A construction of the free digroup

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Abstract
We give a construction of the free digroup on a set $X$ and describe the halo and the group parts of it. We prove that the free digroup on $X$ is isomorphic to the free digroup on a set $Y$ if and only if $\text{card}(X) = \text{card}(Y)$.

Keywords Digroup · Dialgebra · Free digroup · Gröbner–Shirshov basis

1 Introduction and preliminaries
In the theory of Leibniz algebras, one of the prominent open problems is to find an appropriate generalization of Lie’s third theorem, which associates a (local) Lie group to any (real or complex) Lie algebra. A key and difficult aspect of this problem is to find the appropriate analogue of Lie group for Leibniz algebras, that is, to determine what should be the correct generalization of the notion of a group. So little is known about what properties these group-like objects should have that Loday dubbed them “coquecigrues” in [4]. And so the problem has come to be known as the “coquecigrue” problem (for Leibniz algebras).

The notion of a digroup first implicitly appeared in Loday’s work [5]. Further, it was proposed independently by Kinyon [2], Felipe [1] and Liu [3], to provide a partial...
solution to the coquecigrue problem for Leibniz algebras. Kinyon [2] gave a much cleaner definition of a digroup as follows.

**Definition 1** A digroup is a pair \((G, 1)\) equipped with two binary operations \(\triangleright\) and \(\triangleleft\), a unary operation \(\dagger\), and a nullary operation 1, where 1 is called the unit of \(G\), satisfying each of the following six axioms:

\[
\begin{align*}
G1. \; (G, \triangleright) \text{ and } (G, \triangleleft) \text{ are both semigroups,} \\
G2. \; a \triangleleft (b \triangleright c) &= a \triangleleft (b \triangleright c), \\
G3. \; (a \triangleleft b) \triangleright c &= (a \triangleright b) \triangleleft c, \\
G4. \; a \triangleright (b \triangleleft c) &= (a \triangleright b) \triangleleft c, \\
G5. \; 1 \triangleright a &= a = a \triangleleft 1, \\
G6. \; a \triangleright a^\dagger &= 1 = a^\dagger \triangleleft a.
\end{align*}
\]

Using semigroup theory, Kinyon showed that every digroup is a product of a group and a “trivial” digroup, in the service of his partial solution to the coquecigrue problem. Of course, digroups are a generalization of groups and play an important role in this open problem from the theory of Leibniz algebras. Recently the study of algebraic properties on digroups has attracted considerable attention. In [7], Phillips gave a simple basis of independent axioms for the variety of digroups. Salazar-Díaz, Velásquez and Wills-Toro [8] studied a further generalization of the digroup structure and showed analogues to the first isomorphism theorem. Lately Ongay, Velásquez and Wills-Toro [6] discussed the notion of normal subdigroups and quotient digroups, and then established the corresponding analogues of the classical Isomorphism Theorems. Different examples of digroups can be found in [12]. For further investigation of the structure of digroups, A.V. Zhuchok proposed some open problems at the end of his paper [11]. One of the open problems is: “Construct the free digroup”.

In this paper, we apply the method of Gröbner–Shirshov bases for dialgebras in [9] to give a construction of the free digroup on a set and solve the above problem by Zhuchok.

In [9], a Composition-Diamond lemma for dialgebras was established and a method to find normal forms of elements of an arbitrary disemigroup was given. Clearly, the variety of digroups is a subvariety of the variety of disemigroups. Then the free digroup on a set is a quotient of a free disemigroup modulo certain relations.

The paper is organized as follows. In Sect. 2, we recall the definitions of the Gröbner–Shirshov bases and Composition-Diamond Lemma for dialgebras in [9]. In Sect. 3, we give a Gröbner–Shirshov basis for the free digroup \(F(X)\) on a set \(X\) and thus obtain explicit normal forms of elements of \(F(X)\). Moreover, we also describe the halo and the group parts of \(F(X)\) and show that \(F(X) \cong F(Y)\) if and only if \(\text{card}(X) = \text{card}(Y)\).

## 2 Composition-diamond lemma for dialgebras

As is known to all, the method of Gröbner–Shirshov bases is a powerful tool to solve the normal form problem in various categories, including dialgebras [9]. In this section,
we review some of concepts and results on Gröbner–Shirshov bases for dialgebras, see [9].

Let $k$ be a field. A dialgebra (disemigroup) is a $k$-module (set) equipped with two binary operations $\vdash$ and $\triangleright$, satisfying axioms G1–G4 in (1). Recall that for every dialgebra (disemigroup) $D$, for all $x_1, \ldots, x_n$ in $D$, every parenthesesizing of $x_1 \vdash \cdots \vdash x_m \triangleright \cdots \triangleright x_t$ gives the same element in $D$, which we denote by $[x_1 \ldots x_t]_m$. In particular, the notation $[x_1]_1$ means $x_1$. Let $Di\langle X \rangle$ be the free dialgebra over $k$ generated by a set $X$, $X^+$ the set of all nonempty associative words on $X$ and $X^* = X^+ \cup \{\epsilon\}$ the free monoid on $X$, where $\epsilon$ is the empty word. Write

$$[X^+]_o := \{[u]_m \mid u \in X^+, m \in \mathbb{Z}^+, 1 \leq m \leq |u|\},$$

where $|u|$ is the length of $u$. Note that $[x]_1 = x$ if $x \in X$. It is well known from [5,9] that $[X^+]_o$ is the free disemigroup on $X$ and a $k$-basis of $Di\langle X \rangle$, where for all $[u]_m, [v]_n \in [X^+]_o$,

$$[u]_m \vdash [v]_n = [uv]_{|u|+n}, \ [u]_m \triangleright [v]_n = [uv]_m.$$

Free disemigroups were studied in [10]. For any $h = [u]_m \in [X^+]_o$, we call $u$ the associative word of $h$, and $m$, denoted by $p(h)$, the position of center of $h$.

Let $X$ be a well-ordered set. For any $u = x_{i_1}x_{i_2}\cdots x_{i_n}, v = x_{j_1}x_{j_2}\cdots x_{j_m} \in X^+$, where $x_{i_1}, x_{j_1} \in X$, define

$$u > v \iff ([u], x_{i_1}, x_{i_2}, \cdots, x_{i_n}) > ([v], x_{j_1}, x_{j_2}, \cdots, x_{j_m}) \text{ lexicographically.}$$

Then $>$ is a well ordering on $X^+$ and we call it the \textit{deg-lex ordering}.

A well ordering $>$ on $X^+$ is \textit{monic} if for any $u, v \in X^+$, we have

$$u > v \implies w_1uw_2 > w_1vw_2 \text{ for all } w_1, w_2 \in X^*.$$

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Clearly, the deg-lex ordering is monomial. Let $X^+$ be endowed with a monomial ordering. We define the \textit{monic-center ordering} on $[X^+]_o$ as follow: for any $[u]_m, [v]_n \in [X^+]_o$,

$$[u]_m > [v]_n \iff (u, m) > (v, n) \text{ lexicographically.} \text{ (2)}$$

In particular, if $X^+$ is endowed with the deg-lex ordering, we call the ordering defined by (2) the \textit{deg-lex-center ordering} on $[X^+]_o$. It is clear that a monomial-center ordering is a well ordering on $[X^+]_o$. Here and subsequently, the monomial-center ordering on $[X^+]_o$ will be used, unless otherwise stated.

For convenience we assume that $[u]_m > 0$ for any $[u]_m \in [X^+]_o$. Then every nonzero polynomial $f \in Di\langle X \rangle$ has the leading monomial $\overline{f}$. We denote the associative word of $\overline{f}$ by $\mathcal{f}$ and the leading term of $f$ by $lt(f)$. If $\mathcal{f} > \mathcal{f}_f$, where $r_f := f - lt(f)$, then $f$ is called strong. If the coefficient of $\overline{f}$ in $f$ is equal to 1, then $f$ is called \textit{monic}. For a subset $S$ of $Di\langle X \rangle$, $S$ is \textit{monic} if $s$ is monic for all $s \in S$.\textcopyright Springer
Let $s$ be a monic polynomial. If

$$g = [x_1 \cdots x_k \cdots x_n]_{x_{i_k} \to x} := [x_1 \cdots x_{i_k-1} x_{i_k} x_{i_k+1} \cdots x_n]_m,$$

where $1 \leq k \leq n$, $x_i \in X$, $1 \leq l \leq n$, then we call $g$ an $s$-diword. For simplicity, we denote the $s$-diword of the form (3) by $(asb)$, where $a, b \in X^*$. Moreover, if either $k = m$ or $s$ is strong, then we call the $s$-diword $g$ is normal. Let $(asb)$ be a normal $s$-diword, then $(asb) = [a \tilde{a} b]_l$ for some $l \in P((asb))$, where

$$P((asb)) := \begin{cases} \{n \in \mathbb{Z}^+ | 1 \leq n \leq |a|\} \cup \{|a| + p(\tilde{a})\} & \text{if } s \text{ is strong}, \\ \cup \{n \in \mathbb{Z}^+ | |a\tilde{a}| < n \leq |a\tilde{a}b|\} & \text{if } s \text{ is not strong}. \end{cases}$$

If this is so, we denote the normal $s$-diword (or normal $S$-diword) $(asb)$ by $[asb]_l$.

Let $f, g$ be monic polynomials in $Di(X)$.

1) If $f$ is not strong, then we call $x \vdash f$ the composition of left multiplication of $f$ for all $x \in X$ and $f \vdash [u]_{[u]}$ the composition of right multiplication of $f$ for all $u \in X^+$.

2) Suppose that $w = \tilde{f} = a\tilde{a}b$ for some $a, b \in X^*$ and $(agb)$ is a normal $g$-diword.

2.1 If $p(\tilde{f}) \in P([agb])$, then we call

$$(f \cdot g)_{\tilde{f}} = f - [agb]_{p(\tilde{f})}$$

the composition of inclusion of $f$ and $g$.

2.2 If $p(\tilde{f}) \notin P([agb])$ and both $f$ and $g$ are strong, then for any $x \in X$ we call

$$(f \cdot g)_{[xw]}_l = [xf]_l - [xagb]_l$$

the composition of left multiplicative inclusion of $f$ and $g$, and

$$(f \cdot g)_{[wx]}_w = [fx]_w - [agbx]_{[wx]}$$

the composition of right multiplicative inclusion of $f$ and $g$.

3) Suppose that there exists $w = \tilde{f}b = a\tilde{a}$ for some $a, b \in X^*$ such that $|\tilde{f}| + |\tilde{a}| > |w|$, $(fb)$ is a normal $f$-diword and $(ag)$ is a normal $g$-diword.

3.1 If $P([fb]) \cap P([ag]) \neq \emptyset$, then for any $m \in P([fb]) \cap P([ag])$ we call

$$(f \cdot g)_{[w]}_m = [fb]_m - [ag]_m$$

the composition of intersection of $f$ and $g$.

3.2 If $P([fb]) \cap P([ag]) = \emptyset$ and both $f$ and $g$ are strong, then for any $x \in X$ we call

$$(f \cdot g)_{[xw]}_l = [xfb]_l - [xag]_l$$
the composition of left multiplicative intersection of $f$ and $g$, and

$$(f \cdot g)_{[wx]} = [fx]_{[wx]} - [gx]_{[wx]}$$

the composition of right multiplicative intersection of $f$ and $g$.

For any composition $(f \cdot g)_n$ mentioned above, we call $[u]_n$ the ambiguity of $f$ and $g$.

**Definition 2** ([9] Definition 3.7) Let $S$ be a monic subset of $Di \langle X \rangle$. A polynomial $h \in Di \langle X \rangle$ is called trivial modulo $S$, if $h = \sum_i \alpha_i [a_i s_i b_i]_{m_i}$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, and $[a_i s_i b_i]_{m_i} \leq \overline{h}$ if $\alpha_i \neq 0$.

A monic set $S$ is called a Gröbner–Shirshov basis in $Di \langle X \rangle$ if any composition of polynomials in $S$ is trivial modulo $S$.

For convenience, for any $f, g \in Di \langle X \rangle$ and $[w]_m \in [X^+]_\omega$, we write

$$f \equiv g \mod (S, [w]_m)$$

which means that $f - g = \sum_i \alpha_i [a_i s_i b_i]_{m_i}$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, and $[a_i s_i b_i]_{m_i} < [w]_m$ if $\alpha_i \neq 0$.

Note that for a monic set $S$, $S$ is a Gröbner–Shirshov basis in $Di \langle X \rangle$ if and only if

(i) for any composition $h$ of the form 1) in $S$, $h$ is trivial modulo $S$;
(ii) for any $f, g \in S$ and any composition $(f \cdot g)_{[w]_m}$ of the form 2) or 3), $(f \cdot g)_{[w]_m} \equiv 0 \mod (S, [w]_m)$.

The following lemma is Composition-Diamond lemma for dialgebras in [9].

**Lemma 1** ([9] [Theorem 3.18]) Let $S$ be a monic subset of $Di \langle X \rangle$, $> a$ monomial-center ordering on $[X^+]_\omega$ and $Id(S)$ the ideal of $Di \langle X \rangle$ generated by $S$. Then the following statements are equivalent.

(i) $S$ is a Gröbner–Shirshov basis in $Di \langle X \rangle$.
(ii) $f \in Id(S) \Rightarrow \overline{f} = [ab]_m$ for some normal S-diword $[ab]_m$.
(iii) $Irr(S) = \{[u]_m \in [X^+]_\omega \mid [u]_m \neq [ab]_m \}$ for any normal S-diword $[ab]_m$ is a $k$-basis of the quotient dialgebra $Di \langle X \mid S \rangle := Di \langle X \rangle / Id(S)$.

Let $Disgp(X) := [X^+]_\omega$ be the free disemigroup on $X$. For an arbitrary disemigroup $D$, $D$ has an expression

$$D = Disgp(X \mid S) := [X^+]_\omega / \rho(S)$$

for some generator set $X$ with defining relations $S \subseteq [X^+]_\omega \times [X^+]_\omega$, where $\rho(S)$ is the congruence on $([X^+]_\omega, \cdot, \cdot, \cdot)$ generated by $S$. For convenience, we let $[u]_m - [v]_n$ or $[u]_m = [v]_n$ stand for the pair $([u]_m, [v]_n)$ in $Disgp(X)$.

**Lemma 2** ([9] [Theorem 5.1]) Let $D = Disgp(X \mid S)$ be the disemigroup generated by $X$ with defining relations $S$ and $> a$ be a monomial-center ordering on $[X^+]_\omega$. If $S \subseteq Di \langle X \rangle$ is a Gröbner–Shirshov basis, then $Irr(S) = \{[u]_m \in [X^+]_\omega \mid [u]_m \neq [ab]_m \}$ for any normal S-diword $[ab]_m$ is a set of normal forms of elements of $D$. 
3 Free digroup

In this section, we first introduce the notion of the free digroup, and then provide a disemigroup presentation for it. We will apply Lemma 2 to obtain the set of normal forms of elements of the free digroup which gives a construction of the free digroup.

Loday [5] used the term “dimonoid” to refer to what we have called a disemigroup. We have made a slight change in the terminology to be more consistent with standard usage in semigroup theory. Let \((D, \triangleright, \triangleleft)\) be a disemigroup. An element \(e\) in \(D\) is called a \textit{bar-unit} if it satisfies \(e \triangleright a = a \triangleleft e = a\) for all \(a \in D\). The following lemma shows that Definition 1 is equivalent to Kinyon’s definition ([2], Definition 4.1).

**Lemma 3** Let \(G\) be a disemigroup with a bar-unit \(e\). Then \((G, e)\) is a digroup with the unit \(e\) if

\[ \forall a \in G, \exists b \in G \text{ such that } a \triangleright b = b \triangleleft a = e. \]

**Proof** It suffices to prove that for all \(a \in G\) there exists a unique \(b \in G\) such that \(a \triangleright b = b \triangleleft a = e\). Suppose that there exist \(b, b'\) such that \(a \triangleright b = a \triangleright b' = b \triangleleft a = e\). Then \(b = b \triangleleft e = (a \triangleright b) \triangleleft a = (a \triangleright b') \triangleleft a = e \triangleleft b' = e \triangleleft b', \)

\[ b = e \triangleleft b' = e \triangleleft (e \triangleleft e) = (e \triangleleft e) \triangleleft b = e \triangleleft b = b'. \]

\[ \square \]

A map \(f\) from a disemigroup \(G\) to a disemigroup \(H\) is called a \textit{disemigroup homomorphism} if for any \(a, b \in G\), \(f(a \triangleright b) = f(a) \triangleright f(b)\), \(f(a \triangleleft b) = f(a) \triangleleft f(b)\). Let \((G, 1_G)\) and \((H, 1_H)\) be digroups. A disemigroup homomorphism \(f\) from \((G, 1_G)\) to \((H, 1_H)\) is called a \textit{digroup homomorphism} if \(f(1_G) = 1_H\).

Note that if \(f\) is a digroup homomorphism from \((G, 1_G)\) to \((H, 1_H)\), then for any \(a \in G\), we have \(f(a^\dagger) = f(a)^\dagger\).

**Definition 3** Let \(X\) be a subset of a digroup \((D, 1_D)\). Then \((D, 1_D)\) is called a \textit{free digroup} on \(X\) provided the following holds: for any digroup \((G, 1_G)\) and any function \(\varphi\) from the set \(X\) into \((G, 1_G)\), there exists a unique extension of \(\varphi\) to a digroup homomorphism \(\varphi^*\) from \((D, 1_D)\) into \((G, 1_G)\).

In the paper [7], it is shown that the free digroup on a set \(X\) exists. Following from universal properties of free digroups, it is easy to see that is uniquely determined up to isomorphism. In this case, we denote the free digroup on \(X\) by \((F(X), e)\).

We now turn to a construction of the free digroup. Let a set \(X\) be given. We write \(X^{-1}\) which is a set disjoint from \(X\) with a one-to-one correspondence, i.e. \(X^{-1} = \{x^{-1} \mid x \in X\}\) and \(X \cap X^{-1} = \emptyset\), and write \(X^{\pm 1} = X \cup X^{-1}\).

The following lemma gives a disemigroup presentation to the free digroup on \(X\). Let \(Y\) be a set, \(S\) a set of relations in \(\text{Disgp}(Y)\) and \(\rho(S)\) the congruence on \(\text{Disgp}(Y)\) generated by \(S\). For simplicity of notation, for all \([u]_m, [v]_n \in \text{Disgp}(Y)\), we write \([u]_m = [v]_n\) in \(\text{Disgp}(Y)[S]\) which means that \(([u]_m, [v]_n) \in \rho(S)\). We denote the equivalent class of \([u]_m\) in \(\text{Disgp}(Y)\) by \([u]_m\rho(S)\).
Lemma 4 Let \( X \) be a set and \( e \notin X^\pm \) be a symbol. Then \((\text{Disgp}(X^\pm \cup \{e\}|S), e\rho(S))\) is the free digroup on \( X \), where \( S \) consists of the following relations:

\[
x^\pm 1 = e = x^\pm 1 \vdash x, \quad y \vdash e = y = e \vdash y, \quad x \in X, \ y \in X^\pm \cup \{e\}.
\]

Proof We first show that \((\text{Disgp}(X^\pm \cup \{e\}|S), e\rho(S))\) is a digroup. It is easy to see that \( e\rho(S) \) is a bar-unit in \((\text{Disgp}(X^\pm \cup \{e\}|S))\).

Now, in \( \text{Disgp}(X^\pm \cup \{e\}|S) \), the following holds:

(i) \( y \vdash e = e \vdash y \) for all \( y \in X^\pm \cup \{e\} \). In fact, \( e \vdash e = e \vdash e \). For any \( x \in X \), \( x \vdash e = x \vdash (x^\pm 1 \dashv x) = (x \vdash x^\pm 1) \dashv x = e \vdash x \). For any \( z^\pm 1 \in X^\pm \), we have

\[
z^\pm 1 \vdash e = z^\pm 1 \vdash (z \vdash z^\pm 1) = (z^\pm 1 \vdash z) \vdash z^\pm 1 = e \vdash z^\pm 1 = z^\pm 1
\]

and

\[
e \vdash z^\pm 1 = (z^\pm 1 \vdash z) \vdash z^\pm 1 = z^\pm 1 \vdash (z \vdash z^\pm 1) = z^\pm 1 \vdash e = z^\pm 1.
\]

It follows that \( z^\pm 1 \vdash e = e \vdash z^\pm 1 \).

(ii) Define \( x^0 = e \) for all \( x \in X \). Then for any \( [u]_n \in \text{Disgp}(X^\pm \cup \{e\}) \), we may assume that \( u = x^e_1 \cdots x^e_n \), \( x_i \in X \), \( e_i \in \{-1, 0, 1\} \). Then there exists \( [x^e_1 \cdots x^e_n]_{n+1} \) such that

\[
[x^e_1 \cdots x^e_n]_m \vdash [x^e_1 \cdots 1^e_1]_{n+1} = (x^e_1 \vdash \cdots \vdash x^e_n \vdash x^{-e}_n \vdash \cdots \vdash x^1_1) \vdash e
\]

\[
e \vdash e = e
\]

and

\[
[x^{-e}_n \cdots x^{-e}_1]_{n+1} \vdash [x^e_1 \cdots x^e_n]_m = x^{-e}_n \vdash \cdots \vdash x^{-e}_1 \vdash e \vdash x^e_1 \vdash \cdots \vdash x^e_n
\]

\[
e \vdash (x^{-e}_n \vdash \cdots \vdash x^{-e}_1 \vdash x^e_1 \vdash \cdots \vdash x^e_n)
\]

\[
e \vdash e \vdash e.
\]

By Lemma 3, \((\text{Disgp}(X^\pm \cup \{e\}|S), e\rho(S))\) is a digroup.

We now turn to show that \((\text{Disgp}(X^\pm \cup \{e\}|S), e\rho(S))\) is free. For any digroup \((G, 1_G)\) and function \( \varphi : X \to G \), define a map

\[
\phi : X^\pm \cup \{e\} \to G, \ x \mapsto \varphi(x), \ x^{-1} \mapsto (\varphi(x))^\dagger, \ e \to 1_G, \ x \in X.
\]

Then we obtain a disemigroup homomorphism

\[
\phi^* : \text{Disgp}(X^\pm \cup \{e\}) \to G, \ [x^e_1 \cdots x^e_n]_m \mapsto [\phi(x^e_1) \cdots \phi(x^e_n)]_m
\]

induced by \( \phi \). Since

\[
\phi^*(x \vdash x^{-1}) = \varphi(x) \vdash (\varphi(x))^\dagger = 1_G = (\varphi(x))^\dagger \vdash \varphi(x) = \phi^*(x^{-1} \vdash x),
\]
\[
\phi^*(e) = 1_G, \\
\phi^*(y \cdot e) = \phi^*(y) \cdot 1_G = \phi^*(y) = 1_G \triangleright \phi^*(y) = \phi^*(e \triangleright y),
\]
we have a disemigroup homomorphism

\[
\varphi^* : \text{Disgp}(X^{\pm 1} \cup \{e\}|S) \rightarrow G, \quad [x_1^{e_1} \cdots x_n^{e_n}]_m \rho(S) \mapsto [\phi(x_1^{e_1}) \cdots \phi(x_n^{e_n})]_m.
\]

As \(\varphi^*(e \rho(S)) = \varphi(e) = 1_G\), it follows that \(\varphi^*\) is a digroup homomorphism. It is easy to see that \(\varphi^*|_X = \varphi\) and such \(\varphi^*\) is unique. \(\square\)

Now we give an explicit construction of the free digroup \(\text{Disgp}(X^{\pm 1} \cup \{e\}|S), e \rho(S))\) by using the Gröbner–Shirshov bases method.

**Lemma 5** Let \(X\) be a set and \(e \not\in X^{\pm 1}\) be a symbol. Let

\[
S = \{(x x^{-1})_2 - e, [x x^{-1}]_1 - e, [y x^{-1}] - y, [e y]_2 - y \mid x \in X, y \in X^{\pm 1} \cup \{e\}\}
\]

be a subset of \(\text{Di}(X^{\pm 1} \cup \{e\})\). Then the following statements hold.

(i) \(\text{Disgp}(X^{\pm 1} \cup \{e\} | S) = \text{Disgp}(X^{\pm 1} \cup \{e\} | S \cup T)\), where \(T\) consists of the following relations:

\[
t_1 = [x^{-1} e]_2 - x^{-1}, \quad t_2 = [e x_i \cdots x_n]_1 - [x_i \cdots x_n x^{-1}]_m + 1, \\
t_3 = [xe]_2 - [ex]_1, \quad t_4 = [x^{-1} x_i \cdots x_{lm} z^{-1}]_m + 2 - [x^{-1} x_i \cdots x_{lm} z^{-1}]_1,
\]

where \(m \geq 0, n \geq 0, x, z, x_i, x_j \in X, 1 \leq k \leq m, 1 \leq j \leq n\).

(ii) Let \(X^{\pm 1}\) be a well-ordered set. Define an ordering on \(X^{\pm 1} \cup \{e\}\) by \(e < z\) for any \(z \in X^{\pm 1}\) and the ordering on \(((X^{\pm 1} \cup \{e\})^+)\) is given in the deg-lex-center way. Then \(S \cup T\) is a Gröbner–Shirshov basis in \(\text{Di}(X^{\pm 1} \cup \{e\})\).

**Proof** (i) Let \(\rho(S)\) be the congruence of \(\text{Disgp}(X^{\pm 1} \cup \{e\})\) generated by \(S\). We only need to show that \(T \subseteq \rho(S)\). By the proof of Lemma 4, we obtain that the relations \(t_1, t_3\) and \(t_2\) with \(n = 0\) hold in \(\text{Disgp}(X^{\pm 1} \cup \{e\} | S)\). In the disemigroup \(\text{Disgp}(X^{\pm 1} \cup \{e\} | S)\), we also have

\[
\begin{align*}
(e \triangleright x_i \triangleright \cdots \triangleright x_{lm}) \triangleright x^{-1} &= (x_i \triangleright \cdots \triangleright x_{lm} \triangleright e) \triangleright x^{-1} \\
&= x_i \triangleright \cdots \triangleright x_{lm} \triangleright (e \triangleright x^{-1}) \\
&= x_i \triangleright \cdots \triangleright x_{lm} \triangleright x^{-1} \text{ if } n > 0; \\
x^{-1} \triangleright z^{-1} &= x^{-1} \triangleright (e \triangleright z^{-1}) = (x^{-1} \triangleright e) \triangleright z^{-1} = x^{-1} \triangleright z^{-1}; \\
x^{-1} \triangleright x_i \triangleright \cdots \triangleright x_{lm} \triangleright z^{-1} &= x^{-1} \triangleright x_i \triangleright \cdots \triangleright x_{lm} \triangleright (e \triangleright z^{-1}) \\
&= x^{-1} \triangleright (x_i \triangleright \cdots \triangleright x_{lm} \triangleright e) \triangleright z^{-1} \\
&= x^{-1} \triangleright (e \triangleright x_i \triangleright \cdots \triangleright x_{lm}) \triangleright z^{-1} \\
&= (x^{-1} \triangleright e) \triangleright x_i \triangleright \cdots \triangleright x_{lm} \triangleright z^{-1} \text{ if } m > 0.
\end{align*}
\]

It follows that \(t_2, t_4 \in \rho(S)\). Thus the result holds.
(ii) Let \( R = S \cup T \). We will show that all compositions in \( R \) are trivial modulo \( R \).

Here we show that some compositions in \( R \), as examples, are trivial modulo \( R \).

It is evident that all possible compositions of left (right) multiplication are ones related to \( t_4 \) and they are equal to zero. Let

\[
s_1 = [xx^{-1}]_2 - e, \quad s_2 = [x^{-1}x]_1 - e, \quad s_3 = [ye]_1 - y, \quad s_4 = [ey]_2 - y,
\]

where \( x \in X \), \( y \in X^{\pm 1} \cup \{e\} \).

Here and subsequently, we denote by, for example, \(" f \wedge g, [w]_m \)" the composition of the polynomials of \( f \) and \( g \) with ambiguity \([w]_m\).

1) Compositions of inclusion and left (right) multiplicative inclusion.

By noting that in \( R \),

\[
s_3 \wedge s_4, \quad w = ee, \quad P(s_3) \cap P(s_4) = \emptyset; \quad s_3 \wedge t_1, \quad w = x^{-1}e, \quad P(s_3) \cap P(t_1) = \emptyset;
\]

\[
s_3 \wedge t_3, \quad w = xe, \quad P(s_3) \cap P(t_3) = \emptyset; \quad s_4 \wedge t_2, \quad w = ex^{-1}, \quad P(s_4) \cap P(t_2) = \emptyset;
\]

\[
t_2 \wedge s_1, \quad w = ex_i \cdots x_{i_n}x_{i_n}^{-1}, \quad P(t_2) \cap P([ex_i \cdots x_{i_{n-1}}]) = \{1\};
\]

\[
t_2 \wedge s_4, \quad w = ex_i \cdots x_{i_n}x_{i_n}^{-1}, \quad P(t_2) \cap P([s_4x_i \cdots x_{i_n}x_{i_n}^{-1}]) = \emptyset;
\]

\[
t_4 \wedge s_1, \quad w = x^{-1}x_i \cdots x_{i_m}x_{i_m}^{-1}, \quad P(t_4) \cap P(x^{-1}x_i \cdots x_{i_{m-1}}x_{i_1}) = \{m + 2\};
\]

\[
t_4 \wedge s_2, \quad w = x_i^{-1}x_i \cdots x_{i_m}z^{-1}, \quad P(t_4) \cap P(s_2x_i \cdots x_{i_m}z^{-1}) = \{m + 2\},
\]

all possible of compositions of inclusion in \( R \) are:

\[
t_2 \wedge s_1, \quad [ex_i \cdots x_{i_n}x_{i_n}^{-1}]_1; \quad t_4 \wedge s_1, \quad [x^{-1}x_i \cdots x_{i_m}x_{i_m}^{-1}]_{m+2};
\]

\[
t_4 \wedge s_2, \quad [x_i^{-1}x_i \cdots x_{i_m}z^{-1}]_{m+2},
\]

and all possible of compositions of left (right) multiplicative inclusion in \( R \) are:

\[
s_3 \wedge s_4, \quad [yee]_1, [eey]_3; \quad s_3 \wedge t_1, \quad [yx^{-1}e]_1, [x^{-1}ey]_3;
\]

\[
s_3 \wedge t_3, \quad [yx]_1, [xey]_3; \quad s_4 \wedge t_2, \quad [yex^{-1}]_1, [ex^{-1}y]_3;
\]

\[
t_2 \wedge s_4, \quad [yex_i \cdots x_{i_n}x_{i_n}^{-1}]_1, [ex_i \cdots x_{i_n}x_{i_n}^{-1}y]_{n+3},
\]

where \( m \geq 0, n \geq 0, y \in X^{\pm 1} \cup \{e\} \), \( x, x_i, x_{i_j}, z \in X \), \( 1 \leq k \leq m, 1 \leq j \leq n \).

It is easy to check that all above compositions are trivial modulo \( R \). Here, for example, we just check \( t_4 \wedge s_1 \) with ambiguity \([w]_{m+2} = [x^{-1}x_i \cdots x_{i_m}x_{i_m}^{-1}]_{m+2}\).

\[
(t_4, s_1)_{[w]_{m+2}} = x^{-1} \uparrow x_i \uparrow \cdots \uparrow x_{i_m-1} \uparrow e - x^{-1} \uparrow x_i \uparrow \cdots \uparrow x_{i_m-1} \uparrow x_{i_m} \uparrow x_{i_m}^{-1}
\]

\[
\equiv x^{-1} \uparrow e \uparrow x_i \uparrow \cdots \uparrow x_{i_m-1} \uparrow x^{-1} \uparrow x_i \uparrow \cdots \uparrow x_{i_m-1} \uparrow (x_{i_m} \uparrow x_{i_m}^{-1})
\]

\[
\equiv x^{-1} \uparrow x_i \uparrow \cdots \uparrow x_{i_m-1} \uparrow x^{-1} \uparrow x_i \uparrow \cdots \uparrow x_{i_m-1} \uparrow e
\]

\[
\equiv 0 \mod (R, [w]_{m+2}).
\]

2) Compositions of intersection and left (right) multiplicative intersection.
By noting that in $R$,

\begin{align*}
\text{s}_1 \land \text{s}_2, \quad &w = xx^{-1}x, \quad P([s_1 x]) \cap P([xs_2]) = \{2\}; \\
\text{s}_1 \land \text{s}_3, \quad &w = xx^{-1}e, \quad P([s_1 e]) \cap P([xs_3]) = \{2\}; \\
\text{s}_1 \land t_1, \quad &w = xx^{-1}e, \quad P([s_1 e]) \cap P([xt_1]) = \{3\}; \\
\text{s}_1 \land t_4, \quad &w = xx^{-1}x_{i_1} \cdots x_{i_m}z^{-1}, \quad P([s_1 x_{i_1} \cdots x_{i_m}z^{-1}]) \cap P([xt_4]) = \{m + 3\}; \\
\text{s}_2 \land \text{s}_1, \quad &w = x^{-1}xx^{-1}, \quad P([s_2 x^{-1}]) \cap P([x^{-1}s_1]) = \{1, 3\}; \\
\text{s}_2 \land \text{s}_3, \quad &w = x^{-1}xe, \quad P([s_2 e]) \cap P([x^{-1}s_3]) = \{1\}; \\
\text{s}_2 \land t_3, \quad &w = x^{-1}xe, \quad P([s_2 e]) \cap P([x^{-1}t_3]) = \{1, 3\}; \\
\text{s}_3 \land \text{s}_3, \quad &w = yee, \quad P([s_3 e]) \cap P([ys_3]) = \{1\}; \\
\text{s}_3 \land \text{s}_4, \quad &w = yey', \quad P([s_3 e']) \cap P([ys_4]) = \{1, 3\}; \\
\text{s}_3 \land t_2, \quad &w = yex_{i_1} \cdots x_{i_n}x^{-1}, \quad P([s_3 x_{i_1} \cdots x_{i_n}x^{-1}]) \cap P([yt_2]) = \{1\}; \\
\text{s}_4 \land \text{s}_1, \quad &w = eex^{-1}, \quad P([s_4 x^{-1}]) \cap P([es_1]) = \{3\}; \\
\text{s}_4 \land \text{s}_2, \quad &w = ex^{-1}x, \quad P([s_4 x]) \cap P([es_2]) = \{2\}; \\
\text{s}_4 \land \text{s}_3, \quad &w = eye, \quad P([s_4 e]) \cap P([es_3]) = \{2\}; \\
\text{s}_4 \land \text{s}_4, \quad &w = eee, \quad P([s_4 e]) \cap P([es_4]) = \{3\}; \\
\text{s}_4 \land t_1, \quad &w = ex^{-1}e, \quad P([s_4 e]) \cap P([et_1]) = \{3\}; \\
\text{s}_4 \land t_2, \quad &w = eex_{i_1} \cdots x_{i_n}x^{-1}, \quad P([s_4 x_{i_1} \cdots x_{i_n}x^{-1}]) \cap P([et_2]) = \{2\}; \\
\text{s}_4 \land t_3, \quad &w = exe, \quad P([s_4 e]) \cap P([et_3]) = \{3\}; \\
\text{s}_4 \land t_4, \quad &w = ex^{-1}x_{i_1} \cdots x_{i_m}z^{-1}, \quad P([s_4 x_{i_1} \cdots x_{i_m}z^{-1}]) \cap P([et_4]) = \{m + 3\}; \\
\text{t}_1 \land \text{s}_3, \quad &w = x^{-1}ee, \quad P([t_1 e]) \cap P([x^{-1}s_3]) = \{2\}; \\
\text{t}_1 \land \text{s}_4, \quad &w = x^{-1}ey, \quad P([t_1 e]) \cap P([x^{-1}s_4]) = \{3\}; \\
\text{t}_1 \land t_2, \quad &w = x^{-1}ex_{i_1} \cdots x_{i_n}z^{-1}, \quad P([t_1 x_{i_1} \cdots x_{i_n}z^{-1}]) \cap P([x^{-1}t_2]) = \{2\}; \\
\text{t}_2 \land \text{s}_2, \quad &w = ex_{i_1} \cdots x_{i_n}x^{-1}x, \quad P([t_2 x]) \cap P([ex_{i_1} \cdots x_{i_n}s_2]) = \{1\}; \\
\text{t}_2 \land \text{s}_3, \quad &w = ex_{i_1} \cdots x_{i_n}x^{-1}e, \quad P([t_2 e]) \cap P([ex_{i_1} \cdots x_{i_n}s_3]) = \{1\}; \\
\text{t}_2 \land \text{t}_1, \quad &w = ex_{i_1} \cdots x_{i_n}x^{-1}e, \quad P([t_2 e]) \cap P([ex_{i_1} \cdots x_{i_n}t_1]) = \{1, n + 3\}; \\
\text{t}_2 \land t_4, \quad &w = ex_{i_1} \cdots x_{i_n}x^{-1}x_{i_1} \cdots x_{i_m}z^{-1}, \\
&\quad P([t_2 x_{i_1} \cdots x_{i_m}z^{-1}]) \cap P([ex_{i_1} \cdots x_{i_n}t_4]) = \{n + m + 3\}; \\
\text{t}_3 \land \text{s}_3, \quad &w = xee, \quad P([t_3 e]) \cap P([xs_3]) = \{2\}; \\
\text{t}_3 \land \text{s}_4, \quad &w = xey, \quad P([t_3 e]) \cap P([xs_4]) = \{3\}; \\
\text{t}_3 \land t_2, \quad &w = xex_{i_1} \cdots x_{i_n}z^{-1}, \quad P([t_3 x_{i_1} \cdots x_{i_n}z^{-1}]) \cap P([xt_2]) = \{2\}; \\
\text{t}_4 \land \text{s}_2, \quad &w = x^{-1}x_{i_1} \cdots x_{i_m}z^{-1}z, \quad P([t_4 z]) \cap P([x^{-1}x_{i_1} \cdots x_{i_m}s_2]) = \{m + 2\}; \\
\text{t}_4 \land \text{s}_3, \quad &w = x^{-1}x_{i_1} \cdots x_{i_m}z^{-1}e, \quad P([t_4 e]) \cap P([x^{-1}x_{i_1} \cdots x_{i_m}s_3]) = \{m + 2\}; \\
\text{t}_4 \land \text{t}_1, \quad &w = x^{-1}x_{i_1} \cdots x_{i_m}z^{-1}e, \quad P([t_4 e]) \cap P([x^{-1}x_{i_1} \cdots x_{i_m}t_1]) = \emptyset;
\end{align*}
A construction of the free digroup

It is easy to check that all compositions of intersection are trivial modulo R. Therefore, for example, we check

\[ t_4 \wedge t_4, \quad w = x^{-1}x_1 \cdots x_{i_m}z^{-1}x_1 \cdots x_{i_n}z'^{-1}, \]

\[ P([t_4x_1 \cdots x_{i_n}z']) \cap P([x^{-1}x_1 \cdots x_{i_m}t_4]) = \emptyset, \]

there is no composition of left (right) multiplicative intersection and all possible compositions of intersection in R are:

\[
\begin{align*}
  &s_1 \wedge s_2, [xx^{-1}x]_2; \quad s_1 \wedge s_3, [xx^{-1}e]_2; \\
  &s_1 \wedge t_1, [xx^{-1}e]_3; \quad s_1 \wedge t_4, [xx^{-1}x_1 \cdots x_{i_m}z^{-1}]_{m+3}; \\
  &s_2 \wedge s_1, [x^{-1}xx^{-1}]_1, [x^{-1}xx^{-1}]_3; \quad s_2 \wedge s_3, [x^{-1}xe]_1; \\
  &s_2 \wedge t_3, [x^{-1}xe]_1, [x^{-1}xe]_3; \quad s_3 \wedge s_3, [yee]_1; \\
  &s_3 \wedge s_4, [yey']_1, [yey']_3; \quad s_3 \wedge t_2, [yex_1 \cdots x_{i_n}x^{-1}]_1; \\
  &s_4 \wedge s_1, [exx^{-1}]_3; \quad s_4 \wedge s_2, [ex^{-1}x]_2; \\
  &s_4 \wedge s_3, [eye]_2; \quad s_4 \wedge s_4, [eе]_3; \\
  &s_4 \wedge t_1, [ex^{-1}e]_3; \quad s_4 \wedge t_2, [eeex_1 \cdots x_{i_n}x^{-1}]_2; \\
  &s_4 \wedge t_3, [exe]_3; \quad s_4 \wedge t_4, [ex^{-1}x_1 \cdots x_{i_m}z^{-1}]_{m+3}; \\
  &t_1 \wedge s_2, [x^{-1}ee]_2; \quad t_1 \wedge s_4, [x^{-1}ey]_3; \\
  &t_1 \wedge t_2, [x^{-1}ex_1 \cdots x_{i_n}z^{-1}]_2; \quad t_2 \wedge s_2, [ex_1 \cdots x_{i_n}x^{-1}]_1; \\
  &t_2 \wedge s_3, [ex_1 \cdots x_{i_n}x^{-1}e]_1; \quad t_2 \wedge t_1, [ex_1 \cdots x_{i_n}x^{-1}]_1. \\
  &t_2 \wedge t_1, [ex_1 \cdots x_{i_n}x^{-1}]_{n+3}; \quad t_2 \wedge t_4, [ex_1 \cdots x_{i_n}x^{-1}x_1 \cdots x_{i_m}z^{-1}]_{n+m+3}; \\
  &t_3 \wedge s_3, [xee]_2; \quad t_3 \wedge s_4, [xey]_3; \\
  &t_3 \wedge t_2, [xeex_1 \cdots x_{i_n}z^{-1}]_2; \quad t_4 \wedge s_2, [x^{-1}x_1 \cdots x_{i_m}z^{-1}]_{m+2}; \\
  &t_4 \wedge s_3, [x^{-1}x_1 \cdots x_{i_m}z^{-1}e]_{m+2},
\end{align*}
\]

where \( m \geq 0, n \geq 0, \ y, y' \in X^\pm_1 \cup \{ e \}, \ x, z, z', x_k, x_{ij} \in X, \ 1 \leq k \leq m, \ 1 \leq j \leq n. \)

It is easy to check that all compositions of intersection are trivial modulo R. For example, we check \( t_4 \wedge s_2 \) with ambiguity \( [w]_{m+2} = [x^{-1}x_1 \cdots x_{i_m}z^{-1}z]_{m+2}. \)

\[
(t_4, s_2)[w]_{m+2} = x^{-1} \vdash x_1 \vdash \cdots \vdash x_{i_m} \vdash e - x^{-1} \vdash x_1 \vdash \cdots \vdash x_{i_m} \vdash z^{-1} \vdash z \\
\equiv x^{-1} \vdash e - x^{-1} \vdash x_1 \vdash \cdots \vdash x_{i_m} - x^{-1} \vdash x_1 \vdash \cdots \vdash x_{i_m} \vdash e \\
\equiv x^{-1} \vdash x_1 \vdash \cdots \vdash x_{i_m} - x^{-1} \vdash x_1 \vdash \cdots \vdash x_{i_m} \\
\equiv 0 \mod (R, [w]_{m+2}).
\]

Therefore \( R = S \cup T \) is a Gröbner–Shirshov basis in \( D_i \langle X^\pm_1 \cup \{ e \} \rangle. \) □

Let \( X \) be a set and
Theorem 1 Let $X$ be a set and $123$ where $G. Zhang, Y. Chen$ $X$ be the free group on $[51x325]$. For example, if $u \in (X^{\pm 1})^*$, there is a unique reduced word $\hat{u} \in gp(X)$. Let $\tau : (X^{\pm 1} \cup \{ e \})^* \rightarrow (X^{\pm 1})^*$ be the semigroup homomorphism which is deduced by

$$x \mapsto x, \quad x^{-1} \mapsto x^{-1}, \quad e \mapsto e, \quad x \in X.$$ Let $v \in (X^{\pm 1} \cup \{ e \})^*$. For abbreviation, we let $\hat{v}$ stand for $\tau(v)$.

Let $u \in (X^{\pm 1})^*$ and $u = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, n \geq 0$ with $x_i \in X, \varepsilon_j \in \{-1, 1\}$. Write $u^{-1} = x_n^{-\varepsilon_n} \cdots x_1^{-\varepsilon_1}$ for $n > 0$ and $e^{-1} = e$. Define

$$\lambda : (X^{\pm 1})^* \setminus X^* \rightarrow \mathbb{N}, \quad x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \mapsto \min\{j \mid \varepsilon_j = -1, 1 \leq j \leq n\}.$$ For example, if $u = x_1x_2^{-1}x_3x_4^{-1}, x_i \in X$, then $\lambda(u) = 2$.

The following theorem gives a characterization of the free digroup

$$(F(X), e) = (Disgp(X^{\pm 1} \cup \{ e \}|S), e\rho(S)).$$

**Theorem 1** Let $X$ be a set and $(F(X), e)$ the free digroup on $X$ with the unit $e$. Then

$$F(X) = \Omega_e \cup \Omega_X \cup \Omega_{X^{-1}}$$

where

- $\Omega_e = \{ [eu]_1 \mid u \in X^* \}$,
- $\Omega_X = \{ [uxv]_{|u|+1} \mid x \in X, u, v \in gp(X) \}$,
- $\Omega_{X^{-1}} = \{ [ux^{-1}v]_{|u|+1} \mid x \in X, u \in X^*, v \in (X^{\pm 1})^*, ux^{-1}v \in gp(X) \}$.

Moreover, in $F(X)$, the operations $\vdash, \dashv$ and $\dagger$ are as follows: for any $[q]_n \in F(X), [p]_m \in \Omega_e \cup \Omega_{X^{-1}}, [uxv]_{|u|+1} \in \Omega_X$,

- $[q]_n \vdash [p]_m = \begin{cases} [eq\hat{p}]_1 & \text{if } \hat{p} \in X^*, \\ [q\hat{p}]_{\lambda(\hat{p})} & \text{otherwise,} \end{cases}$
- $[p]_m \dashv [q]_n = \begin{cases} [e\hat{p}q]_1 & \text{if } \hat{p} \in X^*, \\ [\hat{p}q]_{\lambda(\hat{p})} & \text{otherwise,} \end{cases}$
- $[q]_n \vdash [uxv]_{|u|+1} = [\hat{u}vx\hat{v}]_{|\hat{u}|+1},$ $[uxv]_{|u|+1} \dashv [q]_n = [uxv\hat{q}]_{|u|+1},$
- $([q]_n)^\dagger = \begin{cases} [e(\hat{q})^{-1}]_1 & \text{if } \hat{q}^{-1} \in X^*, \\ [(\hat{q})^{-1}]_{\lambda(\hat{q})^{-1}} & \text{otherwise.} \end{cases}$
Proof By Lemmas 2 and 5, we know that $Irr(S \cup T) = \Omega_e \cup \Omega_x \cup \Omega_{x-1}$ is a set of normal forms of elements of $F(X) = Disgp(X^{\pm 1} \cup \{e\} \mid S)$. The formulas for the operations follow from relations in Lemma 5.

Let $(D, 1_D)$ be a digroup, $J$ be the set of all inverses in $D$ and $E$ be the halo of $D$, that is,

$$J := \{a^\dagger \mid a \in D\}, \quad E := \{e \in D \mid e \vdash a = a \vdash e = a, a \in D\}.$$

Kinyon [2] showed that $J = \{a \vdash 1_D \mid a \in D\}$, $J$ is a group in which $\vdash = \vdash$ and $(D, 1_D)$ is isomorphic to the digroup $(E \times J, (1_D, 1_D))$ where $\vdash$ and $\vdash$ are defined by

$$(u, h) \vdash (v, k) = (h \vdash v \vdash h^\dagger, h \vdash k), \quad (u, h) \vdash (v, k) = (u, h \vdash k).$$

We call $J$ the group part and $E$ the halo part of the digroup $(D, 1_D)$.

Now we consider the group and halo parts of the free digroup $(F(X), e)$. By Theorem 1, we immediately have

Corollary 1 Let $X$ be a set, $J$ and $E$ be the group and halo parts of $(F(X), e)$. Then $J = \Omega_e \cup \Omega_{x-1}$ and $E = \{e\} \cup \{(uxv)[u]_1 + 1 \in \Omega_x \mid u\bar{x}v = e \textrm{ in } gp(X)\}$.

Let $X = \{x\}$. For simplicity of notation, we write $x^{-n}$ instead of $(x^{-1})^n$ for any $n \in \mathbb{N}$. Then for any $i \in \mathbb{Z}$, the length of $x^i$ is the absolute value $|i|$ of $i$. The following corollary gives the free digroup generated by a single element.

Corollary 2 Let $X = \{x\}$. Then $F(X) = \Omega_e \cup \Omega_x \cup \Omega_{x-1}$, where

$$\Omega_e = \{[ex^n]_1 \mid n \geq 0\}, \quad \Omega_x = \{[(x^i xx^j)[i]_1 + 1 \mid i, j \in \mathbb{Z}\}, \quad \Omega_{x-1} = \{[x^{-m}]_1 \mid m \geq 1\},$$

and the operations $\vdash$, $\vdash$ and $\dagger$ are as follows: for any $[ex^n]_1, [ex^n]_1 \in \Omega_e, [x^i xx^j][i]_1 + 1 \in \Omega_x$ and $[x^{-m}]_1, [x^{-m}]_1 \in \Omega_{x-1}, t = i + 1 + j, t' = i' + 1 + j', p = n - m', p' = -m + n', q = t - m', q' = -m + t', s = t + n', s' = n + t'$.
\begin{align*}
\left( [e x^n]_1 \right)^\dagger = [x^{-m}]_1, \quad \left( [x^i x x^j]_{i+1} \right)^\dagger = & \begin{cases} 
[x^{-m}]_1 & \text{if } t \leq 0, \\
[x^{-t}]_1 & \text{if } t > 0,
\end{cases} \\
\left( [e x^n]_1 \right)^\dagger = [e x^n]_1, \quad \left( [x^i x x^j]_{i+1} \right)^\dagger = & \begin{cases} 
[x^{-m}]_1 & \text{if } s' \geq 0, \\
[x^s]_1 & \text{if } s' < 0,
\end{cases} \\
\left( [e x^n]_1 \right)^\dagger = [e x^n]_1, \quad \left( [x^i x x^j]_{i+1} \right)^\dagger = & \begin{cases} 
[x^{-m}]_1 & \text{if } p' \geq 0, \\
[x^p]_1 & \text{if } p' < 0.
\end{cases}
\end{align*}

Moreover, \( J = \{ [e x^n]_1 \mid n \geq 0 \} \cup \{ [x^{-m}]_1 \mid m \geq 1 \} \) is the group part of \((F(X), e)\) and \( E = \{ e \} \cup \{ [x^{-m}]_1 \mid m \geq 1 \} \) is the halo part of \((F(X), e)\).

Let \((G, 1_G)\) be a digroup and \( H \) be a subset of \( G \). A pair \((H, 1_G)\) is called a subdigroup of \((G, 1_G)\) if \( 1_G \in H \) and \((H, 1_G)\) is a digroup under the operations of \((G, 1_G)\). For a subset \( U \) of \( G \), we write \( \langle U \rangle \) for the subdigroup of \( G \) generated by \( U \).

Let \((A, 1_A)\) and \((B, 1_B)\) be two digroups. The set of all digroup homomorphisms from \((A, 1_A)\) to \((B, 1_B)\) will be denoted by \( Hom(A, B) \). The set of all maps from a set \( X \) to a set \( Y \) will be denoted by \( Y^X \) and the cardinality of the set \( X \) will be denoted by \( card(X) \). In the following theorem, we are going to give a characterization of two isomorphic free digroups.

**Theorem 2** Let \( X \) and \( Y \) be sets. Then \((F(X), e)\) is isomorphic to \((F(Y), e')\) if and only if \( card(X) = card(Y) \).

**Proof** Following from universal properties of free digroups, it is easy to check that \((F(X), e) \cong (F(Y), e')\) if \( card(X) = card(Y) \). Let \( \sigma \) be a digroup isomorphism from \((F(X), e)\) to \((F(Y), e')\) and \( B \) be a digroup with \( card(B) = 2 \). Then \( card(Hom(F(X), B)) = card(Hom(F(Y), B)) \). By Definition 3, we have

\[
\begin{align*}
\text{card}(Hom(F(X), B)) &= \text{card}(B^X) = 2^{\text{card}(X)}, \\
\text{card}(Hom(F(Y), B)) &= \text{card}(B^Y) = 2^{\text{card}(Y)}.
\end{align*}
\]

It follows that \( 2^{\text{card}(X)} = 2^{\text{card}(Y)} \). If \( X \) is finite, then so is \( Y \) and \( card(X) = card(Y) \). Otherwise, \( Y \) is infinite and \( card(X) \geq \aleph_0 \). For any \( x \in X, \sigma(x) \in F(Y) \), there exists a finite subset \( V_x \) of \( Y \) such that \( \sigma_x \in (V_x) \). Let \( W = \bigcup_{x \in X} V_x \). Clearly \( W \subseteq Y \). We claim that \( W = Y \). Otherwise, let \( y \in Y \setminus W \). Then \( \sigma^{-1}(y) \in F(X) \) and there exists a finite subset \( U = \{ x_1, \ldots, x_m \} \) of \( X \) such that \( \sigma^{-1}(y) \in (U) \). Thus \( y = \sigma(\sigma^{-1}(y)) \in \sigma((U)) \subseteq (W) \). But \( y \in Y \setminus W \). This contradicts the fact that \((F(Y), e')\) is the free digroup on \( Y \). Therefore, \( card(Y) = card(W) = card(\bigcup_{x \in X} V_x) \leq card(X)\aleph_0 = card(X) \). The proof for \( card(X) \leq card(Y) \) is similar and we have done. \( \square \)
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