Principal Bundles and the Dixmier Douady Class

Alan L. Carey¹, Diarmuid Crowley², Michael K. Murray¹

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Abstract

A systematic consideration of the problem of the reduction and extension of the structure group of a principal bundle is made and a variety of techniques in each case are explored and related to one another. We apply these to the study of the Dixmier-Douady class in various contexts including string structures, $U_{res}$ bundles and other examples motivated by considerations from quantum field theory.

1 Introduction

This paper develops the theory of principal bundles with the aim of studying various manifestations of the Dixmier-Douady class. The motivating example is that of principal bundles whose structure group is an infinite dimensional Lie group much studied by many authors in connection with string theory (it is the restricted unitary group in the terminology of Pressley and Segal [22]). Our results include a demonstration that there are interesting examples of such bundles and we relate them to string structures. We also discuss obstructions (or characteristic classes) arising from them. Finally we connect our work to the notion of bundle gerbe, bundles with structure group the projective unitaries and to infinite dimensional Clifford bundles. A statement of the main results of the paper is given later in this introduction.

We now digress a little to explain the history which lead to this paper. Some time ago Gross [13] suggested that quantum electrodynamics lends itself to a formulation in terms of infinite dimensional Clifford bundles. It was Segal [22] who showed that using bundles with fibre the projective space of a Fock space (carrying a representation of the Clifford algebra) in non-abelian gauge theories one could explain the origin of Hamiltonian anomalies (in particular that discovered by Faddeev and Mickelsson [11]). Mickelsson, in his study of anomalies and gauge theories, found it useful to introduce the idea of a Fock bundle. These are also related to bundles whose fibre is an infinite dimensional Clifford algebra. In this paper we approach the study of these bundles through the theory of infinite dimensional principal bundles whose structure group is the restricted unitary group.

In [6] two of the authors began a related study: that of string structures. Our ideas were partly influenced by the history above. An abstract version of the problem discussed in [6] is to start with a principal bundle $P$ over a manifold $M$ with structure group $G$. Let $\hat{G}$ be a central extension of $G$ by $U(1)$. Then one can ask when there exists
a principal bundle $\hat{P}$ with structure group $\hat{G}$ such that $\hat{P}/U(1) = P$ (we call this the extension problem). Brylinski [3] observed that the obstruction to the existence of $\hat{P}$, may be identified with a class in $H^3(M, \mathbb{Z})$ (Cech cohomology) studied in a different context by Dixmier and Douady [10]. Finally in [7] and [8] the connection between Hamiltonian anomalies, the characteristic classes arising in the Atiyah-Singer families index theorem and the Dixmier-Douady class was established.

In this paper we attempt to unify some of these various manifestations of the Dixmier-Douady class. We start by showing (in theorem 4.1) that if $G$ is simply connected the Dixmier-Douady class of a principal bundle $P$ is the transgression of the Chern class of the $U(1)$ principal bundle $\hat{G} \to G$. Next we develop an obstruction theory for the extension problem showing (Section 5) that it too leads to the Dixmier-Douady class.

We find that the physical examples discussed above can all be related to principal bundles with structure group the restricted unitary group. To explain this let $H$ be a complex Hilbert space and $P_+$ an orthogonal projection on $H$ with infinite dimensional kernel and co-kernel. Denote by $U_{res}$ the group of unitary operators $U$ on $H$ such that $UP_+ - P_+U$ is Hilbert-Schmidt and by $PU$ the projective unitary group. The existence of $U_{res}$ bundles with non-trivial Dixmier-Douady class and their relation to the work of Brylinski et al is covered in Sections 7 and 8. This is handled by exploiting the existence of an embedding of the smooth loop group $L_dG$ of a compact Lie group $G$ into $U_{res}$. There are canonical central extensions of both $L_dG$ ([22]) and $U_{res}$ which are compatible with the embedding of the former in the latter. Now Killingback [15] argued that the obstruction to extending a principal $L_dG$ bundle over the space of smooth loops in $M$ to a principal bundle having fibre equal to this extension transgresses to half the Pontrjagin class of $M$. On the other hand it was shown by Brylinski [3] that the obstruction is the Dixmier-Douady class. Following McLaughlin [17] and [6] we can prove equality of these establishing as a corollary the existence of principal $U_{res}$ bundles with non-trivial Dixmier-Douady class.

Our next result concerns the connection between $U_{res}$ bundles and $PU$ bundles. There is a standard inclusion of $U_{res}$ into $PU$ which we review in Section 8. Let $P(M, U_{res})$ be a principal $U_{res}$–bundle over $M$. We use the the prefix $\Sigma$ to denote the reduced suspension of a space and $\Sigma^q$ to denote the suspension isomorphism on cohomology

$$\Sigma^q : H^q(M, \mathbb{Z}) \cong H^{q+1}(\Sigma M, \mathbb{Z}).$$

One of our main results (Section 11) is that for $M$ compact, there is an associated $U(\infty)$-bundle, $\Sigma P(\Sigma M, U(\infty))$ (an element of $K^1(M)$) over $\Sigma M$ with

$$\Sigma^3(D(P)) = c_2(\Sigma P)$$

(the right hand side being the second Chern class of $\Sigma P$). We deduce from this that the structure group of a $PU$–bundle, $Q$ reduces to $U_{res}$ if and only if there is a $U_{res}$ bundle, $P$ whose Dixmier-Douady class coincides with that of $Q$. This happens if and only if there is a $U(\infty)$–bundle, $\Sigma P(\Sigma M, U(\infty))$ over $\Sigma M$ such that $c_2(\Sigma P) = \Sigma^3(D(Q))$.

There are interesting connections between this paper and a number of other recent results. For example another way of viewing the extension of a principal $G$ bundle to a
principal $G$ bundle is to use the recently introduced notion of a bundle gerbe \[20\]. In this exposition we have avoided use of that viewpoint although it has partly motivated our arguments in section 4 and we discuss it briefly in Section 12. The original construction of the Dixmier-Douady class \[10\] was in connection with bundles of $C^*$-algebras with fibre the compact operators and hence with principal bundles whose fibre is $PU$. In the case of principal $U_{res}$ bundles the associated $C^*$-algebra bundles have fibre the infinite dimensional Clifford algebra. Specifically, in Section 12 we associate to any principal $U_{res}$ bundle over $M$ a bundle whose fibre is the $C^*$-algebra of the canonical anticommutation relations (CAR) over $H$ (an algebra isomorphic to the infinite dimensional Clifford algebra). The vanishing of the Dixmier-Douady class allows us to construct an associated Hilbert bundle over $M$ whose fibre is a representation space for the CAR-algebra (in fact it is a Fock space) such that the sections of the CAR-bundle over $M$ act on sections of the Fock bundle in the obvious way. Finally, one of the most interesting by-products of our investigation is the explicit construction in Section 6 of the classifying space of $PU$.

2 Preliminary material on principal $G$-bundles

We recall some facts about principal $G$ bundles starting with the definition. A (topological) principal $G$ bundle over a topological space $M$ is a triple $P(M, G)$ where $G$ is a topological group (the structure group) and $P$ (the total space) and the base $M$ are topological spaces with a continuous surjection $\pi: P \rightarrow M$. The group $G$ acts continuously and freely on the right of $P$ and the orbits of this action are precisely the fibres of the map $\pi$. We require that the bundle is locally trivial in the sense that there is a locally finite cover $\{U_\alpha | \alpha \in A\}$ of $M$ with the property that if $P_\alpha = \pi^{-1}(U_\alpha)$ then there are homeomorphisms $P_\alpha \rightarrow U_\alpha \times G$ which send $p$ to $(\pi(p), s_\alpha(p))$ and which commute with the action of $G$ so that $s_\alpha(pg) = s_\alpha(p)g$. Note that the trivial bundle $M \times G$ is naturally a principal bundle $G$ bundle over $M$ if we define the obvious right action $(m, h)g = (m, hg)$. Two principal bundles $P(M, G)$ and $Q(M, G)$ are said to be isomorphic if there is a homeomorphism $f: P \rightarrow Q$ commuting with the $G$ action and the projection map so that the induced action on $M$ is the identity. We will be interested in isomorphism classes of principal bundles which may be classified in two ways that we detail in the next sections.

All that we have said so far holds also in the category of manifolds and smooth maps with the corresponding modifications to the definitions. In particular many of the principal bundles that we discuss below arise as the quotient of a Lie group $G$ by a closed subgroup $H$. To show that $G \rightarrow G/H$ is a principal $H$ bundle over $G/H$ one needs to demonstrate that this fibration is locally trivial in the topological sense. In all the cases which arise in this paper both $G$ and $H$ are Banach Lie groups and the result follows by a theorem of E. Michael (\[15\]) on the existence of local continuous sections for the fibration $G \rightarrow G/H$. 

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2.1 Principal bundles and non-abelian cohomology

Notice that the function \( s_\alpha s_\beta^{-1}: P_\alpha \cap P_\beta \to G \) is constant on fibres and hence descends to define the transition functions of \( P \) with respect to the cover by 

\[
g_{\alpha\beta}: U_\alpha \cap U_\beta \to G.
\]

It is straightforward to check that the transition functions \( g_{\alpha\beta} \) form a Cech cocycle for the sheaf \( G \) of continuous \( G \)-valued functions on \( M \). It is also straightforward to check that if the trivialisations are changed then the cocycle changes by a coboundary. Hence a principal bundle defines a class in \( H^1(M, G) \). Moreover it is possible to show by the standard ‘clutching construction’ (see for example [14]) that every cohomology class arises in this way. We have:

**Proposition 2.1.** The isomorphism classes of principal \( G \) bundles over \( M \) are in bijective correspondence with the elements of \( H^1(M, G) \).

It is important to note that the cohomology space \( H^1(M, G) \) is not a group. It is a pointed set, pointed by the equivalence class of the identity cocycle which corresponds under the isomorphism from 2.1 to the trivial \( G \) bundle.

2.2 Classifying spaces for principal \( G \) bundles

Another way of describing the isomorphism classes of principal bundles is to use classifying spaces. If \( f: N \to M \) is a map and \( P(M, G) \) is principal bundle then there is a pull-back bundle \( f^*(P)(N, G) \) defined by

\[
f^*(P) = \{(p, n): \pi(p) = f(n)\} \subset P \times N.
\]

We make \( f^*(P) \) a topological space or manifold by its definition as a subspace or submanifold of \( P \times N \). The action of \( G \) is \((p, n)g = (pg, n)\).

A principal \( G \) bundle \( EG(BG, G) \) is called a classifying space for principal \( G \) bundles if it has the property that for any principal bundle \( P(M, G) \) there is a map \( f \), unique up to homotopy, such that \( f^*(EG) \) is isomorphic to \( P \). The map \( f \) is called a classifying map for \( P \). A standard fact, see for example, [14], is that classifying spaces exist and are unique up to homotopy equivalence.

It is sufficient for our purposes to work in the category of spaces with the homotopy type of a CW–complex, denoted \( CW \) (see, for example, [25] pp. 400). Any map between two CW-complexes whose associated maps on the homotopy groups are all isomorphisms (a weak homotopy equivalence) is, in fact, a homotopy equivalence ([25] pp. 405). For example, differentiable manifolds have the homotopy type of a CW-complex and \( CW \) is closed under the operation of forming loop spaces. An extremely useful characterisation of classifying spaces within the category \( CW \) is the fact that a principal \( G \)-bundle, \( P(M, G) \) is a classifying space if and only if \( P \) is weakly contractible (i.e. \( \pi_q(P) = 0 \) for all \( q \)). Recall Kuiper’s Theorem [16] which states that \( U(H) \), the full unitary group of a separable Hilbert space is contractible in the uniform topology. This makes \( U(H) \) a candidate for the total space of \( CW \)-universal bundles.
If $EG(BG, G)$ is a classifying space for $G$-bundles we may summarise our discussion as:

**Proposition 2.2.** The set of isomorphism classes of principal $G$-bundles over $M$ is in bijective correspondence with the set of homotopy classes of maps from $M$ to $BG$.

### 2.3 Characteristic classes of principal $G$ bundles

A characteristic class, $c$, for principal $G$ bundles assigns to any principal $G$ bundle $P(M, G)$ an element $c(P)$ in $H^\ast(M)$, the cohomology of $M$. This assignment is required to be *natural* in the sense that if $f : N \to M$ and $P$ is a $G$ bundle over $M$ then

$$c(f^\ast(P)) = f^\ast(c(P)).$$

Note that, among other things, this implies that $c(P)$ depends only on the isomorphism class of $P$. The results above on classifying spaces give us a complete characterisation of all characteristic classes. If $c$ is a characteristic class we can apply it to $EG$ and obtain an element $\xi = c(EG) \in H^\ast(BG)$. Conversely if $\xi \in H^\ast(BG)$ then we can define a characteristic class by defining $c(P) = f^\ast(\xi)$ where $f$ is a classifying map for $P$. So characteristic classes are in bijective correspondence with the cohomology of $BG$.

### 2.4 Associated fibrations

We shall need to consider other fibrations that arise as *associated fibrations* to a principal bundle. If $P(M, G)$ is a principal bundle and $G$ acts on the left of a space $X$ then $G$ acts on $P \times X$ by $(p, x)g = (pg, g^{-1}x)$ and the quotient $(X \times G)/G$ is a fibration over $M$ with fibre isomorphic to $X$.

### 3 Changing the structure group

Let $\phi : H \to G$ be a topological group homomorphism. If $Q(M, H)$ is an $H$ bundle consider the problem of finding a $G$ bundle $P$ and a $\tilde{\phi} : Q \to P$ such that

1. $\tilde{\phi}(Q_m) \subset P_m$ for all $m$ in $M$, and
2. $\tilde{\phi}(qh) = \tilde{\phi}(q)\phi(h)$ for all $q$ in $Q$ and $h$ in $H$.

This problem can be always solved in a canonical way. To define $P$ we let $H$ act on the left of $G$ by $hg = \phi(h)g$ and define $P$ to be the associated fibration to this action. The group $H$ acts on $Q \times H$ by $(p, g)g' = (p, gg')$. The action of $G$ commutes with the action of $H$ and makes $P$ into a principal $G$ bundle. We denote it by $\phi_\ast(Q)$.

It is straightforward to show that if we choose local trivialisations of $Q$ with transition functions $h_{\alpha \beta}$ they define local trivialisations of $P$ with transition functions $\phi \circ h_{\alpha \beta}$. In other words $P$ is the image of $Q$ under the induced map

$$\phi : H^1(M, H) \to H^1(M, G).$$

In terms of classifying spaces we have the following theorem:
Theorem 3.1. Let \( \phi: H \to G \) be a group homomorphism. Then there is a map
\[
B\phi: BH \to BG
\]
with the property that if \( f: M \to BH \) is a classifying map for an \( H \) bundle \( Q \) then \( B\phi \circ f: M \to BG \) is a classifying map for the \( G \) bundle \( \phi^* (Q) \).

Proof. This follows from the standard constructions of the classifying map and the classifying space (see for example [14]). \( \square \)

More interesting is the ‘inverse’ problem to this. If \( P(M, G) \) is a principal bundle can we find a principal \( H \) bundle \( Q \) such that \( \phi^*(Q) \) is isomorphic to \( P \)? A number of ways of deciding when this is possible are known.

First, in terms of Céch cohomology: a bundle \( Q \) exists if the bundle \( P(M, G) \) lies in the image of
\[
\phi: H^1(M, H) \to H^1(M, G).
\]
Second, in terms of classifying spaces we have

Theorem 3.2. Let \( \phi: H \to G \) be a group homomorphism. Then if \( f: M \to BG \) is a classifying map for \( P \) then a \( Q \) bundle \( H \) exists with \( \phi^*(Q) \simeq P \) if and only if \( f \) lifts to a map \( \hat{f}: M \to BH \) such that \( B\phi \circ \hat{f} = f \).

Proof. This follows from Theorem 3.1. \( \square \)

The third method, which will be explained in the examples below, is to formulate the problem as that of finding a section of a fibration and to employ obstruction theory.

We are interested in two particular cases of this general problem:

1. \( H \) is a closed Lie subgroup of \( G \)
2. \( \hat{G} \to G \) is a central extension with kernel \( U(1) \).

In the first of these cases we say that the structure group \( G \) reduces to \( H \) and in the second that it lifts to \( \hat{G} \).

3.1 Reducing the structure group

Let \( H \) be a closed Banach Lie subgroup of a Banach Lie group \( G \). If \( Q(M, H) \) is a principal bundle with a bundle map from \( Q(M, H) \) to \( P(M, G) \) then it identifies \( H \) with its image inside \( P \). This image is a reduction of \( P \) to \( H \). That is, it is a submanifold of \( P \) which is stable under \( H \) and forms, with this \( H \) action, a principal \( H \) bundle over \( M \). It is clear that the problem of reducing \( P \) to \( H \) is equivalent to the problem of finding a reduction to \( H \). Given a bundle \( P(M, G) \), consider a fibre \( P_m \). A reduction of \( P \) involves selecting an \( H \) orbit in \( P_m \) for each \( m \). The set of all \( H \) orbits in \( P_m \) is \( P_m/H \) and a reduction of \( P \) therefore corresponds to a section of the fibering \( P/H \to M \) whose fibre at \( m \) is \( P_m/H \).

Applying this to the classifying space of \( G \) we see that \( EG \to EG/H \) is a principal \( H \) bundle with contractible total space and hence a classifying space for \( H \). The map \( H \subset G \) induces a map \( BH \to BG \) which under these identifications is the map \( EG/H \to BG \). It is now straightforward to show that the following theorem holds.
Theorem 3.3. Let $P(M, G)$ be a principal $G$–bundle with classifying map $f : M \to BG$ then the following conditions are equivalent to the structure group of $P$ reducing to $H$:

1. The fibration $P/H \to M$ has a global section.

2. The classifying map, $f$, has a lift, $\hat{f}$, to $BH = EG/H$.

If, in addition, $H$ is normal in $G$, then a final equivalent condition is $\rho[P] = 0$ where $\rho$ is the map in first cohomology induced by the canonical projection $G \to G/H$

$$\rho : H^1(M, G) \to H^1(M, G/H).$$

Proof. (1) Defining a reduction of $P$ to $H$ means picking out, for each $m$ in $M$ an orbit of $H$ inside $P_m$ or equivalently an element of $P_n/H$. But the latter defines a section of $P/H$.

(2) Theorem 3.2.

(3) If $P$ has a reduction to $H$ then we can always choose our local trivialisations so that the transition functions take values in $H$. Hence $\rho(P) = 0$. Conversely if the transition functions are $g_{\alpha\beta}$ and $\rho(P) = 0$ then we must have

$$g_{\alpha\beta} = g_{\beta} h_{\alpha\beta} g_{\alpha}^{-1}$$

where $g_{\alpha} : U_\alpha \to G$ and $h_{\alpha\beta} : U_\alpha \cap U_\beta \to H$. Let the transition functions be defined by local trivialisations $p \mapsto (\pi(p), s_\alpha(p))$ so that $g_{\alpha\beta} \circ \pi = s_\beta s_\alpha^{-1}$. If we modify these by letting $s'_\alpha = s_\alpha g_{\alpha}$ and $s'_\beta = s_\beta g_{\beta}$ then we find that the new transition functions are $h_{\alpha\beta}$ as required. 

Before we can apply Theorem 3.3 usefully we need the obstructions to lifting maps from the base space of a fibre bundle to the total space (loc cit Steenrod pp. 177 — 181). Briefly, assume that $M$ is a CW complex and that we are trying to lift a map $f : M \to B$ to the total space of the fibre bundle $\pi_E : E \to B$ with fibre $F$ such that the lift, $\hat{f}$ satisfies $f = \pi_E \circ \hat{f}$. We define $\hat{f}$ over the zero skeleton of $M$ by lifting $f$ arbitrarily. Extending over the 1–skeleton of $M$ is only a problem if the fibre, $F$, is not connected. In general, there is no difficulty in extending a map from the $n$–skeleton to the $(n+1)$–skeleton of $M$ if $\pi_n(F)$ is zero. We will be interested in the case that $F$ has non-vanishing homotopy only in one dimension, that is, it is an Eilenberg-Maclane space. Recall that if $A$ is a group and $n>0$ then we denote by $K(A, n)$ the Eilenberg-Maclane space whose only non-vanishing homotopy in a dimension greater than zero occurs in dimension $n$ where $\pi_n(K(A, n)) = A$. In this case the general theorem from [28] page 302 becomes:

Theorem 3.4. Let $f : M \to B$ be a continuous map where $M$ is a CW complex and let $\pi_E : E \to B$ be a fibration over $M$ with fibre $F = K(A, n)$ (i.e. an Eilenberg MacLane space) with $n>0$ and $A$ abelian. Then there exists a cohomology class, $o(f, E) \in H^{n+1}(M, A)$, which depends only on the homotopy class of $f$ and which has the property that $f$ has a lift, $\hat{f} : M \to E$ if and only if $o(f, E) = 0$.

Moreover if $g : M' \to M$ is continuous, then

$$o(f \circ g, E) = g^*(o(f, E)) \in H^{n+1}(M', A)).$$
Note 3.1. Notice that it suffices to define $o(id, E)$ where $id: B \to B$ is the identity map. Then $o(f, E) = f^*(o(id, E))$.

Note 3.2. We use the notation $H^{n+1}(M, A)$ to denote the fact that the cohomology may takes values, not simply in $\pi_n(F) = A$ but in a possibly twisted $A$ bundle over $B$. However, when this bundle is trivial we recover standard cohomology and this is the case precisely when the action of $\pi_1(B)$ on the fibre is trivial. Fibrations of this sort may be called principal $K(A, n)$-fibrations and it is easy to check that the pull-back of a principal $K(A, n)$-fibration is itself a principal $K(A, n)$-fibration. It follows that when $K(A, n)$ is realised as a topological group, principal $K(A, n)$-bundles are principal $K(A, n)$-fibrations (since $\pi_1(BK(A, n)) = 0$).

The following lemma allows one to compute the homotopy groups of the fibre of the map $B\phi: BH \to BG$ in the case that $\phi$ is an inclusion.

Lemma 3.1. Let $i: H \hookrightarrow G$ be an inclusion of topological groups. Then there is a commutative diagram of homotopy groups for all $q \geq 0$.

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_q(BH) & \delta & \pi_{q-1}(H) & \longrightarrow & 1 \\
\downarrow & & Bi_* & \downarrow i_* & \downarrow & \\
1 & \longrightarrow & \pi_q(BG) & \delta & \pi_{q-1}(G) & \longrightarrow & 1
\end{array}
$$

Proof. Setting $B := (EH \times G)/H = B(BH, G, Bi)$ let $Bi'$ be the bundle morphism $Bi': B \to EG$ covering $Bi$ and let $I$ be the obvious bundle morphism $I: EH \to B$ covering $id_H$. Then $Bi' \circ i: EH \to EG$ is a bundle morphism covering $Bi$. The commutative diagram above is just the commutative diagram of the long exact sequences of the fibrations $EG(BG, G)$ and $EH(BH, H)$ with the map of fibre bundles $Bi' \circ i$ (including the weak contractibility of $EG$ and $EH$).

3.2 Obstruction and transgression

Recall the spectral sequence of a fibration [13]. If $E \xrightarrow{p} B$ is a fibration with fibre $F$ there is a spectral sequence with

$$E_2^{p,q} = H^p(B, H^q(F, \mathbb{Z}))$$

converging to a grading of the total cohomology of $E$. If $H^1(F, \mathbb{Z}) = 0$ then the differential $d_3$ of this spectral sequence defines a map

$$\tau = d_3 : H^2(F, \mathbb{Z}) \to H^3(B, \mathbb{Z})$$

called the transgression [13]. Note that this is a different transgression from that mentioned in the introduction.

A useful fact we will use later is
**Proposition 3.1.** If \([x] \in H^3(B, \mathbb{Z})\) and \(\pi^*(x) = dy\) for a two class \(y\) on \(E\) then \(\tau([y_F]) = [x]\).

**Proof.** ([19] page 81.)

Then we have

**Theorem 3.5.** Let \(f : M \to B\) be a continuous map where \(M\) is a CW complex and let \(\pi_E : E \to B\) a fibre bundle over \(M\) with fibre \(F\). Suppose further that \(\pi_2(F) = \mathbb{Z}\) is the only non-vanishing homotopy group. Let \(\mu\) generate \(H^2(F, \mathbb{Z}) = \mathbb{Z}\). Then \(o(id, E)\) is the transgression of \(\pm \mu\).

**Proof.** See [19] pages 103 and 109.

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### 4 Extending the structure group

Let

\[
1 \to U(1) \to \hat{G} \xrightarrow{\rho} G \to 1
\]

be a short exact sequence of Lie groups with \(U(1)\) central. If \(P(M, G)\) is a principal bundle we are interested in the problem of finding a lift of \(P\) to a \(\hat{G}\) bundle \(\hat{P}\) over \(M\). We present two methods of defining a characteristic class: the Dixmier-Douady class and the obstruction class, both of which are obstructions to finding such a lift. We then show that they are, in fact, equal.

#### 4.1 The obstruction class

**Proposition 4.1 ([9]).** We can realise \(B\hat{G}\) as a principal \(BU(1)\)-bundle over \(BG\).

**Proof.** Steenrod [27] showed that Milgram’s realisation of the classifying space makes \(E\) a functor \(\tilde{\iota}\) from the category of topological spaces and continuous homomorphisms to itself. In fact we have the following commutative diagram where the vertical arrows are the inclusion of a fibre.

\[
\begin{array}{cccc}
1 & \longrightarrow & U(1) & \longrightarrow & \hat{G} & \xrightarrow{\rho} & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & EU(1) & \xrightarrow{Bi} & E\hat{G} & \xrightarrow{B\rho} & EG & \longrightarrow & 1
\end{array}
\]

Functorality allows us to move from \(U(1)\) central in \(\hat{G}\) to \(EU(1)\) central in \(E\hat{G}\) and thus \(U(1)\) is normal in \(E\hat{G}\). Since we have a closed inclusion, \(U(1) \hookrightarrow \hat{G}\), \(E\hat{G}/U(1)\) is a realisation of \(BU(1)\) as a topological group. Moreover, \(E\hat{G}/U(1)\) is a principal \(G\)-bundle over \(B\hat{G}\) with \(G\) canonically identified as a subgroup. We may form the associated bundle \(B := (E\hat{G} \times E\hat{G}/N)/G \to BG\) which is a principal \(E\hat{G}/U(1)\)-bundle over \(BG\). However, we may also project \(B\) onto \(B\hat{G}\) and the fibre is \(EG\) which is contractible. So \(B\) has the homotopy type of \(B\hat{G}\) and hence is another realisation of \(B\hat{G}\) proving the result.
From Theorem 3.2 we see that if \( P(M, G) \) is a principal \( G \)-bundle with classifying map \( f: M \to BG \) then \( P \) lifts to \( \hat{G} \) if and only if there is a lift of \( f \) to \( B\hat{G} \). To find when such a lift occurs we can use Theorem 3.4 from obstruction theory. The classifying space \( BU(1) \) is an Eilenberg-Maclane space whose only non-vanishing homotopy is \( \pi_2(BU(1)) = \mathbb{Z} \). Since we have realised \( B\hat{G} \) as a principal \( BU(1) \)-bundle it follows from 3.2 it follows that there is no twisting in the co-efficient group and that the obstruction to lifting \( f \) is a class \( O(f) \in H^3(M, \mathbb{Z}) \). The results of 3.4 imply that this defines a characteristic class in \( H^3(M, \mathbb{Z}) \). To get an exact normalisation for this class we choose the generator \( \mu \in H^2(BU(1), \mathbb{Z}) \) to be the Chern class and then define \( O(P) = f^*(\tau(\mu)) \).

4.2 The Dixmier-Douady class

Because \( U(1) \) is central in \( \hat{G} \) it is possible to show that there is a short exact sequence of pointed sets

\[
H^1(M, \mathbb{U}(1)) \to H^1(M, \hat{G}) \to H^1(M, G) \xrightarrow{\delta} H^1(M, \mathbb{U}(1)),
\]

The definition of an exact sequence of pointed sets is that if \( X, Y \) and \( Z \) are sets with points \( x, y \) and \( z \) and

\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]

is a sequence of pointed maps (that is \( f(x) = y \) and \( g(y) = z \)) then this sequence is exact at \( Y \) if \( f(X) = g^{-1}(z) \). This clearly agrees with the definition for groups if the point of a group is the identity.

The map \( \delta \) is defined as follows. Choose a Leray cover \( \{U_\alpha\} \) and local sections \( s_\alpha : U_\alpha \to P \). Then the transition functions of the bundle are defined by \( s_\alpha = s_\beta g_{\alpha \beta} \). We can lift these to maps

\[
\hat{g}_{\alpha \beta} : U_\alpha \cap U_\beta \to \hat{G}.
\]

Of course these may not be transition functions for a \( \hat{G} \) bundle. Their failure to be so is measured by the cocycle

\[
e_{\alpha \beta \gamma} = \hat{g}_{\beta \gamma} \hat{g}_{\alpha \gamma}^{-1} \hat{g}_{\alpha \beta}
\]

which takes values in \( U(1) \). Because \( U(1) \) is central it can be shown that \( e_{\alpha \beta \gamma} \) defines a class in \( H^2(M, \mathbb{U}(1)) \) which vanishes precisely when we can lift the bundle \( P \) to \( \hat{G} \).

We can use the short exact sequence of groups

\[
0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0
\]

to define an isomorphism

\[
H^2(M, \mathbb{U}(1)) \simeq H^3(M, \mathbb{Z}).
\]

The result of applying this isomorphism to \( e_{\alpha \beta \gamma} \) defines a characteristic class \( D(P) \in H^3(M, \mathbb{Z}) \) called the Dixmier-Douady class. Explicitly if we choose \( w_{\alpha \beta \gamma} \) so that \( e_{\alpha \beta \gamma} = \exp(2\pi iw_{\alpha \beta \gamma}) \) then the Dixmier-Douady class has a representative

\[
d_{\alpha \beta \gamma \delta} = w_{\beta \gamma \delta} - w_{\alpha \gamma \delta} + w_{\alpha \beta \delta} + w_{\alpha \beta \gamma}.
\]
Note that if \( p \) is a point in the fibre \( P_m \) above \( m \) then there is a homeomorphism \( G \to P_m \) defined by \( g \mapsto pg \). If \( G \) is connected then changing \( p \) gives a homotopic homeomorphism and hence there is a unique identification of the cohomology of \( G \) with the cohomology of \( P \). We want to prove

**Theorem 4.1.** Let \( P \to M \) be a principal \( G \) bundle with \( G \) one-connected. Let \( \hat{G} \to G \) be a central extension of \( G \) by \( U(1) \). Let \([\mu]\) be the cohomology class (the Chern class of \( \hat{G} \to G \)) that the central extension defines on \( G \) and hence also on any fibre of \( P \to M \). Then the transgression of \([\mu]\) is the Dixmier-Douady class of the bundle \( P \to M \).

**Proof.** To do this we need an alternative definition of transgression from [25]. We start with \( \pi : P \to M \) as above and \( P_{m_0} = \pi^{-1}(m_0) \) the fibre above some fixed \( m_0 \). Then there are homomorphisms:

\[
H^q(P_{m_0}, \mathbb{Z}) \xrightarrow{\delta} H^{q+1}(P, P_{m_0}, \mathbb{Z}) \xrightarrow{\pi^*} H^{q+1}(M, \{m_0\}, \mathbb{Z}) \xrightarrow{\tau} H^{q+1}(M, \mathbb{Z}).
\]

The transgression is then the map

\[
\tau : \delta^{-1}(\text{Im}(\pi^*)) \to H^{q+1}(M, \mathbb{Z})/\pi^*(\text{Ker}(\pi^*)).
\]

defined by \( \tau(u) = j^*(\pi^*)^{-1}\delta(u) \) where \( u \in H^q(F, \mathbb{Z}) \) is such that \( \delta(u) \in p^*(H^{q+1}(B, b_0, \mathbb{Z})) \).

For our purposes it is most useful to realise the cohomology here using Cech cocycles. Recall that if \( X \) is a topological space and \( A \) is a subspace we can define the relative integral cohomology \( H(X, A, \mathbb{Z}) \) as follows. We take a cover \( \mathcal{U}(\mathcal{T}) \) of \( X \) and a subcover \( \mathcal{U}(\mathcal{T})' \subset \mathcal{U}(\mathcal{T}) \) of \( A \) and consider the induced map of complexes of Cech cocycles for the group \( \mathbb{Z} \):

\[
C^p(X, \mathcal{U}(\mathcal{T})) \to C^p(A, \mathcal{U}(\mathcal{T})').
\]

We define \( C^p((X, A), (\mathcal{U}(\mathcal{T}), \mathcal{U}(\mathcal{T})')) \) to be the kernel of this map and define

\[
H^p((X, A), (\mathcal{U}(\mathcal{T}), \mathcal{U}(\mathcal{T})'), \mathbb{Z})
\]

to be the cohomology of the complex \( C^p((X, A), (\mathcal{U}(\mathcal{T}), \mathcal{U}(\mathcal{T})')) \). The cohomology group \( H^p(X, A, \mathbb{Z}) \) is now defined in the usual way by taking the direct limit as the covers are refined.

In the particular case of interest choose a cover \( \mathcal{U}(\mathcal{T}) \) of \( M \) with respect to which the Dixmier-Douady class can be represented by a cocycle \( d_{\alpha\beta\gamma\delta} \in C^3(M, \mathcal{U}(\mathcal{T})) \) as in (1.4). Choose \( \alpha_0 \) so that \( m_0 \in U_{\alpha_0} \) and let \( \mathcal{U}(\mathcal{T})' = \{U_{\alpha_0}\} \). Then the restriction of any cocycle in \( C^3(M, \mathcal{U}(\mathcal{T})) \) to \( C^3(m_0, \mathcal{U}(\mathcal{T})') \) is automatically zero.

Consider the transition functions \( g_{\alpha\beta} \). The pullback of \( g_{\alpha\beta} \) to \( P \) is trivial because it satisfies

\[
\pi^*g_{\alpha\beta} = \sigma_\beta \sigma_\alpha^{-1}
\]

where \( \sigma_\alpha(p) \) is defined by \( \sigma_\alpha(s_\alpha(x)g) = g \).

Restricted to any fibre the maps \( \sigma_\alpha : P_m \to G \) are homeomorphisms. Cover \( G \) by open sets \( V_\alpha \) over which \( \hat{G} \to G \) has transition functions \( h_{ab} \) relative to local sections \( r_\alpha : V_\alpha \to \hat{G} \). Then we can use the maps

\[
\begin{align*}
\pi^{-1}(U_\alpha) & \to U_\alpha \times G \\
p & \mapsto (\pi(p), \sigma(p))
\end{align*}
\]
to pull the $V_a$ back to $P$ to define open sets $W_{(a,a)} \subset \pi^{-1}(U_a)$. The cover $\mathcal{W} = \{W_{(a,a)}\}$ is a refinement of the cover $\{\pi^{-1}(U_a)\}$. If $\rho_{a_1,a_2,\ldots,a_d}$ is a cocycle for $\{\pi^{-1}(U_a)\}$ we denote by $\rho_{(a_1,a_1),(a_2,a_2),\ldots,(a_d,a_d)}$ its restriction to $\{W_{(a,a)}\}$.

In particular consider the $G$ valued cocycle $\sigma_{(a,a)}$. This can be lifted to $\hat{G}$ by defining $\hat{\sigma}_{(a,a)} = r_a \circ \sigma_{(a,a)}$. Then $\pi^* \hat{g}_{(a,a)}(\beta,b)$ and $\hat{\sigma}_{(\beta,b)} \hat{\sigma}_{(\beta,b)^{-1}}^{-1}$ are both lifts of $\pi^* g_{(a,a)}(\beta,b)$ so that we must have

$$\hat{\sigma}_{(\beta,b)} \hat{\sigma}_{(a,a)^{-1}}^{-1} h_{(a,a)}(\beta,b) = \pi^* \hat{g}_{(a,a)}(\beta,b)$$  \hspace{1cm} (4.5)

for a cocycle

$$h_{(a,a)}(\beta,b) : U_{(a,a)} \cap U(\beta,b) \to U(1).$$

Hence we have

$$\pi^* e_{(a,a)}(\beta,b)(\gamma,c) = h_{(\beta,b)}(\gamma,c) h_{(a,a)}(\gamma,c)^{-1} h_{(a,a)}(\beta,b).$$

Letting $h_{(a,a)}(\beta,b) = \exp(2\pi i v_{(a,a)}(\beta,b))$ gives

$$\pi^* w_{(a,a)}(\beta,b)(\gamma,c) = v_{(\beta,b)}(\gamma,c) - v_{(a,a)}(\gamma,c) + v_{(a,a)}(\beta,b) + n_{(a,a)}(\beta,b)(\gamma,c)$$  \hspace{1cm} (4.6)

for $n_{(a,a)}(\beta,b)(\gamma,c)$ some integer valued co-cycle. Finally we deduce that

$$\pi^* d_{(a,a)}(\beta,b)(\gamma,c)(\delta,d) = n_{(\beta,b)}(\gamma,c)(\delta,d) - n_{(a,a)}(\gamma,c)(\delta,d) + n_{(a,a)}(\beta,b)(\delta,d) - n_{(a,a)}(\beta,b)(\gamma,c).$$  \hspace{1cm} (4.7)

Consider now the cohomology on the fibre $P_{m_0}$. We define a cover $\mathcal{W}'$ which covers $P_{m_0}$ by

$$\mathcal{W}' = \{W_a = W_{(a_0,a)}\}.$$

We make corresponding notational changes to indicate restriction of cocycles from $\mathcal{W}$ to $\mathcal{W}'$. For example the restriction of $n_{(a,a)}(\beta,b)(\gamma,c)$ is $n_{abc} = n_{(a_0,a_0)(a_0,b)(a_0,c)}$. We then have from equation (4.6)

$$0 = \pi^* w_{(a_0,a)}(a_0,b)(a_0,c) = v_{bc} - v_{ac} + v_{ab} + n_{abc}$$

so that

$$n_{abc} = -v_{bc} + v_{ac} - v_{ab}$$  \hspace{1cm} (4.8)

Using equation (4.5) we see that

$$\hat{\sigma}_{(a_0,a)^{-1}}^{-1} = \hat{\sigma}_{(a_0,a)} \hat{\sigma}_{(a_0,b)}^{-1} = \exp(-2\pi i v_{ab}).$$

Finally note that $\hat{\sigma}$ is defined by $\sigma_{(a,a)} = r_a \circ \sigma_{a} \mid U_{(a,a)}$ so that

$$\exp(-2\pi i v_{ab}) = (r_b r_a^{-1}) \circ \sigma_{a_0}$$

where $r_a r_b^{-1}$ are the transition functions of $\hat{G} \to G$.

Finally we can calculate the transgression of the Chern class. It follows from (4.8) that the Chern class in $H^2(P_{m_0},\mathbb{Z})$ is represented by the cocycle $n_{abc}$. We want to apply the coboundary map in relative cohomology to this to obtain a class in $H^3(P,P_{m_0},\mathbb{Z})$. We do this by first extending $n_{abc}$ to a class on all of $P$ and then applying the Céch coboundary to it. But we obtained $n_{abc}$ by restricting $n_{(a,a)}(\beta,b)(\gamma,c)$ so this is an obvious extension and then (4.7) shows that if we apply the Céch coboundary to $n_{(a,a)}(\beta,b)(\gamma,c)$ we obtain the class $\pi^* d_{(a,a)}(\beta,b)(\gamma,c)(\delta,d)$ which is the pullback of the Dixmier-Douady class as required. \hfill \Box
5 Equality of the two classes

This Section is devoted to the proof of the following fact.

**Theorem 5.1.** The obstruction and the Dixmier-Douady classes are equal.

**Proof.** Notice first that the universal bundle for $U(1)$, $EU(1) \to BU(1)$, can be realised as $E\hat{G} \to E\hat{G}/U(1)$. Also we have that $G$ acts on $E\hat{G}/U(1)$ and hence we can form the associated fibration

$$(EG \times E\hat{G}/S^1)/G \to BG.$$ 

The fibres of this are therefore $BU(1)$. Notice also that if we project onto $B\hat{G}$ that fibering has contractible fibres and $(EG \times E\hat{G}/S^1)/G$ is homotopy equivalent to $B\hat{G}$.

Consider the diagram

$$\begin{array}{ccc}
\hat{G} & \to & E\hat{G} \\
\downarrow & & \downarrow \\
G & \to & E\hat{G}/U(1).
\end{array}$$

It follows that the bottom arrow must be the classifying map. Let $\mu$ be the generator of $H^2(BU(1), \mathbb{Z})$. Let $f$ be the classifying map. Then $f^*(\mu)$ is the class of the bundle $\hat{G} \to G$.

We now have a commuting diagram of fibrations:

$$\begin{array}{ccc}
EG & \xrightarrow{\tilde{f}} & B\hat{G} \\
\downarrow & & \vee \\
BG & & \\
\end{array}$$ (5.1)

where the map $\tilde{f}$ restricted to fibres is the classifying map $f$.

Let us denote by $[\mu]$ the class on a fibre of $B\hat{G} \to BG$ which is the fundamental class in $H^2(BU(1), \mathbb{Z}) = \mathbb{Z}$. Then by Theorem 3.5 we have that this transgresses to the obstruction class. Also by Theorem 4.1 the class $\tilde{f}^*([\mu])$ restricted to a fibre transgresses to the Dixmier-Douady class. But for the commuting diagram (5.1) of fibrations the transgression maps will commute with $\tilde{f}^*$ and hence the obstruction and Dixmier-Douady class coincide.

6 The classifying space of the projective unitaries

Given the importance of $\mathcal{P}U$ and principal $\mathcal{P}U$–bundles in the following theory we remark that there is a simple construction of a $B\mathcal{P}U$ which is a homogenous, infinite dimensional smooth manifold and will allow us to obtain a $BG$ when $G \hookrightarrow \mathcal{P}U$ is a closed embedding of Banach Lie groups. Throughout this section all groups are equipped with their natural Banach Lie group topologies (in the case of $\mathcal{P}U$ this arises from the norm topology on the unitary group).

**Proposition 6.1.** There exists a closed inclusion of $QU(H) = U(H)/U(1)$ in $U(\mathcal{T})$, the unitary group of the Hilbert space of Hilbert–Schmidt operators $\mathcal{T}$ on $H$ ($U(\mathcal{T})$ is equipped with the norm topology).
Proof. Given \([a] \in \mathcal{PU}\), choose a representative \(a \in U\). Then define

\[
    i : \mathcal{PU} \to U(T)
\]

\[
    [a] \mapsto \text{Ad}(a)
\]

where

\[
    \text{Ad}(a) : T \to T
\]

\[
    t \mapsto a.t.a^*
\]

Clearly \(i\) is well defined and is injective. To prove continuity of \(i\) considering any convergent sequence \((a_n)_{n=1}^{\infty} \to [1]\) in \(\mathcal{P}U\). By taking \(n\) large enough we may assume that the \([a_n]\) lie in a neighbourhood over which \(U (\mathcal{P}U, S^1)\) is locally trivial. Hence we may assume that there is a sequence \((a_n)_{n=N}^{\infty} \to 1\) in \(U\). Then it is straightforward to see that \(\|\text{Ad}(a_n) - \text{Ad}(1)\|_{B(T)} \to 0\) as \(n \to \infty\). To see that the image of \(i\) is closed consider a sequence \(i([a_n]) \to b\) where \(b \in U(T)_u\). Define a \(*\)-automorphism of \(T\) by

\[
    b'(t) = \lim_{n \to \infty} \text{Ad}(a_n)t.
\]

One can verify that \(b'\) is a \(*\)-automorphism of \(T\). Since \(T\) is uniformly dense in \(K(H)\), the compact operators on \(H\), \(b'\) defines a \(*\)-automorphism of \(K(H)\) and is thus of the form \(\text{Ad}(a)\) for some \(a \in U(H)_u\). Hence \(b = \text{Ad}(a) = i([a])\) and the image of \(i\) is closed.

Finally, to see that \(i\) defines a homeomorphism we begin with the metric, \(\rho\), which defines the topology on \(\mathcal{P}U(H)\).

\[
    \rho([a], [b]) = \inf(\lambda \in S^1)\|a - \lambda b\|_{B(H)}
\]

Now let \(\Theta_{u,v}\) be the rank one operator \(\Theta_{u,v} : H \to H\) given by \(w \mapsto (v, w)u\). Then the map \(u \otimes v \mapsto \Theta_{u,v}\) extends to an isomorphism of \(\overline{H} \otimes H\) with \(T\). Here the bar denotes the complex conjugate Hilbert space. The operator \(\text{Ad}(a)\) becomes \(\overline{a} \otimes a\) where \(\overline{a}\) denotes the action of \(a\) on the conjugate space. To prove our result it suffices to work in a neighbourhood of the identity in \(U(T)\). Now for \(\overline{a} \otimes a\) to be close to the identity operator the spectrum of \(a\) must contain a gap (for if the spectrum is the whole circle then it is not possible for \(\overline{a} \otimes a\) to be close to the identity). That being the case we can assume \(-1\) is not in the spectrum of \(a\) by multiplying by a phase if necessary. Assume we have a sequence \(a_n \in U(H)\) with

\[
    \|\text{Ad}(1) - \text{Ad}(a_n)\|_{B(T)} \to 0.
\]

Then there is a sequence of self adjoint operators \(K_n\) on \(T\) with \(a_n = \exp(iK_n)\) and the spectrum of \(K_n\) is \([\gamma_n, \delta_n] \subset [-\pi, \pi]\). In fact we may assume

\[
    \gamma_n = \inf_{\|u\|_H = 1} \{(u, K_n u)\},
\]

\[
    \delta_n = \sup_{\|u\|_H = 1} \{(u, K_n u)\}.
\]
\[
||\text{Ad}(a_n) - \text{Ad}(1)|| = \sup\{| \exp i(\lambda - \mu) - 1 | \mid \lambda, \mu \in [\gamma_n, \delta_n]\}
\]

\[
= \exp i(\delta_n - \gamma_n) - 1
\]

On the other hand

\[
\inf_{\lambda} ||a_n - \lambda 1|| = | \exp[\frac{i(\delta_n - \gamma_n)}{2}] - 1 |
\]

\[
= ||\text{Ad}(a_n) - \text{Ad}(1)||.
\]

Thus, if \( \| \text{Ad}(1) - \text{Ad}(a_n) \|_{B(H)} \to 0 \) as \( n \to \infty \) then \( \rho([1], [a_n]) \to 0 \). Hence \( i^{-1} : i(\mathcal{PU}(H) \to \mathcal{PU}(H)) \) is continuous and thus \( i \) is a homeomorphism.

This result shows that \( \mathcal{PU} \) is a Banach Lie subgroup of \( \mathcal{U}(\mathcal{T}) \). The contractibility of \( \mathcal{U}(\mathcal{T}) \) (Kuiper’s theorem) means that (after identifying \( i(\mathcal{PU}) \) and \( \mathcal{PU} \)), we have that

\[
\mathcal{U}(\mathcal{T})/(\mathcal{U}(\mathcal{T})/\mathcal{PU}, \mathcal{PU})
\]

is a locally trivial (by \([13]\)) universal \( \mathcal{PU} \)–bundle and that \( \mathcal{U}(\mathcal{T})/\mathcal{PU} \) is a \( B\mathcal{PU} \). More generally, if \( G \) is a closed sub-Banach Lie Group of \( \mathcal{PU} \), then \( \mathcal{U}(\mathcal{T})/(\mathcal{U}(\mathcal{T})/G, G) \) is a universal \( G \)–bundle.

7 String structures

We start with a principal \( G \)–bundle, \( P(M, G) \) where \( G \) is a compact Lie group and form the bundle \( L_dP(L_dM, L_dG, f) \) where, in general, \( L_dM \) denotes the space of differentiable loops into a finite dimensional manifold \( M \). It is well known ([22] Ch 6) that \( L_dG \) has a canonical central extension by \( S^1 \), \( \tilde{L_dG} \), induced from an embedding of \( L_dG \) in the restricted unitary group which in turn embeds in the projective unitaries of a second Hilbert space \( H_\pi \). Henceforth \( U \) and \( \mathcal{PU} \) will refer respectively to the unitaries and projective unitaries over \( H_\pi \).

\[
L_dG \hookrightarrow \mathcal{U}_{\text{res}} \hookrightarrow \mathcal{PU}(H_\pi),
\]

\[
\tilde{L_dG}(L_dG, S^1) = i^*U(H_\pi)(\mathcal{PU}(H_\pi), S^1),
\]

[\( \tilde{L_dG} \)] generates \( H^2(L_dG, \mathbb{Z}) \).

The idea of a string structure arises as follows. Starting with a principal \( SO(n) \)–bundle,

\[
P(M, SO(n), f)
\]

\((n > 2)\), which is usually the frame bundle of a tangent bundle, \( TM \), and which has a \( \text{Spin}(n) \) structure \( Q(M, \text{Spin}(n), f) \) with classifying map \( f \) one forms the loop bundle

\[
L_dQ(L_dM, L_d\text{Spin}(n), L_d\tilde{f}).
\]
The bundle $P$ is said to have a *string structure* if and only if the structure group of $L_d Q$ extends to $\tilde{L}_d \text{Spin}(n)$. Of course, the Dixmier-Douady class of $L_d Q$, $D[L_d Q]$, is the obstruction to the existence of a string structure. Killingback proposed that twice $D[L_d Q]$ was in fact the transgression of the Pontryagin class of $P$. Since then MacLaughlin [17] and Carey and Murray [6] have produced rigorous proofs of Killingback’s result.

### 7.1 Loop spaces, groups and bundles

Henceforth, let $X$ be a topological space, $H$ a topological group, $M$ a finite dimensional manifold and $G$ a compact Lie group. By $(\Omega_d M, m_0)$ we denote the based, differentiable loops into $M$.

$$\Omega_d(M, m_0) := \{ \gamma \in L_d(M) : \gamma(0) = \gamma(1) = m_0 \}$$

When the base point is unimportant we shall suppress it. $L_c X$ and $\Omega_c X$ shall denote the spaces of continuous loops and continuous based loops respectively, both with the compact open topology. Whereas, $L_s M$ and $\Omega_s M$ shall denote the loop spaces used by Carey and Murray [6] consisting of free or based continuous loops differentiable except perhaps at $m_0$. $L_d X$ and $\Omega_d X$ have the structure of differentiable Frechet manifolds when given the Frechet topology (see [6]). Moreover, in the case where the spaces are groups, $L_c H$ and $L_s,d G$ have, respectively, the structure of a topological group or a Lie group under pointwise multiplication of loops. (When considering based loops into a group, the base point is taken to be the identity of the group.) We shall next show that all three loop spaces are homotopic and hence they share many properties. When dealing with facts and properties equally applicable to either the differentiable, piecewise differentiable or continuous loops we shall drop the subscripts and use $L_X$ and $\Omega_X$ where it is understood that $X$ is a manifold if the loop functor in question is any of $L_d$, $L_s$, $\Omega_d$ or $\Omega_s$.

**Proposition 7.1.** Let $M$ be a differentiable manifold of finite or infinite dimension, then $\Omega_c M$, $\Omega_s M$ and $\Omega_d M$ have the same homotopy type.

**Proof.** We shall show that the obvious inclusions $i : \Omega_d M \hookrightarrow \Omega_s M$, $j : \Omega_s M \hookrightarrow \Omega_c M$ and $j \circ i$ are weak homotopy equivalences. Then, since $\Omega_c M$, $\Omega_s M$ and $\Omega_d M \in \text{CW}$, it will follow that they are of the same homotopy type. Firstly, we start with some standard notation and the case of $j \circ i$:

$$I^n := \{(y_0, \ldots, y_{n-1}) \in \mathbb{R}^n : 0 \leq y_i \leq 1\},$$

$$dI^n := \{(y_0, \ldots, y_{n-1}) \in \mathbb{R}^n : y_i = 0 \text{ or } 1 \text{ for some } i\},$$

$$C((X, A), (Y, B)) = \{ f \in C(X, Y) : f(A) \subset B\}.$$

Then $\pi_q(M) = [(I^q, dI^q), (M, m_0)]$. Recall the 1–1 correspondence between the sets of maps

$$\phi : C((I^n, dI^n), (\Omega_c M, m_0)) \rightarrow C((I^{n+1}, dI^{n+1}), (M, m_0)).$$

$$\phi(f)(y_0, y_1, \ldots, y_n) = f(y_1, \ldots, y_n)(y_0)$$
(Here \( m_0 \) denotes both the base point of \( M \) and the constant loop onto it.) It is well known that \( \phi \) descends to an isomorphism on the homotopy groups

\[
\phi_* : \pi_n(\Omega_c M) \cong \pi_{n+1}(M).
\]

Observe also that if \( g \in C((I^{q+1}, dI^{q+1}), (M, m_0)) \) is differentiable then

\[
\phi^{-1}(g) \in C((I^q, dI^q), (\Omega_d M, m_0)).
\]

So now we can show that

\[
(j \circ i)_* : \pi_q(\Omega_d M) \rightarrow \pi_q(\Omega_c M)
\]

is bijective. From 17.8 and 17.8.1 of Bott and Tu, it follows that there is a differentiable map, \( g \), in the homotopy class of \( \phi(f) \) (surjectivity of \( (j \circ i)_* \)) and that any two differentiable maps, \( \phi(f_0) \) and \( \phi(f_1) \) which are continuously homotopic are homotopic via a path of differentiable maps (injectivity of \( (j \circ i)_* \)). This argument also shows that \( j \) is a weak homotopy equivalence and thus so too is \( i \).

7.2 The loop map

If \( X \) and \( Y \) are two spaces (manifolds) and \( f \) is a continuous (differentiable) map \( f : X \rightarrow Y \) then there is a continuous (differentiable) map, the loop of \( f \), denoted \( Lf : LX \rightarrow LY \) where \( \gamma \mapsto f \circ \gamma \). If \( P(M, G, f) \) is a locally trivial principal \( G \)-bundle then \( LP(LM, LG) \) is a locally trivial principal \( LG \)-bundle. Now, we may realise \( EG(BG, G) \) as a smooth principal \( G \)-bundle via the inclusion of \( G \) in \( O(n) \) for some \( n \) and the realisation of the classifying space of \( O(n) \) as the infinite dimensional Steifel manifold (see [28]). It follows that \( LEG \) makes sense for differentiable loops and since \( LEG \) is also a contractible space that

\[ BLG = LBG. \]

Since the homotopy class of a continuous map between manifolds always contains a differentiable map we may take the classifying map of any principal \( G \)-bundle to be differentiable and hence \( LP(LM, LG) \) has classifying map \( Lf \). All of this holds mutatis mutandis for the based loops.

7.3 Transgression

Given two spaces, \( X \) and \( Y \), the slant product (see [28] p. 287) is the product in general (co)homology theories which corresponds to integration over the fibre of \( X \times Y \) in de Rham theory. Let \( \omega \in H^q(X \times Y, \mathbb{Z}) \), \( a \in H_p(X, \mathbb{Z}) \) and \( b \in H_{q-p}(Y, \mathbb{Z}) \) then the slant product:

\[
/ : H^q(X \times Y, \mathbb{Z}) \times H_p(Y, \mathbb{Z}) \rightarrow H^{q-p}(X, \mathbb{Z})
\]

is given by

\[
(\omega/a)(b) = \omega(a \otimes b)
\]
We shall need the following functorial property. Given $f : X \to X', g : Y \to Y'$ and $\omega' \in H^q(X' \times Y', \mathbb{Z})$ then
\[ ((f \times g)^* \omega')/a = f^*(\omega'/g_*a) \tag{7.1} \]
Let $ev : \Omega X \times S^1 \to X$ be the evaluation map and let $i$ be the fundamental class of $H_1(S^1, \mathbb{Z})$. Then the transgression homomorphism between the cohomologies of a space and its loop space is defined as follows.
\[ t^q : H^{q+1}(X, \mathbb{Z}) \to H^q(\Omega X, \mathbb{Z}) \]
\[ \omega \mapsto ev^*(\omega)/i \]
One can easily check that the following diagram commutes.
\[
\begin{array}{ccc}
\Omega X \times S^1 & \xrightarrow{ev} & X \\
\downarrow \Omega f \times Id & & \downarrow \text{id} \\
\Omega X' \times S^1 & \xrightarrow{ev} & X'
\end{array}
\]
By applying (7.1) to $\Omega f \times \text{Id}$ and $\text{Id}$ one sees that
\[ t^q(f^* \omega) = (ev^*(f^* \omega))/i \]
\[ = (\Omega f \times \text{Id})* (ev^*) (\omega)/i \]
\[ = (\Omega f)^*((ev^*) (\omega)/\text{Id}_*i) \]
\[ = (\Omega f)^*t^q(\omega). \tag{7.2} \]
In simple cases, McLaughlin ([17] p 147) has noted that transgressions can be computed using the Hurewicz homomorphism as follows. Given any spaces $X$ and $Y$, let $[X, Y]_0$ denote the set of based homotopy classes of continuous based maps from $X$ to $Y$, then there is a well known bijective, adjoint correspondence (closely related to the correspondence mentioned in Proposition 7.1)
\[ [\Sigma X, Y]_0 \xrightarrow{\Delta} [X, \Omega_c Y]_0 \tag{7.3} \]
which descends in the case that $X = S^{q-1}$ to the isomorphism between the homotopy groups of a space and its loop space,
\[ \delta_q : \pi_q(Y) \cong \pi_{q-1}(\Omega_c Y). \]
In fact, $\delta_q = \partial_q$, the boundary map in the long exact sequence of the continuous path fibration, $P_c Y \to Y$. Now let $\pi : S^{q-1} \times S^1 \to \Sigma S^{q-1}$ be the projection defined by the equivalence relation $(\theta, 1) \sim (\theta', 1)$ and $(\theta_0, t) \sim (\theta_0, t')$ for all $\theta, \theta' \in S^{q-1}$ and for all $t, t' \in S^1$ where $\theta_0$ is the base point of $S^{q-1}$. Then, by the definition of $\Delta$, the following diagram commutes.
\[
\begin{array}{ccc}
S^{q-1} \times S^1 & \xrightarrow{\Delta \times \text{Id}} & \Omega_c X \times S^1 \\
\downarrow \pi & & \downarrow \text{id} \\
\Sigma S^{q-1} & \xrightarrow{f} & X
\end{array}
\]
If \( j \in H_{q-1}(S_{q-1}, \mathbb{Z}) \) is a generator then \( \pi_*(j \otimes i) := k \) generates \( H_q(S_q, \mathbb{Z}) \). Thus for \( \omega \in H^q(X, \mathbb{Z}) \),

\[
\omega(f_*(k)) = \omega(f_*\pi_*(j \otimes i)) = \omega(ev_*(\Delta f)_*j \otimes i) = t^q(\omega)((\Delta f)_*j) \tag{7.4}
\]

In cases where the Hurewicz homomorphism, \( \phi : \pi_{q-1}(\Omega_c X) \to H_{q-1}(\Omega_c X, \mathbb{Z}) \) is surjective and \( H_{q-1}(\Omega_c X, \mathbb{Z}) \) is torsion free, (7.4) will allow us to compute \( t^q \) since in this case a cohomology class \( \omega' \in H_{q-1}(\Omega_c X, \mathbb{Z}) \) is determined by the value it takes on \( \Delta f \) as \( \Delta f \) runs through \( \pi_{q-1}(\Omega_c X) \). We can also use the fact that the continuous and differentiable loop spaces are homotopic (Proposition 7.1) to gain the same result when \( X = M \) is a manifold and we consider differentiable loops (now we must consider a differentiable map, \( g : S^{q-1} \to \Omega_d M \) which is homotopic to \( \Delta f \)).

We can apply this to interpret the transgression homomorphism as the looping of maps when we regard \( H^q(X, \mathbb{Z}) \) as \([X, K(\mathbb{Z}, q)]\). In this case the Hurewicz homomorphism is an isomorphism and if \( 1 \in H^q(K(\mathbb{Z}, q), \mathbb{Z}) \) is a generator then \( \tau^q(1) := 1' \) generates \( H^{q-1}(K(\mathbb{Z}, q-1), \mathbb{Z}) = H^{q-1}(\Omega K(\mathbb{Z}, q-1), \mathbb{Z}) \) and

\[
\tau^q(f^*(1)) = (\Omega f)^*(1'). \tag{7.5}
\]

8 Killingback’s result

In this section we confine our attention to cases where \( G \) is a compact, connected and simply connected Lie group and we consider string structures for smooth bundles with fibre \( \Omega_s G \). We can consider \( \Omega_s G \) and \( \Omega_d G \) interchangeably since the obvious inclusion

\[
\Omega_d G \hookrightarrow \Omega_s G
\]

is a homotopy equivalence. This means that, for a Lie group \( G \), isomorphism classes of \( \Omega_d G \)-bundles, \( \Omega_s G \)-bundles and \( \Omega_c G \)-bundles bundles are in 1–1 correspondence via the obvious bundle inclusions. Thus, the problem of finding a string structure is identical in the case of \( \Omega_d G \) and \( \Omega_s G \) as the following commutative diagram makes clear.

\[
\begin{array}{ccc}
H^1(M, \Omega_s G) & \cong & H^1(M, \Omega_d G) \\
D \downarrow & & D \downarrow \\
H^2(M, S^1) & \cong & H^2(M, S^1)
\end{array}
\]

We see that for a principal \( G \)-bundle, \( P(M, G) \), over a manifold, \( M \), \( D[\Omega_s P] = 0 \) if and only if \( D[\Omega_d P] = 0 \). This links the work of Carey and Murray [3] and McLaughlin [17]. Moreover since \( LG \) is homeomorphic to \( \Omega G \times G \) we need only consider based loops when
$G$ is simply connected for then $H^i(G,\mathbb{Z}) = 0$ for $i = 1, 2$ and the canonical projection
\( \phi: LG = \Omega G \times G \to \Omega G \), induces an isomorphism
\[
\phi^*: H^2(\Omega G,\mathbb{Z}) \cong H^2(LG,\mathbb{Z}).
\]
The correspondence between circle bundles and second integral cohomology entails,
\[
\hat{L}_s(L_sG, S^1) = \phi^*\Omega_sG(\Omega_sG, S^1) = \Omega_sG(\Omega_sG, S^1) \times G.
\]
Now note that for any topological groups, $G$ and $H$, $H^1(M, G \times H) = H^1(M, G) \times H^1(M, H)$. The following commutative diagram shows that $L_sP$ has a string structure if and only if $\Omega_sP$ has one (the first two vertical arrows are the obvious projections and $\rho$ is the map induced on cohomology from the projection $\rho: \Omega_sG \to \Omega_sG$).

\[
\begin{array}{ccc}
H^1(M, \Omega_sG) \times H^1(M, G) & \xrightarrow{\rho \times \text{Id}} & H^1(M, \Omega_sG) \times H^1(M, G) \\
\downarrow & & \downarrow \\
H^1(M, \Omega_sG) & \xrightarrow{\rho} & H^1(M, \Omega_sG)
\end{array}
\]

Let us now turn to the general situation for $\Omega_sG$. Start with a principal $SO(n)$–bundle, $P(M, SO(n), f)$ ($n > 4$), (typically $P$ is the frame bundle of the tangent bundle of a Spin manifold $M$) that has a $Spin(n)$–structure $Q(M, Spin(n), \hat{f})$ and form the loop bundle $\Omega_sQ(\Omega_sM, \Omega_sSpin(n), \Omega_s\hat{f})$. Now, realise $B\Omega_sSpin(n)$ as $\Omega_sBSpin(n)$. Since $Spin(n)$ is two-connected with $\pi_3(Spin(n)) \cong \mathbb{Z}$, $BSpin(n)$ is three-connected and $H^4(BSpin(n),\mathbb{Z}) \cong \mathbb{Z}$. Thus (7.4) gives us that
\[
t^4: H^4(BSpin(n), \mathbb{Z}) \to H^3(\Omega_sBSpin(n), \mathbb{Z})
\]
is an isomorphism so choose $\omega \in H^4(BSpin(n), \mathbb{Z})$, a generator, so that $t^4(\omega) = \mu$, the universal Dixmier-Douady class. So,
\[
D[L_sQ] = (\Omega_s\hat{f})^*\mu \\
= (\Omega_s\hat{f})^*t^4(\omega) \\
= t^4(\hat{f}^*\omega) \text{ by (7.2)}. 
\]
McLaughlin [17] in his Lemma 2.2 shows by analysing the spectral sequence of the bundle $BSpin(n)(BSO(n), B\mathbb{Z}_2)$ that for $n > 4$
\[
2\hat{f}^*(\omega) = P_1(P),
\]
where $P_1(P)$ is the first Pontryagin class of $P$. Thus
\[
2D[L_sQ] = t^4(P_1(P))
\]
which is Killingback’s result. Now (7.4) entails that $t^4$ is injective for $M$ ($q-2$)-connected and hence the vanishing of $(1/2)P_1(P)$ is a necessary and sufficient for the existence of a string structure if $M$ is two-connected, and merely a sufficient condition in general.
9 The restricted unitary group

We start with a separable Hilbert space $H = H^+ + H_-$ decomposed by infinite dimensional subspaces $H^+$ and $H_-$ which are the range of the self adjoint projections $P^+$ and $P^-$ respectively, $Id_H = P^+ + P^-$. The restricted unitary group relative to a polarisation is defined by

$$U_{res}(H, P^+) = \{ u \in U(H) : P^\pm u P^\mp \text{ is Hilbert Schmidt} \}.$$ 

Now $U_{res}$ is not equipped with the subspace topology from $U(H)$ but with its own topology coming from the metric $\rho$.

$$\rho(u_1, u_2) = ||P^+(u_1 - u_2)P^+|| + ||P^-(u_1 - u_2)P^-|| + |P^+(u_1 - u_2)P^-|_{HS} + |P^-(u_1 - u_2)P^+|_{HS}$$

Where $| |_{HS}$ denotes the symmetric norm on the Hilbert–Schmidts. Typically the Hilbert space and polarisation are understood and omitted from the notation. If $(\cdot, \cdot)$ denotes the inner product on $H$, then the CAR (canonical anti-commutation relations) algebra over $H$, $CAR(H)$ is the $C^*$–algebra generated by the set

$$\{ a(f), a^*(f), f \in H \}$$

whose elements satisfy the canonical anti-commutation relations

$$a(f).a(g) + a(g)a(f) = 0$$

$$a(f).a^*(g) + a(g^*).a(f) = (f, g).$$

Any unitary $u \in U(H)$ allows one to define an automorphism of $CAR(H)$ (called a Bogoliubov transformation) by

$$\alpha_u((a(f)) = a(u.f) \quad \alpha_u((a^*(f)) = a^*(u.f)$$

An irreducible (Fock) representation $\pi$ of $CAR(H)$ is determined via the GNS construction from the state $\omega$ defined by

$$\omega(a^*(f_1)\ldots a^*(f_M)a(g_N)\ldots a(g_1) = \delta_{MN}\det(g_i, P^-f_j).$$

The result we need, due originally to Friedrichs, is the theorem (see [24]) that, given a Bogoliubov transformation $\alpha_u$, there exists a unitary $W(u) \in U(H_\pi)$ such that

$$\pi(\alpha_u(a(f))) = \pi(a(u.f)) = Ad(W(u))(\pi(a(f)) = W(u)\pi(a(f))W(u)^*$$

iff $u \in U_{res}(H)$. Since $\pi$ is irreducible, $W(u)$, is uniquely defined up to a scalar which is killed by the adjoint. Hence the above defines an embedding

$$i : U_{res} \hookrightarrow \mathcal{PU}(H_\pi)$$

of the restricted unitaries of $H$ in the projective unitaries on $H_\pi$. It is a corollary of a proof of (Carey 1984 Lemma 2.10) that this embedding is closed in $\mathcal{PU}(H_\pi)$. Furthermore
we shall see below that $H^2(U_{res}, \mathbb{Z}) = \mathbb{Z}$ and the canonical central extension of $U_{res}$, $\hat{U}_{res}$, defined by the generator of $H^2(U_{res}, \mathbb{Z})$ is given by

$$U_{res}(U_{res}, S^1) = i^*U(H_{\pi})(PU(H_{\pi}), S^1).$$

Hence the assumptions of Section 3 are fulfilled. Finally, note that $U_{res}$ is a disconnected group with connected components labelled by the Fredholm index of $P^+UP^+$. We denote the connected component of the identity by $U_{res}^0$. Henceforth we drop reference to the different Hilbert spaces over which $U_{res}$ and $PU$ are defined and it shall be understood that $PU$ refers to the projective unitaries on $H_{\pi}$ and not $H$.

We now summarise the homotopy properties of $U_{res}$ and its role as a classifying space for $U(\infty)$ and the relation between $U_{res}$ and $PU$ bundles.

## 10 $U_{res}$ as a classifying space

The group of unitaries with determinant, say $T$, consists of those operators of the form $1+\text{trace class}$. By considering $T(H^+)$, Pressley and Segal [22] (see Ch 6) show that there is a principal $T$–bundle over $U_{res}^0$, the connected component of $U_{res}$ with contractible total space and hence $U_{res}^0$ is a $BT$. It is known that $T$ has homotopy type of the direct limit of the finite unitaries.

$$T \simeq U(\infty) = \lim_{n \to \infty} (U(n))$$

So $U_{res}^0$ is a $CW$–classifying space for $T$ and thus $U(\infty)$. So we have $U_{res}^0 \simeq BT$. Since the homotopy groups of $U(\infty)$ are well known by Bott periodicity we have that

$$\pi_q(U_{res}) = \begin{cases} \mathbb{Z}, & q \text{ even,} \\ 0, & q \text{ odd.} \end{cases}$$

(This result has elsewhere been proven via methods more closely tied to $U_{res}$’s structure as a group of operators, see Carey (1983).) Now $U(\infty)$ and $BU(\infty)$ are classifying spaces for reduced $K$–theory and we have:

**Proposition 10.1.**

$$BU_{res} \simeq U(\infty) \quad U_{res} \simeq \Omega_cU(\infty).$$

**Proof.** It is known that the embedding of $\Omega_dU(n) \hookrightarrow U_{res}$ extends to a map $i: \Omega_dU(\infty) \hookrightarrow U_{res}$ and one can check that this is a weak homotopy equivalence and hence a homotopy equivalence. By Proposition 7.1, $\Omega_dU(\infty) \simeq \Omega_cU(\infty)$ and thus, remembering that via the path fibration $B\Omega_cG \simeq G^0$,

$$BU_{res} \simeq B\Omega_cU(\infty) = U(\infty).$$

If we loop this equation we find,

$$\Omega_cU(\infty) \simeq \Omega_cBU_{res} \simeq B\Omega_cU_{res} \simeq U_{res}.$$
Note 10.1. Over the category of CW-complexes of dimension less than a given integer, $CW_n$, and over the category of finite CW-complexes, $CW_{fin}$, the functors of reduced $K$-theory have $BU(\infty)$ as a classifying space (see [14] p118). If follows that isomorphism classes of $U_{res}$-bundles correspond bijectively with elements of reduced $K$-theory. Specifically, $\tilde{K}^1(X)$ of a base is defined to be the stable isomorphism classes of vector bundles over the reduced suspension of $X$, $\Sigma X$. For $X \in CW_n$ or $CW_{fin}$

\[
\tilde{K}^1(X) = [\Sigma X, BU(\infty)] \\
= [X, \Omega_c BU(\infty)] \text{ apply } \Delta \\
= [X, BO, U(\infty)] \\
= [X, BU_{res}] \\
= Bun_X(U_{res})
\]

where $Bun_X(U_{res})$ denotes the set of all isomorphism classes of $U_{res}$ bundles over $X$. Now elements of $\tilde{K}$ correspond bijectively with $U(\infty)$-bundles, $U_{res}$-bundles correspond with $\Omega_c U(\infty)$ bundles. So our correspondence can be seen as a mapping between $\Omega_c U(\infty)$-bundles over a space and $U(\infty)$-bundles over the reduced suspension of that space which is attained by applying $\Delta$ or $\Delta^{-1}$ to the classifying maps of the bundles. We exploit this in the next subsection.

11 The Dixmier-Douady class and the second Chern class

Regarding $U_{res}$ as a subgroup of $PU$ via the inclusion mentioned in Section [6], we may ask when can we reduce the structure group of a $PU$–bundle, $P(\mathcal{PU}, M, f)$ to $U_{res}$? By Theorem [3] we translate this question into a search for maps $\hat{f}$ such that $f = g \circ \hat{f}$. Where we take $g : BU_{res} \to BPU$ to be a fibration with fibre $F$.

\[
BU_{res} \simeq U(\infty) \\
g \downarrow \\
X \xrightarrow{f} BPU \simeq K(\mathbb{Z}, 3)
\]

In general we know that if there were a section of $\pi$, say $s$, then this would entail the existence of group homomorphisms

\[
g^* : H^*(BPU, \mathbb{Z}) \to H^*(BU_{res}, \mathbb{Z}) \\
s^* : H^*(BU_{res}, \mathbb{Z}) \to H^*(BPU, \mathbb{Z})
\]

such that

\[
s^* \circ g^* = (g \circ s)^* = id.
\]

It is a group theoretic result that this implies that $H^*(BPU, \mathbb{Z})$ would be a direct summand of $H^*(BU_{res}, \mathbb{Z})$. But we know (See Bott and Tu pp 245–246) that $H^*(BPU, \mathbb{Z})$ has torsion
whereas $H^*(BU_{res}, \mathbb{Z})$ is a free group. Therefore the sought after section cannot exist and the structure group of some $PU$–bundles does not reduce to $U_{res}$.

The situation in specific instances depends in part on the homotopy groups of the fiber, which we can compute in this case by noting that

$$i_\ast : \pi_q(U_{res}) \rightarrow \pi_q(\mathcal{P}U)$$

is an isomorphism for $q = 2$ and null otherwise. It follows by Lemma 3.1 that $g_\ast : \pi_q(BU_{res}) \rightarrow \pi_q(BPU)$ is an isomorphism for $q = 3$ and null otherwise. By considering the long exact homotopy sequence of the fibration

$$F \rightarrow BH \xrightarrow{g} BG$$

we see that

$$\pi_q(F) = \begin{cases} \mathbb{Z}, & q \text{ odd} \\ 0, & q \text{ even or 3.} \end{cases}$$

Now the cohomology, $H^n(K(\mathbb{Z}, 3))$, of $K(\mathbb{Z}, 3)$ is zero for $n = 1$ and torsion for $n > 3$ (see Bott and Tu pp 245–246). Hence obstructions to lifting $f$ can lie only in $H^{2n+4}(M, \mathbb{Z})$ ($n \geq 1$). So the structure group of any $PU$–bundle over a space with free, even (greater than fourth) cohomology groups reduces to $U_{res}$.

We recast this problem in a more general setting by exploiting the correspondence between $U_{res}$-bundles over a space $x$ and $\tilde{K}^1(X)$. There is a suspension isomorphism on cohomology,

$$\Sigma^q : H^q(X, \mathbb{Z}) \cong H^{q+1}(\Sigma X, \mathbb{Z})$$

which one can obtain from the Mayer-Vietoris sequence for $(\Sigma X, CX, CX)$ (where “$CX$” denotes the reduced cone of $X$) or by using the adjoint relation, $\Delta$ (see 7.3) between $\Sigma$ and $\Omega_c$ considered as functors on $CW$:

$$X \xrightarrow{\Delta f} \Omega_c K(\mathbb{Z}, q + 1) = K(\mathbb{Z}, q) \longleftrightarrow \Sigma X \xrightarrow{f} K(\mathbb{Z}, q + 1).$$

If $1$ and $1'$ are as in (7.5) then

$$\Sigma^q((\Delta f)^*(1')) = f^*(1). \quad (11.1)$$

The next proposition uses the suspension isomorphism and the transgression homomorphism to link characteristic classes of principal $U_{res}$–bundles over with the characteristic classes of the associated $U(\infty)$–bundles over $\Sigma X$.

**Proposition 11.1.** Let $c$ be a characteristic class for principal $U(\infty)$-bundles defined by its universal class $c^* \in H^{q+1}(BU(\infty), \mathbb{Z})$ and let $t^q(c)$ be the characteristic class for principal $U_{res}$-bundles with universal class $t^q(c^*)$. If $P(X, U_{res}, \Delta f)$ is a principal $U_{res}$–bundle over $X$ and

$$\Sigma P(\Sigma X, U(\infty), f))$$

is the associated $U(\infty)$–bundle over $\Sigma X$, $\xi(\text{element of } \tilde{K}^1(X))$ then

$$c(\Sigma P) = \Sigma^q(t^q(c)(P)).$$
Proof. Let $c$ also denote a map $c : BU(\infty) \to K(\mathbb{Z}, q+1)$ which pulls back $1 \in K(\mathbb{Z}, q+1)$ to $c^*$. We must show that $f^*(c^*) = \Sigma^q((\Delta f)^*(t^q(c^*)))$. By applying Proposition \[10.1\], realise $BU_{res}$ as $\Omega_c BU(\infty)$. Thus we may exploit the adjoint pairing $\Delta$ between the functors $\Omega_c$ and $\Sigma$. We have the following maps

$$\begin{align*}
\Sigma X & \xrightarrow{f} BU(\infty) \xrightarrow{c} K(\mathbb{Z}, q+1) \\
X & \xrightarrow{\Delta f} \Omega_c BU(\infty) \xrightarrow{\Omega_c c} \Omega_c K(\mathbb{Z}, q+1)
\end{align*}$$

with $\Delta(c \circ f) = \Omega_c c \circ \Delta f$. Now by \[7.3\], $(\Omega_c c)^*(1') = t^q(c^*)$ and thus

$$\begin{align*}
\Sigma^q((\Delta f)^*(t^q(c^*))) &= \Sigma^q((\Delta f)^*(\Omega_c c)^*(1')) \\
&= \Sigma^q(\Delta(c \circ f)^*(1')) \\
&= (c \circ f)^*(1) \text{ by } \[1.1\] \\
&= f^*(c^*)
\end{align*}$$

and the proposition is proved.

\[\square\]

**Proposition 11.2.** Let $D$ be the Dixmier-Douady class for principal $U_{res}$-bundles, let $c_2$ be the second Chern class for $U(\infty)$-bundles and let $P$ and $\Sigma P$ be as above. Then

$$\Sigma^q(D(P)) = c_2(\Sigma P)$$

Proof. Let $D^* \in H^3(BU_{res}, \mathbb{Z})$ and $c_2^* \in H^4(BU(\infty, \mathbb{Z}))$ denote the universal classes of $D$ and $c_2$ respectively, then by Proposition \[1.1\] it suffices to show that

$$t^3(c_2^*) = D^*.$$ 

Using \[14\] (Ch 20, Corollary 9.8) one deduces that there is a $U(\infty)$-bundle over $S^4$, $P(S^4, U(\infty), f)$, with $c_2(P)$ a generator of $H^4(S^4, \mathbb{Z})$. Let $k$ denote the generator of $H_3(S^4, \mathbb{Z})$ such that $1 = c_2(P)(k) = f^*(c_2^*)(k)$ and let $j$ be the corresponding generator of $H_3(S^3, \mathbb{Z})$ (in the sense of \[7.4\]). Then by \[7.4\]

$$t^3(c_2^*)((\Delta f)_*j) = f^*(c_2^*)(k) = 1.$$ 

But one can show by considering long exact sequence of the fibration

$$U(n)(S^{2n-1}, U(n-1))$$

(n large) that the Hurewicz homomorphism is an isomorphism on

$$\pi_3(BU_{res}) \cong \pi_3(U(\infty)) \cong \mathbb{Z}.$$ 

Hence, $t^3(c_2^*)$ generates $H^3(BU_{res}, \mathbb{Z})$ (as it evaluates to 1 on the generator of $H_3(BU_{res}, \mathbb{Z})$) and so $t^3(c_2^*) = D^*$ as required. \[\square\]

In summary, the structure group of a $PU$-bundle, $Q(X, PU)$ reduces to $U_{res}$ if and only if there is a $U_{res}$ bundle, $P(X, U_{res})$ whose Dixmier-Douady class coincides with that of $Q$. This, we have just seen, happens if and only if there is a $U(\infty)$-bundle, $\Sigma P(S^X, U(\infty))$ over $\Sigma X$ such that $c_2(\Sigma P) = \Sigma^3(D(P))$. We know from above that one cannot, in general, construct a $U(\infty)$-bundle with an arbitrary second Chern class on any given space. This differs from the case for the first Chern class where one can always find a line bundle, and hence a $U(\infty)$-bundle, for any given element of $H^2(M, \mathbb{Z})$. 

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12 Connections with other viewpoints

12.1 Bundle gerbes

An alternative method of defining the obstruction to lifting a bundle to a central extension is to use the notion of bundle gerbes \[\text{[20]}.\] We will sketch the theory here and refer the reader to \[\text{[20]}\] for details. If \(Y \to M\) is a fibration define \(Y^{[p]}\) to be the \(p\)th fibre product of \(Y\) with itself. Then a bundle gerbe over \(M\) is a pair \((J, Y)\) where \(\pi: Y \to M\) is a fibration and \(J \to Y^{[2]}\) is a \(U(1)\) bundle. Furthermore for any \(x, y\) and \(z\) in \(Y\) we require the existence of a bundle morphism, called the bundle gerbe product,

\[J_{(x,y)} \otimes J_{(y,z)} \to J_{(x,z)}\]

depending continuously or smoothly on \(x, y\) and \(z\). Moreover this composition is required to be associative. Note that for \(U(1)\) principal bundles there is a natural notion of tensor product and dual, see \[\text{[20]}\] for details.

If \(L \to Y\) is a \(U(1)\) bundle then we can define a bundle gerbe \((Y, \delta(L))\) by

\[\delta(Y)_{(x,y)} = L_x \otimes L_y^*.\]

A bundle gerbe is called trivial if it is isomorphic to a bundle gerbe of the form \(\delta(L)\). The obstruction to a bundle gerbe \((J, Y)\) over \(M\) being trivial is a three class in \(H^3(M, \mathbb{Z})\) called the Dixmier-Douady class of the bundle gerbe. Its definition can be found in \[\text{[20]}\].

The connection with our work is the bundle gerbe arising as the obstruction to extending the structure group of a \(G\) bundle \(P\) to \(\hat{G}\) where

\[0 \to U(1) \to \hat{G} \to G \to 0\]

is a central extension. Note that if we form the fibre product \(P^{[2]}\) there is a map \(\sigma: P^{[2]} \to G\) defined by \(p = q \sigma(p, q)\). We define \(J = \sigma^*(\hat{G})\) where here we think of \(\hat{G}\) as a \(U(1)\) bundle over \(G\). It is easy to check that the group multiplication in \(\hat{G}\) defines the required bundle gerbe product. It is shown in \[\text{[20]}\] that

**Theorem 12.1** \([\text{[20]}]\). The bundle gerbe \(L\) is trivial if and only if the bundle \(P\) lifts to \(\hat{G}\). The Dixmier-Douady class of \(L\) is the same Dixmier-Douady class which is the obstruction to the bundle \(P\) lifting to \(\hat{G}\).

12.2 The Dixmier-Douady class and Clifford bundles

We now interpret the Dixmier-Douady class as an obstruction in a different setting which is closer in spirit to that of the original (cf \[\text{[10]}\]). Suppose we have a principal fibre bundle \(P(M, \mathcal{U}_{res})\) and a locally finite cover \(\{U_\beta | \beta \in A\}\) of \(M\). The transition functions \(g_{\beta \gamma}\) may be used to define the transition functions for a locally trivial bundle over \(M\) with fibre the CAR algebra. This is achieved by defining automorphisms of the CAR algebra by \(u_{\beta \gamma}(a(v)) = a(g_{\beta \gamma}v)\) \((v \in \mathcal{H})\) and using the \(u_{\alpha \beta}\) as transition functions for a fibre bundle \(C(M, CAR(\mathcal{H}))\). If the Dixmier-Douady class of \(P(M, \mathcal{U}_{res})\) is trivial then one can find unitaries

\[\{W(u_{\beta \gamma}) | \beta, \gamma \in A\}\]
acting on the Hilbert space $H_\pi$ of $\pi$ which form a Cech 2-cocycle with values in the unitaries on $H_\pi$. Using these as transition functions one defines a 'Fock bundle' over $M$ with fibre the Fock space $H_\pi$. Thus the Dixmier-Douady class of $P(M,\mathcal{U}_{res})$ is an obstruction to the existence of a locally trivial bundle over $M$ with fibre the Fock space and on sections of which the Clifford bundle (as a field of $C^*$-algebras) acts. This is analogous to the original introduction of the Dixmier-Douady class as an obstruction to the triviality of a bundle of $C^*$-algebras with fibre the compact operators.

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References

[1] R. BOTT and L. TU: Differential Forms in Algebraic Topology, Springer–Verlag New York (1982).

[2] J.-L. BRYLINSKI: Loop spaces, Characteristic classes and Geometric Quantization. Birkhäuser, Berlin, (1992).

[3] A.L.CAREY: Infinite Dimensional Groups and Quantum Field Theory, Acta Applicandae Mathematicae 1 (1983) 321-332.

[4] A.L.CAREY and D.M.O’BRIEN: Automorphisms of the Infinite Dimensional Clifford Algebra and The Atiyah-Singer Mod 2 Index, Topology 22(1983) 437-448.

[5] A.L.CAREY: Some Infinite Dimensional Groups and Bundles, Publications of the Research Institute for Mathematical Sciences Kyoto University 20 (1984) 1103-1117.

[6] A.L.CAREY and M.K.MURRAY: String Structures and the Path Fibration of a Group, Commun Math Phys 141,(1991) 441-452.

[7] A.L.CAREY and M.K.MURRAY: Faddeev’s anomaly and bundle gerbes, Letters in Math. Phys. 37 (1996), 29-36

[8] A.L.CAREY, J. MICHELSSON and M.K.MURRAY: Index theory, Gerbes and Hamiltonian quantisation” Commun Math Phys to appear (1997)

[9] R.COUEREAUX and K.PILCH: String Structures on Loop Bundles, Commun Math PhysS 120 (1989) 353-378.

[10] J.DIXMIER: $C^*$–algebras, North Holland Amsterdam (1977).

[11] L. FADDEEV and S. SHATASVILI: Algebraic and hamiltonian methods in the theory of nonabelian anomalies. Theoret. Math. Phys. 60, 770 (1985) and J. MICHELSSON: Chiral anomalies in even and odd dimensions. Commun. Math. Phys. 97, 361 (1985)
[12] J. FRENKEL: Cohomologie non Abeliene et Espaces Fibres, Bulletin de la Societe de Mathematique Francais (1957) 85 135-220.

[13] L. GROSS, On the formula of Mathews and Salam, J. Funct Analysis. 25 (1977) 162 - 209.

[14] D. HUSEMOLLER: Fibre Bundles, McGraw-Hill, New York (1966).

[15] T. P. KILLINGBACK: World-sheet Anomalies and Loop Geometry, Nuclear Physics (1987) B288 578-588.

[16] N. KUIPER: Contractibility of the Unitary Group in Hilbert Space, Topology 3 (1964) 19-30.

[17] D. A. MCLAUGHLIN: Orientation and String Structures on Loop Space, Pac. J. Math. (1992) 155 1-31.

[18] E. MICHAEL: in Set-valued mappings, selections and topological properties of $2^X$; proceedings of the conference held at the State University of New York at Buffalo, May 8-10, 1969. Edited by W. M. Fleischman, Springer-Verlag, (1970).

[19] R. E. MOSHER and M. C. TANGORA: Cohomology operations and applications in homotopy theory, Harper & Row, New York, 1968.

[20] M. K. MURRAY: Bundle Gerbes J. London Math. Soc. (2) 54 (1996) 403–416.

[21] M. K. MURRAY and D. STEVENSON: A universal bundle gerbe (preprint).

[22] A. PRESSLEY and G. SEGAL: Loop Groups, Clarendon Press, Oxford (1986).

[23] G. SEGAL: Faddeev’s anomaly in Gauss’ law, unpublished preprint (1985)

[24] D. SHALE and W. F. STINESPRING: Spinor Representations of Infinite Orthogonal Groups, J. Math. Mech. (1965) 14, 315-322.

[25] E. H. SPANIER: Algebraic Topology, McGraw-Hill (1966).

[26] N. STEENROD: The Topology of Fibre Bundles, Princeton University Press (1951).

[27] N. STEENROD: Milgram's Classifying Space of a Topological Group, Topology 7 (1968), 349-68.

[28] G. W. WHITEHEAD: Elements of Homotopy Theory, Springer-Verlag (1978).

1. Department of Pure Mathematics, University of Adelaide, Adelaide, South Australia 5005, Australia. acrey@maths.adelaide.edu.au, mmurray@maths.adelaide.edu.au

2. Department of Mathematics, Indiana University, Bloomington, IN 47405-4301, USA. dcrowley@iu-math.math.indiana.edu