A multivariate extension of the Misspecification-Resistant Information Criterion.

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Abstract

The Misspecification-Resistant Information Criterion (MRIC) proposed in [H.-L. Hsu, C.-K. Ing, H. Tong: On model selection from a finite family of possibly misspecified time series models. The Annals of Statistics. 47 (2), 1061–1087 (2019)] is a model selection criterion for univariate parametric time series that enjoys both the property of consistency and asymptotic efficiency. In this article we extend the MRIC to the case where the response is a multivariate time series and the predictor is univariate. The extension requires novel derivations based upon random matrix theory. We obtain an asymptotic expression for the mean squared prediction error matrix, the vectorial MRIC and prove the consistency of its method-of-moments estimator. Moreover, we prove its asymptotic efficiency. Finally, we show with an example that, in presence of misspecification, the vectorial MRIC identifies the best predictive model whereas traditional information criteria like AIC or BIC fail to achieve the task.

Keywords: information criteria, model selection, multivariate time series, Mean Square Prediction Error.
1 Introduction

The model selection step is a fundamental task in statistical modelling and its implementation typically depends upon the objective of the exercise. In the time series framework the focus is on either forecasting future values or describing/controlling the process that has generated the data (DGP). A good model selection criterion must feature a good ability to identify the model with the “best” fit to future values, in a specified sense. In particular, in the parametric time series framework, we can identify two main properties. The first one is consistency, i.e., the ability to select the true DGP with probability one as the sample size diverges. This assumes that a true model exists and is among the set of candidate models. If either the set of candidate models does not contain the true DGP, or, for some reason, a true model cannot be postulated, then a selection criterion should be asymptotically efficient, for instance, in the mean square sense, i.e. it minimizes the mean squared prediction error as the sample size diverges. Starting from the seminal work of Akaike, Akaike [1973] a plethora of model selection criteria has been proposed. These include Akaike’s AIC Akaike [1973, 1974], Schwarz’s Bayesian Information Criterion (BIC) Schwarz [1978], and Rissanen’s Minimum Description Length (MDL) Rissanen [1978]. Such criteria paved the way for various extensions dealing with different unsolved issues. For instance, the AIC is efficient but not consistent (i.e. it leads to select overfitting models), whereas the BIC is consistent but not efficient, see Hsu et al. [2019a] for a discussion.

A recent development for model selection in possibly misspecified parametric time series models in the fixed-dimensionality setting is given by the Misspecification-Resistant Information Criterion (hereafter MRIC) Hsu et al. [2019a]. Fixed-dimensionality means that the number of observations increases to infinity while the number of ‘true’ parameters is finite. In this respect, the MRIC provides a solution to the original research question of Akaike: it enjoys both consistency, in case the true model is included as a candidate, and asymptotic efficiency when a true model either cannot be assumed or is not included. Moreover, when the number of variables in the model grows with the sample size, the MRIC can achieve asymptotic efficiency, without the need for additional criteria. Finally, in the high-dimensional setting, the MRIC can be used together with appropriate model selection criteria to identify the best predictive models. The MRIC is based upon the addi-
tive decomposition of the mean squared prediction error in a term that depends upon the misspecification level and a term that measures the sampling variability of the predictor. The idea is to select the model with smallest variability among those that minimize the misspecification index.

The appealing properties of the MRIC make it an ideal tool for omnibus time series model selection but, to date, only the univariate response case has been studied Hsu et al. [2019a]. In this work we extend the MRIC to multivariate time series with a single regressor as to obtain the vectorial MRIC (hereafter VMRIC). As it will be clear, such an extension does not easily derive from the univariate case since it requires dealing with the dependence structure within the components of the vector of forecasting error and hence relies upon random matrix theory. Such multivariate extension can be used in all those models where many time series depend upon a single regressor, like for instance, in econometrics, where many interest rates depend upon a single macroeconomic indicator, such as inflation. Other possible applications include dimension reduction and hedging, which is intimately connected to the problem of model selection Bessler et al. [2016].

The rest of the paper is organized as follows: in Section 2 we introduce the notation and in Section 2.1 summarize the available results for the univariate case; in Section 3 we extend the MRIC approach to multivariate time series with a single regressor. In particular, in Section 3.1 we obtain the asymptotic decomposition of the Mean Squared Prediction Error (hereafter MSPE) matrix into two parts: the first one is linked to the goodness of fit of the model and the second one depends upon the prediction variance. In Section 3.2 we present the VMRIC and derive a consistent estimator for it, whereas in Section 3.3, we prove the asymptotic efficiency of the VMRIC. Section 4 presents an example to assess the effect of misspecification in the VMRIC framework. All the proofs are detailed in Section 5. A contains auxiliary technical lemmas.

2 Notation and preliminaries

For each $t$, let $\{x_t\}$ and $\{y_t\}$, with $x_t = (x_{t,1}, \ldots, x_{t,m})^\top$ and $y_t = (y_{t,1}, \ldots, y_{t,w})^\top$, be two weakly stationary stochastic processes defined over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $m = 1$ ($w = 1$, respectively) we write $x_t$ ($y_t$). Given a vector $v$ and a matrix
\(M\), we use \(\|v\|\) and \(\|M\|\) to refer to the \(L_2\) vectorial norm and the matrix norm induced by the Euclidean norm, respectively. We write \(o(1)\) \((o_p(1))\) to indicate a sequence that converges (in probability) to zero and \(O(1)\) \((O_p(1))\) to indicate a sequence that is bounded (in probability). Moreover, let \(\{c_n\}\) be a sequence of scalar random variables whereas \(\{v_n\}\) and \(\{M_n\}\) are sequences of random vectors and random matrices, respectively. We adopt the following notation: \(v_n = o_p(c_n)\) if \(\|v_n\|/c_n = o_p(1)\); \(v_n = O_p(c_n)\) if \(\|v_n\|/c_n = O_p(1)\); \(M_n = o_p(c_n)\) if \(\|M_n\|/c_n = o_p(1)\); \(M_n = O_p(c_n)\) if \(\|M_n\|/c_n = O_p(1)\). For further details on matrix algebra see Seber \([2008]\), Horn and Johnson \([2013]\), Oden and Demkowicz \([2018]\), for multivariate time series see Reinsel \([1993]\), Lütkepohl \([2005]\), Tsay \([2014]\), and for asymptotic tools for vector and matrices, see Jiang \([2010]\).

Let \(\{(x_t, y_t), t \in \{1, \ldots, n\}\}\) be the observed sample, and divide the interval \([1, n]\) into the training set \([1, N]\) and the test set \([N + 1, N + h]\), with \(h\) being the forecasting horizon. Note that \(x_t\) can contain both endogenous and exogenous variables, therefore, Model \((1)\) encompasses many different models including, inter alia, VAR and VARX models. We denote \(\bar{x} = n^{-1}\sum_{t=1}^{n} x_t\) and \(\bar{y} = n^{-1}\sum_{t=1}^{n} y_t\), i.e. the two sample means. Without loss of generality assume \(E[x_t] = E[y_t] = 0\). In order to forecast \(y_{n+h}, h \geq 1\), we adopt the following \(h\)-step ahead forecasting Model:

\[
y_{t+h} = B_h x_t + \varepsilon_t^{(h)},
\]

where \(B_h\) is a \((w \times m)\) matrix and \(\varepsilon_t^{(h)}\) is the vector containing the \(w\) \(h\)-step ahead forecast errors; as before, if \(w = 1\) we write \(\varepsilon_t^{(h)}\).

**Remark 1.** Since the model can possibly be misspecified, the prediction error vector \(\varepsilon_t^{(h)}\) can be serially correlated, and also correlated with \(x_s, s \neq t\). Moreover, the multivariate framework differs from Hsu et al. \([2019a]\) in different key aspects. For instance, (i) the components of the error vector can be cross-correlated, and (ii) \(x_t \varepsilon_t^{(h)}\) and \(x_k \varepsilon_k^{(h)}\), for \(t \neq k\), can also be both serially and cross correlated.

Define

\[
\hat{R} = N^{-1} \sum_{t=1}^{N} x_t x_t^\top \quad \text{and} \quad R = E[x_1 x_1^\top].
\]

4
Then, the ordinary least squares estimator (hereafter OLS) of $B_h$ results:

$$\hat{B}_n(h) = \hat{R}^{-1} \left( N^{-1} \sum_{t=1}^{N} x_t y_{t+h}^\top \right). \quad (3)$$

When $w = 1$, $R$ and $B$ become $R$ and $\beta$, respectively. The prediction of $y_{n+h}$, $h \geq 1$, is given by

$$\hat{y}_{n+h} = \hat{B}_n(h)x_n \quad (4)$$

and the corresponding Mean Squared Prediction Error matrix is

$$\text{MSPE}_h = E \left[ (y_{n+h} - \hat{y}_{n+h})(y_{n+h} - \hat{y}_{n+h})^\top \right]. \quad (5)$$

### 2.1 The MRIC for parametric univariate time series models

In Hsu et al. [2019a], the authors focused on the case $w = 1$ and $m \geq 1$. Under appropriate conditions, they obtained the following asymptotic decomposition of MSPE:

$$\text{MSPE}_h = E \left[ (y_{n+h} - \hat{y}_{n+h})^2 \right] = \text{MI}_h + n^{-1}(\text{VI}_h + o(1)), \quad (6)$$

with

$$\text{MI}_h = E \left[ \left( \varepsilon_n^{(h)} \right)^2 \right], \quad \text{VI}_h = \text{tr} \left( \hat{R}^{-1} \hat{C}_{h,0} \right) + 2 \sum_{s=1}^{h-1} \text{tr} \left( \hat{R}^{-1} \hat{C}_{h,s} \right),$$

where $C_{h,s} = E \left[ x_1 x_{1+s}^\top \varepsilon_1^{(h)} \varepsilon_{1+s}^{(h)} \right]$, $s \geq 0$, is the cross-covariance matrix between the regressors and the $h$-step ahead prediction error at lag $s$.

**Remark 2.** The first part of Eq. (6) is the Misspecification Index (MI), linked to the goodness-of-fit of the model and coincides with the $h$-step ahead prediction error variance. The second component is the Variability Index (VI), which depends upon the variance of the $h$-step ahead predictor, $\hat{y}_{n+h} = \hat{\beta}_n^\top(h)x_n$, and is also linked to the bias of the estimator of $\beta_h$.

Based upon the above decomposition, the MRIC is defined as follows:

$$\text{MRIC}_h = \hat{\text{MI}}_h + \frac{\alpha_n}{n} \hat{\text{VI}}_h, \quad (7)$$

with $\hat{\text{MI}}_h$ and $\hat{\text{VI}}_h$ being the estimators of $\text{MI}_h$ and $\text{VI}_h$ respectively, i.e.:

$$\hat{\text{MI}}_h = N^{-1} \sum_{t=1}^{N} \left( \varepsilon_t^{(h)} \right)^2, \quad \hat{\text{VI}}_h = \text{tr} \left( \hat{R}^{-1} \hat{C}_{h,0} \right) + 2 \sum_{s=1}^{h-1} \text{tr} \left( \hat{R}^{-1} \hat{C}_{h,s} \right),$$
where 
\[ \hat{C}_{h,s} = (N - s)^{-1} \sum_{t=1}^{N-s} x_t x_{t+s}^\top \hat{\varepsilon}_t^{(h)} \hat{\varepsilon}_t^{(h)} \] and 
\[ \hat{\varepsilon}_t^{(h)} = y_{t+h} - \hat{\beta}_n(h) x_t \] is the estimated forecast error; \( \alpha_n \) is a penalization term sequence such that, as \( n \) increases:

\[ \frac{\alpha_n}{\sqrt{n}} \to +\infty \quad \text{and} \quad \frac{\alpha_n}{n} \to 0. \]  

(8)

It is shown that \( \hat{MI}_h \) and \( \hat{VI}_h \) are consistent estimators of \( MI_h \) and \( VI_h \), moreover the asymptotic efficiency of the MRIC is proved. By minimizing this criterion, the model which minimizes \( VI \) among those with minimum \( MI \) is selected. Among other features, the MRIC is particularly helpful in situations where competing models present the same goodness-of-fit and the same number of parameters.

**Remark 3.** The type of penalty considered in Hsu et al. [2019a] is similar to that used in Shibata, 1989, p. 230] for the correctly specified case.

### 3 A multivariate extension of the MRIC framework

In this section we extend the MRIC approach to the case where the response is a multivariate time series \( (w \geq 2) \) and the predictor is univariate \((m = 1)\), for a generic \( h \)-step ahead forecast. Hence, Model (1) reduces to \( y_{t+h} = \beta_h x_t + \varepsilon_t^{(h)} \), namely:

\[
\begin{align*}
  y_{t+h,1} &= \beta_{h,1} x_t + \varepsilon_t^{(h)} \\
  y_{t+h,2} &= \beta_{h,2} x_t + \varepsilon_t^{(h)} \\
  &\vdots \\
  y_{t+h,w} &= \beta_{h,w} x_t + \varepsilon_t^{(h)} 
\end{align*}
\]  

(9)

#### 3.1 Asymptotic decomposition of the MSPE matrix

We extend the asymptotic representation of the MSPE\(_h\) defined in (6) which is the key step to derive the VMRIC in this multivariate framework. We rely upon the following assumptions, which are the natural multivariate extensions of those in Hsu et al. [2019a].
Assumptions 1.

(C1) \( \exists q_1 > 5, 0 < K_1 < \infty : \text{ for any } 1 \leq n_1 < n_2 \leq n, \)
\[
E \left[ \left( (n_2 - n_1 + 1)^{-1/2} \sum_{t=n_1}^{n_2} x_t^2 - E x_t^2 \right)^{q_1} \right] \leq K_1.
\]

(C2) 1. \( C_{h,s} = E \left[ \varepsilon_t^{(h)} x_t \left( \varepsilon_{t+s}^{(h)} x_{t+s} \right)^\top \right] \perp t, \)
\[
2. E \left[ x_1 x_n \varepsilon_{1,i}^{(h)} \varepsilon_{n,j}^{(h)} \right] = o(n^{-1}) \forall i, j \in \{1, \ldots, w\}.
\]

(C3) 1. \( \sup_{-\infty < t < \infty} E [ |x_t|^{10} ] < \infty, \)
\[
2. \sup_{-\infty < t < \infty} E \left[ \| \varepsilon_t^{(h)} \|^6 \right] < \infty.
\]

(C4) \( \exists 0 < K_2 < \infty : \text{ for } 1 \leq n_1 < n_2 \leq n, E \left[ \left( (n_2 - n_1 + 1)^{-1/2} \sum_{t=n_1}^{n_2} \varepsilon_t^{(h)} x_t \right)^5 \right] < K_2. \)

(C5) For any \( q > 0, E \left[ |\hat{R}^{-1}|^q \right] = O(1). \)

(C6) \( \exists \mathcal{F}_t \subseteq \mathcal{F}, \mathcal{F}_t \) an increasing sequence of \( \sigma \)-fields such that:
\[
1. x_t \text{ is } \mathcal{F}_t\text{-measurable}
\]
\[
2. \sup_{-\infty < t < \infty} E \left[ |E x_t^2 | \mathcal{F}_{t-k} - R|^3 \right] = o(1), \text{ as } k \to \infty,
\]
\[
3. \sup_{-\infty < t < \infty} E \left[ \left\| E \left[ \varepsilon_t^{(h)} x_t | \mathcal{F}_{t-k} \right] \right\|^3 \right] = o(1), \text{ as } k \to \infty.
\]

Theorem 1. Under the regularity conditions (C1) – (C6), the asymptotic expression of the MSPE\(_h\) defined in (5) results
\[
N \left\{ E \left[ (Y_{n+h} - \hat{Y}_{n+h}) (Y_{n+h} - \hat{Y}_{n+h})^\top - E \left[ \varepsilon_n^{(h)} \varepsilon_n^{(h)} \right] \right] \right\} = R^{-1} E \left[ \left( \varepsilon_1^{(h)} x_1 \right) \left( \varepsilon_1^{(h)} x_1 \right)^\top \right] + \ldots
\]
\[
+ R^{-1} E \left[ \sum_{s=1}^{h-1} \left\{ \left( \varepsilon_1^{(h)} x_1 \right) \left( \varepsilon_s^{(h)} x_{s+1} \right)^\top + \left( \varepsilon_s^{(h)} x_{s+1} \right) \left( \varepsilon_1^{(h)} x_1 \right)^\top \right\} \right] + o(1).
\]
3.2 VMRIC and its consistent estimation

In this section we introduce the VMRIC. Let \( \{\alpha_n\} \) be the penalization term sequence defined as in Eq. (8).

\[
\text{VMRIC}_h = \| \text{MI}_h \| + \left\| \frac{\alpha_n}{n} \text{VI}_h \right\|, \tag{11}
\]

where \( \text{MI}_h = E \left[ \left( \varepsilon_t^{(h)} \varepsilon_t^{(h)\top} \right) \right], \quad \text{VI}_h = R^{-1} \left( C_{h,0} + \sum_{s=1}^{h-1} \left( C_{h,s} + C_{h,s}^\top \right) \right), \]

\( C_{h,s} = E \left[ \left( x_t \varepsilon_t^{(h)} \right) \left( x_t \varepsilon_t^{(h)} \right)^\top \right]. \)

The VMRIC can be estimated via the method of moments as to obtain:

\[
\hat{\text{VMRIC}}_h \equiv \| \hat{\text{MI}}_h \| + \left\| \frac{\alpha_n}{n} \hat{\text{VI}}_h \right\|, \tag{12}
\]

where \( \hat{\text{MI}}_h = N^{-1} \sum_{t=1}^N \left( \hat{\varepsilon}_t \hat{\varepsilon}_t^\top \right) , \quad \hat{\text{VI}}_h = \hat{R}^{-1} \left[ \hat{C}_{h,0} + \sum_{s=1}^{h-1} \left( \hat{C}_{h,s} + \hat{C}_{h,s}^\top \right) \right], \)

and \( \hat{C}_{h,s} = (N-s)^{-1} \sum_{t=1}^{N-s} x_t x_{t+s} \hat{\varepsilon}_t \hat{\varepsilon}_{t+s}^\top, \) and \( \hat{\varepsilon}_t = y_{t+h} - \hat{\beta}_n(h)x_t \) is the estimated forecast error vector.

In Theorem 2 we prove that \( \hat{\text{MI}}_h \) and \( \hat{\text{VI}}_h \) are consistent estimators of \( \text{MI}_h \) and \( \text{VI}_h \), respectively. Theorem 2 relies upon the following assumptions, that are less restrictive with respect to (C1) – (C6). For further discussions on the assumptions see [Hsu et al., 2019a, Remark 1–3, p. 1073].

**Assumptions 2.** For each \( 0 \leq s \leq h - 1 \), we assume the following:

\[
(A1) \quad n^{-1} \sum_{t=1}^n \left( \varepsilon_t^{(h)} \varepsilon_t^{(h)\top} \right) = E \left[ \varepsilon_t^{(h)} \varepsilon_t^{(h)\top} \right] + O_p \left( n^{-1/2} \right)
\]

\[
(A2) \quad n^{-1} \sum_{t=1}^n \left( x_t \varepsilon_t^{(h)} \right) \left( x_{t+s} \varepsilon_{t+s}^{(h)} \right)^\top = C_{h,s} + o_p(1),
\]

\[
(A3) \quad n^{-1/2} \sum_{t=1}^n x_t \varepsilon_t^{(h)} = o_p(1).
\]

\[
(A4) \quad n^{-1} \sum_{t=1}^n x_t^2 = R + o_p(1),
\]

\[
(A5) \quad \sup_{-\infty < t < \infty} E \left[ \left\| \varepsilon_t^{(h)} \right\|^4 \right] + \sup_{-\infty < t < \infty} E \left[ |x_t|^4 \right] < \infty.
\]
Theorem 2. If Assumptions (A1) – (A5) hold, then for the case \( w \geq 2 \), and \( m = 1 \) we obtain:

\[
\hat{MI}_h = MI_h + O_p(n^{-1/2}), \\
\hat{VI}_h = VI_h + o_p(1).
\]

3.3 Asymptotic efficiency

In this section we prove the asymptotic efficiency of the VMRIC in the fixed dimensionality framework. To this end, let \( \mathcal{M} \) be the set of \( K \) candidate models; each model is indicated either by \( \ell \) or \( \kappa \), \( 1 \leq \ell, \kappa \leq K \). Define the subsets \( M_1 \) and \( M_2 \) as follows:

\[
M_1 = \left\{ \kappa : 1 \leq \kappa \leq K, \|MI_h(\kappa)\| = \min_{1 \leq \ell \leq K} \|MI_h(\ell)\| \right\}
\]

(13)

\[
M_2 = \left\{ \kappa : \kappa \in M_1, \|VI_h(\kappa)\| = \min_{\ell \in M_1} \|VI_h(\ell)\| \right\}.
\]

(14)

In short, for a given forecast horizon \( h \), \( M_1 \) contains the models with the minimum \( MI_h \) whereas in \( M_2 \) we are minimizing \( VI_h \) among the candidates models in \( M_1 \). The definition of efficiency used in our framework is the same as that of Hsu et al. [2019a]:

Definition 1. Given a sample of size \( n \), a model selection criterion is said to be asymptotically efficient if it selects the model \( \hat{\ell}_h \) such that

\[
\lim_{n \to \infty} \Pr \left( \hat{\ell}_h \in M_2 \right) = 1.
\]

Remark 4. Alternative definitions of asymptotic efficiency for model selection are available. For instance, in the framework of linear stationary processes, Shibata [1980] defines the Mean Efficiency when a criterion attains asymptotically a lower bound for the sum of squared prediction errors. Also, the notion of Approximate Efficiency is given in Shibata [1984]. In Li [1987], a criterion that depends upon the ratio between loss functions is introduced. This latter definition is similar to the Loss Efficiency proposed in Shao [1997].

The VMRIC selects the model with the smallest variability index among those that achieve the best goodness of fit. Hence, the selected model \( \hat{\ell}_h \) is such that:

\[
VMRIC_h \left( \hat{\ell}_h \right) \equiv \min_{1 \leq \ell \leq K} \|\hat{MI}_h(\ell)\| + \min_{\ell \in M_1} \left\| \frac{C_n}{n} \hat{VI}_h(\ell) \right\|.
\]

(15)
In the next Theorem we show that the VMRIC is an asymptotic efficient model selection criterion in the sense of Definition 1.

**Theorem 3.** Assume that for each $1 \leq \ell \leq K$, $0 \leq s \leq h - 1$, Theorem 2 holds and let $\hat{\ell}_h$ be the model selected by the VMRIC. Then we have that:

$$\lim_{n \to \infty} \Pr\left(\hat{\ell}_h \in M_2\right) = 1,$$

namely, the VMRIC is asymptotically efficient in the sense of Definition 1.

### 4 Example: a misspecified bivariate AR(2) model

The aim of this section is twofold. First, we assess the goodness of the theoretical derivations and the finite sample behaviour of the method of moments estimator for the VMRIC. Second, we show that in presence of misspecification, the VMRIC leads to selecting the best predictive model (i.e. is asymptotically efficient) whereas both the AIC and the BIC fail to do so. In order to achieve the goals we consider a bivariate AR(2) DGP and use two misspecified predictive models for it: in Model 1 there is one omitted lagged predictor, whereas Model 2 uses only one non-informative predictor. We derive theoretically the Mean Square Prediction Error matrix and the VMRIC for both models and these show that Model 1 is a better predictive model over Model 2. Based on this, we assess the ability of the VMRIC, and of the multivariate versions of the AIC and BIC to select the best model (Model 1) in finite samples and for different parameterizations.

We start by providing the definition of misspecification. Consider an increasing sequence of $\sigma$-fields, $\{\mathcal{G}_t\}$ such that $\sigma(\mathbf{x}_s, s \leq t) \subseteq \mathcal{G}_t \subseteq \mathcal{F}$, where $\{\mathbf{x}_t\}$ is an $m$-dimensional weakly stationary process defined over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.** The $h$-step ahead forecasting model:

$$\mathbf{y}_{t+h} = \beta_h^\top \mathbf{x}_t + \mathbf{e}_t^{(h)},$$

is correctly specified with respect to an increasing sequence of $\sigma$-fields, $\{\mathcal{G}_t\}$ if

$$E [\mathbf{y}_{t+h} \mid \mathcal{G}_t] = \beta_h^\top \mathbf{x}_t \ a.s., \ \forall - \infty < t < \infty.$$  

(17)

Otherwise, it is misspecified.
Remark 5. The presence of misspecification implies that: $E[\varepsilon_t^{(h)} x_t] = 0$, while it is possible to have $E[\varepsilon_t^{(h)} x_s] \neq 0$, for $s \neq t$, i.e. we have null simultaneous correlation and non-null cross correlation between the forecasting error vector and the regressor.

Consider the following DGP:

$$y_{t+1} = aw_t + \varepsilon_{t+1},$$

where $a \neq 0$, $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (hereafter i.i.d.) bivariate random vectors with $E[\varepsilon_1] = 0$, $E[\varepsilon_1\varepsilon_1'] > 0$ and $w_t$ is the following scalar AR(2) process:

$$w_t = \phi_1 w_{t-1} + \phi_2 w_{t-2} + \delta_t,$$

where $\phi_1 \phi_2 \neq 0$, $\{\delta_t\}$ a sequence of i.i.d. random variables independent of $\{\varepsilon_t\}$ such that $E[\delta_1] = 0$ and $E[\delta_1^2] = 1 - \phi_2^2 - \phi_1^2 \frac{1 + \phi_2}{1 - \phi_2}$. Hence, we obtain $E[w_t^2] \equiv \gamma_w(0) = 1$, where $\gamma_w(j) = E[w_tw_{t+j}]$ is the $j$-th lag autocovariance of $w$.

We consider the correctly specified 2-step ahead forecasting model:

$$y_{t+2} = aw_{t+1} + \varepsilon_{t+1},$$

which leads to

$$y_{t+2} = a \phi_1 w_t + a \phi_2 w_{t-1} + \varepsilon_t^{* (2)},$$

where $\varepsilon_t^{* (2)} = \varepsilon_{t+2} + a \delta_{t+1}$. It can be easily proved that $E[\varepsilon_t^{* (2)} w_{t-j}] = 0$ for $j \geq 0$.

Now, consider the following misspecified model, Model 1:

$$y_{t+2} = \beta w_t + \varepsilon_t^{(2)},$$

with $\beta = \frac{E[y_{t+2}w_t]}{V[w_T]} = a \left( \frac{\phi_1 + \phi_1 \phi_2}{1 - \phi_2} \right)$. The forecasting error results:

$$\varepsilon_t^{(2)} = \varepsilon_t^{* (2)} - a \phi_2 \left[ \frac{\phi_1}{1 - \phi_2} w_t - w_{t-1} \right].$$

Remark 6. In presence of misspecification $E[\varepsilon_t^{(2)} w_t] = 0$, whereas $E[\varepsilon_t^{(2)} w_{t-j}] \neq 0$ for $j \neq 0$. We show that this occurs in our case:

$$E[\varepsilon_t^{(2)} w_{t-j}] = -a \frac{\phi_2}{1 - \phi_2} \left\{ \phi_1 E[w_tw_{t-j}] - (1 - \phi_2) E[w_{t-1}w_{t-j}] \right\}$$

$$= -a \frac{\phi_2}{1 - \phi_2} \left\{ \gamma_w(j + 1) - \gamma_w(j - 1) \right\},$$

which is zero if $j = 0$, otherwise this is generally not the case.
We compute the theoretical value of the VMRIC by using Eq. (11). After some routine algebra, we get:

\[
\text{MI} = E \left[ \epsilon_n^{(2)} \epsilon_n^{(2)\top} \right] = \sigma_\varepsilon^2 + aa^\top \left[ \sigma_\delta^2 + \phi_2^2 (1 - \gamma_w^2(1)) \right],
\]

(22)

which highlights how the variance-covariance matrix of the 2-step ahead forecast vector is equal to the DGP’s variance-covariance plus a bias term that depends upon the misspecification considered.

Now we focus on the variability index VI. We get

\[
C_{2,0} = \sigma_\varepsilon^2 + aa^\top \left\{ \sigma_\delta^2 + \phi_2^2 \left( \gamma_w(1)^2 E \left[ w_t^4 \right] - 2\gamma_w(1) E \left[ w_t^3 w_{t-1} \right] + E \left[ w_t^2 w_{t-1}^2 \right] \right) \right\}
\]

(23)

and

\[
C_{2,1} = aa^\top \gamma_w(1) \left( b_1 E \left[ w_{t-1}^3 w_{t-2} \right] + b_2 E \left[ w_{t-1} w_{t-2}^3 \right] + b_3 E \left[ w_{t-1}^2 w_{t-2}^2 \right] \right),
\]

(24)

where

\[
b_1 = 2\phi_1 \phi_2 \gamma_w(1) - \phi_2, \quad b_2 = -\phi_2^2, \quad b_3 = \phi_2 \left( \phi_2 \gamma_w(1) - 2\phi_1 + \gamma_w(1)^{-1} \right).
\]

Following Eq. (11), the results from Eq. (22), (23), and (24), deliver the VMRIC for this case.

Now we consider a second misspecified model, Model 2:

\[
y_{t+2} = \rho z_t + \eta_t^{(2)},
\]

(25)

where \( z_t \) is a weakly stationary linear AR(1) process independent of \( w_t \):

\[
z_t = \psi_1 z_{t-1} + v_t
\]

(26)

with \( \psi_1 \in (-1, 1) \), and \( \{v_t\} \) is a sequence of i.i.d. random variables independent of both the error terms \( \{\delta_t\} \) and \( \{\varepsilon_t\} \) such that \( E[v_t] = 0 \) and \( E[v_t^2] = 1 - \psi_1^2 \), delivering \( E[z_t] = 0 \) and \( E[z_t^2] = 1 \). Thus, \( z_t \) is uncorrelated with both \( w_t \) and \( y_t \), therefore \( \rho = 0 \). The forecasting error in this case results \( \eta_t^{(2)} = aw_{t+1} + \varepsilon_{t+2} \). Following similar arguments as above we obtain MI and VI for Model 2:

\[
\text{MI} = \sigma_\varepsilon^2 + aa^\top
\]

(27)

\[
\text{VI} = \sigma_\varepsilon^2 + aa^\top (1 + 2\psi_1 \gamma_w(1))
\]

(28)

As mentioned above, Model 1 is misspecified since it omits the lagged predictor \( w_{t-1} \), while Model 2 only includes the non-informative predictor \( z_t \).
Table 1: Parameters’ combinations for the DGP of Eq. (18), (19), and (26).

| Case | $\phi_1$ | $\phi_2$ | $a_1$ | $a_2$ | $\psi_1$ |
|------|---------|---------|-------|-------|---------|
| 1    | 0.4     | -0.75   | 1.50  | -2.00 | 0.80    |
| 2    | -0.4    | -0.45   | -0.75 | 1.25  | -0.65   |
| 3    | 0.3     | -0.80   | 1.00  | 0.50  | -0.75   |

4.1 Finite sample performance

First, we compare the above theoretical derivations with their sample counterpart. We consider three different parameterizations, presented in Table 1. Also, $\alpha_n = n^\alpha$ with $\alpha = 0.85$. Note that, in order for Eq. (8) to hold, $\alpha$ must range in $(0, 1)$. Further experiments showed that results are fairly robust if reasonable values of $\alpha$ are selected. For an empirical method to determine it, see [Hsu et al., 2019b, Section 5]. We take the following variance/covariance matrix for the innovations:

$$E[\varepsilon_t \varepsilon_t^\top] = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$ 

We compute both the VMRIC for Model 1 and Model 2, and estimate the VMRIC and VMRIC on a large sample of $n = 10^6$ observations. The results are shown in Table 2 for the two models, where the theoretical VMRIC (rows 1 and 3) is compared with the estimated one (rows 2 and 4). The results seem to confirm the consistency of the estimator shown in Eq. (12). Clearly, the VMRIC of Model 1 is consistently smaller than that of Model 2 and indicates its superior predictive capability. The finite sample behaviour of the method of moments estimator of the VMRIC can be further appreciated in Table 3 where we show their bias and Mean Squared Error (MSE). The results are based upon 1000 Monte Carlo replications and seem to indicate a rate of convergence of the order of $n^{-1}$.

In Table 4, we show the percentages of correct model selection by the VMRIC, compared with the multivariate version of the AIC and BIC for the three parameterizations of Table 1. For a sample size of $n = 100$, both the AIC and BIC select the best predictive model in about 50% of the cases and relying upon them is tantamount to tossing a fair coin. In such a case, the VMRIC selects the correct model in about 80% of the cases and reaches 100%
Table 2: Theoretical and estimated VMR I C of Models 1 and 2, for the three parameterizations of Table 1, computed on a data set of \( n = 10^6 \) observations.

| Case | VMRIC | VMRIC | VMRIC | VMRIC |
|------|-------|-------|-------|-------|
| 1    | 6.671 | 6.636 | 7.914 | 7.902 |
| 2    | 2.777 | 2.768 | 3.164 | 3.168 |
| 3    | 2.801 | 2.784 | 2.994 | 2.993 |

for \( n = 1000 \). On the contrary, for Case 3, both the AIC and BIC cannot go above 64% for a sample size as large as \( n = 10000 \) observations and this is a general indication of their lack of asymptotic efficiency.

5 Proofs

In this section we detail the proofs of the three theorems. Hereafter all the derivations hold for any fixed \( h \geq 1 \); for the sake of presentation we write \( \epsilon_t \) instead of \( \epsilon_t^{(h)} \). Remember that \( \{l_n\} \) indicates an increasing sequence of positive integers such that:

\[
l_n \to \infty, \quad \frac{l_n}{\sqrt{n}} = o(1)
\]

and define \( a = n - l_n - h \) and \( b = n - l_n - h + 1 \).

5.1 Proof of Theorem 1

The proof of Theorem 1 relies upon four propositions.

Proposition 1. Under assumptions of Theorem 1, it holds that:

\[
N(1) = (III) + o(1),
\]

where

\[
(I) = -E \left[ x_n \hat{R}^{-1} \left( \hat{\Sigma} \epsilon_n^T + \epsilon_n \hat{\Sigma}^T \right) \right], \quad (III) = -E \left[ x_n R^{-1} \left( \hat{\Sigma}_A \epsilon_n^T + \epsilon_n \hat{\Sigma}_A^T \right) \right],
\]

with \( \hat{\Sigma} = \left( N^{-1} \sum_{t=1}^N x_t \epsilon_t \right) \) and \( \hat{\Sigma}_A = \sum_{t=1}^N \epsilon_t x_t \).
Table 3: Bias and Mean-Squared Error (MSE) for the (method of moments) estimator of the VMRIC for the three parameterizations, $\alpha = 0.85$ and different sample size $n$. The results are based upon 1000 Monte Carlo replications.

| $n$  | Case 1   |   | Case 2   |   | Case 3   |   |
|------|----------|---|----------|---|----------|---|
|      | Bias     | MSE | Bias     | MSE | Bias     | MSE |
| 100  | 0.227    | 1.137 | 0.063    | 0.306 | 0.030    | 0.182 |
| 250  | 0.117    | 0.455 | 0.022    | 0.107 | 0.032    | 0.076 |
| 500  | 0.061    | 0.225 | 0.015    | 0.048 | 0.004    | 0.032 |
| 1000 | 0.019    | 0.109 | 0.010    | 0.023 | 0.002    | 0.015 |
| 2500 | 0.008    | 0.044 | 0.001    | 0.009 | 0.001    | 0.006 |
| 5000 | 0.009    | 0.023 | 0.001    | 0.004 | 0.003    | 0.003 |
| 10000| 0.001    | 0.012 | 0.003    | 0.002 | 0.001    | 0.002 |
| 15000| 0.004    | 0.008 | 0.001    | 0.001 | 0.002    | 0.001 |
| 30000| 0.002    | 0.004 | 0.001    | 0.001 | 0.001    | 0.001 |

Table 4: Percentages of correctly selected models by the three information criteria for the three parameterizations and varying sample size $n$.

| $n$  | Case 1 |          | Case 2 |          | Case 3 |          |
|------|--------|----------|--------|----------|--------|----------|
|      | VMRIC  | AIC      | BIC    | VMRIC  | AIC      | BIC    |
| 100  | 85.9   | 52.5     | 52.5   | 84.6   | 56.2     | 56.2   |
| 1000 | 99.9   | 65.6     | 65.6   | 99.9   | 73.7     | 73.7   |
| 10000| 100    | 88.0     | 88.0   | 100    | 97.8     | 97.8   |
|      | 72.1   | 49.0     | 49.0   | 97.0   | 56.8     | 56.8   |
|      | 100    | 63.8     | 63.8   | 100    | 63.8     | 63.8   |
Proof. Let $A_1 = \sum_{t=1}^{N} (\varepsilon_t x_t)\varepsilon_n^\top$ and note that

$$\| (I) - (III) \| = \left\| E \left[ x_n \left( \hat{R}^{-1} - R^{-1} \right) (A_1 + A_1^\top) \right] \right\|. \quad (31)$$

By using standard properties of the norm, (30) follows upon proving that

$$\left\| E \left[ x_n \left( \hat{R}^{-1} - R^{-1} \right) A_1^\top \right] \right\| = o(1). \quad (32)$$

Let

$$\tilde{R} = (n - l_n)^{-1} \sum_{t=1}^{n-l_n} x_t^2. \quad (33)$$

By adding and subtracting $\varepsilon_n x_n \left( \hat{R}^{-1} \left[ \sum_{t=1}^{N} (\varepsilon_t x_t) \right]^\top \right)$, we have

$$E \left[ x_n \left( \hat{R}^{-1} - R^{-1} \right) A_1^\top \right] = E \left[ \varepsilon_n x_n \left( \hat{R}^{-1} - \tilde{R}^{-1} \right) \sum_{t=1}^{N} \varepsilon_t^\top x_t \right]$$

$$+ E \left[ \varepsilon_n x_n \left( \hat{R}^{-1} - R^{-1} \right) \sum_{t=1}^{N} \varepsilon_t^\top x_t \right]$$

$$= E \left[ \varepsilon_n x_n \left( \hat{R}^{-1} - \tilde{R}^{-1} \right) \left( \sum_{t=1}^{N} \varepsilon_t x_t \right)^\top \right] \quad (34)$$

$$+ E \left[ \varepsilon_n x_n \left( \hat{R}^{-1} - R^{-1} \right) \left( \sum_{t=b}^{N} \varepsilon_t x_t \right)^\top \right] \quad (35)$$

$$+ E \left[ \varepsilon_n x_n \left( \tilde{R}^{-1} - R^{-1} \right) \left( \sum_{t=1}^{a} \varepsilon_t x_t \right)^\top \right]. \quad (36)$$

We show below that the norms of (34), (35) and (36) are asymptotically negligible. Focus on the first one: by combining conditions (C3), (C4), Lemma 1, and Hölder’s inequality, it follows that $\| (34) \|$ is bounded by

$$E \left[ \left\| \varepsilon_n x_n \left( \hat{R}^{-1} - \tilde{R}^{-1} \right) \left( \sum_{t=1}^{N} \varepsilon_t x_t \right)^\top \right\| \right] \leq E \left[ \| \varepsilon_n \|^6 \right]^\frac{1}{6} E \left[ \| x_n \|^6 \right]^\frac{1}{6} E \left[ \left\| \hat{R}^{-1} - \tilde{R}^{-1} \right\|^3 \right]^\frac{1}{3}$$

$$\times E \left[ \left\| N^{\frac{1}{2}} N^{-\frac{1}{2}} \sum_{t=1}^{N} \varepsilon_t x_t \right\|^3 \right]^\frac{1}{3} = O \left( \frac{l_n}{n^{1/2}} \right).$$
which converges to zero due to the definition of \( l_n \) in \((29)\). Similarly, we have that \( \| (35) \| \) is bounded by

\[
E \left[ \| x_n \|^{6/7} E \left[ \left| \hat{R}^{-1} - R^{-1} \right|^{2} \right]^{3/7} \left( N - b + 1 \right)^{\frac{3}{2}} \left( N - b + 1 \right)^{-\frac{1}{2}} \sum_{t=b}^{N} \hat{e}_t x_t \right]^{\frac{3}{7}}.
\]

which is an \( O \left( n^{-1/2} l_n \right) \) thereby vanishing asymptotically. Lastly, Condition (C6), Lemma 1, and Hölder’s inequality imply that \( \| (35) \| \) is bounded by

\[
E \left[ \| \hat{e}_t x_t \|_{F_{t-l_n}} \|^{3} \right]^{\frac{1}{3}} E \left[ \left| \hat{R}^{-1} - R^{-1} \right|^{3} \right]^{\frac{1}{3}} E \left[ \left\| a^{1/2} a^{-1/2} \sum_{t=1}^{a} \hat{e}_t x_t \right\|^{3} \right]^{\frac{1}{3}} = o (1)
\]

and this completes the proof.

**Proposition 2.** Under assumptions of Theorem 1, it holds that:

\[
N(\text{II}) = (\text{IV}) + o(1), \quad (37)
\]

where

\[
(\text{II}) = E \left[ \hat{R}^{-1} \hat{\Sigma} x_n x_n \hat{\Sigma}^{\top} \hat{R}^{-1} \right], \quad (\text{IV}) = E \left[ \hat{\Sigma}_B R^{-1} \hat{\Sigma}_B^{\top} \right],
\]

with \( \hat{\Sigma} \) being defined in Proposition 1 and \( \hat{\Sigma}_B = N^{-\frac{1}{2}} \sum_{t=1}^{N} \hat{e}_t x_t \).

**Proof.** Let \( M_1 = x_n \left( \hat{R}^{-1} - R^{-1} \right) \hat{\Sigma}_B \) and \( M_2 = x_n R^{-1} \hat{\Sigma}_B \). Since

\[
N(\text{II}) = E \left[ (M_1 + M_2) (M_1 + M_2)^\top \right] = E \left[ M_1 M_1^\top \right] + E \left[ M_2 M_2^\top \right] + E \left[ M_1 M_2^\top \right] + E \left[ M_2 M_1^\top \right]
\]

the proof of \((37)\) reduces to show that the following conditions hold:

\[
\| E \left[ M_1 M_1^\top \right] \| = o (1), \quad (38)
\]

\[
\| E \left[ M_1 M_2^\top \right] \| = o (1), \quad (39)
\]

\[
\| E \left[ M_2 M_2^\top \right] - (\text{IV}) \| = o (1). \quad (40)
\]
Conditions (38) and (39) readily follow from Assumptions (C3) and (C4), Lemma 1, the non singularity of \( R \) and Hölder’s inequality:

\[
\mathbb{E} \left[ \| M_1 M_1^\top \| \right] = \mathbb{E} \left[ \left\| x_n^2 \left( \hat{R}^{-1} - R^{-1} \right)^2 \hat{\Sigma}_B \hat{\Sigma}_B^\top \right\| \right] \leq \left( \mathbb{E} \left[ |x_n|^{10} \right] \right)^{\frac{1}{5}} \left( \mathbb{E} \left[ \| \hat{R}^{-1} - R^{-1} \|^{5} \right] \right)^{\frac{2}{5}} \times \left( \mathbb{E} \left[ \| \hat{\Sigma}_B \|^{5} \right] \right) = o(1);
\]

\[
\mathbb{E} \left[ \| M_1 M_2^\top \| \right] = \mathbb{E} \left[ \left\| x_n^2 \left( \hat{R}^{-1} - R^{-1} \right) R^{-1} \hat{\Sigma}_B \hat{\Sigma}_B^\top \right\| \right] \leq \left( \mathbb{E} \left[ |x_n|^{10} \right] \right)^{\frac{1}{5}} \left( \mathbb{E} \left[ \| \hat{R}^{-1} - R^{-1} \|^{5} \right] \right)^{\frac{1}{5}} \times \left( \mathbb{E} \left[ \| \hat{\Sigma}_B \|^{5} \right] \right)^{\frac{2}{5}} = o(1).
\]

As concerns (40), decompose the vector \( \hat{\Sigma}_B \) as:

\[
\hat{\Sigma}_B = N^{-\frac{1}{2}} \sum_{t=1}^{N} \varepsilon_t x_t = u + w, \quad \text{with} \quad u = N^{-\frac{1}{2}} \sum_{t=1}^{a} \varepsilon_t x_t \quad \text{and} \quad w = N^{-\frac{1}{2}} \sum_{t=b}^{N} \varepsilon_t x_t.
\]

Hence, we have that

\[
\mathbb{E} \left[ M_2 M_2^\top \right] - (IV) = \mathbb{E} \left[ u R^{-1} x_n x_n R^{-1} u^\top \right] - \mathbb{E} \left[ u R^{-1} R R^{-1} u^\top \right] + \mathbb{E} \left[ u R^{-1} x_n x_n R^{-1} w^\top \right] - \mathbb{E} \left[ u R^{-1} R R^{-1} w^\top \right] + \mathbb{E} \left[ w R^{-1} x_n x_n R^{-1} u^\top \right] - \mathbb{E} \left[ w R^{-1} R R^{-1} u^\top \right] + \mathbb{E} \left[ w R^{-1} x_n x_n R^{-1} w^\top \right] - \mathbb{E} \left[ w R^{-1} R R^{-1} w^\top \right].
\]

The law of iterated expectations implies that:

\[
\| \mathbb{E} \left[ M_2 M_2^\top \right] - (IV) \|
\leq \| \mathbb{E} \left[ u R^{-1} \left( \mathbb{E} \left[ x_n^2 \mid \mathcal{F}_{n-l_n} \right] - R \right) R^{-1} u^\top \right] \|
+ \| \mathbb{E} \left[ u R^{-1} \left( \mathbb{E} \left[ x_n^2 \mid \mathcal{F}_{n-l_n} \right] - R \right) R^{-1} w^\top \right] \|
+ \| \mathbb{E} \left[ w R^{-1} \left( \mathbb{E} \left[ x_n^2 \mid \mathcal{F}_{n-l_n} \right] - R \right) R^{-1} u^\top \right] \|
+ \| \mathbb{E} \left[ w R^{-1} \left( \mathbb{E} \left[ x_n^2 \mid \mathcal{F}_{n-l_n} \right] - R \right) R^{-1} w^\top \right] \|. \tag{41}
\]

By using arguments previously developed, it is easy to see that, under Assumptions (C4) and (C6), (41) – (44) asymptotically vanish. Therefore, conditions (38) – (40) are fulfilled and the proof is completed. \( \square \)
Proposition 3. Under assumptions of Theorem 1, it holds that:

\[(III) = -(D) + o(1),\]  \hspace{1cm} (45)

where

\[(D) = E \left[ R^{-1} \left\{ \sum_{j=h}^{N-1} \left\{ (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top + (\varepsilon_{j+1} x_{j+1}) (\varepsilon_1 x_1)^\top \right\} \right\} \right]\]

Proof. The result readily follows upon noting that, under Assumption (C2) and the weakly stationarity of the process \(\{ x_t \} \), it holds that:

\[(III) = - \sum_{t=1}^{N} E \left[ R^{-1} \left\{ (\varepsilon_t x_t) (\varepsilon_n x_n)^\top + (\varepsilon_n x_n) (\varepsilon_t x_t)^\top \right\} \right]
= - \sum_{j=h}^{n-1} E \left[ R^{-1} \left\{ (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top + (\varepsilon_{j+1} x_{j+1}) (\varepsilon_1 x_1)^\top \right\} \right]
= - E \left[ R^{-1} \left\{ \sum_{j=h}^{N-1} \left\{ (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top + (\varepsilon_{j+1} x_{j+1}) (\varepsilon_1 x_1)^\top \right\} \right\} \right] + o(1).\]

Proposition 4. Under assumptions of Theorem 1, it holds that:

\[(IV) = (1) + (Q) + (D) + o(1),\]  \hspace{1cm} (46)

where

\[(1) = N^{-1} E \left[ R^{-1} \left\{ \sum_{t=1}^{N} (\varepsilon_t x_t) (\varepsilon_t x_t)^\top \right\} \right],\]
\[(Q) = E \left[ R^{-1} \left\{ \sum_{s=1}^{h-1} \left\{ (\varepsilon_1 x_1) (\varepsilon_{s+1} x_{s+1})^\top + (\varepsilon_{s+1} x_{s+1}) (\varepsilon_1 x_1)^\top \right\} \right\} \right],\]
\[(D) = E \left[ R^{-1} \left\{ \sum_{j=h}^{N-1} \left\{ (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top + (\varepsilon_{j+1} x_{j+1}) (\varepsilon_1 x_1)^\top \right\} \right\} \right].\]

Proof. Let

\[(2) = N^{-1} E \left[ R^{-1} \left\{ \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} (\varepsilon_j x_j) (\varepsilon_k x_k)^\top \right\} \right],\]

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and note that (IV) − (1) = (2) + (2) ⊤. Moreover
\[
(2) = N^{-1} E \left[ R^{-1} \left\{ \sum_{j=1}^{N-1} (N - j) (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top \right\} \right]
\]

\[
= E \left[ R^{-1} \left\{ \sum_{j=1}^{N-1} (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top \right\} \right] - N^{-1} E \left[ R^{-1} \left\{ \sum_{j=1}^{N-1} j (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top \right\} \right].
\]

Assumptions (C2) implies that (48) is o(1). Since (47) can be written as
\[
E \left[ R^{-1} \left\{ \sum_{s=1}^{h-1} (\varepsilon_1 x_1) (\varepsilon_{s+1} x_{s+1})^\top \right\} \right] + E \left[ R^{-1} \left\{ \sum_{j=h}^{N-1} (\varepsilon_1 x_1) (\varepsilon_{j+1} x_{j+1})^\top \right\} \right],
\]
then (47) + (47) ⊤ = (Q) + (D) and this completes the proof.

**Proof of Theorem 1**

We prove that:
\[
N \left\{ E \left[ (y_{n+h} - \hat{y}_{n+h}) (y_{n+h} - \hat{y}_{n+h})^\top - E \left[ \varepsilon_n^{(h)} \varepsilon_n^{(h)\top} \right] \right] \right\}
\]

\[
= R^{-1} E \left[ (\varepsilon_1^{(h)} x_1) (\varepsilon_1^{(h)} x_1)^\top \right] + R^{-1} E \left\{ \sum_{s=1}^{h-1} (\varepsilon_1^{(h)} x_1) (\varepsilon_{s+1}^{(h)} x_{s+1})^\top + (\varepsilon_{s+1}^{(h)} x_{s+1}) (\varepsilon_1^{(h)} x_1)^\top \right\}
\]

\[
+ o(1).
\]

Since
\[
(\hat{\beta} - \beta) = \hat{R}^{-1} \left( N^{-1} \sum_{t=1}^{N} x_t y_{t+h} \right) - \beta = \hat{R}^{-1} \left( N^{-1} \sum_{t=1}^{N} x_t \varepsilon_t \right),
\]
routine algebra implies that:
\[
E \left[ (y_{n+h} - \hat{y}_{n+h}) (y_{n+h} - \hat{y}_{n+h})^\top \right] - E \left[ \varepsilon_n \varepsilon_n^\top \right] = (I) + (II).
\]

By applying Propositions 1 – Propositions 4, we have:
\[
N \left\{ E \left[ (y_{n+h} - \hat{y}_{n+h}) (y_{n+h} - \hat{y}_{n+h})^\top - E \left[ \varepsilon_n^{(h)} \varepsilon_n^{(h)\top} \right] \right] \right\} = N(I) + N(II)
\]

\[
= (III) + (IV) + o(1) = (1) + (Q) + o(1).
\]

The proof is completed upon noting that (1) = (49) and (Q) = (50).
5.2 Proof of Theorem 2

We start proving that
\[
\hat{\text{MI}}_h = \text{MI}_h + O_p(n^{-1/2}).
\] (52)

Note that
\[
\hat{\text{MI}}_h = N^{-1} \left( \sum_{t=1}^{N} \epsilon_t \epsilon_t^\top \right) - \left( N^{-1} \sum_{t=1}^{N} x_t \epsilon_t \right) \hat{R}^{-1} \left( N^{-1} \sum_{s=1}^{N} x_s \epsilon_s \right)^\top
\]

hence, it holds that \( \hat{\text{MI}}_h - \text{MI}_h \) equals
\[
N^{-1} \left\{ \sum_{t=1}^{N} \left( \epsilon_t \epsilon_t^\top - \mathbb{E} [\epsilon_1 \epsilon_1^\top] \right) \right\}
\] (53)
\[- \left( N^{-1} \sum_{t=1}^{N} x_t \epsilon_t \right) \hat{R}^{-1} \left( N^{-1} \sum_{t=1}^{N} x_t \epsilon_t \right)^\top .
\] (54)

Assumption (A1) implies that (53) = \( O_p(n^{-1/2}) \) whereas, by combining Assumptions (A3) and (A4) with the non-singularity of \( R \) and Hölder’s inequality, it can be shown that (54) = \( O_p(n^{-1}) \) and hence the proof of (52) is complete.

Next, we prove that
\[
\hat{\text{VI}}_h = \text{VI}_h + o_p(1).
\]

It suffices to show that
\[
\hat{C}_{h,s} = C_{h,s} + o_p(1).
\] (55)

It holds that \( \hat{C}_{h,s} \) is equal to
\[
(N-s)^{-1} \sum_{t=1}^{N-s} \left(x_t \epsilon_t^\top \right)^\top \left(x_{t+s} \epsilon_{t+s}^\top \right)
\] (56)
\[- (N-s)^{-1} \sum_{t=1}^{N-s} x_t^2 x_{t+s} \left( \hat{\beta}_n(h) - \beta_h \right) \epsilon_{t+s}^\top
\] (57)
\[- (N-s)^{-1} \sum_{t=1}^{N-s} x_t x_{t+s}^2 \epsilon_t \left( \hat{\beta}_n(h) - \beta_h \right)^\top
\] (58)
\[+ (N-s)^{-1} \sum_{t=1}^{N-s} x_t^2 x_{t+s} \left( \hat{\beta}_n(h) - \beta_h \right) \left( \hat{\beta}_n(h) - \beta_h \right)^\top.
\] (59)
We prove that (57) is \( o_p(1) \) componentwise. To this end consider:

\[
E \left[ (N - s)^{-1} \sum_{t=1}^{N-s} x_t^2 x_{t+s} \varepsilon_{t+s,i} \right],
\]

with \( \varepsilon_{t+s,i} \) being the \( i \)-th component of the vector \( \varepsilon_{t+s} \). The triangular inequality and Hölder’s inequality imply that:

\[
E \left[ (N - s)^{-1} \sum_{t=1}^{N-s} x_t^2 x_{t+s} \varepsilon_{t+s,i} \right] \leq (N - s)^{-1} \sum_{t=1}^{N-s} E \left[ x_t^2 x_{t+s} \varepsilon_{t+s,i} \right] \leq (N - s)^{-1} \sum_{t=1}^{N-s} \left\{ (E [x_t^4])^{1/2} (E [x_{t+s} \varepsilon_{t+s,i}^2])^{1/2} \right\}.
\]

Since \( \hat{\beta}_n(h) - \beta_h = \hat{R}^{-1} \left( N^{-1} \sum_{j=1}^N x_j \varepsilon_{j,h} \right) \), by combining Assumptions (A3), (A4) and (A5) with Chebyshev’s inequality we obtain that (57) is \( o_p(1) \). Similarly, we can verify that (58) and (59) are \( o_p(1) \). Lastly, Condition (A2) implies that (56) = \( C_{h,s} + o_p(1) \), hence (55) is verified and the whole proof is complete.

### 5.3 Proof of Theorem 3

By Theorem 2 the VMRIC\(_h\) defined in (15) can be written as:

\[
\text{VMRIC}_h \left( \hat{\ell}_h \right) = \min_{1 \leq \ell \leq K} \| \text{MI}_h + O_p(n^{-1/2}) \| + \min_{\ell \in M_1} \left\| \frac{\alpha_n}{n} \text{VI}_h + o_p \left( \frac{\alpha_n}{n} \right) \right\|.
\]

Therefore,

\[
\lim_{n \to \infty} \text{VMRIC}_h \left( \hat{\ell}_h \right) = \min_{1 \leq \ell \leq K} \| \text{MI}_h \|
\]

and hence

\[
\lim_{n \to +\infty} \Pr \left( \hat{\ell}_h \in M_1 \right) = 1.
\]

Now, consider two models \( \ell_1 \) and \( \ell_2 \) in the candidates set \( J_{\ell_1}, J_{\ell_2} \in M_1 \) such that \( \text{VI}_h(\ell_1) \neq \text{VI}_h(\ell_2) \). We show that

\[
\lim_{n \to \infty} \Pr \left\{ \text{VMRIC}_h(\ell_1) - \text{VMRC}_h(\ell_2) \right\} = \text{sign} \left\{ \| \text{VI}_h(\ell_1) \| - \| \text{VI}_h(\ell_2) \| \right\}
\]

is 1.

By defining \( \text{MI}_h^* \) to be the minimum value of \( \text{MI}_h \) over the family of candidate models, we have:

\[
\text{VMRIC}_h(\ell_1) = \| \text{MI}_h^* + O_p(n^{-1/2}) \| + \left\| \frac{\alpha_n}{n} \text{VI}_h(\ell_1) + o_p \left( \frac{\alpha_n}{n} \right) \right\|,
\]

\[
\text{VMRIC}_h(\ell_2) = \| \text{MI}_h^* + O_p(n^{-1/2}) \| + \left\| \frac{\alpha_n}{n} \text{VI}_h(\ell_2) + o_p \left( \frac{\alpha_n}{n} \right) \right\|.
\]
Therefore, for sufficiently large \( n \), it holds that:

\[
\text{VMRIC}_h(\ell_1) - \text{VMRIC}_h(\ell_2) = \left\| \frac{\alpha_n}{n} \right\| (\| \text{VI}_h(\ell_1) \| - \| \text{VI}_h(\ell_2) \|).
\]

Thus

\[
\text{sign} \{ \text{VMRIC}_h(\ell_1) - \text{VMRIC}_h(\ell_2) \} = \text{sign} \{ \| \text{VI}_h(\ell_1) \| - \| \text{VI}_h(\ell_2) \| \},
\]

and (63) is verified and implies that

\[
\lim_{n \to \infty} \text{Pr} \left( \hat{\ell}_h \in M_2 \right) = 1.
\]

(64)

This completes the proof.

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### A Technical Lemma

The following lemma is a general result that holds for the multivariate predictor case (i.e., \( m \geq 1 \)). It relies upon Assumption (C1) of [Hsu et al., 2019a, p. 1068], which reduces to our Assumption (C1) when \( m = 1 \).

**Lemma 1.** Let \( R \) and \( \tilde{R} \) be defined in (2) and \( \tilde{R} \) be the multivariate version of Eq. (33). Then, for \( 0 < \gamma \leq 5 \), it holds that:

\[
\text{E} \left[ \left\| \tilde{R}^{-1} - \tilde{R}^{-1} \right\| \right] = O \left[ \left( \frac{l_n}{n} \right)^{\gamma} \right],
\]

(65)

\[
\text{E} \left[ \left\| R^{-1} - \tilde{R}^{-1} \right\| \right] = O \left[ n^{-\gamma/2} \right],
\]

(66)

\[
\text{E} \left[ \left\| R^{-1} - \tilde{R}^{-1} \right\| \right] = O \left[ n^{-\gamma/2} \right],
\]

(67)

with \( l_n \) being defined in (29).
Proof. Triangle inequality implies that
\[
\|\hat{R}^{-1} - \tilde{R}^{-1}\| \leq \|\hat{R}^{-1}\| \|\hat{R} - \tilde{R}\| \|\tilde{R}^{-1}\|,
\]
\[
\|R^{-1} - \hat{R}^{-1}\| \leq \|\hat{R}^{-1}\| \|\hat{R} - R\| \|R^{-1}\|,
\]
\[
\|R^{-1} - \tilde{R}^{-1}\| \leq \|\tilde{R}^{-1}\| \|\tilde{R} - R\| \|R^{-1}\|.
\]
Since $R$ is invertible we have that $\|R^{-1}\| = O(1)$; under Assumption (C5), $\|\hat{R}^{-1}\| = O(1)$.
Moreover, it can be easily proved that also $\|\tilde{R}^{-1}\| = O(1)$. Therefore, by deploying Hölder’s inequality, the results will be verified if we prove the following three conditions:
\[
E \left[ \|\hat{R} - \tilde{R}\| \right] = O \left( \frac{ln}{n} \right), \tag{68}
\]
\[
E \left[ \|\hat{R} - R\| \right] = O \left( n^{-1/2} \right), \tag{69}
\]
\[
E \left[ \|\tilde{R} - R\| \right] = O \left( n^{-1/2} \right). \tag{70}
\]
Let $a = n - l_n$ and $b = a + 1$. As for (68) note that
\[
E \left[ \|\hat{R} - \tilde{R}\| \right] \leq E \left[ \left\| \left( \frac{1}{N} - \frac{1}{a} \right) \sum_{t=1}^{a} x_t x_t^\top \right\| \right] + E \left[ \left\| \frac{1}{N} \sum_{t=b}^{N} x_t x_t^\top \right\| \right] = O \left( \frac{ln}{n} \right).
\]
Conditions (69) and (70) readily derive from Assumption (C1) and hence the proof is completed.

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