A COMPARISON OF $q$-DECOMPOSITION NUMBERS IN THE $q$-DEFORMED FOCK SPACES OF HIGHER LEVELS

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ABSTRACT. The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The $q$-decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Ugllov in the $q$-deformed Fock space. In this paper, we show that parts of $q$-decomposition matrices of level $\ell$ coincides with that of level $\ell - 1$ under certain conditions of multi charge.

1. Introduction

The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For a multi charge $s = (s_1, \ldots, s_r) \in \mathbb{Z}^r$, the $q$-deformed Fock space $F_q[s]$ of level $\ell$ is the $\mathbb{Q}(q)$-vector space whose basis are indexed by $\ell$-tuples of Young diagrams, i.e. $\{[\lambda; s] | \lambda \in \Pi^\ell \}$, where $\Pi$ is the set of Young diagrams. Heisenberg group (resp. quantum group $U_q(\widehat{sl}_n)$) acts on $F_q[s]$ as level $q^\ell$ (resp. level $q^\ell$). Both actions commute on $F_q[s]$.

The canonical bases $\{G^+(\lambda; s) | \lambda \in \Pi^\ell \}$ and $\{G^-(\lambda; s) | \lambda \in \Pi^\ell \}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $-\Pi_{gl}$. Define matrices $\Delta^+(q) = (\Delta^+_{\lambda, \mu}(q))_{\lambda, \mu}$ and $\Delta^-(q) = (\Delta^-_{\lambda, \mu}(q))_{\lambda, \mu}$ by

$$G^+(\lambda; s) = \sum_{\mu} \Delta^+_{\lambda, \mu}(q) [\mu; s], \quad G^-(\lambda; s) = \sum_{\mu} \Delta^-_{\lambda, \mu}(q) [\mu; s].$$

We call $\Delta^+_{\lambda, \mu}(q)$ and $\Delta^-_{\lambda, \mu}(q)$ $q$-decomposition numbers. These $q$-decomposition matrices play an important role in representation theory. However it is difficult to compute $q$-decomposition matrices.

In the case of $\ell = 1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+(1)$ coincides with the decomposition matrix of $v$-Schur algebra. Ariki defined a $q$-analogue of decomposition numbers of $v$-Schur algebra by using Khovanov-Lauda’s grading, and proved that it coincides with the $q$-decomposition numbers [Ari]. For $\ell \geq 2$, Yvonne [Yvo06] conjectured that the matrix $\Delta^+(q)$ coincides with the $q$-analogue of the decomposition matrices of cyclotomic Schur algebras at a primitive $n$-th root of unity under a suitable condition on multi charge.

Let $O_s(\ell, 1, m)$ be the category $O$ of rational Cherednik algebra of $(\mathbb{Z}/\ell \mathbb{Z}) \times \mathbb{Z}_m$ associated with multicharge $s$. Rouquier [Rou08, Theorem 6.8, §6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in $O_s(\ell, 1, m)$ are equal to the corresponding coefficients $\Delta^+_{\lambda, \mu}(q)$, where $m = |\lambda| = |\mu|$. It is expected that $\oplus_{m \geq 0} O_s(\ell, 1, m)$ should categorify $F_1[s]$. (see [Sha] for the details.) More generally, it is expected that, together with a suitable grading, $\oplus_{m \geq 0} O_s(\ell, 1, m)$ should categorify $F_q[s]$. For the detail of correspondence between the charges of $O_s(\ell, 1, m)$ and the charges of Fock spaces, see [Rou08].

Now, we state our main theorems. We say that the $j$-th component $s_j$ of the multi charge is sufficiently large for $[\lambda; s]$ if $s_j - s_i \geq \lambda_i^{(0)}$ for any $i = 1, 2, \ldots, \ell$, and that $s_j$ is sufficiently small for $[\lambda; s]$ if $s_i - s_j \geq |\lambda| = |\lambda(1)| + \cdots + |\lambda(\ell)|$ for any $i = 1, 2, \ldots, \ell$ (see Definition 3.1). More generally, for a positive integer $N$ we say that $s_j$ is sufficiently small for $N$ if $s_i - s_j \geq N$ for all $i \neq j$. If $s_j$ is sufficiently large,
for $|\lambda; s|$ and $|\lambda; s| > |\mu; s|$, then the $j$-th components of $\lambda$ and $\mu$ are both the empty Young diagram $\emptyset$ (Lemma 3.2). On the other hand, if $s_j$ is sufficiently small for $|\lambda; s|$ and $|\lambda; s| \geq |\mu; s|$, then $\mu^{(j)} = \emptyset$ implies $\lambda^{(j)} = \emptyset$. (Lemma 3.3).

Our main results are as follows:

**Theorem A.** (Theorem 3.4)
Let $\varepsilon \in \{+, -\}$. If $s_j$ is sufficiently large for $|\lambda; s|$, then

$$\Delta_{\lambda, \mu; s}(q) = \Delta_{\lambda, \mu; s}(q),$$

where $\bar{\lambda}$ (resp. $\bar{\mu}, \bar{s}$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu, s$), $\Delta_{\lambda, \mu; s}(q)$ is the $q$-decomposition number of level $\ell$ and $\Delta_{\lambda, \mu; s}(q)$ is the $q$-decomposition number of level $\ell - 1$.

**Theorem B.** (Theorem 3.5)
Let $\varepsilon \in \{+, -\}$. If $s_j$ is sufficiently small for $|\mu; s|$ and $\mu^{(j)} = \emptyset$, then

$$\Delta_{\lambda, \mu; s}(q) = \Delta_{\lambda, \mu; s}(q),$$

where $\bar{\lambda}$ (resp. $\bar{\mu}, \bar{s}$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu, s$).

Shoji and Wada proved some product formulae of $q$-decomposition numbers [SW09, Theorem 2.9]. There are some overlaps between our results and their product formula. [SW09] has some assumptions “dominance” on the multi charge while our results don’t. On the concluding facts, [SW09] has a flexibility of embedding of $q$-decomposition matrices while our results don’t.

Our results are related to category $O$ in the following sense. In the category $O$, Chuang and Miyachi conjectured the following:

**Conjectures.** [CM, §5]

(A') Let $\lambda' \in \Pi^c$. If $s_1$ is sufficiently large for any $|(\emptyset, \lambda'); s|$, there exists an embedding

$$O_s(\ell, 1, m) \hookrightarrow O_s(\ell + 1, 1, m).$$

(B') If $s_\ell$ is sufficiently small for $m$, there exists a quotient functor

$$O_s(\ell + 1, 1, m) \twoheadrightarrow O_s(\ell, 1, m),$$

where $s$ is obtained by omitting the $j$-th component of $s$.

We see that Conjecture (A') (resp. (B')) is consistent with Theorem A (resp. Theorem B) by taking into account the conjecture that $\oplus_{m \geq 0} O_s(\ell, 1, m)$ should categorify $F_q[s]$. Theorem A (resp. Theorem B) gives a strong support to the conjecture (A') (resp. (B')).

This paper is organized as follows. In Section 2, we review the $q$-deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results. In Section 4, we review the straightening rules in the $q$-deformed Fock spaces. Theorem A(Theorem 3.4) and Theorem B(Theorem 3.5) are proved in Section 5 and 6 respectively.

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**Notations.** For a positive integer $N$, a partition of $N$ is a non-increasing sequence of non-negative integers summing to $N$. We write $|\lambda| = N$ if $\lambda$ is a partition of $N$. The length $l(\lambda)$ of $\lambda$ is the number of non-zero components of $\lambda$. And we use the same notation $\lambda$ to represent the Young diagram corresponding to $\lambda$. For an $\ell$-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(\ell)})$ of Young diagrams, we put $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(\ell)}|$.
2. The $q$-deformed Fock spaces of higher levels

2.1. $q$-wedge products and straightening rules. Let $n$, $\ell$, $s$ be integers such that $n \geq 2$ and $\ell \geq 1$. We define $P(s)$ and $P^{++}(s)$ as follows:

\begin{align*}
(1) \quad P(s) &= \{ k = (k_1, k_2, \cdots) \in \mathbb{Z}^\infty \mid k_r = s - r + 1 \text{ for any sufficiently large } r \} \\
(2) \quad P^{++}(s) &= \{ k = (k_1, k_2, \cdots) \in P(s) \mid k_1 > k_2 > \cdots \}.
\end{align*}

Let $\Lambda^s$ be the $\mathbb{Q}(q)$ vector space spanned by the $q$-wedge products

\begin{equation}
\tag{3} \label{eq:uk}
 u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots, \quad (k \in P(s))
\end{equation}

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on $n$ and $\ell$. [Ugl00, Proposition 3.16] (The precise description will be given in §4.)

Example 2.1. \begin{enumerate}[\text{(i)}]
\item For every $k_1 \in \mathbb{Z}$, \text{ } u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1}$. Therefore $u_{k_1} \wedge u_{k_1} = 0$.
\item Let $n = 2$, $\ell = 2$, $k_1 = -2$, and $k_2 = 4$. Then
\[ u_{-2} \wedge u_4 = q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0. \]
\item Let $n = 2$, $\ell = 2$, $k_1 = -1$, $k_2 = -2$ and $k_3 = 4$. Then
\[ u_{-1} \wedge u_{-2} \wedge u_4 = u_{-1} \wedge (u_{-2} \wedge u_4) = u_{-1} \wedge \left( q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0 \right) = q u_{-1} \wedge u_4 \wedge u_{-2} + (q^2 - 1) u_{-1} \wedge u_2 \wedge u_0 \]
\end{enumerate}

By applying the straightening rules, every $q$-wedge product $u_k$ is expressed as a linear combination of so-called ordered $q$-wedge products, namely $q$-wedge products $u_k$ with $k \in P^{++}(s)$. The ordered $q$-wedge products $\{u_k \mid k \in P^{++}(s)\}$ form a basis of $\Lambda^s$ called the standard basis.

2.2. Abacus. It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with $N$ runners labeled $1, 2, \cdots N$ from left to right. The positions on the $i$-th runner are labeled by the integers having residue $i$ modulo $N$.

\[
\begin{array}{c@{\quad}c@{\quad}c@{\quad}c@{\quad}c@{\quad}c@{\quad}c}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
-N + 1 & -N + 2 & \cdots & -1 & 0 & \phantom{-1} & \\
1 & 2 & \cdots & N - 1 & N & \phantom{1} & \\
N + 1 & N + 2 & \cdots & 2N - 1 & 2N & \phantom{1} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

Each $k \in P^{++}(s)$ (or the corresponding $q$-wedge product $u_k$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions $k_1, k_2, \cdots$. We call this configuration the abacus presentation of $u_k$.

Example 2.2. If $n = 2$, $\ell = 3$, $s = 0$, and $k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots)$, then the abacus presentation of $u_k$ is
We use another labeling of runners and positions. Given an integer $k$, let $c, d$ and $m$ be the unique integers satisfying

\[ k = c + n(d - 1) - n\ell m, \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell. \]

Then, in the abacus presentation, the position $k$ is on the $c + n(d - 1)$-th runner (see the previous example). Relabeling the position $k$ by $c - nm$, we have $\ell$ abaci with $n$ runners.

**Example 2.3.** In the previous example, relabeling the position $k$ by $c - nm$, we have

\[
\begin{array}{cccccccc}
d = 1 & d = 2 & d = 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
5 & -4 & 5 & -4 & \cdots & m = 3 \\
-3 & -2 & -3 & -2 & -1 & 0 & \cdots & m = 1 \\
1 & 2 & 1 & 2 & 1 & 2 & \cdots & m = 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c = 1 & c = 2 & c = 1 & c = 2 & c = 1 & c = 2 \\
\end{array}
\]

We assign to each of $\ell$ abacus presentations with $n$ runners a $q$-wedge product of level 1. In fact, straightening rules in each “sector” are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see also [Ugl00] and §4.1 for the detail)

We introduce some notation.

**Definition 2.4.** For an integer $k$, let $c, d$ and $m$ be the unique integers satisfying (4), and write

\[ u_k = u_{c - nm}^{(d)}. \]

Also we write $u_{c_1 - nm_1}^{(d_1)} > u_{c_2 - nm_2}^{(d_2)}$ if $k_1 > k_2$, where $k_i = c_i + n(d_i - 1) - n\ell m_i$, $(i = 1, 2)$.

We regard $u_{c - nm}^{(d)}$ as $u_{c - nm}$ in the case of $\ell = 1$.

**Example 2.5.** If $n = 2$, $\ell = 3$, then we have

\[ u_{-10} \wedge u_1 = -q^{-1} u_1 \wedge u_{-10} + (q^{-2} - 1) u_{-4} \wedge u_{-5}, \]

that is,

\[ u_{-2}^{(1)} \wedge u_1^{(1)} = -q^{-1} u_1^{(1)} \wedge u_{-2}^{(1)} + (q^{-2} - 1) u_0^{(1)} \wedge u_{-1}^{(1)}. \]
On the other hand, in the case of \( n = 2, \ell = 1 \),
\[
    u_{-2} \wedge u_{1} = -q^{-1} u_{1} \wedge u_{-2} + (q^{-2} - 1) u_{0} \wedge u_{-1}.
\]

2.3. \( \ell \)-tuples of Young diagrams. Another indexation of the ordered \( q \)-wedge products is given by the set of pairs \((\lambda, s)\) of \( \ell \)-tuples of Young diagrams \( \lambda = (\lambda^{(1)}, \cdots, \lambda^{(\ell)}) \) and integer sequences \( s = (s_{1}, \cdots, s_{\ell}) \) summing up to \( s \). Let \( k = (k_{1}, k_{2}, \cdots) \in P^{+}(s) \), and write
\[
k_{r} = c_{r} + n(d_{r} - 1) - n\ell m_{r} - q, \quad 1 \leq c_{r} \leq n, \quad 1 \leq d_{r} \leq \ell, \quad m_{r} \in \mathbb{Z}.
\]
For \( d \in \{1, 2, \cdots, \ell\} \), let \( k_{1}^{(d)}, k_{2}^{(d)}, \cdots \) be integers such that
\[
    \beta_{(d)} = \{c_{r} - nm_{r} \mid d_{r} = d\} = \{k_{1}^{(d)}, k_{2}^{(d)}, \cdots\} \quad \text{and} \quad k_{1}^{(d)} > k_{2}^{(d)} > \cdots
\]
Then we associate to the sequence \((k_{1}^{(d)}, k_{2}^{(d)}, \cdots)\) an integer \( s_{d} \) and a partition \( \lambda^{(d)} \) by
\[
k_{r}^{(d)} = s_{d} - r + 1 \quad \text{for sufficiently large} \ r \quad \text{and} \quad \lambda_{r}^{(d)} = k_{r}^{(d)} - s_{d} + r - 1 \quad \text{for} \ r \geq 1.
\]
In this correspondence, we also write
\[
(6) \quad u_{k} = |\lambda; s\rangle \quad (k \in P^{+}(s)).
\]

Example 2.6. If \( n = 2, \ell = 3, s = 0, \) and \( k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots) \), then
\[
    k_{1} = 6 = 2 + 2(3 - 1) - 6 \cdot 0, \quad k_{2} = 3 = 1 + 2(2 - 1) - 6 \cdot 0, \\
    k_{3} = 2 = 2 + 2(1 - 1) - 6 \cdot 0, \quad \cdots \quad \text{and so on.}
\]

Hence,
\[
    \beta^{(1)} = \{2, 1, 0, -1, -2, \cdots\} \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \cdots\} \quad \beta^{(3)} = \{2, -3, -4, -5, \cdots\}.
\]
Thus, \( s = (2, 0, -2) \) and \( \lambda = (1, (0, 1, (4)) \).

Note that we can read off \( s = (2, 0, -2) \) and \( \lambda = (1, (0, 1, (4)) \) from the abacus presentation. (see Example 2.3)

2.4. The \( q \)-deformed Fock spaces of higher levels.

Definition 2.7. For \( s \in \mathbb{Z}^{\ell} \), we define the \( q \)-deformed Fock space \( F_{q}[s] \) of level \( \ell \) to be the subspace of \( \Lambda^{s} \) spanned by \( |\lambda; s\rangle \) \( (\lambda \in \Pi^{s}) \):
\[
(7) \quad F_{q}[s] = \bigoplus_{\lambda \in \Pi^{s}} \mathbb{Q}(q) |\lambda; s\rangle.
\]
We call \( s \) a multi charge.

2.5. The bar involution.

Definition 2.8. The involution \( \overline{\cdot} \) of \( \Lambda^{s} \) is the \( \mathbb{Q} \)-vector space automorphism such that \( \overline{q} = q^{-1} \) and
\[
(8) \quad \overline{u_{k}} = u_{k_{1}} \wedge \cdots \wedge u_{k_{r}} \wedge u_{k_{r+1}} \wedge \cdots = (-q)^{\kappa(c_{1}, \cdots, c_{r})} q^{-\kappa(c_{1}, \cdots, c_{r})} (u_{k_{1}} \wedge \cdots \wedge u_{k_{r}}) \wedge u_{k_{r+1}} \wedge \cdots,
\]
where \( c_{i}, d_{i} \) are defined by \( k_{i} \) as in (4), \( r \) is an integer satisfying \( k_{r} = s - r + 1 \). And \( \kappa(a_{1}, \cdots, a_{r}) \) is defined by
\[
\kappa(a_{1}, \cdots, a_{r}) = \# \{(i, j) \mid i < j, a_{i} = a_{j}\}.
\]
Remarks (i) The involution is well defined. i.e. it doesn’t depend on $r$ [Ug00].
(ii) The involution comes from the bar involution of affine Hecke algebra $H_r$. (see §4 for more detail.)
(iii) The involution preserves the $q$-deformed Fock space $F_q[s]$ of higher level.

2.6. The dominance order. We define a partial ordering $|\lambda; s| \geq |\mu; s|$. For $|\lambda; s|$ and $|\mu; s|$, we define multi-sets $\tilde{\lambda}$ and $\tilde{\mu}$ as

\[
\tilde{\lambda} = \{\lambda^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \operatorname{max}(l(\lambda^{(d)}), l(\mu^{(d)})) \}\,
\tilde{\mu} = \{\mu^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \operatorname{max}(l(\lambda^{(d)}), l(\mu^{(d)})) \}.
\]

We denote by $\tilde{\lambda}$ (resp. $\tilde{\mu}$) the sequence obtained by rearranging the elements in the multi-set $\tilde{\lambda}$ (resp. $\tilde{\mu}$) in decreasing order.

Definition 2.9. Let $|\lambda; s| = L_{k_1} \wedge L_{k_2} \wedge \cdots$ and $|\mu; s| = L_{g_1} \wedge L_{g_2} \wedge \cdots$. We define $|\lambda; s| \geq |\mu; s|$ if $|\lambda| = |\mu|$ and

\[
\begin{cases}
(a) & |\lambda| \neq |\mu|, \quad \sum_{j=1}^{r} \lambda_j \geq \sum_{j=1}^{r} \mu_j \quad \text{(for all } r = 1, 2, 3, \cdots) \quad , \\
(b) & |\lambda| = |\mu|, \quad \sum_{j=1}^{r} k_j \geq \sum_{j=1}^{r} g_j \quad \text{(for all } r = 1, 2, 3, \cdots) .
\end{cases}
\]

Remark. The order in Definition 2.9 is different from the order in [Ug00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

Example 2.10. Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1, 1), 0)$, and $\mu = (0, (2))$. Then, $|\lambda; s| = L_{2} \wedge L_{1} \wedge L_{-2} \wedge L_{-1} \wedge L_{2} \wedge \cdots$ and $|\mu; s| = L_{3} \wedge L_{1} \wedge L_{-1} \wedge L_{-2} \wedge L_{2} \wedge \cdots$. In Uglov’s order, $|\mu; s|$ is greater than $|\lambda; s|$. However, $|\lambda; s| > |\mu; s|$ under our order since $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\} = \{2, 2, -1\}$ and $\{\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\} = \{1, 1, 1\}$.

We define a matrix $(a_{\lambda, \mu}(q))_{\lambda, \mu}$ by

\[
|\lambda; s| = \sum_{\mu} a_{\lambda, \mu}(q) |\mu; s|.
\]

Then the matrix $(a_{\lambda, \mu}(q))_{\lambda, \mu}$ is unitriangular with respect to $\geq$, that is

\[
\begin{cases}
(a) & \text{if } a_{\lambda, \mu}(q) \neq 0 , \text{ then } |\lambda; s| \geq |\mu; s|, \\
(b) & a_{\lambda, \lambda}(q) = 1.
\end{cases}
\]

(see the identity (27) for the detail.)

Thus, by the standard argument, the unitriangularity implies the following theorem.

Theorem 2.11. [Ug00] There exist unique bases $\{G^+(\lambda; s) | \lambda \in \Pi^f\}$ and $\{G^-(\lambda; s) | \lambda \in \Pi^f\}$ of $F_q[s]$ such that

\[
\begin{align*}
(\lambda) & G^+(\lambda; s) = G^+(\lambda; s) , & G^-(\lambda; s) = G^-(\lambda; s) \\
(ii) & G^+(\lambda; s) \equiv |\lambda; s| \mod q \mathcal{L}^+ , & G^-(\lambda; s) \equiv |\lambda; s| \mod q^{-1} \mathcal{L}^-
\end{align*}
\]

where $\mathcal{L}^+ = \bigoplus_{\lambda \in \Pi^f} \mathbb{Q}[q] |\lambda; s|$, $\mathcal{L}^- = \bigoplus_{\lambda \in \Pi^f} \mathbb{Q}[q^{-1}] |\lambda; s|$. 
Define matrices $\Delta^+(q) = (\Delta^+_{\lambda\mu}(q))_{\lambda\mu}$ and $\Delta^-(q) = (\Delta^-_{\lambda\mu}(q))_{\lambda\mu}$ by

\[
G^+(\lambda; s) = \sum_{\mu} \Delta^+_{\lambda\mu}(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_{\mu} \Delta^-_{\lambda\mu}(q) |\mu; s\rangle.
\]

The entries $\Delta^\pm_{\lambda\mu}(q)$ are called $q$-decomposition numbers. Note that $q$-decomposition numbers $\Delta^\pm(q)$ depend on $n$, $\ell$, and $s$. The matrices $\Delta^+(q)$ and $\Delta^-(q)$ are also unipotent with respect to $\geq$.

It is known [Ug100, Theorem 3.26] that the entries of $\Delta^-(q)$ are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type $A$, and that they are polynomials in $q$ with non-negative integer coefficients (see [KT02]).

3. A Comparison of $q$-Decomposition Numbers

3.1. Sufficiently large and sufficiently small.

Definition 3.1. Let $s = (s_1, s_2, \cdots, s_\ell) \in \mathbb{Z}^\ell$ be a multi charge and $1 \leq j \leq \ell$.

(i). We say that the $j$-th component $s_j$ of the multi charge $s$ is sufficiently large for $|\lambda; s\rangle \in F_q[\ell]$ if

\[
s_j - s_i \geq \lambda^{(i)}_1 \quad \text{for all} \quad i = 1, 2, \cdots, \ell.
\]

More generally, we say that $s_j$ is sufficiently large for a $q$-wedge $u_k$ if

\[
s_j \geq c_r - nm_r \quad \text{for all} \quad r = 1, 2, \cdots,
\]

where $k_r = c_r + n(d_r - 1) - n\ell m_r$, ($r = 1, 2, \cdots$), $1 \leq c \leq n$ and $1 \leq d \leq \ell$ (see §2).

(ii). We say that $s_j$ is sufficiently small for $|\lambda; s\rangle$ if

\[
s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all} \quad i \neq j.
\]

Note that the definition of sufficiently small depends only on the size of $\lambda$ and the multi charge $s$. When we fix the multi charge $s$, we say that $s_j$ is sufficiently small for $N$ if

\[
s_i - s_j \geq N \quad \text{for all} \quad i \neq j.
\]

Remark. If $|\lambda; s\rangle$ is 0-dominant in the sense of [Ug100], that is

\[
s_i - s_{i+1} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all} \quad i = 1, 2, \cdots, \ell - 1,
\]

then $s_i$ is sufficiently large for $|\lambda; s\rangle$ and $s_\ell$ is sufficiently small for $|\lambda; s\rangle$.

Lemma 3.2. If $s_j$ is sufficiently large for $|\lambda; s\rangle$ and $|\lambda; s\rangle \geq |\mu; s\rangle$, then

(i) $\lambda^{(j)} = 0$,

(ii) $s_j$ is also sufficiently large for $|\mu; s\rangle$. In particular, $\mu^{(j)} = 0$.

Proof. It is clear that $\lambda^{(j)} = 0$ by the definition.

Note that

\[
s_j \text{ is sufficiently large for } |\lambda; s\rangle \iff s_j - s_i \geq \lambda^{(i)}_1 \quad \text{for all } i = 1, 2, \cdots, \ell
\]

\[
\iff s_j \geq \max\{\lambda^{(1)}_1 + s_1, \cdots, \lambda^{(\ell)}_1 + s_\ell\} = \tilde{\lambda}_1.
\]

If $|\lambda; s\rangle \geq |\mu; s\rangle$, then $\tilde{\lambda}_1 \geq \tilde{\mu}_1$ and so $s_j \geq \tilde{\mu}_1$. It means that $s_j$ is sufficiently large for $|\mu; s\rangle$. \qed
Lemma 3.3. Suppose that \( s_j \) is sufficiently small for \(|\lambda; s\rangle\). If \(|\lambda; s\rangle \geq |\mu; s\rangle\) and \( \mu^{(j)} = \emptyset \), then \( \lambda^{(j)} = \emptyset \).

Proof. Suppose that \( l(\lambda^{(j)}) \geq 1 \). Then \( s_j \) is the minimal integer in the set \( \{ \mu^{(d)}_d + s_d \mid 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)})) \} \) because \( \mu^{(j)} = \emptyset \) and \( s_j \) is the minimal integer in \( s \). On the other hand, the minimal integer in the set \( \{ \lambda^{(d)}_d + s_d \mid 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)})) \} \) is greater than \( s_j \) because \( s_j \) is sufficiently small for \(|\lambda; s\rangle\). Therefore \(|\lambda; s\rangle \not\geq |\mu; s\rangle\). This is a contradiction. \( \square \)

3.2. Main results. Now, we are ready to state our main theorems. We will prove the theorems in §5 and §6 respectively.

Theorem 3.4. Let \( \varepsilon \in \{+, -\} \). If \( s_j \) is sufficiently large for \(|\lambda; s\rangle\), then

\[
\Delta^\varepsilon_{\lambda, \mu, s}(q) = \Delta^\varepsilon_{\lambda, \mu, 3}(q),
\]

where \( \tilde{\lambda} \) (resp. \( \tilde{\mu}, \tilde{s} \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

Theorem 3.5. Let \( \varepsilon \in \{+, -\} \). If \( s_j \) is sufficiently small for \(|\mu; s\rangle\) and \( \mu^{(j)} = \emptyset \), then

\[
\Delta^\varepsilon_{\lambda, \mu, s}(q) = \Delta^\varepsilon_{\lambda, \mu, 3}(q),
\]

where \( \tilde{\lambda} \) (resp. \( \tilde{\mu}, \tilde{s} \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

Example 3.6. (i) If \( n = \ell = 2, s = (3, -3) \) and \( \lambda = (\emptyset, (6)), \mu = (\emptyset, (5, 1)) \), then \( s_1 \) is sufficiently large for \(|\lambda; s\rangle\). Hence

\[
\Delta^-_{\lambda, \mu, s}(q) = \Delta^-_{\lambda, \mu, 3}(q) = \Delta^-_{\lambda, \mu, 3}(q) = -q^{-1}.
\]

(ii) If \( n = \ell = 2, s = (3, -3) \) and \( \lambda = ((6), \emptyset), \mu = ((5, 1), \emptyset) \), then \( s_2 \) is sufficiently small for \(|\mu; s\rangle\). Hence

\[
\Delta^-_{\lambda, \mu, s}(q) = \Delta^-_{\lambda, \mu, 3}(q) = \Delta^-_{\lambda, \mu, 3}(q) = -q^{-1}.
\]

4. \( q \)-wedges and straightening rules

In this section, we review the straightening rules \([\text{Ugl00}]\) to prove our main results.

4.1. affine Hecke algebra and straightening rules. In this paragraph, we review the affine Hecke algebra of type \( A_1 \) and straightening rules. We treat only the case of type \( A_1 \). Indeed for our proof we only need the straightening rule of \( q \)-wedge whose length is equal to two. In fact, the straightening rules for a \( q \)-wedge which length is greater than 2 is obtained from the straightening rules for two adjacent element. (see Example [5.4] More general case, see \([\text{Ugl00}]\).

The Hecke algebra \( H \) of type \( A_1 \) is the algebra over \( \mathbb{Q}(q) \) with generator \( T_1 \) and relation

\[
(T_1 - q^{-1})(T_1 + q) = 0.
\]

The affine Hecke algebra \( \hat{H} \) is the tensor space \( H \) and the polynomial ring \( \mathbb{Q}(q)[X_1^+, X_2^+] \) with relations

\[
(19)
\]
Proposition 4.2.

(20) \[ X^1 T_1 = T_1 X^{s_1(\lambda)} + (q - q^{-1}) \frac{X^{s_1(\lambda)} - X^1}{1 - X_1 X_2^{-1}}, \]

(21) \[ T_1 X^1 = X^{s_1(\lambda)} T_1 + (q - q^{-1}) \frac{X^{s_1(\lambda)} - X^1}{1 - X_1 X_2^{-1}}, \]

where \( \lambda \in \mathbb{Z}^2 \) and \( s_1 \) is the transposition.

Let \( P_1 \) be the \( \mathbb{Q}(q) \)-vector space whose basis is \( \{ (c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}, 1 \leq c_1 \leq n, 1 \leq c_2 \leq n \} \). Define the right action of \( H \) on \( P_1 \) as

(22) \[ (c_1, c_2) \cdot T_1 = \begin{cases} (c_2, c_1) & \text{if } c_1 < c_2, \\ q^{-1} (c_1, c_1) & \text{if } c_1 = c_2, \\ (c_2, c_1) - (q - q^{-1}) (c_1, c_2) & \text{if } c_1 > c_2. \end{cases} \]

Let \( P_2 \) be the \( \mathbb{Q}(q) \)-vector space whose basis is \( \{ |d_1, d_2\rangle \mid d_1, d_2 \in \mathbb{Z}, 1 \leq d_1 \leq \ell, 1 \leq d_2 \leq \ell \} \). Define the left action of \( H \) on \( P_2 \) as

(23) \[ T_1 \cdot |d_1, d_2\rangle = \begin{cases} |d_2, d_1\rangle & \text{if } d_1 < d_2, \\ -q |d_1, d_1\rangle & \text{if } d_1 = d_2, \\ |d_2, d_1\rangle - (q - q^{-1}) |d_1, d_2\rangle & \text{if } d_1 > d_2. \end{cases} \]

Define a vector space \( \Lambda \) by

\[ \Lambda = P_1 \otimes_H \hat{H} \otimes_H P_2. \]

**Definition 4.1.** [Ugl100] For \( (c_1, c_2) \in P_1, |d_1, d_2\rangle \in P_2, \) and \( m_1, m_2 \in \mathbb{Z} \), put \( k_j = c_j + n(d_j - 1) - n\ell m_j, (j = 1, 2) \). Denote \( (c_1, c_2) \otimes X_1^{m_1} X_2^{m_2} \otimes |d_1, d_2\rangle \in \Lambda \) by

(24) \[ u_{k_1} \wedge u_{k_2}. \]

**Proposition 4.2.** [Ugl100] For integers \( k_1, k_2 \), let \( c_j, d_j, m_j \) be the unique integers satisfying \( k_j = c_j + n(d_j - 1) - n\ell m_j, 1 \leq c_j \leq n \) and \( 1 \leq d_j \leq \ell, \) \( (j = 1, 2) \). Then,

(25) \[ u_{k_2} \wedge u_{k_1} = (-q^{-1})^{\delta_{d_1=d_2}} \left\{ q^\alpha u_{k_1} \wedge u_{k_2} + \text{sgn}(m) (q - q^{-1}) \sum_{j=\beta}^{\lceil m_2 - m_1 \rceil - \gamma} u_{k_1 - c_1 + c_2 - \text{sgn}(m)n\ell j} \wedge u_{k_2 + c_1 - c_2 + \text{sgn}(m)n\ell j} \right\}, \]

where

\[ \text{sgn}(m) = \begin{cases} 1 & \text{if } m_1 < m_2, \\ -1 & \text{if } m_1 > m_2, \\ 0 & \text{if } m_1 = m_2, \end{cases} \]

\[ \alpha = \begin{cases} 1 & \text{if } c_1 = c_2 \text{ and } k_1 > k_2, \\ -1 & \text{if } c_1 = c_2 \text{ and } k_1 < k_2, \\ 0 & \text{if } c_1 \neq c_2, \end{cases} \]

\[ \delta_{d_1=d_2} = \begin{cases} 1 & \text{if } d_1 = d_2, \\ 0 & \text{if } d_1 \neq d_2, \end{cases} \]

\[ \beta = \begin{cases} 0 & \text{if } c_1 > c_2, m_1 < m_2 \text{ or } c_1 < c_2, m_1 > m_2, \\ 1 & \text{if otherwise}. \end{cases} \]
Corollary 4.3. \( \gamma = \begin{cases} 1 & \text{if } d_1 < d_2, m_1 < m_2 \text{ or } d_1 > d_2, m_1 > m_2, \\ 0 & \text{if } d_1 > d_2, m_1 < m_2 \text{ or } d_1 < d_2, m_1 > m_2. \end{cases} \)

Proof. We only show the statement in the case \( c_1 = c_2, m_1 < m_2, \) and \( d_1 < d_2. \) The other case can be treated similarly.

In this case, \( k_1 > k_2, \delta_{d_1,d_2} = 0, \) \( \text{sgn}(m) = 1, \alpha = 1, \beta = 1, \) and \( \gamma = 1. \) Note that \( X_1X_2 \) and \( T_1 \) commute each other thanks to the relation (20), that is \( X_1X_2T_1 = T_1X_1X_2. \) From the relation (20), for any positive integer \( N \) we have

\[
X_1^N T_1 = T_1^{-1} X_2^N + (q-q^{-1})(X_1 X_2^{N-1} + X_1^2 X_2^{N-2} + \cdots + X_1^{N-1} X_2). \tag{26}
\]

Hence

\[
u_{k_2} \land u_{k_1} \]

\[
= (c_1, c_1) \otimes X_1^{m_2} X_2^{m_1} \otimes |d_2, d_1\rangle \quad \text{(by Definition 4.1)}
\]

\[
= (c_1, c_1) \otimes (X_1 X_2)^{m_1} X_1^{m_2-m_1} T_1 \otimes |d_1, d_2\rangle \quad \text{(by (23))}
\]

\[
= (c_1, c_1) \otimes (X_1 X_2)^{m_1} \left\{ T_1^{-1} X_2^{m_2-m_1} + (q-q^{-1})(X_1 X_2^{m_2-m_1} + X_1^2 X_2^{m_2-m_1-2} + \cdots + X_1^{m_2-m_1-1} X_2) \right\} \otimes |d_1, d_2\rangle \quad \text{(by (26))}
\]

\[
= q (c_1, c_1) \otimes X_1^{m_2} X_2^{m_1} \otimes |d_1, d_2\rangle + (q-q^{-1}) (c_1, c_1) \otimes \left( X_1^{m_1+1} X_2^{m_2-1} + X_1^{m_1+2} X_2^{m_2-2} + \cdots + X_1^{m_2-1} X_2^{m_1+1} \right) \otimes |d_1, d_2\rangle \quad \text{(by (23))}
\]

\[
= q u_{k_1} \land u_{k_2} + (q-q^{-1}) \left( u_{k_1-n\ell} \land u_{k_1+n\ell} + u_{k_2-n\ell} \land u_{k_2+n\ell} + \cdots + u_{k_1-n\ell(m_2-m_1-1)} \land u_{k_2+n\ell(m_2-m_1-1)} \right) 
\]

\[
= q u_{k_1} \land u_{k_2} + (q-q^{-1}) \sum_{j=1}^{m_2-m_1-1} u_{k_1-n\ell j} \land u_{k_2+n\ell j}. 
\]

The identity (25) is rewritten in terms of the notation of Definition 2.4 as follows.

Corollary 4.3. Under the same notations in Proposition 4.2 we have

\[
u_{e_2-nm_2} \land u_{e_1-nm_1}^{(d_2)} = (-q^{-1})^{|d_1-d_2|} q^{|d_1|} u_{e_1-nm_1}^{(d_1)} \land u_{e_2-cm_2}^{(d_2)} \]

\[
+ \text{sgn}(m) (-q^{-1})^{|d_1-d_2|} (q-q^{-1}) \sum_{j=1}^{m_1-m_2-\gamma} u_{e_2-nm_1-\text{sgn}(m)nj}^{(d_1)} \land u_{e_1-nm_2+\text{sgn}(m)nj}^{(d_2)}. \tag{27}
\]

Remarks. (i) Note that the identity (27) depends only on the inequality relationship between \( d_1 \) and \( d_2 \) \((c_1, c_2)\). It is independent of \( \ell. \)

Let \( \ell \) and \( j \) be integers such that \( 1 \leq j \leq \ell + 1. \) Let

\[
u = u_{k_1}^{(d_1)} \land u_{k_2}^{(d_2)} \land \cdots
\]
be a $q$-wedge product of level $\ell$. Define $d'_1, d'_2, \cdots$ as

$$d'_r = \begin{cases} d_r & \text{if } d_r < j, \\ d_r + 1 & \text{if } d_r \geq j, \end{cases} \quad (r = 1, 2, \cdots).$$

Then,

$$u' = u^{(d'_1)}_{k_1} \wedge u^{(d'_2)}_{k_2} \wedge \cdots$$

is the $q$-wedge product of level $\ell + 1$. In this way, we regard a $q$-wedge product $u$ of level $\ell$ as the $q$-wedge product of level $\ell + 1$.

(ii). Let $k'_1 = k_1 - c_1 + c_2 - \text{sgn}(m)n\ell j$ and $k'_2 = k_2 + c_1 - c_2 + \text{sgn}(m)n\ell j$. That is to say, $u_{k'_1} \wedge u_{k'_2}$ appears in the summation of (25). Then, $k'_1$ and $k'_2$ satisfy following properties.

(a). $k'_1$ and $k'_2$ are in between $k_1$ and $k_2$, i.e. $k_1 < k'_1 < k_2$, ($i = 1, 2$) or $k_1 > k'_1 > k_2$, ($i = 1, 2$).

(b). $k'_1$ and $k'_2$ swap the c-part with $k_1$ and $k_2$. That is, there exist $m'_1, m'_2 \in \mathbb{Z}$ such that $k'_1 = c_2 + n(d_1 - 1) - n\ell m'_1$ and $k'_2 = c_1 + n(d_2 - 1) - n\ell m'_2$.

(c). $k'_1 + k'_2 = k_1 + k_2$.

In abacus presentation, the positions of $k'_1, k'_2$ and $k_1, k_2$ look like

$$\begin{array}{c|c}
  d = d_1 & d = d_2 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  k_2 & k_2 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  k_1 & k'_1 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  c = c_1 & c = c_2 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  c = c_2 & c = c_2 \\
\end{array}$$

4.2. Several properties of $q$-wedge products. In this paragraph, we summarize other properties of $q$-wedge products which will be needed in the proof of our main theorems.

Lemma 4.4 ([Ugl00]). If $k \geq t$, then

(i). $u_t \wedge u_k \wedge u_{k-1} \wedge \cdots \wedge u_t = 0$,

(ii). $u_k \wedge u_{k-1} \wedge \cdots \wedge u_t \wedge u_k = 0$.

More generally, we have

Corollary 4.5. If $k \geq m \geq t$, then

(i). $u_m \wedge u_k \wedge u_{k-1} \wedge \cdots \wedge u_t = 0$,

(ii). $u_k \wedge u_{k-1} \wedge \cdots \wedge u_t \wedge u_m = 0$.

Proof. The first assertion immediately follows from Lemma 4.4 (i). We prove (ii) by induction on $m - t$. If $m = t$, then the assertion follows from Lemma 4.4 (ii).

Let $m - t > 0$. From the identity (25), we know that there exist $b_0(q), \cdots, b_{m-t}(q)$ such that

$$u_t \wedge u_m = \sum_{j=0}^{m-t} b_j(q) u_{m-j} \wedge u_{t+j},$$

Then,
Here, by the induction hypothesis, \( u_k \land \cdots \land u_{t+1} \land u_t \land u_m = 0 \) for all \( 0 \leq j \leq m-t \). Therefore \( u_k \land \cdots \land u_{t+1} \land u_t \land u_m = 0 \).

The next corollary follows from the above corollary and Corollary 4.3

**Corollary 4.6.** If \( k \geq m \geq t \) and \( 1 \leq j \leq \ell \), then

(i) \( u_{m}^{(j)} \land u_{k}^{(j)} \land u_{k-1}^{(j)} \land \cdots \land u_{t}^{(j)} = 0 \),

(ii) \( u_{k}^{(j)} \land u_{k-1}^{(j)} \land \cdots \land u_{t}^{(j)} \land u_{m}^{(j)} = 0 \).

**Definition 4.7.** Let

\[
u = u_{d_1}^{(j)} \land u_{d_2}^{(j)} \land \cdots \land u_{d_r}^{(j)} , \quad \nu_a = c_a - nm_a , \quad (a = 1, 2, \ldots, r) \quad \text{and}
\]
\[
ob = g_b - nm_b , \quad (b = 1, 2, \ldots, t).
\]

and suppose that \( d_a \neq d_b \) for all \( a \in \{1, \ldots, r\} \) and \( b \in \{1, \ldots, t\} \). Then we define \( \xi(u, v) \) as

\[
\xi(u, v) = \# \{(a, b) \mid c_a = c_b', \ u_{k_a}^{(d_a)} < u_{k_b}^{(d_b)} \}.
\]  

**Lemma 4.8.** Let \( a \in \mathbb{Z}, \ t \in \mathbb{Z}_{\geq 0}, \ 1 \leq i \leq \ell, \) and \( 1 \leq j \leq \ell \).

(i) Let \( u_{k}^{(j)} \) be the maximal element such that \( u_{k}^{(j)} < u_{m}^{(j)} \). Let \( u_{k-1}^{(j)} = u_{k}^{(j)} \land u_{k-1}^{(j)} \land u_{k-2}^{(j)} \). Then,

\[
u_a = q^{-\xi(u_{k-1}^{(j)}, u_{k}^{(j)})} u_{k-1}^{(j)} \land u_{k}^{(j)}.
\]

(ii) Let \( u_{k}^{(j)} \) be the minimal element such that \( u_{g+t}^{(j)} > u_{k}^{(j)} \). Let \( u_{g+t}^{(j)} = u_{g+t}^{(j)} \land u_{g+t}^{(j)} \land u_{g+t}^{(j)} \). Then,

\[
u_a = q^{\xi(u_{g+t}^{(j)}, u_{g+t}^{(j)})} u_{g+t}^{(j)} \land u_{g+t}^{(j)}.
\]

In the abacus presentation, \( u_{d_1}^{(j)}, u_{d_2}^{(j)}, u_{d_3}^{(j)}, \) and \( u_{d_4}^{(j)} \), look as follows.

(i) \( d = i \)

\[
\begin{array}{c}
\xi \\
\vdots \\
\xi \\
\end{array}
\]

(ii) \( d = j \)

where the boxed region means that all positions are occupied by beads.

**Proof.** We only show (i) by induction on \( t \). If \( t = 0 \), then the assertion follows from the identity (27).

Let \( t \geq 1 \). Then, from the identity (27), we have
We only prove Theorem 3.4 in the case of $\varepsilon = -$. The proof in the case of $\varepsilon = +$ is similar. Through this section we fix $j$ ($1 \leq j \leq \ell$).
5.1. **Preliminary for the proof.** Fix an sufficiently large integer \( r \) so that for every ordered \( q \)-wedge product appearing in our argument, all of the components after \( r \)-th factor are consecutive. We are able to truncate \( q \)-wedge products at the first \( r \) parts. See \[Ugl00\] for detail. Then \(|\lambda; s\rangle\) can be identified with \( v_\lambda \) defined by

\[
|\lambda; s\rangle = v_\lambda \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots .
\]

First, we extend the definition of "sufficiently large" on the finite \( q \)-wedge products and introduce some notations.

**Definition 5.1.** Let \( u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_q} \) be an ordered \( q \)-wedge product and write \( k_a = c_a + n(d_a - 1) - n\ell d_a \) for \( a = 1, 2, \cdots, r \) as in (4). Then define \( \tilde{u}_k \) to be the \( q \)-wedge obtained from \( u_k \) by removing all factors \( u^{(d_a)} \) with \( d_a = j \).

**Lemma 5.2.** Suppose that \( s_j \) is sufficiently large for \(|\lambda; s\rangle\) and \( \Delta_{\lambda, \mu}^-(q) \neq 0 \). Let \( v_\lambda = |\lambda; s\rangle \), \( v_\mu = |\mu; s\rangle \) and \( r \) as above. Then,

\[
\xi(\emptyset^{\{j\}}, \tilde{v}_\lambda) = \xi(\emptyset^{\{j\}}, v_\mu).
\]

**Proof.** Let \( u_{k_1} \wedge u_{k_2} \) be a \( q \)-wedge product. Suppose that \( s_j \) is sufficiently large for \( u_{k_1} \wedge u_{k_2} \). Let \( u_{k_1}' \wedge u_{k_2}' \) be a \( q \)-wedge product which appears in the linear expansion of the straightening of \( u_{k_1} \wedge u_{k_2} \).

Put \( \xi = \xi(\emptyset^{\{j\}}, u_{k_1} \wedge u_{k_2}) \), \( \xi_1 = \xi(\emptyset^{\{j\}}, u_{k_1}) \), \( \xi_2 = \xi(\emptyset^{\{j\}}, u_{k_2}) \), \( \xi' = \xi(\emptyset^{\{j\}}, u_{k_1}' \wedge u_{k_2}') \), \( \xi'_1 = \xi(\emptyset^{\{j\}}, u_{k_1}') \) and \( \xi'_2 = \xi(\emptyset^{\{j\}}, u_{k_2}') \) (see Definition 4.9). Note that \( \xi = \xi_1 + \xi_2 \) and \( \xi' = \xi'_1 + \xi'_2 \). Then, from the abacus presentation below, we obtain \( \xi = \xi' \). That is, the straightening rule preserves \( \xi \) if \( s_j \) is sufficiently large.

\[
\begin{array}{c|c|c}
\xi_1 & \xi_2 & d = j \\
\hdashline
k_1 & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
& k_2 & d = j \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\xi_1' & \xi_2' & d = j \\
\hdashline
k_1 & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
& k_2 & d = j \\
\end{array}
\]

straightening rule

where beads are filled in the boxed region.

If \( \Delta_{\lambda, \mu}^-(q) \neq 0 \), then \( v_\mu \) appears in the linear expansion of the straightening of \( \tilde{v}_\lambda \). Therefore, the above argument assures the assertion. \(\square\)

From Lemma 4.8, we have

**Corollary 5.3** (see \[Ugl00\], Lemma 5.19). If \( s_j \) is sufficiently large for an ordered \( q \)-wedge product \( u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_q} \). Then

\[
u_k = q^{-\xi(\emptyset^{\{j\}}, u_k)} \emptyset^{\{j\}} \wedge \tilde{u}_k.
\]
Example 5.4. Let \( n = 2, \ell = 3, s = (0, 2, -2) \) and \( \lambda = ((1, 1), \emptyset, (3)) \). Then \( s_2 \) is sufficiently large for \( |\lambda; s\rangle \). Take \( r = 7 \), then

\[
\begin{align*}
    u_k &= u_5 \land u_4 \land u_3 \land u_1 \land u_{-2} \land u_{-3} \land u_{-4} \land u_{-7} \\
    &= u_1^{(3)} \land u_2^{(2)} \land u_1^{(1)} \land u_0^{(2)} \land u_0^{(1)} \land u_1^{(1)} \land u_0^{(1)} \land u_{-3}^{(3)} \\
    &= q^{-1} u_1^{(3)} \land u_2^{(2)} \land u_1^{(1)} \land u_0^{(2)} \land u_0^{(1)} \land u_1^{(1)} \land u_0^{(1)} \land u_{-3}^{(3)} \\
    &= q^{-3} u_2^{(2)} \land u_1^{(1)} \land u_0^{(2)} \land u_{-1}^{(2)} \land u_1^{(3)} \land u_1^{(1)} \land u_0^{(1)} \land u_{-3}^{(3)} \\
    &= q^{-3} \emptyset^{[2]} \land \bar{u}_k.
\end{align*}
\]

Lemma 5.5. If \( s_j \) is sufficiently large for an ordered \( q \)-wedge product \( u_k = u_{k_1} \land u_{k_2} \land \cdots \land u_{k_r} \). Then,

\[
    \overline{u_k} = q^{-\xi(0^{[j]}, \bar{u}_k)} \emptyset^{[j]} \land \overline{u_k}.
\]

Proof. Let \( \xi = \xi(0^{[j]}, \bar{u}_k) \) and \( \eta = \xi(\bar{u}_k, \emptyset^{[j]}) \). By Corollary 5.3, we have

\[
    u_k = q^{-\xi} \emptyset^{[j]} \land \bar{u}_k.
\]

Thus, we have

\[
\begin{align*}
    \overline{u_k} &= q^\xi q^{-\xi-\eta} \overline{u_k} \land \emptyset^{[j]} \quad \text{(Definition of bar involution (8))} \\
    &= q^{-\eta} \overline{u_k} \land \emptyset^{[j]} \quad \text{(} (\emptyset^{[j]} = \emptyset^{[j]} \text{)} \\
    &= q^{-\eta} q^{\eta-\xi} \emptyset^{[j]} \land \overline{u_k} \quad \text{(By Corollary 4.10)} \\
    &= q^{-\xi} \emptyset^{[j]} \land \overline{u_k}
\end{align*}
\]

5.2. Proof of Theorem 3.4. Let \( \bar{\Pi}^\ell \) be the subset of \( \Pi^\ell \) whose \( j \)-th component is the empty Young diagram, i.e.

\[
\Pi^\ell = \{ \lambda \in \Pi^\ell | \lambda^{(j)} = \emptyset \}.
\]

Theorem 3.4 is a direct consequence of the next proposition.

Proposition 5.6. Suppose that \( s_j \) is sufficiently large for \( |\lambda; s\rangle \). Then,

\[
G^+(\lambda; s) = \sum_{\mu \in \Pi^\ell} \Delta^+_{\lambda, \mu, 3}(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_{\mu \in \Pi^\ell} \Delta^-_{\lambda, \mu, 3}(q) |\mu; s\rangle,
\]

where \( \lambda \) (resp. \( \mu, s \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

Proof. We only show the statement in for \( G^- \). The case of \( G^+ \) is treated similarly.

Take a sufficiently large integer \( r \). Put \( F = \sum_{\mu \in \Pi^\ell} \Delta^-_{\lambda, \mu, 3}(q) |\mu; s\rangle \). We prove \( \overline{F} = F \) and \( F \equiv |\lambda; s\rangle \) mod \( q^{-1} \mathcal{L}^- \).

The second statement is clear since \( \bar{\lambda} = \bar{\mu} \) if and only if \( \lambda = \mu \). We show \( \overline{F} = F \). Let \( \xi = \xi(0^{[j]}, \bar{\nu}_\lambda) \).
\[
\overline{F} = \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q^{-1}) \overline{u}_\mu \\
= \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q^{-1}) q^{-\epsilon} \theta [J] \wedge \overline{u}_\mu \\
= q^{-\epsilon} \left( \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q^{-1}) \theta [J] \wedge \overline{u}_\mu \right) \\
= q^{-\epsilon} \theta [J] \wedge \left( \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q) \overline{u}_\mu \right) \\
= q^{-\epsilon} \theta [J] \wedge \left( \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q) \overline{v}_\mu \right) \\
= \overline{F}.
\]

Note that \( G^-(\lambda; \overline{s}) = \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q) \overline{u}_\mu \) and \( \overline{G}^-(\lambda; \overline{s}) = G^-(\lambda; \overline{s}) \). Therefore,

\[
\overline{F} = q^{-\epsilon} \theta [J] \wedge \left( \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q) \overline{v}_\mu \right) \quad \text{(By Corollary 5.3 & Lemma 5.2)}
\]

\[
= \sum_{\mu \in \Pi'} \Delta^-_{\lambda, \mu, \beta}(q) \overline{v}_\mu \\
= F.
\]

\[\square\]

6. Proof of Theorem 5.5

Throughout this section, we fix \( j (1 \leq j \leq \ell) \).

6.1. The quotient space \( \overline{F}_q[s]_{\leq N} \). In this paragraph, we fix a positive integer \( N \) and assume that \( s_j \) is sufficiently small for \( N \), i.e. \( s_i - s_j \geq N \) for all \( i \neq j \).

We define \( \overline{F}_q[s]_{\leq N} \) to be the subspace spanned by \( \{ \lambda; s | A^{(j)} = \emptyset, |\lambda| \leq N \} \). We also define a map \( \pi: F_q[s] \to \overline{F}_q[s]_{\leq N} \) (quotient map) by

\[
\pi(\lambda; s) = \begin{cases} 
|\lambda; s| & \text{if } A^{(j)} = \emptyset \text{ and } |\lambda| \leq N \\
0 & \text{otherwise}
\end{cases}
\]

We import the bar involution on \( \overline{F}_q[s]_{\leq N} \) from \( F_q[s] \), that is

\[
(32) \quad \overline{\pi}(\lambda; s) = \pi(\lambda; s), \quad (\lambda; s) \in \overline{F}_q[s]_{\leq N}.
\]

The unitriangularity of the bar involution \( (11) \) and Lemma 3.3 assure that the bar involution \( \overline{F}_q[s]_{\leq N} \) is well-defined.

It is clear that the following two property hold from the definition of \( \overline{F}_q[s]_{\leq N} \).

**Proposition 6.1.** Let \( \epsilon \in \{+, -\} \). There is a unique basis \( \{ \overline{G}^\epsilon(\lambda; s) | \lambda \in \Pi', |\lambda| \leq N \} \) of \( \overline{F}_q[s]_{\leq N} \) such that

...
 Proposition 6.3. \( \overline{G}(\lambda; s) = G^e(\lambda; s) \).

\[ G^e(\lambda; s) \equiv |\lambda; s\rangle \mod q^e \bar{L}^e, \quad \bar{L}^e = \bigoplus_{\lambda \in \Pi^f} \mathbb{Q}[q^e]|\lambda; s\rangle \]

and \( \bar{\Pi}^f = \{ \lambda \in \Pi^f | \lambda^{(j)} = \emptyset \} \).

**Definition 6.2.** Let \( \varepsilon \in \{+, -\} \). Suppose that \( \lambda^{(j)} = \mu^{(j)} = \emptyset, |\lambda| \leq N \) and \( |\mu| \leq N \). Define \( \Delta^e_{\lambda, \mu}(q) \) by

\[
G^e(\mu; s) = \sum_{\lambda \in \bar{\Pi}^f} \Delta^e_{\lambda, \mu}(q)|\lambda; s\rangle, \quad G^e(\lambda; s) = \sum_{\mu \in \bar{\Pi}^f} \Delta^e_{\lambda, \mu}(q)|\mu; s\rangle.
\]

**Proposition 6.3.** Let \( \varepsilon \in \{+, -\} \). If \( \lambda^{(j)} = \mu^{(j)} = \emptyset, |\lambda| \leq N \) and \( |\mu| \leq N \), then

\[
\Delta^e_{\lambda, \mu}(q) = \Delta^e_{\lambda, \mu}(q).
\]

Note that if \( N \geq |\lambda| \) and \( N \geq |\mu| \), then \( \Delta^e_{\lambda, \mu}(q) \) is independent of the choice of \( N \).

6.2. **Proof of Theorem 3.5**. As in §4, we only prove Theorem 3.5 in the case of \( \varepsilon = -\).

In this paragraph, we assume that \( s_j \) is sufficiently small for \( |\lambda; s\rangle \). Let \( N = |\lambda| \) and we fix a sufficient large integer \( r \).

The structure of our proof of Theorem 3.5 is similar to that of Theorem 3.4. Lemma 6.4 and Lemma 6.5 play roles similar to Lemma 5.2 and Corollary 5.3 respectively.

**Lemma 6.4.** Let \( \lambda, \mu \in \Pi^f \) such that \( \Delta^e_{\lambda, \mu}(q) \neq 0 \). If \( s_j \) is sufficiently small for \( |\lambda; s\rangle \) and \( \lambda^{(j)} = \mu^{(j)} = \emptyset \), then

\[ \xi(\emptyset^{[j]}, \tilde{v}_s) = \xi(\emptyset^{[j]}, \tilde{v}_\mu). \]

**Proof.** Let \( u_{k_1}^{(d_1)} \land u_{k_2}^{(d_2)} \) be a \( q \)-wedge product such that \( d_i \neq j \) and \( k_i \geq s_j \) for \( i = 1, 2 \). Let \( u_{k_1}^{(d_1')} \land u_{k_2}^{(d_2')} \) be a \( q \)-wedge product which appears in the linear expansion of the straightening of \( u_{k_1} \land u_{k_2} \).

We put \( \xi = \xi(\emptyset^{[j]}, u_{k_1}^{(d_1)} \land u_{k_2}^{(d_2)}), \xi_1 = \xi(\emptyset^{[j]}, u_{k_1}^{(d_1)}), \xi_2 = \xi(\emptyset^{[j]}, u_{k_2}^{(d_2)}), \xi' = \xi(\emptyset^{[j]}, u_{k_1}^{(d_1')} \land u_{k_2}^{(d_2')}), \xi_1' = \xi(\emptyset^{[j]}, u_{k_1}^{(d_1')}), \xi_2' = \xi(\emptyset^{[j]}, u_{k_2}^{(d_2')}). \) Note that \( \xi = \xi_1 + \xi_2 \) and \( \xi' = \xi_1' + \xi_2' \).

Then, from the abacus presentation below, we obtain \( \xi = \xi' \).

\[
\begin{array}{c|c|c|c|c|c|c|c}
| & | & d = d_1 & d = d_2 & d = d_1 & d = d_2 & d = j \\
\xi_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\xi_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

straightening rule

\[
\begin{array}{c|c|c|c|c|c|c|c}
| & | & d = d_1 & d = d_2 & d = d_1 & d = d_2 & d = j \\
\xi'_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\xi'_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k'_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
where beads are filled in the boxed region.

Since \( s_j \) is sufficiently small for \( v, \) for each \( i \neq j, \)
\[
\lambda_i^{(i)} + s_i > \lambda_2^{(i)} - 1 + s_i > \cdots > \lambda_l^{(i)} - l + s_i \geq s_j
\]
where \( l = l(\lambda^{(i)}) \) is the length of \( \lambda^{(i)}. \)

If \( \Delta_{\lambda, \mu(q)} \neq 0, \) then \( v_{\mu} \) appears in the linear expansion of the straightening of \( v \). Therefore, the above argument assures the assertion.

\[ \square \]

Lemma 6.5. Let \( \lambda \in \Pi^{\ell}. \) If \( \lambda^{(j)} = \emptyset, \) then
\[
v_{\lambda} = q^{-\xi^{(j)}(\emptyset), \emptyset^{(j)}} v_{\lambda} \wedge \emptyset^{(j)}.
\]
See Definition 4.7, Definition 5.1, and Definition 4.9 for the definition of \( \xi, v_{\lambda} \) and \( \emptyset^{(j)} \) respectively.

Proof. The proof follows from Lemma 4.8 and Lemma 6.4 (see also Example 5.4.) \[ \square \]

Lemma 6.6. Let \( 1 \leq j \leq \ell, m > 0 \) and \( \lambda \) be a partition of length at most \( m. \) Let \( 1 \leq d \leq \ell \) and \( k \) be an integer satisfying \( \lambda_1 + s_j \leq k. \) If \( j \neq d, \) then \( \lambda^{(j)} \wedge u_k^{(d)} \) is expanded as
\[
\lambda^{(j)} \wedge u_k^{(d)} = q^{-\xi^{(j)}(u_k^{(d+1)}, \lambda^{(j)}) + \xi(\emptyset, u_k^{(d)})} u_k^{(d)} \wedge \lambda^{(j)} + \sum_{\mu \neq \emptyset} b_{\mu}(q) u_k^{(d)} \wedge \mu^{(j)} + \mu^{(j)}
\]
where \( \lambda^{(j)} \) is defined in Definition 4.7.

Moreover, if \( b_{\mu}(q) \neq 0, \) then \( \mu_1 + s_j \leq k. \)

Proof. Applying the identity (27) repeatedly, we expand \( \lambda^{(j)} \wedge u_k^{(d)} \) as a linear combination of \( u_k^{(d')} \wedge \mu^{(j)} \) such that \( k' \leq k. \) \[ \square \]

Corollary 6.7. Let \( 1 \leq j \leq \ell, m > 0, t > 0, \)
\[
\emptyset^{(j)} = u_s^{(j)} \wedge u_s^{(j-1)} \wedge \cdots \wedge u_s^{(j)} \quad \text{and} \quad u = u_g^{(d_1)} \wedge u_g^{(d_2)} \wedge \cdots \wedge u_g^{(d_t)}.
\]

If \( d_b \neq j \) for all \( b = 1, 2, \cdots, t \) and \( s_j - r \leq g_1 \leq g_2 \leq \cdots \leq g_t, \) then \( \emptyset^{(j)} \wedge u \) can be written in the form
\[
\emptyset^{(j)} \wedge u = q^{-\xi(u, \emptyset)} u \wedge \emptyset^{(j)} + \sum_{\mu \neq \emptyset} v_{\mu}(q) \wedge \mu^{(j)},
\]
where \( v_{\mu}(q) \) is a linear combination of \( q \)-wedge products.

Proof. Apply Lemma 6.6 repeatedly. \[ \square \]

In the proof of Theorem 3.5, the next two lemmas (Lemma 6.8 and Lemma 6.9) play roles similar to Corollary 4.10 and Lemma 5.6 in the proof of Theorem 3.4.

Lemma 6.8. Let \( \lambda \in \Pi^{\ell}. \) If \( \lambda^{(j)} = \emptyset, \) then
\[
\pi(\emptyset^{(j)} \wedge v_{\lambda}) = q^{-\xi(\emptyset, \emptyset^{(j)})} \pi(v_{\lambda} \wedge \emptyset^{(j)}).
\]
Proof. Let $\xi = \xi(\emptyset^{[j]}, \nu^{(i)})$, $\eta = \xi(\nu^{(i)}, \emptyset^{[j]})$ and

$$\nu^{(i)} = u^{(d_1)}_{g_1} \wedge u^{(d_2)}_{g_2} \wedge \ldots \wedge u^{(d_i)}_{g_i}.$$  

From the definition of the bar involution (8),

$$\overline{\nu^{(i)}} = (-q)^{\kappa(\emptyset^{[j]})} q^{-\kappa(\emptyset^{[j]})} u^{(d_1)}_{g_1} \wedge u^{(d_2)}_{g_2} \wedge \ldots \wedge u^{(d_i)}_{g_i},$$

where $(-q)^{\kappa(\emptyset^{[j]})}$ and $q^{-\kappa(\emptyset^{[j]})}$ are suitable constants (see (8)). Then, from Corollary 6.7

$$\emptyset^{[j]} \wedge \overline{\nu^{(i)}} = (-q)^{\kappa(\emptyset^{[j]})} q^{-\kappa(\emptyset^{[j]})} \emptyset^{[j]} \wedge u^{(d_1)}_{g_1} \wedge u^{(d_2)}_{g_2} \wedge \ldots \wedge u^{(d_i)}_{g_i} = q^{-\kappa(\emptyset^{[j]})} \emptyset^{[j]} \wedge \sum_{\mu \neq 0} \nu_{\mu}(q) \wedge \mu^{[j]},$$

where $\nu_{\mu}(q)$ is a linear combination of $q$-wedge products.

Finally, we shall prove $\pi(\nu_{\mu}(q) \wedge \mu^{[j]}) = 0$ if $\mu \neq 0$. To do it, it is enough to prove the next claim.

Claim. Let $\nu \neq \emptyset$ and $\nu \in \Pi^f$ such that $\nu^{(i)} = \emptyset$. Then,

$$\pi(\nu \wedge \mu^{[j]}) = 0.$$

(Proof of Claim)

Define $\nu_{\mu} \in \Pi^f$ as $\nu_{\mu}^{(i)} = \mu$ and $\nu_{\mu}^{(i)} = \nu^{(i)} (i \neq j)$. From the straightening rule (25 or 27), any $|\nu; s|$ appearing in the linear expansion of the straightening of $\nu \wedge \mu^{[j]}$ is less than or equal to $|\nu_{\mu}; s|$. Thus, from Lemma 3.3, the $j$-th component is not empty. $\square$

Lemma 6.9. Let $\lambda \in \Pi^f$. If $\lambda^{(j)} = \emptyset$, then

$$\pi(\overline{\nu^{(i)}}) = q^{-\xi(\emptyset^{[j]}, \emptyset^{[j]})} \pi(\overline{\nu^{(i)} \wedge \emptyset^{[j]}}).$$

Proof. The proof of this proposition is similarly argued to the proof of Lemma 5.5.

Let $\xi = \xi(\emptyset^{[j]}, u^{(i)})$ and $\eta = \xi(u^{(i)}, \emptyset^{[j]})$. Then,

$$\overline{\nu^{(i)}} = q^{-\xi} \overline{\nu^{(i)}} \wedge \emptyset^{[j]}$$

$$= q^{\xi} q^{-\kappa(\emptyset^{[j]})} \overline{\nu^{(i)}} \wedge \emptyset^{[j]}$$

$$= q^{\kappa(\emptyset^{[j]})} \emptyset^{[j]} \wedge \overline{\nu^{(i)}}.$$

Thus, from Lemma 6.8

$$\pi(\overline{\nu^{(i)}}) = q^{-\eta} q^{\kappa(\emptyset^{[j]})} \pi(\overline{\nu^{(i)} \wedge \emptyset^{[j]}}) = q^{-\xi} \pi(\overline{\nu^{(i)} \wedge \emptyset^{[j]}}).$$

$\square$

Now Theorem 3.5 is an immediate consequence of the next proposition and Proposition 6.3.
Proposition 6.10. Let $\lambda \in \tilde{\Pi}^r$. Suppose that $s_j$ is sufficiently small for $|\lambda; s)$. Then,

$$G^+(\lambda; s) = \sum_{\mu \in \Pi^r} \Delta^+_{\lambda, \mu^3}(q) \pi(|\mu; s)) \quad G^-(\lambda; s) = \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q) \pi(|\mu; s))$$

where $\tilde{\lambda}$ (resp. $\tilde{\mu}, \tilde{s}$) is obtained by omitting the j-th component of $\lambda$ (resp. $\mu, s$).

In particular,

$$\Delta^+_{\lambda, \mu^3}(q) = \Delta^+_{\lambda, \mu^3}(q) \quad \Delta^-_{\lambda, \mu^3}(q) = \Delta^-_{\lambda, \mu^3}(q).$$

Proof. The proof of this proposition is similarly to that of Proposition 5.6.

We only show the statement in the case of $G^-$. The case of $G^+$ is treated similarly.

Take a sufficiently large $r$. Put $F = \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q) \pi(|\mu; s))$. We prove $\overline{F} = F$ and $F \equiv |\lambda; s)$ mod $q^{-1} \mathcal{L}^-$. The second statement is clear since $\tilde{\lambda} = \tilde{\mu}$ if and only if $\lambda = \mu$. We show $\overline{F} = F$. Let $\xi = \xi(v, \emptyset^{[j]}).$

$$\overline{F} = \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q^{-1}) \pi(u_\mu)$$

$$= \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q^{-1}) \pi(u_\mu) \quad \text{ (By the definition of bar involution for } \overline{F}_{q[s]}_{=N})$$

$$= \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q^{-1}) q^{-\varepsilon} \pi(u_\mu \wedge \emptyset^{[j]}) \quad \text{ (By Lemma 6.9 & Lemma 6.4)}$$

$$= q^{-\varepsilon} \left( \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q^{-1}) \pi(u_\mu \wedge \emptyset^{[j]}) \right)$$

$$= q^{-\varepsilon} \pi \left( \left( \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q^{-1}) u_\mu \right) \wedge \emptyset^{[j]} \right)$$

$$= q^{-\varepsilon} \pi \left( \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q) u_\mu \wedge \emptyset^{[j]} \right)$$

Note that $G^-(\tilde{\lambda}; \tilde{s}) = \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q) u_\mu$ and $G^-(\tilde{\lambda}; \tilde{s}) = G^-(\tilde{\lambda}; \tilde{s})$. Therefore,

$$\overline{F} = q^{-\varepsilon} \pi \left( \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q) u_\mu \wedge \emptyset^{[j]} \right)$$

$$= \sum_{\mu \in \Pi^r} \Delta^-_{\lambda, \mu^3}(q) \pi(v_\mu) \quad \text{ (By Corollary 6.5 & Lemma 6.4)}$$

$$= F.$$

References

[Ari] S. Ariki, Graded $q$-Schur algebras, mathArXiv 0903.3453.

[CM] Chuang and H. Miyachi, Hidden Hecke Algebras and Duality, in preparation.
[JMMO91] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado, Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$, Comm. Math. Phys. 136 (1991), no. 3, 543–566. MR1099695 (93a:17015)

[KT02] M. Kashiwara and T. Tanisaki, Parabolic Kazhdan-Lusztig polynomials and Schubert varieties, J. Algebra 249 (2002), no. 2, 306–325, DOI 10.1006/jabr.2000.8690. MR1901161 (2004a:14049)

[Rou05] R. Rouquier, Representations of rational Cherednik algebras, Infinite-dimensional aspects of representation theory and applications, Contemp. Math., vol. 392, Amer. Math. Soc., Providence, RI, 2005, pp. 103–131. MR2189874 (2007d:20006)

[Rou08] ———, q-Schur algebras and complex reflection groups, Mosc. Math. J. 8 (2008), no. 1, 119–158, 184. MR2422270 (2010b:20081)

[Sha] P. Shan, Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras, math.arXiv:0811.4549

[SW09] T. Shoji and K. Wada, Product formulas for the cyclotomic v-Schur algebra and for the canonical bases of the Fock space, J. Algebra 321 (2009), no. 11, 3527–3549, DOI 10.1016/j.jalgebra.2008.03.011. MR2510060 (2010m:20074)

[VV99] M. Varagnolo and E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J. 100 (1999), no. 2, 267–297, DOI 10.1215/S0012-7094-99-10010-X. MR1722955 (2001c:17029)

[VV08] ———, Cyclotomic double affine Hecke algebras and affine parabolic category $O, I$, math.arXiv:0810.5000 (2008).

[Ugl00] D. Uglov, Canonical bases of higher-level $q$-deformed Fock spaces and Kazhdan-Lusztig polynomials, Physical combinatorics (Kyoto, 1999), Progr. Math., vol. 191, Birkhäuser Boston, Boston, MA, 2000, pp. 249–299. MR1768086 (2001k:17030)

[Yvo06] X. Yvonne, A conjecture for $q$-decomposition matrices of cyclotomic v-Schur algebras, J. Algebra 304 (2006), no. 1, 419–456, DOI 10.1016/j.jalgebra.2006.03.048. MR2256400 (2008d:16051)

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