ASYMPTOTIC EXPRESSIONS OF EIGENVALUES AND FUNDAMENTAL SOLUTIONS OF A DISCONTINUOUS FOURTH-ORDER BOUNDARY VALUE PROBLEM

ERDOĞAN ŞEN, SERKAN ARACI, AND MEHMET ACIKGOZ

Abstract. In the present paper, we deal with a fourth-order boundary value problem problem with eigenparameter dependent boundary conditions and transmission conditions at an interior point. A self-adjoint linear operator $A$ is defined in a suitable Hilbert space $H$ such that the eigenvalues of such a problem coincide with those of $A$. We obtain asymptotic formulae for its eigenvalues and fundamental solutions. Our applications possess a number of interesting properties for studying in boundary value problems which we state in this paper.

2010 Mathematics Subject Classification. 34L20, 35R10.

Keywords and phrases. asymptotics of eigenvalues and eigenfunctions; transmission conditions; fourth-order differential operator.

1. Introduction

It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researchers are interested in the discontinuous Sturm-Liouville problem (see [1-6]). Various physics applications of this kind problem are found in many literatures, including some boundary value problem with transmission conditions that arise in the theory of heat and mass transfer (see [8,9]). The literature on such results is voluminous and we refer to [1-11].

Fourth-order discontinuous boundary value problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point of discontinuity have been investigated in [12,13]. Note that discontinuous Sturm-Liouville problems with eigen-dependent boundary conditions and with four supplementary transmission conditions at the points of discontinuity have been investigated in [3].

In this study, we shall consider a fourth-order differential equation

\[(1.1) \quad Lu := (a(x)u''(x))'' + q(x)u(x) = \lambda u(x)\]

on $I = [-1, 0) \cup (0, 1]$, with boundary conditions at $x = -1$

\[(1.2) \quad L_1u := u(-1) = 0, \]
\[(1.3) \quad L_2u := \beta_1u'(-1) + \beta_2u''(-1) = 0, \]

with the six transmission conditions at the points of discontinuity $x = 0$,
linear operator $A$ on Krein spaces. A fundamental symmetry on the Krein space is given by $Z$. Then

$$L_7 u := \lambda u (1) + u'''(1) = 0,$$

and the eigen-dependent boundary conditions at $x = 1$

$$L_7 u := \lambda u (1) + u'''(1) = 0,$$

where $a (x) = a_1^2$, for $x \in [-1, 0)$, $a (x) = a_2^2$, for $x \in (0, 1]$, $a_1 > 0$ and $a_2 > 0$ are given real numbers, $q(x)$ is a given real-valued function continuous in $[-1, 0) \cup (0, 1]$ and has a finite limit $q(0\pm) = \lim_{x \to 0 \pm} q(x)$; $\lambda$ is a complex eigenvalue parameter; $\beta_i, \delta_i$ ($i = 1, 2$) are real numbers and $|\beta_1| + |\beta_2| \neq 0$, $|\delta_1| + |\delta_2| \neq 0$.

2. Preliminaries

Firstly we define the inner product in $L^2$ for every $f, g \in L^2 (I)$ as

$$\langle f, g \rangle_1 = \frac{1}{a_1^2} \int_{-1}^{0} f(x) g(x) \, dx + \frac{1}{a_2^2} \int_{0}^{1} f(x) g(x) \, dx,$$

where $f_1(x) = f(x)|_{[-1, 0)}$, $f_2(x) = f(x)|_{(0, 1]}$. It is easy to see that $(L^2 (I), [\cdot, \cdot])$ is a Hilbert space. Now we define the inner product in the direct sum of spaces $L^2 (I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}$ by

$$[F, G] := \langle f, g \rangle_1 + \langle h_1, k_1 \rangle + \langle h_2, k_2 \rangle + \langle h_3, k_3 \rangle + \langle h_4, k_4 \rangle$$

for $F := (f, h_1, h_2, h_3, h_4)$, $G := (g, k_1, k_2, k_3, k_4) \in L^2 (I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}$. Then $Z := (L^2 (I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}, [\cdot, \cdot])$ is the direct sum of modified Krein spaces. A fundamental symmetry on the Krein space is given by

$$J := \begin{bmatrix}
J_0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & sgn\delta_1 & 0 \\
0 & 0 & 0 & 0 & sgn\delta_2
\end{bmatrix},$$

where $J_0 : L^2 (I) \to L^2 (I)$ is defined by $(J_0 f) (x) = f (x)$. We define a linear operator $A$ in $Z$ by the domain of definition

$$D (A) := \{ f, h_1, h_2, h_3, h_4 \} \in \mathbb{Z} \mid f^{(i)}_1 \in AC_{loc} ((-1, 0)), f^{(i)}_2 \in AC_{loc} ((0, 1)), i = 0, 3,$n

$$Lf \in L^2 (I), Lk f = 0, k = \frac{1}{6}, h_1 = f (1), h_2 = f' (1), h_3 = -\delta_1 f'' (0), h_4 = -\delta_2 f' (0), AF = (Lf, -f''' (1), -f'' (1), f'' (0) - f'' (0), f''' (0) - f''' (0)), \quad F = (f, f' (1), f' (1), -\delta_1 f'' (0), -\delta_2 f' (0)) \in D (A).$$

Consequently, the considered problem (1.1)-(1.9) can be rewritten in operator form as

$$L_3 u := u (0+) - u (0-) = 0,$$

$$L_4 u := u' (0+) - u' (0-) = 0,$$

$$L_5 u := u'' (0+) - u'' (0-) + \lambda \delta_1 u' (0-) = 0,$$

$$L_6 u := u''' (0+) - u''' (0-) + \lambda \delta_2 u (0-) = 0,$$
After defining this solution, we may define the solution \( \phi \) by virtue of Lemma 3.1, after defining this solution, we may define the solution \( \phi \) on \((-1, 0)\) by means of the solution

\[
\begin{align*}
\phi_{11} (-1) &= 0, \quad \phi'_{11} (-1) = \phi''_{11} (-1) = 0, \\
\phi_{11}'' (-1) &= -1.
\end{align*}
\]

By virtue of Lemma 3.1, after defining this solution, we may define the solution \( \phi_{12} (x, \lambda) \) of Eq. (1.1) on \([0, 1]\) by means of the solution \( \phi_{11} (x, \lambda) \) by the initial conditions

\[
\phi_{12} (0) = \phi_{11} (0), \quad \phi'_{12} (0) = \phi'_{11} (0), \quad \phi''_{12} (0) = \phi''_{11} (0) - \lambda \delta_1 \phi'_{11} (0), \quad \phi'''_{12} (0) = \phi'''_{11} (0) - \lambda \delta_2 \phi_{11} (0).
\]

After defining this solution, we may define the solution \( \phi_{21} (x, \lambda) \) of Eq. (1.1) on \([-1, 0)\) which satisfies the initial conditions

\[
\phi_{21} (-1) = 0, \quad \phi'_{21} (-1) = \beta_2, \quad \phi''_{21} (-1) = -\beta_1, \quad \phi'''_{21} (-1) = 0.
\]

After defining this solution, we may define the solution \( \phi_{22} (x, \lambda) \) of Eq. (1.1) on \([0, 1]\) by means of the solution \( \phi_{21} (x, \lambda) \) by the initial conditions

\[
\begin{align*}
\phi_{22} (0) &= \phi_{21} (0), \quad \phi'_{22} (0) = \phi'_{21} (0), \quad \phi''_{22} (0) = \phi''_{21} (0) - \lambda \delta_1 \phi'_{21} (0), \\
\phi'''_{22} (0) &= \phi'''_{21} (0) - \lambda \delta_2 \phi_{21} (0).
\end{align*}
\]

Analogically we shall define the solutions \( \chi_{11} (x, \lambda) \) and \( \chi_{12} (x, \lambda) \) by the initial conditions

\[
\begin{align*}
\chi_{12} (1) &= -1, \quad \chi'_{12} (1) = \chi''_{12} (1) = 0, \quad \chi'''_{12} (1) = \lambda, \quad \chi_{11} (0) = \chi_{12} (0), \\
\chi_{11} (0) &= \chi_{12} (0), \quad \chi''_{11} (0) = \chi''_{12} (0) + \lambda \delta_1 \chi_{12} (0), \\
\chi'''_{11} (0) &= \chi'''_{12} (0) + \lambda \delta_2 \chi_{12} (0).
\end{align*}
\]
Moreover, we shall define the solutions $\chi_{21} (x, \lambda)$ and $\chi_{22} (x, \lambda)$ by the initial conditions
\[
\begin{align*}
\chi_{22} (1) &= 0, \quad \chi'_{22} (1) = -1, \quad \chi''_{22} (1) = \lambda, \quad \chi'''_{22} (1) = 0, \quad \chi_{21} (0) = \chi_{22} (0), \\
\chi_{21} (0) &= \chi'_{22} (0), \quad \chi''_{21} (0) = \chi''_{22} (0) + \lambda \delta_1 \chi_{22} (0), \quad \chi'''_{21} (0) = \chi'''_{22} (0) + \lambda \delta_2 \chi_{22} (0).
\end{align*}
\]
Let us consider the Wronskians
\[
W_1 (\lambda) := \begin{vmatrix}
\phi_{11} (x, \lambda) & \phi_{21} (x, \lambda) & \chi_{11} (x, \lambda) & \chi_{21} (x, \lambda) \\
\phi_{12} (x, \lambda) & \phi_{22} (x, \lambda) & \chi_{12} (x, \lambda) & \chi_{22} (x, \lambda) \\
\phi_{11} (x, \lambda) & \phi_{21} (x, \lambda) & \chi_{11} (x, \lambda) & \chi_{21} (x, \lambda) \\
\phi_{12} (x, \lambda) & \phi_{22} (x, \lambda) & \chi_{12} (x, \lambda) & \chi_{22} (x, \lambda)
\end{vmatrix}
\]
and
\[
W_2 (\lambda) := \begin{vmatrix}
\phi_{11} (x, \lambda) & \phi_{21} (x, \lambda) & \chi_{11} (x, \lambda) & \chi_{21} (x, \lambda) \\
\phi_{12} (x, \lambda) & \phi_{22} (x, \lambda) & \chi_{12} (x, \lambda) & \chi_{22} (x, \lambda) \\
\phi_{11} (x, \lambda) & \phi_{21} (x, \lambda) & \chi_{11} (x, \lambda) & \chi_{21} (x, \lambda) \\
\phi_{12} (x, \lambda) & \phi_{22} (x, \lambda) & \chi_{12} (x, \lambda) & \chi_{22} (x, \lambda)
\end{vmatrix}
\]
which are independent of $x$ and entire functions. This sort of calculation gives $W_1 (\lambda) = W_2 (\lambda)$. Now we may introduce in consideration the characteristic function $W (\lambda)$ as $W (\lambda) = W_1 (\lambda)$.
\[\square\]

**Theorem 3.2.** The eigenvalues of the problem (1.1)-(1.9) are the zeros of the function $W (\lambda)$.

**Proof.** Let $W (\lambda) = 0$. Then the functions $\phi_{11} (x, \lambda), \phi_{21} (x, \lambda)$ and $\chi_{11} (x, \lambda), \chi_{21} (x, \lambda)$ are linearly dependent, i.e.,
\[
k_1 \phi_{11} (x, \lambda) + k_2 \phi_{21} (x, \lambda) + k_3 \chi_{11} (x, \lambda) + k_4 \chi_{21} (x, \lambda) = 0
\]
for some $k_1 \neq 0$ or $k_2 \neq 0$ and $k_3 \neq 0$ or $k_4 \neq 0$. From this, it follows that $k_3 \chi_{11} (x, \lambda) + k_4 \chi_{21} (x, \lambda)$ satisfies the boundary conditions (1.2)-(1.3). Therefore
\[
\begin{align*}
k_3 \chi_{11} (x, \lambda) + k_4 \chi_{21} (x, \lambda), & \quad x \in [-1, 0], \\
k_3 \chi_{12} (x, \lambda) + k_4 \chi_{22} (x, \lambda), & \quad x \in (0, 1]
\end{align*}
\]
is an eigenfunction of the problem (1.1)-(1.9) corresponding to eigenvalue $\lambda$.

Now we let $u (x)$ be any eigenfunction corresponding to eigenvalue $\lambda$, but $W (\lambda) \neq 0$. Then the functions $\phi_{11}, \phi_{21}, \chi_{11}, \chi_{21}$ would be linearly independent on $(0, 1]$. Therefore $u (x)$ may be represented as
\[
u (x) = \begin{cases}
c_1 \phi_{11} (x, \lambda) + c_2 \phi_{21} (x, \lambda) + c_3 \chi_{11} (x, \lambda) + c_4 \chi_{21} (x, \lambda), & x \in [-1, 0] \\
c_5 \phi_{12} (x, \lambda) + c_6 \phi_{22} (x, \lambda) + c_7 \chi_{12} (x, \lambda) + c_8 \chi_{22} (x, \lambda), & x \in (0, 1]
\end{cases}
\]
where at least one of the constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and $c_8$ is not zero.

Considering the equations
\[
L_u (u (x)) = 0, \quad v = \overline{1, 8}
\]
as a system of linear equations of the variables $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and taking (3.1)-(3.5) into account, it follows that the determinant of this
system is

\[
\begin{pmatrix}
0 & 0 & L_1 \chi_11 & 0 & 0 & 0 \\
0 & 0 & L_2 \chi_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L_3 \phi_{12} & 0 \\
0 & 0 & 0 & 0 & L_4 \phi_{12} & 0 \\
- \phi_{12} (0) & - \phi_{22} (0) & - \chi_{12} (0) & - \chi_{22} (0) & \phi_{12} (0) & \phi_{22} (0) \\
- \phi_{12} (0) & - \phi_{22} (0) & - \chi_{12} (0) & - \chi_{22} (0) & \phi_{12} (0) & \phi_{22} (0) \\
- \phi_{12} (0) & - \phi_{22} (0) & - \chi_{12} (0) & - \chi_{22} (0) & \phi_{12} (0) & \phi_{22} (0) \\
- \phi_{12} (0) & - \phi_{22} (0) & - \chi_{12} (0) & - \chi_{22} (0) & \phi_{12} (0) & \phi_{22} (0)
\end{pmatrix}
\begin{pmatrix}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{pmatrix}
\begin{pmatrix}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{pmatrix}
= -W (\lambda)^3 \neq 0.
\]

Therefore, the system (3.6) has only the trivial solution \( c_i = 0 \ (i = 1, 8) \).

Thus we get a contradiction, which completes the proof. \( \square \)

4. Asymptotic formulae for eigenvalues and fundamental solutions

We start by proving some lemmas.

Lemma 4.1. Let \( \phi (x, \lambda) \) be the solution of Eq. (1.1) defined in Section 3, and let \( \lambda = s^4 \), \( s = \sigma + it \). Then the following integral equations hold for \( k = 0, 3 \):

\[
\frac{d^k}{dx^k} \phi_{11} (x, \lambda) = \frac{a^3}{2s^3} \frac{d^k}{dx^k} \sin \frac{s (x + 1)}{a_1} \frac{1}{a_1} + \frac{a^3}{4s^3} \frac{d^k}{dx^k} e^{\frac{s (x + 1)}{a_1}} + \frac{a^3}{8s^3} \frac{d^k}{dx^k} e^{-\frac{s (x + 1)}{a_1}}
\]

\[
+ \frac{a^3}{2s^3} \int_{-1}^{x} \frac{d^k}{dx^k} \left( \sin \frac{s (x - y)}{a_1} - e^{\frac{s (x - y)}{a_1}} + e^{-\frac{s (x - y)}{a_1}} \right) q (y) \phi_{11} (y, \lambda) dy.
\]

\[
\frac{d^k}{dx^k} \phi_{12} (x, \lambda) = \left( \frac{\phi_{12} (0)}{2} - \frac{a^3 \phi_{12}'' (0)}{2s^2} \right) \frac{d^k}{dx^k} \cos \frac{sx}{a_2} + \left( \frac{a_2 \phi_{12}' (0)}{2s} + \frac{a^3 \phi_{12}'' (0)}{2s^3} \right)
\]

\[
\times \frac{d^k}{dx^k} \sin \frac{sx}{a_2} + \left( \frac{\phi_{12} (0)}{4} - \frac{a^3 \phi_{12}''' (0)}{4s^2} \right) \frac{d^k}{dx^k} e^{\frac{sx}{a_2}} + \left( \frac{a_2^3 \phi_{12}''' (0)}{4s} + \frac{a^3 \phi_{12}''' (0)}{4s^3} \right) \frac{d^k}{dx^k} e^{-\frac{sx}{a_2}}
\]

\[
+ \frac{a^3}{2s^3} \int_{0}^{x} \frac{d^k}{dx^k} \left( \sin \frac{s (x - y)}{a_2} - e^{\frac{s (x - y)}{a_2}} + e^{-\frac{s (x - y)}{a_2}} \right) q (y) \phi_{12} (y, \lambda) dy.
\]

\[
\]
Then differentiating it with respect to $x$

\[ \frac{d^k}{dx^k} \phi_{21} (x, \lambda) = \frac{\beta_1 a_1^2}{2s^2} \frac{d^k}{dx^k} \cos \frac{s(x + 1)}{a_1} + \frac{\beta_2 a_1}{2s} \frac{d^k}{dx^k} \sin \frac{s(x + 1)}{a_1} \]

\[ + \left( \frac{\beta_2 a_1}{4s} - \frac{\beta_1 a_1^2}{4s^2} \right) \frac{d^{s+1}}{dx^{s+1}} e^{-\frac{s(x+1)}{a_1}} - \left( \frac{\beta_2 a_1^2}{4s^2} + \frac{\beta_1 a_1^3}{4s^3} \right) \frac{d^k}{dx^k} e^{-\frac{s(x+1)}{a_1}} \]

(4.3) + $\frac{a_1^3}{2s^3} \int_{-1}^{x} \frac{d^k}{dx^k} \left( \sin \frac{s(x-y)}{a_1} - e^{-\frac{s(x-y)}{a_2}} + e^{-\frac{s(x-y)}{a_2}} \right) q(y) \phi_{21} (y, \lambda) dy.

\[ \frac{d^k}{dx^k} \phi_{22} (x, \lambda) \]

\[ = \left( \frac{\phi_{22} (0)}{2} - \frac{a_2^3 \phi_{22}'' (0)}{2s^2} \right) \frac{d^k}{dx^k} \cos \frac{x}{a_2} + \left( \frac{2a_2 \phi_{22}' (0)}{2s} - \frac{a_2^3 \phi_{22}''' (0)}{2s^3} \right) \]

\[ \times \frac{d^k}{dx^k} \sin \frac{x}{a_2} + \left( \frac{\phi_{22} (0)}{4} - \frac{a_2 \phi_{22}' (0)}{4s} + \frac{a_2^3 \phi_{22}''' (0)}{4s^3} \right) \frac{d^k}{dx^k} \cos \frac{x}{a_2} \]

\[ \times \frac{d^k}{dx^k} e^{-\frac{x}{a_2}} + \left( \frac{\phi_{22} (0)}{4} - \frac{a_2 \phi_{22}' (0)}{4s} + \frac{a_2^3 \phi_{22}''' (0)}{4s^3} \right) \frac{d^k}{dx^k} e^{-\frac{x}{a_2}} \]

(4.4) + $\frac{a_1^3}{2s^3} \int_{0}^{x} \frac{d^k}{dx^k} \left( \sin \frac{s(x-y)}{a_2} - e^{-\frac{s(x-y)}{a_2}} + e^{-\frac{s(x-y)}{a_2}} \right) q(y) \phi_{22} (y, \lambda) dy.$

**Proof.** Regard $\phi_{11} (x, \lambda)$ as the solution of the following non-homogeneous Cauchy problem:

\[ \left\{ \begin{array}{l}
- \left( a(x) u'' (x) \right)'' - s^4 u (x) = q(x) \phi_{11} (x, \lambda), \\
\phi_{11} (-1, \lambda) = 1, \quad \phi_{11}' (-1, \lambda) = 0, \\
\phi_{11}'' (-1, \lambda) = 0, \quad \phi_{11}' (-1, \lambda) = 0.
\end{array} \right. \]

Using the method of constant changing, $\phi_{11} (x, \lambda)$ satisfies

(4.1) \[ \phi_{11} (x, \lambda) = \frac{a_1^3}{2s^3} \sin \frac{s(x+1)}{a_1} + \frac{a_1^3}{4s^3} e^{-\frac{s(x+1)}{a_1}} + \frac{a_1^3}{4s^3} e^{-\frac{s(x+1)}{a_1}} \]

\[ + \frac{a_1^3}{2s^3} \int_{-1}^{x} \left( \sin \frac{s(x-y)}{a_1} - e^{-\frac{s(x-y)}{a_1}} + e^{-\frac{s(x-y)}{a_1}} \right) q(y) \phi_{11} (y, \lambda) dy. \]

Then differentiating it with respect to $x$, we have (4.1). The proof for (4.2), (4.3) and (4.4) is similar. \( \square \)

**Lemma 4.2.** Let $\lambda = s^4$, $s = \sigma + it$. Then the following integral equations hold for $k = 0, 3$:

(4.5) \[ \frac{d^k}{dx^k} \phi_{11} (x, \lambda) = O \left( |s|^{k-1} e^{|s| \frac{s(x+1)}{a_1}} \right). \]
ON A DISCONTINUOUS FOURTH-ORDER BOUNDARY VALUE PROBLEM

\[ \frac{d^k}{dx^k} \phi_{12} (x, \lambda) = \frac{a_2^2 s^2}{2} \phi'_{11} (0) \frac{d^k}{dx^k} \cos \frac{sx}{a_2} + \frac{a_2^3 s^2}{2} \phi_{11} (0) \frac{d^k}{dx^k} \sin \frac{sx}{a_2} \]

\[ \text{(4.6)} \quad - \frac{a_2^2 s^2}{4} \phi'_{11} (0) \frac{d^k}{dx^k} \left( e^{\frac{sx}{a_2}} + e^{-\frac{sx}{a_2}} \right) - \frac{a_2^3 s^2}{4} \phi_{11} (0) \frac{d^k}{dx^k} \left( e^{\frac{sx}{a_2}} - e^{-\frac{sx}{a_2}} \right) + O \left( e^{\left| \frac{s^2}{a_2} \right|} \right) \]

Each of these asymptotic formulae holds uniformly for \( x \) as \( |\lambda| \to \infty \).

**Proof.** Let

\[ F_{11} (x, \lambda) = e^{-|\frac{s^2}{a_2}|} \phi_{11} (x, \lambda) . \]

It is easy to see that \( F_{11} (x, \lambda) \) is bounded. Therefore \( \phi_{11} (x, \lambda) = O (e) \).

Substituting it into (4.1) and differentiating it with respect to \( x \) for \( k = 0, 3 \), we obtain (4.5). According to transmission conditions (1.4)-(1.7) as \( |\lambda| \to \infty \), we get

\[ \phi_{12} (0) = \phi_{11} (0) , \quad \phi'_{12} (0) = \phi'_{11} (0) , \quad \phi''_{12} (0) = -s^4 \delta_1 \phi'_{11} (0) , \]

\[ \phi'_{12} (0) = -s^4 \delta_2 \phi_{11} (0) . \]

Substituting these asymptotic formulae into (4.2) for \( k = 0 \), we obtain

\[ \phi_{12} (x, \lambda) = \frac{a_2^2 s^2}{2} \phi'_{11} (0) \cos \frac{sx}{a_2} + \frac{a_2^3 s^2}{2} \phi_{11} (0) \sin \frac{sx}{a_2} \]

\[ - \frac{a_2^2 s^2}{4} \phi'_{11} (0) \left( e^{\frac{sx}{a_2}} + e^{-\frac{sx}{a_2}} \right) - \frac{a_2^3 s^2}{4} \phi_{11} (0) \left( e^{\frac{sx}{a_2}} - e^{-\frac{sx}{a_2}} \right) + \frac{a_2^3}{2s^2} \int_0^x \sin \frac{s (x - y)}{a_2} \left( e^{\frac{sx}{a_2}} - e^{-\frac{sx}{a_2}} \right) q (y) \phi_{12} (y, \lambda) dy \]

\[ \text{(4.7)} \quad + O \left( e^{\left| \frac{s^2}{a_2} \right|} \right) . \]
Multiplying through by \(|s|^{-3}e^{-|s|(a_{11}+a_{22})/s_1}\), and denoting

\[ F_{12}(x, \lambda) := O \left( |s|^{-3}e^{-|s|(a_{11}+a_{22})/s_1} \right) \phi_{12}(x, \lambda). \]

Denoting \(M := \max_{x \in [0,1]} |F_{12}(x, \lambda)|\) from the last formula, it follows that

\[ M(\lambda) \leq \frac{M(\lambda)}{2|s|^3} \int_0^x q(y) \, dy + M_0 \]

for some \(M_0 > 0\). From this, it follows that \(M(\lambda) = O(1)\) as \(|\lambda| \to \infty\), so

\[ \phi_{12}(x, \lambda) = O \left( |s|^3 e^{\left|s\right|(a_{11}+a_{22})/s_1} \right). \]

Substituting this back into the integral on the right side of (4.7) yields (4.6) for \(k = 0\). The other cases may be considered analogically. \(\square\)

Similarly one can establish the following lemma. for \(\chi_{ij}(x, \lambda) (i = 1, 2, j = 1, 2).\)

**Lemma 4.3.** Let \(\lambda = s^4, s = \sigma + it\). Then the following integral equations hold for \(k = 0, 3:\)

\[
\frac{dk}{dx} \chi_{11}(x, \lambda) = -\frac{a_1^2 s^2 \delta_1 \chi_{12}(0)}{2} \frac{dk}{dx} \cos \frac{sx}{a_1} + \frac{a_1^3 s^2 \delta_2 \chi_{12}(0)}{2} \frac{dk}{dx} \sin \frac{sx}{a_1} + \frac{a_1^3 s \delta_1 \chi_{12}(0)}{4} \frac{dk}{dx} \left( e^{sx/a_1} + e^{-sx/a_1} \right) + O \left( |s|^{k+1} e^{\left|s\right|(a_{11}+a_{22})/s_1} \right). 
\]

\[
\frac{dk}{dx} \chi_{12}(x, \lambda) = -\frac{a_1^2 s \delta_2 \chi_{12}(0)}{2} \frac{dk}{dx} \sin \frac{sx}{a_2} + \frac{a_1^3 s \delta_2 \chi_{12}(0)}{4} \frac{dk}{dx} \left( e^{sx/a_2} - e^{-sx/a_2} \right) + O \left( |s|^{k+1} e^{\left|s\right|(a_{11}+a_{22})/s_2} \right). 
\]

\[
\frac{dk}{dx} \chi_{21}(x, \lambda) = -\frac{a_1^2 s^2 \delta_1 \chi_{22}(0)}{2} \frac{dk}{dx} \cos \frac{sx}{a_1} + \frac{a_1^3 s^2 \delta_2 \chi_{22}(0)}{2} \frac{dk}{dx} \sin \frac{sx}{a_1} + \frac{a_1^2 s \delta_1 \chi_{22}(0)}{4} \frac{dk}{dx} \left( e^{sx/a_1} + e^{-sx/a_1} \right) + O \left( |s|^{k+2} e^{\left|s\right|(a_{11}+a_{22})/s_1} \right). 
\]

\[
\frac{dk}{dx} \chi_{22}(x, \lambda) = -\frac{a_1^2 s^2 \delta_2 \chi_{22}(0)}{2} \frac{dk}{dx} \cos \frac{sx}{a_2} + \frac{a_1^2 s^2 \delta_2 \chi_{22}(0)}{4} \frac{dk}{dx} \left( e^{sx/a_2} - e^{-sx/a_2} \right) + O \left( |s|^{k+1} e^{\left|s\right|(a_{11}+a_{22})/s_2} \right), 
\]

where \(k = 0, 3\). Each of these asymptotic formulae holds uniformly for \(x\).
Theorem 4.4. Let \( \lambda = s^4, s = \sigma + it \). Then the characteristic functions \( W_i(\lambda) \) \((i = 1, 2)\) have the following asymptotic formulae:

\[
W_1(\lambda) = W_2(\lambda) = O \left( |s|^{11} e^{2|s| \left( \frac{\beta_1 + \beta_2}{a_1 a_2} \right)} \right).
\]

Proof. Substituting the asymptotic equalities \( \frac{d^k}{dx^k} \chi_{11}(-1, \lambda) \) and \( \frac{d^k}{dx^k} \chi_{21}(-1, \lambda) \) into the representation of \( W_1(\lambda) \), we get

\[
W_1(\lambda) = \begin{vmatrix}
0 & 0 & \chi_{11}(-1, \lambda) & \chi_{21}(-1, \lambda) \\
0 & \beta_2 & \chi_{11}(-1, \lambda) & \chi_{21}(-1, \lambda) \\
0 & -\beta_1 & \chi_{11}(-1, \lambda) & \chi_{21}(-1, \lambda) \\
-1 & 0 & \chi_{11}(-1, \lambda) & \chi_{21}(-1, \lambda)
\end{vmatrix}
\]

\[
= \frac{a_1^3 \delta_1 \delta_2 s^3}{8} \left( \chi_{12}'(0) \chi_{22}(0) - \chi_{12}(0) \chi_{22}'(0) \right)
\]

\[
\times \left( \begin{vmatrix}
0 & 0 & \cos \frac{\beta_2}{a_1} & e^{-\frac{\beta_1}{a_1}} - e^{\frac{\beta_1}{a_1}} \\
0 & \beta_2 & -\frac{\beta_1}{a_1} \sin \frac{\beta_2}{a_1} & \frac{\beta_1}{a_1} e^{-\frac{\beta_1}{a_1}} - e^{\frac{\beta_1}{a_1}} \\
0 & -\beta_1 & -\frac{\beta_1}{a_1} \cos \frac{\beta_2}{a_1} & \frac{\beta_1}{a_1} e^{\frac{\beta_1}{a_1}} - e^{-\frac{\beta_1}{a_1}} \\
-1 & 0 & -\frac{\beta_1}{a_1} \cos \frac{\beta_2}{a_1} & \frac{\beta_1}{a_1} e^{\frac{\beta_1}{a_1}} - e^{-\frac{\beta_1}{a_1}}
\end{vmatrix} \\
+ \begin{vmatrix}
0 & 0 & \sin \frac{\beta_1}{a_1} & e^{-\frac{\beta_1}{a_1}} + e^{\frac{\beta_1}{a_1}} \\
0 & \beta_2 & \frac{\beta_1}{a_1} \cos \frac{\beta_2}{a_1} & \frac{\beta_1}{a_1} e^{-\frac{\beta_1}{a_1}} + e^{\frac{\beta_1}{a_1}} \\
0 & -\beta_1 & -\frac{\beta_1}{a_1} \sin \frac{\beta_2}{a_1} & \frac{\beta_1}{a_1} e^{\frac{\beta_1}{a_1}} + e^{-\frac{\beta_1}{a_1}} \\
0 & 0 & -\frac{\beta_1}{a_1} \sin \frac{\beta_2}{a_1} & \frac{\beta_1}{a_1} e^{\frac{\beta_1}{a_1}} + e^{-\frac{\beta_1}{a_1}}
\end{vmatrix}
\right)
\]

\[
+ O \left( |s|^{15} e^{2|s| \left( \frac{\beta_1 + \beta_2}{a_1 a_2} \right)} \right) = 0.
\]

Analogically, we can obtain the asymptotic formulae of \( W_2(\lambda) \). □

Corollary 4.5. The real eigenvalues of the problem (1.1)-(1.9) are bounded below.

Proof. Putting \( s^2 = it^2 \) \((t > 0)\) in the above formulas, it follows that

\[
W(\lambda) \rightarrow \infty \text{ as } t \rightarrow \infty.
\]

Therefore, \( W(\lambda) \neq 0 \) for \( \lambda \) negative and sufficiently large in modulus. □

Corollary 4.6. The non-real eigenvalues of the problem (1.1)-(1.9) are bounded below and above.

Now we can obtain the asymptotic approximation formulae for the eigenvalues of the considered problem (1.1)-(1.9).

Since the eigenvalues coincide with the zeros of the entire function \( W(\lambda) \), it follows that they have no finite limit. Moreover, we know from Corollary 4.5 that all real eigenvalues are bounded below. Hence, we may renumber them as \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \), listed according to their multiplicity.
Theorem 4.7. The eigenvalues $\lambda_n = s_n^4$, $n = 0, 1, 2, \ldots$ of the problem (1.1)-(1.9) have the following asymptotic formulae for $n \to \infty$:

$$\sqrt[4]{\lambda'_n} = a_1 \pi \left(\frac{2n - 1}{2}\right) + O\left(\frac{1}{n}\right) \quad \text{and} \quad \sqrt[4]{\lambda''_n} = a_2 \pi \left(\frac{2n + 1}{2}\right) + O\left(\frac{1}{n}\right).$$

Proof. By applying the well-known Rouché’s theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour $C$, and $|g(s)| < |f(s)|$ on $C$, then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside $C$ provided that each zero is counted according to their multiplicity, we can obtain these conclusions. □

References

[1] T. Kim, Identities involving Frobenius-Euler polynomials arising from non-linear differential equations, *Journal of Number Theory*, Volume 132, Issue 12, December 2012, Pages 2854-2865.

[2] M. Demirci, Z. Akdoğan, O.Sh. Mukhtarov, Asymptotic behavior of eigenvalues and eigenfunctions of one discontinuous boundary-value problem, *International Journal of Computational Cognition* 2(3) (2004) 101–113.

[3] D. Buschmann, G. Stolz, J. Weidmann, One-dimensional Schrödinger operators with local point interactions, *Journal für die Reine und Angewandte Mathematik* 467 (1995) 169–186.

[4] M. Kadakal, O.Sh. Mukhtarov, Sturm-Liouville problems with discontinuities at two points, *Comput. Math. Appl.* 54 (2007) 1367-1379.

[5] Q. Yang, W. Wang, Asymptotic behavior of a differential operator with discontinuities at two points, *Mathematical Methods in the Applied Sciences* 34 (2011) 373-383.

[6] O. Sh. Mukhtarov, E. Tunç, Eigenvalue problems for Sturm Liouville equations with transmission conditions, *Israel Journal of Mathematics* 144(2) (2004) 367-380.

[7] E. Şen, A. Bayramov, Calculation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary condition, *Mathematical and Computer Modelling* 54 (2011) 3090-3097.

[8] C.T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* 77 (1977) 293-308.

[9] I. Titeux, Y. Yakubov, Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients, *Math. Models Methods Appl.* 7 (7) (1997) 1035–1050.

[10] P.A. Binding, P.J. Browne, Oscillation theory for indefinite Sturm-Liouville problems with eigen-parameter-dependent boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* 127 (1997) 1123–1136.

[11] N.B. Kerimov, Kh.R. Mamedov, On a boundary value problem with a spectral parameter in the boundary conditions, *Sibirsk. Mat. Zh.* 40 (2) (1999) 325–335 (English translation: *Siberian Math. J.* 40 (2) (1999) 281–290).

[12] Q. Yang, W. Wang, A class of fourth-order differential operators with transmission conditions, *Iranian Journal of Science and Technology Transaction A* A4(2011) 323-332.

[13] Q. Yang, Spectrum of a fourth order differential operator with discontinuities at two points, *International Journal of Modern Mathematical Sciences* 1(3)(2012) 134-142.
ON A DISCONTINUOUS FOURTH-ORDER BOUNDARY VALUE PROBLEM

Department of Mathematics, Faculty of Science and Letters, Namık Kemal University, 59030 Tekirdağ, TURKEY
E-mail address: erdogan.math@gmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: mtarink@hotmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: acikgoz@gantep.edu.tr