Asymptotic normality of a linear threshold estimator in fixed dimension with near-optimal rate

Debarghya Mukherjee, Moulinath Banerjee, Ya’acov Ritov

Department of Statistics, University of Michigan, Ann Arbor

Abstract

Linear thresholding models postulate that the conditional distribution of a response variable in terms of covariates differs on the two sides of a (typically unknown) hyperplane in the covariate space. A key goal in such models is to learn about this separating hyperplane. Exact likelihood or least square methods to estimate the thresholding parameter involve an indicator function which make them difficult to optimize and are, therefore, often tackled by using a surrogate loss that uses a smooth approximation to the indicator. In this note, we demonstrate that the resulting estimator is asymptotically normal with a near optimal rate of convergence: \( n^{-1} \) up to a log factor, in a classification thresholding model. This is substantially faster than the currently established convergence rates of smoothed estimators for similar models in the statistics and econometrics literatures.

Keywords: Binary response, Change plane models, Linear thresholding models.

1. Introduction and Assumptions

The simple linear regression model assumes a uniform linear relationship between the covariate and the response, in the sense that the regression parameter \( \beta \) is the same over the entire covariate domain. In practice, the situation can be more complicated: for instance, the regression parameter may differ from sub-population to sub-population within a large (super-) population. Some common techniques to account for such heterogeneity include mixed linear models, introducing an interaction effect, or fitting different models among each sub-population which corresponds to a supervised classification setting where the true groups (sub-populations) are \textit{a priori} known.

A more difficult scenario arises when the sub-populations are unknown, in which case regression and classification must happen simultaneously. Consider the scenario where the conditional mean of \( Y \) given \( X \) is different for different unknown sub-groups. A well-studied treatment of this problem – the so-called change point problem – considers a simple thresholding model where membership in a sub-group is determined by whether a real-valued observable \( X \) falls to the left or right of an unknown parameter \( \gamma \). More recently, there has been work for multi-dimensional covariates, namely when the membership is determined by which side a random vector \( X \) falls with respect to an hyperplane with unknown normal vector \( \theta_0 \). A con-
and studied computational algorithms and consistency of the same. This model and others with similar structure, called change plane models, are useful in various fields of research, e.g. modeling treatment effect heterogeneity in drug treatment (3), modeling sociological data on voting and employment (3), or cross country growth regressions in econometrics (9).

Other aspects of this model have also been investigated. 11 examined the change plane model from the statistical testing point of view, with the null hypothesis being the absence of a separating hyperplane. They proposed a test statistic, studied its asymptotic distribution and provided sample size recommendations for achieving target values of power. 6 extended the change point detection problem in the multi-dimensional setup by considering the case where \(X^T \theta_0\) forms a multiple change point data sequence. The key difficulty with change plane type models is the inherent discontinuity in the optimization criteria involved where the parameter of interest appears as an argument to some indicator function, rendering the optimization extremely hard. To alleviate this, one option is to kernel smooth the indicator function, an approach that was adopted by Seo and Linton 9 in a version of the change-plane problem, motivated by earlier results of Horowitz 2 that dealt with a smoothed version of the maximum score estimator. Their model has an additive structure of the form:

\[
y_t = \beta^T x_t + \delta^T \tilde{x}_t \mathbb{1}\{q^T \psi > 0\} + \epsilon_t,
\]

where \(\psi\) is the (fixed) change-plane parameter, and \(t\) can be viewed as a time index. Under a set of assumptions on the model (Assumptions 1 and 2 of their paper), they showed asymptotic normality of their estimator of \(\psi\) obtained by minimizing a smoothed least squares criterion that uses a differentiable distribution function \(\mathcal{K}\). The rate of convergence of \(\hat{\psi}\) to the truth was shown to be \(\sqrt{n}/\sigma_n\) where \(\sigma_n\) was the bandwidth parameter used to smooth the least squares function. As noted in their Remark 3, under the special case of i.i.d. observations, their requirement that \(\log n / (n \sigma_n^2) \rightarrow 0\) translates to a maximal convergence rate of \(n^{3/4}\) up to a logarithmic factor. The work of 6 who considered multiple parallel change planes (determined by a fixed dimensional normal vector) and high dimensional linear models in the regions between consecutive hyperplanes also builds partly upon the methods of 6 and obtains the same (almost) \(n^{3/4}\) rate for the normal vector (as can be seen by putting Condition 6 in their paper in conjunction with the conclusion of Theorem 3).

In this note, we focus on a canonical change plane estimation problem with binary response and i.i.d. observations. The model can be briefly described as follows: The covariate \(X \sim P\) where \(P\) is distribution on \(\mathbb{R}^d\). The conditional distribution of \(Y\) given \(X = x\) is modeled as follows:

\[
P(Y = 1|X = x) = P_0 \mathbb{1}(x^T \theta_0 \leq 0) + P_1 \mathbb{1}(x^T \theta_0 > 0)
\]

for some parameters \(\alpha_0, \beta_0 \in (0, 1)\) and \(\theta_0 \in \mathbb{R}^d\), the latter being of primary interest for estimation. This model is identifiable up to a permutation of \((\alpha_0, \beta_0)\) and the scale of \(\theta_0\). Henceforth we assume \(\alpha_0 < \beta_0\) and \(\theta_0 \in \Theta\) where \(\Theta = \{\theta \equiv (\theta_1, \theta_2, \ldots, \theta_d) : \theta_1 = 1\}\). Our goal is to show that a kernel-smoothed version of an appropriate loss function in this problem can produce estimates of \(\theta_0\) that converge at a rate \(\text{logarithmically close to } 1/n\), and are asymptotically normal under a set of reasonable assumptions on the underlying model. Our message here is that while smoothing the indicator function does compromise the rate of convergence relative to the exact least squares/most maximum likelihood estimator, a fact also noted in 6, the compromise can be quite minimal (i.e. to within a logarithmic factor of the best rate). We note that the rate obtained here is faster than the one obtained by 6 in their model, which while being different in certain aspects – ours is a classification problem and theirs a regression, and the model assumptions are also not fully comparable,
but both have the same fundamental structure of a jump discontinuity at a separating hyperplane. We expect that our convergence rate should arise in many different manifestations of the change-point model under fairly reasonable assumptions. At a more technical level, we are able to obtain a faster convergence rate for the normal vector to the hyperplane in comparison to [9] and [6] as our techniques allow us to impose a weaker restriction on our smoothing bandwidth \( \sigma_n \). We expect that our convergence rate should arise in many different incarnations of the change-plane model under fairly mild conditions (discussed later) \( \theta_0 = \text{arg min}_{\theta \in \Theta} M(\theta) \), uniquely. Under the standard classification paradigm, when we know a priori that \( \alpha_0 < 1/2 < \beta_0 \), we can take \( \gamma = 1/2 \), and in the absence of this constraint, \( \hat{Y} \), which converges to some \( \gamma \) between \( \alpha_0 \) and \( \beta_0 \), may be substituted in the loss function. In the rest of the paper, we confine ourselves to a known \( \gamma \), and for technical simplicity, we take \( \gamma = \frac{\theta_0 + \theta_{N0}}{2} \), but this assumption can be removed quite easily at the cost of a non-zero mean in the limiting normal distribution derived in Theorem 1.1. Thus, \( \theta_0 \) is estimated by:

\[
\theta_n = \text{arg min}_{\theta \in \Theta} M_n(\theta) = \text{arg min}_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} (Y_i - \gamma) \mathbb{I}(X_i^\top \theta \leq 0).
\]

(3)

It is not difficult to establish that under mild conditions \( \| \theta_n - \theta_0 \|_2 = O_p(n^{-1}) \) but inference is difficult as the limit distribution is unknown, and in any case, would be a highly non-standard distribution. Recall that even in the one-dimensional change point model with fixed jump size, the least squares change point estimator converges at rate \( n \) to the truth with a non-standard limit distribution, namely a minimizer of a two-sided compound Poisson process (see [5] for more details). We resort to a smooth approximation of the indicator function in (3) using a distribution kernel with suitable bandwidth. The smoothed version of the population score function then becomes:

\[
M_n(\theta) = \mathbb{E} \left( (Y - \gamma) \left( 1 - K \left( \frac{X^\top \theta}{\sigma_n} \right) \right) \right)
\]

(4)

where we take \( K \) to be a twice-differentiable probability distribution function supported on \([-1, 1]\) with symmetric density (see Assumption 4), and the corresponding empirical version is:

\[
\hat{M}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \gamma) \left( 1 - K \left( \frac{X_i^\top \theta}{\sigma_n} \right) \right).
\]

Define \( \hat{\theta}_n \) and \( \theta_n \) to be the minimizer of the smoothed version of the empirical and population score function respectively. Before stating our main results, we articulate our assumptions. From now on, we define \( \bar{x} = (x_2, \ldots, x_p) \) for any vector \( x \in \mathbb{R}^p \).

**Assumption 1.** The below assumptions pertain to the parameter space and the distribution of \( X \):

1. The parameter space \( \Theta \) is a compact subset of \( \mathbb{R}^p \).
2. The support of the distribution of $X$ contains an open subset around origin of $\mathbb{R}^p$ and the distribution of $X_1$ conditional on $\bar{X} = (X_2, \ldots, X_p)$ has, almost surely, everywhere positive density with respect to Lebesgue measure.

3. $\bar{X}$ has a sub-gaussian distribution with sub-gaussian parameter $\sigma^2$ (See Remark at the end of the roadmap of the proof for a brief discussion on this assumption).

For notational convenience, define the following:

1. Define $f_\theta(\cdot | \bar{X})$ to the conditional density of $X^\top \theta$ given $\bar{X}$ for $\theta \in \Theta$. Note that the following relation holds:
   \[ f_\theta(\cdot | \bar{X}) = f_{X_1}(\cdot - \bar{\theta}^\top \bar{X} | \bar{X}), \]
   where we define $f_{X_1}(\cdot | \bar{X})$ is the conditional density of $X_1$ given $\bar{X}$.

2. Define $f_0(\cdot | \bar{X}) = f_{\theta_0}(\cdot | \bar{X})$ where $\theta_0$ is the unique minimizer of the population score function $M(\theta)$.

3. Define $f_{\bar{X}}(\cdot)$ to be the marginal density of $\bar{X}$.

The rest of the assumptions are as follows:

**Assumption 2.** $f_0(y|\bar{X})$ is at-least once continuously differentiable almost surely for all $\bar{X}$. Also assume that there exists $\delta$ and $t$ such that
   \[ \inf_{|y| \leq \delta} f_0(y|\bar{X}) \geq t \]
   for all $\bar{X}$ almost surely.

   This assumption can be relaxed in the sense that one can allow the lower bound $t$ to depend on $\bar{X}$, provided that some further assumptions are imposed on $\mathbb{E}(t(\bar{X}))$. As this does not add anything of significance to the import of this paper, we use Assumption 2 to simplify certain calculations.

**Assumption 3.** Define $m(\bar{X}) = \sup_t f_{X_1}(t|\bar{X}) = \sup_\theta \sup_t f_\theta(t|\bar{X})$. Assume that $\mathbb{E}(m(\bar{X})^2) < \infty$.

**Assumption 4.** The below assumptions pertain to the smoothing kernel $K$.

- $K$ is a twice continuously differentiable probability distribution function supported on $[-1, 1]$ and symmetric around 0.
- Both $k$ and $k'$ are uniformly bounded on $[-1, 1]$, where $k$ is the derivative of the kernel $K$.
- $k'$ is a Lipschitz function.

**Assumption 5.** Define $h(\bar{X}) = \sup_t f_\theta'(t|\bar{X})$. Assume that $\mathbb{E}\left(h^2(\bar{X})\right) < \infty$.

**Assumption 6.** Assume that $f_{\bar{X}}(0) > 0$ and also that the minimum eigenvalue of $\mathbb{E}\left(\bar{X} \bar{X}^\top f_0(0|\bar{X})\right) > 0$.

---

1 Recall that a vector $X$ follows a sub-gaussian distribution with parameter $\sigma^2$ if for any vector $\nu \in S^{p-1}$ the distribution of $\nu^\top X$ follows a univariate sub-gaussian distribution with parameter $\sigma^2$. 

**Sufficient conditions for above assumptions:** We now demonstrate some sufficient conditions for the above assumptions to hold. If the support of $X$ is compact and both $f_1(\cdot | X)$ and $f'_1(\cdot | X)$ are uniformly bounded in $\tilde{X}$, then Assumptions (1, 3, 4, 5) follow immediately. The first part of Assumption (6) i.e. the assumption $f_0(0) > 0$ is also fairly general and satisfied by many standard probability distributions. The second part of Assumption (6) is satisfied when $f_0(0 | \tilde{X})$ has some lower bound independent of $\tilde{X}$ and $\tilde{X}$ has non-singular dispersion matrix. Assumption (4) is a standard assumption on the kernel and is satisfied for a large class of distribution functions, e.g. $k(x) = (3/4)(1 - x^2)$ on $[-1, 1]$.

**Remark 1.** Our analysis remains valid in presence of an intercept term. Assume, without loss of generality, that the second co-ordinate of $X$ is 1 and let $\tilde{X} = (X_3, \ldots, X_p)$. It is not difficult to check that all our calculations go through under this new definition of $\tilde{X}$. We, however, avoid this scenario for simplicity of exposition.

Below we state our main theorem. In the next section, we first provide a roadmap of our proof and then fill in the corresponding details. For the rest of the paper, we choose our bandwidth $\sigma_n$ to satisfy $\frac{\log n}{n \sigma_n} \rightarrow 0$.

**Theorem 1.1: The main theorem**

Under Assumptions (1 - 6), 
\[ \sqrt{n/\sigma_n} (\hat{\theta}_n - \theta_0) \Rightarrow N(0, \Gamma) \] for some non-stochastic matrix $\Gamma$, which will be defined subsequently.

**Remark 2.** As our procedure requires the weaker condition $(\log n)/(n \sigma_n) \rightarrow 0$, it is easy to see from the above theorem that the rate of convergence can be almost as fast as $n/ \sqrt{\log n}$.

2. Roadmap of the proof

The proof of the theorem is relatively long, so we break it into several lemmas. We provide a roadmap of the proof in this section while the elaborate technical derivations of the supporting lemmas are relegated to section 4. Define the following:

\[
T_n(\theta) = \nabla \hat{h}_n(\theta) = -\frac{1}{n \sigma_n} \sum_{i=1}^{n} (Y_i - \gamma) k \left( \frac{X_i^\top \theta}{\sigma_n} \right) \tilde{X}_i
\]

\[
Q_n(\theta) = \nabla^2 \hat{h}_n(\theta) = -\frac{1}{n \sigma^2_n} \sum_{i=1}^{n} (Y_i - \gamma) k' \left( \frac{X_i^\top \theta}{\sigma_n} \right) \tilde{X}_i \tilde{X}_i^\top
\]

As $\hat{h}_n$ minimizes $\hat{h}_n(\theta)$ we have $T_n(\hat{h}_n) = 0$. Using one step Taylor expansion we have:

\[
T_n(\hat{h}_n) = T_n(\theta_0) + Q_n(\hat{h}_n) (\hat{h}_n - \theta_0) = 0
\]

or:

\[
\sqrt{n/\sigma_n} (\hat{h}_n - \theta_0) = -\left( \sigma_n Q_n(\hat{h}_n) \right)^{-1} \sqrt{n \sigma_n} T_n(\theta_0)
\]

for some intermediate point $\bar{h}_n$ between $\hat{h}_n$ and $\theta_0$. The following lemma establishes the asymptotic properties of $T_n(\theta_0)$:
Lemma 2.1: Asymptotic Normality of $T_n$

For a kernel $K$ satisfying the conditions of Assumption 4 and with $n\sigma^3_n \to \lambda$, then
\[
\sqrt{n\sigma_n} T_n(\theta_0) \Rightarrow N(\mu, \Sigma)
\]
where
\[
\mu = -\sqrt{\lambda} \beta_0 - \alpha_0 \left[ \int_{-1}^1 k(t) |t| \, dt \right] \int_{\mathbb{R}^p} \ddot{x} f'(0|\ddot{x}) \, dP(\ddot{x})
\]
and
\[
\Sigma = \left[ a_1 \int_{-1}^1 (k(t))^2 \, dt + a_2 \int_0^1 (k(t))^2 \, dt \right] \int_{\mathbb{R}^p} \ddot{x} \ddot{x}^\top f(0|\ddot{x}) \, dP(\ddot{x})
\]
Here $a_1 = (1 - \gamma)^2 \alpha_0 + \gamma^2 (1 - \alpha_0)$, $a_2 = (1 - \gamma)^2 \beta_0 + \gamma^2 (1 - \beta_0)$ and $\alpha_0, \beta_0, \gamma$ are model parameters defined around equation (2).

In the case that $n\sigma^3_n \to 0$, which holds when $\log n / (n\sigma_n) \to 0$ as assumed prior to the statement of the theorem, $\lambda = 0$ and we have:
\[
\sqrt{n\sigma_n} T_n(\theta_0) \to N(0, \Sigma)
\]

Next, we analyze the convergence of $Q_n(\hat{\theta}_n)^{-1}$ which is stated in the following lemma:

Lemma 2.2: Convergence in Probability of $Q_n$

Under Assumptions (1 - 6), for any random sequence $\tilde{\theta}_n$ such that $\|\tilde{\theta}_n - \theta_0\|/\sigma_n \xrightarrow{P} 0$,
\[
\sigma_n Q_n(\tilde{\theta}_n) \xrightarrow{P} Q = \frac{\beta_0 - \alpha_0}{2} \left( \int_{-1}^1 -k'(t) \sgn(t) \, dt \right) \mathbb{E} \left( \dddot{x} \ddot{x}^\top f(0|\dddot{x}) \right)
\]

It will be shown later that the condition $\|\tilde{\theta}_n - \theta_0\|/\sigma_n \xrightarrow{P} 0$ needed in Lemma 2.2 holds for the (random) sequence $\theta_n$. Then, combining Lemma 2.1 and Lemma 2.2, we conclude from equation 5 that:
\[
\sqrt{n/\sigma_n} (\tilde{\theta}_n - \theta_0) \Rightarrow N(0, Q^{-1} \Sigma Q^{-1})
\]

This concludes the proof of our Theorem 1.1 with $\Gamma = Q^{-1} \Sigma Q^{-1}$.

Observe that, to show $\|\tilde{\theta}_n - \theta_0\| = o_P(\sigma_n)$, it suffices to to prove that $\|\tilde{\theta}_n - \theta_0\| = o_P(\sigma_n)$. Towards that direction, we have following lemma:

Lemma 2.3: Rate of convergence

Under Assumptions (1 - 6),
\[
\frac{n}{\log n} d_n^2 (\hat{\theta}_n, \theta_n) = O_P(1)
\]
where
\[ d_n(\theta, \theta_n) = \sqrt{\frac{\|\theta - \theta_n\|^2}{\sigma_n} \mathbb{I}(\|\theta - \theta_n\| \leq K\sigma_n) + \|\theta - \theta_n\| \mathbb{I}(\|\theta - \theta_n\| \geq K\sigma_n)} \]
for some specific constant \( K \). (This constant will be mentioned precisely in the proof).

The lemma immediately leads to the following corollary:

**Corollary 2.1**

If \((\log n)/(n\sigma_n) \to 0\) then \( \frac{\|\hat{\theta}_n - \theta_0\|}{\sigma_n} \overset{P}{\to} 0 \)

Finally, to establish \( \|\hat{\theta}_n - \theta_0\|/\sigma_n \overset{P}{\to} 0 \), all we need is that \( \|\theta_n - \theta_0\|/\sigma_n \to 0 \) as demonstrated in the following lemma:

**Lemma 2.4: Convergence of population minimizer**

If the smoothing kernel satisfies Assumption 4 then for any sequence of \( \sigma_n \to 0 \), we have: \( \|\theta_n - \theta_0\|/\sigma_n \to 0 \).

Hence the final roadmap is the following: Using Lemma 2.4 and Corollary 2.1 we establish that \( \|\hat{\theta}_n - \theta_0\|/\sigma_n \overset{P}{\to} 0 \) if \( \log n/(n\sigma_n) \to 0 \). This, in turn, enables us to prove that \( \sigma_n Q(\bar{\theta}_n) \overset{P}{\to} Q \), which, along with Lemma 2.1, establishes the main theorem.

**Remark 3.** One may wonder whether the subgaussian assumption on the distribution of \( X \) can be relaxed. We use this condition to prove Lemma 2.2. The fact that \( \bar{\theta}_n \) defined in this Lemma is a random sequence necessitates the use of a Glivenko-Cantelli type result. This is facilitated by the subgaussianity assumption on the distribution of \( X \) which allows us to apply a certain maximal inequality. It may be possible to relax the assumption if one is willing to use more sophisticated inequalities, but we have not explored this direction.

3. Discussion

In this note we have established that, under some mild assumptions, the kernel-smoothed change plane estimator is asymptotically normal with near optimal rate \( n^{-1} \). To the best of our knowledge, the state of the art result in this genre of problems is due to [9], where they demonstrate a best possible rate about \( n^{-3/4} \). The main difference between their approach and ours are the proofs of Corollary 2.1 and Lemma 2.4. Our techniques are based upon modern empirical process theory which allow us to consider much smaller bandwidths \( \sigma_n \) compared to those in [9], who appear to require larger values to achieve the result of Corollary 2.1 possibly owing to their reliance on the techniques developed in [2].
4. Details

4.1. Some technical results

Below we state two technical results from empirical process theory which are essential for the rest of the analysis. The first theorem (Theorem 3.4.1 of [10]) analyzes the rate of convergence of an estimator obtained via minimizing an empirical risk function:

**Theorem 4.1: Rate theorem**

For each \( n \), let \( M_n \) and \( M_n \) be stochastic processes indexed by a set \( \Theta \). Let \( \theta_n \in \Theta \) be a possibly random sequence, and let \( \theta \to d_n(\theta, \theta_n) \) be a possibly random map from \( \Theta \) to \([0, \infty)\). Let \( 0 \leq \delta_n \leq \eta \) be arbitrary, and suppose that, for every \( n \) and \( \delta_n < \delta \leq \eta \),

\[
\begin{align*}
\sup_{\delta/2 < d_n(\theta, \theta_n) \leq \delta, \theta_n \in \Theta} M_n(\theta) - M_n(\theta_n) &\geq \delta^2, \\
\mathbb{E} \left( \sup_{d_n(\theta, \theta_n) \leq \delta} \sqrt{n} d_n(\theta, \theta_n) - (M_n - M_n)(\theta_n) \right) &\leq \phi_n(\delta),
\end{align*}
\]

for functions \( \phi_n \) such that \( \delta \to \phi_n(\delta)/\delta^\alpha \) is decreasing on \((\delta_n, \eta)\), for some \( \alpha < 2 \). Let \( r_n \leq \delta_n^{-1} \) satisfy

\[
r_n^2 \phi_n \left( \frac{1}{r_n} \right) \leq \sqrt{n}
\]

for every \( n \). If the sequence \( \hat{\theta}_n \) takes value in \( \Theta_n \) and satisfies \( M_n(\hat{\theta}_n) \leq M_n(\theta_n) + O_p(r_n^{-2}) \) and \( d_n(\hat{\theta}_n, \theta_n) \) converges to 0 in outer probability, then \( r_n d_n(\hat{\theta}_n, \theta_n) = O_p(1) \). If all the above conditions are valid for \( \eta = \infty \), then the requirement of consistency is unnecessary.

The next one is a maximal inequality which enables us to bound the expected deviation of an empirical process around its center using a bound on the covering number of the underlying class of functions:

**Theorem 4.2: Maximal inequality (Theorem 8.7 of [8])**

Let \( \mathcal{F} \) be a measurable class of functions with a constant envelope \( U \) (i.e. \( \|f\|_\infty \leq U \) for all \( f \in \mathcal{F} \)) such that for some constant \( A > e^{2} \) and \( V \geq 2 \) and for every finitely supported probability measure \( Q \)

\[
N(\epsilon U, \mathcal{F}, L_2(Q)) \leq \left( \frac{A}{\epsilon} \right)^V 0 \leq \epsilon < 1
\]

where \( N(\epsilon U, \mathcal{F}, L_2(Q)) \) is the \( \epsilon U \) covering number of \( \mathcal{F} \) with respect to the \( L_2(Q) \) norm. Then for all \( n \),

\[
\mathbb{E} \left\| \sum_{i=1}^n (f(X_i) - P f) \right\|_F \leq L \left( \sigma \sqrt{n V \log \frac{AU}{\sigma}} \vee VU \log \frac{AU}{\sigma} \right)
\]

where \( L \) is a universal constant and \( \sigma \) is such that \( \sup_{f \in \mathcal{F}} P(f - P f)^2 \leq \sigma^2 \). In particular if \( n \sigma^2 \geq
Since the first summand is just \( E \) observation:

\[
\frac{\log (AU/\sigma)}{\sigma} \text{ then the above result shows that }
\]

\[
\mathbb{E} \left\| \sum_{i=1}^{n} (f(X_i) - Pf) \right\|_F \leq \sqrt{n\sigma^2 V \log (AU/\sigma)}.
\]

### 4.2. Variant of quadratic loss function

In this sub-section we argue why the loss function in (2) is a variant of the quadratic loss function for any \( \gamma \in (\alpha_0, \beta_0) \). Assume that we know \( \alpha_0, \beta_0 \) and seek to estimate \( \theta_0 \). We start with an expansion of the quadratic loss function:

\[
\begin{align*}
\mathbb{E} (Y - \alpha_0 \mathbb{1}_{X^\top \theta \geq 0} - \beta_0 \mathbb{1}_{X^\top \theta < 0})^2 &= \mathbb{E} (Y - \alpha_0 \mathbb{1}_{X^\top \theta \geq 0} - \beta_0 \mathbb{1}_{X^\top \theta < 0})^2 |X) \\
&= \mathbb{E}_X (\mathbb{E} (Y^2 | X) + \mathbb{E}_X (\alpha_0 \mathbb{1}_{X^\top \theta \geq 0} + \beta_0 \mathbb{1}_{X^\top \theta < 0})^2 - 2 \mathbb{E}_X (\alpha_0 \mathbb{1}_{X^\top \theta \geq 0} + \beta_0 \mathbb{1}_{X^\top \theta < 0}) \mathbb{E}(Y | X)) \\
&= \mathbb{E}_X (\mathbb{E} (Y|X)) + \mathbb{E}_X (\alpha_0 \mathbb{1}_{X^\top \theta \geq 0} + \beta_0 \mathbb{1}_{X^\top \theta < 0})^2 - 2 \mathbb{E}_X (\alpha_0 \mathbb{1}_{X^\top \theta \geq 0} + \beta_0 \mathbb{1}_{X^\top \theta < 0}) \mathbb{E}(Y | X)
\end{align*}
\]

Since the first summand is just \( \mathbb{E}Y \), it is irrelevant to the minimization. A cursory inspection shows that it suffices to minimize

\[
\mathbb{E} ((\alpha_0 \mathbb{1}_{X^\top \theta \geq 0} + \beta_0 \mathbb{1}_{X^\top \theta < 0}) - \mathbb{E}(Y | X))^2 = (\beta_0 - \alpha_0)^2 \mathbb{P} (\text{sgn}(X^\top \theta) \neq \text{sgn}(X^\top \theta_0))
\]

(6)

On the other hand the loss we are considering is \( \mathbb{E} ((Y - \gamma) \mathbb{1}_{X^\top \theta < 0}): \)

\[
\mathbb{E} ((Y - \gamma) \mathbb{1}_{X^\top \theta < 0}) = (\beta_0 - \gamma) \mathbb{P}(X^\top \theta_0 > 0, X^\top \theta \leq 0) + (\alpha_0 - \gamma) \mathbb{P}(X^\top \theta_0 \leq 0, X^\top \theta \leq 0),
\]

(7)

which can be rewritten as:

\[
(\alpha_0 - \gamma) \mathbb{P}(X^\top \theta_0 \leq 0) + (\beta_0 - \gamma) \mathbb{P}(X^\top \theta_0 > 0, X^\top \theta \leq 0) + (\gamma - \alpha_0) \mathbb{P}(X^\top \theta_0 > 0, X^\top \theta > 0).
\]

By Assumption [1] for \( \theta \neq \theta_0 \), \( \mathbb{P} (\text{sgn}(X^\top \theta) \neq \text{sgn}(X^\top \theta_0)) > 0 \). As an easy consequence, equation (6) is uniquely minimized at \( \theta = \theta_0 \). To see that the same is true for (7) when \( \gamma \in (\alpha_0, \beta_0) \), note that the first summand in the equation does not depend on \( \theta \), that the second and third summands are both non-negative and that at least one of these must be positive under Assumption [1].

### 4.3. Linear curvature of the population score function

Before going into the proofs of the Lemmas and the Theorem, we argue that the population score function \( M(\theta) \) has linear curvature near \( \theta_0 \), which is useful in proving Lemma 2.3. We begin with the following observation:

**Lemma 4.1: Curvature of population risk**

Under Assumption [2] we have:

\[
u_\pm ||\theta - \theta_0||_2 \leq M(\theta) - M(\theta_0) \leq u_\pm ||\theta - \theta_0||_2
\]

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for some constants $0 < u_- < u_+ < \infty$, for all $\theta \in \Theta$.

**Proof.** First, we show that

$$M(\theta) - M(\theta_0) = \frac{(\beta_0 - \alpha_0)}{2} \mathbb{P}(\text{sgn}(X^T\theta) \neq X^T(\theta_0))$$

which follows from the calculation below:

$$M(\theta) - M(\theta_0)$$
$$= \mathbb{E}((Y - \gamma)\mathbb{I}(X^T\theta \leq 0)) - \mathbb{E}((Y - \gamma)\mathbb{I}(X^T\theta_0 \leq 0))$$
$$= \frac{\beta_0 - \alpha_0}{2} \mathbb{E}\left(\mathbb{I}(X^T\theta \leq 0) - \mathbb{I}(X^T\theta_0 \leq 0)\right) \left(\mathbb{I}(X^T\theta_0 \geq 0) - \mathbb{I}(X^T\theta_0 \leq 0)\right)$$
$$= \frac{\beta_0 - \alpha_0}{2} \mathbb{E}\left(\mathbb{I}(X^T\theta \leq 0, X^T\theta_0 \geq 0) - \mathbb{I}(X^T\theta \leq 0, X^T\theta_0 \leq 0) + \mathbb{I}(X^T\theta \geq 0, X^T\theta_0 \leq 0)\right)$$
$$= \frac{\beta_0 - \alpha_0}{2} \mathbb{E}(\text{sgn}(X^T\theta) \neq X^T(\theta_0)).$$

We now analyze the probability of the wedge shaped region, the region between the two hyperplanes $X^T\theta = 0$ and $X^T\theta_0 = 0$. Note that,

$$\mathbb{P}(X^T\theta > 0 > X^T\theta_0)$$
$$= \mathbb{P}(-\bar{X}^T\bar{\theta} < X_1 < -\bar{X}^T\bar{\theta}_0)$$
$$= \mathbb{E} \left( \left| F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}_0) - F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}) \right| \mathbb{I}(\bar{X}^T\bar{\theta}_0 \leq \bar{X}^T\bar{\theta}) \right) \tag{8}$$

A similar calculation yields

$$\mathbb{P}(X^T\theta < 0 < X^T\theta_0) = \mathbb{E} \left( F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}) - F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}_0) \right) \mathbb{I}(\bar{X}^T\bar{\theta}_0 \geq \bar{X}^T\bar{\theta}) \tag{9}$$

Adding both sides of equation\[8\] and \[9\] we get:

$$\mathbb{P}(\text{sgn}(X^T\theta) \neq \text{sgn}(X^T\theta_0)) = \mathbb{E} \left| F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}) - F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}_0) \right|$$

Define $\Theta_{\text{max}} = \sup_{\|\theta\|} \|\theta\|$, which is finite by Assumption\[1\] Below, we establish the lower bound:

$$\mathbb{P}(\text{sgn}(X^T\theta) \neq \text{sgn}(X^T\theta_0))$$
$$= \mathbb{E} \left| F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}) - F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}_0) \right|$$
$$\geq \mathbb{E} \left| F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}) - F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}_0) \right| \mathbb{I}(\|\bar{X}\| \leq \delta/\Theta_{\text{max}})$$
$$\geq \mathbb{E} \left| F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}) - F_{X_1|\bar{\theta}} (-\bar{X}^T\bar{\theta}_0) \right| \mathbb{I}(\|\bar{X}\| \leq \delta/\Theta_{\text{max}})$$
$$\geq r \mathbb{E} \left| X^T(\theta - \theta_0) \right| \mathbb{I}(\|X\| \leq \delta/\Theta_{\text{max}})$$
$$= r\|\theta - \theta_0\| \mathbb{E} \left| X^T(\frac{\theta - \theta_0}{\|\theta - \theta_0\|}) \right| \mathbb{I}(\|X\| \leq \delta/\Theta_{\text{max}})$$

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The proof is based on Taylor expansion of the conditional density:

\[ \geq t|\theta - \theta_0| \inf_{\gamma \in S^{p-1}} \mathbb{E}[|X^T \gamma| \mathbf{I}(\|X\| \leq \delta/\Theta_{\text{max}})] \]

\[ = u_-|\theta - \theta_0|. \]

At the very end, we have used the fact that

\[ \inf_{\gamma \in S^{p-1}} \mathbb{E}[|X^T \gamma| \mathbf{I}(\|X\| \leq \delta/\Theta_{\text{max}})] > 0 \]

To prove this, assume that the infimum is 0. Then, there exists \( \gamma_0 \in S^{p-1} \) such that

\[ \mathbb{E}[|X^T \gamma_0| \mathbf{I}(\|X\| \leq \delta/\Theta_{\text{max}})] = 0, \]

as the above function continuous in \( \gamma \) and any continuous function on a compact set attains its infimum. Hence, \( |X^T \gamma_0| = 0 \) for all \( \|X\| \leq \delta/\Theta_{\text{max}} \), which implies that \( X \) does not have full support, violating Assumption \( 1(2) \). This gives a contradiction.

Establishing the upper bound is relatively easier. Going back to equation (10), we have:

\[ \mathbb{P}(\text{sgn}(X^T \theta) \neq \text{sgn}(X^T \theta_0)) \]

\[ = \mathbb{E}\left[|F_{X|X^T \theta}(X^T \theta) - F_{X|X^T \theta_0}(X^T \theta_0)|\right] \]

\[ \leq \mathbb{E}\left[m(X) \|X\| |\theta - \theta_0|\right] \quad \text{[}m(\cdot) \text{ is defined in Assumption 3]} \]

\[ \leq u_+|\theta - \theta_0|, \]

as \( \mathbb{E}[m(\tilde{X})|X|] < \infty \) by Assumption 3 and the sub-Gaussianity of \( \tilde{X} \).

\[ \Box \]

4.4. Proof of Lemma 2.7

**Proof.** We first prove that under our assumptions \( \sigma_n^{-1} \mathbb{E}(T_n(\theta_0)) \xrightarrow{n \to \infty} A \) where

\[ A = -\frac{\beta_0 - \alpha_0}{2!} \int_{-1}^{1} k(t) |t| dt \int_{\mathbb{R}^{p-1}} \bar{x} f_0'(0|\bar{x}) \, dP(\bar{x}) \]

The proof is based on Taylor expansion of the conditional density:

\[ \sigma_n^{-1} \mathbb{E}(T_n(\theta_0)) \]

\[ = -\sigma_n^{-2} \mathbb{E}\left(Y - \gamma k\left(\frac{X^T \theta_0}{\sigma_n}\right) \bar{X}\right) \]

\[ = -\sigma_n^{-2} \mathbb{E}\left(k\left(\frac{X^T \theta_0}{\sigma_n}\right) \bar{X}(I(X^T \theta_0 \geq 0) - I(X^T \theta_0 \leq 0))\right) \]

\[ = -\sigma_n^{-2} \int_{\mathbb{R}^{p-1}} \bar{x} \left[\int_{-\infty}^{0} k\left(\frac{z}{\sigma_n}\right) f_0(z|\bar{x}) \, dz - \int_{0}^{\infty} k\left(\frac{z}{\sigma_n}\right) f_0(z|\bar{x}) \, dz \right] \, dP(\bar{x}) \]

\[ = -\frac{\beta_0 - \alpha_0}{2} \sigma_n^{-1} \int_{\mathbb{R}^{p-1}} \bar{x} \left[\int_{0}^{1} k(t) f_0(\sigma_n t|\bar{x}) dt - \int_{-1}^{0} k(t) f_0(\sigma_n t|\bar{x}) dt \right] \, dP(\bar{x}) \]

\[ = -\frac{\beta_0 - \alpha_0}{2} \sigma_n^{-1} \int_{\mathbb{R}^{p-1}} \bar{x} \left[\int_{0}^{1} k(t) f_0(0|\bar{x}) dt - \int_{-1}^{0} k(t) f_0(0|\bar{x}) dt \right] \, dP(\bar{x}) \]
Finally, suppose $\sqrt{\lambda} \rightarrow \lambda$. Define $W_n = \sqrt{n\sigma_n} [T_n(\theta) - \mathbb{E}(T_n(\theta))]$. Using Lemma 6 of Horowitz [2], it is easily established that $W_n \Rightarrow N(0, \Sigma)$. Also, we have:

$$
\sqrt{n\sigma_n} \mathbb{E}(T_n(\theta)) = \sqrt{n\sigma_n} \mathbb{E}(T_n(\theta)) = \sqrt{\lambda} \mathbb{E} = \mu
$$

As $\sqrt{n\sigma_n} T_n(\theta) = W_n + \sqrt{n\sigma_n} \mathbb{E}(T_n(\theta))$, we conclude that $\sqrt{n\sigma_n} T_n(\theta) \Rightarrow N(\mu, \Sigma)$. \hfill \Box

4.5. Proof of Lemma 2.2

Proof. Let $\epsilon_n \downarrow 0$ be a sequence such that $\mathbb{P}(||\hat{\theta} - \theta|| \leq \epsilon_n \sigma_n) \rightarrow 1$. Define $\Theta_n = \{\theta : ||\theta - \theta|| \leq \epsilon_n \sigma_n\}$. We show that

$$
\sup_{\theta \in \Theta_n} ||\sigma_n Q_n(\theta) - Q||_F \rightarrow 0
$$

where $||\cdot||_F$ denotes the Frobenius norm of a matrix. Sometimes, we omit the subscript $F$ when there is no ambiguity. Define $G_n$ to be collection of functions:

$$
G_n = \left\{ g_{\theta}(x,y) = -\frac{1}{\sigma_n} (y - \gamma) \tilde{x} \tilde{x}^T k'(\frac{x^T \theta}{\sigma_n}), \theta \in \Theta_n \right\}
$$
We can break the expression in two terms:

\[
\sup_{\theta \in \Theta_n} |\sigma_n Q_n(\theta) - Q| \leq \sup_{\theta \in \Theta_n} |\sigma_n Q_n(\theta) - \mathbb{E}(\sigma_n Q_n(\theta))| + \sup_{\theta \in \Theta_n} |\mathbb{E}(\sigma_n Q_n(\theta)) - Q|
\]

\[
= \| (P_n - P) \|_{g_1} + \sup_{\theta \in \Theta_n} |\mathbb{E}(\sigma_n Q_n(\theta)) - Q|
\]

\[
= T_{1,n} + T_{2,n} \quad \text{[Say]}
\]

We first show that \(T_{1,n} \xrightarrow{p} 0\) by arguing that \(E(T_{1,n}) \rightarrow 0\). To that end, we invoke the following concentration inequality:

**Theorem 1** (Concentration inequality for a sub-Gaussian random variable). Suppose \(X\) is a sub-Gaussian random vector with sub-gaussian parameter \(\sigma^2\), i.e. \(v^T X\) is a sub-Gaussian random variable with parameter \(\sigma^2\) for all \(v\) such that \(\|v\| = 1\). Then for all \(t > 0\):

\[
\mathbb{P}(\|X\| > t) \leq 2 e^{-\frac{t^2}{2d}}
\]

where \(d\) is the dimension of the support of \(X\).

A discussion and the proof of the theorem can be found in ([7]). Now, for some \(K_n > 0\) we can write the individuals functions in \(G_n\) as the sum of two functions:

\[
g_\theta(x, y) = -\frac{1}{\sigma_n} (y - \gamma) \bar{x}^T \bar{k}' \left( \frac{x^T \theta}{\sigma_n} \right) \mathbb{I} (\| \bar{x} \| \leq K_n) - \frac{1}{\sigma_n} (y - \gamma) \bar{x}^T \bar{k}' \left( \frac{x^T \theta}{\sigma_n} \right) \mathbb{I} (\| \bar{x} \| > K_n)
\]

\[A = g_{\theta,1} + g_{\theta,2}\]

We will choose \(K_n\) judiciously in the proof. As this function is a matrix-valued function of fixed dimension, we show the convergence in probability coordinate wise. Fix an entry of the matrix (say \((i, j)\)) and consider the corresponding functions:

1. \(g_{\theta,1}^{(i,j)}(x, y) = \frac{1}{\sigma_n} (y - \gamma) \bar{x}_i \bar{k}' \left( \frac{x^T \theta}{\sigma_n} \right) \mathbb{I} (\| \bar{x} \| \leq K_n)\).
2. \(g_{\theta,2}^{(i,j)}(x, y) = -\frac{1}{\sigma_n} (y - \gamma) \bar{x}_i \bar{k}' \left( \frac{x^T \theta}{\sigma_n} \right) \mathbb{I} (\| \bar{x} \| > K_n)\).
3. \(g_{\theta}^{(i,j)}(x, y) = g_{\theta,1}^{(i,j)}(x, y) + g_{\theta,2}^{(i,j)}(x, y)\).

and analogously the corresponding collections:

1. \(G_{n,1}^{(i,j)} = \{g_{\theta,1}^{(i,j)}(x, y) : \theta \in \Theta_n\}\).
2. \(G_{n,2}^{(i,j)} = \{g_{\theta,2}^{(i,j)}(x, y) : \theta \in \Theta_n\}\).
3. \(G_{n}^{(i,j)} = \{g_{\theta,1}^{(i,j)}(x, y) + g_{\theta,2}^{(i,j)}(x, y) : \theta \in \Theta_n\}\).

Using the above notation we have:

\[
\mathbb{E} \| (P_n - P) \|_{g_{\theta}^{(i,j)}} \leq \mathbb{E} \sup_{\theta \in \Theta_n} \| (P_n - P) g_{\theta,1}^{(i,j)} \| + \mathbb{E} \sup_{\theta \in \Theta_n} \| (P_n - P) g_{\theta,2}^{(i,j)} \|
\]

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For the first summand, we have a constant envelope for the collection \( G_{n,1} \) as \( \| G_{n,1} \|_\infty \leq \frac{M_2 k_1^2}{\sigma_n} \). It is immediate that \( P(G_{n,1}^{(i,j)})^2 \leq \frac{C}{\sigma_n} \leq v_n^2 \) for some constant \( C > 0 \). Also,:

\[
\sup_Q N(\epsilon U_n, G_{n,1}^{(i,j)}, L_2(Q)) \leq \left( \frac{A}{\epsilon} \right)^V
\]

for some fixed constant \( V < \infty \), where \( N(\epsilon U_n, G_{n,1}^{(i,j)}, L_2(Q)) \) denote the \( \epsilon \) covering number of the collection \( G_{n,1}^{(i,j)} \) with respect to the \( L_2(Q) \) norm, which follows because the functions \( (x, y) \to (x^T \theta)/\sigma_n \) are a VC class of functions and \( k' \) is Lipschitz (Assumption \( \text{[H]} \)). Using Theorem 8.7 of [8] we have:

\[
\frac{v_n}{\sqrt{n}} \sqrt{V \log \frac{A U_n}{V_n}} = \frac{C}{\sqrt{n} \sigma_n} \sqrt{V \log \frac{A M_2 K_1^2}{C \sqrt{n}}} = \frac{C}{\sqrt{n} \sigma_n} \sqrt{V \log \frac{A M_2 C_1^2 \log n}{C \sqrt{n}}},
\]

which goes to 0 as \( n \to \infty \). On the other hand,

\[
\frac{U_n}{n} \sqrt{V \log \frac{A U_n}{v_n}} = \frac{M_2 C_1 \log n}{n \sigma_n} \sqrt{V \log \frac{A M_2 C_1 \log n}{C \sqrt{n}}},
\]

which again converges to 0 as \( n \to \infty \). From this we conclude:

\[
\mathbb{E} \left[ \sup_{\theta \in \Theta_k} \left\| (P_n - P) G_{n,1}^{(i,j)} \right\| \right] \xrightarrow{n \to \infty} 0.
\]

To control the other summand, we use Theorem 2.14.1 of [10]. The following function serves as an envelope for the collection \( G_{n,2}^{(i,j)} \):

\[
G_n(x, y) = \frac{M_2}{\sigma_n} |\tilde{x}_i \tilde{x}_j| \mathbb{I}_{(\| \tilde{x} \|_2 > C_1 \sqrt{\log n})}
\]

Hence, we have

\[
\sqrt{P G_n^{(i,j)}} \leq \frac{M_2}{\sigma_n} \sqrt{\mathbb{E} \left( \tilde{x}_i \tilde{x}_j \mathbb{I}_{(\| \tilde{x} \|_2 > C_1 \sqrt{\log n})} \right)} \\
\leq \frac{M_2}{\sigma_n} \sqrt{\mathbb{E}^{1/2} \left( |\tilde{x}_i \tilde{x}_j|^2 \right) \mathbb{P}(\| \tilde{x} \|_2 > C_1 \sqrt{\log n})} \\
\leq \frac{M_3}{\sigma_n} \left( S^d e^{c_1 \log n} \right)^{1/4} \\
= \frac{M_3 S^{d/4}}{n \sigma_n} e^{c_1 \log n} \leq \frac{M_3 S^{d/4}}{n \sigma_n},
\]
for any choice of $C_1 > 4 \sqrt{2} \sigma$. Similar to equation (11), we can establish that:

$$ \sup_Q N(\epsilon\|G_n\|_{Q,2}, G_n^{(i,j)}, L_2(Q)) \leq \left(\frac{A}{\epsilon}\right)^V $$

for some fixed constant $V > 0$ (may be different from previous $V$), which implies

$$ \mathcal{J}(1, G_n^{(i,j)}) = \sup_Q \int_0^1 \sqrt{1 + \log N(\epsilon\|G_n\|_{Q,2}, G_n^{(i,j)}, L_2(Q))} < \infty. $$

Hence, using Theorem 2.14.1 of [10], we conclude:

$$ \mathbb{E} \left[ \sup_{\theta \in \Theta_n} \left\| (P_n - P)G_n^{(i,j)} \right\|_{\mathbb{C}} \right] \leq \mathcal{J}(1, G_n^{(i,j)}) \sqrt{PG_n^2} \leq \frac{M_1 5^{d/2}}{n^{3/2} \sigma_n} \to 0 \quad (13) $$

as $n \sigma_n \to \infty$. Combining equations (12) and (13), we conclude that $\mathbb{E} \left[ \left\| (P_n - P)G_n \right\|_{\mathbb{C}} \right] \to 0$ as $n \to \infty$. This concludes the proof of convergence of the first summand.

For uniform convergence of the second summand $T_{n,2}$, define $\chi_n = \{ \bar{x} : \|\bar{x}\| \leq 1 / \sqrt{\sigma_n} \}$. Then $\chi_n \uparrow \mathbb{R}^{p-1}$. Also for any $\theta \in \Theta_n$, if we define $\gamma_n = \gamma_n(\theta) = (\theta - \theta_n)/\sigma_n$, then $|\gamma_n \bar{x}| \leq \sqrt{\sigma_n}$ for all $n$ and for all $\theta \in \Theta_n, \bar{x} \in \chi_n$.

Now,

$$ \sup_{\theta \in \Theta_n} \| \mathbb{E}(\sigma_n Q_n(\theta) - Q) \| = \sup_{\theta \in \Theta_n} \| \mathbb{E}(\sigma_n Q_n(\theta)(\chi_n)) - Q_1 + (\mathbb{E}(\sigma_n Q_n(\theta)(\chi_n)) - Q_2) \| $$

where

$$ Q_1 = \frac{\beta_0 - \alpha_0}{2} \left( \int_{-1}^1 -k'(t) \text{sgn}(t) \, dt \right) \mathbb{E}(\bar{X} \bar{X}^T f(0) \bar{X})1(\chi_n) $$

$$ Q_2 = \frac{\beta_0 - \alpha_0}{2} \left( \int_{-1}^1 -k'(t) \text{sgn}(t) \, dt \right) \mathbb{E}(\bar{X} \bar{X}^T f(0) \bar{X})1(\chi_n). $$

Note that

$$ \| \mathbb{E}(\sigma_n Q_n(\theta)(\chi_n)) - Q_1 \| $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$

$$ = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{\chi_n} \bar{x} \bar{x}^T \left[ \int_{-1}^1 k'(|t - \bar{x}^T \gamma_n|) \, dt \right] f(0) \, dP(\bar{x}) \right\| $$

$$ - \frac{\beta_0 - \alpha_0}{2} \left[ \int_{\chi_n} \left[ \int_{-1}^1 -k'(t) \, dt \right] f(0) \, dP(\bar{x}) \right] $$
4.6. Proof of Lemma 2.4

Combining equations 14 and 15, we conclude the proof.

\[ \rightarrow 0 \quad \text{[As } n \rightarrow \infty], \quad (14) \]

by DCT and Assumptions 1 and 5. For the second part:

\[ \| \mathbb{E}(\sigma_n Q_n(\theta) 1(\gamma_n^{\ast})) - Q_2 \| = \left\| \frac{\beta_0 - \alpha_0}{2} \int_{X_0} z \mathbb{E}^{\gamma} \left[ \int_{-1}^{t} k' (t) f_0(\sigma_n(t - \tilde{\gamma}_n) \tilde{x}) \right] dt \right\| \]

\[ \leq \frac{\beta_0 - \alpha_0}{2} \left\| \int_{X_0} z \mathbb{E}^{\gamma} \left[ \int_{-1}^{t} F_n(0, \tilde{\gamma}_n) \right] dt \right\| \rightarrow 0 \quad \text{[As } n \rightarrow \infty], \quad (15) \]

again by DCT and Assumptions 1 and 5. Combining equations 14 and 15, we conclude the proof.

4.6. Proof of Lemma 2.4

Here we prove that \( \| \theta_n - \theta_0 \| / \sigma_n \rightarrow 0 \) when \( \theta_n \) is the minimizer of \( M_n(\theta) \) and \( \theta_0 \) is the minimizer of \( M(\theta) \).

**Proof.** Define \( \eta = (\theta_n - \theta_0) / \sigma_n \). At first we show that, \( \| \eta \|_2 \) is \( O(1) \), i.e. there exists some constant \( \Omega_1 \) such that \( \| \eta \|_2 \leq \Omega_1 \) for all \( n \):

\[ \| \theta_n - \theta_0 \|_2 \leq \frac{1}{\mu_n} (M(\theta_n) - M(\theta_0)) \quad \text{[Follows from Lemma 4.1]} \]

\[ \leq \frac{1}{\mu_n} (M(\theta_n) - M_n(\theta_n) + M_n(\theta_n) - M_n(\theta_n) - M_n(\theta_n) - M(\theta_0)) \]

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Hence:

\[ \frac{1}{\mu_1} (M(\theta_n) - M_n(\theta_n) + M_n(\theta_0) - M(\theta_0)) \quad \because M_n(\theta_n) - M_n(\theta_0) \leq 0 \]

\[ \leq \frac{2K_1}{\mu_1 \sigma_n} \quad \text{[from equation (17)]} \]

As \( \theta_n \) minimizes \( M_n(\theta) \):

\[ \nabla M_n(\theta_n) = E \left( (Y - \gamma) \bar{X} k \left( \frac{X^T \theta_n}{\sigma_n} \right) \right) = 0 \]

Hence:

\[ 0 = E \left( (Y - \gamma) \bar{X} k \left( \frac{X^T \theta_n}{\sigma_n} \right) \right) \]

\[ = \frac{(\beta_0 - \alpha_0)}{2} E \left( \bar{X} k \left( \frac{X^T \theta_0}{\sigma_n} \right) \left( \mathbb{I}(X^T \theta_0 \geq 0) - \mathbb{I}(X^T \theta_0 < 0) \right) \right) \]

\[ = \frac{(\beta_0 - \alpha_0)}{2} \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( \frac{z}{\sigma_n} + \bar{\eta}^T \bar{x} \right) f_0(z|\bar{x}) \, dz \, dP(\bar{x}) \]

\[ - \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( \frac{z}{\sigma_n} + \bar{\eta}^T \bar{x} \right) f_0(z|\bar{x}) \, dz \, dP(\bar{x}) \]

\[ = \sigma_n \frac{(\beta_0 - \alpha_0)}{2} \left[ \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( t + \bar{\eta}^T \bar{x} \right) f_0(\sigma_n t|\bar{x}) \, dt \, dP(\bar{x}) \right] \]

\[ - \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( t + \bar{\eta}^T \bar{x} \right) f_0(\sigma_n t|\bar{x}) \, dt \, dP(\bar{x}) \]

As \( \sigma_n^{(\beta_0-\alpha_0)} > 0 \), we can forget about it and continue. Also, as we have proved \( ||\tilde{\eta}|| = O(1) \), there exists a subsequence \( \eta_n \) and a point \( c \in \mathbb{R}^{p-1} \) such that \( \eta_n \to c \). Along that sub-sequence we have:

\[ 0 = \left[ \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( t + \bar{\eta}_n^T \bar{x} \right) f_0(\sigma_n t|\bar{x}) \, dt \, dP(\bar{x}) \right] \]

\[ - \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( t + \bar{\eta}_n^T \bar{x} \right) f_0(\sigma_n t|\bar{x}) \, dt \, dP(\bar{x}) \]

Taking limits on both sides and applying DCT (which is permissible as the loss is bounded) we conclude:

\[ 0 = \left[ \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( t + c^T \bar{x} \right) f_0(0|\bar{x}) \, dt \, dP(\bar{x}) \right] \]

\[ - \int_{\mathbb{R}^{p-1}} \bar{X} \int_0^\infty k \left( t + c^T \bar{x} \right) f_0(0|\bar{x}) \, dt \, dP(\bar{x}) \]

\[ = \left[ \int_{\mathbb{R}^{p-1}} \bar{X} f_0(0|\bar{x}) \int_{c^T \bar{x}}^\infty k(t) \, dt \, dP(\bar{x}) \right] \]

\[ - \int_{\mathbb{R}^{p-1}} \bar{X} f_0(0|\bar{x}) \int_{-\infty}^{c^T \bar{x}} k(t) \, dt \, dP(\bar{x}) \]
Now, taking the inner-products of both sides with respect to \( c \), we get:

\[
\mathbb{E} \left( c^\top \tilde{X} \left( 2K(c^\top \tilde{X}) - 1 \right) f_0(0|\tilde{X}) \right) = 0. \tag{16}
\]

By our assumption that \( K \) is symmetric kernel and that \( K(t) > 0 \) for all \( t \in (-1, 1) \), we easily conclude that \( c^\top \tilde{X} \left( 2K(c^\top \tilde{X}) - 1 \right) \geq 0 \) almost surely in \( \tilde{X} \) with equality iff \( c^\top X = 0 \), which is not possible unless \( c = 0 \). Hence we conclude that \( c = 0 \). This shows that any convergent subsequence of \( \eta_n \) converges to 0, which completes the proof. \( \square \)

4.7. Proof of Lemma 2.3

Proof. To obtain the rate of convergence of our kernel smoothed estimator we use Theorem 4.1: There are three key ingredients that one needs to take care of in order to apply this theorem:

1. Consistency of the estimator (otherwise the conditions of the theorem needs to be valid for all \( \eta \)).
2. The curvature of the population score function near its minimizer.
3. A bound on the modulus of continuity in a vicinity of the minimizer of the population score function.

Below, we establish the curvature of the population score function (item 2 above) globally, thereby obviating the need to establish consistency separately. Recall that the population score function was defined as:

\[
M_n(\theta) = \mathbb{E} \left( (Y - \gamma) \left( 1 - K \left( \frac{X^\top \theta}{\sigma_n} \right) \right) \right)
\]

and our estimator \( \hat{\theta}_n \) is the argmin of the corresponding sample version. Consider the set of functions \( \mathcal{H}_n = \{ h_\theta : h_\theta(x, y) = (y - \gamma) \left( 1 - K \left( \frac{x^\top \theta}{\sigma_n} \right) \right) \} \). Next, we argue that \( \mathcal{H}_n \) is a VC class of functions with fixed VC dimension. We know that the function \( (x, y) \mapsto x^\top \theta/\sigma_n : \theta \in \Theta \) has fixed VC dimension (i.e. not depending on \( n \)). Now, as a finite dimensional VC class of functions composed with a fixed monotone function or multiplied by a fixed function still remains a finite dimensional VC class, we conclude that \( \mathcal{H}_n \) is a fixed dimensional VC class of functions with bounded envelope (as the functions considered here are bounded by 1).

Now, we establish a lower bound on the curvature of the population score function \( M_n(\theta) \) near its minimizer \( \theta_n \):

\[
M_n(\theta) - M_n(\theta_n) \geq d_n^2(\theta, \theta_n)
\]

where

\[
d_n(\theta, \theta_n) = \sqrt{\frac{||\theta - \theta_n||^2}{\sigma_n^2} \mathbb{I} (||\theta - \theta_n|| \leq K\sigma_n) + ||\theta - \theta_n|| \mathbb{I} (||\theta - \theta_n|| > K\sigma_n)}
\]

for some constant \( K > 0 \). The intuition behind this compound structure is following: When \( \theta \) is in \( \sigma_n \) neighborhood of \( \theta_n \), \( M_n(\theta) \) behaves like a smooth quadratic function, but when it is away from the truth, \( M_n(\theta) \)
Combining, we have

\[ W \text{ we bound each summand separately:} \]

\[ \text{uniform over } \theta \]

\[ K \]

Here, the constant \( \epsilon \) that for any pair of positive constants \((\epsilon_1, \epsilon_2)\):

\[ \text{as small as possible, as we have } \]

\[ \epsilon_1 \text{ as small as possible, as we have } \]

\[ \| \theta - \theta_0 \| \leq (u_+ / 4) \| \theta - \theta_0 \| \]

\[ \text{where the last inequality holds for all large } n \text{ as proved in Lemma 2.4. Using Lemma 2.4 again, we conclude}\]

\[ \| \theta - \theta_0 \| \geq \left( \frac{2(2K_1 + \epsilon_1)}{u_+} + \epsilon_2 \right) \sigma_n \Rightarrow \| \theta - \theta_0 \| \geq \frac{2(2K_1 + \epsilon_1)}{u_+} \sigma_n \]
for all large $n$, which implies:

$$M_n(\theta) - M_n(\theta_0) \geq \frac{(u_- / 4)\|\theta - \theta_0\|}{\sigma_n} \left( \frac{2(2K_1 + \epsilon_1)}{u_-} + \epsilon_2 \right) \sigma_n$$

$$\geq \frac{(u_- / 4)\|\theta - \theta_0\|}{\sigma_n} \left( \frac{7K_1}{u_-} \right) \tag{18}$$

For the quadratic curvature, we perform a two step Taylor expansion: Define $\eta = (\theta - \theta_0)/\sigma_n$. We have:

$$\nabla^2 M_n(\theta) = \frac{\beta_0 - \alpha_0}{\sigma_n^2} \mathbb{E} \left[ \tilde{X} \tilde{X}^\top \kappa \left( \frac{X^\top \eta}{\sigma_n} \right) \left( \mathbb{I}(X^\top \theta_0 \leq 0) - \mathbb{I}(X^\top \theta_0 \geq 0) \right) \right]$$

$$= \frac{\beta_0 - \alpha_0}{\sigma_n^2} \mathbb{E} \left[ \tilde{X} \tilde{X}^\top \kappa \left( \frac{X^\top \theta_0 + \tilde{X}^\top \eta}{\sigma_n} \right) \left( \mathbb{I}(X^\top \theta_0 \leq 0) - \mathbb{I}(X^\top \theta_0 \geq 0) \right) \right]$$

$$= \frac{\beta_0 - \alpha_0}{\sigma_n^2} \mathbb{E} \left[ \tilde{X} \tilde{X}^\top \kappa \left( \int_{-\infty}^0 \left( \frac{z}{\sigma_n} + \tilde{X}^\top \eta \right) f_0(z|\tilde{X}) \, dz - \int_0^\infty \kappa \left( \frac{z}{\sigma_n} + \tilde{X}^\top \eta \right) f_0(z|\tilde{X}) \, dz \right) \right]$$

$$= \frac{\beta_0 - \alpha_0}{\sigma_n^2} \mathbb{E} \left[ \tilde{X} \tilde{X}^\top \kappa \left( \int_{-\infty}^0 \left( t + \tilde{X}^\top \eta \right) f_0(\sigma_n t|\tilde{X}) \, dt - \int_0^\infty \kappa \left( t + \tilde{X}^\top \eta \right) f_0(\sigma_n t|\tilde{X}) \, dt \right) \right]$$

$$= \frac{\beta_0 - \alpha_0}{\sigma_n^2} \mathbb{E} \left[ \tilde{X} \tilde{X}^\top \kappa \left( \int_{-\infty}^0 \left( t + \tilde{X}^\top \eta \right) dt - \int_0^\infty \kappa \left( t + \tilde{X}^\top \eta \right) dt \right) \right] + R$$

$$= \frac{\beta_0 - \alpha_0}{\sigma_n^2} \mathbb{E} \left[ \tilde{X} \tilde{X}^\top \kappa \left( \int_{-\infty}^0 \left( t + \tilde{X}^\top \eta \right) dt - \int_0^\infty \kappa \left( t + \tilde{X}^\top \eta \right) dt \right) \right] + R \tag{19}$$

For further analysis, define

$$\Lambda(\kappa) = \inf_{|v_1|=1,|v_2|=1} \mathbb{E}_{\tilde{X}} \left[ |v_1^\top \tilde{X}|^2 f(0|\tilde{X}) \mathbb{I}((|v_2^\top \tilde{X}| \leq \kappa) \right]$$

We show that $\Lambda(\kappa) > 0$ for all $\kappa > 0$ via contradiction. On the contrary, suppose $\Lambda(\kappa) = 0$ for some $\kappa > 0$. Then we have:

$$\inf_{|v_1|=1,|v_2|=1} \mathbb{E} \left[ |v_1^\top \tilde{X}|^2 f(0|\tilde{X}) \mathbb{I}((|v_2^\top \tilde{X}| \leq \kappa) \right] = 0$$

Hence, there exists $\tilde{v}_1, \tilde{v}_2 \in S^{p-1}$ such that $\mathbb{E} \left[ |\tilde{v}_1^\top \tilde{X}|^2 f(0|\tilde{X}) \mathbb{I}((|\tilde{v}_2^\top \tilde{X}| \leq \kappa) \right] = 0$, which further implies $|\tilde{v}^\top \tilde{X}| = 0$ almost surely on the set $|\tilde{v}_2^\top \tilde{X}| \leq \kappa$. This violates Assumption 4. Hence, our claim is demonstrated. Now, fix a $\kappa$ (to be chosen later) and assume $|\tilde{\eta}| \leq 1/2\kappa$. Then, for any vector $v$ with $|v| = 1$,

$$\nabla^2 M_n(\theta) \geq \frac{(u_- / 4)\|\theta - \theta_0\|}{\sigma_n} \left( \frac{7K_1}{u_-} \right) \tag{18}$$
\[= \mathbb{E} \left[ |v^\top X|^2 f_0(0) k(X^\top \eta) 1(|\eta^\top X| \leq 1) \right] \]
\[\geq \mathbb{E} \left[ |v^\top X|^2 f_0(0) k(X^\top \eta) 1(|\eta^\top X| \leq 1/2) \right] \]
\[\geq k(1/2) \mathbb{E} \left[ |v^\top X|^2 f_0(0) X 1(|\eta^\top X| \leq 1/2|\eta|) \right] \quad [\zeta = \eta/|\eta|] \]
\[\geq k(1/2) \mathbb{E} \left[ |v^\top X|^2 f_0(0) X 1(|\eta^\top X| \leq \kappa) \right] \quad [\because |\eta| \leq 1/2k] \]
\[\geq k(1/2) \Lambda(\kappa). \quad (20) \]

To analyze the behavior of the residual term \( R \), fix \( v \in \mathbb{R}^{n-1} \). Then:
\[
\left| v^\top R v \right| 
= \frac{1}{\sigma_n} \mathbb{E} \left[ (v^\top X)^2 \left[ \int_{-\infty}^{0} k'(t + X^\top \eta) (f_0(\sigma_n t X) - f_0(0 X)) \, dt - \int_{0}^{\infty} k'(t + X^\top \eta) (f_0(\sigma_n t X) - f_0(0 X)) \, dt \right] \right] 
\leq \mathbb{E} \left[ (v^\top X)^2 h(X) \int_{-\infty}^{0} |k'(t + X^\top \eta)||t| \, dt \right] 
\leq \mathbb{E} \left[ (v^\top X)^2 h(X) \int_{-1}^{1} |k'(t)| |t - X^\top \eta| \, dt \right] 
\leq \mathbb{E} \left[ (v^\top X)^2 h(X)(1 + ||X||/2k) \int_{-1}^{1} |k'(t)| \, dt \right] = C_1 \quad \text{[say]} \quad (21) 
\]
by Assumption [1] and Assumption [5].

By a two-step Taylor expansion, we have:
\[ M_n(\theta) - M_n(\theta_n) = \frac{1}{2} (\theta - \theta_n)^\top \nabla^2 M_n(\theta_n) (\theta - \theta_n) \]
for some intermediate point \( \theta^*_n \). Plugging in \( \theta = \theta^*_n \) in (19), and using the bounds (20) and (21), we conclude that for \( ||\theta^*_n - \theta_n|| \leq \sigma_n/2k \):
\[ M_n(\theta) - M_n(\theta_n) \geq \frac{(\beta_0 - \alpha_0) k(1/2)}{2\sigma_n} \Lambda(\kappa) ||\theta - \theta_n||^2 + \frac{1}{2} (\theta - \theta_n)^\top R(\theta - \theta_n) \]
\[ \geq \frac{(\beta_0 - \alpha_0) k(1/2)}{2} \Lambda(\kappa) ||\theta - \theta_n||^2 - C_1 \sigma_n ||\theta - \theta_n||^2 \sigma_n \]
\[ \geq \frac{(\beta_0 - \alpha_0) k(1/2)}{4} \Lambda(\kappa) \frac{||\theta - \theta_n||^2}{\sigma_n} \quad \text{[For all large } n \text{]} \]

Hence we conclude that:
\[ M_n(\theta) - M_n(\theta_n) \geq \frac{||\theta - \theta_n||^2}{\sigma_n} 1(||\theta_n^* - \theta_n|| \leq 2K \sigma_n) , \]
with \( K = 1/4\kappa \) where \( \kappa \) is the same constant as in equation [20].

Now, for \( n \) large enough it is the case that \( ||\theta_n - \theta_n|| \leq K \sigma_n \), since \( ||\theta_n - \theta_n||/\sigma_n \to 0 \) (see Lemma [2.4]). So, if \( \theta \) satisfies \( ||\theta - \theta_n|| \leq K \sigma_n \), then \( ||\theta_n^* - \theta_n|| \leq K \sigma_n \) (as \( \theta_n^* \) is an intermediate point between \( \theta \) and \( \theta_n \)) by the triangle inequality \( ||\theta_n^* - \theta_n|| \leq 2K \sigma_n \). This implies that for all large \( n \), \( 1(||\theta - \theta_n|| \leq K \sigma_n) \leq 1(||\theta_n^* - \theta_n|| \leq 2K \sigma_n) \), which further implies:
\[ M_n(\theta) - M_n(\theta_n) \geq \frac{||\theta - \theta_n||^2}{\sigma_n} 1(||\theta - \theta_n|| \leq K \sigma_n) \quad (22) \]
for all large $n$. Combining the conclusions from equation [18] and [22] and by choosing $\kappa = u_\tau / (28K_1)$, we derive the lower bound on the curvature of our M-estimation problem with $\mathcal{K} = 7K_1 / u_\tau$.

Finally, we bound the modulus of continuity to apply Theorem 4.1.

$$\mathbb{E} \left( \sup_{d_n(\theta, \hat{\theta}) \leq 6} |(M_n - M_n)(\theta) - (M_n - M_n)(\hat{\theta}_n)| \right).$$

To that end, we use Theorem 4.2. In our case, the envelope $U$ can be taken as 1. To apply the maximal inequality in the theorem we need a bound on $\sup_{f \in F} P(f - Pf)^2 \leq \sup_{f \in F} P f^2$, where, in our case

$$\mathcal{F} = \mathcal{F}_\delta = \left\{ f_\theta : f_\theta(x, y) = (y - \gamma) \left( K \left( \frac{x^\top \theta_n}{\sigma_n} \right) - K \left( \frac{x^\top \hat{\theta}}{\sigma_n} \right) \right), d_n(\theta, \theta_n) \leq \delta \right\}.$$

The collection $\mathcal{F}$ is also a VC class of functions with dimension free of $n$, which can be deduced using the same line of argument as was employed for $\mathcal{H}_n$. To bound the variance of the functions in this class we rely on $f^{(n)}$, the conditional density of $X^\top \theta_n$ given $\tilde{X}$.

$$\mathbb{E}(f_\theta^2) \leq \mathbb{E} \left[ K \left( \frac{x^\top \theta_n}{\sigma_n} \right) - K \left( \frac{x^\top \hat{\theta}}{\sigma_n} \right) \right]^2
= \mathbb{E} \left[ \int_{-\infty}^{\infty} \left( K \left( \frac{z}{\sigma_n} \right) - K \left( \frac{z}{\sigma_n} + \tilde{\eta}^\top \tilde{X} \right) \right)^2 f_\theta(z|\tilde{X}) \, dz \right]
= \sigma_n \mathbb{E} \left[ \int_{-\infty}^{\infty} \left( K(t) - K(t + \tilde{\eta}^\top \tilde{X}) \right)^2 f_\theta(\sigma_n|\tilde{X}) \, dt \right]
= \sigma_n \int_{\mathbb{R}^+} \left[ \int_{-\infty}^{\infty} \left( K(t) - K(t + \tilde{\eta}^\top \tilde{x}) \right)^2 f_\theta(\sigma_n|\tilde{x}) \, dt \right] \, dP(\tilde{x})
\leq \sigma_n \int_{\mathbb{R}^+} m(\tilde{x}) \left[ \int_{-\infty}^{\infty} \left( K(t) - K(t + \tilde{\eta}^\top \tilde{x}) \right)^2 \, dt \right] \, dP(\tilde{x}), \quad [m(\tilde{x}) \text{ is defined in Assumption 3}] (23)$$

where the last inequality follows from Assumption 3.

First assume that $0 \leq \tilde{\eta}^\top \tilde{x} \leq 2$. Then we have:

$$K(t + \tilde{\eta}^\top \tilde{x}) - K(t) = \begin{cases} 0 & t \leq -1 - \tilde{\eta}^\top \tilde{x} \\ K(t + \tilde{\eta}^\top \tilde{x}) - K(t) & -1 - \tilde{\eta}^\top \tilde{x} \leq t \leq -1 \\ K(t + \tilde{\eta}^\top \tilde{x}) - K(t) & -1 \leq t \leq 1 - \tilde{\eta}^\top \tilde{x} \\ 1 - K(t) & 1 - \tilde{\eta}^\top \tilde{x} \leq t \leq 1 \\ 0 & t \geq 1 \end{cases}$$

Hence:

$$\int_{-\infty}^{\infty} \left( K(t) - K(t + \tilde{\eta}^\top \tilde{x}) \right)^2 \, dt
\leq \int_{-1 - \tilde{\eta}^\top \tilde{x}}^{1 - \tilde{\eta}^\top \tilde{x}} K^2(t + \tilde{\eta}^\top \tilde{x}) \, dt + \int_{-1}^{1 - \tilde{\eta}^\top \tilde{x}} \left( K(t + \tilde{\eta}^\top \tilde{x}) - K(t) \right)^2 \, dt + \int_{1 - \tilde{\eta}^\top \tilde{x}}^{1} (1 - K(t))^2 \, dt$$

22
From equation 23, we have:

\[
\begin{aligned}
&= \int_{-1}^{1} K^2(t) \, dt + \int_{-1}^{1} (K(t + \eta^T \tilde{x}) - K(t))^2 \, dt + \int_{-1}^{1} (1 - K(t))^2 \, dt \\
&= 2 \int_{-1}^{1} (1 - K(t))^2 \, dt + \int_{-1}^{1} (K(t + \eta^T \tilde{x}) - K(t))^2 \, dt \\
&= 2 \int_{-1}^{1} (K(t + \eta^T \tilde{x}) - K(t))^2 \, dt + \int_{-1}^{1} (1 - K(t))^2 \, dt \\
&= 2 \int_{-1}^{1} (1 - t)^2 (k(t))^2 \, dt + \int_{-1}^{1} (\eta^T \tilde{x})^2 (k(t))^2 \, dt \\
&\leq K_+^2 [2 \int_{-1}^{1} (1 - t)^2 \, dt + \int_{-1}^{1} (\eta^T \tilde{x})^2 \, dt] \\
&= K_+^2 [2 \int_{0}^{\eta^T \tilde{x}} t^2 \, dt + (\eta^T \tilde{x})^2 (2 - \eta^T \tilde{x})] \\
&= K_+^2 [2(\eta^T \tilde{x})^2 - (\eta^T \tilde{x})^3 / 3] \\
&\leq 2K_+^2 (\eta^T \tilde{x})^2
\end{aligned}
\]

Now suppose \( \eta^T \tilde{x} \geq 2 \). Then:

\[
K(t + \eta^T \tilde{x}) - K(t) = \begin{cases} 
0 & t \leq -1 - \eta^T \tilde{x} \\
K(t + \eta^T \tilde{x}) - 1 - \eta^T \tilde{x} & -1 - \eta^T \tilde{x} \leq t \leq -1 \\
1 & 1 - \eta^T \tilde{x} \leq t \leq 1 \\
1 - K(t) & -1 \leq t \leq 1 \\
0 & t \geq 1
\end{cases}
\]

From equation 23 we have:

\[
\begin{aligned}
&\int_{-\infty}^{\infty} (K(t) - K(t + \eta^T \tilde{x}))^2 \, dt \\
&= \int_{-1}^{1} K^2(t) \, dt + \int_{-1}^{1} dt + \int_{-1}^{1} (1 - K(t))^2 \, dt \\
&= 2 \int_{-1}^{1} K^2(t) \, dt + (\eta^T \tilde{x} - 2) \\
&\leq 2\eta^T \tilde{x} \quad \therefore \int_{-1}^{1} K^2(t) \, dt \leq 2
\end{aligned}
\]

A similar calculation produces analogous bounds for \( \eta^T \tilde{x} \leq 0 \). Therefore,

\[
P_f \leq C_1 \sigma_n \mathbb{E}[m(\tilde{X}) \{(\eta^T \tilde{X})^2 \mathbb{1} (|\eta^T \tilde{X}| \leq 2) + |\eta^T \tilde{X}| \mathbb{1} (|\eta^T \tilde{X}| \geq 2)\}]
\]

for some constant \( C_1 \). To analyze the term inside the expectation, note:

\[
\mathbb{E}[m(\tilde{X}) \{(\eta^T \tilde{X})^2 \mathbb{1} (|\eta^T \tilde{X}| \leq 2) + |\eta^T \tilde{X}| \mathbb{1} (|\eta^T \tilde{X}| \geq 2)\}] \leq 2\mathbb{E}(m(\tilde{X})|\eta^T \tilde{X}|) \leq 2\|\eta\|\mathbb{E}(m(\tilde{X})||\tilde{X}|). \quad (24)
\]
4.8. Proof of Corollary 2.1

Combining these two terms, we conclude:

\[ \mathbb{E}\left[n(\hat{X}) \mathbb{I}(n^{\top} \hat{X} \leq 2) + n(\eta^{\top} \hat{X})(n^{\top} \eta \geq 2)\right] \leq \frac{1}{2} \mathbb{E}(m(\hat{X})(\eta^{\top} \hat{X})^{2}) \leq \frac{1}{2} \|\eta\|^{2} \mathbb{E}(m(\hat{X})\|\hat{X}\|^{2}). \]  

(25)

Combining equations 24 and 25, we conclude that:

\[ P_{f_{\delta}}^{2} \leq C_1 \sigma_n \left[ 2\|\eta\|\mathbb{E}(m(\hat{X})\|\hat{X}\|) + \frac{1}{2} \|\eta\|^{2} \mathbb{E}(m(\hat{X})\|\hat{X}\|^{2}) \right] \leq C_2 \sigma_n \left[ \|\eta\|\wedge \|\eta\|^{2} \right] \quad \text{[By Assumption 1 and 3]} \leq C_2 \sigma_n \left[ \|\eta\| \mathbb{I}(\|\hat{X}\| \leq \mathcal{K}) + \|\eta\| \mathbb{I}(\|\hat{X}\| > \mathcal{K}) \right] \leq C_2 d_{f_n}(\theta, \theta_n). \]  

(26)

It follows directly from equation 25 that \( \sup_{f \in \mathcal{F}} P f^{2} \leq C_2 d_n^2 \). Hence, a valid choice for \( \sigma_n \) in Theorem 4.2 is \( \sqrt{C_2 \delta} \). Using this in the maximal inequality yields:

\[ \mathbb{E}\left( \sup_{\delta_n(\theta, \theta_n) \leq \delta} \|\delta_n(\theta, \theta_n) - M_n(\theta, \theta_n)\| \right) \leq \delta \frac{\sqrt{\log 1/\delta}}{\sqrt{n}}. \]

Thus, we can choose \( \phi_n(\delta) = \delta \frac{\sqrt{\log 1/\delta}}{\sqrt{n}} \). Hence, we have \( r_n d_n(\theta, \theta_n) = O_P(1) \) where \( r_n \) satisfies

\[ r_n^2 \phi_n\left(\frac{1}{r_n}\right) \leq \sqrt{n}. \]

A valid choice for \( r_n = \sqrt{n/\log n} \) which concludes the proof.

4.8. Proof of Corollary 2.1

Proof. In Lemma 2.3, we have established:

\[ \frac{n}{\log n} \left[ \frac{\|\hat{\theta}_n - \theta_n\|^{2}}{\sigma_n} \mathbb{I}(\|\hat{\theta}_n - \theta_n\| \leq \mathcal{K} \sigma_n) + \|\hat{\theta}_n - \theta_n\| \mathbb{I}(\|\hat{\theta}_n - \theta_n\| > \mathcal{K} \sigma_n) \right] = O_P(1) \]  

(27)

From equation 27, we have

1. \[ \sqrt{n} \frac{1}{\sigma_n \log n} \|\hat{\theta}_n - \theta_n\| \mathbb{I}(\|\hat{\theta}_n - \theta_n\| \leq \mathcal{K} \sigma_n) = O_P(1) \]

2. \[ \frac{n}{\log n} \|\hat{\theta}_n - \theta_n\| \mathbb{I}(\|\hat{\theta}_n - \theta_n\| > \mathcal{K} \sigma_n) = O_P(1) \]

Combining these two terms, we conclude:

\[ \sqrt{n} \frac{1}{\sigma_n \log n} \|\hat{\theta}_n - \theta_n\| \mathbb{I}(\|\hat{\theta}_n - \theta_n\| \leq \mathcal{K} \sigma_n) + \frac{n}{\log n} \|\hat{\theta}_n - \theta_n\| \mathbb{I}(\|\hat{\theta}_n - \theta_n\| > \mathcal{K} \sigma_n) = O_P(1) \]

which further implies:

\[ t_n \|\hat{\theta}_n - \theta_n\| = O_P(1) \]

where \( t_n = \sqrt{n/(\sigma_n \log n)} \wedge (n/\log n) \). Now suppose \( \log n/\sigma_n \to 0 \). Then, it is immediate that:

\[ \text{...} \]
1. \[
\frac{\sqrt{n}}{\log(n)\sigma_n} \gg \frac{1}{\sigma_n} \text{ as } \sigma_n \frac{\sqrt{n}}{\log(n)\sigma_n} = \sqrt{\frac{n\sigma_n}{\log(n)}} \to \infty.
\]

2. \[
\frac{n}{\log n} \gg \frac{1}{\sigma_n} \text{ as } \frac{\log n}{\sigma_n} \to 0.
\]

Hence \( t_n \gg \sigma_n^{-1} \), which implies \( \|\hat{\theta}_n - \theta_n\|/\sigma_n \overset{p}{\to} 0 \). This concludes the proof.

\[\square\]

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