ON THE DIMENSION OF SPACES OF ALGEBRAIC CURVES PASSING THROUGH n-INDEPENDENT NODES

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Introduction. Denote the space of all bivariate polynomials of total degree \( \leq n \) by \( \Pi_n \), i.e., \( \Pi_n = \{ \sum_{i+j \leq n} a_{ij}x^iy^j \} \). We have that
\[
N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).
\]

Consider a set of \( s \) distinct nodes \( \mathcal{X} = \{ \mathcal{X}_s \} = \{ (x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s) \} \). The problem of finding a polynomial \( p \in \Pi_n \) which satisfies the conditions
\[
p(x_i, y_i) = c_i, \quad i = 1, \ldots, s,
\]
is called interpolation problem.

A polynomial \( p \in \Pi_n \) is called a fundamental polynomial for a node \( A \in \mathcal{X} \) if \( p(A) = 1 \) and \( p|_{\mathcal{X} \setminus \{A\}} = 0 \), where \( p|_{\mathcal{X}} \) means the restriction of \( p \) on \( \mathcal{X} \). We denote the fundamental polynomial by \( p^*_A \). Sometimes we call fundamental also a polynomial that vanishes at all nodes of \( \mathcal{X} \) but one, since it is a nonzero constant times a fundamental polynomial.

Definition 1. The interpolation problem with a set of nodes \( \mathcal{X}_s \) and \( \Pi_n \) is called n-poised if for any data \( (c_1, c_2, \ldots, c_s) \) there is a unique polynomial \( p \in \Pi_n \) satisfying the interpolation conditions (1).
A necessary condition of poisedness is \(|X_s| = s = N\).

**Proposition 1.** A set of nodes \(X_N\) is \(n\)-poised if and only if
\[
p \in \Pi_n \quad \text{and} \quad p|_{X_N} = 0 \implies p = 0.
\]

Next, let us consider the concept of \(n\)-independence (see [1][2]).

**Definition 2.** A set of nodes \(X\) is called \(n\)-independent, if all its nodes have \(n\)-fundamental polynomials. Otherwise, it is called \(n\)-dependent.

Fundamental polynomials are linearly independent. Therefore a necessary condition of \(n\)-independence of \(X_s\) is \(s \leq N\).

**Some Properties of \(n\)-Independent Nodes.** Let us start with the following simple (see Lemma 2.2 [3]):

**Lemma 1.** Suppose that a node set \(X\) is \(n\)-independent and a node \(A \notin X\) has \(n\)-fundamental polynomial with respect to the set \(X \cup \{A\}\). Then the latter node set is \(n\)-independent too.

Denote the distance between the points \(A\) and \(B\) by \(\rho(A, B)\). Let us recall the following (see [4][5])

**Lemma 2.** Suppose that \(X_s = \{A_i\}_{i=1}^s\) is an \(n\)-independent set. Then there is a number \(\varepsilon > 0\) such that any set \(X'_s = \{A'_i\}_{i=1}^s\), with the property that \(\rho(A_i, A'_i) < \varepsilon, i = 1, \ldots, s\), is \(n\)-independent too.

Next result concerns the extension of \(n\)-independent sets (see Lemma 2.1 [2]).

**Lemma 3.** Any \(n\)-independent set \(X\) with \(|X| < N\) can be enlarged to an \(n\)-poised set.

In the sequel we will need the following modification of the above result.

**Lemma 4.** Given \(n\)-independent sets \(X_i, i = 1, \ldots, m, \text{ where } |X_i| = s_i < N\), a node \(A\) and any number \(\varepsilon > 0\), then there is a node \(A'\) such that \(\rho(A, A') < \varepsilon\) and each set \(X_i \cup \{A'\}, i = 1, \ldots, m\), is \(n\)-independent.

**Proof.** Let us use induction with respect to the number of sets: \(m\). Suppose that we have one set \(X_s\). Since \(s < N\), there is a nonzero polynomial \(p \in \Pi_n\) such that \(p|_{X_s} = 0\). Now evidently there is a node \(B \notin X\) such that \(\rho(A, B) < \varepsilon\) and \(p(B) \neq 0\). Thus \(p\) is an \(n\)-fundamental polynomial of the node \(B\) with respect to the set \(X \cup \{B\}\). Hence, in view of Lemma 1 the set \(X_s \cup \{B\}\) is \(n\)-independent. Then, assume that Lemma is true in the case of \(m - 1\) sets, i.e., there is a node \(B\) such that \(\rho(A, B) < (1/2)\varepsilon\) and each set \(X_i \cup \{B\}, i = 1, \ldots, m - 1\), is \(n\)-independent. In view of Lemma 2 there is a number \(\varepsilon' < (1/2)\varepsilon\) such that for any \(C\) with \(\rho(C, B) < \varepsilon'\) each set \(X_i \cup \{C\}, i = 1, \ldots, m - 1\), is \(n\)-independent. Next, in view of first step of induction there is a node \(A'\) such that \(\rho(A', B) < (1/2)\varepsilon\) and the set \(X_m \cup \{A'\}\) is \(n\)-independent. Now, it is easily seen that \(A'\) is a desirable node. \(\square\)

Denote the linear space of polynomials of total degree at most \(n\) vanishing on \(X\) by
\[
\mathcal{P}_{n,X} = \{p \in \Pi_n : p|_{X} = 0\}.
\]

The following two propositions are well-known [2].

**Proposition 2.** For any node set \(X\) we have that
\[
\dim\mathcal{P}_{n,X} = N - |Y|,
\]
where \(Y\) is a maximal \(n\)-independent subset of \(X\).
Proposition 3. If a polynomial \( p \in \Pi_n \) vanishes at \( n+1 \) points of a line \( \ell \), then we have that \( p = \ell r \), where \( r \in \Pi_{n-1} \).

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter \( p \), say, to denote the polynomial \( p \) of degree \( \geq 1 \) and the curve given by the equation \( p(x,y) = 0 \).

Set \( d(n,k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k) \). The following is a generalization of Proposition 3 (see Prop. 3.1 [6]).

Proposition 4. Let \( q \) be an algebraic curve of degree \( k \leq n \) without multiple components. Then the following hold:

i) any subset of \( q \) containing more than \( d(n,k) \) nodes is \( n \)-dependent;

ii) any subset \( X_s \) of \( q \) containing exactly \( d = d(n,k) \) nodes is \( n \)-independent if and only if the following condition holds:

\[
p \in \Pi_n \quad \text{and} \quad p|\chi_q = 0 \implies p = qr, \quad \text{where} \quad r \in \Pi_{n-k}.
\] (2)

Thus, according to Proposition 4(i), at most \( d(n,k) \) nodes of \( X \) can lie in the curve \( q \) of degree \( k \leq n \). This motivates the following definition (see Def. 3.1 [6]).

Definition 3. Given an \( n \)-independent set of nodes \( \chi_s \) with \( s \geq d(n,k) \), a curve of degree \( k \leq n \) passing through \( d(n,k) \) points of \( \chi_s \) is called maximal.

We say that a node \( A \) of an \( n \)-poised set \( \chi \) uses a line \( \ell \), if the latter divides the fundamental polynomial of \( A \), i.e., \( p^*_A = \ell q \) for some \( q \in \Pi_{n-1} \).

Let us bring a characterization of maximal curves (see Prop. 3.3 [6]):

Proposition 5. Let a node set \( \chi \) be \( n \)-poised. Then a curve \( \mu \) of degree \( k, \ k \leq n \), is a maximal curve if and only if it is used by any node in \( \chi \setminus \mu \).

Next result concerns maximal independent sets in curves (see Prop. 3.5 [5]).

Proposition 6. Assume that \( \sigma \) is an algebraic curve of degree \( k \) without multiple components and \( \chi_s \subset \sigma \) is any \( n \)-independent node set of cardinality \( s, s < d(n,k) \). Then the set \( \chi_s \) can be extended to a maximal \( n \)-independent set \( \chi_d \subset \sigma \) of cardinality \( d = d(n,k) \).

Finally, let us bring a well-known

Lemma 5. Suppose that \( m \) linearly independent curves pass through all the nodes of \( \chi \). Then for any node \( A \notin \chi \) there are \( m-1 \) linearly independent curves in the linear span of given curves, passing through \( A \) and all the nodes of \( \chi \).

Main Result. Let us start with (see Theorem 1 [7]).

Theorem 1. Assume that \( \chi \) is an \( n \)-independent set of \( d(n,k-1) + 2 \) nodes lying in a curve of degree \( k \) with \( k \leq n \). Then the curve is determined uniquely by these nodes.

Next result in this series is the following (see Theorem 4.2 [5])

Theorem 2. Assume that \( \chi \) is an \( n \)-independent set of \( d(n,k-1) + 1 \) nodes with \( k \leq n-1 \). Then two different curves of degree \( k \) pass through all the nodes of \( \chi \) if and only if all the nodes of \( \chi \) but one lie in a maximal curve of degree \( k-1 \).

Now let us present the main result of this paper:

Theorem 3. Assume that \( \chi \) is an \( n \)-independent set of \( d(n,k-2) + 2 \) nodes with \( k \leq n-1 \). Then four linearly independent curves of degree less than or equal
to $k$ pass through all the nodes of $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but two lie in a maximal curve of degree $k - 2$.

Let us mention that the inverse implication here is evident. Indeed, assume that $d(n, k - 2)$ nodes of $\mathcal{X}$ are located in a curve $\mu$ of degree $k - 2$. Therefore, the curve $\mu$ is maximal and the remaining two nodes of $\mathcal{X}$, denoted by $A$ and $B$, are outside of it: $A, B \notin \mu$. Hence we have that

$$\mathcal{P}_{k, \mathcal{X}} = \{ p : p \in \Pi_k, p(A) = p(B) = 0 \} = \{ q \mu : q \in \Pi_2, q(A) = q(B) = 0 \}.$$ 

Thus we readily get that $\dim \mathcal{P}_{k, \mathcal{X}} = \dim \{ q \in \Pi_2 : q(A) = q(B) = 0 \} = \dim \mathcal{P}_{2, \{A, B\}} = 6 - 2 = 4$. In the last equality we use the fact that any two nodes are 2-independent.

We get also that there can be at most 4 linearly independent curves of degree $\leq k$ passing through all the nodes of $\mathcal{X}$.

Before starting the proof of Theorem 6 let us present two lemmas.

**Lemma 6.** Assume that $\mathcal{X}$ is an $n$-independent node set and a node $A \in \mathcal{X}$ has an $n$-fundamental polynomial $p_A^*$ such that $p_A^*(A') \neq 0$. Then we can replace the node $A$ with a node $A'$ which belongs only to one component of $\sigma$; in particular, such replacement can be done in the following two cases:

1. if a node $A \in \mathcal{X}$ belongs to several components of $\sigma$, then we can replace it with a node $A'$, which belongs to only one component of $\sigma$;

2. if a curve $q$ is not a component of an $n$-fundamental polynomial $p_A^*$ then we can replace the node $A$ with a node $A'$ lying in $q$.

**Proof.** Indeed, notice that $p_A^*(A') \neq 0$ means that $p_A^*$ is a fundamental polynomial for the node $A'$ with respect to the set $\mathcal{X}'$. Next, for i) note that a fundamental polynomial of a node $A$ differs from 0 in a neighborhood of $A$. Finally, for ii) note that $q$ is not a component of $p_A^*$ means, that there is a point $A' \in q$ such that $p_A^*(A') \neq 0$. □

**Lemma 7.** Assume that the hypotheses of Theorem 6 hold and assume additionally that there is a curve $q_{k-1} \in \Pi_{k-1}$ passing through all the nodes of $\mathcal{X}$. Then all the nodes of $\mathcal{X}$ but two lie in a maximal curve $\mu$ of degree $k - 2$.

**Proof.** First note that the curve $q_{k-1}$ is of exact degree $k - 1$, since it passes through more than $d(n, k - 2)$ $n$-independent nodes. This implies also that $q_{k-1}$ has no multiple component. Therefore, in view of Proposition 6 we can extend the set $\mathcal{X}$ till a maximal $n$-independent set by adding $n - k + 1$ nodes, i.e.,

$$\mathcal{Y} = \mathcal{X} \cup \mathcal{A},$$

where $\mathcal{A} = \{ A_0, \ldots, A_{n-k} \}$.

In view of Lemma 6 i), we may suppose that the nodes from $\mathcal{A}$ are not intersection points of the components of the curve $q_{k-1}$.

Next, we are going to prove that these $n - k + 1$ nodes are collinear together with $m \geq 2$ nodes from $\mathcal{X}$. To this end denote the line through the nodes $A_0$ and $A_1$ by $\ell_{01}$. Then for each $i = 2, \ldots, n - k$ choose a line $\ell_i$ passing through the node $A_i$, which is not a component of $q_{k-1}$. We require also that each line passes through only one of the mentioned nodes and therefore the lines are distinct.

Now suppose that $p \in \Pi_k$ vanishes on $\mathcal{X}$. Consider the polynomial $r = p\ell_{01}\ell_2 \cdots \ell_{n-k}$. We have that $r \in \Pi_n$ and $r$ vanishes on the node set $\mathcal{Y}$, which is
a maximal \( n \)-independent set in the curve \( q_{k-1} \). Therefore, we obtain that \( r = q_{k-1}s \), where \( s \in \Pi_{n-k+1} \). Thus we have that

\[
p\ell_{01} \ell_2 \cdots \ell_{n-k} = q_{k-1}s.
\]

The lines \( \ell_i, \ i = 2, \ldots, n - k \), are not components of \( q_{k-1} \). Therefore, they are components of the polynomial \( s \). Thus we obtain that

\[
p\ell_{01} = q_{k-1}\beta, \text{ where } \beta \in \Pi_2.
\]

Now let us verify that \( \ell_{01} \) is a component of \( q_{k-1} \). Indeed, otherwise it is a component of the conic \( \beta \) and we get that

\[
p \in \Pi_k, \quad p|_X = 0 \implies p = q_{k-1}\ell, \text{ where } \ell \in \Pi_1.
\]

Therefore, we get \( \dim P_{k,\ell} = 3 \), which contradicts the hypothesis.

Thus we conclude that

\[
q_{k-1} = \ell_{01}q_{k-2}, \text{ where } q_{k-2} \in \Pi_{k-2}.
\]

The curve \( q_{k-2} \) passes through at most \( d(n, k - 2) \) nodes from \( X \). Hence we get that at least 2 nodes from \( X \) belong to the line \( \ell_{01} \).

Next we will show that exactly 2 nodes from \( X \) belong to \( \ell_{01} \), which will prove Lemma. Assume by way of contradiction that at least 3 nodes from \( X \) lie in \( \ell_{01} \). First let us show that all the nodes of \( A \) belong to \( \ell_{01} \). Suppose conversely that a node from \( A \), say \( A_2 \), does not belong to the line \( \ell_{01} \). Then in the same way as in the case of the line \( \ell_{01} \) we get that \( \ell_{02} \) is a component of \( q_{k-1} \). Thus the node \( A_0 \) is an intersection point of two components of \( q_{k-1} \), i.e., \( \ell_{01} \) and \( \ell_{02} \), which contradicts our assumption.

Next let us verify that in the beginning we could choose a non-collinear \( n \)-independent set \( A \subset q_{k-1} \), which will be a contradiction and will complete the proof. To this end let us prove that one can move any node of \( A \), say \( A_0 \), from \( \ell_{01} \) to the other component \( q_{k-2} \) such that the resulted set \( A \) remains \( n \)-independent.

In view of Lemma \( \ref{lemma:independent_set} \) \( ii \), for this we need to find an \( n \)-fundamental polynomial of \( A_0 \), for which \( q_{k-2} \) is not a component. Let us show that any fundamental polynomial of \( A_0 \) has this property. Indeed, suppose conversely that for an \( n \)-fundamental polynomial \( p_{A_0}^{k+1} \in \Pi_n \) the curve \( q_{k-2} \) is a component, i.e., \( p_{A_0}^{k+1} = q_{k-2}r \), where \( r \in \Pi_{n-k+2} \). We get from here that \( r \) vanishes at all the nodes in \( Y \cap \ell_{01} \) except \( A_0 \). Thus \( r \) vanishes at \( \geq 3 + (n - k + 1) - 1 = n - k + 3 \) nodes in \( \ell \). Therefore, in view of Proposition \( \ref{proposition:vanishing} \) \( r \) vanishes at all the points of \( \ell_{01} \) including \( A_0 \), which is a contradiction.

Now we are in a position to present

**Proof of Theorem \( \ref{theorem:dimension} \)** Recall that it remains to prove the direct implication. Let \( \sigma_1, \ldots, \sigma_4 \) be the four curves of degree \( \leq k \) that pass through all the nodes of the \( n \)-independent set \( X \) with \( |X| = d(n, k - 2) + 2 \). First we will consider

**Case \( n \geq k + 2 \).** Let us start by choosing three nodes \( B_1, B_2, B_3 \notin X \) such that the following four conditions are satisfied:

\( i \) the set \( X \cup \{B_1, B_2, B_3\} \) is \( n \)-independent;

\( ii \) the nodes \( B_1, B_2, B_3 \) are non-collinear;

\( iii \) each line through \( B_i \) and \( B_j \), \( 1 \leq i < j \leq 3 \), does not pass through any node from \( X \);
iv) for any subset \( A \subset \mathcal{X} \), \( |A| = 3 \) the set \( A \cup \{ B_1, B_2, B_3 \} \) is 2-poised.

Let us verify that one can find such nodes. Indeed, in view of Lemma \( \mathcal{L} \) we can start by choosing some nodes \( B'_i, i = 1, 2, 3 \), satisfying the condition i). Then, according to Lemma \( \mathcal{L}_2 \) for some positive \( \varepsilon \) all the nodes in \( \varepsilon \) neighborhoods of \( B'_i, i = 1, 2, 3 \), satisfy the condition i). Next, by using Lemma \( \mathcal{L}_4 \) three times, for the nodes \( B'_i, i = 1, 2, 3 \), consecutively, we obtain that there are nodes \( B''_i, i = 1, 2, 3 \), satisfying the condition iv) and \( \rho(B''_i, B'_i) < (1/2)\varepsilon, i = 1, 2, 3 \). Now notice that both conditions i) and iv) are satisfied for \( B''_i, i = 1, 2, 3 \). Then, according to Lemma \( \mathcal{L}_2 \) for some positive \( \varepsilon' > 0 \) all the nodes in \( \varepsilon' \) neighborhoods of \( B''_i, i = 1, 2, 3 \), satisfy the conditions i) and iv). Finally, from these \( \varepsilon' \) neighborhoods we can choose the nodes \( B_i, i = 1, 2, 3 \), satisfying the conditions ii), iii), too.

Note that, in view of Proposition \( \mathcal{P} \) the condition iv) means that

v) any conic through the triple \( B_1, B_2, B_3 \) passes through at most two nodes from \( \mathcal{X} \).

Next, in view of Proposition \( \mathcal{P}_5 \) there is a curve of degree at most \( k \), denoted by \( \sigma \), which passes through all the nodes of \( \mathcal{X} : = \mathcal{X} \cup \{ B_1, B_2, B_3 \} \).

Now notice that the curve \( \sigma \) passes through more than \( d(n, k - 2) \) nodes and, therefore, its degree equals either to \( k - 1 \) or \( k \). By taking into account Lemma \( \mathcal{L}_7 \) we may assume that the degree of the curve \( \sigma \) equals to \( k \). Evidently, in view of Lemma \( \mathcal{L}_7 \) we may assume also that \( \sigma \) has no multiple component.

Therefore, by using Proposition \( \mathcal{P}_6 \) we can extend the set \( \mathcal{X} \) till a maximal \( n \)-independent set \( \mathcal{X}'' \subset \sigma \). Notice that, since \( |\mathcal{X}''| = d(n, k) \), we need to add a set of \( d(n, k) - (d(n, k - 2) + 2) - 3 = 2(n - k) \) nodes to \( \mathcal{X}' \), denoted by \( \mathcal{A} : = \{ A_1, \ldots, A_{2(n-k)} \} : \mathcal{X}'' := \mathcal{X} \cup \{ B_1, B_2, B_3 \} \cup \mathcal{A} \).

Thus the curve \( \sigma \) becomes maximal with respect to this set. In view of Lemma \( \mathcal{L}_6 \) i), we require that each node of \( \mathcal{A} \) may belong only to one component of the curve \( \sigma \). Then, by using Lemma \( \mathcal{L}_5 \) we get a curve \( \sigma_0 \) of degree at most \( k \), different from \( \sigma \) that passes through all the nodes of \( \mathcal{X} \) and two more arbitrary nodes, which will be specified below.

We intend to divide the set of nodes \( \mathcal{A} \) into \( n - k \) pairs such that the lines \( \ell_1, \ldots, \ell_{n-k-1} \) through \( n - k - 1 \) pairs from them, respectively, are not components of \( \sigma \). The remaining pair we associate with the curve \( \sigma_0 \). More precisely, we require that \( \sigma_0 \) passes through the two nodes of the last pair.

Before establishing the mentioned division of \( \mathcal{A} \), let us verify how we can finish the proof by using it. Denote by \( \beta \) the conic through the triple of the nodes \( B_1, B_2, B_3 \) and the pair of nodes associated with the line \( \ell_{n-k-1} \). Notice that the following polynomial \( \sigma_0 \beta \ell_1 \ell_2 \ldots \ell_{n-k-2} \) of degree \( n \) vanishes at all the \( d(n, k) \) nodes of \( \mathcal{X}'' \subset \sigma \). Consequently, according to Proposition \( \mathcal{P}_4 \) \( \sigma \) divides this polynomial:

\[
\sigma_0 \beta \ell_1 \ell_2 \ldots \ell_{n-k-2} = \sigma q, \quad q \in \Pi_{n-k}.
\]  

The distinct lines \( \ell_1, \ell_2, \ldots, \ell_{n-k-2} \) do not divide the polynomial \( \sigma \in \Pi_k \), therefore, all they have to divide \( q \in \Pi_{n-k} \). Therefore, we get from \( (3) \):

\[
\sigma_0 \beta = \sigma \beta', \text{ where } \beta' \in \Pi_2.
\]
Now, suppose first that the conic $\beta$ is irreducible. Since the curves $\sigma$ and $\sigma_0$ are different the conics $\beta$ and $\beta'$ also are different. Therefore, the conic $\beta$ has to divide $\sigma \in \Pi_k$: $\sigma = \beta r, \; r \in \Pi_{k-2}$.

Now, we derive from this relation that the curve $r$ passes through all the nodes of the set $\mathcal{X}$ but two. Indeed, $\sigma$ passes through all the nodes of $\mathcal{X}$. Therefore, these nodes are either in the curve $r$ or in the conic $\beta$. But the latter conic passes through the triple of nodes $B_1, B_2, B_3$, and according to the condition $v)$, it passes through at most two nodes of $\mathcal{X}$. Thus $r$ passes through at least $d(n,k-2)$ nodes of $\mathcal{X}$. Since $r$ is a curve of degree $k-2$, we conclude that $r$ is a maximal curve and passes through exactly $d(n,k-2)$ nodes of $\mathcal{X}$.

Next suppose that the conic $\beta$ is reducible. Consider first the case when the pair of nodes associated with the line $\ell_{n-k-1}$ is collinear with a node from the triple $B_1, B_2, B_3$, say with $B_1$. Thus we have that $\beta = \ell_{n-k-1} \ell$, where the line $\ell$ passes through the nodes $B_2, B_3$.

The line $\ell_{n-k-1}$ does not divide the polynomial $\sigma \in \Pi_k$, therefore it has to divide $\beta'$. Therefore we get from the relation (4) that

$$\sigma_0 \ell = \sigma \ell', \text{ where } \ell' \in \Pi_2. \tag{5}$$

Now, the lines $\ell$ and $\ell'$ are different, so $\ell$ has to divide $\sigma \in \Pi_k$:

$$\sigma = \ell r, \; r \in \Pi_{k-1}. \tag{6}$$

In view of above condition $iii)$, the line $\ell$ does not pass through any node of $\mathcal{X}$. Therefore, the curve $r$ of degree $k-1$ passes through all the nodes of $\mathcal{X}$. Thus the proof of Theorem is completed in view of Lemma $[\text{7}]$.

Observe that we may conclude from here that any line component of the curve $\sigma$, as well as of the curve $\sigma_0$, passes through at least a node from $\mathcal{X}$. Thus, in view of $iii)$ the (three) lines through two nodes from $\{B_1, B_2, B_3\}$ are not a component of $\sigma$. Hence, in view of Lemma $[\text{6}]$ we may assume that the nodes of $\mathcal{A}$ do not belong to these three lines. Consequently, no extra case of a reducible $\beta$ is possible.

Next let us establish the above mentioned division of the node set $\mathcal{A}$ into $n-k$ pairs such that the lines $\ell_1, \ldots, \ell_{n-k-1}$ through $n-k-1$ pairs from them, respectively, are not components of $\sigma$. Thus we need to have pairs of nodes not belonging to the same line component of $\sigma$.

Recall that the nodes of $\mathcal{A}$ belong only to one component of the curve $\sigma$. Therefore, the line components do not intersect at the nodes of $\mathcal{A}$. By using induction on $n-k$, it can be proved easily that the mentioned division of $\mathcal{A}$ is possible if and only if no $n-k$ nodes of $\mathcal{A}$, not counting those two associated with the curve $\sigma_0$, are located in a line component. Observe also that any two nodes of the set $\mathcal{A}$ may be considered as associated with $\sigma_0$.

Now note that there can be at most two undesirable line components of the curve $\sigma$, each of which contains $n-k$ nodes from $\mathcal{A}$. In this case one node from each of the two components we associate with $\sigma_0$.

Suppose that there is only one undesirable line component with $n-k$ or $n-k+1$ nodes. Then one or two nodes from here we associate with $\sigma_0$, respectively.
Finally consider the case of one undesirable line component \( \ell \) with \( m \geq n - k + 2 \) nodes. Recall that each line component passes through at least a node from \( X \). We have that \( \sigma = \ell q \), where \( q \in \Pi_{k-1} \) is a component of \( \sigma \). Now, in view of Lemma 6 \( ii \), we will move \( m - n + k - 1 \) nodes, one by one, from \( \ell \) to the component \( q \). For this it suffices to prove that during this process each node \( A \in \ell \cap \mathcal{A} \) has no fundamental polynomial, for which the curve \( q \) is a component. Suppose conversely that \( p_A^k = qr \), \( r \in \Pi_{n-k+1} \). Now we have that \( r \) vanishes at \( \geq n - k + 1 \) nodes in \( \ell \cap \mathcal{A} \setminus \{ A \} \), and at least at a node from \( \ell \cap X \) mentioned above. Thus \( r \) together with \( p_A^k \) vanishes at the whole line \( \ell \), including the node \( A \), which is a contradiction. It remains to note that there will be no more undesirable line, except \( \ell \), in the resulted set \( \mathcal{A} \) after the described movement of the nodes, since we keep exactly \( n - k + 1 \) nodes in \( \ell \cap \mathcal{A} \).

Finally let us consider

**Case** \( n = k + 1 \). Consider three collinear nodes \( B_1, B_2, B_3 \notin X \) such that the following two conditions are satisfied:

\( \ell' \) the set \( X \cup \{ B_1, B_2, B_3 \} \) is \( n \)-independent; 

\( \ell'' \) the line through \( B_i \), \( i = 1, 2, 3 \), does not pass through any node from \( X \).

Let us verify that one can find such nodes \( B_1, B_2, B_3 \), or the conclusion of Theorem 3 holds. Indeed, in view of Lemma 3 we can start by choosing some two nodes \( B_i, i = 1, 2 \), such that

\( \ell'' \) the set \( X \cup \{ B_1, B_2 \} \) is \( n \)-independent.

Then, according to Lemma 3, for some positive \( \varepsilon \) all the nodes in \( \varepsilon \) neighborhoods of \( B_i, i = 1, 2 \), satisfy \( \ell'' \). Thus, from this neighborhoods we can choose the nodes \( B_i, i = 1, 2 \), such that the line through them \( \ell_0 \) does not pass through any node from \( X \). Now it remains to prove Theorem 3 under the assumption that there is no node \( B_3 \in \ell_0 \) such that the condition \( \ell' \) holds.

Indeed, this means that any polynomial \( p \in \Pi_{k} \) vanishing on \( X \cup \{ B_1, B_2 \} \) vanishes identically on \( \ell_0 \). In view of Lemma 3 we may choose a such polynomial \( p \) from the linear span of four linearly independent curves of the hypothesis. Then we get that \( p \in \Pi_{k}, p|_{\ell_0} = 0 \). Thus we have \( p = \ell_0 q \), where \( q \in \Pi_{k-1} \). Now, in view of \( \ell'' \) we readily deduce that the curve \( q \) of degree \( \leq k - 1 \) passes through all the nodes of \( X \). Thus the proof of Theorem is completed in view of Lemma 7.

Now we may assume that we have three collinear nodes \( B_1, B_2, B_3 \notin X \), satisfying the conditions \( \ell' \) and \( \ell'' \).

Next, as in the previous case, we get a curve of degree \( k \), denoted by \( \sigma \), which has no multiple component and passes through all the nodes of the set \( X' := X \cup \{ B_1, B_2, B_3 \} \). Then, by using Proposition 3 we extend the set \( X' \) till a maximal \( n \)-independent set \( X'' = X' \cup \mathcal{A} \subset \sigma \). Note that \( |\mathcal{A}| = 2 \) in this case.

Then, as in the previous case, we get a curve \( \sigma_0 \) of degree \( k \) different from \( \sigma \), passing through all the nodes of the set \( X \) and two nodes of \( \mathcal{A} \). Now observe that the polynomial \( \sigma_0 \ell_0 \in \Pi_{k+1} \) vanishes on the maximal \( n = (k + 1) \)-independent set \( X'' \subset \sigma \). Therefore we have that \( \sigma_0 \ell_0 = \sigma \ell \) where \( \ell \in \Pi_1 \). Since \( \sigma_0 \) and \( \sigma \) are different so are also \( \ell_0 \) and \( \ell \). Thus \( \ell_0 \) is a component of \( \sigma \), i.e., \( \sigma = \ell_0 r \), where \( r \in \Pi_{k-1} \). Now, in view of above condition \( \ell'' \), the line \( \ell_0 \) does not pass through any
An Application to the Gasca-Maeztu Conjecture. Recall that a node \( A \in \mathcal{X} \) uses a line \( \ell \) means that \( \ell \) is a factor of the fundamental polynomial \( p = p_A \), i.e., \( p = \ell r \) for some \( r \in \Pi_{n-1} \).

A \( GC_n \)-set in the plane is an \( n \)-poised set of nodes, where the fundamental polynomial of each node is a product of \( n \) linear factors. The Gasca-Maeztu conjecture states that any \( GC_n \)-set possesses a subset of \( n + 1 \) collinear nodes.

It was proved in [5], that any line passing through exactly 2 nodes of a \( GC_n \)-set \( \mathcal{X} \) can be used at most by one node from \( \mathcal{X} \).

Below we consider the case of lines passing through exactly 4 nodes.

**Corollary.** Let \( \mathcal{X} \) be an \( n \)-poised set of nodes and \( \ell \) be a line, which passes through exactly 4 nodes. Suppose \( \ell \) is used by at least four nodes from \( \mathcal{X} \). Then it is used by exactly six nodes from \( \mathcal{X} \). Moreover, if it is used by six nodes, then they form a 2-poised set. Furthermore, in the latter case, if \( \mathcal{X} \) is a \( GC_n \) set, then the six nodes form a \( GC_3 \) set.

**Proof.** Assume that \( \ell \cap \mathcal{X} = \{A_1, \ldots, A_4\} =: A \). Assume also that the four nodes in \( B := \{B_1, \ldots, B_4\} \in \mathcal{X} \) use the line \( \ell \), that is,

\[
p_{B_i}^\ast = \ell q_i, \quad i = 1, \ldots, 4, \quad \text{where} \quad q_i \in \Pi_{n-1}.
\]

The polynomials \( q_1, \ldots, q_4 \) vanish at \( N - 8 \) nodes of the set \( \mathcal{X}' := \mathcal{X} \setminus (A \cup B) \). Hence through these \( N - 8 = d(n, n - 3) + 2 \) nodes pass four linearly independent curves of degree \( n - 1 \). By Theorem [5] there exists a maximal curve \( \mu \) of degree \( n - 3 \) passing through \( N - 10 \) nodes of \( \mathcal{X}' \) and the remaining two nodes denoted by \( C_1, C_2 \) are outside of it. Now, according to Proposition [5] the nodes \( C_1, C_2 \) use \( \mu : p_{C_i}^\ast = \mu r_i, \quad r_i \in \Pi_3, \quad i = 1, 2. \)

These polynomials \( r_i \) have to vanish at the four nodes of \( A \subset \ell \). Hence \( q_i = \ell \beta_i, \quad i = 1, 2, \) with \( \beta_i \in \Pi_2 \). Therefore, the nodes \( C_1, C_2 \) use the line \( \ell : p_\ell^{C_i} = \mu \ell \beta_i, \quad i = 1, 2. \)

Hence, if four nodes in \( B \subset \mathcal{X} \) use the line \( \ell \), then there exist two more nodes \( C_1, C_2 \in \mathcal{X} \) using it and all the nodes of \( Y := \mathcal{X} \setminus (A \cup B \cup \{C_1, C_2\}) \) lie in a maximal curve \( \mu \) of degree \( n - 3 : Y \subset \mu \).

Next, let us show that there is no seventh node using \( \ell \). Assume by way of contradiction that except of the six nodes in \( S := \{B_1, \ldots, B_4, C_1, C_2\} \), there is a seventh node \( D \) using \( \ell \). Of course we have that \( D \in Y \).

Then we have that four nodes \( B_1, B_2, B_3 \) and \( D \) are using \( \ell \), therefore, as it was proved above, there exist two more nodes \( E_1, E_2 \in \mathcal{X} \) (which may coincide or not with \( B_4 \) or \( C_1, C_2 \)) using it and all the nodes of \( Y' := \mathcal{X} \setminus (A \cup \{B_1, B_2, B_3, D, E_1, E_2\}) \) lie in a maximal curve \( \mu' \) of degree \( n - 3 \). We have also that

\[
p_{B_i}^\ast = \mu' q_i, \quad q_i \in \Pi_3.
\]

Now, notice that both the curves \( \mu \) and \( \mu' \) pass through all the nodes of the set \( \mathcal{Z} := \mathcal{X} \setminus (A \cup B \cup \{C_1, C_2, D, E_1, E_2\}) \) with \( |\mathcal{Z}| \geq N - 13. \)
Then, we get from Theorem 1 with \( k = n - 4 \), that \( N - 13 = d(n, n - 4) + 2 \) nodes determine the curve of degree \( n - 3 \) passing through them uniquely. Thus \( \mu \) and \( \mu' \) coincide. Therefore, in view of \( \mathcal{Y} \subset \mathcal{M} \) and (6), \( p_D \) vanishes at all the nodes of \( \mathcal{Y} \), which is a contradiction since \( D \in \mathcal{Y} \).

Now let us verify the last "moreover" statement. Suppose the six nodes in \( S \subset X \) use the line \( \ell \). Then, as we obtained earlier, the nodes \( Y = X \setminus (A \cup B \cup \{C_1, C_2\}) \) are located in a maximal curve \( \mu \) of degree \( n - 3 \). Therefore, the fundamental polynomial of each \( A \in S \) uses \( \mu : p_A = \mu q_A \), where \( q_A \in \Pi_2 \). It is easily seen that \( q_A \) is a 2-fundamental polynomial of \( A \in S \). □

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