A SQUARE ROOT VELOCITY FRAMEWORK FOR CURVES OF BOUNDED VARIATION

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Abstract. The square root velocity transform is a powerful tool for the efficient computation of distances between curves. Also, after factoring out reparametrisations, it defines a distance between shapes that only depends on their intrinsic geometry but not the concrete parametrisation. Though originally formulated for smooth curves, the square root velocity transform and the resulting shape distance have been thoroughly analysed for the setting of absolutely continuous curves using a relaxed notion of reparametrisations. In this paper, we will generalise the square root velocity distance even further to a class of discontinuous curves. We will provide an explicit formula for the natural extension of this distance to curves of bounded variation and analyse the resulting quotient distance on the space of unparametrised curves. In particular, we will discuss the existence of optimal reparametrisations for which the minimal distance on the quotient space is realised.

1. Introduction

The mathematical analysis of shapes is a topic that is required in a large number of different applications ranging from mathematical image processing and inverse problems over computational biology to different problems in computer vision; see [4, 6, 8, 14, 21] to name but a few examples, see also [19] for a larger overview. In order to perform tasks like classification and for studying the geometry of datasets of shapes, it is necessary to define a suitable distance on the shape space. Here we will focus on one-dimensional shapes, that is, parametrised curves \( c : I \to \mathbb{R}^d \), \( I = [0,1] \), where we identify two curves if they only differ by a translation or a reparametrisation. In this setting, a particularly useful distance is the square root velocity (SRV) distance introduced in [15, 20]. Given a sufficiently smooth, regular curve \( c \), we define its square root velocity transform \( q := R(c) = \dot{c}/\sqrt{|\dot{c}|} \). Then, the distance between two curves \( c_1 \) and \( c_2 \) with square root velocity transforms \( q_1 \) and \( q_2 \) is defined as \( d(c_1, c_2) = \|q_1 - q_2\|_{L^2} \). It is also possible to regard the space of all smooth regular curves as a manifold with the Riemannian structure inherited from \( L^2(I; \mathbb{R}^d) \) via the mapping \( R \). This differential geometric point of view of shape analysis has been studied and discussed for instance in [2, 3, 15, 19].

With this definition, one does not yet obtain a distance on the shape space, as \( d(c_1, c_2) \) depends on the parametrisation of the curves \( c_1 \) and \( c_2 \). For that, one needs to consider instead the quotient distance

\[
 d^S([c_1], [c_2]) = \inf_{\varphi_1, \varphi_2} d(c_1 \circ \varphi_1, c_2 \circ \varphi_2),
\]

where the infimum is taken over all orientation preserving smooth diffeomorphisms of the unit interval \( I \). Various approaches for the efficient numerical solution of (discretisations of) this optimisation problem have been suggested, ranging from...
gradient based optimisation methods [12] over dynamical programming [7, 15] and the reformulation as a Hamilton–Jacobi–Bellman equation [22] to machine learning methods [11, 16]. There exists also an analytic algorithm for the case where the curves \(c_1\) and \(c_2\) are piecewise linear [13].

In addition to providing a distance between shapes, the actual solutions \((\bar{\varphi}_1, \bar{\varphi}_2)\) of the optimisation problem (1) can be used to define correspondences, or best matches, between the curves \(c_1\) and \(c_2\): The best match on \(c_2\) for the point \(c_1(t)\) is the point \(c_2(\bar{\varphi}_2(\bar{\varphi}_1^{-1}(t)))\). To that end, however, it is necessary that the optimisation problem (1) actually admits a solution. In [5], it has been shown that this is the case provided that the two curves \(c_1\) and \(c_2\) are continuously differentiable and the reparametrisations \(\varphi_1\) and \(\varphi_2\) are allowed to be merely absolutely continuous and non-decreasing instead of being smooth and strictly increasing. At the same time, an example of two Lipschitz curves was provided, where the infimum in (1) is not attained. A closer inspection of the results of [5], however, reveals that a convex relaxation of the problem (1) admits its minimum for arbitrary absolutely continuous curves \(c_1\) and \(c_2\). Using this relaxed distance measure, one can thus generalise shape distances to absolutely continuous curves and still obtain best matches between arbitrary curves. Because the reparametrisations are merely non-decreasing and not necessarily strictly increasing, it is possible, though, that a single point on one of the curves corresponds to a whole line segment on the other curve.

In this paper, we will generalise the analysis of [5] to discontinuous curves, or, more specifically, to curves of bounded variation. There it is no longer possible to define the square root velocity transform \(R(c)\) in a meaningful manner, as this would involve taking the square root of the derivative of the curve \(c\), which is a Radon measure. Instead, we work directly with the SRV distance and provide an explicit formula for a relaxation of this distance from absolutely continuous curves to curves of bounded variation, see Theorem 1 below. The main tool here is a generalisation of the Reshetnyak continuity and lower semi-continuity theorems [17]. Next, we discuss how the relaxed SRV distance can be used for defining a shape distance on curves of bounded variation. The main challenge here is the fact that a composition of a discontinuous curve with a non-decreasing reparametrisation is not necessarily well-defined: If the reparametrisation \(\varphi\) maps a whole interval \([a, b]\) to a single point \(t\) where the curve \(c\) has a jump, then it is \(a\)-priori not clear how the composition \(c \circ \varphi\) should be interpreted on that interval. In Definition 2 we provide such an interpretation, which we show to be natural in the context we are working in, see Proposition 3. Finally, we consider the particular setting of special curves of bounded variation, where we show that our approach gives rise to a shape distance, for which optimal matchings exists for each pair of curves, see Theorem 6.

In Section 2 we will recall the main definitions and results concerning the SRV transform for absolutely continuous curves, which are relevant for this paper. The generalisation to curves and shapes of bounded variation is presented in Section 3. Finally, the proofs of all the results are collected in Sections 4 and 5.

2. The SRV framework for absolutely continuous curves

In the following, we will provide a brief introduction into the square root velocity framework for absolutely continuous curves following the results of [5].

2.1. Square root velocity transform. Denote by \(I = [0, 1]\) the unit interval and by \(AC(I; \mathbb{R}^d)\) the space of absolutely continuous curves in \(\mathbb{R}^d\). Moreover, let \(AC_0(I; \mathbb{R}^d)\) the subspace of absolutely continuous curves satisfying \(c(0) = 0\). The square root velocity-transform (SRVT) of a curve \(c \in AC(I; \mathbb{R}^d)\) is defined as
\[ R: AC(I; \mathbb{R}^d) \to L^2(I; \mathbb{R}^d), \]
\[
R(c) = \frac{\dot{c}}{\sqrt{|\dot{c}|}}
\]
Here the fraction \(\dot{c}/\sqrt{|\dot{c}|}\) is set to be zero at points where \(\dot{c} = 0\). The mapping \(R\) is a bijection from \(AC_0(I; \mathbb{R}^d)\) to \(L^2(I; \mathbb{R}^d)\) with inverse
\[
R^{-1}(q)(x) = \int_0^x q|q| \, dy.
\]
Moreover, the SRVT is norm-preserving in the sense that
\[
\|R(c)\|_{L^2}^2 = \|\dot{c}\|_1 = \text{len}(c).
\]

Given two curves \(c_1, c_2 \in AC_0(I; \mathbb{R}^d)\), we define their (squared) SRV distance as
\[
d(c_1, c_2)^2 = \|R(c_1) - R(c_2)\|_{L^2}^2.
\]
This can be rewritten as
\[
d(c_1, c_2)^2 = \text{len}(c_1) + \text{len}(c_2) - 2S(c_1, c_2)
\]
with
\[
S(c_1, c_2) = \langle R(c_1), R(c_2) \rangle_{L^2} = \int_I \left( \frac{\dot{c}_1}{|\dot{c}_1|} - \frac{\dot{c}_2}{|\dot{c}_2|} \right) \sqrt{|\dot{c}_1||\dot{c}_2|} \, dx.
\]

2.2. **Shape space distance.** With the definition above one obtains a distance on the space of absolutely continuous curves. However, we are also interested in the case where one identifies curves if they are equal up to parametrisation. Following [5], we define
\[
\bar{\Gamma} := \{ \gamma \in AC(I; I) : \gamma(0) = 0, \gamma(1) = 1, \gamma' > 0 \text{ a.e.} \},
\]
the set of all absolutely continuous reparametrisations of the unit interval. Moreover, we define
\[
\check{\Gamma} := \{ \gamma \in AC(I; I) : \gamma(0) = 0, \gamma(1) = 1, \gamma' \geq 0 \text{ a.e.} \}.
\]
Then \(\check{\Gamma}\) is the closure of \(\Gamma\) in \(AC(I; I)\) (with respect to the norm topology).

We say that two curves \(c_1, c_2 \in AC_0(I; \mathbb{R}^d)\) are equivalent, if there exist \(\varphi_1, \varphi_2 \in \check{\Gamma}\) and a curve \(\dot{c} \in AC_0(I; \mathbb{R}^d)\) such that \(c_1 = \dot{c} \circ \varphi_1\) and \(c_2 = \dot{c} \circ \varphi_2\). It has been shown in [5] Prop. 12 that this defines an equivalence relation \(\sim\) on \(AC_0(I; \mathbb{R}^d)\). In the following, we will denote the equivalence class of a curve \(c\) by \([c]\). Moreover, the quotient space of unparametrised curves is denoted by \(B(I; \mathbb{R}^d) = AC_0(I; \mathbb{R}^d)/\sim\).

One can show that
\[
B(I; \mathbb{R}^d) = \{ c \circ \varphi : c \in AC_0(I; \mathbb{R}^d), \dot{c} \neq 0 \text{ a.e.} \} \cup \{0\}.
\]
Moreover, two curves are equivalent, if and only if they have the same constant speed parametrisation.

On the quotient space \(B(I; \mathbb{R}^d)\) we now consider the induced distance
\[
d^\mathcal{S}([c_1], [c_2]) = \inf_{\varphi_1, \varphi_2 \in \check{\Gamma}} d(c_1 \circ \varphi_1, c_2 \circ \varphi_2).
\]
Note here that, for \(c \in AC(I; \mathbb{R}^d)\) and \(\varphi \in \check{\Gamma}\), the composition \(c \circ \varphi\) is again absolutely continuous, since \(\varphi\) is a non-decreasing function [10] Prop. 225C].

Since the length of a curve is invariant under reparametrisation, we can also write
\[
d^\mathcal{S}([c_1], [c_2])^2 = \text{len}(c_1) + \text{len}(c_2) - \sup_{\varphi_1, \varphi_2 \in \check{\Gamma}} S(c_1 \circ \varphi_1, c_2 \circ \varphi_2)
\]
with
\[
S(c_1 \circ \varphi_1, c_2 \circ \varphi_2) = \int_I \left( \frac{\dot{c}_1 \circ \varphi_1}{|\dot{c}_1 \circ \varphi_1|} - \frac{\dot{c}_2 \circ \varphi_2}{|\dot{c}_2 \circ \varphi_2|} \right) \sqrt{|\dot{c}_1 \circ \varphi_1||\dot{c}_2 \circ \varphi_2|} dx.
\]
It has been shown in [5] Prop. 15 that the supremum in (3) is attained for some \( \varphi_1, \varphi_2 \in \Gamma \) provided that \( c_1, c_2 \in C^1(I; \mathbb{R}^d) \) satisfy \( \dot{c}_i \neq 0 \) almost everywhere.

Note that the functional \( S(c_1, c_2) \) is invariant under simultaneous reparametrisations of the curves \( c_1 \) and \( c_2 \), that is, \( S(c_1 \circ \varphi, c_2 \circ \varphi) = S(c_1, c_2) \) for all \( \varphi \in \Gamma \). This property is crucial for the geometric properties of the quotient space \( B(I; \mathbb{R}^d) \), and is also often exploited in numerical discretisations of (3).

2.3. Scale invariant SRV distance. It is also possible to define a scale invariant SRV distance. Here one defines, for \( c \in AC_0(I; \mathbb{R}^d) \setminus \{0\} \),
\[
\tilde{R}(c) = \frac{1}{\sqrt{|\text{len}(c)|}} \frac{\dot{c}}{|\dot{c}|} = R(c/\text{len}(c)).
\]
This gives a mapping from \( AC_0(I; \mathbb{R}^d) \setminus \{0\} \) to the unit sphere in \( L^2(I; \mathbb{R}^d) \). The corresponding scale invariant SRV distance between two non-zero curves is therefore defined as the spherical distance between their scale invariant SRVTs, that is,
\[
d(c_1, c_2) = \arccos(\langle R(c_1/\text{len}(c_1)), R(c_2/\text{len}(c_2)) \rangle_{L^2}) = \arccos(S(c_1/\text{len}(c_1), c_2/\text{len}(c_2))).
\]
Moreover, we can define a scale invariant shape distance by
\[
\tilde{d}([c_1], [c_2])^2 = \inf_{\varphi_1, \varphi_2 \in \Gamma} \tilde{d}(c_1 \circ \varphi_1/\text{len}(c_1), c_2 \circ \varphi_2/\text{len}(c_2))^2
= \arccos\left( \sup_{\varphi_1, \varphi_2 \in \Gamma} S(c_1 \circ \varphi_1/\text{len}(c_1), c_2 \circ \varphi_2/\text{len}(c_2)) \right).
\]
In this article, we will focus on the unscaled variant of the SRV distance. However, all our results are based solely on the properties of the function \( S \), which is central both for the scaled and the unscaled variants. Thus all our results hold mutatis mutandis also for the scaled SRV framework.

3. Generalisation to BV curves

We now want to generalise the SRV distance to discontinuous curves, or, more specifically, to curves of bounded variation. Note here that we will not generalise the SRVT itself, since the definition of \( R(c) \) involves the square root of the derivative of \( c \), which is a measure if \( c \in BV(I; \mathbb{R}^d) \) is a general function of bounded variation. However, we will see that the resulting distance can still be defined.

3.1. Curves of bounded variation. In the following, we collect some results concerning functions of bounded variation that will be needed throughout the paper. For more details, we refer to [1 Sec. 3.2].

For every \( c \in BV(I; \mathbb{R}^d) \) and \( x \in I \) the one-sided essential limits \( c^-(x) := \text{ess lim}_{y \to x^-} c(y) \) and \( c^+(x) := \text{ess lim}_{y \to x^+} c(y) \) are well-defined. Thus we can define the subspace
\[
BV_0(I; \mathbb{R}^d) := \{ c \in BV(I; \mathbb{R}^d) : c^-(0) = 0 \}.
\]
For \( c \in BV_0(I; \mathbb{R}^d) \) and every \( x \in I \) we have that \( c'(x) = Dc((0, x)) \) and \( c''(x) = Dc((0, x)) \).

We say that a pointwise defined function \( \tilde{c} : I \to \mathbb{R}^d \) is a good representative of \( c \in BV(I; \mathbb{R}^d) \), if for all \( x \in I \) we have \( \tilde{c}(x) \in [c^-(x), c^+(x)] \). That is, \( \tilde{c}(x) = \text{ess lim}_{y \to x} c(y) \) whenever \( c \) is essentially continuous, and \( \tilde{c}(x) \) lies on the line segment from \( c^-(x) \) to \( c^+(x) \) whenever \( c \) has a jump at \( x \). We will always identify \( c \) with any of its good representatives. Finally, for good representatives the
measure theoretic total variation of \( c \) coincides with the pointwise total variation, that is,
\[
|Dc|(I) = \sup \left\{ \sum_k |c(x_{k+1}) - c(x_k)| : 0 < x_1 < x_2 < \ldots < x_N < 1 \right\}.
\]

In a slight abuse of notation, we denote by \( \text{len}(c) := |Dc|(I) \) the length of a discontinuous curve, including all of its jumps. We say that a sequence \( \{c^{(k)}\}_{k \in \mathbb{N}} \subset \text{BV}(I; \mathbb{R}^d) \) converges strictly to \( c \in \text{BV}(I; \mathbb{R}^d) \), denoted
\[
c^{(k)} \rightharpoonup^* c,
\]
if \( \|c^{(k)} - c\|_{L^1} \to 0 \) and \( \text{len}(c^{(k)}) \to \text{len}(c) \). Equivalently, we have that \( c^{(k)} \to^s c \), if and only if \( c^{(k)}(0) \to c(0) \), \( Dc^{(k)} \to^s Dc \) in the sense of weak convergence of Radon measures, and \( \text{len}(c^{(k)}) \to \text{len}(c) \).

Assume that \( c \in \text{BV}(I; \mathbb{R}^d) \). Then its weak derivative \( Dc \in \mathcal{M}(I; \mathbb{R}^d) \) can be decomposed as
\[
Dc = \dot{c} \mathcal{L}^1 + D^j c + D^c c,
\]
where \( \dot{c} \) is the classical derivative of \( c \) (which exists almost everywhere), \( D^j c \) is a purely atomic measure—the jump part of \( Dc \)—, and \( D^c c \) is a non-atomic, singular measure—the Cantor part of \( Dc \). We denote for \( c \in \text{BV}(I; \mathbb{R}^d) \) by
\[
\Sigma(c) := \{x \in I : c^\ell(x) \neq c^r(x)\}
\]
the jump set of \( c \) and define
\[
[c](x) := c^r(x) - c^\ell(x) \quad \text{for } x \in \Sigma(c).
\]
Then the jump part \( D^j c \) of \( Dc \) can be written as
\[
D^j c = \sum_{x \in \Sigma(c)} [c](x) \delta_x
\]
with \( \delta_x \) denoting the Dirac delta centered at \( x \).

Furthermore, we can decompose every function \( c \in \text{BV}_0(I; \mathbb{R}^d) \) uniquely as
\[
c = c^{(a)} + c^{(j)} + c^{(c)}
\]
such that \( c^{(a)} \) is absolutely continuous and \( (c^{(a)})' = \dot{c} \), \( Dc^{(j)} = D^j c \), and \( Dc^{(c)} = D^c c \). The set of functions \( c \) where the Cantor part \( c^{(c)} \) equals zero is denoted \( \text{SBV}(I; \mathbb{R}^d) \).

3.2. Extension of the SRV distance. We define
\[
\hat{S}(c_1, c_2) := \limsup_k S(c_1^{(k)}, c_2^{(k)}) : c_i^{(k)} \in \text{AC}(I; \mathbb{R}^d), c_i \to^* c_i
\]
and
\[
\hat{d}(c_1, c_2) = \text{len}(c_1) + \text{len}(c_2) - 2\hat{S}(c_1, c_2).
\]
That is, \( \hat{d} \) is the strictly lower semi-continuous hull of the function that is equal to \( d \) on \( \text{AC}(I; \mathbb{R}^d)^2 \) and equal to +\( \infty \) else.

Our first main theorem provides an explicit expression for \( \hat{S} \), and consequently also for \( \hat{d} \).

**Theorem 1.** Assume that \( c_1, c_2 \in \text{BV}(I; \mathbb{R}^d) \). Then
\[
\hat{S}(c_1, c_2) = \int_1 \left( \frac{d(Dc_1)}{|Dc_1|} + \frac{d(Dc_2)}{|Dc_2|} \right) \sqrt{\frac{d(Dc_1)}{d(|Dc_1| + |Dc_2|)} \frac{d(Dc_2)}{d(|Dc_1| + |Dc_2|)}} d(|Dc_1| + |Dc_2|).
\]
Here the integrand is set to zero at points where either \( d|Dc_1|/d(|Dc_1| + |Dc_2|) = 0 \) or \( d|Dc_2|/d(|Dc_1| + |Dc_2|) = 0 \). Moreover \((\cdot)^+ := \max\{\cdot, 0\}\) denotes the positive part of the argument.

In particular, if \( c_1, c_2 \in \text{SBV}(I; \mathbb{R}^d) \), then

\[
\hat{S}(c_1, c_2) = \int_I \left( \frac{\dot{c}_1}{|\dot{c}_1|} \cdot \frac{\dot{c}_2}{|\dot{c}_2|} \right)^+ \sqrt{|\dot{c}_1| |\dot{c}_2|} \, dx \\
+ \sum_{x \in \Sigma(c_1) \cap \Sigma(c_2)} \left( \frac{|c_1'(x)|}{||c_1'(x)||} \cdot \frac{|c_2'(x)|}{||c_2'(x)||} \right)^+ \sqrt{|c_1'(x)||c_2'(x)|}.
\]

**Proof.** See Section 4. \(\square\)

### 3.3. Reparametrisations of BV curves.

Again, we are interested in the case of distances modulo reparametrisations. However, in a setting with discontinuous curves \( c \) and reparametrisations \( \varphi \) that are merely non-decreasing, but not necessarily strictly increasing, the expression \( c \circ \varphi \) does not always make sense: If \( \varphi \) is constant on a non-trivial interval, say \( \varphi(x) = y \) for all \( x \in [a, b] \) with \( b > a \) and \( c \) has a jump at \( y \), then \( c \circ \varphi \) is not well-defined on the interval \([a, b]\). Thus we have to use a generalised definition of reparametrisations.

**Definition 2.** Let \( c \in \text{BV}(I; \mathbb{R}^d) \) and \( \varphi \in \Gamma \). We define

\[
[c, \varphi] := \{g \in \text{BV}(I; \mathbb{R}^d) : g(x) \in [c'(\varphi(x)), c'(\varphi(x))] \text{ for all } x \in I \}
\]

and \( \text{len}(c) = \text{len}(g) \).

Here \([c'(\varphi(x)), c'(\varphi(x))] \subset \mathbb{R}^d\) denotes the line segment from \(c'(\varphi(x))\) to \(c'(\varphi(x))\). In particular, if \( x \in I \) is such that \( c \) is continuous at \( \varphi(x) \) and \( g \in [c, \varphi] \), then \( g(x) = c(\varphi(x)) \) and \( g \) is continuous at \( x \). Moreover, note that \([c, \varphi]\) consists of the single element \( c \circ \varphi \) if either \( c \) is continuous or \( \varphi \) injective.

With a slight abuse of notation, we thus can define the distance between the sets \([c_1, \varphi_1]\) and \([c_2, \varphi_2]\) as

\[
\hat{d}([c_1, \varphi_1], [c_2, \varphi_2]) := \inf_{g_1 \in [c_1, \varphi_1]} \hat{d}(g_1, g_2) = \text{len}(c_1) + \text{len}(c_2) - \sup_{g_1 \in [c_1, \varphi_1]} \hat{S}(g_1, g_2).
\]

Our next result shows that the same distance function is obtained as the lower semi-continuous extension of the mapping \((c_1, c_2, \varphi_1, \varphi_2) \mapsto d(c_1 \circ \varphi_1, c_2 \circ \varphi_2)\) with respect to strict convergence of the curves \( c_i \) and uniform convergence of the reparametrisations \( \varphi_i \).

**Proposition 3.** Define the functional \( D : \text{BV}(I; \mathbb{R}^d)^2 \times \bar{\Gamma}^2 \to \mathbb{R} \cup \{+\infty\} \),

\[
D(c_1, c_2; \varphi_1, \varphi_2) := \begin{cases} \\
d(c_1 \circ \varphi_1, c_2 \circ \varphi_2) & \text{if } c_i \in \text{AC}(I; \mathbb{R}^d) \text{ and } \varphi_i \in \Gamma, \\
+\infty & \text{else}.
\end{cases}
\]

Then the lower semi-continuous hull of \( D \) with respect to strict convergence on \( \text{BV}(I; \mathbb{R}^d) \) and uniform convergence on \( \bar{\Gamma} \) is the functional

\[
\hat{D}(c_1, c_2; \varphi_1, \varphi_2) = \inf_{g_i \in [c_i, \varphi_i]} \hat{d}(g_1, g_2) = \hat{d}([c_1, \varphi_1], [c_2, \varphi_2]).
\]

**Proof.** See Section 5.\(\square\)

The next result shows that the resulting distance function is invariant under simultaneous reparametrisations:

**Proposition 4.** Assume that \( c_1, c_2 \in \text{BV}(I; \mathbb{R}^d) \) and \( \varphi_1, \varphi_2 \in \bar{\Gamma} \). Then we have for all \( \psi \in \bar{\Gamma} \) that

\[
\hat{d}([c_1, \varphi_1 \circ \psi], [c_2, \varphi_2 \circ \psi]) = \hat{d}([c_1, \varphi_1], [c_2, \varphi_2]).
\]
3.4. A shape distance on SBV curves. We now consider the particular setting where the involved curves are special functions of bounded variation.

Assume that \( c \in \text{SBV}(I; \mathbb{R}^d) \setminus \{0\} \). Then we can construct an “equivalent” function \( G(c) \in \text{AC}(I; \mathbb{R}^d) \) in the following way: Define the function \( \xi: I \to I \)

\[
\xi(x) = \frac{|D^c e|(0,x)}{2\text{len}(c)} + (1-\alpha)x, \quad \text{with } \alpha = \frac{|D^c e|(I)}{2\text{len}(c)}.
\]

Since \( 0 \leq \alpha \leq 1/2 \), it follows that \( \xi \) is a strictly increasing function with \( \xi(0) = 0 \), \( \xi(1) = 1 \), and \( \Sigma(\xi) = \Sigma(e) \). Denote now by \( \zeta: I \to I \) the (unique) non-decreasing left inverse of \( \xi \). Then \( \zeta: I \to I \) is Lipschitz continuous with Lipschitz constant \( 1/(1-\alpha) \leq 2 \). Now define the function \( G(c) \in \text{AC}(I; \mathbb{R}^d) \) by setting \( G(c)(x) = c(\zeta(x)) \) if \( c \) is continuous at \( \zeta(x) \), and

\[
G(c)(x) = c'(\zeta(x)) + \frac{x - \xi'(\zeta(x))}{\xi'(\zeta(x)) - \xi'(\zeta(x))} [c'(\zeta(x)) - c'(\zeta(x))]
\]

if \( c \) and thus also \( \xi \) is discontinuous at \( \zeta(x) \). That is, the jumps of \( c \) are replaced by a linear interpolation between the left and right limits of \( c \) at the jump points. In particular, we have that \( G(c) \in [c, \zeta] \).

**Definition 5.** We say that two curves \( c_1, c_2 \in \text{SBV}(I; \mathbb{R}^d) \) are equivalent, denoted \( c_1 \sim c_2 \), if the curves \( G(c_1) \) and \( G(c_2) \) have the same constant speed parametrisation. By \([c]\) we denote the equivalence class of the curve \( c \).

Moreover, we define the **shape distance**

\[
\hat{d}^S([c_1], [c_2]) = \inf_{g_1 \in [c_1]} \hat{d}(g_1, g_2) = \text{len}(c_1) + \text{len}(c_2) - \sup_{g_2 \in [c_2]} \hat{S}(g_1, g_2)
\]

on the set of equivalence classes with respect to \( \sim \).

The following result shows that the distance between the shapes \([c_1]\) and \([c_2]\) can be computed by minimising the curve distance over all reparametrisations of \( c_1 \) and \( c_2 \) in the sense of Definition 2.

**Theorem 6.** Assume that \( c_1, c_2 \in \text{SBV}(I; \mathbb{R}^d) \) satisfy \( \dot{c}_i(x) \neq 0 \) for a.e. \( x \in I \). Then

\[
\hat{d}^S([c_1], [c_2]) = \inf_{\varphi, \psi \in \mathcal{G}} \hat{d}([c_1, \varphi_1], [c_2, \varphi_2]) = \inf_{\psi \in \mathcal{G}} \hat{d}(G(c_1) \circ \psi_1, G(c_2) \circ \psi_2).
\]

Moreover, the infima in (4) are attained at some \( \varphi_1, \varphi_2 \in \mathcal{G} \) and \( \psi_1, \psi_2 \in \mathcal{G} \).

**Proof.** See Section 5.2.

In addition, this result shows that it is possible to compute optimal reparametrisations for arbitrary SBV curves. In particular, it is possible to define optimal matchings between SBV curves.

4. Extension of the SRV distance to BV curves

In this section, we will prove Theorem 1, which provides an explicit form for the extension of \( S \) to curves of bounded variation. In order to do so, we will make use of the fact that \( S \) only depends on the derivatives of the involved curves. This allows us to reformulate Theorem 1 as a result concerning the extension of integral functionals from \( L^1(I; \mathbb{R}^d) \) to \( \mathcal{M}(I; \mathbb{R}^d) \), the space of \( \mathbb{R}^d \)-valued finite Radon measures on \( I \). To that end, we define the functional \( F: L^1(I; \mathbb{R}^d)^2 \to \mathbb{R} \),

\[
F(u, v) := -\int_I \left( \frac{u}{|u|} \cdot \frac{v}{|v|} \right) |u||v| \, dx,
\]
and consider its extension \( \hat{F} \) to \( \mathcal{M}(I; \mathbb{R}^d) \), defined as
\[
\hat{F}(\mu, \nu) = \inf \left\{ \liminf_k F(u^{(k)}, v^{(k)}) : u^{(k)} \rightharpoonup^* \mu, \ v^{(k)} \rightharpoonup^* \nu, \right. \\
\left. \|u^{(k)}\|_{L^1} \rightarrow |\mu|(I), \ \|v^{(k)}\|_{L^1} \rightarrow |\nu|(I) \right\}
\]
for \( \mu, \nu \in \mathcal{M}(I; \mathbb{R}^d) \). Then
\[
S(c_1, c_2) = -\hat{F}(\hat{c}_1, \hat{c}_2)
\]
for all \( c_1, c_2 \in AC(I; \mathbb{R}^d) \). Moreover, due to the definition of strict convergence on \( BV(I; \mathbb{R}^d) \) and since we can, up to translations, identify a curve with its derivative, we have that
\[
\hat{S}(c_1, c_2) = -\hat{F}(Dc_1, Dc_2)
\]
for all \( c_1, c_2 \in BV(I; \mathbb{R}^d) \). Thus it is sufficient to derive an explicit formula for the functional \( \hat{F} \).

To do so, we will prove a generalisation of Reshetnyak’s continuity and lower semi-continuity theorems [17]. These theorems essentially state that a positively homogeneous integral functional on \( \mathcal{M}(I; \mathbb{R}^d) \) is continuous with respect to strict convergence, and that it is weakly* lower semi-continuous, if and only if the integrand is convex. This result is not immediately applicable to our situation, as we are dealing with a functional depending on two measures, and we require lower semi-continuity with respect to separate strict convergence. Thus, in Section 4.1, we will formulate a generalisation of Reshetnyak’s continuity theorem that provides a lower bound for the functional \( F \). Then, in Section 4.2, we will show that this lower bound is actually sharp. Finally, we will conclude the proof of Theorem 1 in Section 4.3.

4.1. Lower semi-continuity of integral functionals. Let \( \Omega \subset \mathbb{R}^n \) be open and bounded, and let
\[
f : \Omega \times \Sigma \times S^{d-1} \times S^{d-1} \rightarrow \mathbb{R},
\]
where \( \Sigma \) denotes the one-dimensional unit simplex and \( S^{d-1} \) the \( d - 1 \)-dimensional unit sphere. For simplicity, we identify \( \Sigma \) with the interval \([0, 1]\). Assume that \( f \) is lower semi-continuous and bounded and that for every \( x \in \Omega \) the mappings
\[
(\xi, \zeta) \mapsto f(x, 0, \xi, \zeta)
\]
and
\[
(\xi, \zeta) \mapsto f(x, 1, \xi, \zeta)
\]
are constant.

Then we can define the functional \( F: \mathcal{M}(\Omega; \mathbb{R}^d)^2 \rightarrow \mathbb{R} \),
\[
F(\mu, \nu) = \int_{\Omega} f(x, \frac{d|\mu|}{d(|\mu| + |\nu|)}, \frac{d\mu}{d|\mu|}, \frac{d\nu}{d|\nu|}) \ d(|\mu| + |\nu|).
\]
Since \( f(x, \tau, \cdot, \cdot) \) is assumed to be constant for \( \tau \in \{0, 1\} \), the integrand is independent of the choice of \( d\mu/d|\mu| \) and \( d\nu/d|\nu| \) outside of the supports of \( |\mu| \) and \( |\nu| \), respectively, and thus the integral is well-defined.

Theorem 7. Assume that \( f \) is lower semi-continuous and bounded and that the mapping
\[
\tau \mapsto f(x, \tau, \xi, \zeta)
\]
is convex for every \( x \in \Omega \) and \( \xi, \zeta \in S^{d-1} \). Then \( F \) is lower semi-continuous with respect to strict convergence in both components. That is, assume that \( \mu_k \rightharpoonup^* \mu \) and \( \nu_k \rightharpoonup^* \nu \) in \( \mathcal{M}(\Omega; \mathbb{R}^d) \) such that \( |\mu_k|(\Omega) \rightarrow |\mu|(\Omega) \) and \( |\nu_k|(\Omega) \rightarrow |\nu|(\Omega) \). Then
\[
F(\mu, \nu) \leq \liminf_k F(\mu_k, \nu_k).
\]
Proof. We follow the proof of Reshetnyak's continuity theorem as presented in [18 Thm. 10.3] (see also [11 Thm. 2.38, 2.39]).

For simplicity, we write \( m_k = d\mu_k/d|\mu_k|, \quad n_k = d\nu_k/d|\nu_k|, \quad t_k = d|\mu_k|/(|\mu_k| + |\nu_k|), \) and similarly \( m = d\mu/d|\mu|, \quad n = d\nu/d|\nu|, \quad t = d|\mu|/(|\mu| + |\nu|). \) Next we define the measures \( \sigma_k \) on \( \Omega \times \Sigma \times S^{d-1} \times S^{d-1} \) by

\[
\sigma_k = (|\mu_k| + |\nu_k|) \otimes (\delta_{t_k}(x) \otimes \delta_{m_k}(x) \otimes \delta_{n_k}(x)),
\]

that is,

\[
\int_{\Omega \times \Sigma \times S^{d-1} \times S^{d-1}} \varphi(x, \tau, \xi, \zeta) \, d\sigma_k = \int_{\Omega} \varphi(x, t_k(x), m_k(x), n_k(x)) \, d(|\mu_k| + |\nu_k|)
\]

for every \( \varphi \in C_0(\Omega \times \Sigma \times S^{d-1} \times S^{d-1}) \). Since the measures \(|\mu_k|\) and \(|\nu_k|\) are uniformly bounded, it follows that the sequence \( \sigma_k \) is bounded as well. After possibly passing to a subsequence, we may assume without loss of generality that \( \sigma_k \rightharpoonup^* \sigma \) for some \( \sigma \in M(\Omega \times \Sigma \times S^{d-1} \times S^{d-1}) \).

Denoting by \( \pi: \Omega \times \Sigma \times S^{d-1} \times S^{d-1} \to \Omega \) the projection onto the first component, we obtain that \( \pi_\# \sigma_k = |\mu_k| + |\nu_k|. \) Since \( \mu_k \) and \( \nu_k \) converge strictly to \( \mu \) and \( \nu \), it follows that \(|\mu_k|\) and \(|\nu_k|\) converge weakly* to \(|\mu|\) and \(|\nu|\) (see [18 Cor. 10.2]). Thus \( \pi_\# \sigma_k \rightharpoonup^* \pi_\# \sigma = |\mu| + |\nu|. \)

Now (see [18 Thm. 4.4]) there exists a weakly* measurable family \( \rho_x \in M(\Sigma \times S^{d-1} \times S^{d-1}) \) such that \( \rho_x(\Sigma \times S^{d-1} \times S^{d-1}) = 1 \) and \( \sigma = (|\mu| + |\nu|) \otimes \rho_x. \)

Let now \( \psi \in C_0(\Omega) \) be continuous and define \( \varphi \in C_0(\Omega \times \Sigma \times S^{d-1} \times S^{d-1}; \mathbb{R}^d), \)

\[
\varphi(x, \tau, \xi, \zeta) = \psi(x) \tau \xi.
\]

Then

\[
\int_{\Omega} \psi(x) \left( \int_{\Sigma \times S^{d-1} \times S^{d-1}} \tau \xi \, d\rho_x(\tau, \xi, \zeta) \right) d(|\mu| + |\nu|)
= \int_{\Omega \times \Sigma \times S^{d-1} \times S^{d-1}} \varphi(x, \tau, \xi, \zeta) \, d\sigma
= \lim_k \int_{\Omega \times \Sigma \times S^{d-1} \times S^{d-1}} \varphi(x, t_k(x), m_k(x), n_k(x)) \, d(|\mu_k| + |\nu_k|)
= \lim_k \int_{\Omega} \psi(x) t_k(x) m_k(x) \, d(|\mu_k| + |\nu_k|)
= \lim_k \int_{\Omega} \psi(x) \, d\mu_k(x)
= \int_{\Omega} \psi(x) \, dm(x)
= \int_{\Omega} \psi(x) t(x) m(x) \, d(|\mu| + |\nu|).
\]

Since \( \psi \in C_0(\Omega) \) was arbitrary, we obtain that

\[
\int_{\Sigma \times S^{d-1} \times S^{d-1}} \tau \xi \, d\rho_x(\tau, \xi, \zeta) = t(x) m(x)
\]

for \(|\mu| + |\nu|\)-a.e. \( x. \) Similarly, using a function \( \varphi(x, \tau, \xi, \zeta) = \psi(x) (1 - \tau) \zeta, \) one obtains that

\[
\int_{\Sigma \times S^{d-1} \times S^{d-1}} (1 - \tau) \zeta \, d\rho_x(\tau, \xi, \zeta) = (1 - t(x)) n(x)
\]
As a consequence, we can write
\[ \tau d\rho_x(\tau, \xi, \zeta) = t(x) \]
for \(|\mu| + |\nu|\)-a.e. \(x\). Moreover, using a function \(\phi(x, \tau, \xi, \zeta) = \psi(x)\tau\) and recalling that \(|\mu_k| \to^* |\mu|\), we obtain that
\[ \int_{\Sigma \times S^{d-1} \times S^{d-1}} \tau d\rho_x(\tau, \xi, \zeta) = t(x) \]
for \(|\mu| + |\nu|\)-a.e. \(x\).

In particular, we obtain that
\[ \frac{1}{2} \int_{\Sigma \times S^{d-1} \times S^{d-1}} \tau|\xi - m(x)|^2 + (1 - \tau)|\zeta - n(x)|^2 d\rho_x(\tau, \xi, \zeta) \]
\[ = \int_{\Sigma \times S^{d-1} \times S^{d-1}} \tau(1 - \langle \xi, m(x) \rangle) + (1 - \tau)(1 - \langle \zeta, n(x) \rangle) d\rho_x(\tau, \xi, \zeta) \]
\[ = 1 - (t(x)m(x), m(x)) - ((1 - t(x))n(x), n(x)) \]
\[ = 0. \]

This shows that the measure \(\rho_x\) is concentrated on the set
\[(0, 1) \times \{m(x)\} \times \{n(x)\} \cup \{0, 1\} \times S^{d-1} \times S^{d-1}.\]

As a consequence, we can write
\[ \rho_x = T_x \otimes \delta_{m(x)} \otimes \delta_{n(x)} + \delta_0 \otimes A_x + \delta_1 \otimes B_x, \]
where \(T_x \in \mathcal{M}(\Sigma)\) satisfies \(T_x((0, 1)) = 0\), and \(A_x, B_x \in \mathcal{M}(S^{d-1} \times S^{d-1})\). Note moreover that
\[ T_x(\Sigma) + A_x(S^{d-1} \times S^{d-1}) + B_x(S^{d-1} \times S^{d-1}) = \rho_x(\Sigma \times S^{d-1} \times S^{d-1}) = 1. \]

Since \(f\) is lower semi-continuous and bounded, and \(\sigma_k \to^* \sigma\), we now obtain that
\[ \liminf_k F(\mu_k, \nu_k) = \liminf_k \int_{\Omega} f(x, t_k(x), m_k(x), n_k(x)) d(|\mu_k| + |\nu_k|) \]
\[ = \liminf_k \int_{\Omega \times \Sigma \times S^{d-1} \times S^{d-1}} f(x, \tau, \xi, \zeta) d\sigma_k \]
\[ \geq \int_{\Omega \times \Sigma \times S^{d-1} \times S^{d-1}} f(x, \tau, \xi, \zeta) d\sigma \]
\[ = \int_{\Omega} \left( \int_{\Sigma \times S^{d-1} \times S^{d-1}} f(x, \tau, \xi, \zeta) d\rho_x(\tau, \xi, \zeta) \right) d(|\mu| + |\nu|) \]
\[ = \int_{\Omega} \left( \int_{\Sigma} f(x, \tau, m(x), n(x)) dT_x(\tau) \right) d(|\mu| + |\nu|) \]
\[ + \int_{\Omega} \left( \int_{S^{d-1} \times S^{d-1}} f(x, 0, \xi, \zeta) dA_x(\xi, \zeta) \right) d(|\mu| + |\nu|) \]
\[ + \int_{\Omega} \left( \int_{S^{d-1} \times S^{d-1}} f(x, 1, \xi, \zeta) dB_x(\xi, \zeta) \right) d(|\mu| + |\nu|). \]

Next we use that \(f(x, 0, \xi, \zeta)\) and \(f(x, 1, \xi, \zeta)\) are constant and obtain that
\[ \int_{S^{d-1} \times S^{d-1}} f(x, 0, \xi, \zeta) dA_x(\xi, \zeta) = f(x, 0, m(x), n(x)) A_x(S^{d-1} \times S^{d-1}), \]
\[ \int_{S^{d-1} \times S^{d-1}} f(x, 1, \xi, \zeta) dB_x(\xi, \zeta) = f(x, 1, m(x), n(x)) B_x(S^{d-1} \times S^{d-1}). \]

Moreover, as the mapping \(\tau \mapsto f(x, \tau, \xi, \zeta)\) is convex, we can use Jensen’s inequality and estimate
\[ \int_{\Sigma} f(x, \tau, m(x), n(x)) dT_x(\tau) \geq T_x(\Sigma) f \left( x, \frac{1}{T_x(\Sigma)} \int_{\Sigma} \tau dT_x(\tau), \xi, \zeta \right). \]
Thus we see that
\[
\int_{\Sigma \times S^{d-1} \times S^{d-1}} f(x, \tau, \xi, \zeta) \, d\rho_x(\tau, \xi, \zeta) \geq A_x(S^{d-1} \times S^{d-1}) f(x, 0, m(x), n(x)) + B_x(S^{d-1} \times S^{d-1}) f(x, 1, m(x), n(x)) + T_x(\Sigma) f(x, \frac{1}{T_x(\Sigma)} \int_{\Sigma} \tau \, dT_x(\tau), \xi, \zeta).
\]

Now recall that
\[
A_x(S^{d-1} \times S^{d-1}) + B_x(S^{d-1} \times S^{d-1}) + T_x(\Sigma) = 1
\]
and
\[
0 \cdot A_x(S^{d-1} \times S^{d-1}) + 1 \cdot B_x(S^{d-1} \times S^{d-1}) + T_x(\Sigma) \frac{1}{T_x(\Sigma)} \int_{\Sigma} \tau \, dT_x(\tau)
\]
\[
= \int_{\Sigma \times S^{d-1} \times S^{d-1}} \tau \, d\rho_x(\tau, \xi, \zeta) = m(x)
\]
for \((|\mu| + |\nu|)\)-a.e. \(x\). Thus we can use the convexity of \(f\) w.r.t. \(\tau\) and further estimate
\[
\int_{\Sigma \times S^{d-1} \times S^{d-1}} f(x, \tau, \xi, \zeta) \, d\rho_x(\tau, \xi, \zeta) \geq f(x, t(x), m(x), n(x)).
\]
Combining these estimates, we see that
\[
\liminf_k F(\mu_k, \nu_k) \geq \int_{\Omega} \left( \int_{\Sigma \times S^{d-1} \times S^{d-1}} f(x, \tau, \xi, \zeta) \, d\rho_x(\tau, \xi, \zeta) \right) \, d(|\mu| + |\nu|)
\]
\[
\geq \int_{\Omega} f(x, t(x), m(x), n(x)) \, d(|\mu| + |\nu|) = F(\mu, \nu),
\]
which concludes the proof. \(\square\)

4.2. Relaxation of integral functionals. Let again \(\Omega \subset \mathbb{R}^n\) and let \(f: \bar{\Omega} \times \Sigma \times S^{d-1} \times S^{d-1} \to \mathbb{R}\) be such that the mappings \((\xi, \zeta) \mapsto f(x, 0, \xi, \zeta)\) and \((\xi, \zeta) \mapsto f(x, 1, \xi, \zeta)\) are constant for every \(x\). Consider moreover the functional \(F: L^1(\Omega; \mathbb{R}^d)^2 \to \mathbb{R}\),
\[
F(u, v) = \int_{\Omega} f(x, \frac{|u|}{|u| + |v|}, \frac{u}{|u|}, \frac{v}{|v|}) (|u| + |v|) \, dx.
\]
In the following result, we will compute the lower semi-continuous extension of \(F\) to \(\mathcal{M}(\Omega; \mathbb{R}^d)^2\) with respect to strict convergence of measures.

**Theorem 8.** Assume that \(f: \bar{\Omega} \times \Sigma \times S^{d-1} \times S^{d-1} \to \mathbb{R}\) is continuous and bounded. Define
\[
\hat{F}(\mu, \nu) := \inf \left\{ \liminf_k F(u_k, v_k) : u_k \mathcal{L}^n \rightharpoonup^* \mu, v_k \mathcal{L}^n \rightharpoonup^* \nu, \right\}
\]
\[
\|u_k\|_{L^1} \to |\mu|(\Omega), \|v_k\|_{L^1} \to |\nu|(\Omega).
\]
Then
\[
\hat{F}(\mu, \nu) = F_c(\mu, \nu) := \int_{\Omega} f_c(x, \frac{d|\mu|}{d(|\mu| + |\nu|)}, \frac{d\mu}{d|\mu|}, \frac{d\nu}{d|\nu|}) \, d(|\mu| + |\nu|)
\]
for every \(\mu, \nu \in \mathcal{M}(\Omega; \mathbb{R}^d)\), where \(f_c\) denotes the lower semi-continuous convex hull of \(f\) with respect to the second variable.

**Proof.** In view of Theorem 7, we see that \(F_c\) is lower semi-continuous with respect to strict convergence in both components, which implies that \(F_c \leq \hat{F}\). Thus it is enough to find for each \(\mu, \nu \in \mathcal{M}(\Omega; \mathbb{R}^d)\) sequences \(u_k \mathcal{L}^n \rightharpoonup^* \mu, v_k \mathcal{L}^n \rightharpoonup^* \nu\) with \(|u_k|_{L^1} \to |\mu|(\Omega)\) and \(|v_k|_{L^1} \to |\nu|(\Omega)\) and \(F(u_k, v_k) \to F_c(\mu, \nu)\).
Assume first that \( \mu = u\mathcal{L}^n, \nu = v\mathcal{L}^n \) with \( u, v \in C(\bar{\Omega}; \mathbb{R}^d) \). We can write

\[
F_c(u, v) = \int_{\Omega} f_c(x, \frac{|u|}{|u| + |v|}, \frac{u}{|u|}, \frac{v}{|v|})(|u| + |v|) \, dx = \int_{\Omega} \left( \alpha(x) f(x, a(x), \frac{u}{|u|}, \frac{v}{|v|}) + \beta(x) f(x, b(x), \frac{u}{|u|}, \frac{v}{|v|}) \right)(|u| + |v|) \, dx
\]

for some \( 0 \leq \alpha(x), \beta(x) \leq 1 \) with \( \alpha(x) + \beta(x) = 1 \) and

\[
\alpha(x)a(x) + \beta(x)b(x) = \frac{|u(x)|}{|u(x)| + |v(x)|}.
\]

Now consider for \( k \in \mathbb{N} \) the family of cubes \( \hat{Q}_k^i = \frac{1}{2^k} \prod_{t=1}^n [i_t, i_t + 1], i \in \mathbb{Z}^d \) and define \( Q_k^i = \hat{Q}_k^i \cap \Omega \). Then we obtain finite partitions

\[
\Omega = \bigcup_{i \in I_k} Q_k^i
\]

for finite index sets \( I_k \subset \mathbb{Z}^d \). Choose moreover for all \( k \) and \( i \) in \( I_k \) some \( x_k^i \in Q_k^i \) and a partition \( Q_k^i = A_k^i \cup B_k^i \) with disjoint sets \( A_k^i, B_k^i \) such that \( |A_k^i| = \alpha(x_k^i)|Q_k^i| \) and \( |B_k^i| = \beta(x_k^i)|Q_k^i| \).

Define now functions \( u_k, v_k \) by

\[
\begin{align*}
    u_k(x) &= a(x_k^i)\frac{|u(x_k^i)| + |v(x_k^i)|}{|u(x_k^i)|}u(x_k^i), & x \in A_k^i, \\
    v_k(x) &= (1 - a(x_k^i))\frac{|u(x_k^i)| + |v(x_k^i)|}{|v(x_k^i)|}v(x_k^i), & x \in A_k^i, \\
    u_k(x) &= b(x_k^i)\frac{|u(x_k^i)| + |v(x_k^i)|}{|u(x_k^i)|}u(x_k^i), & x \in B_k^i, \\
    v_k(x) &= (1 - b(x_k^i))\frac{|u(x_k^i)| + |v(x_k^i)|}{|v(x_k^i)|}v(x_k^i), & x \in B_k^i.
\end{align*}
\]

Then

\[
\frac{|u_k(x)|}{|u_k(x)| + |v_k(x)|} = \begin{cases} 
    a(x_k^i), & \text{if } x \in A_k^i, \\
    b(x_k^i), & \text{if } x \in B_k^i.
\end{cases}
\]

Moreover we have for all \( \ell \in \mathbb{N} \) and \( k \geq \ell \) that

\[
\int_{Q_k^i} u_k(x) \, dx = \sum_{Q_k^i \subset Q_\ell^j} \frac{|u(x_k^i)| + |v(x_k^i)|}{|u(x_k^i)|}(\alpha(x_k^i)|A_k^i| + \beta(x_k^i)|B_k^i|)u(x_k^i)
\]

\[
= \sum_{Q_k^i \subset Q_\ell^j} \frac{|u(x_k^i)| + |v(x_k^i)|}{|u(x_k^i)|}((\alpha(x_k^i)a(x_k^i) + \beta(x_k^i)b(x_k^i))u(x_k^i)|Q_k^i|
\]

\[
= \sum_{Q_k^i \subset Q_\ell^j} u(x_k^i)|Q_k^i|.
\]

This shows that \( u_k \mathcal{L}^n \rightharpoonup u \mathcal{L}^n \). Similarly, we obtain that \( v_k \mathcal{L}^n \rightharpoonup v \mathcal{L}^n \). Also, we have that \( \|u_k\|_{L^1} \to \|u\|_{L^1} \) and \( \|v_k\|_{L^1} \to \|v\|_{L^1} \).
In addition,
\[
F(u_k, v_k) = \sum_{i \in I_k} \int_{\Omega^i} f(x, \frac{|u_k(x)|}{|u_k(x)| + |v_k(x)|}, \frac{u_k(x)}{|u_k(x)|}, \frac{v_k(x)}{|v_k(x)|}) (|u_k(x)| + |v_k(x)|) \, dx
\]
\[
= \sum_{i \in I_k} \int_{A_i^k} f(x, a(x^i_k), \frac{u(x^i_k)}{|u(x^i_k)|}, \frac{v(x^i_k)}{|v(x^i_k)|}) (|u(x^i_k)| + |v(x^i_k)|) \, dx
\]
\[
+ \sum_{i \in I_k} \int_{B_i^k} f(x, b(x^i_k), \frac{u(x^i_k)}{|u(x^i_k)|}, \frac{v(x^i_k)}{|v(x^i_k)|}) (|u(x^i_k)| + |v(x^i_k)|) \, dx.
\]
Because of the continuity of \(f\) and the fact that \(|A_i^k| = \alpha(x^i_k)\) and \(|B_i^k| = \beta(x^i_k)\), it follows that
\[
\lim_{k} F(u_k, v_k) = \int_{\Omega} \alpha(x) f\left(x, a(x), \frac{u(x)}{|u(x)|}, \frac{v(x)}{|v(x)|}\right) (|u(x)| + |v(x)|) \, dx
\]
\[
+ \int_{\Omega} \beta(x) f\left(x, b(x), \frac{u(x)}{|u(x)|}, \frac{v(x)}{|v(x)|}\right) (|u(x)| + |v(x)|) \, dx
\]
\[
= F_\ast(u, v).
\]
This proves the assertion for \(u, v \in C(\overline{\Omega}; \mathbb{R}^d)\).

Now let \(\mu, \nu \in \mathcal{M}(\Omega; \mathbb{R}^d)\) be arbitrary. Then we can define the regularised measures \(\mu_\varepsilon = \varphi_\varepsilon \ast \mu\) and \(\nu_\varepsilon = \varphi_\varepsilon \ast \nu\), where \(\varphi_\varepsilon\) is a (scaled) standard mollifier. Then \(\mu_\varepsilon \rightharpoonup \mu\) and \(\nu_\varepsilon \rightharpoonup \nu\). In addition, we have that \([[(\mu_\varepsilon, \nu_\varepsilon)](\Omega) \to ||(\mu, \nu)||(\Omega)\) (see [1] Thm. 2.2). As a consequence, the Reshetnyak continuity theorem [8] Thm. 10.3 implies that \(F_\ast(\mu_\varepsilon, \nu_\varepsilon) \to F_\ast(\mu, \nu)\). Now note that \(\mu_\varepsilon\) and \(\nu_\varepsilon\) are of the form \(\mu_\varepsilon = u_\varepsilon \mathcal{L}^n\) and \(\nu_\varepsilon = v_\varepsilon \mathcal{L}^n\) with continuous functions \(u_\varepsilon, v_\varepsilon\). Thus we can use the first part of this proof to find, for every \(\varepsilon > 0\), sequences \(u_\varepsilon^k, v_\varepsilon^k\) with \(u_\varepsilon^k \mathcal{L}^n \rightharpoonup u_\varepsilon \mathcal{L}^n, v_\varepsilon^k \mathcal{L}^n \rightharpoonup v \mathcal{L}^n\) and \(|u_\varepsilon^k|_{L^1} \to |u_\varepsilon|_{L^1}, |v_\varepsilon^k|_{L^1} \to |v_\varepsilon|_{L^1}\), such that \(F(u_\varepsilon^k, v_\varepsilon^k) \to F_\ast(u_\varepsilon, v_\varepsilon)\). By choosing an appropriate diagonal sequence, the claim of the theorem is proven. \(\Box\)

4.3. Proof of Theorem [1] We will now apply the results of the previous sections to the particular functional \(F: L^1(I; \mathbb{R}^d)^2 \to \mathbb{R}\),
\[
F(u, v) = -\int_I \left(\frac{u}{\sqrt{|u|}} + \frac{v}{\sqrt{|v|}}\right) \sqrt{|u||v|} \, dx.
\]
This can be written in the form required by Theorems [7] and [8] by defining \(f: \Sigma \times S^{d-1} \times S^{d-1} \to \mathbb{R}\),
\[
f(t, \xi, \zeta) = -\langle \xi, \zeta \rangle \sqrt{t(1-t)}.
\]
Note that we do not have any dependence on the \(x\)-variable.

According to Theorem [8] we require the convex hull \(f_\ast\) of the function \(f\) with respect to the \(t\) variable for fixed \(\xi\) and \(\zeta\). However, if \(\langle \xi, \zeta \rangle \geq 0\), then the mapping \(t \mapsto f(t, \xi, \zeta)\) is already convex and thus \(f(t, \xi, \zeta) = f_\ast(t, \xi, \zeta)\) in this case. Conversely, if \(\langle \xi, \zeta \rangle < 0\), then the convex hull of the mapping \(t \mapsto f(t, \xi, \zeta)\) is the constant function \(f_\ast(t, \xi, \zeta)\). This can be summarised to
\[
f_\ast(t, \xi, \zeta) = -\langle \xi, \zeta \rangle^+ \sqrt{t(1-t)}
\]
for every \((t, \xi, \zeta) \in \Sigma \times S^{d-1} \times S^{d-1}\). As a consequence, we have that
\[
\hat{F}(\mu, \nu) = -\int_I f_\ast\left(\frac{d|\mu|}{d(|\mu| + |\nu|)}, \frac{d\mu}{d|\mu|}, \frac{d\nu}{d|\nu|}\right) d(|\mu| + |\nu|)
\]
\[
= -\int_I \left(\frac{d\mu}{d|\mu|} + \frac{d\nu}{d|\nu|}\right) \sqrt{\frac{d|\mu|}{d(|\mu| + |\nu|)}} \sqrt{\frac{d|\nu|}{d(|\mu| + |\nu|)}} d(|\mu| + |\nu|).
\]
As discussed in the beginning of Section 4, this then implies that
\[ \hat{S}(c_1, c_2) = -\hat{F}(Dc_1, Dc_2) \]
\[ = \int I \frac{d\mu_k}{d\mu} \frac{d\nu_k}{d\nu} + \frac{d\mu}{d\mu} \frac{d\nu}{d\nu} d\mu d\nu, \]
which concludes the proof of Theorem 1.

5. The shape distance on \( BV(I; \mathbb{R}^d) \) and \( SBV(I; \mathbb{R}^d) \)

5.1. Distance between reparametrised BV functions. In this section, we will prove Propositions 3 and 4. To that end, we will need some results concerning the pointwise convergence of strictly convergent sequences of BV-functions. The basis for these is the following result on the properties of strictly convergent measures.

Lemma 9. Let \( \mu \in \mathcal{M}(I; \mathbb{R}^d) \) and assume that \( \mu_k \rightharpoonup \mu \) such that \( \mu_k(I) \to \mu(I) \). If \( U \subseteq I \) is open with \( \mu(\partial U) = 0 \), then \( \mu_k(U) \to \mu(U) \).

Proof. By [18, Cor. 10.2] we obtain that \( |\mu_k| \to |\mu| \). Now we can use [9, Cor. 1.204] to obtain the assertion. \( \square \)

Lemma 10. Let \( c \in BV(I; \mathbb{R}^d) \), and assume that the sequence of functions \( c_k \in BV(I; \mathbb{R}^d) \) converges strictly to \( c \). Then for every \( x \in I \) the set of accumulation points of the sequence \( c_k(x) \) is contained in the interval \([c^l(x), c^r(x)]\). In particular, if \( c \) is continuous at \( x \), then \( c_k(x) \to c(x) \).

Proof. Let \( z \in \mathbb{R}^d \) be any accumulation point of the sequence \( c_k(x) \). After possibly passing to a subsequence, we may then assume that \( c_k(x) \to z \). Assume now that \( z \notin [c^l(x), c^r(x)] \) and denote
\[ \epsilon := |z - c^l(x)| + |z - c^r(x)| - |c^l(x) - c^r(x)| > 0. \]
Since \( c_k \to c \) weakly* in \( BV(I; \mathbb{R}^d) \), it follows that \( c_k(y) \to c(y) \) for almost every \( y \in I \). We can therefore choose \( x_0 < x \) and \( x_1 > x \) such that \( c_k(x_0) \to c(x_0) \), \( c_k(x_1) \to c(x_1) \), all the functions \( c \) and \( c_k \) are continuous at \( x_0 \) and \( x_1 \), \( |Dc|(x_0, 0) + |Dc|(x_1, 1) < \epsilon/4 \), \( |c(x_0) - c^l(x)| < \epsilon/4 \), and \( |c(x_1) - c^r(x)| < \epsilon/4 \). Then
\[ |Dc|(I) = |Dc|(0, x_0) + |Dc|(x_0, x_1) + |Dc|(x_1, 1). \]
We have that \( |Dc|(0, x_0) = \lim_k |Dc_k|(0, x_0) \) and \( |Dc_2|(x_1, 1) \). Moreover,
\[ |Dc|(x_0, x_1) = |Dc|(x_0, x) + |Dc|(x, x_1) + |c^l(x) - c^r(x)| \]
\[ < |z - c^l(x)| + |z - c^r(x)| - \frac{3\epsilon}{4} < |z - c(x_0)| + |z - c(x_1)| - \frac{\epsilon}{4}. \]
However, we have that
\[ \lim_k \inf |Dc_k|(x_0, x_1) \geq \lim_k \inf |c_k(x_0) - c_k(x)| + |c_k(x) - c_k(x_1)| \]
\[ = |z - c(x_0)| + |z - c(x_1)| > |Dc|(x_0, x_1) + \frac{\epsilon}{4}. \]
Combining the results above, we obtain that \( \lim \inf_k |Dc_k|(I) > |Dc|(I) + \epsilon/4 \), which contradicts the assumption that \( c_k \) converges strictly to \( c \). \( \square \)

Lemma 11. Let \( c \in BV(I; \mathbb{R}^d) \), and assume that the sequence of functions \( c_k \in BV(I; \mathbb{R}^d) \) converges strictly to \( c \). Assume moreover that \( x \in I \) is such that \( c \) is continuous at \( x \). Then we have for every sequence \( x_k \to x \) that \( c_k(x_k) \to c(x) \).
Proof. We can estimate
\[ |c_k(x_k) - c(x)| \leq |c_k(x_k) - c_k(x)| + |c_k(x) - c(x)|. \]
The function \( c \) is continuous at \( x \) and thus, in view of Lemma 11, the last term tends to zero as \( k \to \infty \). For the first term, we can estimate
\[ |c_k(x_k) - c_k(x)| \leq |Dc_k|(x_k, x). \]
Let now \( \varepsilon > 0 \) be such that \( |Dc([x - \varepsilon, x + \varepsilon])| = 0 \). Then we have for sufficiently large \( k \) that \( [x_k, x] \subset (x - \varepsilon, x + \varepsilon) \) and thus
\[ \limsup_k |Dc_k|(x_k, x) \leq \limsup_k |Dc_k|(x - \varepsilon, x + \varepsilon) = |Dc|(x - \varepsilon, x + \varepsilon). \]
Since this holds for almost every \( \varepsilon > 0 \) we obtain that
\[ \limsup_k |Dc_k|(x_k, x) \leq |Dc|(x) = 0. \]
Combining all the estimates, we arrive at the claim. \( \Box \)

Proof of Proposition 3. Let \( c_i \in BV(I; \mathbb{R}^d) \) and \( \varphi_i \in \bar{\Gamma} \) be fixed.

Assume that \( \{ c_i^{(k)} \}_{k \in \mathbb{N}} \subset AC(I; \mathbb{R}^d) \) converge strictly to \( c_i \) and that \( \{ \varphi_i^{(k)} \}_{k \in \mathbb{N}} \subset \Gamma \) converge uniformly to \( \varphi_i \). Since \( \text{len}(c_i^{(k)} \circ \varphi_i^{(k)}) = \text{len}(c_i^{(k)}) \to \text{len}(c_i) \), the functions \( c_i^{(k)} \circ \varphi_i^{(k)} \) are uniformly bounded in \( BV(I; \mathbb{R}^d) \). After possibly passing to a subsequence, we may therefore assume that \( c_i^{(k)} \circ \varphi_i^{(k)} \to^* g_i \) for some \( g_i \in BV(I; \mathbb{R}^d) \). Now let \( \varepsilon > 0 \). Then there exist \( 0 < x_1 < \ldots < x_N < 1 \) such that \( c_i \) is continuous at \( \varphi_1(x_\ell) \) for each \( 1 \leq \ell \leq N \), and
\[ \text{len}(c_i) \leq \sum_{\ell=1}^{N-1} |c_i(\varphi_1(x_{\ell+1})) - c_i(\varphi_1(x_\ell))| + \varepsilon. \]
In addition, we can choose the points \( x_\ell \) in such a way that \( c_i^{(k)}(\varphi_i^{(k)}(x_\ell)) \to g_i(x_\ell) \).
Since \( \varphi_1^{(k)}(x_\ell) \to \varphi_1(x_\ell) \) for all \( \ell \) and \( c_i^{(k)} \to^* c_i \), we obtain from Lemma 11 that
\[ \text{len}(g_i) \geq \sum_{\ell=1}^{N-1} |g_i(x_{\ell+1}) - g_i(x_\ell)| = \lim_{k \to \infty} \sum_{\ell=1}^{N-1} |c_i^{(k)}(\varphi_1^{(k)}(x_{\ell+1})) - c_i^{(k)}(\varphi_1^{(k)}(x_\ell))| \]
\[ = \sum_{\ell=1}^{N-1} |c_i(\varphi_1(x_{\ell+1})) - c_i(\varphi_1(x_\ell))| \geq \text{len}(c_i) - \varepsilon. \]
Since \( \varepsilon \) was arbitrary, this shows that \( \text{len}(g_i) \geq \text{len}(c_i) = \lim_k \text{len}(c_i^{(k)} \circ \varphi_i^{(k)}) \), which in turn shows that, actually, \( c_i^{(k)} \circ \varphi_i^{(k)} \) converges strictly to \( g_i \). Similarly, we obtain that \( c_2^{(k)} \circ \varphi_2^{(k)} \to^* g_2 \). Because \( \hat{d} \) is strictly lower semi-continuous, it follows that
\[ \hat{d}(g_1, g_2) \leq \liminf_{k \to \infty} \hat{d}(c_1^{(k)} \circ \varphi_1^{(k)}, c_2^{(k)} \circ \varphi_2^{(k)}) \leq \liminf_{k \to \infty} D(c_1^{(k)}, c_2^{(k)}; \varphi_1^{(k)}, \varphi_2^{(k)}). \]
Next note that it follows from Lemma 11 that \( g_i \in [c_i, \varphi_i] \), and thus
\[ \inf_{g_i \in [c_i, \varphi_i]} \hat{d}(g_1, g_2) \leq \liminf_{k \to \infty} D(c_1^{(k)}, c_2^{(k)}; \varphi_1^{(k)}, \varphi_2^{(k)}). \]
Since the sequences \( \{ c_i^{(k)} \}_{k \in \mathbb{N}} \) and \( \{ \varphi_i^{(k)} \}_{k \in \mathbb{N}} \) converging to \( c_i \) and \( \varphi_i \), respectively, were arbitrary, and \( \hat{D} \) is the lower semi-continuous hull of \( D \), it follows that
\[ \inf_{g_i \in [c_i, \varphi_i]} \hat{d}(g_1, g_2) \leq \hat{D}(c_1, c_2; \varphi_1, \varphi_2). \]
Now assume that \( g_i \in [c_i, \varphi_i] \) and that \( \{g_i^{(k)}\}_{k \in \mathbb{N}} \subset AC(I; \mathbb{R}^d) \) converge strictly to \( g_i \). Define
\[
\varphi_i^{(k)} = \frac{1}{k} \text{Id} + \left(1 - \frac{1}{k}\right) \varphi_i,
\]
and let
\[
c_i^{(k)} := g_i^{(k)} \circ (\varphi_i^{(k)})^{-1}.
\]
Then \( \varphi_i^{(k)} \in \Gamma, c_i^{(k)} \in AC(I; \mathbb{R}^d), \) and \( \varphi_i^{(k)} \to \varphi \) uniformly. Moreover, by definition of \( c_i^{(k)} \), and since \( \text{len}(g_i) = \text{len}(c_i) \), we have that \( \text{len}(c_i^{(k)}) = \text{len}(g_i^{(k)}) \to \text{len}(g_i) = \text{len}(c_i). \) Next, we show that \( c_i^{(k)}(y) \to c_i(y) \) at every point \( y \) where \( c_i \) is continuous. To that end, let \( y \) be such that \( \varphi_i^{(k)}(y) = \varphi_i(y) \). Then \( c_i^{(k)}(y) = g_i^{(k)}((\varphi_i^{(k)})^{-1}(y)) \).

Since the sequence \( (\varphi_i^{(k)})^{-1}(y) \) is bounded, it has a convergent subsequence, say \( (\varphi_i^{(k)})^{-1}(y) \to z \). Since \( \varphi_i^{(k)} \) converges pointwise to \( \varphi_i \), it follows that \( y = \varphi_i(z) \).

As a consequence, we have that \( g_i(z) \in [\varphi_i^{(k)}(z), \varphi_i(\varphi_i(z))] = [c_i(y), c_i^{(k)}(y)] = \{c_i(y)\} \), which implies in particular that \( g_i \) is continuous at \( z \). Since \( g_i^{(k)} \to g_i \) and \( (\varphi_i^{(k)})^{-1}(y) \to z \), it follows from Lemma 11 that
\[
c_i^{(k)}(y) = g_i^{(k)}((\varphi_i^{(k)})^{-1}(y)) \to g_i(z) = c_i(y).
\]

Since this holds for every convergent subsequence, we now obtain that \( c_i^{(k)}(y) \to c_i(y) \). Together with the convergence \( \text{len}(c_i^{(k)}) \to \text{len}(c_i) \), this implies that \( c_i^{(k)} \to c_i \). Similarly we obtain that \( c_2^{(k)} \to c_2 \).

Since \( D \) is the lower semi-continuous hull of \( D \) with respect to strict convergence in the \( c_i \) components and uniform convergence in the \( \varphi_i \) components, it follows that
\[
\hat{D}(c_1, c_2; \varphi_1, \varphi_2) \leq \liminf_k D(c_1^{(k)}, c_2^{(k)}; \varphi_1^{(k)}, \varphi_2^{(k)})
= \liminf_k d(c_1^{(k)} \circ \varphi_1^{(k)}, c_2^{(k)} \circ \varphi_2^{(k)}) = \liminf_k d(g_1^{(k)}, g_2^{(k)}).
\]

Since \( g_i \in [c_i, \varphi_i] \) were arbitrary, and \( \{g_i^{(k)}\}_{k \in \mathbb{N}} \subset AC(I; \mathbb{R}^d) \) were arbitrary sequences converging strictly to \( g_i \) it follows from the definition of \( \hat{D} \) that
\[
\hat{D}(c_1, c_2; \varphi_1, \varphi_2) \leq \inf_{g_i \in [c_i, \varphi_i]} \inf_{\{g_i^{(k)}\}_{k \in \mathbb{N}} \subset AC(I; \mathbb{R}^d)} d(g_1^{(k)}, g_2^{(k)}) = \inf_{g_i \in [c_i, \varphi_i]} \hat{d}(g_1, g_2),
\]
which concludes the proof.

Proof of Proposition 4 Define
\[
\psi^{(k)} := \frac{1}{k} \text{Id} + \left(1 - \frac{1}{k}\right) \psi.
\]
Then \( \|\psi^{(k)} - \psi\|_\infty \leq 2/k \). Moreover, \( \psi^{(k)} \) satisfies \( (\psi^{(k)})' \geq 1/k \) almost everywhere, and thus \( \psi^{(k)} \) is invertible with inverse \( \vartheta^{(k)} := (\psi^{(k)})^{-1} \in \Gamma \).

Now assume that the sequences \( \{c_i^{(k)}\}_{k \in \mathbb{N}} \subset AC(I; \mathbb{R}^d) \) converge strictly to \( c_i \) and that \( (\varphi_i^{(k)})_{k \in \mathbb{N}} \subset \Gamma \) converge uniformly to \( \varphi_i \). Then
\[
\|\varphi_i^{(k)} \circ \psi^{(k)} - \varphi_i \circ \psi\|_\infty \leq \|\varphi_i^{(k)} \circ \psi^{(k)} - \varphi_i \circ \psi^{(k)}\|_\infty + \|\varphi_i \circ \psi^{(k)} - \varphi_i \circ \psi\|_\infty.
\]

Now the first term on the right hand side converges to zero because of the uniform convergence of \( \varphi_i^{(k)} \) to \( \varphi_i \), and the second term converges to zero because of the uniform continuity of \( \varphi_i \) and the uniform convergence of \( \psi^{(k)} \to \psi \). Thus \( \varphi_i^{(k)} \circ \psi^{(k)} \) converges uniformly to \( \varphi_i \circ \psi \). Since \( D \) is lower semi-continuous with respect to strict
Thus, \( \gamma \) can thus assume without loss of generality that all the functions there exist sequence, we may therefore assume without loss of generality that continuous with Lipschitz constant at most 2. After possibly passing to a subsequence, we have

\[
\bar{\gamma}(x) = \lim_{k \to \infty} \gamma_{i_k}(x)
\]

for some \( i_k \) such that \( \gamma_{i_k} \to \bar{\gamma} \). Since by construction the functional \( \bar{\gamma} \) is invariant under simultaneous reparametrisations, it follows that \( \bar{\gamma} \) is a minimising sequence for \( \bar{\gamma} \).

Conversely, assume that \( \gamma_{i}(k) \to \gamma_i \) \( \gamma_i \) := \( \gamma_i \circ \psi \) uniformly. Then we can similarly estimate

\[
\| \gamma_i(k) \circ \theta - \gamma_i \|_{\infty} \leq \| \gamma_i(k) \circ \theta - \gamma_i \circ \theta \|_{\infty} + \| \gamma_i \circ \theta - \gamma_i \circ \theta \|_{\infty}.
\]

Again, the first term converges to zero because of the uniform convergence of \( \gamma_i(k) \) to \( \gamma_i \). For the second term, let \( x \in I \) and let \( y \in I \) with \( \psi(y) = x \). Then

\[
| \gamma_i(\theta(k)(x)) - \gamma_i(x) | = | \gamma_i(\psi(\theta(k)(\psi(y)))) - \gamma_i(\psi(y)) |
\]

Since \( \| \psi - \theta(k) \| \leq 2/k \), there exists \( \varepsilon_k \) with \( \| \varepsilon_k \| \leq 2/k \) such that

\[
| \psi(\theta(k)(\psi(y))) - \psi(k)(\psi(y)) | = \| \psi(k)(\psi(y)) + \varepsilon_k - \psi(y) + \varepsilon_k |.
\]

Thus we have that

\[
| \gamma_i(\theta(k)(x)) - \gamma_i(x) | = | \gamma_i(\psi(y) + \varepsilon_k) - \gamma_i(\psi(y)) |
\]

Because of the uniform continuity of \( \gamma_i \), this implies that the second term in (6) converges to zero, and therefore \( \| \gamma_i(k) \circ \theta - \gamma_i \|_{\infty} \to 0 \). With the same argumentation as before, this now implies that

\[
\tilde{D}(c_1, c_2, \varphi_1, \varphi_2) \leq \tilde{D}(c_1, c_2, \varphi_1 \circ \psi, \varphi_2 \circ \psi).
\]

With Proposition 3 we now arrive at the assertion.

5.2. Proof of Theorem 6. We start with showing that the infima in (4) are attained.

**Proposition 12.** Assume that \( c_1, c_2 \in BV(I; \mathbb{R}^d) \). Then the optimisation problem

\[
\inf_{(\varphi_1, \varphi_2) \in \Gamma} \tilde{d}([c_1, \varphi_1], [c_2, \varphi_2])
\]

admits a solution \( (\varphi_1, \varphi_2) \). Moreover, there exist \( \varphi_i \in [c_1, \varphi_i] \) such that

\[
\tilde{d}([c_1, \varphi_1], [c_2, \varphi_2]) = \tilde{d}(\varphi_1, \varphi_2).
\]

**Proof.** We follow the proof of [5, Proposition 15].

Assume that \( (\varphi_1(k), \varphi_2(k))_{k \in \mathbb{N}} \) is a minimising sequence for \( \tilde{d}([c_1, \cdot], [c_2, \cdot]) \). Then there exist \( \psi(k) \in \Gamma \) such that \( \varphi_i(k) = \gamma_i(k) \circ \psi(k) \), where \( \gamma_i(k) \in \Gamma \) are such that \( (\varphi_1(k))' \circ \psi(k) = \gamma_i(k) \circ \psi(k) \) for a.e. \( x \in I \). By Proposition 3 we have that

\[
\tilde{d}([c_1, \varphi_1(k)], [c_2, \varphi_2(k)]) = \tilde{d}([c_1, \gamma_1(k) \circ \psi(k)], [c_2, \gamma_2(k) \circ \psi(k)]) = \tilde{d}((c_1, \gamma_1(k)], [c_2, \gamma_2(k)])
\]

Thus \( (\gamma_1(k), \gamma_2(k)) \) is a minimising sequence as well. After replacing \( \psi(k) \) by \( \gamma_i(k) \), we can thus assume without loss of generality that all the functions \( \varphi_i(k) \) are Lipschitz continuous with Lipschitz constant at most 2. After possibly passing to a subsequence, we may therefore assume without loss of generality that \( \varphi_i(k) \to \bar{\varphi}_i \), for some \( \bar{\varphi}_i \in \Gamma \). Since by construction the functional \( (\varphi_1, \varphi_2) \to \tilde{d}([c_1, \varphi_1], [c_2, \varphi_2]) \) is lower semi-continuous with respect to uniform convergence, it follows that \( (\varphi_1, \varphi_2) \) is a minimiser of \( \tilde{d}([c_1, \cdot], [c_2, \cdot]) \).
Now, by definition,
\[ d([c_1, \bar{\varphi}_1], [c_1, \bar{\varphi}_2]) = \inf_{g \in [c_1, \bar{\varphi}_1]} \bar{\mathcal{S}}(g_1, g_2). \]

Since \( \bar{\mathcal{S}} \) is strictly lower semi-continuous and the sets \([c_1, \bar{\varphi}_1]\) are compact with respect to strict convergence, the existence of \( \bar{g}_1 \) and \( \bar{g}_2 \) follows. \( \square \)

Next we will show that the functions \( \bar{g}_1 \) in Proposition [12] can be chosen in \( \text{SBV}(I; \mathbb{R}^d) \), if \( c_i \in \text{SBV}(I; \mathbb{R}^d) \). For that, we need two preparatory results.

**Lemma 13.** Assume that \( c \in \text{SBV}(I; \mathbb{R}^d) \), that \( \varphi \in \bar{\Gamma} \), and that \( g \in [c, \varphi] \). Then the singular part \( D^s g \) of the measure \( Dg \) is concentrated on \( \varphi^{-1}(\Sigma(c)) \).

**Proof.** Write \( c = c^{(a)} + c^{(j)} \). Then \( g = c^{(a)} \circ \varphi + h \) for some \( h \in [c^{(j)}, \varphi] \). We have that \( c^{(a)} \circ \varphi \in AC(I; \mathbb{R}^d) \), which implies that \( D^s c = D^s h \). We may therefore assume without loss of generality that \( c^{(a)} = 0 \), \( c = c^{(j)} \), and \( g \in [c^{(j)}, \varphi] \).

Let \( x \in I \) and write \( \varphi^{-1}(x) = [a, b] \) with \( a \leq b \). Then there exists a sequence \( a_k \to a^- \) such that \( c \) is continuous at each \( \varphi(a_k) \). In particular, we have that \( g(a_k) = c(\varphi(a_k)) \) for each \( k \). Since \( g^\prime \) and \( c^\prime \) are left continuous and \( \varphi(a_k) \to \varphi(a) = x \), it follows that \( g^\prime(a) = \lim_{k \to \infty} g(a_k) = \lim_{k \to \infty} c(\varphi(a_k)) = c^\prime(x) \). Similarly, we obtain that \( g^\prime(b) = c^\prime(x) \). This implies in particular that

\[ |Dg|(|\varphi^{-1}(x)|) \geq |g^\prime(b) - g^\prime(a)| = |c^\prime(x) - c^\prime(x)| \]

for every \( x \in I \). Thus we have that

\[ |Dc|(I) = |Dg|(I) \geq |Dg|(|\varphi^{-1}(\Sigma(c))|) = \sum_{x \in \Sigma(c)} |Dg|(|\varphi^{-1}(x)|) \geq \sum_{x \in \Sigma(c)} |c^\prime(x) - c^\prime(x)| = |D^s c|(I) = |Dc|(I). \]

This shows that \( |Dg|(|\varphi^{-1}(\Sigma(c))|) = |Dg|(I) \), which in turn implies that \( |Dg|(I \setminus \varphi^{-1}(\Sigma(c))) = 0 \). \( \square \)

**Lemma 14.** Assume that \( U \subset \mathbb{R}^d \) is a Borel set, \( \nu \in \mathcal{M}_+(U) \setminus \{0\} \) is a non-trivial positive Radon measure on \( U \), and \( g: U \to \mathbb{R}_{\geq 0} \) is a non-negative Borel function on \( U \) with \( \int_U g(x) \, d\nu > 0 \). Assume moreover that \( \mu \in \mathcal{M}_+(U) \) solves the optimisation problem

\[ (7) \quad F(\mu) := \int_U g(x) \sqrt{\frac{d\mu}{d(\mu + \nu)} \frac{d\nu}{d(\mu + \nu)}} \, d(\mu + \nu) \to \max_{\mu \in \mathcal{M}_+(U)} \frac{\mu(B_\varepsilon(x))}{\mu(B\varepsilon(x)) + \nu(B\varepsilon(x))} = 1 \]

Then \( \mu \ll \nu \).

**Proof.** Decompose \( \mu = \mu^a + \mu^s \) with \( \mu^a \ll \nu \) and \( \mu^s \perp \nu \). Then \( \mu^s \) is concentrated on the set

\[ E := \left\{ x \in U : \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{\nu(B_\varepsilon(x))} = \infty \right\}. \]

Moreover, we have for \( (\mu + \nu) \)-a.e. \( x \in E \) that

\[ \frac{d\nu}{d(\mu + \nu)}(x) = \lim_{\varepsilon \to 0} \frac{\nu(B_\varepsilon(x))}{\mu(B_\varepsilon(x)) + \nu(B_\varepsilon(x))} = 0. \]

As a consequence, \( F(\mu) = F(\mu^a) \). Now denote by \( u \in L^1(U; \nu) \) the density of \( \mu^a \) with respect to \( \nu \), that is, \( \mu^a = u \nu \). Then

\[ \frac{d\mu}{d(\mu + \nu)} = \frac{u}{u + 1} \quad \text{and} \quad \frac{d\nu}{d(\mu + \nu)} = \frac{1}{u + 1}. \]
Proposition 16. Since the sets $\{\text{jump points}\}$ of $g$ is in this case the trivial functional $F \equiv 0$. Moreover, we have is in the proof of Lemma 14 that $\hat{\mu} \in P$. Let $\nu \in \mathcal{M}(\mathbb{R}^d)$, where the maximum is taken over all $\nu \in \mathcal{M}(\mathbb{R}^d)$, it follows that $\mu^a \ll \nu$ and $\int_U g(x) d\nu > 0$ we have that $F(\mu^a) = 0$. Thus the functional $F$ is in this case the trivial functional $F(\mu) \equiv 0$.

Remark 15. Let $F$ be as in (4), but assume that $\int_U g(x) d\nu = 0$. Let moreover $\mu \in \mathcal{P}(U)$. Then we can again decompose $\mu = \mu^a + \mu^s$ with $\mu^a \ll \nu$ and $\mu^s \perp \nu$. Moreover, we have is in the proof of Lemma 14 that $F(\mu) = F(\mu^a)$. However, because $\mu^a \ll \nu$ and $\int_U g(x) d\nu = 0$ we have that $F(\mu^a) = 0$. Thus the functional $F$ is in this case the trivial functional $F(\mu) \equiv 0$.

Proposition 16. Assume that $c_1 \in \text{SBV}(I; \mathbb{R}^d)$ and that $\varphi_i \in \hat{\Gamma}$ satisfy $\varphi_1^+ + \varphi_2^+ = 2$ almost everywhere. Then there exist functions $\hat{g}_i \in [c_1, \varphi_i]$ such that $\hat{g}_i \in \text{SBV}(I; \mathbb{R}^d)$ and

$$d([c_1, \varphi_1], [c_2, \varphi_2]) = d(g_1, g_2).$$

Proof. Since the sets $[c_1, \varphi_1]$ are strictly compact and $d([c_1, \cdot], [c_2, \cdot])$ is strictly lower semi-continuous, there exist $g_1 \in [c_1, \varphi_1]$ such that $d([c_1, \varphi_1], [c_2, \varphi_2]) = d(g_1, g_2)$.

We will show that it is possible to replace $g_1$ and $g_2$ by functions $\hat{g}_1, \hat{g}_2 \in \text{SBV}(I; \mathbb{R}^d)$ in such a way that $d(g_1, g_2) = d(\hat{g}_1, \hat{g}_2)$, or, equivalently, $\hat{S}(g_1, g_2) = \hat{S}(\hat{g}_1, \hat{g}_2)$.

According to Lemma 13 the singular part of $Dg_1$ is concentrated on $\varphi_1^{-1}(\Sigma(c_1))$. Now assume that $y \in \Sigma(c_1)$. Since $\varphi_1$ is continuous and non-decreasing it follows that $\varphi_1^{-1}(y)$ is either a single point or a closed interval. Denote now by $R \subset \Sigma(c_1)$ the set of jump points $y$ of $c_1$ for which $\varphi_1^{-1}(y)$ is a non-degenerate interval. Then the non-atomic part $D^0 g_1$ of $Dg_1$ is concentrated on the set $E := \bigcup_{y \in R} \text{int} \varphi_1^{-1}(y)$, since $\varphi_1^{-1}(\Sigma(c_1)) \setminus E$ is at most countable.

Let now $y \in R$ and denote $[a, b] := \varphi_1^{-1}(y)$. By assumption, the function $g_1$ solves the optimisation problem $\max_{\tilde{S}(g_1, g_2)}$.

Moreover, we have that

$$Dg_1 \mathbf{L}(a, b) = v \int Dg_1 \mathbf{L}(a, b) \quad \text{with} \quad v = \frac{c_1^1(y) - c_1^2(y)}{c_1^2(y) - c_1^1(y)} \in \mathbb{R}^d.$$

Now let $\mu \in M_+(a, b)$ be a positive Radon measure satisfying $\mu(a, b) = |Dg_1|(a, b)$. Then the function $\hat{g}_1$ defined by

$$\hat{g}_1 = \begin{cases} g_1(x) & \text{if } x \notin (a, b), \\ g_1^a(a) + v\mu(a, x) & \text{if } x \in (a, b), \end{cases}$$

satisfies $\hat{g}_1 \in [c_1, \varphi_1]$, and $\hat{Dg_1} \mathbf{L}(a, b) = v\mu$ and $\hat{Dg_1} \mathbf{L}(I \setminus (a, b)) = Dg_1 \mathbf{L}(I \setminus (a, b))$. Thus $\hat{Dg_1} \mathbf{L}(a, b)$ solves the optimisation problem

$$\int_{(a, b)} \left\langle v, \frac{dDg_2}{d[Dg_2]} \right\rangle^+ \sqrt{\frac{d\mu}{d(\mu + |Dg_2|)}} \frac{d[Dg_2]}{d(\mu + |Dg_2|)} \rightarrow \max$$

where the maximum is taken over all $\mu \in M_+(a, b)$ with $\mu(a, b) = |Dg_1|(a, b)$.

Assume now that $\int_{(a, b)} \langle v, dDg_2/d[Dg_2] \rangle^+ d[Dg_2] > 0$. Then we obtain from Lemma 13 that $\hat{Dg_1} \mathbf{L}(a, b) \ll |Dg_2| \mathbf{L}(a, b)$. Since $\varphi_1$ is constant on $[a, b]$ and $\varphi_1^+ + \varphi_2^+ = 2$ almost everywhere, it follows that $\varphi_2'$ is strictly increasing on $(a, b)$. Thus $\varphi_2^{-1}(\Sigma(c_1)) \cap (a, b)$ is an at most countable union of single points. Since by Lemma 13 the measure $D^0 g_2 \mathbf{L}(a, b)$ is concentrated on $\varphi_2^{-1}(\Sigma(c_1)) \cap (a, b)$,
it is purely atomic. Thus $|D^*g_1| \mathbb{L}(a,b)$ is purely atomic as well and therefore $|D^*g_1| \mathbb{L}(a,b) = 0$.

Now assume that $\int_{(a,b)} |v| \, dDg_2/d|Dg_2| + d|Dg_2| = 0$. As seen in Remark 17, we may assume without loss of generality that the integral in (8) is equal to zero for all choices of $\mu$. We can therefore replace $g_1$ on the interval $(a,b)$ by the function

$$g_1(x) = g_1^c(a) + \frac{x-a}{b-a}(g_1^c(b) - g_1^c(a)),$$

corresponding to a choice of $\mu = s\mathcal{L}^1$ with $s = |Dg_1|(a,b)/(b-a)$ and have that $\hat{S}(g_1,g_2) = S(g_1,g_2)$.

Repeating this procedure first for each $y \in R$ and then for the function $g_2$, we arrive at the claim. \hfill \Box

**Proposition 17.** Let $c_1, c_2 \in SBV(I; \mathbb{R}^d) \setminus \{0\}$. Then

$$\min_{(\varphi_1, \varphi_2) \in \Gamma} \hat{d}([c_1, \varphi_1], [c_2, \varphi_2]) = \min_{(\psi_1, \psi_2) \in \Gamma} \hat{d}(G(c_1) \circ \psi_1, G(c_2) \circ \psi_2).$$

**Proof.** We start by recalling the construction of the functions $G(c_i)$ in Section 3.4. We set

$$\xi_i(x) = \frac{|D^*c_i|(0,x)}{2 \text{len}(c_i)} + (1 - \alpha_i)x \quad \text{with} \quad \alpha_i := \frac{|D^*c_i|(I)}{2 \text{len}(c_i)}.$$

Moreover, we denote by $\zeta_i : I \to I$ the non-decreasing left inverse of $\xi_i$. Then the functions $G(c_i)$ satisfy $G(c_i)(x) = c_i(\zeta_i(x))$ for $x \notin \zeta(\Sigma(c_i))$ and $G(c_i)(\xi_i(x)) = c_i(x)$ for $x \notin \Sigma(c_i)$. Moreover, we have $G(c_i) \in [c_i, \zeta_i]$. Assume now that $(\psi_1, \psi_2) \in \Gamma$. Define $\varphi_i = \zeta_i \circ \psi_i$. Since $G(c_i) \in [c_i, \zeta_i]$, it follows that $G(c_i) \circ \psi_i \in [c_i, \varphi_i]$. Thus

$$\hat{d}([c_1, \varphi_1], [c_2, \varphi_2]) \leq \hat{d}(G(c_1) \circ \psi_1, G(c_2) \circ \psi_2).$$

Since $(\psi_1, \psi_2)$ was arbitrary, this shows that the inequality $\leq$ holds in (9).

Now assume that the maximum of $\hat{d}([c_1, ], [c_2, ])$ is attained at $(\varphi_1, \varphi_2) \in \Gamma^2$. In view of the proof of Proposition 12, we may assume without loss of generality that $\varphi_1 + \varphi_2 = 2$ almost everywhere in $I$. Since $c_1, c_2 \in SBV(I; \mathbb{R}^d)$, there exist by Proposition 16 functions $g_1, g_2 \in SBV(I; \mathbb{R}^d)$ such that $g_i \in [c_i, \varphi_i]$ and $\hat{d}([c_1, \varphi_1], [c_2, \varphi_2]) = \hat{d}(g_1, g_2)$.

We will now construct functions $h_1, h_2 \in AC(I; \mathbb{R}^d)$ in such a way that $\hat{d}(g_1, g_2) = \hat{d}(h_1, h_2)$. The construction is similar as for $G(c_i)$, but we have to be careful to keep the function value of $\hat{d}$ unchanged. We denote therefore

$$\beta = \frac{|D^*g_1|(I) + |D^*g_2|(I)}{2(\text{len}(c_1) + \text{len}(c_2))},$$

and define the function $\gamma : I \to I$,

$$\gamma(x) = \frac{|D^*g_1|(0,x) + |D^*g_2|(0,x)}{2(\text{len}(c_1) + \text{len}(c_2))} + (1 - \beta)x.$$

Next we denote by $\vartheta : I \to I$ the non-decreasing left inverse of $\gamma$, and define the functions $h_i \in AC(I; \mathbb{R}^d)$ by $h_i(x) = g_i(\vartheta(x))$ if $\gamma$ (and thus $g_i$) is continuous at $\vartheta(x)$ and

$$h_i(x) = g_i^c(\vartheta(x)) + \frac{x - \gamma^c(\vartheta(x))}{\gamma^c(\vartheta(x)) - \gamma^c(\vartheta(x))}(g_i^c(\vartheta(x)) - g_i^c(\vartheta(x)))$$

else.
Now assume that $x \in I$ is such that $y := \vartheta(x) \in \Sigma(g_1) \cup \Sigma(g_2)$. Then we have by construction of $h_i$ that
\[
\hat{h}_i(x) = \frac{g_i^*(y) - g_i^*(y)}{\gamma^*(y) - \gamma^*(y)} = \frac{|g_i|(y)}{|\gamma|(y)}.
\]
Thus
\[
\hat{S}(h_1, h_2) = \int_\gamma \left( \frac{\hat{h}_1}{|\hat{h}_1|} \hat{h}_2 + \sqrt{|\hat{h}_1||\hat{h}_2|} \right) dx
\]
\[
= \int_\gamma \left( \frac{\hat{g}_1 \circ \vartheta \circ \vartheta}{|\hat{g}_1||\hat{g}_2|} + \frac{\hat{g}_2 \circ \vartheta}{|\hat{g}_2|} \right) + \frac{\sqrt{|\gamma^*(y)|}}{|\gamma^*(y)|} \gamma^*(y) \right) dx
\]
\[
= \int_\gamma \left( \frac{\hat{g}_1}{|\hat{g}_1|} \frac{\hat{g}_2}{|\hat{g}_2|} + \frac{\sqrt{|\gamma^*(y)|}}{|\gamma^*(y)|} \gamma^*(y) \right) dx
\]
\[
= \hat{S}(g_1, g_2).
\]
Since $\text{len}(g_i) = \text{len}(h_i)$, this implies that also $\hat{d}(h_1, h_2) = \hat{d}(g_1, g_2)$. We will next construct functions $\psi_i \in \Gamma$ such that $h_i = G(c_i) \circ \psi_i$.

We start by defining the function $\hat{\psi}_i := \xi_i \circ \varphi_i \circ \vartheta$. Since $\vartheta$ is Lipschitz, $\varphi_i$ is absolutely continuous, and $\xi_i \in \text{SBV}(I; \mathbb{R})$, it follows that $\hat{\psi}_i \in \text{SBV}(I; \mathbb{R})$ as well. Moreover we have that $G(c_i) (\hat{\psi}_i(x)) = h_i(x)$ for every $x \in I$ such that $\varphi_i \circ \vartheta(x) \notin \Sigma(c_i)$. We may thus define $\psi_i := \hat{\psi}_i$ for $x \in I \setminus (\varphi_i \circ \vartheta)^{-1}(\Sigma(c_i))$.

Now let $y \in \Sigma(c_i)$ and denote $[a, b] := (\varphi_i \circ \vartheta)^{-1}(y)$. Let moreover $x \in [a, b]$. Then $h_i(x)$ lies on the line segment $[c_i^*(y), c_i^*(y)]$, that is, we can write
\[
h_i(x) = \lambda(x) c_i^*(y) + (1 - \lambda(x)) c_i^*(y) = c_i^*(y) + \lambda(x) [c_i](y)
\]
for some $0 < \lambda(x) < 1$. Moreover, since $h_i$ is absolutely continuous, it follows that the mapping $x \mapsto \lambda(x)$ is absolutely continuous. Define now
\[
\psi_i(x) = \xi_i^*(y) + \lambda(x) [\xi_i](y).
\]
Then $\xi_i(\psi_i(x)) = y$ and thus
\[
G(c_i)(\psi_i(x)) = c_i^*(y) + \frac{\psi_i(x) - \xi_i^*(y)}{[\xi_i](y)} = c_i^*(y) + \lambda(x) [c_i](y) = h_i(x).
\]

By construction we have that $\psi_i : I \to I$ is non-decreasing and $G(c_i) \circ \psi_i = h_i$. Since the restriction of $\psi_i$ to $(\varphi_i \circ \vartheta)^{-1}(y)$ is absolutely continuous for each $y \in \Sigma(c_i)$ and the restriction of $\psi_i$ to $I \setminus (\varphi_i \circ \vartheta)^{-1}(\Sigma(c_i))$ is in $\text{SBV}(I; \mathbb{R})$, it follows that $\psi_i \in \text{SBV}(I; \mathbb{R})$ as well. Since in addition $\psi_i$ is continuous, it follows that it is actually absolutely continuous and therefore contained in $\Gamma$.

This proves the assertion.

\begin{proof}[Proof of Theorem 6] By Proposition 12 both of the infima in (4) are attained at some $\varphi_i, \psi_i \in \Gamma$. Moreover, by Proposition 17 we have that
\[
\inf_{\varphi_i \in \Gamma} \hat{d}(c_1, \varphi_1, [c_2, \varphi_2]) = \inf_{\psi_i \in \Gamma} \hat{d}(G(c_1) \circ \psi_1, G(c_2) \circ \psi_2).
\]
It remains to show that this is further equal to
\[
\hat{d}^S([c_1], [c_2]) := \inf_{g_i \in \Gamma} \hat{d}(g_1, g_2).
\]

\end{proof}
Assume therefore that $g_i \in [c_i]$. Denote by $h_i \in \mathcal{AC}(I; \mathbb{R}^d)$ the constant length parametrisation of $G(g_i)$. Then we can write $G(g_i) = h_i \circ \vartheta_i$ for some $\vartheta_i \in \Gamma$. Since $c_i \sim g_i$, it follows that $h_i$ is also the constant length parametrisation of $G(c_i)$. Now, since $\dot{c}_i \neq 0$ almost everywhere, it follows that also $G(c_i) \not= 0$ almost everywhere, and thus we can write $h_i = G(c_i) \circ \eta_i$ for some $\eta_i \in \Gamma$. Thus $G(g_i) = G(c_i) \circ \eta_i \circ \vartheta_i$ and thus

$$\hat{d}(g_1, g_2) \geq \inf_{\psi_i \in \Gamma} \hat{d}(G(c_1) \circ \psi_1, G(c_2) \circ \psi_2).$$

Since this holds for every $g_i \in [c_i]$ and since $G(c_i) \circ \psi_i \in [c_i]$, the assertion follows.

\[ \square \]

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