SOME PROPERTIES OF NON-COMPACT COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we study the volume growth property of a non-compact complete Riemannian manifold $X$. We improve the volume growth theorem of Calabi (1975) and Yau (1976), Cheeger, Gromov and Taylor (1982). Then we use our new result to study gradient Ricci solitons. We also show that on $X$, for any $q \in (0, \infty)$, every non-negative $L^q$ subharmonic function is constant under a natural decay condition on the Ricci curvature.

1. INTRODUCTION

The motivation for this paper comes from the interest in the understanding the Ricci solitons [8]. However, at this moment, almost all works in this direction are about Gradient Ricci Solitons. See [8], [2], [4], and [11]. Generally speaking, a non-compact Ricci soliton may not be a gradient Ricci soliton. So it may be interesting to consider problems related to Ricci solitons.

In this paper, we consider the volume growth properties of the non-compact complete Riemannian manifold $(X, g)$ under a natural Ricci curvature condition. We can improve the volume growth theorem of Calabi [3] and Yau [14]. Then we use our new result to study gradient Ricci soliton. We also show that on $X$, for any $q \in (0, \infty)$, every non-negative $L^q$ subharmonic function is constant under a natural decay condition on the Ricci curvature.

Result about Gradient Solitons is stated in section four. In section two, we generalize the result of Calabi [8] and Yau [14] on infinite volume property for Riemannian manifolds with non-negative Ricci curvature. Calabi and Yau’s result was generalized by Cheeger-Gromov-Taylor (see Theorem 4.9 in [6]) to Riemannian manifolds.
with lower bound like
\[ Rc \geq -\frac{\nu_n}{r^2(x)}, \]
for \( r(x) \gg 1 \) and some restricted dimensional constant \( \nu_n \), where \( Rc \) is the Ricci tensor of the metric \( g \), and \( r(x) \) is the distance function from some fixed point \( x_0 \). We can remove this restriction to the dimensional constant \( \nu_n \). Our result is

**Theorem 1.** Assume \( (X, g) \) be a complete non-compact Riemannian manifold. Let \( C(r) \) be a continuous function satisfying
\[ C(r) \geq -C r^{-2}(x) \]
for \( r \) large. Assume its Ricci curvature has the lower bound
\[ Rc \geq C(r). \]
Then for any \( x \in X \) and \( r > 1 \), there is a constant \( C(n, VolB_1(p)) \) such that it holds
\[ VolB_r(x) \geq C(n, VolB_1(x)) r. \]

In section three, we give some remarks on Yau’s gradient estimate and vanishing properties for subharmonic functions.

2. Proof of Theorem 1

We let \( D \) and \( R(.,.) \) be the the Levi-Civita derivative and Riemannian curvature of the metric \( g \) respectively. Let \( x \) be a point which is inside the cut locus of \( p \in X \). Let \( \gamma(t) \) be the geodesic from \( p \) to \( x \). Choose a fixed point \( x_0 \) in the curve \( \gamma \). Let \( r = r(x) \) be the distance function to the fixed point \( x_0 \in X \). For any \( Y \in T_x X \) and \( g(X, \frac{\partial}{\partial t}) = 0 \). Then we can get an Jacobi field \( \hat{Y} \) by extending \( X \) along \( \gamma \) (see [5] or [II]). Let \( I_0(.,.) \) be the index form along \( \gamma \). Then the hessian of \( r \) at \( x \)
\[ H(r)(Y,Y) = \hat{Y}\hat{Y}r - D\hat{Y}\hat{Y}r \]
can be written as
\[ \int_0^r (|D_t\hat{Y}|^2 - g(R(\hat{Y}, D_t)D_t, \hat{Y}))dt \]
which is the index form \( I_0(\hat{Y},\hat{Y}) \). We now extend \( Y \) along \( \gamma \) and get a parallel vector field \( E \). Then by the minimizing property of the index form \( I_0^r \) we have that
\[ I_0^r(\hat{Y},\hat{Y}) \leq I_0^r(tE, tE), \]
and the right side of the above inequality is
\[
\frac{1}{r} - \frac{1}{r^2} \int_0^r t^2 g(R(E, D_t) D_t, E) dt
\]
To compute the Laplacian \( \Delta r \) of \( r \), we choose vector fields
\[
\left\{ \frac{\partial}{\partial t}, E_1, \ldots, E_{n-1} \right\}
\]
as an orthonormal basis of \( T_{\gamma(t)} X \) and parallel along \( \gamma \). Then we have
\[
\Delta r = \sum_{i=1}^{n-1} H(r)(E_i, E_i),
\]
which is bounded above by
\[
\frac{n - 1}{r} - \frac{1}{r^2} \int_0^r t^2 Rc(D_t, D_t) dt.
\]
Using the assumption that \( Rc \geq C(r) \) we get that
\[
\Delta r \leq \frac{n - 1}{r} - \frac{1}{r^2} \int_0^r t^2 C(t) dt.
\]
Since
\[
- \frac{1}{r^2} \int_0^r t^2 C(t) dt \leq \frac{C}{r}.
\]
we have
\[
\Delta r \leq \frac{n - 1 + C}{r}.
\]
Now it is standard to verify (see also page 7 in [12] or [7]) that in the distributional sense, it holds on \( X \)
\[
\Delta r \leq \frac{n - 1 + C}{r}.
\]
Then it holds in the distributional sense that
\[
\Delta r^2 = 2r \Delta r + 2 \leq 2(n + 1 + C).
\]
That is, for any non-negative function \( \phi \in C_0^\infty(X) \),
\[
(1) \quad \int_X r^2 \Delta \phi \leq 2(n + 1 + C) \int_X \phi.
\]
We now follow the argument of Schoen and Yau (see [12]). By approximation, we can let \( \phi \) in (1) be a Lipschitz function with compact support. Choose \( \phi(x) = \xi(r(x)) \), where \( \xi(r) = 1 \) for \( r \leq R - 1 \), \( = 0 \) for \( r \geq R + 1 \), and \( \xi'(r) = -\frac{1}{2} \) for \( R - 1 \leq r \leq R + 1 \). By direct computation we have
\[
\int_X r^2 \Delta \phi = -2 \int_{B_{R+1}(p)} \xi' r |Dr|^2
\]
Note that $|Dr| = 1$, then we get
\[ \int_X r^2 \Delta \phi = \int_{B_{R+1}(p) - B_{R-1}(p)} r \]
From this we clearly have,
\[ \int_X r^2 \Delta \phi \geq (R - 1) \text{Vol}(B_{R+1}(p) - B_{R-1}(p)). \]
Note that
\[ \int_X \phi \leq \text{Vol}B_{R+1}(p) \]
and
\[ B_1(x) \subset B_{R+1}(p) - B_{R-1}(p). \]
Then by (1) we have
\[ (R - 1) \text{Vol}B_1(x) \leq 2(n + 1 + C) \text{Vol}B_{R+1}(p). \]
Since
\[ B_{R+1}(p) \subset B_{2(R+1)}(x), \]
we obtain that
\[ (R - 1) \text{Vol}B_1(x) \leq 2(n + 1 + C) \text{Vol}B_{2(R+1)}(x). \]
This implies Theorem 1.

3. Remarks on harmonic functions

In his beautiful work [13], Yau proved that

**Theorem 2.** Let $X$ be an $n$ ($\geq 2$) dimensional complete Riemannian manifold with $Rc(X) \geq -(n-1)K$, where $K \geq 0$ is a constant. Assume $u$ is a positive harmonic function on $X$. Let $B_R$ be a geodesic ball in $X$. Then it holds on $B_{R/2}$ that
\[ |D\log u| \leq C_n \left( \frac{1 + R\sqrt{K}}{R} \right) \]
where $C_n$ is a constant depending only on $n$.

We now use this gradient estimate to study the positive solution of the equation on the manifold $X$:
\[ \Delta u = -c^2 u, \]
where $c \geq 0$ is a constant.
Proposition 3. Let $X$ be an $n \geq 2$ dimensional complete Riemannian manifold with $Rc(X) \geq -(n-1)K$, where $K \geq 0$ is a constant. Assume $u$ is a positive solution of (2) on $X$. Let $B_R$ be a geodesic ball in $X$. Then it holds on $B_{R/2}$ that

$$|D\log u| \leq C_n \left( \frac{1 + R\sqrt{K}}{R} \right)$$

where $C_n$ is a constant depending only on $n$.

Proof. Let

$$N = X \times R$$

have the product metric, so that we still have $Rc(N) \geq -(n-1)K$ provided $Rc(X) \geq -(n-1)K$. We write $D_N$ as the covariant derivative on $N$. Let

$$w(x, t) = e^{ct}u(x).$$

Then

$$\Delta_N w = 0.$$

By Yau’s estimate we have that

$$(3) \quad |D\log u| = |D\log w| \leq |D_N \log w| \leq C_n \left( \frac{1 + R\sqrt{K}}{R} \right).$$

The important matter for us is the constant $C_n$ independent of the constant $c$. This is an important thing for us to study the Ricci soliton.

We also need to study the positive solution of the following equation on the manifold $X$:

$$(4) \quad \Delta u = c^2 u \geq 0,$$

where $c \geq 0$ is a constant. Note that non-negative solutions to (3) are non-negative subharmonic functions. So we now use our Theorem 1 to study $L^q$ non-negative subharmonic functions on complete Riemannian manifolds. As in the proof of Theorem 2.5 in [9], we have

Proposition 4. Suppose $X$ is a complete Riemannian manifold of dimension $n$. Let $C(r)$ be a continuous function satisfying

$$C(r) \geq -Cr^{-2}(x)$$

for $r$ large. Assume that there is a constant $C > 0$ such that the Ricci curvature has the bound

$$Rc \geq C(r).$$

Then for any $q \in (0, \infty)$, $X$ has no $L^q$ non-negative subharmonic function except the constants.
4. Ricci soliton with Ricci curvature quadratic decay

In this section, we assume that the non-compact complete Riemannian manifold \((X, g)\) is a gradient Ricci soliton, that is, there is a smooth function \(f\) such that

\[ Rc(g) = D^2 f, \]

on \(X\). The classification for Ricci solitons is important to the research for Ricci flow, see [8], [2], [4], [10] and [11]. Generally speaking, a non-compact Ricci soliton may not be a gradient Ricci soliton.

For a gradient Ricci soliton, as showed by R. Hamilton [8], we have a constant \(M\) such that

\[ |Df|^2 + s = M \]

where \(s\) is the scalar curvature of \(G\). We assume that \(f\) is not a constant, so \(X\) is not Ricci flat.

We now come to the question: Whether the constant \(M\) is bounded under a nice curvature condition?

From the definition of the gradient Ricci soliton, it is clear that

\[ s = \Delta f. \]

Set

\[ u = e^f, \]

which is a positive function on \(X\). Then we have [11] that

\[ \Delta u = Mu, \]

If \(M\) is negative, then we can write it as \(M = -c^2\), and then we can use the gradient estimate in Proposition 3. Note that

\[ D\log u = Df. \]

Then in this case, we have the bound

\[ |Df| \leq C_n \left( \frac{1 + R\sqrt{K}}{R} \right), \]

for every \(R > 0\), provided \(Rc(X) \geq -(n - 1)K\) on the ball \(B_R\) and

\[ -n(n - 1)K \leq M \leq 0. \]

If \(M = c^2\) for some constant \(c \geq 0\), we can use our Proposition 4. In conclusion we have

**Theorem 5.** Assume \((X, g)\) is a complete non-compact Gradient Ricci soliton such that

\[ Rc = D^2 f. \]
Suppose $X$ is not Ricci flat. Let $C(r)$ be a continuous function satisfying
\[ C(r) \geq -Cr^{-2}(x) \]
for $r$ large. Assume the Ricci curvature has the bound
\[ Rc \geq C(r). \]
Then either (1) we have $M = 0$, and $u := e^{f}$ is a positive harmonic function on $X$; or (2) $M > 0$ and for any $q \in (0, \infty)$, we have
\[ \int_X u^q = \infty. \]

Proof. In fact we have two cases when (i) $M \leq 0$ or (ii) $M > 0$. In case (i), we clearly have that
\[ s < |Df|^2 + s = M \leq 0. \]
The lower bound for $R$ follows easily from the assumption that
\[ s(x) \geq nC(r(x)) \]
for $r(x)$ large. By this we have $s(x) \to 0$ as $r(x) \to \infty$, and $M = 0$. In case (ii), $u$ is a positive subharmonic function, amd we use Proposition 4 to get
\[ \int_X u^q = \infty. \]

References

[1] Th.Aubin, *Non-linear Analysis on manifolds*, Springer, New York, 1982.
[2] R.Bryant, *Gradient Kahler-Ricci solitons*, ArXiv.math.DG/0407453, 2004.
[3] E.Calabi, *On manifolds with non-negative Ricci curvature II*, Notices AMS, 22(1975)205.
[4] H.D.Cao, *Existence of gradient Kahler-Ricci soliton*, Elliptic and parabolic methods in geometry, Eds. B.Chow, R.Gulliver, S.Levy, J.Sullivan, A K Peters, pp.1-6, 1996.
[5] J.Cheeger and D.Ebin, *Comparison Theorems in Riemannian Geometry*, North Holland, Amsterdam, 1975.
[6] J.Cheeger, M.Gromov, M.Taylor, *Finite propagation speed, kernal estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, Journal of Diff. Geom., 17(1982)15-54.
[7] T.Colding and W.Minicozzi II, *An Excursion into Geometric Analysis*, Arxiv.math.DG/0309021, 2003.
[8] R.Hamilton, *The formation of Singularities in the Ricci flow*, Surveys in Diff. Geom., Vol.2, pp7-136, 1995.
[9] P.Li and R.Schoen, *$L^p$ and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math., 153(1984)279-301.

[10] Li Ma, *Ricci-Hamilton flow on Surfaces*, Global scientific publishing, Singapore, 2004.

[11] Li Ma, *Remarks on Ricci solitons*, Arxiv *math.DG/0411426* 2004.

[12] R.Schoen and S.T.Yau, *Lectures on Differential Geometry*, IP, Boston, 1994.

[13] S.T.Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math., 28(1975)201-228.

[14] S.T.Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana University Math. Journal, 25(1976)659-670.

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