SURFACES OF REVOLUTION WITH PRESCRIPTION MEAN AND SKEW CURVATURES IN LORENTZ-MINKOWSKI SPACE

LUIZ C. B. DA SILVA

This is a pre-print version of the article published as [Da Silva, Tohoku Math. J. 73 (2021), 317–339] whose full-text is available in [https://doi.org/10.2748/tmj.20190729].

Abstract. In this work, we investigate the problem of finding surfaces in the Lorentz-Minkowski 3-space with prescribed skew ($S$) and mean ($H$) curvatures, which are defined through the discriminant of the characteristic polynomial of the shape operator and its trace, respectively. After showing that $H$ and $S$ can be interpreted in terms of the expected value and standard deviation of the normal curvature seen as a random variable, we address the problem of prescribed curvatures for surfaces of revolution. For surfaces with a non-lightlike axis and prescribed $H$, the strategy consists in rewriting the equation for $H$, which is initially a nonlinear second order Ordinary Differential Equation (ODE), as a linear first order ODE with coefficients in a certain ring of hypercomplex numbers along the generating curves: complex numbers for curves on a spacelike plane and Lorentz numbers for curves on a timelike plane. We also solve the problem for surfaces of revolution with a lightlike axis by using a certain ODE with real coefficients. On the other hand, for the skew curvature problem, we rewrite the equation for $S$, which is initially a nonlinear second order ODE, as a linear first order ODE with real coefficients. In all the problems, we are able to find the parameterization for the generating curves in terms of certain integrals of $H$ and $S$.

Introduction

The problem of finding surfaces with prescribed mean ($H$) or Gaussian ($K$) curvatures is very important in Differential Geometry. In general, one is led to the study of a non-linear PDE: a nonlinear elliptic PDE of Hessian type for $K$ [20, 31], also known as Monge-Ampère equation; and a nonlinear elliptic PDE of divergent type for $H$ [19]. However, for surfaces invariant by a 1-parameter subgroup of isometries [15] this problem is easier and reduces to that of solving a certain non-linear second order ODE [11, 15, 25]. Similar results are also found for surfaces in Lorentz-Minkowski geometry [2, 21, 22, 24] and in other ambient spaces as well [3, 4, 30, 35]. If we write $H$ and $K$ in terms of the principal curvatures $\kappa_1$ and $\kappa_2$, the problem reduces to finding surfaces with a prescribed sum and product of the principal curvatures. On the other hand, the difference $\kappa_1 - \kappa_2$...
seems to be given less attention. In this work, we are interested in the study of the mean curvature $H$ and skew curvature $S = \sqrt{H^2 - \epsilon K}$ in Lorentz-Minkowski space, where the parameter $\epsilon$ is $-1$ for a space-like surface and $+1$ for a time-like one (in Euclidean space $S$ is just $S = \sqrt{H^2 - K}$). If the shape operator is diagonalizable, then the skew curvature may be written as $S = |\kappa_1 - \kappa_2|$, while the mean curvature is half the sum $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ (in Lorentz-Minkowski space it is possible to have $H^2 - \epsilon K < 0$, in which case the shape operator is no longer diagonalizable [27] and $S$ may be chosen in a way that $iS < 0$).

In the 1960s, B.-Y. Chen obtained global results for Euclidean closed surfaces related with the integrated skew curvature [6] (named by him as the difference curvature). In the 1970s, the skew curvature was independently reintroduced in Euclidean space by T. K. Milnor as an auxiliary tool in the study of open surfaces [28]. Later, she investigated properties of some quadratic forms defined by using the skew curvature [29]. It is worth mentioning that at an umbilic point the skew curvature vanishes and, consequently, it can work as a measure of the surface bend anisotropy. Indeed, we shall see that the mean and skew curvatures can be associated with the expected value and standard deviation of the normal curvature seen as a random variable, see Eq. (24). In addition, the behavior of $S$ can be related to the diagonalizability of the shape operator, a problem that does make sense in non-Riemannian geometry [8, 12, 17, 27]. Moreover, in a 3-dimensional space form of curvature $c$ it is valid the relation $H^2 - K \geq -c$, with equality valid for totally umbilical surfaces only [7]. (A similar expression holds in semi-Riemannian space forms as well. However, for timelike surfaces, i.e., $\epsilon = +1$, the equality $H^2 - K = -c$ does not imply umbilicity in general [17].) Finally, let us mention that the square of the skew curvature appears in the study of the Willmore functional $W = \int (H^2 - K)dA$ [33] and also as a geometry-induced potential in the context of the quantum dynamics of a particle constrained to move on a surface in Euclidean space [9, 11].

Recently, surfaces in Euclidean space with constant skew curvature were investigated [26, 32] and the problem of finding surfaces of revolution with prescribed skew curvature was solved in the context of a quantum constrained dynamics [11] . In this work, we shall address the problem of finding surfaces of revolution with prescribed skew curvature in Lorentz-Minkowski space following similar techniques to those of Ref. [11]. In addition, we also revisit the problem of finding Lorentzian surfaces of revolution with prescribed mean curvature by using the complex numbers and the so-called Lorentz numbers (also known as double numbers, see supplement C of Ref. [34]), which constitutes a natural generalization of the technique employed in Euclidean space [25] and can allow for a better understanding of the results found in Lorentz-Minkowski space [22, 24].

This work is divided as follows. In Section 1 we present the fundamentals of the geometry in Lorentz-Minkowski space along with the classification of rotations, an Euler theorem for the normal curvature $\kappa_n$, and a statistical interpretation of $H$ and $S$ as an expected value and standard deviation of $\kappa_n$, respectively. In Section 2 we address the
problem of finding surfaces of revolution with prescribed mean curvature: the surfaces with a non-lightlike axis are described in Subsections 2.1., 2.2., and 2.3, whose solution of the prescribed \( H \) problem is analyzed in 2.4; and, in Subsection 2.5, we solve the problem for surfaces of revolution with a lightlike axis. Finally, in Section 3, we solve the prescribed skew curvature problem for surfaces of revolution with a non-lightlike axis. In Appendix A, we present the ring of Lorentz numbers which constitute an important tool in Section 2.

The present author would like to thank useful discussions with Renato T. Gomes (from Universidade Federal Rural de Pernambuco, Recife, Brazil).

1. Differential geometric background

We now present some geometric preliminaries and also establish an Euler theorem for the normal curvature. This leads to a statistical interpretation for \( H \) and \( S \) which qualifies them as appropriate quantities in the study of the extrinsic behavior of a surface. In later sections, we shall study all the basic types of surfaces of revolution in \( \mathbb{E}^3_1 \) (see Table 1) and show how to find surfaces of revolution with prescribed mean or skew curvature. Both problems shall be solved by conveniently rewriting the respective curvature equations in terms of certain linear ODE’s.

Let us denote by \( \mathbb{E}^3_1 \) the 3-dimensional Lorentz-Minkowski space, i.e., the vector space \( \mathbb{R}^3 \) equipped with the index one metric
\[
\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3.
\]

On the other hand, we shall denote the usual Euclidean space by \( \mathbb{E}^3 \).

In \( \mathbb{E}^3_1 \) we may introduce the concept of causal character as follows: we say that \( v \in \mathbb{E}^3_1 \) is (i) spacelike, (ii) timelike, or (iii) lightlike if (i) \( \langle v, v \rangle > 0 \) or \( v = 0 \), (ii) \( \langle v, v \rangle < 0 \), or (iii) \( \langle v, v \rangle = 0 \) and \( v \neq 0 \), respectively. Given a regular parameterized curve \( \alpha : I \rightarrow \mathbb{E}^3_1 \), i.e., \( \alpha' \neq 0 \), we say that \( \alpha \) is a spacelike, timelike, or lightlike curve if \( \alpha' \) is spacelike, timelike, or lightlike in \( I \), respectively (for space- or time-like curves we may introduce an arc-length parameter \( s \) as usual: \( s = \int \sqrt{\langle \alpha'(t), \alpha'(t) \rangle} \, dt \)). On the other hand, for a regular surface \( \Sigma \) given by a parameterization \( X : U \rightarrow \Sigma \subset \mathbb{E}^3_1 \), we say that \( \Sigma \) is a spacelike, timelike, or lightlike surface if the induced metric \( p \mapsto \langle \cdot, \cdot \rangle|_{T_p \Sigma} \) is Riemannian, Lorentzian (non-degenerate with index 1), or degenerate with rank 1, respectively.

A rotation in \( \mathbb{E}^3_1 \) is an isometry leaving a certain straight line pointwise fixed, known as the rotation axis \[27\]. In this way, it suffices to consider the three cases below:

(a) **timelike axis**: supposing that the axis is \((0, 0, 1)\), we have

\[
T_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \theta \in \mathbb{S}^1;
\]
(b) spacelike axis: supposing that the axis is \((1, 0, 0)\), we have
\[
S_\theta = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cosh \theta & \sinh \theta \\
0 & \sinh \theta & \cosh \theta
\end{pmatrix}, \quad \theta \in \mathbb{R}; \quad \text{and}
\]
(c) lightlike axis: supposing that the axis is \((0, 1, 1)\), we have
\[
L_\theta = \begin{pmatrix}
1 & \theta & -\frac{\theta^2}{2} \\
-\theta & 1 - \frac{\theta^2}{2} & \frac{\theta^2}{2} \\
-\theta & -\frac{\theta^2}{2} & 1 + \frac{\theta^2}{2}
\end{pmatrix}, \quad \theta \in \mathbb{R}.
\]

Apart from subsection 1.1, in this work we shall be interested in surfaces of revolution only, i.e., surfaces \(\Sigma\) invariant by \(T_\theta, S_\theta, \text{ or } L_\theta\): e.g., for a timelike axis, \(\Sigma = T_\theta(\Sigma)\) for all \(\theta\). Here, the whole surface can be obtained by rotating a curve, the generating curve, that can be assumed to be contained in a plane (which also contains the axis, Figure 1).

The many possibilities for the causal characters of the rotation axis in combination with the causal characters of the generating curve and the plane where it is contained in give rise to various types of surfaces of revolution in \(\mathbb{E}_3^1\), as will become clear in the following (see Table 1 and Figure 1).

| Axis | Plane | Curve | Surface |
|------|-------|-------|---------|
| time | time  | time  | time    |
|      | space | space |         |
| space| time  | time  | time    |
|      | space | space |         |
|      | space | space | time    |
| light| time  | time  | time    |
|      | space | space |         |

Table 1. Causal characters of surfaces of revolution in terms of the causal characters of the rotation axis, the plane that contains the generating curve, and the generating curve (the only lightlike surfaces of revolution are lightlike planes and light cones \([23]\)).

Due to the intimate relationship between the causal characters of a vector subspace \(V \subset \mathbb{E}_3^1\) and its orthogonal complement \(V^\perp\) induced by \(\langle \cdot, \cdot \rangle\), Prop. 1.1 of \([27]\), if a surface \(\Sigma\) admits a unit normal vector field \(N\) (in particular, \(\Sigma\) is not lightlike), then we have
\[
\epsilon = \langle N, N \rangle \Rightarrow \epsilon = \begin{cases} 
-1, & \text{if } \Sigma \text{ is spacelike} \\
+1, & \text{if } \Sigma \text{ is timelike}
\end{cases}
\]
In local coordinates $X : U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{E}^3_1$, $X = X(u, v)$, the normal can be written as
\begin{equation}
N = \frac{X_u \times X_v}{\sqrt{|X_u \times X_v|}},
\end{equation}
where $\times$ is the cross product in $\mathbb{E}^3_1$:
\begin{equation}
(u_1, u_2, u_3) \times (v_1, v_2, v_3) = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), -(u_1v_2 - u_2v_1)).
\end{equation}

The coefficients $g_{ij}$ and $h_{ij}$ of the first and second fundamental forms of $\Sigma$ are defined as $g_{11} = \langle X_u, X_u \rangle$, $g_{12} = \langle X_u, X_v \rangle$, $g_{22} = \langle X_v, X_v \rangle$, and $h_{11} = \langle X_{uu}, N \rangle$, $h_{12} = \langle X_{uv}, N \rangle$, $h_{22} = \langle X_{vv}, N \rangle$, respectively. It is worth mentioning that $-\epsilon = \text{sgn}(g) := \text{sgn}(\det g_{ij})$ and, therefore, $g > 0$ for a spacelike surface and $g < 0$ for a timelike one (if $\Sigma$ is lightlike, then $g = 0$) [27].

Finally, in local coordinates the Gaussian $K$ and mean $H$ curvatures are [27]
\begin{equation}
K = \epsilon \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} \quad \text{and} \quad H = \frac{\epsilon}{2} \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2},
\end{equation}
while the skew curvature $S$ is
\begin{equation}
S = \sqrt{H^2 - \epsilon K}.
\end{equation}

By convention, if $H^2 - \epsilon K < 0$, we may choose $S$ in a way that $iS < 0$. For surfaces of revolution, however, we do not need to worry about such a possibility. Indeed, since the shape operator $A_p = -\text{d}N$ is always diagonalizable for surfaces of revolution [17] (see also the explicit computations in Sect. 2) and since the discriminant of the characteristic polynomial of $A_p$ is precisely $4(H^2 - \epsilon K)$ [27], it follows that $S^2 = H^2 - \epsilon K \geq 0$ here.

Remark 1.1. In [8], the skew curvature is defined to be $\sqrt{H^2 - K}$ and denoted by $H'$. Here, we shall denoted it by $S$ (from skew) instead of $H'$ in order to avoid confusion with the derivative of $H$. In addition, since $4(H^2 - \epsilon K)$ is the discriminant of the characteristic polynomial of the shape operator [27], it seems to be more natural to define $S$ the way we do. Finally, we believe that the results to be presented in the subsection below and also in Section 3, for surfaces of revolution with non-lightlike axis, will show that our definition allows for a suitable use of the skew curvature concept.

1.1. Euler theorem and statistical interpretation of the mean and skew curvatures. Since $S$ vanishes at an umbilic point, it can be thought to be a measure of the surface bend anisotropy. Indeed, we shall show below that the mean and skew curvatures are respectively given in terms of the expected value and standard deviation of the normal curvature $\kappa_n$, when we see $\kappa_n$ as a random variable. This suggests that $H$ and $S$ together are appropriate quantities to offer a glimpse of the extrinsic behavior of a surface.

Let $\Sigma$ be a regular parameterized surface, not necessarily of revolution, and $p \in \Sigma$. The normal curvature $\kappa_n$ at $p$ is a real function over the set of unit tangent vectors, i.e., $\kappa_n : S^1 \subset T_p\Sigma \to \mathbb{R}$ for a spacelike surface or $\kappa_n : S^1 \cup H^1 \subset T_p\Sigma \to \mathbb{R}$ for a timelike one.
In addition, the standard deviation is for the expected value of $\kappa$ ($i = 1, 2$). In particular, we are assuming that $A_p$ is diagonalizable, in which case the mean and skew curvatures can be written as $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $S = \sqrt{(\kappa_1 - \kappa_2)^2}$. (This is the case for surfaces of revolution.)

If $\Sigma$ is spacelike, the induced metric is Riemannian and then we can write any unit tangent vector $v$ at $p$ as

$$v = \cos \phi u_1 + \sin \phi u_2, \phi \in S^1.$$  

This leads to the following Euler theorem

$$\kappa_n(p, v) = \langle A_p v, v \rangle = \cos^2(\phi) \kappa_1 + \sin^2(\phi) \kappa_2.$$  

Notice this is the same expression we would obtain for a surface $\Sigma$ in Euclidean space.

On the other hand, if $\Sigma$ is timelike, the induced metric is Lorentzian and then we can write any unit tangent vector $v$ at $p$ as

$$v = \begin{cases} 
\pm \cosh \phi u_1 + \sinh \phi u_2, & \text{if } \langle v, v \rangle = +1 \\
\sinh \phi u_1 \pm \cosh \phi u_2, & \text{if } \langle v, v \rangle = -1 
\end{cases}, \phi \in \mathbb{R},$$  

where we are assuming for simplicity that $u_1$ is the spacelike eigenvector and $u_2$ is the timelike one. This leads to the following Euler theorem

$$\kappa_n(p, v) = \begin{cases} 
cosh^2(\phi) \kappa_1 - \sinh^2(\phi) \kappa_2, & \text{if } \langle v, v \rangle = +1 \\
\sinh^2(\phi) \kappa_1 - \cosh^2(\phi) \kappa_2, & \text{if } \langle v, v \rangle = -1 
\end{cases}.$$  

Now, pretending $\kappa_n$ is a random variable, there are two important parameters naturally associated with it, namely the expected value $\langle \kappa_n \rangle$ and the standard deviation $\sqrt{\langle (\Delta \kappa_n)^2 \rangle}$. Then, if $\Sigma$ is spacelike ($\epsilon = -1$), we can use Eq. (11) to establish the following relation for the expected value of $\kappa_n$ with respect to the uniform (probability) density $\frac{d\phi}{2\pi}$

$$\langle \kappa_n \rangle = \int_0^{2\pi} \kappa_n(\phi) \frac{d\phi}{2\pi} = \frac{\kappa_1 + \kappa_2}{2} \Rightarrow H = -\langle \kappa_n \rangle.$$  

In addition, the standard deviation is

$$\sqrt{\langle (\Delta \kappa_n)^2 \rangle} = \sqrt{\int_0^{2\pi} [\kappa_n(\phi) - \langle \kappa_n \rangle]^2 \frac{d\phi}{2\pi}} = \sqrt{\frac{(\kappa_1 - \kappa_2)^2}{8} \Rightarrow S = 2\sqrt{2} \sqrt{\langle (\Delta \kappa_n)^2 \rangle}}.$$  

It is worth mentioning that in Euclidean space we would find analogous results for $\kappa_n$, namely $\langle \kappa_n \rangle = H$ and $S = 2\sqrt{2} \sqrt{\langle (\Delta \kappa_n)^2 \rangle}$, as can be easily verified.

The results above for surfaces with an induced metric of Riemannian signature suggest that we can replace the continuous distribution $\kappa_n$ by a discrete one taking the possible values $\kappa_1$ or $\kappa_2$, since in this case

$$\langle \kappa_n \rangle = \frac{1}{2} \sum_{i=1}^{2} \kappa_i = \frac{\kappa_1 + \kappa_2}{2}$$  

and

$$\sqrt{\langle (\Delta \kappa_n)^2 \rangle} = \sqrt{\frac{1}{2 - 1} \sum_{i=1}^{2} (\kappa_i - \langle \kappa_n \rangle)^2} = \sqrt{\frac{1}{2}(\kappa_1 - \kappa_2)^2}.$$  

Finally, the standard deviation of the normal curvature is given by

\[(\kappa_n) = \text{finite part} \left( \lim_{a \to \infty} \frac{1}{2a} \int_{-a}^{+a} \kappa_n(\phi) d\phi \right).\]

We have

\[(\frac{e^{\phi} + e^{-\phi}}{2})^2 = \frac{1}{2a} \int_{-a}^{+a} \frac{e^{2\phi} + 2 + e^{-2\phi}}{4} d\phi = \pm \frac{1}{2} + \frac{\sinh(2a)}{4a} \to \pm \frac{1}{2} + \frac{e^{2a}}{4a}.
\]

Taking into account the finite contributions only, the expected value of \(\kappa_n\) is given by

\[\langle \kappa_n \rangle = \langle \kappa_1 \cosh^2 \phi - \kappa_2 \sinh^2 \phi \rangle = \frac{\kappa_1 + \kappa_2}{2} = H.
\]

On the other hand, to find the standard deviation, we first compute

\[\left\langle \left( \frac{e^{\phi} + e^{-\phi}}{2} \right)^4 \right\rangle_a = \frac{1}{2a} \int_{-a}^{+a} \left( \pm \frac{1}{2} + \frac{\cosh 2\phi}{2} \right)^2 d\phi = \frac{3}{8} + \frac{\sinh 2a}{4a} + \frac{\sinh 4a}{32a} \]

and

\[\langle \cosh^2 \phi \sinh^2 \phi \rangle_a = \frac{1}{2a} \int_{-a}^{+a} \left( \frac{1}{2} + \frac{\cosh 2\phi}{2} \right) \left( -\frac{1}{2} + \frac{\cosh 2\phi}{2} \right) d\phi = -\frac{1}{8} + \frac{\sinh 4a}{32a} \]

Taking into account the finite contributions only,

\[\langle (\Delta \kappa_n)^2 \rangle = \langle \kappa_1^2 \cosh^4 \phi - 2\kappa_1 \kappa_2 \cosh^2 \phi \sinh^2 \phi + \kappa_2^2 \sinh^4 \phi \rangle - \left( \frac{\kappa_1 + \kappa_2}{2} \right)^2 = \frac{1}{8} (\kappa_1 - \kappa_2)^2.
\]

Finally, the standard deviation of the normal curvature is given by

\[\sqrt{\langle (\Delta \kappa_n)^2 \rangle} = \sqrt{\frac{(\kappa_1 - \kappa_2)^2}{8}} = \frac{S}{2\sqrt{2}}.
\]

In short, the results above for both space- and time-like surfaces give that the expected value \(\langle \kappa_n \rangle\) and the standard deviation \(\sqrt{\langle (\Delta \kappa_n)^2 \rangle}\) of \(\kappa_n\) are associated with the mean \(H\) and skew \(S\) curvatures according to

\[H = \epsilon \langle \kappa_n \rangle\] and \[S = 2\sqrt{2} \sqrt{\langle (\Delta \kappa_n)^2 \rangle}.
\]

Remark 1.2. A similar procedure of considering the finite contributions of diverging integrals also appears in the formulation of the Cauchy integral formula for functions over the Lorentz numbers [5]. Indeed, in general \(f(w) \neq \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-w} dz\), where \(\gamma(\theta) = w + Re^{i\theta}\) is an Euclidean circle around \(z = w\). However, considering only the finite contribution of the same integral over the branch of a hyperbola we obtain \(f(w) = \lim_{a \to \infty} \frac{1}{2a} \int_{\gamma} \frac{f(z)}{z-w} dz\).
2. Prescribed mean curvature equation in Lorentz-Minkowski space

In this section we solve the problem of prescribed mean curvature. Following Kenmotsu [25], the strategy for surfaces of revolution with a non-lightlike axis (subsections 2.1, 2.2, and 2.3) consists in considering the generating curve parameterized by arc-length and then write the equation for the mean curvature, which is initially a nonlinear second order ODE, as a linear first order ODE with coefficients in a certain ring of hypercomplex numbers along the generating curves (subsection 2.4): complex number $\mathbb{C}$ for curves on a spacelike plane and Lorentz numbers $\mathbb{L}$ (see Appendix A) for curves on a timelike plane. For a lightlike axis we are still able to solve the prescribed $H$ problem using the real numbers $\mathbb{R}$ (subsection 2.5).

Remark 2.1. The surfaces described in section 2.3, i.e., the ones generated from curves on a spacelike plane rotated around a spacelike axis (Figure 1(c)), furnish a counter-example to the assertion that a surface of revolution in $\mathbb{E}^3_1$ inherits the causal character of its generating curve [24] (there Ishihara and Hara only take into account the revolution of curves on the timelike $yz$-plane).

2.1. Rotation of a curve on a timelike plane around a timelike axis. Let $\alpha : I \to \mathbb{E}^3_1$ be a $C^2$ regular curve in the $xz$-plane, i.e., $\alpha(s) = (x(s), 0, z(s))$ with $s$ arclength parameter and $x > 0$. Considering a rotation of this curve around the $z$-axis gives the
following surface of revolution

(25) \[ Z(s, \theta) = (x(s) \cos \theta, x(s) \sin \theta, z(s)), \]

where \( \theta \in (0, 2\pi) \). Since \( s \) is the arc-length parameter of \( \alpha \), we can write

(26) \[ \eta = \langle \alpha', \alpha' \rangle = x'^2 - z'^2 \in \{-1, 1\}. \]

The first fundamental form \( I \) is given by

(27) \[ I = \eta \, ds^2 + x^2 \, d\theta^2. \]

Since \( g_{11}g_{22} - g_{12}^2 = \eta \, x^2 \Rightarrow \epsilon = -\eta \), it follows that \( Z \) is a spacelike (timelike) surface if and only if \( \alpha \) is a spacelike (timelike) curve.

Writing the normal vector to \( Z \) as

(28) \[ N = (-z' \cos \theta, -z' \sin \theta, -x'), \]

the second fundamental form \( II \) is given by

(29) \[ II = (x'z'' - xx'z') \, ds^2 + xz' \, d\theta^2. \]

Since both \( I \) and \( II \) are diagonal, the shape operator \( A = I^{-1}II \) is diagonalizable.

The mean curvature equation is then written as

(30) \[ 2xH + xx'z'' - xx''z' + \eta z' = 0. \]

Since \( \alpha \) is parametrized by arc-length, we have the additional equation

(31) \[ x'^2 - z'^2 = \eta \Rightarrow x'x'' = z'z''. \]

Multiplying Eq. (30) by \( x' \) gives

(32) \[ 2xx'H + \eta(xz')' = 0. \]

On the other hand, multiplying Eq. (30) by \( z' \) gives

(33) \[ 2xz'H + \eta(xx')' - 1 = 0. \]

By defining \( A(s) = x(s)x'(s) + \tau \, x(s)z'(s) \) in \( L \), we can write the two equations above in a single expression as

(34) \[ A'(s) + 2 \tau \eta \, H(s) \, A(s) - \eta = 0. \]
2.2. Rotation of a curve on a timelike plane around a spacelike axis. Let $\beta : I \to \mathbb{E}_1^3$ be a $C^2$ regular curve in the $xz$-plane, i.e., $\alpha(s) = (x(s), 0, z(s))$ with $s$ arc-length parameter and $z > 0$. Considering a rotation of this curve around the $x$-axis gives the following surface of revolution

$$X_I(s, \theta) = (x(s), z(s) \sin \theta, z(s) \cosh \theta),$$

where $\theta \in (-\infty, +\infty)$. Since $s$ is the arc-length parameter of $\beta$, we can write

$$\eta = \langle \alpha', \alpha' \rangle = x'^2 - z'^2 \in \{-1, 1\}.$$

The first fundamental form $I$ is given by

$$I = \eta \, ds^2 + z^2 \, d\theta^2.$$

Since $g = g_{11}g_{22} - g_{12}^2 = \eta \, z^2 \Rightarrow \epsilon = -\eta$, it follows that $X_I$ is a spacelike (timelike) surface if and only if $\beta$ is a spacelike (timelike) curve.

Writing the normal vector to $X_I$ as

$$N = (-z', -x' \sin \theta, -x' \cosh \theta),$$

the second fundamental form $II$ is given by

$$II = (x'z'' - x''z') \, ds^2 + z x' \, d\theta^2.$$

Since both $I$ and $II$ are diagonal, the shape operator $A = I^{-1}II$ is diagonalizable.

The mean curvature equation is then written as

$$2zH + z x'z'' - z x''z' + \eta x' = 0.$$

Since $\beta$ is parameterized by arc-length, we have the additional equation

$$x'^2 - z'^2 = \eta \Rightarrow x'x'' = z'z''.$$

Multiplying Eq. (40) by $x'$ gives

$$2zx'H + \eta (zz')' + 1 = 0.$$

On the other hand, multiplying Eq. (40) by $z'$ gives

$$2zz'H + \eta (zx')' = 0.$$

By defining $B(s) = z(s)z'(s) + \tau z(s)x'(s)$ in $L$, we can write the two equations above in a single expression as

$$B'(s) + 2 \tau \eta H(s) B(s) + \eta = 0.$$
2.3. Rotation of a curve on a spacelike plane around a spacelike axis. Let \( \gamma : I \to E^3 \) be a \( C^2 \) regular curve in the \( xy \)-plane, i.e., \( \gamma(s) = (x(s), y(s), 0) \) with \( s \) arc-length and \( y > 0 \). Considering a rotation of this curve around the \( x \)-axis gives the following surface of revolution

\[
X_{II}(s, \theta) = (x(s), y(s) \cosh \theta, y(s) \sinh \theta),
\]

where \( \theta \in (-\infty, +\infty) \). Since \( s \) is the arc-length parameter of \( \gamma \), the first fundamental form \( I \) is given by

\[
I = ds^2 - y^2 \, d\theta^2.
\]

Since \( g = g_{11}g_{22} - g_{12}^2 = -y^2 \Rightarrow \epsilon = +1 \), it follows that \( X_{II} \) is a timelike surface (observe that \( \gamma \) is necessarily a spacelike curve).

Writing the normal vector to \( X_{II} \) as

\[
N = (y', -x' \cosh \theta, -x' \sinh \theta),
\]

the second fundamental form \( II \) is given by

\[
II = (x''y' - x'y'') \, ds^2 - x'y \, d\theta^2.
\]

Since both \( I \) and \( II \) are diagonal, the shape operator \( A = I^{-1}II \) is diagonalizable.

The mean curvature equation is then written as

\[
2yH - yy'x'' + yx'y'' - x' = 0.
\]

Since \( \beta \) is parameterized by arc-length, we have the additional equation

\[
x'^2 + y'^2 = 1 \Rightarrow x'x'' = -y'y''.
\]

Multiplying Eq. (49) by \( x' \) gives

\[
2yx'H + (yy')' - 1 = 0.
\]

On the other hand, multiplying Eq. (49) by \( y' \) gives

\[
2yy'H - (yx')' = 0.
\]

By defining \( C(s) = y(s)y'(s) + i\, y(s)x'(s) \) in \( \mathbb{C} \) we can write the two equations above in a single expression as

\[
C'(s) - 2i\, H(s)\, C(s) - 1 = 0.
\]
2.4. Solution of the mean curvature equation for surfaces of revolution with a non-lightlike axis. In this subsection we shall prove three theorems (Theorems 2.2, 2.3, and 2.4) stating that, given a continuous function $H : I \rightarrow \mathbb{R}$, there exists a 3-parameter family of $C^2$ curves whose corresponding surface of revolution has $C^0$ mean curvature $H$ when rotated around a non-lightlike axis as described in Figures 1(a), 1(b), and 1(c). Here, we also comment on the characterization of constant mean curvature surfaces of revolution as Delaunay surfaces (Theorem 2.6).

For surfaces of revolution with a non-lightlike axis the mean curvature equation strongly depends on the causal character of the plane, $\Pi$, that contains the generating curve. Indeed, from Eqs. (34), (44), and (53), the mean curvature equations relate to either

$$(54) \quad A'(s) + 2 \tau \eta H(s) A(s) - \eta = 0 \quad \text{and} \quad B'(s) + 2 \tau \eta H(s) B(s) + \eta = 0$$

if $\Pi$ is timelike or to

$$(55) \quad C'(s) - 2i H(s) C(s) - 1 = 0$$

if $\Pi$ is spacelike. These equations can be solved exactly:

(a) for a curve on a timelike plane rotated around a timelike axis the solution is

$$(56) \quad A(s) = \left[ \int_0^s \eta e^{2\tau \eta \int_0^t H(u) du} dt \right] e^{-2\tau \eta \int_0^t H(t) dt} + A_0 e^{-2\tau \eta \int_0^t H(t) dt},$$

where $A_0$ is a constant;

(b) for a curve on a timelike plane rotated around a spacelike axis the solution is

$$(57) \quad B(s) = -\left[ \int_0^s \eta e^{2\tau \eta \int_0^t H(u) du} dt \right] e^{-2\tau \eta \int_0^t H(t) dt} + B_0 e^{-2\tau \eta \int_0^t H(t) dt},$$

where $B_0$ is a constant; and

(c) for a curve on a spacelike plane (rotated around a spacelike axis) the solution is

$$(58) \quad C(s) = \left[ \int_0^s e^{-2i \int_0^t H(u) du} dt \right] e^{2i \int_0^t H(t) dt} + C_0 e^{2i \int_0^t H(t) dt},$$

where $C_0$ is a constant.

From the solutions above we can find a generating curve $(x(s), 0, z(s))$ or $(x(s), y(s), 0)$ leading to a surface with prescribed mean curvature $H$. Indeed, the Lorentzian variable $A$ in Eq. (34) satisfies

$$(59) \quad \begin{cases} A\bar{A} = \eta x^2 \\ A - \bar{A} = 2\tau xz' \\ \Rightarrow z' = \frac{A - \bar{A}}{2\tau \sqrt{\eta AA}}.\end{cases}$$

On the other hand, the Lorentzian variable $B$ in Eq. (44) satisfies

$$(60) \quad \begin{cases} B\bar{B} = -\eta z^2 \\ B - \bar{B} = 2\tau zx' \\ \Rightarrow x' = \frac{B - \bar{B}}{2\tau \sqrt{-\eta BB}}.\end{cases}$$
Finally, the complex variable $C$ in Eq. (53) satisfies

\begin{equation}
\begin{cases}
C\tilde{C} = y^2 \\
C - \tilde{C} = 2iyx' \\
\Rightarrow x' = \frac{C - \tilde{C}}{2i\sqrt{CC'}}.
\end{cases}
\end{equation}

**Theorem 2.2.** Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a $C^2$ surface of revolution with timelike axis $Oz$ and $C^0$ mean curvature $H(s)$. Then, we write $\alpha(s)$ as

\begin{equation}
\alpha(s; H, \mathbf{a}) = (\sqrt{\eta}[(g_1 + \eta a_1)^2 - (f_1 + \eta a_2)^2], 0, \int_0^s \frac{g_1'(f_1 + \eta a_2) - f_1'(g_1 + \eta a_1)}{\sqrt{\eta}[(g_1 + \eta a_1)^2 - (f_1 + \eta a_2)^2]} dt + a_3),
\end{equation}

where we have introduced the functions

\begin{equation}
\begin{cases}
f_1(s) = \int_0^s \sinh(2\eta \int_0^t H(u)du)dt \\
g_1(s) = \int_0^s \cosh(2\eta \int_0^t H(u)du)dt
\end{cases}
\end{equation}

and the constant vector $\mathbf{a} = (a_1, a_2, a_3)$ satisfies the initial conditions at $s = 0$ given by $\alpha(0) = (\sqrt{\eta}(a_1^2 - a_2^2), 0, a_3)$ and $\alpha'(0) = [\eta(a_1^2 - a_2^2)]^{-1/2}(a_1, 0, a_2)$.

Conversely, given a continuous function $H(s)$, $s \in I$, and a constant vector $(a_1, a_2, a_3) \in S_1 \times \mathbb{R}$, where $S_1 = \bigcup_{s \in I}\{(X,Y) : [X + \eta g_1(s)]^2 - [Y + \eta f_1(s)]^2 \neq 0\}$, then the $C^2$ curve $\alpha(s; H(s), \mathbf{a})$ generates a $C^2$ surface of revolution with $C^0$ mean curvature $H(s)$ when rotated around the (timelike) $z$-axis.

**Proof.** Using the functions introduced in Eq. (63), we can rewrite Eq. (56) as

\begin{equation}
A = \eta[(f_1 + \eta a_2) + \tau(g_1 + \eta a_1)](-f_1' + \tau g_1'),
\end{equation}

where $\tau A_0 = a_2 + \tau a_1$. Consequently,

\begin{equation}
\begin{cases}
A - \tilde{A} = 2\eta \tau [g_1'(f_1 + \eta a_2) - f_1'(g_1 + \eta a_1)] \\
AA = -(f_1 + \eta a_2)^2 + (g_1 + \eta a_1)^2
\end{cases}
\end{equation}

Finally, inserting the relations above in Eq. (59) gives the expressions for $x(s)$ and $z(s)$ (after integration of $x'$) resulting in the expression for $\alpha(s; H(s), \mathbf{a})$ in Eq. (62).

Geometrically, the constants $a_1, a_2, a_3$ are related to the initial conditions, i.e., position and initial velocity of the generating curve at $s = 0$: $\alpha(0) = (\sqrt{\eta}(a_1^2 - a_2^2), 0, a_3)$ and $\alpha'(0) = [\eta(a_1^2 - a_2^2)]^{-1/2}(a_1, 0, a_2)$.

Conversely, given a continuous function $H : I \rightarrow \mathbb{R}$ and $(a_1, a_2, a_3) \in S_1 \times \mathbb{R}$, notice that $S_1$ is a non-empty open subset of $\mathbb{R}^2$, the curve $\alpha(s; H(s), \mathbf{a})$ is a unit speed curve of class $C^2$ that generates a $C^2$ surface of revolution around the (timelike) $z$-axis with continuous mean curvature $H$.

**Theorem 2.3.** Let $\beta(s) = (x(s), 0, z(s))$ be the generating curve of a $C^2$ surface of revolution with spacelike axis $Ox$ and $C^0$ mean curvature $H(s)$. Then, we write $\beta(s)$ as

\begin{equation}
\beta(s; H, \mathbf{b}) = (\int_0^s -\eta \frac{g_2'(f_2 - \eta b_1) - f_2'(g_2 - \eta b_2)}{\sqrt{\eta}[(f_2 - \eta b_1)^2 - (g_2 - \eta b_2)^2]} dt + b_3, 0, \sqrt{\eta}[(f_2 - \eta b_1)^2 - (g_2 - \eta b_2)^2]),
\end{equation}
where we have introduced the functions

\[
\begin{align*}
f_2(s) &= \int_0^s \sinh(2\eta\int_0^t H(u)du)dt \\
g_2(s) &= \int_0^s \cosh(2\eta\int_0^t H(u)du)dt
\end{align*}
\]

and the constant vector \( b = (b_1, b_2, b_3) \) satisfies the initial conditions at \( s = 0 \) given by \( \beta(0) = (b_3, 0, \sqrt{\eta(b_1^2 - b_2^2)}) \) and \( \beta'(0) = [\eta(b_1^2 - b_2^2)]^{-1/2}(b_1, 0, b_2) \).

Conversely, given a continuous function \( H(s), s \in I \), and a constant vector \( (b_1, b_2, b_3) \in S_2 \times \mathbb{R} \), where \( S_2 = \bigcup_{s \in I} \{(X, Y) : [X - \eta f_2(s)]^2 - [Y - \eta g_2(s)]^2 \neq 0\} \), then the \( C^2 \) curve \( \beta(s; H(s), b) \) generates a \( C^2 \) surface of revolution with \( C^0 \) mean curvature \( H(s) \) when rotated around the (spacelike) \( x \)-axis.

**Proof.** The proof is analogous to the previous one, since the solution for the mean curvature equation in terms of \( B \), Eq. (44), is analogous to that of \( A \), Eq. (34). Notice that here we should write the constant of integration \( B_0 \) in Eq. (57) as \( \tau B_0 = b_1 + \tau b_2 \). □

**Theorem 2.4.** Let \( \gamma(s) = (x(s), y(s), 0) \) be the generating curve of a \( C^2 \) surface of revolution with spacelike axis \( Ox \) and \( C^0 \) mean curvature \( H(s) \). Then, we write \( \gamma(s) \) as

\[
\gamma(s; H(s), c) = \left( \int_0^s \frac{F'(G + c_2) - G'(F - c_1)}{\sqrt{(F - c_1)^2 + (G + c_2)^2}} dt + c_3, \sqrt{(F - c_1)^2 + (G + c_2)^2}, 0 \right),
\]

where we have introduced the functions

\[
\begin{align*}
F(s) &= \int_0^s \sin(2\int_0^t H(u)du)dt \\
G(s) &= \int_0^s \cos(2\int_0^t H(u)du)dt
\end{align*}
\]

and the constant vector \( c = (c_1, c_2, c_3) \) satisfies the initial conditions at \( s = 0 \) given by \( \gamma(0) = (c_3, \sqrt{c_1^2 + c_2^2}, 0) \) and \( \gamma'(0) = (c_1^2 + c_2^2)^{-1/2}(c_1, c_2, 0) \).

Conversely, given a continuous function \( H(s), s \in I \), and constants \( (c_1, c_2, c_3) \in T \times \mathbb{R} \), where \( T = \bigcup_{s \in I} \{(X, Y) : [X - F(s)]^2 + [Y + G(s)]^2 \neq 0\} \), then the \( C^2 \) curve \( \gamma(s; H(s), c) \) generates a \( C^2 \) surface of revolution with \( C^0 \) mean curvature \( H(s) \) when rotated around the (spacelike) \( x \)-axis.

**Proof.** Using the functions introduced in Eq. (69), we can rewrite Eq. (58) as

\[
C = [(F - c_1) + i(G + c_2)](F' - iG'),
\]

where \( iC_0 = -c_1 + ic_2 \). Consequently,

\[
\begin{align*}
C - \bar{C} &= 2i[F'(G + c_2) - G'(F - c_1)] \\
C\bar{C} &= (F - c_1)^2 + (G + c_2)^2
\end{align*}
\]

Finally, inserting the relations above in Eq. (61) gives the expressions for \( x(s) \) (after integration of \( x' \)) and \( y(s) \) resulting in the expression for \( \gamma(s; H(s), c) \) in Eq. (68). Geometrically, the constants \( c_1, c_2, c_3 \) are related to the initial conditions, i.e., position and initial velocity of the generating curve at \( s = 0 \): \( \gamma(0) = (c_3, \sqrt{c_1^2 + c_2^2}, 0) \) and \( \gamma'(0) = (c_1^2 + c_2^2)^{-1/2}(c_1, c_2, 0) \).
Conversely, given a continuous function $H : I \to \mathbb{R}$ and $(c_1, c_2, c_3) \in T \times \mathbb{R}$, notice that $T$ is a non-empty open subset of $\mathbb{R}^2$, the curve $\gamma(s; H(s), c)$ is a unit speed curve of class $C^2$ that generates a $C^2$ surface of revolution around the (spacelike) $x$-axis with continuous mean curvature $H$. □

**Remark 2.5.** Notice that the curvature function of the curves $(f_i, g_i)$, $i \in \{1, 2\}$, and $(F, G)$ are precisely $\kappa = 2H$. Indeed, applying the expressions for the curvature function of a curve $c(s)$ in a Lorentzian $(+, -)$ and in an Euclidean $(+, +)$ plane, i.e., $\kappa = -\eta_c \|c''\|$ [10] and $\kappa = \|c''\|$, respectively, gives the desired result (here $\eta_c = \langle c', c' \rangle = \sinh^2(2\eta \int H) - \cosh^2(2\eta \int H) = -1$). A similar result is valid in $\mathbb{E}^3$ [25].

To finish this subsection, let us mention that in Euclidean space a theorem due to Delaunay asserts that surfaces of revolution with constant mean curvature are precisely the undulary, nodary, and catenary [13]. These surfaces are obtained by rotating roulettes of ellipses, hyperbolas, and parabolas, respectively [16]. Delaunay-type theorems were already established for surfaces of revolution with generating curves on a timelike plane [22, 24]. In the next theorem, we shall prove that the same is valid for the situation where the generating curve lies on a spacelike plane.

**Theorem 2.6 (Delaunay-type theorem).** Let $\gamma(s) = (x(s), y(s), 0)$ be the generating curve of a surface of revolution $S_\gamma$ with spacelike axis $Ox$. Then, the surface $S_\gamma$ has constant mean curvature $H$ if, and only if, $\gamma$ is the roulette of a conic in the $xy$-plane.

**Proof.** Since the $xy$-plane is spacelike, its conics and roulettes are the same as in the Euclidean plane. Finally, due to the fact that the mean curvature equations in Euclidean space, Eq. (1) of [25], and in Lorentz-Minkowski space [49] are the same, it follows that constant mean curvature surfaces of revolution with generating curve in a spacelike axis are obtained from the revolution of roulettes of a conic. □

2.5. **Rotation of a curve on a timelike plane around a lightlike axis.** Let $\lambda : I \to \mathbb{E}^3_1$ be a $C^2$ regular curve in the $yz$-plane, i.e., $\lambda(s) = (0, y(s), z(s))$ with $s$ arc-length and $y > z$. Considering a rotation of this curve along a lightlike axis given by $(0, 1, 1)$ results in the following surface of revolution

$$L(s, \theta) = \left( |y(s) - z(s)| \theta, y(s) - \frac{\theta^2}{2} [y(s) - z(s)], z(s) - \frac{\theta^2}{2} [y(s) - z(s)] \right),$$

where $\theta \in (-\infty, +\infty)$. Since $s$ is the arc-length parameter of $\lambda$, we can write

$$y'^2 - z'^2 = \eta \in \{-1, +1\}.$$

The first fundamental form is given by

$$I = \eta \, ds^2 + (y - z)^2 \, d\theta^2.$$
Since \( g = g_{11}g_{22} - g_{12}^2 = \eta(y - z)^2 \Rightarrow \epsilon = -\eta \), it follows that \( L \) is a spacelike (timelike) surface if and only if \( \lambda(s) \) is a spacelike (timelike) curve.

Writing the normal vector to \( L \) as

\[
N = \left( - (y' - z') \theta, z' + \frac{\theta^2}{2}(y' - z'), y' + \frac{\theta^2}{2}(y' - z') \right),
\]

the second fundamental form is given by

\[
\Pi = (y''z' - y'z'') \, ds^2 + (y - z)(y' - z') \, d\theta^2.
\]

Since both \( I \) and \( \Pi \) are diagonal, the shape operator \( A = I^{-1/2} \Pi \) is diagonalizable.

The mean curvature equation is then written as

\[
2(y - z)H + (y - z)(y''z' - y'z'') + \eta(y' - z)' = 0.
\]

Since \( \lambda(s) \) is parametrized by arc-length, we have the additional equation

\[
y'^2 - z'^2 = \eta \Rightarrow y'' = z'z''.
\]

Multiplying Eq. (77) by \((y' - z')\) gives

\[
2\eta (y - z)(y' - z')H + [(y - z)(y' - z')]' = 0.
\]

Solving the above equation gives \((y - z)(y' - z') = a_0 e^{2\eta \int H} \). Then, we have the relation

\[
(y - z)\,d(y - z) = a_0 e^{-2\eta \int H} \, ds
\]

and, consequently,

\[
y(s) - z(s) = \left\{a_1 + a_0 \int_0^s e^{-2\eta \int H(t) \, dt} \, du \right\}^{1/2},
\]

where \( a_0 \) and \( a_1 \) are constants to be specified at \( s = 0 \). On the other hand, multiplying Eq. (77) by \((y' + z')\) gives

\[
2(y - z)(y' + z')H - \eta(y - z)(y'' + z'') + 1 = 0 \Rightarrow 2\eta H \frac{y' + z'}{y - z} - \left( \frac{y' + z'}{y - z} \right)' = 0.
\]

The solution of the above equation gives \( y' + z' = (y - z)b_0 e^{2\eta \int H} \). Then, we have

\[
y(s) + z(s) = b_1 + b_0 \int_0^s a(u) \exp \left( 2\eta \int_0^u H(t) \, dt \right) \, du,
\]

where \( b_0, b_1 \) are constants and \( a(u) = y(u) - z(u) \), Eq. (80). Finally, from the knowledge of \( y + z \) and \( y - z \) we can find the expressions for \( y \) and \( z \):

\[
y(s) = \frac{(b_1 + a_1 + a_0 \int_0^s du e^{2\eta \int_0^u dt \, H} + b_0 \int_0^s du e^{2\eta \int_0^u dt \, H} \sqrt{a_1 + a_0 \int_0^u dt \, e^{2\eta \int_0^u dt \, H}})}{2};
\]

\[
z(s) = -\frac{(b_1 - a_1 + a_0 \int_0^s du e^{2\eta \int_0^u dt \, H} + b_0 \int_0^s du e^{2\eta \int_0^u dt \, H} \sqrt{a_1 + a_0 \int_0^u dt \, e^{2\eta \int_0^u dt \, H}})}{2}.
\]
3. Prescribed skew curvature equation in Lorentz-Minkowski space

We now address the problem of prescribed skew curvature for surfaces of revolution with a non-lightlike axis, as depicted in Figures 1(a), 1(b), and 1(c). Following da Silva et al. [11], the strategy consists in considering the generating curve as a graph and then write the equation for the skew curvature, which is initially a nonlinear second order ODE, as a linear first order ODE (with real coefficients). This approach can be seen as an adaptation of the techniques presented in [1] and [2] for helicoidal surfaces with prescribed mean/Gaussian curvature in Euclidean and Lorentz-Minkowski spaces, respectively. Unfortunately, we were not able to solve the prescribed problem for surfaces of revolution with a lightlike axis, Figure 1(d), with the present technique.

It is worth mentioning that in Ref. [11], the authors point to the fact that a curve which is a graph in the $xz$-plane, say $\alpha(u) = (u, 0, z(u))$, can be rotated around either the $x$- or $z$-axis. Nonetheless, these two possibilities lead to the same answer for the prescribed skew curvature problem in $\mathbb{E}^3$. Notice, however, that a priori these equivalent procedures do not make sense in $\mathbb{E}^3_1$, since distinct choices for the rotation axis lead to distinct types of surfaces (see Table 1). Instead, we should fix the axis and consider a curve as a graph in two ways (see subsections below).

3.1. Rotation of a curve on a timelike plane around a timelike axis. Let $\alpha : I \rightarrow \mathbb{E}^3_1$ be a $C^2$ regular curve in the $xz$-plane, i.e., $\alpha(u) = (x(u), 0, z(u))$ with $x > 0$. Considering a rotation of this curve around the timelike axis given by $(0, 0, 1)$ results in the following surface of revolution

\begin{equation}
Z(u, \theta) = (x(u) \cos \theta, x(u) \sin \theta, z(u)),
\end{equation}

where $\theta \in (0, 2\pi)$. The causal character of $\alpha$ can be denoted through

\begin{equation}
\eta = \text{sgn}(\langle \alpha', \alpha' \rangle) = \text{sgn}(x'^2 - z'^2) \in \{-1, 1\}.
\end{equation}

The first fundamental form $I$ is given by

\begin{equation}
I = (x'^2 - z'^2) \, du^2 + x^2 \, d\theta^2.
\end{equation}

Since $\langle Z_u \times Z_\theta, Z_u \times Z_\theta \rangle = -x^2(x'^2 - z'^2)$, we have $\epsilon = -\eta$ and the normal to $Z(u, \theta)$ is

\begin{equation}
N = -\frac{1}{\sqrt{\eta(x'^2 - z'^2)}} (z' \cos \theta, z' \sin \theta, x').
\end{equation}

The second fundamental form $II$ is given by

\begin{equation}
II = \frac{x'z'' - x''z'}{\sqrt{\eta(x'^2 - z'^2)}} \, du^2 + \frac{xz'}{\sqrt{\eta(x'^2 - z'^2)}} \, d\theta^2.
\end{equation}
3.1.1. Generating curve as a graph with $x$ as independent variable. Let $\alpha(u) = (u, 0, z(u))$ be a graph with the $x$-direction as the independent variable. Here, the mean and Gaussian curvatures are

$$H = -\frac{u z'' + z'(1 - z'^2)}{2u(\eta(1 - z'^2)^{3/2})} \quad \text{and} \quad K = -\frac{z' z''}{u(1 - z'^2)^2},$$

respectively. Defining

$$A = \frac{z'}{u \sqrt{\eta(1 - z'^2)}} \quad \text{and} \quad B = \frac{z'^2}{(1 - z'^2)},$$

the Gaussian and mean curvatures can be respectively written as the linear equations

$$A' + \frac{2}{u} A = -\eta \frac{2}{u} H \quad \text{and} \quad B' = -2uK.$$

In addition, observing that $B = \eta u^2 A^2$, we can write

$$S^2 = H^2 - \epsilon K = \left(A + \frac{u}{2} A'\right)^2 - (A^2 + uA') = \frac{u^2}{4} A'^2.$$

The function $A(u)$ can be written in terms of the skew curvature $S$ as

$$A(u) = \pm 2 \int \frac{S(v)}{v} \, dv + a_0,$$

where $a_0$ is a constant of integration. Now, using the expression for $A$ in Eq. (91), we find that

$$z'^2 = \frac{\eta u^2 A^2}{1 + \eta u^2 A^2} \Rightarrow z(u) = \pm \int \frac{vA(v)}{\sqrt{\eta + v^2 A^2(v)}} \, dv + z_0,$$

where $z_0$ is another constant of integration.

3.1.2. Generating curve as a graph with $z$ as independent variable. Let $\alpha(u) = (x(u), 0, u)$ be a graph with the $z$-direction as the independent variable. Here, the mean and Gaussian curvatures are

$$H = \frac{1 - x'^2 + xx''}{2x[\eta(x'^2 - 1)]^{3/2}} \quad \text{and} \quad K = \frac{x''}{x(x'^2 - 1)^2},$$

respectively. Defining

$$A(u) = \frac{1}{x \sqrt{\eta(x'^2 - 1)}},$$

we can write

$$\pm S = \frac{1 - x'^2 - xx''}{2x[\eta(x'^2 - 1)]^{3/2}} = \eta \frac{x}{2x'} \frac{dA}{du}.$$

Then, we have the following ODE for $A$

$$x \frac{dA}{du} \pm 2 \eta S \frac{dx}{du} = \left(x \frac{dA}{dx} \mp 2 \eta S\right) \frac{dx}{du} = 0.$$
If \(x'(u) \equiv 0\) we have a cylinder. Otherwise, we are led to

\[
\frac{dA}{dx} \mp 2\eta S = 0 \Rightarrow A = \pm 2\eta \int \frac{S(v)}{v} dv + a_1.
\]

Now, using the definition of \(A\) above, we finally find that

\[
\frac{dx}{du} = \pm \sqrt{\frac{\eta + x^2A^2}{x^2A^2}} \Rightarrow u(x) = \pm \int \frac{vA(v)}{\sqrt{\eta + v^2A(v)^2}} dv + u_0.
\]

Observe that the equation above is identical to Eq. (95), but instead of finding \(x(u)\) as a function of \(u\) we found its inverse. This shows that a graph of a solution \(f(u)\) of Eq. (95) gives rise to a surface of revolution with prescribed \(S\) with either \(x\)- or the \(z\)-axis as the independent variable direction, i.e., we can choose either \(\alpha(u) = (u, 0, f(u))\) or \(\alpha(u) = (f(u), 0, u)\) to rotate around the timelike \(z\)-axis. The only difference between these two choices lies in the causal character of \(\alpha\).

### 3.2. Rotation of a curve on a timelike plane around a spacelike axis

Let \(\beta : I \rightarrow \mathbb{E}^3_1\) be a \(C^2\) regular curve in the \(xz\)-plane, i.e., \(\beta(u) = (x(u), 0, z(u))\) with \(z > 0\). Considering a rotation of this curve around the spacelike axis given by \((1, 0, 0)\) results in the following surface of revolution

\[
X_I(u, \theta) = (x(u), z(u) \sinh \theta, z(u) \cosh \theta),
\]

where \(\theta \in (-\infty, \infty)\). The causal character of \(\beta\) can be described through

\[
\eta = \text{sgn}(\langle \beta', \beta' \rangle) = \text{sgn}(x'^2 - z'^2) \in \{-1, 1\}.
\]

The first fundamental form \(I\) is given by

\[
I = (x'^2 - z'^2) \, du^2 + z^2 \, d\theta^2.
\]

Since \(\langle \partial_u X_I \times \partial_\theta X_I, \partial_u X_I \times \partial_\theta X_I \rangle = -z^2(x'^2 - z'^2)\), we have \(\epsilon = -\eta\) and the normal to \(X_I(u, \theta)\) is

\[
N = -\frac{1}{\sqrt{\eta(x'^2 - z'^2)}}(z', x' \sinh \theta, x' \cosh \theta).
\]

The second fundamental form \(II\) is given by

\[
II = \frac{x'z'' - x''z'}{\sqrt{\eta(x'^2 - z'^2)}} \, du^2 + \frac{x'z}{\sqrt{\eta(x'^2 - z'^2)}} \, d\theta^2.
\]

#### 3.2.1. Generating curve as a graph with \(z\) as independent variable

Let \(\beta(u) = (x(u), 0, u)\) be a graph with the \(z\)-direction as the independent variable. Here, the mean and Gaussian curvatures are

\[
H = \frac{ux'' - x'(x'^2 - 1)}{2u[\eta(x'^2 - 1)]^{3/2}} \quad \text{and} \quad K = \frac{x'x''}{u(x'^2 - 1)^2},
\]
respectively. Defining
\begin{equation}
A = \frac{x'}{u \sqrt{\eta (x'^2 - 1)}} \quad \text{and} \quad B = \frac{x'^2}{(x'^2 - 1)},
\end{equation}
the Gaussian and mean curvatures can be respectively written as the linear equations
\begin{equation}
A' + 2 \frac{u}{u} A = -\eta \frac{2}{u} H \quad \text{and} \quad B' = -2uK.
\end{equation}

Observing that the equations above are analogous to those of \( Z \), Eqs. (91) and (92), we have
\begin{equation}
A(u) = \pm 2 \int \frac{S(v)}{v} \, dv + a_0,
\end{equation}
where \( a_0 \) is a constant of integration. Now, using the expression for \( A \) in Eq. (108), we find that
\begin{equation}
x'^2 = -\frac{\eta u^2 A^2}{1 - \eta u^2 A^2} \quad \Rightarrow \quad x(u) = \pm \int \frac{v A(v)}{\sqrt{-\eta + v^2 A^2(v)}} \, dv + x_0,
\end{equation}
where \( x_0 \) is another constant of integration.

### 3.2.2. Generating curve as a graph with \( x \) as independent variable.

Let \( \beta(u) = (u, 0, z(u)) \) be a graph with the \( x \)-direction as the independent variable. Here, the mean and Gaussian curvatures are
\begin{equation}
H = -\frac{1 - z'^2 + zz''}{2z[\eta(1 - z'^2)]^{3/2}} \quad \text{and} \quad K = -\frac{z''}{z(1 - z'^2)^2},
\end{equation}
respectively. Defining
\begin{equation}
A(u) = \frac{1}{z \sqrt{\eta (1 - z'^2)}},
\end{equation}
we can write
\begin{equation}
\pm S = \frac{1 - z'^2 - zz''}{2z[\eta(1 - z'^2)]^{3/2}} = -\eta \frac{z}{2z'} \frac{dA}{du}.
\end{equation}
Then, we have the following ODE for \( A \)
\begin{equation}
z \frac{dA}{du} \pm 2\eta S \frac{dz}{du} = \left( z \frac{dA}{dz} \pm 2\eta S \right) \frac{dz}{du} = 0.
\end{equation}
If \( z'(u) \equiv 0 \) we have a cylinder. Otherwise, we are led to
\begin{equation}
z \frac{dA}{dz} \pm 2\eta S = 0 \quad \Rightarrow \quad A = \pm 2\eta \int \frac{S(v)}{v} \, dv + a_1.
\end{equation}

Now, using the definition of \( A \) above, we finally find that
\begin{equation}
\frac{dz}{du} = \pm \sqrt{-\eta + z^2 A^2} \quad \Rightarrow \quad u(z) = \pm \int \frac{v A(v)}{\sqrt{-\eta + v^2 A(v)^2}} \, dv + u_0.
\end{equation}
Observe that the equation above is identical to Eq. (111), but instead of finding \( z(u) \) as a function of \( u \) we found its inverse. This shows that a graph of a solution \( f(u) \) of
Eq. (111) gives rise to a surface of revolution with prescribed $S$ with either $x$- or the $z$-axis as the independent variable direction, i.e., we can choose either $\beta(u) = (u, 0, f(u))$ or $\beta(u) = (f(u), 0, u)$ to rotate around the spacelike $x$-axis. The only difference between these two choices lies in the causal character of $\beta$.

3.3. Rotation of a curve on a spacelike plane around a spacelike axis. Let $\gamma : I \to \mathbb{E}^3$ be a $C^2$ regular curve in the $xy$-plane, i.e., $\gamma(u) = (x(u), y(u), 0)$ with $y > 0$. Considering a rotation of this curve around the (spacelike) $x$-axis results in the following surface of revolution

$$X_{II}(u, \theta) = (x(u), y(u) \cosh \theta, y(u) \sinh \theta),$$

where $\theta \in (-\infty, \infty)$. The curve $\gamma$ is always spacelike, since

$$\langle \gamma', \gamma' \rangle = (x''(u)^2 + y''(u)^2) > 0.$$

The first fundamental form $I$ is given by

$$I = (x''(u)^2 + y''(u)^2) du^2 - y'^2 d\theta^2,$$

the normal to $X_{II}(u, \theta)$ is

$$N = \frac{1}{\sqrt{x'^2 + y'^2}} (y', -x' \cosh \theta, -x' \sinh \theta),$$

and the second fundamental form $II$ is

$$II = \frac{x''y' - x'y''}{\sqrt{x'^2 + y'^2}} du^2 - \frac{x'y}{\sqrt{x'^2 + y'^2}} d\theta^2.$$

3.3.1. Generating curve as a graph with $y$ as independent variable. Let $\gamma(u) = (x(u), u, 0)$ be a graph with the $y$-direction as the independent variable. Here, the mean and Gaussian curvatures are

$$H = \frac{ux'' + x'(1 + x'^2)}{2u(1 + x'^2)^{3/2}} \text{ and } K = \frac{x'x''}{u(1 + x'^2)^2},$$

respectively. Defining

$$A = \frac{x'}{u\sqrt{(1 + x'^2)}} \text{ and } B = \frac{x'^2}{(1 + x'^2)},$$

the Gaussian and mean curvatures can be respectively written as the linear equations

$$A' + \frac{2}{u} A = \frac{2}{u} H \text{ and } B' = 2uK.$$

In addition, observing that $B = u^2 A'$, we can write

$$S^2 = H^2 - K = \left(A + \frac{u}{2} A'\right)^2 - (A^2 + uAA') = \frac{u^2}{4} A'^2.$$

The function $A(u)$ can be written in terms of the skew curvature $S$ as

$$A(u) = \pm 2 \int \frac{S(v)}{v} dv + a_0,$$
where \( a_0 \) is a constant of integration. Now, using the expression for \( A \) in Eq. (124), we find that

\[
x'{}^2 = \frac{u^2A^2}{1 - u^2A^2} \Rightarrow x(u) = \pm \int \frac{vA(v)}{\sqrt{1 - v^2A^2(v)}} \, dv + x_0,
\]

where \( x_0 \) is another constant of integration.

### 3.3.2. Generating curve as a graph with \( x \) as independent variable.

Let \( \gamma(u) = (u, y(u), 0) \) be a graph with the \( x \)-direction as the independent variable. Here, the mean and Gaussian curvatures are

\[
H = \frac{1 + y'{}^2 - yy''}{2y(1 + y'{}^2)^{3/2}} \quad \text{and} \quad K = -\frac{y''}{y(1 + y'{}^2)^2},
\]

respectively. Defining

\[
A(u) = \frac{1}{y\sqrt{1 + y'{}^2}},
\]

we can write

\[
\pm S = \frac{1 + y'{}^2 + yy''}{2y(1 + y'{}^2)^{3/2}} = -\frac{y}{2y'} \frac{dA}{du}.
\]

Then, we have the following ODE for \( A \)

\[
y \frac{dA}{du} \pm 2S \frac{dy}{du} = \left( y \frac{dA}{dy} \pm 2S \right) \frac{dy}{du} = 0.
\]

If \( y'(u) \equiv 0 \) we have a cylinder. Otherwise, we are led to

\[
y \frac{dA}{dy} \pm 2S = 0 \Rightarrow A(u) = \pm 2 \int \frac{S(v)}{v} \, dv + a_1.
\]

Now, using the definition of \( A \) above, we finally find that

\[
\frac{dy}{du} = \pm \sqrt{\frac{1 - y^2A^2}{y^2A^2}} \Rightarrow u(y) = \pm \int \frac{vA(v)}{\sqrt{1 - v^2A^2(v)}} \, dv + u_0.
\]

Observe that the equation above is identical to Eq. (128), but instead of finding \( y(u) \) as a function of \( u \) we found its inverse. This shows that a graph of a solution \( f(u) \) of Eq. (128) gives rise to a surface of revolution with prescribed \( S \) with either \( x \)- or the \( y \)-axis as the independent variable direction, i.e., we can choose either \( \gamma(u) = (u, f(u), 0) \) or \( \gamma(u) = (f(u), u, 0) \) to rotate around the spacelike \( x \)-axis. Notice that the causal character of \( \gamma \), and the respective surface of revolution, does not depend on this choice.
## Appendix A. Lorentz numbers

The ring of Lorentz numbers $\mathbb{L}$, often named double or hyperbolic numbers \[33\], is $\mathbb{L} = \{a + b\tau : a, b \in \mathbb{R}, \tau \notin \mathbb{R}, \text{ and } \tau^2 = 1\}$, which is isomorphic to $\mathbb{R}[X]/(X^2 - 1)$, where $\mathbb{R}[X]$ is the ring of real polynomials. The sum and product in the (commutative) ring $\mathbb{L}$ is defined as usual: $(a + b\tau) + (a + b\tau) = (a + a) + (b + b)\tau$ and $(a + b\tau)(a + b\tau) = (aa + bb\beta) + (ab + ba)\tau$. Moreover, $\mathbb{L}$ does not form a field, since $(a \pm a\tau)^2 = 0$ for all $a$.

Consequently, if $a^2 - b^2 \neq 0$, then $w^{-1} = \bar{w}/(ww)$, where $ww = a^2 - b^2 \in \mathbb{R}$. A Lorentz number $w$ is space-, time-, or light-like if $ww > 0$ or $w = 0$, $ww < 0$, or $ww = 0$, respectively. The lightlike numbers are precisely the zero divisors of $\mathbb{L}$ and are of the form $a \pm a\tau$. The set of invertible Lorentz numbers is $\mathbb{L}^* = \{w \in \mathbb{L} : \exists w^{-1}\} = \mathbb{L} - \{a \pm a\tau\}$.

In addition, the Lorentz numbers admit the linear representation

\[
(135) \quad a + b\tau \mapsto \begin{pmatrix} a & b \\ b & a \end{pmatrix},
\]

from which we can define a polar form of a Lorentz number $w \in \mathbb{L}^*$ to be

\[
(136) \quad a + b\tau = \begin{cases} r(\cosh \theta + \tau \sinh \theta), & \text{if } a^2 - b^2 > 0 \\ r(\sinh \theta + \tau \cosh \theta), & \text{if } a^2 - b^2 < 0 \end{cases},
\]

where $r = \sqrt{|ww|} = \sqrt{|a^2 - b^2|}$ is the length of $w$. We also define an exponential function

\[
(137) \quad \exp(a + b\tau) = e^{a+b\tau} = e^a \exp(\cosh b + \tau \sinh b).
\]

Given a function $f(s) = a(s) + b(s)\tau$, where $a, b$ are differentiable real functions of a real variable $s$, it is easy to verify using the expressions above that

\[
(138) \quad \frac{d}{ds} e^{f(s)} = [a'(s) + b'(s)\tau] e^{f(s)} = f'(s) e^{f(s)}.
\]

**Example A.1.** The solution of the linear ODE $dw/dt + g(s)\tau w(s) = 0$ with initial condition $w(s_0) = w_0$, where $s$ and $g(s)$ are real, is given by

\[
(139) \quad w(s) = w_0 \exp\left(\tau \int_{s_0}^{s} g(u)du\right).
\]

Notice that this ODE is equivalent to the system

\[
(140) \quad \begin{cases} x' + g(s)y = 0 \\ y' + g(s)x = 0 \end{cases}, \quad x(s_0) = \text{Re}(w_0), \quad y(s_0) = \text{Im}(w_0).
\]

**Remark A.2.** It is possible to define a notion of differentiability for functions $f : \mathbb{L} \to \mathbb{L}$ as done, e.g., in Ref. \[14\]. (In fact, it is possible to introduce a notion of differentiability for functions over any algebra, see e.g. \[18\].) However, the few concepts and formalism introduced in this Appendix suffice for our purposes.
REFERENCES

[1] C. Baikoussis and T. Koufogiorgos, Helicoidal surfaces with prescribed mean or Gaussian curvature, J. Geom. 63 (1998) 25–29.
[2] C. C. Beneki, G. Kaimakamis and B. J. Papantoniou, Helicoidal surfaces in three-dimensional Minkowski space, J. Math. Anal. Appl. 275 (2002) 586–614.
[3] R. Caddeo, P. Piu, and A. Ratto, SO(2)-invariant minimal and constant mean curvature surfaces in three dimensional homogeneous spaces, Manuscripta Math. 87 (1995) 1–12.
[4] R. Caddeo, P. Piu, and A. Ratto, Rotation surfaces in $H_3$ with constant Gauss curvature, Boll. Unione Mat. Ital. 7 (1996) 341–357.
[5] F. Catoni and P. Zampetti, Cauchy-like integral formula for functions of a hyperbolic variable, Adv. Appl. Clifford Algebras 22 (2012) 23–37.
[6] B.-Y. Chen, On the difference curvature of surfaces in Euclidean space, Math. J. Okayama Univ. 14 (1969) 153–157.
[7] B.-Y. Chen, Mean curvature and shape operator of isometric immersions in real-space-forms, Glasgow Math. J. 38 (1996) 87–97.
[8] J. N. Clelland, From Frenet to Cartan: the method of moving frames, American Mathematical Society, Providence, 2017.
[9] R. C. T. Da Costa, Quantum mechanics of a constrained particle, Phys. Rev. A. 23 (1981) 1982.
[10] L. C. B. Da Silva, Moving frames and the characterization of curves that lie on a surface, J. Geom. 108 (2017) 1091–1113.
[11] L. C. B. Da Silva, C. C. Bastos, and F. G. Ribeiro, Quantum mechanics of a constrained particle and the problem of prescribed geometry-induced potential, Ann. Phys. (New York) 379 (2017) 13–33.
[12] L. C. B. Da Silva, The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces, J. Geom. 110 (2019) 31.
[13] C. Delaunay, Sur la surface de révolution dont la courbure moyenne est constante, J. Math. Pure. Appl. Sér. 1 6 (1841) 309–320. With a note appended by M. Sturm.
[14] L. Di Terlizzi, J. J. Konderak and I. Lacirasella, On differentiable functions over Lorentz numbers and their geometric applications, Differ. Geom. Dyn. Syst. 16 (2014) 113–139.
[15] M. P. Do Carmo and M. Dajczer, Helicoidal surfaces with constant mean curvature, Tohoku Math. J. 34 (1982) 425–435.
[16] J. Eells, The surfaces of Delaunay, The Mathematical Intelligencer 9 (1987) 53–57.
[17] A. Fujioka and J. Inoguchi, Timelike surfaces with harmonic inverse mean curvature. In: Surveys on Geometry and Integrable Systems, pp. 113–141, Mathematical Society of Japan, Tokyo, 2008.
[18] P. M. Gadea and J. Muñoz Masqué, $A$-differentiability and $A$-analyticity, Proc. Amer. Math. Soc. 124 (1996) 1437–1443.
[19] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin Heidelberg, 1977.
[20] C. E. Gutierrez, The Monge-Ampère equation, Birkhäuser, Basel, 2001.
[21] J.-I. Hano and K. Nomizu, On isometric immersions of the hyperbolic plane into the Lorentz-Minkowski space and the Monge-Ampère equation of a certain type, Math. Ann. 262 (1983) 245–253.
[22] J.-I. Hano and K. Nomizu, Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space, Tohoku Math. J. 36 (1984), 427–437.
[23] J.-I. Inoguchi and S. Lee, Lightlike surfaces in Minkowski 3-space, Int. J. Geom. Meth. Mod. Phys. 6 (2009), 267–283.
[24] T. Ishihara and F. Hara, Surfaces of revolution in the Lorentzian 3-space, J. Math. Tokushima Univ. 22 (1988), 1–13.
[25] K. Kenmotsu, Surfaces of revolution with prescribed mean curvature, Tohoku Math. J. 16 (1981), 161–177.
[26] F. Li and Z. Guo, Surfaces with closed Möbius form, Differ. Geom. Appl. 39 (2015), 20–35.
[27] R. López, Differential geometry of curves and surfaces in Lorentz-Minkowski space, Int. Electron. J. Geom. 7 (2014), 44–107.
[28] T. K. Milnor, Restrictions on the curvatures of \( \phi \)-bounded surfaces, J. Differ. Geom. 11 (1976), 31–46.
[29] T. K. Milnor, The curvatures of some skew fundamental forms, Proc. Amer. Math. Soc. 62 (1977), 323–329.
[30] S. Montaldo and I. Onnis, Invariant surfaces of a three-dimensional manifold with constant Gauss curvature, J. Geom. Phys. 55 (2005) 440–449.
[31] G. A. C. Smith, Global singularity theory for the Gauss curvature equation, Ensaios Matemáticos 28 (2015) 1–114.
[32] M. Toda and A. Pigazzini, A note on the class of surfaces with constant skew curvature, J. Geom. Symmetry Phys. 46 (2017) 51–58.
[33] T. J. Willmore, A survey on Willmore immersions. In: Geometry and Topology of Submanifolds IV, Leuven, pp. 11–16, 1991.
[34] I. M. Yaglom, A simple non-Euclidean geometry and its physical basis, Springer-Verlag, New York, 1979.
[35] D. W. Yoon, D.-S. Kim, Y. H. Kim, and J. W. Lee, Helicoidal surfaces with prescribed curvatures in \( \text{Nil}_3 \), Int. J. Math. 24 (2013) 1350107.

Departamento de Matemática
Universidade Federal de Pernambuco
Recife, PE 50670-901, Brazil.

Current address: Department of Physics of Complex Systems
Weizmann Institute of Science
Rehovot 7610001, Israel.

Email address: luiz.da-silva@weizmann.ac.il