POINT PROCESSES, COST, AND THE GROWTH OF RANK IN LOCALLY COMPACT GROUPS*

BY

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ABSTRACT
Let $G$ be a locally compact, second countable, unimodular group that is nondiscrete and noncompact. We explore the ergodic theory of invariant point processes on $G$. Our first result shows that every free probability measure preserving (pmp) action of $G$ can be realized by an invariant point process.

We then analyze the cost of pmp actions of $G$ using this language. We show that among free pmp actions, the cost is maximal on the Poisson processes. This follows from showing that every free point process weakly factors onto any Poisson process and that the cost is monotone for weak factors, up to some restrictions. We apply this to show that $G \times \mathbb{Z}$ has fixed price 1, solving a problem of Carderi.

We also show that when $G$ is a semisimple real Lie group, the rank gradient of any Farber sequence of lattices in $G$ is dominated by the cost of the Poisson process of $G$. The same holds for the symmetric space $X$ of $G$. This, in particular, implies that if the cost of the Poisson process of the hyperbolic 3-space $\mathbb{H}^3$ vanishes, then the ratio of the Heegaard genus and the rank of a hyperbolic 3-manifold tends to infinity over arbitrary expander Farber sequences, in particular, the ratio can get arbitrarily large. On the other hand, if the cost of the Poisson process on $\mathbb{H}^3$ does not vanish, it solves the cost versus $L^2$ Betti problem of Gaboriau for countable equivalence relations.

1. Introduction
Let $G$ be a locally compact, second countable, unimodular group that is nondiscrete and noncompact, endowed with a Haar measure $\lambda$. We think of $\lambda$ as an inherent parameter of $G$, as all the notions trivially scale with $\lambda$. Throughout the paper, we will make these assumptions on $G$ except when stated otherwise.

A point process $\Pi$ on $G$ is a random closed and discrete subset $\Pi$ of $G$. More precisely, it is a random variable taking values in the configuration space of $G$:

$$\mathbb{M} = \mathbb{M}(G) = \{\omega \subseteq G \mid \omega \text{ is closed and discrete}\}.$$ 

When the law of $\Pi$ is invariant under the left $G$-action, we call $\Pi$ an invariant point process. We do not assume the reader has any knowledge of point process theory and have made the paper as self-contained as possible.

Invariant point processes are examples of probability measure preserving (pmp) actions. Recall that a pmp action is essentially free (or simply free for short) if the stabilizer of almost every point in the action space is trivial.
In the particular case of point processes, this means that the set of points will almost surely have no symmetries. Our first theorem shows that actually every free pmp action can be realized this way.

**Theorem 1.1:** Every free pmp action of $G$ is isomorphic to a point process on $G$.

Note that freeness is a necessary condition here as can be seen from the action of $\mathbb{R}^2$ on $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{R})$. This action is however a point process on a homogeneous space of $\mathbb{R}^2$.

The proof of Theorem 1.1 exhibits an analogy between point processes of locally compact groups and the symbolic dynamics of countable groups. For a pmp action of a countable group $\Gamma$, every Borel partition of the underlying space gives rise to an invariant random coloring of $\Gamma$ by considering the orbit of a random point of the underlying space. Similarly, every cross section of a free pmp action of $G$, when considering its intersection with the $G$-orbit of a random point, will become a point process on $G$. So point processes serve as stochastic visualizations of pmp actions of locally compact groups, just like invariant random colorings do for countable groups. This paper aims to show that this visualization leads to new and meaningful results.

The correspondence above and the classical theorem of Forrest [For74] on the existence of cross sections (see also [KPV15] and [Kec19, Section 3.B]) immediately yields that every free pmp action factors onto an invariant point process. The factor map can be upgraded to an isomorphism by using a marked point process. These are random discrete subsets where each point carries a mark from some mark space (for example, a finite set of colors). Then Theorem 1.1 is proved by showing that every marked point process is isomorphic to an unmarked one, by “spatially encoding” the marks.

We now introduce the cost of a point process $\Pi$ on $G$. A **factor graph** $\mathcal{G}$ of $\Pi$ is an equivariantly and measurably defined graph $\mathcal{G}(\Pi)$ whose vertex set is $\Pi$. For example, one can define the distance graph for $r > 0$ to be the set of pairs $g, h \in \Pi$ with $d(g, h) < r$, where $d$ is a left-invariant metric on $G$. Informally speaking, the **cost** of $\Pi$ is defined as the infimum of the average degrees over connected factor graphs of $\Pi$, suitably normalized to be an isomorphism invariant. For precise definitions see Section 4. We then define cost for free pmp actions of $G$ via Theorem 1.1, which is well-defined since cost is an isomorphism invariant.
The cost of pmp actions of countable groups has been an active subject in the last twenty years; see Gaboriau’s paper [Gab] and the survey paper [Fur11] for the literature. It has been known in the community that cross sections naturally allow one to extend the notion of cost to free pmp actions of locally compact groups, but due to the lack of results, the notion stayed dormant. The first explicit appearance of the definition can be found in a recent paper of Carderi [Car18]. The definition we work with is essentially equivalent to his, but we develop it intrinsically as a point process theoretic notion.

One of the most important families of processes on a discrete group is Bernoulli percolations $\text{Ber}(p)$. The natural analogue of this family for nondiscrete groups is the Poisson point process of intensity $t > 0$. Here the intensity of an invariant point process is the expected number of points which fall in a set of unit volume. This quantity can be shown to be independent of the choice of set. An explicit description of Poisson point processes will be given later, but one should know that these processes are “completely independent”.

**Theorem 1.2:** Poisson point processes have maximal cost among all free pmp actions on $G$. In particular, the cost of a Poisson point process does not depend on its intensity.

We denote the cost of a Poisson point process on $G$ by $c_P(G)$. The above result can be looked at as a locally compact analogue of a result of Abért and Weiss [AW13] where they show that for a countable group, Bernoulli actions have maximal cost among all pmp actions.

A central open problem in cost theory is the Fixed Price problem of Gaboriau, that asks whether all free pmp actions of a countable group have the same cost. This is also open in the locally compact setting.

**Question 1:** Is it true that all free point processes on $G$ have the same cost?

Gaboriau [Gab02] asks if for a countable pmp equivalence relation, the cost of the relation equals its first $L^2$ Betti number $\beta_1(G)$ plus 1. Note that an affirmative answer for this would imply an affirmative answer to Question 1, as well, using the cross section correspondence.

Since the cost of any free process is at least one, a viable way to prove that a group has fixed price one is by showing that the Poisson point process admits connected factor graphs with average degree $1 + \varepsilon$ for all $\varepsilon > 0$. We succeed in this for the first nontrivial case, answering a question of Carderi [Car18]:
Theorem 1.3: Every free pmp action of $G \times \mathbb{Z}$ has cost one if $G$ is compactly generated.

Our proof is truly a stochastic proof in nature as it essentially uses some special properties of Poisson point processes.

In countable cost theory, it remains an open question if the direct product $\Gamma \times \Delta$ of two infinite countable groups $\Gamma$ and $\Delta$ has fixed price one. It is known to hold if one of the groups contains a fixed price one subgroup. When trying to generalize Theorem 1.3 to arbitrary products, we seem to hit a somewhat similar barrier.

Question 2: Let $G$ and $H$ be compactly generated but noncompact groups. Does $G \times H$ have fixed price one?

Another application of Theorem 1.2 concerns the growth of the minimal number of generators (the rank gradient) for a sequence of lattices in semisimple real Lie groups. Recall that a discrete subgroup $\Gamma \leq G$ is a lattice if it has finite covolume in $G$. Let $d(\Gamma)$ denote the minimal number of generators (also known as the rank) of $\Gamma$. When $G$ is a semisimple Lie group, $d(\Gamma)$ is finite and by a theorem of Gelander [Gel11], we have

$$\frac{d(\Gamma) - 1}{\text{vol}(G/\Gamma)} \leq C$$

for some constant $C$ only depending on $G$.

A sequence of lattices $\Gamma_n$ in $G$ is Farber, if $G/\Gamma_n$ approximates $G$ in the Benjamini–Schramm topology, or, equivalently, if the corresponding invariant random subgroups weakly converge to the trivial subgroup.

Theorem 1.4: Let $G$ be a semisimple real Lie group and let $\Gamma_n$ be a Farber sequence of lattices in $G$. Then

$$\limsup_{n \to \infty} \frac{d(\Gamma_n) - 1}{\text{vol}(G/\Gamma_n)} \leq c_P(G) - 1.$$
Farber sequences. Here a sequence of lattices is uniformly discrete if there exists $C > 0$ such that the infimal injectivity radius is bounded below by $C$ for $G/\Gamma_n$.

**Question 3:** Let $G$ be a semisimple real Lie group that is not a compact extension of $\text{SL}_2(\mathbb{R})$. Is $c_P(G) = 1$?

Note that by work of Conley, Gaboriau, Marks and Tucker-Drob the group $\text{SL}_2(R)$ is treeable and has fixed price greater than 1.

We now showcase three concrete cases for a semisimple Lie group where computing the cost of the Poisson process would solve known problems of a different nature. Note that it is natural to ask about the cost of the Poisson process on the symmetric space $X$ of $G$ rather than on the group $G$ itself. As we show in Theorem 7.7 these two invariants are equal.

**Case $G = \text{SL}_2(\mathbb{C})$ and $X = \mathbb{H}^3$.** If $c_P(G) > 1$, then we get free point processes in $G$ with different cost. Moreover, we also get a countable equivalence relation whose first $L^2$ Betti number is not equal to its cost-1, answering a question of Gaboriau [Gab02]. If, on the other hand, $c_P(G) = 1$, then we get that the Heegaard genus divided by the rank of the fundamental group of a (compact) hyperbolic 3-manifold can get arbitrarily large. In fact, we yield this for any expander Farber sequence of hyperbolic 3-manifolds, which is understood as the typical behavior. Indeed, by the work of Lackenby [Lac06] for expander sequences, the Heegaard genus grows linearly, while using our work, the rank would grow sublinearly in the volume. Note that it is a longstanding open problem whether this ratio is absolutely bounded over all 3-manifolds, and in fact it was only proved recently in the deep paper of Li [Li13] that for compact hyperbolic 3-manifolds, the Heegaard genus can differ from the rank.

**Case when $G$ has higher rank.** For these Lie groups, Fraczyk recently proved in a beautiful paper [Fra22] that the growth of the first mod 2 homology vanishes for Farber sequences in $G$. Surprisingly, his method does not seem to carry over to odd primes, so for primes other than 2, this is still an open problem. As the rank is an upper bound for the first mod $p$ homology of a discrete group, proving $c_P(G) = 1$ would settle this problem.

By a standard induction argument, proving $c_P(G) = 1$ would show that any lattice in $G$ has fixed price 1, a problem of Gaboriau that is still open for cocompact lattices.
CASE WHEN $G$ HAS HIGHER RANK AND PROPERTY (T). Using [ABB$^+$17], for semisimple Lie groups with (T) one can actually omit the Farber condition.

**Corollary 1.5:** Let $G$ be a higher rank semisimple real Lie group with property (T) and let $\Gamma_n$ be any sequence of lattices in $G$ with $\text{vol}(G/\Gamma_n) \to \infty$. Then

$$\limsup_{n \to \infty} \frac{d(\Gamma_n) - 1}{\text{vol}(G/\Gamma_n)} \leq c_p(G) - 1.$$  

In particular, if $c_p(G) = 1$, then we get a totally uniform vanishing theorem for the growth of rank for lattices in these groups, including $\text{SL}(d,\mathbb{R})$ ($d \geq 3$). It is shown in [AGN17] that any Farber sequence in $\text{SL}(3,\mathbb{Z})$ has vanishing rank gradient, but the uniform version is wide open.

Note that in their very recent paper Lubotzky and Slutsky [LS22] showed that in the above situation, every sequence of non-uniform lattices will have rank gradient 0. Their proof uses deep classical results on non-uniform lattices, like arithmeticity and the Congruence Subgroup Property but in turn gives a much stronger upper bound for the number of generators than what we ask, logarithmic in the covolume. In most cases, they can even improve this with a loglog factor. Their methods do not seem to readily generalize to co-compact lattices. Our purely geometric approach may have the potential to be applied more widely but the payoff is that, being a limiting argument, it is not expected to yield such explicit estimates.

The proof of Theorem 1.2 uses the stochastic visualization method to show that every free action is “sufficiently rich” in randomness to “simulate” the Poisson point process. In particular, connected factor graphs of the Poisson point process can be transferred to an arbitrary free process in a way that can at worst decrease the average degree. Simulation here refers to weak factoring, a notion we introduce that is inspired by weak containment of actions; see the survey of Kechris and Burton [BK20].

For invariant point processes $\Pi$ and $\Upsilon$, we say that $\Upsilon$ is a **factor** of $\Pi$ if there exists a $G$-equivariant Borel map $\Phi : \mathbb{M} \to \mathbb{M}$ such that

$$\Phi(\Pi) = \Upsilon.$$  

We say that $\Upsilon$ is a **weak factor** of $\Pi$ or $\Pi$ **weakly factors** onto $\Upsilon$ if there exist factor maps $\Phi_1, \Phi_2, \ldots$ of $\Pi$ such that $\Phi_n(\Pi)$ converges weakly to $\Upsilon$. 
Theorem 1.6: Let $\Pi$ be a free point process on $G$. Then $\Pi$ weakly factors onto the Poisson point process of intensity $t$, for all $t$.

In particular, Poisson processes on $G$ of different intensities weakly factor onto each other. More is known in the amenable case: Ornstein and Weiss showed [OW87] that for a large class of amenable groups, the Poisson point processes of different intensity are in fact isomorphic as actions (see [SW+19] for an alternative construction on $\mathbb{R}^n$ with additional properties).

The proof of Theorem 1.6 revolves around IID-marked processes. Let $[0,1]^{\Pi}$ denote the random $[0,1]$-marked subset of $G$ whose underlying set is $\Pi$ and has independent and identically distributed $\text{Unif}[0,1]$ random variables. We call this the IID of $\Pi$. Once this definition and that of the Poisson point process is understood, one can readily see that the IID of any process factors onto the Poisson point process. We then prove:

Theorem 1.7: Let $\Pi$ be a free point process on $G$. Then $\Pi$ weakly factors onto $[0,1]^{\Pi}$, its own IID.

Somewhat surprisingly, it is not entirely trivial to show that weak factoring is a transitive notion, but we are able to prove it. Thus in particular, Theorem 1.7 implies that free point processes weakly factor onto the Poisson point process.

We next investigate how cost behaves with respect to factor maps. It is easy to see that it can only increase under a factor map: if $\Pi$ factors onto $\Upsilon$, then $\text{cost}(\Pi) \leq \Upsilon$. In particular, this shows that cost is an isomorphism invariant of actions. This monotonicity of cost under factor maps can be pushed further:

Theorem 1.8: Suppose $\Pi$ weakly factors onto $\Upsilon$, as witnessed by a sequence of factor maps $\Phi_n(\Pi)$ weakly converging to $\Upsilon$. Under appropriate tightness conditions on $\Pi, \Upsilon,$ and the sequence $\Phi_n$, we have $\text{cost}(\Pi) \leq \text{cost}(\Upsilon)$.

See Section 5.2 for a precise statement. This cost monotonicity theorem, limited as it is, is powerful enough to prove that the Poisson point process has maximal cost amongst all free processes.

The paper is structured as follows. In Section 1, we give the basic definitions and notations of point processes for those who have never encountered them before, and describe the most important examples of point processes for our work. In Section 2, we introduce the Palm measure of a point process and the rerooting groupoid. In Section 3, we define the cost of an invariant point
process and prove basic properties of it. In Section 4, we define weak factoring of point processes and prove that (in certain circumstances) cost is monotone with respect to weak factoring. We use this to show that the Poisson has maximal cost amongst all free processes. In Section 5, we use the fact that the Poisson has maximal cost to give the first nontrivial examples of nondiscrete groups with fixed price. In Section 6, we connect the rank gradient of sequences of lattices in a group with the cost of the Poisson point process on said group. In Section 7, we discuss the modifications required to extend the above theory to invariant point processes on symmetric spaces. In the appendix, we include a summary of necessary technical facts from point process theory with references for proofs. No originality is claimed for this material.

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2. Point processes and factors of interest

Let $(Z,d)$ denote a complete and separable metric space (a csms). A point process on $Z$ is a random discrete subset of $Z$. We will also study random discrete subsets of $Z$ that are marked by elements of an additional csms $\Xi$. Typically $\Xi$ will be a finite set that we think of as colors.

Definition 2.1: The configuration space of $Z$ is

$$\mathbb{M}(Z) = \{ \omega \subset Z \mid \omega \text{ is locally finite} \},$$

and the $\Xi$-marked configuration space of $Z$ is

$$\Xi^M(Z) = \{ \omega \subset Z \times \Xi \mid \omega \text{ is discrete, and if } (g,\xi) \in \omega \text{ and } (g,\xi') \in \omega \text{ then } \xi = \xi' \}. $$

Note that

$$\Xi^M(Z) \subset \mathbb{M}(Z \times \Xi).$$

We think of a $\Xi$-marked configuration $\omega \in \Xi^M(Z)$ as a locally finite subset of $Z$ with labels on each of the points (whereas a typical element of $\mathbb{M}(Z \times \Xi)$ is a locally finite subset where each point has possibly multiple marks).

If $\omega \in \Xi^M(Z)$ is a marked configuration, then we will write $\omega_z$ for the unique element of $\Xi$ such that $(z,\omega_z) \in \omega$. 
The Borel structure on configuration spaces is exactly such that the following point counting functions are measurable. Let $U \subseteq Z$ be a Borel set. It induces a function $N_U : \mathcal{M}(Z) \rightarrow \mathbb{N}_0 \cup \{\infty\}$ given by

$$N_U(\omega) = |\omega \cap U|.$$ 

We will primarily be interested in point processes defined on locally compact and second countable (lcsc) groups $G$. Such groups admit a unique (up to scaling) left-invariant Haar measure $\lambda$; we fix such a choice. We will further assume that $G$ is unimodular, although it is not strictly necessary for every argument in the paper. Recall:

**Theorem 2.2** (Struble’s theorem, see [CdlH16, Theorem 2.B.4]): *Let $G$ be a locally compact topological group. Then $G$ is second countable if and only if it admits a proper\(^1\) left-invariant metric.*

Such a metric is unique up to coarse equivalence (bilipschitz if the group is compactly generated). We fix $d$ to be any such metric.

We mostly consider the configuration space of a fixed group $G$. So out of notational convenience let us write $\mathcal{M} = \mathcal{M}(G)$ and $\Xi^\mathcal{M} = \Xi^\mathcal{M}(G)$. The latter here is an abuse of notation: formally $\Xi^\mathcal{M}$ ought to denote the set of functions from $\mathcal{M}$ to $\Xi$, but instead we are using it to denote the set of functions from elements of $\mathcal{M}$ to $\Xi$.

Note that the marked and unmarked configuration spaces of $G$ are Borel $G$-spaces. To spell this out, $G \rhd \mathcal{M}$ by

$$g \cdot \omega = g\omega$$

and $G \rhd \Xi^\mathcal{M}$ by

$$g \cdot \omega = \{(gx, \xi) \in G \times \Xi \mid (g, \xi) \in \omega\}.$$ 

**Definition 2.3:** A point process on $G$ is a $\mathcal{M}(G)$-valued random variable

$$\Pi : (\Omega, \mathbb{P}) \rightarrow \mathcal{M}(G).$$

Its law or distribution $\mu_\Pi$ is the pushforward measure $\Pi_* (\mathbb{P})$ on $\mathcal{M}(G)$. It is invariant if its law is an invariant probability measure for the action $G \rhd \mathcal{M}(G)$.

The associated point process action of an invariant point process $\Pi$ is

$$G \rhd (\mathcal{M}(G), \mu_\Pi).$$

\(^1\) Recall that a metric is proper if closed balls are compact.
Some remarks and caveats are in order:

- Point processes which are not invariant are very much of interest, but will only come up when we discuss “Palm processes”. Thus we will sometimes say “point process” when we strictly mean invariant point process.

- Speaking properly, we are discussing simple point processes, that is, those where each point has multiplicity one. We will discuss this more later.

- $\Xi$-marked point processes are defined similarly, with $\Xi^M$ taking the place of $M$. There is not much difference between marked point processes and unmarked ones for our purposes (it is just a case of which is more convenient for the particular problem at hand). Thus “point process” might also mean “marked point process”.

- One could certainly define point processes on a discrete group, but this is essentially percolation theory. We are specifically trying to understand the nondiscrete case, and so will assume $G$ is nondiscrete.

- The other case of interest we will discuss is $\text{Isom}(S)$-invariant point processes on $S$, where $S$ is a Riemannian symmetric space. For instance, one would consider isometry invariant point processes on Euclidean space $\mathbb{R}^n$ or hyperbolic space $\mathbb{H}^n$. We will discuss this case more in Section 7.2.

- Our interest in point processes is almost exclusively as actions. We will therefore rarely distinguish between a point process proper and its distribution. Thus we may use expressions like “suppose $\mu$ is a point process” to mean “suppose $\mu$ is the distribution of some point process”.

- The configuration space of any Polish space will be Polish, so the probability theory of point processes on such spaces is well behaved. The metric properties of configuration spaces that we require are listed in the appendix, with references for proofs.

**Definition 2.4:** The **intensity** of a point process $\mu$ is

$$
\text{int}(\mu) = \frac{1}{\lambda(U)} \mathbb{E}_\mu [N_U],
$$

where $U \subset G$ is any Borel set of positive (but finite) Haar measure, and

$$
N_U(\omega) = |\omega \cap U|
$$

is its point counting function.
To see that the intensity is well-defined (that is, does not depend on our choice of $U$), observe that the function $U \mapsto \mathbb{E}_\mu[N_U]$ defines a Borel measure on $G$ which inherits invariance from the shift invariance of $\mu$. So by uniqueness of Haar measure, it is some scaling of our fixed Haar measure $\lambda$—the intensity is exactly this multiplier. We also see that whilst the intensity depends on our choice of Haar measure, it scales linearly with it.

Note that a point process has intensity zero if and only if it is empty almost surely.

2.1. Examples of point processes.

Example 1 (Lattice shifts): Let $\Gamma < G$ be a lattice, that is, a discrete subgroup that admits an invariant probability measure $\nu$ for the action $G \curvearrowright G/\Gamma$. The natural map $\mathbb{M}(G/\Gamma) \to \mathbb{M}(G)$ given by

$$\omega \mapsto \bigcup_{a\Gamma \in \omega} a\Gamma$$

is left-equivariant, and hence maps invariant point processes on $G/\Gamma$ to invariant point processes on $G$. In particular, we have the lattice shift, given by choosing a $\nu$-random point $a\Gamma$.

Example 2 (Induction from a lattice): Now suppose one also has a pmp action $\Gamma \curvearrowright (X, \mu)$. It is possible to induce this to a pmp action of $G$ on $G/\Gamma \times X$. This can be described as an $X$-marked point process on $G$. To do this, fix a fundamental domain $F \subset G$ for $\Gamma$. Choose $f \in F$ uniformly at random, and independently choose a $\mu$-random point $x \in X$. Let

$$\Pi = \{(f\gamma, \gamma \cdot x) \in G \times X \mid \gamma \in \Gamma\}.$$ 

Then $\Pi$ is a $G$-invariant $X$-marked point process.

In this way one can view point processes as generalised lattice shift actions. Note that there are groups without lattices (for instance Neretin’s group, see [BCGM12]), but every group admits interesting point processes, as we discuss now. The most fundamental of these is known as the Poisson point process. We will define this after reviewing the Poisson distribution:

Recall that a random integer $N$ is Poisson distributed with parameter $t > 0$ if

$$\mathbb{P}[N = k] = \exp(-t) \frac{t^k}{k!}.$$
We write $N \sim \text{Pois}(t)$ to denote this. It is convenient to extend this definition to $t = 0$ and $t = \infty$ by declaring $N \sim \text{Pois}(0)$ when $N = 0$ almost surely and $N \sim \text{Pois}(\infty)$ when $N = \infty$ almost surely.

**Definition 2.5:** Let $X$ be a complete and separable metric space equipped with a non-atomic Borel measure $\lambda$.

A point process $\Pi$ on $X$ is **Poisson with intensity** $t > 0$ if it satisfies the following two properties:

(Poisson point counts) for all $U \subseteq G$ Borel, $N_U(\Pi)$ is a Poisson distributed random variable with parameter $t\lambda(U)$, and

(Total independence) for all $U, V \subseteq G$ disjoint Borel sets, the random variables $N_U(\Pi)$ and $N_V(\Pi)$ are independent.

For reasons that should not be immediately apparent, the above defining properties are in fact equivalent. We will write $\mathcal{P}_t$ for the distribution of such a random variable, or simply $\mathcal{P}$ if the intensity is understood.

We think of the Poisson point process as a completely random scattering of points in the group. It is an analogue of Bernoulli site percolation for a continuous space.

We now construct the process (somewhat) explicitly. Partition $G$ into disjoint Borel sets $U_1, U_2, \ldots$ of positive but finite volume. For each of these, independently sample from a Poisson distribution with parameter $t\lambda(U_i)$. Place that number of points in the corresponding $U_i$ (independently and uniformly at random).

This description can be turned into an explicit sampling rule,\(^2\) if one desires.

For proofs of basic properties of the Poisson point process (such as the fact that it does not depend on the partition chosen above), see the first five chapters of Kingman’s book [Kin93].

**Definition 2.6:** A pmp action $G \curvearrowright (X, \mu)$ is **ergodic** if for every $G$-invariant measurable subset $A \subseteq X$, we have $\mu(A) = 0$ or $\mu(A) = 1$.

The action is **mixing** if for all measurable $A, B \subseteq (X, \mu)$ we have

$$\lim_{g \to \infty} \mu(gA \cap B) = \mu(A)\mu(B).$$

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\(^2\) That is, one can define a measurable function $f : \prod_n X_n \to M$ defined on an appropriate product of probability spaces such that the pushforward measure is the distribution of the Poisson point process.
The action is **essentially free** if \( \text{stab}_G(x) = \{1\} \) for \( \mu \) almost every \( x \in X \). In the case of point process actions we will sometimes use the term **aperiodic** to refer to this.

**Proposition 2.7**: The Poisson point process actions \( G \c (M, \mathcal{P}_t) \) on a noncompact group \( G \) are essentially free and ergodic (in fact, mixing).

A proof of freeness that is readily adaptable to our setting can be found as [ABB+17, Proposition 2.7]. For ergodicity and mixing, see the proof of the discrete case in Proposition 7.3 of the Lyons–Peres book [LP16]. It directly adapts, once one knows the required cylinder sets exist.

Although the subscript \( t \) suggests that the Poisson point processes form a continuum family of actions, this is not always the case:

**Theorem 2.8** (Ornstein–Weiss): Let \( G \) be an amenable group which is not a countable union of compact subgroups. Then the Poisson point process actions \( G \c (M, \mathcal{P}_t) \) are all isomorphic.

The following definition uses notation that does not appear in the literature (the object of course does, but there does not appear to be a symbolic representation for it):

**Definition 2.9**: If \( \Pi \) is a point process, then its **IID version** is the \([0,1]\)-marked point process \([0,1]^\Pi\) with the property that conditional on its set of points, its labels are independent and IID \( \text{Unif}[0,1] \) distributed. If \( \mu \) is the law of \( \Pi \), then we will write \([0,1]^\mu\) for the law of \([0,1]^\Pi\).

One can define the IID of a point process over spaces other than \([0,1]\) (for instance, \([n]=\{1,2,\ldots,n\}\) with the counting measure), but we will only use the full IID.

**Remark 2.10**: As we have mentioned, \([0,1]\)-marked point processes on \( G \) are particular examples of point processes on \( G \times [0,1] \). One can show (see [LP18, Theorem 5.6]) that the Poisson point process on \( G \times [0,1] \) with respect to the product measure \( \lambda \otimes \text{Leb} \) is just the IID version of the Poisson point process on \( G \), a fact which we will make use of later.

**Proposition 2.11**: The IID Poisson point process on a noncompact group is ergodic (and in fact mixing).
This can be seen by viewing the IID Poisson on $G$ as the Poisson point process on $G \times S^1$, restricted to $G$. Note that the restriction of a mixing action to a noncompact subgroup is mixing.

**Remark 2.12:** One can define “the IID” of any probability measure preserving countable Borel equivalence relation, see [BHI18]. This construction is known as the Bernoulli extension, and is ergodic if the base space is ergodic.

**Proposition 2.13:** Let $\Pi$ be a point process on a group $G$ which is non-empty almost surely. Then $|\Pi| = \infty$ almost surely if and only if $G$ is noncompact.

**Proof.** It is immediate that any point process on a compact group must be finite almost surely (as it is a discrete subset of the space).

Now suppose $\Pi$ is a non-empty point process on $G$ which is finite almost surely. Then the IID of this process $[0, 1]^{\Pi}$ still has this property. We define the following $G$-valued random variable:

$$f([0, 1]^{\Pi}) = \text{the unique } x \in \Pi \text{ with maximal label in } [0, 1]^{\Pi}.$$  

The invariance of the point process translates into equivariance of the map $f : [0, 1]^M \rightarrow G$. Thus this random variable’s law is an invariant probability measure on $G$. Such a measure exists exactly when $G$ is compact. ■

**2.2. Factors of point processes.**

**Definition 2.14:** A **point process factor map** is a $G$-equivariant and measurable map $\Phi : M \rightarrow M$. If $\mu$ is a point process and $\Phi$ is only defined $\mu$ almost everywhere, then we will call it a $\mu$ **factor map**.

We will be interested in two monotonicity conditions:

- if $\Phi(\omega) \subseteq \omega$ for all $\omega \in M$, we will call $\Phi$ a **thinning** (and usually denote it by $\theta$), and
- if $\Phi(\omega) \supseteq \omega$ for all $\omega \in M$, we will call $\Phi$ a **thickening** (and usually denote it by $\Theta$).

We use the same terms for marked point processes as well.

**Remark 2.15:** There are two possible ways to interpret the above monotonicity conditions for a $\Xi$-marked point process, depending on what you want to do with the mark space. One can consider

$$\Phi : \Xi^M \rightarrow \Xi^M, \text{ or } \Phi : \Xi^M \rightarrow M.$$
In the former case, the definition above works verbatim. In the latter case, one should interpret a statement like \( \omega \subseteq \Phi(\omega) \) as \( \omega \) is contained in the underlying set \( \pi(\Phi(\omega)) \) of \( \Phi(\omega) \), where \( \pi : \Xi^M \to M \) is the map that forgets labels.

**Example 3** (Metric thinning): Let \( \delta > 0 \) be a tolerance parameter. The \( \delta \)-**thinning** is the equivariant map \( \theta_\delta : M \to M \) given by

\[
\theta_\delta(\omega) = \{ g \in \omega \mid d(g, \omega \setminus \{g\}) > \delta \}.
\]

When \( \theta_\delta \) is applied to a point process, the result is always a \( \delta \)-separated point process\(^3\) (but possibly empty).

We define \( \theta_\delta \) in the same way for marked point processes (that is, it simply ignores the marks).

**Example 4** (Independent thinning): Let \( \Pi \) be a point process. The **independent** \( p \)-**thinning** defined on its IID \([0, 1]^{\Pi}\) is given by

\[
I_p([0, 1]^{\Pi}) = \{ g \in \Pi \mid \Pi_g \leq p \}.
\]

One can show that independent \( p \)-thinning of the Poisson point process of intensity \( t > 0 \) yields the Poisson point process of intensity \( pt \), as one would expect. See [LP18, Chapter 5] for further details.

**Example 5** (Constant thickening): Let \( F \subset G \) be a finite set containing the identity \( 0 \in G \), and \( \Pi \) be a point process which is \( F \)-**separated** in the sense that \( \Pi \cap \Pi_f = \emptyset \) for all \( f \in F \setminus \{0\} \). Then there is the associated thickening \( \Theta^F(\Pi) = \Pi F \). It is intuitively obvious that \( \text{int}(\Theta^F(\Pi)) = |F| \text{int}(\Pi) \). This can be formally established as follows: let \( U \subseteq G \) be of unit volume. Then

\[
\text{int}(\Theta^F(\Pi)) = \mathbb{E}|U \cap F| \quad \text{by definition}
\]

\[
= \sum_{f \in F} \mathbb{E}|U \cap \Pi_f| \quad \text{by } F\text{-separation}
\]

\[
= \sum_{f \in F} \mathbb{E}|U f^{-1} \cap \Pi| \quad \text{by } \text{unimodularity}
\]

\[
= \sum_{f \in F} \mathbb{E}|U \cap \Pi| \quad \text{by } \text{unimodularity}
\]

\[
= \mathbb{E}|F| \text{int}(\Pi).
\]

This is the first real appearance of our unimodularity assumption.

---

\(^3\) Probabilists refer to such processes as **hard-core**.

In particular, we can demonstrate that \( \text{int} \Theta^F(\Pi) = |F| \text{int} \Pi \) is not automatically true without unimodularity. For this, let \( \Pi \) denote the unit intensity Poisson point process on \( G \), and \( F = \{0, f\} \) where \( f \in G \) is chosen such that \( \lambda(Uf^{-1}) < 1 \). Then \( |Uf^{-1} \cap \Pi| \) is Poisson distributed with parameter \( \lambda(Uf^{-1}) \), and so by the above calculation \( \text{int} \Theta^F(\Pi) < 2 \cdot \text{int} \Pi \).

Monotone maps have been investigated in the specific case of the Poisson point process on \( \mathbb{R}^n \). We note the following interesting theorems:

**Theorem 2.16** (Holroyd, Peres, Soo [HLS11]): Let \( s > t > 0 \). Then the Poisson point process on \( \mathbb{R}^n \) of intensity \( s \) can be thinned to the Poisson point process of intensity \( t \). That is, there exists an equivariant and deterministic thinning \( \theta : (\mathcal{M}(\mathbb{R}), \mathcal{P}_s) \to (\mathcal{M}(\mathbb{R}), \mathcal{P}_t) \).

**Theorem 2.17** (Gurel-Gurevich and Peled [GGP13]): Let \( t > s > 0 \) be intensities. Then the Poisson point process on \( \mathbb{R}^n \) of intensity \( s \) cannot be thickened to the Poisson point process of intensity \( t \). That is, there is no equivariant and deterministic thickening \( \Theta : (\mathcal{M}(\mathbb{R}), \mathcal{P}_s) \to (\mathcal{M}(\mathbb{R}), \mathcal{P}_t) \).

We stress in the above theorems the deterministic nature of the maps. If one is allowed additional randomness (that is, one asks for a factor of IID map), then both theorems are easily established.

We note the following fact, which we will use (and prove) later after developing some notation.

**Example 6:** If \( \Pi \) is any non-empty point process, then its IID factors onto the Poisson (in fact, onto the IID Poisson).

**Definition 2.18:** A **factor \( \Xi \)-marking** of a point process is a \( G \)-equivariant map \( \mathcal{C} : \mathcal{M} \to \Xi^\mathcal{M} \) such that the underlying subset in \( G \) of \( \mathcal{C}(\omega) \) is \( \omega \). That is, \( \mathcal{C} \) is a rule that assigns a mark from \( \Xi \) to each point of \( \omega \) in some deterministic way. Again, if \( \mathcal{C} \) is only defined \( \mu \) almost everywhere then we will call it a **\( \mu \)-factor \( \Xi \)-marking**.

**Example 7:** Let \( \theta : \mathcal{M} \to \mathcal{M} \) be a thinning. Then the associated 2-coloring is \( \mathcal{C}_\theta : \mathcal{M} \to \{0, 1\}^\mathcal{M} \) given by

\[
\mathcal{C}_\theta(\omega) = \{(g, 1|g \in \theta(\omega))| g \in \omega\}.
\]

We will see that all markings are built out of thinnings in a similar way.
Remark 2.19: There is a difference between the thinning map \( \theta \) and the resulting thinned process \( \theta_* (\mu) \) that can be a source for confusion. Passing to the thinned process (in principle) can lose information about \( \mu \).

For example, let \( \Pi \) denote a Poisson point process on \( G \) and \( \Upsilon \) an independent random shift of a lattice \( \Gamma < G \). Define the following thinning \( \theta : \mathbb{M} \to \mathbb{M} \) by

\[
\theta(\omega) = \{ g \in \omega \mid g\Gamma \subseteq \omega \}.
\]

Then \( \theta(\Pi \cup \Upsilon) = \Upsilon \), and so the thinning completely loses the Poisson point process.

Definition 2.20: Let \( \Phi : \mathbb{M} \to \mathbb{M} \) be a factor map. We think of its input \( \omega \) as being red, its output \( \Phi(\omega) \) as being blue, and their overlap \( \omega \cap \Phi(\omega) \) as being purple.

For \( g \in \omega \), let \( \text{Color}(g) \in \{\text{Red}, \text{Blue}, \text{Purple}\} \) be

\[
\text{Color}(g) = \begin{cases} 
\text{Red} & \text{if } g \in \omega \setminus \Phi(\omega), \\
\text{Blue} & \text{if } g \in \Phi(\omega) \setminus \omega, \\
\text{Purple} & \text{if } g \in \omega \cap \Phi(\omega).
\end{cases}
\]

Now define \( \Theta^\Phi : \mathbb{M} \to \{\text{Red}, \text{Blue}, \text{Purple}\}^\mathbb{M} \) to be the following input/output thickening of \( \Phi \) (see also Figure 1):

\[
\Theta^\Phi(\omega) = \{(g, \text{Color}(g)) \in G \times \text{Red, Blue, Purple} \mid g \in \omega \}.
\]

Let \( \pi : \{\text{Red, Blue, Purple}\}^\mathbb{M} \to \mathbb{M} \) be the projection map that deletes red points and then forgets colors, that is,

\[
\pi(\omega) = \{ g \in \omega \mid \omega_g \in \{\text{Blue, Purple}\} \}.
\]

Remark 2.21: Observe that \( \Phi = \pi \circ \Theta^\Phi \)—that is, an arbitrary factor map decomposes as the composition of a thinning and a thickening. In this way we can often reduce the study of arbitrary factors to that of monotone factors.

Definition 2.22: The space of graphs in \( G \) is

\[
\text{Graph}(G) = \{(V, E) \in \mathbb{M}(G) \times \mathbb{M}(G \times G) \mid E \subseteq V \times V \}.
\]

This is a Borel \( G \)-space (with the diagonal action).

A factor graph is a measurable and \( G \)-equivariant map \( \Phi : \mathbb{M}(G) \to \text{Graph}(G) \) with the property that the vertex set of \( \Phi(\omega) \) is \( \omega \).

If a factor graph is connected, then we will refer to it as a graphing.
Remark 2.23: The elements of $\text{Graph}(G)$ are technically directed graphs, possibly with loops, and without multiple edges between the same pair of vertices. It is possible to define (in a Borel way) whatever space of graphs one desires (colored, undirected, etc.) by taking appropriate subsets of products of configuration spaces.

Remark 2.24: One might prefer to call factor graphs as above monotone factor graphs. Our terminology follows that of probabilists; see, for instance, [HP05]. We have not found a use for the less restrictive factor graph concept.

Example 8: The distance-$R$ factor graph is the map $\mathcal{D}_R : \mathcal{M} \to \text{Graph}(G)$ given by

$$\mathcal{D}_R(\omega) = \{(g, h) \in \omega \times \omega \mid d(g, h) \leq R\}.$$ 

The connectivity properties of this graph fall under the purview of continuum percolation theory, see for instance [MR96].
3. The rerooting equivalence relation and groupoid

We now introduce a pair of algebraic objects that capture factors of a point process. For exposition’s sake, we will first discuss unmarked point processes on a group $G$.

**Definition 3.1:** The space of rooted configurations on $G$ is

$$M_0(G) = \{ \omega \in M(G) \mid 0 \in \omega \}.$$  

If $G$ is understood, then we will drop it from the notation for clarity.

The rerooting equivalence relation on $M_0$ is the orbit equivalence relation of $G \acts M$ restricted to $M_0$. Explicitly:

$$R = \{ (\omega, g^{-1}\omega) \in M_0 \times M_0 \mid g \in \omega \}.$$  

This defines a countable Borel equivalence relation structure on $M_0$. It is degenerate whenever $\omega \in M_0$ exhibits symmetries: for instance, the equivalence class of $\mathbb{Z}$ viewed as an element of $M_0(\mathbb{R})$ is a singleton. We are usually interested in essentially free actions, where such difficulties will not occur. Nevertheless, we do care about lattice shift point processes and so we will introduce a groupoid structure that keeps track of symmetries.

The space of birooted configurations is

$$\overrightarrow{M}_0 = \{ (\omega, g) \in M_0 \times G \mid g \in \omega \}.$$  

We visualise an element $(\omega, g) \in \overrightarrow{M}_0$ as the rooted configuration $\omega \in M_0$ with an arrow pointing to $g \in \omega$ from the root (i.e., the identity element of $G$).

The above spaces form a groupoid $(\overrightarrow{M}_0, \overrightarrow{M}_0)$ which we will refer to as the rerooting groupoid. Its unit space is $\overrightarrow{M}_0$ and its arrow space is $\overrightarrow{M}_0$. We can identify $M_0$ with $\overrightarrow{M}_0 \times \{0\} \subset \overrightarrow{M}_0$.

The multiplication structure is as follows: we declare a pair of birooted configurations $(\omega, g), (\omega', h)$ in $\overrightarrow{M}_0$ to be composable if $\omega' = g^{-1}\omega$, in which case

$$(\omega, g) \cdot (\omega', h) := (\omega, gh).$$  

Note that if $\Gamma < G$ is a discrete subgroup (so $\Gamma \in M_0(G)$), then the above multiplication on $\{\Gamma\} \times \Gamma \subset \overrightarrow{M}_0(G)$ is just the usual one.

The source map $s: \overrightarrow{M}_0 \to M_0$ and target map $t: \overrightarrow{M}_0 \to M_0$ are

$$s(\omega, g) = \omega \quad \text{and} \quad t(\omega, g) = g^{-1}\omega.$$
Note that the rerooting groupoid is **discrete** in the sense that $s^{-1}(\omega)$ is at most countable for all $\omega \in \mathcal{M}_0$.

**Remark 3.2:** Let $\mathcal{M}_0^{aper}$ denote the set of rooted configurations $\omega$ that are aperiodic in the sense that $\text{stab}_G(\omega) = \{e\}$. Then the groupoid generated by $\mathcal{M}_0^{aper}$ in $\mathcal{M}_0$ is principal.\(^4\)

**Definition 3.3:** If $\Xi$ is a space of marks, then the **space of $\Xi$-marked rooted configurations** is

$$\Xi^{\mathcal{M}_0} = \{\omega \in \Xi^{\mathcal{M}} | \exists \xi \in \Xi \text{ such that } (0, \xi) \in \omega\}.$$

The **$\Xi$-marked rerooting groupoid** is defined as previously, with $\Xi^{\mathcal{M}_0}$ taking the place of $\mathcal{M}_0$.

### 3.1. Borel Correspondences between the Groupoid and Factors

Suppose $\theta : \mathcal{M} \to \mathcal{M}$ is an equivariant and measurable thinning. Then we can associate to it a subset of the rerooting groupoid, namely

$$A_\theta = \{\omega \in \mathcal{M}_0 | 0 \in \theta(\omega)\}.$$

This association has an inverse: given a Borel subset $A \subseteq \mathcal{M}_0$, we can define a thinning $\theta^A : \mathcal{M} \to \mathcal{M}$

$$\theta^A(\omega) = \{g \in \omega | g^{-1} \omega \in A\}.$$

Thus we see that Borel subsets $A \subseteq \mathcal{M}_0$ of the rerooting groupoid correspond to Borel thinning maps $\theta : \mathcal{M} \to \mathcal{M}$.

In the $\Xi$-marked case, one associates to a subset $A \subseteq \Xi^{\mathcal{M}_0}$ a thinning $\theta^A : \Xi^{\mathcal{M}} \to \Xi^{\mathcal{M}}$.

In a similar way, we can see that if $P : \mathcal{M}_0 \to [d]$ is a Borel partition of $\mathcal{M}_0$ into $d$ classes, then there is an associated factor $[d]$-coloring $\mathcal{C}^P : \mathcal{M} \to [d]^{\mathcal{M}}$ given by

$$\mathcal{C}^P(\omega) = \{(g, P(g^{-1}\omega)) \in G \times [d] | g \in \omega\},$$

and given a factor $[d]$-coloring $\mathcal{C} : \mathcal{M} \to [d]^{\mathcal{M}}$ one associates the partition $P^\mathcal{C} : \mathcal{M}_0 \to [d]$ given by

$$P(\omega) = c,$$

where $c$ is the unique element of $[d]$ such that $(0, c) \in \mathcal{C}(\omega)$.

Again, these associations are mutual inverses.

---

\(^4\) Recall that a groupoid is **principal** if its isotropy subgroups are all trivial. That is, the groupoid structure is just that of an equivalence relation.
More generally, we see that Borel factor $\Xi$-markings $C : M \to \Xi^M$ correspond to Borel maps $P : M_0 \to \Xi$.

Now suppose that $\mathcal{G} : M \to \text{Graph}(G)$ is an equivariant and measurable factor graph. Then we can associate to it a subset of the rerooting groupoid’s arrow space, namely
\[
\mathcal{A}_\mathcal{G} = \{(\omega, g) \in \overrightarrow{M_0} \mid (0, g) \in \mathcal{G}(\omega)\}.
\]
In the other direction, we associate to a subset $A \subseteq \overrightarrow{M_0}$ the factor graph $\mathcal{G}_A : M \to \text{Graph}(G)$
\[
\mathcal{G}_A(\omega) = \{(g, h) \in \omega \times \omega \mid (g^{-1}\omega, g^{-1}h) \in A\}.
\]
Thus we see that Borel subsets $A \subseteq \overrightarrow{M_0}$ of the rerooting groupoid’s arrow space correspond to Borel factor (directed) graphs $\mathcal{G} : M \to \text{Graph}(G)$.

**Remark 3.4:** If $\mu$ is a point process, then the correspondence still works in one direction: namely, we can associate subsets $A \subset M_0$ (or $A \subseteq \overrightarrow{M_0}$) to $\mu$-thinnings $\theta^A : (M, \mu) \to M$ (or $\mu$-factor graphs $\mathcal{G}_A : (M, \mu) \to M$ respectively).

We run into trouble in the other direction: suppose $\theta : M \to M$ is a thinning, but only defined $\mu$ almost everywhere. We wish to restrict it to $M_0$, but a priori this makes no sense—that is a subset of measure zero. It turns out that there is a way to make sense of this due to equivariance, but it will require some more theory that we explain in the next section.

### 3.2. The Palm Measure

We will now associate to a (finite intensity) point process $\mu$ a probability measure $\mu_0$ defined on the rerooting groupoid $M_0$. When the ambient space is unimodular, this will turn the rerooting groupoid into a probability measure preserving (pmp) discrete groupoid.

Informally, the Palm measure of a point process $\Pi$ is the process conditioned to contain the root. A priori this makes no sense (the subset $M_0$ has probability zero), but there is an obvious way one could interpret the statement: condition on the process to contain a point in an $\varepsilon$ ball about the root, and take the limit as $\varepsilon$ goes to zero. See [DVJ07, Theorem 13.3.IV] and [LP18, Section 9.3] for further details.

We will instead take the following concept of relative rates as our basic definition:
Definition 3.5: Let \( \Pi \) be a point process of finite intensity with law \( \mu \). Its (normalized) **Palm measure** is the probability measure \( \mu_0 \) defined on Borel subsets of \( \mathbb{M}_0 \) by

\[
\mu_0(A) := \frac{\text{int}(\theta^A(\Pi))}{\text{int}(\Pi)},
\]

where \( \theta^A \) is the thinning associated to \( A \subseteq \mathbb{M}_0 \).

More explicitly,

\[
\mu_0(A) := \frac{1}{\text{int}(\mu)} \frac{\mathbb{E}_\mu[\#\{g \in U \mid g^{-1}\omega \in A\}]}{\mathbb{E}[U \cap \Pi]},
\]

where \( U \subseteq G \) is any measurable set with \( 0 < \lambda(U) < \infty \). To make formulas simpler, we will often choose \( U \) to be of unit volume. Alternatively, note that by the definition of intensity we may write

\[
\mu_0(A) = \frac{\mathbb{E}[\#\{g \in U \mid g^{-1}\Pi \in A\}]}{\mathbb{E}[U \cap \Pi]}.
\]

We also define the Palm measure of a \( \Xi \)-marked point process similarly, with \( \Xi^{\mathbb{M}_0} \) taking the place of \( \mathbb{M}_0 \).

A **Palm version** of \( \Pi \) is any random variable \( \Pi_0 \) with law \( \mu_0 \). That is, we require that for all Borel \( B \subseteq \mathbb{M}_0 \) we have

\[
\mathbb{P}[\Pi_0 \in B] = \mu_0(B).
\]

We now describe some Palm calculus. If \( \Pi \) is a point process with Palm version \( \Pi_0 \) and \( \Phi(\Pi) \) is some factor map, then we wish to express the Palm version \( \Phi(\Pi)_0 \) of \( \Phi(\Pi) \) in terms of \( \Pi_0 \) and \( \Phi \). The Palm calculus tells us how this is done. It will be sufficient for our purposes to compute the Palm measure of factors for factor which are forgettings, thinnings, colored thickenings, and colorings. In each case the answer is more or less obvious, so we will give an informal description of the answer and then verify that it satisfies the required property.

**Example 9** (Forgetting labels): If \( \Pi \) is a labelled point process, then the Palm measure of \( \Pi \) after we forget the labels is the same thing as forgetting the labels from the Palm measure \( \Pi_0 \).

We prove this after the following clarification:

When talking about the Palm measure for a \( \Xi \)-marked point process, it is important in the above to choose the correct thinning. Recall from Remark 2.15
that for a subset $A \subseteq \Xi^{M_0}$ one can discuss two possible kinds of thinnings, namely

$$\theta^A : \Xi^M \to \Xi^M$$

or

$$\pi \circ \theta^A : \Xi^M \to M,$$

where $\pi : \Xi^M \to M$ is the map that forgets labels.

It is the former kind of thinning one should take.

Note that if $\Pi$ is a $\Xi$-marked point process, then its intensity remains the same if you forget the marks, that is, $\text{int} \Pi = \text{int} \pi(\Pi)$. More generally, the operation of taking the Palm version commutes with forgetting labels. That is, $\pi(\Pi_0) = (\pi(\Pi))_0$. To see this, let $B \subseteq M_{0}$, and observe

$$P[\pi(\Pi_0) \in B] = P[\Pi_0 \in \pi^{-1}(B)]$$

$$= \frac{\text{int} \theta^{\pi^{-1}(B)}(\Pi)}{\text{int} \Pi}$$

$$= \frac{\text{int} \pi(\theta^{\pi^{-1}(B)}(\Pi))}{\text{int} \pi(\Pi)}$$

$$= \frac{\text{int} \theta^B(\pi(\Pi))}{\text{int} \pi(\Pi)}$$

$$= P[\pi(\Pi)_0 \in B],$$

where we simply followed our nose.

**Example 10 (Lattice actions):** If $\Gamma < G$ is a lattice, then the Palm measure of the associated lattice shift is just $\delta_{\Gamma}$—that is, the atomic measure on $\Gamma \in M_0(G)$. More generally, if $\Gamma \curvearrowright (X, \mu)$ is a pmp action, then the Palm measure of the associated induced $X$-marked point process is its symbolic dynamics. That is, the map $\Sigma : (X, \mu) \to X^M$ given by

$$\Sigma(x) = \{ (\gamma, \gamma^{-1} \cdot x) \in G \times X \mid \gamma \in \Gamma \},$$

pushes forward $\mu$ to the Palm measure. In words, you sample a $\mu$-random point $x \in X$ and track its orbit under $\Gamma$ (the inverse is an artefact of our left bias).

**Remark 3.6:** Suppose $\Pi$ is a finite intensity point process such that its Palm version is an atomic measure, say $\Pi_0 = \Omega$ almost surely where $\Omega \in M_0$. Then $\Omega$ is a lattice in $G$. Note that $\Omega$ is automatically a discrete subset of $G$, and a simple mass transport argument shows that it is a subgroup. The covolume of this subgroup is the reciprocal of the intensity of $\Pi$. 


Example 11 (Mecke–Slivnyak Theorem): If \( \Pi \) is a Poisson point process, then its Palm measure has the same law as \( \Pi \cup \{0\} \), where \( 0 \in G \) is the identity.

In fact, this is a characterisation of the Poisson point process: if the Palm measure of \( \mu \) is obtained by simply adding the root,\(^5\) then \( \mu \) is the Poisson point process (of some intensity).

The proof of the above fact can be found in [LP18, Section 9.2]. As a consequence, the Palm measure of the IID Poisson is the IID of the Palm measure of the Poisson itself.

Example 12: The Palm version \( \mathcal{C}^A(\Pi)_0 \) of a 2-coloring \( \mathcal{C} : \mathbb{M} \to \{0, 1\}^\mathbb{M} \) determined by a subset \( A \subseteq \mathbb{M}_0 \) (as in Example 7) is simply \( \mathcal{C}(\Pi_0) \).

Example 13 (Thinnings): The Palm version \( \theta(\Pi)_0 \) of a thinning \( \theta = \theta^A \) of \( \Pi \) (determined by a subset \( A \subseteq \mathbb{M}_0 \)) is described in terms of its Palm version \( \Pi_0 \) as a conditional probability as follows:

\[
P[\theta(\Pi)_0 \in B] = P[\theta(\Pi_0) \in B \mid \Pi_0 \in A]
\]

for any \( B \subseteq \mathbb{M}_0 \).

That is, the Palm measure \( \theta(\Pi)_0 \) can be obtained by sampling from \( \Pi_0 \) conditioned that the root is retained in the thinning, and then applying the thinning.

To see this, first one should work from the definitions to show that

\[
\theta^B(\theta^A(\Pi)) = \theta^{A \cap (\theta^A)^{-1}(B)}.
\]

Therefore

\[
P[(\theta(\Pi))_0 \in B] = \frac{\int \theta^B(\theta^A(\Pi))}{\int \theta^A(\Pi)} = \frac{\int \theta^{A \cap (\theta^A)^{-1}(B)}(\Pi)}{\int \theta^A(\Pi)} \quad \text{By the observation}
\]

\[
= \frac{P[\Pi_0 \in A \cap (\theta^A)^{-1}(B)]}{P[\Pi_0 \in A]} = \frac{P[\{\theta(\Pi_0) \in B\} \cap \{\Pi_0 \in A\}]}{P[\Pi_0 \in A]} = \frac{P[\{\theta(\Pi_0) \in B\} \cap \{\Pi_0 \in A\}]}{P[\Pi_0 \in A]},
\]

which is exactly the definition of the desired conditional probability.

\(^5\) More formally, consider the map \( F : \mathbb{M} \to \mathbb{M}_0 \) given by \( F(\omega) = \omega \cup \{0\} \), by “adding the root” we mean the Palm measure \( \mu_0 \) is the pushforward \( F_* \mu \).
Example 14: Let $\Theta = \Theta^F$ be a constant thickening determined by $F \subset G$, as described in Example 5. If $\Pi$ is an $F$-separated process, then the Palm version $\Theta(\Pi)_0$ of the thickening $\Theta(\Pi)$ is as follows: sample from $\Pi_0$, and independently choose to root $\Theta(\Pi_0)$ at a uniformly chosen element $X$ of $F$. That is,

$$\Theta(\Pi)_0 \stackrel{d}{=} X^{-1}\Theta(\Pi).$$

To see this, we compute as follows:

$$\mathbb{P}[\Theta(\Pi)_0 \in B] = \frac{1}{\int \Theta(\Pi)} \mathbb{E}[\#\{g \in U \cap \Pi F \mid g^{-1}\Theta(\Pi) \in B\}]$$

by definition

$$= \frac{1}{|F|} \int \mu \sum_{f \in F} \mathbb{E}[\#\{g \in U \cap \Pi f \mid g^{-1}\Theta(\Pi) \in B\}]$$

by Example 5

$$= \frac{1}{|F|} \int \mu \sum_{f \in F} \mathbb{E}[\#\{g \in U f^{-1} \cap \Pi \mid g^{-1} \Pi \in \Theta^{-1}(B)\}]$$

by equivariance

$$= \frac{1}{|F|} \int \mu \sum_{f \in F} \mathbb{P}[\Pi_0 \in \Theta^{-1}(B)]$$

by unimodularity

$$= \frac{1}{|F|} \sum_{f \in F} \mathbb{P}[\Theta(\Pi)_0 \in B]$$

by definition

$$= \mathbb{P}[X^{-1}\Theta(\Pi)_0 \in B].$$

The Palm measure has an associated integral equation, which we will refer to as “the CLMM”, following the convention of [Bla17]. It is also referred to as “the refined Campbell theorem” in [LP11] and [DVJ03], for example.

**Theorem 3.7 (Campbell–Little–Mecke–Matthes):** Let $\mu$ be a finite intensity point process on $G$ with Palm measure $\mu_0$. Write $\mathbb{E}$ and $\mathbb{E}_0$ for the associated integral operators.

If $f : G \times M_0 \to \mathbb{R}_{\geq 0}$ is a measurable function (not necessarily invariant in any way), then

$$\mathbb{E} \left[ \sum_{x \in \omega} f(x, x^{-1}\omega) \right] = \int \mu \mathbb{E}_0 \left[ \int_G f(x, \omega)d\lambda(x) \right].$$

---

6 When we define the Palm measure of a set $B \subseteq M_0$, we usually write “$g \in U$” rather than “$g \in U \cap \Pi$”, as the latter condition $g^{-1}\Pi \in B$ already implies $g \in \Pi$. For this computation it is better to really spell it out though.
The proof of the above theorem is a standard monotone convergence argument, and as such we will only give the first step of the argument and leave the details to the reader. Observe that by definition of the Palm measure, for any $U \subseteq (G)$ of finite volume and any measurable $A \subseteq \mathbb{M}_0$, we have

$$E[\#\{g \in U \mid g^{-1}\Pi \in A\}] = \text{int}(\mu)\mu_0(A)\lambda(U).$$

Now rewrite the integrand on the left-hand side as a sum, and the right hand side as an integral:

$$E\left[\sum_{g \in \Pi} \mathbb{1}(g, g^{-1}\omega) \in U \times A\right] = \text{int}(\mu) \int_{\mathbb{M}_0} \int_G \mathbb{1}(g, \omega) \in U \times A] d\lambda(g) d\mu_0(\omega),$$

and observe that this is exactly the claimed theorem (in slightly different notation) in the case of $f(x, \omega) = \mathbb{1}[(x, \omega) \in U \times A]$. The theorem follows for arbitrary $f$ by the monotone convergence theorem.

**Remark 3.8:** If $\nu$ is a point process with $\nu_0 = \mu_0$, then $\nu = \mu$, that is, the Palm measure determines the point process.

To see this, we use the existence of a map $\mathcal{V} : [0, 1] \times \mathbb{M}_0 \rightarrow \mathbb{M}$ with the property that if $\mu$ is any point process with Palm measure $\mu_0$, then

$$\mathcal{V}_*(\text{Leb} \otimes \mu_0) = \mu.$$ 

This is a consequence of the Voronoi inversion formula; see [LP18, Section 9.4].

### 3.3. Unimodularity and the Mass Transport Principle

The Mass Transport Principle is a powerful tool in percolation theory; see [LP16] for an introduction and historical context. For the convenience of the reader, we include a proof of it for our context and in our notation, but no originality is claimed. For further generalisations of the mass transport principle see [Kal11], [GL11], and [Kal17, Chapter 7] and for further exposition in the context of point processes see [Bla17].
The source and range maps \( s, t : \mathbb{M}_0 \to \mathbb{M}_0 \) induce a pair of measures on \( \mathbb{M}_0 \) defined by

\[
\mu_0^s(G) = \int_{\mathbb{M}_0} |s^{-1}(\omega) \cap G| d\mu_0(\omega),
\]

and

\[
\mu_0^t(G) = \int_{\mathbb{M}_0} |t^{-1}(\omega) \cap G| d\mu_0(\omega).
\]

In our factor graph interpretation this corresponds to the expected indegree and outdegree of \( G \) respectively, where we view \( G \) as a directed rooted graph. To see this, recall that for a rooted configuration \( \omega \in \mathbb{M}_0 \),

\[
s^{-1}(\omega) = \{(\omega, g) \in \mathbb{M}_0 \times G \mid g \in \omega\}
\]

and

\[
t^{-1}(\omega) = \{(g^{-1}\omega, g^{-1}) \in \mathbb{M}_0 \times G \mid g \in \omega\},
\]

and that there is an edge from \( 0 \) to \( g \) in \( G(\omega) \) exactly when \( (\omega, g) \in G \), and an edge from \( g \) to \( 0 \) exactly when \( (g^{-1}\omega, g^{-1}) \in G \). Thus

\[
\deg_0(G(\omega)) = |s^{-1}(\omega) \cap G(\omega)| \quad \text{and} \quad \deg(G(\omega)) = |t^{-1}(\omega) \cap G(\omega)|.
\]

\textbf{Remark 3.9:} We have had to adapt notation to suit our purposes. Usually a groupoid would be denoted by a letter like \( G \), and that is the set of arrows. Then its units would be denoted \( G_0 \). We have tried to match this up with the necessary notation from point process theory as closely as possible.

We choose to denote outdegree by an expression like \( \deg_0(G(\omega)) \) instead of \( \deg_+(G(\omega))(0) \) as the arrows are more evocative, and the subscript notation becomes very small (as in, for instance, \( \deg_{G(\Pi_0)}(0) \)).

\textbf{Proposition 3.10:} If \( G \) is unimodular, then \( \mu_0^s = \mu_0^t \). That is, \( (\mathbb{M}_0, \mu_0) \) forms a discrete pmp groupoid.

Equivalently, if \( \Pi_0 \) is the Palm version of any point process \( \Pi \) on \( G \), then

\[
\mathbb{E}[\deg_0(G(\Pi_0))] = \mathbb{E}[\deg_0(G(\Pi_0))].
\]

We will denote by \( \mu_0^s \) this common measure \( \mu_0^s = \mu_0^t \).
Proof of Proposition 3.10. Fix $U \subseteq G$ of unit volume. We compute:

\[
\overrightarrow{\mu^s_0}(\mathcal{G}) = \mathbb{E}_{\mu_0} \left[ \sum_{g \in \omega} 1[\omega, g) \in \mathcal{G}] \right] \tag{by definition}
\]

\[
= \mathbb{E}_{\mu_0} \left[ \int_G 1[x \in U] \sum_{g \in \omega} 1[(\omega, g) \in \mathcal{G}] d\lambda(x) \right]
\]

\[
= \frac{1}{\text{int } \mu} \mathbb{E}_{\mu} \left[ \sum_{x \in \omega} 1[x \in U] \sum_{g \in x^{-1}\omega} 1[(x^{-1}\omega, g) \in \mathcal{G}] \right] \tag{by the CLLM}
\]

\[
= \frac{1}{\text{int } \mu} \mathbb{E}_{\mu} \left[ \sum_{h \in \omega} \sum_{g \in h^{-1}\omega} 1[h g^{-1} \in U] 1[(g h^{-1}\omega, g) \in \mathcal{G}] \right]
\]

\[
= \mathbb{E}_{\mu_0} \left[ \int_G \sum_{g^{-1} \in \omega} 1[h g^{-1} \in U] 1[(g \omega, g) \in \mathcal{G}] d\lambda(h) \right] \tag{by the CLLM}
\]

\[
= \mathbb{E}_{\mu_0} \left[ \int_G \sum_{g \in \omega} 1[h \in U] 1[(g^{-1}\omega, g^{-1}) \in \mathcal{G}] d\lambda(h) \right] \tag{by unimodularity}
\]

\[
= \mathbb{E}_{\mu_0} \left[ \sum_{g \in \omega} 1[(g^{-1}\omega, g^{-1}) \in \mathcal{G}] \int_G 1[h \in U g^{-1}] d\lambda(h) \right]
\]

\[
= \mathbb{E}_{\mu_0} \left[ \sum_{g \in \omega} 1[(g^{-1}\omega, g^{-1}) \in \mathcal{G}] \right] \tag{by unimodularity}
\]

\[
= \overrightarrow{\mu^t_0}(\mathcal{G}),
\]

as desired, where the first instance of CLMM is applied with

\[
f_1(x, \omega) = 1[x \in U] \sum_{g \in \omega} 1[(\omega, g) \in \mathcal{G}],
\]

and

\[
f_2(x, \omega) = \sum_{g^{-1} \in \omega} 1[x g^{-1} \in U] 1[(g \omega, g) \in \mathcal{G}]
\]

in the second instance. ■

Definition 3.11: The **Palm groupoid** of a point process $\Pi$ with law $\mu$ is $(\overrightarrow{\mathbb{M}_0}, \overrightarrow{\mu_0})$. If $\Pi$ is free, then this groupoid is principal, and thus we refer to $\Pi$’s **Palm equivalence relation** $(\overrightarrow{\mathbb{M}_0}, \mathcal{R}, \mu_0)$.

Definition 3.12: Let $\Pi$ be a point process and $\mathcal{G}$ an undirected factor graph of $\Pi$. Its **edge density** is $\mathbb{E}[	ext{deg}_0(\mathcal{G}(\Pi_0))]$, where $\Pi_0$ is the Palm version of $\Pi$. 

By the above proposition, if $\mathcal{G}'$ is any orientation of $\mathcal{G}$, then the edge density can be expressed as

$$E[\text{deg}_0(\mathcal{G}(\Pi_0))] = 2E[\text{deg}_0(\mathcal{G}'(\Pi_0))].$$

Speaking properly then, we should be talking of directed factor graphs, but for this reason we will often think of the factor graphs as undirected.

**Theorem 3.13 (The Mass Transport Principle):** Let $G$ be a unimodular group, and $\Pi$ a point process on $G$ with Palm version $\Pi_0$. Suppose $T: G \times G \times \mathbb{M} \rightarrow \mathbb{R}_{\geq 0}$ is a measurable function which is diagonally invariant in the sense that

$$T(gx, gy; g\omega) = T(x, y; \omega) \quad \text{for all } g \in G.$$

Then

$$E \left[ \sum_{y \in \Pi_0} T(0, y; \Pi_0) \right] = E \left[ \sum_{x \in \Pi_0} T(x, 0; \Pi_0) \right].$$

We view $T(x, y; \Pi_0)$ as representing an amount of mass sent from $x$ to $y$ when the configuration is $\Pi_0$. Thus the integrand on the left-hand side represents the total mass sent out from the root, and similarly the integrand on the right-hand side represents the total mass received by the root.

**Proof of Theorem 3.13.** The mass transport principle follows from Proposition 3.10.

First, observe that as in the proof of the CLMM, there is an integral equation implied by Proposition 3.10 and a monotone convergence argument. To wit,

$$E \left[ \sum_{g \in \Pi_0} f(\Pi_0, g) \right] = E \left[ \sum_{g \in \Pi_0} f(g^{-1}\Pi_0, g^{-1}) \right],$$

where $f: \overrightarrow{\mathbb{M}}_0 \rightarrow \mathbb{R}_{\geq 0}$ is a measurable function (defined $\mu_0$ almost everywhere).

We now apply the integral equation with $f(\omega, g) = T(0, g; \omega)$. Note that

$$f(g^{-1}\omega, g^{-1}) = T(0, g^{-1}; g^{-1}\omega) = T(g, 0; \omega),$$

where the second equality is by diagonal invariance. Hence the integral equation for $f$ yields

$$E \left[ \sum_{g \in \Pi_0} T(0, g; \Pi_0) \right] = E \left[ \sum_{g \in \Pi_0} T(g, 0; \Pi_0) \right],$$

which is exactly the mass transport principle expressed with a differently named integrating variable. ■
Remark 3.14: One can use the CLMM formula (see Theorem 3.7) to express \( \mu_0(G) \) without reference to the Palm measure. Let \( U \subseteq G \) be of unit volume, and apply the formula to \( f(x, \omega) = 1[x \in U \rightarrow \deg_0(G(\omega))] \), resulting in

\[
\mu_0(G) = \frac{1}{\text{int}\Pi} \mathbb{E} \left[ \sum_{x \in \Pi} 1[x \in U \rightarrow \deg_x(G(\Pi))] \right]
\]

(note that by equivariance \( \deg_0(G(x^{-1}\omega)) = \deg_x(G(\omega)) \)).

As an application of the CLMM, we will find an expression for the Palm version of general thickenings:

**Example 15 (Palm measures of general thickenings):** Suppose one has for each configuration \( \omega \in \mathcal{M} \) and each \( g \in \omega \) a measurably defined finite subset \( F_\omega(g) \) satisfying the following properties:

- **Monotonicity:** That \( g \in F_\omega(g) \),
- **Separation:** If \( g, h \in \omega \) are distinct then \( F_\omega(g) \cap F_\omega(h) = \emptyset \), and
- **Equivariance:** For all \( \gamma \in G \), we have \( F_{\gamma \omega}(\gamma \omega) = \gamma F_\omega(g) \).

Then one can define a thickening \( \Theta : \mathcal{M} \rightarrow \mathcal{M} \) by

\[
\Theta(\omega) = \bigsqcup_{g \in \omega} F_\omega(g).
\]

That is, each point \( g \in \omega \) looks at the current configuration, and adds points \( F_\omega(g) \) locally to it according to some equivariant rule. Every thickening has this form (see Definition 3.15 and the ensuing discussion). We refer to points of \( \omega \) as **progenitors** and points of \( F_\omega(g) \) as \( g \)'s **spawn** in \( \omega \).

It stands to reason that if \( \Pi \) is a point process satisfying the above rules almost surely, then

\[
\text{int} \Theta(\Pi) = \mathbb{E}|F_{\Pi_0}(0)| \cdot \text{int} \Pi.
\]

Just as in Example 5 though, this will require unimodularity to prove, this time in the form of the Mass Transport Principle.

Let us identify the thickening with its input/output version. Note that if we compute the Palm version of the latter, then we get it for the former by simply forgetting the labels. Our reason for doing this is simple: we need to be able to identify which points were progenitors and which points are spawn. This is only possible if we use the input/output version, but the downside is that that is more notationally cumbersome.
We first verify that \( \text{int}\Theta(\Pi) = \mathbb{E}[F_{\Pi_0}(0)] \cdot \text{int}\Pi \). In order to apply mass transport, we need to know the following fact:

\[
P[\Theta(\Pi)_0 \in A | 0 \text{ is a progenitor}] = P[\Theta(0) \in A].
\]

This follows from the definitions by similar manipulations to those we have already seen.

With this fact in hand, define a transport as follows:

\[
T(x, y, \Theta(\Pi)) = 1[x \text{ spawned } y \text{ in } \Theta(\Pi)].
\]

Then the total mass received by the root is always one (as everyone is spawned by someone), and hence the expected mass received is one.

The expected mass sent out is

\[
\mathbb{E}_0[1 | 0 \text{ is a progenitor}] \cdot \#\{\text{spawn of 0}\} = \mathbb{P}[0 \text{ is a progenitor}] \cdot \mathbb{E}[|F_{\Pi_0}(0)|],
\]

where \( \mathbb{E}_0 \) denotes expectation with respect to the Palm measure of \( \Theta(\Pi) \), and the equality follows from the fact above and the definition of conditional probability.

We have by the definition of progenitor

\[
P[0 \text{ is a progenitor}] = \frac{\text{int}\Pi}{\text{int}\Theta(\Pi)},
\]

so \( \text{int}\Theta(\Pi) = \mathbb{E}[F_{\Pi_0}(0)] \cdot \text{int}\Pi \) by the mass transport principle.

We now express the Palm version of \( \Theta(\Pi) \) in terms of \( \Pi_0 \) and \( \Theta \). Note that for this to be defined we must assume \( \Pi \) has finite intensity and that

\[
\mathbb{E}[|F_{\Pi_0}(0)|] < \infty.
\]

Let

- \( N \) be a random variable with
  \[
  \mathbb{P}[N = n] = \frac{n\mathbb{P}[|F_{\Pi_0}(0)| = n]}{\mathbb{E}[F_{\Pi_0}(0)]} = \frac{n\mathbb{P}[0 \text{ spawns } n \text{ points of } \Theta(\Pi_0)]}{\mathbb{E}[F_{\Pi_0}(0)]},
  \]
- \( \Upsilon^n \) denote \( \Pi_0 \) conditioned on the event \( \{0 \text{ spawns } n \text{ points of } \Theta(\Pi_0)\} \),
- \( X \) be a uniformly chosen element of \( F_{\Upsilon^n}(0) \) (conditional on \( \Upsilon^n \)).

\footnote{An advantage of using the input/output version of the thickening is that we can exactly identify who spawned who in a well-defined way.}
We claim that $X^{-1}\Theta(Y^N)$ is a Palm version of $\Theta(\Pi)$.

In words, we are sampling from the Palm measure $\Pi_0$ biased towards the configurations that spawn more points, and then applying the thickening and rooting at one of the spawns uniformly at random.\(^8\)

Let

$$A \subseteq \{\text{Red, Blue, Purple}\}^{M_0}.$$  

We find an expression for $\mathbb{P}[\Theta(\Pi)_0 \in A]$ by using mass transport: define

$$T(x, y; \Theta(\Pi)) = 1[\{x \text{ spawns } y \text{ in } \Theta(\Pi)\} \cap \{y \in \theta^A(\Theta(\Pi))\}].$$

The expected mass in with respect to $T$ is exactly $\mathbb{P}[\Theta(\Pi)_0 \in A]$. The expected mass out is

$$\mathbb{E}\left[ \sum_{y \in \Theta(\Pi)_0} T(0, y; \Theta(\Pi)_0) \right] = \mathbb{E}\left[ 1[0 \text{ is a progenitor}] \times \#\{0 \text{ spawns } y \text{ with } y \in \theta^A(\Theta(\Pi)_0)\} \right] 
= \mathbb{P}[0 \text{ is a progenitor}] \times \mathbb{E}[\#\{0 \text{ spawns } y \text{ with } y \in \theta^A(\Theta(\Pi)_0)\}|0 \text{ is a progenitor}] 
= \frac{1}{\mathbb{E}[|F_{\Pi_0}(0)|]} \mathbb{E}[\#\{y \in F_{\Pi_0}(0) \mid y^{-1}\Theta(\Pi_0) \in A\}] 
= \frac{1}{\mathbb{E}[|F_{\Pi_0}(0)|]} \mathbb{E}[\#\{y \in F_{\Pi_0}(0) \mid y^{-1}\Theta(\Pi_0) \in A\}| |F_{\Pi_0}(0)| = n] \mathbb{P}[|F_{\Pi_0}(0)| = n].$$

We can now match up this expression with our earlier description of $X^{-1}\Phi(Y^N)$.

\(^8\) To see that some kind of size biasing is required, consider the point process

$$Z + \text{Unif}[0, 1] \subset \mathbb{R},$$

and define a thickening which leaves points marked 0 as they are and adds a thousand points tightly packed around points marked 1. A “typical point” of the resulting process should look more like a configuration with a thousand points near the origin, and the size biasing accommodates for this.
Recall that if \( Y \subseteq [n] \) is a random subset, then \( \mathbb{E}|Y| = n\mathbb{P}[X \in Y] \), where \( X \) is a uniformly chosen element of \([n]\).

\[
\mathbb{E}\left[ \sum_{y \in \Theta(\Pi)_0} T(0, y; \Theta(\Pi)_0) \right]
\]

\[
= \frac{1}{\mathbb{E}|F_{\Pi_0}(0)|} \times \sum_n \mathbb{E}[\#\{y \in F_{\Pi_0}(0) \mid y^{-1}\Theta(\Pi_0) \in A\}|\mathbb{E}|F_{\Pi_0}(0)| = n] \mathbb{P}[\mathbb{E}|F_{\Pi_0}(0)| = n]
\]

\[
= \frac{1}{\mathbb{E}|F_{\Pi_0}(0)|} \sum_n \mathbb{E}[\#\{y \in F_{\gamma^n}(0) \mid y^{-1}\Theta(\gamma^n) \in A\}] \mathbb{P}[\mathbb{E}|F_{\Pi_0}(0)| = n]
\]

\[
= \sum_n n\mathbb{P}[X^{-1}\Theta(\gamma^n) \in A] \mathbb{P}[\mathbb{E}|F_{\Pi_0}(0)| = n] \frac{\mathbb{P}[\mathbb{E}|F_{\Pi_0}(0)| = n]}{\mathbb{E}|F_{\Pi_0}(0)|}
\]

\[
= \sum_n \mathbb{P}[X^{-1}\Theta(\gamma^n) \in A] \mathbb{P}[N = n]
\]

\[
= \mathbb{P}[X^{-1}\Theta(\gamma^N) \in A],
\]
as desired.

In fact, every thickening can be expressed à la Example 15, as we shall now see.

**Definition 3.15:** Let \( \omega \in \mathbb{M} \) be a configuration, and \( g \in \omega \) one of its points. The associated **Voronoi cell** is

\[
V_\omega(g) = \{x \in G \mid d(x, g) \leq d(x, h) \text{ for all } h \in \omega\}.
\]

The associated **Voronoi tessellation** is the ensemble of closed sets \( \{V_\omega(g)\}_{g \in \omega} \).

Left-invariance of the metric \( d \) implies that the Voronoi cells are equivariant in the sense that for all \( \gamma \in G \), we have

\[
V_{\gamma\omega}(\gamma g) = \gamma V_\omega(g).
\]

Note that discreteness of the configuration implies that the Voronoi tessellation forms a locally finite cover of the ambient space by closed sets. We would like to think of these sets as forming a partition of the ambient space, but this is not necessarily true even in the measured sense: the boundaries of the Voronoi cells can have positive volume. For example, let \( \Gamma \) be a discrete group and consider \( \Gamma \times \{0\} \subset \Gamma \times \mathbb{R} \).
Lie groups and Riemannian symmetric spaces essentially avoid this deficiency, as hyperplanes\(^9\) have zero volume.

So depending on the examples one is interested in one can assume that the Voronoi cells are essentially disjoint (that is, that their intersection is Haar null). If this property is necessary then one can make a small modification to ensure it: we introduce a tie breaking function that allows points belonging to multiple Voronoi cells to decide which one they shall belong to. Take any Borel isomorphism \(T : G \to \mathbb{R}\).\(^{10}\) Let us define

\[
V^T_\omega(g) = \{x \in G \mid \text{for all } h \in \omega \setminus \{g\}, \ d(x, g) < d(x, h) \\
\text{or } d(x, g) = d(x, h) \text{ and } T(x^{-1}g) < T(x^{-1}h)\}.
\]

Note that these tie-broken Voronoi cells form a measurable partition of \(G\).

That is, we have traded the Voronoi cells being closed for them being genuinely disjoint. The equivariance property \(V^T_{\gamma\omega}(\gamma g) = \gamma V^T_\omega(g)\) still holds as well.

If \(\Theta : \mathcal{M} \to \mathcal{M}\) is a thickening, then we simply define

\[F_\omega(g) = V_\omega(g) \cap \Theta(\omega).\]

### 3.4. Ergodicity and the Factor Correspondences in the Measured Category

In this section we show how to extend the correspondences of Section 3.1 to the measured category, which connects the distribution \(\mu\) of a point process to its Palm measure \(\mu_0\), and objects understood to be defined \(\mu\) almost everywhere with those defined \(\mu_0\) almost everywhere.

**Definition 3.16:** A subset \(A \subseteq \mathcal{M}\) of unrooted configurations is **shift-invariant** if for all \(\omega \in A\) and \(g \in G\), we have \(g\omega \in A\).

A subset \(A_0 \subseteq \mathcal{M}_0\) of rooted configurations is **rootshift invariant** if for all \(\omega \in A_0\) and \(g \in \omega\), we have \(g^{-1}\omega \in A_0\).

The groupoid \((\stackrel{\rightarrow}{\mathcal{M}_0}, \stackrel{\rightarrow}{\mu_0})\) is **ergodic** if every rootshift invariant subset \(A \subseteq \mathcal{M}_0\) has \(\mu_0(A) = 0 \text{ or } 1\).

Note that if \(A \subseteq \mathcal{M}\) is shift-invariant, then \(A_0 := A \cap \mathcal{M}_0\) is rootshift invariant, and if \(A_0 \subseteq \mathcal{M}_0\) is rootshift-invariant, then \(A := GA_0\) is shift invariant. Thus shift-invariant subsets and rootshift-invariant subsets are in bijective correspondence. Moreover:

---

\(^9\) Sets of the form \(\{x \in X \mid d(x, g) = d(x, h)\}\) for a fixed distinct pair \(g, h \in X\).

\(^{10}\) Recall that standard Borel spaces are isomorphic if they have the same cardinality.
Proposition 3.17: Let $\mu$ be a point process with Palm measure $\mu_0$.

1. If $A \subseteq \mathbb{M}_0$ is rootshift invariant, then $\mu_0(A) = \mu(GA)$.
2. If $A \subseteq \mathbb{M}$ is shift invariant, then $\mu_0(A \cap \mathbb{M}_0) = \mu(A)$.

That is, under the correspondence between rootshift invariant subsets of $\mathbb{M}_0$ and shift invariant subsets of $\mathbb{M}$, the measures $\mu_0$ and $\mu$ coincide.

In particular, $G \acts (\mathbb{M}, \mu)$ is ergodic if and only if $(\mathbb{M}_0, \mu_0)$ is ergodic.

Proof. We assume ergodicity and prove the statements about measures. We then prove the general statement by using the ergodic case.

First, suppose $G \acts (\mathbb{M}, \mu)$ is ergodic, and let $A \subseteq \mathbb{M}_0$ be rootshift invariant. Then for any $U \subseteq G$ of unit volume,

$$\mu_0(A) = \frac{1}{\text{int } \mu} \mathbb{E}_\mu[\# \{ g \in U \mid g^{-1}\omega \in A \}] \quad \text{by definition}$$

$$= \frac{1}{\text{int } \mu} \mathbb{E}_\mu[|\omega \cap U| \mathbb{1}[\omega \in GA]] \quad \text{by rootshift invariance of } A$$

$$= \mu(GA) \quad \text{by ergodicity.}$$

In particular, we see that $\mu_0(A)$ is zero or one, so the equivalence relation is ergodic.

Now suppose $(\mathbb{M}_0, \mathcal{R}, \mu)$ is ergodic, and let $A \subseteq \mathbb{M}$ be shift invariant.

$$\mu_0(A \cap \mathbb{M}_0) = \frac{1}{\text{int } \mu} \mathbb{E}_\mu[\# \{ g \in U \mid g^{-1}\omega \in A \cap \mathbb{M}_0 \}] \quad \text{by definition}$$

$$= \frac{1}{\text{int } \mu} \mathbb{E}_\mu[|\omega \cap U| \mathbb{1}[\omega \in A \cap \mathbb{M}_0]] \quad \text{by shift invariance of } A$$

$$= \mu(A) \quad \text{by ergodicity.}$$

For the general case, we appeal to the ergodic decomposition theorem (see [GG00] for a proof):

Theorem 3.18: Let $G$ be an lcsc group, and $G \acts (X, \mu)$ a pmp action on a standard Borel space. Then there exists a standard Borel space $Y$ equipped with a probability measure $\nu$ and a family $\{p_y \mid y \in Y\}$ of probability measures $p_y$ on $X$ with the following properties:

1. For every Borel $A \subseteq X$, the map $y \mapsto p_y(A)$ is Borel, and

$$\mu(A) = \int_Y p_y(A) d\nu(y).$$
(2) For every \( y \in Y \), \( p_y \) is an invariant and ergodic measure for the action \( G \rhd (X, p_y) \).

(3) If \( y, y' \in Y \) are distinct, then \( p_y \) and \( p'_y \) are mutually singular.

There is an almost identically stated version of the above theorem for pmp cbers as well. These two decompositions are essentially equivalent, in a way that we shall now discuss.

If \((Y, \nu)\) and \( \{p_y \mid y \in Y\} \) is the ergodic decomposition for \( G \rhd (M, \mu) \), then the Palm measures \( (p_y)_0 \) of the \( p_y \) form an ergodic decomposition for \((M_0, \mathcal{R}, \mu_0)\). That is, for all \( A \subseteq M_0 \) we have

\[
\mu_0(A) = \int_Y (p_y)_0(A) d\nu(y).
\]

Applying the previous ergodic case to this yields the general formula.

**Theorem 3.19:** Let \( G \) be a locally compact and second countable group, and \( \Pi \) an invariant point process on \( G \) with law \( \mu \).

Then associated to this data is an \( r \)-discrete probability measure preserving groupoid \((\overrightarrow{M_0}, \mu_0)\) called the Palm groupoid of \( \Pi \). It has the following properties:

- Thinning maps \( \theta : (M, \mu) \to M \) of \( \Pi \) are in correspondence with Borel subsets \( A \) of the unit space \( M_0 \) of the Palm groupoid defined \( \mu_0 \) almost everywhere,
- Factor \( \Xi \)-markings \( \mathcal{C} : (M, \mu) \to \Xi^M \) are in correspondence with Borel \( \Xi \)-valued maps \( P \) defined on the unit space \( M_0 \) of the Palm groupoid defined \( \mu_0 \) almost everywhere, and
- Factor graphs \( \mathcal{G} : (M, \mu) \to \text{Graph}(G) \) of \( \Pi \) are in correspondence with Borel subsets \( \mathcal{A} \) of the arrow space \( \overrightarrow{M_0} \) of the Palm groupoid defined \( \mu_0 \) almost everywhere.

The Palm measure is well studied, but the equivalence relation structure seems to have been mostly overlooked. One can find two direct references to it: Example 2.2 in a paper of Avni [Avn05] and a question of Bowen in [Bow18] (specifically, Questions and comments, item 1).

We now prove Theorem 3.19, building on Section 3.1. The task here is to verify that under the correspondence, objects which are equal almost everywhere with respect to the point process are equal almost everywhere with respect to the Palm measure, and vice versa.
Lemma 3.20: Let \( \mu \) be a point process on \( G \) with Palm measure \( \mu_0 \), and \( X \) a Borel \( G \)-space.

Let \( \Phi, \Phi' : \mathbb{M} \to X \) be an equivariant Borel map. Then

\( \Phi = \Phi' \mu \) almost everywhere if and only if \( \Phi|_{\mathbb{M}_0} = \Phi'|_{\mathbb{M}_0} \mu_0 \) almost everywhere.

Proof. Observe that by equivariance the sets

\( \{ \omega \in \mathbb{M} | \Phi(\omega) = \Phi'(\omega) \} \) and \( \{ \omega \in \mathbb{M}_0 | \Phi(\omega) = \Phi'(\omega) \} \)

are shift invariant and rootshift invariant respectively. So by Proposition 3.17 one is \( \mu \)-sure if and only if the other is \( \mu_0 \)-sure, as desired.

Proof of Theorem 3.19. The method is essentially the same for thinnings and for markings, so we will just prove the thinning statement. To that end, let \( \theta : (\mathbb{M}, \mu) \to \mathbb{M} \) be a thinning. Note that by our assumption that \( \theta \) is equivariant, we have

\( \{ \omega \in \mathbb{M} | \theta(\omega) \subseteq \omega \} \) has \( \mu \) measure one.

This is a shift invariant set, so by Proposition 3.17 we have

\( \{ \omega \in \mathbb{M}_0 | \theta(\omega) \subseteq \omega \} \) has \( \mu_0 \) measure one.

We are now able to define \( A = \{ \omega \in \mathbb{M}_0 | 0 \in \theta(\omega) \} \), and this will be our desired subset of \((\mathbb{M}_0, \mu_0)\).

It follows from equivariance that the thinning \( \theta^A \) associated to \( A \) satisfies

\( \theta^A|_{\mathbb{M}_0} = \theta|_{\mathbb{M}_0} \mu_0 \) almost everywhere,

so by Lemma 3.20 we have \( \theta^A = \theta \) (\( \mu \) almost everywhere).

It remains to verify that if \( A = B \mu_0 \) almost everywhere (that is, that
\( \mu_0(A \Delta B) = 0 \), then \( \theta^A = \theta^B \) (\( \mu \) almost everywhere).

Recall\(^{11} \) that the saturation of \( A \Delta B \)

\[ [A \Delta B] = \{ g^{-1} \omega \in \mathbb{M}_0 | \omega \in A \Delta B \text{ and } g \in \omega \} \]

is \( \mu_0 \) null if \( A \Delta B \) is \( \mu_0 \) null.

Observe that for \( \omega \notin [A \Delta B] \) we have

\( \theta^A(\omega) = \theta^B(\omega) \),

and hence \( \theta^A|_{\mathbb{M}_0} = \theta^B|_{\mathbb{M}_0} \mu_0 \) almost everywhere, and we finish by again applying Lemma 3.20.

---

\(^{11} \) This is a general fact about nonsingular cbers, and it follows from the fact that they can all be generated by actions of countable groups.
If $\mathcal{G}$ is a factor graph of $\mu$, then in the same fashion we see that it has a well-defined restriction to $(M_0, \mu_0)$. We then define

$$\mathcal{A} = \{(\omega, g) \in M_0 \times G \mid (0, g) \in \mathcal{G}(\omega)\}.$$ 

We must verify that if $\mathcal{A}, \mathcal{B} \subseteq \overrightarrow{M_0}$ are subsets with $\overrightarrow{\mu_0}(A \Delta B) = 0$, then their associated factor graphs $\mathcal{G}_\mathcal{A}$ and $\mathcal{G}_\mathcal{B}$ are equal $\mu$ almost everywhere. This assumption states

$$\int_{M_0} \# \{ g \in \omega \mid (\omega, g) \in \mathcal{A} \Delta \mathcal{B} \} d\mu_0(\omega) = 0$$

and hence the integrand is zero $\mu_0$ almost everywhere. By again considering the saturation of sets, we see that

$$\mu_0(\{ \omega \in M_0 \mid \text{for all } g \in \omega, g^{-1} \omega \in \mathcal{A} \Delta \mathcal{B} \}) = 0,$$

from which the argument finishes as in the case of thinnings.  

3.5. EVERY FREE ACTION IS A POINT PROCESS, AND THE CROSS-SECTION PERSPECTIVE. We have taken the perspective that point processes are an intrinsically interesting class of pmp actions of lcsc groups to study. They are also a fairly general class: in this section we will prove Theorem 1.1, that every free and pmp action of a nondiscrete lcsc group $G$ on a standard Borel measure space $(X, \mu)$ is abstractly isomorphic to a finite intensity point process.

This is similar to the following fact: let $\Gamma \curvearrowright (X, \mu)$ be a pmp action of a discrete group $\Gamma$. The **symbolic dynamics** of this action is the map

$$\Sigma : (X, \mu) \to X^\Gamma$$

$$\Sigma_x(\gamma) = \gamma^{-1}x.$$ 

This is an injective and equivariant map, so we may identify the action $\Gamma \curvearrowright (X, \mu)$ with the invariant coloring action $\Gamma \curvearrowright (X^\Gamma, \Sigma_* \mu)$.

In this way, we see that all pmp actions of discrete groups are isomorphic to invariant colorings.$^{12}$

A standard technique in the study of free pmp actions of lcsc groups is to analyze their associated cross-sections. This will gives an analogue of symbolic dynamics for nondiscrete groups.

---

$^{12}$ If desired, one can fix a Borel isomorphism $X \cong [0, 1]$ so that the coloring space is the same for all actions.
Definition 3.21: Let $G \curvearrowright (X, \mu)$ be a pmp action on a standard Borel measure space $(X, \mu)$.

A **discrete cross-section** for the action is a Borel subset $Y \subset X$ such that for $\mu$-every $x \in X$ the set $\{g \in G \mid g^{-1}x \in Y\}$ is a closed and discrete non-empty subset of $G$.

Example 16: The set $\mathbb{M}_0 \subset \mathbb{M}$ is a discrete cross-section for all non-empty point process actions $G \curvearrowright (\mathbb{M}, \mu)$.

There is a sense in which this $\mathbb{M}_0$ is the only cross-section, which we now discuss.

Fix a discrete cross-section $Y$ for $G \curvearrowright X$. We associate to this data two maps

$$V : (X, \mu) \rightarrow \mathbb{M}$$
$$V' : (X, \mu) \rightarrow Y^\mathbb{M}$$

$$V_x = \{g \in G \mid g^{-1}x \in Y\}$$
$$V'_x = \{(g, g^{-1}x) \in G \times Y \mid g^{-1}x \in Y\}.$$

These are equivariant maps, and the second one is always injective. In particular, we see that every action which admits a cross-section also admits a point process factor, and is isomorphic to a marked point process.

Note that

$$V^{-1}(\mathbb{M}_0) = Y.$$

In this way we see that a discrete cross-section is the same thing as an unmarked point process factor.

Remark 3.22 (Terminological discussion): If $\mathcal{P}(\omega)$ is some property of discrete subsets $\omega$ in $G$, then we can investigate discrete cross-sections of actions $G \curvearrowright (X, \mu)$ such that the associated subset $V_x$ satisfies $\mathcal{P}$ for $\mu$ almost every $x \in X$.

For instance, $\mathcal{P}(\omega)$ might be the property “$\omega$ is uniformly discrete” or “$\omega$ is a net” (see Definition 4.15 for the meaning of these terms). We will refer to a discrete cross-section such that $\mathcal{P}(V_x)$ is satisfied for $\mu$ almost every $x \in X$ as a $\mathcal{P}$ cross-section.

Note that if $G \curvearrowright (\mathbb{M}, \mu)$ is the Poisson point process action, then $\mathbb{M}_0$ is not a lacunary cross-section. It is for this reason that we feel the terminology should be modified slightly.

---

13 Recall that an injective map between standard Borel spaces is always a Borel isomorphism onto its image.
Theorem 3.23 (Forrest [For74], see also [KPV15]): Every free and nonsingular\textsuperscript{14} action of an lcsc group on a standard probability space admits a discrete cross-section. Moreover, the cross-section can be chosen to be uniformly separated and even a net.

One sees that Theorem 1.1 is true by applying the above theorem with the unmarking technique of Proposition 4.10.

Remark 3.24: In fact, cross-sections of actions are known to exist in great generality, see [Kec19] for further examples.

Our keen interest in free actions is because it allows us to identify the orbit $Gx$ of any point $x \in X$ with $G$ itself. One can run into issues in the absence of this.

For instance, let $\mathbb{R} \times \mathbb{R}$ act on $\{\bullet\} \times \mathbb{R}/\mathbb{Z}$ diagonally, where $\{\bullet\}$ denotes a singleton with trivial action.

Then $\{(\bullet, 0)\}$ is a lacunary cross-section for the action. If we try to construct a map $\mathcal{V}$ as before, then we would map $(\bullet, x)\in\{\bullet\} \times \mathbb{R}/\mathbb{Z}$ to the subset of $\mathbb{R}^2$

$$\mathcal{V}(\bullet, x) = \mathbb{R} \times \{x + \mathbb{Z}\}.$$  

In this way one has constructed a random closed set as a factor of the action, but it is not a point process. In fact, it is possible to view an arbitrary pmp action as a kind of “bundle” of point processes over the various homogeneous spaces $G/H$, where $H$ ranges over the closed subgroups of $G$, but we will not explore this further.

The following theorem is described as folklore in [KPV15]:

Theorem 3.25 (Folklore theorem, see [KPV15, Proposition 4.3]): Let $G$ be a unimodular lcsc group, and $G \bowtie (X, \mu)$ a pmp action on a standard Borel space. Fix a lacunary cross-section $Y \subset X$ for the action. Then:

1. The orbit equivalence relation of $G \bowtie X$ restricts to a cber $\mathcal{R}$ on $Y$.
2. There exists an $\mathcal{R}$-invariant probability measure $\nu$ on $Y$.
3. The action $G \bowtie (X, \mu)$ is ergodic if and only if the cber $(Y, \mathcal{R}, \nu)$ is ergodic.
4. The group $G$ is noncompact if and only if the cber is aperiodic $\nu$ almost everywhere.
5. The group $G$ is amenable if and only if the cber $(Y, \mathcal{R}, \nu)$ is amenable.

\textsuperscript{14} Recall that an action is nonsingular if it preserves null sets, that is, if $\mu(A) = 0$ then $\mu(gA) = 0$ for all $g \in G$. 

The mathematical content of Theorem 3.19 can be viewed as a rediscovery of the above theorem with different proofs, together with interpretation of factor constructions as objects living on the Palm groupoid.

**Question 4:** Is there a more point process theoretic method to construct discrete cross-sections of free pmp actions?

We have seen that if $G \curvearrowright (X, \mu)$ is a free pmp action, then cross-sections are the same thing as point process factor maps, and that every choice of cross-section gives an isomorphic representation of the action as a marked point process. These ideas can be combined.

Suppose $\Phi : (X, \mu) \to \mathbb{M}$ is an equivariant factor map. Then $Y = \Phi^{-1}(\mathbb{M}_0)$ is a cross-section for the action $G \curvearrowright (X, \mu)$. We also have the isomorphism $\mathcal{Y} : (X, \mu) \to Y^\mathbb{M}$. These can be combined, and we see that the map

$$\Phi \circ \mathcal{Y}^{-1} : Y^\mathbb{M} \to \mathbb{M}$$

is simply the map that forgets labels.

In other words, every extension of a point process is just the point process with an enriched mark space.

4. The cost of a point process

4.1. Definition and monotonicity for factors. Our goal is to extend the notion of cost for pmp cbers to point processes. For further background on cost, see [Gab00], [Gab10], [Gab], and [KM04].

Informally speaking, the cost of a point process is the “cheapest” way to wire it up. We look at all connected factor graphs of the process and compute the expected degree at the origin in the Palm version. This is then suitably normalized to give an isomorphism invariant.

**Definition 4.1:** Let $\Pi$ be a point process on $G$ (possibly marked) with finite but non-zero intensity. Its **groupoid cost** is defined by

$$\text{cost}(\Pi) - 1 = \int \mu \cdot \inf_{\mathcal{G}} \left\{ \frac{1}{2} \mathbb{E} [\deg_0 \mathcal{G}(\Pi_0)] - 1 \right\},$$

where the infimum is taken over all connected factor graphs $\mathcal{G}$ of $\Pi$ and $\Pi_0$ denotes the Palm version of $\Pi$. Equivalently by Remark 3.14,

$$\text{cost}(\Pi) - 1 = \inf_{\mathcal{G}} \left\{ \frac{1}{2} \mathbb{E} \left[ \sum_{x \in U \cap \Pi} \deg_x \mathcal{G}(\Pi) \right] \right\} - \text{int}(\Pi),$$

where $U$ is a set of unit volume in $G$. 
Remark 4.2: The cost respects the ergodic decomposition of a process, and so for this reason it suffices to consider ergodic processes.

Definition 4.3: The cost of a group is the infimum of the cost of all its free point processes.

A group is said to have fixed price if all of its essentially free point processes have the same cost.

At the time of writing there are no groups known that do not have this property.

Remark 4.4: We sometimes refer to Definition 4.1 as groupoid as it can be thought of as the infimal “size” of a generator of the groupoid $(\overrightarrow{M}_0, \mu_0)$, in a way that we now discuss.

Recall that (directed) factor graphs are in correspondence with subsets of $\overrightarrow{M}_0$. We identify objects under this correspondence.

One defines the product of two factor graphs $\mathcal{G}, \mathcal{H} \subset \overrightarrow{M}_0$ by taking all well-defined products. More explicitly,

$$\mathcal{G} \cdot \mathcal{H} = \{ (\omega, gh) \in \overrightarrow{M}_0 | (\omega, g) \in \mathcal{G} \text{ and } (g^{-1}\omega, h) \in \mathcal{H} \}.$$ 

From the factor graph viewpoint, the edges of $\mathcal{G} \cdot \mathcal{H}$ are those pairs of vertices that can be reached by following an edge of $\mathcal{G}$ and then an edge of $\mathcal{H}$.

A Borel generator of $\overrightarrow{M}_0$ is a Borel factor graph $\mathcal{G}$ such that

$$\langle \mathcal{G} \rangle := \bigcup_n \mathcal{G}^n = \overrightarrow{M}_0.$$ 

In other words, it is a connected factor graph.

If $\Pi$ is a point process with law $\mu$, then a generator of the measured groupoid $(\overrightarrow{M}_0, \overrightarrow{\mu}_0)$ is a factor graph $\mathcal{G}$ such that

$$\overrightarrow{\mu}_0(\overrightarrow{M}_0 \setminus \langle \mathcal{G} \rangle) = 0.$$ 

In other words, it is a factor graph which is connected almost surely.

With these definitions, one can equivalently rephase the probabilistic definition of the cost of $\Pi$ as

$$\text{cost}(\Pi) - 1 = \text{int}(\Pi) \cdot \inf_{\mathcal{G}} \{ \overrightarrow{\mu}_0(\mathcal{G}) - 1 \},$$

where $\mathcal{G}$ runs over all generators of $(\overrightarrow{M}_0, \overrightarrow{\mu}_0)$. 


Example 17: If $\Pi$ is the lattice shift corresponding to $\Gamma < G$, then
\[
\text{cost}(\Pi) = 1 + \frac{d(\Gamma) - 1}{\text{covol}(G/\Gamma)},
\]
where $d(\Gamma)$ denotes the rank of $\Gamma$, that is, its minimum number of generators. To see this, observe that by equivariance a factor graph of the lattice shift is determined by a single subset $S \subseteq \Gamma$, and connects $x \in \Pi$ to all $xs \in \Pi$ for $s \in S$. The graph is connected exactly when $S$ generates $\Gamma$. The formula then follows from the definition of cost.

Remark 4.5: In a concurrently appearing work [Mel21] by the second author, it is shown that the Palm equivalence relation of any free point process on an amenable group is hyperfinite almost everywhere. It follows that amenable groups (in particular $\mathbb{R}^n$) have fixed price one.

We will show that all groups of the form $G \times \mathbb{R}$ have fixed price one. This gives an alternative proof that $\mathbb{R}^n$ has fixed price one.

It would be interesting to see a “direct” proof of this fact. That is, to exhibit reasonably explicit connected factor graphs that have cost less than $1 + \varepsilon$ for every $\varepsilon > 0$.

In [CT13] an explicit factor graph of the Poisson point process on $\mathbb{R}^2$ is described and shown to be a connected and one-ended tree. It follows that it has cost one.

Lemma 4.6: Let $\Pi$ be a point process of finite intensity, and $\Phi$ a factor map of $\Pi$ such that $\Phi(\Pi)$ has finite intensity. Then
\[
\text{cost}(\Pi) \leq \text{cost}(\Phi(\Pi)).
\]

Thus cost is monotone for factors.

Corollary 4.7: If $\mu$ and $\nu$ are finite intensity point processes that factor onto each other, then
\[
\text{cost}(\mu) = \text{cost}(\nu).
\]

In particular, the cost of $\mu$ only depends on its isomorphism class as an action.

Proof of Lemma 4.6. Recall from Remark 2.21 that $\Phi$ decomposes as the composition of a thinning $\pi$ and a thickening $\Theta^\Phi$. We prove
\[
\text{cost}(\Pi) \leq \text{cost}(\Theta^\Phi(\Pi)) \leq \text{cost}(\pi(\Theta^\Phi(\Pi))) = \text{cost}(\Phi(\Pi)),
\]
where the last equality holds as $\Phi = \pi \circ \Theta^\Phi$. 
We prove the second inequality first, as it is simpler. For this we use the non-Palm definition of cost.

To that end, let $G$ be a graphing of $\Phi(\Pi)$ that $\varepsilon$-computes the cost, that is, with

$$E \left[ \sum_{x \in U \cap \Phi(\Pi)} \stackrel{\rightarrow}{\deg_x} G(\Phi(\Pi)) \right] - \text{int}(\Phi(\Pi)) \leq \text{cost}(\Phi(\Pi)) - 1 + \varepsilon.$$

We will use it to define a graphing $\mathcal{H}$ of the thickened process $\Theta^\Phi(\Pi)$. Recall that this process has three types of points: red, purple, and blue.

Let $\mathcal{N}$ be the factor graph of $\Theta^\Phi(\Pi)$ that connects each red point $x$ to its nearest blue neighbor. If this is not well-defined, then we use the tie-breaking function $T : G \to \mathbb{R}$ of Section 3.15 to make it so in an equivariant way.

That is, if $y_1, y_2, \ldots, y_n$ are the (finitely many!) blue points of $\Theta^\Phi(\Pi)$ that are closest to $x$, then let $y$ be the element that minimizes $T(x^{-1} y_i)$ and add in a directed edge $x \to y$ to $\mathcal{N}$.

We can view $G$ as defining a factor graph on $\Theta^\Phi(\Pi)$, which lives on the blue and purple points.

Now let $\mathcal{H}(\Theta^\Phi(\Pi)) = \mathcal{G}(\Phi(\Pi)) \cup \mathcal{N}(\Theta^\Phi(\Pi))$. This is connected as an undirected graph, so by the definition of cost:

$$\text{cost}(\Theta^\Phi(\Pi)) - 1 \leq E \left[ \sum_{x \in \Theta^\Phi(\Pi) \cap U} \stackrel{\rightarrow}{\deg_x} \mathcal{H}(\Theta^\Phi(\Pi)) \right] - \text{int}(\Theta^\Phi(\Pi))$$

$$= E \left[ \sum_{x \in U \cap \Pi \setminus \Phi(\Pi)} 1 + \sum_{x \in U \cap \Phi(\Pi)} \stackrel{\rightarrow}{\deg_x} \mathcal{G}(\Phi(\Pi)) \right]$$

$$- \text{int}(\Pi \setminus \Phi(\Pi)) - \text{int}(\Phi(\Pi))$$

$$= E \left[ \sum_{x \in U \cap \Phi(\Pi)} \stackrel{\rightarrow}{\deg_x} \mathcal{G}(\Phi(\Pi)) \right] - \text{int}(\Phi(\Pi))$$

$$\leq \text{cost}(\Phi(\Pi)) - 1 + \varepsilon.$$

As $\varepsilon$ was arbitrary, this proves the second inequality.

For the other inequality, we use the explicit description of the Palm measure as in Example 15 and the Palm definition of cost.

The idea of the proof is: we have a graphing defined on a larger subset, and we must push it onto a smaller subset somehow. We will simply transfer all edges of $\Theta^\Phi(\Pi)$ to $\Pi$ along the Voronoi cells.

For $g \in \Pi$, let $F_\Pi(g) = V_\Pi(g) \cap \Theta^\Phi(\Pi)$. 
Let us call a graphing $G$ of $\Theta^\Phi(\Pi)$ starlike if for all $g \in \Pi$ and $x \in F_\Pi(g)$, we have $(g, x) \in G$. If $G$ is any graphing, then we can perturb it to find a starlike graphing of the same edge measure. Let us take this for granted for now and see how the proof concludes.

Let $G$ be a starlike graphing of $\Theta^\Phi(\Pi)$ that $\varepsilon$-computes the cost. Let us define a graphing $H$ of $\Pi$ as follows: join $x, y \in \Pi$ and $g \in H(\Pi)$ if there exists $x' \in F_\Pi(x)$ and $y' \in F_\Pi(y)$ such that $x'$ and $y'$ are connected by an edge in $\Theta^\Phi(\Pi)$.

When we push $G$ onto $\Pi$, some edges get killed. For instance, if two Voronoi cells have many edges between them, then some get killed. By assuming that the graphing is starlike we are guaranteed to kill enough edges. In particular, we kill $|F_\Pi(g)| - 1$ edges at each $g \in \Pi$.

To make the proof more legible, we write $I_\Pi = \text{int}(\Pi)$ and $I_\Theta = \text{int}(\Theta^\Phi(\Pi))$, so that $I_\Theta = I_\Pi \cdot \mathbb{E}[F_{\Pi_0}(0)]$.

We compute its expected outdegree as follows:

$$I_\Pi \cdot \mathbb{E}[\deg_0^r H(\Pi_0) - 1] \leq I_\Pi \cdot \mathbb{E}\left[ \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) - |F_{\Pi_0}(0)| \right]$$

$$= I_\Pi \cdot \mathbb{E}\left[ \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) \right] - I_\Pi \cdot \mathbb{E}|F_{\Pi_0}(0)|$$

$$= I_\Pi \cdot \mathbb{E}\left[ \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) \right] - I_\Theta.$$  

We now work on this first term:

$$I_\Pi \cdot \mathbb{E}\left[ \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) \right]$$

$$= \frac{I_\Theta}{\mathbb{E}|F_{\Pi_0}(0)|} \mathbb{E}\left[ \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) \right]$$

$$= \sum_{k \geq 1} \frac{I_\Theta}{\mathbb{E}|F_{\Pi_0}(0)|} \mathbb{E}\left[ \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) \right] \mathbb{P}[|F_{\Pi_0}(0)| = k]$$

$$= I_\Theta \sum_{k \geq 1} \mathbb{E} \left[ \frac{1}{|F_{\Pi_0}(0)|} \sum_{x \in F_{\Pi_0}(0)} \deg_x G(\Theta^\Phi(\Pi_0)) \right] \mathbb{P}[|F_{\Pi_0}(0)| = k]$$

$$= I_\Theta \mathbb{E}[\deg_0 G(\Theta^\Phi(\Pi_0))].$$
where we use the explicit description of the Palm measure of a general thickening proven in Example 15. Thus

\[ I_\Pi \cdot \mathbb{E}[\deg_0 \mathcal{H}(\Pi_0) - 1] \leq I_\Theta \mathbb{E}[\deg_0 (\mathcal{G}(\Theta^0(\Pi)_0)) - 1] \]

proving \( \text{cost}(\Pi) \leq \text{cost}(\Theta^0(\Pi)) \), as desired.

At last, we must show how to perturb graphings to be starlike. The idea is simple: if some \( g \in \Pi \) is not starlike, then there is some \( x \in F_\Pi(g) \) such that \( (g, x) \notin \mathcal{G} \). However, there must be some path from \( g \) to \( x \) in \( \mathcal{G} \) so we pinch an edge from that path and thus rob Peter to pay Paul. In this way we can improve a given factor graph to be more starlike. By iterating in an appropriate way we can construct the desired factor graph.

Let

\[ \Pi' = \bigcup_{g \in \Pi} \{ h \in F_\Pi(g) \mid h \neq g \text{ and } (g, h) \notin \mathcal{G} \} \]

denote the subprocess of points that violate starlikeness.

The edges of \( \mathcal{G} \) are of three kinds according to how they interface with the Voronoi cells of \( \Pi \):

- **Starlike** edges, those of the form \( (g, h) \) where \( g \in \Pi \) and \( h \in F_\Pi(g) \),
- **Intracell** edges, those of the form \( (h, h') \) where \( h, h' \in F_\Pi(g) \) for some \( g \in \Pi \) with neither of \( h \) or \( h' \) being \( g \), and
- **Crossing** edges, those of the form \( (h, h') \) with \( h \in F_\Pi(g) \) and \( h' \in F_\Pi(g') \) with \( g, g' \in \Pi \) and \( g \neq g' \).

We consider the space \( \mathcal{G} \) of marked factor graphs of \( \Pi \) with the following properties:\(^{15}\)

- They are simply \( \mathcal{G} \) as an unmarked graph,
- Points \( h \) of \( \Pi' \) receive either the blank mark \( \bullet \),
- or they are marked by a non backtracking path in \( \mathcal{G} \) from \( h \) to \( g \), where \( h \in F_\Pi(g) \), and one crossing or intracell edge of this path is colored red, and
- each red edge appears in at most one of the paths of points of \( \Pi' \).

---

\(^{15}\) By constructing an appropriate subset of a configuration space, one can encode these graphs as a standard Borel space.
These factor graphs are basically rewiring rules for $\mathcal{G}$. If $\mathcal{H} \in \mathcal{G}$, then each point of $\Pi'$ that receives a path label in $\mathcal{H}$ replaces the red crossing edge with its starlike edge (see Figure 3). This is an equivariant, measurable, and deterministic rule, so defines a factor graph of $\Pi$.

Figure 3. A single rewiring move, with the gray edge sliding to the location of the black edge. We stress that this move must be taken at all chosen points simultaneously.
Note that this rewiring does not change the edge measure of the graph (we rob Peter to pay Paul), and it remains connected.

Let
\[ \iota(H') = \text{the intensity of } \Pi' \text{ points that receive path labels in } H \]
and
\[ f(H') = \sup \{ \iota(H') \mid H' \in \mathcal{G} \text{ and } H \preceq H' \}, \]
where we declare \( H \preceq H' \) if every path label in \( H \) is present in \( H' \). That is, \( H' \) has simply replaced \( \bullet \) labelled points in \( H \) by path labels.

**Claim:** There is a maximal element \( \mathcal{G}_\infty \) (with respect to \( \preceq \)) of \( \mathcal{G} \), and the rewiring of \( \mathcal{G}_\infty \) associated to it is starlike.

It is easy to find the maximal element. Choose \( \mathcal{G}_1 \) such that
\[ f(\mathcal{G}) \leq \iota(\mathcal{G}_0) + 1, \]
and then inductively choose \( \mathcal{G}_{n+1} \) such that
\[ f(\mathcal{G}_n) \leq \iota(\mathcal{G}_{n+1}) + \frac{1}{n}. \]
Let \( \mathcal{G}_\infty \) denote the “union” of the \( \mathcal{G}_n \), where we declare that path labels trump \( \bullet \) labels.

If \( H \in \mathcal{G} \) is a factor graph with \( \mathcal{G}_\infty \preceq H \), then \( \mathcal{G}_n \preceq H \) for all \( n \), so
\[ \iota(H) \leq f(\mathcal{G}_n) \leq \iota(\mathcal{G}_n) + \frac{1}{n} \Rightarrow \iota(\mathcal{G}_\infty). \]
We conclude that \( \mathcal{G}_\infty = H \) almost surely since one process is a subset of the other.

We will now prove that the rewiring associated to \( \mathcal{G}_\infty \) is starlike by contradiction. Using the assumption we construct \( H \in \mathcal{G} \) with \( \mathcal{G}_\infty \preceq H \) and \( \iota(\mathcal{G}_\infty) < \iota(H) \). This violates maximality of \( \mathcal{G}_\infty \).

Let \( \Pi_\infty \) denote the result of rewiring \( \mathcal{G}_\infty \). Set
\[ \Pi_\infty = \{ g \in \Pi \mid g \text{ is not starlike in } \overline{\mathcal{G}} \}. \]
We are going to make these points more starlike. For each \( g \in \Pi_\infty \), choose a point \( x_g \in \Theta(\Pi) \) in an equivariant and measurable way. More precisely, we consider the set
\[ \{ y \in F_\Pi(g) \mid (g, y) \notin \overline{\mathcal{G}} \} \]
and choose \( x_g \) to be the element minimizing \( I(g^{-1}y) \).
Fix a non-backtracking path \( P(g, x_g) \) from \( g \) to \( x_g \) in \( \mathcal{G} \). We do this for all \( g \in \Pi \times \) simultaneously, again in an equivariant and measurable way: look at all paths between \( g \) and \( x_g \) of minimal length (as in number of \( \mathcal{G} \) edges used), and choose one using the Borel isomorphism \( I \) in a similar way to before.

Choose \( N \) so large that there is a positive intensity of points \( g \in \Pi \times \) with paths \( P(g, x_g) \) of length at most \( N \).\(^{16}\)

We now construct our desired marked factor graph \( \mathcal{H} \) as follows:

- Every point in \( \mathcal{G} \) that has a path label retains its path label.
- Every point of \( \Pi' \setminus \Pi \times \) is marked .
- Every point \( g \) of \( \Pi \times \) whose path \( P(g, x_g) \) has length greater than \( N \) is marked .

This leaves the points of \( \Pi \times \) whose paths are bounded by \( N \). Note that this is a locally finite family—each edge appears on at most finitely many \( P(g, x_g) \).

Every path in the rewired graph \( \overline{\mathcal{G}} \) can be associated to a path in \( \mathcal{G}_\infty \) itself—every time one of the starlike edges that was added in the rewiring process is added, just go the long way in \( \mathcal{G} \). We refer to this as the detour version of the path.

Now, to construct the remaining labels check if there are any paths \( P(g, x_g) \) which contain an intracell edge \( e = (h, h') \) with \( h, h' \in F_{\Pi}(g) \). If so, then we label \( g \) by the detour version of this path in \( \mathcal{G}_\infty \) with \( e \) colored red.

Observe that the remaining paths must contain at least two crossing edges. Each \( g \) will apply to the first edge crossing edge it sees on the path from \( g \) to \( x_g \).

Each edge \( (h, h') \) receives finitely many applicants \( \{g_1, g_2, \ldots, g_k\} \). It chooses the element of this set which minimizes

\[
\min\{I(g_i^{-1}h), I(g_i^{-1}h')\}.
\]

At last, we finish the construction of \( \mathcal{H} \) by marking the points who were rejected by ., and the remaining ones by the detour version of this path in \( \mathcal{G} \) with their chosen edge colored red.

Then \( \mathcal{G}_\infty \preceq \mathcal{H} \) by construction, but \( \iota(\mathcal{H}) > \iota(\mathcal{G}_\infty) \).

We are therefore able to replace \( \mathcal{G} \) by \( \mathcal{G}_\infty \) and assume our factor graph is starlike, as desired.

---

\(^{16}\) If the process is not ergodic then this \( N \) should be a random variable, in any case one can manage.
Remark 4.8: The groupoid cost can really increase under a factor map: take the example of Remark 2.19 with $\mathbb{Z}^n < \mathbb{R}^n$ for $n > 1$. That is, consider the union of the $\mathbb{Z}^n$ lattice shift with the Poisson point process. As a free process, this has cost one. But it factors onto the lattice shift, which has cost greater than one.

One can also prove cost monotonicity by invoking Gaboriau’s theorem on the cost of complete sections:

**Theorem 4.9** ([Gab00, Proposition II.6], see also [KM04, Theorem 21.1]): If $(X, \mathcal{R}, \mu)$ is a pmp cber and $S \subseteq X$ is a complete section,\(^{17\text{ }}\) then

\[
\text{cost}_\mu(\mathcal{R}) - 1 = \mu(S)(\text{cost}_{\mu|S}(\mathcal{R}|_S) - 1),
\]

where $\mathcal{R}|_S = \mathcal{R} \cap S \times S$ is the restriction and

\[
\mu|S := \frac{\mu(\bullet \cap S)}{\mu(S)}
\]

is the conditional measure.

Suppose $\Phi : (\mathbb{M}, \mu) \to \mathbb{M}$ is a point process factor map with $\Phi_*\mu$ of finite intensity. Then

\[
Y := \Phi^{-1}(\mathbb{M}_0) = \{\omega \in \mathbb{M} \mid 0 \in \Phi(\omega)\}
\]

forms a discrete cross section for the action $G \acts (\mathbb{M}, \mu)$.\(^{18\text{ }}\) One can define a “Palm measure” $\mu_Y$ on $Y$ by replacing all references to $\mathbb{M}_0$ with $Y$, and similarly there is a rerooting equivalence relation $\mathcal{R}_Y$ on $Y$. This again forms a pmp cber. Then we have a morphism $\Phi : (Y, \mathcal{R}_Y, \mu_Y) \to (\mathbb{M}_0, \mathcal{R}, \Phi_*\mu)$ of pmp cbers, so

\[
\text{cost}_{\mu_Y}(\mathcal{R}_Y) \leq \text{cost}_{\Phi_*\mu}(\mathcal{R}).
\]

One can see that $Y \cup \mathbb{M}_0$ also forms a discrete cross section, and both $Y$ and $\mathbb{M}_0$ are complete sections for it. Then two applications of Gaboriau’s theorem shows

\[
\text{cost}_{\mu_Y}(\mathcal{R}_Y) = \text{cost}_\mu(\mathcal{R}),
\]

thus proving cost monotonicity.

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\(^{17\text{ }}\) That is, it meets almost every orbit of $X$.

\(^{18\text{ }}\) See Section 3.5 for the definition and further context.
4.2. Unmarking. We have defined cost of groups by looking at all free unmarked point processes on the group. This is no loss of generality:

**Proposition 4.10:** Every free point process $\Pi$ on a nondiscrete group with marks from a standard Borel space $\Xi$ is equivariantly isomorphic to an unmarked point process. More precisely, if $\Pi$ has marks from a standard Borel space $\Xi$ and $\mu$ is its law, then there is a measurable and equivariant almost everywhere isomorphism $\Phi : (\Xi^M, \mu) \to (M, \Phi_*\mu)$.

Since cost is an isomorphism invariant (even if one process is marked and the other is not), this shows that one cannot find point processes with lower cost by using some tricky mark space.

We refer to Proposition 4.10 as **unmarking**. It should be easy to convince oneself that such a proposition will be true, although the details will necessarily be somewhat messy and ad hoc. We call the technique used local encoding, which is illustrated in the following example:

![Figure 4. Locally encoding labels of a point process.](image)

This is a point process in $\mathbb{R}^2$ labelled by the set $\{+, -\}$, which we have colored as cyan and magenta respectively in the diagram.

The map $\Phi : \{+, -\}^M \to M$ takes the input configuration, and adds a small decoration around each point. In this case we are literally encoding $+$ marks as a plus symbol centered at each point and similarly for $-$ marks.
Barring some exceptional circumstances, you should be able to convince yourself that $\Phi$ is an injective map, and thus is an isomorphism onto its image for many input processes. The proof for general $G$ works along the same lines.

We will employ a general lemma that is no doubt well known to experts. For the convenience of the reader we translate a proof appearing in [Tim04] and attributed to Yuval Peres into our language.

**Lemma 4.11:** Let $\mu$ be a free point process on $G$, and $\mathcal{F}$ a locally finite measurable factor graph of $\mu$. Then one can equivariantly and measurably construct a non-trivial independent subset of $\mathcal{F}$.

To spell this out, this means there exists a map $I : (\mathbb{M}, \mu) \to \mathbb{M}$ with the properties that

- $I(\omega) \subset \omega$ almost surely, and
- if $g, h \in I(\omega)$, then $g$ and $h$ are not connected in $\mathcal{F}(\omega)$.

**Proof.** The key idea in the proof can be illustrated by via the factor labeling $\otimes : \mathbb{M} \to \mathbb{M}_0^{\mathbb{M}}$ given by

$$\otimes(\omega) = \{(g, g^{-1}\omega) \in G \times \mathbb{M}_0(G) \mid g \in \omega\}.$$  

Under $\otimes$, each point $g$ of a configuration $\omega$ looks at how the configuration looks like from its perspective, and records it as a label. That is, it views itself as the center of the universe (this is what the symbol $\otimes$ is meant to represent, we will call the map *egotistical* or *self-centered*).

Observe that $\mu$ is an (essentially) free action if and only if $\otimes(\omega)$ has distinct labels almost surely. For if $g, h \in \omega$ receive the same label under the egotistical map, then $g^{-1}\omega = h^{-1}\omega$, i.e., $gh^{-1} \in \text{stab}_G(\omega)$. Conversely, if $g \in \text{stab}_G(\omega)$ is nontrivial, then for all $x \in \omega$ the label $x^{-1}\omega$ of $x$ is the same as that of $gx$, as

$$(gx)^{-1}\omega = x^{-1}\omega.$$  

Fix a countable dense subset $Q \subset \mathbb{M}_0$. Let us define a thinning $I_q : \mathbb{M} \to \mathbb{M}$ for each $q \in Q$ by

$$I_q(\omega) = \{g \in \omega \mid d(g^{-1}\omega, q) < d(h^{-1}\omega, q) \text{ for all } h \in \omega \text{ adjacent to } g \text{ in } \mathcal{F}(\omega)\}.$$  

Note that each $I_q(\omega)$ is an independent subset of $\mathcal{F}(\omega)$, but it is possibly empty. However, the union over all $q$ of the $I_q$ is $\omega$ by freeness, so at least one such $I_q$ must define a non-empty independent subset, as desired.
In particular, by applying the lemma to the factor graph $\mathcal{D}_R$ of Example 8, one has:

**Corollary 4.12:** Let $\Pi$ be a free point process. Then for all $R > 0$ one deterministically, measurably, and equivariantly selects a subset $\Pi_R \subset \Pi$ that is $R$ uniformly separated, in the sense that if $x$ and $y$ are distinct points of $\Pi_R$, then $d(x, y) > R$.

Our proof will also make use of a technique we refer to as label trickery:

**Proposition 4.13 (Label trickery):** Let $\Pi$ be any free point process (possibly marked), and $\theta(\Pi)$ any nonempty thinning. Then there exists a marked point process $\Upsilon$ such that the underlying point set of $\Upsilon$ is $\theta(\Pi)$, and $\Upsilon$ is isomorphic to $\Pi$ as a pmp action. In particular, $\Upsilon$ is a free action.

The same can be achieved with $\Upsilon$ having marks from the compact space $[0, 1]$.

**Proof.** Let $\Upsilon = \theta(\varnothing(\Pi))$, that is,

$$\Upsilon = \{(g, g^{-1}\Pi) \in \mathbb{M} \times \mathbb{M}_0 \mid g \in \theta(\Pi)\}.$$

Observe that this is an injective map, as one can recover $\Pi$ uniquely from the knowledge of any point $\Upsilon$ and its label, and so $\Upsilon$ is an isomorphic process to $\Pi$.

For the second statement, simply fix a Borel isomorphism $I: \mathbb{M}_0 \to [0, 1]$, and define

$$\Upsilon = \{(g, I(g^{-1}\Pi) \in \mathbb{M} \times [0, 1] \mid g \in \theta(\Pi))\}.$$

**Proof of Proposition 4.10.** Suppose $\Pi$ is a free $\Xi$-marked point process with law $\mu$. We can (and do) assume that $\Pi$ is abstractly isomorphic to a uniformly separated process with a slightly different (but nevertheless standard Borel) mark space by using the previous two propositions.

Let $X$ denote the space

$$X = \{\omega \in \mathbb{M}_0(B(0, \delta/100)) \mid \omega \cap B(0, \delta/200) = \{0\},$$

and $\forall x \in \omega \setminus \{0\}, |B(x, \delta/200)| > 1\}$. This is a Borel subset of a standard Borel space, and hence standard Borel in its own right. One can readily see that it is uncountable, and hence there is a

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19 It exists as $\mathbb{M}_0$ is a Polish space, and thus standard Borel, and all standard Borel spaces of the same cardinality are isomorphic.
Borel isomorphism $I : \Xi \to X$. Define the following factor map:

$$
\Phi : \Xi^M \to \mathbb{M},
$$

$$
\Phi(\omega) = \bigcup_{x \in \omega} xI(\xi_x),
$$

where $\xi_x$ denotes the label of $x$ (that is, $(x, \xi_x) \in \omega$).

This is an injective map: we can recover the underlying set of any input configuration to $\Phi$ by identifying the points which are $\delta/200$-isolated. We can then uniquely recover their labels by applying the inverse of $I$ locally. 

### 4.3. Cost is finite for compactly generated groups.

**Proposition 4.14:** Suppose $G$ is compactly generated by $S \subseteq G$. Then every free point process $\Pi$ on $G$ has finite cost.

Implicitly we are assuming that $\Pi$ has finite intensity, so that its cost is defined.

We recall some definitions and facts from metric geometry, see [CdlH16] for further details in the specific context we are interested in.

**Definition 4.15:** Let $(X, d)$ be a metric space.

- $(X, d)$ is **coarsely connected** if there exists $c > 0$ such that for all $x, x' \in X$ there are points $x_1, x_2, \ldots, x_n \in X$ with $x = x_1$, $x_n = x'$, and $d(x_i, x_{i+1}) \leq c$ for all $i$.
- A subset $\omega \subseteq X$ is **uniformly discrete** if there exists $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for all distinct $x, y \in \omega$.
- A subset $\omega \subseteq X$ is **coarsely dense** if there exists $r > 0$ such that for every $x \in X$, $d(x, \omega) < r$.
- A **Delone set** is a subset $\omega \subseteq X$ which is both uniformly discrete and coarsely dense.
- An $\varepsilon$-**net** is a subset $\omega \subseteq X$ which is $\frac{\varepsilon}{2}$ uniformly discrete and $\varepsilon$ coarsely dense.

**Theorem 4.16** (See [CdlH16, Proposition 1.D.2]): Let $G$ be an lcsc group with a left-invariant proper metric $d$ which generates its topology. Then $G$ is compactly generated if and only if it is coarsely connected.
Note that if $X$ is coarsely connected, then so too is any coarsely dense subset of $X$.

**Definition 4.17:** Let $S \subseteq G$ be a compact and symmetric generating set.

The **Cayley factor graph** associated to $S$ is the map

$$Cay(\bullet, S) : \mathbb{M} \to \text{Graph}(G)$$

given by

$$Cay(\omega, S) = \{(g, gs) \in \omega \times \omega \mid s \in S\}.$$

Note that this graph is not necessarily connected, for instance for the Poisson point process. However, if $\Pi$ is a point process which is almost surely $c$-coarsely-connected for $c$ such that $B(0, c) \subseteq S$ then $Cay(\Pi, S)$ is connected. This condition can always be satisfied by replacing $S$ with an appropriate power of the generating set $S^k$, since $S^k$ exhausts $G$ and in particular must contain $B(0, c)$ for $k$ sufficiently large.

The following can be readily deduced from existing results in the literature (even removing the compact generation assumption), but we include a separate proof for completeness.

**Proposition 4.18:** Suppose $\Pi$ is a free and ergodic point process on a compactly generated group $G$. Then for every $R > 0$ there exists a finite intensity thickening $\Theta$ of $\Pi$ such that $\Theta(\Pi)$ is almost surely $R$-coarsely-dense.

Moreover, if $\Pi$ is $\delta$-separated (with $\delta < 2R$), then $\Theta$ will also be $\delta$-separated.

**Proof.** It suffices to prove the statement for ergodic processes.

Fix $R > 0$. We will construct a factor map $\Phi$ of $\Pi$ such that $\Phi(\Pi)$ is $\frac{R}{2}$ uniformly separated and $\Theta(\Pi) := \Pi \sqcup \Phi(\Pi)$ is $R$-coarsely dense. The uniform separation then implies that this thickening has finite intensity.

The idea of the proof is the following: observe that every uniformly separated subset of a metric space is a subset of a Delone set. You can prove this using the well-ordering principle or Zorn’s lemma (as to your taste). Now consider a sample $\Pi$ from the point process. We know there are some ways to add points to it to get something coarsely dense, the only difficulty is that we are required to make these choices equivariantly. We will select points that see the “frontier” of the process, which will then add points to cover a piece of the frontier. At every stage the frontier gets smaller, and in the limit we cover the whole space.
For configurations $\omega \in M$, let $\omega^t$ denote the following closed set

$$\omega^t = \bigcup_{g \in \omega} B(g, t),$$

that is, the union of all closed balls about the points of $\omega$. If $\Pi$ is a point process, then $\Pi^t$ is a random closed subset of $G$.

We call a point $g \in \Pi$ on the frontier if

$$B(g, c_1 R) \not\subset \Pi^R,$$

where $c_1 > 1$ is some parameter to be chosen later, and let $F(\Pi)$ denote the subset of frontier points of $\Pi$. This is a metrically defined condition, and hence equivariant. We will define a rule $\Phi_1(\Pi)$ that specifies a collection of points such that their $R$-balls cover all the $c_1 R$-balls of the frontier points of $\Pi$. We will then iterate this construction (so that $\Phi_2(\Pi)$’s $R$-balls cover the $c_2 R$-balls of $\Phi_1(\Pi) \cup \Pi$’s frontier points, for some $c_2 > c_1$, and so on). In this way we will find enough points to cover the whole space.

Choose $c_1$ large such that $\mathbb{P}[\Pi c_1 R \setminus \Pi^R \neq \emptyset] = 1$. If this is not possible, then the process is already $R$-coarsely-dense by ergodicity.

One can decompose the frontier points of $\Pi$ as

$$F(\Pi) = \bigsqcup_{n} F_n(\Pi),$$

where each $F_n(\Pi)$ is $10c_1 R$ uniformly separated. This can be done by using the existence of a Borel kernel of the factor graph $\mathcal{G}_{10c_1 R}(\Pi_0)$ defined on the frontier points of $\Pi_0$, see [KST99, Section 4] for further information on Borel kernels. Note that by using the Palm process, the resulting sets $F_n(\Pi)$ are equivariantly defined.

We now fix an auxiliary (deterministic) $R$-net $\mathcal{N} \subset G$. If $W \subseteq G$ is a Borel region and $g \in G$, then let

$$N(g, W) = \{x \in g^{-1} \mathcal{N} \mid B(x, R) \cap W \neq \emptyset\}.$$  

Note that $N(g, W)^R \supseteq W$, as $\mathcal{N}$ is coarsely dense. Define

$$\Phi_1(\Pi) = \bigsqcup_{g \in F_1(\Pi)} N(g, B(g, c_1 R) \setminus \Pi^R),$$

and inductively

$$\Phi_{n+1}(\Pi) = \bigsqcup_{g \in F_n(\Pi)} N\left(g, B(g, c_1 R) \setminus \left(\Pi \cup \bigcup_{i \leq n} \Phi_i(\Pi)^R\right)\right).$$
Then

\[ \Pi^{c_1 R} \subseteq \Pi^R \cup \bigcup_{n \geq 1} \Phi_n(\Pi)^R. \]

We now repeat this procedure as many times as necessary (possibly countably infinitely many times) until we construct the desired thickening \( \Theta \). The only care necessary is that one should choose the parameters \( c_1 < c_2 < \cdots \) so that they tend to infinity, as

\[ G = \bigcup_{n \geq 1} \Pi^{c_n R} \]

for any such sequence.

Remark 4.19: In Section 3.5 we describe the connection between point processes and “cross-sections” of actions. The previous proposition can be deduced from the fact that every free action admits a “cocompact cross-section”. A similar statement to the proposition directly phrased in terms of cross-sections can be found in [Slu17, Section 2], where it is shown that any cross-section can be extended to a cocompact cross-section. That proof works without the compact generation assumption.

Proof of Proposition 4.14. It suffices to consider the case where \( \Pi \) is ergodic; see [KM04] Corollary 18.6.

If \( \Pi \) is \( \delta \)-separated, then the thickening constructed in Proposition 4.18 has finite cost for the Cayley factor graph, which is connected (for a suitable choice of generating set). Cost is monotone for factors, so \( \Pi \) itself has finite cost.

Otherwise, choose \( \delta \) sufficiently small so that the \( \delta \)-thinning of \( \Pi \) is non-empty almost surely. By label trickery (Proposition 4.13), \( \Pi \) is isomorphic to a (marked) \( \delta \)-separated point process, reducing to the previous case.

5. The Poisson point process has maximal cost

We begin with the observation that every IID process factors onto the Poisson:

**Proposition 5.1:** Let \( \Pi \) be a point process on \( G \). Then \([0, 1]^\Pi\) factors onto the Poisson point process.

**Proof.** Fix a map \( F : [0, 1] \to M(G) \) such that if \( \xi \sim \text{Unif}[0, 1] \), then \( F(\xi) \) is a Poisson point process on \( G \) of unit intensity.
We will use the Voronoi tessellation to simply glue independent copies of the Poisson point process in each cell, resulting in a Poisson point process.

Define a factor map $\Phi([0,1]^\Pi)$ by

$$\Phi([0,1]^\Pi) = \bigcup_{g \in [0,1]^\Pi} g \cdot F(\xi_g) \cap V^T_\Pi(g),$$

where $\xi_g$ denotes the label of $g$ in $[0,1]^\Pi$. Then $\Phi([0,1]^\Pi)$ is the Poisson point process.

In particular, the cost of every IID process is at most the cost of the Poisson.

Our goal is to prove an asymptotic version of this statement. We will show that every free point process “weakly” factors onto an IID process, and that cost is monotone for (certain) weak factors. This will prove that the Poisson point process has maximal cost amongst all free point processes.

5.1. Weak factoring and Abért–Weiss for point processes. We have seen that cost is monotone under factor maps. We will now introduce a weaker version of factoring and investigate its relationship to cost:

**Definition 5.2:** Let $\Pi$ and $\Upsilon$ be point processes. Then $\Pi$ weakly factors onto $\Upsilon$ if there is a sequence $\Phi^n$ of factors of $\Pi$ such that $\Phi^n(\Pi)$ weakly converges to $\Upsilon$.\(^{20}\)

The restive reader is advised to take a look at the statements of Theorem 5.5 and Theorem 5.10. These are the tools that will be used to prove the headline theorem of this section. The other results in this section are necessary but have a more routine flavor.

**Theorem 5.3:** Let $\Pi$ and $\Upsilon$ be point processes on an amenable group $G$. If $\Pi$ is free, then $[0,1]^\Pi$ weakly factors onto $\Upsilon$.

The proof of this uses a lemma, a proof of which can be found in a concurrently appearing paper by the second author:

\(^{20}\) For more information on weak convergence in the context of point processes, see Appendix A.
Lemma 5.4: If $\Pi$ is a free point process on an amenable group $G$, then there exist factor partitions $\mathcal{P}_n(\Pi) = \{P^n_g\}_{g \in \Pi}$ with the following properties:

- **Equivariance:** $P^n_{\gamma g} = \gamma P^n_g$.
- **Partitioning:** For each $n$, $G$ is the union of $\{P^n_g\}_{g \in \Pi}$, and if $g, h \in \Pi$ then $P^n_g = P^n_h$ or $P^n_g \cap P^n_h = \emptyset$.
- **Increasing:** For each $n$, $P^n_g \subseteq P^{n+1}_g$.
- **Exhausting:** For all compact $C \subseteq G$, there exists $N$ and $g \in \Pi$ such that $C \subseteq P^N_g$.
- **Finite volume:** For all $n$, $0 < \lambda(P^n_g) < \infty$.
- **Finitariness:** For each $n$, $P^n_g \cap \Pi$ is finite (and contains $g$).

Figure 5. The factor partitions from Lemma 5.4 are “clumpings”, and should be visualised like this.

*Proof of Theorem 5.3.* Let $f : [0, 1] \to \mathbb{M}$ be a measurable map with $f(\xi) \sim \Upsilon$ if $\xi \sim \text{Unif}[0, 1]$.

Choose factor partitions $\mathcal{P}_n(\Pi) = \{P^n_g\}_{g \in \Pi}$ as in Lemma 5.4. Let $\Pi_n$ be an equivariantly defined subprocess of $\Pi$ which consists of one point chosen out of each cell $P^n_g$—we are able to do this by essential freeness. For instance, fix a Borel isomorphism of $\mathbb{M}_0$ with $[0, 1]$ and note that the induced label in $[0, 1]$ at each point of $\Pi$ is distinct for all points (essential freeness), and thus we may choose $\Pi_n$ to consist of the point with maximal label amongst its cell.
Define factors $\Phi_n$ as follows:

$$\Phi_n([0, 1]^{\Pi}) = \bigcup_{g \in \Pi_n} g \cdot (f(\xi_g) \cap P^n_g).$$

That is, in each cell we glue a copy of the process $\Upsilon$ sampled according to the label $\xi_g$ on $g$.

It follows immediately that $\Phi_n([0, 1]^{\Pi})$ weakly converges to $\Upsilon$: if $C \subseteq G$ is any compact stochastic continuity set for $\Upsilon$, then for sufficiently large $N$ is entirely contained in some $P^n_g$, and thus the point counts of $\Phi^n([0, 1]^{\Pi})$ inside $C$ are exactly distributed the same as those of $\Upsilon$.

The following statement is due to Abért and Weiss [AW13] for discrete groups, we extend it to point processes:

**Theorem 5.5:** Let $\Pi$ be an essentially free point process on a noncompact group $G$. Then $\Pi$ weakly factors onto $[0, 1]^{\Pi}$, its own IID.

**Proof.** It suffices to show that $\Pi$ weakly factors onto $[d]^{\Pi}$, where $[d] = \{1, 2, \ldots, d\}$ is equipped with the uniform measure. Here $[d]^{\Pi}$ is thus the finitary IID of $\Pi$. This suffices as $[d]^{\Pi}$ weakly converges to $[0, 1]^{\Pi}$ as $d \to \infty$. We will do this by constructing factor $[d]$-labellings $\mathcal{C}_n$ of $\Pi$ such that $\mathcal{C}_n(\Pi)$ weakly converges to $[d]^{\Pi}$.

To do this, we will use the second moment method, hewing close to the original Abért–Weiss recipe.

The strategy will be as follows. Consider the set of $[d]$-labellings of $\Pi$. We will study a probabilistic model that produces a random element of this space. We will show that this random deterministic coloring satisfies certain constraints with positive probability. In particular, there must exist a $[d]$-labelling satisfying those constraints. By adjusting the parameters of this model, one can produce the desired sequence $\mathcal{C}_n$.

Fix a countable weak convergence determining family $\{V_i\}$ as discussed at Lemma A.21, so that the sets $V_i \subseteq G \times [d]$ are bounded stochastic continuity sets for $[d]^{\Pi}$. We will construct a sequence of factor colorings $\mathcal{C}_n$ of $\Pi$ such that for fixed $k$,

$$N_{V_k}(\mathcal{C}_n \Pi) \text{ converges weakly to } N_{V_k}([d]^{\Pi}),$$

where

$$V_k = (V_1, V_2, \ldots, V_k).$$
Set $W_k = \bigcup_{i \leq k} V_k$ to be the total window. Formally this is a subset of $G \times [d]$, but we view it as a subset of $G$. For $\varepsilon > 0$ arbitrary, we choose $\delta > 0$ so small that the following properties are true, where $\mu$ denotes the law of $\Pi$:

$$\mu(\{ \omega \in \mathcal{M} \mid \text{for all } g, h \in \omega \cap W_k, g \neq h \text{ implies } d(g^{-1} \omega, h^{-1} \omega) > \delta \}) > 1 - \varepsilon$$

and

$$(\mu \otimes \mu)(\{ (\omega, \omega') \in \mathcal{M} \times \mathcal{M} \mid \text{for all } (g, h) \in (\omega \cap W_k) \times (\omega' \cap W_k),
\quad (g^{-1} \omega, h^{-1} \omega') > \delta \}) > 1 - \varepsilon.$$

This is possible by essential freeness (for $A_\delta$) and essential freeness and non-compactness of $G$ (for $B_\delta$). To see this, let us formulate essential freeness in the following way:

$$\mu(\{ \omega \in \mathcal{M} \mid \text{for all } g, h \in \omega, g \neq h \text{ implies } g^{-1} \omega \neq h^{-1} \omega \}) = 1.$$

This remains an almost sure event if we restrict $g$ and $h$ to lie in the window $W_k$.

Now observe that $A_\delta$ increases as $\delta$ tends to zero to this almost sure event.

The argument for $B_\delta$ is similar, but depends on the fact that the set

$$B_0 = \{ (\omega, \omega') \in \mathcal{M} \times \mathcal{M} \mid \text{for all } g \in \omega, h \in \omega', g^{-1} \omega \neq h^{-1} \omega' \}$$

is $\mu \otimes \mu$ almost sure, which is less immediate as we now show.

Observe that $\mu \otimes \mu$ defines a point process of $G \times G$ of intensity $\text{int}(\mu)^2$, and Palm measure $\mu_0 \otimes \mu_0$. By the correspondence of measures between $\mu \otimes \mu$ and $\mu_0 \otimes \mu_0$, we equivalently ask that

$$(\mu_0 \otimes \mu_0)(\{ (\omega, \omega') \in \mathcal{M}_0 \times \mathcal{M}_0 \mid \omega \neq \omega' \}) = 1,$$

or equivalently that $\mu_0$ has no atoms. This is contradicted by essential freeness: if $\omega \in \mu_0$ is an atom, then $\omega$ is shift invariant, that is, $g^{-1} \omega = \omega$ for all $g \in \omega$. This implies that $\omega$ is in fact a subgroup of $G$, and it is discrete by definition. Now the ergodic component of $\mu$ corresponding to $\omega$ is supported on $G\omega$, and thus defines a $G$-invariant probability measure on $G/\omega$, that is, it is a lattice shift. But $\mu$ was assumed to be essentially free.

We now construct a random coloring $C$ of $\Pi$ in the following way: let

$$\mathcal{M}_0 = \bigsqcup_i D_i,$$

where $\text{diam}(D_i) < \delta$,

be a partition of $\mathcal{M}_0$ into small measurable sets. By the correspondences we have described, any $[d]$-coloring of the sets $D_i$ corresponds to a factor
coloring $C : M \to [d]^M$ in the following way:

$$C(\omega) = \{(g,c) \in \omega \times [d] \mid g^{-1}\omega \in M_0 \text{ is colored by } c\}.$$  

We look at such $C$ when the $D_i$ sets are colored uniformly at random by elements of $[d]$. To emphasize: we are considering a distribution on deterministic colorings.

For an integral vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}_0^k$, we set

$$T_\alpha = \{\omega \in [d]^M \mid (N_{V_1}(\omega), \ldots, N_{V_k}(\omega)) = \alpha\}$$

to be the set of configurations whose point/color statistics in $W_k$ are prescribed by $\alpha$.

Note that $C_*\mu(T_\alpha)$ is a random variable (whose source of randomness is $C$). Given $k$ and $M$, we use the second moment method to prove the existence of $C$ such that for all $\alpha \in \mathbb{N}_0^k$ with $||\alpha||_\infty \leq M$,

$$|C_*\mu(T_\alpha) - [d]^{\mu}(T_\alpha)| < \varepsilon.$$  

Then any sequence of such colorings with $k, M$ tending to infinity will witness that $\Pi$ weakly factors onto $[d]^{\Pi}$.

Exchanging order of integration allows us to express the mean of $C_*\mu(T_\alpha)$ as

$$\mathbb{E}[C_*\mu(T_\alpha)] = \mathbb{E}[\mu(C^{-1}(T_\alpha))]$$

$$= \mathbb{E}\left[\int_M 1[C(\omega) \in T_\alpha]d\mu(\omega)\right]$$

$$= \int_M \mathbb{E}[1[C(\omega) \in T_\alpha]]d\mu(\omega).$$

Note that for $\omega \in A_\delta$, all pairs of distinct points $g, h \in \omega$ from the window $W_k$ have the property that $g^{-1}\omega$ and $h^{-1}\omega$ fall into different $D_i$ sets, and are therefore assigned independent colors. Thus for $\omega \in A_\delta$,

$$\mathbb{E}[1[C(\omega) \in T_\alpha]] = [d]^{\mu}(T_\alpha).$$  

As $\mu(A_\delta) > 1 - \varepsilon$, it follows that

$$|\mathbb{E}[C_*\mu(T_\alpha)] - [d]^{\mu}(T_\alpha)| < 2\varepsilon.$$  

We now work on the variance. Again, exchanging order of integration in a similar way to before allows us to express the mean of $(C_*\mu(T_\alpha))^2$ as

$$\mathbb{E}[(C_*\mu(T_\alpha))^2] = \int\int_{M \times M} \mathbb{E}[1[C(\omega) \in T_\alpha]1[C(\omega') \in T_\alpha]]d\mu(\omega)d\mu(\omega').$$
By similar reasoning to before, for \((\omega, \omega') \in (A_\delta \times A_\delta) \cap B_\delta\), the colors one will see at points in \(W_k\) will be independent. Thus for such \((\omega, \omega')\) we have
\[
\mathbb{E}[1[\mathcal{G}(\omega) \in T_\alpha]1[\mathcal{G}(\omega') \in T_\alpha]] = (|d|^\mu(T_\alpha))^2.
\]
Note that
\[
(A_\delta \times A_\delta) \cap B_\delta = (A_\delta \times M_0) \cap (M_0 \times A_\delta) \cap B_\delta,
\]
so by the union bound \((\mu \otimes \mu)((A_\delta \times A_\delta) \cap B_\delta) > 1 - 3\varepsilon\).

Putting this together,
\[
\text{Var}(\mathcal{G}_*(\mu)(T_\alpha)) = \mathbb{E}[(\mathcal{G}_*(\mu)(T_\alpha))^2] - (\mathbb{E}[\mathcal{G}_*(\mu)(T_\alpha)])^2 < 12\varepsilon.
\]

We now apply Chebyshev’s inequality which states that for any \(c > 0\),
\[
\mathbb{P}[|\mathcal{G}_*(\mu)(T_\alpha) - \mathbb{E}[\mathcal{G}_*(\mu)(T_\alpha)]| > c] < \frac{\text{Var}(\mathcal{G}_*(\mu)(T_\alpha))}{c^2}.
\]

Our bounds on the mean and the variance of \(\mathcal{G}_*(\mu)(T_\alpha)\) and the choice \(c = \varepsilon^{1/3}\) yield
\[
\mathbb{P}[|\mathcal{G}_*(\mu)(T_\alpha) - |d|^\mu(T_\alpha)| > \varepsilon^{1/3} + 2\varepsilon] < 12\varepsilon^{1/3}.
\]
Let \(E_\alpha\) denote the event \(\{|\mathcal{G}_*(\mu)(T_\alpha) - |d|^\mu(T_\alpha)| < \varepsilon^{1/3} + 2\varepsilon\}\). Then by the union bound
\[
\mathbb{P}\left[\bigcap_{\alpha \in \mathbb{N}_0^k, \|\alpha\|_\infty \leq M} E_\alpha\right] \geq 1 - 12M^k \varepsilon^{1/3}.
\]

In particular, by choosing \(\varepsilon\) sufficiently small, such a coloring exists. \(\blacksquare\)

**Proposition 5.6:** Suppose \(\Pi\) and \(\Upsilon\) are point processes, with \(\Pi\) weakly factoring onto \(\Upsilon\) and \(\Psi(\Upsilon)\) being a factor of \(\Upsilon\). Then \(\Pi\) weakly factors onto \(\Psi(\Upsilon)\).

It follows that weak factoring is a transitive notion.

**Proof.** We have seen that every factor map decomposes as the composition of a (colored) thickening and a thinning. We are therefore able to reduce the problem to the case where \(\Psi\) is a thinning, a coloring, and a thickening.

We will repeatedly use the following fact: if \(\Pi\) weakly factors onto \(\Psi_m(\Upsilon)\) for a sequence of factors \(\Psi_m\), and \(\Psi_m\) converges to \(\Psi\) pointwise, then \(\Pi\) weakly factors onto \(\Psi(\Upsilon)\).

We begin with the case of thinnings.

Let \(\Phi^\alpha(\Pi)\) weakly converge to \(\Upsilon\). We write \(\mu\) and \(\nu\) for the laws of \(\Pi\) and \(\Upsilon\) respectively. Let \(\theta^A\) be a thinning of \(\Upsilon\), determined by a subset \(A \subseteq (M_0, \nu_0)\) as in Theorem 3.19.
The idea of the proof is this: if $A$ were a $\nu_0$ continuity set, then the corresponding thinning $\theta^A : \mathbb{M} \to \mathbb{M}$ is continuous $\nu$ almost everywhere, and so $\theta^A(\Phi^n(\Pi))$ converges to $\theta^A(\Upsilon)$ by Lemma A.4. We handle the general case by approximating $A$ by $\nu_0$ continuity sets.

**Claim:** If $A$ is a $\nu_0$ continuity set, then $\theta^A : \mathbb{M} \to \mathbb{M}$ is continuous $\nu$ almost everywhere.

To see this, recall the saturation notion we used in the proof of Theorem 3.19. We have assumed $\nu_0(\partial A) = 0$, and hence $\nu(G\partial A) = 0$ too. Then $\theta^A$ is continuous on the complement of this set. Note that if $\omega \not\in G\partial A$, then $g^{-1}\omega \not\in \partial A$ for all $g \in \omega$. One can now see that if $\omega_n$ converges to $\omega$, then $\theta^A(\omega_n)$ restricted to any fixed radius ball is eventually equal to $\theta^A(\omega)$, as desired.

For the general case, let $A_m \subseteq \mathbb{M}_0$ be $\nu_0$-continuity sets such that

\[ \nu_0(A \Delta A_m) < \frac{1}{2^m}. \]

Then for every $m$, we have $\theta^{A_m}(\Phi^n(\Pi)) \to \theta^{A_m}(\Upsilon)$ by our earlier argument.

By Borel–Cantelli, $A \Delta A_m$ does not occur infinitely often. This is true for the saturation, so we see that $\theta^{A_m} \to \theta^A$ pointwise almost surely and hence also in distribution. By choosing an appropriate subsequence of $m$s and $n$s we find our desired sequence of factor maps.

The above proof for thinnings can be immediately adapted to prove that $\Pi$ weakly factors onto $\Psi(\Upsilon)$ if $\Psi$ is any $[d]$-coloring of $\Upsilon$. Since any coloring is a pointwise limit of finitary colorings, we see that $\Pi$ weakly factors onto any coloring of $\Upsilon$.

Finally, suppose $\Psi$ is a thickening of $\Upsilon$. By using the Voronoi tessellation we may express $\Psi$ in the following form:

\[ \Psi(\omega) = \bigcup_{g \in \omega} g F(g^{-1}\omega), \]

where $F : \mathbb{M}_0 \to \mathbb{M}_0$ is a measurable function.

We say that $\Psi$ is a **bounded range** thickening if there exists $C > 0$ such that $F(\Upsilon_0) \subseteq B(0, C)$ almost surely.

It is easy to show that $\Psi$ is the pointwise limit of such thickenings, so we are reduced to this case.
Define $I : \mathbb{M}_0(B(0, C))^{\mathbb{M}} \rightarrow \mathbb{M}$ by

$$I(\omega) = \bigcup_{g \in \omega} g\xi_g,$$

where $\xi_g$ is the label of $g$ in $\omega$, that is, $(g, \xi_g) \in \omega$.

This is the implementation map: it takes a schema for a thickening and implements it.

**Claim:** The implementation map $I$ is continuous.

The task is to show that given $R, \varepsilon > 0$ there exists $S, \delta > 0$ such that if $\omega$ and $\omega'$ are $(S, \delta)$-wobbles of each other, then $I(\omega)$ and $I(\omega')$ are $(R, \varepsilon)$-wobbles of each other.

The idea is simply that the behaviour of $I(\omega)$ in a ball of radius $R$ is determined by $\omega$ restricted to the ball of radius $R + C$, as the thickening is of bounded range $C$. By choosing $\delta$ sufficiently small (depending on the labels of the points in $\omega \cap B(0, R + C)$), we find our desired $S$ and $\delta$.

With the claim in hand, our desired result follows from the claim and Lemma A.4.

**Corollary 5.7:** Let $\Pi$ and $\Upsilon$ be point processes on an amenable group, with $\Pi$ free. Then $\Pi$ weakly factors onto $\Upsilon$.

**Proof.** By Theorem 5.5, $\Pi$ weakly factors onto $[0, 1]^\Pi$, and by Theorem 5.3, $[0, 1]^\Pi$ weakly factors onto $\Upsilon$. Hence the claim follows from Proposition 5.6.

**Lemma 5.8:** Suppose $\Pi_n$ weakly converges to $\Pi$. Then $[0, 1]^{\Pi_n}$ weakly converges to $[0, 1]^\Pi$.

**Proof.** This can be seen, for instance, by verifying that the finite-dimensional distributions of $[0, 1]^{\Pi_n}$ weakly converge to those of $[0, 1]^\Pi$.

Recall that a $[0, 1]$-marked point process on $G$ is just a particular kind of point process on $G \times [0, 1]$. It therefore suffices to check weak convergence of the finite-dimensional distributions against stochastic continuity sets of $[0, 1]^\Pi$ in product form.

To that end, let $V = (V_1, V_2, \ldots, V_k)$ denote a collection of stochastic continuity sets for $\Pi$, and

$$[0, p) = ([0, p_1), [0, p_2), \ldots, [0, p_k))$$
a family of intervals in $[0, 1]$. We denote by

$$V \times [0, p) = (V_1 \times [0, p_1), \ldots, V_k \times [0, p_k))$$

the stochastic continuity set of $[0, 1]^\Pi$. Fix an integral vector $\alpha \in \mathbb{N}_0^k$. We must show that $\mathbb{P}[N_{V \times [0, p)}] [0, 1]^\Pi_n = \alpha]$ converges to $\mathbb{P}[N_{V \times [0, p)}] [0, 1]^\Pi = \alpha]$. We find the following explicit expression simply by conditioning on $\beta$, the total number of points appearing in $V$:

$$\mathbb{P}[N_{V \times [0, p)}] [0, 1]^\Pi_n = \alpha] = \sum_{\beta \geq \alpha} \mathbb{P}[N_{V \times [0, p)}] [0, 1]^\Pi_n \mid N_{V \Pi_n} = \beta] \mathbb{P}[N_{V \Pi_n} = \beta]$$

$$= \sum_{\beta \geq \alpha} \prod_{i=1}^k p_\alpha^\beta_i (1 - p_i)^{\beta_i - \alpha_i} \mathbb{P}[N_{V \Pi_n} = \beta],$$

where by $\beta \geq \alpha$ we mean that $\beta_i \geq \alpha_i$ for each entry.

There is an identical expression for $\Pi$ (simply replace all instances of $\Pi_n$ by $\Pi$). The conclusion follows, as $\mathbb{P}[N_{V \Pi_n} = \beta]$ converges to $\mathbb{P}[N_{V \Pi} = \beta]$ for all $\beta$.

We have seen that all free point processes are able to weakly factor onto their own IID. It is natural to ask if all this hassle was worth it—can a point process always factor directly onto its own IID?

**Theorem 5.9** (Holroyd, Lyons, Soo [HLS11]): *The Poisson point process cannot be split into two independent Poisson point processes of lower intensity without additional randomness.*

More precisely, there does not exist a deterministic two coloring

$$\mathcal{C} : (M, \mathcal{P}) \rightarrow \{0, 1\}^M$$

such that $\mathcal{C} \ast \mathcal{P}$ is the IID Ber$(p)$ labelled Poisson point process for $0 < p < 1$.

**Example 18:** Some point processes can factor onto their own IID, however. Note that taking the IID of a point process is idempotent, in the sense that

$$[0, 1]^/[0, 1]^\Pi \cong ([0, 1]^2)^\Pi \cong [0, 1]^\Pi.$$

For an unlabelled example, one can simply spatially implement $[0, 1]^\Pi$. That is, using the method sketched at Proposition 4.10 one can find an unlabelled point process $\Upsilon$ (abstractly) isomorphic to $[0, 1]^\Pi$, and thus $[0, 1]^\Upsilon \cong \Upsilon$. 
5.2. Cost monotonicity for (certain) weak factors. In this section we will always assume $G$ is compactly generated by $S \subset G$.

Question 5: Suppose $\Pi$ weakly factors onto $\Upsilon$. Is it true that $\text{cost}(\Pi) \leq \text{cost}(\Upsilon)$? That is, is cost monotone for weak factors?

This is the real theorem that we would like to prove. We are able only to prove the following theorem, which implies that cost is monotone for certain weak factors:

**Theorem 5.10:** Suppose $\Pi^n$ is a sequence of point processes that weakly converge to $\Pi$. Then

$$\limsup_{n \to \infty} \text{cost}(\Pi^n) \leq \text{cost}(\Pi)$$

holds in the following cases:

1. If there exists $\delta, R > 0$ such that $\Pi_n$ and $\Pi$ are all $\delta$ uniformly separated and $R$ coarsely connected.
2. If all the $\Pi_n$ are free and $\Pi$ is $\delta$ uniformly separated.

Moreover, the same statements are true if the point processes have labels from a compact mark space $\Xi$.

We will need an auxiliary lemma, which we will use again later:

**Lemma 5.11:** Let $\Pi$ be a point process with law $\mu$. Then for all but countably many $\delta > 0$, the $\delta$-metric-thinning map $\theta^\delta : \mathbb{M} \to \mathbb{M}$ is continuous $\mu$ almost everywhere.

In particular, if $\Pi_n$ weakly converges to $\Pi$, then $\theta^\delta(\Pi_n)$ weakly converges to $\theta^\delta(\Pi)$.

To prove the lemma, simply note that any $\delta$ such that $B_G(0, \delta)$ is a stochastic continuity set for $\Pi_0$ works.

**Proof of Theorem 5.10.** We prove (1), and then show how to reduce (2) to (1).

By increasing $S$ if necessary, we may assume that the Cayley factor graph of $\Pi_n$ and $\Pi$ with respect to $S$ is connected almost surely.

Denote the distributions of $\Pi_n$ and $\Pi$ by $\mu_n$ and $\mu$ respectively.

We call a factor graph $\mathcal{G}$ a $\mu$-continuity factor graph if it has the property that

$$\lim_{n \to \infty} \mu^n_0(\mathcal{G}) = \mu^0_0(\mathcal{G}).$$
The same technique used to prove Proposition A.22 shows that factor graphs of the form $\mathcal{G}_{A,V} = (A \times V) \cap \overrightarrow{M_0}$, where $A \subseteq M_0$ is a $\mu_0$ continuity set and $V \subseteq G$ is a bounded stochastic $\mu$ continuity set, are $\mu$-continuity factor graphs.

The idea of the proof is that we will take a cheap graphing $G$ for the limit process $\mu$, and use it to produce a cheap $\mu_0$-continuity graphing $H$. The continuity property then gives us information about the costs of $\mu_n$, but only if we can ensure $H$ is connected on $\Pi_n$. This is why we assume coarse density.

Note that by outer regularity of the measure $\overrightarrow{\mu_0}$, for every factor graph $G$ and $\varepsilon > 0$ there exists an open factor graph $G' \supseteq G$ such that $\overrightarrow{\mu_0}(G') \leq \overrightarrow{\mu_0}(G) + \varepsilon$.

Therefore in the definition of cost one can replace “measurable graphing” by “open graphing”.

**Claim:** Every open graphing $G$ of $\mu$ contains a $\mu$-continuity factor graph $H_N$ such that its $N$th power satisfies $H^N \supseteq \overrightarrow{M_0} \cap (H_\delta \times S)$.

Here $H_\delta$ denotes the space of $\delta$ separated configurations as in Lemma A.15.

Note that this condition and the assumption on $R$ and $S$ implies that $H$ is connected on any $\delta$-uniformly separated and $R$ coarsely dense input. In particular, $H(\Pi_n)$ is connected for every $n$.

Let us assume that all of $\Pi_n$ and $\Pi$ have unit intensity and finish the proof: for any $\varepsilon > 0$, choose a graphing $G$ of $\Pi$ such that $\overrightarrow{\mu_0}(G) \leq \text{cost}(\Pi) + \varepsilon$. Take $H$ as in the above procedure. Then:

$$\limsup_{n \to \infty} \text{cost}(\Pi_n) \leq \limsup_{n \to \infty} \overrightarrow{\mu_0}^n(\mathcal{H}) \quad \text{as $\mathcal{H}$ is a graphing of $\Pi_n$}$$

$$= \overrightarrow{\mu_0}(\mathcal{H}) \quad \text{since $\mathcal{H}$ is a $\overrightarrow{\mu_0}$-continuity graphing}$$

$$\leq \overrightarrow{\mu_0}(G) \quad \text{as $\mathcal{H} \subseteq G$}$$

$$\leq \text{cost}(\Pi) + \varepsilon \quad \text{by assumption on $G$}.$$

Since $\varepsilon > 0$ was arbitrary, this proves the result for unit intensity processes. In general, the steps are the same except with the more complicated formula for cost of processes with non-unit intensity. One just needs to know that if $\Pi_n$ weakly converges to a finite intensity process $\Pi$, then $\text{int}(\Pi_n)$ converges to

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21 Let us unpack the definition: there is an edge between $g, h \in \mathcal{G}_{A,V}(\omega)$ if $g^{-1} \omega \in A$ and $g^{-1} h \in V$. That is, each point $g$ decides if it will have edges (checks if $g^{-1} \omega \in A$), and then simply connects to all points in $gV$. 
int(Π). This follows by weak convergence and choosing a stochastic continuity set \( U \) for \( \Pi \), noting that the point count function \( N_U \) is thus continuous and bounded as we are taking uniformly separated processes.

It remains to prove the claim about \( \mu \)-continuity factor graphs.

Recall from Lemma A.20 that \( \mathbb{M}_0 \) and \( G \) admit topological bases \( \{ A_i \} \) and \( \{ V_j \} \) consisting of \( \mu_0 \)-continuity sets and \( \mu \) stochastic continuity sets (respectively). So by definition of the subspace topology, \( \mathbb{M}_0 \) admits a topological basis \( \{ \mathcal{G}_{A_i, V_j} \} \) consisting of \( \mu \)-continuity factor graphs.

For each \( k \in \mathbb{N} \) define

\[
\mathcal{H}_k = \bigcup_{i,j \leq k} \mathcal{G}_{A_i, V_j}.
\]

Since \( \mathcal{H}_k \) consists of finitely many continuity factor graphs, it is itself a continuity factor graph. Each \( \mathcal{H}_k \) is also open, and increases to \( \mathcal{G} \) as \( k \) tends to infinity.

As \( \mathcal{G} \) is generating, \( \{ \mathcal{H}_k \}_{k \in \mathbb{N}} \) forms an open cover of the compact space \( \mathbb{M}_0 \cap (H_\delta \times S) \).\(^{22}\) In particular, there exists \( N \) such that

\[
\mathcal{H}^0_N \supseteq \mathbb{M}_0 \cap H_\delta \times S,
\]

proving the claim.

One sees that the essential feature in the above proof strategy was compactness, and therefore it remains true for \( \Xi \)-labelled point processes if \( \Xi \) is compact, as mentioned.

With this observation in hand, we can now deduce statement (2) from (1). We will produce a weakly convergent sequence of separated and coarsely connected point processes, where each term has the same cost as \( \Pi_n \) and the weak limit factors onto \( \Pi \), thus has cost at most the cost of \( \Pi \). This proves the statement.

Choose \( \delta' < \delta \) as in Lemma 5.11 so that the \( \delta' \) metric thinning satisfies

\[
\theta^{\delta'}(\Pi_n) \text{ weakly converges to } \theta^{\delta'}(\Pi) = \Pi.
\]

Now observe by label trickery (see Proposition 4.13) we can find \([0,1]\)-labelled point processes \( \Upsilon_n \) each isomorphic to the respective \( \Pi_n \) and such that their underlying point set is \( \theta^{\delta'}(\Pi_n) \).

\(^{22}\) See Lemma A.15.
Note that $\Upsilon_n$ might not weakly converge, but it will have subsequential weak limits. All such subsequential weak limits will be some kind of (possibly random) labelling of $\Pi$.

To see this, let $\pi : [0, 1]^M \to M$ be the map that forgets labels. Thus

$$\pi(\Upsilon_n) = \theta^{\delta'}(\Pi_n).$$

Since $\pi$ is continuous, it preserves weak limits. Let $\Upsilon$ be any subsequential weak limit of $\Upsilon_n$, along a subsequence $n_k$. Then by continuity

$$\pi(\Upsilon) = \lim_{k \to \infty} \pi(\Upsilon_{n_k}) = \lim_{k \to \infty} \theta^{\delta'}(\Pi_{n_k}) = \Pi.$$ 

Now let $\Theta_n(\Upsilon_n)$ be the input/output versions of the $(\delta', R)$-Delone thickenings that exist from Proposition 4.18. Here we use that $\Upsilon_n$ are free actions. By input/output we mean you keep track of which points of the thickening are input and output, as in Definition 2.20. In particular,

$${\text{cost(}}\Theta_n(\Upsilon_n)\text{)} = \text{cost(}}\Upsilon_n\text{)} = \text{cost(}}\Pi_n\text{)},$$

where the first equality holds because we took the input/output version of the thickening.

Let $\Upsilon'$ denote any subsequential weak limit of $\Theta_n(\Upsilon_n)$. Then $\Upsilon'$ factors onto $\Pi$, by a similar argument to the earlier one about forgetting certain labels. Putting this all together:

$$\limsup_{n \to \infty} \text{cost(}}\Pi_n\text{)} = \text{cost(}}\Theta_n(\Upsilon_n)\text{)} \leq \text{cost(}}\Upsilon'\text{)} \leq \text{cost(}}\Pi\text{)},$$

where the final inequality holds because cost can only increase under factors. 

Remark 5.12: In the second part of the proof, one might want to replace label trickery by something like “each $\Pi_n$ is isomorphic to a random Delone set $\Upsilon_n$, which has subsequential weak limits, so choose one such limit $\Upsilon \ldots$”, but then it is not clear what the cost of $\Upsilon$ has to do with the cost of $\Pi$. One would require the Delone-ification process to preserve weak limits in some sense in order to relate cost(\Upsilon) and cost(\Pi).

Remark 5.13: There is label trickery in [AW13] too: it is always assumed there that the action is continuous on a compact space.

**Theorem 5.14:** If $\Pi$ is a free point process, then its cost is at most the cost of the IID Poisson point process on $G$. 

Proof. We know by Theorem 5.5 that $\Pi$ weakly factors onto $[0, 1]^\Pi$, and $[0, 1]^\Pi$ factors onto the IID Poisson. We would like to say “so $\Pi$ weakly factors onto the IID Poisson, and hence has less cost by the cost monotonicity statement”, but our cost monotonicity statement is too weak for this, so we use a different argument.

Note that $\Pi$ is abstractly isomorphic to a $\delta$ uniformly separated process $\Pi'$ by Proposition 4.10. Then $\Pi'$ is also free and has the same cost as $\Pi$. Now $\Pi'$ weakly factors onto its own IID, more explicitly, there is a sequence of factor labellings $\Phi_n(\Pi')$ weakly converging to $[0, 1]^\Pi'$. Because $\Phi_n(\Pi')$ is a labelling of $\Pi'$ it is itself free and uniformly separated. Putting it all together:

$$\text{cost}(\Pi) = \text{cost}(\Pi') \quad \text{as they are isomorphic actions}$$
$$\leq \limsup_{n \to \infty} \text{cost}(\Phi_n(\Pi')) \quad \text{cost can only increase for factors}$$
$$\leq \text{cost}([0, 1]^\Pi') \quad \text{by Theorem 5.10}$$
$$\leq \text{cost}([0, 1]^{\mathcal{P}}) \quad \text{as } [0, 1]^\Pi' \text{ factors onto } [0, 1]^{\mathcal{P}},$$

as desired. 

6. Some fixed price one groups

Theorem 5.14 is a strategy for proving that groups have fixed price, and we will use it to that end for $G \times \mathbb{Z}$. However, our argument is somewhat indirect and requires taking a further weak limit. It would be instructive to have a more direct argument:

**Question 6:** Can one explicitly construct for every $\varepsilon > 0$ connected factor graphs of the IID Poisson on $G \times \mathbb{Z}$ of edge measure less than $1 + \varepsilon$?

To see what we mean by explicit, one should consider the discrete case: if $\Gamma$ is a finitely generated group, then it is straightforward to construct factor of IID connected graphs of small edge measure on Bernoulli (site) percolation on $\Gamma \times \mathbb{Z}$. We would like a construction in that vein.

So instead we will use the weak factoring strategy to reduce the above problem to a much simpler one, where we can construct such factor graphs.
6.1. Groups of the form $G \times \mathbb{Z}$.

**Definition 6.1:** Let $\Pi$ be a point process on $G$. Its **vertical coupling** on $G \times \mathbb{Z}$ is $\Delta(\Pi) = \Pi \times \mathbb{Z}$.

Here $\Delta : \mathcal{M}(G) \to \mathcal{M}(G \times \mathbb{Z})$ is induced by the diagonal embedding of $G$ into $G^\mathbb{Z}$. For this reason one might prefer to call $\Delta(\Pi)$ the **diagonal coupling**, but this terminology will not be suitable when we go to $G \times \mathbb{R}$.

![Diagram](image)

Figure 6. How one should think of $G \times \mathbb{Z}$. Note that if $\Pi$ is a point process on $G$, then its vertical coupling is simply infinitely many copies of the same points stacked on top of each other.

**Lemma 6.2:** The IID version $[0,1]^{\Delta(\Pi)}$ of a vertically coupled process has cost one.

The proof uses the fact that Bernoulli percolation of a factor graph can be implemented as a factor of IID. This sort of trick will be familiar to many, but we will nevertheless spell it out:
Definition 6.3: Let $\mathcal{G}$ be a factor graph of a point process $\Pi$. Its $\varepsilon$ edge percolation is the factor graph $\mathcal{G}_\varepsilon$ defined on $[0,1]^\Pi$ in the following way: for points $g, h \in [0,1]^\Pi$ let

$$g \sim_{\mathcal{G}_\varepsilon} h \quad \text{whenever} \quad g \sim_{\mathcal{G}} h \text{ and } \xi_g \oplus \xi_h < \varepsilon.$$ 

Here $\oplus$ denotes addition of the labels modulo one.

Observe that if $(g, h_1)$ and $(g, h_2)$ are edges of $\mathcal{G}(\Pi)$, then the random variables $\xi_g \oplus \xi_{h_1}$ and $\xi_g \oplus \xi_{h_2}$ are independent uniform once again.

Remark 6.4: If $\mathcal{G}$ is already a factor graph defined on $[0,1]^\Pi$, then we can implement $\mathcal{G}_\varepsilon$ on $[0,1]^\Pi$, that is, without adding further randomness (via the replication trick, see the discussion below).

Proof of Lemma 6.2. Let $\mathcal{G}$ be any graphing of $\Pi$ with finite edge density. We lift it to a factor graph $\mathcal{G}^\Delta$ of $\Delta(\Pi)$ in the following way:

$$(g, n) \sim_{\mathcal{G}^\Delta(\Pi)} (h, n) \quad \text{if and only if} \quad g \sim_{\mathcal{G}(\Pi)} h,$$

that is, as $\Delta(\Pi)$ is just copies of $\Pi$ stacked on every level of $G \times \mathbb{Z}$, then we simply copy $\mathcal{G}$ onto every level of $G \times \mathbb{Z}$ as well.

Let $\mathcal{V}$ denote the factor graph of $\Delta(\Pi)$ consisting of vertical edges, that is for every $(g, n) \in \Delta(\Pi)$ we have an edge to $(g, n+1)$.

One can see that $\mathcal{V} \cup \mathcal{G}^\Delta$ is a connected factor graph. But this is also true when we percolate the edges level wise, that is, when we consider $\mathcal{V} \cup \mathcal{G}^\Delta_\varepsilon$. This is because if $(g, n) \sim_{\mathcal{G}^\Delta} (h, n)$ is an edge destroyed in the percolation, then we can slide up along vertical edges and consider the edge

$$(g, n + 1) \sim_{\mathcal{G}^\Delta} (h, n + 1)$$

instead. Its chance of survival in the percolation is independent from the previous edge, and hence we get another go to cross over. By sliding up far enough we are guaranteed to be able to cross. Finally, observe that the edge density of $\mathcal{V} \cup \mathcal{G}^\Delta_\varepsilon$ is $1 + \varepsilon$ times the edge density of $\mathcal{G}$.

Lemma 6.5: Suppose $\Pi$ and $\Upsilon$ are point processes, and $\Pi$ factors onto $\Upsilon$. Then $[0,1]^\Pi$ factors onto $[0,1]^\Upsilon$. In particular, if $[0,1]^\Pi$ weakly factors onto $\Upsilon$ then $[0,1]^\Pi$ weakly factors onto $[0,1]^\Pi$ too.
The proof of this uses the following replication trick: note that the randomness in one \( \text{Unif}[0,1] \) random variable \( \xi \) is equivalent to the randomness in an entire IID sequence \( \xi_1, \xi_2, \ldots \) of \( \text{Unif}[0,1] \) random variables.

More precisely, there is an isomorphism (as measure spaces)
\[
I : ([0,1], \text{Leb}) \to ([0,1]^N, \text{Leb}^\otimes N).
\]
So if \( \xi \sim \text{Unif}[0,1] \), then we will write \( I(\xi) = (\xi_1, \xi_2, \ldots) \) for the associated IID sequence of \( \text{Unif}[0,1] \) random variables.

**Proof of Lemma 6.5.** Suppose \( \Upsilon = \Phi(\Pi) \). If \( g \in [0,1]^\Pi \), then we write \( \xi^g \) for its label, and by the replication trick \( \xi_1^g, \xi_2^g, \ldots \) for the associated IID sequence of \( \text{Unif}[0,1] \) random variables.

We define a factor map \( \widetilde{\Phi} \) of \( [0,1]^\Pi \) as follows:
\[
\widetilde{\Phi}(\pi) = \bigcup_{g \in [0,1]^\Pi} \{ (h, \xi^g_i) \in G \times [0,1] | V_g(\Pi) \cap \Phi(\Pi) = (h_1, h_2, \ldots) \},
\]
where we mean that \( (h_1, h_2, \ldots) \) is any enumeration of \( V_g(\Pi) \cap \Phi(\Pi) \) performed in an equivariant way. Note that the intersection \( V_g(\Pi) \cap \Phi(\Pi) \) could be empty.

For instance, look at the elements of \( h \in V_g(\Pi) \cap \Phi(\Pi) \) which are closest to \( g \). Then let \( h_1 \) be the element that minimizes \( T(g^{-1}h) \), where \( T : G \to [0,1] \) is the tie-breaking function of Section 3.15. Then let \( h_2 \) be the next smallest element, and so on, until you exhaust the closest elements. Then look at the batch of next closest elements and so on. One can check that this is an equivariant construction (any construction where you do the same thing at every point will be).

Then \( \widetilde{\Phi}([0,1]^\Pi) = [0,1]^{\Upsilon} \), as desired.

For the second part, simply note that taking the IID is idempotent in the sense that \( [0,1]^{([0,1]^\Pi)} \cong [0,1]^\Pi \), and apply Lemma 5.8.

**Proposition 6.6:** The IID Poisson on \( G \times \mathbb{Z} \) weakly factors onto the vertically coupled Poisson of \( G \).

**Proof.** We will construct factor maps \( \Phi^\pi : [0,1]^\mathcal{M}(G \times \mathbb{Z}) \to \mathcal{M}(G \times \mathbb{Z}) \) that “straighten” the input in the following way: for a given input \( \omega \in [0,1]^\mathcal{M} \), we select a “sparse” subset of its points. At each one of these we propagate them upwards by placing copies of them on the levels above. This will converge to a vertically coupled process for suitable inputs.
More precisely, let $\Pi$ denote the (unit intensity) IID Poisson on $G \times \mathbb{Z}$. We will denote points of $\Pi$ by $(g, l) \in G \times \mathbb{Z}$, and write $\Pi_{g,l}$ for its label.

We now define the factor map $\Phi^n$ in two stages as a thinning and then a thickening to simplify the analysis. Let

$$\Pi^{1/n} = \left\{ (g, l) \in \Pi \mid \Pi_{g,l} \leq \frac{1}{n} \right\},$$

and $F_n = \{0\} \times \{0, 1, \ldots, n - 1\}$. Set

$$\Phi^n \Pi = \Theta^{F_n}(\Pi^{1/n}),$$

where we write $\Phi^n \Pi$ for $\Phi^n(\Pi)$ to conserve parentheses.

Let us explain what this means:

- At the first step $\Pi \mapsto \Pi^{1/n}$, we independently thin $\Pi$ to get a sub-process of intensity $\frac{1}{n}$. By the discussion in Example 4, the resulting process $\Pi^{1/n}$ is simply a Poisson point process on $G \times \mathbb{Z}$ of intensity $\frac{1}{n}$. We refer to the points of $\Pi^{1/n}$ as progenitors.

- Each progenitor $(g, l)$ spawns additional points with the same $G$-coordinate on the next $n - 1$ levels above it. This is the map

$$\Pi^{1/n} \mapsto \Theta^{F_n}(\Pi^{1/n}) = \Phi^n \Pi.$$

- By the discussion at Example 5, $\Phi^n \Pi$ is a process of unit intensity.

We will employ the following strategy to show that $\Phi^n \Pi$ weakly converges to the vertical Poisson:

1. The sequence $(\Phi^n \Pi)$ admits weak subsequential limits, which a priori might be random counting measures,
2. These subsequential limits are actually simple point processes,
3. All of these subsequential limits are vertical processes, and
4. That process is the vertical Poisson.

Recall that if $(x_n)$ is a relatively compact sequence and every subsequential limit of $(x_n)$ is $x$, then $x_n$ converges to $x$.

By this basic fact and the above items, we can conclude that $(\Phi^n \Pi)$ weakly converges to the vertical Poisson.

We now verify that $\{\Phi^n \Pi\}$ is uniformly tight, proving (1). It suffices to verify that the distributions of point counts $N_C(\Phi^n \Pi)$, where $C = B_G(0, r) \times [L]$ denotes a cylinder whose base is a ball of radius $r$ and its height (in levels) is $L$, are uniformly tight.
Let $X_i$ denote the number of points in $B_G(0,r) \times \{i\}$ with label $\Pi_{g,i} \leq \frac{1}{n}$, that is, the number of progenitors on the $i$th level. Thus the $X_i$ are IID Poisson random variables with parameter $\lambda(B_G(0,r))/n$.

One can explicitly describe the random variable $N_C(\Phi^n \Pi)$ in terms of the $X_i$s, but for our purposes it is enough to observe that 

$$N_C(\Phi^n \Pi) \leq L \sum_{i=1}^{n} X_i.$$  

The sum of independent Poisson random variables is again Poisson distributed (with parameter the sum of the parameters of the individual Poissons), so we see that $N_C(\Phi^n \Pi)$ is bounded in terms of a random variable that does not depend on $n$. Therefore $\{\Phi^n \Pi\}$ is uniformly tight.

To prove item (2), note that the above shows that the point counts in $B_G(0,r) \times \{0\}$ for $\Phi^n \Pi$ are exactly Poisson distributed with parameter $\lambda(B_G(0,r))$. Thus if $\Upsilon$ is any subsequential weak limit of $\Phi^n \Pi$ and $r$ is such that $B_G(0,r) \times \{0\}$ is a stochastic continuity set for $\Upsilon$, then $N_{B_G(0,r) \times \{0\}} \Upsilon$ will also be Poisson distributed. In particular, $\Upsilon$ must be a simple point process.

For item (3), let $\Upsilon$ be any subsequential weak limit of $\Phi^n \Pi$. Observe that $\Upsilon$ is vertical if and only if $(g,l) \in \Upsilon$ implies $(g,l+1) \in \Upsilon$. The idea is that this property is satisfied for most points of $\Phi^n \Pi$, and therefore must be preserved in the weak limit. Note that a process is vertical if and only if its Palm measure is vertical almost surely.

We can now explicitly describe the Palm measure of $\Phi^n(\Pi)$. Recall from Theorem 11 that the Palm version $\Pi_0^{1/n}$ of $\Pi^{1/n}$ is simply $\Pi^{1/n} \cup \{(0,0)\}$.

To express the Palm version of the $F_n$-thickening of $\Pi^{1/n}$ (à la Example 14), it will be useful to introduce the following notation. For each $k \in \mathbb{N}$, let 

$$\Pi_k^{1/n} = \Pi_0^{1/n} \cdot (0,k) = \{(g,l+k) \in G \times \mathbb{Z} \mid (g,l) \in \Pi_0^{1/n}\}.$$ 

That is, you simply shift $\Pi_0^{1/n}$ up by $k$ levels. Then 

$$\Phi^n(\Pi_0) = \Pi_0^{1/n} \cup \Pi_1^{1/n} \cup \cdots \cup \Pi_{n-1}^{1/n}.$$ 

Denote by $K$ a random integer chosen uniformly from $\{0,1,\ldots,n-1\}$. Then the Palm version of $\Phi^n(\Pi)$ is 

$$(\Phi^n \Pi)_0 = \Pi_{-K}^{1/n} \cup \Pi_{-K+1}^{1/n} \cup \cdots \cup \Pi_{-K+n-1}^{1/n},$$ 

where we use parentheses to stress that it is the Palm version of $\Phi^n \Pi$, not $\Phi^n$ applied to $\Pi_0$.  

Let us say that a rooted configuration $\omega \in \mathbb{M}_0(G \times \mathbb{Z})$ has an $\varepsilon$-successor if there is a point approximately above the root $(0, 0)$ in $\omega$. More precisely, we define an event

$$\{\omega \text{ has an } \varepsilon\text{-successor} \} := \{\omega \in \mathbb{M}_0 \mid N_{BG(0,\varepsilon) \times \{1\}} \omega > 1\}.$$ 

From this, we see that

$$P[(\Phi^n \Pi)_0 \text{ has an } \varepsilon\text{-successor}] \geq \frac{n - 1}{n},$$

as $(\Phi^n \Pi)_0$ certainly has an $\varepsilon$-successor whenever $K < n - 1$.

Recall that $\Upsilon$ was any subsequential weak limit of $\Phi^n \Pi$. Fix a subsequence $n_i$ such that $\Phi^{n_i} \Pi$ weakly converges to $\Upsilon$.

Choose a sequence $\varepsilon_k$ tending to zero such that $B_G(0, \varepsilon_k) \times \{1\}$ is a stochastic continuity set for $\Upsilon$. This is possible by Lemma A.20. Then for each $k$

$$\frac{n_i - 1}{n_i} \leq P[\Phi^{n_i}(\Pi)_0 \text{ has an } \varepsilon_k\text{-successor}] \to P[\Upsilon_0 \text{ has an } \varepsilon_k\text{-successor}].$$

So $\Upsilon_0$ has $\varepsilon_k$-successors almost surely for every $k$, and hence has 0-successors. That is, $\Upsilon$ is a vertical process, at last proving item (3).

Finally, for item (4) we observe that any vertical process is completely determined by its intersection with $G \times \{0\}$. We observed in the proof of item (2) that $\Upsilon$ is a Poisson point process on the 0th level, so it must be the vertical Poisson, as desired. ■

**Corollary 6.7:** Groups of the form $G \times \mathbb{Z}$ have fixed price one.

**Proof.** By the previous proposition and Lemma 6.5, we know that the IID Poisson weakly factors onto the IID of the vertically coupled Poisson. Explicitly, there exists factor maps $\Phi^n : [0, 1]^M \rightarrow [0, 1]^M$ such that

$$\Phi^n \Pi \text{ weakly converges to } [0, 1]^{\Delta(\mathcal{P})},$$

where $\Pi$ is the IID Poisson on $G$ and $\mathcal{P}$ is the Poisson on $G$.\(^{23}\)

Choose $\delta < 1$ as in Lemma 5.11 such that metric $\delta$-thinning preserves the weak limit. Note that because $\delta < 1$, the thinning commutes with the vertical coupling: that is, $\theta^\delta(\Delta(\mathcal{P})) = \Delta(\theta^\delta \mathcal{P})$. Therefore

$$\theta^\delta(\Phi^n \Pi) \text{ weakly converges to } [0, 1]^{\Delta(\theta^\delta(\mathcal{P}))}.$$\(^{23}\)

This is a slight abuse of notation: we were using $\mathcal{P}$ to denote the law of the Poisson point process, but in the above expression we treat it as if it were a random variable. We do this to prevent the profusion of asterisks representing pushforwards of measures.
Putting this all together,

\[ \text{cost}(\Pi) \leq \limsup_{n \to \infty} \text{cost}(\theta^\delta(\Phi^n \Pi)) \]

as cost can only increase under factors

\[ \leq \text{cost}([0,1]^{\Delta(\theta^\delta(\mathcal{P}))}) \]

by Theorem 5.10

\[ = 1 \]

by Lemma 6.2.

Since the IID Poisson has maximal cost, this proves that \( G \times Z \) has fixed price one.

Remark 6.8: With further percolation-theoretic assumptions on \( G \), one can directly show that \( \text{cost}(\Phi^n(\Pi)) \leq 1 + \varepsilon_n \), where \( \varepsilon_n \) tends to zero. This is by constructing factor graphs on \( \Phi^n(\Pi) \).

By using the Poisson net, one can prove an analogue of the Babson and Benjamini theorem [BB99] and show that the distance \( \mathcal{D}_R \) factor graph on the Poisson point process on a compactly presented and one-ended group has a unique infinite connected component if \( R \) is sufficiently large.

Now on \( \Phi^n(\Pi) \), we construct a factor graph as follows: add in all vertical edges, and the \( \mathcal{D}_R \) edges horizontally. Now percolate the horizontal edges. One can show that by adding a small amount of edges to this, the result is a graph with a unique infinite connected component.

6.2. Groups of the form \( G \times \mathbb{R} \). We now outline the modifications required to extend the \( G \times Z \) case to the following theorem:

**Theorem 6.9:** Groups of the form \( G \times \mathbb{R} \) have fixed price one.

**Proof.** The strategy will be exactly the same as in Proposition 6.6.

We define factor maps \( \Phi^n \) of the IID Poisson \( \Pi \) using the same formula as in the \( G \times Z \) case. We claim these weakly converge to a point process \( \Upsilon \) which is vertical in the sense that \((g,t) \in \Upsilon\) implies \((g,t+n) \in \Upsilon\) for all \( n \in \mathbb{Z} \).

First we show \( \{\Phi^n(\Pi)\} \) is uniformly tight. This works exactly as in the \( G \times Z \) case, except instead of counting progenitors \( X_i \) on \( G \times \{i\} \), we count them on \( G \times [i,i+1) \) for \( i \in \mathbb{Z} \).

Next we show that any subsequential weak limit \( \Upsilon \) of \( \{\Phi^n(\Pi)\} \) is not just a random counting measure, but an actual point process. This follows as in the \( G \times Z \) case, as \( \Phi^n(\Pi) \) has the same distribution in \( G \times [0,1) \) as a Poisson point process on \( G \times \mathbb{R} \).

The proof that \( \Upsilon \) is a vertical point process works the same as in the \( G \times Z \) case.
At this point one can observe that a vertical process is determine by its intersection with $G \times [0,1)$, and therefore $\Phi^n(\Pi)$ weakly converges to a unique point process $\Upsilon$.

We now adapt Lemma 6.2 to this context, and show that if $\Upsilon$ is any vertical point process such that the projection $\pi(\Upsilon)$ has finite cost, then the IID process $[0,1]^\Upsilon$ has cost one.

Let $\pi: G \times \mathbb{R} \to G$ denote the projection map. Observe that if $\Upsilon$ is vertical, then $\pi(\Upsilon)$ is discrete, and hence defines a point process on $G$. For contrast, observe that the projection $\pi(\Pi)$ of the Poisson point process $\Pi$ is almost surely dense, and hence does not define a point process on $G$. In the case of the $\Upsilon$ we construct as a weak limit, its projection $\pi(\Upsilon)$ is just the Poisson point process on $G$.

Figure 7. A portion of a graphing on the projection of a vertical process, and how it might look when lifted. Note that it gets wobbled a bit in the process.
Let $G$ be a finite cost graphing of $\pi(\Upsilon)$. We lift this to a factor graph of $\Upsilon$ in the following way:

$$(g_1, t_1) \sim H(\Upsilon) (g_2, t_2) \text{ when } g_1 \sim G(\pi(\Upsilon)) g_2 \text{ and } |t_1 - t_2| < 1.$$ 

Let $\mathcal{V}(\Upsilon)$ denote the set of vertical edges, that is

$$\mathcal{V}(\Upsilon) = \{((g, t), (g, t + 1)) \mid (g, t) \in \Upsilon\}.$$ 

Then as in Lemma 6.2, the vertical edges $\mathcal{V}(\Upsilon)$ together with an $\varepsilon$-percolation of $H(\Upsilon)$ defines a cheap connected factor graph of $\Upsilon$.

We conclude from this that $G \times \mathbb{R}$ has fixed price one by the same kind of reasoning as in Corollary 6.7.

**Remark 6.10:** The limiting process here can be described as follows: sample from a Poisson on $G \times [0, 1)$, and then simply extend it periodically in the second coordinate.

### 7. Rank gradient of Farber sequences vs. cost

In this section we will discuss how to connect rank gradient to the cost of the Poisson point process in certain situations. We first work with general lcsc groups, and then specialize to semisimple Lie groups and prove a stronger theorem.

**Definition 7.1:** Let $(\Gamma_n)$ denote a sequence of lattices in a fixed group $G$.

The sequence is Farber if for every compact neighborhood of the identity $V \subseteq G$ we have

$$\mathbb{P}[a\Gamma_n a^{-1} \cap V = \{e\}] \to 1 \text{ as } n \to \infty,$$

where $a\Gamma_n$ denotes a coset of $\Gamma_n$ chosen randomly according to the (normalized) finite $G$-invariant measure on $G/\Gamma_n$.

Note that $a\Gamma_n a^{-1}$ is exactly the stabilizer of $a\Gamma_n$ for the action $G \actson G/\Gamma$. Thus the Farber condition says that the action on most points of the quotient is locally injective.

Equivalently, the condition states that $a\Gamma_n \cap V a = \{a\}$ with high probability. It is this second form that we will actually use in the proof below. We think of $a$ as being a point sampled randomly from a fundamental domain for $\Gamma_n$ in $G$, and thus it states that the $V$-neighborhood around this point $a$ meets the lattice shift $a\Gamma_n$ only trivially.

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Definition 7.2: Let \((\Gamma_n)\) denote a sequence of lattices in a fixed group \(G\). Its \textbf{rank gradient} is
\[
\text{RG}(G, (\Gamma_n)) = \lim_{n} \frac{d(\Gamma_n) - 1}{\text{covol} \Gamma_n},
\]
whenever this limit exists.

Remark 7.3: If \(G\) is discrete, then the \(\Gamma_n\) are all finite index subgroups. The Nielsen–Schreier formula
\[
\frac{d(\Gamma_n) - 1}{[G : \Gamma_n]} \leq d(G) - 1
\]
shows that the terms in the rank gradient are at least bounded.

Gelander proved [Gel11] an analogue of this formula for lattices in connected semisimple Lie groups without compact factors.

In the Seven Samurai paper [ABB+17], it is shown that if \(G\) is a center-free semisimple Lie group of higher rank with property (T), then any sequence of irreducible lattices \((\Gamma_n)\) in \(G\) is automatically Farber, as long as \(\text{covol}(\Gamma_n)\) tends to infinity.

In the particular case of a decreasing chain \(\Gamma = \Gamma_1 > \Gamma_2 > \cdots\) of finite index subgroups, Abért and Nikolov showed [AN12] that the rank gradient \(\text{RG}(\Gamma, (\Gamma_n))\) can be described as the groupoid cost of an associated pmp action \(\Gamma \curvearrowright \partial T(\Gamma, (\Gamma_n))\) on the boundary of a rooted tree.

7.1. Cocompact lattices in general groups.

Definition 7.4: We say that a lattice \(\Gamma < G\) is \textbf{\(\delta\)-uniformly discrete} if all of its right cosets \(\Gamma a \in \Gamma \backslash G\) are \(\delta\) uniformly separated as subsets of \(G\). That is, for all distinct pairs \(\gamma_1, \gamma_2 \in \Gamma\), we have \(d(\gamma_1 a, \gamma_2 a) \geq \delta\). Equivalently by left-invariance of the metric, \(d(e, a^{-1} \gamma a) \geq \delta\) for all \(\gamma \in \Gamma\) not equal to the identity \(e \in G\).

If \((\Gamma_n)\) is a sequence of lattices, then we say it is \(\delta\)-uniformly discrete if each \(\Gamma_n\) is \(\delta\) uniformly discrete in the above sense.

Theorem 7.5: Let \((\Gamma_n)\) be a Farber sequence of cocompact lattices. Suppose further that the sequence is uniformly discrete. If its rank gradient exists, then
\[
\text{RG}(G, (\Gamma_n)) \leq c_P(G) - 1,
\]
where \(c_P(G)\) denotes the cost of the Poisson point process on \(G\). In particular, if \(G\) has fixed price one then the rank gradient vanishes.
The above theorem is spiritually the same as one proved independently by Carderi in [Car18], but in a drastically different language (namely, that of ultraproducts of actions). The theorem is therefore his, but we include our own proof as it has a different flavor. In the subsequent section we will discuss a similar theorem which applies for nonuniform lattices, at least with additional assumptions on the group.

**Proof of Theorem 7.5.** Recall that the cost of a lattice shift is

\[
\text{cost}(G \curvearrowright G/\Gamma_n) = 1 + \frac{d(\Gamma_n) - 1}{\text{covol}\Gamma_n},
\]

which is essentially the term appearing in the rank gradient definition. We would therefore like to take a weak limit of these actions to get some free point process, and then appeal to the cost monotonicity result. Of course, this is completely senseless: the intensity of the lattice shift tends to zero, so it weak limits on the empty process.

Therefore we thicken the lattice shifts to get processes \( \Pi_n \) with a nontrivial weak limit. This thickening procedure must be done correctly, so that we can apply our (weak) cost monotonicity result.

We will produce a sequence of \([0, 1]\)-marked point processes \( \Pi_n \) such that

- each \( \Pi_n \) is a \(2\delta\)-net,
- each \( \Pi_n \) is a factor of the lattice shift \( a\Gamma_n \), and so has cost at least

\[
1 + \frac{d(\Gamma_n) - 1}{\text{covol}\Gamma_n},
\]

and
- they have a subsequential weak limit \( \Upsilon \) with IID \([0, 1]\) labels.

Then

\[
\text{RG}(G, (\Gamma_n)) + 1 \leq \limsup_{n \to \infty} \text{cost}(\Pi_n) \leq \text{cost}(\Upsilon) \leq c_P(G)
\]

by the cost monotonicity result, as desired.

View the space of right cosets \( \Gamma_n \backslash G \) as a compact metric space, where the distance between two cosets \( \Gamma_n b_1, \Gamma_n b_2 \in \Gamma_n \backslash G \) is just their distance as closed subsets of \( G \).

Let \( B_n = \{\Gamma_n b_1^n, \Gamma_n b_2^n, \ldots, \Gamma_n b_{k_n}^n\} \) be a collection of \(2\delta\)-nets in \( \Gamma_n \backslash G \), where \( \delta \) is the uniform discreteness parameter. We also choose \( b_1^n = e \) for all \( n \).

This specifies a sequence of thickenings \( \Theta_n \) of the corresponding lattice shifts: that is, \( a\Gamma_n \mapsto aB_n \).
Note that $\Theta_n(a\Gamma_n)$ is a $2\delta$-net: it is true that

$$d(a\Gamma_nb^n_i,a\Gamma_nb^n_j) = d(\Gamma_nb^n_i,\Gamma_nb^n_j) \geq \delta$$

for $i \neq j$ by our choice of $B_n$, and points of $\Gamma_nb^n_i$ are uniformly separated too exactly by our uniform discreteness assumption. It is also $2\delta$-coarsely dense, by the same property for $B_n$.

Since $\{\Theta_n(a\Gamma_n)\}$ is a collection of random $2\delta$-nets, it is automatically uniformly tight, and all subsequential weak limits are $2\delta$-nets (and in particular, simple point processes).

At this point we would like to apply cost monotonicity to $\Theta_n$ by passing to a subsequential weak limit. Our issue here is that one would have to demonstrate that this weak limit is a free action in order to compare its cost to the cost of the Poisson. To do this, one would have to use the Farber condition in an essential way.

We bypass this by a labelling trick: note that the IID of any point process is automatically a free action (as any two points of it will receive distinct values almost surely, killing any possible symmetries). So we will limit on an IID labelled process instead.

Consider the action $G \curvearrowright G/\Gamma_n \times [0,1]$, where the action on the second coordinate is trivial. We refer to this as the periodic IID lattice shift. It is simply the lattice shift, but with every point of it receiving the same IID label. This is of course distinct from the IID of the lattice shift, but note

$$\text{cost}(G \curvearrowright G/\Gamma_n) = G \curvearrowright G/\Gamma_n \times [0,1]$$

as cost is the integral of the cost over the ergodic decomposition (see [KM04, Corollary 18.6]). We write $(a\Gamma_n,\xi)$ for a sample from this space.

Now we thicken as before, but this time distribute labels: let

$$\Theta_n(a\Gamma_n,\xi) = \bigcup_{i=1}^{k_n} a\Gamma_nb^n_i \times \{\xi_i\},$$

where $\xi_1,\xi_2,\ldots$ is an IID sequence of $\text{Unif}[0,1]$ random variables produced as a factor of $\xi$.

In other words, each point of the lattice shift adds points to the space and gives them an IID $[0,1]$ label (but note that each point starts with the same label $\xi$, so this is not the IID of the thickening).
Let $\Upsilon$ denote any subsequential weak limit of $\Theta_n(a\Gamma_n, \xi)$, and pass to that subsequence. Then $\pi(\Theta_n(a\Gamma_n, \xi))$ weakly converges to $\pi(\Upsilon)$, as the projection $\pi : G \times [0, 1] \to G$ that forgets labels is continuous.

Our task is to show that $\Upsilon = [0, 1]^{\pi(\Upsilon)}$.

The idea of the proof is the following: fix $C \subseteq G$ to be a bounded stochastic continuity set for $\Upsilon$. We want to prove that the labels of the points of $\Theta_n(a\Gamma_n, \xi)$ in $C$ are independent and uniform on $[0, 1]$. They are already $\text{Unif}[0, 1]$ by definition, so we must now consider their dependencies. Again, by definition, points of $C$ arising from distinct $a\Gamma_n b_i^n$ are automatically independent. The only dependency issue that can arise is when $a\Gamma_n b_i^n \cap C$ has multiple points. We will show that this is a vanishingly rare event.

This will be achieved via the following lemma:

**Lemma 7.6:** Let $C \subseteq G$ be compact. If $(\Gamma_n)$ is a Farber sequence and $B_n \subseteq G$ is any sequence of finite subsets, then

$$\mathbb{P}[\exists b \in B_n \text{ such that } |a\Gamma_n b \cap C| > 1] \to 0.$$ 

**Proof.** Apply the Farber condition with any set $V$ containing $CC^{-1}$. If $b \in B_n$ is such that there are $a\gamma_1, a\gamma_2$ distinct elements of $a\Gamma_n b \cap C$, then

$$(a\gamma_1 b)(a\gamma_2 b)^{-1} = a\gamma_1 \gamma_2^{-1} a^{-1}$$

is in $CC^{-1}$,

so $a\gamma_1 \gamma_2^{-1} a^{-1} ab = a\gamma_1 \gamma_2^{-1} b \in Vab$, and this element is also in $a\Gamma_n b$. By the Farber condition,

$$a\Gamma_n b \cap Vab = \{ab\}$$

with high probability, and so

$$\mathbb{P}[\exists b \in B_n \text{ such that } |a\Gamma_n b \cap C| > 1] \leq \mathbb{P}[a\Gamma_n a^{-1} \cap V = \{e\}] \to 0,$$

finishing the proof of the lemma.

Let $V = (V_1, V_2, \ldots, V_k)$ denote a collection of bounded stochastic continuity sets for $\Upsilon$, and $[0, p) = ([0, p_1), [0, p_2), \ldots, [0, p_k))$ a family of intervals in $[0, 1]$. We denote by $V \times [0, p) = (V_1 \times [0, p_1), \ldots, V_k \times [0, p_k))$ the stochastic continuity set of $[0, 1]^\Upsilon$.

Let $C$ be a compact set large enough to contain $\bigcup_i V_i$.

On the event from the lemma,

$$N_V(\Theta_n(a\Gamma_n, \xi)) = N_V([0, 1]^{\Theta_n(a\Gamma_n)},$$

where by $\Theta_n(a\Gamma_n)$ we simply mean $(\Theta_n(a\Gamma_n, \xi))$ with the labels erased.

Therefore $\Theta_n(a\Gamma_n, \xi)$ converges weakly to $[0, 1]^\Upsilon$, finishing the proof.
7.2. The rerooting groupoid for homogeneous spaces. One must also investigate point processes on the homogeneous spaces of groups. The setup which we will consider is the action of $G$ on $X = G/K$, where $K \leq G$ is a compact subgroup. This covers our principal case of interest, namely Riemannian symmetric spaces (such as $\text{Isom}(\mathbb{H}^d) \curvearrowright \mathbb{H}^d$).

All of the point process theoretic definitions (such as thinnings and factor graphs) can be readily adapted to this context. There are some minor subtleties that must be addressed, however. Our aim is to define a cost notion for $G$-invariant point processes on $X$, and relate it to cost for $G$-invariant point processes on $G$ itself. We will show:

**Theorem 7.7:** Assume that the Poisson point process on $X = G/K$ is free as a $G$ action.\(^{24}\) Then

$$\sup_{\Pi} \text{cost}_X(\Pi) = \sup_{\Pi} \text{cost}_G(\Pi),$$

where the supremum is taken over the set of free point processes on $X$ and on $G$ respectively. In particular, if $X$ has fixed price one then $G$ itself has fixed price one.

**Remark 7.8:** The appeal of the above theorem is that it allows one to prove fixed price for a group by working on the symmetric space instead, where the geometry is more apparent. As will be evident in the proof, it suffices to prove fixed price for the Poisson point process on $X$, which is a concrete probabilistic object.

Our starting point is to note that such spaces $X$ enjoy the properties we have been using:\(^{25}\) namely, an invariant proper metric that makes $X$ an lcsc space and a $G$-invariant Borel measure $\lambda_X$ on $X$.

For the metric on $X$, one takes the Hausdorff metric:

$$d_X(aK, bK) = \max \{ \sup_{k_1 \in K} \inf_{k_2 \in K} d(ak_1, bk_2), \sup_{k_2 \in K} \inf_{k_1 \in K} d(ak_1, bk_2) \},$$

and for the measure $\lambda_X$ one takes the pushforward $\pi_* \lambda$, where $\pi : G \to X$ is the quotient map $a \mapsto aK$.

---

\(^{24}\) Some assumption is required. For instance, if $K$ contains an element of the center of $G$ then the action will not be free.

\(^{25}\) The limiting factor here is really the metric: a $G$-invariant Borel measure exists on $G/H$ in fairly great generality (it is an imposition on the modular functions of $G$ and $H$), but an invariant metric will only exist if $G/H$ is homeomorphic (in an appropriate sense) to a quotient $G'/H'$ with $H'$ compact in $G$; see [AD13] for further details.
We recall the mapping theorem (See [Kin93, Section 2.3] for further details):

**Theorem 7.9 (Mapping theorem):** Suppose $X$ is a standard Borel space with $\sigma$-finite measure $\lambda$, $\Pi$ is the Poisson point process with intensity measure $\lambda$, and $f : X \to Y$ is a measurable function. Then $f(\Pi)$ is the Poisson point process on $Y$ with intensity measure $f_* \lambda$, assuming this measure has no atoms.

In other words, a map between the base spaces $X$ and $Y$ induces a map from the Poisson point process on $X$ to the Poisson point process on $Y$ (with some intensity measure). It is intuitive that this should lead to an inequality on costs, and we will detail how this works.

Our study splits into two cases, according to whether $K$ is Haar null or not (for $G$’s Haar measure, of course). If $\lambda(K) > 0$, then by the Steinhaus Theorem we have that $K$ is open, and hence a compact clopen subgroup of $G$. It will then also have countable index. This situation occurs for instance in the study of Cayley–Abels graphs of totally disconnected locally compact groups. In this case one is essentially looking at Bernoulli percolation on the quotient space. We will instead focus on the case that $\lambda(K) = 0$.

One can consider $G$-invariant point processes $\Pi$ on $X$, which should be understood as random elements of $\mathcal{M}(X)$ whose distribution is $G$-invariant. We may define the intensity of $\Pi$ as before:

$$\text{int}(\Pi) = \frac{1}{\lambda_X(U)} \mathbb{E}[|\Pi \cap U|],$$

where $U \subseteq X$ is any set of unit volume.

One can further consider notions of thinnings, partitionings, cost, and so on. We follow our previous strategy of studying these by looking at the associated groupoid. Let us write 0 for the element $K \in X$, which we view as the root of the space. Accordingly we will define the space of rooted configurations in $X$ as

$$\mathcal{M}_0(X) = \{\omega \in \mathcal{M}(X) \mid 0 \in \omega\}.$$

Note that the orbit equivalence relation of $G \curvearrowright \mathcal{M}(X)$ no longer restricts to $\mathcal{M}_0(X)$ to an equivalence relation with countable classes. The solution to this problem is to take a quotient:

---

26 It states that if $A \subseteq G$ is a subset of a locally compact group with positive Haar measure, then $AA^{-1}$ contains an identity neighborhood.
Proposition 7.10: Let $K$ act on $\mathbb{M}_0(X)$ by left multiplication. Then the action is smooth, that is, the space of orbits $K \backslash \mathbb{M}_0(X)$ is a standard Borel space.

It is a general fact that compact groups always act smoothly on standard Borel spaces (see [Zim84, Proposition 2.1.12] and its corollary), but it is possible to give a direct proof in this case. The proof is very reminiscent of the section “Extension to more general point processes” of [HP03].

Proof. We will construct a measurable function $F : \mathbb{M}_0(X) \to [0,1]$ with the property that $F(\omega) = F(\omega')$ if and only if $\omega' \in K\omega$. This is a characterization of smoothness.

Fix a family $U_n$ of open subsets of $G$ with the property that it determines elements of $\mathbb{M}_0(X)$ in the sense that $\omega = \omega' \iff \{n \in \mathbb{N} \mid \omega \cap U_n \neq \emptyset\} = \{n \in \mathbb{N} \mid \omega' \cap U_n \neq \emptyset\}$.

Let $f : \{0,1\}^\mathbb{N} \to [0,1]$ be any continuous order-preserving injection, and consider the map $F : \mathbb{M}_0(X) \to [0,1]$ given by

$$F(\omega) = \inf_{k \in K} f((1[\omega \cap k \cdot \omega \neq \emptyset])_n).$$

Note that the component functions $\omega \mapsto 1[\omega \cap \emptyset]$ are lower semi-continuous, so the infimum is attained.

The function is constant on $K$-orbits by definition, but by the separating nature of the family $\{U_n\}$ it also takes distinct values for points in distinct orbits. □

A Borel thinning $\theta : \mathbb{M}(X) \to \mathbb{M}(X)$ corresponds to a Borel subset $A \subseteq K \backslash \mathbb{M}_0(X)$.

Note that the latter can be identified with a subset of $\mathbb{M}_0(X)$ which is $K$-invariant in the sense that $KA = A$, we will make such identifications freely.

To see why this $K$-invariance is required, consider the formula

$$\theta^A(\omega) := \{gK \in \omega \mid g^{-1}\omega \in A\}.$$

For this to be well-defined, we need that the condition does not depend on our choice of coset representative $gK$. This is exactly asking for $K$-invariance of $A$.

In the same way one can see that Borel factor $[d]$-colorings correspond to Borel partitions of $K \backslash \mathbb{M}_0(X)$ indexed by $[d]$, and so on.
If $\Pi$ is a $G$-invariant point process on $X$ with law $\mu$, then we may use the above to define its Palm measure $\mu_0$ on $K\backslash M_0(X)$ as before:

$$
\mu_0(A) := \frac{\int \theta^A(\Pi)}{\int \Pi} = \frac{1}{\int \Pi} \mathbb{E}[\#\{gK \in U \mid g^{-1}\omega \in A\}], \quad \text{where } U \subseteq X \text{ has unit volume.}
$$

We equip $K\backslash M_0(X)$ with the following rerooting equivalence relation:

$$
K\omega \sim_R K\omega' \text{ if and only if } \exists aK \in \omega \text{ such that } K\omega' = Ka^{-1}\omega.
$$

We can also define a groupoid structure. If one defines

$$
\overrightarrow{M_0}(X) = \{(\omega, aK) \in M_0(X) \times X \mid aK \in \omega\},
$$

then there is a natural diagonal action of $K$ on $\overrightarrow{M_0}(X)$. The quotient is again standard Borel, we denote it $K\backslash \overrightarrow{M_0}(X)$.

Then $K\backslash M_0(X)$ and $K\backslash \overrightarrow{M_0}(X)$ form the unit space and arrow space (respectively) of a groupoid, which we call the rerooting groupoid. The source and target maps are defined as before

$$
s,t : K\backslash \overrightarrow{M_0}(X) \to K\backslash M_0(X),
$$

$$
s(K\omega, KaK) = K\omega,
$$

$$
t(K\omega, KaK) = Ka^{-1}\omega.
$$

A pair of arrows $(K\omega, KaK), (K\omega', KbK) \in K\backslash \overrightarrow{M_0}(X)$ are deemed composable if $K\omega' = Ka^{-1}\omega$, in which case

$$(K\omega, KaK) \cdot (K\omega', KbK) := (K\omega, KabK).$$

We can equip this groupoid with the Palm measure, resulting in a $r$-discrete pmp groupoid as before.

**Definition 7.11:** Let $\Pi$ be a $G$-invariant point process on $X$. Its groupoid cost is

$$
cost_X(\Pi) - 1 = \inf_{\mathcal{G}} \left\{ \frac{1}{2} \mathbb{E} \left[ \sum_{x \in U \cap \Pi} \deg_x \mathcal{G}(\Pi) \right] \right\} - \text{int}(\Pi),
$$

where $U$ is a set of unit volume in $X$ and the infimum is taken over all connected equivariantly defined factor graphs of $\Pi$.

The cost of $X$ is

$$
cost(X) := \inf\{\text{cost}_X(\Pi) \mid \Pi \text{ is an invariant free point process on } X\}.$$
Aside from replacing $M_0(G)$ with $K\backslash M_0(X)$, our earlier arguments apply verbatim and prove the following:

- If $\Phi : M(X) \to M(X)$ is a $G$-equivariant factor map, then
  \[ \text{cost}_X(\Pi) \leq \text{cost}_X(\Phi(\Pi)). \]

In particular, $\text{cost}_X(\Pi)$ only depends on the isomorphism class of $\Pi$.

- Every free point process weakly factors onto its own IID.
- The Poisson point process on $X$ has maximal $\text{cost}_X$ amongst free $X$ processes, assuming the Poisson point process is free.

Remark 7.12: In this level of generality, the Poisson point process on $X = G/K$ might not be free and thus must be assumed. For instance, let $G = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and $K = \mathbb{R}/\mathbb{Z}$. Then $K$ acts trivially on the quotient $X$, and thus no $G$-invariant point process on $X$ is free (even their IID will not be free). These examples are rather contrived however.

Theorem 7.13: If $\Pi$ is a free point process on $X$, then its $\text{cost}_X$ is equal to its cost as a $G$-action.

Recall from the introduction that the cost of a free pmp action of $G$ is defined by picking an isomorphic representation of the action as a point process, and taking the cost of that.

This theorem will employ a “whittling” construction. Note that we can view point processes on $X$ as random closed subsets of $G$ (which happen to be unions of cosets of a fixed compact subgroup). We are able to exploit freeness to deterministically whittle this random closed subset to a point process:

Proposition 7.14: If $\Pi$ is a free point process on $X$, then it admits a deterministic lift to $G$: that is, there exists an invariant point process $\Upsilon = \Upsilon(\Pi)$ on $G$ such that

- $\Upsilon \subset \Pi$ almost surely,
- $\Upsilon$ intersects each coset $aK$ at most once, and
- $\pi(\Upsilon) = \Pi$.

In other words, we are able to select one element out of every coset $aK \in \Pi$ in an equivariant and deterministic way.
Proof of Theorem 7.13. Observe that the process $\Upsilon$ from Proposition 7.14 is isomorphic to $\Pi$, so $\text{cost}(\Pi) = \text{cost}(\Upsilon)$. We verify that $\text{cost}(\Upsilon) = \text{cost}_X(\Pi)$. Note that factor graphs of $\Pi$ and $\Upsilon$ can be bijectively identified, and so will have the same edge measures. Finally, they have the same intensity: choose $U \subseteq X$ with volume one, and observe that $\Pi \cap U$ and $\Upsilon \cap UK$ are in bijection, with $UK$ also having volume one.

We will require a simple lemma, which essentially already appears in Lemma 4.11 but we isolate for clarity. It works for point processes on $G$ and on $X$.

**Lemma 7.15:** A point process $\Pi$ is free if and only if it admits a deterministic labelling by $[0, 1]$ such that all of the labels are distinct (almost surely).

**Proof.** Clearly if such a labeling exists, then the process must be free.

For the converse, let $I : \mathbb{M}_0 \rightarrow [0, 1]$ be a Borel isomorphism. Define a labelling by

$$L : \mathbb{M} \rightarrow [0, 1]^\mathbb{M}$$

$$L(\omega) = \{(x, I(x^{-1}\omega) \mid x \in \omega\}.$$ Observe that two distinct points $x, y \in \omega$ receive the same label in $L(\omega)$ exactly when $I(x^{-1}\omega) = I(y^{-1}\omega)$, and so $xy^{-1}\omega = \omega$. If the process is free, then this never occurs, as desired.

To run the proof on $X$, simply replace $\mathbb{M}_0$ by $L\mathbb{M}_0(X)$.

Proof of Proposition 7.14. By virtue of being free, we may use Theorem 1.1 to fix an isomorphism of $\Pi$ with a point process $\Pi'$ on $G$. The desired process $\Upsilon$ will be the result of pushing the points of $\Pi'$ into $\Pi$.

Of course $\Pi'$ is itself a free process, so we may fix a deterministic labelling $L(\Pi')$ of its points à la Lemma 7.15.

Assign each coset $aK$ of $\Pi$ to a point of $x$ of $\Pi'$ in your preferred equivariant way. For instance, note that every such coset intersects some (finite but non-zero) number of Voronoi cells of $\Pi'$. For each $aK \in \Pi$, assign it to whichever of these cells has the germ with the highest label in $L(\Pi')$. We denote by $A_x$ the set of cosets in $\Pi$ that we assign to $x \in \Pi'$ in this way.

Fix a Borel transversal $T \subset G$. Note that $xT$ is another Borel transversal for any $x \in G$, so $xT \cap aK$ selects the unique point representative of $aK$ with respect to this transversal.
Finally, set
\[ \Upsilon = \bigcup_{x \in \Pi'} xT \cap aK. \]
This selects one representative from every coset in \( \Pi \), and at every stage it was performed in an equivariant way, so is our desired invariant point process. \( \blacksquare \)

**Proof of Theorem 7.7.** Let \( \Pi \) denote the Poisson point process on \( G \). Then by the mapping theorem (Theorem 7.9), the image \( \Upsilon \) of \( \Pi \) under the quotient map \( G \to X \) is the Poisson point process on \( X \). Since \( \text{cost}_G \) can only increase under factor maps, we have
\[ \text{cost}_G(\Pi) \leq \text{cost}_G(\Upsilon). \]
But the Poisson point process has maximal cost amongst free \( G \)-actions, so there is equality. By Theorem 7.13,
\[ \text{cost}_X(\Upsilon) = \text{cost}_G(\Upsilon), \]
and as discussed, the Poisson point process has maximal cost amongst all free point processes on \( X \), finishing the proof. \( \blacksquare \)

**Question 7:** It is natural to ask if \( G \) and \( X \) have the same infimal cost as well.

### 7.3. Farber sequences in semisimple Lie groups.

The goal of this section is to prove Theorem 1.4 from the introduction, which we restate here:

**Theorem:** Let \( G \) be a semisimple real Lie group and let \( \Gamma_n \) be a Farber sequence of lattices in \( G \). Then
\[ \limsup_{n \to \infty} \frac{d(\Gamma_n) - 1}{\text{vol}(G/\Gamma_n)} \leq c_P(G) - 1. \]

There would be a more straightforward (and general) proof of the above if a more general form of cost monotonicity were true, however we are unable to prove (or disprove) the following statement: suppose \( \Pi_n \) is a sequence of finite cost point processes that weakly converge to a random net \( \Pi \). Is it true that
\[ \limsup_{n \to \infty} \text{cost}(\Pi_n) \leq \text{cost}(\Pi)? \]

To prove Theorem 1.4 we will use the geometric interpretation of being a Farber sequence—specifically, see [ABB+17, Corollary 3.3]. In brief, it means that for all \( r > 0 \) the injectivity radius of a randomly chosen point of the quotient manifold \( M_n = \Gamma_n \backslash X \) is larger than \( r \) with high probability, where \( X = G/K \).
denotes the symmetric space of $G$. We will also heavily call upon the paper [Gel11]. Additionally, it will be assumed that the reader understands the proof of Theorem 7.5.

**Proof of Theorem 1.4.** First, let us handle the special case of $G = \text{PSL}_2(\mathbb{R})$, for which the theorem is true but for fundamentally different reasons. In this case the $\Gamma_n$ being discussed are fundamental groups of finite volume hyperbolic surfaces, and we only require that $\text{covol}(\Gamma_n)$ tends to infinity. This allows us to eliminate additive constants in the following:

$$\lim_{n \to \infty} \frac{d(\Gamma_n) - 1}{\text{vol}(G/\Gamma_n)} = \lim_{n \to \infty} \frac{b_1(\Gamma_n)}{\text{vol}(G/\Gamma_n)} = \lim_{n \to \infty} -\frac{\chi(G/\Gamma_n)}{\text{vol}(G/\Gamma_n)} = \frac{1}{2\pi} = \beta_1(G)$$

$$\leq c_p(G) - 1$$

where $\beta_1(G)$ is the first $L^2$-Betti number of $G$. We are using the Gauss–Bonnet formula above and Gaboriau’s result that the first $L^2$-Betti number of a relation is dominated by its cost-1.

Note that in [CGMTD21], they prove (in particular) that (cross sections of) actions of $\text{PSL}_2(\mathbb{R})$ are treeable and thus have fixed price equal to their $L^2$-Betti number plus one. Thus we actually have equality all the way above.

We now handle the other cases. Let us denote by $a\Gamma_n$ the lattice shift corresponding to $\Gamma_n < G$. Let us summarize the proof:

1. We produce a sequence of uniformly separated factors $\Phi^n(a\Gamma_n)$ of the lattice shifts $G \curvearrowright G/\Gamma_n$. Note that by equivariance they must be of the form $\Phi^n(a\Gamma_n) = a\Gamma_n F_n$, and

$$\text{cost}(G \curvearrowright G/\Gamma_n) - 1 = \frac{d(\Gamma_n) - 1}{\text{vol}(G/\Gamma_n)} \leq \text{cost}(\Phi^n(a\Gamma_n)) - 1$$

as cost is monotone under factors.

2. We show that $\Phi^n(a\Gamma_n)$ admits subsequential weak limits, and any such subsequential weak limit is a random net.

3. As in the proof of Theorem 7.5, we now use the periodic IID lattice shift and distribute randomness, replacing $\Phi^n(a\Gamma_n)$ by marked processes which converge to an IID-labelled random net.

4. Using results of [Gel11], our desired inequality follows from cost monotonicity.
We will show that the distance-$R$ factor graph $\mathcal{D}_R$ is connected on
\[
\Phi^n(a\Gamma_n) = a\Gamma_n F_n.
\]
Observe that, by left-invariance of the metric, this is true if and only if it is connected on $\Gamma_n F_n$. Observe that this is finitely many right cosets, that is,
\[
\Gamma_n F_n \subset \Gamma_n \setminus G.
\]
We will show that $\mathcal{D}_R$ is connected by appealing to properties of the further quotient
\[
\Gamma_n \setminus G/K = \Gamma_n \setminus X.
\]

The essential result we use from [Gel11] is the following. As long as $G$ is not $\text{PSL}(2, \mathbb{R})$, there exist constants $\varepsilon, \varepsilon' > 0$ and a sequence of connected subsets $U_n \subset X$ such that:

- The projection $\Gamma_n U_n \subset \Gamma_n \setminus X$ contains the $\varepsilon$-thick part of $\Gamma_n \setminus X$, and
- The projection $\Gamma_n U_n \subset \Gamma_n \setminus X$ is contained in the $\varepsilon'$-thick part of $\Gamma_n \setminus X$.

Here $\varepsilon$ is defined in [Gel11, Lemma 2.3] and $\varepsilon' = \varepsilon/(2\mu)$, where $\mu$ is defined after the proof of Lemma 2.4. Crucially, these constants only depend on $G$. In the paper, our $U_n$ is denoted by $\widetilde{\psi}_{\leq 0}$ and it is a level set with respect to a function inversely measuring the injectivity radius.

**Claim:** There exists a sequence $\Phi^n(a\Gamma_n)$ of factors that are uniformly separated and any subsequential weak limit of them is a random net.

**Proof of Claim.** We choose $F_n \subset G$ following [Gel11, Corollary 2.13]. We choose a maximal $\varepsilon'$-separated subset $E_n$ of $\Gamma_n U_n K \subset \Gamma_n \setminus X$. Then the union of $\varepsilon'$-balls around $E_n$ will cover $\Gamma_n U_n$ by maximality, hence, the set of points not covered by these balls lies in the $\varepsilon'$-thin part of $\Gamma_n \setminus X$. By the Farber condition, the density of these points tends to 0 in $n$. This means that for any subsequential weak limit of the point processes $a\Gamma_n F_n$, the probability that the identity is distance more than $\varepsilon'$ from the root of $X$ is 0. That is, the union of $\varepsilon'$-balls for any subsequential weak limit will equal $X$ a.s., that is, the weak limit will be a net. The slight difference is that we now need the same on $G$, not on $X$. In order to do that, we pick a coset representative (randomly or deterministically) with respect to $K$. This can only increase the $X$-distance but only by a bounded amount, so even on $G$, the same argument holds with worse constants. □
Similar to before, we distribute randomness from the $\Gamma_n$-periodic IID processes to $\Phi^n(a\Gamma_n)$. Call the resulting process $\Pi^n$. By passing to a subsequential weak limit, we can assume that $\Pi^n$ weakly converges to some process $\Upsilon$. As before in Theorem 7.5, the Farber condition ensures that $\Upsilon$ is in fact an IID labelled process (and in particular, its cost is at most the cost of the Poisson point process).

Our final task is to relate the cost of the $\Pi^n$ processes to the cost of $\Upsilon$. We write $\mu^n_0$ for the Palm measure of $\Pi^n$ and $\mu_0$ for the Palm measure of $\Upsilon$.

By the proof of Theorem 5.10, any factor graph which $\delta$-computes the cost of $\Upsilon$ contains a $\mu_0$-continuity factor graph $G$ which is connected for $\Upsilon$. Thus

$$\limsup_{n \to \infty} \mu^n_0(G) \leq \mu_0(G).$$

A priori, there is no reason to expect that $G$ is connected for any $\Pi_n$, however. But note that by construction the graphing $G$ has the property that for large enough $R > 0$ there exists a constant $N$ such that $G^n(\omega) \supseteq \mathcal{D}_R(\omega)$ for all $\varepsilon'$-separated configurations $\omega$, where $\mathcal{D}_R$ denotes the distance-$R$ factor graph as usual. In particular, $G$ is connected for the lattice shift factors $a\Gamma_n F_n$, as they are coarsely connected: by left-invariance of the metric, we may simply consider $\Gamma_n F_n$, and recall that its image in $X$ lies in the connected subset $U_n$. Now for any pair of points $x$ and $y$ in $\Gamma_n F_n$, take a path between their images $xK$ and $yK$ lying within $U_n$, and note that it induces a coarse path (with bounded jumps) between $x$ and $y$ themselves. Thus

$$\limsup_{n \to \infty} \frac{d(\Gamma_n) - 1}{\text{vol}(G/\Gamma_n)} \leq \text{cost}(\Pi^n) - 1$$

as desired. □

Appendix A. Metric properties of configuration spaces and weak convergence

In this appendix, we will discuss the necessary technical background to understand the rest of this paper. The aim is that a reader unfamiliar with the theory of point processes will be able to read this section and have the core ideas of what is going on (if not the finer technical details). No originality is claimed for this material, and so explicit references are given.

The following fact is the most basic requirement for a well-behaved probability theory:
Theorem A.1 (See [DVJ03, Theorem A2.6.III]): If $X$ is a complete and separable metric space, then $\mathcal{M}(X)$ is a Borel subset of a complete and separable metric space $\mathcal{M}^\#(X)$, and is thus a standard Borel space.

Note that configurations $\omega \in X$ can be viewed as measures on $X$, by defining $\omega(A) = |\omega \cap A|$. So configurations form particular examples of locally finite measures on $X$, and $\mathcal{M}^\#(X)$ will be the space of such measures. In this language, a point process is a particular example of a random measure. Probabilists are interested in other examples of random measures, and have thus developed a framework suitable to handle all their cases of interest. We adopt their framework with small notational modifications.

We assume the reader is at least passingly familiar with weak converge of measures on metric spaces. Recall:

Definition A.2: Let $\mathcal{M}(X)$ denote the space of totally finite measures $\eta$ on $X$, that is, those with $\eta(X) < \infty$.

The Prokhorov metric $d_{\text{prok}}$ on $\mathcal{M}(X)$ is

$$d_{\text{prok}}(\eta, \eta') = \inf \{ \varepsilon \geq 0 \mid \text{for all Borel } A \subseteq X, \eta(A) \leq \eta'(A^\varepsilon) + \varepsilon \text{ and } \eta'(A) \leq \eta(A^\varepsilon) + \varepsilon \},$$

where $A^\varepsilon$ is the $\varepsilon$-halo of $A$, that is,

$$A^\varepsilon = \{ x \in X \mid d(x, A) < \varepsilon \}.$$

If $\eta$ is a totally finite measure on $X$, then a $\eta$-continuity set is a subset $A \subseteq X$ with the property that $\eta(\partial A) = 0$, where $\partial A$ denotes the topological boundary of $A$.

A sequence of totally finite measures $\eta_n$ weakly converges to $\eta$ if either of the following conditions hold:

(WC1) for all continuous and bounded functions $f : X \to \mathbb{R}$,

$$\lim_{n \to \infty} \int_X f(x) \, d\eta_n(x) = \int_X f(x) \, d\eta(x).$$

(WC2) For all $\eta$-continuity sets $A \subseteq X$,

$$\lim_{n \to \infty} \eta_n(A) = \eta(A).$$

\textsuperscript{27} And, for that matter, non-invariant point processes.
Remark A.3: The Prokhorov metric metrizes this convergence notion (that is, \( \eta_n \) converges weakly to \( \eta \) if and only if \( d_{\text{prok}}(\eta_n, \eta) \) converges to zero).

The equivalence of (WC1) and (WC2) is usually referred to as the Portmanteau theorem.

The definition involving continuity sets will have preeminence for us. To explain the name: note that the indicator function \( 1_A : X \to \{0, 1\} \) is continuous \( \eta \) almost everywhere if and only if \( A \) is an \( \eta \)-continuity set.

We often make use of the following well-known fact:

**Lemma A.4:** If \( \Phi : X \to Y \) is a continuous map of metric spaces, then \( \Phi \) preserves weak limits: if \( \mu_n \) is a sequence of Borel probability measures on \( X \) weakly converging to \( \mu \), then \( \Phi_*\mu_n \) weakly converges to \( \Phi_*\mu \).

Moreover, the same is true if \( \Phi \) is merely continuous \( \mu \) almost everywhere.

**Proof.** The first statement is immediate: if \( f : Y \to \mathbb{R} \) is a continuous and bounded function, then \( f \circ \Phi : X \to \mathbb{R} \) is continuous and bounded as well, so

\[
\lim_{n \to \infty} \int_Y f d\Phi_*\mu_n = \lim_{n \to \infty} \int_X f \circ \Phi d\mu_n = \int_X f \circ \Phi d\mu = \int_Y f d\Phi_*\mu.
\]

For the second statement we work with the definition involving continuity sets. Let \( A \subseteq Y \) be a \( \Phi_*\mu \) continuity set, that is, assume \( \mu(\Phi^{-1}(\partial A)) = 0 \). Let \( D_\Phi = \{x \in X \mid \Phi \text{ is discontinuous at } x\} \) denote the discontinuity set of \( \Phi \). One can show that for any \( A \subseteq Y \) that \( \partial \Phi^{-1}(A) \subseteq \Phi^{-1}(\partial A) \cup D_\Phi \), and so

\[
\mu(\partial \Phi^{-1}(A)) \leq \mu(\Phi^{-1}(\partial A)) + \mu(D_\Phi) = 0,
\]

that is, \( \Phi^{-1}(A) \) is a \( \mu \)-continuity set. Therefore

\[
\lim_{n \to \infty} \Phi_*\mu_n(A) = \lim_{n \to \infty} \mu_n(\Phi^{-1}(A)) = \mu(\Phi^{-1}(A)) = \Phi_*\mu(A),
\]

as desired. \( \blacksquare \)

**Definition A.5:** Let \( \mathcal{M}^\#(X) \) denote the space of **boundedly finite** measures, that is, those Borel measures \( \eta \) on \( X \) that are finite on metrically bounded subsets of \( X \).

Fix a basepoint \( x_0 \in X \). Let

\[
\eta^{(r)}(A) := \eta(A \cap B(x_0; r))
\]

denote the restriction of a boundedly finite measure \( \eta \) to the \( r \)-ball about \( x_0 \). Note that \( \eta^{(r)} \) is therefore an element of \( \mathcal{M}(X) \).
We now define a metric $d^#$ on $\mathcal{M}^#(X)$:

$$d^#(\eta, \eta') = \int_0^\infty e^{-r} \frac{d_{\text{prok}}(\eta^{(r)}, \eta'^{(r)})}{1 + d_{\text{prok}}(\eta^{(r)}, \eta'^{(r)})} dr.$$ 

A sequence of boundedly finite measures $\eta_n$ weak-$#$ converges to $\eta$ if any of the following conditions hold:

(WHC1) For all continuous and bounded functions $f : X \to \mathbb{R}$ which vanish outside a bounded set,

$$\lim_{n \to \infty} \int_X f(x) d\eta_n(x) = \int_X f(x) d\eta(x).$$

(WHC2) For all bounded $\eta$-continuity sets $A \subseteq X$,

$$\lim_{n \to \infty} \eta_n(A) = \eta(A).$$

(WHC3) There exists a sequence $r_k$ of radii increasing to infinity such that for every $k \in \mathbb{N}$,

$$\eta_n^{(r_k)} \text{ converges weakly to } \eta^{(r_k)}.$$ 

Remark A.6: The space we have defined is obviously extremely metrically dependent (recall that every metric is topologically equivalent to a bounded metric). However, our case of interest is proper left-invariant metrics on locally compact groups, which are all coarsely equivalent and thus have a well-defined notion of metrically boundedness.

Defining the metric required us to fix an arbitrary base point $x_0$. If one chooses a different basepoint $x'_0$, then the resulting metrics will be bilipschitz equivalent.

In case $X$ is locally compact, then weak-$#$ convergence is equivalent to vague convergence.

Theorem A.7: The space $\mathcal{M}^#(X)$ equipped with the $d^#$ metric is complete and separable. Its Borel structure is exactly such that the mass measuring functions $N_A : X \to \mathbb{N}_0 \cup \infty$ given by $\eta \mapsto \eta(A)$ are measurable, where $A$ is an arbitrary Borel subset of $X$.\(^{28}\)

\(^{28}\) Earlier we called these “point counting” functions, because that is a more suitable name when the measure is atomic.
Remark A.8: The Borel structure on $\mathcal{M}^\#(X)$ can be generated by an even smaller collection of mass measuring functions: one only needs to look at $N_A$ where $A$ ranges over a semiring of bounded Borel sets that generate the Borel structure on $X$.

We will require the following more explicit explanation of what weak-$\#$ convergence is:

**Definition A.9:** Let $\omega \in \mathcal{M}(X)$ be a configuration. We call another configuration $\omega' \in \mathcal{M}(X)$ a $(\varepsilon, R)$-wobble of $\omega$ (where $\varepsilon, R > 0$ are some parameters) if $\omega^{(R)}$ is in bijection with $\omega'^{(R)}$, and moreover this bijection $\sigma : \omega^{(R)} \to \omega'^{(R)}$ can be chosen in such a way that

$$d(x, \sigma(x)) < \varepsilon \quad \text{for all } x \in \omega^{(R)}.$$

One direction of the following lemma is immediate; the converse is less elementary and can be found in [DVJ03] as Proposition A2.6.II:

**Lemma A.10:** A sequence of configurations $\omega_n$ converges to $\omega$ with respect to $d^\#$ if and only if there are sequences $\varepsilon_n \to 0$ and $R_n \to \infty$ such that each $\omega_n$ is a $(\varepsilon_n, R_n)$-wobble of $\omega$.

We can now discuss weak convergence of point processes. View $\mathcal{M}(X)$ as a subset of $\mathcal{M}^\#(X)$ equipped with the $d^\#$ metric, and recall that a point process is a probability measure on $\mathcal{M}(X)$. This is what we mean by a sequence of point processes weakly converging.

Note that the weak limit of a sequence of point processes $\mu_n$ will (a priori) be a probability measure on $\mathcal{M}^\#(X)$, not on $\mathcal{M}(X)$. That is, a point process might converge to a random measure which is not a point process. It is easy to see that the only thing that can go wrong is mass accumulation: the limit measure will be a random atomic measure, but some atoms might have mass larger than one.

**Definition A.11:** A **counting measure** on $X$ is a measure $\eta$ with $\eta(A) \in \mathbb{N}_0$ for all bounded Borel subsets $A \subseteq X$. A **simple** counting measure is a measure $\eta$ with $\eta(\{x\}) = 0$ or 1 for all $x \in X$.

If $\eta$ is a counting measure, then its **support** is

$$\text{support}(\eta) = \{x \in X \mid \eta(\{x\}) > 0\}.$$ 

That is, $\text{support}(\eta)$ is $\eta$ with the multiplicities removed.
Example 19: Let \( \{X_k\}_{k \in \mathbb{Z}} \) denote an IID sequence of uniform \([0, 1]\) random variables. Consider the following sequence of point processes:

\[
\Pi_n = \mathbb{Z} \cup \left\{ k + \frac{X_k}{n} \mid k \in \mathbb{Z} \right\}.
\]

In words: take two copies of \( \mathbb{Z} \), where you wobble all the points of one copy by smaller and smaller amounts (this is not a point process proper in our sense, as it is not invariant, but one can take a uniform \([0, 1]\) shift of \( \Pi_n \) if one insists).

Then \( \Pi_n \) weakly converges to the deterministic measure \( \mu \) given by

\[
\mu(A) = 2|A \cap \mathbb{Z}|.
\]

Remark A.12: In this language, what we have been calling point processes are random simple counting measures, and the comment above states that the weak limit of random simple counting measures, if it exists, will be a possibly non-simple random counting measure.

In the literature one sometimes sees random counting measures referred to as “point processes”, and random simple counting measures as “simple point processes”.

Definition A.13: Let \((X,d)\) be a csms, and \((\mu_n)\) a sequence of Borel probability measures on \(X\). The sequence is uniformly tight if for every \(\varepsilon > 0\) there exists a compact set \(K \subseteq X\) such that \(\mu_n(X \setminus K) < \varepsilon\) for all \(n \in \mathbb{N}\).

Recall from Prokhorov’s theorem that a sequence \((\mu_n)\) is uniformly tight if and only if its closure \((\overline{\mu_n})\) is compact. To apply this to point processes, we need to know about compact sets in \(\mathcal{M}^\#(X)\), and it is not at all evident what compact sets are given our definition of the metric. It is thus the following more explicit form of uniform tightness that we will use:

Theorem A.14 (See [DVJ07, Proposition 11.1.VI]): Suppose \(X\) is a locally compact csms.\(^{29}\) A sequence of point processes \((\Pi_n)\) on \(X\) is uniformly tight if and only if for every closed ball \(B \subseteq X\) and any \(\varepsilon > 0\) there exists an \(M > 0\) such that

\[
\mathbb{P}[N_B \Pi_n > M] < \varepsilon \quad \text{for all } n \in \mathbb{N}.
\]

\(^{29}\) There is a slightly more complicated version of the theorem for general Polish spaces, but we will not use it, so do not state it.
Lemma A.15: Let \((X, d)\) be a locally compact csms, and
\[
H_\delta = \{ \omega \in \mathbb{M}(X) \mid d(x, y) \geq \delta \text{ for all distinct } x, y \in \omega \}
\]
denote the space of \textit{\(\delta\)-uniformly-separated} configurations. Then \(H_\delta\) is compact in \(\mathbb{M}(X)\).

Probabilists often refer to such configurations as \textit{hard-core}, hence our choice of letter.

The previous lemma is proved using the following basic fact:

Lemma A.16: Let \((X, d)\) denote a compact metric space. Then for all \(\delta > 0\) there exists some \(C > 0\) such that \(|A| \leq C\) for any \(\delta\)-separated subset \(A \subseteq X\).

The above discussion has been rather abstract. We now outline an equivalent interpretation of weak convergence that will be much more useful in certain applications.

Definition A.17: Let \(\Pi\) be a point process with law \(\mu\). A \textit{stochastic continuity set} of \(\Pi\) is a Borel subset \(V \subseteq G\) of the ambient space such that
\[
P[\Pi \cap \partial V \neq \emptyset] = 0.
\]
Equivalently, it is a subset such that its point counting function
\[
N_V : \mathbb{M} \to \mathbb{N}_0 \cup \{\infty\}
\]
is continuous \(\mu\) almost everywhere.

Let \(V = (V_1, V_2, \ldots, V_k)\) denote a collection of stochastic continuity sets for \(\Pi\).

The \textit{finite-dimensional distributions} of \(\Pi\) are the random vectors
\[
N_V(\Pi) = (N_{V_1}\Pi, N_{V_2}\Pi, \ldots, N_{V_k}\Pi),
\]
where \(V\) runs over all possible collections of stochastic continuity sets.

Remark A.18: The sets
\[
\{ \omega \in \mathbb{M} \mid N_V(\omega) = \alpha \}, \quad \text{where } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}_0^k
\]
should be thought of as analogous to the cylinder sets in the space \(\{0, 1\}^\Gamma\), where \(\Gamma\) is a discrete group.

Note that if \(V\) is a family of stochastic continuity sets for \(\Pi\), then \(N_V\) is continuous \(\mu\) almost everywhere. Thus by the earlier fact on weak limits and continuous functions, we see that weak convergence of point processes implies weak convergence of the finite-dimensional distributions. The surprising fact is that the converse is true:
THEOREM A.19 (See [DVJ07, Theorem 11.1.VII]): A sequence $\Pi_n$ of point processes weakly converges to $\Pi$ if and only if for all collections $V = (V_1, V_2, \ldots, V_k)$ of stochastic continuity sets of $\Pi$, the finite-dimensional distributions $N_V(\Pi_n)$ weakly converge to $N_V(\Pi)$.

For this to make any sense at all, it must be the case that the finite-dimensional distributions determine a point process. That is, if two point processes $\Pi$ and $\Pi'$ have $N_V(\Pi) \overset{d}{=} N_V(\Pi')$ for all collections of stochastic continuity sets $V$, then $\Pi \overset{d}{=} \Pi'$. This is proved using the following lemma, which states that there is an abundance of continuity sets:

**Lemma A.20:** Let $\Pi$ be a point process with law $\mu$. Then

- for all $g \in G$, there are at most countably many $r > 0$ such that the open ball $B_G(g, r)$ is not a stochastic continuity set of $\Pi$, and
- for all $\omega \in M$, there are at most countably many $r > 0$ such that the open ball $B_M(\omega, r)$ is not a $\mu$ continuity set.

In particular, both $G$ and $M$ admit topological bases consisting of $\mu$ stochastic continuity sets / $\mu$ continuity sets (respectively).

**Proof.** The method is the same in both cases, so we only write the proof for the first statement. The idea of the proof is that there cannot be so many stochastic continuity sets, else local finiteness will be contradicted. It is enough to prove that for every $r > 0$ and $\varepsilon > 0$ there exists only finitely many $r_1, r_2, \ldots$ in $(0, r)$ such that

$$
\mathbb{P}[\Pi \cap \partial B(0, r_i) \neq \emptyset] > \varepsilon \quad \text{for all } i.
$$

Suppose not. That is, suppose we have $r, \varepsilon > 0$ and infinitely many $\{r_n\} \subset (0, r)$ satisfying the above equation. Then

$$
\varepsilon \leq \limsup_{n \to \infty} \mathbb{P}[\Pi \cap \partial B(0, r_n) \neq \emptyset] \leq \mathbb{P}[\limsup_{n \to \infty} (\Pi \cap \partial B(0, r_n) \neq \emptyset)].
$$

Recall that the lim sup of a sequence of events is the event that they occur infinitely often. So we see that

$$
\{\Pi \cap \partial B(0, r_n) \neq \emptyset \text{ for infinitely many } n\} \subseteq \{|\Pi \cap B(0, r)| = \infty\}.
$$

We have shown that with positive probability, $\Pi$ has infinitely many points in $B(0, r)$, a contradiction by local finiteness. $\blacksquare$
Note that the continuity sets form an algebra, and the cylinder sets
\[ \{ \omega \in \mathbb{M} \mid N_{\mathbf{V}}(\omega) = \alpha \} \]
are continuity sets when \( \mathbf{V} \) is a collection of stochastic continuity sets. As a
measure is determined by its values on any algebra that generates the Borel sigma algebra, we therefore see that point processes are determined by their finite-dimensional distributions. With a bit more work (see [DVJ03, Proposition A2.3.IV] and [DVJ07, Corollary 11.1.III, Theorem 11.I.VII]), one can prove:

**Lemma A.21:** Let \( \Pi \) be a point process. Then there exists a countable family \( \{ V_i \}_{i \in \mathbb{N}} \) of metrically bounded and disjoint Borel subsets \( V_i \subseteq G \) such that \( \Pi_n \) weakly converges to \( \Pi \) if and only if \( N_{\mathbf{V}} \Pi_n \) weakly converges to \( N_{\mathbf{V}} \Pi \) where \( \mathbf{V} \) ranges over all finite subcollections of \( \{ V_i \} \).

In particular, weak convergence can be verified by a countable collection of statements, each of which only requires one to observe the process in compact windows.

The following lemma is a simpler case of [DVJ07, Exercise 13.2.2], and is presumably known with a more elegant proof. The technique will be used for a later proof, so we include it.

**Proposition A.22:** Suppose \( \mu^n \) is a sequence of finite intensity point processes that weakly converge to a finite intensity process \( \mu \), and \( \text{int} \mu_n \) converges to \( \text{int} \mu \). Then the Palm measures \( \mu_0^n \) weakly converge to \( \mu_0 \).

**Proof.** Let \( A \subseteq \mathbb{M}_0 \) be a \( \mu_0 \)-continuity set, and \( U \subseteq G \) a stochastic \( \mu \) continuity set. Recall that
\[ \mu_0(A) = \frac{1}{\text{int} \mu \lambda(U)} \mathbb{E}_\mu[\#\{ g \in U \mid g^{-1} \omega \in A \}]. \]

**Claim:** For every \( k \in \mathbb{N} \), the function
\[ \omega \mapsto \min\{ \#\{ g \in U \mid g^{-1} \omega \in A \}, k \} \]
is continuous \( \mu \) almost everywhere.

This function can only be discontinuous on the boundary of
\[ A^{(l)} = \{ \omega \in \mathbb{M} \mid \#\{ g \in U \mid g^{-1} \omega \in A \} = l \}. \]
Observe that
\[ \mu_0(\partial A) = 0 \] by assumption, so
\[ \mu_0([\partial A]) = 0 \] as saturations of null sets are null in a countable groupoid,
\[ \mu(G\partial A) = 0 \] by Proposition 3.17.

We show that
\[ \partial A^{(l)} \cap \{ N_{\partial U} \omega = 0 \} \subseteq G\partial A \]
for all \( l \geq 1 \), establishing the claim.

Suppose \( \omega \in \partial A^{(l)} \cap \{ N_{\partial U} \omega = 0 \} \). Then we can find two sequences \( \omega_n, \omega'_n \) both converging to \( \omega \) such that \( \omega_n \in A^{(l)} \) and \( \omega'_n \notin A^{(l)} \) for every \( n \in \mathbb{N} \). We take these to be \((\varepsilon_n, R_n)\)-wobbles of \( \omega \).

We see that (for large \( n \)) the configurations \( \omega, \omega_n, \omega'_n \) are all approximately equal inside \( U \). See Figure 8.

We will refer to the points \( g \in U \cap \omega \) such that \( g^{-1}\omega \in A \) as \textbf{A-points of \( \omega \)} and likewise for \( \omega_n \) and \( \omega'_n \).

Now for every (large) \( n \), the number of \( A \) points of \( \omega_n \) in \( U \) is \( l \), and the number of \( A \) points of \( \omega'_n \) in \( U \) is some (bounded) number other than \( l \). Since
the configurations are a small wobble of $\omega$ then, we can find $g_n \in \omega \cap U$ such that the corresponding points $x_n$ of $\omega_n$ and $y_n$ of $\omega'_n$ are an $A$ point and not an $A$ point, respectively.

As $g_n$ ranges over a finite set $\omega \cap U$, we can choose $g \in \omega \cap U$ and a subsequence $(n_k)$ such that $g_{n_k} = g$ for every $k \in \mathbb{N}$. Then

$$x^{-1}_{n_k} \omega_{n_k} \to g^{-1} \omega, \text{ and } y^{-1}_{n_k} \omega'_{n_k} \to g^{-1} \omega,$$

which witnesses that $g^{-1} \omega \in \partial A$, so $\omega \in G\partial A$, as desired. ■

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