Mass angular momentum inequality for axisymmetric vacuum data with small trace

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Abstract: In this paper, we proved the mass angular momentum inequality for axisymmetric, asymptotically flat, vacuum constraint data sets with small trace. Given an initial data set with small trace, we construct a boost evolution spacetime of the Einstein vacuum equations. Then a perturbation method is used to solve the maximal surface equation in the spacetime under certain growing condition at infinity. When the initial data set is axisymmetric, we get an axisymmetric maximal graph with the same ADM mass and angular momentum as the given data. The inequality follows from the known results about the maximal graph.

1 Introduction

Based on the gravitational collapse pictures, it is conjectured that the angular momentum should be bounded by the mass for physically reasonable solutions of the Einstein equations. It is true for Kerr black hole solutions which are stationary. For dynamical, axisymmetric solutions some progresses have been made over the past few years. Dain first proved such an inequality for Brill data (See Definition 2.1 of [16]), which is a class of specialized axisymmetric, maximal, asymptotically flat vacuum data. Later, Chruściel, Li and Weinstein generalized it to a class of axisymmetric, maximal data admitting an Ernst potential with positive mass density and certain asymptotically flatness conditions. Recently R. Schoen and the author gave a simplified proof for more general asymptotic conditions and an $L^6$ norm bound.

All the existing results require the solutions to be maximal, which restricts the data to be a special time-slice in a spacetime. However it should be unnecessary according to the gravitational collapse pictures. It is natural and interesting to study the non-maximal case. In this paper we will prove the mass angular momentum inequality for non-maximal vacuum data with small trace by exploring the Einstein equations and a perturbation method. Using notations in Section 1.2, our main theorem is

Theorem 1.1. (Main Theorem 1) Suppose $(\Sigma, g)$ is a simply connected 3-manifold, which is Euclidean at infinity with two ends and axisymmetric in the sense of Definition 1.3. Given

\[ \text{The axisymmetric condition is indeed necessary, since otherwise vacuum counterexamples were constructed by Huang, Schoen and Wang} \]
an asymptotically flat, axisymmetric vacuum data \((g, k) \in \mathcal{VC}^{s+2,\delta+\frac{3}{2}}(\Sigma)\) (see Definition 1.3) with \(s \in \mathbb{N}, s \geq 7, \delta \in \mathbb{R}, -\frac{3}{2} < \delta < -1, \) if \(\|tr_gk\|_{H_{s-2,\delta+\frac{3}{2}}}(\Sigma) \leq \epsilon\) with \(\epsilon\) given in Theorem 1.6, we have

\[
m \geq \sqrt{|J|}, \tag{1.1}
\]

where \(m\) and \(J\) are the ADM mass (1.6) and angular momentum (1.8) of \((\Sigma, g, k)\) respectively.

Our method comes from a question suggested by R. Schoen:

(Q): Is there a canonical way to deform a non-maximal, axisymmetric, vacuum data to a unique maximal, vacuum data with the same physical quantities, i.e. the mass and angular momentum, which also preserves the axially symmetry?

A definite answer of the above question will imply the mass angular momentum inequality in the non-maximal case. In fact, there are already some works about the deformation of vacuum constraint equations [4][13]. But it is hard to maintain the symmetries and physical quantities at the same time. So the main difficulty is to maintain the symmetries and the physical quantities simultaneously when deforming the vacuum constraint equations. We overcome this difficulty by using certain conversation laws of the Einstein equations.

1.1 General Relativity backgrounds

In Einstein’s theory for General Relativity\(^2\), we use \((\mathcal{V}^{3,1}, \gamma)\) to denote a spacetime, where \(\mathcal{V}^{3,1}\) is a 4-dimensional oriented smooth manifold, and \(\gamma\) is a Lorentzian metric of signature \((3, 1)\). The Einstein equation, which predicts the evolution of the spacetime, is given by

\[
\text{Ric}_{\gamma} - \frac{1}{2}R_{\gamma}\gamma = 8\pi T, \tag{1.2}
\]

where \(\text{Ric}_{\gamma}\) is the Ricci curvature of \(\gamma\), and \(R_{\gamma}\) the scalar curvature of \(\gamma\). \(T\) is the stress-energy tensor. In the vacuum case, \(T \equiv 0\), so the Einstein vacuum equation, abbreviated as (EVE) in the following, reduces to

\[
\text{Ric}_{\gamma} = 0. \tag{1.3}
\]

A vacuum constraint initial data set or abbreviated as vacuum data for the Einstein vacuum equations is a triple \((\Sigma, g, k)\), where \(\Sigma\) is a connected complete 3-dimensional manifold, \(g\) a Riemmanian metric, and \(k\) a symmetric two tensor on \(\Sigma\), satisfying the vacuum constrain equations, abbreviated as (VCE),

\[
\begin{cases}
R_g - |k|_g^2 + (tr_gk)^2 = 0, \\
\text{div}_g(k - (tr_gk)g) = 0.
\end{cases} \tag{1.4}
\]

By the famous initial value formulation for the Einstein equations by Y. Choquet-Bruhat in 1952(see [9][24]), we can always think the vacuum data \((\Sigma, g, k)\) as been embedded in some

\(^2\)We refer to [24] for all the concepts.
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spacetime \((V, \gamma)\) satisfying (EVE), where \(g\) is the restriction of \(\gamma\) to \(\Sigma\), and \(k\) is the second fundamental form of the embedding.

\((\Sigma, e)\) is called Euclidean at infinity, where \(e\) is a Riemannian metric on \(\Sigma\), if there is a compact subset \(\Sigma_{\text{int}} \subset \Sigma\), such that the compliment \(\Sigma_{\text{ext}} = \Sigma \setminus \Sigma_{\text{int}}\) is a disjoint union of finitely many open sets \(\Sigma_{\text{ext}} = \bigcup E_i\), and each \(E_i\) is diffeomorphic to \(\mathbb{R}^3\) cutting off a ball \(B_R\), and on each \(E_i\), \(e\) is the pull back of the standard Euclidean metric on \(\mathbb{R}^3\). Here \(\Sigma_{\text{int}}\) is called the interior region, \(\Sigma_{\text{ext}}\) the exterior region, and each \(E_i\) an end. Each end \(E_i\) has a coordinate system \(\{x_i : i = 1, 2, 3\}\) inherited from \(\mathbb{R}^3\). Let \(r = \sqrt{\sum_i (x_i)^2}\). \((\Sigma, g, k)\) is said to be asymptotically flat, abbreviated as \((\text{AF})\), if \((\Sigma, e)\) is Euclidean at infinity for some \(e\), and there exists an \(\alpha > \frac{1}{2}\), such that under coordinates \(\{x_i : i = 1, 2, 3\}\),

\[
g_{ij} = \delta_{ij} + o_2(r^{-\alpha}), \quad k_{ij} = o_1(r^{-1-\alpha}). \tag{1.5}\]

Under these conditions, the ADM mass is defined as,

\[
m = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu^j d\sigma(r), \tag{1.6}\]

where \(g_{ij,k} = \frac{\partial g_{ij}}{\partial x_k}\) and \(\nu^j\) is the Euclidean unit outer normal of \(S_r\) with \(d\sigma(r)\) the surface element of \(S_r\). The famous positive mass theorem by Schoen and Yau \[21\][22] and Witten \[25\] says that \(m \geq 0\) under the dominant energy condition.

If the initial data set \((\Sigma, g, k)\) is axisymmetric[16][11] under a Killing vector field \(\xi\), i.e.

\[
\mathcal{L}_\xi g = 0, \quad \mathcal{L}_\xi k = 0, \tag{1.7}\]

where \(\mathcal{L}\) denotes the Lie derivative, we also have a well-defined angular momentum \(J\) of a close 2-surface \(S \subset \Sigma\)

\[
J(S) = \frac{1}{8\pi} \int_S \pi_{ij} \xi^j \nu^i d\sigma_g, \tag{1.8}\]

where \(\pi_{ij} = k_{ij} - tr_g(k)g_{ij}\) is divergence free by (1.4), and \(\nu\), \(d\sigma_g\) are, respectively the unit outer normal of \(S\) and surface element w.r.t. \(g\).

1.2 Ideas and main results

In this paper, we will prove the mass angular momentum inequality for certain axisymmetric, AF vacuum data \((\Sigma, g, k)\) with small \(tr_gk\), especially we partially solved the question asked by Schoen. We will use the full Einstein equations and a perturbation method. Given an AF vacuum data \((\Sigma, g, k)\), we will solve the boost problem of (EVE) for \((\Sigma, g, k)\) as [10] to get a spacetime \((V, \gamma)\), where \(V\) is a subset of \(\Sigma \times \mathbb{R}\) which grows linearly at infinity. Given a function \(u\) defined on \(\Sigma\), the graph \(\text{Graph}_u = \{(x, u(x)) \in \Sigma \times \mathbb{R}, x \in \Sigma\}\) of \(u\) lies inside \(V\), when \(|u|\) has roughly sub-linear growth. We want to find a solution to \(H_u = 0\), where \(H_u\) is the mean curvature of \(\text{Graph}_u\) w.r.t. \((V, \gamma)\). Now fix a 3-manifold \((\Sigma, e)\) Euclidean at infinity, we can construct a mapping \(\mathcal{H}\) which takes the triple \((g, k, u)\) to the mean curvature \(H_u\), i.e.
\( \mathcal{H} : (g, k, u) \rightarrow H_u \). Viewing \((g, k)\) as parameters and \(u\) as unknown function, our equation changes to
\[
\mathcal{H}(g, k, u) = 0. \tag{1.9}
\]
When \((g, k)\) is maximal, \(u \equiv 0\) is a solution to (1.9). So we can use the inverse function theorem to solve \(\mathcal{H}(g, k, u) = 0\) when \(\text{tr}_g k\) is small enough. From now on, we always assume \(s \in \mathbb{N}\) and \(\delta \in \mathbb{R}\). Using notations from Section 2, we have

**Definition 1.2.** Fix a 3-dimensional manifold \((\Sigma, e)\) which is Euclidean at infinity.

1. The **vacuum constraint data sets** \(\mathcal{VC}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) is defined to be the set of solutions \((g, k)\) to (1.4), such that \((g - e, k) \in H_{s+1, \delta + \frac{1}{2}}(\Sigma) \times H_{s, \delta + \frac{1}{2}}(\Sigma)\).
2. The **maximal vacuum constraint data sets** \(\mathcal{MV C}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) is defined to be the subset of \(\mathcal{VC}_{s+1, \delta}(\Sigma)\) satisfying \(\text{tr}_g k = 0\).

Inside \(\mathcal{VC}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) and \(\mathcal{MV C}_{s+1, \delta + \frac{1}{2}}(\Sigma)\), we use the topology induced by the Sobolev norms of \(H_{s+1, \delta + \frac{1}{2}}(\Sigma) \times H_{s, \delta + \frac{1}{2}}(\Sigma)\) as in Definition 1.2.

**Definition 1.3.** A simply connected 3-manifold \((\Sigma, e)\) which is Euclidean at infinity is called

axisymmetric, if

1. \(\Sigma\) is diffeomorphic to \(\mathbb{R}^3\) minus some points \(\{a_k\}_{k=1}^{l-1}\) on the \(z\)-axis \(\Gamma = \{(\rho, \varphi, z) \in \mathbb{R}^3 : \rho = 0\}\), with one end modeled by a neighborhood of \(\infty\), and other ends by a neighborhood of \(a_k\) with coordinates given by a Kelvin transformation: \(x' = \frac{x - a_k}{|x - a_k|}\);
2. \(\ell_0 e = 0\), where \(\varphi\) is the azimuth of the cylindrical coordinate \(\rho, \varphi, z\).

**Remark 1.4.** Near \(\infty\), \(e\) is given by the Euclidean metric \(ds_0^2\), and near each pucture \(a_k\), \(e\) is the pull back of the Euclidean metric by the Kelvin transformation, i.e. \(e = \frac{1}{|x|}ds_0^2\). In fact, by Chruściel’s reduction in [11], any simply connected, axisymmetric, AF vacuum data \((\Sigma, g)\) has the underlying topology \(\Sigma\) given by \(\mathbb{R}^3\) minus finitely many points on the \(z\) axis, with the Killing vector field \(\frac{\partial}{\partial \varphi}\).

**Definition 1.5.** Given \((\Sigma, e)\) as in Definition 1.3,

1. An initial data set \((g, k)\) is called axisymmetric, if symmetry conditions (1.7) hold for the Killing vector field \(\xi = \frac{\partial}{\partial \varphi}\).
2. \(\mathcal{VC}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) and \(\mathcal{MV C}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) are the axisymmetric subset of \(\mathcal{VC}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) and \(\mathcal{MV C}_{s+1, \delta + \frac{1}{2}}(\Sigma)\) respectively.

The following Theorem is one of our main results, which is a summarization of Theorem 1.12 Lemma 1.13 Lemma 1.14 and Theorem 1.16.

**Theorem 1.6. (Main Theorem 2)** Given \(s \geq 4, -2 < \delta < -1\).

(i) Let \((\Sigma, e)\) be a 3-dimensional manifold which is Euclidean at infinity. For any \((g, k) \in \mathcal{VC}_{s+2, \delta + \frac{1}{2}}(\Sigma)\), where \(\lambda e \leq g \leq \lambda^{-1} e\) for some \(\lambda > 0\), there exists a small number \(\epsilon\) depending only on \(\lambda\) and \(\|g-e\|_{H_{s+2, \delta + \frac{1}{2}}(\Sigma)} + \|k\|_{H_{s+1, \delta + \frac{3}{2}}(\Sigma)}\), such that if \(\|\text{tr}_g k\|_{H_{s+2, \delta - \frac{1}{2}}(\Sigma)} < \epsilon\), then there exists a spacetime \((\mathcal{V}, \gamma)\) solving the (EVE), and a function \(u \in H_{s+2, \delta - \frac{1}{2}}(\Sigma)\) solving the maximal surface equation (1.5) inside \((\mathcal{V}, \gamma)\). The induced metric \(g_u\) and second fundamental form \(k_u\) of \(\text{Graph}_{u}\) satisfy \((g_u, k_u) \in \mathcal{MV C}_{s+1, \delta + \frac{1}{2}}(\Sigma)\).
(ii) If \(-\frac{3}{2} < \delta < -1\), the ADM mass of \((\Sigma, g_u, k_u)\) is the same as that of \((\Sigma, g, k)\).

(iii) If \((\Sigma, e, g, k)\) is simply connected, axisymmetric, then \(u\) can be chosen to be axisymmetric, hence \((\Sigma, g_u, k_u)\) is axisymmetric, and has the same angular momentum as \((\Sigma, g, k)\).

Remark 1.7. The weight \(\delta\) corresponds the decay \(g \sim e + o(r^{-(\delta+2)})\) and \(k \sim o(r^{-(\delta+3)})\) by the Sobolev embedding lemma 2.5 \((g_u, k_u)\) is always assumed to be pulled back to \(\Sigma\) by the graphical map \(F_u : x \rightarrow (x, u(x))\).

Remark 1.8. The order of regularity of our final solution \((g_u, k_u)\) decreases by 1 than our starting data \((g, k)\). This is due to the fact that the restriction of \(H_s\) Sobolev functions on a spacetime to a hypersurface decreases the regularity by 1 (see Lemma 2.8).

Our main Theorem 1.1 is then a corollary of the above theorem.

Proof of Theorem 1.1: Let \(u\) be the solution given in part (iii) of Theorem 1.6. Then the induced maximal data \((g_u, k_u) \in \mathcal{M}V C^a_{s+1, \delta+\frac{1}{2}}(\Sigma)\), and the ADM mass \(m\) and angular momentum \(J\) of \((g, k)\) and \((g_u, k_u)\) are the same. Now by Sobolev embedding lemma 2.5 \((g_u - e, k_u) \in C^s_{\beta} \times C^s_{\beta+1}(\Sigma)\) for some \(\frac{1}{2} < \beta < \delta+2 < 1\). So \((\Sigma, g_u, k_u)\) is an axisymmetric, maximal vacuum data, with asymptotic conditions \(g_u = \delta + O_{s-1}(\frac{1}{r^s})\) and \(k_u = O_{s-2}(\frac{1}{r^{s+1}})\), so the mass angular momentum inequality in [23] holds on \((\Sigma, g_u, k_u)\). Hence \(m \geq \sqrt{|J|}\). □

The paper is organized as follows: In Section 2 we will review the weighted Sobolev space theories covered by [10][8][7][2] and the geometry of hypersurfaces in 3+1 dimension Lorentzian spaces. In Section 3 we will extend the boost theory in [10][8] to the case of multi-ends. In Section 4 we will set up a perturbation problem for the mean curvature of graphs. We will take initial data sets as parameters and use linear theory in [2][19] and[7] and the Quantitative Inverse Function Theorem 4.10. Finally we will prove the main results in Section 4.4.

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2 Preliminaries

In this section, we give some preliminary results on the weighted Sobolev space theories and the geometry of hypersurfaces in Lorentz spaces.

2.1 Weighted Sobolev space theories

Here we give our definition of the weighted sobolev space. Most of the results here can be found in [7][10] and [8]. We will mainly talk about two types of domains.

Type 1 domain: sub-domain of \(\mathbb{R}^3\).
Let $U$ be an open set in $\mathbb{R}^n$, $\sigma(x) = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^n$, and $V$ a finite dimensional vector space. Given $s \in \mathbb{N}$, $\delta \in \mathbb{R}$.

**Definition 2.1.** $C^s_\delta(U)$ is the Banach space of $C^s$ functions $u : U \rightarrow V$, with finite norm

$$
\|u\|_{C^s_\delta(U)} = \sup_U \left\{ \sum_{|\alpha| \leq s} \sigma^{\delta+|\alpha|} |D^\alpha u| \right\}.
$$

$H_{s,\delta}(U)$ is the class of functions $u : U \rightarrow V$, with weak derivatives up to order $s$, such that $\sigma^{\delta+|\alpha|} D^\alpha u \in L^2(U)$ for all $\alpha \leq s$. $H_{s,\delta}(U)$ is a Hilbert space with inner product:

$$
\langle u_1, u_2 \rangle_{H_{s,\delta}(U)} = \sum_{|\alpha| \leq s} \langle \sigma^{\delta+|\alpha|} D^\alpha u_1, \sigma^{\delta+|\alpha|} D^\alpha u_2 \rangle_{L^2(U)}.
$$

Then the norm is: $\|u\|_{H_{s,\delta}(U)} = \langle u, u \rangle_{H_{s,\delta}(U)}^{1/2}$.

Now we will list some properties of $H_{s,\delta}(U)$, which can be found in [10][7] and [8]. Given $0 < \epsilon \leq 1$, and $\phi_\epsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $\phi_\epsilon(x) = \frac{x}{\sigma(x)^1 + \epsilon}$. An open subset $U \subset \mathbb{R}^3$ is said to have the extended cone property if $\phi_\epsilon(U)$ has the cone property $\overline{B}_R^\epsilon$ for each $0 < \epsilon \leq 1$.

**Lemma 2.2.** Given $U$ satisfying the extended cone property,

(i) (embedding). If $s' < s - \frac{n}{2}$ and $\delta' < \delta + \frac{n}{2}$, the inclusion $H_{s,\delta}(U) \subset C^{s'}_{\delta'}(U)$ is continuous;

(ii) (multiplication). If $s \leq s_1, s_2$, $s < s_1 + s_2 - \frac{n}{2}$ and $\delta < \delta_1 + \delta_2 + \frac{n}{2}$, the multiplication $\langle f_1, f_2 \rangle \rightarrow f_1 f_2$ is continuous: $H_{s_1,\delta_1}(U) \times H_{s_2,\delta_2}(U) \rightarrow H_{s,\delta}(U)$.

Hence $H_{s,\delta}(U)$ is a Banach algebra if $s > \frac{n}{2}$ and $\delta > -\frac{n}{2}$.

**Type 2 domain: manifold which is Euclidean at infinity.**

Let $(\Sigma, e)$ be an $n$ dimensional manifold which is Euclidean at infinity. Let $x = \{x^i\}$ be the local coordinates, where $\{x^i\}$ is the pull back of the standard coordinates from $\mathbb{R}^n \setminus B_R$ when restricted to $E_i$, and $e = ds_0^2 = \sum_{i=1}^n (dx^i)^2$ on $E_i$. Fix a point $O \in \Sigma_{int}$, and define a function on $\Sigma$ by

$$
\sigma_e(x) = (1 + d_0^2(x, O))^{1/2}.
$$

Clearly $\sigma_e(x)$ is equivalent to $\sigma(x) = (1 + |x|^2)^{1/2}$ on each end $E_i$.

When we use $\Sigma$ to model an initial data set, the spacetime should have topology as a sub-domain of $\Sigma \times \mathbb{R}$. Using coordinates $(x^i, t)$ on $\Sigma \times \mathbb{R}$, it has a natural reference metric

$$
\tilde{e} = dt^2 + e.
$$

For $\theta \in (0, 1]$, the boost region $\Omega_\theta$ is defined as,

$$
\Omega_\theta = \{(x, t) \in \Sigma \times \mathbb{R} : |t| \leq \theta \sigma_e(x)\}.
$$

On $\Omega_\theta$, the distance function $d_\theta(\cdot, O)$ is equivalent to $d_e(\cdot, O)$, so we can use $\sigma_e$ to define the weighted Sobolev space on $\Omega_\theta$. Given a smooth tensor bundle $E \rightarrow \Sigma$ or $E \rightarrow \Omega_\theta$ and $s \in \mathbb{N}$, $\delta \in \mathbb{R}$.

\[\text{See the remark under Definition 2.3 of [10]}\]
Lemma 2.5. \( \|u\|_{C^s_b(\Sigma)} = \sup_{\Sigma(\text{or } \Omega)} \left\{ \sum_{|\alpha| \leq s} \sigma_{\epsilon}^{b+|\alpha|} |D^\alpha u|_{e(\text{or } \epsilon)} \right\} \).

\( H_{s,\delta}(\Sigma) \) or \( H_{s,\delta}(\Omega) \) is the class of sections \( u : \Sigma \to E \), or \( u : \Omega \to E \) with finite norm \( \|u\|_{C^s_b} \).

Remark 2.4. In fact, the definitions are independent of the choice of \( e \) on \( \Sigma_{\text{int}} \).

Lemma 2.6. (Lemma 2.4, 2.5 in [7], Appendix 1 in [6])

(i) (embedding). If \( s' < s - \frac{n+1}{2}, \delta' < \delta + \frac{n+1}{2} \), the inclusion \( H_{s,\delta}(\Sigma) \subset C^{s'}_{\delta'}(\Sigma) \) is continuous;

(ii) (multiplication). If \( s \leq s_1, s_2, s < s_1 + s_2 - \frac{n+1}{2}, \delta < \delta_1 + \delta_2 + \frac{n+1}{2} \), the multiplication \( (f_1, f_2) \to f_1f_2 \) is a continuous map: \( H_{s_1,\delta_1}(\Sigma) \times H_{s_2,\delta_2}(\Sigma) \to H_{s,\delta}(\Sigma) \), hence \( H_{s,\delta}(\Sigma) \) is a Banach algebra if \( s > \frac{n}{2}, \delta > -\frac{n}{2} \). Furthermore,

\[
\|f_1f_2\|_{H_{s,\delta}(\Sigma)} \leq C\|f_1\|_{H_{s_1,\delta_1}(\Sigma)}\|f_2\|_{H_{s_2,\delta_2}(\Sigma)},
\]

where \( C \) is a constant depending only on \( \{n, s_1, s_2, \delta_1, \delta_2\} \).

Divide \( \Omega \) as \( \Omega = (\Omega_{\text{int}}) \cup_{i=1}^d (\Omega_i) \), where \( (\Omega_i) = \{(x, t) \in \Omega \mid x \in E_i\} \), and \( (\Omega_{\text{int}}) \) the complement. Now \( (\Omega_{\text{int}}) \) is a compact manifold, and \( (\Omega_i) \subset \mathbb{R}^{n+1} \) satisfies the extended cone property in the above section, and hence Lemma 2.2. By working separately on \( (\Omega_i) \) and \( (\Omega_{\text{int}}) \) as in [7] using Lemma 2.2 we have similar results.

Lemma 2.7. (composition). Given \( \Omega, \Omega' \) as above and \( f : \Omega \to \Omega' \) a differentiable map, such that \( |Df|_c \geq c > 0 \) and \( f - id \in H_{s+1,\delta-1}(\Omega) \) with \( s > \frac{n}{2} \) and \( \delta > -\frac{n}{2} \), then for any \( s' \leq s + 1, \delta' \in \mathbb{R} \), the composition \( u \to u \circ f \) is an isomorphism as a map:

\[
H_{s',\delta'}(f(\Omega)) \to H_{s',\delta'}(\Omega).
\]
Define the function \( \tau(x,t) = \frac{t}{\sigma_e(x)} \). Denote the level surface of \( \tau \) by \( \Sigma_\tau = \{(x,t) \in \Sigma \times \mathbb{R} : \tau(x,t) = \tau \} \). Then \( \Omega_\theta \) has a foliation \( \Omega_\theta = \bigcup_{\tau \in (-\theta,\theta)} \Sigma_\tau \). The restriction norm is defined as:

\[
\|u\|_{H^s,\delta(\Sigma_\tau,\Omega_\theta)} = \left( \sum_{k=0}^{s} \|D_k^t u|_{\Sigma_\tau}\|_{H^{s-k,\delta}(\Sigma)}^2 \right)^{1/2}.
\]

(2.4)

Using ideas similar to the proof of Lemma 3.1 in [8], we can get,

Lemma 2.8. (restriction). \( \forall \tau \in (-\theta,\theta) \), we have the following continuous inclusion:

\[
H_{s+1,\delta}(\Omega_\theta) \subset H_{s,\delta+\frac{1}{2}}(\Sigma_\tau,\Omega_\theta),
\]

for every \( s \in \mathbb{N} \) and \( \delta \in \mathbb{R} \).

### 2.2 Geometry of hypersurface in Lorentzian space

In this section, we will review the geometry of hypersurfaces in a Lorentzian space. We will mainly focus on the mean curvature of the hypersurface. Notation and part of the results here trace back to [1], and all concepts of Lorentzian space can be found in [24]. Let \((V,\gamma)\) be a \((3+1)\) dimensional Lorentzian space, with \(\langle \cdot, \cdot \rangle\) the metric pairing and \(\nabla\) the connection. A smooth function \(t \in C^\infty(V)\) is called a time function if \(\nabla t\) is nonzero, and everywhere timelike, i.e. \(\langle \nabla t, \nabla t \rangle < 0\). We call a hypersurface \(\Sigma\) spacelike if the restriction of \(\gamma\) to \(\Sigma\) is Riemannian. In a local coordinate system \(\{x^i, t\}\), where \(t\) is a time function, the metric can be written as (See equation (2.12) of [1]):

\[
\gamma = -(\alpha^2 - \beta^2)dt^2 + 2\beta_idx^i dt + g_{ij}dx^idx^j,
\]

(2.5)

where \(\alpha\) is the lapse function, i.e. \(\alpha^2 = -\langle \nabla t, \nabla t \rangle\), \(g_{ij}\) a Riemannian metric, and \(\beta = g^{ij}\beta_i\partial_j\) the shift vector. Here we use \(\partial_t = \frac{\partial}{\partial t}\) and \(\partial_i = \frac{\partial}{\partial x^i}\) as coordinate vectors. The inverse metric \(\gamma^{-1}\) is given by:

\[
\gamma^{\mu\nu} = \begin{bmatrix}
-\frac{1}{\alpha^2}
\frac{\beta^i}{\alpha^2}
\frac{\beta^j}{\alpha^2}
\frac{\alpha^2}{\alpha^2}
\end{bmatrix},
\]

(2.6)

under coordinate system \(\{t, x^1, x^2, x^3\}\).

We will denote the level surface of the time function \(t\) by \(\Sigma_t = \{p \in V : t(p) = t\}\). Let \(D\) be the gradient operator on \(\Sigma_t\), and \(\text{div}^0\) the divergence operator on \(\Sigma_t\). The future-directed timelike unite normal \(T\) of \(\Sigma_t\) is given by:

\[
T = -\alpha \nabla t = \alpha^{-1}(\partial_t - \beta),
\]

(2.7)

and the second fundamental form \(k_{ij}^0\) and the mean curvature \(H^0\) of the slice \(\Sigma_t\) are given by,

\[
k_{ij}^0 = \langle \partial_i, \nabla_{\partial_j} T \rangle = \frac{1}{2} \alpha^{-1}\partial_t g_{ij} - \frac{1}{2} \alpha^{-1} L\beta g_{ij},
\]

(2.8)

\[\text{See Chap 10 of [24] for details.}\]

\[\text{See Appendix 5 for details.}\]
\[ H^0 = g^{ij} A^0_{ij} = \frac{1}{2} \alpha^{-1} g^{ij} \partial_t g_{ij} - \alpha^{-1} \text{div}^0(\beta). \] (2.9)

Given a spacelike hypersurface \( \Sigma \), we can always choose local coordinates \( \{x^i, t\} \), such that \( \Sigma \) is locally the \( t = 0 \) level surface \( \Sigma_0 \). Given a smooth function \( u \in C^\infty(\Sigma) \), we can study the graph of \( u \), i.e. \( \text{Graph}_u = \{(x^i, u(x))\} \) in local coordinates. So we call this \( u \) the \textit{height function}. By extending \( u \) parallel to \( V \) requiring that

\[ \partial_t u = 0, \] (2.10)

\( \text{Graph}_u \) can be viewed as level surface of \((u - t) = 0\). The unit normal of \( \text{Graph}_u \) is \( \nu \) well-defined. Define the canonical graphical diffeomorphism \( F: \Sigma \rightarrow \text{Graph}_u \) by \( F(x) = (x, u(t)) \). Then \( \text{Graph}_u \) has a local coordinate system \( \{x^i : i = 1, 2, 3\} \). The coordinate vector frame \( \{\partial_i\} \) on \( \Sigma \) is passed by \( F \) to a local frame

\[ \alpha_i = \partial_i + u_i \partial_t : \quad i = 1, 2, 3, \] (2.13)

on \( \text{Graph}_u \). Now denote \( M = \text{Graph}_u \). Using this local coordinates, the restriction \( \gamma|_M \) of \( \gamma \) to \( \text{Graph}_u \), denoting by \( g_M = (g_M)_{ij}dx^i dx^j \), is given by

\[ (g_M)_{ij} = g_{ij} + \beta_i u_j + u_i \beta_j - (\alpha^2 - \beta^2) u_i u_j. \] (2.14)

Then the inverse metric matrix is calculated in the Appendix 5 by equation (5.12) as:

\[ (g_M)^{ij} = g^{ij} - \frac{1}{\alpha^2} \delta^i \delta^j + \frac{\nu^2}{\alpha^2} (\beta - \alpha U)^i (\beta - \alpha U)^j \]

\[ = \gamma^{ij} + \frac{\nu^2}{\alpha^2} (\beta - \alpha U)^i (\beta - \alpha U)^j. \] (2.15)

So the mean curvature \( H_u \) of the graph \( M \) is given by

\[ H_u = (g_M)^{ij} \langle \nabla_{\alpha_i} N, \alpha_j \rangle. \] (2.16)

### 3 Boost evolution

Fix a 3-manifold \((\Sigma, e)\), which is Euclidean at infinity. Let \( \tilde{e} = dt^2 + e \) be the reference metric on \( \Sigma \times \mathbb{R} \). Given integer \( s \geq 4 \), and real number \( \delta > -2 \), we consider vacuum constraint initial data sets \((\Sigma, g, k)\), such that \((g, k) \in V_{s, \delta + \frac{1}{2}}(\Sigma)\). Here boost evolution means

\(\text{See Appendix 5 for details.}\)
that in the spacetime \((\mathcal{V}, \gamma)\) which is evolved by (EVE) taking \((\Sigma, g, k)\) as initial data set, where \(\mathcal{V} \subset \Sigma \times \mathbb{R}\), both the future and past temporal distance \(\chi_{\pm}(x)\) to the boundary of \(\mathcal{V}\) is proportional to the space distance \(\sigma(x)\) for \(x \in \Sigma\), i.e. \(\chi_{\pm}(x) \geq c \sigma(x)\) for \(c > 0\). We will extend the boost evolution on \(\mathbb{R}^3\) in [10] to the case of \(\Sigma\).

### 3.1 Reduced Einstein equation and results on compact domain

Let us review the reduction using harmonic gauge initially introduced by Y. Choquet-Bruhat (see [6]). Using \(\{x^i : i = 1, 2, 3\}\) as local coordinates on \(\Sigma\), and \(x^\mu = (x^0, x^i)\), with \(x^0 = t\) as coordinates on \(\mathcal{V} \subset \Sigma \times \mathbb{R}\), the Ricci curvature can be expressed as:

\[
\text{Ric}^{\mu\nu} = R^\mu_{h\nu} + \frac{1}{2}(\gamma^{\mu\alpha} D_\alpha \Gamma^\nu + \gamma^{\nu\alpha} D_\alpha \Gamma^\mu),
\]

where \(\Gamma^{\mu}_{\alpha\beta}\) is the Christoffel symbol of \(\gamma\), \(\Gamma^\mu = \gamma^{\alpha\beta} \Gamma^\mu_{\alpha\beta}\), and

\[
R^\mu_{h\nu} = \frac{1}{2}\{\gamma^{\alpha\beta} D_\alpha D_\beta \gamma^{\mu\nu} - B^{\mu\nu}(\gamma, D\gamma)\},
\]

with \(B^{\mu\nu} = P^{\mu\nu,\rho\sigma}_{\alpha\beta,\kappa\lambda} D_\rho \gamma^{\alpha\beta} D_\sigma \gamma^{\kappa\lambda}\), and \(P\) is a rational function of \(\gamma^{\alpha\beta}\). In fact, The Einstein vacuum equation \(\text{Ric}_\gamma = 0\) is a degenerated differential equation system due to its invariance under diffeomorphic transformations. Harmonic gauge is used to fix this gauge freedom by Y. Choquet-Bruhat, which means that we can choose \(\text{id} : (\mathcal{V}, \gamma) \rightarrow (\mathcal{V}, \tilde{e})\) to be a wave map, i.e. \(\Box_{(\gamma, \tilde{e})}\text{id} = 0\).

Denote

\[
f^{\mu} = \Gamma^{\mu} - \gamma^{\alpha\beta} \tilde{\Gamma}^{\mu}_{\alpha\beta},
\]

(3.1)

to be the harmonic gauge vector, where \(\tilde{\Gamma}^{\mu}_{\alpha\beta}\) the Christoffel symbol of \(\tilde{e}\). \(f^\mu\) is the difference of two connections, hence a tensor, then harmonic gauge condition reduces to \(f^\mu = 0\), or:

\[
\Box_{\gamma} x^\mu = -\gamma^{\alpha\beta} \tilde{\Gamma}^{\mu}_{\alpha\beta},
\]

(3.2)

where \(\Box_{\gamma}\) is the Laplacian operator of the Lorentzian metric \(\gamma\), and \(\Box_{\gamma} x^\mu = -\Gamma^\mu\). Now under harmonic gauge (3.2), the (EVE) (1.3) reduced to [10]

\[
\gamma^{\alpha\beta} D_\alpha D_\beta \gamma^{\mu\nu} = B^{\mu\nu}(\gamma, D\gamma) + \frac{1}{2} \gamma^{\alpha\beta}\{\gamma^{\mu\rho} \tilde{R}^{\nu}_{\beta\alpha\rho} + \gamma^{\nu\rho} \tilde{R}^{\mu}_{\beta\alpha\rho}\},
\]

(3.3)

where \(\tilde{R}\) is the curvature of \(\tilde{e}\). The Cauchy data for these equations consist of:

\[
\gamma|_\Sigma = \phi, \ D_t \gamma|_\Sigma = \psi.
\]

(3.4)
For given initial data set \( (g, k) \), we need to construct Cauchy data \((\phi, \psi)\) by requiring \( f^\mu|_\Sigma = (\Gamma^\mu - \gamma^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta})|_\Sigma = 0 \). To fix the freedom in choosing a harmonic gauge, we require the coordinate system of \( V \) is Gaussian on \( \Sigma \), which means:

\[
\phi^{00} = -1, \quad \phi^{0i} = 0, \quad \phi^{ij} = g^{ij}.
\]

The condition \((\Gamma^\mu - \gamma^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta})|_\Sigma = 0\) requires \(\psi\):

\[
\psi^{00} = -4\text{tr}g, \quad \psi^{0i} = -(\Gamma^i - g^{kj}\tilde{\Gamma}^i_{kj}), \quad \psi^{ij} = 2g^{ik}g^{jl}k_{kl}.
\]

Define a reference Lorentzian metric by

\[
\tilde{\eta} = -dt^2 + e.
\]

When the initial data \((g - e, k) \in H_{s, \delta + \frac{1}{2}}(\Sigma) \times H_{s, -1, \delta + \frac{1}{2}}(\Sigma)\), the Cauchy data (3.5) (3.6) satisfy \((\phi - \tilde{\eta}, \psi) \in H_{s, \delta + \frac{1}{2}}(\Sigma) \times H_{s, -1, \delta + \frac{1}{2}}(\Sigma)\). In fact, by multiplication lemma 2.5, \((g - e, k) \rightarrow (\phi - \tilde{\eta}, \psi)\) is a continuous map \(H_{s, \delta + \frac{1}{2}}(\Sigma) \times H_{s, -1, \delta + \frac{1}{2}}(\Sigma) \rightarrow H_{s, \delta + \frac{1}{2}}(\Sigma) \times H_{s, -1, \delta + \frac{1}{2}}(\Sigma)\).

To solve (EVE) (1.3), we can first solve the reduced equation (3.3) by quasilinear theory (see Appendix 3 in [6], and Section 5 in [10]), and then show that the harmonic gauge is preserved.

In fact, Bianchi identity and the reduced equation (3.3) imply that the harmonic gauge vector \(f^\mu\) satisfies a linear equation

\[
\Box\gamma f^\mu + A(\gamma, D\gamma)Df = 0.
\]

So we can use uniqueness of linear equations to show that \(f^\mu \equiv 0\) since we chose \(f^\mu|_\Sigma = 0\), and the constraint equations (1.4) implies that \(\partial_t f^\mu|_\Sigma = 0\).

Now we summarize a local version of the existence and causal uniqueness theorem based on the interior region \(\Sigma_{\text{int}}\) of \((\Sigma, e)\), which has dimension \(n = 3\). We can extend the interior region \(\Sigma_{\text{int}}\) to contain the annuli \(B_{2R} \setminus B_R\) of each end \(E_i\) of \((\Sigma, e)\). Now define a causal set \((V_{\text{int}})_{\theta, \lambda}\) based on \(\Sigma_{\text{int}}\) as follows:

\[
(V_{\text{int}})_{\theta, \lambda} = \{(x, t) \in \Sigma_{\text{int}} \times [-\theta, \theta] : |x| \leq 2R - \lambda|t|, \text{if } x \in E_i\},
\]

where \(\theta \in (0, 1]\) and \(\lambda \geq 2\) is a positive number. Now \((V_{\text{int}})_{\theta, \lambda}\) has a lateral boundary \(L_{\theta, \lambda} = \{(x, t) \in (V_{\text{int}})_{\theta, \lambda} : |x| = 2R - \lambda|t|\}\). When \(\lambda\) is large enough depending only on \(e\), \(L_{\theta, \lambda} = L_{\theta, \lambda} \cap \{t \geq 0\}\) (or \(L_{\theta, \lambda} = L_{\theta, \lambda} \cap \{t \leq 0\}\) is spacelike and ingoing (or outgoing) w.r.t. \(\tilde{\eta}\), hence \((V_{\text{int}})_{\theta, \lambda}, \tilde{\eta})\) is causal.

Combining Theorem 7.4, Theorem 8.3 of Chap 6, and Corollary 4.8, Theorem 4.11, Theorem 4.13 of Appendix 3 in [6], and using a cutoff argument as in Theorem 3.7, we have the following well-known local existence and uniqueness theorem,

\[\text{[11]}\text{See page 164 in [6].}\]
\[\text{[12]}\text{See page 167 in [6] and Section 4 in [10].}\]
\[\text{[13]}\text{See page 167 in [6].}\]
\[\text{[14]}\text{See Definition 2.11 of Appendix 3 in [6].}\]
3 Boost evolution on manifold Euclidean at infinity

We first modify the linear boost theory in [10] to the case based on an Euclidean end $E \cong \mathbb{R}^{n-1} \setminus B_R$. Let us fix a special type of boost regions. Denote $\bar{x} = (x^1, \cdots, x^n) \in \mathbb{R}^n$, such that $x = (\bar{x}, t) \in \mathbb{R}^n$. Later on, we will denote the index for $t$-coordinates as 0, while index for $\bar{x}$ as $i$ with $i = 1, \cdots, n - 1$. Let $\bar{\sigma}(\bar{x}) = (1 + |\bar{x}|^2)^{1/2}$. For $\theta \in (0, 1/2]$, $\lambda \geq 2$ and a given end $E \cong \mathbb{R}^{n-1} \setminus B_R$, the boost region $V_{\theta,\lambda}$ based on $E$ is defined as:

$$V_{\theta,\lambda} = \{(\bar{x}, t) \in \mathbb{R}^n, \frac{|t|}{\sigma(\bar{x})} < \theta, |\bar{x}| \geq R + \lambda|t|\}, \quad (3.10)$$

Define function $\tau$ as $\tau(x) = \frac{t}{\sigma(x)}$. Then the level surface of $\tau$ is $E_{\tau} = \{x \in V_{\theta,\lambda} : \tau(x) = \tau\}$. $V_{\theta,\lambda}$ has a foliation:

$V_{\theta,\lambda} = \cup_{\tau \in (-\theta, \theta)} E_{\tau}$.

The lateral boundary of $V_{\theta,\lambda}$ is defined as,

$$L_{\theta,\lambda} = \{(\bar{x}, t) \in V_{\theta,\lambda} : |\bar{x}| = R + \lambda|t|\}. \quad (3.11)$$

Denote the upper part of $V_{\theta,\lambda}$ by $V_{\theta,\lambda}^+ = \{(\bar{x}, t) \in V_{\theta,\lambda} : t \geq 0\}$, then the boundary $\partial V_{\theta,\lambda}^+$ is constituted by $E$, $E_0$ and the upper lateral boundary $L_{\theta,\lambda}^+ = L_{\theta,\lambda} \cap V_{\theta,\lambda}^+$. Similarly, we have $V_{\theta,\lambda}^- = \{(\bar{x}, t) \in V_{\theta,\lambda} : t \leq 0\}$ and lower lateral boundary $L_{\theta,\lambda}^- = L_{\theta,\lambda} \cap V_{\theta,\lambda}^-$. Clearly $V_{\theta,\lambda}^\pm$ and the slices $E_\tau$ satisfy the extended cone property in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ respectively as in Section 2.1 and hence satisfy Lemma 2.2.

We introduce a class of hyperbolic metrics on $V_{\theta,\lambda}$ using the foliation $\{E_{\tau}\}_{\tau \in (-\theta, \theta)}$. The function $\tau$ is in fact a time function on $(V_{\theta,\lambda}, \eta)$, where $\eta = -dt^2 + \sum_{i=1}^{n-1} (dx^i)^2$ is the Minkowski metric. Let $\tilde{n}_\mu$ be the unit future co-normal of the foliation $\{E_{\tau} : \tau \in (-\theta, \theta)\}$, given by

$$\tilde{n} = \tilde{N} D\tau = \frac{1}{\sqrt{1 - \frac{\tau^2}{\sigma^2}}(x)} \left(\frac{\tau}{\sigma(\bar{x})} x^i dx^i\right), \quad (3.12)$$

where $\tilde{N}$ is the lapse function for the foliation $\{E_\tau\}$, defined by: $\tilde{N}^{-2} = -\langle D\tau, D\tau\rangle_\eta = \frac{1 - \tau^2|\bar{x}|^2/\sigma^2(\bar{x})}{\sigma^2(\bar{x})}$. $\tilde{n}$ can be viewed as a standard calibration for the foliation $V_{\theta,\lambda} = \cup E_{\tau}$, which is used to define the “regularity” of hyperbolicity. Denoting $|\cdot|$ as the standard Euclidean norm for tensors on $V_{\theta,\lambda}$, we have $\|\cdot\|$.

\[\text{See page } 397 \text{ and page } 585 \text{ in [10].}\]

\[\text{See also Definition 4.1 in [10].}\]
Definition 3.2. A $C^0$ covariant symmetric 2-tensor $\gamma^{\mu\nu}$ on $V_{\theta,\lambda}$ is called regularly hyperbolic, if there exist positive numbers $a$, $b$ and $C$ such that in $V_{\theta,\lambda}$:

1. $-\gamma^{\mu\nu}\bar{n}_\mu\bar{n}_\nu \geq a$;
2. for all tangent covectors $\zeta_\mu$ of $E_\tau$, i.e. $\gamma^{\mu\nu}\zeta_\mu\bar{n}_\nu = 0$, we have $\gamma^{\mu\nu}\zeta_\mu\zeta_\nu \geq b|\zeta|^2$;
3. $|\gamma| \leq C$;
4. The upper(or lower) lateral boundary $L^+_{\theta,\lambda}$ (or $L^-_{\theta,\lambda}$) is spacelike and ingoing(or out going) w.r.t. $\gamma$, i.e. every timelike curve entering $V^+_{\theta,\lambda}$ (or every timeline curve exiting $V^-_{\theta,\lambda}$) is past directed.

The coefficient of regular hyperbolicity of $\gamma$ is defined as,

$$h = \max\{\frac{1}{a}, \frac{1}{b}, C\}.$$ \hspace{1cm} (3.13)

Remark 3.3. Condition (4) implies that this type of $V_{\theta,\lambda}$ is a causal subset based on $E$ w.r.t. $\gamma$. Here we briefly talk about the criterion for Condition (4) to be true. We mainly discuss the case $L^+_{\theta,\lambda}$, and $L^-_{\theta,\lambda}$ is similar. The defining function of $L^+$ is given by $l(\bar{x}, t) = \lambda t + R - |\bar{x}|$, so the normal co-vector of $L^+$ is given by $dl = \lambda dt - d\bar{r}$, where $\bar{r} = |\bar{x}|$. Now $dl = \lambda(dt - \frac{\bar{r}^2}{\sigma^2(\bar{x})}d\bar{r}) + (\frac{\bar{r}^2}{\sigma^2(\bar{x})} - 1)d\bar{r} = \lambda\sqrt{1 - \frac{\bar{r}^2 |\bar{x}|^2}{\sigma^2(\bar{x})}}\bar{n} + (\frac{\bar{r}^2}{\sigma^2(\bar{x})} - 1)d\bar{r}$. So using the regularly hyperbolicity, we have $\gamma(dl, dl) \leq \lambda^2(1 - \theta^2)^2(\bar{n}, \bar{n}) + \lambda(\theta(\theta - 1)C \leq -a(1 - \theta^2) + C\lambda(\theta^2 - 1) < 0$, when $\lambda$ is chosen large enough depending only on $a$ and $C$, hence on $h$.

Remark 3.4. The set of regularly hyperbolic metrics on $V_{\theta,\lambda}$ is open in the space $C^0(V_{\theta,\lambda})$ of bounded continuous covariant symmetric 2-tensors. In fact, $\eta$ is regular hyperbolic with $a = 1$, $b = 1 - \theta^2$ and $C = \sqrt{\eta}$, and $L_{\theta,\lambda}$ is space-like and ingoing w.r.t $\eta$ when $\lambda \geq 2$. Since the space-like and ingoing condition for $L_{\theta,\lambda}$ is an open condition, there exist a small $\epsilon > 0$, depending only on $\theta$, $\lambda$ and $n$, such that any $C^0$ covariant symmetric 2-tensor $\gamma$, with $|\gamma - \eta| \leq \epsilon$, is regularly hyperbolic in $V_{\theta,\lambda}$.

Now consider a family of linear differential operators of second order in $V_{\theta,\lambda}$:

$$Lu = \Sigma_{k=0}^2 a_k \cdot D^k u,$$ \hspace{1cm} (3.14)

where $u$ and $Lu$ are $\mathbb{R}^N$ valued functions on $V_{\theta,\lambda}$, and $a_k$ are matrix valued functions. The following hypotheses are required to the existence theorem:

- **Hypothesis** (1)(weak coupling and hyperbolicity). $a_2 = \gamma Id$, i.e. $(a_2)^{\mu\nu I} = \gamma^{\mu\nu}\delta^I_I$, $\mu, \nu = 0, \cdots, n - 1$, $I, J = 1, \cdots, N$, where $\gamma$ is a regularly hyperbolic metric on $V_{\theta,\lambda}$.
- **Hypothesis** (2)(regularity). There exist integers $s_k$ and real numbers $\delta_k$, such that: $s_k > \frac{n}{2} + k - 1$, $\delta_k > 2 - k - \frac{n}{2}$: $0 \leq k \leq 2$, and (1): $a_k \in H_{s_k,\delta_k}(V_{\theta,\lambda})$ for $k = 0, 1$; (2) $\gamma - \eta \in H_{s_2,\delta_2}(V_{\theta,\lambda})$.

\[^{17}\text{See Definition 2.11 of Appendix 3 in [?].}\]
Remark 3.5. Now denote
\[ s' = \min_{0 \leq k \leq 2} \{ s_k \} + 1, \]
\[ m = \| \gamma - \eta \|_{H_{s_2,\delta_2}(V_{0,\lambda})} + \sum_{k=0}^{1} \| a_k \|_{H_{s_k,\delta_k}(V_{0,\lambda})}, \]
\[ \mu = \| \gamma - \eta \|_{H_{s_2-1,\delta_2+1/2}(E,V_{0,\lambda})} + \sum_{k=0}^{1} \| a_k \|_{H_{s_k-1,\delta_k+1/2}(E,V_{0,\lambda})}. \]
By the restriction Lemma 2.8, \( \mu \leq cm \). Using the multiplication Lemma 2.2 the regularity hypothesis (2) implies that
\[ L : H_{s+1,\delta}(V_{0,\lambda}) \rightarrow H_{s-1,\delta+2}(V_{0,\lambda}), \]
is a continuous map for \( 1 \leq s \leq s' \) and \( \delta \in \mathbb{R} \).

Then we have the existence and uniqueness theorem for linear systems.

**Theorem 3.6.** Let \( L \) be a differential operator defined by (3.14) in \( V_{0,\lambda} \), satisfying Hypotheses (1) and (2). Let \( \beta \in H_{s-1,\delta+2}(V_{0,\lambda}), \phi \in H_{s,\delta+\frac{1}{2}}(E) \) and \( \psi \in H_{s-1,\delta+\frac{3}{2}}(E) \), with \( 2 \leq s \leq s' \), \( \delta \in \mathbb{R} \). Then the Cauchy problem:
\[ Lu = \beta, \quad u|_{\Sigma} = \phi, \quad D_t u|_{\Sigma} = \psi, \]
has a unique solution \( u \in H_{s,\delta}(V_{0,\lambda}) \), and satisfies the estimates:
\[ \| u \|_{H_{s,\delta}(V_{0,\lambda})} \leq c \theta^\frac{1}{2} \left\{ \| \phi \|_{H_{s,\delta+\frac{1}{2}}(E)} + \| \psi \|_{H_{s-1,\delta+\frac{3}{2}}(E)} + \| \beta \|_{H_{s-1,\delta+2}(V_{0,\lambda})} \right\}, \]
where \( c \) is a continuous increasing function of \( (\theta,h,m) \), and \( h, m \) are defined by equations (3.13) (3.16) respectively.

**Proof.** It follows from the energy estimates Theorem 5.8 in Appendix 5.2 and similar approximation argument as in the proof of Theorem 5.1 in [8] and Theorem 4.1 in [10]. \( \square \)

Now we extend the existence theory for the boost problem in \( \text{H} \) to \( \Sigma \). Let \( \Omega_\theta \) be the boost region based on \( \Sigma \) as defined in (2.2). We will construct a solution to the reduced EVE (3.3) in \( \Omega_\theta \) under the harmonic gauge. We deal with the boost evolution separately on the interior region \( \Sigma_{int} \) and on each end \( E_i \). On compact set \( \Sigma_{int} \), we can use Theorem 3.6. On each end \( E \), we can complete the initial data \((g,k)|_E \) to \( \mathbb{R}^3 \) and apply the boost theory in \( \text{H} \) to get existence. Then we can cut off the solution in the causal set based on the end \( E \) by our linear Theorem 3.6. Causal uniqueness (see Corollary 4.8 of Appendix 3 in [10]) tells us that the solutions we got based on \( \Sigma_{int} \) and \( E_i \)’s match together to a global solution.

**Theorem 3.7.** For \( s \geq 4, \delta > -2 \). Given vacuum data \((g,k) \in \mathcal{V}C_{s,\delta+\frac{1}{2}}(\Sigma) \), with \( g \geq \lambda_0 e \) for some \( \lambda_0 > 0 \), there exits a \( \theta \in (0,1) \) and a \( C_0 > 0 \) depending only on \( \lambda_0 \), \( \| g - e \|_{H_{s,\delta+\frac{1}{2}}(\Sigma)} + \| k \|_{H_{s-1,\delta+\frac{3}{2}}(\Sigma)} \), and a unique Lorentzian metric \( \gamma \) solving the reduced EVE (3.3) on \( \Omega_\theta \), which has Cauchy data \((\phi,\psi) \) on \( \Sigma \) given by \((g,k) \) in (3.5), such that \( (\gamma - \tilde{\eta}) \in H_{s,\delta}(\Omega_\theta) \), and \( \| \gamma - \tilde{\eta} \|_{H_{s,\delta}(\Omega_\theta)} \leq C_0 \). Furthermore \( \gamma \) is the solution to EVE (1.3) under harmonic gauge.
Proof. We first focus on a fixed end \( E \). In fact, we can extend \( (g,k)|_E \) to \( (\bar{g},\bar{k}) \) on \( \mathbb{R}^3 \) by a cut and paste method, such that \( (\bar{g},\bar{k}) = (g,k) \) on \( E \) with \( \bar{g} \geq \lambda \delta \), where \( \lambda \geq c^{-1} \lambda_0 \) and 
\[
\|\bar{g} - \delta\|_{H^{s,\delta+\frac{3}{2}}(\mathbb{R}^3)} + \|\bar{k}\|_{H^{s,\delta+\frac{3}{4}}(\mathbb{R}^3)} \leq c\|g - e\|_{H^{s,\delta+\frac{3}{2}}(E)} + \|k\|_{H^{s,\delta+\frac{3}{2}}(E)}
\] for some fixed \( c > 1 \).

By Lemma 5.1 and Theorem 6.1 in \([10]\), there exist \( C_1 > 0 \) and \( \theta_1 \in (0,1) \) depending only on \( \lambda \) and 
\[
\|\bar{g} - \delta\|_{H^{s,\delta+\frac{3}{2}}(\mathbb{R}^3)} + \|\bar{k}\|_{H^{s,\delta+\frac{3}{4}}(\mathbb{R}^3)}
\] and a unique solution \( \bar{\gamma} \) to the reduce EVE \((3.3)\) on \( \Omega_{\theta_1} \), taking on \( \mathbb{R}^3 \) the Cauchy data \((\bar{\phi},\bar{\psi})\) given by \((\bar{g},\bar{k})\) as in \((3.5)\) \((3.6)\) where the Christoffel symbol for \( \mathbb{R}^3 \) is \( \bar{\Gamma}|_{\mathbb{R}^3} = 0 \), and \( \|\bar{\gamma} - \eta\|_{H^{s,\delta}(\Omega_{\theta_1})} \leq C_1 \). Here \( \Omega_{\theta_1} \) is the boost region \((2.2)\) when \( \Sigma = \mathbb{R}^3 \). Furthermore, \( \bar{\gamma} \) is regularly hyperbolic \((3.2)\), with the coefficient of regularly hyperbolicity \( h_1 \) depending only on \( \lambda \) and 
\[
\|\bar{g} - \delta\|_{H^{s,\delta+\frac{3}{2}}(\mathbb{R}^3)} + \|\bar{k}\|_{H^{s,\delta+\frac{3}{4}}(\mathbb{R}^3)}
\] is regularly hyperbolic in \( \Omega_{\theta_1} \) (See Definition 4.1 in \([10]\)). Condition (4) is true if we take \( \lambda_1 \) large enough depending only on the regularly hyperbolicity \( h_1 \) of \( \bar{\gamma} \) as discussed in Remark \(3.3\).

Then we claim that \( \bar{\gamma} \) is a solution of \((\text{EVE})\) \((1.3)\) in harmonic gauge inside the causal set \( \Omega_{\theta_1}\lambda_1 \). In fact, since \( (g,k) \) is a solution of \((\text{VCE})\) \((1.1)\) on \( E \), the harmonic gauge condition 
\[
f^{\mu} = \Gamma^{\mu}_{\gamma} = 0 \quad \text{and} \quad \partial_{\gamma} f^{\mu} = 0 \quad \text{on} \quad E
\] are satisfied by the choice of initial conditions \((3.5)\) \((3.6)\). Notice that \( f \) satisfies a linear equation \((3.8)\), which satisfies the requirement of Theorem \(3.6\) by argument on page 293 in \([10]\). Hence the harmonic gauge vector \( f = 0 \) on \( \Omega_{\theta_1}\lambda_1 \) by the estimate \((3.19)\) in Theorem \(3.6\) hence \( \bar{\gamma} \) is a solution of \( \text{EVE} \) \((1.3)\) on \( \Omega_{\theta_1}\lambda_1 \).

Now denote the restriction \( \bar{\gamma} \) to \( \Omega_{\theta_1}\lambda_1 \) by \( \gamma \). We claim that \((g,k)|_E \) is uniquely determined by \((g,k)|_E \) when \( \gamma \) is regularly hyperbolic on \( \Omega_{\theta_1}\lambda_1 \). Suppose \( \gamma_1 \) and \( \gamma_2 \) are two such solutions of reduced EVE \((3.3)\) as above with initial value given by \((3.5)\) \((3.6)\) from vacuum data \((g_1,k_1)\) and \((g_2,k_2)\) respectively. Then 
\[
\|\gamma_i - \eta\|_{H^{s,\delta}(\Omega)} \quad \text{are uniformly bounded by the corresponding norm of } (\eta_i - \eta, k_i).
\] Now subtract the reduced EVE \((3.3)\) satisfied by \( \gamma_1 \) and \( \gamma_2 \):

\[
\gamma_1^{\alpha\beta} D_\alpha D_\beta (\gamma_1^{\mu\nu} - \gamma_2^{\mu\nu}) - (D^2 \gamma_2) (\gamma_2 - \gamma_1) - (B(\gamma_1,D\gamma_1) - B(\gamma_2,D\gamma_2)) = 0,
\] (3.20)

where (see equations \((4.4)(4.5)\) in \([10]\))

\[
B(\gamma_1,D\gamma_1) - B(\gamma_2,D\gamma_2) = P(\gamma_1)(D\gamma_1)^2 - P(\gamma_2)(D\gamma_2)^2
\]

\[
= (P(\gamma_1) - P(\gamma_2))(D\gamma_1)^2 + P(\gamma_2)(D\gamma_1 + D\gamma_2)(D\gamma_1 - D\gamma_2).
\]

Here \( P \) is a rational function of \( \gamma \). Using the multiplication lemma \((2.2)\) \((D\gamma_1)^2\), \( P(\gamma_2)(D\gamma_1 + D\gamma_2) \in H^{s-1,\delta+1}(\Omega) \). Using the mean value inequality, and the Sobolev embedding lemma \((2.2)\) we have the pointwise estimates:

\[
|P(\gamma_1) - P(\gamma_2)| \leq C|\gamma_1 - \gamma_2|,
\]
where \( C \) depends only on \( \| \gamma_i - \eta \|_{H_{s,\delta}(V)} \), \( i = 1, 2 \). Now viewing equation (3.20) as a differential equation for \( (\gamma_1 - \gamma_2) \), and using the first energy estimate Lemma 5.4 in Appendix 5, we have

\[
\| \gamma_1 - \gamma_2 \|_{H_{s,\delta}(V)} \leq C \| \gamma_1 - \gamma_2 \|_{H_{s,\delta}(V)} \leq C(\| g_1 - g_2 \|_{H_{s,\delta}(E,V)} + \| k_1 - k_2 \|_{H_{s,\delta}(E,V)}).
\]

Hence the uniqueness is true.

Combing all the above, we get a unique regularly hyperbolic solution \( \gamma \) to the (EVE) under harmonic gauge on \( V_{\theta_1,\lambda_1} \), where \( \theta_1, \lambda_1 \) and \( \| \gamma - \eta \|_{H_{s,\delta}(V)} \) depend only on \( \lambda_0 \) and \( \| g - e \|_{H_{s,\delta}(E,V)} \). \( \| k \|_{H_{s,\delta}(E,V)} \).

Now extend \( \Sigma_{int} \) to include annuli \( B_r \setminus B_R \subset E_i \), and take the solution \( \gamma \) inside the causal set \( (V_{int})_{\theta_0,\lambda_0} \) based on \( \Sigma_{int} \) by Theorem 3.7. We can combine it with all the solutions \( (V_{\theta_1,\lambda_1}, \gamma) \) on each end \( E_i \). Now causal uniqueness (See Corollary 4.8 of Appendix 3 in [6]) implies that they coincide in the intersection of \( (V_{int})_{\theta_0,\lambda_0} \) and \( (V_{\theta_1,\lambda_1}) \) is a causal set based on \( \Sigma_{int} \cap E \) w.r.t. \( \gamma \) by our construction. So by choosing the smallest \( \theta \), such that \( \Omega_\theta \subset (V_{int})_{\theta_0,\lambda_0} \cup \bigcup_{i=1}^l V_{\theta_1,\lambda_1} \), we get the conclusion. \ \( \square \)

### 4 Perturbation method

Here we will apply the Inverse Function Theorem (See [5] [20]) to get maximal graphs in the spacetime evolution of given AF vacuum data sets with small \( t \) trace. Fix a 3-manifold \( (\Sigma, e) \) which is Euclidean at infinity. We always assume \( s \in \mathbb{N}, s \geq 4, \) and \( \delta > -2 \). Consider the vacuum data sets \( (\Sigma, g, k) \), with \( (g, k) \in VC_{s+1,\delta+\frac{4}{3}}(\Sigma) \). Let \( (V, \gamma) \) be the boost evolution of \( (g, k) \) given by Theorem 3.7, then we will study the graph of given function \( u \) in the spacetime \( (V, \gamma) \). We will take \( (g, k) \) as parameters, and study the perturbation problem for the mean curvature \( H_u \) of this graph. We will show that for appropriately chosen weighted Sobolev spaces, the linearization of \( H_u \) with respect to \( u \) is invertible in certain sense.

#### 4.1 Differentiability of mean curvature operator

Given a vacuum data set \( (g, k) \in VC_{s+1,\delta+\frac{4}{3}}(\Sigma) \), with \( g \geq \lambda e \) for some \( \lambda > 0 \). By Theorem 3.7 there exists a uniform \( \theta \in (0, 1) \) and a uniform \( C > 0 \), depending only on \( \lambda \) and \( \| g - e \|_{H_{s+1,\delta+\frac{4}{3}}(\Sigma)} + \| k \|_{H_{s,\delta+\frac{4}{3}}(\Sigma)} \), and a unique Lorentzian solution \( \gamma \) of the reduced EVE (3.3) on \( \Omega_\theta \), taking \( (g, k) \) as initial data, and \( \| \gamma - \eta \|_{H_{s+1,\delta}(\Omega_\theta)} \leq C \). Moreover, from the proof of Theorem 3.7, the regularly hyperbolic coefficient \( h \) of \( \gamma \) in each boost end \( V_{\theta_1,\lambda_1} \), and the regularly sliced coefficient of \( \gamma \) in \( (V_{int})_{\theta_0,\lambda_0} \) are all uniformly bounded by a constant depending only on \( \lambda \) and the norm of \( (g, k) \). Hence the determinant of \( \gamma^{\mu \nu} \) is bounded away from 0 by a constant depending only on \( \lambda \) and the norm of \( (g, k) \).

Now let us summarize some properties of metric components of \( \gamma \).

---

19 The bound for \( (\gamma - \eta) \) also comes directly by Theorem 3.6.

20 See the constant \( N, A \) and \( B \) in Definition 11.8 in page 397 in [5].
Lemma 4.1. For $s \geq 3$, $\delta > -2$. Given a $(3+1)$ Lorentz metric $\gamma^{\mu\nu}$ of form (2.4) in $\Omega_0$ with $(\gamma - \tilde{\gamma})^{\mu\nu} \in H_{s,\delta}(\Omega_0)$, if the determinant $\det(\gamma^{\mu\nu}) \leq -\tilde{\lambda}$ for some $\tilde{\lambda} > 0$, then $(\gamma - \tilde{\gamma})^{\mu\nu}$ lies in $H_{s,\delta}(\Omega_0)$, and in the metric form (2.5) (2.6) of $\gamma$, the components $(\alpha - 2 - 1), (\alpha - 1), \beta, \gamma^{ij} - e^{ij}, g_{ij} - e_{ij}$ all lie in $H_{s,\delta}(\Omega_0)$. Furthermore, their norms are all bounded by a constant depending only on $\tilde{\lambda}$ and $\|\gamma - \tilde{\gamma}\|_{H_{s,\delta}(\Omega_0)}$.

Proof. The inverse matrix $\gamma_{\mu\nu} = \det(\gamma^{\mu\nu}) \operatorname{adj}(\gamma^{\mu\nu})$, where $\operatorname{adj}(\gamma^{\mu\nu})$ is the adjoint matrix of $\gamma^{\mu\nu}$. Since $\det(\gamma^{\mu\nu})$ is bounded away from 0 by $\tilde{\lambda}$, the Banach algebra property (Lemma 2.6) of $H_{s,\delta}(\Omega_0)$ implies that $\gamma_{\mu\nu} - \tilde{\gamma}_{\mu\nu}$ also lies in $H_{s,\delta}(\Omega_0)$, with $\|\gamma_{\mu\nu} - \tilde{\gamma}_{\mu\nu}\|_{H_{s,\delta}(\Omega_0)}$ bounded by a constant depending only on $\tilde{\lambda}$ and $\|\gamma - \tilde{\gamma}\|_{H_{s,\delta}(\Omega_0)}$. From the expression (2.5) (2.6) of $\gamma$ and the fact that $(\gamma - \tilde{\gamma})^{\mu\nu}, (\gamma - \tilde{\gamma})_{\mu\nu} \in H_{s,\delta}(\Omega_0)$, we know that $(\alpha^2 - 1)$, $(1 - \alpha^{-1})$, $\beta$, $\gamma^{ij}$, $g_{ij} - e_{ij}$, $(g^{ij} - \frac{\beta_i \beta_j}{\alpha^2} - e^{ij}) \in H_{s,\delta}(\Omega_0)$ with their norms bounded by $\|\gamma - \tilde{\gamma}\|_{H_{s,\delta}(\Omega_0)}$. So $\alpha^2$ is bounded both from below and above by certain constant. By Taylor’s expansion $|\alpha - 1| = |\sqrt{1 + (\alpha^2 - 1) - 1}| \leq C|\alpha^2 - 1|$, hence is $L_0^2$ integrable. For higher order derivatives of $(\alpha - 1)$, we can use the multiplication Lemma 2.6 and the bound of $(\alpha^2 - 1)$ to show that $D^{\mu}(\alpha - 1)$ lies in $L_{s|\mu|,\delta + |\mu|}(\Omega_0)$. So $(\alpha - 1)$ lies in $H_{s,\delta}(\Omega_0)$ and has the norm bounded by a constant depending only on $\tilde{\lambda}$ and $\|\gamma - \tilde{\gamma}\|_{H_{s,\delta}(\Omega_0)}$. \qed

So the metric coefficients of out boost solution $\gamma$ satisfy that $\{(\alpha - 1), \beta, \gamma^{ij} - e^{ij}, g_{ij} - e_{ij}\} \in H_{s+1,\delta}(\Omega_0)$ with norms bounded by a constant depending only on the elliptic constant $\lambda$ of $g$ and $\|g - e\|_{H_{s+1,\delta}}(\Sigma) + \|k\|_{H_{s,\delta}}(\Sigma)$, where $\tilde{\lambda}$ is the mean curvature of $\gamma$. By the Soblev embedding $H_{s+1,\delta}(\Omega_0) \subset C^2(\Sigma)$ for some $0 < \kappa < \delta + 2$, all the terms above are uniformly bounded.

Given $s_1 \geq 3$ and $\delta_1 > -2$. Let $B_\rho$ be a ball of radius $\rho$ containing scalar functions in $H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma)$ with $\|u\|_{H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma)} \leq \rho$. We can choose $\rho$ small enough, such that after embedding $\|u\|_{C^2(\Sigma)} \leq C\rho \leq \theta/2$ for some $-1 < \kappa < \delta_1 + 1$, and,

Condition (A): $\|u(x)\|_{L^\infty} \leq (\theta/2)(\sigma(x))^{-\kappa} < (\theta/2)\sigma(x)$.

(4.1)

So $\text{Graph}_u = \{(x, u(x)) : x \in \Sigma\}$ is a submanifold in $\Omega_0$. Furthermore, $|Du|_e \leq C\rho(\sigma(x))^{-(\kappa+1)}$. As $(\alpha - 1), \beta, (g - e)$ are all uniformly bounded, we can then choose $\rho$ small enough satisfying:

Condition (B): $|Du|_e \leq \frac{1}{100}, |\langle \beta, Du \rangle_g| \leq \frac{1}{2}, |U| = \frac{\alpha |Du|_g}{1 + \langle \beta, Du \rangle_g} \leq \frac{1}{2}$.

(4.2)

where $U$ is defined in (2.12). Then $\text{Graph}_u$ is spacelike and $\nu = \sqrt{1 - |U|^2}$ is well-defined. So we can study the operator

$H : u \rightarrow H_u,$

(4.3)

where $H_u$ is the mean curvature of $\text{Graph}_u$ given by (2.10).

Now we will show that composition is continuous as follows,

Lemma 4.2. Given $s_1 \geq 3$, $\delta_1 > -2$ and $\theta \in (0, 1)$. Consider $B_\rho \subset H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma)$ with $\rho$ small enough satisfying Conditions (A) as above for the $\theta$. Then the composition map:

$(f, u) \rightarrow \tilde{f} = f(x, u(x) + t)$,

(4.4)
is a continuous map \( H_{s',\delta'}(\Omega_\theta) \times \mathcal{B}_\rho \to H_{s',\delta'}(\Omega_{\theta/2}) \), for \( s' \leq s_1 + 1 \) and \( \delta' \in \mathbb{R} \). Furthermore, when restricted to \( \text{Graph}_u \),

\[
(f, u) \to f(x, u(x))
\]

is a continuous map \( H_{s',\delta'}(\Omega_\theta) \times \mathcal{B}_\rho \to H_{s'-1,\delta'+\frac{1}{2}}(\Sigma, \Omega_{\theta/2}) \).

**Proof.** Condition(A) \( 4.1 \) implies that \( |u(x)| \leq (\theta_0/2)\sigma(x)-\kappa \) for some \(-1 < \kappa < \delta_1 + 1\), so we can consider a well-defined map \( F : \Omega_\theta \to \Omega_{\frac{\theta}{2}} \), where \( F : (x,t) \to (x,u(x)+t) \). Then \( |DF| = 1 \), so \( F \) is a diffeomorphism \( \Omega_\theta \to F(\Omega_\theta) \). Furthermore, \((F-\text{id})(x,t)=(0,u(x)) \in H_{s_1+1,\delta_1} (\Omega_\theta) \). Now we can apply lemma \( 2.7 \) to the mapping \( F \), so \( f \to \tilde{f} = f \circ F \) is an isomorphism \( H_{s',\delta'}(\Omega_\theta) \to H_{s',\delta'}(F(\Omega_\theta)) \). In fact, by the bound of \( u \), we know that \( F(\Omega_\theta) \) contains \( \Omega_{\theta/2} \), so clearly \( \tilde{f} \) lies in \( H_{s',\delta'}(\Omega_{\theta/2}) \), and we have the continuity for the first factor \( f \). For the second factor \( u \), we only need to show that \( u \to f(x,u(x)+t) \) is continuous \( H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma) \to L^2_{\delta'}(\Omega_{\theta/2}) \) for fixed \( f \in L^2_{\delta'}(\Omega_{\theta/2}) \). Using multiplication lemma \( 2.6 \) recursively to higher derivatives as in the proof of Theorem 2.3 in \( 10 \) gives the continuity in \( H_{s',\delta'} \). Suppose \( u_n \to u \) in \( H_{s_1+1,\delta_1-\frac{1}{2}} \), hence \( u_n \to u \) in \( C^0_\kappa \) for some \(-1 < \kappa < \delta_1 + 1\). To show the \( L^2_{\delta'} \) continuity, we can approximate \( f \) by compactly supported smooth function \( g \) in \( L^2_{\delta'} \), then \( \left| f(x,u_n(x)+t) - f(x,u(x)+t) \right| \leq \left| f(x,u_n(x)+t) - g(x,u_n(x)+t) \right| + \left| g(x,u_n(x)+t) - f(x,u(x)+t) \right| + \left| g(x,u_n(x)+t) - g(x,u(x)+t) \right| \). The first and second terms can be chosen very small in \( L^2_{\delta'} \), and the third one converge to 0 in \( L^2_{\delta'} \). So we get the continuity. For the restriction, we can directly apply the restriction lemma \( 2.8 \) to \( \tilde{f} \). \( \square \)

Moreover, we also have the differentiability w.r.t. \( u \).

**Lemma 4.3.** Given \( s_1 \geq 3 \), \( \delta_1 > -2 \), \( \theta \in (0,1) \), \( \delta' \in \mathbb{R} \) and \( f \in H_{s_1+1,\delta'}(\Omega_{\theta}) \). Consider \( \mathcal{B}_\rho \subset H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma) \) with \( \rho \) chosen to satisfy Condition(A) in \( \{4.1\} \) for the \( \theta \). Then

\[
\mathcal{F} : u \to f(x,u(x)),
\]

is continuous Fréchet differentiable as a map \( \mathcal{B}_\rho \to H_{s_1+1,\delta'+\frac{1}{2}}(\Sigma) \). Furthermore, the Fréchet derivative is given by formal derivatives,

\[
D_u \mathcal{F}(v) = \partial_t f(x,u(x)) \cdot v,
\]

where \( v \in H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma) \).

**Proof.** Using lemma \( 4.2 \) we know that \( f(x,u(x)+t) \) lies in \( H_{s_1+1,\delta'}(\Omega_{\theta/2}) \), and \( f(x,u(x)) \in H_{s_1,\delta'+\frac{1}{2}}(\Sigma, \Omega_{\theta/2}) \). Hence \( \partial_t f(x,t) \in H_{s_1,\delta'+1}(\Omega_{\theta/2}) \) and \( \partial_t f(x,u(x)) \in H_{s_1-1,\delta'+\frac{1}{2}}(\Sigma) \). To show that \( \mathcal{F} \) is Fréchet differentiable(See Definition 1.1.1 in \( 5 \)), we can first show Gateaux differentiable(See Definition 1.1.2 in \( 5 \)), i.e.

\[
\lim_{\tau \to 0} \frac{\| f(x,u(x)+\tau v(x)) - f(x,u(x)) - \partial_t f(x,u(x))(\tau v(x)) \|_{H_{s_1+1,\delta'+\frac{1}{2}}(\Sigma)}}{\tau \| v(x) \|_{H_{s_1+1,\delta_1-\frac{1}{2}}(\Sigma)}} = 0,
\]

(4.6)
for any \( v \in H_{s_1 + 1, \delta_1 - \frac{1}{2}}(\Sigma) \). Using Newton-Leibniz formula,

\[
f(x, u(x) + \tau v(x)) - f(x, u(x)) = \left( \int_{s=0}^{1} \partial_t f(x, u(x) + s\tau v(x))ds \right)(\tau v(x)),
\]

(4.7)

Using the multiplication lemma (2.3) in the case \( H_{s_1 - 1, \delta' + \frac{1}{2}}(\Sigma) \times H_{s_1 + 1, \delta_1 - \frac{1}{2}}(\Sigma) \to H_{s_1 - 1, \delta' + \frac{1}{2}}(\Sigma) \), we only need to show,

\[
\lim_{\tau \to 0} \| \partial_t f(x, u(x) + \tau v(x)) - \partial_t f(x, u(x)) \|_{H_{s_1 - 1, \delta' + \frac{1}{2}}(\Sigma)} = 0.
\]

This convergence follows from the continuity of \((\partial_t f, u) \to \partial_t f(x, u(x))\) as a map \( H_{s_1, \delta' + 1}(\Omega_\theta) \times H_{s_1 + 1, \delta_1 - \frac{1}{2}}(\Sigma) \) in Lemma 4.2. Now the multiplication operator \( L_u : v \to \partial_t f(x, u(x)) \cdot v \) is a bounded linear operator \( L(H_{s_1 + 1, \delta_1 - \frac{1}{2}}(\Sigma), H_{s_1 - 1, \delta' + \frac{1}{2}}(\Sigma)) \) with

\[
\| L_u \|_{L(H_{s_1 + 1, \delta_1 - \frac{1}{2}}(\Sigma), H_{s_1 - 1, \delta' + \frac{1}{2}}(\Sigma))} \leq C \| \partial_t f(x, u(x)) \|_{H_{s_1 - 1, \delta' + \frac{1}{2}}(\Sigma)}
\]

by inequality (2.3). The operator \( L_u \) is also continuous w.r.t \( u \) by Lemma 4.2, so we know that \( F \) is Fréchet differentiable by Theorem 1.1.3 in [5], and \( D_u F(v) = \partial_t f(x, u(x)) \cdot v \).

Now we can prove the differentiability of \( H_u \) w.r.t. \( u \).

**Proposition 4.4.** For \( s \geq 4, \delta > -2 \). Given a vacuum data \((g, k) \in VC_{s+1, \delta+ \frac{1}{2}}(\Sigma) \) and \( \theta \) the boost ratio as in the beginning of this section. If \( B_\rho \subset H_{s, \delta - \frac{1}{2}}(\Sigma) \) with \( \rho \) satisfying Conditions (A)/(B) as in (4.1) (4.2) for the \( \theta \), then the mean curvature operator \( (4.3) \ H : B_\rho \to H_{s-2, \delta+ \frac{3}{2}}(\Sigma) \) is continuous differentiable w.r.t. \( u \), i.e. \( D_u H \in C(B_\rho, L(H_{s, \delta - \frac{1}{2}}(\Sigma), H_{s-2, \delta+ \frac{3}{2}}(\Sigma))) \).

Furthermore, \( D_u H \) is given by the formal variational formula.

**Proof.** By the choice of \( \rho \), \( H \) is well-defined. Write out the expression for \( H_u \) in (2.16) in local coordinates \( \{(t, x^i) : i = 1, 2, 3\} \) of \( \Omega_\theta \) as follows:

\[
H_u = (g_{ij})^{ij} \left\{ \nabla_{\alpha_i} N, \alpha_j \right\}_\gamma = \nu \cdot (g_{ij})^{ij} \left\{ \nabla_{\alpha_i} u_i, \partial_i (U + T) \right\} - u_j \partial_i \gamma
\]

\[
= \nu \cdot (g_{ij})^{ij} \left\{ \left( \partial_i + u_i \partial_i \right) \left( U + T \right) \partial_j + u_j \partial_j \partial_i \right\} - (U + T) \nabla \left( \partial_i + u_i \partial_i \partial_j + u_j \partial_j \partial_i \right) \gamma
\]

\[
+ (U + T) \Gamma_{ij} + U_1 \Gamma_{ij} + u_i \Gamma_{ij} + u_j \Gamma_{ij} \gamma
\]

(4.8)

where \( \Gamma_{\mu\nu,\sigma} \) is the Christoffel symbol for \( \gamma \), and all coefficients of \( \gamma \) are evaluated at \( (x, u(x)) \). Except for the term \( \nu \), \( H_u \) is an algebraic expression containing two type of terms in (4.8). One type of terms are the composition of the coefficients of \( (\gamma - \tilde{\eta}) \) and \( \partial \gamma \) with \( (x, u(x)) \), and the other terms contains \( \partial u \) and \( \partial^2 u \). The only term appears in the denominator is 1 + \( \langle \beta, Du \rangle \), and \( \langle \beta, Du \rangle \) \( \leq \frac{1}{4} \) by the choice of \( \rho \) as in Condition(B).

Since \( (\gamma - \tilde{\eta}) \in H_{s+1, \delta}(\Omega_\theta) \), the composition of the metric coefficients of \( (\gamma - \tilde{\eta}) \) with \( (x, u(x)) \), i.e. \( \{ (\gamma^{\mu\nu} - \tilde{\eta}^{\mu\nu}), (\gamma_{\mu\nu} - \tilde{\eta}_{\mu\nu}), (\alpha - 1), \beta^i, \beta_i, (g^{ij} - e^{ij}), (g_{ij} - e_{ij}) \} \) is
continuous differentiable w.r.t. $u$ as maps $H_{s,\delta-\frac{1}{2}}(\Sigma) \rightarrow H_{s-2,\delta+\frac{3}{2}}(\Sigma)$ by lemma 4.3. Similarly the composition of the coefficients of $\partial^\gamma$ with $(x,u(x))$, i.e. $(\partial^\gamma)(x,u(x))$ are also continuous differentiable w.r.t. $u$ as maps $H_{s,\delta-\frac{1}{2}}(\Sigma) \rightarrow H_{s-2,\delta+\frac{3}{2}}(\Sigma)$ and $L_2 u$ w.r.t. $u$. First we have, 

Lemma 4.5. Let $\Sigma = \cup_{i=1}^L E_i$. Given a function $\chi \in C_0^\infty(\mathbb{R}^3 \setminus B_1)$, such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $\mathbb{R}^3 \setminus B_2$. We can find a partition of unity $\{\chi_i,r\}_{i=0}^l$ of $\Sigma$ for $r > R$, with $\chi_i,r(x) = \chi(|x|/r)$ for $x \in E_i \cong \mathbb{R}^3 \setminus B_R$, and $\chi_i,r(x) = 0$ for $x \in \Sigma \setminus E_i$, and $\chi_0,r(x) = 1 - \sum_{i=1}^l \chi_i,r(x)$. Then $u = \sum_{i=1}^l u_i,r$, with $u_i,r = \chi_i,r u$. Let us fix an end $E_i$ and $u_i,r$ and forget the sub-index $i$ now.

4.2 Linear theory

Given a 3-dimensional manifold $(\Sigma,e)$ which is Euclidean at infinity. Let us give some results about linear elliptic operators which are asymptotic to the Laplacian $\Delta_e$ on $(\Sigma,e)$. Such type of elliptic operators have been widely studied in [2][7][10][19].

Let $L$ be an operator on $(\Sigma,e)$ of form:

$$Lu = \sum_{k=0}^2 a_k \partial^k u,$$

with $u$ and $Lu$ functions on $\Sigma$, satisfying:

$$\lambda e \leq a_2 \leq \lambda^{-1} e \text{ as metrics, with } \lambda \text{ the elliptic coefficient;}$$

$$(a_2 - e) \in H_{s_0+1,\delta_0}(\Sigma), \quad a_1 \in H_{s_0,\delta_0+1}(\Sigma), \quad a_0 \in H_{s_0-1,\delta_0+2}(\Sigma),$$

where $s_0 \geq 4$, $\delta_0 > -\frac{3}{2}$. We will show that in certain weighted spaces, such $L$ have uniformly bounded inverse on the orthogonal compliment of $\text{ker}(L)$ depending only on the norms of the coefficients. First we have,

Lemma 4.5. Given $s \leq s_0$, $-\frac{3}{2} < \delta < -\frac{1}{2}$. There exists a constant $C$ and a large $r > R$, depending only on $s_0$, $\delta_0$, the elliptic coefficient $\lambda$ and the norms $\|a_2 - e\|_{H_{s_0,\delta_0}(\Sigma)}$, $\|a_1\|_{H_{s_0-1,\delta_0+1}(\Sigma)}$ and $\|a_0\|_{H_{s_0-2,\delta_0+2}(\Sigma)}$, such that for any $u \in H_{s,\delta-1}(\Sigma)$,

$$\|u\|_{H_{s,\delta-1}(\Sigma)} \leq C(\|Lu\|_{H_{s-2,\delta+1}(\Sigma)} + \|u\|_{H_{s-2}(\Sigma_{\text{int},2r})}),$$

where $\Sigma_{\text{int},2r}$ is the union of $\Sigma_{\text{int}}$ with all the annuli $B_{2r} \setminus B_R$ inside each end $\Sigma_i$, and $H_{s-2}$ is the standard $L^2$ Sobolev space on $\Sigma_{\text{int},2r}$.

Proof. Let $\Sigma = \cup_{i=1}^l E_i$. Given a function $\chi \in C_0^\infty(\mathbb{R}^3 \setminus B_1)$, such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $\mathbb{R}^3 \setminus B_2$. We can find a partition of unity $\{\chi_i,r\}_{i=0}^l$ of $\Sigma$ for $r > R$, with $\chi_i,r(x) = \chi(|x|/r)$ for $x \in E_i \cong \mathbb{R}^3 \setminus B_R$, and $\chi_i,r(x) = 0$ for $x \in \Sigma \setminus E_i$, and $\chi_0,r(x) = 1 - \sum_{i=1}^l \chi_i,r(x)$. Then $u = \sum_{i=1}^l u_i,r$, with $u_i,r = \chi_i,r u$. Let us fix an end $E_i$ and $u_i,r$ and forget the sub-index $i$ now.
Since $-\frac{3}{2} < \delta < -\frac{1}{2}$ corresponds to non-exceptional value in [2], we can apply Theorem 1.7 in [2] with $p = 2$ here,

$$\|u_r\|_{H_{s,\delta-1}(\mathbb{R}^3)} \leq C_1 \|\Delta u_r\|_{H_{s-2,\delta+1}(\mathbb{R}^3)}$$

$$\leq C_1 \left\{ \|Lu_r\|_{H_{s-2,\delta+1}(E)} + \|(L - \Delta)u_r\|_{H_{s-2,\delta+1}(E)} \right\},$$

(4.11)

where $\Delta$ is the laplacian operator w.r.t. $\delta_{ij}$ and $C_1$ a uniform constant.

$$\|Lu_r\|_{H_{s-2,\delta+1}(E)} \leq \|\chi_1 Lu\|_{H_{s-2,\delta+1}(E)} + \|2a_{ij}\delta_i u\delta_j \chi_r + (a_{ij}^2 \partial^2 \chi_r + a_{ij}^1 \partial \chi_r) u\|_{H_{s-2,\delta+1}(E)}$$

$$\leq C_2(r) (\|Lu\|_{H_{s-2,\delta+1}(E)} + \|u\|_{H_{s-1}(A_r)}),$$

(4.12)

with $A_r = B_{2r} \setminus B_r$, and $C_2(r)$ is a constant depending only on $r$ and $\|a_2 - \epsilon\|_{H^s_{\delta_0}(A_r)}$, $\|a_1\|_{H^s_{\delta_0-\delta_0+1}(A_r)}$. Since $\delta_0 > -\frac{1}{2}$, we can assume $\delta_0 - \epsilon = \delta_1 > -\frac{1}{2}$ for some $\epsilon > 0$. Using multiplication lemma [2,5],

$$\|(L - \Delta)u_r\|_{H_{s-2,\delta+1}(E)} = \|(a_{ij}^1 - \delta_{ij})^2 u_r + a_{ij}^1 \partial u_r + a_0 u_r\|_{H_{s-2,\delta+1}(E)}$$

$$\leq C_3 \left\{ \|a_2 - \epsilon\|_{H_{s,\delta_1}(E_r)} + \|a_1\|_{H_{s-1,\delta_1+1}(E_r)} + \|a_0\|_{H_{s-2,\delta_1+2}(E_r)} \right\} \|u_r\|_{H_{s,\delta-1}(E)},$$

(4.13)

where $E_r = \mathbb{R}^3 \setminus B_r$ and $C_3$ a uniform constant. Now $\|a_2 - \epsilon\|_{H_{s,\delta_1}(E_r)} + \|a_1\|_{H_{s-1,\delta_1+1}(E_r)} + \|a_0\|_{H_{s-2,\delta_1+2}(E_r)} \leq \left( \|a_2 - \epsilon\|_{H_{s,\delta_0}(E_r)} + \|a_1\|_{H_{s-1,\delta_0+1}(E_r)} + \|a_0\|_{H_{s-2,\delta_0+2}(E_r)} \right) \gamma^{-\epsilon}$ for $r$ large enough. So we can always choose a $r > R$, depending only on $\delta_0$ and $\gamma^{-\epsilon}$, such that $\|(L - \Delta)u_r\|_{H_{s-2,\delta+1}(E)} \leq \frac{1}{2C_1} \|u_r\|_{H_{s,\delta-1}(E)}$. Putting them back to inequality (4.11),

$$\|u_r\|_{H_{s,\delta-1}} \leq C_4 \left\{ \|Lu\|_{H_{s-2,\delta+1}(E)} + \|u\|_{H_{s-1}(A_r)} \right\},$$

(4.14)

where $C_4$ depends only on $C_2(r)$. Using an interpolation inequality (see Lemma 2.2 in [7]) to $\|u\|_{H_{s-1}(A_r)}$, we can get the estimate of (4.10) on each end. Applying the standard $L^2$ estimates to $u_{0,r}$ on $\Sigma_{int,2r}$ (See Corollary 2.2 on page 547 in [9]),

$$\|u_{0,r}\|_{H_{s}(\Sigma_{int,2r})} \leq C_5 \left\{ \|Lu_{0,r}\|_{H_{s-2}(\Sigma_{int,2r})} + \|u_{0,r}\|_{H_{s-2}(\Sigma_{int,2r})} \right\},$$

(4.15)

where $C_5$ depends only on $s_0$, the elliptic coefficient $\lambda$ and the norms $\|a_2 - \epsilon\|_{H_{s}(V_{int,2r})}$, $\|a_1\|_{H_{s-1}(V_{int,2r})}$ and $\|a_0\|_{H_{s-2}(V_{int,2r})}$. Combing results on all ends $E_i$ and $V_{int,2r}$ together, we can get (4.10) with $r$ and constant $C$ satisfying the requirement.

Now we can prove a lemma similar to Theorem 1.10 in [2] and Theorem 5.6 in [19].

**Lemma 4.6.** Given $s \leq s_0$, $-\frac{3}{2} < \delta < -\frac{1}{2}$, the operator $L$ is a Fredholm operator:

$$H_{s,\delta-1}(\Sigma) \rightarrow H_{s-2,\delta+1}(\Sigma),$$

i.e. $L$ has finite-dimensional kernel $\ker(L, \delta - 1) = \{v \in H_{s,\delta-1}(\Sigma) : Lv = 0\}$, and finite-dimensional co-kernel $\text{coker}(L, \delta - 1)$. 

Proof. From the multiplication lemma\[2.5\] we know that \( L \) is a bounded linear map \( H_{s,\delta-1}(\Sigma) \to H_{s-2,\delta+1}(\Sigma) \). Standard argument using inequality \[4.10\] as in Theorem 1.10 in \[2\] shows that \( N(L) \) is finite-dimensional and \( L \) has close range. So \( L \) is semi-Fredholm.

To show that \( L \) has finite-dimensional co-kernel, we will borrow the techniques in Theorem 5.6 of \[19\]. First, inequality \[4.13\] shows that the operator norm of \( (L - \triangle) : H_{s,\delta-1}(E_r) \to H_{s-2,\delta+1}(E_r) \) is \( o(1) \) as \( r \to \infty \). So for large enough \( r \), the fact that \( \triangle \) is Fredholm by Theorem 1.7 in \[2\] and that the Fredholm property is open w.r.t operator norms show that \( L_i = \triangle + \chi_{i,r}(L - \triangle) \) is Fredholm on \( \mathbb{R}^3 \), where clearly \( L_i = L \) on \( E_2 \). So there exists a bounded linear operator \( S_i : H_{s-2,\delta+1}(\mathbb{R}^3) \to H_{s,\delta-1}(\mathbb{R}^3) \), such that \( L_i S_i = \text{id} + K_i \) with \( K_i \) a compact operator. Now \( L : H_{s,\delta-1}(\Sigma_{\text{int.8r}}) \to H_{s-2,\delta+1}(\Sigma_{\text{int.8r}}) \) is Fredholm since \( \Sigma_{\text{int.8r}} \) is compact, so there exists a Fredholm inverse \( S_0 : H_{s-2,\delta+1}(\Sigma_{\text{int.8r}}) \to H_{s,\delta-1}(\Sigma_{\text{int.8r}}) \), such that \( LS_0 = \text{id} + K_0 \), for \( K_0 \) compact operator. Define

\[
Su = \chi_{0,4r} S_0 u_{0,8r} + \sum_{i=1}^{l} \chi_{i,2r} S_i u_{i,r},
\]

which is a bounded linear operator \( H_{s-2,\delta+1}(\Sigma) \to H_{s,\delta-1}(\Sigma) \). Then a calculation as in (5.6.5) of \[19\] shows that \( LS = \text{id} + K \) for \( K \) compact operator. So \( L \) has finite-dimensional co-kernel. \( \square \)

The Fredholm index of \( L \) is defined to be:

\[
i(L, \delta - 1) = \dim \ker(L, \delta - 1) - \dim \text{coker}(L, \delta - 1).
\]

By comparing the index of \( L \) to the laplacian \( \triangle_e \) of \( e \), we can show that \( L \) is surjective when \( a_0 \leq 0 \).

Lemma 4.7. Given \( s \leq s_0, -\frac{3}{2} < \delta < -\frac{1}{2} \), Suppose \( a_0 \leq 0 \), then \( L \) is surjective. Furthermore, \( \dim \ker(L, \delta - 1) = d_{\delta - 1} = \dim \ker(\triangle_e, \delta - 1) \). If we denote \( \ker(L, \delta - 1) \perp \) to be the orthogonal compliment of \( \ker(L, \delta - 1) \) w.r.t the \( L_{\delta - 1}^2 \) inner product \( \langle \cdot, \cdot \rangle_{L_{\delta - 1}^2(\Sigma)} \) as in definition \[2.5\] then:

\[
L : \ker(L, \delta - 1) \perp \to H_{s-2,\delta+1}(\Sigma),
\]

is an isomorphism.

Proof. Since \( L \) can be joint continuously to \( \triangle_e \) by \( L_t = tL + (1-t)\triangle_e \), we know \( i(L, \delta - 1) = i(\triangle_e, \delta - 1) \). Theorem 6.2 in \[19\] says that \( \triangle_e \) is surjective when \( \delta - 1 < -\frac{1}{2} \). In order to show \( L \) is surjective, or equivalently \( \dim \text{coker}(L, \delta - 1) = 0 \), we only need to show \( \dim \ker(L, \delta - 1) \leq \dim \ker(\triangle_e, \delta - 1) \). This comes from the asymptotical expansion given in \[2\]. For \( u \in \ker(L, \delta - 1) \), by Theorem 1.17 in \[2\], \( Lu = 0 \) implies that on each end \( E_i \), there exists a harmonic homogenous function \( h_k \) of order \( k \) such that \( h_k \) of order \( k \leq k(\delta) \), where \( k(\delta) = \max\{k \in \mathbb{Z} : k \leq -(\delta + \frac{3}{2})\} \). There exists a harmonic homogenous function \( h_k \) of order \( k \leq k(\delta) \), where \( k(\delta) = \max\{k \in \mathbb{Z} : k \leq -(\delta + \frac{3}{2})\} \). Such that \( u = h_k + o(t^{k-\beta}) \) for \( 0 < \beta < \delta + \frac{3}{2} \). In our case, \( k(\delta) = 0 \). In fact, if \( u \neq 0 \), there must exist at one end, on which \( k \geq 0 \). Or the decay implies \( u = o(1) \) at infinity.

\[21\]See the definition for \( k(\delta) \) in \[2\]. Their \( \delta \) is the same as \( -(\delta + \frac{3}{2}) \) here.
on $\Sigma$, so $u = 0$ by maximum principle since $a_0 \leq 0$. So $\dim \ker(L, \delta - 1)$ is less or equal to the number of linearly independent harmonic polynomials of order $\leq k(\delta)$ multiplied with the number of ends. It is easy to see that the basis of $\ker(\Delta, \delta - 1)$ is consisted just by functions which have main part the harmonic polynomial on one end, and $O(1/r)$ parts in other ends. So the leading terms shows that $\dim \ker(L, \delta - 1) \leq \dim \ker(\Delta, \delta - 1)$. The isomorphism on orthogonal compliment is direct when $L$ is surjective.

In fact, we can show a uniform norm bound for the inverse of $L$ on $\ker(L, \delta - 1)$.  

**Lemma 4.8.** Given $s \leq s_0$, $-\frac{3}{2} < \delta < -\frac{1}{2}$. Suppose $a_0 \leq 0$. Denote the inverse of $L : \ker(L, \delta - 1) \rightarrow H_{s-2,\delta+1}(\Sigma)$ by $L^{-1}$, then there exists a constant $C$ depending only on $s_0$, $\delta_0$, the elliptic coefficient $\lambda$ and the norms $\|a_2 - e\|_{H_{s_0+1,\delta_0}(\Sigma)}$, $\|a_1\|_{H_{s_0,\delta_0+1}(\Sigma)}$ and $\|a_0\|_{H_{s_0-1,\delta_0+2}(\Sigma)}$, such that for any $v \in H_{s-2,\delta+1}(\Sigma)$,

$$\|L^{-1}v\|_{H_{s,\delta-1}(\Sigma)} \leq C\|v\|_{H_{s-2,\delta+1}(\Sigma)}.$$  

(4.17)

**Proof.** We only need to show that for any $u \in \ker(L, \delta - 1)$,

$$\|u\|_{H_{s,\delta-1}(\Sigma)} \leq C \|Lu\|_{H_{s-2,\delta+1}(\Sigma)}$$

for a uniform constant $C_1$ depending only on $s_0$, $\delta_0$, the elliptic coefficient $\lambda$ and the norms $\|a_2 - e\|_{H_{s_0+1,\delta_0}(\Sigma)}$, $\|a_1\|_{H_{s_0,\delta_0+1}(\Sigma)}$ and $\|a_0\|_{H_{s_0-1,\delta_0+2}(\Sigma)}$. By contradiction argument, suppose that the statement is wrong, which means that there exists a sequence of operators $L_i$ with $a_i,0 \leq 0$, uniformly bounded elliptic coefficient $\lambda_i \geq \lambda_0 > 0$ and uniformly bounded coefficients $\|a_{i,2} - e\|_{H_{s_0+1,\delta_0}(\Sigma)}$, $\|a_{i,1}\|_{H_{s_0,\delta_0+1}(\Sigma)}$, $\|a_{i,0}\|_{H_{s_0-1,\delta_0+2}(\Sigma)} \leq C_0$, and a sequence of functions $u_i \in \ker(L_i, \delta - 1)$, such that $\|u_i\|_{H_{s,\delta-1}(\Sigma)} \geq i\|L_iu_i\|_{H_{s-2,\delta+1}(\Sigma)}$. By re-normalizing, we get a sequence of functions $u_i$, with $\|u_i\|_{H_{s,\delta-1}(\Sigma)} = 1$, while $\|L_iu_i\|_{H_{s-2,\delta+1}(\Sigma)} \rightarrow 0$. By weak compactness, there exists a subsequence, which we still denote by $L_i$, such that the coefficients of $L_i$ converges weakly to that of a linear operator $L_\infty$ with $\lambda_0 e \leq a_{\infty,2} \leq \lambda_0^{-1} e$, $a_{\infty,0} \leq 0$ and $\|a_{\infty,2} - e\|_{H_{s_0+1,\delta_0}(\Sigma)}$, $\|a_{\infty,1}\|_{H_{s_0,\delta_0+1}(\Sigma)}$, $\|a_{\infty,0}\|_{H_{s_0-1,\delta_0+2}(\Sigma)} \leq C_0$. Using inequality (4.10), there is a uniform constant $C_2$,

$$\|u_i - u_j\|_{H_{s,\delta-1}(\Sigma)} \leq C_2(\|L_i(u_i - u_j)\|_{H_{s-2,\delta+1}(\Sigma)} + \|u_i - u_j\|_{H_{s-2}(\Sigma_{int,2r})})$$

$$\leq C_2(\|L_iu_i\|_{H_{s-2,\delta+1}(\Sigma)} + \|L_ju_j\|_{H_{s-2,\delta+1}(\Sigma)} + \|u_i - u_j\|_{H_{s-2}(\Sigma_{int,2r})}).$$

(4.18)

Now $\|(L_i - L_j)u_j\|_{H_{s-2,\delta+1}(\Sigma)} \leq C(\|a_{2,i} - a_{2,j}\|_{H_{s_0,\delta'-1}(\Sigma)} + \|a_{1,i} - a_{1,j}\|_{H_{s_0-1,\delta'-1}(\Sigma)} + \|a_{0,i} - a_{0,j}\|_{H_{s_0-2,\delta'+1}(\Sigma)})$ for some $\delta_0 > \delta' > -\frac{3}{2}$ by multiplication lemma [236]. The compact embedding (Lemma 2.1 in [3]) of $\|H_{s_0+1-i,\delta_0+1}(\Sigma) \subset H_{s_0-i,\delta'-1+i}(\Sigma)$ for $i = 0, 1, 2$ imply that $\|(L_i - L_j)u_j\|_{H_{s-2,\delta+1}(\Sigma)} \rightarrow 0$ for a subsequence of $\{L_i\}$. Together with the compactness of $H_{s,\delta-1}(\Sigma) \subset H_{s-2}(\Sigma_{int,2r})$, there exists a subsequence, which we still denote by $u_i$, such that $u_i$ converge strongly in $H_{s,\delta-1}(\Sigma)$ to a function $u_\infty$, with $\|u_\infty\|_{H_{s,\delta-1}(\Sigma)} = 1$. Furthermore we
have that $L_\infty u_\infty = 0$ weakly by the weak convergence, and hence strongly in $H_{s-2,\delta+\frac{1}{2}}(\Sigma)$ by elliptic regularity.

By Lemma 4.7, we know that $\dim \ker(L_i, \delta - 1) = d_{\delta-1}$. We claim that $\ker(L_i, \delta - 1)$ converge to a $d_{\delta-1}$ dimensional linear subspace of $\ker(L_\infty, \delta - 1)$. Let $\{v_{i,a}\}_{a=1}^{d_{\delta-1}}$ be an $L^2_{\delta-1}$ orthogonal basis for $\ker(L_i, \delta - 1)$, with $\|v_{i,a}\|_{H_{s,\delta-1}(\Sigma)} = 1$. By equation (4.18),

$$
\|v_{i,a} - v_{j,a}\|_{H_{s,\delta-1}(\Sigma)} \leq C(\|(L_i - L_j)v_{j,a}\|_{H_{s-2,\delta+1}(\Sigma)} + \|v_{i,a} - v_{j,a}\|_{H_{s-2}(\Sigma_{int,2r})}).
$$

Similar argument as above implies that a subsequence of $v_{i,a}$ converge strongly in $H_{s,\delta-1}(\Sigma)$ to some $v_{\infty,a}$. Hence $v_{\infty,a} \in \ker(L_\infty, \delta - 1)$, and $\{v_{\infty,a}\}_{a=1}^{d_{\delta-1}}$ are also orthogonal in $L^2_{\delta-1}$ with $\|v_{\infty,a}\|_{H_{s,\delta-1}(\Sigma)} = 1$. Since $L_\infty$ satisfies all the requirement of Lemma 4.7, $\dim \ker(L_\infty, \delta - 1) = d_{\delta-1}$. Hence the limit of $\ker(L_i, \delta - 1)$ is exactly the entire $\ker(L_\infty, \delta - 1)$. As $u_i$ is perpendicular to $\ker(L_i, \delta - 1)$ in $L^2_{\delta-1}$, passing to the limit, we know that $u_\infty$ is perpendicular to $\ker(L_\infty, \delta - 1)$ in $L^2_{\delta-1}$ too, which is a contradiction to that $\|u_\infty\|_{H_{s,\delta-1}(\Sigma)} = 1$ and $L_\infty u_\infty = 0$. So we finish the proof.

4.3 Existence of maximal data

Now let us calculate the linearization of $H$ with respect to $u$ at $(g,k,0)$. Fix a vacuum data $(g,k) \in \mathcal{V}C_{s+2,\delta+\frac{1}{2}}(\Sigma)$ with the unique boost solution $(\mathcal{V}, \gamma)$ given by Theorem 3.7. Recall the form (2.5) of $\gamma$ in local coordinates $(x^i, t)$ of $\Omega_\delta$. According to the initial data equations (3.5), (3.6) for $\gamma$, the coefficients restricted to $t = 0$ slice are given by:

$$
\alpha|_{\Sigma} \equiv 1; \quad \beta|_{\Sigma} \equiv 0.
$$

(4.19)

In fact, our choice of $\alpha|_{\Sigma}$ and $\beta|_{\Sigma}$ implies that $\partial_t|_{\Sigma}$ is the unit normal of $\Sigma$. Now recall the second variational formula for the mean curvature in section 2 of [1]. Let $X$ be a vector field in a neighborhood of $\Sigma$ with associated flow $\phi_s : \mathcal{V} \to \mathcal{V}$. Denote $H(s)$ by the mean curvature of $\phi_s(\Sigma)$, then

$$
\partial_s(H(s))|_{s=0} = -\triangle_g \langle X, N \rangle + \langle X, N \rangle (|k|^2_g + \text{Ric}_\gamma(N,N)) + \langle X, \nabla_g H \rangle,
$$

(4.20)

where $N$ is the unit normal of $\Sigma$ inside $\mathcal{V}$, and $\text{Ric}_\gamma$ the Ricci curvature of $\gamma$. In our case, $\text{Ric}_\gamma \equiv 0$ by (1.3) since our $(\mathcal{V}, \gamma)$ is vacuum, and the unit normal $N = \partial_t$ on $\Sigma$. We can choose the vector field to be $X = v\partial_t$, where $v$ is a compactly supported smooth scalar function, so $\langle X, \nabla_g H \rangle = 0$. Then $\partial_s H(s)|_{s=0}$ is the linearization of $H$ w.r.t $u$, and $\langle X, N \rangle = -v$. Now combining all and using Proposition 4.4, we have,

**Lemma 4.9.** Using notations in Proposition 4.4, the Fréchet derivative of $H(g,k,u)$ with respect to factor $u$ at a vacuum data $(g,k,0)$ is a linear operator $L_0 : H_{s,\delta-\frac{1}{2}}(\Sigma) \to H_{s-2,\delta+\frac{1}{2}}(\Sigma)$ given by:

$$
(D_vH)(g,k,0) = L_0 v = (\triangle_g|k|^2_g) v.
$$

(4.21)
Now let us focus on the operator $L$. $L$ is in fact Fredholm and surjective by Lemma 4.6 and Lemma 4.7. By making use the fact that $L$ has finite-dimensional kernel and is surjective, we can get the existence of solutions of $H(g,k,u)=0$ for $(g,k)$ with small trace $tr_0$, by a perturbation method, but no uniqueness due to the existence of non-trivial kernel $ker(L, \delta - \frac{1}{2})$. We will give an existence and uniqueness theorem in the orthogonal compliment of the kernel in order to find symmetry preserving solutions in the following section. Let us first give a Quantitative Inverse Function Theorem motivated by (20).

**Theorem 4.10.** Let $X$, $Y$ be Banach spaces, and $U \subset X$ an open set. Suppose $F: U \to Y$ is a continuous map, and has Fréchet derivative w.r.t $x$, such that \[\frac{\partial F}{\partial x}(x)\] is continuous. For a point $x_0 \in U$, with $F(x_0) = y_0$. Suppose $\frac{\partial F}{\partial x}(x_0): X \to Y$ is invertible, and $\|\left[\frac{\partial F}{\partial x}(x_0)\right]^{-1}\| \leq C$. Assume that we can find $r_0 > 0$, such that for any $x \in B_{r_0}(0) \subset U$,

\[\left\|\frac{\partial F}{\partial x}(x) - \frac{\partial F}{\partial x}(x_0)\right\| \leq \frac{1}{2C}.\]  \hspace{1cm} (4.22)

Then for any $y \in Y$ with

\[|y - y_0|_Y < \frac{r_0}{2C},\]

there exist a unique $x \in B_{r_0}(0)$, such that $F(x) = y$.

**Proof.** Fix a $y \in B_{r_0/2C}(y_0) \subset Y$. Let us consider the map $T: B_{r_0}(0) \subset X \to Y$, defined by

\[T(x) = x - \left[\frac{\partial F}{\partial x}(x_0)\right]^{-1}(F(x_0 + x) - y).\]

$x$ is a fixed point if and only if $F(x_0 + x) = y$. So let us use the fixed point theorem to find a fixed point for $T$ on $B_{r_0}(0)$. first, for any $x_1, x_2 \in B_{r_0}(0)$,

\[|T(x_1) - T(x_2)|_X = |(x_1 - x_2) - \left[\frac{\partial F}{\partial x}(x_0)\right]^{-1}(F(x_0 + x_1) - F(x_0 + x_2))|_X \]

\[\leq \left\|\frac{\partial F}{\partial x}(x_0)\right\|^{-1} \cdot \left|\frac{\partial F}{\partial x}(x_0)(x_1 - x_2) - \frac{\partial F}{\partial x}(x_0 + \bar{x})(x_1 - x_2)\right|_Y \]

\[\leq C \left\|\frac{\partial F}{\partial x}(x_0) - \frac{\partial F}{\partial x}(x_0 + \bar{x})\right\| \cdot |x_1 - x_2|_X \]

\[\leq C \frac{1}{2C} |x_1 - x_2| \leq \frac{1}{2} |x_1 - x_2|,\]  \hspace{1cm} (4.23)

where we used the mean value theorem in the first “≤”, and condition (4.22) in the third “≤”. So $T$ is a contraction map on $B_{r_0}(0)$. Next, for any $x \in B_{r_0}(0)$, and $|y - y_0|_Y < \frac{r_0}{2C}$,

\[|T(x)|_X \leq \left\|\left[\frac{\partial F}{\partial x}(x_0)\right]^{-1}\right\| \cdot \left\|\frac{\partial F}{\partial x}(x_0)x - (F(x_0 + x) - F(x_0))\right\|_Y \]

\[\leq C \left\|\frac{\partial F}{\partial x}(x_0) - \frac{\partial F}{\partial x}(x_0 + \bar{x})\right\| |x|_X + |y - F(x_0)|_Y \]

\[\leq C \left\|\frac{\partial F}{\partial x}(x_0) - \frac{\partial F}{\partial x}(x_0 + \bar{x})\right\| \cdot |x|_X + |y - F(x_0)|_Y \leq C \left(\frac{1}{2C}r_0 + \frac{r_0}{2C}\right) < r_0,\]  \hspace{1cm} (4.24)
where we use condition (4.22) in the last “<”. So T maps \( B_{r_0}(0) \) to \( B_{r_0}(0) \). By applying the Contraction Mapping Theorem to \( T : B_{r_0}(0) \to B_{r_0}(0) \), we finish the proof.

**Remark 4.11.** This can be viewed as a carefully reworking of the proof of Theorem 1.2.1 in [5]. Theorem 3.1 and Theorem 3.2 in [20] also gave a proof about the quantitative inverse function theorem.

**Theorem 4.12.** For \( s \geq 4 \), \(-2 < \delta < -1\). Fix a 3-manifold \((\Sigma, \epsilon)\) which is Euclidean at infinity and a \( \lambda > 0 \). Given a vacuum data \((g, k) \in \mathcal{VC}_{s+2, \delta+\frac{1}{2}}(\Sigma)\), with \( g \geq \lambda \epsilon \), there is an \( \epsilon > 0 \) and a \( \rho' > 0 \) small enough, depending only on the norms \( \|g - e\|_{H^{s+2, \delta+\frac{1}{2}}(\Sigma)} + \|k\|_{H^{s+1, \delta+\frac{3}{2}}(\Sigma)} \) and the elliptic constant \( \lambda \), such that if \( \|tr_k\|_{H^{s-2, \delta+\frac{5}{2}}(\Sigma)} \leq \epsilon \), there exists a unique function \( u \in \text{ker}(L_0, \delta - \frac{1}{2})^\perp \) with \( \|u\|_{H^{s, \delta-\frac{1}{2}}(\Sigma)} \leq \rho' \), such that \( u \) is a solution of the maximal surface equation (1.9).

**Proof.** For the given \((g, k) \in \mathcal{VC}_{s+2, \delta+\frac{1}{2}}(\Sigma)\) with \( \theta \) the boost ratio, we can choose a \( \rho \) ball \( B_\rho \subset H^{s, \delta-\frac{1}{2}}(\Sigma) \), with \( \rho \) small enough depending only on \( \theta \), \( \|g - e\|_{H^{s+2, \delta+\frac{1}{2}}(\Sigma)} + \|k\|_{H^{s+1, \delta+\frac{3}{2}}(\Sigma)} \) and \( \lambda \) as in Proposition 4.9. Then the map \( \mathcal{H} \) is continuously differentiable w.r.t. \( u \) as a map \( B_\rho \cap \text{ker}(L_0, \delta - \frac{1}{2})^\perp \to H^{s-2, \delta+\frac{1}{2}}(\Sigma) \), and the Fréchet derivative is \((D_u \mathcal{H})(g, k, 0) = L_0v = (\Delta g - |k|^2) v \) by Lemma 4.9. The coefficient of \( L_0 \) satisfies the hypothesis (4.9), where \( s_0 = s + 1 \) and \( \delta_0 = \delta + \frac{1}{2} \) by multiplication lemma 2.5, the elliptic constant equals to \( \lambda \) and \( \|a_{0, 2} - e\|_{H^{s+2, \delta+\frac{1}{2}}(\Sigma)} \), \( \|a_{0, 1}\|_{H^{s+1, \delta+\frac{3}{2}}(\Sigma)} \), \( \|a_{0, 0}\|_{H^{s, \delta+\frac{3}{2}}(\Sigma)} \) are bounded from above by a constant depending only on \( \|g - e\|_{H^{s+2, \delta+\frac{1}{2}}(\Sigma)} \) and \( \|k\|_{H^{s+1, \delta+\frac{3}{2}}(\Sigma)} \). So \((D_u \mathcal{H})(g, k, 0)\) is an isomorphism \( \text{ker}(L_0, \delta - \frac{1}{2})^\perp \to H^{s-2, \delta+\frac{1}{2}}(\Sigma) \) by Lemma 4.7 since \( a_{0, 0} = -|k|^2 < 0 \). Now we will show that the conditions in the quantitative Inverse Function Theorem 4.10 are satisfied. By Lemma 4.8 there exists a constant \( C_0 \) depending only on \( \lambda \), \( \|a_{0, 2} - e\|_{H^{s+2, \delta+\frac{1}{2}}(\Sigma)} \), \( \|a_{0, 1}\|_{H^{s+1, \delta+\frac{3}{2}}(\Sigma)} \), \( \|a_{0, 0}\|_{H^{s, \delta+\frac{3}{2}}(\Sigma)} \), such that,

\[
\|L_0^{-1}\|_{L(H^{s-2, \delta+\frac{1}{2}}(\Sigma), \text{ker}(L_0, \delta - \frac{1}{2})^\perp)} \leq C_0.
\]

Abbreviate the operator norm \( \|\cdot\|_{L(H^{s, \delta+\frac{1}{2}}(\Sigma), H^{s-2, \delta+\frac{1}{2}}(\Sigma))} = \|\cdot\| \). Let us study \( \|D_u \mathcal{H}(g, k, u) - D_u \mathcal{H}(g, k, 0)\| \). Fix the boost evolution \((\Omega_\theta, \gamma)\) of \((g, k)\), with \( \|\gamma - \bar{\gamma}\|_{H^{s+2, \delta}(\Omega_\theta)} \) uniformly bounded by a constant depending only on \( \lambda \) and \( \|g - e\|_{H^{s+2, \delta+\frac{1}{2}}(\Sigma)} + \|k\|_{H^{s+1, \delta+\frac{3}{2}}(\Sigma)} \). Then \( D_u \mathcal{H}(g, k, u) \) is the first variation \( D_u(H_u) \) of \( H_u \) w.r.t. \( u \) inside \((\Omega_\theta, \gamma)\). From the formula of \( H_u \) in (4.8), we know that \( D_u(H_u) \) is a second order differential operator. The coefficients of \( D_u(H_u) \) are constituted by algebraic expressions of \( \partial u \), \( \partial^2 u \) and components of \( \gamma \), \( \partial \gamma \), \( \partial^2 \gamma \) evaluated at \((x, u(x))\). Let \( a \) be any component of \( \partial^2 \gamma \) (similar for \( \gamma \) and \( \partial \gamma \)), using the Newton-Leibniz formula,

\[
a(x, u(x)) - a(x, 0) = \left( \int_{\tau=0}^{1} \partial_\tau a(x, \tau u(x)) d\tau \right) u(x),
\]
where \( \partial_t a(x, u(x)) \) has uniform \( H^{s-2, \delta + \frac{1}{2}}(\Sigma) \) norm depending only on \( \| \partial^3 \gamma \|_{H^{s-1, \delta + 3}(\Omega_\theta)} \) and \( \rho \) by Lemma 4.12. So \( \| a(x, u(x)) - a(x, 0) \|_{H^{s-2, \delta + \frac{1}{2}}(\Sigma)} \leq C_3 \| u \|_{H^{s-1, \delta + 3}(\Sigma)} \) by multiplication lemma 2.6, where \( C_3 \) depends only on \( \| \gamma - \tilde{\gamma} \|_{H^{s+2, \delta}(\Omega_\theta)} \) and \( \rho \). Hence by comparing the coefficients of \( D_u \mathcal{H}(g, k, u) \) and \( D_u \mathcal{H}(g, k, 0) \), we can choose \( \| u \|_{H^{s-1, \delta + 3}(\Sigma)} \leq \rho' \) with \( \rho' \) small enough, depending only on \( \| \gamma - \tilde{\gamma} \|_{H^{s+2, \delta}(\Omega_\theta)} \) and \( C_0 \), such that,

\[
\| D_u \mathcal{H}(g, k, u) - D_u \mathcal{H}(g, k, 0) \| \leq \frac{1}{2C_0}.
\]

For the \( \rho' \) chosen above, if we take \( \epsilon < \frac{\rho'}{2C_0} \), then

\[
\| 0 - \mathcal{H}(g, k, 0) \|_{H^{s-2, \delta + \frac{1}{2}}(\Sigma)} = \| \text{tr} g k \|_{H^{s-2, \delta + \frac{1}{2}}(\Sigma)} \leq \frac{\rho'}{2C_0}.
\]

Now by the Quantitative Inverse Function Theorem 4.10 if we choose the \( \epsilon \) and \( \rho' \) as above, where \( \epsilon \) and \( \rho' \) depend only on \( \lambda \), \( \| g - \tilde{g} \|_{H^{s+2, \delta + \frac{1}{2}}(\Sigma)} \) and \( \| k \|_{H^{s+1, \delta + \frac{3}{2}}(\Sigma)} \), there exists a unique \( u \in B_{\rho'} \cap \ker (L_0, \delta - \frac{1}{2}) \), such that \( u \) solves \( \mathcal{H}(u) = 0 \). \( \square \)

### 4.4 Proof of the main Theorems

Here we will study the properties of the maximal graph gotten above. We will improve the regularity of the solution using a bootstrap argument, and show that the ADM mass of the maximal graph is the same as the given data. Moreover the maximal graph can be chosen to be axisymmetric if \((g, k)\) is axisymmetric, and the angular momentum of the maximal graph is the same as \((g, k)\).

In Theorem 4.12 the solution \( u \) has only \( s \) weak derivatives due to the contraction mapping principal. In fact, by exploring the structure of the mean curvature operator (4.3), we can gain more regularity for \( u \).

**Lemma 4.13.** (Regularity analysis). In Theorem 4.12, the solution \( u \in H^{s+2, \delta - \frac{1}{2}}(\Sigma) \). Denote \( M = \text{Graph}_u \), and let \( g_M \) be the metric and \( k_M \) the second fundamental form induced by \( M \subset (\mathcal{V}, \gamma) \), then \( (g_M, k_M) \in \mathbb{VC}_{s+1, \delta + \frac{1}{2}}(\Sigma) \).

**Proof.** In the local coordinates formula (4.8), we can collect together all the terms containing \( \partial^2_{ij} u \), then the maximal surface equation \( \mathcal{H}(u) = 0 \) can be rewritten as a linear second order elliptic equation for \( u \) with \( \partial u \) and \( u \) terms as coefficients:

\[
(g^M)^{ij}(x, u(x)) u_{ij} = f(x),
\]

where \( f(x) \) is a polynomial of \( g^M(x, u(x)), \partial u, \gamma(x, u(x)) \) and \( (\partial \gamma)(x, u(x)) \). First the spacelike property of \( M = \text{Graph}_u \) implies that \( (g^M)^{ij} \) is elliptic. Furthermore, \( (g^M)^{ij}(x, u(x)) - \gamma^{ij}(x) = \gamma^{ij} - \gamma^{ij} + \frac{\nu^2}{\alpha}(\beta^i - \alpha U^i)(\beta^j - \alpha U^j) \in H^{s-1, \delta + \frac{1}{2}}(\Sigma), f(x) \in H^{s-1, \delta + \frac{3}{2}}(\Sigma) \) by the argument in the proof of Proposition 4.4, Lemma 4.2 and the Banach algebra property in
Lemma 2.6 Since \((g^M)^i_j\) lie in \(C^0\) and \(H_{s-1}\) locally, \(u \in (H_{s+1})_{loc}(\Sigma)\) by standard elliptic regularity theory. Furthermore, the linear operator \(Lu = (g^M)^i_j \partial^2_{ij} u\) satisfies the hypothesis of the weighted elliptic regularity Theorem 6.1 in \([7]\) since \(s \geq 4\), hence \(u \in H_{s+1,\delta - \frac{1}{2}}(\Sigma)\) by Theorem 6.1 in \([7]\). Now we can bootstrap this process. In fact, by the composition Lemma 4.2, the right hand side \(f(x)\) lies in at most \(H_{s,\delta + \frac{1}{2}}(\Sigma)\) since there are \(\partial \gamma(x,u(x))\) terms. So bootstrap ends when \(u \in H_{s+2,\delta - \frac{1}{2}}(\Sigma)\).

On the graph \(M\), \((g_M)_{ij} = (g_{ij} + \beta_i u_j + \beta_j u_i - (\alpha^2 - \beta^2) u_i u_j)(x,u(x))\) by \((2.14)\),

\[
\begin{align*}
(k_M)_{ij} &= \nu \cdot \{ (\partial_i + u_i \partial_t)(U^\mu + T^\mu) \cdot (\gamma_{ij} + u_j \gamma_{i\mu}) \\
&\quad + (U^\mu + T^\mu)(\Gamma_{ij\mu} + u_i \Gamma_{\mu j} + u_j \Gamma_{\mu i} + u_i u_j \Gamma_{\mu \nu}) \},
\end{align*}
\]

by formula \((1.8)\). So by the proof of Proposition 4.3 \(((g_M)_{ij} - e_{ij}) \in H_{s+1,\delta + \frac{1}{2}}(\Sigma)\) and \((k_M)_{ij} \in H_{s,\delta + \frac{3}{2}}(\Sigma)\).

In order to define the ADM mass and linear momentum, we need to assume \(-\frac{3}{2} < \delta < -1\), then by the embedding lemma 2.5 \((g_M - e) \in C^{s-1}_\beta(\Sigma)\) and \(k_M \in C^{*\frac{3}{2}}_\beta(\Sigma)\) for some \(\frac{1}{2} < \beta < 1\), which satisfy the conditions \((1.5)\). Similar conditions are also satisfied by \((g - e, k)\). We can define the ADM mass \(m, m_M\) for \((g,k)\) and \((g_M,k_M)\) respectively.

**Lemma 4.14.** For \(-\frac{3}{2} < \delta < -1\), in Theorem 4.12, \(m = m_M\).

**Proof.** We will use the multiplication lemma 2.5 frequently when we multiply two Soblev functions. \((g_M)_{ij}(x) - g_{ij}(x,u(x)) = (\beta_i u_j + \beta_j u_i - (\alpha^2 - \beta^2) u_i u_j)(x,u(x))\) by \((2.14)\). Now \(\beta(x,u(x)) \in H_{s+1,\delta + \frac{1}{2}}(\Sigma)\) and \(\partial u \in H_{s+1,\delta + \frac{1}{2}}(\Sigma)\) imply \((g_M)_{ij}(x) - g_{ij}(x,u(x)) \in H_{s+1,\delta + 1}(\Sigma)\).

\[
g_{ij}(x,u(x)) - g_{ij}(x) = \{ \int_{s=0}^1 \partial_s g_{ij}(x, su(x)) ds \} \cdot u(x),
\]

which shows \({g_{ij}(x,u(x)) - g_{ij}(x)} \in H_{s+1,\delta + 1}(\Sigma)\), since \(\partial_s g_{ij}(x, su(x)) \in H_{s+1,\delta + \frac{3}{2}}(\Sigma)\) and \(u \in H_{s+2,\delta - \frac{1}{2}}(\Sigma)\). Hence \({(g_M)_{ij}(x) - g_{ij}(x)} \in H_{s+1,\delta + 1}(\Sigma) \subset C^{s-1}_\beta(\Sigma)\), for some \(1 < \beta < \delta + \frac{3}{2}\) by the embedding lemma 2.5. By checking the definition \((1.6)\), we know that a error term of decay rate \(o(r^{-1})\) will not change the mass, so \(m = m_M\). \(\square\)

Now we will study the preservation of symmetry by this constructions. We need a lemma about symmetry preserving by the reduced EVE \((3.3)\).

**Lemma 4.15.** Given a vacuum data \((g,k) \in V C^{s+2,\delta + \frac{1}{2}}(\Sigma)\), and \((\Omega_\theta, \gamma)\) the boost evolution of \((g,k)\) given by Theorem 3.7. Suppose that both \((g,k)\) and \(e\) are symmetric under a Killing vector field \(\xi\) on \(\Sigma\), i.e. \((g,k)\) satisfy \((1.7)\), and \(L_\xi e = 0\), where \(e\) is the canonical metric on \(\Sigma\). Then the parallel translation \(\tilde{\xi}\) of \(\xi\) into \(\Omega_\theta\) is a Killing vector field of \(\gamma\).

**Proof.** Now let \(\phi_s : \Sigma \to \Sigma\) be the one parameter group of diffeomorphisms corresponding to \(\xi\). Then \((\phi_s)^* g = g\), \((\phi_s)^* k = k\) and \((\phi_s)^* e = e\). Now let us extend \(\phi_s\) to a diffeomorphism \(\tilde{\phi}_s : \Omega_\theta \to \Omega_\theta\) by

\[
\tilde{\phi}_s(x,t) = (\phi_s(x), t).
\]

(4.27)
Then \((\tilde{\phi}_s)^* \tilde{e} = \tilde{e}\) where \(\tilde{e}\) is defined by (2.1). By the initial conditions (3.5) and (3.6) for \(\gamma\), we know that \(\gamma_s = (\tilde{\phi}_s)^* \gamma\) has the same initial conditions as \(\gamma\) on \(\Sigma\). If we can show that \(\gamma_s\) also solves the reduced (EVE) (3.3), the uniqueness in Theorem 3.7 implies that \(\gamma_s = \gamma\). Since \(\gamma_s\) is Ricci flat, we only need to show that \((\Omega_\theta, \gamma_s)\) is also in a harmonic gauge, or equivalently, \(id : (\Omega_\theta, \gamma_s) \rightarrow (\Omega_\theta, \tilde{e})\) is a wave map. By pulling back the wave map equation \(\Box_{(\gamma_s, \tilde{e})} id = 0\) by \(\tilde{\phi}\), we get \(\Box_{(\tilde{\phi}_s)^* \gamma, (\tilde{\phi}_s)^* \tilde{e})\} id = 0\), which reduces to \(\Box_{(\gamma_s, \tilde{e})} id = 0\). This means that \(\gamma_s\) is also in a harmonic gauge, hence \(\gamma_s = \gamma\). Now the vector field corresponding to \(\tilde{\phi}_s\) is clearly the parallel translation of \(\xi\) into \(\Omega_\theta\).

Now we can prove the preservation of symmetry for the maximal surface.

**Theorem 4.16.** Given \(s \geq 4, -2 < \delta < -1\). Suppose \((\Sigma, e)\) is a 3-manifold, which is Euclidean at infinity and axisymmetric in the sense of Definition 1.3. If \((g, k) \in \mathcal{VC}_{s+2, \delta+\frac{1}{2}} (\Sigma)\) is axisymmetric, and \(\|tr_g k\|_{H_{s-2, \delta+\frac{1}{2}} (\Sigma)} \leq \epsilon\) with \(\epsilon\) given by Theorem 4.12, then the solution \(u\) of the maximal surface equation (1.9) given in Theorem 4.12 can be chosen to be axisymmetric, i.e. \(\partial_\varphi u = 0\). Hence \((\Sigma, g_u, k_u)\) is axisymmetric, and the angular momentum of \((g_u, k_u)\) equals that of \((g, k)\).

**Proof.** By Theorem 4.12 \(\mathcal{H}(g, k, u) = 0\) has a unique solution \(u \in B_\rho \cap ker(L_0, \delta - \frac{1}{2})\). Let \(\phi_s\) be the diffeomorphism corresponding to the Killing vector field \(\xi = \frac{\partial}{\partial \varphi}\) in Definition 1.3 and \(\tilde{\phi}_s\) the extension given in (4.27). When \((g, k)\) is also axisymmetric, the boost solution \((\Omega_\theta, \gamma)\) is invariant under \(\tilde{\phi}_s\) by Lemma 4.15. Now pulling back \(\mathcal{H}(g, k, u) = 0\) by \(\tilde{\phi}_s\), we can see that \(\phi_s^* u\) is a solution of \(\mathcal{H}(\phi_s^* g, \phi_s^* k, \phi_s^* u) = 0\), hence \(\mathcal{H}(g, k, \phi_s^* u) = 0\). Since \((\Sigma, e)\) and \((g, k)\) are all invariant under \(\phi_s\), \(ker(L_0, \delta - \frac{1}{2})\) and hence \(ker(L_0, \delta - \frac{1}{2})\) are also invariant under \(\phi_s\), which means that \(\phi_s^* u \in B_\rho \cap ker(L_0, \delta - \frac{1}{2})\), then uniqueness in Theorem 4.12 implies that \((\phi_s)^* u = u\). So \(u\) is axisymmetric, hence is \((g_u, k_u)\) since \(\gamma\) is also axisymmetric.

For the angular momentum, we have another formular, which is called Komar integral (see section 11.2 in [24] for definition and equivalence with (1.8)),

\[
J(S) = \frac{1}{16\pi} \int_S * d\xi, \tag{4.28}
\]

where \(*\) is the Hodge star operator w.r.t \(\gamma\), and \(\xi\) the killing vector field. Since \(* d\xi\) is a close form, we know that \(J(S)\) is invariant for any two spacelike close surface \(S\) and \(S'\) which are homologous equivalent. So \((\Sigma, g, k)\) and \((Graph_u, g_u, k_u)\) have the same angular momentum.

**5 Appendix**

**5.1 Geometry of hypersurface**

Here we show the detailed calculation of the mean curvature of a level surface. Part of the results here already appeared in [1]. First let us calculate the future-directed timelike unit...
normal vector of \( \Sigma_t \) defined by \( T = -\frac{\nabla t}{|\nabla t|} \), which is given by:
\[
T = -\alpha \nabla t = -\alpha (\gamma^t \partial_t + \gamma^i \partial_i) = \alpha^{-1} (\partial_t - \beta). \tag{5.1}
\]

Graph of \( u \) can be viewed as level surface of \( (u - t) = 0 \), so the unit normal of Graph of \( u \) is \( N = \frac{\nabla (u - t)}{|\nabla (u - t)|} \). Now
\[
\nabla u = \gamma^{ij} u_j \partial_i + \gamma^{ij} u_j \partial_i = \frac{1}{\alpha^2} \langle \beta, Du \rangle \partial_i + Du - \frac{1}{\alpha^2} \langle \beta, Du \rangle \beta^i \partial_i \tag{5.2}
\]

So \( N \) is calculated as
\[
\nabla (u - t) = \frac{1}{\alpha^2} \langle \beta, Du \rangle \partial_t + Du - \frac{1}{\alpha^2} \langle \beta, Du \rangle \beta^i \partial_i + \frac{1}{\alpha^2} \beta^i \partial_i - \frac{1}{\alpha^2} \beta^i \partial_i \tag{5.3}
\]

Writing \( U = \frac{\alpha Du}{1 + \langle \beta, Du \rangle} \), then \( N = \frac{U + T}{|U + T|} \), where \( |U + T| = (1 - |U|^2)^{1/2} \), so we get equation \( (2.11) \).

Denoting \( M = \text{Graph}_u \), let us calculate the mean curvature. For completeness we give the inverse metric matrix \( (g_M)^{-1} \) of \( g_M \) in \( (2.14) \). First we need to calculate the co-frame of \( (2.13) \). Denoting them by \( \alpha^i = a^i_k dx^k + a^i_t dt : i, k = 1, 2, 3 .. \) Then they should satisfy:
\[
\alpha^i (\alpha_k) = \delta^i_k, \quad \alpha^i (N) = 0. \tag{5.4}
\]

The last equation gives
\[
(a^i_k dx^k + a^i_t dt)[\alpha (U + T)] = (a^i_k dx^k + a^i_t dt)(\frac{\alpha^2 Du}{1 + \langle \beta, Du \rangle} + (\partial_t - \beta)) \tag{5.5}
\]

So
\[
a^i_t = a^i_k (\beta - \frac{\alpha^2 Du}{1 + \langle \beta, Du \rangle})^k = a^i_k (\beta - \alpha U)^k. \tag{5.6}
\]

Putting into the first on in \((5.4)\), we have
\[
(a^i_k dx^k + a^i_t (\beta^l - \alpha U^l) dt)(\partial_k + u_k \partial_t) = a^i_k + a^i_t (\beta^l - \alpha U^l) u_k = \delta^i_k. \tag{5.7}
\]

Denoting matrix \( A = (a^i_k) \), then the above equations change to the matrix equation
\[
A \cdot [id + (\beta - \alpha U)(Du)^t] = id. \tag{5.8}
\]
Solving the last matrix equation, we get

\[ a'_{ik} = Id - \frac{(\beta - \alpha U^i)u_k}{1 + \langle \beta - \alpha U, Du \rangle} = Id - \frac{(\beta - \alpha U^i)u_k}{1 + \langle \beta, Du \rangle - (1 + \langle \beta, Du \rangle)|U|^2} \]

\[ = Id - \nu^2 \frac{(\beta - \alpha U^i)u_k}{1 + \langle \beta, Du \rangle} = Id - \nu^2(\beta/\alpha - U)^iU_k, \tag{5.9} \]

where we have used \( U = \frac{\alpha Du}{1 + \langle \beta, Du \rangle} \), and \( \nu^{-2} = 1 - |U|^2 \). Then

\[ a^i_t = a^i_k(\beta - \alpha U)^k = (\delta^i_k - \nu^2 \frac{(\beta - \alpha U^i)u_k}{1 + \langle \beta, Du \rangle})(\beta^k - \alpha U^k) \]

\[ = \beta^i - \nu^2 \frac{(\beta - \alpha U^i)\langle \beta, Du \rangle}{1 + \langle \beta, Du \rangle} - \alpha U^i + \nu^2(\beta - \alpha U^i)|U|^2 \]

\[ = (1 + \nu^2|U|^2)(\beta - \alpha U^i) - \nu^2 \frac{\langle \beta, Du \rangle}{1 + \langle \beta, Du \rangle}(\beta - \alpha U^i) \]

\[ = \nu^2(\beta - \alpha U^i) - \nu^2 \frac{\langle \beta, Du \rangle}{1 + \langle \beta, Du \rangle}(\beta - \alpha U^i) \]

\[ = \nu^2 \frac{\beta - \alpha U^i}{1 + \langle \beta, Du \rangle}. \tag{5.10} \]

So the co-frame is given by

\[ \alpha^i = (\delta^i_k - \nu^2 \frac{(\beta/\alpha - U)^iU_k}{1 + \langle \beta, Du \rangle})dx^k + \frac{\nu^2}{1 + \langle \beta, Du \rangle}(\beta - \alpha U)^i dt. \tag{5.11} \]
Taking inner product of the co-frame with respect to $\gamma^{-1}$, we can calculate $g_M^{-1}$.

\[
(g_M)^{ij} = \langle (\delta_k^i - \nu^2(\beta/\alpha - U)^j U_k)dx^k + \frac{\nu^2(\beta - \alpha U)^i}{1 + \langle \beta, Du \rangle} dt, (\delta_k^j - \nu^2(\beta/\alpha - U)^i U_l)dx^l + \frac{\nu^2(\beta - \alpha U)^j}{1 + \langle \beta, Du \rangle} dt \rangle_{\gamma} \\
= (\delta_k^i - \nu^2(\beta/\alpha - U)^j U_k)(\delta_l^j - \nu^2(\beta/\alpha - U)^i U_l)(g^{kl} - \frac{1}{\alpha^2} \delta^k \delta^l) \\
+ (\delta_k^i - \nu^2(\beta/\alpha - U)^i U_k)\frac{\nu^2(\beta - \alpha U)^j}{1 + \langle \beta, Du \rangle} \frac{\beta^k}{\alpha^2} + (\delta_k^i - \nu^2(\beta/\alpha - U)^i U_k)\frac{\nu^2(\beta - \alpha U)^j}{1 + \langle \beta, Du \rangle} \frac{\beta^k}{\alpha^2} \\
- \frac{1}{\alpha^2} \nu^A(\beta/\alpha - U)^i(\beta - \alpha U)^j \\
= g^{ij} - \frac{1}{\alpha^2} \delta^i \delta^j - \nu^2(\beta/\alpha - U)^j U^i - \nu^2(\beta/\alpha - U)^i U^j + \frac{\nu^2(\beta - \alpha U)^j \beta^i}{\alpha^2(1 + \langle \beta, Du \rangle)} \\
+ \frac{\nu^2(\beta - \alpha U)^i \beta^j}{\alpha^2(1 + \langle \beta, Du \rangle)} + \frac{\nu^A(\beta/\alpha - U)^i(\beta - \alpha U)^j}{\alpha^2(1 + \langle \beta, Du \rangle)} \\
- \frac{\nu^i(\beta - \alpha U)^j}{\alpha^2(1 + \langle \beta, Du \rangle)^2} \langle \beta, Du \rangle + \frac{\nu^i(\beta - \alpha U)^j}{\alpha^2(1 + \langle \beta, Du \rangle)^2} \\
- 2\frac{\nu^A(\beta/\alpha - U)^i(\beta - \alpha U)^j}{\alpha^2(1 + \langle \beta, Du \rangle)^2} - \frac{\nu^i(\beta - \alpha U)^j}{\alpha^2(1 + \langle \beta, Du \rangle)^2} \\
= g^{ij} - \frac{1}{\alpha^2} \delta^i \delta^j - \nu^2(\beta/\alpha - U)^j U^i - \nu^2(\beta/\alpha - U)^i U^j + \nu^i(\beta/\alpha - U)^j(\beta/\alpha - U)^i U^j \\
+ \frac{\nu^2}{\alpha^2} \delta^i \delta^j + \frac{\nu^2}{\alpha^2}(\beta - \alpha U)^i(\beta - \alpha U)^j \\
= g^{ij} - \frac{1}{\alpha^2} \delta^i \delta^j + \frac{\nu^2}{\alpha^2}(\beta - \alpha U)^i(\beta - \alpha U)^j. \\
(5.12)
\]

### 5.2 Linear boost estimates on an end

Here we will give a detailed version of linear boost estimates on an Euclidean end using method in \[8\] and \[10\]. It was also mentioned in \[3\]. We will mainly give the energy estimates needed to prove Theorem 3.6. For convenience, we sometime abbreviate $V_{\theta, \lambda} = V$ in this section. Given a regularly hyperbolic metric $\gamma^{\mu\nu}$ and a $\mathbb{R}^N$-valued function $u$ in $V_{\theta, \lambda}$, we can associate it with the energy-momentum tensor $T^{\mu\nu}$\[24\]

\[
T^{\mu\nu} = G^{\mu\nu\rho\sigma} D_\rho u \cdot D_\sigma u \tag{5.13}
\]

where

\[
G^{\mu\nu\rho\sigma} = \gamma^{\mu\rho} \gamma^{\nu\sigma} + \gamma^{\mu\sigma} \gamma^{\nu\rho} - \gamma^{\mu\nu} \gamma^{\rho\sigma}.
\]

\[\text{\[24\]See also equation (4.6) in \[8\].}\]

\[\text{\[24\]Here the inner product of $D_\rho u \cdot D_\sigma u$ is $\sum_{k=1}^{N} D_\rho u^k D_\sigma u^k$.}\]
5 APPENDIX

Given the unit normal \( \tilde{n} \) of \( \{ E_\tau \} \) defined in (3.12), the momentum vector field relative to \( \tilde{n} \) is

\[
P^\mu = T^{\mu\nu} \tilde{n}_\nu. \tag{5.14}\]

Furthermore, the divergence of \( P^\mu \) is

\[
D_\mu P^\mu = 2(\gamma^\rho\sigma \tilde{n}_\rho D_\sigma u) \cdot \gamma^{\mu\nu} D^2_{\mu\nu} u + Q, \tag{5.15}
\]

where

\[
Q = \Lambda^{\mu\nu} D_\mu u \cdot D_\nu u, \text{ with } \Lambda^{\mu\nu} = D_\rho (G^{\mu\rho\sigma \cdot \tilde{n}}_\sigma).
\]

Let \( N^{-2} = -\langle D\tau, D\tau \rangle_\gamma \) be the lapse function for \( \tau \) w.r.t. \( \gamma \) and \( n = ND\tau \) the unit co-normal of \( \{ E_\tau \} \) w.r.t. \( \gamma \). We introduce an orthonormal frame \( \{ e_0, e_1, \cdots, e_{n-1} \} \) w.r.t. \( \gamma \), such that \( e_0 \) is along the direction of \( \tilde{n}^\mu = \gamma^{\mu\nu} \tilde{n}_\nu \), i.e. \( e_0 = \frac{N}{\sqrt{N}} \tilde{n}^\mu \), where \( \left( \frac{N}{\sqrt{N}} \right)^{-2} = |\tilde{n}|^2_\gamma \), and \( e_i \) perpendicular to \( \tilde{n}^\mu \). According to Section 2 in [8], we know that \( |\tilde{n}|^2_\gamma = \gamma^{\mu\nu} \tilde{n}_\mu \tilde{n}_\nu = (\frac{N}{\sqrt{N}})^{-2} \) is bounded from both above and below by some constants depending only on \( \theta \) and \( h \).

**Lemma 5.1.** When \( \gamma \) is regularly hyperbolic, \( P^\mu \) is past time-like w.r.t. \( \gamma \).

**Proof.** \( T^{\mu\nu} = 2D^\mu u \cdot D^\nu u - |D^\mu u|^2 \gamma^{\mu\nu} \), so \( P^\mu = T^{\mu\nu} \tilde{n}_\nu = 2D^\mu u \cdot D^\nu \tilde{n}_\nu - |D^\mu u|^2 \tilde{n}^\mu \), and

\[
\gamma_{\mu\nu} P^\mu P^\nu = 4\gamma_{\mu\nu} (D^\mu u \cdot D^\rho \tilde{n}_\rho) (D^\nu u \cdot D^\sigma \tilde{n}_\sigma) - 4|D^\mu u|^2 (D^\mu u \tilde{n}_\mu \cdot D^\nu u \tilde{n}_\nu) + |D^\mu u|^4 |\tilde{n}|^2_\gamma
\leq |D^\mu u|^4 |\tilde{n}|^2_\gamma \leq 0.
\]

The first “\( \leq \)” comes from Cauchy-Schwartz inequality, and the second comes from the fact that \( \tilde{n} \) is time-like w.r.t. \( \gamma \).

Take \( l^\mu \) as a future like vector field, then in the orthonormal frame \( \{ e_0, e_1, \cdots, e_{n-1} \} \) as above, \( l^0 > \sqrt{\sum_{i=1}^{n-1} (l^i)^2} \), and

\[
\gamma_{\mu\nu} P^\mu l^\nu = 2[(D^0 u) l^0 + (D_i u) l^i] (D^0 u)(\tilde{N}/N) - [-(D^0 u)^2 + \sum (D_i u)^2] (-l_0) (\tilde{N}/N)
\geq (\tilde{N}/N)[(D^0 u)^2 + \sum (D_i u)^2] (l^0 - \sqrt{\sum (l^i)^2}) \geq 0.
\]

The first “\( \geq \)” comes from the Cauchy-Schwartz inequality. So it shows that \( P \) is past time-like w.r.t. \( \gamma \). \( \square \)

Now we introduce the restriction norm and restriction lemma similar to (2.8) and Lemma 2.8. Given \( u \in H_{s,\delta}(V_{\delta,\lambda}) \), the restriction norm to hypersurface \( E_\tau \) is defined as:

\[
\|u\|_{H_{s,\delta}(E_\tau, V_{\delta,\lambda})} = \left( \sum_{k=0}^{s} \|D_k u\|_{H^k E_\tau}^2 \right)^{1/2}.
\]

The following restriction lemma follows similar from Lemma 3.1 in [8]:
Lemma 5.2. (restriction). \( \forall \tau \in (-\theta, \theta) \), we have the following continuous inclusion:

\[
H_{s+1, \delta}(V_{\theta, \lambda}) \subset H_{s, \delta + \frac{1}{2}}(E_\tau, V_{\theta, \lambda}),
\]

for every \( s \in \mathbb{N} \) and \( \delta \in \mathbb{R} \).

Now we have the first energy estimates.

**Lemma 5.3.** (First Energy Estimates). Assume that \( \gamma^{\mu \nu} \) is regularly hyperbolic, and \( (\gamma - \eta) \in C^\infty \cap C^{1,0}(V) \), \( a_1 \in C^\infty \cap C^{0,1}(V) \) and \( a_0 \in C^\infty \cap C^{0,2}(V) \). For \( L \) defined in (3.14), with \( a_2 = \gamma \text{Id} \), every \( u \in C^\infty_0(V) \) satisfies the fundamental energy estimates:

\[
\|u\|_{H_{1, \delta + \frac{1}{2}}(E_\tau, V)} \leq c(\|u\|_{H_{1, \delta + \frac{1}{2}}(E, V)} + \|\beta\|_{H_{0, \delta + 2}(V)}),
\]

where \( 0 \leq \tau \leq \theta \), \( \beta = Lu \), and \( c \) is a constant depending only on \( \theta \), the coefficients \( h \) of regular hyperbolicity (3.13) of \( \gamma \), and \( \|D\gamma\|_{C^{1,0}} + \|a_1\|_{C^{0,1}} + \|a_2\|_{C^{0,2}} \).

**Proof.** Let \( \tilde{P}^\mu = \sigma^{2(\delta + \frac{1}{2})}P^\mu \). Multiply (5.15) by \( \sigma^{2(\delta + \frac{1}{2})} \), we get:

\[
D_\mu \tilde{P}^\mu = 2\sigma^{2(\delta + \frac{1}{2})}(\gamma^{\rho \sigma} \tilde{n}_\rho D_\sigma u) \cdot \gamma^{\mu \nu} D^2_\mu, u + \tilde{Q},
\]

where

\[
\tilde{Q} = \sigma^{2(\delta + \frac{3}{2})}Q',
\]

with

\[
Q' = Q + 2(\delta + 3/2)x^i / \sigma^2 P^i \simeq (D \gamma \ast \gamma + \sigma^{-1} \gamma \ast \gamma) Du \ast Du.
\]

Plug in \( Lu = \beta \),

\[
D_\mu \tilde{P}^\mu = \sigma^{2(\delta + \frac{3}{2})}[2(\gamma^{\rho \sigma} \tilde{n}_\rho D_\sigma u) \cdot (\beta - a_1 Du - a_0 u) + Q'].
\]

Now we integrate on the upper part \( V_{\tau, \lambda}^+ = \{ x \in V_{\tau, \lambda} : \tau \geq 0 \} \) for \( \tau \leq \theta \). Since \( P \) is compactly supported, the divergence theorem of \( (V_{\tau, \lambda}^+, \eta) \) gives,

\[
\int_{E_\tau} \tilde{P}^\mu \tilde{n}_\mu d\Sigma - \int_{E} \tilde{P}^\mu \tilde{n}_\mu d\Sigma + \int_{L_{\tau, \lambda}^+} \tilde{P}^\mu \tilde{n}_\mu d\sigma = \int_{V_{\tau, \lambda}^+} \tilde{P}^\mu \tilde{n}_\mu dx
\]

\[
= \int_{V_{\tau, \lambda}^+} \sigma^{2(\delta + \frac{3}{2})}[2(\gamma^{\rho \sigma} \tilde{n}_\rho D_\sigma u) \cdot (\beta - a_1 Du - a_0 u) + Q'] dx,
\]

where \( \tilde{n}_\mu \) is the unit outer co-normal of the upper lateral boundary \( L_{\tau, \lambda}^+ = L_{\tau, \lambda} \cap V_{\tau, \lambda}^+ \) under \( \eta \), which is future timelike w.r.t. \( \gamma \) by property (4) of the regular hyperbolicity (3.2). Using the fact that \( P \) is past time-like (Lemma 5.1), we know that

\[
\tilde{P}^\mu \tilde{n}_\mu = \sigma^{2(\delta + \frac{3}{2})}P^\mu \tilde{n}_\mu \geq 0, \text{ on } L_{\tau, \lambda}^+.
\]

Now define:

\[
x_1(\tau) = \int_{E_\tau} |\sigma^{\delta + 3/2} Du|^2 d\Sigma = \|Du\|^2_{H_{0, \delta + \frac{1}{2}}(E_\tau, V)},
\]
Since \( \tilde{n}_\mu = \tilde{N} D_\mu \tau = \frac{\tilde{N}}{\tau} n_\mu \),

\[
P^\mu_{\bar{\nu}} n_\mu = T^\mu_{\nu} n_\mu n_\nu = 2(\gamma^\mu_{\sigma} D_\sigma u \tilde{n}_\mu)^2 - |Du|^2 |n|_\gamma^2
\]

\[
= \left( \frac{\tilde{N}}{\tau} \right)^2 (2n^\mu n^\nu + \gamma^\mu_{\nu}) D_\mu u D_\nu u.
\]

Using Proposition 2.3 in [8], \( \Gamma^\mu_{\nu\sigma} = 2n^\mu n^\nu + \gamma^\mu_{\nu} \) is uniformly elliptic, with the elliptic coefficient depending only on the coefficient of regular hyperbolicity \( h \). Using equation (2.8)(2.13) of [8], \( d\Sigma_\tau \simeq c d\Sigma \), with \( c \) depending only on \( \theta \), so we have:

\[
\int_{E_\tau} \tilde{P}^\mu_{\bar{\nu}} n_\mu d\Sigma_\tau \geq c_1^{-1} x_1(\tau),
\]

\[
\int_{E} \tilde{P}^\mu_{\bar{\nu}} n_\mu d\Sigma_0 \leq c_1 x_1(0),
\]

where \( c_1 \) is a constant depending only on \( \theta \) and the regular hyperbolicity coefficient \( h \). Now using Cauchy-Schwarz inequality and the fact \( dx = \sigma d\tau d\Sigma \) to the right hand side of (5.19),

\[
| \int_{V_{\tau,\lambda}^+} 2\sigma^2 (\gamma^\rho_{\sigma} \tilde{n}_\rho D_\sigma u) \cdot \beta dx | \leq c_1 \int_0^\tau \| Du \|_{H_{0,\delta} + \frac{3}{4}}(E_\tau, V) \| \beta \|_{H_{0,\delta} + \frac{3}{4}}(E_\tau, V) d\tau';
\]

\[
| \int_{V_{\tau,\lambda}^+} 2\sigma^2 (\gamma^\rho_{\sigma} \tilde{n}_\rho D_\sigma u) \cdot a_0 u d dx | \leq c_1 \| a_0 \|_{C^{0,1}} \int_0^\tau \| Du \|_{H_{0,\delta} + \frac{3}{4}}(E_\tau, V) d\tau';
\]

\[
| \int_{V_{\tau,\lambda}^+} 2\sigma^2 (\gamma^\rho_{\sigma} \tilde{n}_\rho D_\sigma u) \cdot a_0 u d dx | \leq c_1 \| a_0 \|_{C^{0,2}} \int_0^\tau \| Du \|_{H_{0,\delta} + \frac{3}{4}}(E_\tau, V) d\tau';
\]

\[
| \int_{V_{\tau,\lambda}^+} 2\sigma^2 (\gamma^\rho_{\sigma} \tilde{n}_\rho D_\sigma u) \cdot Q' d x | \leq c_1 (1 + \| D\gamma \|_{C^{0,1}}) \int_0^\tau \| Du \|_{H_{0,\delta} + \frac{3}{4}}(E_\tau, V) d\tau',
\]

where \( c_1 \) denotes a constant depending only on the regular hyperbolicity coefficient \( h \). Now define:

\[
x_0(\tau) = \int_{E_\tau} |\sigma^{\delta+1/2} u|^2 d\Sigma = \| u \|_{H_{0,\delta} + \frac{3}{4}}^2(E_\tau, V),
\]

then (5.19) can be changed to

\[
x_1(\tau) \leq c_2 \{ x_1(0) + \int_0^\tau \| \beta \|_{H_{0,\delta} + \frac{3}{4}}(E_\tau, V) x_1(\tau')^{1/2} d\tau' + m_1 \int_0^\tau y_1(\tau') d\tau' \},
\]

where \( c_2 \) is a constant depending only on \( \theta \) and the regular hyperbolicity coefficient \( h \), and

\[
m_1 = \| D\gamma \|_{C^{0,1}} + \| a_1 \|_{C^{0,1}} + \| a_0 \|_{C^{0,2}},
\]

\[
y_1(\tau) = x_1(\tau) + x_0(\tau) = \| u \|_{H_{0,\delta} + \frac{3}{4}}^2(E_\tau, V),
\]

Using Cauchy-Schwarz inequality,

\[
(u(\tau) - u(0))^2 = \left( \int_0^\tau \frac{\partial u}{\partial \tau'} d\tau' \right)^2 \leq \tau \int_0^\tau \left( \frac{\partial u}{\partial \tau'} \right)^2 d\tau'.
\]
Consider the projection map \( \pi : V_{\theta, \lambda} \to E \) defined by \( \pi(\tilde{x}, t) = \tilde{x} \), then \( E_r' = \pi(E_r) \subset E_r' \), if \( \tau' < \tau \), then
\[
\int_{E_r'} |\sigma^{\delta+1/2}(u(\tau) - u(0))|^2 d\Sigma \leq \tau \int_0^\tau \{ \int_{E_r'} |\sigma^{\delta+3/2} \frac{\partial u}{\partial t}|^2 d\Sigma \} d\tau' \leq \tau \int_0^\tau x_1(\tau') d\tau'.
\]

So,
\[
x_0(\tau) \leq 2x_0(0) + 2\tau \int_0^\tau x_1(\tau') d\tau'.
\]

Adding (5.22) and (5.25), we can get the integral inequality,
\[
y_1(\tau) \leq c_2 \{ y_1(0) + \int_0^\tau \| \beta \|_{H_{0, \delta+\frac{1}{2}}(E_r', V)} y_1^{1/2}(\tau') d\tau' + m_1 \int_0^\tau y_1(\tau') d\tau' \}
\]

Using the Gronwall lemma,
\[
y_1^{1/2}(\tau) \leq \exp\left(\frac{1}{2} c_2 m_1 \tau\right) \{ y_1^{1/2}(0) + \frac{1}{2} \int_0^\tau e^{\frac{1}{2} c_2 m_1 \tau'} c_2 \| \beta \|_{H_{0, \delta+\frac{1}{2}}(E_r', V)} d\tau' \}.
\]

Hence we finished the proof by using \( y_1^{1/2}(\tau) = \| u \|_{H_{1, \delta+\frac{1}{2}}(E_r, V)}. \)

This result can be weakened to the case of rough coefficients by approximation methods.

**Lemma 5.4.** If \( \gamma \) is regularly hyperbolic on \( V \), \( (\gamma - \eta) \in C^{1,0}(V) \), \( a_1 \in C^{0,1}(V) \) and \( a_0 \in C^{0,2}(V) \), then every \( u \in H_{2, \delta}(V) \) satisfies the fundamental energy estimates [5.17], with \( \beta = Lu \).

**Proof.** This comes from an approximation argument exactly the same as Lemma 4.2 in [8].

Using more differentiability of the coefficients, we can improve the energy estimates containing high order derivatives.

**Lemma 5.5.** (High Order Estimates). Given \( s \leq s' \) with \( s' \) defined in [3.15]. If \( \gamma \) is regularly hyperbolic, \( (\gamma - \eta) \in C^{\infty}(V) \), \( a_1 \in C^{\infty}(V) \) and \( a_0 \in C^{\infty}(V) \), then every \( u \in C_0^\infty(V) \) satisfies the main energy estimates:
\[
\| u \|_{H_{s, \delta+\frac{1}{2}}(E_r, V)} \leq c(\| u \|_{H_{s, \delta+\frac{1}{2}}(E, V)} + \| \beta \|_{H_{s-1, \delta+2}(V)}),
\]

where \( 0 \leq \tau \leq \theta \), \( \beta = Lu \), and \( c \) is a constant depending only on \( \theta \), the coefficient of regular hyperbolicity \( h \) and \( m \) (defined in [3.16]).

**Proof.** Apply \( D^{i-1} \) for \( 2 \leq i \leq s \) to \( Lu = \beta \), we can get
\[
\gamma^{\mu\nu} D^2_{\mu\nu} u^{[i-1]} = \beta^{[i-1]},
\]

where \( u^{[i-1]} = D^{i-1} u \), and
\[
\beta^{[i-1]} = D^{i-1} \beta - \sum_{p=1}^{i-1} \binom{i-1}{p} D^p \gamma D^{i+1-p} u - \sum_{p=0}^{i-1} \binom{i-1}{p} (D^p a_1 D^{i-p} u + D^p a_0 D^{i-1-p} u) .
\]
Now define
\[ x_i(\tau) = \int_{E_\tau} |\sigma^{\delta+i+\frac{1}{2}}D^i u|^2 d\Sigma = \|D^i u\|_{H_{0,\delta+i+1/2}(E_\tau, V)}^2, \]  
and apply (5.22) in Lemma 5.3 to (5.29) with \( x \) replaced by \( x_1 \) and apply (5.22) in Lemma 5.3 to (5.29) with \( x \) replaced by \( x_1 \), then
\[ x_i(\tau) \leq c_1 \left\{ x_i(0) + \int_0^\tau \|\beta^{[i-1]}\|_{H_{0,\delta+i+3/2}(E_\tau, V)} x_i^{1/2}(\tau')d\tau' + m_1 \int_0^\tau x_i(\tau')d\tau' \right\}, \]
with \( c_1 \) depending only on the coefficient of regular hyperbolicity \( h \) and \( m_1 \) defined in (5.23). Now define \( y_i(\tau) = y_1(\tau) + \sum_{j=2}^i x_j(\tau) = \|u\|_{H_{1,\delta+1/2}(E_\tau, V)}^2 \).

We have
\[ x_i(\tau) \leq c_1 \left\{ x_i(0) + \int_0^\tau \|D^{i-1}\beta\|_{H_{0,\delta+i+3/2}(E_\tau, V)} x_i^{1/2}(\tau')d\tau' + c_4 (m + m_1) \int_0^\tau x_i(\tau')d\tau' \right\}. \]
Summing all \( i \) from 1, we can get
\[ y_i(\tau) \leq c_1 \left\{ y_i(0) + \int_0^\tau \|\beta\|_{H_{-1,\delta+5/2}(E_\tau, V)} y_i^{1/2}(\tau')d\tau' + c_4 (m + m_1) \int_0^\tau y_i(\tau')d\tau' \right\}. \]
Using the Gronwall lemma,
\[ y_i^{1/2}(\tau) \leq \exp(c_5 (m + m_1)\tau) \left\{ y_i^{1/2}(0) + c_1 \int_0^\tau e^{c_5 (m + m_1)\tau'} \|\beta\|_{H_{-1,\delta+5/2}(E_\tau, V)} d\tau' \right\}, \]
where \( c_5 = \frac{1}{2} c_1 c_4 \). Hence we finish the proof realizing \( m_1 \leq c_6 m \) by the imbedding lemma 2.2.
Using the equation $Lu = \beta$ and an argument similar to Lemma 4.4 in [S], we can estimate $\|u\|_{H_{s,\delta+1/2}(E, V)}$ by the spatial norms $\|\phi\|_{H_{s,\delta+1/2}(E)}$, $\|\psi\|_{H_{s-1,\delta+3/2}(E)}$ and $\|\beta\|_{H_{s-2,\delta+5/2}(E, V)}$, where $\phi = u|_E$ and $\psi = D_t u|_E$. We need the following technical lemma which says that we can take the division in the Banach algebra $H_{s,\delta}(U)$, when $s > \frac{q}{2}$ and $\delta > -\frac{q}{2}$.

**Lemma 5.6.** Given $U$ satisfied the extended cone property, $s > \frac{q}{2}$, $\delta > -\frac{q}{2}$ and a function $f$, if $(f - 1) \in H_{s,\delta}(U)$, and $|f| \geq c > 0$, then $(f - 1)^{-1} \in H_{s,\delta}(U)$, furthermore, $\|f - 1\|_{H_{s,\delta}(U)}$ is bounded by a constant depending only on $n$, $s$, $\delta$ and $\|f - 1\|_{H_{s,\delta}(U)}$.

**Proof.** Since $|f| \geq c > 0$, $f^{-1}$ is well defined. Since $f^{-1} - 1 = -\frac{f - 1}{f}$ and $|f|^{-1} \leq c^{-1}$ uniformly bounded, $(f - 1)^{-1} \in H_{0,\delta}(U)$. Now $D^\alpha(f^{-1} - 1) = \sum_{\alpha_1 + \cdots + \alpha_i = \alpha} \frac{D^\alpha f - D^\alpha f_0}{f_0^{\alpha_i}}$, where $\alpha$ is multi-indexes, with $1 \leq |\alpha| \leq s$. Since $(f^{\alpha})^{-1}$ is uniformly bounded, and using the multiplication Lemma 2.2, $D^\alpha f \cdots D^\alpha f \in H_{0,\delta+|\alpha|}(U)$, hence $D^\alpha(f^{-1} - 1) \in H_{0,\delta+|\alpha|}(U)$. So $(f^{-1} - 1) \in H_{s,\delta}(U)$. The norm bounds follows from the norm bounds of each $D^\alpha(f^{-1} - 1)$.

**Lemma 5.7.** Given an operator $L$ defined in (3.14) satisfying Hypothesis (1) and (2), then every $u \in H_{s+1,\delta}(V)$ with $2 \leq s \leq s'$, which solves $Lu = \beta$ satisfies:

$$\|u\|_{H_{s,\delta+1/2}(E, V)} \leq c(\|\phi\|_{H_{s,\delta+1/2}(E)} + \|\psi\|_{H_{s-1,\delta+3/2}(E)} + \|\beta\|_{H_{s-2,\delta+5/2}(E, V)}),$$

where $\phi = u|_E$, $\psi = D_t u|_E$ and $c$ is a constant depending only on $s$, $\delta$ and $\mu$ (defined in (3.17)).

**Proof.** By the restriction Lemma 5.2, $u \in H_{s+1,\delta}(V)$ implies that $\phi \in H_{s,\delta+1/2}(E)$ and $\psi \in H_{s-1,\delta+3/2}(E)$. Now define the following functions on $E$:

$$\psi^{[p]} = D^\gamma u, \quad 0 \leq p \leq s.$$

Since

$$\|u\|_{H_{s,\delta+1/2}(E, V)} = \sum_{p=1}^{s} \|\psi^{[p]}\|_{H_{s-p,\delta+p+1/2}(E)}^2,$$

we only need to prove that:

$$\|\psi^{[p]}\|_{H_{s-p,\delta+p+1/2}(E)} \leq c_p(\|\phi\|_{H_{s,\delta+1/2}(E)} + \|\psi\|_{H_{s-1,\delta+3/2}} + \|\beta\|_{H_{s-2,\delta+5/2}(E, V)}).$$

It is true for $p = 0, 1$. Let us use a reduction argument to prove this for all $p \leq s$. Suppose it is true for $0 \leq q \leq p - 1$. Take $D^p u$ to the equation $Lu = \beta$, and move all the terms containing $t$-derivatives of $u$ of order less than $p$, i.e. $D^p_t u$ with $q < p$, to the right hand side, then we get

$$\gamma^{00} \psi^{[p]} = D^p_t \beta - \sum_{q=0}^{p-3} \binom{p-2}{q} (D^{p-q-q_0} \gamma^{00}) \psi^{[q+2]} - \sum_{q=0}^{p-2} \binom{p-2}{q} (D^{p-q-q_0} \gamma^{00}) D^q_t \psi^{[q+1]} + (D^{p-q-q_0} \gamma^{ij}) D^0_t D^q_j \psi^{[q]} + (D^{p-q-q_0} \gamma^{ij}) D^0_t D^q_j \psi^{[q]} + (D^{p-q-q_0} \gamma^{ij}) D^0_t D^q_j \psi^{[q]}.$$

(5.35)
Using the multiplication Lemma 2.2 and Hypothesis (1) in the case:

\[ H_{s_2-1-(p-2)-q}, \delta_2+1/2+(p-2)-q}(E) \times H_{s-(q+2), \delta+1/2+(q+2)}(E) \rightarrow H_{s-p, \delta+1/2+p}(E), \]

we can estimate

\[ \|(D^p_{L, q} \gamma) D \psi \|_{H_{s-p, \delta+1/2+p}(E)} \leq c_3 \| \gamma - \eta \|_{H_{s_2-1, \delta_2+1/2}}(E, V) \| \psi \|_{H_{s-q, \delta+1/2+q}(E)} \]

where \( c_3 \) is a constant depending only on \( s \) and \( \delta \). Now using similar arguments to evaluate the \( H_{s-p, \delta+p+1/2}(E) \) norm of other terms in (5.35), together with our inductive hypothesis, we can get

\[ \| \gamma^{00} \|_{H_{s-p, \delta+p+1/2}(E)} \leq \| D^p_{L, q} \|_{H_{s-p, \delta+1/2+p}(E)} + c_4 \sum_{q=0}^{p-1} \| \psi \|_{H_{s-q, \delta+1/2+q}}(E) \]

\[ \leq c'_p \left( \| \phi \|_{H_{s, \delta+1/2}}(E) + \| \psi \|_{H_{s-1, \delta+3/2}}(E) + \| \beta \|_{H_{s-2, \delta+5/2}}(E, V) \right), \]  

(5.36)

where \( \mu \) is defined in (3.17), \( c_4 \) is a constant depending only on \( s, p \) and \( \delta \), while \( c'_p \) a constant depending only on \( \mu, s, p \) and \( \delta \).

Here

\[ \gamma^{00} = (\gamma^{0\mu} D_\mu D_\nu t) |_{t=0} = \sigma^2 (\gamma^{0\mu} D_\mu t D_\nu \tau) |_{t=0} = -N^{-2} \sigma^2 \leq -c < 0, \]

where \( c > 0 \) is a constant depending only on \( \theta \) and \( h \) according to Section 2 in [8]. Now \( (\gamma - \eta) \in H_{s_2, \delta_2}(V) \) implies that \( (\gamma^{00} + 1) \in H_{s_2-1, \delta_2+1/2}(E) \), hence \((\gamma^{00})^{-1} + 1 \in H_{s_2-1, \delta_2+1/2}(E)\) by Lemma 5.6 and furthermore \( \|(\gamma^{00})^{-1} + 1\|_{H_{s_2-1, \delta_2+1/2}}(E) \) is bounded by a constant depending only on \( n, s_2, \delta_2 \) and \( \| \gamma^{00} + 1 \|_{H_{s_2-1, \delta_2+1/2}}(E) \). Now multiply \( \gamma^{00} \psi \) by \((\gamma^{00})^{-1}\), and use equation (5.36) and the multiplication Lemma 2.2 then we finish the proof.

By combining all the above estimates, we can get the energy estimates in Theorem 3.6.

**Theorem 5.8.** Given \( L \) a differential operator defined by (3.14) in \( V_{0, \lambda} \), satisfying hypotheses (1) and (2). Let \( \beta \in H_{s-1, \delta+2}(V_{0, \lambda}) \), \( \phi \in H_{s, \delta+1/2}(E) \) and \( \psi \in H_{s-1, \delta+1/2}(E) \), with \( 2 \leq s \leq s' \), \( \delta \in \mathbb{R} \). Then every \( u \in H_{s+1, \delta}(V) \), which solves \( Lu = \beta \), with \( u \big|_E = \phi \), \( D_t u \big|_E = \psi \) satisfies the estimates (3.19).

**Proof.** First we can plug in (5.34) to (5.28). Then it follows from an approximation argument similar to the proof of Lemma 4.5 in [8] and an integration of (5.28) w.r.t. \( \tau \) on \([-\theta, \theta] \).

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