The commutator of raising and lowering operators for angular momentum to the free particle’s Hamiltonian

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Abstract. In general some operators in quantum mechanics are not commutable, which means the measurement of two or more operators can’t be done simultaneously. Angular momentum operator and Hamiltonian are examples of operators that can only be measured using a mathematical approach in the form of a commutation relationship. Based on the fifth postulate of quantum mechanics about a constant of motion, if an operator commute with Hamiltonian operator then the operator is a constant of motion. This study was conducted to examine the commutation of the angular momentum operators to the Hamiltonian operator. The commutation of the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z,$ and $\hat{L}$ to the Hamiltonian operator shows that the operators are commute because the values are zero. Otherwise, the results of the component quadratic angular momentum operators commutator $\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2, \hat{L}_x^2, \hat{L}_y^2,$ and $\hat{L}_z^2$ against Hamiltonian operator show that the operators aren’t commute with the Hamiltonian operator because the values are not zero. Based on the results of the commutation relation indicate that the angular momentum operators are the constant of motion otherwise the quadratic angular operator are not the constant of motion.

1. Introduction

The development of quantum physics began with Max Planck’s postulate about energy quanta. This postulate became the basis for the emergence of other quantum theories, such as the photoelectric effect, the Bohr atomic theory, the Compton effect, the de Broglie hypothesis, and others. In addition to Planck’s postulate, de Broglie’s hypothesis about wave dualism is also one of the theories that underlie quantum mechanics and can change the perspective of classical physics into modern physics. Based on his belief that nature is symmetrical, de Broglie states that if waves are like particles, then particles can also be like waves. Furthermore, de Broglie’s hypothesis became the basis of Heisenberg and Schrodinger to formulate the theory of quantum mechanics [1].

In general, the problem of quantum mechanics can be solved using two approaches, namely the differential-integral approach and the formal mathematical approach. The solution to the problem of quantum mechanics must be obedient to some basic postulates of quantum mechanics, which are state representations, observable (dynamic variables) representations, expected values of measurement, and dynamics of quantum systems [2]. The postulate on the representation of dynamic variables states that a dynamic variable in quantum mechanics is represented in the form of an operator. The operator is a mathematical function that can be used to measure dynamic variables in a quantum system [3]. Some examples of operators in quantum mechanics are position operator, linear momentum operator, angular momentum operator, and energy operator. These operators will be meaningful if it’s applied to a wave function.

Angular momentum operators have an important role in quantum mechanics. The amount of angular momentum in quantum mechanics is a finite number of states or called quantization numbers. Angular momentum operators can be used to review quantum systems that move in spherical coordinates for example in the Bohr’s model of hydrogen atom [4]. The Bohr’s model of hydrogen atom is based on the quantization of angular momentum. Besides angular momentum also has an important role in observing magnetic properties, the movement of electrons in atoms, molecular rotations, the movement of nucleons in thermonuclear reactions, and others.

The Hamiltonian operator is a representation of the total energy in a quantum mechanical system. The Hamiltonian operator of free particles is a representation of its kinetic energy only
because free particles move without being influenced by any force. So based on the relationship \( F = -\frac{dV}{dx} \) with \( V \) is potential energy, if the force \( F \) is zero then the potential energy acting on the particle can be a constant or can be zero according to the rules of integration [5].

Measurements in quantum physics are generally not commutable. It means that the measurements of two or more observables cannot be done simultaneously [6]. When the wave behavior of an observable accuracy is higher, then the observable particle's accuracy will be low. Conversely, if the particle nature of an observable accuracy is higher, then the observable wave nature of accuracy will be low so that measurements of two or more observables are carried out using a commutator. The commutator can be meaningful if it is imposed on the wave function. If the commutator between several operators is zero then the operators are commutable. This means that these operators can be measured simultaneously. Measurement of momentum and position is one example of a measurement that is not commutable [7].

Angular momentum operators and Hamiltonian are observables in quantum mechanics that can only be measured using a mathematical approach in the form of a commutation relationship [8]. Therefore research on the commutation relationship between angular momentum operators and Hamiltonian was carried out. This study uses Cartesian coordinates to be general and easily transformed into other coordinates. This research updates are on the operator and the variables used. The previous study used an energy operator which is positive, while this study used the Hamiltonian operator which is negative. The research variables used in the previous study are only angular momentum components, while this research variable not only uses angular momentum components but also uses the raising and lowering operators of angular momentum.

### 2. Method

An angular momentum operator is an operator formed by position operators and linear momentum operators [9]. The linear operator has the following expression:

\[
p = -i\hbar \hat{\nabla}
\]

where

\[
\hat{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}
\]

then the linear momentum component is obtained as follows:

\[
\hat{p}_x = -i\hbar \frac{\partial}{\partial x}; \quad \hat{p}_y = -i\hbar \frac{\partial}{\partial y}; \quad \hat{p}_z = -i\hbar \frac{\partial}{\partial z}
\]

Based on the definition of the linear momentum operator, the angular momentum operator can be obtained by cross-product operation between the position operator \( \hat{r} \) and the linear momentum operator \( \hat{p} \) as follows:

\[
L = r \times p = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
x & y & z \\
p_x & p_y & p_z
\end{vmatrix}
\]

\[
(y p_z - z p_y) \hat{i} + (z p_x - x p_z) \hat{j} + (x p_y - y p_x) \hat{k}
\]

The solution above produces the components of angular momentum as follows:

\[
L_x = y p_z - z p_y
\]

\[
L_y = z p_x - x p_z
\]

\[
L_z = x p_y - y p_x
\]

If the linear momentum operator component is substituted into the angular momentum operator, it will get:

\[
L_x = y \left( -i\hbar \frac{\partial}{\partial z} \right) - z \left( -i\hbar \frac{\partial}{\partial y} \right) = -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)
\]

\[
L_y = z \left( -i\hbar \frac{\partial}{\partial x} \right) - x \left( -i\hbar \frac{\partial}{\partial z} \right) = -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)
\]

\[
L_z = x \left( -i\hbar \frac{\partial}{\partial y} \right) - y \left( -i\hbar \frac{\partial}{\partial x} \right) = -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\]
Use the definition of $L^2 = \mathbf{L} \cdot \mathbf{L}$ we get the operator $\hat{L}^2$ as follows:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \text{(11)}$$

Then do the following way to obtain the components of quadratic angular momentum operator:

$$\hat{L}_x^2 = (yp_x - zp_y)(yp_x - zp_y)$$

$$= y^2p_x - y^2p_y + z^2p_x - z^2p_y - ipy_x + ipy_y$$

$$= -h^2y^2 \frac{\partial^2}{\partial z^2} + h^2y \frac{\partial}{\partial y} + h^2y^2 \frac{\partial^2}{\partial y \partial z} + h^2y^2 \frac{\partial^2}{\partial y \partial z} - h^2z^2 \frac{\partial^2}{\partial y^2}$$

$$\text{by substituting the angular momentum operator component \(\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2\) into the operator \(\hat{L}^2\) we obtain:}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

The angular momentum operator $\hat{L}_{\pm}$ is a combination of the two components of the angular momentum operator [4]. The raising and lowering operators for angular momentum is defined as follows:

$$\hat{L}_+ = \hat{L}_x \pm i \hat{L}_y \quad \text{(17)}$$

by substituting the angular momentum operator component $\hat{L}_x$ dan $\hat{L}_y$ into the operator $\hat{L}_\pm$ we can obtain:

$$\hat{L}_+ = (yp_x - zp_y) + i(zp_x - xp_y) \quad \text{(18)}$$

$$\hat{L}_- = (yp_x - zp_y) - i(zp_x - xp_y) \quad \text{(19)}$$

The raising and lowering operators for quadratic angular momentum are obtained using the following method:

$$\hat{L}_+^2 = (yp_x - zp_y + izp_x - ixp_y)(yp_x - zp_y + izp_x - ixp_y)$$

$$= y^2p_x - y^2p_y + z^2p_x - z^2p_y - ipy_x + ipy_y$$

$$= -h^2y^2 \frac{\partial^2}{\partial z^2} + h^2y \frac{\partial}{\partial y} + h^2y^2 \frac{\partial^2}{\partial y \partial z} - ih^2yz \frac{\partial^2}{\partial x \partial z} + ih^2yz \frac{\partial^2}{\partial y \partial z} + h^2z^2 \frac{\partial^2}{\partial y^2}$$

$$+ h^2yz \frac{\partial^2}{\partial y \partial z} - h^2z^2 \frac{\partial^2}{\partial y^2} + ih^2yz \frac{\partial^2}{\partial x \partial y} - ih^2yz \frac{\partial^2}{\partial x \partial y}$$

$$+ ith^2yz \frac{\partial^2}{\partial x \partial y} - h^2z \frac{\partial^2}{\partial z^2} + h^2z \frac{\partial^2}{\partial z^2}$$

$$- ih^2xz \frac{\partial^2}{\partial y \partial z} + h^2x \frac{\partial^2}{\partial z^2}$$

$$= h^2$$

$$- x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - ix \frac{\partial}{\partial x} + iy \frac{\partial}{\partial y} + x^2 \frac{\partial^2}{\partial z^2} - y^2 \frac{\partial^2}{\partial z^2} + z^2 \frac{\partial^2}{\partial x^2}$$

$$+ 2xyz \frac{\partial^2}{\partial y \partial z} - 2ixy \frac{\partial^2}{\partial z^2} - 2iyz \frac{\partial^2}{\partial x \partial y} + 2iz^2 \frac{\partial^2}{\partial x \partial y}.$$
\[
\hat{H} = \hat{\mathbf{p}}^2 - 2ixyp_x - 2i\mathbf{y}p_x + 2iz^2p_y - 2ixyp_z - 2i\mathbf{y}p_z + 2iz^2p_y(p_y)
\] 

The solution above produces the raising and lowering operators for quadratic angular momentum as follows:

\[
\hat{L}_+^2 = \hat{\mathbf{p}}^2 - 2ixyp_x - 2i\mathbf{y}p_x + 2iz^2p_y - 2ixyp_z - 2i\mathbf{y}p_z + 2iz^2p_y(p_y)
\]
\[
\hat{L}_-^2 = \hat{\mathbf{p}}^2 - 2ixyp_x + ixyp_y + iy p_x + x^2 p_x^2 - y^2p_x^2 + z^2p_x^2 - z^2p_y^2 - 2ixzp_xp_z + 2yzp_y + 2iz^2p_y(p_y)
\]

The next step is to determine the Hamiltonian operator for free particles. The Hamiltonian operator is a representation of the total energy in a quantum mechanical system [10]. The use of the Hamiltonian operator in the Schrodinger equation is written:

\[
\hat{H}\Psi = E\Psi
\]

The Hamiltonian operator is the total energy accumulation that consists of kinetic and potential energy. The form of the operator of energy is as follows:

\[
E = T + V
\]

by using the definition of kinetic energy:

\[
T = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)
\]

where \(m\) is the mass of the particle moving with momentum \(p\) in the \(x, y\) and \(z\) coordinates. Then by substituting equation (25) to equation (24) will be obtained:

\[
E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x,y,z)
\]

The free particle in its movement is not affected by any force so the resultant force is equal to zero. Based on the definition of \(F = -\frac{\partial V}{\partial x}\) it can be said that the potential energy of free particles is zero. If the potential energy of free particles is zero then the Hamiltonian operator for free particles becomes:

\[
\hat{H} = -\frac{\hbar}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)
\]

After the angular momentum operator and the Hamiltonian operator are defined, then the commutation relationship between the two operators can be determined in the following way:

\[
[A, B] = AB - BA
\]

If the commutator calculation result of two operators is zero, then the two operators are commutable. But in general operators in quantum mechanics are not commutable or \([\hat{A}, \hat{B}] \neq 0\) [11]. The following are the properties of commutators in quantum mechanics:

a. \([\hat{A}, \hat{B}] + [\hat{B}, \hat{A}] = 0\) 

b. \([\hat{A}, \hat{A}] = 0\)

c. \([\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]\)

d. \([\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]\)

e. \([\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + \hat{B}[\hat{A}, \hat{C}]\)

f. \([\hat{A}, \hat{B}] = [\hat{A}, \hat{C}] [\hat{B}, \hat{C}]\)

g. \([\hat{A}, [\hat{B}, \hat{C}]] + [\hat{C}, [\hat{A}, \hat{B}]] + [\hat{B}, [\hat{C}, \hat{A}]] = 0\)

Based on the fourth postulate regarding the dynamics of the quantum system, the state function \(\Psi\) which evolves with time at the Schrodinger equation satisfies the relationship:

\[
\frac{\partial \Psi}{\partial t} = i\hbar \hat{H}\Psi
\]

If the expectation value \(\langle A \rangle\) evolves with time, the equation will be obtained:

\[
\frac{\partial}{\partial t} \langle A \rangle = \frac{\partial}{\partial t} \int \Psi^* \hat{A} \Psi dv = \int \Psi^* \hat{\mathbf{p}}^2 \Psi dv + \int \Psi^* \frac{\partial}{\partial t} \hat{\mathbf{p}} \Psi dv + \int \Psi^* \hat{\mathbf{p}} \Psi dv
\]
Then with a mathematical calculation equation (37) will produce the following equation:

\[
\frac{\partial}{\partial t} \langle \hat{A} \rangle = \frac{1}{i \hbar} \langle [\hat{A}, \hat{H}] \rangle + \langle \frac{\partial \hat{A}}{\partial t} \rangle \quad (38)
\]

If the operator \( \hat{A} \) value is constant against time then \( \frac{\partial}{\partial t} \langle \hat{A} \rangle \) is zero so based on the equation (38) we can get

\[
\frac{\partial}{\partial t} \langle \hat{A} \rangle = \frac{1}{i \hbar} \langle [\hat{A}, \hat{H}] \rangle \quad (39)
\]

If the operator \( \hat{A} \) commute with Hamiltonian operator \( \hat{H} \), then the operator \( \hat{A} \) is a constant of motion [12]. For example when the position operator \( \hat{x} \) replaces the operator \( \hat{A} \) equation (39) will be:

\[
\frac{\partial}{\partial t} \langle \hat{x} \rangle = \frac{1}{i \hbar} \langle [\hat{x}, \hat{H}] \rangle = \frac{1}{i \hbar} \langle \hat{x}, \frac{p^2}{2m} + V \rangle \quad (40)
\]

by using mathematical calculations it will be obtained

\[
\frac{\partial}{\partial t} \langle \hat{x} \rangle = \langle \frac{p}{m} \rangle \quad (41)
\]

Equation (41) represents the equation of motion in mechanics, whereas for the operator \( \hat{A} \) which is replaced by the linear momentum operator \( \hat{p} \) equation (39) will be:

\[
\frac{\partial}{\partial t} \langle \hat{p} \rangle = \frac{1}{i \hbar} \langle [\hat{p}, \hat{H}] \rangle = \frac{1}{i \hbar} \langle \hat{p}, \frac{p^2}{2m} + V \rangle \quad (42)
\]

By using the commutation relationship between the linear momentum operator \( \hat{p} \) and the potential energy \( V \) below:

\[
[\hat{p}, V] = -i \hbar \left( \frac{\partial V}{\partial x} \right) \psi \quad (43)
\]

Then equation (43) substituted into equation (42) we obtain

\[
\frac{\partial}{\partial t} \langle \hat{p} \rangle = - \left( \frac{\partial V}{\partial x} \right) = \langle F \rangle \quad (44)
\]

Equation (44) represents the equation of motion in mechanics.

Postulates about the dynamics of the quantum system can be used as a basis for observing the results of the commutator of the angular momentum operator and the free particle’s Hamiltonian. If the angular momentum operator commute with the Hamiltonian operator, then the angular momentum operator is a representation of the constant of motion. However, if the two operators are not commutative, the operator is not a representation of the constant of motion.

### 3. Result and Discussion

Measurement of two or more dynamic variables in quantum mechanics is done using a mathematical approach in the form of a commutator. Commutator is a commutation operation between two or more operators. Commutators can be meaningful if applied to a wave function. Two or more operators can be called commute if the result of the commutator is zero. When two or more operators are commute, they can be measured simultaneously. Commutator can be done using two ways, namely the differential method and the operator method. In this research, the commutator of the angular momentum operators to Hamiltonian operator use both methods so that the calculation results obtained are more accurate. This study uses cartesian coordinates to be more general and more easily changed to other coordinates. The wave function used is not normalized because the focus of this study is not the wave function but on the commutation relationship between the two operators.

Based on the results of mathematical manual calculations using commutator and differential rules the following data are obtained:

**Table 1.** The commutator calculation result of angular momentum operator and free particle’s Hamiltonian

| No. | Commutator | Result |
|-----|------------|--------|
| 1   | \([\hat{L}_x, \hat{H}]\) | 0      |
| 2   | \([\hat{L}_y, \hat{H}]\) | 0      |
| 3   | \([\hat{L}_z, \hat{H}]\) | 0      |
Table 1 shows that the commutator results of the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{L}_+ \hat{L}_-, \hat{L}_{+}^2$ and the Hamiltonian operator $\hat{H}$ are zero. This means that the commutation relationship between the components of the angular momentum operators to the Hamiltonian operator are commute. Likewise, the relationship between the raising and lowering operators of angular momentum to the Hamiltonian operator are commute. The quantum mechanical postulate about the constant of motion states that an operator commanding the Hamiltonian operator is a constant of motion [4]. Two or more commutable operators can be measured simultaneously [5]. Based on the data in Table 3.1, it can be said that the raising and lowering operators of angular momentum are the constants of motion because they are commutable to the Hamiltonian operator.

Table 2. The commutator calculation result of quadratic angular momentum operator and free particle’s Hamiltonian

| No. | Commutator | Result                  |
|-----|------------|-------------------------|
| 1   | $[\hat{L}_x^2, \hat{H}]$ | $2\hbar^2 \hat{H}_{yz}$ |
| 2   | $[\hat{L}_y^2, \hat{H}]$ | $2\hbar^2 \hat{H}_{xx}$ |
| 3   | $[\hat{L}_z^2, \hat{H}]$ | $2\hbar^2 \hat{H}_{xy}$ |
| 4   | $[\hat{L}_+^2, \hat{H}]$ | $4\hbar^2 \hat{H}$     |
| 5   | $[\hat{L}_-^2, \hat{H}]$ | $-2\hbar^2 \left( \hat{H}_x - \hat{H}_y + \frac{i}{m}p_x p_y \right)$ |
| 6   | $[\hat{L}_-^2, \hat{H}]$ | $-2\hbar^2 \left( \hat{H}_x - \hat{H}_y - \frac{i}{m}p_x p_y \right)$ |

Based on Table 2, it can be seen that the commutator of the quadratic angular momentum operator and the free particle’s Hamiltonian is $4\hbar^2 \hat{H}$. The commutator of the raising operator for quadratic angular momentum and the Hamiltonian operator is $-2\hbar^2 \left( \hat{H}_x - \hat{H}_y + \frac{i}{m}p_x p_y \right)$. The commutator of the lowering operator for quadratic angular momentum and the Hamiltonian operators $-2\hbar^2 \left( \hat{H}_x - \hat{H}_y - \frac{i}{m}p_x p_y \right)$. These data show that quadratic angular momentum $\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2, \hat{L}_+^2, \hat{L}_-^2$ do not commute with the Hamiltonian operator $\hat{H}$ because the results are not zero. Two or more non-commutable operators are not a fixed number and cannot be measured together in a quantum system [1]. It means that the raising and lowering operators for angular momentum is not the constant of motion.

In this study, the commutator of angular momentum operators to the Hamiltonian operator is imposed on the free particle wave function. Free particles are particles whose movements are not affected by any force so that the total energy in free particles only consists of kinetic energy [4]. The commutator is imposed on a wave function to have a physical meaning. In addition to using the free particle wave function, commutators can also be applied to other system wave functions, such as the wave function of the hydrographic atom, the wave function of a particle in a box, and so on. The wave function imposed on the commutator will change to another function and the result of its operation can be a constant multiple of the original function. The wave function that changes to a constant multiple of the original function when operated on the commutator is a wave function that satisfies the eigenproblem [13]. Even though the wave function imposed on a commutator from a different system, the commutation operation will still produce the same value if the wave function imposed meets the eigenproblem because the commutator has the role of a catalyst.
The research data obtained in this study when compared with previous studies [9] there are some similarities and some differences. The similarity of the data of this study and previous studies [9] is found in the results of the commutator of the angular momentum operators to the zero value Hamiltonian operator. While the difference is found in the results of the commutator of quadratic angular momentum operators to the Hamiltonian operator. In previous studies [9] the results of the commutator operators $\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2, \hat{L}^2$ to the Hamiltonian operator $\hat{H}$ sequentially valued $-2h^2\hat{H}_{yz}, -2h^2\hat{H}_{xz}, -2h^2\hat{H}_{xy}, -4h^2\hat{H}$ while in this study the results of the commutator operators $\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2, \hat{L}^2$ to the Hamiltonian operator $\hat{H}$ sequentially valued $2h^2\hat{H}_{yz}, 2h^2\hat{H}_{xz}, 2h^2\hat{H}_{xy}, 4h^2\hat{H}$.

Based on these comparisons, it appears that the actual difference between this study and previous studies occurred because the operators used were different. This study uses a negative Hamiltonian operator. Whereas previous studies used positive energy operators. The difference between Hamiltonian operators and energy operators is that Hamiltonian operators are Hermitian operators while energy operators are not Hermitian operators. According to quantum mechanics theories only Hermitian operators whose expectations value are real [12].

4. Conclusion
The commutation of the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{L}_+ , \hat{L}_-$ to the Hamiltonian operator $\hat{H}$ are commutable so that the angular momentum operators are classified as a motion constant. The angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{L}_+ , \hat{L}_-$ can be measured simultaneously with the Hamiltonian operator in a free particle quantum system. The commutation of the angular momentum operators $\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2, \hat{L}^2, \hat{L}_+^2, \hat{L}_-^2$ to the Hamiltonian operator $\hat{H}$ are not commutable, so the quadratic angular momentum operators are not classified as a motion constant. The quadratic angular momentum operators $\hat{L}_x^2, \hat{L}_y^2, \hat{L}_z^2, \hat{L}^2, \hat{L}_+^2, \hat{L}_-^2$ cannot be measured together with the Hamiltonian operator in the free particle quantum system. Further research can be done by changing the research variables using other operators such as position operators and linear momentum operators.

Acknowledgment
The author would like to thank Drs. Bambang Supriadi, M.Sc. who has supported, motivated, guided, and directed the author in writing this article. The author would like to thank Vela Riziqiyah, S.Pd for providing inspiration, motivation, and attention to this study. The author also thanks the research team for the Quantum Group and Physics Theory of Physics Education from the University of Jember for providing knowledge, support, and inspiring writers.

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