Counting the Number of Solutions to Certain Infinite Diophantine Equations

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Abstract. Let $r$, $v$, $n$ be positive integers. This paper investigate the number of solutions $s_{r,v}(n)$ of the following infinite Diophantine equations

\[ n = 1^r \cdot |k_1|^v + 2^r \cdot |k_2|^v + 3^r \cdot |k_3|^v + \cdots \]

for $k = (k_1, k_2, k_3, \ldots) \in \mathbb{Z}^\infty$. For each $(r, v) \in \mathbb{N} \times \{1, 2\}$, a generating function and some asymptotic formulas of $s_{r,v}(n)$ are established.

1. Introduction and statement of results

Let $r$, $n$ be positive integers. A partition into $r$-th powers of an integer $n$ is a sequence of non-increasing $r$-th powers of positive integers whose sum equals $n$. Such a partition corresponds to a solution of the following infinite Diophantine equation:

\[ n = 1^r \cdot k_1 + 2^r \cdot k_2 + 3^r \cdot k_3 + \cdots \]

for $k = (k_1, k_2, k_3, \ldots) \in \mathbb{N}_0^\infty$. Let $p_r(n)$ be the number of partitions of $n$ into $r$-th powers and let $p_r(0) := 1$, we have the generating function

\[ \sum_{n \geq 0} p_r(n) q^n = \prod_{n \geq 1} \frac{1}{1 - q^{nr}}, \]

where $q \in \mathbb{C}$ with $|q| < 1$.

Determining the values of $p_r(n)$ has a long history and can be traced back to the work of Euler. In the famous paper [3], Hardy and Ramanujan proved an asymptotic expansion for $p_1(n)$ as $n \to \infty$. They [3, p. 111] also gave an asymptotic formula for $p_r(n)$, $r \geq 2$, without proof. In [7, Theorem 2], Wright confirmed their asymptotic formula

\[ p_r(n) \sim \frac{c_r n^{\frac{1}{r+1}} - \frac{2}{r}}{\sqrt{2\pi (1 + 1/r)}} e^{(r+1)c_r n^{\frac{1}{r+1}}} \]

as integer $n \to \infty$, where $c_r = \left( r^{-1} \zeta(1 + 1/r) \Gamma(1 + 1/r) \right)^{1+1/r}$, $\zeta(\cdot)$ is the Riemann zeta function and $\Gamma(\cdot)$ is the classical Euler Gamma function.

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In this paper we investigate certain infinite Diophantine equation analogous to (1.1). For given positive integers \(n\) and \(r\), we use \(s_{r,v}(n)\) to denote the number of solutions of the following infinite Diophantine equation

\[
n = 1^r \cdot |k_1|^v + 2^r \cdot |k_2|^v + 3^r \cdot |k_3|^v + \cdots
\]

for \(k = (k_1, k_2, k_3, \ldots) \in \mathbb{Z}^\infty\). The first result of this paper is about the generating function for \(s_{r,v}(n)\).

**Proposition 1.1.** Let \(s_{r,v}(0) := 1\) and \(q \in \mathbb{C}\) with \(|q| < 1\). We have

\[
G_{r,1}(q) := \sum_{n \geq 0} s_{r,1}(n)q^n = \prod_{n \geq 1} \frac{1 + q^{nr}}{1 - q^{nr}}
\]

and

\[
G_{r,2}(q) := \sum_{n \geq 0} s_{r,2}(n)q^n = \prod_{j \geq 1} \prod_{n \geq 1} \frac{1 - (-1)^n q^{njr}}{1 + (-1)^n q^{njr}}.
\]

**Remark 1.2.** From the proof of this proposition (see Subsection 2.1), the above infinite product expansion for \(G_{r,s}(q)\) \((r \in \mathbb{N}, s = 1, 2)\) follows the identities

\[
\sum_{n \in \mathbb{Z}} q^{|n|} = \frac{1 + q}{1 - q} \quad \text{and} \quad \sum_{n \in \mathbb{Z}} q^{|n|^2} = \prod_{n \geq 1} \frac{1 - (-q)^n}{1 + (-q)^n}.
\]

They actually follow from the geometric sequence sum formula and the Jacobi triple product identity. However, any useful expansion for the sum \(\sum_{n \in \mathbb{Z}} q^{|n|^v}\) with each integer \(v > 2\) is still not found yet. Therefore, whether there are infinite product formulas which is similar to Proposition 1.1 for \(s_{r,v}(n)\) \((r \in \mathbb{N}, v \in \mathbb{Z}_{>2})\) is still a question to be settled.

Thanks to the infinite product expansion in Proposition 1.1 we can determine the asymptotic behavior of \(G_{r,v}(q)\) when \(|q| \to 1^-\). From which we can further determine the asymptotics of \(s_{r,v}(n)\) \(((r, v) \in \mathbb{N} \times \{1, 2\}\)} as \(n \to \infty\). More precisely, we prove

**Theorem 1.3.** For any given positive integers \(r\) and \(p\), we have

\[
s_{r,1}(n) = \frac{\kappa_r^{3/2}}{\sqrt{2^{r+1} \pi^r}} \left( \frac{1}{n} \right)^{1+1/r \over 1 + 1/r} W_{r,1/2} \left( \kappa_r n^{1+1/r} \right) \left( 1 + O \left( \frac{1}{n^p} \right) \right)
\]

and

\[
s_{r,2}(n) = \frac{\kappa_r^{5/4}}{\sqrt{2^{r} \pi^{r+1}}} \left( \frac{\eta(1/r)}{n} \right)^{1+1/r \over 1 + 1/r} W_{r,1/3} \left( \kappa_r \eta(1/r) \left( \frac{n}{\eta(1/r)} \right)^{1/r} \right) \left( 1 + O \left( \frac{1}{n^p} \right) \right)
\]

as integer \(n \to \infty\). Here \(\kappa_r > 0\) is given by

\[
\kappa_r^{1+1/r} = 2r^{-1}(1 - 2^{1-1/r}) \zeta(1+1/r) \Gamma(1 + 1/r),
\]
\( \eta(s) = \sum_{n \geq 1} (-1)^{n-1} n^{-s} \) is the Dirichlet eta function, and

\[
W_{\alpha, \beta}(\lambda) = \frac{1}{2\pi} \int_{-1}^{1} (1 + iu)^\beta \exp \left( \lambda \alpha^{-1} (1 + iu)^{-\alpha} + (1 + iu) \right) \, du
\]

for all \( \alpha, \beta, \lambda > 0 \).

Using the standard saddle-point method, such as referring to [5, p. 127, Theorem 7.1], we can derive an asymptotic expansion for \( W_{\alpha, \beta}(\lambda) \) as \( \lambda \to +\infty \). Hence it is possible to derive full asymptotic expansions for \( s_{r,v}(n) \) \((r, v) \in \mathbb{N} \times \{1, 2\}\). In particular, we have the following leading asymptotics.

**Corollary 1.4.** For any given positive integer \( r \), we have

\[
s_{r,1}(n) \sim 2^{-(r+3)/4} \pi^{-(r+1)/2} (1 + 1/r)^{-1/2} \eta(1/r)^{3/4} n^{1/r} \quad \text{as} \quad n \to \infty.
\]

\[
s_{r,2}(n) \sim 2^{-(r+2)/4} \pi^{-(r+3)/4} (1 + 1/r)^{-1/2} \eta(1/r)^{3/4} n^{1/r} \quad \text{as} \quad n \to \infty.
\]

2. Some results of the generating function

2.1. Proof of Proposition 1.1

We shall proceed in a formal manner to prove Proposition 1.1. Formally, using (1.2) we have

\[
\sum_{n \geq 0} s_{r,v}(n) q^n = \sum_{n \geq 0} q^n \sum_{k \in \mathbb{Z}_\infty \atop j \mapsto |k_j| = n} 1 = \sum_{k \in \mathbb{Z}_\infty} q^{\sum_{j \geq 1} j^r |k_j|^s} = \prod_{j \geq 1} \left( \sum_{k \in \mathbb{Z}} q^{j^r |k_j|^s} \right).
\]

Now, for \( q \in \mathbb{C} \) with \( |q| < 1 \), by noting that

\[
\sum_{n \in \mathbb{Z}} q^{n|} = 1 + 2 \sum_{n \geq 1} q^n = \frac{1 + q}{1 - q}
\]

and an identity of Gauss (see Andrews [1, Corollary 2.10])

\[
\sum_{n \in \mathbb{Z}} q^{n^2} = \prod_{n \geq 1} \frac{1 - (-q)^n}{1 + (-q)^n},
\]
we have

\[ G_{r,1}(q) := \sum_{n \geq 0} s_{r,1}(n)q^n = \prod_{n \geq 1} \frac{1 + q^n}{1 - q^n} \]

and

\[ G_{r,2}(q) := \sum_{n \geq 0} s_{r,2}(n)q^n = \prod_{j \geq 1} \prod_{n \geq 1} \frac{1 - (-1)^n q^{nj^r}}{1 + (-1)^n q^{nj^r}}. \]

Clearly, the product for \( G_{r,1}(q) \) is absolute convergence for all \( q \in \mathbb{C} \) with \( |q| < 1 \). For the product for \( G_{r,2}(q) \), since

\[
\left| \prod_{j \geq 1} \prod_{n \geq 1} \frac{1 - (-1)^n q^{nj^r}}{1 + (-1)^n q^{nj^r}} \right| \leq \prod_{j \geq 1} \prod_{n \geq 1} \frac{1 + |q|^{nj^r}}{1 - |q|^{nj^r}} = \prod_{\ell \geq 1} \left( \frac{1 + |q|^{\ell}}{1 - |q|^{\ell}} \right)^{\sigma_{1,r}(\ell)},
\]

where

\[ \sigma_{1,r}(\ell) = \# \{(n, j) \in \mathbb{N}^2 : nj^r = \ell \} \leq \ell; \]

and hence the product is absolute convergence for all \( q \in \mathbb{C} \) with \( |q| < 1 \). This completes the proof of Proposition 1.1.

2.2. Asymptotics of the generating function

To give a proof of Theorem 1.3, we need to determine asymptotics of the generating function in Proposition 1.1 at \( q = 1 \).

**Proposition 2.1.** Let \( r \) be a given positive integer, \( z = x + iy \) with \( x, y \in \mathbb{R} \) and \( \arg(z) \leq \pi/4 \). As \( z \to 0 \),

\[
G_{r,1}(e^{-z}) = \frac{z^{1/2} \exp \left( r \kappa_r^{1+1/r} z^{-1/r} \right)}{\sqrt{2^{r+1} \pi^r}} (1 + O(|z|^p))
\]

and

\[
G_{r,2}(e^{-z}) = \frac{z^{1/4} \exp \left( r \eta_r(1/r) \kappa_r^{1+1/r} z^{-1/r} \right)}{\sqrt{2^{2r} \pi^{2r+1}}} (1 + O(|z|^p))
\]

holds for any given \( p > 0 \). Here \( \kappa_r > 0 \) such that

\[ \kappa_r^{1+1/r} = 2r^{-2}(1 - 2^{-1-1/r})\zeta(1 + 1/r)\Gamma(1/r). \]

**Proof.** The proof of the result for \( G_{r,1}(e^{-z}) \) is similar to \( G_{r,2}(e^{-z}) \), hence we only prove the later one. We shall follow the proof of [1, p. 89, Lemma 6.1]. The series for the Riemann zeta function

\[ \zeta(s) = \sum_{n \geq 1} n^{-s} \]
and the Dirichlet eta function
\[ \eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} \]
converge absolutely and uniformly for \( s \in \mathbb{C} \) when \( \Re(s) \geq c > 1 \). Therefore, by using Mellin’s transform,
\[
\log G_{r,2}(e^{-z}) = 2 \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{j \geq 1} \sum_{n \geq 1} (-1)^{n-1} e^{-n\ell j^r z} \\
= 2 \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{j \geq 1} \sum_{n \geq 1} (-1)^{n-1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (n\ell j^r z)^{-s} \Gamma(s) \, ds \\
= 2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \sum_{\ell \geq 1} \frac{1}{\ell^{s+1}} \sum_{n \geq 1} (-1)^{n-1} \frac{1}{n^s} \sum_{j \geq 1} 1 \right) \Gamma(s) z^{-s} \, ds,
\]
that is
(2.1) \[
\log G_{r,2}(e^{-z}) = \frac{2}{2\pi i} \int_{c-i\infty}^{c+i\infty} (1 - 2^{-1-s}) \zeta(s+1) \eta(s) \zeta(rs) \Gamma(s) z^{-s} \, ds
\]
for all \( z \in \mathbb{C} \) with \( \Re(z) > 0 \). Since the only poles of gamma function \( \Gamma(s) \) are at \( s = -k \) \( (k \in \mathbb{Z}_{\geq 0}) \), and all are simple; \( \eta(s) \) is an entire function on \( \mathbb{C} \); all \( s = -2k \) \( (k \in \mathbb{N}) \) are zeros of zeta function \( \zeta(s) \), and \( s = 1 \) is the only pole of \( \zeta(s) \) and is simple. Thus, it is easy to check that the only possible poles of the integrand
\[
g_r(s) z^{-s} := (1 - 2^{-1-s}) \zeta(s+1) \eta(s) \zeta(rs) \Gamma(s) z^{-s}
\]
are at \( s = 0 \) and \( 1/r \). For all \( \sigma \in [a, b] \), \( a, b \in \mathbb{R} \) and real number \( t \), \( |t| \geq 1 \), we have the well-known classical facts (see [6] p. 38, p. 92) that
\[
\Gamma(\sigma + it) \ll_{a,b} |t|^{|\sigma|-1/2} \exp \left( -\frac{\pi}{2} |t| \right) \quad \text{and} \quad \zeta(\sigma + it) \ll_{a,b} |t|^{|\sigma|+1/2}.
\]
Hence we have \( g_r(s) \ll_{a,b} |t|^{|O(1)} \exp \left( -\frac{\pi}{2} |t| \right) \). Thus, using the residue theorem, moving the line of integration \( \Re(s) = -p \) with any given \( p > 0 \), and taking into account the possible pole at \( s = 0 \) and \( 1/r \) of \( g(s) \), we obtain
(2.2) \[
\log G_{r,2}(e^{-z}) = 2 \sum_{s \in \{0, 1/r\}} \text{Res} \left( g_r(s) z^{-s} \right) + O(|z|^p)
\]
as \( z \to 0 \) with \( |\arg(z)| < \pi/4 \). By Laurent expansion of \( \zeta(s+1) \) and \( \Gamma(s) \) at \( s = 0 \), we have
\[
\zeta(s+1) = 1/s + \gamma + O(|s|) \quad \text{and} \quad \Gamma(s) = 1/s - \gamma + O(|s|)
\]
as \( s \to 0 \). Therefore,

\[
\text{Res}_{s=1/r} \left( g_r(s) z^{-s} \right) = \frac{(1 - 2^{-1-1/r}) \zeta(1/r + 1) \eta(1/r) \Gamma(1/r)}{r z^{1/r}}
\]

and

\[
\text{Res}_{s=0} \left( g_r(s) z^{-s} \right) = \frac{1}{8} \log \left( \frac{z}{2 r^{r+1}} \right).
\]

Combining \[2.2\] with above results, we obtain the proof of this proposition.

We also need the following upper bound results.

**Lemma 2.2.** Let \((r, v) \in \mathbb{N} \times \{1, 2\}\) be given, \(z = x + iy\) with \(x \in \mathbb{R}_+\) and \(y \in (-\pi, \pi] \setminus (-x, x)\). As \(x \to 0\),

\[
\Re \left( \log \frac{G_{r,v}(e^{-x})}{G_{r,v}(e^{-z})} \right) \gg x^{-1/r}.
\]

**Proof.** By using Proposition \[1.1\] with \(q \in \mathbb{C}\) and \(|q| < 1\), we have

\[
\log G_{r,1}(q) = \sum_{j \geq 1} \log \left( \frac{1 + q^{r_j}}{1 - q^{r_j}} \right)
\]

\[=
\sum_{j \geq 1} \left( \sum_{\ell \geq 1} (-1)^{\ell-1} q^{\ell r_j} \frac{\ell}{\ell} + \sum_{\ell \geq 1} q^{\ell r_j} \frac{\ell}{\ell} \right)
\]

\[=
\sum_{\ell \geq 1} \frac{1}{\ell} \sum_{j \geq 1} ((-1)^{\ell-1} + 1) q^{\ell r_j} = 2 \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{j \geq 1} q^{\ell r_j}
\]

and

\[
\log G_{r,2}(q) = \sum_{n,j \geq 1} \log \left( \frac{1 - (-1)^n q^{nj r_j}}{1 + (-1)^n q^{nj r_j}} \right)
\]

\[=
\sum_{n,j \geq 1} \sum_{\ell \geq 1} \frac{1}{\ell} (-1)^n q^{nj r_j} \frac{\ell}{\ell} + (-1)^{\ell} (-1)^n q^{nj r_j} \frac{\ell}{\ell}
\]

\[=
\sum_{\ell,j \geq 1} \frac{(-1)^{\ell-1} - 1}{\ell} \frac{(-q^{r_j})^\ell}{1 - (-q^{r_j})^\ell} = 2 \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{j \geq 1} q^{r_j} \frac{\ell}{1 + q^{r_j} \frac{\ell}{}},
\]

Furthermore,

\[
\Re \left( \log \frac{G_{r,1}(e^{-x})}{G_{r,1}(e^{-z})} \right) = 2 \sum_{\ell \geq 1} \frac{1}{\ell} \sum_{j \geq 1} e^{-j \ell x} \Re \left( 1 - \exp \left( \frac{2 \pi i \ell x^r y}{2 \pi} \right) \right)
\]
and
\[
\Re \left( \log \frac{G_{r,2}(e^{-x})}{G_{r,2}(e^{-z})} \right) = 2 \sum_{\ell \geq 1} \sum_{\ell \text{ odd}} \sum_{j \geq 1} \frac{\Re \left( \frac{e^{-j' \epsilon x}}{1 + e^{-j' \epsilon x}} - \frac{e^{-j' \epsilon z}}{1 + e^{-j' \epsilon z}} \right)}{\cosh(j' \ell x) + \cos(j' \ell y)} \sin^2 \left( \frac{j' \ell y}{2} \right).
\]

By noting that all summand in above sums are nonnegative we have
\[
\Re \left( \log G_{r,1}(e^{-x}) \right) \geq 2 \sum_{j \geq 1} e^{-j' x} \Re \left( 1 - \exp \left( \frac{2\pi i j'r y}{2\pi} \right) \right)
\]
\[\gg \sum_{(2\pi/x)^{1/r} < j \leq 2(2\pi/x)^{1/r}} \Re \left( 1 - \exp \left( \frac{2\pi i j'r y}{2\pi} \right) \right)
\]
and
\[
\Re \left( \log G_{r,2}(e^{-x}) \right) \geq 2 \sum_{j \geq 1} \frac{\tanh \left( \frac{j' \pi}{2} \right)}{\cosh(j' \ell x) + \cos(j' \ell y)} \sin^2 \left( \frac{j' \ell y}{2} \right)
\]
\[\gg \sum_{(2\pi/x)^{1/r} < j \leq 2(2\pi/x)^{1/r}} \Re \left( 1 - \exp \left( \frac{2\pi i j'r y}{2\pi} \right) \right)
\]

Thus by using Lemma 2.3 with \( L = (2\pi/x)^{1/r} \), we find that
\[
\Re \left( \log \frac{G_{r,v}(e^{-x})}{G_{r,v}(e^{-z})} \right) \gg \delta_r (2\pi/x)^{1/r} \gg x^{-1/r}
\]
holds for all sufficiently small \( x > 0 \) and \( v \in \{1, 2\} \). This finishes the proof.

\[\square\]

**Lemma 2.3.** Let \( r \in \mathbb{N}, y \in \mathbb{R} \) and \( L \in \mathbb{R}_+ \) such that \( L^{-r} < |y| \leq 1/2 \). Then there exists a constant \( \delta_r \in (0, 1) \) depending only on \( r \) such that
\[
\left| \sum_{L < n \leq 2L} e^{2\pi \text{i} nr y} \right| \leq (1 - \delta_r)L
\]
holds for all positive sufficiently large \( L \).

**Proof.** The lemma for \( r = 1 \) is easy and we shall focus on the cases of \( r \geq 2 \). By the well-known Dirichlet’s approximation theorem, for any \( y \in \mathbb{R} \) and \( L > 0 \) being sufficiently large, then there exist integers \( d \) and \( h \) with \( 0 < h \leq L^{r-1} \) and \( \gcd(h, d) = 1 \) such that
\[
|y - \frac{d}{h}| < \frac{1}{hL^{r-1}}.
\]
(2.3)
The use of [4, Equation 20.32] implies that

\[ \sum_{L < n \leq 2L} e^{2\pi in^r y} = \frac{1}{h} \sum_{1 \leq j \leq h} e^{2\pi ij^r y} \int_L^{2L} e^{2\pi iu^r (y-d)} \, du + O(h). \]  

(2.4)

If the real number \( y \) satisfies \( L^{-r} < |y| \leq L^{1-r} \), then \( y \) satisfies the approximation (2.3) with \( (h, d) = (1, 0) \). This means that

\[ \left| \sum_{L < n \leq 2L} e^{2\pi in^r y} \right| = \left| \int_L^{2L} e^{2\pi iu^r y} \, du + O(1) \right| \leq 2 \cdot \frac{1}{2\pi r |y| L^{r-1}} (1 + 2^{1-r}) + O(1) \leq \frac{1 + 2^{1-r}}{\pi r} L + O(1). \]

(2.5)

If the real number \( y \) satisfies \( 1/2 \geq |y| \geq L^{1-r} \), then \( y \) satisfies the approximation (2.3) with \( h \geq 2 \). Further, by using [2, Lemma 2.1] in (2.4), we find that there exists a positive constant \( \delta_{r1} \) depending only on \( r \) such that

\[ \left| \sum_{L < n \leq 2L} e^{2\pi in^r y} \right| \leq (1 - \delta_{r1}) L + O(h). \]

(2.6)

On the other hand, the use of Weyl’s inequality (see [4, Lemma 20.3]) implies that

\[ \sum_{L < j \leq 2L} e^{2\pi ij^r y} \ll \varepsilon L^{1+\varepsilon} (h^{-1} + L^{-1} + hL^{-r})^{2^{1-r}} \ll L^{1-2^{-r-1/2}} \]

holds for all integers \( h \in (L^{1/2}, L^{r-1}] \). By using (2.5), (2.6) and (2.7), it is not difficult to obtain the proof of the lemma.

3. Proof of the main theorem

From Proposition 2.1 and Lemma 2.2, we can check that the sequences \( \{s_{r,1}(n)\}_{n \geq 0} \) and \( \{s_{r,2}(n)\}_{n \geq 0} \) satisfy the conditions of Proposition 3.1 below. Therefore, applying the following proposition, Theorem 1.3 and Corollary 1.4 follow.

Proposition 3.1. For a sequence \( \{c_n\}_{n \geq 0} \) of real numbers, we let \( G(q) := \sum_{n \geq 0} c_n q^n \). Suppose that for \( x \in \mathbb{R}_+ \) and \( y \in (-\pi, \pi] \),

\[ G(e^{-x-iy}) - \gamma(x + iy)^{\beta} e^{\kappa (x+iy)^{-\alpha}} \ll x^p G(e^{-x}), \quad x \to 0 \]

holds for any given \( p > 0 \), where \( \kappa, \gamma, \beta, \alpha \in \mathbb{R}_+ \). Then, for any given \( p > 0 \) we have

\[ c_n = \gamma \left( \frac{\kappa}{n} \right)^{1+\beta} W_{\alpha,\beta} \left( \frac{1}{\kappa^{1+\beta} n^{1+\alpha}} \right) \left( 1 + O(n^{-p}) \right) \]

as integer \( n \to \infty \). In particular,

\[ c_n \sim 2^{-1/2} \pi^{-1/2} (1 + \alpha)^{-1/2} \gamma \kappa^{\beta+\alpha/2} n^{-1+\beta+\alpha/2} e^{(1+\alpha^{-1})\kappa^{1+\beta} n^{1+\alpha}}, \quad n \to \infty. \]
Proof. For any given positive integer \( n \) sufficiently large, by using the orthogonality we have
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{-x-iy}) e^{nx+niy} \, dy.
\]

We split the above integral as
\[
c_n = \frac{1}{2\pi} \int_{-x}^{x} \gamma(x+iy)^{\beta} e^{\kappa \alpha^{-1}(x+iy)^{-\alpha}+n(x+iy)} \, dy
\]
\[
+ \frac{1}{2\pi} \int_{-\pi}^{\pi} (G(e^{-x-iy}) - \gamma(x+iy)^{\beta} e^{\kappa \alpha^{-1}(x+iy)^{-\alpha}}) e^{n(x+iy)} \, dy
\]
\[
=: I(n) + E(n).
\]

Let \( x = \left( \frac{\kappa}{n} \right)^{\frac{1}{1+\alpha}} \). For \( E(n) \), we estimate that
\[
E(n) \ll \int_{-\pi}^{\pi} x^{(1+\alpha)p} G(e^{-x}) e^{nx} \, dy
\]
\[
\ll \int_{-\pi}^{\pi} x^{p} e^{\kappa \alpha^{-1}x^{-\alpha}+nx} \, dy \ll n^{-p} e^{(\alpha^{-1}+1)\kappa \frac{1}{1+\alpha} n \frac{\alpha}{1+\alpha}}
\]
holds for any given \( p > 0 \). For \( I(n) \), we compute that
\[
I(n) = \frac{\gamma}{2\pi i} \int_{-\pi}^{\pi} x^{1+i\beta} e^{\kappa \alpha^{-1}x^{-\alpha}+nx} \, dz
\]
\[
= \gamma x^{1+i\beta} \int_{1-i}^{1+i} u^{\beta} e^{\kappa \alpha^{-1}u^{-\alpha}+nu} \, du
\]
\[
= \gamma \left( \frac{\kappa}{n} \right)^{\frac{1+i+\beta}{1+\alpha}} \frac{1}{2\pi i} \int_{1-i}^{1+i} u^{\beta} e^{\kappa \frac{1}{1+\alpha} n \frac{\alpha}{1+\alpha} (\alpha^{-1}u^{-\alpha}+u)} \, du,
\]
that is
\[
I(n) = \gamma \left( \frac{\kappa}{n} \right)^{\frac{1+i+\beta}{1+\alpha}} W_{\alpha,\beta} \left( \kappa \frac{1}{1+\alpha} n \frac{\alpha}{1+\alpha} \right).
\]

By using the standard Laplace saddle-point method (see, for example, [5, p. 127, Theorem 7.1]), since the integral
\[
W_{\alpha,\beta}(\lambda) = \frac{1}{2\pi} \int_{-1}^{1} (1 + iu)^{\beta} \exp \left( \lambda (\alpha^{-1} (1 + iu)^{-\alpha} + (1 + iu)) \right) \, du
\]
has a simple saddle point \( u = 0 \), it is not difficult to prove that
\[
W_{\alpha,\beta}(\lambda) \sim \frac{1}{\sqrt{2\pi(1+\alpha)}} \frac{e^{(1+\alpha^{-1})\lambda}}{\lambda^{1/2}}
\]
as \( \lambda \to +\infty \). The proof of Proposition 3.1 follows from (3.1)–(3.3) and (3.4). This completes the proof.

\( \square \)
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