Abstract. A family of lattice packings of $n$-dimensional cross-polytopes ($\ell_1$ balls) is constructed by using the notion of Sidon sets in finite Abelian groups. The resulting density exceeds that of any prior construction by a factor of at least $2^{\Theta(n \ln n)}$ in the asymptotic regime $n \to \infty$.

1. Introduction

Dense packings of spheres and other bodies in Euclidean spaces have been objects of mathematical research for centuries [3, 5, 10]. Apart from their intrinsic mathematical value, they have also found applications in error correction coding, physics, etc. In this note we consider the problem of efficiently packing cross-polytopes ($\ell_1$ balls) and give a simple and explicit construction of lattice packings in arbitrary dimension whose density is significantly larger than that of any prior construction.

We should also note that dense packings of cross-polytopes induce reasonably dense packings of superballs ($\ell_\sigma$ balls), especially for small values of $\sigma$ (1 ≤ $\sigma$ < 2), by using the trivial method of inscribing a superball inside a cross-polytope, see [12].

An $n$-dimensional cross-polytope $C_n$ is a unit ball in $\mathbb{R}^n$ with respect to the $\ell_1$ metric, $C_n = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$. A cross-polytope of radius $r$ is the body $rC_n = \{r x : x \in C_n\}$ of volume $\frac{(2r)^n}{n!}$. By a discrete cross-polytope of radius $r$ we mean the set $C_n \cap \mathbb{Z}^n$ of cardinality $\sum_{j=0}^{\left\lfloor r \right\rfloor} 2^j \binom{n}{j} \binom{j}{\left\lfloor r \right\rfloor}$.

Rush [12] has shown that, for $n = \frac{p-1}{2}$, where $p$ is an odd prime, the cross-polytope can be constructively lattice packed in $\mathbb{R}^n$ with density $\delta$ satisfying

$$\delta \geq \frac{(2t + 1)^n}{n! (2n + 1)^t}.$$
Here \( t \) is an arbitrary number from the range \( \{1, \ldots, n\} \), and the choice that maximizes the lower bound in (1.1) is

\[
(1.2) \quad t = \frac{n}{\ln(2n + 1)} - \frac{1}{2}.
\]

The idea used in [12] was to obtain the packing lattice from a code with minimum Lee distance \( 2t + 1 \) described in [11, Ch. 9], via the so-called Construction A [3]. Other codes with the desired Lee distance can be used as a basis for such a construction, e.g., the BCH-like code from [11], the resulting density being larger than the one from [12] but still smaller than the density we shall obtain below by a different method.

As pointed out in [12], better lower bounds on the packing density of cross-polytopes may be obtained via non-constructive methods such as the Minkowski–Hlawka theorem. It is desirable, however, both from the mathematical viewpoint and in applications, to have at one’s disposal explicit constructions of packings. As in [12] and most other works, we consider the “constructiveness” of our method self-evident and do not provide a formal definition of this notion (see also [6] for a discussion on this issue).

2. Results

Our construction and the resulting density are given in the statement and the proof of the following theorem.

**Theorem 2.1.** Let \( n \) be a prime power. The cross-polytope can be constructively lattice packed in \( \mathbb{R}^n \) with density

\[
(2.1) \quad \delta > \frac{(2t + 1)^{n-1}}{n! n^t},
\]

where \( t \) is an arbitrary positive integer.

It is easy to show that the choice of \( t \) that maximizes the expression on the right-hand side of (2.1) is

\[
(2.2) \quad t = \frac{n - 1}{\ln n} - \frac{1}{2}.
\]

**Proof.** Let \( b_1, b_2, \ldots, b_n \) be a collection of elements of the cyclic group \((\mathbb{Z}_q, +)\) \((q \) will be specified shortly) having the property that the sums \( b_{i_1} + b_{i_2} + \cdots + b_{i_t} \), where \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq n \), are all different\(^1\). An equivalent way of expressing this condition is that the sums

\[
(2.3) \quad \sum_{i=1}^{n} r_i b_i, \quad \text{where } r_i \in \mathbb{Z}, \ r_i \geq 0, \ \sum_{i=1}^{n} r_i = t, \quad \text{are all different.}
\]

(Here \( r_i b_i \) denotes the sum of \( r_i \) copies of \( b_i \in \mathbb{Z}_q \).) Two elegant constructions of such sets were described in [2], one of which is repeated next for completeness.

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1. Lee distance is essentially the \( \ell_1 \) distance defined on the torus \( \mathbb{Z}_q^n \). A code in \( \mathbb{Z}_q^n \) with minimum Lee distance \( 2t + 1 \) can therefore be seen a packing of discrete cross-polytopes (\( \ell_1 \) balls) of radius \( t \) in the torus, see [4].
2. Such a collection of elements is called a Sidon set (or a Sidon sequence) of order \( t \) [9]. For more on their connection to lattice packing problems, see [7, 8].
For a prime power \( n \), let \( \alpha_1 = 0, \alpha_2, \ldots, \alpha_n \) be the elements of the Galois field \( GF(n) \) and \( \beta \) a primitive element of the extended field \( GF(n^t) \). Let \( b_1, b_2, \ldots, b_n \) be the numbers from the set \( \{1, 2, \ldots, n^t - 1\} \) defined by
\[
\beta^{b_i} = \beta + \alpha_i, \quad i = 1, \ldots, n.
\]
Then the numbers \( b_1 = 1, b_2, \ldots, b_n \) satisfy the condition (2.3), for otherwise, if we had \( b_1 + b_2 + \cdots + b_i = b_{j_1} + b_{j_2} + \cdots + b_{j_i} \), it would follow from (2.3) that
\[
(\beta + \alpha_i)(\beta + \alpha_{i_2}) \cdots (\beta + \alpha_{i_t}) = (\beta + \alpha_{j_1})(\beta + \alpha_{j_2}) \cdots (\beta + \alpha_{j_t})
\]
and, after canceling the \( \beta^t \) terms, that \( \beta \) is a root of a polynomial of degree \( < t \) with coefficients in \( GF(n) \), which is not possible.

Given the above-described elements \( b_1, b_2, \ldots, b_n \) from \( \mathbb{Z}_{n^t - 1} \), define the following lattice:
\[
\mathcal{L} = \left\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \pmod{2t+1}, \quad \sum_{i=1}^{n} x_i b_i = 0 \pmod{n^t-1} \right\}.
\]
We will show that the minimum \( \ell_1 \) distance of the points in this lattice is \( 2t+1 \).

Note that any two points \( x, y \in \mathcal{L} \) with \( \sum_{i=1}^{n} x_i \neq \sum_{i=1}^{n} y_i \) satisfy \( \sum_{i=1}^{n} (x_i - y_i) = k(2t+1) \) for a nonzero \( k \in \mathbb{Z} \). They must be at distance at least \( 2t+1 \) because \( \sum_{i=1}^{n} |x_i - y_i| \geq \sum_{i=1}^{n} |x_i - y_i| = |k|(2t+1) \). Therefore, it suffices to consider the points \( x, y \in \mathcal{L}, x \neq y \), with \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \). Assume that, for two such points, \( \sum_{i=1}^{n} |x_i - y_i| \leq 2t \). Then there exists a point \( z \in \mathbb{Z}^n \) such that \( z = x + r = y + s \), where \( r, s \in \mathbb{Z}^n \) are integer vectors satisfying \( r_i \geq 0, s_i \geq 0, \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i = t \). This, together with the fact that \( \sum_{i=1}^{n} x_i b_i = \sum_{i=1}^{n} y_i b_i \pmod{n^t-1} \) (see (2.6)), implies
\[
\sum_{i=1}^{n} r_i b_i = \sum_{i=1}^{n} s_i b_i \pmod{n^t-1}.
\]
As this contradicts (2.3), our assumption that \( \sum_{i=1}^{n} |x_i - y_i| \leq 2t \) must be wrong. Therefore, as claimed, the minimum \( \ell_1 \) distance of the points in the lattice \( \mathcal{L} \) is \( 2t+1 \), implying that it induces a packing of cross-polytopes of radius \( t + \frac{1}{2} \).

To complete the proof, let us compute the density of the packing just described. The volume of the cross-polytope of radius \( t + \frac{1}{2} \) equals \( \frac{(2t+1)^n}{n!} \), and the determinant of the lattice \( \mathcal{L} \) — the volume of its fundamental cell — equals \( \det \mathcal{L} = (2t+1)(n^t-1) \). The packing density is therefore \( \frac{(2t+1)^n}{n!(n^t-1)} \).

In dimensions \( n \) that are not prime powers, the same construction can be used with \( b_1, b_2, \ldots, b_n \) being, e.g., the first \( n \) of the numbers \( b_1, b_2, \ldots, b_{p(n)} \) from \( \mathbb{Z}_{p(n)^t-1} \) satisfying (2.3), where \( p(n) \) is the smallest prime power greater than or equal to \( n \).

The density of the resulting lattice packing is \( \frac{(2t+1)^n}{n!(p(n))^t} \).

Comparing the densities in (1.1) and (2.1), we see that the latter is larger by a factor of \( \frac{t}{2t+1} \). When \( t \sim \frac{n}{\ln n} \) (the choice that maximizes both, see (1.2) and (2.2)), the improvement is of the order \( 2^{o(n^{1/n})} \).

Compared with \([12]\), our construction has the following advantages: 1.) the packing is defined for every \( n, t \); 2.) the lattice is constructed directly, rather than from
a code with specified minimum distance; 3.) the resulting packing density is larger by a factor of $2^{\Theta(n \ln n)}$ as $n \to \infty$.

**Lattice packings of discrete cross-polytopes in** $\mathbb{Z}^n$. It is evident from the above proof that the lattice $L \subset \mathbb{Z}^n$ from (2.6) defines a packing of discrete cross-polytopes of radius $t$ in $\mathbb{Z}^n$, the density of which is $\frac{|C_{t^c}\cap \mathbb{Z}^n|}{\det L}$ (by density in the discrete case we mean the fraction of points in $\mathbb{Z}^n$ covered by the cross-polytopes). We state this result below as it may be of separate interest.

**Theorem 2.2.** Let $n$ be a prime power and $t$ an arbitrary positive integer. The discrete cross-polytope of radius $t$ can be constructively lattice packed in $\mathbb{Z}^n$ with density

$$\delta > \frac{\sum_{j \geq 0} 2^j \binom{n}{j} \binom{t}{j}}{(2t + 1) n^t}.$$  

(2.8)

For a fixed radius $t$ and $n \to \infty$, the asymptotic value of the expression on the right-hand side of (2.8) is

$$\frac{2^t}{t! (2t + 1)}.$$  

(2.9)

For $t = 1, 2$, this lower bound can be improved. For $t = 1$ the optimal density is in fact equal to 1 for every $n$, as perfect packings of discrete cross-polytopes of radius 1 exist (and are easily constructed) in all dimensions [4]. For $t = 2$, the construction from [12] yields the asymptotic density $\frac{1}{\pi} = \frac{1}{7}$, while the expression in (2.9) equals $\frac{2}{5}$. For $t \geq 3$, the asymptotic density in (2.9) is, to the best of our knowledge, the highest known.

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