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Fractional Supersymmetry
as a Superposition of Ordinary Supersymmetry\(^1\)

A B S T R A C T

It is shown how to derive fractional supersymmetric quantum mechanics of order \(k\) as a superposition of \(k - 1\) copies of ordinary supersymmetric quantum mechanics.

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1 Introduction

In recent years, fractional supersymmetry has been the subject of numerous works. Indeed, $k$-fractional supersymmetry is closely connected to the notion of quantum algebra (deformation theory) and to the concept of intermediate statistics (of anyons [1] and $k$-fermions [2, 3]) interpolating between Bose-Einstein statistics and Fermi-Dirac statistics. Therefore, fractional supersymmetry constitutes a useful tool for dealing with anyonic statistics.

Fractional supersymmetric quantum mechanics of order $k$ can be considered as an extension of ordinary supersymmetric quantum mechanics which corresponds to $k = 2$. An ordinary supersymmetric quantum-mechanical system may be generated from a doublet $(H, Q)_2$ of operators satisfying [4, 5]

\[ Q^2 = 0, \]
\[ QQ^\dagger + Q^\dagger Q = H. \]

The self-adjoint operator $H$ and the operator $Q$ act on a separable Hilbert space. The operator $H$ is referred to as the Hamiltonian and the operator $Q$ as the supersymmetry operator of the ordinary supersymmetric quantum-mechanical system. The operator $Q$ gives rise to $\mathcal{N} = 2$ dependent supercharges $Q_- = Q$ and $Q_+ = Q^\dagger$ connected via Hermitean conjugation. They are nilpotent operators of order $k = 2$ and commute with the Hamiltonian $H$.

The ordinary supersymmetric quantum-mechanical system $(H, Q)_2$ can be extended to a fractional supersymmetric quantum-mechanical system $(H, Q)_k$ with $k \in \mathbb{N} \setminus \{0, 1, 2\}$.
as follows. The system \((H, Q)_k\) may be defined by \([6, 7]\)

\[
\begin{align*}
Q_- &= Q, \quad Q_+ = Q^\dagger, \quad Q_\pm^k = 0, \quad (1a) \\
Q_{-1}^k Q_+ + Q_{-2}^k Q_+ Q_- + \cdots + Q_+ Q_{-1}^k &= Q_{-2}^k H, \quad (1b) \\
[H, Q_\pm] &= 0, \quad H = H^\dagger, \quad (1c)
\end{align*}
\]

where the self-adjoint operator \(H\), the Hamiltonian of the system, and the \(\mathcal{N} = 2\) supercharges \(Q_-\) and \(Q_+\) act on a separable Hilbert space. Of course, the case \(k = 2\) corresponds to an ordinary supersymmetric quantum-mechanical system.

In the present work, we study how it is possible to connect ordinary and \(k\)-fractional supersymmetric quantum-mechanical systems.

## 2 The algebra \(W_k\)

As an interesting question, we may ask: How to construct a fractional supersymmetric quantum-mechanical system of order \(k\) and, thus, fractional supersymmetric quantum mechanics of order \(k\)? This question can be answered through the definition of a generalized Weyl-Heisenberg algebra \(W_k\). We now define the generic algebra \(W_k\) and shall see in the next section how a fractional supersymmetric quantum-mechanical system of order \(k\) may be associated with a given algebra \(W_k\).

For \(k\) given, with \(k \in \mathbb{N} \setminus \{0, 1\}\), the algebra \(W_k\) is generated by four linear operators \(X_-\), \(X_+\), \(N\) and \(K\). The operators \(X_-\) and \(X_+ = X_\dagger\) are shift operators connected via Hermitean conjugation. The operator \(N\), called number operator, is self-adjoint. Finally, the operator \(K\) is a \(Z_k\)-grading unitary operator. The generators \(X_-\), \(X_+\), \(N\) and \(K\) satisfy \([8]\)

\[
\begin{align*}
[X_- , X_+] &= \sum_{s=0}^{k-1} f_s(N) \Pi_s, \\
[N , X_-] &= -X_-, \quad (+\text{h.c.}), \\
[K , X_-] &= 0, \quad (+\text{h.c.}), \\
[K , N] &= 0, \quad K^k = 1.
\end{align*}
\]

The functions \(f_s : N \mapsto f_s(N)\) are such that \(f_s(N)^\dagger = f_s(N)\), \([A, B]_q = AB - qBA\), and the operators \(\Pi_s\) are defined by

\[
\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st} K^t
\]

where

\[
q = \exp \left( \frac{2\pi i}{k} \right)
\]

is a root of unity. To a given set \(\{f_s : s = 0, 1, \ldots, k - 1\}\) corresponds one algebra \(W_k\).

The generalized Weyl-Heisenberg algebra \(W_k\) covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given
set \( \{ f_s : s = 0, 1, \cdots, k - 1 \} \) yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

\[
\forall s \in \{0, 1, \cdots, k - 1\} : f_s(N) = f_s \text{ independent of } N
\]
corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [9]) and the case

\[
\forall s \in \{0, 1, \cdots, k - 1\} : f_s(N) = aN + b, \ (a, b) \in \mathbb{R}^2
\]
to systems with translational shape-invariant potentials (in the sense of Ref. [10]). For instance, the case \((a = 0, b > 0)\) corresponds to the harmonic oscillator potential, the case \((a < 0, b > 0)\) to the Morse potential and the case \((a > 0, b > 0)\) to the Pöschl-Teller potential. For these various potentials, the part of \( W_0 \) spanned by \( X_-, X_+ \) and \( N \) can be identified with the ordinary Weyl-Heisenberg algebra for \((a = 0, b \neq 0)\), with the \( su(2) \) Lie algebra for \((a < 0, b > 0)\) and with the \( su(1,1) \) Lie algebra for \((a > 0, b > 0)\).

3 A \( k \)-fractional system associated with \( W_k \)

In order to associate a \( k \)-fractional supersymmetric quantum-mechanical system associated with a given generalized Weyl-Heisenberg algebra \( W_k \), we must define a supersymmetry operator \( Q \) and an Hamiltonian \( H \). The supersymmetry operator \( Q \) is defined by

\[
Q \equiv Q_- = X_- (1 - \Pi_1) \leftrightarrow Q_-^\dagger \equiv Q_+ = X_+ (1 - \Pi_0).
\]

Then, the Hamiltonian \( H \) associated with \( W_k \) can be deduced from Eq. (1b). This yields

\[
H = (k - 1)X_+X_- - \sum_{s=3}^{k-1} \sum_{t=2}^{k-1} (t - 1) f_t(N - s + t) \Pi_s
\]

\[
- \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t - k) f_t(N - s + t) \Pi_s.
\]

(Note that the summation from \( s = k-2 \) to \( s = k \) appearing in some previous works by the authors [8] should be replaced by a summation from \( s = 3 \) to \( s = k \).) It can be checked that \( H \) is self-adjoint and commutes with \( Q_- \) and \( Q_+ \). As a conclusion, the doublet \((H, Q)_k \) associated to \( W_k \) satisfies Eq. (1) and thus defines a \( k \)-fractional supersymmetric quantum-mechanical system.

4 Connection between fractional supersymmetry and ordinary supersymmetry

In order to establish a connection between fractional supersymmetric quantum mechanics of order \( k \) and ordinary supersymmetric quantum mechanics (corresponding to \( k = 2 \)), it is necessary to construct sub-systems from the doublet \((H, Q)_k \) that correspond to ordinary
supersymmetric quantum-mechanical systems. This may be achieved in the following way \[11\]. The general Hamiltonian $H$ can be rewritten as

$$H = \sum_{s=1}^{k} H_s \Pi_s$$

where

$$H_s \equiv H_s(N) = (k - 1)X_+X_- - \sum_{t=2}^{k-1}(t - 1) f_t(N - s + t)$$

$$+ (k - 1) \sum_{t=s}^{k-1} f_t(N - s + t).$$

It can be shown that the operators $H_{k} \equiv H_0, H_{k-1}, \ldots, H_1$ turn out to be isospectral operators. It is possible to factorize $H_s$ as \[11\]

$$H_s = X(s)_+ X(s)_-.$$

Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \Pi_s, \quad q(s)_+ = X(s)_+ \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- X(s)_+ \Pi_{s-1} + X(s)_+ X(s)_- \Pi_s.$$

It is then a simple matter of calculation to prove that $h(s)$ is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = \{q(s)_-, q(s)_+\}, \quad [h(s), q(s)_\pm] = 0.$$  

Consequently, the doublet $(h(s), q(s))_2$, with $q(s) \equiv q(s)_-$, satisfies Eq. (1) with $k = 2$ and thus defines an ordinary supersymmetric quantum-mechanical system (corresponding to $k = 2$).

The Hamiltonian $h(s)$ is amenable to a form more appropriate for discussing the link between ordinary supersymmetry and fractional supersymmetry. Indeed, we can show that

$$X(s)_- X(s)_+ = H_s(N + 1).$$

Then, we can obtain the important relation

$$h(s) = H_{s-1} \Pi_{s-1} + H_s \Pi_s$$

to be compared with the expansion of $H$ in terms of supersymmetric partners $H_s$.

As a result, the system $(H, Q)_k$, corresponding to $k$-fractional supersymmetry, can be described in terms of $k - 1$ sub-systems $(h(s), q(s))_2$, corresponding to ordinary supersymmetry. The Hamiltonian $h(s)$ is given as a sum involving the supersymmetric partners $H_{s-1}$ and $H_s$. Since the supercharges $q(s)_\pm$ commute with the Hamiltonian $h(s)$, it follows that

$$H_{s-1}X(s)_- = X(s)_-H_s, \quad H_sX(s)_+ = X(s)_+H_{s-1}.$$
As a consequence, the operators $X(s)_+$ and $X(s)_-$ render possible to pass from the spectrum of $H_{s-1}$ and $H_s$ to the one of $H_s$ and $H_{s-1}$, respectively. This result is quite familiar for ordinary supersymmetric quantum mechanics (corresponding to $s = 2$).

For $k = 2$, the operator $h(1)$ is nothing but the total Hamiltonian $H$ corresponding to ordinary supersymmetric quantum mechanics. For arbitrary $k$, the other operators $h(s)$ are simple replicas (except for the ground state of $h(s)$) of $h(1)$. In this sense, fractional supersymmetric quantum mechanics of order $k$ can be considered as a set of $k - 1$ replicas of ordinary supersymmetric quantum mechanics corresponding to $k = 2$ and typically described by $(h(s), q(s))_2$. As a further argument, it is to be emphasized that

$$H = q(2)_- q(2)_+ + \sum_{s=2}^{k} q(s)_+ q(s)_-$$

which can be identified with $h(2)$ for $k = 2$.

5 Conclusions

Starting from a $Z_k$-graded algebra $W_k$, characterized by a set \{ $f_s : s = 0, 1, \cdots, k - 1$ \} of structure functions, it was shown how to associate a $k$-fractional supersymmetric quantum-mechanical system $(H, Q)_k$ characterized by an Hamiltonian $H$ and a supercharge $Q$.

The Hamiltonian $H$ for the system $(H, Q)_k$ was developed as a superposition of $k$ isospectral supersymmetric partners $H_k, H_{k-1}, \cdots, H_1$. It was proved that the system $(H, Q)_k$ can be described in terms of $k - 1$ sub-systems $(h(s), q(s))_2$ which are ordinary supersymmetric quantum-mechanical systems.
References

[1] Goldin, G.A., Menikoff, R., and Sharp, D.H., J. Math. Phys., 1980, vol.21, p.650. Goldin, G.A., and Sharp, D.H., Phys. Rev. Lett., 1996, vol.76, p.1183.

[2] Daoud, M., Hassouni, Y., and Kibler, M., in: Symmetries in Science X, eds. Gruber, B., and Ramek, M., Plenum, New York, 1998; Yad. Fiz., 1998, vol.61, p.1935.

[3] Pan, H.-Y., and Zhao, Z.S., Phys. Lett. A, 2003, vol.312, p.1.

[4] Nicolai, H., J. Phys. A: Math. Gen., 1976, vol.9, p.1497.

[5] Witten, E., Nucl. Phys. B, 1981, vol.188, p.513; 1982, vol.202, p.253.

[6] Rubakov, V.A., and Spiridonov, V.P., Mod. Phys. Lett. A, 1988, vol.3, p.1337.

[7] Khare, A., J. Phys. A, 1992, vol.25, p.L749; J. Math. Phys., 1993, vol.34, p.1277.

[8] Daoud, M., and Kibler, M., Phys. Part. Nuclei (Suppl. 1), 2002, vol.33, p.S43; Int. J. Quantum Chem., 2003, vol.91, p.551.

[9] Sukhatme, U.P., Rasinariu, C., and Khare, A., Phys. Lett. A, 1997, vol.234, p.401.

[10] Junker, G., Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin, 1996.

[11] Daoud, M., and Kibler, M., Phys. Lett. A, 2004, vol.321, p.147.