Generalized KdV Equation for Fluid Dynamics and Quantum Algebras

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Abstract

We generalize the non-linear one-dimensional equation of a fluid layer for any depth and length as an infinite order differential equation for the steady waves. This equation can be written as a q-differential one, with its general solution written as a power series expansion with coefficients satisfying a nonlinear recurrence relation. In the limit of long and shallow water (shallow channels) we reobtain the well known KdV equation together with its single-soliton solution.

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1 Introduction

The Korteweg-de Vries equation (KdV) was proposed by Korteweg and de Vries [1] one century ago to explain the steady translation water waves observed in a channel by Scott Russel in 1834. The main property of this equation consists in the equal occurrence of non-linearity and dispersion. Some of its most important solutions, the solitary waves, are single, localised and non-dispersive entities (no dispersion in shape), having also localised finite energy density. Among these solutions, the solitons are solitary waves with special added requirements concerning their behaviour at infinity \((x \to \infty, t \to \infty)\) and having special properties concerning the scattering between different such single-solitary solutions. Despite this non-linearity, the KdV equation is an infinite dimensional Hamiltonian system [2-5] and, remarkable, when the KdV solution evolves in time, the eigenvalues of the associated Sturm-Liouville operator \(\partial^2 + \eta\), where \(\eta\) is the solution of the KdV equation, remain constant. Several extended and complex methods have been developed to study and to solve the KdV equation and other non-linear wave equations (non-linear Schrödinger equation, NLS, modified-KdV equation, MKdV, sine-Gordon equation, etc.): inverse scattering method, theoretical group methods, etc. (for a review of some of these techniques one can look into the book by Witham [6] or the paper by Scott et al [7].

Due to its properties, the KdV equation was the source of many applications and results in a large area of non-linear physics (For a recent review see [8] and the references herein). For example, the KdV equation and its generalizations (KdV hierarchies, KP equation and its hierarchies and their supersymmetric generalizations [9]) appear in different fields of physics, from plasma physics [10] to string theory [11], or to describe the dynamics of quantum extended objects [12], and nonlinear phenomena in nuclear physics (non-linear liquid drop model) [13,14], and the list is far from being exhausted. In all these applications, the KdV equation (or other non-linear ones) appears as the consequence of certain simplifications of the physical models, especially due to some perturbation techniques involved.
In the present paper we have used only one of the two well-known necessary conditions for obtaining the KdV equation in shallow water channels, i.e. the smallness of the amplitude of the soliton with respect to the depth of the channel, $h$. This is the single condition which we use, and hence, we consider $h$ to be an arbitrary parameter.

In order to obtain the KdV equation from shallow water channel models [3,5], a second condition is imposed, i.e. the depth $h$ of the channel should be smaller than the half-width of the solitary wave. We do not use, in our generalised approach, nor this second condition neither the condition of infinite length of the channel. In our case the length of the channel, $L$, is also considered an arbitrary parameter.

We introduce this generalisation in connection with the last (above mentioned) nuclear physics application. If the height of the perturbation is comparable to the nuclear radius (big clusters formation on the nuclear surface or symmetric nuclear molecules) it is not appropriate to consider that the nuclear dynamics is that of a shallow fluid layer. Therefore, we should like to study, as a first step of our aim, the non-linear dynamics of a fluid in a channel of arbitrary depth, in a bounded domain. This different starting point leads us to a new type of equation which generalizes in some sense the KdV equation and reduces to this in the shallow water case.

As we shall show, this generalized equation (infinite order differential equation) can also be written in terms of a q-differential equation. In this context we introduce here another possible physical interpretation for the q-deformation parameter, in the field of fluid dynamics, as the depth of the fluid layer. We use the connection between the theory of quantum algebras (or q-groups, q-deformed algebras), originally initiated by Jimbo and Drinfeld [15], and the q-differential equations [16].

In the present model we take into account not only the gravity but also the influence of the surface pressure acting on the free surface of the fluid. This implies that the KdV-like structure for the dynamical equation arrises in a lower order (first order in the smallness parameters) and one may not take into account the second order of smallness to obtain a KdV equation. We also note that, despite the general tendency followed by papers concerning applications of q-groups in physics of first constructing the deformed algebra and later searching possible applications, we have obtained a
q-differential equation in a natural way, by starting from a traditional physical one-dimensional hydrodynamical model.

2 The generalized KdV equation

Let us consider a one-dimensional ideal incompressible fluid layer with depth $h$ and constant density $\rho$, in an uniform gravitational field. We suppose irrotational motion and consequently the field of velocities is obtained from a potential function $\Phi(x, y, t)$

$$\vec{V}(x, y, t) = (u, v) = \nabla \Phi(x, y, t).$$  \hspace{1cm} (1)

The continuity equation for the fluid results in the Laplace equation for $\Phi$:

$$\Delta \Phi(x, y, t) = 0.$$  \hspace{1cm} (2)

The above Laplace equation should be solved with appropriate boundary conditions for our physical problem. We take a two-dimensional domain: $x \in [x_0 - L, x_0 + L]$ (the "horizontal" coordinate) and $y \in [0, \xi(x, t)]$ (the "vertical" coordinate) where $x_0$ is an arbitrary parameter, $L$ is an arbitrary length and $\xi(x, t)$ is the equation of the free surface of the fluid. The boundary conditions on the lateral walls, $x = x_0 \pm L$ and on the bottom $y = 0$, consist in the condition of vanishing of the normal velocity component. The free surface fulfills the kinematic condition [3]:

$$v \bigg|_{\Sigma} = (\xi_t + \xi_x u) \bigg|_{\Sigma},$$  \hspace{1cm} (3)

where we denote by $\Sigma$ the free surface of equation $y = \xi(x, t)$ and the lower index indicates the derivative. Eq. (3) expresses the fact that the fluid particles, which belong to the surface, remain during time evolution in this surface. By taking into account the above boundary conditions on the lateral walls and on the bottom, we can write the potential of the velocities in the form:

$$\Phi(x, y, t) = \sum_{k=0}^{\infty} \cosh \left( \frac{k\pi y}{L} \right) \left( \alpha_k(t) \cos \left( \frac{k\pi x}{L} \right) + \beta_k(t) \sin \left( \frac{k\pi x}{L} \right) \right),$$  \hspace{1cm} (4)
where $\alpha_k$ and $\beta_k$ are time dependent coefficients fulfilling the condition

$$\frac{k\pi(x_0 \pm L)}{L} = \frac{\beta_k \cos k\pi(x_0 \pm L)}{L},$$

for any positive integer $k$. This restriction introduces a special time dependence of $\alpha_k$ and $\beta_k$, i.e. $\frac{\alpha_k}{\beta_k} = \gamma_k = \text{const.}$ for any $k$, when $\beta_k \neq 0$. If $\beta_k = 0$ we simply equate with 0 the inverse of the above fraction. It exists also a dependence of the $\alpha_k, \beta_k$ functions of $x_0$. This special time dependence does not affect the generality of the potential $\Phi$, but only the balance between the two terms in the RHS of eq.(4). If we fix all $\beta_k = 0$ we can take arbitrary values for $x_0$ and general form for $\alpha_k$. In the infinite channel limit, $L \to \infty$, there are no more restrictions concerning $\alpha_k$ and $\beta_k$ functions, and we can choose $x_0 = 0$, too.

We introduce the function:

$$f(x, t) = \sum_{k=0}^{\infty} \frac{k\pi}{L} \left(-\alpha_k(t) \sin \left(k\pi \frac{x}{L}\right) + \beta_k(t) \cos \left(k\pi \frac{x}{L}\right)\right),$$

and the velocity field can be written like:

$$u = \Phi_x = \cos(y\partial) f(x, t)$$
$$v = \Phi_y = -\sin(y\partial) f(x, t).$$

where, for simplicity, the operator $\partial$ represents, from now on, the partial derivative with respect to the $x$ coordinate. The equations (5,6) do not depend on $L$ and we equally treat the case $L \to \infty$, when we look for unbounded domains and travelling profiles for the solutions.

In the following we use in the boundary condition (3) the velocities evaluated at $y = \xi(x, t) = h + \eta(x, t)$ in the first order in $\eta$, as we describe the perturbations with small height compared to the depth, but not necessarily with large widths. Eqs.(6) read:

$$u(x, \xi(x, t), t) = [\cos(h\partial) - \eta(x, t)\partial \sin(h\partial)] f(x, t)$$
$$v(x, \xi(x, t), t) = -[\sin(h\partial) + \eta(x, t)\partial \cos(h\partial)] f(x, t).$$

The dynamics of the fluid is described by the Euler equation at the free surface. The Euler equation, written on $\Sigma$, results (after writing it in terms of the potential,
differentiating it with respect to $x$ and imposing the condition $y = \xi(x,t)$ in the form

$$u_t + uu_x + vv_x + gn_x + \frac{1}{\rho}P_x = 0,$$

(8)

where $g$ represents the gravitational acceleration and $P$ the surface pressure. Following the same approach used in the calculation of surface capillary waves [17], we have for our one-dimensional case

$$\left. P \right|_\Sigma = \frac{\sigma}{\mathcal{R}} = \frac{\sigma \eta_{xx}}{(1 + \eta_x^2)^{3/2}} \simeq -\sigma \eta_{xx}, \quad \text{for small } \eta. \quad (9)$$

where $\mathcal{R}$ is the local radius of curvature of the surface (in this case, of the curve $y = \xi(x,t)$) and $\sigma$ is the surface pressure coefficient. In the interior of the fluid the pressure is given by the Euler equation. The nonlinearities appear in the dynamics through the nonlinear terms in eqs. (3), (7), (8) and (9). Consequently, we have a system of two differential equations (3,8) for the two unknown functions: $f(x,t)$ and $\eta(x,t)$, with $u$ and $v$ depending on $\eta$ and $f$ from eqs.(7). With $f$ and $\eta$ determined and introduced in the expressions of $u$ and $v$, we can finally find the coefficients $\alpha_k$ and $\beta_k$. In the following we treat these equations in the approximation of small perturbations of the surface $\Sigma$, with respect to the depth, $a = \max|\eta^{(k)}(x,t)| << h$, where $k = 0, \ldots, 3$ are orders of differentiation.

In the linear approximation eqs. (3) and (8) become, respectively

$$-\sin(h\partial)f = \eta_t$$

$$\cos(h\partial)f_t = -g\eta_x + \frac{\sigma}{\rho}\eta_{xxx}, \quad (10)$$

and we obtain, by eliminating $\xi_t$ from eqs.(10),

$$\cos(h\partial)\eta_{tt} = \sin(h\partial)\left(\frac{c_0^2}{h^2}\eta_x - \frac{\sigma}{\rho}\eta_{xxx}\right), \quad (11)$$

which, in the lowest order of approximation in $(h\partial)$ for the sine and cosine functions, and in the absence of the surface pressure, gives us the familiar wave equation $\eta_{tt} = c_0^2 \eta_{xx}$, where $c_0 = \sqrt{gh}$ is the sound wave velocity. By introducing the solution $\eta = e^{i(kx - \omega t)}$ in the linearized eq.(11) we obtain a non-linear dispersion relation

$$\omega^2 = c_0^2k^2 \left(1 + \frac{\sigma}{\rho g}\right) \tanh(kh) \frac{kh}{k}. \quad (12)$$
In the limit of shallow waters we find, for the dispersion relations, in the case \( \sigma = 0 \) the acoustic waves limit, and, for \( \sigma \gg \rho g \) the surface capilar waves limit \( \omega^2 = \frac{\rho g h^4}{\sigma} \).

In the absence of the surface pressure \( (\sigma = 0) \) the function \( f \) is given, in this linear approximation, at least formally, by:

\[
f_{\text{lin}} = \frac{c_0}{h} \left( \frac{\sin(2h\partial)}{2h\partial} \right)^{-1/2} \eta,
\]

which in the limit of shallow fluid has a particular solution in the form

\[
f^0(x,t) = \frac{c_0}{h} \eta.
\]

For the time derivative of \( f \) we have, from the second equation of eqs.(10),:

\[
f_{t \text{lin}}(x,t) = (\cos(\partial))^{-1} \left( -\frac{c_0^2}{h} \eta_x + \frac{\sigma}{\rho} \eta_{xxx} \right),
\]

which in the limit of a shallow fluid reduces to:

\[
f^0_t(x,t) = -\frac{c_0^2}{h} \eta_x + \frac{\sigma}{\rho} \eta_{xxx}
\]

Following [3], we look for the solution of eqs.(3,8) in the form

\[
f = \frac{a}{h} c_0 \tilde{\eta} + \left( \frac{a}{h} \right)^2 f_2
\]

\[
f_t = -c_0^2 \frac{a}{h} (\cos(h\partial))^{-1} \tilde{\eta}_x + \frac{a\sigma}{\rho} (\cos(h\partial))^{-1} \tilde{\eta}_{xxx} + \left( \frac{a}{h} \right)^2 g_2,
\]

which represents a sort of perturbation technique in \( a/h \), where \( \eta = a \tilde{\eta} \). Of course a functional connection exists between the perturbations \( f_2(x,t) \) and \( g_2(x,t) \). Eq.(3), in the lowest order in \( a/h \), yields:

\[
-c_0 \sin(h\partial) \tilde{\eta} = h \tilde{\eta}_t + ac_0 (\tilde{\eta}_x \cos(h\partial) + \tilde{\eta} \cos(h\partial) \tilde{\eta}_x).
\]

If we approximate \( \sin(h\partial) \simeq h\partial - \frac{1}{6} (h\partial)^3 \), \( \cos(h\partial) \simeq 1 - \frac{1}{2} (h\partial)^2 \), we obtain, from eq.(16), the polynomial differential equation:

\[
a \tilde{\eta}_t + 2c_0 \epsilon^2 h \tilde{\eta} \tilde{\eta}_x + c_0 \epsilon h \tilde{\eta}_x - c_0 \epsilon \frac{h^3}{6} \tilde{\eta}_{xxx} - \frac{c_0 \epsilon^2 h^3}{2} \left( \tilde{\eta}_x \tilde{\eta}_{xx} + \tilde{\eta} \tilde{\eta}_{xxx} \right) = 0,
\]

where \( \epsilon = \frac{a}{h} \). The first four terms in eq.(17) correspond to the zero order approximation terms in \( f \), obtained from the boundary condition at the free surface, in eq.(6.1.15a)
from [3], i.e. the traditional way of obtaining the KdV equation in shallow channels. In this case all the terms are in first and second order in $\epsilon$. If we apply the second restriction with respect to the solutions, i.e. the half-width to be much larger than $h$, we can neglect in eq.(17) the last parenthesis, and we obtain exactly the KdV equation for the free surface boundary condition. In other words we understand the condition $h\partial$ "small" like $(h\partial)f(x,t) \ll 1$ all over the domain of definition of $f$. This means that the spatial extension of the perturbation $f(x,t)$ is large compared to $h$, which is exactly the case in which KdV equation arrises from the shallow water model (see Chapter 6 in [3], $h\partial f(x,t)$ of order $\simeq \frac{h}{l} = \delta \ll 1$).

By using again the approximations given by eq.(7), we can write the Euler eq.(8) in the form:

$$\partial_t \Omega f + \Omega f(\partial_t \Omega f) + \Omega f(\partial_\Omega f) + \omega f(\partial_\omega f) = -g\eta_x + \frac{\sigma}{\rho} \eta_{xxx}, \quad (18)$$

where we used the notations:

$$\Omega = \cos(h\partial) - \eta\partial \sin(h\partial)$$
$$\omega = \sin(h\partial) + \eta\partial \cos(h\partial). \quad (19)$$

We note here an interesting property of the operators given in eq.(19):

$$\Omega + i\omega = e^{iy\partial} \Sigma + O_2(h\partial) + O_2(h\eta). \quad (20)$$

Following the same procedure like for the free surface boundary condition eq.(3), i.e. using the approximation for small $\eta$, we obtain, from eq.(18):

$$\cos(h\partial)f_t = -g\eta_x + \frac{\sigma}{\rho} \eta_{xxx}, \quad (21)$$

which, in the lowest order in $a/h$ and by using eqs.(14,15), reduces to an identity.

Before further analysis, we would like to note that in the shallow water case, following the same addimensional notations like in Chapter 6 of [3], i.e. $\epsilon = \frac{a}{h}$, $\delta = \frac{h}{l}$, where $l$ gives the order of magnitude of the half-width of the perturbation $\eta$, and introducing the new addimensional parameter $\alpha = -\frac{\sigma}{gl^2} \frac{\rho}{g}$, we obtain, for the Euler equation, the form:

$$\tilde{\eta}_t + \frac{3}{2} \tilde{\eta}_x + \alpha \frac{\epsilon}{2} \tilde{\eta}_{xx} = 0, \quad (22)$$
which is again the traditional KdV equation. The primes attached to the lower indexes represent addimensional units, according to [3]. The difference between eq.(22) and the corresponding eq.(6.1.15.b) from [3] is given by the inclusion of the surface pressure effects. If the coefficient $\alpha$ is large enough, one is no more obliged to take into account second order terms, i.e. $\delta^2$, to obtain the KdV equation. Of course, the above approach introduces changes in the differential equations which are involved with higher order perturbations, i.e. $f^{(1)}$ and $f^{(2)}$ from [3] or $f_2$, $g_2$ in our case.

The reduction of eq.(3) to the KdV equation occurs if, in eq.(16) we limit only to third order of magnitude in $h\partial$:

$$
\eta_t + c_0 \eta_x - c_0 \frac{h^2}{6} \eta_{xxx} + \frac{2c_0}{h} \eta \eta_x = 0
$$

In the following we shall investigate this generalised KdV equation (GKdV), obtained from eq.(16), by keeping all terms in sin and cos, namely:

$$
\eta_t + \frac{c_0}{h} \sin(h\partial) \eta + \frac{c_0}{h} (\eta_x \cos(h\partial) \eta + \eta \cos(h\partial) \eta_x) = 0 \quad (23)
$$

Eqs.(3) and (18), in higher orders in $\frac{a}{h}$ and in $h\partial$, yield the corresponding differential equations for the functions $f_2$ and $g_2$, but we shall study here only eq.(23).

### 3 The q-differential form of the GKdV equation

In this Section we limit ourselves to the steady translation waves and consider only solutions of the form $\eta(x, t) = \eta(x + Ac_0 t) = \eta(X)$ where $A \in R$ and $X = x + Ac_0 t$.

Eq.(23) can be written in the form:

$$
Ah \eta_X(X) + \frac{\eta(X + ih) - \eta(X - ih)}{2i} + \eta_X(X) \frac{\eta(X + ih) + \eta(X - ih)}{2} + \eta(X) \frac{\eta_X(X + ih) + \eta_X(X - ih)}{2} = 0,
$$

(24)
if we suppose that $\eta$ is an analytic function in the domain $Re(z) \in (\infty, \infty), Im(z) \in (-h, h)$ of the complex plane. We study the solutions with a rapid decrease at infinity and make a change of variable: $v = e^{BX}$ for $x \in (-\infty, 0)$ and $v = e^{-BX}$ for $x \in (0, \infty)$. Here $B$ is a positive real constant. Putting:

$$\eta(X) = -hA + f(v),$$

we obtain a q-differential equation for the function $f(v)$:

$$f(v)\frac{D_qf^2(v)}{D_qf(v)} + f_v(v)\frac{D_qf^2(v)}{D_qf(v)} + 2\frac{\sin(Bh)}{B}D_qf(v) = 0,$$  \hspace{1cm} (25)

where we denoted $q = e^{iBh}$, $|q| = 1$, and the operator of the q-derivative is defined \cite{16}:

$$D_qf(v) = \frac{f(qv) - f(q^{-1}v)}{qv - q^{-1}v}. \hspace{1cm} (26)$$

We then write the solution of eq.(24) (or eq.(25)) as a power series in $v$:

$$f(v) = \sum_{n=0}^{\infty} a_n v^n, \hspace{1cm} (27)$$

and we must put $a_0 = hA$ to have $\lim_{x \to \pm\infty} \eta(x) = 0$. Equation (25) results in a recurrence non-linear relation for the coefficients $a_n$:

$$\left[Ahk + \frac{1}{B} \sin(Bhk)\right]a_k = -\sum_{n=1}^{k-1} n \left(\cos(Bh(k-n)) + \cos(Bh(k-1))\right)a_n a_{k-n}. \hspace{1cm} (28)$$

By taking $k = 1$ in the above relation we obtain $a_1 \left[Ah + \frac{1}{B} \sin(Bh)\right] = 0$. Without any loss of generality, due to the arbitrariness of $B$ we can write:

$$A = -\frac{\sin(Bh)}{Bh}. \hspace{1cm} (29)$$

This relation fixes the velocity of the envelope of the perturbation if its asymptotic behaviour is fulfilled. To have $A \neq 0$ we need $Bh \neq k\pi$ for $k$ integer. In this condition
$a_1$ is arbitrary at present and, by writing $a_k = \alpha_k a_1^k$ we have $\alpha_1 = 1$ and the recurrence relation:

$$\alpha_k = \frac{2B \cos \frac{Bh(n-1)}{2}}{k \sin(Bh) - \sin(kBh)} \sum_{n=1}^{k-1} n \cos \frac{Bh(2k - n - 1)}{2} \alpha_n \alpha_{k-n}, \quad (30)$$

for $k \geq 2$. This recurrence relation gives the $k$ coefficient if we know the first $k - 1$ ones.

To obtain a smooth behaviour of the solution $\eta(X)$ in $X = 0$, i.e. a continuity of its derivative, we must introduce the condition:

$$f_v(1) = \sum_{n=1}^{\infty} n \alpha_n \alpha_1^{n-1} = 0, \quad (31)$$

or the derivative of the power series $f(v)$ with the coefficients given by eq.(28) is zero in $z \in R, z = a_1$. This condition fixes $a_1$.

We study now a limiting case of the relation (28), by replacing sin and cos expressions with their lowest nonvanishing terms in their power expansions. Thus, we have:

$$\alpha_k = \frac{6}{B^2 h^3 k(k^2 - 1)} \sum_{n=1}^{k-1} n \alpha_n \alpha_{k-n}. \quad (32)$$

It is a straightforward exercise to prove that:

$$\alpha_k = \left( \frac{1}{2B^2 h^3} \right)^{k-1} k \quad (33)$$

is the solution of the above recurrence relation. This can be done by using the mathematical induction and by taking into account the relations:

$$\sum_{n=1}^{k-1} n^2 = \frac{k(k - 1)(2k - 1)}{6},$$

$$\sum_{n=1}^{k-1} n^3 = \left( \frac{k(k - 1)}{2} \right)^2.$$

We can write the power expansion:

$$g(z) = \sum_{k=1}^{\infty} k \left( \frac{1}{2B^2 h^3} \right)^{k-1} z^k. \quad (34)$$
having the radius of convergence $R = 2B^2h^3$ (due to the Cauchy-Hadamard criterium).

The function $g(z)$ can be written in the form

$$g(z) = z \left( 1 - \frac{z}{2B^2h^3} \right) 2B^2h^3 = -\frac{z}{\left( 1 - \frac{z}{2B^2h^3} \right)^2}$$

(35)

The condition (31) results in $a_1 = -2B^2h^3$ and

$$\alpha_k = k \left( \frac{1}{2B^2h^3} \right)^{k-1} (-2B^2h^3)^k = 2B^2h^3(-1)^k k,$$

(36)

which gives:

$$\eta(X) = 2B^2h^3 \sum_{k=1}^{\infty} k \left( -e^{-B|X|} \right)^k = 2B^2h^3 \frac{e^{-B|X|}}{(1 + e^{-B|X|})^2} = \frac{B^2h^3}{2} \frac{1}{(\cosh(BX/2))^2}.$$  (37)

As one expects, the above solution is exactly the single-soliton solution of the KdV equation and it was indeed obtained by assuming $h$ small in the recurrence relation (28).

We stress at this point that the GKdV equation has two general features, expressed in the reduction of eq.(16) to KdV equation and in eq.(23): in the limit $h\partial$ ”small”, both the differential equation GKdV and one of its solutions, $\eta(X)$, go versus the KdV equation and its single-soliton solution, correspondingly.

In the general case we do not obtain a simple solution like the relation (33) but we have all the necessary informations from the relations (28). It seems that the power series $g(z)$ with the coefficients given by the recurrence relations (28) has a nonvanishing radius of convergence. The problem of the existence of a real point $z_0$ in the disk of convergence or on its frontier in which $g'(z_0) = 0$ needs further studies. For the KdV equation this point is on the frontier of the disk of convergence of the power series with coefficients given by the relations (28). We mention that $g'(z_0) = 0$ means that the function is not univalent (it is not injective in the vicinity of this point).

An interesting comment concerning an unexpected connection is appropriate: Bieberbach formulated in 1916 the following conjecture: if $f(z) = z + a_2z^2 + ...$ is analytic and univalent in the unit disk, then $|a_n| \leq n$ for all $n$, with equality occuring only for
rotations of the Koebe function \( k(z) = \frac{z}{(1-z)^2} \) [18]. The form of our solution for the KdV equation, eq. (34), is exactly the Koebe function up to a scaling of the \( z \) variable. This conjecture was demonstrated in 1986 by de Branges [19]. This fact suggests that one could obtain information about the possible zeros of the \( g'(z) \) if it is possible to obtain estimates for the convergence radius of the power series with the coefficients given by (30).

The change of the variable which we used to obtain the non-linear q-differential equation (25) can be used in the time-dependent equation obtained from eq.(23). If \( \eta(x,t) = f(v,t) \) where \( v = e^{Bx} \) or \( v = e^{-Bx} \), then we obtain:

\[
2\frac{h}{c_0} v f^\mp_t(v,t) \pm B f^\mp_v(v) \frac{D_q f^{\mp 2}_v(v,t)}{D_q f^\mp_v(v,t)} + B f^\mp(v) \frac{D_q f^{\mp 2}_v(v,t)}{D_q f^\mp_v(v,t)} \pm 2\sin(Bh)D_q f^\mp(v,t) = 0, \tag{38}
\]

where \( \pm \) refers to the positive or negative real semiaxis. If we try a solution of the form \( f^\pm(v,t) = \sum_{k=0}^{\infty} a_k^\pm v^k \), then we obtain the following infinite system of differential equations:

\[
\frac{h}{c_0} \frac{d}{dt} a_k^\pm(t) = \pm \sin(kBh) a_k^\pm(t) + 2B \cos \frac{Bh(n - 1)}{2} \sum_{n=0}^{k} n a_n^\pm(t) a_{k-n}^\pm(t) \cos \left( \frac{(2k - n - 1)Bh}{2} \right). \tag{39}
\]

We note that if \( a_0(t_0) = 0 \) then this relation is preserved in time. The above system can be resolved step by step as the equation for \( a_n \) involves only the coefficients \( a_k \) with \( k \leq n \). However, the smooth behaviour of the solution in \( x = 0 \) at every moment is a nontrivial problem.

## 4 Conclusions

We have shown that the KdV equation describing the shallow liquids can be generalised for any depths and lengths. Consequently the influence of the layer depth could be
included in the present studies on different fields, like clusterization on the nuclear surface in nuclear physics.

The present paper shows two important results: a generalization of the KdV equation starting from a physical model and an embedding of this non-linear equation into a q-deformed formalism.

By using a non-linear hydrodynamic approach for a fluid layer we obtained a dynamical differential equation, of infinite order, which generalizes the KdV equation for shallow water. Contrary to the later one, the generalised equation is valid for any depth and length of the fluid layer. We succeeded to write this equation in a q-differential formalism, and consequently we have obtained a non-linear recurrence relation for the coefficients of its general solution. We stress the importance of the introduction of the surface pressure term, eqs. (8,9), which provides the necessary dispersion term, i.e. $\eta_{xxx}$. This term is essential in two respects: first, it introduces the dispersion in a smaller order than in the traditional case [3], and second, it represents the single term responsible for dispersion in the case of cylindrical geometry [13,14]. Both the generalized KdV equation and its formal solution reach the KdV system in the shallow water limiting case, which gives confidence in the present approach. We conjecture that exists a deep connection between non-linear differential equations, infinite order linear differential equations (which are connected with finite difference equations) and q-differential equations and their q-deformed symmetries. In this sense, in the present paper we succeeded to show that by starting from a one dimensional model for an ideal incompressible irrotational fluid layer, a more general differential equation, than the KdV equation, was obtained. The equation which we proposed not only generalizes the KdV equation, but also can be written as a q-deformed nonlinear equation with a deformation parameter of modulus one. In our case the deformation parameter $q$ is put into relation with the depth of the fluid. To our knowledge this one of the few cases when one can add a physical interpretation to this parameter.

We stress that the result of the Section 3, i.e. the realisation of the GKdV equation as a q-deformed one, could be the starting point to search for more interesting symmetries. Perhaps, there is a more general connection with the q-deformed algebras.
It would be interesting to interpret the q-GKdV eqs. (24,25) as the Casimir element of a certain q-deformed algebra. This possibility could open the way to construct invariants of this equation and to compare them, in the limiting case of shallow waters, with the KdV invariants. Such work is in progress.

We expect many possible application of such a formalism, especially in the field of nuclear clusterization. If the GKdV equation arises from a Lagrangean formalism or from a Hamiltonian formalism, then, one can apply this result in physical models which involve exotic shapes (cluster decay, spontaneous fission, fragmentation of nuclei, atomic clusters, etc).

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