Classification of limit varieties of $\mathcal{J}$-trivial monoids

S. V. Gusev and O. B. Sapir

Abstract

A variety of algebras is a limit variety if it is non-finitely based but all its proper subvarieties are finitely based.

We present a new pair of limit varieties of monoids and show that together with the five limit varieties of monoids previously discovered by Jackson, Zhang and Luo and the first-named author, there are exactly seven limit varieties of $\mathcal{J}$-trivial monoids.

1 Introduction

A variety of algebras is called finitely based (abbreviated to FB) if it has a finite basis of its identities, otherwise, the variety is said to be non-finitely based (abbreviated to NFB). Much attention is paid to studying of FB and NFB varieties of algebras of various types. In particular, the FB and NFB varieties of semigroups and monoids have been the subject of intensive research (see the survey [24]).

A variety is hereditary finitely based (abbreviated to HFB) if all its subvarieties are FB. A variety is called a limit variety if it is NFB but every its proper subvariety is FB. Limit varieties play an important role because every NFB variety contains some limit subvariety by Zorn’s lemma. It follows that a variety is HFB if and only if it does not contain any limit variety. So, if one manages to classify all limit varieties within some class of varieties, then this classification implies a description of all HFB varieties in this class.

Limit varieties are very rare. Only five explicit examples of limit varieties of monoids are known so far. The first two examples of limit monoid varieties $\mathbf{L}$ and $\mathbf{M}$ were discovered by Jackson [7] in 2005 (the formal definitions of these varieties will be given in Subsection 2.3). In 2013, Zhang found a NBF variety of monoids that does not contain the varieties $\mathbf{L}$ and $\mathbf{M}$ [26] and, therefore, she proved that

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there exists a limit variety of monoids that differs from \( L \) and \( M \). In [27], Zhang and Luo pointed out an explicit example of such variety.

If \( S \) is a semigroup, then the monoid obtained by adjoining a new identity element to \( S \) is denoted by \( S^1 \). If \( M \) is a monoid, then the variety of monoids generated by \( M \) is denoted by \( \text{var} M \). Let \( A^1 = \text{var} A^1 \), where

\[
A = \langle a, b, c | a^2 = a, b^2 = b, ab = ca = 0, ac = cb = c \rangle = \{a, b, c, ba, bc, 0\}.
\]

The semigroup \( A \) was introduced by its multiplication table and shown to be FB in [15, Section 19]. Its presentation was recently suggested by Edmond W. H. Lee. If \( V \) is a monoid variety, then \( \tilde{V} \) denotes the variety dual to \( V \), i.e., the variety consisting of monoids anti-isomorphic to monoids from \( V \). The variety \( A^1 \lor \tilde{A^1} \) is the third example of limit variety of monoids [27] mentioned in the previous paragraph. The fourth and the fifth examples of limit monoid varieties \( J \) and \( \tilde{J} \) are provided in [3] (the formal definition of the variety \( J \) will be given in Subsection 2.3).

In [2], Green introduces five equivalence relations on a semigroup. These relations are collectively referred to as Green’s relations, and play a fundamental role in studying semigroups. We recall the definition of one of them, which we use in present paper. For elements \( a \) and \( b \) of a semigroup \( S \), Green’s relation \( \mathcal{J} \) is defined by

\[
a \mathcal{J} b \text{ if and only if } S^1 a S^1 = S^1 b S^1, \quad \text{i.e., } a \text{ and } b \text{ generate the same ideal.}
\]

A semigroup \( S \) is called \( \mathcal{J} \)-trivial if Green’s relation \( \mathcal{J} \) is the equality relation.

It turns out that all known examples of limit varieties of monoids are varieties of \( \mathcal{J} \)-trivial monoids. In this article, we present a new pair of limit varieties of monoids \( K \) and \( \tilde{K} \) and provide the following classification of limit varieties of \( \mathcal{J} \)-trivial monoids, which is the main result of the article.

**Theorem 1.1.** A variety of \( \mathcal{J} \)-trivial monoids is HFB if and only if it excludes the varieties \( A^1 \lor \tilde{A^1}, J, \tilde{J}, K, \tilde{K}, L \) and \( M \). Consequently, there are precisely seven limit varieties of \( \mathcal{J} \)-trivial monoids.

A monoid is aperiodic if all its subgroups are trivial. We note that several classifications of limit varieties of monoids were obtained earlier in some other classes. Namely, Lee proved in [10] the uniqueness of the limit varieties \( L \) and \( M \) in the class of varieties of finitely generated aperiodic monoids with central idempotents. In [11], Lee generalized the result of [10] and established that \( L \) and \( M \) are the only limit varieties within the class of varieties of aperiodic monoids with central idempotents. Just recently, the first-named author [4] proved that a variety of aperiodic monoids with commuting idempotents is HFB if and only if it excludes the varieties \( J, \tilde{J}, L \) and \( M \).

The article consists of six sections. Section 2 contains definitions, notation, auxiliary results and introduces the varieties \( K \) and \( \tilde{K} \). In Section 3 we provide a sufficient condition under which a monoid variety is HFB. In Section 4 we provide
a classification of aperiodic monoid varieties, which implies that every variety of $\mathcal{J}$-trivial monoids is either HFB or contains one of the varieties $A^1 \lor \overleftarrow{A^1}$, $J$, $\overleftarrow{J}$, $K$, $\overleftarrow{K}$, $L$ or $M$ (Corollary 4.6). In Section 5 we show that $K$ and $\overleftarrow{K}$ are new limit varieties of monoids (Proposition 5.1). Thus, we provide the proof of Theorem 1.1. Finally, Section 6 is devoted to a description of the subvariety lattice of the limit variety $K$.

2 Preliminaries

2.1 Varieties of $\mathcal{J}$-trivial monoids

The following claim is well-known but we provide its proof for the sake of completeness.

**Fact 2.1.** Let $V$ be a variety of $\mathcal{J}$-trivial monoids. Then $V$ satisfies the identities $x^n \approx x^{n+1}$ and $(xy)^n \approx (yx)^n$ for some $n \geq 1$.

*Proof.* Since the Green’s relation $\mathcal{J}$ on any group coincides with the universal relation, all the groups of $V$ are trivial. This implies that $V$ is aperiodic and so satisfies the identity $x^n \approx x^{n+1}$ for some $n \geq 1$.

Let $M$ be a monoid from $V$ and $a, b \in M$. Then $(ab)^n = (ab)^{n+1} = a(ba)^n b$ and similarly, $(ba)^n = b(ab)^n a$. Therefore, $(ab)^n$ and $(ba)^n$ lie in the same $\mathcal{J}$-class of $M$. Since $M$ is $\mathcal{J}$-trivial, $(ab)^n = (ba)^n$. It follows that $M$ and so $V$ satisfy $(xy)^n \approx (yx)^n$. $\square$

2.2 Words, identities and Dilworth-Perkins construction

Let $\mathcal{A}$ be a countably infinite set called an alphabet. As usual, let $\mathcal{A}^+$ and $\mathcal{A}^*$ denote the free semigroup and the free monoid over the alphabet $\mathcal{A}$, respectively. Elements of $\mathcal{A}$ are called letters and elements of $\mathcal{A}^*$ are called words. We treat the identity element of $\mathcal{A}^*$ as the empty word, which is denoted by $1$. Words and letters are denoted by small Latin letters. However, words unlike letters are written in bold. The content of a word $w$, i.e., the set of all letters occurring in $w$, is denoted by $\text{con}(w)$. A letter is called simple [multiple] in a word $w$ if it occurs in $w$ once [at least twice]. The set of all simple [multiple] letters in a word $w$ is denoted by $\text{sim}(w)$ [respectively $\text{mul}(w)$]. We use $W \subseteq$ to denote the closure of a set of words $W \subset \mathcal{A}^*$ under taking subwords.

The following construction was introduced by Perkins [17] to build the first two examples of finitely generated NFB varieties of semigroups (although in essence it appears in [16], where it is attributed to Dilworth). For any set of words $W$, let $M(W)$ denote the Rees quotient monoid of $\mathcal{A}^*$ over the ideal $\mathcal{A}^* \setminus W \subseteq$ consisting of all words that are not subwords of any word in $W$. Given a finite set of words $W$, $M(W)$ is a finite $\mathcal{J}$-trivial monoid with zero.
2.3 Generalized Dilworth-Perkins construction

The following generalization of $M(W)$ construction was introduced by the second-named author in [21, 22]. Given a congruence $\tau$ on the free monoid $A^*$ we use $\circ_\tau$ to denote the binary operation on the quotient monoid $A^*/\tau$. The elements of $A^*/\tau$ are called $\tau$-words or $\tau$-classes and written using lowercase letters in the typewriter style\footnote{We call the elements of $A^*/\tau$ by $\tau$-words when we want to emphasize the relations between them and we refer to $u \in A^*/\tau$ as a $\tau$-class when we are interested in the description of the words contained in $u$.}. The subword relation on $A^*$ can be naturally extended to $\tau$-words as follows: given two $\tau$-words $u, v \in A^*/\tau$ we write $v \leq_\tau u$ if $u = p \circ_\tau v \circ_\tau s$ for some $p, s \in A^*/\tau$. Given a set of $\tau$-words $W$ we define $W^{\leq_\tau}$ as closure of $W$ in quasi-order $\leq_\tau$. If $W$ is a set of $\tau$-words, then $M_\tau(W)$ denotes the Rees quotient of $A^*/\tau$ over the ideal $(A^*/\tau) \setminus W^{\leq_\tau}$. For brevity, if $w_1, w_2, \ldots, w_k \in A^*/\tau$, then we write $M_\tau(w_1, w_2, \ldots, w_k)$ rather than $M_\tau(\{w_1, w_2, \ldots, w_k\})$. If $\tau$ is the trivial congruence on $A^*$, then $M_\tau(W)$ construction coincides with Dilworth-Perkins construction.

Let $\tau_1$ denote the congruence on the free monoid $A^*$ induced by the relations $a = aa$ for each $a \in A^*$. We refine $\tau_1$ into three more congruences as follows. Given $u, v \in A^*$, we define:

- $u \gamma v$ if and only if $u \tau_1 v$ and $\text{mul}(u) = \text{mul}(v)$;
- $u \lambda v$ if and only if $u \gamma v$ and the first two occurrences of each multiple letter are adjacent in $u$ if and only if these occurrences are adjacent in $v$;
- $u \rho v$ if and only if $u \gamma v$ and the last two occurrences of each multiple letter are adjacent in $u$ if and only if these occurrences are adjacent in $v$.

Notice that the relations $\rho$ and $\lambda$ are dual to each other. If $\tau \in \{\tau_1, \gamma, \lambda, \rho\}$, then it is verified in Proposition 8.5 in [22] that the relation $\leq_\tau$ is an order on $A^*/\tau$ and given a finite set of $\tau$-words $W$, $M_\tau(W)$ is a finite $J$-trivial monoid with zero.

We say that a $\tau$-word $u \in A^*/\tau$ is a $\tau$-term for a variety $V$ if, for any word $u$ in $u$, we have $u \tau v$ whenever $V$ satisfies $u \approx v$. The following lemma generalizes Lemma 3.3 in [7].

**Lemma 2.2** ([22] Corollary 3.6). Let $\tau$ be either the trivial congruence or $\tau \in \{\tau_1, \gamma\}$. Let $W$ be a subset of $A^*/\tau$. Then a monoid variety $V$ contains $M_\tau(W)$ if and only if every $\tau$-word in $W$ is a $\tau$-term for $V$. \qed

If $u \in A^*$ and $x \in \text{con}(u)$, then an island formed by $x$ in $u$ is a maximal subword of $u$, which is a power of $x$. For example, the word $ab^2a^5ba^3$ has three islands formed by $a$ and two islands formed by $b$. We say that $u \in A^*$ is 2-island-limited if each letter forms at most 2 islands in $u$. For example, the word $asbtb^5a^7$ is 2-island-limited. Given $\tau \in \{\tau_1, \gamma, \lambda, \rho\}$ we say that $u \in A^*/\tau$ is 2-island-limited if $u$ is 2-island-limited for each $u \in u$. \footnote{We call the elements of $A^*/\tau$ by $\tau$-words when we want to emphasize the relations between them and we refer to $u \in A^*/\tau$ as a $\tau$-class when we are interested in the description of the words contained in $u$.}
**Fact 2.3** ([22] Lemma 6.3 and its dual). Let \( \tau \in \{\lambda, \rho\} \) and \( u \) be a 2-island-limited \( \tau \)-word. If \( u \) is a \( \tau \)-term for a monoid variety \( V \), then every \( \tau \)-word \( v \) with \( v \leq_{\tau} u \) is also a \( \tau \)-term for \( V \).

**Lemma 2.4** ([22] Corollary 6.4 and its dual). Let \( \tau \in \{\lambda, \rho\} \) and \( W \) be a set of 2-island-limited \( \tau \)-words. Then a monoid variety \( V \) contains \( M_{\tau}(W) \) if and only if every \( \tau \)-word in \( W \) is a \( \tau \)-term for \( V \). \( \square \)

Let \( \mathfrak{B} \) denote an alphabet, which consists of symbols \( a^+ \) for each \( a \in \mathfrak{A} \). If \( \tau \in \{\tau_1, \gamma, \lambda, \rho\} \), then it is convenient to represent the elements of \( \mathfrak{A}^*/\tau \) by words in the alphabet \( \mathfrak{A} \cup \mathfrak{B} \). For each \( a \in \mathfrak{A} \) consider the following rewriting rules on \( (\mathfrak{A} \cup \mathfrak{B})^* \):

- \( R^{\tau_1}_{a \rightarrow a^+} \) replaces \( a \) by \( a^+ \);
- \( R^{\gamma}_{a \rightarrow a^+} \) replaces an occurrence of \( a \) in \( u \) by \( a^+ \) only in case that either \( a \) appears in \( u \) at least twice or \( a^+ \) appears in \( u \);
- \( R^{\lambda}_{a \rightarrow a^+} \) replaces an occurrence of \( a \) in \( u \) by \( a^+ \) only in case that either \( a \) or \( a^+ \) appears in \( u \) to the left of this occurrence of \( a \);
- \( R^{\rho}_{a \rightarrow a^+} \) replaces an occurrence of \( a \) in \( u \) by \( a^+ \) only in case that either \( a \) or \( a^+ \) appears in \( u \) to the right of this occurrence of \( a \);
- \( R_{a^+a^+ \rightarrow a^+} \) replaces \( a^+a^+ \) by \( a^+ \);
- \( R_{aa^+ \rightarrow a^+} \) replaces \( aa^+ \) by \( a^+ \);
- \( R_{a^+a \rightarrow a^+} \) replaces \( a^+a \) by \( a^+ \).

It is proved in Lemma 8.1 in [22] that if \( \tau \in \{\tau_1, \gamma, \lambda, \rho\} \), then using the rules

\[
\{R^{\tau_1}_{a \rightarrow a^+}, R_{a^+a^+ \rightarrow a^+}, R_{aa^+ \rightarrow a^+}, R_{a^+a \rightarrow a^+} \mid a \in \mathfrak{A}\} \quad (2.2)
\]

in any order, every word \( u \in (\mathfrak{A} \cup \mathfrak{B})^* \) can be transformed to a unique word \( r_{\tau}(u) \) such that none of these rules is applicable to \( r_{\tau}(u) \). Let \( \mathcal{R}_{\tau} \) denote the set of all words in \( (\mathfrak{A} \cup \mathfrak{B})^* \) to which none of the rewriting rules \( (2.2) \) is applicable. It is verified in Proposition 8.2 in [22] that the \( \tau \)-words in \( \mathfrak{A}^*/\tau \) can be identified with the elements of \( \mathcal{R}_{\tau} \) such that for each pair of words \( u, v \in \mathcal{R}_{\tau} \) we have

\[ u \circ_{\tau} v = r_{\tau}(uv). \]

For example, \((bta^+) \circ_{\lambda}(a^+b^+) = bta^+b^+\). The next example lists all non-zero elements of the 20-element monoid \( M_{\lambda}(bta^+b^+) \).

**Example 2.5.**

\[
\{bta^+b^+\}^{\leq_{\lambda}} = \{1, a, a^+, b, b^+, t, bt, ta^+, ab, ab^+, a^+b, a^+b^+, bta, bta^+, ta^+b, ta^+b^+, bta^+b, bta^+b^+\}.
\]
If \( w_1, w_2, \ldots, w_k \in \mathfrak{A}^*/\tau \), then we use \( M_\tau(w_1, w_2, \ldots, w_k) \) to denote the monoid variety generated by \( M_\tau(w_1, w_2, \ldots, w_k) \). If \( \tau \) is the trivial congruence on \( \mathfrak{A}^* \), then we write \( M(w_1, w_2, \ldots, w_k) \) rather than \( M_\tau(w_1, w_2, \ldots, w_k) \).

It turns out that every limit variety of \( J \)-trivial monoids is generated by a monoid of the form \( M_\tau(w_i) \). In particular, \( L = M(abtasb) \) and \( M = M(abtasb, atsab) \) [7]. The limit variety \( J \) was introduced in [3] as a variety given by some infinite identity system. According to Theorem 7.2(v) in [22] and its dual, \( J = M_\lambda(atba^sb^b) \) and \( J = M_\mu(a^+tb^tasb) \). In view of Theorem 4.3(iv) in [22], the limit variety \( A^1 \lor \overline{A}^1 \) discovered in [27] is generated by the monoid \( M_\lambda(a^+b^+ta^+, a^+tb^+a^+) \).

Put \( K = M_\lambda(bta^+b^+) \). According to Example 2.3, the variety \( K \) is generated by a 20-element monoid. In fact, \( K \) is generated by a 12-element submonoid of \( M_\lambda(bta^+b^+) \) (see Remark 6.6). We are going to prove in Proposition 5.1 that \( K \) and \( \overline{K} \) are new limit varieties of monoids.

### 2.4 Two useful facts

If \( M \) is a monoid or a class of monoids and \( \Sigma \) is an identity or a set of identities, then we write \( M \models \Sigma \) whenever \( M \) satisfies \( \Sigma \). A word \( w \) is an isoterms for \( M \) if \( M \) violates any non-trivial identity of the form \( w \approx w' \). For a word \( w \) and a set of letters \( X \subseteq \con(w) \), let \( w(X) \) be the word obtained from \( w \) by deleting from \( w \) all letters that are not in \( X \). We write \( w(x_1, x_2, \ldots, x_k, X) \) rather than \( w(X \cup \{x_1, x_2, \ldots, x_k\}) \) whenever \( X \subseteq \con(w) \) and \( x_1, x_2, \ldots, x_k \in \con(w) \setminus X \).

The next two facts are well-known. We provide their proofs for the sake of completeness.

**Lemma 2.6.** For a monoid \( M \) the following are equivalent:

(i) \( xy \) is an isoterms for \( M \);

(ii) every identity \( u \approx v \) satisfied by \( M \) has the following properties:

\[
\mul(u) = \mul(v) \quad \text{and} \quad \u Simmons(u) = \v Simmons(v).
\]

**Proof.** (i) \( \Rightarrow \) (ii) Clearly, \( x \) is an isoterms for \( M \). Then \( \mul(u) = \mul(v) \) and \( \SimSim(u) = \SimSim(v) = \{t_1, \ldots, t_m\} \) for some \( m \geq 0 \). Therefore, \( \u Simmons(u) = t_1 \ldots t_m \) and \( \v Simmons(u) = t_{\pi 1} \ldots t_{\pi m} \) for some permutation \( \pi \) on \( \{t_1, \ldots, t_m\} \). If \( \pi \) is not the identical permutation, then \( M \) satisfies \( t_i t_j \approx t_j t_i \) for some \( 1 \leq i < j \leq m \). To avoid contradiction, \( \pi \) must be the identical permutation, that is, \( \u Simmons(u) = \v Simmons(v) \). Implication (ii) \( \Rightarrow \) (i) is evident. \( \square \)

A block of a word \( w \) is a maximal subword of \( w \) that does not contain any letters simple in \( w \). If \( xy \) is an isoterms for a monoid \( M \), then, in view of Lemma 2.6 every identity of \( M \) is of the form

\[
u_0 \prod_{i=1}^m (t_i u_i) \approx v_0 \prod_{i=1}^m (t_i v_i), \quad (2.3)
\]
where \( \text{sim}(u) = \text{sim}(v) = \{t_1, \ldots, t_m\} \) for some \( m \geq 0 \) and \( \text{mul}(u) = \text{con}(u_0 \ldots u_m) = \text{con}(v_0 \ldots v_m) = \text{mul}(v) \). For each \( i = 0, 1, \ldots, m \), we say the blocks \( u_i \) and \( v_i \) are corresponding.

**Lemma 2.7.** Let \( M \) be a monoid that satisfies \( x^n \approx x^{n+1} \) for some \( n \geq 1 \). If \( xy \) is not an isoterm for \( M \), then \( M \) is commutative or idempotent. \( \square \)

*Proof.* If \( x \) is not an isoterm for \( M \), then \( M \) satisfies \( x \approx x^k \) for some \( k > 1 \). Since the identities \( x^n \approx x^{n+1} \) and \( x \approx x^k \) imply \( x \approx x^2 \), the monoid \( M \) is idempotent.

If \( x \) is an isoterm for \( M \) but \( xy \) is not an isoterm for \( M \), then \( M \) satisfies \( xy \approx yx \) and is commutative. \( \square \)

## 3 A sufficient condition under which a monoid variety is HFB

A monoid \( M \) is said to be [hereditary] finitely based if \( \text{var} M \) is \([H]\)FB. The variety of monoids given by an identity system \( \Delta \) is denoted by \( \text{var} \Delta \). We fix the following set of identities for the rest of this section:

\[
\Sigma = \{xtysxy \approx xtxsy, xtysyx \approx xtysxy\}.
\]

The following proposition is the main result of this section and generalizes Proposition 3.1 in [4], which says that \( \text{var}\{xtysxy \approx xtxsy, x^2y^2 \approx y^2x^2\} \) is HFB.

**Proposition 3.1.** Let \( M \) be a monoid such that \( M \models \Sigma \). Then \( M \) is HFB.

To prove Proposition 3.1 we need some auxiliary results. We will often use the following fact without references.

**Fact 3.2.** The identity

\[
xtysyx \approx xtxsyx
\]

implies the identities \( xtx \approx xtx^2 \) and

\[
xtysxy \approx xtysyx.
\]

**Proof.** The identity \( xtx \approx xtx^2 \) can be obtained by erasing letters \( s \) and \( y \) from (3.1). Then (3.2) follows from (3.1) because \( xtysxy \overset{xtxsx^2xy}{\approx} \) \( xtysx^2y \overset{xtys(xy)^2}{\approx} xtysyxy \). \( \square \)

The following lemma generalizes Lemma 3.2 in [4].

**Lemma 3.3.** Let \( w = v_1v_2xv_3 \), where \( v_1, v_2, v_3 \in \mathbb{A}^* \). If \( v_1 \) contains \( x \) and \( v_2 \) does not contain any letters that are simple in \( w \), then \( \Sigma \) implies \( w \approx v_1xv_2xv_3 \).

**Proof.** In view of Fact 3.2, the identity (3.1) implies (3.2). The rest of the proof is similar to the proof of Lemma 3.2 in [4]. The only difference is that instead of \( xtysxy \approx xtysyx \) we use one of the identities (3.1) or (3.2). \( \square \)
Let $u \approx v$ be an identity such that $u(\text{sim}(u)) = v(\text{sim}(v))$. If $x \in \text{mul}(u)$, we say that $u \approx v$ is \emph{x-well-balanced} if $u(x, \text{sim}(u)) = v(x, \text{sim}(u))$. Following Lee [11], we say that the identity is \emph{well-balanced} if $u \approx v$ is x-well-balanced for each $x \in \text{mul}(u)$. The identity $u \approx v$ is \emph{not well-balanced at} $x$ if the number of occurrences of $x$ in some block $b$ of $u$ does not equal to the number of occurrences of $x$ in the corresponding block $b'$ of $v$. If $x \in \text{mul}(u)$ and $b$ is the block of $u$, which contains the first occurrence of $x$, then we say that $u$ is \emph{x-compact} if $b$ contains at most two occurrences of $x$ and every other block of $u$ contains at most one occurrence of $x$. We say that a word $u$ is \emph{compact} if $u$ is x-compact for each $x \in \text{mul}(u)$. We say that an identity $u \approx v$ is \emph{compact} if both $u$ and $v$ are compact and $u(\text{sim}(u)) = v(\text{sim}(v))$. A word that contains at most one multiple letter is called \emph{almost-linear}. An identity $u \approx v$ is called \emph{almost-linear} if both $u$ and $v$ are almost-linear. Given a monoid $M$ we use $\Gamma(M)$ to denote the set of all almost-linear identities of $M$.

**Lemma 3.4.** Let $M$ be a monoid such that $xy$ is an isoterm for $M$ and $M \models \Sigma$. Then every identity of $M$ can be derived from a compact well-balanced identity of $M$ together with $\Sigma \cup \Gamma(M)$.

**Proof.** Let $u \approx v$ be an identity of $M$. In view of Lemma 3.3, $\text{mul}(u) = \text{mul}(v)$ and $u(\text{sim}(u)) = v(\text{sim}(v))$. Let $u \approx v$ be not well-balanced at precisely $k$ different letters. We will use induction on $k$.

**Induction base:** $k = 0$. Then $u \approx v$ is well-balanced. In view of Lemma 3.3 we can remove some occurrences of letters in $u$ and $v$ and obtain the words $u^*$ and $v^*$ such that the identity $u^* \approx v^*$ is compact and well-balanced and $u \approx v$ is equivalent modulo $\Sigma$ to $u^* \approx v^*$, we are done.

**Induction step:** $k > 0$. Then $u' = u(x, \text{sim}(u)) \neq v(x, \text{sim}(v)) = v'$ for some $x \in \text{mul}(u)$.

Suppose that every block of $u$ contains at most one occurrence of $x$. Then we apply the identity $u' \approx v'$ to $u$ and obtain a word $w$ such that the identity $w \approx v$ is x-well-balanced. By the induction assumption, the identity $w \approx v$ can be derived from a compact well-balanced identity $\sigma$ of $M$ together with $\Sigma \cup \Gamma(M)$. Then $u \approx v$ follows from $\{\sigma\} \cup \Sigma \cup \Gamma(M)$ because $u' \approx v' \in \Gamma(M)$, and we are done. By a similar argument we can show that if every block of $v$ contains at most one occurrence of $x$, then $u \approx v$ follows from a compact well-balanced identity of $M$ together with $\Sigma \cup \Gamma(M)$. So, by Lemma 3.3 we may assume that some block $a$ of $u$ contains the first and second occurrences of $x$ and some block $b$ of $v$ also contains the first and second occurrences of $x$. Two cases are possible.

**Case 1:** the blocks $a$ and $b$ are corresponding. Lemma 3.3 allows us to remove some occurrences of letters in $u$ and $v$ and obtain the words $u^*$ and $v^*$ such that the identity $u^* \approx v^*$ is compact well-balanced and $u \approx v$ is equivalent modulo $\Sigma$ to $u^* \approx v^*$. Therefore, we may assume that $u$ and $v$ are compact. Then, by symmetry, we may assume that some block $b'$ of $v$ contains $x$ but the corresponding block $a'$ of $u$ to $b'$ does not contain $x$. We may choose $a'$ and $b'$ so that they are the rightmost
blocks in the words $\mathbf{u}'$ and $\mathbf{v}'$ with such a property. Let $t$ be simple letter that is next to the block $a'$ on the left of it in $\mathbf{u}$.

**Subcase 1.1**: the last occurrence of $x$ precedes the block $a'$ in $\mathbf{u}$. Then the identity $\mathbf{u}(x, t) \approx \mathbf{v}(x, t)$ is equivalent modulo $\Sigma$ to

$$x^2t \approx x^2tx,$$

whence $M$ satisfies

$$xtxs \approx x^2txs \approx x^2tx \approx x^2txs \approx x^2txs.$$

Then we apply the identity $xtxs \approx x^2txs$ to $\mathbf{u}$ and obtain a word $\mathbf{w}$ such that the identity $\mathbf{w} \approx \mathbf{v}$ is $x$-well-balanced. By the induction assumption, the identity $\mathbf{w} \approx \mathbf{v}$ can be derived from a compact well-balanced identity $\sigma$ of $M$ together with $\Sigma \cup \Gamma(M)$. Then $\mathbf{u} \approx \mathbf{v}$ follows from \{\sigma\} $\cup$ $\Sigma \cup \Gamma(M)$ because $xtxs \approx x^2txs \in \Gamma(M)$, and we are done.

**Subcase 1.2**: the last occurrence of $x$ is preceded by the block $a'$ in $\mathbf{u}$. Let $s$ be simple letter that is next to the blocks $a'$ on the right of it in $\mathbf{u}$. Then the identity $\mathbf{u}(x, t, s) \approx \mathbf{v}(x, t, s)$ is equivalent modulo $\Sigma$ to

$$x^2tsx \approx x^2tsx,$$

whence $M$ satisfies

$$xzxtxsx \approx x^2txsx \approx x^2txsx \approx x^2txsx \approx x^2txsx.$$

Then we apply the identity $xzxtxsx \approx x^2txsx$ to $\mathbf{u}$ and obtain a word $\mathbf{w}$ such that the identity $\mathbf{w} \approx \mathbf{v}$ is $x$-well-balanced. By the induction assumption, the identity $\mathbf{w} \approx \mathbf{v}$ can be derived from a compact well-balanced identity $\sigma$ of $M$ together with $\Sigma \cup \Gamma(M)$ because $xzxtxsx \approx x^2txsx \in \Gamma(M)$, and we are done.

**Case 2**: the blocks $a$ and $b$ are not corresponding. Let $a'$ be the corresponding block to $b$. We may assume without loss of generality that the block $a'$ precedes the block $a$ in $\mathbf{u}$. Let $t$ be simple letter that is next to the block $a'$ on the right of it in $\mathbf{u}$. Clearly, the identity $u(x, t)x \approx v(x, t)x$ is equivalent modulo $\Sigma$ to

$$tx^2 \approx x^2tx,$$

whence $M$ satisfies

$$txsx \approx tx^2sx \approx x^2txsx \approx x^2txsx \approx x^2txsx.$$

Then we apply the identity $txsx \approx x^2txsx$ to $\mathbf{u}$ and obtain a word $\mathbf{w}$ such that the corresponding block of $\mathbf{w}$ to the block $b$ of $\mathbf{v}$ contains the first and second occurrences of $x$ in $\mathbf{w}$. By Case 1, the identity $\mathbf{w} \approx \mathbf{v}$ can be derived from a compact well-balanced identity $\sigma$ of $M$ together with $\Sigma \cup \Gamma(M)$. Then $\mathbf{u} \approx \mathbf{v}$ follows from \{\sigma\} $\cup$ $\Sigma \cup \Gamma(M)$ because $txsx \approx x^2txsx \in \Gamma(M)$, and we are done. \qed
The expression \( i_w x \) means the \( i \)th occurrence of a letter \( x \) in a word \( w \). We say that a pair of occurrences \( \{i_x, j_y\} \) of letters \( x \) and \( y \) in a well-balanced identity \( u \approx v \) is \textit{critical} if \( u \) contains \( i_u x j_u y \) as a subword and \( j_v y \) precedes \( i_v x \) in \( v \). Let \( w \) denote the result of replacing \( i_u x j_u y \) by \( j_u y i_u x \) in \( u \). Given a set of identities \( \Delta \) and a well-balanced identity \( u \approx v \), we say that the critical pair \( \{i_x, j_y\} \) is \( \Delta \)-removable in \( u \approx v \) if \( \Delta \) implies \( u \approx w \).

The following special case of Lemma 3.4 in [19] describes the standard method of deriving identities by removing critical pairs. This method traces back to the articles [9, 18].

**Lemma 3.5.** Let \( M \) be a monoid and \( \Delta \) be a set of identities. Suppose that each critical pair in every compact well-balanced identity of \( M \) is \( \Delta \)-removable. Then every compact well-balanced identity of \( M \) can be derived from \( \Delta \). □

Let \( \Phi \) denote the set of compact well-balanced identities with two multiple letters of the form \( p_c \approx q_c \) or \( y s p_c \approx y s q_c \), where \( p \) and \( q \) are words in \( \{x, y\}^+ \) with the property that both \( x \) and \( y \) appear at least once and at most twice in both \( p \) and \( q \) and occur in \( p \) the same number of times as in \( q \), and \( c \) be either the empty word or a word from the set

\[
\{t x y, \prod_{\ell=1}^{k} (t_x e_\ell) \mid k \geq 1, e_1, e_2, \ldots, e_k \in \{1, x, y\}\}.
\]

Notice that if \( u \approx v \) is an identity from \( \Phi \) then the difference between \( u \) and \( v \) is only in the corresponding blocks \( p \) and \( q \). Also notice that at most two blocks of \( u \) (and \( v \)) contain both \( x \) and \( y \). Moreover, if two blocks of \( u \) contain both \( x \) and \( y \) then \( u \) has at most three blocks and the last block of \( u \) is \( xy \). For example, the set \( \Phi \) contains \( y s x y^2 t x y \approx y s x y t x y \). For more examples of identities in \( \Phi \) we refer the reader to look at the cluster of seven identities used in the proof of Proposition 6.5 below.

Given a monoid \( M \) we use \( \Phi(M) \) to denote the set of those identities from \( \Phi \) that hold in \( M \).

**Proof of Proposition 3.1.** If \( xy \) is not an isoterm for \( M \), then \( M \) is either commutative or idempotent by Lemma 2.7. Then \( M \) is FB because each commutative monoid [6] and each idempotent monoid [25] are FB. So, we may assume that \( xy \) is an isoterm for \( M \). According to Lemma 3.4, every identity of \( M \) can be derived from a compact well-balanced identity of \( M \) together with \( \Sigma \cup \Gamma(M) \). By the result of Volkov [23, Corollary 2], every set of almost-linear identities gives a FB variety. Thus, \( \text{var}\Gamma(M) \) is FB. It is easy to see that every subset of \( \Phi \) gives a FB subvariety within \( \text{var}\Sigma \). So, it suffices to prove that every compact well-balanced identity \( u \approx v \) of \( M \) follows from \( \Sigma \cup \Phi(M) \).

Let \( \{i_x, j_y\} \) be a critical pair in \( u \approx v \) and \( w \) denote the result of replacing \( i_u x j_u y \) by \( j_u y i_u x \) in \( u \). We consider four cases and show that in every case the identity \( u \approx v \) together with \( \Sigma \) imply some identity in \( \Phi(M) \) which (together with
can be used to “remove the critical pair \( \{i, j\} \)”, that is, to derive the identity \( u \approx w \).

Since \( u \approx v \) is well-balanced, we have \( u(\text{sim}(u)) = v(\text{sim}(v)) \) and both \( x \) and \( y \) are multiple letters. Let \( a \) denote the block of \( u \), which contains the critical pair \( \{i, j\} \) and \( a' \) denote the corresponding block of \( v \) to \( a \). Then \( a = a_1 u x a_2 \) and \( v = v' a' v'' \) for some \( a_1, a_2, u', u'', v', v'' \in A^* \).

**Case 1**: block \( a \) contains neither the first occurrence of \( x \) nor the first occurrence of \( y \). In this case, \( a(x, y) = xy \) and \( a'(x, y) = yx \) because \( u \approx v \) is compact and well-balanced. Two subcases are possible.

**Subcase 1.1**: some block \( b \) of \( u'' \) contains both an occurrence of \( x \) and an occurrence of \( y \). Let \( u'' = u_1 b u_2 \) for some \( u_1, u_2 \in A^* \). Let \( t \) be the simple letter that is immediately to the right of the block \( a \). Then \( u \approx v \) implies

\[
u'(x, y) x y tu''(x, y) \approx v'(x, y) y x t v''(x, y)
\]

and so

\[
u'(x, y) x y tu''(x, y) x y \approx v'(x, y) y x t v''(x, y) x y.
\]

The last identity is equivalent modulo \( \Sigma \) to

\[
p x y t x y \approx q y x t x y,
\]

where \( p, q \in \{xy, yx\} \) because we can remove all non-first occurrences of \( x \) and \( y \) in \( u'(x, y) \) and \( v'(x, y) \) and the words \( u''(x, y) \) and \( v''(x, y) \) by Lemma 3.3. Clearly, (3.8) \( \in \Phi(M) \). Then \( M \) satisfies

\[
u \approx u'a_1 p xy a_2 u_1 x y b u_2 \quad \text{Lemma 3.3}
\]

and we are done.

**Subcase 1.2**: no block of \( u'' \) contains both an occurrence of \( x \) and an occurrence of \( y \). Let \( \{t_1, t_2, \ldots, t_k\} \) be the possibly empty set of simple letters of \( u \) in \( u'' \). Then \( u \approx v \) implies

\[
u'(x, y) x y u''(x, y, t_1, t_2, \ldots, t_k) \approx v'(x, y) y x v''(x, y, t_1, t_2, \ldots, t_k)
\]

Since the identity \( u \approx v \) is well-balanced and compact,

\[
u''(x, y, t_1, t_2, \ldots, t_k) = v''(x, y, t_1, t_2, \ldots, t_k) = \prod_{\ell=1}^{k} (t_\ell e_\ell),
\]

where \( e_1, e_2, \ldots, e_k \in \{1, x, y\} \). The identity (3.9) is equivalent modulo \( \Sigma \) to

\[
p x y \prod_{\ell=1}^{k} (t_\ell e_\ell) \approx q y x \prod_{\ell=1}^{k} (t_\ell e_\ell),
\]

and we are done.
Evidently, if \( p(3.12) \in 2 \) and so \( i \) the identity \( = 1 \) and so \( j \) and so because we can remove the words \( u \) The last identity is equivalent modulo \( \Sigma \) to \( e \) of \( y \) Further, if \( x \) Clearly, \( (3.13) \)

Subcase 2.1: some block \( b \) of \( u'' \) contains both an occurrence of \( x \) and an occurrence of \( y \). Let \( u'' = u_1 bu_2 \) for some \( u_1, u_2 \in 2^* \). Let \( t \) be the simple letter that is immediately to the right of the block \( a \). Then \( u \approx v \) implies

\[
Pxyqt''u''(x, y) \approx p'yq'xr'tv''(x, y)
\]

and so

\[
Pxyqt''u''(x, y)xy \approx p'yq'xr'tv''(x, y)xy.
\]

The last identity is equivalent modulo \( \Sigma \) to

\[
Pxyqtxy \approx p'yq'xr'txy \tag{3.12}
\]

because we can remove the words \( u''(x, y) \) and \( v''(x, y) \) by Lemma 3.3. Clearly, \( (3.12) \in \Phi(M) \). Then \( M \) satisfies

\[
\begin{align*}
u & \overset{\text{Lemma 3.3}}{\approx} u'a_1pxyqa_2u_1xybu_2 \overset{\Sigma}{\approx} u'a_1p'yq'xr'a_2u_1xybu_2.
\end{align*}
\]

Evidently, if \( p' = 1 \), then \( \text{con}(p') \subseteq \text{con}(a_1) \). Let now \( p' \neq 1 \). If \( x \in \text{con}(p') \), then \( i = 2 \) and so \( x \in \text{con}(p) \). If \( y \in \text{con}(p') \), then \( j = 2 \) and so \( y \in \text{con}(p) \). We see that \( \text{con}(p') \subseteq \text{con}(p) \subseteq \text{con}(a_1) \) in either case. Analogously, \( \text{con}(r') \subseteq \text{con}(a_2) \). Further, if \( x \in \text{con}(q') \), then \( i = 2 \) and so \( x \in \text{con}(a_1) \), and if \( y \in \text{con}(q') \), then \( j = 1 \) and so \( y \in \text{con}(a_2) \) because \( u \) is compact. This implies that \( M \) satisfies

\[
u'a_1p'yq'xr'a_2u_1xybu_2 \approx w \] by Lemma 3.3 and we are done.

Subcase 2.2: no block of \( u'' \) contains both an occurrence of \( x \) and an occurrence of \( y \). Let \( \{t_1, t_2, \ldots, t_k\} \) be the possibly empty set of simple letters of \( u \) in \( u'' \). Since the identity \( u \approx v \) is well-balanced and compact, the equality \( (3.10) \) holds, where \( e_1, e_2, \ldots, e_k \in \{1, x, y\} \). Then \( u \approx v \) implies

\[
Pxyq \prod_{\ell=1}^{k} (t_\ell e_\ell) \approx p'yq'xr' \prod_{\ell=1}^{k} (t_\ell e_\ell). \tag{3.13}
\]

Clearly, \( (3.13) \in \Phi(M) \). Then \( M \) satisfies

\[
u \overset{\text{Lemma 3.3}}{\approx} u'a_1pxyqa_2u'' \overset{\Sigma}{\approx} u'a_1p'yq'xr'a_2u''.
\]
Evidently, if \( p' = 1 \), then \( \text{con}(p') \subseteq \text{con}(a_1) \). Let now \( p' \neq 1 \). If \( x \in \text{con}(p') \), then \( i = 2 \) and so \( x \in \text{con}(p) \). If \( y \in \text{con}(p') \), then \( j = 2 \) and so \( y \in \text{con}(p) \). We see that \( \text{con}(p') \subseteq \text{con}(p) \subseteq \text{con}(a_1) \) in either case. Analogously, \( \text{con}(r') \subseteq \text{con}(a_2) \). Further, if \( x \in \text{con}(q') \), then \( i = 2 \) and so \( x \in \text{con}(a_1) \), and if \( y \in \text{con}(q') \), then \( j = 1 \) and so \( y \in \text{con}(a_2) \) because \( u \) is compact. This implies that \( M \) satisfies \( u'a_1p'yq'r'a_2u'' \approx w \) by Lemma 3.3 and we are done.

**Case 3:** block \( a \) contains the first occurrence of \( x \) but does not contain the first occurrence of \( y \). In this case, letter \( x \) appears at most twice in \( a(x, y) \) but letter \( y \) only once because \( u \) is compact. Let \( a(x, y) = p_{iu}x_juq \) and \( a'(x, y) = p'_{iy}yq'_{iv}xr' \).

Let \( s \) be a simple letter that is next to the block \( a \) on the left of it. Two subcases are possible.

**Subcase 3.1:** some block \( b \) of \( u'' \) contains both an occurrence of \( x \) and an occurrence of \( y \). Let \( u'' = u_1bu_2 \) for some \( u_1, u_2 \in A^* \). Let \( t \) be the simple letter that is immediately to the right of the block \( a \). Then \( u \approx v \) implies

\[
y^m spxyqt'u''(x, y) \approx y^m sp'yq'xr'tv''(x, y)
\]

and so

\[
y^m spxyqt'u''(x, y)xy \approx y^m sp'yq'xr'tv''(x, y)xy,
\]

where \( m \) is the number of occurrences of \( y \) in \( u' \). The last identity is equivalent modulo \( \Sigma \) to

\[
y^m spxyqtxy \approx y^m sp'yq'xr'txy
\]  

(3.14)

because we can remove the words \( u''(x, y) \) and \( v''(x, y) \) by Lemma 3.3. If \( m > 1 \), then \( M \) satisfies

\[
yspxyqtxy \approx ysp'yq'xr'txy \approx ysp'yq'xr'txy \approx ysp'yq'xr'txy.
\]

So, in either case the identity

\[
yspxyqtxy \approx ysp'yq'xr'txy
\]  

(3.15)

holds in \( M \). Clearly, \((3.15) \in \Phi(M)\). Then \( M \) satisfies

\[
u \approx u'a_1p'xyqa_2u_1xybu_2 \approx u'a_1p'yq'xr'a_2u_1xybu_2.
\]

Since \( p'q'r' \in \{1, x\} \), if \( p'q' \neq 1 \), then \( i = 2 \), whence \( \text{con}(p'q') \subseteq \text{con}(p) \) and so \( \text{con}(p'q') \subseteq \text{con}(a_1) \). Analogously, \( \text{con}(r') \subseteq \text{con}(a_2) \). This implies that \( M \) satisfies \( u'a_1p'yq'r'a_2u_1xybu_2 \approx w \) by Lemma 3.3 and we are done.

**Subcase 3.2:** no block of \( u'' \) contains both an occurrence of \( x \) and an occurrence of \( y \). Let \( \{t_1, t_2, \ldots, t_k\} \) be the possibly empty set of simple letters of \( u \) in \( u'' \). Since the identity \( u \approx v \) is well-balanced and compact, the equality \((3.10) \) holds, where \( e_1, e_2, \ldots, e_k \in \{1, x, y\} \). Then \( u \approx v \) implies

\[
y^m spxyqc \approx y^m sp'yq'xr'c,
\]  

(3.16)
where \( m \) is the number of occurrences of \( y \) in \( u' \) and \( c = \prod_{\ell=1}^{k}(t_{\ell}e_{\ell}) \). If \( m > 1 \), then \( M \) satisfies
\[
ysp_{xy}qc \quad \text{Lemma 3.3} \quad \approx \quad ysy'^{m}p_{xy}qc \quad \text{Lemma 3.10} \quad \approx \quad ysy'^{m}p'yq'xr'c \quad \text{Lemma 3.3} \quad \approx \quad ysp'yq'xr'c.
\]
So, in either case the identity
\[
ysp_{xy}qc \approx ysp'yq'xr'c \quad (3.17)
\]
holds in \( M \). Clearly, \((3.17) \in \Phi(M)\). Then \( M \) satisfies
\[
u' \quad \text{Lemma 3.3} \quad \approx \quad u'a_{1}p_{xy}qc_{a_{2}u''} \quad (3.17) \quad \approx \quad u'a_{1}p'yq'x'r'a_{2}u''.
\]

Since \( p'q'r' \in \{1, x\} \), if \( p'q' \neq 1 \), then \( i = 2 \), whence \( \text{con}(p'q') \subseteq \text{con}(p) \) and so \( \text{con}(p'q') \subseteq \text{con}(a_{1}) \). Analogously, \( \text{con}(r') \subseteq \text{con}(a_{2}) \). This implies that \( M \) satisfies
\[
u'a_{1}p'yq'x'r'a_{2}u'' \approx w \quad \text{by Lemma 3.3} \quad \text{and we are done.}
\]

**Case 4**: block \( a \) contains the first occurrence of \( y \) but does not contain the first occurrence of \( x \). This case is similar to Case 3.

So, we have proved that the critical pair \( \{i, j\} \) is \((\Sigma \cup \Phi(M))\)-removable in \( u \approx v \). Now Lemma 3.3 applies, with the conclusion that every compact well-balanced identity \( u \approx v \) of \( M \) can be derived from \( \Sigma \cup \Phi(M) \), and we are done. \( \square \)

**Corollary 3.6.** *Any monoid \( M \) that satisfies \( xtx \approx xtx^{2} \) and \( xy^{2}tx \approx xy^{2}xtx \) (3.18) is HFB.***

**Proof.** The identities
\[
xtxsy^{2}tx \approx xtyx^{2}sy \quad (3.18) \quad \approx \quad xtyx^{2}ysy \quad \approx \quad xtyxysy
\]
\[
xysytx^{2}tx \approx xy^{2}xsyt \approx x^{2}ysytx
\]
hold in \( M \). Therefore, \( M \) is HFB by the dual to Proposition 3.1. \( \square \)

Put
\[
E = \langle a, b, c | a^{2} = ab = 0, ba = ca = a, b^{2} = bc = b, c^{2} = cb = c \rangle = \{a, b, c, ac, 0\}.
\]
The monoid \( E^{1} \) was first investigated in Lee and Li \cite{14, Section 14}, where it was shown to be finitely based by \( \{xtx \approx xtx^{2} \approx x^{2}tx, xy^{2}x \approx x^{2}y^{2}\} \). Let \( \overline{A}^{1} \) denote the dual of the monoid \( A^{1} \) (see Section 1).

**Example 3.7.** *The monoid \( \overline{A}^{1} \times E^{1} \) is HFB.***

**Proof.** It follows from \cite{14, Section 14} that \( E^{1} \models \{ xtx \approx xtx^{2}, xy^{2}tx \approx xy^{2}xtx \} \). According to \cite{27, p. 15}, we also have \( \overline{A}^{1} \models \{ xtx \approx xtx^{2}, xy^{2}tx \approx xy^{2}xtx \} \). Therefore, \( \overline{A}^{1} \times E^{1} \) is HFB by Corollary 3.6. \( \square \)

Example 3.7 generalizes the result from \cite{8} that \( E^{1} \) is HFB and the result from \cite{27} that \( A^{1} \) is HFB.
4 Classification of varieties of aperiodic monoids

Recall from Section 2 that \( L = M(xtysy) \) and \( M = M(xytxsy, xstxy) \) are limit varieties of monoids \([7]\). The following lemma is a combination of \([11, \text{Theorem 3.2}]\) and \([4, \text{Lemma 2.1}]\).

**Sorting Lemma 1.** Let \( V \) be a variety of aperiodic monoids. Then either \( V \) is HFB or one of the following holds:

(i) \( V \) contains either \( L \) or \( M \);

(ii) \( V \) satisfies either \( xtx \approx xttx^2 \) or \( xtx \approx x^2tx \).

The goal of this section is to refine Sorting Lemma 1 (see Sorting Lemma 2 below).

We note that some varieties can be generated by monoids of the form \( M_\tau(u) \) for several congruences \( \tau \). For example, \( M_\gamma(a^+t) \approx M_\lambda(a^+t) \approx M_\rho(a^+t) \) \([22, \text{Fact 6.2}]\).

In such a case, we choose the coarsest congruence \( \tau \) to identify the monoid variety.

**Lemma 4.1.** Let \( V \) be a monoid variety such that \( t \) is an isoterm for \( V \) and \( V \) satisfies \( xtx \approx xttx^2 \). If \( M_\lambda(a^+t) \not\subseteq V \), then \( V \) satisfies \( xtxs \approx xtxsx \).

**Proof.** Since \( V \) does not contain \( M_\lambda(a^+t) \), the \( \gamma \)-word \( a^+t \) is not a \( \gamma \)-term for \( V \) by Lemma 2.2. Since \( t \) is an isoterm for \( V \), the variety \( V \) satisfies \( x^n t \approx x^m t x^k \) for some \( n, m \geq 2 \) and \( k \geq 1 \). In view of \( xtx \approx xttx^2 \), the variety \( V \) satisfies \( x^2t \approx x^2tx \). It is easy to check that the identity \( xtxs \approx xtxsx \) is equivalent to \( \{xtx \approx xttx^2, x^2t \approx x^2tx \} \). 

The next lemma is a reformulation of Lemma 4.1 in \([4]\) using Theorem 7.2 in \([22]\).

**Lemma 4.2.** Let \( V \) be a monoid variety such that \( \mathbb{M}_\lambda(ata^+, a^+t) \subseteq V \) and \( V \models xtx \approx xttx^2 \). Then \( V \) satisfies

\[
xtysy \approx xtxysy
\]

whenever \( J = \mathbb{M}_\lambda(atb^+s^+b^+) \not\subseteq V \).

**Lemma 4.3.** Let \( V \) be a monoid variety that satisfies the identity \( xtx \approx xttx^2 \). If \( V \) contains \( \mathbb{M}_\lambda(ata^+) \) but does not contain \( K = \mathbb{M}_\lambda(bta^+b^+) \), then \( V \) satisfies the identity

\[
xty^2x \approx xty^2yx.
\]

**Proof.** Since \( \mathbb{M}_\lambda(bta^+b^+) \not\subseteq V \), Lemma 2.4 implies that \( bta^+b^+ \) is not a \( \lambda \)-term for \( V \). Since \( ata^+ \) is a \( \lambda \)-term for \( V \), we get that \( V \) satisfies \( xty^p x^q \approx xta \) for some \( p \geq 2, q \geq 1 \) and some word \( a \in \{x, y\}^+ \) such that \( xy \) is a subword of \( a \). Then the identities

\[
\begin{align*}
xtysy \approx xtysy & \quad xty^2x^3x \approx xty^2ax \quad xty^2x \approx xty^2yx
\end{align*}
\]

hold in \( V \), and we are done.
Lemma 4.4. Let \( V \) be a monoid variety that satisfies \( xtx \approx x^2tx \). If \( \mathbb{M}_\lambda(ata^+) \not\subseteq V \), then \( V \) satisfies \( xtx \approx x^2tx \).

Proof. If \( xy \) is not an isoterm for \( V \), then \( V \) is commutative or idempotent by Lemma 2.7. In either case, \( V \) satisfies \( xtx \approx x^2tx \) because \( xtx \approx x^2tx^2 \approx x^2tx \) and \( xtx \approx x^2tx \). So, we may assume that \( xy \) is an isoterm for \( V \).

Since \( \mathbb{M}_\lambda(ata^+) \not\subseteq V \), Lemma 2.4 implies that \( ata^+ \) is not a \( \lambda \)-term for \( V \). Then in view of Lemma 2.6, the variety \( V \) satisfies an identity of the form \( xtx^k \approx x^ptx^q \), where \( k, q \geq 1 \) and \( p \geq 2 \). This identity is equivalent modulo \( xtx \approx x^2tx \) and, we are done.

Lemma 4.5. Let \( V \) be a monoid variety that satisfies the identities \( xtx \approx x^2tx \approx x^2tx \approx x^2tx \). If \( A^1 \not\subseteq V \), then \( V \models xty \approx (xy)^2tx \).

Proof. If \( x \) is not an isoterm for \( V \), then \( V \) satisfies \( x \approx x^2 \) and consequently, \( xy^2tx \approx (xy)^2tx \). So, let us assume that \( x \) is an isoterm for \( V \) and consider two cases.

Case 1: \( V \) does not contain \( \mathbb{M}_\gamma(a^++ta^+) \). According to Lemma 2.2, either \( a^+t \) or \( ta^+ \) is not a \( \gamma \)-term for \( V \). Then the variety \( V \) satisfies either \( x^2t \approx x^mtx^k \) or \( tx^n \approx x^mtx^k \) for some \( n \geq 2 \) and \( m, k \geq 1 \). In view of \( xtx \approx x^2tx \approx x^2tx \), the variety \( V \) satisfies either \( x^2t \approx x^2tx \) or \( tx^2 \approx x^2tx \). Each of these identities implies \( xy^2tx \approx (xy)^2tx \).

Case 2: \( V \) contains \( \mathbb{M}_\gamma(a^++ta^+) \). According to Theorem 4.3(iii) in [22], we have \( A^1 = \mathbb{M}_\gamma(a^+b^+ta^+) \). Then Lemma 2.2 implies that \( a^+b^+ta^+ \) is not a \( \gamma \)-term for \( V \). Since \( a^+t \) and \( ta^+ \) are \( \gamma \)-terms for \( V \), the variety \( V \) satisfies \( xy^2tx \approx atx \) for some \( a \in \{x, y\}^+ \) such that \( a \) contains \( xy \) as a subword. Then

\[
V \models xy^2tx \approx x^2y^3tx \approx xaytx \approx xtx^2tx^2x^2tx \approx (xy)^2tx,
\]

and we are done.

Sorting Lemma 2. Let \( V \) be a variety of aperiodic monoids. Then either \( V \) is HFB or one of the following holds:

(i) \( V \) contains one of the varieties \( A^1 \lor A_1, J, \overline{J}, K, \overline{K}, L \) or \( M \);

(ii) \( V \) satisfies either \( \{xtx \approx x^2tx, xy^2tx \approx (xy)^2tx\} \) or dually, \( \{xtx \approx x^2tx, xty^2tx \approx x(yx)^2\} \);

(iii) \( V \) satisfies either \( xtxs \approx xtssx \) or dually, \( txsx \approx txsx \).

Proof. Suppose that \( V \) is not HFB and does not contain any of the varieties \( A^1 \lor A_1, J, \overline{J}, K, \overline{K}, L \) or \( M \). Sorting Lemma 1 implies that \( V \) satisfies \( xtx \approx x^2tx \) or \( xtx \approx x^2tx \). By symmetry, we may assume that \( V \) satisfies \( xtx \approx x^2tx \). If \( x \) is not an isoterm for \( V \), then \( V \) is idempotent and consequently, satisfies \( xy^2tx \approx (xy)^2tx \).
So, we may assume that \( x \) is an isoterm for \( V \). If \( a^t \) is not a \( \gamma \)-term for \( V \), then \( V \) satisfies \( xtxs \approx xtxsx \) by Lemma 4.1. So, we may assume that \( a^t \) is a \( \gamma \)-term for \( V \). Consider two cases.

**Case 1:** \( V \) contains \( M_\lambda(ata^+) \). Since \( V \) does not contain \( J \), Lemma 4.2 implies that \( V \models (4.1) \). Erasing \( s \) from (4.1) we obtain

\[
xtxy \approx xt(xy)^2.
\]

(4.3)

Since \( K \not\subseteq V \), the variety \( V \) satisfies (4.2) by Lemma 4.3. Consequently, \( V \) satisfies

\[
xsytxy \approx xt(2x) \approx xsyt(xy)^2 (4.3)
\]

Hence \( V \) is HFB by Proposition 3.1.

**Case 2:** \( V \) does not contain \( M_\lambda(ata^+) \). In this case, Lemma 4.4 implies that \( V \models xt \approx x^2t \). Since \( A^1 \lor A^1 \not\subseteq V \), Lemma 4.5 and its dual imply that \( V \) satisfies one of the following identities

\[
x^2tx \approx x^2(2t) \approx (xy)^2tx, \quad xy^2tx \approx (xy)^2tx
\]

we are done.

**Corollary 4.6.** Let \( V \) be a variety of \( J \)-trivial monoids. Then either \( V \) is HFB or \( V \) contains one of the varieties \( A^1 \lor A^1, J, \overleftarrow{J}, K, \overleftarrow{K}, L \) or \( M \).

**Proof.** Suppose that \( V \) is not HFB and does not contain any of the varieties \( A^1 \lor A^1, J, \overleftarrow{J}, K, \overleftarrow{K}, L \) or \( M \). In view of Sorting Lemma 2 two cases are possible.

**Case 1:** \( V \) satisfies \( \{xtx \approx x^2t, xy^2tx \approx (xy)^2tx\} \) or \( \{xtx \approx x^2t, x^2tx \approx xt(xy)^2\} \). By symmetry, we may assume that the first of these identity system holds in \( V \). Since \( V \) is \( J \)-trivial, \( V \) satisfies \( (xy)^2 \approx (yx)^2 \) by Fact 2.1. Hence, \( V \) satisfies

\[
xy^2tx \approx (xy)^2tx \approx (xy)^2tx \approx (xy)^2tx \approx (xy)^2tx \approx xy^2tx.
\]

So, \( V \) satisfies \( xy^2tx \approx xy^2tx \). Consequently, \( V \) is HFB by Corollary 3.6.

**Case 2:** \( V \) satisfies either \( xtxs \approx xtxsx \) or \( txsx \approx txsx \). Since \( V \) is \( J \)-trivial, \( V \) satisfies \( (xy)^2 \approx (yx)^2 \) by Fact 2.1. By symmetry, we may assume that the first of these identities holds in \( V \). It is routine to verify that

\[
\text{var}\{xtsx, (xy)^2 \approx (yx)^2\} = \text{var}\{xtx \approx x^2t, x^2t \approx x^2tx, x^2y^2 \approx y^2x^2\}.
\]

This variety is HFB by Proposition 6.1 in [5].
5  The last two examples of limit varieties of $\mathcal{J}$-trivial monoids

A monoid $M$ is said to be non-finitely based if var $M$ is NFB. As in Section 3 we use $iux$ to refer to the $i$th from the left occurrence of $x$ in a word $u$. We use $iux$ to refer to the last occurrence of $x$ in $u$. If $x$ is simple in $u$ then we use $ux$ to denote the only occurrence of $x$ in $u$. If the $ith$ occurrence of $x$ precedes the $jth$ occurrence of $y$ in a word $u$, we write $(iux) <_u (juy)$. If $u = \xi(s)$ for some endomorphism $\xi$ of $\mathcal{A}*$ and $iux$ is an occurrence of a letter $x$ in $u$ then $\xi^{-1}(iux)$ denotes an occurrence $juz$ of a letter $z$ in $s$ such that $\xi(izu)$ regarded as a subword of $u$ contains $iu$.

Sufficient Condition. Let $M$ be a monoid such that $M$ satisfies the identity

$$u_n = xy_1^2y_2^2 \cdots y_{n-1}^2y_n^2 \approx xy_1^2y_2^2 \cdots y_{n-1}^2y_n^2yx_n = v_n$$  (5.1)

for any $n \geq 1$. If the $\lambda$-word $bta+b^+$ is a $\lambda$-term for $M$ then $M$ is NFB.

Proof. Let $u$ be a word such that $M \models u_n = u$ and $(iuu_n) <_u (2ux)$. Since the last occurrence of $y_n$ succeeds the second occurrence of $x$ in $v_n$, in view of Fact 2.1 in [20], to show that $M$ is NFB, it suffices to establish that if the identity $u \models v$ is directly deducible from some identity $s \models t$ of $M$ in less than $n-2$ variables, i.e. $u = a\xi(s)b$ and $v = a\xi(t)b$ for some words $a, b \in \mathcal{A}*$ and some endomorphism $\xi$ of $\mathcal{A}*$, then $(iuu_n) <_v (2ux)$.

We note that if $a, b \notin \text{con}(st)$ then the identity $s \models t$ is equivalent to $asb \models atb$. Then there exists an endomorphism $\xi$ of $\mathcal{A}*$ such that $\xi(a) = a$, $\xi(b) = b$, $\xi(s) = \xi(s)$ and $\xi(t) = \xi(t)$. It follows that we may assume without any loss that $a = b = 1$ and so $u = \xi(s)$ and $v = \xi(t)$, and $s \models t$ is an identity of $M$ in less than $n$ variables.

Since $a^+b^+$ is a $\lambda$-term for $M$ by Fact 2.3 we have:

$$(1ux) <_u (1uy_1) <_u (1uy_2) <_u (1uy_3) <_u \cdots <_u (1uy_n) <_u (1y_n) <_u (2ux).$$  (5.2)

Clearly, the word $ab$ is an isoterm for $M$. Then $\text{sim}(s) = \text{sim}(t)$, $\text{mul}(s) = \text{mul}(t)$, $\text{sim}(u) = \text{sim}(v)$ and $\text{mul}(u) = \text{mul}(v)$ by Lemma 2.6. We use these facts below without any reference.

Working toward a contradiction, suppose that $(2ux) <_v (1y_n)$. Then $(z^{-1}(2ux)) <_t (z^{-1}(1y_n))$, where $z^{-1}(2ux)$ is either the first or the second occurrence of some letter $z$ in $t$ and $z^{-1}(1y_n)$ is the last occurrence of some letter $y$ in $v$. If $y = z$, then $y = z \in \text{sim}(s)$ because $\xi(y) = \xi(z)$ contains both $x$ and $y$ and would otherwise appear at least twice in $u$ as a subword, contradicting (5.2). Then, in view of (5.2), we have $s = z = (z^{-1}(1ux)$ and $z^{-1}(1y_n)$ is not an occurrence of $y = z$ in $t$. Clearly, $(z^{-1}(1y_n) < (z^{-1}(1y_n)$ but $(z^{-1}(1y_n) < (z^{-1}(1y_n)$. This is impossible, because all the $\lambda$-words in $\{b^+a^+\}^{\leq s}$ are $\lambda$-terms for $M$ by Fact 2.3. So, we may assume that $y \neq z$.

Since all the $\lambda$-words in $\{b^+a^+\}^{\leq s}$ are $\lambda$-terms for $M$, $s(z, y) \neq y^iz$ for any $i, j \geq 0$. Hence $s(z, y) \neq y^iz$. Using (5.2) and that $\xi(y)$ contains $y_n$, we obtain:

$$(1y_n) < s (1ux) \leq (z^{-1}(1ux) < (z^{-1}(1y_n).$$

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Since $\xi(z)$ contains $x$ and $(\xi^{-1}(1u) x) \leq_s (\xi^{-1}(1uy_n))$ by (5.2), we conclude that
\[
\xi^{-1}(1u) x = 1sz, \quad x \in \text{sim}(\xi(z)). \tag{5.3}
\]

In view of (5.2) and the fact that $s$ involves less than $n$ letters, the word $s$ has some letter $c$ such that $\xi(c)$ contains $y_iy_{i+1}$ as a subword for some $1 \leq i < n$. The letter $c$ is simple in $s$ because for each $1 \leq i < n$ the word $y_iy_{i+1}$ appears only once in $u$. Using (5.2), (5.3) and that $\xi(y)$ contains $y_n$, we obtain:
\[
(1sz) \leq_s (sc) \leq_s (\xi^{-1}(1uy_n)) \leq_s (1sz) \leq_s (tsy) \leq_s (\xi^{-1}(1uy_n)) \leq_s (\xi^{-1}(2ux)). \tag{5.4}
\]

Two cases are possible.

\textbf{Case 1: } $\xi^{-1}(2vx) = 2tz$.

In this case, $z$ is multiple in $t$. Then $(\xi^{-1}(1ux)) \leq_s (2sz)$, because $\xi(z)$ contains $x$. Since $c$ is simple in $t$, $z \neq c$. Using (5.4) and that $z \neq y$ we obtain:
\[
(1sz) <_s (sc) \leq_s (1sz) \leq_s (tsy) <_s (2sz).
\]

Since $bta^+b^+$ is a $\lambda$-term for $M$, it is easily to see that $bcb^+$ and $bta^+$ are $\lambda$-terms for $M$ too. Then we have:
\[
(1tz) <_t (tc) \leq_t (1ty) \leq_t (ty) <_t (2tz).
\]

Our assumption that $(2vx) <_v (tv)\ y_n$ implies that
\[
(2tz) = (\xi^{-1}(2vx)) \leq_t (\xi^{-1}(tv)\ y_n) = (ty),
\]
a contradiction.

\textbf{Case 2: } $\xi^{-1}(2vx) = 1tv$.

Then using (5.4) and the fact that all the $\lambda$-words in $\{a^+b^+\} \leq_s$ are $\lambda$-terms for $M$, we obtain:
\[
(1tz) \leq_t (tc) \leq_t (1ty). \tag{5.5}
\]

In view of Fact 2.6 in [20], $\xi^{-1}(1vx) = 1tv$ for some $d \in \text{con}(t) = \text{con}(s)$. In this case, $(1tv) = (\xi^{-1}(1tv)) \leq_t (\xi^{-1}(2vx)) = (1tz)$ and therefore $c \neq d$.

Suppose that $(1sz) <_s (sc)$. Since $(1sz) \leq_s (sc) \leq_s (\xi^{-1}(1uy_n))$ by (5.4), and both $\xi(z)$ and $\xi(d)$ contain $x$, we obtain that $(2ux) \leq_u (1uy_n)$, which contradicts to (5.2).

Suppose that $(sc) <_s (1sz)$. Then $M$ satisfies $s(c, d) = cdp \approx d^p cd^\ell = t(c, d)$ for some $p, q \geq 1$ and $\ell \geq 0$, which contradicts the fact that $a^+b^+$ is a $\lambda$-term for $M$.

Therefore, $(tv) \ y_n <_v (2vx)$. Then the monoid $M$ is NFB by Fact 2.1 in [20]. \qed

\textbf{Proposition 5.1.} The varieties $K$ and $\overline{K}$ are limit and different from the limit varieties $A^1 \lor \overline{A^1}$, $J$, $\overline{J}$, $L$ and $M$. 

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Proof. It is a routine to verify that \( K \) satisfies the identity (5.1) for any \( n \geq 1 \). Then Sufficient Condition and Lemma 2.4 imply that \( K \) is NFB.

It is a routine to check that \( K \) satisfies the identities \( xtx \approx xtx^2 \) and \( xy^2tx \approx yxytx \). Therefore, \( K \) contains none of the varieties \(~\mathcal{J}, \mathcal{K}, L \) and \( M \) because they violate \( xtx \approx xtx^2 \). Further, \( K \) does not contain the varieties \( A^1 \lor \mathcal{A}^1 \) and \( J \) because these varieties do not satisfy \( xy^2tx \approx yxytx \).

Since the variety \( K \) is NFB and does not contain any of the varieties \( A^1 \lor \mathcal{A}^1 \), \( L \), \( M \), \( J \) or \( \mathcal{J} \), Corollary 4.6 implies that each proper subvariety of \( K \) is FB. Thus, \( K \) is a new limit variety.

Proof of Theorem 1.1. Corollary 4.6 and Proposition 5.1 imply that a variety of \( J \)-trivial monoids is HFB if and only if it excludes the varieties \( A^1 \lor \mathcal{A}^1 \), \( J \), \( \mathcal{J} \), \( \mathcal{K} \), \( \mathcal{K} \), \( L \) and \( M \). Consequently, there are precisely seven limit varieties of \( J \)-trivial monoids.

6 The subvariety lattice of \( K \)

This section is devoted to a description of the subvariety lattice of the limit variety \( K = \mathbb{M}_\lambda(bta^+b^+) \). The following statement collects some identities of \( K \) and can be easily verified. We use it sometimes without any reference.

Lemma 6.1. The identities \( xytxsy \approx yxtxsy \), (3.5) and (4.3) hold in the variety \( K \). \( \Box \)

Let \( A^1_0 = \text{var} \{ e, f \mid e^2 = e, f^2 = f, fe = 0 \} = \{ e, f, ef, 0 \} \).

Lemma 6.2. The variety \( A^1_0 \lor \mathbb{M}_\lambda(ata^+) \) satisfies an identity \( u \approx v \) if and only if each of the following holds:

(a) \( \text{sim}(u) = \text{sim}(v) \) and \( \text{mul}(u) = \text{mul}(v) \);

(b) for each \( x, y \in \text{con}(u) \) we have \( (1ux) <_u (1uy) \) iff \( (1v x) <_v (1v y) \);

(c) for each \( t \in \text{sim}(u) \) and \( x \in \text{mul}(u) \) we have

\[
(ut) <_u (1ux) \text{ iff } (vt) <_v (1vx) \text{ and } (ut) <_u (2ux) \text{ iff } (vt) <_v (2vx).
\]

Proof. Put \( F = \text{var}\{xtx \approx xtx^2, x^2t \approx x^2tx, x^2y^2 \approx y^2x^2\} \). According to Theorem 7.1(i) in [22], we have \( F = \mathbb{M}_\lambda(ata^+) \). Proposition 6.9 in [5] implies that \( F \) satisfies an identity \( u \approx v \) if and only if (a) and (c) hold. Proposition 4.2 in [19] implies that \( A^1_0 \) satisfies an identity \( u \approx v \) if and only if (a) and (b) hold. Consequently, \( A^1_0 \lor \mathbb{M}_\lambda(ata^+) \models u \approx v \) if and only if (a), (b) and (c) hold. \( \Box \)

Lemma 6.3.
(i) $A_0^1 = M_{\lambda}(a^+b^+)^+ = M_{\lambda}(a^+b^+)^+ = \text{var}\{xtsx \approx xttxs, (xy)^2 \approx (yx)^2\}$;

(ii) $A_0^1 \vee M_{\lambda}(ata^+) = M_{\lambda}(ata^+)^+ = \text{var}\{xtx \approx xtx^2, xtytxy \approx xtysxy, xytsty \approx yxtsxs, (3.5)\}$.

**Proof.** (i) It follows from [21, Section 7] that the monoid $A_0^1$ is isomorphic to $M_{\tau_1}(a^+b^+)$. According to Theorem 4.1(iv) and Fact 6.2 in [22], the monoid $M_{\lambda}(a^+b^+)$ also generates the variety $A_0^1$. Proposition 3.2(a) in [1] says that $A_0^1$ is finitely based by $\{xtsx \approx xttxs, xtysxy \approx xtysxy, xytsty \approx yxtsxs\}$. It is easy to see that this set of identities is equivalent to $\{xtsx \approx xttxs, (xy)^2 \approx (yx)^2\}$.

(ii) Since $A_0^1 = M_{\lambda}(a^+b^+)$ by Part (i), we have

$A_0^1 \vee M_{\lambda}(ata^+)^+ = M_{\lambda}(ata^+, a^+b^+)^+$.

The inclusion $M_{\lambda}(ata^+, a^+b^+) \subseteq M_{\lambda}(ata^+)^+$ follows from Fact 2.3 and Lemma 2.4. The inclusion $M_{\lambda}(ata^+, a^+b^+) \supseteq M_{\lambda}(ata^+)^+$ holds by the fact that $ata^+b^+$ is a $\lambda$-term for $M_{\lambda}(ata^+, a^+b^+)$ and Lemma 2.4. Therefore, $A_0^1 \vee M_{\lambda}(ata^+) = M_{\lambda}(ata^+)^+$.

It is routine to check that $M_{\lambda}(ata^+)^+$ satisfies $xtx \approx xtx^2$ and $xtysxy \approx xtysxy$. Since $a^+b^+$ is a $\lambda$-term for $K = M_{\lambda}(ata^+)^+$ by Fact 2.3, $M_{\lambda}(a^+b^+) \subseteq K$ by Lemma 2.4. Clearly, $ata^+$ is a $\lambda$-term for $K$. It follows that $M_{\lambda}(ata^+) \subseteq K$ by Lemma 2.4. In view of the above, $M_{\lambda}(ata^+)^+$ is a subvariety of $K$. Hence the variety $M_{\lambda}(ata^+)^+$ satisfies the identities $xytsxy \approx yxtsxs$ and (3.5) by Lemma 6.1.

An identity is called 3-limited if every letter occurs in each side of it at most 3 times. The identity (3.5) allows us to add and delete the occurrences of a letter $x$ between the second and the last occurrences of $x$, while the identity $xtx \approx xtx^2$ allows us to add and delete the occurrences of $x$ next to the non-first occurrence of $x$. Thus, every identity of $M_{\lambda}(ata^+)^+$ can be derived from $xtx \approx xtx^2$, (3.5) and a 3-limited identity of $M_{\lambda}(ata^+)^+$.

Let $u \approx v$ be a 3-limited identity of $M_{\lambda}(ata^+)^+$. Since $u \approx v$ has Properties (a), (b) and (c) in Lemma 6.2, it is well-balanced and is a consequence from $\{xtysxy \approx xtysxy, xytsty \approx yxtsxs\}$ by Lemma 4.1 in [19]. Therefore,

$\{xtx \approx xtx^2, xytsty \approx yxtsxs, xytsty \approx xytsty, (3.5)\}$

is an identity basis for $A_0^1 \vee M_{\lambda}(ata^+) = M_{\lambda}(ata^+)^+$.

**Fact 6.4.** Let $V$ be a variety of $\mathcal{J}$-trivial monoids. Then $V$ does not contain $A_0^1$ if and only if $V$ is a variety of aperiodic monoids with commuting idempotents.

**Proof.** Necessity. Suppose that $A_0^1 \notin V$. Then in view of Lemmas 2.2 and 6.3(i), the variety $V$ satisfies an identity $x^py^k \approx w$, where $p, k \geq 1$ and $w$ contains $yx$ as a subword. Without loss of generality, we may assume that $\text{con}(w) = \{x, y\}$.

If $M \in V$ and $e, f$ are two idempotents of $M$ then $ef = w(e, f)$, where $w$ contains $fe$ as a subword. If $w(e, f) = fe$ then we are done. Otherwise, we multiply this equality by $e$ on the left and $f$ on the right and obtain $ef = (ef)^m$ for some $m \geq 2$. 21
In view of Fact 2.1, \( V \) satisfies the identities (2.1) for some \( n \geq 1 \). The equality \( ef = (ef)^m \) can be iterated to make \( m \geq n \). Since \( e \) and \( f \) are arbitrary idempotents of \( M \), we also have \( fe = (fe)^m \). Therefore,

\[
ef = (ef)^m \overset{2.1}{=} (fe)^m = fe.
\]

**Sufficiency** follows from the fact that the idempotents \( e \) and \( f \) of \( A_{10}^1 \) do not commute. \( \square \)

For a monoid variety \( \mathbb{V} \), we denote its subvariety lattice by \( L(\mathbb{V}) \).

**Proposition 6.5.** The lattice \( L(\mathbb{K}) \) has the form shown in Fig. 1.

**Figure 1: the lattice \( L(\mathbb{K}) \)**

\[
\begin{align*}
K &= M_\lambda(bta^+b^+) \\
N &= M_\lambda(a^+tba^+) \\
M_\lambda(a^+ta^+) &\rightarrow A_0^1 = M_\tau_1(a^+b^+) \\
M_\lambda(ata^+) &\rightarrow M_\gamma(a^+ta^+) \\
M_\gamma(ta^+) &\rightarrow M_\gamma(a^+t) \\
M_1(x) &\rightarrow M(x) \\
M(1) &
\end{align*}
\]

**Proof.** Let \( X \) be a proper subvariety of \( K \). If \( M_\lambda(ata^+) \not\subseteq X \), then \( X \models xttx \approx x^2tx \) by Lemma 4.4. Then \( X \) satisfies

\[
xttx \overset{xttx\approx x^2tx}{\approx} x^2txx \overset{3.5}{\approx} x^2txsx \overset{xttx\approx x^2tx}{\approx} xttxsx.
\]

Since \( \{xttx \approx xttxsx, (xy)^2 \approx (yx)^2\} \) is a basis of identities for \( A_0^1 \) by Lemma 6.3(i), we have \( X \subseteq A_0^1 \). The lattice \( L(A_0^1) \) is as shown in Fig. 1 by [12, Fig. 2]. According to Theorem 4.1 in [22], the monoids \( M_\gamma(a^+ta^+) \), \( M_\gamma(ta^+) \) and \( M_\gamma(a^+t) \), respectively, generate the same varieties as the monoids \( B_0^1, I^1 \) and \( J^1 \) in [12, Fig. 2]. So, we may assume that \( M_\lambda(ata^+) \subseteq X \).
If $A^1_0 \not\subseteq X$, then $X$ is a variety of aperiodic monoids with commuting idempotents by Fact 6.4. Since, for any $M \in X$ and $x, y \in M$, the elements $x^2$ and $y^2$ are idempotents, we have $X \models x^2y^2 \approx y^2x^2$. Hence, in view of Lemma 6.1, $X$ is a subvariety of

$$N = \text{var}\{xtx \approx xtx, ytxsy \approx ytxsy, x^2y^2 \approx y^2x^2, (3.5)\}.$$ 

Put $E = \text{var}\{x^2 \approx x^3, x^2y \approx xyx, x^2y^2 \approx y^2x^2\}$. In view of [3, Lemma 3.2], the lattice $L(N)$ is the set-theoretical union of the lattice $L(F \lor \hat{E})$ and the interval $[F \lor \hat{E}, N]$. The lattice $L(F \lor \hat{E})$ is as shown in Fig. 1 by [3, Fig. 1]. According to Theorem 7.2(ii) in [22], the monoid $M_{\lambda}(a^+ta^+)$ generates the variety $F \lor \hat{E}$. Thus, we may assume that $X \in [M_{\lambda}(a^+ta^+), N]$.

Clearly, $N$ satisfies (4.4). Then Lemmas 3.4 and 3.5 in [4] imply that each variety in $[M_{\lambda}(a^+ta^+), N]$ is defined within $N$ by some (possibly empty) set of the following identities: (3.5),

$$xytxy \approx yxtxy,$$

$$yx^2txy \approx xyxtxy,$$

$$x^2ytxxy \approx xyxtxy,$$

$$xy \prod_{i=1}^{n+1} (t_i e_i) \approx yx \prod_{i=1}^{n+1} (t_i e_i),$$

$$yx^2 \prod_{i=2}^{n+1} (t_i e_i) \approx xyx \prod_{i=1}^{n} (t_i e_i),$$

$$x^2y \prod_{i=1}^{n+1} (t_i e_i) \approx xyx \prod_{i=1}^{n} (t_i e_i),$$

$$x^2y \prod_{i=2}^{n+1} (t_i e_i) \approx xyx \prod_{i=2}^{n} (t_i e_i),$$

where $n \geq 1$ and

$$e_i = \begin{cases} 
  x & \text{if } i \text{ is odd}, \\
  y & \text{if } i \text{ is even}.
\end{cases}$$

The identity (3.5) holds in $N$ by the definition. The rest of these identities except for $x^2ytxy \approx xyxtxy$ follow from the identity $ytxsy \approx ytxsy$, which holds in $N$.

Therefore, each variety in $[M_{\lambda}(a^+ta^+), N]$ is defined within $N$ by the trivial identity (then this variety coincides with $N$) or the identity $x^2ytxy \approx xyxtxy$ (then this variety equals to $M_{\lambda}(a^+ta^+)$). Therefore, $X \in \{M_{\lambda}(a^+ta^+), N\}$.

We have $M_{\lambda}(a^+ta^+) = M_{\lambda}(ata^+, a^+t)$ by Theorem 7.2(ii) in [22]. Since $ata^+$ and $a^+t$ are $\lambda$-terms for $M_{\lambda}(a^+tb^+)$, the variety $M_{\lambda}(a^+tb^+)$ contains $M_{\lambda}(a^+ta^+)$ by Lemma 2.4. Since $M_{\lambda}(a^+ta^+) \models x^2ytxy \approx xyxtxy$, the $\lambda$-word $a^+tb^+$ is not a $\lambda$-term for $M_{\lambda}(a^+ta^+)$. Hence, the inclusion $M_{\lambda}(a^+tb^+) \supset M_{\lambda}(a^+ta^+)$ is proper in view of
Lemma 2.4. It is routine to verify that $M_{\lambda}(a^+btb^+) \subseteq N$. Then, since $M_{\lambda}(a^+btb^+)$ properly contains $M_{\lambda}(a^+ta^+)$, we have $N = M_{\lambda}(a^+btb^+)$. Thus, we may assume that $A_{\lambda}^1 \cap M_{\lambda}(ata^+) \subseteq X$. Then $X$ satisfies the identity (1.2) by Lemma 4.3. Hence, the variety $X$ satisfies the identities

$$xtysyx \approx xty^2 \approx xty^2x \approx xty^2xy \approx xty^2x^2 \approx xty^2xy \approx xty^2x^2$$

and so the identity $xsytxy \approx xsytxy$. Since $X$ is a subvariety of $K$, the variety $X$ satisfies the identities in $\{xtx \approx xtx^2, xtytxsy \approx ytxsyt, (3.5)\}$. Therefore, $X = M_{\lambda}(ata^+b^+)$ by Lemma 6.3(ii).

Remark 6.6. The variety $K$ is generated by the submonoid of $M_{\lambda}(bta^+b^+)$ generated by $\{a^+, b, ta^+\}$, which is isomorphic to the 12-element monoid $S^1$, where

$$S = \langle a, b, c | a^2 = a, b^2 = b^3, abc = ac = ba = b^2c = 0, bcb^2 = bcb, ca = c \rangle$$

$$= \{a, b, c, ab, ab^2, b^2c, bcb, cb, cb^2, 0\}.$$

Proof. It is routine to check that a submonoid of $M_{\lambda}(bta^+b^+)$ generated by the set $\{a^+, b, ta^+\}$ is isomorphic to $S^1$. Evidently, $S^1 \in K$. We note that $S^1$ violates $xtysyx \approx xty^2$ because $bca \cdot 1 \cdot ab = bcb \neq 0$ but $bca \cdot 1 \cdot ba = 0$ in $S^1$. This fact and Lemmas 6.1 and 6.3(ii) imply that $S^1 \notin M_{\lambda}(ata^+b^+)$. According to Proposition 6.5, $K$ is generated by $S^1$. □

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Institute of Natural Sciences and Mathematics, Ural Federal University, Lenina 51, Ekaterinburg, 620000, Russia
*E-mail address*: sergey.gusb@gmail.com

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
*E-mail address*: olga.sapir@gmail.com