Bound systems in an expanding universe

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The Schwarzschild solution insertion in an expanding universe model, the so called “Swiss cheese model,” is shown to possess a very unphysical property. Specifically, in this model some of the trajectories are discontinuous functions of their initial conditions. An alternate metric is proposed as a remedy. It goes smoothly between the Schwarzschild exterior solution and the Friedmann-Lemaître, expanding universe metric. It is further shown that the effects of the expansion on planetary motions in the solar system are too small to be currently observed for this alternate metric.

I. INTRODUCTION AND SUMMARY

Currently there are two general relativistic descriptions of spacetime in popular use. For planetary systems and other gravitationally bound structures which are small on the scale of the universe, there is a static description of the behavior of spacetime. On the other hand, for large-scale behavior, there is a time dependent description which is appropriate as a description of phenomena such as the observed red-shift of distant galaxies.

The classic question is, “How can these two disparate descriptions of spacetime possibly be reconciled with each other?” The current standard answer is that these two are meshed together on a spherical surface surrounding a mass concentration which grows with time in the static metric, but stays at a fixed coordinate radius in the non-static metric. This is the “Schwarzschild solution in a cosmological model” picture, or it is also called the “Swiss cheese model.” This later name refers to the fact that in this picture, the background material is removed inside the spherical boundary. The mass removed depends on the mass of the central concentration and the curvature of space. The replacement of the mass interior to the sphere by a concentration of mass at the center is based on Birkhoff’s theorem which says that in a homogeneous, zero pressure cosmological model, as long as the material inside a sphere is spherically symmetric, we can replace it with a compact mass at the center with no change on its exterior effects. To make this matching work, the pressure of the exterior solution (Friedmann-Lemaître metric class of the general Robertson-Walker line elements) must vanish, as does the pressure of the interior (the exterior Schwarzschild metric) solution. This condition places a restriction on the form of the universal expansion factor of the overall universe. Although the matching conditions can be met, as we shall see in the third section, an additional problem arises when the dynamics are considered.

In the second section, for the convenience of the reader, I gather together the necessary classical equations from general relativity for the study at hand. One non-classical item in this section is, instead of the usual 3 + 1 spacetime split into 3 space and 1 time dimensions, I split spacetime into one radial coordinate, and time plus the two angles of spherical coordinates. The change allows the direct computation of the relevant extrinsic curvatures.

In the third section, I compute the stress-energy tensors and the extrinsic curvatures for both the Schwarzschild and the Friedman-Lemaître metrics used in the “Swiss cheese model.” By a reparameterization of the Schwarzschild metric, both the intrinsic and the extrinsic curvature can be made to be continuous. I remark that the stress-energy tensor, for certain parameter choices, displays no pressure discontinuity but only a cosmic fluid density discontinuity. That discontinuity is in line with the “Swiss cheese model” idea that there are holes in the cosmic fluid.

In the fourth section, I show that it can happen, for trajectories which approach the metric interface an near grazing angles, that the subsequent trajectories are discontinuous functions of their initial conditions. Those which enter the inner or Schwarzschild metric region can be bound in a finite sized, closed orbit, while those which do not, travel on a parabolic trajectory to infinity. To emphasis, this case is not the same as in Newtonian orbit theory where ellipses of progressively larger size blend into parabolas, but here the ellipse is just finite in size!

In the fifth section, I introduce an alternative metric. This metric is basically an adaptation of the Schwarzschild metric in curved space. I compute the stress-energy tensor. It shows an isotropic pressure, and no mass-density flux. It is only second order in magnitude in both the Hubble constant and the inverse radius of curvature of the universe. The same statement is true of the Friedman-Lemaître stress energy tensor. In addition I have computed the extrinsic curvature. Both the stress-energy tensor and the extrinsic curvature are continuous, outside the Schwarzschild radius of course, as they come from infinitely differentiable expressions.

In the sixth section, I compute the equations of motion for a freely moving test particle in the alternative metric. I then transform them to a coordinate system at rest at the center of the mass concentration. The flat space,
slowly moving particle, weak gravitational field limit of these equations of motion are also given. The only correction to Newton’s equations of motion in this limit is a term proportional to the square of Hubble’s constant, $H_0$.

In the final section, I gives some examples of the dynamics found using my alternative metric.

II. METRICS

It is useful to review the properties of a few metrics. The line elements will all be of the general form,

$$ds^2 = -e^\mu(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + e^\nu(dx^4)^2$$

$$= g_{ij}dx^i dx^j,$$  \tag{2.1}

where the Einstein summation convention is used, and

where

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2,$$  \tag{2.2}

appropriate to the non-static, spherically symmetric case.

Eq.\,[2.3] defines the metric tensor $g_{ij}$. It will be of interest to know what Einstein field equation is satisfied for each of these metrics. The equation is,

$$R^l_{ij} - \frac{1}{2} R g^l_{ij} + \Lambda g^l_{ij} = -8\pi T^l_{ij},$$  \tag{2.3}

where $T_{ij}$ is the stress-energy tensor, and $R_{ij}$ is the Ricci tensor, which is a contraction of Riemann’s four index curvature tensor. The Ricci tensor can be expressed in terms of the three index Christoffel symbols $\Gamma$ as,

$$R_{km} = (\Gamma^l_{km})_{,i} - (\Gamma^l_{ki})_{,m} + \Gamma^l_{mn} \Gamma^n_{ki},$$  \tag{2.4}

where the notation $\frac{d}{dx^i}$ means take the partial derivative with respect to $x^i$. In turn, the Christoffel symbols are defined in terms of the metric tensor as,

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk} (g_{ki,j} + g_{kj,i} - g_{ij,k}),$$

where $g^{mk} g_{kj} = \delta_j^m$,  \tag{2.5}

defines $g^{km}$, and $\delta_j^m$ is the Kronecker delta function. Finally,

$$R = g^{ij} R_{ij},$$  \tag{2.6}

is the contraction of the Ricci tensor.

The $T^{4i}$ element is the mass-energy density. The $T^{4\beta}$ is the mass-flux density through an area perpendicular to the direction $\beta$ per unit time. (Greek indices run from 1 to 3 while Roman indices run from 1 to 4.) The $T^{\alpha\alpha}$ element is the pressure in the $\alpha$ direction, and the $T^{\alpha\beta}$ element is the flux density of the $\alpha$ component of momentum in the $\beta$ direction. It is manifest that eq.\,[2.3] allows the direct computation of the stress energy tensor from the metric tensor. Tolman \[4\] gives the results for the form of the line element

$$ds^2 = -e^\mu(dx^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + e^\nu dt^2,$$  \tag{2.7}

which is the spherical coordinate version or eq.\,[2.1]. The non-vanishing elements of the stress-energy tensor are

$$8\pi T^{1\alpha} = -e^{-\mu}\left(\frac{\mu''}{2} + \frac{\nu}{2} \right) + e^{-\nu} \left(\frac{\mu'}{2} - \frac{\nu}{2} \right) - \Lambda$$

$$8\pi T^{2\alpha} = 8\pi T^{3\alpha} = -e^{-\mu}\left(\frac{\mu''}{2} + \frac{\nu}{2} \right) + e^{-\nu} \left(\frac{\mu'}{2} - \frac{\nu}{2} \right) - \Lambda$$

$$8\pi T^{4\alpha} = -e^{-\mu}\left(\frac{\mu''}{2} + \frac{\nu}{2} \right) + e^{-\nu} \left(\frac{\mu'}{2} - \frac{\nu}{2} \right) - \Lambda$$

$$8\pi T^{14} = -e^{-\nu}\left(\frac{\mu'}{2} - \frac{\nu}{2} \right)$$

where $\Lambda$ is the cosmological constant, an over dot denotes the time derivative, and a prime denotes the derivative with respect to $r$.

Another quantity which is useful to consider is the extrinsic curvature in \[6\] This concept arises in the Arnol’d et al. \[7\] splitting of four dimensional spacetime into 3 dimensional space plus one dimensional time. The relationship between the metrics is given by

$$ds^2 = (4) g_{ij} dx^i dx^j$$

$$= (3) g_{\alpha\beta} (dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt) + N^2 dt^2,$$  \tag{2.9}

where the $N^\alpha$ are the three shift functions and $N$ is the lapse (of proper time) function. The intrinsic curvature is the analogue of $R$ [eq.\,[2.3]] in three dimensions. The extrinsic curvature measures the fractional shrinkage and deformation as one advances in time from one space-like hyperplane to the next. The extrinsic curvature tensor is given as

$$K_{\alpha\beta} = \frac{1}{2N} \left[ N_{\alpha|\beta} + N_{\beta|\alpha} - \frac{\partial g_{\alpha\beta}}{\partial t} \right],$$  \tag{2.10}

where the notation $\frac{d}{dx^i}$ means the covariant derivative with respect to $x^\alpha$.

The reason for the interest in the extrinsic curvature in our case is that the necessary and sufficient junction conditions \[8\] to join two metrics in 4 dimensional spacetime across a 3 dimensional hypersurface is that both $g_{\alpha\beta}$ and $K_{\alpha\beta}$ be continuous across the surface. In our case we are concerned with line elements of the class of eq.\,[2.7]. The split is between $r$ instead of $t$ and the other three variables. In this case the shift functions are all zero, and the
lag function is \( N = \exp(0.5\mu) \). The non-zero elements of \( K \) are
\[
K_{22} = \frac{1}{2}(\mu' r^2 + 2r) \exp(0.5\mu), \\
K_{33} = \frac{1}{2}(\mu' r^2 + 2r) \sin^2 \theta \exp(0.5\mu), \\
K_{44} = -\frac{1}{2} \nu' \exp(\nu - 0.5\mu).
\]

Returning to the three-space, constant-time formalism, we give the equations of motion of a free test particle as seen by local co-moving observers along the test particle’s path. That is to say, a set of observers whose coordinates do not change with time. In other words, we need the equations for the geodesic curves in spacetime. For the class of line elements we are considering, it is simplest to start with a Lagrangian formulation. By eq. 2.1 we may write this formulation as,
\[
s = \int L \, dt \Rightarrow L = \frac{ds}{dt} = (g_{ij} \dot{x}^i \dot{x}^j)^{1/2}
\]
The standard Euler-Lagrange equations for an extreme in path length (Here we seek a minimum distance between two fixed endpoints.) are
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0.
\]
Thus, using the diagonal nature of the metric tensor, we obtain for the standard geodesic equations,
\[
\frac{d}{dt} \left( \frac{ds}{dt} \right)^{-1} \dot{x}^i = \frac{1}{2} g_{ij,\alpha} \frac{dx^i}{dt} \frac{dx^j}{dt}.
\]
where, of course \( \frac{dx^4}{dt} = 1 \). From the line element we get
\[
\left( \frac{ds}{dt} \right)^{-1} = \left[ e^\nu + g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \right]^{1/2}
\]
These results display, for the class of line elements we are considering, the three, second-order, non-linear, coupled equations for the three coordinates \( x^\alpha \) of a test particle as a function of time. We have assumed isochronous coordinates everywhere in the three-dimensional, space-like hyper-surface. The square roots can removed from eq. 2.14 by rewriting it as
\[
\frac{d}{dt} \left[ \frac{ds}{dt} \right]^2 \frac{d}{dt} \left[ \frac{ds}{dt} \right]^{-2} \left[ g_{\alpha i} \dot{x}^i \right] + \frac{d}{dt} \left[ g_{\alpha i} \dot{x}^i \right] = \frac{1}{2} g_{ij,\alpha} \frac{dx^i}{dt} \frac{dx^j}{dt}.
\]

III. “SWISS CHEESE MODEL”

As mentioned in the first section, there is a popular cosmological model which in the large has the Friedmann-Lemaître line element,
\[
ds^2 = \frac{a(t)^2}{(1 + \frac{r}{2R})^2} \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + c^2 (dx^4)^2
\]
where in the notation of Sec. II,
\[
e^\mu = \frac{a(t)^2}{(1 + \frac{(a(t)r)}{2a(t)R})^2}, \quad e^\nu = c^2.
\]
where \( c \) is the velocity of light and \( a(t)R \) is the radius of curvature of the model universe.

The non-zero elements of the stress-energy tensor associated with this line element are, by eq. 2.8,
\[
8\pi T_1^1 = \frac{8\pi T_2^2}{\frac{3}{a(t)R} + 3 \left( \frac{\dot{a}}{ac} \right)^2 - \Lambda = 8\pi \rho_0},
\]
\[
8\pi T_4^4 = 3 \left( \frac{\dot{a}}{ac} \right)^2 - \Lambda = 8\pi \rho_{400}.
\]
The extrinsic curvature tensor, as given by eq. 2.11 for this metric has the non-zero elements,
\[
K_{22} = \frac{a(t)r}{1 + (\frac{r}{2R})^2}, \quad K_{33} = \sin^2 \theta K_{22}.
\]

In this cosmological model on scales small compared to that of the universe, as in, for example, the solar system, a quite different metric is used. It is Schwarzschild’s exterior solution. The line element for it is
\[
ds^2 = - \left( 1 + \frac{m}{2r} \right)^4 \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + c^2 \left( \frac{1 - m/2r}{1 + m/2r} \right)^2 (dx^4)^2,
\]
where in the notation of Sec. II,
\[
e^\mu = \left( 1 + \frac{m}{2r} \right)^4, \quad e^\nu = c^2 \left( \frac{1 - m/2r}{1 + m/2r} \right)^2.
\]
Here \( m \) is an abbreviation for \( GM/c^2 \) where \( M \) is the mass concentration. This metric is only valid outside the Schwarzschild radius \( r >> r_0 = 2GM/c^2 \). Inside this radius, Schwarzschild’s interior solution is required but we shall not be concerned with this aspect here.

There are no non-zero elements of the stress-energy tensor associated with this line element, unless the cosmological constant is different from zero. Thus it corresponds to zero pressure and zero density, except for a mass concentration in the center. If \( \Lambda \neq 0 \) then,
\[
T_1^1 = T_2^2 = T_3^3 = T_4^4 = -\Lambda/8\pi
\]
This result would correspond to a uniform density and pressure through out space, rather than the empty space with \( \Lambda = 0 \).

A cross comparison of eq. 3.7 with eq. 3.3 shows that unless,
\[
\frac{1}{(a(t))^2} + \left( \frac{\dot{a}}{ac} \right)^2 = \frac{\dot{a}}{ac^2} = 0, \tag{3.8}
\]
the stress-energy tensor is discontinuous at the boundary. The only solution of these equations, as \( R \) is a constant, is \( \dot{a} \) is a constant, i.e., a linear expansion factor, or a static flat universe.

Before we can us eq. 2.11 to compute extrinsic curvature for the Schwarzschild case, we must first reparameterize the metric so that the “Swiss cheese model” metric boundary is given by one of the coordinates equals a constant. It will be convenient to use spherical coordinates. If we follow the textbook approach, then we should start with a zero pressure cosmological model. By Birkoff’s theorem in this case we may hollow out a sphere and replace it by a mass concentration at the center. Since the density of the “cosmic fluid” is inversely proportional to \( a^3 \), the radius of the sphere is fixed in the Friedmann-Lemaître coordinates. The surface of the sphere is a three dimensional hypersurface parameterized by the time and the two angle variables of spherical coordinates. To proceed, we note that an observer sitting on the boundary is on a geodesic for \( t_{FL} = \) a constant for all time. The relation between the Friedmann Lemaître time \( \bar{t} \) and the Schwarzschild variables as seen by this observer is
\[
(ds)^2 = c^2 (dt_e)^2 \tag{3.9}
\]
\[
= \left[ c^2 \left( \frac{1 - \frac{m}{2r_e}}{1 + \frac{m}{2r_e}} \right)^2 - \left( 1 + \frac{m}{2r_e} \right)^4 \left( \frac{dr_e}{dt_e} \right)^2 \right] (dt_e)^2.
\]
Since the observer is on a geodesic, we may deduce, by means of the time component of eq. 2.14 that he sees
\[
\dot{c} \left( \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right)^2 \frac{dt}{ds} = K^{1/2}, \tag{3.10}
\]
where \( K \) is a constant of integration. Note is taken that, as the integration is over \( t \), \( K \) may depend on \( r \), of course. By combining eq. 3.9 and eq. 3.10, we obtain,
\[
\frac{dr_e}{dt_e} = \frac{c}{\left(1 + \frac{m}{2r_e}\right)^3} \left[ 1 - \frac{1}{K} \left( 1 - \frac{m}{2r_e} \right)^2 \right]^{1/2},
\]
\[
\frac{dt_e}{dt_e} = K^{1/2} \left( \frac{1 + \frac{m}{2r_e}}{1 - \frac{m}{2r_e}} \right)^2, \tag{3.11}
\]
which gives the behavior of \( r_e, t_e \) as a function of the Friedmann-Lemaître time variable, as seen by the comoving observer sitting on the metric boundary in the Swisscheese model.

By equating the coefficients of the angular variables, we obtain,
\[
\frac{1 + \frac{m}{2r_e}}{r_e} \frac{dr_e}{dt_e} = \frac{a(t_e) \bar{r}_e}{1 + \left( \frac{r_e}{2R} \right)^2},
\]
\[
\frac{\dot{a}(t_e)}{a(t_e)} = \frac{1}{r_e} \frac{dr_e}{dt_e} \left( \frac{1 - \frac{m}{2r_e}}{1 + \frac{m}{2r_e}} \right) \frac{dt_e}{dt_e}.
\]
Note is taken that the boundary coordinate \( \bar{r}_e \) in the Friedmann-Lemaître metric depends on the central mass concentration, and on the radius of curvature of the universe, i.e. on \( m \) and \( R \).

Next we combine eqs. 3.11 and 3.12 and comparing the result with the “cosmological equation.” This equation is the equation for the \( T_4^4 \) component of the stress energy tensor derived from the Friedmann-Lemaître line element. The result of the application of this textbook method is that the constant of integration is
\[
K = \left[ 1 - \left( \frac{\bar{r}_e}{2R} \right)^2 \right]^{2} \left[ 1 + \left( \frac{\bar{r}_e}{2R} \right)^2 \right]^{-2}, \tag{3.13}
\]
which is a function of \( m \) and \( R \).

We are now in a position to introduce a reparameterization of the Schwarzschild metric. We choose the new parameters \( t = \bar{t} \) and \( \bar{r} \). The second variable is chosen so as to keep the metric continuous at the interface, and to keep the metric diagonal, if possible. The reparameterized Schwarzschild metric is given by eq. 5.1. Eq. 3.9 insures that \( g_{44} = c^2 \). The equation of continuity, and the vanishing of the elements \( \bar{g}_{14} = \bar{g}_{41} \) yield the conditions,
\[
\left( \frac{dr_e}{dt_e} \right)^2 g_{11} + \left( \frac{dr_e}{dt_e} \right)^2 g_{44} = \bar{g}_{11},
\]
\[
\left( \frac{g_{11}}{dt_e} \right) \frac{dr}{d\bar{r}} + \left( \frac{dt}{dt_e} \right) g_{44} \frac{dt}{d\bar{r}} = 0 \tag{3.14}
\]
The solution of these equations is
\[
\frac{dr}{d\bar{r}} = \frac{K^{1/2} a(t)}{\left(1 - \frac{m^2}{4r_e^2}\right) \left[ 1 + \left( \frac{\bar{r}_e}{2R} \right)^2 \right]},
\]
\[ \frac{dt}{d\tilde{r}} = e \left[ 1 + \frac{K^{1/2} a(t) \left( 1 + \frac{m}{2r_c} \right)^3}{1 + \left( \frac{r_c}{2R} \right)^2} \right]^{1/2} \]  

\times \left[ 1 - \frac{1}{K} \left( 1 + \frac{m}{2r_c} \right) \right] \left( \frac{1 - m^2}{4r_c^2} \right)^2 a(t)^2 \left( \frac{1 - m^2}{4r_c^2} \right)^2 \left[ 1 + \left( \frac{r_c}{2R} \right)^2 \right]^{1/2} \right], \quad (3.15)\]

\[ \hat{g}_{11} = - \left( \frac{1 - m^2}{4r_c^2} \right)^2 \left[ 1 + \left( \frac{r_c}{2R} \right)^2 \right]^{1/2} \]

\[ \hat{g}_{11} = - \frac{\hat{r}}{R^2 + 0.25r^2}, \]

\[ d\log(-\hat{g}_{11}) = -\frac{\hat{r}}{R^2 + 0.25r^2}, \quad d\log(\hat{g}_{11}) = -\frac{a(t) m}{r_c} \left[ 1 - \left( \frac{r_c}{2R} \right)^2 \right] \left( 1 - \frac{m^2}{4r_c^2} \right)^3 \left[ 1 + \left( \frac{r_c}{2R} \right)^2 \right] \]  

To leading order in a flat universe, eq. (3.16) becomes,  

\[ \frac{d\log(-\hat{g}_{11})}{dr} = 0, \quad \frac{d\log(\hat{g}_{11})}{dr} = -a(t) \frac{m}{r_c}. \quad (3.17) \]

That such a discontinuity may occur is well known [1]. I will explore some of the consequences of this discontinuity in the next section.

As was remarked above, it is well known that it is a necessary condition that the pressure equal zero in order for this matching to occur [1]. As the pressure p = −T^\alpha_\alpha for each \alpha (not summed here), the zero-pressure condition means, by eq. (3.7) which gives the pressure for the Schwarzschild case in terms of the T^\alpha_\alpha elements that Λ = 0. Turning to the the Friedmann-Lemaitre case, we see from eq. (3.3) that it must be that

\[ \frac{1}{a(t)R^2} + 2\frac{\ddot{a}}{ac^2} + \left( \frac{\dot{a}}{ac} \right)^2 = 0. \quad (3.18) \]

If use substitute the standard form a(t) = (t/t_0)^\psi in eq. (3.18) then the only solutions are R = ∞ together with \psi = 0 or 2/3. The first solution is the trivial case of a static universe and is of no interest in the present discussion. In the second case, the stress-energy tensor element T^t_t > 0, which means there is a mass discontinuity. This is expected by the structure of the approximation of scooping out a hollow sphere, and then having a uniform density outside.

### IV. DYNAMICS IN THE “SWISS CHEESE MODEL”

Following up on the discontinuity in the derivate of the metric, perpendicular to the metric boundary in the “Swiss Cheese Model,” we investigate the some of the dynamic properties of this model. It suffices for my purposes to consider the simpler case of a flat (R = ∞) universe. The equations of motion are as follows. For the Schwarzschild metric, we get from eq. (2.14) the well known Newtonian equations

\[ \frac{d^2 \rho}{dt^2} = -\frac{GM}{\rho} \hat{\rho} \quad (4.1) \]

The solutions to this equation are the familiar Newtonian conic sections.

For the Friedmann-Lemaitre metric from eq. (2.14) we get a first integral as

\[ \dot{x} = \frac{\dot{a}(t)}{a(t)} \hat{\rho} \quad (4.2) \]

For my purposes, it is more convenient to use a variable more closely equal to the proper distances. Thus for \hat{\rho} = a(t)\hat{\rho}, eq. (4.2) becomes,

\[ \frac{d^2 \rho}{dt^2} = \frac{\dot{a}(t)}{a(t)} \hat{\rho} \quad (4.3) \]

In both metrics \rho is within plotting accuracy for the proper distance in the examples I will consider.

The general solution of eq. (4.3) is

\[ \dot{\rho} = \ddot{A} + \frac{\dot{B}}{t - \psi} = \ddot{A} t^{2/3} + \frac{\dot{B}}{t^{1/3}}, \quad (4.4) \]

when the currently, theoretically favored value of \psi = 2/3 is chosen. Let us take the example in rectangular coordinates where initially \( x = \lambda, \dot{x} = 0, y = 0, \dot{y} = \lambda \) at time \( t = t_0 \). Then eq. (4.4) becomes,

\[ x = \lambda \left[ \left( \frac{t}{t_0} \right)^{2/3} - 2 \left( \frac{t}{t_0} \right)^{1/3} \right], \]

\[ y = 3t_0 \lambda \left[ \left( \frac{t}{t_0} \right)^{2/3} - \frac{1}{\left( \frac{t}{t_0} \right)^{1/3}} \right]. \quad (4.5) \]

It is worth noting that for \( t - t_0 \ll t_0 \) that the motion is almost exactly that of a straight line, as is to be expected in flat empty space. Specifically, direct calculation yields

\[ \ddot{x}(t_0) = -\frac{2\lambda}{9t_0^2}, \quad \dot{y}(t_0) = 0. \quad (4.6) \]
This apparent acceleration is of order $H_0^2$.

The solutions in eq. 4.4 are quadratic in the parameter $t^{1/3}$. Thus we can obtain the equation for the trajectory as a quadratic plus linear expression in $\rho_x$ and $\rho_y$. The form would be

$$\rho_y = c(a\rho_x + b\rho_y)^2 + d(a\rho_x + b\rho_y) \quad (4.7)$$

This form is readily recognized as a parabola. Thus it is the case that a freely moving particle in an Friedmann-Lemaître expanding space always appears to be moving in a parabola. This effect is caused by the small $\ddot{a}$ term which appears in the equation 4.3 as a forcing term.

As an illustration of the behavior of the “Swiss cheese model” I have computed the following trajectories. I use as a unit of time the Hubble time, that is $1/H_0$, which is of the order of $10^{18}$ seconds. As a unit of distance I use $\sqrt{M_\odot G/H_0^2}$ which is about 20 million Astronomical units. $M_\odot$ is the mass of the sun. I set $t_0 = 1$ to switch to our current units. The metric interface is at $r = t^{2/3}$ in these units for flat spacetime, as mentioned above. In order to follow a Friedmann-Lemaître trajectory the test particle’s distance from the origin must be larger at every time than that for the interface. Thus,

$$\lambda^2 \left[ 10t^{4/3} - 22t + 13t^{2/3} \right] > t^{4/3}, \quad (4.8)$$

The trajectory will intersect the interface if eq. 4.8 is an equality. By means of the quadratic formula, an intersection will occur if

$$t^{1/3} = \frac{11 \pm \sqrt{-9 + 13/\lambda^2}}{10 - 1/\lambda^2} \quad (4.9)$$

It will be observed that for $\lambda < \sqrt{13}/3$ there are two real roots. If $\lambda = 1$, then $t^{1/3} = 1, 13/9$. On the other hand, if $\lambda > \sqrt{13}/3$, the roots are imaginary, so there are no intersections. If $\lambda = \sqrt{13}/3$ there is a double root at $t^{1/3} = 13/11$. In this case the parabolic trajectory just grazes the metric interface.

In Fig. 1, I illustrate the two different trajectories when $r_0 = \sqrt{13}/3 + \epsilon$ (the Friedman-Lemaître case), and begin at $x_0 = \sqrt{13}/3 - \epsilon$ (the Schwarzschild case). Here, $\epsilon > 0$ may be chosen as small as one pleases. The initial part of both trajectories is a parabola generated by the Friedmann-Lemaître metric. At time $t = (13/11)^3$, marked in the figure, the two trajectories separate.

Put another way, the Schwarzschild metric permits closed orbits and the Friedmann-Lemaître metric does not. In terms of the latter coordinates, one can choose initial conditions so that the parabolic trajectory just grazes the metric interface (fixed radial coordinate in this metric) and the speed is low enough that for infinitesimally different initial conditions the trajectory crosses the interface and is caught in a bound state, or alternatively misses the interface and proceeds on its parabolic trajectory. All these effects take place in supposedly
empty space of the order of 20 million AU from a mass concentration of size $M_\odot$, and are quite counter to one’s physical intuition that such discontinuities should not occur there.

V. AN ACCEPTABLE METRIC

I propose the following line element to represent a mass condensation in an expanding and curved universe. It is of the form 2.7 where

$$e^\mu = \frac{a(t)^2}{1 + (a(t)r/2a(t)R)^2} \left[ 1 + \frac{m}{2a(t)r} \right]^4, \quad (5.1)$$

$$e^\nu = c^2 \left( \frac{1 - m/2a(t)r}{1 + m/2a(t)r} \right)^2, \quad m \equiv \frac{GM}{c^2} \left[ 1 + \left( \frac{r}{2R} \right)^2 \right]^{1/2}$$

where $G$ is Newton’s constant of gravitation. It is to be noted that this metric is an adaptation of the Schwarzschild metric in curved space. It is not claimed that this metric is unique. Certainly any coordinate transformation of this metric is equivalent. It does however have the property that in the limit where $a(t)$ is a constant, it reduces to the Schwarzschild metric in curved space. Also, when the central mass vanishes it reduces to the Friedmann-Lemaître metric. It is an infinitely differentiable solutions to the Einstein field equations which correspond to values of the stress-energy tensor which are only of second order in $R^{-1}$ and $H_0$, except at the central mass. This latter property is also true of the Friedmann-Lemaître metric in curved space, as may be seen in eq. 2.8. Thus we have, by this example whose properties are globally similar to the Friedmann-Lemaître metric, demonstrated that the Swiss cheese model is not required to match the expansion of the universe observed at large scales and the absence of any such expansion observable in the solar system. As shown in the previous section since the Swiss cheese model has a rather severe and very unphysical deficiency, we think it is time to begin the search for an acceptable metric.

The non-zero elements of the extrinsic curvature is given by (2.11) as

$$K_{22} = a(t)r \frac{[1 - (r/2R)^2]}{[1 + (r/2R)^2]^2} \left[ 1 + \left( m/2a(t)r \right)^2 \right]$$

$$K_{33} = \sin^2 \theta K_{22}, \quad (5.2)$$

$$K_{44} = - \frac{mc^2(1 - m/2a(t)r)}{a(t)^2r^2(1 + m/2a(t)r)^3}$$

In the limit that $m \to 0$ these curvatures reduce to those of eq. 2.7 and in the limit $R \to \infty$ and $\dot{a} = 0 (a = 1)$ they reduce to

$$K_{22} = r \left[ 1 - \left( \frac{m}{2r} \right)^2 \right], \quad K_{33} = \sin^2 \theta K_{22},$$

$$K_{44} = - \frac{mc^2(1 - m/2r)}{r^2(1 + m/2r)^3} \quad (5.3)$$

These curvatures agree with those of the unparameterized Schwarzschild metric. The curvatures in eq. 5.2 differ only in that $r$ is replaced by $a(t)r$, $R$ by $a(t)R$, and there are corrections for the overall curvature of space. Thus, it reflects, as does the metric, the very same behavior, in terms of $a(t)r$ instead of $r$ as was found by Schwarzschild for his metric.

Both the metric and the extrinsic curvature are continuous outside the Schwarzschild radius, which is necessary for a metric to be acceptable.

We now turn to the stress-energy tensor. The current metric shares with both the Friedmann-Lemaître metric, eq. 3.1, and the Schwarzschild metric, eq. 3.5, the property that $T^1_1 = T^4_4 = 0$, so there is no mass-density flux. This property follows by direct computation from eq. 2.8 and eq. 5.1. For the other non-zero components, we find

$$8\pi T^1_1 = \frac{1 + (m/2a(t)r)^2}{a(t)^2R^2(1 + ma(t)r)(1 - ma(t)r)} + 2\frac{a(ma(t)r + 3a^2[1 - m/2a(t)r])}{a(t)^2c^2(1 - ma(t)r) - \Lambda}$$

$$8\pi T^2_2 = 8\pi T^3_3 = 8\pi T^1_1$$

$$8\pi T^4_4 = \frac{3}{a(t)^2R^2(1 + ma(t)r)^3} + 3\frac{a^2}{c^2a^2} - \Lambda$$

In the limit as $m \to 0$ or $r \to \infty$ these tensor elements reduce to those of eq. 3.3 and also in the limit as $R \to \infty$ and $\dot{a} \to 0$ they vanish as with the Schwarzschild metric, unless $\Lambda \neq 0$, in which case the result (3.7) is obtained. The dominate terms in the diagonal elements (pressure and density) are a sum of terms of second order in $1/R$ and terms containing two time derivatives of the universal expansion factor $a(t)$. The metric given by eqs. 5.1 and 2.7 is the solution of Einstein’s field equations eq. 2.3 when the stress-energy tensor, eq. 5.4 is specified.

VI. DYNAMICS

In this section we investigate the equations of motion a test particle in our proposed metric as described by eq. 2.7 and eq. 5.1. Our treatment diverges from that of McVittie [8] at this point as he directly substitutes $\rho = a(t)r$ into the line element rather than using the transformation equations

$$\ddot{g}_{kl} = \frac{\partial x^i}{\partial x^l} \frac{\partial x^j}{\partial x^k} \theta_{ij}, \quad (6.1)$$

The correct change of the line element for this change of variables is

$$ds^2 = -e^{\nu} [(d\rho)^2 + \rho^2(d\phi)^2 + \rho^2 \sin^2 \theta (d\phi)^2]$$

$$+ \left[ e^\nu - e^{\nu} \left( \frac{\rho a}{\dot{a}} \right)^2 \right] (d\tau)^2 + 2e^\nu \left( \frac{\rho a}{\dot{a}} \right) d\rho dr, \quad (6.2)$$
where
\[ e^U = \frac{(1 + m/2\rho)^4}{[1 + (\rho/2a(t)R)]^2} \]
\[ e^V = c^2 \left( \frac{1 - m/2\rho}{1 + m/2\rho} \right)^2 \] (6.3)

Clearly these are non-synchronous coordinates as the coefficient of \( dp/dt \) is non-zero.

We will make this change of variables later after the equations of motion have been derived. The main difference is that McVittie omits the \( \dot{\mu} \) terms from his subsequent equations. The equations of motion then become, using eq. 2.14 for \( \theta \) and \( \phi \),
\[ \left( \frac{ds}{dt} \right)^2 \left[ \left( \frac{ds}{dt} \right)^{-1} e^{\mu} r^2 \dot{\theta} \right] = e^{\mu} r^2 \sin \theta \dot{\phi}^2 \]
\[ \left( \frac{ds}{dt} \frac{d}{dt} \left[ \left( \frac{ds}{dt} \right)^{-1} e^{\mu} r^2 \sin \theta \dot{\phi} \right] \right) = 0 \] (6.4)

One can see by inspection, the motion in the plane \( \theta = \pi/2 \) is a solution. The integral of the second equation gives the result,
\[ r^2 \dot{\phi} = Ae^{-\mu} \left( \frac{ds}{dt} \right), \] (6.5)

where \( A \) is a constant of integration. This equation is the conservation of angular momentum in this coordinate system. To obtain the equation of motion for \( r \), it is convenient to use eq. 2.10. We obtain,
\[ -\frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{d}{dt} \left[ \left( \frac{ds}{dt} \right)^{-2} e^{\mu} \dot{r} \right] = -\frac{1}{2} \left[ \mu \dot{\mu} e^{\mu} r^2 + (\mu r^2 + 2r) e^{\mu} \dot{\theta}^2 \right. \]
\[ \left. + (\mu r^2 + 2r) e^{\mu} \sin^2 \theta \dot{\phi}^2 - \nu' e^{\nu} \right], \] (6.6)

where, from the line element,
\[ \left( \frac{ds}{dt} \right)^2 = -e^{\mu} \left[ \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right] + e^{\nu}. \] (6.7)

At this point, we simplify to motion in the \( \theta = \pi/2 \) plane. Thus \( \dot{\theta} = \dot{\phi} = 0 \), and eq. 6.6 becomes,
\[ \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{d}{dt} \left[ \left( \frac{ds}{dt} \right)^{-2} \right] \dot{r} + e^{-\mu} \frac{d}{dt} \left[ e^{\mu} \dot{r} \right] = \frac{1}{2} \left[ \mu' \dot{r}^2 + (\mu' r^2 + 2r) \dot{\phi}^2 - \nu' e^{\nu - \mu} \right]. \] (6.8)

These formulas yield the equations of motion of a test particle in the reference frame which is at rest with respect to the coordinate system at the location of the test particle. We want to find the motion with respect to an observer at rest at \( r = 0 \). To this end we introduce the change of variables,
\[ r = \frac{\rho}{a(t)}, \quad \dot{r} = \frac{\rho}{a(t)} - \frac{\dot{a}(t)\rho}{a(t)^2} \] (6.9)

This change of variables yields coordinates which are equal to the proper distances, as viewed from the origin, when the mass concentration is absent. With the substitution 6.9 we obtain for eq. 6.3 and eq. 6.7
\[ \rho^2 \dot{\phi} = Aa(t)e^{-\mu} \left( \frac{ds}{dt} \right), \] (6.10)
\[ \left( \frac{ds}{dt} \right)^2 = -e^{\mu} \frac{a(t)^2}{a(t)} \left[ \left( \frac{\rho}{a(t)} - \frac{a(t)}{a(t)} \right)^2 + \rho^2 \dot{\phi}^2 \right] + e^{\nu}. \] (6.11)

Finally, eq. 6.8 becomes,
\[ \frac{\rho - 2\dot{a}\rho + \rho}{a^2} \left[ 2 \left( \frac{\dot{a}}{a} \right)^2 - \frac{\rho}{a^2} \right] + \dot{\mu} \left( \frac{\rho}{a} - \frac{a}{a} \rho \right) \]
\[ = \frac{1}{2} \left[ \frac{\mu'(a^2 - a^2) + 2}{a^2} \left( 2 \mu' a^2 + 2 \right) \right] \left[ A^2 a^4 e^{-2\mu} \right] \left( \frac{ds}{dt} \right)^2 \]
\[ - \frac{1}{2} \rho'(a e^{-\mu} - 1) \frac{ds}{dt} \frac{d}{dt} \left[ \left( \frac{ds}{dt} \right)^{-2} \right] \left( \frac{\rho}{a} \right) \] where we have used eq. 6.10 to eliminate the \( \dot{\phi} \) dependence. There is also a \( \dot{\phi} \) in \( ds/dt \) but it too can be eliminated by the substitution of \( \dot{\phi} \) from eq. 6.10 in eq. 6.11.

To assess the importance of the various terms it is helpful to note the following dimensionless quantities, in “planetary units,”
\[ T_\oplus H_0 \approx 5 \times 10^{-11}, \quad \left( \frac{v_\oplus}{c} \right)^2 \approx 1 \times 10^{-8}, \]
\[ \frac{GM_\oplus}{c^2 R_\oplus} \approx 1 \times 10^{-8}, \quad \left( \frac{R_\oplus}{R_{Hubble}} \right)^2 \approx 0.6 \times 10^{-30}, \] (6.13)

where \( T_\oplus, v_\oplus, R_\oplus \) are the orbital period, velocity, and radius of the earth, and \( M_\oplus \) is the mass of the sun. Some limiting cases are of interest. First, we take the flat-space, slow-speed limit, \( i.e., R = \infty \), and
\[ \left( \frac{ds}{dt} \right)^2 \approx c^2 \left( 1 - \frac{GM}{2c^2 \rho} \right)^2. \] (6.14)

Thus eq. 6.12 reduces to
\[ \frac{\rho}{a} - \frac{\dot{a}}{a} \rho - \left( \frac{\rho}{a} - \frac{\dot{a}}{a} \rho \right) \frac{m(3 - m/\rho)}{a \rho [1 - (m/2\rho)]^2} \]
\[ = -\frac{m(\dot{\rho} - \dot{a}\rho)^2}{\rho^2 (1 + m/2\rho)} + \frac{A^2 c^2 (1 - m/2\rho)^3}{\rho^3 (1 + m/2\rho)^{11}} \frac{mc^2 (1 - m/2\rho)}{\rho^7} \] (6.15)

We may further reduce these equations by discarding terms which are proportional to \( c^{-2} \) for when the velocities are much less than the speed of light. By eq. 6.1 these
are the terms proportional to $m$ alone, but we retain, of course, the terms in $mc^2$. Eq. (6.15) reduces further to

$$\dot{\rho} - \frac{\ddot{a}}{a} \rho = \frac{A^2c^2}{\rho^3} - \frac{GM}{\rho^2} = \frac{GM}{\rho^2} \left( \frac{\rho_0}{\rho} - 1 \right). \tag{6.16}$$

The coefficient of the discarded term on the left-hand side of eq. (6.15) is of the order of $10^{-19}$ as it is the product of two first order corrections. The discarded (first) term on the right-hand side of eq. (6.15) is smaller by a factor of $\rho^2/c^2$ than the last term. The other items discard are factors of $GM/c^3 \rho$ smaller than the dominant terms.

It is to be noticed that in the limit of eq. (6.16), that the term equals that for the sun's mass, $2\pi/\phi$, and Petrosian [9] do take account of this term. The constant $\rho_0$ is just another form of the constant of integration $A$. The magnitude of the $\ddot{a}$ term equals that for the sun’s gravity at about 0.5 kiloparsecs. The effects on the scale of the solar systems are too small to be measured.

**VII. EXAMPLES**

The equations of motion in a flat, Friedmann-Lemaître expanding universe for a slowly moving test particle under no external forces are, by eq. (6.16) and the corresponding reduction of eq. (6.4),

$$\dot{\rho} - \frac{\ddot{a}}{a} \rho = -\frac{\psi(1 - \psi)}{t^2} \rho, \quad \frac{d}{dt} \left( \rho^2 \dot{\phi} \right) = 0, \tag{7.1}$$

for the standard form of the universal expansion factor for the universe, as describe at the end of section III. $t$ is the current age of the universe. One easily recognizes these equations to be just exactly Newton’s equations in a plane in spherical coordinates. If we change to rectangular coordinates, we get exactly eq. (5.4). The behavior of the solutions is discussed in detail in Section IV above.

Next we consider the case where we add gravitational effects to their leading order. For purely radial motion, the $A$ of eq. (6.10) is zero. Thus the equation of motion (6.16) becomes,

$$\dot{\rho} = \frac{\dot{a}(t)}{a(t)} \rho - \frac{GM}{\rho^2} \tag{7.2}$$

This equation differs from the Schwarzschild metric equation, (4.1) by the addition of a term in $\dot{a}$. We will consider this behavior over time periods short compared to the age of the universe. If we multiply by $d\rho$ and integrate, we get,

$$\frac{1}{2} \dot{\rho}^2 = \frac{\ddot{a}}{2a} \rho^2 + \frac{GM}{\rho} + E_0 \tag{7.3}$$

where $E_0$ is the constant of integration. If $\ddot{a} = 0$, then $E_0 = 0$ would correspond to a test particle which had zero velocity at $\rho = \infty$. I choose to examine this special case. Then eq. (7.3) becomes

$$\frac{1}{2} \dot{\rho}^2 = \frac{\ddot{a}}{2a} \rho^2 + \frac{GM}{\rho} + \frac{\rho_0^2}{\rho^2} \tag{7.4}$$

by Pierce’s tables [10]. Thus,

$$\rho^{3/2} = \rho_0^{3/2} + \sqrt{-2GMa/\dot{a}} \sin \left( \frac{1}{2} \left( t - t_0 \right) \sqrt{-\frac{\ddot{a}}{a}} \right) \tag{7.5}$$

$$\approx \rho_0^{3/2} + \frac{3}{2} \sqrt{2GMa} \left( t - t_0 \right) + \frac{3}{8a} (t - t_0)^2 \cdots \right]. \tag{7.6}$$

It is evident from this solution that the leading order corrections due to the expansion of the universe are of the order $H_0^0 (t - t_0)^2$ which is extremely small on the planetary time scale. The solution when $(t - t_0) = O(t_0)$ would take account of the time dependence of $\dot{a}/a$. The more general case, where $E_0 \neq 0$, can also be integrated in terms of elliptic functions of the first and third kinds [11].

Next we investigate bound circular motion. To do so, I set $\dot{\rho} = 0$ in eq. (6.10) which gives,

$$\frac{A^2c^2}{\rho^3} = \frac{GM}{\rho^2} + \frac{\ddot{a}}{a} \rho. \tag{7.6}$$

If I use eq. (6.10) to reintroduce $\dot{\phi}$, and remember that the period $T = 2\pi/\dot{\phi}$, then I find,

$$\frac{A^2c^2}{\rho^3} = 1 + \frac{\ddot{a} \rho^3}{GM}, \tag{7.7}$$

which is Kepler’s law relating the square of the period to the cube of the radius, with a correction caused by the expansion of space. For the case $a(t) \propto t^{2/3}$ the correction term becomes, $-\frac{1}{3} H_0^2 \rho^3 / GM$. If we use the solar mass, then

$$\frac{4\pi^2 \rho^3}{GMT^2} = 1 - 3.307 \times 10^{-23} h_{50}^2 \rho^3, \tag{7.8}$$

where $\rho$ is in astronomical units and $h_{50} = 1$ when $H_0 = 50$ km per second per megaparsec.
These results are illustrated in Fig. 2. In flat space at time $t_0$, the interface between the static Schwarzschild metric and the non-static Friedmann-Lemaître metric is at distance unity from the origin, in the units of section IV. We display the Schwarzschild result when started with unit angular velocity ($\dot{\phi} = 1$). It is a circle of radius unity. Next we display the trajectory just outside the interface. It is, as expected, a parabolic curve. I have started it with unit angular velocity (again $\dot{\phi} = 1$) at a distance of 1.25 from the mass concentration. It is to be noted that after a unit (Hubble) time has passed, the two trajectories are significantly separated.

The reason for starting it further out is, as explained in section IV, that if I were to have started it at a distance between 1.0 and $\sqrt{\frac{13}{3}} \approx 1.2018504$, the trajectory would have been over taken by the expanding spherical interface between the two metrics of the “Swiss cheese model.”

In addition I show the trajectory using the presently considered metric. I have again started with unit distance and unit angular velocity. The result here is that it converges fairly quickly to an elliptical trajectory. In this case I find, using the standard equations,

$$v^2 = GM \left( \frac{2}{r} - \frac{1}{A} \right), \quad \frac{1}{2} ru^2 = A = \frac{1}{2} B \sqrt{\frac{GM}{A}},$$

(7.9)

where $A, B$ are the major and minor semi-axes, and the areal velocity $A$ is a constant by the conservation of angular momentum. In this case I find $A \approx 1.0437$ and $B \approx 1.0216$. In as much as the semilatus rectum parameter $p = 1$, the latus rectum itself is clearly defined by the intersection of this ellipse with the unit circle (Schwarzschild trajectory). The latus rectum is the line through the focus (origin in this case) which is perpendicular to the semimajor axis. The eccentricity is $e \approx 0.20462$.

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