Cherenkov images are multi-fractal in nature. We show that multi-fractal behaviour of Cherenkov images arises due to multiplicative nature of pair production and bremsstrahlung processes in the longitudinal shower development passage.

1. Introduction:

Extensive Air Showers (EAS) [1] are produced when VHE/UHE primary photons, protons and other high Z nuclei enter atmosphere from the top and produce Cherenkov radiation which can be studied by imaging technique or by measuring lateral distribution. Cherenkov images recorded by a TACTIC-like γ-ray telescope represents the two dimensional distribution of Cherenkov light pattern produced in the atmosphere. In the Hillas parameterization approach, image is approximated as a ellipse and parameters like image "shape" and "orientation" are calculated. Hillas parameters like Length, Width Miss, Alpha, Distance, Size etc are essentially a set of second order moments. Among these parameters 'α' and 'Width' are known to be powerful parameters for segregation of γ-ACE from cosmic-ACE background. Using this approach, present day telescopes have been able to reject cosmic ray background events at 98% level, while retaining up to 50% of the γ-ray events from a point source. Hillas parameters are found to be good classifiers for small images (close to telescope threshold energy) but tend to fail for large images (of higher primary energy of the order of 10 Tev or higher) as too many tail pixels are part of the image. Again it is difficult to study fainter compact sources and γ-rays coming non-compact sources or of a diffusive origin. Up to 50% loss of actual image due to parameterization is also a big constraint.

It is interesting to observe that various products of EAS, like Cherenkov photons and particle density distributions (on the ground) have shown multifractal features. Multifractal nature of density fluctuations of particles has been experimentally verified and Lipshitz-Holder exponent distribution of EAS has been found to be sensitive parameter to indentify the nature of individual EAS [2]. We have shown that high energy γ-ray simulated images of TACTIC [3] like γ-ray telescope are multi-fractal in nature [4,5]. Multifracal nature of Cherenkov images have been confirmed experimentally [6,7]. Because of multifractal nature of EAS products, it becomes important to understand the basic multifractal process that takes place during the longitudinal passage of EAS.

Fractals are self-similar objects which look same on many different scales of observations. Fractals are defined in terms of Hausdroff-Bescovitch dimensions. Fractal dimensions characterize the geometric support of a structure but can not provide any information about a possible distribution or a probability that may be part of a given structure. This problem has been solved by defining an infinite set of dimensions known as generalised dimensions which are achieved by dividing the object under study into pieces, each piece is labelled by an index i=0,1,2,...,N. If we associate a probability p_i with each piece of size l_i than partition function can be defined as

\[ \Gamma(q, \tau) = \lim_{l \to 0} \Gamma(q, \tau, l) \]  

where

\[ \Gamma(q, \tau, l) = \sum_{i=1}^{n} \frac{p_i^q}{l_i^q} \]  


Multifractal nature of extensive air showers

A. Razdan

Nuclear Research Laboratory, Bhaba Atomic Research Centre, Mumbai 400 005, India

Presenter: A. Razdan (akrazdan@apsara.barc.ernet.in), ind-razdan-A-abs3-he15-poster
For unique function \( \tau(q) \) it has been shown that

\[
\Gamma(q, \tau) = \infty \quad (3)
\]

for \( \tau < \tau(q) \)

\[
\Gamma(q, \tau) = 0 \quad (4)
\]

for \( \tau > \tau(q) \).

This permits to define generalised dimension \( D_q \)

\[
(q - 1)D_q = \tau(q) \quad (5)
\]

Here \( q \) is a parameter which can take all values between \(-\infty\) to \(\infty\). This formalism is called as multi-fractal formalism which characterizes both the geometry of a given structure and the probability measure associated with it.

Cherenkov images arise due to EAS and have been found to be multifractal in nature. In this paper we will attempt to understand underlying process responsible for multifractal behaviour of Cherenkov images.

2. Shower development as a multifractal process

A ultra relativistic \( \gamma \)-ray enters atmosphere from the top and interacts with air molecule to produce electron-photon cascade. The radiation length(x) for pair production and bremsstrahlung process is equal in UHE/VHE region [1,10]. Particle -photon cascade in the atmosphere is sustained alternately by electron (\( e^- \)) -positron (\( e^+ \)) pair production and bremsstrahlung process till the average energy per particle reaches critical value \( E_c \) below which energy loss process is mainly dominated by ionization process.

Shower development can be visualized a process in which a \( \gamma \)-ray of energy \( E_0 \) (= 1 Tev (say)) after traveling distance \( x \) (on average) produces electron-positron pair each having energy \( \frac{E_0}{2} \). Since energy is getting divided into two equal parts , we can attribute a resolution of energy \( E=2^{-1} \). In the next radiation length both electron and positron lose half of their energy (on average) and each radiates one photon. Thus in this radiation length there are two photons and two particles (\( e^-1 \) and \( e^+1 \)) each having energy \( \frac{E_0}{4} \) which can be attributed to energy resolution \( E=2^{-2} \). At this stage fraction of photons ( \( p_1=\frac{1}{2} \) ) is same as fraction of particles ( \( p_2=\frac{1}{2} \) ). As the shower develops into next radiation length both electron and positron lose half of their energy and produce one photon each. At the same time two photons produced in the previous radiation length interact with air molecule to produce electron (\( e^- \)) -positron (\( e^+ \)) pair. Thus in third radiation length there are two photons and six particles each having energy \( \frac{E_0}{8} \). This stage can be attributed to the energy resolution \( E=2^{-3} \). It is important to note that, at this stage the process of equal division in energy between each photon and each particle continues but the process of unequal measure between photons and particles begins. In this radiation length the fraction of photons is \( p_1=\frac{1}{3} \) and fraction of particles (electrons + positrons) is \( p_2=\frac{2}{3} \). The fourth radiation length corresponds to energy resolution \( E=2^{-4} \) as total of 16 photons and particles are produced each having energy \( \frac{E_0}{8} \). However, there are 10 charged particles (5 positrons + 5 electrons ) and 6 photons, a case of unequal energy division and unequal fraction.

At a distance of \( nx \), the total number of particles and photons is \( 2^n \), each having average energy \( \frac{E_0}{2^n} \) and on an average shower consists of fraction of \( \frac{2}{3} \) particles and \( \frac{1}{3} \) photons even at nth stage. This corresponds to energy resolution of \( E=2^{-n} \). At nth stage each particle or photon can be labelled sequentially with \( i=0,1,2,..... \) The probability or fraction of particles and photons can be written as \( p_i=p_i^0p_i^{n-k} \). The partition function for finite
energy can be written as

$$\Gamma(q, \tau, E) = \sum_{i=1}^{N_k} \frac{N_k p_i^q}{E_i^\tau}$$  \hspace{1cm} (6)$$

where $N_k$ is the number of particles and photons. For the simplicity of calculations, we assume that incoming energy $E$ is equal to unit energy. This will not make any difference to actual results but integrate it with other classical examples of multifractal behaviour, e.g. curdling of cantor set [9]. On the average shower development process has a recursive structure similar to cantor set [9] because both processes are inherently binomial multiplicative in nature. For $\tau=\tau(q)$, we have

$$\Gamma(q, \tau(q), E) = 1.$$  \hspace{1cm} (7)$$

In the limit of $E \to 0$, the most dominant contribution to this partition function will survive when $\tau=\tau(q)$, where $\tau(q)$ is the solution of the equation

$$(p_1^q E^{-\tau(q)} + p_2^q E^{-\tau(q)})^n = 1$$  \hspace{1cm} (8)$$

Above equation can be easily solved to get $\tau(q)$.

3. Case of equal energy and unequal fractions

In each radiation length the measure of photons and particles fluctuates but on the average the shower consists of $\frac{2}{3}$ positrons and electrons and $\frac{1}{3}$ photons. In the present case $E=\frac{1}{2}$, $p_1=\frac{1}{3}$, $p_2=\frac{2}{3}$. Using above equation and Stirling approximation, we have

$$D_q = \frac{1}{q-1} \frac{\ln(p_1^q + p_2^q)}{\ln E}$$  \hspace{1cm} (9)$$

For $q=0$, $D_0$ gives fractal dimension. For $q=1$, $D_1$ is the information dimension which encodes the entropy scaling and $q=2$, $D_2$ is the correlation dimension which measures scaling of two point density correlation. Apart from $D_0,D_1,D_2$ there are infinite set of other exponents from which information can be obtained by constructing an equivalent picture of the system in terms of scaling indices ‘$\alpha$’ for the probability measure defined on a support of fractal dimension $f(\alpha)$. $f(\alpha(q))$ is the fractal dimension of the set. In case all $D_q$’s are equal to fractal dimensions $D_0$ than $f(\alpha(q))$ collapses to a point $f=\alpha=D_0$, indicating that scaling behaviour of a fractal measure is of single type. This is situation which involves neither probability variation nor length variation.

4. Case of unequal energy and unequal fractions

It is possible that as the shower develops deep into atmosphere, division of energy may not remain equal. With the possibility of $E_1 \neq E_2 \neq \frac{1}{2}$ and/or $p_1 \neq \frac{1}{3}$ and $p_2 \neq \frac{2}{3}$ shower development process can still be described as a multifractal process because either the energy or the probability or both need to be different for a multifractal process. The partition function for this case can be written as

$$(p_1^q E_1^{-\tau(q)} + p_2^q E_2^{-\tau(q)})^n = 1.$$  \hspace{1cm} (10)$$

where $E_1 + E_2=E$ and $p_1 + p_2=p$. Above equation can be numerically solved for $\tau(q)$ if values of $E_1,E_2,p_1$ and $p_2$ are known and fixed for each radiation length.
5. Case of loss of energy

As the shower develops deep into atmosphere most realistic situation is that there will be energy loss in all radiation lengths. So a generalized scenario can be described in which $E_1 \neq E_2 \neq \frac{1}{2}$ and $p_1 \neq \frac{1}{3}$ and $p_2 \neq \frac{2}{3}$ with the condition $E_1 + E_2 < E$ in each radiation length. Each shower is unique because shower development process is random and there are no unique values of $E_1, E_2, p_1$ and $p_2$.

6. Simulation studies

Any simulation / experimental arrangement to measure EAS will consist of set of detectors which can measure only a sample or fraction of EAS products. This sample may consist of distribution of charged particles / photons. The set of detectors which measure this distribution represents a geometric support and multifractal measures can be related to this geometric support. This can be done by calculating multifractal moments
of this distribution and obtain generalized dimensions from scaling properties of multifractal moments. The definitions of multifractal properties given in previous section are not defined with respect to any support and hence can not be applied directly to simulated / experimental data.

Simulated cherenkov images were generated for $\gamma$-rays and protons using CORSIKA code for TACTIC configuration [4,5 and references therein]. $\gamma$-rays of energy 50 TeV and protons of energy 100 TeV are considered for multifractal studies. Each simulated image is divided into $M=2^\nu$ where $\nu=2,4,6,8$, is the scale. The multifractal moments are

$$G_q = \sum_{i=1}^{M} \left( \frac{k_j}{N} \right)^q$$

where $k_j$ is the number of photoelectrons in the kth cell and N is the total number of photoelectrons in whole image. $G_q$ shows a power law behaviour with $M$, i.e. $G_q = M^{\tau(q)}$ where $\tau(q)$ is related to generalized multifractal dimension by

$$D_q = \frac{\tau(q)}{q-1}$$

We have calculated average values of $D_q$ for 1000 images each for $\gamma$-rays and protons. We have repeated above studies for 30 TeV $\gamma$-rays and 60 TeV protons. No energy dependence of $D_q$ on q was observed.

7. Results:

As the shower develops deep into the atmosphere, values of $E_1, E_2, p_1$ and $p_2$ in various radiation lengths may not remain fixed but may fluctuate. It is not possible to solve equation (4) for all fluctuating values in each radiation length. Hence we have to take average values. Figure 1 depicts $D_q$ dependence on q for various average values. Continuous curve has been obtained in figure 1 for the case $E_1=E_2=\frac{1}{2}$, $p_1=\frac{1}{3}$ and $p_2=\frac{2}{3}$. Simulated data corresponding to protons is represented by (+) sign and simulated data corresponding to $\gamma$-rays is represented by (x) sign. $D_q$ values of each cherenkov image corresponding to $\gamma$-rays and protons were calculated for 1000 images and average was computed. In figure 1, these average values of $D_q$ for both protons and gamma-rays have been shown. It is clear from figure that theoretically known average values of shower progression does not match with average values of simulated results.

Figure 2 shows average values of $D_q$ versus q behaviour for five different cases. Curve ‘a’ is actually part (of continuous curve) of figure 1, shown here for the purpose of comparison. Average values of $D_q$ corresponding to proton (+ sign shown as curve d) and $\gamma$-rays ( x sign shown as curve e) have been replotted. The continuous corresponding to ‘b’ and ‘c’ have been obtained . Curve ‘b’ corresponds to $E_1=\frac{12}{15}, E_2=\frac{10}{15}, p_1=\frac{2}{5}$ and $p_2=\frac{3}{5}$. Curves ‘b’ correspond to 23 % loss of energy in proton initiated showers in all radiation lengths on the average and curve ‘c’ corresponds to average loss of 18 % energy in all radiation lengths by $\gamma$-ray initiated shower. Curve ‘a’ corresponds to case of no energy loss. It is clear that theoretical model of shower development considered with average loss of energy in each radiation length is a good approximation of showers.

8. Discussion:

In this paper we have theoretically calculated generalized dimension $D_q$ of shower development process based on simple model for the case of average values of energy energy and probability. We have considered the case of no loss of energy and average loss of energy per radiation length in the shower development process. We have also simulated Cherenkov images corresponding to $\gamma$-rays and proton initiated showers. Average values of $D_q$ have been calculated for both types of images. Comparison of average theoretical values and
average simulated values are given in figures 1 and 2. It is clear that theoretical model of shower development
considered with average loss of energy in each radiation length in this paper is good approximation for showers.

It is interesting to observe that initiated EAS progression comes close to the mathematical definition of classic
cantor set. Curves b and c in figure 2, represent two scale fractal measure of cantor set in which a fraction
of energy at each stage gets removed. Such processes have been discussed [15] in detail [page 86]. The
identification of EAS model as a multifractal process is actually experiencing a cantor process or cantor dust
in actual physical world.

In EAS process there is division of energies in terms of charged particles and photons as the appearances of
charged particles, in any radiation length, means production of cherenkov radiation and appearance of photons
in any radiation length means absence of cherenkov radiation. The cherenkov image formed on telescope is ac-
ually superimposition of cherekov photons produced at different heights. Multifractal measures are associated
the distribution of data on a geometric support. This support may be ordinary plane surface, photomultiplier
tube (PMT) camera or a fractal itself.

Figure 2. Variation in the muon (4-layer) rate over
For particle detection, it has been shown by Kempa[13] that multifractal structures of density fluctuations of charged particles near EAS can be used to distinguish between $\gamma$-ray and hadron content of cosmic ray. The method of multifractal moment analysis has been applied to simulated data as observed by KASKADE experiment to distinguish EAS lateral distributions due to various types of primaries [12]. The fractal moments are part of the sample of observables used for KASKADE data analysis [13].

Another important point that is clear from this studies is that so called a ‘toy model’ of EAS can match detailed simulated CORSIKA results just by introducing a small loss term in each radiation length.

The different values of $E_1$, $E_2$, $p_1$ and $p_2$ fitted in figure 2 are not unique values. Many other combinations of energies and probabilities will produce same results. It is because EAS process is random in nature and all combinations of different values of energies and probabilities are equally and uniquely probable. It is important to note that there cannot be any unique values ascribed to energies and probabilities in EAS.

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