Generating novel multi-scroll chaotic attractors via fractal transformation

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Abstract The fractal and chaos are bound tightly, and their relevant researches are well-established. Few of them, however, concentrate on the research of the possibility of combining fractal and chaotic systems for generating multi-scroll chaotic attractors. This paper presents a novel non-equilibrium point chaotic system, exhibiting extremely rich and complex hidden behaviors including chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability, and initial offset-boosting behavior. The proposed system is combined with fractal transformation to observe a new class of multi-scroll attractors such as multi-ring attractors, separated-scroll attractors, and nested attractors. Particularly, the first swallow-like attractors are found. Moreover, another efficient method to generate a different class of chaotic attractors applies parabola transformation and triangle transformation. Additionally, the spectrum entropy (SE) complexity is also employed to discuss the complexity of the proposed system before and after fractal, resulting in chaotic sequences with the fractal transformation that has higher complexity. Finally, we develop a hardware platform to implement the presented attractors before and after fractal in a way to confirm the accuracy of the numerical simulations, providing a theoretical basis for the next application in image encryption.

Keywords Hidden attractors · Multi-ring attractors · Separated attractors · Nested attractors · MCU implementation

1 Introduction

From a computational point of view, the attractors in chaotic systems can be divided into self-excited and hidden attractors. A self-excited [1–5] attractor can be defined as the intersection of the attractor’s basin of attraction with any open neighborhood of the stationary state. Otherwise, it is known as a hidden attractor [6–11]. Hidden attractors cannot be located by applying a standard calculation program. Hence, it is difficult to predict their presence in the system. Hidden attractors have played an important role in theoretical problems and practical engineering applications [12], so it is of great significance to investigate hidden attractors. Meanwhile, hidden attractors are often encountered in dynamic systems with line equilibria [13], no equilibrium [14], or stable equilibria [15]. At present, many chaotic systems with hidden attractors have been reported. In 2020, Gong et al. found a novel 4D chaotic system with infinite equilibria using a linear state feedback controller in the...
Sprott C system [16]. That same year, Deng et al. presented four-wing hidden attractors with one stable equilibrium point [17]. In 2021, Jahanshahi et al. discussed a 4D system with hidden attractors that lacks equilibria [18]. Gu et al. also discovered a novel non-equilibrium memristor-based system with multi-wing attractors and multiple transient transitions [19].

Motivated by research literature on chaotic systems with hidden attractors, we present a novel non-equilibrium point chaotic system in this paper. Although there is no equilibrium point in the system, attractive characteristics, such as chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability, and initial offset-boosting behavior, can be observed.

Multi-scroll chaotic systems have more complexity than general chaotic systems, which plays a significant role in chaos-based communication security [20], chaotic cryptanalysis [21], and image encryption [22]. In the existing methods for generating multi-scroll chaotic attractors, such as piecewise linear function [23], step function [24], sine function [25], switching manifold [26], time delay saturation sequence [27], these function-based methods for generating multi-scroll chaotic attractors lead to non-smooth chaotic systems with very low complexity and the number of scrolls is also not easy to adjust. Fortunately, the use of fractal transformations compensates for these shortcomings. Nowadays, there are relatively few studies on the combination of chaos and fractals to generate multi-scroll attractors, which suggests that further research is worthwhile. Bouallegue [28] et al. proposed a general multi-scroll and multi-wing attractors using the fractal process with a cascade of Lorenz system in 2011, which the fractal algorithm is difficult to understand. Also, Bouallegue [29] et al. combined the Lorentz attractor and the Chua attractor with fractal process, respectively, in which a large number of separated attractors were produced. Guo [30] et al. derived a multi-wing spherical chaotic system by fractal map based on the Qi-3D four-wing chaotic system. In 2018, Xiao [31] et al. combined Julia fractal with the memristor-based chaotic system to generate abundant multi-scroll attractors, providing a new method to construct multi-scroll attractors. In 2021, Yan [32] et al. designed a memristor-based chaotic system, which can be able to generate multi-scroll chaotic attractors. In particular, a large number of multi-scroll attractors with different topologies are produced by combining the system with the fractal process. However, the above-mentioned papers only display multi-scroll attractors without considering both the dynamic analysis and hardware implementation. In 2019, Dai [33] et al. proposed a rotational compound chaotic system that undergoes fractal transformation and is based on simplified the Lorenz multi-wing and the Chua multi-scroll chaotic system. It is noteworthy that dynamic analysis and hardware implementation were given for the first time in this paper; yet, one fractal transformation is exploited while ignoring multiple fractal processes, and the shape of the multi-scroll attractors is single and unclear. To sum up, we have designed a chaotic system without equilibrium points with hidden attractors, and the different scrolls attractors can be obtained simply by adjusting the parameters of the system. Also, we try to combine the chaotic system with multiple fractal transformations to generate multi-ring attractors, separated-scroll attractors, and nested attractors with different topologies. In addition, triangular transformation and parabolic transformation are also employed in fractal systems to present novel attractors.

In summary, the major contributions of this paper are as follows.

1. The multi-scroll attractors are produced by merely modifying the proposed system parameters.
2. The complex dynamical phenomena of the proposed system, such as chaos, hyper-chaos, various coexisting multi-scroll chaotic attractors, and initial offset-boosting behaviors, are observed.
3. Combining the hidden attractor with the fractal transformation generates abundant multi-scroll attractors. Then, by choosing different fractal systems, multi-ring attractors, separated attractors, and nested attractors are generated. Furthermore, swallow-shaped attractors are also observed for the first time.
4. Based on either parabola transformation or triangle transformation, another new class of attractors is discovered.
5. The system after fractal has higher SE complexity, which becomes more suitable for image encryption and information security.
6. Hardware implementations are also considered in order to prove the effectiveness of numerical simulation and theoretical analysis.
The rest of the paper is organized as follows. In Sect. 2, a novel 4D non-equilibrium point system with hidden attractors is presented. Moreover, a large number and complex dynamic phenomena are discovered. In Sect. 3, novel and interesting multi-scroll attractors are presented based on the fractal transformation. In Sect. 4, the microcontroller-based experimental implementation of the proposed system before and after fractal transformation is given to verify its physical existence. Finally, the conclusion is elaborated in Sect. 5.

2 Analytical study

In this section, we present the analytical study of a novel chaotic system without equilibrium. Firstly, the corresponding chaotic system model is proposed and described. Then, the multi-scroll chaotic attractors are generated by analyzing parameters of the system. Finally, the effect of the initial state on the system is also discussed in detail.

2.1 Model of the new system without equilibrium

Based on a 3D chaotic system proposed in reference [34], the mathematical model of a novel 4D autonomous system without equilibrium is investigated in this paper. It is described by

\[
\begin{align*}
\dot{x} &= ay + bxz, \\
\dot{y} &= -cx + yzw, \\
\dot{z} &= d - |x|, \\
\dot{w} &= gz,
\end{align*}
\]

(1)

The model consists of three nonlinear terms and four linear terms. In Eq. (1), \(x, y, z\) and \(w\) are the state variables, while \(b, d, g\) are three positive parameters that are fixed as \(b = 15\), \(d = 1\), and \(g = 5\). At the same time, parameters \(a\) and \(c\) are a variable employed to generate multi-scroll attractors, which will be analyzed in the next section.

It is important to note that the absolute value term is added to the system (1), owing to the nonlinear feature of the absolute term which is a good candidate to design chaotic circuits with special characteristics. From a practical view, an absolute term is easy to be implemented by employing electronic components such as resistors, diodes, and operational amplifiers.

Additionally, symmetric systems may indicate the coexistence of attractors.

The nonlinear system (1) can be expressed in vector notation as:

\[
\frac{d\theta}{dt} = f(\theta) = \begin{bmatrix}
f_1(x, y, z, w) \\
f_2(x, y, z, w) \\
f_3(x, y, z, w) \\
f_4(x, y, z, w)
\end{bmatrix}
\]

(2)

where \(\theta \in \mathbb{R}^4\) is a state vector and \(f(\theta)\) is a smooth function given by:

\[
\begin{align*}
f_1(x, y, z, w) &= ay + bxz, \\
f_2(x, y, z, w) &= -cx + yzw, \\
f_3(x, y, z, w) &= d - |x|, \\
f_4(x, y, z, w) &= gz.
\end{align*}
\]

(3)

The divergence of the vector field \((\text{div}(f) = \nabla \cdot f)\) of the nonlinear system (2) can be calculated as:

\[
\text{div}(f) = \nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w} = z(b + w).
\]

(4)

when the parameter \(b = 15\) is chosen, the initial conditions are \((-0.1, -0.1, -0.1, -0.1, -0.1\), it has \(\nabla \cdot f = z(b + w) < 0\), i.e., the system (1) is dissipative, which means that the attractor may be a chaotic attractor.

It is easy to verify that system (1) has non-equilibrium, and the detailed analysis is as follows:

\[
\begin{align*}
ay + bxz &= 0, \\
-cx + yzw &= 0, \\
d - |x| &= 0, \\
gz &= 0,
\end{align*}
\]

(5)

by observing the second and the fourth term in Eq. (5), it’s obvious that

\[
z = 0, x = 0
\]

(6)

and from the third term in Eq. (5), we have

\[
x = \pm 1
\]

(7)

Equation (6) is inconsistent with Eq. (7). Thus, there is non-equilibrium in the system (1), implying the appearance of hidden attractors.

Furthermore, it is observed that the system (1) is invariant when coordinates are transformed as follows
there are mainly two positive Lyapunov exponents in a depends on the variable transformation between chaos and hyper-chaos speaking, the main change of the system (1) is that the indicate that the system (1) is hyper-chaotic. Generally

\[ P : R^4 \rightarrow R^4, X \rightarrow PX, P = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  

Equation (8) indicates that the system (1) has symmetry on the z-w plane. Therefore, any projection of the attractor has rotational symmetry on the z-w plane, and there may be coexisting attractors in the system (1).

2.2 Hidden multi-scroll chaotic attractors

This section focuses on chaotic attractors with changeable numbers of scrolls. The flexibility of scroll numbers means that only the parameters of the system (1) need to be adjusted for generating multi-scroll chaotic attractors [35]. The initial values of the system (1) are fixed to \((-0.1, -0.1, -0.1, -0.1)\) and the rich dynamic behavior of the system (1) is investigated using the bifurcation diagram and the Lyapunov spectrum by keeping the parameter a varying and setting the parameter c = 1. The Lyapunov exponent [36] is an effective indicator of the dynamic behavior of chaotic systems, which is defined as

\[ \text{LE} = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|M(t)\|}{\|M(0)\|} \]  

where \(M(x)\) is the time-dependent error for different initial conditions and \(t\) is the time for simulation. A convenient method for calculating the Lyapunov exponent is the Wolf algorithm [37], where the global characteristics of the dynamical system are described by the trajectory of the Jacobian matrix. The bifurcation diagram of the state variable \(x\), depicted by the maximum value method, is shown in Fig. 1a, which corresponds to the Lyapunov exponent spectrum in Fig. 1b. According to the Lyapunov exponent spectrum, the system (1) always keeps the chaotic state in the range \(a \in [0, 1]\). Moreover, it is worth-noting that there are mainly two positive Lyapunov exponents in the interval \(a \in [0.131, 0.624) \cup (0.624, 1]\), which indicate that the system (1) is hyper-chaotic. Generally speaking, the main change of the system (1) is that the transformation between chaos and hyper-chaos depends on the variable \(a\). Next, the fourth-order Runge–Kutta (ode45) algorithm is employed to perform the numerical solution of ordinary differential equations. When the simulation step size is 0.01 s and the time length is 1000 s, the typical parameter \(a\) is capable of producing chaotic attractors and hyper-chaotic attractors. The hyper-chaotic attractor at \(a = 0.2\) (LE1 = 0.0344, LE2 = 0.01850, LE3 = −0.0035, LE4 = −0.0286) and the chaotic attractor at \(a = 0.9\) (LE1 = 0.1587, LE2 = 0.0072, LE3 = −0.0043, LE4 = −0.1542) are illustrated in Fig. 2a–b, respectively. Interestingly, the hyper-chaotic system (1) can not only generate double-scroll chaotic attractors but also produce multi-scroll attractors. Furthermore, parameters \(a\) and \(c\) are adjusted, four typical multi-scroll chaotic attractors can be obtained as shown in Fig. 3a–d, respectively. The corresponding Poincaré sections when \(z = 0\) are shown in Fig. 4 which illustrate the existence of multi-scroll chaotic attractors. In addition, the Poincaré section also proves that the system (1) has rich and complex dynamic behaviors due to the existence of dense points with fractal structures.

2.3 Hidden coexisting attractors

Multi-stability [16–18] means the coexistence of multiple attractors, which is an inherent characteristic of many nonlinear dynamical systems. Concurrently, multi-stability can be applied in image processing or as a random source for information engineering applications. Furthermore, multi-stability sometimes leads to difficulties in engineering application; however, the ability to improve the system state by varying initial conditions provides great flexibility. Furthermore, when the number of coexisting attractors approaches infinity, this phenomenon is known as extreme multi-stability. From Eqs. (5)–(7), it has been proved that there is no equilibrium point in the system (1), that is, the attractors generated from the system (1) are all hidden. At present, hidden coexisting multiple chaotic attractors have been observed in many nonlinear circuits; however, hidden multiple coexisting multi-scroll chaotic attractors are rare. In order to analyze the extreme multi-stability of the system (1), the parameters are configured as \(a = 0.2\) and \(c = 1\), the initial conditions are considered as \((-0.1, y(0), -0.1, -0.1)\), and the simulation step size is 0.001 s and the time length is 1000 s. The Lyapunov exponent and the bifurcation diagram are then used to analyze the different motion states of the system (1).
along with the varying initial condition $y(0)$, which are shown in Fig. 5a and b, respectively. From Fig. 5a, it is obvious that one positive Lyapunov exponent or two positive Lyapunov exponents alternate in the interval $[-1, 1]$, which implies the presence of hidden chaotic attractors or hidden hyper-chaotic attractors. When the initial value $y(0)$ varies continuously, the system (1) will produce many hidden coexisting multi-scroll attractors for the simulation step size is 0.01 s and the time length is 1000 s. To avoid drawing similar graphs, we choose several typical initial values to represent the coexistence of multiple hidden multi-scroll attractors, which are illustrated in Fig. 6a. Additionally, the dynamical behaviors are summarized in Table 1.

It is known that the basin of attraction [38] is a convenient tool to discuss coexisting attractors and hidden attractors, and the basin of attraction is defined as a set of points where the trajectory of the system converges to the corresponding attractor at the corresponding set of initial conditions. Thus, the basins of attraction in $x(0) - y(0)$ initial plane with...
\( z(0) = w(0) = -0.1 \) is displayed in Fig. 6b, where the light blue represents the hidden two-scroll attractor corresponding to \( y(0) = 0.1 \). Moreover, the light blue represents the three-scroll hidden attractor at \( y(0) = -0.9 \), and the red and the light yellow regions of the basin represent the different multi-scroll hidden attractors corresponding to \( y(0) = 0.43 \) and \( y(0) = -0.5 \), respectively. The attraction regions of different colors demonstrate different initial states \( y(0) \), exhibiting completely different topologies. From this attraction basin, the probability of obtaining the double-scroll attractor is greater than the probability to obtain the other attractors since the set of initial conditions leading to the double-scroll attractor is rather large than the others. Based on the former analyses, the proposed system (1) has significant initial value sensitivity, which reveals complex extreme multi-stability phenomena.

Additionally, an intriguing phenomenon—known as initial offset-boosting behavior [39, 40]—has been observed from this system (1). The phenomenon is also referred to as the extreme multi-stability, and it
must be noted that an infinite number of attractors are coexisting attractors with the same topology. Fixed $a = 0.2$, $c = 1$, the initial condition is $(-0.1, -0.1, -0.1, w(0))$, $w(0)$ is chosen as the offset-boosted controller so that it gradually increases. From the initial value $w(0) = 0, 0.5, 1, 1.5, 2$, and $2.4$, the phase diagrams of the six hidden chaotic attractors in different planes are shown in Fig. 7a, b.

3 Multi-scroll attractors generated by fractal transformation

In this section, since the system (1) exhibits a wealth of nonlinear dynamic behaviors such as chaos, hyperchaos, hidden extreme multi-stability, and the initial offset-boosting, it is extremely sensitive to system parameters and initial values. Therefore, multi-scroll chaotic attractors after fractal with different topologies will be observed by simply modifying the parameters and initial values of the system (1). Meanwhile, we first introduce the specific fractal transformation

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Fig. 4 Poincaré sections of the system (1) when $z = 0$. a $a = 0.02$ and $c = 1$, b $a = 0.05$ and $c = 1$, c $a = 0.05$ and $c = 1$, d $a = 0.95$ and $c = 0.025$
algorithm while regarding the system (1) as the seed system. Then, by designing different fractal systems, we are able to generate a variety of multi-scroll attractors including multi-ring attractors, separated-scroll attractors, nested multi-ring attractors, and nested and separated attractors. Meanwhile, not only do we observe swallow-like attractors for the first time, but we also find another form of attractors based on fractal transformation, the seed system, parabola transformation, and triangle transformation. Finally, the $SE$ complexity of the seed system before and after fractal is compared. The experiment proves that the multi-scroll chaotic attractors based on fractal are not only better at adjusting the number of scrolls, but also make up for the lack of smoothness of the chaotic system caused by the function method.

Fig. 5 The dynamic behavior corresponds to the initial value $y(0)$. a the Lyapunov exponent spectrum with respect to $y(0)$, b the bifurcation diagram with respect to $y(0)$

Fig. 6 Coexistence of hidden multiple-scroll attractors with different initial conditions and Corresponding basin of attraction in the $x(0)$-$y(0)$ plane. a coexistence of hidden multiple-scroll attractors, b basin of attraction in the $x(0)$-$y(0)$ plane, where $a = 0.2$, $b = 15$, $c = 1$, $d = 1$, and $g = 5$, $z(0) = w(0) = -0.1$
3.1 The fractal algorithm

Boualallegue et al. [28] proposed the fractal algorithm to obtain abundant multi-scroll chaotic attractors where the number of scrolls is the power of 2, so the proposed algorithm can easily generates \(2^n\) scrolls chaotic attractors. The description of the algorithm, however, is extremely difficult to analyze. Thus, we derive and improve the algorithm based on the Julia iterative mapping, which is defined as a quadratic complex \(Z\) mapping with the following expression

\[
Z_{n+1} = Z_n^2 + Z_c
\]  

(10)

Let

\[
\begin{align*}
Z_{n+1} &= x_{n+1} + iy_{n+1} \\
Z_n &= x_n + iy_n \\
Z_c &= 0
\end{align*}
\]

(11)

Taking Eq. (11) into Eq. (10), we can obtain

\[
\begin{align*}
x_{n+1} &= x_n^2 - y_n^2 \\
y_{n+1} &= 2x_ny_n
\end{align*}
\]

(12)

and then the subscripts of variables in Eq. (12) are interchanged to get

\[
\begin{align*}
x_n &= x_{n+1}^2 - y_{n+1}^2 \\
y_n &= 2x_{n+1}y_{n+1}
\end{align*}
\]

(13)

Thus, \(x_{n+1}\) and \(y_{n+1}\) can be represented by \(x_n\) and \(y_n\) as

\[
\begin{align*}
x_{n+1} &= \pm \sqrt{x_n^2 + y_n^2 + x_n} \\
y_{n+1} &= \frac{y_n}{2x_{n+1}}
\end{align*}
\]

(14)

Then, let us assume that the system is defined by

\[
\begin{align*}
\dot{x}_1 &= x_1(x_1, x_2, x_3) \\
\dot{x}_2 &= y_1(x_1, x_2, x_3) \\
\dot{x}_3 &= z_1(x_1, x_2, x_3)
\end{align*}
\]

(15)

We process the system (15) by a fractal process inspired by the Julia process. As a result, we obtain the following two outputs

\[
(x_{n+1}, y_{n+1}) = PJ(x_1(x_1, x_2, x_3), y_1(x_1, x_2, x_3))
\]

(16)

where \(PJ\) represents a switch that is switched in two cases:

| Colored | Dynamics | \(\alpha(0)\) |
|---------|----------|----------------|
| \(0.1\) | Hyper-chaos | \(0.034, 0.0185, -0.003, -0.020\) |
| \(0.9\) | Hyper-chaos | \(0.032, 0.011, -0.013, -0.021\) |
| \(0.1\) | Hyper-chaos | \(0.062, -0.004, -0.006, -0.020\) |
| \(0.9\) | Hyper-chaos | \(0.145, -0.003, -0.007, -0.122\) |

Table 1 The typical attractor types in differently colored regions for \(\alpha = 0.2\)
If \( x_n > 0 \), the outputs are computed as follows

\[
\begin{align*}
x_{n+1} &= \pm \sqrt{x_n^2 + y_n^2 + \frac{x_n}{2}} \\
y_{n+1} &= \frac{y_n}{2x_{n+1}}
\end{align*}
\] (17)

If \( x_n < 0 \), the outputs are computed as follows

\[
\begin{align*}
x_{n+1} &= \frac{y_n}{2y_{n+1}} \\
y_{n+1} &= \pm \sqrt{x_n^2 + y_n^2 - \frac{x_n}{2}}
\end{align*}
\] (18)

By evolving Eqs. (15)–(18) into an algorithm, we propose a schematic block diagram of the fractal transformation, as shown in Fig. 8. The \( x_n \) and \( y_n \) are the input sequences by the Runge–Kutta method (ODE45) to solve the seed system, while \( x_{n+1} \) and \( y_{n+1} \) are regarded as output sequences after the fractal transformation. The algorithm then computes the output sequences \( x_{n+1} \) and \( y_{n+1} \) by different functions depending on the value of the variable \( x_n \). This process is called fractal transformation and is defined as \( PJ \). Finally, the output sequences \( x_{n+1} \) and \( y_{n+1} \) are viewed as the next input sequences \( x_n \) and \( y_n \), which implies that in theory the fractal transformation can loop infinitely. Therefore, the fractal transformation can be rewritten as

\[
(x_{n+1}, y_{n+1}) = PJ(x_n, y_n)
\] (19)

### 3.2 Multi-ring chaotic attractors

This subsection introduces multi-ring chaotic attractors generated by investigating the combination of the fractal transformation with the seed system. The seed system exhibits a wealth of nonlinear dynamic behaviors including chaos, hyper-chaos, hidden multi-stability, and initial offset-boosting behavior, which means that the seed system is very sensitive to parameters and initial values. Therefore, multi-ring attractors will be observed by simply modifying the parameters and initial values of the seed system. All the above-mentioned advantages due to the combination of the seed system and the fractal transformation are of great significance for a variety of practical applications. In this section, the application of the fractal algorithm \( PJ \) proposed in Sect. 3.1 to generate more complex multi-ring attractors is presented. Moreover, the mathematical derivation of the three-time fractal transformations is also presented.

According to the derivation of Eqs. (15)–(18), let \( \phi \) be a system of fractal transformation, which is defined by

\[
\phi : (x_n, y_n) \rightarrow (u_n, v_n)
\] (20)

The system \( \phi \) that undergoes three-time fractal transformations is represented by...
\[
\begin{align*}
\phi : & \quad \begin{cases} 
(u_1, v_1) = PJ(x_1, y_1) \\
(u_2, v_2) = PJ(u_1, v_1) \\
(u_3, v_3) = PJ(u_2, v_2)
\end{cases} \\
& \quad \begin{cases} 
\text{If } x_1 > 0, \text{ the output of the first fractal is calculated as follows} \\
\quad u_1 = \pm \sqrt{x_1^2 + y_1^2 + \frac{x_1}{2}} \\
\quad v_1 = y_1 \\
\end{cases} \\
& \quad \begin{cases} 
\text{If } x_1 < 0, \text{ the output of the first fractal is} \\
\quad u_1 = \frac{y_1}{2v_1} \\
\quad v_1 = \pm \sqrt{x_1^2 + y_1^2 - \frac{x_1}{2}}
\end{cases} \\
& \quad \begin{cases} 
\text{If } u_1 > 0, \text{ the output of the twice fractal transformation is} \\
\quad u_2 = \pm \sqrt{u_1^2 + v_1^2 + \frac{u_1}{2}} \\
\quad v_2 = \frac{v_1}{2u_2}
\end{cases} \\
& \quad \begin{cases} 
\text{Take Eq. (22) into Eq. (24), that is} \\
\quad \text{If } u_1 < 0, \text{ the output of the twice fractal transformations is}
\end{cases}
\end{align*}
\]

Fig. 8 The principle block diagram of the fractal algorithm

where \( x_1 \) and \( y_1 \) represent the input sequences by solving the seed system (1), and \( u_1 \) and \( v_1 \) represent the output sequences with the one-time fractal transformation. Next, the output sequences \( u_1 \) and \( v_1 \) are considered as the input sequences of the second fractal transformation. Thus in the case of the second transformation:

\[
\begin{align*}
\text{If } x_1 > 0, \text{ the output of the first fractal is calculated}
\end{align*}
\]

\[
\begin{align*}
\text{If } x_1 < 0, \text{ the output of the first fractal is}
\end{align*}
\]

\[
\begin{align*}
\text{If } u_1 > 0, \text{ the output of the twice fractal transformations is calculated}
\end{align*}
\]

\[
\begin{align*}
\text{Take Eq. (22) into Eq. (24), that is}
\end{align*}
\]

\[
\begin{align*}
\text{If } u_1 < 0, \text{ the output of the twice fractal transformations is}
\end{align*}
\]
Moreover, when the initial states are 0.001 s and the time length is 2000s, respectively.

The expressions of the initial states are
\[
\left\{\begin{array}{l}
u_1 = \frac{v_1}{2v_2} \\
v_2 = \pm \sqrt{u_1^2 + v_1^2 - \frac{u_1}{2}}
\end{array}\right.
\] (26)

Take Eq. (23) into Eq. (26),

\[
\left\{\begin{array}{l}
u_2 = \frac{y_1}{4v_2 \sqrt{x_1^2 + y_1^2 + \frac{y_1}{2}}} \\
v_2 = \pm \sqrt{\left(\pm 2 \sqrt{x_1^2 + y_1^2 + \frac{y_1}{2}}\right)^2 + \left(\pm \sqrt{x_1^2 + y_1^2 - \frac{x_1}{2}}\right)^2 - \frac{y_1}{2}}
\end{array}\right. 
\] (27)

Finally, the output sequences \( u_2 \) and \( v_2 \) are seen as input sequences of the third fractal transformation. Thus in the case of the third transformation:

If \( u_2 > 0 \), the output of the third fractal transformation is as follows

\[
\left\{\begin{array}{l}
u_3 = \pm \sqrt{u_2^2 + v_2^2 + \frac{u_2}{2}} \\
v_3 = \frac{v_3}{2u_3}
\end{array}\right. 
\] (28)

If \( u_2 < 0 \), the output of the third fractal transformation is as follows

\[
\left\{\begin{array}{l}
u_3 = \frac{v_2}{2v_3} \\
v_3 = \pm \sqrt{u_2^2 + v_2^2 + \frac{u_2}{2}}
\end{array}\right. 
\] (29)

The expressions of \( v_2 \) and \( u_2 \) are then taken into Eq. (29) to obtain the expressions of the initial sequence \( x_1 \) and \( y_1 \). Thus, Eq. (21) is a fractal system with three-time fractal transformations. To represent these three fractal processes more clearly, Fig. 9 shows a schematic diagram of the fractal system (21). Figure 10a–c, respectively, illustrate the multi-scroll chaotic attractors after once, twice, three-time fractal transformations when the seed system parameters are at \( a = 0.005, c = 1 \), the initial conditions are \((-0.1, -0.1, -0.1, -0.1)\), and the simulation step size is 0.001 s and the time length is 2000s, respectively. Moreover, when the initial states are \((0.1, -0.1, -0.1, -0.1)\) for the seed system (1) with parameters configured as \( a = 0.1 \) and \( c = 0.1 \), multi-ring chaotic attractors with different topologies can also be observed after performing all three-times fractal transformations on the fractal system (21), as shown in Fig. 11a–c. The relationship between the number of scrolls and the fractal process can be easily found in Figs. 10 and 11. Suppose the initial number of scrolls is \( C_1 \), and the number of fractal processes is \( n \), then the number of scrolls after the fractal process is \( C_2 = C_1 \times 2^n \). Using this approach, it is easy to generate multi-scroll chaotic attractors.

Furthermore, by adding arctangent functions to the fractal system (21) and constructing different input sequences in each fractal transformation, multi-ring chaotic attractors with different topologies can be observed. Similarly, let \( \phi \) be a system of fractal transformation, and a simple example is designed as follows

\[
\phi : \left\{\begin{array}{l}
(u_1, v_1) = \text{PJ} (\arctan(x_1), \arctan(y_1)) \\
(u_2, v_2) = \text{PJ} (x_1, y_1) \\
(u_3, v_3) = \text{PJ} (v_1 + v_2, u_1 - u_2)
\end{array}\right. 
\] (30)

Firstly, the output sequences \( u_1 \) and \( v_1 \) can be calculated by taking the input sequences \( x_1 \) and \( y_1 \) as the variables of the arctangent function in the first fractal transformation. Then, in the second fractal transformation, the output sequences \( u_2 \) and \( v_2 \) are obtained from the input sequences \( x_1 \) and \( y_1 \). Finally, the third fractal transformation combines the behavior of the first two fractal transformations. The schematic diagram of the fractal system (30) is displayed in Fig. 12. (The mathematical derivation process is similar to the derivation process of the fractal system (21).) The variation of the seed system parameters increases the un-controllability of the shape, while different parameters \( a \) and \( c \) can be used to produce different shapes under the simulation step size is
0.001 s, and the time length is 2000s. The fractal system (30) with different parameters generating the multi-ring attractors with different topologies is verified in Fig. 13a, b, respectively.

To explore the influence of fractal system parameters on the fractal system, another fractal system (31) is designed and considered. The parameters of the seed system are specified as $a = 0.005$, $c = 0.1$, the step size is 0.001 s, and the time length is 2000s. Typical parameters in a fractal system (31) yield different topologies chaotic attractors as the following: $k = 1.6$, $g = -1.7$ and $k = 2$, $g = -1.5$, and those two sets are depicted in Fig. 14a, b, respectively. It has been shown that the dynamic behavior of the attractor can be significantly changed by adjusting the parameters in the fractal system (31). It must be emphasized that the schematic diagram of the fractal system (31) is similar to the schematic diagram of the fractal system (21).
Fig. 12  Schematic diagram of the fractal system (30)

Fig. 13  The multi-scroll chaotic attractor with different parameters for initial conditions \((-0.1, -0.1, -0.1, -0.1)\): \(a = 0.005, c = 2, b = 0.1, c = 0.5\)

Fig. 14  The multi-scroll chaotic attractor with different fractal system parameters. \(a = 1.6, g = -1.7, d = 2, g = -1.5\)
3.3 Chaotic attractors with scrolls separated

It should be pointed out that chaotic attractors have many practical applications in random number generators, image encryptions, and communication security. In order to increase the complexity of chaotic dynamics and improve the security of the system, the chaotic attractors with simple topologies can be replaced by several separated chaotic attractors. In addition, this subsection will introduce the formation process of swallow-like attractors in detail.

First of all, a fractal system that can produce chaotic attractors with scrolls separated is designed as follows

\[
\phi : \begin{cases} 
(u_1, v_1) = PJ (x_1 + \arctan(x_1), y_1 + \arctan(y_1)) \\
(u_2, v_2) = PJ (x_1, y_1) \\
(u_3, v_3) = PJ (ku_1 + gv_2, kv_1 + gu_2) 
\end{cases}
\]

(31)

where \( k \) is the separation factor, and the multi-ring chaotic attractor will be separated as the separation factor \( k \) increases. For the sake of clarity, we can divide the chaotic attractor into three stages: the initial state, the intact state before reaching the threshold, and the separation state. Figure 15 gives a detailed analysis of the separation process for the seed system’s parameters \( a = 0.2, \ c = 0.02, \) initial values \((-0.1, -0.1, -0.1, -0.1)\), the step size is 0.001 s and the time length is 1500 s. The chaotic attractor is at the initial state with \( k = 0 \), and a multi-ring attractor can be observed. Then, the multi-ring attractor starts to separate gradually as \( k \) increases. When \( k = 0.5 \), separation distance reaches the threshold where the chaotic attractor is about to be completely separated. Finally, the attractors are completely separated with \( k = 1 \), and four swallow-like attractors can be observed as shown in Fig. 15c.

Then, to further verify the effectiveness of the fractal system (32), the simulation results are given for the seed system’s parameters \( a = 0.02, \ c = 1 \) and initial values \((-0.1, -0.1, -0.1, -0.1)\), which are illustrated in Fig. 16a–c. In summary, we can conclude that the formation of separated scroll attractors follows the law that the greater the separation distance, the greater the distance between each attractor in a fractal system.

Finally, to obtain a richer and more complex chaotic attractors with scrolls separated, a fractal system undergoing four-time fractal transformations is represented by

\[
\phi : \begin{cases} 
(u_1, v_1) = PJ (x_1 + \arctan(x_1) + \alpha, y_1 + \arctan(y_1) + \alpha) \\
(u_2, v_2) = PJ (x_1 + 55, y_1 - 55) \\
(u_3, v_3) = PJ (2u_1 - 2u_2 + 55, 2v_1 - 2v_2 - 55) \\
(u_4, v_4) = PJ (2u_2 + v_3 + 2u_1, v_2 + u_3 + 2v_1) 
\end{cases}
\]

(33)

For system (33), the novel chaotic attractors with scrolls separated are plotted by setting \( \alpha = 0 \) or \( \alpha = 5 \) for seed system’s parameters \( a = 0.1, \ c = 0.1, \) initial values \((-0.1, -0.1, -0.1, -0.1)\), the simulation step size is 0.001 s, and the time length is 2000s, respectively, which are depicted in Fig. 17a, b. Figure 17a shows eight chaotic attractors with eight scroll, while Fig. 17b draws sixteen chaotic attractors with four-scroll. The chaotic system generated by multi-fractal processes has higher complexity than the seed system (1), which is discussed in Sect. 3.6.

3.4 Nested chaotic attractors

Nested chaotic attractors are rare to be found in previous work. The ability to generate nested chaotic attractors allows us to choose them for different applications. In this subsection, we propose two fractal systems to generate a new class of chaotic attractors with nested scrolls. Moreover, the generated nested chaotic attractors have different types of behaviors.

3.4.1 Nested multi-ring chaotic attractors

For the first example, the newly designed fractal system is as follows
Fig. 15 Different states in the separation process: a initial state \((k = 0)\), b intact state before reaching the threshold \((k = 0.5)\), c separation state \((k = 1)\)

Fig. 16 Different states in the separation process. a initial state \((k = 0)\), b intact state before reaching the threshold \((k = 2)\), c separation state \((k = 3)\)

Fig. 17 The novel chaotic attractors with scrolls separated. a \(\alpha = 0\), b \(\alpha = 5\)
With nested two chaotic attractors and the other with two identically shaped nested and separated attractors.

3.5 A different class of chaotic attractors

In order to obtain more complex and abundant chaotic attractors, two approaches—parabola transformation and triangle transformation [41]—are described in this subsection, which produces chaotic attractors different from subsection 3.3 and subsection 3.2. In our contribution, we explore the combination of parabolic and triangular transformations, respectively, with Julia’s fractal process to generate different classes of attractors.

3.5.1 Triangle transformation

Firstly, the triangle transformation map, which is defined $T$ as follows:

$$T : \begin{cases}
\dot{x}_{n+1} = 6x_n - 6(x_n^2 - y_n^2) \\
\dot{y}_{n+1} = 6y_n - 12x_ny_n
\end{cases} \quad (36)$$

The seed chaotic system (1) with fractal transformation combined with triangular transformation, a new fractal system is designed as follows

$$\phi : \begin{cases}
(u_1, v_1) = PJ(\arctan(x_1), \arctan(y_1)) \\
(u_2, v_2) = PJ(x_1, y_1) \\
(u_3, v_3) = PJ(2u_1 - 2u_2, 2v_1 - 2v_2) \\
(u_4, v_4) = PJ(2u_3 + g, 2v_3 + f) \\
(u_5, v_5) = PJ(u_4 - u_5 + j, v_4 - v_5 + k)
\end{cases} \quad (37)$$

where $T$ represents the triangle transformation. By setting the parameters of the seed system to $a = 0.005$, $c = 0.1$, the initial values are configured as $(-0.1, -0.1, -0.1, -0.1)$, the simulation step size is 0.001 $s$, and the time length is 1500s. Figure 20a shows the result of the chaotic attractor after triangle transformation. To further verify the correctness of the proposed method, by employing different fractal systems and different seed system parameters, attractors with different topologies based on triangle transformation can be found. Here, the parameters of the seed system are configured as $a = 0.1$, $c = 0.1$. The phase diagram is depicted in Fig. 20b. To explain the algorithm more clearly, we give the block diagram form of the fractal system (37), as shown in Fig. 21.
\[
\begin{align*}
\phi : & \quad (u_1, v_1) = \text{PJ} \left( x_1 + \arctan(x_1) - 1, y_1 + \arctan(y_1) - 1 \right) \\
& \quad (u_2, v_2) = \text{PJ} \left( 2x_1, 2y_1 \right) \\
& \quad (X, Y) = T(2v_2, 2u_1)
\end{align*}
\] (38)

3.5.2 Parabola transformation

Similarly, parabolic transformation map \( T \) is defined as

\[
T : \quad \begin{cases}
    x_{n+1} = 6x_n(1-x_n) \\
    y_{n+1} = 6y_n
\end{cases}
\] (39)

The seed chaotic system (1) with fractal transformation combined with parabolic transformation, the attractor can be obtained by using the parabolic transformation applied to the fractal system (37) and the fractal system (38), respectively, as shown in Fig. 22a, b. The two methods—triangle transformation and parabolic transformation—generate attractors with different forms and behaviors.

3.6 Comparison of SE Complexity

The goal of this subsection is to discuss the complexity of the seed system sequences before and after the one-
time fractal transformation mentioned above. Generally known, the higher the chaos degree of the chaotic sequence is, the greater the corresponding complexity holds. The complexity of the algorithm plays an essential role in the dynamic analysis of nonlinear systems. At present, there are several methods to
measure the complexity, such as spectrum entropy (SE) \cite{42}, \(C_0\) entropy \cite{43}, and permutation entropy (PE) \cite{44}. Out of the three, the \(SE\) algorithm is used to calculate the complexity of the chaotic sequence because it has the advantages of simplicity, robustness, and efficiency and without any coarse-grained preprocessing. Therefore, the \(SE\) complexity is defined as follows

\[
SE = - \sum_{j=0}^{N/2-1} P_j \ln P_j
\]  

where \(j\) represents the frequency point and \(P_j\) is the relative power spectrum probability. As the energy spectrum of the series is well uniformly distributed in the conversion region, the original series is more analogous to the random signal, indicating a higher complexity. To make a better comparison, the seed system parameter \(c = 1\) is fixed. Figure 23a illustrates the \(SE\) complexity spectrum of the seed system before and after fractal transformation with respect to the control parameter \(a\). At the same time, in order to compare the influence of the complexity of the initial value seed system before and after fractal transformation, the spectrums of \(SE\) concerning the initial value \(y(0)\) and parameters \(a = 0.005, c = 1\) are drawn accordingly, and the results are illustrated in Fig. 23b. The results display that the complexity of the seed system can be greatly improved by one-time fractal transformation; therefore, the seed system after fractal transformation is more compatible for image encryption and information security. The above phenomena are mainly revealed by numerical simulations on the MATLAB platform.

Table 2 reports the spectral entropy of a number of multi-scroll attractors, including fractal attractors and chaotic systems that have been reported. The \(SE\) of the seed chaotic system is 0.5560. Then, after the seed system undergoes a one-time fractal (Eq. (21)), the complexity of the chaotic system increases to 0.9536, which makes the complexity of the seed system greatly improved. The seed system has higher complexity after going through twice fractal processes and three-time fractal processes, which are 0.9537 and 0.9542, respectively. The complexity of the nested structures (Eq. (35)), the triangle transformation (Eq. (38)), and the parabola transformation (Eq. (38)) is 0.9516, 0.9533, and 0.9525, respectively. In particular, the complexity of the attractor with separated scrolls (Eq. (33)) reaches 0.9579. Usually, the \(SE\) of the multi-scroll attractor is larger than other multi-scroll attractors, such as fractional-order multi-scroll attractor \cite{45}, multi-scroll attractor in memristive Hindmarsh–Rose (HR) neuron model \cite{46}, time-delay multi-scroll attractor \cite{47}, and grid compound multi-scroll attractor \cite{48}. The results demonstrate that the fractal transformation approach is effective in improving the dynamic behavior of chaotic systems.

![Fig. 22](image-url) Parabolic attractor. a the chaotic attractor based on parabolic transformation generated by applying fractal system (37) with \(a = 0.005, c = 0.1\), b the chaotic attractor based on parabolic transformation generated by applying fractal system (38) with \(a = 0.1, c = 0.1\)
Compared with analog hardware circuits [51], the system built upon digital hardware circuits has the advantages of strong control and good stability, which gives the possibility of direct application to the field of digital-voice communication security [52]. For these reasons, a programmable hardware platform based on STM32 is designed to implement different multi-scroll chaotic attractors of the seed system before and after fractal transformation. The flow diagram of MCU implementation is depicted in Fig. 24. This hardware platform includes a high-performance microcontroller named STM32F407ZGT6, which is based on the ARM Cortex-M4 32-bit RISC core, and a 16-bit two-channel D/A converters AD5689. (The STM32F407ZGT6 has two 12-bit D/A converters; however, the accuracy of the D/A converters is

| SE comparison | Chaotic attractors | SE complexity |
|---------------|-------------------|--------------|
| The work of this paper | The seed system (Eq. (1)) | 0.5560 |
| | One-time fractal (Eq. (21)) | 0.9536 |
| | Two-time fractals (Eq. (21)) | 0.9537 |
| | Three-time fractals (Eq. (21)) | 0.9542 |
| | Triangle transformation (Eq. (38)) | 0.9533 |
| | Parabola transformation (Eq. (38)) | 0.9525 |
| | Nested multi-ring (Eq. (35)) | 0.9516 |
| | Separated multi-scroll (Eq. (33)) | 0.9579 |
| Cited references | Fractional-order multi-scroll [45] | 0.6876 |
| | Memristive HR neuron model [46] | 0.5209 |
| | Time-delay multi-scroll [47] | 0.6764 |
| | Grid-compound multi-scroll [48] | 0.7005 |
| | Memristive hidden chaotic system [49] | 0.7900 |
| | Non-Hamiltonian conservative system [50] | 0.9500 |

4 MCU implementation

Fig. 23 The complexity of initial value \( y(0) \) and parameter \( a \) before and after one-time fractal transformation. a the complexity of parameter \( a \) before and after one-time fractal transformation, b the complexity of initial value \( y(0) \) before and after one-time fractal transformation

Table 2 SE comparison of the complexity of different chaotic attractors

![Fig. 23](image-url)
relatively low, so we externally connect a 16-bit D/A converter AD5689. The calculations are carried out on the MCU platform. Firstly, C language is employed to code the expressions into executable programs, and then load the programs into the microcontroller. Simultaneously, all initial states and the control parameters are preloaded on the hardware platform. Next, the signals are converted from digital signals into analog signals via the D/A converter. Finally, the analog signals are sent to the oscilloscope (RIGOL DS2102A).

The first step is to consider the seed system before fractal. Firstly, the Euler algorithm is employed to discretize the seed system (1) proposed in this paper. The expression is given

\[
\begin{align*}
    x(n+1) &= x(n) + h(ay(n) + bx(n)z(n)) \\
    y(n+1) &= y(n) + h(-cx(n) + y(n)z(n)w(n)) \\
    z(n+1) &= z(n) + h(dx - x(n)) \\
    w(n+1) &= w(n) + h(gz(n))
\end{align*}
\]

(41)

In Eq. (41), the discretization step \( h \) is set to 0.0001 and the selection of initial conditions is \((0.1, 0.1, 0.1, 0.1)\). Then, the discrete solutions can be obtained by Eq. (41). Next, the solutions are further sent to the D/A converter for time series conversion. Finally, the phase orbit diagram can be clearly captured on the digital storage oscilloscope (RIGOL DS2102A). Especially, Fig. 26a, b display two multi-scroll hidden attractors with respect to the parameters \((a, c) = (0.02, 1)\) and \((a, c) = (0.05, 1)\), corresponding to Fig. 3a and b, respectively.

The second step is to consider the seed system (1) after fractal. Based on the seed chaotic system in Eq. (1) and the fractal algorithm in Fig. (8), we set the same values of the fractal system parameters and initial values in the MCU experiment. Thus, employing the hardware experimental platform, the above-mentioned typical multi-scroll attractors after fractal can be captured. Screen capture of the hardware platform that generates the ring-scroll chaotic attractor (Eq. (21)) is shown in Fig. 25.

Concurrently, the multi-scroll attractors with one-time fractal transformation and three-time fractal transformations for the parameters \((a, c) = (0.005, 1)\) are exhibited in Fig. 26c, d, corresponding to Fig. 10a and c, respectively. When \((a, c) = (0.2, 0.02)\) and \((a, c) = (0.02, 1)\), separated-scroll attractors for \(k = 1\) and \(k = 3\) in the fractal system (32) are depicted in Fig. 26e, f. The nested attractor in the fractal system (34) is shown in Fig. 26g, and the nested and separated attractors in the fractal system (35) are displayed in Fig. 26h. Similarly, Fig. 26i, j correspond to Figs. 20a, 21 and 22b, respectively. It is noticed that the results of MCU hardware implementation are generally consistent with those of numerical simulations, proving the possibility of physical implementation.
Fig. 26 Chaotic attractors of the seed system before and after fractal process are captured from hardware platform. 

- **a** multi-scroll attractors of the system (1) for $a = 0.02$ and $c = 1$,
- **b** multi-scroll attractors of the system (1) for $a = 0.05$ and $c = 1$,
- **c** once fractal transformation of the fractal system (21) for $a = 0.005$ and $c = 1$,
- **d** three-times fractal transformation of the fractal system (21) for $a = 0.005$ and $c = 1$,
- **e** the fractal system (32) for $k = 1$ with $a = 0.2$ and $c = 0.02$,
- **f** the fractal system (32) for $k = 3$ with $a = 0.02$ and $c = 1$,
- **g** the fractal system (34) with nested attractors for $a = 0.1$ and $c = 0.1$,
- **h** the fractal system (35) with nested and separated attractors for $a = 0.1$ and $c = 0.1$,
- **i** triangle transformation with $a = 0.005$ and $c = 0.1$ for the fractal system (35),
- **j** parabolic transformation with $a = 0.1$ and $c = 0.1$ for the fractal system (36).
5 Conclusion

In this paper, a novel non-equilibrium point system with hidden attractors has been reported and analyzed, which displays rich hidden dynamical properties such as chaos, hyper-chaos, multi-scroll attractors, extreme multi-stability, and initial offset-boosting. The complex hidden dynamical characteristics of the system are investigated by means of Lyapunov exponents, bifurcation diagrams, phase diagrams, and basin of attraction. The next step was to combine the fractal transformation with the seed chaotic system, so a large amount of multi-ring chaotic attractors, scrolls separated attractors, and nested attractors are generated. Then, another new class of attractors was also discovered based on parabola transformation and triangle transformation. Additionally, spectrum entropy (SE) complexity was employed to discuss the complexity of the seed system before and after fractal, and the result indicates that the chaotic sequences with fractal transformation have higher complexity. As a result of the developed algorithm and fractal system’s abilities to generate various multi-scroll chaotic attractors, it allows us to choose a number of attractors for image encryptions and communication security. Finally, the MCU experiments showed coherence with Matlab simulation. By designing different fractal systems, more complex chaotic attractors such as nested chaotic attractors, and separated scrolls chaotic attractors can be observed, and these are potential areas for future research.

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Declarations

Conflict of interest We declare that we have no conflict of interest. All authors contribute equally to this work.

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