VARIATIONAL PROPERTIES OF THE KINETIC SOLUTIONS OF SCALAR CONSERVATION LAWS

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Abstract. We discuss properties of kinetic solutions of scalar conservation laws in the variational approach developed by Panov\cite{Panov2004, Panov2005} and also Brenier\cite{Brenier1989}. Our main result shows that such solutions can be considered as curves in a suitable Hilbert space with tangents that are unique minimizers of an interaction functional.

1. Introduction

We consider a scalar conservation law
\[ u_t + \nabla_x \cdot f(u) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \]
with the flux \( f : \mathbb{R} \to \mathbb{R}^n \). A function \( u(t, x) \) is called an entropy weak solution if for any convex entropy/entropy flux pairs \((\eta(u(t, x)), q(u(t, x)))\) of the flux function \( f \),
\[ \eta_t + \nabla_x \cdot q \leq 0, \quad \mathcal{D}'(\mathbb{R}_+^{n+1}). \]

The existence of unique entropy solutions for\eqref{eq:conservation} with the initial data \( u(t = 0) = u_0 \in L^\infty(\mathbb{R}^n) \), as well as their stability, was obtained in Kruzhkov\cite{Kruzhkov1970}. The solution can be described using the kinetic formulation, as proved by Lions-Perthame-Tadmor\cite{Lions1994}. In this approach, \( u(t, x) \) is a weak entropy solution iff the kinetic density function
\[ Y(t, x, v) = \begin{cases} 1 & v \geq u(t, x), \\ 0 & v < u(t, x), \end{cases} \]
verifies the transport equation
\[ Y_t + f_v(v) \cdot \nabla_x Y = -\partial_v m, \quad \mathcal{D}'(\mathbb{R}_+^{n+2}), \]
for some measure \( m \in \mathcal{M}_+(\mathbb{R}_+^{n+2}) \). To be precise, in\cite{Lions1994}, the kinetic density \( \chi(t, x, v) = H(v) - Y(t, x, v) \), where \( H = 1, v \geq 0, H = 0, v < 0 \), was used, but the result can be expressed through \( Y(t, x, v) \) as well.

Condition\eqref{eq:transport} can be equivalently expressed via a variational form: for any regular test functions \( \tilde{Y}(x, v) \), nondecreasing in \( v \), it holds
\[ \iint (\tilde{Y} - Y)(Y_t + f_v \cdot \nabla_x Y) \, dx \, dv \geq 0, \quad \mathcal{D}'(\mathbb{R}). \]
Indeed, the equivalence holds because,
\[ \iint Y(Y_t + f_v \cdot \nabla_x Y) \, dx \, dv = \partial_t \int |u(t, x)| \, dx = 0, \]

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if, for example, \( u_0 \in L^1(\mathbb{R}^n) \), or periodic. An interesting kinetic formulation for weak solutions were obtained in Panov\cite{Panov2012, Panov2013}, by allowing generic nondecreasing in \( v \) functions \( Y(t, x, v) \) in \( \{ -1, 0, 1 \} \), rather than functions with the range in \{ -1, 0, 1 \}.

Panov\cite{Panov2012} introduced a class of strong measure-valued solutions on \( \mathbf{1} \) as the set of parametrized probability measures \( \nu_{t,x} \in M_+(\mathbb{R}) \), for which the level sets of their distribution functions \( Y(t, x, v) := \nu_{t,x}((-\infty, v]) \) are the graphs of weak entropy solutions of \( \mathbf{1} \), i.e.

\[
\begin{align*}
    u(t, x, \lambda) &= \sup\{ v : Y(t, x, v) \leq \lambda \},
    \\
    \nu_{t,x} &= \delta_{u(t,x)}.
\end{align*}
\]

is an entropy weak solution of \( \mathbf{1} \) for any \( \lambda \in (0, 1) \). It was shown that for each initial data \( \nu_{0,x} \) there is a unique, global strong measure-valued solution. The weak entropy solutions of \( \mathbf{1} \) are naturally contained in this approach as measures

\[
\nu_{t,x} = \delta_{u(t,x)}.
\]

The set strong measure-valued solutions is the subclass of the entropy measure-valued solutions were introduced by Tartar\cite{Tartar1979} in his compensated compactness method. Such solutions were further studied by DiPerna\cite{DiPerna1989}, who showed that the weak entropy solutions are unique in the class of measure-valued solutions and by Schochet\cite{Schochet1993}, who showed that entropy measure-valued solutions with a prescribed initial data are not unique.

An equivalent formulation of a strong measure-valued solution was given in Panov\cite{Panov2013} where it was shown that \( Y(t, x, v) \) is the distribution of a strong measure valued solution iff it verifies \( \mathbf{5} \). Note that the formulation \( \mathbf{5} \) in addition to \( \mathbf{4} \) prescribes a non-trivial, non-linear constraint:

\[
\int\int Y^2(t, x, v) \, dx \, dv = \text{const.}, \quad \text{a.e.} \, t > 0.
\]

This result was later re-discovered by Brenier\cite{Brenier2013} in the following form. He proved that \( Y \) is the solution of \( \mathbf{5} \) iff \( u(t, x, \lambda) \) from \( \mathbf{6} \) is an entropy weak solution of \( \mathbf{1} \) for any \( \lambda \) in the interior of the range of \( Y \). In that paper, the variational formulation \( \mathbf{5} \) is expressed as a differential inclusion on a Hilbert space of \( L^2 \) integrable in \((x, v)\) functions:

\[
Y_t \in -f_v \cdot \nabla_x Y - \partial K(Y),
\]

where \( \partial K(Y) \) is the subdifferential to the indicator function of a convex, closed cone \( K \) consisting of all non-decreasing in \( v \) functions, see section \( \mathbf{3} \) for details. Operator appearing on the right of \( \mathbf{9} \) is monotone, providing the uniqueness of solutions and if in addition it is maximal then the existence follows from the classical results, for example Brezis\cite{Brezis1973}.

Additionally to the existence/uniqueness/stability of solution Brenier\cite{Brenier2013} proves the regularity of solutions of \( \mathbf{1} \): if the initial data \( Y_0(x, v) \) is differentiable, \( \nabla_x Y_0 \in L^2_{x,v} \), then

\[
\partial_t Y, \nabla_x Y \in L^\infty((0, +\infty); L^2_{x,v}).
\]

It can also be shown that \( \partial_t Y \in L^\infty((0, +\infty); L^2_{x,v}) \), if in addition \( \partial_x Y_0 \in L^2_{x,v} \).

The results \cite{Panov2012, Panov2013, Brenier2013} show a remarkable fact that all weak entropy solutions of \( \mathbf{1} \) can be obtained through \( \mathbf{4} \) (or \( \mathbf{7} \)) from a globally stable and regular (if \( \partial_x Y_0 \in L^2_{x,v} \)) kinetic densities \( Y(t, x, v) \) – solutions of \( \mathbf{1} \) (or \( \mathbf{5} \)).

In this paper we further investigate the properties of solution of \( \mathbf{9} \). Such solutions can be considered as curves \( Y(t) \) with values in the admissible cone \( K \) for which the tangent \( \partial_t Y(t) \) belongs to the tangent cone \( T_K(Y(t)) = \text{closure} \{ h(\bar{Y} - Y(t)) : h > 0, \bar{Y} \in K \} \).
Our main result shows that \(Y(t)\) is the solution of \(\Phi\) iff \(\partial_t Y(t)\) minimizes the functional \(\|V + f_u \cdot \nabla_x Y(t)\|_{L^2_x(v)}\) over all directions \(V \in T_K(Y(t))\). This functional can considered as an “instantaneous” interaction functional. Since set \(K\) restrict solutions only in \(v\) direction this functional is local in \(x\), i.e.,

\[
\min_{T_K(Y(t))} \|V + f_u \cdot \nabla_x Y(t)\|_{L^2_x(v)} = \|\min_{T_K(Y(t,x,\cdot))} \|V(\cdot) + f_u(\cdot) \cdot \nabla_x Y(t,x,\cdot)\|_{L^2_x} ||_{L^2_x(v)}
\]

Zero minimal value is attained on solutions \(Y(t)\) that are simply transported, i.e. \(Y_t + f_v \cdot \nabla_x Y = 0\), while remaining in \(K\). At the level of weak entropy solutions of \(\Phi\) such \(Y(t)\) corresponds to classical solutions. For shock waves it is proportional to the shock strength, see section \(3\).

After the prove of this result, which based on the fact that solutions of \(\partial_t Y \in - A(Y(t))\), with maximal monotone operator \(A\) are slow solutions, i.e., the solutions for which \(\|A(Y(t))\|\) is minimal, we show that there are travelling wave solution to \(\Phi\) that correspond to the shock waves of \(\Phi\). Such travelling waves move with the actual shock speed \(\sigma = \Delta f / \Delta u\). The shock speed appears in solving a minimization problem

\[
\min_{V \in T_K(Y(t))} \|V + f_v \cdot \nabla_x Y(t)\|_{L^2_x(v)}
\]

The shock profiles of this type are obtained by smoothing in \(x\) direction the kinetic density \(\Phi\) of the shock wave \(u(t,x)\). This however is rather exceptional case. In the last part of this paper we give an example that shows that generically if \(u(t,x)\) is a weak entropy solution that contain interacting waves and if \(Y(t,x,v)\) its kinetic density, then \(\tilde{Y} = \tilde{Y}(t,x,v) \ast \omega_\varepsilon(x)\), is not a solutions of \(\Phi\). This happens because the constraint \(\Phi\) is non-linear in \(Y\) and does not commute with averaging.

2. General theory

Let \(H\) be the space of \(2\)-periodic in \(x\), functions \(Y(x,u)\) of \((x,u) \in \mathbb{R}^n \times [0,1]\), with the norm

\[
\|Y\|^2 = \langle Y,Y \rangle = \int_{\Pi} \int_0^1 Y^2(x,u) dudx, \quad \Pi = [-L,L]^n.
\]

Let \(K \subset H\) a set of \(Y\)’s non-decreasing in \(u\). \(K\) is a closed cone and for any \(Y \in K\), we denote

\[
T_K(Y) = H - \text{ closure of } \{h(\tilde{Y} - Y) : h \geq 0, \tilde{Y} \in K\}, \quad (10)
\]

the tangent cone to \(K\) at \(Y\) and the normal cone:

\[
\partial K(Y) = \{Z \in H : \langle Z, \tilde{Y} - Y \rangle \leq 0, \forall \tilde{Y} \in K\}. \quad (11)
\]

Also by \(N = \{Z \in H : \langle Z, \tilde{Y} \rangle \leq 0, \forall \tilde{Y} \in K\}\) we denote the polar cone to \(K\). We consider the Cauchy problem

\[
\partial_t Y + f_u \cdot \nabla_x Y \in - \partial K(Y), \quad Y(t = 0) = Y_0. \quad (12)
\]

The flux function \(f \in Lip([0,1])^n\) – Lipschitz continuous on \([0,1]\).

It was shown in Brenier\(^2\) that for any \(Y_0 \in K\), there is a unique solutions of \(\Phi\) \(Y \in C([0, +\infty) ; H)\), for any \(t \geq 0\), \(Y(t) \in K\), and if \(\nabla_x Y_0 \in H\), then also

\[
\partial_t Y, \nabla_x Y \in L^\infty((0, +\infty); H).
\]

Our main result contained in the following theorem.
Theorem 1. Let \( Y_0 \in K \) and \( \nabla_x Y_0 \in \mathbb{H} \). For the solution \( Y \in C([0, +\infty); \mathbb{H}) \) of (12), a.e. \( t > 0 \),
\[
\|\partial_t Y(t)\| = \min_{Z \in \partial K(Y(t))} \|Z + f_v \cdot \nabla_x Y(t)\|, \quad (13)
\]
and
\[
\|\partial_t Y(t) + f_v \cdot \nabla_x Y(t)\| = \min_{V \in T_K(Y(t))} \|V + f_v \cdot \nabla_x Y(t)\|. \quad (14)
\]
Conversely, each of the conditions (13), (14) defines a unique solution of the problem (12).

Proof. Let \( a(v) = f_v(v)/|f_v(v)|, \ v \in [0,1] \). If \( |f_v| > 0 \), we define \( B(Y) = |f_v(v)|\partial_a v \cdot \partial_a Y \) for a.e. \( v \in [0,1] \) and such that \( \partial_a Y \in \mathbb{H} \).

Lemma 1. Let
\[
c_0 = \text{ess inf}_{[0,1]} |f_v(v)| > 0. \quad (15)
\]
Then, \( B \) is a maximal monotone operator.

Proof. Monotonicity of \( B \) follows directly from the definition of \( B \) and periodicity of \( Y \) in \( x \). To show maximality, let \( W \in \mathbb{H} \), and \( \lambda > 0 \) consider a problem:
\[
Y + \lambda |f_v|\partial_a Y = W.
\]
For a.e. \( v \in [0,1] \), \( W_0(v,\cdot) \in L^2(\Pi) \) and the equation can be integrated along the characteristic to obtain a periodic solutions \( Y(v,\cdot) \). The inclusion \( \lambda Y \in D(B) \) follows from the a priori estimates
\[
\|Y\| \leq \|W\|, \quad c_0\|\partial_a Y\| \leq (\sqrt{\lambda})^{-1}\|W\|.
\]

Lemma 2. Under the condition on \( f \) from the previous lemma, \( B + \partial K \) is a maximal monotone operator on \( \mathbb{H} \).

Proof. Consider now a proper, l.s.c., convex function
\[
K(Y) = \begin{cases} 
0, & x \in K, \\
+\infty, & x \notin K.
\end{cases}
\]
The subdifferential \( \partial K \) is a maximal monotone operator. The Yosida approximation of \( \partial K(Y) \), equals \( \nabla K(\lambda) \), where
\[
K_\lambda(Y) = \inf_{\tilde{Y} \in \mathbb{H}} \left( K(\tilde{Y}) + \frac{1}{2\lambda}\|	ilde{Y} - Y\|^2 \right),
\]
see Theorem 4, p. 162 of Aubin-Cellina[1]. It follows that \( K_\lambda(Y) = \inf_{\tilde{Y} \in K} \frac{1}{2\lambda}\|	ilde{Y} - Y\|^2 \), and
\[
\nabla K_\lambda(Y) = \frac{1}{\lambda} \pi_N(Y),
\]
where \( \pi_N(Y) \) is a projection of \( Y \) onto \( N \) – the polar cone to \( K \). Operator \( \frac{1}{\lambda} \pi_N(\cdot) \) is a Lipschitz continuous operator from \( \mathbb{H} \) to \( \mathbb{H} \), with the Lipschitz constant \( \frac{1}{\lambda} \) and is monotone.
From lemma 1 and lemma 2.4 of Brezis [3] it follows that $B + \frac{1}{\lambda} \pi_N(\cdot)$ is a maximal monotone operator. So, for any $W \in H$ and $\alpha > 0$, there is a solution $Y^\lambda$ of
\[ Y + \alpha \left( B(Y) + \nabla K^\lambda(Y) \right) = W. \tag{16} \]
Using monotonicity we get
\[ \|Y^\lambda\| \leq \|W\|. \]
Also, since
\[ \langle \nabla K^\lambda(Y^\lambda), \partial_a Y^\lambda \rangle = \langle \frac{1}{\lambda} \pi_N(Y^\lambda), \partial_a Y^\lambda \rangle = 0, \]
we obtain
\[ \alpha \langle f_v, \partial_a Y^\lambda, \partial_a Y^\lambda \rangle = -\langle W, \partial_a Y^\lambda \rangle, \]
and consequently,
\begin{align*}
\|\partial_a Y^\lambda\| &\leq C(\alpha, c_0)\|W\|, \quad \text{(17)} \\
\|\nabla K^\lambda(Y^\lambda)\| &\leq C(\alpha, \esssup_v |f_v(v)|, c_0)\|W\|. \quad \text{(18)}
\end{align*}
We want to pass to the limit $\lambda \to 0$ in the equation (16). We have shown that all terms in that equation are weakly compact in $H$. It remains to show that the sequence $Y^\lambda$ is strongly compact and use the strong-weak closeness of the maximal monotone operator $\partial K$. Using the equation (16) we compute
\[ \|Y^\lambda - Y^\mu\| \leq \alpha(\lambda\langle \nabla K^\lambda(Y^\lambda), \nabla K^\mu(Y^\mu) \rangle + \mu\langle \nabla K^\lambda(Y^\lambda), \nabla K^\mu(Y^\mu) \rangle) \]
\[ - \lambda\|\nabla K^\lambda(Y^\lambda)\|^2 - \mu\|\nabla K^\mu(Y^\mu)\|^2. \]
This estimate, due to (13), implies that $Y^\lambda$ is Cauchy and converges to some $Y \in H$. Moreover $(I + \lambda\partial K)^{-1}(Y^\lambda) = \pi_K(Y) - \lambda\nabla K^\lambda(Y^\lambda)$ converges to $Y$. Since $\nabla K^\lambda(Y^\lambda) \in \partial K((I + \lambda\partial K)^{-1}(Y^\lambda))$, and $\partial K$ is strongly-weakly closed, it follows that $\nabla K^\lambda(Y^\lambda) \to \partial K(Y)$, and $Y$ is the solution of
\[ Y + \alpha(B(Y) + \partial K(Y)) = W, \]
proving this the maximality of $B + \partial K$.

Consider a Cauchy problem
\[ \partial_t Y \in -B(Y) - \partial K(Y), \quad Y(t = 0) = Y_0. \tag{19} \]
Under the non-degeneracy condition (15), $B + \partial K$ is maximal monotone and the problem (19) has a unique solution with the properties listed in the next theorem, see theorem 1, p.142 of [1].

**Theorem.** Let $Y_0 \in D(B) \cap K$. There is a unique solution $Y(t)$ of (19) for $t \in [0, +\infty)$, with the following properties: $Y(t) \in D(B) \cap K$,
\[ Y \in C([0, T]; H), \quad \forall T > 0, \partial_t Y, \partial_a Y \in L^\infty(0, +\infty; H). \]
Moreover,
\begin{enumerate}
\item If $\nabla_x Y_0 \in H$, then for any $t > 0$, $\|\nabla_x Y(t)\| \leq \|\partial_x Y_0\|$. \item $\partial_t Y(\cdot)$ is continuous from the right on $[0, +\infty)$ and $\|\partial_t Y(t)\| \leq \esssup_u |f_v(v)|\|\partial_x Y_0\|$. \end{enumerate}
(3) For any $t > 0$,
\[
\partial_t Y(t) = \arg \min_{\tilde{Y} \in \{-f_\varepsilon, f_{\varepsilon}Y(t) - \partial K(Y(t))\}} \|\tilde{Y}\|.
\] (20)

Let $f \in \text{Lip}([0,1])^n$ and $f_\varepsilon$ be sequence of Lipschitz continuous vector functions such that: (i) $f_\varepsilon \rightarrow f$ in $C([0,1])^n$; (ii) $f_{\varepsilon,v} \rightarrow f_v$, a.e. $u \in (0,1)$; (iii) $\|f_{\varepsilon,v}\|_{L^\infty((0,1))^n}$ uniformly bounded; (iv) for all $\varepsilon \in (0,\varepsilon_0)$, inf$_{[0,1]} |f_{\varepsilon,v}| > 0$. For each $f_\varepsilon$ and $Y_0 \in D(B) \cap K$, there is a solution $Y_\varepsilon$ that solves (19) with $f_\varepsilon$ and verifies the conclusions of the cited above theorem. It follows from the same theorem and assumptions on $f_\varepsilon$ that norms $\|Y_\varepsilon(t)\|$, $\|\partial_t Y_\varepsilon(t)\|$, $\|\nabla_x Y_\varepsilon(t)\|$ are uniformly bounded in $(t, \varepsilon) \in [0, +\infty) \times (0, \varepsilon_0)$. Moreover, by monotonicity we obtain:
\[
\|Y_{\varepsilon_1}(t) - Y_{\varepsilon_2}(t)\| \leq t(\text{ess sup}_v |f_{\varepsilon_1,v} - f_{\varepsilon_2,v}|)\|\partial_x Y_0\|, \quad \forall \varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0),
\] (21)
i.e. $Y_\varepsilon$ is compact in $C([0, +\infty); \mathbb{H})$. With this and using the fact that $\partial K$, as a maximal monotone operator is strongly-weakly closed, we obtain $Y = \lim Y_\varepsilon$ - the solution of (12) with $\partial_t Y, \nabla_x Y \in L^\infty((0, +\infty); \mathbb{H})$.

Let us prove property (13). Consider operator $A = B + \partial K$. It is monotone and has a maximal extension $\tilde{A}$. Thus $Y(t, x, u)$ is also a solution of $\partial_t Y \in -\tilde{A}(Y)$, and by (20), $\|\partial_t Y(t)\| = \min_{Z \in \tilde{A}(Y(t))} \|Z\|$, for a.e. $t$. Since $\partial_t Y \in A(Y(t))$, and (13) follows.

Now we can prove (14). For the tangent cone $T_K(Y)$, defined in the beginning of this section, we have that $V \in T_K(Y)$ iff $\forall Z \in \partial K(Y)$, $\langle V, Z \rangle \leq 0$, i.e. $T_K(Y)$ is the polar cone to a convex closed cone $\partial K(Y)$. Property (13) states that $\partial_t Y$ equals $(I - \pi_{\partial K})(-B(Y))$, where $\pi_{\partial K}$ is the projector onto $\partial K(Y)$. This can be stated equivalently, that $\partial_t Y$ is the projection of $-f_v \cdot \nabla_x Y$ onto $T_K(Y)$, or
\[
\|\partial_t Y(t) + B(Y(t))\| = \min_{V \in T_K(Y(t))} \|V + B(Y(t))\|,
\]
for a.e. $t$.

3. Examples

Consider a scalar conservation law (11) in one dimension with a convex flux function $f(u)$. We prescribe the initial data
\[
\begin{align*}
u_0(x) = \begin{cases}
u^+, & x \in [-L, 0] \cup [L/2, L], \\\nu^-, & x \in (0, L/2),
\end{cases}
\end{align*}
\]
with $u^+ > u^-$. The weak entropy solution $u(t, x)$ of (11) consists (for small times) of a shock wave propagating from $x = 0$ with the speed
\[
\sigma = (f(u^+) - f(u^-))/(u^+ - u^-)
\] (22)
and a rarefaction wave centred at $x = L/2$. The solution has this structure until the moment the shock wave collides with the r-wave. Let us choose a small $\varepsilon > 0$ and consider the kinetic formulation for this problem. We define
\[
\tilde{Y}(t, x, v) = \begin{cases}0, & v < u(t, x), \\
1, & v \geq u(t, x).
\end{cases}
\] (23)
$Y(t, x, v)$ is the solution of the variational problem (5), but it is not differentiable in $x$, and so we can not test it in the interaction functional (14). We will approximate $Y(t, x, v)$ by

$$Y_\varepsilon(0, x, v) = Y(0, x, v) * \omega_\varepsilon(x),$$

where $\omega_\varepsilon(x)$ is the standard (supported on $[x-\varepsilon, x+\varepsilon]$, non-negative, unit mass) smoothing kernel.

It can be verified that $Y_\varepsilon(t, x, v)$ for all small $t$, for which the shock wave and the r-wave in $u(t, x)$ are separated by the distance larger than $4\varepsilon$, is the solution of (9) (or 22). Indeed, $Y_\varepsilon$ verifies (4) because a convolution in $x$ with a non-negative kernel doesn’t change the structure of that equation, and, moreover it can be checked by a computation that the conservation property (5) holds as well. The structure of $Y_\varepsilon$ is simple; it consists of a smoothed shock wave : for $x \in (\sigma t - 2\varepsilon, \sigma t + 2\varepsilon)$, it equals

$$Y_\varepsilon(t, x, v) = \begin{cases} 1 & v > u^+, \\ \int_{x-\varepsilon}^{x+2\varepsilon} \omega_\varepsilon(y) dy & v \in [u^-, u^+], \\ 0 & v < u^-, \end{cases}$$

and the part that corresponds to the regularization of the rarefaction wave. Next we would like to find the value of the interaction functional. Let us fix time $t = 0$ and for $x \in [-L, L]$, consider

$$\min_{V \in T_K(Y_\varepsilon(0, x, \cdot))} \|V + f_\varepsilon(v)\partial_x Y_\varepsilon(0, x, v)\|^2_{L^2((-1,1))}. \quad (24)$$

In the next lemma we show that the minimal value of is zero when $x$ is in the range of the r-wave and it is proportional to the shock strength $|u^+ - u^-|$ for $x$ in the range of the shock discontinuity. For the $x$’s in the later case, $\partial_t Y_\varepsilon + \sigma \partial_x Y_\varepsilon = 0$, where $\sigma$ from (22).

**Lemma 3.** Let $\varepsilon < L/8$. The minimizer $V_0$ of (24) equals

$$V_0 = \partial_t Y_\varepsilon(0, x, v) = \begin{cases} -\sigma \partial_t Y_\varepsilon(0, x, v) & x \in (-2\varepsilon, 2\varepsilon), v \in [0,1], \\ -f_\varepsilon(v)\partial_x Y_\varepsilon(0, x, v) & x \in (-2\varepsilon + L/2, L/2 + 2\varepsilon), v \in [0,1], \\ 0 & \text{otherwise,} \end{cases} \quad (25)$$

with $\sigma$ from (22). The minimal value is proportional to the strength of the shock wave $|u^+ - u^-|$, for $x \in (-4\varepsilon, 4\varepsilon)$, and is 0 for other values of $x$.

**Proof.** Assume that $\sigma > 0$. The other case it treated similarly. The approximate initial datum $Y_\varepsilon(0, x, v)$, for $x \in (-2\varepsilon, 2\varepsilon)$, equals

$$Y_\varepsilon(0, x, v) = \begin{cases} 1 & v > u^+, \\ \int_{x-\varepsilon}^{x+2\varepsilon} \omega_\varepsilon(y) dy & v \in [u^-, u^+], \\ 0 & v < u^-, \end{cases}$$

and for $x \in (-2\varepsilon + L/2, L/2 + 2\varepsilon)$,

$$Y_\varepsilon(0, x, v) = \begin{cases} 1 & v > u^+, \\ \int_{x-\varepsilon}^{x+2\varepsilon} \omega_\varepsilon(y) dy & v \in [u^-, u^+], \\ 0 & v < u^-.
\end{cases}$$
\[ \partial_x Y_\varepsilon \text{ in the corresponding ranges equals} \]
\[
\partial_x Y_\varepsilon(0, x, v) = \begin{cases} 
0 & v > u^+, \\
\omega_\varepsilon(-x) & v \in [u^-, u^+], \\
0 & v < u^-, 
\end{cases}
\]

for \( x \in (-2\varepsilon, 2\varepsilon) \), and
\[
\partial_x Y_\varepsilon(0, x, v) = \begin{cases} 
0 & v > u^+, \\
-\omega_\varepsilon(L/2 - x) & v \in [u^-, u^+], \\
0 & v < u^-, 
\end{cases}
\]

for \( x \in (-2\varepsilon + L/2, L/2 + 2\varepsilon) \). Then, with
\[
V \in T_K(Y_\varepsilon(0, x, \cdot)) = L^2((0, 1)) - \text{closure of } \{ h(\tilde{Y}(v) - Y_\varepsilon(0, x, v)) : h \geq 0, \partial_x \tilde{Y} \geq 0 \},
\]

\[
\|V + f_v(v)\partial_x Y_\varepsilon(0, x, v)\|^2 = \int_0^{u^-} |V(v)|^2 dv + \int_{u^+}^1 |V(v) - 1|^2 dv + \int_{u^-}^{u^+} |V(v) + f_v(v)\partial_x Y_\varepsilon(0, x, v)|^2 dv.
\]

(26)

Notice that, due to the fact that \( Y_\varepsilon(0, x, v) \) takes only three values, all functions \( V(v) \), such that \( V = 0 \), \( V(v) = 1 \), for \( v \in (u^+, 1) \), and \( V(v) \) is non-decreasing on \([u^-, u^+]\), belong to \( T_K(Y_\varepsilon(0, x, \cdot)) \). Thus,
\[
\min_{V \in T_K(Y_\varepsilon(0, x, \cdot))} \|V + f_v(v)\partial_x Y_\varepsilon(0, x, v)\|^2 = \min_{V(v) \geq 0, v \in [u^-, u^+]} \int_{u^-}^{u^+} |V(v) + f_v(v)\partial_x Y_\varepsilon(0, x, v)|^2 dv.
\]

For \( x \in (-2\varepsilon + L/2, L/2 + 2\varepsilon) \) we can take \( V = \omega_\varepsilon(L/2 - x)f_v(v) \), for \( v \in [u^-, u^+] \). Such \( V \) gives zero value of the functional. For the shock discontinuity range \( x \in (-2\varepsilon, 2\varepsilon) \), because \( f_v(v)\partial_x Y_\varepsilon(0, x, v) \) is non-decreasing in \( v \), minimum will be achieved on constant functions \( V(v) = c : \)
\[
\min_{V'(v) \geq 0, v \in [u^-, u^+]} \int_{u^-}^{u^+} |V(v) + f_v(v)\partial_x Y_\varepsilon(0, x, v)|^2 dv = \min_c \int_{u^-}^{u^+} |c + f_v(v)\partial_x Y_\varepsilon(0, x, v)|^2 dv.
\]

The later functional is minimized for \( c = -\sigma \partial_x Y_\varepsilon(0, x, v) \). This establishes (25). It is easily verified that with such minimizer the value of (24) is proportional to \( |u^+ - u^-| \).

Next we will show that the regularizations \( Y_\varepsilon = Y * \sigma_\varepsilon(x) \) of the kinetic density \( Y \) of a weak entropy solution \( u(t, x) \) are not, in general, solutions of the variational problems (9) (or (5)). For that we consider a conservation law:
\[
u_t + ((u - \frac{1}{2})^2)_x = 0,
\]
(27)
with $2L$ periodic data

$$u_0(x) = \begin{cases} 
1, & x \in [-L, 0], \\
0, & x \in (0, 1), \\
1, & x \in [1, L]. 
\end{cases}$$

The corresponding entropy solution $u(t, x)$ consists of a stationary shock wave at $x = 0$, and a r-wave centred at $x = 1$ that propagates to the left with speed 1. Moreover, the values of $u(t, x)$ in the r-wave depend linearly on $x$. Let, as in the previous example, $Y(t, x, v)$ be the kinetic function of $u(x, t)$ and $Y_\varepsilon(t, x, v) = Y(t, x, v) * \omega_\varepsilon(x)$, where for the definiteness we take $\omega_\varepsilon(x)$ to be smooth, non-negative function, compactly supported on $[-2\varepsilon, 2\varepsilon]$ and equal to 1 on $[-\varepsilon, \varepsilon]$. It was shown in the previous example that for small times $Y_\varepsilon$ is a solution of (9). Consider time $t = 1$ – the moment the r-wave reaches shock. $Y_\varepsilon(1, x, v)$ is a smoothing in $x$ direction of the characteristic function of a triangle $\{(x, v) : x \in (0, 2), v \in (x, 1)\}$, and $Y_\varepsilon(0, x, v)$ is a smoothing in $x$ direction of the characteristic function of a square $\{(x, v) : x \in (0, 1), v \in (0, 1)\}$. For all small $\varepsilon$, one directly verifies that

$$\int_{-L}^{L} \int_{0}^{1} |Y_\varepsilon(1, x, v)|^2 \, dx \, dv > \int_{-L}^{L} \int_{0}^{1} |Y_\varepsilon(0, x, v)|^2 \, dx \, dv,$$

violating the conservation property (5).

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