Spectrum of Strongly Regular Graphs under Graph Operators

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Abstract

The spectrum of a graph $G$ is the collection of all eigen values of the adjacency matrix of $G$. A graph $G$ is strongly regular with parameters $(n, k, a, c)$ if $G$ is neither complete nor empty, any two adjacent vertices of $G$ have a common neighbours and any two non-adjacent vertices of $G$ have $c$ common neighbours. An edge-regular graph with parameters $(n, k, a)$ is a graph on $n$ vertices which is regular of degree $k$ and any two adjacent vertices have exactly $a$ common neighbours. In this paper, we show that if $G$ is strongly regular then the Gallai graph $\Gamma(G)$ and the anti-Gallai graph $\Delta(G)$ of $G$ are edge-regular. Also we find the adjacency spectrum of Gallai and anti-Gallai graph of some strongly regular graphs.

Keywords: Gallai Graph, Anti-Gallai Graph, Strongly Regular Graph, Edge-regular Graph, Adjacency Spectrum

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1 Introduction

Let $G$ be a simple graph on $p$ vertices $\{1, 2, ..., p\}$ with an adjacency matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & \text{if the vertex } i \text{ is adjacent to the vertex } j \\ 0, & \text{otherwise} \end{cases}$$

Then $A(G)$ is a real symmetric matrix with zero as the diagonal entries. The polynomial $\det(\lambda I - A) = 0$ is called the characteristic polynomial of $A$ or $G$ and is denoted by $P_G(\lambda)$. The eigen values of $A$, which are zeros of $|\lambda I - A| = 0$ are called the eigen values of $G$ and form its spectrum denoted by $\text{spec}(G)$ \[8\]. If the distinct eigen values of $G$ are $\lambda_1, \lambda_2, ..., \lambda_m$ with multiplicities $t_1, t_2, ..., t_m$ respectively then $\text{spec}(G)$ is denoted by $[\lambda_1^{t_1}, \lambda_2^{t_2}, ..., \lambda_m^{t_m}]$. Since the adjacency matrix is a real symmetric matrix the eigen values can be ordered as $\lambda_1 \geq \lambda_2, ..., \geq \lambda_n$ \[7\].

The line graph $L(G)$ of a graph \[14\] $G$ has the edges of $G$ as its vertices and two distinct edges of $G$ are adjacent in $L(G)$ if they are adjacent in $G$. The adjacency spectrum of line graph of many classes of graphs has been studied in \[7, 8\] and \[12\]. The Gallai graph $\Gamma(G)$ of a graph $G$ \[15\] has the edges of $G$ as its vertices and two distinct edges are adjacent in $\Gamma(G)$ if they are adjacent in $G$, but do not lie on a common triangle in $G$. The anti-Gallai graph $\Delta(G)$ \[15\] of a graph $G$ has the edges of $G$ as its vertices and two distinct edges of $G$ are adjacent in $\Delta(G)$, if they lie on a common triangle in $G$. Clearly, the Gallai graph and the anti-Gallai graph are spanning subgraphs of the line graph \[15\]. Though the line graphs are extensively studied in literature, there are only a handful of papers in Gallai and anti-Gallai graphs. Some of the interesting papers on the Gallai and anti- Gallai graphs are \[2, 3, 9, 15\] and \[16\].

Let $G$ be a $k$- regular graph with $n$ vertices. The graph $G$ is said to be strongly regular \[5\] with parameters $(n, k, a, c)$ if the following conditions hold:

1. $G$ is neither complete nor empty;
2. any two adjacent vertices of $G$ have 'a' common neighbours;
3. any two non-adjacent vertices of $G$ have 'c' common neighbours.

In literature, by weakening the above definition we have the definition of an edge- regular graph. An edge-regular graph \[17\] with parameters $(n, k, a)$ is a graph on $n$ vertices which is regular of degree $k$ and any two adjacent vertices have exactly $a$ common neighbours.
The anti-Gallai graph find its application in linguistics to identify the polysemous words [1], which motivated us to examine its spectra. The spectrum of some classes of Gallai and anti-Gallai graphs are studied in [10] and [11]. If $G$ is regular then $L(G)$ is also regular and this property was helpful in studying the spectrum of line graphs. Unfortunately, this nice behaviour is not there for the Gallai and the anti-Gallai graphs. But we observed that $\Gamma(G)$ and $\Delta(G)$ of a strongly regular graph $G$ is regular. So in this paper we focus on the Gallai and the anti-Gallai graphs of strongly regular graphs.

1.1 Standard Definitions

**Definition 1.1.** The fan graph denoted by $F_n$ is $n$ copies $K_3$’s with a common vertex. Note that $F_n$ is a planar undirected graph with $2n + 1$ vertices and $3n$ edges [13].

**Definition 1.2.** Two triangles sharing a common edge i.e; $K_4 - e$, is called a diamond [9].

**Definition 1.3.** The semi total point graph $R(G)$ of a graph $G$ is obtained from $G$ by adding a new vertex corresponding to every edge of $G$, then joining each new vertex to the end vertices of the corresponding edge i.e; each edge of $G$ is replaced by a triangle [18].

**Definition 1.4.** Let $G_1$ and $G_2$ be vertex-disjoint graphs. The join, $G_1 \vee G_2$, of $G_1$ and $G_2$ is the supergraph of $G_1 + G_2$ in which each vertex of $G_1$ is adjacent to every vertex of $G_2$ [4].

All graph theoretic notations and terminology not mentioned here are from [7] and [4].

2 Strongly regular Graphs

If $G = (n, k, a, c)$ is a connected strongly regular graph then, we have the following direct observations.

Observation 1: $c \geq 1$.

Observation 2: An edge $e = uv$ and a vertex $w$ must belong to cycle $C_n$ of length at most 5.
Observation 3: Every pair of edges must belong to a cycle of length at most 6.

Observation 4: If $c > 1$ then every pair of non-adjacent vertices must belong to at least one cycle of length 4.

Observation 5: If $c = 1$ then diamonds and $C_4$’s are forbidden in $G$.

Observation 6: A strongly regular graph is 2-connected.

Observation 7: If $a = 1$ then diamonds and $K_4$’s are forbidden in $G$.

Lemma 2.1. Let $G = (n, k, a, c)$ be a connected strongly regular graph with $a = 1$ then every vertex together with its neighbours form a $k/2$-fan.

Proof. Consider an arbitrary vertex $u$ in $G$. Since $G$ is connected, there exists an edge incident on $u$ and by hypothesis that edge belongs to a $K_3$. $G \neq K_3$ and regular implies that there exists another edge incident on $u$. But by Observation 7, diamonds are forbidden in $G$ and hence that edge also belongs to another $K_3$. Proceeding like this we get $u$ together with its neighbours form a $k/2$-fan.$\square$

Lemma 2.2. Let $G = (n, k, a, c)$ be a connected strongly graph with $a = 2$ then every vertex is a central vertex of at least one wheel.

Proof. Consider $u \in V(G)$. As in the above proof an edge $e_1$ incident on $u$ will be a central edge of a diamond. Let $e_2$ and $e_3$ be the other edges of the diamond which are incident on $u$. Since $a = 2$ there exists a diamond with $e_2$ as the central edge. The same argument holds for $e_3$ also. Proceeding like this, the process will terminate only when $u$ together with these edges form a wheel. It may happen that there exists another edge incident on $u$ which is not in the above wheel. In this case by the same argument we can find another wheel with $u$ as the central vertex.$\square$

The following theorem is useful for us.

Theorem 2.3. [5][8] The distinct eigenvalues of a connected strongly regular graph $G = (n, k, a, c)$ are $k, \frac{1}{2}[(a - c) + \sqrt{(a - c)^2 + 4(k - c)}]$ and $k, \frac{1}{2}[(a - c) + \sqrt{(a - c)^2 - 4(k - c)}].$
Theorem 2.4. If $G = (n, k, a, c)$ be a strongly regular graph then $\Gamma(G)$ and $\Delta(G)$ are regular graphs.

Proof. Consider an arbitrary vertex $x$ of $\Gamma(G)$. Let $uv$ be the edge corresponding to this vertex in $G$. Then by the definition of $\Gamma(G)$, the degree of a vertex $x$ is $d_{\Gamma(G)}(x) = d_G(u) + d_G(v) - 2|N(u) \cap N(v)| - 2$, where $d_{\Gamma(G)}(x)$ denotes the degree of $x$ in $\Gamma(G)$ and $d_G(u), d_G(v)$ denote the degrees of $u, v$ in $G$. Hence $\Gamma(G)$ is $2(k - a - 1)$-regular.

Now consider an arbitrary vertex $x$ in $\Delta(G)$. Let $e = uv$ be the corresponding edge in $G$. Then the degree of $x$ in $\Delta(G)$ is $2|N(u) \cap N(v)| = 2a$. Therefore, $\Delta(G)$ is $2a$-regular.

\[\square\]

3 Gallai Graph

As per Theorem 2.4 we can see that $\Gamma(G)$ and $\Delta(G)$ of a strongly regular graph are regular graphs. But, in this section we observe that $\Gamma(G)$ of some special class of strongly regular graphs yield something more than the regularity. i.e., if $G$ is strongly regular then the Gallai graph $\Gamma(G)$ is edge-regular.

Theorem 3.1. Let $G$ be a connected graph. The Gallai graph $\Gamma(G)$ is disconnected if and only if there exists a partition of the edge set into $E_1, E_2, ... E_p$ where $p \geq 2$, such that $e_i \in E_i$ and $e_j \in E_j$ are incident in $G$ implies $e_i$ and $e_j$ span a triangle in $G$.

Proof. Suppose that $\Gamma(G)$ is disconnected and let $\Gamma_1, \Gamma_2, ..., \Gamma_p$ with $p \geq 2$ be the components of $\Gamma(G)$. Let $E_i = \{ e \in G : e \text{ is an edge corresponding to a vertex } v \text{ in } \Gamma_i \}$, where $1 \leq i \leq p$. Clearly, $E_i$ is a partition for $E(G)$. Since the connectedness of $G$ implies the connectedness of $L(G)$, at least one edge of $E_i$ is incident with some $e \in E_j$ for some $j$. But, $\Gamma(G)$ is disconnected and hence if $e_i \in E_i$ is incident with $e_j \in E_j$ then they must span a triangle in $G$.

For the converse part assume that such a partition exists for $E(G)$. Then for any $i$ and $j$, the vertices corresponding to the edges in $E_i$ and $E_j$ are in different components in $\Gamma(G)$.

\[\square\]

Theorem 3.2. Let $G = (n, k, 0, c)$ be a connected strongly regular graph. Then $\Gamma(G)$ is connected and edge-regular. Also it is strongly regular if and only if the
following conditions hold:
1. If $c = 1$ then any two non-adjacent edges belong to a common $C_5$.
2. If $c > 1$ then any two non-adjacent edges belong to a common $C_4$.

Proof. Given $a = 0$ and hence $G$ is $K_3$-free and $\Gamma(G) \cong L(G)$. Since $G$ is connected and $k$-regular $L(G)$ is connected and $2k - 2$ regular.

Consider two adjacent vertices $x$ and $y$ in $L(G)$. Let $e_1$ and $e_2$ be the corresponding edges in $G$, then $e_1$ and $e_2$ have a common vertex in $G$. The number of common vertices of $x$ and $y$ in $L(G)$ is same as the number of edges incident on the common vertex of both $e_1$ and $e_2$. Since the graph is $k$-regular it is equal to $k - 2$. Hence $\Gamma(G)$ is an edge-regular graph with parameters $(nk^2, 2k - 2, k - 2)$.

Now consider two non-adjacent vertices $x$ and $y$ in $L(G)$. Let $f_1$ and $f_2$ be the corresponding edges in $G$. Then $f_1$ and $f_2$ have no common vertex in $G$. The number of common vertices of $x$ and $y$ in $L(G)$ is same as the number of edges adjacent to both $f_1$ and $f_2$.

If $c = 1$, by Observations 3 and 5, $f_1$ and $f_2$ belong to $C_5$ or $C_6$ or both. If $f_1$ and $f_2$ belong to $C_5$ then the number of common vertices of $x$ and $y$ is 1, otherwise it is 0. Since $G$ is strongly regular by Observations 2 and 5 there exist non adjacent edges which belong to a $C_5$. So $L(G)$ is strongly regular if and only if any two non-adjacent edges belong to a $C_5$.

If $c > 1$, $f_1$ and $f_2$ belong to $C_4$, $C_5$ or $C_6$. If $f_1$ and $f_2$ belong to $C_4$ then the number of common vertices of $x$ and $y$ is 2; otherwise it is 1 or 0. Since $c > 1$, in $G$ there are edges which belong to $C_4$. So $\Gamma(G)$ is strongly regular if and only if any two non-adjacent edges belong to a $C_4$.

\[\text{Theorem 3.3.} \quad \text{Let } G = (n, k, 0, 1) \text{ be a connected strongly regular } C_6\text{-free graph then the distinct eigenvalues of } \Gamma(G) \text{ are } 2k - 2, \frac{1}{2}[(k - 3) + \sqrt{(k - 3)^2 + 4(2k - 3)}] \text{ and } \frac{1}{2}[(k - 3) - \sqrt{(k - 3)^2 + 4(2k - 3)}].\]

Proof. Since $c = 1$ by Observations 3 and 5 any two non-adjacent edges must belong to either $C_6$ or $C_5$ or both. Since the graph is $C_6$-free any two edges must belong to $C_5$, hence by Theorem 3.2 $\Gamma(G)$ is a strongly regular graph with parameters $(\frac{nk}{2}, 2k - 2, k - 2, 1)$. Then the result follows from Theorem 2.3. \qed
Theorem 3.4. Let $G = (n, k, 0, c)$, where $c > 1$ be a connected strongly regular graph in which every pair of non-adjacent edges belong to a $C_4$. Then the distinct eigenvalues of $\Gamma(G)$ are $2k - 2, \frac{1}{2}[(k - 4) + \sqrt{(k - 4)^2 + 4(2k - 4)}]$ and $\frac{1}{2}[(k - 4) - \sqrt{(k - 4)^2 + 4(2k - 4)}]$.

Proof. Since $c > 1$, by Theorem 3.2, $\Gamma(G)$ is a strongly regular graph with parameters $(\frac{nk}{2}, 2k - 2, k - 2, 2)$. Then the result follows from Theorem 2.3.

Theorem 3.5. Let $G = (n, k, a, c)$, where $a = 0$ or $1$ be a connected strongly regular graph then $\Gamma(G)$ is 2-connected.

Proof. We have the following cases.

Case 1: $a = 0$
Since $G$ is $K_3$-free, $\Gamma(G) \cong L(G)$. When we consider any two vertices of $L(G)$, by Observation 3, the corresponding edges belong to a $C_4, C_5$ or $C_6$. So in $L(G)$ any two vertices belong to a $C_4, C_5$ or $C_6$. Hence it is 2-connected.

Case 2: $a = 1$
In order to show that $\Gamma(G)$ is 2-connected it is enough to show that any two vertices belong to a cycle. Consider two vertices $x$ and $x'$. The following cases arise.

1. $xx'$ is an edge in $\Gamma(G)$.
2. $xx'$ is not an edge in $\Gamma(G)$.

Subcase A: $xx'$ is an edge in $\Gamma(G)$ implies that the corresponding edges $e$ and $e'$ are incident in $G$ and not belong to a $K_3$. By Lemma 2.1 there exists an edge $l$ such that $e$ and $l$ span a $K_3$ in $G$. Similarly there exists edge $l'$ such that $e'$ and $l'$ span a $K_3$ in $G$. Then the vertices corresponding to $e, l, e', l'$ form a path $P_4$ in $\Gamma(G)$. Then this path together with the edge $xx'$ form a $C_4$ in $\Gamma(G)$.

Subcase B: $xx'$ not an edge in $\Gamma(G)$ means that either the corresponding edges are incident in $G$ and belong to a $K_3$ or are not incident. For the first case, by Lemma 2.1 there exist at-least two edges $l$ and $l'$ which span a $K_3$ in $G$. Then the vertices corresponding to $e, l, e', l'$ span a $C_4$ in $\Gamma(G)$. For the latter case by Observation 3, $e$ and $e'$ belong to a cycle $C_n$ where $3 < n < 7$ in $G$. If it is an induced $C_n$ clearly $x$ and $x'$ belong to a $C_n$ in $\Gamma(G)$. If not, there are edges which belong to a $K_3$ in $G$. Consider two edges $e_1$ and $e_2$ with a common vertex $u$ and span a $K_3$ in $G$. By Lemma 2.1 there exist another edges $l$ and $l'$ incident
on \( u \) which span a \( K_3 \) in \( G \). Since diamonds are forbidden in \( G \), the vertices corresponding to \( e_1, l, e_2 \) form a \( P_3 \) in \( G \). So, if two edges span a \( K_3 \) in \( G \) then by the above explanation we can find an edge such that the vertices corresponding to the edges of \( K_3 \) and the new edge form a \( P_3 \) in \( G \). Also if \( e_1, e_2, e_3 \) are three consecutive edges in \( C_n \) and if \( e_1 \) and \( e_2 \) span a \( K_3 \) in \( G \), since diamonds are forbidden in \( G \), \( e_2 \) and \( e_3 \) cannot span a \( K_3 \) in \( G \). So by the above explanation in \( \Gamma(G) \), we can find a cycle \( C_n \) of length at most 9 containing the vertices \( x \) and \( x' \). That is, in \( \Gamma(G) \) any two vertices belong to at least one \( C_n \). Hence \( \Gamma(G) \) is 2-connected.

\[ \square \]

**Theorem 3.6.** If \( G = (n, k, 1, c) \) is a connected strongly regular then \( \Gamma(G) \) is edge-regular and is strongly regular if and only if \( c > 1 \) and any two non-adjacent edges belong to a \( C_4 \).

**Proof.** Since \( G \) is strongly regular \( \Gamma(G) \) is \( 2(k - 2) \)-regular by Theorem 2.4. Now consider two adjacent vertices \( v_1 \) and \( v_2 \) in \( \Gamma(G) \). Let \( e_1 \) and \( e_2 \) be the corresponding edges in \( G \), then the number of common vertices of \( v_1 \) and \( v_2 \) is same as the number of \( K_{1,3} \)’s in which \( e_1 \) and \( e_2 \) are present. By Lemma 2.1 since the number of such induced \( K_{1,3} \) is \( k - 2 \), any two adjacent vertices have \( k - 2 \) common neighbours. Hence \( \Gamma(G) \) is an edge-regular graph with parameters \((\frac{n}{2}, 2k - 4, k - 2)\).

To prove that \( \Gamma(G) \) is strongly regular, consider two vertices \( v_1 \) and \( v_2 \) which are non-adjacent in \( \Gamma(G) \). Let \( e_1 \) and \( e_2 \) be the corresponding edges in \( G \). Then in \( G \) either \( e_1 \) and \( e_2 \) span a triangle or \( e_1 \) and \( e_2 \) are non-adjacent.

In the first case, the number of common neighbours of \( v_1 \) and \( v_2 \) is same as the number of edges which form \( K_{1,2} \) with both \( e_1 \) and \( e_2 \). By Lemma 2.1 it is same as \( k - 2 \).

In the latter case, the number of common vertices is same as the number of induced \( P_3 \)’s with end edges \( e_1 \) and \( e_2 \). To find the number of induced \( P_3 \)’s we consider the following cases.

1) \( c = 1 \)
2) \( c > 1 \)

If \( c = 1 \) by Observation 3, \( e_1 \) and \( e_2 \) may belong to a \( C_5 \) or \( C_6 \). By Observation 5, \( C_4 \)’s are forbidden in \( G \). Therefore the number of such induced \( P_3 \) is one if \( e_1 \) and \( e_2 \) belong to a \( C_5 \); otherwise it is zero. Since \( G \) contains \( C_5 \), \( \Gamma(G) \) is strongly regular if and only if \( k - 2 = 1 \) and any two non-adjacent edges belong to at least one \( C_5 \). But it is not possible since \( k \) is an even number by Lemma 2.1.
If $c > 1$, by the same argument in the above theorem $\Gamma(G)$ is strongly regular if and only if $k - 2 = 2$ and any two edges belong in a $C_4$. Hence $\Gamma(G)$ is strongly regular if and only if $k = 4, c > 1$ and any two non adjacent edges belong to a $C_4$.

**Theorem 3.7.** Let $G = (n, k, 1, c)$, where $c > 1$ be a connected strongly regular graph with $C_4$ connecting any two edges then the distinct eigenvalues of $\Gamma(G)$ are $4, \sqrt{2}$ and $-\sqrt{2}$.

**Proof.** If $G = (n, k, 1, c)$, where $c > 1$ be a connected strongly regular graph with $C_4$ connecting any two edges, by Theorem 3.6 $\Gamma(G)$ is a connected strongly regular graph with parameters $(2n, 4, 2)$. Then by Theorem 2.3 the distinct eigenvalues of $\Gamma(G)$ are $4, \sqrt{2}$ and $-\sqrt{2}$.

## 4 Anti-Gallai graph

**Theorem 4.1.** Let $G$ be a connected graph. The anti-Gallai graph $\Delta(G)$ is disconnected if and only if there exists a partition of the edge set into $E_1, E_2, ...E_p$ where $p \geq 2$, such that if $e_i \in E_i$ and $e_j \in E_j$ are incident then $e_i$ and $e_j$ not belong to a $K_3$ and there exists at least one pair of this type.

**Proof.** Suppose that $\Delta(G)$ is disconnected and let $\Delta_1, \Delta_2, ...\Delta_p$ with $p \geq 2$ be the components of $\Delta(G)$. As in the proof of Theorem 3.1 consider $E_i = \{e \in G : e$ is an edge corresponding to a vertex $v$ in $\Delta_i\}$, where $1 \leq i \leq p$. Clearly, $E_i$ is a partition for $E(G)$. Since the connectedness of $G$ implies the connectedness of $L(G)$, at least one edge $e_i \in E_i$ is incident with some $e_j \in E_j$. But, $\Delta(G)$ is disconnected and hence if $e_i \in E_i$ is incident with $e_j \in E_j$ then they are not belong to a triangle in $G$.

For the converse part assume that such a partition exists for $E(G)$. Then for any $i$ and $j$, the vertices corresponding to the edges in $E_i$ and $E_j$ induce different components in $\Delta(G)$.

**Theorem 4.2.** If $G = (n, k, 0, c)$ be a connected strongly regular graph then the spectrum of $\Delta(G)$ is $\{0, \frac{kn}{c} \}$.

**Proof.** Since $a = 0$, $G$ is $K_3$-free. Hence $\Delta(G)$ is totally disconnected. Therefore the spectrum consists only zero value.
Theorem 4.3. If $G = (n, k, 1, c)$ be a connected strongly regular graph. Then the spectrum of $\Delta(G)$ is $(-1^{kn}, 2^{kn})$.

Proof. Since $a = 1$, every edge of $G$ belong to exactly one $K_3$ and no two $K_3$ share a common edge. Therefore $\Delta(G)$ is the disjoint union of $\frac{kn}{6}$ triangles. Hence the spectrum of $\Delta(G)$ consists of the spectrum of $K_3$’s.

Theorem 4.4. $\Delta(L(K_{n,n}))$ is cospectral with $2nL(K_n)$.

Proof. $L(K_{n,n})$ contains $2n$ copies of $K_n$ sharing common vertices, where two $K_n$’s have no common edges and the edges of two copies of $K_n$ not belong to a $K_3$. Hence $\Delta(L(K_{n,n}))$ is the disjoint union of $\Delta(K_n)$’s ($2n$ times). Since $K_n$ is a $K_{1,2}$-free graph, $\Delta(K_n) \cong L(K_n)$. Hence the spectrum of $\Delta(L(K_{n,n}))$ is same as the spectrum of $L(K_n)$ repeating $2n$ times.

Theorem 4.5. Let $G = (n, k, 2, c)$ is a connected strongly regular graph where each vertex belong to exactly one wheel then $\Delta(G)$ is connected and edge-regular.

Proof. Suppose $\Delta(G)$ is disconnected. Then by Theorem 4.1 there exists at least two edges $e_i \in E_i$ and $e_j \in E_j$; are incident but not belong to a $K_3$. Let $u$ be the common vertex of both $e_i$ and $e_j$. By assumption since the neighbouring vertices induce a wheel in $G$ there exist edges $e_{i+1}, e_{i+2}, ..., e_{j-1}, e_j$ such that the pairs of edges $(e_i, e_{i+1}), (e_{i+1}, e_{i+2}), ..., (e_{j-1}, e_j)$ belong to a $K_3$ in $G$. Since $e_i \in E_i$, the edges $e_{i+1}, e_{i+2}, ..., e_{j-1}, e_j$ are all belong to $E_i$. Which is a contradiction. Hence $\Delta(G)$ is connected.

By Theorem 2.3 $\Delta(G)$ is a 4-regular graph. Since $G$ is not $K_4$ any two adjacent vertices in $\Delta(G)$ has only one common neighbour. Hence $\Delta(G)$ is edge-regular with parameters $(\frac{nk}{2}, 4, 1)$.

The following lemmas are useful to prove the next theorem.

Lemma 4.6. [6] If $G$ be a graph and $H$ an induced subgraph. Then the eigenvalues of $H$ interlace those of $G$.

Lemma 4.7. [4] Let $G$ be a $k$-regular graph then the $\text{spec}(G) \in [-4, 4]$.

Lemma 4.8. [11] If $G = H \lor K_1$, where $H$ is $K_3$-free then $\Delta(G)$ is the semi total point graph of $H$. 

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Theorem 4.9. Let $G = (n, k, 2, c)$ is a connected strongly regular graph where each vertex belong to exactly one wheel then the $\text{spec} \Delta(G) \in [-4, 4]$ and the eigen values of $R(C_k)$ interlace those of $\Delta(G)$.

Proof. By Theorem 4.5, $\Delta(G)$ is a 4-regular graph. Therefore by Lemma 4.7 the largest eigenvalue of $\Delta(G)$ is 4 and $\text{spec}(G) \in [-4, 4]$. Also by assumption each vertex belong to exactly one wheel $(C_n \lor K_1)$. For $G$, since $a = 2$ a vertex together with its neighbours form an induced $k$-wheel. Then by Lemma 4.8 it is clear that $\Delta(G)$ contains the semi total point graph $R(C_k)$ as an induced subgraph. Therefore by Lemma 4.6 the eigen values of $R(C_k)$ interlace those of $\Delta(G)$.

\[ \square \]

5 Conclusion

$\Gamma(G)$ and $\Delta(G)$ are the spanning subgraphs of the well known operator $L(G)$. $L(G)$ preserves the properties like regularity and connectedness but usually $\Gamma(G)$ and $\Delta(G)$ do not preserve these properties. In this paper, we find that there are some class of strongly regular graphs for which regularity and connectedness are preserved while applying the operators $\Gamma(G)$ and $\Delta(G)$. Using the regularity property, we have obtained the spectrum of these graphs.

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