Complex Langevin method applied to the 2D $SU(2)$ Yang–Mills theory

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The complex Langevin method in conjunction with the gauge cooling is applied to the 2D lattice $SU(2)$ Yang–Mills theory that is analytically solvable. We obtain strong numerical evidence that at large Langevin time the expectation value of the plaquette variable converges, but to a wrong value when the complex phase of the gauge coupling is large.

Subject Index B01, B05, B34, B38
1. Introduction

As a possible approach to the functional integral with complex measure, such as the one encountered in the finite density QCD [1, 2], the complex Langevin method [3–5] has attracted much attention in recent years. This recent interest was triggered mainly by the discovery of sufficient conditions for the convergence of the method to a correct answer [6, 7]. Ref. [8] is a review on recent developments. Roughly speaking, if the probability distribution of configurations generated by the Langevin dynamics damps sufficiently fast at infinity of configuration space, the statistical average over the configurations is shown to be identical to the integration over the original complex measure. It has been observed that, in systems for which the complex Langevin method converges but to a wrong answer (such as the 3D XY model [9]), this requirement of a sufficiently localized distribution is broken, typically in “imaginary directions” in configuration space.

After the above understanding, a prescription in lattice gauge theory that makes the probability distribution well-localized was proposed in Ref. [10]; the prescription is termed “gauge cooling” and it proceeds as follows: The link variables in lattice gauge theory are originally elements of the compact gauge group $SU(N)$. When the (effective) action is complex, however, the corresponding Langevin evolution drives link variables into “imaginary directions” and link variables become elements of $SL(N, \mathbb{C})$, a non-compact gauge group. This evolution tends to make the distribution wide in non-compact directions; in terms of the $SU(N)$ Lie algebra, those non-compact directions are parametrized by imaginary coordinates. At this point, one notes that the definition of a physical observable that is invariant under the original $SU(N)$ gauge transformations can always be tailored so that it is invariant also under the non-compact $SL(N, \mathbb{C})$ gauge transformations. The idea of the gauge cooling is that by applying the $SL(N, \mathbb{C})$ gauge transformations appropriately along the complex Langevin evolution, one squeezes the distribution well-localized so that the prerequisite of the convergence theorem [6, 7] is fulfilled while without changing physical observables.

In a 1D gauge model and in the 4D QCD with heavy quarks, it has been confirmed that the gauge cooling makes the distribution well-localized and the complex Langevin method gives rise to correct answers [10]. More recently, this method was applied to the full QCD at finite density [11]. See also Refs. [12–14]. One should note, however, that the Langevin dynamics itself is defined on gauge non-invariant variables (i.e., link variables) and also that the gauge cooling step cannot be regarded as a Langevin evolution that is induced by a holomorphic action; the latter is assumed in the convergence theorem [6, 7]. Strictly speaking, therefore, the convergence theorem does not apply when the gauge cooling is employed. The method should still be carefully examined in various possible ways.

In the present paper, we apply the complex Langevin method in conjunction with the gauge cooling to the 2D lattice Yang–Mills theory which can be analytically solved [15–17]. By doing this, we examine the validity of the method. The partition function of the 2D Yang–Mills theory on the lattice is given by

$$Z = \int \left[ \prod_{\mathbf{x}, \mu} dU_{\mathbf{x}, \mu} \right] e^{-S},$$  \hspace{1cm} (1.1)

We will shortly describe the Langevin evolution of link variables.
where $U_{x,\mu}$ are link variables defined on a two-dimensional rectangular lattice, $S$ is the lattice action

$$S = -\frac{\beta}{2N} \sum_x \text{Tr} \left[ U_{01}(x) + U_{01}(x)^{-1} \right],$$

and the plaquette variable is defined by

$$U_{\mu\nu}(x) = U_{x,\mu} U_{x,\mu+\nu} U_{x+\nu,\mu} U_{x,\nu}^{-1}. \tag{1.3}$$

For simplicity, we assume that the gauge group is $SU(2)$, that is, $U_{x,\mu} \in SU(2)$ in the original integral (1.1). On the other hand, when the gauge coupling $\beta$ is complex, the corresponding Langevin equation (Eq. (2.1) below) evolves link variables as elements of $SL(2, \mathbb{C})$. Thus, the distinction between $U_{x,\mu}^\dagger$ and $U_{x,\mu}^{-1}$ becomes very important in the complex Langevin dynamics. For the convergence theorem in Refs. [6, 7] to apply, the action $S$ that generates the drift force in the Langevin equation and physical observables must be holomorphic functions of dynamical variables; our above definitions (1.2) and (1.3) that entirely use $U_{x,\mu}^{-1}$ not $U_{x,\mu}^\dagger$ are chosen by this criterion. Note also that the plaquette action (1.2) is invariant under the $SL(2, \mathbb{C})$ lattice gauge transformations (such as the one in Eq. (2.4)).

We will consider the expectation value of the plaquette variable:

$$\langle \text{Tr} [U_{01}(x)] \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_{x,\mu} e^{-S \text{Tr} [U_{01}(x)]}. \tag{1.4}$$

Even if the gauge coupling $\beta$ is complex, this can be exactly computed by the character expansion [13–17]. Under periodic boundary conditions, one yields

$$\langle \text{Tr} [U_{01}(x)] \rangle = \frac{N}{V} \frac{\partial}{\partial \beta} \ln Z, \quad Z = \sum_{n=1}^{\infty} \left[ \frac{2}{\beta} I_n(\beta) \right]^V, \tag{1.5}$$

where $I_n(x)$ denotes the modified Bessel function of the first kind and $V$ is the number of lattice points.

2. Complex Langevin method and the gauge cooling

The following procedures are basically identical to the ones adopted for the 4D lattice QCD in Ref. [11] for example, although our 2D pure-gauge system is much simpler.

For the link variable, the Langevin equation with a discretized Langevin time $t$ with the time step $\epsilon$ is defined by

$$U_{x,\mu}(t + \epsilon) = \exp \left[ i \sum_a \frac{\lambda_a}{\sqrt{\epsilon}} \left( \sqrt{2} \eta_{a,x,\mu}(t) - \epsilon D_{a,x,\mu} S \right) \right] U_{x,\mu}(t), \tag{2.1}$$

where $\lambda_a$ ($a = 1, 2, 3$) are Pauli matrices, $\eta_{a,x,\mu}(t)$ are Gaussian real random numbers of the variant

$$\langle \eta_{a,x,\mu}(t) \eta_{b,y,\nu}(t') \rangle = 2 \delta_{ab} \delta_{xy} \delta_{\mu\nu} \delta_{tt'}, \tag{2.2}$$

and $D_{a,x,\mu} S$ is the drift force generated by the action $S$ in Eq. (1.2); the derivative with respect to the link variable is given by

$$D_{a,x,\mu} f(U) = \frac{\partial}{\partial \xi f(e^{i\xi \lambda_a U_{x,\mu}})} \bigg|_{\xi=0}. \tag{2.3}$$

\(^{2}\)Throughout this paper, $N = 2$.  

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When the gauge coupling $\beta$ is complex, the drift force becomes complex and the Langevin evolution evolves link variables as elements of $SL(2, \mathbb{C})$.

The above complex Langevin dynamics tends to make the probability distribution function of link variables wide in non-compact directions of $SL(2, \mathbb{C})$. To squeeze the distribution well-localized without changing gauge invariant quantities, we apply the following $SL(2, \mathbb{C})$ gauge transformation (this step is the gauge cooling)

$$U_{x,\mu} \rightarrow U'_{x,\mu} = V_x U_{x,\mu} V^{-1}_{x+\hat{\mu}},$$  

(2.4)

where

$$V_x = e^{-\alpha f^x_a \lambda_a}, \quad f^x_a = 2 \text{Tr} \left[ \lambda_a \sum_{\mu} \left( U_{x,\mu} U^\dagger_{x,\mu} - U^\dagger_{x-\hat{\mu},\mu} U_{x-\hat{\mu},\mu} \right) \right],$$  

(2.5)

and $\alpha > 0$ is a real parameter. The distance defined by

$$d = \frac{1}{V} \sum_{x,\mu} \frac{1}{N} \text{Tr} \left( U_{x,\mu} U^\dagger_{x,\mu} - 1 \right) \geq 0$$  

(2.6)

measures how a $SL(2, \mathbb{C})$ gauge field is far away from the subspace of $SU(2)$ gauge fields. It is then straightforward to see that for a sufficiently small $\epsilon$, the gauge cooling (2.4) decreases or does not change the distance $d$. Note that $f^x_a$ in Eq. (2.5) is not a holomorphic function of link variables and thus the step (2.4) cannot be regarded as a part of the complex Langevin dynamics in which the drift force is generated by a holomorphic action; this fact prevents us from applying the convergence theorem [6, 7] to the above procedures.

3. Result of numerical simulations

We numerically solved the Langevin equation with the discretized Langevin time, Eq. (2.1), on a $V = 4^2$ lattice. Periodic boundary conditions are imposed. The maximal size of the time step $\epsilon$ we adopted was 0.001 and, when the drift force becomes large, we further reduced $\epsilon$ “adaptively” according to the prescription in Ref. [19].

As the parameter $\alpha$ in the gauge cooling (2.5), we tried both $\alpha = 1$ and the “adaptive choice” (see Ref. [8])

$$\alpha_{ad} = \frac{1}{D}, \quad D \equiv \frac{1}{V} \sum_{a,x} |f^x_a| + 1.$$  

(3.1)

Our numerical results did not show any notable difference in these two choices and we present the results with the latter choice in what follows.

To determine an appropriate rate of the gauge cooling (2.4) along the Langevin evolution, we observed the time evolution of the distance $d$ (2.6) by changing the number of the gauge cooling steps per one Langevin update (2.1). In Fig. 1 for $\beta = 0.4 + 2.0i$, we plotted the evolution of the distance $d$ as the function of the Langevin time $t$ by changing the number of the gauge cooling steps per one Langevin update as, 10, 30, and 100.$^3$ Since this plot shows the evolution including the Langevin stochastic dynamics, the distance $d$ does not necessarily decrease. Since we do not see much difference for those three choices, we adopted 10 gauge cooling steps per one Langevin update. With this choice, the evolution of the distance $d$ for various complex gauge couplings, $\beta = 0.4 + 0.4i$, $0.4 + 2.0i$, $2.0 + 0.4i$, and $2.0 + 2.0i$, looks as Fig. 2. It appears that the gauge cooling is working perfectly for those complex

$^3$We define the Langevin time $t$ such that it does not elapse during the gauge cooling steps.
Fig. 1 Evolution of the distance $d (2.6)$ for $\beta = 0.4 + 2.0i$ with various numbers of the gauge cooling steps per one Langevin update (2.1), 10, 30, and 100.

Fig. 2 Evolution of the distance $d (2.6)$ for various complex gauge couplings, $\beta = 0.4 + 0.4i$, $0.4 + 2.0i$, $2.0 + 0.4i$, and $2.0 + 2.0i$. The number of the gauge cooling steps per one Langevin update (2.1) is 10.

gauge couplings, suppressing the evolution to non-compact imaginary directions.

Now, we turn to the computation of the expectation value of the plaquette, Eq. (1.4), by the complex Langevin method. Starting from a configuration of random $SU(2)$ matrices, we discarded configurations until the Langevin time $t = 11$ for thermalization. Then 1000 configurations separated by $\Delta t = 1$ from $t = 11$ to $t = 1010$ are used to compute the expectation value. For typical values of the complex gauge coupling, we confirmed that the plaquette values between configurations separated by $\Delta t = 1$ practically have no autocorrelation. In
Figs. 3 and 4, we plotted the real and imaginary parts of the expectation value (1.4) obtained by the complex Langevin (CL) method. The error bars are statistical ones. The horizontal axis is the the complex phase $\theta$ of the gauge coupling with the modulus 1.5:

$$\beta = 1.5 e^{i\theta}, \quad 0 \leq \theta \leq \pi/2.$$  \hspace{1cm} (3.2)

The solid line curves are exact values given by Eq. (1.5). We see that the complex Langevin method reproduces the real part fairly well, while it clearly fails to converge to the correct value of the imaginary part when the complex phase of the gauge coupling is large.

The gradation plot in Fig. 5 shows the relative error

$$\frac{|\langle \text{Tr} [U_{01}(x)] \rangle_{\text{CL}} - \langle \text{Tr} [U_{01}(x)] \rangle_{\text{exact}}|}{|\langle \text{Tr} [U_{01}(x)] \rangle_{\text{exact}}|}$$  \hspace{1cm} (3.3)

on the first quadrant of the complex $\beta$ plane. The (quadrant) circle in the figure is Eq. (3.2) along which Figs. 3 and 4 are plotted. Clearly, the relative error of the complex Langevin method becomes large when the complex phase of the gauge coupling becomes large. Four black crosses in the figure indicate complex gauge couplings we used in Fig. 2; the behavior in Fig. 2 thus suggests that the gauge cooling is correctly working for the region of the complex gauge coupling shown in Fig. 5. Nevertheless, the complex Langevin method shows large deviation from the correct value as in Fig. 4. This is the main result of the present paper.

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4 When the gauge group is $SU(2)$, the partition function (1.1) is invariant under $\beta \rightarrow -\beta$. Because of this invariance and of the complex conjugation, it is sufficient to consider the range of the phase, $0 \leq \theta \leq \pi/2$.

5 The block around the origin $\beta = 0$ is omitted from the plot because $\langle \text{Tr} [U_{01}(x)] \rangle \sim 0$ for $\beta \sim 0$. 

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Fig. 3  Real part of the expectation value (1.4) obtained by the complex Langevin (CL) method and the exact value given by Eq. (1.5). The horizontal axis is the complex phase $\theta$ of the gauge coupling in Eq. (3.2).
Fig. 4 Imaginary part of the expectation value (1.4) obtained by the complex Langevin (CL) method and the exact value given by Eq. (1.5). The horizontal axis is the complex phase $\theta$ of the gauge coupling in Eq. (3.2).

Fig. 5 Gradation plot of the relative error (3.3) on the complex $\beta$ (the gauge coupling) plane. The block around the origin $\beta = 0$ is omitted from the plot. The quadrant is Eq. (3.2) along which Figs. 3 and 4 are plotted. Four black crosses indicate complex gauge couplings we used in Fig. 2.

It is of interest how configurations generated by the Langevin dynamics distribute in configuration space. To give some idea on this point, in Figs. 6 and 7 we present scatter plots of the plaquette variable (averaged over the lattice volume) for each configuration. Both cases, with and without the gauge cooling, are shown. Fig. 6 is for $\beta = 1.5 e^{i(0.3\pi/2)}$ (i.e., $\theta = 0.3\pi/2$) and corresponds to points in Figs. 3 and 4 with a relatively small complex
Fig. 6 Distribution of the plaquette variable averaged over the lattice volume. $\beta = 1.5 e^{i(0.3\pi/2)} (\theta = 0.3\pi/2)$.

Fig. 7 Distribution of the plaquette variable averaged over the lattice volume. $\beta = 1.5 e^{i(0.7\pi/2)} (\theta = 0.7\pi/2)$.

phase. Fig. 7 is, on the other hand, for $\beta = 1.5 e^{i(0.7\pi/2)}$ (i.e., $\theta = 0.7\pi/2$) and corresponds to points in Figs. 3 and 4 with a large complex phase and with large deviation. Although there is a tendency that when the complex phase of the gauge coupling is large the distribution becomes somewhat wider even after the gauge cooling, it is not clear from the scatter plot in Fig. 4 alone whether the distribution is so poor-localized as to break the prerequisite of the convergence theorem [6, 7]. More detailed study is needed on this point.
4. Conclusion

In the present paper, we applied the complex Langevin method in conjunction with the gauge cooling to the 2D lattice SU(2) Yang–Mills theory. Our intention was to examine the validity of the method by using this analytically solvable model. Somewhat unexpectedly, as shown in Figs. 4 and 5, we obtained strong numerical evidence that the method fails to converge to the correct value when the complex phase of the gauge coupling is large. As we emphasized in Introduction, the convergence proof of Refs. 6, 7 does not necessarily apply when the gauge cooling is employed; thus there is no contradiction even if the method leads to a wrong answer. Nevertheless, it is not yet clear what causes the failure for the gauge coupling with a large complex phase. To find the resolution of the problem we found in the present study, first we have to pin down what is the real source of the failure. For this, consideration on the basis of another approach to the functional integral with complex measure, the Lefschetz thimble 20–24, might provide useful insight. See also Ref. 25 for suggestive observation.

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References

[1] P. de Forcrand, PoS LAT 2009, 010 (2009) [arXiv:1005.0539 [hep-lat]].
[2] C. Gattringer, PoS LATTICE 2013, 002 (2014) [arXiv:1401.7788 [hep-lat]].
[3] G. Parisi, Phys. Lett. B 131, 393 (1983).
[4] J. R. Klauder, J. Phys. A 16, L317 (1983).
[5] J. R. Klauder, Phys. Rev. A 29, 2036 (1984).
[6] G. Aarts, E. Seiler and I. O. Stamatescu, Phys. Rev. D 81, 054508 (2010) [arXiv:0912.3360 [hep-lat]].
[7] G. Aarts, F. A. James, E. Seiler and I. O. Stamatescu, Eur. Phys. J. C 71, 1756 (2011) [arXiv:1101.3270 [hep-lat]].
[8] G. Aarts, L. Bongiovanni, E. Seiler, D. Sexty and I. O. Stamatescu, Eur. Phys. J. A 49, 89 (2013) [arXiv:1303.6425 [hep-lat]].
[9] G. Aarts and F. A. James, JHEP 1008, 020 (2010) [arXiv:1005.3468 [hep-lat]].
[10] E. Seiler, D. Sexty and I. O. Stamatescu, Phys. Lett. B 723, 213 (2013) [arXiv:1211.3709 [hep-lat]].
[11] D. Sexty, Phys. Lett. B 729, 108 (2014) [arXiv:1307.7748 [hep-lat]].
[12] A. Mollgaard and K. Splittorff, Phys. Rev. D 88, no. 11, 116007 (2013) [arXiv:1309.4335 [hep-lat]].
[13] K. Splittorff, arXiv:1412.0502 [hep-lat].
[14] A. Mollgaard and K. Splittorff, Phys. Rev. D 91, no. 3, 036007 (2015) [arXiv:1412.2729 [hep-lat]].
[15] R. Balian, J. M. Drouffe and C. Itzykson, Phys. Rev. D 10, 3376 (1974).
[16] A. A. Migdal, Sov. Phys. JETP 42, 413 (1975) [Zh. Eksp. Teor. Fiz. 69, 810 (1975)].
[17] H. J. Rothe, World Sci. Lect. Notes Phys. 43, 1 (1992) [World Sci. Lect. Notes Phys. 59, 1 (1997)] [World Sci. Lect. Notes Phys. 74, 1 (2005)] [World Sci. Lect. Notes Phys. 82, 1 (2012)].
[18] G. Aarts and I. O. Stamatescu, JHEP 0809, 018 (2008) [arXiv:0807.1597 [hep-lat]].
[19] G. Aarts, F. A. James, E. Seiler and I. O. Stamatescu, Phys. Lett. B 687, 154 (2010) [arXiv:0912.0617 [hep-lat]].
[20] M. Cristoforetti et al. [AuroraScience Collaboration], Phys. Rev. D 86, 074506 (2012) [arXiv:1205.3906 [hep-lat]].
[21] M. Cristoforetti, F. Di Renzo, A. Mukherjee and L. Scorzato, Phys. Rev. D 88, no. 5, 051501 (2013) [arXiv:1303.7204 [hep-lat]].
[22] H. Fujii, D. Honda, M. Kato, Y. Kikukawa, S. Komatsu and T. Sano, JHEP 1310, 147 (2013) [arXiv:1309.4371 [hep-lat]].
[23] G. Aarts, L. Bongiovanni, E. Seiler and D. Sexty, JHEP 1410, 159 (2014) [arXiv:1407.2090 [hep-lat]].
[24] T. Kanazawa and Y. Tanizaki, arXiv:1412.2802 [hep-th].
[25] A. Cherman, P. Koroteev and M. Ünsal, arXiv:1410.0385 [hep-th].