On \((2k)\)-Minimal Submanifolds

M.-L. Labbi

Abstract

Recall that a submanifold of a Riemannian manifold is said to be minimal if its mean curvature is zero. It is classical that minimal submanifolds are the critical points of the volume function.

In this paper, we examine the critical points of the total \((2k)\)-th Gauss-Bonnet curvature function, called \((2k)\)-minimal submanifolds. We prove that they are characterized by the vanishing of a higher mean curvature, namely the \((2k+1)\)-Gauss-Bonnet curvature.

Furthermore, we show that several properties of usual minimal submanifolds can be naturally generalized to \((2k)\)-minimal submanifolds.

Mathematics Subject Classification (2000): 53C40, 53C42.

Keywords. Generalized minimal submanifolds, generalized Laplacian, Gauss-Bonnet curvatures.

1 Introduction

Recall that a submanifold \(M\) of a Riemannian manifold \((\tilde{M}, \tilde{g})\) is said to be minimal if its mean curvature vanishes everywhere. It is classical that minimal submanifolds are the critical points of the volume function.

In this paper, we consider the critical points of the total \((2k)\)-th Gauss-Bonnet curvature function, called \((2k)\)-minimal submanifolds.

These generalize ordinary minimal submanifolds obtained for \(k = 0\) and Reilly’s \(r\)-minimal hypersurfaces of the Euclidean space when \(r = 2k\), [4].

We prove that they are characterized by the vanishing of a higher mean curvature, namely the \((2k+1)\)-Gauss-Bonnet curvature. This result generalizes a similar result of Reilly obtained for submanifolds of the Euclidean space [5].

The paper is divided into two parts. In the first part, we first recall useful facts about some operations on double forms, namely the exterior product,
generalized Hodge star operator, contraction map and the inner product. These tools are used in this paper to provide first an alternative elegant approach to symmetric functions and Newton transformations. And secondly, to provide a natural introduction to the $k$–th Gauss-Bonnet curvatures (for $k$ even or odd) and the Einstein-Lovelock tensors.

In the second part of this paper, we first prove the first variation formula for the total $(2k)$-th Gauss-Bonnet curvature function in order to characterize the critical points. Next, we prove several facts about $(2k)$-minimal submanifolds which generalize similar properties about ordinary minimal submanifolds. In particular, we prove that complex submanifolds of a Kahlerian manifold are always $(2k)$-minimal for all $k$. Also, we show that compact, irreducible isotropy homogeneous spaces always admit $(2k)$-minimal immersions in a sphere.

The natural extension of Laplace operator that naturally appears in our context is the operator $\ell_{2k}$. Roughly speaking, it is obtained by "contracting" the Hessian by the $(2k)$-th Einstein-Lovelock tensor, (recall that the usual Laplacian is just the contraction of the Hessian by the metric under consideration).

We prove that for a compact manifold these generalized Laplacians are self adjoint and with zero integral, in fact they can be written as a divergence. Furthermore, if the metric on the manifold has positive (resp. negative) definite $(2k)$-Einstein-Lovelock tensor then the operator $\ell_{2k}$ is elliptic and positive (resp. negative) definite. In particular, we obtain a maximum principle for these operators.

Finally, we study some properties of $(2k)$-minimal immersions in Euclidean space and spheres. For example, we prove that an isometric immersion $F: M \to \mathbb{R}^{n+p}$ is $(2k)$-minimal if and only if the coordinates functions $F_i$ of $F$ are $\ell_{2k}$-harmonic functions. In particular, we prove that there are no non trivial compact $(2k)$-minimal submanifolds in the Euclidean space with positive definite (or negative definite) $(2k)$-th Einstein-Lovelock tensor.

2 Elementary symmetric functions vs. Gauss-Bonnet curvatures

2.1 Double Forms: Algebraic properties

Let $(V, g)$ be an Euclidean real vector space of dimension $n$. In the following we shall identify whenever convenient (via their Euclidean structures), the
vector spaces with their duals. Let $\Lambda V = \bigoplus_{p \geq 0} \Lambda^p V$ denotes the exterior algebra of $p$-vectors on $V$.

A double form on $V$ of degree $(p, q)$ is defined to be a bilinear form $\Lambda^p V \times \Lambda^q V \to \mathbb{R}$. Alternatively, it is a multilinear form defined on $V$ which is skew symmetric in the first $p$-arguments and also in the last $q$-arguments. If $p = q$ and the bilinear form is symmetric we say that we have a symmetric double form.

The usual exterior product of $p$-vectors extends in a natural way to double forms of any degree [2]. In particular, the exterior product of two ordinary bilinear forms coincides with the Kulkarni-Nomizu product. Furthermore, $k$-times the exterior product of a symmetric bilinear form $B$ with itself is a symmetric double form of order $(k, k)$ and is given by

$$B^k(x_1 \wedge \ldots \wedge x_k, y_1 \wedge \ldots \wedge y_k) = k! \det[B(x_i, y_j)].$$

In particular, for $B = g$, $g^k$ is the canonical inner product on $\Lambda^k V$. The former inner product induces a natural inner product of double forms and shall be denoted by $\langle \cdot, \cdot \rangle$. The contraction map $c$ on double forms is the adjoint of the exterior multiplication map by the metric $g$.

Suppose we have chosen an orientation on the vector space $V$. The classical Hodge star operator $\ast : \Lambda^p V \to \Lambda^{n-p} V$ can be extended naturally to operate on double forms by declaring for a $(p, q)$-double form the following:

$$\ast \omega(\ast \cdot, \ast \cdot) = (-1)^{(p+q)(n-p-q)} \omega(\ast \cdot, \ast \cdot).$$

Note that $\ast \omega$ does not depend on the chosen orientation as the usual Hodge star operator is applied twice. The so-obtained operator provides a simple relation between the contraction map $c$ of double forms and the multiplication map by the metric:

$$g \omega = \ast c \ast \omega \text{ and } c \omega = \ast g \ast \omega. \quad (1)$$

Furthermore, we have the following properties for all $\omega, \theta \in D^{p,q}$:

$$\langle \omega, \theta \rangle = \ast(\omega, \ast \theta) = (-1)^{(p+q)(n-p-q)} \ast (\ast \omega, \theta), \quad (2)$$

$$\ast \ast \omega = (-1)^{(p+q)(n-p-q)} \omega. \quad (3)$$

Finally, if $\omega$ is a symmetric $(p,p)$-double form satisfying the first Bianchi identity then we have [2]

$$\ast \left( \frac{g^{n-p} \omega}{(n-p)!} \right) = \frac{1}{p!} c^p \omega \text{ and } \ast \left( \frac{g^{n-p-1} \omega}{(n-p-1)!} \right) = \frac{c^p \omega}{p!} g - \frac{c^{p-1} \omega}{(p-1)!}. \quad (4)$$
2.2 Elementary Symmetric Functions

Let \((V, g)\) be an Euclidean space of dimension \(n\), \(B\) a given symmetric bilinear form on \(V\). We denote by \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\) the eigenvalues of the operator corresponding to \(B\) via \(g\). Let \(s_k = s_k(\lambda_1, ..., \lambda_n)\) be the elementary symmetric functions for \(k = 0, 1, ..., n\), where \(s_0 = 1, s_1 = \sum_{i=1}^{n} \lambda_i, ..., s_n = \lambda_1...\lambda_n\).

The previous operations on double forms provide an alternative nice way to write these invariants as follows.

**Proposition 2.1** Let \((V, g)\) be an Euclidean space, \(B\) a given symmetric bilinear form. If \(s_k\) denotes the \(k\)-th elementary symmetric function in the eigenvalues of the operator corresponding to \(B\) and \(c, \ast, B^k\) denote respectively the contraction map, the generalized Hodge star operator and the exterior product of \(B\) with itself \(k\)-times then

\[
s_k = \frac{1}{(k!)^2} c^k B^k = \frac{1}{k!(n-k)!} \ast (g^{n-k} B^k).
\]

In particular, the trace and determinant of the operator associated to \(B\) via \(g\) are given by

\[
s_1 = tr_g B = \ast \left\{ \frac{g^{n-1}}{(n - 1)!} B \right\} \quad \text{and} \quad s_n = det_g B = \ast B^n \frac{n!}{n!}.
\]

**Proof.** It is not difficult to see that the eigenvalues of \(\frac{B^k}{k!}\) are all possible products \(\lambda_{i_1}\lambda_{i_2}...\lambda_{i_k}\) with \(i_1 < i_2 < \ldots < i_k\). From this it is clear that its complete contraction determines \(s_k\) as in the proposition. The second statement is a direct application of formula (4) above.

**Corollary 2.2** Let \(A, B\) be symmetric bilinear forms, denote by \(s_i(A), s_j(B)\) the elementary symmetric functions in the eigenvalues of the operator corresponding to \(A\) and \(B\) respectively via the scalar product \(g\). If \(A = B + \lambda g\) for some \(\lambda \in \mathbb{R}\), then for each \(k, 0 \leq k \leq n\), we have

\[
s_k(A) = \sum_{i=0}^{k} \frac{k!(n-i)!}{i!(k-i)!(n-k)!} s_i(B) \lambda^{k-i}.
\]

**Proof.** Straightforward, just use the binomial theorem and the previous proposition.
2.3 Newton Transformations

Associated with the elementary symmetric functions are the so-called Newton transformation \([4]\). We reformulate below their definition in terms of the operations introduced above of double forms:

**Definition 2.3** For \(0 \leq k \leq n\), the \(k\)-th Newton transformation of a bilinear form \(B\) on \((V, g)\) is defined to be

\[
t_k(B) = * \left\{ \frac{g^{n-k-1} B^k}{(n-k-1)! \ k!} \right\}.
\]

For \(k = n\), we set \(t_n(B) = 0\).

The following properties of \(t_k\) are known \([4]\):

**Proposition 2.4** For each \(k\), \(0 \leq k \leq n\), we have for \(B\) and \(t_k(B)\) as above

1. \(\langle t_k(B), B \rangle = (k + 1)s_{k+1}(B)\). This property is equivalent to the celebrated Newton’s formula.
2. \(t_k(B) = s_k(B)g - \frac{c_{k-1} B^k}{(k-1)!}\).
3. \(ct_k(B) = (n - k)s_k(B)\).

It is only for the seek to illustrate the elegance of this new approach to symmetric functions and Newton transformations that we are proving the previous proposition below.

**Proof.** To prove the first part, we need just to use formulas \([2]\) and \([3]\) as follows:

\[
\langle t_k(B), B \rangle = *\{ *t_k(B) \}B = * \left\{ \frac{g^{n-k-1} B^{k+1}}{(n-k-1)! \ k!} \right\} = (k + 1)s_{k+1}(B).
\]

The second part results directly from formula \([4]\).

Finally, the third part results from formulas \([1]\) and \([3]\) as follows:

\[
ct_k(B) = g * t_k(B) = * \left\{ \frac{g^{n-k} B^k}{(n-k-1)! \ k!} \right\} = (n - k)s_k(B).
\]

\[\blacksquare\]
2.4 Gauss-Bonnet curvatures and Einstein-Lovelock tensors

Let $(M, g)$ be a hypersurface of the $(n + 1)$-dimensional Euclidean space. The Gauss equation relates the second fundamental form $B$ of $M$ to its Riemann curvature tensor. Precisely, it states that $R = 1/2B^2$, where of course the product in $B^2 = BB$ is the exterior product of double forms. In particular, $B^{2k} = 2^k R^k$ and therefore the even order symmetric functions in the eigenvalues of $B$ and the corresponding Newton transformations are intrinsic invariants of the geometry of the hypersurface and are respectively given by

$$s_{2k} = \frac{2^k}{[(2k)!]^2} 2^k R^k = \ast \left\{ 2^k \frac{g^{n-2k}}{(n-2k)!} \frac{R^k}{(2k)!} \right\}.$$  \hspace{1cm} (5)

$$t_{2k} = \ast \left\{ 2^k \frac{g^{n-2k-1}}{(n-2k-1)!} \frac{R^k}{(2k-1)!} \right\}.$$  \hspace{1cm} (6)

The even order symmetric functions in the eigenvalues of $B$ and the corresponding Newton transformations are no longer intrinsic for hypersurfaces of arbitrary Riemannian manifolds. Instead of that, we consider the following natural intrinsic generalization of these curvatures:

**Definition 2.5** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $k$ be a positive integer such that $0 \leq 2k \leq n$.

1. The $(2k)$-th Gauss-Bonnet curvature, denoted $h_{2k}$, is the function defined on $M$ by

$$h_{2k} = \frac{1}{(n-2k)!} \ast \left( g^{n-2k} R^k \right).$$  \hspace{1cm} (5)

2. The $(2k)$-th Einstein-Lovelock tensor, denoted $T_{2k}$, is defined by

$$T_{2k} = \ast \left( \frac{1}{(n-2k-1)!} g^{n-2k-1} R^k \right).$$  \hspace{1cm} (6)

If $2k = n$, we set $T_n = 0$. For $k = 0$ we have $h_0 = 1$ and $T_0 = g$.

Using formula (4) above, these invariants can alternatively be written as

$$h_{2k} = \frac{\epsilon_{2k} R^k}{(2k)!} \text{ and } T_{2k} = h_{2k} g - \frac{\epsilon_{2k-1} R^k}{(2k-1)!}.$$  \hspace{1cm} (7)

Note that $h_2$ is the half of the usual scalar curvature and $T_2$ is the usual Einstein tensor. Recall that if $n$ is even then $h_n$ is up to a constant the Gauss-Bonnet integrand of $(M, g)$. 

6
An important property of these invariants is that the Einstein-Lovelock tensor $T_{2k}$ is the gradient of the total $(2k)$-th Gauss-Bonnet curvature seen as a functional on the space of Riemannian metrics on $M$, see [3].

In the special case of a hypersurface of a space form with constant $c$, the invariants $h_{2k}$ are related to the symmetric functions $s_{2i}$ of the eigenvalues of the second fundamental form (which are intrinsic in this special case), as follows:

**Proposition 2.6** In a space form of curvature $c$ we have

$$h_{2k} = \frac{1}{2^k(n-2k)!} \sum_{i=0}^{k} \frac{k!(n+2i-2k)!}{i!(k-i)!} s_{2k-2i} c^i,$$

and

$$s_{2k} = \frac{k!}{(n-2k)!} \sum_{i=0}^{k} (-1)^{k-i} \frac{2^i(n-2i)!}{i!(k-i)!} h_{2i} c^{k-i}.$$

**Proof.** For a hypersurface of a space form with constant $c$, the Gauss equation asserts that the Riemann curvature tensor of the hypersurface is determined from the second fundamental form $B$ by $R = c g^2 + B^2$. Inserting this in the formulas defining $h_{2k}$ and $s_{2k}$ we get the desired results.

Similar formulas hold for $T_{2k}$ and $t_{2k}$.

### 3 Generalized minimal submanifolds

Let $(\tilde{M}, \tilde{g})$ be an $(n+p)$-dimensional Riemannian manifold, and let $M$ be an $n$-dimensional submanifold of $\tilde{M}$. We shall denote by $g$ the induced metric on $M$. The purpose of this section is to characterize those submanifolds (endowed with the induced metric) that are critical points of the total Gauss-Bonnet curvature function.

#### 3.1 Gauss-Bonnet curvatures of odd order

Recall that the Gauss-Bonnet curvatures $h_{2k}$ of $(M,g)$ are intrinsic invariants and are defined by [5]. We extend the definition of these curvatures to cover odd orders as follows:
Definition 3.1 For a normal vector $N$ at a point $m \in M$ and for $n \geq 2k+1$, we define the $(2k+1)$ Gauss-Bonnet curvature of the submanifold $(M, g)$ by

$$h_{2k+1}(N) = *\left(\frac{g^{n-2k-1}}{(n-2k-1)!} R^k B_N\right).$$ (8)

For $n = 2k$, set $h_{2k+1}(N) = 0$. Where $B$ denotes the vector valued second fundamental form of $M$ and $B_N(u, v) = \tilde{g}(B(u, v), N)$.

The so obtained invariants $h_{2k+1}$ are normal differential forms on $M$ of degree 1 (duals of normal vector fields). They are tensorial in $N$.

For $k = 0$, using (4) we get

$$h_1(N) = *\left(\frac{g^{n-1}}{(n-1)!} B_N\right) = cB_N.$$ 

That is $h_1$ is nothing but the usual mean curvature of $M$. Furthermore, for a hypersurface of the Euclidean space the invariant $h_{2k+1}$ can be seen as a scalar function on $M$ and

$$h_{2k+1} = *\left(\frac{g^{n-2k-1}}{(n-2k-1)!} \left(\frac{1}{2} B^2\right)^k B\right) = \frac{(2k + 1)!}{2^k} s_{2k+1}.$$ 

That is, up to a constant, the usual $(2k + 1)$-mean curvature of the hyper-surface $M$.

Using formulas (2) and (3), it is straightforward that

$$h_{2k+1}(N) = *\left(\left\langle *T_{2k}\right\rangle B_N\right) = \left\langle T_{2k}, B_N\right\rangle.$$ (9)

3.2 Double Forms: Differential Properties

For the seek of completeness, we recall in this paragraph some useful differential properties of double forms, for more details see [1, 3].

Let $(M, g)$ be a Riemannian manifold of dimension $n$ and $T_M$ its tangent space at $m \in M$. We denote by $D^{p,q}$ the vector bundle over $M$ whose fiber at $m$ is the space of all $(p, q)$-double forms on $T_M$ as in the first part.

Note that the previous algebraic properties are still true for the sections of the bundle $D^{p,q}$.

The second Bianchi map, denoted $D$, maps $D^{p,q}$ into $D^{p+1,q}$. Its restriction to $D^{p,0}$ coincides with $-d$, where $d$ is the operator of exterior differentiation of $p$-forms. There exists a second natural extension of $d$ namely the adjoint second Bianchi map $\tilde{D}$. It sends $D^{p,q}$ into $D^{p,q+1}$.
The operators \( \delta = c\tilde{D} + \tilde{D}c \) and \( \tilde{\delta} = cD + Dc \) generalize the classical \( \delta \) operator on differential forms. Furthermore, they are respectively the formal adjoints (with respect to the integral scalar product on a compact manifold) of the operators \( D \) and \( \tilde{D} \).

The operator \( D\tilde{D} + \tilde{D}D \) sends a \((p, q)\)-double form to a \((p + 1, q + 1)\)-double form and its restriction to functions \(((0,0)\)-double forms) is twice the usual Hessian:

\[
[D\tilde{D} + \tilde{D}D](f) = 2\text{Hess}(f).
\]

Similarly, for \( p, q \geq 1 \), the operator \( \delta\tilde{\delta} + \tilde{\delta}\delta \) sends a \((p, q)\)-double form to a \((p - 1, q - 1)\)-double form and satisfies

\[
\delta\tilde{\delta} + \tilde{\delta}\delta = (-1)^{(p+q)(n-p-q)} \ast (D\tilde{D} + \tilde{D}D) \ast.
\]

Furthermore, with respect to the integral scalar product on a compact manifold, the operator \( \delta\tilde{\delta} + \tilde{\delta}\delta \) is the formal adjoint of the operator \((-1)^{(p+q)}(D\tilde{D} + \tilde{D}D)\).

### 3.3 The first variation formula

Let \( F \) be a local variation of \( M \), that is a smooth map

\[
F : M \times (-\epsilon, \epsilon) \to \tilde{M},
\]

such that \( F(x, 0) = x \) for all \( x \in M \) and with compact support \( \text{supp}F \), where

\[
\text{supp}F = \{x \in M \exists t \in (-\epsilon, \epsilon) \; F(x, t) \neq x\}.
\]

The implicit function theorem implies that there exists \( \epsilon > 0 \) such that for all \( t \) with \( |t| < \epsilon \), the map \( \phi_t = F(\cdot, t) : M \to \tilde{M} \) is a diffeomorphism onto a submanifold \( M_t \) of \( \tilde{M} \).

Denote by \( g_t \) the pull back via \( \phi_t \) of the induced metric on \( M_t \) from \( (\tilde{M}, \tilde{g}) \), precisely \( g_t = \phi_t^* (\tilde{g}) \). Note that \( g_1 = g \).

**Lemma 3.2** If \( \xi = \frac{d}{dt}|_{t=0}\phi_t \) denotes the variation vector field relative to \( F \), then the first variation of \( g_t \) is given by

\[
h(u, v) = \frac{d}{dt}|_{t=0} g_t(u, v) = 2B_N(u, v) + A_{\xi^T}(u, v).
\]

Where \( N, \xi^T \) are respectively the normal and tangent components of the vector field \( \xi \), \( A_{\xi^T}(u, v) = g(\nabla_u \xi^T, v) + g(\nabla_v \xi^T, u) \) is like \( B_N \) a symmetric bilinear form and \( \nabla \) denotes the Levi-Civita connection of \((M, g)\).
Proof. Using for example coordinate vector fields, one can prove without difficulties that
\[ h(u, v) = \tilde{g}(\tilde{\nabla}_u \xi, v) + \tilde{g}(\tilde{\nabla}_v \xi, u). \]
Where \( \tilde{\nabla} \) denotes the Levi-Civita connection of \( \tilde{M} \). So if \( \xi = N + \xi^T \), where \( N = \xi^\perp \), then \( \tilde{g}(\tilde{\nabla}_u \xi^T, v) = g(\nabla_u \xi^T, v) \) and \( \tilde{g}(\tilde{\nabla}_u N, v) = B_N(u, v). \)

**Lemma 3.3** Let \( X \) be a tangent vector field to \( M \), \( T \) a symmetric \((1, 1)\)-double form on \( M \) and let \( \alpha = T(X, \cdot) \) be the corresponding 1-form, then
\[ \langle T, A_X \rangle = \delta \alpha - \delta T. \]  \hspace{1cm} (13)
Where \( A_X(u, v) = g(\nabla_u X, v) + g(\nabla_v X, u) \).

**Proof.** Let \( \{e_i\} \) be local orthonormal vector fields around \( m \in M \) which diagonalize \( T \) at \( m \) and such that \( \nabla e_i = 0 \), then at \( m \) we have
\[ \langle T, A_X \rangle = 2 \sum_i T(\nabla_{e_i} X, e_i) = \sum_i \{\nabla_{e_i}(\alpha(e_i)) - (\nabla_{e_i} T)(X, e_i)\}. \]

We are now ready to state and prove the first variation formula:

**Theorem 3.4** Let \( M \) be a submanifold of the Riemannian manifold \((\tilde{M}, \tilde{g})\). Let \( \xi = \frac{d}{dt} |_{t=0} \phi_t \) denotes the variation vector field relative to a local variation \( F \) of \( M \) with compact support as above.

1. If \( H_{2k}(t) = \int_M h_{2k}(g_t) \mu_{g_t} \) denotes the total \((2k)\)-th Gauss-Bonnet curvature of \( \phi_t(M) \), where \( \mu_{g_t} \) denotes the corresponding Riemannian volume element, then
\[ H'_{2k}(0) = \int_M h_{2k+1}(\xi^\perp) \mu_g. \] \hspace{1cm} (14)
2. The submanifold \( M \) is a critical point for the total \((2k)\)-th Gauss-Bonnet curvature function for all local variations of \( M \) if and only if the \((2k + 1)\)-Gauss-Bonnet curvature \( h_{2k+1}(N) \) of \( M \) vanishes for all normal directions \( N \).
Proof. Since by construction $h$ vanishes outside the compact subset $\text{supp} F$, lemma 3.5 below and Gauss’ theorem imply that

$$H'_{2k}h = \int_M \left( h'_{2k}h + \frac{h_{2k}}{2} \text{tr}_g h \right) \mu_g$$

$$= -\frac{1}{2} \langle \frac{e^{2k-1}}{(2k-1)!} R^k, h \rangle + \frac{h_{2k}}{2} < g, h >$$

$$= \frac{1}{2} < h_{2k}g - \frac{e^{2k-1}}{(2k-1)!} R^k, h >$$

$$= \frac{1}{2} \langle T_{2k}, h \rangle.$$

Where $\langle \cdot, \cdot \rangle$ is the integral scalar product. Next using \cite{12, 13}, Gauss’s theorem and the fact that Einstein Lovelock tensors are divergence free \cite{3} we get

$$\frac{1}{2} \langle T_{2k}, h \rangle = \langle T_{2k}, B_N \rangle.$$

Consequently, equation (9) shows that $H'_{2k}h = h_{2k+1}(N)$ as desired. 

**Lemma 3.5** \cite{2} **The** directional derivative of $h_{2k}$ at $g$ in a given direction $h$ is given by

$$h'_{2k}h = -\frac{1}{2} \langle \frac{e^{2k-1}}{(2k-1)!} R^k, h \rangle + \text{div} W_1 + \text{div} W_2.$$

Where $W_1$ (resp. $W_2$) is the tangent vector field over $M$ corresponding to the 1-form $\delta \left( \frac{k(n-2k)}{4(n-2k)!} R^{k-1}h \right)$ (resp. $\delta \left( \frac{k(n-2k)}{4(n-2k)!} R^{k-1}h \right)$).

### 3.4 (2k)-Minimal submanifolds

With respect to the previous variational formula and by analogy to the case of usual minimal submanifolds we set the following definition:

**Definition 3.6** For $0 \leq 2k \leq n$, An $n$-submanifold $M$ of the Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be $(2k)$-minimal if

$$h_{2k+1} \equiv 0.$$ 

In the extreme cases: the 0-minimal submanifolds are nothing but the usual minimal submanifolds. And if $n$ is even, every submanifold is $n$-minimal (the condition is empty).

We provide below examples of intermediate minimal submanifolds:
1. A flat submanifold is always \((2k)\)-minimal for all \(k > 0\). In fact \(R \equiv 0 \Rightarrow h_{2k+1} \equiv 0\). This shows that \((2k)\)-minimal does not imply the usual minimality condition.

2. A totally geodesic submanifold is always \((2k)\)-minimal for all \(k \geq 0\). In fact \(B \equiv 0 \Rightarrow h_{2k+1} \equiv 0\).

3. A submanifold with constant curvature \(\lambda \neq 0\) is \((2k)\)-minimal if and only if it is minimal in the usual sense. In fact, in this case \(R = \frac{\lambda}{2} g^2\), and therefore

\[
h_{2k+1}(N) = *\left(\frac{g^{n-2k-1}}{(n-2k-1)!} 2^{-k} \lambda^k g^{2k} B_N \right) = \frac{(n-1)! \lambda^k}{(n-2k-1)! 2^k c B_N}.
\]

4. If \(M\) is a hypersurface of the Euclidean space then \((2k)\)-minimality coincides with Reilly’s \((2k)\)-minimality [4]. In fact, as we have seen above, in this case \(h_{2k+1}\) coincides, up to a constant, with the \((2k+1)\)-mean curvature of the hypersurface.

In particular, there are no \((2k)\)-minimal compact hypersurfaces of the Euclidean space. For, it is standard that there always exists one point on the hypersurface where all the mean curvatures are positive and in particular not equal to zero.

5. If \(M\) is a hypersurface of a space form \((\tilde{M}, \tilde{g})\) of constant \(\lambda\) then \(M\) is \((2k)\)-minimal if and only if

\[
\sum_{i=0}^{k} \frac{k!(2k-2i+1)!(n-2k-1+2i)! \lambda^i}{i!(k-i)!(n-2k-1)!2^k} s_{2k-2i+1} = 0.
\]

Where \(s_j\) denotes the symmetric functions in the eigenvalues of the shape operator of the hypersurface. This fact can be proved easily after using the Gauss equation \(R = \lambda/2g^2 + 1/2B^2\) and the binomial theorem. Notice the difference with Reilly’s \(r\)-minimality [4].

6. Any complex submanifold \(M\) of a Kahlerian manifold \((\tilde{M}, \tilde{g})\) is \((2k)\)-minimal for any \(k\).
To prove this fact, note first that $M$ is then itself a Kahlerian manifold. The Riemann curvature tensor of $M$ is therefore invariant under the complex structure $J$, that is $R(J., J., J., J.) = R(., ., ., .)$.

It can be shown without difficulties that the same property is therefore true for the tensors $R^k$ and consequently for the contraction $c^{2k-1}R^k$. Consequently, the Einstein-Lovelock tensors $T_{2k} = h_{2k}g - \frac{c^{2k-1}R^k}{(2k-1)!}$ are $J$-invariant, that is $T_{2k}(J., J.) = T_{2k}(., .)$.

On the other hand, the second fundamental form of $M$ satisfies $B(J x, y) = B(x, J y)$. It is then straightforward that $h_{2k+1}(N) = \langle T_{2k}, B_N \rangle \equiv 0$.

7. Suppose the submanifold $(M, g)$ is $(2k)$-Einstein, that is $T_{2k} = \lambda g$, then:

- If $\lambda = 0$ then $M$ is $(2k)$-minimal.
- If $\lambda \neq 0$ then $M$ is $(2k)$-minimal if and only if it is minimal in the usual sense.

In particular, since irreducible isotropy homogeneous spaces are $(2k)$-Einstein for all $k > 0$ [3], then they admit a $(2k)$-minimal immersion in a sphere for all $k > 0$.

Remark. Since the considerations of theorem 3.4 are of local nature, the theorem remains then true for immersed submanifolds $M \subset \tilde{M}$. Therefore it makes sense to consider also $(2k)$-minimal immersed submanifolds.

3.5 Generalized Laplace Operators

Let $(M, g)$ be a Riemannian manifold and $f$ be a smooth function defined on $M$. Recall that the usual Laplacian of $f$ is given by

$$\Delta f = -cHess\,(f) = -\langle g, Hess\,(f) \rangle$$

where $Hess\,(f)$ denotes the Hessian of the function $f$. Instead of just taking the trace of the Hessian, one can takes the determinant (Monge-Ampère operator) and more generally the elementary symmetric functions in the eigenvalues of $Hess\,(f)$. These generalized operators appear naturally in the context of the $\sigma_k$-Yamabe problem, where they are denoted $\sigma_k(Hess\,(f))$, see [6] and the references therein. Precisely, they are up to a constant equal to

$$c^{2k}Hess^k\,(f) = \langle Hess^k(f), g^k \rangle.$$
In our context of generalized minimal submanifolds, another natural generalization of the Laplace operator appears naturally. Precisely, we set

**Definition 3.7** Let \( f \) be a smooth function on \((M, g)\). We define the \(\ell_{2k}\)-Laplacian operator of \((M, g)\) as

\[
\ell_{2k}(f) = -\langle T_{2k}, \text{Hess}(f) \rangle.
\]  

(15)

Where \( T_{2k} \) denotes the \((2k)\)-th Einstein-Lovelock tensor of \((M, g)\) and \( 0 \leq 2k < n \).

We shall say that the function \( f \) is \(\ell_{2k}\)-harmonic if \(\ell_{2k}(f) = 0\).

For \( k = 0 \) we have \( T_0 = g \) and then \(\ell_0 = \Delta\) is the usual Laplacian. Furthermore, if \((M, g)\) is \((2k)\)-Einstein, that is \( T_{2k} = \lambda g \), then \(\ell(f) = \lambda \Delta(f)\) is, up to a constant, the usual Laplacian.

In particular, the later property holds for manifolds with constant curvature and for isotropy irreducible homogeneous manifolds [3].

We prove below some properties of these operators.

**Proposition 3.8** Let \( D, \tilde{D}, \delta \) and \( \tilde{\delta} \) be as in section 3.2. The \(\ell_{2k}\)-Laplacian can be written in any one of the following equivalent forms:

\[
\ell_{2k}(f) = \star \left\{ \frac{g^{n-2k-1}}{(n-2k-1)!} R^k \text{Hess}(f) \right\}
\]

\[
= \star \left\{ [D \tilde{D} + \tilde{D} D] (\frac{g^{n-2k-1}}{2(n-2k-1)!} f R^k) \right\}
\]

\[
= \frac{1}{2} [\delta \tilde{\delta} + \tilde{\delta} \delta] (f T_{2k}).
\]

In particular, \(\ell_{2k}(f)\) is a divergence and therefore \(\int_M \ell_{2k}(f) dv \equiv 0\) if \( M \) is compact.

**Proof.** The first formula is a direct consequence of the definition of \( T_{2k} \) and formula (2). The second one results from the fact that the metric \( g \) and the Riemann tensor are in \( \ker(D \cap \tilde{D}) \), and the fact that the former is closed under the exterior product of double forms [1]. The last formula results from [3] and [11].

**Proposition 3.9** If for some \( k \) with \( 0 \leq 2k < n \), the Einstein-Lovelock tensor \( T_{2k} \) is positive definite (or negative definite), then the operator \(\ell_{2k}\) is elliptic.
Proof. Recall that in local coordinates we have
\[ \text{Hess}(f) = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^k} \Gamma^k_{ij} \right) dx^i \otimes dx^j. \]
Therefore,
\[ \ell_{2k}(f) = \sum_{i,j} T_{2k} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^k} \Gamma^k_{ij} \right). \]

**Proposition 3.10** If \( M \) is compact, then the operator \( \ell_{2k} \) is self adjoint with respect to the integral scalar product, that is for arbitrary smooth functions \( u, v \) on \( M \) we have
\[ \langle u, \ell_{2k}(v) \rangle = \langle v, \ell_{2k}(u) \rangle. \]
Furthermore, If \( T_{2k} \) is positive definite (resp. negative definite) then \( \ell_{2k} \) is positive definite (resp. negative definite). More precisely we have the following integral formula
\[ \langle \ell_{2k}(f), f \rangle = T_{2k}(df^2, df^2). \]

Proof. First note that
\[ \text{Hess}(uv) = \frac{1}{2} [D\tilde{D} + \tilde{D}D](uv) \]
\[ = v \text{Hess}(u) + u \text{Hess}(v) + (Du\tilde{D}v + \tilde{D}uDv) \]
\[ = v \text{Hess}(u) - u \text{Hess}(v) + D(u\tilde{D}v) + \tilde{D}(uDv). \]
Since, with respect to the integral scalar product, the formal adjoints of \( D \) and \( \tilde{D} \) are divergences, namely \( \delta \) and \( \tilde{\delta} \), see section 3.2 and since the Einstein Lovelock tensors are divergence free [3], it results from the previous formula that
\[ 0 = - \int_M \ell_{2k}(uv) d\text{vol} = \langle T_{2k}, \text{Hess}(uv) \rangle \]
\[ = \langle T_{2k}, v \text{Hess}(u) \rangle - \langle T_{2k}, u \text{Hess}(v) \rangle + 0 \]
\[ = -\langle v, \ell_{2k}(u) \rangle + \langle u, \ell_{2k}(v) \rangle. \]
Next, it results from the formula in the third line of this proof, after taking \( u = v = f \), that
\[ 0 = -2 \langle \ell_{2k}(f), f \rangle + \langle Df\tilde{D}f + \tilde{D}fDf, T_{2k} \rangle. \]
On the other hand, at each point of \( M \) we have
\[
\langle Df \tilde{D}f + \tilde{D}f Df, T_{2k} \rangle = 2 \sum_{i,j} T_{2k}(e_i, e_j) df(e_i) df(e_j)
\]
\[
= 2 \sum_{i,j} T_{2k}(df(e_i)e_i, df(e_j)e_j) = 2 T_{2k}\langle df^2, df^2 \rangle.
\]
This completes the proof.

The proof of the following corollary is straightforward:

**Corollary 3.11** Let \((M, g)\) be a compact manifold of positive definite (or negative definite) Einstein-Lovelock tensor \( T_{2k} \) then every smooth and \( \ell_{2k} \)-harmonic function on \( M \) is constant.

### 3.6 \((2k)\)-minimal submanifolds in Euclidean space and in the spheres

We suppose now that \( \tilde{M} = \mathbb{R}^{n+p} \) is the Euclidean space. For \( v \in \mathbb{R}^{n+p} \), we define the coordinate function \( f_v : M \to \mathbb{R} \) by \( f_v(m) = \langle v, m \rangle \). It is not difficult to see that the Hessian of \( f_v \) is nothing but the second fundamental form of \( M \) in the direction of the normal component of \( v \), that is
\[
\text{Hess}(f_v)(x, y) = -\langle B(x, y), v \rangle.
\]
In particular if \( v \) is normal to \( M \), we get
\[
\ell_{2k}(f_v) = -\langle T_{2k}, \text{Hess}(f_v) \rangle = \langle T_{2k}, B_v \rangle = h_{2k+1}(v). \tag{16}
\]
We have therefore proved the following result:

**Proposition 3.12** A submanifold \( M \) of the Euclidean space is \((2k)\)-minimal if and only if the coordinate functions restricted to \( M \) are \( \ell_{2k} \)-harmonic functions on \( M \).

Let now \( F : M^n \to \mathbb{R}^{n+p} \) be an isometric immersion and let \( F_i(x) = \langle F(x), e_i \rangle \) be the \( i \)-th component of \( F \) in \( \mathbb{R}^{n+p} \). If we look to \( F \) as a function from \( F(M) \) to \( \mathbb{R}^{n+p} \) and since the results are local, we may assume \( F(M) \) a submanifold and then \( \text{Hess}(F_i) = B_i \). In particular, if \( h_{2k+1} \) is the Gauss-Bonnet curvature of \( F(M) \) in \( \mathbb{R}^{n+p} \) then we have (componentwise)
\[
\ell_{2k}(F) = h_{2k+1}. \tag{17}
\]
In particular, \( F \) is \((2k)\)-minimal if and only if \( \ell_{2k}(F_i) = 0 \) for all \( i \).

Using corollary 3.11 and the previous remark we immediately obtain
**Corollary 3.13** Let $0 \leq 2k < n$ and let $(M, g)$ be a compact Riemannian $n$-manifold with positive definite (or negative definite) Einstein-Lovelock tensor $T_{2k}$. Then there is no non trivial isometric $(2k)$-minimal immersion of $M$ in the Euclidean space.

In particular, there are no non trivial compact $2$-minimal submanifolds in the Euclidean space with positive definite (or negative definite) Einstein tensor. In other words, if $M$ is a compact submanifold of the Euclidean space with positive definite (or negative definite) Einstein tensor, then there exist variations of $M$ which increase the total scalar curvature and others which decrease it.

Note that the condition of positive (or negative) definiteness of $T_{2k}$ in the previous corollary is necessary, as the flat torus admits non trivial $(2k)$-minimal isometric immersions in the Euclidean space.

Finally, we prove the following about $(2k)$-minimal immersions in spheres.

**Proposition 3.14** Let $F : M^n \to S^{n+p} \subset \mathbb{R}^{n+p+1}$ be an isometric immersion. Then $F$ is $(2k)$-minimal into $S^{n+p}$ if and only if there is a smooth function $\phi : M \to \mathbb{R}$ such that $\ell_{2k} F = \phi F$ (componentwise).

**Proof.** It is not difficult to see that the two second fundamental forms of the submanifold in the sphere and in the Euclidean space coincide in normal directions that are tangent to the sphere. Consequently, it results from its definition that the $(2k + 1)$th Gauss-Bonnet curvature $h_{2k+1}$ of $F(M)$ in $S^{n+p}$ coincides with the restriction to $T S^{n+p}$ of the $(2k + 1)$th Gauss-Bonnet curvature of $F(M)$ in $\mathbb{R}^{n+p+1}$. The later being equal to $\ell_{2k}(F)$ as above, then $h_{2k+1} \equiv 0$ if and only if the components of $\ell_{2k}(F)$ vanishes for the directions tangent to the sphere. But since $F$ takes its values in the sphere then $F(x)$ are normal vectors to $S^{n+p}$ therefore $F$ is a $(2k)$-minimal if and only if $\ell_{2k}(F) = \phi F$. 

**References**

[1] Kulkarni, R. S., *On Bianchi Identities*, Math. Ann. 199, 175-204(1972).

[2] Labbi, M. L., *Double forms, curvature structures and the $(p,q)$-curvatures*, Transactions of the American Mathematical Society, 357, n10, 3971-3992 (2005).

[3] Labbi, M. L., *Variational properties of the Gauss-Bonnet curvatures*, arXiv:math.DG/0406548.
[4] Reilly, R. C., *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geometry, 8 (1973) 465-477.

[5] Reilly, R. C., *Variational Properties of Mean Curvatures*, Proc. Summer Sem. Canad. Math. Congress. 102-114, (1971).

[6] Viaclovsky, J., *Conformal geometry and fully nonlinear equations*, arXiv:math.DG/0609158.

Labbi M.-L.
Department of Mathematics,
College of Science, University of Bahrain,
P. O. Box 32038 Bahrain.
E-mail: labbi@sci.uob.bh