Emergent gauge field for a chiral bound state on curved surface

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Abstract
Emergent physics is one of the most important concepts in modern physics, and one of the most intriguing examples is the emergent gauge field. Here we show that a gauge field emerges for a chiral bound state formed by two attractively interacting particles on a curved surface. We demonstrate explicitly that the center-of-mass wave function of such a deeply bound state is monopole harmonic instead of spherical harmonic, which means that the bound state experiences a magnetic monopole at the center of the sphere. This emergent gauge field is due to the coupling between the center-of-mass and the relative motion on a curved surface, and our results can be generalized to an arbitrary curved surface. This result establishes an intriguing connection between the space curvature and gauge field, and paves an alternative way to engineer a topological state with space curvature, and may be observed in a cold atom system.

Keywords: two-body problem, curved surface, gauge field, Dirac monopole

1. Introduction
The emergent gauge field in low-energy physics is a very intriguing physics in nature. It means that the original model does not contain a gauge field explicitly, but gauge theory appears in low-energy effective theory when tracing out some high-energy degree of freedoms. The emergent gauge field has appeared, for instance, in the low-energy field theory description of strongly correlated materials such as doped Mott insulators and spin liquids [1] or Dirac fermions in graphene under local distortions [2]. Here we propose a new scenario that when two particles form a chiral bound state on a two-dimensional curved surface, the center-of-mass motion of this pair, as the remaining low-energy degree of freedom, experiences a gauge field that originated from the space curvature. This result brings about an interesting connection between space curvature and a gauge field in a simple system.

We consider a system consisting of two particles interacting via a short-range potential. Such a problem can be treated by partial wave expansion, and generally the s-wave channel is the most dominant for low-energy collisions. Nevertheless, when the potential is tuned to a p-wave or other high partial wave resonances, the interaction in this high partial wave channel can be very strong and a chiral bound state with non-zero relative angular momentum can form. This kind of two-body problem has been extensively studied in various circumstances in cold atom systems, such as in reduced dimensions [3, 4], or in the presence of gauge field or spin–orbit coupling [5]. It has been found that both the dimensionality and the gauge field can strongly affect the low-energy behavior of the scattering phase shift and bound states. However, such a problem has not been considered in curved spaces and the effect of space curvature on bound state behavior has not been addressed before. This will be the main issue to address in this work. We should also stress that our results focus on the strong pairing regime, which is complementary to previous discussions of BCS superfluids [6, 7] or superfluid vortex on curved surfaces [8].

2. Emergent gauge field in quantum-mechanical two-body problem on a sphere
Before considering the general situation of a curved surface, we illustrate the physics by solving a concrete model of two particles confined on a sphere (with radius $R$) with short-range interaction (with range denoted by $r_0$). Following the standard
procedure of solving a two-body problem with short-range interaction, hereafter we solve this non-interacting problem when the distance between two atoms is larger than $R_0$ and then fix the entire wave function by matching the short-range Bethe–Peierls boundary condition [9].

2.1. Defining coordinates

The two-body system can be described by the coordinates of two particles $r_1$ and $r_2$. For the convenience of separating the center-of-mass and the relative coordinates, we introduce another set of coordinates $(\alpha, \beta, \gamma, \theta)$. We define unit vectors $\hat{n}_c$ and $\hat{n}_r$ as $\hat{n}_c = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{|\mathbf{r}_1 + \mathbf{r}_2|}$ and $\hat{n}_r = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = 2R \sin \frac{\theta}{2}$, which represent the directions of the center-of-mass and the relative positions, respectively, and $\theta$ is the angle between the positions of two particles $r_1$ and $r_2$. Since $\hat{n}_c$ and $\hat{n}_r$ are orthogonal to each other, we can set a body-fixed frame $Ox'y'z'$ where $\hat{n}_c$ is the $z'$ axis, $\hat{n}_r$ is the $y'$ axis and the $x'$ axis is determined by the right hand rule. The body-fixed frame $Ox'y'z'$ is related to the space-fixed frame $Oxyz$ by a rotation matrix $R$, i.e. $(\hat{e}_x', \hat{e}_y', \hat{e}_z') = R(\hat{e}_x, \hat{e}_y, \hat{e}_z)$, and the rotation matrix $R$ can be parameterized by three Euler angles $(\alpha, \beta, \gamma)$.

Figure 1 shows the geometric meaning of $(\alpha, \beta, \gamma, \theta)$. We define the center-of-mass point $C$ as the middle point of the geodesic (the great circle), and $\alpha$ and $\beta$ are the azimuthal and polar angles of point $C$. $\gamma$ is the rotational angle of the geodesic along $\mathbf{n}_c$, it specifies the orientation of the geodesic. The angle $\theta$ represents the relative distance between the two atoms, and is proportional to the length of the geodesic. That is, the coordinates $(\alpha, \beta)$ describe the center-of-mass position, while the coordinates $(\gamma, \theta)$ describe the relative motion.

![Figure 1](image-url)

**Figure 1.** A schematic plot of coordinates $(\alpha, \beta, \gamma, \theta)$ for two particles on a sphere. Angle $\theta$ (which is not shown on the plot) is the angle between $r_1$ and $r_2$. The two-body system can be described by the coordinates of two particles $r_1$ and $r_2$. For the convenience of separating the center-of-mass and the relative coordinates, we introduce another set of coordinates $(\alpha, \beta, \gamma, \theta)$. We define unit vectors $\hat{n}_c$ and $\hat{n}_r$ as $\hat{n}_c = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{|\mathbf{r}_1 + \mathbf{r}_2|}$ and $\hat{n}_r = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = 2R \sin \frac{\theta}{2}$, which represent the directions of the center-of-mass and the relative positions, respectively, and $\theta$ is the angle between the positions of two particles $r_1$ and $r_2$. Since $\hat{n}_c$ and $\hat{n}_r$ are orthogonal to each other, we can set a body-fixed frame $Ox'y'z'$ where $\hat{n}_c$ is the $z'$ axis, $\hat{n}_r$ is the $y'$ axis and the $x'$ axis is determined by the right hand rule. The body-fixed frame $Ox'y'z'$ is related to the space-fixed frame $Oxyz$ by a rotation matrix $R$, i.e. $(\hat{e}_x', \hat{e}_y', \hat{e}_z') = R(\hat{e}_x, \hat{e}_y, \hat{e}_z)$, and the rotation matrix $R$ can be parameterized by three Euler angles $(\alpha, \beta, \gamma)$.

2.2. Expressing the kinetic energy operator in the new coordinates

The two-body system $(r_1, r_2)$ can be regarded as a single point moving on the product manifold $S^2 \times S^2$ equipped with metric $ds^2 = dr_1^2 + dr_2^2$. Under the coordinates $(\alpha, \beta, \gamma, \theta)$, the metric tensor becomes $ds^2 = dr_1^2 + dr_2^2 = g_{\alpha\beta} dr^\alpha dr^\beta$, where $(u^1, u^2) = (\alpha, \beta, \gamma, \theta)$, where $g_{\alpha\beta}$ is listed in the appendix. Now the kinetic energy operator becomes (we set $\hbar = m = R = 1$)

$$\hat{T} = \frac{L_1^2}{2} + \frac{L_2^2}{2} = -\frac{1}{\sqrt{g}} \partial_{\gamma} (\sqrt{g} g^{\gamma\theta} \partial_{\theta}),$$

(1)

where $g = \sin^2 \beta \sin^2 \theta$ is the determinant of matrix $g_{\alpha\beta}$, $g^{\gamma\theta}$ is the matrix inverse of the $g_{\alpha\beta}$. After some straightforward simplification, we obtain

$$\hat{T} = \frac{J_1^2}{4} + \frac{J_2^2}{4} + \frac{J_3^2}{4 - \sin^2 \frac{\theta}{2}} \frac{1}{\sin \theta} \partial_{\theta} \left( \sin \theta \partial_{\theta} \right),$$

(2)

where $J_1, J_2, J_3$ are the angular momenta along the three body-fixed axes $x', y'$ and $z'$. It is worth mentioning that $J_3$ only depends on Euler angles $(\alpha, \beta, \gamma)$, and their explicit expressions are listed in the appendix.

In the limit $\theta \to 0$, $\hat{T}$ becomes

$$\hat{T} \approx \frac{J_1^2}{4} + \frac{J_2^2}{4} + \frac{1}{\theta} \partial_{\theta} \left( \theta \partial_{\theta} \right) + \frac{J_3^2}{\theta^2}.$$  

(3)

In this limit, the two-body system is effectively living on the tangent plane of the sphere. By comparing $\hat{T}$ with the kinetic energy operator on a flat 2D plane, we can see that $\hat{T}$ corresponds to the relative distance between two particles, $J_3$ is the relative angular momentum $l$ (therefore, hereafter we use $l$ to denote the quantum number of $J_3$) and $J_1$ and $J_2$ represent the center-of-mass moment. This also suggests that we can still apply the short-range boundary condition for a flat 2D plane to our problem.

2.3. Eigen-wave function of the kinetic energy

To determine the eigenstates of the kinetic energy term, we expand the wave function in terms of Wigner D-matrices, which are often used to solve the spectrum of a rigid rotor [10]. The Wigner D-matrix $D_{\alpha\beta\gamma}(\alpha, \beta, \gamma)$ is a common eigenstate of operators $J_1^2, J_2$, and $J_3$ with eigenvalues $j(j + 1)$, $m$ and $l$, respectively. Here $J_1 = -i \partial_{\theta}$ represents the angular momentum along the space-fixed $z$ axis. $\hat{T}$ has a simple form under basis $D_{\alpha\beta\gamma}$ because

$$\frac{J_1^2}{4} + \frac{J_2^2}{4 \cos^2 \theta/2} + \frac{J_3^2}{4 \sin^2 \theta/2} = A J_1^2 + B J_2^2 - C (J_1^2 + J_2^2),$$

where $J_0 = J_1 \pm i J_2$ are the lowering and raising operators of $J_3$, $A, B$ and $C$ are simple functions of $\theta$ (see appendix). From
the above equation, we can see that the first two terms are diagonal under basis $D_{ij}$, while $J^2$ can couple $l$ to $l \mp 1$.

For instance, if we consider a $p$-wave interaction, $l$ should take the value of $\pm 1$ at short distance and can be coupled to $l=\pm 3, \ldots$ at long distance. Therefore, the wave function can be written as

$$\psi^\pm = \varphi_1^k(\theta) \frac{D_{m1}^{i\pm} \pm D_{m1}^{i\pm}}{\sqrt{2}} + \varphi_3^k(\theta) \frac{D_{m3}^{i\pm} \pm D_{m3}^{i\pm}}{\sqrt{2}} + \ldots \quad (4)$$

Here the superscript $\pm$ stands for two sets of solutions with different parity. Here we note that the expansion only contains a finite number of terms since $l$ cannot exceed the total angular momentum $j$.

2.4. Matching the short-range boundary condition

For a $p$-wave interaction on a 2D plane, the boundary condition is

$$\phi_{2D}(r) \sim \left( \frac{1}{r} - \frac{\pi r}{4s} \right) - \frac{E_{rel}}{2} \log \frac{r}{r_0}, \quad (5)$$

where $s$ is the scattering area, $r_0$ is the effective range for the $p$-wave scattering, and $E_{rel}$ is the energy for relative motion. $s$ can be tuned by a magnetic Feshbach resonance in an ultracold atomic system. The eigenenergy $E$ can then be calculated by expanding the wave functions equation (4) in the short-range limit and comparing them with the boundary condition equation (5).

In figure 2, we plot the wave functions for the total angular momentum $j = 3$. As one can see, it turns out that for a $p$-wave deeply bound state, both $\varphi_1^\pm$ approach the same $\varphi_{2D}$ and $\varphi_3^\pm$ vanish.

![Figure 2.](image-url)

**Figure 2.** The wave functions $\varphi_1^\pm$ (a) and $\varphi_1^\mp$ (b) and $\varphi_3^\pm$ (c) and $\varphi_3^\mp$ (d) for total angular momentum $j = 3$. Solid blue lines are for a shallow bound state with $E = -3.5 \hbar^2/(mR^2)$, while solid red lines are for a deeply bound state with $E = -10 \hbar^2/(mR^2)$. The dashed line is the bound state wave function $\varphi_{2D}(r)$ with $r = 0$ for 2D plane with same short-range boundary condition. This plot shows that for a deeply bound state, both $\varphi_1^+$ and $\varphi_3^+$ approach the same $\varphi_{2D}$ and $\varphi_3^\pm$ vanish.
This is a central result for this part. We see that in the limit of a deeply bound state, the center-of-mass and the relative motion become effectively decoupled and the wave function can be written as a product of the relative and the center-of-mass wave functions. The center-of-mass wave function is the most intriguing and suggestive part. It is represented by a Wigner D-matrix. Mathematically, it is known that the Wigner D-matrices are related to the monopole harmonics \( Y_{l,j,m} \) by a gauge transformation introduced by Wu and Yang \([11, 12]\). They are the eigenstates of a charged particle moving around a magnetic monopole. This suggests that there emerges a gauge field in our system which couples to the center-of-mass motion, and the charge of this monopole is the quantum number \(-l\). That is to say, it exists for \(p\)-wave or other higher partial wave bound states but not for \(s\)-wave. For bound states with opposite angular momentum, \(l\) or \(-l\), they experience an opposite monopole charge.

3. **Gauge field in the classical Hamiltonian**

The classical kinetic energy is given by \( T = \frac{1}{2} g_{ij} \dot{u}^i \dot{u}^j \), it can be separated into \( T_c + T_{rel} \) as

\[
T_c = \frac{1}{2} \{ (\alpha, \beta) \} h_{\alpha \beta} (\dot{\alpha}, \dot{\beta}),
\]

\[
T_{rel} = \frac{1}{2} I_3 (\gamma + \cos \beta \alpha)^2 + \frac{1}{4} \dot{\beta}^2,
\]

where \( I_3 = \sin^2 \frac{\theta}{2} \) is the moment of inertia along the \( z' \) axis, \( h_{\alpha \beta} \) is a two by two matrix whose elements are listed in the appendix. The angular velocity along the \( z' \)-axis is \( \omega_3 = \gamma + \cos \beta \alpha = \gamma + A_\alpha \alpha \), where \( A_\alpha = \cos \beta \). Introducing the conjugate momenta as \( p_3 = \partial T_3 / \partial \dot{u}_3 \) as

\[
\begin{pmatrix} p_3 \\ p_\beta \end{pmatrix} = h_{\alpha \beta} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + I_3 A_\alpha \begin{pmatrix} \dot{\gamma} + A_\alpha \alpha \\ 0 \end{pmatrix},
\]

\[
L_\alpha = p_3 I_3 (\gamma + A_\alpha \alpha), \quad p_\beta = \frac{1}{2} \dot{\beta}.
\]

the classical Hamiltonian can be rewritten as

\[
H = \frac{1}{2} (p_3 - L A_3, p_\beta) h^{\alpha \beta} \begin{pmatrix} p_\alpha - L A_\alpha \\ p_\beta \end{pmatrix} + \frac{L^2}{2 I_3} + p_\beta^2 + ..., \tag{10}
\]

where \( h^{\alpha \beta} \) is the inverse of \( h_{\alpha \beta} \), and ... represents the remaining potential and interaction terms.

This Hamiltonian shows that the center-of-mass effectively moves on a deformed sphere with metric \( h_{\alpha \beta} \). While in the deeply bound limit, we find

\[
\lim_{\rho \to 0} h_{\alpha \beta} = 2 \begin{pmatrix} \sin^2 \beta & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}
\]

which is exactly the metric tensor of a sphere. In addition, there emerges a vector potential \( A \) corresponding to the coupling between the center-of-mass and the relative angular momentum \( L \). If we calculate the corresponding magnetic field of the vector potential \( A \), we obtain

\[
B = \nabla \times A = \frac{1}{R^2 \sin \beta} \frac{\partial A_\alpha}{\partial \beta} \dot{e}_\alpha = -\frac{\dot{e}_\beta}{R^2}. \tag{12}
\]

This is exactly the magnetic field of a Dirac monopole at the center of this sphere. The relative angular momentum plays the role of a monopole charge, which is quantized in a quantum theory. Thus, it is consistent with the wave function of a pair found in the quantum theory.

4. **Physical meaning of the gauge transformation**

It is natural to ask what the meaning of gauge transformation is. To answer this question, we recall that \( \gamma \) is the angle between the geodesic and the local base vector \( \dot{e}_\gamma \). Thus, the gauge transformation simply represents a local rotation of the base vector by \( f(\alpha, \beta) \), and the Hamiltonian is invariant under following gauge transformation, \( \gamma \to \gamma - f(\alpha, \beta) \), \( A_\alpha \to A_\alpha + \delta_\alpha f \) and \( A_\beta \to A_\beta + \partial_\beta f \), where \( f(\alpha, \beta) \) is an arbitrary function defined on the sphere. According to the definition of \( \omega_3 \), we know that \( d \gamma = \omega_3 \dot{r} - A_\gamma \dot{\alpha} \). If the center-of-mass is carried along close path \( C \) on the sphere, the net change of \( \gamma \) should not depend on the gauge choice, and it is given by

\[
\Delta \gamma = \int_C \omega_3 \dot{r} - \oint_C A_\gamma \dot{\alpha} = \int_C \omega_3 \dot{r} - \oint_C A \cdot \dot{\ell}. \tag{13}
\]

Besides the regular \( \int_C \omega_3 \dot{r} \) term, the net change \( \Delta \gamma \) acquires an extra geometric term related to the vector potential \( A \), which only depends on the path \( C \) and is an analog of the geometric phase introduced by Berry \([13]\). Using Stokes’ theorem we can write

\[
\oint_C A \cdot \dot{\ell} = \int_C B \cdot \dot{\ell} \],

which is clearly gauge invariant. That is to say, the emergent gauge field is related to this geometric effect.

A more intuitive understanding of this gauge field is shown in figure 3. For a chiral bound state, two atoms rotate around the normal axis \( \hat{n} \), which is always perpendicular to the local surface. Therefore, when the bound state travels along a closed loop on the surface, the direction of this normal axis varies, which gives rise to a geometric term identical to the solid angle expanded by the direction. This is similar to the Berry’s phase effect of spin varying on the Bloch sphere.

5. **Generalization to arbitrary manifolds**

This emergent gauge field can also be generalized to an arbitrary two-dimensional manifold \( M \) with metric

\[
d s^2_M = g_{ij} \dot{u}^i \dot{u}^j = g_{11} \dot{u}^1 \dot{u}^1 + g_{22} \dot{u}^2 \dot{u}^2. \tag{14}
\]

For simplicity, we have chosen orthogonal coordinates \((u^1, u^2)\) such that \( g_{12} = g_{21} = 0 \). Similar to the sphere case, we introduce coordinates \((q^1, q^2, \gamma, r)\), as shown in figure 4, where \((q^1, q^2)\) are the coordinates of the center-of-mass which is defined as the middle point of the geodesic connecting two particles, \( r \) is the length of the geodesic, and \( \gamma \) is the angle between the geodesic and the tangent vector \( \hat{e}_1 = \frac{\hat{\gamma}}{\sqrt{g_{11}}} \) at point \( C \).
Generally, it is difficult to write the metric tensor $ds^2$ explicitly using coordinates $(q^1, q^2, \gamma, r)$. Nevertheless, if we assume that the two particles form a deeply bound state such that we are only interested in the regime where $r$ is much smaller than any other length scales, we can expand $ds^2$ in the small $r$ limit, and to the leading order, $ds^2$ becomes

$$ds^2 = \frac{1}{2} (r^2 d\phi^2 + dr^2) + 2g_{\mu\nu} dq^\mu dq^\nu + r^2 \sqrt{g} \Gamma^{\mu}_{\nu\rho} \sqrt{g} dq^\nu d\gamma .$$

(15)

Here $\Gamma^{\mu}_{\nu\rho}$ is the Christoffel symbol of the connection. Given the expression of the metric tensor, and performing similar derivation above, we find that the kinetic energy can be expressed as

$$\hat{T} = \frac{1}{2} \frac{L^2}{r^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right),$$

(16)

where $L = -i \partial$ and $A_{\mu} = \frac{r}{8} \Gamma_{\mu}^{\nu\rho}$. Again, we obtain a vector potential $A_{\mu}$ and the corresponding magnetic field is given by

$$B = \nabla \times A = -K \hat{e}_i .$$

(17)

Here one can show that $K$ is exactly the Gaussian curvature of $\mathcal{M}$, and $\hat{e}_i = \hat{e}_1 \times \hat{e}_2$ is a unit vector perpendicular to the surface. That is to say, the gauge field corresponds to a magnetic field perpendicular to the surface with the strength identical to the local Gaussian curvature. It is easy to see that the sphere case discussed above is a special example of this generalized result.

6. Conclusion

We have calculated the two-body problem with short-range interaction on a sphere. By introducing a set of new coordinates, we exactly calculated the bound state problem and demonstrated that the center-of-mass experiences an emergent magnetic monopole field. We discussed the physical origin of this emergent gauge field and generalized this result to an arbitrary curved surface. We add two final remarks before finishing.

First, if one creates a surface with periodically modulated curvature, it gives rise to a magnetic flux lattice to a chiral bound state and creates topological band structure. More interestingly, considering a bound state with either angular momentum $l$ or $-l$, it can be considered as a spin-$\frac{1}{2}$ particle, and since the emergent magnetic fields are opposite for opposite angular momenta, the time-reversal symmetry is recovered which can lead to a time-reversal invariant topological insulator. Our results indicate that topological matter may also arise from nontrivial space curvature.

Secondly, our results can also be directly tested in cold atom experiments [14]. For instance, considering a Rydberg atom whose outermost shell electron is excited to a highly excited state with wave function $|\phi^e(r)\rangle$, the interaction between other ground state atoms and the Rydberg atom is proportional to the density of the excited electron $|\phi^e(r)\rangle^2$ [15, 16], where the most attractive part is a thin shell of sphere centered at the position of the ion. It has been observed that a few atoms can be trapped by this potential shell [15, 16]. These trapped atoms basically live on a two-dimensional sphere and interact via a short-range potential, as our model requires. This will be an ideal system to test our results.

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Appendix

A.1. Coordinates transformation

For two particles on a sphere, the kinetic energy operator can be expressed as \( \hat{T} = \frac{J_i^2}{2} + \frac{J_j^2}{2} = -\frac{1}{2}\Delta_{L,B} \). Here \( \Delta_{L,B} \) is the Laplace–Beltrami operator on the produce manifold \( S^2 \times S^2 \). It is related to the metric tensor \( g_{ij} \) by

\[
\Delta_{L,B} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j).
\]

Recall that the metric tensor is given by

\[
d s^2 = d r^2 + d \theta^2 = \left[ d\left(\cos \frac{\theta}{2} \hat{n}_c + \sin \frac{\theta}{2} \hat{n}_r \right) \right]^2 + \left[ d\left(\cos \frac{\theta}{2} \hat{n}_c - \sin \frac{\theta}{2} \hat{n}_r \right) \right]^2.
\]

It is now straightforward to express \( \hat{n}_c \) and \( \hat{n}_r \) in terms of \((\alpha, \beta, \gamma, \theta)\) and substitute them into above equation. We then obtain \( ds^2 = g_{ij} du^i du^j \), with \( g_{ij} = \)

\[
\begin{pmatrix}
\sin^2 \beta \left( \cos^2 \gamma + \cos^2 \frac{\theta}{2} \sin^2 \gamma \right) + \sin^2 \frac{\theta}{2} \cos^2 \beta & -\sin^2 \frac{\theta}{2} \sin \beta \sin \gamma \sin \gamma & \sin^2 \gamma \cos \gamma & 0 \\
-\sin^2 \frac{\theta}{2} \sin \beta \sin \gamma & \sin^2 \gamma & 0 & 0 \\
\sin^2 \frac{\theta}{2} \cos \beta & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

while \((u^1, u^2, u^3, u^4)\) refer to \((\alpha, \beta, \gamma, \theta)\). The metric tensor \( h_{\mu \nu} \) mentioned in the main text can also be calculated

\[
h_{\mu \nu} = 2 \begin{pmatrix}
\sin^2 \beta \left( \cos^2 \gamma + \cos^2 \frac{\theta}{2} \sin^2 \gamma \right) - \sin^2 \frac{\theta}{2} \sin \beta \sin \gamma \\
-sin^2 \frac{\theta}{2} \sin \beta \sin \gamma & \sin^2 \gamma + \cos^2 \frac{\theta}{2} \cos^2 \gamma
\end{pmatrix}
\]

A.2. Hamiltonian in new coordinates

Substitute the metric \( g_{ij} \) into equation (18), we then obtain the kinetic energy equation (2) in main text with

\[
J_1 = i \left( \frac{\cos \gamma}{\sin \beta} \frac{\partial}{\partial \alpha} - \sin \gamma \frac{\partial}{\partial \beta} + \cot \beta \cos \gamma \frac{\partial}{\partial \gamma} \right)
\]

\[
J_2 = i \left( \frac{\sin \gamma}{\sin \beta} \frac{\partial}{\partial \alpha} - \cos \gamma \frac{\partial}{\partial \beta} + \cot \beta \sin \gamma \frac{\partial}{\partial \gamma} \right)
\]

\[
J_3 = -i \frac{\partial}{\partial \gamma}.
\]

They are nothing but the total angular momenta along three body-fixed axis. One can calculate the commutation relations directly and find that \([J_k, J_l] = -i \epsilon_{kjl} J_k \). Therefore, we can construct the lowering and raising operators \( J_\pm = J_1 \pm i J_2 \) and rewrite the rotation part as

\[
\frac{J_1^2}{4} + \frac{J_2^2}{4 \cos^2 \theta/2} + \frac{J_3^2}{4 \sin^2 \theta/2} = AJ_1^2 + BJ_2^2 - C(J_1^2 + J_2^2),
\]

with

\[
A = \frac{1}{8} \left( \frac{1}{\cos \theta/2} + 1 \right), \quad B = \frac{1}{8} \left( \frac{2}{\sin \theta/2 \cos \theta/2} - 1 \right), \quad C = \frac{\tan^2 \theta/2}{16}.
\]

A.3. Metric tensor on an arbitrary curved surface

To calculate the metric tensor on an arbitrary curved manifold \( M \), we shall first calculate the geodesic \( u^\lambda(t) \) that connects \( x \) and \( y \) (see figure 1 in main text). It is given by the geodesic equation

\[
\frac{d^2 u^\lambda}{dt^2} + \Gamma^\lambda_{\mu \nu} \frac{du^\mu}{dt} \frac{du^\nu}{dt} = 0,
\]

with initial condition \( u^\lambda(0) = q^\lambda \) and \( \dot{u}^\lambda(0) = v^\lambda \). While \( v^\lambda \) are set by the angle \( \gamma \),

\[
v^\lambda = \frac{\cos \gamma}{\sqrt{\gamma_{11}(q^\gamma)}} \quad \text{and} \quad v^2 = \frac{\sin \gamma}{\sqrt{\gamma_{22}(q^\gamma)}}.
\]

In the tightly bind limit, one can expand \( u^\lambda(t) \) by \( t \) and obtain

\[
u^\lambda(t) = q^\lambda + v^\gamma t - \frac{1}{2} \Gamma^\lambda_{\mu \nu} v^\mu v^\nu t^2 + O(t^3).
\]

Now we can express the old coordinates \( x^\lambda \) and \( y^\lambda \) in new coordinates via \( x^\lambda = u^\lambda(r/2) \) and \( y^\lambda = u^\lambda(-r/2) \). Following the same approach in the spherical case, we obtain the metric tensor in the main text.

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