Many-variable identities for the dilogarithm and the dual resonance model

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The symmetrized dilogarithm \( L(x) \) satisfies certain identities containing an arbitrary number, denoted by \( n - 3 \), of independent variables. It is shown that the general prescription for writing the \( (n-3) \)-variable dilogarithm identity is closely related to that for writing the basic quantities in the \( n \)-point Veneziano amplitude of the dual resonance model, which is a physically unsuccessful model of strong interactions in particle physics. Duality transformations of \( n \)-point tree Feynman graphs of the \( \phi^3 \) theory are represented by birational transformations of \( n-3 \) variables corresponding to the \( n-3 \) internal lines of the tree and then the dilogarithm identities are expressed in terms of those variables. The identities can also be rewritten in a form independent of the choice of a particular tree by introducing three redundant parameters.

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1. The dilogarithm and its identities

The dilogarithm or the Spence function \([1]\) is expressed by a very simple integral, but it is not expressible in terms of elementary functions. It is often encountered in calculating physical quantities (such as the box-graph Feynman amplitude).

The definition of the dilogarithm \( \text{Li}_2(x) \) is

\[
\text{Li}_2(x) = -\int_0^x dx' \frac{\log(1-x')}{x'} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.
\]

In particular, one has

\[
\text{Li}_2(0) = 0, \quad \text{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}.
\]

The symmetrized dilogarithm, \( L(x) \), introduced by Rogers is defined by

\[
L(x) \equiv -\frac{1}{2} \int_0^x dx' \left( \frac{\log(1-x')}{x'} + \frac{\log x'}{1-x'} \right) = \text{Li}_2(x) + \frac{1}{2} \log x \log(1-x).
\]

Since the additional term vanishes at \( x = 0, 1 \), (1.2) implies

\[
L(0) = 0, \quad L(1) = \frac{\pi^2}{6}.
\]

As an analytic function, \( L(z) \) is holomorphic in the complex plane with two cuts, \([1, +\infty)\) and \((-\infty, 0]\).
In any dilogarithm identity, the $\log \cdot \log$ terms do not appear at all if it is expressed in terms of $L(x)$. Thus the symmetrized dilogarithm is quite convenient, and therefore throughout the present paper, we always work with $L(x)$.

The dilogarithm identities containing one or two independent variables are well known. The identities that hold when all variables of $L$ simultaneously belong to $[0, 1]$ are as follows:

- Euler’s identity
  \[ L(x) + L(1 - x) = \frac{\pi^2}{6}. \] (1.5)

- Abel’s identity or pentagon identity
  \[ L(x) + L(y) + L(1 - xy) + L\left(\frac{1 - x}{1 - xy}\right) + L\left(\frac{1 - y}{1 - xy}\right) = \frac{\pi^2}{2}; \] (1.6)

alternatively, the use of (1.5) in the last three terms of the left-hand side of (1.6) yields

\[ L(x) + L(y) = L(xy) + L(\frac{x(1 - y)}{1 - xy}) + L(\frac{y(1 - x)}{1 - xy}). \] (1.7)

Some three-variable identities are known [1], e.g.,

\[ L\left(\frac{(1 - x)(1 - y)}{(1 - v)(1 - w)}\right) + L\left(\frac{1 - x}{1 - vy^{-1}}\right) + L\left(\frac{1 - x}{1 - wy^{-1}}\right) - L\left(\frac{1 - v}{1 - vy^{-1}}\right) - L\left(\frac{1 - w}{1 - wy^{-1}}\right) + L(x) + L(y) - L(v) - L(w) = L(1) \] (1.8)

with a constraint $xy = vw$. Unfortunately, the rule of controlling the expression for (1.8) is unclear; moreover, it does not reduce to a two-variable identity for special values of one variable in an obvious way. Hence it seems quite difficult to extend (1.8) to the case of four or more variables.

We now propose the following three-variable identity as a more natural one:

\[ L(x) + L(y) + L(z) + L(1 - xyz) + L\left(\frac{1 - x}{1 - xy}\right) + L\left(\frac{1 - z}{1 - yz}\right) + L\left(\frac{1 - xy}{1 - xyz}\right) + L\left(\frac{1 - yz}{1 - xyz}\right) + L\left(\frac{(1 - y)(1 - xyz)}{(1 - xy)(1 - yz)}\right) = \pi^2. \] (1.9)

Indeed, if we set any one of three variables equal to either 0 or 1 in (1.9), then it automatically reduces to (1.6). It is quite remarkable that the variables of $L$ appearing in (1.5), in (1.6), and in (1.9) are exactly the same as the basic quantities appearing in the 4-point, 5-point, and 6-point Veneziano amplitudes, respectively, of the dual resonance model, where the dual resonance model is a physically unsuccessful (but mathematically beautiful) model of strong interactions in particle physics developed about forty years ago [2,3]. Once this fact is recognized, it is straightforward to extend the dilogarithm identities to the case of an arbitrary number of variables. In the present paper, we explicitly give the general many-variable dilogarithm identity in three forms.

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2 General many-variable identities are also known; see “Note added” at the end of the paper.
The present paper is organized as follows. In Sect. 2, the proofs of (1.5), (1.6), and (1.9) are presented. In Sect. 3, the construction of the basic variables of the \( n \)-point Veneziano amplitude is reviewed. In Sect. 4, the general many-variable dilogarithm identity in the multiperipheral form is proposed and proved. In Sect. 5, the duality transformation and its representation as a birational transformation are reviewed, and then the many-variable dilogarithm identity in tree form is presented. In Sect. 6, the dilogarithm identity is rewritten into cross-ratio form. The final section is devoted to a conclusion.

2. Proofs of one-, two-, and three-variable identities

It is instructive to present the proofs of (1.5), (1.6), and (1.9) explicitly. We use \( v_1, v_2, v_3 \) instead of \( x, y, z \).

One-variable case

\[
S_4(v_1) \equiv L(v_1) + L(1 - v_1) = L(1). \tag{2.1}
\]

**Proof** Differentiating the left-hand side of (2.1), we have

\[
-2S'_4(v_1) = \frac{1}{v_1} \log(1 - v_1) + \frac{1}{1 - v_1} \log v_1 - \frac{1}{1 - v_1} \log v_1 - \frac{1}{v_1} \log(1 - v_1) = 0. \tag{2.2}
\]

Thus \( S_4(v_1) \) is a constant, which is given by \( S_4(0) = L(1) \). \[\Box\]

Two-variable case

\[
S_5(v_1, v_2) \equiv L(v_1) + L(v_2) + L(1 - v_1 v_2) + L \left( \frac{1 - v_1}{1 - v_1 v_2} \right) + L \left( \frac{1 - v_2}{1 - v_1 v_2} \right) = 3L(1). \tag{2.3}
\]

**Proof** Differentiating the left-hand side of (2.3), we have

\[
-2 \frac{\partial}{\partial v_1} S_5(v_1, v_2) = \frac{1}{v_1} \log(1 - v_1) + \frac{1}{1 - v_1} \log v_1 - \frac{1}{v_1} \log(1 - v_1 v_2) - \frac{v_2}{1 - v_1 v_2} \log(v_1 v_2)
\]

\[
+ \left( \frac{v_2}{1 - v_1 v_2} - \frac{1}{1 - v_1} \right) \log v_1(1 - v_2) - \left( \frac{v_2}{1 - v_1 v_2} + \frac{1}{v_1} \right) \log \frac{1 - v_1}{1 - v_1 v_2}
\]

\[
+ \frac{v_2}{1 - v_1 v_2} \log \frac{v_2(1 - v_1)}{1 - v_1 v_2} + \left( -\frac{v_2}{1 - v_1 v_2} + \frac{1}{1 - v_1} \right) \log \frac{1 - v_2}{1 - v_1 v_2}. \tag{2.4}
\]

Bringing together the coefficients of each of \( 1/v_1 \), \( 1/(1 - v_1) \), and \( v_2/(1 - v_1 v_2) \), we find that all of them vanish. Thus \( S_5(v_1, v_2) \) is independent of \( v_1 \).

With \( v_1 = 1 \), the integration constant becomes

\[
S_5(0, v_2) = L(0) + L(v_2) + L(1) + L(1) + L(1 - v_2) = S_4(v_2) + 2L(1) = 3L(1). \tag{2.5}
\]

\[\Box\]

Three-variable case

\[
S_6(v_1, v_2, v_3) \equiv L(v_1) + L(v_2) + L(v_3) + L(1 - v_1 v_2 v_3) + L \left( \frac{1 - v_1}{1 - v_1 v_2} \right) + L \left( \frac{1 - v_3}{1 - v_1 v_2} \right)
\]

\[
+ L \left( \frac{1 - v_1 v_2}{1 - v_1 v_2 v_3} \right) + L \left( \frac{1 - v_2 v_3}{1 - v_1 v_2 v_3} \right) + L \left( \frac{1 - v_3}{1 - v_1 v_2 v_3} \right) \left( \frac{1 - v_2}{(1 - v_1 v_2)(1 - v_2 v_3)} \right) = 6L(1). \tag{2.6}
\]
Proof. Differentiating the left-hand side of (2.6), we have

\[-2 \frac{\partial}{\partial v_1} S_6(v_1, v_2, v_3) = \frac{1}{v_1} \log(1 - v_1) + \frac{1}{1 - v_1} \log v_1\]

\[-\frac{1}{v_1} \log(1 - v_1 v_2 v_3) - \frac{v_2 v_3}{1 - v_1 v_2 v_3} \log(v_1 v_2 v_3)\]

\[+ \left( \frac{v_2}{1 - v_1 v_2} - \frac{1}{1 - v_1} \right) \log v_1(1 - v_2) - \left( \frac{v_2}{1 - v_1 v_2} + \frac{1}{v_1} \right) \log v_1 v_2(1 - v_3)\]

\[\times \log \frac{1 - v_1}{1 - v_1 v_2} + \left( \frac{v_2 v_3}{1 - v_1 v_2 v_3} - \frac{v_2}{1 - v_1 v_2 v_3} \right) \log \frac{v_2(1 - v_1)(1 - v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)}\]

\[\times \log \frac{1 - v_2}{1 - v_1 v_2} + \left( \frac{v_2}{1 - v_1 v_2} - \frac{1}{1 - v_1} \right) \log \frac{(1 - v_2)(1 - v_1 v_2 v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)}.\]

(2.7)

The coefficient of \(1/v_1\):

\[\log(1 - v_1) - \log(1 - v_1 v_2 v_3) - \log \frac{1 - v_1}{1 - v_1 v_2} - \log \frac{1 - v_1 v_2}{1 - v_1 v_2 v_3} = 0.\]  

(2.8)

The coefficient of \(1/(1 - v_1)\):

\[\log v_1 - \log \frac{v_1(1 - v_2)}{1 - v_1 v_2} + \log \frac{1 - v_2 v_3}{1 - v_1 v_2 v_3} + \log \frac{(1 - v_2)(1 - v_1 v_2 v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)} = 0.\]  

(2.9)

The coefficient of \(v_2/(1 - v_1 v_2)\):

\[\log \frac{v_1(1 - v_2)}{1 - v_1 v_2} - \log \frac{1 - v_1}{1 - v_1 v_2} - \log \frac{v_1 v_2(1 - v_3)}{1 - v_1 v_2 v_3}\]

\[\times \log \frac{v_2(1 - v_1)(1 - v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)} - \log \frac{(1 - v_2)(1 - v_1 v_2 v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)} = 0.\]  

(2.10)

The coefficient of \(v_2 v_3/(1 - v_1 v_2 v_3)\):

\[\log \frac{v_1 v_2 v_3}{1 - v_1 v_2 v_3} + \log \frac{v_1 v_2 v_3(1 - v_3)}{1 - v_1 v_2 v_3}\]

\[\log \frac{v_2(1 - v_1)(1 - v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)} - \log \frac{v_2(1 - v_1)(1 - v_3)}{(1 - v_1 v_2)(1 - v_2 v_3)} = 0.\]  

(2.11)

Thus \(S_6(v_1, v_2, v_3)\) is independent of \(v_1\).
With $v_1 = 0$, the integration constant is given by

$$S_6(0, v_2, v_3) = L(0) + L(v_2) + L(v_3) + L(1) + L(1) + L\left(\frac{1 - v_3}{1 - v_2 v_3}\right)$$

$$+ L(1) + L(1 - v_2 v_3) + L\left(\frac{1 - v_2}{1 - v_2 v_3}\right)$$

$$= S_5(v_2, v_3) + 3L(1) = 6L(1). \quad (2.12)$$

3. From the dual resonance model

In this section, we summarize the facts known in the dual resonance model that become relevant to the discussion of the dilogarithm identities.

We consider a tree Feynman graph of the $\phi^3$ theory, which consists of $n \geq 3$ external lines, $n - 3$ internal lines, and $n - 2$ vertices; every vertex is incident to three lines. Of course, any tree Feynman graph can be drawn on a plane without crossing, though the ways of drawing them are not unique. The number of ways of realizing it on a plane is that of the non-equivalent cyclic permutations of $n$ external lines, i.e.,

$$N_V(n) = \frac{1}{2} (n - 1)! \quad (3.1)$$

If we want to emphasize the discrimination of the cyclic ordering, we call the Feynman graph of a particular cyclic ordering of external lines a “cyclically ordered Feynman graph”.

The number of tree Feynman graphs with $n$ external lines is given by

$$N_T(n) = (2n - 5)!! = \frac{(2n - 4)!}{2^{n-2} (n - 2)!} \quad (3.2)$$

where the symbol $!!$ denotes the factorial of odd integers only. For example, as shown in Fig. 1, we have $N_T(4) = 3$.

![Fig. 1. $n = 4$ tree Feynman graphs.](image-url)

The number of cyclically ordered tree Feynman graphs corresponding to a particular tree Feynman graph $T$ is given by

$$\Omega_T(n) = 2^{n-3}. \quad (3.3)$$

---

3 This is proved by mathematical induction. Since $N_T(3) = 1$ is evident, we have only to show $N_T(n + 1) = (2n - 3)N_T(n)$. A tree Feynman graph with $n + 1$ external lines is obtained from a tree Feynman graph with $n$ external lines by attaching a new external line to any one of its external and internal lines. Therefore, the number of ways of doing so is $n + (n - 3) = 2n - 3$.

4 Again, proof is given by mathematical induction. Since $\Omega_T(3) = 1$ is evident, we have only to show $\Omega_T(n + 1) = 2\Omega_T(n)$. Since the $(n + 1)$th external line is drawn on a plane on either side of the line of the Feynman graph, the number of ways of drawing it is 2.
Let the number of tree Feynman graphs realizable in a particular cyclic ordering of external lines be $\mathcal{N}_V(n)$. The total number of cyclically ordered tree Feynman graphs with $n$ external lines is expressed by $N_V(n)\mathcal{N}_V(n)$ as well as by $N_T(n)\mathcal{N}_T(n)$; hence, both numbers must be equal, i.e.,

$$N_V(n)\mathcal{N}_V(n) = N_T(n)\mathcal{N}_T(n). \tag{3.4}$$

Substituting (3.1)–(3.3) into (3.4), we find [4]

$$\mathcal{N}_V(n) = \frac{(2n - 4)!}{(n - 1)!(n - 2)!}. \tag{3.5}$$

This number $\mathcal{N}_V(n)$ is equal to the number of triangulations of a convex $n$-polygon, as is seen by considering the dual graphs of Feynman graphs. It is also equal to the Catalan number for $N = n - 2$.

Hereafter, suppose that the cyclic ordering of external lines is fixed to a particular one. We denote $n - 3$ internal lines by $l_j$ ($j = 1, 2, \ldots, n - 3$) and the set of $n$ external lines by $A = \{A_1, A_2, \ldots, A_n\}$. If $P_+$ and $P_-$ are two disjoint subsets of $A$ whose union is $A$, the division $P = (P_+|P_-) = (P_-|P_+)$ is called a “channel”. If $P_+ = \{A_j, A_{j+1}, \ldots, A_k\}$ and $P_- = \{A_{k+1}, \ldots, A_n, A_1, A_2, \ldots, A_{j-1}\}$, we write the corresponding channel as $P_{j,k+1}$. Note that $P_{j,k+1} = P_{k+1,j}$.

Two channels $P$ and $\overline{P}$ are said to be “crossing” each other if none of the four sets $P_+ \cap \overline{P}_+', \overline{P}_+ \cap P_-', \overline{P}_- \cap P_+', \overline{P}_- \cap \overline{P}_-$ is empty. $P_{j,k+1}$ and $P_{j',k'+1}$ cross each other if and only if either $j < j' < k + 1 < k' + 1$ or $j' < j < k + 1 < k' + 1$. If either $P_+$ or $P_-$ consists of 0 or 1 external line, then there is no channel crossing with $P = (P_+|P_-)$. Hereafter we consider only the channels that have at least one cross channel, and the totality of such channels is denoted by $\mathcal{P}$. The number of the channels belonging to $\mathcal{P}$ is equal to that of the diagonal lines of an $n$-polygon, namely, $\frac{1}{2}n(n - 3)$. For $P_{j,k+1} \in \mathcal{P}$, we have $2 \leq (k + 1) - j \leq n - 2$.

With each channel $P \in \mathcal{P}$, we associate a non-negative real quantity $u_P$, which we call a “channel variable”. The channel variables must satisfy the following simultaneous nonlinear algebraic equations$^5$, called “crossing-symmetric equations”:

$$u_P = 1 - \prod_{\overline{P}} u_{\overline{P}}, \tag{3.6}$$

where the product goes over all channels $\overline{P}$ that cross $P$. For simplicity, $u_{P_{j,k+1}}$ is abbreviated as $u_{j,k+1}$ hereafter.

It is known that the crossing-symmetric equations are exactly solvable in closed form. The solutions for $u_P$ are expressed as rational functions of $n - 3$ arbitrary non-negative real parameters. The dependence on those parameters is determined by giving a tree $T$; the $n - 3$ parameters correspond to the $n - 3$ internal lines of $T$.

In this section, we adopt a multiperipheral graph as $T$, where a “multiperipheral graph” is a tree Feynman graph in which all its internal lines lie on a straight line $L$ and all its external lines can be drawn on a plane on one side of $L$ without contradicting the prescribed cyclic ordering. Figure 2 shows a multiperipheral graph. For $n \geq 5$, there are $n$ multiperipheral graphs corresponding to the cyclic permutation of the names of external lines.

$^5$ This is a physical requirement of the dual resonance model.
Fig. 2. A multiperipheral graph.

If an internal line of a tree Feynman graph $T$ is cut, $T$ splits into two connected parts. Correspondingly, the set $\mathcal{A}$ of external lines also decomposes into two subsets uniquely. For instance, if an internal line $l_j$ is cut in Fig. 2, $\mathcal{A}$ decomposes into $\{A_n, A_1, A_2, \ldots, A_j\}$ and $\{A_{j+1}, A_{j+2}, \ldots, A_{n-1}\}$. Thus, cutting $l_j$ induces a channel $P_{j+1,n}$. Of course, this correspondence depends on the choice of $T$. In general, the two channels corresponding to two internal lines of a tree do not cross each other.

Now, we give the solution to (3.6) corresponding to the above multiperipheral graph. Since the channels $P_{j+1,n}$ corresponding to the internal lines $l_j$ ($j = 1, 2, \ldots, n - 3$) do not cross each other, we can choose $u_{j+1,n}$ as independent parameters $v_j$ (the admissible region is $0 \leq v_j \leq 1$), i.e., we set

$$u_{j+1,n} = v_j \quad \text{for} \quad j = 1, 2, \ldots, n - 3. \quad (3.7)$$

Then it is known [5] that the other channel variables $u_{j,k+1}$ ($1 \leq j < k \leq n - 2$) are given by

$$u_{j,k+1} = \frac{(1 - \prod_{i=j}^{k-1} v_i)(1 - \prod_{i=j}^{k} v_i)}{(1 - \prod_{i=j}^{k-1} v_i)(1 - \prod_{i=j}^{k} v_i)} \quad \text{with} \quad v_0 \equiv v_{n-2} \equiv 0. \quad (3.8)$$

The important point is the existence of supplementary conditions: The condition $v_0 \equiv 0$ implies that, if $j = 1$, the second factor of the numerator and the first factor of the denominator are equal to 1, and the condition $v_{n-2} \equiv 0$ implies that, if $k = n - 2$, the second factors of both numerator and denominator are equal to 1.

For illustration, we present concrete examples.

The case of $n = 4$

$$u_{24} = v_1 = 1 - u_{13},$$
$$u_{13} = 1 - v_1 = 1 - u_{24}. \quad (3.9)$$

The case of $n = 5$

$$u_{25} = v_1 = 1 - u_{13}u_{14},$$
$$u_{35} = v_2 = 1 - u_{14}u_{24},$$
$$u_{14} = 1 - v_1v_2 = 1 - u_{25}u_{35},$$
$$u_{13} = \frac{1 - v_1}{1 - v_1v_2} = 1 - u_{24}u_{25},$$
$$u_{24} = \frac{1 - v_2}{1 - v_1v_2} = 1 - u_{13}u_{35}. \quad (3.10)$$
The case of \( n = 6 \)

\[
\begin{align*}
    u_{26} &= v_1 = 1 - u_{13} u_{14} u_{15}, \\
    u_{36} &= v_2 = 1 - u_{14} u_{15} u_{24} u_{25}, \\
    u_{46} &= v_3 = 1 - u_{15} u_{25} u_{35}, \\
    u_{15} &= 1 - v_1 v_2 v_3 = 1 - u_{26} u_{36} u_{46}, \\
    u_{13} &= \frac{1 - v_1}{1 - v_1 v_2} = 1 - u_{24} u_{25} u_{26}, \\
    u_{35} &= \frac{1 - v_3}{1 - v_2 v_3} = 1 - u_{14} u_{24} u_{46}, \\
    u_{14} &= \frac{1 - v_1 v_2}{1 - v_1 v_2 v_3} = 1 - u_{25} u_{26} u_{35} u_{36}, \\
    u_{25} &= \frac{1 - v_2 v_3}{1 - v_1 v_2 v_3} = 1 - u_{13} u_{14} u_{36} u_{46}, \\
    u_{24} &= \frac{(1 - v_2)(1 - v_1 v_2)}{(1 - v_1 v_2)(1 - v_2 v_3)} = 1 - u_{13} u_{35} u_{36}.
\end{align*}
\]

(3.11)

Looking at (3.9), (3.10), and (3.11), we find that they are closely related to (2.1), (2.3), and (2.6), respectively.

4. Many-variable dilogarithm identity

**Theorem 1.** We set

\[
S_n(v_1, v_2, \ldots, v_{n-3}) \equiv \sum_{j=1}^{n-3} L(v_j) + \sum_{1 \leq j < k \leq n-2} L(u_{j,k+1})
\]

(4.1)

with

\[
u_{j,k+1} \equiv \frac{\left(1 - \prod_{i=j}^{k-1} v_i\right) \left(1 - \prod_{i=j}^{k} v_i\right)}{\left(1 - \prod_{i=j-1}^{k-1} v_i\right) \left(1 - \prod_{i=j}^{k} v_i\right)} \text{ with } v_0 \equiv v_{n-2} \equiv 0;
\]

(4.2)

then the following identity holds:

\[
S_n(v_1, v_2, \ldots, v_{n-3}) = \frac{(n-2)(n-3)}{12} \pi^2.
\]

(4.3)

**Proof.** We employ mathematical induction with respect to \( n \). For \( n = 4 \), (4.3) reduces to (2.1). Hence we assume \( n \geq 5 \). We rewrite (4.1) as

\[
S_n(v_1, v_2, \ldots, v_{n-3}) = \sum_{j=1}^{n-4} L(v_j) + L(v_{n-3}) + \sum_{1 \leq j < k \leq n-4} L(u_{j,k+1}) + \sum_{j=1}^{n-4} L(u_{j,n-2}) + \sum_{j=1}^{n-3} L(u_{j,n-1}).
\]

(4.4)

We compare it with \( S_{n-1}(v_1, v_2, \ldots, v_{n-4}) \). For \( k \leq n - 4 \), \( u_{j,k+1} \) coincides with that of \( S_{n-1} \). But since the supplementary condition for \( S_{n-1} \) is not \( v_{n-2} \equiv 0 \) but \( v_{n-3} \equiv 0 \), the \( u_{j,n-2} \) of \( S_{n-1} \) is
not the same as that of $S_n$. In order to discriminate them, we write the former as $\tilde{u}_{j,n-1}$. From the definition (4.2), we have

$$
\tilde{u}_{j,n-2} = \frac{1 - \prod_{i=j}^{n-4} v_i}{1 - \prod_{i=j-1}^{n-4} v_i}.
$$

(4.5)

Comparing (4.5) with

$$
\begin{align*}
\begin{aligned}
\tilde{u}_{j,n-1} &= \frac{1 - \left(\prod_{i=j}^{n-4} v_i\right) v_{n-3}}{1 - \left(\prod_{i=j-1}^{n-4} v_i\right) v_{n-3}}, & \\
\end{aligned}
\end{align*}
$$

(4.6)

we find

$$
\tilde{u}_{j,n-2} = u_{j,n-1} \bigg|_{v_{n-3}=1}.
$$

(4.7)

Hence we have

$$
S_{n-1}(v_1, v_2, \ldots, v_{n-4}) = \sum_{j=1}^{n-4} L(v_j) + \sum_{1 \leq j < k \leq n-4} L(u_{j,k+1}) + \sum_{j=1}^{n-4} L(u_{j,n-1} \bigg|_{v_{n-3}=1}).
$$

(4.8)

Since $u_{n-3,n-1} \big|_{v_{n-3}=1} = 0$, the summation over $j$ in the last term of (4.8) may be extended to $n - 3$. Comparing (4.4) with (4.8), we find

$$
\begin{align*}
\begin{aligned}
S_n(v_1, v_2, \ldots, v_{n-3}) &= S_{n-1}(v_1, v_2, \ldots, v_{n-4}) + L(v_{n-3}) \\
&\quad + \sum_{j=1}^{n-3} \left[ L(u_{j,n-1}) - L(u_{j,n-1} \bigg|_{v_{n-3}=1}) \right] + \sum_{j=1}^{n-4} L(u_{j,n-2}). & \\
\end{aligned}
\end{align*}
$$

(4.9)

The induction assumption implies $\partial S_{n-1} / \partial v_1 = 0$; moreover, for $j \geq 3$, we see $\partial u_{j,k+1} / \partial v_1 = 0$. Accordingly, we obtain

$$
\frac{\partial}{\partial v_1} S_n(v_1, v_2, \ldots, v_{n-3}) = \frac{\partial}{\partial v_1} \sum_{j=1}^{2} \left[ L(u_{j,n-1}) - L(u_{j,n-1} \bigg|_{v_{n-3}=1}) + L(u_{j,n-2}) \right].
$$

(4.10)

For simplicity, we write $v_1 = v$, $\prod_{i=2}^{n-4} v_i = a$, $\prod_{i=2}^{n-3} v_i = av_{n-3} = b$; then from (4.2) we have

$$
\begin{align*}
\begin{aligned}
u_{1,n-1} &= 1 - bv, & \\
u_{2,n-1} &= \frac{1 - b}{1 - bv}, & \\
u_{1,n-1} \bigg|_{v_{n-3}=1} &= 1 - av, & \\
u_{2,n-1} \bigg|_{v_{n-3}=1} &= \frac{1 - a}{1 - av}, & \\
u_{1,n-2} &= \frac{1 - av}{1 - bv}, & \\
u_{2,n-2} &= \frac{(1 - a)(1 - bv)}{(1 - b)(1 - av)}. & \\
\end{aligned}
\end{align*}
$$

(4.11)
Therefore

\[-2 \frac{\partial S_n}{\partial v} = -\frac{1}{v} \log(1 - bv) - \frac{b}{1 - bv} \log(bv)\]
\[+ \frac{b}{1 - bv} \log \frac{b(1 - v)}{1 - bv} + \left( -\frac{b}{1 - bv} + \frac{1}{1 - v} \right) \log \frac{1 - b}{1 - bv}\]
\[+ \frac{1}{v} \log(1 - av) + \frac{a}{1 - av} \log(av)\]
\[- \frac{a}{1 - av} \log \frac{a(1 - v)}{1 - av} - \left( -\frac{a}{1 - av} + \frac{1}{1 - v} \right) \log \frac{1 - a}{1 - av}\]
\[+ \left( \frac{b}{1 - bv} - \frac{a}{1 - av} \right) \log \frac{(a - b)v}{1 - bv} + \left( -\frac{b}{1 - bv} - \frac{1}{v} \right) \log \frac{1 - av}{1 - bv}\]
\[+ \left( \frac{a}{1 - av} - \frac{b}{1 - bv} \right) \log \frac{(a - b)(1 - v)}{(1 - b)(1 - av)} + \left( -\frac{a}{1 - av} + \frac{1}{1 - v} \right)\]
\[\times \log \frac{(1 - a)(1 - bv)}{(1 - b)(1 - av)}.\]

(4.12)

Rewriting (4.12), we find

\[-2 \frac{\partial S_n}{\partial v} = \frac{1}{v} \left[ -\log(1 - bv) + \log(1 - av) - \log \frac{1 - av}{1 - bv} \right]\]
\[+ \frac{1}{1 - v} \left[ \log \frac{1 - b}{1 - bv} - \log \frac{1 - a}{1 - av} + \log \frac{(1 - a)(1 - bv)}{(1 - b)(1 - av)} \right]\]
\[+ \frac{a}{1 - av} \left[ \log(av) - \log \frac{a(1 - v)}{1 - av} + \log \frac{1 - a}{1 - av} - \log \frac{(a - b)v}{1 - bv} \right]\]
\[+ \log \frac{(a - b)(1 - v)}{(1 - b)(1 - av)} - \log \frac{(1 - a)(1 - bv)}{(1 - b)(1 - av)}\]
\[+ \frac{b}{1 - bv} \left[ -\log(bv) + \log \frac{b(1 - v)}{1 - bv} - \log \frac{1 - b}{1 - bv} + \log \frac{(a - b)v}{1 - bv} \right]\]
\[\quad - \log \frac{1 - av}{1 - bv} - \log \frac{(a - b)(1 - v)}{(1 - b)(1 - av)} \right]\]
\[= 0.\]  

(4.13)

Thus, \(S_n(v_1, v_2, \ldots, v_{n-3})\) is independent of \(v_1\).

Next, to determine the integration constant, we set \(v_1 = 0\) in (4.1); then, as shown below, we have

\[S_n(0, v_2, \ldots, v_{n-3}) = S_{n-1}(v_2, v_3, \ldots, v_{n-3}) + (n - 3)L(1) + L(0).\]  

(4.14)

As before, the quantities of \(S_{n-1}(v_2, v_3, \ldots, v_{n-3})\) (its supplementary conditions are \(v_1 \equiv v_{n-2} \equiv 0\)) are expressed by affixing a tilde. For \(j \geq 3\), we have \(u_{j,k+1} = \tilde{u}_{j,k+1}\). As for \(j = 1\) and \(j = 2\), we have

\[u_{1,k+1} \Big|_{v_1=0} = \frac{1 - v_1 \prod_{i=2}^{k-1} v_i}{1 - v_1 \prod_{i=2}^{k} v_i} \Big|_{v_1=0} = 1\]  

(4.15)
and
\[ u_{2,k+1}\big|_{v_1=0} = \left( 1 - \prod_{i=2}^{k-1} v_i \right) \left( 1 - v_1 \prod_{i=2}^{k} v_i \right) = \frac{1 - \prod_{i=2}^{k-1} v_i}{1 - \prod_{i=2}^{k} v_i} = \tilde{u}_{2,k+1} \quad (4.16) \]

respectively. Then (4.14) follows.

The induction assumption implies
\[ S_{n-1}(v_2, v_3, \ldots, v_{n-3}) = \frac{1}{2} (n-3)(n-4)L(1) \]

Hence (4.14) yields
\[ S_n(v_1, v_2, \ldots, v_{n-3}) = \frac{1}{2} (n-2)(n-3)L(1). \]

\section{5. Representation of duality transformations}

Recently, the pentagon identity has become of renewed interest in connection with the investigation of the quantum dilogarithm \cite{6}, in which the following property is of central importance: The five variables of \( L \) appearing in (1.7) can be described by the transformations, called “mutations”, of two parameters \( y_1(t) \) and \( y_2(t) \), where \( t \) is a discrete time. By mutations, the quantities \( x = y_1/(1 + y_1) \) and \( y = y_2(1 + y_1)/(1 + y_2 + y_1y_2) \) generate the variables of \( L \), and return to the original ones by repeating mutations five times, though \( y_1 \) and \( y_2 \) are interchanged. From the standpoint of the dual resonance model, however, the mutation may be understood as a modified version of the duality transformation. The property that all variables of \( L \) appearing in the dilogarithm identity are encountered without doubling by repeating the same kind of transformations is the specialty of the \( n = 5 \) case. For the general cyclically ordered tree Feynman graphs, certain birational transformations of the parameters corresponding to their internal lines provide a representation of duality transformations. In this section, we reformulate the present author’s old work \cite{4,7} on the duality transformation considered in the dual resonance model, restricting ourselves to the discussions relevant to the dilogarithm identities.

First, we define the duality transformation. Given any internal line \( l_j \), there are four lines, internal or external, adjacent to it. Without changing their cyclic ordering, there exists a unique way of nontrivially changing the incidence of \( l_j \). This transformation of a graph is called the “duality transformation” with respect to \( l_j \) and denoted by \( \sigma_j \); see Fig. 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3}
\caption{Duality transformation with respect to \( l_j \).}
\end{figure}

Now, we investigate the properties of the duality transformation \( \sigma_j \). Evidently, if it acts twice, the graph returns to the original one; therefore
\[ \sigma_j^2 = e, \quad (5.1) \]

where \( e \) denotes the identity transformation. As for two different internal lines \( l_j \) and \( l_k \) (\( j \neq k \)), if they are not adjacent, they are commutative, i.e.,
\[ \sigma_j \sigma_k = \sigma_k \sigma_j \quad \text{for } l_j \cap l_k = \emptyset, \quad (5.2) \]

\footnote{If we consider the dual graph, a duality transformation corresponds to changing the way of triangulation.}
where $l_j \cap l_k$ stands for the set of the vertices common to $l_j$ and $l_k$. On the other hand, if they are adjacent, i.e., if $l_j \cap l_k \neq \emptyset$, then $\sigma_j$ and $\sigma_k$ are non-commutative; see Fig. 4.

![Duality transformations performed for two adjacent lines $l_j$ and $l_k$ alternately.](image)

As is seen from Fig. 4, the cyclically ordered Feynman graph returns to the original one if the transformations are performed five times, but the positions of $l_j$ and $l_k$ have been interchanged. Denoting the interchange of $l_j$ and $l_k$ by $\pi_{jk}(=\pi_{kj})$, we have

$$\sigma_j \sigma_k \sigma_j \sigma_k \sigma_j = \pi_{jk}. \quad (5.3)$$

Since the right-hand side of (5.3) is invariant under the interchange of $j$ and $k$, we obtain

$$\sigma_j \sigma_k \sigma_j \sigma_k \sigma_j = \sigma_k \sigma_j \sigma_k \sigma_j \sigma_k. \quad (5.4)$$

We call the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-3}$ the “duality transformation group”\(^7\), and write it as $D_n$. In this group, internal lines are discriminated, and hence it is natural to consider the factor group $V_n = D_n/S_{n-3}$ by dividing it by the symmetric group, $S_{n-3}$, of the permutations of $n-3$ internal lines. Since any two tree Feynman graphs of the same cyclic ordering can be mutually transmuted into each other by repeatedly applying duality transformations, the order of the group $V_n$ equals $\mathcal{N}_V(n)$ given by (3.5).

The group $V_n$ can be represented as a graph $\mathcal{G}_n$ in the following way: Each element of $V_n$ is represented as a vertex of $\mathcal{G}_n$ and if there is a duality transformation connecting two elements of $V_n$, the corresponding two vertices are connected by a line. Then $\mathcal{G}_n$ is a degree-$(n-3)$ homogeneous graph having $\mathcal{N}_V(n)$ vertices. Evidently, $\mathcal{G}_4$ consists of only one line and $\mathcal{G}_5$ is a pentagon. For $n \geq 6$, $\mathcal{G}_n$ is a patchwork of squares and pentagons. The pentagon of $\mathcal{G}_5$ corresponds to the fact stated in the beginning of this section: The mutation of $y_1(t)$ and $y_2(t)$ can be regarded as a representation of the duality transformation. However, the property that all elements of $\mathcal{G}_n$ are encountered without doubling by repeating duality transformations is limited to the case of $n=5$ only, i.e., $\mathcal{G}_n$ with $n \geq 6$ has no Hamilton circuit.

We can construct a natural representation of the duality transformations by means of birational transformations of $n-3$ parameters in such a way that all channel variables $u_P$ remain invariant, as shown below. According to this result, the solution to the crossing-symmetric equations (3.6) can be constructed in terms of the parameters $x_j$ of any tree Feynman graph.

\(^7\)It is noteworthy that (5.2) and (5.4) are similar to the identities for the generators in the braid group.
For illustration, we first consider the case of $n = 5$. Figure 5 shows an $n = 5$ tree Feynman graph, called $G^{(5)}$, and its duality-transformed graphs $\sigma_1 G^{(5)}$ and $\sigma_2 \sigma_1 G^{(5)}$, where $\sigma_1$ and $\sigma_2$ are the duality transformations corresponding to $l_1$ and $l_2$, respectively. In order to keep the positions of $l_1$ and $l_2$, the central graph $\sigma_1 G^{(5)}$ is drawn in the reversed form.

![Fig. 5. Tree Feynman graphs $G^{(5)}$, $\sigma_1 G^{(5)}$, and $\sigma_2 \sigma_1 G^{(5)}$.](image)

Comparing the expressions for the channel variables $u_P$ of $G^{(5)}$ with those of $\sigma_1 G^{(5)}$, we find that corresponding to $\sigma_1$, the parameters are transformed as

$$v'_1 = \frac{1-v_1}{1-v_1 v_2}, \quad v'_2 = v_2.$$  \hfill (5.5)

Of course, corresponding to $(\sigma_1)^{-1} = \sigma_1$, the inverse transformation of the parameters is in the same form as the above. Concretely, suppose that the channel variables of $G^{(5)}$ are expressed in terms of the parameters $v_1$ and $v_2$ as in (3.10), then (5.5) implies

$$u'_{13} = v'_1 = \frac{1-v_1}{1-v_1 v_2} = u_{13},$$

$$u'_{35} = v'_2 = v_2 = u_{35},$$

$$u'_{24} = 1 - v'_1 v'_2 = \frac{1-v_2}{1-v_1 v_2} = u_{24},$$

$$u'_{25} = \frac{1-v'_1}{1-v'_1 v'_2} = v_1 = u_{25},$$

$$u'_{14} = \frac{1-v'_2}{1-v'_1 v'_2} = 1-v_1 v_2 = u_{14},$$  \hfill (5.6)

where $u'_{j,k+1}$ is the channel variable of the central graph in Fig. 5. We confirm $u'_P = u_P$; i.e., the channel variables are invariant under $\sigma_1$.

Furthermore, when $\sigma_2$ is applied to $\sigma_1 G^{(5)}$, i.e., if we perform the transformation

$$v''_1 = v'_1, \quad v''_2 = \frac{1-v'_2}{1-v'_1 v'_2},$$  \hfill (5.7)

then we have

$$u''_{13} = v''_1 = v'_1 = u'_{13} = u_{13},$$

$$u''_{14} = v''_2 = \frac{1-v'_2}{1-v'_1 v'_2} = u'_{14} = u_{14},$$

$$u''_{25} = 1 - v''_1 v''_2 = \frac{1-v'_1}{1-v'_1 v'_2} = u'_{25} = u_{25},$$

$$u''_{24} = \frac{1-v''_1}{1-v''_1 v''_2} = 1-v'_1 v'_2 = u'_{24} = u_{24},$$

$$u''_{35} = \frac{1-v''_2}{1-v''_1 v''_2} = v'_2 = u'_{35} = u_{35}.$$  \hfill (5.8)
Thus we obtain the expression for the channel variables of the rightmost graph in Fig. 5. In this way, the birational transformations of the parameters present a representation of duality transformations.

In the above example, any vertex is “external”, i.e., for any vertex there exists at least one external line that is incident to it. For \( n = 6 \), there is a tree Feynman graph in which an internal vertex exists, where a vertex is said to be “internal” if all lines incident to it are internal lines.

![Diagram](https://example.com/diagram.png)

**Fig. 6.** Duality transformation into a tree Feynman graph containing an internal vertex.

In an \( n = 6 \) multiperipheral graph, if the duality transformation corresponding to the central line \( l_2 \) of the left graph in Fig. 6 is performed, we encounter a tree Feynman graph containing an internal vertex, as is drawn as the right graph in Fig. 6. In such a case, the transformation of parameters is not restricted to \( l_2 \), i.e., we must also transform the parameters corresponding to the lines adjacent to it. Furthermore, the parameter \( x_2 \) corresponding to \( l_2 \) is no longer equal to the channel variable corresponding to \( l_2 \).

The duality transformation \( \sigma_2 \) shown in Fig. 6 is represented as follows:

\[
\begin{align*}
    x_1 &= \frac{v_1(1 - v_2v_3)}{1 - v_3}, \\
    x_2 &= \frac{1 - v_2}{(1 - v_1v_2)(1 - v_2v_3)}, \\
    x_3 &= \frac{v_3(1 - v_1v_2)}{1 - v_1};
\end{align*}
\]

or solving (5.9) inversely,

\[
\begin{align*}
    v_1 &= \frac{x_1(1 - x_2x_3)}{1 - x_1x_2x_3}, \\
    v_2 &= \frac{(1 - x_2)(1 - x_1x_2x_3)}{(1 - x_2x_3)(1 - x_1x_2)}, \\
    v_3 &= \frac{x_3(1 - x_1x_2)}{1 - x_1x_2x_3}.
\end{align*}
\]

The reason why the inverse transformation does not have the same form as the original one is, of course, that both graphs are not topologically equivalent. If we set \( v_3 = x_3 = 0 \), then (5.9) reduces to (5.6). Thus the parameter corresponding to an external line should be understood as 0. By (5.9), the admissible region \( \{0 \leq v_j \leq 1 \ (j = 1, 2, 3)\} \) is mapped onto a larger region lying in \( \{x_j \geq 0 \ (j = 1, 2, 3)\} \).

Substituting (5.10) into (3.11), we obtain

\[
\begin{align*}
    u_{26} &= \frac{x_1(1 - x_2x_3)}{1 - x_1x_2x_3}, \\
    u_{24} &= \frac{x_2(1 - x_3x_1)}{1 - x_1x_2x_3},
\end{align*}
\]
\[ u_{46} = \frac{x_3(1 - x_1 x_2)}{1 - x_1 x_2 x_3}, \]
\[ u_{13} = \frac{1 - x_1 x_2}{1 - x_1 x_2 x_3}, \]
\[ u_{35} = \frac{1 - x_2 x_3}{1 - x_1 x_2 x_3}, \]
\[ u_{15} = \frac{1 - x_3 x_1}{1 - x_1 x_2 x_3}, \]
\[ u_{14} = \frac{(1 - x_1)(1 - x_1 x_2 x_3)}{(1 - x_1 x_2)(1 - x_3 x_1)}, \]
\[ u_{36} = \frac{(1 - x_2)(1 - x_1 x_2 x_3)}{(1 - x_2 x_3)(1 - x_1 x_2)}, \]
\[ u_{25} = \frac{(1 - x_3)(1 - x_1 x_2 x_3)}{(1 - x_3 x_1)(1 - x_2 x_3)}. \]

(5.11)

We see that (5.11) is certainly consistent with the symmetry under the cyclic permutations \( l_1 \mapsto l_2 \mapsto l_3 \mapsto l_1; \ A_1 \mapsto A_3 \mapsto A_5 \mapsto A_1, \ A_2 \mapsto A_4 \mapsto A_6 \mapsto A_2 \).

Next, as an example of the tree Feynman graph containing an internal line whose end vertices are both internal, we consider the following \( n = 8 \) one (Fig. 7). It is obtained from a multiperipheral graph by applying two duality transformations \( \sigma_2 \) and \( \sigma_4 \) successively; since they are commutative, the ordering of their action is irrelevant.

![Fig. 7. Tree Feynman graph containing an internal line connecting two internal vertices.](https://academic.oup.com/ptep/article-abstract/2013/2/023A01/1500661)

Since \( \sigma_4 \sigma_2 \) is a combination of the duality transformations corresponding to \( l_2 \) and \( l_4 \), we should have

\[ v_1 = \frac{x_1(1 - x_2 x'_3)}{1 - x_1 x_2 x'_3}, \]
\[ v_2 = \frac{(1 - x_2)(1 - x_1 x_2 x'_3)}{(1 - x_2 x'_3)(1 - x_1 x_2)}, \]
\[ v_3 = \frac{x'_3 (1 - x_1 x_2)}{1 - x_1 x_2 x'_3} = \frac{x''_3 (1 - x_4 x_5)}{1 - x_4 x_5 x''_3}, \]
\[ v_4 = \frac{(1 - x_4)(1 - x_4 x_5 x'_3)}{(1 - x_4 x'_3)(1 - x_4 x_5)}, \]
\[ v_5 = \frac{x_5 (1 - x_4 x''_3)}{1 - x_4 x_5 x''_3}, \]

(5.12)
but the expression for $v_3$ would become twofold if we set $x'_3 = x''_3 = x_3$. To be consistent, we set
\[
x'_3 = \frac{x_3(1 - x_4x_5)}{1 - x_4x_5x_3}, \quad x''_3 = \frac{x_3(1 - x_1x_2)}{1 - x_1x_2x_3}.
\]

Indeed, (5.13) implies
\[
\frac{x'_3(1 - x_1x_2)}{1 - x_1x_2x'_3} = \frac{x_3(1 - x_1x_2)(1 - x_4x_5)}{1 - x_1x_2x_3 - x_3x_4x_5 + x_1x_2x_3x_4x_5} = \frac{x''_3(1 - x_4x_5)}{1 - x_4x_5x''_3}.
\]

This is a representation of the commutativity between $\sigma_2$ and $\sigma_4$.

Substituting (5.12) into (3.7) and (3.8) for $n = 8$, we obtain
\[
\begin{align*}
    u(l_1) &= u_{28} = \frac{x_1(1 - x_2x'_3)}{1 - x_1x_2x'_3}, \\
    u(l_2) &= u_{24} = \frac{x_2(1 - x_1x'_3)}{1 - x_1x_2x'_3}, \\
    u(l_3) &= u_{48} = \frac{x'_3(1 - x_1x_2)}{1 - x_1x_2x'_3} = \frac{x''_3(1 - x_4x_5)}{1 - x_4x_5x''_3}, \\
    u(l_4) &= u_{46} = \frac{x_4(1 - x_3x''_3)}{1 - x_4x_5x''_3}, \\
    u(l_5) &= u_{68} = \frac{x_5(1 - x_4x''_3)}{1 - x_4x_5x''_3},
\end{align*}
\]

where the channel variable corresponding to $l_j$ is denoted by $u(l_j)$. We can likewise calculate the other 15 channel variables. This tree Feynman graph is symmetric under two kinds of reversion; the interchange of $\{A_8, A_1, A_2, A_3; l_1, l_2\}$ and $\{A_7, A_6, A_5, A_4; l_5, l_4\}$ and the interchange of $\{A_2, A_3, A_4, A_5; l_2, l_4\}$ and $\{A_1, A_8, A_7, A_6; l_1, l_5\}$. We can confirm that the expressions for $u_P$ are consistent with those symmetries.$^8$

After the above consideration, it is not difficult to find the prescription for how to express $u(l_j)$ in terms of the parameters $x_1, \ldots, x_{n-3}$ in the case of the general tree Feynman graph $T$.

First, we note that any tree graph is 2-colorable; i.e., all its vertices can be colored by two colors in such a way that the colors of any two adjacent vertices are different. Those two colors are called “prime” and “double-prime”. Then any internal line $l_j$ is incident to one prime vertex and to one double-prime one. We denote the two adjacent lines incident to the prime vertex by $l_p$ and $l_q$ and those incident to the double-prime vertex by $l_r$ and $l_s$. Furthermore, we denote one of these five lines by $l_k$ generically, i.e., $k$ stands for any one of $j, p, q, r, s$. Of course, $l_k$ may be an external line; if so, we set $x_k \equiv 0$. Then the channel variable, $u(l_j)$, corresponding to $l_j$ has the following two mutually equivalent expressions in terms of the parameters:
\[
u(l_j) = \frac{x'_j(1 - x''_jx''_j)}{1 - x'''_jx'''_j} = \frac{x''_j(1 - x'_jx''_j)}{1 - x'''_jx'''_j}, \quad (5.16)
\]

$^8$Note that the reversion of $u_{j,k+1}$ is not $u_{R(j),R(k+1)}$ but $u_{R(k),R(j)+1}$, where $R$ denotes the reversion.
where $x'_k$ and $x''_k$ are the parameters corresponding to $l_k$ incident to the prime vertex and to the double-prime one, respectively. For $l_j$, they are given by

$$
x'_j = \frac{x_j(1 - x''_r x''_s)}{1 - x''_r x''_s x_j},
$$

$$
x''_j = \frac{x_j(1 - x'_p x'_q)}{1 - x'_p x'_q x_j},
$$

(5.17)

where $x_k = 0$ if $l_k$ is an external line. The inductive use of (5.17) is sufficient to present the explicit expression for $u(l_j)$ in terms of $x_1, \ldots, x_{n-3}$. Indeed, if $l_p$ is an external line, then we have $x'_p = 0$, and therefore (5.16) and (5.17) reduce to

$$
u(l_j) = x_j = \frac{x_j(1 - x''_r x''_s)}{1 - x''_r x''_s x_j}.
$$

(5.18)

Moreover, if $l_r$ is also an external line, i.e., if both vertices incident to $l_j$ are external, then we have

$$
u(l_j) = x_j.
$$

(5.19)

If $T$ is a multiperipheral graph, all its vertices are external, and therefore (5.19) holds for any internal line of $T$.

An arbitrary cyclically ordered tree Feynman graph $T$ can be transformed into a multiperipheral graph, $M_T$, of the same cyclic ordering by repeated applications of duality transformations. Using the above parametrization, we can find the expressions for the channel variables, $u_P$, of $T$ from those of $M_T$. The transformation from $x_1, \ldots, x_{n-3}$ to $v_1, \ldots, v_{n-3}$ used here is a birational transformation between those parameters. Furthermore, if $0 \leq x_j \leq 1$ for all $j$, then we have $0 \leq u_P \leq 1$ for $T$.

The proof of the above proposition is given by using mathematical induction with respect to $n$ [7].

From the above proposition, we see the following facts: The transformation between any two tree Feynman graphs of the same cyclic ordering is represented by a birational transformation between their parameters. If a cyclically ordered tree Feynman graph has some topological symmetries, the expressions for the channel variables have corresponding symmetry properties.

From the consideration made above, Theorem 1 can be extended to the general tree Feynman graph.

**Theorem 2.** Let $T$ be an arbitrary cyclically ordered tree Feynman graph with $n$ external lines and $P$ be a channel of $T$. Then there exist $\frac{1}{2}n(n - 3)$ quantities $u_P$, which are rational functions of $n - 3$ parameters $x_1, x_2, \ldots, x_{n-3}$ (their explicit expressions are calculable by the method stated above) and they satisfy the crossing-symmetric equations (3.6). For the symmetrized dilogarithm $L(x)$, the following $(n - 3)$-variable identity holds:

$$
\sum_{P \in \mathcal{P}} L(u_P) = \frac{(n - 2)(n - 3)}{12} \pi^2.
$$

(5.20)

Since it is an identity, it can be analytically continued; i.e., the restriction to $0 \leq u_P \leq 1$ is unnecessary, but then, of course, the Riemann sheet on which each $u_P$ lies must be determined by following the path of analytic continuation.

---

9 We have written $v_j$ instead of $x_j$ in Sect. 3.
As a concrete example, we consider the \( n = 6 \) non-multiperipheral graph presented in Fig. 6. Since this graph has cyclic permutation symmetry, the corresponding dilogarithm identity also has the same symmetry. From Theorem 2 and (5.11), we have

\[
\sum_{i,j,k: \text{cyclic}} \left[ L \left( \frac{x_i(1-x_j x_k)}{1-x_i x_j x_k} \right) + L \left( \frac{1-x_j x_k}{1-x_i x_j x_k} \right) + L \left( \frac{(1-x_i)(1-x_i x_j x_k)}{(1-x_i x_j)(1-x_i x_k)} \right) \right] = \pi^2. \tag{5.21}
\]

In particular, if we set \( x_1 = x_2 = x_3 = x \), then we have

\[
L \left( \frac{x + x^2}{1 + x + x^2} \right) + L \left( \frac{1 + x + x^2}{1 + x + x^2} \right) + L \left( \frac{1 + x + x^2}{(1 + x)^2} \right) = \frac{\pi^2}{3}. \tag{5.22}
\]

6. Dilogarithm identities in the Koba–Nielsen form

As long as we construct the expressions for the channel variables on the basis of a tree, their dependence on the parameters is necessarily asymmetric. However, if one employs the Koba–Nielsen representation of parameters \([8,9]\), then their expressions become of the same form.

We consider a circle \( C \) in a complex plane, and place \( n \) independent complex parameters \( z_1, z_2, \ldots, z_n \) \((z_{n+j} \equiv z_j)\) in this cyclic ordering on \( C \). A channel \( P \) is a division of their set into two subsets containing at least two elements without obstructing the cyclic ordering, i.e., \( P_{j,k+1} \equiv (z_j, z_{j+1}, \ldots, z_k|z_{k+1}, z_{k+2}, \ldots, z_{j-1}) \) \((1 \leq j < k \leq n + j - 3, \ k \leq n - 1)\). Then any channel variable is given by a cross ratio\(^{10}\)

\[
u_{j,k+1} = \frac{(z_j - z_k)(z_{k+1} - z_{j-1})}{(z_j - z_{k+1})(z_k - z_{j-1})} \tag{6.1}.
\]

Although three parameters are redundant, they correspond to the freedom of determining \( C \). In particular, if we set \( z_1 = 0, \ z_{n-1} = 1, \ z_n = \infty^{11}\), then \( C \) becomes the real axis and \( 0 \leq z_2 \leq z_3 \leq \cdots \leq z_{n-2} \leq 1 \). If we set \( v_j = z_{j+1}/z_j \) \((n = 1, 2, \ldots, n - 3)\), then we have \( 0 \leq v_j \leq 1, \) and (6.1) reduces to either (3.7) or (3.8). Conversely, by setting \( v_j = u_{j+1,n} \) \((j = 1, 2, \ldots, n - 3)\), (6.1) is reproduced from the formula (3.8) for a multiperipheral graph. Thus, from Theorem 1 together with analytic continuation, we obtain the following theorem.

**Theorem 3.** For \( n \geq 4 \) complex variables \( z_1, z_2, \ldots, z_n \), the following identity holds:

\[
\sum_{1 \leq j < k \leq \min(n-1, n+j-3)} L \left( \frac{(z_j - z_k)(z_{k+1} - z_{j-1})}{(z_j - z_{k+1})(z_k - z_{j-1})} \right) = \frac{(n-2)(n-3)}{12} \pi^2, \tag{6.2}
\]

where an appropriate Riemann sheet should be chosen for each term.

If we apply Euler’s identity (1.5) to each term of the left-hand side of (6.2), the following dilogarithm identity is obtained:

\[
\sum_{1 \leq j < k \leq \min(n-1, n+j-3)} L \left( \frac{(z_j - z_{j-1})(z_{k+1} - z_k)}{(z_j - z_{k+1})(z_{j-1} - z_k)} \right) = \frac{n-3}{6} \pi^2. \tag{6.3}
\]

\(^{10}\) One can easily prove \( \arg(u_{j,k+1}) \geq 0 \) and \( |u_{j,k+1}| \leq 1 \) by geometrical consideration.

\(^{11}\) Alternatively, we may make the transformation \( z_j' = \frac{(z_j - z_1)(z_n - z_{n-1})}{(z_j - z_n)(z_1 - z_{n-1})} \).
7. Conclusion

In the present paper, we have found that the dilogarithm identities can be extended to general many-variable ones. Our main results are the identity in multiperipheral form (Theorem 1 in Sect. 4), that in general tree form (Theorem 2 in Sect. 5) and that in cross-ratio form (Theorem 3 in Sect. 6). Of course, those identities can be rewritten in various forms by combining them with those involving fewer variables, as was done in deriving (6.3).

It is quite remarkable that the dilogarithm identities are closely related to the channel variables of the dual resonance model. But, at present, it is not clear whether this fact is merely accidental or has a deep meaning. Since, as is well known, the dual resonance model is reproduced by the bosonic string theory, there might exist some relation between the dilogarithm and the string theory. It is also unclear whether or not our results are relevant to the quantum version of the dilogarithm. It is desirable to make further investigations from various points of view.

Note added One of the referees kindly made the following comment:

There is a well known dilogarithm identity of \( n \) (\( n \) corresponds to \( n - 3 \) of the present paper) independent variables called the dilogarithm identity, associated with the \( Y \)-sides). As an evident example, for

This was also first proved by Frenkel–Szenes [11].

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\[ Y_i(u - 1)Y_i(u + 1) = (1 + Y_{i-1}(u))(1 + Y_{i+1}(u)), \]

where \( Y_0(u) = Y_{n+1}(u) = 0 \). (It is also identified with the \( Y \)-system of type \( A_1 \) at higher level by the well known level–rank duality.)

Any solution of the above \( Y \)-system has the following periodicity:

\[ Y_i(u + n + 3) = Y_{n+1-i}(u). \]  

This was conjectured by Al. B. Zamolodchikov [10], and proved by Frenkel–Szenes [11] and Gliozzi–Tateo [12]. Having this (then conjectured) periodicity in mind, Gliozzi–Tateo ([13], Eq. (2.14)) proposed the conjecture of the dilogarithm identities:

\[
\sum_{u = 0}^{n} \sum_{i = 1}^{n} \frac{Y_i(u)}{1 + Y_i(u)} = \frac{\pi^2}{6} n(n + 1). 
\]

This was also first proved by Frenkel–Szenes [11].

This identity (7.3) is equivalent to the author’s identity (doubled in both the left- and right-hand sides). As an evident example, for \( n = 3 \), the identity (7.3) has the following explicit form ([13], Eq. (2.20)):

\[
H(x) + H(y) + H(z) + H\left(\frac{1 + y}{x}\right) + H\left(\frac{1 + x + y + z + xz}{yz}\right) \\
+ H\left(\frac{(1 + x + y)(1 + y + z)}{xyz}\right) + H\left(\frac{(1 + x)(1 + z)}{y}\right) \\
+ H\left(\frac{1 + y}{z}\right) + H\left(\frac{1 + x + y + z + xz}{xy}\right) = \pi^2, 
\]

where \( H(x) = L(x/(1 + x)) \). By identifying

\[
\frac{1 + y}{1 + x + y}, \quad \frac{(1 + x + y)(1 + y + z)}{(1 + y)(1 + x + y + z + xz)}, \quad \frac{1 + x + y + z + xz}{(1 + y + z)(1 + x)} 
\]

(7.5)
in (7.4) with $x, y, z$ of (1.9), it is easy to see that (7.4) is equivalent to (1.9). For the general $n$-variable case, it was shown by Gliozzi–Tateo ([12], Eq. (2.26)) that the general solution of the $Y$-system (7.1) is given by a cross ratio of $n + 3$ points. (In fact, this is their proof of the periodicity (7.2).) It should be easy for the author to check and confirm that, upon the substitution of the cross-ratio solution, the identity (7.3) coincides with the third expression (6.2). More directly, the alternative form of the third expression (6.3) also appeared in the paper by Bridgeman ([14], Eq. (6)).

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