We discuss the perturbative approach à la Dyson to a quantum field theory with nonlocal self-interaction \( \phi \star \cdots \star \phi \); according to Doplicher, Fredenhagen and Roberts (DFR). In particular, we show that the Wick reduction of non locally time-ordered products of Wick monomials can be performed as usual, and we discuss a very simple Dyson diagram.

Dedicated to Jacques Bros on the occasion of his 70th birthday.

1 Introduction

During the XXth century, locality has been so valuable a principle in the development of high energy physics, that it is strongly encoded in our minds. Sometimes we advocate locality even when it is not strictly necessary.

I once heard Daniel Kastler telling this story, that a sixty or so pages long paper appeared, about a certain topic in commutative algebra; the astonishing fact was that the only necessary change in order to generalize the results of that paper to the non abelian case was the removal of every occurrence of the word “abelian”. The conclusion of Kastler’s tale was that commutativity comes so natural to our mind, that sometimes even an expert might overlook the generality of some arguments.

Something similar happens with locality. We will see that some aspects and methods of the perturbation theory of a certain class of nonlocal theories are exactly the same as in the local case.

Indeed, we will consider the nonlocal theory which naturally arises when attempting a perturbative quantum field theory à la Dyson on the flat DFR quantum spacetime. It could have been easily recognized that the usual diagrammatic representation of the correction terms to the trivial scattering matrix has nothing to do with locality. The interesting point, however, is that it has not
been easy to recognize this, probably because of some psychological obstruction of the kind mentioned by Daniel Kastler.

In the next section, we will give a short description of the machinery underlying the DFR model of a flat quantum spacetime, and of a quantum field theory built on top of it. We will take the occasion to clarify the relations between the original DFR notations and those which are now current in the literature (see also the appendix).

In section 3, we will describe how to derive the Dyson diagrams for a (possibly) nonlocal perturbation theory using precisely the same methods which were developed in the late 40’s. An analogous discussion can be done for the Feynman diagrams, see [1] for details.

We will draw some conclusions in section 4.

2 DFR Quantum Spacetime, and All That

2.1 The Underlying Philosophy

In their seminal paper [2], Doplicher Fredenhagen and Roberts proposed to derive a simple model of spacetime coordinates quantization, stemming from first principles endowed with an operational meaning.

Indeed, the idea of quantizing the coordinates was quite old [3]. The idea that non-commuting coordinates would produce a coarse grained spacetime also was far from new. However, the spirit of the DFR proposal was rather original. While Snyder’s space coordinates have discrete spectra so to induce a covariant analogue of lattice discretization, the DFR coordinates all have continuous spectra, so that no limitations arise on the precision of the localization in one coordinate. Limitations arise instead on the precision of simultaneous localization in two or more non-commuting coordinates.

It is common folk lore that limitations on localization in the small should arise since, according to our understanding of high energy physics, the localization process requires that a certain amount of energy is transferred to the geometric background: the smaller the localization region, the higher the energy density induced in the localization region. If the localization process reaches a sufficiently small length scale (typically the Planck length scale $\lambda_P = \sqrt{\sqrt{G} \hbar c^{-3}}$), a closed horizon might trap the region under observation, preventing any information to escape from it.

This classical argument has been invoked for example to claim that it is not possible to localize with a precision below the Planck length scale (see e.g. [6, 4]). A moment’s thought, however, would make it clear that such a statement is not really supported by the above argument. Indeed, one might envisage to localize below the Planck length scale in one space dimension, at the cost of

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1 Apparently the first who made this remark (in a slightly different form, namely revisiting the Heisenberg’s microscope gedankenexperiment) was C. Alden Mead as early as 1959, but his paper underwent referee troubles and was published in 1964 [4]. See the interesting letter of Mead to Physics Today [5].
sufficiently delocalizing in the remaining space dimensions. The resulting admissible localization region would be very large (compared with $\lambda_P$) and very thin; so thin to obtain localization much below the Planck length (in one direction), and so large to keep the energy density sufficiently low to avoid black hole formation. It was precisely this remark that led DFR to cast limitations on the admissible localization regions in the form of uncertainty relations among the coordinates. In the lack of more direct motivations, they decided to reproduce the path which, long ago, led to canonical quantization; namely to find commutation relations inducing precisely the required uncertainty relations.

2.2 The DFR Basic Model

A heuristic analysis led DFR to postulate a very simple toy model of a flat, fully covariant quantum spacetime, described by four quantum coordinates, i.e. four selfadjoint operators $q^\mu$ on the infinite dimensional, separable Hilbert space $\mathcal{H}$. Setting $\lambda_P = 1$ (in suitable units) and

$$Q^{\mu\nu} = -i[q^\mu, q^\nu],$$

the commutation relations are $[q^\mu, Q^{\rho\sigma}] = 0$, or equivalently

$$\exp(ik_\mu q^\mu) \exp(ih_\mu q^\mu) = \exp\left(-\frac{i}{2} k_\mu Q^{\mu\nu} h_\nu\right) \exp(i(h + k)_\mu q^\mu),$$

(1)

to be complemented with the statement that the joint spectral values $\sigma^{\mu\nu}$ of the pairwise commuting operators $Q^{\mu\nu}$ define precisely the set $\Sigma$ of the antisymmetric matrices $\sigma$ fulfilling

$$\sigma_{\mu\nu} \sigma^{\mu\nu} = 0, \quad (\sigma^{\mu\nu} (\ast \sigma)_{\mu\nu})^2 = 16.$$

The requirement that $[q^\mu, Q^{\rho\sigma}] = 0$ is a simplifying, otherwise arbitrary ansatz; once this ansatz is accepted, the limitations on the set $\Sigma$ stem out of the DFR stability condition of spacetime under localization, together with the quest for covariance. We will not give the details; the interested reader is referred to the original paper.

The coordinates are covariant: there is a unitary representation $U$ of the Poincaré group $\mathcal{P}$, such that

$$U(\Lambda, a) q^\mu U(\Lambda, a)^{-1} = \Lambda^\mu_\nu q^\nu + a^\mu, \quad (\Lambda, a) \in \mathcal{P}.$$

In strict analogy with Weyl quantization, one may consider the quantization of an ordinary function $f = f(x)$ of $\mathbb{R}^4$ defined by

$$f(q) = \int_{\mathbb{R}^4} dk \, \hat{f}(k) e^{ikq},$$

(2)

---

2More precisely, equation (1) gives the formal relations $[q^\mu, Q^{\rho\sigma}] = 0$ the precise mathematical status of regular, strong commutation relations.

3We recall that the Hodge dual $\ast \sigma$ of an antisymmetric 2-tensor $\sigma$ is given by $(\ast \sigma)_{\mu\nu} = (1/2) \epsilon_{\mu\nu\lambda\sigma} \sigma^{\lambda\rho}$. 

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where \( kq = k_\mu q^\mu \), and
\[
\tilde{f}(k) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} dx \, f(x)e^{-ikx}.
\]

At this point, one might wish to follow the suggestion of Weyl (for example like von Neumann did \[8\]) and describe the operator product \( f_1(q)f_2(q) \) in terms of a suitable product of the ordinary functions \( f_1, f_2 \). Unfortunately, the set of all the operators of the form \( f(q) \) is not closed under operator products. It is then necessary to preliminarily extend the DFR quantization to a larger class of functions, namely the functions \( F = F(\sigma, x) \) of \( \Sigma \times \mathbb{R}^4 \). The full DFR quantization of a function of the form
\[
F(\sigma, x) = \sum_i g_i(\sigma) f_i(x)
\]
(a DFR symbol) is given by
\[
F(Q, q) = \sum_i g_i(Q) f_i(q);
\]
above, \( f_i(q) \) is understood as in \[2\], while \( g_i(Q) \) is the joint functional calculus of the pairwise commuting operators \( Q^{\mu\nu} \).

We may now define a covariant product \( \star \) of two DFR symbols (the DFR twisted product) by requiring that
\[
F_1(Q, q)F_2(Q, q) = (F_1 \star F_2)(Q, q);
\]
By standard computations, one easily finds
\[
(F_1 \star F_2)(\sigma, x) =
\]
\[
= \frac{1}{\pi^4} \int_{(\mathbb{R}^4)^2} da db \, F_1(\sigma, x + a)F_2(\sigma, x + b) \exp\left( -2i\sigma^{-1}_{\mu\nu} a_\mu b_\nu \right).
\]
An asymptotic expansion of the (reduced) DFR product is widely known as the Moyal product. See the appendix for more details.

Following \[2\], we may define two maps, which, for self evident reasons, we will denote by \( \int_{\{q^0=t\}} d^3q \) and \( \int d^4q \), respectively:
\[
\int_{\{q^0=t\}} d^3q \, F(Q, q) = \int_{\mathbb{R}^3} d^3x \, F(Q, (t, x)), \quad \int d^4q \, F(Q, q) = \int_{\mathbb{R}^4} d^4x \, F(Q, x).
\]
These maps are positive: for all \( F \)'s and \( t \)'s,
\[
\int_{\{q^0=t\}} d^3q \, F(Q, q)F(Q, q)^* \geq 0, \quad \int d^4q \, F(Q, q)F(Q, q)^* \geq 0.
\]
In particular, the positivity of the map \( \int_{\{q^0=t\}} d^3q \) is compatible with the uncertainty relations, since the latter allow for exact localization in \( q^0 \), at the cost of total delocalization in the remaining coordinates. Note also that, for any \( x \in \mathbb{R}^4 \) fixed, the map \( F(Q, q) \mapsto F(Q, x) \) is not positive.
2.3 Field Theory

A (Wightman, say) quantum field $\phi(x)$ on ordinary Minkowski spacetime is a (generalized) function taking values (morally) in the field algebra $\mathcal{F}$, namely it is (morally) in $C(\mathbb{R}^4, \mathcal{F}) \simeq C(\mathbb{R}^4) \otimes \mathcal{F}$. Here $C(\mathbb{R}^4)$ is the localization algebra.

Hence, it is natural to replace the classical localization algebra $C(\mathbb{R}^4)$ with its quantized counterpart, the algebra $\mathcal{E}$ generated by the quantum coordinates $q^\mu$. In this “semiclassical” model, it is natural to seek for quantum fields on quantum spacetime as elements of (morally) the algebra $\mathcal{E} \otimes \mathcal{F}$. By analogy with the quantization of ordinary functions, DFR proposed the following quantization of the free Klein–Gordon field:

$$\phi(q) := \int_{\mathbb{R}^4} dk \, e^{ikq} \otimes \hat{\phi}(k).$$

Then they made the following, fundamental remark. Let $\mathcal{H}_0(\phi(x), \partial_\mu \phi(x))$ be the free Hamiltonian density. It is well known that the full free Hamiltonian

$$H_0 = \int d^3x \, \mathcal{H}_0(\phi(t, x), \partial_\mu \phi(t, x))$$

does not depend on the time $t$. Then, it was shown in [2] that

$$\int_{\{q^0=t\}} d^3q \, \mathcal{H}_0(\phi(q), \partial_\mu \phi(q)) = H_0$$

(as a constant function of $\sigma$). To put it in a more explicit way, with

$$\mathcal{H}_0(\phi(x), \partial_\mu \phi(x)) = \frac{1}{2} \left( \sum_\mu (\partial_\mu \phi)^2(x) + m^2 \phi^2(x) \right),$$

we have

$$\int_{\{q^0=t\}} d^3q \, \mathcal{H}(\phi(q), \partial_\mu \phi(q)) =$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} d^3x \, \left( \sum_\mu ((\partial_\mu \phi) \ast (\partial_\mu \phi))(t, x) + m^2(\phi \ast \phi)(t, x) \right) = H_0$$

(as a constant function of $\sigma$).

This remark is the starting point for defining an effective perturbation theory on the ordinary Minkowski quantum spacetime, in the so called interaction

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4. This result is independent from the well known general fact that, for any two admissible functions $f, g$, not necessarily solutions of the Klein–Gordon equation, then $\int d^4x \, (f \ast g)(x) \equiv \int d^4x \, (fg)(x)$. The latter result is commonly taken as the starting point for a perturbative approach to nonlocal theories in the Euclidean setting, since it implies that the free action is unchanged by replacing the ordinary pointwise product with some twisted product $\ast$. Note however that the Wick rotation of a twisted product is ill defined (namely no well defined $\ast$ may be reached by Wick rotating the DFR product $\star$), so that at present (and to the best of the author’s knowledge) the only known relation between Euclidean and Minkowskian “twisted theories” is a weak, indirect formal analogy.
representation; of course, this ansatz should be taken with a grain of salt. Since
the Hamiltonian is unaffected by the replacement of ordinary pointwise products
with twisted products, we may consider perturbations of the usual free fields.
The only remaining difficulty is the dependence on $\sigma$; to get rid of it in view of
an effective theory on ordinary spacetime, we need to integrate it out by means
of some measure on $\Sigma$. Unfortunately, there is no Lorentz invariant measure on
$\Sigma$; the best we can find is a rotation invariant measure $d\sigma$. Quite unfortunately,
this will destroy the covariance of this class of models under Lorentz boosts.

By analogy with the case of the local $\phi^n$ interaction, we may consider the
interaction Hamiltonian

$$H_I(t) = \int_{q^2=t} :\phi(q)^n: = \int da : (\phi \star \cdots \star \phi)(Q,(t,a)) :,$$

where $:AB\cdots:$ denotes the normal (i.e. Wick) ordering of operator products.
To obtain an effective interaction Hamiltonian on the classical Minkowski space-
time, we integrate the $\sigma$ dependence out, getting

$$H_{I}^{\text{eff}}(t) = \int d\sigma \int da : (\phi \star \cdots \star \phi)(\sigma,(t,a)) : = \int da \int_{\mathbb{R}^3} dx_1 \cdots dx_n W_{t,a}(x_1,\ldots,x_n) :\phi(x_1) \cdots \phi(x_n):$$

for a suitable kernel $W_{x}$. Following Dyson [9], the (formal) $S$-matrix is given
by

$$S = I + \sum_{N=1}^{\infty} \frac{(-ig)^N}{N!} \int_{\mathbb{R}^N} dt_1 \cdots dt_N T[H_{I}^{\text{eff}}(t_1),\ldots,H_{I}^{\text{eff}}(t_N)],$$

where

$$T[H_{I}^{\text{eff}}(t_1),\ldots,H_{I}^{\text{eff}}(t_N)] = \sum_{\pi} H_{I}^{\text{eff}}(t_{\pi(1)}) \cdots H_{I}^{\text{eff}}(t_{\pi(N)}) \prod_{k=1}^{N-1} \theta(t_{\pi(k+1)} - t_{\pi(k)})$$

5If $C_{\sigma}$ is the kernel such that

$$(f_1 \star \cdots \star f_n)(\sigma,x) = \int_{(\mathbb{R}^4)^n} dx_1 \cdots dx_n C_\sigma(x-x_1,\ldots,x-x_n)f_1(x_1) \cdots f_n(x_n),$$

then

$$W_\sigma(x_1,\ldots,x_n) = \int_{\Sigma} d\sigma C_\sigma(x-x_1,\ldots,x-x_n).$$
is the product of the $H^H_\ell(t_j)$'s taken in the order of decreasing times (the sum over $\pi$ running over all permutations of $(1, \ldots, N)$). Note that, contrary to the usual conventions, we wrote $T[A, B, \ldots]$ instead of the traditional $T[AB\cdots]$.

As pointed out in [2], the time ordering acts on the overall times $t_1, \ldots, t_n$ of the interactions Hamiltonians, not on the time arguments of the fields which appear in the definition of the interaction Hamiltonian. Since the perturbation theory described above is built on top of a Hamiltonian model, the $S$-matrix of the DFR $\phi^n$ interaction is (formally) unitary by construction. More recent concerns about possible unitarity violation were the consequence of a too naive way of performing the time ordering prescription (see [10], and references therein; see also [11] [12]). The ultraviolet regularity of a $\phi^3$ DFR model has recently been proved by Bahns [13], under a weaker prescription for averaging over $\sigma$.

At this point it might be natural to convince ourselves that the trick of absorbing the time ordering into some analogue of the Stueckelberg–Feynman propagator is not possible any more. Indeed, in the local case the Stueckelberg–Feynman propagators allow us to consider one diagram, describing at once all possible arrangements of the time of the vertices [14] [15]; this feature might seem apparently lost in the present nonlocal case. We will see in the next section how, on the contrary, things are bound to go exactly the same way as in the local case.

Before closing this section, let us recall that different perturbative approaches, which are equivalent in the local case, may well fail to be such in the nonlocal case. As an example of this situation we mention the noncommutative analogue of the Yang–Feldman equation proposed in [10], and developed in [16] [11]. Finally, see [17] [18] for a different generalization of the Wick product, based on optimal localization; the resulting model is free of ultraviolet divergences.

3 Nonlocal Dyson Diagrams

In [19], Denk and Schweda showed that the above mentioned concerns were wrong; indeed, they were able to show that it was possible to absorb the time ordering in the definition of a simple generalization of the Stueckelberg–Feynman propagator, namely

$$\mathcal{D}(x; \tau) = \frac{1}{i} \left( \Delta_+ (x) \theta(\tau) + \Delta_+ (-x) \theta(-\tau) \right).$$

When the theory is local, then the above general propagator (the “contractor”, according to Denk and Schweda) is always evaluated at $\tau = x^0$, in which case it reproduces the usual Stueckelberg–Feynman propagator:

$$\mathcal{D}(x; x^0) = \Delta_{SF}(x).$$

The original argument (which is casted in the case of non Wick ordered interactions, and worked out in the framework of the Gell–Mann & Low formula) is rather involved: it relies on a clever, though very tricky manipulation consisting of Wick reducing the ordinary pointwise products of fields, and recombining the products of two–point functions and $\theta$ functions by hand.
There is, however, a profound reason why such a clever rearrangement of terms leads to the desired result: indeed, one can instead reproduce exactly the same line of reasoning which can be found in any standard textbook on local quantum field theory, since the Wick reduction of time ordered products of Wick monomials can be performed in the nonlocal setting considered here, too [1]. In other words, the second Wick theorem is not local; this is so deeply true that even the original proof of Wick does not rely on locality [20].

Let us see this in the case of the Dyson diagrams. Consider the $N^{th}$ order contributions to the S-matrix:

$$S^{(N)} = \frac{(-ig)^N}{N!} \int_{\mathbb{R}^N} dt_1 \cdots dt_N \, T[H^\text{eff}_1(t_1), \ldots, H^\text{eff}_N(t_N)]$$

We introduce the following shorthands:

$$x_j = (x_{j1}, \ldots, x_{jn}) \in \mathbb{R}^{4n}, \quad dx_j = \prod_{k=1}^n dx_{jk}, \quad \phi^{(n)}(x_j) = \phi(x_{j1}) \cdots \phi(x_{jn}),$$

so that

$$H^\text{eff}_I(t) = \int_{\mathbb{R}^3} da \int_{\mathbb{R}^{4n}} d\bar{x} \, W(t,a) :\phi^{(n)}(\bar{x}) :,$$

and a short, standard computation gives

$$S^{(N)} = \frac{(-ig)^N}{N!} \int_{(\mathbb{R}^3)^N} da_1 \cdots da_N \int_{(\mathbb{R}^{4n})^N} d\bar{x}_1 \cdots d\bar{x}_N \, W_{a_1}(\bar{x}_1) \cdots W_{a_N}(\bar{x}_N) \times \prod_{j=1}^{N-1} \theta(\tau_{\pi(j+1)} - \tau_{\pi(j)}) \times T^{\tau_{\pi(1)}, \ldots, \tau_{\pi(N)}},$$

where we introduced the following, natural notation:

$$T^{\tau_1, \tau_2, \ldots, \tau_k}[A_1, A_2, \ldots, A_k] = \sum_{\pi} A_{\pi(1)} \cdots A_{\pi(k)} \prod_{j=1}^{k-1} \theta(\tau_{\pi(j+1)} - \tau_{\pi(j)}),$$

namely the product of the $A_j$'s is taken in the order of decreasing $\tau_j$'s. Note that this definition may be given in general; there is no need for any a priori relation between the factors $A_j$ and the parameters $\tau_j$ which we may wish to attach to those factors; we will call the above a general time ordered product, to highlight this fact.

The key remark here is that the mechanism for the Wick reduction of a general Time ordered product works as usual. The only difference with respect to the local case is that, here, we have to keep in mind that to each field there corresponds a parameter driving its position in the time ordered product; this was implicit in the local case, where the time parameter corresponding to each field $\phi(x^0, x)$ was precisely $x^0$.

---

6We rename $t_j$ as $a_j^0$, so that $\int_{a_1} dt_1 \int_{a_2} da_2 = \int_{a_3} da.$
In view of this remark, we need for a notation which indicates this correspondence explicitly, e.g.

\[ \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 \cdots \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 : \phi(x_{11}) \cdots \phi(x_{1n}) \phi(x_{21}) \cdots \phi(x_{2n}) \cdots \phi(x_{N1}) \cdots \phi(x_{Nn}) : = \]  

Wick contractions\(^7\), then, may be defined in the obvious way with respect to the above correspondence:

\[ \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 \cdots \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 : \phi(x_{11}) \cdots \phi(x_{iu}) \cdots \phi(x_{jv}) \cdots \phi(x_{Nn}) : = \]

= \[ \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 \cdots \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 : \phi(x_{11}) \cdots \phi(x_{iu}) \cdots \phi(x_{jv}) \cdots \phi(x_{Nn}) : \mathcal{D}(x_{iu} - x_{jv}; a_i^0 - a_j^0), \]

where a caret indicates omission. The reader is invited to read the beautiful original paper of Wick [20], to convince herself that the proof does not rely on the requirement that the time associated to the field is the same as the time argument of the field (see also [1] for a more detailed, “non local” discussion).

Hence, we still may state the second general Wick theorem, according to which

The general time ordered product

\[ T^{a_0^1 \cdots a_N^0} [\phi^{(1)}(x_1); \cdots; \phi^{(n)}(x_N)]; \]

equals the sum of the terms obtained by applying all possible choices of any number (including none) of allowed general contractions to \[ \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 \cdots \Phi^\alpha_0 \Phi^\beta_0 \Phi^\gamma_0 \], where no contraction is allowed, which involves two fields associated with the same time parameter \( a_j^0 \).

By applying the Wick reduction to (4), we obtain a certain sum of integrals, which we may label by means of Dyson diagrams, namely diagrams consisting of \( N \) (= the order in perturbation theory) vertices, each of which is the origin of no more than \( n \) (possibly none) lines for a \( \phi^{*n} \) interaction; each line connects two distinct vertices (no loops, no external lines). This is possible because a Wick monomial \( \phi(x_1) \cdots \phi(x_n) \) is totally symmetric in the arguments \( x_1, \ldots, x_n \), so that we may safely replace each kernel \( W_a(x_1, \ldots, x_n) \) by its totally symmetric part in (4). To see how to proceed, let us consider the following second order contribution in the case of a \( \phi^{*3} \) interaction:

\[ \frac{-g^2}{2} \int_{(R^4)^2} \int_{(R^{4n})^2} dxdy \, W_a(x)W_b(y) : \phi(x_1) \phi(x_2) \phi(x_3) \phi(y_1) \phi(y_2) \phi(y_3) : = \]

= \[ \frac{-g^2}{2} \int_{(R^4)^2} \int_{(R^{4n})^2} dxdy \, W_a(x)W_b(y) : \phi(x_3) \phi(y_1) : \times \]

\[ \times \mathcal{D}(x_1 - y_2; a^0 - b^0) \mathcal{D}(x_2 - y_3; a^0 - b^0). \]

\(^7\)Some readers might be surprised (I was surprised) to discover that the standard graphic notation for contractions is not due to Wick; it was first introduced by Houriet and Kind in Helv. Phys. Acta 22, 319 (1949). In his paper, Wick politely complains for he was forced to abandon that very convenient notation for typographical reasons. Hence, he writes e.g.

: \( UVW^*XY^*Z^* \) instead of \( \underbrace{UVWXYZ} \).
It is clear that, up to renaming the integration variables and using the total symmetry of $W_a, W_b$, and $\phi(x_3)\phi(y_1)$; the above integral is exactly the same that we would obtain from the contribution

$$\frac{-g^2}{2} \int_{(R^4)^2} \int_{(R^n)^2} dadb \int dx dy W_a(x) W_b(y) \phi(x_1)\phi(x_2)\phi(x_3)\phi(y_1)\phi(y_2)\phi(y_3):$$

Hence, the only informations which are needed to write it down are that there are two Wick monomials and two contractions between those two monomials. To give this information, it is sufficient to draw a diagram with two vertices (the two Wick monomials) and two lines connecting them (the contractions). Of course, one also has to count the multiplicity of a diagram, namely the number of different contraction schemes that would lead to the same integral up to dummy integration variables.
4 Conclusions

We have seen that the usual diagrammatic expansion of the Dyson series is not special to local interactions. Indeed, diagrams are a consequence of two nonlocal tools, namely the Dyson perturbation series and the normal (Wick) ordering of products of creation and annihilation parts. Both these tools are completely unrelated to locality, hence this result should have not come out as a surprise. Feynman diagrams require an additional tool, the Gell–Mann & Low formula [21], which also has nothing to do with locality; indeed, it was shown in [1] that the dear old Feynman diagrams also arise naturally in the reduction of the nonlocal Green functions of the DFR perturbative model.

Actually, we did not introduce any really new argument, everything was already virtually contained in the papers of Dyson, Wick, and Gell–Mann & Low. The case of the usual local $\phi^n$ interaction can be reobtained as a special case, by setting $W_a(x_1,\ldots,x_n) = \prod_j \delta^{(4)}(x_j - a)$ in (4).

The unified treatment of both local and nonlocal $\phi^n$ interactions may be of some practical interest, since it allows to study the convergence of the large scale limit diagramwise. Moreover, it may allow for developing a renormalization scheme for nonlocal theories with a strict correspondence between local and nonlocal subtractions. In the large scale limit, it might be natural to expect that nonlocal, possibly finite subtractions converge to the infinite subtractions of the local renormalized theory. However, it should be kept in mind that a different point of view may well be taken. For example, in [16], a different prescription for the admissible subtractions was investigated, in order to only select those subtractions which are divergent already at the Planck length scale.

We close with the following remark. Many years ago, Caianiello raised some concerns about what he regarded as a too naive interpretation of Feynman diagrams as pictorial representations of actual scattering processes (see [22]). He remarked that, even if we were inclined to accept such a view prior to renormalization, some paradox might arise after implementing some subtraction prescription (e.g. in the fermionic case, strong interference among free field modes belonging to different diagrams might produce violations of the exclusion principle). Here, we found that, even prior to renormalization, diagrams seem to be nothing more than a graphic representation of the CCR algebra combined with ordinary quantum mechanical perturbation theory; it might be misleading to try to see more than that.
Appendix. Twisted Products

Actually, for both historical and technical reasons, the DFR twisted product was laid down in Fourier space. To avoid confusion, in this appendix we will reserve the symbol $\times$ to indicate the ordinary convolution product, and $\hat{\times}$ to indicate the twisted convolution product (twisted product in Fourier space).

Setting

$$
(\varphi_1 \hat{\times} \varphi_2)(\sigma, k) = \int \mathbb{R}^4 dh \, \varphi_1(\sigma, h) \varphi_2(\sigma, k-h) \exp \left( \frac{i}{2} k_\mu \sigma^{\mu\nu} h_\nu \right),
$$

$$
\varphi_j(\sigma, \cdot) \in L^1(\mathbb{R}^4),
$$
one easily finds

$$
F_1 \star F_2 = \hat{F}_1 \hat{\times} \hat{F}_2, \quad F_j(\sigma, \cdot) \in L^1(\mathbb{R}^4) \cap \hat{L}^1(\mathbb{R}^4).
$$

A (formal) asymptotic expansion of the product $\star$ can be easily obtained by standard Fourier theory. Indeed, by replacing the exponential factor $\exp[(i/2)k_\sigma h] = \exp[-(i/2)h_\sigma(k-h)]$ with the corresponding exponential series and (formally) exchanging the sum and the integration, one obtains

$$
(\hat{F}_1 \hat{\times} \hat{F}_2)(\sigma, x) = \int \mathbb{R}^4 dk e^{ikx} \int \mathbb{R}^4 dh \, \hat{F}_1(\sigma, h) \hat{F}_2(\sigma, k-h) \exp \left\{ \frac{i}{2} \sigma^{\mu\nu} k_\mu(h-k)_\nu \right\} = F_1(\sigma, x) F_2(\sigma, x) + \sum_{n=1}^{\infty} \frac{(-i/2)^n}{n!} \sigma^{\mu_1\nu_1} \cdots \sigma^{\mu_n\nu_n} \left( (\partial_{\mu_1} \cdots \partial_{\mu_n} F_1)(\sigma, x) \partial_{\nu_1} \cdots \partial_{\nu_n} F_2)(\sigma, x) \right).
$$

The above formal series gives a precise meaning to the more compact definition

$$
\mathcal{A}[F_1 \hat{\times} F_2](\sigma, x) = \left( \exp \left\{ \frac{i}{2} \sigma^{\mu\nu} \partial_\mu \otimes \partial_\nu \right\} F_1 \otimes F_2 \right)(\sigma, x) \quad (6)
$$

\footnote{Twisted products first arose in the framework of canonical quantization, in the late 1920’s. Their use was first advocated by Weyl [7], who however did not publish explicit equations; following Weyl’s suggestion, von Neumann [8] laid down the twisted product in Fourier space (twisted convolution). The twisted product in position space first appeared (in the form of an asymptotic expansion) in a paper by Grönwold; the integral form was first used by Baker and explicitly written down by Pool. The first rigorous results on asymptotic expansions of twisted products are probably due to Antonet, and a comprehensive investigation can be found in [23], to which we also refer for the bibliographical coordinates missing in this footnote. The seminal work of Weyl and von Neumann inspired Wigner to define the so called Wigner transform; Wigner’s work in turn led Moyal to define the so called Moyal bracket or sine–bracket $\{ f, g \}_{\star} = f \star g - g \star f$; the Moyal bracket then plaid a fundamental role in a seminal paper by Bayen et al about geometric quantization of phase manifolds. The covariant version of the twisted product was first introduced by DFR in order to quantize the spacetime. For some strange reason, in the current literature about QFT on noncommutative spacetime, the DFR variant of the Weyl–von Neumann twisted product is widely known as the Moyal product.

\footnote{With our conventions, we have $(\partial_\mu f)(k) = -ik_\mu f(k)$ and $f \times g = (2\pi)^4 f g$.}
of the asymptotic expansion $\mathcal{A}[F_1 \ast F_2]$ of $F_1 \ast F_2$; here, $\otimes$ is a tensor product of functions defined fibrewise over $\sigma$ by

$$(F_1 \otimes F_2)(\sigma, x, y) = F_1(\sigma, x)F_2(\sigma, y),$$

and $m$ is the fibrewise multiplication map

$$m(F_1 \otimes F_2)(\sigma, x) = F_1(\sigma, x)F_2(\sigma, x).$$

Some authors take (6) as their definition of twisted product.

This asymptotic expansion, however, may be rather misleading. Assume that, for a fixed value of $\sigma$, the functions $F_j(\sigma, x)$ of $x$ are $C^\infty$ and have compact support, $j=1,2$; moreover, assume that the supports are disjoint. Since derivatives cannot enlarge the supports, we have $\mathcal{A}[F_1 \ast F_2](\sigma, \cdot) \equiv 0$. In this precise sense, the asymptotic expansion of the twisted product defines (in the sense of formal power series) a local, non commutative product. This is rather unsatisfactory from the point of view of spacetime quantization, since on general grounds we expect the quantum geometry to be non local.

Note also that, in the special case envisaged right above, there is no reason why we should have $F_1 \ast F_2 \equiv 0$, and there are counterexamples, indeed. In other words, to naively rely on the above asymptotic expansion amounts to work with a completely different algebra than that of “true” twisted products. This also shows that the series may fail to converge (if it converges at all) to the actual twisted product $F_1 \ast F_2$.

While for large classes of functions the asymptotic expansion truncated at order $n$ agrees with the twisted product up to terms of order $\lambda_p^{2(n+1)}$, the issue of convergence is rather delicate, and there are very few general results (see [23] and references therein). Certainly, the asymptotic expansion converges to (the Antonet extension of) $F_1 \ast F_2$ if $\hat{F}_1, \hat{F}_2$ have compact supports; in this case, $F_1, F_2$ are real-analytic, i.e. they are nonlocal in the sense that they cannot be deformed locally while preserving analyticity. In a sense, we may equivalently describe the nonlocality of quantized spacetime using either (3) as a nonlocal product of local objects ($L^1$ functions), or (6) as a local product of nonlocal objects (analytic functions). As a (formal) product of smooth functions, (6) is irremediably local, and should be dismissed.

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