Abstract. In this paper we describe the geometry of distributions by their symmetries, and present a simplified proof of the Frobenius theorem and some related corollaries. Then, we study the geometry of solutions of $F$–Gordon equation; A PDE which appears in differential geometry and relativistic field theory.

1. Introduction

We begin this paper with the geometry of distributions. The main idea here is the various notions of symmetry and their use in solving a given differential equation. In section 2, we introduce the basic notions and definitions.

In section 3, we describe the relation between differential equations and distributions. In section 4, we present the geometry of distributions by their symmetries, and find out the symmetries of $F$–Gordon equation by this machinery. In section 5, we introduce a simplified proof of the Frobenius theorem and some related corollaries. In section 6, we describe the relations between symmetries and solutions of a distribution.

In all steps, we study the $F$–Gordon equation as an application; a partial differential equation which appears in differential geometry and relativistic field theory. It is a generalized form the Klein-Gordon equation "$u_{tt} - u_{xx} + u = 0$". It is a relativistic version of Schrodinger equation, which is used to describe spinless particles. It was named after Walter Gordon and Oskar Klein [BEM, KRA].

Mathematics Subject Classification: 35Q40, 35Q80, 53D10.
Key words and phrases: Distribution, Lie symmetry, Contact geometry, Klein-Gordon equation.
2. Tangent and Cotangent Distribution

Let $M$ be an $(m + n)$-dimensional smooth manifold.

**Definition 1.** A map $D : M \to TM$ is called an $m$-dimensional tangent-distribution on $M$, or briefly $\text{Tan}^m$-distribution, if

$$D_x := D(x) \subseteq T_x M \quad (x \in M)$$

is an $m$-dimensional subspace of $T_x M$. The smoothness of $D$ means that:
For each $x \in M$, there exists an open neighborhood $U$ of $x$; and smooth vector fields $X_1, \ldots, X_m$ such that:

$$D_y = \langle X_1(y), \ldots, X_m(y) \rangle$$

$$:= \text{span}_\mathbb{R} \{X_1(y), \ldots, X_m(y)\} \quad (y \in U)$$

**Definition 2.** A map $D : M \to T^*_x M$ is called an $n$-dimension cotangent-distribution on $M$, or briefly $\text{Cot}^n$-distribution, if

$$D_x := D(x) \subseteq T^*_x M \quad (x \in M)$$

is an $n$-dimensional subspace of $T^*_x M$. The smoothness of $D$ means that:
For each $x \in M$, there exists an open neighborhood $U$ of $x$ and smooth 1-forms $\omega_1, \ldots, \omega_n$ such that:

$$D_y = \langle \omega_1(y), \ldots, \omega_n(y) \rangle$$

$$:= \text{span}_\mathbb{R} \{\omega_1(y), \ldots, \omega_n(y)\} \quad (y \in U)$$

In the sequel, without loss of generality, we can assume these definitions are globally satisfied.

There is a correspondence between these two types of distributions. For $\text{Tan}^m$-distribution $D$, there exist nowhere zero smooth vector fields $X_1, \ldots, X_m$ on $M$ such that: $D = \langle X_1, \ldots, X_m \rangle$; and similarly, for $\text{Cot}^n$-distribution $D$, there exist global smooth 1-forms $\omega_1, \ldots, \omega_n$ on $M$ such that $D = \langle \omega_1, \ldots, \omega_n \rangle$.

**Example 1.** (Cartan distribution) Let $M = \mathbb{R}^{k+1}$. Denote the coordinates in $M$ by $x, p_0, p_1, \ldots, p_k$ and given a function $f(x, p_0, \ldots, p_{k-1})$ consider the following differential 1-forms:

$$\omega^0 = dp_0 - p_1 \, dx, \quad \omega^2 = dp_1 - p_2 \, dx, \quad \cdots$$

$$\omega^{k-2} = dp_{k-2} - p_{k-1} \, dx, \quad \omega^{k-1} = dp_{k-1} - f(x, p_0, \ldots, p_{k-1}) \, dx,$$
and the distribution \( D = \langle \omega^0, \cdots, \omega^{k-1} \rangle \). This is the 1-dimensional distribution, called the Cartan distribution. This distribution can also be described by a single vector field \( X, D = \langle X \rangle \), where

\[
X = \partial_x + p_1 \partial_{p_0} + p_2 \partial_{p_1} + \cdots + p_{k-1} \partial_{p_{k-2}} + f(x, p_0, \cdots, p_{k-1}) \partial_{p_{k-1}}.
\]

**Example 2. (F–Gordon equation)** Let \( F : \mathbb{R}^5 \to \mathbb{R} \) be a differentiable function. The corresponding F–Gordon PDE is \( u_{xy} = F(x, y, u, u_x, u_y) \). We construct a \( 7 \)-dimensional sub-manifold \( M \) defined by

\[
J^2(\mathbb{R}^2, \mathbb{R}) = \{x, y, u, p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy}\}.
\]

Consider the 1-forms

\[
\omega^1 = du - p \, dx - q \, dy, \quad \omega^2 = dp - r \, dx - F \, dy, \quad \omega^3 = dq - F \, dx - t \, dy.
\]

This distribution can also be described by the following vector fields

\[
X_1 = \partial_x + p \partial_u + r \partial_p + F \partial_q, \quad X_2 = \partial_y + q \partial_u + F \partial_p + t \partial_q, \quad X_3 = \partial_r, \quad X_4 = \partial_t.
\]

**Definition 3.** Let \( D : M \to TM \) be a \( \text{Tan}^n \)-distribution and set

\[
\text{Ann} \, D_x := \{ \omega_x \in T^*_x M \mid \omega_x|_{D_x} = 0 \}.
\]

It is clear that \( \dim \text{Ann} \, D_x = n \). An 1-form \( \omega \in \Omega^1(M) \) annihilates \( D \) on a subset \( N \subset M \), if and only if \( \omega_x \in \text{Ann} \, D_x \) for all \( x \in M \).

The set of all differential 1-forms on \( M \) which annihilates \( D \), is called annihilator of \( D \) and denoted by \( \text{Ann} \, D \).

Therefore, for each \( \text{Tan}^n \)-distribution

\[
D : M \to TM, \quad D : x \mapsto D_x,
\]

we can construct a \( \text{Cot}^n \)-distribution

\[
D : M \to T^* M, \quad D : x \mapsto D_x = \text{Ann} \, D_x;
\]

and vice versa; In the other words, for each \( \text{Tan}^n \)-distribution \( D = \langle X_1, \cdots, X_m \rangle \),
we can construct a \( \text{Cot}^n \)-distribution \( D = \text{Ann} \, D = \langle \omega^1, \cdots, \omega^n \rangle \), and vice versa.

**Theorem 1.** a) \( D \) and its annihilator, are modules over \( C^\infty(M) \).

b) Let \( X \) be a smooth vector field on \( M \) and \( \omega \in \text{Ann} \, D \), then

\[
L_X \omega \equiv -\omega \circ L_X \mod D
\]

**Proof:** (a) is clear, and for (b), if \( Y \) belongs to \( D \), then \( \omega(Y) = 0 \), and

\[
(L_X \omega)Y = X.(\omega(Y)) - \omega[X, Y] = -\omega[X, Y] = -(\omega \circ L_X)Y.
\]
3. Integral Manifolds, Maximal Integral Manifolds.

Definition 4. Let $D$ be a distribution. A bijective immersed sub-manifold $N \subset M$, is called an integral manifold of $D$ if one of the following equivalence conditions is satisfied:
1) $T_xN \subseteq D_x$, for all $x \in N$.
2) $N \subseteq \bigcap_{i=1}^{n} \ker \omega^i$.

$N \subset M$ is called maximal integral manifold if for each $x \in N$, there exists an open neighborhood $U$ of $x$ such that there is no integral manifold $N'$ containing $N \cap U$.

It is clear that, the dimension of maximal integral manifold does not exceed, the dimension of the distribution.

Definition 5. $D$ is called a completely integrable distribution, or briefly CID, if for all maximal integral manifold $N$, one of the following equivalence conditions is satisfied:
1) $\dim N = \dim D$.
2) $T_xN = D_x$ for all $x \in N$.
3) $N \subseteq \bigcap_{i=1}^{n} \ker \omega^i$, and if $N'$ be an integral manifold with $N \cap N' \neq \emptyset$, then $N' \subseteq N$.

In the sequel, the set of all maximal integral manifolds is denoted by $N$.

Theorem 2. $N = \bigcap_{i=1}^{n} \ker \omega^i$; that is $\omega^i |_{N} = 0$ for $i = 1, \ldots, n$.

Example 3. (Continuation of Example 1) If $N$ is an integral curve of the distribution then $x$ can be chosen as a coordinate on $N$, and therefore

$N = \{(x, h_0(x), h_1(x), \ldots, h_{k-1}(x)) | x \in \mathbb{R}\}$.

Conditions $\omega^0|_{N} = 0, \ldots, \omega^{k-1}|_{N} = 0$ imply that $h_1 = h'_0, h_2 = h'_1, \ldots, h_{k-1} = h'_{k-1}$ or that

$N = J^{k-1}h = \{(x, h(x), h'(x), \ldots, h^{(k-1)}(x) | x \in \mathbb{R}\}$

for some function $h : \mathbb{R} \to \mathbb{R}$.

The last equation $\omega^{k-1}|_{N} = 0$ gives us an ordinary differential equation

$h^{(k)}(x) = f(x, h(x), h'(x), \ldots, h^{(k-1)}(x))$.

The existence theorem shows us once more that the integral curves do exist, and therefore the Cartan distribution is a CID.
Example 4. (Continuation of Example 2) This distribution is not CID, because there is no 4-dimensional integral manifold, and \( \dim \mathbf{D} = 4 \). For, if \( N \) be a 4-dimensional integral manifold of the distribution, then \( (x, y, u, p) \) can be chosen as coordinates on \( N \), and therefore

\[
N : \begin{cases} 
q = h(x, y, u, p), \\
r = l(x, y, u, p), \\
t = m(x, y, u, p), \\
s = F(x, y, u, p, h).
\end{cases}
\]

Condition \( \omega^1|_N = 0 \) imply that \(-p \, dx - h(x, y, u, p) \, dy + du = 0\), which is impossible.

By the same reason, we conclude that there is not any 3-dimensional integral manifold.

Now, if \( N \) be a 2-dimensional integral manifold of the distribution, then \( (x, y) \) can be chosen as coordinates on \( N \), and therefore

\[
N : \begin{cases} 
u = h(x, y), \\
p = l(x, y), \\
r = m(x, y), \\
t = a(x, y), \\
s = F(x, y, u, p, q).
\end{cases}
\]

Conditions \( \omega^1|_N = 0 \) and \( \omega^2|_N = 0 \) imply that \( l = h_x, m = h_y, n = l_x = h_{xx}, \) and \( a = m_y = h_{yy} \).

The last equation \( \omega^3|_N = 0 \) implies that \( h_{xy} = F(x, y, h, h_x, h_y) \). This distribution is not CID.

4. Symmetries

In this section, we consider a distribution \( \mathbf{D} = \langle X_1, \ldots, X_m \rangle = \langle \omega^1, \ldots, \omega^n \rangle \) on manifold \( M^{n+m} \).

**Definition 6.** A diffeomorphism \( F : M \to M \) is called a symmetry of \( \mathbf{D} \) if \( F_\ast \mathbf{D}_x = \mathbf{D}_{F(x)} \) for all \( x \in M \).

Therefore,

**Theorem 3.** The following conditions are equivalent:

1) \( F \) is a symmetry of \( \mathbf{D} \);
2) \( F_\ast \omega^i \)'s determine the same distribution \( \mathbf{D} \); that is \( \mathbf{D} = \langle F_\ast \omega^1, \ldots, F_\ast \omega^n \rangle \);
3) \( F_\ast \omega^i \wedge \cdots \wedge \omega^n = 0 \) for \( i = 1, \ldots, n \);
4) \( F_\ast \omega^i = \sum_{j=1}^n a_{ij} \omega^j \), where \( a_{ij} \in C^\infty(M) \);
5) \( (F_\ast X_i)|_x \in \mathbf{D}_{F(x)} \) for all \( x \in M \) and \( i = 1, \ldots, n \); and
6) \( F_* X_i = \sum_{j=1}^{n} b_{ij} X_j \), where \( b_{ij} \in C^\infty(M) \).

**Theorem 4.** If \( F \) be a symmetry of \( D \) and \( N \) be an integral manifold, then \( F(N) \) is an integral manifold.

*Proof:* \( F \) is a diffeomorphism, therefore \( F(N) \) is a sub-manifold of \( M \). From other hand, if \( x \in N \), then \( \omega^i|_{F(x)} = (F^*\omega^i)|_x = 0 \) for all \( i = 1, \ldots, n \), therefore \( F(N) = \{F(x) \mid x \in N\} \) is an integral manifold.

**Theorem 5.** Let \( N \) be the set of all maximal integral manifolds, and \( F : M \to M \) be a symmetry, then \( F(N) = N \).

*Proof:* If \( x \in N \), then \( \omega^i|_{F(x)} = (F^*\omega^i)|_x = 0 \) for all \( i = 1, \ldots, n \), therefore \( F(x) \in N \); and \( F(N) \subset N \).

Now if \( y \in N \), then there exists \( x \in M \) such that \( F(x) = y \), since \( F \) is a diffeomorphism. Therefore \( (F^*\omega^i)|_x = \omega^i|_{F(x)} = \omega^i|_y = 0 \) for all \( i = 1, \ldots, n \); thus \( x \in N \) and \( N \subseteq F(N) \).

**Definition 7.** A vector field \( X \) on \( M \) is called an infinitesimal symmetry of a distribution \( D \), or briefly a symmetry of \( D \), if the flow \( \text{Fl}_t^X \) of \( X \) be a symmetry of \( D \) for all \( t \).

**Theorem 6.** A vector field \( X \in \mathfrak{X}(M) \) is a symmetry if and only if

\[
L_X \omega^i|_D = 0 \quad \text{for all} \quad i = 1, \ldots, n.
\]

*Proof:* Let \( X \) be a symmetry. If \( \Omega = \omega^1 \wedge \cdots \wedge \omega^n \), then \( \{(\text{Fl}_t^X)^*\omega^i\} \wedge \Omega = 0 \), by the (3) of Theorem 3; Moreover by the definition \( L_X \omega^i := \frac{d}{dt}|_0 (\text{Fl}_t^X)^*\omega^i \), one gets

\[
(L_X \omega^i) \wedge \Omega = \lim_{t \to 0} \frac{1}{t} ((\text{Fl}_t^X)^*\omega^i - \omega^i) \wedge \Omega
= \lim_{t \to 0} \frac{1}{t} (\{(\text{Fl}_t^X)^*\omega^i\} \wedge \Omega - \omega^i \wedge \Omega^1) = 0.
\]

Therefore \( L_X \omega^i|_D = 0 \).

In converse, let \( L_X \omega^i|_D = 0 \) or \( L_X \omega^i = \sum_{j=1}^{n} b_{ij} \omega^j \) for \( i = 1, \ldots, n \) and \( b_{ij} \in C^\infty(M) \). Now, if \( \gamma_i(t) := \{(\text{Fl}_t^X)^*\omega^i\} \wedge \Omega \), then

\[
\gamma_i(0) = \{(\text{Fl}_0^X)^*\omega^i\} \wedge \Omega = 0,
\]

and

\[
\gamma_i'(t) = \frac{d}{dt} \{(\text{Fl}_t^X)^*\omega^i\} \wedge \Omega = ((\text{Fl}_t^X)^*L_X \omega^i) \wedge \Omega
= (\text{Fl}_t^X)^* \left( \sum b_{ij} \omega^j \right) \wedge \Omega = \sum B_{ij} \{(\text{Fl}_t^X)^*\omega^j\} \wedge \Omega
\]
where $B_{ij} = (Fl_X)^*b_{ij} = b_{ij} \circ Fl_X$, and
\[
\gamma_i'(t) = \sum B_{ij} \gamma_i(t), \quad i = 1, \cdots, n. \tag{2}
\]
Therefore, $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a solution of the linear homogeneous system of ODEs (2) with initial conditions (1), and $\gamma$ must be identically zero.

**Theorem 7.** $X$ is symmetry if and only if for all $Y \in D$ then $[X, Y] \in D$

**Proof:** By above theorem. $X$ is symmetry if and only if; for all $\omega \in \text{Ann} D$ then $L_X \omega \in \text{Ann} D$.

The Theorem comes from the Theorem 1-(b). $L_X \omega = -\omega \circ L_X$ on $D$. In other words, $(L_X \omega)Y = -\omega[X, Y]$ for all $Y \in D$.

Denote by $\text{Sym}_D$ the set of all symmetries of a distribution $D$.

**Example 5.** (Continuation of Example 3) Let $k = 2$. A vector field $Y = a \partial_x + b \partial_{p_0} + c \partial_{p_1}$ is an infinitesimal symmetry of $D$ if and only if $L_Y \omega^i \equiv 0 \mod D$, for $i = 1, 2$. These give two equations

\[
c = Xb - p_1 Xa, \quad Xc = f Xa + Yf,
\]

**Example 6.** (Continuation of Example 4) We consider the point infinitesimal transformation:

\[
Z = X(x, y, u) \partial_x + Y(x, y, u) \partial_y + U(x, y, u) \partial_u \\
+ P(x, y, u, p, q, r, t) \partial_p + Q(x, y, u, p, q, r, t) \partial_q \\
+ R(x, y, u, p, q, r, t) \partial_r + T(x, y, u, p, q, r, t) \partial_t.
\]

Then, $Z$ is an infinitesimal symmetry of $D$ if and only if $L_Z \omega^i \equiv 0 \mod D$, for $i = 1, 2, 3$. These give ten equations

\[
P_r = P_t = Q_r = Q_t = 0, \\
p^2 X_u + q p Y_y + q U_x + pQ = p q X_x + q P + q^2 Y_x + p U_y, \\
r p X_y + F Y_y + q P_x + (qr - pF)P_y + (q F - pt)Q_y + p X F_x + p Y F_y + p U F_u \\
+ p P F_p + p Q F_q = q r X_x + q F Y_x + p P_y + q R, \\
p F X_y + p t Y_y + (qr - p F)Q_p + (q F - pt)Q_q + q Q_x + p T \\
= q F X_x + q t Y_x + p Q_y + q P F_p + q Y F_y + q X F_x + q F Q_q + q U F_u, \\
Q_y + q Q_u + F Q_p + t Q_q = t Y_y + t q Y_u + F X_y + q X F_x + U, \\
U_u + q U_u = p X_y + p q X_u + q Y_y + q^2 Y_u + Q, \\
P_y + q P_u + F P_p + t P_q = r X_y + q r X_u + F Y_y + q Y F_u \\
+ (X F_x + Y F_y + U F_u + P F_p + Q F_q).
Complicated computations using Maple, shows that:

\[
P = -pX_x - p^2 X_u - qY_x - pqY_u + U_x + pU_u,
\]

\[
Q = \frac{1}{p} \left( pqX_x - p^2 X_y + q^2 Y_x - pqY_y - qU_x + pU_y + qP \right),
\]

\[
R = \frac{1}{q} \left( pqX_x - p^2 X_y + q^2 Y_x - pqY_y - qU_x + pU_y + qP \right) F_q
\]

+ ((qr - pF) P_p + (qF - pt) P_q + qP_x - pP_y

+ pXF_x + pYF_y + pUF_u + pPF_p + (pq - qF) F)

+ pF_q P_p + pYF_y + pUF_u + pPF_p + qPF_q

- pq^2 P_x + p^2 qP_y + pq(pF - qr) P_p + pq(pt - qF) P_q

+ pq(pXF_x + pYF_y + pUF_u + pPF_p + qPF_q)

- p^2 q^3 + 2p^3 qX_{xy} - p^4 Y_{xy} - pq^3 Y_{xx} + 2p^2 q^2 Y_{xy} - p^3 qY_{yy}

+ pq^2 U_{xx} - 2p^2 qU_{xy} + p^3 U_{yy}
\]

and \(X = X(x, u - qy), Y = Y(y, u - px)\) and \(U(x, y, u)\) must satisfy in PDE:

\[
(pF_p - F) X_x + p(pF_p - 2F) X_u + (qF_q - F) Y_y + q(qF_q - 2F) Y_u
\]

\[-F_p U_x - F_q U_y + (F - pF_p - qF_q) U_u + U_{xy} + qU_{xx} + pU_{yu} + pqU_{uu}
\]

\[= XF_x + YF_y + UF_u.
\]

5. A proof of Frobenius Theorem

**Theorem 8.** Let \(X \in \text{Sym}_D \cap \mathcal{D}\), and \(N\) maximal integral manifold. Then \(X\) is tangent to \(N\).

**Proof:** Let \(X(x) \notin T_xN\), then exists open set \(U\) of \(x\) and sufficiently small \(\varepsilon\). such that \(\bar{N} := \bigcup_{-\varepsilon < t < \varepsilon} \text{Fl}_t^X(N \cap U)\) is a smooth sub manifold of \(M\).

Since \(X \in \mathcal{D}\), So \(\bar{N}\) is an integral manifold.

Since \(X \in \text{Sym}_D\), So tangent to \(\text{Fl}_t^X(N \cap U)\) belongs to \(\mathcal{D}\), for all \(-\varepsilon < t < \varepsilon\).

On the other hand, tangent spaces to \(\bar{N}\) are sums of tangent spaces to \(\text{Fl}_t^X(N \cap U)\) and the 1–dimensional subspace generated by \(X\), but both of them belongs to \(\mathcal{D}\), and there means \(\bar{N} \subset N\).
Theorem 9. If $X \in D \cap \text{Sym}_D$ and $N$ be a maximal integral manifold, then $\text{Fl}_t^X(N) = N$ for all $t$.

Theorem 10 (Frobenious). A distribution $D$ is completely integrable, if and only if it is closed under Lie bracket. In other words, $[X,Y] \in D$ for each $X,Y \in D$.

Proof: Let $N$ be a maximal integral manifold with $T_x N = D_x$. Therefore, for all $X,Y \in D$ there $X$ and $Y$ are tangent to $N$ and so $[X,Y]$, is also tangent to $N$.

On the other hand, Let for all $X,Y \in D$ there $[X,Y] \in D$. By Theorem, all $x \in D$ is symmetry too. and so all $X \in D$ is tangent to $N$. and this means $T_x N = D_x$, for all $x \in N$.

Theorem 11. A distribution $D$ is completely integrable if and only if $D \subset \text{Sym}_D$.

Theorem 12. Let $D = \langle \omega^1, \ldots, \omega^n \rangle$ be completely integrable distribution and $X \in D$. Then the differential 1–forms $(\text{Fl}_t^X)^* \omega^1, \ldots, (\text{Fl}_t^X)^* \omega^n$ vanish on $D$ for all $t$.

Proof: $D$ is completely integrable, then $X$ is symmetry. Hence

$$(\text{Fl}_t^X)^* \omega^i = \sum_j a_{ij} \omega^j.$$ 

6. Symmetries and Solutions

Definition 8. If an (infinitesimal) symmetry $X$ belongs to the distribution $D$, then it is called a characteristic symmetry. Denote by $\text{Char}(D) := S_D \cap D$ the set of all characteristic symmetries.

It is shown that $\text{Char}(D)$ is an ideal of the Lie algebra $S_D$, and is a module on $C^\infty(M)$. Thus, we can define the quotient Lie algebra

$$\text{Shuf}(D) := \text{Sym}_D / \text{Char}(D).$$

Definition 9. Elements of $\text{Shuf}(D)$ are called shuffling symmetries of $D$.

Any symmetry $X \in \text{Sym}_D$ generates a flow on $N$ (the set of all maximal integral manifolds of $D$), and, in fact the characteristic symmetries generate trivial flows. In other words, classes $X \mod \text{Char}(D)$ mix or "shuffle" the set of all maximal manifolds.
Example 7. (Continuation of Example 5) Let $k = 2$. In this case
\[ \partial_x \equiv -p_1 \partial_{p_0} - f \partial_{p_1} \mod \text{Char}(D), \]
therefore, Shuf(D) spanned by $Z = (b - ap_1) \partial_{p_0} + (c - af) \partial_{p_1}$, where
\[ c = Xb - p_1 Xa, \quad Xc = f Xa + Yf. \]

Example 8. (Continuation of Example 6) In this case, we have
\[ \partial_x \equiv -p \partial_u - r \partial_p - F \partial_q, \quad \partial_r \equiv 0, \]
\[ \partial_y \equiv -q \partial_u - F \partial_p - t \partial_q, \quad \partial_t \equiv 0, \]
in Shuf(D). Therefore Shuf(D) spanned by
\[ W = (U - pX - qY) \partial_u + (P - rX - FY) \partial_p + (Q - FX - tY) \partial_q, \]
where
\[
\begin{align*}
P &= -pX_x - p^2 X_u - qY_x - pqY_u + U_x + pU_u, \\
Q &= \frac{1}{p} \left( pqX_x - p^2 X_y + q^2 Y_x - pqY_y - qU_x + pU_y + qP \right),
\end{align*}
\]
and $X = X(x, u - qy)$, $Y = Y(y, u - px)$ and $U(x, y, u)$ must satisfy in PDE:
\[
\begin{align*}
(pF_p - F)X_x + &p(pF_p - 2F)X_u + (qF_q - F)Y_y + q(qF_q - 2F)Y_u, \\
- &F_p U_x - F_q U_y + (F - pF_p - qF_q)U_u + U_{xy} + qU_{xu} + pU_{yu} \\
+ &pqU_{uu} = XF_x + YF_y + UF_u.
\end{align*}
\]

Example 9. (Quasilinear Klein-Gordon Equation) In this example, we find the shuffling symmetries of Quasilinear Klein-Gordon Equation:
\[ u_{tt} - \alpha^2 u_{xx} + \gamma^2 u = \beta u^3, \]
, as an application of previous example, where $\alpha$, $\beta$, and $\gamma$ are real constants. The equation can be transformed by defining $\xi = \frac{1}{2}(x - \alpha t)$ and $\eta = \frac{1}{2}(x + \alpha t)$. Then, by the chain rule, we obtain $\alpha^2 u_{\xi\eta} + \gamma^2 u = \beta u^3$. This equation reduce to
\[ u_{xy} = au + bu^3, \]
by $t = y$, $a = -(\gamma/\alpha)^2$ and $b = \beta/\alpha^2$. 
By solving the PDE (3), we conclude that Shuf(D) spanned by the three following vector fields:

\[ \begin{align*}
X_1 &= (px - qy) \partial_u - (p + yu^2(a + bu) - rx) \partial_p + (q + xu^2(a + bu) - ty) \partial_q, \\
X_2 &= q \partial_u + u^2(a + bu) \partial_p + t \partial_q, \\
X_3 &= p \partial_u + r \partial_p + u^2(a + bu) \partial_q.
\end{align*} \]

For example, we have

\[ F_l^{X_3}(x, y, u, p, q, r, t) = \left( x, y, u + sp + \frac{s^2}{2}r, p + sr, q + s^3u^2(a + bu) + \frac{s^4}{40}up(2a + 3bu) + \frac{s^3}{42}(14ap^2 + 42bup^2 + 14aur + 21bu^2r) + \frac{s^4}{4}p(bp^2 + ar + 3bur) + \frac{s^5}{20}r(6bp^2 + ar + 3bur) + \frac{s^6}{8}bpr^2 + \frac{s^7}{56}br^3, r, t \right) \]

and if \( u = h(x, y) \) be a solution of (4), then \( F_l^{X_3}(x, y, h, h_x, h_y, h_{xx}, h_{yy}) \) is also a new solution of (4), for sufficiently small \( s \in \mathbb{R} \).

References

[ALV] D.V. Alekseevskij and V.V. Lychagin and A.M. Vinogradov, Basic Ideas and Concepts of Differential Geometry, Geometry vol. I, Springer-Verlag, New York, Heidelberg, Berlin, 1991.

[BEM] A. Barone and F. Esposito and C.J. Magee and A.C. Scott, Theory and applications of the Sine-Gordon equation, Rev. Nuovo Cim. 1 (1971), 227-267.

[KRA] H. Kragh and, Equation with the many fathers. The Klein-Gordon equation in 1926, American Journal of Physics 52, Issue 11, (1984), 1024-1033.

[KLR] A. Kushner and V.V. Lychagin and V. Rubtsov, Contact Geometry and Non-linear Differential Equations, Cambridge University Press, 2007.

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Geometry of Distributions and $F$–Gordon equation

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Abstract. In this paper we describe the geometry of distributions by their symmetries, and present a simplified proof of the Frobenius theorem and some related corollaries. Then, we study the geometry of solutions of $F$–Gordon equation; A PDE which appears in differential geometry and relativistic field theory.

Keywords: Distribution, Lie symmetry, Contact geometry, Klein-Gordon equation.

2000 Mathematics Subject Classification: 35Q40, 35Q80, 53D10.

1. Introduction

We begin this paper with the geometry of distributions. The main idea here is the various notions of symmetry and their use in solving a given differential equation. In section 2, we introduce the basic notions and definitions.

In section 3, we describe the relation between differential equations and distributions. In section 4, we present the geometry of distributions by their symmetries, and find out the symmetries of $F$–Gordon equation by this machinery. In section 5, we introduce a simplified proof of the Frobenius theorem and some related corollaries. In section 6, we describe the relations between symmetries and solutions of a distribution.

In all steps, we study the $F$–Gordon equation as an application; a partial differential equation which appears in differential geometry and relativistic field theory. It is a generalized form the Klein-Gordon equation "$u_{tt} - u_{xx} + u = 0"$. It is a relativistic version of Schrodinger equation, which is used to describe spinless particles. It was named after Walter Gordon and Oskar Klein [?, ?].