An Affine Model of a Riemann Surface Associated to a Schwarz–Christoffel Mapping

Richard Cushman

Department of Mathematics and Statistics, University of Calgary, Calgary, AB T2N 1N4, Canada; r.h.cushman@gmail.com

Abstract: In this paper, we construct an affine model of a Riemann surface with a flat Riemannian metric associated to a Schwarz–Christoffel mapping of the upper half plane onto a rational triangle. We explain the relation between the geodesics on this Riemann surface and billiard motions in a regular stellated $n$-gon in the complex plane.

Keywords: Schwarz–Christoffel; Riemann surface; discrete subgroup

MSC: 30C30; 30F10

1. Introduction

We give a section by section summary of the contents of this paper.

In §1 we define the Schwarz–Christoffel conformal map $F_0$ (2) of the complex plane less $\{0, 1\}$ onto a quadrilateral $Q$, which is formed by reflecting a rational triangle $T_{n_0 n_1 n_\infty}$ in the real axis.

In §2, following Aurell and Itzykson [1] we associate to the map $F_0$ the affine Riemann surface $S$ in $\mathbb{C}^2$ defined by $\eta^n = \bar{\xi}^{n-n_0}(1-\bar{\xi})^{n-n_1}$, where $\mathbb{C}^2$ has coordinates $(\xi, \eta)$ and $n = n_0 + n_1 + n_\infty$. Thinking of $S$ as a branched covering

$$\pi : S \to \mathbb{C} \setminus \{0, 1\} : (\xi, \eta) \mapsto \bar{\xi}$$

with branch points at $(0, 0)$, $(1, 0)$ and $\infty$ corresponding to the branch values 0, 1, and $\infty$, respectively, we show that $S$ has genus $\frac{1}{2}(n + 2 - (d_0 + d_1 + d_\infty))$, where $d_j = \gcd(n, n_j)$ for $j = 0, 1, \infty$. Let $S_{\text{reg}}$ be the set of nonsingular points of $S$. The map $\bar{\pi} = \pi|_{S_{\text{reg}}} : S_{\text{reg}} \to \mathbb{C} \setminus \{0, 1\}$ is a holomorphic $n$-fold covering map with covering group the cyclic group generated by

$$R : S_{\text{reg}} \subseteq \mathbb{C}^2 \to S_{\text{reg}} \subseteq \mathbb{C}^2 : (\xi, \eta) \mapsto (\xi, e^{2\pi i/n} \eta).$$

In §3 we build a model $\tilde{S}_{\text{reg}}$ of the affine Riemann surface $S_{\text{reg}}$. The quadrilateral $Q$ is holomorphically diffeomorphic to a fundamental domain $D$ of the action of the covering group $S_{\text{reg}}$. Rotating $Q$ by

$$R : \mathbb{C} \to \mathbb{C} : z \mapsto e^{2\pi i/n} z$$

gives a regular stellated $n$-gon $K^*$, which is invariant under the dihedral group $G$ generated by the mappings $R$ and $U : \mathbb{C} \to \mathbb{C} : z \mapsto z$. We study the group theoretic properties of $K^*$. We show that $K^*$ is invariant under the reflection $S_{\text{reg}}^{(j)} = R^j U$ in the ray $\{ t e^{2\pi i n_j/n} \in \mathbb{C} | t \geq 0 \}$ for $j = 0, 1, \infty$. To construct the model $\tilde{S}_{\text{reg}}$ of the affine Riemann surface $S_{\text{reg}}$ from the regular stellated $n$-gon $K^*$ we follow Richens and Berry [2]. We identify two nonadjacent closed edges of $\text{cl}(K^*)$, the closure of $K^*$, if one edge is obtained from the other by a reflection $S_{\text{reg}}^{(j)} = R^j S_{\text{reg}} R^{-k}$ for some $j = 0, 1, \infty$. The identification space $(\text{cl}(K^*) \setminus O)^{-}$, where $O$ is the center of $K^*$, is a complex manifold except at points...
corresponding to O or a vertex of $\text{cl}(K^*)$, where it has a conical singularity. The action of $G$ on $K^* \setminus O$ induces a free and proper action on the identification space $(K^* \setminus O)^\sim$, whose orbit space $\tilde{S}_{\text{reg}}$ is a complex manifold with compact closure in $\mathbb{CP}^2$, with genus $\frac{1}{2}(n^2 + \cdots + d_{n0})$. Moreover $\tilde{S}_{\text{reg}}$ is holomorphically diffeomorphic to the affine Riemann surface $S_{\text{reg}}$.

In §4, we construct an affine model $\tilde{S}_{\text{reg}}$ of the Riemann surface $S_{\text{reg}}$. In other words, we find a discrete subgroup $\mathfrak{G}$ of the 2-dimensional Euclidean group $E(2)$, which acts freely and properly on $C \setminus V^+$ such that after forming an identification space $(C \setminus V^+)\sim$ the $\mathfrak{G}$ orbit space $(C \setminus V^+)\sim / \mathfrak{G}$ is holomorphically diffeomorphic to $S_{\text{reg}}$. We now describe the group $\mathfrak{G}$. Reflect the regular stellated $n$-gon $K^*$ in its edges, and then in the edges of the reflected regular stellated $n$-gons, et cetera. We obtain a group $\mathfrak{T}$ generated by $2n$ translations $\tau_k$ so that $\tau_1 \cdots \tau_{2n}$ sends the center $O$ of $K^*$ to the center of a repeatedly reflected reflected $n$-gon. The set $V^+$ is the union of the image under $\tau_1 \cdots \tau_{2n}$ of a vertex of $\text{cl}(K^*)$ and its center $O$ for every $(\ell_1, \ldots, \ell_{2n}) \in (\mathbb{Z}_{\geq 0})^{2n}$. Let $\mathfrak{G}$ be the semi-direct product $G \times \mathfrak{T}$. The fundamental domain of the $\mathfrak{G}$ action on $C \setminus V^+$ is $\text{cl}(K^*)$ less its vertices and center. Identifying equivalent open edges of $K^* \setminus O$ and then taking $G$ orbits, it follows that the affine model $\tilde{S}_{\text{reg}}$ of the affine Riemann surface $S_{\text{reg}}$ is the $\mathfrak{G}$ orbit space $(C \setminus V^+)\sim / \mathfrak{G}$.

In §5 we show that the mapping

$$\delta_Q : \mathcal{D} \subseteq S_{\text{reg}} \subseteq \mathbb{C}^2 \to Q \subseteq C : (\xi, \eta) \mapsto (F_Q \circ \hat{\gamma})(\xi, \eta) = z$$

straightens the nowhere vanishing holomorphic vector field $X$ (11) on $S_{\text{reg}}$, that is, $T_{(\xi, \eta)} \delta_Q X(\xi', \eta') = \frac{d}{d\xi} \mid_{\xi = \delta_Q(\xi', \eta')}$ for every $(\xi', \eta') \in \mathcal{D}$. We pull back the flat metric $\gamma = dz \circ d\bar{z}$ on $C$ by $\delta_Q$ to the metric $\Gamma$ on $S_{\text{reg}}$. So $\delta_Q$ is a local developing map. Since $\frac{d}{d\xi}$ is the geodesic vector field on $(Q, \gamma|_{Q})$, it follows that $X$ is a holomorphic geodesic vector field on $(S_{\text{reg}}, \Gamma)$.

In §6 we study the geometry of the developing map $\delta_Q$. The dihedral group $\mathfrak{G}$ generated by $\mathcal{R}$ and $U : S_{\text{reg}} \to S_{\text{reg}} : (\xi, \eta) \mapsto (\overline{\xi}, \eta)$ is a group of isometries of $(S_{\text{reg}}, \Gamma)$. The group $\mathfrak{G}$ generated by $\mathcal{R}$ and $U : C \to C : z \mapsto \overline{z}$ is a group of isometries of $(Q, \gamma|_{Q})$.

Extend the holomorphic map $\delta_Q$ to a holomorphic map $\delta_K : S_{\text{reg}} \to K^*$ by requiring that $\mathcal{R} \circ \delta_K = \delta_Q \circ \mathcal{R}^\sim$ on $\mathcal{R}^{-1}(\mathcal{D})$. This works since $\mathcal{D}$ is a fundamental domain of the action of the covering group on $S_{\text{reg}}$, which implies $S_{\text{reg}} = \Pi_{0 \leq j \leq n} \mathcal{R}^\sim(\mathcal{D})$. Thus, the local holomorphic diffeomorphism $\delta_K$, intertwines the $\mathfrak{G}$ action on $(S_{\text{reg}}, \Gamma)$ with the $G$ action on $(K^*, \gamma|_{K^*})$ and intertwines the local geodesic flow of the holomorphic geodesic vector field $X$ with the local geodesic flow of the holomorphic vector field $\frac{d}{d\xi}$.

Following Richens and Berry [2] we impose the condition: when a geodesic, starting at a point in $\text{int}(\text{cl}(K^*) \setminus O)$, meets $K^*$ it undergoes a reflection in the edge of $K^*$ that it meets. Such geodesics never meet a vertex of $\text{cl}(K^*)$. Thus, this type of geodesic becomes a billiard motion in $K^* \setminus O$, which is defined for all time. Billiard motions in polygons have been extensively studied. For a nice overview see Berger ([3], chpt. XI) and references therein. An argument shows that $\mathfrak{G}$ invariant geodesics on $(S_{\text{reg}}, \Gamma)$ correspond under the map $\delta_K \mid_{K^* \setminus O}$ to billiard motions on $(K^* \setminus O, \gamma|_{K^* \setminus O})$.

Repeatedly reflecting a billiard motion in an edge of $K^* \setminus O$ and suitable edges of suitable $\mathfrak{T}$ translations of $K^* \setminus O$ gives a straight line motion $\lambda$ on $C \setminus V^+$. The image of the segment of a billiard motion, where $\lambda$ intersects $K^* \setminus O$, in the orbit space $(C \setminus V^+)\sim / \mathfrak{G} = \tilde{S}_{\text{reg}}$, is a geodesic. Here we use the flat Riemannian metric $\hat{\gamma}$ on $\tilde{S}_{\text{reg}}$, which is induced by the $\mathfrak{G}$ invariant Euclidean metric $\gamma$ on $C \setminus V^+$ restricted to $K^* \setminus O$. Consequently, $(\tilde{S}_{\text{reg}}, \hat{\gamma})$ is an affine analogue of the affine Riemann surface $S_{\text{reg}}$ thought of as the orbit space of a discrete subgroup of $\text{PGL}(2, \mathbb{C})$ acting on $C$ with the Poincaré metric, see Weyl [4].
2. A Schwarz–Christoffel Mapping

Consider the conformal Schwarz–Christoffel mapping

$$F_T : \mathbb{C}^+ \rightarrow T = T_{n_0 n_1 n_\infty} \subseteq \mathbb{C} : \xi \mapsto \int_{0}^{\xi} \frac{dw}{w^{1 - \frac{m}{n}} (1 - w)^{1 - \frac{m}{n}}} = z \quad (1)$$

of the upper half plane $\mathbb{C}^+$ to the rational triangle $T = T_{n_0 n_1 n_\infty}$ with interior angles $\frac{n_0}{n} \pi$, $\frac{n_1}{n} \pi$, and $\frac{n_\infty}{n} \pi$, see Figure 1. Here $n_0 + n_1 + n_\infty = n$ and $n_j \in \mathbb{Z}_{\geq 1}$ for $j = 0, 1$ and $\infty$ with $1 \leq n_0 \leq n_1 \leq n_\infty$. Because $n_\infty$ is greater than or equal to either $n_0$ or $n_1$, it follows that the corresponding side $OC$ is the longest side of the triangle $T = \triangle OCD$.

In the integrand of (1) we use the following choice of complex $\zeta$-th root. Suppose that $\zeta \in \mathbb{C} \setminus \{0, 1\}$. Let $\zeta = r_0 e^{i \theta_0}$ and $1 - \zeta = r_1 e^{i \theta_1}$ where $r_0, r_1 \in \mathbb{R}_{>0}$ and $\theta_0, \theta_1 \in [0, 2\pi)$. For $\zeta \in (0, 1)$ on the real axis we have $\theta_0 = \theta_1 = 0$, $\zeta = r_0 > 0$, and $1 - \zeta = r_1 > 0$. So $(\zeta^n - r_0) (1 - \zeta)^{n-1})^{1/n} = (r_0^{n-1} r_1^{1/n})^{1/n} e^{i((n-n_0) \theta_0 + (n_1-n_0) \theta_1) / n}$.

From (1) we get

$$F_T(0) = 0, \quad F_T(1) = C, \quad \text{and} \quad F_T(\infty) = D,$$

where $C = \int_{0}^{1} \frac{dw}{w^{1 - \frac{m}{n}} (1 - w)^{1 - \frac{m}{n}}} \in \mathbb{C}$ and $D = e^{\frac{m}{n} \pi i} \left(0^{n-1} r_1^{1/n} \right) C$. Consequently, the bijective holomorphic mapping $F_T$ sends $\text{int}(\mathbb{C}^+ \setminus \{0, 1\})$, the interior of the upper half plane less 0 and 1, onto $T$, the interior of the rational triangle $T = T_{n_0 n_1 n_\infty}$, and sends the boundary of $\mathbb{C}^+ \setminus \{0, 1\}$ to the edges of $\partial T$ less their end points $O, C$ and $D$, see Figure 1. Thus, the image of $\mathbb{C}^+ \setminus \{0, 1\}$ under $F_T$ is $\text{cl}(T) \setminus \{O, C, D\}$. Here $\text{cl}(T)$ is the closure of $T$ in $\mathbb{C}$.

![Figure 1](https://via.placeholder.com/150)

**Figure 1.** The rational triangle $T = T_{n_0 n_1 n_\infty}$.

Because $F_T|_{[0, 1)}$ is real valued, we may use the Schwarz reflection principle to extend $F_T$ to the holomorphic diffeomorphism

$$F_Q : C \setminus \{0, 1\} \rightarrow Q = T \cup \overline{T} \subseteq C : \xi \mapsto z = \begin{cases} F_T(\xi), & \text{if } \xi \in \mathbb{C}^+ \setminus \{0, 1\} \\ \overline{F_T(\xi)}, & \text{if } \xi \in \mathbb{C}^+ \setminus \{0, 1\}. \end{cases} \quad (2)$$

Here $Q = Q_{n_0 n_1 n_\infty}$ is a quadrilateral with internal angles $2\pi \frac{n_0}{n}$, $\pi \frac{n_1}{n}$, $2\pi \frac{n_\infty}{n}$, and vertices at $O, D, C,$ and $\overline{D}$, see Figure 2. The conformal mapping $F_Q$ sends $C \setminus \{0, 1\}$ onto $\text{cl}(Q) \setminus \{O, D, C, D\}$. 
3. The Geometry of an Affine Riemann Surface

Let \( \xi \) and \( \eta \) be coordinate functions on \( \mathbb{C}^2 \). Consider the affine Riemann surface \( S = S_{n_0, n_1, n_\infty} \) in \( \mathbb{C}^2 \), associated to the holomorphic mapping \( F \), defined by

\[
g(\xi, \eta) = \eta^n - \xi^{n-n_0}(1-\xi)^{n-n_1} = 0,
\]

see Aurell and Itzykson [1]. We determine the singular points of \( S \) by solving

\[
0 = dg(\xi, \eta) = -(n-n_0)\xi^{n-n_0-1}(1-\xi)^{n-n_1-1}(1-\frac{2n-n_0-n_1}{n-n_0}\xi)\,d\xi + n\eta^{n-1}\,d\eta
\]

(4)

For \( (\xi, \eta) \in S \), we have \( dg(\xi, \eta) = 0 \) if and only if \( (\xi, \eta) = (0,0) \) or \( (1,0) \). Thus, the set \( S_{\text{sing}} \) of singular points of \( S \) is \( \{ (0,0), (1,0) \} \). So the affine Riemann surface \( S_{\text{reg}} = S \setminus S_{\text{sing}} \) is a complex submanifold of \( \mathbb{C}^2 \). Actually, \( S_{\text{reg}} \subseteq \mathbb{C}^2 \setminus \{ \eta = 0 \} \), for if \( (\xi, \eta) \in S \) and \( \eta = 0 \), then either \( \xi = 0 \) or \( \xi = 1 \).

Lemma 1. Topologically \( S_{\text{reg}} \) is a compact Riemann surface \( \overline{S} \subseteq \mathbb{C}P^2 \) of genus \( g = \frac{1}{2} (n + 2 - (d_0 + d_1 + d_\infty) - 3) \) less three points: \( [0 : 0 : 1], [1 : 0 : 1], \) and \( [0 : 1 : 0] \). Here \( d_j = \gcd(n_j, n) \) for \( j = 0, 1, \infty \).

Proof. Consider the (projective) Riemann surface \( \overline{S} \subseteq \mathbb{C}P^2 \) specified by the condition \( [\xi : \eta : \zeta] \in \overline{S} \) if and only if

\[
G(\xi, \eta, \zeta) = \xi^{n-n_0-n_1}\eta^n - \xi^{n-n_0}(\zeta - \xi)^{n-n_1} = 0.
\]

(5)

Thinking of \( G \) as a polynomial in \( \eta \) with coefficients which are polynomials in \( \xi \) and \( \zeta \), we may view \( \overline{S} \) as the branched covering

\[
\pi : \overline{S} \subseteq \mathbb{C}P^2 \rightarrow \mathbb{C}P : [\xi : \eta : \zeta] \mapsto [\xi : \zeta].
\]

(6)

When \( \zeta = 1 \) we get the affine branched covering

\[
\pi = \pi|S : S = S \cap \{ \zeta = 1 \} \subseteq \mathbb{C}^2 \rightarrow \mathbb{C} = \mathbb{C}P \cap \{ \zeta = 1 \} : (\xi, \eta) \mapsto \xi.
\]

(7)

From (3) it follows that \( \eta = \omega_k(\xi^{n-n_0}(1-\xi)^{n-n_1})^{1/n} \), where \( \omega_k \) for \( k = 0, 1, \ldots, n-1 \) is an \( n \)th root of unity with and \( ( \cdot )^{1/n} \) is the complex \( n \)th root used in the definition of the mapping \( F \) (1). Thus, the branched covering mapping \( \overline{\pi} (6) \) has \( n \) “sheets” except at its branch points. Since

\[
\eta = \xi^{1 - \frac{n_0}{n}} (1 - \xi)^{1 - \frac{n_1}{n}} = \xi^{1 - \frac{n_0}{n}} (1 - (1 - \frac{n_1}{n})\xi + \cdots)
\]

(8a)

and

\[
\eta = (1 - \xi)^{1 - \frac{n_1}{n}} (1 - (1 - \xi))^{1 - \frac{n_0}{n}}
\]

(8b)
(8b)

\[ (1 - \xi)^{\frac{1}{2}} \pi (1 - (1 - \frac{u_0}{\pi})(1 - \xi) + \cdots), \]

it follows that \( \xi = 0 \) and \( \xi = 1 \) are branch points of the mapping \( \pi \) of degree \( \frac{d}{\pi} \) and \( \frac{d}{\pi} \), since \( d_j = \gcd(n, m_j) = \gcd(n - m_j, m_j) \) for \( j = 0, 1 \), see McKean and Moll ([5], p. 39). Because

\[
\eta = \left( \frac{1}{\xi} \right)^{-1}(\frac{1}{\xi} - \frac{1}{\xi^2})^{1-\frac{u_0}{\pi} + \frac{d}{\pi}}(1 - (1 - \frac{u_0}{\pi})\frac{1}{\xi} + \cdots),
\]

(8c)

\( \infty \) is a branch point of the mapping \( \pi \) of degree \( \frac{d}{\pi} \), where \( d_\infty = \gcd(n, m_\infty) \). Hence the ramification index of 0, 1, \( \infty \) is \( d_0(n - 1) = n - d_0, n - d_1, \) and \( n - d_\infty \), respectively. Thus, the map \( \pi \) has \( d_\infty \) fewer sheets at 0, \( d_1 \) fewer at 1, and \( d_\infty \) fewer at \( \infty \) than an \( n \)-fold covering of \( \mathbb{C} \mathbb{P} \). Thus, the total ramification index \( r \) of the mapping \( \pi \) is \( r = (n - d_0) + (n - d_1) + (n - d_\infty) \). By the Riemann–Hurwitz formula, the genus \( g \) of \( \mathcal{S} \) is

\[ g = \frac{1}{2} (n + 2 - (d_0 + d_1 + d_\infty)). \]

Consequently, the affine Riemann surface \( \mathcal{S} \) is the compact connected surface \( \mathcal{S} \) less the point at \( \infty \), namely, \( \mathcal{S} = \mathcal{S} \setminus \{0 : 1 : 0\} \). So \( \mathcal{S}_{\text{reg}} \) is the compact connected surface \( \mathcal{S} \) less three points: \{0 : 0 : 1\}, \{1 : 0 : 1\}, and \{0 : 1 : 0\}. \( \square \)

Examples of \( \mathcal{S} = \mathcal{S}_{n_0,n_1,n_\infty} \)

1. \( n_0 = 1, n_0 = 1, n_\infty = 4; n = 6 \). So \( d_0 = 1, d_1 = 1, d_\infty = 2 \). Hence
   \[ 2g = 8 - 4 = 4. \] So \( g = 2 \).

2. \( n_0 = 2, n_1 = 2, n_\infty = 3; n = 7 \). So \( d_0 = d_1 = d_\infty = 1 \). Hence
   \[ 2g = 9 - 3 = 6. \] So \( g = 3 \).

Table 1 below lists all the partitions \( \{n_1,n_0,n_\infty\} \) of \( n \), which give a low genus Riemann surface \( \mathcal{S} = \mathcal{S}_{n_0,n_1,n_\infty} \).

Table 1. Based on the table in Aurell and Itzykson ([1], p. 193).

| \( g \) | \( n_0, n_1, n_\infty; n \) | \( g \) | \( n_0, n_1, n_\infty; n \) |
|---|---|---|---|
| 1 | 1, 1, 1; 3 | 3 | 2, 2, 3; 7 |
| 1 | 1, 1, 2; 4 | 3 | 1, 3, 3; 7 |
| 1 | 1, 2, 3; 6 | 3 | 1, 1, 5; 7 |
| 2 | 1, 2, 2; 5 | 3 | 2, 3, 3; 8 |
| 2 | 1, 1, 3; 5 | 3 | 1, 2, 5; 8 |
| 2 | 1, 1, 4; 6 | 3 | 1, 1, 6; 8 |
| 2 | 1, 3, 4; 8 | 3 | 2, 3, 4; 9 |
| 2 | 2, 3, 5; 10 | 3 | 1, 3, 5; 9 |
| 2 | 1, 4, 5; 10 | 3 | 1, 2, 6; 9 |
| 3 | 3, 4, 5; 12 |
| 3 | 1, 5, 6; 12 |
| 3 | 1, 3, 8; 12 |
| 3 | 2, 5, 7; 14 |
| 3 | 1, 6, 7; 14 |

Corollary 1. If \( n \) is an odd prime number and \( \{n_1,n_0,n_\infty\} \) is a partition of \( n \) into three parts, then the genus of \( \mathcal{S} \) is \( \frac{1}{2} (n - 1) \).

Proof. Because \( n \) is prime, we get \( d_0 = d_1 = d_\infty = 1 \). Using the formula \( g = \frac{1}{2} (n + 2 - (d_0 + d_1 + d_\infty)) \) we obtain \( g = \frac{1}{2} (n - 1) \). \( \square \)
Corollary 2. The singular points of the Riemann surface $\mathcal{S}$ are $[0 : 0 : 1]$, $[1 : 0 : 1]$, and if $n_\infty > 1$ then also $[0 : 1 : 0]$.

Proof. A point $[\xi : \eta : \zeta] \in \mathcal{S}_{\text{sing}}$ if and only if $[\xi : \eta : \zeta] \in \mathcal{S}$, that is,

$$0 = G(\xi, \eta, \zeta) = \xi^{n-(n_0+n_1)}\eta^n - \xi^{n-n_0}(\zeta - \xi)^{n-n_1}$$  \hspace{1cm} (10a)

and

$$(0,0,0) = DG(\xi, \eta, \zeta) = -\frac{1}{2} \xi^{n-n_0-1}(\xi - \zeta)^{n-n_1-1}((n - n_0)(\zeta - \xi) - (n - n_1)\zeta),$$

$$n\eta^{n-1}\xi^{n-(n_0+n_1)}, (n - (n_0 + n_1))\eta^n\xi^{n-n_0-n_1-1} - (n - n_1)\xi^{n-n_0}(\zeta - \xi)^{n-n_1-1}$$  \hspace{1cm} (10b)

We need only check the points $[0 : 0 : 1]$, $[1 : 0 : 1]$ and $[0 : 1 : 0]$. Since the first two points are singular points of $\mathcal{S} = \mathcal{S} \setminus \{[0 : 1 : 0]\}$, they are singular points of $\mathcal{S}$. Thus, we need to see if $[0 : 1 : 0]$ is a singular point of $\mathcal{S}$. Substituting $(0,1,0)$ into the right hand side of (10b) we get $\{0 \, 0, 0, \# \text{reg} \neq - (n_0 + n_1) = 1$. Thus, $[0 : 1 : 0]$ is a singular point of $\mathcal{S}$ only if $n_\infty > 1$.  \hfill $\square$

Lemma 2. The mapping \( \widehat{\pi} = \pi|_{\mathcal{S}_{\text{reg}}} : \mathcal{S}_{\text{reg}} \subseteq \mathbb{C}^2 \rightarrow \mathbb{C} \setminus \{0, 1\} : (\xi, \eta) \mapsto \xi \) is a surjective holomorphic local diffeomorphism.

Proof. Let $(\xi, \eta) \in \mathcal{S}_{\text{reg}}$ and let

$$X(\xi, \eta) = \eta \frac{\partial}{\partial \xi} + \frac{n-n_0}{n} \frac{\xi^{n-n_0-1}(1-\xi)^{n-n_1-1}}{\eta^{n-2}} \frac{\partial}{\partial \eta}. \hspace{1cm} (12)$$

By hypothesis $(\xi, \eta) \in \mathcal{S}_{\text{reg}}$ implies that $\eta \neq 0$. The vector $X(\xi, \eta)$ is defined and is nonzero. From $(X_{\xi} \partial g)(\xi, \eta) = 0$ and $T_{(\xi,\eta)}\mathcal{S}_{\text{reg}} = \ker dG(\xi, \eta)$, it follows that $X(\xi, \eta) \in T_{(\xi,\eta)}\mathcal{S}_{\text{reg}}$. Using the definition of $X(\xi, \eta)$ (12) and the definition of the mapping $\pi$ (7), we see that the tangent of the mapping $\widehat{\pi}$ (11) at $(\xi, \eta) \in \mathcal{S}_{\text{reg}}$ is given by

$$T_{(\xi,\eta)}\widehat{\pi} : T_{(\xi,\eta)}\mathcal{S}_{\text{reg}} \rightarrow T_{\xi}(\mathbb{C} \setminus \{0, 1\}) = \mathbb{C} : X(\xi, \eta) \mapsto \eta \frac{\partial}{\partial \xi}. \hspace{1cm} (13)$$

Since $X(\xi, \eta)$ and $\eta \frac{\partial}{\partial \xi}$ are nonzero vectors, they form a complex basis for $T_{(\xi,\eta)}\mathcal{S}_{\text{reg}}$ and $T_{\xi}(\mathbb{C} \setminus \{0, 1\})$, respectively. Thus, the complex linear mapping $T_{(\xi,\eta)}\widehat{\pi}$ is an isomorphism. Hence $\widehat{\pi}$ is a local holomorphic diffeomorphism.  \hfill $\square$

Corollary 3. $\widehat{\pi}$ (11) is a surjective holomorphic n to 1 covering map.

Proof. We only need to show that $\widehat{\pi}$ is a covering map. First we note that every fiber of $\widehat{\pi}$ is a finite set with $n$ elements, since for each fixed $\xi \in \mathbb{C} \setminus \{0, 1\}$ we have $\widehat{\pi}^{-1}(\xi) = \{\xi, \eta \in \mathcal{S}_{\text{reg}}|\eta = \omega_k(\xi^{n-n_0}(1 - \xi)^{n-n_1})^{1/n}\}$. Here $\omega_k$ for $k = 0, 1, \ldots, n-1$, is an $n$th root of 1 and $\xi^{1/n}$ is the complex $n$th root used in the definition of the Schwarz–Christoffel map $F_{Q}(\xi)$. Hence the map $\widehat{\pi}$ is a proper surjective holomorphic submersion, because each fiber is compact. Thus, the mapping $\widehat{\pi}$ is a presentation of a locally trivial fiber bundle with fiber consisting of $n$ distinct points. In other words, the map $\widehat{\pi}$ is a $n$ to 1 covering mapping.  \hfill $\square$
Consider the group $\hat{G}$ of linear transformations of $\mathbb{C}^2$ generated by
\[
R : \mathbb{C}^2 \to \mathbb{C}^2 : (\xi, \eta) \mapsto (\xi, e^{2\pi i/n} \eta).
\]

Clearly $R^n = \text{id}_{\mathbb{C}^2} = e$, the identity element of $\hat{G}$ and $\hat{G} = \{e, R, \ldots, R^{n-1}\}$. For each $(\xi, \eta) \in S$ we have
\[
(e^{2\pi i/n} \eta)^n - \xi^n - n_0 (1 - \xi)^{n-n_1} = \eta^n - \xi^n - n_0 (1 - \xi)^{n-n_1} = 0.
\]

So $R(\xi, \eta) \in S$. Thus, we have an action of $\hat{G}$ on the affine Riemann surface $S$ given by
\[
\Phi : \hat{G} \times S \to S : (g, (\xi, \eta)) \mapsto g(\xi, \eta). \tag{14}
\]

Since $\hat{G}$ is finite, and hence is compact, the action $\Phi$ is proper. For every $g \in \hat{G}$ we have $\Phi_\xi(0,0) = (0,0)$ and $\Phi_\xi(1,0) = (1,0)$. So $\Phi_\xi$ maps $S_{\text{reg}}$ into itself. At $(\xi, \eta) \in S_{\text{reg}}$ the isotropy group $\hat{G}(\xi, \eta)$ is $\{e\}$, that is, the $\hat{G}$-action $\Phi$ on $S_{\text{reg}}$ is free. Thus, the orbit space $S_{\text{reg}} / \hat{G}$ is a complex manifold.

**Corollary 4.** Consider the holomorphic mapping
\[
\rho : S_{\text{reg}} \subseteq \mathbb{C}^2 \to S_{\text{reg}} / \hat{G} \subseteq \mathbb{C}^2 : (\xi, \eta) \mapsto [(\xi, \eta)],
\]
where $[(\xi, \eta)]$ is the $\hat{G}$-orbit $\Phi_\xi(\xi, \eta) \in S_{\text{reg}} | \xi \in \hat{G}$ of $(\xi, \eta)$ in $S_{\text{reg}}$. The $\hat{G}$ principal bundle presented by the mapping $\rho$ is isomorphic to the bundle presented by the mapping $\hat{\pi}$ (11).

**Proof.** We use invariant theory to determine the orbit space $S / \hat{G}$. The algebra of polynomials on $\mathbb{C}^2$, which are invariant under the $\hat{G}$-action $\Phi$, is generated by $\pi_1 = \xi$ and $\pi_2 = \eta^n$. Since $(\xi, \eta) \in S$, these polynomials are subject to the relation
\[
\pi_2 - \pi_1^{n-n_0} (1 - \pi_1)^{n-n_1} = \eta^n - \xi^n - n_0 (1 - \xi)^{n-n_1} = 0. \tag{15}
\]

Equation (15) defines the orbit space $S / \hat{G}$ as a complex subvariety of $\mathbb{C}^2$. This subvariety is homeomorphic to $\mathbb{C}$, because it is the graph of the function $\pi_1 \mapsto \pi_1^{n-n_0} (1 - \pi_1)^{n-n_1}$. Consequently, the orbit space $S_{\text{reg}} / \hat{G}$ is holomorphically diffeomorphic to $\mathbb{C} \setminus \{0,1\}$.

It remains to show that $\hat{G}$ is the group of covering transformations of the bundle presented by the mapping $\hat{\pi}$ (11). For each $\xi \in \mathbb{C} \setminus \{0,1\}$ look at the fiber $\hat{\pi}^{-1}(\xi)$. If $(\xi, \eta) \in \hat{\pi}^{-1}(\xi)$, then $R^{\pm 1}(\xi, \eta) = (\xi, e^{\pm 2\pi i/n} \eta) \in S_{\text{reg}}$, since $(\xi, e^{\pm 2\pi i/n} \eta) \neq (0,0)$ or $(1,0)$ and $(\xi, e^{2\pi i/n} \eta) \in S$. Thus, $\Phi_{R^{\pm 1}}(\hat{\pi}^{-1}(\xi)) \subseteq \hat{\pi}^{-1}(\xi)$. So $\hat{\pi}^{-1}(\xi) \subseteq \Phi_R(\hat{\pi}^{-1}(\xi)) \subseteq \hat{\pi}^{-1}(\xi)$. Hence $\Phi_R(\hat{\pi}^{-1}(\xi)) = \hat{\pi}^{-1}(\xi)$. Thus, $\Phi_R$ is a covering transformation for the bundle presented by the mapping $\hat{\pi}$. So $\hat{G}$ is a subgroup of the group of covering transformations. These groups are equal because $G$ acts transitively on each fiber of the mapping $\hat{\pi}$. ⧫

**4. Another Model for $S_{\text{reg}}$**

We construct another model $\hat{S}_{\text{reg}}$ for the smooth part $S_{\text{reg}}$ of the affine Riemann surface $S$ (3) as follows. Let $D \subseteq S_{\text{reg}}$ be a fundamental domain for the $\hat{G}$ action $\Phi$ (14) on $S_{\text{reg}}$. So $(\xi, \eta) \in D$ if and only if for $\xi \in \mathbb{C} \setminus \{0,1\}$ we have $\eta = (\xi^n - n_0 (1 - \xi)^{n-n_1})^{1/n}$. Here $\sqrt[n]{\cdot}$ is the $n$th root used in the definition of the mapping $F_Q$ (2). The domain $D$ is a connected subset of $S_{\text{reg}}$ with nonempty interior. Its image under the map $\bar{\pi}$ (11) is $\mathbb{C} \setminus \{0,1\}$. Thus, $D$ is one “sheet” of the covering map $\bar{\pi}$. So $\bar{\pi}|_D$ is one to one.

Using the extended Schwarz–Christoffel mapping $F_Q$ (2), we give a more geometric description of the fundamental domain $D$. Consider the mapping
\[
\delta : D \subseteq S_{\text{reg}} \to Q \subseteq \mathbb{C} : (\xi, \eta) \mapsto F_Q(\bar{\pi}(\xi, \eta)), \tag{16}
\]
where the map $\pi$ is given by Equation (11). The map $\delta$ is a holomorphic diffeomorphism of $\text{int} \, D$ onto $\text{int} \, Q$, which sends $\partial D$ homeomorphically onto $\partial Q$. Look at $\text{cl}(Q)$, which is a closed quadrilateral with vertices $O$, $D$, $C$, and $\overline{D}$. The set $\delta(D)$ contains the open edges $OD$, $DC$, and $CD$ but not the open edge $OD$ of $\text{cl}(Q)$, see Figure 3 above.

![Figure 3](image)

**Figure 3.** The image $Q$ of the fundamental domain $D$ under the mapping $\delta$. The open edges $OD$, $CD$, and $CD$ of the quadrilateral are included; while the open edge $OD$ of $\text{cl}(Q)$ is excluded.

Let $K^* = K_{n_0,n_1,n_\infty} = \bigcup_{0 \leq j \leq n-1} R^j(\delta(D))$ be the region in $\mathbb{C}$ formed by repeatedly rotating $Q = \delta(D)$ through an angle $2\pi/n$. Here $R$ is the rotation $z \mapsto e^{2\pi i/n} z$. We say that the quadrilateral $Q = Q_{2n_0,n_\infty,2n_1,n_\infty}$ forms $K^*$, see Figure 4 above.

![Figure 4](image)

**Figure 4.** The regular duodecagon $K$ and the stellated regular duodecagon $K^* = K^*_{4,4,4}$ formed by rotating the quadrilateral $Q_{4,4,4}$ through an angle $2\pi/12$ around the origin.

**Theorem 1.** The connected set $K^*$ is a regular stellated $n$-gon with its $2n$ vertices omitted, which is formed from the quadrilateral $Q' = OD'CD'$, see Figure 5.

![Figure 5](image)

**Figure 5.** The dart in the figure is the quadrilateral $Q' = OD'CD'$, which is the union of the triangles $T = \Delta OD'C$ and the triangle $T'$.

**Proof.** By construction the quadrilateral $Q' = OD'CD'$ is contained in the quadrilateral $Q = ODC\overline{D}$. Note that $Q \subseteq \bigcup_{j=-n_1+1}^{n-1} R^j(Q')$. Thus,

$$K^* = \bigcup_{j=0}^{n} R^j(Q) \subseteq \bigcup_{j=0}^{n} R^j(Q') \subseteq \bigcup_{j=0}^{n} R^j(Q) = K^*.$$
So $K^*=\bigcup_{j=0}^r R^j(Q')$. Thus, $K^*$ is the regular stellated $n$-gon less its vertices, one of whose open sides is the diagonal $D^TD'$ of $Q'$. □

We would like to extend the mapping $\delta$ (16) to a mapping of $S_{\text{reg}}$ onto $K^*$. Let

$$\delta_{\Phi_{\gamma}} : \Phi_{\gamma}(D) \subseteq S_{\text{reg}} \to R^J(\delta(D)) \subseteq K^* : (\xi, \eta) \mapsto R^J(\Phi_{\gamma}(\xi, \eta)),$$

where $\Phi$ is the $\tilde{G}$ action defined in Equation (14). So we have a mapping

$$\delta_{K^*} : S_{\text{reg}} \subseteq \mathbb{C}^2 \to K^* \subseteq \mathbb{C}$$

(17)
defined by $(\delta_{K^*})|_{\Phi_{\gamma}(D)} = \delta|_{\Phi_{\gamma}(D)}$. The mapping $\delta_{K^*}$ is defined on $S_{\text{reg}}$, because $S_{\text{reg}} = \Pi_{0 \leq j \leq n-1} \Phi_{\gamma}(D)$, since $D$ is a fundamental domain for the $\tilde{G}$-action $\Phi$ (14) on $S_{\text{reg}}$. Because $K^* = \Pi_{0 \leq j \leq n-1} R^j(\delta(D))$, the mapping $\delta_{K^*}$ is surjective. Hence $\delta_{K^*}$ is holomorphic, since it is continuous on $S_{\text{reg}}$ and is holomorphic on the dense open subset $\Pi_{0 \leq j \leq n-1} R^j(\text{int } D)$ of $S_{\text{reg}}$. Let $U : \mathbb{C} \to \mathbb{C} : z \mapsto \zeta$ and let $G$ be the group generated by the rotation $R$ and the reflection $U$ subject to the relations $R^n = U^2 = \varepsilon$ and $RU = UR^{-1}$. Shorthand $G = \langle U, R \rangle$, $R^p = e = R^n$, and $RU = UR^{-1}$. Then $G = \{ e, R^p U^\ell, \ell = 0, 1 & p = 0, 1, \ldots, n-1 \}$. The group $G$ is the dihedral group $D_{2n}$. The closure $\text{cl}(K^*)$ of $K^* = \Pi_{0 \leq j \leq n-1} R^j(Q)$ in $\mathbb{C}$ is invariant under $\tilde{G}$, the subgroup of $G$ generated by the rotation $R$. Because the quadrilateral $Q$ is invariant under the reflection $U : z \mapsto \zeta$, and $UR^j = R^{-j}U$, it follows that $\text{cl}(K^*)$ is invariant under the reflection $U$. So $\text{cl}(K^*)$ is invariant under the group $G$.

We now look at some group theoretic properties of $K^*$.

**Lemma 3.** If $F$ is a closed edge of the polygon $\text{cl}(K^*)$ and $g|_F = \text{id}|_F$ for some $g \in G$, then $g = e$.

**Proof.** Suppose that $g \neq e$. Then $g = R^p U^\ell$ for some $\ell \in \{ 0, 1 \}$ and some $p \in \{ 0, 1, \ldots, n-1 \}$. Let $g = R^p U$ and suppose that $F$ is an edge of $\text{cl}(K^*)$ such that $\text{int}(F) \cap \mathbb{R} \neq \emptyset$, where $\mathbb{R} = \{ \text{Re } z \mid z \in \mathbb{C} \}$. Then $U(F) = F$, but $U|_F \neq \text{id}|_F$. So $g|_F = R^p U|_F \neq \text{id}|_F$. Now suppose that $\text{int}(F) \cap \mathbb{R} = \emptyset$. Then $U(F) \neq F$. So $U|_F \neq \text{id}|_F$. Hence $g|_F \neq \text{id}|_F$. Finally, suppose that $g = R^p$ with $p \neq 0$. Then $g(F) \neq F$. So $g|_F \neq \text{id}|_F$. □

**Lemma 4.** For $j = 0, 1, \infty$ put $S(i) = R^j U$. Then $S(i)$ is a reflection in the closed ray $\ell^i = \{ te^{i \pi n / n} \mid t \in \mathbb{C} \} \cap \mathbb{C}$. The ray $\ell^0$ is the closure of the side $OD$ of the quadrilateral $Q = ODCD$ in Figure 5.

**Proof.** $S(i)$ fixes every point on the closed ray $\ell^i$, because

$$S(i) \{ te^{i \pi n / n} \mid t \in \mathbb{C} \} = R^i \{ te^{-i \pi n / n} \mid t \in \mathbb{C} \} = \{ te^{i \pi n / n} \mid t \in \mathbb{C} \}.$$

Since $(S(i))^2 = (R^j U)(R^j U) = R^0 (LUU) = e$, it follows that $S(i)$ is a reflection in the closed ray $\ell^i$. □

**Corollary 5.** For every $j = 0, 1, \infty$ and every $k \in \{ 0, 1, \ldots, n-1 \}$ let $S(k)^i = R^k S(i) R^{-k}$. Here $S(k)^i = S(k)^i = S(i)$, because $R^n = e$. Then $S(k)^i$ is a reflection in the closed ray $R^k \ell^i$.

**Proof.** This follows because $(S(k)^i)^2 = R^k (S(i))^2 R^{-k} = e$ and $S(k)^i$ fixes every point on the closed ray $R^k \ell^i$, for

$$S(k)^i (R^k \{ te^{i \pi n / n} \mid t \in \mathbb{C} \}) = R^k S(i) \{ te^{i \pi n / n} \mid t \in \mathbb{C} \}$$

$$= R^k \{ te^{i \pi n / n} \mid t \in \mathbb{C} \}.$$ □
Corollary 6. For every \( j = 0, 1, \infty \), every \( S^{(i)}_k \) with \( k = 0, 1, \ldots, n-1 \), and every \( g \in G \), we have \( gS^{(i)}_kg^{-1} = S^{(i)}_{k+1} \) for a unique \( r \in \{0, 1, \ldots, n-1\} \).

**Proof.** We compute. For every \( k = 0, 1, \ldots, n-1 \) we have

\[
RS^{(i)}_k R^{-1} = R(R^kS^{(i)}_k R^{-k})R^{-1} = R^{(k+1)}S^{(i)}_k R^{-k+1} = S^{(i)}_{k+1} \tag{18}
\]

and

\[
US^{(i)}_k U^{-1} = U(R^{(k+n)}UR^{-(k+n)})U = R^{-(k+n)}UR^{(k+n)} = S^{(i)}_{(k+2n_j)} = S^{(i)}_j, \tag{19}
\]

where \( t = -(k + 2n_j) \) mod \( n \). Since \( R \) and \( U \) generate the group \( G \), the corollary follows. \( \square \)

Corollary 7. For \( j = 0, 1, \infty \) let \( G_j \) be the group generated by the reflections \( S^{(i)}_k \) for \( k = 0, 1, \ldots, n-1 \). Then \( G_j \) is a normal subgroup of \( G \).

**Proof.** Clearly \( G_j \) is a subgroup of \( G \). From Equations (18) and (19) it follows that \( gS^{(i)}_kg^{-1} \in G_j \) for every \( g \in G \) and every \( k = 0, 1, \ldots, n-1 \), since \( G \) is generated by \( R \) and \( U \). However, \( G_j \) is generated by the reflections \( S^{(i)}_k \) for \( k = 0, 1, \ldots, n-1 \), that is, every \( g' \in G_j \) may be written as \( S^{(i)}_{i_1} \cdots S^{(i)}_{i_p} \), where for \( \ell \in \{1, \ldots, p\} \) we have \( i_\ell \in \{0, 1, \ldots, n-1\} \). So \( gS^{(i)}_kg^{-1} = gS^{(i)}_{i_1} \cdots S^{(i)}_{i_p}g^{-1} = (gS^{(i)}_{i_1}g^{-1}) \cdots (gS^{(i)}_{i_p}g^{-1}) \in G_j \) for every \( g \in G \), that is, \( G_j \) is a normal subgroup of \( G \). \( \square \)

As a first step in constructing the model \( \tilde{S}_{\text{reg}} \) of \( S_{\text{reg}} \) from the regular stellated \( n \)-gon \( K^* \) we look at certain pairs of edges of \( \text{cl}(K^*) \). For each \( j = 0, 1, \infty \), we say two distinct closed edges \( E \) and \( E' \) of \( \text{cl}(K^*) \) are adjacent if and only if they intersect at a vertex of \( \text{cl}(K^*) \). For \( j = 0, 1, \infty \) let \( E_j \) be the set of unordered pairs of equivalent closed edges \( E \) and \( E' \) of \( \text{cl}(K^*) \), that is, the edges \( E \) and \( E' \) are not adjacent and \( E' = S^{(i)}_m(E) \) for some generator \( S^{(i)}_m \) of \( G_j \). Recall that for \( x \) and \( y \) in some set, the unordered pair \( [x, y] \) is precisely one of the ordered pairs \( (x, y) \) or \( (y, x) \). Note that \( \bigcup_{j=0,1,\infty} E_j \) is the set of all unordered pairs of nonadjacent edges of \( \text{cl}(K^*) \). Geometrically, two nonadjacent closed edges \( E' \) and \( E \) of \( \text{cl}(K^*) \) are equivalent if and only if \( E' \) is obtained from \( E \) by reflection in the line \( R^mE \) for some \( m \in \{0, 1, \ldots, n-1\} \) and some \( j = 0, 1, \infty \). In Figure 6, where \( K^* = K^*_1, A^* \), parallel edges of \( K^* \), which are labeled with the same letter, are \( G^\infty \)-equivalent. This is no coincidence.

![Figure 6](image-url)

**Figure 6.** The triangulation \( T_{ij}(K^*) \) of the regular stellated hexagon \( K^* \). The vertices of \( \text{cl}(K^*) \) are labeled \( X_i = R^iX \) for \( X = A, B, C \) and equivalent edges by \( a, b, c, d, e, f \).
**Theorem 2.** Let $K^*$ be formed from the quadrilateral $Q = T \cup \overline{T}$, where $T$ is the isosceles rational triangle $T_{\text{right}}$, less its vertices. Then nonadjacent edges of $\partial \text{cl}(K^*)$ are $G^0$-equivalent if and only if they are parallel, see Figure 7.

**Proof.** In Figure 7, let $OAB$ be the triangle $T$ with $\angle AOB = \alpha$, $\angle OAB = \beta$, and $\angle ABO = \gamma$. Let $OABA''$ be the quadrilateral formed by reflecting the triangle $OAB$ in its edge $OB$. The quadrilateral $OABA''$ reflected it its edge $OA$ is the quadrilateral $OAB'A'$. Let $AC'$ be perpendicular to $A'B'$ and $AC$ be perpendicular to $A''B'$, see Figure 7. Then $CAC'$ is a straight line if and only if $\angle C'AB' + \angle B'AB + \angle BAC = \pi$. By construction $\angle C'AB' = \angle BAC = \pi/2 - 2\gamma$ and $\angle B'AB = 2\pi - 2\beta$. So

$$
\pi = 2(\frac{\pi}{2} - 2\gamma) + 2(\pi - \beta) = 3\pi - 2(\beta + \gamma) - 2\gamma
$$

$$
= 3\pi - 2(\alpha + \beta + \gamma) + 2(\alpha - \gamma) = \pi + 2(\alpha - \gamma),
$$

if and only if $\alpha = \gamma$. Hence the edges $A''B$ and $A'B'$ are parallel if and only if the triangle $OAB$ is isosceles. \(\Box\)

![Figure 7. The geometric configuration.](image_placeholder)

**Theorem 2.** Let $K^*$ be the regular stellated $n$-gon formed from the rational quadrilateral $Q_{d_0 \ldots d_n}$ with $d_j = \gcd(n_j, n)$ for $j = 0, 1, \ldots, 5$. The $G$ orbit space formed by first identifying equivalent edges of the regular stellated $n$-gon $K^*$ formed from $Q$ less $O$ and then acting on the identification space by the group $G$ is $\tilde{S}_{\text{reg}}$ which is a smooth 2-sphere with $g$ handles, where $g = \frac{1}{2}(n + 2 - (d_0 + d_1 + d_\infty))$, less some points corresponding to the image of the vertices of $\text{cl}(K^*)$.

**Example 1.** Before we begin proving Theorem 2 we consider the following special case. Let $K^* = K^*_{1,1,4}$ be a regular stellated hexagon formed by repeatedly rotating the quadrilateral $Q' = OD'C'D'$ by $R$ through an angle $2\pi/6$, see Figure 6.

Let $G^0$ be the group generated by the reflections $S_k^{(0)} = R^k S^{(0)} R^{-k} = R^{2k+1} U$ for $k = 0, 1, \ldots, 5$. Here $S^{(0)} = RU$ is the reflection which leaves the closed ray $\ell^0 = \{te^{i\phi}/6 \mid t \in OD'\}$ fixed. Define an equivalence relation on $\text{cl}(K^*)$ by saying that two points $x$ and $y$ in $\text{cl}(K^*)$ are equivalent, $x \sim y$, if and only if 1) $x$ and $y$ lie on $\partial \text{cl}(K^*)$ with $x$ on the closed edge $E$ and $y = S_m^{(0)}(x) \in S_m^{(0)}(E)$ for some reflection $S_m^{(0)} \in G^0$ or 2) if $x$ and $y$ lie in the interior of $\text{cl}(K^*)$ and $x \sim y$. Let $\text{cl}(K^*)$ be the space of equivalence classes and let

$$
\rho : \text{cl}(K^*) \rightarrow \text{cl}(K^*)^\sim : p \mapsto [p]
$$

be the identification map which sends a point $p \in \text{cl}(K^*)$ to the equivalence class $[p]$, which contains $p$. Give $\text{cl}(K^*)$ the topology induced from $\mathbb{C}$. Placing the quotient topology on $\text{cl}(K^*)$ turns it into a connected topological manifold without boundary, whose closure is compact. Let $K^*$ be $\text{cl}(K^*)$ less its vertices. The identification space $(K^* \setminus O)^\sim = \rho(K^* \setminus O)$ is a connected 2-dimensional smooth manifold without boundary.
Let $G = \langle R, U \mid R^6 = e = U^2 \& RU = UR^{-1} \rangle$. The usual $G$-action

$$G \times \text{cl}(K^*) \subseteq G \times C \to \text{cl}(K^*) \subseteq C : (g, z) \mapsto g(z)$$

preserves equivalent edges of $\text{cl}(K^*)$ and is free on $K^* \setminus O$. Hence it induces a $G$ action on $(K^* \setminus O)^\sim$, which is free and proper. Thus, its orbit map

$$\sigma : (K^* \setminus O)^\sim \to (K^* \setminus O)^\sim / G = \tilde{S}_{\text{reg}} : z \mapsto zG$$

is surjective, smooth, and open. The orbit space $\tilde{S}_{\text{reg}} = \sigma((K^* \setminus O)^\sim)$ is a connected 2-dimensional smooth manifold. The identification space $(K^* \setminus O)^\sim$ has the orientation induced from an orientation of $K^* \setminus O$, which comes from $C$. So $\tilde{S}_{\text{reg}}$ has a complex structure, since each element of $G$ is a conformal mapping of $C$ into itself.

Our aim is to specify the topology of $\tilde{S}_{\text{reg}}$. The regular stellated hexagon $K^* \setminus O$ less the origin has a triangulation $\mathcal{T}_{K^* \setminus O}$ made up of 12 open triangles $R'(\triangle OCD')$ and $R'(\triangle OMC')$ for $j = 0, 1, \ldots, 5$; 24 open edges $R'(OC)$, $R'(CD')$, and $R'(CD)$ for $j = 0, 1, \ldots, 5$; and 12 vertices $R'(D')$ and $R'(C)$ for $j = 0, 1, \ldots, 5$, see Figure 6. Consider the set $E^0$ of unordered pairs of equivalent closed edges of $\text{cl}(K^*)$, that is, $E^0$ is the set $[E, S_k^{(0)}(E)]$ for $k = 0, 1, \ldots, 5$, where $E$ is a closed edge of $\text{cl}(K^*)$. Table 2 lists the elements of $E^0$. $G$ acts on $E^0$, namely, $g : [E, S_k^{(0)}(E)] = [g(E), gS_k^{(0)}g^{-1}(g(E))]$, for $g \in G$. Since $G^0$ is the group generated by the reflections $S_k^{(0)}$, $k = 0, 1, \ldots, 5$, it is a normal subgroup of $G$. Hence the action of $G$ on $E^0$ restricts to an action of $G^0$ on $E^0$ and the $G$ action permutes $G^0$-orbits in $E^0$. Thus, the set of $G^0$-orbits in $E^0$ is $G$-invariant.

**Table 2.** The set $E^0$. Here $D'_k = R^k(D')$ and $\overline{D}'_k = R^k(\overline{D}')$ for $k = 0, 2, 4$ and $C_k = R^k(C)$ for $k = 0, 1, \ldots, 5$, see Figure 6.

| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|-----|-----|-----|-----|-----|-----|
| $[\overline{D}'_2C_1, S_0^{(0)}(\overline{D}'C)] = \overline{D}'_2C_1$ | $[D'_2C_1, S_0^{(0)}(D'C)] = D'_2C_2$ | $[D'_2C_2, S_0^{(0)}(D'_4C_2)] = D'_2C_3$ | $[\overline{D}'_2C_2, S_0^{(0)}(\overline{D}'_2C)] = \overline{D}'_2C_2$ | $[\overline{D}'_2C_4, S_0^{(0)}(\overline{D}'_2C_4)] = \overline{D}'_2C_5$ | $[D'_2C_3, S_0^{(0)}(D'_4C_3)] = D'_2C_4$ |

We now look at the $G^0$-orbits on $E^0$. We compute the $G^0$-orbit of $d \in E^0$ as follows. We have

$$(UR) \cdot d = [UR(\overline{D}'_2C_2), UR(S_2^{(0)}(\overline{D}'_2C))] = [UR(\overline{D}'_2C_2), UR(\overline{D}'_2C_3)]$$

$$= [U(D'_2C_2), U(D'_4C_4)] = [\overline{D}'_2C_5, \overline{D}'_2C_2] = d.$$

Since

$$R^2 \cdot d = R^2 \cdot [\overline{D}'_2C_2, S_2^{(0)}(\overline{D}'_2C)] = [R^2(\overline{D}'_2C_2), R^2S_2^{(0)}R^{-2}(R^2(\overline{D}'_2C))]$$

$$= [\overline{D}'_2C_4, S_4^{(0)}(\overline{D}'_2C_4)] = [\overline{D}'_2C_4, \overline{D}'_2C_5] = e$$

and

$$R^4 \cdot d = [R^4(\overline{D}'_2C_2), R^4S_2^{(0)}R^{-4}(R^4(\overline{D}'_2C))]$$

$$= [\overline{D}'_2C_6, S_0^{(0)}(\overline{D}'C)] = [\overline{D}'_2C_6, S_0^{(0)}(\overline{D}'C)] = [\overline{D}'_2C_6, \overline{D}'_2C_1] = a,$$

the $G^0$ orbit $G^0 \cdot d$ of $d \in E^0$ is $(G^0/\langle UR \rangle (UR)^2 = e) \cdot d = H^0 \cdot d = \{a, d, e\}$. Here $H^0 = \langle V = R^2 \mid V^3 = e \rangle$, since $G^0 = \langle V = R^2, UR \mid V^3 = e = (UR)^2 \& V(UR) = (UR)V^{-1} \rangle$. Similarly, the $G^0$-orbit $G^0 \cdot f$ of $f \in E^0$ is $H^0 \cdot f = \{b, c, f\}$. Since $G^0 \cdot a \cup G^0 \cdot f = E^0$, we have found all $G^0$-orbits on $E^0$. The $G$-orbit of $OC$ is $R'(OC)$ for $j = 0, 1, \ldots, 5$, since
Suppose that $B$ is an end point of the closed edge $E$ of $\text{cl}(K^*)$. Then $E$ lies in a unique $[E, S^m_{\sigma_m}(E)]$ of $\mathcal{E}^0$. Let $G^0 \cdot [E, S^m_{\sigma_m}(E)]$ be the $G^0$-orbit of $[E, S^m_{\sigma_m}(E)]$. Then $g' \cdot B$ is an end point of the closed edge $g'(E)$ of $g' \cdot [E, S^m_{\sigma_m}(E)] \in \mathcal{E}^0$ for every $g' \in G^0$. So $\mathcal{O}(B) = \{g' \cdot B | g' \in G^0\}$ the $G^0$-orbit of the vertex $B$. It follows from the classification of $G^0$-orbits on $\mathcal{E}^0$ that $\mathcal{O}(E') = \{E', D'_{\sigma}, D'_1\}$ and $\mathcal{O}(\overline{E}) = \{\overline{D}, \overline{D'_2}, \overline{D'_4}\}$ are $G^0$-orbits of the vertices of $\text{cl}(K^*)$, which are permuted by the action of $G$ on $\mathcal{E}^0$. Furthermore, $\mathcal{O}(C) = \{C, C_1, \ldots, C_5\}$ and $\mathcal{O}(\text{D} \& \overline{D}) = \{\text{D}, \text{D'}_1, \text{D'}_2, \text{D'}_4\}$ are $G$-orbits of vertices of $\text{cl}(K^*)$, which are end points of the $G$-orbit of the rays $OC$ and $OD'$, respectively.

To determine the topology of the $G$ orbit space $\tilde{S}_{\text{reg}}$ we find a triangulation of $\tilde{S}_{\text{reg}}$. Note that the triangulation $\mathcal{T}_{K^* \setminus O}$ of $K^* \setminus O$, illustrated in Figure 6, is $G$-invariant. Its image under the identification map $\rho$ is a $G$-invariant triangulation $\mathcal{T}_{(K^* \setminus O)^\sim}$ of $(K^* \setminus O)^\sim$. After identification of equivalent edges, each vertex $\rho(v)$, each open edge $\rho(E)$, having $\rho(O)$ as an end point, or each open edge $\rho([F, F'])$, where $[F, F']$ is a pair of equivalent edges of $\text{cl}(K^*)$, and each open triangle $\rho(T)$ in $\mathcal{T}_{(K^* \setminus O)^\sim}$ lies in a unique $G$ orbit. It follows that $\sigma(\rho(v))$, $\sigma(\rho(E))$ or $\sigma(\rho([F, F']))$, and $\sigma(\rho(T))$ is a vertex, an open edge, and an open triangle, respectively, of a triangulation $\mathcal{T}_{\tilde{S}_{\text{reg}}} = \sigma(\mathcal{T}_{(K^* \setminus O)^\sim})$ of $\tilde{S}_{\text{reg}}$. The triangulation $\mathcal{T}_{\tilde{S}_{\text{reg}}}$ has 4 vertices, corresponding to the $G$ orbits $\sigma(\rho(\text{D}'))$, $\sigma(\rho(\overline{\text{D}}))$, $\sigma(\rho(O(C)))$, and $\sigma(\rho(\text{D} \& \overline{\text{D}})))$; 18 open edges corresponding to $\sigma(\rho(R(OC)))$, $\sigma(\rho(R(\text{OD})))$, and $\sigma(\rho(R(\text{CD})))$ for $j = 0, 1, \ldots, 5$; and 12 open triangles $\sigma(\rho(R(\triangle OCD)))$ and $\sigma(\rho(R(\triangle OCD')))$. Thus, the Euler characteristic $\chi(\tilde{S}_{\text{reg}})$ of $\tilde{S}_{\text{reg}}$ is $4 - 18 + 12 = -2$. Since $\tilde{S}_{\text{reg}}$ is a 2-dimensional smooth real manifold, $\chi(\tilde{S}_{\text{reg}}) = 2 - 2g$, where $g$ is the genus of $\tilde{S}_{\text{reg}}$. Hence $g = 2$. So $\tilde{S}_{\text{reg}}$ is a smooth 2-sphere with 2 handles, less a finite number of points, which lies in a compact topological space $\tilde{S} = \text{cl}(K^*)^\sim / G$, that is its closure, see Figure 8. This completes the example.

Figure 8. The G-orbit space $\tilde{S}_{\text{reg}}$ is 2-sphere with two handles.

Proof of Theorem 2. We now begin the construction of $\tilde{S}_{\text{reg}}$ by identifying equivalent edges of $\text{cl}(K^*)$. For each $j = 0, 1, \infty$ let $[E, S^m_{\sigma_m}(E)]$ be an unordered pair of equivalent closed edges of $\text{cl}(K^*)$. We say that $x$ and $y$ in $\text{cl}(K^*)$ are equivalent, $x \sim y$, if 1) $x$ and $y$ lie in $\partial \text{cl}(K^*)$ with $x \in E$ and $y = S^m_{\sigma_m}(x) \in S^m_{\sigma_m}(E)$ for some $m \in \{0, 1, \ldots, n - 1\}$ and some $j = 0, 1, \infty$ or 2) $x$ and $y$ lie in $\text{int} \text{cl}(K^*)$ and $x = y$. The relation $\sim$ is an equivalence relation on $\text{cl}(K^*)$. Let $\text{cl}(K^*)^\sim$ be the set of equivalence classes and let

$$\rho : \text{cl}(K^*) \to \text{cl}(K^*)^\sim : p \mapsto [p]$$

be the map which sends $p$ to the equivalence class $[p]$, that contains $p$. Compare this argument with that of Richens and Berry [2]. Give $\text{cl}(K^*)$ the topology induced from $\mathbb{C}$ and put the quotient topology on $\text{cl}(K^*)^\sim$. □

Theorem 3. Let $K^*$ be $\text{cl}(K^*)$ less its vertices. Then $(K^* \setminus O)^\sim = \rho(K^* \setminus O)$ is a smooth manifold. Furthermore, $\text{cl}(K^*)^\sim$ is a topological manifold.

Proof. To show that $(K^* \setminus O)^\sim$ is a smooth manifold, let $E_+ \text{ be an open edge of } K^*$. For $p_+ \in E_+$ let $D_{p_+}$ be a disk in $\mathbb{C}$ with center at $p_+$, which does not contain a vertex of
cl(\(K^*\)). Set \(D^+_{p_+} = K^* \cap D_{p_+}\). For each \(j = 0, 1, \infty\) let \(E_+\) be an open edge of \(K^*\), which is equivalent to \(E_+\) via the reflection \(S_m^{(j)}\), that is, \(\{cl(E_+), cl(E_-) = S_m^{(j)}(cl(E_+))\} \in E\) is an unordered pair of \(S_m^{(j)}\) equivalent edges. Let \(p_- = S_m^{(j)}(p_+)\) and set \(D^-_{p_-} = S_m^{(j)}(D^+_{p_+})\). Then \(V[p] = \rho(D^+_{p_+} \cup D^-_{p_-})\) is an open neighborhood of \([p] = [p_+] = [p_-]\) in \((K^* \setminus O)^-\), which is a smooth 2-disk, since the identification mapping \(\rho\) is the identity on int \(K^*\). It follows that \((K^* \setminus O)^-\) is a smooth 2-dimensional manifold without boundary.

We now handle the vertices of \(cl(K^*)\). Let \(v_+\) be a vertex of \(cl(K^*)\) and set \(D_{v_+} = \overline{D} \cap cl(K^*), \) where \(D\) is a disk in \(C\) with center at the vertex \(v_+\). Then \(\rho = re^{i\pi\theta}\). The map

\[
W_{v_+} : D_+ \subseteq C \to D_{v_+} \subseteq C : re^{i\pi\theta} \mapsto |r - r_0|e^{i\pi\theta} \theta
\]

with \(r \geq 0\) and \(0 \leq \theta \leq 1\) is a homeomorphism, which sends the wedge with angle \(\pi\) to the wedge with angle \(\pi r\). The latter wedge is formed by the closed edges \(E_+\) and \(E_-\) of \(cl(K^*)\), which are adjacent at the vertex \(v_+\) such that \(e^{i\pi\theta}E_+ = E_+\) with the edge \(E_+\) being swept out through int \(cl(K^*)\) during its rotation to the edge \(E_+\). Because \(cl(K^*)\) is a rational regular stellated \(n\)-gon, the value of \(n\) is a rational number for each vertex of \(cl(K^*)\). For each \(j = 0, 1, \infty\) let \(E_+ = S_m^{(j)}(E_+)\) be an edge of \(cl(K^*)\), which is equivalent to \(E_+\) and set \(v_+ = S_m^{(j)}(v_+)\). Then \(v_+\) is a vertex of \(cl(K^*)\), which is the center of the disk \(D_{v_+} = S_m^{(j)}(D_{v_+})\). Set \(D_- = D_{v_+} \cap D_{v_+}\). Then \(D = D_+ \cup D_-\) is a disk in \(C\). The map \(W : D \to \rho(D_{v_+} \cup D_{v_+})\), where \(W|D_+ = \rho\circ W_{v_+}\) and \(W|D_- = \rho\circ S_m^{(j)}\circ W_{v_+}\), is a homeomorphism of \(D\) into a neighborhood \(\rho(D_{v_+} \cup D_{v_+})\) of \([v] = [v_+] = [v_-]\) in \((K^*)^-\). Consequently, the identification space \(cl(K^*)^-\) is a topological manifold. \(\square\)

We now describe a triangulation of \(K^* \setminus O\). Let \(T' = T_{1/n_{1}, n_{1} - (1/n_{1})}\) be the open rational triangle \(\triangle OCD'\) with vertex at the origin \(O\), longest side \(OC\) on the real axis, and interior angles \(\frac{1}{n_{1}}\pi, \frac{n_{1}}{n_{1}}\pi\), and \(\frac{n_{1} - 1}{n_{1}}\pi\). Let \(Q'\) be the quadrilateral \(T' \cup \overline{T}\). Then \(Q'\) is a subset of the quadrilateral \(Q = ODC\overline{D}\) and is an edge of \((K^*)^-\). The 2n triangles \(cl(R'(T')) \setminus \{O\}\) and \(cl(R'(T')) \setminus O\) with \(k = 0, 1, \ldots, n - 1\) form a triangulation \(T_{K^* \setminus O}\) of \(K^* \setminus O\) with 2n vertices \(R^k(C)\) and \(R^k(D')\) for \(k = 0, 1, \ldots, n - 1\); for open edges \(R^k(OC), R^k(OD'), R^k(CD'),\) and \(R^k(CD)\) for \(k = 0, 1, \ldots, n - 1\); and 2n open triangles \(R^k(T'), R^k(T')\) with \(k = 0, 1, \ldots, n - 1\). The image of the triangulation \(T_{K^* \setminus O}\) under the identification map \(\rho(21)\) is a triangulation \(T_{(K^*)^-}\) of the identification space \((K^*)^-\).

The action of \(G\) on \(cl(K^*)\) preserves the set of unordered pairs of \(S_m^{(j)}\) equivalent edges of \(cl(K^*)\) for each \(j = 0, 1, \infty\). Hence \(G\) induces an action on \((K^*)^-\), which is proper, since \(G\) is finite. The \(G\) action is free on \(K^* \setminus O\) and thus on \((K^*)^-\) by Lemma A2. We have proved

**Lemma 6.** The \(G\)-orbit space \(\tilde{S} = cl(K^*)^- / G\) is a compact connected topological manifold with \(\tilde{S}_{reg} = (K^* \setminus O)^- / G\) being a smooth manifold. Let

\[
\sigma : cl(K^*)^- \to \tilde{S} = cl(K^*)^- / G : z \mapsto zG.
\]

Then \(\sigma\) is the \(G\) orbit map, which is surjective, continuous, and open. The restriction of \(\sigma\) to \(K^* \setminus O\) has image \(\tilde{S}_{reg}\) and is a smooth open mapping.

We now determine the topology of the orbit space \(\tilde{S}_{reg}\). For each \(j = 0, 1, \infty\) and \(\ell_j = 0, 1, \ldots, d_j - 1\) let \(A_{\ell_j}^{(j)}\) be an end point of a closed edge \(E\) of \(cl(K^*)\), which lies on the unordered pair \([E, S_m^{(j)}(E)] \in E\). Then \(H^j \cdot A_{\ell_j}^{(j)}\) is an end point of the edge \(H^j \cdot E\) of the unordered pair \([H^j \cdot E, S_m^{(j)}(E)] \in E\). See Appendix A for the definition of the group \(H_j\). The sets \(\set{A_{\ell_j}^{(j)}}\) with \(\ell_j = 0, 1, \ldots, d_j - 1\) are permuted by \(G\). The action of \(G\) on \(K^* \setminus O\) preserves the set of open edges of the triangulation \(T_{K^* \setminus O}\). There are
3n-orbits: $R^k(OC); R^k(OD')$, since $OD' = R(OD)$; and $R^k(CD)$, since $C\overline{D} = U(CD)$ for $k = 0, 1, \ldots, n - 1$. So the image of the triangulation $T_{K'^{-}\cup} \cup$ under the continuous open map

$$\mu = \sigma_0 \pi \mid_{K'^{-}\cup} : K' \setminus \cup \rightarrow \tilde{S}_{\text{reg}}$$

is a triangulation $T_{\tilde{S}_{\text{reg}}}$ of the $G$-orbit space $\tilde{S}_{\text{reg}}$ with $d_0 + d_1 + d_\infty$ vertices $\mu(\cup(A_{\ell_j}^{(j)}))$, where $j = 0, 1, \infty$ and $\ell_j = 0, 1, \ldots, d_j - 1; 3n$ open edges $\mu(R^k(OC)), \mu(R^k(OD'))$, and $\mu(R^k(CD))$ for $k = 0, 1, \ldots, n - 1$; and $2n$ open triangles $\mu(R^k(T'))$ and $\mu(R^k(\overline{T}))$ for $k = 0, 1, \ldots, n - 1$. Thus, the Euler characteristic $\chi(\tilde{S}_{\text{reg}})$ of $\tilde{S}_{\text{reg}}$ is $d_0 + d_1 + d_\infty - 3n + 2n = d_0 + d_1 + d_\infty - n$. However, $\tilde{S}_{\text{reg}}$ is a smooth manifold. So $\chi(\tilde{S}_{\text{reg}}) = 2 - 2g$, where $g$ is the genus of $\tilde{S}_{\text{reg}}$. Hence $g = 1/2 (n + 2 - (d_0 + d_1 + d_\infty))$. Compare this argument with that of Weyl ([4], p. 174). This proves Theorem 2. □

Since the quadrilateral $Q$ is a fundamental domain for the action of $G$ on $K'$, the $G$ orbit map $\pi = \sigma_0 \pi : K' \subseteq \subseteq C \rightarrow \tilde{S}$ restricted to $Q$ is a bijective continuous open mapping. However, $\delta_{\text{reg}} : D \subseteq S \rightarrow Q \subseteq C$ is a bijective continuous open mapping of the fundamental domain $D$ of the $G$ action on $S$. Consequently, the $G$ orbit space is homeomorphic to the $G$ orbit space $\tilde{S}$. The mapping $\pi$ is holomorphic except possibly at 0 and the vertices of $\text{cl}(K')$. So the mapping $\pi = \delta_{K'} : S_{\text{reg}} \rightarrow \tilde{S}_{\text{reg}}$ is a holomorphic diffeomorphism.

5. An Affine Model of $S_{\text{reg}}$

We construct an affine model of the affine Riemann surface $S_{\text{reg}}$ as follows. Return to the regular stellated $n$-gon $K' = K_{n_0 n_1 n_\infty}$, which is formed from the quadrilateral $Q = Q_{n_0 n_1 n_\infty}$ less its vertices. Repeatedly reflecting in the edges of $K'$ and then in the edges of the resulting reflections of $K'$ et cetera, we obtain a covering of $\mathbb{C} \setminus V^+$ by certain translations of $K'$. Here $V^+$ is the union of the translates of the vertices of $\text{cl}(K')$ and its center O. Let $\Sigma$ be the group generated by these translations. The semidirect product $\mathfrak{G} = G \ltimes \Sigma$ acts freely, properly and transitively on $\mathbb{C} \setminus V^+$. It preserves equivalent edges of $\mathbb{C} \setminus V^+$ and it acts freely and properly on $(\mathbb{C} \setminus V^+)\setminus$, the space formed by identifying equivalent edges in $\mathbb{C} \setminus V^+$. The orbit space $\mathbb{C} \setminus V^+/\mathfrak{G}$ is holomorphically diffeomorphic to $S_{\text{reg}}$ and is the desired affine model of $S_{\text{reg}}$. We now justify these assertions.

First we determine the group $T$ of translations.

**Lemma 7.** Each of the 2n sides of the regular stellated n-gon $K'$ is perpendicular to exactly one of the directions

$$e^{i\left[\frac{1}{2} - \frac{n_0}{2} + 2k\frac{1}{2}\pi\right]} \text{ or } e^{i\left[-\frac{1}{2} - \frac{n_0}{2} + (2k + 1)\frac{1}{2}\pi\right]}, \quad \text{for } k = 0, 1, \ldots, n - 1.\tag{23}$$

**Proof.** From Figure 9 we have $\angle D'CO = \frac{n_0}{2}\pi$. So $\angle COH = \frac{1}{2}\pi - \frac{n_0}{2}\pi$. Hence the line $\ell_0$, containing the edge $CD'$ of $K'$, is perpendicular to the direction $e^{i\left[\frac{1}{2} - \frac{n_0}{2}\right]\pi}$. Since $\triangle CO\overline{D}'$ is the reflection of $\triangle CO\overline{D}$ in the line segment $OC$, the line $\ell_1$, containing the edge $CD'$ of $K'$, is perpendicular to the direction $e^{i\left[-\frac{1}{2} + \frac{n_0}{2}\right]\pi}$. Because the regular stellated n-gon $K'$ is formed by repeatedly rotating the quadrilateral $Q' = OD'\overline{CD}$ through an angle $\frac{2\pi}{n_0}$, we find that Equation (23) holds. □
Since $\angle COH = \frac{1}{2}\pi - \frac{n+1}{n}\pi$, it follows that $|H| = |C| \sin \pi \frac{n}{n} \frac{n+1}{n}$ is the distance from the center $O$ of $K^*$ to the line $\ell_0$ containing the side $CD'$, or to the line $\ell_1$ containing the side $CD$. So $u_0 = (|C| \sin \pi \frac{n}{n} \frac{n+1}{n}) e^{i \frac{1}{2} - \frac{n+1}{n}\pi}$ is the closest point $H$ on $\ell_0$ to $O$ and $u_1 = (|C| \sin \pi \frac{n}{n} \frac{n+1}{n}) e^{i \frac{1}{2} + \frac{n}{n} \pi}$ is the closest point $\overline{H}$ on $\ell_1$ to $O$. Since the regular stellated $n$-gon $K^*$ is formed by repeatedly rotating the quadrilateral $Q'$ = $OD'C\overline{D}$ through an angle $\frac{2\pi}{n}$, the point

$$u_{2k} = R^k u_0 = (|C| \sin \pi \frac{n}{n} \frac{n+1}{n}) e^{i \frac{1}{2} - \frac{n+1}{n} + 2k\frac{1}{n}\pi}$$

lies on the line $\ell_{2k} = R^k \ell_0$, which contains the edge $R^k (CD')$ of $K^*$, while

$$u_{2k+1} = R^k u_1 = (|C| \sin \pi \frac{n}{n} \frac{n+1}{n}) e^{i \frac{1}{2} + \frac{n}{n} \pi + (2k+1)\frac{1}{n}\pi}$$

lies on the line $\ell_{2k+1} = R^k \ell_1$, which contains the edge $R^k (CD')$ of $K^*$ for every $k \in \{0,1,\ldots,n-1\}$. Furthermore, the line segments $Ou_{2k}$ and $Ou_{2k+1}$ are perpendicular to the line $\ell_{2k}$ and $\ell_{2k+1}$, respectively, for $k \in \{0,1,\ldots,n-1\}$.

**Corollary 8.** For $k = 0,1,\ldots,n-1$ we have

$$\overline{p_{2k}} = u_{2(n-k)+1} \quad \text{and} \quad \overline{p_{2k+1}} = u_{2(n-k)}.$$  

**Proof.** We compute. From (24) it follows that

$$\overline{p_{2k}} = U(u_{2k}) = UR^k(u_0) = R^{-k}(U(u_0))$$

$$= R^{-k}(u_1) = R^{n-k}(u_1) = u_{2(n-k)+1},$$

using (25); while from (25) we get

$$\overline{p_{2k+1}} = U(u_{2k+1}) = UR^k(u_1) = R^{-k}(U(u_1)) = R^{n-k}(u_0) = u_{2(n-k)}.$$  

\[\Box\]

**Corollary 9.** For $k, \ell \in \{0,1,\ldots,2n-1\}$ we have

$$u_{(k+2\ell) \mod 2n} = R^\ell u_k.$$  

**Proof.** If $k = 2i$, then $u_k = R^i u_0$, by definition. So

$$R^\ell u_k = R^{\ell+i} u_0 = u_{(2i+2\ell) \mod 2n} = u_{(k+2\ell) \mod 2n}.$$  

If $k = 2i+1$, then $u_k = R^i u_1$, by definition. So

$$R^\ell u_k = R^{\ell+i} u_1 = u_{(2(i+\ell)+1) \mod 2n} = u_{(k+2\ell) \mod 2n}.$$  

\[\Box\]
For \( k = 0, 1, \ldots, 2n - 1 \) let \( \tau_k \) be the translation
\[
\tau_k : \mathbb{C} \to \mathbb{C} : z \mapsto z + 2u_k.
\] (28)

**Corollary 10.** For \( k, \ell \in \{0, 1, \ldots, 2n - 1\} \) we have
\[
\tau_{(k+2\ell)} \circ 2n^\circ \circ R^\ell = R^\ell \circ \tau_k.
\] (29)

**Proof.** For every \( z \in \mathbb{C} \), we have
\[
\tau_{(k+2\ell)} \mod 2n^\circ \circ R^\ell(z) = z + 2\mu_{(k+2\ell)} \mod 2n^\circ \quad \text{using (28)}
\]
\[
= z + 2R^\ell u_k \quad \text{by (27)}
\]
\[
= R^\ell (R^{-\ell}z + 2u_k) = R^\ell \circ \tau_k(R^{-\ell}z).\]

\( \square \)

Reflecting the regular stellated \( n \)-gon \( K^* \) in its edge \( CD' \) contained in \( \ell_0 \) gives a congruent regular stellated \( n \)-gon \( K_0^* \) with the center \( O \) of \( K^* \) becoming the center \( 2u_0 \) of \( K_0^* \).

**Lemma 8.** The collection of all the centers of the regular stellated \( n \)-gons, formed by reflecting \( K^* \) in its edges and then reflecting in the edges of the reflected regular stellated \( n \)-gons et cetera, is
\[
\{\tau_{0^\circ}^{\ell_0} \cdots \tau_{2n^\circ - 1}^{\ell_0} (0) \in \mathbb{C} \mid (\ell_0, \ldots, \ell_{2n-1}) \in (\mathbb{Z}_{\geq 0})^{2n}\} = \{2 \sum_{\ell_0, \ldots, \ell_{2n-1} = 0}^{\infty} \ell_0 u_0 + \cdots + \ell_{2n-1} u_{2n-1}\},
\]
where for \( k = 0, 1, \ldots, 2n - 1 \) we have
\[
\tau_k^\ell = \frac{\ell_k}{\tau_{0^\circ}^{\ell_0} \cdots \tau_k^0} : \mathbb{C} \to \mathbb{C} : z \mapsto z + 2\ell^\circ u_k.
\]

**Proof.** For each \( k_0 = 0, 1, \ldots, 2n - 1 \) the center of the \( 2n \) regular stellated congruent \( n \)-gon \( K_{k_0}^* \) formed by reflecting in an edge of \( K^* \) contained in the line \( \ell_{k_0} \) is \( \tau_{k_0}(0) = 2u_{k_0} \). Repeating the reflecting process in each edge of \( K_{k_0}^* \) gives \( 2n \) congruent regular stellated \( n \)-gons \( K_{k_0k_1}^* \) with center at \( \tau_{k_0}(0) = 2(u_{k_1} + u_{k_0}) \), where \( k_1 = 0, 1, \ldots, 2n - 1 \). Repeating this construction proves the lemma. \( \square \)

The set \( \mathcal{V} \) of vertices of the regular stellated \( n \)-gon \( K^* \) is
\[
\{ V_{2k} = Ce^{2k(\frac{1}{2} \pi i)}, \, V_{2k+1} = D'e^{2(k+1)(\frac{1}{2} \pi i)} \quad \text{for} \, \ 0 \leq k \leq n - 1 \},
\]
see Figure 5. Clearly the set \( \mathcal{V} \) is \( G \) invariant.

**Corollary 11.** The set
\[
\mathcal{V}^+ = \{ v_{\ell_0 \cdots \ell_{2n-1}} = \tau_{0^\circ}^{\ell_0} \cdots \tau_{2n^\circ - 1}^{\ell_0^\circ} (V) \mid V \in \mathcal{V} \cup \{O\} \land (\ell_0, \ldots, \ell_{2n-1}) \in (\mathbb{Z}_{\geq 0})^{2n}\}
\] (30)

is the collection of vertices and centers of the congruent regular stellated \( n \)-gons \( K^*, K_{k_1}^*, K_{k_0k_1}^*, \ldots \).

**Proof.** This follows immediately from Lemma 8. \( \square \)
Corollary 12. The union of $K_s^* K_{k_0}^* K_{k_1}^* \cdots K_{k_{k-1}}^*$, where $0 \leq k \leq 2n-1$, covers $\mathbb{C} \setminus \mathbb{V}^+$, that is,

$$K^* \cup \bigcup_{\ell \geq 0} \bigcup_{0 \leq j \leq \ell} \bigcup_{0 \leq k \leq 2n-1} K_{k_{k^j-1}}^* = \mathbb{C} \setminus \mathbb{V}^.$$ 

Proof. This follows immediately from $K_{k_{k^j-1}}^* = \tau_{k^j} \cdots \tau_{k_0}(K^*)$. □

Let $T$ be the abelian subgroup of the 2-dimensional Euclidean group $E(2)$ generated by the translations $\tau_k$ (28) for $k = 0, 1, \ldots 2n-1$. It follows from Corollary 12 that the regular stellated $n$-gon $K^*$ with its vertices and center removed is the fundamental domain for the action of the abelian group $T$ on $\mathbb{C} \setminus \mathbb{V}^+$. The group $T$ is isomorphic to the abelian subgroup $\mathfrak{T}$ of $(\mathbb{C}, +)$ generated by $\{2\ell_0 k_{n-1}\}$.

Next we define the group $\Phi$ and show that it acts freely, properly, and transitively on $\mathbb{C} \setminus \mathbb{V}^+$. Consider the group $\Phi = G \rtimes \mathfrak{T} \subseteq G \times \mathfrak{T}$, which is the semidirect product of the dihedral group $G$, generated by the rotation $R$ through $2\pi/n$ and the reflection $U$ subject to the relations $R^n = e = U^2$ and $RU = UR^{-1}$, and the abelian group $\mathfrak{T}$. An element $(R^i U^j, 2u_k)$ of $\Phi$ is the affine linear map

$$(R^i U^j, 2u_k) : \mathbb{C} \to \mathbb{C} : z \mapsto R^i U^j z + 2u_k.$$ 

Multiplication in $\Phi$ is defined by

$$(R^i U^j, 2u_k) \cdot (R^i' U^j', 2u_{k'}) = (R^{i+i'} U^{j+j'}, (R^i U^j)(2u_k) + 2u_{k'}),$$

which is the composition of the affine linear map $(R^i' U^j, 2u_{k'})$ followed by $(R^i U^j, 2u_k)$. The mappings $G \to \Phi : R \mapsto (R^i U^j, 0)$ and $\mathfrak{T} \to \Phi : 2u_k \mapsto (e, 2u_k)$ are injective, which allows us to identify the groups $G$ and $\mathfrak{T}$ with their image in $\Phi$. Using (31) we may write an element $(R^i U^j, 2u_k)$ of $\Phi$ as $(e, 2u_k) \cdot (R^i U^j, 0)$. So

$$(e, 2u_{(j+2k) \mod 2n}) \cdot (R^i U^j, 0) = (R^i U^j, 2u_{(j+2k) \mod 2n}),$$

For every $z \in \mathbb{C}$ we have

$$R^i U^j z + 2u_{(j+2k) \mod 2n} = R^i U^j z + R^i U^j (2u_k),$$

that is,

$$(R^i U^j, 2u_{(j+2k) \mod 2n}) = (R^i U^j, R^i U^j (2u_k)) = (R^i U^j, 0) \cdot (e, 2u_k).$$

Hence

$$(e, 2u_{(j+2k) \mod 2n}) \cdot (R^i U^j, 0) = (R^i U^j, 0) \cdot (e, 2u_k),$$

which is just Equation (29). The group $\Phi$ acts on $\mathbb{C}$ as $E(2)$ does, namely, by affine linear orthogonal mappings. Denote this action by

$$\psi : \Phi \times \mathbb{C} \to \mathbb{C} : ((g, \tau), z) \mapsto \tau(g(z)).$$

Lemma 9. The set $\mathbb{V}^+$ (30) is invariant under the $\Phi$ action.

Proof. Let $v \in \mathbb{V}^+$. Then for some $(\ell_0, \ldots, \ell_{2n-1}) \in \mathbb{Z}_{\geq 0}^{2n}$ and some $w \in \mathbb{V} \cup \{O\}$

$$v = \tau_{\ell_0} \cdots \tau_{\ell_{2n-1}}(w) = \psi_{(e, 2u')}(w),$$

where $u' = \sum_{k=0}^{2n-1} \ell_k u_k$. For $(R^i U^j, 2u) \in \Phi$ with $j = 0, 1, \ldots, n-1$ and $\ell = 0, 1$ we have

$$\psi_{(R^i U^j, 2u)} v = \psi_{(R^i U^j, 2u)} \cdot \psi_{(e, 2u')} (w) = \psi_{(R^i U^j, 2u) \cdot (e, 2u')} (w).$$
where \(w' = \psi((R/U')^\ell(w)) = R/U^\ell(w) \in \bigvee \cup \{\emptyset\}\). If \(\ell = 0\), then
\[
R^\ell U' = R^\ell \left( \sum_{k=0}^{2n-1} \ell_k^1 u_k \right) = \sum_{k=0}^{2n-1} \ell_k^1 R^\ell (u_k) = \sum_{k=0}^{2n-1} \ell_k^1 u_{(k+2j) \mod 2n};
\]
while if \(\ell = 1\), then
\[
R^\ell U'(u') = \sum_{k=0}^{2n-1} \ell_k^1 R^\ell (u_{(k+2j) \mod 2n});
\]
Here \(k'(k) = \left\{ \begin{array}{ll} \frac{m-k+1}{2} & \text{if } k \text{ is even} \\
\frac{m-k-1}{2} & \text{if } k \text{ is odd} \end{array} \right.\). So \((e, 2(R/U^\ell u' + u)) \in \mathcal{T}\), which implies \(\psi((e, 2(R/U^\ell u' + u)))(w')) \in \bigvee\), as desired. \(\square\)

**Lemma 10.** The action of \(G\) on \(C \setminus \bigvee\) is free.

**Proof.** Suppose that for some \(v \in C \setminus \bigvee\) and some \((R/U^\ell, 2u) \in G\) we have \(v = \psi((R/U^\ell, 2u))(v)\). Then \(v\) lies in some \(K^*_{k_0 \cdots k_\ell}\). So for some \(v' \in K^*\) we have
\[
v = \tau_0^\ell \circ \cdots \circ \tau_{2n-1}^{n-1}(v') = \psi((e, 2u'))(v'),
\]
where \(u' = \sum_{j=0}^{2n-1} \ell_j^1 u_j\) for some \((\ell_0^1, \ldots, \ell_{2n-1}^1) \in (\mathbb{Z}_{>0})^{2n}\). Thus,
\[
\psi((e, 2u'))(v') = \psi((R/U^\ell, 2u))(v') = \psi((R/U^\ell 2R/U^\ell + 2u))(v').
\]
This implies \(R/U^\ell = e\), that is, \(j = \ell = 0\). So \(2u' = 2R^\ell U' u' + 2u = 2u' + 2u\), that is, \(u = 0\). Hence \((R/U^\ell, u) = (e, 0)\), which is the identity element of \(G\). \(\square\)

**Lemma 11.** The action of \(T\) (and hence \(G\)) on \(C \setminus \bigvee\) is transitive.

**Proof.** Let \(K^*_{k_0 \cdots k_\ell}\) and \(K^*_{k'_0 \cdots k'_\ell'}\) lie in
\[
C \setminus \bigvee = K^* \cup \bigcup_{\ell \geq 0} \bigcup_{0 \leq j \leq \ell} \bigcup_{0 \leq k_j \leq 2n-1} K^*_{k_0 \cdots k_\ell},
\]
Since \(K^*_{k_0 \cdots k_\ell} = \tau_{k_0} \circ \cdots \circ \tau_{k_\ell}(K^*)\) and \(K^*_{k'_0 \cdots k'_\ell'} = \tau_{k'_0} \circ \cdots \circ \tau_{k'_\ell}(K^*)\), it follows that
\[
(\tau_{k'_0} \circ \cdots \circ \tau_{k'_\ell}) \circ (\tau_{k_0} \circ \cdots \circ \tau_{k_\ell})^{-1}(K^*_{k_0 \cdots k_\ell}) = K^*_{k'_0 \cdots k'_\ell}.
\]
The action of \(G\) on \(C \setminus \bigvee\) is proper because \(G\) is a discrete subgroup of \(E(2)\) with no accumulation points.

We now define an edge of \(C \setminus \bigvee\) and what it means for an unordered pair of edges to be equivalent. We show that the group \(G\) acts freely and properly on the identification space of equivalent edges.

Let \(E\) be an open edge of \(K^*\). Since \(E_{k_0 \cdots k_\ell} = \tau_{k_0} \cdots \tau_{k_\ell}(E) \in K^*_{k_0 \cdots k_\ell}\) it follows that \(E_{k_0 \cdots k_\ell}\) is an open edge of \(K^*_{k_0 \cdots k_\ell}\). Let
\[
\mathcal{E} = \{E_{k_0 \cdots k_\ell} | \ell \geq 0, 0 \leq j \leq \ell \& 0 \leq k_j \leq 2n - 1\}.
\]
Then \(\mathcal{E}\) is the set of open edges of \(C \setminus \bigvee\) by 12. Since \(\tau_{k_0} \circ \cdots \circ \tau_{k_\ell}(0)\) is the center of \(K^*_{k_0 \cdots k_\ell}\), the element \((e, \tau_{k_0} \circ \cdots \circ \tau_{k_\ell}) \cdot (g, (\tau_{k_0} \circ \cdots \circ \tau_{k_\ell})^{-1})\) of \(G\) is a rotation-reflection of
Lemma 13. Let $\mathfrak{g} \in \mathfrak{g}^{\prime}$ be the set of unordered pairs $[E_{k_{0} \cdots k_{r}} E'_{k_{0} \cdots k_{r}}]$ of equivalent open edges of $K_{k_{0} \cdots k_{r}}$, so that $E_{k_{0} \cdots k_{r}} \cap E'_{k_{0} \cdots k_{r}} = \emptyset$, and let $E_{k_{0} \cdots k_{r}} = \tau_{k_{0}} \cdots \tau_{k_{r}}(E)$ and $E'_{k_{0} \cdots k_{r}} = \tau_{k_{0}} \cdots \tau_{k_{r}}(E')$ of $\mathfrak{c}(K_{k_{0} \cdots k_{r}})$ are not adjacent, which implies that the open edges $E$ and $E'$ of $K^{*}$ are not adjacent, and for some generator $S_{m}^{(j)}$ of the group $G^{l}$ of reflections with $j = 0, 1, \infty$ we have

$$E_{k_{0} \cdots k_{r}}' = (\tau_{k_{0}} \cdots \tau_{k_{r}})(S_{m}^{(j)}((\tau_{k_{0}} \cdots \tau_{k_{r}})^{-1}(E_{k_{0} \cdots k_{r}}))).$$

Let $\mathfrak{e}^{j} = \cup_{\ell \geq 0} \cup_{0 \leq j \leq \ell} \cup_{0 \leq k_{j} \leq 2\pi} \mathfrak{e}^{j}_{k_{0} \cdots k_{r}}$. So $\mathfrak{e}^{j}_{k_{0} \cdots k_{r}}$ is the set of unordered pairs of equivalent edges of $\mathfrak{C} \setminus \mathfrak{V}^{+}$. Define an action $\ast$ of $\mathfrak{e}^{j}$ on $\cup_{j = 0, 1, \infty} \mathfrak{e}^{j}$ by

$$(g, \tau) \ast [E_{k_{0} \cdots k_{r}}, E'_{k_{0} \cdots k_{r}}] = \left( (\tau' \circ \tau)(g(\tau')^{-1}(E_{k_{0} \cdots k_{r}})), (\tau' \circ \tau)(g(\tau')^{-1}(E'_{k_{0} \cdots k_{r}})) \right)$$

where $\tau' = \tau_{k_{0}} \cdots \tau_{k_{r}}\tau_{k_{0}} \cdots \tau_{k_{r}}$. Define a relation $\sim$ on $\mathfrak{C} \setminus \mathfrak{V}^{+}$ as follows. We say that $x$ and $y$ in $\mathfrak{C} \setminus \mathfrak{V}^{+}$ are related, $x \sim y$, if 1) $x \in F = \tau(E) \in \mathfrak{e}^{j}$ and $y \in F' = \tau(E') \in \mathfrak{e}^{j}$ such that $[F, F'] = [\tau(E), \tau(E')] \in \mathfrak{e}^{0}$, where $[E, E'] \in \mathfrak{e}^{j}$ with $E' = S_{m}^{(j)}(E)$ for some $S_{m}^{(j)} \in \mathfrak{g}$ and $y = \tau(S_{m}^{(j)}(\tau^{-1}(x)))$ for some $j = 0, 1, \infty$, or 2) $x, y \in (\mathfrak{C} \setminus \mathfrak{V}^{+}) \setminus \mathfrak{e}$ and $x = y$. Then $\sim$ is an equivalence relation on $\mathfrak{C} \setminus \mathfrak{V}^{+}$. Let $(\mathfrak{C} \setminus \mathfrak{V}^{+})\sim$ be the set of equivalence classes and let $\Pi$ be the map

$$\Pi: \mathfrak{C} \setminus \mathfrak{V}^{+} \to (\mathfrak{C} \setminus \mathfrak{V}^{+})\sim$$

which assigns to every $p \in \mathfrak{C} \setminus \mathfrak{V}^{+}$ the equivalence class $[p]$ containing $p$.

Lemma 12. $\Pi_{|K^{*}}$ is the map $\rho$ (20).

Proof. This follows immediately from the definition of the maps $\Pi$ and $\rho$. \square

Lemma 13. The usual action of $\mathfrak{e}$ on $\mathfrak{C}$, restricted to $\mathfrak{C} \setminus \mathfrak{V}^{+}$, is compatible with the equivalence relation $\sim$, that is, if $x, y \in \mathfrak{C} \setminus \mathfrak{V}$ and $x \sim y$, then $(g, \tau)(x) \sim (g, \tau)(y)$ for every $(g, \tau) \in \mathfrak{e}$.

Proof. Suppose that $x \in F = \tau'(E)$, where $\tau' \in T$. Then $y \in F' = \tau'(E')$, since $x \sim y$. So for some $S_{m}^{(j)} \in \mathfrak{g}$ with $j = 0, 1, \infty$, we have $\tau^{-1}(y) = S_{m}^{(j)}(\tau^{-1}(x))$. Let $(g, \tau) \in \mathfrak{e}$. Then

$$(g, \tau)(\tau^{-1}(y)) = g(\tau^{-1}(y)) + u_{\tau} = g(S_{m}^{(j)}(\tau^{-1}(x)) + u_{\tau}.$$So $(g, \tau)(y) \in (g, \tau)\ast F'$. However, $(g, \tau)(x) \in (g, \tau)\ast F$ and $[(g, \tau)\ast F, (g, \tau)\ast F'] = (g, \tau)\ast [F, F']$. Hence $(g, \tau)(x) \sim (g, \tau)(y)$. \square

Because of Lemma 13, the usual $\mathfrak{e}$-action on $\mathfrak{C} \setminus \mathfrak{V}^{+}$ induces an action of $\mathfrak{e}$ on $(\mathfrak{C} \setminus \mathfrak{V}^{+})\sim$.

Lemma 14. The action of $\mathfrak{e}$ on $(\mathfrak{C} \setminus \mathfrak{V}^{+})\sim$ is free and proper.

Proof. The following argument shows that it is free. Using Lemma A2 we see that an element of $\mathfrak{e}$, which lies in the isotropy group $\mathfrak{e}_{[F, F']} \subset \mathfrak{e}_{F \ast F'}$ for $[F, F'] \in \mathfrak{e}_{F}$, interchanges the edge $F$ with the equivalent edge $F'$ and thus fixes the equivalence class $[p]$ for every $p \in F$. Hence the $\mathfrak{e}$ action on $(\mathfrak{C} \setminus \mathfrak{V}^{+})\sim$ is free. It is proper because $\mathfrak{e}$ is a discrete subgroup of the Euclidean group $E(2)$ with no accumulation points. \square

Theorem 4. The $\mathfrak{e}$-orbit space $(\mathfrak{C} \setminus \mathfrak{V}^{+})\sim / \mathfrak{e}$ is holomorphically diffeomorphic to the $G$-orbit space $(K^{*} \setminus O)^{\ast}/G = \tilde{S}_{\text{reg}}$. 


Proof. The claim follows because the fundamental domain of the $\mathcal{G}$-action on $\mathbb{C} \setminus \mathbb{V}^+$ is $K^* \setminus O$ is the fundamental domain of the $G$-action on $K^* \setminus O$. Thus, $\Pi(\mathbb{C} \setminus \mathbb{V}^+)$ is a fundamental domain of the $\mathcal{G}$-action on $(\mathbb{C} \setminus \mathbb{V}^+)^\sim$, which is equal to $\rho(K^* \setminus O) = (K^* \setminus O)^\sim$ by Lemma 12. Hence the $\mathcal{G}$-orbit space $(\mathbb{C} \setminus \mathbb{V}^+)^\sim / \mathcal{G}$ is equal to the $G$-orbit space $\tilde{S}_{\text{reg}}$. So the identity map from $\Pi(\mathbb{C} \setminus \mathbb{V}^+)$ to $(K^* \setminus O)^\sim$ induces a holomorphic diffeomorphism of orbit spaces. $\square$

Because the group $\mathcal{G}$ is a discrete subgroup of the 2-dimensional Euclidean group $E(2)$, the Riemann surface $(\mathbb{C} \setminus \mathbb{V}^+)^\sim / \mathcal{G}$ is an affine model of the affine Riemann surface $\tilde{S}_{\text{reg}}$.

6. The Developing Map and Geodesics

In this section, we show that the mapping

$$\delta : \mathcal{D} \subseteq \mathcal{S}_{\text{reg}} \to Q \subseteq \mathbb{C} : (\xi, \eta) \mapsto (F_Q \circ \tilde{\pi})(\xi, \eta)$$

(35)

straightens the holomorphic vector field $X$ (12) on the fundamental domain $\mathcal{D} \subseteq \mathcal{S}_{\text{reg}}$, see [6] and Flaschka [7]. We also verify that $X$ is the geodesic vector field for a flat Riemannian metric $\Gamma$ on $\mathcal{D}$.

First we rewrite Equation (13) as

$$T_{(\xi, \eta)} \tilde{\pi} (X(\xi, \eta)) = \eta \frac{\partial}{\partial \xi}, \quad \text{for } (\xi, \eta) \in \mathcal{D}. \quad (36)$$

From the definition of the mapping $F_Q$ (2) we get

$$\mathrm{d} z = \mathrm{d} F_Q = \frac{1}{(\xi^{n-n_0} (1 - \xi)^{n-n_1})^{1/n}} \mathrm{d} \xi = \frac{1}{\eta} \mathrm{d} \xi,$$

where we use the same complex $n$th root as in the definition of $F_Q$. This implies

$$\frac{\partial}{\partial z} = T_\xi F_Q \left( \eta \frac{\partial}{\partial \xi} \right), \quad \text{for } (\xi, \eta) \in \mathcal{D} \quad (37)$$

For each $(\xi, \eta) \in \mathcal{D}$ using (36) and (37) we get

$$T_{(\xi, \eta)} \delta(X(\xi, \eta)) = (T_\xi F_Q \circ T_{(\xi, \eta)} \tilde{\pi}) (X(\xi, \eta)) = T_\xi F_Q (\eta \frac{\partial}{\partial \xi}) = \frac{\partial}{\partial z} |_{z=\delta(\xi, \eta)}$$

So the holomorphic vector field $X$ (12) on $\mathcal{D}$ and the holomorphic vector field $\frac{\partial}{\partial z}$ on $Q$ are $\delta$-related. Hence $\delta$ sends an integral curve of the vector field $X$ starting at $(\xi, \eta) \in \mathcal{D}$ onto an integral curve of the vector field $\frac{\partial}{\partial z}$ starting at $z = \delta(\xi, \eta) \in Q$. Since an integral curve of $\frac{\partial}{\partial z}$ is a horizontal line segment in $Q$, we have proved

**Theorem 5.** The holomorphic mapping $\delta$ (35) straightens the holomorphic vector field $X$ (12) on the fundamental domain $\mathcal{D} \subseteq \mathcal{S}_{\text{reg}}$.

We can say more. Let $u = \Re z$ and $v = \Im z$. Then

$$\gamma = \mathrm{d} u \circ \mathrm{d} u + \mathrm{d} v \circ \mathrm{d} v = \mathrm{d} z \circ \overline{\mathrm{d} z}\quad (38)$$

is the flat Euclidean metric on $\mathbb{C}$. Its restriction $\gamma|_{\mathbb{C} \setminus \mathbb{V}^+}$ to $\mathbb{C} \setminus \mathbb{V}^+$ is invariant under the group $\mathcal{G}$, which is a subgroup of the Euclidean group $E(2)$.

Consider the flat Riemannian metric $\gamma|_{\mathbb{C}}$ on $Q$, where $\gamma$ is the metric (38) on $\mathbb{C}$. Pulling back $\gamma|_{\mathbb{C}}$ by the mapping $F_Q$ (2) gives a metric

$$\tilde{\gamma} = F_Q^* (\gamma|_{\mathbb{C}}) = |\xi^{n-n_0} (1 - \xi)^{n-n_1}|^{-2/n} \mathrm{d} \xi \circ \overline{\mathrm{d} \xi}$$
on $C \setminus \{0,1\}$. Pulling the metric $\hat{\gamma}$ back by the projection mapping $\hat{\pi} : C^2 \to C : (\xi, \eta) \mapsto \xi$
gives
\[
\hat{\Gamma} = \hat{\pi}^* \hat{\gamma} = |\xi^{n-n_0}(1-\xi)^{n-n_1}|^{-2/n} \partial_\xi \circ \hat{\partial}_\xi
\]
on $C^2$. Restricting $\hat{\Gamma}$ to the affine Riemann surface $S_{\text{reg}}$ gives $\Gamma = \frac{1}{\eta} \partial_\xi \circ \frac{1}{\eta} \hat{\partial}_\xi$.

**Lemma 15.** $\Gamma$ is a flat Riemannian metric on $S_{\text{reg}}$.

**Proof.** We compute. For every $(\xi, \eta) \in S_{\text{reg}}$ we have
\[
\Gamma(\xi, \eta)(X(\xi, \eta), X(\xi, \eta)) = \frac{1}{\eta} \partial_\xi (\eta \frac{d}{d\xi} + \frac{u-n_0}{n} (1-\xi) \frac{2n-n_0-n_1}{n_1-2} \frac{d}{d\eta}) + \frac{1}{\eta} \hat{\partial}_\xi (\eta \frac{d}{d\eta} + \frac{u-n_0}{n} (1-\xi) \frac{2n-n_0-n_1}{n_1-2} \frac{d}{d\eta}) = \frac{1}{\eta} \partial_\xi (\eta \frac{d}{d\xi}) + \frac{1}{\eta} \hat{\partial}_\xi (\eta \frac{d}{d\eta}) = 1.
\]
Thus, $\Gamma$ is a Riemannian metric on $S_{\text{reg}}$. It is flat by construction.

Because $D$ has nonempty interior and the map $\delta$ (35) is holomorphic, it can be analytically continued to the map
\[
\delta_Q : S_{\text{reg}} \subseteq C^2 \to Q \subseteq C : (\xi, \eta) \mapsto F_Q(\hat{\pi}(\xi, \eta)),
\]
since $\delta = \delta_Q|D$. By construction $\delta_Q^* (\gamma|Q) = \Gamma$. So the mapping $\delta_Q$ is an isometry of $(S_{\text{reg}}, \Gamma)$ onto $(Q, \gamma|Q)$. In particular, the map $\delta$ is an isometry of $(D, \Gamma|D)$ onto $(Q, \gamma|Q)$. Moreover, $\delta$ is a local holomorphic diffeomorphism, because for every $(\xi, \eta) \in D$, the complex linear mapping $T_{(\xi, \eta)} \delta$ is an isomorphism, since it sends $X(\xi, \eta)$ to $\frac{\partial}{\partial \xi}$. Thus, $\delta$ is a developing map in the sense of differential geometry, see Spivak ([8], p. 97) note on §12 of Gauss [9].

The map $\delta$ is local because the integral curves of $\frac{\partial}{\partial \xi}$ on $Q$ are only defined for a finite time, since they are horizontal line segments in $Q$. Thus, the integral curves of $X$ (12) on $D$ are defined for a finite time. Since the integral curves of $\frac{\partial}{\partial \xi}$ are geodesics on $(Q, \gamma|Q)$, the image of a local integral curve of $\frac{\partial}{\partial \xi}$ under the local inverse of the mapping $\delta$ is a local integral curve of $X$. This latter local integral curve is a geodesic on $(D, \Gamma|D)$, since $\delta$ is an isometry. Thus, we have proved

**Theorem 6.** The holomorphic vector field $X$ (12) on the fundamental domain $D$ is the geodesic vector field for the flat Riemannian metric $\Gamma|D$ on $D$.

**Corollary 13.** The holomorphic vector field $X$ on the affine Riemann surface $S_{\text{reg}}$ is the geodesic vector field for the flat Riemannian metric $\Gamma$ on $S_{\text{reg}}$.

**Proof.** The corollary follows by analytic continuation from the conclusion of Theorem 6, since $\text{int } D$ is a nonempty open subset of $S_{\text{reg}}$ and both the vector field $X$ and the Riemannian metric $\Gamma$ are holomorphic on $S_{\text{reg}}$. □

**7. Discrete Symmetries and Billiard Motions**

Let $\mathcal{G}$ be the group of homeomorphisms of the affine Riemann surface $S$ (3) generated by the mappings
\[
\mathcal{R} : S \to S : (\xi, \eta) \mapsto (\xi, e^{2\pi i/n} \eta) \quad \text{and} \quad \mathcal{U} : S \to S : (\xi, \eta) \mapsto (\xi, \eta).
\]
Clearly, the relations $\mathcal{R}^n = \mathcal{U}^2 = e$ hold. For every $(\xi, \eta) \in S$ we have
\[
\mathcal{U} \mathcal{R}^{-1}(\xi, \eta) = \mathcal{U}(\xi, e^{-2\pi i/n} \eta) = (\xi, e^{2\pi i/n} \eta) = \mathcal{R}(\xi, \eta) = \mathcal{R}(\xi, \eta).
\]
So the additional relation $\mathcal{U} \mathcal{R}^{-1} = \mathcal{R} \mathcal{U}$ holds. Thus, $\mathcal{G}$ is isomorphic to the dihedral group.
Lemma 16. \(\mathcal{G}\) is a group of isometries of \((\mathcal{S}_{\text{reg}}, \Gamma)\).

Proof. For every \((\zeta, \eta) \in \mathcal{S}_{\text{reg}}\) we get

\[
\mathcal{R}^j \Gamma(\xi, \eta)(X(\xi, \eta), X(\xi, \eta)) = \Gamma(\mathcal{R}(\zeta, \eta))(T_{(\xi, \eta)} \mathcal{R}(X(\xi, \eta)), T_{(\xi, \eta)} \mathcal{R}(X(\xi, \eta)))
\]

\[
= \Gamma(\zeta, e^{2\pi i/n} \eta) \left( e^{\frac{2\pi i}{n} \eta} \left( n \frac{d}{d\xi} + \frac{n-1}{n} \xi \right) \right) = \Gamma(\zeta, \eta) \left( X(\xi, \eta), X(\xi, \eta) \right)
\]

and

\[
\mathcal{U}^j \Gamma(\xi, \eta)(X(\xi, \eta), X(\xi, \eta)) = \Gamma(\mathcal{U}(\zeta, \eta))(T_{(\xi, \eta)} \mathcal{U}(X(\xi, \eta)), T_{(\xi, \eta)} \mathcal{U}(X(\xi, \eta)))
\]

\[
= \frac{1}{|\eta|^2} \frac{d\xi}{d\xi} \left( e^{\frac{2\pi i}{n} \eta} \frac{\partial}{\partial \xi} \right) \left( e^{\frac{2\pi i}{n} \eta} \frac{\partial}{\partial \xi} \right) = \Gamma(\zeta, \eta) \left( X(\xi, \eta), X(\xi, \eta) \right).
\]

\(\square\)

Recall that the group \(G\), generated by the linear mappings

\(R : \mathbb{C} \to \mathbb{C} : z \mapsto e^{2\pi i/n} z\) and \(U : \mathbb{C} \to \mathbb{C} : z \mapsto \bar{z}\),

is isomorphic to the dihedral group.

Lemma 17. \(G\) is a group of isometries of \((\mathbb{C}, \gamma)\).

Proof. This follows because \(R\) and \(U\) are Euclidean motions. \(\square\)

We would like the developing map \(\delta_Q^n (39)\) to intertwine the actions of \(\mathcal{G}\) and \(G\) and the geodesic flows on \((\mathcal{S}_{\text{reg}}, \Gamma)\) and \((\mathcal{Q}, \gamma|_\mathcal{Q})\). There are several difficulties. The first is: the group \(G\) does not preserve the quadrilateral \(Q\). To overcome this difficulty we extend the mapping \(\delta_Q^n (39)\) to the mapping \(\delta_K^n (17)\) of the affine Riemann surface \(\mathcal{S}_{\text{reg}}\) onto the regular stellated \(n\)-gon \(K^n\).

Lemma 18. The mapping \(\delta_K^n (17)\) intertwines the action \(\Phi (14)\) of \(G\) on \(\mathcal{S}_{\text{reg}}\) with the action

\[
\Psi : G \times K^n \to K^n : (g, z) \mapsto g(z)
\]

of \(G\) on the regular stellated \(n\)-gon \(K^n\).

Proof. From the definition of the mapping \(\delta_K^n\) we see that for each \((\zeta, \eta) \in \mathcal{D}\) we have \(\delta_K^n(\mathcal{R}(\zeta, \eta)) = \mathcal{R} \delta_K^n(\zeta, \eta)\) for every \(j \in \mathbb{Z}\). By analytic continuation we see that the preceding equation holds for every \((\zeta, \eta) \in \mathcal{S}_{\text{reg}}\). Since \(F_Q(\zeta) = \mathcal{F}_Q(\zeta)\) by construction and \(\hat{\pi}(\zeta, \eta) = \hat{\zeta} (11)\), from the definition of the mapping \(\delta (35)\) we get \(\delta(\zeta, \eta) = \delta(\zeta, \eta)\) for every \((\zeta, \eta) \in \mathcal{D}\). In other words, \(\delta_K^n(\mathcal{U}(\zeta, \eta)) = \mathcal{U} \delta_K^n(\zeta, \eta)\) for every \((\zeta, \eta) \in \mathcal{D}\). By analytic continuation we see that the preceding equation holds for all \((\zeta, \eta) \in \mathcal{S}_{\text{reg}}\). Hence on \(\mathcal{S}_{\text{reg}}\) we have

\[
\delta_K^n \circ \Phi_s = \Psi \circ (\phi_s)^n \circ \delta_K^n \quad \text{for every } g \in \mathcal{G}.
\]

The mapping \(\phi : \mathcal{G} \to G\) sends the generators \(\mathcal{R}\) and \(\mathcal{U}\) of the group \(\mathcal{G}\) to the generators \(R\) and \(U\) of the group \(G\), respectively. So it is an isomorphism. \(\square\)
There is a second more serious difficulty: the integral curves of $\frac{\partial}{\partial z}$ run off the quadrilateral $Q$ in finite time. We fix this by requiring that when an integral curve reaches a point $P$ on the boundary $\partial Q$ of $Q$, which is not a vertex, it undergoes a specular reflection at $P$. (If the integral curve reaches a vertex of $Q$ in forward or backward time, then the motion ends). This motion can be continued as a straight line motion, which extends the motion on the original segment in $Q$.

To make this precise, we give $Q$ the orientation induced from $C$ and suppose that the incoming (and hence outgoing) straight line motion has the same orientation as $\partial Q$. If the incoming motion makes an angle $\alpha$ with respect to the inward pointing normal $N$ to $\partial Q$ at $P$, then the outgoing motion makes an angle $\alpha$ with the normal $N$, see Richens and Berry [2]. Specifically, if the incoming motion to $P$ is an integral curve of $\frac{\partial}{\partial z}$, then the outgoing motion, after reflection at $P$, is an integral curve of $R^{-1} \frac{\partial}{\partial z} = e^{-2\pi i/n} \frac{\partial}{\partial z}$. Thus, the outward motion makes a turn of $-2\pi/n$ at $P$ towards the interior of $Q$, see Figure 10 (left). In Figure 10 (right) the incoming motion has the opposite orientation from $\partial Q$. This extended motion on $Q$ is called a billiard motion. A billiard motion starting in the interior of $\text{cl}(Q) \setminus (\text{cl}(Q) \cap R)$ is defined for all time and remains in $\text{cl}(Q)$ less its vertices, since each of the segments of the billiard motion is a straight line parallel to an edge of $\text{cl}(Q)$ and does not hit a vertex of $\text{cl}(Q)$, see Figure 11.

![Figure 10. Reflection at a point $P$ on $\partial Q$.](image)

![Figure 11. A periodic billiard motion in the equilateral triangle $T = T_{1,1,1}$ starting at $P$. First, extended by the reflection $U$ to a periodic billiard motion in the quadrilateral $Q = T \cup U(T)$. Second, extended by the reflection $S$ to a periodic billiard motion in $Q \cup S(Q)$. Third, extended by the reflection $SR$ to a periodic billiard motion in the stellated equilateral triangle $H = K_{1,1,1}^{SR} = Q \cup S(Q)SR(S(Q))$.](image)

We can do more. If we apply a reflection $S$ in the edge of $Q$ in its boundary $\partial Q$, which contains the reflection point $P$, to the initial reflected motion at $P$, see Figure 12.

![Figure 12. Continuation of a billiard motion in the quadrilateral $Q$ to a billiard motion in the quadrilateral $S(Q)$ obtained by the reflection $S$ in an edge of $Q$.](image)

The motion in $S(Q)$ when it reaches $\partial S(Q)$, et cetera, the extended motion becomes a billiard motion in the regular stellated $n$-gon $K^* = Q \cup \bigcup_{0 \leq k \leq n-1} SR^k(Q)$, see Figure 11. So we have verified
Theorem 7. A billiard motion in the regular stellated $n$-gon $K^*$, which starts at a point in the interior of $K^* \setminus O$ and does not hit a vertex of $\text{cl}(K^*)$, is invariant under the action of the isometry subgroup $\hat{\mathcal{G}}$ of the isometry group $\mathcal{G}$ of $(K^*, \gamma_{|K^*})$ generated by the rotation $R$.

Let $\hat{\mathcal{G}}$ be the subgroup of $\mathcal{G}$ generated by the rotation $R$. We now show

Lemma 19. The holomorphic vector field $X$ (12) on $S_{\text{reg}}$ is $\hat{\mathcal{G}}$-invariant.

Proof. We compute. For every $(\xi, \eta) \in S_{\text{reg}}$ and for $R \in \hat{\mathcal{G}}$ we have

$$T_{(\xi, \eta)} \Phi_R (X(\xi, \eta)) = e^{2\pi i/n} \left[ \eta \frac{\partial}{\partial \xi} + \frac{n - n_0}{n} \xi \frac{1 - \xi}{\eta} \right]$$

$$= (e^{2\pi i/n} \eta) \frac{\partial}{\partial \xi} + \frac{n - n_0}{n} \xi \frac{1 - \xi}{(e^{2\pi i/n} \eta)^{n-2}} \frac{\partial}{\partial (e^{2\pi i/n} \eta)}$$

$$= X(\xi, e^{2\pi i/n} \eta) = X \circ \Phi_R (\xi, \eta).$$

Hence for every $j \in \mathbb{Z}$ we get

$$T_{(\xi, \eta)} \Phi_{R^j} (X(\xi, \eta)) = X \circ \Phi_{R^j} (\xi, \eta)$$  \hspace{1cm} (42)

for every $(\xi, \eta) \in S_{\text{reg}}$. In other words, the vector field $X$ is invariant under the action of $\hat{\mathcal{G}}$ on $S_{\text{reg}}$. \hfill \Box

Corollary 14. For every $(\xi, \eta) \in \mathcal{D}$ we have

$$X|_{\Phi_{R^j}(\mathcal{D})} = T \Phi_{R^j} X|_{\mathcal{D}}.$$  \hspace{1cm} (43)

Proof. Equation (43) is a rewrite of Equation (42). \hfill \Box

Corollary 15. Every geodesic on $(S_{\text{reg}}, \Gamma)$ is $\hat{\mathcal{G}}$-invariant.

Proof. This follows immediately from the lemma. \hfill \Box

Lemma 20. For every $(\xi, \eta) \in S_{\text{reg}}$ and every $j \in \mathbb{Z}$ we have

$$T_{\Phi_{R^j}(\xi, \eta)} \delta_{K^*} (X(\xi, \eta)) = \left. \frac{\partial}{\partial z} \right|_{z = \delta_{K^*}(\Phi_{R^j}(\xi, \eta))}.$$

(44)

Proof. From Equation (41) we get $\delta_{K^*} \circ \Phi_R = \Psi_R \circ \delta_{K^*}$ on $S_{\text{reg}}$. Differentiating the preceding equation and then evaluating the result at $X(\xi, \eta) \in T_{(\xi, \eta)} S_{\text{reg}}$ gives

$$(T_{\Phi_R(\xi, \eta)} \delta_{K^*} \circ T_{(\xi, \eta)} \Phi_R) X(\xi, \eta) = (T_{\delta_{K^*}(\xi, \eta)} \Psi_R \circ T_{(\xi, \eta)} \delta_{K^*}) X(\xi, \eta)$$

for all $(\xi, \eta) \in S_{\text{reg}}$. When $(\xi, \eta) \in \mathcal{D}$, by definition $\delta_{K^*}(\xi, \eta) = \delta(\xi, \eta)$. So for every $(\xi, \eta) \in S_{\text{reg}}$

$$T_{(\xi, \eta)} \delta_{K^*} (X(\xi, \eta)) = T_{(\xi, \eta)} \delta (X(\xi, \eta)) = \left. \frac{\partial}{\partial z} \right|_{z = \delta(\xi, \eta)} = \left. \frac{\partial}{\partial z} \right|_{z = \delta_{K^*}(\xi, \eta)}.$$

Thus,

$$T_{\Phi_R(\xi, \eta)} \delta_{K^*} (T_{(\xi, \eta)} \Phi_R X(\xi, \eta)) = T_{\delta_{K^*}(\xi, \eta)} \Psi_R \left( \left. \frac{\partial}{\partial z} \right|_{z = \delta_{K^*}(\xi, \eta)} \right).$$  \hspace{1cm} (45)

for every $(\xi, \eta) \in \mathcal{D}$. By analytic continuation (45) holds for every $(\xi, \eta) \in S_{\text{reg}}$. Now $T_{(\xi, \eta)} \Phi_R$ sends $T_{(\xi, \eta)} S_{\text{reg}}$ to $T_{\Phi_R(\xi, \eta)} S_{\text{reg}}$. Since $T_{(\xi, \eta)} \Phi_R X(\xi, \eta) = e^{2\pi i/n} X(\xi, \eta)$ for every
(ξ, η) ∈ S_{reg}, it follows that $e^{2\pi i/\eta}X(\xi, \eta)$ is in $T_{\Phi(\xi, \eta)}S_{\text{reg}}$. Furthermore, since $T_{\delta \kappa}(\xi, \eta)\Psi_R$ sends $T_{\delta \kappa}(\xi, \eta)K^*$ to $T_{\Psi_R(\delta \kappa(\xi, \eta))}K^*$, we get

$$T_{\delta \kappa(\xi, \eta)}\Psi_R \left( \frac{\partial}{\partial z} |_{-\delta \kappa(\xi, \eta)} \right) = K \frac{\partial}{\partial z} |_{-\Psi_R(\delta \kappa(\xi, \eta))}.$$ 

For every $(\xi, \eta) \in S_{\text{reg}}$ we obtain

$$T_{\Phi(\xi, \eta)}\delta \kappa^* (X(\xi, \eta)) = \frac{\partial}{\partial z} |_{-\Psi_R(\delta \kappa(\xi, \eta))},$$  \hspace{1cm} \text{(46)}$$

that is, Equation (44) holds with $j = 0$. A similar calculation shows that Equation (46) holds with $R$ replaces by $R^C$. This verifies Equation (44). \hfill \Box

We now show

**Theorem 8.** The image of a $\tilde{G}$ invariant geodesic on $(S_{\text{reg}}, \Gamma)$ under the developing map $\delta \kappa^*$ (17) is a billiard motion in $K^*$, see Figure 13.

![Figure 13](image)

**Proof.** Because $\Phi_R$ and $\Psi_R$ are isometries of $(S_{\text{reg}}, \Gamma)$ and $(K^*, \gamma|_{K^*})$, respectively, it follows from equation (41) that the surjective map $\delta \kappa^*: (S_{\text{reg}}, \Gamma) \rightarrow (K^*, \gamma|_{K^*})$ (17) is an isometry. Hence $\delta \kappa^*$ is a local developing map. Using the local inverse of $\delta \kappa^*$ and Equation (44), it follows that a billiard motion in $\text{int}(K^* \setminus O)$ is mapped onto a geodesic in $(S_{\text{reg}}, \Gamma)$, which is possibly broken at the points $(\xi_i, \eta_i) = \delta \kappa^*_C(p_i)$. Here $p_i \in \partial K^*$ are the points where the billiard motion undergoes a reflection. However, the geodesic on $S_{\text{reg}}$ is smooth at $(\xi_i, \eta_i)$ since the geodesic vector field $X$ is holomorphic on $S_{\text{reg}}$. Thus, the image of the geodesic under the developing map $\delta \kappa^*$ is a billiard motion. \hfill \Box

**Theorem 9.** Under the restriction of the mapping

$$v = \sigma_{\circ} \Pi : \mathbb{C} \setminus \mathbb{V}^+ \rightarrow (\mathbb{C} \setminus \mathbb{V}^+)^*/\mathcal{G} = \tilde{S}_{\text{reg}}$$  \hspace{1cm} \text{(47)}$$
to $K^* \setminus O$ the image of a billiard motion $\lambda_z$ is a smooth geodesic $\check{\lambda}_v(z)$ on $(\tilde{S}_{\text{reg}}, \check{\gamma})$, where $v^* (\check{\gamma}) = \gamma|_{\mathbb{C} \setminus \mathbb{V}^+}$.

**Proof.** Since the Riemannian metric $\gamma$ on $\mathbb{C}$ is invariant under the group of Euclidean motions, the Riemannian metric $\gamma|_{K^* \setminus O}$ on $K^* \setminus O$ is $\mathcal{G}$-invariant. Hence $\gamma|_{K^* \setminus O}$ is invariant under the reflection $S_m$ for $m \in \{0, 1, \ldots, n - 1\}$. So $\gamma|_{K^* \setminus O}$ pieces together to give a Riemannian metric $\gamma^*$ on the identification space $(K^* \setminus O)^*$. In other words, the pull back of $\gamma^*$ under the map $\Pi|_{K^* \setminus O}: K^* \setminus O \rightarrow (K^* \setminus O)^*$, which identifies equivalent edges of $K^*$, is the metric $\gamma|_{K^* \setminus O}$. Since $\Pi|_{K^* \setminus O}$ intertwines the $G$-action on $K^* \setminus O$ with the $G$-action on $(K^* \setminus O)^*$, the metric $\gamma^*$ is $\check{G}$-invariant. It is flat because the metric $\gamma$ is flat. So $\gamma^*$ induces a flat Riemannian metric $\check{\gamma}$ on the orbit space $(K^* \setminus O)^*/\mathcal{G} = \tilde{S}_{\text{reg}}$. Since the billiard motion $\lambda_z$ is a $\check{G}$-invariant broken geodesic on $(K^* \setminus O, \check{\gamma}|_{K^* \setminus O})$, it gives rise to a continuous broken
geodesic $\lambda^\sim_{(z)}$ on $((K^+ \setminus O)^\sim, \gamma^\sim)$, which is $\hat{G}$-invariant. Thus, $\hat{\lambda}_v(z) = v(\lambda_z)$ is a piecewise smooth geodesic on the smooth $G$-orbit space $((K^+ \setminus O)^\sim / G = \hat{S}_{\text{reg}}$).

We need only show that $\hat{\lambda}_v(z)$ is smooth. To see this we argue as follows. Let $s \subset K^*$ be a closed segment of a billiard motion $\gamma_z$, that does not meet a vertex of $\text{cl}(K^*)$. Then $s$ is a horizontal straight line motion in $\text{cl}(K^*)$. Suppose that $E_{k_0}$ is the edge of $K^*$, perpendicular to the direction $u_{k_0}$, which is first met by $s$ and let $P_{k_0}$ be the meeting point. Let $S_{k_0}$ be the reflection in $E_{k_0}$. The continuation of the motion $s$ at $P_{k_0}$ is the horizontal line $RS_{k_0}(s)$ in $K^*_{k_0}$. Recall that $K^*_0$ is the translation of $K^*$ by $T_{k_0}$. Using a suitable sequence of reflections in the edges of a suitable $K^*_{k_0\cdots k_1}$ each followed by a rotation $R$ and then a translation in $\mathcal{T}$ corresponding to their origins, we extend $s$ to a smooth straight line $\lambda$ in $\mathbb{C} \setminus \mathbb{V}^+$, see Figure 14. The line $\lambda$ is a geodesic in $(\mathbb{C} \setminus \mathbb{V}^+, \gamma|_{\mathbb{C} \setminus \mathbb{V}^+})$, which in $K^*$ has image $\hat{\lambda}_v(z)$ under the $\mathfrak{g}$-orbit map $v$ (47) that is a smooth geodesic on $(\hat{S}_{\text{reg}}, \hat{\gamma})$. The geodesic $v(\lambda)$ starts at $v(z)$. Thus, the smooth geodesic $v(\lambda)$ and the geodesic $\hat{\lambda}_v(z)$ are equal. In other words, $\hat{\lambda}_v(z)$ is a smooth geodesic. 

Figure 14. The billiard motion $\gamma_z$ in the stellated regular 3-gon $K^*_{1,1,1}$ meets the edge 0, is reflected in this edge by $S_0$, and then is rotated by $R$. This gives an extended motion $RS_{0} \gamma_z$, which is a straight line that is the same as reflecting $\gamma_z$ by $U$ and then translating by $\tau_0$.

Thus, the affine orbit space $\hat{S}_{\text{reg}} = (\mathbb{C} \setminus \mathbb{V}^+) / \mathfrak{g}$ with flat Riemannian metric $\hat{\gamma}$ is the affine analogue of the Poincaré model of the affine Riemann surface $\hat{S}_{\text{reg}}$ as an orbit space of a discrete subgroup of $\text{PGL}(2, \mathbb{C})$ acting on the unit disk in $\mathbb{C}$ with the Poincaré metric.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data was used in this research.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Group Theoretic Properties

In this appendix we discuss some group theoretic properties of the set of equivalent edges of $\text{cl}(K^*)$, which we use to determine the topology of $\hat{S}_{\text{reg}}$.

Let $E$ be the set of unordered pairs $[E, E']$ of nonadjacent edges of $\text{cl}(K^*)$. Define an action $\cdot$ of $G$ on $E$ by 

$$g \cdot [E, E'] = [g(E), g(E')]$$

for every unordered pair $[E, E']$ of nonadjacent edges of $\text{cl}(K^*)$. For every $g \in G$ the edges $g(E)$ and $g(E')$ are nonadjacent. This follows because the edges $E$ and $E'$ are nonadjacent and the elements of $G$ are invertible mappings of $\mathbb{C}$ into itself. So $\mathfrak{g} = g(E \cap E') = g(E) \cap g(E')$. Thus, the mapping $\cdot$ is well defined. It is an action because for every $g$ and $h \in G$ we have 

$$g \cdot (h \cdot [E, E']) = g \cdot [h(E), h(E')] = [g(h(E)), g(h(E'))]$$

$$= [(gh)(E), (gh)(E')] = (gh) \cdot [E, E'].$$
Since \( E = \bigcup_{j=0,1,\infty} E_j \), the action \( \cdot \) of \( G \) on \( E \) induces an action \( \cdot \) of the group \( G' \) of reflections on the set \( E' \) of equivalent edges of \( \text{cl}(K^*) \), which is defined by
\[
g_j \cdot [E, S_k^{(j)}(E)] = [g_j(E), g_j(S_k^{(j)}(E))] = [g_j(E), (g_j S_k^{(j)} g_j^{-1})(g_j(E))],
\]
for every \( g_j \in G_j \), every edge \( E \) of \( \text{cl}(K^*) \), and every generator \( S_k^{(j)} \) of \( G_j \), where \( k = 0, 1, \ldots, n - 1 \). Since \( g_j S_k^{(j)} g_j^{-1} = S_{k'}^{(j)} \) by Corollary 6, the mapping \( \cdot \) is well defined.

**Lemma A1.** The group \( G \) action \( \cdot \) sends a \( G' \)-orbit on \( E' \) to another \( G' \)-orbit on \( E' \).

**Proof.** Consider the \( G' \)-orbit of \( [E, S_m^{(j)}(E)] \in E' \). For every \( g \in G \) we have
\[
g_j \cdot [E, S_m^{(j)}(E)] = (g G_j g^{-1}) \cdot [E, S_m^{(j)}(E)] = g_j \cdot [E, S_m^{(j)}(E)],
\]
because \( G' \) is a normal subgroup of \( G \) by Corollary 7. Since
\[
g_j \cdot [E, S_m^{(j)}(E)] = [g(E), g(S_m^{(j)}(E))] = [g(E), g S_m^{(j)} g^{-1}(g(E))]
\]
and \( g S_m^{(j)} g^{-1} = S_{r}^{(j)} \) by Corollary 6, it follows that \( g_j \cdot [E, S_m^{(j)}(E)] \in E' \). \( \square \)

**Lemma A2.** For every \( j = 0, 1, \infty \) and every \( k = 0, 1, \ldots, n - 1 \) the isotropy group \( G_{e_k}^{(j)} \) of the \( G' \)
action on \( E' \) at \( e_k = [E, S_k^{(j)}(E)] \) is \( \langle S_k^{(j)} \mid (S_k^{(j)})^2 = e \rangle \).

**Proof.** Every \( g \in G_{e_k}^{(j)} \) satisfies
\[
e_k = [E, S_k^{(j)}(E)] = g \cdot e_k = g \cdot [E, S_k^{(j)}(E)]
\]
if and only if
\[
[E, S_k^{(j)}(E)] = [g(E), g S_k^{(j)} g^{-1}(g(E))] = [g(E), S_k^{(j)}(g(E))]
\]
if and only if one of the statements 1) \( g(E) = E \& S_k^{(j)}(E) = S_k^{(j)}(g(E)) \) or 2) \( E = g(S_k^{(j)}(E)) \& g(E) = S_k^{(j)}(E) \) holds. From \( g(E) = E \) in 1) we get \( g = e \) using Lemma 3. To see this we argue as follows. If \( g \neq e \), then \( g = R^p S_k^{(j)} \ell \) for some \( \ell = 0,1 \) and some \( p \in \{0,1,\ldots,n-1\} \), see Equation (A1). Suppose that \( g = R^p \) with \( p \neq 0 \). Then \( g(E) \neq E \), which contradicts our hypothesis. Now suppose that \( g = R^p S_k^{(j)} \). Then \( E = g(E) = R^p S_k^{(j)}(E) \), which gives \( R^{-p}(E) = S_k^{(j)}(E) \). Let \( A \) and \( B \) be end points of the edge \( E \). Then the reflection \( S_k^{(j)} \) sends \( A \) to \( B \) and \( B \) to \( A \), while the rotation \( R^{-p} \) sends \( A \) to \( A \) and \( B \) to \( B \). Thus, \( R^{-p}(E) \neq S_k^{(j)}(E) \), which is a contradiction. Hence \( g = e \). If \( g(E) = S_k^{(j)}(E) \) in 2), then \( (S_k^{(j)} g)(E) = E \). So \( S_k^{(j)} g = e \) by Lemma 3, that is, \( g = S_k^{(j)} \). \( \square \)

For every \( j = 0, 1, \infty \) and every \( m_j = 0, 1, \ldots, \frac{n}{d_j} - 1 \) let \( G_{e_m d_j}^{(j)} = \{ g_j \in G' \mid g_j \cdot \}
\]
e_m d_j = e_m d_j \} be the isotropy group of the \( G' \) action on \( E' \) at \( e_m d_j = [E, S_m^{(j)}(E)]. \) Since \( G_{e_m d_j}^{(j)} = \langle S_k^{(j)} \mid (S_k^{(j)})^2 = e \rangle \) is an abelian subgroup of \( G' \), it is a normal subgroup. Thus,
\[
H' = G' / G_{e_m d_j}^{(j)}
\]
is a subgroup of \( G' \) of order \( 2n / d_j / 2 = n / d_j \). This proves
Lemma A3. For every $j = 0, 1, \infty$ and each $m_j = 0, 1, \ldots, \frac{n}{d_j} - 1$ the $G^j$-orbit of $e_{m_j}^{j}$ in $E^j$ is equal to the $H^j$-orbit of $e_{m_j}^{j}$ in $E^j$.

Lemma A4. For $j = 0, 1, \infty$ we have $H^j = \langle V = R^d | V^{n/d_j} = e \rangle$.

Proof. Since

$$S_k^{(j)} = R_k S^{(j)} R^{-k} = R_k (R^n U) R^{-k} = R^{2k + n_j} U = R^{2k} S^{(j)}, \quad (A1)$$

we get $S_k^{(j)} = (R_k)^{m_j} U = (R^d)^{m_j} S^{(j)}$. Because the group $G^j$ is generated by the reflections $S^{(j)}_k$ for $k = 0, 1, \ldots, n - 1$, it follows that

$$G^j \subseteq \langle V = R^d, S^{(j)}_k | V^{n/d_j} = e = (S^{(j)}_k)^2 \rangle \subseteq \langle V S^{(j)}_k | V^{n/d_j} = S^{(j)}_k, V^{-1} = K_j \rangle.$$ 

$K_j$ is a subgroup of $G$ of order $2n/d_j$. Clearly the isotropy group $G^j = \langle S^{(j)}_k | (S^{(j)}_k)^2 = e \rangle$ is an abelian subgroup of $G^j$. Hence $H^j = G^j / G^j \subseteq K^j / G^j = L^j$, where $L^j$ is a subgroup of $K^j$ of order $(2n/d_j)^2 = n/d_j$. Thus, the group $L^j$ has the same order as its subgroup $H^j$. So $H^j = L^j$. However, $L^j = \langle V = R^d | V^{n/d_j} = e \rangle$. \qed

Let $f^j_\ell = R^\ell \cdot e^0$. Then

$$f^j_\ell = R^\ell \cdot e^0 = R^\ell \cdot [E, S^{(j)}(E)]$$

$$= [R^\ell(E), R^\ell S^{(j)} R^{-\ell}(R^\ell(E))] = [R^\ell(E), S^{(j)}_\ell (R^\ell(E))].$$

So

$$V^m \cdot f^j_\ell = V^m \cdot [R^\ell(E), R^\ell S^{(j)} R^{-\ell}(R^\ell(E))]$$

$$= [V^m(R^\ell(E)), V^m S^{(j)}_\ell V^{-m}(V^m(R^\ell(E)))]$$

$$= [R^{md_j + \ell(E)}, S^{(j)}_{md_j + \ell}(E)] = e^{j}_{md_j + \ell}.$$ 

This proves

$$d_j - 1 \bigcup_{\ell_j = 0}^{d_j - 1} H^j \cdot f^j_{\ell_j} = \bigcup_{\ell_j = 0}^{d_j - 1} \bigcup_{m_j = 0}^{n_j - 1} V^{m_j} \cdot f^j_{\ell_j} = \bigcup_{k=0}^{n-1} e^j_k \quad (A2)$$

since every $k \in \{0, 1, \ldots, n - 1\}$ may be written uniquely as $m_j d_j + \ell_j$ for some $m_j \in \{0, 1, \ldots, \frac{n}{d_j} - 1\}$ and some $\ell_j \in \{0, 1, \ldots, d_j - 1\}$.

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