Supplemental Appendix

“Characteristic-Sorted Portfolios: Estimation and Inference”

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A  Optimal Choice of $J_t$

A.1 Theoretical Quantities

Here we provide explicit formulas for the main terms of the MSE expansion given in Theorem 3. First let us define $q_{jt} = P(z \in P_{jt} | \mathcal{F}_t)$. Then, we have that:

$$B_t(z) = JT^{-1}n_t^{-1}\mu'(z)\sum_{i=1}^{n_t}J_t^d\sum_{j=1}^{J_t}1_{jt}(z)q_{jt}^{-1}1_{jt}(z_{it} - z)$$

where $\mu'(z) = \frac{\partial \mu(z)}{\partial z}|_{z = z_0}$ and

$$V^{(1)}_t(z) = nJ^{-d}T^{-1}n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t}1_{jt}(z)q_{jt}^{-2}\{1_{jt}(z_{it})\sigma_{it}^2 - \mathbb{E}[1_{jt}(z_{it})\sigma_{it}^2 | \mathcal{F}_t]\},$$

$$V^{(2)}_t(z) = n^2J^{-2d}T^{-1}n_t^{-3}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t}1_{jt}(z)q_{jt}^{-3}\{1_{jt}(z_{it})\sigma_{it}^2 - \mathbb{E}[1_{jt}(z_{it})\sigma_{it}^2 | \mathcal{F}_t]\}.$$\[\text{Finally, we have } \mathcal{C} = \sum_{t=1}^{T}C_t(z_L) + \sum_{t=1}^{T}C_t(z_H) \text{ where,}\]

$$C_t(z) = n^{3/2}J^{-3d/2}T^{-1/2}n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t}1_{jt}(z)q_{jt}^{-2}\{1_{jt}(z_{it})\sigma_{it}^2 - \mathbb{E}[1_{jt}(z_{it})\sigma_{it}^2 | \mathcal{F}_t]\}$$

- $2n^{3/2}J^{-3d/2}T^{-1/2}\sum_{t=1}^{T}n_t^{-3}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t}1_{jt}(z)q_{jt}^{-3}(1_{jt}(z_{it}) - q_t)1_{jt}(z_{it})\sigma_{it}^2.$

A.2 Empirical Implementation

As we discussed in Section 5 we base our choice of the optimal number of portfolios in our empirical applications based on equation (11). To do so let $t_{\max} = \arg\max_{1 \leq t \leq T} n_t$, $n = n_{\max}$ and $J = J_{\max}$. For all other time periods we scale $J_t$ as $J_t = J(n_t/n)^{\frac{1}{J_{\max}}}$ (see discussion in Section 4). We then choose a grid of values for $J$ as $J = (n_{\min}/n)^{\frac{1}{J_{\max}}}, \ldots, J_{\max})$ where $t_{\min} = \arg\min_{1 \leq t \leq T} n_t$. In our empirical applications we set $J_{\max} = 400$.

To estimate the MSE in practice we have the following estimator,

$$\hat{\text{MSE}}(\hat{\mu}(z_H) - \hat{\mu}(z_L); J_1, \ldots, J_T)$$

$$= \left(\hat{\mu}'(z_H) \cdot T^{-1} \sum_{i=1}^{n_t}n_t \sum_{j=1}^{J_t}1_{jt}(z_{it})\hat{\mu}_{jt}(z_H)\hat{\mu}_{jt}(z_{it} - z_H)\right)\left(\hat{\mu}'(z_L) \cdot T^{-1} \sum_{i=1}^{n_t}n_t \sum_{j=1}^{J_t}1_{jt}(z_{it})\hat{\mu}_{jt}(z_L)\hat{\mu}_{jt}(z_{it} - z_L)\right)^2$$

$$+ T^{-2} \sum_{t=1}^{T} (\hat{\mu}_t(z_H) - \hat{\mu}_t(z_L) - (\hat{\mu}_t(z_H) - \hat{\mu}_t(z_L)))^2$$  \hspace{1cm} (A.1)

where

$$\hat{\mu}_t(z) = \sum_{j=1}^{J_t}N_t^{-1/2}n_t \sum_{i=1}^{n_t}1_{jt}(z_{it})\hat{\mu}_{jt}(z_{it})(R_{it} - X_{it}\hat{\beta}_t),$$

$$\hat{\mu}(z) = T^{-1} \sum_{t=1}^{T} \hat{\mu}_t(z).$$

Here $\omega_{it}$ is the weight applied to the returns in each portfolio which satisfies $\sum_{t=1}^{n_t}1_{jt}(z_{it})\omega_{it} = 1$ for each
$j = 1, \ldots, J_t^d$ and at each time $t$. As is common, we use lagged market equity to weight the returns in each portfolio in our empirical applications. The plug-in estimate of $\hat{V}^{(2)}_{\frac{J_t^d}{n_T}}$ implicit in the above expression utilizes the logic of the Fama-MacBeth variance estimator applied to the higher-order variance term. As a plug-in estimator of $\mu'(z)$ we use the time-series average of the estimated slope coefficient from a local regression using the 40 closest points to $z$ (ties included) at each point in time.

**Remark 1.** As discussed in Remark 9 of the main text, when we are interested in point estimation, the optimal choice is $J_t^{**}$ rather than $J_t^*$. In analogy with equation (A.1) we can utilize the following estimator,

$$
\hat{\text{MSE}}^{**}(\hat{\mu}(z_H) - \hat{\mu}(z_L); J_1, \ldots, J_T) = \left( \mu'(z_H) \cdot T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_t^d} \sum_{i=1}^{n_t} \omega_{it} \hat{1}_{j_t} \hat{1}_{j_t}(z_H) \hat{1}_{j_t}(z_{it}) (z_{it} - z_H) \\
- \mu'(z_L) \cdot T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_t^d} \sum_{i=1}^{n_t} \omega_{it} \hat{1}_{j_t} \hat{1}_{j_t}(z_L) \hat{1}_{j_t}(z_{it}) (z_{it} - z_L) \right)^2 + \hat{V}_{FM}(z).
$$

In this case, we would scale all other time periods as $J_t = J (n_t/n)^{\frac{1}{d+2}}$. We then choose a grid of values for $J$ as $J = ([n_{t_{\text{min}}}/n]^{\frac{1}{d+2}}, \ldots, J_{\text{max}})$.

□
Figure A.1: **Cross-Sectional Sample Sizes**

The top chart shows the monthly cross-section sample sizes over time, $n_t$, for the primary data set from the Center for Research in Security Prices (CRSP). The bottom chart shows the cross-section sample sizes over time for those stocks listed on the New York Stock Exchange (NYSE).

*All*

![Chart showing cross-sectional sample sizes over time for all stocks.]

*NYSE Only*

![Chart showing cross-sectional sample sizes over time for NYSE listed stocks.]

4
B Proofs

B.1 Notation

In this Supplementary Appendix we use a generalized notation relative to the manuscript. Note that \( N_{jt} \) satisfies \( N_{jt} = n_t \hat{q}_{jt} \) where \( \hat{q}_{jt} \) is defined below. The other mappings from the manuscript to remainder of this supplement are as follows:

- \( \hat{1}_{jt} \mapsto 1_{jt} \)
- \( \hat{1}_{jt}(z) \mapsto \hat{1}_{jt}(z) \)
- \( d \mapsto d \)

We also abstain from bold symbols in the remainder of the supplement for simplicity of notation.

B.2 Model, Setup and Assumptions

Let \( R_{it} \in \mathbb{R} \) be the return of asset \( i \) at time \( t \) with regressor of interest, \( z_{it} \in \mathbb{R}^{d_z} \) and additional controls, \( x_{it} \in \mathbb{R}^{d_x} \). The model is

\[
R_{it} = \mu(z_{it}) + x_{it}' \beta_t + \varepsilon_{it}, \quad i = 1, \ldots, n_t, \quad t = 1, \ldots, T,
\]

where \( \beta_t \in \mathbb{R}^{d_x} \forall t \) and \( \mu(\cdot) \) is a time-invariant function. We make the following assumptions:

**Assumption 1.** Let the sigma fields \( \mathcal{F}_t = \sigma(f_t) \) be generated by a sequence of unobserved (possibly dependent) random vectors \( \{f_t : t = 0, 1, \ldots, T\} \). For \( t = 1, \ldots, T \), the following conditions hold.

1. Conditional on \( \mathcal{F}_t \), \( \{(R_{it}, z_{it}', x_{it}') : i = 1, 2, \ldots, n_t\} \) are iid satisfying Model (\(*\)).

2. \( \mathbb{E}[\varepsilon_{it}| z_{it}, x_{it}, \mathcal{F}_t] = 0 \); uniformly in \( t \), \( \Omega_{uu,t} = \mathbb{E}[\mathbb{V}(x_{it}| z_{it}, \mathcal{F}_t)|\mathcal{F}_t] \) is bounded and its minimum eigenvalue is bounded away from zero; \( \sigma_{it}^2 = \mathbb{E}[\varepsilon_{it}^2| z_{it}, x_{it}, \mathcal{F}_t] \) is bounded and bounded away from zero, and \( \mathbb{E}[|\varepsilon_{it}|^{2+\phi}| z_{it}, x_{it}, \mathcal{F}_t] \) is bounded for some \( \phi > 0 \); \( \mathbb{E}[a'x_{it}| z_{it}, \mathcal{F}_t] \) is sub-Gaussian for all \( a \in \mathbb{R}^{d_x} \).

3. Conditional on \( \mathcal{F}_t \), \( z_{it} \) has time-invariant support, denoted \( \mathcal{Z} \), and continuous Lebesgue density bounded away from zero.

4. \( \mu(z) \) is twice continuously differentiable; \( |\mathbb{E}[x_{it,\ell}| z_{it} = z, \mathcal{F}_t] - \mathbb{E}[x_{it,\ell}| z_{it} = z', \mathcal{F}_t]| \leq C \|z - z'\| \) for all all \( z, z' \in \mathcal{Z} \) where \( x_{it,\ell} \) is the \( \ell \)th element of \( x_{it} \) and the constant \( C > 0 \) is not a function of \( t \) or \( \mathcal{F}_t \).

**Assumption 2.** The cross-sectional sample sizes diverge proportionally for a sequence \( n \to \infty \), \( n_t = \kappa_t n \) with \( \kappa_t \leq 1 \) and uniformly bounded away from zero.

**Assumption 3.** The sequences \( n, T, \) and \( J \) obey: (a) \( n^{-1}J^{d_z} \log(n) \log(J^{d_z} \lor T) \to 0 \), (b) \( \sqrt{nT}J^{-(d_z/2+1)} \to 0 \), and, if \( d_z \geq 1 \), (c) \( T/n \to 0 \).

Finally, let \( \|A\| = \text{tr}(A'A) \) for a matrix \( A \). If \( A \) is square we denote the minimum eigenvalue by \( \lambda_{\min}(A) \).

Define for two sequences \( a_{n,T} \asymp b_{n,T} \) if \( \limsup_{n,T \to \infty} |a_{n,T}/b_{n,T}| < \infty \) and \( \limsup_{n,T \to \infty} |b_{n,T}/a_{n,T}| < \infty \).
B.3 Estimation Approach

We approximate the unknown function $\mu(\cdot)$ at fixed time $t$ by a partitioning estimator. At each point in time $t$, the number of partitions may depend on $(n_t, n, T)$. Let $J_t^{d_s}$ be the number of partitions for time $t$ and by assumption we have that, uniformly in $t$, $J_t \approx J$ for some sequence $J = J_{n,T} \to \infty$ as $n, T \to \infty$. Throughout the Appendix, for simplicity of notation, we will suppress any dependence and just refer to $J_t$ and $J$.

If we write

$$
\hat{\mu}_t^0 (z) = B_t (z)' \gamma_t^0, \quad B_t (z) = \left( \hat{I}_1 (z), \ldots, \hat{I}_{J_t} (z) \right)',
$$

where

$$
\hat{I}_{jt} (z) = \hat{I}_{jt,1} (z_1) \hat{I}_{jt,2} (z_2) \cdots \hat{I}_{jt,d} (z_d),
$$

and

$$
\begin{align*}
\{ \hat{I}_{jt,t} (z_t) = 1 \} & \iff \{ \hat{b}_{(j,t-1),t} \leq z_t < \hat{b}_{jt,t} \}, & 1 \leq j_t < J, \\
\{ \hat{I}_{jt,t} (z_t) = 1 \} & \iff \{ \hat{b}_{(j,t-1),t} \leq z_t \leq \hat{b}_{jt,t} \}, & j_t = J,
\end{align*}
$$

where $\hat{b}_{jt,t} = \hat{F}_{jt,t}^{-1} (j_t/J)$ and $\hat{F}_{jt,t}^{-1} (\cdot)$ is the empirical quantile function of $z_{it,t}$ for the cross-section at time $t$. Our estimator of $\mu(\cdot)$ at time $t$ is

$$
\hat{\mu}_t (z) = B_t (z)' \hat{\gamma}_t, \quad \hat{\gamma}_t = (B_t' M_X t B_t)^{-1} B_t' M_X t R_t,
$$

where $B_t = (B_t (z_{11t}), \ldots, B_t (z_{nt}))' = n_t \times J_t^{d_s}$, $M_X t = I_{n_t} - X_t (X_t' X_t)^{-1} X_t'$, and $X_t$ is the $n_t \times d_x$ matrix of the stacked $x_{it}$.

Furthermore our estimator of $\mu(\cdot)$ based on the full sample is

$$
\hat{\mu} (z) = \frac{1}{T} \sum_{t=1}^T \hat{\mu}_t (z).
$$

Corresponding to $\hat{\mu}_t (z)$, our estimator of $\beta_t$ at time $t$ is,

$$
\hat{\beta}_t = (X_t' M_{B_t} X_t)^{-1} X_t' M_{B_t} R_t,
$$

where $M_{B_t} = I_{n_t} - B_t (B_t' B_t)^{-1} B_t'$.

It will also be useful to introduce some additional definitions. First, let $I_{jt} (z)$ be defined similar as above so that

$$
I_{jt} (z) = I_{jt,1} (z_1) I_{jt,2} (z_2) \cdots I_{jt,d} (z_d),
$$

and

$$
\begin{align*}
\{ \hat{I}_{jt,t} (z_t) = 1 \} & \iff \{ b_{(j,t-1),t} \leq z_t < b_{jt,t} \}, & 1 \leq j_t < J, \\
\{ \hat{I}_{jt,t} (z_t) = 1 \} & \iff \{ b_{(j,t-1),t} \leq z_t \leq b_{jt,t} \}, & j_t = J,
\end{align*}
$$

and $b_{jt,t} = F_{jt,t}^{-1} (j_t/J)$ and $F_{jt,t}^{-1} (\cdot)$ is the quantile function for $z_{it,t}$ for the cross-section at time $t$. Then, recall that,

$$
q_{jt} = \mathbb{E} (I_{jt} (z_{it}) | F_t).
$$
The sample analog is
\[ \tilde{q}_{jt} = \frac{1}{n_t} \sum_{i=1}^{n_t} \mathbb{I}_{jt}(z_{it}). \]

Further define
\[ \hat{q}_{jt} = \frac{1}{n_t} \sum_{i=1}^{n_t} \hat{\mathbb{I}}_{jt}(z_{it}). \]

It will also be useful to define 1 as
\[ 1_{jt} = 1_{q,jt}1_{jt,t} = 1 \{ \hat{q}_{jt} \geq q_{jt}/2 \} \times 1 \left\{ \lambda_{\min} \left( \hat{\Omega}_{uu,t} \right) \geq C_{un}/2 \right\}, \]

where \( C_{un} \) is the lower bound, uniformly in \( t \), introduced in Assumption 1(2).

Finally define
\[ V(z) = T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} 1_{jt} \hat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \hat{\mathbb{I}}_{jt}(z_{it}) \sigma_{jt}^2, \]
\[ \hat{V}(z) = T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} 1_{jt} \hat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \hat{\mathbb{I}}_{jt}(z_{it}) \varepsilon_{it}^2, \]

along with
\[ \hat{V}_{FM}(z) = T^{-2} \sum_{t=1}^{T} (\hat{\mu}_t(z) - \hat{\mu}(z))^2, \]
\[ \hat{V}_{PI}(z) = T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} 1_{jt} \hat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \hat{\mathbb{I}}_{jt}(z_{it}) \varepsilon_{it}^2. \]

**B.4 Lemmas**

Our first lemma is a generalization of Cattaneo and Farrell (2013, Lemma A.2) to allow for random partitions.

**Lemma 1.** Let Assumptions 1-3 hold. Then, (i) there exists a \( \gamma_{jt}^0 \) such that
\[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t} \sup_z \left| \hat{\mathbb{I}}_{jt}(z) \mu(z) - \hat{\mathbb{I}}_{jt}(z) \gamma_{jt}^0 \right| = O_p \left( J^{-1} \right), \]
and
\[ E \left[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t} \sup_z \left| \hat{\mathbb{I}}_{jt}(z) \mu(z) - \hat{\mathbb{I}}_{jt}(z) \gamma_{jt}^0 \right|^2 \right] = O \left( J^{-2} \right). \]

(ii) If we define \( h_{t,\ell}(z) = h_{t,\ell}(z, \mathcal{F}_t) = E \left[ x_{it,\ell} | \mathcal{F}_t, z_{it} = z \right] \) where \( x_{it,\ell} \) is the \( \ell \)th element of \( x_{it} \), there exists a \( \pi_{jt,\ell}^0 \) such that
\[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t} \sup_z \left| \hat{\mathbb{I}}_{jt}(z) h_{t,\ell}(z) - \hat{\mathbb{I}}_{jt}(z) \pi_{jt,\ell}^0 \right| = O_p \left( J^{-1} \right), \]
and
\[ E \left[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t} \sup_z \left| \hat{\mathbb{I}}_{jt}(z) h_{t,\ell}(z) - \hat{\mathbb{I}}_{jt}(z) \pi_{jt,\ell}^0 \right|^2 \right] = O \left( J^{-2} \right). \]
Stack $\gamma_t^0 = \left(\gamma_{1,t}^0, \ldots, \gamma_{d_t,t}^0\right)'$ and

$$\Pi_t^0 = \begin{bmatrix}
\pi_{1t,1}^0 & \cdots & \pi_{1t,d_t}^0 \\
\vdots & \ddots & \vdots \\
\pi_{d_t,t,1}^0 & \cdots & \pi_{d_t,t,d_t}^0
\end{bmatrix}$$

We also stack the $h_{t,t} (\cdot)$’s as $h_t (\cdot) = (h_{t,1} (\cdot), \ldots, h_{t,d_t} (\cdot))'$ and then stack again as the $n_t \times d_x$ matrix $H_t = (h_t (z_{1t}), \ldots, h_t (z_{n_t}))'$. Finally, define $U_t = X_t - H_t$. Recall from above that $\Omega_{uu,t} = \text{plim}_{n \to \infty} U_t' U_t / n_t = E \left[ (x_{it} - E [x_{it} | z_{it}, F_t]) (x_{it} - E [x_{it} | z_{it}, F_t])' \right] F_t = E [V (x_{it} | z_{it}, F_t) | F_t].$

**Lemma 2.** Let Assumptions 1-3 hold. Then,

$$\max_{1 \leq t \leq T} \max_{1 \leq j \leq d_x} |\hat{q}_{jt} - q_{jt}|^2 = O_p \left( \frac{\log (J^{d_x} \vee T)}{J^{d_x} n} \right).$$

**Lemma 3.** Let Assumptions 1-3 hold. Define,

$$\hat{\Omega}_{uu,t} = X_t' M_{B_r} X_t / n_t.$$

Then,

$$\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\Omega}_{uu,t} - \Omega_{uu,t} \right\|^2 = O_p (n^{-1}) + O_p (J^{-4}) + O_p \left( n^{-2} J^{2d_x} \right),$$

and

$$\max_{1 \leq t \leq T} \left\| \hat{\Omega}_{uu,t} - \Omega_{uu,t} \right\| = O_p \left( \log (T) n^{-1/2} \right) + O_p (J^{-2}) + O_p \left( n^{-1} J^{d_x} \right).$$

**Lemma 4.** Let Assumptions 1-3 hold. Then,

$$\left| T^{-1} \sum_{t=1}^{T} 1_{\beta_t} s_t' \left( \hat{\beta}_t - \beta_t \right) \right|^2 = O_p \left( n^{-1} T^{-1} \right) + O_p (J^{-4}) + O_p \left( J^{2d_x} n^{-3} \right) + O_p \left( J^{d_x - 4} n^{-2} \right)$$

and

$$T^{-1} \sum_{t=1}^{T} 1_{\beta_t} \left( s_t' \left( \hat{\beta}_t - \beta_t \right) \right)^2 = O_p (n^{-1}) + O_p (J^{-4}),$$

where $\|s_t\| \leq C$ almost surely and $s_t$ is nonrandom conditional on $z_t$ and $F_t$.

**Lemma 5.** Let Assumptions 1-3 hold. Then, $C_1 n^{-1} T^{-1} J^{d_x} \left[ 1 + o_p (1) \right] \leq V (z) \leq C_2 n^{-1} T^{-1} J^{d_x} \left[ 1 + o_p (1) \right]$ for constants $C_1$ and $C_2$ bounded and bounded away from zero.

**Lemma 6.** Under Assumptions 1-3 and if $J^d \log (T)^2 \log (T \wedge J^d)^{-1} = O (n)$ then

$$V (z)^{-1} T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{d_x} J_{j,t} \bar{\xi}_{j,t} (z) \bar{q}_{jt} (z_{it}) (z_{it}^2 - \sigma_{jt}^2) = o_p (1).$$

Before proceeding note that by Lemma 2 and 3 we have that $1_{q,jt} \to 1$ and $1_{\beta,t} \to 1$ with probability approaching one.
B.5 Proof of Theorem 1

Recall that our estimator is
\[ \hat{\mu}(z) = T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it})(\mu(z_{it}) - \mu(z)), \]
which can be decomposed as
\[ \hat{\mu}(z) - \mu(z) = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \]
where
\[
\mathcal{L}_1 = T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it})(\mu(z_{it}) - \mu(z)), \\
\mathcal{L}_2 = T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it}) \varepsilon_{it}, \\
\mathcal{L}_3 = -T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it}) x_{it}' \left( \hat{\beta}_t - \beta_t \right), \\
\mathcal{L}_4 = T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_z} (1_{jt} - 1) \hat{\mu}_{jt}(z) \mu(z). 
\]

We will work with the re-scaled estimator:
\[ V(z)^{-1/2}(\hat{\mu}(z) - \mu(z)) = V(z)^{-1/2} \mathcal{L}_1 + V(z)^{-1/2} \mathcal{L}_2 + V(z)^{-1/2} \mathcal{L}_3 + V(z)^{-1/2} \mathcal{L}_4, \]
and show that
\[ V(z)^{-1/2}(\hat{\mu}(z) - \mu(z)) = V(z)^{-1/2} \mathcal{L}_2 + o_p(1), \quad V(z)^{-1/2} \mathcal{L}_2 \longrightarrow_d \mathcal{N}(0,1). \]

B.5.1 Term: $\mathcal{L}_1$

By Lemma 5 we need only show that
\[
\left| \epsilon^{d/2} T^{-1/2} \sum_{t=1}^{T} n_t^{1/2} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it})(\mu(z_{it}) - \mu(z)) \right| = o_p(1). 
\]

We have,
\[
\begin{align*}
T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it})(\mu(z_{it}) - \mu(z)) & \\
\leq T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it}) \mu(z_{it}) - \gamma_{jt}^{0} \\
& + T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it}) \mu(z) - \gamma_{jt}^{0}. 
\end{align*}
\]

The first term is
\[
T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it}) \mu(z_{it}) - \gamma_{jt}^{0} 
\]
\[
\leq \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_z} \sup_{z} \left| \hat{\mu}_{jt}(z) \mu(z) - \hat{\mu}_{jt}(z) \gamma_{jt}^{0} \right| \times T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_z} 1_{jt} \hat{\mu}_{jt}(z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt}(z_{it}) 
\]
\[
\leq \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_z} \sup_{z} \left| \hat{\mu}_{jt}(z) \mu(z) - \hat{\mu}_{jt}(z) \gamma_{jt}^{0} \right|, 
\]
which is $O_p\left(J^{-1}\right)$ by Lemma 1. The second term follows by the same steps so that

$$
|J^{-d/2}T^{-1/2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{r_{is}^d} 1_{j_t \in \mathcal{J}_t}(z) \hat{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it}) (\mu(z_{it}) - \mu(z))| = O_p\left(J^{-(d/2+1)}T^{1/2}n_t^{-1}\right),
$$

which is $o_p(1)$ under our rate assumptions.

### B.5.2 Term: $\mathcal{L}_2$

For $\mathcal{L}_2$ define the sigma field, $\mathcal{G}_s = \sigma(z_1, \ldots, z_T, x_1, \ldots, x_T, \mathcal{F}_1, \ldots, \mathcal{F}_T, \varepsilon_1, \ldots, \varepsilon_s)$ and the variable

$$
\xi_s = V(z)^{-1/2} T^{-1} n_s^{-1} \sum_{i=1}^{n_s} \sum_{j=1}^{r_{is}^d} 1_{j_s \in \mathcal{J}_s}(z) \hat{q}_{js}^{-1} \mathbb{I}_{js}(z_{is}) \varepsilon_{is}.
$$

Note first that

$$
\sum_{t=1}^{T} \mathbb{E}\left[|\xi_t^2| \mid \mathcal{G}_{t-1}\right] = V(z)^{-1/2} T^{-1} n_s^{-1} \sum_{i=1}^{n_s} \sum_{j=1}^{r_{is}^d} 1_{j_s \in \mathcal{J}_s}(z) \hat{q}_{js}^{-1} \mathbb{I}_{js}(z_{is}) \mathbb{E}\left[\varepsilon_{is} \mid \mathcal{G}_{s-1}\right],
$$

with $\mathbb{E}\left[\varepsilon_{is} \mid \mathcal{G}_{s-1}\right] = 0$. Thus, $(\xi_s, \mathcal{G}_s)$ is a martingale difference sequence with $\sum_{t=1}^{T} \mathbb{E}\left[|\xi_t|^2 \mid \mathcal{G}_{t-1}\right] = 1$. By Hall and Heyde (1980, Corollary 3.1) we need only show that

$$
\sum_{t} \mathbb{E}\left[|\xi_t|^{2+\delta} \mid \mathcal{G}_{t-1}\right] = o_p(1) \quad \text{for all} \; \epsilon > 0.
$$

This is implied by showing that

$$
\sum_{t} \mathbb{E}\left[|\xi_t|^{2+\delta} \mid \mathcal{G}_{t-1}\right] = o_p(1),
$$

for some $\delta > 0$. To show this note that,

$$
\sum_{t} \mathbb{E}\left[|\xi_t|^{2+\delta} \mid \mathcal{G}_{t-1}\right] = V(z)^{-(1+\delta/2)} T^{-(2+\delta)} \sum_{t} \mathbb{E}\left[\left|n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{r_{is}^d} 1_{j_t \in \mathcal{J}_t}(z) \hat{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it}) \varepsilon_{it}\right|^{2+\delta} \mid \mathcal{F}_t, x_t, z_t, \mathcal{G}_{t-1}\right] \mathbb{E}\left[|\varepsilon_t|^{2+\delta} \mid \mathcal{G}_{t-1}\right].
$$

Then note that

$$
\mathbb{E}\left[\left|n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{r_{is}^d} 1_{j_t \in \mathcal{J}_t}(z) \hat{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it}) \varepsilon_{it}\right|^{2+\delta} \mid \mathcal{F}_t, x_t, z_t, \mathcal{G}_{t-1}\right] \leq C \sum_{i=1}^{n_t} \mathbb{E}\left[\left|n_t^{-1} \sum_{j=1}^{r_{is}^d} 1_{j_t \in \mathcal{J}_t}(z) \hat{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it}) \varepsilon_{it}\right|^{2+\delta} \mid \mathcal{F}_t, x_t, z_t, \mathcal{G}_{t-1}\right] \vee
$$

$$
\sum_{j=1}^{r_{is}^d} \mathbb{E}\left[\left|n_t^{-1} 1_{j_t \in \mathcal{J}_t}(z) \hat{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it}) \varepsilon_{it}\right|^{2+\delta} \mid \mathcal{F}_t, x_t, z_t, \mathcal{G}_{t-1}\right].
$$
\[
C \left( \sum_{i=1}^{n_t} E \left[ \left| n_t^{-1} \sum_{j=1}^{J_t^{ds}} 1_{jt} \hat{\mu}_{jt} (z) \hat{\varphi}_{jt}^{-1} \hat{\nu}_{jt} (z_{it}) \varepsilon_{it} \right|^2 \right| F_t, x_t, z_t, \mathcal{G}_{t-1} \right) \right]^{1+\delta/2}.
\]

The first term is
\[
\sum_{i=1}^{n_t} E \left[ \left| n_t^{-1} \sum_{j=1}^{J_t^{ds}} 1_{jt} \hat{\mu}_{jt} (z) \hat{\varphi}_{jt}^{-1} \hat{\nu}_{jt} (z_{it}) \varepsilon_{it} \right|^2 \right| F_t, x_t, z_t, \mathcal{G}_{t-1} \right] \leq C \sum_{i=1}^{n_t} \left| n_t^{-1} \sum_{j=1}^{J_t^{ds}} 1_{jt} \hat{\mu}_{jt} (z) \hat{\varphi}_{jt}^{-1} \hat{\nu}_{jt} (z_{it}) \right|^{2+\delta} \leq C n^{-2+\delta} J^d (2+\delta) \sum_{i=1}^{n_t} \left| \sum_{j=1}^{J_t^{ds}} \hat{\mu}_{jt} (z) \hat{\nu}_{jt} (z_{it}) \right|^{2+\delta} = C n^{-2+\delta} J^d (2+\delta) \sum_{j=1}^{J_t^{ds}} \hat{\mu}_{jt} (z) \sum_{i=1}^{n_t} \hat{\nu}_{jt} (z_{it}) = C \left( n^{-1} J^d \right)^{1+\delta}.
\]

The second term is
\[
\left( \sum_{i=1}^{n_t} E \left[ \left| n_t^{-1} \sum_{j=1}^{J_t^{ds}} 1_{jt} \hat{\mu}_{jt} (z) \hat{\varphi}_{jt}^{-1} \hat{\nu}_{jt} (z_{it}) \varepsilon_{it} \right|^2 \right| F_t, x_t, z_t, \mathcal{G}_{t-1} \right) \right]^{1+\delta/2} \leq C \left( J^d n_t^{-1} \right)^{-(1+\delta/2)} T^{-1-\delta} \left( n_t^{-1} J^d (1+\delta) \right)^{1+\delta/2} \leq C T^{-\delta/2},
\]

and the result follows.

**B.5.3 Term: \( L_3 \)**

We have
\[
-V (z)^{-1/2} L_3 = V (z)^{-1/2} T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{j=1}^{n_t} \sum_{i=1}^{J_t^{ds}} 1_{jt} \hat{\mu}_{jt} (z) \hat{\varphi}_{jt}^{-1} \hat{\nu}_{jt} (z_{it}) x'_it \left( \hat{\beta}_t - \beta_t \right)
\]
\[
= V (z)^{-1/2} T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \hat{h}_t (z)' \left( \hat{\beta}_t - \beta_t \right),
\]
where
\[
\hat{h}_t (z) = \sum_{j=1}^{J_t^{ds}} \hat{\mu}_{jt} (z) \hat{\pi}_{jt}, \quad \hat{\pi}_{jt} = 1_{q,j} \hat{\varphi}_{jt}^{-1} n_t^{-1} \sum_{i=1}^{n_t} \hat{\nu}_{jt} (z_{it}) x_{it}.
\]

Thus,
\[
-V (z)^{-1/2} L_3 = V (z)^{-1/2} T^{-1} \sum_{t=1}^{T} 1_{\beta,t} h_t (z)' \left( \hat{\beta}_t - \beta_t \right)
\]
\[ +V (z)^{-1/2} T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \left( \hat{h}_t (z) - h_t (z) \right)' \left( \hat{\beta}_t - \beta_t \right) \]

\[ = V (z)^{-1/2} \mathcal{L}_{31} + V (z)^{-1/2} \mathcal{L}_{32}. \]

First we have,

\[ |\mathcal{L}_{31}|^2 = \left| T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \left( \hat{h}_t (z) - h_t (z) \right)' \left( \hat{\beta}_t - \beta_t \right) \right|^2 \]

\[ = O_p \left( n^{-1} T^{-1} \right) + O_p \left( J^{-4} \right) + O_p \left( J^{2d} n^{-3} \right) + O_p \left( J^{d-4} n^{-2} \right), \]

by Lemma 4. Thus,

\[ \frac{nT}{Jd} |\mathcal{L}_{31}|^2 = O_p \left( n^{-1} T^{-1} \right) + O_p \left( \frac{nT}{Jd+2} J^{-2} \right) + O_p \left( \frac{Jd}{n} T \right) + O_p \left( \frac{T}{nJ^2} \right), \]

which is \( o_p (1) \) under Assumption 3. Next, by the Cauchy-Schwartz inequality

\[ |\mathcal{L}_{32}|^2 \leq \frac{1}{T} \sum_{t=1}^{T} \left| \hat{h}_t (z) - h_t (z) \right|^2 \times \frac{1}{T} \sum_{t=1}^{T} 1_{\beta,t} \left| \hat{\beta}_t - \beta_t \right|^2. \]

The order of the first factor follows by exactly the same steps as in the proof of Theorem 2 for the consistency of the Fama-MacBeth style variance estimator (ignoring the \( S_{12}^{FM} \) term). That is, we show below that

\[ \frac{1}{T^2} \sum_{t=1}^{T} (\hat{\mu} (z) - \mu (z))^2 = O_p \left( \frac{Jd}{nT} \right). \]

Thus, the first factor is \( O_p \left( n^{-1} T^{-1} \right) \). By Lemma 4, the second factor is \( O_p \left( n^{-1} \right) + O_p \left( J^{-4} \right) \). Thus,

\[ \frac{nT}{Jd} |\mathcal{L}_{32}|^2 = O_p \left( n^{-1} T \right) + O_p \left( T J^{-4} \right), \]

which is \( o_p (1) \) under our assumptions.

**B.5.4 Term: \( \mathcal{L}_4 \)**

Finally consider \( \mathcal{L}_4 \):

\[ |V (z)^{-1/2} \mathcal{L}_4| = |V (z)^{-1/2} T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J} 1_{j,t} (\hat{I}_{jt} - 1) \hat{I}_{jt} (z) \mu (z)| \]

\[ \leq CV (z)^{-1/2} T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J} 1_{j,t} (|1_{j,t} - 1|) \hat{I}_{jt} (z) \]

\[ \leq Cn^{1/2} T^{1/2} J^{-d/2} \times \max_{1 \leq t \leq T} \max_{1 \leq j \leq J} |1_{j,t} - 1|, \]

Thus, \( |V (z)^{-1/2} \mathcal{L}_4| = o_p (1) \) by Lemmas 2 and 3.
B.6 Proof of Theorem 2

B.6.1 Proof of Consistency of Variance Estimators (FM-style Variance Estimator)

We need to show that
\[
\frac{n_T}{Jd_z} \left( \hat{V}_\text{FM}(z) - V(z) \right) = o_p(1), \quad \hat{V}_\text{FM}(z) = T^{-2} \sum_{t=1}^{T} (\hat{\mu}_t(z) - \mu(z))^2.
\]

First note that
\[
\hat{V}_\text{FM}(z) = T^{-2} \sum_{t=1}^{T} (\hat{\mu}_t(z) - \mu(z))^2 = T^{-2} \sum_{t=1}^{T} (\hat{\mu}_t(z) - \mu(z))^2 - T^{-1} (\hat{\mu}(z) - \mu(z))^2.
\]

Recall that,
\[
\hat{\mu}_t(z) - \mu(z) = n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) \left( R_{i_1 t_1} - x_{i_1 t_1} \hat{\beta}_{t_1} \right) - \mu(z)
\]
\[
= n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) \varepsilon_{i_1 t_1}
\]
\[
+ n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) (\mu(z_{i_1 t_1}) - \mu(z))
\]
\[
- n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) x_{i_1 t_1}' \left( \hat{\beta}_{t_1} - \beta_{t_1} \right)
\]
\[
+ \sum_{j_1} (1_{j_1 t_1} - 1) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) \mu(z) .
\]

Thus, since we have already shown that \( \frac{n_T}{Jd_z} \left( \hat{V}(z) - V(z) \right) = o_p(1) \) by Lemma 6 then by the CS inequality it is sufficient to show that \( |S_{11}^{\text{FM}}| = o_p(1), |S_{12}^{\text{FM}}| = o_p(1), |S_{13}^{\text{FM}}| = o_p(1) \), and \( |S_2^{\text{FM}}| = o_p(1) \) where
\[
S_{11}^{\text{FM}} = \frac{n}{T J d_z} \sum_{t_1} \left[ n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) (\mu(z_{i_1 t_1}) - \mu(z)) \right]^2
\]
\[
S_{12}^{\text{FM}} = \frac{n}{T J d_z} \sum_{t_1} \left[ n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) x_{i_1 t_1}' \left( \hat{\beta}_{t_1} - \beta_{t_1} \right) \right]^2
\]
\[
S_{13}^{\text{FM}} = \frac{n}{T J d_z} \sum_{t_1} \left( \sum_{j_1} (1_{j_1 t_1} - 1) \widehat{I}_{j_1 t_1}(z) \mu(z) \right)^2
\]
\[
S_2^{\text{FM}} = \frac{n}{J d_z} (\hat{\mu}(z) - \mu(z))^2 .
\]

First consider, \( S_2^{\text{FM}} \). We have already shown that \( \hat{\mu}(z) - \mu(z) = O_p \left( \sqrt{J d_z n^{-1} T^{-1}} \right) \), so that \( S_2 \) satisfies
\[
S_2^{\text{FM}} = \frac{n}{J d_z} (\hat{\mu}(z) - \mu(z))^2 = O_p \left( T^{-1} \right) = o_p(1)
\]

Next, consider \( S_{11}^{\text{FM}} \),
\[
S_{11}^{\text{FM}} = \frac{n}{T J d_z} \sum_{t_1} \left[ n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) (\mu(z_{i_1 t_1}) - \mu(z)) \right]^2
\]
\[
\leq \frac{n}{T J d_z} \sum_{t_1} \left[ n_{t_1}^{-1} \sum_{j_1} q_{j_1 t_1}^{-1} \sum_{i_1} 1_{j_1 t_1} \widehat{\mu}_{j_1 t_1}(z) \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) (|\mu(z_{i_1 t_1}) - \mu(0)| + |\mu(z) - \mu(0)|) \right]^2
\]
\[
\leq C \max_{1 \leq t \leq T} \max_{1 \leq j \leq p_{t_1}} \sup_z \left| \hat{I}_{j t}(z) \mu(z) - \hat{I}_{j t}(z) \gamma^0_{j t_1} \right|^2 \times \frac{n}{T J d_z} \sum_{t_1} \left[ \sum_{j_1} q_{j_1 t_1}^{-1} 1_{j_1 t_1} \widehat{I}_{j_1 t_1}(z) n_{t_1}^{-1} \sum_{i_1} \widehat{I}_{j_1 t_1}(z_{i_1 t_1}) \right]^2
\]
\[
\leq C \max_{1 \leq t \leq T} \max_{1 \leq j \leq p_{t_1}} \sup_z \left| \hat{I}_{j t}(z) \mu(z) - \hat{I}_{j t}(z) \gamma^0_{j t_1} \right|^2 \times \frac{n}{J d_z} .
\]
and so $S_{11}^{FM} = O_p\left(nJ^{-(d_z+2)}\right)$ which is $o(1)$ under our rate assumptions.

Next consider $S_{12}^{FM}$,

\[
S_{12}^{FM} = \frac{n}{TJ^{d_z}} \sum_{t_1} \left[ \sum_{j_1} \hat{q}_{j_1 t_1} \sum_{\beta_1} 1_{\beta_1} \left( \hat{h}_{t_1} (z) - h_{t_1} (z) \right) ' (\hat{\beta}_1 - \beta_1) \right]^2
\]

\[
= \frac{n}{TJ^{d_z}} \sum_{t_1} \left[ \sum_{\beta_1} 1_{\beta_1} \left( \hat{h}_{t_1} (z) - h_{t_1} (z) \right) ' (\hat{\beta}_1 - \beta_1) \right]^2
\]

\[
\leq C \frac{n}{TJ^{d_z}} \sum_{t_1} \sum_{\beta_1} 1_{\beta_1} \left( \hat{h}_{t_1} (z) - h_{t_1} (z) \right) ' (\hat{\beta}_1 - \beta_1) \right]^2
\]

\[
= S_{121}^{FM} + S_{122}^{FM}.
\]

$S_{121}^{FM}$ follows by exactly the same steps as in the proof for $L_3$ so that we have

\[
\frac{n}{TJ^{d_z}} \sum_{t_1} 1_{\beta_1} \left( \hat{h}_{t_1} (z) - h_{t_1} (z) \right) ' (\hat{\beta}_1 - \beta_1) \right]^2 = \frac{n}{J^{d_z}} T \cdot (O_p\left(n^{-1}T^{-1}\right) + O_p\left(T^{-1}J^{-4}\right))
\]

\[
= O_p\left(J^{-d_z}\right) + O_p\left(\frac{nT}{J^{d_z+2}} \times \frac{1}{TJ^2}\right),
\]

which is $o_p(1)$ under our Assumptions 3. Next consider $S_{122}^{FM}$,

\[
\frac{n}{TJ^{d_z}} \sum_{t_1} 1_{\beta_1} \left( \hat{h}_{t_1} (z) - h_{t_1} (z) \right) ' (\hat{\beta}_1 - \beta_1) \right]^2 \leq \left( \frac{n}{J^{d_z}} \right) T^{-1} \sum_{t_1} \left\| \hat{h}_{t_1} (z) - h_{t_1} (z) \right\|^2 \left( \sum_{t_1} 1_{\beta_1} \right) \left\| \hat{\beta}_1 - \beta_1 \right\|^2.
\]

The first factor is $O_p(1)$. To see this note that we can show that $\left( \frac{n}{J^{d_z}} \right) T^{-1} \sum_{t_1} (\hat{\mu}_t (z) - \mu (z))^2 = O_p(1)$ by showing $|S_{11}^{FM}| = o_p(1)$, $|S_{12}^{FM}| = o_p(1)$, and $|S_{2}^{FM}| = o_p(1)$. We can then follow the same steps to show that $\left( \frac{n}{J^{d_z}} \right) T^{-1} \sum_{t_1} \left\| \hat{h}_{t_1} (z) - h_{t_1} (z) \right\|^2 = O_p(1)$. The second factor is $O_p\left(Tn^{-1}\right) + O_p\left(TJ^{-4}\right)$ which is $o_p(1)$ by our rate assumptions and since $TJ^{-4} = TJ^{-2} \cdot J^{-2}$ which is $o(1)$ under Assumption 3.

Finally consider $S_{13}^{FM}$,

\[
S_{13}^{FM} = \frac{n}{TJ^{d_z}} \sum_{t_1} \sum_{j_1} (1_{j_1 t_1} - 1) \hat{\beta}_{j_1 t_1} (z) \mu (z)^2
\]

\[
= \frac{n}{TJ^{d_z}} \sum_{t_1} \sum_{j_1} |1_{j_1 t_1} - 1| \hat{\beta}_{j_1 t_1} (z) \mu (z)^2
\]

\[
\leq \sup_z \mu (z)^2 \times \frac{n}{TJ^{d_z}} \sum_{t_1} \sum_{j_1} \max_{1 \leq t \leq T, 1 \leq j \leq d_z} |1_{j_1 t_1} - 1|
\]

\[
\leq C \left( \frac{n}{J^{d_z}} \max_{1 \leq t \leq T, 1 \leq j \leq d_z} \right),
\]

so that $S_{13}^{FM} = o_p(1)$.

**B.6.2 Proof of Consistency of Variance Estimators (Plug-in Variance Estimator)**

We need to show that

\[
\frac{nT}{J^{d_z}} \left( \hat{V}_{PI} (z) - V (z) \right) = o_p(1), \quad \hat{V}_{PI} (z) = T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{d_z} 1_{ji} \hat{\beta}_{ji t} (z) \hat{q}_{ji t} \hat{z}_{ji t} (z) \hat{z}_{it}^2.
\]
First note that
\[
\frac{nT}{Jd_z} \left( \tilde{V}_{pi}(z) - V(z) \right) = \frac{nT}{Jd_z} \left( \tilde{V}(z) - V(z) \right) + S_{1}^{PI} + S_{2}^{PI} + S_{3}^{PI},
\]
where
\[
S_{1}^{PI} = -\frac{n}{Jd_z T} \sum_{t=1}^{T} n_{t}^{-2} \sum_{i_{1} \neq i_{2}} \sum_{j_{1}} J^{st}_{ij} \tilde{h}_{j_{1}t} (z) \tilde{q}_{j_{1}t}^{2} \tilde{h}_{jt} (z_{i_{1}t}) \tilde{h}_{jt} (z_{i_{2}t}) \varepsilon_{i_{1}t} \varepsilon_{i_{2}t}
\]
\[
S_{2}^{PI} = \frac{2n}{Jd_z T} \sum_{t=1}^{T} n_{t}^{-2} \sum_{i_{1}} \sum_{i_{2}} \sum_{j_{1}} J^{st}_{ij} \tilde{h}_{j_{1}t} (z) \tilde{q}_{j_{1}t}^{2} \tilde{h}_{jt} (z_{i_{1}t}) \varepsilon_{it} (\varepsilon_{it} - \varepsilon_{it})
\]
\[
S_{3}^{PI} = \frac{n}{Jd_z T} \sum_{t=1}^{T} n_{t}^{-2} \sum_{i_{1}} \sum_{i_{2}} \sum_{j_{1}} J^{st}_{ij} \tilde{h}_{j_{1}t} (z) \tilde{q}_{j_{1}t}^{2} \tilde{h}_{jt} (z_{i_{1}t}) (\varepsilon_{it} - \varepsilon_{it})^{2}.
\]

Again, we have already shown that \( \frac{nT}{Jd_z} \left( \tilde{V}(z) - V(z) \right) = o_p (1) \).

**Term:** \( S_{1}^{PI} \)

First consider \( S_{1}^{PI} \):
\[
E \left| S_{1}^{PI} \right|^{2} = E \left[ \frac{n}{Jd_z T} \sum_{t=1}^{T} n_{t}^{-2} \sum_{i_{1} \neq i_{2}} \sum_{j_{1}} J^{st}_{ij} \tilde{h}_{j_{1}t} (z) \tilde{q}_{j_{1}t}^{2} \tilde{h}_{jt} (z_{i_{1}t}) \tilde{h}_{jt} (z_{i_{2}t}) \varepsilon_{i_{1}t} \varepsilon_{i_{2}t} \right]^{2}
\]
\[
= \left( \frac{n}{Jd_z T} \right)^{2} \sum_{t_{1}, t_{2}} n_{t_{1}}^{-2} n_{t_{2}}^{-2} \sum_{i_{1} \neq i_{2}} \sum_{i_{3} \neq i_{4}} \sum_{j_{1}, j_{2}} E \left[ 1_{j_{1}t_{1}} 1_{j_{2}t_{2}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{h}_{j_{2}t_{2}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \tilde{q}_{j_{2}t_{2}}^{2} \times \tilde{h}_{j_{1}t_{1}} (z_{i_{1}t_{1}}) \tilde{h}_{j_{1}t_{1}} (z_{i_{2}t_{1}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{1}t_{2}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{2}t_{2}}) \varepsilon_{i_{1}t_{1}} \varepsilon_{i_{2}t_{1}} \varepsilon_{i_{1}t_{2}} \varepsilon_{i_{2}t_{2}} \right].
\]

The expectation is nonzero only when \( (t_{1} = t_{2}) \) and either \((i_{1} = i_{3}) , (i_{2} = i_{4}) \) or \((i_{1} = i_{4}) , (i_{2} = i_{3}) \). This yields
\[
E \left| S_{1}^{PI} \right|^{2} \leq C \left( \frac{n}{Jd_z T} \right)^{2} \sum_{t} n_{t}^{-4} \sum_{i_{1}, i_{2}} \sum_{j_{1}} E \left[ 1_{j_{1}t_{1}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \tilde{h}_{j_{1}t_{1}} (z_{i_{1}t}) \tilde{h}_{j_{1}t_{1}} (z_{i_{2}t}) \varepsilon_{i_{1}t} \varepsilon_{i_{2}t} \right]^{2}
\]
\[
\leq C \left( \frac{n}{Jd_z T} \right)^{2} \sum_{t} n_{t}^{-4} \sum_{i_{1}, i_{2}} \sum_{j_{1}} E \left[ 1_{j_{1}t_{1}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \tilde{h}_{j_{1}t_{1}} (z_{i_{1}t}) \tilde{h}_{j_{1}t_{1}} (z_{i_{2}t}) \right]
\]
\[
= C \left( \frac{n}{Jd_z T} \right)^{2} \sum_{t} n_{t}^{-4} \sum_{j_{1}} E \left[ 1_{j_{1}t_{1}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \right]
\]
\[
\leq C J^{2d_{z}} \left( \frac{n}{Jd_z T} \right)^{2} T n^{-2}
\]
\[
= CT^{-1}
\]
so that \( S_{1}^{PI} = o_p (T^{-1/2}) \) by Markov’s inequality.

**Term:** \( S_{2}^{PI} \)

This term is
\[
S_{2}^{PI} = \frac{2n}{Jd_z T} \sum_{t=1}^{T} n_{t}^{-2} \sum_{i_{1}} \sum_{j_{1}} J^{st}_{ij} \tilde{h}_{j_{1}t} (z) \tilde{q}_{j_{1}t}^{2} \tilde{h}_{jt} (z_{i_{1}t}) \varepsilon_{it} \varepsilon_{it}
\]
\[
= -\frac{2n}{Jd_z T} \sum_{t_{1}, t_{2}} n_{t_{1}}^{-2} n_{t_{2}}^{-1} \sum_{i_{1}, i_{2}} \sum_{j_{1}, j_{2}} 1_{j_{1}t_{1}} 1_{j_{2}t_{2}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \tilde{h}_{j_{1}t_{1}} (z_{i_{1}t_{1}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{1}t_{2}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{2}t_{2}}) \varepsilon_{i_{1}t_{1}} \varepsilon_{i_{2}t_{2}}
\]
\[
- \frac{2n}{Jd_z T} \sum_{t_{1}, t_{2}} n_{t_{1}}^{-2} n_{t_{2}}^{-1} \sum_{i_{1}, i_{2}} \sum_{j_{1}, j_{2}} 1_{j_{1}t_{1}} 1_{j_{2}t_{2}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \tilde{h}_{j_{1}t_{1}} (z_{i_{1}t_{2}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{1}t_{2}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{2}t_{2}}) \times \varepsilon_{i_{1}t_{1}} (\mu (z_{i_{2}t_{2}}) - \mu (z_{i_{1}t_{1}}))
\]
\[
+ \frac{2n}{Jd_z T} \sum_{t_{1}, t_{2}} n_{t_{1}}^{-2} n_{t_{2}}^{-1} \sum_{i_{1}, i_{2}} \sum_{j_{1}, j_{2}} 1_{j_{1}t_{1}} 1_{j_{2}t_{2}} \tilde{h}_{j_{1}t_{1}} (z) \tilde{q}_{j_{1}t_{1}}^{2} \tilde{h}_{j_{1}t_{1}} (z_{i_{1}t_{2}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{1}t_{2}}) \tilde{h}_{j_{2}t_{2}} (z_{i_{2}t_{2}}) \times
\]
\[ S_{21} = S_{22} + S_{23} + S_{24} \]

**Term: \( S_{21} \)** First consider \( S_{21} \) which can be decomposed as

\[
S_{21} = -\frac{2n}{Jd_zT^2} \sum_{t_1,t_2} n_{t_1}^{-2} \sum_{i_1,i_2} \sum_{j_1,j_2} 1_{j_1t_1} 1_{j_2t_2} \overline{\mu}_{j_1t_1} (z_1) \overline{q}_{j_1t_1} \overline{q}_{j_2t_2} (z_2) \varepsilon_{i_1t_1} \varepsilon_{i_2t_2} \]

The first term is \( S_{211} \) which satisfies

\[
E \left| S_{211} \right| = \frac{2n}{Jd_zT^2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1t_1} \overline{\mu}_{j_1t_1} (z) \overline{q}_{j_1t_1} \overline{q}_{j_1t_1} (z_1) \varepsilon_{i_1t_1} \varepsilon_{i_1t_1} \leq C \frac{Jd_z}{nT}
\]

so that \( S_{211} = o_p(1) \) by Markov’s inequality and our rate assumptions. Following similar steps, we can show that \( S_{212} = o_p(Jd_z/n^{-3/2}T^{-3/2}) \), \( S_{213} = o_p(Jd_z^2/n^{-3/2}T^{-1}) \), and \( S_{214} = o_p(Jd_z/n^{-1}T^{-1}) \) which are all \( o_p(1) \) under our rate assumptions.

**Term: \( S_{22} \)** Next consider \( S_{22} \)
Now consider,

\[
E \left| S_{221}^{\text{PI}} \right| \leq \frac{2n}{Jd_2 T^2} \sum_{t_1,t_2} n_{t_1}^{-1} n_{t_2}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \hat{\mu}_{j_2 t_2} (z_{i_1 t_1}) \hat{\mu}_{j_2 t_2} (z_{i_2 t_2}) \times |\varepsilon_{i_1 t_1} (\mu (z_{i_2 t_2}) - \mu (z_{i_1 t_1}))| \right]
\]

\[
\leq C \frac{n}{Jd_2 T^2} \sum_{t_1,t_2} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{\mu}_{j_2 t_2} (z_{i_1 t_1}) \hat{\mu}_{j_2 t_2} (z_{i_2 t_2}) \times \varepsilon_{i_1 t_1} \left[ x_{i_2 t_2}' (\hat{\beta}_t - \beta_t) - x_{i_1 t_1}' (\hat{\beta}_t - \beta_t) \right] \right]
\]

\[
\leq C \frac{1}{T}.
\]

Thus, \( S_{221}^{\text{PI}} = o_p (1) \) by Markov’s inequality. Following similar steps it can be shown that \( S_{222}^{\text{PI}} = O_p \left( Jd_2 n^{-1} T^{-1/2} \right) \) and \( S_{223}^{\text{PI}} = O_p \left( Jd_2 n^{-1/2} T^{-1/2} \right) \) which are \( o_p (1) \) under our rate assumptions.

**Term: \( S_{23}^{\text{PI}} \)** Next consider \( S_{23}^{\text{PI}} \)

\[
S_{23}^{\text{PI}} = \frac{2n}{Jd_2 T^2} \sum_{t_1,t_2} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \times \varepsilon_{i_1 t_1} \left[ x_{i_2 t_2}' (\hat{\beta}_t - \beta_t) - x_{i_1 t_1}' (\hat{\beta}_t - \beta_t) \right] \]

\[
= S_{231}^{\text{PI}} + S_{232}^{\text{PI}}.
\]

First consider \( S_{231}^{\text{PI}} \):

\[
S_{231}^{\text{PI}} = - \frac{2n}{Jd_2 T^2} \sum_{t_1,t_2} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \times \varepsilon_{i_1 t_1} \left[ x_{i_2 t_2}' (\hat{\beta}_t - \beta_t) - x_{i_1 t_1}' (\hat{\beta}_t - \beta_t) \right]
\]

\[
= - \frac{2n}{Jd_2 T^2} \sum_{t_1} n_{t_1}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \times \varepsilon_{i_1 t_1} \left[ x_{i_2 t_2}' (\hat{\beta}_t - \beta_t) - x_{i_1 t_1}' (\hat{\beta}_t - \beta_t) \right]
\]

so that by the CS inequality

\[
|S_{231}^{\text{PI}}|^2 \leq \left( \frac{2n}{Jd_2 T^2} \right)^2 \times \sum_{t_2} \left( \sum_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \hat{\mu}_{j_1 t_1} (z_{i_1 t_1}) \hat{\mu}_{j_2 t_2} (z_{i_2 t_2}) \varepsilon_{i_1 t_1} x_{i_2 t_2}' \right) \times \left( \sum_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1,i_2} \sum_{j_1,j_2} 1_{j_1 t_1} 1_{j_2 t_2} \hat{\mu}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \hat{\mu}_{j_1 t_1} (z_{i_1 t_1}) \hat{\mu}_{j_2 t_2} (z_{i_2 t_2}) \varepsilon_{i_1 t_1} x_{i_2 t_2}' \right)'
\]

\[
\sum_{t_2} \left( \hat{\beta}_t - \beta_t \right)' \left( \hat{\beta}_t - \beta_t \right) = O_p (T n^{-1}) + O_p (T J^{-4})
\]
The second factor has expectation

\[ \mathbb{E} \sum_{t_2} \left( \sum_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1, i_2} \sum_{j_1, j_2} 1_{j_1 t_1} 1_{j_2 t_2} \hat{I}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-1} \hat{I}_{j_2 t_2} (z_{i_1 t_1}) \hat{I}_{j_2 t_2} (z_{i_2 t_2}) \epsilon_{i_1 t_1} x_{i_2 t_2}^{j_1 t_2} \right) \times \left( \sum_{t_3} n_{t_3}^{-2} n_{t_2}^{-1} \sum_{i_3, i_4} \sum_{j_3, j_4} 1_{j_3 t_3} 1_{j_4 t_4} \hat{I}_{j_3 t_3} (z) \hat{q}_{j_3 t_3}^{-2} \hat{q}_{j_4 t_4}^{-1} \hat{I}_{j_4 t_4} (z_{i_3 t_3}) \hat{I}_{j_4 t_4} (z_{i_4 t_4}) \epsilon_{i_3 t_3} x_{i_4 t_4}^{j_2 t_4} \right) \]

\[ = \sum_{t_1, \ldots, t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-2} \sum_{i_1, \ldots, i_4} \sum_{j_1, \ldots, j_4} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} 1_{j_4 t_4} \hat{I}_{j_1 t_1} (z) \hat{I}_{j_3 t_3} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_3 t_3}^{-1} \hat{q}_{j_2 t_2}^{-1} \hat{q}_{j_4 t_4}^{-1} \right] \]

The expectation is zero unless \((t_1 = t_3, i_1 = i_3)\) so we have

\[ \sum_{t_1, t_2} n_{t_1}^{-4} n_{t_2}^{-2} \sum_{i_1, i_2} \sum_{j_1, j_2} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} 1_{j_4 t_4} \hat{I}_{j_1 t_1} (z) \hat{I}_{j_3 t_3} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_3 t_3}^{-1} \hat{q}_{j_2 t_2}^{-1} \hat{q}_{j_4 t_4}^{-1} \right] \]

\[ \leq C \sum_{t_1, t_2} n_{t_1}^{-4} n_{t_2}^{-2} \sum_{i_1, i_2} \sum_{j_1, j_2} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} 1_{j_4 t_4} \hat{I}_{j_1 t_1} (z) \hat{I}_{j_3 t_3} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_3 t_3}^{-1} \hat{q}_{j_2 t_2}^{-1} \hat{q}_{j_4 t_4}^{-1} \right] \]

\[ \leq C T^2 n^{-3} J^{3d_3} . \]

Thus,

\[ |S_{231}^{PI}|^2 \leq O \left( \left( \frac{n}{J d_s T^2} \right)^2 \right) \times O_p \left( T^2 n^{-3} J^{3d_3} \right) \times \left[ O_p (T n^{-1}) + O_p (T J^{-4}) \right] , \]

by Markov’s inequality and \(S_{231}^{PI} = o_p (1)\) by our rate assumptions. By similar steps we can show that \(|S_{232}^{PI}|^2 \leq O \left( \left( \frac{n}{J d_s T^2} \right)^2 \right) O_p \left( J^{2d_s} n^{-2} T^3 \right) \left[ O_p (T n^{-1}) + O_p (T J^{-4}) \right] \) and so \(S_{232}^{PI} = o_p (1)\) under our rate assumptions.

**Term: \(S_{24}^{PI}\)** Finally, consider \(S_{24}^{PI}\). This term satisfies,

\[ |S_{24}^{PI}| \leq \frac{2 n}{J d_s T^2} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum_{i_1} \sum_{j_1, j_2} 1_{j_1 t_1} 1_{j_2 t_2} - 1 \hat{I}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-2} \hat{I}_{j_2 t_2} (z_{i_1 t_1}) \hat{I}_{j_2 t_2} (z_{i_1 t_1}) \epsilon_{i_1 t_1} \mu (z_{i_1 t_1}) \]

\[ \leq \max_{1 \leq i \leq 1} \max_{1 \leq j \leq J^{d_s}} |1_{j_1 t_1} - 1| \times C \frac{n}{J d_s T^2} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum_{i_1} \sum_{j_1, j_2} 1_{j_1 t_1} 1_{j_2 t_2} (z) \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2} (z_{i_1 t_1}) \hat{I}_{j_2 t_2} (z_{i_1 t_1}) \epsilon_{i_1 t_1} . \]

The second factor has expectation

\[ C \frac{n}{J d_s T^2} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-1} \sum_{i_1} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} (z) \hat{q}_{j_1 t_1}^{-2} \hat{I}_{j_2 t_2} (z_{i_1 t_1}) \hat{I}_{j_2 t_2} (z_{i_1 t_1}) \epsilon_{i_1 t_1} \right] \]

\[ \leq C \frac{n}{J d_s T^2} \sum_{t_1, t_2} n_{t_1}^{-1} \sum_{j_1} \mathbb{E} \left[ 1_{j_1 t_1} \hat{I}_{j_1 t_1} (z) \hat{q}_{j_1 t_1}^{-1} \right] \]

\[ \leq C. \]

Thus this term is \(o_p (1)\) by Lemma 2.

**Term: \(S_3^{PI}\)**
We have that

\[ S^{\text{PI}}_3 = \left( \frac{nT}{Jd^2} \right) T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{Jd^r} 1_{jt} \hat{I}_{jt} (z) \hat{q}^{\text{PI}}_{jt} (z_i) (\hat{e}_{it} - e_{it})^2 \]

\[ = \left( \frac{nT}{Jd^2} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \times \]

\[ \left[ T^{-1} \sum_{t_2} n_{t_2}^{-1} \sum_{j_2} \sum_{i_2} \hat{q}^{\text{PI}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) (\hat{e}_{i_2 t_2} e_{i_2 t_2})^2 \right]^2 \]

\[ + \left( \frac{nT}{Jd^2} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \times \]

\[ \left[ T^{-1} \sum_{t_2} n_{t_2}^{-1} \sum_{j_2} \hat{q}^{\text{PI}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) (\hat{e}_{i_2 t_2} (\hat{\beta}_{t_2} - \beta_{t_2}) - x'_{i_2 t_2} (\hat{\beta}_{t_1} - \beta_{t_1}))]^2 \]

\[ + \left( \frac{nT}{Jd^2} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \times \]

\[ \left[ T^{-1} \sum_{t_2} n_{t_2}^{-1} \sum_{j_2} (1_{j_2 t_2} - 1) \overline{\hat{I}}_{j_2 t_2} (z_i) \mu (z_i) \right]^2 \]

\[ = S^{\text{PI}}_3 + S^{\text{PI}}_3^2 + S^{\text{PI}}_3^3 + S^{\text{PI}}_3^4. \]

**Term: \( S^{\text{PI}}_{31} \)** Now consider \( S^{\text{PI}}_{31} \). It is

\[ S^{\text{PI}}_{31} = \left( \frac{nT}{Jd^2} \right) T^{-4} \sum_{t_1,t_2,t_3} n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} \times \]

\[ \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_3 t_3} (z_i) \hat{e}_{i_2 t_2} \hat{e}_{i_3 t_3} \]

\[ = \sum_{\ell=1}^{5} S^{\text{PI}}_{31} \ell, \]

where

\[ S^{\text{PI}}_{311} = \left( \frac{nT}{Jd^2} \right) T^{-4} \sum_{t_1=t_2=t_3} n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} \times \]

\[ \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_3 t_3} (z_i) \hat{e}_{i_2 t_2} \hat{e}_{i_3 t_3} \]

\[ S^{\text{PI}}_{312} = \left( \frac{nT}{Jd^2} \right) T^{-4} \sum_{t_1 \neq t_2 \neq t_3} n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} \times \]

\[ \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_3 t_3} (z_i) \hat{e}_{i_2 t_2} \hat{e}_{i_3 t_3} \]

\[ S^{\text{PI}}_{313} = \left( \frac{nT}{Jd^2} \right) T^{-4} \sum_{t_1=t_2 \neq t_3} n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} \times \]

\[ \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_3 t_3} (z_i) \hat{e}_{i_2 t_2} \hat{e}_{i_3 t_3} \]

\[ S^{\text{PI}}_{314} = \left( \frac{nT}{Jd^2} \right) T^{-4} \sum_{t_1 \neq t_2 \neq t_3} n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} \times \]

\[ \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_3 t_3} (z_i) \hat{e}_{i_2 t_2} \hat{e}_{i_3 t_3} \]

\[ S^{\text{PI}}_{315} = \left( \frac{nT}{Jd^2} \right) T^{-4} \sum_{t_1=t_2 \neq t_3} n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1 t_1} 1_{j_2 t_2} 1_{j_3 t_3} \times \]

\[ \hat{I}_{j_1 t_1} (z) \hat{q}^{\text{PI}}_{j_1 t_1} (z_i) \overline{\hat{I}}_{j_2 t_2} (z_i) \overline{\hat{I}}_{j_3 t_3} (z_i) \hat{e}_{i_2 t_2} \hat{e}_{i_3 t_3} \]
Term: $S_{311}^{PI}$ First consider $S_{311}^{PI}$:

\[
S_{311}^{PI} = \left( \frac{n T}{J d_s} \right) T^{-4} \sum_{t_1=t_2=t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1t_1} 1_{j_2t_2} 1_{j_3t_3} \times
\]

\[
\hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-1} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \prod_{t_1} (z_{i_1t_1}) \prod_{t_2} (z_{i_2t_2}) \prod_{t_3} (z_{i_3t_3}) \epsilon_{i_2t_2} \epsilon_{i_3t_3}
\]

\[
= \left( \frac{n T}{J d_s} \right) T^{-4} \sum_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} 1_{j_1t_1} 1_{j_2t_2} 1_{j_3t_3} \times
\]

\[
\hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-1} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \prod_{t_1} (z_{i_1t_1}) \prod_{t_2} (z_{i_2t_2}) \prod_{t_3} (z_{i_3t_3}) \epsilon_{i_2t_2} \epsilon_{i_3t_3}
\]

\[
= \left( \frac{n T}{J d_s} \right) T^{-4} \sum_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1} 1_{j_1t_1} \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-1} \hat{g}_{j_1t_1} (z) \hat{g}_{j_1t_1} (z) \prod_{t_1} (z_{i_1t_1}) \prod_{t_2} (z_{i_2t_2}) \prod_{t_3} (z_{i_3t_3}) \epsilon_{i_2t_2} \epsilon_{i_3t_3}
\]

which satisfies

\[
E \left| S_{311}^{PI} \right| \leq \left( \frac{n T}{J d_s} \right) T^{-4} \sum_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum_{i_1,i_2,i_3} \sum_{j_1} E \left[ 1_{j_1t_1} \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-1} \hat{g}_{j_1t_1} (z) \hat{g}_{j_1t_1} (z) \prod_{t_1} (z_{i_1t_1}) \prod_{t_2} (z_{i_2t_2}) \prod_{t_3} (z_{i_3t_3}) \epsilon_{i_2t_2} \epsilon_{i_3t_3} \right]
\]

\[
\leq C \left( \frac{n T}{J d_s} \right) T^{-4} \sum_{t_1} n_{t_1}^{-1} \sum_{j_1} E \left[ 1_{j_1t_1} \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-1} \right]
\]

\[
= C \left( \frac{n T}{J d_s} \right) T^{-4} \sum_{t_1} n_{t_1}^{-1} \sum_{j_1} E \left[ \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-1} \right]
\]

\[
\leq C \left( \frac{n T}{J d_s} \right) T^{-4} J d_s \sum_{t_1} n_{t_1}^{-1} \sum_{j_1} E \left[ \hat{g}_{j_1t_1} (z) \right]
\]

\[
\leq CT^{-2},
\]

so that $S_{311}^{PI} = O_p (T^{-2}) = o_p (1)$ by Markov’s inequality. By similar steps we can show that $S_{312}^{PI} = O_p (T^{-1}) = S_{313}^{PI} = O_p (T^{-1})$, $S_{314}^{PI} = O_p (T^{-1})$, and $S_{315}^{PI} = O_p (J d_s n^{-1})$ which are $o_p (1)$ under our rate assumptions.

Term: $S_{32}^{PI}$ Now consider $S_{32}^{PI}$:

\[
S_{32}^{PI} = \left( \frac{n T}{J d_s} \right) T^{-2} \sum_{t_2} n_{t_1}^{-2} \sum_{j_1} 1_{j_1t_1} \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{g}_{j_1t_1} (z) \hat{g}_{j_1t_1} (z) \prod_{j_1} (z_{i_1t_1}) \prod_{j_2} (z_{i_2t_2}) \prod_{j_3} (z_{i_3t_3}) (\mu (z_{i_2t_2}) - \mu (z_{i_1t_1}))
\]

\[
\leq C \left[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J d_s} \sup_{z} \left| \prod_{t} (z) \mu (z) - \prod_{j} (z) \mu (z) \right| \right]^2 \left( \frac{n T}{J d_s} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{j_1} 1_{j_1t_1} \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{g}_{j_1t_1} (z) \hat{g}_{j_1t_1} (z) \prod_{j_1} (z_{i_1t_1}) \prod_{j_2} (z_{i_2t_2}) \prod_{j_3} (z_{i_3t_3}) (\mu (z_{i_2t_2}) - \mu (z_{i_1t_1}))
\]

\[
\times \left[ T^{-1} \sum_{j_2} n_{j_2}^{-1} \sum_{j_3} 1_{j_2t_2} \hat{g}_{j_2t_2} (z) \hat{g}_{j_2t_2} (z) \prod_{j_2} (z_{i_2t_2}) \prod_{j_3} (z_{i_3t_3}) \right]^2
\]

The first factor is $O_p (J^{-2})$ by Lemma 1. The second factor is

\[
\leq \left( \frac{n T}{J d_s} \right) T^{-2} \sum_{t_2} n_{t_1}^{-2} \sum_{j_1} 1_{j_1t_1} \hat{g}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{g}_{j_1t_1} (z) \hat{g}_{j_1t_1} (z) \prod_{j_1} (z_{i_1t_1}) \prod_{j_2} (z_{i_2t_2}) \prod_{j_3} (z_{i_3t_3}) (\mu (z_{i_2t_2}) - \mu (z_{i_1t_1}))
\]

\[
\leq \left( \frac{n T}{J d_s} \right) T^{-2} \sum_{t_2} n_{t_2}^{-1} \sum_{j_2} 1_{j_2t_2} \hat{g}_{j_2t_2} (z) \hat{g}_{j_2t_2} (z) \prod_{j_2} (z_{i_2t_2}) \prod_{j_3} (z_{i_3t_3}) \right]^2
\]

\[
\leq \left( \frac{n T}{J d_s} \right) T^{-2} \sum_{t_2} n_{t_2}^{-1} \sum_{j_2} 1_{j_2t_2} \hat{g}_{j_2t_2} (z) \hat{g}_{j_2t_2} (z) \prod_{j_2} (z_{i_2t_2}) \prod_{j_3} (z_{i_3t_3}) \right]^2
\]
\[
\begin{align*}
S_{32}^\pi &= O_p(J^{-2}) = o(1).
\end{align*}
\]

**Term: \(S_{33}^\pi\) Now consider \(S_{33}^\pi\)**

\[
S_{33}^\pi = \left(\frac{nT}{J} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \left( n_{t_1}^{-1} \sum_{i_1} \tilde{T}_{j_1 t_1} (z_{i_1 t_1}) \right) \\
&\leq C.
\]

Thus \(S_{32}^\pi = O_p(J^{-2}) = o(1)\).

First consider \(S_{33}^\pi\). By the CS inequality we have,

\[
S_{33}^\pi = 2 \left(\frac{nT}{J} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \left( n_{t_1}^{-1} \sum_{i_1} \tilde{T}_{j_1 t_1} (z_{i_1 t_1}) \right) \\
&\leq 2 \left(\frac{nT}{J} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \left( n_{t_1}^{-1} \sum_{i_1} \tilde{T}_{j_1 t_1} (z_{i_1 t_1}) \right) \\
&\leq 2 \left(\frac{nT}{J} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1 t_1} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \left( n_{t_1}^{-1} \sum_{i_1} \tilde{T}_{j_1 t_1} (z_{i_1 t_1}) \right) \\
&\leq C \left(\frac{nT}{J} \right) T^{-4} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum_{i_1, i_2, j_1, j_2} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{q, j_1 t_1} 1_{q, j_2 t_2} 1_{q, j_1 t_3} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \right. \\
&\leq C \left(\frac{nT}{J} \right) T^{-4} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum_{i_1, i_2, j_1, j_2} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{q, j_1 t_1} 1_{q, j_2 t_2} 1_{q, j_1 t_3} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \right].
\]

The last factor is \(O_p(Tn^{-1}) + O_p(TJ^{-4})\) The first and second factor are then bounded by

\[
2 \left(\frac{nT}{J} \right) T^{-4} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum_{i_1, i_2, j_1, j_2} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{q, j_1 t_1} 1_{q, j_2 t_2} 1_{q, j_1 t_3} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \right. \\
&\leq C \left(\frac{nT}{J} \right) T^{-4} \sum_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum_{i_1, i_2, j_1, j_2} \sum_{j_1, j_2} \mathbb{E} \left[ 1_{q, j_1 t_1} 1_{q, j_2 t_2} 1_{q, j_1 t_3} \tilde{T}_{j_1 t_1} (z) \frac{\hat{q}_{j_1 t_1} - 2}{\tilde{T}_{j_1 t_1} (z_{i_1 t_1})} \right].
\]

Thus, \(S_{33}^\pi = O_p(n^{-1}) + O_p(J^{-4})\). By similar steps we can show that \(S_{33}^\pi = O_p(n^{-1}) + O_p(T^{-1}J^{-2})\) which is \(o_p(1)\) under our rate assumptions.
Term: $S_{34}^{PI}$ Now consider $S_{34}^{PI}$:

$$S_{34}^{PI} = \left( \frac{nT}{j_{dz}} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1t_1 \hat{\mu}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{\mu}_{j_1t_1} (z_i_{t_1})} \times$$

$$\left[ T^{-1} \sum_{t_2} \sum_{j_2} (1_{j_2t_2} - 1) \hat{\mu}_{j_2t_2} (z_i_{t_1}) \mu (z_i_{t_1}) \right]^2 ,$$

which satisfies

$$|S_{34}^{PI}| \leq \max_{1 \leq t \leq T} \max_{1 \leq j \leq j_{dz}} |1_{jt} - 1| \times C \left( \frac{nT}{j_{dz}} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1t_1 \hat{\mu}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{\mu}_{j_1t_1} (z_i_{t_1})} .$$

The second factor is

$$\left( \frac{nT}{j_{dz}} \right) T^{-2} \sum_{t_1} n_{t_1}^{-2} \sum_{i_1} \sum_{j_1} 1_{j_1t_1 \hat{\mu}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{\mu}_{j_1t_1} (z_i_{t_1})}$$

$$= \frac{n}{j_{dz} T} \sum_{t_1} n_{t_1}^{-1} \sum_{i_1} \sum_{j_1} 1_{j_1t_1 \hat{\mu}_{j_1t_1} (z) \hat{q}_{j_1t_1}^{-2} \hat{\mu}_{j_1t_1} (z_i_{t_1})}$$

$$\leq C \frac{n}{T} \sum_{t_1} n_{t_1}^{-1} \sum_{i_1} \sum_{j_1} \hat{\mu}_{j_1t_1} (z)$$

$$\leq C .$$

Thus, $S_{34}^{PI} = o_p (1)$ by Lemma 2.

### B.7 Proof of Theorem 3

As discussed in the main text we will work with a modified version of $L_2$ and $L_3$ where we assume that the conditional quantiles are known. We start with

$$L_1 = T^{-1} \sum_{t=1}^{T} n_{t}^{-1} \sum_{i=1}^{n_{t}} \sum_{j=1}^{j_{dz}} 1_{jt} \hat{\mu}_{jt} (z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_i t) (\mu (z_i t) - \mu (z)) ,$$

$$L_{21} = T^{-1} \sum_{t=1}^{T} n_{t}^{-1} \sum_{i=1}^{n_{t}} \sum_{j=1}^{j_{dz}} 1_{jt} \hat{\mu}_{jt} (z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_i t) \varepsilon_{it}$$

$$L_{22} = -T^{-1} \sum_{t=1}^{T} n_{t}^{-1} \sum_{i=1}^{n_{t}} \sum_{j=1}^{j_{dz}} 1_{jt} \hat{\mu}_{jt} (z) \hat{q}_{jt}^{-1} q_{jt}^{2} (\hat{q}_{jt} - q_{jt}) \hat{\mu}_{jt} (z_i t) \varepsilon_{it}$$

$$L_{23} = T^{-1} \sum_{t=1}^{T} n_{t}^{-1} \sum_{i=1}^{n_{t}} \sum_{j=1}^{j_{dz}} 1_{jt} \hat{\mu}_{jt} (z) \hat{q}_{jt}^{-1} q_{jt}^{2} (\hat{q}_{jt} - q_{jt})^{2} \hat{\mu}_{jt} (z_i t) \varepsilon_{it}$$

$$L_{3} = -T^{-1} \sum_{t=1}^{T} n_{t}^{-1} \sum_{i=1}^{n_{t}} \sum_{j=1}^{j_{dz}} 1_{jt} \hat{\mu}_{jt} (z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_i t) x_{it} ' (\hat{\beta}_t - \beta_t) ,$$

$$L_{4} = T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{j_{dz}} (1_{jt} - 1) \hat{\mu}_{jt} (z) \mu (z) ,$$

and recall that $\hat{q}_{jt} = n_{t}^{-1} \sum_{i=1}^{n_{t}} \hat{\mu}_{jt} (z_i t)$ so that

$$\hat{\mu} (z) - \mu (z) = L_1 + L_{21} + L_{22} + L_{23} + L_{3} + L_{4} .$$

Thus,

$$E \left[ |\hat{\mu} (z) - \mu (z)|^2 \right] = M_1 + M_2 + M_3 + M_4 + M_5 + o_p \left( J^{-2} + \frac{J_{dz}^{2}}{n^2 T} \right) ,$$

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where

\[ \mathcal{M}_1 = \mathcal{L}_1^2 \]
\[ \mathcal{M}_2 = \mathbb{E} \left[ (\mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{L}_{23})^2 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \]
\[ \mathcal{M}_3 = \mathbb{E} \left[ \mathcal{L}_3^2 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \]
\[ \mathcal{M}_4 = 2\mathcal{L}_1 \mathbb{E} \left[ \mathcal{L}_3 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \]
\[ \mathcal{M}_5 = 2\mathbb{E} \left[ (\mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{L}_{23}) \mathcal{L}_3 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \]

since \( \mathbb{E} \left[ \mathcal{L}_1 (\mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{L}_{23}) \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] = 0 \) and by Lemmas 2 and 3 all terms involving \( \mathcal{L}_4 \) are of smaller order.

### B.7.1 Term: \( \mathcal{M}_1 \)

First, consider, \( \mathcal{L}_1^2 \). For this term we work with estimated quantiles. We have,

\[
\mathcal{L}_1 = T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \frac{\hat{\mu}_{jt}}{\hat{\mu}_{jt}} (z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_{it}) (\mu (z_{it}) - \mu (z)) \\
= \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \frac{\hat{\mu}_{jt}}{\hat{\mu}_{jt}} (z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_{it}) (z_{it} - z) \\
+ T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \frac{\hat{\mu}_{jt}}{\hat{\mu}_{jt}} (z) \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_{it}) (z_{it} - z), \quad \frac{\partial \mu (z)}{\partial z^j} \bigg|_{z = \hat{z}} (z_{it} - z) \\
= \mathcal{L}_{11} + \mathcal{L}_{12},
\]

where \( \hat{z} = \alpha \cdot z + (1 - \alpha) \cdot z_{it} \), \( \alpha \in (0, 1) \). Thus, we need only show that \( \mathcal{L}_{12} \) is \( O_p (J^{-1}) \). We have,

\[
|\mathcal{L}_{12}| \leq T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_{it}) (z_{it} - z), \quad \frac{\partial \mu (z)}{\partial z^j} \bigg|_{z = \hat{z}} (z_{it} - z) \\
\leq C T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_{it}) \| z_{it} - z \|^2 \\
\leq C T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \hat{q}_{jt}^{-1} \hat{\mu}_{jt} (z_{it}) \sum_{s=1}^{d_z} \left( \hat{b}_{jst} - \hat{b}_{js-1}, t, s \right)^2 \\
\leq \max_{1 \leq t \leq T} \max_{1 \leq j \leq J} \max_{1 \leq s \leq d_z} \left| \hat{b}_{jst} - \hat{b}_{js-1}, t, s \right|^2 \times C T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{l_{jz}^{z}} 1_{jt} \hat{\mu}_{jt} (z),
\]

and so \( \mathcal{L}_{12} = O_p (J^{-2}) \) by Lemma 1 and the result follows.

### B.7.2 Term: \( \mathcal{M}_2 \)

We have

\[
\mathcal{M}_{21} = \mathbb{E} \left[ |\mathcal{L}_{21}|^2 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \\
\mathcal{M}_{22} = \mathbb{E} \left[ |\mathcal{L}_{22}|^2 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \\
\mathcal{M}_{23} = \mathbb{E} \left[ |\mathcal{L}_{23}|^2 \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \\
\mathcal{M}_{24} = 2\mathbb{E} \left[ \mathcal{L}_{21} \mathcal{L}_{22} \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \\
\mathcal{M}_{25} = 2\mathbb{E} \left[ \mathcal{L}_{21} \mathcal{L}_{23} \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \\
\mathcal{M}_{26} = 2\mathbb{E} \left[ \mathcal{L}_{22} \mathcal{L}_{23} \mid \mathcal{F}_1, \ldots, \mathcal{F}_T \right].
\]
Squared Terms

First note that $M_{21}$ is

$$M_{21} = -2 \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-2} \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$= M_{211} + M_{212} + o_p \left( \frac{J^{2d_z} n^{-2} T^{-1}}{n} \right),$$

by Lemma 2 where

$$M_{211} = -2 \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-2} \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 \mathbb{E} \left[ \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 | \mathcal{F}_t \right]$$

$$M_{212} = -2 \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} (z) q_{jt}^{-2} \{ \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 - \mathbb{E} \left[ \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 | \mathcal{F}_t \right] \}.$$

Note that $M_{211} = O_p \left( J^{d_z} n^{-1} T^{-1} \right)$ as given in the proof of Theorem 1. Next, $M_{212}$ is mean zero with variance,

$$\mathbb{E} \left[ |M_{212}|^2 \right] = -2 \sum_{i=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} (z) q_{jt}^{-4} \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 \mathbb{E} \left[ \mathbb{1}_{jt} (z) q_{jt}^{-4} \left( \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 - \mathbb{E} \left[ \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 | \mathcal{F}_t \right] \right)^2 \right]$$

$$\leq C T^{-3} n^{-3} J^{3d_z},$$

where the first equality follows by Assumption 1. Thus, $M_{212} = O_p \left( J^{3d_z} n^{-3/2} T^{-3/2} \right)$.

Next, we have

$$M_{22} = -2 \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-4} (\tilde{q}_{jt} - q_{jt})^2 \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$= -2 \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-4} (\mathbb{1}_{jt} (z_i t) - q_{jt}) (\mathbb{1}_{jt} (z_i t) - q_{jt}) \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$= M_{221} + M_{222} + M_{223} + M_{224},$$

where

$$M_{221} = -2 \sum_{t=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-4} (\mathbb{1}_{jt} (z_i t) - q_{jt}) (\mathbb{1}_{jt} (z_i t) - q_{jt}) \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$M_{222} = -2 \sum_{t=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-4} (\mathbb{1}_{jt} (z_i t) - q_{jt}) (\mathbb{1}_{jt} (z_i t) - q_{jt}) \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$M_{223} = 2T^{-2} \sum_{t=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-4} (\mathbb{1}_{jt} (z_i t) - q_{jt}) (\mathbb{1}_{jt} (z_i t) - q_{jt}) \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$M_{224} = -2 \sum_{t=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-4} (\mathbb{1}_{jt} (z_i t) - q_{jt}) (\mathbb{1}_{jt} (z_i t) - q_{jt}) \mathbb{1}_{jt} (z_i t) \sigma_{it}^2.$$

However,

$$\mathbb{E} |M_{221}| = -2 \sum_{t=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) \mathbb{E} \left[ q_{jt}^{-4} (1 - q_{jt})^2 \mathbb{1}_{jt} (z_i t) \sigma_{it}^2 \right] \leq C T^{-1} n^{-3} J^{3d},$$

so that $M_{221} = O_p \left( T^{-1} n^{-3} J^{3d} \right) = o_p \left( J^{2d_z} n^{-2} T^{-1} \right)$ by Markov’s inequality and Assumption 3. Next,

$$M_{222} = -2 \sum_{t=1}^{T} n_t^{-4} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t} \mathbb{1}_{jt} \mathbb{1}_{jt} (z) q_{jt}^{-2} (\mathbb{1}_{jt} (z_i t) - q_{jt})^2 \mathbb{1}_{jt} (z_i t) \sigma_{it}^2$$

$$= M_{2221} + M_{2222} + o_p \left( \frac{J^{2d_z} n^{-2} T^{-1}}{n} \right),$$

where
by Lemma 2 where

\[ \mathcal{M}_{2221} = T^{-2} \sum_{t=1}^{T} n_t^{-4} (n_t - 1) \sum_{i_t}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{E} \left[ (\mathbb{I}_{jt}(z) q_{jt}^{-3}) \sigma_{i_t}^{2} | \mathcal{F}_t \right] \]

\[ \mathcal{M}_{2222} = T^{-2} \sum_{t=1}^{T} n_t^{-4} \sum_{i_t \neq i_2}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{I}_{jt}(z) q_{jt}^{-4} \times \]

\[ \left\{ (\mathbb{I}_{jt}(z) - q_{jt})^2 \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} - \mathbb{E} \left[ (\mathbb{I}_{jt}(z) - q_{jt})^2 \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} | \mathcal{F}_t \right] \right\}. \]

\[ \mathcal{M}_{2221} = O_p \left( J^{2d_z} n^{-2} T^{-1} \right). \]

Next note that \( \mathcal{M}_{2222} \) is mean zero with variance,

\[ \mathbb{E} \left[ |\mathcal{M}_{2222}|^2 \right] \leq T^{-4} \sum_{t=1}^{T} n_t^{-8} \sum_{i_t \neq i_2,i_3 \neq i_4}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{E} \left[ (\mathbb{I}_{jt}(z) q_{jt}^{-8}) \left\{ (\mathbb{I}_{jt}(z) - q_{jt})^2 \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} - \mathbb{E} \left[ (\mathbb{I}_{jt}(z) - q_{jt})^2 \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} | \mathcal{F}_t \right] \right\} \right] \]

\[ \left\{ (\mathbb{I}_{jt}(z) - q_{jt})^2 \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} - \mathbb{E} \left[ (\mathbb{I}_{jt}(z) - q_{jt})^2 \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} | \mathcal{F}_t \right] \right\}. \]

There are six nonzero terms and by Markov’s inequality and following similar steps as above we can show that \( \mathcal{M}_{2222} = O_p \left( T^{-3/2} n^{-5/2} J^{2d_z}/2 \right) = o_p \left( J^{2d_z} n^{-2} T^{-1} \right) \) by Assumption 3. Next,

\[ \mathcal{M}_{223} = 2T^{-2} \sum_{t=1}^{T} n_t^{-4} \sum_{i_t \neq i_2}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{I}_{jt}(z) q_{jt}^{-4} (1 - q_{jt}) (\mathbb{I}_{jt}(z) - q_{jt}) \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} \]

is mean zero with variance

\[ \mathbb{E} \left[ |\mathcal{M}_{223}|^2 \right] = 4T^{-4} \sum_{t=1}^{T} n_t^{-8} \sum_{i_t \neq i_2,i_3 \neq i_4}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{E} \left[ (1 - q_{jt})^2 (\mathbb{I}_{jt}(z) - q_{jt}) (\mathbb{I}_{jt}(z) - q_{jt}) \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} \sigma_{i_t}^{2} \right]. \]

There are three nonzero terms and by Markov’s inequality and following similar steps as above we can show that \( \mathcal{M}_{223} = O_p \left( J^{2d_z}/2 n^{-5/2} T^{-3/2} \right) = o_p \left( J^{2d_z} n^{-2} T^{-1} \right) \) by Assumption 3.

Finally, we have \( \mathcal{M}_{224} \),

\[ \mathcal{M}_{224} = T^{-2} \sum_{t=1}^{T} n_t^{-4} \sum_{i_t \neq i_2,i_3 \neq i_4}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{I}_{jt}(z) q_{jt}^{-4} (\mathbb{I}_{jt}(z) - q_{jt}) (\mathbb{I}_{jt}(z) - q_{jt}) \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} \]

which is mean zero with variance

\[ \mathbb{E} \left[ |\mathcal{M}_{224}|^2 \right] = T^{-4} \sum_{t=1}^{T} n_t^{-8} \sum_{i_t \neq i_2,i_3 \neq i_4}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{E} \left[ (1 - q_{jt})^2 (\mathbb{I}_{jt}(z) - q_{jt}) (\mathbb{I}_{jt}(z) - q_{jt}) (\mathbb{I}_{jt}(z) - q_{jt}) \mathbb{I}_{jt}(z) \sigma_{i_t}^{2} \sigma_{i_t}^{2} \right]. \]

There are four nonzero terms and by Markov’s inequality and following similar steps as above we can show that \( \mathcal{M}_{224} = O_p \left( T^{-3/2} n^{-2} J^{2d_z} \right) = o_p \left( J^{2d_z} n^{-2} T^{-1} \right) \) by Markov’s inequality and Assumption 3.

Next we have

\[ \mathcal{M}_{23} = \mathbb{E} \left[ \mathcal{L}_2 \mathcal{F}_1, \ldots, \mathcal{F}_T \right] = T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i_t}^{n_t} \sum_{j_t=1}^{J_t} \mathbb{I}_{jt}(z) \mathbb{Q}_{jt}^{-2} \mathbb{I}_{jt}(z) \mathbb{Q}_{jt}^{-4} (\mathbb{I}_{jt}(z) - q_{jt}) \mathbb{I}_{jt}(z) \sigma_{i_t}^{2}. \]
and note that
\[
T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-2} q_{jt}^{-4} (\hat{q}_{jt} - q_{jt})^4 \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 \\
\leq CT^{-2} \sum_{t=1}^{T} n_t^{-1} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-1} q_{jt}^{-4} (\hat{q}_{jt} - q_{jt})^4 \\
\leq CT^{-2} \cdot \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_d} q_{jt}^{-4} |\hat{q}_{jt} - q_{jt}|^4 \cdot J^d n_t^{-1},
\]
so that \(\mathcal{M}_{23} = O_p \left( T^{-2} n^{-3} J^3 d_s \log(J^d \vee T)^2 \right) = o_p \left( J^{2d_s n^{-2} T^{-1}} \right).

### Cross-Product Terms

The first cross-product term is:
\[
\mathcal{M}_{24} = 2 \mathbb{E} [L_{21} L_{22} | 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T] \\
= -2T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-3} (\hat{q}_{jt} - q_{jt}) \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 \\
= -2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i_1,i_2}^{n_t} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-3} (\mathbb{I}_{jt}(z_i t) - q_{jt}) \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 \\
= \mathcal{M}_{241} + \mathcal{M}_{242} + \mathcal{M}_{243} + o_p \left( J^{2d_s n^{-2} T} \right)
\]
by Lemma 2 where
\[
\mathcal{M}_{241} = -2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i_1} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-3} \mathbb{E} \left[ \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 | \mathcal{F}_t \right] \\
\mathcal{M}_{242} = -2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i_1} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-3} \left\{ \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 - \mathbb{E} \left[ \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 | \mathcal{F}_t \right] \right\} \\
\mathcal{M}_{243} = -2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i_1 \neq i_2} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-3} (\mathbb{I}_{jt}(z_i t) - q_{jt}) \mathbb{I}_{jt}(z_i t) \sigma_{it}^2.
\]
Note that \(\mathcal{M}_{241} = O_p \left( J^{5d_s n^{-5/2} T^{-3/2}} \right).\) For \(\mathcal{M}_{242}\) it is mean zero with variance,
\[
\mathbb{E} |\mathcal{M}_{242}|^2 = 4T^{-4} \sum_{t=1}^{T} n_t^{-2} \sum_{i_1} \sum_{j=1}^{J_d} \mathbb{E} \left[ \mathbb{I}_{jt}(z) q_{jt}^{-6} \left\{ \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 - \mathbb{E} \left[ \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 | \mathcal{F}_t \right] \right\} \right]^2 \\
\leq CT^{-4} \sum_{t=1}^{T} n_t^{-2} \sum_{i_1} \sum_{j=1}^{J_d} \mathbb{E} \left[ \mathbb{I}_{jt}(z) q_{jt}^{-5} \right] \\
\leq CT^{-3} n^{-5} J^{5d},
\]
and so \(\mathcal{M}_{242} = o_p \left( J^{5d_s n^{-5/2} T^{-3/2}} \right) = o_p \left( J^{2d_s n^{-2} T^{-1}} \right)\) by Markov’s inequality and Assumption 3. Next, \(\mathcal{M}_{243}\)
\[
\mathcal{M}_{243} = -2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i_1 \neq i_2} \sum_{j=1}^{J_d} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-3} (\mathbb{I}_{jt}(z_i t) - q_{jt}) \mathbb{I}_{jt}(z_i t) \sigma_{it}^2,
\]
is conditionally mean zero with variance
\[
\mathbb{E} \left[ |\mathcal{M}_{243}|^2 \right] = 4T^{-4} \sum_{t=1}^{T} n_t^{-2} \sum_{i_1 \neq i_2} \sum_{j=1}^{J_d} \mathbb{E} \left[ \mathbb{I}_{jt}(z) q_{jt}^{-6} (\mathbb{I}_{jt}(z_i t) - q_{jt}) (\mathbb{I}_{jt}(z_i t) - q_{jt}) \mathbb{I}_{jt}(z_i t) \mathbb{I}_{jt}(z_i t) \sigma_{it}^2 \sigma_{it}^2 \right] \\
= \mathcal{N}_{243}^{(1)} + \mathcal{N}_{243}^{(2)} + \mathcal{N}_{243}^{(3)}.
\]
Then,

\[
\mathcal{N}_{243}^{(1)} = 4T^{-4} \sum_{t=1}^{T} n_t^{-6} \sum_{i_1 \neq i_2, i_3 \neq i_4} n_t^{J_{d_t}^d} \sum_{j=1}^{j_{d_t}^d} \mathbb{E}\left[ \Pi_{jt}(z) q_{jt}^{-6} \Pi_{jt}(z_{i_2t}) - q_{jt} \right] \Pi_{jt}(z_{i_1t}) \Pi_{jt}(z_{i_3t}) \sigma_{i_1t}^2 \sigma_{i_3t}^2
\]

\[
\mathcal{N}_{243}^{(2)} = 4T^{-4} \sum_{t=1}^{T} n_t^{-6} \sum_{i_1 \neq i_3} n_t^{J_{d_t}^d} \sum_{j=1}^{j_{d_t}^d} \mathbb{E}\left[ \Pi_{jt}(z) q_{jt}^{-6} \Pi_{jt}(z_{i_2t}) - q_{jt} \right] \Pi_{jt}(z_{i_1t}) \Pi_{jt}(z_{i_3t}) \sigma_{i_1t}^2 \sigma_{i_3t}^2
\]

\[
\mathcal{N}_{243}^{(3)} = 4T^{-4} \sum_{t=1}^{T} n_t^{-6} \sum_{i_1 \neq i_2, i_3 \neq i_4} n_t^{J_{d_t}^d} \sum_{j=1}^{j_{d_t}^d} \mathbb{E}\left[ \Pi_{jt}(z) q_{jt}^{-6} \Pi_{jt}(z_{i_2t}) - q_{jt} \right] \Pi_{jt}(z_{i_1t}) \Pi_{jt}(z_{i_3t}) \sigma_{i_1t}^2 \sigma_{i_3t}^2
\].

Then,

\[
\mathcal{N}_{243}^{(1)} = 4T^{-4} \sum_{t=1}^{T} n_t^{-6} \sum_{i_1 \neq i_2, i_3 \neq i_4} n_t^{J_{d_t}^d} \sum_{j=1}^{j_{d_t}^d} \mathbb{E}\left[ \Pi_{jt}(z) q_{jt}^{-6} \Pi_{jt}(z_{i_2t}) - q_{jt} \right] \Pi_{jt}(z_{i_1t}) \Pi_{jt}(z_{i_3t}) \sigma_{i_1t}^2 \sigma_{i_3t}^2 \leq C T^{-3} n^{-3} J^{3d_t}
\]

Next

\[
\mathcal{N}_{243}^{(2)} = 4T^{-4} \sum_{t=1}^{T} n_t^{-6} \sum_{i_1 \neq i_2} n_t^{J_{d_t}^d} \sum_{j=1}^{j_{d_t}^d} \mathbb{E}\left[ \Pi_{jt}(z) q_{jt}^{-6} \Pi_{jt}(z_{i_2t}) - q_{jt} \right] \Pi_{jt}(z_{i_1t}) \sigma_{i_1t}^4 \leq C T^{-3} n^{-4} J^{4d_t}
\]

and

\[
\mathcal{N}_{243}^{(3)} = 4T^{-4} \sum_{t=1}^{T} n_t^{-6} \sum_{i_1 \neq i_2} n_t^{J_{d_t}^d} \sum_{j=1}^{j_{d_t}^d} \mathbb{E}\left[ \Pi_{jt}(z) q_{jt}^{-6} \Pi_{jt}(z_{i_2t}) - q_{jt} \right] \Pi_{jt}(z_{i_1t}) \sigma_{i_1t}^2 \sigma_{i_2t}^2 \leq C T^{-3} n^{-4} J^{4d_t}
\]

Thus, \( \mathcal{N}_{243} \) is mean zero and of order \( O_p \left( T^{-3/2} n^{-3/2} J^{3d_t/2} \right) \).

The next cross product term is

\[
\mathcal{M}_{25} = \mathbb{E}\left[ \mathcal{L}_{21} \mathcal{L}_{23} | \mathbf{3}, \mathbf{X}, \mathbf{F}_1, \ldots, \mathbf{F}_T \right]
\]

\[
= T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{d_t}^d} 1_{jt} \Pi_{jt}(z) q_{jt}^{-1} \Pi_{jt}(z_{i_2t}) - q_{jt} \Pi_{jt}(z_{i_1t}) \sigma_{i_1t}^2
\]

\[
= \mathbb{E}\left[ |\mathcal{L}_{21}|^2 \right] \mathbb{E}\left[ \mathbf{3}, \mathbf{X}, \mathbf{F}_1, \ldots, \mathbf{F}_T \right]
\]

\[
- T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{d_t}^d} 1_{jt} \Pi_{jt}(z) q_{jt}^{-4} \Pi_{jt}(z_{i_2t}) - q_{jt} \Pi_{jt}(z_{i_1t}) \sigma_{i_1t}^2
\]

The second term satisfies

\[
\left| - T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{d_t}^d} 1_{jt} \Pi_{jt}(z) q_{jt}^{-4} \Pi_{jt}(z_{i_2t}) - q_{jt} \Pi_{jt}(z_{i_1t}) \sigma_{i_1t}^2 \right|
\]

\[
\leq C T^{-2} \sum_{t=1}^{T} n_t^{-1} \sum_{j=1}^{J_{d_t}^d} 1_{jt} \Pi_{jt}(z) q_{jt}^{-4} |\Pi_{jt}(z_{i_1t}) - q_{jt}|^3
\]

\[
\leq C J^{2d} \frac{J^{d/2} \log (J^d \vee T)^{3/2}}{n^{1/2}}
\]
and so

\[ \mathcal{M}_{25} = \mathbb{E} \left[ \left| \mathcal{L}_{22} \right|^2 \left| 3, \mathcal{X}_t, \mathcal{F}_1, \ldots, \mathcal{F}_T \right| \right] + o_p \left( J^{2d_z n^{-2} - T^{-1}} \right). \]

Finally,

\[ \mathcal{M}_{26} = \mathbb{E} \left[ \mathcal{L}_{22} \mathcal{L}_{23} \left| 3, \mathcal{X}_t, \mathcal{F}_1, \ldots, \mathcal{F}_T \right| \right] = -T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) \tilde{q}_{jt}^{-4} \tilde{q}_{jt}^{-4} (\tilde{q}_{jt} - q_{jt})^3 \mathbb{I}_{jt} (z_{it}) \sigma_{it}^2. \]

But

\[ \left| T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) \tilde{q}_{jt}^{-4} \tilde{q}_{jt}^{-4} (\tilde{q}_{jt} - q_{jt})^3 \mathbb{I}_{jt} (z_{it}) \sigma_{it}^2 \right| \]
\[ \leq CT^{-2} \sum_{t=1}^{T} n_t^{-1} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) q_{jt}^{-4} (|\tilde{q}_{jt} - q_{jt}|)^3 \]
\[ \leq CT^{-1} \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_{dz}} q_{jt}^{-3} |\tilde{q}_{jt} - q_{jt}|^3 \cdot J^{d_z - n^{-1}}, \]

so that \( \mathcal{M}_{26} = O_p \left( T^{-1} n^{-5/2} J^{5d_z/2} \log (J \vee T)^{3/2} \right) = o_p \left( J^{2d_z n^{-2} T^{-1}} \right) \) by Lemma 2 and Assumption 3. Thus,

\[ \mathcal{M}_2 = \mathbb{E} \left[ (\mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{L}_{23})^2 \left| 3, \mathcal{X}_t, \mathcal{F}_1, \ldots, \mathcal{F}_T \right| \right] = 3 \mathbb{E} \left[ (\mathcal{L}_{22})^2 \left| 3, \mathcal{X}_t, \mathcal{F}_1, \ldots, \mathcal{F}_T \right| \right] + 2 \mathbb{E} \left[ (\mathcal{L}_{21} \mathcal{L}_{22}) \left| 3, \mathcal{X}_t, \mathcal{F}_1, \ldots, \mathcal{F}_T \right| \right] + o_p \left( J^{2d_z n^{-2 T}} \right), \]

where

\[ \mathcal{V}_t^{(1)} (z) = n J^{-d_z} T^{-1} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) q_{jt}^{-2} \mathbb{E} \left[ \mathbb{I}_{jt} (z_{it}) \sigma_{it}^2 \right], \]
\[ \mathcal{V}_t^{(2)} (z) = n^2 J^{-2d_z} T^{-1} n_t^{-3} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) q_{jt}^{-3} \mathbb{E} \left[ \mathbb{I}_{jt} (z_{it}) \sigma_{it}^2 \right], \]
\[ \mathcal{C}_t (z) = n^{3/2} J^{-3d_z/2} T^{-1/2} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) q_{jt}^{-2} \mathbb{E} \left[ \mathbb{I}_{jt} (z_{it}) \sigma_{it}^2 \right] \]
\[ + 2n^{3/2} J^{-3d_z/2} T^{-1/2} n_t^{-3} \sum_{i_1 \neq i_2} \sum_{j=1}^{J_{dz}} 1_{jt} \mathbb{I}_{jt} (z) q_{jt}^{-3} \mathbb{E} \left[ \mathbb{I}_{jt} (z_{it}) \sigma_{it}^2 \right] \]

B.7.3 Term: \( \mathcal{M}_3 \)

We have,
\[ M_3 = \mathbb{E} \left[ T^{-1} \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{\text{obs}}^t} 1_{\beta_t} (z) \hat{q}_{ij} (z) (z_i) x_i' (\hat{\beta}_t - \beta_t)^2 \right] 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \]
\[ = \mathbb{E} \left[ T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (z)' (\hat{\beta}_t - \beta_t)^2 \right] 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \]
\[ \leq 2\mathbb{E} \left[ T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (z)' (\hat{\beta}_t - \beta_t)^2 \right] 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \]
\[ + 2\mathbb{E} \left[ T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (\hat{h}_t (z) - h_t (z))' (\hat{\beta}_t - \beta_t)^2 \right] 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \]
\[ = M_{31} + M_{32} \]

However, following similar steps as in the proof of Lemma 4

\[ M_{31} = O_p \left( n^{-1} T^{-1} \right) + O_p \left( J^{-4} \right) + O_p \left( J^{d_z} n^{-3} \right) + O_p \left( J^{d_z - 4} n^{-2} \right). \]

The \( O_p \left( n^{-1} T^{-1} \right) \) term is not a function of \( J \) and the remaining terms are \( o_p \left( J^{d_z} n^{-2} T^{-1} \right) \). By the CS inequality, the second term satisfies

\[ M_{32} \leq T^{-1} \sum_{t=1}^{T} \| \hat{h}_t (z) - h_t (z) \| ^2 \times T^{-1} \sum_{t=1}^{T} \mathbb{E} \left[ \| \hat{\beta}_t - \beta_t \| ^2 \right] 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T. \]

The first factor is \( O_p \left( n^{-1} J^{d_z} \right) \) by similar steps as in the proof of Theorem 1 and the second factor, following similar steps as in the proof of Lemma 4, so that

\[ M_{32} = O_p \left( n^{-1} J^{d_z} \right) \times O_p \left( J^{-4} + n^{-1} J^{-2} \right) = o_p \left( J^{d_z} n^{-2} T^{-1} \right), \]

and the result follows.

**B.7.4 Term: \( M_4 \)**

We have,

\[ M_4 = 2\mathcal{L}_1 \mathbb{E} \left[ L_3 | 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \]
\[ = 2\mathcal{L}_1 \mathbb{E} \left[ T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (z)' \hat{\Omega}_{uu,t}^{-1} X_t M_{B_t} (\mu (z_t) + \varepsilon_t) / n_t | 3, \mathcal{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \right] \]
\[ = 2\mathcal{L}_1 \times T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (z)' \hat{\Omega}_{uu,t}^{-1} X_t M_{B_t} (z_t) / n_t, \]

where, with some abuse of notation, we define \( \mu (z_t) \) as the \( n_t \times 1 \) vector \( \mu (z_t) = (\mu (z_1), \mu (z_2), \ldots, \mu (z_{n_t}))' \).

The first factor is \( O_p \left( J^{-1} \right) \) by the proof of Theorem 1 and the second factor satisfies

\[ T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (z)' \hat{\Omega}_{uu,t}^{-1} X_t M_{B_t} \mu (z_t) / n_t = M_{41} + M_{42}, \]

where

\[ M_{41} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} h_t (z)' \hat{\Omega}_{uu,t}^{-1} (H_t + U_t)' M_{B_t} \mu (z_t) / n_t \]
\[ M_{42} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} (\hat{h}_t (z) - h_t (z))' \hat{\Omega}_{uu,t}^{-1} X_t M_{B_t} \mu (z_t) / n_t. \]
Following similar steps as in the proof of Lemma 4 we have that,

\[ |M_{41}|^2 = O_p \left( J^{-4} \right) + O_p \left( n^{-1}T^{-1}J^{-2} \right) + O_p \left( n^{-2}J^{-2} \right) + O_p \left( n^{-1}J^{-6} \right) + O_p \left( J^{2d_s-2}n^{-3} \right). \]

For \( M_{42} \), by the CS inequality,

\[ |M_{42}|^2 \leq T^{-1} \sum_{t=1}^{T} \left\| h_t(z) - h_t(z) \right\|^2 \times T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \left\| \tilde{\Omega}_{u,u,t}^{-1} X_t'M_{B_t,\mu}(z_t)/n_t \right\|^2. \]

The first factor is \( O_p \left( n^{-1}J^{d_s} \right) \) by the same steps as in the proof of Theorem 1. The second factor is

\[ T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \left\| \tilde{\Omega}_{u,u,t}^{-1} X_t'M_{B_t,\mu}(z_t)/n_t \right\|^2 \leq T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \lambda_{\max} \left( \tilde{\Omega}_{u,u,t}^{-1} \right)^2 \left\| X_t'M_{B_t,\mu}(z_t)/n_t \right\|^2 \leq CT^{-1} \sum_{t=1}^{T} \left\| X_t'M_{B_t,\mu}(z_t)/n_t \right\|^2. \]

Following similar steps as in the proof of Lemma 4 we have that,

\[ |M_{42}|^2 = O_p \left( n^{-1}J^{d_s} \right) O_p \left( J^{-4} + n^{-1}J^{-2} \right) = o_p \left( J^{2d_s-2}n^{-2}T^{-1} \right). \]

Thus,

\[ |M_4|^2 = O_p \left( J^{-6} \right) + O_p \left( n^{-1}T^{-1}J^{-4} \right) + O_p \left( n^{-2}J^{-4} \right) + O_p \left( n^{-1}J^{-8} \right) + O_p \left( J^{2d_s-4}n^{-3} \right) + O_p \left( n^{-1}J^{d_s}J^{-6} \right) + O_p \left( J^{d_s}n^{-2} \right), \]

so that \( M_4 = o_p \left( J^{-2} + J^{2d_s}n^{-2}T^{-1} \right) \) by Assumption 3.

**B.7.5 Term: \( M_5 \)**

Finally, we have

\[
M_5 = 2E \left[ (L_{21} + L_{22} + L_{23}) L_3 \left| \mathbf{3}, \mathbf{x}, \mathbf{F}_1, \ldots, \mathbf{F}_T \right. \right] \\
= 2E \left[ T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{d_s} 1_{jt} I_{jt}(z) \tilde{q}_{jt}^{-1} I_{j(tz)}(z_{it}) \varepsilon_{it}\varepsilon_{it}^t \left| \mathbf{3}, \mathbf{x}, \mathbf{F}_1, \ldots, \mathbf{F}_T \right. \right] \\
= 2E \left[ T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{d_s} 1_{jt} I_{jt}(z) \tilde{q}_{jt}^{-1} I_{j(tz)}(z_{it}) \varepsilon_{it}\varepsilon_{it}^t \tilde{\Omega}_{u,u,t}^{-1} X_t'M_{B_t}(\mu(z_t) + \varepsilon_t)/n_t \left| \mathbf{3}, \mathbf{x}, \mathbf{F}_1, \ldots, \mathbf{F}_T \right. \right] \\
= 2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i=1}^{n_t} \sum_{j=1}^{d_s} 1_{jt} I_{jt}(z) \tilde{q}_{jt}^{-1} I_{j(tz)}(z_{it}) \varepsilon_{it}\varepsilon_{it}^t \tilde{\Omega}_{u,u,t}^{-1} X_t'M_{B_t}(H_t + U_t) \tilde{\Omega}_{u,u,t}^{-1} x_{it} \\
= \mathcal{M}_{51} + \mathcal{M}_{52},
\]

where \( \Sigma_t = \text{diag} (\sigma^2_{1t}, \ldots, \sigma^2_{nt}) \) and

\[
\mathcal{M}_{51} = 2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i=1}^{n_t} \sum_{j=1}^{d_s} 1_{jt} I_{jt}(z) \tilde{q}_{jt}^{-1} I_{j(tz)}(z_{it}) \varepsilon_{it}\varepsilon_{it}^t \tilde{\Sigma}_t M_{B_t}(H_t + U_t) \tilde{\Omega}_{u,u,t}^{-1} x_{it} \\
\mathcal{M}_{52} = 2T^{-2} \sum_{t=1}^{T} n_t^{-3} \sum_{i=1}^{n_t} \sum_{j=1}^{d_s} 1_{jt} I_{jt}(z) \tilde{q}_{jt}^{-1} I_{j(tz)}(z_{it}) \varepsilon_{it}\varepsilon_{it}^t \tilde{\Sigma}_t M_{B_t} U_t \tilde{\Omega}_{u,u,t}^{-1} x_{it}.
\]
Thus,

\[ \mathcal{M}_{51} = 2T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \left| \epsilon_{t} \sum_{t} M_{B_{t}} \left( H_{t} - B_{t} \Pi_{t}^{0} \right) \hat{\Omega}_{u_{x}, t}^{-1} x_{i t} \right| \]

\[ \leq 2T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \left| \epsilon_{t} \sum_{t} M_{B_{t}} \left( H_{t} - B_{t} \Pi_{t}^{0} \right) \right| \left| \hat{\Omega}_{u_{x}, t}^{-1} x_{i t} \right| \]

\[ \leq \max_{1 \leq t \leq T} \left\| H_{t} - B_{t} \Pi_{t}^{0} \right\| C J^{d_{x}} T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \left\| x_{i t} \right\| \]

However,

\[ \max_{1 \leq t \leq T} \left\| H_{t} - B_{t} \Pi_{t}^{0} \right\| ^{2} = \max_{1 \leq t \leq T} \left( \left\| H_{t} - B_{t} \Pi_{t}^{0} \right\| ^{2} \right) \]

\[ = \max_{1 \leq t \leq T} \sum_{i=1}^{n} \left\| h_{t} (z_{i}) - B_{t} (z_{i}) \right\| _{\pi_{t}^{0}} ^{2} \]

\[ = \sum_{i=1}^{n} \sum_{t=1}^{d_{x}} \max_{1 \leq t \leq T} \left\| \sum_{j=1}^{d_{x}} \mathbb{I}_{j} (z_{i}) h_{t, j} (z_{i}) - \bar{q}_{j} (z_{i}) \pi_{j, t}^{0} \right\| ^{2} \]

\[ \leq \sum_{i=1}^{n} \sum_{t=1}^{d_{x}} \max_{1 \leq t \leq T} \sup_{z} \left\| \bar{q}_{j} (z) h_{t, j} (z) - \bar{q}_{j} (z) \pi_{j, t}^{0} \right\| ^{2} , \]

so that \( \max_{1 \leq t \leq T} \left\| H_{t} - B_{t} \Pi_{t}^{0} \right\| ^{2} = O_{p} \left( n J^{-2} \right) \) so that, by Markov’s inequality, \( \mathcal{M}_{51} = O_{p} \left( J^{-1} T^{-1} n^{-3/2} \right) = o_{p} \left( J^{2 d_{x} n^{-2} T^{-1}} \right) \) by Assumption 3.

\[ \mathcal{M}_{52} = 2T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \left| \epsilon_{t} \sum_{t} M_{B_{t}} U_{t} \hat{\Omega}_{u_{x}, t}^{-1} x_{i t} \right| \]

But

\[ \left| \mathcal{M}_{52} \right| \leq 2T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \left| \epsilon_{t} \sum_{t} M_{B_{t}} U_{t} \hat{\Omega}_{u_{x}, t}^{-1} x_{i t} \right| \]

\[ \leq C J^{d_{x}} T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \left\| U_{t} \right\| \left\| x_{i t} \right\| , \]

and

\[ J^{d_{x}} T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} \mathbb{E} \left[ 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \mathbb{E} \left[ \left\| U_{t} \right\| \left\| x_{i t} \right\| \mid z_{t}, F_{t} \right] \right] \]

\[ \leq J^{d_{x}} T^{-2} \sum_{t=1}^{T} n_{t}^{-3} \sum_{i=1}^{n_{T}} \sum_{j=1}^{d_{x}} \mathbb{E} \left[ 1_{j} \mathbb{I}_{j} (z) \bar{q}_{j}^{-1} \mathbb{I}_{j} (z_{i}) \sqrt{\mathbb{E} \left[ \left\| U_{t} \right\| ^{2} \mid z_{t}, F_{t} \right] \mathbb{E} \left[ \left\| x_{i t} \right\| ^{2} \mid z_{t}, F_{t} \right]} \right] \]

\[ \leq C T^{-1} n^{-3/2} , \]

Thus, \( \mathcal{M}_{52} = O_{p} \left( T^{-1} n^{-3/2} \right) \) which is \( o_{p} \left( J^{2 d_{x} n^{-2} T^{-1}} \right) \) under Assumption 3.
B.8 Proofs of Lemmas

Proof of Lemma 1. We would like to show that there exists a $\gamma_{jt}^0$ such that

$$\max_{1 \leq t \leq T} \max_{1 \leq j \leq J} \sup_z |\bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z)| \gamma_{jt}^0 = O_p(J^{-1}).$$

Let $\gamma_{jt}^0 = (\gamma_{jt}^0(z_{1t}, \ldots, z_{nt})) = \mu(\hat{b}_t)$ where $\hat{b}_t = (\hat{b}_{(j_1-1)/2t,1}, \ldots, \hat{b}_{(jd_s-1)/2t,d_s})'$. We have

$$P \left( \max_{1 \leq t \leq T} \max_{1 \leq j \leq J} \sup_z |\bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z)| > \frac{C}{J} \right) \leq \sum_{t=1}^T \sum_{j=1}^J P \left( \sup_z |\bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z)| > \frac{C}{J} \right).$$

Let us focus on the summand,

$$\mathbb{P} \left( \sup_z \left| \bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right| > \frac{C}{J} \right).$$

where $\hat{\gamma}_{jt} = \alpha z + (1 - \alpha) \hat{b}_t$, $\alpha \in (0, 1)$. Now, choose $C_1$ sufficiently large such that $\max_{1 \leq s \leq d_s} \sup_z \left| \frac{\partial \mu(z)}{\partial z} \right|_{z=\hat{\gamma}_{jt}} < C_1$. Then,

$$\mathbb{P} \left( \sup_z \left| \bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right| > \frac{C}{J} \right) \leq \mathbb{P} \left( \sup_z \left| \bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right| > \frac{C}{J} \right).$$

Thus, we can focus on

$$\mathbb{P} \left( \sup_z \left| \bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right| > \frac{C}{J} \right) \leq \sum_{s=1}^{d_s} \mathbb{P} \left( \hat{b}_{j_s,t,s} - \hat{b}_{(j_s-1),t,s} > \frac{C_1}{J} \right).$$

Recall that we can define the empirical quantile function in terms of the order statistics of the $z_{it}$'s:

$$\hat{F}_{t,s}^{-1}(p) = z_{(k)_{t,s}}, \quad \frac{k-1}{n} < p \leq \frac{k}{n},$$

where $z_{(k)_{t,s}}$ is the $k$th order statistic of $z_{it,s}$. Thus we have

$$\mathbb{P} \left( \sup_z \left| \bar{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right| > \frac{C}{J} \right) \leq \sum_{s=1}^{d_s} \mathbb{P} \left( \hat{b}_{j_s,t,s} - \hat{b}_{(j_s-1),t,s} > \frac{C_1}{J} \right).$$

where

$$\frac{k_{1,s} - 1}{n} < \frac{j_s - 1}{J} \leq \frac{k_{1,s}}{n}, \quad \frac{k_{2,s} - 1}{n} < \frac{j_s}{J} \leq \frac{k_{2,s}}{n},$$

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for \( s = 1, \ldots, d_2 \) and \( u(k)_t, s \) are order statistics of (conditionally) independent standard uniform random variables. The last line uses the fact that, conditional of \( \mathcal{F}_t \), \( z_{it} \) are iid with CDF \( F_{z_{it}|\mathcal{F}_t}(z) \) and so we can use the probability integral transform to map to the uniform order statistics from a sample of \( n_1 \) standard uniform random variables (conditional on \( \mathcal{F}_t \)). Using another mean-value expansion we have

\[
\begin{align*}
P \left( F_{z_{it}, s|\mathcal{F}_t}^{-1}(u(k)) - F_{z_{it}, s|\mathcal{F}_t}^{-1}(u(k)) > \frac{C_1}{J} \mid \mathcal{F}_t \right) \\
= P \left( q_{z_{it}, s|\mathcal{F}_t}(\tilde{u}) (u(k)_{t, s} - u(k)) > \frac{C_1}{J} \mid \mathcal{F}_t \right) \\
= P \left( q_{z_{it}, s|\mathcal{F}_t}(\tilde{u}) (u(k)_{t, s} - u(k)) - E \left[ u(k)_{t, s} - u(k) \mid \mathcal{F}_t \right] > \frac{C_1}{J} - q_{z_{it}, s|\mathcal{F}_t}(\tilde{u}) \mid \mathcal{F}_t \right)
\end{align*}
\]

where \( \tilde{u} = \alpha u(k), s + (1 - \alpha) u(k)_{t, s} \) and \( \alpha \in (0, 1) \), and

\[
q_{z_{it}, s|\mathcal{F}_t}(u) = \frac{\partial}{\partial u} F_{z_{it}, s|\mathcal{F}_t}^{-1}(u) = \begin{cases} \frac{1}{f_{z_{it}, s|\mathcal{F}_t}} & \text{if } f_{z_{it}, s|\mathcal{F}_t}(u) > 0 \\ 0 & \text{otherwise} \end{cases}
\]

Under our assumptions \( q_{z_{it}, s|\mathcal{F}_t}(u) \) is positively bounded from above and below. Conditional on \( \mathcal{F}_t \), \( u(k)_{t, s} \mid \mathcal{F}_t \sim \text{Beta}(k, n - k) \) so that

\[
E \left[ u(k)_{t, s} - u(k) \mid \mathcal{F}_t \right] = \frac{k_2 - k_1}{n + 1}.
\]

The inequalities defining \( k_1 \) and \( k_2 \) imply that

\[
\frac{1}{J} - \frac{1}{n} \leq \frac{k_2 - k_1}{n} \leq \frac{1}{J} + \frac{1}{n}.
\]

Thus,

\[
\begin{align*}
P \left( F_{z_{it}, s|\mathcal{F}_t}^{-1}(u(k)) - F_{z_{it}, s|\mathcal{F}_t}^{-1}(u(k)) > \frac{C_1}{J} \mid \mathcal{F}_t \right) \\
= P \left( (u(k)_{t, s} - u(k)) - E \left[ u(k)_{t, s} - u(k) \mid \mathcal{F}_t \right] > \frac{1}{J} \frac{C_1}{J} - E \left[ u(k)_{t, s} - u(k) \mid \mathcal{F}_t \right] \mid \mathcal{F}_t \right) \\
\leq P \left( (u(k)_{t, s} - u(k)) > \frac{C_1}{J} - \left( \frac{1}{J} \frac{C_1}{J} - 1 \right) E \left[ u(k)_{t, s} - u(k) \mid \mathcal{F}_t \right] \mid \mathcal{F}_t \right) \\
\leq 2 \exp \left\{ -C_2 (n + 1) \left( \frac{1}{J} \frac{C_3}{J} - 1 \right)^2 \right\},
\end{align*}
\]

where the last line uses the fact that, conditional of \( \mathcal{F}_t \), \( u(k)_{t, s} - u(k) \) are the sum of individual uniform spacings and are distributed as \( \left( u(k)_{t} - u(k)_{t} \mid \mathcal{F}_t \sim \text{Beta}(k, n - 1 - (k_2 - k_1)) \right) \) and Bobkov and Ledoux (2016, Proposition B.10). We can put all this together to obtain

\[
P \left( \sup_z \left| \hat{\gamma}_{jt}(z) \mu(z) - \hat{\gamma}_{jt}(z) \gamma_{jt} \right| > C \right) \leq \sum_{t=1}^{T} \sum_{j=1}^{J} \sum_{j=1}^{J'} \left[ 1 \wedge 2 \exp \left\{ -C_2 (n + 1) \left( \frac{1}{J} \frac{C_3}{J} - 1 \right)^2 \right\} \right]
\]

\[
\leq E \left[ 1 \wedge 2JT \exp \left\{ -C_2 (n + 1) \left( \frac{1}{J} \frac{C_3}{J} - 1 \right)^2 \right\} \right] = o(1)
\]
under our assumptions. Next we would like to show,
\[
\mathbb{E} \left[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_{d_z}^t} \sup_z \left| \tilde{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right|^2 \right] = O(J^{-2}).
\]
This follows immediately from the previous result since,
\[
\mathbb{E} \left[ \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_{d_z}^t} \sup_z \left| \tilde{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right|^2 \right] \leq C_1 \mathbb{P} \left[ \left( \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_{d_z}^t} \sup_z \left| \tilde{\mu}_{jt}(z) - \hat{\mu}_{jt}(z) \right|^2 > \frac{C_2}{J^2} \right) \right] + \frac{C_2}{J^2}.
\]

Proof of Lemma 2. By the proof of Lemma 1 we have that
\[
\mathbb{P} \left( \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_{d_z}^t} \max_s \left| \hat{b}_{jt,s} - \hat{b}_{jt,s} \right| > \frac{C}{J} \right) = o(1).
\]
By Einmahl and Ruymgaart (1987, Theorem 3.1) for all sequences \( \delta_n = O(J^{-d_z}) \),
\[
\max_{1 \leq t \leq T} \max_{1 \leq j \leq J_{d_z}^t} \left| \tilde{\mu}_{jt} - \hat{\mu}_{jt} \right|^2 = O_p \left( \frac{\log (J_{d_z} \lor T)}{J_{d_z} n} \right)
\]
provided that
\[
\frac{J_{d_z} \log (n)}{n} \to 0, \quad \text{and,} \quad \frac{J_{d_z} \log (n)}{n} \to \infty,
\]
which are satisfied under our assumptions. □

Proof of Lemma 3. We will first find the order of
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| \Omega_{u,t} - \Omega_{u,t} \right\|^2 = \frac{1}{T} \sum_{t=1}^{T} \left\| (X_t' MB_t X_t/ n_t) - \Omega_{u,t} \right\|^2 \\
\leq C \cdot \frac{1}{T} \sum_{t=1}^{T} \left\| X_t' MB_t X_t/ n_t - U_t U_t' / n_t \right\|^2 + C \cdot \frac{1}{T} \sum_{t=1}^{T} \left\| U_t U_t' / n_t - \Omega_{u,t} \right\|^2.
\]
For the second term we have that
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} \left\| U_t U_t' / n_t - \Omega_{u,t} \right\|^2 \right] = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \left\| \frac{1}{n_t} \sum_{i=1}^{n_t} u_{it} u_{it}' - \Omega_{u,t} \right\|^2 \right] \\
= \frac{1}{T} \sum_{t=1}^{T} \text{tr} \left( \frac{1}{n_t^2} \sum_{i_1,i_2} \mathbb{E} \left[ (u_{i_1t} u_{i_2t}' - \Omega_{u,t}) (u_{i_2t} u_{i_1t}' - \Omega_{u,t})' \right] \right) \\
\leq C \cdot \frac{1}{n},
\]
so this term is \( O_p \left( n^{-1} \right) \) by Markov’s inequality. For the first term note that,
\[
X_t' MB_t X_t/ n_t - U_t U_t' / n_t = (U_t + H_t)' MB_t (U_t + H_t) / n_t \\
= H_t' MB_t H_t / n_t - U_t' (I_{n_t} - MB_t) U_t / n_t + U_t' MB_t H_t / n_t + H_t' MB_t U_t / n_t,
\]

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so that
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| X_t' M_B X_t / n_t - U_t U_t' / n_t \right\|^2 \leq C \cdot \frac{1}{T} \sum_{t=1}^{T} \left\| H_t' M_B H_t / n_t \right\|^2 \\
+ C \cdot \frac{1}{T} \sum_{t=1}^{T} \left\| U_t' (I_{n_t} - M_B) U_t / n_t \right\|^2 \\
+ C \cdot \frac{1}{T} \sum_{t=1}^{T} \left\| U_t' M_B H_t / n_t \right\|^2
\]
For the first term,
\[
\left\| H_t' M_B H_t / n_t \right\| = n_t^{-1} \left\| (H_t - B_t \Pi_t^0)' M_B (H_t - B_t \Pi_t^0) \right\| \\
\leq C \cdot n_t^{-1} \sum_{i=1}^{n_t} \left\| h_t (z_{it}) - B_t (z_{it} \Pi_t^0) \right\|^2 \\
= C \cdot n_t^{-1} \sum_{i=1}^{n_t} \sum_{\ell=1}^{d_x} \sum_{j=1}^{J_{ds}} \left( \hat{\mathbf{J}}_{jt} (z_{it}) h_{t,\ell} (z_{it}) - \bar{\mathbf{J}}_{jt} (z_{it}) \right)^2 \\
\leq C \cdot n_t^{-1} \sum_{i=1}^{n_t} \sum_{\ell=1}^{d_x} \max_{1 \leq j \leq J_{ds}} \sup_z \left( \hat{\mathbf{J}}_{jt} (z) h_{t,\ell} (z) - \bar{\mathbf{J}}_{jt} (z) \right)^2 \\
\leq C \cdot \max_{1 \leq j \leq T} \max_{1 \leq \ell \leq J_{ds}} \max_z \left( \hat{\mathbf{J}}_{jt} (z) h_{t,\ell} (z) - \bar{\mathbf{J}}_{jt} (z) \right)^2,
\]
and so \( \left\| H_t' M_B H_t / n_t \right\| = O_p (J^{-2}) \) by Lemma 1. Next let \( P_{B_t} = I_{n_t} - M_B \) with elements \( [P_{B_t}]_{i,j} = p_{i,j} \) and note that,
\[
\left\| U_t' P_{B_t} U_t / n_t \right\|^2 = n_t^{-2} \mathbb{E} \left[ \text{tr} \left( (U_t' P_{B_t} U_t)' U_t' P_{B_t} U_t \right) \right] \\
= n_t^{-2} \mathbb{E} \left[ \text{tr} \left( P_{B_t} U_t U_t' P_{B_t} U_t' \right) \right] \\
= n_t^{-2} \mathbb{E} \left[ \text{tr} \left( P_{B_t} \mathbb{E} \left[ U_t U_t' P_{B_t} U_t' \right] \right) \right].
\]
Then,
\[
\text{tr} \left( P_{B_t} \mathbb{E} \left[ U_t U_t' P_{B_t} U_t' \right] \right) = \sum_{i=1}^{n_t} \sum_{\ell_0=1}^{n_t} \sum_{\ell_1=1}^{d_x} \sum_{\ell_2=1}^{n_t} \sum_{\ell_3=1}^{d_x} p_{i,\ell_0,\ell_1,\ell_2,\ell_3} \mathbb{E} \left[ u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} z_t, \mathcal{F}_t \right].
\]
This expectation is nonzero only when \( \{\ell_0 = \ell_2, \ell_3 = i\} \) or \( \{\ell_0 = \ell_3, \ell_2 = i\} \) or \( \{\ell_0 = i, \ell_2 = \ell_3\} \). These correspond to the following three terms:
\[
\text{tr} \left( P_{B_t} \mathbb{E} \left[ U_t U_t' P_{B_t} U_t' \right] \right) \leq \sum_{i=1}^{n_t} \sum_{\ell_0=1}^{n_t} \sum_{\ell_1=1}^{d_x} p_{i,\ell_0,\ell_1,\ell_2,\ell_3} \mathbb{E} \left[ u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} \right] z_t, \mathcal{F}_t \]
\[
+ \sum_{i=1}^{n_t} \sum_{\ell_0=1}^{n_t} \sum_{\ell_1=1}^{d_x} \sum_{\ell_2=1}^{n_t} p_{i,\ell_0,\ell_1,\ell_2,\ell_3} \mathbb{E} \left[ u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} \right] z_t, \mathcal{F}_t \]
\[
+ \sum_{i=1}^{n_t} \sum_{\ell_0=1}^{n_t} \sum_{\ell_1=1}^{d_x} \sum_{\ell_2=1}^{n_t} \sum_{\ell_3=1}^{d_x} p_{i,\ell_0,\ell_1,\ell_2,\ell_3} \mathbb{E} \left[ u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} u_{i,\ell_0,\ell_1,\ell_2,\ell_3} \right] z_t, \mathcal{F}_t \]
\[
\leq C \cdot \text{tr} \left( P_{B_t}^2 \right) + C \cdot \text{tr} \left( P_{B_t}^2 \right) + C \cdot \left[ \text{tr} \left( P_{B_t} \right) \right]^2 \\
\leq C \cdot (J_{ds} + J_{ds}^2).
\]
Thus,
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left\| U_t' P_{B_t} U_t / n_t \right\|^2 \leq C \cdot n^{-2} \left( J_{ds} + J_{ds}^2 \right),
\]
so this term is \( O_p(n^{-2}J_{ds}^2) \) by Markov’s inequality. Finally, note that
\[
\max_{1 \leq t \leq T} \mathbb{E} \left[ \left\| U_t' M_B H_t / n_t \right\|^2 \right]
\]
We will now find the order of max
\[ \max_{1 \leq t \leq T} E \left[ \| U_t' M_B, H_t \|^2 \right] \]
\[ = C \cdot n^{-2} \max_{1 \leq t \leq T} E \left[ \text{tr} \left( E \left[ U_t U_t' z_t, F_t \right] M_B H_t H_t' M_B \right) \right] \]
\[ \leq C \cdot n^{-2} \max_{1 \leq t \leq T} E \left[ \text{tr} \left( (H_t - B_t \Pi_t')' M_{B_t} (H_t - B_t \Pi_t') \right) \right] \]
\[ \leq C \cdot n^{-2} \sum_{i=1}^{n} \sum_{t=1}^{d} \max_{1 \leq t \leq T} E \left[ \left| h_{t,1} (z_{it} - B_t (z_{it})' \pi_{t,i}^0 \right|^2 \sum_{j=1}^{J_{d,2}} \tilde{I}_{jt} (z) \right] \]
\[ \leq C \cdot n^{-1} J^{-2}. \]

Thus,
\[ \frac{1}{T} \sum_{t=1}^{T} \left\| X_t' M_B, X_t / n_t - U_t U_t' / n_t \right\|^2 = O_p \left( J^{-4} \right) + O_p \left( n^{-2} J^{2d} \right) + O_p \left( n^{-1} J^{-2} \right), \]
and
\[ \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\Omega}_{uu,t} - \Omega_{uu,t} \right\|^2 = O_p \left( n^{-1} \right) + O_p \left( J^{-4} \right) + O_p \left( n^{-2} J^{2d} \right) + O_p \left( n^{-1} \right). \]

We will now find the order of max\(_{1 \leq t \leq T} \left\| \hat{\Omega}_{uu,t} - \Omega_{uu,t} \right\|. \) Recall that

\[ \hat{\Omega}_{uu,t} - \Omega_{uu,t} = H_t' M_B, H_t / n_t \]
\[ - U_t' (I_{n_t} - M_B) U_t / n_t \] (B.2)
\[ + U_t' M_B, H_t / n_t + H_t' M_B U_t / n_t \] (B.2)
\[ + (U_t' U_t / n_t - \Omega_{uu,t}). \] (B.2)

The first term (equation (B.2)) satisfies max\(_{1 \leq t \leq T} \| H_t' M_B, H_t / n_t \| = O_p \left( J^{-2} \right) \) by the derivations above. Now consider equation (B.2). Let a \( \in \mathbb{R}^{d_x} \), \( a \neq 0 \). Then by the CS inequality it is sufficient to show that max\(_{1 \leq t \leq T} |a' U_t' P_B, U_t a| / n_t = O_p \left( J^{d_x n^{-1}} \right). \) First note that

\[ \max_{1 \leq t \leq T} \left| a' U_t' P_B, U_t a \right| \leq \max_{1 \leq t \leq T} \left| a' U_t' P_B, U_t a - a' E \left[ U_t' P_B, U_t \right] z_t, F_t \right| a \right| + \max_{1 \leq t \leq T} \left| a' E \left[ U_t' P_B, U_t \right] z_t, F_t \right| a \right|. \]

Define \( \tilde{U}_t = U_t a \) which is a \( n_t \times 1 \) vector and note that conditional on \( F_t \), the elements of \( \tilde{U}_t \) are independent (and mean zero). We will deal with the second term first,

\[ \left| a' E \left[ U_t' P_B, U_t \right] z_t, F_t \right| a \right| / n_t \leq C n^{-1} \left| \text{tr} \left( P_B, \tilde{U} \right) \right| / n_t \]
\[ \leq C n^{-1} J^{d_x}. \]

By our assumption of sub-Gaussianity on \( x_{it} \) and that sums of sub-Gaussian variables are also sub-Gaussian, the Hanson-Wright inequality (see, for example, Rudelson and Vershynin (2013) yields

\[ \mathbb{P} \left( \left| \tilde{U}_t' P_B, \tilde{U}_t - E \left[ \tilde{U}_t' P_B, \tilde{U}_t \right] z_t, F_t \right| \geq \delta \right| z_t, F_t \right) \]
\[ \leq 2 \exp \left\{ -C \min \left( \frac{\delta^2}{2 K^4 \| P_B \|^2}, \frac{\delta}{K^2 \sup_{\| y \|=1} \| P_B y \|} \right) \right\} \]
\[ = 2 \exp \left\{ -C \min \left( \frac{\delta^2}{2 K^4 J^{d_x}}, \frac{\delta}{K^2} \right) \right\}, \]

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if we map \( \delta \mapsto \delta \log (T)^{1/2} J^{d_z/2} \) we have that
\[
\Pr \left( \max_{1 \leq t \leq T} J^{-d_z/2} \log (T)^{-1/2} \left| \tilde{U}_t' P_B \tilde{U}_t - \mathbb{E} \left[ \tilde{U}_t' P_B \tilde{U}_t \right] \right| \geq \delta \right| z_t, F_t) \\
\leq 2 \exp \left\{ -C \min \left( \frac{\delta^2 \log (T)}{2K^4}, \frac{\delta \log (T)^{1/2}}{K^2} J^{d_z/2} \right) \right\} \\
\leq 2 \exp \left\{ -C \frac{\delta^2 \log (T)}{2K^4} \right\},
\]
for sufficiently large \( n \) and \( T \). Thus,
\[
\Pr \left( \max_{1 \leq t \leq T} J^{-d_z/2} \log (T)^{-1/2} \left| \tilde{U}_t' P_B \tilde{U}_t - \mathbb{E} \left[ \tilde{U}_t' P_B \tilde{U}_t \right] \right| \geq \delta \right| z_t, F_t) \\
\leq T \max_{1 \leq t \leq T} \Pr \left( J^{-d_z/2} \log (T)^{-1/2} \left| \tilde{U}_t' P_B \tilde{U}_t - \mathbb{E} \left[ \tilde{U}_t' P_B \tilde{U}_t \right] \right| \geq \delta \right| z_t, F_t) \\
\leq 2T \exp \left\{ -C \frac{\delta^2 \log (T)}{2K^4} \right\} \\
= \exp \left\{ \log (2) + \log (T) \left[ 1 - C \frac{\delta^2}{2K^4} \right] \right\},
\]
which can be made arbitrarily small for \( \delta \) sufficiently large. Thus,
\[
\max_{1 \leq t \leq T} \left| \tilde{U}_t' P_B \tilde{U}_t - \mathbb{E} \left[ \tilde{U}_t' P_B \tilde{U}_t \right] \right| / n_t = O_p \left( \log (T)^{1/2} J^{d_z/2} n^{-1} \right),
\]
and \( \max_{1 \leq t \leq T} |a' U'_t P_B a| / n_t = O_p \left( J^{d_z-1} n^{-1} \right) = O_p \left( J^{d_z-1} n^{-1} \right) \). By similar steps we may show that equation \((B.2)\) satisfies
\[
\max_{1 \leq t \leq T} |a' U'_t a - a' \mathbb{E} \left[ U'_t \right] a| / n_t = O_p \left( \log (T)^{1/2} n^{-1/2} \right)
\]
Finally, we need to deal with the term \( U_t' M_B H_t / n_t \). First note that \( \| U_t' M_B H_t / n_t \|^2 = n_t^{-2} \operatorname{tr} (U_t' M_B H_t H'_t M_B U_t) \) so that we may focus on
\[
|a' U_t' M_B H_t H'_t M_B U_t a| \leq |U_t' M_B H_t H'_t M_B U_t - \mathbb{E} \left[ U_t' M_B H_t H'_t M_B U_t \right]| + \mathbb{E} \left[ U_t' M_B H_t H'_t M_B U_t \right] z_t, F_t).
\]
The second term has expectation
\[
\max_{1 \leq t \leq T} \mathbb{E} \left[ U_t' M_B H_t H'_t M_B U_t \right] z_t, F_t) = \max_{1 \leq t \leq T} \mathbb{E} \left[ \operatorname{tr} \left( M_t H_t H'_t M_B, z_t, F_t, \mathbb{E} \left[ U_t' \right] \right) \right] \\
\leq C \max_{1 \leq t \leq T} \mathbb{E} \left[ \operatorname{tr} \left( (H_t - B_t \Pi^0_t)' M_B (H_t - B_t \Pi^0_t) \right) \right] \\
\leq Cn \sum_{\ell=1}^{d_z} \max_{1 \leq t \leq T} \mathbb{E} \left[ \sup_{1 \leq j \leq d_z} \left\| h_{t, \ell} (z) H_{t, \ell} (z) - \tilde{h}_{t, \ell} (z) \right\|_{\ell, z} \right]^2 \\
\leq Cn J^{-2}.
\]
Thus, by Markov’s inequality it is \( O_p (n J^{-2}) \). For the first term consider we can again utilize the Hanson-Wright inequality which yields
\[
\Pr \left( \max_{1 \leq t \leq T} |U_t' M_B H_t H'_t M_B U_t - \mathbb{E} \left[ U_t' M_B H_t H'_t M_B U_t \right]| > \delta \right| z_t, F_t) \\
\leq 1 \land 2 \sum_{t=1}^T \exp \left\{ -C \min \left( \frac{\delta^2}{2K^4 \|M_B H_t H'_t M_B\|^2}, \frac{\delta}{K^2 \sup_{\|y\|=1} \|M_B H_t H'_t M_B y\|} \right) \right\}
\]
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\[
\leq 1 \land 2T \max_{1 \leq t \leq T} \exp \left\{ -C \min \left( \frac{\delta^2}{2K^4 \| M_{Bt} H_t H_t' M_{Bt} \|^2}, \frac{\delta}{K^2 \| M_{Bt} H_t H_t' M_{Bt} \|} \right) \right\}
\]
by properties of matrix norms. If we map \( \delta \mapsto \delta \log (T) nJ^{-2} \) then

\[
\mathbb{P} \left( \max_{1 \leq t \leq T} \left| \hat{U}_t' M_{Bt} H_t H_t' M_{Bt} \hat{U}_t - \mathbb{E} \left[ \hat{U}_t' M_{Bt} H_t H_t' M_{Bt} \hat{U}_t \right] \right| > \delta \log (T) nJ^{-2} \right) 
\leq \mathbb{E} \left( 1 \land 2 \sum_{t=1}^{T} \exp \left\{ -C \min \left( \frac{\delta^2 \log (T) ^2 n^2 J^{-4} \| M_{Bt} H_t H_t' M_{Bt} \|^2}{2K^4 \| M_{Bt} H_t H_t' M_{Bt} \|^2}, \frac{\delta \log (T) nJ^{-2}}{K^2 \| M_{Bt} H_t H_t' M_{Bt} \|} \right) \right\} 
\leq C \cdot \mathbb{P} \left( \max_{1 \leq t \leq T} \| M_{Bt} H_t H_t' M_{Bt} \| \geq C_2 nJ^{-2} \right) 
+ \left( 1 \land 2 \sum_{t=1}^{T} \exp \left\{ -C \min \left( \frac{\delta^2 \log (T) ^2 \| M_{Bt} H_t H_t' M_{Bt} \|^2}{2K^4 C_2^2}, \frac{\delta \log (T) nJ^{-2}}{K^2 C_2} \right) \right\} \right)
\]
The first term is \( o(1) \) by Lemma 1 and the second term can be made arbitrarily small for sufficiently large \( \delta \) so that

\[
\max_{1 \leq t \leq T} \left| \hat{U}_t' M_{Bt} H_t H_t' M_{Bt} \hat{U}_t - \mathbb{E} \left[ \hat{U}_t' M_{Bt} H_t H_t' M_{Bt} \hat{U}_t \right] \right| = O_p \left( \log (T) nJ^{-2} \right).
\]

Thus,

\[
\max_{1 \leq t \leq T} \| U_t' M_{Bt} H_t / n_t \| = O_p \left( \log (T) n^{-1} J^{-2} \right) + O \left( n^{-1} J^{-2} \right) = O_p \left( \log (T) n^{-1} J^{-2} \right),
\]
which is of smaller order than the term in equation (B.2).

**Proof of Lemma 5.** We have

\[
V (z) = T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{j_{ds}} \mathbf{1}_{j_{tj}} \hat{u}_{jt} (z) \hat{q}_{jt}^{-2} \hat{\eta}_{jt} (z_{it}) \sigma_{it}^2
\leq CT^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{j_{ds}} \mathbf{1}_{j_{tj}} \hat{u}_{jt} (z) \hat{q}_{jt}^{-2} \hat{\eta}_{jt} (z_{it})
= CT^{-2} \sum_{t=1}^{T} n_t^{-1} \sum_{j=1}^{j_{ds}} \mathbf{1}_{j_{tj}} \hat{u}_{jt} (z) \hat{q}_{jt}^{-1}
\leq C J_t d n^{-1} T^{-2} \sum_{t=1}^{T} \sum_{j=1}^{j_{ds}} \mathbf{1}_{j_{tj}} \hat{u}_{jt} (z)
\leq C J_t d n^{-1} T^{-1} + C J_t d n^{-1} T^{-2} \sum_{t=1}^{T} \sum_{j=1}^{j_{ds}} \mathbf{1}_{j_{tj}} (1 - 1) \hat{u}_{jt} (z)
\leq C J_t d n^{-1} T^{-1} + C J_t d n^{-1} T^{-1} \max_{1 \leq t \leq T} \max_{1 \leq j \leq j_{ds}} \mathbf{1}_{j_{tj}} (1 - 1),
\]
and so the second term is \( o_p (1) \) by Lemma 2. The lower bound follows by similar steps.

**Proof of Lemma 4.** We have that

\[
\mathbf{1}_{\beta_t} \left( \hat{\beta}_t - \beta_t \right) = \mathbf{1}_{\beta_t} \hat{\Omega}_{\mu u, t}^{-1} X_t' M_{Bt} \left( \mu (z_t) + \varepsilon_t \right) / n_t
\]
Next recall that $X_t = U_t + H_t$ so we can decompose $1_{\beta,t} (\hat{\beta}_t - \beta_t)$ as

$$T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} X'_t M_{B_t} (\mu (z_t) + \varepsilon_t) / n_t$$

$$= T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} (U_t + H_t)' M_{B_t} (\mu (z_t) + \varepsilon_t) / n_t. $$

For the first result it is then sufficient to consider $\sum_t |\mathcal{K}_{1t}|^2$ where

$$\mathcal{K}_{11} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U_t' \varepsilon_t / n_t$$

$$\mathcal{K}_{12} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U'_t (I_{nt} - M_{B_t}) \varepsilon_t / n_t$$

$$\mathcal{K}_{13} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} H'_t M_{B_t} \mu (z_t) / n_t$$

$$\mathcal{K}_{14} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} H'_t M_{B_t} \varepsilon_t / n_t$$

$$\mathcal{K}_{15} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U'_t M_{B_t} \mu (z_t) / n_t$$

For the second result, by the CS inequality, it is sufficient to show that $\sum_t \mathcal{K}_{2t} = O_p (n^{-1}) + O_p (J^{-1})$ where,

$$\mathcal{K}_{21} = T^{-1} \sum_{t=1}^{T} \left( 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U'_t \varepsilon_t / n_t \right)^2$$

$$\mathcal{K}_{22} = T^{-1} \sum_{t=1}^{T} \left( 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U'_t (I_{nt} - M_{B_t}) \varepsilon_t / n_t \right)^2$$

$$\mathcal{K}_{23} = T^{-1} \sum_{t=1}^{T} \left( 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} H'_t M_{B_t} \mu (z_t) / n_t \right)^2$$

$$\mathcal{K}_{24} = T^{-1} \sum_{t=1}^{T} \left( 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} H'_t M_{B_t} \varepsilon_t / n_t \right)^2$$

$$\mathcal{K}_{25} = T^{-1} \sum_{t=1}^{T} \left( 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U'_t M_{B_t} \mu (z_t) / n_t \right)^2$$

We will prove the second result, first. Consider $\mathcal{K}_{21}$

$$|\mathcal{K}_{21}| = T^{-1} \sum_{t=1}^{T} \left| 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} U'_t \varepsilon_t / n_t \right|^2$$

$$\leq T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \lambda_{\max} \left( \hat{\Omega}_{u,t}^{-1} \right) \| s_t \|^2 \| U'_t \varepsilon_t / n_t \|^2$$

$$\leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \text{tr} \left( (U'_t \varepsilon_t \varepsilon_t' U_t) \right).$$

Taking expectations we obtain,

$$T^{-1} \sum_{t=1}^{T} n_t^{-2} \mathbb{E} \left[ \text{tr} (U'_t E [\varepsilon_t \varepsilon_t' | z_t, x_t, \mathcal{F}_t] U_t) \right] \leq C n^{-1}. $$

Thus, $\mathcal{K}_{21} = O_p (n^{-1})$ by Markov’s inequality. By similar steps we can show that $\mathcal{K}_{22} = O_p (J^{d_x} n^{-2})$.

Next consider $\mathcal{K}_{23}$

$$\mathcal{K}_{23} = T^{-1} \sum_{t=1}^{T} n_t^{-2} \left( 1_{\beta,t} s_t^i \hat{\Omega}_{u,t}^{-1} H'_t M_{B_t} \mu (z_t) / n_t \right)^2$$

$$\leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \left( 1_{\beta,t} \left[ \max \left( \hat{\Omega}_{u,t}^{-1} \right) \right] \| M_{B_t} (H_t - B_t \Pi'_t) \|^2 \| M_{B_t} (\mu (z_t) - B_t \gamma'_t) \|^2 \right)$$

$$\leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \left( \sum_{s=1}^{n_t} \| h_t (z_{it}) - B_t (z_{it})' \pi'_t \|^2 \right) \left( \sum_{i=1}^{n_t} \| \mu (z_{it}) - B_t (z_{it})' \gamma'_t \|^2 \right)$$

$$\leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \left( \sum_{i=1}^{n_t} \sum_{t=1}^{J_{i,t}} \sum_{j=1}^{d_x} \left( J_{i,t,j} (z_{it}) h_{t,j} (z_{it}) - \bar{J}_{i,t,j} (z_{it}) \pi'_{j,t} (z_{it}) \right)^2 \right) \times$$
Thus, $K_{23} = O_p(J^{-4})$. Now consider $K_{24}$

$$K_{24} = T^{-1} \sum_{t=1}^{T} \left| 1_{\beta,t} s_{t}^l \tilde{\Omega}_{u,t}^{-1} H_t^t M_{B_t} \varepsilon_t / n_t \right|^2 \leq CT^{-1} \sum_{t=1}^{T} \left\| H_t^t M_{B_t} \varepsilon_t / n_t \right\|^2.$$

Taking expectations gives

$$T^{-1} \sum_{t=1}^{T} n_t^{-2} \mathbb{E} \left[ \text{tr} \left( H_t^t M_{B_t} \mathbb{E} \left[ \varepsilon_t \varepsilon_t' \big| z_t, x_t, \mathcal{F}_t \big] M_{B_t} H_t \right) \right] \leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \mathbb{E} \left[ \text{tr} \left( \left( H_t - B_t \gamma_t^0 \right)^' \left( H_t - B_t \gamma_t^0 \right) \right) \right] \leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{t=1}^{d_x} \mathbb{E} \left[ \max_{1 \leq j \leq d_x} \sup_{z} \left| \tilde{i}_{j,t} (z) \mu_t (z) - \tilde{i}_{j,t} (z) \gamma_t^0 \right|^2 \right] \sum_{j=1}^{d_x} \tilde{i}_{j,t} (z) \right] \leq C n^{-1} J^{-2}.$$

Thus $K_{24} = O_p(n^{-1} J^{-2})$ by Markov’s inequality. Now consider $K_{25}$

$$K_{25} = T^{-1} \sum_{t=1}^{T} \left| 1_{\beta,t} s_{t}^l \hat{\Omega}_{u,t}^{-1} U_t^t M_{B_t} \mu (z_t) / n_t \right|^2 \leq CT^{-1} \sum_{t=1}^{T} \left\| U_t^t M_{B_t} \mu (z_t) / n_t \right\|^2.$$

Taking expectations gives

$$T^{-1} \sum_{t=1}^{T} n_t^{-2} \mathbb{E} \left[ \left\| U_t^t M_{B_t} \mu (z_t) \right\|^2 \right] \leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \mathbb{E} \left[ \text{tr} \left( \mathbb{E} \left[ U_t^t U_t^t' \big| z_t, \mathcal{F}_t \right] M_{B_t} \mu (z_t) M_{B_t} \right) \right] \leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \mathbb{E} \left[ \text{tr} \left( \left( \mu (z_t) - B_t \gamma_t^0 \right)^' \left( \mu (z_t) - B_t \gamma_t^0 \right) \right) \right] \leq CT^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \mathbb{E} \left[ \max_{1 \leq j \leq d_x} \sup_{z} \left| \tilde{i}_{j,t} (z) \mu_t (z) - \tilde{i}_{j,t} (z) \gamma_t^0 \right|^2 \right] \leq C n^{-1} J^{-2}.$$

Thus, $K_{25} = O_p(n^{-1} J^{-2})$ and

$$\sum_{\ell} K_{2\ell} = O_p \left( n^{-1} + O_p \left( J^{d_x} T^{n^{-2}} \right) + O_p \left( J^{-4} \right) + O_p \left( n^{-1} J^{-2} \right) \right) = O_p \left( n^{-1} + O_p \left( J^{-4} \right) \right),$$

where the second equality follows by Assumption 3. Now consider the first result. We have that $K_{11}$ satisfies $K_{11} = K_{111} + K_{112}$ where

$$K_{111} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_{t}^l \Omega_{u,t}^{-1} U_t^t \varepsilon_t / n_t$$

and

$$K_{112} = T^{-1} \sum_{t=1}^{T} 1_{\beta,t} s_{t}^l \Omega_{u,t}^{-1} \left( \Omega_{u,t} - \hat{\Omega}_{u,t} \right) \hat{\Omega}_{u,t}^{-1} U_t^t \varepsilon_t / n_t.$$

For $K_{111}$ we have

$$\mathbb{E} |K_{111}|^2 = T^{-2} \sum_{t_1, t_2} n_{t_1}^{-1} n_{t_2}^{-1} \mathbb{E} \left[ 1_{\beta,t_1} 1_{\beta,t_2} s_{t_1}^l s_{t_2}^l \Omega_{u,t_1}^{-1} U_{t_1}^t \mathbb{E} \left[ \varepsilon_{t_1} \varepsilon_{t_2}' \big| \mathcal{F}_{t_1}, \mathcal{F}_{t_2}, z_{t_1}, z_{t_2}, x_{t_1}, x_{t_2} \right] U_{t_2}^t \Omega_{u,t_2}^{-1} s_{t_2}^l \right] = T^{-2} \sum_{t_1} n_{t_1}^{-2} \mathbb{E} \left[ 1_{\beta,t_1} s_{t_1}^l \Omega_{u,t_1}^{-1} U_{t_1}^t \varepsilon_{t_1} \varepsilon_{t_1}' U_{t_1}^t \Omega_{u,t_1}^{-1} s_{t_1}^l \right] \leq C n^{-1} T^{-1},$$

and
following similar steps as for the term $\mathcal{K}_{21}$. Then, by the CS inequality
\[
|\mathcal{K}_{112}|^2 \leq T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \left| s_t' \Omega_{uu,t} \left( \Omega_{uu,t} - \hat{\Omega}_{uu,t} \right) \hat{\Omega}_{uu,t}^{-1} \right|^2 \times T^{-1} \sum_{t=1}^{T} \left| U'e_{t}/ n_t \right|^2 \\
\leq T^{-1} \sum_{t=1}^{T} 1_{\beta,t} \left[ \lambda_{\max} \left( \Omega_{uu,t}^{-1} \right) \right]^2 \left[ \lambda_{\max} \left( \hat{\Omega}_{uu,t}^{-1} \right) \right]^2 \left| \Omega_{uu,t} - \hat{\Omega}_{uu,t} \right|^2 \times T^{-1} \sum_{t=1}^{T} \left| U'e_{t}/ n_t \right|^2 \\
\leq CT^{-1} \sum_{t=1}^{T} \left| \hat{\Omega}_{uu,t} - \Omega_{uu,t} \right|^2 \times T^{-1} \sum_{t=1}^{T} \left| U'e_{t}/ n_t \right|^2.
\]
The first factor is $O_p \left( n^{-1} \right) + O_p \left( J^{-4} \right) + O_p \left( J^{2d} n^{-2} \right)$ by Lemma 3 and by similar steps as for $\mathcal{K}_{21}$ the second factor has expectation,
\[
T^{-1} \sum_{t=1}^{T} \mathbb{E} \left[ \left| U'e_{t}/ n_t \right|^2 \right] \leq C n^{-1}.
\]
Thus, $|\mathcal{K}_{11}|^2 = O_p \left( n^{-1} T^{-1} \right) + O_p \left( n^{-2} \right) + O_p \left( n^{-1} J^{-4} \right) + O_p \left( J^{2d} n^{-3} \right)$ by Markov’s inequality. By similar steps we can show that \\
\[
|\mathcal{K}_{12}|^2 = O_p \left( T^{-1} n^{-2} J^{d} \right) + O_p \left( J^{d} n^{-3} \right) + O_p \left( J^{d} n^{-4} \right) + O_p \left( J^{3d} n^{-4} \right) \\
|\mathcal{K}_{13}| = O_p \left( J^{-2} \right) \\
|\mathcal{K}_{14}|^2 = O_p \left( n^{-1} J^{-2} \right) + O_p \left( n^{-2} J^{-2} \right) + O_p \left( n^{-1} J^{-6} \right) + O_p \left( J^{2d} n^{-3} \right) \\
|\mathcal{K}_{15}|^2 = O_p \left( n^{-1} T^{-1} J^{-2} \right) + O_p \left( n^{-2} J^{-2} \right) + O_p \left( n^{-1} J^{-6} \right) + O_p \left( J^{2d} n^{-3} \right).
\]
Thus, we have that \\
\[
\sum_{\ell} |\mathcal{K}_{1\ell}|^2 = O_p \left( n^{-1} T^{-1} \right) + O_p \left( J^{-4} \right) + O_p \left( J^{2d} n^{-3} \right) + O_p \left( J^{d} n^{-4} \right).
\]

Proof of Lemma 6. We would like to show that \\
\[
V^{-1} \left( z \right)^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{it}} \mathbb{I}_{\beta} \hat{\eta}_{it}^{(z)} \left( z \right) \hat{q}_{jt}^{-2} \hat{\eta}_{jt}^{(z)} \left( z \right) \left( \varepsilon_{it}^2 - \sigma_{it}^2 \right) = o_p \left( 1 \right).
\]
By Lemma 5 we need only show that \\
\[
J^{-d} n^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{it}} \mathbb{I}_{\beta} \hat{\eta}_{it}^{(z)} \left( z \right) \hat{q}_{jt}^{-2} \hat{\eta}_{jt}^{(z)} \left( z \right) \left( \varepsilon_{it}^2 - \sigma_{it}^2 \right) = J_1 + J_2,
\]
where \\
\[
\hat{\eta}_{it}^{\pm} = \epsilon_{it}^2 \mathbb{I} \{ \epsilon_{it}^2 > t_{nT} \} - \mathbb{E} \left[ \epsilon_{it}^2 \mathbb{I} \{ \epsilon_{it}^2 > t_{nT} \} \right] \mathcal{F}_t, z_{it}, x_{it} \\
\hat{\eta}_{it}^{-} = \epsilon_{it}^2 \mathbb{I} \{ \epsilon_{it}^2 \leq t_{nT} \} - \mathbb{E} \left[ \epsilon_{it}^2 \mathbb{I} \{ \epsilon_{it}^2 \leq t_{nT} \} \right] \mathcal{F}_t, z_{it}, x_{it}
\]
First consider $J_1$, \\
\[
\mathbb{P} \left( \left| J^{-d} n^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{it}} \mathbb{I}_{\beta} \hat{\eta}_{it}^{(z)} \left( z \right) \hat{q}_{jt}^{-2} \hat{\eta}_{jt}^{(z)} \left( z \right) \eta_{it}^{\pm} \right| > \delta_{nT} \right) \leq \frac{n^2}{\delta_{nT}^2 J^{2d} T^2} \times
\]

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\[
\sum_{t_1,t_2=1}^T n_t^{-2} n_t^{-2} \sum_{1 \leq i,j \leq n_t} \sum_{j_1,j_2=1}^{J_{ds}} \mathbb{E} \left[ 1_{j_1 t_1} 1_{j_2 t_2} \tilde{q}_{j_1 t_1} (z) \tilde{q}_{j_2 t_2} (z) \eta_{j_1 t_1}^+ \eta_{j_2 t_2}^+ \right] = \frac{n^2}{\delta_n T J^{2d} \tilde{T}^2} \times \sum_{t=1}^T n_t^{-4} n_t^{-4} \sum_{j=1}^{J_{dz}} \mathbb{E} \left[ 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t})^2 \right] \\
\leq \frac{n^2}{\delta_n T J^{2d} \tilde{T}^2} \sum_{t=1}^T n_t^{-4} n_t^{-4} \sum_{j=1}^{J_{dz}} \mathbb{E} \left[ 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}) \mathbb{E} \left[ \varepsilon_{t t}^i 1 \{ \varepsilon_{t t} > t_n \} | \mathcal{F}_t, z_{it}, x_{it} \right] \right] \\
\leq \frac{n^2}{\delta_n T n_t T^{2d} T} \sum_{t=1}^T n_t^{-4} n_t^{-4} \sum_{j=1}^{J_{dz}} \mathbb{E} \left[ 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}) \mathbb{E} \left[ \varepsilon_{t t}^4 \varepsilon_{t t} \mathbb{E} \left[ \varepsilon_{t t}^4 \varepsilon_{t t} | \mathcal{F}_t, z_{it}, x_{it} \right] \right] \right] \\
\leq \frac{C J^{2d} n^2}{\delta_n T n_t T^{2d} \tilde{T}^2} \sum_{t=1}^T n_t^{-3} \sum_{j=1}^{J_{dz}} \mathbb{E} \left[ 1_{j t} \tilde{q}_{j t} (z) n_t^{-1} \sum_{i=1}^{n_t} \tilde{q}_{j t} (z) \right] \\
\leq \frac{C J^{2d} n^2}{\delta_n T n_t T^{2d} \tilde{T}^2} \sum_{t=1}^T n_t^{-3} \sum_{j=1}^{J_{dz}} \mathbb{E} \left[ 1_{j t} \tilde{q}_{j t} (z) \right] \\
\leq \frac{C J^d}{\delta_n T n_t T \tilde{T}^2}.
\]

Now consider \( J_2 \):

\[
\left| J^{-d} n T^{-1} \sum_{t=1}^T n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right| \\
\leq J^{-d} n T^{-1} \sum_{t=1}^T \left| n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right| \\
\leq J^{-d} n \max_{1 \leq t \leq T} \left| n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right|
\]

and

\[
\mathbb{P} \left( J^{-d} n \max_{1 \leq t \leq T} \left| n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right| > \delta_n \right) \\
\leq \sum_{t=1}^T \mathbb{P} \left( \left| n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right| > \delta_n \right).
\]

This is a mean-zero, bounded random variable. The summands are bounded by

\[
\left| J^{-d} n T^{-1} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right| \\
\leq J^{-d} n T^{-1} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \\
\leq I_{n T}^2 J^d n T^{-1} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \\
\leq C I_{n T}^2 J^d.
\]

The rescaled sum of the variances are

\[
\frac{1}{n_t} \sum_{i=1}^{n_t} \mathbb{E} \left[ \left| J^{-d} n T^{-1} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \right|^2 \right] \mathbb{E} \left[ \mathcal{F}_t, z_{it}, x_{it} \right] \\
\leq \frac{C}{n_t J^{2d}} \sum_{i=1}^{n_t} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 (z, \eta_{j t}^-) \mathbb{E} \left[ \left| \eta_{j t}^- \right|^2 \right] \mathbb{E} \left[ \mathcal{F}_t, z_{it}, x_{it} \right] \\
\leq \frac{C}{J^{2d}} \sum_{j=1}^{J_{dz}} 1_{j t} \tilde{q}_{j t} (z) \tilde{q}_{j t}^2 \left( n_t^{-1} \sum_{i=1}^{n_t} \tilde{q}_{j t} (z) \eta_{j t}^- \right)
\]

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\[ \leq C J^d \sum_{j=1}^{J^d_t} \mathbb{1}_{j t} \hat{I}_{j t} (z) \]

\[ \leq C J^d. \]

Thus,

\[ \mathbb{P} \left( \sum_{t=1}^{T} n_t^{-1} \sum_{i=1}^{n_t} n_t^{-1} \sum_{j=1}^{J^d_t} \mathbb{1}_{j t} \hat{I}_{j t} (z) \eta_{i t} \right) \leq \delta_{n T} \]

\[ \leq 1 \land \sum_{t=1}^{T} \mathbb{P} \left( \sum_{i=1}^{n_t} n_t^{-1} \sum_{j=1}^{J^d_t} \mathbb{1}_{j t} \hat{I}_{j t} (z) \eta_{i t} \right) \leq \delta_{n T} \]

\[ \leq C \sum_{t=1}^{T} \mathbb{P} \left( \sum_{i=1}^{n_t} n_t^{-1} \sum_{j=1}^{J^d_t} \mathbb{1}_{j t} \hat{I}_{j t} (z) \eta_{i t} \right) \leq \delta_{n T} \]

\[ \leq 1 \land \sum_{t=1}^{T} \mathbb{P} \left( \sum_{i=1}^{n_t} n_t^{-1} \sum_{j=1}^{J^d_t} \mathbb{1}_{j t} \hat{I}_{j t} (z) \eta_{i t} \right) \leq \delta_{n T} \]

\[ \leq C \sum_{t=1}^{T} \exp \left\{ - \frac{n_t \delta_{n T}^2}{C J^d + C \delta_{n T} J^d t_{n T}^2} \right\} \]

\[ = 1 \land C \exp \left\{ \log (T) \left( 1 - \frac{C n \delta_{n T}^2 J^d \log (T)^{-1}}{1 + \delta_{n T} t_{n T}^2} \right) \right\}. \]

Thus, we need to find conditions such that

\[ t_{n T} \rightarrow \infty \]

\[ \delta_{n T} \rightarrow 0 \]

\[ \frac{t_{n T}^2 n T}{J^d n \delta_{n T}} = O (1) \]

\[ \frac{J^d \log (T)}{n \delta_{n T}} \not\rightarrow \infty \]

\[ \frac{J^d \log (T)}{n \delta_{n T}} \not\rightarrow \infty. \]

Let \( t_{n T} = \log (T \lor J^d)^{1/4} \log (T \land J^d)^{-1/4} \) and \( \delta_{n T}^2 = J^d n^{-1} \log (T \lor J^d). \) Then, in reverse order, we have

\[ \frac{J^d \log (T)}{n \delta_{n T}} = \frac{J^d / 2 \log (T)}{n^{1/2}} \log (T \land J^d)^{1/2} = \sqrt{\frac{J^d \log (T)^2 \log (T \land J^d)^{-1}}{n}}, \]

which is \( O (1) \) by assumption. Then

\[ \frac{J^d \log (T)}{n \delta_{n T}^2} = \frac{\log (T)}{\log (T \lor J^d)} = O (1), \]

and

\[ \frac{J^d}{\delta_{n T}^2 t_{n T} n T} = \frac{1}{\log (T \lor J^d) \log (T \lor J^d)^{e/4} \log (T \land J^d)^{-e/4} T} = o (1). \]
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