Infinite towers of supertranslation and superrotation memories

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Abstract. A framework that structures the gravitational memory effects and which is consistent with gravitational electric-magnetic duality is presented. A correspondence is described between memory observables, particular subleading residual gauge transformations, associated overleading gauge transformations and their canonical surface charges. The existence of an infinite tower of subleading soft graviton theorems is argued to imply an infinite number of conservation laws at spatial infinity and, in turn, an infinite number of memory effects at null infinity. It is shown that the leading order mutually commuting supertranslations and (novel) superrotations are both associated with a leading displacement memory effect, which suggests the existence of new boundary conditions.

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Memory effects can be formulated as the difference between two observables defined in the initial and final non-radiative regions of spacetime after energy has escaped to null infinity. Distinct classes of memory effects have been found: the displacement memory effect \[1,2\], the velocity kick memory effect \[3\] and, more recently, the spin and center-of-mass memory effects \[4–6\]. The aim of this essay is to propose a framework for describing memory effects and illustrate some of its features. This construction will extend the relationship between memories, gauge transformations and soft theorems \[7\].

The memory ↔ gauge transformation map. I consider non-local gauge-invariant observables \(O = O(\theta, \phi)\) defined by integration of a functional of the gauge-invariant fields defined in the asymptotic null region \(I^+\) (i.e. the Weyl tensor for pure gravity) between infinite past \(I^+\) and infinite future \(I^+_\), retarded times. Such an observable \(O\) is a memory observable if and only if it can be written as the difference between a local memory field \(\varphi_+\) defined at \(I^+\) and a local memory field \(\varphi_-\) defined at \(I^-\), \(O = \varphi_+ - \varphi_-\). The fields \(\varphi_{\pm}\) might not exist in arbitrary gauges. The definition only requires that one gauge exists where \(\varphi_{\pm}\) can be defined. The observable \(O\) is gauge-invariant and it vanishes if all gauge-invariant fields asymptotically vanish at null infinity.

I only consider field configurations that are non-radiative at future and past retarded times. For field configurations where the gauge-invariant fields asymptote to zero at future and past retarded times, two residual gauge transformations have to exist that shift each local memory field to zero. Since the memory observable is gauge-invariant, the residual gauge transformation \(\delta\) shifts identically \(\varphi_-\) and \(\varphi_+, \delta \varphi_- = \delta \varphi_+\). I assume that physical field configurations obey fall-off boundary conditions at infinity. It implies that \(\varphi_{\pm}\) are subleading components of the fields and the associated gauge transformations \(\delta\) are subleading. Memory observables are therefore in one-to-one correspondence with particular subleading residual gauge transformations in the gauge where the local memory fields are defined. Boundary conditions on the physical fields restrict the possible gauge transformations that they asymptote to.

The subleading towers. Motivated by several independent results \[6,8,11\], I will only consider memory observables whose local memory fields can be defined in harmonic gauge. The solutions to \(\Box \xi^\mu = 0\) form 2 scalar and 1 vector representations of \(SO(3)\). It exists for each representation an infinite tower of gauge transformations \(\delta(N)\) labelled by an integer \(N \geq 0\). The generators \(\{\xi_T(N), \xi_W(N) , \xi_R(N)\}\) depend upon arbitrary functions \(T = T(\theta, \phi), W = W(\theta, \phi)\) and \(R^A = R^A(\theta, \phi)\). The supertranslation and superrotation towers are generated by

\[
\begin{align*}
\xi_T(N) &= r^{-N} (T \partial_u - \frac{1 + \delta_{N,0}}{2r} \nabla^A T \partial_A) + \ldots , \\
\xi_R(N) &= r^{-N} R^A \frac{1}{r} \partial_A + \ldots
\end{align*}
\]

Some cases are familiar: \(\xi_T(0)\) are the BMS asymptotic supertranslation symmetries and \(\xi_R(1)\) are the subleading Diff(\(S^2\)) transformations found in \[6\]. The novel leading superrotations \(\xi_R(0)\) are on the same footing as the supertranslations since asymptotically
\[ \partial_u \sim r^{-1} \partial_A. \] They are \(1/r\) subleading with respect to the Lorentz transformations. For generic \(N\), one can map a subleading gauge transformation \(\delta_{(N)}\) to an overleading gauge transformation \(\delta_{(-N)}\), which transforms the metric at leading order or above, by analytic continuation \(N \mapsto -N\). Examples of overleading transformations are the superrotations defined in [12,13] which are combinations of \(\xi_{(-1)}\) and \(\xi_{(-1)}^{T=\nabla R}\). Finally, the generators \(\xi_{(N)}^{W}\) are Weyl transformations given for \(N = 0\) by

\[ \xi_{(0)}^{W} = W(r \partial_r + u \partial_u) + u \nabla^A W \frac{1}{r} \partial_A + \ldots \] (2)

The vectors form an algebra which is determined by their leading order commutator. The supertranslations \(\xi_{s(0)}^{T}\) and the superrotations \(\xi_{s(0)}^{R}\) commute.

Gravitational electric-magnetic duality. The set of vectors \(N \geq 1\) is complete under a notion of gravitational electric-magnetic duality [4]. The structure of electromagnetic duality is richer in gravity as compared to electromagnetism thanks to the property that the gauge parameters themselves can be dualized, which dispense with the need for a dual potential formulation to describe all relevant canonical charges [10,14]. The electric-magnetic dual of an infinitesimal gauge transformation \(\xi^{\mu}\) is defined as \((\ast \xi)^{\mu} \equiv \epsilon^{\mu \alpha \beta \gamma} n_\alpha \partial_\beta \xi_\gamma\). Here \(n_\alpha \partial_\alpha = r \partial_r + u \partial_u\) is normal to \(dS_3\) slices of Minkowski spacetime outside the lightcone at the origin. It is also the zero mode of the Weyl transformation \(\xi_{(0)}^{W}\). The duality acts as

\[ \ast \xi_{s(N)}^{T} = \frac{N+1}{2} \xi_{s(N)}^{R=\nabla T}, \quad \ast \xi_{s(N)}^{R} = (1-N) \xi_{s(N)}^{R=\nabla \hat{R}} - N \xi_{s(N+1)}^{W}, \quad \ast \xi_{s(N)}^{W} = 0, \] (3)

for all \(N \geq 1\), which proves the claim. Here \(\hat{R}^A \equiv \gamma^{AB} \epsilon_{BC} R^C\) where \(\epsilon_{AB}\) is the volume form and \(\gamma_{AB}\) the unit metric on the sphere.

The memory observables. What are the observables? The metric perturbation asymptote to \(\delta_{(N)} g_{\mu \nu}\) at early and late retarded times, which allows by construction to identify the memory fields in relevant components \(\varphi_{\pm} = \{C_{(N)}^u(\theta, \phi), C_{(N)}^{r+}(\theta, \phi), C_{(N)}^{A}(\theta, \phi)\}\), which are associated with \(\xi_{s(N)}^{T}, \xi_{s(N)}^{W}\) and \(\xi_{s(N)}^{R}\) at \(\mathcal{I}_+\) and \(\mathcal{I}_-^+\), respectively. The two scalar observables are derived quantities in terms of

\[ \Delta_{u(N)}^{+} = C_{(N)}^{u} \bigg|_{\mathcal{I}_+}, \quad \Delta_{r(N)}^{+} = C_{(N)}^{r} \bigg|_{\mathcal{I}_+}. \] (4)

A derived observable of \(\Delta_{u(N)}^{+}\) is the displacement memory effect described in [15].

The vectorial memory observables can be defined as a retarded time delays \(\Delta^+ u\), \(\Delta_{+} u\) defined for 2 counter-propagating null rays along a ring \(R\) of circumference \(2\pi L\) defined in the asymptotic future null region,

\[ \Delta_{+} u = \frac{1}{2\pi L} \oint_R \gamma_{AB} C_{(N)}^{B} d\mathbf{x} \bigg|_{\mathcal{I}_+}, \quad \Delta_{+} u = \frac{1}{2\pi L} \oint_R \epsilon_{AB} C_{(N)}^{B} d\mathbf{x} \bigg|_{\mathcal{I}_+}. \] (5)

For \(N = 1\) the first observable exactly reduces to the spin memory [4,6]. I expect that the second observable for \(N = 1\) relates to the center-of-mass memory [5].

† Gravitational duality for \(N = 0\) generates unphysical NUT charges and is therefore not enforced.
All other memory effects are new, but many may become trivial depending on the radiative boundary conditions. In particular, the superrotations $\xi^R_{(0)}$ are associated with a displacement memory effect at the same order as the one associated with supertranslations. Indeed, using a cut-and-paste procedure, the linearized shockwave metric that encodes a superrotation transition is $\eta_{\mu\nu} + \Theta(u)\mathcal{L}_{\xi^R_{(0)}}\eta_{\mu\nu}$ and its Riemann tensor is $R_{\mu\nu\rho\sigma} \sim r\partial^2_u \Theta(u)$, which leads to a leading displacement memory effect.

In the second part of this essay, I will describe how such towers of memories are associated to towers of subleading soft theorems following a sequence of equivalences:

Associated Noether charges. Gauge transformations are associated with canonical surface charges, which are integrable in non-radiative regions. Overleading gauge transformations are usually discarded since they are not tangent to the phase space (except if one enlarges the phase space and renormalizes the theory, which also leads to overleading memory observables [16]). However, as it is emphasized in several works [9][10][13][16][17], overleading gauge transformations $\delta_{(-N)}$ lead to finite Noether charges $Q_{(-N)}$ of the standard phase space that are conserved at spatial infinity, after implementing a subtraction procedure. Given the chain of correspondences presented above, we conclude that subleading transformations $\delta_{(N)}$ are in one-to-one correspondence with finite conserved Noether charges $Q_{(-N)}$ at spatial infinity that depend upon the subleading orders of the field. For example, $\xi^T_{(0)}$ is associated with the Bondi mass aspect and $\xi^R_{(1)}$ is associated with the Bondi angular momentum aspect.

Conserved charges at spatial infinity. How to define the towers of conserved charges? The simplest answer uses the representation of spatial infinity as $dS_3$ [18]. The dynamics of gravity can be re-expressed in terms of an infinite tower of symmetric transverse traceless fields $T_{ab}^{(N)}$ on $dS_3$ of the form $(\square + N^2 - 3)T_{ab}^{(N)} = s_{ab}$ where $a, b$ are indices on $dS_3$, $\square$ is the d’Alembertian and $s_{ab}$ are non-linear sources depending on lower subleading fields $(0, \ldots, N-1)$ [19] (we ignore here possible logarithmic branches). The initial and final boundary conditions are related by a junction condition, which is necessary to define scattering between $I^+$ and $I^-$. What is left as independent initial data is a tower of symmetric traceless transverse tensors $C_{ab}^{(N)}(\theta, \phi)$. For $N \geq 0$ such tensors encode 2 arbitrary functions on the sphere as shown e.g. in [14]. I expect that they will precisely encode the conserved charges $Q_{(-N)}$. Such a construction has been performed in electromagnetism [20]. The non-linearities of gravity are not expected to prevent its generalization.

Existence from soft theorems. Conserved charges at spatial infinity lead to Ward identities, which are precisely the soft theorems. Their existence go hand-in-hand. Now, the existence of at least one tower of soft theorems was recently established [11]. More precisely, the soft graviton identities written in [11] can be recognized as the Ward
identities of momentum multipole symmetries, which are associated with conserved charges at spatial infinity [10]. Though the relationship between such multipole symmetries and the residual transformations considered here remains to be found, it demonstrates the existence of an infinite tower of conserved charges and, therefore, by the above sequence of equivalences, of memories.

Conclusion. Memory effects are naturally organized in infinite towers, which are mapped by gravitational electric-magnetic duality. The leading supertranslation and leading (novel) superrotations are both associated with a leading displacement memory effect. Consequently, I conjecture the existence of new boundary conditions for asymptotically flat spacetimes that admit the Lorentz group together with mutually commuting supertranslations and superrotations as asymptotic symmetry group. The existence of an infinite number of subleading soft graviton theorems points to the existence of infinite towers of conserved charges at spatial infinity, and, of memories at null infinity. These considerations are preliminary and many gaps remain to be filled in. In addition, many extensions can be explored such as loop corrections, higher dimensional generalizations, coupling to matter and alternative gauge theories.

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