MAXIMAL SURFACES AND THE UNIVERSAL TEICHMÜLLER SPACE

FRANCESCO BONSANTE AND JEAN-MARC SCHLENKER

Abstract. We show that any element of the universal Teichmüller space is realized by a unique minimal Lagrangian diffeomorphism from the hyperbolic plane to itself. The proof uses maximal surfaces in the 3-dimensional anti-de Sitter space. We show that, in \( AdS^{n+1} \), any subset \( E \) of the boundary at infinity which is the boundary at infinity of a space-like hypersurface bounds a maximal space-like hypersurface. In \( AdS^3 \), if \( E \) is the graph of a quasi-symmetric homeomorphism, then this maximal surface is unique, and it has negative sectional curvature. As a by-product, we find a simple characterization of quasi-symmetric homeomorphisms of the circle in terms of 3-dimensional projective geometry.

1. Introduction

1.1. The universal Teichmüller space. We consider here the universal Teichmüller space \( T \), which can be defined as the space of quasi-symmetric homeomorphisms from \( S^1 \) to \( S^1 \) up to projective transformations, see e.g. [19]. The quasi-symmetric homeomorphisms of \( S^1 \) to \( S^1 \) are precisely the homeomorphisms which are the boundary value of a quasi-conformal diffeomorphism from \( \mathbb{H}^2 \) to \( \mathbb{H}^2 \), so that the universal Teichmüller space \( T \) can be defined as the space of quasi-conformal diffeomorphisms from \( \mathbb{H}^2 \) to \( \mathbb{H}^2 \), up to composition with a hyperbolic isometry and up to the equivalence relation which identifies two quasi-conformal diffeomorphisms if they have the same boundary value.

It was conjectured by Schoen that any element in the universal Teichmüller space can be uniquely realized as a quasi-conformal harmonic diffeomorphism:

Conjecture 1.1 (Schoen [28]). Let \( \phi : S^1 \to S^1 \) be a quasi-symmetric homeomorphism. There is a unique quasi-conformal harmonic diffeomorphism \( \psi : \mathbb{H}^2 \to \mathbb{H}^2 \) such that \( \partial \psi = \phi \).

A number of partial results were obtained towards this conjecture, proving the uniqueness of \( \psi \) and its existence if \( \phi \) is smooth enough, see [29, 3, 25] and the references there.

1.2. Minimal Lagrangian diffeomorphisms. Our first goal here is to prove an analog of Conjecture 1.1 with harmonic maps replaced by close relatives: minimal Lagrangian diffeomorphisms.

Definition 1.2. Let \( \Phi : S \to S' \) be a diffeomorphism between two hyperbolic surfaces. \( \Phi \) is minimal Lagrangian if it is area-preserving, and its graph is a minimal surface in \( S \times S' \).

The relationship between harmonic maps and minimal Lagrangian maps is as follows.

Proposition 1.3. Let \( S_0 \) be a Riemann surface, and let \( \psi : S_0 \to S \) be a harmonic diffeomorphism from \( S_0 \) to a hyperbolic surface \( S \). Let \( q \) be the Hopf differential of \( \psi \). There is a unique harmonic diffeomorphism \( \psi' : S_0 \to S' \) from \( S_0 \) to another hyperbolic surface \( S' \) with Hopf differential \( -q \). Then \( \psi' \circ \psi^{-1} : S \to S' \) is a minimal Lagrangian map.

Conversely, let \( \Phi : S \to S' \) be a minimal Lagrangian map between two (oriented) hyperbolic surfaces, and let \( S_0 \) be the graph of \( \Phi \), considered as a Riemann surface with the complex structure defined by its induced metric in \( S \times S' \). Then the natural projections from \( S_0 \) to \( S \) and to \( S' \) are harmonic maps, and the sum of their Hopf differentials is zero.

Thus minimal Lagrangian maps are a kind of “symmetric squares” of harmonic maps.

It is known that any diffeomorphism between two closed hyperbolic surfaces can be deformed to a unique harmonic diffeomorphism, see e.g. [21, 22]. In the same manner, it was proved by Schoen and by Labourie that any such diffeomorphism can be deformed to a unique minimal Lagrangian diffeomorphism [24, 28].

Our first result is an extension of this existence and uniqueness result to the universal Teichmüller space.
**Theorem 1.4.** Let \( \phi : S^1 \rightarrow S^1 \) be a quasi-symmetric homeomorphism. There is a unique quasi-conformal minimal Lagrangian diffeomorphism \( \Phi : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \) such that \( \partial \Phi = \phi \).

The result of Schoen and Labourie on closed hyperbolic surfaces obviously follows from this. The proof of Theorem 1.4 can be found in Section 6. Note that partial results in this direction were obtained previously by Aiyama, Akutagawa and Wan [2], who proved the existence part of Theorem 1.4 when \( \phi \) has small dilatation. Recently, Brendle has obtained results on the existence and uniqueness of minimal Lagrangian diffeomorphisms between two convex domains of the same, finite area in hyperbolic surfaces, see [13].

1.3. **The anti-de Sitter space.** The proof of Theorem 1.4 is essentially based on the geometry of maximal surfaces in the anti-de Sitter (AdS) 3-dimensional space, as already in [2]. This follows a pattern in some recent works (see [26] 11 [12] [14] and also [2]), where results on Teichmüller theory were proved using 3-dimensional AdS geometry, although mostly in a somewhat different direction. The relationship between maximal surfaces in 3-dimensional AdS manifolds and minimal Lagrangian maps between closed hyperbolic surfaces was also used recently in [23].

The 3-dimensional AdS space can be considered as a Lorentzian analog of the 3-dimensional hyperbolic space. It can be defined as the quadric

\[ \text{AdS}_3 = \{ x \in \mathbb{R}^{2,2} | (x, x) = -1 \}, \]

where \( \mathbb{R}^{2,2} \) is \( \mathbb{R}^4 \) endowed with bilinear symmetric form of signature \((2, 2)\). It is a geodesically complete Lorentz manifold of constant curvature \(-1\). Another way to define it is as the Lie group \( SL(2, \mathbb{R}) \), endowed with its bi-invariant Killing metric. More details are given in Section 2. The key point for us, basically discovered by Mess [26] and used in the references mentioned above, is that space-like surfaces in \( \text{AdS}_3^* \) naturally give rise to area-preserving diffeomorphisms from the hyperbolic plane to itself. In this way, Theorem 1.4 is proved below to be equivalent to an existence and uniqueness statement for maximal space-like surfaces in \( \text{AdS}_3^* \), and it is in this form that it is proved.

The anti-de Sitter space can of course be defined in higher dimensions. The existence part of the result on maximal surfaces is actually stated (and proved) below in the more general context of maximal hypersurfaces in \( \text{AdS}_{n+1}^* \), see Theorem 1.6. The uniqueness part, however, is considered here only in \( \text{AdS}_3^* \) (and it needs hypothesis that are more interesting in dimension \( 2 + 1 \)), see Theorem 1.10.

1.4. **Maximal surfaces and minimal Lagrangian diffeomorphisms.** For closed hyperbolic surfaces, the existence of a minimal Lagrangian diffeomorphism is equivalent to the existence of a maximal space-like surface in a 3-dimensional globally hyperbolic AdS manifold, see [23]. This relation extends to maximal surfaces in \( \text{AdS}_3^* \) and the universal Teichmüller space as follows.

One way to consider the bridge between Teichmüller theory and AdS geometry is through the asymptotic boundary of \( \text{AdS}_3^* \) – denoted by \( \partial_{\infty} \text{AdS}_3^* \) – that, as for the hyperbolic space, furnishes a natural compactification of \( \text{AdS}_3^* \). As in the hyperbolic case, a conformal Lorentzian structure is defined on \( \partial_{\infty} \text{AdS}_3^* \). There is a natural projection \( \partial_{\infty} \text{AdS}_3^* \rightarrow S^1 \times S^1 \) that is a 2-to-1 covering (see Section 2.6 for details). The graph of any homeomorphism of \( S^1 \) lifts to a spacelike closed curve in \( \partial_{\infty} \text{AdS}_3^* \).

**Proposition 1.5.**

- Let \( S \subset \text{AdS}_3^* \) be a maximal space-like graph with uniformly negative sectional curvature. Then there is a minimal Lagrangian diffeomorphism \( \Phi_S : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \) associated to \( S \), and the graph of \( \partial \Phi_S : \partial \mathbb{H}^2 \rightarrow \partial \mathbb{H}^2 \) is the projection of the boundary at infinity of \( S \) in \( \partial_{\infty} \text{AdS}_3^* \).
- Conversely, to any quasi-conformal minimal Lagrangian diffeomorphism \( \Phi : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \) is associated a maximal surface \( S \) with uniformly negative sectional curvature and with boundary at infinity equal to the lifting of the graph of \( \partial \Phi \) in \( \partial_{\infty} \text{AdS}_3^* \).

It is this proposition which provides the bridge between Theorem 1.4 and the existence and uniqueness of maximal surfaces in \( \text{AdS}_3^* \).

1.5. **Existence and regularity of maximal hypersurfaces in \( \text{AdS}_{n+1}^* \).** We can state an existence result for maximal hypersurfaces in the AdS space with fixed boundary values. The regularity conditions on the boundary values are quite weak, since we only demand that it bounds some space-like surface in \( \text{AdS}_{n+1}^* \).

**Theorem 1.6.** Let \( \Gamma \) be a closed acausal \( C^{0,1} \) graph in \( \partial_{\infty} \text{AdS}_{n+1}^* \) (\( n \geq 2 \)). If \( \Gamma \) does not contain lightlike segments, then there is a maximal space-like hypersurface \( S_0 \) such that \( \partial S_0 = \Gamma \).

We provide in Section 4 a direct proof of this result, where the maximal surface is obtained as a limit of bigger and bigger maximal disks.

This existence result can be improved insofar as the regularity of the hypersurface is concerned. To state this improvement, we need a definition. Let \( \Gamma \) be a nowhere time-like graph in \( \partial_{\infty} \text{AdS}_{n+1}^* \). Using the projective
model of \( AdS_{n+1}^* \) which is also recalled in Section 2.5, we can consider the convex hull of \( \Gamma \), it is a convex subset of \( AdS_{n+1}^* \) with boundary at infinity containing \( \Gamma \), we use the notation \( CH(\Gamma) \). We denote by \( C(\Gamma) \) the intersection with \( AdS_{n+1}^* \) of \( CH(\Gamma) \) (considered as a subset of projective space). The boundary of \( C(\Gamma) \) is the disjoint union of two nowhere-time-like hypersurfaces, which we call \( \partial_+ C(\Gamma) \) and \( \partial_- C(\Gamma) \).

**Definition 1.7.** The *width* of \( C(\Gamma) \) (or by extension of \( \Gamma \)), denoted by \( w(C(\Gamma)) \) (resp. \( w(\Gamma) \)) is the supremum of the (time) distance between \( \partial_- C(\Gamma) \) and \( \partial_+ C(\Gamma) \).

It is proved below (Lemma 4.16) that \( w(\Gamma) \) is always at most equal to \( \pi/2 \).

**Theorem 1.8.** Suppose that \( w(\partial_\infty S) < \pi/2 \) in Theorem 1.6. Then \( S_0 \) can be taken to have bounded second fundamental form.

The proof is also in Section 6.

1.6. **The mean curvature flow.** We also give in the appendix another proof of Theorem 1.6. It is based on the mean curvature flow for hypersurfaces in the anti-de Sitter space.

**Theorem 1.9.** Let \( S \subset AdS_{n+1}^* \) be a space-like graph. There exists a long-time solution of the mean curvature flow with initial value \( S \) with fixed boundary at infinity, defined for all \( t > 0 \).

This flow converges, as \( t \to \infty \), to a maximal surface. When \( w(\partial_\infty S) < \pi/2 \), we also have bounds on the second fundamental form of the hypersurfaces occuring in the flow.

1.7. **Uniqueness of maximal surfaces in \( AdS_3^* \).** We do not know whether maximal hypersurfaces with given boundary at infinity are unique in \( AdS_{n+1}^* \). We can however state a result for surfaces in \( AdS_3^* \), under a regularity assumption on the boundary at infinity.

**Theorem 1.10.** Let \( S \) be a space-like graph in \( AdS_3^* \). Suppose that the boundary at infinity of \( S \) is the graph of a quasi-symmetric homeomorphism from \( S_1 \) to \( S_1^1 \). Then there is a unique maximal surface in \( AdS_3^* \) with boundary at infinity \( \partial_\infty S \) and with bounded second fundamental form, and it has negative sectional curvature.

The proof, which can be found in Section 6, is based on the following proposition.

**Proposition 1.11.** Let \( S_0 \subset AdS_3^* \) be a maximal space-like graph with bounded principal curvatures. Then either it is flat, or its sectional curvature is uniformly negative (bounded from above by a negative constant).

Those results should be compared to the existence and uniqueness of a maximal surface in a maximal globally hyperbolic AdS 3-dimensional manifold, see [5]. Theorem 1.6 applies to this case, with \( S_0 \) the lift of a closed surface in the globally hyperbolic manifold \( M \). In this case the boundary at infinity of \( S \) is the limit set of \( M \), which is the graph of a quasi-symmetric homeomorphism (see [26] [4]). Theorem 1.12 then shows that \( w(\partial_\infty S) < \pi/2 \), so that Theorem 1.10 also applies.

1.8. **A characterization of quasi-symmetric homeomorphisms of the circle.** Consider a homeomorphism \( u : S^1 \to S^1 \), let \( \Gamma_u \subset S^1 \times S^1 \simeq \partial_\infty AdS_3^* \) be its graph.

**Theorem 1.12.** \( w(\Gamma_u) \) is at most \( \pi/2 \). It is strictly less than \( \pi/2 \) if and only if \( u \) is quasi-symmetric.

The first part here is just Lemma 4.10 already mentioned above. The second part is proved in Section 6.1.

This statement can be considered in a purely projective way, because the fact that a point of \( \partial_\infty C(\Gamma_u) \) is at distance strictly less than \( \pi/2 \) from \( \partial_\infty C(\Gamma_u) \) corresponds to a purely projective property, stated in terms of the duality between points and space-like planes in \( AdS_3^* \), see Section 2.4. This duality is itself a projective notion, see Section 2.5.

The proof uses the considerations explained above on the properties of maximal surfaces bounded by \( \Gamma_u \), it can be found in Section 6.1. It is based on Theorem 1.8 and to a partial converse, in dimension 3 only: if an acausal graph in \( \partial_\infty AdS_3^* \) is the boundary of a maximal surface with bounded second fundamental form which is not a “horosphere” (as described in Section 5.2), then \( \Gamma \) is the graph of a quasi-symmetric homeomorphism from \( S^1 \) to \( S^1 \).

1.9. **What follows.** Section 2 contains a number of basic notions on the anti-de Sitter (AdS) space and some of its basic properties. It is included here for completeness, in the hope of making the paper reasonably self-contained for reader not yet familiar with AdS geometry. Section 3 similarly contains some basic facts (presumably less well-known) on space-like hypersurfaces in the AdS space.

Section 4 is perhaps the heart of the paper. After some preliminary statements on maximal space-like hypersurfaces in AdS, it contains both an existence theorem for maximal hypersurfaces with given boundary
data at infinity, and a statement on the regularity of those hypersurfaces under a geometric condition on the boundary at infinity. This condition is later translated (for surfaces in the 3-dimensional AdS space) in terms of quasi-symmetric regularity of the data at infinity.

In Section \ref{sec:regularity} we further consider this regularity issue, with emphasis on surfaces in \( \text{AdS}_3 \), and we prove a uniqueness result for maximal surfaces with regular enough data at infinity. Finally we prove Theorem \ref{thm:maximal}.

Appendix A contains an alternative proof of the existence of a maximal hypersurface with given data at infinity, based on the mean curvature flow. This approach also yields some regularity results.

2. The Anti de Sitter space

This section contains a number of basic statements on AdS geometry, which are necessary in the proof of the main results. Readers who are already familiar with AdS geometry will find little interest in it, we have however decided to include it to make the paper self-contained, hoping that it is useful for readers interested in Teichmüller theory but not yet in AdS geometry.

2.1. Definitions. We consider the hyperboloid model of the hyperbolic space: the hyperbolic space \( \mathbb{H}^n \) is identified with the set of future-pointing unit timelike vectors in \((n + 1)\)-dimensional Minkowski space \( \mathbb{R}^{n,1} \). In this work, if it is not specified differently, we always use this identification. In particular points of \( \mathbb{H}^n \) are identified with elements \((x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\) such that \( \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1 \). We also fix the point \( x^0 = (0, \ldots, 0, 1) \in \mathbb{H}^n \).

Let \( \mathbb{R}^{n,2} \) be \( \mathbb{R}^{n+2} \) equipped with the symmetric 2-form
\[
(x, y) = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} - x_{n+2} y_{n+2}.
\]
The \((n + 1)\)-dimensional anti de Sitter space is the set
\[
\text{AdS}_{n+1}^* = \{ x \in \mathbb{R}^{n,2} \mid (x, x) = -1 \}.
\]
The tangent space at a point \( x \in \text{AdS}_{n+1}^* \) is the linear hyperplane orthogonal to \( x \) with respect to \( (\cdot, \cdot) \). The restriction of \( (\cdot, \cdot) \) to \( T_x \text{AdS}_{n+1}^* \) is a Lorentzian scalar product.

Remark 2.1. With this definition of \( \text{AdS}_{n+1}^* \), its isometry group is immediately seen to be \( O(n, 2) \). In particular, this isometry group acts transitively on the points of \( \text{AdS}_{n+1}^* \). More precisely, it acts simply transitively on the set of couples \((x, e)\) where \( x \in \text{AdS}_{n+1}^* \) and \( e \) is an orthonormal basis of \( T_x \text{AdS}_{n+1}^* \). It is also clear (using the action of \( O(n, 2) \) by isometries) that the geodesics in \( \text{AdS}_{n+1}^* \) are precisely the intersections of \( \text{AdS}_{n+1}^* \) with the linear planes in \( \mathbb{R}^{n,2} \) containing 0.

There is a map
\[
\Phi : \mathbb{H}^n \times \mathbb{R} \rightarrow \text{AdS}_{n+1}^*
\]
defined by
\[
(1) \quad \Phi((x_1, \ldots, x_{n+1}), t) = (x_1, x_2, \ldots, x_n, x_{n+1} \cos t, x_{n+1} \sin t).
\]
\( \Phi \) is a covering map, so topologically \( \text{AdS}_{n+1}^* \cong \mathbb{H}^n \times S^1 \). It will often be convenient to consider the universal cover \( \tilde{\text{AdS}}_{n+1} \) of \( \text{AdS}_{n+1}^* \), that is \( \mathbb{H}^n \times \mathbb{R} \), equipped with the pull-back of the metric on \( \text{AdS}_{n+1}^* \).

It is easy to see that this metric at a point \(((x_1, \ldots, x_{n+1}), t)\) takes the form
\[
(2) \quad g_{\mathbb{H}^n} - x_{n+1}^2 dt^2.
\]
If we consider the Poincaré model of \( \mathbb{H}^n \), the metric can be written as
\[
(3) \quad \frac{4}{(1-r^2)^2} (dy_1^2 + \cdots + dy_n^2) - \left( \frac{1 + r^2}{1 - r^2} \right)^2 dt^2,
\]
where \( r = \sqrt{y_1^2 + \cdots + y_n^2} \) and \( y_1, \ldots, y_n \) are the Cartesian coordinates on the ball \( \{ y \in \mathbb{R}^n \mid r(y) < 1 \} \).

By (2) we see that the time translations
\[
(x, t) \rightarrow (x, t + a)
\]
are isometries of \( \text{AdS}_{n+1}^* \). The coordinate field \( \frac{\partial}{\partial t} \) is a Killing vector field and the slices \( \mathbb{H}^n \times \{ t \} \) are totally geodesic.

We denote by \( \nabla \) the Levi-Civita connections of both \( \text{AdS}_{n+1}^* \) and \( \mathbb{H}^n \). Since \( \mathbb{H}^n \times \{ t \} \) is totally geodesic, the restriction of \( \nabla \) on this slice coincides with its Levi-Civita connection.

We say that a vector \( v \in T_x \text{AdS}_{n+1}^* \) is horizontal if it is tangent to the slice \( \mathbb{H}^n \times \{ t \} \). Analogously it is vertical if it is tangent to the line \( \{ x \} \times \mathbb{R} \).
The lapse function $\phi$ is defined by

$$\phi^2 = -\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle$$

The gradient of $t$ is a vertical vector at each point and it equal to

$$\nabla t = -\frac{1}{\phi^2} \frac{\partial}{\partial t},$$

so its squared norm is equal to $-\frac{1}{\phi^2}$.

2.2. The asymptotic boundary and the causal structure. We denote by $\overline{\text{AdS}}_{n+1}$ the manifold with boundary $\overline{\mathbb{H}}^n \times \mathbb{R}$, where $\overline{\mathbb{H}}^n$ is the usual compactification of $\mathbb{H}^n$ (obtained for instance in the projective model of $\mathbb{H}^n$). Another way to consider $\overline{\text{AdS}}_{n+1}$ is as the universal cover of the compactification of $\text{AdS}_{n+1}^*$ defined by adding the projectivization of the cone of vectors $x \in \mathbb{R}^{n,2}$ such that $\langle x, x \rangle = 0$.

Clearly $\text{AdS}_{n+1}^*$ is the interior part of $\overline{\text{AdS}}_{n+1}$, whereas its boundary, $\partial \overline{\mathbb{H}}^n \times \mathbb{R}$ is called the asymptotic boundary of $\text{AdS}_{n+1}^*$ and is denoted by $\partial_\infty \text{AdS}_{n+1}^*$. The following statement is clear when considering the definition of $\text{AdS}_{n+1}^*$ as a quadric.

**Lemma 2.2.** Every isometry $f$ of $\text{AdS}_{n+1}$ extends to a homeomorphism of $\overline{\text{AdS}}_{n+1}$.

The asymptotic boundary of a set $K \subset \text{AdS}_{n+1}$ — denoted by $\partial_\infty K$ — is the set of the accumulation points of $K$ in $\partial_\infty \text{AdS}_{n+1}$.

By (3) it is clear that the conformal structure on $\text{AdS}_{n+1}$ extends to the boundary. This means that in the conformal class of the metric $g$ there is a metric $g^*$ that extends to the boundary. We can for instance put $g^* = \frac{1}{\cos \phi} g$.

A vector $v$ tangent at some point in $\partial_\infty \text{AdS}_{n+1}$ is timelike (lightlike, spacelike) if $g^*(v, v) < 0$ ($= 0$, $> 0$). Notice that the definition makes sense since the sign of $g^*(v, v)$ depends only on the conformal class of $g^*$.

**Lemma 2.3.** Let $c : (-1, 1) \to \text{AdS}_{n+1}^*$ be an inextensible timelike path. If the function $t$ is bounded from above on $c$, there exists the limit $p_1 = \lim_{s \to -1} c(s) \in \partial_\infty \text{AdS}_{n+1}^*$.

**Proof.** The vertical component of $\dot{c}$ is

$$\dot{c}_V = \langle \dot{c}, \nabla t \rangle \frac{\partial}{\partial t} = \dot{t} \frac{\partial}{\partial t}.$$

Since the norm of $\frac{\partial}{\partial t}$ for $g^*$ is 1, we have $|\dot{c}_V|_{g^*} = \dot{t}$. On the other hand, the fact that $c$ is timelike implies

$$|\dot{c}_H|_{g^*} \leq |\dot{c}_V|_{g^*} = \dot{t}.$$

Since the function $t$ is increasing along $c$, the bound on $t$ along $c$ implies that $\dot{c}$ is bounded in a neighbourhood of 1. It follows that the path $c_H$ obtained by projecting $c$ to $\mathbb{H}^n$, has finite length with respect to the metric $\frac{1}{\cos \phi} g|_{\mathbb{H}^n}$. This implies that there exists the limit $x_1 = \lim_{s \to -1} c_H(s)$. On the other hand, since $t$ is increasing along $c$ there exists the limit $t_1 = \lim_{s \to -1} t(c(s))$. The point $p_1 = (x_1, t_1)$ is the limit point of $c$. Since we assume that $c$ is inextensible in $\text{AdS}_{n+1}^*$, $p_1 \in \partial_\infty \text{AdS}_{n+1}^*$.

The point $p_1$ is an asymptotic end-point of $c$.

An inextensible path is without end-points if and only if the function $t$ takes all the real values along $c$, or equivalently, if $c$ does not admit any asymptotic end-point. Vertical lines are instances of inextensible paths without end-points.

2.3. Geodesics and geodesic hyperplanes in $\text{AdS}_{n+1}^*$. The next statement, which is classical, describes the geodesics in $\text{AdS}_{n+1}^*$, considered as a quadric in $\mathbb{R}^{n,2}$.

**Lemma 2.4** (see [10]). Geodesics in $\text{AdS}_{n+1}^*$ are the intersection $\text{AdS}_{n+1}^*$ with linear 2-planes in $\mathbb{R}^{n,2}$ containing 0.

In particular given a tangent vector $v$ at some point $p \in \text{AdS}_{n+1}^*$ we have

$$\exp_p(sv) = \begin{cases} \cos(s)p + \sin(s)v & \text{if } \langle v, v \rangle = -1; \\ p + sv & \text{if } \langle v, v \rangle = 0; \\ \cosh(s)p + \sinh(s)v & \text{if } \langle v, v \rangle = 1. \end{cases}$$

**Remark 2.5.** Totally geodesics $k$-planes in $\text{AdS}_{n+1}^*$ are the intersection of $\text{AdS}_{n+1}^*$ with $(k + 1)$-linear planes of $\mathbb{R}^{n,2}$ containing 0.
Spacelike and lightlike geodesics are open simple curves. Homotopically, timelike geodesics are simple closed non-trivial curve. Moreover every complete timelike geodesic starting at \( p \) passes through \(-p\) at time \( (2k + 1)\pi\) and at \( p \) at time \( 2k\pi \) for \( k \in \mathbb{Z} \).

Passing to the universal cover, we get the following statement.

**Lemma 2.6.** Given a point \( p = (x, t) \in AdS_{n+1} \) there is a discrete set \( \{p_n \mid n \in \mathbb{Z}\} \) such that every timelike geodesic \( \gamma \) starting through \( p \) passes through \( p_n \) at time \( n\pi \).

Moreover, \( p_{2k} = (x, t + 2k\pi) \) and \( p_{2k+1} = (y, t + (2k + 1)\pi) \) where \( y \) is some point in \( \mathbb{H}^n \) independent of \( k \).

In what follows we will often use the points \( p_1 \) and \( p_{-1} \). To simplify the notation we will denote these points by \( p_+ \) and \( p_- \).

Timelike geodesics are timelike paths without end-points. On the other hand since spacelike geodesics are conjugated by some isometry to horizontal ones by some isometry, they have 2 asymptotic end-points. Using the projection \( \Phi \) one can check that the path \( c(s) = (x(s), \arccos\left(\frac{1}{\sqrt{1 + s^2}}\right)) \) where \( x(s) = (s, 0, \ldots, 0, \sqrt{1 + s^2}) \) is a lightlike geodesic. Since \( c \) has two asymptotic end-points, the same property holds for every lightlike geodesic.

**Remark 2.7.** Points in \( \partial_{\infty}AdS_{n+1} \) related by a timelike arc in \( \partial_{\infty}AdS_{n+1} \) are not joined by a geodesic arc in \( AdS_{n+1} \). Indeed by the above description if a geodesic connects two points in the asymptotic boundary of \( AdS_{n+1} \) then it is either space-like or light-like (and in this case it is contained in the boundary).

Totally geodesic \( n \)-planes in \( AdS_{n+1} \) are distinguished by the restriction of the ambient metric on them. They can be timelike, spacelike or lightlike according as whether this restriction has Lorentzian, Euclidean or degenerate signature.

Spacelike hyperplanes are conjugated by some isometry to horizontal planes. Timelike hyperplanes are conjugated by some isometry to the hyperplane \( P_0 \times \mathbb{R} \), where \( P_0 \) is a totally geodesic hyperplane in \( \mathbb{H}^n \).

For lightlike hyperplanes we will need a more precise description.

**Lemma 2.8.** Let \( P \) be a lightlike hyperplane. There are two points \( \zeta_- \) and \( \zeta_+ \) in \( \partial_{\infty}AdS_{n+1} \) such that \( P \) is foliated by lightlike geodesics with asymptotic end-points \( \zeta_- \) and \( \zeta_+ \).

The foliation of \( P \) by lightlike geodesics extends to a foliation of \( \overline{\mathcal{P}} \setminus \{\zeta_-, \zeta_+\} \) by lightlike geodesics, where \( \mathcal{P} \) denotes the closure of \( P \) in \( AdS_{n+1} \).

**Proof.** It is sufficient to prove the statement for a specific lightlike plane. Consider the hypersurface \( P_0 = \{(x, t) \in AdS_{n+1} \mid t = \arccos\left(\frac{x_n}{\sqrt{n+1}}\right)\} \). Using the projection \( \Phi \) one see that \( P_0 \) is a totally geodesic plane, indeed \( \Phi(P_0) \) is a connected component of the intersection of \( AdS_{n+1}^* \) with the linear plane defined by the equation \( y_1 - y_{n+2} = 0 \).

We consider the natural parameterization \( \sigma : \mathbb{H}^n \to P_0 \) defined by \( \sigma(x) = (x, \arccos(\frac{\sqrt{n}}{x_n})) \). Since the function \( \frac{x_n}{\sqrt{n+1}} \) extends to the boundary of \( \mathbb{H}^n \), the map \( \sigma \) extends to \( \mathbb{H}^n \) and gives a parameterization of the closure \( \overline{P_0} \) of \( P_0 \) in \( AdS_{n+1} \).

The level surfaces \( H_a = \{x_n = a\} \) are totally geodesic hyperplanes orthogonal to the geodesic \( c = \{x_2 = \ldots = x_n = 0\} \). Let \( N \) be the unit future-oriented vector field on \( \mathbb{H}^n \) orthogonal to \( H_a \) for all \( a \). A simple computation shows that

- for all \( a \), \( \sigma|_{H_a} \) is an isometric embedding;
- \( \dot{N} = \sigma_\ast(N) \) is a lightlike field;
- \( \dot{N} \) is orthogonal to \( \sigma(H_a) \).

It follows that \( P_0 \) is a lightlike plane. The integral lines of \( \dot{N} \) produce a foliation of \( P_0 \) by lightlike geodesics. Notice that integral lines of \( \dot{N} \) are the images of integral lines of the field \( N \). By standard hyperbolic geometry, all these lines join the endpoints, say \( x_- \), \( x_+ \), of the geodesic \( c \). We conclude that lightlike geodesics of \( P_0 \) join \( \sigma(x_-) \) to \( \sigma(x_+) \). Since the foliation of \( \mathbb{H}^n \) by integral lines of \( N \) extends to a foliation of \( \overline{\mathbb{H}^n} \setminus \{x_- , x_+\} \), the foliation given by \( \dot{N} \) extends to a foliation of \( \overline{P_0} \setminus \{\zeta_-, \zeta_+\} \). By continuity we conclude that the leaves of this foliation are lightlike.

For a lightlike plane \( P \) the points \( \zeta_- \) and \( \zeta_+ \) are called respectively the past and the future end-points of the plane.

Spacelike and lightlike hyperplanes disconnect \( AdS_{n+1} \) in two connected components, that coincide with the past and the future of them. Their asymptotic boundary is a no-where timelike closed curve. On the other hand the asymptotic boundary of a timelike plane is the union of two inextensible timelike curves.
2.4. The causal structure of $AdS_{n+1}$. If $c : [0, 1] \to AdS_{n+1}$ is a timelike path, its length is defined in this way:

$$\ell(c) = \int_0^1 (-\langle \dot{c}(s), \dot{c}(s) \rangle)^{1/2} \, ds.$$ 

Given $p \in AdS_{n+1}$ we consider the set $P_-(p)$ (resp. $P_+(p)$) defined respectively as the set of points that can be joined to $p$ through a past-directed (resp. future-directed) timelike geodesic of length $\pi/2$.

**Remark 2.9.** For a point $x \in AdS_{n+1}$ we can identify the set of unit timelike vectors at $x$ with the geodesic plane $P^*_x = x^\perp \cap AdS_{n+1}$ (where $x^\perp$ is the linear plane orthogonal to $x$). $P^*_x$ has two connected components. Equation (4) shows that these components are the images of $P_+(p)$ and $P_-(p)$, where $p$ is any preimage of $x$ in $AdS_{n+1}$.

The following properties of $P_-(p)$ and $P_+(p)$ are a direct consequence of Remark 2.9.

**Lemma 2.10.** The sets $P_-(p)$ and $P_+(p)$ are complete, space-like totally geodesic planes. Every timelike geodesic starting at $p$ meets $P_-(p)$ and $P_+(p)$ orthogonally.

**Remark 2.11.** For the point $p_0 = (x^0, 0)$, a direct computation (still using the projection $\Phi$) shows that $P_-(p_0)$ and $P_+(p_0)$ are level curves of the time function $t$ corresponding to values $-\pi/2$ and $\pi/2$ respectively.

The planes $P_-(p)$ and $P_+(p)$ are disjoint and bound an open precompact domain $U_p$ in $\overline{AdS_{n+1}}$. For instance, for $p = (x^0, 0)$ we have $U_p = \{(x, t) \in \overline{AdS_{n+1}} \mid -\pi/2 < t < \pi/2\}$.

By definition the interior of $U_p$ (denoted by $\text{int}(U_p)$) is the intersection of $U_p$ with $AdS_{n+1}$. Notice that

$$\text{int}(U_p) = I^+(P_-(p)) \cap I^+(P_+(p)).$$

Notice that $P_+(p_k) = P_-(p_{k+1})$ for every $k \in \mathbb{Z}$. In particular $U_{p_i} \cap U_{p_j} = \emptyset$ if $|i - j| > 1$ and $\overline{U_{p_i} \cap U_{p_{i+1}}} = P_+(p_i)$.

Given $p \in AdS_{n+1}$ we denote by $C_p$ the set of points joined to $p$ through a timelike geodesic of length less than $\pi/2$.

**Proposition 2.12.**

- $C_p \subset U_p$.
- spacelike and lightlike geodesics join $p$ to points in $U_p \setminus C_p$, whereas timelike geodesics are contained in $\bigcup_{n \in \mathbb{Z}} C_{p_n}$.
- $I^+(p) \subset C_p \cup I^+(P_+(p)) = C_p \cup \bigcup_{k>0} U_{p_k}$.
- $\partial C_p \cap U_p$ is the lightlike cone through $p$, whereas $\partial_c C_p$ is the union of the asymptotic boundary of $P_+(p)$ and the asymptotic boundary of $P_-(p)$.

This proposition can be easily proved using the projection $\Phi$ and the explicit formula (4).

It is worth noticing that $AdS_{n+1}$ is not geodesically convex. Indeed the set of points in $AdS_{n+1}$ that can be joined to $p$ by a geodesic is $\text{int}(U_p) \cup \bigcup C_{p_k}$. 
Corollary 2.13. The set $I^-(p_+) \cap I^+(p_-)$ is the maximal star neighbourhood of $p$.

Given $p \in AdS_{n+1}$ and $q \in I^+(p)$, the distance between them is defined as

$$\delta(p, q) = \sup \{ \ell(c) | c \text{ timelike path joining } p \text{ to } q \}.$$ 

The next statement is true in a rather general context and can be proved by classical arguments.

Lemma 2.14. If $U$ is a star neighbourhood of $p$, then the distance from $p$

$$\delta : U \cap I^+(p) \ni q \mapsto \delta(p, q) \in \mathbb{R}$$

is smooth. For $q \in U \cap I^+(p)$ the distance $\delta(p, q)$ is realized by the unique geodesic joining $p$ to $q$ contained in $U$.

Remark 2.15. The definition of the distance shows that for $q \in I^+(p) \cap U$ and $r \in I^+(q)$, the reverse of the triangle inequality holds

$$\delta(p, r) \geq \delta(p, q) + \delta(q, r).$$

2.5. The projective model. As noted in the proof of Lemma 2.6 the geodesics in $AdS^*_{n+1}$ are obtained as the intersection of $AdS^*_{n+1}$ with the linear planes of $\mathbb{R}^{n+2}$ containing 0.

For this reason the projection map

$$\pi : AdS_{n+1} \to \mathbb{R}P^{n+1}$$

is projective: it sends geodesics of $AdS_{n+1}$ to projective segments. The image of this projective map is the interior of a quadric $Q \subset \mathbb{R}P^{n+1}$ of signature $(n-1,1)$.

Notice for $p \in AdS_{n+1}$ the domain $\Phi(\int(U_p))$ is a connected component of $AdS^*_{n+1} \setminus P^*_{\Phi(p)}$. Thus the domain $\pi(\Phi(U_p))$ is contained in some affine chart of $\mathbb{R}P^{n+1}$.

In this way we construct a projective embedding

$$\pi^* : \int(U_p) \to \mathbb{R}^{n+1}.$$

The map $\pi^*$ can be easily computed assuming $p = (x^0, 0)$. In this case $U_p = \{(x, t) | t \in (-\pi/2, \pi/2)\}$ so $\Phi(\int(U_p)) = \{(y_1, \ldots, y_n, y_{n+1}, y_{n+2}) \in AdS^*_{n+1} | y_{n+1} > 0\}$ and

$$\pi^* (x_1, \ldots, x_{n+1}, t) = \left( \frac{x_1}{x_{n+1} \cos t}, \frac{x_2}{x_{n+1} \cos t}, \ldots, \frac{x_n}{x_{n+1} \cos t}, \tan t \right)$$

for every $(x_1, \ldots, x_n) \in \mathbb{H}^n$ and $t \in (-\pi/2, \pi/2)$.

Notice that the map extends continuously on $U_p$ to a map, still denoted by $\pi^*$. From (6), the image $\pi^*(U_p)$ is the set

$$\{(z_1, \ldots, z_{n+1}) \big| \sum_{i=1}^n z_i^2 \leq z_{n+1}^2 + 1\}.$$

In particular we deduce that every point $q \in U_p$ (even on the boundary) can be joined to $p$ by a unique geodesic and that this geodesic continuously depend on $q$.

We have seen above how to associate to a point $p \in AdS_{n+1}$ two totally geodesic space-like hyperplanes $P_- (p)$ and $P_+(p)$. Both planes are sent by $\pi$ to the intersection with $\pi(AdS^*_{n+1})$ of the same projective plane $P$, and $P$ has a purely projective definition. Indeed the light-cone of $p$ is tangent to $Q$ along a circle $C$, and the image by $\pi$ of the boundary at infinity of $P_- (p)$ is precisely $C$. One way to see this is by using the fact that in the projective model of $AdS_{n+1}$ (as for the hyperbolic space) the distance between two points can be defined in terms of the Hilbert distance of the quadric $Q$, see e.g. [27].

This duality extends to a duality between totally geodesic (space-like) $k$-planes in $\pi(AdS^*_{n+1})$, with the dual of a $k$-plane $P$ being a $(n-k)$-plane $P^*$. Then $P^*$ can be defined as the intersection between the hyperplanes dual to the points of $P$, and conversely. Then $P^*$ can be characterized as the set of points at distance $\pi/2$ from $P$ along a time-like segment, and conversely.

2.6. The 3-dimensional AdS space. The general description of the $n$-dimensional anti-de Sitter space $AdS^*_{n+1}$ above can be refined when $n = 2$, and $AdS^*_{2+1}$ has some quite specific properties.

One such specificity is that $AdS^*_{2+1}$ is none other than the Lie group $SL(2, \mathbb{R})$, with its Killing metric. This point of view, which is important in itself (see [26, 4]), will not be used explicitly here.

Another feature which is specific of $AdS_3$ is the fact that the boundary of $\pi(AdS_3)$ in $\mathbb{R}P^3$ is a quadric of signature $(1, 1)$ which, as is well known, is foliated by two families of projective lines, which we will call $L_l$ and $L_r$ ($l$ and $r$ stand for “left” and “right” here). Those projective lines correspond precisely to the isotropic curves in the Lorentz-conformal structure on $\partial_\infty AdS_3$. Each line of one family intersects each line of the other family
at exactly one point, this provides an identification of \( \partial \pi(AdS_3^*) \) with \( S^1 \times S^1 \), with each copy of \( S^1 \) identified with one of the two families of lines foliating \( \partial \pi(AdS_3^*) \).

This has interesting consequences, in particular those explained in Section 3.4. Another consequence is that the isometry group of \( AdS_3 \) can be naturally identified (up to finite index) with the product of two copies of \( PSL(2, \mathbb{R}) \). Indeed any isometry of \( AdS_3 \) in the connected component of the identity acts on the two families of lines foliating \( \partial_\infty AdS_3 \) by permuting those lines, and this action is projective on each family of lines. Conversely, any couple of elements of \( PSL(2, \mathbb{R}) \) can be obtained in this manner.

3. Spacelike graphs in \( AdS_{n+1} \)

This section continues the description of the geometry of the AdS space, with emphasis on space-like surfaces. Readers already familiar with AdS geometry might not be very surprised by most of the results, but several notations and lemmas will be used in the next section.

3.1. Definitions. A smooth embedded hypersurface \( M \) in \( AdS_{n+1} \) is \emph{spacelike} if for every \( x \in M \) the restriction of the inner product \( \langle \cdot , \cdot \rangle \) on \( T_xM \) is positive definite. It turns out that a Riemannian structure is induced on every spacelike hypersurface by the ambient metric.

We say that a spacelike surface \( M \) in \( AdS_{n+1} \) is a \emph{graph} if there is a function \( u : \mathbb{H}^n \to \mathbb{R} \) such that \( M \) coincides with the graph of \( u \).

First let us check which functions correspond to spacelike graphs. The function \( u \) induces a function on \( \mathbb{H}^n \times \mathbb{R} \) \( \hat{u}(x,t) = u(x) \).

The gradient of \( \hat{u} \) at a point \((x,t)\) is the horizontal vector that projects to the gradient of \( u \) at \( x \).

The graph of \( u \), say \( M_u \), is defined by the equation \( \hat{u} - t = 0 \). Thus the tangent space \( T_{(x,u(x))}M = \ker(dt - d\hat{u})(x,u(x)) \). In particular the normal direction of \( M \) at \((x,u(x))\) is generated by the vector

\[
\bar{\nu} = \bar{\nabla}t - \bar{\nabla}\hat{u}
\]

whose norm is

\[
|\bar{\nabla}\hat{u}|^2 - \frac{1}{\phi^2}.
\]

Since \( |\bar{\nabla}\hat{u}| = |\bar{\nabla}u| \) we deduce that \( M \) is spacelike if and only if

\[
1 - \phi^2|\bar{\nabla}u|^2 < 0,
\]

and the future-pointing normal vector is

\[
\nu = \frac{\phi}{\sqrt{1 - \phi^2|\bar{\nabla}u|^2}}(\bar{\nabla}\hat{u} - \bar{\nabla}t).
\]

It is interesting to express (9) using the Poincaré model of hyperbolic space. In that case we have

\[
\bar{\nabla}u = \left(1 - r^2\right)^{2/4} \left(\frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_n}\right)
\]

so

\[
|\bar{\nabla}u|^2 = \left(1 - r^2\right)^{-1} \sum \left(\frac{\partial u}{\partial y_j}\right)^2
\]

and condition (9) becomes

\[
\sum \left(\frac{\partial u}{\partial y_j}\right)^2 < \frac{4}{(1 + r^2)^2}.
\]

In particular the function \( u \) is 2-Lipschitz with respect to the Euclidean distance of the ball.

**Lemma 3.1.** Let \( M = M_u \) be a smooth spacelike graph in \( AdS_{n+1} \). Then the function \( u \) extends to a continuous function

\[
\bar{u} : \mathbb{H}^n \to \mathbb{R}.
\]

In particular the closure of \( M \) in \( \overline{AdS_{n+1}} \) is still a graph.
3.2. Acausal surfaces. A $C^{0,1}$ hypersurface $M$ in $AdS_{n+1}$ is said to be weakly spacelike if for every $p \in M$ there is a neighbourhood $U$ of $p$ in $AdS_{n+1}$ such that $U \setminus M$ is the disjoint union $I^+_U(M) \cup I^-_U(M)$.

A neighbourhood satisfying the above property will be called a good neighbourhood of $p$.

It is not hard to see that a spacelike surface is weakly spacelike. On the other hand a $C^1$ weakly spacelike surface is characterized by the property that no tangent plane is timelike.

A weakly spacelike graph is a weakly spacelike surface that is the graph of some function $u$. Weakly spacelike graphs correspond to Lipschitz functions $u$ such that the inequality
\[ 1 - \phi^2|\nabla u|^2 \leq 0 \]
holds almost everywhere.

As for spacelike graphs it is still true that the closure of acausal graphs in $\overline{AdS_{n+1}}$ is a graph.

First we provide an intrinsic characterization of weakly spacelike graphs.

Proposition 3.2. Let $M$ be a connected weakly spacelike hypersurface. The following statements are equivalent:

1. $M$ is a weakly spacelike graph;
2. $AdS_{n+1} \setminus M$ is the union of 2 connected components;
3. every inextensible timelike curve without end-points meets $M$ exactly in one point.

Proof. The implication (1) $\Rightarrow$ (2) is clear.

Assume (3) holds. Then every vertical line meets $M$ exactly in one point. This shows that the projection $\pi : M \rightarrow \mathbb{H}^n$ is one-to-one. Since $M$ is a topological manifold, the Invariance of Domain Theorem implies that $\pi$ is a homeomorphism. Thus $M$ is a graph.

Finally suppose that (2) holds. We consider the equivalence relation on $M$ such that $p \sim q$ if there are good neighbourhoods $U$ and $V$ of $p$ and $q$ respectively such that $I^+_U(p)$ and $I^-_V(q)$ are contained in the same component of $AdS_{n+1} \setminus M$. Equivalence classes are open. Since $M$ is connected, all points are equivalent. We deduce that there is a component, say $\Omega_+$, of $AdS_{n+1} \setminus M$ such that if $c = c(s)$ is a future-directed timelike path hitting $M$ for $s = 0$, then there is $\epsilon > 0$ such that $c(s) \in \Omega_+$ for $0 < s < \epsilon$. In the same way, there is a component, say $\Omega_-$ such that $c(s) \in \Omega_-$ for $-\epsilon < s < 0$.

If $U$ is a good neighbourhood of some point $p \in M$, then $U \subset \Omega_+ \cup M \cup \Omega_-$, so $\Omega_+ \cup \Omega_- \cup M$ is an open neighbourhood of $M$. Since the closure of every component of $AdS_{n+1} \setminus M$ contains points in $M$, by the assumption (2), $\Omega_+$ and $\Omega_-$ are different components of $AdS_{n+1} \setminus M$ and $AdS_{n+1} = \Omega_- \cup M \cup \Omega_+$.

It follows that no future-directed timelike curve starting at a point of $\Omega_+$ can end at some point of $M$. Since any future-directed timelike curve that starts on $M$ intersects $\Omega_+$, points of $M$ are not related by timelike curves and $I^+(M) \subset \Omega_+$ and $I^-(M) \subset \Omega_-$.

In particular, given a point $p \in M$, the surface $M$ is contained in $U_p$. It follows that the restriction of the time-function $t$ on $M$ is bounded in some interval $[a, b]$. Moreover $\Omega_+$ contains the region $\{(x,t) | t > b\}$, instead $\Omega_-$ contains the region $\{(x,t) | t < a\}$.

Since the restriction of $t$ on any inextensible timelike curve without end-points $c$ takes all the values of the interval $(-\infty, +\infty)$ we have that $c$ contains points of $\Omega_+$ and points of $\Omega_-$. Thus it must intersect $M$. Since points of $M$ are not related by timelike arcs, such intersection point is unique. \hfill \Box

Remark 3.3. Proposition 3.2 implies that spacelike graphs are intrinsically described in terms of the geometry of $AdS_{n+1}$. In particular, if $M$ is a spacelike graph, and $\gamma$ is an isometry of $AdS_{n+1}$, then $\gamma(M)$ is still a spacelike graph.

Remark 3.4. Given a point $p \in AdS_{n+1}$ we have that $\partial I^+(p)$ is a weakly spacelike graph. Indeed we can assume $p = (x^0, 0)$. In that case it turns out that $\partial I^+(p)$ is the graph of the function $\arccos\left(\frac{1}{x^{n+1}}\right)$.

An important feature of weakly spacelike graphs is that they are acausal as the following proposition states.

Proposition 3.5. Let $M = M_a$ be a weakly spacelike graph in $AdS_{n+1}$, and let $\overline{M}$ denote its closure in $\overline{AdS_{n+1}}$. Given $p \in M$, then, for every $q \in \overline{M}$, $p$ and $q$ are connected by a geodesic $[p, q]$ that is not timelike. Moreover, if this geodesic is lightlike, then it is contained in $M$.

Proof. Proposition 3.2 implies that $M \cap I^+(p) = \emptyset$ and $M \cap I^-(p) = \emptyset$. In particular, $M \subset U_p$ that is a star-neighbourhood of $p$. It follows that any point $q$ of $\overline{M}$ is connected to $p$ by some geodesic that continuously depends on $p$. Since points of $M$ cannot be connected to $p$ by a timelike geodesic, the same holds for points in $\partial \overline{M}$.

Finally, let us prove that if $[p, q]$ is lightlike, then it is contained in $M$. 

Let \( u_\pm : \mathbb{H}^n \to \mathbb{R} \) be such that \( \partial I^\pm(p) \) is the graph of \( \Gamma_{u_\pm} \). Let us set \( p = (x_0, t_0) \) and \( q = (x_1, t_1) \). Consider the geodesic arc of \( \mathbb{H}^n \), say \( x(s) \), starting from \( x_0 \) and ending at \( x_1 \) defined for \( s \in [0, T] \) (\( T \) can be \( +\infty \) if \( x_1 \in \partial \mathbb{H}^n \)). Notice that the function of \( s \) defined by \( u_+(s) = u_+(x(s)) \) satisfies
\[
\dot{u}_+ = \frac{1}{\phi(x(s))}, \quad u_+(0) = t_0.
\]

On the other hand the function \( u(s) = u(x(s)) \) satisfies
\[
\dot{u} = (\nabla u, \frac{dx}{ds}) \leq \frac{1}{\phi(x(s))}, \quad u(0) = t_0.
\]
Comparing (12) and (13) we deduce that
\[ u(s) \leq u_+(s), \]
and the equality holds at some \( s_0 \) if and only if \( \dot{u}(s) = \frac{1}{\phi(x(s))} \) on the interval \( [0, s_0] \), that is equivalent to say that the light-like segment joining \( p = (x_0, t_0) \) to \( q = (x(s_0), u(x(s_0))) \) is contained in \( M \).

In an analogous way we show that \( u_-(s) \leq u(s) \).

**Remark 3.6.** The hypothesis that \( M \) is a graph is essential in Proposition 3.5. It is not difficult to construct a spacelike surface \( M \) containing points \( p, q \) that are related by a vertical segment.

For a weakly spacelike surface \( M \), a point \( p \in M \) is *singular* if it is contained in the interior of some lightlike segment contained in \( M \). The singular set of \( M \) is the set of singular points.

Analogously we define the singular set of the asymptotic boundary \( \Sigma \) of \( M \). Notice that the singular set of \( \Sigma \) can be non-empty even if \( M \) does not contain singular points.

### 3.3. The domain of dependence of a spacelike graph

Let \( M \) be a spacelike graph in \( AdS_{n+1} \), and let \( \Sigma \) denote its asymptotic boundary. We will suppose that \( M \) does not contain any singular point.

The domain of dependence of \( M \) is the set \( D \) of points \( x \in AdS_{n+1} \) such that *every inextensible causal path through \( x \) intersects \( M \).*

It can be easily shown that this property is equivalent to requiring that \( (I^+(x) \cup I^-(x)) \cap M \) is precompact in \( AdS_{n+1} \).

There is an easy characterization of \( D \) in terms of \( \Sigma \).

**Lemma 3.7.** With the notations of Section 2.3 a point \( p \) lies in \( D \) if and only if \( \Sigma \) is contained in \( U_p \).

**Proof.** Suppose that \( p \in D \). Without loss of generality we can suppose that \( p \in I^- (M) \). By the hypothesis, \( I^+(p) \cap M \) is precompact in \( AdS_{n+1} \) (whereas \( I^- (p) \cap M = \emptyset \)). Thus there is a compact ball \( B \subset \mathbb{H}^n \) such that \( I^+(p) \cap M \) is contained in the cylinder above \( B \). In particular, \( M \setminus (B \times \mathbb{R}) \) is contained in \( U_p \). It follows that \( \Sigma \subset \overline{U_p} \).

If some point \( x \) of \( \Sigma \) were contained in \( \partial_{\infty} P_+(p) \) then the geodesic joining \( p \) to \( x \) would be lightlike and would intersects \( M \) in some point \( q \). Then by Proposition 3.5, the lightlike geodesic segment joining \( q \) to \( x \) would be contained in \( M \) and this would contradict the hypothesis that \( M \) does not contain any singular point.

Let us consider now a point \( p \) such that \( \Sigma \subset U_p \). Again we can suppose that \( p \in I^- (M) \). By the assumption the asymptotic boundary of \( M \) and the asymptotic boundary of \( I^+(p) \) are disjoint. It follows that \( I^+(p) \cap M \) is pre-compact in \( AdS_{n+1} \).

**Corollary 3.8.** Two spacelike surfaces share the boundary at infinity if and only if their domains of dependence coincide.

**Proposition 3.9.** The domain \( D \) is geodesically convex and its closure at infinity is precisely \( \Sigma \).

The boundary of \( D \) is the disjoint union of two weakly spacelike graphs \( \partial_{\pm} D = \mathcal{M}_{u_\pm} \) whose boundary at infinity is \( \Sigma \).

Every point \( p \in \partial D \) is joined to \( \Sigma \) by a lightlike ray.

To prove this proposition we need a technical lemma of AdS geometry.

**Lemma 3.10.** Given two points \( p, q \in AdS_{n+1} \) connected along a geodesic segment \( [p, q] \) and given any point \( r \) lying on such a segment, we have that
\[ U_p \cap U_q \subset U_r. \]
Proof. Let $u_p$ (resp. $v_p$) be the real function on $\mathbb{H}^n$ such that $P_+(p)$ (resp. $P_-(p)$) is the graph of $u_p$ (resp. $v_p$).

Analogously define $u_q, v_q, u_r, v_r$.

We have that
\[
U_p = \{(x,t)|v_p(x) < t < u_p(x)\}, \quad U_q = \{(x,t)|v_q(x) < t < u_q(x)\}, \quad U_r = \{(x,t)|v_r(x) < t < u_r(x)\}.
\]
In particular, $U_p \cap U_q = \{(x,t)|\max\{v_p(x), v_q(x)\} < t < \min\{u_p(x), u_q(x)\}\}$. Then, the statement turns out to be equivalent to the inequalities
\[
v_r \leq \max\{v_p, v_q\} \quad \min\{u_p, u_q\} \leq u_r.
\]

If the segment $[p, q]$ is timelike, then, up to isometry, we can suppose that $p = (x^0, 0), q = (x^0, a), r = (x^0, b)$ with $0 \leq b \leq a$. In this case we have $u_p(x) = \pi/2, u_q(x) = a + \pi/2, u_r(x) = b + \pi/2$ so the statement easily follows.

Suppose now that the geodesic $[p, q]$ is spacelike. Up to isometry, we can suppose that $p = (x_p, 0), q = (x_q, 0), r = (x_r, 0)$ where $x_p, x_q, x_r$ are the following points in (the hyperboid model of) $\mathbb{H}^n$:
\[
x_p = (-\sinh \epsilon, 0, \ldots, 0, \cosh \epsilon), \quad x_q = (\sinh \eta, 0, \ldots, 0, \cosh \eta), \quad x_r = (0, \ldots, 0, 1),
\]
where $\eta$ and $\epsilon$ are respectively the distance from $p$ and $q$ to $r$.

The corresponding points $p^*, q^*, r^* \in AdS^*_{n+1}$ are
\[
p^* = (-\sinh \epsilon, 0, \ldots, 0, \cosh \epsilon), \quad q^* = (\sinh \eta, 0, \ldots, 0, \cosh \eta, 0), \quad r^* = (0, \ldots, 0, 1, 0).
\]

By Remark 2.9 $\Phi(P_+(p))$ is a component of the intersection of $AdS^*_{n+1}$ with the hyperplane defined by the equation
\[-y_1 \sinh \epsilon - y_{n+1} \cosh \epsilon = 0.
\]
In particular, pulling-back this equation, we deduce that the set $P_+(p)$ is a component of the set
\[
\{(x_1, \ldots, x_{n+1}, t) \in \mathbb{H}^n \times \mathbb{R} | -x_1 \sinh (\epsilon) - x_{n+1} \cos t \cosh (\epsilon) = 0\}.
\]
Since the function $t$ takes value in $(0, \pi)$ on $P_+(p)$ we deduce that
\[
u_p(x_1, \ldots, x_{n+1}) = \arccos \left(\frac{-x_1 \sinh \epsilon}{x_{n+1} \cosh \epsilon}\right).
\]
Analogously, we derive
\[
u_r(x_1, \ldots, x_{n+1}) = \pi/2 \quad u_q(x_1, \ldots, x_{n+1}) = \arccos \left(\frac{x_1 \sinh \eta}{x_{n+1} \cosh \eta}\right).
\]
Notice that $u_p \leq \pi/2$ if $x_1 \leq 0$, whereas $u_q \leq \pi/2$ if $x_1 \geq 0$. It follows that $\min\{u_p, u_q\} \leq u_r$.

When $[p, q]$ is lightlike, the computation is completely analogous. \qed

Remark 3.11. From the proof of the lemma we have that $P_+(p)$ and $P_-(q)$ are disjoint in $AdS_{n+1}$ if $p$ and $q$ are joined by a timelike segment, while they meet along a $(n - 1)$-dimensional geodesic plane if $p$ and $q$ are connected by a spacelike geodesic. Finally in the lightlike case, they meet at the asymptotic end-points of the geodesic through $p$ and $q$.

Proof of Proposition 3.9. Let $p$ be a point contained in $D$ and consider the nearest conjugate points $p_{\pm}$ to $p$ as defined in Section 2.3. First we show that $D$ is contained in the star neighbourhood $I^-(p_{\pm}) \cap I^+(p_{\pm})$ of $p$. Let $q \notin I^-(p_{\pm})$. If $q \in I^+(p_{\pm})$ then $I^-(p_{\pm}) \subset I^-(q)$. Since $\Sigma$ is contained in the asymptotic boundary of the past of $P_+(p) = P_-(p_{\pm})$ that in turn coincides with the asymptotic boundary of $I^-(p_{\pm})$, we see that $\Sigma \subset \partial_{\infty} I^-(q)$, so that $\Sigma \cap U_q = \emptyset$. Suppose now that $q$ is related to $p_{\pm}$ by a spacelike geodesic. Remark 3.11 shows that $\partial_{\infty} P_+(p_{\pm}) \cap \partial_{\infty} P_-(q)$ contains a point $(\xi, t)$. Since $\Sigma$ is a graph on $\partial_{\infty} \mathbb{H}^n$, there is a point in $\Sigma$ of the form $(\xi, t')$ and since $\Sigma \subset I^-(P_-(p_{\pm}))$ we get $t' < t$. It follows that $(\xi, t')$ is not contained in $U_q$. Eventually we obtain that $q \notin D$. The same argument shows that any point in $D$ must be contained in $I^+(p_{\pm})$ so $D$ is contained in $I^-(p_{\pm}) \cap I^+(p_{\pm})$.

We deduce from this that given two points $p, q \in D$, the geodesic segment $[p, q]$ joining them exists and does not contain any point conjugate to $p$. Given a point $r \in [p, q]$ the region $U_r$ contains $U_p \cap U_q$, so that $U_r$ contains $\Sigma$. By Lemma 3.7, it follows that $r \in D$. This shows that $D$ is convex.

Clearly $\Sigma$ is contained in the boundary of $D$. On the other hand, given any other point $q \in \partial_{\infty} AdS_{n+1}$, the vertical line through $q$ meets $\Sigma$ at a point $q'$. By Remark 2.7, there is no geodesic arc in $AdS_{n+1}$ joining $q$ to $q'$. Since $D$ is convex, $q'$ cannot lie on $D$. In particular, the asymptotic boundary of $D$ coincides with $\Sigma$.

To prove that the boundary of $D$ has two components, we notice that every timelike geodesic, say $c$, through a point $p \in M$ must intersect $\partial D$ in two points which are contained in the future and in the past of $M$ respectively.
Indeed, since $D$ is contained in some compact region of $\overline{AdS_{n+1}}$, it turns out that $c \cap D$ is precompact without asymptotic points. By the convexity of $D$, we have that $c \cap D$ is a compact segment and clearly there is an end-point in the future of $M$ and another end-point in the past of $M$.

Let us define $\partial_+ D = \partial D \cap I^+(M)$. The previous argument proves that no timelike geodesic can join points of $\partial_+ D$. Since $D$ is convex, points of $\partial_+ D$ are joined by lightlike or spacelike geodesic arcs. In particular $\partial_+ D$ is an acusal set. By general results (see e.g. [9]) it is a weakly spacelike surface (in particular it is a $C^{0,1}$-embedded surface).

In addition, every inextensible timelike path without endpoints must intersect $\partial_+ D$ at some point. By Proposition 5.2 we deduce that $\partial_+ D$ is a weakly spacelike graph.

To conclude we have to prove that points in $\partial D$ are connected to $\Sigma$ by some lightlike ray. By the characterization of $D$ given by Lemma 3.7 we have that $\partial D$ is the set of points $p$ such that $\Sigma \subset \overline{U_p}$ and $\Sigma \cap \partial_\infty(P_-(p) \cup P_+(p)) \neq \emptyset$. Take a point $y$ in this intersection. By the convexity of $D$, the segment $c$ joining $x$ to $y$ (that is lightlike) is contained in $\overline{D}$. Points on $c$ are joined to $y \in \Sigma$ by a lightlike geodesic, so they cannot be contained in $D$. In particular $c \subset \partial D$.

Remark 3.12. Since timelike arcs in $D$ do not contain conjugate points, their length is less than $\pi$. In particular, the length of any timelike geodesic segment joining a point of $\partial_- D$ and a point of $\partial_+ D$ is less than $\pi$. If there exists a point $q_+ \in \partial_+ D$ and $q_- \in \partial_- D$ such that $\delta(q_-, q_+) = \pi$, then we have $P_-(q_+) = P_+(q_-) = P$ and $U_{q_+} \cap U_{q_-} = \emptyset$. Since $\Sigma$ is contained in $U_{q_+} \cap U_{q_-}$, we conclude that $\Sigma = \partial_\infty P$. In this case $D = I^-(q_+) \cap I^+(q_-)$.

Remark 3.13. The closure of $D$ in $\overline{AdS_{n+1}}$ is compact.

Lemma 3.14. For every $p \in D$ the intersection $I^+(p) \cap \overline{D}$ is compact in $AdS_{n+1}$.

Proof. Since the closure of $D$ in $\overline{AdS_{n+1}}$ is compact, it is sufficient to show that no point in $\partial_\infty AdS_{n+1}$ is an accumulation point for $\overline{D} \cap I^+(p)$. However the set of boundary accumulation points of $I^+(p)$ is disjoint from $U_p$, whereas the set of boundary accumulation points for $D$ is $\Sigma$, that is contained in $U_p$.

Lemma 3.15. There is a point $p \in D$ such that $D \subset U_p$.

Proof. We consider first the case there are points $q_+ \in \partial_+ D$ and $q_- \in \partial_- D$ such that $\delta(q_-, q_+) = \pi$. By Remark 3.12 we deduce that $D = I^-(q_+) \cap I^+(q_-)$ and any point on the plane $P_-(q_+) = P_+(q_-)$ satisfies the statement.

Now we consider the case where $\delta(q, q') < \pi$ for $q \in \partial_- D$ and $q' \in \partial_+ D$. We define two functions on $D$

$$\tau_+(p) = \sup_{q \in D \cap I^+(p)} \delta(p, q), \quad \tau_-(p) = \sup_{q \in D \cap I^-(p)} \delta(q, p)$$

that are Lipschitz-continuous (see [10]). By Lemma 3.14 for $p \in D$ there is $q_+ \in \overline{D}$ such that $\tau_+(p) = \delta(p, q_+(p))$ and analogously there is a point $q_-(p)$ such that $\tau_-(p) = \delta(q_-(p), p)$. Clearly $q_+(p) \in \partial_+ D$ and $q_-(p) \in \partial_- D$.

Notice that by the reverse of triangle inequality we have $\tau_+(p) + \tau_-(p) \leq \delta(q_-(p), q_+(p)) < \pi$. In particular the open sets $\Omega_- = \{\tau_- < \pi/2\}$ and $\Omega_+ = \{\tau_+ < \pi/2\}$ cover $D$. Since they are not empty, it follows that there exists a point $p$ such that $\tau_-(p) < \pi/2$ and $\tau_+(p) < \pi/2$, so $D \subset U_p$.

3.4. From space-like graphs in $AdS_3$ to diffeomorphisms of $\mathbb{H}^2$. There is a relation between some space-like surfaces in $AdS_3$ (satisfying some specific properties) and diffeomorphisms from $\mathbb{H}^2$ to $\mathbb{H}^2$. More specifically, there is a one-to-one relation between maximal graphs in $AdS_3$ with negative sectional curvature and minimal Lagrangian diffeomorphisms from the hyperbolic disk to itself. The quasi-conformal minimal Lagrangian diffeomorphisms correspond precisely to the maximal graphs with uniformly negative sectional curvature.

This relation, which is well-known (see [2]), is at the heart of the proof of Theorem 1.4, so we outline its construction and its main properties here, referring to [23] [4] [13] [6] for more details.

Let $S \subset AdS_3$ be a space-like graph. Let $I$ be its induced metric, $B$ its shape (or Weingarten) operator, and let $E$ be the identity map from $TS$ to $TS$ at each point. Denote by $J$ the complex structure of $I$ on $S$. We can then define two metrics $\mu_1, \mu_r$ as:

$$\mu_1 = I((E + JB), (E + JB)),$$ 

$$\mu_r = I((E - JB), (E - JB)) .$$

It is then not difficult to show that both $\mu_1$ and $\mu_r$ are hyperbolic metrics (see [23] [6]) – the reason for this being that $E \pm JB$ satisfies the Codazzi equation, $d^\gamma(E \pm JB) = 0$ on $S$, and that $\det(E \pm JB) = 1 + \det(B)$ is equal to minus the sectional curvature of the induced metric $I$ on $S$, which by the Gauss equation in $AdS_3$ is equal to $-1 - \det(B)$. 


However $\mu_l$ and $\mu_r$ are not necessarily smooth metrics, they might have singularities when $E \pm JB$ is singular, that is – by the determinant computation just mentioned – when $1 + \det(B) = 0$. This means that $\mu_l$ and $\mu_r$ are smooth hyperbolic metrics whenever the induced metric on $S$ has negative sectional curvature.

There is a nice geometric interpretation of metrics $\mu_l$ and $\mu_r$ that is based on a specific feature of $AdS_3$.

Every leaf of the left (right) foliation of $\partial_\infty AdS_3$ meets the boundary of any spacelike planes exactly at one point. Consider a fixed totally geodesic plane $P_0$. Given any other plane $P$ there are two natural identifications $\Phi_{P,l}, \Phi_{P,r} : \partial_\infty P \to \partial_\infty P_0$ obtained by following each of the families of lines $L_l, L_r$.

By means of the projective model, it can be easily seen that maps $\Phi_{P,l}$ and $\Phi_{P,r}$ extend uniquely to isometries of $AdS_3$ – still denoted by $\Phi_{P,l}$, $\Phi_{P,r}$ – sending $P$ to $P_0$ (see 26 [12] for details).

It is also not difficult to check that replacing $P_0$ by another geodesic plane does not change $\Phi_{P,l}$ and $\Phi_{P,r}$ up to left composition by some isometry of $AdS_3$ preserving respectively $L_l$ and $L_r$.

Now given any spacelike surface $S$ we can define two maps $\Phi_l, \Phi_r : S \to P_0$ as

$$\Phi_l(x) = \Phi_{P(x),l}(x) \quad \Phi_r(x) = \Phi_{P(x),r}(x),$$

where $P(x)$ is the geodesic plane tangent to $S$ at $x$. Still in this case, replacing $P_0$ does not change $\Phi_l$ and $\Phi_r$, up to left composition with some isometry of $AdS_3$ that preserves respectively $L_l$ and $L_r$.

The following is a basic remark, see e.g. [23] for a proof – it can actually be checked by a direct computation, by choosing $P_0$ as the tangent plane at the point $x$.

**Lemma 3.16.** The pull-backs by $\Phi_l$ (resp. $\Phi_r$) of the hyperbolic metric on $P_0$ is precisely the metric $\mu_l$ (resp. $\mu_r$).

A consequence is that $\Phi_l$ and $\Phi_r$ are non-singular when $\mu_l, \mu_r$ are non-degenerate metrics, and we have seen that this is the case when $\det(B) \neq -1$. We are therefore lead to consider surfaces with negative sectional curvature (the Gauss formula indicates that the sectional curvature of $S$ is $K = -1 - \det(B)$).

Lemma 3.16 which is a local statement, can be improved, under the condition that $S$ is a space-like maximal graph with negative curvature. Here we call $\pi_l$ (resp. $\pi_r$) the map from $\partial_\infty AdS_3$ to $P_0$ sending a point $x \in \partial_\infty AdS_3$ to the intersection with $P_0$ of the line of $L_l$ (resp. $L_r$) containing $x$.

**Proposition 3.17.** Suppose that $S$ is a maximal space-like graph with sectional curvature bounded from above by some negative constant. Then $\Phi_l$ (resp. $\Phi_r$) is a global diffeomorphism from $S$ to $P_0$. $\Phi_l$ (resp. $\Phi_r$) extends continuously to the closure of $S$ in $AdS_3$, and its boundary value is the restriction of $\pi_l$ (resp. $\pi_r$) to $\partial_\infty S$.

The difficult part to prove is the extension result. We need the following technical lemma that gives a condition for the extension. Unfortunately this lemma does not apply directly to $S$, but to the surface $S^+$ of points whose distance from $S$ is $\pi/4$. We then factorize the map $\Phi_l$ as the composition of the corresponding map $\Phi_l^* : S^+ \to P_0$ and a diffeomorphism $\sigma : S \to S_+$ that is given by the normal evolution and that is the identity on the boundary.

**Lemma 3.18.** Let $S$ be a spacelike surface in $AdS_3$ with negative curvature whose boundary curve $\Gamma$ does not contain singular points (that is, $\partial_\infty S$ does not contain any lightlike segment). Consider the maps $\Phi_l, \Phi_r : S \to P_0$ described above. Suppose that there is no sequence of points $x_n$ on $S$ such that the totally geodesic planes $P_n$ tangent to $S$ at $x_n$ converge to a lightlike plane $P$ whose past end-point and future end-point are not on $\Gamma$.

Then for any sequence of points $x_n \in S$ converging to $x \in \partial_\infty S$ we have that $\Phi_l(x_n) \to \pi_l(x)$ (resp. $\Phi_r(x_n) \to \pi_r(x)$) in $P_0$.

**Proof.** We prove that for any sequence $x_n \to x \in \partial_\infty S$ there is a subsequence such that $\Phi_l(x_{n_k})$ converges to $\pi_l(x)$.

Indeed, up to passing to a subsequence we can suppose that the totally geodesic plane $P_n$ tangent to $S$ at $x_n$ converges to a plane $P_\infty$. Since $x$ is the limit of points on $P_n$, it belongs to $\partial_\infty P_\infty$.

We distinguish two cases

1. $P_\infty$ is spacelike;
2. $P_\infty$ is lightlike.

First we deal with the first case. We have that $\Phi_l(x_n) = \Phi_{P_n,l}(x_n)$. Since $P_n \to P_\infty$ it can be checked that $\Phi_{P_n,l} \to \Phi_{P,l}$ uniformly on $AdS_3$ (see [12]). So we have

$$\Phi_l(x_n) \to \Phi_{P_\infty,l}(x) = \pi_l(x).$$

Consider now the case where $P_\infty$ is lightlike. By the assumption either the past or the future end-point of $P_\infty$ is contained in $\Gamma = \partial_\infty S$. Since points on $\Gamma$ are not joined by lightlike segments, the intersection between
Along with $P_\infty$ is only this point. Since $x \in \Gamma \cap P_\infty$, we conclude that $x$ is either the past endpoint or the future end-point of $P$. Up to reversing the time-orientation we can suppose that $x$ is the past end-point of $P_\infty$.

Up to some isometry of $AdS_3$ preserving the leaves of $L_l$ we can suppose that $x \in P_0$ so it is sufficient to prove that $\Phi_l(x_n) \to x$.

Consider any geodesic $l$ on $P_0$ and let $U$ be the half-plane bounded by $l$ containing the point $x$. We will show that for $n$ large enough $\Phi_l(x_n) \in U$.

The four leaves of $L_l$ and $L_r$ passing through the end-points of $l$ bound a rhombus $R$ in $\partial_\infty AdS_3$ containing $x$ in its interior (see Figure 1). The end-points of $l$ are two opposite vertices of $R$ and there are two other opposite vertices $z_-$ and $z_+$ such that $z_-$ is the past end-point of both edges adjacent to it and $z_+$ is the future end-point of both edges adjacent to $z_+$. Since $x$ is the past endpoint of $P_\infty$, this plane intersects the frontier of $R$ in two points, one for each edge with vertex $z_+$. In particular also $P_n \cap R$ is for $n$ large enough an arc $c_n$ joining two points on the edges adjacent to $z_+$.

Let $L_-$ be the lightlike plane whose past end-point is $z_-$ and $L_+$ be the lightlike plane whose future end-point is $z_+$. Notice that $V = I^-(L_+) \cap I^+(L_-)$ is a neighbourhood of $x$ in $AdS_3$ and the asymptotic boundary of $V$ is exactly $R$. In particular, for $n$ large enough, $x_n \in V$.

The boundary of $L_+$ is the union of the two past-directed lightlike rays starting from $z_+$ and $L_-$ is the union of two future-directed lightlike rays starting from $z_-$. It turns out that $H_n = P_n \cap I^-(L_+)$ is the half-plane on $P_n$ that is the convex hull of $c_n$. Since $c_n$ is contained in the future of $\partial_\infty L_-$ we have that $H_n \subset I^+(L_-)$. And we conclude that

$$P_n \cap V = H_n.$$  

Since for $n$ large enough $x_n \in P_n \cap V$, we have that

$$\Phi_l(x_n) = \Phi_{P_n,l}(x_n) \in \Phi_{P_n,l}(H_n).$$

Now $\Phi_{P_n,l}(H_n)$ is the half-plane of $P_0$ whose asymptotic boundary is $\pi_l(c_n)$.

Notice that $\pi_l(c_n)$ is contained in $\partial_\infty U$ so we have $\Phi_l(x_n) \in \Phi_{P_n,l}(H_n) \subset U$. 

\[ \square \]

Remark 3.19. If $S$ is a future-convex graph and its boundary does not contain singular points then the condition required in Lemma 3.18 is satisfied. Indeed totally geodesic planes tangent to $S$ are support planes so if we take a sequence of such planes $P_n$ that converges to some lightlike plane $P_\infty$, we have that $P_\infty$ cannot intersects $S$ transversally. In particular $S$ is contained in the past of $P_\infty$. This implies that either the boundary of $S$ is disjoint from the boundary of $P_\infty$ or that the past end-point of $P_\infty$ is contained in the boundary of $S$.

Now if the tangency points $x_n$ of $P_n$ with $S$ converge to some asymptotic point $x$, clearly $x \in S \cap P_\infty$. Thus, in this case we have that the past end-point of $P_\infty$ is contained in the boundary of $S$. Since the boundary of $S$ does not contain lightlike segments, the point $x$ must coincide with the past end-point of $P_\infty$. 

\[ \square \]
Lemma 3.20. Let $S$ be a maximal spacelike graph with sectional curvature bounded from above by some negative constant. The asymptotic boundary of $S$ does not contain any lightlike segment.

The proof is based on some simple preliminary claims.

Claim 3.21. Let $S \subset \text{AdS}_3$ be a space-like graph with principal curvatures in $(-1, 1)$. Then the equidistant surfaces $S_r$ at (oriented) time-like distance $r$ from $S$, for all $r \in (-\pi/4, \pi/4)$, are smooth, space-like graphs. If the principal curvatures of $S$ are in $(-1 + \epsilon, 1 - \epsilon)$, then, for $r$ close enough to $\pi/4$, $S_r$ is past-convex, and $S_{-r}$ is future-convex.

Proof. If $(S_r)_{r \in I}$ is a non-singular foliation of a neighborhood of $S$ by space-like surfaces at constant distance $r$ from $S$, then the shape operator $B_r$ of $S_r$ satisfies a Riccati type equation relative to $r$:

$$
\frac{dB_r}{dr} = B_r^2 - I,
$$

where $I$ is the identity. It follows that the principal curvatures of $S$ evolve as $\tan(r - r_0)$, where $r_0$ is chosen so that $\tan(r_0)$ is the principal curvature of $S$ at the corresponding point and in the corresponding direction.

Suppose now that $S$ has principal curvatures $k \in (-1 + \epsilon, 1 - \epsilon)$ at each point, for some $\epsilon > 0$. This implies that, at each point and in each principal direction, $r_0 \in (-\pi/4 + \alpha, \pi/4 - \alpha)$, where $\alpha > 0$ is another constant. As a consequence, the equidistant foliation $(S_r)$ is well-defined for $r \in [-\pi/4, \pi/4]$, and moreover the surfaces $S_{\pi/4 - \alpha}$ and $S_{-\pi/4 + \alpha}$ are smooth and respectively strictly concave and strictly convex, so that the domain

$$
\Omega = \bigcup_{r \in [-\pi/4 + \alpha, \pi/4 - \alpha]} S_r
$$

is convex with smooth boundary, with principal curvatures bounded from below by a strictly positive constant.

Corollary 3.22. Let $S$ be a space-like maximal surface, with sectional curvature bounded from above by a negative constant. Then $w(S) < \pi/2$.

Proof. This follows from the claim because the convex hull of $S$ is contained in $\Omega$, and $w(\Omega) \leq \pi/2 - 2\alpha < \pi/2$.

Claim 3.23. Suppose that there is a light-like segment in $\partial_\infty S$. Then $w(S) = \pi/2$.

Proof. The boundary at infinity of $S$ is the graph of a map $u : S^1 \to S^1$. If $\partial_\infty S$ contains a light-like segment then $u$ is not continuous, and its graph has a “jump”, as in the left-hand side of Figure 2. Composing $u$ on the left with a sequence of projective transformations, we can make its graph as close as wanted (in the Hausdorff topology) from the standard 2-step graph shown on the right-hand side of Figure 2 (This is achieved by composing $u$ on the right with a sequence of powers of a projective transformation having as attracting fixed point the point where the “jump” occurs.) We call $\Gamma_0$ this 2-step graph, considered as a subset of $\partial \pi(\text{AdS}_3)$ (here $\pi$ is the map in the projective model of $\text{AdS}_3$).

Now $\Gamma_0$, as a subset of $\partial \pi(\text{AdS}_3)$, is composed of four light-like segments. It has four vertices, and it is not difficult to check that the lines $\Delta$ and $\Delta^*$ connecting the two pairs of opposite points are two dual space-like lines in $\pi(\text{AdS}_3)$. In particular, if $\text{CH}(\Gamma_0)$ denotes the convex hull of $\Gamma_0$, then $w(\text{CH}(\Gamma_0)) = \pi/2$.

Since $\partial_\infty S$ can be made arbitrarily close to $\Gamma_0$ by applying AdS isometries (corresponding to composing $u$ on the left and on the right with projective transformations of $S^1$), it follows that $w(S) = \pi/2$.

Figure 2. Deforming a graph to the standard 2-step graph.
Proof of Lemma 3.20. The statement follows directly from Corollary 3.22 and Claim 3.28.

Let us come back to Proposition 3.17.

Proof of Proposition 3.17. We consider again the surface $S_\pm$ of points in the future of $S$ at distance $\pi/4$ from $S$. We have seen that $S_\pm$ is smooth and past-convex. Moreover a diffeomorphism $\sigma : S \to S_\pm$ is uniquely determined so that the Lorentzian distance between $x$ and $\sigma(x)$ is exactly $\pi/4$.

Since the distance between points on $S_\pm$ and points on $S$ is bounded, they share the same boundary. Moreover, since the boundary of $S$ does not contain lightlike segments, it can easily seen that the map $\sigma$ extends to the identity at the boundary.

We claim that the map $\Phi_t$ can be factorized as the composition of $\sigma$ and $\Phi_t^+$, where $\Phi_t^+ : S_\pm \to P_0$ is the map constructed in the same way as $\Phi_t$. The claim and Remark 3.19 imply that $\Phi_t$ extends to the boundary.

Let us prove the claim. Given any point $x \in S$, we have to check that $\Phi_t(x) = \Phi_t^+(\sigma(x))$. Up to isometry we can suppose that:

- $P_0$ is the plane tangent to $S$ at $x$,
- $x = (x^0, 0)$ and $P_0$ is the horizontal plane.

With this assumption clearly $\Phi_t(x) = x$.

Since the segment joining $x$ to $\sigma(x)$ is orthogonal to both $S$ and $S_\pm$, it follows that $\sigma(x) = (x^0, \pi/4)$ and the plane $P_\pm$ tangent to $S_\pm$ at $\sigma(x)$ is the horizontal plane.

In this case the map $\Phi_{P_\pm,l}$ can be explicitly computed. In particular it is given by $\Phi_{P_\pm,l}(y,t) = (R(y), t - \pi/4)$ where $R \in \text{Isom}(\mathbb{H}^2)$ is a rotation of angle $\pi/4$ around $x^0$. It easily follows that $\Phi_t^+(\sigma(x)) = \Phi_{P_\pm,l}(\sigma(x)) = x$, and this proves the claim.

Notice that the map $\Phi_t$ and $\Phi_r$ turn to be proper maps. On the other hand, under the hypothesis that $S$ has negative sectional curvature, $\Phi_t$ and $\Phi_r$ are local diffeomorphisms from $S$ to $P_0$, so that, by the Dependence of Domain Theorem, they are global diffeomorphism from $S$ to $P_0$.

Definition 3.24. Suppose that $S$ has negative sectional curvature. We call $\Phi_S : \Phi_t^{-1} \circ \Phi_r : \mathbb{H}^2 \to \mathbb{H}^2$. $\Phi_S$ is a global diffeomorphism, well-defined up to composition by a hyperbolic isometry.

By construction the differential of $\phi_S$ is given at each point by $(E + JB)^{-1}(E - JB)$. It follows that, as long as the principal curvatures of $S$ are in $[-1 + \epsilon, 1 - \epsilon]$ for some $\epsilon > 0$, the diffeomorphism $\phi_S$ is quasi-conformal (and conversely).

Lemma 3.25. The map $\Phi_S$ extends to a homeomorphism from $\mathbb{H}^2$ to $\mathbb{H}^2$, and the graph of $\partial \Phi_S : S^1 \to S^1$ in (the image by $\pi$) of $\partial S_3$ is the boundary at infinity of $S$ in $\partial_\infty \mathbb{H}^2$.

Proof. The extension of $\Phi_S$ to the boundary is a direct consequence of its definition and of the extension to the boundary of $\Phi_t$ and $\Phi_r$. It is then clear that the graph of $\partial \Phi_S$ is equal to $\partial \infty S$, since the restrictions of $\pi_t$ and $\pi_r$ to $\partial \infty S$ are equal to the boundary values of $\Phi_t$ and $\Phi_r$.

We have now proved the first two points in Proposition 1.6. To prove the third point it is necessary to construct, given a quasi-conformal minimal Lagrangian diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$, a maximal space-like $S$ such that $\Phi = \Phi_S$. One way to do this is through the identification of $\mathbb{H}^2 \times \mathbb{H}^2$ with the space of time-like geodesics in $AdS_3$ (see [6]). We rather use here local arguments (as in [23]).

Let $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ be a minimal Lagrangian diffeomorphism. Call $\rho_l$ and $\rho_r$ the hyperbolic metrics on the two copies of $\mathbb{H}^2$ (this underlines the relationship with the construction in the previous paragraphs). The fact that $\Phi$ is minimal Lagrangian is equivalent (see [23]) to the fact that

$$\Phi^* \rho_r = \rho_l(b, b) ,$$

where $b$ is self-adjoint (for $\rho_l$), of determinant 1, and satisfies the equation

$$d^{\nabla^l} b = 0 ,$$

where $\nabla^l$ is the Levi-Civita connection of $\rho_l$ and $d^{\nabla^l} b$ is defined (see [11]) as

$$(d^{\nabla^l} b)(x,y) = \nabla^l_x(by) - \nabla^l_y(bx) - b([x,y]) .$$

We can then define a metric $I$ on $S$ by

$$4I = \rho_l((E + b)\cdot, (E + b)\cdot) .$$

(14)
Since \( b \) is non-singular and has positive eigenvalues, \( I \) is a metric on \( \mathbb{H}^2 \). Since \( d^\nabla b = 0 \) we also have \( d^\nabla (E + b) = 0 \), it follows from standard arguments (see e.g. \[22\]) that the Levi-Civita connection of \( I \) is

\[
\nabla_x y = (E + b)^{-1} \nabla_x ((E + b)y) ,
\]

and therefore that the curvature \( K \) of \( I \) is equal to

\[
K = \frac{K_I}{\det((E + b)/2)} = -\frac{4}{\det(E + b)} = -\frac{4}{2 + \text{tr}(b)} .
\]

Let \( J \) be the complex structure of \( I \), we now define \( B : T\mathbb{H}^2 \to T\mathbb{H}^2 \) as follows:

\[
(15) \quad JB = (E + b)^{-1}(E - b) .
\]

Then \( JB \) has some remarkable properties.

1. \( d^\nabla JB = 0 \). This follows from a direct computation, because \( d^\nabla (E - b) = 0 \). Since \( J \) is parallel for \( \nabla \), it follows that \( d^\nabla B = 0 \).
2. \( JB \) is self-adjoint for \( I \), because \( E - b \) is self-adjoint for \( \rho_I \). It follows that \( B \) is traceless.
3. \( JB \) is traceless – this follows from a direct computation in a basis where \( b \) is diagonal, using the fact that \( \det(b) = 1 \). It follows that \( B \) is self-adjoint.
4. \( \det(JB) = \frac{\det(E - b)}{\det(E + b)} = \frac{2 - \text{tr}(b)}{2 + \text{tr}(b)} \). It follows that \( K = -1 - \det(B) \).

In other terms, setting \( \mathbb{I} = I(B, \cdot , \cdot ) \), we see that \( \mathbb{I} \) satisfies the Gauss and Codazzi equation relative to \( I \). It follows that there exists a (unique) isometric embedding of \( (\mathbb{H}^2, I) \) in \( AdS_3 \) with second fundamental form \( \mathbb{II} \) (and shape operator \( B \)).

Equation (15) then shows that \( E + JB = 2(2 + b)^{-1} \), so that \( \mu_I = \rho_I \), and a direct computation shows also that \( \mu_r = \rho_r \). If \( \Phi \) is quasi-conformal then \( b \) is bounded, so that the sectional curvature of \( S \) is uniformly negative. The first part of this section shows that the graph of \( \partial \Phi \) in \( S^1 \times S^1 \simeq \partial_\infty AdS_3 \) is equal to the boundary at infinity of \( S \), and this finishes the proof of Proposition [13].

4. The existence and regularity of maximal graphs

Given a smooth spacelike surface \( M \) in \( AdS_{n+1} \) we consider the future-oriented normal vector field \( \nu \). The gradient function with respect to the field \( T = -\phi \nabla t \) is

\[
v_M = - (\nu, T) .
\]

It measures the angle between the hypersurface \( M \) and the horizontal slice. Notice that \( v_M(x) \geq 1 \) for every \( x \in M \). If \( M \) is the graph of a function \( u \) then

\[
v_M = \frac{1}{\sqrt{1 - \phi^2 |\nabla u|^2}}.
\]

In that case the normal field \( \nu \) is equal to \( \nu = \phi v_M(\nabla u - \nabla t) \).

The shape operator of \( M \) is the linear operator of \( TM \) defined by

\[
B(v) = \nabla_v \nu
\]

whereas the second fundamental form is defined by \( \mathbb{II}(v, w) = (v, B(w)) \). The mean curvature, denoted by \( H \), is the trace of \( B \).

In [7] a general formula for the mean curvature of a spacelike graph is given. If \( M \) is the spacelike graph of a function \( u \) we have

\[
(16) \quad H = \frac{1}{v_M} \left( \text{div}_M(\phi \text{grad}_M u) + \text{div}_M T \right),
\]

where \( \text{div}_M \) is the operator on \( M \) defined

\[
\text{div}_M X = \sum \langle e_i, \nabla_{e_i} X \rangle, \quad X \in \Gamma(TAdS_{n+1})
\]

where \( e_i \) is any orthonormal basis.

A spacelike surface \( M \) is maximal if its mean curvature vanishes.
4.1. **Maximal hypersurfaces and convex subsets.** We concentrate here on convexity properties of maximal hypersurfaces in $AdS_{n+1}$.

**Lemma 4.1.** Let $M$ be a compact maximal graph. Suppose that there exists a spacelike plane $P$ such that $\partial M$ is contained in $I^-(P)$. Then $M$ is contained in $I^-(P)$.

**Proof.** Suppose by contradiction that a point $p_0$ of $M$ lies in the future of $P$. Without loss of generality we can suppose that $P$ is the horizontal plane $\{ t = 0 \}$ and $p_0 = (x^0, a)$ with $a > 0$. Since $M$ is contained in $I^+((p_0)_{-}) \cap I^-((p_0)_{+})$, by our assumption on the boundary we have that $0 < a < \pi$ and $\partial M$ is contained in the region of points with $-\pi < t < 0$.

Consider the function $u : AdS_{n+1} \to \mathbb{R}$ defined at the point $p = (x, t)$ as

$$u(p) = x_{n+1} \sin(t).$$

By our assumption,

$$u(p) < 0 \quad \text{for every } p \in \partial M.$$

We compute now $\Delta u$, where $\Delta$ is the Beltrami-Laplace operator of $M$. Notice that $u$ is the pull-back of the function $u^*$ defined on $AdS_{n+1}^*$ as

$$u^*(y) = \langle y, e \rangle,$$

where $e = (0, \ldots, 0, -1)$. Thus we can suppose that $M$ is immersed in $AdS_{n+1}^*$ and compute $\Delta u^*$. Notice that the gradient of $u^*$ is the orthogonal projection of $e$ on $M$, that is,

$$\nabla u(y) = e + \langle e, y \rangle y + \langle e, \nu^* \rangle \nu^* = e + uy + \langle e, \nu^* \rangle \nu^*,$$

where $\nu^*$ is the normal field of $M$ in $AdS_{n+1}^*$. Since for $v \in T_y M$, $\nabla_v (\nabla u)$ is the tangential part of $\nabla_v (\nabla u)$ (where $\nabla$ is the standard connection in $\mathbb{R}^{2,2}$) we have

$$\nabla_v (\nabla u^*) = u^* v + \langle e, \nu^* \rangle B(v).$$

Taking the trace we get $\Delta u^* = nu^* + \langle e, \nu^* \rangle H = nu^*$, where the last equality holds since $M$ is maximal. Eventually we have

$$\Delta u = nu.$$

In particular if the maximum of the function $u$ is achieved at some interior point of $M$, then it must be negative. Since $u(p_0) > 0$ we get a contradiction.

**Definition 4.2.** A convex slub of $AdS_{n+1}$ is a convex domain in $AdS_{n+1}$ whose boundary is the union of two acausal graphs.

Let $K$ be a convex slub and $M_v$ and $M_u$ be its boundary components with $v < u$. The domain $K$ is

$$\{(x, t) | u(x) \leq t \leq v(x) \}.$$

The component $M_v$ (resp. $M_u$) is called the past (resp. future) boundary of $K$. Notice that the future boundary is past-convex: this means that points of $M_v$ are related by a spacelike geodesic that lies in the past of $M_v$. Analogously $M_u$ is future convex.

Since points of a convex slub $K$ are connectible by geodesics, Remark 2.7 implies that the asymptotic boundary of $K$ can intersect each vertical line in $\partial_{\infty} AdS_{n+1}$ at most one point. So we have

**Corollary 4.3.** If $K$ is a convex slub then its boundary components share the same asymptotic boundary.

**Remark 4.4.** Let $u$ and $v$ be two spacelike functions defined on $\mathbb{H}^n$ such that $M_u$ is past convex, $M_v$ is future convex and $v(x) < u(x)$. Corollary 4.3 implies that in general the domain $\Omega = \{(x, t) | v(x) < t < u(x) \}$ is not convex. On the other hand it is not difficult to see that if the functions $u$ and $v$ coincide on $\partial \mathbb{H}^n$, then $\Omega$ is a convex slub.

**Remark 4.5.** Let $K$ be a convex slub and $D$ be the domain of dependence of its asymptotic boundary. Then $K$ is contained in $D$.

An important property of convex slubs is that a maximal surface whose boundary is contained in a convex slub is completely contained in the slub.

**Proposition 4.6.** Let $\Omega$ be a convex slub. If $M$ is a compact maximal surface such that $\partial M$ is contained in $\Omega$. Then $M$ is contained in $\Omega$.

Proposition 4.6 is a direct consequence of Lemma 4.1 and the following lemma.
Lemma 4.7. Let \( \Omega \) be a convex slab and let \( S_- , S_+ \) denote respectively its past and future boundary. For every \( p \in S_- \) (resp. \( p \in S_+ \)) there is a spacelike geodesic plane \( P_p \) passing through \( p \) such that \( \Omega \subset I^+(P_p) \) (resp. \( \Omega \subset I^-(P_p) \)).

Moreover we have
\[
\Omega = \bigcap_{p \in S_-} I^+(P_p) \cap \bigcap_{p \in S_+} I^-(P_p) .
\]

Proof. Since \( \Omega \) is contained in the domain of dependence of \( D \) of its asymptotic boundary, there is a point \( p \) such that \( \Omega \subset U_p \). Up to isometry we can suppose that \( p = (x^0, 0) \) and consider the projective map
\[
\pi^* : U_p \to \mathbb{R}^{n+1}
\]
constructed in Section 2.3. Since \( \pi^* \) is a projective map, the set \( \pi^*(\Omega) \) is convex in \( \mathbb{R}^{n+1} \).

Given a point \( q \in S_+ \) the point \( q^* = \pi^*(q) \) lies on the boundary of \( \pi^*(\Omega) \), so there is a support plane \( P^* \) passing through it. We can consider the plane in \( U_p \) equal to \( P_q = (\pi^*)^{-1}(P^*) \). This plane passes through \( q \) and does not meet the interior of \( \Omega \). Since any timelike arc passing through \( q \) meet the interior of \( \Omega \), the plane \( P_q \) is not timelike. In particular \( P \) disconnects \( AdS_{n+1} \) in two components that are the future and the past of \( P_q \). Since \( q \in S_+ \) it turns out that \( \Omega \subset I^-(P_q) \). Analogously for \( q \in S_- \) we find a plane \( P_q \) such that \( \Omega \subset I^+(P_q) \).

In particular the inclusion
\[
\Omega \subset \bigcap_{p \in S_-} I^+(P_p) \cap \bigcap_{p \in S_+} I^-(P_p)
\]
is proved. Now take a point \( q \notin \Omega \). Consider a timelike geodesic arc contained in \( AdS_{n+1} \backslash \Omega \) such that \( q \) is an end-point and the other end-point, say \( p \), lies on \( \partial \Omega \). Without loss of generality we can assume \( p \in S_+ \). In that case it turns out that \( q \in I^+(P_p) \), so the reverse inclusion is also proved. \( \square \)

Lemma 4.8. Let \( \Sigma \) be a spacelike graph in \( \partial_\infty AdS_{n+1} \). There is a convex slab \( K(\Sigma) \), called the convex hull of \( \Sigma \), such that :

- The asymptotic boundary of \( K(\Sigma) \) is \( \Sigma \).
- Every convex slab with boundary \( \Sigma \) contains \( K(\Sigma) \).

Proof. Let \( D \) be the domain of dependence of \( \Sigma \) and take \( p \in D \).

Consider the image \( \Sigma^* \) of \( \Sigma \) through the projective map
\[
\pi^* : U_p \to \mathbb{R}^{n+1} .
\]
Clearly \( \Sigma^* \) is contained in the image, say \( D^* \), of \( D \). In particular the convex hull in \( \mathbb{R}^{n+1} \) of \( \Sigma^* \), say \( K \), is contained in \( D^* \).

We denote by \( K(\Sigma) \) the convex set \( (\pi^*)^{-1}(K) \). It is clear that \( \Sigma \) is contained in the asymptotic boundary of \( K(\Sigma) \). By Corollary 4.3 \( \Sigma \) coincides with the asymptotic boundary of \( K(M) \).

Clearly no support plane of \( K(\Sigma) \) can be timelike. Indeed timelike planes disconnect the asymptotic boundary of \( K(\Sigma) \). This implies that the boundary of \( K(\Sigma) \) in \( AdS_{n+1} \) is locally achronal. Moreover it has two components, and each of them disconnects \( AdS_{n+1} \) in two components. It follows easily that \( K(\Sigma) \) is a convex slab. \( \square \)

Remark 4.9. The same proof shows that: for a spacelike graph \( M \) in \( AdS_{n+1} \), there is convex slab, say \( K(M) \), such that

- \( K(M) \) contains \( M \).
- If \( K \) is a convex slab containing \( M \), then \( K(M) \subset K \).

The slab \( K(M) \) is called the convex hull of \( M \).

Clearly if \( D \) is the domain of dependence of \( \Sigma \) we have \( K(\Sigma) \subset \overline{D} \). An important technical point for what follows is the following statement. Recall that singular points of \( \Sigma \) are points contained in some light-like segment contained in \( \Sigma \).

Lemma 4.10. If \( \Sigma \) is spacelike graph in \( \partial_\infty AdS_{n+1} \) without singular points, then the boundary components of \( K = K(\Sigma) \) do not contain singular points. Moreover, in this case, no point of \( K \) is contained in \( \partial D \).

Proof. Suppose that a lightlike segment \( c \) is contained in \( \partial_+ K \). Take a support plane \( P \) of \( \partial_+ K \) at some point of \( c \). Clearly \( P \) is lightlike and contains \( c \). For every \( p \in c \) notice that
\[
I^+(P) \cap \partial_+ K = \emptyset , \quad \Sigma \subset \overline{U_p} .
\]
Let \( p^- \) be the past end-point of the lightlike geodesic through \( p \) contained in \( P \). Let \( l \) be the vertical line through \( p^- \). Since \( \Sigma \) is a graph, it must intersect \( l \) at some point. Notice that one component of \( l \setminus \{ p \} \) is contained in \( I^+(P) \) whereas the other component is contained in \( I^-(p) \). This remark and (18) show that \( \Sigma \) must intersect \( l \) at \( p^- \), that is, \( p^- \in \Sigma \).

By a classical theorem on convex sets in Euclidean space (still using the projective map \( \pi^* \) as in Lemma 4.7), \( P \cap K(\Sigma) \) is the convex hull of \( P \cap \Sigma \). Thus there is another point \( q \in P \cap \Sigma \). By Lemma 2.8, we conclude that \( p^- \) and \( q \) are connected by a lightlike segment and this contradicts the assumption that \( \Sigma \) does not contain any singular point.

Eventually, segments joining points of \( \partial^+ K(\Sigma) \) to \( \Sigma \) are spacelike. By Proposition 3.9 we conclude that no point of \( \partial^+ K(\Sigma) \) is contained in \( D \). □

4.2. Existence of entire maximal graph with given boundary condition. Let \( \Sigma \) be a spacelike graph in \( \partial_\infty AdS_{n+1} \) without singular points. In this section we prove the main theorem on the existence of a maximal graph with given asymptotic boundary.

**Theorem 4.11.** There is a maximal graph \( M \) in \( AdS_{n+1} \) whose boundary at infinity coincides with \( \Sigma \).

Let us consider the following notation that we will use through this section:
- \( D \) is the domain of dependence of \( \Sigma \);
- \( K \) is the convex hull of \( \Sigma \);
- \( S \) is the future boundary of \( K \);
- \( B_r \) is the ball in \( \mathbb{H}^n \) centered at \( x^0 \) of radius \( r \);
- \( S_r \) is the intersection of \( S \) with the cylinder \( B_r \times \mathbb{R} \).

In [8] (Theorem 4.1) it is shown that there is a maximal surface \( M_r \) such that \( \partial M_r = \partial S_r \). Moreover \( M_r \) is homotopic to \( S_r \) (rel. \( \partial S_r \)) in the sense that there exists a family of spacelike embeddings

\[
h_s : S_r \to AdS_{n+1}
\]

such that

1. \( h_0 = Id, h_1(S_r) = M_r \);
2. \( h_s(x) = x \) for \( x \in \partial S_r \) and \( s \in [0,1] \);
3. the map \( s \mapsto h_s(x) \) is a vertical path for every \( x \in S_r \).

It easily follows that \( M_r \) is the graph of some function defined on \( B_r \). Putting the previous results together we obtain the following lemma.

**Lemma 4.12.** For every \( r > 0 \), there is a maximal surface \( M_r \) such that \( \partial M_r = \partial S_r \). Moreover, the surface \( M_r \) is a graph of a function \( u_r \) defined on \( B_r \) and is contained in \( K \).

4. The basic idea of the proof of Theorem 4.11 is to construct a sequence \( r_k \to +\infty \) such that \( u_{r_k} \) converges \( C^2 \) on compact subset of \( \mathbb{H}^n \). The proof is based on an a-priori gradient estimate, that is a particular case of an estimate proved by Bartnik [8]. Given a point \( p \in AdS_{n+1} \) and \( \epsilon > 0 \) we denote by \( I^+_\epsilon(p) \) the set of points in the future of \( p \) whose distance from \( p \) is at least \( \epsilon \).
Lemma 4.13. Let \( p \in AdS_{n+1} \) and \( \epsilon > 0 \), and let \( H \subset I^-(p_+) \) be a compact domain (where \( p_+ \) is defined in Section 2.3). There is a constant \( C = C(p,\epsilon,H) \) such that, for every maximal surface \( M \) that verifies the following conditions:
- \( \partial M \cap I^+(p) = \emptyset \),
- \( M \cap I^+(p) \) is contained in \( H \),
we have that
\[
\sup_{M \cap I^+(p)} v_M < C
\]
where \( v_M \) is the gradient function of \( M \).

Proof. Let us consider the time-function
\[
\tau(x) = \delta(x,p) - (\epsilon/2)
\]
where \( \delta(x,p) \) is the Lorentzian distance between \( x \) and \( p \). This function is smooth on the domain \( V = H \cap I^+(p) \).

Notice that by the assumption on \( M \), the region \( M \cap V \) contains the region of \( M \) where \( \tau \geq 0 \) and \( M \cap I^+(p) \) is contained in \( V \).

We can apply Theorem 3.1 of \cite{RS} and conclude that
\[
\sup_{M \cap I^+(p)} v_M < C
\]
where \( C \) depends on the \( C^2 \)-norms of \( t \) and \( \tau \) and on the \( C^0 \) norm of \( \text{Ric} \), taken on the domain \( V_{\tau \geq 0} \) with respect to a reference Riemannian metric. \( \square \)

We can prove now Theorem 4.11.

Proof of Theorem 4.11. For every point \( p \in D \cap I^-(\partial_-K) \) we choose \( \epsilon = \epsilon(p) \) such that the family \( \{ I^+(p) \cap K \}_{p \in D \cap I^-(\partial_-K)} \) is an open covering of \( K \).

Given a number \( R \), the intersection \( (B_R \times \mathbb{R}) \cap K \) is compact, so there is a finite numbers of points \( p_1, \ldots, p_{k_0} \in D \cap I^-(\partial_-K) \) such that for all \( k \in \{1, \ldots, k_0\} \), there exists \( \epsilon_k = \epsilon(p_k) \) such that
\[
(B_R \times \mathbb{R}) \cap K \subset \bigcup_{k=1}^{k_0} I^+_k(p_k) .
\]

For all \( k \in \{1, \ldots, k_0\} \), \( p_k \in D \), so that the intersection \( I^+_k(p_k) \cap D \) is compact. Moreover, \( D \subset I^-((p_k)_+) \).

It follows that the set \( H_k = I^+_k(p_k) \cap K \) is compact and contained in \( I^-((p_k)_+) \).

By Lemma 4.13 there is a constant \( C_k \), such that
\[
\sup_{M \cap I^+_k(p_k)} v_M < C_k
\]
for every maximal surface \( M \) that satisfies the following requirements:
- \( \partial M \cap I^+(p_k) = \emptyset \);
- \( M \cap I^+(p_k) \) is contained in \( H_k \).

By the compactness of \( I^+(p_k) \cap D \), there is \( r_0 > 0 \) such that
\[
I^+(p_k) \subset B_{r_0} \times \mathbb{R}
\]
for \( k = 1, \ldots, k_0 \).

Let \( \{ M_r \} \) be the family of maximal surfaces constructed in Lemma 4.12. Then \( M_r \subset K \). Moreover there exists \( r_0 > 0 \) such that, for \( r > r_0 \), \( \partial M_r \cap I^+(p_k) = \emptyset \) for \( k = 1, \ldots, k_0 \).

It follows that \( \sup_{M_r \cap I^+_k(p_k)} v_{M_r} \leq C_k \) for \( k = 1, \ldots, k_0 \). Since \( M_r \cap (B_R \times \mathbb{R}) \subset \bigcup_k I^+_k(p_k) \) we conclude that
\[
\sup_{M_r \cap (B_R \times \mathbb{R})} v_{M_r} \leq \max\{C_1, \ldots, C_{k_0}\}
\]
for every \( r > r_0 \).

Eventually we deduce that for every \( R \) there is a constant \( C(R) \) such that the gradient function of \( v_{M_r} \) is bounded by \( C(R) \) for \( r \) uniformly bounded. Take now any divergent sequence \( r_i \). Let \( u_i \) be the function defined on \( B_{r_i} \) such that \( M_{r_i} = M_{u_i} \). By comparing Equation (16) with estimate (19), we see that the restriction of \( u_i \) on \( B_R \) is solution of a uniformly elliptic quasi-linear operator on \( B_R \), with bounded coefficients.

Since \( |u_i| \) and \( |\nabla u_i| \) are uniformly bounded on \( B_R \), by elliptic regularity theory (see e.g. [20]) the norms of \( u_i \) in \( C^{2,\alpha}(B_{R-1}) \) are uniformly bounded. It follows that the family \( u_i \) is precompact in \( C^2(B_{R-1}) \).
4.3. Regularity of maximal hypersurfaces. We will now show that if the distance between $K$ and the past boundary of $D$ is strictly positive, then any maximal surface contained in $K$ has bounded second fundamental form.

**Theorem 4.14.** Suppose that there exists $\epsilon > 0$ such that, for every $y \in \partial_- K$, there exists a point $x \in \partial_- D$ such that $\delta(x,y) \geq \epsilon$. Then there exists a constant $C > 0$, depending on $\epsilon$, such that the second fundamental form of any maximal graph contained in $K$ is bounded by $C$.

To prove this theorem we will need the following relation between the boundaries of $D$ and $K$. The first part of the lemma will be used in the proof of Theorem 4.14, while the second part will be necessary below.

**Lemma 4.15.** Let $\Sigma \subseteq \partial_\infty T^*\{-\infty\}$ be space-like graph, let $K = K(\Sigma)$ be its convex hull, and let $D = D(\Sigma)$ be its domain of dependence. Then:

1. For all $q \in K$ and $p \in \partial_- D \cap I^-(q)$ we have that $\delta(p,q) \leq \pi/2$.
2. For all $q \in \partial_+ K$ there exists $p \in \partial_- D \cap I^-(q)$ such that $\delta(p,q) = \pi/2$.

The proof of the first point in dimension $2 + 1$ can be found in [10]. That argument actually applies in every dimension. For the sake of completeness we sketch the argument here.

**Proof.** Since $p \in \partial_- D$, $\Sigma$ is contained in $\overline{U}_p$ and $\Sigma \cap (P_+(p) \cup P_-(p)) \neq \emptyset$.

Notice that the plane $P_+(p)$ does not disconnect $\Sigma$, so, it is a support plane for $K$. In particular $K \subset \overline{T^-(P_+(p))}$. This implies that the distance of every point of $K \cap I^+(p)$ from $p$ is bounded by $\pi/2$, and proves the first point. Moreover, since $P_+(p)$ is a support plane of $K$, its intersection with $\partial_+ K$ is non-empty. But for any point $q \in P_+(p)$ we have $\delta(p,q) = \pi/2$, and this proves the second point.

As a consequence we find a bound on the width of the boundary at infinity of a space-like graph in $AdS_{n+1}$. This estimate is improved for $n = 2$ when the boundary at infinity is the graph of a quasi-symmetric homeomorphism, see Theorem [11,12]

**Lemma 4.16.** Let $M \subset AdS_{n+1}$ be a space-like graph. Then $w(\partial_\infty M) \leq \pi/2$.

We can now prove Theorem 4.14.

**Proof of Theorem 4.14.** We consider $q_0 = (x^0,0)$ and consider the horizontal plane $P_0$ passing through $(x^0,\pi/2 - \epsilon/2)$, and define $H_0 = I^+(q_0) \cap \overline{I^-}(P_0)$.

From Lemma 4.13 we find a constant $C$ (depending on $\epsilon$) such that

$$\sup_{N \cap \overline{I^+}_{\partial_+}(q_0)} |v_N| < C$$

for every maximal surface $N$ such that

1. $\partial N \cap I^+(q_0) = \emptyset$,
2. $N \cap I^+(q_0) \subset H_0$.

Moreover, by applying the elliptic regularity theory as in the proof of Theorem 4.11 we see that there is another constant, still denoted by $C$, such that

$$\sup_{N \cap \overline{I^+}_{\partial_+}(q_0)} |A|^2 < C$$

for the same class of maximal surfaces.

Now consider a point $p$ on the maximal surface $M$. By the assumption there is a point $p_0 \in \partial_- D$ such that $\delta(p,p_0) > \epsilon$. We can fix a point $q$ on the segment $[p_0,p]$ such that $\delta(p,q) > \epsilon/2$.

Since $I^+(q) \cap K$ is compact, there is a point $r \in \partial_- K$ that maximizes the distance from $q$. Lemma 4.16 and the reverse triangle inequality imply that $\hat{s} := \delta(q,r) < \pi/2 - \epsilon/2$.

Moreover the plane passing through $r$ and orthogonal to the segment $[q,r]$ is a support plane $P$ for $K$ (that is $K \subset \overline{T^-(P)}$).
Now consider an isometry $\gamma$ of $AdS_{n+1}$ such that $\gamma(q) = (x^0, 0)$ and $\gamma(r) = (x^0, \bar{s})$. We have that $\gamma(P)$ is the horizontal plane through $(x^0, \bar{s})$. Since $\bar{s} < \pi/2 - \epsilon/2$, $\gamma(P) \subset I^-(P_0)$. Thus, $\gamma(K) \subset I^-(P_0)$, and $\gamma(M) \cap I^+(q_0) \subset H_0$.

In particular $\gamma(M)$ satisfies the conditions (1), (2) above and we conclude that

$$\sup_{\gamma(M) \cap I^+_r(q_0)} |\tilde{A}|^2 < C.$$  

where $\tilde{A}$ denotes the second fundamental form of $\gamma(M)$.

Since $\gamma(p) \in I^+_{\epsilon/2}(q_0)$ we conclude that

$$|A|^2(p) = |\tilde{A}|^2(\gamma(p)) < C.$$  

where the constant $C$ is independent of the point $p$. \hfill \Box

**Corollary 4.17.** Suppose that $w(K) < \pi/2$. Then there exists $C > 0$ such that any maximal space-like graph in $K$ has second fundamental form bounded by $C$.

**Proof.** Let $\epsilon = \pi/2 - w(K)$, so that $\epsilon > 0$. Let $y \in \partial_- K$. Consider a point $z \in \partial_+ K \cap I^+(y)$ for which $\delta(y, z)$ is maximal. Then $\delta(y, z) \leq w(K)$ by definition of $w$.

Let now $\Delta$ be the past-oriented time-like geodesic ray starting from $z$ and containing $y$, and let $x$ be its intersection with $\partial_- D$. By the definition of $z$, the space-like plane orthogonal to $\Delta$ at $z$ is a support plane of $K$ (otherwise $z$ would not maximize $\delta(y, \cdot)$ on $\partial_+ K$).

This shows that $z$ is also a critical point of $\delta(x, \cdot)$ on $\partial_+ K$ and, since $K$ is convex, it is a maximum of this function on $\partial_+ K$. Therefore $\delta(x, y) = \pi/2$ by the second point of Lemma 4.13. Therefore $\delta(x, y) \geq \epsilon$. So we can apply Theorem 4.14 which yields the result. \hfill \Box

5. Uniqueness of maximal surfaces in $AdS_3$

We consider in this section the uniqueness of maximal graphs with given boundary at infinity and bounded second fundamental form in $AdS_3$. The argument has two parts. The first is to show that those surfaces have negative sectional curvature. The second part is to show that the existence of such a negatively curved maximal space-like graph forbids the existence of any other maximal graph with the same boundary. Both parts use a version “at infinity” of the maximum principle, for which a compactness argument is needed. For the first part we need a simple compactness statement on sequences of maximal surfaces.

5.1. A compactness result for sequences of maximal hypersurfaces. The following statement is useful to use “at infinity” the maximum principle.

**Lemma 5.1.** Choose $C > 0$, a point $x_0 \in AdS_{n+1}$, and a future-oriented unit time-like vector $n_0 \in T_{x_0} AdS_{n+1}$. There exists $r_0 > 0$ as follows. Let $P_0$ be the space-like hyperplane orthogonal to $n_0$ at $x_0$, let $D_0$ be the disk of radius $r_0$ centered at $x_0$ in $P_0$, and let $(S_n)_{n \in \mathbb{N}}$ be a sequence of maximal space-like graphs containing $x_0$ and orthogonal to $n_0$, with second fundamental form bounded by $C$. After extracting a sub-sequence, the restrictions of the $S_n$ to the cylinder above $D_0$ converge $C^\infty$ to a maximal space-like disk with boundary contained in the cylinder over $\partial D_0$.

The proof given here applies with a few modifications to the more general context of maximal (resp. minimal) immersions of hypersurfaces in any Lorentzian (resp. Riemannian) manifold with bounded geometry, we state the lemma in $AdS_{n+1}$ for simplicity.

**Proof.** For all $n$, the surface $S_n$ is the graph of a function $f_n$ over $P_n$. The bound on the second fundamental form of $S_n$, along with the fact that the $S_n$ are orthogonal to $n_0$, indicates that, for some $r > 0$, the derivative of $f_n$ is bounded on the disk of center $x_0$ and radius $r$, more precisely there exists $\epsilon > 0$ such that

$$\phi \| \nabla f_n \| < 1 - \epsilon$$

on this disk of center $x_0$ and radius $r$.

This, along with the bound on the second fundamental form of $S_n$ (again) shows that the Hessian of $f_n$ is bounded by a constant depending on $r$ (for $r$ small enough). Thus we can extract from $(f_n)_{n \in \mathbb{N}}$ a subsequence which is $C^{1,1}$ converging to a function $f_\infty$ on the disk of center $x_0$ and radius $r$. Moreover the gradient of $f_\infty$ is uniformly bounded, so that $f_\infty$ is a disk which is uniformly space-like.

By definition the $f_n$ are solutions of Equation (10), which just translates analytically the fact that their graphs are maximal surfaces. Since $f_\infty$ is a $C^{1,1}$-limit of the $f_n$, it is itself a weak solution of (10). Since Equation (10) is quasi-linear, it then follows from elliptic regularity that $f_\infty$ is $C^\infty$, and that $(f_n)$ is $C^\infty$-converging to
Lemma 5.3. Let $f_\infty$ (see [23]). This means that the restriction of the $S_\infty$ to the cylinder above the disk of radius $r_0$ in $P_0$, for some $r_0 > 0$ (depending only on $C$) converge to a limit which is a maximal surface, the graph of $f_\infty$ over the disk of radius $r_0$.

5.2. Maximal surfaces with bounded second fundamental form. The first proposition of this section is the following, its proof is based on Lemma 5.1

Proposition 5.2. Let $S$ be a complete maximal surface in $AdS_3$. Suppose that the norm of the fundamental form of $S$ is bounded. Then $S$ either has negative sectional curvature, or $S$ is flat. If the supremum of the sectional curvature of $S$ is $0$, then $w(\partial_\infty S) = \pi/2$.

The completeness mentioned here is with respect to the induced metric on $S$. The proof uses two preliminary statements. The first is taken from [23], where it can be found in the proof of Lemma 3.11, p. 214. Note that the sign of the Laplacian used here is defined so that $\Delta$ is negative as an operator acting on $L^2$.

Lemma 5.3. Let $\Sigma$ be a maximal space-like surface in a 3-dimensional $AdS$ manifold. Let $B$ be its shape operator, and let $\chi = \log(-\det(B))/4$. Then $\chi$ satisfies the equation

$$\Delta \chi = e^{4\chi} - 1.$$  

As a consequence, we can apply the maximum principle to $\chi$, it shows that $\chi$ cannot have a positive local maximum. This can be translated into a statement on $K$, using the Gauss formula, which shows that $K = -1 + e^{4\chi}$.

Lemma 5.4. Suppose that $K$ has a local maximum at a point where it is non-negative. Then $K = 0$ at that point, and on the whole surface $S$, so that $S$ is flat (in the intrinsic sense).

We need another elementary statement, characterizing the maximal surfaces with flat induced metric in $AdS_3$. We include the proof for the reader’s convenience.

Lemma 5.5. Let $\Sigma$ be a space-like maximal surface in $AdS_3$, with zero sectional curvature. Then $\Sigma$ is a subset of a “horosphere”, that is, its principal curvatures are $-1$ and 1, and its lines of curvature form two orthogonal foliations by parallel lines. If $\Sigma$ is a space-like graph, then its boundary at infinity is the union of four light-like segments in $\partial_\infty AdS_3$.

Proof. Since $\Sigma$ is maximal, its principal curvatures are at each point two opposite numbers, $k$ and $-k$. The Gauss formula asserts that the sectional curvature of $\Sigma$ is $K = -1 + k^2$, so $k = 1$. Let $(e_1, e_2)$ be an orthonormal frame of unit principal vectors on $\Sigma_0$, and let $II$ be the second fundamental form of $\Sigma$. The Codazzi equation can be written as follows, at any point $m \in \Sigma$, for any vector field $x$ on $\Sigma$ such that $\nabla x = 0$ at $m$:

$$I((d^\Sigma B)(e_1, e_2), x) = e_1.I(e_2, x) - e_2.I(e_1, x) - II([e_1, e_2], x) = 0.$$  

Since the first two terms clearly vanish and $II$ is non-degenerate, $[e_1, e_2] = 0$, so that, if $\omega$ is the connection form of the frame $(e_1, e_2)$,

$$\nabla e_1, e_2 - \nabla e_2, e_1 = -\omega(e_1)e_1 - \omega(e_2)e_2 = 0.$$  

Therefore $e_1$ and $e_2$ are both parallel vector fields, and the first part of the statement follows.

There is a simple way to describe such a horosphere. Consider a space-like line $\Delta$ in $AdS_3$, and the set $\Sigma_0$ of endpoints of the future-oriented time-like segments of length $\pi/4$ starting from $\Delta$. An explicit computation (as in the proof of Proposition 5.2 below) shows that $\Sigma_0$ is precisely a horosphere as described above. The action of the isometry group of $AdS_3$ shows that there exists a unique surface of this type passing through each point $x$ of $AdS_3$, with fixed (time-like) normal and fixed principal direction at $x$ for the principal curvature +1, so any maximal graph with zero sectional curvature is of this type.

Let $\Delta^*$ be the line dual to $\Delta$, that is, the set of endpoints of future-oriented time-like segments of length $\pi/2$ starting from $\Delta$ (see Section 2.5). Now let $\partial \Sigma_0$ be the boundary of infinity of $\Sigma_0$. Considering the projective model of $AdS_3$ shows that $\partial \Sigma_0$ contains the endpoints at infinity $\Delta_-$ and $\Delta_+$ of $\Delta$, and also the endpoints at infinity of $\Delta_0^*$ and $\Delta_0^*$ of $\Delta^*$. Since $\partial \Sigma_0$ is a nowhere time-like curve in $\partial_\infty AdS_3$, it is necessarily made of the four segments from $\Delta_+$ to $\Delta_0^*$, from $\Delta_0^*$ to $\Delta_+$, from $\Delta_-$ to $\Delta_0^*$, and from $\Delta_0^*$ to $\Delta_-$, which are all light-like. This proves the last part of the lemma.

Proof of Proposition 5.2. Since $S$ has bounded second fundamental form, its sectional curvature $K$ is bounded, we call $K_S$ the upper bound of $K$ on $S$. Lemma 5.3 already shows that if this upper bound is attained on $S$, then it is non-positive, and if it is equal to 0 then $S$ is flat. We will use Lemma 5.1 to extend this argument to the case where the upper bound $K_S$ is not attained.
Consider a sequence \((s_n)_{n \in \mathbb{N}}\) of points in \(S\) such that \(K_S - 1/n < K(s_n) < K_S\), and apply to \(S\) a sequence of isometries \((\phi_n)_{n \in \mathbb{N}}\) which sends \(s_n\) to a fixed point \(x_0\) and the oriented unit normal vector to \(S\) at \(s_n\) to a fixed vector \(n_0\). Since \(S\) has bounded second fundamental form, Lemma \ref{lem:boundedness} shows that we can extract from the sequence \((\phi_n(S))_{n \in \mathbb{N}}\) a subsequence which converges, in the neighborhood of \(x_0\), to a maximal space-like graph \(S_0\). By construction the curvature of \(S_0\) has a local maximum at \(x_0\), and this local maximum is equal to \(K_S\). Lemma \ref{lem:maximality} therefore shows that \(K_S \leq 0\).

Suppose now that \(K_S = 0\). Then the sequence \(\phi_n(S)\) converges, in a neighborhood of \(x_0\), to a “horosphere” \(\Sigma_0\), as described in Lemma \ref{lem:horospheres}. Lemma \ref{lem:boundedness} shows that the convergence is \(C^\infty\) in compact subsets of \(\partial_\infty \text{AdS}_3\). Let \(E_0\) be the boundary at infinity of \(\phi_n(S)\). Since \(\phi_n(S)\) is space-like, \(E_0\) is a nowhere time-like curve in \(\partial_\infty \text{AdS}_3\). By construction, \(E_n = (\rho_{l,n}, \rho_r,n)E\), where \(E = \partial_\infty S\), \((\rho_{l,n})\) and \((\rho_r,n)\) are two sequences of elements of \(\text{PSL}_2(\mathbb{R})\), and, for all \(n \in \mathbb{N}\), \((\rho_{l,n}, \rho_r,n)\) is considered as an isometry acting on \(\text{AdS}_3\) through the natural identification (see Section \ref{sec:horospheres} or \cite{26,4}).

By Lemma \ref{lem:boundedness} (more precisely the fact that space-like hypersurfaces in \(\text{AdS}_{n+1}\) are the graphs of 2-Lipschitz functions), \(\phi_n(S)\) converges on compact subsets of \(\text{AdS}_3\) to \(\Sigma_0\), \(E_n\) converges to the boundary at infinity of \(\Sigma_0\), which we call \(E_0\). In particular, using the notations in the proof of Lemma \ref{lem:horospheres} for each \(n \in \mathbb{N}\) there are four points \(x_+^n, x_-^n, x_+^{n*}, x_-^{n*} \in E_n\) which can be chosen so that \(x_+^n \to \Delta_+, x_-^n \to \Delta_-, x_+^{n*} \to \Delta_*^+\) and \(x_-^{n*} \to \Delta_*^-\). Therefore, for \(n\) large enough, there are points \(y_n, z_n\) which are arbitrarily close to \(\Delta\) and to \(\Delta^*\) respectively, with \((y_n)\) and \((z_n)\) converging to limits respectively in \(\Delta\) and to \(\Delta^*\). The distance between the limits is \(\pi/2\), so that the distance between \(y_n\) and \(z_n\) goes to \(\pi/2\) as \(n \to \infty\), this shows that \(w(K) = \pi/2\).

5.3. **Quasi-symmetric homeomorphisms and the width.** There is another important relation which is valid only in \(\text{AdS}_3\), as stated in the next proposition.

**Proposition 5.6.** Let \(E\) be a weakly space-like graph in \(\partial_\infty \text{AdS}_3\) (that is, \(E\) is a space-like curve). Let \(K\) be the convex hull of \(E\). Suppose that \(w(K) = \pi/2\). Then \(E\) is not the graph of a quasi-symmetric homeomorphism from \(S^1\) to \(S^1\).

**Proof.** We suppose that \(w(K) = \pi/2\), it follows that there exist two sequences of points \((x_n)\) in \(\partial_\infty K\) and \((y_n)\) in \(\partial_\infty K\) such that \(\delta(x_n, y_n) \to \pi/2\). We can suppose (replacing \(x_n\) and \(y_n\) by points in the same face of \(\partial_\infty K\) if necessary) that \(x_n\) is contained in a space-like geodesic \(\Delta_n \subset \partial_\infty K\), and that \(y_n\) is contained in a space-like geodesic \(\Delta'_n \subset \partial_\infty K\).

We can find a sequence \((\phi_n)\) of isometries of \(\text{AdS}_3\) such that \(\phi_n(x_n) \to x\), \(\phi_n(y_n) \to y\), with \(\delta(x, y) = \pi/2\). Moreover, \(\phi_n(K)\) is the convex hull \(\phi_n(E)\). Since the \(\phi_n(K)\) are convex, they converge (perhaps after extracting a subsequence) in the Hausdorff topology to some \(K_0\), which is the convex hull of \(E_0 = \lim \phi_n(E)\). Moreover, extracting a subsequence again if necessary, we can suppose that \(\phi_n(\Delta_n) \to \Delta\) and that \(\phi_n(\Delta'_n) \to \Delta'\). Since \(x \in \Delta, y \in \Delta'\), and \(\delta(x, y) = \pi/2\), \(\Delta' = \Delta^*\), otherwise the width of \(K_0\) would have to be strictly larger than \(\pi/2\), contradicting Lemma \ref{lem:boundedness}.

Then \(E_0\) contains the endpoints \(\Delta_-, \Delta_+\) of \(\Delta\), and the endpoints \(\Delta_*^-, \Delta_*^+\) of \(\Delta^*\). Since \(E\) is weakly space-like, so is \(E_0\), so it is the union of four light-like segments joining those four points.

Since \(E_0\) is composed of four light-like segments (with endpoints \(\Delta_+, \Delta_*^+, \Delta_-\) and \(\Delta_*^-\)) there are points \(u, v, u', v'\) in \(\mathbb{R}P^1\), with \(u \neq v\) and \(u' \neq v'\), such that, in the identification of \(\partial_\infty \text{AdS}_3\) with \(\mathbb{R}P^1 \times \mathbb{R}P^1\), \(\Delta_+ = (u, u')\), \(\Delta_*^- = (u, v'), \Delta_- = (v, v')\), and \(\Delta_*^+ = (v, u')\).

So \(E_0\) is the graph of the function \(f_0: \mathbb{R}P^1 \to \mathbb{R}P^1\) sending \((u, v)\) to \((u', v')\). After composing on the right and on the left with projective transformations, we can suppose that it is the graph of the function \(f_0: \mathbb{R}P^1 \to \mathbb{R}P^1\) sending \((0, 2)\) to \((1, 2)\) and \((2, \infty)\) to \((\infty, 0)\).

Consider the points \(-3, -1, 1, \infty \in \mathbb{R}P^1\). A direct computation shows that their cross-ratio is \([-3, -1; 1, \infty] = 2\), while the cross-ratio of their images by \(f_0\) is \([0, 0; 1, 1] = 1\).

It follows that there are 4-tuples of points on \(\phi_n(E)\) whose projection by \(p_1\) are 4-tuples of points with cross-ratio arbitrarily to 2 and whose projection by \(p_2\) are 4-tuples of points with cross-ratio arbitrarily close to 1. This means precisely, by definition of a quasi-symmetric homeomorphism, that \(E\) is not the graph of a quasi-symmetric homeomorphism.

5.4. **Uniqueness of negatively curved maximal surfaces.** We now turn to the second proposition of this section, the fact that maximal space-like graphs with negative sectional curvature are uniquely determined, among all maximal space-like graphs, by their boundary at infinity.

**Proposition 5.7.** Let \(S\) be a maximal graph in \(\text{AdS}_3\), with sectional curvature bounded from above by a negative constant. Then \(S\) is unique among complete maximal graphs with given boundary curve at infinity and bounded second fundamental form.
We first state a preliminary lemma (see also Lemma 5.8). Note that from this point on we will often consider space-graphs in the projective model of $AdS_{n+1}$.

**Lemma 5.8.** Let $u : S^1 \to S^1$ be a homeomorphism, and let $E_u \subset S^1 \times S^1 \simeq \partial(AdS_3)$ be its graph. Let $C(E_u)$ be defined as in the paragraph before Definition 1.7. Then any maximal surface in $AdS_3$ with boundary at infinity $E_u$ is contained in $C(E_u)$.

**Proof.** Let $S \subset AdS_3$ be a maximal surface, with boundary at infinity $E_u$. The image of $S$ in the projective model of $AdS_3$ is a saddle surface, that is, a surface which has opposite principal curvatures at each point. A characterization of saddle surfaces (see [16, Section 6.5.1]) is that, for any relatively compact subset $S$ of the distance to $S$, interior of $S$ surfaces equidistant to $S$ contains the convex hull of $\partial G$. Then $G$ is contained in the convex hull of $\partial G$. This property, applied to an exhaustion of the image of $S$ in the projective model by compact subsets, is precisely what we need.

**Proof of Proposition 5.7.** We consider the domain $\Omega$ introduced in the proof of Claim 5.2. Let $\Omega$ be defined as in the set of points at time-like distance at most $\pi/4$ from $S$. Claim 5.2 shows that $\Omega$ is convex, with smooth, space-like boundary.

Consider now another maximal graph $S' \subset AdS_3$, complete, with the same boundary at infinity as $S$, and with bounded second fundamental form. By construction the boundary of $\Omega$ is equal to $E$. Since $\Omega$ is convex, it contains the convex hull of $E$ and therefore, by Lemma 5.8, it contains $S'$. Let $r_1$ be the supremum over $S'$ of the distance to $S$. The argument above shows that $r_1 \in [0, \pi/4 - \alpha)$, and the maximum principle shows that, if $r_1 > 0$, then it cannot be attained at an interior point of $S'$, since then $S'$ would have to be tangent from the interior of $S_\alpha$, which would contradict the maximality of $S'$.

Since $S'$ is complete, there exists a sequence $(x_n)_n \in \mathbb{N}$ of points in $S'$ such that $d(x_n, S) \to r_1$ and that the norm of the differential at $x_n$ of the restriction to $S'$ of the distance to $S$ goes to zero as $n \to \infty$ (this is a very weak form of a lemma appearing e.g. in [30]).

Consider a sequence of isometries $(\phi_n)_{n \in \mathbb{N}}$ chosen such that $\phi_n(x_n)$ is equal to a fixed point $x_0$, and that the normal to $\phi_n(S')$ at $\phi_n(x_n)$ is a fixed vector $n_0$. Lemma 5.1 shows that, after extracting a sub-sequence, $(\phi_n(S'))_{n \in \mathbb{N}}$ converges in a neighborhood of $x_0$ to a smooth, maximal surface $S'_{\infty}$. Moreover, since the differential at $x_n$ of the distance to $S$ goes to zero, the images by $\phi_n$ of $S$ also converge to a limit $S_{\infty}$, in a neighborhood of its intersection with the normal to $S'_{\infty}$ at $x_0$.

We can now apply the maximum principle to the distance to $S'_{\infty}$ as a maximal surface in the foliation by the surfaces equidistant to $S_{\infty}$, and obtain a contradiction if $r_1 > 0$. So $r_1 = 0$, and $S' = S$.

Together with Proposition 5.2 and Proposition 5.6, Proposition 5.7 leads directly to a simple consequence.

**Corollary 5.9.** Let $S$ be a maximal graph in $AdS_3$, with bounded second fundamental form. Suppose that the boundary at infinity of $S$ is the graph $E$ of a quasi-symmetric homeomorphism from $S^1$ to $S^1$. Then $S$ is the unique maximal surface with boundary at infinity $E$ and bounded second fundamental form.

**6. Proof of the main results**

6.1. **A characterization of quasi-symmetric homeomorphisms.** We now prove Theorem 1.12. Let $u : S^1 \to S^1$ be a homeomorphism, and let $E_u$ be its graph. We already know, from Lemma 1.10, that $w(E_u) \leq \pi/2$. Moreover Proposition 5.6 shows that if $u$ is quasi-symmetric, then $w(E_u) < \pi/2$.

Suppose conversely that $w(E_u) < \pi/2$. We can apply Theorem 1.11 to $E_u$, and obtain a maximal graph $M$ in $AdS_3$ with boundary at infinity equal to $E_u$. Corollary 1.17 shows that $M$ has bounded second fundamental form.

Proposition 5.2 then shows that $M$ has sectional curvature bounded from above by a negative constant. Therefore we obtain through Proposition 1.5 a minimal Lagrangian quasi-conformal diffeomorphism $\phi$ with boundary value equal to $u$. Since $\phi$ is quasi-conformal, $u$ is quasi-symmetric, as claimed.

6.2. **Theorems 1.4 and 1.10.** Theorem 1.4 clearly follows, through Proposition 1.5, from Theorem 1.10, so we now concentrate on this last statement.

**Proof of Theorem 1.10.** Let $E = \partial\infty S \subset \partial\infty AdS_3$, and let $M$ be the maximal graph with boundary at infinity $E$ which is provided by Theorem 4.11. Since $E$ is the graph of a quasi-symmetric homeomorphism, Proposition 5.6 shows that $w(E) < \pi/2$.

The argument in the previous paragraph then shows that $E$ is the boundary at infinity of a maximal graph $M$ in $AdS_3$, which has bounded second fundamental form by Theorem 1.14. Then Proposition 5.2 shows that $M$ has sectional curvature bounded from above by a negative constant. Proposition 5.7 can therefore be used to obtain that $M$ is unique among maximal graphs with boundary at infinity $E$ and bounded second fundamental form.  

□
APPENDIX A. MEAN CURVATURE FLOW FOR SPACELIKE GRAPHS

In this section we prove a longtime existence solution for the mean curvature flow of spacelike graphs in \(\text{AdS}_{n+1}\). The proof is based on Ecker’s estimates [17], that are the parabolic analogous of Bartnik’s estimates we have used in Lemma 4.13. This argument provides an alternate proof of the existence and regularity of maximal surfaces with given asymptotic boundary already proved in Section 4.

We recall that a mean curvature flow of a spacelike surface is a family of spacelike embeddings \(\sigma_s : M \rightarrow \text{AdS}_{n+1}\) such that

\[
\frac{\partial \sigma}{\partial s}(x, s) = H(x, s)\nu(x, s)
\]

where \(H(x, s)\) and \(\nu(x, s)\) are respectively the mean curvature and the normal vector of the surface \(M_s = \sigma_s(S)\) at point \(\sigma_s(x)\).

We also consider the case where \(M\) is compact with boundary. In that case we always consider the Dirichlet condition

\[
\sigma_s(x) = \sigma_0(x) \quad \text{for all } x \in \partial M.
\]

**Lemma A.1.** Let \((M_s)_{s \in [0, s_0]}\) be a family of spacelike surfaces moving by mean curvature flow. If \(M_0\) is a graph of a function \(u_0\) defined on some domain \(\Omega\) of \(\mathbb{H}^n\) with smooth boundary, then so is \(M_s\) for every \(s \in [0, s_0]\).

Moreover, if \(u_s : \Omega \rightarrow \mathbb{R}\) is the function defining \(M_s\) then

\[
\frac{\partial u}{\partial s} = \phi^{-1} v^{-1} H
\]

where \(v\) is the gradient function on \(M_s\).

**Proof.** Since \(M_s\) is homotopic to \(M_0\) through a family of spacelike surfaces with fixed boundary, then \(M_s\) is contained in the domain of dependence of \(M_0\) that, in turn, is contained in \(\Omega \times \mathbb{R}\).

Moreover, \(M_s\) disconnects \(\Omega \times \mathbb{R}\) in two regions. The same argument as in Proposition 3.2 shows that \(M_s\) is a graph on \(\Omega\) of a function \(u_s\).

The evolution equation of \(u_s\) is computed in [15]. \(\square\)

**Remark A.2.**

1. Notice that \(\frac{\partial (t \sigma_s)}{\partial s} = \phi^{-1} v H\), that is different from (22). The reason is that the curve \(\sigma(x, \cdot)\) at some point \(s\) is tangential to the normal of \(M_s\), so in general it is not a vertical line. This implies that the function \(u_s\) agrees with \(t|_{M_s}\) only up some tangential diffeomorphism of \(M_s\).

2. Equation (22) is equivalent, up to tangential diffeomorphisms, to equation (20). This means that if \((u_s)_{s \in [0, s_0]}\) is a solution of (22), there is a time-dependent field \(X_s\) on \(\Omega\) such that the map \(\sigma : \Omega \times [0, s_0] \rightarrow \text{AdS}^{n+1}\) defined by

\[
\sigma(x, s) = (\psi_s(x), u_s(\psi_s(p)))
\]

is a solution of (20), where \(\psi_s\) is the flow of \(X_s\).

**Proposition A.3.** [17] Let \(M_0\) be a spacelike \(C^{0,1}\) compact graph in \(\text{AdS}^{n+1}\). Then there is a smooth solution of (20) for \(s \in (0, +\infty)\) such that

- \(\partial M_s = \partial M_0\) for every \(s\);
- \(M_s \rightarrow M_0\) in the Hausdorff topology as \(s \rightarrow 0\);
- \(M_s \rightarrow M_\infty\) in the \(C^\infty\)-topology as \(s \rightarrow +\infty\), where \(M_\infty\) is the unique maximal spacelike surface with the property that \(\partial M_\infty = \partial M_0\);
- if \(H_s\) denotes the mean curvature on \(M_s\) we have

\[
H_s^2(x) \leq \frac{n}{2} s.
\]

A.1. MEAN CURVATURE FLOW AND CONVEX SUBSETS. To show the convergence of the mean curvature flow, we need to remark that, under suitable hypothesis, it does not leave convex subsets of \(\text{AdS}_{n+1}\).

**Lemma A.4.** Let \(M_s\) be a compact solution of (20). Suppose that there exists a spacelike plane \(P\) such that \(M_0\) is contained in \(\overline{I^-(P)}\) and \(\partial M_0 \subset I^-(P)\). Then \(M_s\) is contained in \(I^-(P)\) for every \(s > 0\).

**Proof.** Without loss of generality we can suppose that \(P\) is the horizontal plane. We consider the function \(u : \text{AdS}^{n+1} \rightarrow \mathbb{R}\) defined, as in the proof of Lemma 4.1, by \(u(x, t) = x_{n+1} \sin t\).

By our assumption

\[
\begin{align*}
 u(p) &\leq 0 & \text{for every } p \in M_0, \\
 u(p) &< 0 & \text{for every } p \in \partial M_s.
\end{align*}
\]
On the other hand the computation in Lemma A.1 shows that
\[
\left( \frac{d}{ds} - \Delta \right) u = -nu
\]
where \(\Delta\) is the Laplace-Beltrami operator on \(M_s\).

In particular if the maximum of the function \(u\) is achieved at some interior point of \(M_s\) we have
\[
\frac{du_{\text{max}}}{ds} \leq nu_{\text{max}}.
\]
By (24), we deduce that \(u_{\text{max}}(s) < 0\) for every \(s > 0\). In particular \(M_s\) is contained in the region \(\{(x,t)|0 < t < \pi\}\) for every \(s > 0\).

Lemma A.3 and Lemma A.7 imply the following property.

**Proposition A.5.** If \(M_s\) be a compact solution of (20) such that \(M_0\) is contained in the closure of some convex sub \(\Omega\), and \(\partial M_0\) is contained in \(\Omega\), then \(M_s\) is contained in \(\Omega\) for every \(s > 0\).

Let \(M = \Gamma_u\) be a weakly spacelike graph and \(\Sigma\) be its asymptotic boundary. We will assume that neither \(M\) nor \(\Sigma\) contains any singular point. Finally we denote by \(D\) the domain of dependence of \(M\) and by \(K\) its convex hull, introduced in Remark 4.9. The same argument as in Lemma A.10 shows that \(\overline{K} \cap \partial D = \emptyset\).

For every \(r > 0\) let \(u^r\) be the restriction of \(u\) on \(B_r\) (that is the ball in \(\mathbb{H}^n\) of center at \(x^0\) and radius \(r\)).

We consider the mean curvature flow with Dirichlet condition of the compact graph of \(u^r\), that is, a map
\[
\sigma^r : \overline{B}_r \times (0, +\infty) \to AdS^{n+1}
\]
that verifies (20) and satisfies
- \(\sigma^r(x,0) = (x,u(x))\) for every \(x \in B_r\),
- \(\sigma^r(x,s) = (x,u(x))\) for every \(x \in \partial B_r\).

Let us denote by \(M^r_s\) the image of \(B_r\) through the map \(\sigma(\cdot,s)\).

By Lemma A.1 and Proposition A.3 there is a family of spacelike functions
\[
u^r_s : \overline{B}_r \to \mathbb{R}
\]
such that \(M^r_s\) is the graph of \(u^r_s\) and the family \((u^r_s)\) satisfies (22).

**Proposition A.6.** For every \(R > 0\), \(\eta > 0\) there is \(\bar{r} > 0\) and constants \(C, C_0, C_1, \ldots\) such that for every \(r > \bar{r}\) and every \(s > \eta\) we have
\[
\sup_{B_R \times \mathbb{R}} \nabla^m A < C
\]
\[
\sup_{M^r_s \cap B_R \times \mathbb{R}} |\nabla^m A| < C_m \quad \text{for } m = 0, \ldots .
\]

**Proof.** The scheme of the proof is the same as for Theorem 4.13. In particular we use the notations introduced there.

We choose points \(p_1, \ldots, p_{k_0} \in D \cap \partial^-(\partial_+ K)\) and numbers \(\epsilon_1, \ldots, \epsilon_k\) such that
\[
(B_R \times \mathbb{R}) \cap K \subset \bigcup_{1}^{k_0} I^+_{\epsilon_k}(p_k).
\]

On \(I^+_{\epsilon_k}(p_k)\) we consider the time function \(\tau_k = \tau_{p_k} - \epsilon_k\) where \(\tau_{p_k}\) denote the Lorentzian distance from \(p_k\) and is a time function on \(I^+(p_k)\). Notice that \(\tau_k\) is smooth on the domain \(\mathcal{Y} = I^+(p_k) \cap I^-(\langle p_k \rangle_+)\).

Moreover \(K \cap I^+_{\epsilon_k/2}(p_k)\) is a compact domain in \(\mathcal{Y}\).

Since \(M^r_s\) is contained in \(K\) for every \(r > 0\) and \(s > \eta\), we deduce that there exists \(r_0\) such that for \(r \geq r_0\) and \(k = 1, \ldots, k_0\)
\[
\partial M^r_s \cap I^+(p_k) = \emptyset
\]
and \(M^r_s \cap \{\tau_k \geq 0\} = M^r_s \cap \overline{I^+_{\epsilon_k/2}(p_k)}\) is compact.

Thus we are in the hypothesis of Theorem 2.1 of [17], there is a constant \(A_k\)
\[
(25) \quad \sup_{M^r_s \cap I^+_{\epsilon_k}(p_k)} v_{M^r_s} \leq A_k(1 + \frac{1}{s}).
\]

where \(A_k\) depends on the \(C^2\) norm of \(\tau_k\) and \(t\) and the \(C^0\) norm of \(Ric\) taken on the domain \(\overline{K \cap I^+_{\epsilon_k/2}(p_k)}\) with respect to a reference Riemannian metric.
In particular for \( s > \eta \) we have
\[
\sup_{M^r_s \cap I^+_k} v_{M^r_s} \leq A_k(1 + \frac{1}{\eta}).
\]
By Theorem 2.2 of [17] we also have that for every \( m = 0, 1, \ldots \) there are constants \( A_{k,m} \) such that
\[
\sup_{M^r_s \cap I^+_k(\rho_k)} |\nabla^m A|^2 \leq A_{k,m}.
\]
In particular, the constants \( C = \sup\{A_1, \ldots A_{k_0}\}, C_m = \sup\{A_{1,m}, \ldots, A_{k,m}\} \) satisfy the statement.

\[\square\]

**Theorem A.7.** There is a family of spacelike functions
\[
\tilde{u}_s : \mathbb{H}^n \to \mathbb{R}
\]
for \( s \in (0, +\infty) \) that verifies (22) such that
- \( \tilde{u}_s \to u \) as \( s \to 0 \) in the compact open topology,
- \( \{\tilde{u}_s\}_{s>1} \) is a relatively compact family in \( C^\infty(\mathbb{H}^n) \).
- the graph \( M_s \) of \( \tilde{u}_s \) is contained in \( K \) for every \( s > 0 \).
- the mean curvature of \( M_s \) satisfies \( H_s(x)^2 < \frac{\pi}{2} \).

**Proof.** For any \( R > 0 \) and \( \varepsilon > 0 \) we consider the restriction of \( u^r \) on \( B_R \times [-\varepsilon, +\infty) \). Proposition [A.6] implies that such restrictions form a pre-compact family in \( C^\infty(B_R \times [-\varepsilon, +\infty)) \).

By a diagonal process, we can construct a sequence \( r_n \to +\infty \) such that \( (u^r_n) \) converges to \( \bar{u} \) in the \( C^\infty \)-topology on compact subsets of \( \mathbb{H}^n \times (0, +\infty) \).

Notice that by construction \( (\tilde{u}_s)_{s>1} \) is precompact in \( C^\infty(\mathbb{H}^n) \).

By the uniform estimate on the gradient function of \( u^r_n \) on \( B_R \) we get that the graph \( M_s \) of \( \tilde{u}_s \) is spacelike.

Clearly \( \tilde{u}_s \) verifies equation (22).

Since (23) holds for every \( u^r_n \), we get that \( H(\bar{u})^2 < \frac{\pi}{2} \).

Analogously, passing to the limit in the inclusion \( M^r_s \subset K \), we get that \( M_s \) is contained in \( K \).

Comparing (22) with (23), it results that
\[
|u^r_n(x) - u(x)| \leq \sqrt{ns}.
\]

Taking the limit for \( r \to +\infty \) we get
\[
|\tilde{u}_s(x) - u(x)| \leq \sqrt{ns}
\]
that shows that \( \tilde{u}_s \to u \) in the compact open topology.

\[\square\]

**Remark A.8.** Taking the limit of \( M_{s_k} \) for a suitable sequence \( s_k \to +\infty \) we obtain a maximal surface contained in \( D \). Thus Theorem [A.7] furnishes another proof of Theorem [3.11].

**References**

1. L. V. Ahlfors, *Lectures on quasiconformal mappings*, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966, Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10.

2. Reiko Aiyama, Kazuo Akutagawa, and Tom Y. H. Wan, *Minimal maps between the hyperbolic discs and generalized Gauss maps of maximal surfaces in the anti-de Sitter 3-space*, Tohoku Math. J. (2) 52 (2000), no. 3, 415–429. MR MR1772805 (2002e:58025)

3. Kazuo Akutagawa, *Harmonic diffeomorphisms of the hyperbolic plane*, Trans. Amer. Math. Soc. 342 (1994), no. 1, 325–342. MR MR1147398 (94e:58029)

4. Lars Andersson, Thierry Barbot, Riccardo Benedetti, Francesco Bonsante, William M. Goldman, François Labourie, Kevin P. Scannell, and Jean-Marc Schlenker, *Notes on: “Lorentz spacetimes of constant curvature” [Geom. Dedicata 126 (2007), 3–45; mr2328921] by G. Mess*, Geom. Dedicata 126 (2007), 47–70. MR MR2328922

5. Thierry Barbot, François Béguin, and Abdelghani Zeghib, *Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3*, Geom. Dedicata 126 (2007), 71–129. MR MR2328923 (2008j:53041)

6. Thierry Barbot, Francesco Bonsante, and Jean-Marc Schlenker, *Collisions of particles in locally AdS spacetimes*, arXiv:0905.1823., 2009.

7. R. Bartnik, *Existence theorems for maximal hypersurfaces in asymptotically flat spacetimes*, Comm. Math. Physics 94 (1984), 155–175.

8. Robert Bartnik, *Regularity of variational maximal surfaces*, Acta Math. 161 (1988), no. 3–4, 145–181. MR MR971795 (90b:58255)

9. Robert Bartnik and Leon Simon, *Spacelike hypersurfaces with prescribed boundary values and mean curvature*, Comm. Math. Phys. 87 (1982/83), no. 1, 131–152. MR MR680653 (84j:58126)

10. Riccardo Benedetti and Francesco Bonsante, *Canonical Wick rotations in 3-dimensional gravity*, Memoirs of the American Mathematical Society 198 (2009), 164pp. math.DG/0508485.

11. A. Besse, *Einstein manifolds*, Springer, 1987.
12. Francesco Bonsante, Kirill Krasnov, and Jean-Marc Schlenker, *Multi black holes and earthquakes on Riemann surfaces with boundaries*, math.GT/0610429., 2006.
13. Francesco Bonsante and Jean-Marc Schlenker, *AdS manifolds with particles and earthquakes on singular surfaces*, math.GT/0609116. Geom. Funct. Anal. 19:1 (2009), 41-82., 2006.
14. ..., Fixed points of compositions of earthquakes, arXiv:0812.3471, 2009.
15. Simon Brendle, *Minimal Lagrangian diffeomorphisms between domains in the hyperbolic plane*, J. Differential Geom. 80 (2008), no. 1, 1–22. MR MR2434257 (2009h:53181)
16. Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 285, Springer-Verlag, Berlin, 1988, Translated from the Russian by A. B. Sosinskiî, Springer Series in Soviet Mathematics. MR MR936419 (89b:53020)
17. Klaus Ecker, *Mean curvature flow of spacelike hypersurfaces near null initial data*, Comm. Anal. Geom. 11 (2003), no. 2, 181–205. MR MR2014875 (2004k:53098)
18. Klaus Ecker and Gerhard Huisken, *Immersed hypersurfaces with constant Weingarten curvature*, Math. Ann. 283 (1989), no. 2, 329–332.
19. F. P. Gardiner and W. J. Harvey, *Universal teichmüller space*, 2000.
20. David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR MR1814364 (2001k:35004)
21. Jürgen Jost, *Harmonic maps between surfaces*, Lecture Notes in Mathematics, vol. 1062, Springer-Verlag, Berlin, 1984. MR MR754769 (85j:58046)
22. ..., *Compact Riemann surfaces*, third ed., Universitext, Springer-Verlag, Berlin, 2006, An introduction to contemporary mathematics. MR MR2247485 (2007b:32024)
23. Kirill Krasnov and Jean-Marc Schlenker, *Minimal surfaces and particles in 3-manifolds*, Geom. Dedicata 126 (2007), 187–254. MR MR2328927
24. F. Labourie, *Surfaces convexes dans l’espace hyperbolique et CP1-structures*, J. London Math. Soc., II. Ser. 45 (1992), 549–565.
25. Vladimir Markovic, *Harmonic diffeomorphisms of noncompact surfaces and Teichmüller spaces*, J. London Math. Soc. (2) 65 (2002), no. 1, 103–114. MR MR1875138 (2002k:32015)
26. Geoffrey Mess, *Lorentz spacetimes of constant curvature*, Geom. Dedicata 126 (2007), 3–45. MR MR2328921
27. Jean-Marc Schlenker, *Métriques sur les polyèdres hyperboliques convexes*, J. Differential Geom. 48 (1998), no. 2, 323–405. MR MR1630178 (2000a:52018)
28. Richard M. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Complex geometry (Osaka, 1990), Lecture Notes in Pure and Appl. Math., vol. 143, Dekker, New York, 1993, pp. 179–200. MR MR1201611 (94g:58055)
29. Luen-Fai Tam and Tom Y. H. Wan, *Quasi-conformal harmonic diffeomorphism and the universal Teichmüller space*, J. Differential Geom. 42 (1995), no. 2, 368–410. MR MR1366549 (96j:32024)
30. Shing Tung Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201–228. MR MR0431040 (55 #4042)

**Dipartimento di Matematica, Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy**

**E-mail address:** francesco.bonsante@unipv.it

**Institut de Mathématiques de Toulouse, UMR CNRS 5219, Université Toulouse III, 31062 Toulouse cedex 9, France**

**E-mail address:** schlenker@math.univ-toulouse.fr