Degeneracy of the quarks, shell structure in the chiral soliton

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Abstract

We obtain multi-soliton solutions with discrete symmetries in the chiral quark soliton model using the rational map ansatz. The solutions exhibit degenerate bound spectra of the quark orbits depending on the background of pion field configurations. It is shown that resultant baryon densities inherit the same discrete symmetries as the chiral fields. Evaluating the radial component of the baryon density, shell-like structure of the valence quark spectra is also observed.

Key words: Topological soliton; Quark model; Nuclear structure model

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1 Introduction

The chiral quark soliton model (CQSM) was developed in 1980’s as a low-energy effective theory of QCD. Since it includes the Dirac sea quark contribution and explicit valence quark degrees of freedom, the model interpolates between the constituent quark model and the Skyrme model [1,2,3,4,5]. The CQSM incorporates the non-perturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. It has been shown that the $B = 1$ solution provides correct observables such as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet SU(3) baryon spectra. For $B = 2$, the stable axially symmetric soliton solution was found in Ref. [6]. The solution exhibits doubly degenerate bound spectrum of the quark orbits in the background of axially symmetric chiral fields with winding number two. Upon quantization, various dibaryon spectra were obtained, showing that the quantum numbers of the ground state coincide with
those of physical deuteron [7,8]. For $B > 2$, the Skyrme model predicts that minimum energy solutions have discrete, crystal-like symmetries [9,10,11]. According to this prediction, we studied the CQSM with $B = 3$ tetrahedrally symmetric chiral fields and obtained triply degenerate spectrum of the quark orbits [12]. Its large degeneracy indicates that the tetrahedrally symmetric solution may be the lowest-lying configuration. Thus, for $B > 3$, one can also expect that the lowest-lying solutions in the CQSM inherits the discrete symmetries predicted in the Skyrme model and have large degeneracy.

In this paper we shall study soliton solutions with $B \geq 3$ in the CQSM using the rational map ansatz for the background of pion fields. The procedure of the numerical computation performed here is shown in Sec.3. In Sec.4, we show obtained classical self-consistent soliton solutions with $B = 3 - 9$. These solutions exhibit various degenerate spectra of the quark orbits depending on the symmetry of the background chiral fields. Such degeneracy generates large shell gaps, which suggests that the solutions are stable local minima. Evaluating the radial component of the baryon density, shell-like structure of the valence quarks can be observed. The results and discussion are summarized in Sec.5.

2 The model

The CQSM incorporates the nonperturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking with the vacuum functional defined by [1]

$$Z = \int \mathcal{D}\pi \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp \left[ i \int d^4 x \bar{\psi} \left( i \partial - MU^\gamma_5 \right) \psi \right], \quad (1)$$

where the SU(2) matrix

$$U^\gamma_5 = \frac{1 + \gamma_5}{2} U + \frac{1 - \gamma_5}{2} U^\dagger \quad \text{with} \quad U = \exp \left( i \tau \cdot \pi / f_\pi \right)$$

describes chiral fields, $\psi$ is quark fields and $M$ is the constituent quark mass. $f_\pi$ is the pion decay constant and experimentally $f_\pi \sim 93\text{MeV}$.

The $B = 1$ soliton solution has been studied in detail at classical and quantum level in Refs. [1,2,3,4,5]. To obtain solutions with $B > 1$, we shall employ the chiral field configuration with winding number $B$ obtained in the Skyrme model as the background of quarks, which can be justified as follows. In Eq. (1), performing the functional integral over $\psi$ and $\psi^\dagger$ fields, one obtains
the effective action
\[ S_{\text{eff}}[U] = -iN_c \text{Sp} \ln iD = -iN_c \log \det iD, \tag{2} \]
where \( iD = i \phi - MU^\gamma_5 \) is the Dirac operator. The classical solutions can be derived by imposing an extremum condition on the effective action with respect to \( U \). For this purpose, let us consider the derivative expansion of the action \([5,13,14]\). Up to quartic terms, we have
\[
S_{\text{eff}} = \int d^4x \left[ -C \text{tr}(L_\mu L^\mu) \\
+ \frac{N_c}{32\pi^2} \text{tr}\left\{ \frac{1}{12}[L_\mu, L_\nu]^2 - \frac{1}{3}(\partial_\mu L^\mu)^2 + \frac{1}{6}(L_\mu L^\mu)^2 \right\} \right], \tag{3}
\]
where \( L_\mu = U^\dagger \partial_\mu U \). Suitably adjusting the coefficients, one can identify the first two terms of Eq. (3) with the Skyrme model action. However, the 4th order terms tend to destabilize solutions and no stable classical solution can be obtained from the above action \([13,15]\). Nevertheless, because of their similarity, it will be justified to adopt the configurations of the solutions in the Skyrme model to chiral fields in the CQSM.

In the CQSM, the number of valence quark is associated with the baryon number such that a soliton with baryon number \( B \) consists of \( N_c \times B \) valence quarks. If the quarks are strongly bound inside the soliton, their binding energy become large and the valence quarks can not be observed as positive energy particles \([16,17]\). Thus, one gets the picture of the topological soliton model in the sense that the baryon number coincides with the winding number of the background chiral field when the valence quarks occupy all the levels diving into negative energy region.

Let us rewrite the effective action in Eq. (2) as
\[
S_{\text{eff}} \rightarrow -iN_c \log \det(i\partial_t - H(U^\gamma_5)), \tag{4}
\]
and introduce the eigenequation
\[
H(U^\gamma_5)\phi_\mu(\mathbf{x}) = E_\mu \phi_\mu(\mathbf{x}), \tag{5}
\]
\[
H(U^\gamma_5) = -i\vec{\alpha} \cdot \nabla + \beta MU^\gamma_5. \tag{6}
\]

The effective action \( S_{\text{eff}}(U) \) is ultraviolet divergent and hence must be regularized. Using the proper-time regularization scheme \([18]\), one can write
\[ S_{\text{eff}}[U] = \frac{i}{2} N_c \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp}\left(e^{-D_0^1 D_{\tau}} - e^{-D_0^1 D_{0\tau}}\right) \]
\[ = \frac{i}{2} N_c T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp}\left[e^{-\tau(H^2 + \omega^2)} - e^{-\tau(H_0^2 + \omega^2)}\right], \tag{7} \]

where \( T \) is the Euclidean time separation, \( D_0 \) and \( H_0 \) are operators with \( U = 1 \). At \( T \to \infty \), we have \( e^{iS_{\text{eff}}} \sim e^{-iE_{\text{field}} T} \). The total energy is then given by
\[ E_{\text{static}}[U] = E_{\text{val}}[U] + E_{\text{field}}[U] - E_{\text{field}}[U = 1], \tag{8} \]

where
\[ E_{\text{val}} = N_c \sum_i E_{\text{val}}^{(i)} \tag{9} \]

is the valence quark contribution with the valence energy \( E_{\text{val}}^{(i)} \) of the \( i \)th valence quark, and the vacuum part is
\[ E_{\text{field}} = N_c \sum_{\mu} \left\{ \mathcal{N}(E_{\mu})|E_{\mu}| + \frac{\Lambda}{\sqrt{4\pi}} \text{exp}\left[ -\left( \frac{E_{\mu}}{\Lambda} \right)^2 \right] \right\}, \tag{10} \]

with
\[ \mathcal{N}(E_{\mu}) = -\frac{1}{\sqrt{4\pi}} \Gamma\left( \frac{1}{2}, \left( \frac{E_{\mu}}{\Lambda} \right)^2 \right). \]

\( \Lambda \) is a cutoff parameter evaluated by the condition that the derivative expansion of Eq. (7) reproduces the pion kinetic term with the correct coefficient,
\[ f_{\pi}^2 = \frac{N_c M^2}{4\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} e^{-\tau M^2}. \tag{11} \]

The contribution to the total energy in the absence of the chiral fields \( (U = 1) \) can be estimated using the eigenstates of the eigenequation,
\[ H_0 \phi_0^{(0)}(x) = E_0^{(0)} \phi_0^{(0)}(x), \tag{12} \]
\[ H_0 = -i\alpha \cdot \nabla + \beta M. \tag{13} \]
In the Skyrme model it is known that solitons with $B \geq 3$ have discrete crystal-like symmetries [9]. Therefore, we expect that soliton solutions of the CQSM inherits the same discrete symmetry as skyrmions. However, it is too complicated to perform a numerical computation if one imposes such discrete symmetries directly on the chiral fields. Thus Houghton, Manton and Sutcliffe proposed remarkable ansatz for the chiral fields, rational map ansatz [11]. According to this ansatz, the chiral fields are expressed in a rational map as

$$U(r, z) = \exp(iF(r)\hat{n}_R \cdot \tau),$$ \hspace{1cm} (14)

where

$$\hat{n}_R = \frac{1}{1 + |R(z)|^2}(2\text{Re}[R(z)], 2\text{Im}[R(z)], 1 - |R(z)|^2)$$

and $R(z)$ is the rational map. The complex coordinate $z$ is given by $z = \tan(\theta/2)e^{i\phi}$ via stereographic projection.

Rational maps are maps from $CP(1)$ to $CP(1)$ (equivalently, from $S^2$ to $S^2$) classified by winding number. In Ref. [11] Manton et al. showed that $B = N$ skyrmions can be well-approximated by rational maps with winding number $N$. The rational map with winding number $N$ possesses $(2N + 1)$ complex parameters whose values can be determined by imposing the symmetry of the skyrmion. We shall use this ansatz for the background chiral fields in the CQSM. Their explicit forms are presented in Appendix A. Since the chiral fields in Eq. (14) is parameterized by polar coordinates, one can apply the numerical technique developed for $B = 1$ to find solutions with higher $B$. Detailed calculations will be presented in the subsequent section.

Field equations for the chiral fields can be obtained by demanding that the total energy in Eq. (8) be stationary with respect to variation of the profile function $F(r)$,

$$\frac{\delta}{\delta F(r)} E_{\text{static}} = 0,$$

which produces

$$S(r) \sin F(r) = P(r) \cos F(r),$$ \hspace{1cm} (15)

where

$$S(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu})\mathcal{N}(E_{\mu})) \langle \mu | \gamma^0 \delta(|x| - r) | \mu \rangle,$$ \hspace{1cm} (16)
\[ P(r) = N_c \sum_{\mu} (n_\mu \theta(E_\mu) + \text{sign}(E_\mu)\mathcal{N}(E_\mu)) \langle \mu | i\gamma^0 \gamma^5 \mathbf{n}_R \cdot \mathbf{\tau} \delta(|x| - r) | \mu \rangle. \] 

(17)

The procedure to obtain self-consistent solutions of Eq. (15) is that 1) solve the eigenvalue equation in Eq. (6) under an assumed initial profile function \( F_0(r) \), 2) use the resultant eigenfunctions and eigenvalues to calculate \( S(r) \) and \( P(r) \), 3) solve Eq. (15) to obtain a new profile function, 4) repeat 1) – 3) until the self-consistency is attained.

The baryon density \( b(x) \) is defined by the zeroth component of the baryon current [2]:

\[ b(x) = \frac{1}{N_c} \langle \bar{\psi} \gamma_0 \psi \rangle = b_{\text{val}}(x) + b_{\text{field}}(x), \]

where

\[ b_{\text{val}}(x) = \sum_i b_{\text{val}}^{(i)}(x) = \frac{1}{N_c} \sum_i \phi_i(x) \dagger \phi_i(x), \]

\[ b_{\text{field}}(x) = \frac{1}{N_c} \left[ \sum_{\mu} \text{sign}(E_\mu) \mathcal{N}(E_\mu) \phi_\mu(x) \dagger \phi_\mu(x) \right. \]

\[ \left. - \sum_{\mu} \text{sign}(E^{(0)}_\mu) \mathcal{N}(E^{(0)}_\mu) \phi^{(0)}_\mu(x) \dagger \phi^{(0)}_\mu(x) \right]. \]

(19)

To examine the shell structure of the quarks, we evaluate the radial density for the \( i \)th valence quark \( \rho^{(i)}(r) \) in which the angular degrees of freedom are integrated out, via,

\[ \rho^{(i)}(r) = \int d\varphi \int \sin \theta d\theta \ b^{(i)}_{\text{val}}(r, \theta, \varphi) \]

(20)

with the baryon number

\[ B = \sum_i \int dr r^2 \rho^{(i)}(r). \]

(21)

3 Numerical technique

The numerical method that is widely used in this model is based on the expansion of the Dirac spinor in the appropriate orthogonal basis. However, since
the symmetries with $B \geq 3$ are discrete and it is hard to find any symmetric operator commuting with the hamiltonian, we shall expand the Dirac field in the Kahana-Ripka basis as in the case of $B = 1$ [16]. The Kahana-Ripka basis which was originally constructed for diagonalizing the hamiltonian with the chiral fields of $B = 1$ hedgehog ansatz is a plane-wave finite basis. The basis is discretized by imposing an appropriate boundary condition on the radial wave functions at the radius $r_{\text{max}}$ chosen to be sufficiently larger than the soliton size. The basis is then made finite by including only those states with the momentum $k$ as $k < k_{\text{max}}$. The results should be, however, independent on $r_{\text{max}}$ and $k_{\text{max}}$.

The hamiltonian with hedgehog ansatz commutes with the parity and the grandspin operator given by

$$K = j + \tau/2 = l + \sigma/2 + \tau/2,$$

where $j, l$ are respectively total angular momentum and orbital angular momentum. Accordingly, the angular basis can be written as

$$|(lj)KM\rangle = \sum_{j_3 l_3} C_{jj_3 l_3}^{KM} \left( \sum_{m_3} C_{l_3 m_3}^{j_3} |lm\rangle |\frac{1}{2} \sigma_3\rangle \right) |\frac{1}{2} \tau_3\rangle.$$  \hspace{1cm} (22)

With this angular basis, the normalized eigenstates of the free hamiltonian in a spherical box with radius $r_{\text{max}}$ can be constructed as follows:

$$u_{KM}^{(1)} = N_k \begin{pmatrix} ij_K(kr)|(KK + \frac{1}{2})KM\rangle \\
\Delta_k j_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \end{pmatrix},$$

$$u_{KM}^{(2)} = N_k \begin{pmatrix} ij_K(kr)|(KK - \frac{1}{2})KM\rangle \\
-\Delta_k j_{K-1}(kr)|(K - 1K - \frac{1}{2})KM\rangle \end{pmatrix},$$

$$u_{KM}^{(3)} = N_k \begin{pmatrix} i\Delta_k j_K(kr)|(KK + \frac{1}{2})KM\rangle \\
-j_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \end{pmatrix},$$

$$u_{KM}^{(4)} = N_k \begin{pmatrix} i\Delta_k j_K(kr)|(KK - \frac{1}{2})KM\rangle \\
j_{K-1}(kr)|(K - 1K - \frac{1}{2})KM\rangle \end{pmatrix},$$

$$v_{KM}^{(1)} = N_k \begin{pmatrix} ij_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \\
-\Delta_k j_K(kr)|(KK + \frac{1}{2})KM\rangle \end{pmatrix},$$
\[ v_{KM}^{(2)} = N_k \begin{pmatrix} i j_{K-1}(k r) |(K - 1 K - \frac{1}{2})K M \rangle \\ \Delta_k j_k(k r) |(K K - \frac{1}{2})K M \rangle \end{pmatrix} , \]

\[ v_{KM}^{(3)} = N_k \begin{pmatrix} i \Delta_k j_{K+1}(k r) |(K + 1 K + \frac{1}{2})K M \rangle \\ j_k(k r) |(K K + \frac{1}{2})K M \rangle \end{pmatrix} , \]

\[ v_{KM}^{(4)} = N_k \begin{pmatrix} i \Delta_k j_{K-1}(k r) |(K - 1 K - \frac{1}{2})K M \rangle \\ -j_k(k r) |(K K - \frac{1}{2})K M \rangle \end{pmatrix} , \]  

(23)

with

\[ N_k = \left[ \frac{1}{2} r_{\text{max}}^3 \left( j_{K+1}(k r_{\text{max}}) \right)^2 \right]^{-1/2} \]  

(24)

and \( \Delta_k = k/(E_k + M) \).

The momenta are discretized by the boundary condition \( j_k(k r_{\text{max}}) = 0 \). The \( u, v \) correspond to the “natural” and “unnatural” components of the basis which stand for parity \((-1)^K\) and \((-1)^{K+1}\) respectively.

Let us construct the trial function using the Kahana-Ripka basis to solve the eigenequations in Eq. (6),

\[ \phi_\mu(x) = \lim_{K_{\text{max}} \to \infty} \sum_{K=0}^{K_{\text{max}}} \sum_{M=-K}^{K} \sum_{j=1}^{4} \left[ \alpha_{KM,\mu}^{(j)} u_{KM}^{(j)}(r, \theta, \varphi) \\
+ \beta_{KM,\mu}^{(j)} v_{KM}^{(j)}(r, \theta, \varphi) \right] . \]  

(25)

According to the Rayleigh-Ritz variational method [19], the upper bound of the spectrum can be obtained from the secular equation

\[ \begin{vmatrix} A^{n,n} - EC^{n,n} & B^{n,u} \\
B^{u,n} & A^{u,u} - EC^{u,u} \end{vmatrix} = 0 , \]  

(26)

where

\[ A^{n,n}_{j,l}(K'M', KM) = \int d^3x u^{(j)}_{K'M'} H^{(l)}_{KM} , \]

\[ B^{n,u}_{j,l}(K'M', KM) = \int d^3x u^{(j)}_{K'M'} H^{(l)}_{KM} , \]

\[ C^{n,n}_{j,l}(K'M', KM) = \int d^3x u^{(j)}_{K'M'} v^{(l)}_{KM} , \]  

(27)
Table 1
A schematic picture of the matrix elements $A(K'M',KM)$ for the case of $B = 3$ (and $B = 5$), up to $K,K' = 2$. $S, P_1, P_0$ and $P_{-1}$ refer to the elements coupled with $(K,M) = (0,0), (1,1), (1,0)$ and $(1,-1)$ respectively. Other elements are all 0.

|        | (0 0) | (1 1) | (1 0) | (1-1) | (2 2) | (2 1) | (2 0) | (2-1) | (2-2) |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (0 0)  | $S$   |       |       |       |       |       |       |       |       |
| (1 1)  |       | $P_1$ |       |       |       |       |       |       |       |
| (1 0)  |       | $P_0$ | $P_0$ |       |       |       |       |       |       |
| (1-1)  |       |       | $P_{-1}$ | $P_{-1}$ |       |       |       |       |       |
| (2 2)  |       | $P_0$ | $P_0$ |       |       |       |       |       |       |
| (2 1)  |       |       | $P_{-1}$ | $P_{-1}$ |       |       |       |       |       |
| (2 0)  | $S$   |       |       |       |       |       |       |       |       |
| (2-1)  |       | $P_1$ |       |       |       |       |       |       |       |
| (2-2)  |       | $P_0$ | $P_0$ |       |       |       |       |       |       |

Table 2
Matrix elements $B^{n,u}(K'M',KM)$ for $B = 3$ (and $B = 5$) up to $K,K' = 2$. $S, P_1, P_0$ and $P_{-1}$ refer to the elements coupled with $(K,M) = (0,0), (1,1), (1,0)$ and $(1,-1)$ respectively. Other elements are all 0.

|        | (0 0) | (1 1) | (1 0) | (1-1) | (2 2) | (2 1) | (2 0) | (2-1) | (2-2) |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (0 0)  | $S$   |       |       |       |       |       |       |       |       |
| (1 1)  |       | $P_1$ |       |       |       |       |       |       |       |
| (1 0)  |       | $P_0$ |       |       |       |       |       |       |       |
| (1-1)  |       |       | $P_{-1}$ |       |       |       |       |       |       |
| (2 2)  |       | $P_0$ |       |       |       |       |       |       |       |
| (2 1)  |       |       | $P_{-1}$ |       |       |       |       |       |       |
| (2 0)  | $S$   |       |       |       |       |       |       |       |       |
| (2-1)  |       | $P_1$ |       |       |       |       |       |       |       |
| (2-2)  |       | $P_0$ |       |       |       |       |       |       |       |

$A^{u,u}, B^{u,u}, C^{u,u}$ are given by interchanging $u$ and $v$ in Eq. (27). For $K_{max} \to \infty$, the spectrum $E$ becomes exact. Eq. (26) can be solved numerically.

The angular part of the Kahana-Ripka basis consists of the spherical harmonics, spin and isospin wave functions. Thus, introducing complex basis, one can rewrite the chiral fields in Eq. (14) in the isospin space as

$$\boldsymbol{\tau} \cdot \hat{n}_R = \tau_1 \hat{n}_1 + \tau_2 \hat{n}_2 + \tau_3 \hat{n}_3 \to \tau_+ \hat{n}_- + \tau_- \hat{n}_+ + \tau_3 \hat{n}_3,$$
\[\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2),\]
\[\hat{n}_+ = \frac{2R}{1 + |R|^2}, \quad \hat{n}_- = \frac{2R^*}{1 + |R|^2}, \quad \hat{n}_3 = \frac{1 - |R|^2}{1 + |R|^2}.\]  
(28)

\(\hat{n}_+\) and \(\hat{n}_3\) can be expanded by the spherical harmonics

\[\hat{n}(\theta, \phi) = \sum_{K=0}^{K_{max}} \sum_{M=-K}^{K} a_{KM} Y_{KM}(\theta, \phi),\]  
(29)

\[a_{KM} = \int \hat{n}(\theta, \phi) Y^*_M(\theta, \phi) \sin \theta d\theta d\phi.\]  
(30)

Then we can perform the angular integration analytically with help of integral formula for the spherical harmonics given by

\[\int d\phi \int \sin \theta d\theta Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) Y^*_{l_3 m_3}(\theta, \phi) = \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l_3 + 1)} C_{l_1 l_2 0}^{l_3 0} C_{l_1 m_1 l_2 m_2}^{l_3 m_3}.\]  
(31)

Since the chiral fields in Eq. (14) is less symmetric than the \(B = 1\) hedgehog, the hamiltonian has no grand spin symmetry. As a result, the states with different grand spin couple strongly, and level splitting within the \(K\) blocks occur. In Tables 1, 2 are schematic pictures of the matrix elements \(A(K'M', KM)\) and \(B(K'M', KM)\) for \(B = 3\). Although the size of the matrix becomes quite large, the functional space can be rearranged and reduced in size owing to the symmetry of the chiral fields. For \(B = 3\), the space is divided with four blocks for each parity.

Numerically, we have to truncate the expansion of the size by cutoff \(k_{max}\) and \(K_{max}\). In the tables, each element consists of a \(4k_{max} \times 4k_{max}\) size of matrix. This block is spanned by \(2 \times \sum_{K=0}^{K_{max}} (2K + 1)/4\) for \(B = 3\) and hence the total matrix size is estimated as

\[\sim (K_{max}(K_{max} + 2)/2 \times 4k_{max})^2.\]  
(32)

The typical parameter values that we employ in our analysis are \(k_{max} = 24\) and \(K_{max} = 8\) for \(B = 3\), giving rather large matrix size \(3800 \times 3800\) approximately. In addition, our analysis is based on the self-consistent scheme and requires computational time much more. Obviously the situation gets worse for higher \(B\). Therefore, in some cases such as \(B = 6, 17\), we oblige to cut the size of the matrix, producing a few % uncertainty in results. In the next section we
Table 3
Mass spectra for $B = 1 - 9, 17$ also for some excited states $B = 5^*, 9^*(\text{in MeV})$. The data for $B = 2$ are taken from Ref. [7]. The ratio of the mass $E_{\text{static}}$ to $B \times E_{\text{static}}^{(B=1)}$ are compared to that of the Skyrme model [11].

| $B$ | $E_{\text{val}}^{(i)}$ | $E_{\text{field}}$ | $E_{\text{static}}$ | $E_{\text{static}}/BE_{\text{static}}^{(B=1)}$ |
|-----|---------------------|------------------|------------------|------------------|
| 1   | 173                 | 674              | 1192             | 1.00             |
| 2   | 173 173             | 1166             | 2204             | 0.92             |
| 3   | 210 210 210         | 1633             | 3522             | 0.98             |
| 4   | 144 146 146 146     | 2628             | 4378             | 0.92             |
| 5   | 123 131 131 139 210 | 3265             | 5467             | 0.92             |
| 6   | 120 124 150 150 206 206 | 3740          | 6603             | 0.92             |
| 7   | 115 120 120 120 166 166 166 | 4554        | 7478             | 0.90             |
| 8   | 97 97 115 120 139 139 203 203 | 5229        | 8565             | 0.90             |
| 9   | 69 101 104 104 107 166 166 179 179 | 6046        | 9573             | 0.89             |
| 17  | 83 95 95 95 153 156 157 173 175 | 177         | 178              | 0.89             |
| 5*  | 157 157 157 232 232 | 10586           | 18650            | 0.93             |
| 9*  | 99 105 105 121 142 142 210 210 210 | 5700        | 9742             | 0.91             |

will discuss the convergence properties of our numerical computation in more detail.

4 Results and discussions

4.1 Numerical results

Let us first show the results of the spectral flow analysis. For convenience we shall take

$$F(r) = -\pi + \pi r/X \quad \text{for} \quad r < X$$

$$= 0 \quad \text{otherwise}$$

(33)
Fig. 1. Spectral flow of $B = 3 - 9$ and $B = 5$ excited state solutions with the occupation number.
as a trial function for the profile function. In Fig. 1 we show the spectral flows for $B = 3 - 9$. As can be seen, the number of $B$ positive energy levels are diving into negative energy region and thus we obtain the baryon number $B$ soliton solutions. Putting the three quarks so as to be colour blind on each valence orbits as well as all on negative energy sea levels, we performed the
Fig. 4. Angular averaged baryon number density of $i$th valence quarks $\rho^{(i)}(r)$ of $B = 3 - 9, 17$, with the occupation number and the eigenvalue (in MeV).
The convergence properties of the solution with self-consistent calculations. The profile functions for $m$:

(i) a large energy gap between the ground state $G$ will be attained.

(ii) a large energy gap between the ground state labeled $A_1$, $A_2$, etc., and the first excited level $A_1$. Small dispersions of the spectra are observed in the results. In some cases they are caused by the finite size effect of the basis (ex. $B = 4$). Growing the size $r_{\text{max}}$ and increasing the number of the basis, more accurate degeneracy will be attained.

Fig. 5. The convergence properties of the solution with $B = 3$. 

Fig. 6. The convergence properties of the solution with $B = 5$. $K_{\text{max}}$ is fixed to be 8 and $m_{\text{max}}$ is varied until convergence is attained.

self-consistent calculations. The profile functions for $B = 3 - 9, 17$ are plotted in Fig. 2. In Table 3 are the results for the valence quark levels as well as the vacuum sea contributions. The valence quark spectra show various degenerate patterns depending on the background configuration. In Table 3, we also show the results of the ratio of the mass $E_{\text{static}}$ to $B \times E_{\text{static}}^{B=1}$, and comparison to that of the Skyrme model. The results are qualitatively in agreement.

The valence quark spectra for various $B$ are shown in Fig. 3. It is interesting that the results strongly suggest the existence of shell structure for the valence quarks. The spectra show (i) four fold degeneracy of the ground state labeled by $G$ and various degenerate pattern for excited levels labeled by $A_1, A_2, \cdots$, (ii) a large energy gap between the ground state $G$ and the first excited level $A_1$. Small dispersions of the spectra are observed in the results. In some cases they are caused by the finite size effect of the basis (ex. $B = 4$). Growing the size $r_{\text{max}}$ and increasing the number of the basis, more accurate degeneracy will be attained.
All our solutions are local minima obtained under the rational map ansatz in Eq. (14). Of course there may exist other stable solutions with lower energy outside the ansatz. We, however, suspect that the large degeneracy caused by the symmetry of the chiral fields would give a strong contribution to the minimization of the total energy.

In Fig. 4 are the results of $\rho^{(i)}(r)$ for $B = 3 - 9, 17$. The behaviour of the density near the origin confirms the existence of three shells ($G, A_1, A_2$). $G$ behaves like “S-wave” and others like “P-,D-wave” in a hydrogen-like atom. However most of the densities are nearly on the same surface and very small (not zero) near the origin. The plateau in the density observed at the center of the nucleus [20] can not be attained in our solutions. Therefore one may need to employ the multi-shell ansatz [21] even in the case of light nuclei.

Our numerical results depend on the basis parameters: radius $r_{\text{max}}$, the number of discretized momenta $k_{\text{max}}$, and the maximum value of the grandspin $K_{\text{max}}$. In Fig. 5 we demonstrate the accuracy of our numerical computation for $B = 3$ with $r_{\text{max}} = 6$ [fm]. The convergence of the solution and the valence energy $E_{\text{static}}, E_{\text{val}}$ are shown as a function of (a) $k_{\text{max}}$ with fixed $K_{\text{max}}$ and (b) $K_{\text{max}}$ with fixed $k_{\text{max}}$. This confirms that for $K_{\text{max}} = 8$ and $k_{\text{max}} = 20$ the solution with $B = 3$ is sufficiently converging.

As stated above, dispersions appear in the spectra, and in some cases it will be eliminated by taking a larger size of the spherical box and increasing the number of basis. However, for most of the solutions obtained here, these dispersions can not be ascribed to numerical errors. Let us examine the relation of the energy and the soliton size $r_{\text{max}}$ for $B = 5$ (see Fig. 6). For each value of $r_{\text{max}}$, we employ sufficiently large number of the basis to attain convergence. As can be seen, the dispersion does not disappear even for larger value of $r_{\text{max}}$. We therefore conclude that these dispersions are not due to numerical errors nor uncertainty but are the inherent feature of the solutions.

4.2 Symmetry and the degeneracy of the quarks

The bunch of valence spectra due to the potential with discrete symmetries has been observed in the study of heavier nuclear systems. In Ref. [22], the valence spectra are highly degenerate because the deformation of the spherically symmetric shell produces large shell gaps. Thus the nuclei can be considered to be more stable than the spherical one. As discussed in Ref. [11], the group theory should predict the level structure of pion fluctuations. However, our problem is more complicated due to the presence of quarks. Before discussing it in detail, let us show how the shell deformation is related to the degeneracy of the spectrum.
In general, if an eigenequation given by

\[ H\psi_\mu = E_\mu \psi_\mu \]  

(34)

is invariant under a symmetric operation \( \hat{R} \in g \), the equation transforms as

\[ \hat{R}H\psi_\mu = H(\hat{R}\psi_\mu) = E_\mu \psi_\mu . \]  

(35)

Therefore the states \( \{ \psi_\mu, \hat{R}\psi_\mu \} \) are degenerate in energy with \( E_\mu \). The set of \( d_\mu \) eigenfunctions \( \{ \psi^{(\mu)}_i \} (i = 1, \ldots, d_\mu) \) belonging to a given eigenvalue \( E_\mu \) will provide the basis for an irreducible representation of the group \( g \) of the hamiltonian [23]:

\[ \hat{R}\psi^{(\mu)}_j = \sum_i \psi^{(\mu)}_i D^{(\mu)}_{ij}(\hat{R}) . \]  

(36)

The operator \( \hat{R} \) are constructed as follows. If chiral fields have some particular point group symmetry i.e., \( U(x') = \hat{G}(\hat{a})U(x)\hat{G}(\hat{a})^\dagger \) (\( \hat{G}(\hat{a}) \in SU(2)_I \), and \( \hat{a} \) denotes the matrix of the following spatial rotation \( a_\nu^\nu \)), the Dirac equation is invariant under the Lorentz transformation

\[ (x')^\nu = a_\mu^\nu x^\mu \text{ or } x' = \hat{a}x \]  

(37)

with

\[ x' = \begin{pmatrix} t \\ x' \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{a} \end{pmatrix}, \quad x = \begin{pmatrix} t \\ x \end{pmatrix}, \]  

(38)

accompanying a corresponding iso-rotation

\[ (i\gamma^\mu \partial_\mu - MU^{\gamma_5}(x))\psi(x) = 0 \Rightarrow (i\gamma^\nu \partial'_\nu - MU^{\gamma_5}(x'))\psi'(x') = 0 , \]  

(39)

with

\[ \psi'(x') = (\hat{S}(\hat{a}) \times \hat{G}(\hat{a}))\psi(x) , \]  

(40)

where \( \hat{S}(\hat{a}) \) is a \( 4 \times 4 \) matrix which is a function of the parameters of the Lorentz transformation \( \hat{a} \), satisfying \( a_\mu^\nu \gamma_5 = \hat{S}^{-1}\gamma^\nu \hat{S} \). The operator \( \hat{R} \) corresponding to this rotation is thus defined by
\[ \psi'(x) \equiv_{def} \hat{R}\psi(x) = \hat{S}(\hat{a}) \times \hat{G}(\hat{a})\psi(\hat{a}^{-1}x). \] (41)

One can easily check that \( R \) commutes with the hamiltonian in Eq. (6). Constructing \( \hat{R} \) for each symmetry of the hamiltonian, one should be able to deduce the degeneracy structure of the spectra occurring in the valence level. The construction of \( \hat{R} \) in the case of \( B = 3 \) is shown in the appendix B, which confirms that the valence level has triply degenerate spectra.

Our numerical results indicate that the winding number strongly couple the elements with different \( K \) and hence correlated valence spectra occur (see Table 1-2). As can be seen from the operator \( K \) of the \( B = 2 \), the degeneracy of the valence spectra are determined by the shape deformation (symmetry) as well as the winding number of the chiral fields [6]. The four-fold degeneracy of the lowest states may be ascribed to the chiral symmetry \( SU(2)_L \times SU(2)_R \) of the hamiltonian. The degenerate structure for \( B \geq 3 \) will be well understood if symmetric operators of the hamiltonian which consist of the angular momentum, spin, isospin and winding number, are explicitly constructed.

5 Summary

In this paper we investigated the multi-soliton solutions in the chiral quark soliton model, using the rational map ansatz as a background chiral fields for quarks. The chiral fields with multi-winding number have particular discrete symmetries and it was shown that the baryon densities inherit the same discrete symmetries as the chiral fields. For the quark levels we observed various degenerate bound spectra depending on the background of chiral field configurations. Evaluating the radial component of the baryon density, shell-like structure of the valence quark spectra was also observed. The group theory should predict these level structures resulting from the symmetry of the background potential. In fact the degeneracy of the valence spectra are determined by the winding number of the chiral fields as well as the shape deformation (symmetry) of solitons. The four-fold degeneracy of the lowest states may be ascribed to the chiral symmetry \( SU(2)_L \times SU(2)_R \) of the hamiltonian. To get better understanding of the relation between the quark level structure and the winding number or the shape deformation, further analysis will be worth to be done in future.

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A The rational maps

In this appendix, we present the explicit forms of the rational maps which we used in our analysis as a background chiral fields (from Refs. [11,24]).

\[
\begin{align*}
R_3 &= \frac{\sqrt{3}iz^2 - 1}{z(z^2 - \sqrt{3}i)} , \\
R_4 &= \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1} , \\
R_5 &= \frac{z(z^4 + 3.94z^2 + 3.07)}{3.07z^4 - 3.94z^2 + 1} , \\
R_6 &= \frac{z^4 + 0.16i}{z^2(0.16z^4i + 1)} , \\
R_7 &= \frac{7/\sqrt{5}z^6 - 7z^4 - 7/\sqrt{5}z^2 - 1}{z(z^6 + 7/\sqrt{5}z^4 + 7z^2 - 7/\sqrt{5})} , \\
R_8 &= \frac{z^6 - 0.14}{z^2(0.14z^6 + 1)} , \\
R_9 &= \frac{z(-3.38 - 11.19iz^4 + z^8)}{1 - 11.19iz^4 - 3.38z^8} , \\
R_9^* &= \frac{z(z^4 - 5)}{-5z^4 + 1} , \\
R_9 &= \frac{5i\sqrt{3}z^6 - 9z^4 + 3i\sqrt{3}z^2 + 1 - 1.98z^2(z^6 - i\sqrt{3}z^4 - z^2 + i\sqrt{3})}{z^3(-z^6 - 3i\sqrt{3}z^4 + 9z^2 - 5i\sqrt{3} - 1.98(-i\sqrt{3}z^6 + z^4 + i\sqrt{3}z^2 - 1))} , \\
R_{17} &= \frac{17z^{15} - 187z^{10} + 119z^5 - 1}{z^2(z^{15} + 119z^{10} + 187z^5 + 17)}. 
\end{align*}
\]

(B.1)

B The Lorentz transformation with the chiral fields

In this appendix, we briefly show the evaluation of the rotation operator \( \hat{R} \) for \( B = 3 \) tetrahedron and also present the results of the transformation law for the numerical basis (23). The \( B = 3 \) tetrahedral soliton is characterized by two symmetry operations [11]: \( Z_2 \times Z_2 \) and \( T_d \). The Lorentz transformation operators \( \hat{S} \) and the operators for the chiral fields \( \{g, h\} \) corresponding to the symmetry operations \( (x')^\nu = a_\mu^\nu x^\mu \) are given by
\[ Z_2 \times Z_2 : \]
\[
(a_g)_\mu^\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \quad \Rightarrow \quad \hat{S}_g = i\gamma^0\gamma^5\gamma^2,
\]
\[ U(\hat{a}_g x) = \hat{g}U(x)\hat{g}^\dagger, \quad \hat{g} = \exp[-i\frac{\pi}{2} \tau_2]. \]
\[ T_d : \]
\[
(a_h)_\mu^\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix} \quad \Rightarrow \quad \hat{S}_h = \exp[i\frac{\pi}{3\sqrt{3}}(\sigma_{23} + \sigma_{31} + \sigma_{12})],
\]
\[ U(\hat{a}_h x) = \hat{h}U(x)\hat{h}^\dagger, \quad \hat{h} = \exp[-i\frac{\pi}{3\sqrt{3}}(\tau_1 + \tau_2 + \tau_3)]. \]

The \( \hat{R} \) is defined by the direct product of these rotation operators together with the inverse spatial rotation for the spinor, that is

\[ \hat{R}_g \psi(x) \equiv (\hat{S}_g \times \hat{g})\psi(\hat{a}_g^{-1} x), \quad (B.1) \]
\[ \hat{R}_h \psi(x) \equiv (\hat{S}_h \times \hat{h})\psi(\hat{a}_h^{-1} x). \quad (B.2) \]

We apply these operators to the Kahana-Ripka basis \( \phi \equiv \{u, v\} \) and finally obtain the following transformation law:

\[ \hat{R}_g \phi_{KM} = (-1)^{K-M} \phi_{K-M}. \quad (B.3) \]
\[ \hat{R}_h \phi_{00} = \phi_{00}. \quad (B.4) \]
\[ \hat{R}_g \begin{pmatrix}
\phi_a \\
\phi_b \\
\phi_c
\end{pmatrix} = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\phi_a \\
\phi_b \\
\phi_c
\end{pmatrix}, \quad (B.5) \]
\[ \phi_a \equiv \frac{1}{\sqrt{2}}(\phi_{11} + \phi_{1-1}), \quad \phi_b \equiv i\phi_{10}, \quad \phi_c \equiv \frac{i}{\sqrt{2}}(\phi_{11} - \phi_{1-1}). \]
\[
\begin{pmatrix}
\phi_\xi \\
\phi_\eta \\
\phi_\zeta \\
\phi_u \\
\phi_v
\end{pmatrix}
= 
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
\phi_\xi \\
\phi_\eta \\
\phi_\zeta \\
\phi_u \\
\phi_v
\end{pmatrix}.
\]

(B.6)

\[
\phi_\xi = \frac{i}{\sqrt{2}} (\phi_{21} + \phi_{2-1}), \quad \phi_\eta = \frac{1}{\sqrt{2}} (\phi_{21} - \phi_{2-1}),
\]
\[
\phi_\zeta = \frac{i}{\sqrt{2}} (\phi_{22} - \phi_{2-2}), \quad \phi_u = \phi_{20}, \quad \phi_v = \frac{1}{\sqrt{2}} (\phi_{22} + \phi_{2-2}).
\]

Thus we confirm that the \( B = 3 \) tetrahedron exhibits triply degenerate spectra.

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