely, a sum of squares of fields and their derivatives; and this is also the focus of the present work. However, it is important to recall that the energy density of a Dirac field is not expressed in this way; accordingly different techniques are required to obtain quantum energy inequalities in this case (see [FV02, FM03, DF06, Smi07] for spin-1/2 and [YW04, HLZ06] for spin-3/2).

### 1.2 Perturbative versus nonperturbative approaches

While QIs were first studied for free fields on Minkowski space, it is now known – as mentioned above – that the concept is compatible at least with some simple types of interaction, specifically the coupling to an external gravitational field and those in conformal field theories. However, on the technical side, the existing results typically rely on the rather simple structure of linear quantum fields fulfilling $c$-number commutation relations. For dealing with general, possibly self-interacting quantum fields, this is far too restrictive. Instead, our aim here is to derive inequalities from general principles of quantum field theory that are not restricted to linear fields.

To date, self-interacting quantum field theories have generally been established in a perturbative setting only, usually without any control on the convergence of the perturbation series. It would seem natural to investigate QIs in this context. However, severe conceptual difficulties arise here. In order to give any reasonable meaning to quantum inequalities in perturbation theory, we need to determine when a formal power series, say $P[g] = \sum_{k=0}^{\infty} c_k g^k$ with $c_k \in \mathbb{C}$, and with the formal variable $g$ being interpreted as a “coupling constant”, should be considered positive. Understanding the set of formal power series as a $*$-algebra, the natural notion of positivity is as follows [DF99]: $P$ is considered positive if and only if

$$P[g] = Q^* [g] Q[g]$$

for some formal power series $Q$.

$$\tag{1.3}$$
It turns out that this condition is equivalent to the following one:

\[ P[g] = g^{2n} \sum_{k=0}^{\infty} d_k g^k \quad \text{with } n \in \mathbb{N}_0, d_k \in \mathbb{R}, d_0 > 0. \quad (1.4) \]

[Here (1.3) ⇒ (1.4) is immediate; the converse follows by inserting \( x = (d_0^{-1} g^{-2n} P[g] - 1) \) into the power series of \( \sqrt{1 + x} \) around \( x = 0 \).] Now Eq. (1.4) shows that this notion of positivity is not useful in our context, since it roughly says that positivity of \( P \) is determined by its lowest-order coefficient. (See [BW98] for a slight variant.) The order-0 coefficient however is supposed to be the contribution from free field theory. So—with this definition—QIs would hold at finite coupling if and only if they hold at coupling \( g = 0 \); the effects of interaction on inequalities cannot be captured in this approach.

Let us illustrate these difficulties in a simple example: Should one consider the following formal power series positive?

\[ P[g] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} g^{2k} \quad (1.5) \]

Considering \( P \) as a convergent series, it would be positive for small, but not for all \( g \). Forgetting all convergence properties, the only information that remains is positivity at \( g = 0 \), i.e., of the zero-order coefficient. The question of interest, however, would be whether the \textit{physical} value of \( g \) falls into the convergence radius of \( \sqrt{P} \); this question is not accessible in formal perturbation theory.

It is therefore necessary to conduct our investigation in a nonperturbative formulation of quantum field theory, such as the Wightman setting [SW64] or the \( C^* \) algebraic formulation of Haag and Kastler [Haa96]. (We shall actually use a combination of both; the technical details will be recalled in Sec. 3.) This is, in a way, a very strong assumption to start with, since we assume that our QFT models have been fully constructed and are under complete topological control. Indeed, the rigorous construction of interacting models in physical space-time still remains an open challenge, while the situation is better in simplified low-dimensional models [GJS87]. The virtues of our axiomatic approach, however, are of a different nature: Within the framework of algebraic quantum field theory, we can formulate physically motivated, qualitative properties of quantum field theories, which can explicitly be verified in simple models such as free field theory, but which appear general enough to be postulated for the interacting situation. We can then show how observable consequences, such as quantum inequalities, follow from these postulated properties.

1.3 Phase space conditions

The specific qualitative properties we will employ are known as phase space conditions. Semi-classical considerations (originating with Bohr and Sommerfeld) suggest that only finitely many independent states (or, dually, observables) are required to describe a quantum system which is restricted to a finite volume in phase space e.g., by cut-offs in configuration space and energy. In quantum field theory, this picture can certainly persist only qualitatively and in an approximate sense. However, it is possible to give a precise meaning to the aforementioned concepts, expressed as the compactness or nuclearity of certain maps; see e.g. [HS65, BW86, BP90]. These phase space conditions have physically interesting consequences: for example, they imply the existence of thermal equilibrium states [BJ89] and are important for the particle interpretation of quantum field theories [Por04].
The role of phase space conditions for QIs has been partially investigated before. Even in the free field situation described above, one may see the need for some restrictions on the phase space behaviour of the theory \cite{Few06}: instead of one field of mass $m$, consider an infinite number of fields with masses $m_j$ (for simplicity, in four-dimensional Minkowski space). The total energy density will obey a QI

$$
\int dt \omega(x(t,0))|g(t)|^2 \geq -\frac{1}{16\pi^3} \int_0^\infty du \, u^4 N(u)|\tilde{g}(u)|^2,
$$

where

$$
N(u) = \sum_j \vartheta(u - m_j)
$$

counts the number of species with masses below energy $u$. If $N$ grows no faster than polynomially with $u$, the lower bound is finite for all $g \in C_0^\infty(\mathbb{R})$; the same condition is known to guarantee that this theory obeys nuclearity in the sense of Buchholz and Wichmann \cite{BW86}. Other ideas concerning the relationship between QEI and nuclearity conditions are discussed in \cite{FOP05}, while connections with thermodynamic stability are described in \cite{FV03} and \cite{SV08}. However, the results presented here are the first in which QIs have been derived as a consequence of phase space criteria.

For our purposes, we will use a microscopic phase space condition recently introduced by one of us (HB) in \cite{Bos05b}; we shall recall its formulation and consequences in Sec. 3. Compared with other similar conditions, it is specifically sensitive in the short-distance regime, the realm which is of most interest for QIs. Indeed, one heuristically expects \cite{HO96} that at short distances and finite energies, the theory may be well-approximated in terms of finitely many observables corresponding to pointlike quantum fields.

This approximation of bounded observables by quantum fields can indeed be made precise \cite{Bos05b} and plays a central role in our approach. Its use is twofold. First, it tells us how our primary objects—local algebras of bounded operators—relate to the quantum fields for which inequalities are formulated. Second, it serves to establish an additional structure for the quantum fields, namely a rigorous version of the operator product expansion \cite{Bos05a}. We can understand this OPE, which describes the “structure constants” of the “improper algebra” of quantum fields, as containing all relevant information about the interaction, and in this sense as a replacement for the Lagrangian \cite{Wil69}. In fact, it is the OPE from which our inequalities will be computed. In particular, the OPE allows us to generalize the notion of normal ordering that has a key role for QIs of linear fields, replacing it with normal products in the sense of Zimmermann \cite{Zim70}.

The remainder of the paper is organized as follows. We start with a non-technical account of our main methods and results in Sec. 2. Then, in Sec. 3 we introduce the framework of nonperturbative quantum field theory that we work in, including the phase space condition mentioned above. Section 4 presents some technical preliminaries from distribution theory. In Sec. 5 we establish the rigorous operator product expansion in the variant that we require. This expansion will be the base of our quantum inequalities, derived in Sec. 6. Dilation covariance as a special case is covered in Sec. 7. We end with a brief outlook in Sec. 8.

## 2 Overview

We now give a non-technical overview of our main techniques and results, postponing rigorous arguments to later sections. Throughout, we work in Minkowski space of dimension $2 + 1$ or
more (possible generalizations are discussed in Sec. 8). For simplicity, we shall always pick a fixed Lorentz frame, and hence a fixed time axis; all quantum fields \( \phi(t) = \phi(t, \vec{x} = 0) \) will be restricted to this time axis, and smeared expressions \( \phi(f) = \int dt f(t) \phi(t) \) will refer to one-dimensional integration only. This is sufficient for regularizing Wightman fields [Bor64]; due to the symmetry properties of Minkowski space, it covers the essential features of the inequalities we wish to consider.

To illustrate our approach we begin by sketching the derivation of a QI for the Wick square of the free real scalar field, essentially following the argument of [Fre00] but in a form which is amenable to our generalization. We will then indicate which changes are necessary to deal with the general situation.

Accordingly, let \( \phi \) denote the free field and let \( \sigma \) be a normal state in the vacuum sector with sufficiently regular high-energy behaviour that the expectation values in the following are finite. The distributional integration kernel

\[
F(t, t') = \sigma(\phi(t)\phi(t')).
\]

is positive-definite, in the sense that for any test function \( g \), we have

\[
\int dt dt' F(t, t') g(t)g(t') = \sigma(\phi(g)^* \phi(g)) \geq 0.
\]

Then, also \( F(t, t')/i\pi(t - t' - i0) \) is positive-definite; namely we have by Fourier analysis,

\[
\int dt dt' F(t, t') \frac{g(t)g(t')}{i\pi(t - t' - i0)} = \int \frac{dp}{\pi} \int dt dt' F(t, t') \frac{g(t)g(t')\theta(p)e^{-ip(t-t')}}{t - t' + i0} = \int_0^\infty \frac{dp}{\pi} \int dt dt' F(t, t')e^{ip(t-t')}g(t') \geq 0.
\]

We now use Wick ordering and introduce new variables \( s = (t + t')/2, s' = t - t' \) in order to rewrite the kernel \( F \):

\[
F(t, t') = \sigma(\phi^2(\cdot(s))) + \Delta_+(s') + \sigma(R(s, s')),
\]

where \( \Delta_+(t - t') = \omega(\phi(t)\phi(t')) \) is the vacuum two-point function of \( \phi \), and the remainder \( R \) is given by

\[
R(s, s') = \phi(t)\phi(t') - \phi\left(\frac{t + t'}{2}\right)^2:
\]

\[
= U(s)\left(\phi(s'/2)\phi(-s'/2) - \phi^2(0)\right)U(s)^*;
\]

it is a smooth function when evaluated in \( \sigma \). Inserting this into Eq. (2.3), we obtain

\[
\sigma(\phi^2(\cdot) + c_g \mathbf{1}) \geq -R_{\sigma, g},
\]

where

\[
f(s) := \int ds' \frac{g(s + s'/2)g(s - s'/2)}{i\pi(s' - i0)},
\]

\[
c_g := \int ds ds' \Delta_+(s') \frac{g(s + s'/2)g(s - s'/2)}{i\pi(s' - i0)},
\]

\[
R_{\sigma, g} := \int ds ds' \sigma(R(s, s')) \frac{g(s + s'/2)g(s - s'/2)}{i\pi(s' - i0)}.
\]
It seems plausible that $R_{\sigma,g}$ becomes small as $\text{supp} \, g$ shrinks to a point. We will give more quantitative estimates in that respect later. Here, let us consider the special case where $g$ is real-valued. Then both $g(s + s'/2)g(s - s'/2)$ and $R(s, s')$ are even functions in $s'$. Hence, in Eqs. (2.7) and (2.9), we can replace the factor $(s' - a_0)^{-1}$ with its even part,

$$\frac{1}{2} \left( \frac{1}{s' - a_0} + \frac{-1}{s' + a_0} \right) = i\pi \delta(s').$$

(2.10)

Since $R(s, 0) = 0$, this results in $R_{\sigma,g} = 0$ and $f(s) = g(s)^2$. Thus Eq. (2.6) gives the more usual inequality for the Wick square,

$$\phi^2: (g^2) \geq -c_2 \mathbf{1}.$$

(2.11)

We now aim at a generalization beyond free field theory. So let $\phi$ be a general, possibly self-interacting local quantum field. (The term "quantum field" is used here in a generic fashion, and may include derivatives of fields as well as composite fields or suitably defined powers of fields.) The main difficulty we face in applying the above construction is that no concept of normal ordering is available; we cannot use Wick ordering to split the product into higher-order and lower-order terms, as in Eq. (2.4). Instead, we shall use an operator product expansion for the product $\phi(t)^* \phi(t')$,

$$\phi^*(t)\phi(t') = \sum_{j=1}^{n} C_j (t - t')\phi_j((t + t')/2) + R_n(t, t').$$

(2.12)

Here $R_n$ is a remainder term, which is "small" where $t$ and $t'$ are close, while the $\phi_j$ are composite fields. Smearing against $\overline{g(t)g(t')}$, where $g \in \mathcal{D}(\mathbb{R})$, the left-hand side is then a positive operator, and this remains true if we multiply (2.12) with any positive-type kernel $K(t - t')$, which takes the role of $K(t - t') = 1/\pi(t - t' - a_0)$ above. In other words, Eqs. (2.2) and (2.3) remain valid. We can then rearrange and obtain as analogue to Eq. (2.6),

$$\sum_{j=1}^{n} \phi_j(f_j) \geq -\int dt \, dt' \overline{g(t)g(t')} K(t - t') R_n(t, t'),$$

(2.13)

where the test functions $f_j$ are given in terms of $g$, $K$, and the OPE coefficients $C_j$ by

$$f_j(s) = \int ds' K(s') C_j(s') g(s + s'/2)g(s - s'/2).$$

(2.14)

Note that there is no guarantee that these functions are necessarily pointwise positive (the issues here are related to Hudson’s theorem [Hud74] and the ‘choice of basis’ invoked in the OPE). We will return to this below.

In order to establish our results rigorously, the main task is to establish the OPE and to control the remainder term on the right-hand side. We will show in Theorem 6.1 that, given $\alpha \geq 0$, one may find $n$, $m$ and $\ell$ so that for all $d > 0$ and $g \in \mathcal{D}(-d, d)$,

$$\sum_{j=1}^{n} \phi_j(f_j) \geq -\epsilon(d)\|g\|^2_{d,m}(1 + H)^{2\ell},$$

(2.15)

where $H$ is the Hamiltonian, $\epsilon(d) = o(d^\alpha)$ as $d \to 0$ and $\| \cdot \|^2_{d,m}$ is equivalent to the Sobolev norm on $W^{m,1}_0(-d, d)$. (Of course, finite sums of field products can be, and are, accommodated by our result.)
The relationship with the QIs described in Sec. 1.1 is most apparent in the case where one of the composite fields, say $\phi_1$, is of higher order than the others, in the sense that there exists $\ell'$ for which $(1 + H)^{-\ell'} \phi_j(f)(1 + H)^{-\ell'}$ is bounded for $j \geq 2$, while $(1 + H)^{-\ell'} \phi_1(f)(1 + H)^{-\ell'}$ is unbounded. Then we may rearrange to write

$$\phi_1(f_1) \geq -\sum_{j=2}^n \phi_j(f_j) - \epsilon(d)\|g\|_{2d,m}^2(1 + H)^{2\ell}. \quad (2.16)$$

In cases where the remainder term vanishes \((2.16)\), it is then a nontrivial QI in the sense of [Few07]: namely, one cannot find constants $C, C'$ such that

$$|\sigma(\phi_1(f_1))| \leq C \sum_{j=2}^n |\sigma(\phi_j(f_j))| + C' \quad (2.17)$$

for all (sufficiently regular) states $\sigma$ because $\phi_1$ is of higher energetic order than the fields on the right-hand side. Examples include the QI \((2.11)\), where the only composite field on the right-hand side is the identity, and the QEI \((1.2)\) on the nonminimally coupled field, where both the identity and Wick square appear on the right-hand side.

This simple situation does not persist in general, however. First, it does not seem guaranteed that a unique choice of a highest-order field $\phi_1$ exists. For interacting fields, one would expect $\phi_1$ to be the normal product of $\phi^* \phi$ in the sense of Zimmermann [Zim70]; but there are indications from perturbation theory that in some cases, this normal product might not be unique [Joh61]. Second, the remainder term cannot be expected to vanish in general—this reflects that the OPE is a controlled approximation, rather than an exact formula. Third, the remainder term is not of lower energetic order than the fields: in fact, $\ell$ is chosen so that each $(1 + H)^{-\ell} \phi_j(f_j)(1 + H)^{-\ell}$ is bounded.

Although the inequalities in Eq. (2.15) remain valid, it is necessary to adapt the criterion of nontriviality to our setting. Our approach is to focus on the short-distance behaviour, in which the remainder term vanishes as $o(d^\alpha)$. By contrast, we will show in Sec. 6.2 that, for the bounds we obtain,

$$\sup_{g \in \mathcal{D}(\mathcal{D}, d)} \|g\|_{2d,m}^2 \left\| (1 + H)^{-\ell} \sum_{j=1}^n \phi_j(f_j)(1 + H)^{-\ell} \right\| \quad (2.18)$$

is not $o(d^\alpha)$ as $d \to 0$. Thus the remainder term cannot dominate the contribution of the composite fields in the small. In this context, it turns out to be crucial to formulate the OPE, and correspondingly the inequalities, in a “basis-independent” fashion, that is, in a way that is independent of a possible arbitrariness in the choice of composite fields.

For practical purposes, it is still important to understand the more specific question as to whether there is a normal product of strictly higher energetic order than the other fields in the OPE. At present it seems to us that this must be discussed in the light of particular examples.

Last but not least, one would like to gain more insight in the properties of the sampling functions $f_j$, given by Eq. (2.14), in particular for the function $f_1$ corresponding to a “highest-order” composite field, which generalizes $f_1(s) = g(s)^2$ from the free field case. In general, it is certainly not expected that $f_1$ depends on $g$ in a simple pointwise fashion. However, one may ask whether $\phi_1$ can be chosen so that $f_1$ retains other properties that are apparent.
in the free-field situation, for example whether \( f_1 \geq 0 \), either pointwise or in an averaged sense. This will depend crucially on the form of the OPE coefficients \( C_j \), which are however unknown in general. We will give two approaches to this problem. The first, in Sec. 6.3, indicates conditions under which one may simultaneously tune the leading sampling function to a given positive form, while also reducing the remainder term. These conditions are broadly met under the assumption that the OPE coefficients have scaling limits in the sense of [FH87]. Our argument here is essentially to form a Riemann sum of QIs over small distance scales, in which the remainder term is suppressed. This approach is, however, tied to basis representations of the OPE; it would appear to be most useful in the context of particular models. Second, in Sec. 7, we discuss the particular case of dilation covariant theories. This is of interest since we can expect that our theory is approximated by a dilation covariant “scaling limit theory” in the ultraviolet. In this restricted situation, we will derive explicit criteria on \( g \) that guarantee positivity of \( f_1 \). Since positivity of \( f_1 \) also fixes the sign of the composite field \( \phi_1 \), this gives us a means of distinguishing the positive “normal square” of \( \phi \) from its negative.

3 Algebras of observables and pointlike quantum fields

As a mathematical basis of quantum field theory, we adopt the framework of local quantum physics [Haa96]. Specifically, for describing pointlike quantum fields, we use the methods set forth in [Bos05b]. For the convenience of the reader, we will collect the relevant notions and results below, and introduce some notations that are useful in our context.

We set out from a local net of algebras, \( \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \), in the vacuum sector. That is, for each bounded open region \( \mathcal{O} \) of Minkowski space, we have an algebra \( \mathfrak{A}(\mathcal{O}) \) of bounded operators; we take these to be von Neumann algebras acting on a common Hilbert space \( \mathcal{H} \). Further, we have a strongly continuous unitary representation \((x, \Lambda) \mapsto U(x, \Lambda)\) of the proper orthochronous Poincaré group on \( \mathcal{H} \), with a common invariant unit vector \( \Omega \in \mathcal{H} \). We write the translation subgroup as \( U(x, 1) = \exp iP_\mu x^\mu \). Together these objects are supposed to fulfill the following axioms:

(i) **Isotony:** \( \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \) if \( \mathcal{O}_1 \subset \mathcal{O}_2 \).

(ii) **Locality:** \([A_1, A_2] = 0\) if \( \mathcal{O}_1, \mathcal{O}_2 \) are two spacelike separated regions, and \( A_i \in \mathfrak{A}(\mathcal{O}_i) \).

(iii) **Covariance:** \( U(x, \Lambda)\mathfrak{A}(\mathcal{O})U(x, \Lambda)^* = \mathfrak{A}(\Lambda \mathcal{O} + x) \) for all Poincaré transformations \((x, \Lambda)\).

(iv) **Positivity of energy:** The joint spectrum of the \( P_\mu \) falls into the closed forward light cone.

(v) **Uniqueness of the vacuum:** \( \Omega \) is unique (up to a phase) as an invariant vector for all \( U(x, 1) \).

We are primarily interested in the algebras associated with standard double cones \( \mathcal{O}_r \) of radius \( r \) centred at the origin, and use \( \mathfrak{A}(r) \) as shorthand for \( \mathfrak{A}(\mathcal{O}_r) \). Also, for most parts we only use the time-translation subgroup of \( U(x, \Lambda) \), which we denote as \( t \mapsto U(t) \), with positive generator \( H = P_0 \geq 0 \). We write the spectral projectors of \( H \) for the interval \([0, E]\) as \( P(E) \).
Let $\Sigma$ be the set of ultraweakly continuous functionals on $\mathcal{B}(\mathcal{H})$. We consider for $\ell > 0$ the subspaces

$$
\mathcal{C}^\ell(\Sigma) = \{ \sigma \in \Sigma \mid \| \sigma \|^{(\ell)} := \| \sigma (1 + H)^\ell \cdot (1 + H)^\ell \| < \infty \},
$$

which are Banach spaces in the norm $\| \cdot \|^{(\ell)}$. Their duals $\mathcal{C}^\ell(\Sigma)^*$ consist of linear forms $\phi$ for which the dual norm $\| \phi \|^{(-\ell)} = \| (1 + H)^{-\ell}\phi (1 + H)^{-\ell} \|$ is finite. [More precisely, $\phi$ are quadratic forms on a dense subspace of $\mathcal{H} \times \mathcal{H}$, for which the form $(1 + H)^{-\ell}\phi (1 + H)^{-\ell}$, with the multiplication defined in the weak sense, is bounded.]

We also introduce the space of smooth functionals, $\mathcal{C}^\infty(\Sigma) = \cap_{\ell>0} \mathcal{C}^\ell(\Sigma)$, and equip it with the Fréchet topology induced by all norms $\| \cdot \|^{(\ell)}$. The dual space $\mathcal{C}^\infty(\Sigma)^*$ is then given by $\cup_{\ell>0} \mathcal{C}^\ell(\Sigma)^*$, and will be considered with the weak* topology. Further, we define for $E > 0$ the set of energy-bounded functionals, $\Sigma(E) = \{ \sigma (P(E) \cdot P(E)) \mid \sigma \in \Sigma \}$. Then $\cup_{E>0} \Sigma(E)$ is dense in $\mathcal{C}^\infty(\Sigma)$ and weakly dense in $\Sigma$. Each space $\mathcal{C}^\ell(\Sigma)$ is invariant under the natural action of hermitean conjugation, i.e. $\sigma^*(A) = \bar{\sigma(A^*)}$, and this structure transfers to the dual spaces; so we can speak of hermitean elements in $\mathcal{C}^\infty(\Sigma)$ and $\mathcal{C}^\infty(\Sigma)^*$.

With respect to pointlike fields, we assume that the theory fulfills a specific type of phase space condition [Bos05b], sensitive in the ultraviolet. To formulate this, consider the inclusion map $\Xi : \mathcal{C}^\infty(\Sigma) \hookrightarrow \Sigma$. We assume that $\Xi$ can be approximated with finite-rank maps in the following sense.

**Definition 3.1** (Microscopic phase space condition). A net $O \mapsto \mathfrak{A}(O)$ is said to satisfy the microscopic phase space condition if for every $\gamma \geq 0$, there exists a linear continuous map $\psi : \mathcal{C}^\infty(\Sigma) \to \Sigma$ of finite rank such that for sufficiently large $\ell > 0$,

$$
r^{-\gamma}\| (\Xi - \psi)(\mathfrak{A}(r)) \|^{(\ell)} \to 0 \quad \text{as } r \to 0.
$$

Here the restriction $|\mathfrak{A}(r)|$ is applied to the image points of the maps, which are functionals in $\Sigma$. This phase space condition is known to be fulfilled in free field theory in at least $3 + 1$ space-time dimensions, for massive free fields also in $2 + 1$ dimensions [Bos00].

The consequences of this condition are as follows [Bos05b]. While the maps $\psi$ are not uniquely fixed by the property above, the image of their dual maps, $\text{img} \psi^* = \Phi_\gamma$, is actually unique at fixed $\gamma$, provided that the rank of $\psi$ is chosen minimal. These finite-dimensional spaces $\Phi_\gamma$ form an increasing sequence $\Phi_0 \subset \Phi_1 \subset \Phi_2 \ldots$, and their union $\cup_{\gamma} \Phi_\gamma = \Phi_{FH}$ is precisely the field content of the theory as defined by Fredenhagen and Hertel [FH81]. After smearing with test functions, the elements $\phi \in \Phi_\gamma$ are local Wightman fields. Actually it suffices for regularizing $\phi$ to smear it along the time axis; that is, for $f \in \mathcal{S}(\mathbb{R})$ and $\phi \in \Phi_{FH}$, the quadratic form $\phi(f) = \int dt f(t) U(t) \phi U(t)^*$ can be continued to an unbounded, but closable operator on the dense invariant domain $\mathcal{C}^\infty(\mathcal{H}) = \cap_{\ell>0} (1 + H)^{-\ell}\mathcal{H}$. Further, $\phi \in \Phi_\gamma$ can be approximated with bounded operators in a controlled way; cf. [Bos05b, Lemma 3.5] and the remark following it:

**Theorem 3.2.** Let $\phi \in \Phi_{FH}$. One can find constants $\ell > 0$, $k > 0$ and operators $A_r \in \mathfrak{A}(r)$ for each $r > 0$ such that, as $r \to 0$,

$$
\| \phi \|^{(-\ell)} < \infty, \quad \| A_r - \phi \|^{(-\ell)} = O(r), \quad \| A_r \|^{(-\ell)} = O(1),
$$

$$
\| A_r \| = O(r^{-k}), \quad \forall n \in \mathbb{N} : \| \frac{d^n}{dt^n} U(t) A_r U(t)^* \| = O(r^{-k-n}).
$$
Moreover, the spaces $\Phi_\gamma$ are related to the approximation of bounded operators in the short distance limit; see [Bos05b, Eq. (4.4)]:

**Theorem 3.3.** Let $p_\gamma : \mathcal{C}^\infty(\Sigma)^* \to \Phi_\gamma \subset \mathcal{C}^\infty(\Sigma)^*$ be a continuous projection onto $\Phi_\gamma$. Then, for sufficiently large $\ell > 0$, 

$$\| \Xi p_\gamma - \Xi \| \mathfrak{A}(r) \|^{(\ell)} = o(r^\gamma).$$

Here $p_\gamma^* : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$ is the pre-dual map to $p_\gamma$, which always exists due to its finite rank. Of course, such projections $p_\gamma$ exist in abundance. Since the spaces $\Phi_\gamma$ are invariant under conjugation, it is possible to choose $p_\gamma$ hermitean, i.e., such that $p_\gamma(A^*) = p_\gamma(A)^*$.

It was shown in [Bos05a] that due to the properties explained above, operator product expansions exist in a rigorous sense. In fact, [Bos05a] established the expansion of $\phi(x)\phi'(y)$ for spacelike separated points $x$ and $y$. A similar scheme can be applied for arbitrary $x$ and $y$, in the sense of distributions, as sketched in [Bos05a] and worked out in more detail in [Bos00, Ch. 5.5]. (See also [BDM09, Sec. 4].) For our purposes, we will need a specific variant of this product expansion, which will be established in Sec. 5.2.

### 4 Distributions as boundary values of analytic functions

If $\sigma \in \Sigma$ is energy-bounded and $\phi$ a Wightman field with sufficiently regular high-energy behaviour, then the distribution $\sigma(\phi^*(t)\phi(t'))$ is the boundary value of an analytic function in the half plane $\text{Im} (t - t') < 0$. If further $\sigma$ is positive, then the distribution is positive-definite. These types of distributions have certain well-known characterizations [SW64, RS75].

Since we will need specific quantitative estimates in our context, we will repeat some of those arguments in detail.

First of all, for each $d > 0$ and $m \in \mathbb{N}$ we define a norm on $\mathcal{D}(-d,d)$ by

$$\| f \|_{d,m} := \max_{0 \leq n \leq m} d^n \| f^{(n)} \|_1.$$ \hspace{1cm} (4.1)

This norm is equivalent to the Sobolev norm defining the space $W^{m,1}_0(-d,d)$ [AF03], but it is convenient to use the above norms owing to their behaviour under scaling. Namely, if $f \in \mathcal{D}(-d,d)$ and $\lambda > 0$, and we set $f_\lambda(t) = \lambda^{-1}f(t/\lambda)$, then $f_\lambda \in \mathcal{D}(-\lambda d, \lambda d)$ and

$$\forall m \in \mathbb{N} : \| f_\lambda \|_{\lambda d, m} = \| f \|_{d,m}.$$ \hspace{1cm} (4.2)

Let us now define the class of analytic functions that is of interest.

**Definition 4.1.** We say that an analytic function $F : \mathbb{R} - i\mathbb{R}_+ \to \mathbb{C}$ is regular at the boundary if there exists $\ell > 0$ such that

$$\| F \|^{(-\ell)} := \sup_{-1 \leq \text{Im} z < 0} |F(z)| |\text{Im} z|^\ell$$

is finite. The space of all such functions for given $\ell$ is denoted as $\mathcal{K}_\ell$; and $\mathcal{K} := \cup_{\ell > 0} \mathcal{K}_\ell$.

As the name suggests, functions in $\mathcal{K}$ have distributional boundary values on the real line.

---

1Throughout the paper, we will write distributions in terms of their formal integration kernels, such as $\int K(x)f(x)dx$ for the evaluation of a distribution $K \in \mathcal{S}'(\mathbb{R})$ on a test function $f \in \mathcal{S}(\mathbb{R})$, even if $K$ does not arise from an integrable function or measure. This is merely a notational convention.
Proposition 4.2. Let $F \in K^\ell$. Then the limit $\lim_{y \to 0^+} F(x - iy)$ exists as a tempered distribution in $x$. The limit distribution $F(x - i0)$ satisfies the following estimate for $f \in \mathcal{D}(-d, d)$, $d > 0$:

$$| \int f(x) F(x - i0) \, dx | \leq 4^{\ell+2} (\ell + 3)(1 + d^{-\ell-2}) \| F \|_{(-\ell)} \| f \|_{d, \ell+2}. $$

Proof. For fixed $f \in \mathcal{S}(\mathbb{R})$, consider the function

$$g(y) := \int f(x) F(x - iy) \, dx, \quad 0 < y \leq 1. \quad (4.3)$$

Since $F$ is analytic in $z = x - iy$, we can obtain the derivatives of $g$ using integration by parts:

$$\forall j \in \mathbb{N}_0 : \quad \frac{d^j g}{dy^j}(y) = \int f(x)(-i)^j \frac{dz}{dz} F(x - iy) \, dx = i^j \int f^{(j)}(x) F(x - iy) \, dx. \quad (4.4)$$

Thus we have the estimate

$$\forall j \in \mathbb{N}_0 : \quad \left| \frac{d^j g}{dy^j}(y) \right| \leq y^{-\ell} \| F \|_{(-\ell)} \| f^{(j)} \|_1. \quad (4.5)$$

We now want to deduce the following improved estimate.

$$\left| \frac{d^j g}{dy^j}(y) \right| \leq 4^{\ell-j+2} (1 + y^{3/2-j}) \| F \|_{(-\ell)} \sum_{k=0}^{\ell+2} \| f^{(k)} \|_1 \quad \text{for } j \in \{0, \ldots, \ell + 2\}. \quad (4.6)$$

In fact, for $j = \ell + 2$, this directly follows from Eq. (4.5). Now suppose that Eq. (4.6) holds for $j + 1$ in place of $j$. We compute:

$$\left| \frac{d^j g}{dy^j}(y) \right| \leq \left| \frac{d^j g}{dy^j}(1) \right| + \int_y^1 y' \left| \frac{d^{j+1} g}{dy^{j+1}}(y') \right| \leq \| F \|_{(-\ell)} \| f^{(j)} \|_1 + 4^{\ell-j+1} \| F \|_{(-\ell)} \sum_{k=0}^{\ell+2} \| f^{(k)} \|_1 \int_y^1 y' (1 + (y')^{1/2-j})$$

$$\leq 4^{\ell-j+1} \| F \|_{(-\ell)} \sum_{k=0}^{\ell+2} \| f^{(k)} \|_1 (4 + 2y^{3/2-j}). \quad (4.7)$$

This proves Eq. (4.6). In particular, the case $j = 1$ shows that $dg/dy$ is bounded as $y \to 0$; thus $g(y)$ converges in this limit. Setting $j = 0$ in Eq. (4.6) then shows that $g(0^+) =: \int f(x) F(x - i0) \, dx$ defines a tempered distribution. Also, if $f \in \mathcal{D}(-d, d)$, we can combine the estimate

$$\sum_{k=0}^m \| f^{(k)} \|_1 \leq (m + 1) \max\{1, d^{-m}\} \| f \|_{d,m} \quad (4.8)$$

with Eq. (4.6), where $j = 0$ and $m = \ell + 2$, in order to show the proposed estimate for the limit distribution. \qed
It is clear from Definition 4.1 that, for two functions which are regular at the boundary, their product inherits this property. More explicitly, for $F \in \mathcal{K}^\ell$ and $G \in \mathcal{K}^m$, we have

$$\|FG\|^{(-\ell-m)} \leq \|F\|^{(-\ell)}\|G\|^{-m}. \quad (4.9)$$

Thus the product of the boundary distributions is well-defined by multiplying the analytic functions. On the other hand, the Fourier transforms of the boundary distributions have support in $[0, \infty)$. This allows for an alternative definition of the distribution product by convolution in Fourier space. The two definitions are in fact equivalent [RS75, Ch. IX.10, Example 4].

Apart from our distributions being boundary values of analytic functions, we also need to consider questions of positivity. We remind the reader of the definitions (the terminology is not completely consistent in the literature). For $g_1, g_2 \in \mathcal{D}(\mathbb{R})$, we introduce the abbreviation $g_1 \circ g_2(s,s') := g_1(s + s'/2)g_2(s - s'/2)$.

**Definition 4.3.** A distribution $K \in \mathcal{S}(\mathbb{R}^2)'$ is called positive-definite if for all $g \in \mathcal{S}(\mathbb{R})$, one has $\int ds \, ds' \, K(s,s') \hat{g} \circ g(s,s') \geq 0$. If here $K$ depends on the second variable only, so $K \in \mathcal{S}(\mathbb{R})'$, it is called a distribution of positive type. With $\mathcal{K}_+ \subset \mathcal{K}$ we denote the subset of positive type distributions.

The Bochner-Schwartz Theorem asserts that distributions of positive type are precisely the Fourier transforms of positive, polynomially bounded measures. We now show that the product of distributions, as discussed further above, preserves positivity if both factors are positive, at least in a special situation that is of interest to us.

**Proposition 4.4.** Let $F \in \mathcal{K}_+$. Let $G : \mathbb{R} \times (\mathbb{R} - i\mathbb{R}_+) \to \mathbb{C}$ such that $G(s, \cdot) \in \mathcal{K}^\ell$ for some $\ell$ and every fixed $s$, where the map $\mathbb{R} \to \mathcal{K}^\ell$, $s \mapsto G(s, \cdot)$ is bounded and continuous in $\|\cdot\|^{(-\ell)}$. Suppose further that $G(s, s' - \epsilon) \circ \hat{g} \o g(s,s')$ is positive-definite. Then the product distribution $P(s,s') = F(s' - \epsilon)G(s,s' - \epsilon)$ is continuous in $s$ and positive-definite.

**Proof.** First, due to Prop. 4.2, the boundedness and continuity of $s \mapsto G(s, \cdot)$ implies that $\int P(s,s')f(s')ds'$ is continuous and bounded in $s$; in particular $P \in \mathcal{S}(\mathbb{R}^2)'$ is well-defined. Now let $\mu$ be the positive measure that arises by Fourier transform of $F(x - \epsilon)$. Since $F(x - \epsilon)$ is a boundary value, we know $\text{supp} \, \mu \subset [0, \infty)$. Therefore we have for $g \in \mathcal{S}(\mathbb{R})$,

$$\int ds \, ds' \, P(s,s') \hat{g} \circ g(s,s')$$

$$= \lim_{\epsilon \to 0^+} \int_0^\infty dp \int ds \, ds' \, e^{-ip(s' - \epsilon)}G(s,s' - \epsilon) \hat{g} \circ g(s,s'). \quad (4.10)$$

Supposing for a moment that the integrand $I(\epsilon, p)$ has an integrable bound in $p$, uniform in $\epsilon$, we can apply the dominated convergence theorem and obtain

$$\int ds \, ds' \, P(s,s') \hat{g} \circ g(s,s') = \int_0^\infty dp \int ds \, ds' \, G(s,s' - \epsilon) \hat{g}_p \circ g_p(s,s'), \quad (4.11)$$

where $g_p(t) = e^{ipt}g(t)$. This is clearly non-negative, since $G$ is positive-definite.
It remains to prove appropriate bounds for \( I(\epsilon, p) \). To that end, choose \( n \in \mathbb{N} \) so large that \( \int d\mu(p)(1 + p)^{-n} < \infty \). We use integration by parts in \( s' \) to obtain

\[
I(\epsilon, p) = (1 + p)^{-n} \int ds \, ds' \left( 1 + p \right)^n e^{-\epsilon(s'-u)p} G(s, s' - u) \tilde{g} \odot g(s, s')
= (1 + p)^{-n} \int ds \, ds' e^{-\epsilon(s'-u)p} \left( 1 - \epsilon \frac{\partial}{\partial s'} \right)^{n} G(s, s' - u) \tilde{g} \odot g(s, s').
\]

(4.12)

Via the Leibniz rule, we can distribute the derivatives \( \partial/\partial s' \) to \( G(s, s') \) and to the test function. Now note that with \( G \), also the derivatives \( \partial^k G/\partial z^k \) fulfil polynomial bounds when \( \text{Im} \, z \to 0 \); namely, we can use the Cauchy integral formula for a circle of radius \( |\text{Im} \, z/2| \) around \( z \) in order to obtain the estimate

\[
\left| \frac{\partial^k G(s, z)}{\partial z^k} \right| \leq 2^k k! \sup_x \|G(x, \cdot )\|_{(\ell)} |\text{Im} \, z|^{\ell - k} \quad \text{for} \quad -\frac{1}{2} \leq \text{Im} \, z < 0.
\]

(4.13)

This implies that \( e^{-\epsilon z/2} \partial^k G(s, z/2) \) belongs to \( K^{\ell + k} \) with norm uniform in \( s \) and \( p \). Applying Proposition 4.2, we can then obtain finite bounds on the integral in (4.12) as \( \epsilon \to 0 \), so

\[
|I(\epsilon, p)| \leq c \left( 1 + p \right)^{-n} \quad \text{for small} \, \epsilon,
\]

(4.14)

with a constant \( c \) depending on \( G \) and \( g \). This is a bound of the required form. \( \square \)

5 \ Products

Our next aim is to describe products of quantum fields that are of interest to us, and derive an operator product expansion for them. Specifically, we are interested in the products of two quantum fields \( \phi, \phi' \), displaced to different points \( t, t' \) on the time axis; this product then exists as a distribution in the difference variable \( s' = t - t' \). In addition, we wish to multiply this distribution with a \( c \)-number distributional kernel in \( t - t' \), and also consider sums of such expressions. The operator product expansion we use is derived by means of techniques described in \( \text{[Bos05a]} \); however, we need to generalise the construction both to include the weighting factors and also to obtain more detailed estimates on OPEs at timelike-separated points.

We can formally describe the products of interest as elements of the algebraic tensor product space \( \Phi_{\text{prod}} := \mathcal{K} \otimes \mathcal{C}^\infty(\Sigma)^* \otimes \mathcal{C}^\infty(\Sigma)^* \). Any element \( \Pi \in \Phi_{\text{prod}} \) has the form of a finite sum,

\[
\Pi = \sum_j K_j \otimes \phi_j \otimes \phi'_j, \quad K_j \in \mathcal{K}, \ \phi_j, \phi'_j \in \mathcal{C}^\infty(\Sigma)^*.
\]

(5.1)

For \( \ell > 0 \), we set \( \Phi_{\text{prod}}^\ell = \mathcal{K}^\ell \otimes \mathcal{C}^\ell(\Sigma)^* \otimes \mathcal{C}^\ell(\Sigma)^* \subset \Phi_{\text{prod}} \); clearly, \( \Phi_{\text{prod}} = \cup_{\ell \geq 0} \Phi_{\text{prod}}^\ell \). Further we consider the subspace \( \Phi_{\text{prod,loc}} = \mathcal{K} \otimes \Phi_{\text{FH}} \otimes \Phi_{\text{FH}} \subset \Phi_{\text{prod}} \), the space of products of pointlike fields. To each product \( \Pi \in \Phi_{\text{prod}} \), we can associate a distribution \( T_\Pi \), heuristically given by

\[
U \left( \frac{t + t'}{2} \right) T_\Pi(t - t') U \left( \frac{t + t'}{2} \right)^* = \sum_j K_j(t - t' - \delta) \phi_j(t) \phi'_j(t').
\]

(5.2)

We shall first discuss in which sense these product distributions exist, before deriving an operator product expansion for them, in the case where \( \phi_j \) and \( \phi'_j \) are local fields. Then we will introduce certain convolutions of these distributions with test functions, generalize the OPE for them, and single out a minimal set of composite fields that will be of use to us.
5.1 Operator products

Before considering our operator products, let us first define the set of distributions of interest.

**Definition 5.1.** A $C^\infty(\Sigma)^*$-valued distribution is a linear map $T: D(\mathbb{R}) \rightarrow C^\infty(\Sigma)^*$ such that there exist constants $\ell > 0$ and $m \in \mathbb{N}_0$, and, for each $d > 0$, a constant $c_d$, with the property that

$$\forall f \in D(-d,d) : \|T(f)\|^{(-\ell)} \leq c_d \|f\|_{d,m}.$$  

Equivalently, we might say that $T[D(-d,d)]$ extends to a map of $W_0^{m,1}(-d,d)$ to $C^\ell(\Sigma)^*$, with finite norm $\|T\|_{d,m}^{(-\ell)} \leq c_d$. In more standard terms, $T$ might be called a distribution of finite order, but since we will not use other distributions in this context, we drop the extra qualifier. As before, we shall denote these distributions using their formal kernels:

$$T(f) = \int dx T(x)f(x).$$

Their expectation values $\sigma(T(x))$, for fixed $\sigma \in C^\ell(\Sigma)$, are then distributions in $D'(\mathbb{R})$ in the usual sense. We shall call a $C^\infty(\Sigma)^*$-valued distribution skew-hermitean if $T(x)^* = T(-x)$.

We shall now clarify in which precise sense the distributions $T_\Pi$ in Eq. (5.2) exist.

**Proposition 5.2.** Let $\ell > 0$. To each $\Pi = \sum_j K_j \otimes \phi_j \otimes \phi'_j \in \Phi_{\text{prod}}$, there exists a unique $C^\infty(\Sigma)^*$-valued distribution $T_\Pi$ such that for any $\sigma \in \bigcup_{E>0} \Sigma(E)$,

$$\sigma(T_\Pi(f)) = \sum_j \int ds' f(s')K_j(s' - \lambda_0)\phi_j(s'/2 - \lambda_0)\phi'_j(-s'/2 + \lambda_0)).$$

The map $\Pi \mapsto T_\Pi$ is linear. Further, there is a constant $c > 0$ such that for any $d \leq 1$,

$$\|T_\Pi\|_{d,3\ell+2}^{(-\ell)} \leq c d^{(3\ell+2)} \sum_j \|K_j\|^{(-\ell)} \|\phi_j\|^{(-\ell)} \|\phi'_j\|^{(-\ell)}.$$

**Proof.** Without loss of generality, we can assume that $\Pi$ is of the form $\Pi = K \otimes \phi \otimes \phi'$. Let $\sigma \in \Sigma(E)$, where $E > 0$ is fixed for the moment. Then, due to the spectrum condition, the distribution $\sigma(\phi(s'/2)\phi'(-s'/2))$ is indeed the boundary value of an analytic function, namely of

$$F(s' - is'') := \sigma(e^{(s'' + is')H/2} \phi e^{-(s'' + is')H} \phi') e^{(s'' + is')H/2}, \quad s'' > 0. \quad (5.3)$$

This function fulfills the bounds

$$|F(s' - is'')| \leq \|\sigma\| e^{E s'''(1 + E)^{2\ell}} \|\phi\|^{(-\ell)} \|\phi'\|^{(-\ell)} \sup_{\lambda > 0} e^{-\lambda s'''} (1 + \lambda)^{2\ell} \leq \|\sigma\| \|\phi\|^{(-\ell)} \|\phi'\|^{(-\ell)} (1 + E)^{2\ell} e^{(1 + E)s''(2\ell)^2(\lambda_{s''})^{-2\ell}}. \quad (5.4)$$

So $F$ is regular at the boundary in the sense of Definition 4.1. Rescaling its argument, we explicitly have

$$\|F(\frac{z}{1 + E})\|^{(-2\ell)} \leq c \|\sigma\| \|\phi\|^{(-\ell)} \|\phi'\|^{(-\ell)} (1 + E)^{4\ell}, \quad (5.5)$$

where the constant $c$ depends on $\ell$ only. The distributional product $K(s' - \lambda_0)F(s' - \lambda_0)$ therefore exists. Rescaling also $K$, and applying Proposition 4.2 and Eq. (4.9), we obtain for any $g \in D(-d,d)$ and with another constant $c'$,

$$\left| \int ds' g(s')K(s' - \lambda_0)F(\frac{s' - \lambda_0}{1 + E}) \right| \leq c' \|\sigma\| \|K\|^{(-\ell)} \|\phi\|^{(-\ell)} \|\phi'\|^{(-\ell)} (1 + E)^{5\ell} (1 + d^{-3\ell - 2}) \|g\|_{d,3\ell+2}. \quad (5.6)$$
Now let \( f \in \mathcal{D}(-d,d), \ d \leq 1 \). We set \( g(s') = (1+E)^{-1} f(s'/(1+E)) \in \mathcal{D}(- (1+E)d, (1+E)d) \) and obtain using Eq. (5.6) with \( (1+E)d \) in place of \( d \),

\[
| \int ds' f(s') K(s' - x) F(s' - x) | \leq c'' \| \sigma \| \| K \| (-\ell) \| \phi \| (-\ell) \| \phi' \| (-\ell) (1 + E)^{5\ell} d^{-3\ell-2} \| f \|_{d,3\ell+2}. \tag{5.7}
\]

This serves to define \( T_\Pi(f) \) on \( \Sigma(E) \) for any \( E \). Using [BDM09] Lemma 2.6, we can extend this linear form to \( \mathcal{C}^{5\ell+1}(\Sigma) \), and obtain another constant \( c''' \) such that

\[
\| T_\Pi \|_{d,3\ell+2}^{(-5\ell-1)} \leq c''' \| K \| (-\ell) \| \phi \| (-\ell) \| \phi' \| (-\ell) d^{-3\ell-2}. \tag{5.8}
\]

The extension is unique by density. It is also clear by construction that \( T_\Pi(f) \) is linear in \( f \) and in \( \Pi \), i.e. multilinear in \( \phi, \phi' \), and \( K \). Then, the estimate (5.8) shows that \( T_\Pi \) is a \( \mathcal{C}^{\infty}(\Sigma)^* \)-valued distribution in the sense of Def. 5.1

### 5.2 Product expansions

We now prove the operator product expansion for a product of pointlike fields, \( \Pi \in \Phi_{\text{prod,loc}} \), in the following form.

**Theorem 5.3.** Let \( \Pi \in \Phi_{\text{prod,loc}} \), and let \( \alpha \geq 0 \). There exist \( \ell > 0, m \in \mathbb{N}, \gamma \geq 0 \), and a hermitean projector \( p_\gamma : \mathcal{C}^{\infty}(\Sigma)^* \rightarrow \Phi_\gamma \) onto \( \Phi_\gamma \) such that

\[
\| T_\Pi - p_\gamma T_\Pi \|_{d,m}^{(-\ell)} = o(d^\alpha) \quad \text{as} \ d \to 0.
\]

This is a variant of [Bos05a, Theorem 3.2]. Note that the approximation emerges into a more familiar form of operator product expansion if \( p_\gamma \) is written in a basis.

**Proof.** Again, we can assume \( \Pi = K \odot \phi \odot \phi' \in \Phi_{\text{prod}}^\ell \) for some \( \ell > 0 \), where now \( \phi, \phi' \in \Phi_{\text{FH}} \). Further, after possibly increasing \( \ell \), we choose \( k > 0 \) and approximating sequences \( A_r, A_r' \) for \( \phi, \phi' \) as in Theorem 5.2. Set \( B_r = K \odot A_r \odot A_r' \). We define \( m := 3\ell + 2 \) and \( \gamma := (2k + \ell + 3)(\alpha + m + 1) \), and choose a hermitean projector \( p_\gamma \) onto \( \Phi_\gamma \). Now we estimate for an as yet unspecified \( \ell \),

\[
\| T_\Pi - p_\gamma T_\Pi \|_{d,m}^{(-\ell)} \leq \| T_\Pi - T_{B_r} \|_{d,m}^{(-\ell)} + \| T_{B_r} - p_\gamma T_{B_r} \|_{d,m}^{(-\ell)}
\]

\[
+ \| p_\gamma \| (-\ell) \| T_{(\Pi-B_r)} \|_{d,m}^{(-\ell)}. \tag{5.9}
\]

Here \( \| p_\gamma \| (-\ell,\ell) \) is a constant independent of \( r \) and \( d \), finite if \( \ell \) is large. We will show below that for large \( \ell \),

\[
\| T_{(\Pi-B_r)} \|_{d,m}^{(-\ell)} = O(rd^{-m}), \tag{5.10}
\]

\[
\| T_{B_r} - p_\gamma T_{B_r} \|_{d,m}^{(-\ell)} = O(r^{-2k-\ell-2}d^{-\ell-2}(r + d)^\gamma). \tag{5.11}
\]

Setting \( r(d) = d^{\alpha + m + 1} \), and using \( \gamma = (2k + \ell + 3)(\alpha + m + 1) \), both terms above are of order \( O(d^{\alpha + 1}) \), of which the theorem follows.
To show Eq. (5.10), we write

$$\Pi - B_r = K \otimes (\phi - A_r) \otimes \phi' + K \otimes A_r \otimes (\phi' - A'_r).$$

(5.12)

For the first summand, we estimate by Proposition 5.2:

$$\| T_{K \otimes (\phi - A_r) \otimes \phi} \|_{d,m}^{(-5\ell-1)} \leq O(d^{-m}) \| K \|^{(-\ell)} \| \phi - A_r \|^{(-\ell)} \| \phi' \|^{(-\ell)} = O(rd^{-m}),$$

(5.13)

as proposed. The second summand of Eq. (5.12) has a similar estimate, which combined gives Eq. (5.10).

For Eq. (5.11), we use the short-distance approximation of Theorem 3.3 on the operator $A_r^P(s') := A_r(s'/2)A'_r(-s'/2) \in \mathfrak{A}(O_{r+d})$, where $|s'| \leq d$, and on its derivatives in $s'$. Using the estimates on the derivatives of $A_r(t)$ and $A'_r(t)$ provided by Theorem 3.2, this entails that for large $\ell$,

$$\frac{d^n}{(ds')^n} \left( A_r^P(s') - p_rA_r^P(s') \right) \|^{(-\ell)} = O((r + d)^\gamma)O(r^{-2k-n}).$$

(5.14)

Now we compute $T_{B_r} - p_rT_{B_r}$, first on a fixed test function $f \in D(-d,d)$, $d \leq 1$, and on a fixed functional $\sigma \in C^\infty(\Sigma)$. By Prop. 5.2 we have

$$\sigma(T_{B_r}(f) - p_rT_{B_r}(f)) = \int ds'h(s')K(s' - 0),$$

where $h(s') = f(s')g(s')$, $g(s') = \sigma(A_r^P(s') - p_rA_r^P(s'))$. (5.15)

(Note that here $h$ is smooth, the only divergent factor is $K$. Therefore, also, sharp energy-bounds of $\sigma$ do not play a role.) Using Proposition 4.2 it follows that

$$|\sigma(T_{B_r}(f) - p_rT_{B_r}(f))| \leq c\|K\|^{(-\ell)}d^{-\ell-2}\|h\|_{d,\ell+2}$$

(5.16)

with a numerical constant $c$. For the Sobolev norm of $h$, we can derive the following estimate by the Leibniz formula.

$$\|h\|_{d,\ell+2} = \|fg\|_{d,\ell+2} \leq 2\ell+2\|f\|_{d,\ell+2} \max_{0 \leq n \leq \ell+2} d^n \sup_{t \in [-d,d]} |g^{(n)}(t)|.$$  

(5.17)

The derivatives of $g$ can be estimated by Eq. (5.14). For $t \in [-d,d]$ one has

$$|g^{(n)}(t)| \leq \|\sigma\|^{(-\ell)}O((r + d)^\gamma)O(r^{-2k-n}),$$

(5.18)

where the $O(\ldots)$ estimates are uniform in $\sigma$. Combining Eqs. (5.16)–(5.18), we obtain

$$\|T_{B_r} - p_rT_{B_r}\|_{d,\ell+2} \leq O(d^{-\ell-2})O((r + d)^\gamma)O(r^{-2k-\ell-2}),$$

(5.19)

which gives Eq. (5.11).

The bounds established are certainly not strict, in particular regarding the value of $\gamma$ (i.e., the number of approximation terms needed in the OPE). They might be improved at the price of extra computational effort, but this is not relevant for our purposes. Note however that the kernels $K$ introduce an extra divergence that might make more OPE terms necessary than in the “ordinary” OPE version with $K = 1$. 

[17]
5.3 Convolutions

In order to establish the existence of quantum inequalities, we need to analyse distributions evaluated on certain convolutions of test functions, similar to Eqs. (2.7)–(2.9) in the free field case. Let us define them, and establish their well-definedness. We remind the reader of the abbreviation \( g_1 \circ g_2(s, s') = g_1(s + s'/2)g_2(s - s'/2) \), and of the notion of skew-hermitean \( C^\infty(\Sigma)^* \)-valued distributions, which fulfill \( T(s')^* = T(-s') \).

**Lemma 5.4.** Let \( T \) be a \( C^\infty(\Sigma)^* \)-valued distribution. Then the bilinear map

\[
\kappa_0[T] : \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R}, C^\infty(\Sigma)^*)
\]

is well-defined; indeed, if \( g_1, g_2 \in \mathcal{D}(-d, d) \), then supp\( \kappa_0[T](g_1, g_2) \subset (-d, d) \). Further,

\[
\kappa[T] : \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \to C^\infty(\Sigma)^*,
\]

is well-defined as a weak integral. Both \( \kappa_0[T] \) and \( \kappa[T] \) are linear in \( T \). If \( T \) is skew-hermitean, then \( \kappa[T](\tilde{g}, g) \) is hermitean for arbitrary \( g \in \mathcal{D}(\mathbb{R}) \). For any \( m \in \mathbb{N} \) and \( d > 0 \), one has the estimate

\[
\|\kappa[T]\|_{d,m}^{(-\ell)} \leq 2^{m+1}\|T\|_{2d,m}^{(-\ell)}.
\]

The Sobolev norms of the bilinear maps are understood here with respect to a product of identical Sobolev norms on the two arguments.

**Proof.** First, \( \kappa_0[T](g_1, g_2)(s) \) is well-defined since \( g_1 \circ g_2(s, \cdot) \) lies in \( \mathcal{D}(-2d, 2d) \) for each fixed \( s \); and it is (weakly) smooth in \( s \) since \( s \mapsto g_1 \circ g_2(s, \cdot) \) is smooth in the \( \mathcal{D}(\mathbb{R}) \) topology. The support properties are clear. Further, one sees that

\[
\|\kappa_0[T](g_1, g_2)(s)\|_{2d,m}^{(-\ell)} \leq \|T\|_{2d,m}^{(-\ell)} \|g_1 \circ g_2(s, \cdot)\|_{2d,m},
\]

which is locally bounded in \( s \). Therefore, for each \( \sigma \in C^\infty(\Sigma) \), the map

\[
\mathbb{R} \to \mathbb{C}, \quad s \mapsto \sigma(U(s) \kappa_0[T](g_1, g_2)(s) U(s)^*)
\]

is continuous. Hence \( \kappa[T] \) is well-defined as a weak integral. Using the Leibniz rule and a change of variables, one finds

\[
\int ds \|g_1 \circ g_2(s, \cdot)\|_{2d,m} \leq \sum_{n=0}^{m} \sum_{r=0}^{n} \binom{n}{r} d^r \|g_1^{(r)}\|_1 d^{n-r} \|g_2^{(n-r)}\|_1 \leq (2^{m+1} - 1) \|g_1\|_{d,m} \|g_2\|_{d,m}.
\]

Together with Eq. (5.20), this yields the estimate

\[
\|\kappa[T]\|_{d,m}^{(-\ell)} \leq 2^{m+1}\|T\|_{2d,m}^{(-\ell)}
\]

as proposed. Also, it is clear in matrix elements that both \( \kappa_0[T](g) \) and \( \kappa[T](g) \) are linear in \( T \). If \( T \) is skew-hermitean, one uses the identity \( \overline{\tilde{g} \circ g(s, s')} = \tilde{g} \circ g(s, -s') \) to conclude \( \kappa_0[T](\tilde{g}, g)(s)^* = \kappa_0[T](\tilde{g}, g)(s) \) and, in consequence, \( \kappa[T](\tilde{g}, g)^* = \kappa[T](\tilde{g}, g) \). □
The estimates above show that our operator product expansion for $T_{\Pi}$, as established in Theorem 5.3, can be transferred to $\kappa[T_{\Pi}]$. This is in fact the form of OPE we shall use for establishing quantum inequalities.

**Corollary 5.5.** Let $\Pi \in \Phi_{\text{prod,loc}}$, and let $\alpha \geq 0$. There exist $\ell > 0$, $m \in \mathbb{N}$, $\gamma \geq 0$, and a hermitean projector $p_{\gamma} : C^\infty(\Sigma)^* \to \Phi_{\gamma}$ onto $\Phi_{\gamma}$ such that

$$\|\kappa[T_{\Pi} - p_{\gamma}T_{\Pi}]\|_{d,m}^{(-\ell)} = o(d^\alpha) \quad \text{as} \quad d \to 0.$$ 

### 5.4 Minimal approximating projectors

The operator product expansion allows us to approximate a given product $\Pi$ with a finite number of composite fields. It is important for our applications to choose the minimal number of composite fields needed, so that none of the approximation terms can be considered “redundant”.

Let us introduce that notion of approximation by finitely many terms more abstractly. This is similar, but not identical to the analysis of normal products in [Bos05a, Sec. IV].

**Definition 5.6.** Let $\Pi \in \Phi_{\text{prod,loc}}$, and $\alpha \geq 0$. A hermitean projector $p$ in $C^\infty(\Sigma)^*$ with finite-dimensional image in $\Phi_{\text{FH}}$ is called $\alpha$-approximating for $\Pi$ if there are constants $\ell > 0$ and $m \in \mathbb{N}$ such that

$$\|\kappa[T_{\Pi} - pT_{\Pi}]\|_{d,m}^{(-\ell)} = o(d^\alpha) \quad \text{as} \quad d \to 0.$$ 

The operator product expansion in Corollary 5.5 tells us that for any given $\alpha$, we can choose $\gamma$ large enough such that any hermitean projector $p$ onto $\Phi_{\gamma}$ is $\alpha$-approximating for $\Pi$. However, this is in a way an “upper estimate” to the OPE, since $\Phi_{\gamma}$ may contain elements that are not actually needed for approximating the given product. We will therefore minimize the approximating projector in a well-defined sense.

This is done as follows. On the family of all $\alpha$-approximating projectors for a given product $\Pi$, we introduce a partial order by

$$p_1 \leq p_2 \iff (\text{img } p_1 \subset \text{img } p_2) \wedge (\ker p_1 \supset \ker p_2).$$

Minimal elements with respect to this partially ordered set will be called minimal $\alpha$-approximating projectors. By dimensional arguments, any decreasing sequence in the set must eventually become constant; so minimal elements certainly exist, and can be constructed below each given $\alpha$-approximating projector. However, there seems to be no reason why they should be unique.

This is in contrast to the situation for normal product spaces [Bos05a Sec. IV], where the approximation property depends on $\text{img } p$ only, i.e., any other projector onto the same space would also be $\alpha$-approximating. In that case, one finds a unique minimal approximating space of fields. In our situation, these stronger results do not seem to follow, the main difficulty being that the convolution $\kappa[\cdot]$ does not commute with projectors. This turns out not to be a problem however: Each minimal $\alpha$-approximating projector will give us a nontrivial quantum inequality.

Let us summarize the main point of the above discussion:

**Proposition 5.7.** Let $\alpha \geq 0$ and $\Pi \in \Phi_{\text{prod,loc}}$. There exists at least one minimal $\alpha$-approximating projector $p$ for $\Pi$. 

---

2Projectors in this space will always be assumed as continuous.
6 Quantum inequalities

We are now going to establish quantum inequalities as a consequence of the operator product expansion above, and prove that they are nontrivial as discussed in Sec. 2.

6.1 Existence of inequalities

In order to establish inequalities, we define a set of products $\Phi_{\text{pos}} \subset \Phi_{\text{prod}, \text{loc}}$ which are “classically positive”, namely a finite sum of absolute squares with positive-type coefficients:

$$\Phi_{\text{pos}} := \{ \sum_j K_j \otimes \phi_j^* \otimes \phi_j \mid K_j \in \mathcal{K}_+, \phi_j \in \Phi_{\text{FH}} \}. \quad (6.1)$$

For any $\Pi \in \Phi_{\text{pos}}$, the distribution $T_\Pi$ is then skew-hermitean. [One verifies this in matrix elements by the integral formula in Prop. 5.2 using the relation $K_j(z) = K_j(-\bar{z})$ for the positive-type kernels $K_j$.]

Products from $\Phi_{\text{pos}}$ now give rise to quantum inequalities. To formulate these, we use the abbreviation $R := (1 + H)^{-1}$.

**Theorem 6.1.** Let $\Pi \in \Phi_{\text{pos}}$ and $\alpha \geq 0$. Let $p$ be an $\alpha$-approximating projector for $\Pi$. There exist $\ell > 0$, $m \in \mathbb{N}$, and a function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ of order $\epsilon(d) = o(d^\alpha)$ such that the following inequality between bounded operators holds.

$$\forall d > 0, \ g \in D(-d, d) : \quad R^\ell \kappa[pT_\Pi](\bar{g}, g) R^\ell \geq -\epsilon(d)(\|g\|_{d,m})^2 \mathbf{1}. \quad (6.2)$$

**Proof.** By Def. 5.6 there exist $\ell$, $m$ and $\epsilon(d) = o(d^\alpha)$ such that

$$\forall d > 0, \ g \in D(-d, d) : \quad \|\kappa[pT_\Pi - T_\Pi](\bar{g}, g)\|^{(-\ell)} \leq \epsilon(d)(\|g\|_{d,m})^2. \quad (6.3)$$

Note here that, since $T_\Pi$ is skew-hermitean, $\kappa[T_\Pi](\bar{g}, g)$ is guaranteed to be hermitean by Lemma 5.4. Since $p$ is hermitean, the same is true for $\kappa[pT_\Pi](\bar{g}, g)$. The expectation values of these expressions in positive functionals are therefore real. Thus, for any $\rho \in \cup_E \Sigma(E)$, $\rho \geq 0$, we obtain from Eq. (6.2),

$$\rho(\kappa[pT_\Pi](\bar{g}, g)) - \rho(\kappa[T_\Pi](\bar{g}, g)) \geq -\|\rho\|^{(\ell)} \epsilon(d)(\|g\|_{d,m})^2. \quad (6.4)$$

Now $\Pi \in \Phi_{\text{pos}}$ is of the form $\Pi = \sum_j K_j \otimes \phi_j^* \otimes \phi_j$. Due to energy-boundedness of $\rho$, we have by Prop. 5.2 and Lemma 5.4,

$$\rho(\kappa[T_\Pi](\bar{g}, g)) = \sum_j \int ds \int ds' \bar{g} \circ g(s, s') K_j(s' - t0) \\ \times \rho(U(s) \phi_j^*(s'/2) \phi_j(-s'/2) U(s)^*). \quad (6.5)$$

Here

$$G_j(s, s' - t0) := \rho(U(s) \phi_j^*(s'/2) \phi_j(-s'/2) U(s)^*) = \rho(\phi_j^*(s + s'/2) \phi_j(s - s'/2)) \quad (6.5)$$

are positive-definite distributions, as they give $\rho(\phi_j^*(g) \phi_j(g))$ when integrated with $\bar{g} \circ g$. Also, a similar estimate as in Eq. 5.4 shows that $s \mapsto G_j(s, \cdot)$ is uniformly bounded in $\| \cdot \|^{(-\ell)}$, and also continuous since the energy-bounded state $\rho$ is analytic for $U(s)$. Thus
the products of $G_j$ with the positive-type kernels $K_j$ are positive-definite as well (Prop. 4.4). Therefore, the expression in Eq. (6.4) is non-negative. Setting $\hat{\rho} = \rho(R^{-2} \cdot R^{-2})$, we can thus reduce Eq. (6.3) to

$$\hat{\rho}(R^\ell \kappa[pT_{\Pi}](\bar{\rho}, g) R^\ell) \geq -\epsilon(d) (\|g\|_{d,m})^2 \hat{\rho}(1). \quad (6.6)$$

Here $R^\ell \kappa[pT_{\Pi}](\bar{\rho}, g) R^\ell$ can be extended to a bounded operator by Eq. (6.2). Since $\hat{\rho}$ can be chosen from a dense subset in the set of all positive functionals, the theorem now follows. \[\square\]

The connection of the theorem with more usual forms of quantum inequalities becomes clear when we write the projector $p$ in a basis:

$$p = \sum_{j=1}^{n} \sigma_j(\cdot) \phi_j, \quad \text{where} \, \sigma_j \in C^\infty(\Sigma), \, \phi_j \in \Phi_{FH}, \, \sigma_j(\phi_k) = \delta_{jk}. \quad (6.7)$$

Here we choose $\phi_j$ and $\sigma_j$ hermitean, which is possible since $p$ is hermitean. Then, the inequality in the theorem can be rewritten as

$$\sum_{j=1}^{n} R^\ell \phi_j(f_j) R^\ell \geq -\epsilon(d) (\|g\|_{d,m})^2 1, \quad (6.8)$$

where the functions $f_1, \ldots, f_n$ are given by

$$f_j(s) = \int ds' \bar{g} \cdot g(s, s') \sigma_j(T_{\Pi}(s')). \quad (6.9)$$

These $f_j$ are actually of compact support, namely $\text{supp} f_j \subset (-d, d)$ if $g \in \mathcal{D}(-d, d)$, see Lemma 5.4. They are also smooth, since $s \mapsto \bar{g} \cdot g(s, \cdot)$ is differentiable in the $S$-topology; so they are indeed proper test functions in $\mathcal{D}(\mathbb{R})$. Further, the $f_j$ are real-valued, which follows from hermiticity of $\sigma_j$ and skew-hermiticity of $T_{\Pi}$.

The inequality (6.8) is of an asymptotic nature, inasmuch as only the asymptotic behaviour of the remainder, $\epsilon(d) = o(d^3)$, is known. For the sake of concreteness, we may choose a fixed test function $g \in \mathcal{D}(-1, 1)$, and define a family of scaled functions $g_d(t) = d^{-1} g(t/d)$. For these, $\|g_d\|_{d,m}$ is independent of $d$, so that the right-hand side of Eq. (6.8) simplifies; the inequality then valid as the parameter $d$ of the family goes to 0.

While the functions $f_j$ are real-valued, they are not guaranteed to be pointwise positive, in contrast to the free field situation [FEL98]. That the positivity properties of $f_j$ are a delicate issue is apparent since Eq. (6.9) has a strong analogy to Weyl quantization. With $\tilde{C}_j$ being the Fourier transform of $C_j(s') = \sigma_j(T_{\Pi}(s'))$, one has

$$f_j(s) = \int \frac{dp}{2\pi} \tilde{C}_j(p) \int ds' e^{ips'} \bar{g} \cdot g(s, s') = \int \frac{dp}{2\pi} \tilde{C}_j(p) W_g(s, p), \quad (6.10)$$

where $W_g$ is the Wigner function associated with the “state” $g$,

$$W_g(s, p) = \int ds' e^{ips'} \bar{g} \cdot g(s + s'/2) g(s - s'/2). \quad (6.11)$$

Now the Wigner function cannot be pointwise positive for compactly supported $g$ [Hud74], so positivity of $f_j$ can only be expected in special situations; see e.g., Prop. 7.3.

Note that Eq. (6.8) is a far-reaching generalization of the usual inequalities for squares of fields in free field theory. In particular, the estimate will in general not be restricted to
two fields, such as the Wick square and the identity in Eq. (2.11), but will involve a possibly large number of fields smeared with different sampling functions. One of the \( \phi_j \) will typically be the identity operator, and another \( \phi_j \) will typically be a normal product in the sense of Zimmermann [Zim70, Bos05a]. This term will usually be distinguished as a highest-order field, relating e.g. to scaling dimensions. But there seems to be no guarantee that such highest-order field exists uniquely, and even less that only two fields \( \phi_1, \phi_2 \) appear in the inequality.

Compared with the usual free-field situation, we also encounter a remainder term \( \epsilon(d) \) which seems unavoidable in this context, but is of negligible order compared with the contributions of the field operators, as we shall see below.

6.2 Nontriviality

While Thm. 6.1 asserts that our construction yields a large variety of valid quantum inequalities, there remains the concern that they could be trivial in the sense that the lower bound could also serve as an upper bound, cf. [F ew07]. In particular, an inequality for a bounded operator \( A \) of the form \( A \geq -\|A\|_1 \) would be considered trivial. Since the exponent \( \ell \) in Eq. (6.8) is so large that all \( R^\ell \phi_j R^\ell \) are bounded, we might well encounter this situation: The left-hand side of Eq. (6.8) might be dominated in norm by the remainder \( \epsilon(d) \). More generally, since Thm. 6.1 puts no further restrictions on the projector \( p \), it might also be possible that \( pT_\Pi \) contains single “redundant terms” that are individually dominated by \( \epsilon(d) \), and are thus essentially irrelevant.

We shall show now that if the approximating projector is chosen minimal, these problems do not occur, and in this sense the inequality is nontrivial.

**Theorem 6.2.** Let \( \Pi \in \Phi_{pos} \) and \( \alpha \geq 0 \). Let \( p \) be a minimal \( \alpha \)-approximating projector for \( \Pi \). Let \( V := \text{img} \ p \). For sufficiently large \( \ell > 0 \) and \( m \in \mathbb{N} \), and for any hermitean projector \( q : V \to V, \ q \neq 0 \), it holds that

\[
d^{-\alpha} \sup_{g \in D(-d,d)} \frac{\|\kappa[qpT_\Pi](\bar{g}, g)\|^{(-\ell)}}{\|g\|_{d,m}^2} \not\to 0 \quad \text{as } d \to 0.
\]

**Proof.** Suppose that \( m, \ell \) and a hermitean projector \( q : V \to V \) are given such that

\[
d^{-\alpha} \sup_{g \in D(-d,d)} (\|g\|_{d,m})^{-2} \|\kappa[qpT_\Pi](\bar{g}, g)\|^{(-\ell)} \to 0 \quad \text{as } d \to 0.
\]  (6.12)

We will show \( q = 0 \). First, we can use the polarization identity for the quadratic form \( \kappa[qpT_\Pi](g_1, g_2) \) in order to show

\[
d^{-\alpha} \|\kappa[qpT_\Pi]\|_{d,m}^{(-\ell)} \to 0.
\]  (6.13)

The triangle inequality then yields

\[
d^{-\alpha} \|\kappa[(1-q)pT_\Pi - T_\Pi]\|_{d,m}^{(-\ell)} \leq d^{-\alpha} \|\kappa[pT_\Pi - T_\Pi]\|_{d,m}^{(-\ell)} + d^{-\alpha} \|\kappa[qpT_\Pi]\|_{d,m}^{(-\ell)} \to 0,
\]  (6.14)

since \( p \) is \( \alpha \)-approximating; we suppose here that \( m, \ell \) are sufficiently large. Now (6.14) shows that \( (1-q)p \) is also \( \alpha \)-approximating for \( \Pi \). It is clear that \( (1-q)p \leq p \). Since however \( p \) is minimal, this implies \( (1-q)p = p \). Thus \( q = 0 \). \( \square \)
Again, let us illustrate the content of the theorem by passing to a basis representation of \( p \), as in Eq. (6.7). For the case \( q = 1 \), the theorem precisely shows that the left-hand side of Eq. (6.8) does not vanish in norm as fast as \( \epsilon(d) = o(d^{\alpha}) \). Further, choose \( q \) specifically as \( q = \sigma_k \phi_k \) with fixed \( k \). Then one obtains \( \kappa[qpT_\Pi](\tilde{g}, g) = \phi_k(f_k) \), with \( f_k \) as in Eq. (6.9). Thus, Thm. 6.2 provides us with a null sequence \( (d_i)_{i \in \mathbb{N}} \), a constant \( c > 0 \), and a sequence of functions \( g^{(i)} \in \mathcal{D}(-d_i, d_i) \) with \( \|g^{(i)}\|_{d_i,m} = 1 \) such that

\[
\|R^\ell \phi_k(f_k(s)) R^\ell\| \geq c (d_i)^\alpha \quad \text{for all } i \in \mathbb{N}, \tag{6.15}
\]

where

\[
f_k^{(i)}(s) = \int ds' \tilde{g}^{(i)} \circ g^{(i)}(s, s') \sigma_k(T_\Pi(s')). \tag{6.16}
\]

So the field \( \phi_k \) in the inequality (6.8) gives a contribution that is large compared to the remainder \( \epsilon(d) \). Theorem 6.2, in full generality, shows that this conclusion is true independent of the choice of basis.

We have argued in Prop. 5.7 that minimal \( \alpha \)-approximating projectors \( p \) exist for any product \( \Pi \), and any \( \alpha \geq 0 \). So we always obtain nontrivial quantum inequalities in the sense above. One might suspect here that the minimization of the approximating projector \( p \) could lead to \( p = 0 \), which might again be seen as trivial. While this is not the case even in a simple free field example, we shall give a general argument that shows that \( p = 0 \) cannot occur, under a mild extra assumption.

**Theorem 6.3.** Let \( \alpha \geq 0 \), and \( \Pi \in \mathcal{F}_{\text{FH}} \setminus \{0\} \). Suppose that the vacuum vector \( \Omega \) is separating for the smeared fields \( \phi(f) \), with \( \phi \in \mathcal{F}_{\text{FH}} \) and \( f \in \mathcal{D}(\mathbb{R}) \). If \( p \) is an \( \alpha \)-approximating projector for \( \Pi \), then \( p \neq 0 \).

We note that the condition of a separating vacuum vector is indeed a rather weak one. It would suffice, for example, that there exists a wedge region \( \mathcal{W} \) such that \( \Omega \) is cyclic for \( \mathfrak{A}(\mathcal{W}) \).

**Proof.** Suppose that \( \alpha \) and \( \Pi \) are given such that \( p = 0 \) is \( \alpha \)-approximating for \( \Pi \). We will show \( \Pi = 0 \). To that end, we choose \( \ell \) and \( m \) sufficiently large, and pick a fixed positive test function \( g \in \mathcal{D}(-1, 1) \). Then \( g_d := d^{-1}g(d^{-1} \cdot) \) lies in \( \mathcal{D}(-d, d) \), and \( \|g_d\|_{d,m} = \|g\|_{1,m} \). Employing Def. 5.6 we obtain

\[
\|\kappa[T_\Pi](\tilde{g}_d, g_d)\|^{(\ell)} \to 0 \quad \text{as } d \to 0. \tag{6.17}
\]

Evaluating the convolution integral in the vacuum state \( \omega \) yields due to translation invariance,

\[
\int ds \, ds' \, \tilde{g}_d \circ g_d(s, s') \omega(T_\Pi(s')) \to 0 \quad \text{as } d \to 0. \tag{6.18}
\]

As argued in the proof of Thm. 6.1, the distribution \( \omega(T_\Pi(s')) \) is of positive type. Hence it is the Fourier transform of a polynomially bounded positive measure \( \mu \). With this information, we can rewrite Eq. (6.18) as

\[
\int d\mu(p) |\tilde{g}_d(p)|^2 \to 0 \quad \text{as } d \to 0. \tag{6.19}
\]

However, as \( d \to 0 \), we have \( |\tilde{g}_d(p)|^2 \to |\tilde{g}(0)|^2 > 0 \) locally uniformly. Since \( \mu \) is positive, we can conclude here that \( \mu \) is the zero measure. So \( \omega(T_\Pi(s')) = 0 \) as a distribution. Using
Since all summands are of positive type, each of them must vanish individually; and clearly,  
also their analytic continuations must vanish. Thus, for any $j$, we have either $K_j = 0$ or  
$\omega(\phi_j^s U(-s')\phi_j) = 0$. But the latter implies $\|\phi_j(f)\Omega\| = 0$ for any $f$ of compact support; thus  
$\phi_j(f) = 0$ by assumption, and ultimately $\phi_j = 0$ by passing to a delta sequence. In total, this  
means $\Pi = 0$. \hfill \box

One might also be concerned that $p$ might project only onto multiples of the identity. Again, this does not occur in the simple example of the Wick square of the free field, as discussed in Sec. 2. In general, we conjecture, but have not proved, that in this case all fields  
appearing in the product $\Pi$ must be multiples of the identity. At the very least, one can show  
that the projector may be taken to be of the form $p = \omega(\cdot)1$, where $\omega$ is the vacuum state.  
If this $p$ is indeed $\alpha$-approximating for $\Pi$, the normal product of $\Pi$ can be defined by point  
splitting, and vanishes identically. So this does not seem to be a case of great interest.

### 6.3 Mesoscopic bounds

The inequalities derived above involve a remainder term that vanishes in the small distance  
limit. Here, we discuss how the remainder can be reduced for test functions of fixed supports,  
especially by forming a Riemann integral of the bounds at short distance.

Let $\chi \in \mathcal{D}(-1, 1)$ and $f \in \mathcal{D}(-d, d)$ be fixed nonnegative functions. We set $\chi_s(s) = \lambda^{-1}\chi(s/\lambda)$ for $\lambda \in (0, 1]$. As in Thm. 6.1, we suppose $p$ to be an $\alpha$-approximating projector for $\Pi \in \Phi_{\text{pos}}$, with $\alpha \geq 0$. The basic inequality of Thm. 6.1 applied to $\chi_s$, entails  
$$R^\ell \kappa[p T_{\Pi}](\chi_\lambda, \chi_\lambda) R^\ell \geq -\epsilon(\lambda)(\|\chi_\lambda\|_{1,m})^2 1 = -\epsilon(\lambda)(\|\chi\|_{1,m})^2 1 \tag{6.21}$$  
for suitable $\ell > 0$ and $m \in \mathbb{N}$, where $\epsilon(\lambda) = o(\lambda^\alpha)$. Applying a time-translation through $\lambda k$, multiplying by $\lambda f(\lambda k)$ and summing, we find  
$$\sum_{k \in \mathbb{Z}} \lambda f(\lambda k) U(\lambda k) \kappa[p T_{\Pi}](\chi_\lambda, \chi_\lambda) U(\lambda k)^* \geq -\epsilon(\lambda)(\|\chi\|_{1,m})^2 \sum_{k \in \mathbb{Z}} \lambda f(\lambda k) R^{-2\ell} \geq -\epsilon(\lambda)(\|\chi\|_{1,m})^2(\|f\|_1 + \lambda \|f'\|_1) R^{-2\ell} \tag{6.22}$$

Passing to a basis representation, we may rewrite this inequality in the form  
$$\sum_{j=1}^n \phi_j(F_{j,\lambda}) \geq -2\epsilon(\lambda)(\|\chi\|_{1,m})^2 \|f\|_{d,1} R^{-2\ell} \tag{6.23}$$

for $\lambda \leq d$ where  
$$F_{j,\lambda}(s) = \sum_{k \in \mathbb{Z}} \lambda f(\lambda k) \int ds' \sigma_j(T_{\Pi}(s')) \chi_\lambda \circ \chi_\lambda(s - \lambda k, s'). \tag{6.24}$$

Owing to the support properties of $\chi_\lambda$, at most two terms contribute to the sum on $k$ for each  
fixed $s$; moreover, $F_{j,\lambda} \in \mathcal{D}(-d, d)$.  

\newpage

24
In any fixed state in $\mathcal{C}^\infty(\Sigma)$ the expectation value of the right-hand side of (6.23) can be made arbitrarily small by reducing $\lambda$, while the behaviour of the terms on the right-hand side is determined by the asymptotic behaviour of the $F_{j,\lambda}$, regarded as compactly supported distributions. In the unlikely event that each $F_{j,\lambda}$ converged to a limit in the weak-* topology on $\mathcal{E}'(\mathbb{R})$, we would have established a quantum inequality without remainder term. It may be useful to give two examples. If the OPE coefficient $\sigma_j(T_\Pi(s'))$ is smooth, then convergence does occur, with

$$F_{j,\lambda} \to \sigma_j(T_\Pi(0))(\|\chi\|_1)^2 f \quad \text{in } \mathcal{E}'(\mathbb{R}) \text{ as } \lambda \to 0. \tag{6.25}$$

(To see this, one integrates against $u(s)$ and observes that the $k$'th summand is subject to only an $O(\lambda^2)$ error if $u(s_\sigma_j(T_\Pi(s'))) \to u_\sigma(T_\Pi(0))$; as there at most $O(\lambda^{-1})$ nonzero summands the result follows by a simple calculation.) On the other hand, if $\sigma_j(T_\Pi(s')) = (\imath \pi)^{-1}/(s' - \imath 0)$, we find

$$\lambda F_{j,\lambda} \to (\|\chi\|_2)^2 f \quad \text{in } \mathcal{E}'(\mathbb{R}) \text{ as } \lambda \to 0. \tag{6.26}$$

(Note that it is the $L^2$-norm that appears here, in contrast to the first example.)

In general, therefore, it cannot be expected that all of the $F_{j,\lambda}$ converge as $\lambda \to 0$. Nonetheless, as in the second example, its leading order behaviour in $\lambda$ can be identified as follows.

**Proposition 6.4.** Let $q$ be the order of the germ of $\sigma_j(T_\Pi(s'))$ at $s' = 0$ and define

$$\eta_j(\lambda) = \int ds' \sigma_j(T_\Pi(s'))(\chi \ast \hat{\chi})(s'), \tag{6.27}$$

where $\hat{\chi}(s') = \chi(-s')$. If $\lambda^{-q} \eta_j(\lambda)^{-1} = o(1)$ as $\lambda \to 0$ then

$$F_{j,\lambda}/\eta_j(\lambda) \to f \quad \text{in } \mathcal{E}'(\mathbb{R}) \text{ as } \lambda \to 0. \tag{6.28}$$

In particular, this is satisfied if $\sigma_j(T_\Pi(s'))$ has a scaling limit of degree $\beta < 0$ and $q = [-1 - \beta]$.

Here, the order of the germ of $\sigma(T_\Pi(s'))$ at $s' = 0$ is the minimal $q \in \mathbb{N}_0$ for which there are $\lambda_0 > 0$ and $C > 0$ such that $|\int ds' \sigma(T_\Pi(s'))u(s')| \leq C \sum_{r=0}^q \sup |u^{(r)}|$ for all $u \in \mathcal{D}(-\lambda_0, \lambda_0)$. The notion of scaling limit is taken from [FH87]: namely, the scaling limit exists if there exists a monotone positive function $N(\lambda)$ for which

$$N(\lambda) \int ds' \sigma_j(T_\Pi(s'))u_\lambda(s') \to S(u) \tag{6.29}$$

for all $u \in \mathcal{D}(\mathbb{R})$, with a nonzero limit for at least one $u$. Under these circumstances, $S$ is a homogeneous distribution, i.e., $S(u_\lambda) = \lambda^\beta S(u)$, with degree $\beta \in \mathbb{R}$ determined by

$$\lim_{\lambda \to 0} \frac{N(\lambda/\lambda')}{N(\lambda)} = \lambda^\beta. \tag{6.30}$$

(Our definition of the degree coincides with that of [GS68, Ch. I Sec. 1.6.] and differs from [FH87].) If $\beta < 0$, for example, the distribution $(s' - \imath 0)^\beta (\log s' - \imath 0)^\gamma$ has a scaling limit of degree $\beta$ and (germ) order $[-1 - \beta]$, and therefore meets the criteria stated.
Proof (of Prop. 6.4). We choose \( \lambda_0 \in (0, 1] \) sufficiently small that \( \sigma(T_{\Pi}(s')) \) has order \( q \) on \((-2\lambda_0, 2\lambda_0)\), and assume henceforth that \( 0 < \lambda < \lambda_0 \). As in the second example above, we integrate \( F_{j,\lambda} \) against \( u \in E(\mathbb{R}) \) and approximate \( u(s) \) by \( u(\lambda k) \) in the \( k \)’th summand, to obtain

\[
\int ds F_{j,\lambda}(s)u(s) = \int ds' \sigma_j(T_{\Pi}(s')) \sum_{k \in \mathbb{Z}} \lambda f(\lambda k) \int ds \chi(\lambda(s, s'))u(s + \lambda k) = \eta_j(\lambda) \sum_{k \in \mathbb{Z}} \lambda f(\lambda k)u(\lambda k) + R_{j,\lambda},
\]

(6.31)

where

\[
R_{j,\lambda} = \int ds' \sigma_j(T_{\Pi}(s')) \sum_{k \in \mathbb{Z}} \lambda f(\lambda k) \int ds \chi(\lambda(s, s'))[u(s + \lambda k) - u(\lambda k)].
\]

(6.32)

Now \( R_{j,\lambda} \) is, at worst, of order \( O(\lambda^{-q}) \) as \( \lambda \to 0 \), as is easily seen using the estimate

\[
\sup_s \left| \int ds' \sigma_j(T_{\Pi}(s')) \chi(\lambda(s, s')) \right| \leq \frac{C}{\lambda^{q+2}},
\]

(6.33)

and the facts that (i) the sum contains at most \( O(\lambda^{-1}) \) nonzero terms; (ii) the \( s \)-integral extends over the region \([-\lambda, \lambda]\). This establishes

\[
\int ds F_{j,\lambda}(s)u(s) = \eta_j(\lambda) \left( \int ds f(s)u(s) + O(\lambda) \right) + O(\lambda^{-q})
\]

(6.34)

as \( \lambda \to 0 \), from which (6.28) follows immediately.

Now suppose that \( \sigma_j(T_{\Pi}(s')) \) has a scaling limit of degree \( \beta < 0 \). It is easy to see that (6.29) implies \( N(\lambda)\eta_j(\lambda) \to S(\chi \ast \hat{\chi}) \). The spectrum condition entails that \( S = C(\nu_{(- \infty, 0)})^\beta \), where the nonzero constant \( C \) is real owing to hermiticity (cf. the proof of Prop. 7.2 below). As \( \beta < 0 \), we may verify directly that \( S(\chi \ast \hat{\chi}) \neq 0 \), that \( N(\lambda) \) is necessarily monotone decreasing and vanishing as \( \lambda \to 0 \). Thus \( \eta_j(\lambda) \to \pm \infty \) depending on the sign of \( C \). Moreover, Eqs. (6.29) and (6.31) entail

\[
\lim_{\lambda' \to 0} \frac{(\lambda')^q \eta_j(\lambda')}{(\lambda' \eta_j(\lambda')^2) = \lambda^{-\beta - q}}.
\]

(6.35)

By hypothesis, \( \sigma_j(T_{\Pi}(s')) \) has order \( q = \left[ -1 - \beta \right] \) (as does \( S \)). Thus \(-\beta - q > 0\) and we deduce that \( \lambda^{-q} \eta_j(\lambda)^{-1} \to 0 \) as \( \lambda \to 0 \).

The significance of this result becomes clear in the situation where one of the composite fields, say \( \phi_1 \), is identified as a field of particular interest, e.g., the normal product. By hermiticity of the projection \( p, \eta_1 \) is real-valued; the hypothesis of Prop. 6.4 requires that \( |\eta_1(\lambda)| \to \infty \) as \( \lambda \to 0 \). If, in fact, \( \eta_1(\lambda) \to +\infty \), we may divide the quantum inequality (6.28) by this factor to obtain a bound

\[
\phi_1(F_{1,\lambda}/\eta_1(\lambda)) + \frac{1}{\eta_1(\lambda)} \sum_{j=2}^{n} \phi_j(F_{j,\lambda}) \geq -\frac{2e(\lambda)}{\eta_1(\lambda)}(\|\lambda\|_{1,m})^2 \|f\|_{d,1} R^{-2\ell}
\]

(6.36)

for \( \lambda < d \). (If \( \eta_1(\lambda) \to -\infty \) we simply reverse the sign of \( \phi_1 \) and hence \( \sigma_1(T_{\Pi}(s')) \) and \( \eta_1 \) to obtain the same result; the possibility that \( \eta_1 \) oscillates in sign as \( \lambda \to 0 \) can be excluded if
the scaling limit exists.) In this form, it is clear that the remainder term may be diminished by reducing $\lambda$, at the possible cost of increasing the magnitude of the terms in composite fields with $j \geq 2$ (if $\eta_j(\lambda)$ grows more rapidly than $\eta_1(\lambda)$). Moreover, the expectation value of the first term tends to that of $\phi_1(f)$ as $\lambda \to 0$ for any state in $C^\infty(\Sigma)$.

Further progress is only possible with more detailed information regarding the (germs of the) OPE coefficient distributions. Nonetheless, we expect that the results presented here will be of use in the context of particular models.

7 Scaling limits and dilation covariance

For a concrete interpretation of our quantum inequalities, it is of particular interest to investigate the detail structure of the sampling functions with which the composite fields are smeared, e.g. the functions $f_j$ in Eq. (6.9). For example, one is interested whether they are pointwise positive, or at least “mostly positive” in a well-defined sense. Of course, these properties depend crucially on the structure of the OPE coefficients involved, about which little is known in the general case. The most reasonable approach therefore seems to investigate those properties under more restrictions on the theory.

In the preceding section, our approach was to approximate a given sampling function with a Riemann sum; this relied on some assumptions on the behavior of the OPE coefficients in the small, and was tied to a choice of basis in the field spaces. In the following, we want to take a different approach: We investigate the structure of sampling functions in a restricted class of quantum field theories, namely in the presence of dilation symmetries. While for a realistic description of microphysics, one would not consider dilation covariant quantum field theories, this case is still important as an idealization at short scales. Namely, in the short-distance regime, quantum field theories should be approximated by a scaling limit theory, which indeed possesses a dilation symmetry.

Let us briefly sketch how the scaling limit of quantum field theories fits into our context. It has been shown by Buchholz and Verch [BV95] that scaling limits can be formulated very naturally on the level of local algebras. Every quantum field theory possesses a scaling limit in this sense, although it might not be unique. The limit theory is, under a suitable choice of limit states, covariant under a strongly continuous unitary representation of the dilation group [BDM09]. However, the structure of these dilation unitaries may be very intricate, acting on a nonseparable Hilbert space. (See also [BDM08].)

In [BDM09], it was shown that this picture is compatible with the usual notion of field renormalization: If the original algebraic theory fulfils a slightly sharpened version of Def. 3.1 then the limit theory fulfils Def. 3.1 too; and pointlike fields in the original theory converge, under a multiplicative renormalization scheme, to pointlike fields in the limit theory. In a certain sense, the projectors $p_\gamma$ onto $\Phi_\gamma$ converge to corresponding projectors $p_\gamma^{(0)}$ in the limit theory. Also, this scheme is compatible with products of pointlike fields and operator product expansions. Thus one can expect that the structures exhibited in Sec. 6 properly converge in the scaling limit, and yield quantum inequalities in the limit theory.

Our aim here is neither to describe this passage to the limit theory in detail, nor to treat all possible cases of dilation group representations that may appear in the limit. Rather, we take the above as a motivation to investigate quantum inequalities in dilation covariant theories, and to show in certain simple cases that stricter classification results on the form of quantum inequalities can be achieved.
In the remainder of this section, we will therefore assume that our theory $\mathfrak{A}$ has a dilation symmetry; i.e., that there exists a strongly continuous unitary representation $\lambda \mapsto U(\lambda)$ of the dilation group on $\mathcal{H}$, which is compatible with the Poincaré group representation, and acts on the local algebras in the usual geometric way. The adjoint action of $U(\lambda)$ can then be extended to $C^\infty(\Sigma)^*$, where we write $\delta_\lambda \phi = U(\lambda)\phi U(\lambda)^*$ in the weak sense. The spaces $\Phi_\gamma$ are invariant under $\delta_\lambda$ [Bos05b, Sec IV]. We shall now consider the action of $\delta_\lambda$ on the structures considered so far, and introduce some definitions for convenience.

**Definition 7.1.** A quadratic form $\phi \in C^\infty(\Sigma)^*$ is called dilation covariant if, with some $\beta \in \mathbb{R}$,

$$
\delta_\lambda \phi = \lambda^\beta \phi \quad \text{for all } \lambda > 0.
$$

A product $\Pi \in \Phi_{\text{prod}}$ is called dilation covariant if, with some $\beta \in \mathbb{R}$,

$$
\delta_\lambda \Pi(s) = \lambda^\beta \Pi(\lambda s) \quad \text{for all } \lambda > 0, \text{ in the sense of distributions.}
$$

A projector $p$ in $C^\infty(\Sigma)^*$ is called dilation covariant if

$$
\delta_{1/\lambda} \circ p \circ \delta_\lambda [\mathfrak{A}(\mathcal{O}_1)] = p[\mathfrak{A}(\mathcal{O}_1)] \quad \text{for all } 0 < \lambda \leq 1,
$$

where $\mathcal{O}_1$ is the standard double cone of radius 1.

Note that the restriction to $\mathfrak{A}(\mathcal{O}_1)$ in the definition of dilation covariant projectors is unavoidable if we want $p$ to be norm-bounded on $\mathfrak{B}(\mathcal{H})$. Namely, suppose that $\delta_{1/\lambda} \circ p \circ \delta_\lambda (A) = p(A)$ for all $A \in \mathfrak{B}(\mathcal{H})$ and $0 < \lambda \leq 1$, and hence for all $\lambda$ by the group relation. Since $\delta_\lambda$ acts as a norm isomorphism on $\mathfrak{B}(\mathcal{H})$, norm-boundedness of $p$ would lead to $\delta_\lambda$ being uniformly bounded on the finite dimensional space $\text{img} \, p$, both for $\lambda \to 0$ and for $\lambda \to \infty$, which would exclude that $\text{img} \, p$ contains fields with nonzero scaling dimension.

Dilation covariant products can easily be constructed, e.g. by choosing dilation covariant fields $\phi_1$, $\phi_2$, and setting $\Pi = (iz)^{-\beta'} \otimes \phi_1 \otimes \phi_2$ with some $\beta' \geq 0$. If $\Pi$ and $p$ are both dilation covariant, Def. 7.1 implies that

$$
\delta_\lambda p T_\Pi(s') = \lambda^\beta p T_\Pi(\lambda s') \quad \text{for } 0 < \lambda \leq 1 \text{ and for } s' \in [-1, 1]; \quad (7.1)
$$

that is, the equation holds when evaluated on test functions with support in $[-1, 1]$. This follows by approximating $T_\Pi$ with sequences of bounded local operators, as in the proof of Thm. 5.3.

We will now consider the form of quantum inequalities in our case, that is, investigate the structure of minimal approximating projectors $p$ and their subprojectors. We shall restrict here to the simplest case, where one deals with one-dimensional subrepresentations of $\delta_\lambda$. In this case, we can find a full classification of our quantum inequality terms.

**Proposition 7.2.** Let $\Pi \in \Phi_{\text{pos}}$ be dilation covariant. Let $p$ be a one-dimensional dilation covariant projector in $C^\infty(\Sigma)^*$. Then, there exist a dilation covariant field $\phi \in \Phi_{\text{FH}}$ and $\beta \in \mathbb{R}$ such that

$$
p T_\Pi(s') = (i(s' - s0))^\beta \phi \quad \text{on the interval } (-1, 1).
$$

**Proof.** We choose $\phi \in \Phi_{\text{FH}}$ and $\sigma \in C^\infty(\Sigma)$ such that $p = \sigma(\cdot) \phi$. Since $\sigma(\phi) = 1$, and since $\phi$ can be approximated by bounded operators as in Thm. 3.2, we can find $A \in \mathfrak{A}(\mathcal{O}_1)$ such that $\sigma(A) = 1$. Using that $p$ is dilation covariant, we obtain

$$
\sigma(\delta_\lambda A) \delta_{1/\lambda} \phi = \sigma(A) \phi = \phi \quad \text{for all } 0 < \lambda \leq 1, \quad (7.2)
$$

28
and thus
\[ \delta \lambda \phi = \sigma(\delta \lambda A) \phi = c(\lambda) \phi \quad \text{for all } 0 < \lambda \leq 1. \] (7.3)

Here the \( \mathbb{C} \)-valued function \( c(\lambda) \) is continuous in \( \lambda \) and fulfils \( c(1) = 1, c(\lambda)c(\lambda') = c(\lambda\lambda') \) if \( \lambda, \lambda' \in (0,1] \). This suffices to conclude that there exists a \( \beta_1 \in \mathbb{C} \) such that
\[ c(\lambda) = \lambda^{\beta_1} \quad \text{for all } 0 < \lambda \leq 1. \] (7.4)

Due to the group relation, we then obtain for all \( \lambda \in \mathbb{R}_+ \),
\[ \delta \lambda \phi = \lambda^{\beta_1} \phi. \] (7.5)

Splitting \( \phi = \phi_R + i\phi_I \) into real and imaginary parts, we note that \( \delta \lambda \) preserves this splitting, which means that \( \beta_1 \) must be real. So \( \phi \) is dilation covariant. Inserting into Eq. (7.1), we arrive at
\[ \sigma(T_{\Pi}(s')) = \lambda^{\beta_2 - \beta_1}(T_{\Pi}(\lambda s')) \quad \text{in the sense of } \mathcal{D}(-1,1)', \] (7.6)

where \( \beta_2 \in \mathbb{R} \) is the exponent relating to \( \Pi \). Using the right-hand side as a definition for \( |s'| > 1 \), we can construct a homogeneous distribution \( D \in \mathcal{D}(\mathbb{R}) \) of degree \( \beta := \beta_1 - \beta_2 \) such that
\[ D(s') = \sigma(T_{\Pi}(s')) \quad \text{in the sense of } \mathcal{D}(-1,1)'. \] (7.7)

The homogeneous distributions of one variable are however fully classified (cf. GS68 Ch. I Sec. 3.11.): They are of the form
\[ D(s') = c_+(s' + i0)^\beta + c_-(s' - i0)^\beta \quad \text{with } c_\pm \in \mathbb{C}. \] (7.8)

We can further restrict the possible form of \( D \). Since \( \sigma \) can be approximated by energy-bounded functionals \( \sigma_E \), and \( \sigma_E(T_{\Pi}(s')) \) has an analytic continuation to the lower half-plane, the only singular direction (in the sense of wave front sets) of \( \sigma(T_{\Pi}(s')) \) at 0 can be the positive half-line. Since the wave front set is determined locally, Eq. (7.7) entails that \( c_+ = 0 \). Absorbing a factor \( i^{-\beta}c_- \) into the field \( \phi \), we finally obtain
\[ pT_{\Pi}(s') = (i(s' - i0))^{\beta} \phi \quad \text{on the interval } (-1,1), \] (7.9)
as proposed.

Now in the above situation, we can easily describe the quantum inequality terms that arise. One finds for any \( g \in \mathcal{D}(-1,1) \),
\[ \kappa[pT_{\Pi}](\bar{g}, g) = \phi(f) \quad \text{with } f(s) = \int ds' (i(s' - i0))^{\beta} \bar{g} \circ g(s, s'). \] (7.10)

This expression would not represent the entire quantum inequality, as approximating projectors will typically not be one-dimensional. Rather, (7.10) would represent one of the summands of the inequality in Eq. (6.8). In typical cases, one may expect that there exists a distinguished highest-order term in the operator product expansion, which corresponds to the “normal product” part of \( \Pi \), and which is described by a one-dimensional dilation covariant projector as above.

\footnote{As mentioned in Sec. 6.3, alternative conditions that force a distribution in the scaling limit to be homogeneous are discussed in FH87.}
Note that Prop. 7.2 determines the field \( \phi \) uniquely. In particular, for \( \beta \leq 0 \), requiring the distributional factor to be of positive type fixes the phase factor of \( \phi \). While other conditions might be used to restrict this phase factor, such as demanding that \( \phi \) be hermitean, the quantum inequalities give a stronger restriction that even fixes a \( \pm \) sign in \( \phi \). In this sense, our quantum inequalities can be used to distinguish the normal square of a field from its negative; squares of fields retain certain aspects of positivity in the quantum case.

Let us further investigate the structure of the smearing function \( f \) obtained in Eq. (7.10). We assume for a moment that \( g \) is real-valued, and thus \( g \circ g(s,s') \) is symmetric in \( s' \). By a standard computation [GS68 Ch. I §3 Nr. 8], one obtains the following simplified expressions in terms of convergent integrals:

\[
f(s) = 2 \cos \frac{\beta \pi}{2} \int_0^{\infty} ds' (s')^\beta g \circ g(s,s') \quad \text{for } \beta > -1, \tag{7.11}
\]

\[
f(s) = 2 \cos \frac{\beta \pi}{2} \int_0^{\infty} ds' (s')^\beta \left( g \circ g(s,s') - \frac{1}{(2k)!} \frac{\partial^{2k}g \circ g}{(\partial s')^{2k}} \bigg|_{s'=0} s'^{2k} \right) \quad \text{for } \beta < -1, |\beta| \not\in 2\mathbb{N} + 1, \tag{7.12}
\]

\[
f(s) = \frac{(-1)^k \pi}{(2k)!} \frac{\partial^{2k}g \circ g}{(\partial s')^{2k}} \bigg|_{s'=0} \quad \text{for } \beta = -2k - 1, k \in \mathbb{N}_0. \tag{7.13}
\]

Using these explicit characterizations, we can directly investigate the positivity properties of the function \( f \). For reasons of simple interpretation, it would be convenient if the \( f(s) \) are positive at each \( s \). We can give some sufficient conditions to this end.

**Proposition 7.3.** Let \( g \in \mathcal{D}(\mathbb{R}) \), \( \beta \in \mathbb{R} \), and \( f \) be given as in Eq. (7.10). If any of the following conditions is fulfilled, it follows that \( f(s) \geq 0 \) for all \( s \in \mathbb{R} \).

(i) \(-1 < \beta \leq 1 \), and \( g(t) \geq 0 \) for all \( t \in \mathbb{R} \).

(ii) \( \beta = -1 \), and \( g \) is real-valued.

(iii) \(-3 < \beta < -1 \), \( \text{supp } g \) is a connected interval \( I \), and \( g \) is logarithmically concave within \( I \).

**Proof.** The case (i) follows immediately from Eq. (7.11). In case (ii) we obtain \( f(s) = \pi g(s)^2 \) from Eq. (7.13), which yields the result. For (iii) observe that in this case Eq. (7.12) reads

\[
f(s) = 2 \left| \cos \frac{\beta \pi}{2} \int_0^{\infty} ds' (s')^\beta \left( g(s)^2 - g(s + s'/2)g(s - s'/2) \right) \right. \tag{7.14}
\]

Now the concavity of \( t \mapsto \log g(t) \) precisely implies that \( g(s)^2 \geq g(s + s'/2)g(s - s'/2) \) for any \( s \) and \( s' \).

The case \( \beta = -1 \) corresponds to the leading order of the OPE in the Wick square of a massless free field theory, as discussed in Sec. 2. Our main interest is therefore in the case where \( \beta \) is near \(-1 \), which might be expected in asymptotically free theories. This realm is covered in the above proposition. In models, it might be possible to exploit the choice of positive-type kernels \( K_j \) in the definition of \( \Pi \) in order to arrive at precisely the case \( \beta = -1 \), so that the function \( f = g^2 \) has a simple interpretation. We do however not investigate this possibility in detail here.
In more generality, for any $\beta \leq 0$, we can at least state the following more qualitative result: Since $(\nu(s' - x))^{\beta}$ is of positive type, one finds

$$\int ds f(s) \geq 0, \quad (7.15)$$

so $f$ has at least a non-negative average, regardless of the choice of $g$. A bit more generally, one can deduce Gårding inequalities for $f$, similar to those familiar from quantum mechanics [EFV05]: For suitable test functions $\chi$, one has

$$\int \chi(s)f(s)ds \geq -c\chi(\|g\|_2)^2. \quad (7.16)$$

Thus positivity of the test function $f$ is preserved at least in a generalized sense.

8 Conclusions and Outlook

We have shown that quantum field theories obeying the microscopic phase space condition of [Bos05b] admit a large class of nontrivial quantum inequalities: to every classically positive expression, i.e., a sum of absolute squares, we find a combination of composite fields that is positive up to an error obeying defined estimates and vanishing in the short distance limit. The composite fields appearing in such QIs are smeared with test functions derived from OPE coefficients as well as a choice of test function $g$. In the free field case, these smearing functions bore a simple relationship to $g$, at least for the normal product; here, the relationship is less direct, although we have succeeded in classifying their structure under simplifying assumptions within dilation covariant theories. Our inequalities are primarily valid in the short-distance limit, when the support of the test functions shrinks to a point. However, we also discussed how to obtain inequalities for smearing functions with extended (mesoscopic) support, in which the remainder term can be reduced at the expense of increasing the contributions from other composite fields.

To conclude we mention a number of open questions and avenues for further investigation. First, more progress can be made in understanding the sampling functions arising. For example, in the dilation covariant setting, one could also allow general finite-dimensional irreducible representations of the dilation group. Second, it would probably not be hard to generalise our bounds from smearing along a fixed timelike inertial curve to smearing along arbitrary smooth timelike curves in Minkowski space. The structure of inequalities is not expected to change significantly under this generalization. Third, one would also like to establish OPE-based quantum inequalities in curved spacetime. Here, the situation is complicated by the lack of a global Hamiltonian to specify scales of spaces of states and fields. A replacement for the topologies thus induced might be found in the detailed microlocal structure of $n$-point functions, for example, using wave-front sets modulo Sobolev regularity (see, e.g., [JS02]). An alternative approach would be to use the stress-energy tensor as the basis for estimates of high-energy behaviour. Hollands has recently established an OPE on curved spacetime for perturbatively constructed theories [Hol07]; however, the generalization of the nonperturbative methods used here presently remains a challenging problem.

Fourth, it would be desirable to obtain results that directly constrain the energy density of a quantum field theory, returning to the original motivation for quantum inequalities. One may heuristically expect from perturbation theory that the energy density in purely bosonic
theories does arise from such a sum of squares (although a generalization would be needed to cater for theories with fermionic fields) and would therefore be amenable to our approach. However, more direct connections to the energy density are unknown at present; in fact, the very concept of energy density is not well established in a nonperturbative context in purely Minkowski space quantum field theory. More generally, no general nonperturbative version of the Noether theorem has been found to date. In the Wightman framework, only very few results about pointlike Noether currents are available [Orz70, Lop91], in particular an existence proof is missing. In the algebraic framework, partial results have been achieved [BDL86] on the base of the so-called split property of the local algebras [Dop82, DL83]. In effect, it is possible to construct “local” energy operators $H_{\mathcal{O},\hat{\mathcal{O}}}$, which are associated with the observable algebra $\mathfrak{A}(\mathcal{O})$ of a bounded region $\mathcal{O}$ and act like the global Hamiltonian on $\mathfrak{A}(\hat{\mathcal{O}})$ for a slightly smaller region $\hat{\mathcal{O}} \subset \subset \mathcal{O}$. These operators fulfil $H_{\mathcal{O},\hat{\mathcal{O}}} \geq 0$, which may be interpreted as a very weak form of energy inequality: Starting from local integrals of the energy density, it seems always possible to add appropriate “boundary terms”, associated with $\mathcal{O} \cap \hat{\mathcal{O}}'$, such that the resulting operator $H_{\mathcal{O},\hat{\mathcal{O}}}$ is positive. However, there is no explicit control on these boundary terms, not even a means of separating them from a “main term”, so that this approach does not yet lead to a meaningful interpretation in terms of quantum energy inequalities.

In curved spacetime, however, the situation is better. Brunetti, Fredenhagen and Verch have shown the existence of a stress-tensor in locally covariant quantum field theories obeying the time-slice axiom [BFV03]. This stress-energy tensor is obtained by functional differentiation with respect to metric perturbations. This prevents an immediate identification of the energy density as a sum of absolute squares of basic fields. Nevertheless, this may serve as a starting point for future study.

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