Space charge limited 2-d electron flow between two flat electrodes in a strong magnetic field

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An approximate analytic solution is constructed for the 2-d space charge limited emission by a cathode surrounded by non emitting conducting ledges of width Λ. An essentially exact solution (via conformal mapping) of the electrostatic problem in vacuum is matched to the solution of a linearized problem in the space charge region whose boundaries are sharp due to the presence of a strong magnetic field. The current density growth in a narrow interval near the edges of the cathode depends strongly on Λ. We obtain an empirical formula for the total current as a function of Λ which extends to more general cathode geometries.

PACS: 52.27.Jt; 52.59.Sa; 52.59.Wd; 85.45.Bz

The study of space charge limited (SCL) current, initiated in the beginning of the last century [1,2], continues to be of great current interest [3-9]. These works are important for the design of high power diodes, techniques of charged particles beams, physics of non-neutral plasmas including plasma sheath, and other numerous applications. The modelling of SCL ionic flow in cylindrical and spherical geometry [3] is also necessary for the inertial-electrostatic confinement of fusion plasmas. Unfortunately only the planar 1-d case permits an analytic solution [1,2] and as pointed out in a recent review [5] even “the seeming simple problem of 2-d planar emission remains unresolved”. This motivated the present work which provides a semi-analytical solution for a prototype 2-d model similar to that studied in [6]. We obtain for the first time, we believe, a reasonable analytic approximation for the currents at the edge of the cathode - an important (though usually undesirable) feature of SCL diodes [6,7]. An extension of our methods should facilitate dealing with this problem to any desirable accuracy thus providing an alternative to PIC simulations.

Model. We consider the current between two conducting flat electrodes where the anode, whose potential is V, is an infinite plane separated by a distance D from the grounded cathode which is an infinitely long strip parallel to the anode. Our assumptions are: 1) The cathode upper surface, of width 2a, has infinite emissivity while the lower face and the ledges of widths Λ do not emit (see Fig.1). 2) A very strong strong magnetic field perpendicular to the electrodes inhibits the transversal components of electron velocities [6,8,9], but almost does not affect the total current [6,8,9]. 3) The emitted electrons leave the cathode with zero velocity [1,2,6].

If the cathode is in the (X,Y) plane and the magnetic field in the Y-direction the velocities v of electrons are parallel to the Y-axis with \( m v^2(X,Y) = 2eU(X,Y) \), where \( U(X,Y) \) represents the potential field while \( m, e \) are the electron mass and charge. The current density \( J(X) \), which clearly is \( Y \)-independent, determines together with \( v(X,Y) \) the density of electrons. Using the dimensionless variables

\[
\begin{align*}
X &= \frac{X}{D}, \quad Y = \frac{Y}{D}, \quad a = \frac{a}{D}, \quad \lambda = \frac{\lambda}{D}, \\
\phi(x,y) &= \frac{U(X,Y)}{V}, \quad j(x) = \frac{\sqrt{m \pi D^2}}{2e V^{3/2}} J(X),
\end{align*}
\]

the nonlinear Poisson equation for the potential then takes the form

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -4\pi \rho(x,y) = \frac{4j(x)}{9\sqrt{\phi(x,y)}} \tag{2}
\]

The electron density \( \rho(x,y) \) and current \( j(x) \) are different from zero only in the shaded rectangle \( Q \) of Fig.1 which shows a two dimensional cross section of our system.

![FIG. 1. Geometry of the system](image)

Eq.(2), subject to the boundary conditions (BC),

\[
\phi(x,0) = 0 \text{ for } |x| < a + \lambda, \quad \phi(x,1) = 1 \text{ for } |x| < \infty,
\]

\[
\frac{\partial \phi}{\partial y}(x,+0) = 0 \text{ for } |x| < a,
\]

is to be solved in the half-plane \( y \leq 1 \) to produce both functions \( \phi(x,y) \) and \( j(x) \) which are non-negative and symmetric about the \( y \)-axis. To do this we first solve eq.(2) approximately in the current region \( Q \) on a rather intuitive level. The problem is nonlinear here and it is not well posed if one disregards the field at \( |x| > a \). Consequently our solution will have a set of free parameters

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which specify \( j(x) \) and \( \phi(x, y) \): in particular \( \phi(a, y) \) and \( \frac{\partial \phi}{\partial x}(a^-, y) \). In the second step the potential \( \phi(\pm a, y) \) is used as the BC and we obtain a Dirichlet problem for the Laplace eq. (2) in the outer region of the half-plane where \( j(x) = 0 \). We solve this problem using conformal mapping techniques and evaluate \( \frac{\partial \phi}{\partial x}(a^+, y) \). If one excludes the points \( x = \pm a, y = 0 \) the electron density \( \rho(a^+, y) \) is finite and \( \rho(a^-, y) = 0 \), therefore the second derivative of \( \phi(x, y) \) has a finite jump at \( x = a \), while the first derivative must be continuous, i.e.

\[
\frac{\partial \phi}{\partial x}(a^-, y) = \frac{\partial \phi}{\partial x}(a^+, y), \quad 0 < y < 1. \tag{4}
\]

In the last step we satisfy approximately the matching condition (4) by adjusting the free parameters mentioned above using the least squares technique. This will give an approximate explicit form for \( j(x) \).

**The space charge region Q.** We want to solve approximately eq. (2) where the function \( j(x) \) is not known nor are the BC for \( \phi \) at \( x = \pm a \). When \( \alpha = \infty \) we have no \( x \) dependence and (2) becomes an ordinary equation which was solved in [1,2] yielding \( \phi_1(y) = y^{4/3}, j_1(x) = 1 \). This gives the Child-Langmuir formula [1], \( J_1 = (2e/m)^{1/2}V^{3/2}/9\pi D^2 \). For \( \alpha > 1 \) it is reasonable to assume that \( j(x) \sim j_1 = 1 \) when \( a - |x| \gg 1 \) and use also a stronger assumption that the difference \( \phi(x, y)/\phi_1(y) - 1 \) is small almost everywhere (i.e. it does not exceed \( \sim 1 - 1.5 \) even near the edges of region). This difference however is not small at the cathode edges, \( x = \pm a \), where the electric field must match the field outside. The large gradients in the field lead to the acceleration of electrons and thus to a strong rise of the current density \( j(x) \) near the boundary of the SCL flow.

We represent \( \phi(x, y) \) in the form \( y^{4/3}[1 + \mu(x, y)] \) and linearize the square root as \( (1 + \mu(x, y))^{-1/2} \approx 1 - \beta \mu(x, y) \), where the number \( \beta \) is chosen to minimize the integral of \( 1 - \beta \mu(x, y) \) on the interval \( 0 \leq \mu \leq 1 \). This yields \( \beta \approx 0.328 \) with relative average error of approximation around 2.2%. For \( \mu = 0.2, 1 \), 1.5 the error is 2.36%, 4.96%, 6.25%, 19.6% respectively. We shall see later that for all \( \lambda \geq 0.1 \mu < 1 \). Substituting in (2) we obtain a linear equation

\[
y^2 \left( \frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} \right) + \frac{8}{3} \frac{\partial \mu}{\partial y} + 4 \frac{1 + \beta}{9} - \mu = 4 \frac{1}{9} [j(x) - 1], \tag{5}
\]

where we dropped a nonlinear term in the right side. The error due to this and to the linearization of the square root is negligible for small \( \mu \) and decreases the right side by at most a factor \( \sim 0.7 \), in all the cases considered (see Table 1) including even \( \mu \approx 2 \).

Using the method of separation of variables we write

\[
\mu(x, y) = \sum_l q_l f_l(x) u_l(y), \quad j(x) = 1 + \frac{9}{4} \sum_l q_l f_l(x), \tag{6}
\]

with

\[
f_l(x) = e^{-k_l(a-x)} + e^{-k_l(a+x)}, \quad |x| \leq a. \tag{7}
\]

Substituting (6) and (7) into (5) and assuming that (5), (3) are satisfied separately for each \( l = 1, 2 \ldots \) gives a set of inhomogeneous equations

\[
y^2 \frac{d^2 u_l}{dy^2} + \frac{8}{3} \frac{du_l}{dy} + \left( k_l^2 y^2 + 4 \frac{1 + \beta}{9} \right) u_l = 1, \tag{8}
\]

with the common BC \( u_l(1) = 0 \). The parameters \( k_l \) and \( q_l \) will be determined later. The potential can be written in the form

\[
\phi(x, y) = y^{4/3} + y^{4/3} \sum_l q_l f_l(x) u_l(y), \tag{9}
\]

where the first term is the Child-Langmuir potential \( \phi_1 \) and the \( u_l(y) \) are assumed finite. The relevant particular solutions of (8), which can be expressed in terms of Lommel’s functions \( s_{-1/6,\alpha}(k_1 y), \nu = \sqrt{9 - 16k_1^2} \) [10], is given by the power series expansion

\[
u(k_1 y) = \frac{9}{4(1 + \beta)} \sum_{n=0}^{\infty} (-1)^n a_n \left( \frac{k_1 y}{2} \right)^{2n}, \tag{10}
\]

where all \( u_l(1) = 0 \) the parameters \( k_l \) are the increasing roots of (14): \( 3.881, 6.675, 10.065, 13.003, 16.316, 19.306, 22.582, 25.600, 28.855, 31.891 \) for \( 1 \leq l \leq 10 \). They can be easily evaluated due to the rapid convergence of (10), asymptotically \( k_l \rightarrow l\pi \). The free parameters \( q_l \) will be used to satisfy (4).

The 2-d mean current density over the whole cathode, which in terms of our scheme is given by

\[
j_2 = \frac{1}{a} \int_0^a j(x) dx \approx 1 + \frac{9}{4a} \sum_l \frac{q_l}{k_l} (1 - e^{-2k_1 a}), \tag{11}
\]

is usually presented [8] as the 1-d current density \( j_1 = 1 + \alpha/2a \). Thus in the original units the mean current has the form

\[
J_2 = J_1 \left( 1 + \frac{\alpha D}{W} \right), \tag{12}
\]

where \( W = 2A \) is the width of the cathode. Using (11) the parameter \( \alpha \) is defined here by

\[
\alpha = 9 \sum_l \frac{q_l}{2k_l} (1 - e^{-2k_1 a}). \tag{13}
\]

**Electrostatic region.** It seems clear that for \( \alpha \gg 1 \) the electric field in the vicinity of the boundary \( x \approx a, 0 \leq y \leq 1 \) is not affected much by the region \( x \leq -a, 0 \leq y \leq 1 \), see Fig.1. This allows us to study a simpler electrostatic problem for a plane which is split according to Fig.2(a). We modified a conformal transform in [11] to the form

\[
z = 2\pi^{-1} \ln(\sqrt{w} + \sqrt{W - 1}) - c\sqrt{w^2 - w}, \tag{14}
\]
which maps the shaded half-plane \( z = x + iy \) on Fig. 2(a) onto the upper half-plane \( w = u + iv \) in Fig. 2(b).

\[
\Phi(u, 0) = \begin{cases} 
1, & \text{if } -\infty < u \leq 0 \\
\phi(a, y(u)), & \text{if } 0 < u < 1 \\
0, & \text{if } u \geq 1,
\end{cases}
\]

has the solution

\[
\Phi(u, v) = \frac{v}{\pi} \int_{-\infty}^{\infty} \frac{\Phi(s, 0)}{(u-s)^2 + v^2} ds
\]

in the upper half-plane \( w \). Here by (14) \( y(u) = 2\pi^{-1} \arccos(\sqrt{u - c}/\sqrt{1-u}) \). The potential \( \Phi(u, v) \) expressed in variables \( x, y \) represents \( \phi(x, y) \) outside the space charge zone. Our task now is to match the inside electric field \( \frac{\partial \phi}{\partial x}(a, y(u)) \) in the interval \( 0 < u < 1 \) with the field outside

\[
\frac{\partial \Phi}{\partial x}(u, v = 0) = \frac{\pi \sqrt{u(1-u)}}{1+c(1-2a)} \frac{\partial \Phi}{\partial v}(u, 0).
\]

Continuity of the electric field. The matching condition (4) guarantees continuity of the electric field at the boundary between the space charge region \( Q \) with the vacuum. Using (7) and (9) we have at \( x = a \) inside the space charge region the field intensity,

\[
\frac{\partial \phi}{\partial x} = y^{4/3} \sum_{l} q_{l} k_{l} (1 - e^{-2k_{l}a}) u_{l}(y)
\]

which should be equal to the vacuum field (17). The exponentially small terms \( e^{-2k_{l}a} \) can be dropped as \( a \) is assumed large. Both terms \( \frac{\partial \Phi}{\partial x} \) and \( \frac{\partial \Phi}{\partial v} \) depend on all parameters \( q_{l} \), but in a different way. One cannot expect an exact equality because of the approximations made. We rewrite the matching condition (4) as

\[
G[y^{4/3}] + \sum_{l} q_{l} \{G[y^{4/3} u_{l}(y)] - k_{l} y^{4/3} u_{l}(y)\} \approx 0,
\]

where the functionals \( G \) can be written explicitly as integrals with a logarithmic singularity.

We minimize the least square divergence from zero of the expression (19) on the interval \( 0.15 < y < 0.85 \) without approaching the endpoints where our treatment is not entirely adequate. A standard procedure yields a set of linear algebraic equations for \( q_{l} \). We did not go further than \( l_{\text{max}} = 10 \). After the \( q_{l} \) are computed one can find the current density (6) and the parameter \( \alpha \) (13).

The accuracy of this method can be evaluated to some degree by determining the relative average discrepancy \( \Delta \) of electric fields at the boundary of the space charge region \( Q \) on the chosen interval of \( y \). The results of our computations are shown in the Table 1, where for different values of the ladder \( \alpha \) one can see also \( \alpha(\lambda) \), parameters \( q_{l}, \mu_{\text{max}} \) (near \( y = 0 \) and \( x = \pm a \)), and the relative height (see Fig. 3) of the current wings \( j_{\text{max}} \), at \( x = \pm a \). When we extend the interval of matching the electric fields up to \( (0.01, 0.99) \) the quantities in the table stay approximately the same, only \( \Delta \) increases. This confirms the general validity of our method and simultaneously shows that the computation of electric fields near the corners of the rectangle \( Q \) is not very good. In particular, in the worst case (the most severe cathode regime, see also \([6,7,12]\)) when \( \lambda = 0 \), the electric field is singular at the cathode edges. The computation becomes unstable, we cannot therefore the data of Table 1 to be accurate there when the linearization fails too.

| \( \lambda \) |
| 0     | 0.1  | 0.3  | 0.5  | 1    | \infty |
| \( \alpha \) |
| 0.6887 | 0.5311 | 0.3463 | 0.2665 | 0.2067 | 0.1905 |
| \( \mu_{\text{max}} \) |
| 1.955 | 1.432 | 0.804 | 0.605 | 0.497 | 0.461 |
| \( j_{\text{max}} \) |
| 3.597 | 2.902 | 2.068 | 1.804 | 1.661 | 1.612 |
| \( \Delta \) |
| 0.0121 | 0.0055 | 0.0037 | 0.0028 | 0.0059 | 0.0044 |
| \( q_{1} \) |
| 0.2448 | 0.2339 | 0.1891 | 0.1530 | 0.1140 | 0.1032 |
| \( q_{2} \) |
| 0.2225 | 0.1743 | 0.0926 | 0.0616 | 0.0465 | 0.0443 |
| \( q_{3} \) |
| 0.1525 | 0.0969 | 0.0389 | 0.0280 | 0.0260 | 0.0246 |
| \( q_{5} \) |
| 0.1184 | 0.0760 | 0.0327 | 0.0253 | 0.0232 | 0.0222 |
| \( q_{6} \) |
| 0.0914 | 0.0500 | 0.0192 | 0.0143 | 0.0148 | 0.0134 |
| \( q_{7} \) |
| 0.0595 | 0.0342 | 0.0142 | 0.0110 | 0.0109 | 0.0100 |
| \( q_{8} \) |
| 0.0439 | 0.0216 | 0.0087 | 0.0061 | 0.0072 | 0.0061 |
| \( q_{10} \) |
| 0.0203 | 0.0108 | 0.0046 | 0.0033 | 0.0037 | 0.0032 |

**TABLE 1**

When \( l \) runs from 1 to 10 the values of \( q_{l} \) decrease approximately by a factor 20-50. Therefore \( \alpha \) is evaluated very well by (13) where the \( k_{l} \) increase from \( \sim 4 \) to 32. The accuracy of \( j(x) \) and \( \mu(x) \) might be improved if one truncates (9) at a larger \( l_{\text{max}} \) though we would not expect dramatic changes. The cases \( \lambda = \infty \) (when the mapping is exact) and \( \lambda = 1 \) are very close which means that the parameter \( \alpha(a, \lambda) \) as well as \( q_{l} \) are approximately independent of \( a \) when \( a > 1 \). Keeping the exponential terms in \( \mu(x, y) \) in the matching conditions (20) does not complicate the calculation and it will give only insignificant corrections. The current density \( j(x) \) at \( x = 0 \) increases in this case by about \( 2q_{1}e^{-3.9a} \). If \( a \) is smaller, but \( 2\lambda + 2a > 1 \), the scheme of computation is the same though the \( q_{l} \) become functions of \( a \) and one cannot decrease \( a \) too much because the first term in (9) needs corrections.

An important part of our analysis is the form (11) of \( f_{l}(x) \) which implies that the current density (10) in a narrow region of width \( \sim 1 \) (\( D \) in the original units) at the cathode edges has a sharp peak which decays faster.
than \( \exp[-3.88(a - |x|)] \). Everywhere else \( j(x) \) is close to the 1-d current \( j_1(x) = 1 \) with the exponentially small corrections. For illustration plots of the current density distribution (6) are shown in Fig.3 for different widths of the cathode.

\[
\text{Fig. 3. The current densities when } \lambda = 0.1, c_1 - c_1, c_2 - c_2, c_3 - c_3 \text{ and } c_4 \text{ correspond to } a = 8, 4, 2, 1 \text{ respectively.}
\]

We can compare our curve \( c_3 \) for \( a = 1 \) with PIC simulations presented in [7]. There for zero cathode recess \( (dx = 0) \), \( A = 50 \text{ mm} \), and unfortunately unspecified width of the shroud a reasonable fit would be \( j_{\text{max}} = 3.9 \) in our Table for \( \lambda = 0.1 \) versus 3.2 in [7]. We get the half-width of the current density peak \( \sim 1.2 \text{ mm} \) while in [7] it was 1 mm. Our magnetic field is stronger and we think also that PIC simulations with finer grids are closer to our computation, but diverge from experimental results because the real cathodes with their finite thickness and roundness do not have the very strong electric field intensities present however in the models.

**Generalization.** We expect that this pattern of narrow wings of the current density holds also for finite flat cathodes with perimeter \( P \) where the boundary region will have an area \( \sim PD \) if we assume reasonable restrictions on the curvature and return back to the original units. In the general case of a cathode with area \( S \gg PD \) the total current can be written as the sum \( J = SJ_1 + PDJ \). The "edge" current \( J \), which is assumed here to be independent of geometry, can be viewed as distributed over the edges of the cathode of width equal to the distance between the electrodes. The ratio \( J/J_1 \) can be evaluated in terms of the parameter \( \alpha \) defined in (13). Comparing \( J = J_1(1 + PDJ_1S) \) with eq. (12) for our geometry, where \( J_1 = J_2, S = 2AL \) and \( P = 2L \) (the length \( L \) of the cathode segment is arbitrary), we have \( J = J_1 \alpha/2 \) and finally

\[
J = J_1 \left( 1 + \frac{PD}{2S} \right),
\]

(20)

which should be applicable in general situations. In particular the factor \( PD/2S \) in (20) becomes \( D/R \) for a circular cathode of the radius \( R \) and \( 2DE(I - C)B^2)/\pi C \) for an elliptical cathode with the half-axes \( B > C \), where \( E(k) \) is the complete elliptical integral. For a rectangular cathode with the sides \( L \) and \( H \) it is equal to \( D(L^{-1} + H^{-1}) \).

**Conclusions.** 1) The current wings, Fig.3, resemble simulated ones [6,7,12]. They are high when the width of ledges \( \lambda \) is small and the vacuum electric field near the cathode edges is strong. Their form becomes practically constant when the ledges are wider than the distance \( D \) between electrodes. 2) The shape of the current wings, which is determined by eigenvalues \( k_l \) of (8), is roughly exponential and the 1-d current is restored up to a few percents at the distance \( D \) from the edges. 3) The parameter \( \alpha \), which defines the net current density, depends on the width of ledges. An approximate empirical formula

\[
\alpha(\lambda) \approx 0.19 + 0.48 \text{e}^{-3.7\lambda},
\]

agrees with the data in Table 1 within \( \sim 3.3\% \). (For a different model with the constant current density \( \alpha \) was estimated in [8] as close to 0.3.) 4) Our techniques of matching the electric fields at the boundary of the space charge region and using rather modest variations of the potential in the \( x \) direction is effective for approximate modelling the 2-d and 3-d flows of charged particles.

**Acknowledgments.** We thank R.Barker, R.J.Umstattd, and O.Costin for inspiration and useful comments. Research supported by AFOSR Grant # F49620-01-0154.

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