Satellite Conjunction Analysis and the False Confidence Theorem

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Satellite conjunction analysis is the assessment of collision risk during a close encounter between a satellite and another object in orbit. A counterintuitive phenomenon has emerged in the conjunction analysis literature: probability dilution, in which lower quality data paradoxically appear to reduce the risk of collision. We show that probability dilution is a special case of a broader structural deficiency in epistemic probability distributions. In probabilistic representations of statistical inference, there are always false propositions that have a high probability of being assigned a high degree of belief. This is the false confidence theorem. As a practical matter, its manifestation in satellite conjunction analysis is particularly detrimental. Under ordinary operating conditions, satellite navigators using epistemic probability of collision as their decision-motivating risk metric are rendered incapable of detecting an impending collision.

An explicit remedy for false confidence can be found in the Martin–Liu theory of inferential models. In satellite conjunction analysis, we show that \( K_\sigma \) uncertainty ellipsoids satisfy the Martin–Liu validity criterion. Performing collision avoidance maneuvers based on ellipsoid overlap will ensure that operational collision risk is capped at the user-specified level.

1 Introduction

A satellite conjunction is an event in which two satellites or a satellite and a piece of debris are estimated to pass near each other. The goal of satellite conjunction analysis is to determine whether the risk of collision is high enough to necessitate a collision avoidance maneuver. Avoiding collision is not just important to the individual satellite operator. Everyone in the aerospace industry has a stake in minimizing the number of in-orbit collisions, because aggregate orbital debris has the potential to make low Earth orbit unnavigable (Kessler et al. 2010; Liou and Johnson 2006, 2008). The situation is further exacerbated by the fact that the number of active satellites is expected to precipitously increase over the next few years (Barry 2017; Davenport 2016). So, while some of the issues involved in conjunction analysis may initially seem arcane, correctly resolving those issues is of existential importance to the aerospace industry.
1.1 A Modern Problem

Over the past 15 years, aerospace researchers have recognized a counterintuitive phenomenon in the collision risk numbers obtained in satellite conjunction analysis. It is called probability dilution (Alfano 2003). Past a certain point, as uncertainty in the satellite trajectories increases, the epistemic probability of collision decreases. The seemingly absurd implication of probability dilution is that lower quality data reduce the risk of collision.

Several researchers have attempted to synthesize an ad hoc fix to probability dilution (e.g., Alfano 2003; Balch 2016; Frigm 2009; Plakalović et al. 2011; Sun et al. 2014). Most of them have defined an alternative collision risk metric that is increasing, or at least non-decreasing, as a function of trajectory uncertainty. However, none of these researchers has offered a rigorous explanation for why probability dilution is supposedly wrong or, at least, how it is misleading. They all proceed on the basis that its counterintuitiveness is proof of its inappropriateness.

1.2 A Centuries-Old Argument

Satellite trajectory estimation is fundamentally a problem of statistical inference, and the confusion over probability dilution cuts to the heart of the long-standing Bayesian–frequentist debate in statistics (for a review, see Barnett 1999). Satellite orbits are inferred using radar data, optical data, GPS data, etc. which are subject to random and bias errors (i.e., noise). The resulting probability distributions used in conjunction analysis represent epistemic uncertainty, rather than aleatory variability. That is to say, it is the trajectory estimates that are subject to random variation, not the satellite trajectories themselves. In the Bayesian view, that is a distinction without a difference; it is considered natural and correct to assess the probability of an event of interest, such as a possible collision between two satellites, based on an epistemic probability distribution (e.g., de Finetti 1935; Laplace 1814; Robert 2007). In the frequentist view, though, it is considered an anathema to try to compute the probability of a non-random event (e.g., Boole 1854; Mayo 1996; Neyman 1937). This prohibition is expressed in some corners of the uncertainty quantification community as the requirement that epistemic uncertainty be represented using non-probabilistic mathematics (e.g., Ferson and Ginzburg 1996; Oberkampf and Roy 2010; Salicone 2007). However, as discussed in Section 2.1, the dynamics and control community, of which satellite navigators are a part, appears to have inherited an implicitly Bayesian view of uncertainty.

1.3 Our Contribution

In Section 3, we show that epistemic probabilities are subject to a phenomenon we call false confidence. That is, in any additive belief function used to represent epistemic uncertainty, there are false propositions that have a high probability of being assigned a high belief value. Probability dilution in satellite conjunction analysis is a manifestation of this more general problem.

In the Martin–Liu theory of inferential models, we find a framework for eliminating false confidence. It is a frequentist–fiducial approach to statistical inference (Martin and Liu 2015),
in which results are described using a non-additive belief function, as in Dempster–Shafer
evidence theory (Shafer 1976). What sets the Martin–Liu theory apart is its validity criterion.
For a statistical inference to be valid in the Martin–Liu sense, only a true proposition may
have a high probability of being assigned a high belief value. In Section 4, we show how the
Martin–Liu validity criterion provides the necessary guidance to resolve probability dilution.

1.4 Paper Outline

This paper explores the problem of false confidence in satellite conjunction analysis and
shows how it can be avoided using a technique for which supporting software is already
widely available. Section 2 reviews the standard computation of collision probability and
explores the root cause of probability dilution. Section 3 states and proves the false confidence
theorem. Section 4 proves that $K\sigma$ ellipsoid overlap detection quantifies collision risk in a
way that is free from false confidence. Section 5 reviews our findings and reiterates the
importance of computing collision risk correctly.

2 Satellite Conjunction Analysis

So far as most satellite navigators are concerned, conjunction analysis starts with a
message from the Joint Space Operations Center (JSpOC), the United States military group
in charge of tracking Earth satellites and orbital debris. JSpOC sends a “conjunction data
message” to the satellite operator whose asset is about to be involved in a conjunction event (Laporte 2014). The conjunction data message includes the estimated positions and
velocities of the two conjuncting objects at the estimated time of closest approach, along
with the uncertainty in those estimates and JSpOC’s estimate of the probability of collision
(Consultative Committee for Space Data Systems 2013). The navigator whose satellite is
involved in the conjunction event may try to improve on those estimates using data obtained
from sources other than JSpOC.

Once the best feasible estimate for the two satellite trajectories has been obtained, the
navigator attempts to compute the collision risk. That calculation of collision risk is the heart
of conjunction analysis. The results help determine whether or not the satellite operator
plans and executes a collision avoidance maneuver. Satellite navigators currently quantify
collision risk using an epistemic probability of collision (Alfano and Oltrogge 2016; Frigm
et al. 2015; Newman et al. 2014).

This part of the paper explores the issues surrounding that epistemic probability of
collision. Section 2.1 describes the standard computation of collision probability. Section 2.2
describes how probability dilution arises from those mathematics. Section 2.3 describes a
more dangerous operational defect in epistemic collision probabilities, of which probability
dilution is a symptom. Section 2.4 explains that the low actuarial probability of collision
has kept disaster at bay, despite the current defects in standard conjunction analysis. That
situation will not endure; the actuarial probability of collision is growing as low Earth orbit
becomes more crowded.
2.1 Epistemic Probability of Collision

The inferred parameter of interest in a conjunction analysis is the vector of positions and velocities of the two satellites at time of closest approach. That is,

\[ \theta = (u_1, v_1, w_1, \dot{u}_1, \dot{v}_1, \dot{w}_1, u_2, v_2, w_2, \dot{u}_2, \dot{v}_2, \dot{w}_2) \]

where \( u_1, v_1, w_1 \) are the position of one satellite; \( \dot{u}_1, \dot{v}_1, \dot{w}_1 \) is its velocity; \( u_2, v_2, w_2 \) are the position of the other satellite; \( \dot{u}_2, \dot{v}_2, \dot{w}_2 \) are its velocity. The true value of this vector is unknown; it is estimated via an inferential algorithm called a “filter” that delivers an estimate, \( \hat{\theta} \), along with its uncertainty, expressed as a \( 12 \times 12 \) covariance matrix, \( C_\Theta \).

Uncertainty in the trajectory estimates is usually assumed to have a multivariate normal distribution (e.g., Alfano 2003; Alfriend et al. 1999; Patera 2001). The epistemic probability density for \( \theta \) is therefore taken to be

\[ f_x(\theta) = \left\{ \frac{1}{(2\pi)^{12}} \det(C_\Theta) \right\}^{-1/2} \exp \left\{ -\frac{1}{2} (\theta - \hat{\theta})^\top C_\Theta^{-1} (\theta - \hat{\theta}) \right\} \]

where \( \det(C_\Theta) \) is the determinant of \( C_\Theta \), and \( x \) represents all of the data used to infer \( \theta \).

Given this representation of uncertainty, it has usually been taken for granted that the correct way to measure collision risk is to compute the epistemic probability of collision (e.g., Alfriend et al. 1999; Chan 2008; McKinley 2006; Newman 2010; Newman et al. 2014; Patera 2001). This reflects the broader view, originating in the signals processing community, that “the estimation problem belongs to the realm of probability theory” (Kalman et al. 1960). The dynamics and control community, in adopting the filtering approach to statistical estimation problems, has inherited an exclusively probabilistic view of uncertainty. While the word “Bayesian” is often missing from the dynamics and control estimation literature, it operates from a fundamentally Bayesian perspective. In fact, the Kalman filter itself can be alternatively derived using Bayesian inference (Meinhold and Singpurwalla 1983). As mentioned in Section 1 however, a frequentist statistician would assiduously caution against using “probability of collision” as a collision risk metric, on the grounds that it is improper to treat an epistemic probability distribution as though it represents genuine aleatory variability (pages 66-67, Cox 2006).

Putting that potential objection aside for a moment, the standard probability of collision calculation proceeds in three steps. First, uncertainty in \( \theta \) is propagated to uncertainty in the relative offset between the two satellites at closest approach. That is,

\[ \Delta u = u_2 - u_1 \quad \Delta v = v_2 - v_1 \quad \Delta w = w_2 - w_1. \]

Since the linear transform of a multivariate normal distribution is itself a multivariate normal distribution, the triple \((\Delta u, \Delta v, \Delta w)\) has a multivariate normal distribution with the following \( 3 \times 3 \) covariance matrix:

\[ C_\Delta = A C_\Theta A^\top \]

where

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0
\end{bmatrix}
\]
or more compactly
\[ C_\Delta = C_{\Theta;1:3,1:3} + C_{\Theta;7:9,7:9} - 2C_{\Theta;1:3,7:9} \]
where \( C_{\Theta;m_1:m_2,n_1:n_2} \) is the sub-matrix of all entries in rows \( i \) and columns \( j \) such that \( m_1 \leq i \leq m_2 \) and \( n_1 \leq j \leq n_2 \). That is, \( C_\Delta \) is the sum of the covariance matrices for the positions of the two satellites minus the covariance between the positions of the two satellites.

In the second step, uncertainty along the axis parallel to the relative velocity vector is integrated out. That is, instead of computing probability of collision as the probability of the true value of the displacement falling within the set of \((\Delta u, \Delta v, \Delta w)\) values indicative of collision, probability of collision is computed as the probability accorded to an extruded ellipse containing that three-dimensional set. The theoretical rationale behind this move is that it captures uncertainty in the timing of closest approach (Patera 2001). The primary practical effect of this step is to reduce a three-dimensional integration problem to a two-dimensional integration problem. More nuanced ways of treating uncertainty in the timing of closest approach are available (Analytical Graphics, Inc. 2018b; Hall et al. 2017), but the standard two-dimensional treatment is sufficient for our analysis.

This transformation is executed in a few sub-steps. It starts with the unit vector in the relative velocity direction, \( i_{\Delta V} \), defined as
\[ \Delta V = (\dot{u_2}, \dot{v_2}, \dot{w_2}) - (\dot{u_1}, \dot{v_1}, \dot{w_1}) \quad \text{and} \quad i_{\Delta V} = \frac{\Delta V}{\|\Delta V\|}. \]
Next, another direction in the plane perpendicular to \( i_{\Delta V} \) can be defined arbitrarily. For example it could be taken as in the same direction as the cross-product of \( i_{\Delta V} \) and \((\Delta u, \Delta v, \Delta w)\), assuming that they are not aligned. Denote this arbitrary direction vector as \( i_w \). The final direction vector is simply the cross-product, \( i_v = i_{\Delta V} \times i_w \). Once this system has been established, define the rotated displacement vector as
\[ \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = M \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix} \quad \text{where} \quad M = \begin{bmatrix} i_{\Delta V}^T \\ i_v^T \\ i_w^T \end{bmatrix}. \]
is the rotation matrix. The covariance matrix for the rotated \((u', v', w')\) vector is
\[ C_{\Delta'} = MC_\Delta M^T. \]
Since \((u', v', w')\) has a multivariate normal distribution, uncertainty in the \( \Delta V \) direction can be integrated out by simply dropping \( w' \). That completes the extrusion step, reducing the problem to the two-dimensional \((u', v')\) space, with point estimate \((\hat{u}', \hat{v}')\) and covariance matrix \( C_{\Delta';1:2,1:2} \).

The third and final step in the standard calculation of collision probability is to define the probability of collision as the probability of having \( u' \) and \( v' \) such that \( u'^2 + v'^2 \leq R^2 \), where \( R \) is the sum of the characteristic radii of the two satellites in the conjunction event. In other words, the satellite shapes are ignored; they are approximated as spherical objects. Collision occurs if and only if, at closest approach, the distance between the two satellites is less than their combined size.
Balch (2016) adds a fourth step to this standard formulation. By the Karhunen-Loève theorem, a multivariate normal distribution can be expressed as a linear transform of a unit normal space of equal or lesser dimension [Ghanem and Spanos 1991]. Therefore, \((u', v')\) can be expressed as

\[
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \hat{u}' \\ \hat{v}' \end{bmatrix} + E_S \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}
\]

where \(S_1^2\) and \(S_2^2\) are the eigenvalues of \(C_{\Delta';1:2,1:2}\); \(E_S\) is a \(2 \times 2\) matrix whose columns are the normalized eigenvectors of \(C_{\Delta';1:2,1:2}\); and finally, \(\xi_1\) and \(\xi_2\) are independent unit normal variables, with mean zero and variance one. Note that, since \(E_S\) is the eigendecomposition of a symmetric matrix, \(E_S^\top E_S\) returns the identity matrix. Defining \(u''\) and \(v''\) as the \(E_S^\top\) rotation of \(u'\) and \(v'\), the magnitude of \((u'', v'')\) is equal to the magnitude of \((u', v')\). That is,

\[
\begin{bmatrix} u'' \\ v'' \end{bmatrix} = E_S^\top \begin{bmatrix} u' \\ v' \end{bmatrix} \Rightarrow u''^2 + v''^2 = u'^2 + v'^2, \quad u'' = \hat{u}' + S_1 \xi_1, \quad \text{and} \quad v'' = \hat{v}' + S_2 \xi_2.
\]

Thus, the following equivalences hold for the integration condition:

\[
u'^2 + v'^2 \leq R^2 \iff u''^2 + v''^2 \leq R^2 \iff (S_1 \xi_1 + \hat{u}'')^2 + (S_2 \xi_2 + \hat{v}'')^2 \leq R^2.
\]

What has been accomplished is that now, instead of integrating a circle in a \((u', v')\) space that will vary from problem to problem, we are integrating an ellipse in a standardized unit normal \((\xi_1, \xi_2)\) space. The centroid of that ellipse is located at \((-\hat{u}'', S_1; -\hat{v}'', S_2)\), and its semi-major and semi-minor axes are \(R/S_1\) and \(R/S_2\).

Following the logic of Patera (2005), the epistemic probability of collision is computed using a contour integral transformation, as follows:

\[
\text{Bel}(C) = \iint \frac{1}{2 \pi} \exp \left( -\frac{1}{2} (\xi_1^2 + \xi_2^2) \right) d\xi_1 d\xi_2
\]

\[
= \int_0^{2\pi} \left( 1 - \frac{e^{-r_\psi^2/2}}{2\pi r_\psi^2} \right) \left( \frac{R^2}{S_1 S_2} - \frac{\hat{u}'' R}{S_1 S_2} \cos \psi - \frac{\hat{v}'' R}{S_1 S_2} \sin \psi \right) d\psi
\]

where

\[
r_\psi^2 = \left( -\frac{\hat{u}''}{S_1} + \frac{R}{S_1} \cos \psi \right)^2 + \left( -\frac{\hat{v}''}{S_2} + \frac{R}{S_2} \sin \psi \right)^2.
\]

This integral can be approximated to negligible error using the trapezoidal rule, provided that at least \(10 \times \max(S_1/S_2, S_2/S_1)\) evenly spaced quadrature points are used [Balch 2016].

Because our goal is to understand and explore how the epistemic probability of collision changes for different levels of uncertainty, we pursue the special case where \(S = S_1 = S_2\). Defining the estimated displacement as \(D = \sqrt{\hat{u}''^2 + \hat{v}''^2} = \sqrt{u'^2 + v'^2}\), this simplification reduces the epistemic probability of collision to a function of two ratios: estimated relative displacement, \(D/R\), and relative uncertainty, \(S/R\). Figure 1 illustrates probability of collision as a function of \(S/R\) for several values of \(D/R\).
Figure 1: Epistemic probability of collision as a function of estimated relative displacement, $D/R$, and relative uncertainty, $S/R$.

2.2 Probability Dilution

The collision probability curves in Figure 1 follow a common pattern. So long as $D/R > 1$, for small uncertainties, the probability of collision is small. That much makes intuitive sense. If the satellites are estimated to miss each other, and the uncertainty in those trajectory estimates is small, then it is natural that the analyst have a high confidence that the satellites are not going to collide. Similarly, it makes sense that as uncertainty grows, the risk of collision also grows. However, something odd happens in the probability curves. Probability of collision eventually hits a maximum, and past that maximum, as relative uncertainty rises, the probability of collision decreases.

This decrease is called probability dilution, and it has an odd implication. Since the uncertainty in the estimates of the trajectories reflects the limits of data quality, probability dilution seems to imply that lowering data quality makes satellites safer. That implication is counterintuitive in the extreme. Balch (2016) described it as an “Ostrich Paradox”. As a rule, lowering the data quality makes any engineering system less safe, and to claim that ignorance somehow reduces collision risk seems foolish on its face.

That being said, the mathematics of probability dilution are ironclad and relatively simple. Recall that, when expressed in unit normal space, the integration region determining the probability of collision is an ellipse whose centroid is at $(-\hat{u}/S_1, -\hat{v}/S_2)$ and whose semi-major and semi-minor axes are $R/S_1$ and $R/S_2$. Figure 2 illustrates how the integration region changes as a function of $S$, when $S = S_1 = S_2$. Combined satellite size, $R$, and estimated distance at closest approach, $D$, are held fixed with $D/R = 5$. For four different
values of $S/R$, the integration region is illustrated, along with the corresponding probability of collision.

Increasing the level of uncertainty has two effects on the integration region. The integration region shrinks, and it draws closer to the origin. The probability density of a unit normal space is highest at the origin, decreasing exponentially as a function of the square of distance from the origin. If $D/R > 1$, the positive effect due to getting closer to the origin, where there is more probability mass, initially outweighs the negative effect due to shrinking the integration region. Near the origin, though, the probability density curve flattens out. So, after a certain point, there is not much more probability mass to be gained by moving closer to the origin, and the negative effect of shrinking the integration region overtakes the positive effect due to shifting the location of the integration region. Past that point, probability of collision decreases as uncertainty increases. That is why probability dilution happens, from a purely mathematical perspective; it is due to a simple and straightforward shrinkage of the integration region.

Even if the normality assumptions and shape assumptions were relaxed, the integration region would still undergo the same shrinkage phenomenon that is illustrated in Figure 2. Each satellite represents a bounded set of points. The displacement values indicative of collision are therefore also a bounded set of points. If one were to relax the assumptions outlined in Section 2.1 instead of a shrinking ellipse, one would have a shrinking irregular shape in a standardized three-dimensional probability space. Probability of collision would
still have the same qualitative behavior as a function of relative uncertainty. Mathematically speaking, the key factor underpinning probability dilution is that the integration region (i.e., failure domain) is bounded.

There is no way of reframing satellite conjunction analysis so that the failure domain is not bounded. Therefore, if we take the view that epistemic probability is valid, then it would seem that probability dilution is fundamental to conjunction analysis. Why, then, do aerospace researchers (e.g., Alfano 2003; Balch 2016; Frigm 2009; Plkalović et al. 2011; Sun et al. 2014) find this supposed mathematical inevitability so counterintuitive?

Interestingly, were the uncertainty in the satellite trajectories aleatory, rather than epistemic, probability dilution would actually make sense intuitively. For example, suppose two satellites were known, with certainty, to be on a collision course, and the satellite operator could only impart an impulse of random magnitude in a random direction, say, via a poorly controlled maneuvering thruster on a tumbling satellite. If the mean of this distribution were the null vector, due to a lack of control over thrust direction, then the higher the variance of the added impulse, the bigger the resulting perturbation and hence the smaller the resulting probability of collision. In this example, the mean of the resulting trajectory distribution would still have the satellites on a collision trajectory, but if one applies a big enough impulse to one of the satellites in a random direction, that poorly controlled collision avoidance maneuver still has a high probability of success. In that context, higher variance in a satellite’s trajectory really does reduce the risk of collision.

However, in this hypothetical example, it is a variance in the trajectory itself that makes the satellite safer, not a variance in the estimate of that trajectory. It should go without saying that, given two satellites on a sure collision trajectory, simply recomputing the trajectories with lower quality data does not make them safer. There is, therefore, an operational distinction to be made between genuine aleatory variability and epistemic uncertainty.

To summarize, the problem with probability dilution is not the mathematics. The mathematics are incontrovertible, and they even make sense in the right context. Probability dilution is only counterintuitive when orbit “variance” reflects the limits of data quality, rather than genuine aleatory variability in the orbits themselves. So, if probability dilution is inappropriate, that inappropriateness must be rooted in a mismatch between the fundamental mathematics of probability theory and the epistemic uncertainty to which they are applied in conjunction analysis.

2.3 Operational Implications

The final clue indicating the inappropriateness of probability dilution is the fact that, for a fixed $S/R$ ratio, there is a maximum possible computable probability of collision. Whether or not the two satellites are on a collision course, no matter what results are obtained, the analyst will have a minimum confidence that the satellites will not collide. That minimum confidence is determined purely by the data quality.

For all $S/R$, the maximum collision probability is obtained when $D/R = 0$; that is, when the best estimate of the two satellite paths indicates that they are on a collision course. Therefore, the curve for $D/R = 0$ in Figure 1 yields the maximum computable probability
of collision as a function of $S/R$. For example, if the uncertainty in the distance between two satellites at closest approach is ten times the combined size of the two satellites, the analyst will always compute at least a 99.5% confidence that the satellites are safe, even if, in reality, they are not.

A relative uncertainty of ten may sound high, but it may help the reader to keep in mind the magnitude of the trajectories being estimated, relative to the size of the objects on those trajectories. Satellite trajectories are measured in thousands of kilometers; satellites are measured in meters. Trajectory uncertainties in conjunction analysis are usually measured in hundreds of meters (Sabol et al. 2010). For conjunction analysis done more than a week in advance, those uncertainties can grow to kilometers (Ghrist and Plakalovic 2012). As a consequence, $S/R$ ratios greater than ten are the rule, not the exception. Under those circumstances, even if two satellites are on a collision course, because of probability dilution, the epistemic probability of collision is guaranteed to be small.

The biggest problem with probability dilution is not its counterintuitiveness; the biggest problem with probability dilution is the false confidence that it imbues. Due to probability dilution, satellite navigators credulously using epistemic probability of collision as their collision risk metric will always have a high confidence that their satellites are safe, regardless of whether or not they really are. The only rare exceptions are conjunctions in which both satellites are extremely well-tracked and at least one of them is relatively large (for an example, see Newman 2010). From an operational perspective, epistemic probability of collision is a dangerously misleading risk metric.

2.4 Actuarial Probability of Collision

One might ask why there have not been more in-orbit collisions, if satellite navigators are being misled by probability dilution. The answer is simple. For any given satellite over any given span of time, there is an actual aleatory probability of collision, and for most of the past 60 years, it has been extremely low. This actuarial probability of collision has nothing to do with conjunction analysis or data quality; it is driven purely by orbital crowding. The main factors in play are the number of objects in orbit, the size of those objects, and the distribution of those objects; that is, whether they are evenly spread or clustered together (Chobotov 1983; Hechler and van der Ha 1981; Jenkin 1996; Kessler 1991; Kessler et al. 1989; McKnight and Anz-Meador 1993; Smirnov 2001).

For most of the history of aerospace, orbital crowding has been extremely low. Imagine driving a dune buggy off-road out in the New Mexico desert, twenty miles from the nearest highway. What are the odds that you would collide with someone else doing the same thing? That is what low Earth orbit was like from 1957 to roughly 1987. Since then, the situation has been deteriorating. A decade ago, a couple of high-profile collisions alerted satellite navigators to the need for regular conjunction analysis (Newman et al. 2009). Nevertheless, more than anything satellite navigators have done, relatively low crowding and the consequently low actuarial probability of collision are responsible for the low number of major collisions that the aerospace community has enjoyed so far.

It may be tempting to try to rationalize probability dilution in terms of the low actuarial
probability of collision. However, the fact that the actuarial probability of collision has long been low has nothing to do with the fact that probability dilution causes the epistemic probability of collision to also be low. As the number of satellites in orbit grows over the coming years, the actuarial probability of collision will also grow and, eventually, cease to be small (Kessler et al., 2010). In contrast, probability dilution will continue to make collision risk look artificially small, effectively blinding satellite navigators to danger.

3 The False Confidence Theorem

What causes probability dilution and false confidence? As explored in Section 2, it is not an error in the mathematics. Rather, it appears to be a mismatch between the mathematics of probability theory and the subject matter to which those mathematics are applied in satellite conjunction analysis. As mentioned in Section 1, questions over the appropriateness of epistemic probability have never been settled. In satellite conjunction analysis, those questions take on an undeniably practical dimension.

This part of the paper introduces the false confidence theorem. Section 3.1 suggests that probability dilution and false confidence are rooted in the axioms of probability theory. Section 3.2 posits a formal definition for false confidence. Section 3.3 proves that all epistemic probability distributions assign copious amounts of false confidence. Section 3.4 places this theorem in the context of the Bayesian–frequentist debate.

3.1 Motivating Insight

Given highly uncertain data, it makes intuitive sense that one would not be certain of collision. With low-quality data, an analyst should be reluctant to assign a high degree of belief to any proposition. The problem is that, within the confines of probability theory, general non-commitment of belief is not an option. One of the axiomatic properties of probability functions is that they are additive. So, any belief not assigned to “collision” is automatically assigned to “not-collision,” i.e.,

\[ \text{Bel}(\neg C) = 1 - \text{Bel}(C). \]

While a low assignment of belief to “collision” might seem natural when uncertainty is high, the consequently high assignment of belief to “not-collision” seems equally unnatural.

3.2 Definition of False Confidence

Strictly speaking, one could characterize any confidence or belief assigned to a false proposition as “false confidence.” However, since the whole point of statistical inference is that one does not know \textit{a priori} the exact true value of the parameter being inferred, it is to be expected that, in the course of any given inference, some false propositions will be assigned some amount of belief. The problem described in Section 2.3 is more severe: There are fixed false propositions that are guaranteed or nearly guaranteed to be assigned a high belief value.
This suggests a formal definition for false confidence. An inferential method can formally be said to suffer from false confidence if, for any arbitrarily specified fixed level of belief and probability of assignment, short of unity, there is at least one false proposition to which the method will have at least the prescribed probability of assigning at least the prescribed level of belief. Next we show that most epistemic probability distributions used in practice suffer from false confidence. In other words, a procedure that draws inferences based on an additive belief function leads to “systematically misleading conclusions” and, hence, is “unacceptable” in the words of Reid and Cox (2015).

3.3 Theorem

Most statistical inference in science and engineering involves one or more real-valued continuous parameter(s) inferred from real-valued observables. Let the observable be \( X \in \mathbb{R}^n \), and let the parameter be \( \theta \in \Omega_\theta \subseteq \mathbb{R}^m \) where \( m \) is the number of parameters being inferred. The epistemic probability distributions (e.g., posteriors) commonly used in practice are, for all values of the observable, continuous distributions over the parameter being inferred. That is, belief assigned by the additive belief function \( \text{Bel}_{\Theta|x} \) can be represented via an epistemic probability density function, say, \( f_x(\theta) \), with respect to Lebesgue measure \( \lambda \) on \( \Omega_\theta \), depending on the observable \( x \). We also assume that \( \sup_\theta f_x(\theta) < \infty \) for \( \text{Pro}_{X|\theta} \)-almost all \( x \), for all \( \theta \). These assumptions describe an ordinary epistemic probability distribution in the sense that most Bayesian posteriors, confidence distributions (Schweder and Hjort 2002, 2016; Xie and Singh 2013), and fiducial distributions (Fisher 1935; Hannig et al. 2016) used in practice satisfy all of these assumptions. The false confidence theorem below states that all epistemic probability distributions satisfying these assumptions suffer from false confidence.

**False Confidence Theorem.** Consider an additive belief function \( \text{Bel}_{\Theta|x} \) characterized by an epistemic probability density function \( f_x(\theta) \) on \( \Omega_\theta \) satisfying the assumptions above. Then for any \( \theta \in \Omega_\theta \), any \( \alpha \in (0, 1) \), and any \( p \in (0, 1) \), there exists a set \( A \subseteq \Omega_\theta \) such that

\[
A \not\ni \theta \quad \text{and} \quad \text{Pro}_{X|\theta}(\{x : \text{Bel}_{\Theta|x}(A) \geq 1 - \alpha\}) \geq \eta.
\]

**Proof.** For fixed \( x \), set \( \bar{f}_x = \sup_\theta f_x(\theta) \). Then for any bounded set \( B \subseteq \Omega_\theta \),

\[
\text{Bel}_{\Theta|x}(B) = \int_B f_x(\theta) d\theta \leq \bar{f}_x \lambda(B). \tag{1}
\]

As a function of \( X \), for fixed \( \theta \), \( \bar{f}_x \) has a distribution; let \( \eta = \eta_{p, \theta} \in (0, \infty) \) satisfy

\[
\text{Pro}_{X|\theta}(\{x : \bar{f}_x \leq \eta\}) \geq \eta.
\]

Choose a neighborhood \( B \ni \theta \) with measure \( \lambda(B) = \alpha/\eta \), so that \( \lambda(B) \eta = \alpha \). Define \( A \) as the complement of \( B \), i.e., \( A = \{\theta \in \Omega_\theta : \theta \not\in B\} \). For any belief function satisfying the Kolmogorov axioms, we have

\[
\text{Bel}_{\Theta|x}(A) = 1 - \text{Bel}_{\Theta|x}(B). \tag{2}
\]
By definition, \( A \neq \theta \), and by (1) and (2), we have

\[
\hat{f}_x \leq \eta \implies \text{Bel}_{\theta|x}(B) \leq \lambda(B) \eta \iff \text{Bel}_{\theta|x}(B) \leq \alpha \iff \text{Bel}_{\theta|x}(A) \geq 1 - \alpha.
\]

By definition of \( \eta \), the left-most event occurs with \( \text{Pro}_{X|\theta} \)-probability at least \( p \) and, therefore, the right-most event occurs with at least \( \text{Pro}_{X|\theta} \)-probability \( p \), proving the claim.

Mathematical formalism belies the simplicity of this proof. Given the continuity assumptions outlined above, one can always define a neighborhood around the true parameter value that is so small that its complement—which, by definition, represents a false proposition—is all but guaranteed to be assigned a high belief value, simply by virtue of its size. That is the entirety of the proof. Further, “sample size” plays no role in this proof; it holds no matter how much or how little information is used to construct the epistemic probability distribution in question. The false confidence theorem applies anywhere that probability theory is used to represent epistemic uncertainty resulting from a statistical inference.

### 3.4 Broader Implications

False confidence is the inevitable result of treating epistemic uncertainty as though it were aleatory variability. Any probability distribution assigns high probability values to large sets. This is appropriate when quantifying aleatory variability, because any realization of a random variable has a high probability of falling in any given set that is large relative to its distribution. Statistical inference is different; a parameter with a fixed value is being inferred from random data. Any proposition about the value of that parameter is either true or false. It is a bad inference that treats a false proposition as though it were true, by consistently assigning it high belief values. That is the defect we see in satellite conjunction analysis, and the false confidence theorem establishes that this defect is universal.

This finding opens a new and decisive front in the debate between Bayesian and frequentist schools of thought in statistics. Traditional arguments over epistemic probability have a distinctively philosophical flavor. They focus on issues like the ontological inappropriateness of epistemic probability distributions (e.g., Boole 1854; Venn 1866), the unjustified use of prior probabilities (Fisher 1930), and the hypothetical logical consistency of personal belief functions in highly abstract decision-making scenarios (e.g., de Finetti 1935; Savage 1972).

In contrast, the false confidence theorem speaks to the practical consequences of treating epistemic uncertainty probabilistically. And on the question of epistemic probability, it appears that the frequentists are right and the Bayesians are wrong. It is not generally safe or sensible or harmless to try to compute the probability of a non-random event.

In satellite conjunction analysis, we have a clear real-world example of epistemic probability run amok, in which the deleterious effects of false confidence are too large and too important to be overlooked. In other applications, there will be propositions similarly afflicted by false confidence. The question that one must resolve on a case-by-case basis is whether the affected propositions are of practical interest. It is a lurking problem for which one must check in any Bayesian posterior, fiducial distribution, or confidence distribution.

For now, we focus on identifying an approach to satellite conjunction analysis that is free from false confidence. Probability theory is not the only framework for representing...
epistemic uncertainty. A cornucopia of alternative axiomatic frameworks exist, most of which are non-additive (Klir 2006). While the false confidence theorem definitively establishes that probability theory is an inappropriate framework for representing epistemic uncertainty, it is not the end of the world. We only need to identify a more appropriate framework.

4 Computing Collision Risk Safely

The Martin–Liu validity criterion points the way to a safe and easily implementable way to assess collision risk between satellites during a conjunction event. The epistemic probability distribution for the satellite trajectories introduced in Section 2.1 can be interpreted as an approximate confidence distribution in the sense of Balch (2012). This means that $K\sigma$ uncertainty ellipsoids drawn from those distributions can be treated as approximate confidence regions. Those confidence regions can be used to rigorously assess collision risk. So long as the uncertainty ellipsoids around the positions of each satellite do not overlap, collision is statistically implausible. Otherwise, a collision avoidance maneuver may be required.

This part of the paper explores how collision risk can be assessed using $K\sigma$ ellipsoids. Section 4.1 provides proof that confidence regions satisfy the Martin–Liu validity criterion and are consequently free from false confidence. Section 4.2 makes the case that $K\sigma$ uncertainty ellipsoids can usually be interpreted as confidence regions. Section 4.3 provides a simple formula relating the $(1-\alpha)$ confidence level associated with each uncertainty ellipsoid to a cap on the operational probability of collision. Section 4.4 hints at the possibility of a more flexible but equally safe approach.

4.1 Martin–Liu Validity of Confidence Regions

The Martin–Liu validity criterion states that no false proposition should have a high probability of being assigned a high belief value (Martin and Liu 2015). More precisely, the probability with which a false proposition will be assigned some amount of belief (i.e., $1-\alpha$) is at most one minus that level of belief (i.e., $\alpha$). Stated mathematically,

$$\text{Pr}_{X|\theta}(\{x : \text{Bel}_{\Theta|x}(A) \geq 1 - \alpha\}) \leq \alpha, \quad \forall \alpha \in [0, 1], \ A \subset \Omega_{\theta}, \ s.t. \ A \notin \theta. \quad (3)$$

By definition, any inferential method that satisfies this criterion will not suffer from the false confidence phenomenon described in Sections 2–3.

Frequentist confidence intervals and confidence regions have traditionally been rationalized via coverage probability. That is, a $(1-\alpha) \times 100\%$ confidence interval or confidence region is accepted as valid because, over repeated independent draws of the data, it has a probability of at least $(1-\alpha)$ of covering the true parameter value. However, if evaluated as tools for assigning belief to propositions (i.e., sets), confidence intervals and confidence regions also satisfy the Martin–Liu validity criterion.

Let $\Gamma_{\alpha}(x)$ be the $(1-\alpha)$ confidence region for $\theta$ given the realized data $x$. The natural assignment of belief and plausibility made by this confidence region are

$$\text{Bel}_{\Theta|x}(A) = \begin{cases} 1 - \alpha, & A \supset \Gamma_{\alpha}(x) \\ 0, & A \not\supset \Gamma_{\alpha}(x) \end{cases} \quad \text{Pls}_{\Theta|x}(A) = \begin{cases} 1, & A \cap \Gamma_{\alpha}(x) \neq \emptyset \\ \alpha, & A \cap \Gamma_{\alpha}(x) = \emptyset \end{cases}.$$
That is to say, a confidence region represents the simple assertion that we are $1 - \alpha$ confident that the true value of $\theta$ is somewhere inside $\Gamma_\alpha(x)$. Any sets containing $\Gamma_\alpha(x)$ inherit that confidence; all other sets accrue no positive confidence. It is a coarse way to represent a statistical inference, perhaps more so than is necessary, but it is also safe in the sense that the Martin–Liu validity criterion holds. Indeed, by the coverage probability definition of a confidence region, the event $\{\Gamma_\alpha(x) \ni \theta\}$ has at least $\text{Pro}_{X|\theta}$-probability $1 - \alpha$. So, for any set $A \not\ni \theta$, the event $\{\Gamma_\alpha(x) \subset A\}$ has $\text{Pro}_{X|\theta}$-probability at most $\alpha$. Therefore, for any false proposition, i.e., any set $A$ such that $A \not\ni \theta$, the probability that said proposition will be assigned a confidence of at least $1 - \alpha$ is less than or equal to $\alpha$, hence (3) holds.

When interpreted this way, a confidence region represents a coarse consonant confidence structure; that is, a possibility distribution with coverage probability properties (Balch 2012). Denoeux and Li (2018) prove that all consonant confidence structures satisfy the Martin–Liu validity criterion. The reason that we have included the special case proof for confidence regions is that most readers are likely to be unfamiliar with possibility distributions and confidence structures. In contrast, simple confidence intervals and confidence regions are well-established, well-disseminated tools of statistical inference. As proven above, they do not suffer from false confidence.

4.2 Uncertainty Ellipsoids as Confidence Regions

It can be shown that $K\sigma$ uncertainty ellipsoids drawn using the filter-derived satellite position estimate and covariance are confidence regions. While the filtering community fosters an implicitly Bayesian treatment of epistemic uncertainty, traditional filter derivations are accomplished in terms of minimizing the error between the estimate and the true “signal,” i.e., $\hat{\theta}$ and $\theta$, respectively (e.g., Kalman et al. 1960). Spall and Wall (1984) proved that an estimate obtained using a linear Kalman filter will have approximately normal errors, even if the observation errors are drawn from non-normal distributions. Del Moral and Guionnet (1999) extended this result to sampling-based filters, such as the unscented Kalman filter. With the exception of conjunction analysis for satellite launch, the trajectory estimates used in conjunction analysis are the result of copious data assimilation. It would seem that we can safely assume that these central limit theorem results apply.

However, there is a complication. These central limit theorems only apply to an estimate of the satellite’s current position based on current data (Sabol et al. 2010). If one is predicting a satellite location several days in advance based on its current trajectory, as is done in conjunction analysis, the predicted position errors are not guaranteed to be normal. This is due to the non-linearity of orbital dynamics. As a rule of thumb, the normality assumption breaks down when position uncertainty in at least one direction is more than a kilometer (Ghrist and Plakalovic 2012). For simplicity, we continue in the assumption that the position errors at closest approach are normal.

Suppose that $\theta(X) - \theta$ has a multivariate normal distribution with zero mean and covariance matrix $C_\theta$. Once again, by the Karhunen-Loève theorem,

$$\hat{\theta}(X) - \theta = E_{C_\theta} \sqrt{\Lambda_{C_\theta}} \xi$$
where $X$ is a (hypothetical) random realization drawn from the same distribution as the data used to estimate the orbits, $E_{C_{\Theta}}$ is a matrix whose columns are the eigenvectors of $C_{\Theta}$, $\Lambda_{C_{\Theta}}$ is a diagonal matrix whose entries are the corresponding eigenvalues of $C_{\Theta}$, and $\xi$ is a unit normal vector of the same dimensionality as $\theta$. This can easily be rearranged into a pivot, as

$$\sqrt{1/\Lambda_{C_{\Theta}}} E_{C_{\Theta}}^T (\hat{\theta}(X) - \theta) = \xi$$

where $1/\Lambda_{C_{\Theta}}$ is a diagonal matrix whose entries are the inverse of the eigenvalues of $C_{\Theta}$. Note that $E_{C_{\Theta}}^T$ and $\sqrt{1/\Lambda_{C_{\Theta}}}$ are simply the inverses of $E_{C_{\Theta}}$ and $\sqrt{\Lambda_{C_{\Theta}}}$, respectively.

A pivot is a function of the parameter being inferred and the random data used to infer it whose output is a random variable with a known fixed distribution (Casella and Berger 2002); in this case, $\xi$ is a unit normal vector. Pivots are special because they can be used to derive confidence regions and confidence distributions (Balch 2012). For example, the following is a confidence distribution for $\theta$:

$$\Theta(x, \xi) = \hat{\theta}(x) - E_{C_{\Theta}} \sqrt{\Lambda_{C_{\Theta}}} \xi$$

where $x$ represents the actual data obtained and $\xi$ serves as a “seed-variable”. Any region, $A \in \mathbb{R}^{\dim(\xi)}$, in $\xi$-space mapped to $\theta$-space as $\Theta(x, A)$ will have a coverage probability equal to the measure of $A$. That is,

$$\text{Pro}_{X|\theta}(\{x : \Theta(x, A) \ni \theta \}) = \text{Pro}_{\xi}(A) = \int_{\xi \in A} \frac{1}{(\sqrt{2\pi})^{\dim(\xi)}} \exp \left[-\frac{1}{2} \sum_{i=1}^{\dim(\xi)} \xi_i^2 \right] d\xi_1...d\xi_{\dim(\xi)}.$$

It should be remembered that the full additive belief function that this confidence distribution could support is subject to the false confidence theorem. However, as proven in Section 4.1, a single consistently derived confidence region will assign confidence and plausibility in a way that is valid in the Martin–Liu sense and therefore not subject to the false confidence theorem. Note also that, to the extent that normality is only approximately satisfied, the resulting confidence region will also be approximate.

A $K\sigma$ uncertainty ellipsoid on $\theta$ is generated by projecting a sphere in $\xi$-space centered at the origin. Let $K > 0$ be some constant. A confidence region of the form

$$\Theta(x, A) \quad \text{where} \quad A = \left\{ \xi : \sum_{i=1}^{\dim(\theta)} \xi_i^2 \leq K^2 \right\}$$

will map to $\theta$ as a $\dim(\theta)$-dimensional ellipsoid. The confidence, $(1 - \alpha)$, associated with this uncertainty ellipsoid can be computed as

$$1 - \alpha = F_{\chi^2, \dim(\theta)}(K^2)$$

where $F_{\chi^2, \dim(\theta)} : \mathbb{R}^+ \rightarrow [0, 1]$ is the cumulative distribution function for a $\chi^2$ distribution with $\dim(\theta)$ degrees of freedom. This choice of confidence region has two intuitively appealing properties. First, the ellipsoid center is, by design, the maximum likelihood estimate for $\theta$. Second, the ellipsoid boundary is an equilikelihood surface. Given the normality assumptions, one can reasonably expect that this is the most efficient confidence region for $\theta$, given a specified confidence level.
4.3 Ellipsoid Overlap Detection

Ellipsoid overlap approaches treat each satellite separately, deriving a separate uncertainty ellipsoid for the position of each satellite. So, \( \dim(\theta) = 3 \), but now there are two \( \theta \)'s, and the operational question is whether or not the uncertainty ellipsoids for \( \theta_1 \) and \( \theta_2 \) overlap. Software for computing these uncertainty ellipsoids and determining whether or not they overlap are commercially available [Alfano et al. 2004, Analytical Graphics, Inc. 2018a]. If the two ellipsoids overlap, a collision avoidance maneuver may be needed. If they do not, the two satellites are considered safe, to within the confidence level associated with the uncertainty ellipsoids.

There is a caveat. As traditionally conceptualized, ellipsoid overlap approaches fail to account for the physical size of the two satellites (e.g., Alfano and Greer 2003). To correct for this, effective ellipsoid overlap should be defined as occurring when the minimum distance between the two uncertainty ellipsoids is less than the combined radius of the two satellites. So long as the position uncertainties are much larger than the satellite size, the distinction is negligible. However, satellite size should not be ignored in problems with relatively small position uncertainties.

In either case, because the two satellites are treated separately, the joint confidence associated with having captured both satellite positions in their respective uncertainty ellipsoids is potentially lower than the confidence individually assigned to each ellipsoid. If each ellipsoid has a \( 1 - \alpha \) confidence attached to it, assuming independence between the data used to infer the two satellite trajectories, the confidence attached to simultaneous coverage is

\[
1 - \alpha' = \Pr_{X_1, X_2}(\{x_1, x_2 : \Theta_1(x_1, A) \ni \theta_1, \Theta_2(x_2, A) \ni \theta_2\}) = (1 - \alpha)^2.
\]

In reality, the errors in the data used to estimate the two satellite positions may not be entirely independent, but the specter of non-independence is easily resolved. Using Fréchet bounds [Ferson et al. 2004], even under totally unknown dependence between the data,

\[
\max(0, 1 - 2\alpha) \leq 1 - \alpha'.
\]

Note that if \( \alpha \) is small, \( (1 - \alpha)^2 = 1 - 2\alpha + \alpha^2 \approx 1 - 2\alpha \). So, using \( 1 - 2\alpha \) to represent the joint confidence associated with the two uncertainty ellipses is both conservative and extremely close to the independence case. Thanks to the result about confidence regions presented in Section 4.1, so long as one performs a maneuver whenever the two uncertainty ellipsoids intersect, the rate at which collisions occur during a conjunction event—i.e., the operational probability of collision—will be capped at \( \alpha' = 2\alpha \).

For example, suppose one were to use a pair of \( 4\sigma \) ellipsoids to represent uncertainty in the positions of the two conjuncting satellites. This represents a 99.9% confidence region around each satellite position. Moreover, one can say with 99.8% confidence that both satellite positions are contained within their respective ellipsoids. And, more importantly, if a satellite operator performs a collision avoidance maneuver whenever those \( 4\sigma \) ellipsoids overlap, the operational probability of collision will be capped at 0.227%.

All of that being said, ellipsoid overlap does not mean that two satellites are definitely going to collide. Rather, it means that collision is still plausible in light of the data. Performing a collision avoidance maneuver is one way of driving down the plausibility of collision,
but it is not the only way. For conjunctions predicted several days in advance, prudent navigators will opt to gather more data, which will allow them to shrink the uncertainty ellipsoids before finally deciding whether or not to make a collision avoidance maneuver. So long as they do not wait too long, this is usually the most reasonable course of action.

4.4 Future Work

Ellipsoid overlap detection is safe, but it is also over-conservative. That is, it indicates a need for more collision avoidance maneuvers than are strictly necessary. As mentioned in Section 4.3, while confidence regions do support valid inferences in the Martin–Liu sense, those inferences are also coarse, more so than is necessary. It should be possible to quantify collision risk using a plausibility of collision derived from a more nuanced possibility distribution. However, no such tool has yet been made commercially available. If any of the proposed non-dilutive collision risk metrics currently in the literature (i.e., Alfano 2003; Balch 2016; Frigm 2009; Plakalović et al. 2011; Sun et al. 2014) satisfy the Martin–Liu validity criterion, it has not been demonstrated. For now, ellipsoid overlap detection is the most efficient readily available approach for assessing conjunction risk that is known to cap the operational probability of collision at a guaranteed rate.

5 Conclusions

In this paper, we set out to explain and, if possible, resolve probability dilution in satellite conjunction analysis. We found that the counterintuitive nature of probability dilution was a clue pointing towards a much deeper and more dangerous operational problem with computed collision probabilities. It turns out that, under ordinary operating conditions for conjunction analysis, a satellite navigator measuring collision risk using epistemic probability of collision will always be certain that their satellite is safe, regardless of whether or not it is. With rare exception, epistemic probability of collision is worse than useless as a risk metric.

The satellite industry has survived this far thanks to the low underlying actuarial probability of collision, which currently plays no role in standard satellite conjunction analysis. As low Earth orbit gets more crowded over the coming years, that real aleatory probability of collision is going to keep growing. Meanwhile, due to probability dilution, satellite navigators are essentially flying blind. If standards of practice in satellite conjunction analysis do not change radically and soon, valuable orbital assets will be lost. Whether fairly or unfairly, a reasonable person could attribute those losses to the negligence of satellite operators and certain key supporting contractors.

One way of explaining this situation is that satellite navigators wandered into the middle of an enduring argument between Bayesians and frequentists, and not even realizing that there was an argument taking place, satellite navigators picked a side. Specifically, they picked the wrong side. In fact, Bayesianism is so ill-suited to satellite collision risk assessment that the errors generated are big enough and undeniable enough to essentially settle the argument. While the numerical methods traditionally associated with Bayesian inference will remain a source of great utility for statisticians and uncertainty quantifiers, the general
uncritical use of posterior probability is no longer plausibly tenable as a way to represent uncertainty about a non-random event. If the peculiarities of satellite conjunction analysis left any room for doubt, the generality of the false confidence theorem eliminates that doubt. Epistemic probabilities in general are not safe for use.

Fortunately, a well-established and safe alternative for assessing satellite collision risk is available. Any method that satisfies the Martin–Liu validity criterion is tautologically free from false confidence, and $K\sigma$ uncertainty ellipsoids used to represent satellite position uncertainty satisfy the Martin–Liu validity criterion. So long as a collision avoidance maneuver is performed whenever the $(1 - \alpha)$ uncertainty ellipsoids of two satellites are projected to overlap, the operational risk of collision in any given conjunction will be capped at $2\alpha$. That means, over a large number of conjunctions, an impending collision will be overlooked in at most a $2\alpha$ fraction of those conjunctions. However, this guarantee is obtained only if ellipsoid overlap is defined as occurring whenever the minimum distance between the two uncertainty ellipsoids is smaller than the combined size of the two satellites.

Finally, it is our opinion that while the ellipsoid overlap criterion is safe, it can be improved upon using a more nuanced possibilistic approach, which will help reduce the false alarm rate. Developing that approach is an on-going topic of work for us.

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