Bayesian Estimation and Prediction of Discrete Gompertz Distribution

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Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, Bayesian inference is used to estimate the parameters, survival, hazard and alternative hazard rate functions of discrete Gompertz distribution. The Bayes estimators are derived under squared error loss function as a symmetric loss function and linear exponential loss function as an asymmetric loss function. Credible intervals for the parameters, survival, hazard and alternative hazard rate functions are obtained. Bayesian prediction (point and interval) for future observations of discrete Gompertz distribution based on two-sample prediction are investigated. A numerical illustration is carried out to investigate the precision of the theoretical results of the Bayesian estimation and prediction on the basis of simulated and real data. Regarding the results of simulation seems to perform better when the sample size increases and the level of censoring decreases. Also, in most cases the results under the linear exponential loss function is better than the corresponding results under squared error loss function. Two real lifetime data sets are used to insure the simulated results.

Keywords: Gompertz distribution; Bayes estimators; squared error loss function; linear exponential loss function; credible intervals; Bayesian prediction; Monte Carlo simulation.

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1 Introduction

Although the life length in the real world may be associated with continuous non-negative lifetime distributions, it is sometimes difficult to get samples from a continuous distribution in real life. In many practical situations, the reliability data are measured in terms of the numbers of runs, cycles or shocks the device sustains before it fails. For example, the number of times the devices are switched on/off, the lifetime of the switch is a \textit{discrete random variable} (drv). Also, the number of voltage fluctuations; which an electrical or electronic item can withstand before its failure, is a drv, the life of equipment is measured by the number of completed cycles or the number of times it operated before failure, or the life of weapon is measured by the number of rounds fired prior to failure. Similarly, in survival analysis the \textit{survival function} (sf) may be a function of drv that is considered as a discrete version of the analogue \textit{continuous random variable} (crv). Such as the length of stay in observation ward; when it is measured by the number of days, or the survival time that the leukemia patients survived since therapy may be counted by number of days or weeks.

The geometric and negative binomial distributions are the discrete versions for the exponential and gamma distributions, respectively, but they have monotonic hazard rate functions and thus they are unsuitable for some situations. Also, there are few discrete distributions which can provide accurate models for both count and times. As Poisson distribution, is used to model counts but not times. Also the binomial distribution is not considered to be popular model for reliability, failure times and counts. It can be approximated to Poisson distribution under suitable conditions. In addition to that, these discrete distributions only cater to positive integers along with zero, but in some analysis the variable of interest can take either zero, positive or negative values. In many situations the interest may be in the difference of two drvs each having integer support \((0, \infty)\). The resulting difference will be another drv with integer support \((-\infty, \infty)\), see Chakraborty and Chakravorty [1]. Thus, there is a need to derive appropriate discrete distributions by discretizing the continuous distributions to fit various types of data. In the light of the above contexts, the study of the discretization of continuous is meaningful.

Several discrete lifetime distributions are constructed by discretizing there conjugate continuous models using several methods. Some of important discrete lifetime models are introduced by Nakagawa and Osaki [2], Khan et al. [3], Roy [4,5], Inusah and Kozubowski [6], Krishna and Pundir [7], Jazi et al. [8]. Also, Gomez-Deniz and Calderin-Ojeda [9], Nekoukhou et al. [10,11], AL-Huniti and AL-Dayian [12], Lekshmi and Sebastian [13], Para and Jan [14], Hussain and Ahmad [15], Hussain et al. [16] and Alamatsaz et al. [17].

Migdadi [18] obtained Bayes estimators for the scale parameter of discrete Rayleigh distribution based on \textit{squared error} (SE) and general entropy loss functions. They also considered prediction for the future ordered observation. Kamari et al. [19] studied Bayesian analysis of discrete Burr distribution, they used the Metropolis-Hastings method to estimate the parameters numerically under two loss functions, SE and absolute error loss functions.

The \textit{discrete Gompertz distribution} (DGD) with two parameters \(a\) and \(\theta\) was introduced by Hegazy et al. [20]. They discussed its properties and estimation of its parameters using the methods of moments and \textit{maximum likelihood} (ML).

A lifetime \textit{random variable} (rv), \(X\), has \textit{Gompertz distribution} (GD) with parameters \(a\) and \(b\), if its \textit{probability density function} (pdf) is as follows:

\[ g(x; a, b) = be^{-ax} \frac{e^{ax-1}}{a}, \quad x > 0; \quad a, b > 0. \]

The corresponding \textit{sf} is given by

\[ S(x; a, b) = e^{-\frac{b}{a} (e^{ax} - 1)}, \quad x > 0; \quad a, b > 0. \]
If the times are grouped into unit intervals, the discrete variable \(X (dX) = [X]\), which is the largest integer less than or equal to \(X\), will have the probability mass function (pmf)

\[
P (x) = P [X = x] = P [x \leq X < x + 1]
\]

\[
= S (x) - S (x + 1), \ x = 0, 1, 2, ...
\]

(1)

The pmf of rv \(dX\) can be viewed as discrete concentration of the pdf of \(X\).

Hegazy et al. [20] constructed the DGD by using (1) and re-parameterization \(\theta = e^{-b}\).

A continuous random variable \(X\) with sf, \(S(x)\) is commonly said to have DGD with parameters \(a\) and \(\theta\); denoted by DGD \((a, \theta)\), if its pmf \(P(X = x)\) is given by

\[
P(x) \equiv P(x; a, \theta) = \theta a^{-1} (e^{ax} - 1) - \theta a^{-1} (e^{a(x+1)} - 1), \ x = 0, 1, 2, ..., \ 0 < \theta < 1, \ a > 0.
\]

(2)

The corresponding cumulative density function (cdf), sf, hazard rate function (hrf) and alternative hrf (ahrf), respectively, are as follows:

\[
F(x; a, \theta) = P(X \leq x) = 1 - \theta a^{-1} (e^{ax} - 1), \ x = 0, 1, 2, ...
\]

(3)

\[
S(x) \equiv S(x; a, \theta) = P(X \geq x) = \theta a^{-1} (e^{ax} - 1), \ x = 0, 1, 2, ...
\]

(4)

\[
h(x; a, \theta) = \frac{\theta a^{-1} (e^{ax} - 1) - \theta a^{-1} (e^{a(x+1)} - 1)}{\theta a^{-1} (e^{ax} - 1)}, \ x = 0, 1, 2, ...
\]

(5)

and

\[
h_1(x; a, \theta) = \ln \left[ \frac{S(x)}{S(x+1)} \right] = \ln(\theta) \cdot a^{-1} e^{ax} (1 - e^{-a}), \ x = 0, 1, 2, ...
\]

(6)

It is important to note that \(S(x)\) of DGD \((a, \theta)\) has the same functional form of GD \((a, b)\).

The rest of this paper is organized as follows: Bayes estimators for the parameters, sf, hrf and ahfr of DGD \((a, \theta)\) are derived based on Type II censored samples. Bayesian prediction based on two-sample prediction is considered. The precision of the theoretical results of Bayesian estimation and prediction on the basis of simulated and real data are investigated.

2 Bayesian Estimation

The Bayesian approach is considered, under SE and linear exponential (LINEX) loss functions to estimate the parameters, sf, hrf and ahfr of the DGD \((a, \theta)\) based on Type II censored samples, using non-informative prior for the parameter \(a\) and conjugate prior for the parameter \(\theta\). Also credible intervals for the parameters, sf, hrf and ahfr are obtained.

2.1 Bayesian estimation for the parameters

Suppose that \(X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}\) is a Type II censored sample of size \(r\) obtained from a life-test on \(n\) items whose lifetimes have a DGD \((a, \theta)\). Then the likelihood function is

\[
L(a, \theta | x) \propto \left[ \prod_{i=1}^{n} p(x_{(i)}) \right] S(x_{(r)})^{n-r},
\]

(7)
where \( P(x) \) and \( S(x) \) are given, respectively, by (2) and (4). The \( x(i) \)’s are ordered times for \( i = 1,2, \ldots, r \).

\[
L(a,\theta|x) \propto \left( \prod_{i=1}^{r} \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r},
\]

where

\[
w_{i1} = a^{-1} (e^{a x(i)} - 1), \quad w_{i2} = a^{-1} \left( e^{a(x(i)+1)} - 1 \right) \quad \text{and} \quad w_r = a^{-1} (e^{a x(r)} - 1).
\]

Assuming that both of the parameters \( a \) and \( \theta \) are independent, \( \theta \) has Beta \((c, d)\) as a prior distribution, and \( a \) has a non-informative prior, the joint prior for \( a \) and \( \theta \) is

\[
\pi(a, \theta) \propto \frac{1}{a} \theta^{c-1} (1 - \theta)^{d-1}, \quad a > 0, \quad 0 < \theta < 1.
\]

The joint posterior distribution for \( a \) and \( \theta \) can be obtained using (8) and (10) as follows:

\[
\pi(a, \theta|x) \propto L(a, \theta|x) \pi(a, \theta)
\]

\[
= k_1 \left( \prod_{i=1}^{r} \left( \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r} \right) \frac{1}{a} \theta^{c-1} (1 - \theta)^{d-1},
\]

where \( k_1^{-1} = \int_{0}^{\infty} \int_{0}^{\infty} \left( \prod_{i=1}^{r} \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r} \frac{1}{a} \theta^{c-1} (1 - \theta)^{d-1} d\theta da, \)

which is a normalizing constant and \( w_{i1}, w_{i2} \) and \( w_r \) are given in (9).

The marginal posterior distributions, \( \pi(a|x) \) and \( \pi(\theta|x) \), are given, respectively, as

\[
\pi(a|x) = k_1 \frac{1}{a} \int_{0}^{\infty} \left( \prod_{i=1}^{r} \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r} \theta^{c-1} (1 - \theta)^{d-1} d\theta,
\]

and

\[
\pi(\theta|x) = k_2 \theta^{c-1} (1 - \theta)^{d-1} \int_{0}^{\infty} \frac{1}{a} \left( \prod_{i=1}^{r} \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r} \theta^{c-1} (1 - \theta)^{d-1} d\theta da.
\]

2.1.1 Point estimation

The Bayesian point estimation of the parameters under the SE and LINEX loss functions are derived.

I. Bayesian estimation under squared error loss function

The Bayes estimators under the SE loss function of the parameters \( a \) and \( \theta \) are the means of their marginal posterior distributions in (13) and (14), respectively, as follows:

\[
a^*_\text{(SE)} = E(a|x)
\]

\[
= \int_{0}^{\infty} \int_{0}^{\infty} k_1 \left( \prod_{i=1}^{r} \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r} \theta^{c-1} (1 - \theta)^{d-1} d\theta da,
\]

and

\[
\theta^*_\text{(SE)} = E(\theta|x)
\]

\[
= \int_{0}^{\infty} \int_{0}^{\infty} k_2 \left( \prod_{i=1}^{r} \theta^{w_{i1}} - \theta^{w_{i2}} \right) [\theta^{w_r}]^{n-r} \theta^{c} (1 - \theta)^{d-1} d\theta da.
\]
II. Bayesian estimation under linear exponential loss function

Under the LINEX loss function, the Bayes estimators for the parameters $a$ and $\theta$ are given, respectively, by

$$
a^*_a (\text{LINEX}) = \frac{-1}{\hat{\theta}} \ln E \left( e^{-a} \big| \bar{x} \right),
$$

(17)

where $\hat{\theta}$ is a constant and $\hat{\theta} \neq 0$.

$$
E \left( e^{-a \bar{x}} \right) = \int_0^\infty \frac{1}{a} e^{-a \bar{x}} \int_0^1 k_1 \prod_{i=1}^{r} \left( \theta^{w_{i1}} - \theta^{w_{i2}} \right) \left[ \theta^{w_r} \right]^{n-r} \theta^{c-1}(1 - \theta)^{d-1} d\theta da,
$$

(18)

and

$$
\theta^*_a (\text{LINEX}) = \frac{-1}{\hat{\theta}} \ln E \left( e^{-\theta} \big| \bar{x} \right),
$$

(19)

where

$$
E \left( e^{-\theta \bar{x}} \right) = \int_0^1 e^{-\theta \bar{x}} \int_0^\infty \frac{1}{a} k_1 \prod_{i=1}^{r} \left( \theta^{w_{i1}} - \theta^{w_{i2}} \right) \left[ \theta^{w_r} \right]^{n-r} \theta^{c-1}(1 - \theta)^{d-1} d\theta d\theta.
$$

(20)

To obtain the Bayes estimates of the parameters, Equations (15), (16), (17) and (19) can be evaluated numerically.

2.1.2 Credible intervals for the parameters

In general, $L(\bar{x})$, $U(\bar{x})$, is a 100 (1 - $\omega$) % credible interval for $\varphi$ if

$$
P \left[ L(\bar{x}) < \varphi < U(\bar{x}) \right] = \int_{L(\bar{x})}^{U(\bar{x})} \pi(\varphi | \bar{x}) d\varphi = 1 - \omega,
$$

where $L(\bar{x})$ and $U(\bar{x})$, are the lower limit (LL) and upper limit (UL) limit.

Since, the marginal posterior distributions are given by (13) and (14), then a 100 (1 - $\omega$) % credible interval for $a$ is $(L(\bar{x}), U(\bar{x}))$, which can be obtained by

$$
P \left[ a > L(\bar{x}) \right] = \int_{L(\bar{x})}^\infty \frac{1}{a} \int_0^1 k_1 \prod_{i=1}^{r} \left( \theta^{w_{i1}} - \theta^{w_{i2}} \right) \left[ \theta^{w_r} \right]^{n-r} \theta^{c-1}(1 - \theta)^{d-1} d\theta da = 1 - \frac{\omega}{2},
$$

(21)

and

$$
P \left[ a > U(\bar{x}) \right] = \int_{U(\bar{x})}^\infty \frac{1}{a} \int_0^1 k_1 \prod_{i=1}^{r} \left( \theta^{w_{i1}} - \theta^{w_{i2}} \right) \left[ \theta^{w_r} \right]^{n-r} \theta^{c-1}(1 - \theta)^{d-1} d\theta da = \frac{\omega}{2},
$$

(22)

Also, a 100 (1 - $\omega$) % credible interval for $\theta$ is $(L(\bar{x}), U(\bar{x}))$ can be derived by

$$
P \left[ \theta > L(\bar{x}) \right] = k_1 \int_{L(\bar{x})}^\theta \theta^{c-1}(1 - \theta)^{d-1} \int_0^\infty \frac{1}{a} \prod_{i=1}^{r} \left( \theta^{w_{i1}} - \theta^{w_{i2}} \right) \left[ \theta^{w_r} \right]^{n-r} da d\theta = 1 - \frac{\omega}{2},
$$

(23)
Similarly, hence, the posterior density function for $sf$ is given by

$$P[\theta > U(x) | x] = k_1 \int_{U(x)} \theta^{c-1} (1 - \theta)^{d-1} \int_0^z \left[ \prod_{i=1}^r \theta^{w_{t1i}} - \theta^{w_{t2i}} \right] \theta^{w_{t2i}} d\theta = \frac{a}{z}. \quad (24)$$

To obtain the credible intervals for the parameters, Equations (21)-(24) should be solved numerically.

### 2.2 Bayesian estimation for survival and hazard rate functions

The posterior distributions of $sf$, $\pi(s|z)$, and $hrf$, $\pi(q|x)$, are obtained as follows:

Considering $\pi(a, \theta|x)$ is the joint posterior distribution of $a$ and $\theta$, given by (12), and

$$ss \equiv S(x) = \theta^{a^{-1}} e^{a(x-1)}, \quad a = z,$$

then $\theta = \frac{x}{(exz-1)}$, and

$$\left| J \right| = \begin{vmatrix} \frac{\partial \theta}{\partial z} & \frac{\partial \theta}{\partial a} \\ \frac{\partial a}{\partial z} & \frac{\partial a}{\partial a} \end{vmatrix} = \frac{x}{(exz-1)^2} S (e^{xz-1})^{-1}. \quad (25)$$

The joint posterior distribution of $ss$ and $z$ is

$$\pi(ss, z | x) = k_1 \left[ \prod_{i=1}^r \left( \frac{z}{(exz-1)^{(c-1)}} \left( 1 - S (e^{xz-1})^{-1} \right) \right) \left[ S (e^{xz-1})^{-1} \right] \right]^{n-r} \times \left[ \prod_{i=1}^r \left( \frac{z}{(exz-1)^{(w_{t1i} - w_{t2i})}} - \frac{z}{(exz-1)^{(w_{t2i})}} \right) \right]^{d-1} \left[ \frac{z}{(exz-1)^{(w_{t2i})}} \right]^{n-r} \left[ \frac{z}{(exz-1)^{(c-1)}} \left( 1 - S (e^{xz-1})^{-1} \right) \right]^{d-1} \left[ \frac{z}{(exz-1)^{(w_{t2i})}} \right]^{n-r} \quad (26)$$

Where

$$w_{t1i} = z^{-1} (e^{xz_i} - 1), \quad w_{t2i} = z^{-1} (e^{xz_i} - 1) \quad \text{and} \quad w_{t1i} = z^{-1} (e^{xz_r} - 1).$$

Hence, the posterior density function for $sf$ is given by

$$\pi(ss | x) = k_1 \int_0^\infty \left[ \prod_{i=1}^r \left( \frac{z}{(exz-1)^{(c-1)}} \left( 1 - S (e^{xz-1})^{-1} \right) \right) \right]^{n-r} \times \left[ \prod_{i=1}^r \left( \frac{z}{(exz-1)^{(w_{t1i} - w_{t2i})}} - \frac{z}{(exz-1)^{(w_{t2i})}} \right) \right]^{d-1} \left[ \frac{z}{(exz-1)^{(w_{t2i})}} \right]^{n-r} \times \left[ \frac{z}{(exz-1)^{(c-1)}} \left( 1 - S (e^{xz-1})^{-1} \right) \right]^{d-1} \left[ \frac{z}{(exz-1)^{(w_{t2i})}} \right]^{n-r} d\theta, 0 < ss < 1. \quad (27)$$

Similarly,

Considering $q \equiv h(x) = \frac{\theta^{a^{-1}} e^{a(x-1)} - \theta}{\theta^{a^{-1}} e^{a(x-1)}}$

$$= 1 - \frac{\theta^{a^{-1}} e^{a(x-1)}}{\theta^{a^{-1}} e^{a(x-1)}} = 1 - \theta^{a^{-1}} e^{a(x-1)}, \quad (28)$$
where $h(x)$ is defined in (5).

Also, let $\theta = (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}$, $a = Q$.

Then

$$|J| = \left| \begin{array}{cc} \frac{\partial q}{\partial a} & \frac{\partial q}{\partial g} \\ \frac{\partial q}{\partial g} & \frac{\partial q}{\partial a} \end{array} \right| = \frac{q}{e^{\theta}(e^{\theta}-1)}(1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}^{-1}. \quad (30)$$

The joint posterior distribution of $q$ and $Q$ is given by

$$\pi(q, Q|x) = k_1 \left[ \prod_{i=1}^{r} \left( (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}w_{1i}^*- (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}w_{2i}^* \right) \right]$$

$$\times \left[ (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}w_{1r}^* \right]^{n-r} \frac{1}{Q} (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}(c-1)^{d-1}$$

$$\times \left( 1 - (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}} \right)^{d-1} \times \frac{q}{e^{\theta}(e^{\theta}-1)}(1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}^{-1},$$

$$0 < q < 1, Q > 0. \quad (31)$$

where

$$w_{1i}^* = Q^{-1}(e^{\theta-x_i} - 1), \quad w_{2i}^* = Q^{-1}(e^{\theta-x_{i+1}} - 1) \quad \text{and} \quad w_{1r}^* = Q^{-1}(e^{\theta-r} - 1).$$

Hence, the posterior density function for hrf is given by

$$\pi(q|x) = k_1 \int_0^\infty \frac{1}{Q} \left[ \prod_{i=1}^{r} \left( (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}w_{1i}^* - (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}w_{2i}^* \right) \right]$$

$$\times \left[ (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}w_{1r}^* \right]^{n-r} \frac{1}{Q} (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}(c-1)^{d-1}$$

$$\times \left( 1 - (1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}} \right)^{d-1} \times \frac{q}{e^{\theta}(e^{\theta}-1)}(1 - q)^{\frac{q}{e^{\theta}(e^{\theta}-1)}}^{-1}dq,$$

$$0 < q < 1, Q > 0. \quad (32)$$

### 2.2.1 Point estimation for the survival and hazard rate functions

The Bayesian point estimation of the sf and hrf under SE and LINEX loss functions are obtained.

#### I. Bayesian estimation under squared error loss function

Under SE loss function, the Bayes estimators of the sf and hrf are the means of their marginal posterior distributions in (27) and (32), respectively, as shown below

$$S_{(SE)}'(x) = E(s|X) = \int_0^1 ss \pi(s|x) dss$$

$$= k_1 \int_0^1 ss \int_0^\infty \left[ \prod_{i=1}^{r} \left( ss^{\frac{x}{e^{\theta}(e^{\theta}-1)}}w_{1i}^* - ss^{\frac{x}{e^{\theta}(e^{\theta}-1)}}w_{2i}^* \right) \right]^{n-r}$$

$$\times ss^{\frac{x}{e^{\theta}(e^{\theta}-1)}}(c-1)^{d-1} \left( 1 - ss^{\frac{x}{e^{\theta}(e^{\theta}-1)}} \right)^{d-1} \frac{x}{e^{\theta}(e^{\theta}-1)} ss^{\frac{x}{e^{\theta}(e^{\theta}-1)}}^{-1}dzdss,$$

$$0 < ss < 1, z > 0. \quad (33)$$
and

\[ h_{(SE)}(x) = E(q|x) \]

\[ = k_1 \int_0^1 q \int_0^1 \frac{1}{q} \left\{ \prod_{i=1}^n \left( (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)^{\alpha_i}} - (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)^{\alpha_i+1}} \right) \right\} \]

\[ \times \left( (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)^{\alpha_i}} \right)^{n-r} \left( 1 - q \right) (c-1) \]

\[ \times \left( 1 - (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)} \right) \]

\[ = \frac{Q}{e^{\theta_i (\epsilon - 1)}} (1 - q) \frac{Q}{e^{\theta_i (\epsilon - 1)}} - 1 \quad dQdq, \quad 0 < q < 1, \quad Q > 0. \] (34)

II. Bayesian estimation under linear exponential loss function

Under the LINEX loss function, the Bayes estimators for the sf and hrf are given, respectively, by

\[ S_{(LINEX)}(x) = \frac{-1}{q} \ln E\left( e^{-\alpha s} | x \right), \] (35)

where

\[ E\left( e^{-\alpha s} | x \right) = k_1 \int_0^1 e^{-\alpha s} \int_0^1 \frac{1}{q} \left\{ \prod_{i=1}^n \left( \frac{z}{ss} (z^\alpha (\epsilon - 1)^{\alpha_i}) - \frac{z}{ss} \right) \right\} \]

\[ \times \left( \frac{z}{e^{\theta_i (\epsilon - 1)}} \right)^{n-r} \left( 1 - \frac{z}{ss} \right) (c-1) \]

\[ \times \left( 1 - (1 - q) \frac{Q}{e^{\theta_i (\epsilon - 1)}} \right) \]

\[ = \frac{z}{ss} (z^\alpha (\epsilon - 1)^{\alpha_i}) - \frac{z}{ss} \quad dzds, \quad 0 < ss < 1, \quad z > 0, \] (36)

and

\[ h_{(LINEX)}(x) = \frac{-1}{q} \ln E\left( e^{-\beta h} | x \right), \] (37)

where

\[ E\left( e^{-\beta h} | x \right) = k_1 \int_0^1 e^{-\beta h} \int_0^1 \frac{1}{q} \left\{ \prod_{i=1}^n \left( (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)^{\alpha_i+1}} - (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)^{\alpha_i+2}} \right) \right\} \]

\[ \times \left( (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)^{\alpha_i}} \right)^{n-r} \left( 1 - q \right) (c-1) \]

\[ \times \left( 1 - (1 - q) \frac{Q}{\epsilon \theta_i (\epsilon - 1)} \right) \]

\[ = \frac{Q}{e^{\theta_i (\epsilon - 1)}} (1 - q) \frac{Q}{e^{\theta_i (\epsilon - 1)}} - 1 \quad dQdq, \quad 0 < q < 1, \quad Q > 0. \] (38)

To obtain the Bayes estimates of the sf and hrf, Equations (33)-(35), (37) and (38) can be solved numerically.

2.2.2 Credible intervals for the survival and hazard rate functions

A 100 (1- \(\alpha\)) % credible interval for \(S(x)\) is \((L(x), U(x))\),
where

\[ P[L(x) < S(x) < U(x)|x] = \int_{L(x)}^{U(x)} \pi(|ss|) ds = 1 - \omega, \]

(39)

where \( \pi(|ss|) \) is given by (27).

Then,

\[
P[S(x) > L(x)|x] = k_1 \int_{L(x)}^{x} \left( \prod_{i=1}^{n-1} \left( \frac{ss}{(e^{x_{i-1}}-1)} \right) \right) \left( 1 - \frac{ss}{(e^{x_{i-1}}-1)} \right)^{d-1} \left[ \frac{ss}{(e^{x_{i-1}}-1)} \right] \right] \right] dzs = 1 - \frac{\omega}{2} \]

(40)

and

\[
P[S(x) > U(x)|x] = k_1 \int_{L(x)}^{x} \left( \prod_{i=1}^{n-1} \left( \frac{ss}{(e^{x_{i-1}}-1)} \right) \right) \left( 1 - \frac{ss}{(e^{x_{i-1}}-1)} \right)^{d-1} \left[ \frac{ss}{(e^{x_{i-1}}-1)} \right] \right] \right] dzs = \frac{\omega}{2} \]

(41)

Also, a 100 (1 - \( \omega \)) \% credible interval for \( h(x) \) is \((L(x), U(x))\),

\[
P[h(x) > L(x)|x] = k_1 \int_{L(x)}^{x} \left[ \prod_{i=1}^{n-1} \left( 1 - q \right)^{\frac{ss}{(e^{x_{i-1}}-1)}} \left( 1 - q \right)^{\frac{ss}{(e^{x_{i-1}}-1)}} \right] \right] \right] dzs = 1 - \frac{\omega}{2} \]

(42)

and

\[
P[h(x) > U(x)|x] = k_1 \int_{L(x)}^{x} \left[ \prod_{i=1}^{n-1} \left( 1 - q \right)^{\frac{ss}{(e^{x_{i-1}}-1)}} \left( 1 - q \right)^{\frac{ss}{(e^{x_{i-1}}-1)}} \right] \right] \right] dzs = \frac{\omega}{2} \]

(43)

Lower and upper credible intervals for sf and hrf can be obtained by solving (40) - (43) numerically.

**3 Bayesian Prediction Based on Two-sample Prediction**

The prediction of the future observation on the basis of the past and present knowledge without any doubt is one of the most important problems in statistics. The order statistics play an important role in prediction
methods. In this section Bayesian two-sample prediction (point and interval) for a future observation of DGD based on Type II censored sample is considered.

Assuming that \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)} \) are the first \( r \) ordered life times in a random sample of \( n \) components (Type II censoring) whose failure times are identically distributed as DGD \((a, \theta)\), informative sample, and \( Y_{(1)}, Y_{(2)}, \ldots, Y_{(m)} \) is a second independent random sample of size \( m \) of future observables from the same distribution; the future sample. Our aim is to predict the \( s^{th} \) order statistic in the future sample based on the informative sample.

For the future sample of size \( m \), let \( Y_{(s)} \) denotes the \( s^{th} \) order statistic, \( 1 \leq s \leq m \).

The conditional pmf of \( Y_{(s)} \) can be obtained as follows:

\[
P^*\left(Y_{s,m} = y_s | \varphi\right) = \frac{m!}{(s-1)! (m-s)!} \int_{(y_{s-1})}^{(y_{s})} v^{s-1} (1-v)^{m-s} dv
\]

\[
= \frac{m!}{(s-1)! (m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j}
\times \left[ (F(y_j))^s - [F(y_{j-1})]^s \right]
\]

\[
= \frac{m!}{(s-1)! (m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j}
\times \left[ 1 - \theta^{a^{-1} (e^{a(y_j+1)}-1)} \right]^s
\times \left[ 1 - \theta^{a^{-1} (e^{a(y_j)}-1)} \right]^s,
\]

\( s = 1,2, \ldots, m. \) \hfill (45)

where \( \varphi = (a, \theta) \) , \( a \) and \( \theta \) are independent, hence the Bayesian predictive mass function (BPMF) of \( Y_{(s)} \) given \( \varphi \) is given by

\[
P(\varphi | y_{(s)}) = \int P^* \left( y_{(s)} | \varphi \right) \pi \left( \varphi | x \right) d\varphi, \quad y_{(s)} = 0,1, \ldots, s = 1,2,3, \ldots, m. \] \hfill (46)

\[P^* \left( y_{(s)} | \varphi \right) \] and \( \pi \left( \varphi | x \right) \) are, respectively, the conditional pmf of the \( s^{th} \) future order lifetime and the joint posterior distribution of \( \varphi \), given by (12).

Therefore, the BPMF of \( Y_{(s)} \) given \( x \) is given by substituting (45) and (12) in (46), then the BPMF of the future order statistic \( Y_{(s)} \) \( s = 1,2, \ldots, m \) is given by

\[
P(\varphi | y_{(s)}) = \int_0^\infty \int_0^1 \frac{m!}{(s-1)! (m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j}
\times \left[ 1 - \theta^{a^{-1} (e^{a(y_{s+1})-1})} \right]^s
\times \left[ 1 - \theta^{a^{-1} (e^{a(y_s)}-1)} \right]^s
\times k_1 \prod_{i=1}^r \left( \theta^{w_{i1}} - \theta^{w_{id}} \right) \left( \theta^{w_r} \right)^{n-r} \frac{1}{a} \theta^{\varepsilon-1} (1-\theta)^{d-1} d\theta da,
\]

\( y_{(s)} = 0,1,2, \ldots; \ a > 0, \ 0 < \theta < 1, \ s = 1,2, \ldots, m. \) \hfill (47)

3.1 Point prediction

Bayesian Point prediction is considered under two types of loss functions; the SE loss function, as a symmetric loss function and the LINEX loss function, as an asymmetric loss function.

I. Bayesian prediction under squared error loss function

The Bayes predictor (BP) for the future observation \( Y_{(s)} \), under SE loss function can be derived using (45) as follows:
$$\hat{\gamma}_{(s)}(\text{SE}) = E\left(y_{(s)}\mid \mathbf{x}\right)$$

$$= \sum_{y_{(s)}=0}^{\infty} y_{(s)} \left( \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \sum_{j=0}^{m-1} \left( \begin{array}{c} m-s \ 0 \end{array} \right) \left( -1 \right)^{j} \frac{x^{j}}{s+j} \right\} \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(s)}+1})^{-1}} \right] \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(s)}+1})^{-1}} \right]^{s+j} \times k_{1} \left( \prod_{i=1}^{s} (\theta_{w_{i1}} - \theta_{w_{i2}}) \right) \left( \theta_{w_{s}} \right)^{n-r} \frac{1}{a} \theta^{-c-1} (1-\theta)^{d-1} d\theta d\alpha \right) \right)$$

$$y_{(s)} = 0, 1, 2, \ldots, \text{and } s=1, 2, \ldots, m. \quad (48)$$

II. Bayesian prediction under linear-exponential loss function

The BP for the future observation $Y_{(s)}$, under LINEX loss function can be obtained using (47) as given below:

$$\hat{\gamma}_{(s)}(\text{LINX}) = \frac{-1}{\theta} \ln E\left(y_{(s)}\mid \mathbf{x}\right),$$

where

$$E\left(y_{(s)}\mid \mathbf{x}\right) = \sum_{y_{(s)}=0}^{\infty} \exp\left(-\theta y_{(s)}\right) \times \left( \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \sum_{j=0}^{m-1} \left( \begin{array}{c} m-s \ 0 \end{array} \right) \left( -1 \right)^{j} \frac{x^{j}}{s+j} \right\} \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(s)}+1})^{-1}} \right] \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(s)}+1})^{-1}} \right]^{s+j} \times k_{1} \left( \prod_{i=1}^{s} (\theta_{w_{i1}} - \theta_{w_{i2}}) \right) \left( \theta_{w_{s}} \right)^{n-r} \frac{1}{a} \theta^{-c-1} (1-\theta)^{d-1} d\theta d\alpha \right) \right)$$

$$y_{(s)} = 0, 1, 2, \ldots, \text{and } s=1, 2, \ldots, m. \quad (49)$$

Special cases:

1. If $s = 1$, in (48) and (49), one can predict the minimum observable, $Y_{(1)}$, which represents the first failure time in the future sample of size $m$, given that $(n-r)$ components had already failed in the informative sample of size $n$. Hence, the BPE of the future $Y_{(1)}$ under SE and LINEX loss functions, respectively, are given by:

$$\hat{\gamma}_{(1)}(\text{SE}) = \sum_{y_{(1)}=0}^{\infty} y_{(1)} \times \left( \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \sum_{j=0}^{m-1} \left( \begin{array}{c} m-1 \ 0 \end{array} \right) \left( -1 \right)^{j} \frac{x^{j}}{1+j} \right\} \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(1)}+1})^{-1}} \right] \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(1)}+1})^{-1}} \right]^{1+j} \times k_{1} \left( \prod_{i=1}^{s} (\theta_{w_{i1}} - \theta_{w_{i2}}) \right) \left( \theta_{w_{s}} \right)^{n-r} \frac{1}{a} \theta^{-c-1} (1-\theta)^{d-1} d\theta d\alpha \right) \right)$$

$$y_{(1)} = 0, 1, 2, \ldots, \text{and } s=1, 2, \ldots, m. \quad (50)$$

and

$$\hat{\gamma}_{(1)}(\text{LINX}) = \frac{-1}{\theta} \left\{ \ln \sum_{y_{(1)}=0}^{\infty} \exp\left(-\theta y_{(1)}\right) \times \left( \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \sum_{j=0}^{m-1} \left( \begin{array}{c} m-1 \ 0 \end{array} \right) \left( -1 \right)^{j} \frac{x^{j}}{1+j} \right\} \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(1)}+1})^{-1}} \right] \times \left[ 1 - \theta^{-a^{-1}(e^{y_{(1)}+1})^{-1}} \right]^{1+j} \times k_{1} \left( \prod_{i=1}^{s} (\theta_{w_{i1}} - \theta_{w_{i2}}) \right) \left( \theta_{w_{s}} \right)^{n-r} \frac{1}{a} \theta^{-c-1} (1-\theta)^{d-1} d\theta d\alpha \right) \right) \right)$$

$$y_{(1)} = 0, 1, 2, \ldots. \quad (51)$$
2. If \( s = m \) in (48) and (49), one can predict the maximum observable, \( Y_{(m)} \), which represents the largest failure time in the future sample of size \( m \), given that \( r \) components had already failed in the informative sample of size \( n \). Hence, the BPE of the future \( Y_{(m)} \) under SE and LINEX loss functions, respectively, are as follows:

\[
\hat{y}_{(m)(SE)} = \sum_{y_{(m)=0}}^{\infty} y_{(m)} \\
\times \int_{0}^{1} \int_{0}^{1} \left\{ (-1)^{j} \frac{1}{m+j} \left[ 1 - \theta^{-1} \left( \frac{a(y_{(m)}+1)}{a} \right) \right]^{m+j} - \left[ 1 - \theta^{-1} \left( \frac{a(y_{(m)}-1)}{a} \right) \right]^{m+j} \right\} \\
\times k_{1} \left( \prod_{i=1}^{\infty} (\theta^{w_{1i}} - \theta^{w_{2i}}) \right)[\theta^{w_{1}}]^{n-r} \frac{1}{a} \theta^{c^{-1}}(1 - \theta)^{-1} d\theta da, y_{(m)} = 0,1,2, \ldots.
\]

and

\[
\hat{y}_{(m)(LINX)} = - \frac{1}{\theta} \ln \sum_{y_{(m)=0}}^{\infty} \exp(-\theta y_{(m)}) \\
\times \int_{0}^{1} \int_{0}^{1} \left\{ (-1)^{j} \frac{1}{m+j} \left[ 1 - \theta^{-1} \left( \frac{a(y_{(m)}+1)}{a} \right) \right]^{m+j} - \left[ 1 - \theta^{-1} \left( \frac{a(y_{(m)}-1)}{a} \right) \right]^{m+j} \right\} \\
\times k_{1} \left( \prod_{i=1}^{\infty} (\theta^{w_{1i}} - \theta^{w_{2i}}) \right)[\theta^{w_{1}}]^{n-r} \frac{1}{a} \theta^{c^{-1}}(1 - \theta)^{-1} d\theta da, y_{(m)} = 0,1,2, \ldots.
\]

3.2 Bayesian prediction bounds

A 100(1 - \( \omega \)) \% Bayesian predictive bounds (BPB) for the future observation \( Y_{(s)} \), such that

\[
P(\mathcal{L}(x) < Y_{(s)} < U(x) | x) = 1 - \omega, \text{ are as follows:}
\]

\[
P(Y_{(s)} \geq \mathcal{L}(x) | x) = \\
\sum_{y_{(s)=l(x)}}^{\infty} \sum_{j=0}^{m} \frac{m!}{(s-1)! (m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^{j} \frac{1}{s+j} \\
\times \left[ 1 - \theta^{-1} \left( \frac{a(y_{(s)}+1)}{a} \right) \right]^{s+j} - \left[ 1 - \theta^{-1} \left( \frac{a(y_{(s)}-1)}{a} \right) \right]^{s+j} \\
\times k_{1} \left( \prod_{i=1}^{\infty} (\theta^{w_{1i}} - \theta^{w_{2i}}) \right)[\theta^{w_{1}}]^{n-r} \frac{1}{a} \theta^{c^{-1}}(1 - \theta)^{-1} d\theta da = 1 - \frac{\omega}{\pi}
\]

and

\[
P(Y_{(s)} \geq U(x) | x) = \\
\sum_{y_{(s)=u(x)}}^{\infty} \sum_{j=0}^{m} \frac{m!}{(s-1)! (m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^{j} \frac{1}{s+j} \\
\times \left[ 1 - \theta^{-1} \left( \frac{a(y_{(s)}+1)}{a} \right) \right]^{s+j} - \left[ 1 - \theta^{-1} \left( \frac{a(y_{(s)}-1)}{a} \right) \right]^{s+j} \\
\times k_{1} \left( \prod_{i=1}^{\infty} (\theta^{w_{1i}} - \theta^{w_{2i}}) \right)[\theta^{w_{1}}]^{n-r} \frac{1}{a} \theta^{c^{-1}}(1 - \theta)^{-1} d\theta da = \frac{\omega}{\pi}
\]

4 Numerical Illustration

Our aim is to investigate the precision of the theoretical results of Bayesian estimation and prediction on the basis of simulated and real data.
4.1 Simulation study

A simulation study is conducted to illustrate the performance of the presented Bayes estimates on the basis of generated data from the DGD. Bayes averages of the parameters, sf, hrf and ahrf based on Type II censoring are computed. Moreover, credible intervals of the parameters, sf, hrf and ahrf are calculated. Also, the Bayes predictors (point and interval) for a future observation from the DGD based on Type II censored data are computed for the two-sample prediction case. All the computations are performed using R programming language.

- Tables 1 and 2 present the Bayes averages, relative absolute biases (RABs) and relative errors (REs) of the Bayes estimates and 95% credible intervals of the parameters $a$ and $\theta$ based on Type II censoring under SE and LINEX loss functions $Nu = 10000$, mission time ($t_0$) = 1 and 10, $a = 20, \theta = 0.99, c = 2$ and $d = 3$.
- Table 3 displays the Bayes averages, RABs, REs of the Bayes estimates and 95% credible intervals of the sf, hrf and ahrf based on Type II censoring under SE and LINEX loss functions ($NR = 10000, t_0 = 1, a = 0.016, \theta = 0.99, c = 2$ and $d = 3$).
- Table 4 presents the Bayes averages, RABs, REs of the Bayes estimates and 95% credible intervals of the parameters, sf, hrf and ahrf based on Type II censoring under two-sample prediction case. All the computations are performed using R programming language.
- Tables 5 and 6 present the Bayes averages, RABs, REs of the Bayes estimates and 95% credible intervals of the survival, hazard and alternative hazard rate functions based on Type II censoring for real data ($NR = 10000, c = 2$ and $d = 3$).
- Table 7 displays the Bayes averages, RABs, REs of the Bayes estimates and 95% credible intervals of the survival, hazard and alternative hazard rate functions based on Type II censoring for real data ($NR = 10000, n = 100, r = 80\%, m = 51, t_0 = 10, a = 20, \theta = 0.99, c = 2$ and $d = 3$).

4.2 Applications

This subsection aims to demonstrate how the proposed methods can be used in practice. Two real lifetime data sets are analyzed to illustrate the theoretical results.

- The first application is a real data set obtained from Srivastava [21]. It represents the number of revolution before failure of each of 23 ball bearing in the life tests and it is as follows:
  17.80, 28.92, 33.41, 52.42, 12.45, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4
- The second application is given by Srivastava [21]. The data are 100 observations on breaking stress of carbon fibers and are as follows:
  3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

To check the validity of the fitted model, Kolmogorov-Smirnov goodness of fit test is performed for each data set. The p values are given, respectively, 0.6478 and 0.281. The p value given in each case showed that the model fits the data very well.

- Table 5 displays the Bayes Averages, RABs, REs of the Bayes estimates and 95% credible intervals of the parameters $a$ and $\theta$ based on Type II censoring for real data ($NR = 10000, c = 2$ and $d = 3$).
- Table 6 presents the Bayes Averages, RABs, REs of the Bayes estimates and 95% credible intervals of the survival, hazard and alternative hazard rate functions based on Type II censoring for real data ($NR = 10000, c = 2$ and $d = 3$).
- Table 8 displays the Bayes predictive estimates and bounds of the future observation based on Type II censoring under two-sample prediction; using SE and LINEX loss functions for real data,
respectively. ( \( NR = 10000, n = 100, r = 80\%, m = 51, t_0 = 10, a = 20, \theta = 0.99, c = 2 \) and \( d = 3 \)).

### 4.3 Concluding remarks

- Regarding the RABs and REs for the Bayes averages, sf, hrf and ahrf under LINEX loss function seems to perform better than the results under SE loss function.
- It is observed, from Table 4 that the estimated value of the sf decreases when the time increases. While the estimated value of the hrf and ahrf increases as the time increases.
- As expected, it is observed that better estimates are obtained when the sample size increases and level of censoring decreases, which is obvious from comparing the RABs and REs of the estimates. This confirms that more provided information by the sample increases the accuracy of the estimates.
- The two-sided 95\% credible intervals distribution become narrower as the sample size increases.
- Regarding the results in Tables 7 and 8, both Bayes predictive estimates and credible intervals of the future observations; for simulated and real data, under SE and LINEX loss functions are very close. Also, the lengths of the credible intervals under LINEX loss function are shorter than the lengths under SE loss function.

### Competing Interests

Authors have declared that no competing interests exist.

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### Appendix

Table 1. Averages, relative absolute biases, relative errors of the Bayes estimates and 95% credible intervals of the parameters $\alpha$ and $\theta$ based on Type II censoring

(NR = 10000, $t_0 = 1, \alpha = 0.016, \theta = 0.9, c = 2$ and $d = 3$)

| N  | r   | Par | Average | RAB | RE | UL | LL | Length | Average | RAB | RE | UL | LL | Length |
|----|-----|-----|---------|-----|----|----|----|--------|---------|-----|----|----|----|--------|
| 30 | 60% | $\alpha$ | 0.0188  | 0.1721 | 3.7154E-10 | 0.0212 | 0.0156 | 0.0056 | 0.0140 | 0.1227 | 1.4994E-09 | 0.0157 | 0.0115 | 0.0042 |
| 18 | $\beta$ | 0.9932 | 0.0032 | 6.4554E-10 | 0.9950 | 0.9900 | 0.0050 | 0.9931 | 0.0031 | 5.4849E-09 | 0.9944 | 0.9899 | 0.0045 |
| 80% | $\alpha$ | 0.0136 | 0.1478 | 2.8361E-04 | 0.0160 | 0.0104 | 0.0056 | 0.0144 | 0.1090 | 1.3160E-04 | 0.0162 | 0.0121 | 0.0041 |
| 24 | $\beta$ | 0.9972 | 0.0027 | 4.2419E-02 | 0.9896 | 0.9849 | 0.0047 | 0.9923 | 0.0023 | 3.5398E-02 | 0.9945 | 0.9905 | 0.0040 |
| 100% | $\alpha$ | 0.0179 | 0.1195 | 2.0337E-02 | 0.0207 | 0.0154 | 0.0053 | 0.0143 | 0.1039 | 1.1215E-02 | 0.0150 | 0.0110 | 0.0040 |
| 30 | $\beta$ | 0.9880 | 0.0021 | 2.5366E-02 | 0.9899 | 0.9860 | 0.0039 | 0.9919 | 0.0019 | 2.2128E-02 | 0.9929 | 0.9892 | 0.0035 |
| 60 | 60% | $\alpha$ | 0.0137 | 0.1434 | 2.2666E-02 | 0.0158 | 0.0116 | 0.0042 | 0.0139 | 0.1500 | 1.4516E-02 | 0.0148 | 0.0108 | 0.0040 |
| 36 | $\beta$ | 0.9932 | 0.0033 | 5.8651E-02 | 0.9943 | 0.9908 | 0.0043 | 0.9876 | 0.0024 | 3.5480E-02 | 0.9892 | 0.9852 | 0.0040 |
| 80% | $\alpha$ | 0.0179 | 0.1216 | 2.0220E-02 | 0.0206 | 0.0166 | 0.0040 | 0.0143 | 0.1062 | 1.1186E-02 | 0.0159 | 0.0120 | 0.0039 |
| 48 | $\beta$ | 0.9925 | 0.0025 | 4.2014E-02 | 0.9940 | 0.9898 | 0.0042 | 0.9922 | 0.0023 | 2.9185E-02 | 0.9935 | 0.9900 | 0.0035 |
| 100% | $\alpha$ | 0.0176 | 0.1022 | 1.0786E-02 | 0.0189 | 0.0150 | 0.0039 | 0.0174 | 0.0854 | 0.1052E-02 | 0.0188 | 0.0150 | 0.0038 |
| 60 | 60% | $\beta$ | 0.9920 | 0.0020 | 2.3787E-02 | 0.9991 | 0.9899 | 0.0038 | 0.9885 | 0.0016 | 1.6868E-02 | 0.9900 | 0.9867 | 0.0034 |
| 120 | 60% | $\alpha$ | 0.0180 | 0.1249 | 1.4516E-02 | 0.0191 | 0.0160 | 0.0031 | 0.0142 | 0.1147 | 1.3974E-02 | 0.0161 | 0.0120 | 0.0026 |
| 72 | $\beta$ | 0.9932 | 0.0026 | 4.5775E-02 | 0.9936 | 0.9895 | 0.0041 | 0.9921 | 0.0021 | 3.2726E-02 | 0.9931 | 0.9897 | 0.0034 |
| 80% | $\alpha$ | 0.0172 | 0.0761 | 5.6996E-02 | 0.0178 | 0.0156 | 0.0022 | 0.0163 | 0.0037 | 3.2854E-02 | 0.0177 | 0.0155 | 0.0022 |
| 96 | $\beta$ | 0.9923 | 0.0023 | 3.1958E-02 | 0.9937 | 0.9897 | 0.0040 | 0.9880 | 0.0021 | 2.4680E-02 | 0.9893 | 0.9860 | 0.0032 |
| 100% | $\alpha$ | 0.0151 | 0.0535 | 3.2347E-02 | 0.0161 | 0.0141 | 0.0020 | 0.0164 | 0.0244 | 1.2950E-02 | 0.0173 | 0.0155 | 0.0019 |
| 120 | $\beta$ | 0.9883 | 0.0017 | 2.1343E-02 | 0.9904 | 0.9867 | 0.0037 | 0.9903 | 0.0003 | 1.3380E-02 | 0.9920 | 0.9895 | 0.0025 |
Table 2. Averages, relative absolute biases, relative errors of the Bayes estimates and 95% credible intervals of the parameters $\alpha$ and $\theta$ based on Type II censoring

$(NR = 10000, t_a = 10, \alpha = 2 \ 0\theta = 0.9 \ c = 2 \ and \ d = 3)$

| n | r | Par | Average | RAB | RE | UL | LL | Length  | Average | RAB | RE | UL | LL | Length  | Length |
|---|---|-----|---------|------|----|----|----|--------|---------|------|----|----|----|----|--------|--------|
| 30 | 60% | $\alpha$ | 19.9971 | 1.48 x 10^{-4} | 2.49 x 10^{-7} | 19.9996 | 19.9956 | 0.0030 | 19.9978 | 1.12 x 10^{-4} | 1.66 x 10^{-7} | 0.0015 | 1.86 x 10^{-7} | 20.0025 | 19.9995 | 0.0030 |
| 18 | $\theta$ | 0.9878 | 0.022 | 2.81 x 10^{-4} | 0.9896 | 0.9865 | 0.003 | 0.9885 | 0.0015 | 0.9899 | 0.9869 | 0.003 |
| 80% | $\alpha$ | 19.9981 | 9.44 x 10^{-4} | 9.93 x 10^{-8} | 19.9991 | 19.9963 | 0.0029 | 20.0015 | 7.71 x 10^{-7} | 7.13 x 10^{-8} | 0.0024 | 7.52 x 10^{-8} | 7.13 x 10^{-7} | 0.0005 | 0.9996 | 0.0028 |
| 24 | $\theta$ | 0.9884 | 0.016 | 1.98 x 10^{-4} | 0.9898 | 0.9869 | 0.0029 | 0.9893 | 0.0010 | 0.9899 | 0.9869 | 0.0023 |
| 100% | $\alpha$ | 20.0013 | 6.62 x 10^{-4} | 6.19 x 10^{-8} | 20.0029 | 19.9999 | 0.0029 | 20.0010 | 5.16 x 10^{-7} | 5.31 x 10^{-8} | 0.0019 | 5.68 x 10^{-8} | 5.45 x 10^{-7} | 0.0011 | 0.9993 | 0.0026 |
| 30 | $\theta$ | 0.9892 | 0.0008 | 9.25 x 10^{-7} | 0.9909 | 0.9873 | 0.0036 | 0.9899 | 0.0011 | 0.9891 | 0.9888 | 0.0023 |
| 60% | $\alpha$ | 20.0019 | 9.39 x 10^{-4} | 1.08 x 10^{-7} | 20.0032 | 20.0000 | 0.0032 | 19.9993 | 3.42 x 10^{-7} | 2.17 x 10^{-8} | 0.0010 | 6.97 x 10^{-9} | 19.9994 | 0.0026 |
| 36 | $\theta$ | 0.9886 | 0.0014 | 1.05 x 10^{-6} | 0.9895 | 0.9874 | 0.0021 | 0.9910 | 0.0010 | 1.40 x 10^{-7} | 6.97 x 10^{-8} | 0.0026 |
| 60% | $\alpha$ | 20.0013 | 6.70 x 10^{-4} | 7.14 x 10^{-8} | 20.0023 | 19.9998 | 0.0025 | 19.9993 | 3.26 x 10^{-7} | 1.93 x 10^{-8} | 0.0002 | 1.89 x 10^{-8} | 2.07 x 10^{-7} | 0.0009 | 0.9885 | 0.0024 |
| 48 | $\theta$ | 0.9909 | 0.0009 | 5.22 x 10^{-7} | 0.9918 | 0.9897 | 0.0020 | 0.9898 | 0.0019 | 0.9890 | 0.9885 | 0.0024 |
| 100% | $\alpha$ | 20.0004 | 1.83 x 10^{-4} | 1.36 x 10^{-8} | 20.0014 | 19.9992 | 0.0022 | 20.0002 | 1.13 x 10^{-7} | 6.83 x 10^{-8} | 0.0009 | 5.65 x 10^{-8} | 1.33 x 10^{-7} | 0.0005 | 0.9992 | 0.0017 |
| 60% | $\theta$ | 0.9904 | 0.0004 | 1.79 x 10^{-7} | 0.9910 | 0.9893 | 0.0017 | 0.9899 | 0.0017 | 0.9895 | 0.9890 | 0.0015 |
| 120 | 80% | $\alpha$ | 20.0010 | 5.02 x 10^{-4} | 3.28 x 10^{-8} | 20.0018 | 19.9997 | 0.0021 | 19.9994 | 2.94 x 10^{-7} | 2.32 x 10^{-8} | 0.0005 | 1.40 x 10^{-8} | 2.64 x 10^{-7} | 0.0015 | 19.9985 | 0.0020 |
| 72 | $\theta$ | 0.9900 | 0.0010 | 5.92 x 10^{-7} | 0.9896 | 0.9873 | 0.0023 | 0.9901 | 0.0015 | 0.9903 | 0.9900 | 0.0022 |
| 96 | $\alpha$ | 20.0009 | 4.73 x 10^{-4} | 2.74 x 10^{-8} | 20.0017 | 19.9988 | 0.0018 | 19.9998 | 9.69 x 10^{-8} | 1.44 x 10^{-8} | 0.0015 | 1.03 x 10^{-7} | 1.77 x 10^{-8} | 0.0011 | 19.9995 | 0.0020 |
| 80% | $\theta$ | 0.9894 | 0.0010 | 2.20 x 10^{-7} | 0.9880 | 0.9855 | 0.0015 | 0.9903 | 0.0015 | 0.9903 | 0.9900 | 0.0021 |
| 120 | $\alpha$ | 20.0002 | 9.48 x 10^{-6} | 7.94 x 10^{-8} | 20.0010 | 19.9993 | 0.0017 | 20.0000 | 1.66 x 10^{-6} | 4.98 x 10^{-8} | 0.0007 | 2.24 x 10^{-8} | 9.18 x 10^{-7} | 0.0004 | 0.9904 | 0.0013 |
| 100% | $\theta$ | 0.9896 | 0.0003 | 1.42 x 10^{-7} | 0.9902 | 0.9889 | 0.0014 | 0.9900 | 0.0014 | 0.9900 | 0.9900 | 0.0013 |
Table 3. Bayes averages, relative absolute biases, relative errors of the Bayes estimates and 95% credible intervals of the $S(t_0)$, $h(t_0)$ and $h_1(t_0)$ based on Type II censoring
($NR = 10000, t_0 = 1, \alpha = 0.016, \theta = 0.9, c = 2$ and $d = 3$)

| n   | r  | Par     | SE     | Average | RAB    | RE   | UL   | LL   | Length | Average | RAB    | RE   | UL   | LL   | Length |
|-----|----|---------|--------|---------|--------|------|------|------|--------|---------|--------|------|------|------|--------|---------|
| 30  | 80%| $S(t_0)$ | 0.9879 | 0.0092 | 3.5197x10^{-4} | 0.9905 | 0.0010 | 0.9910 | 0.0107 | 7.4219x10^{-3} | 0.9919 | 0.9892 | 0.0027 |
| 100%| 30 | $h_0(t_0)$ | 0.9893 | 0.0100 | 1.0800x10^{-3} | 0.9906 | 0.0006 | 0.9892 | 0.0010 | 9.1223x10^{-6} | 0.9891 | 0.9891 | 0.0017 |
| 60  | 80%| $h_0(t_0)$ | 0.9881 | 0.0089 | 6.4644x10^{-3} | 0.9901 | 0.0009 | 0.9890 | 0.0009 | 0.9894 | 0.0106 | 0.0080 | 0.0026 |
| 100%| 60 | $h_0(t_0)$ | 0.9888 | 0.0011 | 8.8012x10^{-3} | 0.9898 | 0.0002 | 0.9894 | 0.0006 | 4.3950x10^{-7} | 0.9902 | 0.9877 | 0.0023 |
| 120 | 80%| $h_0(t_0)$ | 0.9893 | 0.0000 | 1.0800x10^{-3} | 0.9906 | 0.0002 | 0.9898 | 0.0008 | 3.5739x10^{-6} | 0.9890 | 0.9882 | 0.0026 |
| 100%| 120| $h_0(t_0)$ | 0.9893 | 0.0010 | 1.0800x10^{-3} | 0.9906 | 0.0002 | 0.9898 | 0.0007 | 9.1223x10^{-6} | 0.9891 | 0.9891 | 0.0017 |

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Table 4. Bayes averages of the survival, hazard and alternative hazard rate functions at different times

| n  | r   | t₀  | S(t₀)  | SE  | h(t₀)  | SE  | h₁(t₀)  | SE  | S(t₀)  | SE  | h(t₀)  | SE  | h₁(t₀)  | SE  |
|----|-----|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|
| 30 | 80% | 1   | 0.9890 | 0.0107 | 0.0095 | 0.9892 | 0.0107 | 0.0096 |
|    | 24  | 5   | 0.9400 | 0.0130 | 0.0114 | 0.9460 | 0.0114 | 0.0108 |
|    | 100%| 1   | 0.9919 | 0.0109 | 0.0112 | 0.9908 | 0.0124 | 0.0103 |
|    | 60  | 5   | 0.9491 | 0.0111 | 0.0111 | 0.9502 | 0.0108 | 0.0113 |
|    | 80% | 1   | 0.9902 | 0.0115 | 0.0103 | 0.9881 | 0.0110 | 0.0105 |
|    | 48  | 5   | 0.9488 | 0.0123 | 0.0133 | 0.94870| 0.0115 | 0.0117 |
|    | 100%| 1   | 0.9879 | 0.0102 | 0.0102 | 0.9881 | 0.0080 | 0.0100 |
|    | 60  | 5   | 0.9486 | 0.0107 | 0.0111 | 0.9497 | 0.0083 | 0.0112 |
|    | 120 | 80% | 1   | 0.9913 | 0.0102 | 0.0115 | 0.9901 | 0.0081 | 0.0121 |
|    | 96  | 5   | 0.9495 | 0.0116 | 0.0120 | 0.9484 | 0.0111 | 0.0133 |
|    | 100%| 1   | 0.9913 | 0.0113 | 0.0103 | 0.9900 | 0.0104 | 0.0091 |
|    | 120 | 5   | 0.9513 | 0.0114 | 0.0107 | 0.9510 | 0.0114 | 0.0100 |

Table 5. Bayes Averages, relative absolute biases, relative errors of the Bayes estimates and 95% credible intervals of the parameters α and θ based on Type II censoring for real data (NR = 10000, c = 2 and d = 3)

| Real Data | n  | r   | Par | Average | RAB | SE  | UL | LL | Length | Average | RAB | SE  | UL | LL | Length |
|-----------|----|-----|-----|---------|-----|-----|----|----|--------|---------|-----|-----|----|----|--------|
| Application I | 23 | 60% | a   | 0.0178 | 0.00907| 9.2252×10⁻⁷| 0.0190 | 0.0157 | 0.0033 | 0.0176 | 0.0078 | 6.7135×10⁻⁷| 0.0187 | 0.0157 | 0.0030 |
|            | 14 |     | θ   | 0.9868 | 0.0014 | 1.4385×10⁻⁵| 0.9903 | 0.9871 | 0.0032 | 0.9907 | 0.0007 | 5.3227×10⁻⁷| 0.9916 | 0.9890 | 0.0026 |
|            | 80%| a   | 0.0150 | 0.00839| 7.7007×10⁻⁷| 0.0160 | 0.0133 | 0.0027 | 0.0164 | 0.0054 | 9.1816×10⁻⁷| 0.0173 | 0.0154 | 0.0019 |
|            | 18 |     | θ   | 0.9896 | 0.0004 | 2.0661×10⁻⁶| 0.9905 | 0.9886 | 0.0019 | 0.9896 | 0.0003 | 1.5516×10⁻⁷| 0.9903 | 0.9886 | 0.0016 |
|            | 100%| a  | 0.0166 | 0.00166| 6.6437×10⁻⁶| 0.0174 | 0.0156 | 0.0018 | 0.0163 | 0.0012 | 6.2985×10⁻⁷| 0.0169 | 0.0152 | 0.0017 |
|            | 23 |     | θ   | 0.9901 | 0.0002 | 1.5491×10⁻⁷| 0.9909 | 0.9891 | 0.0018 | 0.9902 | 0.0002 | 9.1842×10⁻⁸| 0.9909 | 0.9894 | 0.0015 |
| Application II | 100 | 60% | a   | 0.7855 | 0.0032 | 4.6276×10⁻⁴| 0.7897 | 0.7860 | 0.0037 | 0.7897 | 0.0018 | 1.6245×10⁻⁴| 0.7898 | 0.7882 | 0.0026 |
|            | 60 |     | θ   | 0.9195 | 0.0069 | 2.7283×10⁻⁷| 0.9216 | 0.9176 | 0.0039 | 0.9214 | 0.0049 | 1.1392×10⁻⁷| 0.9225 | 0.9199 | 0.0026 |
|            | 80%| a   | 0.7891 | 0.0026 | 2.9282×10⁻⁶| 0.7902 | 0.7877 | 0.0025 | 0.7903 | 0.0010 | 6.5970×10⁻⁷| 0.7914 | 0.7890 | 0.0023 |
|            | 80 |     | θ   | 0.9212 | 0.0052 | 9.224×10⁻⁷| 0.9249 | 0.9217 | 0.0027 | 0.9222 | 0.0041 | 8.5947×10⁻⁸| 0.9231 | 0.9206 | 0.0025 |
|            | 100%| a  | 0.7901 | 0.0012 | 1.2709×10⁻⁷| 0.7908 | 0.7890 | 0.0016 | 0.7906 | 0.0006 | 2.8990×10⁻⁸| 0.7913 | 0.7896 | 0.0017 |
|            | 100 |     | θ   | 0.9226 | 0.0036 | 6.9554×10⁻⁶| 0.9233 | 0.9208 | 0.0025 | 0.9201 | 0.0033 | 4.8719×10⁻⁸| 0.9211 | 0.9187 | 0.0024 |
| Real Data | n  | r  | Par | SEL | LINEX (v = 0.1) |
|----------|----|----|-----|-----|----------------|
|          |    |    |     | Average | RAB | RE | UL | LL | Length | Average | RAB | RE | UL | LL | Length |
| I        | 23 | 60%| $S(t_0)$ | 0.8190 | 0.0027 | 3.758E-10 | 0.8211 | 0.8171 | 0.0040 | 0.8219 | 0.0008 | 3.3232E-10 | 0.8224 | 0.8211 | 0.0012 |
|          | 14 |    | $h(t_0)$ | 0.0121 | 0.0883 | 6.5555E-10 | 0.0131 | 0.0112 | 0.0019 | 0.0133 | 0.0021 | 2.3107E-10 | 0.0136 | 0.0125 | 0.0011 |
|          | 18 |    | $h(t_0)$ | 0.0145 | 0.0965 | 7.5851E-10 | 0.0156 | 0.0132 | 0.0024 | 0.0134 | 0.0111 | 2.3588E-10 | 0.0137 | 0.0127 | 0.0009 |
|          | 23 |    | $h(t_0)$ | 0.8227 | 0.0017 | 2.2370E-10 | 0.8248 | 0.8210 | 0.0038 | 0.8208 | 0.0004 | 1.1609E-10 | 0.8211 | 0.8203 | 0.0007 |
|          | 30 |    | $h(t_0)$ | 0.0135 | 0.0167 | 6.7812E-10 | 0.0142 | 0.0129 | 0.0013 | 0.0132 | 0.0060 | 7.5552E-10 | 0.0135 | 0.0132 | 0.0003 |
|          | 48 |    | $h(t_0)$ | 0.0132 | 0.0054 | 3.5661E-10 | 0.0137 | 0.0125 | 0.0012 | 0.0133 | 0.0019 | 7.5519E-10 | 0.0135 | 0.0132 | 0.0003 |
|          | 64 |    | $h(t_0)$ | 0.8220 | 0.0009 | 1.3087E-10 | 0.8223 | 0.8213 | 0.0020 | 0.8215 | 0.0004 | 9.6627E-10 | 0.8218 | 0.8212 | 0.0006 |
|          | 80 |    | $h(t_0)$ | 0.0133 | 0.0019 | 8.0379E-10 | 0.0133 | 0.0131 | 0.0001 | 0.0132 | 0.0007 | 1.7524E-10 | 0.0133 | 0.0132 | 0.0001 |
|          | 100|    | $h(t_0)$ | 0.0133 | 0.0040 | 1.5309E-10 | 0.0134 | 0.0132 | 0.0002 | 0.0133 | 0.0004 | 7.4415E-10 | 0.0133 | 0.0131 | 0.0001 |
|          | 80 | 60%| $S(t_0)$ | 0.8917 | 0.0015 | 1.0885E-10 | 0.8928 | 0.8901 | 0.0027 | 0.8925 | 0.0005 | 1.9674E-10 | 0.8931 | 0.8916 | 0.0015 |
|          | 48 |    | $h(t_0)$ | 0.2301 | 0.0048 | 2.9621E-10 | 0.2307 | 0.2289 | 0.0017 | 0.2284 | 0.0025 | 1.0493E-10 | 0.2289 | 0.2276 | 0.0012 |
|          | 64 |    | $h(t_0)$ | 0.2614 | 0.1244 | 0.00020 | 0.2620 | 0.2600 | 0.0020 | 0.2605 | 0.1210 | 0.0019 | 0.2608 | 0.2599 | 0.0008 |
|          | 80 |    | $h(t_0)$ | 0.8925 | 0.0005 | 1.0251E-10 | 0.8934 | 0.8907 | 0.0024 | 0.8925 | 0.0004 | 2.1494E-10 | 0.8932 | 0.8918 | 0.0014 |
|          | 64 |    | $h(t_0)$ | 0.2283 | 0.0032 | 1.6494E-10 | 0.2290 | 0.2274 | 0.0016 | 0.2293 | 0.0012 | 3.4001E-10 | 0.2297 | 0.2287 | 0.0010 |
|          | 80 |    | $h(t_0)$ | 0.2607 | 0.1216 | 0.0019 | 0.2613 | 0.2593 | 0.0020 | 0.2605 | 0.1210 | 0.0017 | 0.2609 | 0.2601 | 0.0008 |
|          | 100|    | $h(t_0)$ | 0.8925 | 0.0004 | 1.4248E-10 | 0.8930 | 0.8918 | 0.0011 | 0.8931 | 0.0002 | 4.3074E-10 | 0.8934 | 0.8926 | 0.0008 |
|          | 80 |    | $h(t_0)$ | 0.2286 | 0.0018 | 1.5313E-10 | 0.2288 | 0.2274 | 0.0014 | 0.2289 | 0.0004 | 2.2123E-10 | 0.2293 | 0.2281 | 0.0012 |
|          | 100|    | $h(t_0)$ | 0.2600 | 0.1192 | 0.0018 | 0.2615 | 0.2597 | 0.0018 | 0.2598 | 0.1183 | 0.0017 | 0.2603 | 0.2596 | 0.0007 |

Table 6. Average, relative absolute biases, relative error of the Bayes estimates and 95% credible intervals of the $S(t_0), h(t_0)$ and $h_1(t_0)$ based on Type II censoring for real data ($NR = 10000, r = 2$ and $d = 3$)
Table 7. Bayes predictive estimates and bounds of the future observation based on Type II censoring under two-sample prediction $NR = 10000, n = 100, r = 8 \% , m = 5 (t_0 = 10, a = 2 \theta = 0.9 \%), c = 2$ and $d = 3$)

| s  | $\hat{y}_s$ | SE   | UL   | LL   | Length | $\hat{y}_s$ | SE   | UL   | LL   | Length |
|----|-------------|------|------|------|--------|-------------|------|------|------|--------|
| 1  | 4.0005      | 4.0016| 3.9992| 0.0023| 4.0000 | 4.0011 | 3.9991| 0.0020 |
| 23 | 14.9988     | 14.999| 14.9975| 0.0019| 15.0008| 15.0014| 15.0000| 0.0014 |
| 35 | 28.9995     | 29.0014| 28.9977| 0.0036| 28.9998| 29.0007| 28.9985| 0.0021 |
| 51 | 39.9999     | 40.0009| 39.9989| 0.0020| 39.9995| 40.0000| 39.9984| 0.0016 |

Table 8. Bayes predictive estimates and bounds of the future observation based on Type II censoring under two-sample prediction for real data ($NR = 10000, c = 2$ and $d = 3$)

| Real data     | s   | $\hat{y}_s$ | SE   | UL   | LL   | Length | $\hat{y}_s$ | SE   | UL   | LL   | Length |
|---------------|-----|-------------|------|------|------|--------|-------------|------|------|------|--------|
| Application I | 1   | 2.0002      | 2.0009| 1.9992| 0.0017| 2.0000 | 2.0004 | 1.9994| 0.0011 |
|               | 10  | 6.9998      | 7.0012| 6.9971| 0.0040| 7.0011 | 7.0018 | 6.9999| 0.0019 |
|               | 20  | 15.0014     | 15.0033| 14.9996| 0.0037| 15.0007| 15.0016| 14.9996| 0.0021 |
| Application II| 1   | 2.9992      | 3.0007| 2.9938| 0.0049| 2.9987| 2.9998| 2.9973| 0.0025 |
|               | 25  | 12.9980     | 13.0007| 12.9945| 0.0062| 13.0018| 13.0035| 12.9999| 0.0036 |
|               | 50  | 20.0016     | 20.0027| 19.9996| 0.0031| 20.0003| 20.0011| 19.9992| 0.0019 |