Global existence of solutions to the Einstein-massive scalar field equations with a cosmological constant for a perfect fluid on the flat Robertson-Walker space-time

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Abstract

In many cases a massive nonlinear scalar field can lead to accelerated expansion in cosmological models. This paper contains mathematical results on this subject for flat Robertson-Walker space-time. Global existence to the coupled Einstein-massive scalar field system which rules the dynamics of a kind of pure matter in the presence of a massive scalar field and cosmological constant is proved, under the assumption that the scalar field \( \phi \) is a non-decreasing function, in Robertson-Walker space-time; asymptotic behaviour are investigated in the case of a cosmological constant bounded from below by a strictly negative constant depending only on the massive scalar field.

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1 Introduction

Scalar fields currently play a remarkable role in the construction of cosmological scenarios aiming to describe the structure and evolution of the early universe. Cosmological models with accelerated expansion are presently of great astrophysical interest. Two main themes are inflation, which concerns the very early universe,
and the accelerated cosmological expansion at the present epoch as evidenced, for instance, by supernova observations. Generally the simplest way of obtaining accelerated expansion within General Relativity is a positive cosmological constant. A more sophisticated variant is the presence of a nonlinear scalar field. In this paper we consider both cosmological constant and scalar field and this leads to interesting new results. For the two themes mentioned above this scalar field is known as the inflaton and quintessence, respectively. A cosmological constant alone may not be an alternative for describing the real universe if the observational data indicate different amounts of acceleration at different times. Evidence of this kind based on supernovae-Ia data is presented in [1]. There is at present no good physical understanding of the exact nature of the inflaton or quintessence and it may even be that both are the same field, producing different phenomena at different times.

Global dynamics of various matter fields coupled to the Einstein equations remain an active research area in General Relativity. The Robertson-Walker space-time is considered to be the basic space-time in cosmology, where homogeneous phenomena such as the one we consider in the present paper are relevant. There are several reasons why it is of interest to consider the Einstein equations with cosmological constant and to couple the equations to a massive nonlinear scalar field.

- Astrophysical observations have made evident the fact that, even in the presence of material bodies, the gravitational field can propagate through space at the speed of light, analogously to electromagnetic waves. A mathematical way to model this phenomenon is to couple a scalar field to the Einstein equations. Let us recall that the Nobel Prize of physics 1993 was awarded for works on this subject. More details on this question can be found in [3], [17]. In fact, several authors realized the interest of coupling scalar field to other fields equations; see for instance [2], [8], [18], [17], [21], [11].

- Now our motivation for considering the Einstein equations with cosmological constant $\Lambda$ is due to the fact that astrophysical observations, based on luminosity via redshift plots of some far away objects such as Supernovae-Ia [1], have made evident the fact that the expansion of the universe is accelerating. A classical mathematical tool to model this phenomenon is to include the cosmological constant $\Lambda$ in the Einstein Equations. In recent years, cosmological models with accelerated expansion have become a particularly active research topic; see for instance [7], [13], [16], [14], [19], [20].

Also recall that the recent Nobel prize of Physics, 2011, was awarded to three Astrophysicists for their advanced research on this phenomenon of accelerated expansion of the universe.
In fact, we must point out that the notion of "dark energy" was introduced in order to provide a physical explanation to universe expansion phenomenon, but the physical structure of this hypothetical form of energy which is unknown in the laboratories remains an open question in modern cosmology; so is the question of "dark matter". Also notice that the scalar fields are considered to be a mechanism producing accelerated models, not only in "inflation", which is a variant of the Big-Bang theory including now a very short period of very high acceleration, but also in the primordial universe. Formally, the Robertson-Walker model with scalar field is obtained if we add to the matter content of the classical Friedmann universe a perfect fluid with energy density \( \epsilon := \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 \) and pressure \( q := \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 \). However, this fluid violates the strong energy condition, i.e., \( \epsilon + 3q = 2 \dot{\phi}^2 - m^2 \phi^2 \) may be negative. It is precisely this violation that leads to inflation in the very early universe [6].

In General Relativity, the evolution of Robertson-Walker models of ordinary matter with a massive scalar are governed, following [4], by equations:

\[
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi (T_{\alpha\beta} + \tau_{\alpha\beta}) \tag{1}
\]

\[
T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{\alpha\beta} \tag{2}
\]

\[
\tau_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} (\nabla^\lambda \phi \nabla_\lambda \phi + m^2 \phi^2), \tag{3}
\]

what we call in the present paper the Einstein-massive scalar field system; where:

- (1) are the Einstein equations\(^2\) for the metric tensor \( g = (g_{\alpha\beta}) \), which represents the gravitational field. \( g \) in this case depends on a unknown single positive real-valued function \( a = a(t) \), called the cosmological expansion factor. \( R_{\alpha\beta} \) is the Ricci tensor, contracted of the curvature tensor. \( R = g^{\alpha\beta} R_{\alpha\beta} \) is the scalar curvature, contracted of the Ricci tensor.

- The ordinary matter is modeling by (2), which represents the general expression of the stress-matter-energy tensor of a relativistic perfect fluid in the chosen signature of \( g \); in which \( \rho \geq 0 \) and \( p \geq 0 \) are unknown functions of the single variable \( t \), representing respectively the matter density and pressure. We consider a perfect fluid of pure radiation type which means that \( p = (1 - \gamma) \rho \), with \( 0 \leq \gamma \leq 2 \). \(^3\) In order to simplify, we consider a comoving fluid: this implies that \( u_i = u^i = 0 \) where \( u = (u_\alpha) \) is a time-like unit vector represents the velocity.

\(^2\) Units have been chosen so that \( c = 1 = G \).

\(^3\) In this work we consider the case where \( \gamma = \frac{4}{3} \).
- (3) represents the stress-matter-energy tensor associated to a non-decreasing massive scalar field $\phi$, which is as $\rho$ a real-valued function of $t$ and finally $\nabla_\alpha$ is the covariant derivative in $g$.

Now, recall that, solving the Einstein equations is determining both the gravitational field and its sources: this means that we have to determine every unknown function introduced above, namely: $a$, $\rho$ and $\phi$. Notice that the spatially homogeneous coupled Einstein-massive scalar field system turns out to be a non-linear second differential system in $a$, $\phi$ and $\rho$. Also notice that, what we call global solution in the present paper, is a solution defined all over the interval $[0, +\infty]$.

In this paper, we prove that if the initial value of the Hubble variable $u$ is strictly negative and if $\Lambda > -\alpha^2$, where $\alpha > 0$ is a constant depending only on the potential of the scalar field, then the coupled Einstein-massive scalar field system, with cosmological constant, has a global in time-solution. This result extends and completes those of [11]. We investigate the asymptotic behaviour which reveals an exponential growth of the gravitational potentials, confirming the accelerated expansion of the universe.

The paper is organized as follows:

- In section 2, we write the Einstein-massive scalar field system in explicit terms.
- In section 3, we introduce the Cauchy problem and we prove the local and global existence of solutions.
- In section 4, we study the asymptotic behaviour.

## 2 Einstein-massive scalar field equations in $a$, $\rho$, $\phi$

In this section we are going to write the equations (1) in explicit terms, next we proceed by a change of unknown functions. The evolution of Robertson-Walker models with massive scalar field $\phi$, cosmological constant $\Lambda$ and ordinary matter, which described by a perfect fluid with matter density $\rho$, are governed following [10], [11] and [12], by the constraint equation named Hamiltonian equation,

$$3\left(\frac{\dot{a}}{a}\right)^2 - \Lambda = 4\pi((\dot{\phi}^2 + m^2\phi^2 + 2\rho), \quad (4)$$

---

4 That is : $\phi \geq 0$ where overdot denotes differentiation with respect to time $t$.

5 The potential of the scalar field is defined here by $V(\phi) = \frac{1}{2}m^2\phi^2$. 

the evolution equation,
\[ 2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 - \Lambda = -4\pi(\dot{\phi}^2 - m^2\phi^2 + \frac{2\rho}{3}), \] (5)

the equations in \(\phi\) and \(\rho\) resulting from conservation equation,
\[ \ddot{\phi} - \frac{3}{a}\dot{\phi}^2 + m^2\phi = 0, \] (6)
\[ \dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0. \] (7)

In next paragraphs, we study the global existence of solutions \(a, \rho\) and \(\phi\) to the coupled system (5), (6), (7) under constraint (4). For this aim, we make a change of unknown functions in order to deduce an equivalent first order differential system to which standard theory applies. We set:
\[ u = \frac{\dot{a}}{a}; \quad v = \frac{1}{a^2}; \quad \psi = \frac{1}{2}\frac{\dot{\phi}^2}{2}. \] (8)

\(u\) is named the Hubble variable; we deduce from (8):
\[ \ddot{u} = \dot{u} + u^2; \quad \dot{v} = -2uv. \] (9)

We choose to look for a \(C^2\)-non-decreasing massive scalar field, \(\dot{\phi} \geq 0\), (8) then gives:
\[ \dot{\phi} = \sqrt{2}\psi^\frac{1}{2}. \] (10)

According to (4) and (8) we deduce from (5), (6), (7) (8) and (9) the equivalent first order differential system:
\[
\begin{align*}
\dot{u} &= -\frac{3}{2}u^2 + \frac{\Lambda}{2} - 4\pi\left(\psi - \frac{1}{2}m^2\phi^2 + \frac{1}{3}\rho\right) \\
\dot{v} &= -2uv \\
\dot{\psi} &= -6uv\psi - \sqrt{2}m^2\Phi\psi^\frac{1}{2} \\
\dot{\phi} &= \sqrt{2}\psi^\frac{1}{2} \\
\dot{\rho} &= -4u\rho,
\end{align*}
\] (11) (12) (13) (14) (15)

to study, subject to the constraint:
\[ 3u^2 - \Lambda = 8\pi(\psi + \frac{1}{2}m^2\phi^2 + \rho) \] (16)
3 Global existence of solutions

3.1 Cauchy problem and constraints

Let $a_0 > 0$, $\rho_0 \geq 0$, $b_0$, $\dot{\phi}_0 > 0$, $\phi_0 \in \mathbb{R}$ be given real numbers. We look for solutions $a$, $\rho$ and $\phi$ of the Einstein-massive scalar field system over $[0, T]$, $T \leq +\infty$, satisfying:

$$a(0) = a_0 ; \dot{a}(0) = b_0 ; \phi(0) = \phi_0 ; \dot{\phi}(0) = \dot{\phi}_0 ; \rho(0) = \rho_0. \quad (17)$$

Our objective now is to prove the existence of global solutions $a$, $\rho$ and $\phi$ defined over the whole interval $[0, +\infty]$, and satisfying (17) called initial conditions, the given numbers $a_0$, $b_0$, $\phi_0$, $\dot{\phi}_0$, $\rho_0$ being the initial data.

It is well known that equation (4) called the Hamiltonian constraint is satisfied all over the domain of the solutions of equation (5) called evolution equation, if and only if equation (4) is satisfied at $t = 0$ i.e. given (17) if the initial data satisfy:

$$3\left(\frac{b_0}{a_0}\right)^2 - \Lambda = 8\pi(\dot{\phi}_0^2 + m^2 \phi_0^2 + 2\rho_0), \quad (18)$$

called the initial constraint. Notice that if $a_0$, $b_0$, $\phi_0$, $\rho_0$, $\dot{\phi}_0$ and $\Lambda$ are given such that $8\pi(\dot{\phi}_0^2 + m^2 \phi_0^2 + 2\rho_0) + \Lambda \geq 0$, then (18) gives two possible choices of $b_0$, namely $b_0 \geq 0$ and $b_0 < 0$. As we will see, this choice of the sign of $b_0 = \dot{a}_0$ called the initial velocity of expansion, will play a key role as far as the global existence of solutions is concerned. In what follows, we suppose that the initial constraint (18) holds. Now we are going to study the equivalent first order differential system (11)-(15), under the constraint (16) and with the initial conditions at $t = 0$, provided by (17) and (8):

$$u(0) := u_0 = \frac{b_0}{a_0} ; \dot{v}(0) := v_0 = \frac{1}{a_0^2} ; \phi(0) = \phi_0 ; \psi(0) := \psi_0 = \dot{\phi}_0 ; \rho(0) = \rho_0. \quad (19)$$

3.2 Local and global existence of solutions

Notice that, by standard theory on the first order differential systems, the Cauchy problem for the (autonomous) system (11)-(15) always admits a unique local solution. What we want to know now is wether or not, this solution is global. Now, also following the standard theory on the first order differential systems, to show that the solution is global, it will be enough if we prove that any solution of the cauchy problem remains uniformly bounded.
**Theorem 3.1**

Let $\Lambda > -4\pi m^2 \phi_0^2$ be given and suppose $\phi_0 > 0$ and $u_0 > 0$. Then the initial value problem for the Einstein-massive scalar field system (11)-(15) has a unique global solution defined all over the interval $[0, +\infty[$.

**Proof 3.1**

It will be enough if we could prove, given the evolution system (11)-(15) that, if each of the functions : $u, v, \psi, \phi$ and $\rho$ is uniformly bounded over every bounded interval.(for instance $[0, T^*]$, where $T^* < +\infty$)

- **Case of $u$.**
  
  Using the evolution equation (11) in $u$, we obtain:
  
  $$\dot{u} = -\frac{3}{2} u^2 + \Lambda \frac{2}{2} + 2\pi m^2 \phi^2 - 4\pi \psi - \frac{4}{3}\pi \rho.$$  
  
  (20)

  But since $\psi \geq 0$, $\rho \geq 0$ (20) gives:

  $$\dot{u} \leq -\frac{3}{2} u^2 + \Lambda \frac{2}{2} + 2\pi m^2 \phi^2.$$  
  
  (21)

  Considering now the Hamiltonian constraint (16) and since $\psi \geq 0$, $\rho \geq 0$ we have:

  $$-3u^2 + \Lambda + 4\pi m^2 \phi^2 \leq 0,$$  
  
  (22)

  (21) then implies:

  $$\dot{u} \leq 0$$  
  
  (23)

  so that $u$ is decreasing. We also deduce from (22) since by assumption we have $\dot{\phi} \geq 0$, then $\phi \geq \phi(0) > 0$, that:

  $$u^2 \geq \frac{\Lambda}{3} + \frac{4}{3}\pi m^2 \phi_0^2.$$  
  
  (24)

  But by hypothesis the r.h.s of (24) is strictly positive. So, since $u$ is continuous, by the mean value theorem, (24) implies:

  $$u \leq -\left(\frac{\Lambda}{3} + \frac{4}{3}\pi m^2 \phi_0^2\right)^{\frac{1}{2}}$$  
  
  (25)

  or

  $$u \geq \left(\frac{\Lambda}{3} + \frac{4}{3}\pi m^2 \phi_0^2\right)^{\frac{1}{2}} > 0.$$  
  
  (26)

  Also by hypothesis, $u(0) > 0$, then only (26) holds : moreover (23) implies $u \leq u(0)$ and we have inequalities:

  $$\left(\frac{\Lambda}{3} + \frac{4}{3}\pi m^2 \phi_0^2\right)^{\frac{1}{2}} \leq u \leq u(0),$$  
  
  (27)

  which show that $u$ is uniformly bounded.
• Case of \(v\).
Now by (12), and given (27) and \(v > 0\), we have \(\dot{v} < 0\) so that \(v\) is a decreasing and positive function, hence:
\[
0 < v < \frac{1}{a_0^2}.
\] (28)

Therefore \(v\) is also uniformly bounded.

• Case of \(\phi\).
We deduce at once from (22) and using (27) that \(\phi\) is bounded.

• Case of \(\psi\).
Let us consider once more the Hamiltonian constraint on the following form:
\[
3u^2 - \Lambda - 4\pi m^2 \phi^2 - 8\pi \rho = 8\pi \psi;
\] (29)
since the l.h.s of (29) is bounded then \(\psi\) is also bounded.

• Case of \(\rho\).
(15) is a linear first order 0.d.e in \(\rho\), which solves at once over \([0, t], t > 0\) to gives:
\[
\rho(t) = \rho(0) \exp(-4 \int_0^t u(s)ds).
\] (30)

(31) shows that, since by (27), \(u > 0\), that \(|\rho(t)| \leq |\rho_0|\).
This completes the proof of Theorem 3.1.

4 Asymptotic behaviour

The mean curvature \(H\) of the space-time is defined by \(H = -g^{ij}k_{ij}\), where \(k_{ij}\), is the fundamental form of the hypersurfaces of constant time defined in the present case by \(k_{ij} = -\frac{1}{2} \partial_t g_{ij}\); Hence in our paper:
\[
H = -g^{ij}k_{ij} = \frac{3\dot{a}}{a} = 3u.
\] (31)

Now we consider the global solution over \([0, +\infty[\) and we investigate the asymptotic behaviour of the different elements at late time. We introduce as in [11] the following quantity which plays a key role:
\[
Q = H^2 - 24\pi T_{00} - 3\Lambda.
\] (32)

Notice that expression (32) of \(Q\) then shows, using Hamiltonian constraint (16) that \(Q \geq 0\). Let us point out first of all that, the quantity \(Q\) plays a key role in General
Relativity, and in the presence of the massive scalar field, it stands for the quantities $S$ in [8], $Z$ in [5], $\tilde{S}$ in [9], $\overline{S}$ in [18] and reduces to $H^2 \pm 3\Lambda$ in [22] which deals with the case of zero scalar field.

At late time we have the following asymptotic behaviour:

**Theorem 4.1**

*On the assumptions of theorem 3.1:*

\[
Q = \mathcal{O}(e^{-3\nu t}) \quad (33)
\]
\[
\rho = \mathcal{O}(e^{-3\nu t}) \quad (34)
\]
\[
\dot{\phi}^2 = \mathcal{O}(e^{-3\nu t}) \quad (35)
\]
\[
\phi^2 \to L > 0 \quad (36)
\]
\[
T_{00} \to \frac{m^2 - L}{2} \quad (37)
\]
\[
H \to (3C_0)^{\frac{1}{2}} \quad (38)
\]
\[
a \to +\infty. \quad (39)
\]

Where:

\[
\nu = \left[\frac{1}{3}(\Lambda + 4\pi m^2 \phi_0^2)\right]^{\frac{1}{2}} \quad ; \quad C_0 = \Lambda + 4\pi m^2 L.
\]

**Proof 4.1**

- We have, using the expression

\[
T_{00} = \psi + \frac{1}{2}m^2 \phi^2, \quad (40)
\]

then, the evolution equations (13) and (14) in $\psi$ and $\phi$ give:

\[
\dot{T}_{00} = -2H\psi. \quad (41)
\]

Expression (32) of $Q$ then gives, using (41)

\[
\dot{Q} = 2H(\dot{H} + 24\pi \psi) \quad (42)
\]

then using (11) and (31) we obtain:

\[
\dot{Q} = -H(H^2 - 3\Lambda - 24\pi T_{00} + 8\pi \rho). \quad (43)
\]

But since $8\pi \rho \geq 0$ and $-H < 0$ and given the expression of $Q$, (43) leads to:

\[
\dot{Q} \leq -HQ. \quad (44)
\]
Integrating (44) over \([0, t], \ t > 0\) yields:

\[0 \leq Q \leq Q_0 \exp\left(\int_0^t -H ds\right)\]

and (27) gives:

\[0 \leq Q \leq Q_0 \exp\left(-3t\left(\frac{1}{3}(\Lambda + 4\pi m^2 \phi_0^2)\right)^\frac{1}{2}\right)\]

and (33) follows.

- Using the Hamiltonian constraint, the results (34) and (35) (since \(\dot{\phi}^2 = 2\psi\)) are direct consequences of (33).

- The evolution equations in \(\phi\) and \(\psi\) give:

\[m^2 \dot{\phi} + \dot{\psi} = \frac{d}{dt}[\psi + \frac{1}{2}m^2 \phi^2] = -2H \psi. \tag{45}\]

But since \(H > 0\) and \(\psi > 0\), (45) implies that the quantity \(\psi + \frac{1}{2}m^2 \phi^2\) is an decreasing function. We then deduce that:

\[\frac{m^2}{2} \phi^2 \leq \frac{m^2}{2} \phi^2 + \psi \leq \frac{m^2}{2} \phi_0^2 + \psi_0. \tag{46}\]

Hence \(\dot{\phi}^2\) is bounded. But the evolution equation in \(\dot{\phi}\) shows that \(\dot{\phi} > 0\); then \(\dot{\phi} \geq \phi_0 > 0\) and since \(\frac{d}{dt}(\phi^2) = 2\phi \dot{\phi} > 0\), \(\phi^2\) is an increasing function. \(\phi^2\) being positive, increasing and bounded has a strictly positive limit, i.e. there exists \(L > 0\) such that:

\[\phi^2 \rightarrow L\]

with:

\[\phi^2 \leq L,\]

hence (36) follows.

- (37) follows from (35), (36) and (40).

- To prove (38) since by (33) \(Q \rightarrow 0\), its expression (32) shows, using (33) that:

\[H^2 - 3\Lambda \rightarrow 12\pi m^2 L. \tag{47}\]

Hence:

\[H^2 - (3\Lambda + 12\pi m^2 L) = \left[H - (3\Lambda + 12\pi m^2 L)^\frac{1}{2}\right]\left[H + (3\Lambda + 12\pi m^2 L)^\frac{1}{2}\right] \rightarrow 0.\]
But by (27), $H > 0$; so:

$$H + (3\Lambda + 12\pi m^2 L)^{\frac{1}{2}} > (3\Lambda + 12\pi m^2 L) > 0,$$

we then deduce that:

$$H \rightarrow (3C_0)^{\frac{1}{2}}. \quad (48)$$

- **We deduce directly from (16) and (26) since $\dot{a} > 0$ that**: $u = \frac{\dot{a}}{a} \geq C_1 > 0$, where $C_1$ is a constant. Then integrating over $[0, t]$, $t > 0$, we have:

$$a(t) \geq a_0 \exp(C_0t). \quad (49)$$

This completes the proof of Theorem 4.1

**Concluding remarks**

- It follows from (49) that $a(t) \rightarrow +\infty$ as $t \rightarrow +\infty$ which shows an exponential growth of the cosmological expansion factor and (49) also confirms mathematically the acceleration phenomenon of the expansion of the universe.

- Theorem 3.1 extends strictly the results of [11] which proved global existence in the case $\Lambda > \Lambda_0 > 0$. This new interesting result is due to the presence of the massive scalar field. In work in progress we aim to study the cases of positively and negatively curved Robertson-Walker spacetimes and geodesic completeness.

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