HOMOGENEOUS AND NONLINEAR GENERALIZED MASTER EQUATIONS ACCOUNTING FOR INITIAL CORRELATIONS

Victor F. Los
Institute of Magnetism, Nat. Acad. of Sci. of Ukraine, 36-b Vernadsky Blvd., 03142 Kiev, Ukraine

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Abstract

Deriving the kinetic (irreversible) equations from the reversible microscopic dynamics of the many-body systems remains one of the principal tasks of statistical physics. One of the key assumptions used to deriving the Boltzmann equation is the Bogoliubov principle of weakening of initial correlations which implies that on a sufficiently large time scale all initial correlations (existing at the initial instant) are damped. This approach leads to the evolution equations that are not valid on all time scales and do not allow considering all the stages of the system evolution, in particular, the short-term and transient regimes, which are of scientific and practical interest for considering the ultrafast relaxation and non-Markovian processes, decoherence phenomena and onset of irreversibility. To take initial correlations into account, a method, based on the conventional time-independent projection operator technique, that allows converting the conventional linear inhomogeneous (containing a source caused by initial correlations) time-convolution generalized master equation (TC-GME) and time-convolutionless GME (TCL-GME) into the homogeneous form exactly, is proposed. This approach results in the exact linear time-convolution and time-convolutionless homogeneous generalized master equations (TC-HGME and TCL-HGME) which take the dynamics of initial correlations into account via modified memory kernels governing the evolution of the relevant part of a distribution function of a many-particle system. These equations describe the evolution of the relevant part of a distribution function on all time scales including the initial stage when the initial correlations matter. The obtained TC-HGME is applied to the spatially homogeneous dilute gas of classical and quantum particles. However, to derive the desired nonlinear equations (the Boltzmann equation in particular) from this actually linear equation, we should make an additional approximation neglecting the time-retardation.
of a one-particle distribution function which restricts the time scale by
times much smaller than the relaxation time for a one-particle distribu-
tion function. To obtain the actually nonlinear evolution equations and
avoid any restrictions on the time scales, we develop a new method, based
on using a time-dependent operator (generally not a projection operator)
converting a distribution function of a total system into the relevant form,
that allows deriving the new exact nonlinear generalized master equations.
The derived inhomogeneous nonlinear GME is a generalization of the lin-
ear Nakajima-Zwanzig TC-GME and can be viewed as an alternative to
the BBGKY chain. To include the initial correlations into consideration,
we convert the obtained inhomogeneous nonlinear GME into the homoge-
neous form by the method which we used for conventional linear GMEs.
The obtained exact homogeneous nonlinear GME describes all evolution
stages of the system of interest and treats initial correlations on an equal
footing with collisions by means of the modified memory kernel. As an
application, we obtain a new homogeneous nonlinear equation retaining
initial correlations for a one-particle distribution function of the spatially
inhomogeneous nonideal gas of classical particles. We show that on the
kinetic time scale, the time-reversible terms resulting from initial correla-
tions vanish (if the particle dynamics have the ergodic mixing property)
and this equation can be converted into the Vlasov-Landau and Boltz-
mann equations without any additional commonly used approximations.

1 INTRODUCTION

While the effects of equilibrium statistical mechanics seem to be well under-
stood and described, nonequilibrium statistical physics does not enjoy such an
opinion. One of the basic tasks of the latter remains deriving the appropriate
evolution equations for the measurable values (statistical expectations) charac-
terizing a nonequilibrium state of a many-particle system. It is expected that
these equations should generally be evolution equations converting into kinetic
(or other irreversible) equations on some time scale. The principal questions are
how to derive such irreversible equations rigorously from underlying microscopic
(reversible) classical or quantum dynamical equations and what is the reason
for the observable irreversibility. Several approaches are usually used to address
these problems, commonly starting with the linear Liouville-von-Neuman equation
for a distribution function (statistical operator) of the $N$-particle ($N \gg 1$)
system under consideration.

One approach leads to the chain of coupled first-order differential equations
for $s$-particle distribution functions $F_s(x_1, \ldots, x_s, t)$ ($1 \leq s \leq N$) known as
the BBGKY hierarchy [1]. The most consistent approach to decoupling the
BBGKY hierarchy was developed by Bogoliubov [1]. He suggested the prin-
ciple of weakening of initial correlations, which implies that at a sufficiently
large time $t - t_0 \gg t_{cor}$ ($t_{cor}$ is the correlation time due to interparticle inter-
action), all initial correlations (existing at the initial instant $t_0$) are damped
and the time-dependence of multiparticle distribution functions is consequently
determined completely by the time-dependence of a one-particle distribution
function (Bogoliubov’s ansatz). This assumes the existence of the time interval (time scale)
\[ t_{\text{cor}} \ll t - t_0 \ll t_{\text{rel}} \]  
(1.1)
(where \( t_{\text{rel}} \) is the relaxation time for a one-particle distribution function) and leads to an approximate conversion (valid only on the large time scale indicated above) of the inhomogeneous equation (including two-particle correlations) for a one-particle distribution function (of the BBGKY hierarchy) into a homogeneous equation. The idea behind this is that the perturbation expansions of the homogeneous equations are much more effective than that for the inhomogeneous ones (like the equations of the BBGKY chain), where the expansions of the functions, rather than equations, are involved. The Bogoliubov ansatz allows thus to avoid the so-called ”secular terms”, i.e the terms growing with time, which emerge after integration of the BBGKY chain inhomogeneous equation for \( s \)-particle distribution function \( F_s(x_1, \ldots, x_s, t) \) \( (1 \leq s < N) \) over time and make the perturbation expansion of these functions ineffective on the time scale of interest in kinetic theory. The Bogoliubov principle of weakening of initial correlations provides a solid mathematical foundation to the Boltzmann hypothesis of the molecular chaos (Stosszahlansatz) which leads to the coarse-grained (smoothed) over the time interval (1.1) evolution of the one-particle distribution function (described by the famous Boltzmann kinetic equation).

Using this approach, Bogoliubov successfully derived the kinetic (Markovian) equations for a one-particle distribution function, the classical and quantum Boltzmann equations in particular, which describe the time-evolution with the characteristic time \( t_{\text{rel}} \) but is unsuitable for taking the initial evolution stage \( t_0 \leq t \leq t_{\text{cor}} \) into account.

The Bogoliubov principle of weakening of initial correlations is introduced as the boundary (initial) condition for the BBGKY chain. For example, in classical physics, it can be written as

\[ \lim_{t - t_0 \to \infty} U(t, t_0) \left[ F_N(x_1, \ldots, x_N, t_0) - f_r(x_1, \ldots, x_N, t_0) \right] = 0, \]

\[ f_r(x_1, \ldots, x_N, t) = \prod_{i=1}^{N} F_1(x_i, t), \quad (1.2) \]

where \( F_N(x_1, \ldots, x_N, t_0) \) is an \( N \)-particle distribution function at \( t = t_0 \), \( x_i = (x_i, p_i) \) is the coordinate of the \( i \)th particle in the phase space, \( f_r(x_1, \ldots, x_N, t) \) is the relevant distribution function with no correlations (the general definition of the relevant distribution function will be given later), and \( U(t, t_0) \) is the operator determining the evolution of all coordinates \( x_i \) in time. The first inequality in (1.1) implies that all correlations between the particles vanish for \( t - t_0 \gg t_{\text{cor}} \), and the second inequality in (1.1), \( t - t_0 \ll t_{\text{rel}} \), allows neglecting the time retardation and replacing \( U(t, t_0)f_r(t_0) \) with \( U(t, t_0)f_r(t) \) (in the first approximation in the small density of particles \( n \)). Thus, decoupling the BBGKY chain is nonlinear procedure (1.2), which results in a closed nonlinear equation (like the Boltzmann equation) for a one-particle distribution function \( F_1(x_i, t) \) and introduces a nonlinearity into the process originally described by
the linear reversible Liouville-von-Neumann equation. It is important to note, that shifting $t_0$ to $-\infty$ in the Bogoliubov limiting procedure (1.2) (in order to get rid of dealing with a singled out instant of time, when a distribution function $F_N(t)$ is replaced with its relevant part) leads to an irreversibility of the relaxation process.

But we note only the small-scale initial correlations with $t_{cor} \ll t_{rel}$ can in fact be neglected on the kinetic time scale. Trying to obtain the collision integral in the kinetic equation with collisions of three particles (in the two-dimensional case) or more particles (the terms of the second and higher orders in $n$), we should handle the divergencies, in particular, resulting from the large-scale initial correlations with $t_{cor} \gtrsim t_{rel}$ (see, e.g., [2], [3]). Hence, the correct expansion of the collision integral in powers of $n$ and consideration of kinetic fluctuations are only possible if we account for initial correlations with $t_{cor} \gtrsim t_{rel}$ and $l_{cor} \gtrsim l$, where $l_{cor} = \overline{v} t_{cor}$, $l = \overline{v} t_{rel}$, $\overline{v}$ is the characteristic (mean) particle velocity, and $l$ is the particle mean-free path [4]. Moreover, there can be correlations which do not damp, e.g. due the conservation laws, such as the collective plasma excitations; also, the initial quantum correlations, caused by the particle statistics, do not damp with time.

Attempts to obtain the Boltzmann equation rigorously (especially in quantum physics) continue. This is important from both the conceptual and practical standpoints. For example, following [5], where the classical Boltzmann equation was rigorously obtained for a system of hard spheres at very low densities and for short times, the authors of [6] analogously (but not rigorously) obtained the quantum Boltzmann equation but using the factorized initial condition, i.e., the absence of initial correlations or random phase approximation (RPA) at $t = t_0$, which is doubtful in principle [7].

On the other hand, we note that, as to the best of our knowledge, there is not yet a way to take initial correlations into account exactly when dealing with the BBGKY chain.

In another approach, leading to the so called generalized master equations (GMEs), an $N$-particle distribution function (or statistical operator) is divided into the essential (relevant) and inessential (irrelevant) parts using time-independent projection operators $P$ and $Q$

$$F_N(x_1, \ldots, x_N, t) = f_r(x_1, \ldots, x_N, t) + f_i(x_1, \ldots, x_N, t),$$
$$f_r(x_1, \ldots, x_N, t) = PF_N(x_1, \ldots, x_N, t), f_i(x_1, \ldots, x_N, t) = QF_N(x_1, \ldots, x_N, t),$$

$$P + Q = 1.$$  
(1.3)

We note, that the relevant and irrelevant parts depend on coordinates (and/or momenta) of all $N$ particles in contrast to the reduced distribution functions (like $F_1(x_1)$). The relevant part, which is usually of main interest, is, as a rule, a vacuum (slowly changing) part of the distribution function (statistical operator), i.e., the part with no correlations (like $f_r(x_1, \ldots, x_N, t)$ in (1.2)). Applying the projectors $P$ and $Q$ to the Liouville-von-Neumann equation, we can obtain the time-convolution (non-Markovian) GME (TC-GME) [8], [9], [10] and time-convolutionless (time-local) GME (TCL-GME) [11], [12], which are the exact
inhomogeneous equation for the relevant part of a distribution function with a source (irrelevant part) containing all many-particle correlations at the initial instant $t_0$. We stress that the time-independent (linear) projection operators $P$ and $Q$ used in this approach commute with the time-derivative operator in the Liouville-von-Neumann equation for the distribution function (statistical operator) of the whole system. Therefore, procedure (1.3) is linear and transforms the linear Liouville-von-Neumann equation into the linear GMEs. Bogoliubov’s principle of weakening of initial correlations or simply the factorized initial conditions (RPA) are commonly used to exclude initial many-particle correlations (a source), which results in an approximate closed linear homogeneous GMEs for the relevant part of the distribution function (statistical operator). The latters are then used to obtain the linear evolution (master) equations for an interesting reduced distribution function (especially like that for an open subsystem interacting with a thermal bath).

There is also an approach based on the Kadanoff-Baym equations for the two-time correlations functions [13]. The original Kadanoff-Baym equations do not contain initial correlations and follow from the exact equations for correlation functions under the assumption of Bogoliubov’s principle of weakening of initial correlations. In the framework of this approach, several attempts were made to include into account the initial correlations. A review of these works can be found in [14].

All kinetic equations which follow from that principle are unable to describe the initial stage of the evolution, $t_0 \leq t \leq t_{cor}$. Therefore, the influence of initial correlations, which can be important for, e.g., the ultrafast relaxation processes, is not described by these equations. Moreover, the time-scale $t \lesssim t_{cor}$ may be of significant interest, because we can expect that on this time-scale the transition from the short-term ”non-kinetic” evolution of a multi-particle system to the long-term ”kinetic” behavior occurs.

In particular, what we have said above means that the proper handling of the initial correlations is crucial in obtaining the closed evolution equations. Neglecting the initial correlations results in the evolution equations which are not valid on all time scales (but, e.g., only on the time scale (1.1)) and do not allow consecutive considering the all stages of the system of interest evolution, in particular, the stage when a system switches from the initial short-term (reversible) evolution (when initial correlations play a role) to the long-term (irreversible) kinetic regime. The outlined problem is common for both principal methods of deriving the kinetic equations: the BBGKY hierarchy and the GMEs. Moreover, the initial correlations can be important for considering the ultrafast relaxation and non-Markovian processes and also for describing the decoherence phenomenon (see, e.g., [15]).

A progress achieved in the research on dynamics of the many-particle systems makes it possible, in principle, to study the entire evolution process of systems in statistical mechanics on any time scale and to determine the conditions under which a system evolves towards the equilibrium state. To do that, the dynamics of initial correlations should be included into consideration, and our goal is to obtain new evolution equations exactly accounting the initial correlations.
To take the initial correlations into account, we first suggest a method, based on the conventional time-independent projection operator technique (specified by (1.3)), that allows converting the linear inhomogeneous TC-GME and TCL-GME into the homogeneous form exactly [16, 17]. This approach leads to the exact linear time-convolution and time-convolutionless homogeneous GMEs (TC-HGME and TCL-HGME), which take the dynamics of initial correlations into account via modified memory kernels governing the evolution of the relevant part of the distribution function (statistical operator) of a many-particle system. These equations describe the evolution (influenced by initial correlations) of the relevant part of a distribution function on all time scales including the initial stage $t - t_0 \lesssim t_{\text{cor}}$, when the transition from the reversible evolution to the irreversible kinetic behavior (resulting from the stochastic instability of the system dynamics) is expected to occur.

We analyze how the obtained linear TC-HGME works for a spatially homogeneous dilute gas of classical and quantum particles. But to obtain the desired nonlinear equations (the Boltzmann equation in particular) describing the evolution of a classical or quantum gas of particles from the linear TC-HGME, we should make an additional approximation neglecting the time retardation of the one-particle distribution function (statistical operator). Hence, in this approach, we are not restricted by the first inequality in (1.1), $t_{\text{cor}} \ll t - t_0$, but still need the second inequality. The obtained (in the linear approximation in the gas density) new homogeneous equations for a one-particle distribution function (statistical operator) include initial correlation caused by interaction between particles (as well as the quantum initial correlations in the quantum case) and describe the evolution process on the time scale which includes the initial stage $t - t_0 \lesssim t_{\text{cor}}$. We show that on the appropriate time scale these equations convert to the classical and quantum Boltzmann equations if all initial correlations (due to interparticle interactions) vanish on this time scale.

We note, the obtained linear homogeneous GMEs, when applied to an open quantum system interacting with a thermal bath, are not subject to the time interval restriction $t - t_0 \ll t_{\text{rel}}$ (here $t_{\text{rel}}$ is the relaxation time due to the system-bath interaction) and describe all stages of a system of interest evolution including the initial stage $t - t_0 \lesssim t_{\text{cor}}$.

We see, that, strictly speaking, it is impossible to obtain a nonlinear master equation (like the Boltzmann equation) in the framework of the linear time-independent projection operator formalism without additional approximations (like discussed above neglect of time retardation which is justified by the second inequality in (1.1)). Thus, a nonlinear equation for a density matrix is often simply postulated in the form of a Lindblad-type master equation whose generator depends parametrically on this density matrix (see, e.g., [18]).

To overcome the mentioned restriction, we suggest an approach allowing to obtain a new class of evolution equations for the relevant part of a distribution function (statistical operator) that suitable for deriving both the nonlinear and linear evolution equations for the reduced distribution functions (statistical operators) of interest and that can hold on any time scale [19]. The class contains exact nonlinear inhomogeneous and homogeneous GMEs deduced from the
Liouville-von-Neumann equation using a (generally) nonlinear time-dependent operator $P(t)$ that converts the many-particle distribution function (statistical operator) into the appropriate relevant form. The time-dependent operator $P(t)$ is not generally a projection operator. The nonlinear inhomogeneous GME can be considered as an alternative to the BBGKY chain. In order to include the initial correlations into consideration, this equation is converted into nonlinear homogeneous GME by the method used for the linear GMEs. In the case of a time-independent linear projection operator $P(1.3)$, the obtained equations reduce to the conventional linear GME and obtained TC-HGME. In contrast to these linear equations, the obtained nonlinear equations are more general and therefore more convenient, in particular, for obtaining Boltzmann-like equations and studying the spatially inhomogeneous nonideal gas of particles for which the spatial nonlocality and time retardation are essential (see, e.g., [4]).

We apply the derived nonlinear GMEs to the spatially inhomogeneous nonideal diluted gas of classical particles. In particular, from the obtained homogeneous nonlinear GME in the linear approximation in the density parameter, we obtain a new homogeneous nonlinear equation for a one-particle distribution function that holds on all time scales (no restrictions like (1.1) are needed) and takes the influence of the dynamics of initial correlations into account in all stages of the evolution process. In this approximation (linear in $n$), the evolution of collisions and initial correlations (in the memory kernel) is governed by the exact two-particle propagator, and the obtained equation is closed in this sense. If the gas dynamics have the necessary property (ergodic mixing flow in the phase space), then all initial correlations vanish at the kinetic time scale, and this equation can be converted into the irreversible nonlinear Boltzmann equation with a self-consistent nonlinear Vlasov term. No additional commonly used assumption, like special initial conditions (factorized ones) or a principle (weakening of initial correlations), is needed.

## 2 Time-Convolution Homogeneous GME

We start with the Liouville-von-Neumann equation for a distribution function (statistical operator) $F_N(t)$ of $N$ $(N \gg 1)$ particles

$$\frac{\partial}{\partial t} F_N(t) = LF_N(t).$$

(2.1)

Here, $F_N(t)$ is a symmetric function of $N$ variables $x_i = (r_i, p_i)$ $(i = 1, ..., N)$ representing the coordinates and momenta of the particles (classical physics) or a statistical operator (quantum mechanics), which satisfy the normalization conditions

$$\int dx_1 \ldots \int dx_N F_N(x_1, \ldots x_N, t) = 1, \quad Tr F_N = 1,$$

(2.2)

$L$ is the Liouville operator acting on $F_N(t)$ in the case of classical physics as

$$LF_N(t) = \{H, F_N(t)\}_P = \sum_{i=1}^{N} \left( \frac{\partial H}{\partial r_i} \frac{\partial F_N(t)}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial F_N(t)}{\partial r_i} \right),$$

(2.3)
where \( \{ H, F_N(t) \}_P \) is the Poisson bracket and \( H \) is the Hamilton function for the system under consideration, while in the quantum physics case \( L \) is the superoperator acting on a statistical operator as

\[
LF_N(t) = \frac{1}{i\hbar} [H, F_N(t)],
\]

(2.4)

where \([,] \) is a commutator and \( H \) is the Hamiltonian. Note, that for simplicity we consider the case of a closed isolated system when \( H \) does not depend on time \( t \).

The formal solution to Eq. (2.1) is

\[
F_N(t) = e^{L(t-t_0)}F_N(t_0),
\]

(2.5)

where \( F_N(t_0) \) is a distribution function (statistical operator) at an initial instant \( t_0 \), when the initial condition for the Liouville-von-Neumann equation (2.1) should be set.

Employing the projection operator technique [8], [9], [10], and applying the projection operators \( P = P^2 \) and \( Q = Q^2 = 1 - P \) to Eq. (2.1), it is easy to obtain the equations for the relevant \( f_r(t) = P F_N(t) \) and irrelevant \( f_i(t) = Q F_N(t) \) (see (1.3)) parts of \( F_N(t) \)

\[
\frac{\partial}{\partial t} f_r(t) = PL[f_r(t) + f_i(t)],
\]

(2.6)

\[
\frac{\partial}{\partial t} f_i(t) = QL[f_r(t) + f_i(t)].
\]

(2.7)

A formal solution to Eq. (2.7) has the form

\[
f_i(t) = \int_{t_0}^{t} \exp[QL(t-t_1)]QLf_r(t_1)dt_1 + \exp[QL(t-t_0)]f_i(t_0).
\]

(2.8)

Inserting this solution into (2.6), we obtain the conventional exact time-convolution generalized master equation (TC-GME) known as the Nakajima-Zwanzig equation for the relevant part of the distribution function (statistical operator)

\[
\frac{\partial}{\partial t} f_r(t) = PLf_r(t) + \int_{t_0}^{t} PL\exp[QL(t-t_1)]QLf_r(t_1)dt_1
+ PL\exp[QL(t-t_0)]f_i(t_0).
\]

(2.9)

It is important to stress that \( f_r(t) \) and \( f_i(t_0) \) are the basic quantities we are dealing with in Eq. (2.9). All functions of dynamical variables, the average values of which we can calculate with the help of \( f_r(t) \) by multiplying equation (2.9) with the corresponding functions (operators) from the right and integrating over the relevant variables (taking a trace), are dependent only on the variables which are not projected off by \( P \) (\( P \) integrates off all excessive variables in
where the projection operator $Q$ (see below), use that $P$ and $Q$ in these expressions act only on $F_N(t)$ but not on the functions of dynamical variables of interest to the right of them. This is the essence of the reduced description method, when, in order to calculate the average values of the functions dependent on a much smaller number of variables than the whole distribution function $F_N(t)$, we actually need only the reduced distribution function (density matrix) like the one-particle distribution functions determining the relevant distribution function $f_r(t)$ in (1.2).

Serving as a basis for many applications, Eq. (2.9), nevertheless, contains the undesirable and in general non-negligible inhomogeneous term (the last term in the right hand side of (2.9)), which depends (via $f_i(t_0)$) on the same large number of variables as the distribution function $F_N(t_0)$ at the initial instant ($f_i(t_0)$ is not composed of the reduced distribution functions). Therefore, Eq. (2.10) does not provide for a complete reduced description of a multiparticle system in terms of relevant (reduced) distribution function. Applying Bogoliubov’s principle of weakening of initial correlations (allowing to eliminate the influence of $f_i(t_0)$ on the large enough time scale $t-t_0 \gg t_{cor}$ or using a factorized initial condition (RPA), when $f_i(t_0) = QF_N(t_0) = 0$, one can achieve the above-mentioned goal and obtain the homogeneous GME for $f_r(t)$, i.e. Eq. (2.9) with no initial condition term. However, obtained in such a way homogeneous GME is either approximate and valid only on a large enough time scale (when all initial correlations vanish) or applicable only for a rather artificial (actually unreal, as pointed in [7]) initial conditions (no correlations at an initial instant of time).

In order to obtain a homogeneous equation for the relevant part of the distribution function (statistical operator) we try to transfer the inhomogeneous initial correlations (irrelevant) term in the right hand side of Eq. (2.9), which depends (via $f_i(t_0)$) on the large enough time scale $t-t_0 \gg t_{cor}$ or using a factorized initial condition (RPA), to the (super)operator (kernel) acting on the relevant part $f_r(t)$. To achieve this goal we suggest to present the initial (irrelevant) term $f_i(t_0) = QF_N(t_0)$ as a following exact identity

$$f_i(t_0) = [QF_N(t_0)]F_N^{-1}(t_0)e^{-L(t-t_0)}(P + Q)e^{L(t-t_0)}F_N(t_0) = C_0 \exp[-L(t-t_0)][f_r(t) + f_i(t)].$$

(2.10)

We assume here that the inverse operator $F_N^{-1}(t_0) = [f_r(t_0) + f_i(t_0)]^{-1}$ exists (see below), use that $P+Q = 1$ and that the backward propagator of the system is $\exp[-L(t-t_0)]$. In (2.10) the initial correlation parameter $C_0$ is introduced

$$C_0 = [QF_N(t_0)]F_N^{-1}(t_0) = f_i(t_0)[f_r(t_0) + f_i(t_0)]^{-1} = f_i(t_0)F_r^{-1}(t_0)[1 + f_i(t_0)F_r^{-1}(t_0)]^{-1} = (1 - C_0)f_i(t_0)F_r^{-1}(t_0),$$

(2.11)

where the projection operator $Q$ acts only on $F_N(t_0)$, which is indicated by enclosing $QF_N(t_0)$ in brackets.
Thus, the additional identity (2.10) has been obtained by multiplying the irrelevant part by the unity $F_{N}^{-1}(t_0)F_{N}(t_0)$ (which implies the existence of $F_{N}^{-1}(t_0)$) and inserting the unities $\exp[-L(t-t_0)]\exp[L(t-t_0)] = 1$ and $P+Q = 1$. Therefore, neither divergency (due to possible vanishing of $F_{N}(t_0)$) nor indetermination of the 0/0 type (behaviors of the numerator and denominator in $F_{N}(t_0)/F_{N}(t_0) = 1$ are similar) can happen. This holds over all further (identical) manipulations (see below). As it is seen from (2.11), the correlation parameter is a series in $f_{i}(t_0)f_{r}^{-1}(t_0)$ and, therefore, one may only need a formal existence of the function (operator) $f_{r}^{-1}(t_0)$ (see also (6.23)), which is invert to the relevant distribution function (statistical operator) chosen with the help of the appropriate projection operator $P$ (generally, it can provide some restriction on the class of appropriate projectors). The relevant part, which is mainly of interest, is, as a rule, a vacuum (relatively slowly changing) part of a distribution function (statistical operator), i.e. the part with no correlations (e.g., a product of the one-particle distribution functions as in (1.2)). It seems plausible that the invert relevant part of the distribution function (statistical operator) defined in a pointed above sense (uncorrelated part) can always be constructed. The examples of construction of the appropriate invert distribution function (operator) $f_{r}^{-1}(t_0)$ will be given below.

As a result of introducing the additional identity (2.10), we have two equations, (2.8) and (2.10), connecting $f_{i}(t)$ with $f_{i}(t_0)$. Finding $f_{i}(t_0)$ from these equations and inserting it into (2.9), we obtain the following exact equation for the relevant part of a distribution function (statistical operator):

$$\frac{\partial}{\partial t} f_{r}(t) = PLR(t-t_0)f_{r}(t) + \int_{t_0}^{t} PLR(t-t_0) \exp[QL(t-t_1)]QLf_{r}(t_1)dt_1, \quad (2.12)$$

where function $R(t-t_0)$ is defined as

$$R(t-t_0) = 1 + C(t-t_0) = \frac{1}{1 - C_0(t-t_0)},$$

$$C(t-t_0) = e^{QL(t-t_0)} \frac{1}{1 - C_0 e^{-L(t-t_0)} e^{QL(t-t_0)}},$$

$$C_0(t-t_0) = e^{QL(t-t_0)} C_0 e^{-L(t-t_0)}. \quad (2.13)$$

We have derived the time-convolution homogeneous generalized master equation (TC-HGME) (2.12) for the relevant part of the distribution function (statistical operator). This equation holds in both the classical and quantum physics cases if the proper redefinition of the symbols is done and all (super)operators exist (we will address the latter problem below). We have not removed any information while deriving equation (2.12), and, therefore, it is exact integro-differential equation which takes into account the initial correlations and their dynamics via the modification of the (super)operator (memory kernel) in GME (2.9) acting on the relevant part of the distribution function (statistical operator) $f_{r}(t)$. The obtained exact kernel of TC-HGME (2.12) can serve as a starting
point for consecutive perturbation expansions. In many cases such expansions of the homogeneous equations (like (2.12)) have much broader range of validity than that for the inhomogeneous ones (like (2.9)).

The initial correlations are entering the TC-HGME (2.14) through the function $R(t)$ (2.13). This function represents a sum of the infinite series of expansion in the properly defined time dependent parameter $C_0(t - t_0)$ which describes the influence of initial correlations in time. Presence of the function $R(t) \neq 1$ in Eq. (2.12) has led to the modification of the first term in the right-hand side of the GME (2.9) as well as the second (collision) term. This modification reflects the influence of initial correlations on the corresponding processes (flow and relaxation) with time. The TC-HGME also allows treating the correlations arising from collisions and initial correlations on an equal footing by expanding the memory kernel in series in some small parameter.

However, the problem of the existence (convergence) of $R(t - t_0)$ can be raised. The function $R(t - t_0)$ behaves properly at all times. Moreover, the expansion of the kernel in (2.12) can result in canceling the pole in function $R(t - t_0)$ as we will see below. In such cases there is no problem with the existence of $R(t - t_0)$.

3 TIME-CONVOLUTIONLESS HOMOGENEOUS GME

Now, let us turn to the case of the so called time-convolutionless (time-local) GME (TCL-GME). It is believed that such TCL-GME can be more easily solved and be more convenient for studying the non-Markovian processes than the integral TC-GME (2.9) (see [18], [11], [12]).

Using the identity $F_N(\tau) = \exp[-L(t - \tau)][f_r(t) + f_i(t)]$, we turn Eq. (2.8) into the form

\[
\frac{\partial}{\partial t} f_r(t) = PL [1 + A(t - t_0)]^{-1} \left[ f_r(t) + e^{QL(t-t_0)} f_i(t_0) \right],
\]

Inserting Eq. (3.1) for $f_i(t)$ into equation (2.6), we obtain the TCL-GME [11], [12]

\[
\frac{\partial}{\partial t} f_r(t) = PL [1 + A(t - t_0)]^{-1} \left[ f_r(t) + e^{QL(t-t_0)} f_i(t_0) \right],
\]

which contains the undesirable inhomogeneous initial condition term $\propto f_i(t_0)$.

Solving Eqs. (3.1) and (2.10), we get for $f_i(t_0)$

\[
\chi^{-1} f_i(t_0) = f_i(t_0) = \chi^{-1} C_0 e^{-L(t-t_0)} (1 + A(t - t_0))^{-1} f_i(t),
\]

where

\[
\chi^{-1} = \left[ 1 - C_0 e^{-L(t-t_0)} (1 + A(t - t_0))^{-1} e^{QL(t-t_0)} \right]^{-1}.
\]
The inverse operator \( (3.4) \), needed for obtaining \( f_i(t_0) \) from Eq. \( (3.3) \), is defined by expanding it into the series in the term \( \propto C_0 \). At \( t = t_0 \), \( A(t - t_0) = 0 \), and there is no singularity at the initial moment of time if \( |C_0| < 1 \). From the existence of \( (1 + A(t - t_0))^{-1} \) \cite{11,12}, it follows that \( \chi^{-1} \) exists at any time if \( |C_0| < 1 \). As for the function \( R(t - t_0) \) \((2.13)\) in TC-HGME \((2.12)\), an expansion in some small parameter can result in canceling the pole in \((3.4)\) and, therefore, the condition \( |C_0| < 1 \) can be not necessary.

Inserting Eq. \((3.3)\) into Eq. \((3.2)\) and using \((3.4)\), we obtain a desired time-convolutionless (time-local) homogeneous generalized master equation (TCL-HGME) for the relevant part of the distribution function (statistical operator) \( f_r(t) \)

\[
\frac{\partial}{\partial t} f_r(t) = PL \left[ 1 + A(t - t_0) \right]^{-1} \left\{ 1 - C_0(t - t_0) \left[ 1 + A(t - t_0) \right]^{-1} \right\}^{-1} f_r(t).
\]

\((3.5)\)

Based on the above mentioned arguments, one can expect that all operators entering equation \((3.5)\) exist.

Thus, using the introduced identity \((2.10)\), we have transferred the inhomogeneous initial correlations term of equation \((3.2)\) into the operator acting on the relevant part of the distribution function (statistical operator). Equation \((3.5)\) is exact homogeneous time-local equation for the relevant part of a distribution function (statistical operator) which accounts for initial correlations via the time-dependent parameter of initial correlations \( C_0(t - t_0) \) defined by \((2.13)\). This equation is expected to work on any time scale and to describe the entire evolution process. Equation \((3.5)\) can also appear more convenient for studying the non-Markovian processes than the TC-HGME \((2.12)\).

4 EVOLUTION EQUATION RETAINING INITIAL CORRELATIONS FOR A HOMOGENEOUS DILUTE GAS OF CLASSICAL PARTICLES

In this section we consider a gas of \( N \) \((N \gg 1)\) interacting identical classical particles. It is supposed that the Hamiltonian of the system can be split into two parts

\[
H = H^0 + H',
\]

\[
H^0 = \sum_{i=1}^{N} H_i^0, \quad H_i^0 = \frac{p_i^2}{2m},
\]

\[
H' = \sum_{i<j=1}^{N} V_{ij}, \quad V_{ij} = V(|x_i - x_j|).
\]
Here, \( H^0 \) describes an ideal gas of noninteracting particles having the momenta \( p_i \) and mass \( m \), and the Hamiltonian of interaction between particles \( H' \) is assumed to be a sum of potentials \( V_{ij} \) dependent only on the difference of coordinates of two particles. We do not suppose here that the interaction is weak and assume that all usual necessary requirements to the properties of forces, by which particles interact with each other, are met. In particular, we suppose that the bound states are not formed (an interaction is repulsive).

To obtain the evolution equation in this case, we presume that the density of particles \( n = N/V \) (\( V \) is the volume of the system) is small enough and use a perturbation theory with the small parameter

\[
\gamma = r_0^3 n \ll 1, \tag{4.2}
\]

where \( r_0 \) is the effective radius of the interparticle interaction. This small parameter guarantees that the collisions between particles are the well separated events and defines the time hierarchy, because \( t_{\text{cor}}/t_{\text{rel}} \sim \gamma \ll 1 \).

First of all, we have to define the projection operator \( P \) selecting the relevant part \( f_r(x_1, \ldots, x_N, t) \) of the \( N \)-particle distribution function. It is convenient to deal with the \( N \)-particle distribution function defined as

\[
f_N(x_1, \ldots, x_N, t) = V^N F_N(x_1, \ldots, x_N, t), \tag{4.1}
\]

which satisfies the same Liouville equation (2.1). Then, the required projection operator is

\[
P = \left[ \prod_{i=2}^N f_1(x_i) \right] \frac{1}{V^{N-1}} \int dx_2 \cdots \int dx_N. \tag{4.3}
\]

Here, \( f_1(x_i) \) is a one-particle distribution function

\[
f_1(x_i, t) = V \int dx_1 \cdots \int dx_{i-1} \int dx_{i+1} \cdots \int dx_N F_N(x_1, \ldots, x_N, t) \tag{4.4}
\]

taken at the initial instant, \( f_1(x_i) = f_1(x_i, t_0) \). In the definition (4.4), a symmetry of a distribution function relatively to a permutation of space-phase coordinates \( x_i = (x_i, p_i) \) is taken into account.

In general, the \( s \)-particle \((s \leq N)\) distribution function is defined as

\[
f_s(x_1, \ldots, x_s, t) = V^s \int dx_{s+1} \cdots \int dx_N F_N(x_1, \ldots, x_N, t). \tag{4.5}
\]

From (2.2) we have the normalization conditions for the reduced distribution functions \( f_s \)

\[
\int dx_1 \cdots \int dx_s f_s(x_1, \ldots, x_s, t) = V^s. \tag{4.6}
\]

Applying the projection operator (4.3) to the \( N \)-particle distribution function \( f_N(t) = V^N F_N(t) \), we obtain the following relevant part \( f_r \)

\[
f_r = P f_N(t) = \left[ \prod_{i=2}^N f_1(x_i) \right] f_1(x_1, t). \tag{4.7}
\]
We suppressed in (4.7) the arguments of the $f_{r}(x_{1}, \ldots, x_{N}, t)$ for brevity.

The irrelevant part comprising initial correlations, $f_{i}(t_{0}) = (1 - P)f_{N}(t_{0}) = f_{N}(t_{0}) - f_{r}(t_{0})$, can always be represented by the following cluster expansion in terms of multiparticle correlation functions (see, for example, [20])

\[
\begin{align*}
    f_{i}(t_{0}) &= \sum_{i,j=1,i<j}^{N} g_{2}(x_{i}, x_{j}) \prod_{k=1,k \neq i,j}^{N-2} f_{1}(x_{k}) + \sum_{i,j,k=1,i<j<k}^{N} g_{3}(x_{i}, x_{j}, x_{k}) \prod_{l=1,l \neq i,j,k}^{N-3} f_{1}(x_{l}) + \ldots, \\
    \text{where} \quad \prod_{k=1,k \neq i,j}^{N-2} f_{1}(x_{k}) \text{and} \quad \prod_{l=1,l \neq i,j,k}^{N-3} f_{1}(x_{l}) \text{stand for the products of} \quad N - 2 \quad \text{and} \quad N - 3 \text{one-particle distribution functions with} \quad k \neq i, j \quad \text{and} \quad l \neq i, j, k, \quad \text{respectively}, \quad \text{whereas} \quad g_{2}(x_{i}, x_{j}) \quad \text{and} \quad g_{3}(x_{i}, x_{j}, x_{k}) \quad \text{are the irreducible two-particle and three-particle correlation functions (further terms in (4.8) are defined in the same way).}
\end{align*}
\]

We suppose, as usual, that all functions $\Phi(x_{1}, \ldots, x_{N})$, defined on the phase space, and their derivatives vanish at the boundaries of the configurational space and at $p_{i} = \pm \infty$. These properties of distribution functions and the form of the Liouville operators (4.9) lead to the following relations

\[
\begin{align*}
    \int dx_{i} L^{0}_{i} \Phi(x_{1}, \ldots, x_{N}, t) &= 0, \\
    \int dx_{i} \int dx_{j} L'_{ij} \Phi(x_{1}, \ldots, x_{N}, t) &= 0. 
\end{align*}
\] (4.10)

These equations also follow, in general, from the Liouville-von-Neumann equation (2.11) and conditions (2.2).

Let us consider the first term in the right-hand side of Eq. (2.12). Applying operators (4.9) to the relevant distribution function (4.7) and taking into account
(4.10), we have
\[
PL^0 f_r = \left[ \prod_{i=2}^{N} f_1(x_i) \right] L_1^0 f_1(x_1, t),
\]
\[
PL' f_r = \left[ \prod_{i=2}^{N} f_1(x_i) \right] n \int dx_2 L_1' f_1(x_2) f_1(x_1, t). \tag{4.11}
\]

To calculate the additional term (hereinafter we put \( t_0 = 0 \) because Eq. (2.12) is valid for any initial instant \( t_0 \))
\[
C(t) = P(L^0 + L') e^{Q(L^0 + L')t} \frac{1}{1 - C_0 e^{-(L^0 + L')t} e^{Q(L^0 + L')t} C_0 e^{-(L^0 + L')t} f_r(t)}, \tag{4.12}
\]
arising due to initial correlations, we use the following relations
\[
e^{(A+B)t} = e^{At} + \int_0^t d\theta e^{A(t-\theta)} B e^{(A+B)\theta}, \tag{4.13}
\]
\[
e^{L_0^0 t} \Phi(x_1, \ldots, x_N, t) = \left[ \prod_{i=1}^{N} e^{L_i^0 t} \right] \Phi(x_1, \ldots, x_N, t)
= \Phi(x_1 - \mathbf{v}_1 t, \mathbf{p}_1, \ldots, x_N - \mathbf{v}_N t, \mathbf{p}_N, t), \tag{4.14}
\]
where \( A \) and \( B \) are the arbitrary operators. Equation (4.14) follows from the definition (4.9) of the Liouvillian \( L^0 \) of the free propagating particles. We also note, that the parameter of initial correlations (2.11) \( C_0 = f_i(0)/[f_r(0) + f_i(0)] \) can be expanded into the power series in \( f_i(0)/f_r(0) \). Using (4.7) and (4.8), we have
\[
\frac{f_i(0)}{f_r(0)} = \sum_{i<j=1}^{N} \frac{g_2(x_i, x_j)}{f_1(x_i) f_1(x_j)} + \sum_{i<j<k} \frac{g_3(x_i, x_j, x_k)}{f_1(x_i) f_1(x_j) f_1(x_k)} + \ldots \tag{4.15}
\]
If we use of (4.13) and expand (4.12) in \( PL^0, L',QL' \) and \( f_i(0)/f_r(0) \), then, it is easy to see that (4.12) can be presented as a sum of products of operators \( PL^0, L', PL', \exp(L_0^0 \theta) \), one-particle distribution functions and correlations functions (4.15). Applying the projection operator (4.3) to such products and accounting for (4.7), (4.9), (4.10), (4.14) and (4.15), we find that (4.12) can be presented as a series expansion in the density parameter \( \gamma \) (4.2).

In what follows, we restrict ourselves to the linear approximation in the density parameter \( \gamma \) (4.2). Also, in order to simplify formulae, a spatially homogeneous case will be considered. In this case a one-particle distribution function does not depend on a particle coordinate, \( f_1(x_j, t) = f_1(\mathbf{p}_j, t) \) (\( \int f(\mathbf{p}, t) d\mathbf{p} = 1 \)), and the multiparticle correlation functions depend only on the differences of coordinates, e.g., \( g_2(x_i, x_j) = g_2(x_i - x_j, \mathbf{p}_i, \mathbf{p}_j) \). A spatial homogeneity allows us to cancel all terms with \( PL^0 \) (such as the first equation (4.11)), because, according to (4.9) and (4.10), all these terms contain a derivative with respect to
a variable $x_1$, which is not under the integration. The second equation \(4.11\) also vanishes in a space homogeneous case, because we consider a potential $V_{ij}$ \(4.1\) dependent of the particle coordinates difference. Taking also into account, that the terms containing $P_L'$ result in the expressions proportional at least to the first power of $n$, we can replace operators $Q = 1 - P$ in \((4.12)\) by the unity and simplify it to

$$C(t) = PL' e^{(L'' + L')t} \frac{1}{1 - C_0} e^{-(L'' + L')t} f_r(t).$$  \(4.16\)

Expanding formally $\frac{1}{1 - C_0}$ and $C_0$ in the power series in $f_i/f_r$, restricting ourselves to the first order in $n$ and then converting the series, we obtain the following expression for the contribution of initial correlations \((4.12)\)

$$C(t) = \left[ \prod_{i=2}^{N} f_1(p_i) \right] n \int dx_2 L_{12} e^{L_{12}t} \frac{g_2(x_1, x_2)}{f_1(p_1) f_1(p_2)} e^{-L_{12}t} f_1(p_2) f_1(p_1, t),$$  \(4.17\)

where $L_{12} = L_{12}^0 + L_{12}'$, $L_{12}^0 = L_1^0 + L_2^0$, i.e. $L_{12}$ is the two-particle Liouvillian \(4.9\), and $f_1(p_i) = f_1(p_i, t_0)$. The parameter of initial correlations $C_0(x_1, \ldots, x_N)$ in \((4.11)\) under the accepted linear approximation in $n$ (binary collisions) reduces to the parameter $C_0^{12}(x_1, x_2)$, which contains only a two-particle correlation function $g_2(x_1, x_2) = g_2(x_1 - x_2, p_1, p_2)$

$$\frac{1}{1 - C_0^{12}(x_1, x_2)} C_0^{12}(x_1, x_2) = \frac{g_2(x_1, x_2)}{f_1(p_1) f_1(p_2)}.  \tag{4.18}$$

Deriving \(4.17\), we have also taken into account that each additional integration over $x_3, \ldots$ adds an additional power of $n$ and, therefore, in the linear approximation in $n$, all formulae can contain no more than one integration over the phase space.

Consider now the second (collision) term of the TC-HGME \((2.12)\). Initial correlations also contribute to this term through the second term of function $R(t) = 1 + [R(t) - 1] \tag{2.13}$. We start with the conventional collision term of the TC-GME taking $R(t) = 1$. To simplify this term, we note, that the following relations hold in general

$$PL^0 e^{L^0 t} Q = 0, \tag{4.19}$$

$$e^{QL^0 t} Q = e^{L^0 t} Q .$$

The second equation \(4.19\) follows from the first one, which in its turn follows from \((4.3), (4.6), (4.9), (4.10), (4.13)\) and \((4.14)\). The term in the considered collision integral, which is proportional to $QL^0 f_r(t)$, gives zero contribution in the case of spatial homogeneity. Thus, the kinetic term under consideration acquires the form

$$K(t) = \int_0^t dt_1 PL' e^{(L'' + QL')t_1} QL' f_r(t - t_1).  \tag{4.20}$$
Using the procedure, described above, we have the collision term (4.20) in the accepted linear approximation in the density parameter \( \gamma \) as

\[
K(t) = \prod_{i=2}^{N} f_1(p_i) n \int_0^t dt_1 \int dx_2 L_1^{t} e^{L_1^{t}f_1(p_2)} f_1(p_1, t - t_1). \tag{4.21}
\]

In the same way we can consider the contribution of initial correlations to the collision integral. Accounting for the second term in \( R(t) \) (2.13) in the kinetic term of equation (2.12) leads to the modification of the collision integral (4.21). In the linear approximation in \( \gamma \) we have for the kinetic term of equation (2.12) (instead of (4.21))

\[
\tilde{K}(t) = \prod_{i=2}^{N} f_1(p_i) n \int_0^t dt_1 \int dx_2 L_1^{t} R_{12}(t) e^{L_1^{t}f_1(p_2)} f_1(p_1, t - t_1), \tag{4.22}
\]

\[
R_{12}(t) = 1 + e^{L_1^{t}} \frac{g_2(x_1, x_2)}{f_1(p_1)f_1(p_2)} e^{-L_1^{t}}. \tag{4.23}
\]

Now, collecting all obtained formulae (4.17), (4.22) and (4.23) and accounting for the definition of the relevant distribution function (4.7), we obtain from the TC-HGME (2.12) the following equation for a one-particle momentum distribution function

\[
\frac{\partial}{\partial t} f_1(p_1, t) = n \int dx_2 L_1^{t} G_{12}(t) f_1(p_2) f_1(p_1, t)
+ n \int dx_2 \int_0^t dt_1 L_1^{t} f_1(p_2) f_1(p_1, t - t_1), \tag{4.24}
\]

where

\[
G_{12}(t) = R_{12} - 1
= e^{L_1^{t}} \frac{g_2(x_1, x_2)}{f_1(p_1)f_1(p_2)} e^{-L_1^{t}}. \tag{4.25}
\]

is the parameter of initial correlations in the linear in \( \gamma \) approximation which is determined by the time evolution of a two-particle correlation function. It is interesting to note that the expression like (4.25) appears as an additional term in the (correlation) entropy caused by two-particle correlation function and obtained by means of quasi-equilibrium statistical operator (see, e.g. [21]). This additional contribution to entropy is essential when a two-particle correlation function damps slowly.

Equation (4.24) is a new equation for a one-particle distribution function obtained from TC-HGME (2.12) in the linear approximation in \( n \) and which accounts for both initial correlations and collisions in this approximation exactly.
It has been obtained not using the Bogoliubov principle of weakening of initial correlations or any other hypothesis. First term in the right-hand side of this equation (linear in $L'_{12}$) is exclusively defined by initial correlations evolving with time. Initial correlations also modify the collision integral (second term in the right-hand side of this equation). It contains, except the term related to an interaction between particles (determined by the Liouvillian $L'_{12}$), also time-dependent initial correlations determined by the two-particle correlation function (4.25). Evolution in time of initial correlations as well as of the collision term is determined by the exact two-particle propagator $\exp(L_{12}t)$ only, and in this sense Eq. (4.24) is closed. As it follows from (4.13), the propagator $\exp(L_{12}t)$ satisfies the integral equation

$$\exp(L_{12}t) = \exp(L^0_{12}t) + \int_0^t dt_1 \exp[L^0_{12}(t-t_1)]L'_{12} \exp(L_{12}t_1), \quad (4.26)$$

where $\exp(L^0_{12}t)$ is the propagator for noninteracting particles. In the case of a weak inter-particle interaction, we can iterate the integral equation (4.26) in $L'_{12}$ and obtain equation (4.24) in the desired approximation in the small interaction parameter (note, that the correlation function $g_2(x_1,x_2)$ also depends on an interaction between particles).

The obtained homogeneous integro-differential equation (4.24) differs from that which could be found from the conventional GME (2.9) using the projection operator (4.3) and the principle of weakening of initial correlations or RPA. In the latter case we would have equation (4.24) in which the time-dependent correlation function factor $G_{12}(t)$ determining the dynamics of initial correlations would be set zero. Moreover, such an equation would be valid only at at the time scale $t \gg t_{cor}$, whereas equation (4.24) is valid on any time scale.

It should be noted, that there can be the large-scale correlations associated with, e.g., the quantities that are conserved. These correlations can only vanish on the time scale of the order of the relaxation time $t_{rel}$ of a distribution function $f_1(p_i,t)$. If at $t \gg t_{cor}$ ($t_{cor} \ll t_{rel}$) the correlation function $G_{12}(t)$ was set zero, this would signify that all initial correlations vanish on this time scale, i.e. the large-scale correlations do not matter.

Equation (4.24) describes the entire evolution process and is expected to switch from initial (reversible) regime into the kinetic (irreversible) one, automatically. It will be shown below, that Eq. (4.24) converts on the appropriate time scale into the conventional Boltzmann equation if all correlations (initial and caused by collisions) damp with time.

5 CONNECTION TO THE CLASSICAL BOLTZMANN EQUATION

Obtained closed equation (4.24) for a one-particle distribution function is not a kinetic equation in the conventional sense. It does not describe generally an
irreversible evolution with time because it is time-reversible if the parameter of initial correlations \(C_{12}^0\) remains unchanged when all the particle velocities are reversed (see below). Thus, a time-asymmetric behaviour can be secured by the special choice of an initial condition. If we suppose that

\[
C_{12}^1(x_1 - x_2, p_1, p_2) = C_{12}^0(x_1 - x_2, -p_1, -p_2),
\]

i.e.,

\[
g_2(x_1 - x_2, p_1, p_2) = g_2(x_1 - x_2, -p_1, -p_2) \quad \text{and} \quad f_1(p_i) = f_1(-p_i),
\]

then equation (4.24) is invariant under the transformation \(t \to -t\), because such a transformation results in changing the sign of all Liouvillians \(L^0_i\) and \(L^0_{ij}\). Such reversibility is understandable on the very early stage of the system evolution, when \(t \lesssim t_{cor}\) and memory of the initial state has not been lost. On the time scale \(t \gg t_{cor}\) the situation can change.

It can be seen from (4.24) that, in order to enter the kinetic (irreversible) stage of the evolution, the reversible terms connected with initial correlations should vanish on some time scale. Moreover, irreversibility of collision integral may be realized if it is possible to extend the integral over \(t_1\) to infinity. Thus, we have to consider the behaviour of initial correlations and the integral over \(t_1\) in Eq. (4.24) in time.

It should be noted that the irreversible evolution can go "forwards" or "backwards". The question as to which of these evolutions should be chosen cannot be answered on the basis of the present consideration of the time-reversible Liouville equation only, and we consider the evolution in the future \((t > 0)\).

In the accepted approximation of the binary collisions, the initial correlations are described by a correlation function \(g_2(x_1, x_2)\) depending on coordinates and momenta of only two particles. The time evolution of this binary correlation function is governed by the two-particle propagator (4.26). In the zero approximation in the interaction \(L_{12}\), the action of the propagator \(\exp(L^0_{12}t)\) on a correlation function \(g_2(x_1, x_2)\) (and on all functions of coordinates) results in shifting the coordinates \(x_1\) and \(x_2\) with time according to (4.14)

\[
\exp(L^0_{12}t)g_2(x_1, x_2) = g_2(r - gt, p_1, p_2),
\]

where \(r = x_1 - x_2\) is the initial distance between particles 1 and 2 and \(g = v_1 - v_2\) is the relative velocity. Therefore, in this case a binary correlation function changes with time according to a change of a distance between particles \(|v_1 - v_2|t\), which is linear in time. In general, an evolution of a two-particle correlation function \(g_2(x_1, x_2)\) under the action of the exact two-particle propagator \(\exp(L_{12}t)\) can be found from the well-known solution to the two-body problem.

It is important to know how a correlation function behaves as the time passes. If there is a big fraction of a "parallel motion" and a distance between particles (or, more generally, a distance between points in the phase space) does not change significantly, a correlation function does not vanish, i.e. the initial correlations do not disentangle with time. It can be so even in the case of the ergodic flow in the phase space. If a distance between particles grows with time and the radius of correlation between particles \(r_{cor} \sim r_0\) is limited, the correlation function \(g_2(x_1, x_2)\) fades with time. But for existence of a limited relaxation time, i.e. a finite time for vanishing a correlation function, this function should
diminish with time exponentially. As it is well known, such an exponential disentangling of correlations with time can be guaranteed by a mixing ergodic flow in the phase space. The mixing ergodic flow is closely connected with the property of the dynamical systems known as the local (stochastic) instability (see, e.g., [22]). If we assume that the dynamics of the system of classical particles under consideration have the property of mixing flow (and this is usually the case), then the initial correlations weak with time and disappear after a short correlation time $t_{\text{cor}}$. In this case, at $t \gg t_{\text{cor}}$, we can neglect the contribution of initial correlations to equation (4.24) and put $G_{12}(t) = 0$.

This cannot, however, be done if there are the correlations associated with the large-scale fluctuations such as the collective excitations, for example, the plasma oscillations. A relaxation time for such fluctuations can be of the order of the kinetic relaxation time $t_{\text{rel}} \gg t_{\text{cor}}$, and, therefore, they are essential on the kinetic and other large time scale regimes. The conventional kinetic equations do not take into account such large-scale correlations and do not describe the fluctuations of the distribution functions. A kinetic theory of fluctuations can be found in, e.g., [4].

Dependence of the initial correlation function $g_2(x_1, x_2)$ and a potential of an interparticle interaction $V_{ij} = V(|x_i - x_j|)$ on a distance between particles is essential for the behaviour of integrals over $x_2$ in (4.24) with time. Let us consider an example, when this dependence is defined as

$$g_2(x_1, x_2) = g_0 \exp\left(-\frac{r^2}{r_{\text{cor}}^2}\right)\phi(p_1, p_2),$$

$$V(r) = V_0 \exp\left(-\frac{r^2}{r_0^2}\right),$$

where $g_0$ and $V_0$ are the constant parameters and $\phi(p_1, p_2)$ is a properly normalized function of the particle momenta. We will estimate the time dependence of the terms in Eq. (4.24), determined by initial correlations, in the case of a weak interparticle interaction, when the time evolution is governed by the "free" propagator $\exp(L_{12}^0t)$. The correlation function $g_2(x_1, x_2)$, under the action of $\exp(L_{12}^0t)$ transforms according to (5.1) as

$$\exp(L_{12}^0t)g_2(x_1, x_2) = g_2(r - gt, p_1, p_2) = g_0 \exp\left(-\frac{|r - gt|^2}{r_{\text{cor}}^2}\right)\phi(p_1, p_2).$$

The function (5.3) goes to zero at $t \to \infty$ and any fixed distance $r$ and velocity $g$.

Using (5.2) and (5.3), the first right-hand side term of Eq. (4.24) can be presented as

$$n \int dp_2 \int dr \left[ \frac{\partial}{\partial r} V_0 \exp\left(-\frac{r^2}{r_0^2}\right) \right] \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] g_2(r - gt, p_1, p_2) f_1(p_1)f_1(p_2)f_1(p_1, t),$$

where $g_2(r - gt, p_1, p_2)$ is defined by (5.3)
It is easy to see, that the integral over \( r \) \( (5.4) \) is not equal to zero only if \( t < t_{\text{cor}} \), where \( t_{\text{cor}} \sim r_{\text{cor}}/\vec{v} \sim r_0/\bar{v} \) and \( \bar{v} \) is a typical (mean) particle velocity. At \( t \gg t_{\text{cor}} \) the integral \( (5.4) \) practically vanishes because of the finite range \( r_0 \) of an interparticle interaction. Of course, such a behaviour is possible if a contribution of the "parallel motion" with small \( g \) to \( (5.4) \) is negligible. The term, related to initial correlations and contributing to the second right-hand side term of Eq. \( (4.24) \) (collision integral), displays the same behaviour with time. Thus, in the considered example, the terms with initial correlations in Eq. \( (4.24) \) vanish at \( t \gg t_{\text{cor}} \) if the particles dynamics is characterized by the ergodic mixing flow in the phase space.

If the effective interaction between particles vanishes at a distance between particles \( r \gg r_{\text{cor}} \), then at \( t_1 \gg t_{\text{cor}} \) the action of \( \exp(L_{12}^{0} t_1) \) on the interaction Liouvillian \( L_{12}^{0} \) \( (4.9) \) under the integral over \( t_1 \) in \( (4.24) \) results in increasing the distance between particles beyond the radius of inter-particle interaction \( r_0 \sim r_{\text{cor}} \) and, therefore, in vanishing the integrand. As follows from Eq. \( (4.24) \), the initial moment \( t = 0 \) refers to the time just before the collision. Given that the time between collisions \( t_{\text{rel}} \), which determines the time of an essential change of the one-particle distribution function \( f_1(p_1, t) \), exceeds considerably \( t_{\text{cor}} \) due to the condition \( (4.2) \), \( f_1(p_1, t - t_1) \) does not practically change within the interval \( 0 \leq t_1 \leq t_{\text{cor}} \). Thus, \( f_1(p_1, t - t_1) \) under the integral in \( (4.24) \) can be substituted by \( f_1(p_1, t) \) with the accuracy of \( t_{\text{cor}}/t_{\text{rel}} \sim \gamma \ll 1 \). Moreover, at \( t \gg t_{\text{cor}} \) the upper limit of integral over \( t_1 \) can be extended to infinity. After such a transformation the collision integral becomes Markovian and irreversible.

Therefore, the transition from the microscopic time scale \( 0 \leq t \leq t_{\text{cor}} \) to the macroscopic one \( t \gg t_{\text{cor}} \) is essential for obtaining an irreversible Markovian kinetic equation. On the macroscopic time scale the reversible-in-time terms of Eq. \( (4.24) \) caused by initial correlations may vanish (if the particle dynamics have the necessary properties) and the collision term acquires the desired form. The extension of the upper limit of the integral in Eq. \( (4.24) \) to infinity \( (t \to \infty) \) and the existence of this limit are also essential for obtaining the irreversible equation from reversible Eq. \( (4.24) \). Existence of the limiting value of such an integral over time (at \( t \to \infty \)) was proved in \( [23] \) in the thermodynamic limit \( (V \to \infty, N \to \infty, n = V/N \text{ is finite}) \) and when the time scales of collisions and of relaxation are widely separated \( (t_{\text{cor}} \ll t_{\text{rel}}) \).

In order to obtain the desired nonlinear equation for the one-particle distribution function \( f_1(p_1, t) \), we should make an additional approximation neglecting the time retardation of the one-particle distribution function which is possible on the time scale \( t \ll t_{\text{rel}} \). In this case, we can approximate the function \( f_1(p_2) = f_1(p_2, 0) \) in Eq. \( (4.24) \) by \( f_1(p_1, t) \) with the accuracy of \( \gamma \ll 1 \) (for more details see below).

Summing up, we can rewrite Eq. \( (4.24) \) on the considered macroscopic time scale \( (1.1) \) as

\[
\frac{\partial}{\partial t} f_1(p_1, t) = n \int dx_2 \int_0^\infty dt_1 L_{12}' \exp(L_{12}' t_1) L_{12}' f_1(p_1, t) f_1(p_2, t) . \tag{5.5}
\]
This equation is Markovian, time-irreversible and describes the relaxation of the system with the characteristic time $t_{\text{rel}}$. Using the two-particle scattering theory, it can be shown that Eq. (5.5) is equivalent to the conventional Boltzmann equation (see, e.g., [20]). In the case of a classical gas with a weak interparticle interaction, the exact two-particle propagator $\exp(L_{12}t)$ in (5.5) can be replaced by the "free" propagator $\exp(L_{01}t)$ according to (4.26). In this case (5.5) is equivalent to the Landau equation (see, [20]).

It is worth noting, that we have managed to obtain the Boltzmann equation (5.5), which is nonlinear as it should be, using the linear theory of projection operators. Actually, Eq. (4.24), which has been used for deriving the Boltzmann equation (5.5), is linear with respect to the distribution function $f_1(p_1,t)$ but contains a one-particle distribution function $f_1(p_2)$ at the initial moment $t = 0$ (this follows from using the time-independent projection operator (4.3) resulting in the relevant distribution function (4.7)). Change in time of the momentum distribution function can be presented as $f_1(p_i, t) = f_1(p_i) + (\partial f_1/\partial t)_0 t + \ldots$, where the derivative can be estimated as $\partial f_1/\partial t \sim f_1/t_{\text{rel}}$ and the relaxation time $t_{\text{rel}}$ is determined by the collision integral in (4.24). On the time scale (1.1), there is no difference in Eq. (4.24) between $f_1(p_2)$ and $f_1(p_2, t)$ and that makes this equation nonlinear. Thus, the second inequality in (1.1), $t \ll t_{\text{rel}}$, is essential for obtaining a nonlinear evolution equation from the linear equation (2.12), which accounts for initial correlations, i.e. derived with no restriction on the time scale defined by the first inequality in (1.1), $t \gg t_{\text{cor}}$.

The described procedure of obtaining the time-irreversible Eq. (5.5) from Eq. (4.24) clearly indicates that irreversibility emerges on the macroscopic time scale (1.1) as a result of the loss with time of the information about both the initial correlations and correlations that emerge due to collisions.

6 EVOLUTION EQUATION RETAINING INITIAL CORRELATIONS FOR A HOMOGENEOUS DILUTE GAS OF QUANTUM PARTICLES

Let us now test the general scheme outlined in Section 2 by applying it to the system of interacting quantum particles. In the quantum physics case the derived HGMEs hold in the forms obtained above but the Liouville operator $L$ in Eq. (2.1) (and elsewhere) as well as the operator $\exp(Lt)$ are now the superoperators acting according to (2.4) on any operator $A(t)$ (particularly on the statistical operator $F_N(t)$ of $N$ interacting quantum particles) as

$$LA(t) = \frac{1}{i\hbar}[H, A(t)], e^{Lt}A(t) = e^{1/\hbar H_{t1}}A(t)e^{-1/\hbar H_{t1}},$$

(6.1)

respectively. Accordingly, Eqs. (2.12) and (3.5) for the relevant part of the statistical operator $f_r(t) = PF_N(t)$ hold with the appropriate redefinitions of
the symbols and operations used in these equations. The order of operators in these equations also matters.

In the space representation, the system of $N$ quantum particles is described by the density matrix $F_N(r_1, \ldots, r_N, r_1', \ldots, r_N', t)$. This matrix should satisfy the symmetry conditions which reflect the statistics of the particles under consideration, i.e.

$$P_{ij}F_N = F_N P_{ij} = \theta F_N,$$

$$\theta = \pm 1,$$  \hspace{1cm} (6.2)

where $P_{ij}$ is the operator of transmutation of any two variables $r_i$ and $r_j$, when it stands to the left of $F_N$, or of $r_i'$ and $r_j'$, when it acts from the right side of $F_N$ (this rule holds for any operators acting on the matrixes). The sign plus is applicable for bosons and minus should be used for fermions.

We focus on the derivation of the evolution equation for a one-particle density matrix $f_1(r_1, r_1', t)$, which is defined according to the following definition of the $s$-particle ($s \leq N$) density matrix

$$f_s(r_1, \ldots, r_s, r_1', \ldots, r_s', t) = V^s Tr(s+1, \ldots, N) F_N(r_1, \ldots, r_N, r_1', \ldots, r_N', t),$$  \hspace{1cm} (6.3)

where $Tr(s+1, \ldots, N)$ denotes the trace taken over the coordinates of $N-s$ particles ($s+1, \ldots, N$). From the normalization condition $\text{Tr} F_N = 1$ it follows that $\Delta Tr f_1 = 1$.

It is convenient to introduce the following projection operator (compare with the definition (4.3))

$$P = \left[ \prod_{i=2}^{N} f_1(r_i, r_i') \right] \frac{1}{V^{N-1}} Tr(2, \ldots, N),$$  \hspace{1cm} (6.4)

where $f_1(r_i, r_i') = f_1(r_i, r_i', t_0)$ is the one-particle density matrix taken at the initial moment of time $t_0$. Therefore, the chosen relevant density matrix under consideration is

$$P f_N(t) = f(t) = \left[ \prod_{i=2}^{N} f_1(r_i, r_i') \right] f_1(r_1, r_1', t)$$  \hspace{1cm} (6.5)

in accordance with the definition (6.3) for the $N$-particle density matrix $f_N = V^N F_N$. As it is seen from equation (6.5), the selected relevant part of the $N$-particle density matrix does not contain any correlations. In order to define the irrelevant part of the density matrix $f_i(t_0) = (1 - P)f_N(t_0)$, which enters equations (2.12) and (3.5) and comprise all initial correlations, we have also to take into account the quantum correlations arising due to the symmetry condition (6.2). These correlations exist independently of the interaction between
particles which we introduce by means of the following Hamiltonian

\[
H = H^0 + H',
\]

\[
H^0 = \sum_{i=1}^{N} K_i, \quad H' = \sum_{i<j=1}^{N} \Phi_{i,j},
\]

\[
K_i = -\frac{\hbar^2}{2m} \nabla_{\mathbf{r}_i}^2, \quad \Phi_{i,j} = \Phi(|\mathbf{r}_i - \mathbf{r}_j|).
\]

Thus,

\[
f_N(t_0) = f_r(t_0) + f_l(t_0)
\]

Thus, \(f_N(t_0) = f_r(t_0) + f_l(t_0)\) can be presented in the form of the cluster expansion with

\[
f_r(t_0) = \prod_{i=1}^{N} f_1(i)
\]

and

\[
f_l(t_0) = \sum_{i,j=1, i < j}^{N} \bar{g}_2(i,j) \prod_{k=1, k \neq i,j}^{N-2} f_1(k) + \sum_{i,j,k=1, i < j < k}^{N} \bar{g}_3(i,j,k) \prod_{l=1, l \neq i,j,k}^{N-3} f_1(l) + \ldots,
\]

where we have used the short notation \(f_1(i) = f_1(\mathbf{r}_i, \mathbf{r}_i')\) (i.e. \(i = \mathbf{r}_i, \mathbf{r}_i', i = 1, \ldots, N\)) and, e.g., \(\prod_{k=1, k \neq i,j}^{N-2} f_1(k)\) stands for the product of \(N-2\) one-particle functions \(f_1(k)\) with \(k\) taking values \(1, \ldots, N\) but \(k \neq i, j\) (compare with \(6.8\)).

A two-particle \(\bar{g}_2(i,j)\) and a three-particle \(\bar{g}_3(i,j,k)\) correlation functions are defined as (the correlation functions for more particles can be written down in the same way)

\[
\bar{g}_2(i,j) = \theta P_{ij} f_1(i) f_1(j) + g_2(i,j),
\]

\[
\bar{g}_3(i,j,k) = (P_{ij} P_{jk} + P_{ij} P_{ik} f_1(i) f_1(j) f_1(k) + (\theta P_{ik} + \theta P_{jk}) g_2(i,j) f_1(k) + (\theta P_{ij} + \theta P_{jk}) g_2(i,j,k) f_1(i) + (\theta P_{ij} + \theta P_{jk}) g_2(i,k) f_1(j) + g_3(i,j,k).
\]

The terms in the right-hand sides of Eqs. \(6.9\) with one, two, etc. permutation operators \(P_{ij}\) represent a two-, three-particle, etc. quantum correlations emerging due to the proper symmetry properties of \(f_N(t_0)\) guaranteed by the symmetrization operators, e.g.,

\[
f_2(i,j) = \theta P_{ij} f_2(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_i', \mathbf{r}_j').
\]

The irreducible two-, three-, etc. correlation functions, \(g_2(i,j), g_3(i,j,k), \) etc., are due to the interaction between particles \(\Phi_{i,j}\) and are proportional to an interaction parameter \(\varepsilon\) in some power. Note, that in contrast to the classical physics case (see \(1.8\)), each correlation function in \(6.9\) contains correlations of all possible orders in \(\varepsilon\), e.g. \(\bar{g}_2(i,j)\) includes two-particle quantum correlations (existing even in the absence of interaction, when \(\varepsilon = 0\)) and two-particle correlations of the first order in \(\varepsilon (g_2(i,j))\). The same is valid for \(\bar{g}_3(i,j,k)\), which
contains a three-particle correlations of the zero, first and second \((g_3(i, j, k))\) orders in \(\varepsilon\).

The following useful relations hold for the permutation operators

\[
P_{ij} P_{jk} = P_{ik} P_{ij},
P_{jk} P_{ij} = P_{ij} P_{ik}.
\] (6.11)

Let us now derive the evolution equation for a gas of interacting quantum particles in the lowest (first) order in the small density of particles parameter \(n\). The corresponding dimensionless small parameter is again (see (4.2))

\[
\gamma = r_0^3 n \sim \frac{t_{cor}}{t_{rel}} \ll 1.
\] (6.12)

We will use the TC-HGME given by Eqs.(2.12) and (2.13) with appropriate redefinitions of the symbols, as it was mentioned above.

The Liouville superoperator \(L = L^0 + L'\), corresponding to the Hamiltonian (6.6), is defined by equation (6.1) as

\[
L^0 = \sum_{i=1}^{N} L^0_i, L^0_i = \frac{1}{i\hbar} \left( -\frac{\hbar^2}{2m} \left( \nabla^2 r_i - \nabla^2 r_i' \right) \right),
\]

\[
L' = \sum_{i<j=1}^{N} L'_{ij}, L'_{ij} = \frac{1}{i\hbar} \left[ \Phi(|r_i - r_j|) - \Phi(|r_i' - r_j'|) \right],
\] (6.13)

where we have used the abovementioned rule, that an operator standing on the right side (in the commutator) acts on the primed variables.

Using the definitions (6.13), one can easily show that the following relations (similar to the classical physics case) hold

\[
Tr_{(i)} L^0_i \varphi_N(1, \ldots, i, \ldots, N) = 0,
Tr_{(i,j)} L'_{ij} \varphi_N(1, \ldots, i, \ldots, j, \ldots, N) = 0,
\] (6.14)

where \(\varphi_N(1, \ldots, i, \ldots, N)\) is some matrix, defined for the \(N\)-particle system, which in the space representation takes the form \(\varphi_N(1, \ldots, i, \ldots, N) = \varphi_N(r_1, \ldots, r_i, \ldots, r_N, r'_1, \ldots, r'_N)\) and satisfies the necessary boundary conditions.

Thus, using the projection operator (6.4) and the relevant density matrix (6.5), the first two terms in TC-HGME (2.12) \(P(L^0 + L') f_r(t)\) (which coincide with those in TC-GME (2.9)) can be presented as

\[
PL^0 f_r(t) = \prod_{i=2}^{N} f_1(i) \left( \frac{1}{i\hbar} - \frac{\hbar^2}{2m} \left( \nabla^2 r_i - \nabla^2 r_i' \right) \right) f_1(r_1, r'_1, t),
\]

\[
PL' f_r(t) = \prod_{i=2}^{N} f_1(i) \left( \frac{n}{\hbar} \int dr_2 \left[ \Phi(|r_1 - r_2|) - \Phi(|r_1' - r_2'|) \right] ight) f_1(r_2, r_2) f_1(r_1, r'_1, t).
\] (6.15)
To simplify the derivation, we restrict ourselves to the spatially homogeneous case, when all matrices are invariant under translations and, in particular, the matrix $f_1(\mathbf{r}_i, \mathbf{r}_i')$ should be of the form

$$f_1(\mathbf{r}_i, \mathbf{r}_i', t) = f_1(\mathbf{r}_i - \mathbf{r}_i', t).$$

(6.16)

Then, it is easy to see that for functions (6.16), $H^0$ commutes with $f_r(t)$ and Eqs. (6.15) vanish, i.e.

$$L^0 f_r(t) = 0, PL^r f_r(t) = 0.$$  

(6.17)

Also, in this spatially homogeneous case, for any matrix $\varphi_N(1, \ldots, N)$

$$PL^0 \varphi_N(1, \ldots, N) = 0.$$  

(6.18)

Let us now consider the terms in TC-HGME (2.12) caused by initial correlations. We note, that in the considered case of identical interacting quantum particles, it is impossible to simply disregard the inhomogeneous term in the exact TC-GME (2.9) (assuming that correlations damp with the distance between particles) due to always existing (at any distances) quantum initial correlations (see (6.9)). Bogoliubov [1] tackled this problem by introducing a special boundary (initial) condition to the BBGKY chain at $t_0 \to -\infty$ implying that at this limiting moment of time the particles are located at distances beyond the correlation radius $r_0$, which means that correlation functions in (6.9), $g_2(i, j)$, $g_3(i, j, k)$, etc., caused by interaction between particles (and proportional to some power of $\varepsilon$), vanish, and

$$\lim_{t_0 \to -\infty} f_i(t_0) = \gamma_N \prod_{i=1}^N f_1(i),$$

(6.19)

where

$$\gamma_N = (1 + \theta P_{1N} + \ldots + \theta P_{N-1, N}) \ldots (1 + \theta P_{12})$$

(6.20)

is the symmetrization operator for $N$ particles. Therefore, in the Bogoliubov approach, the multi-particle density matrix at $t_0 \to -\infty$ is the properly symmetrized product of the one-particle density matrixes (compare with (6.7)-(6.9)). As in the classical physics case, this boundary condition introduces irreversibility into the system evolution and thus allows obtaining the quantum kinetic equations [24]. Also, as in the case of a dilute gas of classical particles, in order to obtain the closed kinetic equation for a one-particle density matrix, Bogoliubov used the specific form of time-dependence of the multi-particle density matrix (valid only at $t - t_0 \gg t_{cor}$), i.e. he assumed that on this time scale the dependence on time of the multi-particle density matrixes is determined by the time-dependence of a one-particle density matrix.

Here, we do not use any of the abovementioned Bogoliubov’s assumptions. Thus, let us consider the (super)operator (see (2.13))

$$C(t) = R(t) - 1 = e^{QLt} \frac{1}{1 - C_0 e^{-Lt} e^{QLt} C_0 e^{-Lt}}$$

(6.21)
determining the influence of initial correlations in Eq. (2.12) on the evolution process (hereinafter we put $t_0 = 0$, because the HGME is valid for any initial moment $t_0$). In order to expand exponential (super)operators in the power series, we use the relation (4.13) which is also valid for any (super)operators $A$ and $B$.

Using (4.13) and expanding (6.21) in $PL^0$ and $QL'$, one can see that all terms with $PL^0$ vanish due to Eq. (6.18) and, therefore, all $QL$ in (6.21) can be replaced by $L^0 + QL'$. Taking into account that the terms containing $PL'$ result in the expressions proportional to at least the first power of $n$ (see, e.g., (6.15)) and looking for the lowest (first) order approximation in $n$ of Eq. (2.12), we can substitute $Q = 1 - P$ in (6.21) with the unity and simplify this expression to

$$C(t) = e^{Lt} \frac{1}{1 - C_0} C_0 e^{-Lt}. \quad (6.22)$$

Using expression (2.11) for $C_0$, we have

$$\frac{1}{1 - C_0} C_0 = f_i(0) f_r^{-1}(0). \quad (6.23)$$

This is a remarkable result showing that in this approximation there is no problem with the existence (convergence/invertability) of $R(t)$ (2.13) (the same has been shown in Sec. 4 for the classical physics case). In connection with that it is worth reminding, that if in some part of the definition space $f_r(0)$ goes to zero, so should do $f_i(0)$ (at the same rate or more promptly than the relevant part). It follows from the possibility of breaking the statistical operator into the uncorrelated ($f_r$) and correlated ($f_i$) parts (see (6.7)-(6.9)), which in turn caused by the fact that if an interaction between particles is switched off ($\varepsilon = 0$), the statistical operator should be given by symmetrized product of one-particle operators. Moreover, in the equations for the relevant part of statistical operator obtained below, as a result of approximation (6.23), the factor $f_i(0) f_r^{-1}(0)$ is always multiplied by $f_i(0)$ (see (6.41) below and take into account that in the approximation under consideration the relevant part changes slowly). This fact additionally guarantees an absence of any divergencies (in other words, any manipulations with initial condition term should result in terms behaving like $f_i(0)$ in accordance with the identity (2.10)).

Now, it is necessary to define $f_r^{-1}(0)$. In the spatially homogeneous case under consideration, it is convenient to work in the momentum representation. The relation between the space and momentum representations for a one-particle density matrix of interest is

$$f_1(r, r', t) = \frac{1}{(2\pi \hbar)^3} \int f_1(p, p', t) e^{ipr/\hbar} e^{-ip'r'/\hbar} dp dp', \quad (6.24)$$

where a function $w(p)$ (with the dimensionality $[p^{-3}]$) is normalized to the unity and corresponds to a classical momentum distribution function. According to
the definition

\[ \int f_1(p, p', t)f_1^{-1}(p'', p', t)dp'' = \delta(p - p'), \tag{6.25} \]

we can take for \( f_1^{-1}(p, p') \) the following matrix

\[ f_1^{-1}(p, p', t) = (2\pi \hbar)^{-3}w^{-1}(p, t)\delta(p - p'). \tag{6.26} \]

Note, that the \( \delta \)-functions in (6.24)-(6.26) (related to the fact that a one-particle distribution function is diagonal in the spatially homogeneous case) always stay under the integrals or are divided out.

In accordance with (6.24) and (6.26), an invert one-particle matrix in the space representation may be defined as

\[ f_1^{-1}(r, r', t) = \frac{1}{(2\pi \hbar)^3} \int f_1^{-1}(p, p', t)e^{ipr/\hbar}e^{-ip'r'/\hbar}dpdp' \]

\[ = \frac{1}{(2\pi \hbar)^6} \int w^{-1}(p, t)e^{ip(r-r')}dp. \tag{6.27} \]

It satisfies the necessary relation

\[ \int f_1(r, r'', t)f_1^{-1}(r'', r', t)dr'' = \delta(r - r'). \tag{6.28} \]

Now, we can define an operator, invert to the relevant operator (6.7), as

\[ f_r^{-1}(0) = \prod_{i=1}^{N} f_1^{-1}(i), \tag{6.29} \]

because, due to (6.25) and (6.28),

\[ [f_r(0)f_r^{-1}(0)] = \prod_{i=1}^{N} I(i). \tag{6.30} \]

Here and further, \( (i) \) stands for \( (r, r_i) \) or \( (p, p_i) \) depending on the representation used, and \( I(i) \) is the unity matrix in the space of \( i \)-particle

\[ I(i) = I(r_i, r'_i) = \delta(r_i - r'_i), \]

\[ I(i) = I(p_i, p'_i) = \delta(p_i - p'_i) \tag{6.31} \]

in the space and momentum representations, respectively.

Applying the introduced definitions for invert matrixes and using Eq. (6.8) for the irrelevant density matrix, the correlation parameter (6.23) can be rewrit-
Using (6.22), (6.23) and (6.32), (6.33), one can obtain in this approximation

\[ P_L \]

space and momentum representations as

\[ P \]

and the matrix elements of the permutation operator \( P_{ij} \) are defined in the space and momentum representations as

\[ P_{ij}(r_i, r_j, r_i', r_j') = \delta(r_i - r_j')\delta(r_j - r_i'), \]

\[ P_{ij}(p_i, p_j, p_i', p_j') = \delta(p_i - p_j')\delta(p_j - p_i'). \] (6.34)

Let us consider the additional (to the flow) term in TC-HGME (2.12), \( F(t) = P_L'C(t)f_r(t) \), caused by initial correlations, in the linear approximation in \( n \). Using (6.22), (6.23) and (6.32), (6.33), one can obtain in this approximation

\[ F(t) = P_L'e^{Lt} \frac{1}{1 - C_0 e^{-Lt}} f_r(t) \]

\[ = \prod_{i=2}^{N} f_1(i) \sum_{i,j=1,i<j}^{N} \tilde{g}(i,j)f_1^{-1}(i)f_1^{-1}(j) \prod_{k=1,k\neq i,j}^{N-2} I(k) \]

\[ + \sum_{i,j,k=1,i<j<k}^{N} \tilde{g}(i,j,k)f_1^{-1}(i)f_1^{-1}(j)f_1^{-1}(k) \prod_{l=1,l\neq i,j,k}^{N-3} I(l) + \ldots, \] (6.32)

where, e.g., the two-particle part of (6.32), according to the expressions for correlation functions (6.9), is

\[ \tilde{g}(i,j)f_1^{-1}(i)f_1^{-1}(j) = \theta P_{ij} + g(i,j)f_1^{-1}(i)f_1^{-1}(j), \] (6.33)

and the matrix elements of the permutation operator \( P_{ij} \) are defined in the space and momentum representations as

\[ P_{ij}(r_i, r_j, r_i', r_j') = \delta(r_i - r_j')\delta(r_j - r_i'), \]

\[ P_{ij}(p_i, p_j, p_i', p_j') = \delta(p_i - p_j')\delta(p_j - p_i'). \] (6.34)

Deriving (6.35), we have taken into consideration that each additional integration over the coordinate or momentum space adds an additional power of \( n \) and, therefore, in the linear approximation in \( n \), all formulae can contain no more than one integration (as in (6.33)). Thus, Eq. (6.35) is determined only by the two-particle dynamics described by the Hamiltonian \( H_{12} = H_{12}^0 + H_{12}' \) (and the corresponding Liouvillian \( L_{12} = L_{12}^0 + L_{12}' \)). We note, that (6.35) is different from the corresponding additional (to flow) term (caused by initial correlations) obtained in Sec. 4 for the classical physics case: except a two-particle correlation function, Eq. (5.35) contains a symmetrization operator \( P_{12} \) responsible for quantum correlations.

Taking into account (6.34) and invariance of the Hamiltonian \( H_{12} \) with regard to renaming the particles, i.e. \( H_{12}(p_1, p_2, p_1', p_2') = H_{12}(p_2, p_1, p_2', p_1) \), it can be shown that in the operator form we have (see, e.g. [25])

\[ P_{12}H_{12}P_{12}^{-1} = H_{12} \] (6.36)
a symmetrization operator commutes with the Hamiltonian) and, therefore,
\[ e^{-\frac{i}{\hbar} (H_{12}^0 + H_{12}') t} P_{12} e^{\frac{i}{\hbar} (H_{12}^0 + H_{12}') t} = P_{12}. \]
Now, it is easy to show that the first term in (6.34), caused by the symmetrization operator \( P_{12} \), vanishes in the spatially homogeneous case (for non-symmetrized \( f_r(t) \) it is demonstrated by Eq. (6.17)). Thus,
\[
PL'_{12} P_{12} f_r(t) = nT \prod_{i=2}^{N} f_1(i) \int dr_2 \frac{1}{\hbar} \Phi(|r_1 - r_2|) \\
- \Phi(|r_1' - r_2'|) f_1(r_1 - r_2) f_1(r_2 - r_1', t) = 0. \quad (6.37)
\]
This can be easily proven, if we substitute the integration variable \( r_2 \) in (6.37)
by \( \rho = r_1 - r_2 \) and \( \rho' = r_2 - r_1' \) in the first and second integrals, respectively,
and take into consideration, that in the adopted approximation (first order in \( n \)) the time-dependence of a one-particle density matrix can be disregarded due
to the fact that the change with time is described by the (super)operator which
is already of the first order in \( n \) for the spatially homogeneous system.
Hence, the contribution of initial correlations to the flow term of Eq. (2.12)
in the first order in \( n \) and for the spatially homogeneous system is
\[
PL' C(t) f_r(t) = nT \prod_{i=2}^{N} f_1(i) \int dt_1 T r_{(2)} L'_{12} G_{12}^q(t) f_1(2) f_1(1, t), \quad (6.38)
\]
where
\[
G_{12}^q(t) = e^{-\frac{i}{\hbar} H_{12} t} g(1, 2) f_1^{-1}(1) f_1^{-1}(2) e^{\frac{i}{\hbar} H_{12} t} \quad (6.39)
\]
is a time-dependent quantum two-particle correlation function. This result is
similar to that obtained for the classical physics case (see (4.24) and (4.25)): the
initial correlations lead to the appearance of the additional, linear in interaction
between particles \( (H_{12}') \), term which is determined by the evolution in time of a
two-particle correlation function. Moreover, this result is closed in the sense that
the time-evolution is also completely determined by the two-particle dynamics
with the Hamiltonian \( H_{12} \).
The collision term of Eq. (2.12) in the adopted approximation can be ob-
tained in the same way. Accounting for equations (6.17), (6.18), (??), (6.22),
(6.23), (6.32), (6.33), (6.36) and remembering that any additional power of \( PL' \)
and any additional integration (taking a trace) over the coordinate or momentum
space adds an additional power of density \( n \), it is not difficult to show that in the linear approximation in \( n \) the collision integral in (2.12) reduces to
\[
K^q(t) = \prod_{i=2}^{N} f_1(i) \int_0^t dt_1 T r_{(2)} L_{12} [1 + \theta P_{12} + G_{12}^q(t)] e^{L_{12} t_1} \\
\times L_{12} f_1(2) f_1(1, t - t_1). \quad (6.40)
\]
(6.40)
This result is different from that obtained for the classical physics case, \( K(t) + \tilde{K}(t) \), given by Eqs. (1.21), (1.22), due to the presence of quantum correlations stipulated by a symmetrization operator \( P_{12} \).

Collecting the obtained results (formulae (6.38) and (6.40)) and accounting for the definition of the relevant statistical operator (6.5), we obtain from the TC-HGME (2.12) the following equation for a one-particle density matrix in the linear approximation in \( n \) for the spatially homogeneous case

\[
\frac{\partial}{\partial t} f_1(1,t) = n \text{Tr}_{(2)} L'_{12} G_{12}^q(t) f_1(2)f_1(1,t) \\
+ n \int_0^t dt_1 \text{Tr}_{(2)} L'_{12} [1 + \theta P_{12} + G_{12}^q(t)] e^{L_{12}^{-1} L'_{12} t_1} f_1(2)f_1(1,t-t_1).
\]

(6.41)

We have obtained a new closed homogeneous integro-differential equation for a one-particle density matrix retaining initial correlations which is valid on any time scale (i.e. we can use equation (6.41) for considering in detail all stages of evolution of a one-particle density matrix). No approximation like the Bogoliubov principle of weakening of initial correlations (valid only on a large time scale \( t \gg t_{\text{cor}} \)) has been used for deriving equation (6.41). This equation is exact in the linear approximation in \( n \) and, therefore, accounts for initial correlations and collisions in this approximation exactly. The first term in the right-hand side of (6.41), linear in interaction \( L'_{12} \), is determined by initial correlations caused by interaction between particles (this term is absent if we apply the boundary condition (6.19)). Initial correlations, both quantum and due to interaction, also modify the second (collision) term of (6.41). Note, that the symmetrization operator \( \theta P_{12} \) (accounting for the particle statistics) appears in this term as a result of a regular procedure outlined above but not as a result of using the boundary condition (6.19), as it is the case when one applies the Bogoliubov principle of weakening of initial correlations [1]. Also, when this principle is applied, the correlation function \( G_{12}^q(t) \) does not show up in the second term of the right-hand side of (6.41)). Evolution with time in (6.41) is completely determined by the exact two-particle dynamics, described by the Hamiltonian \( H_{12} \) (which is quite natural for the considered linear in \( n \) approximation), and in this sense the obtained equation is closed.

7 CONNECTION TO THE QUANTUM BOLTZMANN EQUATION

Equation (6.41) is not a kinetic equation in the conventional sense because it is time reversible (if the correlation function \( g_2(i,j)f_1^{-1}(i)f_1^{-1}(j) \) does not change when the particle velocities are reversed). Thus, one of the possibilities to secure a time-asymmetric behaviour can be the special choice of an initial condition (e.g., the factorized one, like (6.19)).
It is instructive to consider the transition from Eq. (6.41) to an irreversible kinetic equation describing the evolution, say, into the future \((t > 0)\). As it is seen from (6.41), in order to enter the kinetic (irreversible) stage of the evolution, the reversible terms caused by initial correlation function, \(G_{12}^0(t)\), should vanish on some time scale. Let us suppose that this is the case, on the time scale \(t \gg t_{\text{cor}}\), i.e.

\[
G_{12}^0(t) = 0, t \gg t_{\text{cor}}. \quad (7.1)
\]

Here, the correlation time can be estimated as \(t_{\text{cor}} \sim r_0/v\), where \(r_0\) is the radius of inter-particle interaction and \(v\) is a typical mean particle velocity. Thus, if the dynamics of the system of particles under consideration is such (e.g., a mixing ergodic flow in the phase space) that all initial correlations caused by inter-particle interaction vanish on the time scale \(t \gg t_{\text{cor}}\), as it is supposed by condition (7.1), then Eq. (6.41) reduces to

\[
\frac{\partial}{\partial t} f_{1}(1, t) = n \int_0^t dt_1 Tr_{(2)} L_{12}^t e^{t_1 L_{12}} L_{12}^t (1 + \theta P_{12}) f_{12}^{t_1}(t-t_1),
\]

\[
f_{12}^{t_1}(t-t_1) = f_{1}(1) f_{1}(2, t-t_1), t \gg t_{\text{cor}}, \quad (7.2)
\]

where (6.36) is also taken into account. Making use of the definitions (6.1), this equation can be rewritten as

\[
\frac{\partial f_{1}(1, t)}{\partial t} = -\frac{n}{\hbar^2} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \int_0^t dt_1 e^{i(E-E')t_1} Tr_{(2)} \Phi_{12}(E, E', t-t_1), t \gg t_{\text{cor}},
\]

\[
\Phi_{12}(E, E', t-t_1) = \left[ H_{12}^t, \delta(E-H_{12}) \left[ H_{12}^t, (1 + \theta P_{12}) f_{12}^{t_1}(t-t_1) \right] \delta(E' - H_{12}) \right]. \quad (7.3)
\]

In the adopted first approximation in \(n\), we can consider \(f_{12}^{t_1}(t-t_1)\) under the integral in the zero approximation in \(n\), in which it does not change with time. Thus, one can approximate the two-particle relevant part of a density matrix under the integral in (7.3) as \(f_{12}^{t_1}(t-t_1) = f_{1}(1) f_{1}(2, t)\). This approximation is valid on the time scale \(t \ll t_{\text{rel}}\) (see inequality (1.1) at \(t_0 = 0\)) allowing the neglect of time retardation and allows to convert the linear equation for \(f_{1}(1, t)\) (6.41) (or (7.3)) into the nonlinear one. Now, the integral in (7.3) over \(t_1\) can be calculated. Note, that for finite \(t\) equation (7.3) is time reversible and integral over \(t_1\) is a periodic function of \(t\) for a finite volume of a system \(V\). This equation becomes time irreversible if the upper limit of integral can be extended to infinity, \(t \to \infty\), and this limiting value of the integral over \(t_1\) exists. However, this limiting value of the integral can evidently exist only for a large enough volume of the system under consideration. In other words, in order to exclude the long Poincaré’s cycles, the thermodynamic limiting procedure \((V \to \infty, N \to \infty, n = \frac{N}{V} \text{ is finite})\) is needed before calculating the integral. The existing of this integral in the thermodynamic limit was proved in [23].

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If the mentioned limiting value of the integral exists, then the integral can be considered in the Abel’s sense, i.e.

\[
\lim_{t \to \infty} \int_0^t dt_1 e^{i\mathcal{H}(E-E')t_1} = \lim_{\epsilon \to 0^+} \lim_{t \to \infty} \int_0^t dt_1 e^{-\epsilon t_1} e^{i\mathcal{H}(E-E')t_1} = -i\hbar \frac{\text{Pr}}{E-E'} + \pi \hbar \delta(E - E'), \tag{7.4}
\]

where the order of taking limits matters and \(\text{Pr} \) is a symbol of taking a principal value of an integral.

Thus, we consider the system evolution on the time interval, which satisfies two inequalities: \( t \gg t_{\text{cor}} \) (to exclude the influence of initial correlations) and \( t \ll t_{\text{rel}} \) (to go from linear Eq. (6.41) to nonlinear equation). This is the interval \( [134, 280] \) which existence is guaranteed by the condition \( [134, 280] \). Using (7.4), Eq. (7.3) can be rewritten as

\[
\frac{\partial f_1(1,t)}{\partial t} = \frac{\pi n}{\hbar} \int dE T_{T(2)} \Phi_{12}(E,t),
\]

\[
\Phi_{12}(E,t) = \left[ H'_{12}, \delta(E - H_{12}) \left[ H'_{12}, (1 + \theta P_{12}) f_1(1,t) f_1(2,t) \right] \right] \delta(E - H_{12}).
\]

(7.5)

Obtaining (7.5), we have taken into account that Eq. (7.3) contains only the diagonal matrix element \( \Phi_{12}(E,E',\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1,\mathbf{p}_2,t) \) as it should be because of taking a trace over \( \mathbf{p}_2 \) and because the left hand side of (7.3) is diagonal relatively to \( \mathbf{p} \) and \( \mathbf{p} \) (see (6.21)). The latter property of \( T_{T(2)} \Phi_{12}(E,t) \) can be also proven explicitly. This diagonal element has the symmetry property \( \Phi_{12}(E,E',\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1,\mathbf{p}_2,t) = \Phi_{12}(E',E,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1,\mathbf{p}_2,t) \), and, therefore, the imaginary (principal) part of the integral over \( E \) and \( E' \) vanishes, as it follows from (7.3) and (7.4). The mentioned symmetry of \( \Phi_{12}(E,E',\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1,\mathbf{p}_2,t) \) (7.3) is a consequence of the symmetry of matrix elements of \( H'_{12} \) and \( H_{12} \) (and, therefore, of \( H_{12} \)). The latter symmetry follows from the definition (6.6) and means that \( H'_{12}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2') = H'_{12}(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_1,\mathbf{p}_2) \) and \( H_{12}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2') = H_{12}(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_1,\mathbf{p}_2), \)

where

\[
H'_{12}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2') = [K(\mathbf{p}_1) + K(\mathbf{p}_2)] \delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2'), K(\mathbf{p}) = \frac{\mathbf{p}_1^2}{2m},
\]

\[
H_{12}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2') = \frac{1}{(2\pi\hbar)^3} \nu(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2'),
\]

\[
\nu(\mathbf{p}) = \int \Phi(|\mathbf{r}|) \exp(-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{r}) d\mathbf{r}, \nu(-\mathbf{p}) = \nu(\mathbf{p}). \tag{7.6}
\]

The (operator) \( \delta \)-function of the Hamiltonian \( H \) (entering equation (7.5)) with a two-particle Hamiltonian \( H_{12} \) can be expressed via the imaginary part
of Green’s function $G(E)$:

$$
\delta(E - H) = \mp \frac{1}{\pi} \text{Im} \, G(E^\pm),
G(E^\pm) = \frac{1}{E^\pm - H},
$$

$$
E^\pm = E \pm i\varepsilon, \varepsilon = 0^+.
$$

(7.7)

On the other hand, there is the so-called optical theorem relating $\delta(E - H)$ (the imaginary part of Green’s function $G(E^\pm)$ for the full Hamiltonian $H = H^0 + H'$) and the interaction Hamiltonian $H'$ to $\delta(E - H^0)$ (the imaginary part of Green’s function $G^0(E^\pm) = (E^\pm - H^0)^{-1}$) and the $T$-matrix (see, e.g., [21]):

$$
H'\delta(E - H)H' = T^+\delta(E - H^0)T^- = T^-\delta(E - H^0)T^+,
$$

(7.8)

where the $T$-matrix is defined as

$$
T^\pm = T(E^\pm) = H'G(E^\pm)G^{0^{-1}}(E^\pm)
$$

(7.9)

and satisfies the equations

$$
T^\pm = H' + H'G^0(E^\pm)T^\pm = H' + T^\pm G^0(E^\pm)H'.
$$

(7.10)

The relations (7.7) - (7.9) allow to convert $\Phi_{12}(E, t)$ (7.6) into the following form

$$
\Phi_{12}(E, t) = \left[ H^{-1}_{12}, T_{12}^{-1}\delta(E - H^0_{12})T_{12}^{-1}\left[ H^{-1}_{12}, (1 + \theta P_{12})f_1(1, t)f_1(2, t) \right] T_{12}^{-1}\delta(E - H^0_{12})T_{12}^{-1} \right].
$$

(7.11)

where $T_{12}^\pm$ refers to the $T$-matrix (7.9) defined for the two-particle Hamiltonian $H_{12} = H^0_{12} + H'_{12}$. Using (7.10), it is not difficult to show that

$$
T_{12}^{-1}\left[ H^{-1}_{12}, (1 + \theta P_{12})f_1(1, t)f_1(2, t) \right] T_{12}^{-1} = -\left[ T_{12}^{-1}, (1 + \theta P_{12})f_1(1, t)f_1(2, t) \right],
$$

(7.12)

where we have also taken into account that $G^0_{12}(E^\pm) = (E^\pm - H^0_{12})^{-1}$ commutes with $(1 + \theta P_{12})f_1(1, t)f_1(2, t)$ for the spatially homogeneous case (see also (6.17)).

Using (7.12), (7.11), (7.10), and diagonality of $G^0(E^\pm)$ and of both sides of Eq. (7.5) in the momentum representation, we can transform this equation into the following form

$$
\frac{\partial f_1(1, t)}{\partial t} = -\frac{\pi n}{\hbar} \int dET_{(2)} \left[ T_{12}^{+}\delta(E - H^0)[T_{12}^{-}, (1 + \theta P_{12})f_1(1, t)f_1(2, t)]\delta(E - H^0) \right] .
$$

(7.13)

To calculate the trace (integral over $p_2$) in (7.13), one needs to have the matrix elements of $T$-matrix in the $p$-representation. Iterating equation (7.10) and calculating the matrix elements in the momentum representation, one can show that the $T_{12}$ matrix (describing the particle collisions) guarantees the momentum conservation, i.e. $T_{12}^{\pm}(p_1, p_2, p'_1, p'_2) \sim \delta(p_1 + p_2 - p'_1 - p'_2)$. For example, in the first approximation in $H_{12}$, we have $T_{12}^{(1)}(p_1, p_2, p'_1, p'_2) = H_{12}(p_1, p_2, p'_1, p'_2)$, where the matrix element of the interaction Hamiltonian is given by (7.6).
general, it is not difficult to obtain from (7.10) and (6.6) the following expression for the matrix element of $T_{12}$ in the momentum representation:

$$T_{12}^{\pm}(p_1, p_2, p'_1, p'_2) = \frac{1}{(2\pi\hbar)^3} t^{\pm}(p_1, p_2, p'_1, p'_2) \delta(p_1 + p_2 - p'_1 - p'_2).$$

$$t^{\pm}(p_1, p_2, p'_1, p'_2) = \nu(p_1 - p'_1)$$

$$+ \frac{1}{(2\pi\hbar)^3} \int dp'' \int dp'' \nu(p_1 - p_1'') \nu(p'' - p_1') \delta(p_1 + p_2 - p_1'' - p_2')$$

$$+ \ldots$$

(7.14)

This $T$-matrix obeys the following symmetry conditions

$$T_{12}^{\pm}(p_1, p_2, p'_1, p'_2) = T_{12}^{\pm}(p_2, p_1, p'_2, p'_1),$$

$$T_{12}^{\pm}(p_1, p_2, p'_1, p'_2) = T_{12}^{\pm}(p'_1, p'_2, p_1, p_2).$$

(7.15)

Calculating the trace (integral over $p_2$) in (7.13) with the use of (7.14), (7.15), (6.24) and (6.34), we obtain the following equation for the momentum distribution function

$$\frac{\partial w(p_1, t)}{\partial t} = \frac{\pi n}{(2\pi\hbar)^3} \int dp_2 dp'_2 \delta \left[ K(p_1') + K(p_2) - K(p_1') - K(p_2) \right]$$

$$\times \delta(p_1 + p_2 - p_1' - p_2') \left| t^+(p_1, p_2, p'_1, p'_2) + \theta t^+(p_1, p_2, p'_1, p'_2) \right|^2$$

$$\times \left[ w(p_1', t) w(p_2', t) - w(p_1, t) w(p_2, t) \right].$$

(7.16)

Equation (7.16) is the quantum Boltzmann equation in the linear approximation in the particle density $n$ (only the binary collisions have been accounted for). It includes (as it should be) the quantum mechanical processes of particle exchange at scattering (the term proportional to $\theta$), which are due to the quantum statistics of particles. This irreversible kinetic equation (7.10) has been obtained with no use of a factorized initial condition (which introduces irreversibility) for the density matrix of the system under consideration. In the suggested here approach, the kinetic equation (7.10) follows from the evolution equation (6.11), which is exact in the linear approximation in $n$ and describes an evolution process on any time scale treating the initial correlations and correlations due to collisions on the equal footing. The described procedure of obtaining the time-irreversible Eq. (7.10) from Eq. (6.11) clearly indicates that irreversibility emerges on the macroscopic time scale (1.1) as a result of the damping of both the initial correlations caused by interaction and correlations caused by collisions (see also [26]). The latter can be secured by the appropriate properties of the system’s dynamics (e.g., the ergodic mixing flow in the phase space). One can, therefore, expect that Eq. (6.11) would switch automatically from a reversible behaviour to an irreversible one (described by the Boltzmann equation (7.10)) if all correlations vanish with time while going from microscopic to a large enough time scale. Thus, the influence of initial correlations on an evolution process can be revealed.
Before going to the next Section, we would again like to stress, that the obtained above TC-HGME and TCL-HGME are linear equations for the relevant (reduced) statistical operator because they have been derived from the linear Liouville-von-Neumann equation by the conventional linear time-independent projection operator technique. These equations, when applied, e.g., to the (sub)system interacting with a thermal bath, describe the (sub)system’s evolution on all time scales (no restrictions like those determined by (1.1)). But we have seen, that in order to obtain from these equations the nonlinear evolution equations accounting for initial correlations, the second inequality in (1.1) ($t - t_0 \ll t_{rel}$) should still be satisfied. Therefore, we have to turn to a different approach in order to obtain from the Liouville-von-Neumann equation the nonlinear evolution equations valid on all time scales, i.e. to lift all restrictions imposed by inequalities (1.1).

8 NONLINEAR INHOMOGENEOUS GME

We are considering the evolution equations for many-particle systems. It is natural to expect that these equations should generally be nonlinear evolution equations on any time scale due to interaction between particles. Strictly speaking, it is impossible to obtain a nonlinear evolution equation (like the Boltzmann equation) in the framework of the linear time-independent projection operator formalism without additional approximations (as we have seen above). The problem is that the conventional time-independent projection operator (4.3) (or (6.4)) leads to the relevant part of an $N$-particle distribution function (1.7) (or (6.5)) which is linear in the time-dependent one-particle distribution function of interest. That is why, a nonlinear equation for a density matrix is often simply postulated in the form of a Lindblad-type master equation whose generator depends parametrically on this density matrix (see, e.g., [18]).

Therefore, we need a new approach which would allow to obtain, e.g., a nonlinear relevant distribution function (statistical operator) like $f_r(x_1, \ldots, x_N, t) = \prod_{i=1}^{N} F_1(x_i, t)$ in (1.2).

We start again with the Liouville-von-Neumann equation (2.1) describing the evolution of a distribution function (statistical operator) $F_N(t)$ of a system of $N$ ($N \gg 1$) classical or quantum particles but in more general case, when the Hamiltonian and the corresponding Liouville (evolution) (super)operator $L(t)$, acting on $F_N(t)$ according to Eqs. (2.3), (2.4), may depend on time

$$\frac{\partial}{\partial t} F_N(t) = L(t) F_N(t). \quad (8.1)$$

The distribution function (or statistical operator) is subject to the normalization condition (2.2).
The formal solution to equation (8.1) can be written as

\[ F_N(t) = U(t, t_0)F_N(t_0), U(t, t_0) = T \exp \left[ \int_{t_0}^{t} ds L(s) \right] , \]  

where \( T \) denotes the chronological time-ordering operator, which orders the product of time-dependent operators such that their time arguments increase from right to left and \( F_N(t_0) \) is the value of a distribution function (statistical operator) at some initial time \( t_0 \).

We introduce the generally time-dependent operator \( P(t) \) converting the distribution function \( F_N(t) \) into a relevant distribution function \( f_r(t) \), which as a rule is a vacuum (without correlations) many-particle distribution function, i.e., a product of the reduced distribution functions that are necessary and sufficient for describing the evolution of the measurable values (statistical expectations). The problem is to obtain the closed evolution equation for such a reduced distribution function in view of the existence of correlations between the elements of the considered system including initial correlations at the initial time \( t_0 \).

We therefore define the relevant distribution function (statistical operator) as

\[ f_r(t) = P(t)F_N(t) = P(t)U(t, t_0)F_N(t_0) . \]  

(8.3)

We note that the operator \( P(t) \) is generally not a projection operator. Moreover, the action of \( P(t) \) on \( F_N(t) \) is no longer a linear operation (in contrast to applying the conventional time-independent projection operator [9]) because \( P(t) \) can depend on the time-dependent distribution function of interest (see below). Therefore, by applying \( P(t) \) to the linear Liouville equation (8.1), we generally obtain a nonlinear equation.

Using Liouville-von-Neumann equation (8.1), the equation of motion for \( f_r(t) \) (8.3) may be written as

\[ \frac{\partial f_r(t)}{\partial t} = \left[ \frac{\partial P(t)}{\partial t} + P(t)L(t) \right] [f_r(t) + f_i(t)] , \]  

(8.4)

where \( f_i(t) = [F_N(t) - f_r(t)] \) and \( \frac{\partial P(t)}{\partial t} \) is the operator obtained by taking the derivative of the operator \( P(t) \), i.e., we assume that the operator \( \frac{\partial P(t)}{\partial t} \) exists. To make sense of \( \frac{\partial P(t)}{\partial t} \), we can define \( P(t) \) as

\[ P(t) = C(t)D , \]  

(8.5)

where \( C(t) \) is a well defined (operator) function of time and \( D \) is a time-independent (super)operator acting on \( F_N(t) \) and integrating over all unnecessary variables in the spirit of the reduced description method (more details of the specification of \( P(t) \) are given below).

As it follows from (8.4), we can obtain the equation for the irrelevant part of a distribution function \( f_i(t) \) by using Eqs. (8.1) and (8.4) and splitting \( F_N(t) \) into the relevant and irrelevant parts. Thus,
\[ \frac{\partial f_i(t)}{\partial t} = \left[ Q(t)L(t) - \frac{\partial P(t)}{\partial t} \right] \left[ f_i(t) + f_r(t) \right], \quad (8.6) \]

where \( Q(t) = 1 - P(t) \) and therefore \( f_i(t) = Q(t)F_N(t) \).

The formal solution of equation (8.6) is

\[ f_i(t) = \int_{t_0}^{t} dt' S(t, t') \left[ Q(t')L(t') - \frac{\partial P(t')}{\partial t'} \right] f_r(t') + S(t, t_0)f_i(t_0), \quad (8.7) \]

where

\[ S(t, t_0) = T \exp \left\{ \int_{t_0}^{t} ds \left[ Q(s)L(s) - \frac{\partial P(s)}{\partial s} \right] \right\}. \quad (8.8) \]

We can represent the latter operator as the series

\[ S(t, t_0) = 1 + \int_{t_0}^{t} dt_1 \left[ Q(t_1)L(t_1) - \frac{\partial P(t_1)}{\partial t_1} \right] \]
\[ + \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 \left[ Q(t_2)L(t_2) - \frac{\partial P(t_2)}{\partial t_2} \right] \]
\[ \times \left[ Q(t_1)L(t_1) - \frac{\partial P(t_1)}{\partial t_1} \right] + \ldots. \quad (8.9) \]

We note that if the operators \( P \) and \( L \) are independent of time, then \( S(t, t_0) = \exp \left[ Q(t) - Q(t_0) \right] \).

Substituting (8.7) in (8.4), we obtain

\[ \frac{\partial f_r(t)}{\partial t} = \left[ P(t)L(t) + \frac{\partial P(t)}{\partial t} \right] \{ f_r(t) \}
\[ + \int_{t_0}^{t} dt' S(t, t') \left[ Q(t')L(t') - \frac{\partial P(t')}{\partial t'} \right] f_r(t') \]
\[ + S(t, t_0)f_r(t_0) \}. \quad (8.10) \]

Equation (8.10) is a generalization of the Nakajima-Zwanzig time-convolution generalized master equation (TC-GME) (see [8, 9, 10]) for the relevant part of the distribution function (statistical operator) to the case of time-dependent operators \( P \) and \( Q \). As previously noted, \( P(t) \) and \( Q(t) \) are generally not projection operators in the usual sense, and in deriving (8.10) we do not use such a property of the projection operators as \( P^2 = P, Q^2 = Q \) (for \( P(t)P(t') \) and \( Q(t)Q(t') \) such a property holds only for \( t = t' \), as is shown below). The obtained equation, like the conventional TC-GME, is an exact inhomogeneous integro-differential equation for the relevant part of the distribution function (statistical operator) containing the irrelevant part of the distribution function (a source) at the initial instant \( f_i(t_0) = F_N(t_0) - f_r(t_0) \). But, Eq. (8.10) is generally equivalent to the nonlinear equation for the reduced distribution
function of interest (e.g., for the one-particle distribution function considered in the next section) and is therefore also convenient for studying the evolution of many-particle systems described by nonlinear Boltzmann-type equations (in the kinetic regime) where the nonlinearity results from particle collisions.

For time-independent $P$ and $Q$ equation, Eq. 8.10 reduces to the conventional linear TC-GME. This type of equation is more suitable, for example, for studying the evolution of a subsystem interacting with a large system in thermal equilibrium (a thermal bath). Although, as we have seen above, it is possible to obtain from the linear TC-GME the nonlinear evolution equation but on the time scale $t - t_0 \ll t_{rel}$, as we have seen in the previous sections.

To simplify Eq. (8.10) and make sense of the derivative $\frac{\partial P(t)}{\partial t}$, we consider an operator $P(t)$ of form (8.5). This representation of $P(t)$ is suitable for studying many cases of interest (see Sec. 9). Also, splitting a distribution function (statistical operator) into the relevant $P(t)F_N(t)$ and irrelevant $Q(t)F_N(t)$ parts makes sense if the operator $P(t)$ satisfies the relation

$$P(t)P(t) = P(t)$$

for any $t$. In this case, if $P(t) = C(t)D$, then Eq. (8.11) implies

$$DC(t) = 1$$

at any instant $t$. Equation (8.12) generally represents the normalization condition for the distribution function(s) making up $C(t)$ (see (2.2) and (4.6)) and implies that $C(t)$ depends on the variables that are removed from $F_N(t)$ by the operator $D$ ($f_r(t) = P(t)F_N(t)$ depends on the complete set of variables of the distribution function $F_N(t)$ as indicated by (1.3)). From condition (8.12) we have

$$P(t)P(t') = P(t), Q(t)Q(t') = Q(t'), P(t)Q(t') = 0,$$

$$Q(t')P(t) = P(t) - P(t').$$

(8.13)

Condition (8.12) also leads to the relations

$$D\frac{\partial P(t)}{\partial t} = D\frac{\partial C(t)}{\partial t}D = 0,$$

(8.14)

for the time-derivative of $P(t)$, where we assume that the operators $D$ and $\partial/\partial t$ commute ($D$ is independent of time).

We also introduce the reduced distribution function (statistical operator)

$$f_{red}(t) = DF_N(t) = Df_r(t),$$

(8.15)

which is actually needed for calculating the expectation values of interest in a nonequilibrium state. By (8.12), we have

$$D\frac{\partial f_r(t)}{\partial t} = \frac{\partial f_{red}(t)}{\partial t}, Df_r(t) = DQ(t)F_N(t) = 0.$$

(8.16)
Hence, applying the operator $D$ to equation (8.10) from the left, we obtain

$$\frac{\partial f_{\text{red}}(t)}{\partial t} = DL(t)\{f_r(t) + \int_{t_0}^t dt' S(t, t') \left[ Q(t')L(t') - \frac{\partial P(t')}{\partial t'} \right] f_r(t') \} + S(t, t_0)f_r(t_0),$$

(8.17)

where we use (8.12), (8.14), and (8.16).

Without loss of generality, we can split the Hamiltonian or the Liouvillian of a system into two terms as $H(t) = H^0(t) + H'(t)$ or $L(t) = L^0(t) + L'(t)$, where $H^0$ or $L^0$ is related to the energy of the noninteracting particles (subsystems) and $H'$ or $L'$ describes the interaction between particles (subsystems). Then, in addition to satisfying condition (8.12), the operator $D$, which selects the reduced distribution function, should commute with $L^0$ as

$$DL^0(t) = L^0_{\text{red}}(t)D,$$

(8.18)

where $L^0_{\text{red}}(t)$ is the reduced Liouvillian $L^0$ depending only on the variables of the reduced distribution function (statistical operator) $f_{\text{red}}(t)$. Expression (8.18) means that both sides of this relation act on the arbitrary function (operator) defined on the phase (Hilbert) space. Taking into account that

$$C(t')L^0_{\text{red}}(t) = L^0_{\text{red}}(t)C(t') \quad (C(t') \text{ depends on the set of variables that are removed from } F(t) \text{ by } D, \text{ and } L^0_{\text{red}}(t) \text{ is independent of them), we can write Eq. (8.18) in the more general form

$$P(t')L^0(t) = L^0_{\text{red}}(t)P(t').$$

(8.19)

Commutation relation (8.18) follows from a self-consistency argument because if $L'(t) = 0$, then the evolution of a "free" subsystem (not interacting with the rest of the system) should be described in Eq. (8.17) only by $L^0_{\text{red}}(t)\{DL^0(t)f_r(t) = L^0_{\text{red}}(t)f_{\text{red}}(t) = L^0_{\text{red}}(t)Df_r(t); \text{ also see [1]. It can be seen from the relations

$$DL^0(t)S(t, t_1)Q(t_2) = 0,$$

$$DL^0S(t, t_1)\frac{\partial P(t_2)}{\partial t_2} = 0,$$

(8.20)

which follow from (8.18), (8.8), (8.14), and (8.16), that this is the case. Hence, all terms in (8.17) are proportional to $L'(t)$ except the first term, $L^0_{\text{red}}(t)f_{\text{red}}(t)$.

Analogously, using (8.18), (8.8), and relations that follow from (8.12), we can prove the equations

$$S(t, t_1)Q(t_2) = \mathcal{U}(t, t_1)Q(t_2),$$

$$S(t, t_1)\frac{\partial P(t_2)}{\partial t_2} = \mathcal{U}(t, t_1)\frac{\partial P(t_2)}{\partial t_2},$$

(8.21)

where

$$\mathcal{U}(t, t_1) = T \exp \left\{ \int_{t_1}^t ds \left[ L^0(s) + Q(s)L' \right] \right\}.$$

(8.22)
Hence, using (8.20) and (8.21) in (8.17), we can finally rewrite Eq. (8.10) as the equation for the relevant part of the distribution function (statistical operator)

\[
\frac{\partial D_f(t)}{\partial t} = DL(t)f_r(t) + DL'(t) \int_{t_0}^{t} dt' U(t, t') \left[ Q(t')L(t') - \frac{\partial P(t')}{\partial t'} \right] f_r(t') + DL'(t)U(t, t_0)f_r(t_0),
\]

where \(D_f(t) = f_{red}(t)\) and (8.18) holds.

We demonstrate that basic conditions (8.12) and (8.18) are satisfied for a gas of interacting classical particles in the next section.

9 NONLINEAR INHOMOGENEOUS EVOLUTION EQUATION FOR A NONIDEAL INHOMOGENEOUS DILUTE GAS OF CLASSICAL PARTICLES

In this section, we apply Eq. (8.23) to the case of a gas of \(N (N \gg 1)\) identical classical particles. The Liouville operator \(L\) for such a system can be represented in the form (4.9), where \(L^0\) corresponds to the kinetic energy \(H^0 = \sum_{i=1}^{N} p_i^2/2m\) of particles with the momenta \(p_i\) and mass \(m\) and \(L'\) corresponds to the particle interaction \(H' = \sum_{i<j=1}^{N} V_{ij}\) with the pair potential \(V_{ij} = V(|x_i - x_j|)\). Hence, \(L\) is independent of time in the considered case. We do not assume here that the interaction is weak, but we do assume that all necessary requirements for the properties of forces by which the particles interact are satisfied (in particular, bound states do not form). As usual, we also assume that all functions \(\Phi(x_1, \ldots, x_N)\) defined on the phase space and their derivatives vanish at the boundaries of the configuration space and at \(p_i = \pm \infty\). These boundary conditions and the explicit form of the Liouville operators (4.9) lead to relations (4.10) (see Sec. 4).

We seek an evolution equation for a one-particle distribution function (4.4)

\[
f_1(x_i, t) = V \int dx_1 \cdots \int dx_{i-1} \int dx_{i+1} \cdots \int dx_N F_N(x_1, \ldots, x_N, t).
\]

For this, it is convenient to define the operator \(P(t)\) as

\[
P(t) = C(t)D, C(t) = \prod_{i=2}^{N} f_1(x_i, t), D = \frac{1}{V^{N-1}} \int dx_2 \cdots \int dx_N
\]

(compare with (4.3)).
We apply the projection operator (9.2) to \( f_N(t) = V^N F_N(t) \), which satisfies the Liouville-von-Neumann equation (8.1), and obtain the following relevant part of the distribution function

\[
f_r(t) = P(t) f_N(t) = \prod_{i=1}^{N} f_1(x_i, t).
\]

From definition (9.3) we can see an advantage of the time-dependent operator \( P(t) \) (9.2): applying it to an \( N \)-particle distribution function \( f_N(t) \), we can define relevant part (9.3) of the distribution function, which seems more natural and symmetric than (4.7) used in Sec. 4 and obtained using the time-independent projection operator (4.3) (we had to use the time-independent projection operator because the TC-HGME considered in Sec. 2 holds only for such operators). Moreover, relevant part (9.3) has a clear physical meaning because it represents a part of an \( N \)-particle distribution function without correlations which is often of most interest.

Hence, the irrelevant part

\[
f_i(t) = Q(t) f_N(t) = f_N(t) - \prod_{i=1}^{N} f_1(x_i, t)
\]

(9.4)
describes all interparticle correlations, in particular, the initial ones (existing at the initial instant \( t_0 \)), which compose the inhomogeneous term in (8.23). The irrelevant part of the \( N \)-particle distribution function at \( t_0 \), \( f_i(t_0) = f_N(t_0) - f_r(t_0) \), can always be represented (and also at any other time \( t \)) by the cluster expansion (4.8).

Using normalization condition (4.6) for a one-particle distribution function \( \int f_1(x_i, t)dx_i = V \), we can easily see that by applying the integral operator \( D \) to \( C(t) \) (both are defined by (9.2)), we obtain \( DC(t) = 1 \), i.e., condition (8.12) and its consequences given by (8.13) and (8.14) are satisfied. Also, in this case,

\[
\begin{align*}
f_{red}(t) &= D f_r(t) = f_1(x_1, t), \\
DQ(t) &= 0, D f_i(t) = 0
\end{align*}
\]

(9.5)
in accordance with (8.15) and (8.16).

Commutation rules (8.18) and (8.19) also follow from (4.9), (4.10), and (9.2), because

\[
\begin{align*}
DL^0 &= L^0 D \\
P(t)L^0 &= L^0 P(t)
\end{align*}
\]

(9.6)
i.e., \( L^0_{red} = L^0_1 \) in the case under consideration. Therefore, the identities (8.20) - (8.22) are also satisfied, where \( L^0 \) and \( L' \) are independent of time in the considered case.

Using (4.10), (9.5), and (9.6), we can represent Eq. (8.23) in the case under consideration as
\[
\frac{\partial f_1(x_1,t)}{\partial t} = L_1^0 f_1(x_1,t) + n \int dx_2 L_{12}^t f_1(x_2,t) f_1(x_1,t)
\]
\[
+ DL \int_{t_0}^t dt' \mathcal{U}(t,t') \left[ (L_0^0 - L_{10}^t) + Q(t') L' - \frac{\partial P(t')}{\partial t} \right] f_r(t')
\]
\[
+ DL \mathcal{U}(t,t_0) f_1(t_0),
\]
(9.7)

where \( n = N/V \) is the density of particles and \( D, f_r(t), \) and \( f_i(t_0) \) are respectively defined in (9.2), (9.3), and (4.8). Evolution operator (8.22) can be represented by the series similar to (8.9)

\[
\mathcal{U}(t,t') = 1 + \int_{t'}^t dt_1 \left[ L_0^0 + Q(t_1) L' \right]
\]
\[
+ \int_{t'}^{t_1} dt_2 \left[ L_0^0 + Q(t_2) L' \right] \left[ L_0^0 + Q(t_1) L' \right] + \ldots.
\]
(9.8)

We consider equation (9.7) in the first approximation in the particle density \( n \). The corresponding dimensionless small parameter of the perturbation expansion is given by (4.2). From the second equation in (4.10), we can easily see that all terms in expansion (9.8) containing \( P(t) L' \) result in expressions proportional to at least the first power of \( n \) (like the second term in the right-hand side of (9.7)). Therefore, in the first approximation in \( n \), all terms of (9.8) with \( P(t) L' \) can be neglected because the terms of Eq. (9.7) containing \( \mathcal{U}(t,t') \) are already of the first order in \( n \) (applying \( DL' \) leads to an expression of at least the first order in \( n \)). In this approximation, all terms under the integrands in (9.8) are independent of time, and the operator \( \mathcal{U}(t,t') \) reduces to

\[
\mathcal{U}(t,t') = \exp \left[ (L_0^0 + L')(t-t') \right].
\]
(9.9)

It is now convenient to use the expansion for the exponential operator (9.9) (see (4.13))

\[
e^{(L_0^0+L')(t-t')} = e^{L_0^0(t-t')} + \int_{t'}^t d\theta e^{L_0^0(t-\theta)} L' e^{(L_0^0+L')\theta}.
\]
(9.10)

Using this expansion and relations (4.10), we obtain the following evolution equation for a one-particle distribution function in the linear approximation in
the density parameter (4.2) from (9.7):
\[
\frac{\partial f_1(x_1, t)}{\partial t} = L_{0}^0 f_1(x_1, t) + n \int dx_2 \left[ \int_0^{t-t_0} dt_1 e^{(L_{0}^0 + L_{12}')t_1} [(L_{12}^0 + L_{12}')f_1(x_2, t - t_1)f_1(x_1, t - t_1) + \frac{\partial f_1(x_2, t - t_1)}{\partial t}] \right] \\
+ n \int dx_2 L_{12}' e^{(L_{0}^0 + L_{12}')t_0} g_2(x_1, x_2),
\]
(9.11)
where \(L_{12}^0 = L_{1}^0 + L_{2}^0\). In obtaining (9.11), we also use the definition (4.8) for the irrelevant part of the distribution function and take into account that each additional integration over \(x_3, \ldots\) adds an additional power of \(n\) (therefore, in the linear approximation in \(n\), all formulae contain no more than one integration over the phase space).

Equation (9.11) is the main result in this section. We must stress that (9.11) is a nonlinear inhomogeneous time-convolution (non-Markovian) master equation for a one-particle distribution function containing initial correlations (a source). We note, that this new equation is rigorously derived here (the desired nonlinear master equation is usually postulated [18]). Equation (9.11) is exact in the linear approximation in small density parameter (4.2) and holds on any time scale and for any spatial inhomogeneity of the system under consideration. The first term in the right-hand side is a conventional flow term. The second term, a nonlinear Vlasov term, represents the self-consistent field acting on the given (first) particle and determined by all particles of the system. We note that it is impossible (without additional approximations) to obtain this term (appearing only in a spatially inhomogeneous case) from the GME (or HGME) containing time-independent projection operator (4.3) (see Sec. 4). The term quadratic in \(L_{12}'\) describes particle collisions and can lead to dissipation in the system. The terms with \(L_{0}^0\) (the space derivative) and with the time derivative take the change in space and time of the one-particle distribution function on the respective microscopic scales (also see below) of the orders of \(r_0\) and \(t_{cor} \sim r_0/\nu\) into account. These two terms determine the contribution of the pair collisions to the nondissipative characteristics (thermodynamic functions) of the nonideal gas (see, e.g., [4]) and are usually absent in the standard derivation (from the BBGKY chain) of the kinetic equations in the approximation linear in \(n\). Taking these terms into account poses a certain problem because they usually appear in the next (second) approximation in \(n\) (see, e.g., [27]), which is inconsistent with the dissipative three-particle collision term appearing in this approximation. The last (irrelevant) term in (9.11) takes initial correlations (at \(t = t_0\)) into account and is given by the two-particle correlation function \(g_2(x_1, x_2)\) in the linear approximation in \(n\).

We examine the terms in (9.11) more closely, especially their behavior in time. The evolution in time is governed by the exact two-particle propagator \(G_{12}(t) = \exp(L_{12}t)\) \((L_{12} = L_{12}^0 + L_{12}'\) (this is natural for the considered dilute
gas in the lowest approximation in density), which satisfies the integral equation (4.26). The action of "free" propagator $G^0(t) = \exp(L^0 t)$ on any function defined on the phase space is given by Eq. (4.14).

If the particle dynamics have the necessary properties such as a mixing ergodic flow in the phase space resulting from the local stochastic instability, then the distances between particles rapidly (exponentially) increase with time under the action of $G_{12}(t)$ (the contribution of a "parallel motion" is negligible). It was proved that such behavior is the case, for example, for the Boltzmann dilute gas of hard spheres and is now viewed as common for the most real systems considered in statistical physics (see, e.g., [4]). The mixing ergodic flow in the phase space results in the exponential damping of correlations and is the reason for irreversibility. Hence, if the effective particle interaction $V(|r_i - r_j|)$ vanishes at a distance $|r_i - r_j| > r_0$, then the integrand of the integral over $t_1$ in (9.11) vanishes for $t - t_0 > t_{\text{cor}}$ and the same holds (under the action of $G_{12}(t-t_0)$) for the initial correlation term determined by the function $g_2(x_1, x_2)$, which also depends on the particle interaction (we discuss this in more detail below, and it was considered in the spatially homogeneous case in Sec. 5).

Condition (4.12) implies the existence of the time hierarchy

$$t_{\text{cor}} \ll t_{\text{rel}}, \quad (9.12)$$

where $t_{\text{rel}} \sim \gamma^{-1} t_{\text{cor}}$ is the relaxation time for a one-particle distribution function $f_1(x_i, t)$ ($1/m_0^2$ is the particle mean free path). In the initial evolution stage $t_0 \leq t \leq t_{\text{cor}}$, which is very interesting and essential for understanding the irreversibility problem and for studying the non-Markovian processes (the memory effects), decoherence phenomena, and the ultrafast relaxation effects, the initial correlations can be important. But if we consider the time scale $t - t_0 \gg t_{\text{cor}}$, then the initial correlations are damped (as a result of mixing flow), and the source in Eq. (9.11) containing $g_2(x_1, x_2)$ can be neglected. But as Bogoliubov noted in [1], this can be done only on time interval (1.1), which exists because of (9.12). The latter conclusion follows because the corrections to the solution of (9.11) due to initial correlations turn out to be proportional to $|t - t_0|$ (secular terms) and are small only for $|t - t_0| \ll t_{\text{rel}}$. Therefore, the expansion of the irrelevant initial condition term in $n$ (as in passing from (9.7) to (9.11)) is ineffective on the time scale of interest in the kinetic theory $t - t_0 \gtrsim t_{\text{rel}}$.

To get rid of the undesirable initial condition terms, it is often assumed that $f_i(t_0) = 0$, which corresponds to RPA and is incorrect in principle (see [7]). Therefore (as mentioned in Sec. 1), to study the kinetic evolution stage $t \gtrsim t_{\text{rel}}$, a more sophisticated approach like that based on the principle of weakening of initial correlations (Bogoliubov’s ansatz) [1] is needed. But the Bogoliubov approach does not allow considering the initial evolution stage $t_0 \leq t \leq t_{\text{cor}}$ and the situations where the large-scale correlations with $t_{\text{cor}} \gtrsim t_{\text{rel}}$ play an essential role.

If $f_i(x_i, t)$ changes significantly on the macroscopic (hydrodynamic) length $l_h$ with the corresponding time scale $t_h = l_h/\mathcal{V}$ (for the spatially homogeneous gas, there are only the characteristic scales $l$ and $t_{\text{rel}}$ on which $f_i(x_i, t)$ changes), then
the terms with $L_2^0$ and $\frac{\partial F_i(x_2, t-t')}{\partial t}$ in Eq. (9.11) can be neglected because they are both at least of the order of $\gamma$, i.e., their contribution is of the second order in $n$ (but these terms should not be omitted if we wish to take the nondissipative characteristics of the nonideal gas into account self-consistently). On the kinetic time scale $t - t_0 \gtrsim t_{rel}$, the upper limit of integration over $t_1$ can be extended to infinity ($t_0 \to -\infty$) because of the damping of correlations resulting from collisions (as noted above). If we also accept the RPA (or principle of weakening of initial correlations) for $t_0 \to -\infty$, i.e., neglect the initial condition term with $g_2(x_1, x_2)$, then Eq. (9.11) reduces to

$$\frac{\partial f_1(x_1, t)}{\partial t} = L_1^0 f_1(x_1, t) + n \int dx_2 L_2^1 f_1(x_2, t) f_1(x_1, t) + n \int dx_2 L_2^1 \int dt_1 e^{(L_2^0 + L_2^1)t_1} L_2^1 f_1(x_2, t) f_1(x_1, t),$$

(9.13)

where we take into account that for $t \gtrsim t_{rel} \gg t_{cor}$ in the adopted approximation, we can replace $F_1(x_1, t-t_1)$ with $F_1(x_1, t)$ (the integration over $t_1$ in (9.11) gives an essential contribution only up to $t_1 \sim t_{cor}$). In the last (collision) term in (9.13), the distribution functions of the colliding particles can be taken at the same space point $r_1 = r_2$ with the adopted accuracy $\sim r_0/r_h < 1$ (or $r_0/l < 1$).

If the last integral in Eq. (9.13) exists (see [23]), then Eq. (9.13) represents the nonlinear Markovian kinetic equation irreversible in time. In the case of a weak particle interaction, this equation in the first approximation in the small interaction parameter $\varepsilon$ gives the reversible Vlasov equation (Eq. (9.13) with neglect of the third collision term in the right-hand side, which is of the second order in $\varepsilon$). In the second approximation in $\varepsilon$, it coincides with the Vlasov-Landau kinetic equation (see, e.g., [20]). In the spatially homogeneous case, where a one-particle distribution function is independent of the particle coordinate, $f_1(x_1, t) = f_1(p_1, t) (\int f_1(p_1) dp = 1)$, the second (Vlasov) term in (9.13) vanishes because we consider a potential $V_{ij}$ given by (4.9) that depends on the particle coordinate difference. In this case, Eq. (9.13) is equivalent to the nonlinear Boltzmann equation (see, e.g., [20]). We stress that the nonlinearity of (9.13) appears quite naturally in this approach (based on the time-dependent $P(t)$ given by (9.2)) in contrast to the time-independent projection operator approach presented in Secs. 2-7. In the latter case, we must approximate $f_1(x_2, t_0)$ by $f_1(x_2, t)$ (which is possible on the time scale given by (1.1)) in the evolution equation in order to obtain a nonlinear equation from the linear GME or HGME (in the case of the BBGKY chain approach, the nonlinearity is introduced by principle of weakening of initial correlations (1.2)).

10 NONLINEAR HOMOGENEOUS GME

In Sec. 8, we obtained the nonlinear GME for the relevant part of a distribution function (statistical operator). But Eq. (8.10) is not closed in the sense that it is inhomogeneous and contains an irrelevant term generally containing all
multiparticle correlations. As noted above, to eliminate this irrelevant part and obtain a homogeneous equation (like (9.13)), for example, for a one-particle distribution function, the Bogoliubov principle of weakening of initial correlations or the RPA is usually used.

To take initial correlations into account and to obtain the evolution equations applicable on all time scales, we use here a method for converting conventional linear inhomogeneous GMEs into the homogeneous GMEs presented in Secs. 2 and 3. We now apply this approach to equation (8.23). Our goal is to convert Eq. (8.23) into a homogeneous equation containing the irrelevant part of the distribution function (statistical operator) \( f_i(t_0) \) in the “mass” (super)operator acting on the relevant part of the distribution function (statistical operator) \( f_r(t) \). Such an equation allows treating the initial correlations (contained in \( f_i(t_0) \)) on an equal footing with all other correlations (collisions) and is applicable on any time scale including the initial evolution stage when the initial correlations are certainly essential. This is so because there is no need for restrictions given by inequalities (1.1) that allow neglecting the initial correlations and avoiding the “secular terms” in the solution of the evolution equation (the perturbation expansion of the “mass” (super)operator does not lead to such terms on the large (kinetic) time scale). In contrast to the HGME obtained in Sec. 2, the equation derived below is nonlinear and is also applicable to the nonconservative systems (with time-dependent Hamiltonians).

Following Sec. 2, we identically represent the irrelevant part of the distribution function (statistical operator) in the form

\[
f_i(t_0) = F_N(t_0) - f_r(t_0) = Q(t_0)F_N(t_0)
= [Q(t_0)F_N(t_0)] F_N^{-1}(t_0)U^{-1}(t, t_0)[P(t) + Q(t)]U(t, t_0)F_N(t_0)
= C_0U^{-1}(t, t_0)[f_r(t) + f_i(t)],
\]

\( C_0 = [Q(t_0)F_N(t_0)] F_N^{-1}(t_0), U^{-1}(t, t_0) = T_ - \exp[-\int_{t_0}^t dsL(s)], \)  \( 10.1 \)

where \( U^{-1}(t, t_0) \) is the backward-in-time evolution operator for the density matrix \( F_N(t) \) (compare with (8.2)), \( U^{-1}(t, t_0)U(t, t_0) = 1 \), \( T_- \) is the antichronological time-ordering operator arranging the time-dependent operators \( L(s) \) such that the time arguments increase from left to right, \( F_N^{-1}(t_0) \) is the inverse of \( F_N(t_0) \), \( F_N^{-1}(t_0)F_N(t_0) = 1 \), \( P(t) + Q(t) = 1 \). Hence, additional identity \( 10.1 \) is obtained by multiplying the irrelevant part by the unity \( F_N^{-1}(t_0)F_N(t_0) \) (we assume that \( F_N^{-1}(t_0) \) exists) and inserting the unities \( U^{-1}(t, t_0)U(t, t_0) = 1 \) and \( P(t) + Q(t) = 1 \). Therefore, neither a divergence (due to \( F_N(t_0) \) possible vanishing ) nor an indeterminacy of the 0/0 type (the behaviors of the numerator and denominator in \( F_N(t_0)/F_N(t_0) = 1 \) are similar) can occur. This holds for all further (identical) manipulations (see below).
In (10.1) we introduced the parameter of initial correlations
\[ C_0 = [Q(t_0)F_N(t_0)]F_N^{-1}(t_0) = f_i(t_0)[f_r(t_0) + f_i(t_0)]^{-1} \]
\[ = f_i(t_0)f_r^{-1}(t_0)[1 + f_i(t_0)f_r^{-1}(t_0)]^{-1} \]
\[ = (1 - C_0)f_i(t_0)f_r^{-1}(t_0). \] (10.2)

It is important that operator \( Q(t_0) \) acts only on \( F_N(t_0) \) in (10.1) and (10.2), which is indicated by enclosing \( Q(t_0)F_N(t_0) \) in brackets. This follows because \( f_r(t) \) and \( f_i(t_0) \) are the basic quantities we are dealing with in Eq. (8.23). All functions of dynamical variables, whose average values we can calculate using \( f_r(t) \) (or \( f_{\text{red}}(t) \)) by multiplying equation (8.23) by the corresponding functions (operators) from the right and taking an average value (a trace), depend only on the variables that are not integrated off by \( D \) (\( D \) integrates over all excess variables in \( F_N(t) \)). Therefore, if we correspondingly represent \( f_r(t) \) and \( f_i(t_0) \) in (8.23) as \( f_r(t) = P(t)F_N(t) \) and \( f_i(t_0) = Q(t_0)F_N(t_0) \), then the projection operators \( P(t) = C(t)D \) and \( Q(t_0) = 1 - P(t) \) in these expressions act only on \( F_N(t) \) but not on the functions (if any) to the right of them. This is the essence of the reduced description method: to calculate the average values of the functions that depend on a much smaller number of variables than the whole distribution function \( F_N(t) \) does, we in fact need only the reduced distribution function (density matrix) \( f_{\text{red}}(t) \).

As can be seen from (10.2), the correlation parameter is a series in \( f_i(t_0)f_r^{-1}(t_0) \), and we may therefore only need the formal existence of the function (operator) \( f_r^{-1}(t_0) \), which is the inverse of the relevant distribution function (statistical operator) chosen using the appropriate operator \( P(t_0) \). It seems plausible that the inverse of the relevant part of the distribution function (statistical operator) defined in the sense noted above (uncorrelated part) can be constructed (see Secs. 4.6, and the next section).

We now have two equations, (9.1) and (10.1), relating \( f_i(t) \) and \( f_i(t_0) \). Using relations (8.21), finding \( f_i(t_0) \) from the indicated equations as a function of \( f_r(t) \), and substituting it in (8.23), we obtain the equation

\[ \frac{\partial Df_r(t)}{\partial t} = DL^0(t)f_r(t) + DL'(t)R(t, t_0)f_r(t) \]
\[ + DL'(t)R(t, t_0) \int_0^t dt' \Theta(t, t') \left[ Q(t')L(t') - \frac{\partial P(t')}{\partial t'} \right] f_r(t'), \] (10.3)

where the operator \( R(t, t_0) \) is defined as
\[ R(t, t_0) = 1 + C(t, t_0), \]
\[ C(t, t_0) = \Theta(t, t_0) [1 - C_0(t, t_0)]^{-1} C_0U^{-1}(t, t_0), \]
\[ C_0(t, t_0) = C_0U^{-1}(t, t_0) \Theta(t, t_0). \] (10.4)

\( Df_r(t) = f_{\text{red}}(t) \) and relation (8.18) holds.

Equation (10.3) is the central result of this section. We have derived the desired homogeneous generalized evolution equation for the relevant part of the
distribution function (statistical operator). This equation differs from the linear TC-HGME obtained in Sec. 2 from the conventional Nakajima-Zwanzig TC-GME using the time-independent projection operator technique. Equation (10.3) is generally nonlinear and holds for systems with a time-dependent Hamiltonian. It holds in both the classical and quantum physics cases if the symbols are properly redefined and all (super)operators exist (we address the latter question below). We have not made any approximations in deriving Eq. (10.3), and, it is therefore an exact integro-differential equation that takes the initial correlations and their dynamics into account via the modified (super)operator (memory kernel) of (8.23) acting on the relevant part of the distribution function (statistical operator) \( f_r(t) \).

The obtained exact kernel of (10.3) can serve as a starting point for effective perturbation expansions. In many cases, such expansions of homogeneous equations (like (10.3)) have a much broader applicability domain than those of the inhomogeneous equations (like (8.23)) when the expansions of the functions \( (f_r, f_i) \), rather than equations, are involved (see also [1]). Particularly, we can expect that the divergencies appearing in the higher-order terms in the gas density expansion [2] are tackled more conveniently using the ”mass” (super)operator expansion. As can be seen from (10.3), the time evolution of initial correlations \( (C(t, t_0)) \) influences the collision integral (and also the second term) and therefore leads to an additional dependence on the density \( n \) (when the expansion in \( n \) is used). It is known, that initial correlations in the higher orders in \( n \) become more long-lived and are damped only at \( t - t_0 \sim t_{rel} \) because of successive pair collisions (see [2, 3]). While the initial correlations in the lowest approximation in \( n \) (pair collision approximation) are typically the short-scale ones with the characteristic \( t_{cor} \sim r_0/\eta \ll t_{rel} \) and \( (\eta t_{cor}) \ll l \), they are large-scale ones in the next approximations in \( n \) and therefore cannot be neglected on the kinetic time scale. Hence, the initial correlations in the higher orders terms (than linear) of the expansion of Eq. (4.20) in \( n \) may result in the cut-off of the discussed divergencies of the kinetic equation at \( t \sim t_{rel} \) (this was noted in [28]). The foregoing means that the expansion of the ”mass” (super)operator governing the evolution of \( f_r(t) \) in (10.3) provides a new possibility for expanding the kinetic coefficients in \( n \).

Equation (10.3) reduces to the linear TC-HGME obtained in Sec. 2 if the (projection) operator \( P \) and the Hamiltonian \( H \) are independent of time and relations (8.18) and (8.19) hold.

As in Sec. 2, the problem of the existence (convergence) of \( R(t, t_0) \) can be raised. The function \( R(t, t_0) \) behaves properly at all times. Moreover, expanding the kernel of (10.3) may result in canceling the pole in the function \( R(t, t_0) \). For the linear TC-HGME, this correspondingly shown in Secs. 4 and 6 in the linear approximation in the small density for a spatially homogeneous dilute gas of classical and quantum particles. In such cases there is no problem with the existence of \( R(t, t_0) \). Examining (10.2) (from which the relation \( (1 - C_0)^{-1} C_0 = f_i(t_0)/f_r(t_0) \) follows), we can expect that the same is true for \( R(t, t_0) \) given by (10.4) (see below).
NONLINEAR EVOLUTION EQUATION FOR A NONIDEAL INHOMOGENEOUS DILUTE GAS OF CLASSICAL PARTICLES WITH INITIAL CORRELATIONS TAKEN INTO ACCOUNT

We now apply Eq. (10.3) to a system of \(N\) \((N \gg 1)\) identical interacting classical particles described by the Liouvillian (4.9) and considered in Sec. 4. To obtain an equation for the one-particle distribution function \(f_1(x_1, t)\) \((i = 1, \ldots, N)\), we use the operator \(P(t)\) given by (9.2). Hence, taking (4.10), (9.5), and (9.6) into account, we obtain the equation for the one-particle distribution function from (10.3):

\[
\frac{\partial f_1(x_1, t)}{\partial t} = L^0_1 f_1(x_1, t) + n \int dx_2 L'_{12} f_1(x_2, t) f_1(x_1, t) + DL' C(t, t_0) f_1(t) + DL' \int_{t_0}^t dt' R(t, t_0) \left[ (L^0 - L^0_1) + Q(t') L' - \frac{\partial P(t')}{\partial t'} \right] f_i(t').
\] (11.1)

We consider Eq. (11.1) in the linear approximation in density parameter (4.2). Given that the action of \(P(t)\) or \(D\) results in expressions proportional to at least the first power in the density \(n\) and that all terms on the right-hand side of (11.1) (except the first flow term) are already proportional to \(n\) (or \(D\)), in the first approximation in \(n\), we can neglect all terms with \(P(t)\) in (11.1) including such terms in \(U(t, t')\). Then \(U(t, t') = U(t, t')\) in this approximation and is given by (9.9). Therefore, we can approximate \(U^{-1}(t, t_0) U(t, t_0)\) in (10.4) by unity and \(C_0(t, t_0)\) by \(C_0\). Hence, in the adopted approximation, the correlation parameter is

\[
C(t, t_0) = U(t, t_0)[f_i(t_0) f_j^{-1}(t_0)] U^{-1}(t, t_0),
\] (11.2)

as follows from (10.2) and (10.4). To calculate contribution (11.2) of initial correlations to evolution equation (11.1), we should use definitions (9.3) and (4.8) of \(f_i(t_0)\) and \(f_j(t_0)\).

Proceeding as in Sec. 9, i.e., using (4.10), (9.5), (9.10), and the fact (noted above) that each additional integration over the phase space leads to an additional power of \(n\), we can represent Eq. (11.1) in the first approximation in \(n\) as

\[
\frac{\partial f_1(x_1, t)}{\partial t} = L^0_1 f_1(x_1, t) + n \int dx_2 L'_{12}[1 + C_1(t - t_0)] f_1(x_2, t) f_1(x_1, t) + n \int dx_2 L'_{12}[1 + C_1(t - t_0)] \int_{t_0}^t dt_1 e^{L_1' t_1}[L^0_1 + L'_{12}] f_1(x_2, t - t_1) f_1(x_1, t - t_1) + \frac{\partial f_1(x_2, t - t_1)}{\partial t_1} f_1(x_1, t - t_1),
\] (11.3)
where
\[ C_{12}(t - t_0) = e^{L_{12}(t-t_0)} \frac{g_2(x_1,x_2)}{f_1(x_1)f_1(x_2)} e^{-L_{12}(t-t_0)}, \]  
(11.4)

\[ L_{12} = L_{12}^0 + L_{12}', \quad L_{12}^0 = L_{12}' + \text{g} \quad \text{and} \quad f_1(x_i) = f_1(x_i, t_0). \]

Homogeneous nonlinear equation (11.3) is the central result of this section. This is a new equation that (in the linear approximation in \( n \)) exactly describes the evolution of a one-particle distribution function of a nonideal classical gas of particles with arbitrary spatial inhomogeneity and on any time scale. This equation takes initial correlations into account exactly (in the adopted approximation in \( n \)) and treats them on an equal footing with the particle collisions and other processes. The initial correlations enter the memory kernel (mass operator) governing the evolution of the one-particle distribution function and do not result in any undesirable inhomogeneous term in evolution equation (11.3) (cf. (9.11)). The contribution of initial correlations to the evolution process is given by time-dependent correlation parameter (11.4), which describes the evolution in time of the normalized two-particle correlation function \( \frac{x_1(x_1,x_2)}{f_1(x_1)f_1(x_2)} \).

The time-evolution of all terms in (11.3) is determined by the exact two-particle propagator \( G_{12}(t) = \exp(L_{12}t) \).

We consider the terms in the right-hand side of (11.3) and their evolution in time in more detail. The first term is a conventional flow term resulting from the gas inhomogeneity. The second term represents the Vlasov self-consistent field modified by initial correlations. If \( C_{12}(t - t_0) \) vanishes (under the action of \( G_{12}(t - t_0) \)) with time for \( t - t_0 \geq t_{\text{cor}} \) (see below), then we obtain the conventional Vlasov term. The third term in the right-hand side of (11.3) describes the particle collisions and the influence on the collisions of the change in the one-particle distribution function on the microscopic scales in space and time. All these processes are modified by initial correlations given by \( C_{12}(t - t_0) \).

Equation (11.3) thus takes the influence of all pair collisions and correlations on the dissipative and nondissipative characteristics of the nonideal spatially inhomogeneous gas of particles into account (this is not the case in the conventional approaches [23]).

Noting that the term with \( L_{12}^0 \) is of the order of \( r_0/r_h \) and the term with the time derivative of \( f_1(x_2,t) \) is \( \sim t_{\text{cor}}/t_{\text{rel}} \) (\( \sim \gamma \)), we can neglect these terms (if we are not interested in the nondissipative corrections to the nonideal gas dynamics) in the adopted approximation (linear in \( n \)). Equation (11.3) then takes the simpler form
\[
\frac{\partial f_1(x_1,t)}{\partial t} = L_{12}' f_1(x_1,t) + n \int dx_2 L_{12}' [1 + C_{12}(t-t_0)] f_1(x_2,t) f_1(x_1,t) \\
+ n \int dx_2 L_{12}' [1 + C_{12}(t-t_0)] \\
\times \int_0^{t-t_0} dt_1 e^{L_{12} t_1} L_{12} f_1(x_2,t-t_1) f_1(x_1,t-t_1),
\]  
(11.5)

where the distribution functions of the colliding particles in the last (collision) term can be taken at the same space point \( r_1 = r_2 \) with the adopted accuracy
we can represent the term determining the contribution of initial correlation to the second term in the right-hand side of (11.5) as

\[ g_2(x_1, x_2) = g_0 \exp\left(-\frac{r^2}{r_{\text{cor}}^2}\right)\phi(x_1, x_2), \]

where \( r = x_1 - x_2 \) is the distance between particles 1 and 2, \( r_{\text{cor}} \sim r_0 \), \( g_0 \) and \( V_0 \) are constant parameters, and \( \phi(x_1, x_2) \) is a properly normalized function of \( x_1 \) and \( x_2 \). We estimate the time dependence of the terms in (11.5) determined by initial correlations in the case of a weak interparticle interaction when the time evolution is governed by the "free" propagator \( G_{12}(t) = \exp(L_{12}t) \). Then we have (see (11.4))

\[ G_{12}^0(t) g_2(x_1, x_2) = g_0 \exp\left(-\frac{|r - gt|^2}{r_{\text{cor}}^2}\right) \phi(x_1 - v_1t, p_1, x_2 - v_2t, p_2) F_1(x_1 - v_1t, p_1) F_1(x_2 - v_2t, p_2), \]

(11.7)

where \( g = v_1 - v_2 \) is the relative velocity. Function (11.7) tends to zero at \( t \to \infty \) with any fixed distance \( r \) and velocity \( g \). Using (11.9), (11.6), and (11.7), we can represent the term determining the contribution of initial correlation to the second term in the right-hand side of (11.5) as

\[ n \int dp_2 \int dr \left[ \frac{\partial}{\partial r} V_0 \exp\left(-\frac{r^2}{r_0^2}\right) \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) g_0 \exp\left(-\frac{|r - gt|^2}{r_{\text{cor}}^2}\right) \right. \]

\[ \times \phi(x_1 - v_1t, p_1, x_1 - r - v_2t, p_2) F_1(x_1 - v_1t, p_1) F_1(x_1 - r - v_2t, p_2) F_1(x_1 - r, p_2, t) F_1(x_1, p_1, t) \]

(11.8)

(where we set \( t_0 = 0 \) for brevity). It is easy to see that the integral over \( r \) in (11.8) is nonzero only if \( t < t_{\text{cor}} \), where \( t_{\text{cor}} \sim r_{\text{cor}}/\bar{v} \sim r_0/\bar{v} \). For \( t \gg t_{\text{cor}} \), the integral (11.8) practically vanishes because of the finite interparticle interaction range \( r_0 \). Physically, this means that the initial correlations propagate outward from the region of interest determined by \( r_0 \) and do not return in the approximation linear in \( n \) under consideration (also see [20]). Of course, such behavior is possible if the contribution to (11.8) of the "parallel motion" with small \( g \) is negligible. Therefore, the mixing flow in the phase space is necessary to ensure that the initial correlation term vanishes. The same behavior of the initial correlation term in the master equation was found in [29, 30] in the case of a quantum mechanical system interacting with a thermal bath with the mixing property in the van Hove limit. The term related to initial correlations and contributing to the third term in right-hand side of Eq. (11.5) (collision integral) displays the same behavior with time. In the considered example, the
terms with initial correlations in Eq. (11.5) thus vanish for $t \gg t_{\text{cor}}$ (in the considered first approximation in $n$).

If we pass to the kinetic time scale $t - t_0 \gtrsim t_{\text{rel}} \gg t_{\text{cor}}$ in Eq. (11.5), then the initial correlation terms determined by the two-particle correlation function $C_{12}(t - t_0)$ given by (11.4) vanish on this time scale because of the mixing flow in the phase space (as in the example considered above), and we can rewrite Eq. (11.5) in the form of the irreversible Markovian kinetic equation

$$
\frac{\partial f_1(x_1,t)}{\partial t} = L_1^0 f_1(x_1,t) + n \int dx_2 L_{12} f_1(x_2,t) f_1(x_1,t)
+ n \int dx_2 L_{12} \int_0^\infty dt e^{L_{12} t} L_{12} f_1(x_2,t) f_1(x_1,t).
$$

(11.9)

For an interparticle interaction with the small parameter $\varepsilon \ll 1$, the two-particle propagator $\exp(L_{12}t)$ in the collision term in (11.9) can be replaced with the "free" propagator $\exp(L_{12}^0t)$ (see (4.26)), and Eq. (11.9) reduces to the Vlasov-Landau equation in the second approximation in $\varepsilon$ (see, e.g., [20]). In the spatially homogeneous case, Eq. (11.9) coincides with the nonlinear Boltzmann equation for the momentum distribution function $f_1(p_1,t)$.

We note, that all stages of the evolution of the system under consideration can be followed (in the adopted approximation) using Eq. (11.3) beginning from the initial reversible regime $t_0 \leq t \leq t_{\text{cor}}$. If the reversible terms in (11.3), related to the initial correlations $C_{12}(t - t_0)$, vanish with time and the last integral $\int_0^\infty$ in (11.9) exists (see Secs. 5 and 7), then the evolution of the system can switch automatically from the reversible to the irreversible stage described by kinetic equation (11.9).

12 CONCLUSION

In the preceding sections we have introduced a new type of evolution equations: the homogeneous generalized master equations (HGMEs) and nonlinear GMEs. The idea behind this is that we tried to obtain an evolution equations which are capable of describing all stages of the (sub)system of interest evolution in time beginning from the initial stage dependent of the initial system state (initial correlations). Although the BBGKY chain or conventional Nakajima-Zwanzig TC-GME and TCL-GME are formally valid for all $t > t_0$, the actual consideration of the transient (from reversible initial $0 < t - t_0 < t_{\text{cor}}$ to irreversible kinetic $t_{\text{cor}} < (t - t_0) \sim t_{\text{rel}}$) regime and of the influence of initial correlations on the evolution process poses essential difficulties (see, e.g., [14] and references therein).

In the framework of the conventional time-independent projection operator approach, we have derived the exact time-convolution homogeneous generalized master equation (TC-HGME) (2.12) and the exact time-convolutionless generalized master equation (TCL-HGME) (3.5) for the relevant part of a distribution function (statistical operator). In the derivation we have not used
any approximation (like a factorizing initial condition or RPA) or principle (like
the Bogoliubov principle of weakening of initial correlations). These equations
have several advantages as compared to the conventional generalized master
equations (GMEs). The HGMEs contain the parameter of initial correlations
depending on time in the “mass” (super)operator acting on the relevant part
of a distribution function (statistical operator). These equations allow treating
the initial correlations consistently and on an equal footing with the collision
integral by expanding the “mass” (super)operator into the series in the appro-
priate small parameter. The obtained equations are valid on any time scale,
in particular on the initial stage of evolution $t_0 \leq t - t_0 \leq t_{cor}$, which can be
important for studying the irreversibility problem, the non-Markovian and the
ultrafast relaxation processes. The HGMEs enable the consideration of the en-
tire evolution process of the relevant part of the distribution function (statistical
operator) and of the influence of initial correlations on this process. These equa-
tions can automatically switch, in principle, from the initial (reversible) regime
into the kinetic (irreversible) one if the particle dynamics is characterized by
the ergodic mixing flow in the phase space leading to the damping of initial
correlations and correlations caused by collisions.

By appropriate selection of the projection operator, one can rewrite the
obtained HGMEs for the system under consideration. We have considered a
spatially homogeneous dilute gas of classical and quantum particles with an
arbitrary repulsive inter-particle interaction and obtained in the linear approxi-
mation in the small density parameter (4.2) the closed equation for a one-particle
distribution function (4.24) and a one-particle density matrix (6.41) retaining
initial correlations. In this approximation, the evolution equation for a one-
particle momentum distribution function contains only binary collisions and a
two-particle time-dependent (via only two-particle dynamics) correlation func-
tion in the parameter accounting for initial correlations. We have shown as
on the macroscopic time-scale (1.1) these equations may become equivalent to
the Boltzmann classical and quantum equations if all correlations caused by
inter-particle interaction vanish on this time scale.

We have also derived (not postulated as it is usually done) the nonlinear
GMEs describing the evolution of the relevant part of a distribution function
(statistical operator). The new approach leading to these equations is based
on using the nonlinear time-dependent operator $P(t)$ determining the relevant
part of the distribution function (statistical operator). This operator is gener-
ally not a projection operator. The obtained inhomogeneous nonlinear GME
(8.10) (and (8.23)) can be viewed as a generalization of the Nakajima-Zwanzig
linear TC-GME. In the case of time-independent projection operators $P$ and
$Q$, the obtained equation reduces to the conventional TC-GME. This inhomo-
genous nonlinear GME is equally useful for deriving both the nonlinear and
linear evolution equations for the reduced distribution function (statistical op-
erator) of interest in contrast to the linear TC-GME, which is naturally more
convenient for deriving linear evolution equations, for example, for a subsystem
interacting with a thermal bath. Using the obtained equation, we derived new
inhomogeneous nonlinear equation (9.11) for a one-particle distribution function
in the linear approximation in the small density parameter of a gas of classical particles. This equation, which holds for an arbitrary spatial inhomogeneity, contains the space and time changes of the one-particle distribution function on the microscopic time scale (contributing to the nondissipative characteristics of a nonideal gas and thus eliminating the inconsistency in the standard approach) and an irrelevant part (a source) given by the initial two-particle correlation function. Although the obtained inhomogeneous nonlinear GME allows deriving the closed nonlinear equation for a one-particle distribution function, an additional assumption like the RPA or the Bogoliubov principle of weakening of initial correlations is still needed in order to obtain either the Vlasov-Landau or the Boltzmann kinetic equation.

To take initial correlations into account, we used the method suggested in Sec. 2 to convert the inhomogeneous nonlinear GME into the homogeneous form. The obtained exact homogeneous nonlinear GME (10.3) for the relevant part of the distribution function (statistical operator) describes all evolution stages of the (sub)system of interest including the initial stage where the initial correlations play a role. In deriving this equation, which can be used to obtain both the nonlinear and linear evolution equations for the reduced distribution functions, we used no approximation like the RPA or the principle of weakening of initial correlations. The initial correlations are treated in this equation on an equal footing with collisions via the modified memory kernel, which is a starting point for effective perturbation expansions. To test this equation, we used it for a spatially inhomogeneous nonideal gas of classical particles and obtained new homogeneous nonlinear evolution equation (11.3) for a one-particle distribution function retaining initial correlations in the memory kernel in the approximation linear in the density. Equation (11.3) describes all evolution stages, takes all two-particle correlations (collisions) into account, and converts into the nonlinear Boltzmann or Vlasov-Landau equation on the appropriate time scale when all initial correlations (and the ones due to collisions) vanish. It is important to note, that in order to obtain the nonlinear evolution (kinetic) equation, we are not now restricted by the second inequality (1.1) \((t - t_0 < t_{rel})\) as it was in the case of linear TC-HGME. The irreversible Boltzmann and Vlasov-Landau equations can be thus obtained from the Liouville equation without conventional additional approximations (like the RPA or principle of weakening of initial correlations) if the system dynamics have the necessary properties (such as an ergodic mixing flow in the phase space).

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