Non- (quantum) differentiable $C^1$-functions in the spaces with trivial Boyd indices

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**Abstract.** If $E$ is a separable symmetric sequence space with trivial Boyd indices and $S_E$ is the corresponding ideal of compact operators, then there exists a $C^1$-function $f_E$, a self-adjoint element $W \in S_E$ and a densely defined closed symmetric derivation $\delta$ on $S_E$ such that $W \in \text{Dom } \delta$, but $f_E(W) \notin \text{Dom } \delta$.

1. Introduction.

This paper studies properties of infinitesimal generator $\delta^\mathcal{S}$ of a strongly continuous group $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ in Banach algebras $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, given by $\alpha_t(y) = e^{itX}ye^{-itX}$, $y \in \mathcal{S}$, where $X$ is an unbounded self-adjoint operator in the Hilbert space $\mathcal{H}$. The generator $\delta^\mathcal{S}$ is a densely defined closed symmetric derivation on $\mathcal{S}$ and we are concerned with the question when its domain $\text{Dom } \delta^\mathcal{S}$ satisfies the following condition

$$x = x^* \in \text{Dom } \delta^\mathcal{S} \Rightarrow f(x) \in \text{Dom } \delta^\mathcal{S},$$

for every $C^1$-function $f : \mathbb{R} \to \mathbb{C}$. In the recent paper [3], it is shown that there are $C^*$-algebras $\mathcal{S}$ and operators $X$ for which the implication above fails (see also [9]). In this paper, we consider the case when the Banach algebra $\mathcal{S}$ is a symmetrically normed ideal of compact operators on $\mathcal{H}$ (see e.g. [4] and Section 3 below). It is immediately clear that for every self-adjoint operator $X$, the group $\alpha$ acts isometrically on such an ideal $\mathcal{S}$ and, in fact, is a $C_0$-group on $\mathcal{S}$, provided that $\mathcal{S}$ is separable (see e.g. [2]). It is an interesting problem to determine the class of ideals $\mathcal{S}$ in which $\text{Dom } \delta^\mathcal{S}$ is closed with respect to the $C^1$-functional calculus. Note, that the class of such ideals is non-empty. For example it contains the Hilbert-Schmidt ideal. The proof of the latter claim may be found in [10]. On the other hand, it is unclear whether this class contains the Schatten-von
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Neumann ideals $\mathcal{G}$ when $1 < p < \infty$, $p \neq 2$. In this paper, we however show that the class of all symmetrically normed ideals $\mathcal{G}$ whose Boyd indices are trivial fail the implication above. For various geometric characterizations of the latter class we refer to [1] (see also Section 3 below). Our methods are built upon and extend those of [3, 9]. Our results also contribute to the study of commutator bounded operator-functions initiated in [5–7].

2. Schur multipliers.

Let $\mathbb{M}_n(\mathbb{C})$ be the $C^*$-algebra of all $n \times n$ complex matrices, let $B \in \mathbb{M}_n(\mathbb{C})$ be a diagonal matrix $\text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The Schur multiplier $M_f(B)$ associated with the diagonal matrix $B$ and the function $f : \mathbb{R} \mapsto \mathbb{C}$ is defined as follows. For every matrix $X = \{\xi_{jk}\}_{j,k=1}^n \in \mathbb{M}_n(\mathbb{C})$, the matrix $M_f(B)(X) \in \mathbb{M}_n(\mathbb{C})$ has $(j,k)$ entry given by

$$
[M_f(B)(X)]_{jk} = \psi_f(\lambda_j, \lambda_k) \xi_{jk}, \quad 1 \leq j, k \leq n,
$$

where

$$
\psi_f(\lambda, \mu) = \begin{cases} 
\frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \lambda \neq \mu, \\
0, & \lambda = \mu.
\end{cases}
$$

Alternatively, if $\{P_j\}_{j=1}^n$ is the collection of one-dimensional spectral projections of the matrix $B$ then $B = \sum_{j=1}^n \lambda_j P_j$, and

$$
M_f(B)X = \sum_{1 \leq j, k \leq n} \psi_f(\lambda_j, \lambda_k) P_j XP_k. \tag{2.1}
$$

For every matrix $X \in \mathbb{M}_n(\mathbb{C})$ the following equation outlines the interplay between the Schur multiplier $M_f(B)$ and the commutator $[B, X] = BX - XB$

$$
M_f(B)([B, X]) = [f(B), X]. \tag{2.2}
$$

Indeed,

$$
M_f(B)([B, X]) = \sum_{1 \leq j, k \leq n} \psi_f(\lambda_j, \lambda_k) P_j \left[\sum_{s=1}^n \lambda_s P_s, X\right] P_k
= \sum_{1 \leq j, k \leq n} \psi_f(\lambda_j, \lambda_k)(\lambda_j - \lambda_k) P_j XP_k
= \sum_{1 \leq j, k \leq n} (f(\lambda_j) - f(\lambda_k)) P_j XP_k
= \sum_{1 \leq j, k \leq n} P_j \left[\sum_{s=1}^n f(\lambda_s) P_s, X\right] P_k = [f(B), X].
$$
3. Symmetric spaces with trivial Boyd indices.

Let $E = E(0, \infty)$ be a symmetric Banach function space, i.e. $E = E(0, \infty)$ is a rearrangement invariant Banach function space on $(0, \infty)$ (see [8]) with the additional property that $f, g \in E$ and $g \prec f$ imply that $\|g\|_E \leq \|f\|_E$. Here $g \prec f$ denotes submajorization in the sense of Hardy, Littlewood and Polya, i.e.

$$\int_0^t g^*(s) \, ds \leq \int_0^t f^*(s) \, ds, \quad t > 0,$$

where $f^*$ (respectively, $g^*$) stands for the decreasing rearrangement of the function $f$ (respectively, $g$).

Let us consider the group of dilations $\{\sigma_\tau\}_{\tau > 0}$ defined on the space $S = S(0, \infty)$ of all Lebesgue measurable functions on $(0, \infty)$. The operator $\sigma_\tau$, $\tau > 0$ is given by

$$(\sigma_\tau f)(t) = f(\tau^{-1} t), \quad t > \infty.$$

If $E$ is a symmetric Banach function space, then the lower (respectively, upper) Boyd index $\alpha_E$ (respectively, $\beta_E$) of the space $E$ is defined by

$$\alpha_E := \lim_{\tau \to +0} \frac{\log \|\sigma_\tau\|_{E \to E}}{\log \tau} \quad \text{(respectively,} \quad \beta_E := \lim_{\tau \to +\infty} \frac{\log \|\sigma_\tau\|_{E \to E}}{\log \tau}.\quad$$

We say that the space $E$ has the trivial lower (resp. upper) Boyd index when $\alpha_E = 0$ (respectively, $\beta_E = 1$). It is known that, if $\alpha_E = 0$ (respectively, $\beta_E = 1$), then the space $E$ is not an interpolation space in the pair $(L_1, L_\infty)$ for every $p < \infty$ (respectively, $(L_q, L_\infty)$ for every $1 < q$, [8, Section 2.b].

**Proposition 3.1.** ([8, Proposition 2.b.7]) If $E$ be a symmetric sequence space and $\alpha_E = 0$ (respectively, $\beta_E = 1$), then for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exist $n$ disjointly supported vectors $\{x_j\}_{j=1}^n$ in $E$, having the same distribution, such that for every scalars $\{a_j\}_{j=1}^n$ the following holds

$$\max_{1 \leq j \leq n} |a_j| \leq \left(1 + \varepsilon\right) \max_{1 \leq j \leq n} |a_j|$$

(respectively, $1 - \varepsilon$)

$$\sum_{j=1}^n |a_j| \leq \sum_{j=1}^n |a_j|.$$

(3.1)

If $E$ is separable then $x_j$ can be chosen finitely supported.

$\mathcal{E}^E$ denotes the corresponding symmetric ideal of compact operators on the Hilbert space $\ell_2 = \ell_2(\mathbb{N})$, i.e. the space of all compact operators $x$ such that $s(x) \in E$, where $s(x)$ is the step function such that

$$s(x)(t) = s_k, \quad k < t \leq k + 1, \quad k \geq 0$$

and $\{s_k\}_{k \geq 0}$ the sequence of singular numbers (counted with multiplicities) of the operator $x$ (see e.g. [4]). The norm in the space $\mathcal{E}^E$ is given by $\|x\|_{\mathcal{E}^E} := \|s(x)\|_E$. In particular, if $E = L_p$, then the ideal $\mathcal{E}^p = \mathcal{E}^{L_p}$, $1 \leq p < \infty$ stands for the...
Schatten-von Neumann ideals of compact operators and $\mathcal{S}^\infty$ stands for the ideal of all compact operators equipped with the operator norm, see [4].

Let $\ell_2^n$ be the subspace in $\ell_2$ spanned by the first $n$ standard unit vector basis. If an element $B \in \mathcal{S}^E$ is such that $B = B|_{\ell_2^n}$, then we identify $B$ with its matrix from $M_n(\mathbb{C})$.

**Proposition 3.2.** Let $E$ be a separable symmetric function space and $\alpha_E = 0$ (respectively, $\beta_E = 1$). For every scalar $\varepsilon > 0$ and every positive integer $n \in \mathbb{N}$ there exist linear operators $\Phi_n$ and $\Psi_n$ such that

(i) $\Phi_n, \Psi_n : M_n(\mathbb{C}) \rightarrow M_{k_n}(\mathbb{C})$, where $\{k_n\}_{n\geq 1}$ is a sequence of positive integers;

(ii) the operators $\Phi_n, \Psi_n$ map diagonal (respectively, self-adjoint) matrices to diagonal (respectively, self-adjoint) matrices;

(iii) if $M_f(B), M_f(\Phi_n(B))$ are the Schur multiplier associated with the diagonal matrices $B \in M_n(\mathbb{C}), \Phi_n(B) \in M_{k_n}(\mathbb{C})$ and the function $f$, then

$$
\Psi_n(M_f(B)X) = M_f(\Phi_n(B))\Psi_n(X) \quad \text{for every matrix } X \in M_n(\mathbb{C});
$$

(iv) $\|X\|_{\mathcal{S}^\infty} \leq \|\Psi_n(X)\|_{\mathcal{S}^\infty} \leq (1 + \varepsilon)\|X\|_{\mathcal{S}^\infty}$ (respectively, $(1 - \varepsilon)\|X\|_{\mathcal{S}^1} \leq \|\Psi_n(X)\|_{\mathcal{S}^1} \leq \|X\|_{\mathcal{S}^1})$ for every matrix $X \in M_n(\mathbb{C})$.

**Proof.** Let $n$ be a fixed positive integer and $\varepsilon > 0$ be a fixed positive scalar. Let $\{x_j\}_{j=1}^n$ be a sequence of finitely and disjointly supported vectors, having the same distribution such that [5.1] holds. Let $X_0$ be the matrix given by $X_0 = diag\{x^*_1(k)\}_{k \geq 1}$, i.e. $X_0$ is the finite diagonal matrix in $\mathcal{S}^E$ that corresponds to the decreasing rearrangement $x^*_1$ in $E$. Let $I$ be the identity matrix of the same size as $X_0$. We define the linear operators $\Phi_n$ and $\Psi_n$ by

$$
\Phi_n(X) := X \otimes I, \quad \text{and} \quad \Psi_n(X) := X \otimes X_0, \quad X \in M_n(\mathbb{C}).
$$

The claims (i) (ii) now, follow immediately from the definition of $\Phi_n$ and $\Psi_n$ and the claim (iii) follows from (2.1).

Let us prove (iv). For every matrix $X \in M_n(\mathbb{C})$ there exist unitary matrices $U, V$ such that

$$
UXV = diag\{s_1, s_2, \ldots, s_n\}.
$$

Now, it follows from elementary properties of tensors, that

$$
\Phi_n(U)\Psi_n(X)\Phi_n(V) = (U \otimes I)(X \otimes X_0)(V \otimes I) = (UXV) \otimes X_0 = \Psi_n(UXV) = diag\{s_jX_0\}_{j=1}^n,
$$

and so

$$
\|\Psi_n(X)\|_{\mathcal{S}^E} = \|\Phi_n(U)\Psi_n(X)\Phi_n(V)\|_{\mathcal{S}^E} = \|diag\{s_jX_0\}_{j=1}^n\|_{\mathcal{S}^E} = \left\| \sum_{j=1}^n s_jx_j \right\|_{E}.
$$
Now, the claim in (iv) for $\alpha_E = 0$ (respectively, $\beta_E = 1$) follows from combining the equality above with the first estimate in (3.1) (respectively, the second estimate in (3.1)).

The operators $\Phi_n$, $\Psi_n$ are very similar to those, constructed in the proof of [1, Theorem 4.1].

### 4. Commutator estimates.

From now on let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a function with the following properties

1. $h(t) \in C^1(\mathbb{R} \setminus \{0\})$;
2. $h(t) = h(-t)$ when $t \neq 0$, $h(0) \geq 0$;
3. $h(\cdot)$ is increasing function on $(0, \infty)$;
4. $h(\pm \infty) = +\infty$;
5. $0 \leq h'(t)/h(t) \leq 1$ when $t \in (0, \infty)$.

**Proposition 4.1.** Let $h(t)$ be a function that satisfies the conditions (a)–(e) above. If $f$ is a function defined as follows

$$f(t) = \begin{cases} |t|(h(\log |t|))^{-1}, & \text{if } |t| < 1, \ t \neq 0, \\ 0, & \text{if } t = 0. \end{cases} \quad (4.1)$$

then $f(t) \in C^1(-1, 1)$ and $f'(t) \geq 0$ for every $t \in (0, 1)$.

**Proof.** The function given in (4.1) is even so it is sufficient to consider only the case $t \geq 0$. It follows from the definition of the function $f$ that for every $t \in (0, 1)$ function $f$ is continuously differentiable. To calculate the derivative at zero, we use the definition

$$f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} (h(\log t))^{-1} = 0.$$ 

In order to verify that $f'(t) \to 0$ when $t \to +0$, we note first that

$$f'(t) = (h(\log t))^{-1} \left(1 - \frac{h'(\log t)}{h(\log t)}\right), \quad 0 < t < 1.$$ 

Since $h(t) \geq 0$ for every $t \in \mathbb{R}$, together with the property (e) it now follows that for every $t \in (0, 1)$

$$0 \leq f'(t) \leq 2(h(\log t))^{-1} \to 0, \text{ as } t \to +0.$$ 

□
Let matrices $D, V \in \mathbb{M}_m(\mathbb{C})$ and $A, B \in \mathbb{M}_{2m}(\mathbb{C})$ be defined as follows

$$D = \text{diag}\{e^{-1}, e^{-2}, \ldots, e^{-m}\},$$
$$V = \{v_{jk}\}_{j,k=1}^m,$$
$$v_{jk} = \begin{cases} (k-j)^{-1}(e^{-j} + e^{-k})^{-1}, & \text{if } j \neq k, \\ 0, & \text{if } j = k. \end{cases}$$

(4.2)

and

$$A = \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix}, \quad B = \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}.$$

(4.3)

The following proposition provides commutator estimates in the norm of the ideal of compact operators which are very similar to those established in [3] and [9].

**Proposition 4.2.** For any function $f : \mathbb{R} \to \mathbb{R}$ given by (4.1), there exists an absolute constant $K_0$ such that for every $m \geq 3$ and for every scalar $0 < p \leq 1$ the following estimates hold

(i) $\| [B, A] \|_{\mathcal{S}_\infty} \leq \pi$,

(ii) $\| [f(pB), A] \|_{\mathcal{S}_\infty} \geq \frac{pK_0}{h(m - \log p)} \log \frac{m}{2}$.

Proof. The proof of the first claim is based on the norm estimates of the Hilbert matrix, see [3, the proof of Lemma 3.6]. Hence, we need to establish only the second one. Let us first note, since the function $f$ is even, it follows from definition of matrices $A$, $B$ that

$$f(pB)A - Af(pB) = \begin{bmatrix} 0 & f(pD)V - Vf(pD) \\ f(pD)V - Vf(pD) & 0 \end{bmatrix}.$$  

so

$$\| [f(pB), A] \|_{\mathcal{S}_\infty} = \| [f(pD), V] \|_{\mathcal{S}_\infty}.$$  

(4.4)

If $S = \{s_{jk}\}_{j,k=1}^m = f(pD)V - Vf(pD) \in \mathbb{M}_m(\mathbb{C})$, then

$$s_{kj} = s_{jk} = \frac{f(pe^{-j}) - f(pe^{-k})}{(e^{-j} + e^{-k})(k-j)} \geq 0, \quad 1 \leq j, k \leq m.$$  

If $1 \leq j < k \leq m$, then, since functions $h(t)$ and $e^t$ are monotone, we have

$$s_{jk} = \left( \frac{pe^{-j}}{h(j - \log p)} - \frac{pe^{-k}}{h(k - \log p)} \right) (e^{-j} + e^{-k})^{-1}(k-j)^{-1} \geq \frac{p(e^{-j} - e^{-k})}{h(k - \log p)} (2e^{-j}(k-j))^{-1} \geq \frac{p(1 - e^{-1})}{2h(k - \log p)(k-j)} \geq \frac{p(1 - e^{-1})}{2h(m - \log p)(k-j)}.$$  


Now, using \( \sum_{j=1}^{k-1} \frac{1}{j} \geq \log k \), we have
\[
\sum_{j=1}^{m} s_{jk} \geq \sum_{j=1}^{k-1} s_{jk} \geq \frac{p(1-e^{-1})}{2h(m - \log p)} \sum_{j=1}^{k-1} \frac{1}{k-j} \geq \frac{p(1-e^{-1})}{2h(m - \log p)} \log k.
\]

Finally, letting \( x = (m^{-1/2}, m^{-1/2}, \ldots, m^{-1/2}) \in \mathbb{C}^m \), we obtain
\[
\|S\|_{\infty} \geq \langle Sx,x \rangle \geq \frac{1}{m} \sum_{j,k=1}^{m} s_{jk} \geq \frac{1}{m} \frac{p(1-e^{-1})}{2h(m - \log p)} \sum_{k=1}^{m} \log k \geq \frac{1}{m} \frac{p(1-e^{-1})}{2h(m - \log p)} \frac{m}{2} \log \frac{m}{2}.
\]

Setting \( K_0 = \frac{(1-e^{-1})}{4} \), we have
\[
\|f(pD),V]\|_{\infty} \geq \|S\|_{\infty} \geq \frac{pK_0}{h(m - \log p)} \log \frac{m}{2}.
\]

which, together with (4.4), completes the proof. \( \square \)

Together with (2.2), Proposition 4.2 provides a lower estimate for the operator norm of Schur multiplier associated with the function \( f \), given by (4.1), and diagonal matrix \( pB \) given by (4.2) and (4.3) for every scalar \( 0 < p \leq 1 \) and every integer \( m \geq 3 \). Now we extend that lower estimate to a larger class of ideals.

**Proposition 4.3.** Let \( E \) be a separable symmetric function space with trivial Boyd indices. For every \( m \geq 3 \), let \( A_m, B_m \in M_{2m}(\mathbb{C}) \) be given by (4.2) and (4.3), \( \Phi_{2m}, \Psi_{2m} \) be the operators from the Proposition 3.2 for the \( \varepsilon = \frac{1}{2} \). There exists an absolute constant \( K_1 \) such that for every scalar sequence \( 0 < p_m \leq 1 \), and for the sequence of the diagonal matrices \( W_m = \Phi_{2m}(p_m B_m) \in M_{k_m}(\mathbb{C}) \) the following estimate holds
\[
\|M_f(W_m)\|_{\infty E \rightarrow \infty E} \geq \frac{K_1}{h(m - \log p_m)} \log \frac{m}{2}, \quad m \geq 3,
\]
where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an arbitrary function given by (4.1).

**Proof.** Letting \( X_m^\infty = [p_m B_m, \frac{1}{p_m} A_m], \quad m \geq 3, \)

we infer from Proposition 4.2 and from (2.2) that for every \( m \geq 3 \)
\[
\|X_m^\infty\|_{\infty} \leq \pi,
\]
\[
\|M_f(p_m B_m)(X_m^\infty)\|_{\mathcal{S}^\infty} = \|[f(p_m B_m), \frac{1}{p_m} A_m]\|_{\mathcal{S}^\infty} \\
\geq \frac{K_0}{h(m - \log p_m) \log \frac{m}{2}}.
\]

It follows from the definition of Schur multiplication and duality that
\[
\|M_f(p_m B_m)\|_{\mathcal{S}^1 \rightarrow \mathcal{S}^1} = \|M_f(p_m B_m)\|_{\mathcal{S}^\infty \rightarrow \mathcal{S}^\infty} \\
\geq \frac{K_0}{\pi h(m - \log p_m) \log \frac{m}{2}}, \quad m \geq 3.
\]
The last estimate implies that there exists a sequence of \(X_m^1 \in M_{2m}(\mathbb{C})\) such that
\[
\|M_f(p_m B_m)(X_m^1)\|_{\mathcal{S}^1} \geq \|X_m^1\|_{\mathcal{S}^1} \geq \frac{K_0}{2\pi h(m - \log p_m) \log \frac{m}{2}}, \quad m \geq 3.
\]
Suppose now, that \(\alpha_E = 0\) and set \(X_m = \Psi_{2m}(X_m^\infty)\) for every \(m \geq 3\). It follows from Proposition 3.2 that, for every \(m \geq 3\), \(W_m\) is a finite diagonal self-adjoint matrix such that
\[
\|M_f(W_m)(X_m)\|_{\mathcal{S}^E} \geq \frac{\|M_f(p_m B_m)(X_m^1)\|_{\mathcal{S}^1}}{\|X_m^1\|_{\mathcal{S}^1}} = \frac{\|\Psi_{2m}(M_f(p_m B_m)(X_m^\infty))\|_{\mathcal{S}^E}}{\|\Psi_{2m}(X_m^\infty)\|_{\mathcal{S}^E}} \\
\geq \frac{2}{3} \frac{\|M_f(p_m B_m)(X_m^\infty)\|_{\mathcal{S}^\infty}}{\|X_m^\infty\|_{\mathcal{S}^\infty}} \geq \frac{2K_0}{3\pi h(m - \log p_m) \log \frac{m}{2}}.
\]

If we put \(K_1 = 2K_0/(3\pi)\), that completes the proof of the case \(\alpha_E = 0\). The only difference in treating the case \(\beta_E = 1\) is that we need to use \(X_m^1\) instead of \(X_m^\infty\) in the above estimates. \(\square\)

The following proposition proves that if a function \(f : \mathbb{R} \rightarrow \mathbb{R}\) is given by (4.1) and the multipliers \(M_f(W_m)\) are not uniformly bounded in \(\mathcal{S}^E\), then this function is not commutator bounded in the sense of [6].

**Proposition 4.4.** Let \(E\) be a separable symmetric function space. If \(f\) is a \(C^1\)-function and \(W_m \in M_{k_m}(\mathbb{C})\) is a sequence of diagonal matrices \((m \geq 3)\) such that
\[
\|M_f(W_m)\|_{\mathcal{S}^E} \rightarrow \infty,
\]then there exist self-adjoint operators \(W, X\), acting on \(\ell_2\), such that
\[
[W, X] \in \mathcal{S}^E, \quad [f(W), X] \notin \mathcal{S}^E.
\]

If, in addition, the norms \(\|W_m\|_{\mathcal{S}^\infty}\) are uniformly bounded, then \(W(Dom X) \subseteq Dom X\), and if the following series converges
\[
\sum_{m \geq 3} \|W_m\|_{\mathcal{S}^E},
\]
then operator $W$ belongs to $S^E$ and
\[ \|W\|_E \leq \sum_{m \geq 3} \|W_m\|_E. \]

**Proof.** It follows from (4.5) that there exists a subsequence of positive integers $m_r$ ($r \geq 1$) and a sequence of self-adjoint matrices $X_r(i) \in M_{k'_r}(\mathbb{C})$ such that
\[ \|M_f(W_r')(X_r(i))\|_E \geq 2r^3\|X_r(i)\|_E, \quad r \geq 1, \] (4.6)
where we let, for brevity, $k'_r = k_{m_r}$ and $W_r' = W_{m_r} \in M_{k'_r}(\mathbb{C})$. Let $r \geq 1$ be fixed, let $\{\lambda_j\}_{j=1}^{k'_r}$ be the sequence of eigenvalues of the matrix $W_r'$, and let $\{P_j\}_{j=1}^{k'_r}$ be the collection of corresponding one-dimensional spectral projections. For $\lambda \in \mathbb{R}$, we set
\[ Q_{\lambda} = \sum_{1 \leq j \leq k'_r, \lambda_j = \lambda} P_j. \]
There are only a finite number of non-zero projections among $\{Q_{\lambda}\}_{\lambda \in \mathbb{R}}$, let us denote them as $\{Q_s\}_{s=1}^{s'}$, $1 \leq s \leq k'_r$ and the corresponding sequence of eigenvalues as $\{\lambda'_s\}_{s=1}^{s'}$, the scalars $\lambda'_s$ are mutually distinct. We consider the self-adjoint matrices
\[ \hat{X}_r = \sum_{j=1}^{s'} Q_j X_r(i) Q_j, \quad \text{and} \quad X_r^{(2)} = X_r(i) - \hat{X}_r. \]
It follows from (2.1) that (recall that $\psi_f(\lambda, \lambda) = 0$)
\[ M_f(W_r')(\hat{X}_r) = \sum_{1 \leq j, l \leq k'_r} \psi_f(\lambda_j, \lambda_l) P_j \hat{X}_r P_l \]
\[ = \sum_{t=1}^{s} \sum_{1 \leq j, l \leq k'_r} \psi_f(\lambda_j, \lambda_l) P_j Q_t X_r(i) P_l \]
\[ = \sum_{t=1}^{s} \sum_{1 \leq j, l \leq k'_r} \psi_f(\lambda_t, \lambda_t) Q_t X_r(i) Q_t = 0, \]
and so
\[ M_f(W_r')(X_r^{(2)}) = M_f(W_r')(X_r(i)). \] (4.7)
Now, noting that $\|\hat{X}_r\|_E \leq \|X_r(i)\|_E$ (see [4, Theorem III.4.2]) and hence $\|X_r^{(2)}\|_E \leq 2\|X_r(i)\|_E$, $X_r^{(2)}$, we infer from (4.7) and (4.6)
\[ \|M_f(W_r')(X_r^{(2)})\|_E \geq r^3\|X_r^{(2)}\|_E. \] (4.8)
We set
\[ X_r^{(3)} = \sum_{1 \leq j, l \leq k'_r} \lambda_l P_j X_r^{(2)} P_l, \]
where
\[
\lambda_{jl} = \begin{cases} 
0, & \lambda_j = \lambda_l, \\
-i \frac{\lambda_j - \lambda_l}{\lambda_j - \lambda_l}, & \lambda_j \neq \lambda_l.
\end{cases}
\]

The matrix \(X_r^{(3)}\) is self-adjoint and
\[
X_r^{(2)} = \sum_{1 \leq j, l \leq k_r'} P_j X_r^{(2)} P_l = i \sum_{1 \leq j, l \leq k_r'} \lambda_{jl}(\lambda_j - \lambda_l) P_j X_r^{(2)} P_l
\]
\[
= i \sum_{1 \leq j, l \leq k_r'} \lambda_{jl} P_j (W_r' X_r^{(2)} - X_r^{(2)} W_r') P_l
\]
\[
= i \left[ W_r', \sum_{1 \leq j, l \leq k_r'} \lambda_{jl} P_j X_r^{(2)} P_l \right] = i [W_r', X_r^{(3)}].
\]

Finally, we let
\[
X_r = r^{-2} \|X_r^{(2)}\|_{\mathcal{B}(E)}^{-1} X_r^{(3)}.
\]

For every \(r \geq 1\) we have constructed so far the finite self-adjoint matrices \(W_r', X_r\) such that
\[
\|[W'_r, X_r]\|_{\mathcal{B}(E)} \leq ||W'_r, X_r||_{\mathcal{B}(E)} \leq r^{-2} \|X_r^{(2)}\|_{\mathcal{B}(E)}^{-1} \|[W'_r, X_r^{(3)}]\|_{\mathcal{B}(E)} \leq r^{-2} \|X_r^{(2)}\|_{\mathcal{B}(E)}^{-1} \|X_r^{(2)}\|_{\mathcal{B}(E)} = \frac{1}{r^2}
\]

and
\[
\|[f(W'_r), X_r]\|_{\mathcal{B}(E)} \leq r^{-2} \|X_r^{(2)}\|_{\mathcal{B}(E)}^{-1} \|[f(W'_r), X_r^{(3)}]\|_{\mathcal{B}(E)} \leq r^{-2} \|X_r^{(2)}\|_{\mathcal{B}(E)}^{-1} \|M_f(W'_r)([W'_r, X_r^{(3)}])\|_{\mathcal{B}(E)} \leq r^{-2} \|X_r^{(2)}\|_{\mathcal{B}(E)}^{-1} \|M_f(W'_r)(X_r^{(2)})\|_{\mathcal{B}(E)} \geq r \|X_r^{(2)}\|_{\mathcal{B}(E)} \geq r.
\]

Now, we set \(\mathcal{H} = \bigoplus_{r \geq 1} C_k', X = \bigoplus_{r \geq 1} X_r\) and \(W = \bigoplus_{r \geq 1} W_r'\). Recall, that by the definition we have
\[
\mathcal{H} = \{\{\xi_r\}_{r \geq 1} : \xi_r \in C_k', \sum_{r \geq 1} \|\xi_r\|^2 < \infty\},
\]
\[
\text{Dom } X = \{\xi = \{\xi_r\}_{r \geq 1} \in \mathcal{H} : X(\xi) = \{X_r(\xi_r)\}_{r \geq 1} \in \mathcal{H}\},
\]
\[
\text{Dom } W = \{\xi = \{\xi_r\}_{r \geq 1} \in \mathcal{H} : W(\xi) = \{W_r(\xi_r)\}_{r \geq 1} \in \mathcal{H}\}.
\]
\( W, X \) are self-adjoint operators, acting on the separable Hilbert space \( \mathcal{H} \) and

\[
\| [W, X] \|_{\mathcal{E}} \leq \sum_{r \geq 1} \| [W'_r, X_r] \|_{\mathcal{E}} \leq \sum_{r \geq 1} \frac{1}{r^2} < \infty, (4.11)
\]

\[
\| [f(W), X] \|_{\mathcal{E}} \geq \max_{r \geq 1} \| [f(W'_r), X_r] \|_{\mathcal{E}} \leq \infty. (4.12)
\]

If we assume that

\[
\sum_{m \geq 3} \| W_m \|_{\mathcal{E}} < \infty,
\]

then

\[
\| W \|_{\mathcal{E}} \leq \sum_{r \geq 1} \| W'_r \|_{\mathcal{E}} \leq \sum_{m \geq 3} \| W_m \|_{\mathcal{E}} < \infty.
\]

If we assume that \( \sup_{m \geq 3} \| W_m \|_{\mathcal{E}} \leq M < \infty \), then, by (4.11), for every \( \xi = [\xi_r]_{r \geq 1} \in \text{Dom } X \),

\[
\left( \sum_{r \geq 1} \| X_r(W'_r(\xi_r)) \|^{2} \right)^{\frac{1}{2}} = \left( \sum_{r \geq 1} \| W'_r(X_r(\xi_r)) - [W'_r, X_r](\xi_r) \|^{2} \right)^{\frac{1}{2}} \leq M \left( \sum_{r \geq 1} \| X_r(\xi_r) \|^{2} \right)^{\frac{1}{2}} + \sup_{r \geq 1} \| [W'_r, X_r] \|_{\mathcal{E}} \left( \sum_{r \geq 1} \| \xi_r \|^{2} \right)^{\frac{1}{2}} < \infty.
\]

Hence \( W(\xi) \in \text{Dom } X \). The claim is proved. \( \Box \)

It follows from Propositions 4.3 and 4.4 that any function \( f : \mathbb{R} \to \mathbb{R} \) given by (4.1) with the function \( h \) satisfying the condition \( \frac{\log(m/2)}{h(m \log p_m)} \to \infty \), as \( m \to \infty \) (here \( \{p_m\}_{m \geq 0} \) is some scalar sequence satisfying \( 0 < p_m \leq 1 \)) is not commutator bounded in any separable symmetrically normed ideal with trivial Boyd indices. In other words, there exist self-adjoint operators \( W, X \), acting on a separable Hilbert space \( \mathcal{H} \), such that \( [W, X] \in \mathcal{E} \) but \( [f(W), X] \notin \mathcal{E} \). We shall now show how further adjustments to the choice of the function \( h \) and the sequence \( \{p_m\}_{m \geq 0} \) can be made in order to guarantee that the operator \( W \) above belongs to \( \mathcal{E} \). First, we need the following auxiliary results.

**Proposition 4.5.** For every \( \varepsilon > 0 \) there exists a function \( \chi_\varepsilon \) such that

(i) \( \chi_\varepsilon \in C^1(\mathbb{R}) \),
(ii) \( \chi_\varepsilon(t) = 0, \) if \( t \leq 0 \),
(iii) \( \chi_\varepsilon(t) = 1, \) if \( t \geq 1 \),
(iv) \( 0 \leq \chi'_\varepsilon \leq 1 + \varepsilon \).
Proof. Let \( \xi(t) \) be the continuous function such that \( \xi(t) = 0 \), if \( t \leq 0 \) or \( t \geq 1 \), \( \xi(t) = 1 + \epsilon \), if \( \epsilon/(1 + \epsilon) \leq t \leq 1/(1 + \epsilon) \) and linear elsewhere. It then follows, that the function
\[
\chi(\epsilon)(t) = \int_{-\infty}^{t} \xi(\tau) \, d\tau, \quad t \in \mathbb{R},
\]
satisfies the assertion. □

Proposition 4.6. Let \( s_m, q_m \) \( (m \geq 0) \) be two increasing sequences such that

(i) \( s_m \to +\infty \), \( s_0 = 0 \),
(ii) \( q_m \to +\infty \), \( q_0 = 1 \),
(iii) \( \alpha = \sup_{m \geq 1} \frac{\log q_m - \log q_{m-1}}{s_m - s_{m-1}} < 1 \).

Then there exists a function \( h \) that satisfies the conditions (a)–(e) (preceding Proposition 4.1) and such that \( h(s_m) = q_m \) for every \( m \geq 0 \).

Proof. Let \( \epsilon = 1/\alpha - 1 \), and \( \chi(\epsilon) \) be the function from Proposition 4.5. For every \( t \geq 0 \) we define
\[
H(t) = \sum_{m \geq 1} \chi(\epsilon) \left( \frac{t - s_{m-1}}{s_m - s_{m-1}} \right) \left( \log q_m - \log q_{m-1} \right). \tag{4.13}
\]

We have that \( x \geq s_{m-1} \) (respectively, \( x < s_m \)) if and only if
\[
\frac{x - s_{m-1}}{s_m - s_{m-1}} \geq 0 \quad \text{(respectively, } \frac{x - s_{m-1}}{s_m - s_{m-1}} < 1 \).
\]

Now, it follows from above that for every fixed \( t \geq 0 \) the sum (4.13) is finite,
\[
H(s_0) = H(0) = 0 = \log q_0,
\]
and for every \( k \geq 1 \)
\[
H(s_k) = \sum_{m \geq 1} \chi(\epsilon) \left( \frac{s_k - s_{m-1}}{s_m - s_{m-1}} \right) \left( \log q_m - \log q_{m-1} \right)
= \sum_{m=1}^{k} \left( \log q_m - \log q_{m-1} \right) = \log q_k.
\]

We set \( h(t) := \exp(H(t)) \) for every \( t \geq 0 \) and \( h(t) = h(-t) \) for every \( t < 0 \), then \( h(s_m) = q_m \) for every \( m \geq 0 \). Let us check the conditions (a)–(e)

(a) The function \( H \) is a \( C^1 \)-function as a finite sum of \( C^1 \)-functions, so \( h \) is a \( C^1 \)-function for every \( t \neq 0 \);
(b) this item holds by the definition of \( h(t) \), and \( h(0) = \exp(H(0)) = 1 \);
(c) the function \( H(t) \) is increasing, for every \( t \geq 0 \), as it is the sum of increasing functions, so the function \( h \) is increasing also;
(d) since the sequence \( h(s_m) = q_m \) tends to infinity and since \( h(\cdot) \) is an increasing even function, we have \( h(\pm \infty) = +\infty \);
(e) for every $t \geq 0$ there exists an integer $k \geq 1$ such that $s_{k-1} \leq t < s_k$, thus it follows from Proposition 4.3.3 that

$$H'(t) = \sum_{m \geq 1} \chi'_\varepsilon \left( \frac{t - s_{m-1}}{s_m - s_{m-1}} \right) \frac{\log q_m - \log q_{m-1}}{s_m - s_{m-1}}$$

$$= \chi'_\varepsilon \left( \frac{t - s_{k-1}}{s_k - s_{k-1}} \right) \frac{\log q_k - \log q_{k-1}}{s_k - s_{k-1}}$$

$$\leq \alpha(1 + \varepsilon) = 1,$$

and so

$$0 \leq H'(t) = \frac{h'(t)}{h(t)} \leq 1.$$  \hspace{1cm} \Box$$

Now, we are in a position to prove our main result.

**Theorem 4.7.** For every separable symmetric function space $E$ with trivial Boyd indices, there exists a $C^1$-function $f_E$, self-adjoint operators $W, X$, acting on a separable Hilbert space $H$ such that

$$W \in \mathcal{G}^E, \quad [W, X] \in \mathcal{G}^E, \quad W(Dom X) \subseteq Dom X, \quad [f_E(W), X] \notin \mathcal{G}^E.$$  

**Proof.** Let $m \geq 0, q_m := (\log(m + e))^{1/2}, B_m$ be the diagonal matrices, given by (4.2) and (4.3), $\Phi_{2m}, \Psi_{2m}$ be the operators from Proposition 3.2 for $\varepsilon = 1/2$. Let $\{p_m\}_{m \geq 0}$ be a sequence that satisfies the following five conditions

(i) $p_m$ is decreasing to zero;
(ii) $0 < p_m \leq 1$;
(iii) $p_0 = 1$;
(iv) $\frac{1}{p_m} \geq m^2 \|\Phi_{2m}(B_m)\|_{\mathcal{E}}, m \geq 1$;
(v) $\frac{1}{p_m} \geq \frac{q_m^2}{\varepsilon q_{m-1}} : \frac{1}{p_m - 1}, m \geq 1$.

We construct such a sequence by induction. If the numbers $p_0, p_1, \ldots, p_{m-1}$ satisfy the conditions above, then $p_m$ can be taken to be any positive number for which

$$\frac{1}{p_m} \geq \max \left\{ 1, \frac{1}{p_{m-1}}, m^2 \|\Phi_{2m}(B_m)\|_{\mathcal{E}}, \frac{q_m^2}{\varepsilon q_{m-1}} : \frac{1}{p_m - 1} \right\}.$$  

It follows from (v) above, that

$$\frac{ep_{m-1}}{p_m} \geq \frac{q_m^2}{q_{m-1}} ,$$

and, taking logarithms,

$$1 + \log \frac{1}{p_m} - \log \frac{1}{p_{m-1}} \geq 2(\log q_m - \log q_{m-1}).$$
Putting \( s_m = m - \log p_m \), we have
\[
0 \leq \frac{\log q_m - \log q_{m-1}}{s_m - s_{m-1}} \leq \frac{1}{2}, \quad m \geq 0.
\]
Thus, we have verified that the sequences \( \{q_m\}_{m \geq 0} \) and \( \{s_m\}_{m \geq 0} \) satisfy the conditions of Proposition 4.6 and so there exists a function \( h_E(t) \) such that
\[
h_E(m - \log p_m) = h_E(s_m) = q_m = (\log(e + m))^{1/2}.
\]
If \( f_E \) is the function, given by (4.1), with the above choice of \( h_E \), \( W_m := \Phi_{2m}(p_mB_m) \in M_{k_m}(\mathbb{C}) \), where \( B_m \) given by (4.2) and (4.3), is the finite diagonal matrix then it follows from Proposition 4.3
\[
\|M_f(W_m)\|_{\mathcal{E} \to \mathcal{E}} \geq \frac{K_1}{h(m - \log p_m)} \log \frac{m}{2} = \frac{K_1}{(\log(m + e))^{1/2}} \to \infty.
\]
On the other hand, by our choice of \( p_m \) (see (iv) above) we have that
\[
\|W_m\|_{\mathcal{E} \to \mathcal{E}} \leq p_m\|\Phi_{2m}(B_m)\|_{\mathcal{E} \to \mathcal{E}} \leq \frac{1}{m^2}, \quad m \geq 3.
\]
Now the assertion of Theorem 4.7 follows from Proposition 4.6. \( \square \)

5. Domains of generators of automorphism flows.

Let \( E \) be a separable symmetric function space with trivial Boyd indices, let \( \mathcal{S}^E \) be the corresponding symmetrically normed ideal and let \( X \) be a self-adjoint operator (may be unbounded), acting on a separable Hilbert space \( \mathcal{H} \). We consider a group \( \alpha = \{\alpha_t\}_{t \in \mathbb{R}} \) of automorphisms on \( \mathcal{S}^E \) given by
\[
\alpha(t)T = e^{itX} Te^{-itX}, \quad T \in \mathcal{S}^E, \quad t \in \mathbb{R}.
\]
It follows from separability of \( \mathcal{S}^E \) that \( \alpha \) is a \( C_0 \)-group, [2, Corollary 4.3]. The infinitesimal generator \( \delta \) of \( \alpha(t) \) is defined by
\[
\text{Dom } \delta := \left\{ T \in \mathcal{S}^E : \text{ there exists } \| \cdot \|_{\mathcal{S}^E} - \lim_{t \to 0} \frac{\alpha(t)T - T}{t} \right\},
\]
\[
\delta(T) := \| \cdot \|_{\mathcal{S}^E} - \lim_{t \to 0} \frac{\alpha(t)T - T}{t} \quad \text{ for every } T \in \text{Dom } \delta.
\]
\( \delta \) is a closed densely defined symmetric derivation on \( \mathcal{S}^E \), i.e. densely defined closed linear operator such that \( \delta(T^*) = \delta(T)^* \) and \( \delta(TS) = \delta(T)S + T\delta(S) \), for all \( T, S \in \text{Dom } \delta \), see e.g. [2, Proposition 4.5]. It is proved in [3, Proposition 2.2], that
\[
\text{Dom } \delta = \{ T \in \mathcal{S}^E : \ T(\text{Dom } X) \subseteq \text{Dom } X, \ [T, X] \in \mathcal{S}^E \}
\]
and
\[
\delta(T) = i[T, X], \quad T \in \text{Dom } \delta.
\]
Now, Theorem 4.7 yields
Corollary 5.1. For every separable symmetric function space $E$ with trivial Boyd indices, there exists a $C^1$-function $f_E$, a self-adjoint operator $W \in \mathcal{S}^E$ and closed densely defined symmetric derivation $\delta$ on $\mathcal{S}^E$, such that $W \in \text{Dom} \, \delta$ but 

$$f_E(W) \notin \text{Dom} \, \delta.$$ 

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