A meshfree formulation for CFD and linear elasticity problems

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Abstract. This paper documents the development and application of a meshless procedure for computational fluid dynamics and linear elasticity problems. The meshfree nature of this method gives the advantage of dealing with highly distorted domains and even fragmentations without the need of using computationally expensive remeshing approaches and with a very simple implementation. A description of the implementation of this method and also the solutions of some benchmark problems will be presented in order to demonstrate the potential of this formulation for dealing with problems in these fields and to introduce promising future fields of application.

1. Introduction

The ideal for solving many engineering problems is to obtain the best performance at the lowest possible cost. This goal can be reached with a deep knowledge and understanding of the nature that surrounds the problem to be solved. The numerical simulation, widely used nowadays in engineering, is a suitable technology to know in depth many physical phenomena since it provides a large amount of information that cannot be obtained through other methods. Furthermore, in many situations it is the most efficient, profitable, and powerful technique to achieve this purpose [1]. This tool involves the derivation of a mathematical model for the problem, which is frequently expressed in the form of partial differential equations (PDEs) whose analytical solution is generally not available, so numerical methods executed on a computer are used to approximate their solutions.

Numerical mesh-based methods such as the Finite Element Method (FEM), Finite Difference Method (FDM), Finite Volume Method (FVM) are universally accepted numerical techniques for solving the PDEs in different engineering problems, and that is why during the last years many numerical simulation tools are based on them. However, mesh-based methods are not suitable for problems associated with very large mesh deformations, moving discontinuities or problems requiring constant remeshing during the solution process, since it implies a high computational cost. To overcome some of the drawbacks arising when mesh-based methods are used, mesh-free methods have recently been proposed and developed as alternative [2].

The advantages of meshless methods over mesh-based methods are that they use a set of nodes to discretize the problem domain and its boundaries without requiring any information about the relationship between nodes such that they do not form an element mesh which lets to deal with problems involving discontinuities and deformations in the domain without the problems of mesh-based methods. Moreover, they have the flexibility to remove or add points whenever and wherever needed which promote the developing of adaptive schemes [3].

A truly Lagrangian strong-form meshfree method is the so called Finite Pointset Method (FPM) developed by J. Kuhnert [4]. It has proven to be far superior to traditional mesh-based and some other meshfree methods in several computational mechanics and engineering fields, such as flows on manifolds [5], additive manufacturing processes [6], fluid mechanics [7, 8], fluid structure interactions [9], free surface flows [10], radiative heat transfer [11], heat transfer with moving heat sources [12] and fluid flow coupled with heat transfer considering phase changes [13]. It uses the weighted least-squares (WLSM) interpolation scheme to approximate the spatial derivatives [14]. It has many advantages over
other methods since it is able to naturally and easily incorporate any kind of boundary conditions without requiring any special treatment or stabilization and it is really simple to implement.

Therefore, a novel meshless formulation based on the finite point set method for linear elasticity problems is introduced in this work for computational fluid dynamics and linear elasticity problems. In order to get some insight on its performance we compute the numerical solution of some benchmark tests. The structure of the paper is as follows: Section 2 shortly describes the governing equations. Section 3 shows the main ideas behind the novel formulation of FPM for computational fluid dynamics and linear elasticity problems followed by the numerical tests presented in Section 4 with its corresponding results. Finally, some conclusions are given in the last section.

2. Mathematical model

Many physical phenomena such as linear elasticity, the resulting pressure Poisson and the velocity equations arising when the Chorin’s projection method is applied in order to numerically solve the Navier-Stokes equations are governed by an elliptical partial differential equation whose general form reads

$$A_1 \psi + B \cdot \nabla \psi + C_1 \Delta \psi - F = 0$$

with prescribed Dirichlet, Neumann ($\partial \psi / \partial n = \varphi$) or Robin ($A_2 \psi + C_2 \Delta \psi / \partial n = \varphi$) boundary conditions, where $A_i, B, C_i, F$ and $\Pi$ are known.

3. FPM formulation for general elliptic partial differential equations

Consider now the numerical solution of the elliptic partial differential equation (1). A novel FPM approach is proposed to numerically solve general elliptic PDEs and this is shortly introduced next. This approach is based on the FPM formulation proposed in [7] and considered in [10, 12, 13, 15], and it differs from it in the fact that it takes the discretization of the PDE directly in the sparse global linear system of equations and not in the local system of the weighted least squares solution. To give a general idea of this new approach, consider the numerical solution of the general elliptic PDE (1).

Let $\Omega$ be a given domain with boundary $\partial \Omega$ and suppose that the set of points $x_1, x_2, ..., x_N$ are distributed with corresponding function values $\psi(x_1), \psi(x_2), ..., \psi(x_N)$. The problem is to find an approximate value of $\psi$ at some arbitrary location $\psi(x)$ using its discrete values at particles positions inside a neighbourhood of $x$. To define the set of nodes and the neighbourhood of $x$, a weight function $w(x - x_i)$ is introduced

$$w_i = w(x - x_i) = \begin{cases} e^{-y \|x-x_i\|^2/h^2}, & \|x-x_i\| \leq h \\ 0 & \text{else} \end{cases}$$

where $h$ is the smoothing length, $y$ is a positive constant whose value is considered to be 6.5, and $x_i$ is the position of the $i$-th point inside the neighbourhood.

The Taylor expansion of $\psi(x_i)$ around a particle $x$ is

$$\psi(x_i) = \psi(x) + \sum_{k=1}^{2} \frac{\partial \psi}{\partial x_k}(x_{k,i} - x_k) + \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial^2 \psi}{\partial x_k \partial x_j}(x_{k,i} - x_k)(x_{j,i} - x_j) + e_{1,i}$$

for $i = 1, ..., m$, where $m$ is the number of nodes that lie within the neighbourhood of $x$.

In contrast to the approach studied in [7] where the linear system obtained from the Taylor’s expansion is taken together with the general elliptic PDE (1) with the corresponding boundary conditions, in this approach and at this step only the linear system obtained from the Taylor’s expansion is considered which in matrix form reads

$$e = Ma_\psi - b$$

where
\[ M = \begin{pmatrix} 1 & h_{1,1} & \frac{1}{2} h_{1,1}^2 & h_{1,1} h_{2,1} & \frac{1}{2} h_{1,1} h_{2,2} & \cdots \\ 1 & h_{1,2} & \frac{1}{2} h_{1,2}^2 & h_{1,2} h_{2,1} & \frac{1}{2} h_{1,2} h_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & h_{1,m} & \frac{1}{2} h_{1,m}^2 & h_{1,m} h_{2,1} & \frac{1}{2} h_{1,m} h_{2,2} & \cdots \end{pmatrix} \]  

with \( h_{1,j} = (x_{1,j} - x_1) \) and \( h_{2,j} = (x_{2,j} - x_2) \).

\[ a_{\psi} = \begin{pmatrix} \psi_1 \\ \frac{\partial \psi}{\partial x_1} \\ \frac{\partial \psi}{\partial x_2} \\ \frac{\partial^2 \psi}{\partial x_1^2} \\ \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \psi}{\partial x_2^2} \end{pmatrix}^t \]

\[ b = (\psi_1, \psi_2, \ldots, \psi_m)^t \]

\[ e = (e_1, e_2, \ldots, e_m)^t \]

After a minimization of \( e e \) through the weighted least squares method, the unknown vector \( a_{\psi} \) is obtained as

\[ a_{\psi} = (M^t W M)^{-1} (M^t W) b \]

where \( W \) is

\[ W = \begin{bmatrix} w(x - x_1) & 0 & \cdots & 0 & 0 \\ 0 & w(x - x_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & w(x - x_n) & 0 \\ 0 & 0 & \cdots & 0 & w(x - x_1) \end{bmatrix} \]

Thus, the general elliptic PDE (1) can be expressed in terms of \( a_{\psi} \) as

\[ A_1 a_{\psi_1} + B_1 a_{\psi_2} + B_2 a_{\psi_3} + C_1 (a_{\psi_4} + a_{\psi_5}) = F \]

while the Robin boundary condition reads

\[ A_2 a_{\psi_1} + C_2 \frac{\partial \psi_2}{\partial n} + C_2 \frac{\partial \psi_3}{\partial n} = \varphi \]

If \( Q \) is defined as

\[ Q = (M^t W M)^{-1} \]

and its \( k \)-th row is denoted by \( q_k = (q_{k,1}, q_{k,2}, \ldots, q_{k,m}) \), the moving least squares solution reads

\[ a_{\psi} = Q (M^t W) b, \]

and each element in \( a_{\psi} \) can be expressed as

\[ a_{\psi_{ij}} = \sum_{i=1}^{m_j} w_{ij} p_{ki} \psi_{ij} \]

where \( m_j \) denotes the number of \( j \)-th particle neighbours within the shape function support, \( \psi_{ij} \) the unknown function value in the \( i \)-th neighbour particle position and \( p_{ki} \) is defined as

\[ p_{ki} = q_{k,1} + q_{k,2} h_{1,i} + q_{k,3} h_{2,i} + q_{k,4} \frac{h_{1,i}^2}{2} + q_{k,5} h_{1,i} h_{2,i} + q_{k,6} \frac{h_{2,i}^2}{2} \]

It couples each node with its neighbouring nodes yielding to the coupling between inner and boundary nodes. Therefore, the discretization of equation (1) under this setting reads

\[ \sum_{i=1}^{m_j} w_{ij} [A_1 p_{1i} + B_1 p_{2i} + B_2 p_{3i} + C_1 (p_{4i} + p_{5i})] \psi_{ij} = F \]

and if the particle is on a Robin boundary, (16) becomes

\[ \sum_{i=1}^{m_j} w_{ij} (A_2 p_{1i} + C_1 p_{1i} + C_2 p_{3i}) \psi_{ij} = \varphi \]

Since equation (16) is valid for \( j = 1, 2, \ldots, N \), this can be arranged in a global sparse linear system \( K \Psi = f \) which is solved with iterative methods. Thus, general elliptic PDEs such as (1) can be solved in this way, just adding appropriate entries in the systems of equations (16).
4. Numerical Examples

In this section the suitability and feasibility of this FPM formulation in order to solve computational fluid dynamics and linear elasticity problems will be evaluated.

4.1. Computational fluid dynamics

4.1.1. Filling Benchmark.

In order to demonstrate the suitability of this approach for computational fluid dynamics problem, the filling of a circular disk mould with a core in the center will be considered as a benchmark example since the experimental filling pattern is available in the literature [16]. The geometry of this example is shown in the figure 1. In this example, the problem domain was discretized with approximately 6000 points. The inlet velocity was taken as \( v = [0,0,18] \) m/s. The pressure in all particles and the atmospheric pressure were considered as zero. The surface tension forces and the gravitational acceleration vector were neglected. The density and viscosity of the fluid were considered as \( \rho = 1000 \text{ kg/m}^3 \) and \( \mu = 0.01 \text{ Pa.s} \), respectively. Finally, the smoothing length used in the simulation was \( h = 3.25 \text{ mm} \), a no-slip boundary condition was used at solid walls and the time step was chosen as \( \Delta t = 0.004 \text{ s} \).

The numerical results predicted with this formulation are compared with experimental results in figure 2. It indicates that this method perfectly reproduces the evolution of the domain over time, and also remains stable during the course of the filling simulation. Therefore, it reliable to simulate complex filling processes.

![Figure 1. Geometry of the circular disk mould with a core in the center.](image-url)
4.1.2. Filling in casting.

The second numerical test considered here corresponds to the free surface flow problem of the filling of a pump cover. In this example, the problem domain was discretized with approximately 180000 points with a mean spacing of 0.0015 m. The inlet velocity was taken as \( \mathbf{v} = [-0.1,0,0]^t \) m/s. The pressure in all particles as well as the atmospheric pressure were considered as zero. The surface tension forces and the gravitational acceleration vector were neglected. The density and viscosity of the fluid were considered as \( \rho = 6964 \text{ kg/m}^3 \) and \( \mu = 0.0143 \text{ Pa.s} \), respectively. These parameters correspond to the physical parameters of the molten cast iron. Finally, the smoothing length used in the simulation was \( h = 0.0045 \) m, a slip boundary condition was used at solid walls and the time step was chosen as \( \Delta t = 0.004 \) s.

Figure 2. Filling profiles for the circular disk mould with a core in the center at different time steps. 
First column: FPM. Second column: experimental profile in [16].
Two perspective views of the filling patterns at different time steps are depicted in Figure 3. There, the picture on the left shows the view exactly from the top whilst the second one shows the view from the top at some angle to the right. As it is shown in this figure, the leading material is divided in four liquid fronts when it impacts the two annular central sections of the die. Two central jets partially merge forming a single liquid front which is split again when it impacts the central cylindrical obstacle. The emerging jets move backwards and starts filling the rear part of the mould. Splashing droplets and liquid fragmentations are visible in this part. The remaining two jets flow around the curved outsides of the die until they collide with the fronts coming from the rear and central parts of the mould. In the two annular central sections of the mould, the liquid flow up into the upper extensions. At this point the rear part of the mould is substantially filled and the fluid flow is towards the front part of the mould. Afterwards, almost all the mould cavity is filled and the biggest voids are principally behind two of the cylindrical obstacles near the inflow jets. They are uniformly filled until the filling process finishes.

These pictures show the robustness of this FPM formulation for dealing with complex computational fluid dynamics problems.

4.2. Linear Elasticity

Sections should be numbered as follows:
The numerical benchmark in linear elasticity corresponds to the very well known test example of a cantilever beam subjected to an end load, as geometrically shown in Figure 4. The beams has height $D$ and a length $L$. A parabolic tangential load is considered on the right hand side and the left hand side is kept fixed. The parameters for this example are set as follows: $L = 8$ m, $D = 1$ m, $E = 10$ GPa, $\nu = 0.3$ and $P = 1000$ Pa. The analytical solution for the stresses and displacements is found in [17] and reads...
Figure 4. Geometrical description of the two-dimensional cantilever beam problem.

Figure 5. Displacement magnitude distribution

\[
\sigma_{xx} = -\frac{P(L-x)y}{l}, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = \frac{P}{2l} \left(\frac{D^2}{4} - y^2\right) \\
u = -\frac{P}{6EI} \left[(6L-3x)x + (2 + \nu) \left(y^2 - \frac{D^2}{4}\right)\right] \\
v = \frac{P}{6EI} \left[3\nu y^2(L-x) + \frac{D^2}{4}x(4 + 5\nu) + (3L-x)x^2\right]
\]

where \(P\) is the total load force, \(E\) is the Young’s modulus, \(\nu\) is the Poisson’s ratio and \(I = \frac{1}{12}D^3\) is the moment of inertia around the horizontal axis. The displacements are imposed as essential boundary conditions on the left-hand side, traction free boundary conditions are defined on the top and bottom of the domain and traction boundary conditions on the right-hand side are prescribed. The corresponding numerical results regarding the displacement magnitude \(\|\mathbf{u}\|\) and the stresses \(\sigma_{xx}, \sigma_{xy}\) and \(\sigma_v\) are shown in Figures 5 and 6, respectively, where the von Misses stress is defined as

\[
\sigma_v = \sqrt{\sigma_{xx}^2 - \sigma_{xy} \sigma_{yy} + \sigma_{yy}^2 + 3\sigma_{xy}^2}
\]

There the displacement has been scaled by a factor of 50\(e^3\) for visualization purposes and they were obtained considering a discretization with 1936 particles with a mean spacing of 0.0667 m and a smoothing length of 0.2 m leading to an average number of neighbouring particles of 28. These figures show smooth and stable stress patterns. The \(\sigma_{xx}\) and \(\sigma_v\) are compatible with the physical configuration. The approximating solution error regarding the obtained displacements and the corresponding stresses are shown in Figure 7.
These pictures indicate the robustness of this FPM formulation for the solution of this kind of problems since its numerical prediction matches very well with the exact solution, with a more numerically simple management of the boundary conditions.

**Figure 6.** Stresses distributions.

**Figure 7.** Left: displacement error with respect to the exact solution. Right: stress error with respect to the exact solution

**5. Conclusions**

Based on the numerical performance shown in the numerical examples we can conclude that the current approach is suitable and feasible for computational fluid dynamics and linear elasticity problems. It is stable and has enough accuracy in order to capture the physical behaviour observed in the governed processes. Since this formulation is a truly meshfree method it could be used for the study and analysis of complex problems involving high deformations and domain fragmentations with a great
computational advantage since it does not need to compute any numerical quadrature and it does not need remeshing approaches. Further, it is able to naturally and easily handle any kind of boundary conditions without requiring any special treatment or stabilization and it is really simple to implement. Therefore, it could be a promising numerical tool for the simulation of engineering processes involving other phenomena described by elliptic partial differential equations as heat transfer. Consequently, it depicts a rich source of research opportunities.

6. References

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