Non Commutative Geometry of Tilings and Gap Labelling

Johannes Kellendonk∗

Department of Mathematics, King’s College London, Strand, London WC2R 2LS

Abstract

To a given tiling a non commutative space and the corresponding $C^*$-algebra are constructed. This includes the definition of a topology on the groupoid induced by translations of the tiling. The algebra is also the algebra of observables for discrete models of one or many particle systems on the tiling or its periodic identification. Its scaled ordered $K_0$-group furnishes the gap labelling of Schrödinger operators. The group is computed for one dimensional tilings and Cartesian products thereof. Its image under a state is investigated for tilings which are invariant under a substitution. Part of this image is given by an invariant measure on the hull of the tiling which is determined. The results from the Cartesian products of one dimensional tilings point out that the gap labelling by means of the values of the integrated density of states is already fully determined by this measure.

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∗email: johannes@mth.kcl.ac.uk
Introduction

The spatial structure of crystal solids is usually described by point lattices, with each point representing an elementary cell. This cell encodes the local arrangement of the atoms. To get insight into the physical properties of the crystal one may study discrete models in which a finite dimensional Hilbert space is assigned to each elementary cell. In particular the typical features of particle spectra arising from translational symmetry already show up in these models. More generally the spatial structure of a solid may be described by a tiling (its three dimensional analog), which is a covering of the Euclidean space with tiles which do not overlap. In this more general setting, too, it is expected that characteristic features of the solid which depend on the spatial structure show up in the analysis of discrete models over the tiling provided there is some sort of order in the tiling.

The most famous tilings lacking translational symmetry are the non periodic self-similar ones where in many cases the order is manifested in a substitution procedure, which is a kind of self-similarity transformation. Amongst these the Penrose tilings are very...
well known not only because they have nice mathematical properties but also since a three dimensional generalization of them \[2, 3, 4\] serves as an idealized model for certain quasicrystals, which have first been observed in 1984 \[5\]. Quasicrystals have a long range orientational order with a point symmetry which is not compatible with a periodic structure. Up to now quasicrystals with 8-, 10-, and 12-fold symmetry have been observed and correspondingly the interest in tilings with that kind of symmetry is particularly high.

The present work concerns the non commutative geometry of a tiling and as a direct application the gap labelling of discrete Schrödinger operators of quantum mechanical systems on the tiling. Gap labelling is meant here as a characterization of the spectrum of the Schrödinger operator which does not take the specific properties (such as the strengths of the bindings) into account but rather the order in the solid. It is based on the expectation that the spatial structure of the tiling influences the spectrum if it is implemented in the potential in a very general way, namely the value of the potential on a tile shall only depend on the local pattern of tiles around it. If, for instance, the tiling is self-similar and non periodic one typically expects a singular continuous spectrum which is a Cantor set of measure zero and which is self-similar, too, c.f. \[6\] and references therein as well as related articles from \[7\]. Rather generally speaking gap labelling is an the assignment of labels, as elements of a countable set, to the gaps of a Schrödinger operator. A useful candidate for this set is given by the values of the integrated density of states, since it is an ordered set and allows to assign a relative location in the spectrum to different gaps. Also here the spatial structure shows up. Whereas periodic tilings lead to a finite set of gaps, the non periodic self-similar ones are expected to yield integrated densities of states with the values on gaps being dense in \([0, 1]\) and forming a self-similar set. Being interested in the generic features of the spectrum which depend on the spatial structure only we shall not choose a specific operator and compute its spectrum (which might in fact be to difficult to carry out) but rather use the abstract form of the gap labelling and construct an Abelian group from the non commutative geometry of the tiling. The gap labelling is then a map from the gaps of a Schrödinger operator into that group. This group is the scaled ordered \(K_0\)-group (or its image under a homomorphism) of the \(C^{*}\)-algebra which is constructed from the tiling and which may be understood as the algebra of observables. It should at best be in bijective correspondence to the gaps of a Schrödinger operator.

The above-mentioned self-similar tilings may also be of interest in other areas of physics where self-similarity is of importance, e.g. in conformal field theory. Their structure may be analysed by methods of non commutative geometry. In fact, the non commutative space of a Penrose tiling is the first example in A. Connes book \[8\] for the new concept of space. In this example Connes uses a map found by Robinson \[9\] from the set of all Penrose tilings to the set of 0,1-sequences satisfying the constraint \(\forall n : a_n + a_{n+1} \leq 1\) to construct a non commutative space and the corresponding \(C^{*}\)-algebra of a Penrose tiling. This has been the guiding line of our investigations, although the algebra obtained in \[8\] does not contain all translation operators and consequently also not the discrete Laplacian. A more careful look at the equivalence relation induced by translation leads to algebras which contain the discrete Laplacian.
However they are not finitely approximated and their ordered $K_0$-groups are not in all interesting cases (including the Penrose tilings) known to us. In order to obtain at least part of the gap labelling we have to rely on the $AF$-algebra which is determined by the self-similarity structure. It allows us in particular to determine a measure for the frequency of patterns in the tiling. For Cartesian products of one dimensional tilings this measure already determines the gap labelling by means of the values of the integrated density of states, and we expect this to hold for more general tilings, too.

The first section contains the definition of a tiling and its role as an underlying space for a discrete model of a solid. Furthermore the gap labelling as developed by Bellissard \cite{10, 11} is briefly explained.

The non commutative geometry of a tiling is discussed in section two. The non commutative space of a tiling is the quotient of its hull by the equivalence relation induced by translations of a tiling as a whole. It is non Hausdorff in the non periodic case. The algebra $A_T$ of the tiling $T$ is the groupoid-$C^*$-algebra of the groupoid defined by the equivalence relation. For its definition a topology on the groupoid is introduced which is of subtle importance and which is well known if the tiling can be identified with a mapping from a discrete group into the set of (congruence classes of) tiles. The invariants of $A_T$, e.g. the $K$-groups, may be used to characterize the tiling and to distinguish it from others.

As a first application some general considerations about one dimensional tilings and Cartesian products thereof are made. We are mainly interested in non periodic tilings and just briefly touch on periodic ones. The results concerning the algebra and $K$-groups of Cartesian products may be expressed entirely in terms of the one dimensional components.

To obtain more concrete results and treat also tilings having 8-, 10-, and 12-fold orientational symmetry we restrict in the fourth section to tilings which are invariant under a substitution. This is used to construct the analog of a Robinson map $\Xi$ which provides us with an $AF$-algebra $A_\Sigma$ also naturally assigned to the tiling. It allows us to determine the invariant measure on the hull of a tiling which measures the relative frequency of patterns. Two different kind of generalizations of the Fibonacci chain and the Penrose tilings follow as examples.

In the last section we compare the two $C^*$-algebras obtained. This not only sheds some light onto the role of symmetry axes which appear as an obstruction to approximating the Laplacian by elements of $A_\Sigma$ in the von Neumann topology but also allows us to prove that Shubin’s formula holds for the tilings under investigation. We include a paragraph on substitution sequences (one dimensional substitution tilings), for which the ordered $K_0$-groups of $A_T$ and $A_\Sigma$ coincide. This is shown by establishing the Vershik transform \cite{12, 13} which turns out to be particularly simple if constructed using the path space determined by the substitution.
1 Gap labelling of Schrödinger operators on tilings

A $d$ dimensional tiling $\mathcal{T}$ of $\mathbb{R}^d$ is a complete covering of $\mathbb{R}^d$ by tiles which do not overlap. A tile shall here be polyhedron in which a point is distinguished which we call the puncture of the tile. Occasionally we allow a tile to be marked, e.g. by arrows, to break its symmetries. The congruence class of a tile under Euclidean transformations is usually called a prototile and the term oriented prototile will be used for an equivalence class under translations only. Several tiles of a tiling make up a pattern. Correspondingly there are pattern classes, i.e. congruence classes under all Euclidean transformations, and oriented pattern classes which are equivalence classes under translations only. The set of punctures of $\mathcal{T}$ is a subset of $\mathbb{R}^d$ and will be denoted by $\mathcal{T}_{pct}$. The set of punctures of an oriented pattern class $M$ is denoted by $M_{pct}$, and it may be identified with a subset of $\mathbb{R}^d$ once a representative for $M$ has been chosen. In fact $(M, x) \subset (T, y)$ shall denote the representative of the oriented pattern class $M$ which occurs in $T$ such that its puncture $x \in M_{pct}$ is at $y \in T_{pct}$.

In the case that $\mathcal{T}$ is one dimensional its tiles may be ordered so that $\mathcal{T}$ may be understood as a two-sided sequence over $\mathbb{Z}$ having values in the set of oriented prototiles. We simply write $\mathcal{T}_{pct} \cong \mathbb{Z}$ (isomorphic as ordered sets) in that case which however does not mean that $\mathcal{T}$ is periodic. Correspondingly, for Cartesian products of one dimensional tilings we would write $\mathcal{T}_{pct} \cong \mathbb{Z}^d$. Apart from these kind of tilings those with nontrivial orientational symmetry will be considered, since they are of particular importance as idealized models for the spatial structure of quasicrystals.

1.1 Discrete models of solids

As already mentioned in the introduction, the gap labelling is a qualitative characterization of the spectrum of a Schrödinger operator which incorporates the structure of the underlying space. The spatial structure of a solid may be described by a tiling. The atoms are located in the center of the tiles their type and distribution being encoded in the local pattern surrounding them. Since a tiling is to reflect only this spatial structure but not the details as e.g. the bindings – these would be taken care of in a specification of the Schrödinger operator – it should be expected that two tilings whose local patterns may be derived from each other by inspection of only finite patches lead to the same results for the gap labelling.

The Schrödinger operators for one or many particle problems are considered here in a discrete version. They are assumed to be local and to depend only on the pattern classes not exceeding a given finite size. In particular they are selfadjoint operators on $\ell^2(\mathcal{T}_{pct})$ and the above dependence on the pattern classes more precisely means that the action of such an operator $H$ with interaction radius $r$ is of the form (slightly

\footnote{Abusing the language we shall call higher and lower dimensional analogs of a tiling or a tile by the same name.}

\footnote{Actually, limits with respect to a certain $C^*$-norm could also be allowed.}
simplifying the notation):

\[
(H\psi)(x) = \sum_{x' \in M_{\text{pct}}} H_{xx'}\psi(x') \tag{1}
\]

where \((M, x) \subset (T, x)\) is the largest pattern of \(T\) at \(x\) which is covered by an \(r\)-ball around \(x\), and \(H_{xx'} \in \mathbb{C}\) depends only on the pattern class of \(M, x, \text{ and } x'\). The simplest tight binding model is \(H = -\Delta + V\) where \(\Delta_{xx'}\) is nonzero and equal to 1 if and only if the tile at \(x\) and the tile at \(x'\) have a common hypersurface and \(V\) is diagonal \(V_{xx}\) only depending upon the prototile which is represented at \(x\).

### 1.2 The gap labelling

Let us briefly describe how to assign elements of a countable Abelian group (the labels) which are insensitive under certain kind of perturbations to the possible gaps of the spectrum of a Schrödinger operator. This may be motivated by Shubin’s formula which formulates the integrated density of states (IDS) as a trace per volume evaluated on spectral projections of the operator; but this formula is not a necessary prerequisite. A more detailed description of the gap labelling may be found in the works of Bellissard [10, 11, 14].

Let \(H\) be a bounded selfadjoint operator on \(\ell^2(T_{\text{pct}})\) of the kind described above and \(\chi(H \leq E)\) its spectral projection onto the space spanned by eigenfunctions of eigenvalues smaller or equal to \(E\). Denote by \(\chi_\Lambda\) the orthogonal projection onto \(\ell^2(\Lambda_{\text{pct}})\), the space of wavefunctions vanishing outside a finite pattern \(\Lambda\) of \(T\) and set \(H_\Lambda = \chi_\Lambda H \chi_\Lambda\). Let \(|\Lambda_{\text{pct}}|\) denote the number of tiles in \(\Lambda\). To define the notion of an IDS on infinite dimensional Hilbert spaces one makes the ansatz

\[
N_H(E) = \lim_{\Lambda \to T} \frac{1}{|\Lambda_{\text{pct}}|} \text{Tr}(\chi(H_\Lambda \leq E)) \tag{2}
\]

the limit being defined by an increasing chain \(\Lambda_n \subset \Lambda_{n+1} \subset \cdots\) of patterns of \(T\) which approximate \(T\) and \(\text{Tr}\) being the usual operator trace on \(\mathcal{B}(\ell^2(T_{\text{pct}}))\); hence \(\text{Tr}(\chi(H_\Lambda \leq E))\) equals the number of eigenstates of \(H_\Lambda\) to eigenvalues smaller or equal than \(E\). This notion of an IDS is very important in solid state physics and is at least in principal accessible by experiments, c.f. [10].

Questions of existence of the above limit and of its independence of the chain of patterns approximating \(T\) as well as of the boundary conditions chosen for the finite approximant \(H_\Lambda\) (here Dirichlet boundary conditions are used) have not yet been answered in full generality. However at least for the case that \(T_{\text{pct}}\) may be identified with an amenable group and \(E\) lies in a gap may the r.h.s. of (2) be understood as image of a trace evaluated on \(\chi(H \leq E)\). More precisely does

\[
\text{tr}(a) = \lim_{\Lambda \to T} \frac{1}{|\Lambda_{\text{pct}}|} \text{Tr}(\chi_\Lambda a), \tag{3}
\]
define in that case a normalized trace on a suitable subalgebra of $B(\ell^2(T^{\text{pct}}))$, which is proved by Birkhoff’s ergodicity theorem, and amenability implies \[10\]

$$\lim_{\Lambda \to T} \frac{1}{|\Lambda|} \text{Tr}(\chi(H_\Lambda \leq E) - \chi_{\Lambda \chi}(H \leq E)) = 0. \quad (4)$$

A discrete group $G$ is amenable if there exists a sequence of bounded subsets $\{\Lambda_n\}_{n \geq 1}$ approximating $G$, i.e. $\bigcup_{n \geq 1} \Lambda_n = G$ such that $\forall g \in G : \frac{|(\Lambda_n \setminus g\Lambda_n) \cup (g\Lambda_n \setminus \Lambda_n)|}{|\Lambda_n|} \to 0$ as $n \to \infty$. What is important here is not so much the group structure but the fact that the volume of the border of $\Lambda$, i.e. the number of elements which are just at the border of $\Lambda_n$, is negligible compared to the total volume of $\Lambda_n$ if $n$ increases. Therefore Bellissard’s proof of \[4\] carries over to any tiling of the kind considered further down. \[3\] together with \[4\] yield Shubin’s formula

$$\mathcal{N}_H(E) = \text{tr}(\chi(H \leq E)). \quad (5)$$

Clearly $\mathcal{N}_H(E)$ is a monotonically increasing function of $E$ which is constant on gaps. The unital $C^*$-algebra $C(H)$ generated by the selfadjoint $H$ is isomorphic to $C(\sigma(H))$, the complex continuous functions over the spectrum of $H$. If $E$ lies in a gap, i.e. if $E \notin \sigma(H)$, then the characteristic function $\chi_{\{\lambda \in \sigma(H) : \lambda \leq E\}} \in C(\sigma(H))$ and therefore $\chi(H \leq E) = \chi_{\{\lambda \in \sigma(H) : \lambda \leq E\}}(H) \in C(H)$, whereas for $E \in \sigma(H)$ the spectral projection $\chi(H \leq E)$ may in general only be found in the von Neumann closure of $C(H)$. Now if $A$ is $C^*$-algebra with a normalized trace $\text{tr}$ having a representation $\pi$ on $\ell^2(T^{\text{pct}})$ such that $H = \pi(h)$ for some selfadjoint $h \in A$ and Shubin’s formula holds with that $\text{tr}$, then for $E \notin \sigma(H)$

$$\mathcal{N}_H(E) = \text{tr}(\chi_{\{\lambda \in \sigma(H) : \lambda \leq E\}}(h)) \in \text{tr(Proj}(A)) \quad (6)$$

as the trace of a projection depends only on its equivalence class. Two projections are equivalent $p \sim q$ whenever $\exists u \in A : p = uu^*, q = u^*u$, and $\text{Proj}(A)$ denotes the set of equivalence classes. This beautiful result shows a remarkable property of the IDS: the values of the IDS on gaps are stable under perturbations of $H$ which are connected to $H$ by a continuous path in $A$ (continuity is meant here in the norm resolvent sense, c.f. \[15\]). This together with the countability of $\text{tr(Proj}(A))$ for separable $A$ follow from the fact that two projections $p, q$ which are close to each other in the sense of $\|p - q\| < 1$ are equivalent. Hence the values of the IDS on gaps furnish a countable set of labels for the gaps which is stable under certain perturbations.

To compute the possible values of the IDS on gaps, first, a suitable $C^*$-algebra has to be constructed. The set of its projection classes should be as small that at least most elements of $\text{tr(Proj}(A))$ will actually occur as labels for gaps. Of course, $A$ should be related to $H$, but since $\text{Proj}(A)$ does not refer to a specific operator it will only be possible to obtain statements for ”generic” operators. Actually, it is not only the gap structure of a specific discrete Schrödinger operator which is of interest but the generic features of a whole family. At least in simple models the spectrum of an operator over a space continuum may be locally determined by a whole family of discrete operators which act on the Hilbert spaces of the tight binding approximation \[10\].

Concerning now the choice of this algebra we take the point of view that it should be constructed purely from the spatial structure of the underlying tiling. In mathematical terms this is the \( C^* \)-algebra which is assigned to the non commutative space of the tiling, in physical terms it is the algebra of observables namely it is generated by translations and characteristic functions. In \([10]\), where tilings are considered which may be ordered like \( \mathbb{Z}^d \), a \( C^* \)-algebra is constructed starting from a generic Schrödinger operator and its translates. This approach is very closely related to the one considered here – e.g. the algebraic hull of the Schrödinger operator defined in \([10]\) is homeomorphic to the geometric hull discussed below – and leads in the comparable cases to the same algebra.

\( \text{Proj}(\mathcal{A}) \) is in general difficult to compute as it does not carry enough structure. If instead the stabilized algebra \( M_\infty(\mathcal{A}) \) is used the structure of an addition may be defined on its projection classes leading via Grothendieck construction to an Abelian group, the \( K_0 \)-group of \( \mathcal{A} \), c.f. the appendix. This group is naturally ordered, the projection classes of \( M_\infty(\mathcal{A}) \) being the positive elements, and scaled, the class of the unit \( 1 \in \mathcal{A} \) serving as order unit. The trace on \( \mathcal{A} \) induces a state \( \text{tr}_* \) on that group, i.e. a positive homomorphism into \( \mathbb{R} \) normalized to \( \text{tr}_*[1] = 1 \). This allows for a formulation of (6) which may be slightly weaker:

\[
\mathcal{N}_H(E) \in \text{tr}_*(K_0(\mathcal{A})) \cap [0,1],
\]

if \( E \) lies in a gap. This is part of the abstract gap labelling theorem of Bellissard. In many interesting cases the r.h.s. may be computed yielding a set of possible gap labels. In some cases one can show by different methods – being more and less rigorous – that any element of \( \text{tr}_*(K_0(\mathcal{A})) \cap [0,1] \) actually occurs as a value of the IDS on a gap of a generic operator \([10]\text{[17, 18]}\). The period doubling substitution is an example where this agreement could be shown rigorously \([17]\). In case \( K_0^+(\mathcal{A}) \) is already given by \( \{ z \in K_0(\mathcal{A}) | \text{tr}_*(z) > 0 \} \cup \{0\} \) and \( \mathcal{A} \) has cancellation \([7]\) is in fact equivalent to \([1]\) namely \( \text{tr}(\text{Proj}(\mathcal{A})) = \text{tr}_*(K_0(\mathcal{A})) \cap [0,1] \).

As \( \text{tr}_* \) is a homomorphism of groups, \( \text{tr}_*(K_0(\mathcal{A})) \) is a subgroup of \( \mathbb{R} \) which is countable, if \( \mathcal{A} \) is separable. However it should be noted that the elements of \( K_0(\mathcal{A}) \) themselves yield a set of possible labels that are stable under perturbations as described above and which are ordered. This way of labelling the gaps is independent of the validity of the Shubin formula, but without such a formula a gap label cannot be related to the location of the gap in the spectrum. On the other hand, if \( \text{tr}_* \) is not injective, the gap labelling by elements of \( K_0(\mathcal{A}) \) is finer as the one by values of the IDS.

Remark: Even if the elements of \( \text{Proj}(\mathcal{A}) \) would lead to a finer gap labelling than the ones of \( K_0(\mathcal{A}) \) the stabilization of the algebra may be advantageous. In \([11]\) it is shown that for the case \( \mathcal{T}^{\text{pct}} \cong \mathbb{Z}^d \) the algebra assigned to \( \mathcal{T} \) has the same stabilization as the one which would be constructed for the non discrete problem. In other words, the spectrum of a generic operator defined on the space continuum is as well characterized by the elements of \( K_0(\mathcal{A}) \) as its tight binding approximation.
2 The non commutative geometry of a tiling

2.1 The non commutative space of a tiling

For a tiling \( T \) of the Euclidean space \( \mathbb{R}^d \) with distinguished origin \( 0 \in \mathbb{R}^d \) let \( M_r(T) \) denote the smallest pattern of \( T \) covering \( B_r(0) \), the \( r \)-ball of \( 0 \). \( T \) is called locally finite if \( M_r(T) \) contains for finite \( r \) only finitely many tiles. The Euclidean group of isometries, the semi-direct product of \( O(d) \) with the translations, acts on the set \( \mathcal{K} \) of all locally finite tilings of \( \mathbb{R}^d \) in a natural way. This action is denoted by \( x \cdot T \) or, if \( x \) is a translation, by \( T - x \). Define the coincidence radius of two tilings by \( r(T,T') := \sup \{ r' \geq 0 | M_{r'}(T) = M_{r'}(T') \} \). Then \( r(T,T'') \geq \min \{ r(T,T'), r(T',T'') \} \) for any third \( T'' \), and

\[
d(T,T') := \exp(-r(T,T'))
\]

defines a metric on \( \mathcal{K} \) which is invariant under \( O(d) \) but not under translations.

\( \mathcal{K} \) is complete: Let \( \{ T^{(i)} \}_{i \geq 1} \) be a Cauchy sequence in \( \Omega \). Then \( \forall r > 0 \exists N_r \forall n,m \geq N_r : d(T^{(n)}, T^{(m)}) < \exp(-r) \) and hence \( \forall n \geq N_r : M_r(T^{(n)}) = M_r(T^{(N_r)}) \). The increasing chain of patterns \( M_r(T^{(N_r)}) \subset M_r(T^{(N_r+1)}) \subset \cdots, r < r' \cdots \) defines a tiling of \( \mathcal{K} \) which is the limit of the Cauchy sequence.

\( \mathcal{K} \) is totally disconnected (zero dimensional): The metric topology is generated by the open \( \epsilon \)-balls

\[
U_\epsilon(T) = \{ T' \in \mathcal{K} | d(T,T') < \epsilon \} = \{ T' \in \mathcal{K} | r(T,T') > -\ln \epsilon \}.
\]

Local finiteness of \( T \) implies that the image of the function \( d(T,\cdot) : \mathcal{K} \to [0,\infty), T' \mapsto d(T,T') \) is discrete so that \( U_\epsilon(T) \) is as well a closed \( \epsilon + \delta \)-ball for sufficiently small \( \delta \). Hence the topology is generated by open and closed sets which is the definition of a zero dimensional space and implies total disconnectness.

Now consider a fixed tiling \( T \in \mathcal{K} \) having the puncture of one of its tiles sitting on \( 0 \in \mathbb{R}^d \), i.e. \( 0 \in T^{\text{pct}} \).

**Definition 1** The set

\[
\Omega := \{ T - x | x \in T^{\text{pct}} \}
\]

is called the hull of \( T \).

Eventually, we shall be more precise and write \( \Omega_T \) for the hull of \( T \). Clearly \( \Omega \) is finite if and only if \( T \) is periodic in \( d \) independent directions. As \( \Omega \) is the closure of a subset of \( \mathcal{K} \), it is complete. By construction any element \( T \) of \( \Omega \) can be approximated by translates of \( T \), i.e.:

\[
\forall \epsilon > 0 \exists x \in \mathbb{R}^d : d(T - x, T) < \epsilon.
\]

As a consequence, for any \( T \in \Omega_T \) and any pattern \( (M,x) \subset (T,y) \) there is a \( y' \in T^{\text{pct}} \) such that \( (M,x) \subset (T,y') \). In this sense \( T \) cannot be distinguished from \( T \) by inspection of its finite patterns only and is consequently called locally homomorphic to \( T \). If \( T \in \Omega_T \) then also \( T - x \in \Omega_T \) for all \( x \in T^{\text{pct}} \). Completeness of \( \Omega_T \) therefore implies that \( \Omega_T \subset \Omega_T \). If the converse is true as well, i.e. if \( \forall (M,x) \subset (T,y) \exists y' \in T^{\text{pct}} \)
meaning in that homeomorphic hulls are defined by locally equivalent tilings: Let
Although this topology might at first sight appear a little bit unusual, it has a physical
which for complete spaces is equivalent to compactness.

\[ U_{\exp(-r)}(T) \cap \Omega = U_{M_r(T),0} := \{ T' \in \Omega | M_r(T) = M_r(T') \}. \] (11)

More generally, the following sets which are indexed by oriented pattern classes and
one of their punctures

\[ U_{M,x} := \{ T \in \Omega | (M, x) \subset (T, 0) \} \] (12)

are open and closed, they will be of importance later on. It is clear that either
\( U_{M_r(T),0} = U_{M_r(T')},0 \) – namely if \( M_r(T) = M_r(T') \) – or \( U_{M_r(T),0} \cap U_{M_r(T'),0} = \emptyset \). We shall restrict
ourselves to tilings which lead to hulls that satisfy the following condition.

**B1** For any \( r \geq 0 \) there are only finitely many tilings \( T_i \in \Omega \) such that \( M_r(T_i) \) are
pairwise distinct.

This implies that \( \mathcal{T} \) has only finitely many oriented prototiles.

**Lemma 1** If \( \Omega \) satisfies B1 it is compact.

**Proof:** According to condition B1, to a given \( r \geq 0 \) there is a maximal finite family
\( \{ T_i \}_{i \in I} \) of tilings such that \( M_r(T_i) \) are pairwise distinct. Hence

\[ \Omega = \bigcup_{i \in I} U_{M_r(T_i),0} = \bigcup_{i \in I} \{ T' \in \Omega | d(T_i, T') < \exp(r_0 - r) \} \] (13)

for some \( 0 \leq r_0 < r \) so that \( \Omega \) has a finite cover of open \( \exp(r_0 - r) \) balls. By varying \( r \)
and \( r_0 \) the number \( \exp(r_0 - r) \) may take any small positive value so that \( \Omega \) is precompact
which for complete spaces is equivalent to compactness.

Although this topology might at first sight appear a little bit unusual, it has a physical
meaning in that homeomorphic hulls are defined by locally equivalent tilings: Let \( \mathcal{T}_i, i = 1, 2 \) be two tilings satisfying B1, write \( \Omega_i^0 = \{ \mathcal{T}_i - x \in \mathcal{T}_i^{\text{pct}} \} \), i.e. \( \Omega_{\mathcal{T}_i} = \Omega_i^{0} \), and
let \( Y : \mathcal{T}_1^{\text{pct}} \to \mathcal{T}_2^{\text{pct}} \) be a map. \( Y \) then defines a mapping \( Y : \Omega_1^0 \to \Omega_2^0 \)

\[ Y(\mathcal{T}_1 - x) := \mathcal{T}_2 - Y(x) \] (14)

which extends to the closures, if it is continuous. But continuity at \( T \in \Omega_1^0 \) means that

\[ \forall r_2 > 0 \exists r_1 > 0 : Y(U_{M_{r_1}(T),0} \cap \Omega_1^0) \subset U_{M_{r_2}(Y(T)),0} \cap \Omega_2^0, \] (15)

i.e. the pattern around 0 of \( Y(T) \) covering \( B_{r_2}(Y(0)) \) is already determined by the
pattern of \( T \) which covers \( B_{r_1}(0) \). One is used to say that \( Y(T) \) is locally derivable
from \( T \). If moreover \( Y \) is bijective and therefore a homeomorphism, then \( T \) and \( Y(T) \)
are called \textit{locally equivalent}. The spatial structure described by them should then lead
to the same physics \[19\] which is reflected here by the fact that they have homeomorphic hulls.

A special case of local equivalence arrises if one has two tilings which differ only by the puncture of their prototiles, as long as they are both generic. We call a puncture generic if it maximally breaks the symmetry of the prototile. Changing a generic puncture to a non generic one may result in a change of the hull as a topological space. Namely if a puncture \( x \in M^{\text{pct}} \) lies on a symmetry axis or centre of some symmetry \( g \in O(d) \) of the pattern \( M \) then \( U_{g \cdot M, g \cdot x} = U_{M, x} \), whereas otherwise \( U_{g \cdot M, g \cdot x} \cap U_{M, x} = \emptyset \).

**Lemma 2** The hulls defined by tilings which differ only in their puncture are homeomorphic, if both punctures are generic.

**Proof:** Let \( T_1, T_2 \) denote the tiling with the first resp. the second puncture and \( Y : T_1^{\text{pct}} \to T_2^{\text{pct}} \) be the map assigning the first puncture of a tile to the second. Since the punctures are generic to any oriented pattern class \( M_1 \) of \( T_1 \) corresponds a unique oriented pattern class \( M_2 \) of \( T_2 \) which differs only by the puncture. Therefore \( Y(U_{M_1, x} \cap \Omega_1^0) = U_{M_2, Y(x)} \cap \Omega_2^0 \) which shows that \( Y \) extends to a homeomorphism. \( \square \)

Punctures are always assumed to be generic.

Everything that will be treated in this article depends only on the topological structure of \( \Omega \). So we might in principle talk about the non commutative space resp. the algebra of a local equivalence class of a tiling, but we prefer the simpler expressions.

The action of the subgroup of translations restricted to \( \Omega \) yields an equivalence relation which we denote by \( R = \{(T, T') \in \Omega \times \Omega | \exists x \in \mathbb{R}^d : T' = T - x\} \).

**Definition 2** The non commutative space of the tiling \( T \) is the quotient

\[ \Psi := \Omega / \sim_R. \]

In other words, \( \Psi \) is the space of all tilings which are locally homomorphic to \( T \) modulo what may be called globally isomorphic ones. If \( T \) is periodic in \( d \) independent directions then \( \Psi \) is one point. However otherwise it is non Hausdorff with respect to the quotient topology, as any tiling \( T \) is arbitrary close to some translate of \( T \), cf. \([10]\). In a sense \( \Psi \) is then a point with an additional structure which is undetectable by commutative geometry \([3]\). A. Connes’ proposal to analyse such spaces leads via the construction of non commutative \( C^* \)-algebras to the study of their invariants, e.g. their \( K \)-groups. This will be the guiding line below.

A subclass of aperiodic tilings which are of particular interest for us are those which have an orientational symmetry \([20]\). An element \( g \in O(d) \) is called an orientational symmetry of \( T \) if \( \forall r \geq 0 \exists x \in T^{\text{pct}} : M_r(g \cdot T) = M_r(T - x) \). An element \( T \in \Omega \) is called symmetric under \( g \in O(d) \) if \( \exists x \in T^{\text{pct}} : g \cdot T = T - x \). The orientational symmetries yield a subgroup of \( O(d) \) which we shall denote by \( G \). If \( T \) satisfies condition B1 \( G \) is a finite group. If \( T \in \Omega \) then also \( g \cdot T \in \Omega \), provided \( g \in G \). Hence \( G \) acts on \( \Omega \), and this action is continuous as \( O(d) \) preserves the metric.
Remark: The notion of the local isomorphism class of a tiling has been introduced for the description of quasicrystals. A subclass thereof, the closed subset
\[ L_T = \{ T \in \mathcal{K} | \forall r \geq 0 \exists x, y \in \mathbb{R}^d : M_r(T-x) = M_r(T) \text{ and } M_r(T) = M_r(T-y) \} \]
of \( \mathcal{K} \) is closely related to the hull of \( T \). It may be shown that any element of the quotient space \( L_T/\text{transl.} \) contains a representative in \( \Omega_T \).

2.2 The algebra of a tiling

We now define a \( C^* \)-algebra \( \mathcal{A}_T \) to a given tiling \( T \). As already mentioned this algebra will be constructed from the non commutative space \( \Psi_T = \Omega_T/\sim_R \), namely it will be the groupoid-\( C^* \)-algebra of the equivalence relation. From a physicists point of view it is the algebra of observables. This algebra has been introduced in a similar context by J. Bellissard [10, 11]. It yields an application of the theory of groupoid \( C^* \)-algebras developed by Renault [21] which can be carried through once a suitable topology for the groupoid has been found.

A topological groupoid \( \Gamma \) is a topological space together with a continuous product map \((x, y) \mapsto xy: \Gamma^2 \to \Gamma \) which is however only defined on a subset \( \Gamma^2 \subset \Gamma \times \Gamma \) and a continuous inversion map \( x \mapsto x^{-1}: \Gamma \to \Gamma \), such that the following relations hold:

1. \((x^{-1})^{-1} = x\),

2. if \((x, y), (y, z) \in \Gamma^2 \) then \((xy, z), (x, yz) \in \Gamma^2 \) and \((xy)z = x( yz)\),

3. \((x^{-1}, x), (x, x^{-1}) \in \Gamma^2 \); moreover if \((x, y) \in \Gamma^2 \) resp. \( (z, x) \in \Gamma^2 \), then \( x^{-1}(xy) = y \) resp. \( (zx)x^{-1} = z\).

Consequently the elements of \( \Gamma^0 = \{ x^{-1}x | x \in \Gamma \} \) are called units.

Like any equivalence relation \( R \) is a groupoid, namely \( R^2 = \{ ((T, T'), (T'', T''')) | T \sim T' \sim T'' \in \Omega \} \) and the product is given by transitivity, \( (T, T')(T', T''') = (T, T''') \), and inversion by reflexivity \( (T, T')^{-1} = (T', T) \). Units are elements of the form \( (T, T) \), hence \( R^0 \cong \Omega \). The topology of \( R \) is defined to be the one generated by the sets
\[ U_{M,x,x'} := \{(T, T- (x'-x)) | T \in U_{M,x}\} \quad (16) \]
where \( M \) is an oriented pattern class of \( T \) and \( x, x' \in M^{\text{pet}} \). In this topology \( R \) is locally compact, \( R^0 \) open and \( R^T = \{ (T, T-x) | x \in T^{\text{pet}} \} \) discrete. Thus in the terminology used in [21] \( R \) is a principal \( r \)-discrete groupoid. Note that the above topology does not coincide with the relative topology inherited from \( \Omega \times \Omega \! \! \!. \)

On the space of all continuous complex functions with compact support \( C_c(R) \) one introduces a product and an involution by
\[ f \ast g (T, T') = \sum_{T'' \sim_T} f(T, T'') g(T'', T'), \quad (17) \]
\[ f^* (T, T') = \overline{f(T', T)} \quad (18) \]
thus obtaining a topological *-algebra. Restriction to functions with compact support
leads to a sum in [17] which contains only finitely many nonzero terms. $C_c(\mathcal{R})$ is
generated as a *-algebra by the functions $e_{M,x,x'}: \mathcal{R} \to \mathbb{C}$
\[ e_{M,x,x'}(T,T') := \begin{cases} 
1 & \text{if } (M,x) \subset (T,0) \text{ and } T' = T - (x' - x) \\
0 & \text{else}
\end{cases} \tag{19} \]
which correspond to the characteristic functions on $U_{M,x,x'}$. The $C^*$-algebra $\mathcal{A}_T$ associated to the tiling $\mathcal{T}$ is defined to be the completion of $C_c(\mathcal{R})$ with respect to the reduced norm. For the definition of this norm one considers the family of *-representations of $C_c(\mathcal{R})$, which we call physical representations, and which are labelled by a tiling $T \in \Omega$: $\pi_T$ acts on the Hilbert space $\ell^2(\mathcal{R}^T)$ via
\[ (\pi_T(f)\psi)(T') := \sum_{T'' \sim T'} f(T',T'')\psi(T''), \tag{20} \]
$\psi \in \ell^2(\mathcal{R}^T)$ and writing shorter $\psi(T')$ for $\psi(T,T')$. In such a representation $e_{M,x,x'}$ becomes translation by $x' - x$ in the following sense:
\[ (\pi_T(e_{M,x,x'})\psi)(T') = \begin{cases} 
\psi(T' - (x' - x)) & \text{if } (M,x) \subset (T',0) \\
0 & \text{else}
\end{cases}. \tag{21} \]
These representations are irreducible, in fact the commutant of $\pi_T(C_c(\mathcal{R}))$ in $\mathcal{B}(\ell^2(\mathcal{R}^T))$ may be seen to be trivial. The reduced norm of an element of $C_c(\mathcal{R})$ is the supremum of its operator norms in these representations, $\|f\|_{\text{red}} = \sup_T \|\pi_T(f)\|$, and the reduced groupoid-$C^*$-algebra $C^*_{\text{red}}(\mathcal{R})$ is the completion of $C_c(\mathcal{R})$ with respect to this norm. $\mathcal{A}_T = C^*_{\text{red}}(\mathcal{R})$ may be considered as the algebra of observables of a quantum mechanical system, as it is generated by translations.

**Remark:** The product on $C^*_{\text{red}}(\mathcal{R})$ which looks like matrix multiplication is in fact a convolution. To define it in the general case one has to introduce a Haar system on $\Gamma$. A left-Haar system on $\Gamma$ is a family of measures $\{\lambda^u, u \in \Gamma^0\}$ on $\Gamma$ that satisfy

1. the support of $\lambda^u$ is $\Gamma^u = \{x \in \Gamma | xx^{-1} = u\}$,
2. for every $f \in C_c(\Gamma)$ the map $u \mapsto \int f(x) \, d\lambda^u(x)$ is continuous,
3. (left-invariance) $\int f(xy) \, d\lambda^{x^{-1}}(y) = \int f(y) \, d\lambda^{xx^{-1}}(y)$ for all $x \in \Gamma$ and $f \in C_c(\Gamma)$.

A convolution and an involution may then be defined on $C_c(\Gamma)$ by
\[ f * g(x) = \int f(xy) g(y^{-1}) \, d\lambda^{x^{-1}}(y) \tag{22} \]
\[ f^*(x) = \frac{f(x^{-1})}{\|f\|_1} \tag{23} \]
In our case $\mathcal{R}^{(T,T)} = \mathcal{R}^T$ is discrete so that left invariance forces the measure $\lambda^T = \lambda^{(T,T)}$ on $\mathcal{R}^T$, if it is normalized to $\lambda^{(T,T)}(T,T) = 1$, to be the point measure $\{2\}$, i.e. $\lambda^T(T',T'') = \delta_{TT'}$. Then $\|T \mapsto T \mapsto 1\|$ becomes $\|T \mapsto T \mapsto 1\|$.

\(^3\) It is more convenient here to formulate this representation for wavefunctions over $\mathcal{R}^T$ the connection to those on $T^\text{pet}$ being obvious.
A trace on $\mathcal{A}_T$

With the help of an $\mathcal{R}$-invariant measure on $\Omega$ a trace on $\mathcal{A}_T$ can be defined. A measure $\mu$ on $\Omega$ is called invariant (or more precisely $\mathcal{R}$-invariant) if $\mu(U_{M,x})$ is independent of the choice of $x \in M^{\text{pct}}$. Let $P : C^*_\text{red}(\mathcal{R}) \to C(\Omega)$ be the restriction map. $P$ is a generalized conditional expectation, and provided $\mu$ is invariant

$$\text{tr}(f) := \int_{\Omega} P(f) \, d\mu$$

defines a trace on $C^*_\text{red}(\mathcal{R})$. Its cyclicity is a direct consequence of the invariance of $\mu$,

$$\text{tr}(f * g) = \int_{\Omega} (f * g)(T,T) \, d\mu(T) = \int_{\Omega} \sum_{T' \sim T} f(T,T') \, g(T',T) \, d\mu(T).$$

Concerning the gap-labelling we are of course interested in traces that coincide with the one appearing in the r.h.s. of the Shubin formula. If $T^{\text{pct}} \cong \mathbb{Z}$ and $\mu$ is ergodic, then one can show by Birkhoff’s ergodic theorem that the r.h.s. of (3) converges for $\mu$-a.e. $T$ to the trace defined by $\mu^{[1]}$. Together with (4) this implies the validity of the Shubin formula. In the general case we merely can establish the following: In a representation of the kind (20) $P$ is the projection onto the diagonal and therefore $\text{Tr} \circ P = \text{Tr}$. This implies for arbitrary $a \in \mathcal{A}_T$

$$\lim_{\Lambda \to T} \frac{1}{|\Lambda^{\text{pct}}|} \text{Tr}(\chi_{\Lambda} \pi_T(a)) = \lim_{\Lambda \to T} \frac{1}{|\Lambda^{\text{pct}}|} \text{Tr}(\chi_{\Lambda} \pi_T(P(a)))$$

and hence (3) holds for arbitrary $a \in \mathcal{A}_T$, if for all $f \in C(\Omega)$

$$\lim_{\Lambda \to T} \frac{1}{|\Lambda^{\text{pct}}|} \text{Tr}(\chi_{\Lambda} \pi_T(f)) = \mu(f) := \int_{\Omega} f \, d\mu.$$  

We shall construct measures satisfying this equality.

The $K_0$-group of $\mathcal{A}_T$ is not known to us in the general case neither the complete image of the tracial state. But such a state induced by an invariant measure $\mu$ satisfies always

$$\mu(C(\Omega, \mathbb{Z})) \subset \text{tr}_*(K_0(C(\Omega))) \subset \text{tr}_*(K_0(\mathcal{A}_T))$$

$C(\Omega, \mathbb{Z})$ denoting the continuous functions over $\Omega$ with values in $\mathbb{Z}$. Hence part of the values of the tracial state is already been given by the image of an invariant measure on $\Omega$, the normalization of which is determined by the order unit: $\mu(\Omega) = \text{tr}_*[1] = 1$. However we emphasize that in all cases known to us where $\text{tr}_*(K_0(\mathcal{A}_T))$ can be computed even equality holds in (28) which is not obvious from the $K_0$-group.

Remark: For completeness let us briefly describe the notion of $\mathcal{R}$-invariance in the general case [27]. Let $\mu$ be a measure on $\Gamma^0$ and $\{\lambda^u | u \in \Gamma^0\}$ a left Haar system and set $\lambda_u(x) = \lambda^u(x^{-1})$. Then $\nu = \int_{\Gamma^0} \lambda^u \, d\mu(u)$ as well as $\nu^{-1} = \int_{\Gamma^0} \lambda_u \, d\mu(u)$ are in general two different measures on $\Gamma$. $\mu$ is called invariant, if for the Radon-Nikodym derivative $\frac{d\nu}{d\nu^{-1}} = 1$. Applied to our case this one obtains

$$\nu(U_{M,x,x'}) = \int_{\Omega} d\mu(T) \int_{U_{M,z}} d\lambda^T(T', T' - (x' - x)) = \mu(U_{M,x})$$

(29)
as well as
\[ \nu^{-1}(U_{M,x,x'}) = \int_\Omega d\mu(T) \int_{U_{M,x}} d\lambda^T(T' - (x' - x), T') = \mu(U_{M,x'}) \quad (30) \]
leading to the description of invariance of above. The conditional expectation satisfies
\[ \int_\Omega P(g^* f) d\mu = \int_\mathcal{R} f g d\nu^{-1} \quad (31) \]
which coincides with the scalar product \( <f, g> \) of elements in \( L^2(\mathcal{R}, \nu^{-1}) \) so that the latter can also be seen as \( L^2(C^*_{red}(\mathcal{R}), \text{tr}) \) where \( C^*_{red}(\mathcal{R}) \) is the completion of \( C^*_{red}(\mathcal{R}) \) with respect to the scalar product defined by the trace.

### 3 Cartesian products of one dimensional tilings

#### general considerations

In this section a general analysis for the gap labelling of the simplest class of tilings is carried out. The tilings of this class may be understood as mappings from \( \mathbb{Z}^d \) into the set of oriented prototiles, i.e. \( \mathcal{T}^{\text{pct}} \cong \mathbb{Z}^d \). The results obtained are not yet very explicit and extra structure is needed to compute the gap labelling.

#### 3.1 One dimensional tilings

A one dimensional tiling in our sense is a covering of \( \mathbb{R} \) by punctured intervals and is therefore ordered. A prototile is determined by the length of the interval and the position of its puncture. As usual the oriented prototiles are denoted by letters of a (finite) alphabet \( \mathcal{B} = \{a_1, \cdots, a_n\} \). A tiling is therefore a two-sided sequence of letters \( T = \{T_i\}_{i \in \mathbb{Z}} \) and a pattern class of \( T \) is a word in the alphabet which appears in \( T \). We shall fix the letter whose puncture sits on \( 0 \in \mathbb{R} \) to be \( T_0 \).

Also \( \mathcal{T}^{\text{pct}} \) may be ordered and identified with \( \mathbb{Z} \), so that the equivalence relation may be expressed with the help of a homeomorphism \( \varphi \) which acts on a sequence by shifting it to the left, \( (\varphi T)_n = T_{n+1} \), i.e.
\[ \mathcal{R} = \{(T, \varphi^{-k}T) \in \Omega \times \Omega | k \in \mathbb{Z}\}. \quad (32) \]

Moreover, via identification \( (T, \varphi^{-k}T) \cong (T, k) \) one obtains \( \mathcal{R} \cong \Omega \times \mathcal{G} \) the topology being the product topology, where \( \mathcal{G} = \mathbb{Z}_h \) if \( \mathcal{T} \) is \( h \)-periodic, and \( \mathcal{G} = \mathbb{Z} \) if it is nonperiodic. Through the above identification \([17]\) becomes
\[ f \ast g(T, k) = \sum_{m \in \mathcal{G}} f(T, m) g(\varphi^{-m}T, k - m). \quad (33) \]

Equivalently, the functions \( \hat{f} : \mathcal{G} \longrightarrow C(\Omega) \) where \( \hat{f}(k)(T) = f(T, k) \) yield the algebra \( \mathcal{A}_T \) the product being
\[ \hat{f} \ast \hat{g}(k) = \sum_{m \in \mathcal{G}} \hat{f}(m) \varphi^*_m \hat{g}(k - m) \quad (34) \]
the involution \( \hat{f}^* (k) = \varphi_k^* \hat{f} (-k) \). Here \( \varphi_k^* \hat{f} = \hat{f} \circ \varphi^{-m} \) is the pullback action of \( \mathcal{G} \) on \( C(\Omega) \). Summarizing \( \mathcal{A}_T \cong C(\Omega) \times_\varphi \mathcal{G} \) is a crossed product. (In particular, the reduced norm coincides with the usual one, since \( \mathcal{G} \) is amenable.) It may also be obtained as the \( C^* \)-hull of the \( * \)-algebra which is generated by \( C(\Omega) \) together with the function \( u : \mathcal{G} \to C(\Omega), u(n) = \delta_{1n} \) - with \( h \)-periodic delta symbol for \( \mathcal{G} = \mathbb{Z}_h \). \( u \) is unitary, \( u^{-1}(n) = \delta_{-1n} \), and generates the action of \( \varphi \) by conjugation: \( u \star \hat{f} \star u^{-1} = \varphi^* \hat{f} \). It is given by

\[
u = \sum_{a_i a_j \text{appears in } T} \hat{e}_{a_i a_j,1,2} \tag{35}\]

1 resp. 2 abbreviating the puncture of the first resp. second letter in \( a_i a_j \). In a representation \( u \) will be represented as a translation operator and \( u + u^* \) as the discrete Laplacian.

The pair \( (\Omega, \varphi) \) is a topological dynamical system which is topologically transitive as the orbit of \( \mathcal{T} \) is dense. If \( \mathcal{T} \) is homogeneous, i.e. \( \forall T \in \Omega_T : \Omega_T = \Omega_T \), then any orbit is dense in which case the dynamical system is called minimal. In the latter case \( \mathcal{A}_T \) is a simple \( C^* \)-algebra.

### Periodic tilings

Let \( \mathcal{T} \) be a tiling of period \( h \), i.e. it is given by periodic repetition of a word \( a_1 \cdots a_h \). Then the shift action \( \varphi \) is transitive and \( \varphi^h \mathcal{T} = \mathcal{T} \). Hence \( \Omega \) consists of one orbit of \( h \) elements, \( \Omega \cong \mathcal{R}^T \), and \( \mathcal{A}_T = \mathbb{C}^h \times_\varphi \mathbb{Z}_h \cong M_h(\mathbb{C}) \). Next to \( K_1 (M_h(\mathbb{C})) = \{0\} \) one obtains

\[
K_0 (M_h(\mathbb{C})) = \mathbb{Z} \tag{36}
\]

\[
tr_* K_0 (M_h(\mathbb{C})) = \frac{1}{h} \mathbb{Z} \tag{37}
\]

for the \( K_0 \)-group and the values of the state induced by the unique (normalized) trace on \( M_h(\mathbb{C}) \). The representations on \( \ell^2 (\mathcal{R}^T) \cong \mathbb{C}^h \) are for different \( T \) unitary equivalent, and \( e_{a_1 a_2,1,2} = e_{a_1 \cdots a_h,1,1} \). Hence \( u = \sum_{i=1}^h \hat{e}_{a_i a_{i+1},1,2}, a_{h+1} = a_1 \), is the translation operator in this representation which is a representation on the closed chain with \( h \) sites being obtained from \( \mathcal{T} \) by periodic identification. \( \mathcal{A}_T \) is the algebra of observables on this chain.

If one would like to construct an algebra which has representations in which a translation operator on the tiling itself is represented, one has to choose in place of \( u : \mathbb{Z}_h \to C(\Omega) \) an operator the \( h \)-th power of which is not equal to the identity operator. This is guaranteed if one uses instead the \( C^* \)-algebra \( C(\Omega) \times_\varphi \mathbb{Z} \) (with the same \( \varphi \)), namely the operator \( \tilde{u} : \mathbb{Z} \to C(\Omega), \tilde{u}(n) = \delta_{1n} \) corresponds to the translation operator on \( \mathcal{T} \) the \( h \)-th power of it lying in the centre of the algebra. Now

\[
C(\Omega) \times_\varphi \mathbb{Z} \cong M_h(C(S^1)) \tag{38}
\]

coincides with the result following from Bloch’s theorem for one dimensional periodic models over \( \mathbb{Z} \): As the Schrödinger operator does only depend on the local oriented
pattern classes it commutes with translation over $h$ lattice sites. Therefore the Hilbert space of a representation is decomposed into the eigenspaces of the $h$'th power of the translation operator. These eigenspaces are $h$ dimensional and contain wavefunctions which are periodic up to a phase which is the eigenvalue of the eigenspace. If this phase is absorbed into the Schrödinger operator the operator can be understood as an $h \times h$ matrix having coefficients depending on $S^1$.

The $K_0$-group of (38) is simply

$$K_0(M_h(C(S^1))) = \mathbb{Z}.$$  \hspace{1cm} (39)

The choice of a normalized trace on $M_h(C(S^1))$ amounts to the choice of a normalized measure $\mu$ on $S^1$,

$$\text{tr}(a) = \frac{1}{h} \int_{S^1} \text{Tr}(a(z)) \, d\mu(z).$$  \hspace{1cm} (40)

But for projections $p \in M_h(C(S^1))$, i.e. $p(z) \in M_h(C)$, is $\text{Tr}(p(z))$ constant so that for all choices

$$tr_* K_0(M_h(C(S^1))) = \frac{1}{h} \mathbb{Z}.$$  \hspace{1cm} (41)

As a result the gap labelling of periodic systems in one dimension is independent of whether one periodically identifies the tiling or not. The difference shows up in the $K_1$-group. This group is generated by the class of $\tilde{u}$:

$$K_1(M_h(C(S^1))) = \mathbb{Z}.$$  \hspace{1cm} (42)

Non periodic tilings

Focussing now on non periodic tilings $\mathcal{G}$ becomes $\mathbb{Z}$. As the action of $\varphi$ on $\Omega$ is topologically transitive the $K$-groups of $A_T$ may be computed with the help of the exact Pimsner-Voiculescu sequence [22]. This has been carried out in [14, 23].

**Theorem 1** Let $\Omega$ be a totally disconnected compact metric space, $\varphi$ a topologically transitive homeomorphism on $\Omega$ and $\text{tr}$ a normalized trace on $C(\Omega) \times_\varphi \mathbb{Z}$. Then

1. $K_0(C(\Omega) \times_\varphi \mathbb{Z}) \cong C(\Omega, \mathbb{Z}) / E_\varphi$ where $E_\varphi := \{ f - f \circ \varphi^{-1} \mid f \in C(\Omega, \mathbb{Z})\}$

2. $K_1(C(\Omega) \times_\varphi \mathbb{Z}) \cong \mathbb{Z}$ its generator being $[u]$.

3. $tr_* K_0(C(\Omega) \times_\varphi \mathbb{Z}) = \mu(C(\Omega, \mathbb{Z}))$ where $\mu$ is the $\varphi$-invariant measure on $\Omega$ obtained by restriction of $\text{tr}$ to $C(\Omega)$.

4. $K_1(C(\Omega) \times_\varphi \mathbb{Z})$ has no nilpotent elements.

**Proof:** Proofs of the first three statements are given in [14] or [23] so we restrict here to the last one.

For $K_1(C(\Omega) \times_\varphi \mathbb{Z})$ the statement is clear. Let $n[f] = [0], 0 \neq n \in \mathbb{Z}, f \in C(\Omega, \mathbb{Z})$ and $[f]$ being its equivalence class. Hence $\exists h \in C(\Omega, \mathbb{Z}) : nf = h - h \circ \varphi^{-1}$. Let $\tilde{h} := \frac{1}{n}(h - h(\omega))$ for some $\omega \in \Omega$. Then $f = \tilde{h} - \tilde{h} \circ \varphi^{-1}$, and since $\tilde{h}(\omega) = 0$ which implies $\tilde{h}(\varphi^{-n-1}(\omega)) = \tilde{h}(\varphi^{-n}(\omega)) - f(\varphi^{-n}(\omega)) \in \mathbb{Z}$ the function $\tilde{h}$ takes integer values...
on the orbit of \( \omega \). By the transitivity of \( \varphi \) we may choose \( \omega \) in a way that its orbit is dense in \( \Omega \). Continuity of \( \tilde{h} \) implies then \( \tilde{h} \in C(\Omega, \mathbb{Z}) \) and hence \( f \in E_\varphi \), i.e. \( [f] = [0] \). \( \square \)

It should be noted that this theorem also applies to the algebra \( \mathbb{Z} \). In the theorem a \( \varphi \)-invariant measure on \( \Omega \) was determined by restricting the trace to \( f \in C(\Omega) \): \( \mu(f) = \int_\Omega f d\mu = \text{tr}(f) \). Conversely any \( \varphi \)-invariant measure on \( \Omega \) defines a trace on the crossed product by \( \text{tr}(\hat{f}) = \int \hat{f}(0) d\mu \). As there are no periodic orbits this is the only trace restricting to the measure \( \mu \) \( \text{[24]} \) and there is a one to one correspondence between \( \varphi \)-invariant measures on \( \Omega \) and traces on \( \mathcal{A}_\varphi \). Concerning the gap-labelling it is therefore important to know, under which conditions such a trace coincides with the one of the r.h.s. of the Shubin formula. Using Birkhoff’s ergodicity theorem one can show that this is the case if the corresponding measure is normalized and ergodic \( \text{[14]} \). In case \( \mathcal{A}_\varphi \) has a unique normalized trace the corresponding measure is ergodic. Hence the values of the IDS on gaps are determined by an ergodic probability measure on \( \Omega \). To explicitly determine such a measure or the group \( C(\Omega, \mathbb{Z})/E_\varphi \) an additional structure is necessary. This additional structure will later on be the self-similarity of a tiling. However before coming to that the Cartesian product case is discussed.

3.2 Cartesian products of one dimensional tilings

Among the simplest higher dimensional tilings are those which may be regarded as Cartesian products of one dimensional tilings. Their basic feature is that \( \mathcal{F} \) may still be ordered as \( \mathbb{Z}^d \), \( d \) being the dimension of the tiling. By use of deep theorems from \( K \)-theory the gap labelling of this case may be traced back to the one dimensional case and, if the values of the IDS are used as labels, it is again determined by an invariant measure on the hull. We shall restrict here to the non periodic case. In fact, the periodic one may be treated in the same way if one takes (32) as the relevant algebra.

First observe that through the identification of \( \mathcal{T} \) with \( \mathbb{Z}^d \) the groupoid-\( C^* \)-algebra \( \mathcal{A}_\varphi \) becomes in the non periodic case a crossed product with \( \mathbb{Z}^d \): \( \mathcal{A}_\varphi = C(\Omega) \times_\varphi \mathbb{Z}^d \) \( \text{(43)} \) where \( \Omega = \Omega_1 \times \cdots \times \Omega_d \) is the Cartesian product of the one dimensional hulls and \( \varphi = \varphi_1 \times \cdots \times \varphi_d \) is the homeomorphism whose pullback gives the action of \( \mathbb{Z}^d \).

**Lemma 3** Let \( \mathcal{T}_k \), \( k = 1, \cdots, d \) be one dimensional non periodic tilings. Then \( \mathcal{A}_\varphi \cong \mathcal{A}_{\mathcal{T}_1} \otimes \cdots \otimes \mathcal{A}_{\mathcal{T}_d} \) \( \text{(44)} \)

**Proof:** As crossed products of nuclear \( C^* \)-algebras with \( \mathbb{Z}^d \) are nuclear we only have to show that the algebraic tensor product \( C(\Omega_1) \times_\varphi_1 \mathbb{Z} \otimes \cdots \otimes C(\Omega_d) \times_\varphi_d \mathbb{Z} \) is *-isomorphic to a norm-dense subalgebra of \( C(\Omega) \times_\varphi \mathbb{Z}^d \). In fact, the *-homomorphism \( a_1 \otimes \cdots \otimes a_d \mapsto a_1 \ast \cdots \ast a_d \) from the algebraic tensor product onto the crossed product with \( \mathbb{Z}^d \) is injective. Moreover it has a dense image, since \( C(\Omega) \cong C(\Omega_1) \otimes \cdots \otimes C(\Omega_d) \).
and therefore $C(\Omega) \times_{\varphi} \mathbb{Z}^d$ is generated as a $C^*$-algebra by elements of the form $f_1 \delta_{n_1} \ast \cdots \ast f_d \delta_{n_d}$, $f_k \in C(\Omega_k)$. \hfill \Box

For the computation of the $K$-groups of $\mathcal{A}_T$ we may just apply the Künneth formula \cite{106,107}. If $\mathcal{T}, \mathcal{T}'$ are two tilings which are Cartesian products of one dimensional ones this formula yields:

$$K_0(\mathcal{A}_T \times \mathcal{T}') \cong K_0(\mathcal{A}_T) \otimes K_0(\mathcal{A}_{T'}) \oplus K_1(\mathcal{A}_T) \otimes K_1(\mathcal{A}_{T'})$$

(45)

$$K_1(\mathcal{A}_T \times \mathcal{T}') \cong K_0(\mathcal{A}_T) \otimes K_1(\mathcal{A}_{T'}) \oplus K_1(\mathcal{A}_T) \otimes K_0(\mathcal{A}_{T'}).$$

(46)

In fact as the $K$-groups of $C(\Omega_k) \times_{\varphi_k} \mathbb{Z}$ have no nilpotent elements \cite{105,106} is certainly the correct result for $d = 2$, but it also shows that $K_i(\mathcal{A}_T)$ and $K_i(\mathcal{A}_{T'})$ do not contain nilpotent elements and therefore \cite{105,106} holds in any dimension. The result becomes more and more complex for higher $d$. The same result could be obtained by iterative application of the Pimsner-Voiculescu sequence onto the r.h.s. of (47), provided this sequence splits at a certain position. This in general true for $d = 2$ and has been used for the computation in \cite{25} and \cite{23}; for $d = 3$ the splitting property has been explicitly checked in \cite{27}. The latter approach is also useful in cases where $\Omega$ is not of the form of a Cartesian product.

Despite of the complicated structure of the $K$-group for higher dimensions the image of a state on the $K_0$-group which is induced from a tracial product state on $\mathcal{A}_T$ is a simple expression. A tracial product state is a state on $\mathcal{A}_\mathcal{T}_k$. In the non periodic case $\mathcal{A}_T$ is isomorphic to an iterated crossed product by $\mathbb{Z}$:

$$\mathcal{A}_T = C(\Omega) \times_{\varphi} \mathbb{Z}^d = (C(\Omega) \times_{\varphi_1} \mathbb{Z}) \cdots \times_{\varphi_d} \mathbb{Z}.$$  

(47)

By a little abuse of notation the corresponding $\mathbb{Z}$-action on $(C(\Omega) \times_{\varphi_1} \mathbb{Z}) \cdots \times_{\varphi_k-1} \mathbb{Z}$ is also denoted by $\varphi_k$. This form for $\mathcal{A}_T$ is useful for the proof of the following theorem.

**Theorem 2** Let $\mathcal{T}$ be a $d$-fold Cartesian product of one dimensional non periodic tilings $\mathcal{T}_k$ and $\text{tr}$ a tracial product state on $\mathcal{A}_T$. Then

$$\text{tr}_* K_0(\mathcal{A}_T) = \mu(C(\Omega, \mathbb{Z}),$$

(48)

$\mu = \mu_1 \times \cdots \times \mu_d$ being the invariant measure determined by the trace.

*Proof:* This theorem is an application of Theorem 3 of \cite{23} which we first cite: Let $\text{tr}$ be a trace on the crossed product $A \times_{\alpha} \mathbb{Z}$ of $\mathbb{Z}$ with an arbitrary $C^*$-algebra. It restricts to a trace on $A$ which is also denoted by $\text{tr}$. Then the map $\Delta_{\text{tr}}^\alpha : \ker(id - \alpha_s) \subset K_1(A) \rightarrow IR/\text{tr}_*(K_0(A))$ defined by

$$\Delta_{\text{tr}}^\alpha([u]) = \frac{1}{2\pi i} \int_0^1 \text{tr}(\xi(t)\xi^{-1}(t)) \; dt / \text{tr}_*(K_0(A)),$$

(49)

$\xi : [0, 1] \rightarrow GL(A)$ being a piecewise smooth path from 1 to $u \alpha(u^{-1})$, is a well defined group homomorphism, and moreover the sequence

$$0 \rightarrow \text{tr}_*(K_0(A)) \rightarrow \text{tr}_*(K_0(A \times_{\alpha} \mathbb{Z})) \xrightarrow{q} \Delta_{\text{tr}}^\alpha(ker(id - \alpha_s)) \rightarrow 0,$$

(50)
$q : \mathbb{R} \to \mathbb{R}/\text{tr}_s(K_0(A))$ being the canonical projection, is exact. Hereby the restriction of tr to A is also denoted by tr.

We shall apply that theorem to $A = \mathcal{A}_\mathcal{T} \otimes C(\Omega_d)$ and $\alpha = \varphi_d$ where $\mathcal{T}' = \mathcal{T}_1 \times \cdots \times \mathcal{T}_{d-1}$, because $\mathcal{A}_{\mathcal{T}'} \otimes C(\Omega_d) \times \varphi_d \mathbb{Z} = \mathcal{A}_\mathcal{T}$. For this we need to determine $\ker(id - \varphi_d) \subset K_1(A)$.

Representatives of elements of $K_1(A)$ are continuous functions $u : \Omega_d \to GL(\mathcal{A}_{\mathcal{T}'})$ satisfying $\varphi_d(u) = [u \circ \varphi_d^{-1}]$. Hence $(id - \varphi_d)(u) = 0$ whenever $u \sim u \circ \varphi_d^{-1}$. By transitivity of $\varphi_d$ this implies $\forall x, y \in \Omega_d : u(x) \sim_{\mathcal{A}_{\mathcal{T}'}} u(y)$, the equivalence here being homotopy equivalence in $GL(\mathcal{A}_{\mathcal{T}'})$. Since the fundamental group of totally disconnected spaces is trivial, the homotopies in $GL(\mathcal{A}_{\mathcal{T}'})$ at different points may be put together to yield a homotopy in $GL(A)$ between $u$ and the constant function $\tilde{u}(x) = u(x_0)$ for some $x_0$, i.e. $\tilde{u} \sim u$. Hence any element of $\ker(id - \varphi_d)$ may be represented by a constant function over $\Omega_d$ and $\ker(id - \varphi_d) \cong K_1(\mathcal{A}_{\mathcal{T}'})$. Now $\tilde{u} \varphi_d(\tilde{u}^{-1}) = 1$ implies $\Delta_{\text{tr}}^2([u]) = 0$ so that with \(50\)

$$\text{tr}_sK_0(\mathcal{A}_{\mathcal{T}'}) \otimes C(\Omega_d) \times \varphi_d \mathbb{Z} = \text{tr}_sK_0(\mathcal{A}_{\mathcal{T}'}) \otimes C(\Omega_d) \quad \text{(51)}$$

Now let $[x] \in K_1^+(\mathcal{A}_{\mathcal{T}'}) \otimes C(\Omega_d)$. As $[x]$ is represented by a projection valued continuous function $x$ over $\Omega_d$ there is a partition $\{U_i\}_{i \in I}$ of $\Omega_d$ by disjoint closed and open subsets (which has to be finite) such that $x$ is equivalent to a projection of the form $\sum_{i \in I} p_i \otimes \chi_{U_i}$, $p_i^2 = p_i \in M_\text{tr} \langle \mathcal{A}_{\mathcal{T}'}, \chi_{U_i} \rangle$ being the characteristic functions. Therefore

$$\text{tr}_s([x]) = \sum_{i \in I} \int_{U_i} d\mu \text{tr}(p_i) = \sum_{i \in I} \mu(p_i) \text{tr}_s([p_i]) \quad \text{(52)}$$

where the trace on $M_\text{tr} \langle \mathcal{A}_{\mathcal{T}'}, \chi_{U_i} \rangle$ has also been denoted by tr. Hence

$$\text{tr}_sK_0(\mathcal{A}_{\mathcal{T}'}) \otimes C(\Omega_d) = \mu_\text{tr}(C(\Omega_d, \mathbb{Z})) \text{tr}_sK_0(\mathcal{A}_{\mathcal{T}'}) \quad \text{(53)}$$

and successively the theorem follows. \( \square \)

Again, if $\mu_k$ are ergodic measures, the tracial product state on $\mathcal{A}_{\mathcal{T}}$ satisfies the Shubin formula.

### 4 Substitution tilings

In section 3 the results obtained for $K_0(\mathcal{A}_{\mathcal{T}})$ resp. $\text{tr}_s(K_0(\mathcal{A}_{\mathcal{T}}))$ have still been rather abstract. To compute explicitly these groups or an invariant measure on the hull of $\mathcal{T}$ extra structure is needed. Such a structure is given for certain non periodic self-similar tilings by a substitution, the corresponding tilings called substitution tilings (or similarity tilings). Substitutions are not only useful for Cartesian products of one dimensional tilings but also for tilings with nontrivial orientational symmetry $G$. In these cases, in which $K_0(\mathcal{A}_{\mathcal{T}})$ is unknown to us, the substitutions are used to determine the part of $\text{tr}_s(K_0(\mathcal{A}_{\mathcal{T}}))$ which is given by an invariant measure on the hull. For the substitution to be well defined none of its tiles should be invariant under $G$. This might require additional markings for breaking the symmetries.
A deflation $\rho_t$ of a tiling $\mathcal{T}$ is a local prescription by which each tile of $\mathcal{T}$ is replaced by a pattern, made from representatives of the original prototiles rescaled by $t^{-1}$, in such a way that these patterns fit together to form a new tiling $\rho_t(\mathcal{T})$ which differs from $t^{-1}\mathcal{T}$ only by a translation. The tiling $t^{-1}\mathcal{T}$ is $\mathcal{T}$ rescaled by $t^{-1}$, $t > 1$, where $0 \in \mathbb{R}^d$ is the point to be kept fixed, so it has the same prototiles as $\rho_t(\mathcal{T})$, and local refers to the following requirement which a deflation has to satisfy.

1) If the tiles at $x \in \mathcal{T}_{\text{pct}}$ and at $x' \in \mathcal{T}_{\text{pct}}$ differ only by a translation and the one at $x$ is replaced by the pattern $(M, y) \subset (\rho_t(\mathcal{T}), z)$ then the one at $x'$ is to be replaced by $(M, y) \subset (\rho_t(\mathcal{T}), z - (x - x'))$.

The pattern by which tile $a$ of $\mathcal{T}$ is replaced is denoted by $\rho_t(a)$. Furthermore it will be required that $\rho_t$ is $G$-covariant, i.e. that for a tile $a$

2) $\rho_t(g \cdot a)$ and $g \cdot \rho_t(a)$ are in the same oriented pattern class.

A deflation of $\mathcal{T}$ is then determined by the pattern classes of the $\rho_t(a)$ together with finitely many positions, namely at most one for one representative of each oriented prototile. In particular $\rho_t$ may be understood as a map from the set of pattern classes of $\mathcal{T}$ into the set of rescaled pattern classes. In fact the position of $\rho_t(\mathcal{T})$ in $\mathbb{R}^d$ is irrelevant for the sequel. Note that we do not require the pattern $\rho_t(a)$ to fill out exactly the space of the original tile $a$ – if it does it is a decomposition in the sense of [6] – but 1) implies that $\lim_{r \to \infty} \frac{\text{vol} \rho_t(M_r(\mathcal{T}))}{\text{vol} M_r(\mathcal{T})} = 1$, $\text{vol} M$ denoting the volume of the subset of $\mathbb{R}^d$ covered by the pattern $M$. Thus the volume of $\rho_t(a)$ is in a certain sense of average equal to the volume of $a$. This fixes the scaling factor $t$.

It is not difficult to extend the action of $\rho_t$ to all elements of $\mathcal{L}'_\mathcal{T} = \{\mathcal{T} - x | x \in \mathbb{R}^d\}$. Setting $\rho_t(\mathcal{T} - x) = \rho_t(\mathcal{T}) - x$ one obtains a surjective map $\rho_t : \{\mathcal{T} - x | x \in \mathbb{R}^d\} \to \{t^{-1}\mathcal{T} - x | x \in \mathbb{R}^d\}$ which is continuous by 1) and may therefore be extended to the closures $\mathcal{L}'_\mathcal{T}$ resp. $\mathcal{L}'_{t^{-1}\mathcal{T}}$. Clearly, not every tiling admits a deflation but its existence is a special property which in particular expresses a kind of self-similarity (for this reason substitution tilings are called similarity tilings in [6]).

A substitution $\rho$ of a tiling $\mathcal{T}$ is a deflation followed by a rescaling by the factor of $t$ so that the tiles of $\rho(\mathcal{T})$ have original size. Hence, if a tiling $\mathcal{T}$ is a substitution tiling, i.e. if it allows for a substitution $\rho$ – it will also be called invariant under the substitution – then $\exists x \in \mathbb{R}^d : \rho(\mathcal{T}) = \mathcal{T} - x$. We call a pattern $\rho(a)$ of $\rho(\mathcal{T})$ which is the replacement of the tile $a$ of $\mathcal{T}$ a substitute.

The right inverse of $\rho_t$ as a map $\rho_t^{-1} : \mathcal{L}'_{\mathcal{T}} \to \mathcal{L}'_{t\mathcal{T}}$, $\rho_t \circ \rho_t^{-1} = \text{id}$, is called an inflation and correspondingly, the inverse of a substitution is an inflation followed by a rescaling by $t^{-1}$. However a specific deflation may have several right inverses, if it is not injective. Uniqueness of the right inverse enforces non periodicity as is shown in [6].

**Lemma 4** If a deflation $\rho_t$ of $\mathcal{T}$ has a unique right inverse then $\mathcal{T}$ is non periodic.  

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4 If $\mathcal{T}$ is homogeneous then $\mathcal{L}'_{\mathcal{T}} = \mathcal{L}_{\mathcal{T}}$.

5 Here we follow the terminology established in the physical literature which unfortunately differs from the one in [6] where the term inflation is used for what is called substitution here.
Proof: Assume that \( T - x = T \) for a \( x \in \mathbb{R}^d \) such that \( |x| \) is minimal. If \( \rho_t^{-1} \) is a right inverse of \( \rho \) then \( \rho_t^{-1} \) followed by a translation by \( x \) is as well a right inverse. Uniqueness then implies that \( \rho_t^{-1}(T) \) is invariant under translation by \( x \) and therefore \( T - t^{-1}x = T \) which contradicts the minimality of \( |x| \). \( \square \)

It is not at all clear that an inflation can be formulated as a local procedure, since it has to be determined whether a given pattern of \( \rho(T) \) is a substitute of some tile \( a \) of \( T \); this is not insured by the pattern being congruent to \( \rho(a) \), since e.g. all its neighbouring tiles have to form substitutes, too. We call an inflation \emph{local} and correspondingly a substitution \emph{locally invertible}, if the determination of whether or not a pattern of \( \rho(T) \) is a substitute of a tile of \( T \) may be uniquely carried out by inspection of a patch of given finite size containing that pattern. More technically a substitution is locally invertible, if there is an \( r \) such that for all \( x \in \rho(T)^{\text{per}} \) the substitute of which \( x \) is a puncture is uniquely determined by the pattern \( M_r(\rho(T) - x) \). It will be henceforth required that

3) \( \rho \) is locally invertible.

In particular a locally invertible substitution has a unique right inverse, so that we are dealing with non periodic tilings. The important point of local invertibility of a substitution is that we can for any \( T \in \Omega \) uniquely determine the substitutes by inspection of finite patches and moreover repeat this process and determine the \( n \)-fold substitutes, i.e. the patterns which correspond to \( n \)-fold replacements \( \rho^n(a) \) of tiles \( a \in \rho^{-n}(T) \). Of course, the size of the pattern needed to determine the \( n \)-fold substitute to which a tile belongs has to grow exponentially with \( n \). This may be used to construct a continuous map \( \Xi \) from the hull of \( T \) onto the path space of a graph \( \Sigma \) being related to the substitution as follows.

The substitution matrix \( \sigma \) of the substitution \( \rho \) has entries

\[ \sigma_{ij} = \text{number of representatives of } [a_j] \text{ in } \rho(a_i) \] (54)

\([a]\) denoting the equivalence class under all Euclidean transformations of the tile \( a \). These entries are all positive and integer and therefore \( \sigma \) may be interpreted as the connectivity matrix of a graph \( \Sigma \). The vertices \( \Sigma^{(0)} \) of this graph are in one to one correspondence with the prototiles and vertex \( i \), which corresponds to \([a_i]\), is linked to vertex \( j \) by \( \sigma_{ji} \) oriented edges. Hence an (oriented) edge \( \kappa \) has a source \( s(\kappa) \) and a range \( r(\kappa) \). Two edges \( \kappa_1, \kappa_2 \) may be concatenated, denoted by \( \kappa_1 \circ \kappa_2 \) or for short by \( \kappa_1 \kappa_2 \), if they fit together, i.e. if \( r(\kappa_1) = s(\kappa_2) \). A (finite or infinite) path over \( \Sigma \) is a (finite or infinite) sequence of concatenated edges which fit together. One may then analogously define the source and (if the path is finite) the range of a path as the source of the first and the range of the last edge as well as concatenation of two paths. We denote by \( \mathcal{P}_\Sigma^{(n)} \) the space of all paths of length \( n \) and by \( \mathcal{P}_\Sigma \) all infinite paths over \( \Sigma \). \( \mathcal{P}_\Sigma \) is also called the path space over \( \Sigma \). It becomes a metric space if one defines the distance of two paths to be \( d(\gamma, \gamma') = \exp(-l(\gamma, \gamma')) \) with \( l(\gamma, \gamma') = \sup\{i \geq 0| \forall j \leq i : \gamma_j = \gamma'_j\} \) where \( \gamma_i \) denotes the \( i \)th edge of \( \gamma \). The metric topology is generated by sets which are labelled by a finite path \( \xi \)

\[ U_\xi = \{\xi \circ \gamma| \gamma \in \mathcal{P}_\Sigma, s(\gamma) = r(\xi)\} \] (55)
These sets are open and closed which implies that $P_\Sigma$ is totally disconnected. An argument similar to the one above for the hull shows that $P_\Sigma$ is compact.

The substitution is called *primitive* if $\sigma$ is primitive (or aperiodic), i.e. if some power of $\sigma$ has only strictly positive entries. In this case $\Sigma$ is connected and by the Perron-Frobenius theorem $\sigma$ has a non degenerate largest eigenvalue (the Perron-Frobenius-eigenvalue) whose corresponding left and right eigenvectors (Perron-Frobenius-eigenvectors) may be chosen to have only strictly positive entries. The $i$’th component of the left Perron-Frobenius-eigenvector $\nu$, normalized to $\sum_j \nu_j = 1$, furnishes the relative frequency of representatives of prototile $[a_i]$ in $T$. Therefore the Perron-Frobenius-eigenvalue is $\tau = t^d$, the scaling factor to the power of the dimension of the tiling by which the substitution differs from the deflation, i.e. the volume of the pattern $\rho(a)$ is in average $\tau$ times larger than that of $a$. We consider only primitive substitutions.

### 4.1 The map $\Xi$ onto the path space

The substitution $\rho$ can be used to construct a continuous surjective mapping from the hull $\Omega$ onto the path space $P_\Sigma$. The $\sigma_{ji}$ edges linking vertex $i$ and $j$ may be assigned to the $\sigma_{ji}$ different positions a representative $a_i$ of $[a_i]$ may have in $\rho(a_j)$; and for this to be well defined it is crucial that no tile is invariant under $\sigma$. Thus such an edge uniquely encodes that position and by interpreting the $l$’th edge uniquely encodes the $l$’th edge of $\sigma$ as encoding the position of a representative of $[a_{s(i_1)}]$ in $\rho(a_{r(i_1)})$ the whole path encodes the position of a representative of $[a_{s(i_1)}]$ in $\rho^l(a_{r(i_1)})$. For a Penrose tiling this is illustrated in figure 2. Now let $T \in \Omega$, $a_{i_0}$ be the tile on 0, and choose an increasing chain $a_{i_0} \in \eta_1 \subset \eta_2 \subset \cdots$ of patterns which approximate $T$ such that $\eta_n$ is sufficiently large to determine the $n$-fold substitute in $T$ which covers 0. This is possible by the local invertibility of $\rho$ the size of $\eta_n$ having to grow exponentially. The $n$-fold substitute, to which in particular $a_{i_0}$ belongs, is the $n$-fold substitute of some tile $a_{i_n}$. Let $\Xi_n(T)$ be the path which encodes the position of $a_{i_0}$ in that $n$-fold substitute $\rho^n(a_{i_0})$. It is clear that the first $n-1$ edges of $\Xi_n(T)$ yield $\Xi_{n-1}(T)$ so that we may define $\Xi(T)$ to be the path whose $n$’th edge is given by the $n$’th edge of $\Xi_n(T)$.

Decompose $\Xi(T) = \Xi_n(T) \circ \gamma$. Denote by $M_n(T)$ the oriented pattern class of $\rho^n(a_{i_n})$ with $a_{i_n}$ as above and by $x_n(T) \in M_n(T)^{pct}$ the puncture of the tile that is encoded by $\Xi_n(T)$. Thus $M_n(T)$ depends only on $r(\Xi_n(T)) = i_n$ and the orientation of $a_{i_n}$ which is determined by $T$. Furthermore let $\xi \in P_\Sigma^{(n)}$ be any other path with $r(\xi) = i_n$ and let $x_\xi \in M_n(T)^{pct}$ denote the puncture of the tile in $M_n(T)$ which is encoded by $\xi$. Then $(M_n(T), x_n(T)) \subset (T, 0)$ and $(M_n(T), x_\xi) \subset (T - (x_\xi - x_n(T)), 0)$, so that

$$
\Xi(T - (x_\xi - x_n(T))) = \xi \circ \gamma.
$$

**Theorem 3** $\Xi : \Omega \to P_\Sigma$ is a continuous surjective map.

**Proof:** $\Xi$ is continuous if $\forall \epsilon > 0 \exists \delta > 0 : d(T, T') < \delta \Rightarrow d(\Xi(T), \Xi(T')) < \epsilon$, which here amounts to $\forall L > 0 \exists R > 0 : r(T, T') > R \Rightarrow l(\Xi(T), \Xi(T')) > L$. But this follows from the local invertibility of $\rho$, since $l(\Xi(T), \Xi(T')) > L$ if and only if $\Xi_{L+1}(T) = \Xi_{L+1}(T')$. 

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Surjectivity of $\Xi$ follows from compactness of $\Omega$: Given $\gamma \in \mathcal{P}_\Sigma$ let $\{\gamma^{(n)}\}_{n \geq 0}$ be a sequence of infinite paths such that the first $n$ edges of $\gamma^{(n)}$ coincide with the first $n$ edges of $\gamma$ but the $m$-th edges for $m > n + m_0$ with the one of $\Xi(T)$ – by the primitivity of the substitution this is possible for some finite $m_0$. By (26) there is a sequence of tilings $\{T^{(n)}\}_{n \geq 0}$ which are translates of $T$ such that $\Xi(T^{(n)}) = \gamma^{(n)}$. Since $\Omega$ is compact $\{T^{(n)}\}_{n \geq 0}$ has a convergent subsequence and its limit is a preimage of $\gamma$.  

Certainly, $\Xi$ cannot be injective if the tiling has a nontrivial orientational symmetry, because of $\Xi(g \cdot T) = \Xi(T)$. But if we pass over to the quotient space (with quotient topology) and define $\tilde{\Xi} : \Omega/G \to \mathcal{P}_\Sigma$ by $\tilde{\Xi}([T]) = \Xi(T)$ a criterion for the invertibility of $\tilde{\Xi}$ may be given. For this to become clear the notion of a singular tiling is useful.

The inner radius of a pattern $(M, x)$ is the radius of the largest ball around $x$ that is covered by $(M, x)$. Hence, if $x$ is close to the boundary of $M$, the inner radius will be relatively small no matter how large $M$ is. A tiling $T \in \Omega$ is called regular, or more precisely $\rho$-regular, if the inner radius of $(M_n(T), x_n(T))$ diverges with $n$, otherwise it is called singular. As $M_{n+1}(T)$ is congruent to $\rho^n(\rho(a_{n+1}))$ it contains $M_n(T)$ leading to the increasing chain of finite patterns

$$(M_n(T), x_n(T)) \subset (M_{n+1}(T), x_{n+1}(T)) \subset \cdots \subset (T, 0)$$

which is called the $\rho$-chain of $T$. Up to the orientation the $\rho$-chain of $T$ is completely determined by $\Xi(T)$. If $T$ is regular its $\rho$-chain is an approximation of it in the sense that any tile of it lies in some $(M_n(T), x_n(T))$. In contrast singular tilings may or may not be determined by their $\rho$-chains, as one needs to know, what happens outside the inner radius of the limit of the $\rho$-chains. Singular tilings do in fact exist:

**Lemma 5** If $T$ is symmetric then it is singular.

**Proof:** Assume $\exists x \in T^{\text{pct}} : g \cdot T = T - x$, hence $\Xi(T) = \Xi(T - x)$. Let us for a moment identify $M_n(T)^{\text{pct}}$ via (57) with a subset of $T^{\text{pct}}$. Assume $x \in M_n(T)^{\text{pct}}$. Then $x$ is the puncture encoded by $\Xi_n(T - x)$ in $M_n(T)$. But $\Xi_n(T - x) = \Xi_n(T)$ implies that $x = x_n(T - x) = x_n(T) = 0$ contradicting the requirement that the oriented prototiles do not have symmetries. Hence $x \notin M_n(T)^{\text{pct}}$ implying that the inner radius of $(M_n(T), x_n(T))$ is smaller than $|x|$.  

Given a pattern $M$ of some tiling $T$ we define the border of $M$ in $T$ to be the pattern of tiles in $T$ which touch $M$ but do not belong to it. If $T$ contains some underlying structure one expects an oriented pattern class of $T$ to force an occasionally larger oriented pattern class $M$ in the sense that whenever $(M, x) \subset (T, y)$ then also $(M, x) \subset (T, y)$. We say that $\rho$ forces its border if there is an $N$ such that whenever the $N$-fold substitute $\rho^N(a_i)$ occurs in $T$ its border is always the same; we denote the oriented pattern class of $\rho^N(a_i)$ together with the border occurring in the above case by $F(a_i)$. This requirement is weaker than the condition that the oriented pattern classes of the $\rho^N(a_i)$ all force their borders, since we require its border only to be the same if the pattern is an actual $N$-fold substitute. Now if the $N+1$-fold substitute $\rho^{N+1}(a_i)$ occurs in $T$ then even $\rho(F(a_i))$ has to occur at that place. This simply follows from the fact that in that case $\rho^N(a_i)$ occurs in $\rho^{-1}(T)$ as a $N$-fold substitute and forces $F(a_i)$ to occur there. It will be shown that if
B2 $\rho$ forces its border

$\Xi$ is indeed a homeomorphism.

**Lemma 6** If $\rho$ forces its border then all $T \in \Omega$ are uniquely determined by their $\rho$-chain.

**Proof:** Let $T \in \Omega$ perhaps be singular and consider its $\rho$-chain. As above set $i_n = r(\Xi_n(T))$. The pattern $(M_N(T), x_N(T)) \subset (T, 0)$ is by construction the $N$-fold substitute which covers $0$ so that by B2 for all $m \geq 0$

$$(M_{N+m}(T), x_{N+m}(T)) \subset \left(\rho^m(F(a_{i_{N+m}})), x_{N+m}(T)\right) \subset (T, 0)$$

(58)

where the orientation of $a_{i_{N+m}}$ is determined by $M_{N+m}(T)$. Therefore the $\rho$-chain determines in fact the increasing chain

$$(\rho^m(F(a_{i_{N+m}})), x_{N+m}(T)) \subset (\rho^{m+1}(F(a_{i_{N+m+1}})), x_{N+m+1}(T)) \subset \cdots \subset (T, 0).$$

(59)

Let $y \in \rho(F(a_{i_{N}}))$ and take the translate $T_y = T - (y - x_N(T))$, i.e. $(F(a_{i_{N}}), y) \subset (T_y, 0)$. Then $(M_{N+m}(T_y), x_{N+m}(T_y)) \subset (\rho^m(F(a_{i_{N+m}})), x_{N+m}(T - y))$ showing that the $\rho$-chain of $T_y$ is already determined by (59) and hence by the $\rho$-chain of $T$. Since $(F(a_{i_{N}}), x_N(T)) \subset (T, 0)$ contains at least all tiles touching $a_{i_0}$, the tile on $0$, we may show by induction, namely repeating the same argument with $T_y$ in place of $T$, that the $\rho$-chain of $T$ already determines the $\rho$-chains of all its translates, and hence all of $T$. \hfill \Box

**Theorem 4** If $\rho$ forces its border then $\tilde{\Xi} : \Omega/G \to \mathcal{P}_\Sigma$ is a homeomorphism.

**Proof:** Continuity and surjectivity of $\tilde{\Xi}$ follow from the continuity and surjectivity of $\Xi$, and since $\Omega/G$ as well as $\mathcal{P}_\Sigma$ are compact we have to show that $\tilde{\Xi}$ is invertible.

Let $\Xi(T) = \Xi(T')$. In particular $\Xi_n(T) = \Xi_n(T')$ which by covariance of $\rho$ implies that there is a $g \in G$ such that $g \cdot a_{k_0} = a'_{k_0}$ and $g \cdot \rho^n(a_{k_n}) = \rho^n(a'_{k_n})$. Hence $(g \cdot M_n(T), g \cdot x_n(T)) = (M_n(T'), x_n(T'))$ implying by Lemma $\tilde{\Xi} g \cdot T = T'$. \hfill \Box

The last two theorems show that for a tiling which is invariant under a locally invertible substitution that forces its border not only $\mathcal{A}_T$ is naturally assigned to them but also the $AF$-algebra $\mathcal{A}_G$ of the path space; c.f. the appendix for its definition. One advantage of this formulation is that it allows us to compute the image of an $\mathcal{R}$-invariant $G$-invariant measure on $\Omega$. But before coming to that a sufficient criterion for $T$ to be homogeneous will be given.

We call a substitution $G$-primitive if there is an $N$ such that for any oriented prototile $a$ any other oriented prototile $b$ occurs in $\rho^N(a)$. This implies the existence of an $r$ such that for any $x \in T^{\text{pet}}$ any oriented prototile occurs in $M_r(T - x)$. Hence if $T$ allows for a $G$-primitive substitution then any oriented prototile occurs in any $T \in \Omega_T$ and consequently also any $n$-fold substitute in any orientation.

**Lemma 7** Let $T$ be invariant under a $G$-primitive substitution which satisfies B2. Then $T$ is homogeneous.

25
Proof: Let \( T \in \Omega_T \) and \( M \) be an oriented pattern class of \( \mathcal{T} \), we have to show that \( \exists y' \in T^{pct} : (M, x) \subset (T, y') \). Under the hypothesis \( \Xi \) is a homeomorphism and therefore exists an \( n \) and \( I \subset \mathcal{P}_\Sigma^{(n)} \) such that \( \Xi(U_{M,x}) = \bigcup_{\xi \in I} U_{\xi} \). Finiteness of the union of the r.h.s. follows from compactness of \( \Xi(U_{M,x}) \) and disjointness of \( U_{\xi} \) can then always be achieved so that one can find such an \( n \). The statement \( \exists g \in G : (g \cdot M, g \cdot x) \in (T, 0) \) is thus equivalent to \( \Xi_n(T) \in I \). Primitivity of the substitution now implies that for all \( \xi \in I \) there is a \( y \in T^{pct} \) such that \( \Xi_n(T - y) = \xi \), and hence \( \exists g \in G : (g \cdot M, g \cdot x) \in (T, y) \). But we need \( G \)-primitivity to insure that there is as well a \( y' \in T^{pct} \) such that \( M_n(T - y') = g^{-1} \cdot M_n(T - y) \) and hence \((M, x) \in (T, y')\). \(\) 

4.2 The \( G \)-invariant measure on \( \Omega \)

Since the operators of the form \([\cdot]\) depend only on the pattern classes and not on their orientation they are invariant under \( G \), i.e. they are elements of \( \mathcal{A}_T^G := \{ f \in \mathcal{A}_T | f(g \cdot T, g \cdot S) = f(T, S) \} \). A trace on \( \mathcal{A}_T^G \) may be seen as a \( G \)-invariant trace on \( \mathcal{A}_T \) and restricts to a \( \mathcal{R} \)-invariant \( G \)-invariant measure on \( \Omega \) and it is this measure which may be determined using the structure of the path space \( \mathcal{P}_\Sigma \).

Remember that by the lack of symmetry of the prototiles

\[ U_{M,x} \cap U_{g \cdot M, g \cdot x} = \emptyset. \tag{60} \]

For this reason a one to one correspondence of measures \( \mu \) on \( \Omega/G \) with \( G \)-invariant measures \( \mu \) on \( \Omega \) is given on the images of sets \( U_{M,x} \) under the natural projection \( [\cdot] : \Omega \to \Omega/G \) by

\[ \tilde{\mu}([U_{M,x}]) = \mu(\bigcup_{g \in G} U_{g \cdot M, g \cdot x}) = |G| \mu(U_{M,x}). \tag{61} \]

The factor \(|G|\) appears, as the topology on \( \Omega \) was defined through oriented patterns and not through their \( G \)-orbits. In fact, the restriction to \( G \)-invariant operators is not crucial here and we could in principle allow for any operator being represented by an element of \( \mathcal{A}_T \) but one might expect that traces of \( G \)-invariant projections take their values in \(|G|\mu(C(\Omega, \mathbb{Z}))\).

\( \mathcal{A}_\Sigma \) is actually a groupoid-\( C^* \)-algebra, too. The groupoid \( \mathcal{R}_\Sigma \) is defined by the equivalence relation \( \gamma \sim_{\mathcal{R}_\Sigma} \gamma' \) whenever \( \exists n_0 \forall n \geq n_0 : \gamma_n = \gamma'_n \) and its topology is generated by \( U_{\xi,\xi'} := \{ (\xi \circ \gamma, \xi' \circ \gamma) | \gamma \in \mathcal{P}_\Sigma, s(\gamma) = r(\xi) \} \) where \( \xi \) and \( \xi' \) have the same finite length and \( r(\xi') = r(\xi) \). It leads as well to discrete orbits \( \mathcal{R}_\Sigma^\gamma \).

The primitivity of \( \sigma \) implies that \( \mathcal{A}_\Sigma \) has a unique (normalized) trace and hence there is a unique (normalized) \( \mathcal{R}_\Sigma \)-invariant measure \( \mu_\Sigma \) on \( \mathcal{P}_\Sigma \). \( \mathcal{R}_\Sigma \)-invariance in this case means that \( \mu_\Sigma(U_\xi) \) depends only on \( r(\xi) \), c.f. \( (24, 30) \).

**Theorem 5** Let \( \mu \) be a (normalized) \( \mathcal{R} \)-invariant \( G \)-invariant measure on \( \Omega \), the hull of a substitution tiling and \( \text{tr} \) the unique (normalized) trace on the corresponding \( \mathcal{A} \)-algebra \( \mathcal{A}_\Sigma \). Then

\[ \text{tr}_s K_0(\mathcal{A}_\Sigma) \subset \tilde{\mu}(C(\Omega/G, \mathbb{Z})) \tag{62} \]

\( \tilde{\mu} \) being given by \( (62) \), and if the substitution satisfies \( B2 \) both sets are even equal.
Proof: $\tilde{\mu}$ induces a measure $\hat{\mu}$ on $P_{\Sigma}$ by

$$
\hat{\mu}(U_{\xi}) = \tilde{\mu}(\tilde{T}^{-1}(U_{\xi})) = \mu(\{T \in \Omega | \Xi_n(T) = \xi\}),
$$

(63)

$n = |\xi|$. Whereas $T \sim \rho T'$ does not imply $\Xi(T) \sim \rho \Xi(T')$, the converse is true by (50). Fix one $T' \in \Omega$ such that $\Xi_n(T') = \xi$ and let $\xi'$ be a path of length $n$ with $r(\xi') = r(\xi)$ then again using (56)

$$
\mu(\{T | \Xi_n(T) = \xi\}) = |G| \mu(\{T | (M_n(T'), x_n(T')) \subset (T, 0) \land \Xi_n(T) = \xi\}) = \mu(\{T | (x_{\xi'} - x_n(T')) \subset \Xi_n(T) = \xi'\})
$$

the last equality following from the $\mathcal{R}$-invariance of $\mu$, and hence $\hat{\mu}$ is $\mathcal{R}_{\Sigma^{-}}$-invariant. But since the trace on $A_{\Sigma}$ is unique up to normalization $\hat{\mu} = \mu_{\Sigma}$. As for AF-algebras $\mu_{\Sigma}(C(P_{\Sigma}, \mathbb{Z}))$ the inclusion is proven. Clearly $\mu_{\Sigma}(C(P_{\Sigma}, \mathbb{Z})) = \tilde{\mu}(C(\Omega/G, \mathbb{Z}))$ if $\Xi$ is a homeomorphism.

As an application we compute the invariant measure on the hulls of certain generalizations of the Fibonacci chain and of a Penrose tiling.

4.2.1 Examples: Generalizations of the Fibonacci chain and Penrose tilings

We first give examples for one dimensional substitutions which have been extensively studied over the past. Their invariant measure on the hull may as well be computed by methods developed in [29]. But we prefer to use the $K_0$-group of $A_{\Sigma}$ since it not only provides us with the values of the invariant measure but also yields a finer labelling in case $tr$ is not injective. This is based on the equality of $K_{r}(A_{\mathbb{T}})$ and $K_{r}(A_{\Sigma})$ in the one dimensional case the proof of which is postponed to subsection 3.3.1. As a two dimensional example we determine the invariant measure on the hull of a Penrose tiling.

A rather efficient way of thinking of an $n$-letter substitution is as a homomorphism $\rho$ of the free group of $n$ generators $F_n$ [34]. The letters represent the generators and a tiling is a map $T : \mathbb{Z} \to F_n$ such that formally the two-sided infinite product $\prod_{i \in \mathbb{Z}} T_i$ is up to a finite shift of the indices invariant under $\rho$. Denote by $Ad_w$ the conjugation by word $w$, i.e. $Ad_w(a) = waw^{-1}$. If $\rho(a)$ ends resp. begins for all $a$ on word $w$ then $Ad_w$ resp. $Ad_w^{-1}$ act like a left resp. right shift on $\rho(a)$ and consequently $Ad_w \circ \rho$ resp. $Ad_w^{-1} \circ \rho$ lead to the same invariant sequences and the same hull as $\rho$. This shall be used below, because the substitutions considered satisfy B2 only after being iterated a few times and then composed with an inner automorphism as above. In fact we shall obtain substitutions of the form $\tilde{\rho} = Ad_w \circ \rho^n$ which have the property that

B2' the first letter as well as the last letter of $\tilde{\rho}(a)$ are independent of $a$.

This is sufficient for $\tilde{\rho}$ to satisfy B2.

Let us mention here that local invertibility of a substitution is insured by recognizability, a concept introduced in [28]. Local invertibility of $\rho$ implies local invertibility of $\tilde{\rho}$.
Moreover, since the substitution matrix of \( \tilde{\rho} \) equals the \( n \)'th power of the substitution matrix \( \sigma \) of \( \rho \), the AF-algebra defined by it has the same scaled ordered \( K_0 \)-group as \( A_\Sigma \) – both algebras are in fact isomorphic – and its determination may be carried out just using the \( \sigma \). Here, the order unit is always \((1, \ldots, 1)\) so that all left-Perron-Frobenius-eigenvectors have to be normalized to satisfy \( \sum_i \nu_i = 1 \). Different letters are supposed to represent different congruence classes of punctured intervals so that the orientational symmetry is trivial.

**A-type generalizations of the Fibonacci chain**

This family of substitutions is defined on the alphabet \( B = \{a_1, \ldots, a_k\}, k \geq 2 \), by

\[
\rho:\begin{cases}
a_i \mapsto a_{k-i} \cdot a_{k-i} & \text{if } i \leq k-i
a_i \mapsto a_{k-i} \cdot a_{k-i+1} & \text{if } k-i < i < k
a_k \mapsto a_1
\end{cases}
\tag{64}
\]

and \( k = 2 \) yields the Fibonacci chain. An inverse procedure of \( \rho \) may be locally defined as follows: Starting with \( i = 1 \) successively replace any word of \( T \) which is congruent to \( \rho(a_i) \) by \( \tilde{a}_i \). This is a well defined procedure, since after the \( i \)'th such replacement the result will not contain any more letter \( a_{k-i} \). After the \( k \)'th such replacement one ends up with a sequence \( \{\tilde{a}_i\}_{i \in \mathbb{Z}} \) which yields after removing the tildes a preimage of \( T \) under \( \rho \).

\( \rho \) does in this form not obey B2. However if one considers only the sequence of letters at the right end of \( \rho^n(a_k) \) for \( n \geq 0 \) one obtains \( a_k \to a_1 \to a_{k-1} \to a_2 \to a_{k-2} \cdots \to a_{\lfloor \frac{k+1}{2} \rfloor} \to a_{\lfloor \frac{k+2}{2} \rfloor} \cdots \). Hence \( \rho^k(a_i) \) ends for all \( i \) on \( \rho(a_{\lfloor \frac{k}{2} \rfloor+1}) = a_{\lfloor \frac{k}{2} \rfloor+1} \). Therefore the substitution \( \tilde{\rho} = Ad_{a_{\lfloor \frac{k}{2} \rfloor+1}, \rho^k} \) satisfies B2'. The substitution matrix of \( \tilde{\rho} \) is \( \sigma^k \) where

\[
\sigma = \begin{pmatrix}
0 & \cdots & 0 & 1 & 1 \\
0 & \cdots & 1 & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\end{pmatrix}.
\tag{65}
\]

The corresponding graph is a tadpole graph. A twofold covering of it is the Coxeter graph of the Coxeter group \( A_{2k} \), hence the name of the substitution. It is invertible over \( \mathbb{Z} \) so that

\[
K_0(A_\Sigma) = \mathbb{Z}^k.
\tag{66}
\]

The Perron-Frobenius-eigenvalue of \( \sigma \) is \( \tau = 2 \cos \frac{\pi}{2k+1} \) and the components of its normalized left-Perron-Frobenius-eigenvector are \( \nu_l = (2 - \tau) \frac{\sin \frac{t_k}{2k+1}}{\sin \frac{t_k}{2k+1}} \). Since \( 2 - \sigma \) is invertible over \( \mathbb{Z} \),

\[
\text{tr}_* K_0(A_\Sigma) = \left\{ \sum_{l=1}^{k} n_l \frac{\sin \frac{t_k}{2k+1}}{\sin \frac{\pi}{2k+1}} | n_l \in \mathbb{Z} \right\}.
\tag{67}
\]

Moreover, \( \text{tr}_* \) is injective if and only if \( 2k + 1 \) is prime.
Metallic type generalizations of the Fibonacci chain

The so called metallic means furnish the Perron-Frobenius-eigenvalues of the substitution matrices of the following class of 2-letter substitutions given by

\[
\rho : \begin{cases} 
  a \mapsto b^k a^l \\
  b \mapsto a 
\end{cases}
\]

and parametrized by integers \(k \geq 1, l \geq 1\). They may be considered as alternative generalizations of the case \(k = 1, l = 1\) which leads to the Fibonacci chains. They are locally invertible: just replace any \(b^k a^l\) by \(a\) and the remaining \(a\)'s by \(b\)'s. But again they do not satisfy B2 in this form. However if we iterate them twice we obtain \(a \mapsto a^k (b^k a^l)^l, b \mapsto b^k a^l\) so that \(\tilde{\rho} = Ad(b^k a^l) \circ \rho^2\) satisfies B2'. The substitution matrix of \(\tilde{\rho}\) is \(\sigma^2\) where

\[
\sigma = \begin{pmatrix} l & k \\ 1 & 0 \end{pmatrix}
\]

\(\sigma\) has Perron-Frobenius-eigenvalue \(\tau = \frac{l + \sqrt{l^2 + 4k^2}}{2}\) and normalized left-Perron-Frobenius-eigenvector \(\nu = \frac{1}{\tau + k}(1, \tau, k + l - \tau)\). Hence

\[
\text{tr}_*K_0(A_\Sigma) = \left\{ \tau^{1-n} \left( \frac{\tau - 1}{k + l - 1} + n_2 \right) | n > 0, (n_1, n_2) \in \mathbb{Z}^2 \right\},
\]

and since \(\tau\) is irrational \(\text{tr}_*\) is injective so that \(K_0(A_\Sigma)\) may be identified with its image under \(\text{tr}_*\). It should be noted that \(k \geq 2\) yields examples of substitutions having substitution matrices which are not invertible over \(\mathbb{Z}\) but which still contain all information about the invariant measure on the hull.

Penrose tilings

Penrose tilings belong to the best known two dimensional tilings. In the form most suitable for us they consists of two unilateral triangles that have angles which are multiples of \(\frac{\pi}{5}\). A part of such a tiling is shown in figure 1. These tilings were invented by R. Penrose before the discovery of quasicrystals \([5]\). Today a three dimensional generalization of a Penrose tiling \([2, 3, 4]\) serves as an idealized model for the spatial structure of quasicrystals which yield tenfold symmetric Bragg reflexes. Among the vast literature on these tilings we refer to \([9]\) and, in relation to the physics of quasicrystals, to the diverse articles collected in \([7]\).

Originally a Penrose tiling was defined as a "puzzle" by so-called matching conditions. Later on various methods to obtain such a tiling from \(\mathbb{Z}^5\) or the root lattice \(A_4\) of \(SU(5)\) were found. For us it is important that some Penrose tilings, e.g. the fivefold symmetric ones, are invariant under the substitution displayed in figure 2. The set of all Penrose tilings is the hull of such an invariant tiling. In fact, as the substitution is \(G\)-primitive such a tiling is homogeneous so that any Penrose tiling defines the same hull. The substitution is locally invertible and the corresponding map \(\Xi\) has been first given by Robinson though he did not view it as a map between topological spaces \([9]\). Connes used this map to derive the algebra \(A_\Sigma\) and computed the ordered \(K_0\)-group.
and its image under $\text{tr}_*$ [8]. His results are justified by confirming that the substitution satisfies B2. In fact, a Penrose tiling has the property that the boundaries of the $n$-fold substitutes of the triangles furnish for $n \geq 2$ local mirror axes. This means that any tile inside an $n$-fold substitute which has an edge lying on that boundary does also occur in the tiling reflected at that edge. To illustrate this, figure 8 shows the 2-fold substitute of the smaller triangle together with part of the tiles which are forced by it. Further repetition of the substitution procedure leads to local centres of fivefold symmetries at the corners of the $n+2$-fold substitutes, i.e. there are fivefold symmetric patterns at these corners. Hence the substitution determines its border ($N$ can be taken to be 4).

The orientational symmetry is generated by a rotation around $\pi/5$ together with a reflection on axes along the boundaries (edges) of the triangles, hence $|G| = 20$. However there are no tenfold symmetric patterns.

As may be seen from figure 2 which also contains $\Sigma$ and the embedding graph of the substitution its substitution matrix is given by

$$\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and has Perron-Frobenius-eigenvalue $\tau = t^2$ with $t = \frac{1+\sqrt{5}}{2}$ (the inverse of the golden mean) and normalized left-Perron-Frobenius-eigenvector $\nu = (t-1, 2-t)$. This yields

$$\text{tr}_* K_0(A_\Sigma) = \mathbb{Z} + t\mathbb{Z}$$

and $K_0(A_\Sigma) = \mathbb{Z}^2$, as $\sigma$ is invertible over $\mathbb{Z}$.

5 $A_T$ versus $A_\Sigma$

The $AF$-algebra $A_\Sigma$ constructed with the help of the substitution could be used to determine an invariant measure on the hull. However $A_\Sigma$ differs from $A_G^G$. In fact due to the existence of singular tilings the image of $R_\Sigma$ under $\tilde{\Xi}^{-1} \times \tilde{\Xi}^{-1}$ is not closed in the topology of $R$. Nevertheless $A_\Sigma$ may be used to show that the measure $\mu$ yields a trace on $A_T$ which coincides with the one from the Shubin formula. This is so far only clear for one dimensional tilings with trivial $G$ and Cartesian products thereof: we actually determined the unique ergodic measure above of which has been shown by Bellissard [10] that it leads to the right trace.

Let us come back to the formulation of $A_\Sigma$ as groupoid-$C^*$-algebra. For $\xi, \xi' \in P_{\Sigma,i}^{(n)}$ the functions $e_{\xi,\xi'} : R_\Sigma \to \mathbb{C}$

$$e_{\xi,\xi'}(\gamma, \gamma') = \begin{cases} 1 & \text{if } \exists \gamma'' : \gamma = \xi \circ \gamma'' \text{ and } \gamma' = \xi' \circ \gamma'' \\ 0 & \text{else} \end{cases}$$

(73)

generate a topological $*$-algebra, the $C^*$-hull of which is $A_\Sigma$. In the representations corresponding to (20) the generators $\pi_\alpha(e_{\xi,\xi'})$ act on wavefunctions $\phi \in l^2(R_\Sigma)$ by (again writing shorter $\phi(\gamma) = \phi(\alpha, \gamma)$)

$$(\pi_\alpha(e_{\xi,\xi'}))\phi(\gamma) = \begin{cases} \phi(\xi' \circ \gamma'') & \text{if } \exists \gamma'' : \gamma = \xi \circ \gamma'' \\ 0 & \text{else} \end{cases}$$

(74)
The restriction on $\mathcal{R}^T$ of the mapping $(T, T') \mapsto (\Xi(T), \Xi(T')) : \mathcal{R}^T \to \mathcal{R}_{\Sigma}^{\Xi(T)}$ is bijective, if $T$ is regular. Therefore we may represent $\mathcal{A}_\Sigma$ also on $\ell^2(\mathcal{R}^T)$:

$$\pi_T(e_{\xi,\xi'}\psi)(T') := (\pi_{\Xi(T)}(e_{\xi,\xi'}\psi \circ \Xi^{-1})(\Xi(T')))$$

$$= \begin{cases} 
\psi(\Xi^{-1}(\xi' \circ \gamma'')) & \text{if } \exists \gamma'' : \Xi(T') = \xi \circ \gamma'' \\
0 & \text{else}
\end{cases}$$

$$= \begin{cases} 
\psi(S - (x_{\xi'} - x_n(T'))) & \text{if } \Xi_n(T') = \xi \\
0 & \text{else}
\end{cases}$$

(75)

(76)

(77)

where, as for (76), $x_{\xi'} \in M_n(T')^{\text{pet}}$ denotes the puncture of the tile encoded by $\xi'$. For singular tilings (77) shall be the definition of a representation of $\mathcal{A}_\Sigma$. Condition $\Xi_n(T') = \xi$ implies that $\pi_T(e_{\xi,\xi'})$ is only a translation by $(x_{\xi'} - x_n(T'))$ in case both, initial and final point of the translation lie in a common $n$-fold substitude. In particular a translation crossing a boundary of an $n$-fold substitute may only be in $\pi_T(\mathcal{A}_{\Sigma}^{(m)})$ for $m > n$. This shows that the elements of $\pi_T(\mathcal{A}_{\Sigma}^{(n)})$, which is generated by

$$\tilde{e}_{M,x,x'} = \sum_{g \in G} e_{g \cdot M,g \cdot g \cdot x, x'},$$

(78)

may not be approximated in norm by elements of $\pi_T(\mathcal{A}_{\Sigma})$, i.e. the inclusion

$$\pi_T(\mathcal{A}_{\Sigma}) \subset \pi_T(\mathcal{A}_{\Sigma}^{(n)})$$

(79)

which is guaranteed by the local invertibility of $\rho$ is a proper one. But one has:

**Theorem 6** For regular $T$

$$\pi_T(\overline{\mathcal{A}_{\Sigma}}^s) = \overline{\pi_T(\mathcal{A}_{\Sigma}^{(n)})}$$

(80)

where $\overline{\cdot}^s$ denotes strong closure in the algebra of bounded operators on a Hilbert space.

**Proof:** For $x, x' \in M^{\text{pet}}$ let

$$\mathcal{P}_{M, x, x'}^{(n)} := \{ (\xi, \xi') \in \bigcup_i \mathcal{P}_{\Sigma, i}^{(n)} \times \mathcal{P}_{\Sigma, i}^{(n)} | \exists T' \in \Xi_n^{-1}(\xi) : (M, x, x') \subset (M_n(T'), x_n(T'), x_{\xi'}) \}$$

$x_{\xi'} \in M_n(T')^{\text{pet}}$ denoting the puncture of the tile encoded by $\xi'$ and define

$$\pi_T(\tilde{e}_{M,x,x'})^{(n)} := \sum_{(\xi, \xi') \in \mathcal{P}_{M, x, x'}^{(n)}} \pi_T(e_{\xi,\xi'})$$

(81)

the $n$'th approximant of $\pi_T(\tilde{e}_{M,x,x'})$. Denote by $\psi_{T'}$, $T' \sim T$ with $\psi_{T'}(T'') = \delta_{T'T''}$ the usual orthonormal basis of $\ell^2(\mathcal{R}^T)$ and by $r_n$ the inner radius of $(M_n(T), x_n(T))$. Define subspaces $\mathcal{H}^{(n)}$ of $\ell^2(\mathcal{R}^T)$ to be generated by $\{ \psi_{T-x} | x \in M_n(T)^{\text{pet}}, |x| < r_n - r_0 \}$ where $r_0$ shall be the radius of the smallest ball that covers $M$. Then regularity of $T$ is equivalent to the statement that $\forall T' \sim T \exists n : \psi_{T'} \in \mathcal{H}^{(n)}$. Now on all $\psi \in \mathcal{H}^{(n)}$ the $n$'th approximant acts exact: $\pi_T(\tilde{e}_{M,x,x'})^{(n)} \psi = \pi_T(\tilde{e}_{M,x,x'}) \psi$. Thus $\pi_T(\tilde{e}_{M,x,x'})^{(n)}$ strongly converges to $\pi_T(\tilde{e}_{M,x,x'})$. Therefore contains $\overline{\pi_T(\mathcal{A}_{\Sigma})}$ the generators of $\overline{\pi_T(\mathcal{A}_{\Sigma}^{(n)})}$. ☐

The regularity of $T$ is an essential condition for the theorem. In the strong closure of a representation of the AF-algebra on a symmetric tiling there are no translations which
over cross a symmetry axis. In particular the Laplace operator is not even in the strong closure. One might hope that the influence of the symmetry axes on the spectrum of a generic \(G\)-invariant operator may be show up in a more thorough analysis of the relation between \(\mathcal{A}_T\) and \(\mathcal{A}_\Sigma\) and their \(K\)-groups.

The above discussed representation may be used to show that the trace on \(\mathcal{A}_\Sigma\) does satisfy the Shubin formula if a representation on a regular tiling is considered. Let \(T\) be a regular tiling and consider its \(\rho\)-chain. Writing shorter \(\Lambda_m = M_m(T)\) one has

\[
\chi_{\Lambda_m} \pi_T(\mathcal{A}_T^G) \chi_{\Lambda_m} = \pi_T(\mathcal{A}_m^{(m)})
\]

where \(i_m = r(\Xi_m(T))\) and \(\mathcal{A}_m^{(m)}\) is the \(i_m\)’th direct summand of the \(m\)’th approximation of \(\mathcal{A}_\Sigma\), c.f. [108]. For brevity we write \(h_n^{(m)} = h_{n+m-1} \circ \cdots \circ h_n : \mathcal{A}^{(n)} \rightarrow \mathcal{A}^{(n+m)}\) for the embedding of the \(n\)’th approximant into the \(n+m\)’th and identify the approximants with their image under the embeddings \(h_n\) in \(\mathcal{A}_\Sigma\). In particular \(\text{tr}|_{\mathcal{A}^{(n)}} = \text{tr}^{(n)}\).

**Theorem 7** For \(a \in \mathcal{A}_\Sigma\)

\[
\lim_{m \to \infty} \frac{1}{|\Lambda_m^{(m)}|} \text{Tr}(\chi_{\Lambda_m} \pi_T(a)) = \text{tr}(a),
\]

\(\text{tr}\) being the unique normalized trace on \(\mathcal{A}_\Sigma\).

**Proof:** Let \(\chi_{i_m}^{(m)} \in \mathcal{A}_\Sigma^{(m)}\) be the projection on \(\mathcal{A}_m^{(m)}\). Since any two traces on finite dimensional simple \(C^*\)-algebras are proportional

\[
\frac{1}{|\Lambda_m|} \text{Tr}(\chi_{\Lambda_m} \pi_T(a)) = \frac{\text{Tr}(\pi_T(\chi_{i_m}^{(m)} a))}{\text{Tr}(\pi_T(\chi_{i_m}^{(m)}))} = \frac{\text{tr}(m)(\chi_{i_m}^{(m)} a)}{\text{tr}(m)(\chi_{i_m}^{(m)})},
\]

and we need to show that the r.h.s. converges for \(n \to \infty\) to \(\text{tr}(a)\), once \(a \in \mathcal{A}_\Sigma\). As both sides of (83) are continuous in norm we may restrict to \(a \in \mathcal{A}_\Sigma^{(n)}\) for large enough \(n\). Decompose \(a = \sum a_i\) with \(a_i \in \mathcal{A}_i^{(n)}\). Then, c.f. [110],

\[
\text{tr}^{(n+m)}(\chi_{i_{n+m}}^{(n+m)} h_n^{(n+m)}(a)) = \tau^{1-n} \nu_{i_{n+m}} \sum_j s_{j} \text{Tr}(a_j),
\]

which asymptotically behaves like \(\tau^{1-n} \nu_{i_{n+m}} \sum_j s_{j} \text{Tr}(a_j)\), \(S\) here denoting the transformation that diagonalizes \(\sigma\), i.e. \(S \sigma S^{-1} = \text{diag}(\tau, \cdots)\), c.f. [114]. Therefore

\[
\lim_{m \to \infty} \frac{\text{tr}(m)(\chi_{i_m}^{(m)} a)}{\text{tr}(m)(\chi_{i_m}^{(m)})} = \lim_{m \to \infty} \frac{\text{tr}^{(n+m)}(\chi_{i_{n+m}}^{(n+m)} h_n^{(n+m)}(a))}{\text{tr}^{(n+m)}(\chi_{i_{n+m}}^{(n+m)})} = \frac{\sum_j S_{ij} \text{Tr}(a_j)}{\sum_j S_{ij}} = \text{tr}(a),
\]

from which the statement follows. \(\Box\)

This theorem not only shows that the r.h.s. of (43) coincides on elements of \(\pi_T(\mathcal{A}_\Sigma)\) with the trace on \(\mathcal{A}_\Sigma\) but by (27) it even implies that this trace extends to \(\mathcal{A}_T\). Thus to show that Shubin’s formula holds (43) has to be verified. As already mentioned, the proof of (43) given in the appendix of [11] for cases where \(\mathcal{T}^{\text{rec}}\) may be identified with
an amenable group does not rely on the group structure but only on the fact that there
is an increasing chain \( \{ \Lambda_n \}_{n \geq 1} \) approximating \( \mathcal{T} \) such that
\( \lim_{n \to \infty} |b(\Lambda_n)|_{\text{pc}} \to 0 \). Hereby \( b(\Lambda_n) \) denotes the border of \( \Lambda_n \). For the convenience of the reader we present the proof here:
Let \( \{ \Lambda_n \}_{n \geq 1} \) be as for the above theorem, in particular \( \lim_{n \to \infty} |\Lambda_n|_{\text{pc}} \to 0 \). If \( E \) lies in a gap \( \chi(H \leq E) \in C(H) \) and by the Stone-Weierstraß theorem it suffices to proof that
\[
\lim_{n \to \infty} \frac{1}{|\Lambda_n|_{\text{pc}}} \text{Tr}(\chi_{\Lambda_n} H^k - (\chi_{\Lambda_n} H)^k) = 0,
\]
for any natural \( k \). First observe that
\[
\chi_{\Lambda_n}(H^k - (\chi_{\Lambda_n} H)^k) = \chi_{\Lambda_n} H(1 - \chi_{\Lambda_n}) H^{k-1} + \chi_{\Lambda_n} H \chi_{\Lambda_n}(H^{k-1} - (\chi_{\Lambda_n} H)^{k-1})
= \sum_{j=1}^{k-1} (\chi_{\Lambda_n} H)^j (1 - \chi_{\Lambda_n}) H^{k-j}.
\]
Since
\[
\frac{1}{|\Lambda_n|_{\text{pc}}} |\text{Tr}(\chi_{\Lambda_n} H^j (1 - \chi_{\Lambda_n}) H^{k-j})| \leq \frac{1}{|\Lambda_n|_{\text{pc}}} \text{Tr}(\chi_{\Lambda_n}) \| H(\chi_{\Lambda_n} H)^j (1 - \chi_{\Lambda_n}) H^{k-j} \| \leq \| H \|^k
\]
\( H \mapsto \lim_{n \to \infty} \frac{1}{|\Lambda_n|_{\text{pc}}} \text{Tr}(\chi_{\Lambda_n} H^k - (\chi_{\Lambda_n} H)^k) \) is continuous and we may restrict to \( H \in \pi_T(C_c(\mathcal{R})) \). Then \( C := \max\{|M_{\text{pc}}|(M,x) \subset (T,y), \forall x' \in M_{\text{pc}} : H_{xx'} \neq 0\} \) is finite, and
\[
|\text{Tr}(\chi_{\Lambda_n} H^j (1 - \chi_{\Lambda_n}) H^{k-j})| \leq |b(\Lambda_n)|_{\text{pc}} C \max_{x \in T_{\text{pc}}} |x' \in T_{\text{pc}} \sum_{x' \in T_{\text{pc}}} (\chi_{\Lambda_n} H)^j_{xx'} H^{k-j}_{x'x'}|.
\]
Therefore
\[
\lim_{n \to \infty} \frac{1}{|\Lambda_n|_{\text{pc}}} \text{Tr}(\chi_{\Lambda_n} H^k - (\chi_{\Lambda_n} H)^k) \leq \lim_{n \to \infty} \frac{|b(\Lambda_n)|_{\text{pc}}}{|\Lambda_n|_{\text{pc}}} C \sum_{j=1}^{k-1} \max_{x,x' \in T_{\text{pc}}} |H^j_{xx'}| |H^{k-j}_{x'x'}| \]
and \((87)\) follows from \( \lim_{n \to \infty} \frac{|b(\Lambda_n)|_{\text{pc}}}{|\Lambda_n|_{\text{pc}}} \to 0 \).
\[
\square
\]
\subsection{5.1 Substitution sequences}
A one dimensional substitution tiling, examples of which were already given in \([4, 2, 1]\),
is also called a substitution sequence or automatic sequence. It is well known how
to obtain the ergodic measure on the hull of such a tiling even in cases where the
substitution is neither locally invertible nor satisfies B2 \([29, 14]\). However, in case the
hull may be identified with a path space, the induced homeomorphism yields a nice example of a Vershik transform and one may easily show that \( K_0(\mathcal{A}_\mathcal{T}) = K_0(\mathcal{A}_\Sigma) \) as
scaled ordered groups. In fact the equality as ordered groups is already established
once one has shown that \( \varphi \) induces a Vershik transform on the path space: it is given by
Theorem 8.3 in \([13]\). In fact substitution sequences of locally invertible substitutions
satisfying B2 yield concrete examples of minimal dynamical systems – recall that by
Lemma 7 ($\Omega, \varphi$) is minimal – in which the Vershik transform is rather simple and the $K_0$-groups them self, too. In particular for the A-type generalizations of the Fibonacci chain with $2k + 1$ being non prime this is of importance as there the elements of the $K_0$-group yield a finer gap labelling. Hereby again different letters are supposed to represent different congruence classes of punctured intervals so that $G$ is trivial.

To compute $K_0(A_T)$ in case $\rho$ is locally invertible and satisfies B2 so that $\Omega \cong \mathcal{P}_\Sigma$ the minimal dynamical system $(\mathcal{P}_\Sigma, \varphi)$ – the induced homeomorphism $\tilde{\Xi} \circ \varphi \circ \tilde{\Xi}^{-1}$ on $\mathcal{P}_\Sigma$ is simply denoted by $\varphi$, too – is periodically approximated. $\mathcal{P}_\Sigma$ carries the structure of the self-similarity in form of a chain of partitions of itself which become finer and finer: $\{U_\xi\}_{\xi \in \mathcal{P}_\Sigma^{(n)}}$ is a partition of $\mathcal{P}_\Sigma$ into closed disjoint subsets, and it is finer than $\{U_\xi\}_{\xi \in \mathcal{P}_\Sigma^{(n-1)}}$ in the sense that any element of the latter is a disjoint union of elements of the former. This allows us to approximate $\mathcal{P}_\Sigma$ by $\mathcal{P}_\Sigma^{(n)}$ and the shift operation by some map $\varphi_n$ operating cyclically on $\mathcal{P}_\Sigma^{(n)}$ and to obtain the ordered $K_0$-group of $A_T$ as an inductive limit of the direct system of the $K_0$-groups related to the periodic approximants.

Denote by $\mathcal{P}_\Sigma^{(n)}_{i,j}$ the set of paths of length $n$ which end at $j$ and by $\Sigma_{j, i}^{(1)}$ the set of edges from vertex $i$ to $j$. $\mathcal{P}_\Sigma^{(n)}_{i,j}$ may be identified with a subset of $\mathcal{P}_\Sigma^{(n+1)}_{i,j}$ through concatenation, in case $\Sigma_{j, i}^{(1)} \neq \emptyset$: Any edge $\kappa \in \Sigma_{j, i}^{(1)}$ defines by $\xi \mapsto \xi \circ \kappa$ an injective map from $\mathcal{P}_\Sigma^{(n)}_{i,j}$ into

\[
\mathcal{P}_\Sigma^{(n+1)}_{i,j} = \bigcup_{i \in \Sigma^{(0)}_{j, i}} \bigcup_{\kappa \in \Sigma_{j, i}^{(1)}} \mathcal{P}_\Sigma^{(n)}_{i,j} \circ \kappa, \tag{88}
\]

where $\mathcal{P}_\Sigma^{(n)}_{i,j} \circ \kappa = \{\xi \circ \kappa | \xi \in \mathcal{P}_\Sigma^{(n)}_{i,j}\}$. This structure of embeddings is preserved by $\varphi$ in the following sense. Let $\gamma \in \mathcal{P}_\Sigma$ be any path starting at $i$. Recall that a path $\xi \in \mathcal{P}_\Sigma^{(n)}_{i,j}$ encodes the location of the letter $a_s(\xi)$ in $\rho^n(a_r(\xi))$ and that $\varphi$ acts on tilings as left shift. Thus the first $n$ edges of $\varphi(\xi \circ \gamma)$ encode the location of the letter to the right of that $a_s(\xi)$ which was encoded by $\xi$. Moreover, there is a unique path for which $\varphi(\xi \circ \gamma) \notin \mathcal{P}_\Sigma^{(n)}_{i,j} \circ \gamma$. It is the path which encodes the location of the last letter in $\rho^n(a_i)$; it shall be denoted by $\xi^{(n)}_{> \gamma}$. Define $\varphi_n : \mathcal{P}_\Sigma^{(n)} \to \mathcal{P}_\Sigma^{(n)}$ on $\mathcal{P}_\Sigma^{(n)} \setminus \{\xi^{(n)}_{> \gamma}|i \in \Sigma^{(0)}\}$ by

\[
\varphi_n(\xi) \circ \gamma := \varphi(\xi \circ \gamma) \tag{89}
\]

and on $\{\xi^{(n)}_{> \gamma}|i \in \Sigma^{(0)}\}$ by

\[
\varphi_n(\xi^{(n)}_{> \gamma}) := \xi^{(n)}_{> \gamma}, \tag{90}
\]

$\xi^{(n)}_{> \gamma}$ being the path of $\mathcal{P}_\Sigma^{(n)}_{i,j}$ which encodes the position of the first letter in $\rho^n(a_i)$. As a result $\varphi_n|_{\mathcal{P}_\Sigma^{(n)}_{i,j}}$ is cyclic of order $N_i^{(n)} = |\mathcal{P}_\Sigma^{(n)}_{i,j}|$, the number of paths in $\mathcal{P}_\Sigma^{(n)}_{i,j}$.

Stated differently $\varphi$ defines an order on $\mathcal{P}_\Sigma^{(n)}_{i,j}$, namely

\[
\xi^{(n)}_{< \gamma} < \varphi_n^{(n)}(\xi^{(n)}_{< \gamma}) < \cdots < \varphi_n^{N_i^{(n)}-1}(\xi^{(n)}_{< \gamma}) = \xi^{(n)}_{> \gamma}
\]

which corresponds to the order of the letters in $\rho^n(a_i)$. The self-similarity of the tiling now reflects in the fact that these orders are already determined by the orders on
\( \mathcal{P}_{\Sigma,i}^{(1)} = \bigcup_j \Sigma_{ij}^{(1)} \). To see this consider the action of \( \varphi_{n+1} : \mathcal{P}_{\Sigma}^{(n+1)} \to \mathcal{P}_{\Sigma}^{(n+1)} \), i.e. on elements of the form \( \xi \circ \kappa \) with \( \xi \in \mathcal{P}_{\Sigma,s(\kappa)}^{(n)} \). If \( \xi \neq \xi_{>},s(\kappa) \) then the letter encoded by \( \xi \circ \kappa \) is not the last letter of an \( n \)-fold substitute and hence neither the last of an \( n+1 \)-fold substitute. Thus for \( \xi \neq \xi_{>},s(\kappa) \)

\[
\varphi_{n+1}(\xi \circ \kappa) \circ \gamma = \varphi(\xi \circ \kappa \circ \gamma) = \varphi_n(\xi) \circ \kappa \circ \gamma.
\]  

(91)

Now, because of \( \rho^{n+1}(a) = \rho^n(\rho(a)) \) the last letter of \( \rho^n(\rho(a)_\nu) \) – here \( \rho(a)_\nu \) denotes the \( \nu \)th letter of \( \rho(a) \) – is mapped by \( \varphi_{n+1} \) onto the first letter of \( \rho(a)_\nu \) where \( \nu = 1 \) if \( \nu = N_{r(\kappa)} \) and \( \nu' = \nu + 1 \) otherwise. In formula

\[
\varphi_{n+1}(\xi_{>},s(\kappa)) \circ \kappa = \xi_{<},s(\varphi_1(\kappa)) \circ \varphi_1(\kappa)
\]  

(92)

which shows by induction that \( \varphi_n \) is determined by the order on the \( \mathcal{P}_{\Sigma,i}^{(1)} \). Such a transformation is called Vershik transform as it has been introduced by Vershik [12]. It is used in [13] for the analysis of arbitrary minimal dynamical systems (over totally disconnected compact spaces). A chain of partitions becoming finer and finer may then always be constructed but it does in general not lead to a directed system of \( K_0 \)-groups which is stationary.

Let us assume for a moment that the substitution \( \rho \) satisfies B2 in the possibly stronger version – as for the examples given – for which the first letter \( a_f \) as well as the last letter \( a_t \) of \( \rho(a) \) are independent of \( a \). Then there is a unique minimal and a unique maximal path \( \xi_{\min}, \xi_{\max} \) in the sense of [13] which is also characterized by the property \( \varphi(\xi_{\max}) = \xi_{\min} \) but \( (\xi_{\min}, \xi_{\max}) \notin \mathcal{R}_{\Sigma} \). In fact \( \xi_{\min} = \xi_{<},a_f \circ \xi_{<},a_f \circ \cdots \) and \( \xi_{\max} = \xi_{>,a_t} \circ \xi_{>,a_t} \circ \cdots \). The singular tilings are precisely the elements of the \( \mathbb{Z} \)-orbit of \( \Xi^{-1}(\xi_{\min}) \), and \( U_{a_t,a_t,1,2} \) contains \( (\Xi^{-1}(\xi_{\max}), \Xi^{-1}(\xi_{\min})) \) as accumulation point of points of \( (\Xi^{-1} \times \Xi^{-1})/\mathcal{R}_{\Sigma} \). Hence \( (\Xi^{-1} \times \Xi^{-1})/\mathcal{R}_{\Sigma} \) is not a closed subset of \( \mathcal{R} \) in the topology of the latter. \( \mathcal{R} \) is in fact generated as an equivalence relation by \( (\Xi^{-1} \times \Xi^{-1})/\mathcal{R}_{\Sigma} \) and the element \( (\Xi^{-1}(\xi_{\max}), \Xi^{-1}(\xi_{\min})) \). Theorem 8.3 of [13] implies \( K_0(\mathcal{A}_\Sigma) \cong K_0(\mathcal{A}_\Sigma) \) as ordered groups. But let us add a purely \( K \)-theoretic computation which shows that the above \( K_0 \)-groups are isomorphic as groups their order isomorphism then following from the results of the last section. It only requires \( \rho \) to satisfy B2.

Identifying the characteristic functions \( \chi_\xi \) on \( \xi \in \mathcal{P}_{\Sigma,i}^{(n)} \), which are the generators of \( C(\mathcal{P}_{\Sigma,i}^{(n)}, \mathbb{Z}) \), with the standard basis \( \{ e_i \} \) of \( \mathbb{Z}^{N_1^{(n)}} \) furnishes an isomorphism

\[
C(\mathcal{P}_{\Sigma,i}^{(n)}, \mathbb{Z}) \cong \mathbb{Z}^{N_1^{(n)}}.
\]  

(93)

Let \( \Lambda_{N_1^{(n)}} \) be the \( N_1^{(n)} -1 \) dimensional sublattice of \( \mathbb{Z}^{N_1^{(n)}} \) generated by \( \alpha_i = e_i - e_{i+1} \).

Then by the same identification \( E_n := \{ f - f \circ \varphi_n^{-1} | f \in C(\mathcal{P}_{\Sigma}^{(n)}, \mathbb{Z}) \} \cong \Lambda_{N_1^{(n)}} \) and therefore \( C(\mathcal{P}_{\Sigma,i}^{(n)}, \mathbb{Z})/E_n \cong \mathbb{Z}^{N_1^{(n)}}/\Lambda_{N_1^{(n)}} \cong \mathbb{Z} \) or

\[
C(\mathcal{P}_{\Sigma}^{(n)}, \mathbb{Z})/E_n \cong \mathbb{Z}^{\vert \Sigma^{(0)} \vert}.
\]  

(94)
As \( \chi_\xi \sim_{E_n} \chi_{\xi'} \) whenever \( r(\xi) = r(\xi') \), the embedding of groups \( \iota^{(n)} : C(\mathcal{P}_\Sigma^{(n)}, \mathbb{Z})/E_n \to C(\mathcal{P}_\Sigma^{(n+1)}, \mathbb{Z})/E_{n+1} \)

\[
\iota^{(n)}[\chi_\xi] := [\chi_{\{\xi \in \sigma \mid r(\xi) = s(\kappa)\}}] = \sum_{\kappa \in \Sigma^{(1)}_i, r(\xi)} \sum_{\kappa \in \Sigma^{(1)}_i, r(\xi)} [\chi_{\xi \in \kappa}] \tag{95}
\]

is well defined. Hence \((C(\mathcal{P}_\Sigma^{(n)}, \mathbb{Z})/E_n, \iota^{(n)})\) is a directed system of groups. Using the isomorphism of \((93)\) one obtains the commuting diagram

\[
\begin{array}{ccc}
C(\mathcal{P}_\Sigma^{(n)}, \mathbb{Z})/E_n & \xrightarrow{\iota^{(n)}} & C(\mathcal{P}_\Sigma^{(n+1)}, \mathbb{Z})/E_{n+1} \\
\oplus_{i \in \Sigma^{(0)}} \mathbb{Z} N_i^{(n)} / \Lambda N_i^{(n)} & \xrightarrow{\sigma} & \oplus_{i \in \Sigma^{(0)}} \mathbb{Z} N_i^{(n+1)} / \Lambda N_i^{(n+1)}
\end{array}
\tag{96}
\]

which shows that \((C(\mathcal{P}_\Sigma^{(n)}, \mathbb{Z})/E_n, \iota^{(n)})\) is as a directed system isomorphic to \((\mathbb{Z}^{\Sigma^{(0)}}, \sigma)\) the latter having algebraic limit \( K_0(\mathcal{A}_\Sigma) \). Recall that \( K_0(A_T) = C(\mathcal{P}_\Sigma, \mathbb{Z})/E_\varphi \).

**Theorem 8** The direct algebraic limit of the directed system \((C(\mathcal{P}_\Sigma^{(n)}, \mathbb{Z})/E_n, \iota^{(n)})\) is

\[
\lim_{\longrightarrow} C(\mathcal{P}_\Sigma^{(n)}, \mathbb{Z})/E_n = C(\mathcal{P}_\Sigma, \mathbb{Z})/E_\varphi \tag{97}
\]

where \( \iota_n[\chi_\xi] = [\chi_{U_\xi}] \) for \( \xi \in \mathcal{P}_\Sigma^{(n)} \) and hence \( K_0(A_T) \cong K_0(\mathcal{A}_\Sigma) \) as scaled ordered groups.

**Proof:** \( \chi_\xi \sim_{E_n} \chi_{\xi'} \) implies that \( \exists k \in \mathbb{Z} : \varphi^k(\xi) = \xi' \) and hence \( \varphi^k(U_\xi) = U_{\xi'} \). But \( \exists k \in \mathbb{Z} : \varphi^k(U_\xi) = U_{\xi'} \) implies \( \chi_{U_\xi} \sim_{E_\varphi} \chi_{U_{\xi'}} \). Thus \( \iota_n \) is well defined on equivalence classes. Moreover, as

\[
\iota_{n+1} \circ \iota^{(n)}[\chi_\xi] = \iota_{n+1}[\chi_{\{\xi \in \sigma \mid r(\xi) = s(\kappa)\}}] = \sum_{\kappa \in \Sigma^{(1)}, r(\xi) = s(\kappa)} [\chi_{\xi \in \kappa}] = [\chi_{U_\xi}] \tag{98}
\]

diagram \((94)\) commutes, namely \( \iota_{n+1} \circ \iota^{(n)} = \iota_n \). Finally \( [\chi_{U_\xi}] \mapsto \left[[0, \cdots, 0, [\chi_\xi], 0, \cdots] \right] \) is an isomorphism onto the standard realization of the inductive limit, c.f. \((111)\). This shows that \( K_0(A_T) \cong K_0(\mathcal{A}_\Sigma) \) as groups.

To see that \( K_0^+(A_T) = K_0^+(\mathcal{A}_\Sigma) \) recall that \( \text{tr}_* K_0(A_T) = \text{tr}_* K_0(\mathcal{A}_\Sigma) \) by Theorem \(3\). In view of \((115)\) it has to be shown that \( K_0^+(A_T) = \{ z \in K_0(A_T) | \text{tr}_*(z) > 0 \} \cup \{0\} \). The inclusion \( \subset \supset \) follows from the faithfulness and positivity of \( \text{tr} \). To prove the opposite inclusion let \( t = \text{tr}_*(x) \) for some \( x \in K_0(A_T) \). Then there is as well a \( z \in K_0(\mathcal{A}_\Sigma) \) with \( \text{tr}_*(z) = t \) and by the properties of an AF-algebra c.f. \((113)\) and the preceding remark a projection \( p \in C(\mathcal{P}_\Sigma) \otimes M_\infty(\mathbb{C}) \), such that \( \text{tr}_*(z) = \text{tr}(p) \). By the identification \( C(\Omega) \cong C(\mathcal{P}_\Sigma) \) of maximal commuting subalgebras of \( A_T \) and \( \mathcal{A}_\Sigma \), which preserves the restrictions of the traces, the class of \( p \) identifies with a positive element of \( K_0(A_T) \) whose image under \( \text{tr}_* \) is \( t \).

Finally \([1_{A_T}] = [1_{\mathcal{A}_\Sigma}] \) is directly seen from the commuting diagram \((96)\), since \([1_{A_T}] = [\chi_\xi] = [\chi_{P_\Sigma}] \). \( \square \)
Summary

An algebra which is well suited for the gap labelling of Schrödinger operators on a non periodic tiling has been defined purely from the geometrical data of the tiling. For its definition we discussed the hull of the tiling, its non commutative space which is a non Hausdorff set, and the groupoid induced by the translations. A topology on the groupoid was found so that the algebra $A_T$ of the tiling $T$ could be defined as the corresponding (reduced) groupoid $C^*$-algebra. We restricted to a class of tilings which lead to compact hulls.

It was shown that the gap labelling by means of the values of the integrated density of states is partly determined by an invariant measure on the hull of the tiling. More specifically, for Cartesian products of one dimensional systems the abstract gap labelling was solved by expressing the $K$-groups of $A_T$ as well as the image of $K_0(A_T)$ under $\text{tr}_*$ through the ones of their one dimensional components. It turned out that the invariant measure on the hull already fully determines that image under $\text{tr}_*$.

To obtain concrete results we specified to selfsimilar tilings which are invariant under a locally invertible substitution satisfying an extra condition B2. For these a homeomorphism between the hull and a space of paths on a graph could be constructed, which displays the topological structure in a very clear way. Such a path space naturally defines a topological groupoid $R_\Sigma$ which identifies with a subset of the original groupoid $R$ and induces the same invariant measure. This measure was determined by the $K$-theory of the $AF$-algebra of the path space $A_\Sigma$, which is the groupoid $C^*$-algebra of $R_\Sigma$. Moreover the $AF$-algebra was used to show the validity of the Shubin formula for substitution tilings.

Although $R_\Sigma$ cannot be identified with a closed subset of $R$ both algebras, $A_\Sigma$ and $A_T$, have a lot in common. We conjecture that $\text{tr}_* K_0(A_T) = \text{tr}_* K_0(A_\Sigma)$ does not only hold for Cartesian products of one dimensional substitution tilings but also for general ones. One dimensional substitution tilings (sequences) are particularly simple: by a purely $K$-theoretic calculation it was shown that the scaled ordered $K_0$-groups of $A_T$ and $A_\Sigma$ coincide, their difference being given by their $K_1$-groups.

There are two dimensional substitution tilings with 8-, and 12-fold orientational symmetry, e.g. the Ammann-Beenker and Socolar tilings, which are of great interest, but which cannot be treated yet in the above manner, as their substitutions do not satisfy B2. An extension of the analysis of this paper to substitutions satisfying weaker conditions is presently under investigation and will be the subject of a future publication.
A Some $K$-theory and $AF$-algebras

We briefly define the $K$-groups of a unital $C^*$-algebra referring the reader to [31] and [32] for more general treatments. We apply it to $AF$-algebras as an example which is of importance in the main text. But first we recall the definition of the direct algebraic limit of a directed system [33, 34, 32].

A directed system is a family $(A^{(n)}, h^{(n)})$, $n \in \mathbb{Z}^\geq 0$, of objects and morphisms of a category

$$A^{(1)} \xrightarrow{h^{(1)}} A^{(2)} \xrightarrow{h^{(2)}} \ldots.$$  \hspace{1cm} (99)

Its (direct or inductive) algebraic limit is categorically defined as universal repelling object $(A, h_n)$ for which the diagram

$$A^{(n)} \xrightarrow{h^{(n)}} A^{(n+1)}$$

$$h_n \searrow \swarrow h_{n+1}$$

commutes. It is often simply denoted by $A = \varinjlim A^{(n)}$. For the category of Abelian groups the algebraic limit exists and a realization is given through the direct sum of all $A^{(n)}$ modulo the equivalence relation which is generated by

$$(0, \ldots, 0, a, 0, \ldots) \sim (0, \ldots, 0, h^{(n)}(a), 0, \ldots).$$  \hspace{1cm} (101)

For $C^*$-algebras it is the $C^*$-algebraic limit which is of interest. This is the $C^*$-envelope of the above algebraic limit in which case we write instead $A = \varprojlim A^{(n)}$. Any element of $A$ may be approximated by elements of the approximants, i.e. elements of the form $h_n(a)$ with $a \in A^{(n)}$ for some $n$.

A.1 Preliminary $K$-theory of unital $C^*$-algebras

Let $A$ be a unital $C^*$-algebra. $Proj(A)$ denotes the set of equivalence classes of projections of $A$ under $p \sim q$ whenever $\exists u \in A : p = uu^*, q = u^*u$. In order to introduce the structure of addition on $Proj(A)$ the algebra $A$ has to be stabilized: Embed $M_k(A)$ (non-unitally) by $\ast$-homomorphisms

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (102)

into $M_{k+1}(A)$. The stabilization of $A$ is the algebraic limit of the directed system so obtained. It is denoted by $M_\infty(A)$ and isomorphic to $A \otimes M_\infty(\mathbb{C})$. The completion of $M_\infty(\mathbb{C})$ is isomorphic to the algebra of compact operators on a separable Hilbert space. Now the sum of two projection classes $[p], [q]$ of $V(A) := Proj(M_\infty(A))$ may be defined by

$$[p] + [q] = \left[ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]$$  \hspace{1cm} (103)
thereby $V(\mathcal{A})$ becoming a monoid. $K_0(\mathcal{A})$, the $K_0$-group of $\mathcal{A}$, is obtained by Grothendieck’s construction. It is the quotient $K_0(\mathcal{A}) = V(\mathcal{A}) \times V(\mathcal{A})/\sim$ under $([p],[q]) \sim ([p'],[q'])$ whenever $\exists [r] \in V(\mathcal{A}) : [p] + [q'] + [r] = [q] + [p'] + [r]$.

In the cases of interest for us in which $\mathcal{A}$ is unital and carries a (positive) faithful normalized trace $K_0^+(\mathcal{A}) = \{([p],[0])| [p] \in V(\mathcal{A})\}$ defines a positive cone of $K_0(\mathcal{A})$, i.e. a subset which is closed under addition and satisfies $K_0^+(\mathcal{A}) - K_0^+(\mathcal{A}) = K_0(\mathcal{A})$ as well $K_0^+(\mathcal{A}) \cap -K_0^+(\mathcal{A}) = \{0\}$. In other words $K_0(\mathcal{A})$ is an ordered group. To simplify the notation one usually writes $[p]$ in place of $([p],[0])$.

In order to distinguish different algebras having the same stabilization one keeps track of the unit of $\mathcal{A}$, i.e. its $K_0$-class $[1]$ serves as distinguished order unit. The set $\{[p] \in K_0^+(\mathcal{A})| [0] \leq [p] \leq [1]\}$ defines a scale. It coincides with the image of $Proj(\mathcal{A})$ in $K_0^+(\mathcal{A})$ if $\mathcal{A}$ has cancellation, i.e. if $p \sim q, p \perp p', q \perp q' \Rightarrow p' \sim q'$ holds for projections in $V(\mathcal{A})$. $K_0(\mathcal{A})$ together with the above ordering and the distinguished order unit $[1]$ is referred to as the scaled ordered $K_0$-group of $\mathcal{A}$.

To define the $K_1$-group of $\mathcal{A}$ one considers the groups of invertible elements $GL_n(\mathcal{A})$ of $M_n(\mathcal{A})$. These form under the embedding

$$a \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

a directed system of groups the algebraic limit of which is denoted by $GL(\mathcal{A})$. Let $GL(\mathcal{A})_0$ be the connected component of 1 (in the inductive limit topology). Then $K_1(\mathcal{A}) := GL(\mathcal{A})/GL(\mathcal{A})_0$ is as well an Abelian group the multiplication being

$$[u][v] = [uv] \sim \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$  

The computability of $K$-groups of $C^*$-algebras is based on the fact that both, $K_0$ and $K_1$ are functors from the category of $C^*$-algebras to the category of Abelian groups which preserve direct sums and algebraic limits. Any $\ast$-homomorphism $h : \mathcal{A} \to \mathcal{B}$ induces a $\ast$-homomorphism $h \otimes 1 : \mathcal{A} \otimes M_\infty(\mathbb{C}) \to \mathcal{B} \otimes M_\infty(\mathbb{C})$ which is usually also denoted by $h$ and a group homomorphism $h_* = K_i(h) : K_i(\mathcal{A}) \to K_i(\mathcal{B})$ through $h_*[x] = [h(x)]$.

Similarly a normalized trace $\text{tr}$ on $\mathcal{A}$ induces a trace $\text{tr} \otimes \text{Tr} : \mathcal{A} \otimes M_\infty(\mathbb{C}) \to \mathbb{C}$, again usually denoted simply by $\text{tr}$, and a state on $K_0(\mathcal{A})$ by $\text{tr}_*[x] = \text{tr}(x)$. Moreover for a large class of $C^*$-algebras the behavior of the functors under tensor products is known. Namely if one restricts to nuclear $C^*$-algebras, for which a unique $C^*$-tensor product exists, the Künneth formula, which we cite in a simplified way from [31], yields:

**Theorem 9** Let $\mathcal{A}, \mathcal{B}$ nuclear $C^*$-algebras having $K$-groups which do not contain nilpotent elements. Then

$$K_0(\mathcal{A} \otimes \mathcal{B}) \cong K_0(\mathcal{A}) \otimes K_0(\mathcal{B}) \oplus K_1(\mathcal{A}) \otimes K_1(\mathcal{B})$$

$$K_1(\mathcal{A} \otimes \mathcal{B}) \cong K_0(\mathcal{A}) \otimes K_1(\mathcal{B}) \oplus K_1(\mathcal{A}) \otimes K_0(\mathcal{B}).$$

Commutative $C^*$-algebras and AF-algebras are nuclear as well as crossed products of $\mathbb{Z}^d$ with nuclear $C^*$-algebras.
**AF-algebras**

AF-algebras belong to the simplest non-commutative algebras of infinite dimension. They are the norm closure of the algebraic limit of a directed system of finite dimensional $C^*$-algebras $\mathcal{A}^{(n)}$ with $*$-homomorphisms $h^{(n)}$. Any finite dimensional $C^*$-algebra is semi-simple, i.e. it is a direct sum of simple $C^*$-algebras, and its direct summands (simple components) are isomorphic to matrix algebras $M_m(\mathbb{C})$. Hence the approximants $\mathcal{A}^{(n)}$ of $\mathcal{A}$ are of the form

$$\mathcal{A}^{(n)} = \bigoplus_{i=1}^{k_n} \mathcal{A}_i^{(n)}, \quad \mathcal{A}_i^{(n)} \cong M_{N_i^{(n)}}(\mathbb{C}). \quad (108)$$

By the functoriality of $K_1$ one immediately obtains a directed system of $K$-groups $(K_i(\mathcal{A}^{(n)}), h_i^{(n)})$ whose algebraic limits furnish the $K$-groups of $\mathcal{A}$. Because of $K_1(\mathbb{C}) = 0$, the system of $K_1$-groups is trivial as well as its algebraic limit. On the other hand $K_0(\mathbb{C}) = \mathbb{Z}$ implies that $K_0(\mathcal{A}^{(n)}) = \mathbb{Z}^{k_n}$, and $h_\star^{(n)}$ is a positive integer $k_{n+1} \times k_n$ matrix, its $ji$-coefficient being given by the number of times component $M_{N_i^{(n)}}(\mathbb{C})$ is mapped by $h^{(n)}$ into $M_{N_j^{(n+1)}}(\mathbb{C})$. The positive cone of $K_0(\mathcal{A})$ is then given by

$$K_0^+(\mathcal{A}) = \bigcup_{n \geq 1} h_{\star}^+(K_0^+(\mathcal{A}^{(n)})). \quad (109)$$

The directed system of algebras may be encoded in a Bratteli diagram. A Bratteli diagram consists of a set of weighted vertices $V$ which are grouped into floors $V_n$, i.e. $V = \cup_{n \geq 1} V_n$, and of oriented edges between successive floors. The $i$'th vertex of $V_n$ is identified with the generator of $K_0(\mathcal{A}_i^{(n)})$ and has as weight the square root of the dimension of $\mathcal{A}_i^{(n)}$, that is $N_i^{(n)}$. It is joint to the $j$-th vertex of $V_{n+1}$ by $(h_\star^{(n)})_{ji}$ oriented edges. These edges together with $V_n$ and $V_{n+1}$ furnish the (bipartite) graph of the embedding of $\mathcal{A}^{(n)}$ into $\mathcal{A}^{(n+1)}$ and the matrix $h_\star^{(n)}$ is called the embedding matrix.

A Bratteli diagram, which hence also may be seen as a sequence of embedding graphs, determines the directed system of algebras (up to isomorphism) and may be considered as an invariant of it. Hence it also determines the AF-algebra, however it may happen that different Bratteli diagrams which are not comparable by a simple algorithm yield the same AF-algebra. For this reason Bratteli diagrams are unsuitable to classify AF-algebras. This is achieved by $K$-theory. Elliot proved $[58]$ that the scaled ordered $K_0$-group is a complete invariant for AF-algebras.

The directed system of $K_0$-groups is already determined through the unweighted Bratteli diagram alone. If the homomorphisms $h^{(n)}$ are all unital embeddings, then the dimensions of the $\mathcal{A}_j^{(n+1)}$ are already determined by the ones of the $\mathcal{A}_i^{(n)}$ together with the matrix $h_\star^{(n)}$. In this case all weights are already determined by the weights of $V_1$. The weights of $V_1$ yield exactly the information of the order unit, namely $[1] = w$ where $w_i = N_i^{(1)}$.

We now specify to systems for which $h^{(n)}$ are unital embeddings and $K_0(\mathcal{A}^{(n)})$ and $h_\star^{(n)}$ independent of $n$, i.e. the directed system is of the form $(\mathbb{Z}^+, \sigma)$ (stationary). In that case, which is the one occurring in the main text, the floors of the Bratteli diagram may
all be identified with $V_1$ and the edges between them with the edges between $V_1$ and $V_2$ so that the whole diagram may be encoded in a finite weighted graph $\Sigma$. Namely $\Sigma$ has vertices $\Sigma^{(0)} = V_1$ and (oriented) edges $\Sigma^{(1)}$ equal to the edges between $V_1$ and $V_2$ and the weights are the weights of $V_1$. In particular, $\sigma$ is the connectivity matrix of $\Sigma$. We denote the AF-algebra which is defined by the Bratteli diagram by $A_{\Sigma,w}$ where $w$ is the weight vector or, if $w_i = 1$ for all $i$, simply by $A_{\Sigma}$.

The faithful normalized traces on unital AF-algebras do exactly correspond to the states on the scaled ordered $K_0$-group. If in addition $\sigma$ is primitive the trace on $A_{\Sigma}$ is up to normalization unique and correspondingly there is only one state on that group [18,20]: Let $\tau$ be the Perron-Frobenius-eigenvalue and $\nu$ be the left-Perron-Frobenius-eigenvector of $\sigma = h^{(n)}$ normalized to $(\nu,z) = 1$ (Euclidean scalar product), then

$$\text{tr}^{(n)}(a) = \tau^{1-n} \sum_i \nu_i \text{Tr}(a_i),$$

defines a system of faithful normalized traces on the directed system $(A^{(n)}_{\Sigma,w}, h^{(n)})$ where $\text{Tr}$ is the usual trace of matrices. This system is compatible with the embedding structure, i.e. $\text{tr}^{(n+1)} \circ h^{(n)} = \text{tr}^{(n)}$ and may therefore be extended to the algebraic limit (and its norm closure). It induces a state $\text{tr}_{\ast}$ on the scaled ordered $K_0$-group, namely for $z \in K_0(A^{(n)}) \cong \mathbb{Z}^r$:

$$\text{tr}_{\ast}(h_{n\ast}(z)) = \tau^{1-n}(\nu,z)$$

and the image of that state is given by

$$\text{tr}_{\ast}(K_0(A_{\Sigma,w})) = \{\tau^{1-n}(\nu,z) | n > 0, z \in \mathbb{Z}^r\}$$

which is an additive subgroup of $\mathbb{R}$. This group is a self-similar subset of $\mathbb{R}$ as it is invariant under multiplication by $\tau$ and by $\tau^{-1}$. As any element of $A_{\Sigma}$ is approximated by elements of the finite dimensional approximants, every projection of $A_{\Sigma} \otimes M_\infty(\mathbb{C})$ is equivalent to a projection of $C(\mathcal{P}_{\Sigma}^{(n)}) \otimes M_m(\mathbb{C})$ for some $n, m$. Hence, if $x \in \text{tr}_{\ast}(K_0^+(A_{\Sigma}))$ there exists a projection $p \in C(\mathcal{P}_{\Sigma}) \otimes M_\infty(\mathbb{C})$ such that $x = \text{tr}(p)$, and

$$\text{tr}_{\ast}(K_0(A_{\Sigma})) = \mu(C(\mathcal{P}_{\Sigma}, \mathbb{Z}))$$

$\mu$ being the invariant measure corresponding to $\text{tr}$. Let $S$ be a base transformation diagonalizing $\sigma$, i.e. $S \sigma S^{-1} = \text{diag}(\tau, \ldots)$. Then $S_{ij}$ resp. $S^{-1}_{ij}$ are the components of the unnormalized left- resp. right-Perron-Frobenius-eigenvector and may therefore be chosen to be strictly positive. As $\tau$ exceeds all other eigenvalues of $\sigma$ in modulus one obtains for $z \in \mathbb{Z}^r$

$$\lim_{n \to \infty} \tau^{-n}(\sigma^n(z))_i = S_{i1}^{-1} \sum_j S_{1j} z_j$$

and therefore: $\sum_j S_{1j} z_j > 0$ whenever $\exists n : \sigma^n(z) \in (\mathbb{Z}_{>0})^r$. As a result

$$K_0^+(A_{\Sigma}) = \{z \in K_0(A_{\Sigma}) | \text{tr}_{\ast}(z) > 0\} \cup \{0\}.$$
The algebraic limit of the $K_0$-group itself is easily obtained if $\sigma$ is an automorphism of $\mathbb{Z}^r$. In this case $\sigma^{1-n}$ identifies $K_0(A^{(n)}_\Sigma)$ with $K_0(A^{(1)}_\Sigma)$ and therefore $\lim_{n \to \infty} K_0(A^{(n)}_\Sigma) = \mathbb{Z}^r$. If $\sigma$ is not over $\mathbb{Z}$ invertible but still $\frac{1}{\det \sigma} \sigma^r \in \text{Aut}(\mathbb{Z}^r)$, then $\sigma^{-rn} = (\frac{1}{\det \sigma})^n (\frac{1}{\det \sigma} \sigma^r)^{-n}$ which leads to $K_0(A_{\Sigma,w}) = (\mathbb{Z}[\frac{1}{\det \sigma}])^r$ as algebraic limit. In case $\sigma$ is neither degenerate nor $\frac{1}{\det \sigma} \sigma^r$ an automorphism we cannot give such an explicit form for the $K_0$-group. Finally, in case $\sigma$ is degenerate so that the dimension of $F = \text{im}(h^r)$ strictly smaller than $r$, one may use the directed system $(F, h^r|_F)$. It is isomorphic to $(\mathbb{Z}^r, h)$, since

$$
\begin{align*}
\mathbb{Z}^r & \xrightarrow{h^r} \mathbb{Z}^r \\
F & \xrightarrow{h^r|_F} F
\end{align*}
$$

commutes. Hence $K_0(A_{\Sigma})$ coincides with the algebraic limit of $(F, h^r|_F)$ which may, if possible, be computed as described above.

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Figure 1: Part of a Penrose tiling.

Figure 2: Deflation of the Penrose tilings together with an illustration for the Robinson map. The graph $\Sigma$ is shown below. Its (oriented) edges, which are also the edges of the embedding graph, are denoted by $a,b,c,d,e$. The paths of length 2 encoding the positions of the tiles in the 2-fold-substitutions are indicated in the tiles.

Figure 3: This is a pattern of a Penrose tiling which is determined by the 2-fold substitution of the smaller triangle. The boundaries of this 2-fold substitution appear as mirror axes.
