Projections of Gibbs measures on self-conformal sets

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Abstract
We show that for Gibbs measures on self-conformal sets in $\mathbb{R}^d$ ($d \geq 2$) satisfying certain minimal assumptions, without requiring any separation condition, the Hausdorff dimension of orthogonal projections to $k$-dimensional subspaces is the same and is equal to the maximum possible value in all directions. As a corollary we show that Falconer’s distance set conjecture holds for this class of self-conformal sets satisfying the open set condition.

Keywords: Hausdorff dimension, Gibbs measure, self-conformal sets, projections, CP-chain, group extension

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1. Introduction

Let $d \geq k \geq 2$ be integers, let $K \subset \mathbb{R}^d$ be Borel or analytic, and let $\Pi_{d,k}$ be the set of orthogonal projections from $\mathbb{R}^d$ to its $k$-dimensional subspaces, with natural Haar measure $\xi$. Then

$$\dim_H \pi K = \min \{ k, \dim_H K \} \text{ for } \xi\text{-almost every } \pi \in \Pi_{d,k}.$$ 

This famous result, due to Marstrand [14] in the plane and Mattila [15] in $\mathbb{R}^d$, has been the basis for a great deal of work in the field of fractal geometry. Until fairly recently, most of this work concerned general Borel sets $K$ and almost all projections $\pi \in \Pi_{d,k}$. However, Furstenburg’s innovative CP-chain method [11] enabled Hochman and Shmerkin [12] to show that for self-similar sets and measures with dense rotations and which satisfy the strong separation condition, the result holds for all $\pi \in \Pi_{d,k}$. Since then, their work has been followed up by many mathematicians, see the recent survey papers [6, 16, 22] and the references therein. In particular, Falconer and Jin [7] extended their result to random cascade measures (including self-similar measures as special cases) without requiring any separation condition.
In [12] Hochman and Shmerkin also considered the projections of products of Gibbs measures on one-dimensional (1D) nonlinear Cantor sets. The authors used the so-called limit diffeomorphisms of 1D nonlinear iterated function systems developed by Sullivan [25] and Bedford and Fisher [1] to transfer the problem back to the affine case. In higher dimensions, Fraser and Pollicott [10] showed that for Gibbs measures on self-conformal sets with the strong separation condition there exists a limit conformal map under which the Gibbs measures generate a CP-chain. But the strong projection theorem for self-conformal measures cannot be directly proved from this result since the dimension of projections is not preserved under conformal maps, and the dense rotations condition is not clear in their setting.

The main difficulty of studying CP-chains/scenery flows of self-conformal measures comes from the fact that linear ‘zooming-in’ operators do not fit well with the nonlinear iterated function systems. In this paper we use the methods from Falconer and Jin [7], along with those from Hochman and Shmerkin [12], to overcome this difficulty. The main idea (lemma 3.4) is first to zoom-in on measures on the symbolic space, or in other words, to zoom-in with conformal mappings, in order to generate a CP-chain, then zoom-in on the conditional measures in the CP-chain with linear scale functions to estimate the entropy distortions. We also make clear how to formulate an analogue of the dense rotations condition for the minimality of the underlining dynamical system. The methods from [7] also remove the requirement of any separation condition on the underlying sets.

Here we will consider an iterated function system (IFS) of conformal $C^{1+\epsilon}$-maps $\mathcal{I} = \{f_i\}_{i=1}^m$ in $\mathbb{R}^d$ satisfying the following assumptions:

(A0) There is a bounded, convex open set $U \subset \mathbb{R}^d$ such that each $f_i : U \to U$ is an injective conformal map, that is $f_i(x) \in U$, the derivative $f'_i(x)$ exists for every $x \in U$ and is a scalar times a rotation matrix, which we may write as

$$f'_i(x) = r_i(x)O_i(x),$$

where $r_i(x) \in (0, \infty)$ and $O_i(x) \in SO(d, \mathbb{R})$.

(A1) There exists a constant $0 < r^* < 1$ such that $r_i(x) \leq r^*$ for every $1 \leq i \leq m$ and $x \in U$.

**Remark 1.1.** Here we assume the open set $U$ to be convex just for simplicity. It is sufficient to assume that $U$ is connected. The only proof that will be affected by this is lemma 3.2, where instead of connecting two points in $U$ by a line segment, we can connect two points by smooth curves within $U$, see [19] for example.

These assumptions imply that the IFS $\mathcal{I}$ is uniformly contractive on $U$, therefore it defines a unique attractor $K \subset U$, i.e. a non-empty compact set such that

$$K = \bigcup_{i=1}^m f_i(K).$$

Such a $K$ is called a self-conformal set. The set $K$ has a natural symbolic representation: let $\Lambda = \{1, \ldots, m\}$ be the alphabet and let $\Lambda^\mathbb{N}$ be the symbolic space with $m$ letters. For each $i = i_1 \cdots i_n \in \Lambda^n$ denote by

$$f_i = f_{i_1} \circ \cdots \circ f_{i_n}$$

and for $\bar{i} = i_1i_2 \cdots \in \Lambda^\mathbb{N}$ and $n \geq 1$ denote by $\bar{i}|_n = i_1 \cdots i_n$. Fix a point $x_0 \in U$. Then we may define a map $\Phi : \Lambda^\mathbb{N} \to U$ by
\[ \Phi(i) = \lim_{n \to \infty} f_i(x_0). \]

Since all \( f_i \) are injective, the above limit always exists and it does not depend on the choice of \( x_0 \). Its image is the self-conformal set \( K \) and \( \Phi \) is called the canonical map.

For \( i = i_1 \cdots i_n \in \Lambda^\mathbb{N} \) denote by \( [i] = \{ i \in \Lambda^\mathbb{N} : \overline{i}_n = i \} \) the cylinder in \( \Lambda^\mathbb{N} \) encoded by \( i \). Let \( \mathcal{B} \) denote the \( \sigma \)-algebra generated by cylinders. Let \( \sigma \) denote the left-shift operator on \( \Lambda^\mathbb{N} \). Let \( \varphi : \Lambda^\mathbb{N} \to \mathbb{R} \) be a Hölder potential on \( \Lambda^\mathbb{N} \) and let \( \mu_\varphi \) denote its Gibbs measure (see section 2.1 for precise definition). We are interested in the orthogonal projections of the push-forward measure

\[ \Phi \mu_\varphi = \mu_\varphi \circ \Phi^{-1} \]
on the self-conformal set \( K \). Before stating our main result, we shall present an analogue of the dense rotations condition in the self-conformal case. Let \( \mathcal{G} = SO(d, \mathbb{R}) \) and define a map \( \phi : \Lambda^\mathbb{N} \to \mathcal{G} \) as follows:

\[ \phi(i) = O_{i_1} \Phi(\sigma i) \]for \( i = i_1 \cdots \).

The rotation \( \phi(i) \) corresponds exactly to the change of directions at \( \Phi(i) \) while zooming-in using \( f_{i_n}^{-1} \). We may define the skew product \( \sigma_\varphi : \Lambda^\mathbb{N} \times \mathcal{G} \to \Lambda^\mathbb{N} \times \mathcal{G} \) as

\[ \sigma_\varphi (i, O) = (\sigma i, O \phi(i)). \]

We assume

(A2) \( \sigma_\varphi \) has a dense orbit in \( \Lambda^\mathbb{N} \times \mathcal{G} \).

By a compact group extension theorem (see section 2.5), this implies that the dynamical system

\[ (\Lambda^\mathbb{N} \times \mathcal{G}, \mathcal{B} \otimes \mathcal{B}_G, \sigma_\varphi, \mu_\varphi \times \xi) \]is ergodic, where \( \xi \) is the normalised right-invariant Haar measure on \( G \), and \( \mathcal{B}_G \) is its Borel \( \sigma \)-algebra.

Remark 1.2. The assumption (A2) is used only to prove that the dynamical system (1.3) is ergodic. In this dynamical system, (A2) is equivalent to topological transitivity, that is, for any non-empty open sets \( U, V \in \Lambda^\mathbb{N} \times \mathcal{G} \), there exists \( n > 1 \) such that \( \sigma_\varphi^n(U) \cap V \) is non-empty. See [24] for details. We shall prove (see lemma 2.2), that if there exists a dense orbit \( \{ O_{i_n} \Phi(\sigma i) \}_{n \geq 1} \) in \( \mathcal{G} \), then this implies topological transitivity.

Now we are ready to state our main result:

**Theorem 1.3.** Under assumptions (A0)–(A2), for all \( \pi \in \Pi_{d,k} \) we have

\[ \dim_{\mathcal{H}} \pi \Phi \mu_\varphi = \min\{ k, \dim_{\mathcal{H}} \Phi \mu_\varphi \}. \]

With the same approach as in [12] we can also prove the following.

**Corollary 1.4.** Under assumptions (A0)–(A2), for all \( C^1 \)-maps \( h : K \to \mathbb{R}^k \) without singular points,

\[ \dim_{\mathcal{H}} h \Phi \mu_\varphi = \min\{ k, \dim_{\mathcal{H}} \Phi \mu_\varphi \}. \]

It is well-known that if the self-conformal set \( K \) satisfies the open set condition (OSC) then there exists a Gibbs measure \( \mu \) of a Hölder potential on \( \Lambda^\mathbb{N} \) such that \( \dim_{\mathcal{H}} \Phi \mu = \dim_{\mathcal{H}} K \).
and $\mu$ is equivalent to $\dim_H K$-dimensional Hausdorff measure (see [4, 20] for example). This implies the following.

**Corollary 1.5.** Under assumptions (A0)–(A2), as well as the OSC, for all $C^1$-maps $h: K \to \mathbb{R}^k$ without singular points,

$$\dim_H h(K) = \min\{k, \dim_H K\}.$$  \hfill (1.4)

**Remark 1.6.** We note that the OSC is only required when using theorem 1.3 to state a corresponding result for self-conformal sets. The OSC in the above corollary can be relaxed to the so-called strong variational principle: there exists a Hölder potential $\varphi$ such that the corresponding Gibbs measure $\mu_\varphi$ satisfies $\dim_H \mu_\varphi = \dim_H K$. It is known due to Feng and Hu [9] that variational principle holds for conformal IFSs: $\dim_H K$ is reached over the supreme of the push-forward measures of ergodic measures on $(\Lambda^N, \sigma)$. Whether strong variational principle holds for conformal IFSs without separation condition is still open.

By applying corollary 1.5 to $C^1$-maps $h(x) = |x - a|$ outside a neighborhood of $a \in K$ we deduce that Falconer’s distance set conjecture (see [23] and the references therein for most recent developments), is true for this family of self-conformal sets:

**Corollary 1.7.** Under assumptions (A0)–(A2), as well as the OSC, if $\dim_H K \geq 1$, then for $a \in K$,

$$\dim_H \{|x - a| : x \in K\} = \dim_H \{|x - y| : x, y \in K\} = 1.$$  

**Remark 1.8.** The method of using a strong projection theorem to prove a result about distance sets is originally due to Orponen, [17]. Corollary 1.7 in the case of $d = 2$ and $\dim_H K > 1$ was proved in [10], theorem 2.7. The case when $d > 2$ or $\dim_H K = 1$ is new to the best of our knowledge.

Now we give an example of self-conformal sets which satisfy all of our assumptions. According to theorem 14.15 in [5], when $|c| > \frac{1}{2}(5 + 2\sqrt{6}) = 2.475...$, the Julia set $J_f$ defined by the quadratic polynomial $f(z) = z^2 + c$ is totally disconnected, and is the attractor of the conformal IFS

$$\{f_1(z) = \sqrt{z - c}, f_2(z) = -\sqrt{z - c}\}.$$  

We may take $U = \{z : |z| < |2c|^{1/2}\}$. It is easy to see that $U$ is bounded, open and convex, and for all $z \in U$,

$$|f'_1(z)| = \frac{1}{2}|z - c|^{-1/2} \leq \frac{1}{2}(|c| - |2c|^{1/2})^{-1/2} < 1.$$  

This verifies assumptions (A0) and (A1), as well as the OSC. A consequence of lemma 2.2 is that to achieve (A2) it is sufficient to show that there exists a finite word $u$ such that $O_u(x)$ is an irrational rotation, where $u = uu \cdots$ denotes the periodic infinite word of $u$. We prove that such a $u$ exists for $J_f$. Take a fixed point $\alpha = \frac{1 + \sqrt{1 - 4c}}{2} \in J_f$ so that $f_1(\alpha) = \alpha$. We have

$$f'_1(\alpha) = \frac{1}{2\sqrt{\alpha - c}} = \frac{1}{2\alpha} = \frac{1}{1 + \sqrt{1 - 4c}}.$$  

Therefore, if
\[
\frac{\arg f'(\alpha)}{\pi} = \frac{\arg(1 + \sqrt{1 - 4c})}{\pi}
\] is irrational, \[ (1.5) \]
then \( \{O_{p_i}(x, \sigma_p)\}_{n \geq 1} \) is dense in \( SO(2, \mathbb{R}) \) for \( i = 111 \ldots \). This verifies assumption (A2).

Remark 1.9. The above example is a particular case of hyperbolic Julia sets. It is worth mentioning that in [2] Bedford, Fisher and Urbański showed that the scenery flow of hyperbolic Julia sets (a geometric realization of our dynamical system \( (\Lambda^\mathbb{N} \times G, \mathcal{B} \otimes B_G, \mu \times \xi, \sigma_\mu) \)) is ergodic in all cases with the exception of the following:

(i) the Julia set \( J_f \) is a geometric circle and \( f \) is biholomorphically conjugate to a finite Blaschke product, or

(ii) the Julia set \( J_f \) is totally disconnected and \( J_f \) is contained in a real-analytic curve with self-intersections (if any) lying outside the Julia set.

Our proof requires us to change the alphabet of the symbolic space from \( \Lambda \) to \( \Lambda_q \) for \( q \geq 1 \), see (3.1). We need the ergodicity for each \( \Lambda_N^q \), which can be deduced from assumption (A2) (lemma 3.3). It is not clear to us whether Bedford, Fisher and Urbański’s criteria would imply this.

The rest of the paper is organised as follows. In section 2 we will first go through some background on symbolic space and self-conformal sets. We present the bounded distortion property which is satisfied in our setting. We will briefly mention the thermodynamic formalism which defines the Gibbs measure, through which we obtain an ergodic dynamical system. From this ergodicity we know from [9] that our Gibbs measure \( \Phi_{\mu, \phi} \) is exact dimensional (see section 2.6 for the definition). A theorem on compact group extension will show us that the skew product of this dynamical system with \( G \) is also ergodic. From here, having stated some definitions of entropy and dimension, we move on to the dimension of the projections of the self-conformal Gibbs measures. Section 3 uses the methods of [7, 12], to prove that the dimension of these measures takes the ‘expected’ value for \( \xi \)-almost all \( \pi \in \Pi_{\xi} \). Then following a similar argument of Hochman and Shmerkin [12] we may extend the value to all \( \pi \in \Pi_{\xi} \) and to all \( C^1 \)-maps without singular points.

2. Preliminaries

2.1. Symbolic space

Let \( \Lambda = \{1, \ldots, m\} \) be the alphabet on \( m \geq 2 \) symbols. Let \( \Lambda^* = \bigcup_{n \geq 1} \Lambda^n \) be the set of finite words. For \( i \in \Lambda^* \) let \( |i| \) denote the length of the word. Let \( \Lambda^\mathbb{N} \) be the symbolic space of infinite sequences from the alphabet. For \( i \in \Lambda^\mathbb{N}, n \geq 1 \), let \( \bar{i}_n \in \Lambda^n \) be the first \( n \) digits of \( i \). For \( i \in \Lambda^n \), let \( \bar{i} = \{i \in \Lambda^\mathbb{N} : \bar{i}_n = i\} \) be the cylinder rooted at \( i \). We may endow \( \Lambda^\mathbb{N} \) with the standard metric \( d_\rho \) with respect to a real number \( \rho \in (0, 1) \), that is, for \( \vec{i}, \vec{j} \in \Lambda^\mathbb{N} \),

\[
d_\rho(\vec{i}, \vec{j}) = \rho^{|\{n \geq 0 : \bar{i}_n \neq \bar{j}_n\}|},
\]

with the convention that \( \bar{\vec{i}}_0 = \emptyset \) for all \( \vec{i} \in \Lambda^\mathbb{N} \). Then \( (\Lambda^\mathbb{N}, d_\rho) \) is a compact metric space. Let \( \mathcal{B} \) be its Borel \( \sigma \)-algebra. Define the left shift map \( \sigma \) by \( \sigma(\vec{i}) = (i_{n+1})_{n \geq 1} \) for \( \vec{i} = (i_n)_{n \geq 1} \in \Lambda^\mathbb{N} \).
2.2. Self-conformal sets

Let $\mathcal{I}$ be an iterated function system (IFS) as in (1.1) of conformal maps defined on a bounded open connected subset $U \subseteq \mathbb{R}^d$ with non-empty compact attractor $K \subseteq \mathbb{R}^d$ satisfying (1.2). Since $\mathcal{I}$ is uniformly contractive on $U$, one can find a connected open set $V$ such that $K \subset V \subset \mathbb{V} \subset U$ and $\min \{\text{dist}(K, \partial V), \text{dist}(V, \partial U)\} > 0$. Recall that $\Phi : \Lambda^N \to K$ is the canonical projection, that is, $\Phi(i) = \lim_{n \to \infty} f_{i_1} \cdots f_{i_n}(x_0)$ for some $x_0 \in U$. We shall also use the notation $x_i = \Phi(i)$ for $i \in \Lambda^N$.

2.3. Bounded distortion

For $x \in V$ and $i = i_1 \cdots i_n \in \Lambda^n$ we may write

$$f_i'(x) = r_i(x)O_i(x),$$

(2.1)

where

$$r_i(x) = r_{i_1}(f_{i_2} \circ \cdots \circ f_{i_n}(x)) \cdot r_{i_2}(f_{i_3} \circ \cdots \circ f_{i_n}(x)) \cdots r_{i_n}(x),$$

$$O_i(x) = O_{i_1}(f_{i_2} \circ \cdots \circ f_{i_n}(x)) \cdot O_{i_2}(f_{i_3} \circ \cdots \circ f_{i_n}(x)) \cdots O_{i_n}(x).$$

It is well-known that in a simply-connected complex domain every holomorphic function is analytic. Therefore we have the following bounded distortion property: there exists a constant $C_1 > 0$ such that for all $i \in \Lambda^*$ and $x, y \in V$,

$$\frac{r_i(x)}{r_i(y)} \leq C_1.$$  

(2.2)

To see this, for $i = i_1 \cdots i_n \in \Lambda^n$ we have

$$\log \frac{r_i(x)}{r_i(y)} = \sum_{k=1}^{n} \log r_k(f_{i_{k+1}} \cdots f_{i_n}(x)) - \log r_k(f_{i_{k+1}} \cdots f_{i_n}(y)).$$

(2.3)

By the smoothness of $\log r_i$, one can find a constant $\tilde{C}_1$ such that for all $x, y \in V$ and $i \in \Lambda$,

$$|\log r_i(x) - \log r_i(y)| \leq \tilde{C}_1|x - y|.$$  

On the other hand, by (A1), one can find another constant $\tilde{C}_1'$ such that for all $x, y \in V$, $n \geq 1$ and $i = i_1 \cdots i_n \in \Lambda^n$,

$$|f_i(x) - f_i(y)| \leq \tilde{C}_1'(r^*)^n.$$  

Combining these two inequalities we get from (2.3) that

$$\log \frac{r_i(x)}{r_i(y)} \leq \tilde{C}_1 \tilde{C}_1' \sum_{k=1}^{n} (r^*)^{n-k} \leq \tilde{C}_1 \tilde{C}_1' \frac{1}{1 - r^*} := \log C_1.$$  

The bounded distortion also implies the following fact: there exists a constant $C_2 > 0$ such that for all $i \in \Lambda^*$ and all $x, y \in V$,

$$C_2^{-1}r_i|x - y| \leq |f_i(x) - f_i(y)| \leq C_2 r_i|x - y|,$$  

(2.4)

where for $i \in \Lambda^*$ we denote by $\overline{r}_i = \sup\{r_i(x) : x \in V\}$. For a proof see [19, lemma 2.2] for example (note that the proof does not require any separation condition).
2.4. Gibbs measures

Let $\varphi$ be a Hölder potential defined on $\Lambda^N$. This means there exist constants $\kappa > 0$ and $\beta \in (0, 1)$ such that for $n \geq 1$,

$$\text{Var}_n(\varphi) := \sup_{i \in N} \sup_{j \in [i]} |\varphi(i) - \varphi(j)| \leq \kappa \beta^n. \quad (2.5)$$

For $n \geq 1$ the $n$th-order Birkhoff sum of $\varphi$ over $\sigma$ is defined as

$$S_n \varphi(i) = \sum_{k=0}^{n-1} \varphi \circ \sigma^k(i),$$

for $i \in \Lambda^N$. The topological pressure of $\varphi$ on $\Lambda^N$ is given by

$$P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in \Lambda^N} \exp \left( \max_{j \in [i]} S_n \varphi(i) \right),$$

where the existence of the limit can be proved using the sub-additive property of the logarithm

$$e^{-\text{Var}_n(\varphi)} \leq \frac{\mu_\varphi([i_n])}{\exp(S_n \varphi(j) - nP(\varphi))} \leq e^{\text{Var}_n(\varphi)}. \quad (2.6)$$

Also, $\mu_\varphi$ possesses the quasi-Bernoulli property:

$$e^{-\kappa |j-i|} \mu_\varphi([i]) \mu_\varphi([j]) \leq \mu_\varphi([i]) \leq e^{\kappa |j-i|} \mu_\varphi([j]) \mu_\varphi([i]) \quad (2.7)$$

for all $i, j \in \Lambda^*$. (Here we have been more precise on the quasi-Bernoulli constant $e^{\kappa |j-i| - \kappa |j' - i'|}$ in terms of the length of $i$ and $j$.) In particular, when the potential function $\varphi : \Lambda^N \to \mathbb{R}$ takes the values $\varphi(i) = \log p_i$, for a fixed vector $p = (p_i)_{i \in \Lambda}$, such that $0 < p_i < 1$ and $\sum_{i \in \Lambda} p_i = 1$, noting that it is Lipschitz on $(\Lambda^N, d_\rho)$, then we can define a Gibbs measure $\mu_\varphi$ on $\Lambda^N$ by:

$$\mu_\varphi([i_n]) = \exp S_n \varphi(i)$$

$$= p_{i_1} \cdots p_{i_n},$$

for $i \in \Lambda^N$. This is simply the Bernoulli measure on $\Lambda^N$.

2.5. The compact group extension

We will now deal with the system $(\Lambda^N, \mathcal{B}, \sigma, \mu)$ and its compact group extensions, where $\mu = \mu_\varphi$ is a Gibbs measure with respect to a Hölder potential $\varphi$. Recall that $G = SO(d, \mathbb{R})$ is a compact Lie group with Borel $\sigma$-algebra $\mathcal{B}_G$ and we have defined the map $\phi : \Lambda^N \to G$ as

$$\phi(i) = O_n(\Phi(\sigma^i))$$

for $i = i_1 i_2 \cdots$. By the smoothness of conformal maps it is easy to see that $\phi$ is Hölder on $(\Lambda^N, d_\rho)$. We may define the skew product $\sigma_\phi : \Lambda^N \times G \to \Lambda^N \times G$ as follows:

$$\sigma_\phi(i, O) = (\sigma I, O \phi(i)).$$

It is easy to verify that the product measure $\mu \times \xi$ is $\sigma_\phi$-invariant, where $\xi$ is the right-invariant normalised Haar measure on $G$. 609

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Proposition 2.1. The dynamical system $(\Lambda^N \times G, \mathcal{B} \otimes \mathcal{B}_G, \mu \times \xi, \sigma_\phi)$ is ergodic.

Proof. Under (A2) we have that $\sigma_\phi$ has a dense orbit in $\Lambda^N \times G$. The result then follows directly from [18, corollary 4.5]. □

Here we give a sufficient assumption (equivalent to the dense rotations condition when the conformal functions are similarities) to achieve topological transitivity, and therefore (A2).

Lemma 2.2. Assume that for some $i \in \Lambda^N$, 
\[
\{O_{ij}(x_{\sigma i})\}_{n \geq 1} \text{ is dense in } G.
\]
Then the skew product $\sigma_\phi$ is topologically transitive.

Proof. Recall the notation $x_i = \Phi(i)$ for $i \in \Lambda^N$. Fix $i \in \Lambda^N$ so that $\{O_{ij}(x_{\sigma i})\}_{n \geq 1}$ is dense in $G$. Recall the definition of the skew product $\sigma_\phi$.
\[
\sigma_\phi : \Lambda^N \times G \to \Lambda^N \times G, \quad (i, O) \mapsto (\sigma_i, O \cdot O_{ij}(x_{\sigma i})).
\]
Take $U, V$ open sets in $\Lambda^N \times G$. Then there exist finite words $u, v$ such that $[u] \subseteq \pi_X(U)$, and $[v] \subseteq \pi_X(V)$, where $\pi_X$ denotes the projection onto $X$. For $O \in \pi_G(V)$, there exists $n \geq 1$ such that
\[
O \cdot O_{u}([v]|_{[u]})(x_{\sigma |v|++(u)}) = O \cdot O_{u}(x_{\sigma |v|++(u)}) \cdots O_{u}(x_{\sigma |v|+h+2 \cdots})
\]
This follows from the fact that $O$, $O_u(x_i)$ are fixed and the orbit of $O_{ij}(x_{\sigma i})$ is dense. Now consider an infinite word
\[
\tilde{k} = u[v>,...
\]
where the symbols following $v$ are arbitrary. Then $(\tilde{k}, O \cdot O_{u}([v]|_{[u]})(x_{\sigma |v|++(u)})) \in U$, and
\[
\sigma_\phi^{[v]|[u]}(\tilde{k}, O \cdot O_{u}([v]|_{[u]})(x_{\sigma |v|++(u)})) = (v, O) \in V.
\]
In particular, if there exists a finite word $u$ such that $O_u(x_i)$ is an irrational rotation, where $\pi = uu \cdots$ denotes the periodic infinite word of $u$, then (2.8) is true.

2.6. Dimension and entropy

Let $g : Y \to Z$ be a continuous mapping between two metric spaces $Y$ and $Z$. For a Borel measure $\nu$ on $Y$, write
\[
g_\nu = \nu \circ g^{-1},
\]
for the pull-back measure of $\nu$ on $Z$ through $g$. For a measure $\nu$ and $x \in \text{supp}(\nu)$, let
\[
D_\nu(x) = \lim_{r \to 0} \frac{\log \nu(B(x, r))}{\log r},
\]
whenever the limit exists, where $B(x, r)$ is the closed ball of centre $x$ and radius $r$. If for some $\alpha \geq 0$, we have $D_\nu(x) = \alpha$ for $\nu$-a.e. $x$, we say that $\nu$ is exact dimensional.

For $0 < r < 1$ and $\nu$, a probability measure supported by a compact subset $A$ of $\mathbb{R}^2$, let
\[ H_r(\nu) = -\int_A \log \nu(B(x, r))\nu(\mathrm{d}x) \] (2.9)

be the \(r\)-scaling entropy of \(\nu\). Note that, writing \(\mathcal{M}\) for the probability measures supported by \(A\), the map \(H_r : \mathcal{M} \to \mathbb{R} \cup \{\infty\}\) need not be continuous in the weak-\(*\) topology. However, \(H_r\) is lower semicontinuous as it may be expressed as the limit of an increasing sequence of continuous functions of the form \(\nu \to \int \max\{k, \log(1/\int f_k(x - y)\nu(\mathrm{d}y)\nu(\mathrm{d}x))\}\), where \(f_k\) is a decreasing sequence of continuous functions approximating \(\chi_{B(0,r)}\). The lower entropy dimension of \(\nu\) is defined as
\[
\dim_e \nu = \liminf_{r \to 0} \frac{H_r(\nu)}{-\log r},
\]
and the Hausdorff dimension of \(\nu\) is \(\dim_H \nu = \inf\{\dim_H A : \nu(A) > 0\}\). Then
\[
\dim_H \nu \leq \dim_e \nu,
\]
with equality when \(\nu\) is exact dimensional, for details see [8]. From [9] we have that the self-conformal measure \(\Phi\) is exact dimensional.

3. Dimension of projections

Let \(B = B(0,R)\) be the closed ball of radius \(R\), where \(R = \max\{|x| : x \in V\}\). Denote by \(\mathcal{M}\) the family of probability measures on \(B\) and let \(\mathcal{B}_r\) be its weak-\(*\) topology. Denote by \(C(\mathcal{M})\) the family of all continuous functions on \(\mathcal{M}\). We use the separability of \(C(\mathcal{M})\) in \(\|\cdot\|_{\infty}\) to obtain convergence of ergodic averages for all \(h \in C(\mathcal{M})\).

**Proposition 3.1.** We have that for \(\xi\)-a.e. \(O \in G\) and \(\mu\)-a.e. \(i \in \Lambda^N\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} h(O \cdot O_{\phi^n}(\Phi(\sigma^n i))\Phi(\mu)) = \int_G \mathbb{E}_\mu(h(O\Phi\mu))\xi(\mathrm{d}O)
\]
for all \(h \in C(\mathcal{M})\).

**Proof.** Let \(\{h_k\}_{k \geq 1}\) be a countable dense sequence in \(C(\mathcal{M})\). If we write
\[
\mathcal{M} : \Lambda^N \times G \ni (i, O) \to O\Phi\mu \in \mathcal{M},
\]
then it is easy to verify that for \(n \geq 0\),
\[
\mathcal{M} \circ \sigma^N(i, O) = O \cdot O_{\phi^n}(\Phi(\sigma^n i))\Phi(\mu).
\]
Since we know that \((\Lambda^N \times G, \mathcal{B} \otimes \mathcal{B}_G, \mu \times \xi, \sigma_{\phi})\) is ergodic, we have that for \(\xi\)-a.e. \(O\) and \(\mu\)-a.e. \(i\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} h_k(O \cdot O_{\phi^n}(\Phi(\sigma^n i))\Phi(\mu)) = \int_G \mathbb{E}_\mu(h_k(O\Phi\mu))\xi(\mathrm{d}O),
\]
for all \(k \geq 1\). For any \(h \in C(\mathcal{M})\), take a subsequence \(\{h_k'\}_{k \geq 1}\) of \(\{h_k\}_{k \geq 1}\) that converges to \(h\). On the one hand, since \(\mathcal{M}\) is compact, \(h\) is bounded, so by the uniform convergence in \(\|\cdot\|_{\infty}\),
\[
\lim_{k \to \infty} \int_G \mathbb{E}_{\mu}(h'_k(\mathbf{O}\Phi_\mu)) \xi(d\mathbf{O}) = \int_G \mathbb{E}_{\mu}(h(\mathbf{O}\Phi_\mu)) \xi(d\mathbf{O}).
\]

On the other hand, for each \(N\),
\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} h'_n(O \cdot O_{1,n}(\Phi(\sigma^n_t))] \mathbb{P}_{\mu} - \frac{1}{N} \sum_{n=0}^{N-1} h(O \cdot O_{1,n}(\Phi(\sigma^n_t))] \mathbb{P}_{\mu} \right| \leq \|h'_n - h\|_\infty.
\]
Thus the limit
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} h(O \cdot O_{1,n}(\Phi(\sigma^n_t))] \mathbb{P}_{\mu}
\]
exists and equals \(\lim_{k \to \infty} \int_G \mathbb{E}_{\mu}(h'_k(\mathbf{O}\Phi_\mu)) \xi(d\mathbf{O}) = \int_G \mathbb{E}_{\mu}(h(\mathbf{O}\Phi_\mu)) \xi(d\mathbf{O})\), for \(\xi\text{-a.e. } O \in G\) and \(\mu\text{-a.e. } \mathbf{O} \in \Lambda^N\).

3.1. Auxiliary lemmas

We now present three lemmas necessary in order for us to find a lower bound for the dimension of the projections of our Gibbs measure. Lemma 3.2 is regarding distortion of conformal maps under projections. This helps us understand what the images of balls look like under our conformal maps, which is used in lemma 3.4 to control the entropy while zooming in on the measure. Lemma 3.3 guarantees that the ergodicity of the skew product dynamical system is preserved while we change the alphabet. Lemma 3.4 is to estimate the entropy distortions while zooming in with conformal maps. The proof is technical but the main idea is that at very small scales, conformal maps look more and more like similarities.

**Lemma 3.2.** There exists a constant \(C_3\) such that for all \(\pi \in \Pi_{d,k}\), \(n \geq 1\), \(i \in \Lambda^n\) and \(x, y, z \in V\) one has
\[
C_1^{-1} \tau_i |\pi O_i(z)(x - y)| - C_3 \tau_i (|z - x| + |z - y|)|x - y| \leq |\pi f_i(x) - \pi f_i(y)|
\]
\[
\leq \tau_i |\pi O_i(z)(x - y)| + C_3 \tau_i (|z - x| + |z - y|)|x - y|,
\]
where \(C_1\) is as in (2.2).

**Proof.** For \(\pi \in \Pi_{d,k}\) one can find a rotation \(O_{\pi} \in SO(d, \mathbb{R})\) such that
\[
O_{\pi}(z) = (P_j(O_{\pi} z))_{1 \leq j \leq k}.
\]
for all \(z \in \mathbb{R}^d\), where \(P_j : \mathbb{R}^d \to \mathbb{R}\) is the coordinate function \(P_j(x_1, ..., x_d) = x_j\). We will view a function \(f : \mathbb{R}^d \to \mathbb{R}\) as \(f(x_1, ..., x_d) = (f^1(x), ..., f^d(x))\). We fix \(1 \leq j \leq k\) and consider \(f_j : \mathbb{R}^d \to \mathbb{R}\). For \(i \in \Lambda^n\) and \(x, y \in V\) we have
\[
P_j(O_{\pi} f_i(x - f_i(y))) = P_j(O_{x} f_i(x) - O_{y} f_i(y)).
\]
For \(t \in [0, 1]\) define
\[
g(t) = P_j(t(O_{x} f_i(x) - O_{y} f_i(y)) - O_{y} f_i(y) + t(x - y)).
\]
Since \( g(0) = g(1) = -P_i(O \sigma f_i(y)) \), by Rolle’s theorem there exists \( t \in (0, 1) \) such that 
\[
g'(t) = 0, \text{ which means that } 
P_j(O \sigma f_i(x) - O \sigma f_i(y)) = P_j(O \sigma f_i'(z^{i}_{xy})(x - y)),
\]
where \( z^{i}_{xy} = y + t(x - y) \) is some point lying in the line segment between \( x \) and \( y \). Writing 
\[
f_i'(z^{i}_{xy}) = r_i(z^{i}_{xy})O_i(z^{i}_{xy}), \text{ for } z \in V \text{ we have}
\]
\[
P_j(O \sigma \pi f_i(x) - O \sigma f_i(y)) = P_j(O \sigma f_i'(z^{i}_{xy})(x - y)) = r_i(z^{i}_{xy})P_j(O \sigma O_i(z^{i}_{xy})(x - y)) = r_i(z^{i}_{xy})P_j(O \sigma O_i(z^{i}_{xy}) - O_i(z)(x - y)) + r_i(z^{i}_{xy})P_j((O_i(z^{i}_{xy}) - O_i(z))(x - y))).
\]
This holds for all \( 1 \leq j \leq k \), and so 
\[
(O \sigma \pi f_i(x) - f_i(y)) = (r_i(z^{i}_{xy})P_1(\pi O_i(x)(x - y))) + r_i(z^{i}_{xy})P_1(\pi (O_i(x^{i}_{xy}) - O_i(z))(x - y))).
\]
By the smoothness of \( O_i \) and (2.2), as well as the fact that \( O_{\pi} \) is isometric, one can find a constant \( \tilde{C}_3 \) such that 
\[
|O_{\pi} \pi f_i(x) - f_i(y))| \leq \tilde{C}_3 \tilde{r}_i \max_{1 \leq j \leq k} |z^{i}_{xy} - y||x - y|
\]
as well as 
\[
C_1^{-1} \tilde{r}_i |O_i(x)(x - y)| \leq |\pi f_i(x) - \pi f_i(y)| + \tilde{C}_3 \tilde{r}_i \max_{1 \leq j \leq k} |z^{i}_{xy} - z||x - y|.
\]
Finally note that \( z_{xy} \) lies in the line segment between \( x \) and \( y \), therefore 
\[
\max_{1 \leq j \leq k} |z^{i}_{xy} - y| \leq |z - y| + |z - y|.
\]

Let \( \rho = \max \{r_i(x) : i \in \Lambda, x \in K \} \). By (A1) we have \( \rho < 1 \). Recall that for \( i \in \Lambda^* \) we denote by \( \tilde{r}_i = \sup \{r_i(x) : x \in V \} \). For each \( q \geq 1 \) we redefine the alphabet used for symbolic space to obtain one in which the contraction ratios do not vary too much:
\[
\Lambda_q = \{i \in \Lambda^* : \tilde{r}_i \leq \rho^q < \tilde{r}_i^{-q} \},
\]
where \( i^{-q} = i_2 \cdots i_m \). Define \( \tau = \inf \{r_i(x) : i \in \Lambda, x \in V \} \). Since \( f_i \) are conformal and \( V \subset U \) is compact, we have \( \tau > 0 \). By definition, for \( i \in \Lambda_q \) one has 
\[
\rho^q \tau \leq r_i \leq \rho^{|i|} \text{ and } \rho^{|i|} \leq \tilde{r}_i \leq \rho^q.
\]
This implies that for \( i \in \Lambda_q \),
\[
q \frac{\log \rho}{\log \tau} \leq |i| \leq q + \frac{\log \tau}{\log \rho}.
\]
We shall use the same notation \( \sigma : \Lambda_q^N \to \Lambda_q^N \) to denote the left-shift operator according to \( \Lambda_q \). Let \( \mathcal{I}_\Lambda = \{ f \}_j \in \Lambda \) be the conformal IFS over \( \Lambda_q \). By (2.4) we can deduce that the canonical mapping \( \Phi_q : (\Lambda_q^N, d_{\rho}) \to K \) is \( C_2 \cdot R \)-Lipschitz, where \( R = \text{diam}(V) \). Indeed, for \( i, j \in \Lambda_q^N \) with \( d_{\rho}(i, j) = (\rho^q)^n \) so that \( \rho_n = \tilde{\rho}_n \), one has

\[
|\Phi_q(i) - \Phi_q(j)| \leq C_2 \cdot r_{n} \cdot |x_{\sigma^n} - x_{\sigma^n}|
\]

\[
\leq C_2 R \cdot r_{n} \cdot |x_{\sigma^n} - x_{\sigma^n}|
\]

\[
\leq C_2 R \cdot d_{\rho}(i, j).
\]

We consider the Gibbs measure \( \mu_q \) on \( \Lambda_q^N \) with respect to the potential \( \varphi \). Observe that it is the same Gibbs measure as \( \mu \) on embedding \( \Lambda_q^N \) into \( \Lambda^N \). To show that the compact group extension \( (\Lambda_q^N \times G, B \otimes B_G, \mu_q \times \xi, \sigma_{\varphi_q}) \) is also ergodic, where

\[
\phi_q(i) = O_n(\Phi_\sigma(i)) \text{ for } i = i_1 i_2 \cdots \in \Lambda_q^N
\]

and \( \sigma_{\varphi_q} \) is the skew product of \( \phi_q \) with respect to the left shift \( \sigma \) on \( \Lambda_q \), we need the following lemma.

**Lemma 3.3.** Under \((A2)\), \( \sigma_{\varphi_q} \) has a dense orbit in \( \Lambda_q^N \times G \) for each \( q \geq 1 \).

**Proof.** Let \( i = i_1 i_2 \cdots \in \Lambda_q^N \) and \( O \in G \) be such that \( \{ \sigma^n_q(i) \cap O : n \geq 0 \} \) is dense in \( \Lambda_q^N \times G \). For \( q \geq 1 \), denote by \( \Lambda_{<q} = \{ j \in \Lambda^* : r_{j} \geq \rho^q \} \). Then \( \bigcup_{j \in \Lambda_{<q}} \{ T_j \sigma^n_q(i) \cap O \} = \{ \sigma^n_q(i) \cap O : n \geq 0 \} \) is dense in \( \Lambda_q^N \times G \), where for \( j \in \Lambda_{<q} \), \( T_j \sigma^n_q(i) \cap O \) is the unique element in \( \Lambda_q^N \times G \) such that \( \sigma^n_q(T_j \sigma^n_q(i)) = \sigma^n_q(i) \). Since \( \Lambda_{<q} \) is finite, by Baire’s category theorem, there exists a \( j \in \Lambda_{<q} \) such that \( \{ T_j \sigma^n_q(i) \cap O : n \geq 0 \} \) has non-empty interior in \( \Lambda_q^N \times G \), that is it contains a set \( H = [u] \times I \), where \( u \in \Lambda^* \) is finite word and \( I \subseteq G \) has non-empty interior. Take an element \( g \) in the interior of \( I \), then \( \sigma_q[H g^{-1}] = \{ (\sigma_q u) h^{-1} g^{-1} : (u, h) \in H \} \) contains the full set \( \Lambda_q^N \) times a set \( V \) containing a neighbourhood of the identity in \( G \). Since a closed group generated by a set of elements coincides with the closed semigroup generated by them and a compact connected Lie group is generated by any neighbourhood of its identity, we have \( \sigma_q[H g^{-1}] = \Lambda_q^N \times G \). \( \square \)

Using again [18], corollary 4.5, we have that \( (\Lambda_q^N \times G, B \otimes B_G, \mu_q \times \xi, \sigma_{\varphi_q}) \) is ergodic for all \( q \geq 1 \). Let \( n_q = \max \{|i| : i \in \Lambda_q^N \} \). By (3.3) one has \( n_q \to \infty \) as \( q \to \infty \). By (2.7) we have for \( i, j \in \Lambda_q^N \),

\[
c^{-1}_q \mu_q([i]) \mu_q([j]) \leq \mu_q([i]) \mu_q([j]) \leq c_q \mu_q([i]) \mu_q([j]).
(3.4)
\]

where \( c_q = e^{\delta_q \cdot n_q} \). This is due to the fact that for \( i, j \in \Lambda_q^N \), if either \( i \) or \( j \) is the empty word \( \emptyset \) then we have \( \mu_q([i]) = \mu_q([j]) \mu_q([\emptyset]) \) and if both \( i, j \neq \emptyset \) then \( \min \{|i|, |j|\} \geq n_q \). Since \( n_q \to \infty \) as \( q \to \infty \), one has \( c_q \to 1 \) as \( q \to \infty \). For a measure \( \nu \) and a measurable set \( B \) with \( \nu(B) > 0 \) denote by

\[
\nu_B = \frac{1}{\nu(B)} \nu|_B.
\]

In particular for \( \nu = \mu_q \) we shall use the notation \( \mu_{q,B} := (\mu_q)_B \). By (3.4) we have for \( n \geq 1 \) and \( i \in \Lambda_q^N \) that

\[
c^{-1}_q(\sigma^n \mu_q)_i \leq \mu_q \leq c_q(\sigma^n \mu_q)_i
(3.5)
\]
since for any \( j \in \Lambda^* \),
\[
(s^n \mu_q)_{|\{\jmath\}} = \frac{1}{s^n \mu_q(\{\jmath\})} (s^n \mu_q)_{|\{\jmath\}} = \frac{1}{\mu_q(\{\jmath\})} \sum_{k_1 \cdots k_d \in \Lambda^{(d-n)}_p} \mu_q([k_1 \cdots k_d] \cap \{\jmath\}) = \frac{1}{\mu_q(\{\jmath\})} \mu_q([\jmath \}].
\]

As before, \( \Pi_{d,k} \) is the set of orthogonal projections from \( \mathbb{R}^d \) onto its \( k \)-dimensional subspaces, and \( G = SO(d, \mathbb{R}) \) is the rotation group. We shall need the following lemma.

**Lemma 3.4.** For all \( q \geq 1, \pi \in \Pi_{d,k}, O \in G, \tilde{\iota} \in \Lambda_q^N \) and \( n \geq 1 \),
\[
H_{C^{q,n}}(O \cdot \Omega_{\tilde{\iota}}(x;\sigma)) \leq \log c_q + c_q H_{C^{q,n+1}}(O \Phi_{\tilde{\iota}}(\mu_{\tilde{\iota}})).
\]

The proof of this lemma is technical. The estimates would only work when the points under consideration are restricted to the self-conformal set, so we only deal with points mapped from the symbolic space. We also use an auxiliary scaling function \( S_\tau \) at a given point to control the change of diameters. The idea is that when \( r \to 0 \), since conformal maps are twice differentiable, the error terms of order \( r^2 \) are negligible.

**Proof.** Recall the definition (2.9):
\[
H_r(\nu) = \int_{\text{supp}(\nu)} -\log \nu(B(x, r)) \, d\nu dx.
\]

Shortly denote by \( g = \pi O \cdot \Omega_{\tilde{\iota}}(x;\sigma) \Phi_{\tilde{\iota}} \). By (3.5) we have
\[
H_{C^{q,n}}(g \mu_q) = \int_{\Lambda_q^N} -\log \mu_q(\{ \jmath \in \Lambda_q^N : |g(\jmath) - g(\kappa)| \leq C_1^q \rho^q \}) \, d\mu_q(d\kappa)
\leq \log c_q + c_q \int_{\Lambda_q^N} -\log (s^n \mu_q)_{|\{\jmath\}}(\{ \jmath \in \Lambda_q^N : |g(\jmath) - g(\kappa)| \leq C_1^q \rho^q \}) (s^n \mu_q)_{|\{\jmath\}}(d\kappa)
= \log c_q + c_q \int_{\Lambda_q^N} -\log \mu_q(\{ \jmath \in \Lambda_q^N : |g(\sigma^n \jmath) - g(\sigma^n \kappa)| \leq C_1^q \rho^q \}) \, d\mu_q(d\kappa),
\]
(3.6)

We claim that for each \( \jmath, \kappa \in [\tilde{\iota} \{n\}], if \)
\[
|\pi O \xi_{\tilde{\iota}} - \pi O \xi_{\kappa}| \leq \tau_{\kappa, \rho^q},
\]
then
\[
|g(\sigma^n \jmath) - g(\sigma^n \kappa)| \leq C_1^q \rho^q.
\]

Since, by (3.2),
\[
\tau_{\kappa, \rho^q} \geq (r^q)^n \rho^q \geq (r^q)^n \rho^q + 1,
\]
this implies that
\[ \{ j \in \Lambda^N_\nu : |\pi O x_j - \pi O x_k| \leq (r \rho^p)^{n+1} \} \subset \{ j \in \Lambda^N_\nu : |g(\sigma^n j) - g(\sigma^n k)| \leq C_1^2 \rho^p \}. \]

Then by (3.6) we may deduce that
\[
H_{c_{\rho}}(g\mu_{\nu}) \leq \log c_{\rho} + c_{\rho} \int_{\Lambda^N_\nu} -\log \mu_{\nu}(\{ j \in \Lambda^N_\nu : |\pi O x_j - \pi O x_k| \leq (r \rho^p)^{n+1} \}) \cdot \mu_{\nu}(d\delta) = \log c_{\rho} + c_{\rho} H_{(r \rho^p)^{n+1}}(\pi O \Phi_{\rho} \mu_{\nu}(d\delta)).
\]

Now we prove our claim. Fix \( j, k \in [0, 1] \). For \( r > 0 \) write \( S_r(x) = r(x - \sigma^n j) + \sigma^n j \) for \( x \in V \). Note that \( |S_r(x_{\sigma^n j}) - S_r(x_{\sigma^n k})| \leq R \cdot r \) as well as
\[
|S_r(x_{\sigma^n j}) - x_{\sigma^n j}| \cap |S_r(x_{\sigma^n k}) - x_{\sigma^n k}| \leq R \cdot r,
\]
where we recall that \( R = \text{diam}(V) \). First, by lemma 3.2 and the fact that \( S_r \) only consists of scaling and translation, we have
\[
|\pi O f_{\nu}^j(S_r(x_{\sigma^n j})) - \pi O f_{\nu}^k(S_r(x_{\sigma^n k}))| \\
\leq r \tau_{\nu} |\pi O \cdot O f_{\nu}^j(S_r(x_{\sigma^n j}) - S_r(x_{\sigma^n k}))| + 2C_3 R^2 r^2 \tau_{\nu} \\
= r(\tau_{\nu} |\pi O \cdot O f_{\nu}^j(x_{\sigma^n j} - x_{\sigma^n k})| + 2C_3 R^2 r \tau_{\nu}) \\
\leq r(C_1 |\pi O f_{\nu}^j(x_{\sigma^n j}) - \pi O f_{\nu}^k(x_{\sigma^n k})| + 2(C_1 + 1)C_3 R^2 r \tau_{\nu}).
\]

This implies that if
\[
|\pi O x_j - \pi O x_k| = |\pi O f_{\nu}^j(x_{\sigma^n j}) - \pi O f_{\nu}^k(x_{\sigma^n k})| \leq \tau_{\nu} \rho^p,
\]
then for any \( \epsilon > 0 \), for all \( r \in (0, \frac{c_{\rho}}{C(1 + \epsilon) c_{\rho}^p}) \),
\[
|\pi O f_{\nu}^j(S_r(x_{\sigma^n j})) - \pi O f_{\nu}^k(S_r(x_{\sigma^n k}))| \leq (C_1 + \epsilon) r \tau_{\nu} \rho^p.
\]

On the other hand, note that
\[
r(g(\sigma^n j) - g(\sigma^n k)) = \pi O \cdot O f_{\nu}^j(S_r(x_{\sigma^n j}) - S_r(x_{\sigma^n k})).
\]

Thus, by lemma 3.2 again, for all \( r \in (0, \frac{c_{\rho}}{C(1 + \epsilon) c_{\rho}^p}) \),
\[
r(g(\sigma^n j) - g(\sigma^n k)) \leq C_1 ((C_1 + \epsilon) r \rho^p + 2C_3 R^2 r^2),
\]
which implies that
\[
|g(\sigma^n j) - g(\sigma^n k)| \leq C_1 ((C_1 + \epsilon) \rho^p + 2C_3 R^2 r).
\]

This yields that
\[
|g(\sigma^n j) - g(\sigma^n k)| \leq C_1 (C_1 + \epsilon) \rho^p
\]
holds for all \( \epsilon > 0 \), therefore
\[ |g(\sigma^n \tilde{x}) - g(\sigma^n \tilde{x}')| \leq C_1^q \rho^q. \]  

3.2. Lower bound for the dimension of projections

For \( \pi \in \Pi_{d,k}, q \in \mathbb{N} \) and \( \nu \) a measure on \( \mathbb{R}^d \), define

\[
e_q(\pi, \nu) = \frac{1}{-\log C_1^q + q \log(1/\rho)} H_{\xi_{\mu}}(\pi \nu).
\]

So \( e_q : \Pi_{d,k} \times \mathcal{M} \rightarrow [0,1] \) is lower semicontinuous. Define

\[
E_q(\pi) = E_{\mu_q \times \xi}(e_q(\pi, \Omega \Phi_q \mu_q)).
\]

**Proposition 3.5.** For all \( q \geq 1 \), for \( \xi \)-a.e. \( O \in \mathcal{G} \) and \( \mu_q \)-a.e. \( \xi \in \Lambda_q^N \),

\[
\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} H_{\xi_{\mu_q}}(\pi \Omega \Phi_q \mu_q | \xi_{\mu_q}) \geq -\frac{\log C_1^q + q \log(1/\rho)}{c_q} E_q(\pi) - \frac{\log c_q}{c_q},
\]

for all \( \pi \in \Pi_{d,k} \).

**Proof.** Since \( e_q \) is lower semicontinuous, applying proposition 3.1 under the dynamical system \((\Lambda_q^N \times \mathcal{G}, \mathcal{B} \otimes \mathcal{B}_Q, \mu_q \times \xi, \sigma_{\mu_q})\), which is ergodic by lemma 3.3, to a sequence of continuous functions approximating \( e_q \) from below and using the monotone convergence theorem, we have that for \( \xi \)-a.e. \( O \) and \( \mu_q \)-a.e. \( i \),

\[
\lim_{N \rightarrow \infty} \inf \frac{1}{N} \sum_{n=0}^{N-1} e_q(\pi, O \cdot O_j \cdot (x, \sigma_{\mu_q})(\Phi_q \mu_q)) \geq E_q(\pi)
\]

for all \( \pi \in \Pi_{d,k} \), which yields the conclusion.

**Theorem 3.6.** We have for all \( q \geq 1 \), for \( \xi \)-a.e. \( O \in \mathcal{G} \),

\[
\dim_q(\pi \Omega \Phi_q) \geq -\frac{\log C_1^q + q \log(1/\rho)}{c_q(q \log(1/\rho) - \log \rho)} E_q(\pi) - \frac{\log C_2}{q \log(1/\rho) - \log \rho} - \frac{\log c_q}{c_q(q \log(1/\rho) - \log \rho)},
\]

for all \( \pi \in \Pi_{d,k} \), where \( C_2 \) is a constant depending only on \( C_2 : \mathcal{G} \rightarrow \mathbb{R} \), and \( c_q \) is given in (3.4).

**Proof.** First we pick a \( \xi \)-typical \( O \in \mathcal{G} \) such that the statement of proposition 3.5 holds. The mapping \( f = \pi \Omega \Phi_q : (\Lambda_q^N \times \mathcal{G}, h) \rightarrow \mathcal{G} \) is \( C_2 : \mathcal{G} \)-Lipschitz. By [12, theorem 5.4], there exist a \( \rho \)-tree \((X, d_{\rho})\) and maps \( \Lambda_q^N \xrightarrow{h} X \xrightarrow{f'} \mathbb{R}^k \) such that \( f = f' h \), where \( h \) is a tree morphism and \( f' \) is \( C \)-faithful (see [12, definition 5.1]) for some constant \( C \) depending only on \( C_2 : \mathcal{G} \). Then applying [12, proposition 5.3] to the \( \rho \)-tree \((X, d_{\rho})\) (for which \( f' \) is \( \xi^{-1} C \)-faithful), there is
a $C'$ depending only on $L^{-1} C$ such that for all $n \geq 1$,
\[
\left| H(W^p;\Pi_t) - H(W^p;\Pi_t) \right| \leq C'.
\] (3.8)

Then, using proposition 3.5, for $\mu_q$-a.e. $\xi$,
\[
- \log C_1^q + q \log(1/\rho) \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} H(W^p;\Pi_t) \geq \frac{1}{c_q} E_q(\pi) - \log C' - \frac{\log c_q}{q \log(1/\rho) - \log \rho} - \frac{\log c_q}{c_q(q \log(1/\rho) - \log \rho)}
\]
for all $\pi \in \Pi_{d,k}$. Now, using [12, theorem 4.4], it follows that
\[
\dim_h h\mu_q \geq - \log C_1^q + q \log(1/\rho) E_q(\pi) - \frac{\log C'}{q \log(1/\rho) - \log \rho} - \frac{\log c_q}{c_q(q \log(1/\rho) - \log \rho)}
\]
for all $\pi \in \Pi_{d,k}$. Since $f'$ is $C$-faithful and $f' h\mu_q = f\mu_q = \pi \Phi_q \mu_q = \pi \Phi \mu$, the conclusion follows from [12, proposition 5.2].

3.3. Projection theorems

With the approach used in section 3.2, we avoid the need for any separation condition in our projection results. For $\pi \in \Pi_{d,k}$ we have
\[
E_q(\pi) = \mathbb{E}_{\mu_q \times \xi}(e_q(\pi, \Omega \Phi_q \mu_q))
\]
\[
= \int_G \frac{1}{- \log C_1^q + q \log(1/\rho)} H_{C_1^q}(\pi \Omega \Phi_q \mu_q) \xi(\text{d}O)
\]
\[
= \int_G \frac{1}{- \log C_1^q + q \log(1/\rho)} H_{C_1^q}(\pi \Omega \Phi \mu) \xi(\text{d}O),
\] (3.9)
where we have used the fact that $\Phi_q \mu_q = \Phi \mu$ for all $q \geq 1.$

**Theorem 3.7.** The limit
\[
E(\pi) := \lim_{q \to \infty} E_q(\pi)
\]
exists for every $\pi \in \Pi_{d,k}$. Moreover:

(i) For a fixed $\pi \in \Pi_{d,k}$, for $\xi$-a.e. $O$,
\[
\dim_h \pi \Omega \Phi \mu = \dim_h \pi \Omega \Phi \mu = E(\pi).
\]

(ii) For $\xi$-a.e. $O$,
\[
\dim_h \pi \Omega \Phi \mu \geq E(\pi) \quad \text{for all } \pi \in \Pi_{d,k}.
\]

**Proof.** For all integers $q \geq 1$, by theorem 3.6 we have that for $\xi$-a.e. $O \in G$,
\[
\dim_h(\pi \Omega \Phi \mu) \geq - \log C_1^q + q \log(1/\rho) E_q(\pi) - \frac{\log C'}{q \log(1/\rho) - \log \rho} - \frac{\log c_q}{c_q(q \log(1/\rho) - \log \rho)}
\]
for all \( \pi \in \Pi_{d,k} \). As integers \( \{q \geq 1\} \) are countable, we may take a \( \xi \)-full set such that the above statement is true for all \( q \geq 1 \). Since \( C_1, C' \) and \( \ell \) do not depend on \( q \) and \( c_q \to 1 \) as \( q \to \infty \), we obtain for \( \xi \)-a.e. \( O \in G \),

\[
\dim_H(\pi \Phi O \mu) \geq \limsup_{q \to \infty} E_q(\pi)
\]

(3.10)

for all \( \pi \in \Pi_{d,k} \). On the other hand, we also know that \( \dim_e(\nu) \geq \dim_H(\nu) \) for any Borel probability measure \( \nu \). Thus, applying Fatou’s lemma to (3.9), we have

\[
\limsup_{q \to \infty} E_q(\pi) \leq E_\xi(\dim_H(\pi \Phi O \mu)) \leq E_\xi(\dim(\pi \Phi O \mu)) \leq \liminf_{q \to \infty} E_q(\pi).
\]

This shows that \( \lim_{q \to \infty} E_q(\pi) \) exists for all \( \pi \in \Pi_{d,k} \), and (i) and (ii) follow directly. □

Write \( \beta = \min\{k, \dim_H \Phi \mu\} \).

**Corollary 3.8.** (i) \( E(\pi) = \beta \) for all \( \pi \in \Pi_{d,k} \); (ii) \( \dim_H(\pi \Phi \mu) = \beta \) for all \( \pi \in \Pi_{d,k} \).

**Proof.**

(i) For a fixed \( \pi \in \Pi_{d,k} \), since \( \Phi \mu \) is exact-dimensional (see [9]), using theorem 3.7 (i) and applying the Marstrand’s projection theorem for measures (see [13]), for \( \xi \)-a.e. \( O \in SO(d, \mathbb{R}) \),

\[
E(\pi) = \dim_H \pi \Phi O \mu = \min\{k, \dim_H \Phi \mu\} = \beta.
\]

This implies that \( E(\pi) = \beta \) for all \( \pi \in \Pi_{d,k} \).

(ii) By (i) and theorem 3.7(ii), we have for \( \xi \)-a.e. \( O \in SO(d, \mathbb{R}) \),

\[
\dim_H \pi \Phi O \mu \geq \beta \text{ for all } \pi \in \Pi_{d,k}.
\]

Then we get the conclusion from the fact that \( \beta \geq \dim_H \pi \Phi O \mu \) for any \( \pi \in \Pi_{d,k} \) and \( O \in SO(d, \mathbb{R}) \).

For \( C^1 \)-images, we need the following proposition:

**Proposition 3.9.** Let \( \pi \in \Pi_{d,k} \). For all \( C^1 \)-maps \( h : K \to \mathbb{R}^k \) such that \( \sup_{x \in K} \|D_h x - \pi\| < c \rho^q \), we have that for all \( q \geq 1 \), for \( \xi \)-almost every \( O \),

\[
\dim_H h \Phi O \mu \geq
\]

\[
- \frac{\log C_1^2 + q \log(1/\rho)}{c_q(q \log(1/\rho) - \log \ell)} E_q(\pi) - \frac{\log C'}{q \log(1/\rho) - \log \ell} - \frac{\log c_q}{c_q(q \log(1/\rho) - \log \ell)} - O(1/q),
\]

where the constant \( O(1/q) \) only depends on \( \rho, c \) and \( k \).

**Proof.** The proof is similar to that of [12, proposition 8.4] together with proposition 3.5 and theorem 3.6. □

Corollary 3.8 and proposition 3.9 imply the following.
Corollary 3.10. For all \( C^1 \)-maps \( h : K \to \mathbb{R}^k \) without singular points,
\[
\dim_h h \Phi \mu = \min\{k, \dim_h \Phi \mu\}.
\]

Proof. Since \( h \) is a \( C^1 \)-map, \( \dim_h h \Phi \mu \leq \min\{k, \dim_h \Phi \mu\} \). The lower bound follows from proposition 3.9 applied to the restricted measures \( \mu|_{\{i\}} \) for \( i \in \Lambda^\mathbb{N} \) and \( n \geq 1 \) sufficiently large. □

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References

[1] Bedford T and Fisher A M 1997 Ratio geometry, rigidity and the scenery process for hyperbolic Cantor sets Ergod. Theor. Dyn. Syst. 17 531–64
[2] Bedford T, Fisher A M and Urbański M 2002 The scenery flow for hyperbolic Julia sets Proc. London Math. Soc. 85 467–92
[3] Bowen R 1975 Equilibrium States and The Ergodic Theory of Anosov Diffeomorphisms vol 470 (Berlin: Springer)
[4] Bowen R 1979 Hausdorff dimension of quasicircles Publ. Math. IHES 50 11–25
[5] Falconer K 2014 Fractal Geometry: Mathematical Foundations and Applications 3rd edn (New York: Wiley)
[6] Falconer K, Fraser J and Jin X 2015 Sixty years of fractal projections Fractal geometry and stochastics V (Basel: Birkhäuser) pp 3–25
[7] Falconer K and Jin X 2014 Exact dimensionality and projections of random self-similar measures and sets J. Lond. Math. Soc. 90 388–412
[8] Fan A-H, Lau K-S and Rao H 2002 Relationships between different dimensions of a measure Mon. Hefte Math. 135 191–201
[9] Feng D-J and Hu H 2009 Dimension theory of iterated function systems Commun. Pure Appl. Math. 62 1435–500
[10] Fraser J M and Pollicott M 2015 Micromeasure distributions and applications for conformally generated fractals Mathematical Proc. of the Cambridge Philosophical Society vol 159 (Cambridge: Cambridge University Press) pp 547–66
[11] Furstenberg H 2008 Ergodic fractal measures and dimension conservation Ergod. Theor. Dynam. Syst. 28 405–22
[12] Hochman M and Shmerkin P 2012 Local entropy averages and projections of fractal measures Ann. Math. 175 1001–59
[13] Hunt B R and Kaloshin V Y 1997 How projections affect the dimension spectrum of fractal measures Nonlinearity 10 1031
[14] Marstrand J M 1954 Some fundamental geometrical properties of plane sets of fractional dimensions Proc. Lond. Math. Soc. 3 257–302
[15] Mattila P 1975 Hausdorff dimension, orthogonal projections and intersections with planes Ann. Acad. Sci. Fenn. AI Math. 1 227–44
[16] Mattila P 2017 Hausdorff dimension, projections, intersections, and Besicovitch sets (arXiv:1712.09199)
[17] Orponen T 2012 On the distance sets of self-similar sets Nonlinearity 25 1919–29
[18] Parry W 1997 Skew products of shifts with a compact Lie group J. Lond. Math. Soc. 2 395–404
[19] Patzschke N 1997 Self-conformal multifractal measures Adv. Appl. Math. 19 486–513
[20] Peres Y, Rams M, Simon K and Solomyak B 2001 Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets Proc. Am. Math. Soc. \textbf{129} 2689–99

[21] Ruelle D 2004 \textit{Thermodynamic Formalism} 2nd edn (New York: Cambridge University Press)

[22] Shmerkin P 2015 Projections of self-similar and related fractals: a survey of recent developments \textit{Fractal Geometry and Stochastics V} (Basel: Birkhäuser) pp 53–74

[23] Shmerkin P 2018 On the Hausdorff dimension of pinned distance sets (arXiv:1706.00131)

[24] Silverman S 1992 On maps with dense orbits and the definition of chaos Rocky MT J. Math. \textbf{22} 353–75

[25] Sullivan D 1987 Differentiable structures on fractal-like sets, determined by intrinsic scaling functions on dual Cantor sets \textit{Nonlinear Evolution and Chaotic Phenomena} (New York: Springer) pp 101–10