Strain Tensors and Matching Property on Degenerated Hyperbolic Surfaces

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Abstract We prove the regularity of solutions to the strain tensor equation on degenerated hyperbolic surfaces $S$ where the Gauss curvature is zero on a part of boundary. Furthermore, we obtain the density property that smooth infinitesimal isometries are dense in the $W^{2,2}(S, \mathbb{R}^3)$ infinitesimal isometries. Finally, the matching property is established. Those results are important tools in obtaining recovery sequences ($\Gamma$-lim sup inequality) for dimensionally-reduced shell theories in elasticity.

Keywords shell, nonlinear elasticity, Riemannian geometry, tensor analysis

Mathematics Subject Classifications (2010) 74K20(primary), 74B20(secondary).

1 Introduction and Main Results

Let $M \subset \mathbb{R}^3$ be a surface with a normal $\vec{n}$ and let the middle surface of a shell be an open set $S \subset M$. Let $T^k S$ denote all the $k$-order tensor fields on $S$ for an integer $k \geq 0$. Let $T^2_{\text{sym}} S$ be all the 2-order symmetrical tensor fields on $S$. For $y \in W^{1,2}(S, \mathbb{R}^3)$, we decompose it into $y = W + w\vec{n}$, where $w = \langle y, \vec{n} \rangle$ and $W \in TS$. For $U \in T^2_{\text{sym}} S$ given, linear strain tensor of a displacement $y \in W^{1,2}(S, \mathbb{R}^3)$ of the middle surface $S$ takes the form

$$\text{sym} \, DW + w\Pi = U \quad \text{for} \quad x \in S,$$

This work is supported by the National Science Foundation of China, grants no. 12071463 and Key Research Program of Frontier Sciences, CAS, no. QYZDJ-SSW-SYS011.
where $D$ is the connection of the induced metric in $M$, $2\text{sym}\, DW = DW + D^TW$, and $\Pi$ is the second fundamental form of $M$. Equation (1.1) plays a fundamental role in the theory of thin shells, see [2, 4, 3, 5, 7, 9] and many others. When $U = 0$, a solution $y$ to (1.1) is referred to as an \textit{infinitesimal isometry}.

The type of equation (1.1) depends on the sign of the curvature on the region $S$: It is elliptic if $S$ has positive curvature; it is parabolic if the curvature is zero but $\Pi \neq 0$ on $S$; it is hyperbolic if $S$ has negative curvature.

Here we establish the regularity of solutions to (1.1) when $S$ is a non-characteristic region where its curvature is zero on a part of boundary, that will be specified below. Then it is proved that smooth infinitesimal isometries are dense in the $W^{2,2}(S, \mathbb{R}^3)$ infinitesimal isometries on the region $S$. Finally, the matching property is derived that smooth enough infinitesimal isometries can be matched with higher order infinitesimal isometries. Those results are important tools in obtaining recovery sequences ($\Gamma$-lim sup inequality) for dimensionally-reduced shell theories in elasticity, when the elastic energy density scales like $h^\beta$, $\beta \in (2, 4)$, that is, intermediate regime between pure bending ($\beta = 2$) and the von-Kármán regime ($\beta = 4$). Such results have been obtained for elliptic surfaces [5], developable surfaces [2], and hyperbolic surfaces [7, 9]. A survey on this topic is presented in [3].

In this paper we study the degenerated hyperbolic equation (1.1), which is equivalent to a degenerated hyperbolic scalar equation of the form

$$\langle D^2w, Q^\ast\Pi \rangle + \frac{1}{\kappa}\langle Dw, X_0 \rangle + \kappa(\text{tr} g\Pi)w = \kappa f \quad \text{on} \quad S,$$

(1.2)

for $x \in S$, $\kappa \neq 0$, where $w \in L^2(S)$ is the unknown, $f \in L^2(S)$ and $F \in L^2(S, TS)$ are given, and $\kappa$ is the Gauss curvature. When $\kappa = 0$, there are two terms to degenerate in (1.2): The coefficient of $\langle Dw - F, X_0 \rangle$ becomes infinite; the second derivative of $w$ along the direction of the nonzero principal curvature is zero in $\langle D^2w, Q^\ast\Pi \rangle$. Those situations challenge the analysis of (1.2).

Here we employ the Bochner technique and the tensor analysis to cope with the degenerates in (1.2), where some priori estimates near the zero curvature curve in Section 2 play the key role.

We state our main results as follows. Let $S \subset M$ be given by

$$S = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (0, b) \}, \quad a > 0, b > 0,$$

where $\alpha : [0, a) \times [0, b] \to M$ is an imbedding map which is a family of regular curves with two parameters $t$, $s$ such that

$$\Pi(\alpha_t(t, s), \alpha_t(t, s)) > 0 \quad \text{for all} \quad (t, s) \in [0, a) \times [0, b],$$

(1.3)

where $\alpha(\cdot, s)$ is a closed curve with period $a$ for each $s \in [0, b]$ and

$$\{ \alpha(t, 0) \mid t \in [0, a) \}$$
is a closed curve or just one point.

**Curvature assumptions** Let $\kappa$ be the Gaussian curvature function on $M$. We assume that $S$ satisfies the following curvature conditions:

$$\kappa(x) < 0 \quad \text{for} \quad x \in S \cup \Gamma_0; \quad (1.4)$$

$$\kappa = 0, \quad D\kappa(x) \neq 0 \quad \text{for} \quad x \in \Gamma_b,$$  

(1.5)

where

$$\Gamma_b = \{ \alpha(t, b) \mid t \in [0, a) \}, \quad \Gamma_0 = \{ \alpha(t, 0) \mid t \in [0, a) \}.$$

Our main results are the following.

**Theorem 1.1.** Let $S$ be of class $C^{2,1}$. For $U \in W^{2,2}(S, T^2_{\text{sym}} S)$, there exists a solution $y = W + w\tilde{n} \in W^{1,2}(S, \mathbb{R}^3)$ to equation (1.1) satisfying the bounds

$$\|W\|^2_{W^{2,1}(S,T)S} + \|w\|^2_{W^{1,2}(S)} \leq C\|U\|^2_{W^{2,2}(S, T^2_{\text{sym}} S)}.$$  

(1.6)

If, in addition, $S \in C^{m+1,1}$, $U \in W^{m+1,2}(S, T^2_{\text{sym}} S)$ for some $m \geq 2$, then

$$\|W\|^2_{W^{m+2,2}(S,T)S} + \|w\|^2_{W^{m,2}(S)} \leq C\|U\|^2_{W^{m+1,2}(S, T^2_{\text{sym}} S)}.$$  

By the imbedding theorem [1, P. 158], the following corollary is immediate.

**Corollary 1.1.** Let $m \geq 0$ be an integer and let $S$ be of $S \in C^{m+2,1}$. Then problem (1.1) admits a solution $y = W + w\tilde{n} \in C^{m}_{B}(S, \mathbb{R}^3)$ satisfying

$$\|W\|_{C^{m+1}_{B}(S,T)S} + \|w\|_{C^{m}_{B}(S)} \leq C\|U\|_{C^{m+3}_{B}(S, T^2_{\text{sym}} S)},$$

where

$$C^{m}_{B}(S, \mathbb{R}^3) = \{ y \in C^{m}(S, \mathbb{R}^3) \mid D^\alpha y \in L^{\infty}(S, \mathbb{R}^3) \quad \text{for} \quad |\alpha| \leq m \}.$$  

For $y \in W^{1,2}(S, \mathbb{R}^3)$, we denote the left hand side of equation (1.1) by $\text{sym} \nabla y$. Let

$$\mathcal{V}(S, \mathbb{R}^3) = \{ y \in W^{2,2}(S, \mathbb{R}^3) \mid \text{sym} \nabla y = 0 \}.$$  

**Theorem 1.2.** Let $S$ be of class $C^{m+3,1}$ for some integer $m \geq 0$. Then, for every $y \in \mathcal{V}(S, \mathbb{R}^3)$ there exists a sequence $\{ y_k \} \subset \mathcal{V}(S, \mathbb{R}^3) \cap C^{m}_{B}(S, \mathbb{R}^3)$ such that

$$\lim_{k \to \infty} \|y - y_k\|_{W^{2,2}(S, \mathbb{R}^3)} = 0.$$  

A one parameter family $\{ y_\varepsilon \}_{\varepsilon > 0} \subset C^{1}_{B}(\overline{S}, \mathbb{R}^3)$ is said to be a (generalized) $m$th order infinitesimal isometry if the change of metric induced by $y_\varepsilon$ is of order $\varepsilon^{m+1}$, that is,

$$\|\nabla^T y_\varepsilon \nabla y_\varepsilon - g\|_{L^{\infty}(S,T^2)} = O(\varepsilon^{m+1}) \quad \text{as} \quad \varepsilon \to 0,$$
where $g$ is the induced metric of $M$ from $\mathbb{R}^3$, see [2]. A given $m$th order infinitesimal isometry can be modified by higher order corrections to yield an infinitesimal isometry of order $m_1 > m$, a property to which we refer to by matching property of infinitesimal isometries, [2, 5]. This property plays an important role in the construction of a recover sequence in the $\Gamma$-limit for thin shells.

**Theorem 1.3.** Let $S$ be of class $C^{4m,1}$. Given $y \in \mathcal{V}(S, \mathbb{R}^3) \cap C^{4m-2}_{B}(S, \mathbb{R}^3)$, there exists a family $\{ z_\varepsilon \}_{\varepsilon > 0} \subset C^{2}_{B}(S, \mathbb{R}^3)$, equi-bounded in $C^{2}_{B}(S, \mathbb{R}^3)$, such that for all small $\varepsilon > 0$ the family:

$$y_\varepsilon = \text{id} + \varepsilon y + \varepsilon^2 z_\varepsilon$$

is a $m$th order infinitesimal isometry of class $C^2_{B}(S, \mathbb{R}^3)$.

2 Some priori estimates near the zero curvature curve

Let $\nabla$ and $D$ denote the connection of $\mathbb{R}^3$ in the Euclidean metric and the one of $M$ in the reduced metric, respectively. We have to treat the relationship between $\nabla$ and $D$ carefully.

Let $m \geq 1$ be an integer. Let $T \in T^mM$ be a $m$th order tensor field on $M$. We define a $m-1$th order tensor field by

$$i_Y T(Y_1, \cdots, Y_{m-1}) = T(Y, Y_1, \cdots, Y_{m-1}) \quad \text{for} \quad Y_1, \cdots, Y_{m-1} \in TM,$$

which is called an inner product of $T$ with $Y$. For any $T \in T^2S$ and $\alpha \in T_xM$,

$$\text{tr}_g i_\alpha DT$$

is a linear functional on $T_xM$, where $\text{tr}_g i_\alpha DT$ is the trace of the 2-order tensor field $i_\alpha DT$ in the induced metric $g$. Thus there is a vector, denoted by $\text{div}_g T$, such that

$$\langle \text{div}_g T, \alpha \rangle = \text{tr}_g i_\alpha DT \quad \text{for} \quad \alpha \in T_xM, \ x \in M.$$

Clearly, the above formula defines a vector field $\text{div}_g T \in TM$.

We need a linear operator $Q$ ([7], [9]) as follows. For each point $p \in M$, the Riesz representation theorem implies that there exists an isomorphism $Q : T_pM \to T_pM$ such that

$$\langle \alpha, Q\beta \rangle = \det (\alpha, \beta, \vec{n}(p)) \quad \text{for} \quad \alpha, \beta \in T_pM. \quad (2.1)$$

Let $e_1, e_2$ be an orthonormal basis of $T_pM$ with positive orientation, that is,

$$\det \left( e_1, e_2, \vec{n}(p) \right) = 1.$$

Then $Q$ can be expressed explicitly by

$$Q\alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all} \quad \alpha \in T_pM. \quad (2.2)$$
Clearly, $Q$ satisfies
\[ Q^T = -Q, \quad Q^2 = -\text{Id}. \]

The operator $Q$ plays an important role in our analysis.

In the present section, we consider problem
\[
\begin{aligned}
\text{div}_g Q \nabla \vec{n} V &= f_1, \\
\text{div}_g V &= f_2,
\end{aligned}
\]
(2.3)

where $V \in TS$ and $f_i$ are functions on $S$.

For given $\varepsilon > 0$, set
\[
S_{b-\varepsilon} = \{ \alpha(t,s) \mid (t,s) \in [0,a) \times (b-\varepsilon, b) \}, \quad \Gamma_{b-\varepsilon} = \{ \alpha(t,b-\varepsilon) \mid t \in [0,a) \}.
\]

The main results of this section are the following.

**Theorem 2.1.** Let $S_{b-\varepsilon}$ be of class $C^{3,1}$. For given $\varepsilon > 0$ small, there is $\sigma_\varepsilon > 0$ such that for any solution $V \in TS_{b-\varepsilon}$ to (2.3), the following estimates hold true.

\[
\sigma_\varepsilon \| V \|^2_{L^2(S_{b-\varepsilon},TS_{b-\varepsilon})} \leq \| f_1 \|^2_{L^2(S_{b-\varepsilon})} + \| f_2 \|^2_{L^2(S_{b-\varepsilon})} + \| V \|^2_{L^2(\Gamma_{b-\varepsilon},TM)}; \quad (2.4)
\]

\[
\sigma_\varepsilon \| V \|^2_{W^{1,2}(S_{b-\varepsilon},TS_{b-\varepsilon})} \leq \| f_1 \|^2_{W^{1,2}(S_{b-\varepsilon})} + \| f_2 \|^2_{W^{1,2}(S_{b-\varepsilon})} + \| i_\nu DV \|^2_{L^2(\Gamma_b)} + \| V \|^2_{L^2(\Gamma_{b-\varepsilon},TM)} + \| DV \|^2_{L^2(\Gamma_{b-\varepsilon},\mathcal{T}M)}; \quad (2.5)
\]

where $\nu$ is the outside normal of $S_{b-\varepsilon}$.

We make some preparations first. The proof of Theorem 2.1 will be given in the end of this section.

**Lemma 2.1.** We have
\[
QD_X Y = D_X (QY) \quad \text{for} \quad X, Y \in TM.
\]
(2.6)

**Proof.** It follows from (2.1) that
\[
\langle Z, QD_X Y \rangle = \det(Z, \nabla_X Y, \vec{n})
= X \det(Z, Y, \vec{n}) - \det(\nabla_X Z, Y, \vec{n}) - \det(Z, Y, \nabla_X \vec{n})
= X \langle Z, QY \rangle - \langle \nabla_X Z, QY \rangle = \langle Z, \nabla_X (QY) \rangle \quad \text{for} \quad X, Y, Z \in TM,
\]
this gives (2.6). \qed

**Lemma 2.2.** The following formulas are true.
\[
\langle X, Y \rangle Z = \langle Z, Y \rangle X + \langle Z, QX \rangle QY \quad \text{for} \quad X, Y, Z \in TS.
\]
(2.8)
Proof. Let \( p \in S \) be given. If \( Y = 0 \), then (2.8) holds. We assume that \( |Y| = 1 \). Then \( QY, Y \) forms an orthonormal basis of \( T_pS \). Thus

\[
\langle (Z, Y)X + (Z, QX)QY, Y \rangle = \langle (Z, Y)X, Y \rangle = \langle (X, Y)Z, Y \rangle,
\]

\[
\langle (Z, Y)X + (Z, QX)QY, QY \rangle = \langle (Z, Y)(X, QY) + (Z, Y)(Y, QX) + (Z, QY)(QY, QX) \rangle = \langle (X, Y)Z, QY \rangle.
\]

Thus (2.8) follows.

Lemma 2.3. Let \( P \in T^2S \). Let \( X \) and \( Y \) be vector fields and \( f \) be a function. Then

\[
\text{div}_g(PX) = \langle PX, DX \rangle + \langle \text{div}_g P, X \rangle,
\]

(2.9)

\[
\text{div}_g(fP) = f \text{div}_g P + P^T Df.
\]

Proof. Let \( \{e_1, e_2\} \) be a normal frame field at \( p \). We have

\[
\text{div}_g(PX) = \sum_i \langle D_{e_i} PX, e_i \rangle = \sum_i e_i P(X, e_i) = \sum (D_{e_i} P)(X, e_i) + \langle PD_{e_i} X, e_i \rangle
\]

\[
= \sum_i DP(X, e_i, e_i) + \sum_i (P e_j, e_i) DX(e_j, e_i)
\]

\[
= \langle P, DX \rangle + \langle \text{div}_g P, X \rangle \text{ at } p,
\]

and

\[
\langle \text{div}_g(fP), X \rangle = \sum_i D(fP)(X, e_i, e_i) = \sum D_{e_i}(fP)(X, e_i)
\]

\[
= \sum [e_i(f) P(X, e_i) + f(D_{e_i} P)(X, e_i)]
\]

\[
= \langle X, P^T Df + f \text{div}_g P \rangle \text{ at } p.
\]

Lemma 2.4. Let \( P \in T^2S \) and let \( p \in S \) be given. Then the following identities hold.

\[
(Qv, w)P - \langle P, w \rangle Q = Qv \otimes Pw - Pw \otimes Qw,
\]

(2.10)

\[
(Qv, w)QP + \langle P, w \rangle \text{id} = Qv \otimes QPw + Pw \otimes w \text{ for } v, w \in T_pS.
\]

(2.11)

Proof. Set

\[
\Lambda(v, w, P) = (Qv, w)P - \langle P, w \rangle Q = Qv \otimes Pw + Pw \otimes Qw.
\]

Let \( \{e_1, e_2\} \) be an orthonormal basis of \( T_pS \) with positive orientation. Since \( \Lambda(v, w, P) \) is linear with respect to \( v, w, \) and \( P, \) respectively, for (2.10) it suffices to prove \( \Lambda(v, w, P) = 0 \) for \( v = e_i, w = e_j, \) and \( P = e_k \otimes e_l. \) From (2.2), we obtain

\[
\langle Qe_i, e_j \rangle = -\langle e_1 \wedge e_2, e_i \otimes e_j \rangle = \langle e_1 \wedge e_2, e_j \otimes e_i \rangle \text{ for } 1 \leq i, j \leq 2.
\]
Then we compute
\[
\Lambda(e_i, e_j, e_k \otimes e_l)(e_m, e_n) = \langle \Lambda(e_i, e_j, e_k \otimes e_l)e_m, e_n \rangle
\]
\[
= \langle Qe_i, e_j \rangle \delta_{km} \delta_{ln} - \delta_{ki} \delta_{lj} \langle Qe_m, e_n \rangle - \langle Qe_i, e_m \rangle \delta_{kj} \delta_{ln} + \delta_{ki} \delta_{lm} \langle Qe_j, e_n \rangle
\]
\[
= -\langle e_1 \wedge e_2, (\delta_{km} \delta_{ln} e_i \otimes \delta_{lj} e_m - \delta_{kj} \delta_{lm} e_i \otimes e_n - \delta_{kj} \delta_{lm} e_i \otimes e_m + \delta_{lk} \delta_{lm} e_j \otimes e_n) \rangle
\]
\[
= -\langle e_1 \wedge e_2, [\delta_{ln} e_i \otimes (\delta_{km} \delta_{lj} e_m) - \delta_{kj} \delta_{lm} e_i \otimes (\delta_{kn} \delta_{lj} e_m)] \rangle.
\] (2.12)

One observes (2.12) to have
\[
\Lambda(e_i, e_j, e_k \otimes e_l)(e_m, e_n) = -\Lambda(e_i, e_m, e_k \otimes e_l)(e_j, e_n).
\]
It immediately follows that
\[
\Lambda(e_i, e_j, e_k \otimes e_l)(e_j, e_n) = 0 \quad \text{for } j = m.
\]
Let \(i = j \neq m\). From (2.12), we have
\[
\Lambda(e_i, e_i, e_k \otimes e_l)(e_m, e_n) = -e_1 \wedge e_2[\delta_{ln} e_i \otimes (\delta_{km} \delta_{lj} e_m) - \delta_{kj} \delta_{lm} e_i \otimes (\delta_{kn} \delta_{lj} e_m)]
\]
\[
= \delta_{ki} e_1 \wedge e_2[\delta_{ln} e_i \otimes e_m + e_n \otimes (\delta_{kn} \delta_{lj} e_m)].
\]

If \(n = i\), then
\[
\Lambda(e_i, e_i, e_k \otimes e_l)(e_m, e_i) = \delta_{ki} \delta_{ln} e_1 \wedge e_2(e_i \otimes e_i) = 0.
\]

If \(n \neq i\), then \(n = m\) and
\[
\Lambda(e_i, e_i, e_k \otimes e_l)(e_m, e_m) = \delta_{ki} e_1 \wedge e_2[(\delta_{ln} e_i \otimes e_m + e_m \otimes e_i) - \delta_{li} e_m \otimes e_m] = 0.
\]

Let \(i \neq j\) and \(j \neq m\). Then \(i = m\). It follows from (2.12) that
\[
\Lambda(e_i, e_j, e_k \otimes e_l)(e_m, e_n)
\]
\[
= -\delta_{km}(e_1 \wedge e_2, \delta_{ln} e_m \otimes e_j - e_n \otimes (\delta_{lm} e_j - \delta_{lj} e_m))
\]
\[
= \left\{
\begin{aligned}
&-\delta_{km} \delta_{lj} \langle e_1 \wedge e_2, e_m \otimes e_j + e_j \otimes e_m \rangle = 0 \quad \text{if } n = j, \\
&-\delta_{km} \delta_{lm} \langle e_1 \wedge e_2, e_m \otimes e_j - e_m \otimes e_j \rangle = 0 \quad \text{otherwise } n = m.
\end{aligned}
\right.
\]

Using (2.10), we have
\[
\Lambda(Qv, Qw, -QPQ) = -\langle Qv, w \rangle QPQ - \langle Pv, w \rangle Q + v \otimes QPw - QPv \otimes w = 0.
\]

Thus (2.11) follows from \(\Lambda(Qv, Qw, -QPQ)Q = 0\). \(\square\)

We further assume that
\[
\det(\alpha_t, \alpha_s, \bar{n}) > 0 \quad \text{for } \mathcal{S}.
\] (2.13)
For otherwise, we replace $\alpha(t, s)$ with $\alpha(-t, s)$.

Let $x \in \Gamma_b$. Since $\kappa(x) = 0$ and $D\kappa(x) \neq 0$, from [8, Lemma 2.6], there exist vector fields $X_1, X_2$ in a neighborhood of $x$ satisfying $\nabla \bar{n}X_i = \lambda_iX_i$, where $\lambda_i$ are the principal curvatures. Clearly we may extend the vector fields $X_i$ to the region $\overline{S}_{b-\varepsilon}$ when $\varepsilon > 0$ is given small. We assume that $X_i$ are vector fields such that

$$\nabla \bar{n}X_i = \lambda_iX_i, \quad |X_i| = 1, \quad \langle X_1, X_2 \rangle = 0 \quad \text{for} \quad x \in \overline{S}_{b-\varepsilon}, \quad (2.14)$$

where

$$\lambda_1 > 0 \quad \text{for} \quad x \in \overline{S}_{b-\varepsilon},$$

$$\lambda_2 < 0 \quad \text{for} \quad x \in S_{b-\varepsilon}, \quad \lambda_2 = 0 \quad \text{for} \quad x \in \Gamma_b.$$

**Lemma 2.5.** For given $\varepsilon > 0$ small

$$X_2(\lambda_2) \neq 0 \quad \text{for} \quad x \in S_{b-\varepsilon}.$$

**Proof.** It will suffice to prove

$$X_2(\lambda_2) \neq 0 \quad \text{for} \quad x \in \Gamma_b.$$  

First, we claim that

$$\langle Q\alpha_t, X_2 \rangle(x) \neq 0 \quad \text{for} \quad x \in \Gamma_b.$$  

If not, then $\langle Q\alpha_t, X_2 \rangle(x) = 0$ implies that $X_2 = \eta\alpha_t$ with $\eta \neq 0$, and thus

$$\eta^2 \Pi(\alpha_t, \alpha_t) = \langle \nabla X_2 \bar{n}, X_2 \rangle = \lambda_2(x) = 0,$$

which contradicts the assumption $\Pi(\alpha_t, \alpha_t) \neq 0$.

In addition, assumption (1.5) implies

$$\langle D\kappa, \alpha_t \rangle = 0, \quad D\kappa = \eta Q\alpha_t \quad \text{for} \quad x \in \Gamma_b, \quad (2.15)$$

for some $\eta \neq 0$. Thus we obtain

$$X_2(\lambda_2) = \frac{1}{\lambda_1}X_2(\kappa) = \frac{\eta}{\lambda_1}\langle X_2, Q\alpha_t \rangle \neq 0 \quad \text{for} \quad x \in \Gamma_b. \quad (2.16)$$

We assume that

$$X_2(\lambda_2) > 0 \quad \text{for} \quad x \in \overline{S}_{b-\varepsilon}. \quad (2.17)$$

For otherwise, we replace $X_2$ with $-X_2$. Furthermore, we assume that $X_1, X_2$ has positive orientation. For otherwise, we replace $X_1$ with $-X_1$. Thus

$$QX_2 = X_1, \quad QX_1 = -X_2 \quad \text{for} \quad x \in \overline{S}_{b-\varepsilon}. \quad (2.18)$$
Let $Y \in TS_{b{-}\varepsilon}$ be given. Define

$$L_Y V = e^{-s\lambda_2}[(\text{div}_g Q \nabla \bar{n} V + \langle V, Y \rangle) X_1 + \lambda_2(\text{div}_g V) X_2],$$

(2.19)

for $V \in TS_{b{-}\varepsilon}$ and $s > 0$. For given $V, W \in TS_{b{-}\varepsilon}$, we have

$$\langle W, L_Y V \rangle = e^{-s\lambda_2}[(\text{div}_g Q \nabla \bar{n} V + \langle V, Y \rangle)X_1 + \lambda_2(\text{div}_g V)X_2]$$

$$= \text{div}_g(e^{-s\lambda_2}W, X_1)\nabla \bar{n} V + e^{-s\lambda_2}\lambda_2(\text{div}_g W)X_2 - Q\nabla \bar{n} V[e^{-s\lambda_2}W, X_1]$$

$$- V[e^{-s\lambda_2}\lambda_2(W, X_2) + e^{-s\lambda_2}V, Y](W, X_1)$$

$$= \langle L_Y W, V \rangle + \text{div}_g(e^{-s\lambda_2}W, X_1)\nabla \bar{n} V + e^{-s\lambda_2}\lambda_2(\text{div}_g W)X_2)$$

(2.20)

for $x \in S_{b{-}\varepsilon}$, where

$$e^{s\lambda_2}\langle L_Y W, V \rangle = -\langle DW, X_1 \otimes Q \nabla \bar{n} V + \lambda_2 X_2 \otimes V \rangle + s\langle Q \nabla \bar{n} V, D\lambda_2 \rangle \langle W, X_1 \rangle$$

$$+ s\lambda_2 \langle W, D\lambda_2 \rangle \langle W, X_2 \rangle - \langle W, DQ \nabla \bar{n} V, X_1 \rangle - \langle V, D\lambda_2 \rangle \langle W, X_2 \rangle$$

$$- \lambda_2 \langle W, D\lambda_2 \rangle \langle W, X_2 \rangle + \langle V, Y \rangle \langle W, X_1 \rangle$$

(2.21)

for $x \in S_{b{-}\varepsilon}$,

where the following formulas have been used

$$DW(Z_1, Z_2) = \langle DW, Z_1 \otimes Z_2 \rangle \quad \text{for} \quad Z_1, Z_2 \in TS_{b{-}\varepsilon}.$$ 

On the other hand, from (2.19) and Lemma 2.3, we obtain

$$e^{s\lambda_2}\langle V, L_Y W \rangle = \langle V, (\text{div}_g Q \nabla \bar{n} W + \langle W, Y \rangle)X_1 + \lambda_2(\text{div}_g W)X_2 \rangle$$

$$= \langle DW, \langle V, X_1 \rangle Q \nabla \bar{n} V + \lambda_2(\text{div}_g W)X_2 \rangle \text{id} \rangle$$

$$+ \langle \text{div}_g Q \nabla \bar{n} W + Y, W \rangle \langle V, X_1 \rangle$$

(2.22)

for $x \in S_{b{-}\varepsilon}$.

**Proposition 2.1.** For given $\varepsilon > 0$ small, the following identities hold true.

(i) For any $x \in S_{b{-}\varepsilon}$,

$$\langle W, - L_Y V \rangle = \langle - L_Y ^* W, V \rangle - \text{div}_g(e^{-s\lambda_2}W, X_1)\nabla \bar{n} V + e^{-s\lambda_2}\lambda_2(\text{div}_g W)X_2)$$

(2.23)

(ii) For any $X \in TS_{b{-}\varepsilon}$ with $|X| = 1$,

$$\langle X_2, X \rangle \langle (W, X_1) Q \nabla \bar{n} V + \lambda_2(\text{div}_g W), X \rangle$$

$$= \lambda_2 \langle V, X \rangle (W, X) - \Pi(Q X, Q X) \langle V, X_1 \rangle \langle W, X_1 \rangle$$

(2.24)

for $x \in S_{b{-}\varepsilon}$.

**Proof.** (2.23) follows from (2.20).

Next, we prove (2.24). Using (2.8) where $X = X_2, Y = X$, and $Z = V$, we have

$$\langle X_2, X \rangle V = \langle V, X \rangle X_2 + \langle V, X_1 \rangle Q X$$

for $x \in S_{b{-}\varepsilon}$. 

9
Thus we obtain

\[ \langle X_2, X \rangle ((W, X_1)\nabla\bar{n}V + \lambda_2(W, X_2)V) \]
\[ = \langle W, X_1 \rangle \nabla\bar{n}((V, X)X_2 + (V, X_1)QX) + \lambda_2(W, X_2)((V, X)X_2 + (V, X_1)QX) \]
\[ = \lambda_2(W, X_1)(V, X)X_1 + \langle W, X_1 \rangle \nabla\bar{n}QX + \lambda_2(W, X_2)((V, X)X_2 + (V, X_1)QX) \]
\[ = \lambda_2(V, X)W + \langle W, X_1 \rangle(V, X_1)\nabla\bar{n}QX + \lambda_2(W, X_2)(V, X_1)QX \quad \text{for } x \in S_{b-\varepsilon}. \]

It follows by \( \langle QX, X \rangle = 0 \) that

\[ \langle X_2, X \rangle ((W, X_1)\nabla\bar{n}V + \lambda_2(W, X_2)V), X \]
\[ = \lambda_2(V, X)(W, X) - \Pi(QX, QX)(V, X_1)(W, X_1) \quad \text{for } x \in S_{b-\varepsilon}. \]

\[ \square \]

**Proposition 2.2.** Let \( Y \in TS \) and let \( \mathcal{L}_Y \) be given in (2.19). Then there exists a constant \( \sigma_s > 0 \) such that

\[ -e^{s\lambda_2}\langle \mathcal{L}_YW + L^*_YW, W \rangle \geq \sigma_s|W|^2 \quad \text{for } W \in TS_{b-\varepsilon}, \quad x \in S_{b-\varepsilon} \quad (2.25) \]

for \( s > 0 \) large and \( \varepsilon > 0 \) small enough.

**Proof.** Using Lemma 2.3, (2.19), (2.18), and (2.11), we obtain

\[ e^{s\lambda_2}\langle \mathcal{L}_YW, W \rangle = (\div g\nabla\bar{n}W + \langle W, Y \rangle)(W, X_1) + \lambda_2(\div gW)(W, X_2) \]
\[ = \langle DW, (W, X_1)\nabla\bar{n} + \lambda_2(W, X_2)\text{id} \rangle + \langle W, \div g(\nabla\bar{n}) + Y \rangle(W, X_1) \]
\[ = \langle DW, X_1 \otimes Q\nabla\bar{n}W + \lambda_2X_2 \otimes W \rangle + \langle W, \div g(\nabla\bar{n}) + Y \rangle(W, X_1). \quad (2.26) \]

It follows from (2.21) and (2.26) that

\[ e^{s\lambda_2}\langle \mathcal{L}_YW + L^*_YW, W \rangle = s(\nabla\bar{n}W,\div \lambda_2)(W, X_1) + s\lambda_2(W, D\lambda_2)(W, X_2) \]
\[ -\langle W, DQ\nabla\bar{n}X_1 \rangle - \langle W, D\lambda_2 \rangle(W, X_2) - \lambda_2(W, DWX_2) + \langle W, Y \rangle(W, X_1) \]
\[ + \langle W, \div g(\nabla\bar{n}) + Y \rangle(W, X_1) \]
\[ = -sX_2(\lambda_2)[\lambda_1(W, X_1)^2 - \lambda_2(W, X_2)^2] - X_2(\lambda_2)(W, X_2)^2 \]
\[ + 2s\lambda_2X_1(\lambda_2)(W, X_1)(W, X_2) - X_1(\lambda_2)(W, X_1)(W, X_2) \]
\[ - \langle W, DQ\nabla\bar{n}X_1 \rangle - \lambda_2(W, DX_2) + \langle W, Y \rangle(W, X_1) \]
\[ + \langle W, \div g(\nabla\bar{n}) + Y \rangle(W, X_1). \quad (2.27) \]

Since \( \lambda_2 = O(\varepsilon) \) on \( S_{b-\varepsilon} \), (2.25) follows from (2.17) and (2.27), when \( s > 0 \) is large enough and \( \varepsilon > 0 \) is small enough, respectively. \( \square \)
Lemma 2.6. For any $X, Y \in TS$, the following holds.

\begin{align}
\nabla \vec{n}[X, Y] &= D_X \nabla \vec{n}Y - D_Y \nabla \vec{n}X, \quad (2.28) \\
\text{div}_g[X, Y] &= X \text{div}_g Y - Y \text{div}_g X, \quad (2.29)
\end{align}

Proof. A direct calculation yields

\[ D_X \nabla \vec{n}Y - D_Y \nabla \vec{n}X = \nabla_X \nabla_Y \vec{n} - \nabla_Y \nabla_X \vec{n} - \langle \nabla_X \nabla_Y \vec{n} - \nabla_Y \nabla_X \vec{n}, \vec{n} \rangle = \nabla [X, Y] \vec{n}. \]

For $v \in C^1_0(S)$, we have

\[ -\int_S v \text{div}_g[X, Y] dg = \int_S [X, Y] v dg = \int_S (XY v - YX v) dg = \int_S (\text{div}_g X - X \text{div}_g Y) dx. \]

Then (2.29) follows. \qed

Let $\Phi \in TS$ be given such that $\Pi(\Phi, \Phi) \neq 0$ for $p \in \overline{S}_{b-\varepsilon}$.

Define

\[ RV = D_V Q \nabla \vec{n} \Phi - D_Q \nabla \vec{n} V \Phi \quad \text{for} \quad V \in TS_{b-\varepsilon}. \]

Set

\[ h_1 = \frac{\langle R \Phi, Q \Phi \rangle}{\Pi(\Phi, \Phi)}, \quad h_2 = \frac{\langle R Q \Phi, Q \Phi \rangle - h_1 \langle \nabla \vec{n} Q \Phi, \Phi \rangle}{|\Phi|^2}, \quad (2.31) \]

\[ Z = \frac{1}{|\Phi|^2} (R - h_1 Q \nabla \vec{n} - h_2 \text{id})^{T} \Phi. \quad (2.32) \]

Lemma 2.7. The following formula is true.

\[ R = h_1 Q \nabla \vec{n} + h_2 \text{id} + Z \otimes \Phi. \quad (2.33) \]

Proof. We have

\[ \langle (R - h_1 Q \nabla \vec{n} - h_2 \text{id}) \Phi, Q \Phi \rangle = h_1 \Pi(\Phi, \Phi) - h_1 \Pi(\Phi, \Phi) = 0. \]

Since $\frac{Q \Phi}{|\Phi|}, \frac{\Phi}{|\Phi|}$ forms an orthonormal frame, it follows that

\begin{align}
(R - h_1 Q \nabla \vec{n} - h_2 \text{id})W &= \frac{1}{|\Phi|^2} \langle (R - h_1 Q \nabla \vec{n} - h_2 \text{id}) W, \Phi \rangle \\
+ \frac{1}{|\Phi|^4} \langle (R - h_1 Q \nabla \vec{n} - h_2 \text{id}) (W, \Phi) \Phi + \langle W, Q \Phi \rangle Q \Phi, Q \Phi \rangle Q \Phi \\
= \langle W, Z \rangle \Phi \quad \text{for} \quad W \in TS_{b-\varepsilon},
\end{align}

where $\langle \Phi, Q \Phi \rangle = 0$. Thus (2.33) follows. \qed
Proposition 2.3. Let $V \in TS_{b-\varepsilon}$ be a solution to problem (2.3). Let $\Phi \in TS_{b-\varepsilon}$ be given. Then

$$L_Z[\Phi, V] = e^{-s\lambda_2}[(\Phi f_1 - h_1 f_1 - h_2 f_2 + \langle H, V \rangle)X_1 + \lambda_2(\Phi f_2 - \langle D \div g \Phi, V \rangle)X_2],$$

where $Z$ is given in (2.32) and

$$H = \nabla \bar{n} Q D \div g \Phi + h_1 \div g Q \nabla \bar{n} - \div g R - i_Z D \Phi.$$ 

Proof. From (2.28), (2.30), (2.29), (2.9), and (2.33), we have

$$\div g Q \nabla \bar{n}[\Phi, V] = \div g(Q(D_\Phi \nabla \bar{n} V - D_V \nabla \bar{n} \Phi) = \div g(D_\Phi Q \nabla \bar{n} V - D_V Q \nabla \bar{n} \Phi)$$

$$= \Phi \div g Q \nabla \bar{n} V - Q \nabla \bar{n} V \div g \Phi - (h_1 Q \nabla \bar{n} + h_2 \id + Z \otimes \Phi, DV)$$

$$- \langle \div g R, V \rangle$$

$$= \Phi f_1 - Q \nabla \bar{n} V \div g \Phi - h_1((Q \nabla \bar{n}, DV) + \langle \div g Q \nabla \bar{n}, V \rangle) - h_2 \div g V$$

$$- \langle D_\Phi V - D_V \Phi, Z \rangle + \langle h_1 \div g Q \nabla \bar{n}, V \rangle - D \Phi(Z, V)$$

$$= -\langle [\Phi, V], Z \rangle + \langle \nabla \bar{n} Q D \div g \Phi + h_1 \div g Q \nabla \bar{n} - i_Z D \Phi - \div g R, V \rangle$$

$$+ \Phi f_1 - h_2 f_2,$$

and

$$\div g[\Phi, V] = \Phi \div g V - V \div g \Phi = \Phi f_2 - \langle D \div g \Phi, V \rangle.$$ 

Thus (2.34) follows. \qed

Lemma 2.8. $\langle X_2, Q \alpha_t \rangle < 0$ for $x \in \Gamma_b$, where $X_2$ is given in (2.14).

Proof. Since $Q \alpha_t/|\alpha_t|, \alpha_t/|\alpha_t|$ forms an orthonormal vector basis long $\Gamma_0$ with positive orientation, the assumption (2.13) and the curvature conditions (1.4) and (1.5) imply

$$D \kappa = \eta Q \alpha_t, \quad \eta \neq 0, \quad \langle D \kappa, -Q \alpha_t \rangle > 0 \quad \text{for} \quad x \in \Gamma_b.$$ 

Then

$$\eta = \langle D \kappa, Q \alpha_t \rangle/|\alpha_t|^2 < 0 \quad \text{for} \quad x \in \Gamma_b.$$ 

It follows from (2.17) that

$$\langle X_2, Q \alpha_t \rangle = \frac{1}{\eta} \langle X_2, D \kappa \rangle = \frac{\lambda_1}{\eta} X_2(\lambda_2) < 0 \quad \text{for} \quad x \in \Gamma_b.$$ 

\qed
Proof of Theorem 2.1. Step 1. Let $V \in TS_{b-\varepsilon}$ satisfy (2.3). Taking $Y = 0$ and $W = V$ in (2.19), (2.23), and (2.24), respectively, we have, by (2.3),
\[
\mathcal{L}_0 V = e^{-s\lambda_2}(f_1X_1 + \lambda_2 f_2 X_2),
\]
\[
\langle V, -\mathcal{L}_0 V \rangle = \langle -\mathcal{L}_0^* V, V \rangle - \text{div}_g(e^{-s\lambda_2}\langle V, X_1 \rangle Q\nabla \bar{n}V + e^{-s\lambda_2}\lambda_2\langle V, X_2 \rangle V),
\]
and
\[
\langle X_2, X \rangle \langle (\langle V, X_1 \rangle Q\nabla \bar{n}V + \lambda_2\langle V, X_2 \rangle V), X \rangle
\]
\[= \lambda_2\langle V, X \rangle^2 - \Pi(QX, QX)\langle V, X \rangle^2 \text{ for } X \in TS_{b-\varepsilon},
\]
respectively. Since $Q\alpha_t/|\alpha_t|$, $\alpha_t/|\alpha_t|$ forms an orthonormal vector basis along $\Gamma_{b-\varepsilon}$ and $\Gamma_b$, respectively, it follows from (2.13) that
\[
\nu = -Q\alpha_t/|\alpha_t| \text{ for } x \in \Gamma_b; \quad \nu = Q\alpha_t/|\alpha_t| \text{ for } x \in \Gamma_{b-\varepsilon}.
\]
When given $\varepsilon > 0$ is small, from Lemma 2.8, we have
\[
\langle X_2, \nu \rangle > 0 \text{ for } x \in \Gamma_b; \quad \langle X_2, \nu \rangle < 0 \text{ for } x \in \Gamma_{b-\varepsilon}.
\]
We integrate (2.36) over $S_{b-\varepsilon}$ and use (2.37), where $X = \nu$, Proposition 2.2, and (2.38) to obtain
\[
2\langle V, -\mathcal{L}_0 V \rangle_{L^2(S_{b-\varepsilon},TS_{b-\varepsilon})} = \langle V, -\mathcal{L}_0^* V - \mathcal{L}_0 V \rangle_{L^2(S_{b-\varepsilon},TS_{b-\varepsilon})}
\]
\[+ \int_{\Gamma_{b-\varepsilon} \cup \Gamma_b} \frac{e^{-\lambda_2 s}}{\langle X_2, \nu \rangle} \Pi(Q\nu, Q\nu)\langle V, X \rangle^2 - \lambda_2\langle V, \nu \rangle^2 d\Gamma
\]
\[\geq \sigma_s\|V\|_{L^2(S_{b-\varepsilon},TS_{b-\varepsilon})}^2 + \int_{\Gamma_b} \frac{e^{-\lambda_2 s}}{\langle X_2, \nu \rangle} \Pi(Q\nu, Q\nu)\langle V, X \rangle^2 d\Gamma
\]
\[+ \int_{\Gamma_{b-\varepsilon}} \frac{e^{-\lambda_2 s}}{\langle X_2, \nu \rangle} \Pi(Q\nu, Q\nu)\langle V, X \rangle^2 d\Gamma - \lambda_2\langle V, \nu \rangle^2 d\Gamma.
\]
Thus (2.4) follows.

Step 2. We prove (2.5). Let
\[
\Phi_1 = \alpha_t, \quad \Phi_2 = \Phi_1 + \eta Q\Phi_1,
\]
where $\eta > 0$ is given small such that
\[
\Pi(\Phi_i, \Phi_i) > 0 \text{ for } x \in S_{b-\varepsilon}, \quad i = 1, 2.
\]
Clearly, $\Phi_1$, $\Phi_2$ forms a basis of vector fields. It follows from (2.34) that
\[
\mathcal{L}_{Z_i}[\Phi_i, V] = e^{-s\lambda_2}[\Phi_i f_1 - h_{i1} f_1 - h_{i2} f_2 + (H_i, V)]X_1 + \lambda_2(\Phi_i f_2 - \langle D \text{div}_g \Phi_i, V \rangle)X_2,
\]
where $Z_i$ is given in (2.32) with $\Phi = \Phi_i$, and
\[
H_i = \nabla \bar{n}QD \text{div}_g \Phi_i + h_{i1} \text{div}_g \nabla \bar{n} - \text{div}_g R_i - i_{Z_i} D \Phi_i.
\]
We repeat the produce in Step 1 with $\mathcal{L}_0 V$ replaced by $\mathcal{L}_{Z_i}[\Phi_i, V]$ to obtain (2.5). \qed
3 The degenerated hyperbolic regions

In the sequel we assume that \( \Gamma_0 = \{ \alpha(t, 0) \mid t \in [0, a) \} \) is a non-degenerated curve, that does not contain just one point. For otherwise, we may replace \( \Gamma \) with any curve \( \Gamma_s = \{ \alpha(t, s) \mid t \in [0, a) \} \) for \( s \in (0, b) \).

Let
\[
X_0 = Q\nabla nQD\kappa \quad \text{for} \quad x \in S.
\]
Let \( F \in W^{1,2}(S, TS) \) and \( f \in L^2(S) \). We consider a degenerated hyperbolic problem
\[
\begin{cases}
\langle D^2 w, Q^* \Pi \rangle + \frac{1}{\kappa} X_0 w + \kappa (\text{tr} g \Pi) w = \kappa f + \frac{1}{\kappa} \langle X_0, F \rangle + \langle DF, Q^* \Pi \rangle & x \in S, \\
w = q_0, & \langle Dw, Q\nabla n\alpha_t \rangle = q_1, \quad x \in \Gamma_0.
\end{cases}
\tag{3.1}
\]

**Theorem 3.1.** Let \( m \geq 0 \) be an integer. Let \( S \) be of class \( C^{m,1} \). Let \( F \in W^{m+1,2}(S, TS) \), \( f \in W^{m,2}(S) \), \( q_0 \in W^{m+1,2}(\Gamma_0) \), and \( q_1 \in W^{m,2}(\Gamma_0) \). Then problem (3.1) admits a unique solution \( w \in W^{m+1,2}(S) \) satisfying
\[
\|w\|^2_{W^{m+1,2}(S)} + \|w\|^2_{W^{m+1,2}(\Gamma_0)} \leq C(\|f\|^2_{W^{m,2}(S)} + \|F\|^2_{W^{m+1,2}(S, TS)} + \|q_0\|^2_{W^{m+1,2}(\Gamma_0)} + \|q_1\|^2_{W^{m,2}(\Gamma_0)}).
\tag{3.2}
\]

Let
\[
\Gamma_2(w, S) = \int_0^a (|D^2 w|^2 + |Dw|^2 + |w|^2) \circ \alpha(t, 0) dt.
\tag{3.4}
\]

**Theorem 3.2.** Let \( S \) be of class \( C^{2,1} \). Let \( w \) solve problem (3.1). Then there are constants \( C > c > 0 \) such that
\[
c\Gamma_2(w, S) \leq \|w\|^2_{W^{2,2}(S)} + \|f\|^2_{W^{1,2}(S)} + \|F\|^2_{W^{2,2}(S, TS)} \leq C(\Gamma_2(w, S) + \|f\|^2_{W^{1,2}(S)} + \|F\|^2_{W^{2,2}(S, TS)}).
\tag{3.5}
\]

The proofs of Theorems 3.1 and 3.2 will be given in the end of this section.

Fix \( \varepsilon_0 > 0 \) small, such that (2.17), Proposition 2.2, and
\[
\langle X_2, Q\alpha_t \rangle < 0 \quad \text{for} \quad x \in \overline{S}_{b-\varepsilon_0} \quad \text{(by Lemma 2.8)},
\tag{3.6}
\]
hold true. The curve \( \Gamma_{b-\varepsilon_0} \) divides \( S \) into two regions:
\[
S = \Sigma_1 \cup \Sigma_2 \cup \Gamma_{b-\varepsilon_0},
\]
where
\[
\Sigma_1 = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (0, b - \varepsilon_0) \}, \quad \Sigma_2 = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (b - \varepsilon_0, b) \}.
\]
We shall obtain solutions to the boundary-value problems on the regions \(\Sigma_1\) and \(\Sigma_2\), separately. Then paste them together to have solutions to problem (3.1) on the region \(S\).

To apply some existence results in [7, 9] to the boundary-value problem on the regions \(\Sigma_i\), we recall some boundary operators in [7]. Let \(x \in \Gamma_{b-\varepsilon_0}\) be given. \(\mu \in T_x S\) with \(|\mu| = 1\) is said to be the noncharacteristic normal outside \(\Sigma_2\) if there is a curve \(\zeta: (0, \varepsilon) \to \Sigma_2\) such that
\[
\zeta(0) = x, \quad \zeta'(0) = -\mu, \quad \Pi(\mu, \beta) = 0 \quad \text{for} \quad \beta \in T_x \Gamma_{b-\varepsilon_0}.
\]
Let \(\mu\) be the the noncharacteristic normal field along \(\Gamma_0\) outside \(\Sigma_2\). We define the boundary operators \(T_i: T_x M \to T_x M\) by
\[
T_i \beta = \frac{1}{2} \left[ \beta + (-1)^i \chi(\mu, \beta) \rho(\beta) Q \nabla \tilde{n} \beta \right] \quad \text{for} \quad \beta \in T_x M, \quad i = 1, 2,
\]
where
\[
\chi(\mu, \beta) = \text{sign det} \left( \mu, \beta, \tilde{n} \right), \quad \rho(\beta) = \frac{1}{\sqrt{-\kappa}} \text{sign} \Pi(\beta, \beta),
\]
and \text{sign} is the sign function. Noting that
\[
\mu = \frac{Q \nabla \tilde{n} \alpha_t}{|\nabla \tilde{n} \alpha_t|} \quad \text{for} \quad x \in \Gamma_{b-\varepsilon_0},
\]
it follows from (1.3) and (2.13) that
\[
\chi(\mu, \alpha_t) \circ \alpha(t, b - \varepsilon_0) = 1, \quad \rho(\alpha_t) \circ \alpha(t, b - \varepsilon_0) = \frac{1}{\sqrt{-\kappa}} \quad \text{for} \quad t \in [0, a).
\]
Thus
\[
(T_2 - T_1) \alpha_t \circ \alpha(t, 0) = -\frac{1}{\sqrt{-\kappa}} Q \nabla \tilde{n} \alpha_t \quad \text{for} \quad t \in [0, a).
\]
Similarly, as \(\Gamma_0\) is a part of the boundary of the region \(\Sigma_1\), then
\[
(T_2 - T_1) \alpha_t \circ \alpha(t, 0) = \frac{1}{\sqrt{-\kappa}} Q \nabla \tilde{n} \alpha_t \quad \text{for} \quad t \in [0, a).
\]
Since \(\Sigma_1\) is non-degenerated hyperbolic region, by similar arguments as for [7, Theorems 4.2 and 4.3] (or [9]), we have the following. The details are omitted.

**Proposition 3.1.** Problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
D^2 w, Q^* \Pi + \frac{1}{\kappa} X_0 w + \kappa (\text{tr}_g \Pi) w = \kappa f + \frac{1}{\kappa} (X_0, F) + (DF, Q^* \Pi), & x \in \Sigma_1, \\
\langle Dw, Q \nabla \tilde{n} \alpha_t \rangle = q_1 & \text{for} \quad x \in \Gamma_0
\end{array} \right.
\end{aligned}
\]

admits a unique solution \(w\) satisfying
\[
\|w\|_{W^{m+1,2}(\Sigma_1)} + \sum_{i=0}^{m+1} \|D^i w\|_{L^2(\Gamma_{b-\varepsilon_0})} \leq C(\|f\|^2_{W^{m,2}(\Sigma_1)} + \|F\|^2_{W^{m+1,2}(\Sigma_1, T\Sigma_1)} + \|q_0\|^2_{W^{m+1,2}(\Gamma_0)} + \|q_1\|^2_{W^{m,2}(\Gamma_0)}).
\]
Consider a degenerated problem

\[
\begin{aligned}
\begin{cases}
\langle D^2w, Q^*\Pi \rangle + \frac{1}{\kappa}X_0w + \kappa(\text{tr}_g\Pi)w = \kappa f + \frac{1}{\kappa}\langle X_0, F \rangle + \langle DF, Q^*\Pi \rangle, & x \in \Sigma_2, \\
w = q_0, & \langle Dw, Q\nabla\tilde{n}\alpha \rangle = q_1 \quad \text{for} \quad x \in \Gamma_{b-\varepsilon_0}.
\end{cases}
\end{aligned}
\] (3.8)

Consider the regions

\[
\Sigma_{2\varepsilon} = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (b - \varepsilon_0, b - \varepsilon) \} \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0. \] (3.9)

Since \( \Sigma_{2\varepsilon} \) are also non-degenerated hyperbolic regions for all given \( 0 < \varepsilon < \varepsilon_0 \), for the same reasons as for \( \Sigma_1 \), problem (3.8) has solutions in \( W^{m+1,2}(\Sigma_{2\varepsilon}) \) on the regions \( \Sigma_{2\varepsilon} \).

Because \( 0 < \varepsilon < \varepsilon_0 \) can be arbitrarily small, problem (3.8) actually admits a unique solution \( w \) on the region \( \Sigma_2 \), which satisfies \( w \in W^{m+1,2}(\Sigma_{2\varepsilon}) \) for all \( 0 < \varepsilon < \varepsilon_0 \). Then Theorem 3.1 follows from Propositions 3.1 and 3.2 later.

**Lemma 3.1.** We have

\[-\kappa(\nabla\tilde{n})^{-1}\beta = Q\nabla\tilde{n}Q\beta \quad \text{for} \quad \beta \in T_xM, \quad x \in M, \quad \kappa(x) \neq 0.\]

*Proof.* Let \( \kappa(x) \neq 0 \). Let \( e_1, e_2 \) be an orthonormal basis of \( T_xS \) with positive orientation such that

\[\nabla\tilde{n}e_i = \lambda_i e_i \quad \text{for} \quad x,\]

where \( \lambda_i \) are the principal curvatures. We have

\[
\begin{aligned}
Q\nabla\tilde{n}Q\beta &= Q\nabla\tilde{n}Q(\langle \beta, e_1 \rangle e_1 + \langle \beta, e_2 \rangle e_2) = Q\nabla\tilde{n}(-\langle \beta, e_1 \rangle e_2 + \langle \beta, e_2 \rangle e_1) \\
&= Q(-\langle \beta, e_1 \rangle \lambda_2 e_2 + \langle \beta, e_2 \rangle \lambda_1 e_1) = -\langle \beta, e_1 \rangle \lambda_2 e_1 - \langle \beta, e_2 \rangle \lambda_1 e_2 \\
&= -\kappa(\nabla\tilde{n})^{-1}\beta.
\end{aligned}
\]

**Lemma 3.2.** For \( V \in W^{1,2}(S, TS) \), we have

\[
\langle D(\nabla\tilde{n}V), Q^*\Pi \rangle = \text{div}_g\kappa V \quad \text{for} \quad x \in S.
\]

*Proof.* Let \( x \in S \) be fixed. Let \( e_1, e_2 \) be an orthonormal basis of \( T_xS \) with positive orientation such that

\[\nabla\tilde{n}e_1 = \lambda_i e_i.\]

Suppose that \( E_1, E_2 \) be a frame field normal at \( x \) such that

\[E_i = e_i \quad \text{at} \quad x.\]
We have
\[
\langle D(\nabla \bar{n} V), Q^* \Pi \rangle = \lambda_2 E_1 \langle \nabla \bar{n} V, E_1 \rangle + \lambda_1 E_2 \langle \nabla \bar{n} V, E_2 \rangle
\]
\[
= \lambda_2 E_1 \langle V, \nabla \bar{n} E_1 \rangle + \lambda_1 E_2 \langle V, \nabla \bar{n} E_2 \rangle
\]
\[
= \lambda_2 (D_{E_1} V, \nabla \bar{n} E_1) + \lambda_1 (D_{E_2} V, \nabla \bar{n} E_2) + \lambda_2 \langle V, D_{E_1} (\nabla \bar{n} E_1) \rangle + \lambda_1 \langle V, D_{E_2} (\nabla \bar{n} E_2) \rangle
\]
\[
= \kappa \text{div}_g V + (\Pi_{122} \Pi_{111} + \Pi_{11} \Pi_{122}) \langle V, E_1 \rangle + (\Pi_{222} \Pi_{112} + \Pi_{11} \Pi_{222}) \langle V, E_2 \rangle
\]
\[
= \kappa \text{div}_g V + (\Pi_{122} \Pi_{111} + \Pi_{11} \Pi_{222} - 2 \Pi_{12} \Pi_{222}) \langle V, E_1 \rangle + (\Pi_{222} \Pi_{112} + \Pi_{11} \Pi_{222} - 2 \Pi_{12} \Pi_{222}) \langle V, E_2 \rangle
\]
\[
= \kappa \text{div}_g V + E_1(\kappa) \langle V, E_1 \rangle + E_2(\kappa) \langle V, E_2 \rangle = \text{div}_g \kappa V \quad \text{at} \quad x,
\]
where \(\Pi_{ij} = \Pi(E_i, E_j)\) and \(\Pi_{ijk} = D\Pi(E_i, E_j, E_k)\). \(\square\)

Lemma 3.3. For \(v, w \in W^{2,2}(S)\), we have
\[
\langle D^2 v, Q^* \Pi \rangle = \text{div}_g i_{Dv} Q^* \Pi,
\]
\[
\Pi(QDv, QDw) + w\langle D^2 v, Q^* \Pi \rangle = \text{div}_g (w i_{Dv} Q^* \Pi).
\]

Proof. Let \(x \in S\) be fixed. Suppose that \(E_1, E_2\) is a frame field normal at \(x\) with positive orientation such that
\[
\Pi(E_1, E_2) = 0 \quad \text{at} \quad x.
\]
Then
\[
QE_1 = -E_2, \quad QE_2 = E_1 \quad \text{in a neighborhood of} \quad x.
\]
Using the above formulas, we compute
\[
\text{div}_g i_{Dv} Q^* \Pi = E_1 \langle i_{Dv} Q^* \Pi, E_1 \rangle + E_2 \langle i_{Dv} Q^* \Pi, E_2 \rangle = E_1 [\Pi(QDv, QE_1)] + E_2 [\Pi(QDv, QE_2)]
\]
\[
= E_1 (v_1 \Pi_{122} - v_2 \Pi_{112}) + E_2 (-v_1 \Pi_{112} + v_2 \Pi_{111})
\]
\[
= v_1 \Pi_{122} + v_1 \Pi_{222} - v_2 \Pi_{122} - v_1 \Pi_{122} + v_2 \Pi_{111} + v_2 \Pi_{112}
\]
\[
= \langle D^2 v, Q^* \Pi \rangle,
\]
where \(v_i = E_i v, \Pi_{ijk} = D\Pi(E_i, E_j, E_k)\), and the following formulas have been used
\[
\Pi_{122} = \Pi_{112}, \quad \Pi_{121} = \Pi_{112}.
\]
Since \(\langle i_{Dv} Q^* \Pi, Dw \rangle = \Pi(QDv, QDw)\), (3.11) follows from (3.10). \(\square\)

Proposition 3.2. Problem (3.8) admits a unique solution \(w\) satisfying
\[
\|w\|_{W^{1,2}(\Sigma_2)}^2 + \|w\|_{W^{1,2}(\Gamma_h)}^2 \leq C (\|f\|_{L^2(\Sigma_2)}^2 + \|F\|_{W^{1,2}(\Sigma_2, T\Sigma_2)}^2 + \|g_0\|_{W^{1,2}(\Gamma_{h-c_0})}^2 + \|g_1\|_{L^2(\Gamma_{h-c_0})}^2),
\]
\[
(3.12)
\]
\[ \langle Dw, X_0 \rangle = \langle F, X_0 \rangle \quad \text{for} \quad x \in \Gamma_0. \] 

(3.13)

Furthermore, for \( m \geq 1 \), the following estimates hold.

\[
\|w\|_{W^{m+1,2}(\Sigma_2)}^2 + \|w\|_{W^{m+1,2}(\Gamma_b)}^2 \leq C(\|f\|_{W^{m,2}(\Sigma_2)}^2 + \|F\|_{W^{m+1,2}(\Sigma_2, T\Sigma_2)}^2 + \|q_0\|_{W^{m,2}(\Gamma_{b-\varepsilon_0})}^2 + \|q_2\|_{W^{m,2}(\Gamma_{b-\varepsilon_0})}^2).
\]

(3.14)

Proof. Our task is to establish (3.12)-(3.14). Let \( w \) solve problem (3.8). As before, we define

\[ W = (\nabla \tilde{n})^{-1}(Dw - F) \quad \text{for} \quad x \in \Sigma_2. \]

(3.15)

Then

\[ Dw = \nabla \tilde{n}W + F \quad \text{for} \quad x \in \Sigma_2. \]

Let \( E_1, E_2 \) be an orthonormal frame on \( S_{b-\varepsilon_0} \) with positive orientation. From Lemma 2.1, we have

\[
\text{div}_g QDw = \langle D_{E_1}(QDw), E_1 \rangle + \langle D_{E_2}(QDw), E_2 \rangle = \langle QD_{E_1}Dw, E_1 \rangle + \langle QD_{E_2}Dw, E_2 \rangle = -\langle D_{E_1}Dw, QE_1 \rangle - \langle D_{E_2}Dw, QE_2 \rangle = \langle D_{E_1}Dw, E_2 \rangle - \langle QD_{E_2}Dw, E_1 \rangle = 0.
\]

Thus we obtain

\[ \text{div}_g Q\nabla \tilde{n}W = -\text{div}_g QF \quad \text{for} \quad x \in \Sigma_2. \]

On the other hand, from Lemma 3.1, we have

\[ X_0 = -\kappa(\nabla \tilde{n})^{-1}D\kappa \quad \text{for} \quad x \in \Sigma_2. \]

Then

\[ \langle W, D\kappa \rangle = \langle Dw - F, (\nabla \tilde{n})^{-1}D\kappa \rangle = \frac{1}{\kappa} \langle F - Dw, X_0 \rangle \quad \text{for} \quad x \in \Sigma_2. \]

In addition, it follows from Lemma 3.2 and the first equation in (3.8) that

\[
\kappa \text{div}_g W + \langle W, D\kappa \rangle = \text{div}_g \kappa W = \langle D(\nabla \tilde{n}W), Q^*\Pi \rangle = \langle D^2w - DF, Q^*\Pi \rangle = \kappa f - (\text{tr}_g\Pi)w + \frac{1}{\kappa} \langle X_0, F - Dw \rangle \quad \text{for} \quad x \in \Sigma_2.
\]

Thus we obtain

\[ \text{div}_g W = f - (\text{tr}_g\Pi)w \quad \text{for} \quad x \in \Sigma_2. \]

That is, \( W \in T\Sigma_2 \) solves problem

\[
\begin{aligned}
\text{div}_g Q\nabla \tilde{n}W &= -\text{div}_g QF \quad \text{for} \quad x \in \Sigma_2, \\
\text{div}_g W &= -(\text{tr}_g\Pi)w + f \quad \text{for} \quad x \in \Sigma_2.
\end{aligned}
\]

(3.16)

**Step 1.** Consider the case of \( m = 0 \).
We first prove (3.13). Let $L_0 W$ be given by (2.19) where $V = W$ and $Y = 0$. It follows from (3.16) that

$$L_0 W = e^{-s\lambda_2} \{ -(\div_g Q F) X_2 + \lambda_2 [-(\tr_g \Pi) w + f] X_2 \}. \quad (3.17)$$

From (2.23) and (2.25), we have

$$2 \langle W, -L_0 W \rangle \geq \sigma_{\varepsilon_0} |W|^2 - \div_g(e^{-s\lambda_2} \langle W, X_2 \rangle Q \nabla \tilde{n} W + e^{-s\lambda_2} \lambda_2 \langle W, X_2 \rangle W), \quad (3.18)$$

for $x \in \Sigma_2$. Let $\Sigma_{2\varepsilon}$ be given in (3.9) for $0 < \varepsilon < \varepsilon_0$. We integrate (3.18) over $\Sigma_{2\varepsilon}$ and use (2.24) to have

$$2 \langle W, -L_0 W \rangle_{L^2(\Sigma_{2\varepsilon}, T\Sigma_{2\varepsilon})} \geq \sigma_{\varepsilon_0} \|W\|^2_{L^2(\Sigma_{2\varepsilon}, T\Sigma_{2\varepsilon})}$$

for $0 < \varepsilon < \varepsilon_0$.

Next, we deal with the first boundary integration in the right hand side of (3.19). Since

$$\langle W, \nu \rangle = \langle \nu, X_2 \rangle \langle W, X_1 \rangle + \frac{\langle \nu, X_2 \rangle}{\lambda_2} \langle Dw - F, X_2 \rangle \quad \text{for} \quad x \in \Sigma_{2\varepsilon},$$

it follows that

$$\frac{\Pi(\alpha_t, \alpha_t)}{\|\alpha_t\|^2} \langle W, X_1 \rangle^2 + |\lambda_2| \langle W, \nu \rangle^2$$

$$\geq \frac{\langle \nu, X_2 \rangle^2}{2|\lambda_2|} \langle Dw - F, X_2 \rangle^2 + \left[ \frac{\Pi(\alpha_t, \alpha_t)}{\|\alpha_t\|^2} - |\lambda_2| \langle \nu, X_1 \rangle^2 \right] \langle W, X_1 \rangle^2 \quad \text{for} \quad x \in \Gamma_{b - \varepsilon}.$$

Noting that

$$|\lambda_2| = \mathcal{O}(\varepsilon), \quad \langle X_2, \nu \rangle > 0 \quad \text{for} \quad x \in \Gamma_{b - \varepsilon},$$

we have

$$\int_{\Gamma_{b - \varepsilon}} \frac{e^{-s\lambda_2}}{\langle X_2, \nu \rangle} \frac{\Pi(\alpha_t, \alpha_t)}{\|\alpha_t\|^2} \langle W, X_1 \rangle^2 + |\lambda_2| \langle W, \nu \rangle^2 d\Gamma$$

$$\geq \sigma \int_{\Gamma_{b - \varepsilon}} \left( \frac{1}{|\lambda_2|} \langle Dw - F, X_2 \rangle^2 + \langle W, X_1 \rangle^2 \right) d\Gamma \quad \text{for} \quad \varepsilon > 0 \quad \text{small}, \quad (3.20)$$

which imply that

$$\langle Dw, X_2 \rangle = \langle F, X_2 \rangle \quad \text{for} \quad x \in \Gamma_b. \quad (3.21)$$

On the other hand, from (2.35),

$$X_0 = Q \nabla \tilde{n} Q D\kappa = -\eta Q \nabla \tilde{n} \alpha_t = \eta \langle \nabla \tilde{n} \alpha_t, Q X_1 \rangle X_1 + \eta \langle \nabla \tilde{n} \alpha_t, Q X_2 \rangle X_2$$

$$= -\eta \langle \nabla \tilde{n} \alpha_t, X_2 \rangle X_1 + \eta \langle \nabla \tilde{n} \alpha_t, X_1 \rangle X_2 = \eta \lambda_1 \langle \alpha_t, X_1 \rangle X_2 \quad \text{for} \quad x \in \Gamma_b. \quad (3.22)$$
Thus (3.13) follows.

We claim that

\[ \langle X_1, \tau \rangle \neq 0 \quad \text{for} \quad x \in \Gamma_b, \quad (3.23) \]

where \( \tau = \alpha_t/|\alpha_t| \). For otherwise, \( \langle X_1, \tau \rangle = 0 \) implies that \( \tau = \pm X_2 \) and

\[ \Pi(\tau, \tau) = \Pi(X_2, X_2) = 0, \]

which contradicts (1.3). Thus

\[ \langle Dw, \tau \rangle = \langle X_1, \tau \rangle \langle Dw, X_1 \rangle + \langle X_2, \tau \rangle \langle Dw, X_2 \rangle, \]

that is

\[ \langle Dw, X_1 \rangle = \frac{1}{\langle X_1, \tau \rangle} (\langle Dw, \tau \rangle - \langle X_2, \tau \rangle \langle F, X_2 \rangle) \quad \text{for} \quad x \in \Gamma_b. \quad (3.24) \]

We now integrate (3.18) over \( \Sigma \) and use (2.24) to obtain

\begin{align*}
2(W, \mathcal{L}_0 W)_{L^2(\Sigma_2, T \Sigma_2)} &\geq \sigma_0 \|W\|^2_{L^2(\Sigma_2, T \Sigma_2)} + \int_{\Gamma_b} e^{-s\lambda_2} \Pi(\alpha_t, \alpha_t) \langle W, X_1 \rangle^2 d\Gamma \\
&\quad + \int_{\Gamma_b-\varepsilon_0} e^{-s\lambda_2} \frac{\Pi(\alpha_t, \alpha_t)}{|\alpha_t|^2} (W, X_1)^2 - \lambda_2 (W, \nu)^2 |d\Gamma. \quad (3.25) \end{align*}

It follows from (3.25), (3.17), and (3.24) that

\begin{align*}
\|Dw\|^2_{L^2(\Sigma_2, T \Sigma_2)} + \int_{\Gamma_b} \langle Dw, \tau \rangle^2 d\Gamma &\leq C(\|w\|^2_{L^2(\Sigma_2)} + \|f\|^2_{L^2(\Sigma_2)} + \|F\|^2_{W^{1,2}(\Sigma_2, T \Sigma_2)} \\
&\quad + \|F\|^2_{L^2(\Gamma_b, T M)} + \|F\|^2_{L^2(\Gamma_b-\varepsilon_0, T M)} + \|Dw\|^2_{L^2(\Gamma_b-\varepsilon_0, T M)}) \\
&\leq C(\|w\|^2_{L^2(\Sigma_2)} + \|f\|^2_{L^2(\Sigma_2)} + \|F\|^2_{W^{1,2}(\Sigma_2, T \Sigma_2)} + \|Dw\|^2_{L^2(\Gamma_b-\varepsilon_0, T M)}). \quad (3.26) \end{align*}

Thus (3.12) follows from (3.26) and the boundary data in (3.8).

**Step 2.** We prove (3.14) in the case of \( m = 1 \). The case of \( m \geq 2 \) can be treated by the similar arguments.

Let \( W \) be given in (3.15). Suppose that \( \Phi_2 \) and \( \Phi_2 \) are given in (2.40) such that

\[ \Pi(\Phi_i, \Phi_i) > 0 \quad \text{for} \quad x \in \Sigma_2 \]
and $\Phi_2$, $\Phi_2$ forms a basis of vector fields on $\Sigma_2$. By a similar computation as in (2.42), we have
\[
\mathcal{L}_{Z_i}[\Phi_i, W] = e^{-s\lambda_2}[(\Phi_i f_2 - h_{i1} f_2 - h_{i2} f_2 + \langle H_i, W \rangle) X_2 \\
+ \lambda_2 (\Phi_i f_2 - \langle D \div g \Phi_i, W \rangle) X_2],
\]
where $Z_i$, $h_{2i}$, and $H_i$ are the same as in (2.42), and
\[
f_2 = -\div g Q F, \quad f_2 = -(\tr_g \Pi) w + f \quad \text{for} \quad x \in \Sigma_2.
\]
By a similar argument as in Step 1 yields
\[
2\langle [\Phi_i, W], -\mathcal{L}_{Z_i}[\Phi_i, W] \rangle_{L^2(\Sigma_2, T\Sigma_2)} \geq \sigma_0 \|[\Phi_i, W]\|^2_{L^2(\Sigma_2, T\Sigma_2)} + \int_{\Gamma_b \cup \Gamma_{b-\varepsilon_0}} e^{-s\lambda_2} P_i(W_\varepsilon) d\Gamma,
\]
where
\[
P_i(W) = \frac{1}{\langle X_2, \nu \rangle} [\Pi(\tau, \tau) ([\Phi_i, W], X_1)^2 - \lambda_2 ([\Phi_i, W], \nu)^2], \quad i = 1, 2.
\]
By a similar argument as in Step 1 yields
\[
\|\Phi_i, W\|^2_{L^2(\Sigma_2, T\Sigma_2)} \geq \sigma_0 \|\Phi_i, W\|^2_{L^2(\Sigma_2, T\Sigma_2)} + \int_{\Gamma_b \cup \Gamma_{b-\varepsilon_0}} e^{-s\lambda_2} P_i(W_\varepsilon) d\Gamma;
\]
for $i = 1, 2$, which imply that
\[
W \in W^{1,2}(\Sigma_2, T\Sigma_2).
\]
Thus, by the trace theorem,
\[
W \in L^2(\Gamma_b, TM).
\]
It is easy to check that $\tau = \alpha_t/|\alpha_t|$, $X_2$ forms a vector basis along $\Gamma_b$ by (3.23). From (3.21), we have
\[
Dw = p_1 \tau + p_2 X_2 \quad \text{for} \quad x \in \Gamma_b,
\]
where
\[
p_1 = \frac{1}{1 - \langle \tau, X_2 \rangle^2} [\tau(w) - \langle \tau, X_2 \rangle \langle F, X_2 \rangle], \quad p_2 = \frac{1}{1 - \langle \tau, X_2 \rangle^2} [-\langle \tau, X_2 \rangle \tau(w) + \langle F, X_2 \rangle].
\]
Thus

\[
|\langle D_\tau Dw, X_1 \rangle| \geq |\langle \tau, X_1 \rangle| \tau(p_1) - C(|Dw| + |F|) \\
\geq |\langle \tau, X_1 \rangle| |\tau(w)| - C(|\tau(w)| + |F| + |DF|) \quad \text{for} \quad x \in \Gamma_b. \quad (3.32)
\]

Next, we compute

\[
\langle D\Phi_1(\nabla n W), X_1 \rangle = \Phi_1(\lambda_1 \langle W, X_1 \rangle) - \langle \nabla n W, D\Phi_1 X_1 \rangle \\
= \Phi_1(\lambda_1) \langle W, X_1 \rangle + \langle \nabla n D\Phi_1 W, X_1 \rangle + \lambda_1 \langle W, D\Phi_1 X_1 \rangle - \langle \nabla n W, D\Phi_1 X_1 \rangle.
\]

Thus

\[
\lambda_1 \langle D\Phi_1 W, X_1 \rangle = \langle \nabla n D\Phi_1 W, X_1 \rangle = \langle D\Phi_1(Dw - F), X_1 \rangle \\
- \Phi_1(\lambda_1) \langle W, X_1 \rangle - \lambda_1 \langle W, D\Phi_1 X_1 \rangle + \langle \nabla n W, D\Phi_1 X_1 \rangle
\]

which yields, by (3.31) and (3.32),

\[
|\langle D\Phi_1 W, X_1 \rangle| \geq |\alpha_t| \lambda_1 \langle Dw, X_1 \rangle - C(|W| + |F| + |DF|) \\
\geq \sigma |\tau(w)| - C(|\tau(w)| + |F| + |DF|) \quad \text{for} \quad x \in \Gamma_b,
\]

since \( \Phi_1 = |\alpha_t|\tau \). Thus (3.14) follows from (3.30) in the case of \( m = 1 \).

**Proof of Theorem 3.1** We solve problem (3.7) on the region \( \Sigma_1 \) to have a solution \( w_2 \) and then solve problem (3.8) on \( \Sigma_2 \) with the data

\[
q_0 = w_2, \quad q_2 = \langle Dw_2, Q\nabla n\alpha_t \rangle \quad \text{for} \quad x \in \Gamma_{b-\varepsilon_0},
\]

to obtain the solution \( w_2 \). Furthermore, paste two solutions together to have a solution to problem (3.1) on the region \( S \). Clearly, the solution meets our needs.

**Proof of Theorem 3.2** From Theorem 3.1 and [7, Theorem 4.3], we have

\[
c \Gamma_2(w, S) \leq \|w\|^2_{W^{2,2}(\Sigma_1)} + \|f\|^2_{W^{1,2}(\Sigma_1)} + \|F\|^2_{W^{2,2}(\Sigma_1 \cup T\Sigma_1)} \\
\leq \|w\|^2_{W^{2,2}(S)} + \|f\|^2_{W^{1,2}(S)} + \|F\|^2_{W^{2,2}(S \cup T\Sigma_1)} \\
\leq C(\Gamma_2(w, S) + \|f\|^2_{W^{1,2}(S)} + \|F\|^2_{W^{2,2}(S \cup T\Sigma_1)}).
\]

**4 Proofs of Theorems 1.1-1.3 in Section 1**

Let

\[
U \in T_{\text{sym}}^2 S.
\]

22
Consider problem
\[
\begin{aligned}
Dv &= \nabla \bar{n}V + F \quad \text{for } x \in S, \\
\text{div}_g V + \text{tr}_g \Pi & = f \quad \text{for } x \in S,
\end{aligned}
\]  
(6.1)
where \((V, v) \in W^{1,2}(S, T) \times W^{1,2}(S)\) is the unknown and
\[
F = Q[D(\text{tr}_g U) - \text{div}_g U], \quad f = -\text{tr}_g U(Q\nabla \bar{n}, \cdot) \quad \text{for } x \in S.
\]  
(6.2)

For \(y \in W^{1,2}(S, \mathbb{R}^3)\), let
\[
2v = \nabla y(e_2, e_2) - \nabla y(e_2, e_2) \quad \text{for } x \in S,
\]  
(6.3)
\[
V = (\nabla \bar{n})^{-1}(Dv - F) \quad \text{for } x \in S,
\]  
(6.4)
where \(e_2, e_2\) is an orthonormal basis of \(T_x S\) with positive orientation.

By [7, Section 2], there is a \(y \in W^{1,2}(S, \mathbb{R}^3)\) to solve problem (1.1) if and only if \((v, V)\), being given in (6.3) and (6.4), solves problem (6.1). In that case, we have
\[
\begin{aligned}
\nabla_{e_1} y &= U(e_1, e_1)e_1 + [v + U(e_1, e_2)]e_2 - (QV, e_1)\bar{n}, \\
\nabla_{e_2} y &= [-v + U(e_1, e_2)]e_1 + U(e_2, e_2)e_2 - (QV, e_2)\bar{n},
\end{aligned}
\]  
(6.5)
Moreover, from [7, Theorem 2.1], \((V, v)\) is a solution to problem (6.1) if and only if \(v\) solves problem
\[
\langle D^2 v, Q^*\Pi \rangle + \frac{1}{\kappa}X_0 v + \nu \text{tr}_g \Pi = \kappa f + \frac{1}{\kappa}\langle X_0, F \rangle + \langle DF, Q^*\Pi \rangle \quad \text{for } x \in S,
\]  
(6.6)
where \(f\) and \(F\) are given in (6.2).

**Proof of Theorem 1.1** We solve the degenerated hyperbolic problem
\[
\begin{aligned}
\langle D^2 v, Q^*\Pi \rangle + \frac{1}{\kappa}X_0 v + \nu (\text{tr}_g \Pi)v &= \kappa f + \frac{1}{\kappa}\langle X_0, F \rangle + \langle DF, Q^*\Pi \rangle \quad x \in S, \\
v &= \langle Dv, Q\nabla \bar{n}\alpha_t \rangle = 0 \quad \text{for } x \in \Gamma_0,
\end{aligned}
\]
b by Theorem 3.1, to have the solution \(v \in W^{m+1,2}(S)\). Then, set
\[
V = (\nabla \bar{n})^{-1}(Dw - F) \quad \text{for } x \in S.
\]
It follows from [7, Theorem 2.1] that problem (1.1) admits a solution \(y \in W^{m,2}(S, \mathbb{R}^3)\) that satisfies (6.5) on the region \(S\).

Let \(y = W + w\bar{n}\), where \(w = \langle y, \bar{n} \rangle \in W^{m,2}(S)\). Since
\[
\text{sym} DW = U - w\Pi \quad \text{for } x \in S,
\]
it follows from [6, Lemma 4.3] that
\[
W \in W^{m+1,2}(S, TS).
\]
Proof of Theorem 1.2 Let \( y \in W^{2,2}(S, \mathbb{R}^3) \) be an infinitesimal isometry. Let
\[
2v = \nabla y(e_2, e_2) - \nabla y(e_2, e_2), \quad V = (\nabla \vec{n})^{-1}Dv \quad \text{for} \quad x \in S,
\]
where \( e_2, e_2 \) is an orthonormal basis of \( T_xS \) with positive orientation. It follows from (6.5) and (6.6) that
\[
\begin{align*}
\nabla e_2 y &= ve_2 - \langle QV, e_2 \rangle \vec{n}, \\
\nabla e_2 y &= -ve_2 - \langle QV, e_2 \rangle \vec{n}
\end{align*}
\quad \text{for} \quad x \in S. \tag{6.7}
\]
and
\[
\langle D^2v, Q^*\Pi \rangle + \frac{1}{\kappa} X_0 v + \kappa tr_g \Pi = 0 \quad \text{for} \quad x \in S. \tag{6.8}
\]
From (6.7), (6.8), Theorems 3.1, and 3.2, we have
\[
\|y\|_{W^{2,2}(S)} \leq C(\|v\|_{W^{1,2}(S)} + \|V\|_{W^{1,2}(S,T S)}) \leq C\Gamma_2(v, S) \tag{6.9}
\]
where \( \Gamma_2(v, S) \) is given in (3.4) with \( w \) replaced by \( v \).
For given \( \varepsilon > 0 \), we take \( q_{0\varepsilon} \in W^{m+2,2}(\Gamma_0) \) and \( q_{1\varepsilon} \in W^{m+1,2}(\Gamma_0) \) such that
\[
\|v - q_{0\varepsilon}\|_{W^{2,2}(\Gamma_0)} + \|\langle Dv, Q\nabla \vec{n} \alpha_t \rangle - q_{1\varepsilon}\|_{W^{1,2}(\Gamma_0)} \leq \varepsilon. \tag{6.10}
\]
Then we solve problem
\[
\begin{align*}
\langle D^2v, Q^*\Pi \rangle + \frac{1}{\kappa} X_0 v + \kappa (tr_g \Pi)v &= 0 \quad \text{for} \quad x \in S, \\
v &= q_{0\varepsilon}, \quad \langle Dw, Q\nabla \vec{n} \alpha_t \rangle = q_{1\varepsilon} \quad x \in \Gamma_0
\end{align*}
\]
to obtain the solution \( v_\varepsilon \). Thus there is \( y_\varepsilon \in \mathcal{V}(S, \mathbb{R}^3) \) with \( 2v_\varepsilon = \nabla y_\varepsilon(e_2, e_2) - \nabla y_\varepsilon(e_2, e_2) \).
By Theorem 3.1, \( y_\varepsilon \in W^{m+2,2}(S, \mathbb{R}^3) \), and, thus, by the imbedding theorem (see \[1, \text{P. 158}\])
\[
y_\varepsilon \in \mathcal{V}(S, \mathbb{R}^3) \cup C_B^m(S, \mathbb{R}^3).
\]
Finally, it follows from Theorem 3.2, (6.9), and (6.10) that
\[
\|y - y_\varepsilon\|_{W^{2,2}(S)} \leq C\varepsilon.
\]
The proof is complete. \( \square \)

Proof of Theorem 1.3 As in \[2\] we conduct in \( 2 \leq i \leq m \). Let
\[
y_\varepsilon = \sum_{j=0}^{i-1} \varepsilon^j z_j
\]
be an \((i-1)\)th order isometry of class \( C^{2+4(m-i+1)}_B(S, \mathbb{R}^3) \), where \( z_0 = \text{id} \) and \( z_2 = y \) for some \( i \geq 2 \). Then
\[
\sum_{j=0}^{k} \nabla^T z_j \nabla z_{k-j} = 0 \quad \text{for} \quad 1 \leq k \leq i - 1.
\]
Next, we shall find out $z_i \in C_B^{2+4(m-i)}(S, \mathbb{R}^3)$ such that

$$\phi_\varepsilon = y_\varepsilon + \varepsilon^i z_i$$

is an $i$th order isometry. By Corollary 1.1 there exists a solution $z_i \in C_B^{2+4(m-i)}(S, \mathbb{R}^3)$ to problem

$$\text{sym} \nabla z_i = -\frac{1}{2} \text{sym} \sum_{j=1}^{i-1} \nabla^T z_j \nabla z_{i-j}$$

which satisfies

$$\|z_i\|_{C_B^{2+4(m-i)}(S, \mathbb{R}^3)} \leq C \left\| \sum_{j=1}^{i-1} \text{sym} \nabla^T z_j \nabla z_{i-j} \right\|_{C_B^{2+4(m-i)+3}(S, \mathbb{R}^3)}$$

$$\leq C \sum_{j=1}^{i-1} \|z_j\|_{C_B^{2+4(m-i+1)}(S, \mathbb{R}^3)} \|z_i-j\|_{C_B^{2+4(m-i+1)}(S, \mathbb{R}^3)}.$$ 

The conduction completes.

Compliance with Ethical Standards

Conflict of Interest: The author declares that there is no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by the authors.

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