The integrals in Gradshteyn and Ryzhik.
Part 15: Frullani integrals

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Abstract. The table of Gradshteyn and Ryzhik contains some integrals that can be reduced to the Frullani type. We present a selection of them.

1. Introduction

The table of integrals [3] contains many evaluations of the form

\[ \int_0^\infty \frac{f(ax) - f(bx)}{x} \, dx = \left[ f(0) - f(\infty) \right] \ln \left( \frac{b}{a} \right). \]

Expressions of this type are called Frullani integrals. Conditions that guarantee the validity of this formula are given in [1] and [4]. In particular, the continuity of \( f' \) and the convergence of the integral are sufficient for (1.1) to hold.

2. A list of examples

Many of the entries in [3] are simply particular cases of (1.1).

Example 2.1. The evaluation of 3.434.2 in [3]:

\[ \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln \left( \frac{b}{a} \right) \]

corresponds to the function \( f(x) = e^{-x} \).

Example 2.2. The change of variables \( t = e^{-x} \) in Example 2.1 yields

\[ \int_0^1 \frac{t^{b-1} - t^{a-1}}{\ln t} \, dt = \ln \left( \frac{a}{b} \right). \]

This is 4.267.8 in [3].

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Example 2.3. A generalization of the previous example appears as entry 3.476.1 in [3]:

\[ \int_0^\infty \left( e^{-ux} - e^{-ux^p} \right) \frac{dx}{x} = \frac{1}{p} \ln \left( \frac{u}{v} \right). \]

This comes from Frullani’s result with a simple additional scaling.

Example 2.4. The choice

\[ f(x) = \frac{e^{-qx} - e^{-px}}{x}, \]

with \( p, q > 0 \) satisfies \( f(\infty) = 0 \) and

\[ f(0) = \lim_{x \to 0} \frac{e^{-qx} - e^{-px}}{x} = p - q. \]

Then Frullani’s theorem yields

\[ \int_0^\infty \frac{\left( e^{-aqx} - e^{-apx} \right)}{ax} - \frac{\left( e^{-bqx} - e^{-bpx} \right)}{bx} \frac{dx}{x} = (p - q) \ln \left( \frac{b}{a} \right), \]

that can be written as

\[ \int_0^\infty \frac{\left( e^{-aqx} - e^{-apx} \right)}{a} - \frac{\left( e^{-bqx} - e^{-bpx} \right)}{b} \frac{dx}{x^2} = (p - q) \ln \left( \frac{b}{a} \right). \]

This is entry 3.436 in [3].

Example 2.5. Now choose

\[ f(x) = \frac{x}{1 - e^{-x}} \exp(-cx). \]

Then Frullani’s theorem yields entry 3.329 of [3], in view of \( f(0) = e^{-c} \) and \( f(\infty) = 0 \):

\[ \int_0^\infty \frac{a \exp(-cx)}{1 - e^{-ax}} - \frac{b \exp(-cx)}{1 - e^{-bx}} \frac{dx}{x} = e^{-c} \ln \left( \frac{b}{a} \right). \]

Example 2.6. The next example uses

\[ f(x) = (x + c)^{-\mu}, \]

with \( c, \mu > 0 \), to produce

\[ \int_0^\infty \frac{(ax + c)^{-\mu} - (bx + c)^{-\mu}}{x} \frac{dx}{x} = e^{-\mu} \ln \left( \frac{b}{a} \right). \]

This is 3.232 in [3].

Example 2.7. Entry 4.536.2 in [3] is

\[ \int_0^\infty \frac{\tan^{-1}(px) - \tan^{-1}(qx)}{x} \frac{dx}{x} = \frac{\pi}{2} \ln \left( \frac{p}{q} \right). \]

This follows directly from (1.1) by choosing \( f(x) = \tan^{-1} x \).
Example 2.8. The function \( f(x) = \ln(a + be^{-x}) \) gives the evaluation of entry 4.319.3 of [3]:

\[
\int_0^\infty \frac{\ln(a + be^{-px}) - \ln(a + be^{-qx})}{x} \, dx = \ln\left(\frac{a}{a + b}\right) \ln\left(\frac{p}{q}\right).
\]

Example 2.9. The function \( f(x) = ab \ln(1 + x)/x \) produces entry 4.297.7 of [3]:

\[
\int_0^\infty \frac{b \ln(1 + a x) - a \ln(1 + b x)}{x^2} \, dx = ab \ln\left(\frac{b}{a}\right).
\]

Example 2.10. Entry 3.484:

\[
\int_0^\infty \left[\left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^x\right] \, dx = \frac{a + b}{c + g + h} \ln\left(\frac{q}{p}\right).
\]

3. A separate source of examples

The list presented in this section contains integrals of Frullani type that were found in volume 1 of Ramanujan’s Notebooks [2].

Example 3.1.

\[
\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} \, dx = \frac{\pi}{2} \ln\frac{a}{b}
\]

Example 3.2.

\[
\int_0^\infty \frac{\ln\left(\frac{p + qe^{-ax}}{p + qe^{-bx}}\right)}{x} \, dx = \ln\left(1 + \frac{q}{p}\right) \ln\frac{b}{a}
\]

Example 3.3.

\[
\int_0^\infty \left[\left(\frac{ax + p}{ax + q}\right)^n - \left(\frac{bx + p}{bx + q}\right)^n\right] \, dx = \left(1 - \frac{p^n}{q^n}\right) \ln\frac{a}{b}
\]

where \( a, b, p, q \) are all positive.

Example 3.4.

\[
\int_0^\infty \frac{\cos ax - \cos bx}{x} \, dx = \ln\frac{b}{a}
\]

Example 3.5.

\[
\int_0^\infty \sin\left(\frac{(b - a)x}{2}\right) \sin\left(\frac{(b + a)x}{2}\right) \, dx = \int_0^\infty \frac{\cos ax - \cos bx}{2x} \, dx = \frac{1}{2} \ln\frac{b}{a}
\]
Example 3.6. 
\[
\int_0^\infty \sin px \sin qx \frac{dx}{x} = \int_0^\infty \frac{\cos[(p-q)x] - \cos[(p+q)x]}{2x} \, dx = \frac{1}{2} \ln \frac{p+q}{p-q}
\]

Example 3.7. The evaluation of 
\[
\int_0^\infty \ln \left( \frac{1 + 2n \cos ax + n^2}{1 + 2n \cos bx + n^2} \right) \frac{dx}{x} = \begin{cases} 
\ln \left( \frac{b}{a} \right) \ln(1 + n) & n^2 < 1 \\
\ln \left( \frac{b}{a} \right) \ln \left( 1 + \frac{1}{a} \right) & n^2 > 1
\end{cases}
\]
is more delicate and is given in detail in the next section.

Example 3.8. The value 
\[
\int_0^\infty \frac{e^{-ax} \sin ax - e^{-bx} \sin bx}{x} \, dx = 0
\]
follows directly from (1.1) since, in this case \( f(x) = e^{-x} \sin x \) satisfies \( f(\infty) = f(0) = 0 \).

Example 3.9. 
\[
\int_0^\infty \frac{e^{-ax} \cos ax - e^{-bx} \cos bx}{x} \, dx = \ln \frac{b}{a}
\]

4. A more delicate example

Entry 4.324.2 of [3] states that 
\[
(4.1) \quad \int_0^\infty \frac{[\ln(1 + 2a \cos px + a^2) - \ln(1 + 2a \cos qx + a^2)]}{x} \, dx = \\
\begin{cases} 
2 \ln \left( \frac{a}{p} \right) \ln(1 + a) & -1 < a \leq 1 \\
2 \ln \left( \frac{a}{q} \right) \ln(1 + 1/a) & a < -1 \text{ or } a \geq 1.
\end{cases}
\]

This requires a different approach since the obvious candidate for a direct application of Frullani’s theorem, namely \( f(x) = \ln(1 + 2a \cos x + a^2) \), does not have a limit at infinity.

In order to evaluate this entry, start with 
\[
(4.2) \quad \int_0^1 x^y \, dx = \frac{1}{y+1},
\]
so 
\[
(4.3) \quad \int_0^1 dy \int_0^1 x^y \, dx = \int_0^1 dx \int_0^1 x^y \, dy = \int_0^1 x \frac{1}{\ln x} \, dx = \int_0^1 \frac{dy}{y+1} = \ln 2.
\]
This is now generalized for arbitrary symbols \( \alpha \) and \( \beta \) as 
\[
(4.4) \quad \int_0^\infty \frac{e^{\alpha t} - e^{\beta t}}{t} \, dt = \ln \left( \frac{\beta}{\alpha} \right).
\]
To prove (4.4), make the substitution \( u = e^{-t} \) that turns the integral into

\[
\int_0^1 \frac{u^{1-\beta} - u^{1-\alpha}}{\ln u} \, du = \int_0^1 \frac{u^{1-\beta}}{1-\alpha} \, dw \\
= \int_{1-\alpha}^{-1} \frac{dw}{u} \int_0^1 u^w \, du \\
= \int_{1-\alpha}^{-1} \frac{dw}{w+1} \\
= \ln \left( \frac{\beta}{\alpha} \right).
\]

Now observe that \( \left| \frac{2a \cos(rx)}{1+a^2} \right| \leq 1 \), therefore it is legitimate to expand the logarithmic terms as infinite series using \( \ln(1 + z) = \sum_{k \geq 1} \frac{(-1)^{k-1} z^k}{k} \). The outcome reads

\[
\int_0^\infty \frac{dx}{x} \sum_{k \geq 1} \frac{(-1)^{k-1} A^k (\cos^k px - \cos^k qx)}{k} = \\
\sum_{k \geq 1} \frac{(-1)^{k-1} A^k}{2^k k} \int_0^\infty \frac{(e^{ipx} + e^{-ipx})^k - (e^{iqx} + e^{-iqx})^k}{x} \, dx;
\]

where \( A = 2a/(1+a^2) \). The inner integral is evaluated using some binomial expansions. That is,

\[
(4.5) \int_0^\infty \frac{dx}{x} \sum_{k \geq 1} \frac{(-1)^{k-1} A^k (\cos^k px - \cos^k qx)}{k} = \\
\sum_{k \geq 1} \frac{(-1)^{k-1} A^k}{2^k k} \int_0^\infty \frac{(e^{ipx} + e^{-ipx})^k - (e^{iqx} + e^{-iqx})^k}{x} \, dx.
\]

It is time to employ equation (4.4). A closer look at (4.5) shows that care must be exercised. The integrals are sensitive to the parity of \( k \). More precisely, the quantity \( 2r - k \) vanishes if and only if \( k \) is even and \( r = k/2 \), in which case there is a zero contribution to summation. Otherwise, the second integral in (4.5) is always equal to \( \ln(q/p) \). Therefore,

\[
\sum_{r=0}^k \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} \, dx = \begin{cases} 
2^k \ln \left( \frac{\alpha}{\beta} \right) & \text{if } k \text{ is odd}, \\
\left( \frac{2^k - \binom{k}{k/2}}{2^k} \right) \ln \left( \frac{\alpha}{\beta} \right) & \text{if } k \text{ is even}.
\end{cases}
\]
Combining the results obtained thus far yields

\[(4.6)\]
\[I = \int_0^\infty \frac{\ln(1 + 2a \cos(px) + a^2) - \ln(1 + 2a \cos(qx) + a^2)}{x} \, dx\]
\[= \int_0^\infty \frac{dx}{x} \sum_{k \geq 1} \frac{(-1)^{k-1} A^k (\cos^k px - \cos^k qx)}{k} = \sum_{k \geq 1} \frac{(-1)^{k-1} A^k}{k} \sum_{r=0}^{k} \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} \, dx\]
\[= \ln \left( \frac{q}{p} \right) \sum_{k \text{ odd}} \frac{(-1)^{k-1} A^k}{k} + \ln \left( \frac{q}{p} \right) \sum_{k \text{ even}} \frac{1}{2k} \left( \frac{A^2}{2} \right)^k \cdot \left( \frac{2k}{k} \right) = \ln \left( \frac{q}{p} \right) \ln(1 + A) + \frac{1}{2} \ln \left( \frac{q}{p} \right) \sum_{k \geq 1} \left( \frac{2}{k} \right) \left( \frac{A^2}{2} \right)^k \cdot \left( \frac{2k}{k} \right) = \ln \left( \frac{q}{p} \right) \ln(1 + A) + \frac{1}{2} \ln \left( \frac{q}{p} \right) \sum_{k \geq 1} \left( \frac{2}{k} \right) \left( \frac{A^2}{2} \right)^k \cdot \left( \frac{2k}{k} \right)
\]

The last step utilizes the Taylor series

\[(4.7)\]
\[\sum_{k \geq 1} \left( \frac{2k}{k} \right) \frac{Q^k}{k} = -2 \ln \left( \frac{1}{2} \left[ 1 + \sqrt{1 - 4Q} \right] \right)
\]

This follows from the binomial series \[\sum_{k \geq 0} \left( \frac{2k}{k} \right) R^k = 1/\sqrt{1 - 4R}\] after rearranging in the manner

\[\sum_{k \geq 1} \left( \frac{2k}{k} \right) R^{k-1} = \frac{1}{R\sqrt{1 - 4R}} - \frac{1}{R} = \frac{4}{\sqrt{1 - 4R(1 + \sqrt{1 - 4R})}},\]

and then integrating by parts (from 0 to Q)

\[\sum_{k \geq 1} \left( \frac{2k}{k} \right) Q^k = \int_0^Q \frac{4 \cdot dR}{\sqrt{1 - 4Q(1 + \sqrt{1 - 4R})}} = \int_1^{\sqrt{1 - 4Q}} \frac{-2 \cdot du}{1 + u} = -2 \ln \left( \frac{1}{2} \left[ 1 + \sqrt{1 - 4Q} \right] \right) \cdot \frac{4}{\sqrt{1 - 4R(1 + \sqrt{1 - 4R})}}
\]

Formula (4.7) applied to equation (4.6) leading to

\[I = \ln \left( \frac{q}{p} \right) \ln(1 + A) + \frac{1}{2} \ln \left( \frac{q}{p} \right) \sum_{k \geq 1} \left( \frac{2}{k} \right) \left( \frac{A^2}{2} \right)^k \cdot \left( \frac{2k}{k} \right)\]

It remains to replace \[Q = A^2/2^2 = a^2/(1 + a^2)^2\] and use the identity

\[1 - 4Q = \frac{(a^2 - 1)^2}{(a^2 + 1)^2}.
\]

Observe that the expression for \[\sqrt{1 - 4Q}\] depends on whether \(|a| > 1\) or not. The proof is complete.
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