Thermodynamics of Conformal Field Theories and Cosmology\textsuperscript{1}

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Abstract: We study the ratio of the entropy to the total energy in conformal field theories at finite temperature. For the free field realizations of $\mathcal{N} = 4$ super Yang-Mills theory in $D = 4$ and the $(2, 0)$ tensor multiplet in $D = 6$, the ratio is bounded from above. The corresponding bounds are less stringent than the recently proposed Verlinde bound. For strongly coupled CFTs with AdS duals, we show that the ratio obeys the Verlinde bound even in the presence of rotation. For such CFTs, we point out an intriguing resemblance in their thermodynamic formulas with the corresponding ones of two-dimensional CFTs. The discussion is based on hep-th/0101076 [1].

1 Introduction

The Bekenstein bound [2] for the ratio of the entropy $S$ to the total energy $E$ of a closed physical system that fits in a sphere in three spatial dimensions reads [3]

$$\frac{S}{2\pi RE} \leq 1,$$  \hspace{1cm} (1)

where $R$ denotes the radius of the sphere. Despite many efforts, the microscopic origin of the bound remains elusive. A recent interesting development is Verlinde’s observation [4] that CFTs possessing AdS duals satisfy a version of the bound (1). One firstly observes that for general CFTs on $\mathbb{R} \times S^{D-1}$, with the radius of $S^{D-1}$ being $R$, the product $ER$ is independent of the total spatial volume $V$. If one defines the sub-extensive part of the total energy through

$$E_C = DE - (D - 1)TS,$$  \hspace{1cm} (2)

then for strongly coupled CFTs with AdS duals the entropy is given by a generalized Cardy formula

$$S = \frac{2\pi R}{D - 1}\sqrt{E_C(2E - E_C)}.$$  \hspace{1cm} (3)

From (3) one obtains a bound similar to (1), namely

$$\frac{S}{2\pi RE} \leq \frac{1}{D - 1}.$$  \hspace{1cm} (4)

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In view of the above developments, a natural question arising is whether there exists a microscopic derivation of Verlinde’s formula (3) within the thermodynamics of CFTs. This question could be checked in the context of CFTs whose microscopic thermodynamics is well understood, such as free CFTs on $\mathbb{R} \times S^{D-1}$ [5]. We shall discuss free CFTs in dimensions $D = 4, 6$ as well as strongly coupled CFTs with AdS duals.

## 2 CFTs at weak coupling

In this section, we discuss the thermodynamics of conformal field theories. In two dimensions, the entropy and energy of $C$ free bosons read, respectively

$$S = \frac{\pi C}{6} \delta^{-1}, \quad ER = \frac{C}{24} \left( \delta^{-2} + 1 \right).$$

Eq. (5) implies the Cardy formula [6]

$$S = 2\pi \sqrt{\frac{C}{6} \left( E - \frac{C}{24} \right)},$$

and the Bekenstein bound (1) for the ratio

$$\frac{S}{2\pi ER} = \frac{2\delta}{1 + \delta^2} \leq 1.$$  

The above results also hold for fermions.

In four dimensions, the ratio $S/E$ in general diverges. Remarkably, for the $\mathcal{N} = 4$ SYM model, we have

$$\frac{S}{2\pi ER} = \frac{2}{3} \frac{2\delta}{1 + \delta^2}.$$  

There is a critical point at which both $S$ and $ER$ vanish, $\delta_c^2 = 1/3$. For $\delta \leq \delta_c$, we obtain

$$\frac{S}{2\pi ER} \leq \frac{\sqrt{3}}{3},$$

which is weaker than the Verlinde bound (4), $S/(2\pi ER) \leq 1/3$.

It is perhaps worth mentioning that if one imposes periodic boundary conditions on the gaugino, as suggested by Tseytlin to account for the disagreement on the number of degrees of freedom between the weak and strong coupling regimes, the above results still hold.

For the $(2,0)$ tensor multiplet in $D = 6$, we obtain

$$\frac{S}{2\pi ER} = \frac{2\delta}{1 + \delta^2} \left( \frac{3}{2} - \frac{10}{3}\delta^2 + 19\delta^4 \right).$$

This is a well-behaved function of $\delta$. We obtain conclude that

$$\frac{S}{2\pi ER} \leq 0.824,$$

which is less stringent than the Verlinde bound (4).
3 CFTs at strong coupling

In this section we turn our attention to strongly coupled CFTs in $D$-dimensions, possessing AdS duals. The thermodynamics of such theories follows quite generally from the thermodynamics of $(D + 1)$-dimensional AdS black holes, through holography. We consider the rotating Kerr-AdS (KAdS) black hole in $(D + 1)$-dimensions. According to the AdS/CFT duality conjecture [7], the thermodynamical quantities are associated to a strongly coupled $D$-dimensional CFT residing on the conformal boundary of spacetime, i.e. on a rotating Einstein universe.

Defining $\Delta = R/r_+$, where $R$ is the AdS throat size and $r_+$ is the horizon radius, and the Bekenstein entropy $S_B = 2\pi ER/(D - 1)$, we obtain after some algebra

$$\frac{S}{S_B} = \frac{2\Delta}{1 + \Delta^2} \leq 1. \tag{12}$$

The Bekenstein bound is saturated at the Hawking-Page transition point.[8] We further note that we can write

$$2ER = \frac{D - 1}{2\pi} S_B \frac{R}{r_+} [\Delta^{-2} + 1], \tag{13}$$

which, for arbitrary $D$, is exactly the behavior of a two-dimensional CFT (5) with characteristic scale $R$, temperature $\tilde{T} = 1/(2\pi R\Delta) = r_+/(2\pi R^2)$, and central charge proportional to $S_C = SR/r_+$ (Casimir entropy). Thus, $S_C$ is proportional to the number of degrees of freedom coupled at the critical point.

Finally, if we define the ”Bekenstein-Hawking energy” $E_{BH}$ as the energy for which the black hole entropy $S$ and the Bekenstein entropy $S_B$ are equal, one checks that $E_{BH} \leq E$. Furthermore, above the Hawking-Page transition point,

$$E_C \leq E_{BH} \leq E, \quad S_C \leq S \leq S_B, \tag{14}$$

where equality holds when the HP phase transition is reached. As the entropy $S$ is a monotonically increasing function of $E_C$ (or, equivalently, of $r_+$), the maximum entropy is reached when $E_C = E_{BH}$, i.e. at the HP phase transition. It is quite interesting to observe that at this point the central charge $c/12 = S_C/(2\pi)$ takes e. g. for $D = 4$, $\mathcal{N} = 4$ $U(\mathcal{N})$ SYM theory the value\(^3\) $c = 6\mathcal{N}^2$. This is exactly the central charge of a two-dimensional free CFT containing the $6\mathcal{N}^2$ scalars of $D = 4$, $\mathcal{N} = 4$ SYM.

4 Cosmological implications

In this section, we discuss the implications of our results for cosmology. The metric in a radiation dominated closed Friedman-Robertson-Walker (FRW) universe,

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega_{D-1}^2, \tag{15}$$

\(^3\)Here we used the AdS/CFT dictionary $N^2 = \frac{\pi R^3}{2\mathcal{N}c}$.\]
where $R(\tau)$ represents the radius of the universe at a given time $\tau$, is conformally equivalent to

$$d\tilde{s}^2 = -dt^2 + R^2 d\Omega^2_{D-1},$$

(16)

where $dt = R\,d\tau/R(\tau)$. If the radiation is described by a CFT, one can equally well use (16) instead of (15). If, in addition, this CFT admits an AdS dual, it can be described by a Schwarzschild-AdS black hole at some temperature $T$, because (16) is precisely the metric on the conformal boundary of spacetime. The observations made by Verlinde [4] concerning entropy, energy and temperature bounds in a radiation dominated universe then fit nicely into this AdS black hole description. In particular, the universe is weakly (strongly) self-gravitating if $HR \leq 1$ ($HR \geq 1$), where $H = \dot{R}/R$ denotes the Hubble constant, and the dot refers to differentiation with respect to $\tau$. One has $HR = 1$ iff the Bekenstein-Hawking entropy $S$ equals the Bekenstein entropy $S_B$. We saw above that this happens precisely at the HP transition point $r_+ = R$, so the borderline between the weakly and strongly self-gravitating regime is the Hawking-Page phase transition temperature $T_{HP} = (D-1)/2\pi R$. This identification makes indeed sense, because below $T_{HP}$ (weakly gravitating) one has AdS space filled with thermal radiation which collapses above $T_{HP}$ ($r_+ \geq R$, strongly gravitating) to form a black hole. Furthermore, Verlinde [4] found a limiting temperature for the early universe,

$$T \geq T_H = -\frac{\dot{H}}{2\pi H} \quad \text{for} \quad HR \geq 1.$$  

(17)

We conclude that Verlinde’s limiting temperature $T_H$ corresponds to the temperature $T_{HP}$ where the HP phase transition takes place.

5 Conclusions

Concerning further developments of our ideas, it might be interesting to further study our generalized central charge $S_C$, which intriguingly resembles a standard two-dimensional central charge. Such an interpretation leads to the conjecture that there might exist a two-dimensional CFT model whose dynamics in the presence of irrelevant operators underlies the dynamics of the $D$-dimensional CFTs possessing AdS duals. Such a conjecture might explain the fact that the latter theories share unexpectedly many of the properties of two-dimensional CFTs.

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