On the Liouville type theorem for stationary compressible Navier-Stokes-Poisson equations in $\mathbb{R}^N$

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Abstract

In this paper we prove Liouville type result for the stationary solutions to the compressible Navier-Stokes-Poisson equations (NSP) and the compressible Navier-Stokes equations (NS) in $\mathbb{R}^N$, $N \geq 2$. Assuming suitable integrability and the uniform boundedness conditions for the solutions we are led to the conclusion that $v = 0$. In the case of (NS) we deduce that the similar integrability conditions imply $v = 0$ and $\rho =$ constant on $\mathbb{R}^N$. This shows that if we impose the non-vacuum boundary condition at spatial infinity for (NS), $v \to 0$ and $\rho \to \rho_\infty > 0$, then $v = 0$, $\rho = \rho_\infty$ are the solutions.

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1 Introduction

We are concerned on the stationary Navier-Stokes-Poisson equations in $\mathbb{R}^N$, $N \geq 2$.

\begin{align}
\text{div}(\rho v) &= 0, \\
\text{div}(\rho v \otimes v) &= -\nabla p + k \rho \nabla \Phi + \mu \Delta v + (\mu + \lambda) \nabla \text{div} v, \\
\Delta \Phi &= \rho - \rho_0, \\
p &= a \rho^\gamma, \quad \gamma > 1.
\end{align}

The system (NSP) describes compressible gas flows, and $\rho, v, \Phi$ and $p$ denote the density, velocity, the potential of the underlying force and the pressure respectively. The constant $\rho_0 \geq 0$ is called the background state. The viscosity constants $\lambda, \mu$ satisfy $\mu > 0, \lambda + \mu > 0$. Here $k$ is also a physical constant, which signifies the property of the forcing, repulsive if $k > 0$ and attractive if $k < 0$. In particular, in the case $k = 0$ (and without (1.3)) the system reduces to the following stationary compressible Navier-Stokes equations (NS).

\begin{align}
\text{div}(\rho v) &= 0, \\
\text{div}(\rho v \otimes v) &= -\nabla p + \mu \Delta v + (\mu + \lambda) \nabla \text{div} v, \\
p &= a \rho^\gamma, \quad \gamma > 1.
\end{align}

For the general treatment of the system (NSP) and (NS), including the Cauchy problem of the time dependent equations we refer e.g. [2, 4, 5, 6]. Here we prove a Liouville type theorem of the stationary systems (NSP) and (NS) as follows.

**Theorem 1.1** Let $N \geq 2$.

(i) (Navier-Stokes Poisson equations): Suppose $(\rho, v, \Phi)$ is a smooth solution to (NSP) with $k \neq 0$, satisfying

\begin{equation}
\|\rho\|_{L^\infty} + \|\nabla \rho\|_{L^\infty} + \|\nabla v\|_{L^2} + \|v\|_{L^\infty}^{\frac{N}{N-1}} < \infty,
\end{equation}

and the additional condition

\begin{equation}
v \in L^\infty(\mathbb{R}^N)
\end{equation}

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if $N \geq 7$. Then, $v = 0$ on $\mathbb{R}^N$. The functions $\Phi$ and $\rho$ are determined by solving the system:

$$
\begin{align*}
\rho \nabla \Phi &= \frac{a}{k} \nabla \rho^\gamma, \\
\Delta \Phi &= \rho - \rho_0.
\end{align*}
$$

(ii) (Navier-Stokes equations): Suppose $(\rho, v)$ is a smooth solution to (NS) satisfying

$$
\|\rho\|_{L^\infty} + \|\nabla v\|_{L^2} + \|v\|_{L^{\frac{N}{N-1}}} < \infty,
$$

and the additional condition (1.9) if $N \geq 7$. Then, $v = 0$ and $\rho =$ constant on $\mathbb{R}^N$.

Remark 1.1 Let us compare the above theorem with the previous result in [1], which says that a solution to (NS) satisfying

$$
\|\sqrt{\rho}v\|_{L^2} + \|v\|_{L^{\frac{N}{N-1}}} + \|\rho^\gamma\|_{L^1} < \infty
$$

is vacuum, $\rho = 0$. The main difference between (1.12) and (1.11) is that in (1.12) we are assuming the spatial infinity is vacuum,

$$
\rho(x) \to 0 \text{ as } |x| \to \infty,
$$

while in (1.11) it is allowed to have

$$
\rho(x) \to \rho_\infty \text{ as } |x| \to \infty
$$

for a constant $\rho_\infty > 0$. Even with such non-vacuum condition at infinity the Liouville type result holds. In particular for the 2D stationary Navier-Stokes equations if $v \in H^1(\mathbb{R}^2)$ and $\rho$ satisfies (1.14), then $v = 0$ and $\rho = \rho_\infty$ on $\mathbb{R}^2$.

2 Proof of Theorem 1.1

Proof of Theorem 1.1 Let us consider a radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$
\sigma(|x|) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 2,
\end{cases}
$$

(2.1)
and \(0 \leq \sigma(x) \leq 1\) for \(1 < |x| < 2\). Then, for each \(R > 0\), we define
\[
\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).
\] (2.2)

We multiply (1.2) by \(v\sigma_R(x)\), and integrate over \(\mathbb{R}^N\), and integrate by part to obtain,
\[
\mu \int_{\mathbb{R}^N} |\nabla v|^2 \sigma_R \, dx + (\mu + \lambda) \int_{\mathbb{R}^N} (\text{div } v)^2 \sigma_R \, dx
\]
\[=
-\mu \int_{\mathbb{R}^N} v \cdot (\nabla \sigma_R \cdot \nabla) v \, dx - (\mu + \lambda) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \text{div } v(v \cdot \nabla) \sigma_R \, dx
\]
\[+ k \int_{\mathbb{R}^N} \sigma_R \rho(v \cdot \nabla) \Phi \, dx - \int_{\mathbb{R}^N} \sigma_R (v \cdot \nabla) p \, dx - \int_{\mathbb{R}^N} \sigma_R v \cdot \text{div } (\rho v \otimes v) \, dx
\]:= \(I_1 + \cdots + I_5\). (2.3)

Let us estimate \(I_1, \cdots, I_5\) term by term.
\[
|I_1| \leq \frac{\mu}{R} \int_{\{R \leq |x| \leq 2R\}} |\nabla \sigma| |v| |\nabla v| \, dx
\]
\[\leq \frac{\mu \|\nabla \sigma\|_{L^\infty}}{R} \left( \int_{\{R \leq |x| \leq 2R\}} dx \right)^{\frac{1}{N}} \|v\|_{L^\infty (R \leq |x| \leq 2R)} \|\nabla v\|_{L^2 (R \leq |x| \leq 2R)}
\]
\[\leq C \|\nabla \sigma\|_{L^\infty} \|\nabla v\|_{L^2} \|\nabla v\|_{L^2 (R \leq |x| \leq 2R)} \rightarrow 0\] (2.4)
as \(R \rightarrow \infty\), where we used the Sobolev inequality, \(\|v\|_{L^\infty} \leq C \|\nabla v\|_{L^2}\). Estimate of \(I_2\) is similar to \(I_1\) and we have
\[
|I_2| \leq \frac{\mu + \lambda}{R} \int_{\{R \leq |x| \leq 2R\}} |\nabla \sigma| |v| |\nabla v| \, dx
\]
\[\leq C \|\nabla \sigma\|_{L^\infty} \|\nabla v\|_{L^2} \|\nabla v\|_{L^2 (R \leq |x| \leq 2R)} \rightarrow 0\] (2.5)
as \(R \rightarrow \infty\). In order to estimate \(I_3\) we first integrate by part to obtain
\[
I_3 = -k \int_{\mathbb{R}^N} \sigma_R \text{div } (\rho v) \Phi \, dx - k \int_{\mathbb{R}^N} \Phi \rho(v \cdot \nabla) \sigma_R \, dx
\]
\[= -k \int_{\mathbb{R}^N} \Phi \rho(v \cdot \nabla) \sigma_R \, dx,
\]
where we used (1.1). Therefore,

\[
|I_3| \leq \frac{k}{R} \|\nabla \sigma\|_{L^{\infty}} \|\Phi\|_{L^{\infty}} \|\rho\|_{L^{\infty}} \int_{\{R \leq |x| \leq 2R\}} |v| \, dx \\
\leq C \|\nabla \sigma\|_{L^{\infty}} \|\Phi\|_{L^{\infty}} \|\rho\|_{L^{\infty}} \|v\|_{L^{\frac{N}{N-1}}(R \leq |x| \leq 2R)} \rightarrow 0 \quad (2.6)
\]
as \( R \rightarrow \infty \). In order to estimate \( I_4 \) we write the pressure term in the following form:

\[
\nabla p = a \nabla \rho^\gamma = \frac{a \gamma}{\gamma - 1} \rho \nabla \rho^\gamma - 1.
\]

Then, we have

\[
I_4 = -a \frac{\gamma}{\gamma - 1} \int_{\mathbb{R}^N} \sigma_R \rho \nabla \rho^\gamma \, dx
\]

\[
= \frac{a \gamma}{\gamma - 1} \int_{\mathbb{R}^N} \sigma_R \nabla \rho \rho^\gamma \, dx + \frac{a \gamma}{\gamma - 1} \int_{\mathbb{R}^N} \rho^\gamma (v \cdot \nabla) \sigma_R \, dx
\]

\[
= \frac{a \gamma}{\gamma - 1} \int_{\mathbb{R}^N} \rho^\gamma (v \cdot \nabla) \sigma_R \, dx
\]

thanks to (1.1). Thus, we estimate

\[
|I_4| \leq \frac{a \gamma}{(\gamma - 1)R} \|\rho\|_{L^{\infty}} \int_{\{R \leq |x| \leq 2R\}} |v| \|\nabla \sigma\| \, dx \\
\leq C \|\nabla \sigma\|_{L^{\infty}} \|\rho\|_{L^{\infty}} \|v\|_{L^{\frac{N}{N-1}}(R \leq |x| \leq 2R)} \rightarrow 0 \quad (2.7)
\]
as \( R \rightarrow \infty \). For the term \( I_5 \) we first compute by integration by part

\[
I_5 = \int_{\mathbb{R}^N} \sigma_R |v|^2 \div (\rho v) \, dx + \frac{1}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} \sigma_R \rho (v \cdot \nabla) |v|^2 \, dx
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^N} \sigma_R \div (\rho v) |v|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 (v \cdot \nabla) \sigma_R \, dx
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^N} |v|^2 (v \cdot \nabla) \sigma_R \, dx.
\]

Therefore, we estimate

\[
|I_5| \leq \frac{1}{2R} \int_{\{R \leq |x| \leq 2R\}} |v|^3 |\nabla \sigma| \, dx. \quad (2.8)
\]

We estimate the right hand side of (2.8), depending on \( N \).
(i) For $N = 2$ we use the following Ladyzenskaya’s inequality ([3]),

$$
\|v\|_{L^4} \leq 2\|v\|_{L^2}^{\frac{3}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}
$$

to estimate

$$
|I_5| \leq \frac{\|\nabla \sigma\|_{L^\infty}}{2R} \left( \int_{R \leq |x| \leq 2R} |v|^{\frac{2}{3}} \, dx \right)^{\frac{3}{2}} \left( \int_{\{R \leq |x| \leq 2R\}} \frac{1}{2} \, dx \right)^{\frac{1}{3}}
$$

$$
\leq C\|\nabla \sigma\|_{L^\infty} \|v\|_{L^2}^{\frac{3}{2}} R^{-\frac{1}{2}}
$$

$$
\leq C\|\nabla \sigma\|_{L^\infty} \|v\|_{L^2}^{\frac{3}{2}} \|\nabla v\|_{L^2}^{\frac{1}{2}} R^{-\frac{1}{2}} \to 0 \quad (2.9)
$$
as $R \to \infty$.

(ii) For $N = 3$ we use the $L^p$–interpolation followed by the Sobolev inequality

$$
|I_5| \leq \frac{\|\nabla \sigma\|_{L^\infty}}{2R} \left( \int_{\{R \leq |x| \leq 2R\}} |v|^{\frac{2}{3}} \, dx \right)^{\frac{3}{2}} \left( \int_{\{R \leq |x| \leq 2R\}} \frac{1}{2} \, dx \right)^{\frac{1}{3}}
$$

$$
\leq C\|\nabla \sigma\|_{L^\infty} \|v\|_{L^2}^{\frac{3}{2}} \|\nabla \sigma\|_{L^\infty} \|v\|_{L^6}^{\frac{8}{3}}
$$

$$
\leq C\|\nabla \sigma\|_{L^\infty} \|v\|_{L^2}^{\frac{3}{2}} \|\nabla v\|_{L^2}^{\frac{8}{3}} \to 0 \quad (2.10)
$$
as $R \to \infty$.

(iii) For $4 \leq N \leq 6$ we estimate

$$
|I_5| \leq \frac{\|\nabla \sigma\|_{L^\infty}}{2R} \left( \int_{\{R \leq |x| \leq 2R\}} |v|^{\frac{2N}{N-2}} \, dx \right)^{\frac{3(N-2)}{2N}} \left( \int_{\{R \leq |x| \leq 2R\}} \frac{1}{2} \, dx \right)^{\frac{6-N}{2N}}
$$

$$
\leq C\|\nabla \sigma\|_{L^\infty} \|v\|_{L^\frac{2N}{N-2}}^{\frac{3N}{2N}} R^{2-N} \to 0 \quad (2.11)
$$
as $R \to \infty$, since $\|v\|_{L^\frac{2N}{N-2}} \leq C\|\nabla v\|_{L^2} < \infty$.

(iv) For $N \geq 7$ we use the additional hypothesis $v \in L^{\frac{3N}{N-1}}(\mathbb{R}^N)$.

$$
|I_5| \leq \frac{\|\nabla \sigma\|_{L^\infty}}{2R} \left( \int_{\{R \leq |x| \leq 2R\}} |v|^{\frac{3N}{N-1}} \, dx \right)^{\frac{N-1}{N}} \left( \int_{\{R \leq |x| \leq 2R\}} \frac{1}{2} \, dx \right)^{\frac{1}{N}}
$$

$$
\leq C\|\nabla \sigma\|_{L^\infty} \|v\|_{L^{\frac{3N}{N-1}}}^{\frac{3N}{N-1}} (R \leq |x| \leq 2R) \to 0 \quad (2.12)
$$
as $R \to \infty$.

Thus passing $R \to \infty$ in (2.3), we find from (2.4)-(2.12) that

$$
\mu \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + (\mu + \lambda) \int_{\mathbb{R}^N} (\text{div } v)^2 \, dx = 0,
$$

and $v = \text{constant vector in } \mathbb{R}^N$, which combined with the integrability conditions provides us with $v = 0$. Thus the equation (1.2) is reduced to the first part of (1.10). In the case of (ii) the reduced equation is $\nabla p = 0$, which implies that $\rho = \text{constant}$. □

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