REAL REDUCTIVE CAYLEY GROUPS OF RANK 1 AND 2

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WITH AN APPENDIX BY IGOR DOLGACHEV

Abstract. A linear algebraic group $G$ is over a field $K$ is called a Cayley $K$-group if it admits a Cayley map, i.e., a $G$-equivariant $K$-birational isomorphism between the group variety $G$ and its Lie algebra. We classify real reductive algebraic groups of absolute rank 1 and 2 that are Cayley $R$-groups.

1. Introduction

Let $G$ be a connected linear algebraic group defined over a field $K$, and let $\text{Lie}(G)$ denote its Lie algebra. The following definition is due to Lemire, Popov and Reichstein [LPR]:

Definition 1.1 ([LPR]). A Cayley map for $G$ is a $K$-birational isomorphism $G \stackrel{\sim}{\to} \text{Lie}(G)$ which is $G$-equivariant with respect to the action of $G$ on itself by conjugation and the action of $G$ on $\text{Lie}(G)$ via the adjoint representation. A linear algebraic $K$-group is called a Cayley group if it admits a Cayley map. A linear algebraic $K$-group is called a stably Cayley group if $G \times_k (\mathbb{G}_{m,K})^r$ is Cayley for some $r \geq 0$, where $\mathbb{G}_{m,K}$ denotes the multiplicative group.

Lemire, Popov and Reichstein [LPR] classified Cayley and stably Cayley simple groups over an algebraically closed field $k$ of characteristic 0. Borovoi, Kunyavskii, Lermire and Reichstein [BKL] classified stably Cayley simple $K$-groups over an arbitrary field $K$ of characteristic 0. Clearly any Cayley $K$-group is stably Cayley. In the opposite direction, some of the stably Cayley $K$-groups are known to be Cayley, see [LPR] Examples 1.9, 1.11 and 1.16. For other stably Cayley $K$-groups, it is a difficult problem to determine whether they are Cayley or not. We are interested in classification of reductive groups of rank 1 and 2 that are Cayley groups. By the rank of a reductive $K$-group we always mean the absolute rank.

The case of a simple group of type $G_2$ was settled in [LPR, §9.2] and Iskovskikh’s papers [I3], [I4]. Namely, a simple group of type $G_2$ over an algebraically closed field $k$ of characteristic 0 is not Cayley. Hence no simple $K$-group of type $G_2$ over a field $K$ of characteristic 0 is Cayley.
Popov [P] proved in 1975 that, contrary to what was expected (cf. [L, Remarque, p. 14]), the group $\text{SL}_3$ over an algebraically closed field $k$ of characteristic 0 is Cayley; see [LPR, Appendix] for Popov’s original proof, and [LPR] §9.1 for an alternative proof.

Here we are interested in $\mathbb{R}$-groups. If $G$ is an inner form of a split reductive $\mathbb{R}$-group, and $G_{\mathbb{C}} := G \times_\mathbb{R} \mathbb{C}$ is stably Cayley over $\mathbb{C}$, then by [BKLR, Remark 1.8] $G$ is stably Cayley over $\mathbb{R}$. Similarly, since $\text{SL}_{3,\mathbb{C}}$ is Cayley over $\mathbb{C}$ by Popov’s theorem, one might expect that the split $\mathbb{R}$-group $\text{SL}_{3,\mathbb{R}}$ is Cayley over $\mathbb{R}$. However, this turns out to be false, see Theorem 8.1 of the Appendix. On the other hand, the outer form $\text{SU}_3$ of the split group $\text{SL}_{3,\mathbb{R}}$ is Cayley, see Theorem 7.1 and Corollary 4.4.

In this paper we classify real reductive algebraic groups of rank $\leq 2$ that are Cayley. To be more precise, for each real reductive group of rank 1 or 2 (up to an isomorphism) we determine whether it is Cayley or not:

**Theorem 1.2.** Let $G$ be connected reductive algebraic $\mathbb{R}$-group of absolute rank $\leq 2$ over the field $\mathbb{R}$ of real numbers. If $G$ is simple of type $\text{G}_2$ or is isomorphic to $\text{SL}_{3,\mathbb{R}}$, $\text{PGU}_3$, or $\text{PGU}(2,1)$, then $G$ is not Cayley. Otherwise $G$ is Cayley.

Here, following the Book of Involutions [KMRT, §23], we write $\text{PGU}_n$ rather than $\text{PSU}_n$ for the corresponding adjoint group. We write $\text{PGU}(2,1)$ for the (inner) form of $\text{PGU}_3$ corresponding to the Hermitian form with diagonal matrix $\text{diag}(1,1,-1)$.

Note that by [BKLR] Corollary 7.1 any $K$-group $G$ of absolute rank $\leq 2$ over a field $K$ of characteristic 0 is stably Cayley, that is, there exists $r \geq 0$ such that the group $G \times_K \mathbb{G}_m^r$ is Cayley. The following theorem shows that one can always take $r = 2$.

**Theorem 1.3.** Let $G$ be a connected reductive algebraic $K$-group of absolute rank $\leq 2$ over a field $K$ of characteristic 0. If $G$ is of absolute rank 1, then $G$ is Cayley. If $G$ is of absolute rank 2, then $G \times_K \mathbb{G}_m^2$ is Cayley.

**Question 1.4.** Let $G$ be a reductive $\mathbb{R}$-group of absolute rank 2 which is not Cayley, for example $\text{SL}_{3,\mathbb{R}}$. Is $G \times_\mathbb{R} \mathbb{G}_m$ a Cayley group?

**Question 1.5.** Are the $\mathbb{R}$-groups $\text{PGU}_{2n+1}$ Cayley for $n \geq 2$? (Note that these $\mathbb{R}$-groups are stably Cayley, see [BKLR, Thm. 1.4].)

The plan of the rest of the paper is as follows. In Section 2 we reproduce some examples of Cayley groups from [LPR], and state some known properties of Cayley groups. In Section 3 we treat easy cases of Theorem 1.2. In Section 4 we treat the more difficult case $\text{SU}_3$ of Theorem 1.2 using explicit calculations. In Section 5 we prove Theorem 1.3 (case by case). In the Appendix, Igor Dolgachev treats the most difficult cases $\text{SL}_{3,\mathbb{R}}$ and $\text{PGU}_3$ of Theorem 1.2 (and again the case $\text{SU}_3$), using the theory of elementary links due to Iskovskikh [12], [13], [14]. (Note that, since $\text{PGU}(2,1)$ is an inner...
form of $\text{PGU}_3$, the case $\text{PGU}(2,1)$ reduces to $\text{PGU}_3$, cf. [BKLR] Lemma 5.4(c).)

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2. Preliminary remarks

We reproduce some examples from [LPR]. Note that in [LPR] it is always assumed that the characteristic of $K$ is zero, while we attempt to state these results assuming that $K$ is a field of arbitrary characteristic.

Example 2.1 (Cf. [LPR] Ex. 1.9). Consider a finite-dimensional associative $K$-algebra $A$ with unit element $1$, over a field $K$ of arbitrary characteristic, and the $K$-group $A^\times$ of invertible elements of $A$. Then clearly $A^\times$ is Cayley. In particular, the $K$-group $\text{GL}_{n,K}$ is Cayley.

Example 2.2 (Cf. [LPR] Ex. 1.11). Let $A$ be a central simple $K$-algebra of dimension $n^2$, and assume that $\text{char}(K)$ does not divide $n$. For any element $a \in A$ denote by $\text{tr} a$ the trace of the linear operator $L_a$ of left multiplication by $a$ in $A$. Then $\text{tr} 1 = n^2 \neq 0 \in K$. The argument in [LPR] shows that the quotient group $A^\times / \mathbb{G}_{m,K}$ is Cayley.

We see that when $\text{char}(K)$ does not divide $n$, the group $\text{PGL}_{n,K}$ is Cayley. In particular, if $\text{char}(K) \neq 2$, then $\text{PGL}_{2,K}$ is Cayley, and if $\text{char}(K) \neq 3$, then $\text{PGL}_{3,K}$ is Cayley.

Example 2.3 (Cf. [LPR] Ex. 1.16], [W, p. 599]). Let $K$ and $A$ be as in Example 2.1, and let $\iota$ be a involution (over $K$) of $A$. Set

$$G = \{a \in A^\times \mid a^\prime a = 1\}^0,$$

where $S^0$ denotes the identity component of an algebraic group $S$. Then the $K$-group $G$ is Cayley.

We see that if $L/K$ is a separable quadratic extension, then the group $\text{U}_{n,L/K}$ of $n \times n$ unitary matrices in $M_n(L)$ is Cayley over $K$; that the group $\text{Sp}_{2n,K}$ is Cayley over $K$, in particular, $\text{SL}_{2,K} \simeq \text{Sp}_{2,K}$ is Cayley; that the group $\text{SO}(m,n)$ is Cayley over $K$, in particular, the groups $\text{PGL}_{2,K} \simeq \text{SO}(2,1)$ and $\text{Sp}_{4,K}/\mu_{2,K} \simeq \text{SO}(3,2)$ are Cayley. Here we write $\text{SO}(m,n)$ for the special orthogonal group over $K$ of the diagonal quadratic form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$.

We state some known properties of Cayley groups.

Remark 2.4. If $G_1$ and $G_2$ are Cayley $K$-groups over an arbitrary field $K$, then evidently $G_1 \times_K G_2$ is a Cayley $K$-group.

Remark 2.5. If $L/K$ is a finite separable field extension and $H$ is a Cayley $L$-group, then evidently the Weil restriction $R_{L/K}H$ is a Cayley $K$-group.

Remark 2.6. If $G$ is a Cayley $K$-group over an arbitrary field $K$, and $L/K$ is an arbitrary field extension, then $G \times_K L$ is evidently a Cayley $L$-group.
Proposition 2.7 ([BKLR, Lemma 5.4(c)]). If $G$ is a Cayley $K$-group over an arbitrary field $K$, then all the inner forms of $G$ are Cayley. In particular, if all the automorphisms of $G$ are inner, then all the twisted forms of $G$ are inner forms, hence they all are Cayley $K$-groups.

Proposition 2.8 ([BKLR, Corollary 6.5]). Let $G$ be a reductive $K$-group over a field $K$ of characteristic 0, and let $T \subset G$ be a maximal $K$-torus. Then $G$ is Cayley if and only if there exists a $W(G,T)$-equivariant birational isomorphism $T \to \text{Lie}(T)$ defined over $K$, where the Weyl group $W(G,T)$ is viewed as an algebraic $K$-group.

Note that the proof of this (difficult) result uses [CKLR], where it is assumed that $\text{char}(K) = 0$.

3. Proof of Theorem 1.2, easy cases

We start proving Theorem 1.2 case by case.

Proposition 3.1. Any connected reductive $K$-group $G$ of (absolute) rank 1 over a field $K$ of characteristic $\neq 2,3$ is a Cayley group.

Proof. If $G$ is a torus of rank 1, then $G$ is $K$-rational, see e.g. [V, § 4.9, Example 6], hence it is Cayley over $K$. If $G$ is not a torus, then $G$ is simple of rank 1, hence $G$ is a twisted form of one of the groups $\text{SL}_2,K$, $\text{PGL}_2,K$. Both these groups are Cayley over $K$, see Example 2.3. Since all the automorphisms of $\text{SL}_2,K$ and $\text{PGL}_2,K$ are inner, by Proposition 2.7 $G$ is Cayley. □

Proposition 3.2. Any connected, reductive and not semisimple $K$-group $G$ of absolute rank 2 over a field $K$ of characteristic $\neq 2,3$ is a Cayley group.

Proof. If $G$ is a torus of rank 2, then $G$ is $K$-rational, see [V, § 4.9, Example 7], hence it is Cayley over $K$. If $G$ is not a torus, denote by $R := Z(G)^0$ its radical and by $G^{\text{der}} := [G,G]$ its commutator subgroup. The multiplication in $G$ gives a canonical epimorphism $\pi: R \times_K G^{\text{der}} \to G$ with kernel isomorphic to $R \cap G^{\text{der}}$.

If this epimorphism is an isomorphism, then $G$ is isomorphic to the product of two $K$-groups $R$ and $G^{\text{der}}$ of rank 1. By Proposition 3.1 $R$ and $G^{\text{der}}$ are Cayley over $K$, hence by Remark 2.4 $G$ is Cayley.

If the epimorphism $\pi: R \times_K G^{\text{der}} \to G$ is not an isomorphism, then $R \cap G^{\text{der}} = Z(G^{\text{der}})$, and $G$ is a $K$-form of $\text{GL}_2$. The epimorphism $\det: \text{GL}_2 \to \mathbb{G}_m$ induces a canonical isomorphism

$$\text{Out}(\text{GL}_2) \to \text{Aut}(\mathbb{G}_m),$$

where we write $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ for the group of outer isomorphisms of $G$. This canonical isomorphism induces a bijection between the set of $K$-forms of $\text{GL}_2$ up to inner twisting, and the set of $K$-form of $\mathbb{G}_m$ up to an isomorphism, as follows: $[G] \mapsto [G^{\text{tor}}]$, where $G^{\text{tor}} := G/G^{\text{der}}$. Under this bijection, $\text{GL}_2$ corresponds to $\mathbb{G}_m$, and for any separable quadratic extension $L/K$, the unitary group $U_{2,L/K}$ corresponds to $U_{1,L/K}$. Since we get
all the $K$-forms of $G_m$ in this way, we conclude that we have written down all the $K$-forms of $GL_2$, up to inner twisting. Since all these $K$-groups, $GL_2,K$ and $U_2,L/K$, are Cayley, see Examples 2.1 and 2.3, we conclude, using Proposition 2.7, that all the $K$-forms of $GL_2$ are Cayley, hence $G$ is Cayley.

Proposition 3.3. Any connected semisimple $K$-group $G$ of absolute rank 2 of type $A_1 \times A_1$ over a field $K$ of characteristic $\neq 2, 3$ is a Cayley group.

Proof. In this case the group $G$ decomposes into an almost direct product of two groups of type $A_1$ defined either over $K$ or over a separable quadratic extension $L$ of $K$. If this almost direct product is direct, then $G$ is either a direct product of two simple $K$-groups of type $A_1$, and hence Cayley by Proposition 3.1 and Remark 2.4, or $G$ is of the form $R_{L/K}G'$, where $G'$ is a simple $L$-group of type $A_1$, and we conclude by Proposition 3.1 that $G'$ is Cayley over $L$, and conclude by Remark 2.5 that $G$ is Cayley over $K$. If this almost direct product is not direct, then $G$ is a twisted form of $SO_4$, hence $G$ is an inner form of a special orthogonal group of the form $SO(K^4, q)$ for some nondegenerate quadratic form $q$ in 4 variables, and $G$ is Cayley by Example 2.3 and Proposition 2.7.

Proposition 3.4. Any connected simple $K$-group $G$ of absolute rank 2 of type $B_2 = C_2$ over a field $K$ of characteristic $\neq 2, 3$ is a Cayley group.

Proof. In this case $G$ is an (inner) twisted form of one of the $K$-groups $Sp_{4,K}$ and $Sp_{4,K}/\mu_2,K$. Both these groups are Cayley by Example 2.3 and using Proposition 2.7 we conclude that $G$ is Cayley.

Proposition 3.5. Any connected simple $K$-group of absolute rank 2 of type $G_2$ over a field $K$ of characteristic 0 is not Cayley.

Proof. This case was settled in [LPR, §9.2] and Iskovskikh’s papers [I3], [I4], see our Introduction.

The following two propositions are not easy. We assume that $K = \mathbb{R}$.

Proposition 3.6. Let $G$ be connected simple $\mathbb{R}$-group of absolute rank 2 of type $A_2$. If $G$ is isomorphic to $PGL_{3,\mathbb{R}}$, or $SU_3$, or $SU(2,1)$, then $G$ is a Cayley group.

The group $PGL_{3,\mathbb{R}}$ is Cayley by Example 2.2, while the group $SU_3$ will be treated in the next section, see Corollary 4.3 and also in the Appendix, see Theorem 7.1 Note that, since the group $SU(2,1)$ is an inner form of $SU_3$, the case $SU(2,1)$ reduces to $SU_3$, see Proposition 2.7.

Proposition 3.7. Let $G$ be connected simple $\mathbb{R}$-group of absolute rank 2 of type $A_2$. If $G$ is isomorphic to $SL_{3,\mathbb{R}}$, or $PGU_3$, or $PGU(2,1)$, then $G$ is not Cayley.

These difficult cases will be treated in the Appendix, see Theorems 7.2 and 8.1 Note that, since the group $PGU(2,1)$ is an inner form of $PGU_3$, the case $PGU(2,1)$ reduces to $PGU_3$, see Proposition 2.7.
Proof of Theorem 4.2. Theorem 4.2 follows from Propositions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, and 3.7. Note that this proof uses Corollary 4.4 and Theorem 4.1 via Proposition 3.6 and it uses Theorems 7.2 and 8.1 via Proposition 3.7.

4. The group SU$_3$

4.1. Let $W$ be a finite group. Let $L/K$ be a finite Galois extension with Galois group $\Gamma = \text{Gal}(L/K)$. We shall consider $W$-varieties defined over $K$ and $(W, \Gamma)$-varieties defined over $L$. By a $W$-variety defined over $K$ we mean a $K$-variety $X$ with a $W$-action $W \to \text{Aut}(X)$. By a semilinear action of $\Gamma$ on an $L$-variety $Y$ we mean a homomorphism $\rho: \Gamma \to \text{SAut}_{L/K}(Y)$ into the group $\text{SAut}_{L/K}(Y)$ of $L/K$-semilinear automorphisms of $Y$, such that $\rho(\gamma)$ is a $\gamma$-semilinear automorphism of $Y$ for any $\gamma \in \Gamma$ (see [B, §1.1] and [FSS, §1.2] for definitions of semilinear automorphisms). By a $(W, \Gamma)$-variety defined over $L$ we mean an $L$-variety $Y$ with two commuting actions: an $L$-action of $W$ and a semilinear action of $\Gamma$. One defines morphisms and rational maps of $(W, \Gamma)$-varieties. We have a base change functor $X \mapsto X \times_K L$ from the category of $W$-varieties over $K$ to the category of $(W, \Gamma)$-varieties over $L$, and it is well known that this functor is fully faithful, i.e., the natural map

$$\text{Hom}_W(X, X') \to \text{Hom}_{(W, \Gamma)}(X \times_K L, X' \times_K L)$$

is bijective for any two $W$-varieties $X, X'$ defined over $K$. Similarly, $W$-varieties $X$ and $X'$ over $K$ are $W$-equivariantly birationally isomorphic over $K$ if and only if $X \times_K L$ and $X' \times_K L$ are $(W, \Gamma)$-equivariantly birationally isomorphic over $L$. Note that, by Galois descent (see Serre [S, Ch. V.20, Cor. 2 of Prop. 12]), any quasi-projective $(W, \Gamma)$-variety over $L$ comes from a $W$-variety over $K$; we shall not use this fact, however.

4.2. Let $K$ be a field of characteristic 0. Assume that $K$ does not contain non-trivial roots of unity of order 3. Set $L = K(\zeta)$, where $\zeta^3 = 1$, $\zeta \neq 1$. We can also write $L = K(\sqrt{-3})$. (For example, one can take $K = \mathbb{R}$, $L = \mathbb{R}(\sqrt{-3}) = \mathbb{C}$.) We set $\Gamma = \text{Gal}(L/K)$, $\Gamma = \{\text{id}, \gamma\}$, and we write the action of $\gamma$ on $a \in L$ as $a \mapsto \gamma a$.

Let $G = \text{SU}(3, L/K, H) := \text{SU}(L^3, H)$, the special unitary group of the $L/K$-Hermitian form with matrix $H$, where $H \in M_3(L)$ is a nondegenerate $3 \times 3$ Hermitian matrix. Then $G$ is a simple $K$-group, an outer $L/K$-form of the split $K$-group $\text{SL}_{3,K}$. Note that $G = \text{SU}(3, L/K, H)$ is an inner form of the $K$-group $\text{SU}_{3,L/K} := \text{SU}(3, L/K, I_3)$, where $I_3 = \text{diag}(1,1,1)$.

Theorem 4.3. Let a quadratic field extension $L/K$ and an Hermitian matrix $H \in M_3(L)$ be as in §4.2. Then the $K$-group $G = \text{SU}(3, L/K, H)$ is Cayley.

Theorem 4.3 will be proved below.

Corollary 4.4. The $\mathbb{R}$-groups $\text{SU}_3$ and $\text{SU}(2,1)$ are Cayley. □
4.5. Let $K, L$ be as in §4.2. Consider the torus $G^3_{m,K}$ and write the standard action of $\mathfrak{g}$ on it, given by:

\[(4.1) \quad \sigma(x_1, x_2, x_3) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \quad \text{for} \quad \sigma \in \mathfrak{g}.
\]

We consider the $K$-subtorus

\[T := \{(x_1, x_2, x_3) \in G^3_{m,K} \mid x_1x_2x_3 = 1\}
\]

and we set $t = \text{Lie}(T)$.

We set $T_L = T \times_K L$, $t_L = \text{Lie}(T_L) = t \otimes_K L$, then

\[t_L = \{(x_1, x_2, x_3) \in L^3 \mid x_1 + x_2 + x_3 = 0\}.
\]

The group $\mathfrak{g}$ acts on $T_L$ and $t_L$ by formula (4.1), and $\Gamma$ acts by

\[\gamma(x_1, x_2, x_3) = (\gamma x_1, \gamma x_2, \gamma x_3).
\]

We consider also the $\Gamma$-twisted $(\mathfrak{g}, \Gamma)$-varieties $T'_L$ and $t'_L$: the same $L$-varieties $T_L$ and $t_L$ with the same $\mathfrak{g}$-actions, but with the twisted actions of $\gamma$:

\[(x_1, x_2, x_3) \mapsto (\gamma x_1^{-1}, \gamma x_2^{-1}, \gamma x_3^{-1}) \quad \text{for} \quad T'_L,
\]

\[(x_1, x_2, x_3) \mapsto (-\gamma x_1, -\gamma x_2, -\gamma x_3) \quad \text{for} \quad t'_L.
\]

These $(\mathfrak{g}, \Gamma)$-varieties over $L$ come from some $\mathfrak{g}$-varieties $T'$ and $t'$ defined over $K$ which are easy to describe, see below.

4.6. Let $T_{SU_3}$ denote the diagonal maximal torus of $SU_{3,L/K}$, and let $t_{SU_3}$ denote its Lie algebra. Let $N_{SU_3}$ denote the normalizer of $T_{SU_3}$ in $SU_{3,L/K}$, and set $W = N_{SU_3}/T_{SU_3}$. The finite algebraic group $W$ is canonically isomorphic to the symmetric group $\mathfrak{g}$ with trivial Galois action. We see that $T_{SU_3}$ and $t_{SU_3}$ are $\mathfrak{g}$-varieties over $K$. Furthermore, it is well known that $T_{SU_3} \times_K L$ is canonically isomorphic to $T'_L$ and that $t_{SU_3} \otimes_K L$ is canonically isomorphic to $t'_L$ as $(\mathfrak{g}, \Gamma)$-varieties. Therefore we set

\[T' := T_{SU_3}, \quad t' := t_{SU_3}.
\]

Proposition 4.7. Let $K$ be a field of characteristic 0. We assume that $K$ contains no nontrivial cube root of 1, and we set $L = K(\zeta)$, where $\zeta^3 = 1$, $\zeta \neq 1$. Then the $(\mathfrak{g}, \Gamma)$-varieties $T'_L$ and $t'_L$ are $(\mathfrak{g}, \Gamma)$-equivariantly birationally isomorphic over $L$.

4.8. Reduction of Theorem 4.3 to Proposition 4.7. Since our group $SU(3, L/K, H)$ is an inner form of $SU_{3,L/K}$, by Proposition 2.7 in order to prove that the group $SU(3, L/K, H)$ is Cayley, it suffices to prove that $SU_{3,L/K}$ is Cayley. By Proposition 2.8 the group $SU_{3,L/K}$ is Cayley if and only if the $\mathfrak{g}$-varieties $T' = T_{SU_3}$ and $t' = t_{SU_3}$ are $\mathfrak{g}$-equivariantly birationally isomorphic over $K$. The discussion in §4.1 shows that they are $\mathfrak{g}$-equivariantly birationally isomorphic over $K$ if and only if the $(\mathfrak{g}, \Gamma)$-varieties $T'_L$ and $t'_L$ are $(\mathfrak{g}, \Gamma)$-equivariantly birationally isomorphic over $L$. Therefore, Theorem 4.3 follows from Proposition 4.7.
We give here a proof of Proposition 4.7 which is close to the proof of Proposition 9.1 in [LPR]. For an alternative proof (in the case $K = \mathbb{R}$) see the Appendix, Theorem 7.1.

4.9. We consider the variety $(\mathbb{G}_m^3/\mathbb{G}_m^3)_{m,L}(\mathcal{S}_3, \Gamma)$-twisted, which is just $\mathbb{G}_{m,L}^3/\mathbb{G}_m^3$ (with $\mathbb{G}_{m,L}^3$ imbedded diagonally in $\mathbb{G}_m^3$) with the following (twisted) $\mathcal{S}_3$-action and twisted $\Gamma$-action:

$$
\sigma([x]) = [\sigma(x)^{\text{sign} \sigma}], \quad \gamma([x]) = [\gamma x^{-1}] \quad \text{for} \quad x \in \mathbb{G}_{m,L}^3, \quad \sigma \in \mathcal{S}_3.
$$

Here we write $[x] \in \mathbb{G}_{m,L}^3/\mathbb{G}_m^3$ for the class of $x \in \mathbb{G}_{m,L}^3$. We have an $(\mathcal{S}_3, \Gamma)$-equivariant isomorphism

$$(\mathbb{G}_m^3/\mathbb{G}_m^3)_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted} \sim T', \quad [x_1, x_2, x_3] \mapsto (x_2/x_3, x_3/x_1, x_1/x_2).$$

It remains to prove that $(\mathbb{G}_m^3/\mathbb{G}_m^3)_{m,L}(\mathcal{S}_3, \Gamma)$-twisted is $(\mathcal{S}_3, \Gamma)$-equivariantly birationally isomorphic to $t'_L$.

4.10. Consider the following (twisted) $\mathcal{S}_3$-action and twisted $\Gamma$-action on the set $t_L \times t_L$:

$$
\sigma(x, y) := \begin{cases} 
(\sigma(x), \sigma(y)) & \text{if } \sigma \text{ is even}, \\
(\sigma(y), \sigma(x)) & \text{if } \sigma \text{ is odd},
\end{cases} \quad \text{where } \sigma \in \mathcal{S}_3, \quad x, y \in t_L,
$$

$$
\gamma(x, y) := ([y], [\gamma x]).
$$

These actions of $\mathcal{S}_3$ and $\Gamma$ on $t_L \times t_L$ induce actions on the surface $\mathbb{P}(t_L) \times_L \mathbb{P}(t_L)$, on the tensor product $t_L \otimes_L t_L$ and on the 3-dimensional projective space $\mathbb{P}(t_L \otimes_L t_L)$, and we write

$$(\mathbb{P}(t_L) \times_L \mathbb{P}(t_L))_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted}, \quad (t_L \otimes_L t_L)_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted} \quad \text{and} \quad \mathbb{P}(t_L \otimes_L t_L)_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted}$$

for the corresponding $(\mathcal{S}_3, \Gamma)$-varieties.

4.11. We claim that the $(\mathcal{S}_3, \Gamma)$-varieties $(\mathbb{G}_m^3/\mathbb{G}_m^3)_{m,L}(\mathcal{S}_3, \Gamma)$-twisted and $(\mathbb{P}(t_L) \times_L \mathbb{P}(t_L))_{m,L}(\mathcal{S}_3, \Gamma)$-twisted are $(\mathcal{S}_3, \Gamma)$-equivariantly birationally isomorphic. We write $[t] \in \mathbb{P}(t_L)$ for the class of $t \in t_L$. Consider the rational map

$$
\varphi : (\mathbb{G}_m^3/\mathbb{G}_m^3)_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted} \dashrightarrow (\mathbb{P}(t_L) \times_L \mathbb{P}(t_L))_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted}
$$

where $\varphi([x]) = ([x - \tau(x)\mathbf{1}_3], [x^{-1} - \tau(x^{-1})\mathbf{1}_3])$, and $\varphi$ was constructed in [LPR, Proof of Prop. 9.1, Step 1]. Thus $\varphi$ is an $(\mathcal{S}_3, \Gamma)$-equivariant birational isomorphism.

4.12. Consider the Segre embedding

$$
(\mathbb{P}(t_L) \times_L \mathbb{P}(t_L))_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted} \hookrightarrow \mathbb{P}(t_L \otimes_L t_L)_{m,L}(\mathcal{S}_3, \Gamma)-\text{twisted}
$$

given by $([x], [y]) \mapsto [x \otimes y]$, it is $(\mathcal{S}_3, \Gamma)$-equivariant. Its image is a quadric

$Q$ in $\mathbb{P}(t_L \otimes_L t_L)$ described as follows. Choose a basis $D_1 := \text{diag}(1, \zeta, \zeta^2)$. 

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$D_2 := \text{diag}(1, \zeta^2, \zeta)$ of $t_L$, where $\zeta$ is our primitive cube root of unity. Set $D_{ij} = D_i \otimes D_j$. Then

$$Q = \{(\alpha_{11} : \alpha_{12} : \alpha_{21} : \alpha_{22}) \mid \alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}\},$$

where $(\alpha_{11} : \alpha_{12} : \alpha_{21} : \alpha_{22})$ is the point of $\mathbb{P}(t_L \otimes_L t_L)$ corresponding to

$$\alpha_{11}D_{11} + \alpha_{12}D_{12} + \alpha_{21}D_{21} + \alpha_{22}D_{22} \in t_L \otimes_L t_L.$$

4.13. We denote by $V_{11,22}$ the 2-dimensional subspace in $(t_L \otimes_L t_L)_{(\mathcal{S}_3, \Gamma)}$-twisted with the basis $D_{11}, D_{22}$, and we denote by $V_{12}$ and $V_{21}$ the one-dimensional subspaces generated by $D_{12}$ and $D_{21}$, respectively. An easy calculation shows that the subspace $V_{11,22}$ is $\mathcal{S}_3$-invariant and $\Gamma$-invariant, and that the basis vectors $D_{12}$ and $D_{21}$ are $\mathcal{S}_3$-fixed and $\Gamma$-fixed.

Consider the stereographic projection $Q \rightarrow \mathbb{P}(V_{11,22} \oplus V_{12})$ from the $(\mathcal{S}_3, \Gamma)$-fixed $L$-point $x_{21} := [D_{21}] = (0 : 0 : 1 : 0) \in Q(L)$ to the $(\mathcal{S}_3, \Gamma)$-invariant plane $\mathbb{P}(V_{11,22} \oplus V_{12})$. This stereographic projection is a $(\mathcal{S}_3, \Gamma)$-equivariant birational isomorphism. Furthermore, the embedding

$$V_{11,22} \hookrightarrow \mathbb{P}(V_{11,22} \oplus V_{12}), \quad x \mapsto [x + D_{12}]$$

is an $(\mathcal{S}_3, \Gamma)$-equivariant birational isomorphism. Thus the quadric $Q$ is $(\mathcal{S}_3, \Gamma)$-equivariantly birationally isomorphic to the vector space $V_{11,22}$. Since the 2-dimensional $(\mathcal{S}_3, \Gamma)$-vector spaces $V_{11,22}$ and $t_L$ are isomorphic (the map of bases $D_{11} \mapsto D_2, D_{22} \mapsto D_1$ induces an $(\mathcal{S}_3, \Gamma)$-isomorphism $V_{11,22} \sim t_L$), and $t_L$ is isomorphic to $t'_L$ (an isomorphism is given by $(x_i) \mapsto (\sqrt{-3}x_i)$), we conclude that $Q$ is $(\mathcal{S}_3, \Gamma)$-equivariantly birationally isomorphic to $t'_L$.

Thus $T'_L$ is $(\mathcal{S}_3, \Gamma)$-equivariantly birationally isomorphic to $t'_L$. This completes the proofs of Proposition 5.1, Theorem 5.3 and Corollary 4.4. \qedhere

5. The groups $G \times \mathbb{G}_m^2$

In this section we prove Theorem 1.3. Let $K$ be a field of characteristic 0, and let $\overline{K}$ be a fixed algebraic closure of $K$.

Let $G_{2,K}$ denote the split $K$-group of type $G_2$. Since by [LPR] Proposition 9.10, the group $G_{2,\overline{K}}$ is not Cayley over $\overline{K}$, we see that $G_{2,K}$ is not Cayley.

**Proposition 5.1.** For any field $K$ of characteristic 0, the split $K$-group $G_{2,K} \times_K \mathbb{G}_m^2$ is Cayley.

**Corollary 5.2.** For any $K'$-group $G$ of type $G_2$ over a field $K'$ of characteristic 0, the $K'$-group $G \times_K \mathbb{G}_m^2$ is Cayley.

**Proof.** Since $G \times_K \mathbb{G}_m^2$ is an inner form of $G_{2,K} \times_K \mathbb{G}_m^2$, by Proposition 2.7 the corollary follows from Proposition 5.1. \qedhere

5.3. Let $K$ be a field of characteristic 0. We define a $K$-torus $T$ by

$$T := \{(x_1, x_2, x_3) \in \mathbb{G}_m^3, x_1x_2x_3 = 1\}.$$

We define a $K$-action of $\mathcal{S}_3$ on $T$ by

$$\sigma(x_1, x_2, x_3) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \quad \text{for} \quad \sigma \in \mathcal{S}_3.$$
We define a $K$-action of $S_2$ on $T$ by

$$\varepsilon(t) = t^{-1} \text{ for } t \in T,$$

where $\varepsilon$ is the nontrivial element of $S_2$. We obtain a $K$-action of $S_3 \times S_2$ on $T$. Set $t = \text{Lie}(T)$, then $S_3 \times S_2$ acts on $t$. We may regard $T$ as a split maximal torus of $G_{2,K}$, and $S_3 \times S_2$ as the corresponding Weyl group, then $T \times K G^2_{m,K}$ is a maximal torus of $G_{2,K} \times K G^2_{m,K}$.

**Proposition 5.4** ([LPR]). For an arbitrary field $K$ of characteristic 0, the $K$-varieties $T \times K G^2_{m,K}$ and $t \times K A^2_K$ are $S_3 \times S_2$-equivariantly birationally isomorphic over $K$.

**Proof.** This is proved in [LPR] in the proof of Proposition 9.11. The authors assume that $K$ is an algebraically closed field of characteristic 0, but the proof goes through for any field $K$ of characteristic $\neq 2, 3$. □

**Proof of Proposition 5.1.** By Proposition 2.8, our proposition follows from Proposition 5.4. □

**Corollary 5.5.** The $K$-varieties $T \times K G^2_{m,K}$ and $t \times K A^2_K$ of Proposition 5.4 are $S_3 \times S_2$-equivariantly birationally isomorphic over $K$ (with respect to the standard embedding $S_3 \hookrightarrow S_3 \times S_2$).

**Proof.** The $S_3 \times S_2$-equivariant birational isomorphism of Proposition 5.4 is, in particular, $S_3$-equivariant. □

**Proposition 5.6.** For any field $K$ of characteristic 0, the $K$-group $\text{SL}_{3,K} \times K G^2_{m,K}$ is Cayley.

**Proof.** We regard $T$ as a split maximal torus of $\text{SL}_{3,K}$ and $S_3$ as the corresponding Weyl group, then $T \times K G^2_{m,K}$ is a maximal torus of $\text{SL}_{3,K} \times K G^2_{m,K}$. Now by Proposition 2.8, our proposition follows from Corollary 5.5. □

The following lemma is a version of [BKLR, Lemma 5.4(a)].

**Lemma 5.7.** Let $G$ be a reductive $K$-group and $M$ be a $K$-group acting on $G$, over a field $K$ of characteristic 0. Consider the induced action of $M$ on $\text{Lie}(G)$. Let $L/K$ be a Galois extension, and $c: \text{Gal}(L/K) \to M(L)$ be a cocycle. Assume that there exists an $M$-equivariant birational isomorphism $f: G \dashrightarrow \text{Lie}(G)$ over $K$. Then there exists a $cM$-equivariant birational isomorphism of the twisted varieties $\tilde{c}f: \tilde{c}G \dashrightarrow \text{Lie}(\tilde{c}G)$, where $cM$ is the twisted group.

**Proof.** Similar to that of [BKLR, Lemma 5.4(a)]. □

**5.8.** Let $T$ be the $S_3 \times S_2$-torus over $K$ of Section 5.3. Let $L/K$ be an arbitrary quadratic extension. Write $\Gamma = \text{Gal}(L/K) = \{1, \gamma\}$. Define a cocycle (homomorphism)

$$c: \Gamma \to S_3 \times S_2$$
Let $S$ be an arbitrary quadratic extension of fields of characteristic $0$. Let $G := SU(3, L/K, H) := SU(L^3, H)$, the special unitary group of the $L/K$-Hermitian form with matrix $H$, where $H \in M_3(L)$ is a nondegenerate Hermitian matrix. Then $G$ is a simple $K$-group, an outer $L/K$-form of the split $K$-group $SL_{3,K}$. Note that $G = SU(3, L/K, H)$ is an inner form of the $K$-group $SU_{3,L/K} := SU(3, L/K, I_3)$, where $I_3 = diag(1,1,1)$.

**Proposition 5.13.** Let a quadratic extension $L/K$ and a Hermitian matrix $H \in M_3(L)$ be as in §5.12. Let $G = SU(3, L/K, H)$, then $G \times_K \mathbb{G}_{m,K}^2$ is Cayley.

**Proof.** Since $G$ is an inner form of $SU_3 := SU(3, L/K, I_3)$, by Proposition 2.7 it suffices to consider the case $SU_3$. Let $T_{SU_3}$ denote the diagonal maximal torus of $SU_3$, we can identify it with the torus $SU_3$ of Corollary 5.11. Now our proposition follows from Corollary 5.11 and Proposition 2.8.

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Let $G$ be a group. Then $S$ is, in particular, $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$. Then $G$ exists a birational $\mathbb{G}_{m,L}^2$-isomorphism between the $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-varieties $G_{\mathbb{G}_{m,L}^2}^2$ and $L/K$.

**Proposition 5.9.** There exists a birational $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-isomorphism between the $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-varieties $G_{\mathbb{G}_{m,L}^2}^2$ and $L/K$.

**Proof.** This follows from Proposition 5.4 and Lemma 5.7.

5.10. Let $G$ be a group. Then $S$ is, in particular, $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$. Then $G$ exists a birational $\mathbb{G}_{m,L}^2$-isomorphism between the $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-varieties $G_{\mathbb{G}_{m,L}^2}^2$ and $L/K$.

**Proposition 5.9.** There exists a birational $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-isomorphism between the $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-varieties $G_{\mathbb{G}_{m,L}^2}^2$ and $L/K$.

**Proof.** This follows from Proposition 5.4 and Lemma 5.7.

5.10. Let $G$ be a group. Then $S$ is, in particular, $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$. Then $G$ exists a birational $\mathbb{G}_{m,L}^2$-isomorphism between the $(\mathbb{G}_{m,L}^2, \mathbb{A}_L^2)$-varieties $G_{\mathbb{G}_{m,L}^2}^2$ and $L/K$.
Proposition 5.14. Let a quadratic extension \(L/K\) and a Hermitian matrix \(H \in M_3(L)\) be as in §5.12. Let \(G = \PGU(3, L/K, H)\) be the adjoint \(K\)-group corresponding to the simply connected \(K\)-group \(SU(3, L/K, H)\). Then \(G \times_K \mathbb{G}^2_{m,K}\) is Cayley.

Proof. Since \(G\) is an inner form of \(\PGU_3 := \PGU(3, L/K, I_3)\), by Proposition 2.7 it suffices to consider the case \(\PGU_3\). Let \(T_{\PGU_3} \subset \PGU_3\) denote the image of the diagonal maximal torus of \(SU_3\), we can identify the corresponding \(L\)-torus \(T_{\PGU_3} \times_K L\) with the torus \((\mathbb{G}^3_{m,L}/\mathbb{G}^{m,L})\Gamma\) twisted endowed with the following actions of \(S_3\) and \(\Gamma\):

\[
\sigma([x_1, x_2, x_3]) := [x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}] \quad \text{for } \sigma \in S_3 \\
\gamma[x_1, x_2, x_3] = [\gamma x_1^{-1}, \gamma x_2^{-1}, \gamma x_3^{-1}].
\]

We define a homomorphism \((\mathbb{G}^3_{m,L}/\mathbb{G}^{m,L}) \Gamma\) twisted \(\sim \) \(T_L\) by

\[
[x_1, x_2, x_3] \mapsto (x_2/x_3, x_3/x_1, x_1/x_2).
\]

One checks immediately that we obtain an \((\mathcal{S}_3, \Gamma)\)-equivariant isomorphism

\[(\mathbb{G}^3_{m,L}/\mathbb{G}^{m,L}) \Gamma\) twisted \(\sim \) \(T_L\),

and its differential, which is also an \((\mathcal{S}_3, \Gamma)\)-equivariant isomorphism,

\[\text{Lie}(\mathbb{G}^3_{m,L}/\mathbb{G}^{m,L}) \Gamma\) twisted \(\sim \) \(\text{Lie} T_L\).

By Corollary 5.11 there exists an \((\mathcal{S}_3, \Gamma)\)-equivariant birational isomorphism

\[T_L \times_L \mathbb{G}^2_{m,L} \sim \) \(\text{Lie} T_L \times L \mathbb{A}^2_{L}\).

Combining these birational isomorphisms, we obtain an \((\mathcal{S}_3, \Gamma)\)-equivariant birational isomorphism

\[(\mathbb{G}^3_{m,L}/\mathbb{G}^{m,L}) \Gamma\) twisted \(\times_L \mathbb{G}^2_{m,L} \sim \) \(\text{Lie}(\mathbb{G}^3_{m,L}/\mathbb{G}^{m,L}) \Gamma\) twisted \(\times_L \mathbb{A}^2_{L}\),

that is, an \(\mathcal{S}_3\)-equivariant birational isomorphism

\[T_{\PGU_3} \times_K \mathbb{G}^2_{m,K} \sim \) \(\text{Lie}(T_{\PGU_3}) \times_K \mathbb{A}^2_{K}\).

Now Proposition 5.14 follows from Proposition 2.8. \(\square\)

Proof of Theorem 1.3. If \(G\) is of absolute rank 1, then by Proposition 3.1 the group \(G\) is Cayley (and hence, the group \(G \times_K \mathbb{G}^2_{m,K}\) is Cayley). Now assume that \(G\) is of absolute rank 2. If \(G\) is not semisimple, or is of type \(A_1 \times A_1\), or is of type \(B_2 = C_2\), then by Propositions 3.2, 3.3, and 3.4 the group \(G\) is Cayley, hence the group \(G \times_K \mathbb{G}^2_{m,K}\) is Cayley. Otherwise \(G\) is of type \(G_2\) or \(A_2\), and by Example 2.2 and Propositions 5.1, 5.6, 5.13, and 5.14 the group \(G \times_K \mathbb{G}^2_{m,K}\) is Cayley. \(\square\)
Appendix

ELEMENTARY LINKS

by Igor Dolgachev

In this appendix we will follow the ideas from Iskovskikh’s papers [I2], [I3], [I4] to study the Cayley property of the groups SU$_3$, PGU$_3$ and SL$_3$ over $\mathbb{R}$.

6. Elementary links for $G$-surfaces

Let $X$ be a smooth projective surface over a perfect field $K$ and $G$ be a finite group of $K$-automorphisms of $X$. We say that the pair $(X, G)$ is a $G$-surface. Two $G$-surfaces $(X, G)$ and $(X', G)$ are called birationally (biregularly) isomorphic if there exists a birational (biregular) $G$-equivariant map $\phi : X \to X'$ defined over $K$. A $G$-surface $(X, G)$ is called minimal if any birational $G$-equivariant morphism $X \to X'$ is an isomorphism. Any birational $G$-map between two $G$-surfaces can be factored into a sequence of birational $G$-morphisms and their inverses. A birational $G$-morphism $f : X \to Y$ is isomorphic to the blow-up of a closed $G$-invariant 0-dimensional subscheme $a$ of $Y$. For the future use let us remind that the degree of $a$ is the number $\deg(a) = h^0(\mathcal{O}_a)$. If $a$ is reduced and consists of closed points $y_1, \ldots, y_k$ with residue fields $\kappa(y_i)$, then $\deg(a) = \sum \deg(y_i)$, where $\deg(y_i) = [\kappa(x_i) : K]$. The $G$-invariance of $a$ means that $a$ is the union of $G$-orbits.

The birational classification of $G$-surfaces over $K$ is equivalent to the classification of minimal $G$-surfaces up to birational isomorphisms.

From now on we assume that $X$ is a rational surface, i.e. after a finite base change $L/K$, the surface is birationally isomorphic to $\mathbb{P}^2_L$. It is known (see [I2]) that a minimal rational surface belongs to one of the following two classes:

(\text{D}) X is a del Pezzo surface with $\text{Pic}(X)^G \cong \mathbb{Z}$;
(\text{C}) $X$ is a conic bundle with $\text{Pic}(X)^G \cong \mathbb{Z}^2$.

Recall that $X$ is called a del Pezzo surface if the anti-canonical sheaf $\omega_X^{-1}$ is ample. The self-intersection number $(\omega_X, \omega_X)$ takes its value between 1 and 9 and is called the degree of a del Pezzo surface. Also $X$ is a called a conic bundle if there exists a $K$-morphism $f : X \to C$ such that each fiber is reduced and is isomorphic to a conic over $K$ (maybe reducible).

In the case when $K$ is an algebraically closed field, the problem of birational classification of minimal $G$-surfaces is equivalent to the problem of classification of conjugacy classes of finite subgroups of the Cremona group $\text{Cr}_K(2)$ of birational automorphisms of $\mathbb{P}^2_K$. We refer to [D1] for the results in this direction. When $G = \{1\}$, the problem of classification of rational $K$-surfaces has been addressed in fundamental works of V.A. Iskovskikh [I1].
and Yu.I. Manin [M1]. In both cases a modern approach uses the theory of elementary links [I2].

We will be dealing with minimal del Pezzo $G$-surfaces or minimal conic bundles $G$-surfaces. In the $G$-equivariant version of the Mori theory they are interpreted as extremal contractions $\phi : S \to C$, where $C = \text{pt}$ is a point in the first case and $C$ is a curve in the second case. They are also two-dimensional analogs of rational Mori $G$-fibrations.

A birational $G$-map between Mori fibrations is a diagram of $G$-equivariant rational $K$-maps

\[
\begin{array}{ccc}
S & \xrightarrow{\chi} & S' \\
\phi \downarrow & & \phi' \downarrow \\
C & \quad \quad & C'
\end{array}
\]

which in general do not commute with the fibrations. Such a map is decomposed into elementary links. These links are divided into the four following types.

- Links of type I:

They are commutative diagrams of the form

\[
\begin{array}{ccc}
S & \xrightarrow{\chi} & Z = S' \\
\phi \downarrow & & \phi' \downarrow \\
C = \text{pt} & \xrightarrow{\alpha} & C' = \mathbb{P}^1
\end{array}
\]

Here $\sigma : Z \to S$ is a blow-up of a closed $G$-invariant 0-dimensional subscheme $G$-orbit, $S$ is a minimal Del Pezzo surface, $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle, $\alpha$ is the constant map. For example, the blow-up of a $G$-fixed $K$-rational point on $\mathbb{P}^2$ defines a minimal conic $G$-bundle $\phi' : F_0 \to \mathbb{P}^1$ with a $G$-invariant exceptional section. Here and in the sequel we denote by $F_n$ a $K$-surface which becomes isomorphic over the algebraic closure of $K$ to a minimal ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}_K} \oplus \mathcal{O}_{\mathbb{P}_K}(-n))$.

- Links of type II:

They are commutative diagrams of the form

\[
\begin{array}{ccc}
S & \xrightarrow{\sigma} & Z \xrightarrow{\tau} S' \\
\phi \downarrow & & \phi' \downarrow \\
C & = & C'
\end{array}
\]

Here $\sigma : Z \to S, \tau : Z \to S'$ are the blow-ups of $G$-invariant closed 0-dimensional subschemes such that $\text{rank Pic}(Z)^G = \text{rank Pic}(S)^G + 1 = \text{rank Pic}(S')^G + 1, C = C'$ is either a point or a curve. An example of a link of type II is the a link between $\mathbb{P}^2$ and $\mathbb{P}_0$ where one blows up a $G$-invariant closed subscheme $\mathfrak{a}$ of $\mathbb{P}_K^2$ of degree 2 and then blows down the proper transform of the line spanned by $\mathfrak{a}$. Another frequently used link of
type II is an elementary transformation of minimal ruled surfaces and conic bundles.

- Links of type III:
  These are the birational maps which are the inverses of links of type I.

- Links of type IV:
  They exist when $S$ has two different structures of $G$-equivariant conic bundles. The link is the exchange of the two conic bundle structures

$$(6.4) \quad S \xrightarrow{\phi} C \xleftarrow{\phi'} S'$$

**Theorem 6.1.** Let $f : S \dasharrow S'$ be a birational map of minimal $G$-surfaces. Then $\chi$ is equal to a composition of $G$-equivariant elementary links.

The proof of this theorem is the same as in the arithmetic case considered in [I3], Theorem 2.5.

To start an elementary link, one has to blow up a $G$-invariant subscheme of maximal multiplicity of a linear system defining the birational map.

The classification of possible elementary links can be found in [I2]. It is stated in the case $G = \{1\}$, however it can be extended to the general case in a straightforward fashion. The case $G \neq \{1\}$ but $K$ is algebraically closed is considered in [DI], 7.2.

**Example 6.2.** Assume $X$ is a del Pezzo surface $D_6$ of degree 6 and $X' = \mathbb{P}^2_K$. We want to decompose a birational $G$-equivariant map $X \dasharrow X'$ into a composition of elementary links. From Propositions 7.12 and 7.13 in [DI] we obtain that the only elementary link starting at $(X, G)$ ends either at a del Pezzo surface $Y'$ of degree 6 or at $F_0$. Since we do not want to stay on some $(D_6, G)$, we may assume that $Y = F_0$. Now we need an elementary link starting at $Y$. The same Propositions tell us that the end of the next elementary link is either a conic bundle $Y' \to C$, or $F_0$, or $\mathbb{P}^2_K$, or a del Pezzo surface of degree 5 or 6. Since we do not want to return back to $X$ or $F_0$ we may assume that the end of the link $Y'$ is either a conic bundle or a del Pezzo surface of degree 5 or 6. If $Y' = \mathbb{P}^2_K$, Proposition 7.13, case 2, tells us that $F_0$ must contain a $G$-invariant $K$-rational point. If $Y'$ is a del Pezzo surface of degree 5, then the same Proposition tells us that $Z \to Y'$ is the blow-up of a $G$-invariant subscheme of degree 5. Finally, if $Y'$ is a conic bundle, we may continue to do elementary links staying in the class $C$ and at some point we have to link a conic bundle with a del Pezzo surface $Y''$. Proposition 7.12 tells us that $Y''$ is either a del Pezzo surface of degree 4 or $F_0$. Since we do not want to return back to $F_0$, we may assume that $Y''$ is a del Pezzo surface of degree 4. However, we find from Proposition 7.13, case 5, that we are stuck here since any elementary link relates $Y''$ only with itself.
Assume $X$ is birationally $G$-isomorphic to $\mathbb{P}^2_K$. Then the previous analysis shows that $X$ must have a $G$-invariant rational $K$-point allowing us to find an elementary link with $F_0$. To continue, we need to find either a $K$-rational $G$-equivariant point on $F_0$ to link the latter with $\mathbb{P}^2_K$, or to find a $G$-invariant 0-dimensional subscheme of length 5 to link $F_0$ with a del Pezzo surface $D_5$ of degree 5. The only elementary link which ends not at a del Pezzo surface of degree 5 or $F_0$ is a link connecting to $\mathbb{P}^2_K$. It follows from Proposition 7.13, case 4, that to perform this link we need a $K$-rational $G$-invariant point on $D_5$.

Here we exhibit possible elementary links relating a del Pezzo $G$-surface $(D_6, G)$ with $(\mathbb{P}^2, G)$.

\[
\begin{array}{c}
D_6 \\
\downarrow \quad Z \\
\downarrow \quad F_0 \\
\downarrow \quad Z' \\
\downarrow \quad \mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
D_6 \\
\downarrow \quad Z \\
\downarrow \quad F_0 \\
\downarrow \quad D_5 \\
\downarrow \quad \mathbb{P}^2
\end{array}
\]

This is possible only if $F_0$ has a $G$-invariant $K$-rational point.

\[
\begin{array}{c}
D_6 \\
\downarrow \quad Z \\
\downarrow \quad F_0 \\
\downarrow \quad D_5 \\
\downarrow \quad \mathbb{P}^2
\end{array}
\]

This is possible only if $F_0$ has a $G$-invariant closed subscheme of degree 5, and also $D_5$ has a $K$-rational $G$-invariant point.

7. Maximal tori in $\text{SU}(3), \text{PGU}(3)$

Let $\text{SL}_3$ be the split simply connected simple group of type $A_2$ over the field of real numbers. Let $\text{SU}_3$ be its real form defined by an element of $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{SL}_3(\mathbb{C}))$ represented by the map $A \mapsto A^{-1}$. Its group of real points $\text{SU}_3(\mathbb{R})$ is isomorphic to the group $\text{SU}(3)$ of unitary $3 \times 3$ complex matrices. A maximal torus $T$ in $\text{SU}_3$ is a real form of the standard torus $(\mathbb{C}^*)^2 = \{(z_1, z_2, z_3) \in (\mathbb{C}^*)^3 : z_1 z_2 z_3 = 1\}$. It is defined by the map $(z_1, z_2) \mapsto (z_1^{-1}, z_2^{-1}, z_1 z_2^{-1})$ and it is isomorphic to $(\mathbb{S}^1)^2$, where $\mathbb{S}^1 = \text{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ with the natural structure of an algebraic group over $\mathbb{R}$. The group of real points of $\mathbb{S}^1$ is the circle $\text{SU}(1) = \{z \in \mathbb{C} : |z| = 1\}$. Its complex points are $\{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 + z_2^2 = 1\}$. The isomorphism $\mathbb{S}^1(\mathbb{C}) \to \mathbb{C}^*$ is given by $z_1 = z_1 + i z_2$.

Let $C = \text{Proj} \mathbb{R}[t_0, t_1, t_2]/(t_1^2 + t_2^2 - t_3^2)$ be the standard compactification of $\mathbb{S}^1$. It is a plane nonsingular conic defined over $\mathbb{R}$. Its real points satisfying $t_0 \neq 0$ are identified with $\text{SU}(1)$ via the map $a + bi \mapsto [a, b, 1]$. Let $f : \mathbb{P}^1 \to C, \quad [u, v] \mapsto [u^2 - v^2, 2uv, u^2 + v^2]$

be the rational parameterization of $\mathbb{S}^1$ defined over $\mathbb{R}$. We have $[u, v] \cdot [u', v'] := [uu' - vv', uv' + u'v]$.
is mapped to
\[
[(uu' - vv')^2 - (uv' + u'v)^2, 2(uu' - vv')(uv' + u'v), (uu' - vv')^2 + (uv' + u'v)^2] = 
= [(u^2 - v^2)(u'^2 - v'^2) - 4uvu'v', (u^2 - v^2)2u'v' + (u'^2 - v'^2)2uv, (u^2 + v^2)(u'^2 + v'^2)].
\]
This shows that the restriction of the map \( f \) to the open subset \( D^+(u^2 + v^2) \) is a homomorphism of groups.

Now let us consider the subvariety \( X \) of \( \mathbb{P}^3(1) \) given by the condition that \( x \cdot y \cdot z = (1, 0) \). It is given by the equation
\[
uu'v'' - vv'v'' + uu'u'' + u'vu'' = 0.
\]
This is a compactification of the maximal torus \( T \) in \( \text{SU}_3 \). The equation is given by a trilinear function, hence \( X \) is a hypersurface in \( \mathbb{P}^3(1) \) of type \((1, 1, 1)\). By the adjunction formula,
\[
K_X = (K_{\mathbb{P}^3} + X) \cdot X = -(h_1 + h_2 + h_3).
\]
This shows that \( X \) is a del Pezzo surface, anticanonically embedded in \( \mathbb{P}^7 \) by means of the Segre map \( (\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7 \). Here \( h_i \) are the preimages of \( \mathcal{O}_{\mathbb{P}^1}(1) \) under the projections \( p_i : X \to \mathbb{P}^1 \). The degree of the del Pezzo surface \( X \) is equal to \((h_1 + h_2 + h_3)^3 = 6h_1h_2h_3 = 6 \). Over \( \mathbb{C} \), a del Pezzo surface of degree 6 is isomorphic to the blow-up of three non-collinear points in \( \mathbb{P}^2 \).

The boundary \( X \setminus T \) of the torus \( T \) consists of three irreducible (over \( \mathbb{R} \)) components \( p_i^{-1}(V(u^2 + v^2)) \). Over \( \mathbb{C} \), each such component splits into two disjoint curves isomorphic to \( \mathbb{P}^1 \). The boundary becomes a hexagon of lines in the anticanonical embedding. The opposite sides are the pairs of conjugate lines. The group of automorphisms of the root system of type \( A_2 \) of the group \( \text{SU}_3 \) is isomorphic to the dihedral group \( D_6 \) of order 12 (also isomorphic to the direct product \( \mathbb{S}_3 \times \mathbb{Z}/2\mathbb{Z} \)). Its standard action on \( T \) extends to a faithful action on the compactification \( X \). It acts on the hexagon via its obvious symmetries.

Note that the Picard group \( \text{Pic}(X_{\mathbb{C}}) \) is generated by the classes \( e_0, e_1, e_2, e_3 \), where \( e_0 \) is the class of the preimage of a line under the blow-up \( X_{\mathbb{C}} = X_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}} \), and \( e_i \) are the classes of the exceptional curves. The hexagon of lines on \( X \) consists of the six lines with the divisor classes
\[
e_1, e_2, e_3, f_1 = e_0 - e_2 - e_3, f_2 = e_0 - e_1 - e_3, f_3 = e_0 - e_1 - e_2.
\]
The pairs of opposite sides are \( \{f_i, e_i\} \). The group \( \mathbb{S}_3 \) acts on \( \text{Pic}(X) \) by permuting \( e_1, e_2, e_3 \) and the Galois group acts on \( \text{Pic}(X) \) by \( f_i \mapsto e_i \). Note that \(-K_X = 3e_0 - e_1 - e_2 - e_3 \) and, since \( K_X \) is Galois invariant, the conjugation isometry of \( \text{Pic}(X_{\mathbb{C}}) \) sends \( e_0 \) to \( 2e_0 - e_1 - e_2 - e_3 = -K_X - e_0 \) and \( e_0 - e_i \) to \(-K_X - e_0 - (e_0 - e_j - e_k) = e_0 - e_i \). This shows that the pencil of conics \( |e_0 - e_i| \) defines a map \( p_i : X \to \mathbb{P}^1 \) over \( \mathbb{R} \). This defines our embedding
\[
X \hookrightarrow (\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7.
\]
Also note that the invariant part \( \text{Pic}(X)^{\mathbb{S}_3 \times \text{Gal}(\mathbb{C}/\mathbb{R})} = \mathbb{Z}K_X, \) i.e. \( X \) is a minimal \( \mathbb{S}_3 \)-surface over \( \mathbb{R} \).
Consider the real point \( e \in T(\mathbb{R}) \), the unit element of the torus. The tangent plane to the Segre variety \( s(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \) in \( \mathbb{P}^7 \) is spanned by the images of \( e \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times e \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times e \). Its intersection with \( X \) is the point \( e \). Consider the projection \( \mathbb{P}^7 \to \mathbb{P}^3 \) from the tangent plane of \( X \) at \( e \). Its restriction to \( X \) defines a rational map \( X \to Q \), where \( Q \) is a nonsingular quadric \( Q \) in \( \mathbb{P}^3 \). In fact, the rational map is the composition \( \tau \circ \pi^{-1} \), where \( \pi : X' \to X \) is the blow-up of the point \( e \), and \( \tau : X' \to Q \) is the blow-down of the proper transforms of three conics \( R_i \), the images of \( (\mathbb{P}^1 \times \mathbb{P}^1 \times \{ e \}) \cap X, (\{ e \} \times \mathbb{P}^1 \times \mathbb{P}^1) \cap X, \) and \( (\mathbb{P}^1 \times \{ e \} \times \mathbb{P}^1) \cap X \). Note that, \( R_1^2 = 0 \) on \( X \), and \( R_2^2 = -1 \) on \( X' \). We have \( K_X^2 = K_Q^2 = -1 = 6 - 1 = 5 \), and \( K_Q^2 = 5 + 3 = 8 \), so \( Q \) is a del Pezzo of degree 8, i.e a quadric or \( F_1 \). But the latter is not embedded in \( \mathbb{P}^3 \) as a normal surface.

The surface \( X \) has three \( \mathfrak{S}_3 \)-invariant points \( \eta, \eta, \eta^2 \in T(\mathbb{R}) \) corresponding to the diagonal matrices in \( \text{SU}(3) \). The image of \( \eta \) is \( \mathfrak{S}_3 \)-invariant which is a real point in the real structure of \( Q \) defined by the map \( X \to Q \). Projecting from this point, we see that \( Q \) is birationally trivial over \( \mathbb{R} \) as a \( \mathfrak{S}_3 \)-surface.

Applying Proposition 2.8, we obtain

**Theorem 7.1.** The group \( \text{SU}_3 \) is a Cayley group.

Next we consider the group \( \text{PGU}(3) \). It is the quotient of \( \text{SU}_3 \) by the cyclic group \( \mu_3 \) of order 3. Its group \( \text{PGU}_3(\mathbb{R}) \) of real points is isomorphic to the group \( \text{PSU}(3) \). A maximal torus of \( \text{PGU}_3 \) is isomorphic to \( T/\mu_3 \), where \( T \) is a maximal torus of \( \text{SU}_3 \). In the real picture from the previous section, its action on \( T \) is the multiplication map \( \sigma : (u, v) \mapsto (u, v) \cdot (1/2, \sqrt{3}/2) \). The action of \( \mu_3 \) extends to the compactification \( X \) of the maximal torus \( T \) of \( \text{SU}_3 \). Obviously, it leaves invariant the boundary, and has six isolated fixed points on the boundary; they are the vertices of the hexagon. The automorphism group of the del Pezzo surface \( X \) (over \( \mathbb{C} \)) is \( (\mathbb{C}^*)^2 \times D_6 \), and \( \sigma \) belongs to the connected part, and hence acts identically on \( \text{Pic}(X) \). In particular, it acts identically on the sides of the hexagon of lines. The quotient \( Y = X/\mu_3 \) is a singular compactification of a maximal torus of \( \text{PGU}_3 \). It has six singular quotient singularities of type \( 1/3(1,1) \), a minimal resolution \( Y' \to Y \) has six exceptional curves \( E_i \) with \( E_i^2 = -3 \). The proper transforms of the images of the sides of the hexagon are six disjoint \((-1)\)-curves. Together with \( E_i \)'s they form a 12-gon. All of this is defined over \( \mathbb{R} \), the Galois group switches opposite \((-3)\)-sides and opposite \((-1)\)-sides of the 12-gon. Now, we can blow down the \((-1)\)-sides to get a nonsingular surface \( Z \) with a hexagon of \((-1)\)-curves formed by the images of the \((-3)\)-sides. So, \( Z \) is a del Pezzo surface of degree six again! We have found a nonsingular \( \mathfrak{S}_3 \times \text{Gal}(\mathbb{C}/\mathbb{R}) \)-invariant minimal compactification of a maximal torus of \( \text{PGU}_3 \) which is a del Pezzo surface of degree six.

\(^1\)A \((-n)\)-curve is a smooth rational curve on a nonsingular projective surface with self-intersection equal to \(-n\).

\(^2\)One can also arrive at this 12-gon by first blowing up the vertices of the hexagon, then extend the action of \( \mu_3 \) to the blow-up, and then taking the quotient.
Note that the group $\mathfrak{S}_3 \times \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\text{Pic}(Z)$ in the same way as it acts in the case of $\text{SU}_3$. So, as before, we have a $\mathfrak{S}_3$-invariant embedding $Z \hookrightarrow \mathbb{P}^7$ defined over $\mathbb{R}$ with a rational point equal to the orbit $\bar{e}$ of the origin $e \in X$ which consists of the diagonal matrices of $\text{SU}_3$. This time we have no any other $\mathfrak{S}_3$-invariant rational points on $X$ (they obviously do not lie on the boundary). By projection from the point $\bar{e}$, we obtain a quadric $Q$.

The projection defines a $\mathfrak{S}_3$-equivariant isomorphism over $\mathbb{R}$ between the complement of the three conics on $X$ and the complement of the image of the exceptional curve over $\bar{e}$ in $Q$. The latter curve is a conic section $R'$ of $Q$. The three conics are permuted under $\mathfrak{S}_3$, so $\mathfrak{S}_3$ acts on $R'$ without fixed points. Thus a $\mathfrak{S}_3$-invariant real point on $Q$ must be the projection of a real $\mathfrak{S}_3$-invariant point on the del Pezzo surface $X$. There is none except the point which has been blown up. Thus the quadric $Q$ has no $\mathfrak{S}_3$-invariant real points. It follows from Example 6.2 that there is no birational $\mathfrak{S}_3$-equivariant map from $Z$ to $\mathbb{P}^2_{\mathbb{R}}$ (we are stuck at the first elementary link!).

Using Proposition 2.8 we obtain

**Theorem 7.2.** The group $\text{PGU}_3$ is not Cayley.

### 8. Maximal tori in $\text{SL}_3$

The group $\text{SL}_3$ is a simple algebraic group split over $\mathbb{R}$. Its group of real points $\text{SL}_3(\mathbb{R})$ is the group of unimodular real $3 \times 3$-matrices. Its maximal torus is the standard torus $\mathbb{T} = \text{Spec } \mathbb{R}[z_1, z_2, z_3]/(z_1z_2z_3 - 1)$. The group $\mathbb{T}(\mathbb{R})$ of its real points is naturally isomorphic to $\{(a, b, c) \in (\mathbb{R}^*)^3 : abc = 1\}$ with the $\mathfrak{S}_3$-action defined by permutation of the coordinates. Obviously, a real $\mathfrak{S}_3$-invariant point on $\mathbb{T}$ must be equal to the identity point $(1, 1, 1)$.

A natural $\mathbb{T}$-equivariant compactification of $\mathbb{T}$ is the cubic surface $Y = \text{Proj } \mathbb{R}[t_0, t_1, t_2, t_3]/(t_1t_2t_3 - t_0^3)$. It has three quotient singularities of type $\frac{1}{3}(1, 2)$, rational double points of type $A_2$. They are defined over $\mathbb{R}$. The exceptional curve over each singular point consists of two $(-2)$-curves $E_i + E'_i$ intersecting transversally at one point. The intersection point $E_i \cap E'_i$ is a real point, hence the curves are isomorphic to $\mathbb{P}^1$ over $\mathbb{R}$. The group $\mathfrak{S}_3$ permutes the pairs $(E_i, E'_i)$. After we minimally resolve $Y$ over $\mathbb{R}$, we obtain a surface isomorphic to the blow-up of a del Pezzo surface of degree 6 at three vertices of the hexagon of lines. The boundary consists of a 9-gon with 9 consecutive sides $R_1, \ldots, R_9$, where $R_1, P_2, R_4, R_5, R_7, R_8$ are $(-2)$-curves and the sides $R_3, R_6, R_9$ are $(−1)$-curves. The latter curves are the proper transforms of the three lines on the cubic surface $Y$ that join the pairs of the singular points. After we blow down ($\mathfrak{S}_3$-equivariantly) the $(−1)$ curves, we obtain a del Pezzo surface $X$ of degree 6 with a hexagon of lines at the boundary. The linear system that defines the rational map $Y \dashrightarrow X$ consists of quadric sections of $Y$ passing through the singular point. Note that both $X$ and $Y$ are $\mathfrak{S}_3$-equivariant compactifications of $\mathbb{T}$.

**Theorem 8.1.** $\text{SL}_3$ is not Cayley.
follows from Example 6.2 that the first link must end at $F$ isomorphic to a $(\mathbb{P}^2, \mathcal{G}_3)$. Suppose they are birationally isomorphic. It follows from Example [5.2] that the first link must end at $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ which we identify with a split nonsingular quadric $Q$ in $\mathbb{P}^3_{\mathbb{R}}$. The link consists of blowing up the unique $\mathcal{G}_3$-invariant real point on $X$, namely the point $e$, and then blowing down three $(-1)$-curves. They are the images of the conics on $Y$ that, together with the three lines, are cut out by the quadrics $t_it_j - w^2 = 0$. The conics are left invariant under the conjugation but permuted by $\mathcal{G}_3$. The action of $\mathcal{G}_3$ on $X$ shows easily that the induced action of $\mathcal{G}_3$ on $Q$ permutes the two rulings (i.e. the two projections to $\mathbb{P}^1$). It is easy to see, using the description of automorphisms of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ that the quadric $Q$ has no real $\mathcal{G}_3$-invariant points, so the next elementary link relates $Q$ with a del Pezzo surface $\mathcal{D}_5$ of degree 5. For this we need a $\mathcal{G}_3$-invariant 0-dimensional subscheme $a$ of degree 5. It must consist of a $\mathcal{G}_3$-invariant point of degree 2 and an $\mathcal{G}_3$-orbit of three real points. It is easy to see that the only $\mathcal{G}_3$-invariant point of degree 2 is the image of two conjugate scalar matrices in $\text{SL}_3(\mathbb{C})$. There are plenty of $\mathcal{G}_3$-orbits of three real points. Now we have to apply the elementary link $Q \leftarrow Z \rightarrow \mathcal{D}_5$ with the target equal to a del Pezzo surface $\mathcal{D}_5$ of degree 5. Either we stuck here and hence prove the assertion or we find a real $\mathcal{G}_3$-invariant point on $\mathcal{D}_5$ to make the final elementary link with $(\mathbb{P}^2, \mathcal{G}_3)$. Since $Q$ has no such points, a real $\mathcal{G}_3$-invariant point $q$ on $\mathcal{D}_5$ lies on the image of an exceptional curve of $Z \rightarrow Q$ or on the image of an exceptional curve of $Z \rightarrow \mathcal{D}_5$. The three exceptional curves on $Z$ over real points in $Q$ are permuted by $\mathcal{G}_3$ so $q$ cannot lie on them. Also the exceptional curve on $Z$ over the complex point in $Q$ consists of two disjoint conjugate curves. So, $q$ is not on them too. It follows from the description of the linear system defining the link, that the exceptional curves of $Z \rightarrow \mathcal{D}_5$ are the proper transforms $R_1$ and $R_2$ of the two rational curves $R_1$ and $R_2$ of degree 3 (of bidegrees $(2,1)$ and $(1,2)$) on $Q$. Since $\mathcal{G}_3$ permutes the two rulings on $Q$, it cannot leave $R_1$ or $R_2$ invariant. Thus the images of the exceptional curves $\bar{R}_1$ and $\bar{R}_2$ are not fixed under $\mathcal{G}_3$. Thus the point $q$ cannot be one of these points. This shows that the last elementary link $\mathcal{D}_5 \rightarrow \mathbb{P}^2$ is not possible. □

Remark 8.2. The real split group $\text{PGL}_3$ is known to be a Cayley group (see [LPR, Example 1.11]). Using Proposition 2.8 this fact immediately follows from the existence of a $\mathcal{G}_3$-equivariant compactification of a maximal torus of $\text{PGL}_3$ isomorphic to the projective plane. In fact, consider the cubic surface $X$ from the proof of the previous theorem. The quotient of this surface by the cyclic group generated by the transformation $[t_0, t_1, t_2, t_3] \mapsto [\eta_3t_0, t_1, t_2, t_3]$ is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$ via the projection map from the point $[1, 0, 0, 0] \in \mathbb{P}^3 \setminus X$. Its maximal torus $\mathbb{T}$ is the standard torus in $\mathbb{P}^2_{\mathbb{R}}$. 

Proof. By Proposition 2.8 it suffices to prove that $(X, \mathcal{G}_3)$ is not birationally isomorphic to a $(\mathbb{P}^2_{\mathbb{R}}, \mathcal{G}_3)$. Suppose they are birationally isomorphic. It follows from Example [5.2] that the first link must end at $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ which we identify with a split nonsingular quadric $Q$ in $\mathbb{P}^3_{\mathbb{R}}$. The link consists of blowing up the unique $\mathcal{G}_3$-invariant real point on $X$, namely the point $e$, and then blowing down three $(-1)$-curves. They are the images of the conics on $Y$ that, together with the three lines, are cut out by the quadrics $t_it_j - w^2 = 0$. The conics are left invariant under the conjugation but permuted by $\mathcal{G}_3$. The action of $\mathcal{G}_3$ on $X$ shows easily that the induced action of $\mathcal{G}_3$ on $Q$ permutes the two rulings (i.e. the two projections to $\mathbb{P}^1$). It is easy to see, using the description of automorphisms of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ that the quadric $Q$ has no real $\mathcal{G}_3$-invariant points, so the next elementary link relates $Q$ with a del Pezzo surface $\mathcal{D}_5$ of degree 5. For this we need a $\mathcal{G}_3$-invariant 0-dimensional subscheme $a$ of degree 5. It must consist of a $\mathcal{G}_3$-invariant point of degree 2 and an $\mathcal{G}_3$-orbit of three real points. It is easy to see that the only $\mathcal{G}_3$-invariant point of degree 2 is the image of two conjugate scalar matrices in $\text{SL}_3(\mathbb{C})$. There are plenty of $\mathcal{G}_3$-orbits of three real points. Now we have to apply the elementary link $Q \leftarrow Z \rightarrow \mathcal{D}_5$ with the target equal to a del Pezzo surface $\mathcal{D}_5$ of degree 5. Either we stuck here and hence prove the assertion or we find a real $\mathcal{G}_3$-invariant point on $\mathcal{D}_5$ to make the final elementary link with $(\mathbb{P}^2, \mathcal{G}_3)$. Since $Q$ has no such points, a real $\mathcal{G}_3$-invariant point $q$ on $\mathcal{D}_5$ lies on the image of an exceptional curve of $Z \rightarrow Q$ or on the image of an exceptional curve of $Z \rightarrow \mathcal{D}_5$. The three exceptional curves on $Z$ over real points in $Q$ are permuted by $\mathcal{G}_3$ so $q$ cannot lie on them. Also the exceptional curve on $Z$ over the complex point in $Q$ consists of two disjoint conjugate curves. So, $q$ is not on them too. It follows from the description of the linear system defining the link, that the exceptional curves of $Z \rightarrow \mathcal{D}_5$ are the proper transforms $R_1$ and $R_2$ of the two rational curves $R_1$ and $R_2$ of degree 3 (of bidegrees $(2,1)$ and $(1,2)$) on $Q$. Since $\mathcal{G}_3$ permutes the two rulings on $Q$, it cannot leave $R_1$ or $R_2$ invariant. Thus the images of the exceptional curves $\bar{R}_1$ and $\bar{R}_2$ are not fixed under $\mathcal{G}_3$. Thus the point $q$ cannot be one of these points. This shows that the last elementary link $\mathcal{D}_5 \rightarrow \mathbb{P}^2$ is not possible. □
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