An inverse problem in advection-diffusion

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Abstract. An inverse problem to determine an unknown velocity in two-dimensional, time-
independent advection-diffusion equation from data observed at a depth-level is discussed, motivated by an ocean circulation inverse problem. A procedure by which the velocity is
reconstructed from the observed data is established and, as a consequence, the uniqueness
of the velocity realizing the prescribed data is proved. The procedure involves a modification of
the Marchenko method in the inverse scattering theory, based on the Wiener-Hopf factorization
and the Riesz-Schauder theory.

1. Introduction
In this paper we consider a problem of determining an unknown coefficient in an elliptic equations
from data observed at the boundary. By letting $\Delta$ be the two-dimensional Laplace operator and
letting $L^1(0, \infty)$ denote the space of integrable functions on the interval $(0, \infty)$, the problem is
formulated as follows:

Problem 1.1 Given real-valued functions $f(y), g(y)$, determine a real-valued, continuous
function $v(x) \in L^1(0, \infty)$ so that the (overspecified) elliptic system

\[
\begin{cases}
-\Delta \phi + v(x) \frac{\partial \phi}{\partial y} = 0 & (0 < x < \infty, -\infty < y < \infty), \\
\phi(0, y) = f(y) & (-\infty < y < \infty), \\
-\frac{\partial \phi}{\partial x}(0, y) = g(y) & (-\infty < y < \infty),
\end{cases}
\]

admits a solution $\phi(x, y)$ satisfying

\[
\lim_{x+|y| \to \infty} \phi(x, y) = 0. \tag{1.2}
\]

This problem is motivated by a class of ocean circulation inverse problems, which is to
estimate oceanic fields involving physical quantities such as velocities, diffusivities as unknown
factors from sparse data of observable properties called tracers such as salinity, (potential)
temperature, oxygen, nitrate (see Bennett [3], Wunsch [21]). The steady-state conservation of
the concentration $\phi$ of a tracer can be expressed by the time-independent advection-diffusion
equation

\[v \cdot \nabla \phi = \nabla \cdot (\kappa \nabla \phi),\]

provided that the effects of internal sources and sinks for the tracer are negligible. Here $v$ is the
fluid velocity in the ocean, $\kappa$ is the diffusivity (tensor, in general) of the tracer; the left-hand
side represents the rate of change of the concentration due to advection and the right-hand side denotes the molecular diffusion of the tracer (see Apel [2], Bennett [3]). Since the fluid in the ocean is incompressible, we may assume that the velocity field is nondivergent: \( \nabla \cdot \mathbf{v} = 0 \). In addition we assume that: (1) vertical velocities are so small compared with the horizontal velocities that the vertical velocity component is ignored, (2) the velocities and the concentration of the tracer are horizontally isotropic, (3) the variations of the diffusivity \( \kappa \) are sufficiently small to take it as a constant. Let us denote the depth by \( x \) (for a later convenience, we use the notation uncommon in oceanography), the distance in the horizontal separation direction by \( y \), and the velocity in the \( y \) direction by \( v \). Then, under assumptions (1), (2), the velocity field is written as \( \mathbf{v} = (0, v(x, y), 0) \) and, in view of the incompressibility condition \( \nabla \cdot \mathbf{v} = 0 \), \( v(x, y) \) becomes independent of \( y \) automatically. Therefore, after an appropriate nondimensionalization in the aid of assumption (3), we deduce the two-dimensional equation in (1.1). Without worrying about how one can get a bulk of continuous data, let us suppose that the concentration and the flux (rate of transport) of the tracer are observed at a fixed, constant depth set as \( x = 0 \) that is considered to be appropriate for the steady-state model (see Figure 1). Then a mathematical problem is posed in the form of Problem 1.1.

We say that a function \( v(x) \in L^1(0, \infty) \) is a solution of Problem 1.1 for \( f, g \) if \( v(x) \) is a real-valued, continuous function on \([0, \infty)\) and if system (1.1) admits a solution \( \phi \in C^2(\Omega) \cap C^2(\Omega) \) satisfying the condition (1.2) for \( f, g \), where \( \Omega = (0, \infty) \times (-\infty, \infty) \). Throughout the paper we suppose that the Dirichlet data \( f(y) \) is the Fourier image of a function \( \hat{f}(\lambda) \) with \((1 + |\lambda|)\hat{f}(\lambda) \in L^1(R)\) and the Neumann data \( g(y) \) is the image of a function \( \hat{g}(\lambda) \) in \( L^1(R) \), i.e.,

\[
f(y) = \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda y}d\lambda, \quad g(y) = \int_{-\infty}^{\infty} \hat{g}(\lambda)e^{i\lambda y}d\lambda,
\]

with

\[
(1 + |\lambda|)\hat{f}(\lambda) \in L^1(R), \quad \hat{g}(\lambda) \in L^1(R).
\]

It is evident that if \( f(y) = g(y) = 0 \) for all \( x \in R \) (which is equivalent to \( \hat{f} = \hat{g} = 0 \) in \( L^1(R) \)) then any real-valued, continuous function \( v(x) \) on \([0, \infty)\) belonging to \( L^1(0, \infty) \) is a solution of

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**Figure 1.** Schematic of velocities at various depth
Problem 1.1, because \( \phi \equiv 0 \) is a solution of (1.1) satisfying (1.2). The following theorem that will be established in this paper asserts that, except for this trivial case, Problem 1.1 has at most one solution.

**Theorem 1.2** Let \( f(y) \) and \( g(y) \) be given functions expressed as (1.3) with (1.4). Then Problem 1.1 has at most one solution \( v(x) \) in \( L^1(0, \infty) \) unless \( f(y) \equiv g(y) \equiv 0 \).

The aim of this paper is to discuss how one can restore a solution \( v(x) \in L^1(0, \infty) \) of Problem 1.1 from the observed data \( f, g \), in other words, to find a procedure by which \( v(x) \) is reconstructed from the observed data. Theorem 1.2 will be established as a direct consequence of the procedure.

Problem 1.1 can be recast as a kind of inverse spectral problem. To explain this we mention (see Lemma 3.3) that the Dirichlet problem

\[
-\Delta \phi + v(x) \frac{\partial \phi}{\partial y} = 0 \quad (0 < x < \infty, -\infty < y < \infty),
\]

\[
\phi(0, y) = f(y) \quad (-\infty < y < \infty),
\]

(1.5)
can be uniquely solved under the condition (1.2) and the solution is expressed as

\[
\phi(x, y) = \int_{-\infty}^{\infty} \hat{f}(\lambda) \frac{e^{|x, \lambda|}}{e(0, \lambda)} e^{iy\lambda} d\lambda,
\]

(1.6)
in terms of a solution \( e(x, \lambda) \) of the ordinary differential equation

\[
E'' - \lambda^2 E - i\lambda v(x) E = 0 \quad (\lambda = \frac{d^2}{dx^2}, \ 0 < x < \infty),
\]

(1.7)

with the asymptotic behavior

\[
e(x, \lambda) = e^{\mp \lambda x}[1 + o(1)], \quad e'(x, \lambda) = \mp \lambda e^{\mp \lambda x}[1 + o(1)],
\]

(1.8)

according to \( \pm \lambda \geq 0 \) (to \( \pm \text{Re} \lambda > 0 \) if \( \lambda \) is a complex number), as \( x \to \infty \), uniformly in \( -\infty < \lambda < \infty \). We here note that the solution \( e(x, \lambda) \) of (1.7) with (1.8) is uniquely determined from \( v(x) \in L^1(0, \infty) \), continuous in \( \lambda \), and possesses the properties

\[
e(0, \lambda) \neq 0 \quad (-\infty < \lambda < \infty), \quad \lim_{|\lambda| \to \infty} |e(0, \lambda)| = 1.
\]

Identity (1.6) yields

\[
-\frac{\partial \phi}{\partial x}(0, y) = -\int_{-\infty}^{\infty} |\lambda| \hat{f}(\lambda) \frac{e'(0, \lambda)}{|\lambda| e(0, \lambda)} e^{iy\lambda} d\lambda.
\]

Accordingly, in view of the uniqueness theorem of the Fourier transform, system (1.1) admits a solution \( \phi(x, y) \) satisfying (1.2) if and only if

\[
-|\lambda| \hat{f}(\lambda) \frac{e'(0, \lambda)}{|\lambda| e(0, \lambda)} = \hat{g}(\lambda) \quad (\text{almost every } \lambda \in \mathbb{R}).
\]

(1.9)

This relation presents a necessary and sufficient condition for \( v(x) \in L^1(0, \infty) \) to be a solution of Problem 1.1 for given data \( f(y), g(y) \) as (1.3), (1.4). By setting

\[
Q := \{ \lambda \in \mathbb{R} \mid |\lambda| \hat{f}(\lambda) \neq 0 \}, \quad \rho(\lambda) = \frac{\hat{g}(\lambda)}{|\lambda| \hat{f}(\lambda)} \quad (\lambda \in Q)
\]

and noticing that unless \( f \equiv 0 \) the set \( Q \) has a positive Lebesgue measure, Problem 1.1 is recast as the following problem unless \( f \equiv 0 \):
Problem 1.3 Given a complex-valued function \( \rho(\lambda) \) defined on a set \( Q \) of a positive Lebesgue measure, determine a real-valued, continuous function \( v(x) \in L^1(0,\infty) \) so that the solution \( e(x, \lambda) \) of (1.7) with the asymptotic behavior (1.8) satisfies

\[
- \frac{e'(x, \lambda)}{|\lambda| e(x, \lambda)} = \rho(\lambda) \quad (\lambda \in Q).
\]

Our approach to Problem 1.1 consists in applying to Problem 1.3 the Marchenko method (see Agranovich and Marchenko [1], Chadan and Sabatier [4], Marchenko [13], Newton [15], Ramm [17]) in the inverse scattering problem and its generalization by Jaulent and Jean [8], Jaulent [6, 7] in the inverse scattering problem with energy dependent potentials.

In Section 2 we show that if \( v(x) \in L^1(0,\infty) \) then the solution \( e(x, \lambda) \) of (1.7) satisfying (1.8) can be expressed as

\[
e(x, \lambda) = e^{-\lambda x} + \lambda \int_x^\infty K(t) e^{-\lambda t} dt \quad (\text{Re } \lambda \geq 0)
\]

in terms of a continuous, bounded function \( K(x, t) \) defined in \( 0 \leq x \leq t < \infty \). Representation (1.11) is an integral form of Marchenko’s representation of the Jost solution used in the Marchenko method. We refer to (1.11) as the transformation representation, and the function \( K(x, t) \) in it as the transformation kernel. It turns out that the transformation kernel \( K(x, t) \) is uniquely defined from the coefficient \( v(x) \) of the ordinary differential equation (1.7) and is related with \( v(x) \) by the formula

\[
1 + K(x, x) = e^{\frac{i}{2} \int_x^\infty v(\eta) d\eta},
\]

This formula yields \( |1 + K(x, x)| = 1 \) for each \( x \geq 0 \), because \( v(x) \) is real-valued. In terms of the transformation kernel, \( v(x) \) is expressed as

\[
v(x) = 2i \frac{d}{dx} \log(1 + K(x, x)).
\]

In Section 3, by letting \( e_\pm(x, \lambda) \) be the restrictions of \( e(x, \lambda) \) in the regions \( \pm \text{Re } \lambda > 0 \), we regard the function \( e(x, \lambda) \) as a collection of two functions \( e_+(x, \lambda) \) and \( e_-(x, \lambda) \) of \( \lambda \) for each fixed \( x \geq 0 \). For each \( x \geq 0 \), the functions \( e_+(x, \lambda) \) and \( e_-(x, \lambda) \) are holomorphic in the half plane \( \pm \text{Re } \lambda > 0 \) and continuous in the closure \( \pm \text{Re } \lambda \geq 0 \) respectively. Since \( e_-(x, -\lambda) = e_+(x, \lambda) \) for \( \text{Re } \lambda \geq 0 \), these functions are connected together by the conjugate relation on the imaginary axis: \( e_-(x, \lambda) = \bar{e}_+(x, \bar{\lambda}) \) for \( \text{Re } \lambda = 0 \). Furthermore, for pure-imaginary \( \lambda \neq 0 \), these two functions form a fundamental system of solutions to equation (1.7). This implies that \( e_+(x, \lambda) \neq 0 \) for pure-imaginary \( \lambda \neq 0 \). Since \( e_+(x, 0) = 1 \), this remains true for \( \lambda = 0 \). In particular, \( e_+(0, \lambda) \neq 0 \) on the imaginary axis. On the other hand, by virtue of the transformation representation, \( e_+(0, \lambda) \) can be written as

\[
e_+(0, \lambda) = 1 + \int_0^\infty K_1(0, t) e^{-\lambda t} dt \quad (\text{Re } \lambda \geq 0),
\]

where the partial derivative \( K_1(0, t) \) belongs to the space \( L^1(0,\infty) \). This, together with the Riemann-Lebesgue lemma, implies that \( e_+(0, \lambda) \rightarrow 1 + K(0, 0) \) as \( \lambda \rightarrow \pm i \infty \). Notice that \( 1 + K(0, 0) \neq 0 \). Thus, the function \( e_+(0, \lambda) \) is a nowhere vanishing function on the imaginary axis with a non-zero limit as \( \lambda \rightarrow \pm i \infty \). Hence the winding number of \( e_+(0, \lambda) \), which is denoted by \( \text{ind } e_+(0, \lambda) \), can be defined as an integer given by

\[
\text{ind } e_+(0, \lambda) := -\frac{1}{2\pi} \int_{-i\infty}^{i\infty} d[\arg(e_+(0, \lambda))].
\]
We can now state a key fact (see Theorem 3.2): for any \(v(x) \in L^1(0, \infty)\),
\[
\text{ind} \ e_+(0, \lambda) = 0. \tag{1.14}
\]
This fact is vital; due to it we can show that the data \(e_-(0, \lambda)\) say \(S(\lambda)\) on the imaginary axis is uniquely determined from the data \(\rho(\lambda)\) on the set \(Q\) with a positive Lebesgue measure. More precisely speaking, we can show that there exists a complex-valued function \(F(t) \in L^1(R)\) uniquely determined from \(\rho(\lambda)\) by which \(e_-(0, \lambda) e_+(0, \lambda)\) is expressed as
\[
\frac{e_-(0, \lambda)}{e_+(0, \lambda)} = C + \int_{-\infty}^{\infty} F(t)e^{\lambda t} dt \quad (\text{Re} \lambda = 0). \tag{1.15}
\]
Here \(C\) is a complex number with \(|C| = 1\) determined by \(F(t)\).

As will be stated in Section 4, the transformation kernel \(K(x, t)\) in (1.11) and the function \(F(t)\) in (1.15) are connected by the following relation:
\[
K(x, t) + \int_{x}^{\infty} K(x, r)F(r + t)dr + \int_{x}^{\infty} F(r + t)dr = 0 \quad (x \leq t < \infty), \tag{1.16}
\]
where \(\overline{K(x, t)}\) is the complex conjugate of the transformation kernel \(K(x, t)\). This equation plays a role of the Marchenko equation in the inverse scattering theory. Notice that it is corresponding to an integral form of equation (6) in Jaulent [7]. We call equation (1.16) the modified Marchenko equation. The question of the solvability to it is reduced to showing the uniqueness of the solutions to the homogeneous equation associated with (1.16). The uniqueness can be proved in a similar way to that in [6] by the projection method. Our success is due to that the winding number of the function in (1.15) equals to zero following from fact (1.14). In this way we finally show that (1.16) has a unique solution \(K(x, t)\) in the space of bounded, continuous functions on the interval \([x, \infty)\) for each \(x \geq 0\).

\[
S(\lambda) := \frac{e_-(0, \lambda)}{e_+(0, \lambda)} = C + \int_{-\infty}^{\infty} F(t)e^{\lambda t} dt
\]

\[
\rho(\lambda) := \frac{\hat{g}(\lambda)}{|\lambda|\hat{f}(\lambda)}
\]

\[
v(x) \quad \text{§2} \quad K(x, t) : \text{Transformation Kernel}
\]

Our plan described above is illustrated in Figure 2. In the way outlined in the plan we arrive at the following theorem.
Theorem 2.4 Let \( f(y)(\neq 0) \) and \( g(y) \) be given functions expressed as (1.3) with (1.4). If Problem 1.1 has a solution \( v(x) \in L^1(0,\infty) \) then \( v(x) \) can be reconstructed from \( f, g \) in the following steps:

1. The function \( \frac{v(0,\lambda)}{v(x,\lambda)} \) on the imaginary axis can be determined uniquely from the data \( \sigma(\lambda) \), and is represented as (1.15) in terms of a function \( F(t) \in L^1(R) \).

2. The integral equation (1.16) with the function \( F(t) \) is solved uniquely in the space of bounded, continuous functions on the interval \([x, \infty)\) for each \( x \geq 0 \), and the transformation kernel \( K(x, t) \) is obtained.

3. The function \( v(x) \) is derived from \( K(x, x) \) by (1.12).

Theorem 1.4 will be established in the end of this paper by combining Theorems 3.7, 4.1, 3.2, and 2.2. More detailed statement of the first step in Theorem 1.4 is given in Corollary 3.8. Theorem 1.2 is an immediate consequence of Theorem 1.4.

Inverse problems to determine coefficients in elliptic equations from boundary data are discussed in many works (see Isakov [5], Klibanov and Timonov [10], Lavrentiev, Romanov, Shishatskii [12], Megrabov [14], Ramm [17], Romanov [18], Uhlmann [20] and their references). Theorem 1.4 suggests a validity of the Marchenko method for inverse problems to elliptic equations such as Problem 1.1.

2. The transformation kernel

In this section we shall establish integral representations for the solutions \( e_\pm(x, \lambda) \) of equation (1.7) with the asymptotic behaviors \( e_\pm(x, \lambda) \sim e^{\mp \lambda x} \ (x \to \infty) \) for \( \pm \text{Re} \lambda \geq 0 \) respectively under the assumption \( v(x) \in L^1(0,\infty) \). In what follows we discuss the representation only for \( e_+(x, \lambda) \) because they can be shown for \( e_-(x, \lambda) \) in the same way as for \( e_+(x, \lambda) \). Throughout this section we assume that \( v(x) \) is a continuous function on the interval \([0, \infty)\) and belongs to \( L^1(0,\infty) \), and we use the notation

\[
\sigma(x) := \int_x^\infty |v(\eta)|d\eta \quad (x \geq 0).
\]

The following is obtained by a standard argument:

Lemma 2.1 Let \( v(x) \in L^1(0,\infty) \). Then, for each \( \lambda \) in \( \text{Re} \lambda \geq 0 \), equation (1.7) has a unique solution \( e_+(x, \lambda) \) satisfying

\[
\lim_{x \to \infty} e^{\lambda x} e_+(x, \lambda) = 1. \tag{2.1}
\]

The solution \( e_+(x, \lambda) \) satisfies the integral equation

\[
e_+(x, \lambda) = e^{-\lambda x} - i \int_x^\infty \sinh \lambda(x-s) \cdot v(s)e_+(s, \lambda)ds. \tag{2.2}
\]

For each \( x \geq 0 \), the solution \( e_+(x, \lambda) \) is holomorphic with respect to \( \lambda \) in the right half plane \( \text{Re} \lambda > 0 \) and is continuous in its closure \( \text{Re} \lambda \geq 0 \). Moreover, as \( x \to \infty \),

\[
e_+(x, \lambda) = e^{-\lambda x}[1 + o(1)], \quad e'_+(x, \lambda) = -\lambda e^{-\lambda x}[1 + o(1)], \tag{2.3}
\]

uniformly in \( \text{Re} \lambda \geq 0 \).

The solution \( e_+(x, \lambda) \) admits the following representation in terms of the transformation kernel \( K(x, t) \).

Theorem 2.2 Let \( v(x) \in L^1(0,\infty) \). Then the solution \( e_+(x, \lambda) \) can be represented as

\[
e_+(x, \lambda) = e^{-\lambda x} + \lambda \int_x^\infty K(x, t)e^{-\lambda t}dt \quad (\text{Re} \lambda \geq 0), \tag{2.4}
\]
where the kernel \( K(x, t) \) is a continuous, bounded function defined in \( 0 \leq x \leq t < \infty \). The kernel is uniquely determined from \( v(x) \) and possesses the following properties:

1. \( K(x, t) \) satisfies the inequality

\[
|K(x, t)| \leq M\sigma \left( \frac{x + t}{2} \right)
\]

with some constant \( M \) and the equality

\[
1 + K(x, x) = e^{\frac{i}{2} \int_{x}^{\infty} v(\eta)d\eta}.
\]

2. \( K(x, t) \) has partial derivatives \( K_x(x, t), K_t(x, t) \) with respect to each of its variables. For each \( x \in [0, \infty) \), the derivatives \( K_x(x, t), K_t(x, t) \) are integrable functions of \( t \) in the interval \([x, \infty)\) and

\[
\int_{x}^{\infty} |K_x(x, t)| dt \leq M\sigma(x), \quad \int_{x}^{\infty} |K_t(x, t)| dt \leq M\sigma(x)
\]

with some constant \( M \).

**Proof.** We employ the following representation (see Jaulent and Jean [8, Lemma 4.1]):

\[
e_+(x, \lambda) = e^{\frac{i}{4} \int_{x}^{\infty} v(\eta)d\eta} e^{-\lambda x} + \int_{x}^{\infty} A(x, t)e^{-\lambda t} dt \quad (\text{Re} \lambda \geq 0).
\]

Here the function \( A(x, t) \) is obtained as a solution of the integral equation

\[
A(x, t) = -\frac{i}{4} v \left( \frac{x + t}{2} \right) e^{\frac{i}{2} \int_{x}^{\infty} v(\eta)d\eta} + \frac{i}{2} \int_{x}^{\infty} v(s)A(s, t + s - x)ds - \frac{i}{2} \int_{x}^{\infty} v(s)A(s, t + x - s)ds.
\]

By substituting in (2.2) the right-hand side of (2.8) instead of \( e_+(x, \lambda) \) and observing that

\[
-i \int_{x}^{\infty} \sinh \lambda(x - s) \cdot v(s) e^{\frac{i}{2} \int_{x}^{\infty} v(\eta)d\eta} ds = \frac{i}{2} \lambda \int_{x}^{\infty} e^{-\lambda t} dt \int_{x}^{\infty} v(s) e^{\frac{i}{2} \int_{x}^{\infty} v(\eta)d\eta} ds,
\]

\[
-i \int_{x}^{\infty} \sinh \lambda(x - s) \cdot v(s) ds \int_{s}^{\infty} A(s, t)e^{-\lambda t} dt
\]

\[
= \frac{i}{2} \int_{x}^{\infty} e^{-\lambda t} dt \int_{x}^{\infty} v(s)A(s, t + s - x)ds - \frac{i}{2} \int_{x}^{\infty} e^{-\lambda t} dt \int_{x}^{\infty} \frac{v(s)}{2} A(s, t + x - s)ds,
\]

it follows that \( e_+(x, \lambda) \) defined by (2.8) becomes a solution of (2.2).

To solve (2.9) in \( L^1(x, \infty) \) by the method of successive approximations, we put

\[
A_0(x, t) = -\frac{i}{4} v \left( \frac{x + t}{2} \right) e^{\frac{i}{2} \int_{x}^{\infty} v(\eta)d\eta},
\]

\[
A_n(x, t) = i \int_{x}^{\infty} v(s)A_{n-1}(s, t + s - x)ds - i \int_{x}^{\infty} \frac{v(s)}{2} A_{n-1}(s, t + x - s)ds.
\]

Then, by changing the orders of integrations, it follows that

\[
\int_{x}^{\infty} |A_n(x, t)|dt \leq \int_{x}^{\infty} |v(s)|ds \int_{s}^{\infty} |A_{n-1}(s, t)|dt.
\]
Hence, by induction, we obtain
\[ \int_x^\infty |A_n(x,t)|dt \leq \frac{1}{2} \frac{\sigma(x)^{n+1}}{n!}. \]
This shows that, for each \( x \geq 0 \), the integral equation (2.7) has a solution \( A(x,t) = \sum_{n=0}^{\infty} A_n(x,t) \) in the space \( L^1(x,\infty) \) with
\[ \int_x^\infty |A(x,t)|dt \leq \frac{1}{2} \sigma(x)e^{\sigma(x)}, \quad \text{(2.10)} \]
provided that \( v(x) \in L^1(0,\infty) \). Notice that we require no differentiability assumptions on \( v(x) \) because, unlike in [8], we solve (2.9) in \( L^1(x,\infty) \).

We now define
\[ K(x,t) = -\int_t^\infty A(x,\eta)d\eta. \]
Then an integration by parts shows that
\[ e_+(x,\lambda) = \left( e^{\frac{i}{2} \int_x^\infty v(\eta)d\eta} - K(x,x) \right) e^{-\lambda x} + \lambda \int_x^\infty K(x,t)e^{-\lambda t}dt \quad (\text{Re} \lambda \geq 0). \]
On setting \( \lambda = 0 \) and noting \( e_+(x,0) \equiv 1 \) we have (2.6). Moreover, by (2.10), on letting \( M = \frac{1}{2}e^{\sigma(0)} \), we obtain (2.5).

Integrating (2.9), changing the orders of integrations and using (2.6), one can show that \( K(x,t) \) satisfies the integral equation
\[ K(x,t) = \frac{i}{2} \int_x^{x+t} v(s)ds + \frac{i}{2} \int_x^{\infty} v(s)K(s,t+s-x)ds \]
\[ - \frac{i}{2} \int_x^{x+t} v(s)K(s,t+s-x)ds \quad \text{(2.11)} \]
for \( 0 \leq x \leq t < \infty \).

This, together with (2.10), shows that \( K(x,t) \) has the derivative \( K_x(x,t) \), which satisfies
\[ K_x(x,t) = -\frac{i}{4} v \left( \frac{x+t}{2} \right) (1 + K \left( \frac{x+t}{2}, \frac{x+t}{2} \right)) \]
\[ - \frac{i}{2} \int_x^\infty v(s)K_t(s,t+s-x)ds - \frac{i}{2} \int_x^{x+t} v(s)K_t(s,t+s-x)ds. \]
Hence, by (2.10) and Fubini’s theorem, we obtain \( \int_x^\infty |K_x(x,t)|dt \leq \frac{1}{2} \sigma(x) + \frac{1}{2} \sigma(x)^2 \). Letting \( M = \frac{1}{2}(1 + \sigma(0)) \), we have the estimate for \( K_x(x,t) \) in (2.7).

In order to show that \( K(x,t) \) is unique, we suppose that
\[ e_+(x,\lambda) = e^{-\lambda x} + \lambda \int_x^\infty K_1(x,t)e^{-\lambda t}dt = e^{-\lambda x} + \lambda \int_x^\infty K_2(x,t)e^{-\lambda t}dt \quad (\text{Re} \lambda > 0), \]
where \( K_1(x,t), K_2(x,t) \) are continuous, bounded functions in \( 0 \leq x \leq t < \infty \). Then, for each \( \zeta > 0 \),
\[ \int_x^\infty (K_1(x,t) - K_2(x,t))e^{-\zeta t}e^{-i\xi t}dt = 0 \quad (-\infty < \xi < \infty), \]
where \( (K_1(x,t) - K_2(x,t))e^{-\zeta t} \in L^1(x,\infty) \). Accordingly, by the uniqueness theorem for the Fourier transform, we obtain \( K_1(x,t) = K_2(x,t) \).

For the closed half plane \( \text{Re} \lambda \geq 0 \), the following representation of \( e_+(x,\lambda) \) will be used in the subsequent sections:
Lemma 2.3 Let $v(x) \in L^1(0, \infty)$ and let $K(x, t)$ be the function defined in Theorem 2.2. Then, for any $\lambda$ in the half plane $\text{Re} \lambda \geq 0$, the following equality holds:

$$e_+(x, \lambda) = (1 + K(x, x)) e^{-\lambda x} + \int_x^\infty K_t(x, t) e^{-\lambda t} \, dt.$$  

(2.12)

3. Data on the imaginary axis

In the previous section we have established the existence and the integral representation of the solution $e_+(x, \lambda)$ on the right half plane. The solution $e_-(x, \lambda)$ that is asymptotically equal to $e^{\lambda x}$ on the left half plane can be of course obtained in a similar way. However, in the case where $v(x)$ is real-valued, it is more convenient to define directly as a complex conjugate of $e_+(x, -\lambda)$:

$$e_-(x, \lambda) = \overline{e_+(x, -\lambda)} \quad (\text{Re} \lambda \leq 0).$$  

(3.1)

By definition, $e_+(x, \lambda)$ and $e_-(x, \lambda)$ take conjugate complex values at points lying symmetrically with respect to the imaginary axis (i.e. $\lambda$ and $-\lambda$; see Figure 3). Throughout this and forthcoming sections we shall assume, in addition to that $v(x)$ is continuous on $[0, \infty)$ and belongs to $L^1(0, \infty)$, that $v(x)$ is real-valued. Due to the form of (1.7), the function $e_-(x, \lambda)$ defined by (3.1) is a solution of (1.7) under this assumption. In view of Lemma 2.1 and definition (3.1), the solution $e_-(x, \lambda)$ is holomorphic with respect to $\lambda$ in the left half plane $\text{Re} \lambda < 0$, is continuous up to the imaginary axis, and has the asymptotic behavior

$$e_-(x, \lambda) = e^{\lambda x}[1 + o(1)], \quad e'_-(x, \lambda) = \lambda e^{\lambda x}[1 + o(1)],$$  

(3.2)

as $x \to \infty$. In this section we shall employ the pair of $e_+(x, \lambda)$ and $e_-(x, \lambda)$ to construct data $S(\lambda)$ on the imaginary axis that is determined from the data $\rho(\lambda)$ in (1.10) on the real line.

On the imaginary axis $\text{Re} \lambda = 0$ (equivalent to $-\lambda = \lambda$) we have two solutions $e_\pm(x, \lambda)$. These two solutions are mutually connected by the conjugate relation:

$$e_+(x, \lambda) = \overline{e_-(x, \lambda)} \quad (\text{Re} \lambda = 0).$$  

(3.3)

From the asymptotic behaviors (2.3) and (3.2), their Wronskian

$$W[e_+(x, \lambda), e_-(x, \lambda)] := e_+(x, \lambda)e'_-(x, \lambda) - e'_+(x, \lambda)e_-(x, \lambda)$$

has the asymptotic behavior

$$W[e_+(x, \lambda), e_-(x, \lambda)] = 2\lambda[1 + o(1)] \quad (\text{Re} \lambda = 0, \ x \to \infty).$$
But since equation (1.7) does not contain the first-order derivative, the Wronskian here does not depend on $x$. Thus we have

$$W[e_+(x, \lambda), e_-(x, \lambda)] = 2\lambda \quad (\Re \lambda = 0, \ x \geq 0). \quad (3.4)$$

Consequently, for the pure-imaginary $\lambda \neq 0$, two solutions $e_+(x, \lambda)$ and $e_-(x, \lambda)$ are linearly independent, and hence form a fundamental system of solutions to equation (1.7). Notice that $e_-(x, 0) = e_+(x, 0) \equiv 1$ for $\lambda = 0$.

We now present a key lemma, which asserts that $e_+(0, \lambda)$ has no zeros in $\pm \Re \lambda \geq 0$.

**Lemma 3.1** If $v(x)$ is a real-valued function in $L^1(0, \infty)$, then $e_+(0, \lambda) \neq 0$ for any $\lambda$ in the right half plane $\Re \lambda \geq 0$.

**Proof.** If $e_+(0, \lambda) = 0$ for pure-imaginary $\lambda \neq 0$ then, by (3.3), $e_-(0, \lambda) = 0$. However this contradicts (3.4). Hence $e_+(0, \lambda) \neq 0$ for pure-imaginary $\lambda \neq 0$. This holds also for $\lambda = 0$ because $e_+(x, 0) \equiv 1$. Thus, $e_+(0, \lambda) \neq 0$ on the imaginary axis $\Re \lambda = 0$.

To show that this remains true in $\Re \lambda > 0$ we employ the representation

$$e_+(0, \lambda) = 1 + K(0, 0) + \int_0^\infty K_t(0, t)e^{-\lambda t} dt \quad (\Re \lambda \geq 0), \quad (3.5)$$

which is obtained by putting $x = 0$ in (2.12). It follows from (2.6) in Theorem 2.2 that

$$|1 + K(0, 0)| = \left|e^{\frac{1}{2}\int_0^\infty v(0)dy}\right| = 1. \quad (3.6)$$

Moreover, in view of assertion (3) in Theorem 2.2, the function $K_t(0, t)$ in (3.5) belongs to $L^1(0, \infty)$, and therefore, by the Riemann-Lebesgue lemma,

$$\lim_{|\lambda| \to \infty} \int_0^\infty K_t(0, t)e^{-\lambda t} dt = 0, \quad (3.7)$$

in the half plane $\Re \lambda \geq 0$. This implies that a curve $\gamma$ defined by

$$z = e_+(0, i\xi) \quad (\xi : \infty \to -\infty)$$

is a continuous, oriented, closed curve in the complex plane from the point $1 + K(0, 0)$ on the unit circle to itself. Since $e_+(0, \lambda) \neq 0$ on the imaginary axis as has been shown already, the curve $\gamma$ does not pass through the origin $z = 0$. Accordingly, we can define the index of the origin with respect to $\gamma$ by the increment of its argument along the imaginary axis, i.e.,

$$\text{ind } e_+(0, \lambda) := \frac{1}{2\pi} \int_{-i\infty}^{i\infty} d [\arg (e_+(0, \lambda))] = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}. \quad (3.8)$$

(see e.g., Krein [11]). We shall call the index the winding number of $e_+(0, \lambda)$, because it indicates how many times the closed curve $z = e_+(0, \lambda)$ winds around the origin in the counterclockwise direction. Note that the index is also written as

$$\text{ind } e_+(0, \lambda) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d [\log (e_+(0, \lambda))] = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}.$$

By the argument principle, the index $\text{ind } e_+(0, \lambda)$ gives the number of zeros (counted with multiplicities) of $e_+(0, \lambda)$ in the region $\Re \lambda > 0$, because $e_+(0, \lambda)$ is holomorphic in the region and is continuous in its closure. Therefore, to prove the lemma, it suffices to show that $n := \text{ind } e_+(0, \lambda) = 0$. 

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To show that \( n = 0 \), let \( 0 \leq \theta \leq 1 \) and let \( E = e_+(x, \lambda; \theta) \) be the solution of
\[
E'' - \lambda^2 E - i\lambda \theta v(x)E = 0 \quad (0 < x < \infty)
\]
satisfying the condition
\[
\lim_{x \to \infty} e^{\lambda x} e_+(x, \lambda; \theta) = 1
\]
for each \( \theta \in [0, 1] \) and each \( \lambda \) in \( \text{Re} \lambda \geq 0 \). Replacing \( v(x) \) by \( \theta v(x) \) in the above discussion we see that \( e_+(0, \lambda; \theta) \neq 0 \) on the imaginary axis \( \text{Re} \lambda = 0 \) for any \( \theta \in [0, 1] \). Hence the winding number
\[
n(\theta) := \text{ind} e_+(0, \lambda; \theta) = \frac{1}{2\pi} \int_{-i\infty}^{i\infty} d[\text{arg}(e_+(0, \lambda; \theta)]
\]
can be defined for each \( \theta \in [0, 1] \). Clearly \( n(0) = 0 \), because \( e_+(0, \lambda; 0) \equiv 1 \).

If \( n(1) \) were positive, the set \( \{ \theta \in [0, 1] \mid n(\theta) > 0 \} \) would not be empty. Hence we can define a number \( \theta_0 \) by
\[
\theta_0 := \inf \{ \theta \in [0, 1] \mid n(\theta) > 0 \}. \tag{3.9}
\]
Since \( e_+(0, \lambda; \theta) \neq 0 \) on the imaginary axis \( \text{Re} \lambda = 0 \) for any \( \theta \in [0, 1] \), the curve \( \gamma(\theta_0) \) defined by
\[
z = e_+(0, i\xi; \theta_0) \quad (\xi; \infty \to -\infty)
\]
does not pass through the origin \( z = 0 \). But \( e_+(0, \lambda; \theta) \) is continuous in \( \theta \), and therefore, \( n(\theta) \) is also so. Since \( n(\theta) \) are integers, this implies that the winding numbers \( n(\theta) \) are invariant under small perturbations of \( \theta \), in other words, \( n(\theta) = n(\theta_0) \) for \( \theta \) sufficiently near \( \theta_0 \). This is incompatible with definition \( (3.9) \). Thus we conclude that \( n(1) = 0 \). Since \( n = n(1) \), the proof is complete. \( \square \)

As a consequence of Lemma 3.1, we can draw the following conclusion.

**Theorem 3.2** Let \( v(x) \) be a real-valued function in \( L^1(0, \infty) \) and let \( e_+(x, \lambda) \) be the solution of (1.7) satisfying the condition (2.1). Then \( \text{ind} e_+(0, \lambda) = 0 \).

We now define a function \( e(x, \lambda) \) by
\[
e(x, \lambda) = \begin{cases} 
e_+(x, \lambda), & \text{for } \text{Re} \lambda \geq 0, \\
e_-(x, \lambda), & \text{for } \text{Re} \lambda < 0,
\end{cases} \tag{3.10}
\]
and establish the solution formula (1.6) of the Dirichlet problem (1.5) under the condition (1.2). Note that definition \( (3.10) \) is compatible with definition \( (1.8) \) in the introduction.

For a while, we consider \( e(x, \lambda) \) for real \( \lambda \). By \( (2.3) \) and \( (3.1) \) we have the estimates
\[
|e(x, \lambda)| \leq Me^{-|\lambda|x}, \quad |e'(x, \lambda)| \leq M|\lambda|e^{-|\lambda|x} \quad (-\infty < \lambda < \infty, \ x \geq 0), \tag{3.11}
\]
where \( M \) is independent of \( x, \lambda \). It follows from \( (3.5), (3.6), (3.7), (3.1), (3.10) \) that
\[
\lim_{|\lambda| \to \infty} |e(0, \lambda)| = 1. \tag{3.12}
\]
Since, by Lemma 3.1, \( e(0, \lambda) \neq 0 \) for any \( \lambda \in \mathbb{R} \), asymptotic formula \( (3.12) \) implies that there exists a constant \( \delta > 0 \) independent of \( \lambda \) such that
\[
|e(0, \lambda)| \geq \delta > 0 \quad (-\infty < \lambda < \infty). \tag{3.13}
\]

Let us introduce the notation:
\[
\Omega = \{(x, y) \mid 0 < x < \infty, -\infty < y < \infty\}.
\]

In view of \( (3.11) \) and \( (3.13) \), the integral in the right-hand side of (1.6) converges for each \( (x, y) \in \bar{\Omega} \), provided that \( f \) is written as \( (1.3) \) with \( \hat{f}(\lambda) \in L^1(\mathbb{R}) \).
Lemma 3.3 Let \( v(x) \) be a real-valued, continuous function in \( L^1(0, \infty) \) and let \( f(y) \) be a function expressed as (1.3) where \( \hat{f}(\lambda) \in L^1(\mathbb{R}) \). Then the Dirichlet problem (1.5) under condition (1.2) has a unique solution \( \phi \) in the space \( C^2(\Omega) \cap C(\overline{\Omega}) \). The solution \( \phi \) is expressed in terms of a solution \( e(x, \lambda) \) of (1.7) with behavior (1.8).

Proof. We first show that \( \phi \) defined by (1.6) satisfies (1.5). It follows from (3.11), (3.13), (1.7) and the assumption \( \hat{f}(\lambda) \in L^1(\mathbb{R}) \) that, for \( x > 0 \),

\[
\phi_{x}(x, y) = \int_{-\infty}^{\infty} \hat{f}(\lambda) \frac{e^{i\lambda x}}{e(0, \lambda)} e^{i\lambda y} d\lambda
\]

\[
= \lambda^2 \int_{-\infty}^{\infty} \hat{f}(\lambda) \frac{e(x, \lambda)}{e(0, \lambda)} e^{i\lambda y} d\lambda + i\lambda v(x) \int_{-\infty}^{\infty} \hat{f}(\lambda) \frac{e(x, \lambda)}{e(0, \lambda)} e^{i\lambda y} d\lambda
\]

\[
= -\phi_{yy}(x, y) + v(x) \phi_y(x, y).
\]

Hence \( \phi \) satisfies the equation \(-\Delta \phi + v(x) \frac{\partial \phi}{\partial y} = 0 \) in \( \Omega \). The estimate \(|e(x, \lambda)| \leq M e^{-|\lambda|x} \) and the assumption \( \hat{f}(\lambda) \in L^1(\mathbb{R}) \) enables us to apply Lebesgue’s convergence theorem for (1.6) and to conclude that \( \phi(x, y) \) tends to \( f(y) \) as \( x \to 0 \) for each \( y \in \mathbb{R} \). Thus \( \phi \) defined by (1.6) satisfies (1.5).

We next show that the function \( \phi \) satisfies condition (1.2). It follows from (3.11), (3.13) and the assumption \( \hat{f}(\lambda) \in L^1(\mathbb{R}) \) that, for \( x \geq 0 \),

\[
|\phi(x, y)| \leq \frac{M}{\delta} \int_{-\infty}^{\infty} |\hat{f}(\lambda)| e^{-|\lambda| x} d\lambda.
\]

Hence, by Lebesgue’s convergence theorem, we obtain

\[
\lim_{x \to \infty} |\phi(x, y)| = 0 \tag{3.14}
\]

uniformly with respect to \( y \in \mathbb{R} \). To investigate an asymptotic behavior as \(|y| \to \infty\), we require the identity

\[
e_+(x, \lambda) - e_+ \left( x, \lambda + \frac{\pi}{y} \right) = (1 + K(x, t)) e^{-\lambda x} \left( 1 - e^{-\pi \frac{t}{y}} \right) + \int_{x}^{\infty} K_t(x, t) e^{-\lambda t} \left( 1 - e^{-\pi \frac{t}{y}} \right) dt
\]

that follows from (2.12). A similar identity is obtained for \( e_+(x, \lambda) \) by (3.1). From these identities and (3.6), we have, for \( y \neq 0 \),

\[
|e(x, \lambda) - e\left( x, \lambda + \frac{\pi}{y} \right)| \leq e^{-|\lambda| x} \left| 1 - e^{-\pi \frac{t}{y}} \right| + \int_{x}^{\infty} |K_t(x, t)| e^{-|\lambda| t} \left| 1 - e^{-\pi \frac{t}{y}} \right| dt, \tag{3.15}
\]

provided that \( \lambda \) and \( \lambda + \frac{\pi}{y} \) have the same signature. Remembering a proof of the Riemann-Lebesgue lemma (see e.g., Yosida [19]) we have

\[
|\phi(x, y)| \leq \frac{1}{2} \int_{-\infty}^{\infty} \left| \hat{f}(\lambda) \frac{e(x, \lambda)}{e(0, \lambda)} - \hat{f} \left( \lambda + \frac{\pi}{y} \right) \frac{e(0, \lambda + \frac{\pi}{y})}{e(0, \lambda + \frac{\pi}{y})} \right| d\lambda
\]

\[
\leq \frac{1}{2} \int_{-\infty}^{\infty} \left| \hat{f}(\lambda) e(x, \lambda) \frac{e(0, \lambda + \frac{\pi}{y}) - e(0, \lambda) e(0, \lambda + \frac{\pi}{y})}{e(0, \lambda) e(0, \lambda + \frac{\pi}{y})} \right| d\lambda
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} \left| \hat{f}(\lambda) e(x, \lambda) \frac{e(0, \lambda) - e(0, \lambda + \frac{\pi}{y})}{e(0, \lambda + \frac{\pi}{y})} \right| d\lambda
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} \left| \hat{f}(\lambda) - \hat{f}(\lambda + \frac{\pi}{y}) \right| \frac{e(x, \lambda + \frac{\pi}{y})}{e(0, \lambda + \frac{\pi}{y})} d\lambda. \tag{3.16}
\]
It follows from (3.11), (3.13), and (3.15) that the first term in (3.16) converges to 0 as \(|y| \to \infty\) uniformly with respect to \(x\) in \([0, \infty)\). By (3.13), (3.15), and Fubini’s theorem, the second term in (3.16) is estimated as

\[
\int_{-\infty}^{\infty} \left| \frac{\hat{f}(\lambda)}{e(0, \lambda + \frac{\pi}{2})} - \frac{e(x, \lambda)}{e(0, \lambda + \frac{\pi}{2})} \right| d\lambda
\]

\[
\leq \frac{1}{\delta} \int_{|\lambda| \leq \frac{\pi}{2\delta}} |\hat{f}(\lambda)| \left| \frac{e(x, \lambda)}{e(0, \lambda + \frac{\pi}{2})} \right| d\lambda
\]

\[
+ \frac{1}{\delta} \int_{|\lambda| \geq \frac{\pi}{2\delta}} |\hat{f}(\lambda)| \left| e^{-|\lambda|t} \right| d\lambda
\]

\[
+ \frac{1}{\delta} \int_{|\lambda| \geq \frac{\pi}{2\delta}} |K_t(x, t)| \left| e^{-|\lambda|t} \right| d\lambda
\]

Since

\[
\lim_{|y| \to \infty} \left| 1 - e^{\pi \frac{y}{t}} \right| = 0
\]

uniformly with respect to \(x\) on every compact set in \([0, \infty)\) and

\[
\lim_{t \to \infty} \int_{|\lambda| \geq \frac{\pi}{2\delta}} |\hat{f}(\lambda)| e^{-|\lambda|t} d\lambda = 0,
\]

it turns out by (2.7) that the second term in (3.16) converges to 0 as \(|y| \to \infty\) uniformly with respect to \(x\) on every compact set in \([0, \infty)\). In view of (3.11), (3.13), and Lebesgue’s lemma (i.e., \(\int_{-\infty}^{\infty} h(r + \varepsilon) - h(r) dr \to 0\) as \(\varepsilon \to 0\) for all \(h \in L^1(R)\)), the third term in (3.16) converges to 0 as \(|y| \to \infty\) uniformly with respect to \(x\). Thus we conclude that

\[
\lim_{|y| \to \infty} \phi(x, y) = 0
\]

uniformly with respect to \(x\) on every compact set in \([0, \infty)\). This, together with (3.14), proves that \(\phi\) satisfies condition (1.2).

We finally show that the Dirichlet problem (1.5) under condition (1.2) has at most one solution. Our task is to show that if a function \(\phi\) in the space \(C^2(\Omega) \cap C(\Omega)\) satisfies

\[
\begin{align*}
-\Delta \phi + v(x) \frac{\partial \phi}{\partial y} &= 0 \quad (0 < x < \infty, -\infty < y < \infty), \\
\phi(0, y) &= 0 \quad (-\infty < y < \infty),
\end{align*}
\]

and condition (1.2) then \(\phi \equiv 0\). We set \(\psi(x, y) = (2 - e^{-x})^{-1} \phi(x, y)\). Then an elementary calculation shows that \(\psi(x, y)\) satisfies

\[
\psi_{xx} + \psi_{yy} + 2e^{-x} (2 - e^{-x})^{-1} \psi_x - v(x) \psi_y = e^{-x} (2 - e^{-x})^{-1} \psi.
\]

Moreover \(\psi(x, y)\) satisfies condition (1.2), i.e.,

\[
\lim_{x+|y| \to \infty} \psi(x, y) = 0.
\]

We shall show that \(\phi \equiv 0\) in a standard method (the so-called weak maximum principle). If \(\phi\) were not zero identically then, by definition, \(\psi\) would not be zero identically, and therefore, by (3.19) and \(\psi(0, y) = 0\) for any \(y \in R\), \(\psi\) would achieve a positive maximum or a negative minimum in \(\Omega\). If \(\phi\) achieves a positive maximum at a point \(P \in \Omega\) then

\[
\psi_x(P) = \psi_y(P) = 0; \quad \psi_{xx}(P) \leq 0, \quad \psi_{yy}(P) \leq 0.
\]
But this is impossible because, in view of (3.18), \( \psi(P) \leq 0 \) at such point \( P \). Similarly, \( \psi \) cannot achieve a negative minimum in \( \Omega \). Thus we conclude that if \( \phi \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfies (3.17) and (1.2) then \( \phi \equiv 0 \). The proof of the lemma is complete. \( \square \)

Lemma 3.3 leads to the following lemma, which asserts that (1.9) gives a necessary and sufficient condition for a function \( v(x) \) in \( L^1(0, \infty) \) to be a solution of Problem 1.1, in other words, to be a function with which (1.1) is solvable.

**Lemma 3.4** Let \( f(y) \) and \( g(y) \) be given functions expressed as (1.3) with (1.4). Then, elliptic system (1.1) with a real-valued, continuous function \( v(x) \) in \( L^1(0, \infty) \) admits a solution \( \phi \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfying (1.2) if and only if the solution \( e(x, \lambda) \) defined by (3.10) of equation (1.7) with the function \( v(x) \) satisfies relation (1.9).

**Proof.** In view of Lemma 3.3, if (1.1) has a solution \( \phi \) then \( \phi \) is written as (1.6). By the assumption \( \lambda \hat{f}(\lambda) \in L^1(R) \) and (3.11), for each \( y \in R \), the function \( \phi(x, y) \) has the limit

\[
\lim_{x \to 0} -\phi_x(x, y) = -\int_{-\infty}^{\infty} |\lambda| \hat{f}(\lambda) \frac{e'(0, \lambda)}{|\lambda| e(0, \lambda)} e^{i\lambda y} d\lambda \quad (-\infty < y < \infty).
\]

Hence the condition \( \phi_x(0, y) = g(y) \) yields

\[
\int_{-\infty}^{\infty} -|\lambda| \hat{f}(\lambda) \frac{e'(0, \lambda)}{|\lambda| e(0, \lambda)} e^{i\lambda y} d\lambda = g(y) = \int_{-\infty}^{\infty} \hat{g}(\lambda) e^{i\lambda y} d\lambda \quad (-\infty < y < \infty).
\]

By the uniqueness theorem of the Fourier transform, this implies (1.9). Conversely, if (1.9) holds then \( \phi \) satisfies \( \phi_x(0, y) = g(y) \) and therefore, is a solution of (1.1) under (1.2). \( \square \)

**Remark 3.5** From (2.4) we have

\[
-\frac{e'(0, \lambda)}{\lambda} = 1 + K(0, 0) - \int_0^\infty K_x(0, t) e^{-\lambda t} dt \quad (\text{Re } \lambda \geq 0),
\]

where \( K_x(0, t) \in L^1(0, \infty) \). In view of (3.4) and (3.3), the function \( -\frac{e'(0, \lambda)}{\lambda} \) does not vanish on the imaginary axis. Therefore, by the exactly same argument as in the proof of Lemma 3.1, we can find that \( -\frac{e'(0, \lambda)}{\lambda} \neq 0 \) for any \( \lambda \) in \( \text{Re } \lambda \geq 0 \), and hence that

\[
\frac{e'(0, \lambda)}{|\lambda|} \neq 0 \quad (-\infty < \lambda < \infty).
\]

This, combined with Lemma 3.4, implies that the condition \( |\lambda| \hat{f}(\lambda) = 0 \) is equivalent to saying \( \hat{g}(\lambda) = 0 \) if \( v(x) \) is a solution of Problem 1.1. In other words,

\[
\{ \lambda \in R \mid \hat{g}(\lambda) = 0 \} = \{ \lambda \in R \mid \hat{f}(\lambda) = 0 \}
\]

(except for a set of Lebesgue measure zero) is a necessary condition for \( f, g \) to be functions for which Problem 1.1 has a solution \( v(x) \in L^1(0, \infty) \).

We consider the case where \( f(y) \) is not zero identically. Then the set

\[
Q := \{ \lambda \in R \mid |\lambda| \hat{f}(\lambda) \neq 0 \}
\]

has a positive Lebesgue measure. The following lemma asserts that if \( f(y) \) is not zero identically then Problem 1.1 is equivalent to Problem 1.3.
Lemma 3.6 Let \( f(y) (f(y) \neq 0) \) and \( g(y) \) be given functions expressed as (1.3) with (1.4) and set \( \rho(\lambda) = \frac{\dot{g}(\lambda)}{\lambda f(\lambda)} \) for \( \lambda \in Q \). Then \( v(x) \in L^1(0, \infty) \) is a solution of Problem 1.1 if and only if the solution \( e(x, \lambda) \) defined by (3.10) of equation (1.7) with the function \( v(x) \) satisfies relation (1.10).

Proof. By Remark 3.5, we can assume (3.21). If \( v(x) \in L^1(0, \infty) \) is a solution of Problem 1.1 then, by Lemma 3.4, \( e(x, \lambda) \) satisfies (1.9), and therefore, (1.10). Hence, if (1.10) holds the we obtain (1.9) because the equality holds even for the complement of \( Q \) as zeros in both sides. □

In view of Lemma 3.6, under condition \( f(y) \neq 0 \) or/and \( g(y) \neq 0 \), Problem 1.1 is reduced to Problem 1.3. We confine ourselves to the non-exceptional case, and hereafter focus our attention on Problem 1.3. We say that a function \( v(x) \in L^1(0, \infty) \) is a solution of Problem 1.3 for \( \rho(\lambda) \) if \( v(x) \) is a real-valued, continuous function on \([0, \infty)\) and if the solution \( e(x, \lambda) \) defined as (3.10) of (1.7) with \( v(x) \) satisfies (1.10) for \( \rho(\lambda) \).

The first task in our approach to the problem is to get data on the imaginary axis from the data \( \rho(\lambda) \) on the real line:

Theorem 3.7 Let \( \rho(\lambda) \) be a (given) complex-valued, continuous function on a set \( \Lambda \in \mathbb{R} \) of a positive Lebesgue measure and suppose that Problem 1.3 has a solution \( v(x) \in L^1(0, \infty) \) for \( \rho(\lambda) \). Then, on the imaginary axis, the function \( \frac{e^\iota(0, \lambda)}{e^\iota(0, \lambda)} \) can be determined uniquely from \( \rho(\lambda) \), and it is represented as (1.15) in terms of a complex-valued function \( F(t) \in L^1(\mathbb{R}) \) and a constant \( C \) with absolute value 1. The function \( F(t) \) is uniquely determined from \( \rho(\lambda) \) and the constant \( C \) is determined from \( F(t) \) as \( C = 1 - \int_{-\infty}^{\infty} F(t)dt \).

Proof. Since the Lebesgue measure of \( \Lambda \) is positive, \( \Lambda \) has (at least) one accumulation point other than 0. We assume that there is a positive accumulation point. If there are only nonpositive accumulation points then we shall employ \( e_-(x, \lambda) \) in place of \( e_+(x, \lambda) \) in the subsequence discussions.

We proceed in three steps.

Step 1. From (3.5) and (3.20), we have

\[
- \frac{e'_+(0, \lambda)}{\lambda e_+(0, \lambda)} = \frac{1}{1 + K(0, 0) + \int_0^\infty K_t(0, t)e^{-\lambda t}dt} - \frac{1}{1 + K(0, 0) + \int_0^\infty K_t(0, t)e^{-\lambda t}dt} \quad (\text{Re} \lambda \geq 0).
\]

Since \( e_+(0, \lambda) \neq 0 \) for \( \text{Re} \lambda \geq 0 \) by Lemma 3.1, with the aid of the Paley-Wiener theorem (see Paley and Wiener [16, Theorem XVIII]), there exists a function \( F_0(t) \in L^1(0, \infty) \) such that

\[
\frac{1}{e_+(0, \lambda)} = \frac{1}{1 + K(0, 0) + \int_0^\infty K_t(0, t)e^{-\lambda t}dt} = \frac{1}{1 + K(0, 0)} + \int_0^{\infty} F_0(t)e^{-\lambda t}dt \quad (\text{Re} \lambda \geq 0).
\]

Therefore, by the convolution theorem, we obtain

\[
- \frac{e'_+(0, \lambda)}{\lambda e_+(0, \lambda)} = \left(1 + K(0, 0) - \int_0^{\infty} K_x(0, t)e^{-\lambda t}dt\right)\left(\frac{1}{1 + K(0, 0)} + \int_0^{\infty} F_0(t)e^{-\lambda t}dt\right) = 1 + \int_0^{\infty} F_1(t)e^{-\lambda t}dt \quad (\text{Re} \lambda \geq 0),
\]

where \( F_1(t) \) is a function in \( L^1(0, \infty) \) defined by

\[
F_1(t) := (1 + K(0, 0))F_0(t) - (1 + K(0, 0))^{-1}K_x(0, t) - \int_0^t K_x(0, t-s)F_0(s)ds.
\]
This yields
\[ 1 + \int_0^\infty F_1(t)e^{-\lambda t}dt = \rho(\lambda) \quad (\lambda \in Q \cap [0, \infty)). \] (3.24)

Since the left-hand side of this equality is a holomorphic function in Re $\lambda > 0$, the function $F_1(t)$ is determined uniquely from the data $\rho(\lambda)$ on $\lambda > 0$. Actually, if (3.24) holds for two functions $F_1(t), F_1(t) \in L^1(\mathbb{R})$ then we obtain
\[ \int_0^\infty (F_1(t) - F_2(t))e^{-\lambda t}dt = 0 \quad (\lambda \in Q \cap [0, \infty)). \]

But the left-hand side is a holomorphic function in Re $\lambda > 0$ extended continuously up to the imaginary axis as a continuous function and, by assumption, $Q \cap [0, \infty)$ has an accumulation point in Re $\lambda > 0$. Hence, by the uniqueness theorem for complex functions, we obtain
\[ \int_0^\infty (\bar{F}_1(t) - F_1(t))e^{-\lambda t}dt = 0 \quad (Re \lambda \geq 0). \]

This, with the aid of the uniqueness theorem for the Fourier transform, leads to $\bar{F}_1 - F_1 = 0$. To sum up, the data $\rho(\lambda)$ on $Q \cap [0, \infty)$ is continued analytically in Re $\lambda > 0$ and continuously up to the imaginary axis Re $\lambda = 0$ through the function $F_1(t)$.

Step 2. From (3.3) and (3.4), we have
\[
Re - \frac{e_+^\prime(0, \lambda)}{\lambda e_+(0, \lambda)} = \frac{1}{2} \left\{ - \frac{e_+^\prime(0, \lambda)}{\lambda e_+(0, \lambda)} - \frac{\overline{e_+^\prime(0, \lambda)}}{\lambda \overline{e_+(0, \lambda)}} \right\} = \frac{1}{2} \left\{ - \frac{e_+^\prime(0, \lambda)}{\lambda e_+(0, \lambda)} + \frac{e_-^\prime(0, \lambda)}{\lambda e_-(0, \lambda)} \right\} = \frac{1}{2\lambda} \frac{W[e_+(x, \lambda), e_-(x, \lambda)]}{e_-(0, \lambda)e_+(0, \lambda)} = \frac{1}{|e_+(0, \lambda)|^2} \quad (Re \lambda = 0).
\]

This, together with (3.23), yields
\[
\frac{1}{|e_+(0, \lambda)|^2} = Re - \frac{e_+^\prime(0, \lambda)}{\lambda e_+(0, \lambda)} = 1 + \frac{1}{2} \int_0^\infty F_1(t)e^{-\lambda t}dt + \frac{1}{2} \int_{-\infty}^0 \overline{F_1(-t)}e^{-\lambda t}dt \quad (Re \lambda = 0).
\]

Accordingly, by setting
\[
F_2(t) = \begin{cases} 
\frac{1}{2} F_1(t) & \text{for } t \geq 0, \\
\frac{1}{2} \overline{F_1(-t)} & \text{for } t < 0,
\end{cases} \quad (3.25)
\]

we find that $|e_+(0, \lambda)|^2$ is expressed as
\[
|e_+(0, \lambda)|^2 = \frac{1}{1 + \int_{-\infty}^\infty F_2(t)e^{-\lambda t}dt} \quad (Re \lambda = 0)
\]

in terms of a function $F_2(t) \in L^1(\mathbb{R})$.

Step 3. By (3.3) and the Wiener-Lévy theorem (see e.g., Kreǐn [11, Theorem W]), we can rewrite (3.26) as
\[
e_+(0, \lambda)e_-(-0, \lambda) = 1 + \int_{-\infty}^\infty G(t)e^{-\lambda t}dt \quad (Re \lambda = 0),
\]

where $G(t)$ is a function in $L^1(\mathbb{R})$ uniquely determined from $F_2(t)$. It follows from Theorem 3.2 and (3.3) that $\text{ind } (e_+(0, \lambda)e_-(-0, \lambda)) = 0$. Therefore, by the Wiener-Hopf factorization (see e.g., Kreǐn [11, Theorem 2.1]), the function $e_+(0, \lambda)e_-(-0, \lambda)$ can be decomposed as
\[
e_+(0, \lambda)e_-(-0, \lambda) = \left(1 + \int_0^\infty G_+(t)e^{-\lambda t}dt \right) \left(1 + \int_{-\infty}^0 G_-(t)e^{-\lambda t}dt \right) \quad (Re \lambda = 0), \quad (3.27)
\]
where \(G_+(t) \in L^1(0, \infty), G_-(t) \in L^1(-\infty, 0),\) and moreover the functions \(1 + \int_0^\infty G_+(t)e^{-\lambda t}dt\) and \(1 + \int_{-\infty}^0 G_-(t)e^{-\lambda t}dt\) have no zeros in \(\text{Re} \geq 0\) and in \(\text{Re} \leq 0,\) respectively. Decomposition (3.27) is unique and hence the pair of \(G_+(t), G_-(t)\) is uniquely determined from \(G(t).\) By (3.27) we obtain
\[
\frac{e_+(0, \lambda)}{1 + \int_0^\infty G_+(t)e^{-\lambda t}dt} = \frac{1 + \int_{-\infty}^0 G_-(t)e^{-\lambda t}dt}{e_-(0, \lambda)} (\text{Re } \lambda = 0).
\]

The function on the left-hand (right-hand) side of this identity is holomorphic and bounded in the right (respectively, the left) half plane, and is extended continuously up to the imaginary axis. Hence, by Morera’s theorem, it has a continuation to the whole complex plane as an entire, bounded function. In view of Liouville’s theorem, this implies that the function must be a constant, which we denote by \(C_0.\) We thus have
\[
e_+(0, \lambda) = C_0 \left(1 + \int_0^\infty G_+(t)e^{-\lambda t}dt\right), \quad e_-(0, \lambda) = C_0^{-1} \left(1 + \int_{-\infty}^0 G_-(t)e^{-\lambda t}dt\right).
\]

By (3.12) it is clear that \(|C_0| = 1.\) Moreover, since \(1 = e_+(0, 0) = C_0 \left(1 + \int_0^\infty G_+(t)dt\right),\) the constant \(C_0\) is uniquely determined from \(G_+(t)\) as \(C_0^{-1} = 1 + \int_0^\infty G_+(t)dt.\) From these expressions, with the aid of Wiener-Lévy theorem and the convolution theorem, we arrive at
\[
\frac{e_-(0, \lambda)}{e_+(0, \lambda)} = C_0^{-2} \frac{1 + \int_0^\infty G_-(t)e^{-\lambda t}dt}{1 + \int_0^\infty G_+(t)e^{-\lambda t}dt}
\]
\[
= C_0^{-2} + \int_{-\infty}^\infty F(t)e^{\lambda t}dt \quad (\text{Re } \lambda = 0),
\]

where \(F(t) \in L^1(\mathbb{R}).\) Accordingly, by setting \(C = C_0^{-2},\) we find that \(\frac{e_-(0, \lambda)}{e_+(0, \lambda)}\) is expressed as (1.15). The function \(F(t)\) is uniquely determined from \(G(t).\)

In three steps, the functions \(F_1(t), F_2(t), F(t)\) are uniquely determined from \(\rho(\lambda), F_1(t), F_2(t),\) respectively (see Figure 4) and hence, \(\frac{e_-(0, \lambda)}{e_+(0, \lambda)}\) on the imaginary axis is uniquely determined from \(\rho(\lambda)\) on \(Q.\) The proof is complete.

\[
\begin{array}{c}
\frac{e_-(0, \lambda)}{e_+(0, \lambda)} (\text{Re } \lambda = 0) \\
\end{array}
\]

\[
\begin{array}{c}
F(t) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Step 3 (Wiener-Hopf)} \\
F_2(t) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Step 2 (Wronskian)} \\
F_1(t) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Step 1 (Paley-Wiener)} \\
v(x) \\
\end{array}
\]

\[
\begin{array}{c}
\rho(\lambda) (\lambda \in Q) \\
\end{array}
\]

Figure 4. Determination of the data \(\frac{e_-(0, \lambda)}{e_+(0, \lambda)}\)

The ingredients of the proof of Theorem 3.7 can be summarized as follows:
Corollary 3.8 Under the same assumption as in Theorem 3.7, the function \( \frac{e_{-}(0, \lambda)}{e_{+}(0, \lambda)} \) on the imaginary axis can be determined from \( \rho(\lambda) \) in the three steps:

1. The function \( -\frac{e'_{-}(0, \lambda)}{\lambda e_{+}(0, \lambda)} \) is expressed as (3.23) in terms of a function \( F_{1}(t) \in L^{1}(0, \infty) \) that is uniquely determined from \( \rho(\lambda) \).

2. The function \( |e_{+}(0, \lambda)|^{2} \) is expressed as (3.26) in terms of the function \( F_{2}(t) \) defined by (3.25).

3. By the Wiener-Hopf factorization, the function \( |e_{+}(0, \lambda)|^{2} \) is decomposed as (3.27) in a unique manner, and moreover the function \( \frac{e_{-}(0, \lambda)}{e_{+}(0, \lambda)} \) is written as (3.28) where \( C_{0}^{2} = (1 + \int_{0}^{\infty} G_{+}(t)dt)^{-2} \).

We conclude this section with a remark:

Remark 3.9 It is also possible to determine \( -\frac{e'_{-}(0, \lambda)}{\lambda e_{+}(0, \lambda)} \) for \( \lambda \geq 0 \) from \( \frac{e_{-}(0, \lambda)}{e_{+}(0, \lambda)} \) on \( \text{Re} \lambda = 0 \). Actually, we can obtain (3.26) with a function \( F_{2}(t) \) uniquely determined from \( F(t) \) in (1.15) by use of the Wiener-Hopf factorization and the Wiener-Lévy theorem, and then, \( F_{1}(t) \) in (3.23) is determined from \( F_{2}(t) \) as \( F_{1}(t) = 2F_{2}(t) \). As a consequent, the data \( \frac{e_{-}(0, \lambda)}{e_{+}(0, \lambda)} \) on \( \text{Re} \lambda = 0 \) is corresponding to \( -\frac{e'_{-}(0, \lambda)}{\lambda e_{+}(0, \lambda)} \) neither too much nor too less.

4. Modified Marchenko equation

The function \( F(t) \) in representation (1.15) can be related with the transformation kernel \( K(x, t) \) by modified Marchenko equation (1.16). The derivation of the equation is carried out in a similar manner to that in Marchenko [13, §3.2] and Jaulent and Jean [8, §5]. For the detail and more general statement, refer to Kaminura [9, Lemma 4.1].

Theorem 4.1 Suppose that \( v(x) \) is a real-valued function in \( L^{1}(0, \infty) \) and let \( e_{\pm}(x, \lambda) \) be the solutions of (1.7) defined by (2.1), (3.1). Then the transformation kernel \( K(x, t) \) defined in Theorem 2.2 and a function \( F(t) \in L^{1}(\mathbb{R}) \) defined by (1.15) satisfy (1.16).

The following assures that modified Marchenko equation (1.16) with \( F(t) \) in (1.15) can be uniquely solvable in a space of bounded, continuous functions:

Theorem 4.2 Let \( F \in L^{1}(\mathbb{R}) \) and suppose that a function \( S(\lambda) \) defined by

\[
S(\lambda) = C + \int_{-\infty}^{\infty} F(t)e^{\lambda t} dt \quad (\text{Re} \lambda = 0) \tag{4.1}
\]

with some complex number \( C \) satisfies

(S1) \( |S(\lambda)| = 1 \) for each \( \lambda \) on the imaginary axis;

(S2) \( \text{ind} S(\lambda) = 0 \).

Then equation (1.16) has a unique solution \( K(x, t) \) in \( BC[x, \infty) \) for each \( x \geq 0 \). Here \( BC[x, \infty) \) denotes a space of bounded, continuous complex-valued functions \( \varphi(t) \) defined on the interval \([x, \infty)\).

Notice that the absolute value of \( C \) is necessarily 1 by assumption (S1) and the Riemann-Lebesgue lemma. To prove Theorem 4.2, we introduce some notations: Let \( x \in [0, \infty) \) be fixed and set

\[
\varphi(t) = \overline{K(x, x+t)}, \quad f(t) = -\int_{0}^{\infty} F(s + t + 2x)ds.
\]
Then equation (1.16) is transformed to

$$\varphi(t) + \int_0^\infty F(s + t + 2x)\overline{\varphi(s)}ds = f(t) \quad (0 \leq t < \infty). \quad (4.2)$$

Let $L^\infty(0, \infty)$ be the space of Lebesgue measurable, essentially bounded functions defined almost everywhere on $(0, \infty)$ furnished with the norm $||\varphi||_\infty := \text{ess sup} |\varphi(t)|$.

The following is elementary:

**Lemma 4.3** If $\varphi \in L^\infty(0, \infty)$ satisfies (4.2) then $\varphi$ belongs to $BC[0, \infty)$.

**Proof.** By Lebesgue’s lemma, it follows that, for any $\varphi \in L^\infty(0, \infty),$

$$\left| \int_0^\infty F(s + t + \varepsilon + 2x)\overline{\varphi(s)}ds - \int_0^\infty F(s + t + 2x)\overline{\varphi(s)}ds \right| \leq \int_{-\infty}^{\infty} |F(\eta + \varepsilon) - F(\eta)| |\varphi| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This implies that the function $\int_0^\infty F(s + t + 2x)\overline{\varphi(s)}ds$ belongs to $BC[0, \infty)$. Since $f(t) = -\int_2^\infty F(\tau)d\tau$ belongs to $BC[0, \infty)$, the function

$$\varphi(t) = -\int_0^\infty F(s + t + 2x)\overline{\varphi(s)}ds + f(t)$$

belongs to $BC[0, \infty)$. Hence $\varphi$ belongs to $BC[0, \infty)$. □

By Lemma 4.3, for the proof of Theorem 4.2, it suffices to show that equation (4.2) has a unique solution in $L^\infty(0, \infty)$ for each $x \geq 0$. To do it, we define an operator $A_x : L^1(0, \infty) \to L^1(0, \infty)$ by

$$(A_x \psi)(t) = \int_0^\infty F(s + t + 2x)\overline{\varphi(s)}ds. \quad (4.3)$$

As is easily checked, $A_x$ is a bounded operator on $L^1(0, \infty)$.

In what follows we consider $L^\infty(0, \infty)$ and $L^1(0, \infty)$ as real linear spaces. Then $A_x$ becomes a bounded linear operator on $L^1(0, \infty)$. It follows from a well-known fact (see e.g. Yosida [19, Chapter IV-9]) that $L^1(0, \infty)^* = L^\infty(0, \infty)$ (in the sense that $L^1(0, \infty)^*$ is isomorphic to $L^\infty(0, \infty)$) by the correspondence

$L^\infty(0, \infty) \ni \varphi \mapsto < \psi, \varphi > := \text{Re} \int_0^\infty \psi(t)\overline{\varphi(t)}dt \in L^1(0, \infty)^*$ \quad ($\psi \in L^1(0, \infty)$).

By Fubini’s theorem, we have

$$< A_x \psi, \varphi > = \text{Re} \int_0^\infty \overline{\varphi(t)}dt \int_0^\infty F(s + t + 2x)\overline{\psi(s)}ds = <\psi, \int_0^\infty F(s + t + 2x)\overline{\varphi(s)}ds > .$$

Hence the dual operator $A_x^* : L^\infty(0, \infty) \to L^\infty(0, \infty)$ is given in the same form as in (4.3), and so, equation (4.2) is written as $(I + A_x^*)\varphi = f$. Moreover it is shown (see Marchenko [13, Lemma 3.3.1]) that the operator $A_x$ is a compact linear operator on $L^1(0, \infty)$ for each $x \geq 0$. Therefore, by Schauder’s theorem (see e.g. [19]), the dual operator $A_x^*$ is a compact linear operator on $L^\infty(0, \infty)$ for each $x \geq 0$ and, by the Riesz-Schauder theory, the operator $I + A_x^*$ in $L^\infty(0, \infty)$ is bijective if and only if the operator $I + A_x$ in $L^1(0, \infty)$ is injective. In this way, proving Theorem 4.2 is reduced to the following lemma that is due to Jaulent [6, §3] (see also Kamimura [9, Appendix B]).
Lemma 4.4 Under the same assumption for $F \in L^1(\mathbb{R})$ as in Theorem 4.2, the following holds for each $x \geq 0$: If $\psi \in L^1(0, \infty)$ satisfies $(I + A_x)\psi = 0$, i.e.,

$$
\psi(t) + \int_0^\infty F(s + t + 2x)\psi(s)ds = 0 \quad (0 < t < \infty),
$$

then $\psi = 0$.

We conclude this paper with:

**Proof of Theorem 1.4.** By Theorem 3.7 (see also Corollary 3.8) the function $S(\lambda) := \frac{e^{-i(0, \lambda)}}{e^{i(0, \lambda)}}$ on the imaginary axis can be determined from the data $\rho(\lambda)$ on the real line, and $S(\lambda)$ is represented in terms of a function $F(t) \in L^1(\mathbb{R})$ determined uniquely from $\rho(\lambda)$. In view of Theorem 4.1, the transformation kernel $K(x, t)$ associated with the function $v(x)$ satisfies (1.16) for the function $F(t)$. Since $\text{ind} e_+(0, \lambda) = 0$ by Theorem 3.2, we obtain $\text{ind} S(\lambda) = 0$. Therefore, by Theorem 4.2, equation (1.16) for the function $F(t)$ admits a unique solution in the space to that the transformation kernel belongs. This implies that the solution is no other than the transformation kernel $K(x, t)$ associated with the function $v(x)$. Accordingly, by (2.6) in Theorem 2.2, $v(x)$ is derived from the solution by (1.12). The proof is complete. $\square$

**References**

[1] Agranovich Z S and Marchenko V A 1963 *The Inverse Problem of Scattering Theory*, (New York: Gordon and Breach)

[2] Apel J R 1987 *Principles of Ocean Physics* International Geophysics Series 38 (London: Academic Press)

[3] Bennett A F 1992 *Inverse Methods in Physical Oceanography* (Cambridge: Cambridge University Press)

[4] Chadan K and Sabatier P C 1989 *Inverse Problems in Quantum Scattering Theory* 2nd edition (New York: Springer)

[5] Isakov V 1998 *Inverse Problems for Partial Differential Equations* 2nd edition (New York: Springer)

[6] Jaulent M 1972 On an inverse scattering problem with an energy-dependent potential *Ann. Inst. Henri Poincaré* 17 363–78

[7] Jaulent M 1975 Sur le problème inverse de la diffusion pour l’équation de Schrödinger radiale avec un potentiel dépendant de l’énergie *C. R. Acad. Sc. Paris* 280 1467–70.

[8] Jaulent M and Jean C 1972 The inverse s-wave scattering problem for a class of potentials depending on energy *Commun. Math. Phys.* 28 177–220.

[9] Kaminoura Y An inversion formula in energy dependent scattering *J. Int. Eqs. Appl.* to appear

[10] Klibanov M V and Timonov A 2005 Global uniqueness for a 3D/2D inverse conductivity problem via the modified method of Carleman estimates *J. Inv. Ill-posed problems* 13 149–74

[11] Kreǐn M G 1960 Integral equations on a half-line with kernel depending upon the difference of the arguments *Amer. Math. Soc. Transl.* (2) 22 163–288

[12] Lavrentiev M M, Romanov V G and Shishatskii S P 1986 *Ill-posed Problems of Mathematical Physics and Analysis* (Providence: Amer. Math. Soc.)

[13] Marchenko V A 1986 *Sturm-Liouville Operators and Applications* (Basel: Birkhäuser)

[14] Megrabov A G 2003 *Forward and Inverse Problem for Hyperbolic, Elliptic, and Mixed Type Equations* (Utrecht: VSP)

[15] Newton R 1989 *Inverse Schrödinger Scattering in Three Dimensions* (Berlin: Springer)

[16] Paley R and Wiener N 1934 *Fourier Transforms in the Complex Domain* (Providence: Amer. Math. Soc.)

[17] Ramm A G 1992 *Multidimensional Inverse Scattering Problems* (Essex: Longman)

[18] Romanov V G 1987 *Inverse Problems of Mathematical Physics* (Utrecht: VNU Science Press)

[19] Yosida K 1971 *Functional Analysis* 3rd edition (Berlin: Springer)

[20] Uhlmann G 1999 Developments in inverse problems since Caaderno’s Foundational paper *Harmonic Analysis and Partial Differential Equations* 295–345 (Chicago: University of Chicago Press)

[21] Wunsch C 1996 *The Ocean Circulation Inverse Problem* (Cambridge: Cambridge University Press)