Particle Number Fluctuations in Statistical Model with Exact Charge Conservation Laws

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Abstract. Even though the first momenta i.e. the ensemble average quantities in canonical ensemble (CE) give the grand canonical (GC) results in large multiplicity limit, the fluctuations involving second momenta do not respect this asymptotic behaviour. Instead, the asymptotics are strikingly different, giving a new handle in study of statistical particle number fluctuations in relativistic nuclear reactions. Here we study the analytical large volume asymptotics to general case of multispecies hadron gas carrying fixed baryon number, strangeness and electric charge. By means of Monte Carlo simulations we have also studied the general multiplicity probability distributions taking into account the decay chains of resonance states.

1. Introduction

The surprising success of the statistical hadronization model in describing the multihadron production in high-energy nucleus-nucleus reactions (see e.g. [1] and references therein) as well as in elementary particle collisions [2] gives rise to a question if the fluctuations, especially the second moments of multiplicity distributions, follow the statistical resonance gas results as well as the mean multiplicities (the first momenta of multiplicities). Even though the CE first momenta reach the GC values quite fast with increasing multiplicities [3], we have found that the variance of these values always stay different in the CE [4, 5] due to the lack of charge bath, i.e. due to the lack of fluctuations of the relevant net charges.

In section 2 we pose and solve the theoretical problems involved, and give results for the general hadron gas. The section 3 is devoted to the numerical studies employing the methods developed in previous section, and to the comparison of those with the ones given by the Monte Carlo (MC) procedure explained in Ref. [6].

2. Theoretical Studies

The central quantity in our discussion is the scaled variance of the multiplicity \( N \):

\[
\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}.
\]  

This is a finite measurable parameter even at the infinite volume limit, and we will show it losing its volume dependence in large systems. It is worth stressing that for the Poissonian distribution corresponding to Boltzmann GC ensemble, \( \omega = 1 \). Although we keep the theoretical discussion on the level of Boltzmann approximation for brevity, we show also the full quantum statistics results in the section 3.

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In the CE, the partition function does not factorize to one-species parts, but the different species are bound by the group factors giving delta functions over corresponding exact conservation laws. Recall the form of the partition function for canonical hadron gas carrying baryon number \( B \), strangeness \( S \) and electric charge \( Q \) [7]:

\[
Z_Q(T, V) = \left[ \prod_{i=1}^{3} \frac{1}{2\pi} \int_0^{2\pi} d\phi_i e^{-iQ_i\phi_i} \right] Z_{GC}(T, V, \{\lambda_Q_i\}),
\]

where \( \bar{Q}_i = (Q_1, Q_2, Q_3) = (B, S, Q) \). \( \phi_i \in [0, 2\pi) \) is a \( U(1) \) group parameter and a Wick-rotated fugacity factor \( \lambda_Q_i \rightarrow e^{i\phi_i} \) is introduced for every charge in the grand canonical partition function \( Z_{GC} \).

By the substitution \( x_i = e^{i\phi_i} \) and still integrating over the three unit circles we may write

\[
Z_{\bar{Q}-\bar{Q}}(T, V) = \frac{1}{(2\pi i)^2} \oint dx_B dx_S dx_Q \chi_B^{B-1} \chi_S^{S-1} \chi_Q^{Q-1} \times \exp \left\{ -B \ln x_B - S \ln x_S - Q \ln x_Q + \sum_j z_j^1 \chi_B^{B_j} \chi_S^{S_j} \chi_Q^{Q_j} \right\}.
\]

The first and second momenta of a given multiplicity of a hadron subset \( h \) can be calculated by inserting fictitious fugacity to the generating function \( Z_{GC}(T, V, \{\lambda_Q_i \rightarrow e^{i\phi_i}\}) \) in equation (2) and performing the derivatives

\[
\langle N_h \rangle = \left. \frac{1}{Z_Q} \sum_{j \in h} \frac{\partial Z_Q}{\partial \lambda_j} \right|_{\lambda_j = 1} = \sum_{j \in h} z_j^1 \frac{Z_{\bar{Q}-\bar{Q}}}{Z_{\bar{Q}}},
\]

\[
\langle N_h^2 \rangle = \left. \frac{1}{Z_Q} \sum_{j \in h} \sum_{l \in h} \left[ \frac{\partial}{\partial \lambda_j} \left( \chi \frac{\partial Z_Q}{\partial \lambda_l} \right) \right] \right|_{\lambda_j = 1} = \sum_{j \in h} z_j^1 \frac{Z_{\bar{Q}-\bar{Q}}}{Z_{\bar{Q}}} + \sum_{j \in h} z_j^1 \sum_{l \in h} z_l^1 \frac{Z_{\bar{Q}-\bar{Q}-\bar{Q}}}{Z_{\bar{Q}}}.
\]

Using these, the scaled variance can be written as the sum of Poissonian part and the canonical correction:

\[
\omega_h = 1 + \frac{\sum_{i \in h} \langle N_i \rangle \sum_{j \in h} z_j^1 \left( \frac{Z_{\bar{Q}-\bar{Q}-\bar{Q}}}{Z_{\bar{Q}-\bar{Q}}} \right) - z_j^1 \frac{Z_{\bar{Q}-\bar{Q}}}{Z_{\bar{Q}}}}{\sum_{i \in h} \langle N_i \rangle}.
\]

For the large systems, the canonical correction factors can be solved by means of the asymptotic expansion, where the saddlepoint of the \( f(\bar{x}) \) (3) coincides with the values of GC fugacities [8]. In the general case of multi-hadron gas carrying total charges \( \{Q_k\} = (B, S, Q) \) the asymptotic expansion of the partition function involves diagonalization of the Hessian matrix

\[
H_{Q_k, Q_l}(\bar{x}_0) = \frac{\partial^2 f(\bar{x})}{\partial x_{Q_k} \partial x_{Q_l}} |_{x_0} = \lambda_{Q_k}^{-1} \lambda_{Q_l}^{-1} \left\{ Q_k \delta_{kl} + \sum_j (Q_k)_j [(Q_l)_j - \delta_{kl}] \langle N_j \rangle_{GC} \right\}.
\]

Because this is real and symmetric, the diagonalization is done by the orthogonal matrix \( \Lambda : H' = \text{diag}(h_1, h_2, h_3) = \Lambda^T H \Lambda \), where \( h_k \) are the Hessian eigenvalues, and \( \Lambda \) is constructed of corresponding eigenvectors. Applying these and the notation \( \langle N_j \rangle_{GC} \) for GC average multiplicities, the large volume limit of the Eq. (5) is found to be

\[
\omega_h = 1 - \frac{\sum_{i \in h} \langle N_i \rangle_{GC} \sum_{j \in h} \langle N_j \rangle_{GC} \sum_{k} (Q_k)_i (Q_k)_j \lambda_Q_k \lambda_Q_m \sum_i \frac{\Lambda_{Q_k}^i \Lambda_{Q_l}^i}{h_i}}{\sum_{i \in h} \langle N_i \rangle_{GC}}.
\]
where the summations of $Q_{k,l}$ are over relevant charges $B$, $S$ and $Q$, and the $\tilde{N}_{Q_{b}}^{l}$ are the elements of the transformation matrix $A^T$.

As an easily evaluable approximation, we consider the scaled variance for the number of positively ($h^{+}$) and negatively ($h^{-}$) charged particles as well as the sum of all charged particles neglecting the multi-charged particles and the mixing between different charges (non-diagonals in the Hessian). The relevant Hessian term is $H_{QQ} = \lambda_{Q}^{-2} \left[ Q + \sum_{j} z_{j}^{Q} \lambda_{j} Q_{j} (Q_{j} - 1) \right]$, which leads to $H_{QQ} = \lambda_{Q}^{-2} [Q + 2 \langle h^{-} \rangle_{GC}] = \lambda_{Q}^{-2} \left[ \langle h^{+} \rangle_{GC} + \langle h^{-} \rangle_{GC} \right]$. This yields the terms $\frac{Q_{i} Q_{j}}{\langle h^{+} \rangle_{GC} + \langle h^{-} \rangle_{GC}}$ inside the $Q_{k,m}$ summations in the Eq. (7). Applying this to equation (7) yields

$$\lim_{V \to \infty} \omega_{\pm} \simeq 1 - \frac{\langle h^{\pm} \rangle_{GC}}{\langle h^{+} \rangle_{GC} + \langle h^{-} \rangle_{GC}}; \quad \lim_{V \to \infty} \omega_{ch} \simeq 1 - \left( \frac{\langle h^{+} \rangle_{GC} - \langle h^{-} \rangle_{GC}}{\langle h^{+} \rangle_{GC} + \langle h^{-} \rangle_{GC}} \right)^2. \quad (8)$$

This result serves as a “rule of thumb” for any charge under consideration – it applies trivially to baryons and strange particles too – but in quantitative comparisons one must relax the approximations used above. In case of one-charge system, such as pion gas, Eq. (8) is an exact result for thermodynamical limit. One sees immediately, that for a neutral system $\lim_{V \to \infty} \omega_{\pm} = 0$ instead of GC limit $\omega_{\pm} = 1$. When charge density grows indefinitely, the limiting values are $\lim_{V \to \infty} \omega_{+} = 0$, $\lim_{V \to \infty} \omega_{-} = 1$ and $\lim_{V \to \infty} \omega_{ch} = 0$.

3. Numerical Results

In some special cases, Eq. (5) can be easily evaluated for any volume [4, 5], not only in the asymptotic limit. Thus, in Fig. 1 we quote the neutral pion gas charged particle fluctuations as functions of GC multiplicities [4]. Also in Fig. 1, we show the asymptotic large volume results as functions of baryochemical potential in the nucleon-pion gas in $BQ$-canonical calculation [5] together with the approximation where the $B$ is handled in GC manner, see Eq. (8).
In Fig. 2 we turn to the full BSQ-canonical $\omega$ calculations as functions of baryon density for temperature $T = 160$ MeV. The isospin condition is chosen to correspond to the PbPb and AuAu reactions: $Q/B = 0.4$. We take into account all the resonances up to the mass $\sim 1.9$ GeV. The left panel shows the asymptotic results for scaled variances of primary negative, positive and charged hadrons together with Monte Carlo results for $V = 200$ fm$^3$, which is checked to be large enough to give asymptotic results. Also shown are the values for $\omega_-$ and $\omega_{ch}$ after the strong decay chain. In the right panel we present the scaled variances for primary baryons, antibaryons, strange hadrons and anti-strange hadrons.

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