Leggett mode in a strong-coupling model of iron arsenide superconductors

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Using a two-orbital model of the superconducting phase of the pnictides, we compute the spectrum of the Leggett mode – a collective excitation of the phase of the superconducting gap known to exist in multi-gap superconductors – for different possible symmetries of the superconducting order parameter. Specifically, we identify the small regions of parameter space where the Leggett mode lies below the two-particle continuum, and hence should be visible as a sharp resonance peak. We discuss the possible utility of the Leggett mode in distinguishing different momentum dependencies of the superconducting gap. We argue that the observation of a sharp Leggett mode would be consistent with the presence of strong electron-electron correlations in iron-based superconductors. We also emphasize the importance of the orbital character of the Leggett mode, which can result in an experimental observation of the mode in channels other than $A_{1g}$.

I. INTRODUCTION

The discovery of high-temperature superconductivity in iron arsenide and related compounds at the beginning of 20083 has triggered an enormous interest in the condensed matter physics community and has stimulated a flurry of experimental activity.11,12 Upon electron or hole doping of a magnetically-ordered parent state, most of the iron-based superconductors exhibit transition temperatures $T_c$ beyond the conventional BCS regime, with some extending up to 66 K11, thereby breaking the cuprate monopoly on high-temperature superconductivity. Experimental evidence accompanied by theoretical modeling suggest that the pairing in the iron-pnictides is different from the $d$-wave pairing of the cuprates. Nevertheless, they resemble the cuprates in that it is increasingly clear that the magnetism of the parent state (either long-range or fluctuating order) crucially influences the pairing symmetry of the doped system. A conclusive observation of the pairing symmetry still remains elusive, with both nodal and nodeless order parameters reported in experiments. This provides a strong incentive to identify new experimental probes potentially sensitive to the symmetry of the superconducting gap.

While a wide range of nodal gap functions were initially predicted,11,12 the general theoretical view has now converged to favor an extended $s$-wave order parameter (denoted $s^\pm$ or $s_{x^2-y^2}$) that takes opposite signs on the electron and hole pockets along the multi-band Fermi surfaces. The symmetry of this $s_{x^2-y^2}$ gap matches that of the iron-pnictide Fermi surface: it is maximal around $(0,0),(\pi,0),(0,\pi),(\pi,\pi)$ - the location of the Fermi surfaces in the unfolded one-iron-per-site Brillouin zone. This sign-alternating nodeless gap is consistent with some experimental data and also has broad theoretical support.13,20,21,22 Indeed, both strong and weak coupling theories of the onset of superconductivity predict an extended $s$-wave order parameter.

Experimentally, however, there is no consensus about the nature of the order parameter, with both nodal and nodeless gaps being reported. While most experiments can be explained within the framework of an $s^\pm$ gap,17,18,20–26 several facts, such as the $T^2$ dependence of the NMR relaxation rate over a significant temperature range,27–30 residual finite quasiparticle terms in the thermal conductivity,31,32 as well as the power-law behavior of the penetration depth,33,34 remain unsettled. Some of the experiments on penetration depth and thermal conductivity could be explained by an $s^\pm$ order parameter if there were a large gap anisotropy,23 but this contradicts ARPES data, which reveals very isotropic nodeless gaps on the hole Fermi surface.35,36 of magnitudes matching a strong-coupling form $\Delta(k) = \Delta_0 \cos(k_x) \cdot \cos(k_y)$ in the unfolded Brillouin zone.

A possible resolution of this apparent contradiction, consistent with the theoretical prediction of an $s^\pm$ order parameter, is that the gap anisotropy is doping dependent and that different experiments are done at different dopings. In the strong-coupling mean-field picture,17,19 the gap anisotropy is intrinsically doping dependent: the gap has a form $\cos(k_x) \cdot \cos(k_y)$ which becomes more anisotropic as the doping is increased. In a weak-coupling expansion of Fermi surface interactions, the gap anisotropy can arise from the presence of an $A_{1g}$ term $\cos(k_x) + \cos(k_y)$ (which does not break the crystal symmetry but can create nodes on the $(\pi,0)$ and $(0,\pi)$ electron surfaces) in the band interactions upon renormalization.20 A large gap anisotropy is already present in Functional Renormalization Group studies of orbital models.18,24

In this paper, we analyze another physical phenomenon – the Leggett mode of multi-band superconductors – that depends on the strength of the pairing order parameter
and could also in principle quantitatively distinguish a
sign-changing gap from other gap symmetries. Specifi-
cally, we investigate to what extent the pairing symmetry
of the iron-based superconductors can be deduced by an-
alyzing the behavior of the Leggett mode as a function of
doping and the strength of superconducting order param-
eters. As the iron-based superconductors have multiple
orbitals, the superconducting state exhibits a plethora of
collective modes beyond the usual Goldstone/Higgs plas-
mon. Here we use an effective two-orbital model of the
superconducting state to study one of these – the Leggett
mode associated with anti-symmetric phase fluctuations
between the two superconducting order parameters. This
gapped collective mode can, in the right parameter range,
present a sharp collective mode resonance below the two-
particle continuum which could in principle be detected
experimentally.

To determine whether such a collective mode reso-
nance occurs in the pnictide superconductors, we study
the gap and dispersion of the Leggett mode as a func-
tion of doping and the superconducting order param-
eters. We show that, for a sign-changing gap function, the
Leggett mode can be below the two-particle continuum
for a small regime at low doping. In particular, when
the band renormalization is large, an undamped Leggett
mode can exist in a relatively large parameter region.
Thus, the observation of a sharp Leggett mode will vali-
date the presence of strong electron-electron correlations
in the iron-based superconductors. Moreover, in our two-
orbital model, the Leggett mode is a \( B_{1g} \) mode, instead
of a pure \( A_{1g} \) mode, which is expected in any band-based
model. Therefore, the orbital structure of pairing in the
iron-based superconductors can be validated by identify-
ing the existence of the Leggett mode in channels other
than \( A_{1g} \).

Unfortunately, we find that the Leggett mode cannot
qualitatively distinguish between a sign-changing order
parameter and other gapped order parameters. However,
the sign-changing order parameter will have a degree of
anisotropy which depends on doping. For large doping,
the sign-changing order parameter on the enlarged Fermi
surface will exhibit larger anisotropy. As such, the super-
conducting gap will be small on some parts of the Fermi
surface and the Leggett mode will be overdamped, lying
above the two-particle continuum, and hence unobserv-
able. This presents a testable opportunity if, at moderate
doping (when the gaps should theoretically be isotropic),
the Leggett mode is below the two-particle continuum
and hence observable. If so, the observation of a disap-
ppearing collective mode provides indirect support for a
sign-changing gap function.

II. COMPUTING THE EFFECTIVE ACTION

The Leggett mode is a collective excitation of two-
(or multi-) band superconductors, associated with anti-
symmetric phase oscillations between the two bands. It is
thus a neutral mode associated with oscillations between
the supercurrents of the two bands. Here we present the
effective action for this mode derived from a two-orbital
model appropriate to the pnictides at temperatures well
below the onset of superconductivity. To render our cal-
culations analytically tractable, we focus on a simplified
model of the iron-based superconductors that takes into
account only the \( d_{zx}, d_{yz} \) orbitals. In this case, an intra-
band order parameter can have its phase fluctuate be-
tween the two orbitals in two modes: the usual sym-
metric combination (Goldstone) and the antisymmetric
combination, which is the Leggett mode.

While the conventional Leggett mode involves only the
Fermi surface gaps, our work involves a Leggett mode in
the orbital gaps. We consider the orbital basis rather
than an effective band basis, because neglecting the or-
bital structure of the iron-based superconductors is most
likely incorrect: it was shown\(^\text{19}\) that due to the difference
in mirror symmetry eigenvalues of the electron and one of
the hole bands at the \( \Gamma \) point in the Brillouin zone (BZ),
the spin density wave (SDW) state is gapless with a Dirac
point in both two and five-orbital models of iron-based
superconductors. This highly nontrivial effect, confirmed
by experiments\(^\text{20}\), is lost in the effective band-basis pic-
ture. Details of the derivation of the Leggett mode ef-
fective action in the orbital basis, which differs slightly
from the band-basis result of, e.g., Ref.\(^\text{11}\) are given in
Appendix A.

A. Model Hamiltonian

Using the insight provided by numerical and ana-
lytic studies suggesting that the antiferromagnetic ex-
change coupling between next-nearest-neighbor Fe sites
is strong\(^\text{12,42}\) two of us\(^\text{17}\) studied a \( t-J_1-J_2 \) model with-
out band renormalization and obtained a gap function of
the form \( \cos(k_x) \cdot \cos(k_y) \), which changes sign between
the electron and hole-pockets of the Fermi surface of the
material. It is this type of strong-coupling superconduc-
tivity that we will focus on in this paper, but we point
out that other weak coupling approaches exist and give
a similar sign-changing order parameter\(^\text{15,16,18,20–23}\).

To calculate the effective action for the phase modes
of the superconducting state, we employ a model of the
pnictides which incorporates only the \( d_{zx}, d_{yz} \) orbi-
tals at each site, together with hybridization between
the two. Although this description is only truly valid in
the case of unphysically large crystal field splitting, we
use this model for its analytic simplicity. We adopt the
band structure proposed in Ref.\(^\text{14}\) which at first glance
captures the essence of the Density Functional Theory
results:

\[
H_0 = \sum_{k \sigma} \psi^\dagger_{k \sigma} T(k) \psi_{k \sigma} + H_{int}
\]

\[
T(k) = \begin{pmatrix}
\epsilon_{xx}(k) & \epsilon_{xy}(k) \\
\epsilon_{yx}(k) & \epsilon_{yy}(k)
\end{pmatrix}
\]

(1)
where \( \psi_{k,\sigma}^{\dagger} = (c_{dxz,k,\sigma}^{\dagger}, c_{dx^{2}k,\sigma}^{\dagger}, c_{dz^{2}k,\sigma}^{\dagger}) \) is the creation operator for spin \( \sigma \) electrons in the two orbitals and the kinetic terms read:

\[
\begin{align*}
\epsilon_x(k) &= -2t_1 \cos k_x - 2t_2 \cos k_y - 4t_3 \cos k_x \cos k_y - \mu \\
\epsilon_y(k) &= -2t_2 \cos k_x - 2t_1 \cos k_y - 4t_3 \cos k_x \cos k_y - \mu \\
\epsilon_{xy}(k) &= -4t_4 \sin k_x \sin k_y
\end{align*}
\]

(2)

The hoppings have roughly the same magnitude: \( t_1 = -1.0, t_2 = 1.3, t_3 = -0.85, \) and \( t_4 = -0.85 \) in eV. We find that the half-filled, two electrons per site configuration is achieved when \( \mu = 1.54 \) eV.

The missing ingredient in this two-orbital model is the \( dx^{2}-y^{2} \) orbital, which can be shown to be important to the detailed physics of the iron-based superconductors.\(^{33} \) For example, the kinetic model \(^{1} \) gets the location of the second hole pocket wrong—it situates it at the \((\pi, \pi)\) point in the unfolded BZ, whereas LDA calculations show two hole pockets at the \( \Gamma \) point. However, the two-orbital model gets several of the qualitative characteristics of the iron-based superconductors right: it has a nodal SDW instability and a sign-changing \( s \)-wave superconducting instability.

To describe the superconducting phase, we use the approach of Ref. \(^{17} \), adopting a strong-coupling picture in which the interaction Hamiltonian contains anti-ferromagnetic nearest-neighbor and next-nearest-neighbor coupling between the spins in both identical and opposite orbitals. While not entirely correct at lattice scales, it was shown that this model gives remarkably large overlaps with the interactions obtained through the functional renormalization group method,\(^{34} \) and hence can be considered as an effective interaction model for the iron-based superconductors. Furthermore, for our purposes these interactions are important only insofar as they give, after decoupling in the superconducting channel, the sign-changing \( \cos(k_x) \cdot \cos(k_y) \) superconducting order parameter. In this sense, the interacting spin model we can use is thought of as an effective Ginzburg-Landau description of iron-based superconductors; the precise mechanism driving the transition to the superconducting phase is irrelevant to the effective action we derive here.

Based on the mean-field analysis of Ref. \(^{17} \), we will assume throughout that the superconducting instability is dominated by the intra-orbital interactions, so that the gap is diagonal in the orbital basis. Indeed, at the mean-field level, the inter-orbital pairing is weaker than the intra-orbital pairing by a factor of approximately five.\(^{17} \)

In addition there is a large on-site inter-orbital Hund’s rule coupling which will not enter into the present analysis as it does not alter the nature of the order parameter at mean-field level. We will briefly discuss the impacts of this last term, together with the antiferromagnetic inter-orbital interactions, in Sect. \(^{17} \).

### B. Phase-only effective action

To obtain an effective action for the phase of the superconducting gap, we follow the general protocol of Ref. \(^{19} \). Details of this calculation as applied to the orbital basis are given in Appendix \(^{1} \). In essence, one first decouples the interaction terms in the microscopic model using a Hubbard-Stratonovich transformation. This re-expresses operators quadratic in the fermions as interaction terms between a pair of fermions and the superconducting field \( \Phi \). Deep in the superconducting region, where fluctuations in the magnitude \( \Delta = |\Phi| \) can be neglected, integrating out the fermions then yields an effective action for the phase modes of the system. Since we work with a two-orbital model, there are a priori two superconducting gaps, excluding inter-orbital pairing. Though by symmetry their magnitudes have to be equal, this leads to two independent phase degrees of freedom. As is well known, one of these is a Goldstone mode which, upon including the Coulomb interactions, becomes a plasma mode. The other is the (gapped) Leggett mode, which will be our principle focus here.

For our purposes, the two phase degrees of freedom are most conveniently expressed in the basis

\[
\phi \equiv \frac{1}{\sqrt{2}} (\theta_1 + \theta_2) \quad \varphi \equiv \frac{1}{\sqrt{2}} (\theta_1 - \theta_2)
\]

(3)

where \( \theta_1 \) and \( \theta_2 \) are the phases of the gaps in the \( xz \) and \( yz \) orbitals, respectively. Hence \( \phi \) represents the symmetric phase oscillation, while \( \varphi \) represents the (neutral) antisymmetric phase mode. In this basis, we find the effective action to be (see calculation details in Appendix \(^{1} \)):

\[
S_{eff} = \int d\Omega d^2q \left( \phi(\Omega, q) \varphi(\Omega, q) \right) \left( N_{\phi\phi} \left[ \Omega^2 - \epsilon_{\phi\phi,ij}^2 q_i q_j \right] + N_{\varphi\varphi} \left[ \Omega^2 - \epsilon_{\varphi\varphi,ij}^2 q_i q_j \right] \right) \left( \phi(\Omega, q) \varphi(\Omega, q) \right).
\]

(4)
with momentum-independent coefficients given by:

\[
N_{\phi\phi} = -\int \frac{d^2k}{(2\pi)^2} \left\{ \frac{\Delta^2}{4E_+^{(3)}} + \frac{\Delta^2}{4E_-^{(3)}} \right\}
\]

\[
N_{\phi\varphi} = -\int \frac{d^2k}{(2\pi)^2} \frac{(\epsilon_x - \epsilon_y)^2}{(E_+ - E_-)^2} \left\{ \frac{\Delta^2}{4E_+^{(3)}} + \frac{\Delta^2}{4E_-^{(3)}} \right\}
\]

\[
M = \int \frac{d^2k}{(2\pi)^2} \frac{4\Delta^2 \epsilon_{xy}^2}{E_+^{(3)}E_-^{(3)}(E_+^{(3)} + E_-^{(3)})}
\]

with \( \Omega_0 \equiv \sqrt{\frac{M}{-N_{\phi\varphi}}} \). Here, \( E_\pm \) are the two band energies \( E_\pm = \frac{1}{2} \left( \epsilon_x + \epsilon_y \pm \sqrt{(\epsilon_x - \epsilon_y)^2 + 4\epsilon_{xy}^2} \right) \) of the metallic state, and \( E_\pm^{(3)} = \sqrt{E_\pm^2 + \Delta^2} \) are the quasi-particle energies in the superconducting phase. All \( \epsilon, E, \) and \( \Delta \) are evaluated at the momentum \( k \) to be integrated over. The above equations represent the main result of the paper.

In Eq. (4), terms linear in \( q \), as well as terms bilinear in \( q, \Omega \), all vanish in the limit \( T \to 0 \). As expected, this effective action describes one gapless mode, comprised entirely of symmetric phase fluctuations at \( q = 0 \), and one gapped mode. The latter is the Leggett mode; at \( q = 0 \) it consists purely of antisymmetric phase oscillations between the two superconducting gaps. Here we are principally interested in the Leggett mode gap, \( \Omega_0 \), as this represents the threshold at which the mode becomes experimentally observable. Thus if \( \Omega_0 < 2\Delta \), we expect the Leggett mode to appear as a sharp resonance in the spectrum of the pnictide superconductors.

For terms involving \( q^2 \), the expressions for the coefficients \( c_{ij} \) are somewhat more complicated and are thus given in Appendix A1. We note, however, that for \( i \neq j \), any coefficient of \( q_i q_j \) vanishes due to symmetry. Further, for \( i = j \), symmetry of the coefficients under a 90 degree rotation of the Brillouin zone fully determines their direction dependence in \( q \). Taking these symmetries into account, Eq. (4) has the form:

\[
S_{\text{eff}} = \int d\Omega q^2 \left( \phi, \varphi \right) \left( N_{\phi\phi} \left[ \Omega^2 - c_{\phi\phi}^2 q^2 \right] N_{\phi\varphi} \left[ \Omega^2 - \Omega_0^2 - c_{\phi\varphi}^2 q^2 \right] 0 \right) \left( \phi, \varphi \right).
\]

We should note that the Leggett mode gap is proportional to \( \epsilon_{xy}^2 \) – that is, to the off-diagonal kinetic terms in the orbital basis. This is in contrast to the approach of, for example, Ref. [11] in which the superconducting gap is taken to be diagonal in the band basis of the normal state, and it is the inter-band interactions which couple the phases of the two gaps, and hence generate the Leggett mode. This difference stems from the fact that we take the gap to be diagonal in the orbital basis: \( \Delta_\alpha(k) = \langle c_\alpha^\dagger k c_\alpha k \rangle \), where \( \alpha \) indexes the orbitals and assume that the pairing is defined over the whole Brillouin Zone. Any model in which the interaction is written in orbital space and which aims to respect the point-group symmetries of the lattice will require this type of orbital-basis formalism.

### C. Including Hunds interactions

In light of the fact that our approach is based on an absence of off-diagonal interactions in the superconducting channel (when the orbital basis is used), it is useful to consider in more detail the validity of this assumption in the presence of inter-orbital couplings. In the pnictides the ferromagnetic Hund’s rule interaction

\[
H_H = -J_H \sum_r S_{1r} S_{2r} \equiv -J_H \sum_r c_{1r\sigma\sigma'} c_{1r\sigma'}^\dagger c_{2r\gamma'\gamma}^\dagger c_{2r\gamma\gamma'}
\]

is the principal source of such interactions.

Since spin ordering must be absent in the superconducting phase, generically we may decouple the Hunds interaction in either the particle-particle channel or the particle-hole channel. At lowest loop order, the particle-particle interaction serves only to renormalize the band structure. The particle-hole contribution was, as previously noted, shown to be small by Ref. [17]. Neglecting the small inter-orbital pairing at mean-field, we find that the Hunds interaction affects the effective action for the Leggett mode \( \varphi \) only through higher loop corrections in the fermion propagator.

Further, it is straightforward to include the effect of the small inter-orbital interaction in the superconducting channel. Such a term simply modifies the effective action for the superconducting phase by adding a term \( V_{12} (\Delta_1 \Delta_2^* + \Delta_2 \Delta_1^*) \equiv 2|\Delta|^2 V_{12} \cos(\varphi) \). This modifies
the gap of the Leggett mode according to

$$\Omega^2_0 \to \Omega^2_0 - \frac{V_{12}}{V_{11} V_{22} - V_{12}^2 N_{\phi \phi}} \Delta_0^2. \quad (10)$$

Here $V_{\alpha \beta}$ parametrize the superconducting interaction between orbitals $\alpha$ and $\beta$, as described in Eq. A2 and we have taken $\Delta(k) = \Delta_0 \cos(k_x) \cos(k_y)$. For $0 < V_{12} \ll V_{11}, V_{22}$, the effect of including such a term is always to bring the Leggett mode gap down in energy.

D. Effective action with Coulomb terms

In the above analysis, we ignored the effects of the Coulomb interaction on the phase modes. In a single-band superconductor, including the Coulomb interactions modifies the effective action for the phase $\theta$ of the superconducting gap such that $\theta$ becomes a plasma mode\footnote{Eq. (12) shows that including Coulomb interactions does not alter the mass gap of the Leggett mode, as the plasma mode does not mix with the Leggett mode $\phi$ at $q = 0$. The net effect of the Coulomb terms on $\varphi(\Omega, q)$ will be a modification of the $q^2$ term in the effective action of the Leggett mode. Integrating out $\phi$, we obtain:

$$S_{\phi \phi} = \int d\Omega d^2 \varphi(q) \varphi(-q) N_{\phi \phi} \left[ \begin{array}{c} \Omega^2 - \Omega^2_0 - c^2_{\phi \phi}(q_x^2 - q_y^2) \\
\left( 1 + \frac{c^2_{\phi \phi}}{c^2_{\phi \phi} N_{\phi \phi}} \frac{q^2_x - q^2_y}{1 - N_{\phi \phi} U(q)} \right) + \Delta_0 V_{\alpha \beta}^{-1} \Delta_0 \end{array} \right]. \quad (12)$$

Hence for small $q, \Omega$, in the presence of Coulomb interactions, provided that $\lim_{q \to 0} \frac{\Omega^2}{q^2 (1 - N_{\phi \phi} U(q))}$ is finite, the net effect is a modification of the effective velocity of the mode. The above equation needs to be solved self-consistently to obtain the mode dispersion. However, the limit $q \to 0$, which determines whether the Leggett mode is above or below the particle-particle continuum, is unchanged from the case without the Coulomb interaction.}

In the presence of Coulomb interactions, the phase-only effective action has the form:

$$S_{\phi \phi} = \int d\Omega d^2 \varphi(q) \varphi(-q) N_{\phi \phi} \left[ \begin{array}{c} \Omega^2 - \Omega^2_0 - c^2_{\phi \phi}(q_x^2 - q_y^2) \\
\left( 1 + \frac{c^2_{\phi \phi}}{c^2_{\phi \phi} N_{\phi \phi}} \frac{q^2_x - q^2_y}{1 - N_{\phi \phi} U(q)} \right) + \Delta_0 V_{\alpha \beta}^{-1} \Delta_0 \end{array} \right]. \quad (12)$$

III. RESULTS

Having established the general form of the effective action of the Leggett mode, we now turn to a quantitative evaluation of the coefficients in Eq. (14). Our principle interest will be what potential information the Leggett mode can give about the form of the superconducting gap – in particular, we address the question of whether it can distinguish between the popular extended s-wave gap and other plausible pairing symmetries. Unfortunately, it is clear from our equations that the Leggett mode properties depend on the absolute value of the gap function, thereby preventing any qualitative sensitivity of the mode to a sign-change in the gap function. We find that the clearest signature is the lifetime of the Leggett mode as a function of doping – at low dopings we find the Leggett mode to lie below the two-particle continuum; at higher doping the mode is always at higher energies than the two-particle continuum and hence will give at best a very broad resonance.

A. The Leggett Mode gap

We begin by studying the Leggett mode gap $\Omega_0$ for several different gap functions, with the objective of understanding the qualitative differences expected between these in potential experiments. In each case, the mode is expected to be visible if it lies below the two-particle continuum, which is set by $2 \min |\Delta|$ (where the minimum is taken over the Brillouin zone).

At $q = 0$ and $T = 0$, the symmetric and antisymmetric phase oscillations decouple, and from the effective action
The gap of the Leggett mode $\varphi$ is given by

$$\Omega_0 = \sqrt{-\frac{M}{N_{\varphi\varphi}}}$$

with $M$ and $N_{\varphi\varphi}$ given by Eqs. (5) and (7). Note that $M > 0$ and $N_{\varphi\varphi} < 0$, so that the Leggett mode gap is well defined. We can evaluate the coefficients $N_{\varphi\varphi}$ and $M$ by integrating the expressions (5) and (7) numerically over the Brillouin zone. We use the values of $\epsilon_\alpha$ quoted in Eq. (2).

Fig. 1 shows the expected gap of the Leggett mode for extended $s$-wave, standard $s$-wave, $d$-wave, and $\Delta = \Delta_0 \sin k_x \sin k_y$, gaps, as a function of the filling fraction $\nu$ and the maximum gap magnitude $\Delta_0$. The general form of $\Omega_0$ is similar in all four cases: it increases with the superconducting gap $\Delta_0$, and has its lowest values at a filling of approximately $\nu = 0.4$. For all four order parameters, we also find the gap of the Leggett mode shows an academically interesting chemical potential dependence, dropping sharply between $\nu = 0.4$ and $\nu = 0.5$, independent of the momentum-dependence of the order parameter.

The qualitative features of these plots can be understood by considering the form of Eqs. (5) and (7). First, we see that $\Omega_0$ increases monotonically with $\Delta_0$, at a slightly less than linear rate. Though naively both $M$ and $N_{\varphi\varphi}$ scale quadratically with $\Delta_0$, $N_{\varphi\varphi}$ has divergences if $E_+^{(\Delta)}$ or $E_-^{(\Delta)}$ vanish; these are cut off by the gap but nevertheless contribute the major part of the integral. Consequently, $N_{\varphi\varphi}$ is well approximated by $N_{\varphi\varphi} \sim V_{FS}/\Delta$, with $V_{FS}$ the volume of the Fermi surface. On the other hand, $M$ vanishes at the Fermi surface in the limit of small $\Delta$, scaling approximately as $M \Delta$ in this region. Hence the quantity $\sqrt{M/N_{\varphi\varphi}}$ increases with $\Delta$, with a power close to (but slightly less than) 1.

The non-monotonic dependence on $\nu$, which is similar in all four cases, stems from the dependence of the shape and volume of the Fermi surface on the chemical potential. As stated above, the integral expression for $N_{\varphi\varphi}$ is dominated by contributions near the Fermi surface. $M$,
results in more areas where the expressions for $N$ in the normal state. In the superconducting state this set of Fermi pockets appears at the points ($\pm \nu, \pm \pi$) as a second pair of bands crosses the Fermi level leading to a sudden reduction in $\Omega$ near the new branches of the Fermi surface. These account for the non-monotonic behavior of $\Omega_0$ observed in both extended $s$-wave and $d$-wave order parameters between $\nu = 0.3$ and $0.4$, as the cut-off in the normal-state divergences of $N_{\phi \phi}$ grows smaller. Though the application of our simple two-orbital model at large fillings is not warranted, and the features discussed in this paragraph are model-dependent, we expect them to be accurate for gaps diagonal in the orbital basis inasmuch as the band structure given by the two-orbital model is correct.

### B. Observability of the Leggett mode

In order for the Leggett mode to give a sharp resonance in experiments, it should lie below the two-particle continuum. For the $d$-wave and sine-wave gaps, which are nodal for the iron-based superconductors' Fermi surfaces, this is obviously never the case. For ordinary $s$-wave and extended $s$-wave gaps, the position of the Leggett mode at $q = 0$ relative to the two-particle continuum depends on the values of $\nu$ and $\Delta_0$. Figure 3 plots distance between the gap of the Leggett mode and the minimum energy of the two-particle continuum as a function of $\Delta_0$ for both extended $s$-wave ($\Gamma_k = \cos k_x \cos k_y$) and standard $s$-wave ($\Gamma_k = 1$) gaps.

The principle difference between the two nodeless gaps is the range of dopings over which the Leggett mode is expected to be observable. In the pure $s$-wave case, the two-particle continuum is given by $2\Delta_0$, independent of the shape of the Fermi surface. Hence, as seen in Figure 3(b), the dominant effect here is that the gap of the Leggett mode scales sublinearly in $\Delta_0$, and hence becomes observable only at large values of the gap. Its separation from the two-particle continuum is extremely small at the small values of $\Delta_0$ expected to occur near half-filling. (Away from half-filling, the gap of the Leggett mode lies above the two-particle continuum, as shown in the figure, due to the Fermi-surface effects discussed above). In the extended $s$-wave case, however, the minimum of the gap also depends on how close the Fermi surface comes to the nodes of the gap function. Hence the dependence on filling fraction here is more pronounced (Figure 3(a)); for all values of $\Delta_0$ the Leggett mode sits definitively below the two-particle continuum in the interval $0.45 < \nu < 0.5$. Further from half-filling, where the nodes of the extended $s$-wave gap sit closer to the Fermi surface, the mode is never visible. $\Omega_0$ is smaller overall in the extended $s$-wave case, compensating for the fact that the minimum of the two-particle continuum is de facto smaller than in the standard $s$-wave case. Most notably, for the small values of $\Delta_0$ expected near 1/2 -filling, we expect the Leggett mode to be below the two-particle continuum in the extended $s$-wave case.

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**FIG. 2:** Plots of Fermi surface as a function of chemical potential for $0 < \nu < 0.65$. The relevant filling fractions are shown as $\nu$ in the title of the figure. The electron pockets first appear at approximately $\nu = 0.4$, and cross the nodal lines at $\nu = 0.6$. The nature of the hole pockets does not change substantially over the range shown here.

(a) $\nu = 0.5$, $\nu = 0.50002$, $\nu = 0.50002$

(b) $\nu = 0.6$, $\nu = 0.60009$, $\nu = 0.60009$
C. Dispersion of the Leggett mode for extended s-wave gap

We now return to the general form of the effective action for the phase degrees of freedom, and analyze the structure of its modes at small $q$. In the absence of the Coulomb interaction, the form of the dispersion is effectively characterized by

$$w^2 = \frac{1}{2} \left[ q^2 (c_{\phi\phi}^2 + c_{\phi\phi}^2) + \Omega_0^2 \pm \sqrt{ \left( q^2 (c_{\phi\phi}^2 - c_{\phi\phi}^2) + \Omega_0^2 \right)^2 + \frac{4c_{\phi\phi}^4 q^4 \cos^2 2\theta}{N_{\phi\phi}N_{\phi\phi}} } \right],$$

where $\theta$ is the angle in the $(k_x, k_y)$ plane. If $c_{\phi\phi}$ vanishes, we retrieve the gapless Goldstone mode, and the gapped Leggett mode. From Eq. (12), we find that adding the Coulomb term modifies this according to:

$$w^2 = \frac{1}{2} \left[ \left\{ c_{\phi\phi}^2 (1 - 4N_{\phi\phi}U(q)) + c_{\phi\phi}^2 \right\} q^2 + \Omega_0^2 \right] \pm \frac{1}{2} \sqrt{ \left( q^2 (c_{\phi\phi}^2 - c_{\phi\phi}^2 (1 - 4N_{\phi\phi}U(q))) + \Omega_0^2 \right)^2 + \frac{4c_{\phi\phi}^4 q^4 (1 - 4N_{\phi\phi}U(q))}{N_{\phi\phi}N_{\phi\phi}} \cos^2(2\theta) }. \quad (14)$$

In this case, taking the negative sign for $U(q) \sim q^{-1}$, (the unscreened Coulomb interaction in 2D) results in $\omega^2 < 0$, indicating that the Goldstone mode has been replaced by a plasma mode. The structure of the Leggett mode is, however, largely unchanged by the presence of the Coulomb interaction. In particular, we still have $\lim_{q \to \infty} \omega(q) = \Omega_0$.

Figure 4(a) plots the dispersion relation $\Omega(k)$ for the low-energy modes for several values of $\Delta_0, \mu$. As we have kept only terms to quadratic order in $q$, $\Omega$, we expect this to be valid for small $q$, and have restricted the range of the plots accordingly. The important feature to note is that the velocity anisotropy, due to the off-diagonal terms $c_{\phi\phi}(q_x^2 - q_y^2)$, is relatively small and the dispersion is approximately rotationally invariant. Because of this, the dispersion relation of the Leggett mode is well-characterized by the gap $\Omega_0$ (c.f. Sect. IIIA), and the velocity $v \equiv \lim_{q \to \infty} \omega(q)$. This latter is plotted for the extended s-wave gap in Fig. 4(b).

IV. CONCLUSIONS

We have obtained the fluctuation action for the superconducting phase collective modes of a two-orbital model for iron-based superconductors, with particular empha-
sis on the antisymmetric Leggett mode. By fixing the parameters of the band structure, our calculation has identified the range of doping and superconducting gap magnitude over which the undamped Leggett mode exists below the two-particle continuum. As the Leggett mode’s visibility increases with the magnitude of the superconducting gap, this result also suggests that if the bandwidth is narrower, there is a higher possibility of observing the undamped Leggett mode. Therefore, a strong renormalization of bands could enhance the existence of an undamped Leggett mode. Unfortunately, the mode and its dispersion are insensitive to the sign of the order parameter on Fermi surfaces, and the mode does not qualitatively distinguish between the sign-changed $s$-wave and a normal $s$-wave superconductors. However, we find that quantitative characteristics of the mode can in principle distinguish between such different pairing symmetries. First, we find that the Leggett mode does lie below the two-particle continuum, near half filling and sufficiently deep in the superconducting region, for the extended as well as normal $s$-wave gap. This is distinct from the case of nodal gaps, where low-energy quasi-particles are always expected to broaden the Leggett mode resonance. Second, we find that the difference in signatures between the two kinds of $s$-wave pairing symmetry investigated here is subtle, but that the extended $s$-wave gap is visible over a narrower range in doping, but further below the two-particle continuum over much of its range of detectability. This difference comes from the different doping dependence of the two $s$ wave gap functions: unlike the normal, sign-unchanging $s$-wave gap, the $s^\pm$ order parameter will most likely change upon doping as the Fermi surfaces become closer to the line of zeroes that a sign-changing gap should have in the Brillouin zone. This gap variation upon doping is present in both strong and weak coupling models. In this situation, the Leggett mode will move from a relatively sharp mode below the two-particle continuum into a strongly damped mode above the two-particle continuum as doping is increased. This quantitative change can in principle be observed in experiments.

It has been claimed that the Leggett mode has been observed in MgB$_2$ by Raman scattering and point-contact transport measurement, although the energies of the Leggett mode measured in the two experiments are different. Ref. [50] found that in the weak-coupling treatment of superconductors with an $s^\pm$ gap, however, the $A_{1g}$ Leggett mode does not couple to Raman scattering. The analysis carried out here relies heavily on the fact that iron-based superconductors are more strongly coupled than MgB$_2$, and that the superconducting phase is thus well-described by considering the orbital, rather than the band, basis. This leads to a result which differs from that of the weakly-coupled approach in two ways. First, the strong-coupling approach suggests that the Leggett mode should be observable in Raman spectra. Second, in the strong-coupling treatment, the different orbital symmetries should be kept explicitly when determining the relevant Raman channels. For our model, the Leggett mode is caused by an oscillation between the condensates involving the scattering of a pair of $d_{xz}$-orbital electrons into a pair of $d_{yz}$-orbital electrons. Such a process causes a relative density fluctuation, $\delta n = n_{xz} - n_{yz}$, between two orbitals, which belongs to the $B_{1g}$ irreducible representation of the point group ($D_{4h}$) of the crystals. Therefore, the Leggett mode is a $B_{1g}$ mode in this orbital-based model, and should exist in the $B_{1g}$ channel in Raman scattering experiments. (Without the orbital characters, the Leggett mode should be a pure $A_{1g}$ mode, as is the case in MgB$_2$.) Thus, observing the Leggett mode in channels other than $A_{1g}$ should provide important evidence about the orbital structure of condensed pairs in the iron-based superconductors.
Appendix A: Calculating the phase-only effective action

We begin by deriving the effective phase-only action for a generic Hamiltonian of the form:

\[ H = \sum_{\alpha,\beta, r, r'} \sum_{\sigma} c_{\alpha, r, \sigma} \varepsilon(r, r') c_{\beta, r', \sigma} + V_{\alpha, \beta}(r; r') b_{\alpha, r}^\dagger b_{\beta, r'} + i e U(r - r') \rho_{\alpha} \rho_{\beta} \]  

Here \( \alpha, \beta \) are orbital indices, \( V \) is the superconducting interaction (in our model, a spin-spin antiferromagnetic interaction decoupled in the Cooper channel to give \( \cos(k) \) in the BCS basis. The final line of Eq. (A5) gives a separate contribution to the action, which can be expressed, to quadratic order in \( \theta \), \( \bar{\partial} \), \( \delta \), to absorb all terms involving the phase of the superconducting order parameter into the first term of Eq. (A2). We may decouple the two 4-fermion interactions by means of two Hubbard-Stratonovich transformations. This decoupling gives the action:

\[
S = \sum_{\alpha, \beta, r, r'} \left\{ \sum_{\sigma} e_{\alpha, r, \sigma} \left[ (\partial/\partial \tau - \mu) \delta_{\alpha, \beta} + \varepsilon_{\alpha, r, \sigma} \right] c_{\beta, r', \sigma} - \delta_{\alpha, \beta} \left[ \Phi_{\alpha \beta} b_{\alpha, r} + \Phi_{\alpha \beta}^\dagger b_{\alpha, r}^\dagger \right] \right.\]

\[
- i e \delta_{\alpha, \beta} \left[ \chi_{\alpha} \rho_{\alpha} + \chi_{\beta} \rho_{\beta} \right] - \Phi_{\alpha \beta} V_{\alpha, \beta}^{-1}(r; r') \Phi_{\beta r'} - \chi_{\alpha} U(r - r')^{-1} \chi_{r'} \}
\]

where \( \Phi \) is the Hubbard-Stratonovich field associated with the superconducting interaction, and \( \chi \) is associated with the Coulomb interaction.

In computing this effective action, we follow the method of ref. [47] to isolate the action for the phase degrees of freedom. That is, taking \( \Phi_{\alpha r} = \Delta_{\alpha r} e^{i \theta_{\alpha}(r)} \), we perform the gauge transformation

\[
\varepsilon_{\alpha, r, \sigma} \to \varepsilon_{\alpha, r, \sigma} e^{i \theta_{\alpha}(r)/2} \]

then have:

\[
S = \sum_{\alpha, \beta, r, r'} \left\{ \sum_{\sigma} e_{\alpha, r, \sigma} e^{-i \theta_{\alpha}(r)/2} \left[ (\partial/\partial \tau - \mu) \delta_{\alpha, \beta} + \varepsilon_{\alpha, r, \sigma} \right] c_{\beta, r', \sigma} e^{i \theta_{\alpha}(r')/2} - \delta_{\alpha, \beta} \left[ \Delta_{\alpha r} b_{\alpha, r} + \Phi_{\alpha \beta}^\dagger b_{\alpha, r}^\dagger \right] \right.\]

\[
- i e \delta_{\alpha, \beta} \left[ \chi_{\alpha} \rho_{\alpha} + \chi_{\beta} \rho_{\beta} \right] - \Phi_{\alpha \beta} V_{\alpha, \beta}^{-1}(r; r') \Phi_{\beta r'} - \chi_{\alpha} U(r - r')^{-1} \chi_{r'} \}
\]

It is convenient to re-express the kinetic terms as:

\[
\sum_{\alpha, \beta, r, r'} \left\{ e_{\alpha, r, \sigma} e^{-i \theta_{\alpha}(r)/2} \left[ (\partial/\partial \tau - \mu) \delta_{\alpha, \beta} + \varepsilon_{\alpha, r, \sigma} \right] c_{\beta, r', \sigma} e^{i \theta_{\alpha}(r')/2} \right.\]

\[
- e_{\alpha, r, \sigma} e^{-i \theta_{\alpha}(r')/2} \left[ (\partial/\partial \tau - \mu) \delta_{\alpha, \beta} + \varepsilon_{\alpha, r, \sigma} \right] c_{\beta, r', \sigma} e^{i \theta_{\alpha}(r)/2} \right.\]

\[
+ \delta_{\alpha, \beta} \varepsilon_{\alpha, r, \sigma} e^{-i \theta_{\alpha}(r)/2} \left[ (\partial/\partial \tau - \mu) \delta_{\alpha, \beta} + \varepsilon_{\alpha, r, \sigma} \right] e^{i \theta_{\alpha}(r')/2} \}
\]

The first two terms can now be combined with the rest of the fermionic action can be expressed in matrix form in the BCS basis. The final line of Eq. (A5) gives a separate contribution to the action, which can be expressed, to quadratic order in \( \theta \), \( \bar{\partial} \), \( \delta \),

\[
\int \! \! \! \! d^2 r d\tau \left\{ \frac{i}{2} \frac{\partial}{\partial \tau} \theta_{\alpha}(r) + \int \! \! \! \! d^2 k \left( \frac{1}{2} \frac{\partial}{\partial \tau} \theta_{\alpha}(r) \frac{\partial}{\partial k} \varepsilon_{\beta, \gamma} - \frac{1}{8} \theta_{\alpha}(r) \frac{\partial^2}{\partial r_i \partial r_j} \theta_{\beta}(r) \frac{\partial^2}{\partial k_i \partial k_j} \varepsilon_{\beta, \gamma}(k) \right) \right\}
\]

The first term is a total derivative and will not contribute to the dynamics of the phase-only effective action. The second term in any case vanishes, as \( v \) is odd over the Brillouin zone. Hence only the last term appears in the effective action.
As we are interested in the dynamics of the phase degrees of freedom, we replace $\Delta_{\alpha,r}$ by its mean-field value. For the time being, we drop the Coulomb terms by setting $U = 0, \chi = 0$; we will discuss these further in Sect. [3]. We further consider only the slowly varying phase fluctuations. This allows us to expand the exponentials of the fermionic terms in (A4) using:

$$e^{-\frac{i}{2}(\theta_\alpha(r) - \theta_\alpha(r'))} \approx 1 - \frac{i}{2} (\theta_\alpha(r) - \theta_\alpha(r')) - \frac{1}{4} (\theta_\alpha(r) - \theta_\alpha(r'))^2$$

$$e^{-\frac{i}{2}(\theta_\beta(r) - \theta_\beta(r'))} \approx e^{-\frac{i}{2}\varphi_0} \left\{ 1 - \frac{i}{2} (\theta_\beta(r) - \theta_\beta(r') - \varphi_0) - \frac{1}{4} (\theta_\beta(r) - \theta_\beta(r') - \varphi_0)^2 \right\}$$

(A7)

where we have explicitly separated out the possible background expectation value of the phase difference $\varphi_0$ between the gaps. In practice $\varphi_0 = 0$ is set by the mean-field equations.

Defining $\psi^\dagger = (c_{1,r,\uparrow}^\dagger, c_{1,r,\downarrow}^\dagger, c_{2,r,\uparrow}^\dagger, c_{2,r,\downarrow}^\dagger)$, we may now express the first two terms in Eq. (A5), after Fourier transforming, as

$$S_{\text{Fermi}} = \int d\omega_1 d\omega_2 \frac{d^2 k_1 d^2 k_2}{4\pi^2} \psi^\dagger_{k_1,\omega_1} G^{-1}(k_1, k_2, i\omega_1, i\omega_2) \psi_{k_2,\omega_2}$$

(A8)

with

$$G^{-1}(k_1, k_2, i\omega_1, i\omega_2) = G_0^{-1}(k_1, \omega_1)\delta_{\Omega_0,0} + \Sigma(k_1, k_2, i\omega_1, i\omega_2)$$

(A9)

where here $q \equiv k_1 - k_2, \Omega \equiv \omega_1 - \omega_2$. We have:

$$G_0^{-1}(k, \omega) = \begin{pmatrix} i\omega + \epsilon_x & \Delta_1 & \epsilon_{xy} & 0 \\ \Delta_1 & i\omega - \epsilon_x & 0 & -\epsilon_{xy} \\ \epsilon_{xy} & 0 & i\omega + \epsilon_y & \Delta_2 \\ 0 & -\epsilon_{xy} & \Delta_2 & i\omega - \epsilon_y \end{pmatrix}$$

(A10)

where $\epsilon_\alpha \equiv \epsilon_\alpha(k)$ is the kinetic energy in the orbital basis, and $\Delta_\alpha \equiv \Delta_\alpha(k)$ is the momentum-dependent superconducting gap in each orbital. The second part of (A9) is given by:

$$\Sigma(k, q, i\omega, i\Omega) = -\frac{\Omega}{2} \left( \begin{array}{cccc} \theta_1 \sigma_z & 0 & 0 \\ 0 & \theta_2 \sigma_z \\ 0 & 0 & \theta_1 \sigma_z \\ 0 & 0 & 0 & \theta_2 \sigma_z \end{array} \right) + \frac{i}{2} \left( \begin{array}{ccc} \theta_1 \sigma_y \epsilon_{xy}(k_1) & \theta_2 \sigma_y \epsilon_{xy}(k_2) \\ \theta_2 \sigma_y \epsilon_{xy}(k_1) & \theta_1 \sigma_y \epsilon_{xy}(k_2) \end{array} \right)$$

$$-\frac{1}{8} \sum_{k_3, \omega_3} \left( \begin{array}{ccc} \theta_1 \epsilon_{xy}(k_3, i(\Omega - \omega_3)) & 0 \\ 0 & \theta_2 \epsilon_{xy}(k_3, i(\Omega - \omega_3)) \end{array} \right) \left( \begin{array}{ccc} \delta_{q,k_3}^{(2)} \epsilon_{xy}(k_3,i) & 0 \\ 0 & \delta_{q,k_3}^{(2)} \epsilon_{xy}(k_3) \end{array} \right)$$

$$-\frac{1}{8} \sum_{k_1, \omega_1, \omega_3} \left( C(k_1, i\omega_1, k_2, i\omega_2, k_3, i\omega_3) \sigma_z \\ 0 \right) B(k_1, i\omega_1, k_2, i\omega_2, k_3, i\omega_3) \sigma_z$$

(A11)

where $\theta_\alpha \equiv \theta_\alpha(q, i\Omega)$, and we have defined the discrete derivatives:

$$\delta_\alpha \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}(k_1) - \epsilon_{\alpha\beta}(k_2)$$

$$\delta_\alpha^{(2)} \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}(k_1) - \epsilon_{\alpha\beta}(k_2 + k_3) - \epsilon_{\alpha\beta}(k_1 - k_3) + \epsilon_{\alpha\beta}(k_2)$$

(A12)

The off-diagonal terms quadratic in the phases are:

$$B(k_1, i\omega_1, k_2, i\omega_2, k_3, i\omega_3) = \left( \begin{array}{ccc} \theta_1(k_1, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1) & -\theta_2(k_3, i\omega_3) \theta_1(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \\ -\theta_1(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1 - k_3) + \theta_1(k_3, i\omega_3) \theta_1(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \\ \theta_1(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1 - k_3) + \theta_2(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \\ \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1 - k_3) + \theta_2(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \end{array} \right)$$

$$C(k_1, i\omega_1, k_2, i\omega_2, k_3, i\omega_3) = \left( \begin{array}{ccc} \theta_1(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1) & -\theta_2(k_3, i\omega_3) \theta_1(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \\ -\theta_1(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1 - k_3) + \theta_1(k_3, i\omega_3) \theta_1(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \\ \theta_1(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1 - k_3) + \theta_2(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \\ \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_1 - k_3) + \theta_2(k_3, i\omega_3) \theta_2(q - k_3, i(\Omega - \omega_3)) \epsilon_{xy}(k_2 + k_3) \end{array} \right)$$

(A13)

Thus in our treatment, the block diagonal terms involve only discrete differences of the band energies, which will become derivatives when the momentum of the phase variables is small. The off-diagonal terms contribute, as well as such differences, a term which is finite at $q = 0$ (or $k_1 = k_2$). Hence the gap of the Leggett mode is, in the absence of inter-orbital pairing, generated by the kinetic mixing between the two orbitals.
To obtain the effective action, we integrate out the fermions in Eq. (A8). In practice, we must evaluate the result perturbatively in $\Sigma$. Specifically, we have:

$$S_{\text{eff}} = S_{MF} - Tr \ln(1 - G_0 \Sigma)$$

$$\approx S_{MF} + Tr(G_0 \Sigma) + \frac{1}{2} Tr \left( G_0 \Sigma G_0 \Sigma \right)$$

(A14)

where $S_{MF}$ is the mean-field action, from which we self-consistently determine the values of $\Delta_1, \Delta_2$. Here we will evaluate the low-energy, long-wavelength limit of the effective action (A14) by keeping terms to quadratic order in $q, \Omega,$ and $\theta_\alpha(q, \Omega)$.

1. **Evaluating $Tr G_0 \Sigma$ and $Tr G_0 \Sigma G_0 \Sigma$**

For reference, here we give a more detailed account of the calculation in Sect. II

The separate expressions for the two traces are:

$$Tr(G_0 \Sigma) = \int \frac{d^2 k}{(2\pi)^2} \left\{-\frac{\epsilon_{xy}}{4} (\varphi(q, i\Omega)\varphi(-q, -i\Omega)) \left[ 1 - \frac{\epsilon_{xy}}{(E_+ - E_-)E_+^{(\Delta)} E_-^{(\Delta)}} \left( E_-^{(\Delta)} E_+ - E_+^{(\Delta)} E_- \right) \right] \right\}$$

$$\left\{- \frac{q_i q_j}{8} \left( \theta_1(q) \theta_2(q) \right) \right\}$$

$$\left\{ m_{ij}^{(1)} \left[ 1 - \frac{E_+ (E_+ - \epsilon_y)}{E_+^{(\Delta)} (E_+ - E_-)} + \frac{E_+ (E_+ - \epsilon_y)}{E_-^{(\Delta)} (E_+ - E_-)} \right] \right\}$$

$$\left\{ m_{ij}^{(2)} \left[ 1 - \frac{E_+ (E_+ - \epsilon_y)}{E_+^{(\Delta)} (E_+ - E_-)} + \frac{E_+ (E_+ - \epsilon_y)}{E_-^{(\Delta)} (E_+ - E_-)} \right] \right\}$$

(A15)

where $m_{ij}^{(\alpha)} = \frac{\partial^2 \epsilon_{\alpha\beta}}{\partial k_i \partial k_j}$, and we define $\varphi(q) = \theta_1(q) - \theta_2(q)$. We have dropped the linear term in $\Omega$, because it is a total derivative and hence should not contribute to the action. Here all $\epsilon, E, \Delta$ are evaluated at the momentum $k$ to be integrated over. Note that we have also included the quadratic terms in the last line of Eq. (A5).

Evaluating $Tr(G_0 \Sigma G_0 \Sigma)$ gives:

$$\frac{1}{2} Tr(G_0 \Sigma G_0 \Sigma) = -\frac{\Omega^2}{8} \left( \phi(q) \varphi(q) \right) \left( \begin{array}{cc} N_{\phi\phi} & 0 \\ 0 & N_{\varphi\varphi} \end{array} \right) \left( \begin{array}{c} \phi(-q) \\ \varphi(-q) \end{array} \right)$$

$$+ \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{\epsilon_{xy}}{4} \varphi(q) \varphi(-q) - \frac{E_+^{(\Delta)} E_-^{(\Delta)} \Delta^2 + E_+ E_-}{E_+^{(\Delta)} E_-^{(\Delta)}} \right\}$$

$$+ \frac{1}{8} \left( \frac{E_+^{(\Delta)} + E_-^{(\Delta)}}{(E_+^{(\Delta)} + E_-^{(\Delta)}/(E_+^{(\Delta)} - E_-^{(\Delta)}) \right.}$$

$$\left\{ - \left( \frac{\epsilon_{xy}}{2} + \frac{1}{2} (\epsilon_x - \epsilon_y)^2 \right) v_i^{(xy)} v_j^{(xy)} - \frac{\epsilon_{xy}}{2} v_i^{(xy)} v_j^{(xy)} \right\}$$

$$\left\{ - \left( \frac{\epsilon_{xy}}{2} + \frac{1}{2} (\epsilon_x - \epsilon_y)^2 \right) v_i^{(xy)} v_j^{(xy)} - \frac{\epsilon_{xy}}{2} v_i^{(xy)} v_j^{(xy)} \right\}$$

$$+ \left\{ \frac{\epsilon_{xy}}{4} (\epsilon_x - \epsilon_y) \right\}$$

$$\left\{ \frac{\epsilon_{xy}}{2} (\epsilon_x - \epsilon_y) \right\}$$

$$\left\{ \frac{\epsilon_{xy}}{2} (\epsilon_x - \epsilon_y) \right\}$$

(A16)

where $v_{\phi\varphi} = \partial (\epsilon_x + \epsilon_y) / \partial k_i, v_{\varphi\varphi} = \partial (\epsilon_x - \epsilon_y) / \partial k_i$. Here $\Lambda_{\alpha\beta}$ are terms which come from expanding traces involving $B$ in Eq. (A13) to quadratic order in $q$.

Combining the mass terms from Eqs. (A15) and (A16) gives the total mass term:

$$M = \int \frac{d^2 k}{(2\pi)^2} \frac{4 \Delta^2 \epsilon_{xy}}{E_+^{(\Delta)} E_-^{(\Delta)} (E_+^{(\Delta)} + E_-^{(\Delta)})} - 2 \epsilon_{xy}$$

$$\equiv \int \frac{d^2 k}{(2\pi)^2} \frac{4 \Delta^2 \epsilon_{xy}}{E_+^{(\Delta)} E_-^{(\Delta)} (E_+^{(\Delta)} + E_-^{(\Delta)})}$$

(A17)
relevant contributions: from $\phi, \varphi, \chi$ phase fluctuations.

\[
\left( N_{\phi\phi} c_{\phi\phi, ij}^2 e_{\phi\varphi, ij}^2 N_{\varphi\varphi} c_{\varphi\varphi, ij}^2 \right) = \frac{1}{8} \int \frac{d^3 k}{(2\pi)^2} \left\{ - \frac{1}{(E_+ - E_-)} \left[ \frac{E_+}{E_+^{(\Lambda)}} - \frac{E_-}{E_-^{(\Lambda)}} \right] \right. \\
+ \frac{2}{(E_+^{(\Lambda)} + E_-^{(\Lambda)})(E_+ - E_-)^2} \left( 1 - \frac{\Delta^2 + E_+ E_-}{E_+^{(\Lambda)} E_-^{(\Lambda)}} \right) \right.
\]

\[
\left[ - \left( \epsilon_{xy}^2 + \epsilon_{xy}^2 \right) v_i^{(xy)} v_j^{(xy)} - \epsilon_{xy}^2 v_i v_j \right] + \frac{\epsilon_{xy}^2 (v_i v_j + v_j v_i)}{2} \left( \epsilon_{xy}^2 - \frac{1}{2} (\epsilon_x - \epsilon_y)^2 \right) v_i^{(xy)} v_j^{(xy)} - \epsilon_{xy}^2 v_i v_j \left( v_i v_j + v_j v_i \right) \\
+ \left( \epsilon_{xy}^2 (\epsilon_x - \epsilon_y) \right) \left( v_i^{(xy)} v_j^{(xy)} + v_j^{(xy)} v_i^{(xy)} \right) - \epsilon_{xy}^2 (\epsilon_x - \epsilon_y) \left( v_i^{(xy)} v_i^{(xy)} + v_j^{(xy)} v_j^{(xy)} \right) \right) \\
+ \left( 0 \right) \left( \Lambda_{\phi\varphi} \Lambda_{\varphi\varphi} \right) \right\}.
\]

(A18)

Appendix B: Effective action with Coulomb terms

Including terms generated by the Coulomb repulsion modifies the interaction term $\Sigma$ of the full fermion propagator [A11] according to [B2]:

\[
\Sigma = \tau_3 \otimes 1(i \frac{\Omega \phi}{2} - i e \chi) + \tau_3 \otimes \tau_3 \frac{\Omega \varphi}{2} + \Sigma_{kin}
\]

(B1)

where $\Sigma_{kin}$ involves only spatial derivatives of the phases $\theta_1$ and $\theta_2$. Here $\chi$ is the Hubbard-Stratonovich field associated with the Coulomb interaction. The form of the coupling for $\chi$ to fermions can be deduced from gauge invariance: the phase $\theta_1$ is obviously a gauge-dependent quantity, and the gauge-invariant degrees of freedom are the combinations $\partial \theta_2 - e \chi - e \Lambda_0$ and $\nabla \theta_2 - e / c \mathbf{A}$. Hence the effective action for $\chi$ is the same as that for $\partial \theta_2 / (2 e)$. Eq. (B1) indicates that $\chi$ couples in all cases like the time derivative of the symmetric component of the phase fluctuations.

To obtain the full effective action for the phase only modes in the presence of Coulomb interactions, we first integrate out the fermions, giving an effective action for the 3 Hubbard-Stratonovich fields $\phi, \varphi, \chi$. There are two relevant contributions: from $Tr G \Sigma$, we obtain:

\[
-i e \chi(q) Tr [(\tau_3 \otimes 1) G_{k-q}] = -i e \chi(q) \langle \phi_{k-q} \rangle
\]

(B2)

which cancels the first-order term in $\chi$ in the effective action (A2).

From $Tr G E \Sigma$, we obtain contributions whose coefficients are the same as the contributions from the time derivatives of $\phi$. In particular, as the coefficients of the cross-terms in $q, \Omega$ from traces $G E \Sigma$ all vanish, the couplings between $\chi$ and $\phi, \varphi$ depend only on $\Omega$. Hence the effective action for the fields $\phi, \varphi, \chi$ has the form:

\[
S_{eff} = \int d\Omega d^2 q \left( \phi \varphi \chi \right) \left( N_{\phi\phi} \left[ \Omega^2 - e_{\phi\phi}^2 q^2 \right] - 2 \Omega \Omega_{\phi\phi} \left[ \Omega^2 - e_{\phi\phi}^2 q^2 \right] - 4 \Omega_{\phi\phi} \right) \left( \phi \right) \\
+ \Delta_{\phi} V_{\alpha, \beta}^{-1} \Delta_\beta
\]

(B3)

where the coefficients $N, c$ are given in Eqs (5), (A15), and (A16). The dispersion relation is given by finding the values of $q, \Omega$ at which $\mathcal{M}$ is singular. Depending on the values of the parameters, $\mathcal{M}$ may have one or two modes which are finite as $q \to 0$. One of these is the gapped Leggett mode; the other is a sound-like mode (the Carlson-Goldman mode) which we find to be absent at $T = 0$, consistent with Ref. [17]. The third mode is, of course, the plasma mode, which does not appear in the low-energy spectrum. To study only the phase modes, we may equivalently integrate out $\chi$ and $\phi$ to obtain Eq. (A11).
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