Reconstruction and stability in Gel’fand’s inverse interior spectral problem

Roberta Bosi, Yaroslav Kurylev, and Matti Lassas

December 10, 2019

Abstract

Assume that $M$ is a compact Riemannian manifold of bounded geometry given by restrictions on its diameter, Ricci curvature and injectivity radius. Assume we are given, with some error, the first eigenvalues of the Laplacian $\Delta_g$ on $M$ as well as the corresponding eigenfunctions restricted on an open set in $M$. We then construct a stable approximation to the manifold $(M, g)$. Namely, we construct a metric space and a Riemannian manifold which differ, in a proper sense, just a little from $M$ when the above data are given with a small error. We give an explicit log log-type stability estimate on how the constructed manifold and the metric on it depend on the errors in the given data. Moreover a similar stability estimate is derived for the Gel’fand’s inverse problem. The proof is based on methods from geometric convergence, a quantitative stability estimate for the unique continuation and a new version of the geometric Boundary Control method.

1 Introduction

1.1 Inverse interior spectral data and classes of manifolds

Let $(M, g, p)$ be a pointed compact Riemannian manifold, that is, $(M, g)$ is a compact Riemannian manifold without boundary and $p \in M$ is a point on $M$. Let $\Delta_g$ be the Laplace operator on $(M, g)$, with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$
The inverse problem is to reconstruct, in a stable way, the topology, the differentiable structure, and the metric of an unknown Riemannian manifold \((M, g)\), when one is given an open ball \(B = B(p, r_0) \subset M\), the eigenvalues \(\lambda_j\) and the restrictions \(\varphi_j|_B\) of the eigenfunctions in the ball \(B\). We also study the problem of reconstructing an approximation of \((M, g)\) when only finitely many eigenvalues and eigenfunctions are given with errors.

being its eigenvalues and \(\varphi_j, j = 0, 1, 2, \ldots\) being the complete sequence of \(L^2(M)\)-orthonormal eigenfunctions satisfying \(-\Delta_g \varphi_j = \lambda_j \varphi_j\) on \(M\).

**Definition 1** Let \((M, g, p)\) be an \(n\) dimensional compact pointed manifold with \(n \geq 2\). Let \(r_0 > 0\). Then

(i) The pair, consisting of the ball \((B(p, r_0), g|_{B(p,r_0)})\) on the Riemannian manifold \(M\) and the sequence \(\{(\lambda_j, \varphi_j|_{B(p,r_0)}); j = 0, 1, 2, \ldots\}\) of eigenvalues and eigenfunctions, is called the interior spectral data (ISD) of \((M, g, p)\).

(ii) The pair, consisting of the ball \((B(p, r_0), g|_{B(p,r_0)})\) and a finite collection \(\{(\lambda_j, \varphi_j|_{B(p,r_0)}); j = 0, 1, 2, \ldots, J\}\) of the \(J + 1\) first eigenvalues and eigenfunctions, is called the finite interior spectral data (FISD) of \((M, g, p)\).

The interior Gel’fand inverse spectral problem is that of the reconstruction of \((M, g)\) from its ISD. It was solved in [36], [35]. In this paper we consider the problem of an approximate reconstruction of \((M, g)\) when we know only its FISD, namely, the first eigenvalues, \(\lambda_j < \delta^{-1}\) with some small \(\delta \in (0, 1)\) and the corresponding eigenfunctions of \(\varphi_j|_{B(p,r_0)}\). Furthermore, we assume that we know all these objects with some error. However, due to the well-known ill-posedness of inverse problems, to achieve this goal one needs to assume that the manifold to be approximately reconstructed should lie in a properly chosen class of manifolds. In this paper we concentrate on an appropriate Gromov’s class of pointed manifolds.
Next we define a class of manifolds satisfying geometric bounds, in terms of the constants $R, D, i_0,$ and $n,$ and the radius $r_0$. Those constants have to be considered as global parameters in all calculations.

**Definition 2** (Riemannian manifolds of bounded geometry). For any $n \in \mathbb{Z}_+$ and $R > 0, \ D > 0, \ i_0 > 0$, $M_n := M_n(R, D, i_0)$ consists of $n$-dimensional pointed compact Riemannian manifolds $(M, g, p)$ such that

\begin{align}
   i) & \sum_{j=0}^{3} \| \nabla^j \text{Ric}(M, g) \|_{L^\infty(M, g)} \leq R, \\
   ii) & \text{diam}(M, g) \leq D, \\
   iii) & \text{inj}(M, g) \geq i_0.
\end{align}

Here $\text{Ric}(M, g) = \text{Ric}_{jk}^M$ stands for the Ricci curvature of $M$, $\text{diam}(M, g)$ for the diameter of $M$, and $\text{inj}(M, g)$ for the injectivity radius of $(M, g)$. At last, $\nabla$ stands for the covariant derivative on $(M, g)$.

The norm of $\nabla^j \text{Ric}(M, g)$ is computed using the metric $g$, e.g. $\| \nabla \text{Ric}^M \| = (g^{ii'} g^{j'j} g^{kk'} (\nabla_i \text{Ric}_{jk}^M) (\nabla_{i'} \text{Ric}_{j'k'}^M))^{1/2}$.

We recall that a pointed compact Riemannian manifold $(M, g, p)$ consists of a manifold $M$, its Riemannian metric $g$, and an arbitrary point $p \in M$. This definition is used as we specify the point $p$ near which the values of the eigenfunctions are measured.

In the future, without loss of generality, we assume

$$r_0 < \min \left( \frac{i_0}{2}, \frac{\pi}{2 \sqrt{K}}, 1 \right).$$

Here $K$ is the bound for the sectional curvature on $M_n$. The bound $K$ depends only on $R, D, i_0,$ and $n$, see [17]. This makes it possible to use in $B(p, r_0)$ the Riemannian normal coordinates which allows us to compare interior spectral data of different manifolds in $M_n$. To formalise the above, let $B(r_0) \subset \mathbb{R}^n$ be an Euclidian ball of radius $r_0$ and $h$ be some Riemannian coordinates in $B(r_0)$ making it a ball of radius $r_0$ with respect to $h$. Let $D$ be a collection of elements (Data Sequences)

$$DS = \left( (B(r_0), h), \{ (\mu_j, \psi_j|_{B(r_0)}) \}_{j=0}^{\infty} \right)$$

where $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots, \mu_j \to \infty,$ and $\psi_j \in L^2(B(r_0), h)$. 

3
Consider a self-adjoint operator $A_1$ having a compact inverse and its perturbation $A_2 = A_1 + B$, where the operator norm of the self-adjoint operator $B$ is small. Then the eigenvalues $(\mu^1_j)_{j \in \mathbb{N}}$ of the operator $A_1$ and the eigenvalues $(\mu^2_j)_{j \in \mathbb{N}}$ of $A_2$ are $\delta$-close with some small $\delta$. Note that the eigenvalues change continuously in small perturbations, but the eigenvalues may change order, and several eigenvalues can move together forming an eigenvalue of a higher multiplicity. However, the eigenvalues can be grouped together to clusters contained in separated intervals $[a_p, b_p] \subset \mathbb{R}$. The vector spaces spanned by the eigenvectors in such clusters change continuously in small perturbations. This motivates Definition 3.

**Definition 3 (Interior spectral topology.)** Let $\delta > 0$. For $i = 1, 2$, consider the collections $DS^i \in D$.

We say that $DS^1$ and $DS^2$ are $\delta$-close if the following is valid: There are $P \in \mathbb{Z}^+$ and disjoint intervals

$$I_p = (a_p, b_p) \subset (-\delta, \delta^{-1} + \delta), \quad p = 0, 1, \ldots, P,$$

such that

i) $b_p - a_p < \delta$.

ii) For any $\mu^i_j$, $i = 1, 2$ with $|\mu^i_j| < \delta^{-1}$ there is $p$ such that $\mu^i_j \in I_p$.

iii) For $p = 0$, $n_0^1 = 1$. For any $p \geq 1$, the total number $n^i_p$ of elements in sets $J^i_p = \{j \in \mathbb{Z}^+: \mu^i_j \in I_p\}$ coincide, i.e. $n^1_p = n^2_p = n^i_p$, and satisfies $n^i_0 = n^2_0 \geq 1$.

iv) There is an orthogonal matrix $O \in O(n)$, such that the metrics $O_\ast h_1$ and $h_2$ are Lipschitz $\delta$-close on $B(r_0)$, i.e., for any $x \in B(r_0)$ and $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$, $\xi \neq 0$, we have

$$\left(1 + \delta\right)^{-1} \leq \frac{(O_\ast h_1)_{jk}(x) \xi^j \xi^k}{(h_2)_{jk}(x) \xi^j \xi^k} \leq 1 + \delta.$$

\[4\]
v) For any $p$ there is a unitary matrix

$$A_p = \left[ a_{jk}^{(p)} \right]_{j,k \in J_p} \in U(n_p),$$

such that

$$\| A_p \cdot (O \ast \Psi_1^p) - \Psi_2^p \|_{(L^2(B(\rho_0), h_2))^n_p} \leq \delta,$$  \hspace{1cm} (6)

$$\| A_p^{-1} \cdot (O^{-1} \ast \Psi_1^p) - \Psi_2^p \|_{(L^2(B(\rho_0), h_1))^n_p} \leq \delta.$$  \hspace{1cm} (7)

Here, $\Psi_i^p$ is the vector-function $\{\psi_j\}_{j \in J_p}$. Note that above the number $P$ indicates how many groups of eigenvalues are clustered to satisfy conditions i-iv. Moreover, for two sequences $DS^1$ and $DS^2$, the above conditions i-iv may be valid with several different values of $P$ and intervals $I_p$, $p = 1, 2, \ldots, P$.

**Remark 1** Condition v) can be interpreted as the closedness of the Riesz projectors corresponding to $\Delta g_i$ onto $I_p$.

We note that in a more restricted context of Gelfand’s inverse problem for a Schrödinger operator with simple spectrum in a domain in $\mathbb{R}^n$ a similar topology was introduced in [2].

### 1.2 The main results

To formulate our result on an approximate reconstruction, we use the Gromov-Hausdorff distance.

**Definition 4** (GH-topology, see e.g. [23], [13]). Let $(X^i, d^i, p^i)$, $i = 1, 2$ be pointed compact metric spaces. Then the pointed Gromov-Hausdorff distance $d_{GH}(X^1, X^2)$ is the infimum of all $\varepsilon > 0$ such that there is a metric space $(Z, d_Z)$ and isometric embeddings $i_1 : X^1 \to Z$ and $i_2 : X^2 \to Z$ which satisfy

$$d_H(i_1(X^1), i_2(X^2)) < \varepsilon, \quad d_Z(i_1(p^1), i_2(p^2)) < \varepsilon.$$

Here $d_H$ denotes the Hausdorff distance in $Z$, see [13].

The main result of the paper is:
Theorem 1 Let \( n \geq 2, R, D, i_0 \) and \( r_0 \) satisfying (2) be given. Then there exist \( C_1 > 1 \) and \( C_2 > 0 \), depending only on \( n, R, D, i_0 \) and \( r_0 \), such that the following is true:

Let \((M^{(1)}, g^{(1)}), (M^{(2)}, g^{(2)}, p^{(2)}) \in \overline{M}_n\). Assume that the interior spectral data of the operators \(-\Delta_{g^{(i)}}\) on \( M^{(i)} \) in the balls \( B^{(i)} = B_{M^{(i)}}(p^{(i)}, r_0) \subset M^{(i)} \), that is, the collections

\[
( (B^{(i)}, g^{(i)}), \{ (\lambda_j^{(i)}, \varphi_j^{(i)}|_{B^{(i)}}); j = 0, 1, 2, \ldots \})
\]

are \( \delta \)-close, in the sense of Definition 3, with \( 0 < \delta \leq \exp(-e) \). Then

\[
d_{GH}((M^{(1)}, p^{(1)}), (M^{(2)}, p^{(2)})) \leq C_1 (\ln (\ln \frac{1}{\delta}))^{-C_2}.
\]

The above stability estimate is log—log type. It is not known if this type of result is optimal, but the counterexamples of Mandache [39] for equivalent inverse problem show that the stability result can not be better than logarithmic.

The proof of Theorem 1 is constructive, and is based on the following result on the reconstruction of the manifold from the data. Below, when we state that a manifold \((M^*, g^*)\) can be constructed from the data, we mean that there is a sequence of steps, where we solve a finite number of quadratic minimization problems in finite dimensional spaces, choose elements from finite sets or compute certain explicit functions. Indeed, we do the following steps. First, we solve quadratic minimization problems in finite dimensional vector spaces (that are equivalent to solving linear equations) to find the finite sequences \((d^i(\alpha, i))_{j=0}^{j_1} \in \mathbb{R}^{j_1}\), where \((\alpha, i)\) run over a finite index set, see Theorem 4. Second, we use these sequences to compute approximative volumes \(\text{vol}^i(M^*_M(\beta))\), of subsets of \(M\), where \((i, \beta)\) runs over a finite index set, see Lemma 9. Third, we choose the set of admissible indexes \(\beta\) for which the approximative volumes are larger than a certain threshold value, see Definition 7. The admissible indexes are used in Section 6.1 and Lemma 11 to define a finite set of piecewise constant functions, \(R^*_M\), that approximate the collection of the interior distance functions. Using the finite set \(R^*_M\) and a modified version of the construction given in [30] we construct a finite metric space \((M^*, d^*)\), that approximates the Riemannian manifold \((M, \text{dist}_g)\) in Gromov-Hausdorff sense.

\[\]
Proposition 1  Let \( n \geq 2, R, D, i_0 \) and \( r_0 \) satisfying \([2]\) be given. Then there exists a constant \( \delta^* = \delta^*(n, R, D, i_0, r_0) \) and positive constants \( C_2 < 1 \) and \( C_3 > 1 \), depending only on \( n, R, D, i_0 \) and \( r_0 \), such that, for all \( \delta \) with
\[
0 < \delta \leq \delta^*,
\]
the following is true:
Assume that \((M,g,p) \in \overline{M}_n\) and we are given a collection
\[
\{(B(r_0), g^a), \{\mu_j, \varphi_j^a|_{B(r_0)}; j = 0, 1, 2, \ldots, J\}\}
\]
that is \( \delta \)-close, in the sense of Definition \([3]\) to interior spectral data of the operator \(-\Delta_g\) on \( M \).
Using the data \([11]\) we can construct a pointed metric space \((M^*, d^*, p^*)\) such that
\[
d_{GH}(M, M^*) \leq \varepsilon, \quad \text{where} \quad \varepsilon = C_3 \left( \ln \left( \frac{1}{\delta} \right) \right)^{-C_2}.
\]

We note that in Proposition \([1]\) that value of \( J \) is not fixed, but it just has to be so large that every \( j \), for which the eigenvalue \( \lambda_j \) satisfies \( \lambda_j < \delta^{-1} + \delta \), fulfils the inequality \( j \leq J \), see \([4]\). The relation \( J \), i.e., the number of eigenvalues, and the accuracy parameter \( \delta \) is discussed in Remark \([4]\) below.

**A note on the used constants.** In the main part of the paper, we will make frequent use of constants \( c, C, C_1, C_2, \) etc. These constants will depend only on the geometric bounds \( n, R, D, i_0, r_0 \), see Definition \([2]\), but may change in their value from line to line. The constants that depend only on the geometric bounds \( n, R, D, i_0, r_0 \) will be called ‘uniform constants’. When we define a constant for the first time, we specify whether it is uniform or not and write its further dependencies in parenthesis. For example the constant \( C(s, m) \) (or \( C_{s,m} \)) depends also on \( s \) and \( m \). Before Appendix we have collected a table on the locations where the constants \( C_k \) and \( c_k \) are defined. Conventions for constants in the Appendix are explained in each subsection.

Also \( \overline{M}_n \) is the closure of \( M_n \) in the GH topology. Parameter \( K \) above and in Corollary \([1]\) below is the bound for the sectional curvature which is uniform, see \([17]\), on \( \overline{M}_n \).

**Remark 2** As shown in section 2.1 the class \( \overline{M}_n \) is compact. Thus, when checking condition v) of definition \([3]\) it is sufficient to use the standard \( L^2 \)-norm on \( B(r_0) \).
Recall that the for pointed $C^1$-diffeomorphic manifolds $(M_1, p_1)$ and $(M_2, p_2)$ the Lipschitz distance is

$$
d_L((M_1, p_1), (M_2, p_2)) = \inf_{F: M_1 \to M_2} \left( \ln(\text{Lip}(F)) + \ln(\text{Lip}(F^{-1})) + \right.$$ \begin{align*}
&\left. + d_{M_2}(p_2, F(p_1)) + d_{M_1}(p_1, F^{-1}(p_2)) \right) \tag{13}
\end{align*}

where the infimum is taken over bi-Lipschitz maps $F: M_1 \to M_2$ and $\text{Lip}(F)$ is the Lipschitz-constant of the map $F$, see [22]. Inequality (12) combined with the sectional curvature bound (17) and the solution of the geometric Whitney problem [20, Thm. 1, Cor. 1.9] implies the following stable construction result for the manifold $M$ in the Lipschitz topology.

**Corollary 1** Let $(M, g, p) \in \overline{M}_n$, $\delta > 0$, and the metric space $M^*$ be as in Proposition [7]. Using $M^*$ one can construct a smooth pointed Riemannian manifold $(N, g_N, p_N)$ such that $|\text{Sec}(N)| \leq C_4 K$, $\text{inj}(N) \geq \min\{(C_4 K)^{-1/2}, (1 - C_5 K^{1/3} \sigma_0^{2/3}) i_0\}$, and $M$ and $N$ are diffeomorphic. Moreover,

$$
d_L((M, p), (N, p_N)) \leq C_4 K^{1/3} \sigma_0^{2/3}, \quad \sigma_0 = C_3 \left( \ln \left( \frac{1}{\delta} \right) \right)^{-C_2},
$$

Here $\text{Sec}$ stands for the sectional curvature and $C_4, C_5$ are uniform constants.

**Remark 3** Instead of eigenvalues and eigenfunctions one can deal with the heat kernels $H_M(x, y, t)$ of $\partial_t - \Delta_g$, cf. [8], [28], [35]. Definition 3 can be reformulated e.g. as $\|H_{M^{(1)}} - H_{M^{(2)}}\|_{C(B(p, r_0)^2 \times (\delta, \infty))} < \delta$. An analog of Theorem 7 can be obtained. However, we do not dwell on this issue in the paper.

To complete this section we recall that stability in the corresponding direct spectral problem is well-known, see e.g. [31]. In particular, let $M$ be a compact manifold equipped with metrics $g_\ell$, $\ell = 1, 2, \ldots$, and $g_0$. Let $a, b \notin \sigma(-\Delta_{g_0})$. Denote by $P_\ell, P_0$ the spectral orthoprojectors in $L^2(M, g_\ell), L^2(M, g_0)$ on the interval $[a, b]$. Then it follows from Theorems IV.3.16 and VI.5.12 of [31] that if $\|g_\ell - g_0\|_{L^\infty(M)} \to 0$ as $\ell \to \infty$, then $\|P_\ell - P_0\|_{L^2(M, g_0) \to L^2(M, g_0)} \to 0$. This implies that the ISD of $(M, g_\ell)$ converges to the ISD of $(M, g_0)$. 

8
1.3 Earlier results and outline of the paper

The Gel'fand inverse problem, formulated by I. M. Gel'fand in 50s [21], is the problem of determining the coefficients of a second order elliptic differential operator in a domain $\Omega \subset \mathbb{R}^n$ from the boundary spectral data, that is, the eigenvalues and the boundary values of the eigenfunction of the operator. In the geometric Gel'fand inverse problem, a Riemannian manifold with boundary and a metric tensor on it need to be constructed from similar data. For Neumann boundary value problem for the operator $-\Delta_g$ on manifold $M$, the boundary spectral data consists of the boundary $\partial M$, the eigenvalues $\lambda_j$ and the boundary values of the eigenfunction, $\varphi_j|_{\partial M}$, $j = 1, 2, \ldots$. The uniqueness of the solution of the Gel'fand inverse problem has been considered in [5, 6, 41, 30, 40].

To consider the formulation of the stability of the inverse problems, let us consider first the Gel'fand inverse on a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial \Omega$ and a conformally Euclidian metric $g_{jk}(x) = \rho(x)^{-2} \delta_{jk}$. Here, $\rho(x) > 0$ is a smooth real valued function. Then the problem has the form

$$-\sum_{k=1}^{2} \rho(x) \left( \frac{\partial}{\partial x^k} \right)^2 \varphi_j(x) - \lambda_j \varphi_j(x) = 0, \quad \text{in} \ \Omega, \quad \partial_\nu \varphi_j|_{\partial \Omega} = 0. \quad (14)$$

The problem of determining $\rho(x)$ from the boundary spectral data is ill-posed in sense of Hadamard: The map from the boundary data to the coefficient $\rho(x)$ is not continuous so that small change in the data can lead to huge errors in the reconstructed function $\rho(x)$. One way out of this fundamental difficulty is to assume a priori higher regularity of coefficients, that is a widely used trend in inverse problems for isotropic equations, like (14). This type of results is called conditional stability results (see e.g. [1, 2, 46]).

For inverse problems for general metric this approach bears significant difficulties. The reason is that the usual $C^k$ norm bounds of coefficients are not invariant and thus this condition does not suit the invariance of the problem with respect to diffeomorphisms. Moreover, if the structure of the manifold $M$ is not known a priori, the traditional approach can not be used. The way to overcome these difficulties is to impose a priori constraints in an invariant form and consider a class of manifolds that satisfy invariant a priori bounds, for instance for curvature, second fundamental form, radii of injectivity, etc. Under such kind of conditions, invariant stability results for various inverse problems have been proven in [4, 20, 46]. In particular, for the Gel'fand
inverse problem for manifolds with non-trivial topology, an abstract, i.e., a non-quantitative stability result was proven in [4]. There, it was shown that the convergence of the boundary spectral data implies the convergence of the manifolds with respect to the Gromov-Hausdorff convergence. However, this result was based on compactness arguments and it did not provide any estimate. In this paper our aim is to improve this result and to give explicit estimates for an analogous inverse problem.

In this paper we consider a Gel’fand inverse problem for manifolds without boundary. Then, as explained above, instead of assuming that the boundary and the boundary values of the eigenfunctions are known we assume that we are given a small open ball $B \subset M$ and the eigenfunctions $\varphi_j$ are known on this set. Similar type of formulation of the problem with measurements on open sets have been considered in [17, 18]. We show that the Interior Spectral Data (ISD), that is, an open set $B \subset M$, the eigenvalues $\lambda_j$ and the restrictions of the eigenfunctions $\varphi_j|_B$ determine the whole manifold $(M, g)$ in stable way. Also, we quantify this stability by giving explicit inequalities under a priori assumptions on the geometry of $M$. We emphasise that we assume that the eigenfunctions are known only on an open subset $B$ of $M$ that may be chosen to be arbitrarily small but still e.g. the topology of $M$ is determined in a stable way. We note that this paper is a slightly extended and polished version of our preprint in Arxiv, published on Feb. 25, 2017.

We note that in spectral geometry one has studied similar stability problems where the heat kernel are known on the whole manifold, [8, 28, 29]. This data is equivalent to knowing the eigenvalues and the eigenfunctions and the eigenfunctions on the whole manifold.

Outline of the paper: Ch. 2 introduces the geometric set-up. Ch. 3 formulates the stability of the unique continuation for the solution of the wave equation together with Corollary 2 for its spatial projection $v$. Ch. 4 presents Theorems 3 and 4 proving the construction of the approximate Fourier coefficients of $\chi_{\Omega}v$ in the case of respectively exact and approximate FISD. Ch. 5 shows the related approximate interior distance functions. Ch. 6 collects all the previous inequalities to prove Theorem 1 and Proposition 1.
2 Geometric preliminaries

2.1 Properties of the manifolds of bounded geometry

Here we list some results on the class $\mathcal{M}_n(R,D,i_0)$. These results can be found in or immediately follow from [3, 15] with further improvements in [4]. Namely, the class $\mathcal{M}_n$ is precompact in GH-topology. Its closure, $\overline{\mathcal{M}}_n$ consists of pointed Riemannian manifolds $(M,g,p)$ with $g \in C_0^\infty(M)$ which satisfy (1). Here and later * indicates the Zygmund space.

We define the norm of the space $C^k(M)$ invariantly by

$$\|f\|_{C^k(M)} := \sum_{j=0}^{k} \max_{x \in M} \|\nabla^j f(x)\|_g,$$

where the norm is computed using the metric $g$. Next, for $k \in \mathbb{Z}_+$, $\beta \in (0,1]$, we use the the Zygmund spaces $C^{k,\beta}(M) = [C^{k_1}(M), C^{k_2}(M)]_\theta, \quad k + \beta = \theta k_1 + (1-\theta)k_2 \in \mathbb{R}_+, \quad \theta \in (0,1)$.

Here $[\cdot, \cdot]_\theta$ stands for the interpolation, see e.g [9]. Note that, for $\beta \in (0,1)$, the Hölder spaces fulfill $C^{k,\beta}(M) = C^{k+\beta}(M)$.

To achieve the $C^k$-smoothness of $g$, one needs some special coordinates, e.g. harmonic coordinates. For any number $Q > 1$, that we below choose to be $Q = 2$, there is a constant $r(\text{har})$ depending only on $n,R,D,i_0,r_0$ and $Q$, such that, for any $(M,g,p) \in \overline{\mathcal{M}}_n$, $q \in M$, there are $Q$-harmonic coordinates in $B(q,r(\text{har}))$, that we denote by $Y : B(q, \bar{r}(\text{har})) \to \mathbb{R}^n$, $Y_q = Y(B(q, r(\text{har})))$ that we denote by $y$. For $Q = 2$, in these coordinates the metric tensor $g^{(\text{har})}_{jk}(x) = (Y^{-1})_y$ satisfies

$$2^{-1}I \leq (g^{(\text{har})}_{jk}(y))_{j,k=1}^n \leq 2I, \quad \text{for } y \in U_q = Y(B(q, r(\text{har})))$$

$$\|g^{(\text{har})}_{jk}\|_{C^2(Y_q)} \leq C^{(\text{har})},$$

with some uniform constant $C^{(\text{har})}$, see [3, 15] and [4]. We note that the existence of the harmonic radius $r(\text{har})$ and the constant $C^{(\text{har})}$ for which (16) holds for all $(M,g,p) \in \overline{\mathcal{M}}_n$, $q \in M$, is based on compactness results, and therefore the dependency of $r(\text{har})$ and $C^{(\text{har})}$ on $n,R,D,i_0,r_0$ is not explicit.

Sometimes, with a slight abuse of notation we identify $y \in U_q$ with the corresponding point in $Y(y) \in B(q, r(\text{har}))$. 

11
The inequality (16) immediately implies that the sectional curvature $\text{Sec}$ and the Riemannian curvature tensor $R_M$ satisfies

$$|\text{Sec}(M)| \leq K, \quad \|R_M\| \leq K, \quad \|\nabla R_M\| \leq K,$$

where $K$ is a uniform constant.

For the sake of simplicity, we will work with Hölder rather then Zygmund spaces. It follows from [3, 15], with the terminology described in [43, Sec. 10.3.2], that when $(M_k, g_k, p_k) \to (M, g, p)$ in the GH topology on $\overline{M_n}$, then, for all $\beta \in (0, 1)$, there are $C^{5,\beta}$-smooth diffeomorphism $F_k : M_k \to M$ such that

$$F_*(g_k) \to g \quad \text{in} \quad C^{4,\beta}(M), \quad \text{as} \quad k \to \infty.$$  

Thus, for any $\varepsilon > 0, \beta < 1$, there is $\sigma = \sigma(\varepsilon, \beta)$ such that we have the following: For all $M_1, M_2 \in \overline{M_n}$ such that $d_{GH}(M_1, M_2) < \sigma$, there is a diffeomorphism $F : M^1 \to M^2$ and

$$\|g^h - F_*(g^h)\|_{C^{4,\beta}(M^i)} < \varepsilon, \quad i = 1, 2,$$

cf. [43, Sec. 10.3.2]. Returning to (18), for large $k$, $M_k$ and $M$ are diffeomorphic, so that it is possible to use results from [31], see the end of sec. 1.2. This implies stability of the direct problem in the GH topology on $\overline{M_n}$.

Note that we can solve the ordinary differential equations that define the geodesics in the harmonic coordinates. Then it follows from (16) that there is a uniform constant $C_6 > 1$, such that for any ball $B(x, r) \subset M$, where $(M, g, p) \in \overline{M_n}$, we have

$$\frac{1}{C_6}r^n \leq \text{vol}(B(x, r)) \leq C_6r^n, \quad 0 \leq r \leq D.$$  

Thus, the volume of balls having radius $i_0/2$ is bounded below by a uniform constant $v_0$. Furthermore, by [22], the class of Riemannian manifolds $(M, g)$ that satisfy (17) and conditions $\text{diam}(M, g) \leq D$ and $\text{vol}(M, g) \geq v_0$ are pre-compact with respect to the Lipschitz distance $d_L((M_1, p_1), (M_2, p_2))$, see (13) and the closure of this class consists of $C^\infty$-smooth manifolds with $C^{1,\alpha}$-metric. This implies that there is a uniform constant $C^{\text{(Lip)}}$ such that for all $(M_1, g_1, p_1), (M_2, g_2, p_2) \in \overline{M_n}$ we have

$$d_L((M_1, p_1), (M_2, p_2)) \leq C^{\text{(Lip)}}.$$  

(21)
Moreover, by [32], we have that for any \( \varepsilon > 0 \) there is \( \zeta(\varepsilon) > 0 \), such that for \((M_1, g_1, p_1), (M_2, g_2, p_2) \in \mathcal{M}_n \) we have

\[
\text{if } d_{GH}((M_1, p_1), (M_2, p_2)) < \zeta(\varepsilon) \text{ then } d_L((M_1, p_1), (M_2, p_2)) < \varepsilon.
\] (22)

We turn now to the spectral properties on \((M, g, p) \in \mathcal{M}_n\). By [16], the inequality (21) implies that the \( j \)-th eigenvalue \( \lambda_j(M_i, g_i) \) of the Laplacian on the manifold \((M_i, g_i)\) satisfies

\[
e^{-(n+2)C^{(Lip)}/2} \lambda_j(M_1, g_1) \leq \lambda_j(M_2, g_2) \leq e^{(n+2)C^{(Lip)}/2} \lambda_j(M_1, g_1)
\]

for all \((M_1, g_1, p_1), (M_2, g_2, p_2) \in \mathcal{M}_n\). Since the eigenvalues of the manifold \((M_1, g_1)\) satisfy the Weyl’s asymptotics

\[
\lambda_j(M_1) = cM_1 j^{2/n}(1+o(1)) \quad \text{as } j \to \infty,
\]

then there exists a uniform constant \( C_7 > 1 \) such that,

\[
\frac{1}{C_7} j^{2/n} \leq \lambda_j(M) \leq C_7 j^{2/n}, \quad j \in \mathbb{Z}_+ \text{ for all } (M, g, p) \in \mathcal{M}_n(R, D, i_0).
\] (23)

Note that (23) is valid under a weaker assumption that \( \text{Ric}(M) \) is bounded from below, see [9].

**Remark 4** Assume that the collection of \( g^a|_{B_\varepsilon(r_0)} \) and \( ((\lambda_j^a, \varphi_j^a)|_{B_\varepsilon(r_0)})_j \) is \( \delta \)-close to the FISD \( g|_{B_\varepsilon(r_0)} \) and \( (\lambda_j, \varphi_j|_{B_\varepsilon(r_0)})_j \) of the manifold \((M, g, p) \in \mathcal{M}_n\). Then all intervals \( I_p = (a_p, b_p), \quad p = 0, 1, \ldots, P \) in \([4]\) satisfy \( b_j \leq \delta^{-1} + \delta \), and thus the index \( j \) of any eigenvalue \( \lambda_j \) that is in some of these intervals satisfies by (23) the inequality

\[
C_7^{-1} j^{2/n} \leq \delta^{-1} + \delta \leq 2\delta^{-1}.
\]

On the other hand, if \( j < (C_7^{-1}\delta^{-1})^{n/2} \), then \( \lambda_j < \delta^{-1} \). Thus, without loss of generality, we can always assume that the value of \( J \) in Proposition 7 satisfies

\[
(C_7^{-1}\delta^{-1})^{n/2} \leq J \leq (2C_7\delta^{-1})^{n/2}.
\] (24)

**Remark 5** Below we will assume that \( \delta < (3C_7)^{-1} \). Then for \( j \geq 1 \) we have \( \lambda_j \geq C_7^{-1} \) and \( \lambda_j > 3\delta \). Next, assume that \( \lambda_j \) and \( \lambda_k \) with \( k > j \geq 1 \) belong in the same interval \( I_p = (a_p, b_p) \) with \( b_p - a_k < \delta \). Since \( \lambda_j \geq C_7^{-1} > 3\delta \), we have \( a_p > 2\delta \) so that \( b_p < 2a_p \). Then by (23) we have

\[
C_7^{-1} b_p^{2/n} \leq \lambda_k \leq b_p \leq 2a_p \leq 2\lambda_j \leq 2C_7 j^{2/n},
\]

implying

\[
j < k \leq (2^{1/2}C_7)^n j.
\] (25)
Next, instead of harmonic coordinates, we can use coordinates made of the eigenfunctions \( \varphi_j \). It turns out, cf. [7, 4], that in a neighbourhood of any \( x \in M \) there are \( \varphi_{j(1;x)}, \ldots, \varphi_{j(n;x)} \) which form \( C^6 \)-smooth coordinates. Moreover, by the compactness arguments, there are uniform constants \( r \) and \( C \) so that these coordinates are well defined in any ball \( B(x, r) \subset M \), where \( (M, g, p) \in \overline{M}_n \), and the metric tensor \( g \) in these coordinates satisfies (16). There is also a uniform number \( N \in \mathbb{Z}_+ \), such that we can take \( j(\ell; x) \leq N, \ell = 1, \ldots, n. \)

Next, using \( ((\lambda_j, \varphi_j))_{j=0}^{\infty} \), we introduce the Sobolev spaces \( H^s(M) \), \( s \in \mathbb{R} \),

\[
f(x) = \sum_{j=0}^{\infty} f_j \varphi_j(x) \in H^s(M) \quad \text{iff} \quad \|f\|_{H^s}^2 := \sum_{j=0}^{\infty} \langle \lambda_j \rangle^s |f_j|^2 < \infty, \quad (26)
\]

where \( \langle \lambda \rangle = (1 + \lambda^2)^{1/2} \).

### 2.2 Distance coordinates

Recall that there are harmonic coordinates in \( B(x, r^{(har)}) \) ball near any \( x \in M \in \overline{M}_n \), see (16). In the Proposition below we use such coordinates as background coordinates near \( x \).

Below, we say that a subset \( Y \subset X \) is a \( \tau \)-net in the metric space \( X \) if the union of the balls \( B_X(y, \tau) \), \( y \in Y \), contains the whole space \( X \). Also, we say that \( Z \subset X \) is \( \tau \)-separated, if for all \( z_1, z_2 \in Z \), \( z_1 \neq z_2 \) we have \( d_X(z_1, z_2) \geq \tau \). Observe that if \( Z \subset X \) is a maximal \( \tau \)-separated subset of \( X \) (maximal in the sense that any other \( \tau \)-separated subset of \( X \) that contains \( Z \) has to be equal to \( Z \) ), then it is a \( \tau \)-net in \( X \).

**Proposition 2**

There are uniform constants \( \tau_0, \rho_0 < \min\{r^{(har)}/4, r_0/128\} \) and uniform constants \( L \in \mathbb{Z}_+ \) and \( C_8, C_9, C_{10} > 0 \) depending only on \( n, R, D, i_0 \) and \( r_0 \), such that, for any \( (M, g, p) \in \overline{M}_n(R, D, i_0) \) the following holds true: There is a \( \tau_0 \)-net in \( B(p, r_0/4) \) with at most \( L - 1 \) points. Let \( \{z_1, \ldots, z_{L-1}\} \subset B(p, r_0/4) \) be an arbitrary collection of points that is a \( \tau_0 \)-net in \( B(p, r_0/4) \). Then,

(i) For all \( x \in M \), there are \( n \) points \( z_{j(i)} \in Z \), \( j(i) = j(i; x), i = 1, 2, \ldots, n \) such that the map \( X : B(x, \rho_0) \rightarrow \mathbb{R}^n \),

\[
X : y = (y^1, \ldots, y^n) \mapsto (d_M(y, z_{j(1)}), d_M(y, z_{j(2)}), \ldots, d_M(y, z_{j(n)})), \quad (27)
\]
Figure 3: There are \( L - 1 \) points \( z_1, z_2, \ldots, z_{L-1} \in B(p, r_0/4) \) such that for any \( y \in M \) there are \( n \) points \( z_{ji}(y), \ldots, z_{ji}(y) \) so that the distance functions
\[
X^\ell(x) = d_M(x, z_{ji}(y)), \quad \ell = 1, 2, \ldots, n
\]
define local smooth coordinates \( x \mapsto (X^\ell(x))^n_{\ell=1} \) in a neighbourhood \( U_y \subset M \) of the point \( y \).

for coordinates where \( X : B(x, \rho_0) \rightarrow X(B(x, \rho_0)) \) is a Lipschitz-smooth diffeomorphism and
\[
\|DX\|_{L^\infty(B(x, \rho_0))} + \|DX^{-1}\|_{L^\infty(X(B(x, \rho_0)))} \leq C_8. \tag{28}
\]
where the norms are computed using the metric \( g \) on \( M \) and the Euclidean norm in \( \mathbb{R}^n \). Moreover, \( z_{ji}(i) \) can be chosen so that \( d(x, z_{ji}(i)) > r_0/16 \) and the metric tensor \((g_{ij})^n_{i,j=1} = X^*g\) in these coordinates satisfies
\[
C_8^{-1} I \leq (g_{ij}(z))^n_{i,j=1} \leq C_8 I, \quad z \in X(B(x, \rho_0)). \tag{29}
\]

(ii) The map \( H : M \rightarrow \mathbb{R}^{L-1} \), defined by \( H^L(x) = (d_M(x, z_j))^n_{j=1} \), satisfies
\[
\frac{1}{C_9} \leq \left| \frac{H^L(x) - H^L(y)}{d(x, y)} \right| \leq C_9, \quad \text{for all } x, y \in M, \ x \neq y. \tag{30}
\]

Here as a norm in \( \mathbb{R}^{L-1} \) we can take e.g. the Euclidian norm in \( \mathbb{R}^{L-1} \).

**Proof.** Let us first consider one pointed manifold \((M, g, p) \in \overline{M}_n\). Let us consider the extended exponential map
\[
F : TM \rightarrow M \times M, \quad F(x, \xi) = (x, \exp_x(\xi)).
\]
Inequalities (1) and (17) imply that in the set \( S = \{(x, \xi) \in TM : \|\xi\| \leq 2D\} \) the map \( F \) is \( C^2 \)-smooth and its norm in \( C^2(\overline{S}) \) is bounded by a uniform constant. The proof this is analogous to that of Lemma 2 in [33]. Let
$x_0 \in M$ and $\gamma_{x_0, \xi_0}([0, s_0]), \xi_0 \in S_{x_0}M$ be a shortest geodesic from $x_0$ to $p$ where $s_0 = d(x_0, p)$. When $s_0 \geq r_0/2$, choose $s_1 = s_0 - r_0/5$, and when $s_0 < r_0/2$, choose $s_1 = s_0 + r_0/5$. Then the point $p_1 = \gamma_{x_0, \xi_0}(s_1)$ satisfies $p_1 \in \partial B(p, r_0/5)$, and $d(x_0, p_1) \geq r_0/5$. As $r_0 < s_0$, we see that the geodesic $\gamma_{x_0, \xi_0}([0, s_1 + r_0/5])$ is a length minimising geodesic between its endpoints. In particular, this implies that $\gamma_{x_0, \xi_0}([0, s_1])$ continues behind $p_1$ as a shortest curve between its points. As in [23, Lemma 4], (see also [25] where related results are proven with lower regularity assumptions), we see that (1) and (17) imply that there is a uniform constant $C$ and that

$$
\|\gamma_{x_0, \xi_0} - \gamma_{x_0, \xi_0}([0, s_1])\| \leq C.
$$

Then $s_1 \xi_0 \in F$. Let $z_j = \exp_{x_0}(t_j \xi_j) \in B(p, r_0/4)$. We see that $\nabla d_M(\cdot, z_j)|_{x_0} = -\xi_j$ and $\|dF|_{x_0}^{-1}\| \leq C$.

Inverse function theorem, see e.g. [27], and the facts that $\|dF|_{x_0}^{-1}\| \leq C$ and that $F$ has a uniformly bounded $C^2$-norm in $S$, imply for the map

$$
H_{z_1, \ldots, z_n}(x) = (dM(x, z_j))_{j=1}^n
$$

that there are uniform constants $\rho_0 > 0$ and $c_0 > 0$ such that we have

$$
|H_{z_1, \ldots, z_n}(x) - H_{z_1, \ldots, z_n}(x')| \geq c_0 d_M(x, x'), \quad \text{for all } x, x' \in B_M(x_0, \rho_0).
$$

(32)

Let us now choose $\xi_j^0 \in S_{x_0}M, j = 1, \ldots, n$ such that $s_1 \xi_j^0 \in F$. We satisfy

$$
\|\xi_j^0 - \xi_0\| \leq c_0/(2s_1), \quad \|\xi_j^0 - \xi_k^0\| \geq c_0/(4s_1) \quad \text{for } j \neq k.
$$

Let $z_j^0 = \exp_{x_0}(s_1 \xi_j^0)$. As $\|dF|_{x_0}^{-1}\| \leq C$ in $F(F)$, there is a uniform constant $\tau_0 \in (0, r_0/100)$ such that if $z_j \in B(p, r_0/4), j = 1, \ldots, n$ satisfy $d(z_j, z_j^0) < \tau_0$, then there are $\tilde{z}_j \in S_{x_0}M$ and $\tilde{t}_j > 0$ such that $\tilde{z}_j = \exp_{x_0}(\tilde{t}_j \tilde{\xi}_j)$ and

$$
|\tilde{z}_j - \xi_j| < r_0/(8s_1) \quad \text{and} \quad |\tilde{t}_j - s_1| < r_0/8.
$$
Then $\zeta_j$ and $\tilde{t}_j$ satisfy (31). Thus, (32) implies that the map $H^\#_{z_1,\ldots,z_n}(x)$ satisfies (32). This implies that if $\{\hat{z}_i \in B_M(p,r_0/4), i = 1,2,\ldots,i_M\}$ is any $\tau_*$-net in $B_M(p,r_0/4)$ then for all $j = 1,2,\ldots,n$ there are $i_j \in \{1,2,\ldots,i_M\}$ such that $d_M(\hat{z}_{i_j},z_j^0) \leq \tau_*$. Then the above implies that for $H^\#_{z_1,\ldots,z_n}(x)$ satisfies (32).

Observe that above $(x,s\xi) \in \mathbb{B}$, so that $d_M(x,x_0) < r_* < r_0/100$. Moreover, $d(x_0,p_1) \geq r_0/5$, $z_j^0 \in B(p_1,r_0/100)$, $d_M(\hat{z}_{i_j},z_j^0) \leq \tau_* < r_0/100$ yield $d_M(x,\hat{z}_{i_j}) \geq r_0/5 - 3\tau_0/100 > r_0/8$.

Note that above $c_*, \tau_*$ and $\rho_*$ are uniform constants and the estimate (32) is valid for some points $\hat{z}_{i_j}$ in any $\tau_*$-net $\hat{z}_i$ in $B(p,r_0/4)$, that satisfy $d_M(x,\hat{z}_{i_j}) > r_0/8$, and any $(M,g,p) \in \text{M}_n$.

This proves (28) and (29) in claim (i).

Next we consider the claim (ii). Let us show that there are $h_1 > 0$ and $\tau_1 > 0$ such that for any $(M,g,p) \in \text{M}_n$ and any maximal $\tau_1$-separated set $\{z_1,\ldots,z_{L-1}\} \subset B(g,r_0/4)$ we have

$$
\sup_{x,y \in M, x \neq y} \left( \sup_{j_1,\ldots,j_n} \frac{|(d_M(x,\hat{z}_{j_1}))^n_{j=1} - (d_M(y,\hat{z}_{j_1}))^n_{j=1}|_{\mathbb{R}^n}}{d_M(x,y)} \right) \geq h_1,
$$

where the supremum is taken over all $1 \leq j_1 < j_2 < \cdots < j_n \leq L-1$.

Assume the opposite. Then for all $k \in \mathbb{Z}_+$ there are $h_k > 0$, $(M_k,g_k,p_k) \in \text{M}_n$ and $1/k$-nets $\{z^k_j : j = 1,2,\ldots,L_k\} \subset B(p_k,r_0/4)$ and points $x_k,y_k \in M_k$, $x_k \neq y_k$ so that $h_k \to 0$ and

$$
\sup_{x,y \in M_k, x \neq y} \left( \sup_{j_1,\ldots,j_n} \frac{|(d_{M_k}(x_k,z^k_{j_1}))^n_{j=1} - (d_{M_k}(y_k,z^k_{j_1}))^n_{j=1}|_{\mathbb{R}^n}}{d_{M_k}(x_k,y_k)} \right) < h_k.
$$

Using compactness arguments for $\text{M}_n$ and choosing a suitable subsequence of the manifolds $(M_k,g_k,p_k)$ we can assume that $(M_k,g_k,p_k) \to (M,g,p)$ in the Lipschitz-topology. Then there are diffeomorphisms $F_k : M_k \to M$ such that $F_k(p_k) \to p$ and Lip$(F_k) \to 1$ and Lip$(F_k^{-1}) \to 1$. Moreover, we can assume that $F_k(x_k) \to x$ and $F_k(y_k) \to y$ in $M$ and, after using the Cantor diagonalization procedure, we can assume that there are limits $\lim_{k \to \infty} F_k(z^k_j) = z_j$ in $M$, for all $j = 1,2,\ldots$. Next, using (19), we see that $d_{M_k}(x_k,y_k) \to d_M(x,y)$, $d_{M_k}(x_k,z^k_j) \to d_M(x,z_j)$ and $d_{M_k}(y_k,z^k_j) \to d_M(y,z_j)$. Also $\{z^k_j\}_{j=1}^\infty$ is dense in $B_M(p,r_0/4)$. Therefore, $d_M(x,z) = d_M(y,z)$ for all $z \in B_M(p,r_0/4)$. Then [26] Lemma 13 (see also [30] Lemma 3.30), implies that $x = y$. 

17
Let $k$ be so large that $1/k < \tau_s/2$, and
\[
\begin{align*}
    d_M(F_k(p), p) &< \tau_s/2, \quad \text{Lip}(F_k) \leq 2, \quad \text{Lip}(F_k^{-1}) \leq 2, \\
    d_M(F_k(x), x) &< \rho_s/4, \quad d_M(F_k(y), y) < \rho_s/4, \quad h_k < c_s.
\end{align*}
\]
As $x = y$, these imply $d_M(F_k(x), F_k(y)) < \rho_s/2$ and hence $d_M(x, y) < \rho_s$. As $1/k < \tau_s/2$, the points $z^k_j, j = 1, \ldots, L-1$ form a $\tau_s$-net in $B_{M_k}(p, r_0/4)$. Then the inequality (34) for $x_k$ and $y_k$, with $h_k < c_s$, is in contradiction with the fact that there is a subset of $n$ of the points in $\tau_s$-net $z^k_j, j = 1, \ldots, L_k$ for which (32) holds. This proves (33) with some uniform constants $\tau_1$ and $h_1$.

We observe that a maximal $\tau_1$-separated subset in the ball $B(p, r_0/4)$ has at most $C_s = \text{vol}(B(p, r_0/4))/\text{vol}_{n, R}(B(x, \tau_1))$ points, where $\text{vol}_{n, R}(B(x, \tau_1))$ is the volume of the ball of radius $\tau_1$ on the $n$-dimensional sphere having constant curvature $R$. Hence we see that the number of points for a maximal $\tau_1$-separated subset in $B(p, r_0/4)$ is bounded by a uniform constant $C_s$. Thus we can choose $L$ to be the integer part of $C_s$ and $\tau_0 = \min(\tau_s, \tau_1)$, which makes $L$ and $\tau_0$ uniform constants.

As the number $L - 1$ of points in the $\tau_1$-nets we consider is bounded by a uniform constant, we see that (30) is valid with $C_8 = h_1^{-1} + L$. These prove the claims (i) and (ii).

The above considerations bring about the following result.

**Lemma 1** There exist uniform constant $C_{11} > 0$ and uniform constant $N_F \in \mathbb{Z}_+$ (that is, $C_{11}$ and the integer $N_F$ depend only on $n, R, D, i_0, r_0$) such that

(i) Let $\sigma \in (0, \tau_0]$. Then any maximal $\sigma$-separated set $x_1, \ldots, x_{N(\sigma)}$ in $M$ is such that the number of its elements fulfills the bound
\[
    N(\sigma) \leq \tilde{N}(\sigma) = C_{11}\sigma^{-n}.
\]

Moreover, the balls $B(x_k, 4\sigma)$ satisfy the finite intersection property with at most $N_F$ intersections, that is, any point $x \in M$ belongs to at most $N_F$ balls $B(x_k, 4\sigma)$.

(ii) Let $\sigma \in (0, \tau_0]$. Then any maximal $\sigma$-separated set $z_1, \ldots, z_{N_1(\sigma)}$ in $B(p, r_0/4)$ is such that the number of its elements fulfills the bound $N_1(\sigma) \leq \tilde{N}(\sigma)$, and the balls $B(z_k, 4\sigma)$ satisfy the finite intersection property with at most $N_F$ intersections.
Proof. It remains to prove the finite intersection property. It follows from (20) if we take into the account that $B(x_k, 4\sigma) \cap B(x_j, 4\sigma) = \emptyset$ if $d(x_k, x_j) \geq 9\sigma$ and $B(x_k, \sigma/2) \cap B(x_j, \sigma/2) = \emptyset$. □

3 Wave equation: stability for the unique continuation

Consider the initial-value problem for the wave equation

\[ \partial_t^2 w - \Delta_g w = 0 \text{ in } M \times \mathbb{R}, \]
\[ w|_{t=0} = v, \quad w_t|_{t=0} = 0, \]

on $(M,g,p) \in \mathcal{M}_u(R,D,i_0)$ and denote its solution by $w = W(v)$. Our main interest lies in the case when $v \in \mathcal{H}_\Lambda^s(M)$, $\Lambda > 0$,

\[ \mathcal{H}_\Lambda^s(M) = \{ v \in H^s(M) : \|v\|_{H^s(M)} \leq \Lambda \} \tag{37} \]

and we assume in the following that

\[ \frac{3}{2} < s < 2 \tag{38} \]

and denote $\mathcal{H}_\Lambda^0(M) := \{ v \in L^2(M) : \|v\|_{L^2(M)} \leq \Lambda \}$. Using the Fourier decomposition we show that, if $v \in H^s(M)$, then

\[ \|w\|_{H^s(M \times [-T,T])} \leq 6 \sqrt{T} \|v\|_{H^s(M)} \leq C_{20} \|v\|_{H^s(M)}, \quad T < 2D, \tag{39} \]

where $C_{20} = 6 \sqrt{1 + D^2}$.

Associated to the wave operator are the double cones of influence. To define these, let $V \subset M$ be open, $T \in \mathbb{R}_+$. Denote by

\[ \Gamma(V,T) := V \times (-T,T). \]

Then the double cone of influence is given by

\[ \Sigma(V,T) := \{(x,t); d(x,V) + |t| < T\}. \]

By Tataru’s uniqueness theorem [48, 49], if $u$ is a solution to (36) in $M \times (-T,T)$, which satisfies $u = 0$ in $\Gamma(V,T)$, then $u = 0$ in $\Sigma(V,T)$. However,
Figure 4:  Left: Assume that the function $u(x,t)$ vanishes in $\Gamma(z,T)$. If $u(x,t)$ satisfies the wave equation $Pu = 0$, then $u(x,t)$ vanishes in the double cone $\Sigma_0 = D(z,0,T)$. Theorem 2 states that if $Pu = f$ is small, then $u(x,t)$ is small in the domain $D = D(z,\gamma,T)$ (that has a red, curved boundary). In the figure, we consider also the double cone $\Sigma_\gamma = \Sigma(z,\gamma,T) \subset D$. Right: In Corollary 3 we assume that $u(x,t)$ is a solution to $Pu = 0$ and that $u$ is small in the set $\tilde{\Gamma}(z,T) = B(z,r_0/16 + \gamma) \times (-T + r_0/16, T - r_0/16)$, marked with a dotted black boundary. Note that $\Gamma(z,T) \subset \tilde{\Gamma}(z,T)$. Then we apply Theorem 3 to see that $u$ is small in $D = D(z,\gamma,T)$, interpolation, and trace theorem in the black cylindrical set $K = B(z,T - 2\gamma) \times (-\gamma,\gamma)$ to see that $u|_{t=0}$ is small in $B(z,T - 2\gamma)$ that is the intersection of $\{0\} \times M$ and the domain $\Sigma_{2\gamma} = \Sigma(z,2\gamma,T)$, that is shown in the figure with dotted red lines.
for our purposes we need an explicit estimate which follows from Theorem 3.3 in \[12\]. To formulate the results we introduce, for

$$0 < \gamma \leq \frac{r_0}{32}, \quad \frac{r_0}{8} \leq T < 2D,$$

(40)

with \(r_0\) fulfilling (2) and \(z \in M\), the domains

$$\Gamma = \Gamma(z,T) = B(z,r_0/16) \times (-T + r_0/16, T - r_0/16),$$

(41)

$$D = D(z,\gamma, T) = \{(x,t) : (T - d(x,z))^2 - t^2 \geq \gamma^2, |t| < T - r_0/16\},$$

$$\Omega(T) = M \times (-T + r_0/16, T - r_0/16).$$

Also, let for \(b \in \mathbb{R}\),

$$\Sigma(z,b,\gamma,T) = \{(x,t) \in M \times \mathbb{R} : |t| \leq T - r_0/16, |t| \leq T - b\gamma - d_g(x,z)\}$$

(42)

be the “domain of influence” corresponding to the cylinder \(\Gamma(z,T)\). Observe that \(\Sigma(z,\gamma,T) \subset D(z,\gamma,T) \subset \Sigma(z,0,T)\).

In the following we formulate the stability results for the unique continuation in \[12\]. We note that similar results have been obtained by Luc Robbiano in \[44\] with \(\theta = 1\), but with a loss in the domain of dependence and later by C. Laurent and M. Leautard in \[38\] with \(\theta = 1\), but without an explicit calculation of the constants in the domain of dependence.

**Theorem 2** Let \((M, g, p) \in \mathcal{M}_n(R,D,i_0)\). Let \(P = P(x,D) = \partial_t^2 - \Delta_g\) be the wave operator associated with \(M\). Assume that \(w(x,t) = 0\) for all \((x,t) \in \Gamma\). Then, for any \(0 < \theta < 1\), there is \(c_{206}(\gamma,\theta) \geq 1\), depending only on \(n, R, D, i_0, r_0, \theta\) and \(\gamma\) such that the following stability estimate holds true:

$$\|w\|_{L^2(D(z,\gamma,T))} \leq c_{206}(\gamma,\theta) \frac{\|w\|_{H^1(\Omega(T))}}{\left(\ln \left(1 + \frac{\|w\|_{H^1(\Omega(T))}}{\|Pw\|_{L^2(\Omega(T))}}\right)\right)^{\theta}},$$

where \(c_{206}(\gamma,\theta)\) is such that

$$c_{206}(\gamma,\theta) = c_{205}(\theta) \exp(\gamma^{-c_{200}}), \quad c_{200} = 58(n + 1) + 1,$$

(43)

and \(c_{205}(\theta) \geq 1\) depends on \(\theta, n, R, D, i_0, r_0\). Moreover, for any \(0 \leq m \leq 1\),

$$\|w\|_{H^{1-m}(D(z,\gamma,T))} \leq c_{206}(\gamma,\theta)^m \frac{\|w\|_{H^1(\Omega(T))}}{\left(\ln \left(1 + \frac{\|w\|_{H^1(\Omega(T))}}{\|Pw\|_{L^2(\Omega(T))}}\right)\right)^{\theta_m}}.$$

(44)
Proof. Theorem 2 follows from Theorem 3.3 in [12] with \( \ell = r_0/16 \) and \( \mathcal{D}(z,\gamma,T) = S(z,r_0/16,T,\gamma) \). Using that \( w = 0 \) in \( \Gamma \), the domain \( \Lambda \) in the final equation of Theorem 3.3 can be changed into \( \mathcal{D}(z,\gamma,T) \). Moreover, for \( \theta < 1 \), the function \( f_\theta(a,b), \ a,b > 0 \),

\[
f_\theta(a,b) = a \left( \ln(1 + \frac{a}{b}) \right)^{-\theta}, \tag{45}\]

increases when either \( a \) or \( b \) increases. Thus, we can change \( \|w\|_{H^1(\Omega_1)} \) and \( \|Pw\|_{L^2(\Omega_1)} \) in Theorem 3.3 to \( \|w\|_{H^1(\Omega(T))} \) and \( \|Pw\|_{L^2(\Omega(T))} \). Note that, although the results in [12] are formulated for \( M \subset \mathbb{R}^n \), they can be easily reformulated for an arbitrary compact Riemannian manifold which possess \( C^5 \)-smooth covering by coordinate systems with \( C^4 \)-smooth metric tensors.

To consider parameters (43) (see the Appendix for details), we will fix the value of \( \theta \) to be \( \theta = 1/2 \), (46) for simplicity. In the general case, we write \( c_{205}(\theta) \) as \( \theta \)-dependent. We recall that the constants in [12] (see (3.1)) explicitly depend on parameter \( C_{14} > 1 \) such that

\[
C_{14}^{-1}|\xi|^2 \leq g^{jk}(x)\xi_j\xi_k \leq C_{14}|\xi|^2; \quad \|g^{jk}(x)\|_{C^4(M)} \leq C_{14}. \tag{47}\]

Using harmonic coordinates in balls of radius \( r^{(\text{har})} \), this condition is fulfilled due to (16), which also implies \( d_g(x,z) \in C^3 \).

Our main interest will be an estimate for \( v(\cdot) = w(0,\cdot) \) in (36) in the domain \( B(z,T-2\gamma) \).

Corollary 2 Assume (40) and let \( \theta \in [1/2,1) \). Also, let \( \Lambda_1 > 0 \) and \( \varepsilon_2 \in (0,\Lambda_1] \) and \( v \in \mathcal{H}_{\Lambda_1}(M) \). Denote by \( w = W(v) \) the solution to initial-value problem (36) and assume that,

\[
\|w\|_{L^2(B(z,r_0/16+\gamma) \times (-T+r_0/16,T-r_0/16))} \leq \varepsilon_2. \tag{48}\]

Then, calling \( \beta = \theta^2/2 \) and defining \( \varepsilon_1 := \mathcal{E}_1(\varepsilon_2;\theta,\gamma,\Lambda_1) \), we get

\[
\|v\|_{L^2(B(z,T-2\gamma))} \leq \varepsilon_1 \tag{49}\]
where, for $c_{202} = c_{202}(\theta, \gamma)$, and $C_{12}(\theta)$ depending only on $\theta$ and $n, R, D, i_0, r_0$

\[
E_1(\varepsilon_2; \theta, \gamma, \Lambda_1) = c_{202} \Lambda_1 \gamma^{(2-\theta)/2} \left( \ln \left[ \frac{1 + \gamma \Lambda_1^{(s-1)/s} \varepsilon_2^{(s-1)/s}}{\varepsilon_2} \right] \right)^\beta,
\]

\[
c_{202} = C_{12} \exp \left( \gamma^{-(c_{202}/2)} \right),
\]
\[
C_{12} = C_{12}(\theta) \geq 1.
\]

**Proof.** Let the cut-off function $\eta(x) \in C^2_0(B(z, r_0/16 + \gamma/2))$ be equal to one in $B(z, r_0/16)$ and $\|\eta\|_{C^i(M)} \leq C\gamma^{-i}, i = 0, 1, 2$. Then $w_\eta(x, t) = (1 - \eta(x))w(x, t)$ vanishes in $\Gamma$ and we have $(\partial_t^2 - \Delta)w_\eta(x, t) = F$, where

\[
F(x, t) = (\Delta_g \eta(x)) w(x, t) + 2g(\nabla \eta(x), \nabla_x w(x, t))
\]
\[
= (\Delta_g \eta(x)) \left( \eta(x)w(x, t) \right) + 2g(\nabla \eta(x), \nabla_x (\eta(x)w(x, t))) := F_1 + F_2
\]

Here $\eta(x) \in C^2_0(B(z, r_0/16 + \gamma))$ is equal to one in $B(z, r_0/16 + \gamma/2)$ and $\|\eta\|_{C^i(M)} \leq C\gamma^{-i}, i = 0, 1, 2$. Clearly, by hypothesis

\[
\|F_1\|_{L^2(M \times (-T + r_0/16, T - r_0/16))} \leq C\gamma^{-2}\varepsilon_2.
\]

To estimate $F_2$, observe that $\|\nabla \eta \|_{H^s(M \times (-T + r_0/16, T - r_0/16))} \leq C\gamma^{-s}\Lambda_1$, where we have also used (39). Since $\|\nabla \eta \|_{L^2(M \times (-T + r_0/16, T - r_0/16))} \leq \varepsilon_2$, by interpolation arguments, we get

\[
\|\nabla \eta \|_{H^1(M \times (-T + r_0/16, T - r_0/16))} \leq C\gamma^{-1}\Lambda_1^{1/s}\varepsilon_2^{1-1/s}
\]

Since $\text{supp}(\nabla \eta) \cap \text{supp}(\nabla \eta) = \emptyset$, this implies

\[
\|F_2\|_{L^2(M \times (-T + r_0/16, T - r_0/16))} \leq C\gamma^{-1}\Lambda_1^{1/s}\varepsilon_2^{1-1/s},
\]
\[
\|F\|_{L^2(M \times (-T + r_0/16, T - r_0/16))} \leq C\gamma^{-2}\Lambda_1^{1/s}\varepsilon_2^{1-1/s},
\]

where we used $\varepsilon_2 \leq \Lambda_1$. As $s > 1$, we have

\[
\|w_\eta\|_{H^1(M \times (-T + r_0/16, T - r_0/16))} \leq C\gamma^{-1}\Lambda_1.
\]

Using growth properties of the function $f_\theta$ of form (45), it follows from Theorem 2 that

\[
\|w_\eta\|_{H^{1-\theta/2}(D)} \leq Cc_{206}(\gamma, \theta)^{\theta/2} \left( \frac{\gamma^{-1}\Lambda_1}{\ln \left[ 1 + \gamma \Lambda_1^{(s-1)/s}\varepsilon_2^{(s-1)/s} \right]} \right)^\beta.
\]

23
Now observe that by the trace-theorem, for any $\alpha > 1/2$ there exists $C_{13} = C_{13}(\alpha)$ such that, for $r \geq r_0/16$, $z \in M$:

$$\|w(\cdot, 0)\|_{L^2(B(z, r))} \leq C_{13} \gamma^{-\alpha} \|w\|_{H^\alpha(B(z, r) \times (-\gamma, \gamma))},$$

$$\|w(\cdot, 0)\|_{L^2(B(z, T-2\gamma))} \leq C_{13} \gamma^{-\alpha} \|w\|_{H^\alpha(D(z, \gamma, T))}.$$  \hfill (55) \hfill (56)

It follows from (56) with $\alpha = 1 - \theta/2$ and (54) that,

$$\|w_\eta(\cdot, 0)\|_{L^2(B(z, T-2\gamma))} \leq CC_{13} \frac{c_{206}(\gamma, \theta)^{\theta/2} \Lambda_1}{\gamma^{2-\theta/2} \left(\ln \left[1 + \gamma \Lambda_1^{(s-1)/s} \varepsilon_2^{-(s-1)/s}\right]\right)^\beta}. \hfill (57)$$

Next define $\alpha = (1 - \beta)s + \beta > 1/2$. Then by interpolation,

$$\|\eta w\|_{H^\alpha(B(z, r) \times (-\gamma, \gamma))} \leq c_{201} \|\eta w\|_{L^2(B(z, r) \times (-\gamma, \gamma))}^{(s-\alpha)/s} \|\eta w\|_{H^\alpha(B(z, r) \times (-\gamma, \gamma))}^{\alpha/s}.$$  \hfill (58)

Using the fact that supp($\eta$) $\subset B(z, r_0/16 + \gamma)$, we can apply (55) with $r = r_0/16 + \gamma$, the previous inequality and (48), to obtain

$$\|\eta(\cdot)w(\cdot, 0)\|_{L^2(B(z, T-2\gamma))} \leq C_{13} \gamma^{-\alpha} c_{201} (C_{20} \Lambda_1)^{\alpha/s} \varepsilon_2^{(s-1)/s} \varepsilon_2^{-\alpha/s} \left(1 + \gamma \Lambda_1^{(s-1)/s} \varepsilon_2^{-(s-1)/s}\right)^{-\beta}. \hfill (59)$$

Here at the last step we use the fact that $X \geq \ln(1+X)$ for $X > 0$, with $X = \gamma \Lambda_1^{(s-1)/s} \varepsilon_2^{-(s-1)/s}$. Recall that $v(x) = w_\eta(x, 0) + \eta(x)w(x, 0)$. Comparing (57) and (58), we obtain equation (50). The coefficient $c_{202}$ defined in (51) fulfills the inequality

$$c_{202} \geq CC_{13} c_{206}(\gamma, \theta)^{\theta/2} \gamma^{\theta/2-2} + C_{13} c_{201} C_{20} (1-\beta) + \beta/s \gamma^{(\beta-1)s},$$

by using (43) and a proper multiplicative coefficient $C_{12}$ independent on $\gamma$. \square

### 4 Computation of the projection

#### 4.1 Domains of influence

Let $(M, g, p) \in \mathcal{M}_n(R, D, i_0)$. By Proposition 2, we can choose $L - 1$ points $z_j, j = 1, 2, \ldots, L - 1$ that form a $\tau_0$-net in $B_M(p, r_0/4)$. Here, $L$ is bounded by a uniform constant.
In Lemma 1 we showed that for any $\sigma$ there are $N_1(\sigma)$ points, that we enumerate as $z_L, \ldots, z_{L-1+N_1(\sigma)}$, which form a maximal $\sigma$-separated net in $B(p, r_0/4)$ and the balls $B(z_k, 4\sigma)$, $k = L, \ldots, L - 1 + N_1(\sigma)$, satisfy the finite intersection property with at most $N_F$ intersections. In this section we consider arbitrary $\sigma$, which value will be specified later, and points $z_\ell, \ell = L, \ldots, L - 1 + N_1(\sigma)$ that satisfy the conditions of Lemma 1. Also, below is $3/2 < s < 2$.

Our next goal is to approximately construct the values of the distance functions from a variable point $x \in M$ to all points $z_\ell, \ell = 1, 2, \ldots, L - 1 + N_1(\sigma)$, defined in Lemma 1. The main step is to approximate compute the Fourier coefficients of the functions of form $\chi_{\Omega}(x)v(x)$, where $\chi_{\Omega}(x)$ are the characteristic functions of some special subdomains $\Omega \subset M$ and $v(x)$ has a finite Fourier expansion. These subdomains $\Omega$ are defined using distances to $L$ points $\{z_1, \ldots, z_{L-1}, z_i\}$, where $i \in \{L, \ldots, L - 1 + N_1(\sigma)\}$ is arbitrary. For $i \in \{L, L+1, \ldots, L-1+N_1(\sigma)\}$, let $K_i = \{1, 2, \ldots, L-1\} \cup \{i\}$ and define $A^{(i)}$ to be the set of those $\alpha = (\alpha_\ell)_{\ell=1}^{L-1+N_1(\sigma)} \in \mathbb{R}^{L-1+N_1(\sigma)}$, such that

$$r_0/8 \leq \alpha_\ell \leq 2D, \text{ if } \ell \in K_i,$$

$$\alpha_\ell = 0, \text{ if } \ell \not\in K_i.$$  \hspace{1cm} (59)

Below, we will assume that

$$\gamma \leq \sigma.$$  \hspace{1cm} (60)

We denote

$$\tilde{\Gamma}(z, T) = B(z, r_0/16 + \gamma) \times (-T + r_0/16, T - r_0/16).$$  \hspace{1cm} (61)

Next we fix for a while the index $i \in \{L, \ldots, L - 1 + N_1(\sigma)\}$. To construct subdomains $\Omega$, we start with observation sets $\tilde{\Gamma}(z_\ell, \alpha_\ell)$.

$$\tilde{\Gamma}(\alpha) = \bigcup_{\ell \in K_i} \tilde{\Gamma}(z_\ell, \alpha_\ell).$$  \hspace{1cm} (62)

At last, for $b \in \mathbb{R}$, we define

$$M(\alpha, b\gamma) = \bigcup_{\ell \in K_i} B_M(z_\ell, \alpha_\ell + b\gamma).$$  \hspace{1cm} (63)
Then the corresponding domains of stable unique continuation are
\[ D(\alpha) = \bigcup_{\ell \in K_i} D(z_\ell, \gamma, \alpha_\ell), \quad D(\alpha, b\gamma) = \bigcup_{\ell \in K_i} D(z_\ell, \gamma, \alpha_\ell + b\gamma), \tag{64} \]
and the corresponding double cones of influences are given by
\[ \Sigma(\alpha) = \bigcup_{\ell \in K_i} \Sigma(z_\ell, \gamma, \alpha_\ell), \quad \Sigma(\alpha, b\gamma) = \bigcup_{\ell \in K_i} \Sigma(z_\ell, \gamma, \alpha_\ell + b\gamma), \]

We have the following volume estimate.

Lemma 2  a) Let \( \alpha \in A^{(i)}, i = L, \ldots, L - 1 + N_1(\sigma) \) and
\[ A = A(\alpha, \gamma) = \{ x \in M : d(x, \partial M(\alpha, 3\gamma)) \leq 5\gamma \}. \tag{65} \]
Then, there is a uniform constant \( C_{15} > 0 \), depending only on \( n, R, D, i_0 \) and \( r_0 \), such that
\[ \operatorname{vol}(A) \leq C_{15} L\gamma. \]

b) Consequently, by defining \( b(s) \), for \( \frac{3}{2} < s < 2 \), as \( b(s) = 1/2, n = 2, 3 \) and \( b(s) = s/n, n \geq 4 \), we see that there is a uniform constant \( c_1(s) \), depending only on \( s, n, R, D, i_0 \) and \( r_0 \), such that
\[ \| \chi_{B(z_\ell, \alpha_\ell + 8\gamma)} \backslash B(z_\ell, \alpha_\ell - 2\gamma) v \|_{L^2(M)} \leq c_1(s) \gamma^{b(s)} \| v \|_{H^s(M)}. \tag{66} \]

Proof. a) Let \( x \in A \). Then, for some \( \ell \in K_i \),
\[ x \in B(z_\ell, \alpha_\ell + 8\gamma) \backslash B(z_\ell, \alpha_\ell - 2\gamma). \tag{67} \]
Since \( \| d\exp_{z_\ell} |v| \) is uniformly bounded on \( \overline{M}_{n}(R, D, i_0) \) for \( v \in T_{z_\ell}M, |v| \leq 2D, \operatorname{vol}(B(z_\ell, \alpha_\ell + 8\gamma) \backslash B(z_\ell, \alpha_\ell - 2\gamma)) \leq C\gamma, \) for all \( \ell \in K_i \).

b) Similar to part a), we have \( \operatorname{vol}(B(z_\ell, \alpha_\ell + 8\gamma) \backslash B(z_\ell, \alpha_\ell - 2\gamma)) \leq c\gamma. \) Together with the H"{o}lder inequality and the Sobolev embedding \( H^s(M) \to L^q(M), \frac{1}{q} = \frac{1}{2} - \frac{s}{n}, \) (or \( C^0(M) \) for \( n = 2, 3 \)), this implies (66). Note that \( c_1(s) \) is a uniform constant as the embedding can be done in harmonic coordinates defined in balls with uniform radius. \( \square \)
4.1.1 Cut-off estimates and finite dimensional projections

Let us apply Lemma 1, with $\gamma$ instead of $\sigma$, to obtain points $x_\ell \in M$, $\ell = 1, 2, \ldots, N(\gamma)$ such that the balls $B(x_\ell, 2\gamma)$, $\ell = 1, 2, \ldots, N(\gamma)$ are a covering of $M$.

Let $\psi_\ell : M \to \mathbb{R}_+$, $\psi_\ell \in C^0(M)$ be in harmonic coordinates a partition of unity for the covering $B(x_\ell, 2\gamma)$ that satisfy

$$
\|\psi_\ell\|_{C^k,\beta(M)} \leq c_{k,\beta} \gamma^{-(k+\beta)}, \quad k = 0, 1, 2, \quad 0 \leq \beta < 1; \\
\operatorname{supp}(\psi_\ell) \subset B(x_\ell, 2\gamma), \quad \sum_{\ell=1}^{N(\gamma)} \psi_\ell(x) = 1. \quad (68)
$$

Below, we use $\Lambda_s \geq 1$.

**Lemma 3** For $\frac{3}{2} < s < 2$ there is $c_3(s) \geq 1$, in (70) such that, for any $u \in \mathcal{H}^s_{\Lambda_s}(M)$, $i \in \{L, \ldots, L - 1 + N_1(\sigma)\}$ and $\alpha \in \mathcal{A}^{(\gamma)}$, the following holds true: There

$$
u_\alpha \in \mathcal{H}^s_{C_{17}(s;\gamma)\Lambda_s}(M) \cap \mathcal{H}^0_{\Lambda_s}(M), \quad \nu_\alpha(x) = 0, \text{ if } x \in M(\alpha, \gamma), \quad (69)$$

$$
u_\alpha(x) = u(x), \text{ if } x \in M \setminus M(\alpha, 7\gamma),$$

where

$$C_{17}(s, \gamma) = c_3(s)\gamma^{-s}. \quad (70)$$

**Proof.** Define

$$u_\alpha(x) = \Psi(x)u(x), \quad \Psi(x) = \sum_{\operatorname{supp}(\psi_\ell) \cap M(\alpha, 3\gamma) = \emptyset} \psi_\ell(x). \quad (71)$$

For a general $w \in H^2(M)$ we have the following estimate in Sobolev spaces with $\frac{3}{2} < s < 2$

$$\|\Psi w\|_{H^s(M)} \leq C\|\Psi\|_{C^k(M)}\|w\|_{H^s(M)} \leq C m c_{2,0} \gamma^s\|w\|_{H^s(M)}, \quad (72)$$

where $m$ is the number of elements in the set $\{\ell : \operatorname{supp}(\psi_\ell) \cap M(\alpha, 3\gamma) = \emptyset\}$ satisfying $m \leq N(\gamma)$. Thus the existence of $c_3(s)$ such that the claim holds follows then from the finite intersection property of $B(x_\ell, 2\gamma)$, see Lemma 1 and estimates (68). \qed
4.2 Unique continuation for approximate projections

Corollary 2 implies the following result. Note that the notations \( \varepsilon_2 \) and \( \mathcal{E}_2 \) are introduced in order to distinguish \( \varepsilon_2 \) from its upper bound \( \varepsilon_2 \), written as an expression dependent on \( \varepsilon_1 \). Later, in formula (170) we set \( \varepsilon_1 \) to have a specific value and substitute it in the expression \( \mathcal{E}_2(\frac{\varepsilon_1}{4L}; \theta, \gamma, \Lambda_s) \) of formula (172) to obtain a specific value for \( \varepsilon_2 \).

**Corollary 3** Assume that \( v \) satisfies

\[
\|v\|_{H^s(M)} \leq C_{17}(s, \gamma) \Lambda_s \quad \text{and} \quad \|v\|_{L^2(M)} \leq \Lambda_s,
\]
with \( C_{17}(s, \gamma) \) defined in (70), and assume (40). Let \( \varepsilon_1 < \Lambda_s \) and \( \varepsilon_2 \leq \mathcal{E}_2(\frac{\varepsilon_1}{4L}; \theta, \gamma, \Lambda_s) \) where

\[
\mathcal{E}_2(\frac{\varepsilon_1}{4L}; \theta, \gamma, \Lambda_s) = \frac{\Lambda_s \gamma^s/(s-1)}{(\exp \left( \Lambda_s 4L \varepsilon_1^{-1} \gamma^{-2} \theta/2 \right) C_{12} \exp(\gamma - c_{200})^{1/\beta})_{s/(s-1)}} \quad (74)
\]

Let \( w = W(v) \) satisfy

\[
\|w\|_{L^2(\mathbb{R}(z_\ell, \alpha_\ell))} \leq \varepsilon_2
\]

on the domain (61). Then, for \( \ell \in K_i \),

\[
\|w(0, \cdot\|_{L^2(B(z_\ell, \alpha_\ell-2\gamma))} \leq \frac{\varepsilon_1}{4L}, \quad \|w(0, \cdot\|_{L^2(M(\alpha, -2\gamma))} \leq \frac{1}{4} \varepsilon_1.
\]

**Proof.** From a small modification of the proof of Corollary 2 we still can obtain the estimate (49) in the following way. The main point is to replace the initial condition \( \|v\|_{H^s(M)} \leq \Lambda_1 \) with (73). We then deduce the corresponding estimate for the solution \( w = W(v) \) of the wave equation, with \( T = \alpha_\ell \) and \( z = z_\ell \),

\[
\|w\|_{H^s(M \times [-T, T])} \leq C_{17}(s, \gamma) \Lambda_s \quad \text{and} \quad \|w\|_{L^2(M \times [-T, T])} \leq C \Lambda_s,
\]

Let \( \eta \) and \( \tilde{\eta} \) be the smooth localizers defined in the proof of Corollary 2. Calling again \( w_\eta = (1 - \eta(x))w \) and using the definition of \( C_{17} \) in (70) we get,

\[
\|\eta w\|_{H^s(M \times (-T+r_0/16, T-r_0/16))} \leq C \gamma^{-s} \Lambda_s,
\]

\[
\|\tilde{\eta} w\|_{H^s(M \times (-T+r_0/16, T-r_0/16))} \leq C \gamma^{-s} \Lambda_s,
\]

\[
\|w_\eta\|_{H^s(M \times (-T+r_0/16, T-r_0/16))} \leq C \gamma^{-s} \Lambda_s,
\]
and the intermediate $H^m$ norms follow by interpolation. Here the constant $C$ is dependent of $c_3(s)$ and independent of $\gamma$. Consequently,

$$\|\tilde{\eta}_w\|_{H^1(M \times (-T+r_0/16,T-r_0/16))} \leq C\gamma^{-1} \Lambda_s^{1/s} \varepsilon_2^{1-1/s},$$

$$\|F\|_{L^2(M \times (-T+r_0/16,T-r_0/16))} \leq C\gamma^{-2} \Lambda_s^{1/s} \varepsilon_2^{1-1/s}.$$ 

Using growth properties of the function $f_\theta$ we get (54). Also (57) still holds. Therefore we obtain (50), where the new constant $C_{12}$ in (51) now depends on $c_3(s)$. Next we observe that formula (50) implies that when $\varepsilon_1 = 4L \mathcal{E}_1(\varepsilon_2; \theta, \gamma, \Lambda_s)$, we have

$$\varepsilon_2 = \frac{\Lambda_s \gamma^{s/(s-1)}}{\left(\exp \left[ \left(\Lambda_s 4LC_{12}\varepsilon_1^{-1}\gamma-(2-\theta/2)\exp(\gamma^{-(c_{200}\theta/2)})\right)^{1/\beta} \right] - 1 \right)^{s/(s-1)},}$$

and $\mathcal{E}_2$ is defined by removing $-1$ from the denominator of the expression above, and by replacing $\exp(\gamma^{-(c_{200}\theta/2)})$ with $\exp(\gamma^{-c_{200}})$. This is done to simplify the calculations of the paper. The relation (76) follows by imposing on $\varepsilon_2$ the $\mathcal{E}_2$-bound. □

Under the conditions of the Corollary and from the growth properties of $\mathcal{E}_2(\varepsilon_1)$ it follows that

$$\varepsilon_2 \leq \mathcal{E}_2(\frac{\varepsilon_1}{4L}; \theta, \gamma, \Lambda_s) \leq \frac{\varepsilon_1}{4L}, \quad \varepsilon_1 \in (0, \Lambda_s].$$

### 4.3 Approximate projections

Let $\varepsilon_0, \varepsilon_1, \varepsilon_2$ satisfy

$$\varepsilon_0 \leq \frac{\Lambda_s}{10}, \quad \varepsilon_1 = \frac{\varepsilon_0^2}{10\Lambda_s}, \quad \varepsilon_2 = \mathcal{E}_2 \left( \frac{\varepsilon_1}{4L}, \theta, \gamma, \Lambda_s \right).$$

#### 4.3.1 Finite data with and without errors

Below we will use several parameters, and for the sake of clarity of presentation, we have gathered these parameters in this subsection and tell how those will be used.

Below, we will use $j_1 \in \mathbb{Z}_+$ satisfying

$$j_0 \geq j_0(\frac{\varepsilon_2}{8}; \gamma, \Lambda_s),$$

29
where
\[ \hat{j}_0(\varepsilon; \gamma, \Lambda_s) = C_{16} \gamma^{-n} \left( \frac{\Lambda_s}{\varepsilon} \right)^{\frac{n}{2}} \text{ and } C_{16}(s) = c_3(s)^{\frac{n}{2}} C_7^{\frac{n}{2}} (C_{20} + 1)^{\frac{n}{2}}. \] (80)

We also use \( j_1 \in \mathbb{Z}_+ \) satisfying
\[ j_0 \leq j_1 \leq 2^{n/2} (C_7)^n j_0 \] (81)

Moreover, we use
\[ \delta \leq \hat{\delta}_0(\varepsilon, \gamma, j_1, \Lambda_s) = c_5 \frac{1}{j_1} \frac{\varepsilon}{\Lambda_s}, \] (82)

where \( c_5 = \min(C_7^{-1}, \frac{(1+2C_7)^{-1/2}}{100(1+D)^{3/2}L}) \), and \( J \) satisfying
\[ (C_7^{-1} \delta^{-1})^{n/2} \leq J \leq (2C_7 \delta^{-1})^{n/2}, \] (83)

cf. Remark 4. Note that (83) implies that \( \lambda_J \geq \delta^{-1} \), see Def. 3 (ii) and (23).

The use of the above parameters are the following. We will assume that we are given the ball \( (B_e(r_0), g^n) \) and the pairs \( \{ (\lambda_j^a, \varphi_j^a|_{B_e(r_0)}) ; j = 0, 1, 2, \ldots, J \} \). We assume that these data are \( \delta \)-close to FISD of some manifold \( (M, g, p) \in \overline{\mathcal{M}}_n \), that is, the ball \( (B_e(r_0), g) \) and \( \{ (\lambda_j, \varphi_j|_{B_e(r_0)}) ; j = 0, 1, 2, \ldots, J \} \), where the error size parameter \( \delta \) satisfies (82).

We are going to formulate a minimization algorithm that will be used to compute volumes of the sets (63). We consider this minimization algorithm in the two cases, in the case when we have FISD without errors and the case when we have it with errors.

As we have finite data, we need to consider the projection of the solution of the wave equation to finitely many eigenvectors, and we choose \( j_0 \) so that it is enough to use \( j_0 \) eigenvectors. This requires that we have the data \( (\lambda_j, \varphi_j|_{B_e(r_0)}) \) with \( j = 0, 1, 2, \ldots, j_0 \). However, to consider minimization algorithms both for FISD with and without errors, we need to increase the amount of data and we will consider \( (\lambda_j, \varphi_j|_{B_e(r_0)}) \) with \( j = 0, 1, 2, \ldots, j_1 \), where \( j_1 \) is chosen as follows: In Definition 3 there are intervals \( I_p \subset \mathbb{R} \), \( p = 0, 1, \ldots, P \) covering the spectrum of \( M \) in \([0, \delta^{-1} + \delta]\) each \( I_p \) containing a cluster of \( n_p \) eigenvalues \( \lambda_j^a \) and approximate eigenvalues \( \lambda_j^e \). To consider these clusters of eigenvalues, let \( P_0 \) be the smallest integer \( P_0 \leq P \) such that
\[ \{ \lambda_0, \lambda_1, \ldots, \lambda_{j_0} \} \subset \bigcup_{p=0}^{P_0} I_p \] (84)
and then choose \( j_1 \) such that \( j_0 \leq j_1 \leq J \) and

\[
j \leq j_1 \implies \lambda_j \in \bigcup_{p=0}^{P_0} I_p, \quad j > j_1 \implies \lambda_j \notin \bigcup_{p=0}^{P_0} I_p.
\] (85)

We note that this happens with some \( j_1 \) satisfying (81). We also observe that as \( \delta \) satisfies (82) and \( J \) satisfies (83), and as \( \Lambda_s \geq 1, \varepsilon_2 < 1 \) and \( n \geq 2 \), we have

\[
J \geq J_0(\delta) = (C_7^{-1}\delta^{-1})^{n/2} \geq j_1.
\] (86)

### 4.3.2 Minimisation with FISD without errors

**Theorem 3** Let \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) satisfy (78). There is \( \gamma_0(\varepsilon_0; s, \Lambda_s) \) depending only on \( \varepsilon_0, s, \Lambda_s, n, R, D, i_0 \) and \( r_0 \), with the following properties: Let \( \gamma \leq \gamma_0(\varepsilon_0; s, \Lambda_s) \). Assume that \( j_0 \) satisfies (79) and \( j_1 \) satisfies (81), and

\[
u(x) = \sum_{j=0}^{j_1} a_j \varphi_j(x) \in \mathcal{H}_{\Lambda_s}^s(M),
\]

Let \( i \in \{L, \ldots, L - 1 + N_1(\sigma)\} \) and \( \alpha \in \mathcal{A}^{(i)} \).

Moreover, assume that we are given

\[
(B(p, r_0), g|_{B(p,r_0)}), \quad ((\lambda_j, \varphi_j|_{B(p,r_0)}))_{j=0}^{j_1} \quad \text{and} \quad (a_j)_{j=0}^{j_1}.
\] (87)

The data (87) determine the set \( C_m^s \) and the function \( L_{\Lambda_s} : C_m^s \to \mathbb{R} \), defined in (97) and (100), for which the minimizer of \( L_{\Lambda_s} \) in \( C_m^s \) is a sequence \((d_j)_{j=0}^{j_1} = (d_j(\alpha, i))_{j=0}^{j_1} \in \mathbb{R}^{j_1}\) such that \( v(x) = \sum_{j=0}^{j_1} d_j \varphi_j(x) \) satisfies

\[
v \in \mathcal{H}_{(2C_{17}(s,\gamma)\Lambda_s)}^s(M) \cap \mathcal{H}_{2\Lambda_s}^0(M), \quad \|v - \chi_{M(\alpha, -2\gamma)}u\|_{L^2(M)} < \varepsilon_0,
\] (88)

The above bound \( \gamma_0(\varepsilon_0; s, \Lambda_s) \) for \( \gamma \) is defined in (102).

Note that the sequence \((d_j)_{j=0}^{j_1}\) is not unique and that the theorem states the existence of sequences satisfying (88).

The next subsections are devoted to the proof of Theorem 3. In sec. 4.3.3, 4.3.4 and 4.3.5 we keep the index \( i \in \{L, \ldots, L - 1 + N_1(\sigma)\} \) fixed not referring to this.
4.3.3 Finite dimensional projections

Next we introduce some special sets of the finite-dimensional functions.

**Definition 5** Let \( b = (b_j)_{j=0}^{j_1} \in \mathbb{R}^{j_1+1} \) and \( F^*(b) \) be its Fourier coimage

\[
F^*(b) = \sum_{j=0}^{j_1} b_j \varphi_j \in L^2(M).
\]  

For \( a_1, a_2 > 0 \) the class of Fourier coefficients \( C_{j_1,s}(a_1, a_2) \) is defined as

\[
C_{j_1,s}(a_1, a_2) := \{ b \in \mathbb{R}^{j_1+1} : \sum_{j=0}^{j_1} (1 + \lambda_j^2)^s |b_j|^2 \leq a_1^2, \sum_{j=0}^{j_1} |b_j|^2 \leq a_2^2 \}. \]  

For \( w = W(v) \) being the solution to the problem (36) and \( b \in \mathbb{R}^{j_1+1} \), we denote

\[
W(b) = W(F^*(b)) \in C(\mathbb{R}; L^2(M))
\]  

and, for any \( \varepsilon > 0, \alpha \in A^i \), we denote

\[
C_{j_1,s}(\varepsilon; a_1, a_2, \alpha) := \{ b \in C_{j_1,s}(a_1, a_2) : \|W(F^*(b))\|_{L^2((\varepsilon, \alpha))} \leq \varepsilon, \forall \ell \in K_i \}.
\]

**Lemma 4** (i) Let \( v \in H_{C_{17}(s, \gamma)}^s(M) \) and let \( P_{j'}v = \sum_{j=0}^{j'} \langle v, \varphi_j \rangle_{L^2(M)} \varphi_j \). Then for any \( \alpha \in A^i, \varepsilon_2 > 0 \),

\[
\|P_{j'}v - v\|_{L^2(M)} \leq \frac{1}{8(C_20 + 1)} \varepsilon_2, \quad \text{if } j' \geq j_0 \geq j_0\left(\frac{\varepsilon_2}{8}; \gamma, \Lambda_s\right),
\]  

see (39) for \( C_{20} \) and (80) for \( \tilde{j}_0 = \frac{\varepsilon_2}{8}; \gamma, \Lambda_s\).

(ii) Let \( u \in H_{\Lambda_s}^s(M) \) and \( u_{\alpha} \) be given by (71). Let \( j_1 \) satisfy (79)-(81). Then,

\[
v_{\alpha} = P_{j_1}u_{\alpha} \in F^* \left( C_{j_1,s}(\frac{1}{8}\varepsilon_2; \frac{1}{4}C_{17}(s; \gamma)\Lambda_s, \Lambda_s, \alpha) \right).
\]

**Proof.** (i) For \( v = \sum_{j=0}^{\infty} b_j \varphi_j \), we have

\[
\|P_{j'}v - v\|_{L^2(M)}^2 = \sum_{j > j'} |b_j|^2 \leq |\lambda_{j'}|^{-s} C_{17}(s; \gamma)^2 \Lambda_s^2.
\]
Figure 5: In the proof of Theorem 3 we solve the minimization problem (98) where we consider waves $U(x,t) = W(v)$ with initial data $(U(x,0), U_t(x,0)) = (v(x), 0)$, $v = \sum_j b_j \varphi_j$, such that $U|_{\Gamma(\alpha)}$ is small. By using approximate unique continuation in the domain $D_0 = D(\alpha)$, we see that the wave $U$ is small in $\Sigma_2\gamma = \Sigma(\alpha, 2\gamma)$ and that $v(x)$ is small in $M(\alpha, -2\gamma)$.

Moreover, $u_\alpha$ vanishes in the set $M(\alpha, \gamma)$ and thus the wave $U_\alpha = W(u_\alpha)$ produced by the initial data $(u_\alpha, 0)$ vanishes in the extended double cone $\tilde{\Sigma} = \Sigma(\alpha, -\gamma)$ and thus in $\tilde{\Gamma}(\alpha)$. Using these we see that $v_\alpha = \mathcal{P}_j u_\alpha$ is close to the solution $v^*$ of the minimization problem (98).

Here, $C_{17}(s; \gamma)$ is defined in (70) and (23) with $s = 0$, and these imply the estimate (93).

(ii) The finite propagation speed of waves implies, due to $u_\alpha|_{M(\alpha, \gamma)} = 0$, that $W(u_\alpha)|_{\tilde{\Gamma}(z_\ell, \alpha_\ell)} = 0$. By Lemma 3 and (39)

$$\|W(v_\alpha)|_{L^2(\tilde{\Gamma}(z_\ell, \alpha_\ell))}\| \leq \|W(v_\alpha - u_\alpha)|_{L^2(\tilde{\Gamma}(z_\ell, \alpha_\ell))}\| \leq C_{20}\|v_\alpha - u_\alpha|_{L^2(\tilde{\Gamma}(z_\ell, \alpha_\ell))}\| \leq \frac{1}{8}\varepsilon_2. \quad (95)$$

Since $\|P_{j_1}\|_{H^s(M)} = 1$ for any $s$, the claim (i) of the lemma with $j' = j_1$, (95) together with (69) prove (94).

Remark 6 The condition $\|W(\mathcal{F}^*(b))|_{L^2(\tilde{\Gamma}(z_\ell, \alpha_\ell))}\| \leq \varepsilon_*$, see (94), is equivalent to

$$\left\|\sum_{j=0}^{j_1} b_j \cos(\lambda_j t) \varphi_j(x)\right\|_{L^2(\tilde{\Gamma}(z_\ell, \alpha_\ell))}\| \leq \varepsilon_*, \ell \in K_i. \quad (96)$$

which can be directly verified if we know $\{(\lambda_j, \varphi_j|_{B(p, r_0)})\}_{j=0}^{j_1}$. 

33
4.3.4 Minimisation algorithm

Assume that we are given \(a = (a_j)_{j=1}^{j_2} \in \mathbb{R}^{(j_1+1)}\) and denote \(u = F^*(a) \in \mathcal{H}_{\Lambda_s}(M)\). Our next goal is to use FISD to find a vector \(b \in C_{j_1,s}(C_{17}(s, \gamma) \Lambda_s, \Lambda_s)\) such that \(F^*(b)\) is close to \(\chi_M(a) F^*(a)\). To achieve this goal we will use a minimisation method.

Let \(\varepsilon_0, \varepsilon_1, \varepsilon_2\) satisfy (78). Let \(m \in \{1, 2, 4\}\) be a parameter we will use below, and

\[
U_m := F^*(c^*), \quad \text{where } c^*_m = C_{j_1,s}(\frac{1}{2m} \varepsilon_2; \frac{1}{m} C_{17}(s, \gamma) \Lambda_s, \Lambda_s, \alpha).
\]

(97)

**Definition 6** (i) A function \(v \in U_m\) is called an \(\varepsilon_1\)-minimizer of the minimization problem

\[
\min_{h \in U_m} L_u(h), \quad \text{where } L_u(h) = \|h - u\|_{L^2(M)}^2,
\]

if \(v\) satisfies

\[
\|v - u\|_{L^2(M)} \leq J_{\min}(m) + 5 \Lambda_s \varepsilon_1, \quad J_{\min}(m) := \inf_{h \in U_m} \|h - u\|_{L^2(M)}^2.
\]

(99)

(ii) Equivalently, a vector \(b = (b_j)_{j=0}^{j_1} \in C^*_m\) is an \(\varepsilon_1\)-minimizer of the minimization problem

\[
\min_{\xi \in c^*_m} L_a(c), \quad \text{where } L_a(c) = \|\xi - a\|_{\mathbb{R}^{(j_1+1)}}^2,
\]

if

\[
\|b - a\|_{\mathbb{R}^{(j_1+1)}}^2 \leq J_{\min}(m) + 5 \Lambda_s \varepsilon_1, \quad J_{\min}(m) := \inf_{\xi \in c^*_m} \|\xi - a\|_{\mathbb{R}^{(j_1+1)}}^2.
\]

(101)

Observe that for \(c \in C_{j_1,s}(\frac{1}{m} C_{17}(s, \gamma) \Lambda_s, \Lambda_s)\) we can check, using Remark 6 with \(\varepsilon_* = \varepsilon_2/2m\), that \(c \in C^*_m\) and thus find \(b\) which satisfies (101).

Next we assume that, in addition to \(\varepsilon_2\) satisfying (74), \(\gamma\) satisfies

\[
\gamma \leq \gamma_0 = \gamma_0(\varepsilon_0; s, \Lambda_s) = C_{28} \left(\frac{\varepsilon_1}{\Lambda_s}\right)^{1/(b(s))}, \quad \text{with}
\]

\[
C_{28}(s) = \frac{1}{(2Lc_1(s))^{1/(2b(s)))}}, \quad \varepsilon_1 = \frac{\varepsilon_0^2}{10 \Lambda_s},
\]

where \(b(s)\) and \(c_1(s)\) are defined in Lemma 2 b).
Lemma 5 Let \( u \in \mathcal{H}_M^s(M) \), and let \( \varepsilon_0, \varepsilon_1, \varepsilon_2, j_1, \gamma \) satisfy (78), (79)-(81) and (102). 

(i) For \( m \in \{1, 2, 4\} \) and all \( h \in \mathcal{U}_m \), we have

\[
\mathcal{L}_u(h) \geq \|u\|_{L^2(M(\alpha,-2\gamma))}^2 - 2\Lambda_s \varepsilon_1 + \|h - u\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2. 
\]  

(103)

(ii) The function \( v_\alpha \) defined by (94), (71) satisfies \( v_\alpha \in \mathcal{U}_m \) with \( m = 4 \) and \( \varepsilon_1 \)-minimiser,

\[
\mathcal{L}_u(v_\alpha) \leq \|u\|_{L^2(M(\alpha,-2\gamma))}^2 + 2\Lambda_s \varepsilon_1 + 4\varepsilon_1^2. 
\]  

(104)

Note that here \( v_\alpha \in \mathcal{U}_4 \subset \mathcal{U}_2 \subset \mathcal{U}_1 \).

(iii) For all \( m \in \{1, 2, 4\} \), the function \( v_\alpha \in \mathcal{U}_m \) is an \( \varepsilon_1 \)-minimiser,

\[
\mathcal{L}_u(v_\alpha) \leq \mathcal{L}_{\min}(m) + 5\Lambda_s \varepsilon_1. 
\]  

(105)

(iv) For all \( m \in \{1, 2, 4\} \), we have

\[
\left| \mathcal{L}_{\min}(m) - \|u\|_{L^2(M(\alpha,-2\gamma))}^2 \right| \leq 2\Lambda_s \varepsilon_1 + 4\varepsilon_1^2. 
\]  

(106)

**Proof.** (i) We have, for \( h \in \mathcal{U}_m \),

\[
\|h - u\|_{L^2(M)}^2 = \|h - u\|_{L^2(M(\alpha,-2\gamma))}^2 + \|h - u\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2 \geq (\|u\|_{L^2(M(\alpha,-2\gamma))}^2 - \|h\|_{L^2(M(\alpha,-2\gamma))}^2)^2 + \|h - u\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2. 
\]

Since \( h \in \mathcal{U}_m \), (74), (75) and (76) imply that \( \|h\|_{L^2(M(\alpha,-2\gamma))} \leq \varepsilon_1 \). Thus,

\[
\|h - u\|_{L^2(M)}^2 \geq \|u\|_{L^2(M(\alpha,-2\gamma))}^2 - 2\Lambda_s \varepsilon_1 + \varepsilon_1^2 + \|h - u\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2. 
\]

(ii) With \( u_\alpha, v_\alpha \) defined by (71) and (94), \( v_\alpha \in \mathcal{U}_4 \),

\[
\|u - v_\alpha\|_{L^2(M)}^2 = \|u - v_\alpha\|_{L^2(M(\alpha,-2\gamma))}^2 + \|u - v_\alpha\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2 \geq \|u\|_{L^2(M(\alpha,-2\gamma))}^2 + 2\Lambda_s \varepsilon_2 + \varepsilon_2^2 + 2\|u - v_\alpha\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2, 
\]

(107)

where we use that \( u - v_\alpha = (u - u_\alpha) + (u_\alpha - v_\alpha) \) and \( \|v_\alpha\|_{L^2(M(\alpha,-2\gamma))} \leq \varepsilon_2 \), see Lemma 4. Observe, that by (66), (69) and (102),

\[
\|u - u_\alpha\|_{L^2(M\setminus M(\alpha,-2\gamma))}^2 = \|u\|_{L^2(M(\alpha,\gamma)\setminus M(\alpha,-2\gamma))}^2 \leq c_1(s)\Lambda_s^4 L^2 \gamma^{2s} \leq \frac{1}{2}\varepsilon_1^2. 
\]  

35
where $c_1(s)$ is defined in Lemma 2 b). Using (73) and (69), we see that
\[ \|u_u - v_u\|_{L^2(M \setminus M(\alpha - 2\gamma))} \leq \varepsilon_2^2. \]
Thus, inequality (107) yields that
\[ \mathcal{L}_u(v_u) \leq \|u\|_{L^2(M(\alpha - 2\gamma))}^2 + 2\Lambda_s \varepsilon_2 + 3\varepsilon_2^2 + \varepsilon_1^2. \]
As $\varepsilon_2 \leq \varepsilon_1$, see (77), we get (104).
(iii) The claims (i) and (ii) together with (78) yield that
\[ \mathcal{L}_u(v_u) - J_{\text{min}}(m) = \mathcal{L}_u(v_u) - \min_{h \in \mathcal{U}_n} \mathcal{L}_u(h) \leq \left( \|u\|_{L^2(M(\alpha - 2\gamma))}^2 + 2\Lambda_s \varepsilon_1 + 4\varepsilon_1^2 \right) - \left( \|u\|_{L^2(M(\alpha - 2\gamma))} - 2\Lambda_s \varepsilon_1 \right) \leq 5\Lambda_s \varepsilon_1. \]
(iv) The claim (iv) follows from (i) and (ii).

Lemma 6 Let $m \in \{1, 2, 4\}$ and $u, v \in \mathcal{H}_m^s(M), \varepsilon_0, \varepsilon_1$, and $\varepsilon_2$ satisfy (74) and (78), $j_1$ satisfies (79)-(81) and $\gamma$ satisfies (102). Let $v^* = \sum_{j=0}^{j_1} b_j \varphi_j$ be any $\varepsilon_1$-minimizer of the minimization problem (98), with $b \in \mathcal{C}_m$. Then
\[ \|v^* - \chi(M \setminus M(\alpha - 2\gamma))u\|_{L^2(M)}^2 \leq \varepsilon_0^2. \]

Proof. Since $v_u$ satisfies (104) and (105),
\[ \|v^* - u\|_{L^2(M)}^2 \leq \|v_u - u\|_{L^2(M)}^2 + 5\Lambda_s \varepsilon_1 \leq \|u\|_{L^2(M(\alpha - 2\gamma))}^2 + 2\Lambda_s \varepsilon_1 + 4\varepsilon_1^2. \]
Since $v^* - u$ satisfies (103), this inequality implies that
\[ \|v^* - u\|_{L^2(M, M(\alpha - 2\gamma))}^2 \leq 9\Lambda_s \varepsilon_1 + 4\varepsilon_1^2. \]
Since $v^* \in \mathcal{U}$, $u^* = W(v^*)$ satisfies (96) with $\varepsilon^* = \varepsilon_2$, where $\varepsilon_2$ satisfies (74) and (78). It then follows from Corollary 3 that
\[ \|v^*\|_{L^2(M(\alpha - 2\gamma))}^2 \leq \varepsilon_1. \]
Due to (78), this inequality together with (109), implies (108).

Proof of Theorem 3 Assume that $a := (a_j)_{j=0}^{j_1}$ satisfies the hypothesis.
First determine $(b_j)_{j=0}^{j_1}$ so that $v^* = \sum_{j=0}^{j_1} b_j \varphi_j(x)$ is an $\varepsilon_1$-minimizer of (98), $v^* \in \mathcal{C}_{j_1, s}(\frac{1}{m} C_{17}(s, \gamma) \Lambda_s, \Lambda_s)$ with $m = 1$. Then, by (108),
\[ \|\chi M(\alpha - 2\gamma)u - \sum_{j=0}^{j_1} (a_j - b_j) \varphi_j\|_{L^2(M)} < \varepsilon_0. \]
Take $d_j = a_j - b_j$. Then $v(x) = \sum_{j=0}^{j_1} d_j \varphi_j(x)$ satisfies (88).
4.3.5 Minimisation with finite interior spectral data with errors

In this section we consider an approximate construction when there is a \( \delta \)-error in FISD. We assume that we are given the ball \( (B_e(r_0), g^a) \) and the pairs \((\lambda_j^a, \varphi_j^a|_{B_e(r_0)})\) with \( j = 0, 1, 2, \ldots, J \).

We assume that these data are \( \delta \)-close to ISD of some manifold \((M, g, p) \in \mathcal{M}_\Sigma \) in the sense of Definition 3 with intervals \( I_p \subset \mathbb{R}, p = 0, 1, \ldots, P \) covering the spectrum of \( -\Delta_g \) in \([0, \delta^{-1} + \delta]\). We will use parameters \( j_0, j_1 \in \mathbb{Z}_+ \) and \( P_0 \in \mathbb{Z}_+ \) satisfying (79)-(81), (84), and (85). Note that then \( j_0 \leq j_1 \leq J \) and that below we will use \((\lambda_j^a, \varphi_j^a|_{B_e(r_0)})\) with \( j = 0, 1, 2, \ldots, j_1 \). Denote 

\[ J_p = \{ j \in \mathbb{Z}_+; \lambda_j^a \in I_p \} \]

and \( n_p \) is the number of elements in \( J_p \).

Then, for any \( p \) there is \( A^p \in O(n_p) \), \( p = 1, 2, \ldots, P_0 \) such that, if \( j \in J_p \) then

\[ \tilde{\varphi}_j = \sum_{k \in J_p} A^p_{jk} \varphi_k \] (110)

satisfies \( \| \tilde{\varphi}_j - \varphi_j^a \|_{L^2(M)} < \delta \), where \( \varphi_k \) are the eigenfunctions of \( \Delta_g \). Note that \( \sum_{p=0}^{P_0} n_p = j_1 + 1 \). We use below the matrix \( E \in O(j_1 + 1) \),

\[ E = [e_{jk}]_{j,k=0}^{j_1}, \quad e_{jk} = \langle \tilde{\varphi}_k, \varphi_j \rangle_{L^2(M)} \] (111)

and note that \( e_{jk} = 0 \) if \( \lambda_j, \lambda_k \) do not lie in the same \( I_p \).

Let \( b = (b_0, b_1, \ldots, b_{j_1}) \in \mathbb{R}^{j_1+1} \) then, for \( b^a = E(b), b^a = (b_0^a, b_1^a, \ldots, b_{j_1}^a) \) we have

\[ \sum_{j=0}^{j_1} b_j^a \tilde{\varphi}_j(x) = \sum_{j=0}^{j_1} b_j \varphi_j(x). \] (112)

Also, let \( \omega_j \) be the center point of the interval \( I_p \) containing \( \lambda_j^a \) so that \( |\lambda_j^a - \omega_j| < \delta \).

The main goal of this section is to prove

**Theorem 4** Let \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) satisfy (78). Let \( \gamma \) satisfy (102), \( j_0 \) satisfies (79) and \( j_1 \) satisfy (84), \( \delta \) satisfies (82), and let \( J \) satisfies

\[ J_0(\delta) \leq J \leq 2^{n/2} C_7 n J_0(\delta), \quad \text{where} \quad J_0(\delta) = (2C_7 \delta)^{-n/2}. \] (113)

Then the following is valid:
Let \( z_1, \ldots, z_{L-1+N_1(\sigma)} \in B(p, r_0/4) \) be a \( \sigma \)-net. Assume that \( g^a|_{B_e(r_0)} \) and \( ((\lambda^a_j, \varphi^a_j|_{B_e(r_0)}))_{j=0}^{j_1} \) is \( \delta \)-close to \( \text{FISD} g|_{B(p, r_0)} \) and \( ((\lambda_j, \varphi_j|_{B(p, r_0)}))_{j=0}^{j_1} \) of a manifold \((M, g, p) \in \mathbf{M}_n\). Also, assume that \( \tilde{a} = (\tilde{a}_j)_{j=0}^{j_1} \) satisfies \( \sum_{j=0}^{j_1} (\lambda^a_j)^s |\tilde{a}_j|^2 \leq \Lambda^2_s \), and
\[
\tilde{u}^a(x) = \tilde{F}^s(\tilde{a}) = \sum_{j=0}^{j_1} \tilde{a}_j \varphi_j(x), \quad x \in M.
\]

Let \( \alpha \in A^{(i)} \).

Assume that we are given
\[
g^a|_{B_e(r_0)}, \quad ((\lambda^a_j, \varphi^a_j|_{B_e(r_0)}))_{j=0}^{j_1}, \quad \text{and} \quad (\tilde{a}_j)_{j=0}^{j_1}.
\]

Let \( i \in \{L, \ldots, L-1+N_1(\sigma)\} \) and \( \alpha \in A^{(i)} \).

The data \((114)\) determine the set \( C^a_{0,*} \) and the function \( L_a : C^a_{2,*} \to \mathbb{R} \), defined in \((128)\) and \((131)\), for which the minimizer of \( L_a \) in \( C^a_{2,*} \) is a sequence
\[
d^a = d^a(\alpha, i) = (d^a_j(\alpha, i))_{j=0}^{j_1} \in \mathbb{R}^{j_1},
\]
such that \( \tilde{v}^a(x) = \tilde{F}^s(d^a) = \sum_{j=0}^{j_1} d^a_j \varphi_j(x), \quad x \in M, \) satisfies, cf. \((88)\),
\[
\tilde{v}^a \in H^s_{2C_{17(s,\gamma)}(M)} \cap H^0_{2\Lambda_s}(M), \quad \|\tilde{v}^a - \chi_{M(\alpha,-2\gamma)} \tilde{u}^a\|_{L^2(M)} < \varepsilon_0.
\]

### 4.3.6 Proof of Theorem 4

The rest of this section will be devoted to the proof of Theorem 4. Similar to \((89)\), we introduce
\[
\tilde{F}^s(b^a) = \sum_{j=0}^{j_1} b^a_j \tilde{\varphi}_j(x), \quad x \in M;
\]
and wave-type functions

\[ w^a(x, t) = W^a(b^a) := \sum_{j=0}^{j_1} b_j^a \cos(\sqrt{\lambda_j^a} t) \varphi_j^a(x), \quad x \in B(p, r_0); \quad (116) \]

\[ w(x, t) = \widetilde{W}(b^a)(x, t) = W\left(\widetilde{\mathcal{F}}^*(b^a)\right)(x, t), \quad x \in M; \quad (117) \]

\[ \widetilde{w}(x, t) = \sum_{j=1}^{j_1} b_j^a \cos(\sqrt{\omega_j} t) \widetilde{\varphi}_j(x), \quad x \in M; \quad (118) \]

\[ \widetilde{w}^a(x, t) = \widetilde{W}^a(b^a) := \sum_{j=1}^{j_1} b_j^a \cos(\sqrt{\lambda_j^a} t) \widetilde{\varphi}_j(x), \quad x \in M, \quad (119) \]

where we recall that \( W \) is defined by \( W(v) = w \) where \( w \) satisfies (36), and

\[ C^a_{j_1, s}(a_1, a_2) = \{ b^a \in \mathbb{R}^{(j_1+1)} : \sum_{j=0}^{j_1} (\lambda_j^a)^s |b_j^a|^2 \leq a_1^2, \sum_{j=0}^{j_1} |b_j^a|^2 \leq a_2^2 \} \quad (120) \]

\[ C^a_{j_1, s}(\varepsilon, a_1, a_2, \alpha) = \{ b^a \in C^a_{j_1, s}(a_1, a_2) ; \| W^a(b^a) \|_{L^2(\tilde{\Gamma}(z_\ell, \alpha_\ell))} \leq \varepsilon, \ell \in K_i \}. \]

We note that (see (112) and (91))

\[ \widetilde{\mathcal{F}}^*(b^a) = \mathcal{F}^*(E^{-1}b^a), \quad \widetilde{W}(b^a) = W(E^{-1}b^a). \quad (121) \]

**Lemma 7** Let \( b^a \in C^a_{j_1, s}(C_{17}(\gamma, s)\Lambda_s, \Lambda_s) \). If \( \delta < 1 \) satisfies \((82)\) then,

\[ \| w - w^a \|_{L^2(B(p, r_0) \times (-2D, 2D))} \leq \frac{1}{4} \varepsilon_2. \]

**Proof.** Due to (6) and (23), for \( j, k \in J_p \),

\[ |\sqrt{\lambda_j^a} - \sqrt{\omega_k}| \leq 2 \sqrt{C_{18}\delta}, \quad \| \varphi_j - \varphi_j^a \|_{L^2(B(p, r_0))} \leq \delta, \quad \| \varphi_k^a \|_{L^2(B(p, r_0))} \leq 2. \quad (122) \]

Using this, we obtain for \( |t| \leq 2D \) the following estimates. First, the Schwartz inequality implies that

\[ \| w^a(\cdot, t) - \widetilde{w}^a(\cdot, t) \|_{L^2(B(p, r_0))} \leq \left( \sum_{j=0}^{j_1} |b_j^a| \right) \delta \quad (123) \]

\[ \leq (j_1 + 1)^{1/2} \left( \sum_{j=0}^{j_1} (b_j^a)^2 \right)^{1/2} \delta \leq 2\Lambda_s(j_1)^{1/2} \delta. \]

39
Also, we see that
\[
\| \tilde{w}(\cdot, t) - \bar{w}(\cdot, t) \|^2_{L^2(B(p, r_0))} \leq \sum_{j=0}^{j_1} (\cos(\sqrt{\lambda_j} t) - \cos(\sqrt{\omega_j} t))^2 (b_j^a)^2 \tag{124}
\]
\[
\leq (2D)^2 C_{18}^2 \delta^2 \Lambda_s^2 = 4D^2 C_{18} \Lambda_s^2 \delta^2.
\]
We have
\[
\tilde{w}(x, t) = \sum_{p=0}^{P} \sum_{j, k \in J_p} b_j^p \cos(\sqrt{\omega_k} t) A_{jk}^p \varphi_k(x) = \sum_{j=0}^{j_1} \left( \sum_{j \in J_p} A_{jk}^p b_j^p \right) \cos(\sqrt{\omega_k} t) \varphi_k(x)
\]
and
\[
w(x, t) = \sum_{p=0}^{P} \sum_{j, k \in J_p} b_j^p \cos(\sqrt{\lambda_k} t) A_{jk}^p \varphi_k(x) = \sum_{j=0}^{j_1} \left( \sum_{j \in J_p} A_{jk}^p b_j^p \right) \cos(\sqrt{\lambda_k} t) \varphi_k(x),
\]
and as \( A^p \) are orthogonal matrices and \( |\sqrt{\lambda_k} - \sqrt{\omega_k}| \leq 2C_{18}^{1/2} \delta \), we see similarly to (123) and (124)
\[
\| \tilde{w}(\cdot, t) - w(\cdot, t) \|^2_{L^2(B(p, r_0))} \leq 4D^2 C_{18} \Lambda_s^2 \delta^2.
\] (125)

Combining the above estimates with \( \delta < \hat{\delta}_0(\varepsilon_2, \gamma, j_1, \Lambda_s) = c_5 \frac{1}{j_1} \varepsilon_2 \) and \( c_5 = \min(C_7^{-1}, \frac{(1+C_{18})^{-1/2}}{100(1+D)^{3/2}L}) \), we obtain the claim. \( \square \)

By Definition 5 we have
\[
EC_{j_1, \alpha} \left( \frac{1}{2} a_1, a_2 \right) \subset C_{j_1, \alpha} (a_1, a_2) \subset EC_{j_1, \alpha} (2a_1, a_2).
\] (126)

Note that the \( \ell^2 \)-norms of the sequences \( (b_j^p)_{j=1}^{j_1} \) do not depend on eigenvalues and, therefore, the same holds for the exact and approximate data. Also, the \( \ell^2 \)-norms are invariant with respect to the operations involving orthogonal matrices.

Definitions of the sets of sequences in (92) and (120), Lemma 7 and formula (126) imply that
\[
EC_{j_1, \alpha} (\varepsilon_s - \frac{1}{4} \varepsilon_2; \frac{1}{2} a_1, a_2, \alpha) \subset C_{j_1, \alpha} (\varepsilon_s; a_1, a_2, \alpha) \subset EC_{j_1, \alpha} (\varepsilon_s + \frac{1}{4} \varepsilon_2; 2a_1, a_2, \alpha)
\] (127)
Let us use \( \varepsilon_* = \frac{1}{2} \varepsilon_2 \) and define
\[
C_{m}^{a,*} = C_{j_1,s}^{a} \left( \frac{1}{m} \varepsilon_m, \frac{1}{2} C_{17}(\gamma, s)\Lambda_s, \Lambda_s, \alpha \right), \quad m \in \{1, 2, 4\}
\] (128)

Using the notations in (97), we see that
\[
EC_1^* \subset C_2^{a,*} \subset EC_1^*.
\] (129)

Consider the quadratic function \( L_a : \mathbb{R}^{j_1+1} \to \mathbb{R} \),
\[
L_a(c) = \|c - a\|_2^2, \quad L_{Ea}(c) = \|c - Ea\|_2^2.
\]

Note that \( L_a(c) = L_{Ea}(Ec) \). Let \( b_*^{a} \in C_4^a \) and \( b_*^{a,*} \in C_2^{a,*} \) be minimizers of \( L_a \) and \( L_{Ea} \), respectively, that is
\[
L_a(b_*) = \min_{b \in C_4^a} L_a(b) =: J_{min}(4),
\] (130)

and
\[
L_{Ea}(b_*^{a,*}) = \min_{b_*^{a,*} \in C_2^{a,*}} L_{Ea}(b_*^{a}) =: J_{min}^a(2).
\] (131)

Note that we do not anymore consider \( \varepsilon_1 \)-minimizers, but the minimizers. Since \( C^* \) and \( C_2^{a,*} \) are bounded and closed set in \( \mathbb{R}^{j_1+1} \) such minimizers exist.

When \( \varepsilon_1 < \Lambda_s/8 \), Lemma 5 (iv) implies
\[
|J_{min}(4) - J_{min}(1)| \leq 2(2\Lambda_s \varepsilon_1 + 4\varepsilon_1^2) < 5\Lambda_s \varepsilon_1
\] (132)

Using (129), (132), and the fact that \( E \) is an isometry, we see that
\[
J_{min}(1) \leq J_{min}^a(2) \leq J_{min}(4), \quad J_{min}(2) \leq J_{min}(1) + 5\Lambda_s \varepsilon_1.
\]

These implies that the minimizer \( b_*^{a,*} \) of function \( L_{Ea} \) in the set \( C_2^{a,*} \) satisfies \( b_*^{a,*} \in C_2^{a,*} \subset EC_1^* \) and so we have that \( \bar{b}^* = E^{-1}b_*^{a,*} \) is an \( \varepsilon_1 \)-minimizer of \( L_a \) in the class \( C_1^* \). We denote \( \bar{b}^* = (b^*_j)_{j=1}^{j_1} \). Let \( a = E^{-1}a \) so that
\[
\mathcal{F}^*(a) = \sum_{j=0}^{j_1} a_j \varphi_j(x) = u(x).
\]

Then, by applying Lemma 6 we see that \( v^* = \sum_{j=0}^{j_1} \bar{b}^*_j \varphi_j \) satisfies (108).

Then, choosing \( d_s^a = a_j - b_*^{a,*} \), \( j = 0, 1, \ldots, j_1 \), we see that \( \bar{v}^* = \sum_{j=0}^{j_1} d_s^a \varphi_j \) satisfies (115). This proves Theorem 4.
Remark 7 Similarly to Remark 4 and Theorem 4, we see that if the collection of $g^a|_{B_e(r_0)}$ and $((\lambda_j^a, \varphi_j^a|_{B_e(r_0)}))_{j=0}^J$ is $\delta$-close to FISD of a manifold $(M, g, p) \in M_n$, then without loss of generality, we can assume that $J$ satisfies (113). Indeed, the eigenvalues $\lambda_j$ with index $j > 2^{n/2}C_7^nJ_0(\delta)$ are not used in the proof of Theorem 4.

5 Construction of the approximate interior distance maps.

5.1 Volume estimates

Our next goal is to approximately evaluate the volume of $M(\alpha)$, see (63) with $b = 0$.

Lemma 8 There are uniform constants $\varepsilon_0^* > 0$, $C_{21}(s) > 1$, depending only on $s, n, R, D, i_0$ and $r_0$, such that the following holds:

Let $\varepsilon_0 \leq \varepsilon_0^*$. Let $\varepsilon_1, \varepsilon_2$ be defined by (78) while $\gamma_0, j_1$ be defined by (103) and (79)–(81). Assume that we are given $(g^a|_{B_e(r_0)}; \{(\lambda_j^a, \varphi_j^a|_{B_e(r_0)})\})_{j=0}^J$ that is $\delta$-close to FISD of $(M, g, p) \in M_n$. Here $J$ satisfies (113).

Let also $i \in \{L, \ldots, L - 1 + N_1\}$, where $N_1 = N_1(\sigma)$ is defined as in Lemma 7 (ii). Assume that

$$\sigma \leq \tau_0/2$$

where $\tau_0$ is defined in Proposition 2 and let $\alpha \in A^{(i)}$ satisfy (50).

Then we can compute an approximate volume, $vol^a(M(\alpha))$, of the set $M(\alpha)$ that satisfies

$$|vol^a(M(\alpha)) - vol(M(\alpha))| \leq C_{21}(s)\varepsilon_0.$$  

Proof. Recall that

$$\varphi_0(x) = vol(M)^{-1/2}, \quad \mathcal{F}(\varphi_0) = (1, 0, 0, \ldots), \quad \|\varphi_0\|_{H^s} = 1$$

for $s > 0$. (135)

The interval $I_0 = (a_0, b_0)$ in Definition 3 contains only $\lambda_0 = 0$. Thus $\varphi_0|_{B_e(r_0)}$ is a $\delta$-close to $\varphi_0|_{B_e(r_0)} = \widetilde{\varphi}_0|_{B_e(r_0)}$. These allow us to evaluate $vol^a(M)$ so that $|vol^a(M) - vol(M)| \leq C\delta$. Using Theorem 4 we evaluate the Fourier coefficients $(d_j^a)_{j=0}^{j_1}$ of $v^a(x)$ which satisfies (115) with $\widetilde{u} = \varphi_0$. Let

$$vol^a(M(\alpha)) = vol^a(M) \left(\sum_{j=0}^{j_1} (d_j^a)^2\right)^{1/2}$$

(136)
Figure 6: Slicing of the manifold: The intersection of “slices” \( B(z_1, r_1^+) \setminus B(z_1, r_1^-) \) and \( B(z_2, r_2^+) \setminus B(z_2, r_2^-) \), where \( z_1, z_2 \in B(p, r_0) \), is the set \( M^* = (B(z_1, r_1^+) \setminus B(z_1, r_1^-)) \cap (B(z_2, r_2^+) \setminus B(z_2, r_2^-)) \). When \( r_1^+ = \beta_1 \pm 2\sigma \) and \( \beta = (\beta_1, \beta_2, 0, 0, \ldots, 0) \) we have \( M^* = M^*(\beta) \), see (138). We consider analogous indexes \( \beta \) containing \( L \) non-zero elements and the sets \( M^*(\beta) \) for which the approximate volume \( \text{vol}^a(M^*(\beta)) \) is large enough. Using those we choose the set \( \mathcal{B} \) of the “admissible” indexes \( \beta \) and consider the collection \( X \) that is the union of the points \( x_\beta \) chosen from the sets \( M^*(\beta) \) with \( \beta \in \mathcal{B} \), and the points in a maximal \( \sigma \)-net in \( B(p, r_0/2) \), see (151). The set \( X \) can be considered as a discrete approximation of the manifold \( M \). The intersection of slices is also used to construct a distance function \( d_X \) on the discrete set \( X \) which makes \((X, d_X)\) an approximation of the manifold \((M, d_M)\).

Then, by (115),

\[
|\text{vol}^a(M(\alpha)) - \text{vol}(M(\alpha, -2\gamma))| \leq C(\varepsilon_0 + \delta). \tag{137}
\]

Since \( |\text{vol}(M(\alpha)) - \text{vol}(M(\alpha, -2\gamma))| < C_{15}L\gamma \) (cf. Lemma 2), (137) implies estimate (134), if \( \varepsilon_0 \leq \varepsilon_0^* \) with some uniform constants \( C_{21}(s) \) and \( \varepsilon_0^* \). Here \( \varepsilon_0^* \) is defined so that \( \delta < \varepsilon_0, \gamma < \gamma_0 \) for \( \varepsilon_0 < \varepsilon_0^* \), see (82), (102). □

Next we use FISD with errors to approximately find the distances from various points \( x \in M \) to points \( z \in B(p, r_0/4) \). The main tool is to approximately find the volumes of subdomains of \( M \) obtained by the slicing procedure.

For \( i \in \{L, \ldots, L - 1 + N_1(\sigma)\} \) and \( \beta \in \mathcal{A}^{(i)} \), \( M(\beta) \) are the domains defined in (63) with \( \alpha \) replaced by \( \beta \). We consider the intersection of slices,

\[
M^*_i(\beta) = \bigcap_{\ell \in K_i} (B(z_\ell, \beta_\ell + 2\sigma) \setminus B(z_\ell, \beta_\ell - 2\sigma)) \tag{138}
\]

\[
= \left( \bigcap_{\ell \in K_i} B(z_\ell, \beta_\ell + 2\sigma) \right) \cap \left( \bigcup_{\ell \in K_i} B(z_\ell, \beta_\ell - 2\sigma) \right)^c.
\]

43
Here for $\Omega \subset M$, $\Omega^c = M \setminus \Omega$. Note that
\[
\text{vol} \left( \bigcap_{\ell \in K_i} \Omega_\ell \right) = \sum_{\ell \in K_i} \text{vol}(\Omega_\ell \cup \tilde{\Omega})
\] (139)
\[
- \sum_{\ell \neq \ell'=1}^n \text{vol}(\Omega_\ell \cup \Omega_{\ell'} \cup \tilde{\Omega}) + \cdots + (-1)^{(L+1)} \text{vol} \left( \bigcup_{\ell \in K_i} \Omega_\ell \cup \tilde{\Omega} \right) - \text{vol}(\tilde{\Omega}).
\]

By (138), $M_{(i)}^*(\beta)$ has form (139) with
\[
\Omega_\ell = B(z_\ell, \beta_\ell + 2\sigma), \quad \tilde{\Omega} = \bigcup_{\ell \in K_i} B(z_\ell, \beta_\ell - 2\sigma).
\]

For any $\alpha_1, \alpha_2 \in A^{(i)}$ we have
\[
M(\alpha_1) \cup M(\alpha_2) = M(\alpha_m), \quad \text{where } (\alpha_m)_\ell = \max((\alpha_1)_\ell, (\alpha_2)_\ell).
\]

Therefore all terms in (139) are of form $\text{vol} (M(\alpha))$ for some $\alpha \in A^{(i)}$. Thus, using Lemma 8, we can approximately compute each term of (139) with error $C_{21}\varepsilon_0$. Since there are $2^L + 1$ terms in (139), we obtain the following result.

**Lemma 9** Under the conditions of Lemma 8, there exists $\varepsilon_4(n, R, D, i_0, r_0) > 0$ and $C_7 \in (0, 1)$, depending only on $n$, $R$, $D$, $i_0$ and $r_0$, with the following property:

Let $0 < \varepsilon_4 < \varepsilon_4(n, R, D, i_0, r_0)$. It is possible to evaluate approximate volumes, $\text{vol}^a(M_{(i)}^*(\beta))$, of the sets $M_{(i)}^*(\beta)$ of form (138). Moreover,
\[
\left| \text{vol}^a(M_{(i)}^*(\beta)) - \text{vol}(M_{(i)}^*(\beta)) \right| \leq \varepsilon_4, \quad \text{if } \varepsilon_0 \leq C_7 \varepsilon_4.
\] (140)

### 5.2 Distance functions approximation

A function $r(\cdot) \in C^{0,1}(B(p, r_0/4))$ is an *interior distance function* if there is $x \in M$ such that $r(z) = r_x(z) = d(x, z)$, for any $z \in B(p, r_0/4)$.

The interior distance functions determine the interior distance map
\[
R_M : (M, g) \to L^\infty(B(p, r_0/4)), \quad R_M(x) = r_x(\cdot).
\]

The map $R_M$ or, more precisely, its image
\[
R_M(M) := \{r_x(\cdot), \ x \in M\} \subset L^\infty(B(p, r_0/4)),
\] (141)
may be used to reconstruct \((M, g)\). Namely, in \cite{34, 30} it was shown how to reconstruct \((N, g|_N)\), where \(N = M \setminus B(p, r_0/50)\), from the knowledge of boundary distance functions

\[
R_N(N) = \{r^N_x(\cdot) \in L^\infty(\partial N); x \in N\}, \quad r^N_x(z) = d_N(x, z),
\] (142)

where \(d_N\) is the distance in \(N\). Later, in Section 6.1 we show that a Hausdorff approximation \(R^*_M\) to \(R_M(M)\) makes it possible to construct an approximation \(R^*_N\) to \(R_N(N)\).

Thus, our next goal is to construct a desired approximation \(R^*_M\). To this end, we use the volume approximations of the previous subsection.

First, for \(z, z' \in B(p, r_0/2)\), we define an approximate distance \(d^a(z, z')\) using the metric \(g^a\). Then Definition 3 (iv) together with convexity of \(B(p, r_0)\), see \cite{2}, imply that

\[
|d^a(z, z') - d(z, z')| \leq C_{23}\sigma, \quad \text{if } \delta < C_{24}\sigma.
\] (143)

Recall that above we have used a parameter \(\sigma > 0\) which satisfy \(\sigma \leq \tau_0/2\) and we have chosen points \(\{z_1, \ldots, z_{L-1+N_i(\sigma)}\} \subset B(p, r_0/4)\) such that \(\{z_1, \ldots, z_{L-1}\}\) is a \(\tau_0\)-net in \(B(p, r_0/4)\). Moreover, the set \(\{z_L, \ldots, z_{L-1+N_i(\sigma)}\}\) is a maximal \(\sigma\)-separated set in \(B(p, r_0/4)\), see (35).

For any \(i \in \{L, \ldots, L - 1 + N_1(\sigma)\}\) and \(\beta = (\beta_i)_{i=1}^{L-1+N_1(\sigma)} \in \mathbb{R}^{L-1+N_1(\sigma)}, r_0/8 < \beta_i < 2D, \) cf. (59), we define \(\mathcal{T}^{(i)}(\beta) = \beta^{(i)} \in \mathbb{R}^{L-1+N_i(\sigma)}, \) where

\[
\beta^{(i)} = \beta_i, \quad \text{if } \ell \in K_i,
\]

\[
\beta^{(i)} = 0, \quad \text{if } \ell \not\in K_i.
\] (144)

Then, \(\mathcal{T}^{(i)}(\beta) \in \mathcal{A}^{(i)}\).

Observe that, for any \(x \in M \setminus B(p, 3r_0/8+\sigma)\) and \(\ell = 1, \ldots, L - 1 + N_1(\sigma)\) there is \(\beta_i(x) \in \sigma\mathbb{Z}_+\) such that \(\beta_i(x) - \sigma \leq d_M(x, z_\ell) \leq \beta_i(x) + \sigma\). Therefore, \(B(x, \sigma) \subset B(z_\ell, \beta_i(x) + 2\sigma) \setminus B(z_\ell, \beta_i(x) - 2\sigma)\), so that, due to (20),

\[
\text{vol } (B(z_\ell, \beta_i(x) + 2\sigma) \setminus B(z_\ell, \beta_i(x) - 2\sigma)) \geq \frac{1}{C_6}\sigma^n.
\]

Taking into account this inequality together with (140) we require

\[
\varepsilon_4 \leq \frac{1}{4C_6}\sigma^n.
\] (145)
Thus, for \( i \in \{ L, \ldots, L - 1 + N_1(\sigma) \} \), the volume and the approximate volume of the set \( M^*_i(\beta^{(i)}(x)) \), \( \beta^{(i)}(x) = T^{(i)}(\beta(x)) \) satisfy

\[
\text{vol}(M^*_i(\beta^{(i)}(x))) \geq 4\varepsilon_4, \quad \text{vol}^a(M^*_i(\beta^{(i)}(x))) \geq 3\varepsilon_4,
\]

where we use (140). The above considerations motivate the following definition. In order to use only finitely many indexes \( \beta \), in the following we are going to consider \( \beta = (\beta_i)_{t=1}^{L-1+N_1(\sigma)} \) where \( \beta_i \in \sigma\mathbb{Z}_+, \beta_i \leq 2D \).

**Definition 7** Let \( \beta = (\beta_t)_{t=1}^{L-1+N_1(\sigma)} \in \sigma\mathbb{Z}_+^{L-1+N_1(\sigma)} \subset \mathbb{R}_+^{L-1+N_1(\sigma)} \). Such sequence \( \beta \) is called admissible, if \( r_0/8 \leq \beta_t \leq 2D \) and for all indexes \( i \in \{ L, \ldots, L - 1 + N_1(\sigma) \} \), the modified index \( \beta^{(i)} = T^{(i)}(\beta) \in A^{(i)} \) satisfies

\[
\text{vol}^a(M^*_i(\beta^{(i)})) \geq 3\varepsilon_4. \tag{146}
\]

We define the set \( B = \{ \beta \in \sigma\mathbb{Z}_+^{L-1+N_1(\sigma)} ; \beta \text{ is admissible} \} \).

**Lemma 10** For any \( x \in M \setminus B(p, 3r_0/8 + \sigma) \), there exists an admissible \( \beta \in \sigma\mathbb{Z}_+^{L-1+N_1(\sigma)} \) such that \( |d(x, z_\ell) - \beta_\ell| \leq 2\sigma \), for \( \ell \in \{1, 2, \ldots, L - 1 + N_1(\sigma) \} \).

Conversely, there is \( C_{25} > 0 \) depending only on \( n, R, D, i_0 \) and \( r_0 \), such that, if \( \beta \) is admissible, then there is \( x = x_\beta \in M \setminus B(p, 3r_0/8 - C_{25}\sigma) \) such that, for all \( \ell \in \{1, 2, \ldots, L - 1 + N_1(\sigma) \} \), we have

\[
|\beta_\ell - d(x, z_\ell)| \leq C_{25}\sigma. \tag{147}
\]

**Proof.** The first statement follows from considerations before Definition 7.

On the other hand, assume that \( \beta \in B \). Then equations (140) and (146) guarantee that, for any \( i \in \{ L, \ldots, L - 1 + N_1(\sigma) \} \), there is \( x_i \in M^*_i(T^{(i)}(\beta)) \). Moreover, we have \( |d(x_i, z_\ell) - \beta_\ell| \leq 2\sigma \), for \( \ell \in \{1, \ldots, L - 1\} \cup \{i\} \). Moreover, in view of (30), for \( j, k \in \{ L, \ldots, L - 1 + N_1(\sigma) \} \),

\[
d_M(x_j, x_k) \leq C_9|H^L(x_j) - H^L(x_k)| \leq 4C_9\sqrt{L}\sigma.
\]

Defining

\[
C_{25} = 4C_9\sqrt{L} + 3, \tag{148}
\]

and taking \( x = x_{i_1} \) with arbitrary \( i_1 \), we see that \( x \in M \setminus B(p, 3r_0/8 - C_{25}\sigma) \) and that (147) is satisfied. \( \square \)
For the points \( z_{\ell} : \ell \in \{1, \ldots, L - 1 + N_1(\sigma)\} \subset B(p, r_0/4) \), let 
\[ V_{\ell} = \{ y \in B(p, r_0/4) : z_{\ell} \text{ is the unique closest point to } y \text{ in } \{z_{\ell'}\} \}, \]
where \( \ell = 1, \ldots, L - 1 + N_1(\sigma) \), be the corresponding Voronoi region. With 
any \( \beta \in B \) we then associate a piecewise constant function \( r_{\beta} \in L^{\infty}(B(p, r_0/4)) \) 
by defining \( r_{\beta}(z) = \beta_{\ell}, \) for \( z \in V_{\ell} \). Clearly,
\[ d_{L^{\infty}(M)}(r_{\beta}, r_x) \leq C_{26}\sigma, \quad C_{26} = C_{25} + 2. \] (149)

Let 
\[ R_{M,>}^* = \{ r_{\beta}(\cdot) : \beta \in B \} \subset L^{\infty}(B(p, r_0/4)). \]
Choose a maximal \( \sigma \)-net \( \{x_1, \ldots, x_{N_2(\sigma)}\} \subset B(p, r_0/2) \) by adding to \( z_1, \ldots, z_{N_0(\sigma)} \) a \( \sigma \)-net \( z_{N_0(\sigma)+1}, \ldots, z_{N_2(\sigma)} \) in \( B(p, r_0/2) \setminus B(p, r_0/4) \). Again, using Lemma 1, we see that \( N_2(\sigma) \leq C_{11}\sigma^{-n} \). Next we define
\[ r_k(z) = d^{a}(x_k, z_\ell), \quad \text{for } z \in V_{\ell}, \quad k = 1, \ldots, N_2(\sigma), \quad \ell = 1, \ldots, L - 1 + N_1(\sigma); \]
\[ R_{M,<}^* = \{ r_k(\cdot) : k = 1, \ldots, N_2(\sigma) \} \subset L^{\infty}(B(p, r_0/4)), \]
\[ R_{M}^* = R_{M,>}^* \cup R_{M,<}^*. \] (150)

In Figure 6, we consider the set
\[ X = \{ x_{\beta} : \beta \in B \} \cup \{ x_1, \ldots, x_{N_2(\sigma)} \} \subset M. \] (151)
Thus, denoting \( C_{27} = 2C_{23} + 2C_{26} + 1 \), see (143) and (148), we obtain

**Lemma 11** We have 
\[ d_{H}(R_{M}(M), R_{M}^*) \leq C_{27}\sigma, \] (152)
where \( d_{H} \) is the Hausdorff distance in \( L^{\infty}(B(p, r_0/4)) \).

## 6 Proof of Theorem 1 and Proposition 1

### 6.1 From interior distance functions to boundary distance functions

By standard estimates for the differential of the exponential map, see [43, Ch. 6, Cor. 2.4] the diameter of the sphere \( \partial B(p, r) \), \( r < r_0 \), is bounded
\[ \text{diam } (\partial B(p, r)) \leq \pi r \cdot \frac{\sinh(\sqrt{K}r)}{\sqrt{K}r} \leq \pi r \cosh\left(\frac{r}{2}\right) \leq 10 r, \] (153)
where we use condition (I). Let \( N = M \setminus B(p, r_0/25) \).
Lemma 12 Let \( x \in M \setminus B(p, r_0/4) \) and \( y \in \partial N \) and \( z \in \partial B(p, r_0/4) \), let
\[
\begin{align*}
f(y, x, z) &= d_N(y, z) + d_M(z, x), \quad (154) \\
f(y, x) &= \min_{z_1 \in \partial B(p, r_0/4)} f(x, y, z_1),
\end{align*}
\]
where \( d_N \) and \( d_M \) are the distances in \( N \) and \( M \), respectively. Then,
\[
d_N(y, x) = f(y, x) 
\]
(155)

Proof. Clearly, as \( d_M(z, x) \leq d_N(z, x) \) and a shortest curve in \( N \) from \( y \) to \( x \) intersects the sphere \( \partial B(p, r_0/4) \), we see that \( d_N(y, x) \geq f(y, x) \).

On the other hand let \( z' = \arg\min_z (f(y, x; z)) \) and \( \mu([0, f(y, x)]) \) be the corresponding union of the distance minimizing paths from \( y \) to \( z' \) and from \( z' \) to \( x \) for which the minimum in (154) is achieved. Denote \( s_1 = d_N(y, z') \) and consider \( \mu([s_1, f(y, x)]) \). We show next that \( \mu([s_1, f(y, x)]) \subset N \). If this is not the case, there would exists \( s_1 < s_2 < s_3 < f(y, x) \) such that \( \mu(s_1), \mu(s_3) \in \partial B(p, r_0/4), \mu(s_2) \in \partial B(r_0/25) \) and \( \mu[s_3, f(y, x)] \subset M \setminus B(p, r_0/4) \). Then,
\[
s_1 \geq r_0 \left( \frac{1}{4} - \frac{1}{25} \right), \quad s_2 - s_1 \geq r_0 \left( \frac{1}{4} - \frac{1}{25} \right), \quad s_3 - s_2 \geq r_0 \left( \frac{1}{4} - \frac{1}{25} \right) . \quad (156)
\]

On the other hand, consider a curve \( \mu'([0, l]) \) which is parametrised by the arclength and consists of the radial path from \( \mu(s_3) \) to \( y' \in \partial B(r_0/25) \) followed by a shortest path along \( \partial B(r_0/25) \) from \( y' \) to \( y \). Due to (153) and (156),
\[
l \leq r_0 \left( \frac{10}{25} + \frac{1}{4} - \frac{1}{25} \right) < 3r_0 \left( \frac{1}{4} - \frac{1}{25} \right) \leq s_3 .
\]
Taking the union of the path \( \mu'([0, l]) \), connecting \( \mu(s_3) \) to \( y' \), and the path \( \mu(s_3, f(y, x)) \), connecting \( y' \) to \( x \), we get a contradiction to definition (155). Thus, \( \mu([s_1, f(y, x)]) \subset N \), i.e., \( d_N(y, x) \leq f(y, x) \).

Next, using the already constructed set \( R^* \), see (150) together with Lemmata \( \text{[11]} \) and \( \text{[12]} \) we construct a set \( R^*(N) \subset L^\infty(\partial N) \) which approximates
\[
R^{\partial N}(N) = \{ r^{\partial N}_x \in L^\infty(\partial N) : x \in N \},
\]
(157)
where \( r^{\partial N}_x(z) = d_N(x, z) \), for \( z \in \partial B(p, r_0/25) \).
Lemma 13 Let \( R^* \) be the set given in (150), which satisfies (152) be given. Then it defines a set \( R^*(N) \subset L^\infty(\partial N) \) such that
\[
d_H(R^{\partial N}(N), R^*(N)) \leq C_{29}\sigma, \quad C_{29} = 2C_{27} + 2C_{23} + 1.
\]
(158)

Here \( C_{27} \) is defined in (152) and \( C_{23} \) is defined in (143).

Note that here we assume that \( \delta \) satisfies (82), \( \sigma \) satisfies (145) with the related equations for \( \varepsilon_4, \varepsilon_0 \), etc.

**Proof.** The proof is based on the construction of \( R^*(N) \) which satisfies (158).

Observe first that it follows from the proof of Lemma 12 that, if \( x, y \in B(p, r_0/4) \setminus B(p, r_0/25) \subset N \), then
\[
d_N(x, y) \leq \frac{r_0}{2} + \frac{8r_0}{25},
\]
so that a shortest path in \( N \) connecting \( x \) and \( y \) lies in \( B(p, r_0/4) \). Thus it is possible, using (5), to construct an approximation \( \tilde{r}^{\partial N}_x : \partial N \to \mathbb{R} \) that satisfies
\[
\|r^{\partial N}_x - \tilde{r}^{\partial N}_x\|_{L^\infty(\partial N)} \leq C_{30}\sigma,
\]
(159)

with a uniform constant \( C_{30} \), cf. (143). Denote \( R^*_<(N) = \{\tilde{r}^{\partial N}_x ; x \in B(p, r_0/4) \setminus B(p, r_0/25)\} \), then
\[
d_H(R^{\partial N}(B(p, r_0/4)), R^*_<(N)) \leq C_{30}\sigma,
\]
(160)

for \( \delta < \delta_0 \), cf. construction of \( R^*_< \) in subsection 5.2.

Next, let
\[
R^*_c = \{r \in R^* : \min_{z \in \partial N} (r(z)) \geq \frac{r_0}{8}\}
\]

For \( y, z \in B(p, r_0/4) \setminus B(p, r_0/25) \) denote by \( d^*_N(y, z) \) the distance between \( y \) and \( z \) in the metric \( g^a \) along the curves lying in \( B(p, r_0/2) \setminus B(p, r_0/25) \). For each \( r \in R^*_c \) we define
\[
\tilde{r}^{\partial N} \in L^\infty(\partial N) : \tilde{r}^{\partial N}(y) = \inf_{z \in \partial B(p, r_0/4)} (d^*_N(z, y) + r(z));
\]
\[
R^*_>(N) = \{r^{\partial N}(\cdot) : r \in R^*_c\}.
\]
Then, with $R^*(N) = R_0^*(N) \cup R^*_\infty(N)$, we have that
\[ d_H(R_0^N(N), R^*(N)) \leq (2C_{30} + C_{27})\sigma = C_{31}\sigma. \quad (162) \]
Here $C_{27}\sigma$ error comes from an approximation of $d_N(y, z)$, see (152), and $2C_{30}\sigma$ error comes from approximating $d_M(z, x)$ and $d_N(y, z)$ in formula (160), see also (154)-(155) and (12). At last, we use again that $\delta$ satisfies the uniformly bound (82).

□

Recall that the metric tensor $g$ on $B(p, r_0)$ is a representation of a metric in Riemannian normal coordinates and the $C^{2,\alpha}$-norm of the metric is uniformly bounded. Using the fundamental equations of the Riemannian geometry, [43, Ch. 2, Prop. 4.1 (3)], we have that the shape operator $S$ of the surface $\partial B(p, r)$, $r < r_0$, can be given in the Riemannian normal coordinates centered at $p$ in terms of the metric tensor as $S = g^{-1} \partial_\nu g$, where $\nu$ is the unit normal vector of $\partial B(p, r)$. Taking $r = r_0/25$, we see that the $C^{1,\alpha}$-norm of the shape operator $S$ of $\partial N$ is uniformly bounded. Also, by (1) the boundary injectivity radius of $(N, g|_N)$ is bounded below by $24/25$. As the sectional curvature of $M$ and the second fundamental form (that is equivalent to the shape operator) of its submanifold $\partial N$ are bounded, the Gauss-Codazzi equations imply that the sectional curvature of $\partial N$ is bounded. As the metric tensor of $M$ is bounded in normal coordinates in $B(p, r_0)$, we see that the $(n - 1)$-dimensional volume of $\partial N = \partial B(p, r_0/25)$ is bounded from below by a uniform constant. Thus by Cheeger’s theorem, see [43, Ch. 10, Cor. 4.4], the injectivity radius of $\partial N$ is bounded from below by a uniform constant.

Summarising the above, the Ricci curvature of $(N, g|_N)$ is uniformly bounded in $C^\alpha$, the second fundamental form of $\partial N$ is uniformly bounded in $C^{1,\alpha}$, and the diameter and injectivity radii of $N$ and $\partial N$, and the boundary injectivity radius of $(N, \partial N)$ are uniformly bounded. By [30], using the knowledge of the set, $R^*(N)$ of approximate boundary distance functions, which are $C_{29}\sigma$–Hausdorff close to the set, $R_0^N(N)$ of the boundary distance functions of manifold $(N, g|_N)$, one can construct on the set $R^*(N)$ a new distance function $d_N^* : R^*(N) \times R^*(N) \to \mathbb{R}_+$, such that
\[ d_{GH}((N, d_N), (R^*(N), d_N^*)) \leq C_{32}(C_{29}\sigma)^{1/36}, \quad (163) \]
with a uniform $C_{32} > 0$.

Having constructed $(R^*(N), d_N^*)$ we can now construct an approximate metric space $(M^*, d_M^*)$ which is $C_{32}(C_{29}\sigma)^{1/36}$– close to $(M, d_M)$. Indeed, let
Let \( x, y \in N \) and \( \mu[0, l], l = d_M(x, y) \) be a shortest between \( x \) and \( y \). If \( \mu[0, l] \subseteq N \) then \( d_M(x, y) = d_N(x, y) \). If, however, \( \mu[0, l] \) intersects with \( B(p, r_0/25) \) then, due to the convexity of \( B(p, r_0/25) \), there are \( 0 < s_1 < s_2 < l \) such that

\[
\mu[0, s_1] \subseteq N, \quad \mu[s_1, s_2] \subseteq B(p, r_0/25), \quad \mu[s_2, l] \subseteq N.
\]

Therefore, similar to Lemma 12, we obtain

**Corollary 4** Let \( x, y \in N \). Then

\[
d_M(x, y) = \min \left( d_N(x, y), \min_{z_1, z_2 \in \partial B(p, r_0/25)} \left[ d_N(x, z_1) + d_M(z_1, z_2) + d_N(z_2, y) \right] \right).
\]

Next define, for \( \tilde{r}_{1N}, \tilde{r}_{2N} \in R^*(N) \),

\[
d'_M(\tilde{r}_{1N}, \tilde{r}_{2N}) = \min \left( d'_N(\tilde{r}_{1N}, \tilde{r}_{2N}), \min_{z_1, z_2 \in \partial B(p, r_0/25)} \left[ \tilde{r}_{1N}(z_1) + d^a(z_1, z_2) + \tilde{r}_{2N}(z_2) \right] \right).
\]

Using (164) together with (143), (163) and (5), we see that

\[
d_{GH}(M, d_M), (R^*(N), d'_M) \leq (2C_{32} + 1)(C_{29}\sigma)^{1/36} \text{ if } C_{30}\sigma \leq (C_{29}\sigma)^{1/36}. \tag{166}
\]

Here \((N, d_M)\) is the manifold \( N \) with the distance function inherited from \( M \) and \( \delta < \delta_0 \), cf. (160).

Let us define the disjoint union \( M^* = R^*(N) \cup B(p, r_0/25) \). Next we define a metric \( d'_M \) on this set. To this end, consider first \( \tilde{r}_{1N} \in R^*(N), y \in B(p, r_0/25) \). Recall, see the proof of Lemma 13, that the set \( R^*(N) \) is bijective with \( R^*_c \cup (B(p, r_0/4) \setminus B(p, r_0/25)) \). In the case when \( \tilde{r}_{1N} \) is obtained from \( r \in R^*_c \), we define \( d'_M(\tilde{r}_{1N}, y) = r(y) \). Moreover, in the case when \( \tilde{r}_{1N} \) is obtained from \( x \in B(p, r_0/4) \setminus B(p, r_0/25) \), we define \( d'_M(\tilde{r}_{1N}, y) = d^a(x, y) \).

At last, if \( x, y \in B(p, r_0/25) \), we take \( d'_M(x, y) = d^a(x, y) \).

It follows from (166) together with equations (5), (143), (152) and considerations preceding Lemma 11 that

\[
d_{GH}(M^*, d'_M), (M, d_M) \leq (2C_{32} + 1)(C_{29}\sigma)^{1/36}. \tag{167}
\]

Summarizing, we obtain

**Lemma 14** Let \( R^* \) satisfy (152) and \( M^* = R^*(N) \cup B(p, r_0/25) \) with metric \( d'_M \). Then,

\[
d_{GH}(M, d_M), (M^*, d'_M) \leq C_{33}\sigma^{1/36}, \quad C_{33} = (2C_{32} + 1)C_{29}^{1/36}. \tag{168}
\]
6.2 Proof of Theorem 1 and Proposition 1

Proof of Proposition 1. To prove the statement of the Proposition, we collect all the previous estimates. The aim is to find the relation between the final error $\varepsilon$ (i.e. $d_{GH}((M, d_M), (M^*, d^*_M)) \leq \varepsilon$) and the initial error $\delta$. We proceed by following the chain of relations:

$$\varepsilon \mapsto \sigma \mapsto \varepsilon_4 \mapsto \varepsilon_0 \mapsto \varepsilon_1 \mapsto \gamma \mapsto \varepsilon_2 \mapsto j_0 \mapsto j_1 \mapsto \delta.$$  \hfill (169)

To obtain inequality (12) from (168) we set $\sigma = \left( \frac{\varepsilon}{C_{33}} \right)^{36}$ and use it in (145), (140), (168) and (78) with $\Lambda_s = 1$ to determine values of $\varepsilon_4$, $\varepsilon_0$ and $\varepsilon_1$ by setting

$$\varepsilon_4 = \frac{\varepsilon^{36n}}{4C_6C_{33}^{36n}}, \quad \varepsilon_0 = C_7\varepsilon_4 \leq \frac{1}{10}, \quad \text{and}$$

$$\varepsilon_1 = C_{40}\varepsilon^{72n} \quad \text{with} \quad C_{40} = \frac{C_7^2}{160C_6^2C_{33}^{72n}}.$$

To define $\gamma$ so that (40), (60), and (102) are valid, we set

$$\gamma = C_{41}\varepsilon_1^{1/b(s)}, \quad \text{with} \quad C_{41} = \min \left( C_{40}^{-1/(2n)}C_{33}^{-36}, C_{28}, r_0/32 \right).$$ \hfill (171)

Here we have used that $\sigma = \varepsilon_1^{1/(2n)}C_{40}^{-1/(2n)}C_{33}^{-36}$ and noticed that $b(s) < 2n$. From (78) and (74) we get

$$\varepsilon_2 = \frac{(C_{41}\varepsilon_1^{1/b(s)})^{s/(s-1)}}{\left(\exp\left([4LC_{12}\varepsilon_1^{-1}(C_{41}\varepsilon_1^{1/b(s)})^{-2+\theta/2} \exp(C_{41}^{-c_{200}}\varepsilon_1^{-c_{200}/b(s)})]^{1/\beta}\right)^{(s-1)^{s-1}}}} \quad \text{with} \quad \varepsilon_1 \text{ given by } (170).$$ \hfill (172)

Finally, to choose $j_0$ and $\delta$ so that (79), (80), (81) and (82) are satisfied, we set

$$j_0 = j_0(\frac{\varepsilon_2}{8}; \gamma, 1) = C_{16}C_{41}^{-n}8^{-n/s}\varepsilon_1^{-n/b(s)}\varepsilon_2^{-\frac{n}{s}},$$ \hfill (173)

with $\varepsilon_2$ given by (172), and choose $\delta$ so that

$$\delta \leq 2^{-n/2}C_5C_7^{-n}(j_0)^{-1}\varepsilon_2 = 2^{-n/2}C_5C_7^{-n}C_{16}^{-1}C_{41}^{-n}8^{n/s}\varepsilon_1^{-n/b(s)}\varepsilon_2^{1+\frac{n}{s}} = C_{39}\varepsilon_1^{C_{34}}\exp\left[-C_{35}\varepsilon_1^{-C_{37}}\exp(C_{36}\varepsilon_1^{C_{38}})\right].$$ \hfill (174)
with
\[ C_{34} = \frac{1}{b(s)} \left( n + \frac{s + n}{s - 1} \right), \quad C_{35} = \frac{(s + n)}{(s - 1)} \left( 4LC_{12}C_{41} - (2 - \frac{n}{2}) \right)^{1/\beta}, \]
\[ C_{36} = \frac{C_{41}c_{200}}{\beta}, \quad C_{37} = \frac{1}{\beta} \left( 1 + \frac{1}{b(s)} \left( 2 - \frac{\theta}{2} \right) \right), \quad C_{38} = \frac{c_{200}}{b(s)}, \]
\[ C_{39} = 2^{-n/2}c_5C_7^{-n}C_{16}^{-1}8^{n/s}C_{41}^{-1/\beta}C_{36}^{-1}/s + n. \]  

(175)

We use the inequality \( x \leq \exp(x) \) to bound from below the right hand side of the estimate above to obtain, by calling \( C_{43} = \max(C_{34}, C_{37}, C_{38}, 1/(2n)) \),
\[ \delta \leq \exp\left[-\exp((C_{39}^{-1} + C_{35} + C_{36})\varepsilon_1^{-C_{43}})\right]. \]  

(176)

Notice that (143) is also satisfied, by replacing \( C_{39} \) with \( C_{42} = C_{24}/(C_{33}^{1/2}C_{40}^{1/(2n)}) \) and by including \( 1/(2n) \) in \( C_{43} \). Assuming \( 0 < \delta \leq \exp(-\varepsilon) \), we get
\[ (C_{42}^{-1} + C_{35} + C_{36})/\ln \left( \frac{1}{\varepsilon} \right) \leq \varepsilon_1^{C_{43}}, \]  

(177)

Let \( \tau_0 \) be the uniform constant introduced in Proposition 2 and define
\[ C_{44} = \min \left( 1000^{1-C_{43}}, C_{40}C_{43}(C_{33}^{1/2}C_{43}^{1/2})^{2nC_{43}} \right). \]  

(178)

In this way we can set in (177) the two constraints (133) and \( \varepsilon_1 \leq 1/1000 \) (derived from (178) with \( \Lambda_s = 1 \)) and obtain
\[ \delta \leq \delta^*, \quad \text{with} \quad \delta^* = \min \left( \exp(-\varepsilon), \exp[-\exp[C_{44}^{-1}(C_{42}^{-1} + C_{35} + C_{36})]] \right). \]

Finally by using (170) to rewrite \( \varepsilon_1 \) in (177), and defining \( C_2 = 1/(72nC_{43}) \) and \( C_3 = (C_{42}^{-1} + C_{35} + C_{36})C_2C_{40}^{-1/(72n)} \), we obtain (12). \( \Box \)

**Proof of Theorem 1**. Let \( \delta \leq \delta^* \) and let the ISD of \( M^{(i)}, i = 1, 2 \) be \( \delta \)-close. Take the finite collection
\[ \mathcal{D} = \left( (B^e(r_0), g^{(i)}), \{(\lambda_j^{(i)}, \varphi_j^{(i)})\}_{j=0} \right), \]
where the index (1) is related to the IDS of \( M^{(1)} \). By construction the data \( \mathcal{D}_0 \) are \( \delta \)-close to the ISD of both \( M^{(1)} \) and \( M^{(2)} \). By Proposition 1 the metric space \( (M^*, d_{M}^{*}) \) constructed with these data is \( \varepsilon \)-close to both \( (M^{(i)}, d^{(i)}), i = 1, 2 \), where \( \varepsilon \) is given by the right hand side of (12). We then conclude by triangular inequality, for any \( 0 < \delta \leq \delta^* \),
\[ d_{GH}(M^{(1)}, d^{(1)}), (M^{(2)}, d^{(2)}) \leq 2\varepsilon \]  

(179)
We now extend this estimate to the case $\delta \in (0, \exp(-e)]$, when $\delta^* < \exp(-e)$. To this end, observe that the definition of the GH-topology and \cite{1} imply that: $d_{GH}((M^{(1)}, d^{(1)}), (M^{(2)}, d^{(2)})) \leq D$, for any $\delta$. By combining the latter inequality and \cite{179} we obtain the inequality \cite{9} with $C_1 = \max\left(2C_3, D\left(\ln \left(-\ln \delta^*\right)\right)^{C_2}\right)$.

\textbf{Acknowledgements} RB and ML were partially supported by Academy of Finland, projects 303754, 284715 and 263235. YK was partially supported by EPSRC grant EP/L01937X and Institute Henri Poincare.
Table of constants $C_k$, $c_k$, $\tau_0$ and $s$.

Note that all constant depend on $n, R, D, i_0, r_0$ and variables in brackets.

| Name | Introduced in / Notes | Name | Introduced in / Notes |
|------|-----------------------|------|-----------------------|
| $C_1$ | Thm. 1 | $C_2$ | Thm. 1 |
| $C_3$ | Prop. 1 | $C_4$ | Cor. 1 |
| $C_5$ | Cor. 1 | $C_6$ | (20) |
| $C_{(hav)}$ | (16) | $C_{(Lip)}$ | (21) |
| $C_7$ | (23) | $C_8$ | Prop. 2 |
| $C_9$ | Prop. 2 | $C_{10}$ | Prop. 2 |
| $C_{11}$ | Lemma 1 | $C_{12}(\theta)$ | see (51), we use $\theta = 1/2$ |
| $C_{13}$ | (55), we use $\alpha = 1 - \theta/2$ | $c_{206}$ | (43) |
| $c_{206}(\theta)$ | (43), Appendix | $c_{206}(\gamma, \theta)$ | (43), Appendix |
| $C_{14}$ | (47) | $\tau_0$ | Prop. 2 |
| $C_{15}$ | Lemma 2 | $c_1(s)$ | Lemma 2 |
| $b(n)$ | Lemma 2 | $c_3(s)$ | Lemma 3 |
| $s$ | (38) | $C_{16}(s)$ | (80) |
| $C_{17}(s; \gamma)$ | Lemma 3 | $C_{20}$ | (39) |
| $c_5$ | (82) | $C_{21}(s)$ | Lemma 8 |
| $C_7$ | Lemma 9 | $C_{23}$ | (143) |
| $C_{24}$ | (143) | $C_{25}$ | (10) |
| $C_{26}$ | (149) | $C_{27}$ | (11) |
| $C_{28}$ | (102) | $C_{29}$ | Lemma 13 |
| $C_{30}$ | (159) | $C_{31}$ | (162) |
| $C_{32}$ | (163) | $C_{33}$ | (168) |
| $C_{34}$ | (175) | $C_{35}$ | (175) |
| $C_{36}$ | (175) | $C_{37}$ | (175) |
| $C_{38}$ | (175) | $C_{39}$ | (175) |
| $C_{40}$ | (170) | $C_{41}$ | (171) |
| $C_{42}$ | (176) | $C_{43}$ | (177) |
| $C_{44}$ | (178) | | |
7 Appendix

7.1 Calculation of \( c_{206} (\gamma, \theta) \) in Theorem 2

To prove Theorem 2 we need to show that the solution \( w \) of the wave equation

\[
P(y, D)w(y) = \tilde{q}(y), \quad y \in \Omega(T) \subset \mathbb{R}^{n+1}
\]

can be estimated in the set \( \mathcal{D}(z, \gamma, T) \subset \mathbb{R}^{n+1} \) that is between two double cones of the spacetime, i.e. \( \Sigma(z, \gamma, T) \subset \mathcal{D}(z, \gamma, T) \subset \Sigma(z, 0, T) \). Here, \( \gamma > 0 \) is the parameter that indicates how close the set \( \mathcal{D}(z, \gamma, T) \) is to the optimal double cone \( \Sigma(z, 0, T) \). Theorem 2 is proven by applying a proper iterative procedure and the dependency of the coefficient \( c_{206} \) on \( \gamma \) is crucial for our considerations.

The calculation of \( c_{206} \) can be considered as the final step of a long geometric construction. In order to understand it we summarize the previous steps with related references.

In Section 3 of [11] we calculated the parameters of the inequality associated with a (conormally) pseudo-convex function \( \psi \) with respect to the wave operator \( P(y, D) \). Then we used this property to calculate the coefficients of the Tataru inequality (recalled in the following section 7.1.2)

\[
\| e^{-\epsilon |D_0|^2/2\tau} e^{\tau f} P(y, D) u \|_{0, \tau} \leq c_{155, j} \exp(-c_{132} \mu_j^2), \quad \forall \omega \leq \mu_j^2/(3c_{131}).
\]

and to prove the local stability of the unique continuation for the wave operator.

In [12] we used the previous result to prove the global stability of the unique continuation for the wave operator. As recalled in the following section 7.1.3, the proof is based on the iteration \( N \) times of the local stability for the ‘low temporal frequency’ component of the solution \( u \) of the wave equation:

\[
\| A(D_0/\omega) b ((y - y_j)/r) u_j \|_{H^1} \leq c_{155, j} \exp(-c_{132} \mu_j^2), \quad \forall \omega \leq \mu_j^2/(3c_{131}).
\]

Moreover in Section 3.1. of [12] and Appendix A of [12] we applied the stability result in the domain of influence of a cylinder.

In the case of the present paper, the mentioned domain of influence is called \( \Sigma(z, 0, T) \) in Theorem 2 and, according to the iterative procedure, it contains a covering of the set \( \Lambda = \mathcal{D}(z, \gamma, T) \). The balls of the covering have radius \( 2R \), that depends on the distance to the boundary, on the regularity and pseudo-convexity property of the function \( \psi \), and on extra constraints imposed by
the Tataru inequality. The local stability step holds for smaller balls with radius \( r \), and \( r < R \). In Table 1 we summarize these values and in particular we obtain, up to a multiplicative constant,
\[
  r \sim \gamma^{58}. 
\]
(180)
The \( \sim \) symbol is defined precisely in subsection 7.1.1. By construction and for (180), one can calculate the number of local steps of the iteration (see also Table 2)
\[
  N \sim \gamma^{-58(n+1)}. 
\]
(181)
These two values combined with the calculation of the coefficients for the local and the global stability lead to the following relationship between \( c_{206} \) and \( \gamma \)
\[
  c_{206} = \zeta_1 \exp(\gamma^{-\zeta_2}), \text{ for proper positive numbers } \zeta_1, \zeta_2. 
\]
We will prove that formula (181) plays a big role for the calculation of \( \zeta_2 \).
In both articles [11, 12] we used consistent notations for the geometric quantities and labeled the important coefficient as \( c_h \), with a unique \( h \geq 100 \) in order to be able to follow the construction of the final parameters. One can find them in those papers by searching for the corresponding index \( h \).
Here in this Appendix our focus is the dependency of all the parameters (in particular \( c_{206} \)) on the quantity \( \gamma \), since this reflects the cost of getting close to the cone of dependence. For this reason in the following section 7.1.1 we quickly introduce the main relationship between \( \gamma \) and the used Gevrey function localizers, and in the next sections we recalculate the main coefficients of the above results and we summarize them in the Tables 1 and 2.
We will follow the same notation as in [11, 12]. Unfortunately it was not possible to use an analogous notation in the rest of the present paper. Anyway we will point out the different notations.

### 7.1.1 Gevrey functions and dependency on \( \gamma \)

**Assumption.** Let \( \alpha \in [1/3, 1) \), and let \( T, \ell, \gamma \) be defined as in Assumption A5, [12].
(In the present paper, this corresponds to conditions (40) and (41)).

Gevrey functions are used as smooth localizers in the constructions and their
main properties are outlined in Section 4 of \cite{11}. In particular in our calculations we consider the following Gevrey function (see \cite{45}, Ex 1.4.9 for definition):

\[
\chi_1(t) = \chi(1 + t)\chi(1 - t), \quad \text{with } \chi(s) = \exp(-s^{\frac{1}{\alpha}}) \text{ for } s > 0, \chi(s) = 0 \text{ for } s \leq 0.
\]

One can slightly modify the definition such that \(\chi_1 = 1\) in a ball \(B_1 \subset \mathbb{R}\) (with radius 1), \(\chi_1 = 0\) outside the ball \(B_2\) (with radius 2), and \(0 \leq \chi_1 \leq 1\).

Observe that \(\chi_1 \in G_{1/\alpha}^1(\mathbb{R})\) since

\[
|D^\kappa \chi_1(v)| \leq c_0 X c_1 X^{\frac{1}{\alpha}}, \quad \text{with } c_0 X = O(1), c_1 X = O\left(\frac{1}{1 - \alpha}\right).
\]

Here the symbol \(O\) (big-O) means “comparable up to a an absolute multiplicative constant to” (i.e. \(A = O(B)\) implies \(c_{abs} \leq A / B \leq C_{abs}\), for some positive numbers \(c_{abs}, C_{abs}\)).

Furthermore, define \(\chi_\delta(v) := \chi_1(v/\delta), v \in \mathbb{R}^M\).

Hence, \(\mathcal{F}_{v \rightarrow \zeta} \chi_\delta(v) = \delta^M \mathcal{F}_{v \rightarrow \delta \zeta} \chi_1(v)\) for \(\zeta \in \mathcal{C}\), and calling \(c_{2X} = 1 / (e M c_{1X})^\alpha\) we get

\[
|\mathcal{F}_{v \rightarrow \zeta} \chi_\delta(v)| \leq \delta^M c_0 X \exp(\delta H_{B_2}(\text{Im}\zeta) - c_{2X}^\delta |\text{Re}\zeta|^\alpha) \cdot \text{Vol(supp}(\chi_1), dv).
\]

Product estimate: for \(v \in B_2(\mathbb{R}^M)\), calling \(c_{0X,i}, c_{1X,i}\) (resp. \(c_{0X,m}, c_{1X,m}\)) the coefficients in \cite{182} for \(\chi_i\) (resp. \(\chi_m\)),

\[
|D^\kappa \chi_i(v)\chi_m(v)| \leq c_{0X,i} c_{0X,m} \max\{c_{1X,i}, c_{1X,m}\} \max\{c_{1X,i}, c_{1X,m}\} |\kappa_i| |\kappa_m|^{\alpha/\alpha}.
\]

We start by linking \(\gamma\) and the coefficient \((1 - \alpha)\), since both quantities tend to zero.

Assumption. We assume \(\theta = 1/2\). Next, from now on the symbol \(\sim\) means “comparable up a multiplicative coefficient independent of \(\gamma\) or \((1 - \alpha)\) to”.

(i.e. \(A \sim B\) implies \(c \leq A / B \leq C\), with \(c, C\) independent of \(\gamma\) or \((1 - \alpha)\)).

The multiplicative coefficient is in general a uniform geometric constant, in the sense specified at the beginning of the paper.

We will call \(c_{205}\) the resulting multiplicative coefficient for \(c_{206}\).
θ is the exponent appearing in the global stability of the unique continuation (see Theorem 2 of this paper and the following section 7.1.3), while \(1/α\) is the order of the used Gevrey functions. According to [12] (end of page 6469), by construction these two values are related in the following way:

\[ α^N = θ, \quad \text{that implies} \quad α^{1/(n+1)} = \frac{1}{2} \implies (1 - α) \sim r^{n+1}, \quad \text{as} \quad α \to 1, \quad (184) \]

where \(N = c_{170} \sim γ^{-58(n+1)}\) is reported in the following Table 2 and \(r \sim γ^{58}\) is in Table 1. Consequently, for \(χ_1\) and \(c_{1X}\) defined above, we get

\[ c_{1X} \sim \frac{1}{1 - α} \sim \frac{1}{γ^{58(n+1)}}, \quad |χ'_1|_{C^0(Ω_0)} \sim c_{1X}, \quad |χ''_1|_{C^0(Ω_0)} \sim c_{1X}^2. \quad (185) \]

### 7.1.2 Tataru inequality and Table 1

We consider the wave operator in \(\mathbb{R}^{n+1}\),

\[ P(y, D) = -D_0^2 + \sum_{j,k=1}^n g^{jk}(x)D_jD_k + \sum_{j=1}^n h^j(x)D_j + q(x), \quad (186) \]

where \(y = (t, x) \in \mathbb{R} \times \mathbb{R}^n\) are the time-space variables, \(D_0 = -i∂_t, D_j = -i∂_{x_j}\). The coefficients \(g^{jk} \in C^1(\mathbb{R}^n)\) are real and independent of time, and \([g^{jk}]\) is a symmetric positive-definite matrix. The coefficients \(h^j, q \in C^0(\mathbb{R}^n)\) are complex valued and independent of time. Call \(ξ = (ξ_0, ˜ξ)\) the Fourier dual variable of \(y = (t, x)\). In the next theorem we use the exponential pseudodifferential operator \(e^{-|D_0|^2/2τ}v = F_{ξ_0}^{-1}e^{-ξ_0^2/2τ}Fv \to ξ_0v\), with \(F\) and \(F^{-1}\) representing respectively the Fourier transform and its inverse. Let us also define

\[ f(y) = \sum_{|ν| ≤ 2} (\partial^ν φ)(y_0) (y - y_0)^ν/ν! - σ|y - y_0|^2. \quad (187) \]

In following theorem (called Theorem 2.1 in [1]) we recall the Carleman-type estimate by Tataru, named ‘Tataru inequality’.

**Theorem 5** ([1], Theorem 2.1; or [12], Theorem 2.3.) Let \(Ω\) be an open subset of \(\mathbb{R} \times \mathbb{R}^n\). Let \(P(y, D)\) be the wave operator \((186)\), with \(g^{jk}(x) \in C^1(Ω), h^j, q \in C^0(Ω)\). Let \(y_0 \in Ω\) and \(ψ \in C^2(Ω)\) be real valued, for some fixed \(ρ \in (0, 1)\), such that \(ψ(y_0) ≠ 0\) and \(S = \{y; ψ(y) = 0\}\) being an oriented
hypersurface non-characteristic in \( y_{0} \).

Consequently there is \( \lambda > 1 \) such that \( \phi(y) = \exp(\lambda \psi) \) is a conormally strongly pseudoconvex function with respect to \( P \) at \( y_{0} \).

Then there is a real valued quadratic polynomial \( f \) defined in (187) with proper \( \sigma > 0 \), and a ball \( B_{R_{2}}(y_{0}) \) such that \( f(y) < \phi(y) \) when \( y \in B_{R_{2}} - \{ y_{0} \} \) and \( f(y_{0}) = \phi(y_{0}) \); and \( f \) being a conormally strongly pseudoconvex function with respect to \( P \) at \( y_{0} \).

Assumption: We now consider the ‘hyperbolic function’

\[
\psi(t, x; T, z) = (T - d_{g}(x, z))^{2} - t^{2}
\]  

introduced in Definition 3.1 of [12], and its level set \( \psi(y) - \gamma^{2} = 0 \).

Starting from a general \( \psi \), in section 3 of [11], page 180, we have already calculated all the geometric constants associated either with the related pseudoconvexity estimates of \( \psi \) or with the Tataru inequality. They are summarized in Table 1, page 191 of [11], and are copied in Table A.3. of [12] (with few modifications explained in the related Appendix A). Then in section A.1.1. of [12], page 6487, we have recalculated these quantities for the particular case of the ‘hyperbolic function’ \( \psi \) in (188).

Our aim here is to start from Table A.3. and section A.1.1. of [12] in order to find the \( \gamma \)-dependency of those coefficients.

The following new Table 1 must be read from the top to the bottom, since it starts with the basic inequalities and continues with more complicated expressions.

As said, we assume \( \psi \) as in (188), and calculate all coefficients in the Tataru inequality. The first two values \( C_{l} = \min |\psi'(y)| \) and \( p_{1} = \min p(y, \psi') \) are defined at page 6484 of [12], Section A.1., Assumption b). Their limit value is calculated in [12], formula (A.7): i.e. \( C_{l} = 2\gamma_{I} b_{0}^{-1/2}, p_{1} = 4\gamma_{I}^{2} \). Since \( \gamma_{I} = \gamma/\sqrt{2} \) (see Lemma A.3.a, page 6489), and \( b_{0} \) is defined as a constant (see formula (3.1), page 6452), then the \( \gamma \)-dependency of the two coefficients is respectively \( \gamma \) and \( \gamma^{2} \), as shown in the table. The third value of Table 1 is \( \text{dist}\{\partial\Omega_{0}, \Omega_{a}\} \) (alias \( \text{dist}(\Lambda, \partial\Omega_{0}) \)) and behaves like \( \gamma^{2} \), thanks to the estimates (A.12) and (A.11) in [12]. On the other hand, the following coefficients
until $C_3$ in Table 1 are independent from $\gamma$, because of formula (A.6) and (A.8) in [12].

The next values in Table 1 are obtained by substituting the upper values: $M_P, M_1, ..., R_1$, defined in section 3.1 of [11];

$c_T \sim \lambda^3$ (replacing $4n\lambda\psi|_{\max,\Omega_0}$), see (A.2) and Remark A.1 in [12];

$\tau_0, c_{1.7}, c_{2.7}, c_{1.33}$ defined in section 3.2 of [11];

$r, \delta, R$, defined in section 3.3 of [11], here we have renamed $r_0$ by $r$.

These 3 coefficients are used to prove Proposition 2.5 of [11], which is related to the result of local stability for the unique continuation.

$c_1$ of [11] is not used here.

Note that $\sigma, r, \delta, R, \tau_0, R_1, R_2$ have nothing to do with quantities with the same name used in the rest of this paper (outside from the Appendix).

### 7.1.3 Global stability coefficients and Table 2

This section can be seen as an overview of the proof of Theorem 2, with the final estimate for $c_{206}$.

We introduce the main steps and we always follow the notation of [11, 12] to better follow the calculations.

**Assumption:** Define a net of center points $(t_k, z_k)$ for the translated hyperbolic functions:

$$
\psi(y; T_k, z_k, t_k) = (T_k - d_g(x, z_k))^2 - (t - t_k)^2.
$$

Let $\Upsilon = W(z, T, \ell)$ be the initial cylinder (called $\Gamma$ in [41]) and let $\Sigma(z, \ell, T)$ be the related domain of influence (in the paper called $\Sigma(z, 0, T)$ according to [42]).

We choose the domains for the covering

$\Omega_{0,k} \subset \{y; y \in [-T_k + t_k, T_k + t_k] \times \mathbb{R}^n; \psi(y; T_k, z_k, t_k) \geq \gamma_k^2/2, T_k \geq d_g(x, z_k)\}$

and

$\Lambda_k \subset \{y; y \in [-T_k + t_k, T_k + t_k] \times \mathbb{R}^n; \psi(y; T_k, z_k, t_k) \geq \gamma_k^2, T_k \geq d_g(x, z_k)\}$.

Let $\gamma_k \geq \gamma$, for all $k$.

The construction is similar to the one in Figure 1, page 6470 of [12].

The parameters $(t_k, z_k, T_k, \gamma_k)$ should be chosen such that the $x$–projection
of $\Omega_{0,k}$ is contained in the domain $0 < d_g(z_k, x) \leq \frac{2}{3}i_0$, that is within the injectivity radius $i_0$, in order to guarantee the $C^{2,\rho}$-regularity of $\psi(y, T_k, z_k, t_k)$. Moreover the union $\Lambda = \bigcup_{k=1}^{K} \Lambda_k$ should cover a subset of the domain of influence $\Sigma(z, \ell, T)$. For example, let $\Lambda = S(z, \ell, T, \gamma)$, (alias $D(z, \gamma, T)$ in (41)).

The above construction together with the assumption on the Gevrey-regularity of the localizers let us apply Theorem 1.2 in [12]. The details of Assumptions A2-A3 can be checked in the paper, while $P$ is defined in (186).

**Theorem 6 ([12], Theorem 1.2)** Under the conditions of Assumptions A2-A3, define the open set $\Omega_1 = \bigcup_{k=1}^{K} \Omega_{0,k} \setminus \Upsilon$ containing $\Lambda$. Then for every $0 < \theta < 1$ we have

$$\|u\|_{L^2(\Lambda)} \leq c_{161} \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln \left(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\right)\right)^\theta}.$$  

Moreover, for any $m \in (0, 1]$ we get

$$\|u\|_{H^{1-m}(\Lambda)} \leq c_{161}^m \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln \left(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\right)\right)^m \theta}.$$  

The constant $c_{161}$ is calculated in the proof.

Up to a uniform multiplicative constant (and according to Remark 3.8. of [12]), we can identify the constant $c_{161}$ with our final constant $c_{206}$, even if the first one is defined for a bounded domain of the Euclidean space and the second one is defined for a compact manifold $(M, g)$. Indeed by assumption, in each chart of $M$ holds the inequality

$$a_0 I \leq \left[g_{jk}(x)\right]_{j,k=1}^n \leq b_0 I, \quad \text{and} \quad \|g_{jk}\|_{C^4(M)} \leq b_3, \quad a_0 < 1 < b_0,$$

which let one approximate all spatial subdomains to an Euclidean ball.

Theorem 1.2 is a generalization of Theorem 1.1 in [12] for a more complex domain, but with a similar final estimate where the inverse-log term has a different multiplicative constant. For each $\Omega_{0,k}$ Theorem 1.1 in [12] holds with constant $c_{160}$ in place of $c_{161}$. The number $K$ of the used sets $\Omega_{0,k}$ is by
construction proportional to the number of charts covering the domain. This number depends on the bounds for the diameter, the injectivity radius and the harmonic radius of $M$, called respectively $D$, $i_0$ and $r^{(\text{har})}$ in the notation of the paper. Hence we can also write $c_{206} \sim c_{161} \sim c_{160}$.

The technique used to prove the above Theorem consists in iterating the local stability result, but considering the low temporal frequencies separately from the high temporal frequencies. The pseudodifferential operator $A(D_0/\omega)$ defined below is used to localize the low temporal frequencies of the solution $u$, where the estimate is more complicated.

Assumption: We consider a pseudo-differential operator $A(D_0)$ with symbol $a(\xi_0) \in C_0^{1/\alpha}(\mathbb{R})$, $0 \leq a \leq 1$, supported in $|\xi_0| \leq 2$ and equal to one in $|\xi_0| \leq 1$. Hence we can write $A(\beta|D_0|/\omega)v = \mathcal{F}_{\xi_0 \rightarrow t}^{-1} u(\beta|\xi_0|/\omega) \mathcal{F}_{t \rightarrow \xi_0} v$. We fix $a$ as in (182).

Another complication comes from the fact that the local stability result holds just in small balls $B_r(y_j)$, centered in $y_j$ with radius $r$. It is important for our estimates that the balls $B_r(y_j)$, with $j = 1, \ldots, N$, cover the set $\Lambda$. We will choose the center points $y_j$ in the set $\mathcal{E}$, so that the union of the balls is contained in the domain of influence of the cylinder $\Upsilon$, i.e. $\bigcup_{k=1}^N B_r(y_j) \subseteq \bigcup_{k=1}^K \Omega_{0,k} \subset \Sigma(z, \ell, T)$. Furthermore there are particular conditions on the support of $u$ to be fulfilled, also affecting the set $\mathcal{E}$ and the iteration.

Hence in the final domain $\bigcup_{k=1}^K \Omega_{0,k}$ the local stability result must be applied several times to a sequence $\{u_j\}_{j=2}^N$ of proper cut-offs of the solution $u$. Let $u_j$ be defined as:

$$u_j = \prod_{k=1}^{j-1} (1 - b_k)u, \quad b_k := b\left(\frac{2(y - y_k)}{r}\right).$$

Then we can introduce the following Theorem 2.7 in [12], formulating a local stability estimate (of the unique continuation for the wave operator) of inverse exponential type for the low temporal frequencies of $u_j$.

The exact construction of the radii $r$ and $R$ is in Proposition 2.5 of [11], as intersection of several geometric and analytic constraints. The $\gamma$-dependency of $r$ and $R$ is shown in Table 1. In particular we get $r \sim \gamma^{58}$. The number of balls used in the iteration is $N = c_{170} \sim \gamma^{-58(n+1)}$, as shown in Table 2. The
constant $c_{170}$ is defined in formula (2.5) of \cite{12}.

Notice that at each step we reduce the support of the temporal localizer $A(D_0)$, by defining the term $\mu_j = c_{156} \mu_j^{\alpha} - 1$.

We will show that $c_{155,N} \sim \gamma^{-\zeta_4} c_{155,1} \sim \gamma^{-\zeta_5}$, and that $c_{161} \sim N \gamma^{-\zeta_6} c_{156}^{\alpha}$, for proper positive numbers $\zeta_4, \zeta_5, \zeta_6$.

The details of Assumptions A1-A2-A4 and of the set $\mathcal{E}$ can be checked in the paper.

**Theorem 7** (\cite{12}, Theorem 2.7) Under the Assumptions A1-A2-A4, let $y_k \in \mathcal{E}$ and let $b \in G^{1/\alpha}_{\mathbb{R}^{n+1}}$ be a Gevrey functions of class $1/\alpha$ with compact support, such that $0 < \alpha < 1$.

Then, there exist constants $R, r$ with $R \geq 2r > 0$, and $c_{159}$ such that for $\mu > c_{159}$ there are coefficients $c_{151}, c_{152}, c_{154}, c_{155}, c_{156}, \beta, N$ for which, if

$$
\|u\|_{H^1(\Omega_1)} = 1, \quad \|Pu\|_{L^2(\Omega_1)} < 1, \quad \|A\left(\frac{D_0}{\beta \mu}\right)l(y)Pu\|_{L^2} \leq \exp(-\mu^\alpha),
$$

(190)

then calling $\mu_1 = \mu$ and $\mu_j = c_{156} \mu_j^{\alpha} - 1$ for $2 \leq j \leq N$, we have $\mu_j \geq 1$ and

$$
\|u_j\|_{H^1(B_{2R}(y_j))} \leq c_{152}, \quad \|Pu_j\|_{L^2(B_{2R}(y_j))} \leq c_{153};
$$

(191)

$$
\|A(D_0/\mu_j)b((y - y_j)/R)Pu_j\|_0 \leq c_{154,j} \exp(-\mu_j^\alpha),
$$

(192)

and consequently

$$
\|A(D_0/\omega)b((y - y_j)/r)u_j\|_{H^1} \leq c_{155,j} \exp(-c_{132} \mu_j^{\alpha^2}), \quad \forall \omega \leq \mu_j^{\alpha}/(3c_{131}),(193)
$$

The radii $r$ and $R$ are defined in Table A.3, while the coefficients $c_k$ are calculated in the proof of the Theorem.

In the following Table 2 we show the $\gamma$ dependency of the coefficients used in the proof of Theorem 1.2 and Theorem 2.7 in \cite{12}. The coefficients $c_h$ of the local stability are defined in \cite{11} and recalled also in the proof of Lemma 2.6. of \cite{12}, page 6459. As said, the index $h$ is unique and here we briefly remind the definition of $c_h$ and the relationship with other coefficients and with Table 1.

In (185) we obtained the $\gamma$ dependency for $c_{1X}$, i.e. $c_{1X} \sim \gamma^{-58(n+1)}$. It follows that $c_{2X} \sim 1/(c_{1X}^{\alpha})$, where $c_{2X}$ is the coefficient in (183) (it was
called $c_{102}$ in [11]).

Therefore for simplicity we give below the values in Table 2 in terms of their $c_{1 X}$ or $\gamma$ dependency.

In order to calculate the rest we need to refine some estimates.

First of all we recall and improve the coefficients in Lemma 2.1, [12], for the $L^2$ and $H^m$ norms:

$$
c_{107} = c_3 \left( \frac{8}{\beta_1} \Gamma \left( \frac{1}{\alpha} \right) \frac{1}{\alpha(c_{117})^{\frac{1}{\alpha}}} \right) \frac{1}{(\alpha c_{106})^{\frac{1}{\alpha}}},
$$

(194)

$$
c_{108} = c_{107}(1 + |D_x^m f|_{C^0}) + c_{107}(1 + m) \frac{(m+1)}{(\alpha c_{106})^{\frac{(m+1)}{\alpha}}}
$$

$$
\|A(\beta_1 D_0/\mu)f(1 - A(D_0/\mu)v)\|_1 \leq c_{108} \varepsilon^{-c_{106} \mu^\alpha} \|v_{\text{supp}(f)}\|_m.
$$

Next, following Remark 2.8 (4) in [12], we split each smooth Gevrey localizer in time and space:

$$
b\left(\frac{y - y_0}{R}\right) = b\left(\frac{t - t_0}{R}\right) b\left(\frac{x - x_0}{R}\right),
$$

with $b(t) = \chi_1(t) \in C_0^{1/\alpha}(\mathbb{R})$ (as in (182)) and $b(x) \in C_0^{2}(\mathbb{R}^n)$. Consequently the functions $f_1(y), f_2(y), f_3(y)$ (see formula (2.21) in [12]) can generally be written as: $f_1(y) = f_1(t)f_s(x)$, with $f_s(t) = D_0 b_{j-1}(t) + D_0 b_{j-1}(t) + b_{j-1}(t)$ and $f_s(x) = D_x D_0 b_{j-1}(x) + D_x b_{j-1}(x) + b_{j-1}(x)$, for $b_{j-1}(t) := b(2(t-t_{j-1})/r)$. Let $v = b((y-y_{j-1})/r) u_{j-1}$, then

$$
\|A\left(\frac{3D_0}{\nu}\right)f_s(t)\left(1 - A\left(\frac{D_0}{\nu}\right)\right)f_s(x)v\|_1 \leq \|A\left(\frac{3D_0}{\nu}\right)(D_0 f_s(t))\left(1 - A\left(\frac{D_0}{\nu}\right)\right)f_s(x)v\|_0
$$

$$
+ \|A\left(\frac{3D_0}{\nu}\right)f_s(t)\left(1 - A\left(\frac{D_0}{\nu}\right)\right)(D_0 + D_x + 1)(f_s(x)v)\|_0 \leq c_{108} c_{152} \varepsilon \exp(-c_{106} \nu^\alpha)
$$

with $c_{108}$ calculated as in (194) with $\beta_1 = 3, m = 3, c_3 = (r/2)c_{0 X}$. Moreover, we can recalculate the terms at page 6466 of [12]:

$$
c_{162,j} = 2 c_{162,j-1} + c_{153} c_{164} + c_{155,j-1} - P_2 b_{j-1} + h^s(x) D_x b_{j-1} |C^0
$$

$$
+ c_{107} c_{152} (1 + n^2 |g^k|_{C^0}) + |h^s|_{C^0} + c_{155,j-1} 2 D_0 b_{j-1} |C^1 + c_{152} c_{108}
$$

$$
+ c_{155,j-1} |D_0 (2 D_0 b_{j-1})|_{C^0} + c_{152} c_{107}
$$

$$
+ c_{155,j-1} 2 n^k |D_k b_{j-1}|_{C^1} + c_{152} c_{108} n^2 |g^k|_{C^1}
$$

$$
+ c_{155,j-1} |D_x (2 g^k D_k b_{j-1})|_{C^0} + c_{107} c_{152} n^2 |g^k|_{C^1}
$$
\[ c_{162,j} \sim 2c_{162,j-1} + \left( \frac{N^2 c_1 x}{r^2} \right)^{3/2} c_{1X}^{1/2} + c_{155,j-1} (1 + |g|^{kr} |c_1 + |h|^8 |c_0|) \left( \frac{|b'|_0}{r} + \frac{|b''|_0}{r^2} + \frac{|b'|_0^2}{r^2} \right) + \left( \frac{N c_{1X}}{r} \right) c_{108} (1 + |g|^{kr} |c_1 + |h|^8 |c_0|) \sim c_{162,j-1} + c_{155,j-1} \frac{c_{1X}^2}{r^2} \]

\[ c_{154,j} = c_{162,j} + c_{153} \tilde{c}_{107} \sim c_{162,j} + \frac{N^2 c_1 x}{r^2} R^n \sim c_{162,j} \sim c_{155,j-1} \frac{c_{1X}^2}{r^2} \]

\[ c_{116} \sim \gamma^4 c_{154,j} \left( \frac{N c_{1X}}{r^4} \right)^4. \]

By applying Lemma 2.6 in [12] with \( c_U = c_{152}, c_P = c_{153}, c_A = c_{154,j}, \) one obtains:

\[ c_{155,j} = c_{150} (c_{152}, c_{153}, c_{154,j}) \sim c_{1X}^3 \frac{c_{151}^3}{\gamma 48} \sim \frac{N^2 c_{1X}}{\gamma 46+58+2} c_{155,j-1}, \]

\[ c_{156} = \min \left( \frac{1}{18 \beta c_{131}}, \frac{1}{\alpha}, \frac{c_{156}}{3 c_{131}} \right) = \frac{c_{156}}{3 c_{131}} \sim \gamma^{56 \alpha+58(n+1)(\alpha+1)+28}. \]

Now we can obtain the \( \gamma \) dependency of \( c_{160} \) in Theorem 1.1 of [12]. Recalling (184) (i.e. \( \alpha^N = \theta = 1/2 \) and \( (1 - \alpha) \sim \gamma 58(n+1) \)), we get \( c_{159} = \frac{1}{c_{156}^{\alpha/(1-\alpha)} - 1} > 1 \) and therefore:

\[ c_{159} \sim \left( \frac{1}{\gamma^{56 \alpha+58(n+1)(\alpha+1)+28}} \right) \frac{1}{\gamma^{58(n+1)}} = \exp \left( \frac{-[56 \alpha + 58(n+1)(\alpha+1)+28]}{2 \gamma^{58(n+1)}} \ln(\gamma) \right), \]

\[ c_{158} = N c_{155,n} + 3 N c_{131} c_{152} \left( 1 + \frac{|b'| c_0}{r} \right) c_{156}^{-\alpha/(1-\alpha)} \sim N c_{131} c_{152} c_{158}^{1/2} \]

\[ c_{160} = \left( \ln(1 + e^{c_{159}}) \right)^{1/2} + 2^{1/2} c_{158} \sim c_{158}. \]

Hence, \( c_{160} \) of Theorem 1.1 in [12] (and analogously \( c_{161} \) in Th. 1.2.) fulfills the estimate

\[ c_{160} \leq \exp \left( \frac{1}{c_2 c_{160}} \right), \text{ with } c_2 = 58(n+1) + 2. \]

We know that \( c_{206} \sim c_{160}. \) We denote by \( c_{205} \) the uniform multiplicative constant that depends on the uniform geometric parameters \( T, i_0, D, r_0, R, n, \) named according to the notation of the rest of the paper. The number \( c_{205} \) also depends on \( \theta, \) that for simplicity has been fixed here equal to 1/2. The
above inequality gives an estimate for $c_{160} \sim c_{206}$, and thus we can conclude that $c_{206}(\gamma, \theta) = c_{205}(\theta) \exp(\gamma^{-c_{200}})$.

**Remark.** Please notice that there was a misprint in the paper [12] both in the statements of Theorem 3.3. and in Corollary 3.9. However this misprint did not affect the calculations of the present paper (or the results in [12]).

Namely in Theorem 3.3., we have the following erratum (in the denominator of the final inequality):

$$
\|u\|_{L^2(\Lambda)} \leq c_{163} \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln \left(e + \frac{\|u\|_{H^1(\Omega_1)}}{\|f\|_{L^2(\Omega_1)}}\right)\right)^\theta}.
$$

And this is the corresponding corrigendum (replace $e$ with 1):

$$
\|u\|_{L^2(\Lambda)} \leq c_{163} \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln \left(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|f\|_{L^2(\Omega_1)}}\right)\right)^\theta}.
$$

In Corollary 3.9., we have erratum in (3.27), and corrigendum:

$$
\|w\|_{L^2(\Omega_2 \setminus W_1)} \leq c_{166} \frac{\|w\|_{H^1(\Omega_2 \setminus W_1)}}{\left(\ln \left(1 + \frac{\|w\|_{H^1(\Omega_2 \setminus W_1)}}{C \|w\|_{H^1(\Omega_2)} \|w\|_{W_1}}\right)\right)^\theta}.
$$

**Table 1 and Table 2.** We next present the two tables that summerize the previous calculations. They show the $\gamma$ dependency of the parameters. The name of the constants there is unique. The order of the parameters in Table 1 is always increasing in complexity, that is the parameters down may depend on the upper ones. In general the same principle is followed also in Table 2, even if the relationships are more complex. For simplicity the values in Table 2 are expressed in terms of their $c_{1X}$ or $\gamma$ dependency, where we recall that $c_{1X} \sim \gamma^{-58(n+1)}$. 

67
| Name          | Order with respect to $\gamma$ |
|--------------|---------------------------------|
| $C_l$        | $\sim \gamma$ (12, formula (A.7)) |
| $p_1$        | $\sim \gamma^2$ (12, formula (A.7)) |
| $\text{dist} \{\partial \Omega_0, \Omega_4\}$ | $\sim \gamma^2$ (12, formula (A.12)) |
| $|\psi'|_{C^k}$ | $\sim 1$ (12, formula (A.8)) |
| $d_g(x, z)$  | $\in [\ell, T - \gamma]$ (in $\Gamma \setminus $ cylinder) |
| $|\partial_k d_g|$ | $\sim 1$ (12, formula (A.6)) |
| $C_3$        | $\geq 1$ |
| $M_P$        | $\leq 1$ |
| $M_1$        | $\geq \frac{1}{(p_1)^2} = \frac{1}{\gamma^2}$ |
| $M_2$        | $\geq M_1 = \frac{1}{\gamma^2}$ |
| $\lambda$    | $\geq \max \{M_1, e, \frac{1}{C_l}\} = \frac{1}{\gamma^2}$ |
| $\phi_0$     | $\geq e^{-1}$ |
| $\phi_M$     | $\leq e$ |
| $R_1$        | $\leq \min \{1, \gamma^2, \frac{1}{\gamma}\} = \gamma^4$ |
| $c_T$        | $\sim \lambda^3 = \frac{1}{\gamma^{12}}$ (12, formula (A.2) and Remark A.1) |
| $c_{100}$    | $\geq 1$ |
| $\epsilon_0$| $\leq \frac{1}{(\lambda(1+\lambda)+c_T)} = \frac{1}{\lambda^{3}} = \gamma^{12}$ |
| $R_2$        | $\leq \min \left\{ R_1, \frac{C_l}{(1+\lambda+c_T/\lambda)}, \frac{\lambda^2 C_l^2}{c_T}, \left( \frac{1}{c_T^2 M_1 (1+\lambda^2)} \right)^{\frac{1}{2}}, \frac{60}{M_2}, \frac{\lambda}{c_T (1+\lambda^2+\lambda^2(1+\lambda))} \right\}$, |
|              | $\gamma^{20}$ |
| $\sigma$     | $\geq c_T R_2 = \gamma^{48}$ (12, formula (A.2) and Remark A.1) |
| $\tau_0$     | $\geq M_1 \left( (\lambda^2 + c_T R_2)^{\frac{1}{2}} + |h|_{C^0(\Omega_0)} (1 + (\lambda + c_T R_2)^{2}) + |q|_{C^0(\Omega_0)} \right) = \frac{1}{\gamma^{20}}$ |
| $R$          | $\leq R_2 = \gamma^{20}$ |
| $\delta$     | $\leq c_T R_2 = \gamma^{48}$ |
| $r$          | $\leq \frac{\lambda^2 c_T R_2}{(\lambda+c_T R_2)^2} = \gamma^{58}$ |
| $c_{1,T}$    | $\geq \sqrt{\left( \frac{M_1}{\tau_0} + \frac{1}{\lambda} \right)} = \gamma^2$ |
| $c_{2,T}$    | $\geq \sqrt{M_2 (1 + \frac{|x_1|_{C^0(\Omega_0)}}{\tau_0 R^2})} \sim \frac{1}{\gamma^2} + \frac{1}{\gamma^{32}} (|x_1''|_{C^0(\Omega_0)} + \frac{|x_1|_{C^0(\Omega_0)}}{\gamma^4}) \sim \frac{c_1^2 \lambda}{\gamma^8}$ |
| $c_{133}$    | $\geq \frac{|x_1|_{C^0(\Omega_0)}}{\tau_0 R^2} + \frac{|x_1|_{C^0(\Omega_0)}}{R} (1 + \lambda + c_T R_2 + \frac{|h|_{L^\infty(\Omega_0)}}{\tau_0}) = \frac{1}{\gamma^{32}} (|x_1''|_{C^0(\Omega_0)} + \frac{|x_1|_{C^0(\Omega_0)}}{\gamma^4})$ |
| Name | Value |
|------|-------|
| $c_{2X}$ | $= c_{102} = \frac{1}{(ec_{1X})^\alpha}$ |
| $c_{118}$ | $1 + |\phi'(0)|1 + R_2^2 + 5n|\phi''|_0 R_2^2 \sim \frac{1}{\Gamma}$ |
| $c_{115}$ | $\frac{\gamma'}{c_{2X}} |\phi|^2 |c_{2X}|^2 |\chi_1^2 |c_{2X}|^2 (1 + |\phi'|/|c_{2X}| + 1 + \frac{\gamma'}{c_{2X}} |\phi'|^2/\Gamma^2) \sim \frac{c_{1X}^4}{\gamma_{12} + 3\Gamma}$ |
| $c_{122}$ | $\frac{c_{1X}}{\gamma_{1X}}$ |
| $c_{128}$ | $\frac{c_{1X}}{\gamma_{1X}} \sim c_{123}$ |
| $c_{109}$ | $\min\left(\sqrt{c_{128}^2/2, 1}\right) \sim \frac{\gamma_{1X}^{56} \alpha}{c_{1X}}$ |
| $c_{131}$ | $\max\left(16e^{\frac{\Gamma}{\sqrt{2}}, \frac{e^{\Gamma/2}}{2\sqrt{\theta_0}, \frac{e^{\Gamma/2}}{3\sqrt{2}}}}\right) \sim \frac{c_{1X}^2}{\gamma_{1X}^{56} \alpha - 3\Gamma}$ |
| $c_{137}$ | $\min\left(\frac{1}{2}, c_{102} \frac{\alpha(c_{130})}{(\sqrt{2})^\alpha} + \frac{c_{130}}{\sqrt{2}} \frac{c_{102} \alpha(c_{130})}{(1/\sqrt{2})^\alpha} \right) \sim \frac{\gamma_{1X}^{48} \alpha}{c_{1X}^2 c_{130}}$ |
| $c_{170}$ | $N \sim \frac{\gamma_{1X}^{56} \alpha}{c_{1X}^2}$ |
| $\tilde{c}_{117}$ | $c_{2X} R^{\alpha} = (ec_{1X})^{-\alpha} R^{\alpha}$ |
| $\tilde{c}_{106}$ | $\frac{1}{\beta^3} \sim \frac{R^{\alpha}}{c_{1X}^2}$ |
| $\tilde{c}_{107}$ | $R^{\alpha+1} c_{100} \left(\frac{1}{\beta} \Gamma \frac{1}{\alpha(c_{117})} \right) \sim \frac{c_{1X}^2}{\gamma_{1X}^{106}}$ |
| $c_{154,1}$ | $1 + \tilde{c}_{107} \sim R^{\alpha} c_{1X}$ |
| $c_{151}$ | $\max(c_{134}, c_{136}) \sim \max(c_{1X} \frac{\gamma_{1X}^{58} (n-\frac{1}{2})}{c_{180}}, \frac{c_{100}^2}{\gamma_{1X}^{58}}) = \frac{c_{1X}^2}{\gamma_{1X}^{58}}$ |
| $c_{153}$ | $1 + 2N(1+n)^2 |\phi'(0)|c_{2X} + |h|^2 c_{0X}(\frac{\Gamma}{c_{2X}} \sim \frac{N^2 c_{1X}^2}{\beta^3})$ |
| $c_{162,1}$ | $1 \sim \frac{c_{1X}^{3/2}}{\gamma_{1X}^{3/2}} \sim \frac{\gamma_{1X}^{56} \alpha + 58(n+1)(\alpha+1)+28}{\alpha \pi (r/2)}$ |
| $c_{165}$ | $c_{117} \frac{\alpha}{(3\sqrt{4})} \sim \frac{\gamma_{1X}^{38}}{R^{\alpha}}$ |
| $c_{164}$ | $\frac{r^2 c_{1X} \left( \frac{1}{\alpha} \Gamma_{c_{1X}^{58}} \left( \frac{1}{\alpha(c_{117})} \right) \right) + \frac{1}{\alpha \pi (r/2)} \Gamma_{c_{1X}^{58}} + \frac{1}{\alpha \pi (r/2)} \left( \frac{1}{\alpha \pi (r/2)} \right) \sim \frac{c_{1X}^4}{\gamma_{1X}^{3/2}}$ |
| $c_{107}$ | $c_{107} \sim \frac{c_{1X}^{1/2}}{\gamma_{1X}^{1/2}}$ |
| $c_{108}$ | $\left( \frac{106}{c_{1X}^{106}} + \frac{1}{\gamma_{1X}^{106}} \right) \left( 1 + \frac{1}{\gamma_{1X}^{106}} \right) + \frac{1}{\gamma_{1X}^{106}} \left( 1 + \frac{1}{\gamma_{1X}^{106}} \right) \sim \frac{c_{1X}^4}{\gamma_{1X}^{1/2}}$ |
References

[1] Alessandrini G. Stable determination of conductivity by boundary measurements, Appl. Anal., 27 (1988), 153–172.

[2] Alessandrini G., Sylvester J. Stability for a multidimensional inverse spectral theorem. Comm. Part. Diff. Eq. 15 (1990), 711–736.

[3] Anderson M. Convergence and rigidity of manifolds under Ricci curvature bounds, Invent. Math., 102 (1990), 429-445.

[4] Anderson M., Katsuda A., Kurylev Y., Lassas M., Taylor M. Boundary regularity for the Ricci equation, Geometric Convergence, and Gel’fand’s Inverse Boundary Problem, Invent. Math. 158 (2004), 261-321.

[5] Belishev, M. An approach to multidimensional inverse problems for the wave equation. (Russian) Dokl. Akad. Nauk SSSR, 297 (1987), 524–527

[6] Belishev, M., Kurylev, Y. A nonstationary inverse problem for the multidimensional wave equation "in the large". (Russian) Zap. Nauchn. Sem. LOMI, 165 (1987), 21–30.

[7] Belishev M., Kurylev Y. To the reconstruction of a Riemannian manifold via its spectral data (BC-method), Comm. Part. Diff. Eq., 17 (1992), 767-804.

[8] Berard P., Besson G., Gallot S. Embedding Riemannian manifolds by their heat kernel, Geom. Funct. Anal., 4 (1994), 373-398.

[9] Bergh J., Lofström, J. Interpolation spaces. An introduction. Springer-Verlag, 1976, pp. x+207,

[10] Blagoveščenskii, A. A one-dimensional inverse boundary value problem for a second order hyperbolic equation. (Russian) Zap. Nauchn. Sem. LOMI, 15 (1969), 85–90.

[11] Bosi R., Kurylev Y., Lassas M. Stability of the unique continuation for the wave operator via Tataru inequality: the local case, Journal d’Analyse Mathematique, Vol. 134 (2018), 157 – 199.

[12] Bosi R., Kurylev Y., Lassas M. Stability of the unique continuation for the wave operator via Tataru inequality and applications, J. Differential Equations, 260, 8, (2016), 6451-6492.

[13] Burago D., Burago Y. and Ivanov S. A Course in Metric Geometry. AMS, Providence (2001).

[14] Chavel I. Riemannian geometry. A Modern Introduction, 2nd ed, 2006.
[15] Cheeger J., Finiteness theorems for Riemannian manifolds. *Am. J. Math.* **92** (1970), 61–75.

[16] E. Davies, *Spectral Properties of Compact Manifolds and Changes of Metric.* American Journal of Mathematics 112 (1990), 15-39.

[17] de Hoop M., Holman, S., Iversen, E., Lassas, M., Ursin B. *Recovering the isometry type of a Riemannian manifold from local boundary diffraction travel times.* J. Math. Pures et Appl. **103** (2015), 830-848.

[18] de Hoop M., Holman, S., Iversen, E., Lassas, M., Ursin B. *Recovery of a conformally Euclidean metric from local boundary diffraction travel times.* SIAM Journal on Mathematical Analysis **46** (2014), 3705-3726.

[19] Dos Santos Ferreira D., Kenig C., Salo M., and Uhlmann G. *Limiting Carleman weights and anisotropic inverse problems,* Invent. Math. **178** (2009), 119–171.

[20] Fefferman C., Ivanov S., Kurylev Y., Lassas M., Naranayan H. *Reconstruction and interpolation of manifolds I: The geometric Whitney problem.* Preprint arXiv:1508.00674.

[21] Gelfand I. *Some aspects of functional analysis and algebra.* 1957 Proceed. the Intern. Congr. Mathem., Amsterdam, 1954, 1, 253–276.

[22] Greene, R., Wu, H. *Lipschitz convergence of Riemannian manifolds.* Pacific J. Math. **131** (1988), 119–141.

[23] Gromov M. with appendices by Katz M., Pansu P. and Semmes S., *Metric Structures for Riemannian and Non-Riemannian Spaces,* based on ‘Structures metriques pour les varietes riemanniennes’, (LaFontaine J. and Pansu P. eds), Birkhauser (1999).

[24] Guillarmou C., Sa Barreto A. *Inverse problems for Einstein manifolds,* Inverse Probl. Imaging **3** (2009), 1–15.

[25] Ivanov, S. *Distance difference representations of Riemannian manifolds.* arXiv:1806.05257

[26] Helin, T., Lassas T., Oksanen L., Saksala, T. *Correlation based passive imaging with a white noise source.* J. Math. Pures et Appl. **116** 2018, 132–160.

[27] Hörmander L. *The analysis of linear partial differential operators I.* Springer-Verlag, 1985, viii+525 pp.
[28] Kasue A., Kumura H. *Spectral convergence of Riemannian manifolds*, Tohoku Math. J., **3446** (1994), 147-179.

[29] Kasue A., Kumura H. *Spectral convergence of Riemannian manifolds. II*. Tohoku Math. J. **48** (1996), 71-120.

[30] Katchalov A., Kurylev Y., Lassas M. *Inverse Boundary Spectral Problems*, Chapman/CRC, Boca Raton (2001).

[31] Kato T. *Perturbation Theory for Linear Operators*, Springer, Berlin (1995).

[32] Katsuda, A. *Gromov’s convergence theorem and its application*. Nagoya Math. J. **100** (1985), 11–48.

[33] Katsuda A., Kurylev Y., Lassas M. *Stability of boundary distance representation and reconstruction of Riemannian manifolds*. Inverse Problems and Imaging **1** (2007), 135–157.

[34] Kurylev Y. *Multidimensional Gel’fand inverse boundary problem and boundary distance map*. In: Inv. Probl. Related to Geom. (H. Soga, ed.), 1-15, Ibaraki Univ. Press, Mito, 1997.

[35] Kurylev Y., Lassas M., Yamaguchi T. *Uniqueness and Stability in Inverse Spectral Problems for Collapsing Manifolds*. Preprint [arXiv:1209.5875](http://arxiv.org/abs/1209.5875).

[36] Krupchyk K., Kurylev Y., Lassas M. *Inverse spectral problems on a closed manifold*. J. Math. Pures Appl. **90** (2008), 42–59.

[37] Lassas M., Uhlmann G. *Determining Riemannian manifold from boundary measurements*. Ann. Sci. École Norm. Sup. **34** (2001), 771–787.

[38] Laurent C., Leautard M. *Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves*. (2015), [arXiv:1506.04254](http://arxiv.org/abs/1506.04254) To appear in JEMS.

[39] Mandache N. *Exponential instability in an inverse problem for the Schrodinger equation*. Inverse Problems, **17** (2001):1435.

[40] Nachman, A. Sylvester, J., Uhlmann, G. *An n-dimensional Borg-Levinson theorem*. Comm. Math. Phys. **115** (1988), 595–605

[41] Novikov, R. *A multidimensional inverse spectral problem for the equation* $-\Delta \psi + (v(x) - Eu(x))\psi = 0$. (Russian) Funk. Anal. i Prilozhen. **22** (1988), 11–22.
[42] O’Neill, B. *Semi-Riemannian geometry*. Pure and Applied Mathematics, 103. Academic Press, 1983. xiii+468 pp.

[43] Petersen P. Riemannian Geometry, 1st Ed., Springer, New York (1998).

[44] Robbiano, L. *Fonction de cout et controle des solutions des equations hyperboliques*. Asymptotic Anal. **10** (1995), 95–115.

[45] Rodino L. Linear Partial Differential Operators in Gevrey Spaces, World Scientific, (1993).

[46] Stefanov, P., Uhlmann, G. *Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media*. J. Funct. Anal. **154** (1998), 330–358.

[47] Sylvester J., Uhlmann G. A *global uniqueness theorem for an inverse boundary value problem*. Ann. of Math. (2) **125** (1987), 153–169.

[48] Tataru D. *Unique continuation for solutions to PDE’s: between Hormander’s theorem and Holmgren’s theorem*, Comm. Part. Diff. Eq., **20** (1995), 855-884.

[49] Tataru D. *Carleman estimates, unique continuation and applications*, preprint.