Quantum Conductance and Electrical Resistivity

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The Landauer formula for quantum conductance, based on the modern paradigm: “conduction is transmission”, is generalized to samples of macroscopic size. Two regimes of electrical conduction, namely diffusive and ballistic ones, are studied. In the former regime, Drude’s formula for the electrical resistivity is recovered and it is found a maximum conductivity equal to \( \frac{e^2}{\pi \hbar^2} \), which is of the same order of magnitude as that of good metals at room temperature. In the latter, it is obtained in three dimensions a quantum conductance which is compatible with the one deduced by Sharvin in the ballistic regime. It is also found in this case an electrical conductivity which depends on the size of the sample, in agreement with that measured in very pure metals at the temperature of liquid helium. In two dimensions the result for the conductance in the ballistic regime is consistent with that used to analyse quantum point contacts.

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I. INTRODUCTION

A conventional view of the electrical conductivity attributes the onset of it to the linear response of the free electrons to the applied external electric field. This picture is contemplated both in classical Drude-Sommerfeld-Lorentz and in the quantum mechanical Kubo treatments. On the other hand a modern view of the electrical conductance was proposed by Landauer [1], [2], which states that conduction is transmission (see also van Houten and Beenakker [3]). Both the conventional and modern views are treated in a paper by Rammer [4] in which, among other relevant considerations, discusses the connection between linear response formalism and the Landauer approach by expressing the conductance in terms of scattering properties of the sample.

As pointed out by Batra in [5], it is well known that for an ideal one dimensional conductor under ballistic transport conditions, the conductance is transversely quantized in steps of \( 2e^2/h \) as the constriction width is varied. According to Batra [5], [7], the finite resistance of a perfect conductor, which has been previously understood in terms of contact effects, can also be viewed as having quantum mechanical origin in the uncertainty principle.

Landauer [1], [2] obtained a relation of the conductance for a one dimensional sample connecting two reservoirs at different electrochemical potentials through ideal conductors. In the absence of dissipative scattering the conductance \( G \) is given by

\[
G = \frac{e^2}{\pi \hbar} T, \tag{1}
\]

where \( T \) is the transmission probability of the sample, \( e \) the quantum of electrical charge and \( \hbar \) the reduced Planck’s constant.

Experimental verification of equation (2), comes from the study of quantum point contacts (QPC) [8], [9]. In a two dimensional heterojunction the channel width can be controlled by external applied gate voltages. The conducting channel works approximately as a waveguide. As it is widened, the number of transverse eigenstates below the Fermi level increases. Then conductance steps corresponding to different values of \( N \) in equation (2), are observed [2], [8], [9], [10].

In this letter we aim to extend the validity of equation (2) to the case of macroscopic samples in any spatial dimension, once the number \( N \) of conducting channels is properly interpreted. We intend to make connections of relation (2) with the conduction of electricity in good metals at the room temperature, as well with very high impurity-free samples at very low temperatures. We will see that in the first case it is possible to deduce the Drude formula starting from the Landauer relation for the quantum conductance and also to estimate the “real” conductivity of a good metal in room temperature. In the second case we will obtain an electrical conductivity which depends on the size of the sample. We will also compare these results with those obtained by Sharvin [10] and by Lifshits and Kaganov [11], [12]. Finally the ballistic...
regime will be treated from the point of view of a drag force induced by a turbulent flow.

As a starting point of this work let us take a hypercubic sample of a good conductor of size \( L \). By considering that the free electrons which are able to participate in the electric conduction are those close to the Fermi level with momentum \( p_F \), it is convenient to write the following action in momentum space

\[
A = \int_{(\Delta p)^D} \left( \nabla_p \psi |^2 - \frac{\psi^2}{2mE} \right) d^D \bar{p},
\]

where \( \Delta p \) is the wavepacket width around \( p_F \), \( m \) is the electron mass, \( E \) the energy and \( D \) is the space or the space-time dimension.

Thompson [13] has introduced a heuristic method (of the dimensions) as a tool to deal with the critical behavior of a system undergoing a second order phase transition [20]. One of the basic hypothesis of Thompson is that each term of the action (3), by using Thompson’s prescription we may also write

\[
\tilde{\psi}_p = \pm \frac{i}{\sqrt{2mE}} \psi. \tag{4}
\]

A solution of equation (4) gives

\[
\psi = \psi_0 e^{\pm \sqrt{\frac{\Delta p}{2mE}}} p. \tag{5}
\]

It is possible based on (5) to construct a wavepacket centered on \( p_F \). From the first term in the action (3), by using Thompson’s prescription we may also write

\[
\int_{(\Delta p)^D} \nabla_p \psi |^2 d^D \bar{p} \sim |\psi_{av}|^2 (\Delta p)^{D-2} \sim 1. \tag{6}
\]

Equation (6) implies that

\[
|\psi_{av}|^2 \sim (\Delta p)^{2-D}, \tag{7}
\]

where “av” stands for the averaged quantity. Taking into account the uncertainty relations, namely \( \Delta p L \sim h \), enables us to write equation (7) as

\[
|\psi_{av}|^2 \sim L^{D-2}. \tag{8}
\]

We interpret \( |\psi_{av}|^2 \), above as the transmission probability of the sample and we will consider two possible regimes of conduction, namely the classical regime where \( D \) stands for the \( d \)-spatial dimensions and the quantum regime where \( D = d + 1 \) is the \( d \)-spatial plus one time dimension. Therefore combining equations (2) and (8) we obtain

\[
G = \frac{e^2}{\pi \hbar} \left( \frac{L}{l_0} \right)^{D-2}, \tag{9}
\]

where \( l_0 \) is the size of a typical channel.

In their work about the scaling theory of the localization, Abrahams et al [14] have defined a generalized dimensionless conductance that they called “Thouless number” as

\[
g(L) = \frac{G(L)}{e^2/2\hbar}. \tag{10}
\]

Therefore we can identify \( (L/l_0)^{D-2} \) as the “Thouless number” in the delocalized diffusive regime.

\section{The Classical Regime}

In the classical regime \( D \) coincides with the spatial dimensionality \( d \) and here we are particularly interested in the case \( d = 3 \). Thus from equation (9) we have in three dimensions

\[
G_{D=3} = \frac{e^2}{\pi l_0^2} L. \tag{11}
\]

On the other hand, for large \( g \), macroscopic transport theory is correct and gives [14]

\[
G(L) = \sigma L^{d-2}, \tag{12}
\]

where \( \sigma \) is the electric conductivity. Comparing (12) with (9) we get, in the three dimensional case,

\[
\sigma \approx \frac{e^2}{\pi \hbar l_0}, \tag{13}
\]

where \( l_0 \) can be evaluated through the following reasoning. Suppose that we have \( n \) scatterers per unity of volume and let us consider a cylinder shaped tube with longitudinal size equal to the electron mean free path \( \lambda \) and radius equal to the geometric average of \( l_0 \) and \( \ell_F \), where \( \ell_F = l_F/(2\pi) \) is the reduced Fermi wavelength. It must be stressed that \( n \) is numerically equal to the number of electrons per unity of volume, if we consider that electric conductivity always happens in a regime of charge neutrality. We write

\[
n\pi l_0 \ell_F \lambda = 1. \tag{14}
\]

Inserting \( l_0 \) given by (14) into (13) allows us to obtain

\[
\sigma \approx \frac{e^2 n \lambda}{mv_F}, \tag{15}
\]

where \( v_F \) is the Fermi velocity of the charge carrier. Equation (15) is just the Drude formula for the electrical conductivity, but here it was deduced starting from an expression describing quantum conductance.

It is also worthwhile noticing that if \( l_0 \) is considered as the width of the transmission channel then there should be a lower bound for it, namely the reduced Compton
wavelength of the electron. Inserting $ \lambda = \hbar / (mc) $ in (13) yields

$$
\sigma_{\text{max}} = \frac{e^2 mc}{\pi \hbar^2},
$$

where $ \sigma_{\text{max}} $ means the maximum conductivity in the classical (diffusive) regime. A numerical evaluation of (16) gives $ \sigma_{\text{max}} \sim 10^8 (\text{O} \cdot \text{m})^{-1} $. Indeed electrical conductivities of this order of magnitude are typical of those measured in good metals at room temperature.

Another evaluation of the electrical resistivity of metals was accomplished by Lifshits and Kaganov (LK) is (see also Brandt and Chudinov [17]). The result obtained by Lifshits and Kaganov (LK) is

$$
\sigma_{\text{LK}} = \frac{2}{3} \frac{S_F \lambda}{(2\pi \hbar)^3} = \frac{4 e^2 \lambda}{3 \hbar l_F^2},
$$

where $ S_F = 4\pi p_F^2 $ and $ \lambda $ is the electron mean free path. Comparing (17) with (13) yields

$$
\lambda = \frac{3 l_F^2}{4\pi l_0}.
$$

### III. THE QUANTUM REGIME

In the quantum regime we will take $ D $ to be equal to $ d + 1 $ (d spatial plus one time dimension). In this case equation (9) becomes

$$
G = \frac{e^2}{\pi \hbar} \left( \frac{L}{l_0} \right)^{d-1}.
$$

It would be interesting to make a detailed investigation of (19) in the special cases of two and three spatial dimensions ($ d = 2 $ and $ d = 3 $). If we make the natural choice of identifying in (19) $ l_0 $ with $ l_F $ (the Fermi wavelength), we get in three dimensions

$$
G_{d=3} = \frac{e^2}{\pi \hbar} \left( \frac{L}{l_F} \right)^2.
$$

By comparing (20) with relation (12) in the case $ d = 3 $, enables us to write

$$
\sigma_{d=3} = \frac{e^2}{\pi \hbar} \frac{L}{l_F^2}.
$$

A good description of the electrical conduction for very pure metals at low temperatures seems to be the ballistic transport treatment introduced by Sharvin (see also Brandbyge et al [16]). We have

$$
G_{\text{Sharvin}} = \frac{e^2 L \nu}{p_F},
$$

where $ p_F $ is the Fermi momentum and $ \nu $ is the free electron density. One electron which has an uncertainty in momentum equal to $ p_F $ has an uncertainty in position equal to $ l_F = \hbar / p_F $. Therefore by considering spin degeneracy we can write

$$
\nu = \frac{2}{l_F^2}.
$$

Inserting (21) into (22) leads to

$$
G_{\text{Sharvin}} = \frac{e^2}{\pi \hbar} \left( \frac{L}{l_F} \right)^2,
$$

which coincides with our equation (20) (see also Brandbyge et al [16]).

Now let us analyse the two dimensional case in the ballistic regime. From equation (10) we have

$$
G_{d=2} = \frac{e^2}{\pi \hbar} \left( \frac{L}{l_F} \right).
$$

When the measurement is done at very low temperatures in a very pure two dimensional macroscopic sample, the quantization of the conductance cannot be detected since we have in this case $ L \gg l_F $. However, measurements in Ga As-Al Ga As heterojunctions show that each new channel of transmission is activated only when the width $ w $ increases as $ \Delta w = \Delta L \sim l_F $. This characterizes the onset of the quantum conductance as it can be seen in the work by van Houten and Beenakker (please see also [8] and [9]).

If we consider as before that the size of the transmission channel is limited by the electron reduced Compton wavelength, we will have in the diffusive regime an upper bound to the electron mean free path given by

$$
\lambda_{\text{max}} = \frac{3 mc l_F^2}{4\pi \hbar}.
$$

Order of magnitude estimates of (26) results in $ \lambda_{\text{max}} \sim 10^4 \mu \text{m} $ for charge carriers in semiconductors and $ \lambda_{\text{max}} \sim 10^2 \mu \text{m} $ for electrons in metals.

### IV. AN ALTERNATIVE VIEW OF THE BALLISTIC REGIME

A macroscopic body moving at speeds high enough so that the flow of air behind it is turbulent, is subject to a drag force $ D $ given by (see for instance Halliday and Resnick [13])

$$
D = \frac{1}{2} \rho A v^2.
$$
Here $A$ is the effective cross-sectional area of the body, $\rho$ is the density of the fluid, $v$ is the speed of the body and $C$ a dimensionless coefficient.

We think that in the ballistic regime the electrical resistance of a good conductor can be represented by the collisions of the charge carriers with walls which have the size of the sample, so that the area of the walls are $L^2$. We observe that the ultimate wall is that which separates the sample from the surrounding dielectric medium. Changing the reference frame, we can imagine a wall being dragged by the “fluid” composed by the free electron gas. With these ideas in mind, we can suppose that the dissipation $P$ is given by

$$P = Dv_F = \frac{1}{2}C\rho Av_F^3,$$  \hspace{1cm} (28)

where $\rho = nm$, $n$ and $m$ being respectively the number density and mass of charge carriers. Now impose the equality between the drag power (equation (28)) and the power dissipation due to Joule effect, namely,

$$P_J = GV^2,$$ \hspace{1cm} (29)

where $G$ is the electrical conductance and $V$ is the applied potential difference. Using $n$ given by equation (14) and considering the equality between the powers given by (28) and (29), we can write

$$\frac{Cm^2v_F^3A}{\hbar l_0\lambda} = GV^2.$$ \hspace{1cm} (30)

Finally using that

$$\frac{1}{2}mv_F^2 = eV,$$ \hspace{1cm} (31)

we obtain

$$G = \frac{2e^2}{h} \left( \frac{L^2}{l_0\lambda} \right),$$ \hspace{1cm} (32)

where we have fixed $C = 1/2$ and $A = L^2$. Sharvin’s result can be recovered if we put

$$l_0\lambda = l_F^2.$$ \hspace{1cm} (33)

Although (32) has been deduced specifically for the three dimensional case, it can be easily extended to other dimensions. In two dimensions we could think in terms of a “line wall” of size $L$ whereas the drag power can be written as (compare with (28))

$$P_{d=2} = \frac{1}{2}C\rho_sLv_F^3,$$ \hspace{1cm} (34)

where $\rho_s = n_s m$, \hspace{1cm} (35)

$n_s$ being the surface density of charge carriers. Working in an analogous way we have done before we get, after comparing (24) with (29),

$$G_{d=2} = \left( \frac{2e^2}{h} \right) \left( \frac{L}{l_0} \right)^2,$$ \hspace{1cm} (36)

where we have considered $n_s = \frac{\rho_s}{\rho_F}$ and $C = 1/2$.

Again we recover (24) if we take $l_0 = 1_F$.

All these considerations permit us, after taking into account the drag force, to generalize the formula for the electrical conductance in the ballistic regime, which reproduces equation (19) of this work.

V. CONCLUDING REMARKS

It is interesting to remember that in the paper on the scaling theory of localization, Abrahams et al. [4] have introduced the “Thouless number” (equation (10) of this paper). According to the present work it is possible to distinguish two classes of such quantity, one of them being $g_{diff} = (L/l_0)^{d-2}$, referring to the diffusive regime and the other $g_{ball} = (L/l_0)^{d-1}$ referring to the ballistic regime of the electrical conduction.

Finally it is worth to mention that both Landauer transport theory and Kubo’s formula were used to compute DC conductance in a impurity system in a recent work by Castro-Alvaredo and Fring [15]. They found an identical plateau structure for the DC conductance in the ultraviolet limit, displaying the agreement between the two approaches.

[1] R. Landauer, IBM J. Res. Dev. 1, 223 (1957).
[2] Y. Imry and R. Landauer, Rev. Mod. Phys. 71, S 306 Centenary (1999).
[3] H. van Houten and C. Beenakker, Phys. Today 49, 22 (July 1996).
[4] J. Rammer, Rev. Mod. Phys. 63, 781 (1991).
[5] I. P. Batra, Surf. Sci. 395, 43 (1998).
[6] C. Nassif, P. R. Silva, Mod. Phys. Lett. B 13, 829 (1999); Mod. Phys. Lett. B 15, 33 (2001); Mod. Phys. Lett. B 15, 1205 (2001); Mod. Phys. Lett. B 16, 601 (2002); P. R. Silva, Phys. Stat. Sol. (b) 179, K5 (1993); Phys. Stat. Sol. (b) 165, K79 (1991); Phys. Stat. Sol. (b) 174, 497 (1992); Phys. Stat. Sol. (b) 179, K99 (1993); P. R. Silva and V. B. Kokshenev, Braz. J. Phys. 30, 783 (2000).
[7] I. P. Batra, Sol. State Comm. 124, 463 (2002).
[8] B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel, C. T. Foxon, Phys. Rev. Lett. 60, 848 (1988); Phys. Rev. B 43, 12431 (1991).
[9] J. M. Krans, J. M. van Ruitenbeek, V. V. Fisun, I. K. Yanson, L. J. de Jongh, Nature 375, 767 (1995).
[10] Yu. V. Sharvin, Sov. Phys. JETP 21, 655 (1965).
[11] I. M. Lifshits and M. I. Kaganov, Uspekhi Fiz. Nauk 69, 419 (1969); Uspekhi Fiz. Nauk 78, 411 (1962).
[12] I. M. Lifshits, Sov. Phys. JETP 38, 1569 (1960).
[13] C. J. Thompson, J. Phys. A 9, L25 (1976).
[14] E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
[15] C. Kittel, Introduction to Solid State Physics, Wiley, New York, 1976, p. 169.
[16] M. Brandbyge, J. Schiøtz, M. R. Sørensen, P. Stoltze, K. W. Jacobsen, J. K. Nørskov, L. Olesen, E. Laegsgaard, I. Stensgaard and F. Besenbacher, Phys. Rev. B 52, 8499 (1995).
[17] N. B. Brandt and S. M. Chudinov, Electronic Structure of Metals, Mir Publishers, Moscow, 1975.
[18] D. Halliday, R. Resnick, Fundamentals of Physics, Third Edition Extended, Wiley, New York, 1988, p. 109.
[19] O. Castro-Alvaredo, A. Fring, Nucl. Phys. B649, 449 (2003).
[20] For applications of Thompson’s method see [6].