Privacy Preservation by Local Design in Cooperative Networked Control Systems

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Abstract—In this paper, we study the privacy preservation problem in a cooperative networked control system working for the task of LQG control. The system consists of a user and a server: the user owns the state process, the server provides computation capability, and the user employs the server to compute control inputs for it. To enable the server’s computation, the user needs to provide the measurements of the process states to the server. However, the user regards the states as privacy and wants the server to have deviated knowledge of the state estimates rather than the true value. Regarding that, we propose a novel local design of privacy preservation, which creates a deviation in the server’s knowledge of the state estimates from the true value. Meanwhile, we propose an associated privacy metric to measure the deviation and analyze the privacy performance accordingly. Furthermore, we analyze the service loss in the LQG control performance caused by the privacy scheme. Finally, we study the performance trade-off between the privacy preservation and the LQG control, where the proposed optimization problems are solved by numerical methods efficiently.

Index Terms—privacy preservation, cooperative privacy, local design, cooperative networked control systems

I. INTRODUCTION

The networked control systems, featuring the flexibility of configuring the various components of a control system, have gained numerous applications nowadays [1]. This flexibility could further facilitate the system to be run cooperatively by more than one party: different individuals or organizations own respective parts of it. This could be realized by that various parties unite together, or that the original owner of a system outsources some parts to external parties. We name this type of systems as cooperative networked control systems. This cooperative manner should be a natural trend of economic demand, since higher economic efficiency is usually achieved by deeper work division and cooperation. Practically, we observe the exploding cloud service in industry in recent years [2]–[5], which may be viewed as a sign of this trend, since the cloud service can be expected to provide one way easily realizing the configurations of cooperative networked control systems.

A basic scenario of cooperative networked control systems is presented in Fig. 1, which we name as the user-server system. In this scenario, the cooperative networked control system is formed by two parties, the user and the server. The user employs the server to provide certain service for it, and to achieve this, it provides necessary information to the server.

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The cooperation of the user and the server brings efficiency and economics advantages; however, the increased complexity in system structure will also cause new issues, one of which is the privacy issue. The data provided by the user possibly correspond to certain other information, some of which may be supposed to be private. Meanwhile, the server is motivated to analyze the information data purposely, since more knowledge about clients usually helps more. Hence, the user may face unwilling exposure of privacy. For example, when people shop on-line, the data of items bought and money paid (maybe living address also included) are recorded. These data can be related to the customer’s income level, family structure, hobbies, and so on. The business provider could use the customer’s historical shopping records to depict his/her preference profile, so that better business decisions can be made (such as a personalized recommendation service). However, for certain customers, they may not want their preferences exposed. Nevertheless, the risk seems inevitable. We have to say that the server does not literally betray the cooperation; it can be called “honest but curious” [6], i.e., it still honestly serves the duty in cooperation, only meanwhile intending to discover more knowledge of the user as well.

Consequently, to safely use the cooperative systems, we need to address the privacy issue. One common methodology is that the server provides certain privacy policy in favor of the user. Nevertheless, a careful user may still have this demand: regardless of the server’s privacy policies, it demands a local privacy scheme which is not open to the server, such that the server’s achievable knowledge of the user’s private information is not correct but deviated. However, according to the current literature, this demand has been seldom visited yet. In this paper, we study this problem.

A. Related Studies

This cooperative privacy problem has been visited in many directions, such as cloud security and outsourcing security [7]–[9].

When specifically considering the networked control systems, one may first relate the privacy issue to the study on the security of cyber-physical systems [10]–[13], since both of them corresponds to system safety. However, the study methodology on privacy preservation may differ from that of
the cyber-physical security and have its own particularities, as the two have different natures. 1) In the setup of CPS security, an adversary is excluded in the original healthy system, usually unknown. In privacy, the one to defense is the known cooperator who also belongs to the system. 2) In security setups, the information collection (usually by stealthy) is illegal and may be followed by system damaging, which is unwanted. In privacy, information is collected with right authorized, and the cooperator normally serves the duty in cooperation, which is needed. Hence, though overlapped in some aspects, the methodology of cyber-physical security cannot entirely cover the study on privacy preservation, and the study of privacy preservation needs to develop its own one.

The existing studies related to privacy in cooperative networked control systems have several frameworks, each of which is based on the understanding and modeling of the concept of privacy. The first important one is called differential privacy. This concept was originally proposed in the field of statistical databases [14], [15], where it pointed out that one database item has the risk of being identified by analyzing the differences between certain answering outputs of database queries. To protect the privacy, it proposed the method of adding randomness to query answers. This concept and method were later borrowed by the control field. They were mainly employed in the model who has a centralizing or fusing mechanism with multiple participating parties, which are similar to the model of a statistical database who assembles data from multiple participants. Huang et al. [16] considered the model of consensus in a centralized system and in a distributed system, both with preserved differential privacy. He et al. [17] studied a distributed estimation model under differential privacy requirement. Le Ny and Pappas [18] studied the differential privacy problem for centralized Kalman filtering. Le Ny and Mohammady [19] considered the MIMO system. Koufogiannis and Pappas [20] considered the differential privacy preservation for current state of a running dynamical system. Ye and Barg [21] considered the point distribution estimation from privatization samples. The model of constrained optimization for distributed multiple users was also visited, such as Han et al. [22] and Hale and Egerstedt [23].

The second important framework of privacy preservation should be the information-theoretic approach [24]. In this framework, information entropy is used as a metric of privacy. As the indicator of how much information is carried, information entropy is a natural metric to measure the privacy. Compared with differential privacy, who is only available for centralized models with multiple participants, information-theoretic approach also works for models with one single participant. Moreover, in differential privacy, the preservation is usually realized by injecting Gaussian or Laplacian noise, while in the information-theoretic approach, more sophisticated schemes are also proposed. Nekouei et al. [25] presented two privatized information-sharing schemes in a multi-sensor estimation system. Li et al. [26] proposed the privacy-aware policy for resource distribution by solving an optimization problem. Nekouei et al. [27] designed a novel estimator which guarantees the local parameter’s privacy. Jia et al. [28] studied a closed-loop predictive control system. Tanaka et al. [6] considered a cloud-based LQG control system.

Another framework worth attention is the homomorphic encryption. In the previous two frameworks, privacy is preserved by adding noise. Homomorphic encryption provides an entirely different methodology. It finds proper encryption operators or algorithms, which guarantees the homomorphic nature between the data before and after the encryption operation. Farokhi et al. [29], [30] and Tran et al. [31] considered such problems. Ni et al. [32] proposed a novel privacy scheme for distributed estimation based on the technique of homomorphic encryption.

Moreover, several significant studies are not included in the previous frameworks, such as [33], [34], in which novel schemes of privacy preservation for initial values in consensus systems were proposed.

B. The Study of This Paper

In this paper, we consider the privacy preservation problem in a user-server system. Specifically, we study a cooperative LQG control system, in which the state process of the system is owned by the user, the computation function belongs to the server, and the user employs the server to compute the optimal LQG control inputs for it.

To enable the server’s computation, the user provides necessary information about the states to the server. However, the user takes its state trajectory as private information, and does not want the server to seize the correct knowledge of it. It demands a localized privacy scheme, which is not open to the server, such that it manages to send revised data to the server instead of the original ones, while the server is unaware of the situation. Consequently, since the server’s knowledge is gained based on the received data, its knowledge of the user’s state is deviated and hence is incorrect. In this paper, we consider the design and analysis of this type of privacy schemes.

This study has not been fully covered by existing literature yet. Firstly, in substantial existing studies, the proposed privacy schemes are designed at the server (or the party equivalent to the server), not the user [19], [22], or the server knows the existence of the user’s local privacy scheme and is provided the associated parameters as well [6], [35]. In both cases, the server is able to seize the correct, or not accurate but correct, information about the user’s privacy. Secondly, most existing studies investigated open-loop systems without feedback, while few investigated the closed-loop dynamic control system, as the analysis becomes difficult when a privacy scheme is added in the closed-loop dynamic system. The few studies on closed-loop dynamic system [6], [28], [35], [36] either rely on strong assumptions or system scale limitation and have no general results, or propose privacy schemes also open to the server.

The novelty and main contributions of this paper are summarized as follows.

1) We study the privacy preservation problem in a basic cooperative networked control system, the user-server system, which specifically works for the closed-loop LQG control. We propose a novel local design of privacy
preservation scheme for the user, which makes the server’s knowledge of the privacy information deviate from the true one.

2) We propose a novel privacy metric for the proposed privacy methodology, which is meanwhile simple and compatible with the existing study of networked control systems.

3) We analyze the privacy quality (Theorem 1) and the LQG control performance (Theorem 2) and study the trade-off problems between them (Problem 1).

The remainder of the paper is organized as follows. Section II proposes the study framework of the local design for privacy preservation in the considered system. Section III presents the main results. Section IV considers the privacy schemes which achieves privacy demand while has no loss in service performance. Section V presents simulation examples. At last, Section VI discusses the possible future work directions and Section VII makes the conclusion of this study.

Notations: \( \mathbb{Z}_+ \) is the set of non-negative integers and \( k \in \mathbb{Z}_+ \) is the time index. \( \mathbb{R} \) is the set of real numbers. \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space. \( S^n_+ \) (and \( S^n_{++} \)) is the set of \( n \) positive semi-definite matrices (and positive definite matrices); when \( X \in S^n_+ \) (and \( S^n_{++} \)), it is written as \( X \succeq 0 \) (and \( X > 0 \)). \( X \succeq Y \) if \( X - Y \in S^n_+ \). \( E(\cdot) \) is the expectation of a random variable and \( E(\cdot|\cdot) \) is the conditional expectation. \( \text{tr}(\cdot) \) is the trace of a matrix. For a matrix \( X \), \( \mathcal{N}(X) \) is the kernel space of \( X \).

II. FRAMEWORK OF THE PRIVACY PRESERVATION BY LOCAL DESIGN

In this section, we propose the framework of privacy preservation by local design in a user-server system for the task of cooperative LQG control. In the remainder of this section, we present the setup of the ordinary cooperative system, the methodology of the proposed localized privacy preservation, and feasible problems to study.

A. Ordinary Cooperative System

The ordinary user-server system for cooperative LQG control is illustrated in Fig. 2. The user owns a state process to control, the server owns computation capability, and the user employs the server to compute the optimal LQG control inputs for it.

![Fig. 2. Ordinary cooperative LQG control system with imperfect state information.](image)

The user has the following dynamic process to control:

\[
x_{k+1} = Ax_k + Bu_k + w_k,
\]

where \( x_k \in \mathbb{R}^n \) is the state of the process, \( u_k \in \mathbb{R}^m \) is the control input, and \( w_k \) is the Gaussian white noise with distribution \( \mathcal{N}(0,Q) \). The initial condition of the state is assumed as that \( x_0 \) is Gaussian with \( \mathcal{N}(\bar{x}_0,\Sigma_0) \). The time horizon of the process is assumed to be a finite \( T \). \( (A,B) \) is assumed to be controllable.

The user has no direct access to the value of the state \( x_k \), and instead, it owns a sensor measuring the state as follows:

\[
y_k = Cx_k + v_k,
\]

where \( y_k \in \mathbb{R}^m \) is the measurement of \( x_k \) taken by the sensor and \( v_k \) is the white Gaussian noise corrupting the measurement with distribution \( \mathcal{N}(0,R) \) (\( R > 0 \)). The noise \( \{w_k\} \), the noise \( \{v_k\} \), and the initial state \( x_0 \) are mutually independent of each other. We also assume that \( (A,\sqrt{Q}) \) is stabilizable and \( (C,A) \) is detectable.

The user demands the optimal LQG control for its state process. The considered finite-time quadratic objective for the LQG control is denoted as \( O_{0:T} \) and is defined as

\[
O_{0:T} \triangleq E \left[ \sum_{k=0}^{T-1} \left( x_k^T W x_k + u_k^T U u_k + x_{k+1}^T W x_{k+1} \right) \right],
\]

where \( W \) and \( U \) are weight matrices satisfying \( W \succeq 0 \) and \( U > 0 \), and the expectation is taken with respect to all possible randomness.

In this cooperative system, the server computes the optimal LQG control inputs for the user. Since the process states are unavailable, the server has to also estimate the states before computing the control inputs. That is to say, the server serves as the estimator and the controller in the control loop (Fig. 2).

To enable the server’s computation, before the process, the user shares the following system parameters with the server:

- \( A, B, W, U \) (for LQG control);
- \( C, Q, R \) (for state estimation);
- time horizon \( T \);
- initial condition \( \mathcal{N}(\bar{x}_0,\Sigma_0) \).

When the process begins, the user is also supposed to provide \( y_k \) to the server at each time \( k \).

This is the classic problem of LQG control with imperfect state information. The server works as follows. Firstly, before the process, it does the following computation in favor of the optimal LQG control:

\[
S_T = W, \tag{4}
\]

\[
S_k = A'S_{k+1}A + W - A'S_{k+1}B(B'S_{k+1}B + U)^{-1}B'S_{k+1}A, \tag{5}
\]

\[
L_k = -(B'S_{k+1}B + U)^{-1}B'S_{k+1}A. \tag{6}
\]

Secondly, it estimates the user’s state \( x_k \). When the process begins, at each time \( k \), the user sends \( y_k \) to the server. Let \( Y_k \triangleq \{y_1, y_2, ..., y_k\} \). Then the server computes the \textit{a priori} and \textit{a posteriori} estimates \( \hat{x}_{k|k-1} \) and \( \hat{x}_{k|k} \) defined as follows:

\[
\hat{x}_{k|k-1} \triangleq E[x_k|Y_{k-1}],
\]

\[
\hat{x}_{k|k} \triangleq E[x_k|Y_k].
\]
Meanwhile, let $P_{k|k-1}$ and $P_{k|k}$ be the estimation error covariance matrices associated with $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$, respectively:

$$P_{k|k-1} \triangleq E[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})'|Y_{k-1}],$$

$$P_{k|k} \triangleq E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})'|Y_k].$$

The calculation of the state estimation is standard: by Kalman filtering. Given the initial value $\hat{x}_{0|0} = x_0$ and $P_{0|0} = \Sigma_0$, the server does the following computation:

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1}, \quad (7)$$

$$P_{k|k-1} = AP_{k-1|k-1}A' + Q, \quad (8)$$

$$K_k = P_{k|k-1}C'(CP_{k|k-1}C' + R)^{-1}, \quad (9)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}), \quad (10)$$

$$P_{k|k} = (I - K_kC)P_{k|k-1}. \quad (11)$$

Thirdly, the server computes the optimal control input:

$$u_k^* = L_k\hat{x}_{k|k},$$

and returns $u_k^*$ to the user’s actuator.

Then we evaluate the performance of the optimal LQG control. The optimal objective value, denoted as $O_{0:T}^*$, is given as follows [37]:

$$O_{0:T}^* = E(x_0'S_0x_0) + \sum_{k=0}^{T-1} \text{tr}(S_{k+1}Q) + \sum_{k=0}^{T-1} \text{tr}(\Phi_kP_k), \quad (12)$$

where

$$\Phi_k = A'S_{k+1}B(B'S_{k+1}B + U)^{-1}B'S_{k+1}A.$$

### B. Local Design of Privacy Preservation

In the ordinary cooperative system, the estimates of the user’s states trajectory $\{x_k\}_{k=0}^T$ are transparent to the server. However, the user takes its states trajectory as privacy. It wants to conceal the true values of state estimates and let the server have incorrect state estimates, which are deviated from the true ones. Hence, it decides to employ a local privacy scheme, not open to the server, to achieve the desired privacy preservation.

In this paper, as a start, we consider the simplest design of the privacy scheme for the user: just adding a module of privacy processor to process the data to be sent to the server (Fig. 3). At each time $k$, the user generates a signal $z_k$, which we let be a function of $y_k$:

$$z_k = f(y_k),$$

and sends it instead of $y_k$ to the server. The function $f(\cdot)$ could generally depend on $Y_k$ and $u_0, u_1, \ldots, u_{k-1}$:

$$z_k = f(Y_k, u_0, u_1, \ldots, u_{k-1}).$$

**Remark 1:** In existing related studies [6], [25], the privacy scheme is also located at the user side, but the existence of the privacy scheme is known by the server and the parameters are also shared with the server. Although the server cannot have access to the accurate state values because of the privacy scheme, it is still able to obtain correct estimation based on the shared knowledge of the privacy scheme.

### C. Privacy Metric

Since the server is unaware that the user is using a local privacy scheme, it will take $z_k$ as the user’s measurement $y_k$. Then the signal $z_k$ will create deviations in the server’s state estimates, as the user demands.

To analyze the quality of privacy preservation in this scenario, we clarify the following notations.

- The true estimate of $x_k$ and the associated error covariance under the privacy scheme are still denoted by $\hat{x}_{k|k}$ and $P_{k|k}$.
- The public estimate of $x_k$ and the error covariance, i.e., the ones the user let the server know, are denoted as $\hat{x}_{k|k}^{\text{pub}}$ and $P_{k|k}^{\text{pub}}$ (the upper script stands for “public”). This estimate is not true but deviates from the true $\hat{x}_{k|k}$.

Since the server is planned to compute the state estimates and the associated error covariance by eqn. \((7)\)–\((11)\), the public estimate and error covariance evolve accordingly:

$$\hat{x}_{k|k}^{\text{pub}} = A\hat{x}_{k-1|k-1}^{\text{pub}} + Bu_{k-1}, \quad (13)$$

$$P_{k|k}^{\text{pub}} = AP_{k-1|k-1}^{\text{pub}}A' + Q, \quad (14)$$

$$K_k^{\text{pub}} = P_{k|k}^{\text{pub}}C'(CP_{k|k}^{\text{pub}}C' + R)^{-1}, \quad (15)$$

$$\hat{x}_{k|k}^{\text{pub}} = \hat{x}_{k|k}^{\text{pub}} + K_k^{\text{pub}}(y_k - C\hat{x}_{k|k}^{\text{pub}}), \quad (16)$$

$$P_{k|k}^{\text{pub}} = (I - K_k^{\text{pub}}C)P_{k|k}^{\text{pub}}. \quad (17)$$

where $\hat{x}_{k|k}^{\text{pub}}$, $P_{k|k}^{\text{pub}}$, and $K_k^{\text{pub}}$ are the public versions of corresponding variables.

Then we propose the privacy metric to measure the quality of privacy preservation, the deviation covariance at time $k$, as

$$Q_{\text{privacy}}^k \triangleq E[(\hat{x}_{k|k}^{\text{pub}} - \hat{x}_{k|k})(\hat{x}_{k|k}^{\text{pub}} - \hat{x}_{k|k})'|Y_k], \quad (18)$$

and the average deviation covariance over the process as

$$Q_{\text{privacy}} \triangleq \frac{1}{T} \sum_{k=0}^{T-1} Q_{\text{privacy}}^k. \quad (19)$$

They indicate the deviation of the knowledge of the estimates gained by the server from the true one.

**Remark 2:** The proposed metric has a form similar to the error covariance associated with an estimate. Actually, the privacy problem considered in this paper can be seen as an “anti-estimation” problem. Hence, this metric is proper to indicate the privacy level for this type of problems.

Meanwhile, compared with the existing privacy metrics, the proposed one has its advantages. It works for a system with
only one single user, while differential privacy can be only used in centralized systems with multiple users. Compared with information entropy, the proposed metric works for the state estimate rather than a probability distribution, and is more compatible with the common notions in the study of networked control systems.

D. Problems to Study

Based on the proposed method of privacy preservation, we consider to work on the following problems.

1) The analysis of the quality of privacy preservation $Q_{\text{privacy}}$.
2) The privacy scheme makes the user also suffer relative loss in service quality, the LQG control performance, denoted by $Q_{\text{LQG}}$. We also analyze this loss.
3) The trade-off between the privacy preservation and LQG control performances:

$$\max_{\alpha > 0} \text{tr}(Q_{\text{privacy}})$$

s.t. $Q_{\text{LQG}} \leq \alpha$,

where $\alpha > 0$ represents a given performance level.

III. SCHEME DESIGN AND PERFORMANCE ANALYSIS

A. Privacy Scheme

The function to process $y_k$ at the privacy processor could be various. In this paper, we consider one of the simplest: noise injection. We add to $y_k$ a noise signal $\delta_k$, say,

$$z_k = y_k + \delta_k. \quad (20)$$

The noise $\delta_k$ is assumed to be i.i.d. and to have the Gaussian distribution $\mathcal{N}(0, \Sigma_\delta)$.

Remark 3: Although the scheme of noise injection is frequently seen in the studies of privacy preservation, when used in this proposed framework, the analysis turns out to be more difficult because of the method of local design together with the system’s closed-loop. By the local design, the injected noise creates a deviation in the server’s knowledge, and will further create a deviation in the control input from the original one. The effect of the input deviation will accumulate through the closed-loop as time evolves (presented in Subsection III-C). This feature enhances the difficulty of analysis.

B. Privacy Performance

We have the following result.

Theorem 1: It holds that

$$\hat{x}_{\text{pub}}_{k|k} - \hat{x}_{k|k} = (I - K_k C) A (\hat{x}_{\text{pub}}_{k-1|k-1} - \hat{x}_{k-1|k-1}) + K_k \delta_k \quad (21)$$

and

$$Q_{\text{privacy}} = (I - K_k C) A Q_{\text{privacy}} A' (I - K_k C)' + K_k \Sigma_\delta K_k' \quad (22)$$

where $K_k$ is the Kalman gain defined in eqn. (9). The recursion starts with $\hat{x}_{0|0}^{\text{pub}} = \hat{x}_{0|0} = 0$ and $Q_{\text{privacy}}^0 = 0$.

Proof: In appendix.

We can see that the privacy quality $Q_{\text{privacy}}^k$ is determined by $\Sigma_\delta$. Moreover, we find that a lower bound of $Q_{\text{privacy}}^k$ exists.

Theorem 2: A matrix sequence $\{M_k\}, M_k \in \mathcal{S}^{-}_{+}$, satisfies that $M_0 = 0, M_1 = Q_{\text{privacy}}$, and when $k \geq 2$,

$$M_k^- = AM_{k-1}A', \quad (23)$$

$$M_k = M_k^- - M_k^- C' (C M_k^- C' + \Sigma_\delta) ^{-1} C M_k^- . \quad (24)$$

Then

$$Q_{\text{privacy}}^k \geq M_k, \quad \forall k.$$  

Proof: Firstly, we have $Q_{\text{privacy}}^0 = M_0$ and $Q_{\text{privacy}}^1 = M_1$. Since $M_1 \neq 0$, the sequence $\{M_k\}$ will not be all 0. Assume that $Q_{\text{privacy}}^{k-1} \geq M_{k-1}1$. Consider the operator

$$\phi(X, \Gamma) \triangleq (I - \Gamma C) X (I - \Gamma C)' + \Gamma \Sigma_\delta \Gamma'.$$

Then $Q_{\text{privacy}}^{k-1} = \phi(A Q_{\text{privacy}}^{k-1} A', K_k)$. Let

$$\Gamma_k^M = M_k^- C' (C M_k^- C' + \Sigma_\delta)^{-1},$$

we have $M_k = \phi(M_{k-1}, \Gamma_k^M)$. Furthermore, it is straightforward to verify that

$$\Gamma_k^M = \arg\min \phi(M_{k-1}, \Gamma).$$

Since $Q_{\text{privacy}}^{k-1} \geq M_{k-1}$, we have $A Q_{\text{privacy}}^{k-1} A' \geq M_{k-1}^-$. Notice that $\phi(X, \Gamma)$ is linear in $X$. Then we have

$$\phi(A Q_{\text{privacy}}^{k-1} A', K_k) \geq \phi(M_{k-1}, K_k) \geq \phi(M_{k-1}, \Gamma_k^M),$$

which is $Q_{\text{privacy}}^k \geq M_k$.

From the proof of Theorem 1 we also find that $K_k^{\text{pub}} = K_k$ and $P_{k|k}^{\text{pub}} = P_{k|k}$. Hence, $P_{k|k}^{\text{pub}}$ is not the real error covariance associated with $\hat{x}_{\text{pub}}_{k|k}$. We denote the real error covariance associated with $\hat{x}_{k|k}$ as $P_{k|k}^{\text{pub, real}}$.

Lemma 1: It holds that

$$P_{k|k}^{\text{pub, real}} = P_{k|k} + Q_{\text{privacy}}^k.$$  

Proof: We have

$$P_{k|k}^{\text{pub, real}} = E \left[ (x_k - \hat{x}_{\text{pub}}_{k|k}) (x_k - \hat{x}_{\text{pub}}_{k|k})' | Y_k \right]$$

$$= E \left[ (x_k - \hat{x}_{k|k} + \hat{x}_{\text{pub}}_{k|k} - \hat{x}_{\text{pub}}_{k|k}) (x_k - \hat{x}_{k|k} + \hat{x}_{\text{pub}}_{k|k} - \hat{x}_{\text{pub}}_{k|k})' | Y_k \right]$$

$$= P_{k|k} + Q_{\text{privacy}}^k + E \left[ (x_k - \hat{x}_{k|k}) (\hat{x}_{k|k} - \hat{x}_{\text{pub}}_{k|k})' | Y_k \right]$$

$$+ E \left[ (\hat{x}_{k|k} - \hat{x}_{\text{pub}}_{k|k}) (x_k - \hat{x}_{\text{pub}}_{k|k})' | Y_k \right].$$

Notice that $\hat{x}_{k|k}$ is the linear combination of $Y_k, \hat{x}_{\text{pub}}_{k|k}$ is the linear combination of $Y_k$ and $\{\delta_1, ..., \delta_k\}$, and $x_k - \hat{x}_{k|k}$ is independent of $Y_k$ and $\{\delta_1, ..., \delta_k\}$. Then the two expectation terms in the last equality equals 0.

Remark 4: The local design scheme misleads the server’s knowledge so that $\hat{x}_{k|k}$ is deviated from the true one $\hat{x}_{k|k}$. From another perspective, the deviation is equivalent to that the server’s seized estimate $\hat{x}_{\text{pub}}_{k|k}$ is correct but has a larger error covariance $P_{k|k}^{\text{pub, real}}$, which is unknown to the server.
Remark 5 (A Doubtful Server): In our paper, the server entirely trusts the data provided by the user and has no data analysis mechanism. If the server is “doubtful” and has an additional detector to analyze the received measurement sequence \( z_k \), it will find that the covariance of \( z_k \) is not as it supposed to be the one in the provided parameters. The worst case is that the server successfully recovers \( \Sigma_d \), the covariance of the injected noise, and it also makes the correct guess that it is due to noise injection in the measurement sequence (notice that even the server finds the mismatch of the measurement covariance, there are still many possibilities how the user makes the situation). Then the server could adjust the knowledge of the measurement noise covariance \( R \) by \( R^{\text{svr}} = R + \Sigma_d \), and develop its own filtering process to estimate state \( x_k \):

\[
\begin{align*}
\hat{x}^{\text{svr}}_{k|k-1} &= A\hat{x}^{\text{svr}}_{k-1|k-1} + Bu_{k-1}, \\
P^{\text{svr}}_{k|k-1} &= AP^{\text{svr}}_{k-1|k-1}A^T + Q, \\
K_k &= P^{\text{svr}}_{k|k-1} C' (CP^{\text{svr}}_{k|k-1}C' + R^{\text{svr}})^{-1}, \\
\hat{x}^{\text{svr}}_k &= \hat{x}^{\text{svr}}_{k|k-1} + K_k (z_k - C\hat{x}^{\text{svr}}_{k|k-1}), \\
P^{\text{svr}}_k &= (I - K_k C) P^{\text{svr}}_{k|k-1},
\end{align*}
\]

where the superscript “svr” represents the corresponding variables are the server’s local version. This case is equivalent to that the user shares its privacy parameters to the server, as in many previous studies \([6], [35]\). Nevertheless, the knowledge of the server is still corrupted by the noise and hence the user still preserves certain level of privacy.

Remark 6: A feasible analysis on the state trajectory which can be done by the server is to smooth past estimates after the process ends: the server could compute \( \hat{x}^{\text{pub}}_{k|T} \), where \( T \) is the time horizon of the process. Two problems can be visited: 1) what is smoothed result \( \hat{x}^{\text{pub}}_{k|T} \) like and how it is affected by the future information where noise injection also exists, 2) the comparison of \( \hat{x}^{\text{pub}}_{k|T} \) with both \( \hat{x}^{\text{svr}}_{k|T} \) and \( \hat{x}^{\text{svr}}_{k|T} \), respectively.

C. LQG Control Performance

The server computes the control law based on the knowledge of the user’s state estimate. Under the privacy scheme, the server computes the control input as

\[ u_k = L_k \hat{x}^{\text{pub}}_{k|k}, \]

Define

\[ \hat{d}_k \triangleq \hat{x}^{\text{pub}}_{k|k} - \hat{x}_{k|k}. \]

Hence, the true dynamics now is as follows:

\[ x_{k+1} = Ax_k + Bu_k + w_k, \]

\[ u_k = L_k (\hat{x}_{k|k} + \hat{d}_k). \tag{26} \]

In eqn. (26), we can see the caused effect by the privacy scheme. The injected noise also creates a deviation in the control input which will accumulate through the closed-loop, and hence makes the system performance not straightforward to analyze. The control law set \( \{u_k\} \) is non-optimal now; therefore, the resulting state trajectory \( \{x_k\} \) is also non-optimal. Assume that the noise sequence \( \{w_k\} \) and \( \{v_k\} \) is given. Let the state trajectory under the optimal control law \( \{u_k^*\} \) be \( \{x_k^*\} \) and its estimate be \( \{\hat{x}_{k|k}^*\} \) in the system without privacy scheme. We can see that \( \{u_k\}, \{x_k\} \) and \( \{\hat{x}_{k|k}\} \) are all deviated from \( \{u_k^*\}, \{x_k^*\} \) and \( \{\hat{x}_{k|k}^*\} \), respectively, and hence the LQG objective is also affected.

Then we consider the sacrifice in the LQG performance caused by the privacy scheme. Without misunderstanding, we still use the notation \( \mathcal{O}_{0:T} \) to denote the objective under the privacy scheme, which also includes the effect of the uncertainty of \( \delta_k \) now. Define the metric of the quality loss in LQG control performance as

\[ Q_{LQG} = \frac{1}{T} (\mathcal{O}_{0:T} - \mathcal{O}_{0:T}). \tag{27} \]

Theorem 3: Under the privacy scheme, it holds that

\[ \mathcal{O}_{0:T} = E(x_0' S_0 x_0) + \sum_{k=0}^{T-1} \text{tr}(S_{k+1} Q) + \sum_{k=0}^{T-1} \text{tr}(\Phi_k P_k) + \sum_{k=0}^{T-1} \text{tr}(\Phi_k Q^k_{\text{privacy}}), \]

where

\[ \Phi_k = A'S_{k+1} B'(B'S_{k+1} B + U)^{-1} B'S_{k+1} A. \]

Proof: In appendix.

We find that an additional item in the objective value is produced by the privacy scheme. Hence, the performance loss of the LQG control caused by the privacy scheme is given as

\[ Q_{LQG} = \frac{1}{T} \sum_{k=0}^{T-1} \text{tr}(\Phi_k Q^k_{\text{privacy}}). \tag{28} \]

D. Optimization Problem

After the previous analysis, we can see that the privacy preservation is gained at the price of the LQG control performance. To study the trade-off, we propose the following optimization problem: to maximize the privacy metric \( Q_{\text{privacy}} \) when the loss of LQG control performance \( Q_{LQG} \) is required to be under a given level \( \alpha \). This problem is formulated as follows.

Problem 1:

\[ \max_{\Sigma_{d}; Q^k_{\text{privacy}}} \text{tr}(Q_{\text{privacy}}) \]

s.t.

\[ Q_{LQG} \leq \alpha, \]

\[ Q^k_{\text{privacy}} = (I - K_k C) A Q^{k-1}_{\text{privacy}} A'(I - K_k C)' + K_k \Sigma_d K_k', \]

\[ k = 1, 2, ..., T. \]

We find this problem is a linear programming one and can be solved efficiently by numerical methods.

E. Infinite-Time Horizon

Particularly, we consider the case of infinite-time horizon. When the time horizon \( T \) approaches infinity, according to the property of Kalman filtering, \( K_k \) reaches a steady value:

\[ K = \lim_{k \to \infty} K_k. \]
Then when \( k \to \infty \), eqn. \( 22 \) becomes
\[
\lim_{k \to \infty} Q^k_{\text{privacy}} = (I - KC)A \lim_{k \to \infty} Q_{\text{privacy}}^{k-1} A'(I - KC)' + K \Sigma_\delta K'.
\]

Consider the Lyapunov equation
\[
X = (I - KC)AXA'(I - KC)' + K \Sigma_\delta K'. \quad (29)
\]

Since \( K \) is the Kalman gain, \((I - KC)A\) is stable \([38]\). Hence, the existence of the unique solution to \((29)\) is guaranteed. Then the steady state \( \lim_{k \to \infty} Q^k_{\text{privacy}} \) exists. We have the average deviation covariance in this infinite-time case as follows:
\[
Q_{\text{privacy}} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} Q^k_{\text{privacy}}.
\]

Then we find that
\[
Q_{\text{privacy}} = \lim_{k \to \infty} Q^k_{\text{privacy}},
\]
i.e., \( Q_{\text{privacy}} \) coincides with the steady state of \( Q^k_{\text{privacy}} \). Then we have
\[
Q_{\text{privacy}} = (I - KC)A Q_{\text{privacy}} A'(I - KC)' + K \Sigma_\delta K'. \quad (30)
\]

We also present the lower bound of \( Q_{\text{privacy}} \) in this case.

**Corollary 1:** Let \( M \) and \( M^- \) be the solutions to
\[
M^- = AMA',
\]
\[
M = M^- - M^- C'(CM - C' + \Sigma_\delta)^{-1} CM^-.
\]

Then
\[
Q_{\text{privacy}} \geq M.
\]

**Proof:** We can find \( M = \lim_{k \to \infty} M_k \), where \( M_k \) is defined in Theorem \( 2 \). Since \( Q^k_{\text{privacy}} \geq M_k \) for all \( k \), then we have the result. \( \blacksquare \)

Meanwhile, when \( T \to \infty \), according to the LQG control theory, the matrix \( S_k \) defined in eqn. \( 3 \) converges to a steady state and becomes time-invariant. We define
\[
S = \lim_{T \to \infty} S_k.
\]

Then \( S \) satisfies
\[
S = A'SA + W - A'SB(U + B'SB)^{-1} B'SA. \quad (31)
\]

Meanwhile, the associated LQG gain \( L_k \) also becomes time-invariant.

In the infinite-time case, the formulations is reduced accordingly. The loss of LQG performance is simplified as follows:
\[
Q_{LQG} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \text{tr}(\Phi_k Q_{\text{privacy}}^k) = \text{tr}(\Phi Q_{\text{privacy}}),
\]
where \( \Phi = \lim_{k \to \infty} \Phi_k = A'SB(U + B'SB)^{-1} B'SA. \) Since \( Q_{LQG} \) is finite, the stability of the system dynamics is guaranteed.

Then we have the following optimization problem:

**Problem 2:**
\[
\begin{align*}
\max_{\Sigma_\delta, Q_{\text{privacy}}} & \quad \text{tr}(Q_{\text{privacy}}) \\
\text{s.t.} & \quad \text{tr}(\Phi Q_{\text{privacy}}) \leq \alpha, \\
& \quad Q_{\text{privacy}} = (I - KC)A Q_{\text{privacy}} A'(I - KC)' + K \Sigma_\delta K'.
\end{align*}
\]

**Lemma 2:** In the scalar system, the solution to Problem 2 is given by
\[
\Sigma_\delta = \frac{\alpha}{\Phi K^2} \left[ 1 - A^2(1 - KC)^2 \right]
\]
and
\[
Q^*_{\text{privacy}} = \frac{\alpha}{\Phi}.
\]

**Proof:** The result is solved by direct calculation. \( \blacksquare \)

**IV. PERFECT PRIVACY PRESERVATION WITHOUT PERFORMANCE LOSS**

To the user, the best situation is that the server has deviated knowledge of the states while the user is still able to obtain optimal control inputs. We provide simple results on this direction in this section.

**A. System with Perfect State Information**

We first consider a simple scenario. Assume that the user is able to directly have the value of \( x_k \) and the does not need the sensor. This is a simplified version of the scenario we consider in this paper. Then the privacy scheme becomes
\[
z_k = x_k + \delta_k. \quad (32)
\]

In this case, the server will take \( z_k \) as the user’s state \( x_k \). We denote the trajectory the server sees as \( \{x^{\text{pub}}_k\} \). According to the privacy scheme, \( x^{\text{pub}}_k \) is identical to \( z_k \): \( x^{\text{pub}}_k = x_k + \delta_k \).

Similarly, the server computes the control input according to
\[
u_k = L_k x^{\text{pub}}_k. \quad (34)
\]

Denote the LQG objective in this scenario as \( I_{0:T} \) and the optimal objective as \( I^{\ast}_{0:T} \).

Usually, the added noise \( \delta_k \) in the transmitted signal also causes a corresponding additional deviation in the generated control input from the optimal one. However, this added \( \delta_k \) does not always cause a deviation in the control input. As one example, if \( \delta_k \) lies in \( \mathcal{N}(L_k) \), the kernel space of \( L_k \), i.e., \( L_k \delta_k = 0 \), we have
\[
u_k = L_k x^{\text{pub}}_k = L_k (x_k + \delta_k) = L_k x_k,
\]
i.e., the resulted \( u_k \) is the optimal control input for the current state \( x_k \). We follow this idea and obtain the following result.

**Theorem 4:** Given an arbitrary time \( k \), for the variable \( \delta_k \) in eqn. \( 23 \), if
1) \( \delta_k \in \mathcal{N}(A) \) when \( A \) is not of full rank,
2) or \( S_{k+1} A \delta_k \in \mathcal{N}(B') \),
it holds that \( u_k = L_k x_k \), which is the optimal control input at time \( k \). If for each time \( k \), within 1) and 2), there always exists
one condition which is satisfied, then \( u_k = u_k^* \) and \( x_k = x_k^* \) for all \( k \). Moreover, \( \mathcal{I}_{0:T} = \mathcal{I}_{0:T}^* \).

**Proof:** Notice that \( L_k = -(B'S_{k+1} + U)^{-1}B'S_{k+1}A \) and \( B'S_{k+1} + U \) has full rank. Either condition 1) or 2) leads to that \( L_k\delta_k = 0 \).

In this methodology, the variable \( \delta_k \) is unnecessary to be random. This methodology may create arbitrary deviation in the server’s knowledge of \( x_k \) while still maintaining optimal control performance.

However, the problem is, this methodology needs the user to do complex computation before the process. Usually, the reason for a user to employ a server is that it lacks necessary computation capability. But maybe the user could employ another server to compute this particular set of \( \delta_k \) for it.

### B. System with Imperfect State Information

In this scenario, which is mainly studied in this paper, it becomes more difficult to find the perfect preservation scheme than in the previous scenario. We present a result for the system with the matrix \( A \) without full rank.

**Theorem 5:** If \( A \) is not of full rank, for all time \( k \), if the variable \( \delta_k \) in eqn. (20) satisfies that \( K_k\delta_k \neq 0 \) and \( AK_k\delta_k = 0 \), then \( u_k = u_k^*, x_{k|k} = x_{k|k}^*, \) and \( x_{k|k}^* \neq \hat{x}_{k|k}^* \) for all \( k \). Moreover, \( O_{0:T} = O_{0:T}^* \).

**Proof:**

We have \( \hat{x}_{0|0}^* = \hat{x}_{0|0} = \hat{x}_{0|0}^* \) initially. Then

\[
u_0 = L_0\hat{x}_{0|0}^* = L_0\hat{x}_{0|0} = u_0^*.
\]

Then we have

\[
\hat{x}_{1|1}^* = A\hat{x}_{0|0}^* + Bu_0 = A\hat{x}_{1|0} + Bu_0^* = \hat{x}_{1|0}.
\]

It is simple to find \( \hat{x}_{1|1} = \hat{x}_{1|1}^* \). Moreover, since \( K_1\delta_1 \neq 0 \),

\[
\hat{x}_{1|1}^* = \hat{x}_{1|0} + K_1(\gamma_1 - C\hat{x}_{1|0})
\]

\[
\neq \hat{x}_{1|0} + K_1(\gamma_1 - C\hat{x}_{1|0}) = \hat{x}_{1|1} = \hat{x}_{1|1}.
\]

Consider \( k = 2 \), we have

\[
u_1 = L_1\hat{x}_{1|1} = L_1\hat{x}_{1|0} + K_1(\gamma_1 - C\hat{x}_{1|0})
\]

\[
= L_1\hat{x}_{1|0} + K_1(\gamma_1 - C\hat{x}_{1|0}) + L_1K_1\delta_1
\]

\[
= L_1\hat{x}_{1|0} + L_1K_1\delta_1.
\]

Since \( L_1K_1\delta_1 = -(B'S_2 + U)^{-1}B'S_2AK_1\delta_1 = 0 \), we have \( L_1K_1\delta_1 = 0 \). Then we have the equal

\[
u_1 = L_1\hat{x}_{1|1} = u_1^*.
\]

Further we have

\[
\hat{x}_{2|1}^* = A\hat{x}_{1|1} + Bu_1^* = A\hat{x}_{1|0} + K_1(\gamma_1 - C\hat{x}_{1|0}) + Bu_1^*
\]

\[
= A\hat{x}_{1|0} + K_1(\gamma_1 - C\hat{x}_{1|0}) + AK_1\delta_1 + Bu_1^*
\]

\[
= A\hat{x}_{1|0} + Bu_1^* = \hat{x}_{2|1}.
\]

We extend the derivation to general \( k \) and obtain \( u_k = u_k^* \), \( \hat{x}_{k|k} = \hat{x}_{k|k}^* \), and \( \hat{x}_{k|k}^* \neq \hat{x}_{k|k}^* \) for all \( k \). We also see that \( \hat{x}_{k|k-1} = \hat{x}_{k|k-1} = \hat{x}_{k|k-1}^* \).

### V. Examples

In this section, we provide two examples to demonstrate the performance of the privacy schemes.

**Example 1:** We consider a higher-order cooperative system with infinite-time horizon, whose parameters are given as follows:

\[
A = \begin{bmatrix} 0.19 & 0.86 & 0.10 \\ 0.31 & 0.80 & 0.44 \\ 0.13 & 0.43 & 0.40 \end{bmatrix}, \quad B = \begin{bmatrix} 2.0 & 0.9 \\ 9.1 & 2.0 \\ 1.3 & 8.1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 2.0 & 1.6 & 1.2 \\ 2.0 & 2.0 & 1.1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.9 & 0.9 & 0.4 \\ 0.9 & 2.8 & 2.0 \\ 0.4 & 2.0 & 2.4 \end{bmatrix},
\]

\[
R = \begin{bmatrix} 7.0 & 1.8 \\ 1.8 & 0.8 \end{bmatrix}, \quad W = \begin{bmatrix} 1.8 & 2.0 & 0.5 \\ 2.0 & 9.8 & 0.9 \\ 0.5 & 0.9 & 5.4 \end{bmatrix},
\]

\[
U = \begin{bmatrix} 4.5 & 1.0 \\ 1.0 & 8.8 \end{bmatrix}.
\]

Given a level \( \alpha \) of the LQG control performance \( Q_{LQG} \), the optimal privacy performance \( Q_{privacy} \) is determined by solving Problem [2]. We denote the optimal \( Q_{privacy} \) as \( Q_{privacy}^{*} \). We plot the traces of \( Q_{privacy}^{*} \) when \( \alpha \) varies from 0 to 50 with step length of 1, which is presented in Fig. 4.

![Fig. 4. The plot of tr(Q_{privacy}^*) when \( \alpha \) varies from 0 to 50.](image)

We find the curve is a straight line. Although we cannot explicitly obtain the property of the curve in a higher-order system, it is proved in the case of scalar system, which is shown in Lemma [2].

Furthermore, given a particular value of \( \Sigma_k \), we plot the true estimates \( \hat{x}_{k|k} \) and the public estimates \( \hat{x}_{k|k}^* \). Let \( \Sigma_k = 100I \). The plots are presented in Fig. 5. In the plots, we find that \( \hat{x}_{k|k}^* \) is deviated from \( \hat{x}_{k|k} \).

Moreover, we plot the traces of several covariances in Fig. 6. From the deviation covariance \( Q_{privacy}^{k} \), its lower bound \( M_k \), the true estimation error covariance \( P_{k|k} \), and the real error covariance \( P_{k|k,real} \) associated with \( \hat{x}_{k|k} \). Notice that the curve of \( P_{k|k} \) does not coincide with the one of \( M_k \). Meanwhile, we can find that \( M_k \) is only a loose bound of \( Q_{privacy}^{k} \).

**Example 2:** Consider the heating, ventilation, and air conditioning (HVAC) system of a building, which is rented and occupied by a number of user companies and organizations. The building’s department of daily maintenance has an intelligent operation center to work for the HVAC control of the
Consider that the company chooses a particular value of $\Sigma_d$. Let $\Sigma_d = 1$. Then we plot for it the true estimates $\hat{x}_{k|k}$ and the public estimates $\hat{x}^{pub}_{k|k}$ within 24 hours. The plots are presented in Fig. 7. In the plots, we find that $\hat{x}^{pub}_{k|k}$ is deviated from $\hat{x}_{k|k}$. The estimated temperature floats by around 2 degrees, which is acceptable for people who care about their privacy. One can further tune the parameters to optimize the result.

### VI. Future Work

Possible future work may be considered from the following directions.

**System structures.** In this paper we considered a simple cooperative networked control system. In most current literature, the considered systems also have simple structures, such as the centralized ones. General systems in practical world should have various and more complex structures, such as multiple participating parties and more complex topologies.

**The function to process the original data.** In this paper, we only considered adding a zero-mean white Gaussian noise to the original measurement. More advanced forms of noise and more general functions could be further investigated.

**An extra processor at user’s receiving side.** In this paper we only have one processor (before sending data to the server), aiming at using the simplest equipment to preserve privacy. An extra processor at the receiving side can help process the control input back from the server, and can be expected to further improve the scheme performance. The cooperation of the two processors may work like an encoder and a decoder.

**Co-design with the data transmission.** We may call it strategic communication. For example, the user claims a sampling frequency which doubles the real one. At the odd time instance, the user sends the real value and takes the resulting control input; at the even time instance, the user sends interrupting signals and throws the resulting control law (by the the decoding processor). By doing this, the public information is quite different from the true one.

**The information to protect.** Most of the literature, including our study, only considered certain signals as privacy. A larger range of concerns also could be privacy, such as the system parameters, the system structure, or even the goal of the user. The ideal relationship between the user and server may be just the server give the user whatever it claims wanted.
A doubtful server. The possibility that the user has a local privacy scheme may motivate the curious server to deploy a detection module to check the user’s shared data. Then the user should consider the design of the privacy scheme under that situation.

The trouble of the server. As a result of the user’s local privacy scheme, the server cannot collect correct information, which will hurt its statistical consensus. Better designs benefiting both parties could be considered.

VII. CONCLUSION

In this paper, we propose the methodology of local design for privacy preservation for the user in a cooperative LQG control system. We propose the localized privacy scheme and the corresponding privacy metric. Then we analyze the performance of the privacy scheme under the privacy scheme and analyze the service loss in LQG control performance. At last, we propose and solve optimization problems based on the trade-off between the privacy quality and service loss.

APPENDIX

A. Proof of Theorem 7

We consider the recursion of the estimates. When \( k = 0 \), we have

\[
\begin{align*}
\hat{x}_{0|0} &= \hat{x}_{0|0} = \bar{x}_0, \\
P_{0|0} &= P_{0|0} = \Sigma_0.
\end{align*}
\]

When \( k = 1 \), the system dynamic is as follows:

\[
\begin{align*}
x_1 &= Ax_0 + Bu_0 + w_0, \\
u_0 &= L_0 \hat{x}_{0|0}, \\
y_1 &= Cx_1 + v_1.
\end{align*}
\]

The true estimate evolves as follows:

\[
\begin{align*}
\hat{x}_{1|0} &= \hat{x}_{0|0} + Bu_0, \\
P_{1|0} &= AP_{0|0}A^T + Q, \\
K_1 &= P_{1|0}C'(CP_{1|0}C' + R)^{-1}, \\
\hat{x}_{1|1} &= \hat{x}_{1|0} + K_1(y_1 - C\hat{x}_{1|0}), \\
P_{1|1} &= (I - K_1C)P_{1|0}.
\end{align*}
\]

At the server side, the public estimate evolves as follows:

\[
\begin{align*}
\hat{x}_{1|0}^{\text{pub}} &= \hat{x}_{0|0}^{\text{pub}} + Bu_0, \\
P_{1|0}^{\text{pub}} &= AP_{0|0}^{\text{pub}}A^T + Q, \\
K_1^{\text{pub}} &= P_{1|0}^{\text{pub}}C'(CP_{1|0}^{\text{pub}}C' + R)^{-1}, \\
\hat{x}_{1|1}^{\text{pub}} &= \hat{x}_{1|0}^{\text{pub}} + K_1^{\text{pub}}(z_1 - C\hat{x}_{1|1}^{\text{pub}}), \\
P_{1|1}^{\text{pub}} &= (I - K_1^{\text{pub}}C)P_{1|0}^{\text{pub}}.
\end{align*}
\]

Meanwhile, we can find that

\[
\begin{align*}
\hat{x}_{1|1}^{\text{pub}} &\neq \hat{x}_{1|1}, \\
K_1^{\text{pub}} &= K_1, \\
P_{1|1}^{\text{pub}} &= P_{1|1}.
\end{align*}
\]

For general \( k \), the system dynamic is

\[
\begin{align*}
x_k &= Ax_{k-1} + Bu_{k-1} + w_{k-1}, \\
u_{k-1} &= L_{k-1}^{\text{pub}}x_{k-1|1-k-1}, \\
y_k &= Cx_k + v_k.
\end{align*}
\]

The true estimate evolves according to eqn. (7)-(11) and the public estimate evolves according to eqn. (13)-(17). From the recursion, we can find that

\[
\begin{align*}
\hat{x}_{k|k}^{\text{pub}} &\neq \hat{x}_{k|k}, \\
K_k^{\text{pub}} &= K_k, \\
P_{k|k}^{\text{pub}} &= P_{k|k}.
\end{align*}
\]

Then we compare \( \hat{x}_{k|k}^{\text{pub}} \) and \( \hat{x}_{k|k} \). We have

\[
\begin{align*}
\hat{x}_{k|k}^{\text{pub}} - \hat{x}_{k|k} &= (I - K_kC)(\hat{x}_{k|k}^{\text{pub}} - \hat{x}_{k|k-1}) + K_k(z_k - y_k) \\
&= (I - K_kC)(\hat{x}_{k|k}^{\text{pub}} - \hat{x}_{k|k-1}) + K_k\delta_k.
\end{align*}
\]

Since \( K_k^{\text{pub}} = K_k \),

\[
\begin{align*}
\hat{x}_{k|k-1}^{\text{pub}} &= A\hat{x}_{k-1|k-1}^{\text{pub}} + Bu_{k-1}, \\
\hat{x}_{k|k-1} &= A\hat{x}_{k-1|k-1}^{\text{pub}} + Bu_{k-1}.
\end{align*}
\]

then

\[
\hat{x}_{k|k-1}^{\text{pub}} - \hat{x}_{k|k-1} = A(\hat{x}_{k|k-1}^{\text{pub}} - \hat{x}_{k|k-1}^{\text{pub}}).
\]

Hence we have

\[
\begin{align*}
\hat{x}_{k|k}^{\text{pub}} - \hat{x}_{k|k} &= (I - K_kC)A(\hat{x}_{k|k-1}^{\text{pub}} - \hat{x}_{k|k-1}^{\text{pub}}) + K_k\delta_k
\end{align*}
\]

and

\[
Q_{\text{privacy}}^k = (I - K_kC)AQ_{\text{privacy}}^{k-1}A'(I - K_kC)' + K_k\Sigma_k\delta_k K_k'.
\]

Since \( \hat{x}_{0|0}^{\text{pub}} = \hat{x}_{0|0} \), \( Q_{\text{privacy}}^0 \) is initialized by \( Q_{\text{privacy}}^0 = 0 \).

B. Proof of Theorem 5

In this scenario, the states are unaccessible. When we calculate the LQG objective \( \mathcal{O}_{0:T} \) from backward, the expectation should condition on the information set defined as follows:

\[
\mathbb{I}_k = \{y_1, y_2, ..., y_k, u_0, u_1, ..., u_{k-1}\}. \tag{35}
\]

Define

\[
\begin{align*}
H_k &= x_k'Wx_k + u_k'Uu_k, & k = 0, 1, ..., T - 1, \tag{36} \\
H_T &= x_T'Wx_T. \tag{37}
\end{align*}
\]

Then we define

\[
\mathcal{O}_{k:T} = \mathbb{E}\left( \sum_{i=k}^{T-1} H_i \mid \mathbb{I}_k \right).
\]
where the expectation is taken with respect to all possible uncertainties, i.e.,

- $x_k$,
- $w_1, w_{k+1}, \ldots, w_{T-1}$ and $v_{k+1}, v_{k+2}, \ldots, v_T$,
- $\delta_k, \delta_{k+1}, \ldots, \delta_{T-1}$.

Notice that $O_{k:T}$ is a function of $\mathbb{I}_k$, i.e., $O_{k:T}(\mathbb{I}_k)$. We have

$$O_{k:T} = E(x_kWx_k + u_k^Tuu_k + O_{k+1:T} | \mathbb{I}_k)$$

$$= E \left[ \begin{array}{c} E(x_kWx_k + u_k^Tuu_k + \hat{O}_{k+1:T} | \mathbb{I}_k, \delta_k) \end{array} \right].$$

We calculate the objective in a backward manner. First we have

$$O_{T:T} = E(x_TWx_T | \mathbb{I}_T).$$

Then

$$O_{T-1:T} = E(H_{T-1} + O_{T:T} | \mathbb{I}_{T-1})$$

$$= E \left[ \begin{array}{c} E \left( x_{T-1}Wx_{T-1} + u_{T-1}^Tuu_{T-1} \right. \\
+ x_{T-1}Wx_{T-1} | \mathbb{I}_{T-1}, \delta_{T-1} \end{array} \right] | \mathbb{I}_{T-1}.$$  

To simplify the formulation, define

$$J_{T:1:T} = E(x_{T-1}Wx_{T-1} | \mathbb{I}_{T-1}, \delta_{T-1}) + u_{T-1}^Tuu_{T-1}$$

$$+ E(\hat{O}_{T:1:T} | \mathbb{I}_{T-1}, \delta_{T-1}).$$

Then

$$O_{T-1:T} = E \left( J_{T:1:T} | \mathbb{I}_{T-1} \right).$$

Since $\mathbb{I}_{T-1}$ and $\delta_{T-1}$ are conditioned on, the control input $u_{T-1}$, which is determined by $\mathbb{I}_{T-1}$ and $\delta_{T-1}$, is hence fixed. By straightforward calculation, we have

$$J_{T:1:T}$$

$$= E(x_{T-1}Wx_{T-1} | \mathbb{I}_{T-1}, \delta_{T-1}) + u_{T-1}^Tuu_{T-1}$$

$$+ E(\hat{O}_{T:1:T} | \mathbb{I}_{T-1}, \delta_{T-1}).$$

According to eqn. $\mathcal{H}$ and we have

$$J_{T-1:T} = E(x_{T-1}S_{T-1}x_{T-1} | \mathbb{I}_{T-1}, \delta_{T-1}) + tr(S_{T}Q)$$

$$+ tr[A'WB(U + B'S_{k+1}B)^{-1}B'WA_{T-1}x_{T-1}]$$

$$+ \hat{d}_{T-1}^2.$$
Similarly, according to the definition of $S_k$ and $L_k$, and define

\[ r_k = r_{k+1} + \text{tr}(S_{k+1}Q), \]
\[ t_k = t_T + \text{tr}\left[A'S_{k+1}B(U + B'S_{k+1}B)^{-1}B'S_{k+1}AP_{k|k}\right], \]

we have

\[ \mathcal{J}_{k:T} = E(x_{k}x_{k} | \mathcal{U}_{k}, \tilde{d}_k) + r_k + t_k + \phi_{k+1} + (u_k - L_k x_k)'(U + B'S_{k+1}B)(u_k - L_k x_k). \]

Since

\[ u_k = L_k (\hat{x}_k + \hat{d}_k), \]

we have

\[ \mathcal{J}_{k:T} = E(x_{k}x_{k} | \mathcal{U}_{k}, \tilde{d}_k) + r_k + t_k + \phi_{k+1} + \tilde{d}_k L_k' (U + B'S_{k+1}B) L_k \tilde{d}_k. \]

Back to $\mathcal{O}_{k:T}$, we have

\[ \mathcal{O}_{k:T} = E(x_{k}S_k x_k | \mathcal{U}_k) + r_k + t_k + \phi_{k+1} + \text{tr}\left[L_k' (U + B'S_{k+1}B) L_k Q_k^{\text{privacy}}\right]. \]

Let

\[ \phi_k = \phi_{k+1} + \text{tr}\left[L_k' (U + B'S_{k+1}B) L_k Q_k^{\text{privacy}}\right], \]

then

\[ \mathcal{O}_{k:T} = E(x_{k}S_k x_k | \mathcal{U}_{k}) + r_k + t_k + \phi_k. \]

Hence, ultimately we have

\[ \mathcal{O}_{0:T} = E(x_0S_0 x_0) + r_0 + t_0 + \phi_0 = E(x_0'x_0|S_0) + \sum_{k=0}^{T-1} \text{tr}(S_{k+1}Q) + \sum_{k=0}^{T-1} \text{tr}(\phi_k P_k) + \sum_{k=0}^{T-1} \text{tr}\left[L_k' (U + B'S_{k+1}B) L_k Q_k^{\text{privacy}}\right]. \]

\[ = E(x_0'x_0|S_0) + \sum_{k=0}^{T-1} \text{tr}(S_{k+1}Q) + \sum_{k=0}^{T-1} \text{tr}(\phi_k P_k) + \sum_{k=0}^{T-1} \text{tr}(\phi_k Q_k^{\text{privacy}}). \]

\[ \square \]

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