Teleportation-induced entanglement of two nanomechanical oscillators coupled to a topological superconductor

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(Dated: April 30, 2014)

Abstract

A one-dimensional topological superconductor features a single fermionic zero mode that is delocalized over two Majorana bound states located at the ends of the system. We study a pair of spatially separated nanomechanical oscillators tunnel-coupled to these Majorana modes. Most interestingly, we demonstrate that the combination of electron-phonon coupling and a finite charging energy on the mesoscopic topological superconductor can lead to an effective superexchange between the oscillators via the non-local fermionic zero mode. We further show that this electron teleportation mechanism leads to entanglement of the two oscillators over distances that can significantly exceed the coherence length of the superconductor.

PACS numbers: 03.65.Ud, 85.85.+j, 74.45.+c, 71.10.Pm

I. INTRODUCTION

In a 2001 paper, Kitaev discovered the one-dimensional topological superconductor (1D4SC) – a one-dimensional proximity induced p-wave superconductor hosting a single Majorana quasi particle (MQP) at each of its ends. More recently, experimentally feasible realizations of the 1DTSC phase have been proposed in semiconducting nanowires in proximity to an s-wave superconductor. In these settings, the interplay of Rashba spin-orbit coupling and a magnetic field induced Zeeman splitting in the nanowire gives rise to an effective p-wave pairing. By now, several groups have reported first experimental signatures of MQPs in InSb nanowires. Besides the fundamental interest attached to the experimental discovery of Majorana fermions in nature, MQPs as realized in 1DTSC also have intriguing features relating to various aspects of fundamental quantum physics: On the one hand the non-Abelian anyonic nature of MQPs shows great promise for topological quantum information processing architectures. On the other hand the delocalized pair of MQPs at the ends of a 1D4SC can be viewed as a single ordinary (spinless Dirac) fermionic zero mode leading to electron teleportation mechanisms, i.e., coherent long-range quantum effects. In a hybrid system of a 1DTSC and two single-level quantum dots, ground state entanglement of the occupation number of the quantum dots has been reported in Ref. 11.

The understanding of genuine quantum effects on macroscopic length scales is one of the main motivations to study nano-electromechanical systems and nano-optomechanical systems. In recent years, decisive progress towards cooling nanomechanical resonators to the ground state has been reported. However, long distance entanglement of nanomechanical systems which would be another experimental hallmark in fundamental quantum physics has not been achieved yet although a variety of theoretical proposals have been made. In the interest of quantum coherence, different interaction mechanisms between spatially separated systems have been suggested ranging from coupling to a common optical mode to exploiting the large coherence length of a Bose Einstein condensate and a Cooper-pair condensate respectively. A hybrid system of a 1DTSC and one NEMO was studied in Ref. 29.

The article is organized as follows. In Sec. II, we summarize our main results. We propose the setup, discuss a possible realization of it, and introduce the Hamiltonian of the underlying model in Sec. III. We present and discuss the results of the generated entanglement in Sec. IV. Finally, we summarize in Sec. V.

II. MAIN RESULTS

In this work, we bridge the research fields of topological superconductivity and entanglement in nanomechanical systems by proposing a mechanism to entangle two nano-electromechanical oscillators (NEMOs). More concretely, we demonstrate that the electron teleportation mechanism reported in Ref. 10 can lead to an effective superexchange coupling of two distant NEMOs located in the vicinity of the opposite ends of a mesoscopic 1DTSC. The combination of electron-phonon coupling on the NEMOs and a finite Coulomb charging energy lead to the generation of an effective superexchange coupling of two distant NEMOs. The teleportation mechanism guarantees coherence at length scales that significantly exceed those of the superconducting condensate wave function. In the proposed setup (see Fig. 1) entanglement between two distant conducting NEMOs can be generated by simply driving a current through the device. Using a non-Markovian master equation approach, we demonstrate
FIG. 1. (Color online) Schematics of the proposed setup. Two nano-electromechanical oscillators (blue) each tunnel coupled to one end of a one-dimensional topological superconductor. The one-dimensional topological superconductor is sketched as a nanowire (yellow) placed on top of a mesoscopic superconductor (gray). At each end of the wire a single Majorana quasi particle (orange) is located. A gate voltage \( V_g \) is applied (across a gate capacitance \( C_g \)) to the mesoscopic superconductor resulting in a finite charging energy \( E_c = e^2/2C_g \) of the superconductor. The nano-electromechanical oscillators are modeled as normal metal leads at the chemical potentials \( \mu_{L/R} \).

that for NEMOs in their ground states, switching on a tunneling current induces entanglement that persists over many oscillator periods. In the Markovian limit, we derive a Lindblad master equation which provides an intuitive understanding of how number states of the NEMOs arrive a Lindblad master equation which provides an intuitive understanding of how number states of the NEMOs.

III. MODEL

We will now show how an effective coupling between two NEMOs can be generated via an electron teleportation mechanism involving the MQPs located at the ends of a 1DTSC. The proposed setup is shown in Fig. 1 and is modeled by the following Hamiltonian (we put \( \hbar = c = k_B = 1 \))

\[
H = \sum_{\alpha = L,R} H_{\text{osc}}^{(\alpha)} + H_{\text{lead}}^{(\alpha)} + H_{\text{tun}} + H_c,
\]

where \( H_{\text{osc}}^{(\alpha)} = p_{\alpha}^2/2m_\alpha + m_\alpha \Omega_\alpha^2 x_\alpha^2/2 \) describes the two NEMOs denoted by \( \alpha = L, R \) with effective mass \( m_\alpha \), frequency \( \Omega_\alpha \), and position and momentum operators \( x_\alpha \) and \( p_\alpha \), respectively. For simplicity, we assume that \( m_L = m_R \) and \( \Omega_L = \Omega_R \). The conducting NEMOs act as two independent normal metal leads which are characterized by the Hamiltonians \( H_{\text{lead}}^{(\alpha)} = \sum_k \epsilon_k \psi_{\alpha k}^\dagger \psi_{\alpha k} \) and which are held at the chemical potentials \( \mu_{L/R} \). The tunneling Hamiltonian \( H_{\text{tun}} \) from a normal metal lead into a 1DTSC without charging energy can be written as

\[
H_{\text{tun}} = \sum_k \left[ i T_L (\psi_{Lk}^\dagger \psi_{Lk}^\dagger) + \gamma_L + (L \rightarrow R) \right], \tag{1}
\]

where \( \gamma_L \) and \( \gamma_R \) are the annihilation and creation operators, respectively, of a single spinless Dirac fermion that is delocalized over the two ends of the 1DTSC. Equation (1) contains so called anomalous terms which break particle number conservation in the mean field picture of superconductivity as they microscopically involve the creation or annihilation of a Cooper pair which is not explicitly accounted for at that level of description. In a 1DTSC with zero charging energy \( E_c = 0 \), the NEMOs independently couple locally to the two ends of the 1DTSC and the effective coupling necessary for entangling the oscillators is absent. However, the situation is different in a mesoscopic superconductor with a finite charging energy \( E_c \) which gives rise to an explicit dependence of the energy on the number of electrons. Hence, one has to go beyond the effective description of Eq. (1) and explicitly keep track of the change in the number of Cooper pairs in the condensate during anomalous tunneling processes. The gate voltage \( V_g \) is assumed to be adjusted such that the number of Cooper pairs \( N_C \) in the ground state of the 1DTSC is \( N_0 \) and the occupation number \( n_c = c^\dagger c \) of the delocalized fermionic bound state is zero. The charging Hamiltonian \( H_r \) then reads

\[
H_r = E_c (2N_c + n_c - 2N_0)^2. \tag{2}
\]

We would like to point out that \( n_c = \frac{1}{2} (\gamma_L \gamma_R + 1) \) as appearing in \( H_r \) effectively couples the two MQPs \( \gamma_L \) and \( \gamma_R \) even if the direct overlap of the two bound state wave functions is negligible. This coupling is crucial for the electron teleportation mechanisms as it prevents the dynamical independence of the two MQPs. We would like to focus on the parameter regime \( T_\alpha, V < \Omega_\alpha < E_c < \Delta \rightarrow \infty \). In this limit, non-local tunneling processes involving continuum states of the superconductor (e.g. Crossed Andreev Reflection or electron cotunneling) are suppressed. Moreover, in this scenario, there are no resonant levels in the superconductor for first-order tunneling processes. However, second-order cotunneling processes via virtual states with energies on the order of \( E_c \) are allowed and lead to an effective superexchange coupling between the NEMOs as we will derive now. We neglect processes containing intermediate states with two or more excess electrons on
the superconductor which are suppressed by an energy
denominator of at least $4E_c$ and are hence less relevant.
This approximation excludes all terms where an extra
Cooper pair is created. The only anomalous second order
tunneling process which is then allowed is the anomalous
cotunneling depicted in Fig. 2.

After truncating the Hilbert space of the superconduc-
tor to the eigenstates with $E \leq E_c$, we obtain a three-
dimensional Hilbert space with the basis

$$
|0\rangle = |N_C = N_0, n_c = 0\rangle \quad E_0 = 0
$$

$$
|1\rangle = |N_C = N_0, n_c = 1\rangle \quad E_1 = E_c
$$

$$
|2\rangle = |N_C = N_0 - 1, n_c = 1\rangle \quad E_2 = E_c
$$

In this basis, $H_c$ can be represented as: $H_c = \text{diag}\{0, E_c, E_c\}$. The tunneling Hamiltonian (1) con-
strained to the truncated Hilbert space of the supercon-
ductor reads as

$$
H_{\text{tun}} \approx iT_L \sum_k \left( |1\rangle \langle 0| \psi_{Lk} + |2\rangle \langle 0| \psi_{Lk}^\dagger \right) 
+ Tr \sum_k \left( -|1\rangle \langle 0| \psi_{Rk} + |2\rangle \langle 0| \psi_{Rk}^\dagger \right) + \text{h.c.} \quad (2)
$$

The terms in Eq. (2) which involve the breaking and re-
combination of a Cooper pair, respectively are illustrated in Fig. 2. Assuming that the superconductor is initially in its ground state $|0\rangle$, we can integrate out the first or-
der order tunnel coupling to the excited states $|1\rangle, |2\rangle$. That way, we obtain an effective direct tunneling Hamiltonian between the left and the right lead containing the leading second order cotunneling processes in the original tunnel coupling Eq. (1). Explicitly, we get

$$
H_{\text{tun}}^{\text{eff}} = -T_L^2 + T_R^2 - \frac{2T_LT_R}{E_c} \sum_k \left[ |\psi_{Lk}\rangle \langle \psi_{Rk}| + \text{h.c.} \right]. \quad (3)
$$

Recalling the position dependence $T_\alpha = t_{0\alpha} + t_{x\alpha} x_\alpha$ of the tunnel couplings, it becomes clear that Eq. (3) also con-
tains an effective direct coupling between the NEMOs. This formally mimics the superexchange coupling which could also be achieved using a single quantum dot with a finite charging energy. However, we would like to stress two conceptual advantages of the electron teleportation-induced superexchange coupling. First, it guarantees phase coherent coupling between the NEMOs over dis-
tances where the confinement induced level spacing on a quantum dot would become very small. Second, the tunneling density of states associated with the delocal-
ized fermion $c$ in our setting is spatially strongly peaked around the interface between the NEMO and the 1DTSC.

In a large single level quantum dot in contrast, the same spectral weight would be smeared out all over the "bulk" of the dot. In the following, we will demonstrate how this teleportation-induced superexchange coupling can be em-
ployed to generate entanglement between the oscillators over distances which are not limited by the coherence length of the superconducting condensate.

IV. ENTANGLEMENT

As shown above (see Eq. (3)), tunnel coupling two NEMOs to a 1DTSC leads to an effective direct coupling between the NEMOs. Therefore, we expect the gener-
ation of entanglement in the bipartite continuous vari-
able system consisting of the two NEMOs. We study the time evolution of entanglement between the two NEMOs using the logarithmic negativity as an entan-
glement measure: $E_N(\rho_{\text{osc}}) = \log_2(\|\rho_{\text{osc}}^{\text{t}}\|_1)$. Here, $ho_{\text{osc}}^{\text{t}}$ is the partial transpose of the state of the bipar-
tite system. For a Gaussian state, the logarithmic neg-
itivity can be computed from the covariance matrix

$$
\Gamma_{j,k}(t) = \text{Tr}[\rho_{\text{osc}}(t)\{R_j, R_k\}], \quad \text{where } R = (x_1, p_1, x_2, p_2)^T
$$

is the vector of quadratures. We compute the time de-
pendence of the entries of $\Gamma(t)$ by solving the equation of motion for the system’s density matrix $\rho_{\text{osc}}(t)$ employing a time convolutionless master equation method. Within our effective tunneling Hamiltonian approach (see Eq. (3)), the master equation in the Born approximation is given by

$$
\dot{\rho}_{\text{osc}}(t) = -i [H_{\text{osc}}, \rho_{\text{osc}}(t)] 
- \int_0^t d\tau \text{Tr}_{\text{leads}} \left[ H_{\text{tun}}^{\text{eff}}(t - \tau), \rho_{\text{osc}}(\tau) \otimes \rho_{\text{leads}} \right]. \quad (4)
$$

For the sake of simplicity, we assume in the following identical NEMOs ($\Omega_a = \Omega$ and $m_a = m$). We also chose a symmetric coupling and real tunneling amplitudes ($t_{0\alpha} = t_0$ and $t_{x\alpha} = t_x$).

Up to second order in $t_x$, i.e., only taking into account terms $\sim (t_0 t_x)^2$ in Eq. (4), the time dependence of the covariance matrix $\Gamma(t)$ can be obtained similarly as in Ref. 28, for technical details we refer to the Appendix.

In Fig. 3, we show results for the logarithmic negativity, taking for simplicity the vacuum state as an initial state. The Gaussian character of this initial state is preserved at all times of the dynamics. Figure 3a) shows the time dependence of the logarithmic negativity $E_N$ for a fixed
In both cases, the generated entanglement is higher but decays faster than for large E. This can be interpreted by recognizing that the charging energy \( E \) plays a role here. For smaller values of \( E \) compared to larger values of \( E \), Figure 3b) shows that the stationary state becomes unique and entanglement is unexpectedly lost. However, processes destroying and generating the entanglement now compete with each other. This still allows for the generation of entanglement in a dissipative fashion determined by the microscopic environment of the NEMOs. To summarize, we have shown that entanglement between our work and Ref. 11, where the non-local Markovian limit, the equation of motion reduces to

\[
\rho_{\text{osc}} = \mathcal{L}[\rho_{\text{osc}}] = -\frac{i}{\hbar}[H_{\text{osc}}, \rho_{\text{osc}}] + \gamma \mathcal{D}[O] \rho_{\text{osc}}
\]

with the Liouvillian superoperator \( \mathcal{L} \) and a Lindblad dissipator \( \mathcal{D}[O] \rho = \mathcal{L}[O] \rho = 0 \). In our case, we have \( \gamma = \frac{\pi \gamma}{\hbar} \rho_{\text{osc}} \), \( O = a_R^\dagger a_R + a_L^\dagger a_L \), where \( a_\alpha = (x_\alpha \sqrt{\gamma} + i p_\alpha / \sqrt{\hbar}) / \sqrt{2} \) and \( a_\alpha^\dagger = (x_\alpha \sqrt{\gamma} - i p_\alpha / \sqrt{\hbar}) / \sqrt{2} \) are bosonic annihilation and creation operators, respectively. \( \rho_{\text{osc}} \) is the density matrix of the system \( (\mathcal{L}[\rho_{\text{osc}}] = 0) \) is not unique. However, if the number of excitations \( N_{\text{exc}} = \sum \alpha (a_\alpha^\dagger a_\alpha) \) is kept fixed, the steady state is unique. For instance, the pure initial state \( \Psi = |n_L = 1, n_R = 1 \rangle \) is dissipatively driven to the (mixed) entangled state

\[
\rho_{\text{ss}} = \frac{1}{2} |\Psi \rangle \langle \Psi | + \frac{1}{2} |\Phi \rangle \langle \Phi |,
\]

where \( |\Phi \rangle = \frac{1}{\sqrt{2}} (|2, 0 \rangle + |0, 2 \rangle) \) is a maximally entangled state. The degree of entanglement of \( \rho_{\text{ss}} \) is readily quantified by calculating \( E_N(\rho_{\text{ss}}) = \log_2(3/2) \). In the presence of a finite temperature heat bath, sectors of different particle number will start to couple. Thereby, the stationary state becomes unique and entanglement is unsurprisingly lost. However, processes destroying and generating the entanglement now compete with each other. This still allows for the generation of entanglement in a dissipative fashion. The rates of the entanglement generating and destroying processes (characterized by an independent rate determined by the microscopic environment of the NEMOs) are governed by their respective Liouvillian gaps, for details we also refer to Ref. 28.

V. CONCLUDING DISCUSSION

To summarize, we have shown that entanglement between two distant NEMOs can be achieved by tunnel coupling of the NEMOs to two MQPs residing at the ends of a 1DTSC. A finite charging energy on the 1DTSC leads to an effective superexchange coupling between the NEMOs via the non-local MQPs. This electron teleportation mechanism guarantees phase coherence over length scales \( \sim 1/E_c \) that are significantly larger than the superconducting coherence length \( \sim 1/\Delta \). Our proposal allows for entangling two mesoscopic NEMOs initially cooled to their ground states in an all electronic setup by driving a current through the device. In the Markov approximation, the equation of motion for the system’s density matrix \( \rho_{\text{osc}} \) reduces to a Lindblad master equation. In this limit, NEMOs initially prepared in number states can be entangled by purely dissipative means.

We briefly want to elaborate on the conceptual difference between our work and Ref. 11, where the non-local
nature of a pair of MQPs was exploited to create a charge-entangled ground state of two single level quantum dots in the Coulomb blockade regime. On the contrary, in our proposal, the electron charge degrees of freedom are in fact only used to generate an effective superexchange coupling between two rather macroscopic mechanical degrees of freedom. In our setting, entanglement is not a ground state property of a closed system but is dynamically generated by driving a current between the two metallic leads. Remarkably, the thermalization (decoherence) of the electrons after their tunneling into these reservoirs does not affect the coherence times of the entangled NEMOs.

Our analysis relies crucially on the hierarchy \( V, T_n < \Omega < E_c < \Delta \) of the involved energy scales. Finally, we would like to discuss experimentally relevant energy scales in the proposed setup thereby demonstrating the feasibility of the assumed parameter regime. For an InSb wire proximity coupled to a NbTiN superconductor, experimental data reported in Ref. 4 indicate an induced gap on the order of \( \Delta = 250 \mu eV \). By varying the size of the superconductor, the charging energy \( E_c \) can be adjusted. Here, we assume \( E_c = 20 \mu eV \). Frequencies of doubly clamped NEMOs can be as high as \( \Omega = 500 \text{ MHz} \approx 2 \mu eV \), i.e., one order of magnitude smaller than a typical charging energy. Still, such NEMOs could be passively cooled to their ground state at typical dilution refrigerator temperatures. Taking these estimates, the localization length of the MQPs at the ends of the 1DTSC is about 2 \( \mu m \). For the assumed charging energies, the MQPs could be separated by at least 20 \( \mu m \), hence direct tunneling between them is negligible.

**ACKNOWLEDGMENTS**

We would like to thank Christoph Bruder, Patrik Recher, Thomas Schmidt, and Björn Trauzettel for stimulating discussions. SW acknowledges financial support from the Swiss SNF and the NCCR Quantum Science and Technology. JCB acknowledges financial support from the Swedish Research Council (VR) and the ERC Synergy Grant UQUAM.

**Appendix: Details on the equation of motion**

In this Appendix, we give details on the equation of motion of the two NEMOs. For simplicity, we assume identical NEMOs \( (\Omega_n = \Omega, m_n = m) \) and symmetric coupling \((\Delta_{0}, \Delta_{s} = \frac{\Delta}{2})\). Using the effective tunneling Hamiltonian, Eq. (3) of the main text, the equation of motion, Eq. (4) of the main text, can be written as

\[
\dot{\rho}_{\text{osc}}(t) = -i \left[ H_{\text{osc}} + iG^{(\text{cc})}(t)(x_L + x_R)^2, \rho_{\text{osc}}(t) \right] \\
- G^{(\text{cc})}_c(t) [x_L + x_R, [x_L + x_R, \rho_{\text{osc}}(t)]] \\
+ G^{(\text{cc})}_c(t) [x_L + x_R, [p_L + p_R, \rho_{\text{osc}}(t)]] \\
+ G^{(\text{cc})}_c(t) [x_L + x_R, [p_L + p_R, \rho_{\text{osc}}(t)]] \\
- G^{(\text{cc})}_c(t) [x_L x_R, [x_L x_R, \rho_{\text{osc}}(t)]] \\
- G^{(\text{cc})}_c(t) [x_L x_R, [p_L p_R, \rho_{\text{osc}}(t)]] \\
+ G^{(\text{cc})}_c(t) [x_L x_R, [x_L x_R, \rho_{\text{osc}}(t)]] \\
- G^{(\text{cc})}_c(t) [x_L x_R, [x_L x_R, \rho_{\text{osc}}(t)]] \\
+ G^{(\text{cc})}_c(t) [x_L x_R, [x_L x_R, \rho_{\text{osc}}(t)]] .
\]

Non-Markovian effects are included in the equation of motion by the time-dependent kernels given by

\[
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(1)}(\tau) + G^{(1)}(-\tau) \right) \left( 2 t_0 t_x^2 \cos(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(1)}(\tau) + G^{(1)}(-\tau) \right) \left( \frac{2 t_0 t_x^2}{m} \sin(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(2)}(\tau) - G^{(2)}(-\tau) \right) \left( 2 t_0 t_x^2 \cos(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(2)}(\tau) - G^{(2)}(-\tau) \right) \left( \frac{2 t_0 t_x^2}{m} \sin(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(3)}(\tau) + G^{(3)}(-\tau) \right) \left( 4 t_0^4 \cos(\Omega \tau) \cos(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(3)}(\tau) + G^{(3)}(-\tau) \right) \left( \frac{4 t_0^4}{m^2 \Omega^2} \sin(\Omega \tau) \sin(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(4)}(\tau) - G^{(4)}(-\tau) \right) \left( 4 t_0^4 \cos(\Omega \tau) \cos(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(4)}(\tau) - G^{(4)}(-\tau) \right) \left( \frac{4 t_0^4}{m^2 \Omega^2} \sin(\Omega \tau) \sin(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(5)}(\tau) - G^{(5)}(-\tau) \right) \left( 4 t_0^4 \cos(\Omega \tau) \cos(\Omega \tau) \right), \\
G^{(\text{cc})}_c(t) = \int_0^t d\tau \left( G^{(5)}(\tau) - G^{(5)}(-\tau) \right) \left( \frac{4 t_0^4}{m^2 \Omega^2} \sin(\Omega \tau) \sin(\Omega \tau) \right).
\]

The functions \( G^{(1)}(t) \) and \( G^{(2)}(t) \) are given by

\[
G^{(1)}(t) = \frac{1}{2} \{\{B(t), B^\dagger(t)\} \}, \\
G^{(2)}(t) = \frac{1}{2} \{[B(t), B^\dagger(t)] \},
\]

with

\[
B = -\frac{i}{E_c} \psi_L^\dagger \psi_R .
\]

With this we obtain

\[
G^{(m)}(t) = \frac{1}{2 E_c^2} \int d\xi_L \int d\xi_R J(\xi_L, \xi_R) e^{i(\xi_L - \xi_R)t} \times [n_L(\xi_L)(1 - n_R(\xi_R)) - (1 - n_L(\xi_L))(1 - n_R(\xi_R))],
\]
where $n_x(\varepsilon_x) = \frac{1}{e^{\beta(\varepsilon_x - \mu_x)} + 1}^{-1}$ is the Fermi distribution function (with $\beta = 1/T$, $\mu_x$ being the inverse electronic temperature of the leads; we set $k_B = 1$) and

$$J(\varepsilon_L, \varepsilon_R) = \sum_{k,q} \delta(\varepsilon_L - \varepsilon_k) \delta(\varepsilon_R - \varepsilon_q),$$

is an energy-dependent spectral function. To account for a finite lifetime of quasiparticles in the leads, the $\delta$-functions are smeared out and replaced by Lorentzians of width $L_c$

$$J(\varepsilon_L, \varepsilon_R) = \sum_{k,q} \frac{L_c}{(\varepsilon_L - \varepsilon_k)^2 + L_c^2} \frac{L_c}{(\varepsilon_R - \varepsilon_q)^2 + L_c^2}.$$

Energies close to the Fermi level of each lead will contribute most to each of the independent sums. To keep the number of parameters as low as possible, we restrict ourselves to the regime of low applied bias voltages ($V < L_c$). Then, we can approximate the energy-dependent spectral function as

$$J(\varepsilon_L, \varepsilon_R) \approx \frac{1}{(\varepsilon_L - \varepsilon_R)^2 + L_c^2},$$

which implies that an electron with energy $\varepsilon_L$ in the left lead can tunnel into states of the right lead with energy $\varepsilon_R$, broadened by $L_c$. The limit $L_c \to 0$ resembles a resonant tunneling process with narrow densities of states in the leads. The opposite limit, $L_c \to \infty$, corresponds to the so-called wide-band limit with an energy-independent density of states in the leads, i.e., any electron from the left lead can tunnel into the right lead. With this, all the above kernels can be calculated analytically. The resulting expressions are not very insightful and too lengthy to be stated here.
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