STOCHASTIC NONLINEAR FOKKER-PERK-PLANCK EQUATIONS

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Abstract. The existence and uniqueness of measure-valued solutions to stochastic nonlinear, non-local Fokker-Planck equations is proven. This type of stochastic PDE is shown to arise in the mean field limit of weakly interacting diffusions with common noise. The uniqueness of solutions is obtained without any higher moment assumption on the solution by means of a duality argument to a backward stochastic PDE.

1. Introduction

We consider the following stochastic nonlinear, non-local Fokker-Planck equation on $[0, T] \times \mathbb{R}^d$,

\[
\begin{cases}
\partial_t \mu = \partial_{ij}(a_{ij}(t, x, \mu))\mu - \partial_i(b_i(t, x, \mu))\mu - \partial_j(\sigma_{ik}(t, x, \mu))dW^k_t, \\
\mu|_{t=0} = \mu_0,
\end{cases}
\] (1.1)

where $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is a filtered probability space, $W_t$ is a $d_1$-dimensional $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motion for some $d_1 \in \mathbb{N}$, $a = (a_{ij}): [0, T] \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^{d \times d}$ is a function from the space of measures on $\mathbb{R}^d$ to the space of symmetric and non-negative definite matrices, $\sigma = (\sigma_{ik}): [0, T] \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^{d \times d_1}$ takes values in the space of $d \times d_1$ matrices and $b = (b_i): [0, T] \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d$. We emphasize that the coefficients $a_{ij}, b_i, \sigma_{ik}$ depend non-locally and possibly nonlinearly on the solution $\mu$.

Since stochastic PDE of the type (1.1) describe the evolution of conditional distributions of solutions to McKean-Vlasov SDE with common noise (see below) it is natural to consider solutions $(\mu_t)_{t \in [0, T]}$ to (1.1) taking values in the space of finite non-negative measures on $\mathbb{R}^d$. The main result of this paper is to establish the well-posedness of measure-valued solutions to (1.1).

Theorem (see Theorem 5.4 and 5.3 below). Let $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ be a non-negative measure. If the coefficients $a, b, \sigma$ are regular enough, then there exists a unique solution $\mu \in L^1 C_t \mathcal{M}$ to equation (1.1) in the sense of Definition 5.2, below.

Previously, the uniqueness of solutions to (1.1) was known only in the class of solutions to (1.1) admitting a square-integrable density with respect to the Lebesgue measure (e.g.

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\textsuperscript{1} with Einstein summation convention, and $\partial_i$ being the partial derivative with respect to the space variable $x_i$. 

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Kurtz, Xiong [25]). This is in contrast to the deterministic case, where the uniqueness of measure-valued solutions has recently been shown based on duality arguments by Manita, Romanov, Shaposhnikov in [33, 34]. Following this approach, the existence of regular enough solutions to the dual equation, a parabolic PDE backwards in time, implies the uniqueness of solutions to (1.1) with $\sigma \equiv 0$. This line of argument becomes more challenging in the case of stochastic PDE since the dual equation becomes a backward stochastic PDE (BSPDE) and, therefore, has not previously been put to use in the case of stochastic PDE, such as (1.1). This is the purpose of the present work. In particular, the method employed here can be seen as a proof of principle of using duality arguments to derive the uniqueness of solutions for stochastic PDE.

**Motivation and model.** The stochastic PDE (1.1) is linked to stochastic scalar conservation laws (SSCL) of the form

$$
\begin{cases}
  du + \text{div}(\sigma(x,u)u \circ dW) = 0 & \text{in } \mathbb{R}^d \times (0,T), \\
  u = u_0 & \text{on } \mathbb{R}^d \times \{0\}.
\end{cases}
$$

Indeed, rewriting equation (1.2) in Itô form (cf. Appendix A below), yields

$$
\partial_t u + \partial_i (\sigma^{i,k}(x,u)u)dW^k_t + \partial_i (b^i(x,u)u) = \frac{1}{2} \partial_{i,j}^2 (\sigma^{i,k}(x,u)\sigma^{k,j}(x,u)),
$$

which is of the same type as equation (1.1). In particular, we notice that both first order and second order correction terms appear, and that they are both nonlocal in the variable $u$. For the exact definition of $b$, we refer to (A.3) below.

Stochastic scalar conservation laws and thus stochastic PDE of the type (1.1) arise in several applications. Examples are provided by the theory of mean field systems (see Sznitman [39] for an overview) and mean field games with common noise introduced by Lasry and Lions [27–29], with an extensive treatment given by Carmona, Delarue in [8, 9].

Consider the empirical law $L_N^t := \frac{1}{N} \sum_{j=1}^N \delta_{X^j_t}$ of the solution $(X^1, \ldots, X^N) : [0,T] \times \Omega \rightarrow \mathbb{R}^{dN}$ of the weakly interacting particle system

$$
dX^i_t = \sigma(X^i_t, \frac{1}{N} \sum_{j=1}^N \delta_{X^j_t}) \circ dW, \quad X^i_t|_{t=0} = X^i_0,
$$

with initial conditions $(X^i_0)_{i \geq 1}$ independent and identically distributed. The random measure $L_N^T$ converges, as $N \rightarrow \infty$, to a random measure $u$ which evolves according to (1.2) (cf. Section 3.1 below).

The above mentioned convergence of (random) empirical measures is closely linked to the phenomenon of propagation of chaos and to McKean-Vlasov SDE (cf. e.g. [12, 35, 36, 38]). More precisely, in the limit $N \rightarrow \infty$, solutions to (1.4) converge to the solution to the McKean-Vlasov SDE

$$
dX_t = \sigma(X_t, \mathcal{L}(X_t \mid W)) \circ dW, \quad X_t|_{t=0} = X_0,
$$

where $\mathcal{L}(X \mid W)$ is the conditional law of $X$ with respect to $W$, as explained in detail in (2.3) below. Given a solution $X$ to (1.5) its conditional law $\mathcal{L}(X \mid W)$ then satisfies (1.2).
Notably, all of the particles in (1.4) are subject to the same common noise. For this reason, no averaging effect with respect to this noise is observed and it thus survives in the limit \( N \to \infty \), leading to a stochastic PDE.

**Literature.** Stochastic scalar conservation laws have been the object of several studies. In the case that \( \sigma(x, u) = \sigma(u(x)) \), that is, coefficients \( \sigma \) depending on \( u \) in a local and spatially homogeneous way, this class of stochastic PDE was introduced by Lions, Perthame, Souganidis in \([30]\). For linear, spatially inhomogeneous coefficients, the well-posedness of entropy solutions was shown by Friz, Gess in \([17]\). The case of local, nonlinear coefficients was later generalized to spatially inhomogeneous coefficients by Lions, Perthame, Souganidis in \([31]\), Gess, Souganidis in \([20]\) and to include second order operators by Gess, Souganidis in \([21]\) and Fehrman, Gess in \([16]\). Qualitative properties of solutions, such as regularity and finite speed of propagation has been considered by Gassiat, Gess in \([18]\) and Gassiat, Gess, Lions, Souganidis in \([19]\).

In a recent article \([2]\), Barbu and Röckner treat McKean-Vlasov SDE when the dependence on the law is local, proving, roughly speaking, that if there is a solution to the scalar conservation law (1.2), then there also is a solution to the McKean-Vlasov equation (1.5).

The existence and uniqueness of solutions to deterministic non-linear Fokker-Planck equations of the form (1.1) with \( \sigma \equiv 0 \) has been recently studied by several authors in \([5–7, 33, 34]\).

As mentioned above, to the best of our knowledge, the uniqueness of solutions to non-local stochastic PDE of the type (1.1) is known only in the class of solutions \( \mu \) such that for each \( t > 0 \), \( u(t) \) is absolutely continuous with respect to Lebesgue measure and has a density in \( L^2(\mathbb{R}^d) \) (cf. Kurtz, Xiong \([25\text{, p. 115}]\)). Under more restrictive conditions, either on the class of solutions or on the coefficients of (1.1), the well-posedness of solutions to SPDE of the type (1.1) had been previously considered by Dawson, Vaillancourt in \([13]\), where the uniqueness of solutions has been obtained by several methods, e.g. by constructing a dual process, by coupling arguments and by the Krylov-Rozovskii "variational" approach to SPDE.

Motivated from fluid dynamics in vorticity form, also signed measure-valued solutions to SPDE of the type (1.1) have been considered in the literature. We refer to Rémilliard, Vaillancourt \([37]\), Kotelenez \([23]\), Kotelenez, Seadler \([24]\), Amirdjanova, Xiong \([1]\) and the references therein. Again, uniqueness of solutions was obtained only under more restrictive assumptions.

**Outline of the proof.** The proof of uniqueness of solution to (1.1) put forward in the present work relies on the well-posedness for the Lagrangian characteristics

\[
\begin{aligned}
\left\{\begin{array}{l}
dX_t = b(X_t, \mu_t)dt + \sigma(X_t, \mu_t)dW_t + \alpha(X_t, \mu_t)dB_t, \\
X_{t|t=0} = X_0,
\end{array}\right.
\end{aligned}
\]

where \( W \) and \( B \) are independent Brownian motions, \( X_0 \) is an independent random variable on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and \( \alpha := [2a - \sigma^T \sigma]^\frac{1}{2} \).

The proof of uniqueness of solutions to (1.1) proceeds by freezing the coefficients of the Lagrangian characteristics (1.1) and proving the uniqueness of solutions to the resulting linear equation

\[
\partial_t \bar{\mu} = \partial_{ij}^2 (\bar{a} \bar{u}^j(t, x) \bar{\mu}) - \partial_i (\bar{b}^i(t, x) \bar{\mu}) - \partial_i (\bar{\sigma}^{ik}(t, x) \bar{\mu}) dW_t^k.
\]
At this point, in contrary to the previous work [25], the uniqueness of measure-valued solutions to (1.7) has to be shown, while [25] was restricted to solutions allowing square-integrable densities. Here the above mentioned duality argument comes into play, on which we comment in more detail below. The uniqueness of solutions to (1.7) then implies that each solution \( \mu \) to (1.1) is given as the conditional law \( \mathcal{L}(X \mid W) \) of a solution to (1.6). Therefore, uniqueness to (1.6) implies the uniqueness for (1.1).

In order to prove the uniqueness of solutions to (1.7), we employ a duality argument, which leads to the backward stochastic PDE

\[
\partial_t f = -a^{i,j} \partial_{i,j}^2 f - b^i \partial_i f - \sigma^{i,k} \partial_i v^k + v^k dW^k, \quad f_T = \varphi,
\]

where the terminal condition is a sufficiently smooth random test function. We emphasize that in the case of stochastic scalar conservation laws (1.2), and equivalently (1.3) in Itô form, we have

\[
a^{i,j}(u) = \frac{1}{2} \sigma^{i,k}(u) \sigma^{k,j}(u),
\]

which implies that (1.8) is degenerate. For background on degenerate backward stochastic PDE we refer to [15,22,40,41]. In order to invoke the duality argument for measure-valued solutions, we require classical solutions to (1.8) which can be obtained based on [15] by Du, Tang, Zhang, and Sobolev embedding. It then follows (cf. Lemma 4.7 below) that

\[
\mathbb{E} \langle \mu_t, \varphi \rangle = \mathbb{E} \langle \mu_t, f_t \rangle = \mathbb{E} \langle \mu_0, f_0 \rangle,
\]

which implies the uniqueness of measure-valued solutions to (1.7). We also refer to Zhou [41] and Diehl, Friz, Stannat [14] for results on the duality of stochastic PDE and backward stochastic PDE.

**Structure of the paper.** In Section 2 we will set the notation. In Section 3 we analyze the Lagrangian dynamics. Section 4 is devoted to the proof of well-posedness of linear SPDEs using Holmgren principle. In Section 5 we prove well-posedness for the non-local SPDE (1.1).

### 2. Notations and assumptions

We fix two numbers \( d, d_1 \in \mathbb{N} \). We will use the following notational conventions for the indices: \( i, j \in \{1, \ldots, d\} \) and \( k \in \{1, \ldots, d_1\} \).

For any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \), we set

\[
D^{\alpha} = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \cdots \left( \frac{\partial}{\partial x_d} \right)^{\alpha_d}
\]

and \( |\alpha| = \alpha_1 + \cdots + \alpha_d \).

Let \( C_\infty^\infty \) and \( C^n \) be the set of infinitely differentiable differentiable real-valued functions of compact support defined on \( \mathbb{R}^d \) and the set of \( n \) times continuously differentiable functions on \( \mathbb{R}^d \) such that

\[
\| \varphi \|_{C^n} := \sum_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |D^{\alpha} \varphi| < +\infty.
\]
Let $\text{Lip}_1$ be the space of Lipschitz continuous functions in $C^0$, such that

$$\|\varphi\|_{C^0}, \sup_{x \neq y \in \mathbb{R}^d} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq 1.$$ 

For a function $f \in C([0, T]; \mathbb{R}^d)$ we call $\|f\|_{\infty}$ its supremum norm.

For $p > 1$ and an integer $m \geq 0$, we let $W^{m,p}_d = W^{m,p}(_d \mathbb{R}; \mathbb{R})$ be the Sobolev space of real-valued functions on $\mathbb{R}^d$ with finite norm

$$\|f\|_{W^{m,p}_d} := \left( \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} |D^\alpha f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$ 

In the same way, denote by $W^{m,p}(_d \mathbb{R}^1)$ the Sobolev space of $d_1$-dimensional vector-valued functions on $\mathbb{R}^d$, equipped with the norm

$$\|v\|_{W^{m,p}_d} := \left( \sum_{k=1}^{d_1} \|v^k\|_{W^{m,p}_d}^p \right)^{\frac{1}{p}} < \infty.$$ 

We call $\mathcal{M}^\pm(_d \mathbb{R})$ the space of finite signed measures on $\mathbb{R}^d$. On this space we define the total variation norm

$$\|\mu\|_{TV} = \sup_{\|\varphi\|_{C^0} \leq 1} \langle \mu, \varphi \rangle.$$ 

For $r > 0$, we define $\mathcal{M}^\pm_{\leq r} := \{ \mu \in \mathcal{M}^\pm(_d \mathbb{R}) \mid \|\mu\|_{TV} \leq r \}$. We call $\mathcal{M}(_d \mathbb{R}) \subset \mathcal{M}^\pm(_d \mathbb{R})$ (resp. $\mathcal{P}(_d \mathbb{R})$) the space of finite positive (resp. probability) measures on $\mathbb{R}^d$.

For $r > 0$, we call $\mathcal{M}_r(_d \mathbb{R})$ the space of measures in $\mathcal{M}(_d \mathbb{R})$ with total variation equal to $r$, namely

$$\mathcal{M}_r(_d \mathbb{R}) = \{ \mu \in \mathcal{M}(_d \mathbb{R}) \mid \|\mu\|_{TV} = r \}.$$ 

It is worth mentioning that $\mathcal{M}_1(_d \mathbb{R}) = \mathcal{P}(_d \mathbb{R})$. We endow $\mathcal{M}_r(_d \mathbb{R})$ with the Kantorovich-Rubinstein norm

$$\|\mu\| := \sup_{\varphi \in \text{Lip}_1} \left( \int_{\mathbb{R}^d} \varphi d\mu \right), \quad \forall \nu \in \mathcal{M}_r(_d \mathbb{R})$$ 

and let $\rho$ be the induced metric. On $\mathcal{M}_r$ we consider the Borel $\sigma$-algebra induced by $\rho$.

From [4, Theorem 8.3.2], we have that the metric $\rho$ metrizes the weak convergence of measures. Moreover, the space $(\mathcal{M}(_d \mathbb{R}), \rho)$ is complete and separable, see [4, Theorem 8.9.4].

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, let $W$ be a $d_1$-dimensional Brownian motion. Let $(\mathcal{W}^s_t)_{t \geq s}$ be the completion of the filtration generated by the increments of $W$ starting from $s$, namely the completion of

$$\mathcal{W}^s_t := \sigma(W_r - W_s : s \leq r \leq t), \quad \forall s \leq t \in [0, \infty).$$ 

To simplify the notation, we omit the dependence from the starting time when it is zero, i.e. $\mathcal{W}_t := \mathcal{W}^0_t$. Moreover, we set $\mathcal{W} := \bigvee_{t \geq 0} \mathcal{W}_t$. It follows from the independence of the
For $L^2$-integrable, $(\mathcal{F}_t)_{t \geq 0}$-measurable processes $\mu : [0, T] \times \Omega \to \mathbb{R}^d$, there is a subsequence $(\mu^{n_k})_{k \in \mathbb{N}}$ which is almost surely a Cauchy sequence in $(\mathcal{M}_r, \| \cdot \|)$. Since $\mathcal{M}_r$ is complete, there exists a null set $N \subset \Omega$, such that, for all $\omega \in \mathcal{N}^c$, there exists $\mu(\omega) \in \mathcal{M}_r$ such that $\|\mu^{n_k}(\omega) - \mu(\omega)\| \to 0$ as $k \to \infty$. Adaptedness and joint measurability of $\mu$ follows from the respective properties of $\mu^{n_k}$. Since the norm $\| \cdot \|$ is bounded, dominated convergence concludes the argument.

Denote by $L^1_{\omega \times t} \mathcal{M}_r$ the space of $(\mathcal{W}_t)_{t \geq 0}$-adapted, $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F})$-measurable processes $\mu : [0, T] \times \Omega \to \mathcal{M}_r(\mathbb{R}^d)$ such that

$$\mathbb{E} \left[ \int_0^T \|\mu_t\| dt \right] < +\infty.$$ 

Remark 2.4. The space $L^1_{\omega \times t} \mathcal{M}_r$ is complete. Indeed: Given a Cauchy sequence $(\mu^n)_{n \in \mathbb{N}} \subset L^1_{\omega \times t} \mathcal{M}_r$ there is a subsequence $(\mu^{n_k})_{k \in \mathbb{N}}$ which is almost surely a Cauchy sequence in $(\mathcal{M}_r, \| \cdot \|)$. Since $\mathcal{M}_r$ is complete, there exists a null set $N \subset \Omega$, such that, for all $\omega \in \mathcal{N}^c$, there exists $\mu(\omega) \in \mathcal{M}_r$ such that $\|\mu^{n_k}(\omega) - \mu(\omega)\| \to 0$ as $k \to \infty$. Adaptedness and joint measurability of $\mu$ follows from the respective properties of $\mu^{n_k}$. Since the norm $\| \cdot \|$ is bounded, dominated convergence concludes the argument.

Denote by $L^1_{\omega \times t} \mathcal{C}_t \mathcal{M}_r$ the space of $(\mathcal{W}_t)_{t \geq 0}$-adapted continuous processes $\mu : [0, T] \times \Omega \to \mathcal{M}_r(\mathbb{R}^d)$ such that there exists $\nu \in \mathcal{M}_r$ that satisfies

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \|\mu_t\| \right] < +\infty.$$ 

Notice that $L^1_{\omega \times t} \mathcal{C}_t \mathcal{M}_r \subset L^1_{\omega \times t} \mathcal{M}_r$.

For $p \geq 1$, denote by $L^p_{\omega \times t} \mathcal{F}$ the space of $p$-integrable, $(\mathcal{F}_t)_{t \geq 0}$-adapted stochastic processes on $\mathbb{R}^d$. We denote by $C_t L^1_{\omega} := C([0,T]; L^1(\Omega, \mathbb{R}^d))$. 

This is a consequence of Lemma 2.1. Moreover, we will always assume that the space $(\Omega, \mathcal{F}_0, \mathbb{P})$ is atomless. This implies that, given a metric space $E$ and a probability $\mu \in \mathcal{P}(E)$, we can always construct a random variable $X : \Omega \to E$, with $X = \mu$. See [3, Proposition 9.1.11]
3. McKean-Vlasov stochastic differential equation

In this section we discuss the well-posedness of a McKean-Vlasov SDE. Let \( a, b \) and \( \sigma \) be measurable functions as in the introduction. Throughout this section the following assumptions are in force.

**Assumptions 3.1.** There is an \( r > 0 \) such that

(i) (Uniform Lipschitz continuity) There exists a constant \( K > 0 \) such that
\[
\|a(t,x,\mu) - a(t,x',\mu')\| + \|\sigma(t,x,\mu) - \sigma(t,x',\mu')\| + |b(t,x,\mu) - b(t,x',\mu')| \\
\leq K \left( |x - x'| + \rho(\mu,\mu') \right),
\]
for all \( \mu, \mu' \in \mathcal{M}_r(\mathbb{R}^d) \), \( t \in [0,T] \) and \( x, x' \in \mathbb{R}^d \).

(ii) (Uniform boundedness) There exists a constant \( K > 0 \) such that
\[
\|a(t,x,\mu)\| + \|\sigma(t,x,\mu)\| + |b(t,x,\mu)| \leq K,
\]
for all \( \mu \in \mathcal{M}_r(\mathbb{R}^d) \), \( t \in [0,T] \) and \( x \in \mathbb{R}^d \).

(iii) (Parabolicity) For each \( (t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{M}_r(\mathbb{R}^d) \),
\[
[2a^{ij}(t,x,\mu) - \sigma^i \sigma^j(t,x,\mu)] \xi^i \xi^j \geq 0, \quad \forall \xi \in \mathbb{R}^d.
\]

From now on assume that Assumption 3.1 is satisfied. Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a filtered probability space and \( W \) a \( d \)-dimensional Brownian motion on this space, which is compatible with \( (\mathcal{F}_t)_{t \geq 0} \) in the sense of Definition 2.1. Let \( B \) be a \( d \)-dimensional \( (\mathcal{F}_t)_{t \geq 0} \)-adapted Brownian motion independent of \( W \). Moreover, assume that \( X_0 : \Omega \rightarrow \mathbb{R}^d \) is an \( \mathcal{F}_0 \)-measurable random variable. Notice that \( X_0 \) is independent of \( W \) and \( B \).

We set
\[
(3.1) \quad \alpha(t,x,\mu) := \left[ 2a(t,x,\mu) - \sigma^T(t,x,\mu) \sigma(t,x,\mu) \right]^{\frac{1}{2}}, \quad \forall t \in [0,T], \ x \in \mathbb{R}^d, \ \mu \in \mathcal{M}(\mathbb{R}^d).
\]

It follows from Assumption 3.1(iii) that \( \alpha(t,x,\mu) \) is well defined as a symmetric matrix. Moreover, \( \alpha \) is Lipschitz continuous and bounded in its variables \( \mu \) and \( x \), namely, there exists a constant \( K > 0 \), possibly different than before, such that for all \( t \in [0,T], \ x, x' \in \mathbb{R}^d \) and \( \mu, \mu' \in \mathcal{M}_r(\mathbb{R}^d) \),
\[
\|\alpha(t,x,\mu) - \alpha(t,x',\mu')\| \leq K \left( |x - x'| + \rho(\mu,\mu') \right).
\]

We consider the McKean-Vlasov SDE
\[
(3.2) \quad \begin{cases} 
\displaystyle dX_t = b(t,X_t,\mu_t)dt + \sigma(t,X_t,\mu_t)dB_t + \alpha(t,X_t,\mu_t)dW_t, \\
X_{t=0} = X_0, \\
\mu_t := r\mathcal{L}(X_t \mid W),
\end{cases}
\]

**Definition 3.2.** Let \( X_0 : \Omega \rightarrow \mathbb{R}^d \), be \( \mathcal{F}_0 \)-measurable, \( r > 0 \) and define \( \mu_0 := r\mathcal{L}(X_0) \).

We say that a stochastic process \( (X_t,\mu) : [0,T] \times \Omega \rightarrow \mathbb{R}^d \times \mathcal{M}_r \) is a solution to the McKean-Vlasov equation (3.2) with initial condition \( X_0 \), if

i. \( X \) is \( (\mathcal{F}_t)_{t \in [0,T]} \)-adapted and time-continuous.

ii. \( \mu \in L^1_{\omega,t}\mathcal{M}_r \) and for all \( t \in [0,T] \),
\[
\mu_t = r\mathcal{L}(X_t \mid W), \quad \mathbb{P} - a.s.
\]
iii. The following integral equation is satisfied, namely, \( \forall t \in [0, T], \)
\[
X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dW_s + \int_0^t \alpha(s, X_s, \mu_s)dB_s, \quad P \text{-a.s.}
\]

We obtain the following well-posedness result for equation (3.2).

**Theorem 3.3.** Fix \( r > 0 \) and assume Assumption 3.1. Let \( X_0 : \Omega \rightarrow \mathbb{R}^d \), be \( \mathcal{F}_0 \)-measurable. Then, there exists a unique solution \((X, \mu)\) to equation (3.2) in the sense of Definition 3.2. In addition, the solution satisfies \( \mu \in L^1_{\omega,t} \mathcal{M}_r \).

**Proof.** This well-posedness result is a direct consequence of [25, Theorem 2.3]. However, we provide an alternate proof here avoiding the infinite interacting particle system used in [25], but rather studying equation (3.2) directly. To prove the existence of a solution we start with a stochastic process \( \mu \in L^1_{\omega,t} \mathcal{M}_r \) and we freeze the coefficients in (3.2), to obtain the following equation

\[
(3.3) \quad \begin{cases}
    dX_t^\mu = b(t, X_t^\mu, \mu_t)dt + \sigma(t, X_t^\mu, \mu_t)dW_t + \alpha(t, X_t^\mu, \mu_t)dB_t \\
    X_0^\mu = X_0.
\end{cases}
\]

The coefficients \( b(t, x, \mu_t), \sigma(t, x, \mu_t), \alpha(t, x, \mu_t) \) are progressively measurable, Lipschitz continuous and bounded. Hence, there exists a unique time-continuous \((\mathcal{F}_t)_{t \geq 0}\)-adapted solution \((X_t^\mu)_{t \in [0, T]}\) to equation (3.3), see [32, Theorem 3.1.1].

We define the following operator

\[
(3.4) \quad \Phi : L^1_{\omega,t} \mathcal{M}_r \rightarrow L^1_{\omega,t} \mathcal{M}_r
\]

and we will prove that its iterates \( \Phi^k \) for \( k \) large enough are contractions with respect to the metric

\[
d(\mu, \nu) := \mathbb{E} \left[ \int_0^T \rho(\mu_t, \nu_t)dt \right], \quad \forall \mu, \nu \in L^1_{\omega,t} \mathcal{M}_r.
\]

Let \( s, t \in [0, T], p \in [1, \infty), \) and \( \mu \in L^1_{\omega,t} \mathcal{M}_r, \) we have

\[
\mathbb{E} \rho(\Phi(\mu)_t, \Phi(\mu)_s)^p \leq r \mathbb{E}|X_t^\mu - X_s^\mu|^p.
\]

Standard estimates on the solutions of SDEs and Kolmogorov’s continuity theorem imply that the process \( \Phi(\mu) \) has a modification which is time continuous with respect to the weak topology, which is induced by \( \rho. \) Hence, \( \Phi(\mu) \in L^1_{\omega,C_t} \mathcal{M}_r \subset L^1_{\omega,t} \mathcal{M}_r. \)

We proceed by proving that \( \Phi \) is a contraction on \((L^1_{\omega,t} \mathcal{M}_r, d)\). By the definition of the Kantorovich-Rubinstein metric and using the conditional Jensen inequality, we have for each \( t \in [0, T], \)

\[
\mathbb{E} \int_0^t \rho(\Phi(\mu)_s, \Phi(\nu)_s)ds = \mathbb{E} \left[ \int_0^t \sup_{\varphi \in \text{Lip}_1} \mathbb{E} \left[ r\varphi(X_s^\mu) - r\varphi(X_s^\nu) \mid \mathcal{W} \right] ds \right] \leq r \int_0^t \mathbb{E}|X_s^\mu - X_s^\nu|ds, \quad \forall \mu, \nu \in L^1_{\omega,t} \mathcal{M}_r.
\]
Using standard estimates for the solutions of SDEs, Lemma 3.2, the Burkholder-Davis-Gundy inequality, Assumption 3.1(i) and Gronwall’s Lemma, we have
\[ \mathbb{E}|X^\mu_t - X^\nu_t| \leq e^{CT} \int_0^t \mathbb{E}\rho(\mu_s, \nu_s)ds. \]

Hence,
\[ \mathbb{E} \int_0^t \rho(\Phi(\mu)_s, \Phi(\nu)_s)ds \leq re^{CT} \int_0^t \int_0^s \mathbb{E}\rho(\Phi(\mu)_r, \Phi(\nu)_r)dr ds, \]
where the constant \( C > 0 \) depends only on \( r \) and \( K \) as given in Assumption 3.1. Iterating the operator \( \Phi \) \( k \)-times, yields the following inequality
\[ \mathbb{E} \int_0^T \rho(\Phi^k(\mu)_t, \Phi^k(\nu)_t)dt \leq \int_0^T \mathbb{E}|X_t^{\Phi^{k-1}(\mu)} - X_t^{\Phi^{k-1}(\nu)}|dt \leq e^{CT} \int_0^T \mathbb{E}\rho(\mu_t, \nu_t)dt. \]

If \( k \) is large enough, the coefficient \( e^{kCT}/(k-1)! \) is less then one. Hence, \( \Phi^k \) is a contraction on \( L^1_{\omega, t, \mathcal{M}_t} \) and thus has a unique fixed point. This fixed point is also the unique fixed point of \( \Phi \), see [11, Prop 2.3]. Since solutions to (3.2) are precisely the fixed points of \( \Phi \), this yields the existence and uniqueness of solutions to the McKean-Vlasov equation (3.2). \( \square \)

**Remark 3.4.** We note that, under more restrictive assumptions on the coefficients, the conditional law \( \mu = \mathcal{L}(X | W) \) of a solution \( X \) to (3.2) does not depend on \( X_0 \) but only on \( \mu_0 := \mathcal{L}(X_0) \). This follows from the results proved later in Section 5. Indeed, assuming Assumption 5.1 with \( m > d + 2 \), Theorem 5.3 implies that \( \mu \) is a solution to equation (1.1) in the sense of Definition 5.2. Thanks to Theorem 5.4, this solution is unique, given the initial law \( \mu_0 \). This implies that \( \mu \) only depends on \( X_0 \) via its law \( \mu_0 = \mathcal{L}(X_0) \).

### 3.1. Remarks on the associated interacting particle system.

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space. Let \( W \) be an \((\mathcal{F}_t)_{t \geq 0}\)-compatible Brownian motion and \((X^t_0)_{t \geq 0}\) be a sequence of independent and identically distributed (IID) random variables in \( L^2(\Omega, \mathcal{F}_0; \mathbb{R}^d) \) with law \( \mu_0 \). Moreover, consider a sequence of independent \((\mathcal{F}_t)_{t \geq 0}\)-adapted Brownian motions \((B^i_t)_{t \geq 0}\), which are jointly independent of \( W \) and \((X^t_0)_{t \geq 0}\).

Consider the following system of interacting particles on \( \mathbb{R}^d, \forall t \in [0, T], i = 1, \ldots, N, \)
\[
\begin{cases}
dX^i_{t,N} = b(t, X^i_{t,N}, L^i_t)dt + \sigma(t, X^i_{t,N}, L^i_t)dW_t + \alpha(t, X^i_{t,N}, L^i_t)dB^i_t \\
X^{i, N}_0 = X^i_0, 
\end{cases}
\]

where \( L^i_N := \frac{1}{N} \sum_{i=1}^N \delta X^i_{t,N} \) is the empirical measure of the system.

In this section, we work under the following additional assumption.
Assumptions 3.5. There is a constant $K > 0$ such that, for any IID sequence of random variables $(X^1_i)_{i \geq 0}$ on $\mathbb{R}^d$, with law $\mu$, the following holds, for every $x \in \mathbb{R}^d$,

$$
\mathbb{E} |\sigma(x, L^N) - \sigma(x, \mu)|^2 + \mathbb{E} |b(x, L^N) - b(x, \mu)|^2 + \mathbb{E} |\alpha(x, L^N) - \alpha(x, \mu)|^2 \leq \frac{K^2}{N},
$$

where $L^N := \frac{1}{N} \sum_{i=1}^N \delta_{X^i}$.

It is proved in [26, Theorem 2.3] that each particle $X^i_t$ converges to a solution $X^i_t$ of equation (3.2) with initial condition $X^i_0$ and driving noise $W$ and $B$, in the sense, that, for each $i \geq 0$ and $T > 0$,

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |X^i_{t,N} - X^i_t|^2 \right] \leq \frac{C(T)}{N}.
$$

Moreover, from [26, Corollary 2.4], the empirical measure $L^N_t$ converges, as $N \to \infty$, to the conditional law $\mu_t = \mathcal{L}(X^i_t \mid \mathcal{W})$, that is, for each $\varphi \in \text{Lip}_1$, $t \in [0, T]$,

$$(3.5) \quad \mathbb{E}[\langle L^N_t, \varphi \rangle - \langle \mu_t, \varphi \rangle] \leq \frac{C(t)}{\sqrt{N}}.$$

Notice that here $X^1$ is not special, we could define $\mu_t = \mathcal{L}(X^i_t \mid \mathcal{W})$, for any $i \geq 0$, and have the same result. Moreover, we will see in Section 4 that $\mu$ is the solution to equation (1.1) as given by Theorem 5.3 below.

A result of propagation of chaos, similar to the one stated in [39] can be obtained. In this case, however, the propagation of chaos is conditional to the common noise $W$.

Lemma 3.6. The interacting particles $(X^i_{t,N})_{i=1, \ldots, N}$ are $\mu$ chaotic, conditional to $\mathcal{W}$, in the sense that, for $k \in \mathbb{N}$, and $\varphi^1, \ldots, \varphi^k \in \varphi \in \text{Lip}_1$, we have

$$
\lim_{N \to \infty} \mathbb{E} \left[ \varphi^1(X^1_{t,N}) \cdot \ldots \cdot \varphi^k(X^k_{t,N}) \mid \mathcal{W} \right] - \prod_{i=1}^k \langle \mu_t, \varphi^i \rangle = 0, \quad \forall t \in [0, T].
$$

Proof. Without loss of generality, assume $k = 2$. First notice that, for $i \neq j$, $\forall t \in [0, T]$, $X^i_t$ is independent of $X^j_t$, conditionally to $\mathcal{W}$, which implies,

$$
\langle \mu_t, \varphi^1 \rangle \langle \mu_t, \varphi^2 \rangle = \mathbb{E} \left[ \varphi^1(X^i_t)\varphi^2(X^j_t) \mid \mathcal{W} \right] \quad \mathbb{P} - a.s.,
$$

where we used that, for every $i \geq 1$, $\mu_t = \mathcal{L}(X^i_t \mid \mathcal{W})$. Moreover, the particles are exchangeable, even when conditioned to $\mathcal{W}$, $\forall i \neq j$, $\forall t \in [0, T]$,

$$
\mathcal{L}(X^i_{t,N}, X^j_{t,N}) \mid \mathcal{W} = \mathcal{L}(X^i_{t,N}, X^j_{t,N}) \mid \mathcal{W}, \quad \mathcal{L}(X^i_{t,N}, X^j_{t,N}) \mid \mathcal{W} = \mathcal{L}(X^j_{t,N}, X^i_{t,N}).
$$
Brownian motion

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space, compatible with a \(d_1\)-dimensional Brownian motion \(W\) in the sense of Definition 2.1.

In this section we study the well-posedness of solutions to the linear version of (1.1), that is, the SPDE,

\[
\partial_t \mu = \partial_{ij}^2 \left( a^{ij}(x, \omega) \right) - \partial_i (b^i(x, \omega)) - \partial_i (\sigma^{ik}(x, \omega) dW_k^i),
\]

where \(a(t, x, \omega), \sigma(t, x, \omega)\) and \(b(t, x, \omega)\) satisfy the following assumptions

**Assumptions 4.1.** Let \(m \in \mathbb{N} \cup \{0\}\).

(i) The function \(a(t, x, \omega) := (a^{ij}(t, x, \omega)) : \mathbb{R}^+ \times \mathbb{R}^d \times \Omega \to \mathbb{S}^d\) is measurable and \((W_t)_{t \geq 0}\)-adapted. Moreover, there exists a positive constant \(K_m\) such that for all
(t, \omega) \in \mathbb{R}_+ \times \Omega$, $a(t, \cdot, \omega) \in C^m(\mathbb{R}^d; \mathbb{S}^d)$ (the set of $m$-times bounded differentiable functions on the space of real symmetric $d \times d$ matrices) and
\[
\sup_{t, \omega} \|a(t, \cdot, \omega)\|_{C^m} \leq K_m.
\]

(ii) The function $b(t, x, \omega) := (b^i(t, x, \omega)) : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is measurable and $(\mathcal{W}_t)_{t \geq 0}$-adapted. Moreover, there exists a positive constant $K_m$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $b(t, \cdot, \omega) \in C^m(\mathbb{R}^d; \mathbb{R}^d)$ and
\[
\sup_{t, \omega} \|b(t, \cdot, \omega)\|_{C^m} \leq K_m.
\]

(iii) The function $\sigma(t, x, \omega) := (\sigma^{i,k}(t, x, \omega)) : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}^{d \times d_1}$ is measurable and $(\mathcal{W}_t)_{t \geq 0}$-adapted. Moreover, there exists a positive constant $K_m$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $\sigma(t, \cdot, \omega) \in C^m(\mathbb{R}^d; \mathbb{R}^{d \times d_1})$ and
\[
\sup_{t, \omega} \|\sigma(t, \cdot, \omega)\|_{C^m} \leq K_m.
\]

(iv) (Uniform Lipschitz continuity) There exists a constant $K > 0$ such that, for $t \in [0, T]$, $x, x' \in \mathbb{R}^d$ and $\omega \in \Omega$,
\[
\|a(t, x, \omega) - a(t, x', \omega)\| + \|\sigma(t, x, \omega) - \sigma(t, x', \omega)\| + |b(t, x, \omega) - b(t, x', \omega)| \leq K|x - x'|.
\]

(v) (Parabolicity) For each $(t, x, \omega) \in [0, T] \times \mathbb{R}^d \times \Omega$,
\[
[2a^{ij}(t, x, \omega) - \sigma^{i,k}\sigma^{j,k}(t, x, \omega)]\xi^i \xi^j \geq 0, \quad \forall \xi \in \mathbb{R}^d.
\]

Remark 4.2. Assumption \[4.1 (iv)\] is implied by Assumptions (i)-(iii), if $m \geq 1$.

In the following we fix $r > 0$ and we assume that Assumption \[4.1 (iv)\] is satisfied with $m = 0$.

Definition 4.3. We say that $\mu \in L^1_{\omega,t} \mathcal{M}_r$ is a solution to equation \[4.1\] with initial condition $\mu_0 \in \mathcal{M}_r(\mathbb{R}^d)$, if for every $\varphi \in C^2(\mathbb{R}^d)$ and $t \in [0, T]$ there exists a set of full measure $\Omega' \subset \Omega$ on which the following integral equation is satisfied,
\[
\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, a_{i,j} \partial_{i,j} \varphi \rangle ds + \int_0^t \langle \mu_s, b^i \partial_i \varphi \rangle ds + \int_0^t \langle \mu_s, \sigma^{i,k} \partial_i \varphi \rangle dW^k_s.
\]

Remark 4.4. We note that that all the terms in the right-hand side of \[4.2\] are well-defined, because the coefficients $a, b, \sigma$ are $(\mathcal{W}_t)_{t \geq 0}$-adapted and bounded and each $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F})$-measurable, $(\mathcal{W}_t)_{t \geq 0}$-adapted process has a predictable $dt \otimes \mathbb{P}$-version [10, Theorem 3.8].

We next consider the existence of solutions to the linear equation \[4.1\]. Consider the linear version of system \[3.2\], that is
\[
\left\{
\begin{array}{l}
dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \alpha(t, X_t)dB_t \\
X_{t\mid_{t=0}} = X_0,
\end{array}
\right.
\]

where the coefficients are fixed, $(\mathcal{F}_t)_{t \geq 0}$-adapted stochastic processes satisfying Assumptions \[4.1\] with $m = 0$. $X_0 : \Omega \to \mathbb{R}^d$ is an $\mathcal{F}_0$-measurable random variable. As we noted in the proof of Theorem \[5.3\] there exists a unique time-continuous $(\mathcal{F}_t)_{t \geq 0}$-adapted solution $(X_t)_{t \in [0, T]}$ to equation \[5.3\], see [32, Theorem 3.1.1]. Using this solution, the following lemma can be proved in the same way as Theorem \[5.3\] below.
To prove the uniqueness of solutions to the linear equation (4.1), we introduce the dual BSPDE. We fix $t \in [0, T]$ and we take a test function $\varphi$ which is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$-measurable with $\varphi \in L^\infty(\Omega, \mathcal{F}_t, C_c^\infty(\mathbb{R}^d))$. Consider the following BSPDE

$$
\begin{align*}
\partial_t f &= -a^{ij} \partial_{ij} f - b^i \partial_i f - \sigma^{ik} \partial_i v^k + v^k dW^k
\end{align*}
$$

on $\mathbb{R}^d \times [0,t]$.

**Definition 4.6.** Let $t \in [0, T]$, $m \geq 1$ and $\varphi \in L^\infty(\Omega, \mathbb{W}_t, C_c^\infty(\mathbb{R}^d))$. A process $(f, v) : [0, t] \times \Omega \to W^{2,m} \times W^{1,m}$ is a solution to equation (4.4) with terminal condition $\varphi$, if it is progressively measurable and if there is a process $(\mu, \omega) \in \Omega'$ of full measure such that

$$
f(s, x) = \varphi(x) + \int_s^t \left( \begin{array}{c}
\partial_t f - a^{ij} \partial_{ij} f - b^i \partial_i f + \sigma^{ik} \partial_i v^k + v^k dW^k
\end{array} \right) (r, x) \, dr - \int_s^t v^k(r, x) \, dW^k_r
$$

for all $\omega \in \Omega'$, $s \in [0, t]$ and $x \in \mathbb{R}^d$.

Let $m$ be an integer such that $2(m - 2) > d$. If the coefficients $b, \sigma, a$ satisfy Assumptions 4.1, it follows from [13, Theorem 2.1, Corollary 2.2] that there exists a pair of random fields $(f, v)$ such that

$$
f \in L^2_{\omega,t} C^{2,0}_t W^{m,2}_x, \quad \sigma \cdot \nabla f + v \in L^2_{\omega,t} W^{m,2}_x,
$$

which is jointly continuous in $(t, x)$ and is a strong solution to equation (4.4) in the sense of Definition 4.6.

By the assumptions on $m$ and the Sobolev embedding theorem it follows that

$$
f, \sigma \cdot \nabla f + v \in L^2_{\omega,t} C^2_x(\mathbb{R}^d),
$$

which implies that $v \in L^2_{\omega,t} C^1_x(\mathbb{R}^d)$.

We can now show the duality between equations (4.1) and (4.4).

**Lemma 4.7.** Fix $t \in [0, T]$, let $m$ be an integer such that $2(m - 2) > d$ and let $b, \sigma, a$ be (\(\mathcal{W}_t\))-adapted processes satisfying Assumptions 4.1.

Let $\mu : [0, T] \times \Omega \to \mathcal{M}_{L^2_{\omega,2r}}$ be an (\(\mathcal{W}_t\))-adapted process, such that for every $\varphi \in C^2(\mathbb{R}^d)$ and $t \in [0, T]$ there exists a set of full measure $\Omega' \subset \Omega$ on which $\mu$ satisfies the integral equation (4.2).

If $f$ is a solution to equation (4.4) in the sense of Definition 4.6 with terminal condition $f_t = \varphi \in L^\infty(\Omega, \mathbb{W}_t, C_c^\infty(\mathbb{R}^d))$, then

$$
\mathbb{E}(\mu_t, \varphi) = \mathbb{E}(\mu_0, f_0).
$$

**Proof.** Let $\eta_t$ be a standard mollifier, i.e., $\eta_t(x) := \frac{1}{t} \eta(\frac{x}{t})$, with

$$
\eta \in C_c^\infty(\mathbb{R}^d), \quad \eta \geq 0, \quad \int_{\mathbb{R}^d} \eta(x) \, dx = 1.
$$
Hence, we can apply dominated convergence to conclude

\[ \mu_\epsilon^t(x) = \mu_0^t(x) - \int_0^t \langle \mu_s, \sigma_s \partial_x, \eta_s^\epsilon \rangle dW_s^k + \int_0^t \langle \mu_s, a_s \partial_x, \eta_s^\epsilon - b_s \partial_x, \eta_s^\epsilon \rangle ds, \quad \forall \omega \in \Omega_{x,\epsilon}. \]

Using Itô’s formula we can compute the product \( \mu_\epsilon^t(x)f_t(x) \) and obtain the following equality on a set of full measure \( \Omega_{x,\epsilon} \), possibly different from the previous one,

\[
\begin{align*}
(4.7) \quad & \mu_\epsilon^t(x)f_t(x) = \mu_0^t(x)f_0(x) - \int_0^t \langle \mu_s, \sigma_s \partial_x, \eta_s^\epsilon \rangle f_s(x)dW_s^k + \int_0^t \mu_s(x)v_s^k(x)dW_s^k \\
(4.8) \quad & \quad + \int_0^t \langle \mu_s, a_s \partial_x, \eta_s^\epsilon - b_s \partial_x, \eta_s^\epsilon \rangle f_s(x)ds \\
(4.9) \quad & \quad - \int_0^t \mu_s^\epsilon(x) \left[ a_s \partial_x, f_s \right] (x)ds \\
(4.10) \quad & \quad - \int_0^t \mu_s^\epsilon(x) \left[ \sigma_s \partial_x, v_s^k \right] (x)ds - \int_0^t \mu_s^\epsilon(x) [\sigma_s \partial_x, v_s^k](x)ds.
\end{align*}
\]

For every \( x \), we can take the expectation of both sides of \( (4.7) - (4.10) \) and obtain an equality on all of \( \mathbb{R}^d \). We notice that the Itô integrals in \( (4.7) \) are martingales because their arguments are in \( L^2_\omega \), which is a consequence of \( (4.6) \). It follows that both martingale terms vanish in expectation.

Next, we integrate over \( \mathbb{R}^d \), use Fubini’s theorem to interchange Lebesgue integration and expectation, and take the limit \( \epsilon \to 0 \). It remains to identify the limit as \( \epsilon \to 0 \) of each of the resulting terms:

\[
\mathbb{E}[\mu_\epsilon^t] - \mathbb{E}[\mu_t] = \mathbb{E} \left[ \int_{\mathbb{R}^d} \eta^\epsilon(x) - \eta(x)f_t(x)d\mu_t(y)dx - \int_{\mathbb{R}^d} f_t(x)d\mu_t(x) \right] \\
= \mathbb{E} \int_{\mathbb{R}^d} [(\eta^\epsilon * f_t)(x) - f_t(x)]d\mu_t(x).
\]

From the regularity of \( f \) and \( v \), namely \( (4.6) \), it follows that the integration of \( f_t(x) \) with respect to \( \mu(dx) \) is well defined. The joint time-space continuity of \( f \) together with the maximum principle for the solution of the backward equation \( (4.4) \), which is proved in \( [15] \) Corollary 2.3, imply that there exists a constant \( C > 0 \), such that, for all \( t \in [0,T] \), \( x \in \mathbb{R}^d \) and for almost all \( \omega \in \Omega \),

\[ |f_t(x,\omega)| \leq C. \]

Hence, we can apply dominated convergence to conclude

\[ \mathbb{E} \int_{\mathbb{R}^d} [(\eta^\epsilon * f_t)(x) - f_t(x)]d\mu_t(x) \to 0, \quad \text{as} \ \epsilon \to 0. \]
The same argument can be applied at time $t = 0$. We next study the convergence of (4.11).

$$
-E \int_0^t \langle \mu_s^\varepsilon, \sigma^i_k \partial x_i, v_s^k \rangle ds - E \int_0^t \langle \mu_s^\varepsilon, \sigma^i_k \partial x_i, v_s^k \rangle ds
$$

(4.11)  
where the last equality is satisfied when the initial conditions, $\mu_0^1$ and $\mu_0^2$, are the same. Hence, we have that $\langle \mu_t, \varphi \rangle = 0$ on a set of full measure. Since the function $t \mapsto \langle \mu_t, \varphi \rangle$ is continuous, we can easily see that the set of full measure only depends on $\varphi$ that is,

$$
\forall \varphi \in C^\infty_c(\mathbb{R}^d, \mathbb{R}_+), \exists \Omega_\varphi \subset \Omega, \forall t \in [0, T], \langle \mu_t, \varphi \rangle = 0.
$$

Let $R \in \mathbb{N}$, we call $B_R \subset \mathbb{R}^d$, the ball of radius $R$. We have that the space $C^\infty_c(B_R, \mathbb{R}_+)$ is separable. Hence,

$$
\forall R \in \mathbb{N}, \exists \Omega_R \subset \Omega, \forall \varphi \in C^\infty_c(B_R, \mathbb{R}_+), \forall t \in [0, T], \langle \mu_t, \varphi \rangle = 0.
$$

We conclude the proof by noticing that on the set of full measure $\bigcap_{R \in \mathbb{N}} \Omega_R$, $\langle \mu_t, \varphi \rangle = 0$, $\forall \varphi \in C^\infty_c(\mathbb{R}^d, \mathbb{R}_+), \forall t \in [0, T]$. \qed
5. Non-local Fokker-Planck equations

In this section we study the existence and uniqueness of solutions to equation (1.1). After giving the definition of a solution, we prove the existence via the McKean-Vlasov SDE (3.2). The uniqueness follows from the uniqueness of solutions to the linear equation and well-posedness of the McKean-Vlasov SDE.

We will use the following assumptions.

Assumptions 5.1. Let \( m \in \mathbb{N} \cup \{0\} \), assume

(i) The function \( a(t, x, \mu) := (a^1(t, x, \mu)) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{S}^d \) is measurable. Moreover, there exists a positive constant \( K_m \) such that for all \( (t, \mu) \in \mathbb{R}_+ \times \mathcal{M}(\mathbb{R}^d) \),
\[
\sup_{t,\mu} \|a(t, \cdot, \mu)\|_{C^m} \leq K_m.
\]

(ii) The function \( b(t, x, \mu) := (b^1(t, x, \mu)) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d \) is measurable. Moreover, there exists a positive constant \( K_m \) such that for all \( (t, \mu) \in \mathbb{R}_+ \times \mathcal{M}(\mathbb{R}^d) \),
\[
\sup_{t,\mu} \|b(t, \cdot, \mu)\|_{C^m} \leq K_m.
\]

(iii) The function \( \sigma(t, x, \mu) := (\sigma^{i,k} (t, x, \mu)) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^{d \times d_1} \) is measurable. Moreover, there exists a positive constant \( K_m \) such that for all \( (t, \mu) \in \mathbb{R}_+ \times \mathcal{M}(\mathbb{R}^d) \),
\[
\sup_{t,\mu} \|\sigma(t, \cdot, \mu)\|_{C^m} \leq K_m.
\]

(iv) (Uniform Lipschitz continuity) There exists a constant \( K > 0 \) such that, for each \( t \in [0, T] \) and \( (x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{M}_r(\mathbb{R}^d) \),
\[
\|a(t, x, \mu) - a(t, x', \mu')\| + \|\sigma(t, x, \mu) - \sigma(t, x', \mu')\| + |b(t, x, \mu) - b(t, x', \mu')| \\
\leq K \left( |x - x'| + \rho(\mu, \mu') \right),
\]
for all \( \mu, \mu' \in \mathcal{M}_r(\mathbb{R}^d) \), \( t \in [0, T] \) and \( x, x' \in \mathbb{R}^d \).

(v) (Parabolicity) For each \( (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{M}_r(\mathbb{R}^d) \),
\[
2a^{ij} (t, x, \mu) - \sigma^{ik} \sigma^{kj} (t, x, \mu) |\xi|^2 \geq 0, \quad \forall \xi \in \mathbb{R}^d.
\]

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered atomless probability space, compatible with a \( d_1 \)-dimensional Brownian motion \( W \) in the sense of Definition 2.1.

Assume Assumption 5.1 with any \( m = 0 \).

Definition 5.2. We say that \( \mu \in L^1(\Omega, \mathcal{M}_r) \) is a solution to equation (1.1) with initial condition \( \mu_0 \in \mathcal{M}_r(\mathbb{R}^d) \), if for every \( \varphi \in C^2(\mathbb{R}^d) \) and \( t \in [0, T] \), there exists a set of full measure \( \Omega^t \subset \Omega \) on which the following integral equation is satisfied,
\[
\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, a^{1,j}(s, \mu_s) \partial_{ij}^2 \varphi \rangle ds + \int_0^t \langle \mu_s, b^i(s, \mu_s) \partial_i \varphi \rangle ds \\
+ \int_0^t \langle \mu_s, \sigma^{i,k}(s, \mu_s) \partial_j \varphi \rangle dW^k_s.
\]
Under Assumption 4.1 all the integrals in the previous definition are well defined. Moreover, \((\mu_s, \sigma^{i,k}(t, \mu_s) \partial_i \varphi)\) is \((W_t)_{t \geq 0}\)-adapted and \((B(\mathbb{R}_+) \times \mathcal{F})\)-measurable which is enough for the Itô integral to be defined (see Remark 4.4).

**Theorem 5.3.** Let \(r > 0\) and \(\mu_0 \in \mathcal{M}(\mathbb{R}^d)\). If Assumptions 5.2 are in force with \(m = 0\) and if \((X, \mu)\) is a solution to equation (4.3) in the sense of Definition 5.2, with any initial condition \(X_0 : \Omega \to \mathbb{R}^d\) such that \(\mu_0 = r\mathcal{L}(X_0)\), then \(\mu = r\mathcal{L}(X \mid W) \in L^1_{\omega,t}\mathcal{M}(\mathbb{R}^d)\) is a solution to equation (1.1) in the sense of Definition 5.2, with initial condition \(\mu_0\).

**Proof.** Since the probability space \((\Omega, \mathcal{F}_0, \mathbb{P})\) is atomless, there exists an \(\mathcal{F}_0\)-measurable random variable \(X_0 : \Omega \to \mathbb{R}^d\) such that \(\mu_0 = r\mathcal{L}(X_0)\). Assumption 5.1 imply Assumption 3.1, we can apply Theorem 3.3 to get a solution \((X, \mu)\) to equation (3.2) with initial condition \(X_0\). Using Itô’s formula, we check that \(\mu\) solves equation (1.1) in a distributional sense. We have

\[
\varphi(X_t) = \varphi(X_0) + \int_0^t b^i(s, X_s, \mu_s) \partial_i \varphi(X_s) ds
\]

\[
+ \int_0^t \frac{1}{2} \left[ \alpha^{i,l} \alpha^{l,j} + \sigma^{i,k} \sigma^{k,j} \right] (s, X_s, \mu_s) \partial_{i,j}^2 \varphi(X_s) ds
\]

\[
+ \int_0^t \sigma^{i,k}(s, X_s, \mu_s) \partial_i \varphi(X_s) dW_s^k
\]

\[
+ \int_0^t \alpha^{i,j}(s, X_s, \mu_s) \partial_i \varphi(X_s) dB_s^j, \quad \mathbb{P} \text{- a.s.}
\]

By multiplying by \(r\) and taking the conditional expectation with respect to \(W\), we obtain equation (5.1). This follows from the definition of \(\alpha\) and Lemma 3.2.

It follows from the definition of \(\mu_t\) as solution of the McKean-Vlasov SDE and Theorem 3.4 that \(\mu \in L^1_{\omega,t}\mathcal{M}(\mathbb{R}^d)\).

We are ready to state the uniqueness result in the nonlinear case.

**Theorem 5.4.** Let \(r > 0\) and \(\mu_0 \in \mathcal{M}(\mathbb{R}^d)\). Let Assumptions 5.1 be satisfied with

\[
m > \frac{d}{2} + 2.
\]

Then, the solution \(\mu\) of equation (1.1) in the sense of Definition 5.2 is unique and it is given by \(\mu = (\mu_t)_{t \in [0,T]} := (r\mathcal{L}(X_t \mid W))_{t \in [0,T]}\), where \(X_t\) is a solution to equation (3.2) with initial condition \(X_0 : \Omega \to \mathbb{R}^d\) such that \(\mu_0 = r\mathcal{L}(X_0)\).

**Proof.** Let \(\mu \in L^1_{\omega,t}\mathcal{M}(\mathbb{R}^d)\) be a solution to equation (1.1), and set \(\tilde{a}(t, x) := a(t, x, \mu_t), \tilde{\sigma}(t, x) := \sigma(t, x, \mu_t)\) and \(\tilde{b}(t, x) := b(t, x, \mu_t)\). We have that \(\tilde{a}, \tilde{\sigma}\) and \(\tilde{b}\) are \(B(\mathbb{R}_+) \times B(\mathbb{R}^d) \times \mathcal{F}\)-measurable and \((\mathcal{F}_t)_{t \in [0,T]}\)-adapted processes. It follows from Assumption 5.1 that Assumption 1.1 is satisfied by \(\tilde{a}, \tilde{b}, \tilde{\sigma}\).

Let \(X_0 : \Omega \to \mathbb{R}^d\) be an \(\mathcal{F}_0\)-measurable random variable such that \(\mu_0 = r\mathcal{L}(X_0)\). Let \(X : [0, T] \times \Omega \to \mathbb{R}^d\) be a time-continuous, \((\mathcal{F}_t)_{t \geq 0}\)-adapted solution to equation (4.3) with coefficients given by \((\tilde{b}, \tilde{\sigma}, [2\tilde{a} + (\tilde{\sigma})^T \tilde{\sigma}]^\frac{1}{2})\) and initial condition \(X_0\).
Clearly, $\mu$ is also a solution to the linear equation (4.1) with coefficients $(\bar{b}, \bar{\sigma}, \bar{a})$. Since $(\bar{b}, \bar{\sigma}, \bar{a})$ satisfy Assumption 4.1, Theorem 4.5 implies the uniqueness of solutions for the linear equation (4.1) in $L^1_{\omega,t}M_r$, which implies that $\mu$ corresponds to the solution given by Lemma 4.5, that is, $\mu = rL(X | W) \in L^1_{\omega}C_tM_r$.

This implies that the couple $(X, \mu)$ is a solution to the equation (3.2) in the sense of Definition 3.2.

Hence, the solutions of equation (1.1) are characterized as solutions of the McKean-Vlasov equation (3.2), in the sense that, $\mu \in L^1_{\omega,t}M_r$ is a solution to (1.1) in the sense of Definition 5.2, if and only if there exists an $(F_t)_{t \geq 0}$-adapted stochastic process $(X_t)_{t \in [0,T]}$, such that $\mu_t = rL(X_t | W)$ and the pair $(X, \mu)$ is a solution to the McKean-Vlasov equation (3.2) in the sense of Definition 3.2.

The uniqueness of solutions to (1.1) now follows from the uniqueness of solutions to the McKean-Vlasov SDE proven in Theorem 3.3.

\[ \square \]

**Appendix A. From Stochastic Scalar Conservation Laws to non-linear stochastic Fokker-Planck equations**

There is a rigorous way to rely the SSCL (1.2) to the Fokker-Planck equation (1.1) using the concept of the Lions derivative in the space $P_2(\mathbb{R}^d) := \{ \mu \in P(\mathbb{R}^d) \mid \langle \mu, \cdot, \cdot \rangle < +\infty \}$ of probability measures with finite second moment, endowed with the 2-Wasserstein distance. The results in this section are taken from [8,9]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be an atomless filtered probability space, compatible with a $d_1$-dimensional Brownian motion $W$. We study the following non-local scalar conservation law

\[ (A.1) \quad d\mu_t + \text{div}(\mu_t \sigma(x, \mu_t) \circ dW_t) = 0, \]

in the following sense

**Definition A.1.** We say that a stochastic process $\mu : [0, T] \times \Omega \to P_2(\mathbb{R}^d)$ is a solution to equation (A.1) with initial condition $\mu_0 \in P_2(\mathbb{R}^d)$, if for every $\varphi \in C^2(\mathbb{R}^d)$ the following conditions are satisfied:

i. The process $((\mu_t, \sigma^{i,k}(\mu_t) \partial_i \varphi))_{t \in [0,T]}$ is an $(\mathcal{W}_t)_{t \geq 0}$-adapted semimartingale;

ii. For Lebesgue-a.e. $t \in [0, T]$ there exists a set of full measure $\Omega' \subset \Omega$ on which the following integral equation is satisfied

\[ (A.2) \quad \langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, \sigma^{i,k}(\mu_s) \partial_i \varphi \rangle \circ dW^k_s. \]

**Assumptions A.2.** Let $m \in \mathbb{N}$ and assume

(i) (Lions-differentiability) The function $\sigma(x, \mu) := (\sigma^{i,k}(x, \mu)) : \mathbb{R}^d \times P(\mathbb{R}^d) \to \mathbb{R}^{d \times d_1}$ is measurable and satisfies Assumptions [8] (Joint Chain Rule Common Noise), p.279. In particular, $\sigma$ is twice Lions differentiable in the $\mu$ direction with first derivative $\partial_\mu \sigma : \mathbb{R}^d \times \mathbb{R}^d \times P(\mathbb{R}^d) \to \mathbb{R}^{d \times d_1 \times d_1}$. 
(ii) There exists a positive constant $K_m$ such that for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\sigma(\cdot, \mu) \in C^{m+1}(\mathbb{R}^d; \mathbb{R}^{d \times d})$, $\partial_\mu \sigma(\cdot, \mu) \in C^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^{d \times d \times d})$, and

$$\sup_{\mu} \|\sigma(\cdot, \mu)\|_{C^{m+1}} + \sup_{\mu} \|\partial_\mu \sigma(\cdot, \mu)\|_{C^m} \leq K_m.$$ 

(iii) (Uniform Lipschitz continuity) There exists a constant $K > 0$ such that, for all $(x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{M}_r(\mathbb{R}^d)$,

$$\|\sigma(x, \mu) - \sigma(x', \mu')\| \leq K \left(|x - x'| + \rho(\mu, \mu')\right),$$

Proposition A.3. Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and assume that $\sigma$ satisfies Assumption A.2 for some $m > \frac{d}{2} + 2$ and $\sigma$ is independent of the time variable. Then, there exists a $\mathcal{P}_2(\mathbb{R}^d)$-valued solution $\mu$ to equation (A.1) in the sense of Definition A.1

Moreover, $\mu$ is the solution to equation (1.1) with initial condition $\mu_0$ and coefficients $b$ and $\alpha$ given by

$$b(x, \mu) = (b(x, \mu))^i := \frac{1}{2} \sigma^{jk}(t, x, \mu) \partial_j \sigma^{ik}(x, \mu) + \frac{1}{2} G^i(x, \mu),$$

$$a(x, \mu) = (a^{ij}(x, \mu)) := \frac{1}{2} \sigma^{jk}(x, \mu) \sigma^{ik}(x, \mu),$$

with $G^i(x, \mu) = \langle \mu, \sigma^{jk}(\cdot, \mu) \partial_j \sigma^{ik}(\cdot) \rangle^{i,j,k}$.

Remark A.4. Under Assumption A.2 with $m \geq 1$, and thanks to [8, Remark 5.27] the functions $b, \sigma, a$ satisfy Assumption 5.1 with the same $m$.

Proof. Given a random variable $X_0 \in L^2(\Omega, \mathcal{F}_0; \mathbb{R}^d)$ with $\mu_0 = \mathcal{L}(X_0)$ consider the following McKean-Vlasov SDE

$$\begin{cases}
dX_t = b(X_t, \mu_t) dt + \sigma(X_t, \mu_t) dW_t, \\
X_{t=0} = X_0, \\
\mu_t := \mathcal{L}(X_t | \mathcal{W}).
\end{cases}$$

(A.5)

The function $\sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d \times d}$ is given. The coefficient $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ is constructed from $\sigma$ by (A.3). Since $(b, \sigma, a)$, with $a$ given by (A.4), satisfy Assumption 5.1, equation (A.5) is a special case of equation (B.2), with $\alpha := (2a - \sigma^T \sigma)^{\frac{1}{2}} = 0$.

If $(X, \mu_t := \mathcal{L}(X_t | \mathcal{W}))$ is the solution to the McKean-Vlasov equation (A.3), given by Theorem 8.3 with initial condition $X_0 \in L^2(\Omega, \mathcal{F}_0; \mathbb{R}^d)$, then it is easy to verify that $X_t \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d)$. Hence, $\mu_t(\omega) \in \mathcal{P}_2(\mathbb{R}^d)$, for $t \in [0, T]$ and a.e.-$\omega$.

Given a test function $\varphi \in C^2(\mathbb{R}^d)$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, define

$$u(\nu) = (u^k(\nu))_{k=1, \ldots, d} := \langle \nu, \sigma(\cdot, \nu) \partial_k \varphi(\cdot) \rangle.$$ 

(A.6)

It follows from Assumption A.2 that $u$ satisfies Assumption 9 (Simple $C^2$ Regularity), p.268 and we can apply [9, equation (4.28)]. We have

$$u^k(\mu_t) = u^k(\mu_0) + \int_0^t \langle \mu_s, \sigma^j(\cdot, \mu_s)(\partial_j u(\mu_s)(\cdot))^{jk} \rangle dW^j_s + r^k(t), \quad \mathbb{P} - \text{a.s.}$$

(A.7)

where $r(t)$ is a process of bounded variation.
It is shown in [8, equation (5.37)] that
\[
(\partial_\mu u(\nu)(\cdot))^j_k = \partial_j \sigma^{ik}(\cdot, \nu) \partial_i \varphi(\cdot) + \sigma^{ik}(\cdot, \nu) \partial^2_{ij} \varphi(\cdot) \\
\quad + \int_{\mathbb{R}^d} (\partial_\mu \sigma(x, \nu)(\cdot))^j_k \partial_i \varphi(\cdot) d\nu(x), \quad \forall \nu \in \mathcal{P}_2(\mathbb{R}^d).
\]
(A.8)

Due to Theorem 5.4, \(\mu\) solves equation (1.1), in the sense of Definition 5.2. Rewriting the weak formulation yields, \(\forall [0, T]\),
\[
\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, a^{ij}(\mu_s) \partial^2_{ij} \varphi \rangle ds + \int_0^t \langle \mu_s, b^i(\mu_s) \partial_i \varphi \rangle ds \\
\quad + \int_0^t \langle \mu_s, \sigma^{ik}(\mu_s) \partial_i \varphi \rangle dW^k_s, \quad \mathbb{P} - a.s.
\]

We rewrite the Itô integral in terms of a Stratonovich integral. Computing the quadratic covariation between \(u(\mu)\) and \(W\) (\(u\) is defined in (A.6)), it follows from (A.7) and (A.8) that the correction term is
\[
\frac{1}{2} \left[ u(\mu), W \right]_t = -\frac{1}{2} \langle \mu_t, \sigma^{ik}(\mu_t) \partial^2_{ij} \varphi \rangle - \frac{1}{2} \langle \mu_t, \sigma^{ik}(\mu_t) \partial_j \sigma^{ik}(\mu_t) \partial_i \varphi \rangle - \frac{1}{2} \langle \mu_t, G^i(\mu_t) \partial_i \varphi \rangle.
\]

By the definition of \(b\) and \(a\) and cancellations we obtain that \(\mu\) satisfies (A.2).

\section*{Appendix B. Remarks on conditional expectation}

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a fixed probability space.

\noindent \textbf{Lemma B.1.} If \(\mathcal{F}, \mathcal{G}, \mathcal{H} \subset \mathcal{A}\) are three independent \(\sigma\)-algebras, and \(X\) is a random variable measurable with respect to \(\mathcal{F} \vee \mathcal{G}\), then
\[
\mathbb{E}\left[ X \mid \mathcal{F} \vee \mathcal{G} \right] = \mathbb{E}\left[ X \mid \mathcal{F} \right], \quad \mathbb{P} - a.s.
\]

\textit{Proof.} The \(\sigma\)-algebra \(\mathcal{F} \vee \mathcal{G}\) is generated by the sets of the form \(F \cap H\), with \(F \in \mathcal{F}\) and \(H \in \mathcal{H}\). Hence, the following computation concludes the proof
\[
\mathbb{E}\left[ \mathbb{E}\left[ X \mid \mathcal{F} \right] 1_{F \cap H} \right] = \mathbb{E}\left[ \mathbb{E}\left[ X 1_F \mid \mathcal{F} \right] 1_H \right] = \mathbb{E}\left[ X 1_F \right] \mathbb{E}\left[ 1_H \right] = \mathbb{E}\left[ X 1_F 1_H \right].
\]

\section*{Lemma B.2.} Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space compatible with a Brownian motion \(W\) and let \(B\) be another \((\mathcal{F}_t)_{t \geq 0}\)-adapted Brownian motion, which is independent from \(W\). Let \((Y_t)_{t \geq 0}\) be an \((\mathcal{F}_t)_{t \geq 0}\)-adapted bounded stochastic process. Then, \(\forall t \in [0, T], \mathbb{P} - a.s.,
\[
\mathbb{E}\left[ \int_0^t Y_s dB_s \mid W_t \right] = 0,
\]
\[
\mathbb{E}\left[ \int_0^t Y_s dW_s \mid W_t \right] = \int_0^t \mathbb{E}\left[ Y_s \mid W_s \right] dW_s.
\]
Proof. We start with the first equality. Let \( \pi^n = \{ t^n_i : i = 1, \ldots, n \} \) be a sequence of partitions of \([0, t]\) with mesh size going to zero as \( n \to 0 \), such that

\[
\lim_{n \to \infty} \mathbb{E} \left| \sum_{[t^n_i, t^n_{i+1}] \in \pi^n} Y_{t_i} (B_{t_{i+1}} - B_{t_i}) - \int_0^t Y_s dB_s \right| \to 0.
\]

As a straightforward consequence of Jensen’s inequality, one has that convergence in \( L^1 \) implies convergence in \( L^1 \) of the conditional expectations. We can conclude the proof with the following observation

\[
\mathbb{E} \left[ \sum_{[t_i, t_{i+1}] \in \pi^n} Y_{t_i} (B_{t_{i+1}} - B_{t_i}) \left| W \right. \right] = \sum_{[t_i, t_{i+1}] \in \pi^n} \mathbb{E} [B_{t_{i+1}} - B_{t_i}] \mathbb{E} [Y_{t_i} \left| W \right.] = 0, \quad \mathbb{P}-a.s.
\]

Here we used the following property of the conditional expectation: if \( B \) is independent of \( \sigma(Y, W) \), then \( \mathbb{E}[BY \mid W] = \mathbb{E}[B] \mathbb{E}[Y \mid W] \).

The second part of the lemma can be proved in a similar way, taking into account the additional observation that, for every \( 0 \leq s \leq t \leq T \),

\[
\mathbb{E} [Y_s \mid W_t] = \mathbb{E} [Y_s \mid W_s], \quad \mathbb{P} - a.s.
\]

This follows from Lemma B.1. \( \square \)

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