ON THE STABILITY AND TRANSITION OF THE CAHN-HILLIARD/ALLEN-CAHN SYSTEM

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(Communicated by José A. Langa)

Abstract. In this paper, the main objective is to study the stability and transition of the Cahn-Hilliard/Allen-Cahn system. By using the dynamic transition theory, combining with the spectral theorem for general linear completely continuous fields, we prove that the system undergoes a continuous transition and bifurcates from a trivial solution to an attractor as the control parameter crosses a certain critical value. In addition, for some special cases, i.e., the domain is \(n\)-dimensional box \((n = 1, 2, 3)\), we not only obtain the stability of the singular points of the attractors, the topological structure of the attractors is also illustrated.

1. Introduction. In order to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point, Cahn and Novick-Cohen [2] derived the Cahn-Hilliard/Allen-Cahn system in the following form, denoted as (CH/AC):

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -h^4 \Delta^2 u - \lambda h^2 \Delta u + h^2 \Delta (2u^3 + 6uv^2), \quad x \in \Omega, \\
\frac{\partial v}{\partial t} &= h^2 \Delta v + (\lambda - \alpha)v - 2v^3 - 6u^2v, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} |_{\partial \Omega} &= 0, \quad \frac{\partial \Delta u}{\partial n} |_{\partial \Omega} = 0, \quad \frac{\partial v}{\partial n} |_{\partial \Omega} = 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

(1)

in which the unknown function \(u\) denotes the average concentration of one of the components and is a conserved quantity, \(v\) is an order parameter, \(h\) is a positive parameter which represents the lattice spacing and the parameter \(\alpha\) reflects the location of the system within the phase diagram. Moreover, \(\Delta\) is the Laplace operator, \(\Omega \subset \mathbb{R}^n\) \((1 \leq n \leq 3)\) is a bounded domain and \(\lambda\) is a control parameter. It is well known that the (1) can be considered as a system encompassing both the Cahn-Hillard and the Allen-Cahn equations. In particular, in the case of \(u \equiv \frac{1}{2}\), (1) reduces to the Allen-Cahn equation, and when \(v \equiv 0\), (1) reduces to the Cahn-Hillard equation. Allen-Cahn equation and Cahn-Hillard equation have been intensively studied (see for instance [8, 4, 3, 10, 17, 7]).

2010 Mathematics Subject Classification. Primary: 35G60; Secondary: 35Q56.

Key words and phrases. Cahn-Hilliard/Allen-Cahn system, dynamic transition theory, attractor bifurcation, topological structure.

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Up to now, we find several mathematical results on diffuse interface model for simultaneous order-disorder and phase separation. Brochet, Hilhorst, and Novick-Cohen [1] proved the existence of maximal attractor for the CH/AC system. Long time asymptotical behaviors for the CH/AC system were developed in [16]. Gokieli and Ito [5] studied the CH/AC system with constraints, they obtained the existence of global attractor for the CH/AC system, and proved that each element of the ω-limit set of any initial data is a steady-state solution to the CH/AC system. However, the number of the steady-state solutions and their stabilities were not given. Gokieli and Marcinkowski [6] presented a numerical method for solving the CH/AC system. Their simulation results give a first intuition about the stationary state form and stability. Recently, Li and Yan [9] investigated the steady state equation of the CH/AC system. For special case, i.e., the domain Ω is a n-dimensional box, they proved that the steady state equation bifurcates from the trivial solution to $3^n - 1$ nontrivial steady state solution as the control parameter crosses a certain critical value. Despite the previous researches on the existence of global attractor and numerical simulation can bring some insight into the asymptotic behavior in the CH/AC system, it is still inadequate to understand the variation of asymptotical state due to the varying of control parameters and to know the stability of stationary states.

Inspired by the aforementioned work, in order to derive a complete understanding of the dynamical transition in the CH/AC system led by the varying of parameters, we study the system from a new perspective beyond previous studies. More precisely, we will use the new attractor bifurcation and transition theory recently developed by Ma and Wang [13, 14] to analyse the transition and attractor bifurcation phenomenon of the CH/AC system (1). The new theory is a strong mathematical tool dealing with nonlinear dynamical transition, which has been used to study the type of transition of the system in the absence of $v$, see [15]. Liu [12, 11] extended Ma and Wang’s work [15] to more general case with Onsager mobility, and long-range repulsive interactions. But his research was still confined to the case of $v ≡ 0$ (Cahn-Hillard system). Without doubt, it is mathematically and practically important to general the results on the Cahn-Hillard system to the CH/AC system.

Due to the presence of the interaction between $u$ and $v$, it brings some inherent difficulties to do dynamical transition analysis on the CH/AC system. In this article, we overcome these difficulties and reduce the equation (1) to a system of ODEs, then consider the attractor bifurcation of the reduced equation from the perspective of dynamic phase transition, see [13]. We prove that the system (1) bifurcates from the trivial zero solution to an attractor whose structure is completely determined by the dimension of domain and distribution of its singular points (stationary solutions) as the control parameter $λ$ crosses a certain threshold $λ_0$. Particularly, in the case that the domain $Ω$ is a n-dimensional box, the number of singular points of the bifurcated attractor can be explicitly computed, and the stability properties of these singular points are known, and then the topological structure of the attractor is definitely plotted, shown in Fig.1–3. Hence, in the vicinity of the control parameter $λ_0$, if $λ > λ_0$, the exact number of stationary solutions to the CH/AC system contained in the vicinity of zero solution can be determined.

The rest of the paper is arranged as follows. In Section 2, we will convert the CH/AC system into abstract form. Then in Section 3 we consider eigenvalues and eigenfunctions for the linearized eigenvalue equations. In Section 4, we will state
and prove our main results. The discussion and conclusion are carried out in Section 5.

2. **Abstract equation.** In this section, we convert the CH/AC system (1) into an abstract functional setting which is standard in the framework of dynamic transitions. First, we let $H$ and $X$ be function spaces defined as follows:

$$H = \left\{ u \in L^2(\Omega) \mid \int_{\Omega} u \, dx = 0 \right\}, \quad X = \left\{ (u, v) \in H \times L^2(\Omega) \right\},$$

$$X_1 = \left\{ (u, v) \in \left[ H^4(\Omega) \times H^2(\Omega) \right] \cap X \mid \frac{\partial u}{\partial n}|_{\partial \Omega} = 0, \frac{\partial \Delta u}{\partial n}|_{\partial \Omega} = 0, \frac{\partial v}{\partial n}|_{\partial \Omega} = 0 \right\}.$$

Then, we define the operators $L_\lambda = -A + B_\lambda$ and $G : X_1 \rightarrow X$ by

$$\begin{align*}
-A(U) &= \begin{pmatrix} -h^4 \Delta^2 u \\ h^2 \Delta v \end{pmatrix}, \\
B_\lambda(U) &= \begin{pmatrix} -\lambda h^2 \Delta u \\ (\lambda - \alpha) v \end{pmatrix}, \\
G(U) &= \begin{pmatrix} h^2 \Delta (2u^3 + 6uv^2) \\ -2v^3 - 6u^2v \end{pmatrix},
\end{align*}$$

where $U = (u, v)$.

We thus obtain the equivalent operator equation of the evolution equation (1) as follows

$$\frac{dU}{dt} = L_\lambda U + G(U).$$

It is easy to see that $L_\lambda$ is a completely continuous field and (3) satisfies the necessary hypotheses of [13, 14].

3. **Eigenvalues and eigenfunctions.** The linear eigenvalue equations of (1) are given by

$$\begin{align*}
- h^4 \Delta^2 u_1 - \lambda h^2 \Delta u_1 &= \eta(\lambda) u_1, \\
 h^2 \Delta u_2 + (\lambda - \alpha) u_2 &= \eta(\lambda) u_2, \\
\frac{\partial u_1}{\partial n}|_{\partial \Omega} &= 0, \quad \frac{\partial \Delta u_1}{\partial n}|_{\partial \Omega} = 0, \\
\frac{\partial u_2}{\partial n}|_{\partial \Omega} &= 0, \quad \int_{\Omega} u_1 \, dx = 0.
\end{align*}$$

Let $\rho_k$ ($k = 1, 2, \ldots$) and $e_k$ ($k = 1, 2, \ldots$) be the eigenvalue and eigenfunction of the following eigenvalue problem

$$\begin{align*}
-\Delta e_k &= \rho_k e_k, \quad x \in \Omega, \\
\frac{\partial e_k}{\partial n}|_{\partial \Omega} &= 0, \quad \int_{\Omega} e_k \, dx = 0.
\end{align*}$$

It is well known that the eigenvalues of (6) satisfy

$$0 < \rho_1 \leq \rho_2 \leq \cdots, \quad \rho_k \rightarrow \infty \ (k \rightarrow \infty),$$

and the eigenfunctions $\{e_k\}$ of (6) constitute an orthogonal basis of $H$. Especially, when $\Omega = [0, \pi]^n$ ($1 \leq n \leq 3$), then the eigenvalues and eigenfunctions of (6) are explicit given by

$$\rho_K = |K|^2, \quad e_K = \cos k_1 x_1 \cos k_2 x_2 \cdots \cos k_n x_n,$$
Remark 1. It is easy to straightly verify that \( \rho_1 = 1 \) is the first eigenvalue of (6) and 
\[
e_1 = \cos k_1 x_1 \cos k_2 x_2 \cdots \cos k_n x_n, \ k_1^2 + k_2^2 + \cdots + k_n^2 = 1
\]
is the eigenfunction corresponding to \( \rho_1 \).

Lemma 3.1. Let \( \eta_k^j(\lambda)(j = 1, 2; \ k = 1, 2, \cdots) \) be eigenvalues of (5). Then 
\( \eta_k^1(\lambda)(j = 1, 2; \ k = 1, 2, \cdots) \) are real, and
\[
\eta_k^1(\lambda) = \rho_1 h^2 (\lambda - h^2 \rho_1)
\]
\[
\begin{cases}
> 0, \ & \lambda > \lambda_0, \\
= 0, \ & \lambda = \lambda_0, \ & \lambda_0 = h^2 \rho_1, \\
< 0, \ & \lambda < \lambda_0,
\end{cases}
\]
\[
\eta_k^2(\lambda_0) = h^4 \rho_k (\rho_1 - \rho_k) < 0, \ k = 2, 3, \cdots,
\]
\[
\eta_k^2(\lambda_0) = -h^2 \rho_k + h^2 \rho_1 - \alpha < 0, \ k = 1, 2, \cdots.
\]\(\text{Proof.} \) Let \( M_k(\lambda) \) be the matrix given by
\[
M_k(\lambda) = \begin{pmatrix}
-h^4 \rho_k^2 + \lambda h^2 \rho_k & 0 \\
0 & -h^2 \rho_k + \lambda - \alpha
\end{pmatrix}.
\]
Then, all eigenvalues \( \eta_k^j(\lambda) \) of (5) satisfy
\[
M_k x_k^j = \eta_k^j(\lambda) x_k^j, \ j = 1, 2, \ k = 1, 2, \cdots,
\]
where
\[
\eta_k^1(\lambda) = -h^4 \rho_k^2 + \lambda h^2 \rho_k, \ \eta_k^2(\lambda) = -h^2 \rho_k + \lambda - \alpha,
\]
\[
x_k^1 = \begin{pmatrix}
1 \\
0
\end{pmatrix}, \ x_k^2 = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \ k = 1, 2, \cdots.
\]
Furthermore, the eigenvectors \( e_k^j(x)(j = 1, 2; \ k = 1, 2, \cdots) \) of (5) corresponding to 
\( \eta_k^j(\lambda)(j = 1, 2; \ k = 1, 2, \cdots) \) are
\[
e_k^1(x) = \begin{pmatrix} e_k(x) \end{pmatrix}, \ e_k^2(x) = \begin{pmatrix} 0 \\ e_k(x) \end{pmatrix}, \ k = 1, 2, \cdots,
\]
where \( e_k \) as in (6). Thus the proof is complete.

Remark 1. By Lemma 3.1 we know that all eigenvalues of \( L_\lambda \) are \( \eta_k^j(\lambda)(j = 1, 2; \ k = 1, 2, \cdots) \) which are real, and satisfy (8). It is easy to verify that \( \eta_k^j(\lambda)(j = 1, 2; \ k = 1, 2, \cdots) \) are also all eigenvalues of the conjugate operator \( L_\lambda^* \), and \( e_k^j(x)(j = 1, 2; \ k = 1, 2, \cdots) \) are the eigenvectors of \( L_\lambda^* \) corresponding to \( \eta_k^j(\lambda)(j = 1, 2; \ k = 1, 2, \cdots) \).

Remark 2. When \( \Omega = [0, \pi]^n \ (1 \leq n \leq 3) \), one can straightly verify that the algebraic multiplicity \( m \) of the first eigenvalue \( \rho_1 = 1 \) of \( L_\lambda \) is completely determined by the dimension of domain \( \Omega \), i.e. \( m = n \).
4. Attractor bifurcation of the CH/AC system.

4.1. Attractor bifurcation for general bounded domain case. In this subsection we consider the case that \( \Omega \in \mathbb{R}^n \) (1 \( \leq n \leq 3 \)) is a general bounded domain. For the equation (1) we have the following attractor bifurcation theorem.

**Theorem 4.1.** Assume the first eigenvalue \( \rho_1 \) has algebraic multiplicity \( m \geq 1 \). Then the system (1) bifurcates from \( (U, \lambda) = (0, h^2 \rho_1) \) on \( \lambda > h^2 \rho_1 \) to an attractor \( \sum_\lambda \) which is homologic to \( S^{m-1} \). Furthermore, \( \sum_\lambda \) attracts \( O \setminus \Gamma \) for some neighborhood \( O \subset X \) of \( U = 0 \), where \( \Gamma \) is the stable manifold of \( U = 0 \) with co-dimension \( m \).

**Remark 3.** Combining Lemma 3.1 and Theorem 4.1, a sufficient and necessary condition of attractor bifurcation for the system (1) is the control parameter \( \lambda > h^2 \rho_1 \).

**Remark 4.** It is known that attractor contains all possible asymptotic state of a evolution system. Theorem 4.1 states that the asymptotic state of (1) is not unique.

**Proof.** Based on the Theorem 5.2 and 5.10 in [14], we only need to show that \( U = (0, 0) \) is locally stable at \( \lambda = \lambda_0 \). This can be carried out by the reduced equations, which is derived by the spectral theorem and some computations.

First, by the spectral theorem for general linear completely continuous fields [13, 14], we obtain that the space \( X_1 \) and \( X \) can be decomposed into

\[
\begin{align*}
X_1 &= E_1 \bigoplus E_2, \\
X &= E_1 \bigoplus E_2, \\
E_2 &= \text{closure of } E_2 \text{ in } X,
\end{align*}
\]

(11)

for \( \lambda \) near \( \lambda_0 \), where

\[
E_1 = \text{span}\{e_{11}^1, \cdots, e_{1m}^1\},
\]

\[
E_2 = \text{span}\{e_{12}^1, e_{13}^1, \cdots, e_{1k_1}^1, e_{21}^2, \cdots\},
\]

\( e_{ij}^k (i = 1, 2, \cdots, m) \) are the eigenvectors of \( L_\lambda \) corresponding to \( \eta_1^1(\lambda) \) and which have been unitized, and by (10) they satisfying

\[
< e_i^m, e_j^n > = \begin{cases} 
0 & \text{if } i \neq j \text{ or } m \neq n, \\
1 & \text{if } i = j, m = n.
\end{cases}
\]

(12)

Then, it deduces from (11) that for any \( U \in X_1 \), we have

\[
\begin{align*}
U &= \sum_{i=1}^{m} x_i e_{1i}^1 + h(x), \\
h(x) &= (h_1(x), h_2(x))^T = o(|x|) \text{ is the center manifold function},
\end{align*}
\]

(13)

where \( x = (x_1, \cdots, x_m) \). Therefore, the reduced equations of (4) are given by

\[
\frac{dx_i}{dt} = \eta_1^1(\lambda) x_i + \langle G(U), e_{1i}^1 \rangle_X, \quad i = 1, 2, \cdots, m.
\]

(14)

Second, let us get the approximate expression for the reduction equations (14). Substituting (13) into (14), we can obtain the reduced equations of (4) on the center manifold near \( \lambda = h^2 \rho_1 \) as following

\[
\frac{dx_i}{dt} = \eta_1^1(\lambda) x_i + \langle H(x), e_{1i}^1 \rangle_X, \quad i = 1, 2, \cdots, m,
\]

(15)
where
\[ H(x) = G \left( \sum_{i=1}^{m} x_i e_{1i}^1 + h(x) \right). \]

Note that
\[ \sum_{i=1}^{m} x_i e_{1i}^1 + h(x) = \sum_{i=1}^{m} x_i \begin{pmatrix} e_{1i} \\ 0 \end{pmatrix} + \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} =: \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \]
where
\[ u_1 = \sum_{i=1}^{m} x_i e_{1i} + h_1(x), \quad u_2 = h_2(x), \]
and \( e_{1i} (i = 1, 2, \cdots, m) \) are the eigenfunctions of (6) corresponding to \( \rho_1 \). Thus, straightforward calculation gives
\[ \langle H(x), e_{1i}^1 \rangle = \int_{\Omega} h^2 \Delta (2u_1^3 + 6u_1 u_2^2) e_{1i}(\tilde{x}) d\tilde{x} = -\rho_1 h^2 \int_{\Omega} (2u_1^3 + 6u_1 u_2^2) e_{1i}(\tilde{x}) d\tilde{x} = -\rho_1 h^2 \int_{\Omega} \left[ 2 \left( \sum_{i=1}^{m} x_i e_{1i}(\tilde{x}) + h_1(x) \right)^3 + 6 \sum_{i=1}^{m} x_i e_{1i}(\tilde{x}) + h_1(x) \times (h_2(x))^2 \right] e_{1i}(\tilde{x}) d\tilde{x} = -2\rho_1 h^2 \int_{\Omega} (\sum_{i=1}^{m} x_i e_{1i}(\tilde{x}))^3 e_{1i}(\tilde{x}) d\tilde{x} + o(|x|^3). \] (16)

Now, we infer then from (15) and (16) that the approximate expression of the reduced equations
\[ \frac{dx_i}{dt} = \eta_i^1(\lambda)x_i - 2\rho_1 h^2 \int_{\Omega} (\sum_{i=1}^{m} x_i e_{1i}(\tilde{x}))^3 e_{1i}(\tilde{x}) d\tilde{x} + o(|x|^3), \quad i = 1, 2, \cdots, m. \] (17)

Let
\[ g(x) = \left( -\int_{\Omega} (\sum_{i=1}^{m} x_i e_{1i}(\tilde{x}))^3 e_{11}(\tilde{x}) d\tilde{x}, \cdots, -\int_{\Omega} (\sum_{i=1}^{m} x_i e_{1i}(\tilde{x}))^3 e_{1m}(\tilde{x}) d\tilde{x} \right). \]
One can thus have
\[ \langle g(x), x \rangle = -\sum_{i=1}^{m} x_i \int_{\Omega} (\sum_{i=1}^{m} x_i e_{1i}(\tilde{x}))^3 e_{1i}(\tilde{x}) d\tilde{x} = -\int_{\Omega} (\sum_{i=1}^{m} x_i e_{1i}(\tilde{x}))^4 d\tilde{x} \leq -C|x|^4, \] (18)
for some \( C > 0 \). The inequality (18) means that \( x = 0 \) is locally asymptotically stable for (17). \( \square \)
4.2. Attactor bifurcation for n-dimensional box case. In order to obtain the stability properties of the equilibria contained in the attractor and the topological structure of the attractor, in this subsection we consider the case that the domain \( \Omega \) is a \( n \)-dimensional box, i.e. \( \Omega = [0, \pi]^n \) (\( 1 \leq n \leq 3 \)). For equation (1) we have the following attractor bifurcation theorem.

**Theorem 4.2.** For domain \( \Omega = [0, \pi]^n \) (\( 1 \leq n \leq 3 \)) the following assertions hold true:

1. If \( n = 1 \), then the problem (1) bifurcates from \( (U, \lambda) = (0, h^2) \) on \( \lambda > h^2 \) to an attractor \( \sum_\lambda \), which contains exactly two steady states of (1) and the orbits connecting the equilibria, the two steady states solutions are stable, and the structure of attractor shown in Fig.1.

2. If \( n = 2 \), then the problem (1) bifurcates from \( (U, \lambda) = (0, h^2) \) on \( \lambda > h^2 \) to an attractor \( \sum_\lambda \) homologic to \( S^1 \), which contains exactly eight steady state solutions of (1) and the orbits connecting the equilibria, four of the steady state solutions are stable, the other four steady state solutions are unstable, and the corresponding topological structure of the \( S^1 \) attractor shown in Fig.2.

3. If \( n = 3 \), then the problem (1) bifurcates from \( (U, \lambda) = (0, h^2) \) on \( \lambda > h^2 \) to an attractor \( \sum_\lambda \) homologic to \( S^2 \), which contains exactly twenty six steady state solutions of (1) and the orbits connecting the equilibria, six of the steady state solutions are stable, the other twenty steady state solutions are unstable, and the corresponding topological structure of the \( S^2 \) attractor shown in Fig.3.

**Remark 5.** Theorem 4.1 and 4.2 yield the following physical results:

1. If the domain is 2d-box, the system (1) has four stable nontrivial steady-states in the vicinity of zero solution after undergoing a continuous transition (called attractor bifurcation).
2. If the domain is 3d-box, the system (1) has six stable nontrivial steady-states in the vicinity of zero solution after undergoing a continuous transition. These six steady-states are the most possible asymptotic states.

**Remark 6.** The results of Theorem 4.1, 4.2 clearly depend on the control parameter and correct in close proximity to the bifurcation. When the control parameter is varied further $\lambda_0$ the method used in our paper can not be applied and in this case new ideas or further calculations will be required.

**Proof.** For $\Omega = [0, \pi]^n$ ($1 \leq n \leq 3$), the existence of attractor bifurcation can be obtained from Theorem 4.1. Thus, we only need to determine the topological structure of the bifurcated attractors by relying on determining the distribution of its singular points (equilibria of (1)). The proof is divided into a few steps in the following.

**Step 1.** Reduction equation.

Let

$$U = \sum_{|K| \geq 1} y_K e_K^1 + \sum_{|K| \geq 1} z_K e_K^2. \quad (19)$$

Then

$$U = \left( \begin{array}{c} \sum_{|K| \geq 1} y_K e_K \\ \sum_{|K| \geq 1} z_K e_K \end{array} \right) =: \left( \begin{array}{c} u \\ v \end{array} \right). \quad (19)$$

where $e_K$ as in (7).

The steady equation of (1) can be expressed as

$$\eta^1_i(\lambda)y_i + \frac{2}{\pi^n} \int_{\Omega} h^2 \Delta (2u^3 + 6uv^2) \cos x_i dx = 0, \ i = 1, 2, \cdots, n, \quad (20)$$

$$\eta^2_i(\lambda)z_i + \frac{2}{\pi^n} \int_{\Omega} (-2v^3 - 6u^2v) \cos x_i dx = 0, \ i = 1, 2, \cdots, n, \quad (21)$$

$$\eta^1_k(\lambda)y_K + \frac{1}{||e_K||^2_{L^2}} \int_{\Omega} h^2 \Delta (2u^3 + 6uv^2)e_K dx = 0, \ |K| > 1, \quad (22)$$

$$\eta^2_k(\lambda)z_K + \frac{1}{||e_K||^2_{L^2}} \int_{\Omega} (-2v^3 - 6u^2v)e_K dx = 0, \ |K| > 1. \quad (23)$$
We find that

\[
\int_{\Omega} h^2 \triangle (2u^3 + 6uv^2) e_K \, dx = -h^2 |K|^2 \int_{\Omega} (2u^3 + 6uv^2) e_K \, dx.
\]

Then (20) and (21) can be rewritten as

\[
\eta_i^1(\lambda) y_i - \frac{2h^2}{\pi n} \int_{\Omega} (2u^3 + 6uv^2) \cos x_i \, dx = 0, \quad i = 1, 2, \ldots, n, \tag{24}
\]

\[
\eta_i^2(\lambda) z_i - \frac{2}{\pi n} \int_{\Omega} (2u^3 + 6u^2v) \cos x_i \, dx = 0, \quad i = 1, 2, \ldots, n. \tag{25}
\]

For \( K = (k_1, \ldots, k_n) \), we denote

\[
\eta_1^1 = \eta_{K}, \quad y_{ij} = y_{K} \text{ if } k_j = 2 \text{ and } k_i = 0 \text{ for } i \neq j,
\]

\[
\eta_2^1 = \eta_{K}, \quad z_{ij} = z_{K} \text{ if } k_j = 2 \text{ and } k_i = 0 \text{ for } i \neq j,
\]

\[
\eta_1^2 = \eta_{K}, \quad y_{ij} = y_{K} \text{ if } k_i = k_j = 1 \text{ and } k_l = 0 \text{ for } l \neq i, j,
\]

\[
\eta_2^2 = \eta_{K}, \quad z_{ij} = z_{K} \text{ if } k_i = k_j = 1 \text{ and } k_l = 0 \text{ for } l \neq i, j.
\]

We infer from (22) and (23) that

\[
y_{ij} = \frac{8h^2}{\pi n \eta_{ij}^1(\lambda)} \int_{\Omega} (2u^3 + 6uv^2) \cos x_i \cos x_j \, dx
\]

\[
= \frac{8h^2}{\pi n \eta_{ij}^1(\lambda)} \int_{\Omega} \left[ 2 \left( \sum_{|K| \geq 1} y_{K} e_K \right)^3 + 6 \left( \sum_{|K| \geq 1} y_{K} e_K \right) \times \left( \sum_{|K| \geq 1} z_{K} e_K \right) \right] \cos x_i \cos x_j \, dx
\]

\[
= o(|q|^2),
\]

\[
y_{ij} = \frac{8h^2}{\pi n \eta_{ij}^1(\lambda)} \int_{\Omega} (2u^3 + 6uv^2) \cos x_i \cos x_j \, dx
\]

\[
= \frac{8h^2}{\pi n \eta_{ij}^1(\lambda)} \int_{\Omega} \left[ 2 \left( \sum_{|K| \geq 1} y_{K} e_K \right)^3 + 6 \left( \sum_{|K| \geq 1} y_{K} e_K \right) \times \left( \sum_{|K| \geq 1} z_{K} e_K \right) \right] \cos x_i \cos x_j \, dx
\]

\[
= o(|q|^2),
\]

\[
z_{ij} = \frac{2}{\pi n \eta_{ij}^2(\lambda)} \int_{\Omega} (2u^3 + 6u^2v) \cos 2x_j \, dx
\]

\[
= \frac{2}{\pi n \eta_{ij}^2(\lambda)} \int_{\Omega} \left[ 2 \left( \sum_{|K| \geq 1} z_{K} e_K \right)^3 + 6 \left( \sum_{|K| \geq 1} y_{K} e_K \right) \left( \sum_{|K| \geq 1} z_{K} e_K \right) \right] \cos 2x_j \, dx
\]

\[
= o(|q|^2),
\]
\[ z_{ij} = \frac{4}{\pi^2 n^2} \int_\Omega \left( 2v^3 + 6u^2v \right) \cos x_i \cos x_j \, dx \]
\[ = \frac{4}{\pi^2 n^2} \int_\Omega \left[ 2 \left( \sum_{|K| \geq 1} z_K e_K \right)^3 \right. \]
\[ + 6 \left( \sum_{|K| \geq 1} y_K e_K \right) \left( \sum_{|K| \geq 1} z_K e_K \right) \cos x_i \cos x_j \, dx \]
\[ = o(|q|^2), \]

where \( q = (y_1, \ldots, y_n, z_1, \ldots, z_n) \in \mathbb{R}^{2n} \). In a similar way, one can obtain

\[
\begin{align*}
\begin{cases}
  y_K = o(|q|^2), & \forall |K| > 1, \\
  z_K = o(|q|^2), & \forall |K| > 1.
\end{cases}
\end{align*}
\] (26)

Based on (19) and (26) we have

\[
\int_\Omega u^3 \cos x_i \, dx
\]
\[ = \int_\Omega \left( \sum_{j=1}^n y_j \cos x_j \right)^3 \cos x_i \, dx + o(|q|^3) \]
\[ = y_i^3 \int_\Omega \cos^3 x_i \, dx + \sum_{j \neq i} 3y_i y_j^2 \int_\Omega \cos^2 x_i \cos x_j \, dx + o(|q|^3) \]
\[ = \frac{3\pi^n}{4} \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_i y_j^2 \right) + o(|q|^3), \]

\[
\int_\Omega uv^2 \cos x_i \, dx
\]
\[ = \int_\Omega \left( \sum_{j=1}^n y_j \cos x_j \right) \left( \sum_{j=1}^n z_j \cos x_j \right)^2 \cos x_i \, dx + o(|q|^3) \]
\[ = \sum_{j=1}^n y_i z_j^2 \int_\Omega \cos^2 x_i \cos^2 x_j \, dx \]
\[ + \sum_{j \neq i} 2y_j z_j z_i \int_\Omega \cos^2 x_i \cos x_j \, dx + o(|q|^3) \]
\[ = \frac{3\pi^n}{8} y_i z_i^2 + \frac{\pi^n}{4} \sum_{j \neq i} (y_i z_j^2 + 2y_j z_j z_i) + o(|q|^3), \]

\[
\int_\Omega v^3 \cos x_i \, dx
\]
\[ = \int_\Omega \left( \sum_{j=1}^n z_j \cos x_j \right)^3 \cos x_i \, dx + o(|q|^3) \]
\[ = \frac{3\pi^n}{4} \left( \frac{1}{2} z_i^3 + \sum_{j \neq i} z_i z_j^2 \right) + o(|q|^3), \]
\[ \int_\Omega u^2v \cos x_i \, dx \]
\[ = \int_\Omega \left( \sum_{j=1}^n y_j \cos x_j \right)^2 \left( \sum_{j=1}^n z_j \cos x_j \right) \cos x_i \, dx + o(|q|^3) \]
\[ = \sum_{j=1}^n z_j y_j^2 \int_\Omega \cos^2 x_j \cos^2 x_i \, dx \]
\[ + \sum_{j \neq i} 2z_j y_j y_i \int_\Omega \cos^2 x_i \cos^2 x_j \, dx + o(|q|^3) \]
\[ = \frac{3 \pi n}{8} z_i y_i^2 + \frac{\pi n}{4} \sum_{j \neq i} (z_j y_j^2 + 2z_j y_j y_i) + o(|q|^3). \]

Thus, the reduction equation (24) and (25) can be expressed in form of
\[ \eta_1^1(\lambda) y_i - 3h^2 \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_j y_j^2 \right) - 6h^2 \left( \frac{3}{4} y_i z_i^2 \right) \]
\[ + \sum_{j \neq i} \left( \frac{1}{2} z_j y_j^2 + y_j z_j y_i \right) + o(|q|^3) = 0, \quad i = 1, \ldots, n, \]

(27)
\[ \eta_1^2(\lambda) z_i - 3h^2 \left( \frac{1}{2} z_i^3 + \sum_{j \neq i} z_i z_j^2 \right) - 6h^2 \left( \frac{3}{4} z_i y_i^2 \right) \]
\[ + \sum_{j \neq i} \left( \frac{1}{2} z_i y_j^2 + z_j y_j y_i \right) + o(|q|^3) = 0, \quad i = 1, \ldots, n. \]

(28)

**Step 2.** Approximate equation.

Since \((z_1, \ldots, z_n, y_1, \ldots, y_n) = (0, \ldots, 0, 0, \ldots, 0)\) satisfies (28), and when \(\lambda = \lambda_0\) one can deduce from Remark 3.1 that \(\eta_1^2(\lambda_0) < 0\). Then, by the implicit function theorem, we obtain from (28)
\[ z_i = \Phi_i(y) = o(|y|), \quad i = 1, \ldots, n, \]

(29)

where \(y = (y_1, \ldots, y_n)\). Thus, the reduced equation (27) and (28) are equivalent to the following equation
\[ \eta_1^1(\lambda) y_i - 3h^2 \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_j y_j^2 \right) + o(|q|^3) = 0, \quad i = 1, \ldots, n. \]

(30)

Consider the following approximate equations of (30)
\[ \eta_1^1(\lambda) y_i - 3h^2 \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_j y_j^2 \right) = 0, \quad i = 1, \ldots, n. \]

(31)

It is clear that if all solutions of (31) are regular, then the number of solutions of (31) and (30) are the same.

**Step 3.** Topological structure of the attractors.

When \(n = 1\), (31) reduces to the following equation
\[ \eta_1^1(\lambda) y_1 - 3h^2 \left( \frac{1}{2} y_1^3 \right) = 0. \]
It is easy to see that (32) has the following solutions

\[ Y_1 = 0, Y_{2,3} = \pm \sqrt{\frac{6\eta_1(\lambda)}{3h}}. \]

The Jacobian matrix of the vector field in (32) are given by

\[ M_1(y_1) = \eta_1(\lambda) - \frac{9h^2y_1^2}{2}. \quad (33) \]

Then we derive from (33) that

\[ M_1(Y_1) = \eta_1(\lambda) > 0, \quad (34) \]
\[ M_1(Y_2) = -2\eta_1(\lambda) < 0, \quad (35) \]
\[ M_1(Y_3) = -2\eta_1(\lambda) < 0, \quad (36) \]

which implies that \( Y_1 \) is unstable, but \( Y_2, Y_3 \) are stable.

Therefore, we derive Assertion (1).

When \( n = 2 \), (31) reduces to the following equations

\[
\begin{cases}
\eta_1(\lambda)y_1 - 3h^2\left(\frac{1}{2}y_1^3 + y_1y_2^2\right) = 0, \\
\eta_1(\lambda)y_2 - 3h^2\left(\frac{1}{2}y_2^3 + y_2y_1^2\right) = 0.
\end{cases} \quad (37)
\]

It is easy to see that (37) has the following eight nontrivial solutions

\[ Y_{1,2} = \left(0, \pm \sqrt{\frac{6\eta_1(\lambda)}{3h}}\right), \quad Y_{3,4} = \left(\pm \sqrt{\frac{6\eta_1(\lambda)}{3h}}, 0\right), \]
\[ Y_{5,6,7,8} = \left(\pm \sqrt{\frac{2\eta_1(\lambda)}{3h}}, \pm \sqrt{\frac{2\eta_1(\lambda)}{3h}}\right). \]

The Jacobian matrix of the vector field in (37) are given by

\[ M_2(y_1, y_2) = \begin{pmatrix}
\eta_1(\lambda) - 3h^2\left(\frac{1}{2}y_1^3 + y_1^2 + y_1y_2^2\right) & -6h^2y_1y_2 \\
-6h^2y_1y_2 & \eta_1(\lambda) - 3h^2\left(\frac{1}{2}y_2^3 + y_2^2 + y_2y_1^2\right)
\end{pmatrix}. \quad (38) \]

Then we derive from (38) that the eigenvalues of \( M_2(Y_i) (i = 1, 2, 3, 4) \) are all negative, and \( M_2(Y_i) (i = 5, 6, 7, 8) \) have one positive eigenvalue one negative eigenvalue, which implies that \( Y_i (i = 1, 2, 3, 4) \) are stable, but \( Y_i (i = 5, 6, 7, 8) \) are unstable.

Hence, Assertion (2) is derived.

When \( n = 3 \), (31) reduces to the following equation

\[
\begin{cases}
\eta_1(\lambda)y_1 - 3h^2\left(\frac{1}{2}y_1^3 + y_1y_2^2 + y_1y_3^2\right) = 0, \\
\eta_1(\lambda)y_2 - 3h^2\left(\frac{1}{2}y_2^3 + y_2y_1^2 + y_2y_3^2\right) = 0, \\
\eta_1(\lambda)y_3 - 3h^2\left(\frac{1}{2}y_3^3 + y_3y_1^2 + y_3y_2^2\right) = 0.
\end{cases} \quad (39)
\]

By direct calculation one can find that (39) has the following 26 nontrivial solutions

\[ Y_{1,2} = \left(0, 0, \pm \sqrt{\frac{6\eta_1(\lambda)}{3h}}\right), \quad Y_{3,4} = \left(0, \pm \sqrt{\frac{6\eta_1(\lambda)}{3h}}, 0\right), \quad Y_{5,6} = \left(\pm \sqrt{\frac{6\eta_1(\lambda)}{3h}}, 0, 0\right), \]
\[ Y_{7,8,9,10} = \left(0, \pm \sqrt{\frac{2\eta_1(\lambda)}{3h}}, \pm \sqrt{\frac{2\eta_1(\lambda)}{3h}}\right). \]
\[ Y_{11,12,13,14} = \left( \pm \frac{\sqrt{2\eta_1^2(\lambda)}}{3h}, 0, \pm \frac{\sqrt{2\eta_1^2(\lambda)}}{3h} \right), \]
\[ Y_{15,16,17,18} = \left( \pm \frac{\sqrt{2\eta_1^2(\lambda)}}{3h}, \pm \frac{\sqrt{2\eta_1^2(\lambda)}}{3h}, 0 \right), \]
\[ Y_{19,20,21,22,23,24,25,26} = \left( \pm \frac{\sqrt{30\eta_1^2(\lambda)}}{15h}, \pm \frac{\sqrt{30\eta_1^2(\lambda)}}{15h}, \pm \frac{\sqrt{30\eta_1^2(\lambda)}}{15h} \right). \]

The Jacobian matrix of the vector field in (39) are given by
\[ M_3(y_1, y_2, y_3) = \begin{pmatrix} \eta_1^2(\lambda) - l_1(Y) & -6h^2y_1y_2 & -6h^2y_1y_3 \\ -6h^2y_1y_2 & \eta_1^2(\lambda) - l_2(Y) & -6h^2y_2y_3 \\ -6h^2y_1y_3 & -6h^2y_2y_3 & \eta_1^2(\lambda) - l_3(Y) \end{pmatrix}, \quad (40) \]
where
\[ l_1(Y) = 3h^2 \left( \frac{3}{2} y_1^2 + y_2^2 + y_3^2 \right), \]
\[ l_2(Y) = 3h^2 \left( \frac{3}{2} y_2^2 + y_1^2 + y_3^2 \right), \]
\[ l_3(Y) = 3h^2 \left( \frac{3}{2} y_3^2 + y_1^2 + y_2^2 \right). \]

Then we derive from (40) that the eigenvalues of \( M_3(Y_i)(i = 1, \cdots, 6) \) are all negative, \( M_3(Y_i)(i = 7, \cdots, 18) \) have one positive eigenvalue two negative eigenvalues and \( M_3(Y_i)(i = 19, \cdots, 26) \) have two positive eigenvalues one negative eigenvalue, which implies that \( Y_i(i = 1, \cdots, 6) \) are stable and \( Y_i(i = 7, \cdots, 26) \) are unstable.

Therefore, we derive Assertion (3).

The proof of the theorem is complete. \( \square \)

5. Discussion and conclusion. It follows from Lemma 3.1 that the critical control parameter \( \lambda_0 \) is given by
\[ \lambda_0 = h^2 \rho_1 \quad (41) \]
in which \( h \) is a positive parameter which represents the lattice spacing, and \( \rho_1 \) is the first eigenvalue of Laplace operator \( -\Delta \) in bounded domain \( \Omega \). It is known that \( \rho_1 \) is inversely proportional to the diameter of \( \Omega \). Thus, (41) indicates that the diameter of domain \( \Omega \) is larger, the transition from trivial solution more easily happens. Similarly, the value of \( h \) is also a key factor affecting the transition from the trivial solution. The larger \( h \), the system (1) is harder to undergo a transition.

Theorem 4.1-4.2 express that the system (1) can only undergoes a continuous transition (called attractor bifurcation), in the sense that the system always smoothly varies from a steady-state to another steady-state with the continuous varying in the control parameter \( \lambda \). Theorem 4.1-4.2 also indicate that after undergoing a continuous transition, the new state of the system (1) is not unique. The number of new steady-states is only determined by the domain and its dimension. In the special case of \( \text{dim}(\Omega) = 1 \), the new states are two stable steady-states. Moreover, it is worth mentioning that for domain \( \Omega = [0, \pi]^2 \left( [0, \pi]^3 \right) \), the number of new steady state is 8(26). Although, any one among of the 8(26) steady states can be an asymptotic state of the system (1), only four (six) of them are stable, i.e., the four (six) stable steady-state are the most possible asymptotic states. In fact, which one is the asymptotic state of (1), it is definitely determined by its initial value. Moreover, one can easily see from Theorem 4.1-4.2 that the number of new steady-state is in general proportional to the dimension of domain.
Although we only consider the transition of (1) under one type of physical boundary condition, our theoretical analysis in present work can be applied to other boundary condition, for instance, the periodic boundary condition.

Acknowledgments. The work of Quan Wang was supported by the National Nature Science Foundation of China (NSFC) Grant No.11901408. The work of Dongming Yan was supported by the First Class Discipline of Zhejiang-A (Zhejiang University of Finance and Economics-Statistics).

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Received for publication January 2019.

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