COMMITING MATRICES AND THE HILBERT SCHEME OF
POINTS ON AFFINE SPACES

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ABSTRACT. We give a linear algebraic and a monadic descriptions of the
Hilbert scheme of points on the affine space of dimension n which naturally ex-
tends Nakajima’s representation of the Hilbert scheme of points on the plane.
As an application of our ideas and recent results from the literature on commut-
ing matrices, we show that the Hilbert scheme of c points on (C^3) is irreducible
for c ≤ 10.

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1. INTRODUCTION

The Hilbert scheme Hilb[c](C^n) of c points in the affine space of dimension n
parametrizes 0-dimensional subschemes of C^d of length c. The case of n = 2 is
much studied, though less is known about the higher dimensional cases.

The linear algebraic and monadic descriptions of the n = 2 case given by Naka-
jima in [16, Chapters 1 & 2] is particularly relevant to us. One of the goals of this
paper is to give analogous descriptions of the Hilbert scheme Hilb[c](C^n) of c points
on C^n, naturally extending Nakajima’s representation of the Hilb[c](C^2). This goal
is attained in the first part of the paper, Sections 2 through 7.

More precisely, let V and W be complex vector spaces of dimension c and 1,
respectively. Let B_1, . . . , B_n be operators on V commuting with each other and
consider a map $I : W \to V$. The $(n+1)$-tuple $(B_1, \ldots, B_n, I)$ is said to be stable if there is no proper subspace $S \subset V$ which is invariant under each operator $B_k$ and contains the image of $I$. The group $GL(V)$ acts on the set of all such $(n+1)$-tuple by change of basis on $V$.

We prove that there is a one-to-one corresponding between the following objects:

1. ideals $J$ in the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ whose quotient has dimension $c$;
2. stable $(n+1)$-tuple $(B_1, \ldots, B_n, I)$ with $\dim V = c$, modulo the action of $GL(V)$;
3. complexes of the form, called perfect extended monads:

$$V_{-n} \otimes \mathcal{O}_P(1-n) \overset{\alpha_{-n}}{\longrightarrow} V_{-n} \otimes \mathcal{O}_P(2-n) \overset{\alpha_{-1}}{\longrightarrow} \cdots \overset{\alpha_{2-n}}{\longrightarrow} V_0 \otimes \mathcal{O}_P(n) \overset{\alpha_0}{\longrightarrow} V_1 \otimes \mathcal{O}_P(1)$$

where $V_1 := V$, $V_0 = V^{\oplus n} \otimes \mathbb{C}$ and $V_i = V^{\oplus(i-n)}$ for $i < 0$, which are exact everywhere except at degree 0 (grading of the complex is given by the twisting).

Furthermore, using the above correspondence, we also show that the $\text{Hilb}^{[c]}(\mathbb{C}^n)$ is isomorphic (as a scheme) to a GIT quotient of $\mathcal{C}(n,c) \times \text{Hom}(W,V)$ by $GL(V)$, where $\mathcal{C}(n,c)$ denotes the variety of $n$ commuting $c \times c$ matrices.

The correspondence between items (1) and (2) as well as the isomorphism between Hilbert scheme and the GIT quotient of $\mathcal{C}(n,c) \times \text{Hom}(W,V)$ by $GL(V)$ are already present in the representation theory literature, see for instance [19, 20] Appendix by M. V. Nori and [22], and more recently [6, 9, 23]. However, our presentation is more down-to-earth and closer to Nakajima’s description which is familiar to algebraic geometers.

The correspondence with the so-called perfect extended monads is new. In fact, we introduce in Section 4 a new class of objects, extended monads (cf. Definition 4.1), which generalize the usual monads originally introduced by Horrocks in the 1960’s [11] and much studied by several authors since then. The basic theory of extended monads is developed here, with a focus on what we call perfect extended monads. We provide a cohomological characterization of the sheaves that arise as cohomology of a perfect extended monad on projective spaces (see Proposition 4.8 below), showing, in particular, that ideal sheaves of zero dimensional subschemes do satisfy the required conditions.

We complement our discussion on the parametrization via linear algebra of the Hilbert scheme of points on affine varieties in two ways. First, in Section 8 we provide a description of the Hilbert–Chow morphism from $\text{Hilb}^{[c]}(\mathbb{C}^n)$ to the symmetric product of $c$ copies of $\mathbb{C}^n$ in terms of our linear data. Second, we provide a linear algebraic description of Hilbert scheme $\text{Hilb}^{[c]}(\mathcal{Y})$ of $c$ points on an affine variety $\mathcal{Y} \subset \mathbb{C}^n$. More precisely, suppose that $\mathcal{Y}$ is given by algebraic equations $f_1 = \cdots = f_l = 0$; we show in Section 9 that a point in $\text{Hilb}^{[c]}(\mathcal{Y})$ corresponds to a stable $(n+1)$-tuple $(B_1, \ldots, B_n, I)$ such that $f_k(B_1, \ldots, B_n) = 0$ for each $k = 1, \ldots, l$. Moreover, such correspondence yields a schematic isomorphism between $\text{Hilb}^{[c]}(\mathcal{Y})$ and the variety of commuting matrices satisfying $f_k(B_1, \ldots, B_n) = 0$ plus a vector, modulo $GL(V)$.

Finally, as an application of our ideas, it is not difficult to see that $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible whenever $\mathcal{C}(n,c)$ is irreducible, see details in Section 10 below. It then follows from recent results due to Sivic [21] Theorems 26 & 32, that $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is
irreducible if \( c \leq 10 \), while this was known to be the case only for \( c \leq 8 \), cf. [2, Theorem 1.1].

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2. Commuting matrices and stable ADHM data

In this section we shall introduce the necessary material to our construction: let \( V \) be a complex vector space of dimension \( c \) and let \( B_0, B_1, \ldots, B_{n-1} \in \text{End}(V) \) be \( n \) linear operators on \( V \).

**Definition 2.1.** The variety \( C(n, c) \) of \( n \) commuting linear operators on \( V \) is the subvariety of \( \text{End}(V)^{\oplus n} \) whose points are the set of \( n \)-tuples \((B_0, B_1, \ldots, B_{n-1})\) that commutes two by two, that is,

\[
C(n, c) = \{(B_0, B_1, \ldots, B_{n-1}) \in \text{End}(V)^{\oplus n} \mid [B_i, B_j] = 0, \forall i \neq j \}
\]

The commutation relations can be thought of as a system of \( \binom{n}{2} c^2 \) homogeneous equations of degree 2 in \( nc^2 \) variables.

Let \( W \) be a 1-dimensional complex vector space; one can form the space

\[
\mathcal{B} := \text{End}(V)^{\oplus n} \oplus \text{Hom}(W, V)
\]

whose points are represented by the \((n + 1)\)-tuple \( X = (B_0, B_1, \ldots, B_{n-1}, I) \) that will be called an \textit{ADHM datum}. We then define the variety of ADHM data \( \mathcal{V}(n, c) \) as the subvariety of \( \mathcal{B} \) given by

\[
\mathcal{V}(n, c) := C(n, c) \times \text{Hom}(W, V).
\]

**Definition 2.2.** An ADHM datum \( X = (B_0, B_1, \ldots, B_{n-1}, I) \in \mathcal{B} \) is said to be stable if there is no proper subspace \( S \subsetneq V \) such that

\[
B_0(S), B_1(S), \ldots, B_{n-1}(S), I(W) \subset S.
\]

The set of stable points in \( \mathcal{B} \) will be denoted by \( \mathcal{B}^{st} ; \mathcal{V}(n, c)^{st} := \mathcal{B}^{st} \cap \mathcal{V}(n, c) \) will denote the set of stable points in \( \mathcal{V}(n, c) \).

**Definition 2.3.** The stabilizing subspace \( \Sigma_X \) of an ADHM datum \( X \in \mathcal{B} \) is the intersection of all subspaces \( S \subset V \) such that \( B_0(S), B_1(S), \ldots, B_{n-1}(S), I(W) \subset S \).

It is easy to see that \( X \) is stable if and only if \( \Sigma_X = V \). The restricted ADHM datum \( X|_{\Sigma_X} = (B_0|_{\Sigma_X}, B_1|_{\Sigma_X}, \ldots, B_{n-1}|_{\Sigma_X}, I|_{\Sigma_X}) \) is stable in \( \mathcal{B}|_{\Sigma_X} = \text{End}(\Sigma_X)^{\oplus n} \oplus \text{Hom}(W, \Sigma_X) \). The space \( \Sigma_X \subset V \) is the smallest subspace which makes the datum \( X|_{\Sigma_X} \) stable, hence the name.

For each ADHM datum \( X = (B_0, \ldots, B_{n-1}, I) \in \mathcal{B} \), we consider the linear map

\[
\mathcal{R}_n(X) : W^{\oplus n} \rightarrow V
\]

defined by

\[
\mathcal{R}_n(X) : \bigoplus_{k_0, \ldots, k_{n-1}=0}^{c-1} w_{k_0, \ldots, k_{n-1}} \rightarrow \sum_{k_0, \ldots, k_{n-1}=0}^{c-1} B_0^{k_0} B_1^{k_1} \ldots B_{n-1}^{k_{n-1}} I w_{k_0, \ldots, k_{n-1}}
\]
One might think of $\mathcal{R}_n$ as a regular morphism $\mathbb{B} \to \text{Hom}(W^{\otimes e}, V)$, hence continuous in the Zariski topology.

**Proposition 2.4.** For every $X \in \mathbb{B}$ one has

1. $\text{Im } \mathcal{R}_n(X) \subseteq \Sigma_X$;
2. if $\mathcal{R}_n(X)$ is surjective, then $X$ is stable.

**Proof.** For any $S \subseteq V$ satisfying $B_0(S), B_1(S), \ldots, B_{n-1}(S), I(W) \subseteq S$ we have $\text{Im } \mathcal{R}_n(X) \subseteq S$ which in particular implies our first assertion.

Moreover, if $\mathcal{R}_n(X)$ is surjective then we have $c = \text{rk } \mathcal{R}_n(X) \leq \dim \Sigma_X \leq c$. Therefore $\dim \Sigma_X = c$ and $X$ is stable. \qed

If the ADHM datum $X$ is in $\mathcal{V}(n,c)$, then we obtain the following stronger characterization.

**Proposition 2.5.** For every datum $X = (B_0, \ldots, B_{n-1}, I) \in \mathcal{V}(n,c)$ one has:

1. $\text{Im } \mathcal{R}_n(X) = \Sigma_X$;
2. $\mathcal{R}_n(X)$ is surjective if and only if $X$ is stable.

**Proof.** Note first that the second claim follows easily from Proposition 2.4 and the first claim.

To prove the first claim, we only need to prove the inverse inclusion $\Sigma_X \subseteq \text{Im } \mathcal{R}_n(X)$ for those ADHM data which belong to $\mathcal{V}(n,c)$ since the inclusion $\text{Im } \mathcal{R}_n(X) \subseteq \Sigma_X$ holds for all $X \in \mathbb{B}$.

For this end, it is enough show that $\text{Im } \mathcal{R}_n(X)$ is $B_i$-invariant, for all $i \in \{0, \ldots, n-1\}$, and $I(W) \subseteq \text{Im } \mathcal{R}_n(X)$. It is easy to see that $I(W) \subseteq \text{Im } \mathcal{R}_n(X)$, so the results follows by showing that $\text{Im } \mathcal{R}_n(X)$ is $B_i$-invariant, for all $i \in \{0, \ldots, n-1\}$; let

$$\sum_{k_0, \ldots, k_{n-1}=0}^{c-1} B_0^{k_0} B_1^{k_1} \cdots B_{n-1}^{k_{n-1}} I w_{k_0, \ldots, k_{n-1}} \in \text{Im } \mathcal{R}_n(X).$$

For $X \in \mathcal{V}(n,c)$ one has the following identity

$$B_i \left( \sum_{k_0, \ldots, k_{n-1}=0}^{c-1} B_0^{k_0} B_1^{k_1} \cdots B_{n-1}^{k_{n-1}} I w_{k_0, \ldots, k_{n-1}} \right) = \sum_{k_0, \ldots, k_{n-1}=0}^{c-1} B_i B_0^{k_0} B_1^{k_1} \cdots B_{n-1}^{k_{n-1}} I w_{k_0, \ldots, k_{n-1}}$$

$$= \sum_{k_0, \ldots, k_{n-1}=0}^{c-1} B_0^{k_0} \cdots B_i^c \cdots B_{n-1}^{k_{n-1}} I w_{k_0, \ldots, k_{n-1}}$$

$$+ \sum_{k_0, \ldots, k_{i-1}, k_i=0}^{c-1} \sum_{k_{i+1}, \ldots, k_{n-1}=0}^{c-1} B_0^{k_0} \cdots B_i^k \cdots B_{n-1}^{k_{n-1}} I w_{k_0, \ldots, k_{n-1}}.$$
\( a_1 x + a_0 \). Hence, by Cayley-Hamilton Theorem, it follows that \( B_i^c \) is given by a linear combination of lower powers of \( B_i \), i.e., \( B_i^c = -(a_{c-1} B_i^{c-1} + \ldots + a_1 B_i + a_0) \).

With this, we conclude that \( \sum_{k_0, \ldots, k_n = 0}^{k_0, \ldots, k_n = 0} B_i^{k_0} B_i^{k_1} \ldots B_i^{k_n} \) \( I W_{k_0, \ldots, k_n} \in \text{Im } \mathcal{R}_n(X) \) which, in particular means, that \( \text{Im } \mathcal{R}_n(X) \) is \( B_i \)-invariant. Finally, since \( \Sigma_X \) is the smallest subspace of \( V \) with these properties, it then follows that \( \Sigma_X \subseteq \text{Im } \mathcal{R}_n(X) \).

\[ \square \]

Next, we introduce the action of the linear group \( G := GL(V) \) on \( \mathbb{B} \). For all \( g \in G \) and \( X = (B_0, \ldots, B_{n-1}, I) \in \mathbb{B} \), this action is given by

\[ g \cdot (B_0, \ldots, B_{n-1}, I) = (g B_0 g^{-1}, \ldots, g B_{n-1} g^{-1}, g I). \]

For a fixed ADHM datum \( X \), we will denote by \( G_X \) its stabilizer subgroup:

\[ G_X := \{ g \in G \mid g X = X \} \subseteq G. \]

It is easy to see that \( X \) is stable if and only if \( g X \) is stable, and that \( G \) acts on \( \mathcal{V}(n, c) \).

We conclude this section with two results relating stability in the sense of Definition 2.2 with GIT stability.

**Proposition 2.6.** If \( X \in \mathcal{V}(n, c)^{st} \), then its stabilizer subgroup \( G_X \) is trivial.

**Proof.** Let \( X = (B_0, \ldots, B_{n-1}, I) \) be a stable ADHM datum and suppose that there exists an element \( g \neq 1 \) in \( G \) such that \( g I = I \) and \( g B_i g^{-1} = B_i \) for all \( i \in \{0, \ldots, n-1\} \). Then \( \ker(g - 1) \) is \( B_i \)-invariant, for all \( i \in \{0, \ldots, n-1\} \), and \( \text{Im } I \subseteq \ker(g - 1) \). Since \( X \) is stable, then \( \ker(g - 1) \subseteq V \) must be the trivial subspace. Hence \( g \) must be the identity. \( \square \)

Let \( \Gamma(\mathcal{V}(n, c)) \) be the ring of regular functions on \( \mathcal{V}(n, c) \). Fix \( l > 0 \), and consider the group homomorphism \( \chi : G \to \mathbb{C}^* \) given by \( \chi(g) = (\det g)^l \). This can be used for the construction of a suitable linearization of the \( G \)-action on \( \mathcal{V}(n, c) \), that is, to lift the action of \( G \) on \( \mathcal{V}(n, c) \) to an action on \( \mathcal{V}(n, c) \times \mathbb{C} \) as follows:

\[ g \cdot (X, z) := (g \cdot X, \chi(g)^{-1} z) \]

for any ADHM datum \( X \in \mathcal{V}(n, c) \) and \( z \in \mathbb{C} \). Then one can form the scheme

\[ \mathcal{V}(n, c) \times \chi G := \text{Proj} \left( \bigoplus_{i \geq 0} \Gamma(\mathcal{V}(n, c))^{G, \chi_i} \right) \]

where

\[ \Gamma(\mathcal{V}(n, c))^{G, \chi_i} := \{ f \in \Gamma(\mathcal{V}(n, c)) \mid f(g \cdot X) = \chi(g)^{-1} \cdot f(X), \quad \forall g \in G \}. \]

The scheme \( \mathcal{V}(n, c) \times \chi G \) is projective over the ring \( \Gamma(\mathcal{V}(n, c))^G \) and quasi-projective over \( \mathbb{C} \).

**Proposition 2.7.** The orbit \( G \cdot (X, z) \) is closed, for \( z \neq 0 \), if and only if the ADHM datum \( X \in \mathcal{V}(n, c) \) is a stable.

**Proof.** First, suppose that the orbit \( G \cdot (X, z) \) is not closed, then there is a 1-parameter subgroup \( \lambda : \mathbb{C}^* \to G \) such that the limit \( (L, w) := \lim_{t \to 0} \lambda(t) \cdot (X, z) \) exists but does not belong to the orbit \( G \cdot (X, z) \). The existence of the limit \( (L, w) \) implies that \( \det(\lambda(t)) = t^N \) for some \( N \leq 0 \). If \( N = 0 \) then \( \lambda(t) = I_V \), which contradicts the fact that the limit does not belong to the orbit \( G \cdot (X, z) \). Thus
Now, take the weight decomposition $V = \bigoplus_{m \geq 0} V(m)$ of the space $V$, with respect to $\lambda$. Then the existence of a limit implies that

$$B_0(V(m)), B_1(V(m)), \ldots, B_{n-1}(V(m)) \subseteq \bigoplus_{m \geq 0} V(m) \text{ and } I(W) \subseteq \bigoplus_{m \geq 0} V(m).$$

The space $S := \bigoplus_{n \geq 0} V(m) \in V$ is proper since $N < 0$. Moreover, one has $B_0(S), B_1(S), \ldots, B_{n-1}(S) \subseteq S$ and $I(W) \subseteq S$. Hence $X$ is not stable.

Conversely, suppose that the ADHM datum $X = (B_0, B_1, \ldots, B_{n-1}, I)$ is not stable. Then there exists a subspace $S \subseteq V$ such that $B_0(S), B_1(S), \ldots, B_{n-1}(S) \subseteq S$ and $I(W) \subseteq S$. Let $T \subseteq V$ be a subspace such that $V = S \oplus T$. With respect to this decomposition, one can write the linear maps $B_i$, for $0 \leq i \leq n - 1$ and $I$, as follows

$$B^i = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \text{ for } 0 \leq i \leq n - 1 \quad I = \begin{pmatrix} * \\ 0 \end{pmatrix}.$$ 

Now, define the 1-parameter subgroup as

$$\lambda(t) = \begin{pmatrix} \mathbb{I}_S & 0 \\ 0 & t^{-1}\mathbb{I}_T \end{pmatrix},$$

then

$$\lambda(t)B_i\lambda(t)^{-1} = \begin{pmatrix} * & t* \\ 0 & * \end{pmatrix} \text{ and } \lambda(t)I = I.$$ 

It follows that the limit $L = \lim_{t \to 0} \lambda(t) \cdot X$ exists and $\lim_{t \to 0} \lambda(t) \cdot (X, z) = (L, 0)$, which means that the orbit is closed within $\mathcal{V}(n, c) \times \mathbb{C}^*$. \hfill \Box

From Propositions 2.6 and 2.7 and since the group $G$ is reductive, it follows that the quotient space $\mathcal{M}(n, c) := \mathcal{V}(n, c)/G$ is a good categorical quotient [15, Thm. 1.10]. Furthermore, GIT tells us that the GIT quotient $\mathcal{M}(n, c)$ is the space of orbits $G \cdot X \subseteq \mathcal{V}(n, c)$ such that the lifted orbit $G \cdot (X, z)$ is closed within $\mathcal{V}(n, c) \times \mathbb{C}$ for all $z \neq 0$. We conclude therefore, from Proposition 2.7, that

$$\mathcal{M}(n, c) = \mathcal{V}(n, c)^{\text{ad}}/G.$$ 

3. Parametrization of the Hilbert Scheme of $c$ Points in $\mathbb{C}^n$

As a set, the Hilbert scheme of $c$ points on $\mathbb{C}^n$ is given by:

$$\text{Hilb}^c(\mathbb{C}^n) = \{ I \in \mathbb{C}[z_0, \ldots, z_{n-1}] | \dim_{\mathbb{C}}(\mathbb{C}[z_0, \ldots, z_{n-1}]/I) = c \}.$$ 

The existence of its schematic structure is a special case of the general result of Grothendieck [8]. Another explicit construction of the Hilbert scheme of points on the affine plane is given by Nakajima [16]. The reader may also consult [17] for more general results and examples.

The aim of this section is to prove the following result

**Theorem 3.1.** There exists a set-theoretical bijection between the quotient space $\mathcal{M}(n, c)$ and the Hilbert scheme of $c$ points in $\mathbb{C}^n$.

We remark that this result will be strengthen, in Section [7] below, to a scheme theoretic isomorphism rather than just a bijective correspondence. Before proving the above result it will be useful to, first, establish a few lemmata.
Lemma 3.2. If \( X = (B_0, \ldots, B_{n-1}, I) \in \mathcal{V}(n,c) \) is a stable ADHM datum, then the map:

\[
\Phi_X : \mathbb{C}[Z_0, \ldots, Z_{n-1}] \rightarrow V
\]

\[
p(Z_0, \ldots, Z_{n-1}) \mapsto p(B_0, \ldots, B_{n-1})I(1)
\]

is a surjective linear transformation. In particular, \( \mathbb{C}[Z_0, \ldots, Z_{n-1}] / \ker \Phi_X \) is isomorphic to \( V \).

Proof. Observe that \( \text{Im} \ I \subseteq \text{Im} \ \Phi_X \) since the elements of \( \text{Im} \ I \) consist of vectors of the form \( \alpha I(1) \), for some constant \( \alpha \in \mathbb{C} \). The inverse image of such an element is simply the constant polynomial \( \alpha \) itself. Moreover, the image \( \text{Im} \ \Phi_X \) of the map \( \Phi_X \) is \( B_i \)-invariant, for all \( 0 \leq i \leq n-1 \), since all the \( B_i \)'s commute. By stability of the ADHM datum \( X \) we must have \( \text{Im} \ \Phi_X = V \), and hence \( \Phi_X \) is surjective.

It is clear that \( \ker \Phi_X \subset \mathbb{C}[Z_0, \ldots, Z_{n-1}] \) is an ideal. Now, given any two polynomials \( p(Z_0, \ldots, Z_{n-1}) \in \mathbb{C}[Z_0, \ldots, Z_{n-1}] \) and \( q(Z_0, \ldots, Z_{n-1}) \in \ker \Phi_X \), one has \( \Phi_X(p(Z_0, \ldots, Z_{n-1})q(Z_0, \ldots, Z_{n-1})) = p(B_0, \ldots, B_{n-1})q(B_0, \ldots, B_{n-1})I(1) = 0 \). Hence the isomorphism \( \mathbb{C}[Z_0, \ldots, Z_{n-1}] / \ker \Phi_X \cong V \).

\[ \square \]

Let \( \pi : \mathcal{V}(n,c) \rightarrow \mathcal{M}(n,c) \) be the natural projection onto the orbit space \( \mathcal{M}(n,c) \) and denote by \( [X] = [(B_0, \ldots, B_{n-1}, I)] \) the class of the ADHM datum \( X = (B_0, \ldots, B_{n-1}, I) \) in \( \mathcal{V}(n,c) \), that is, \( [X] = \pi(X) \).

Lemma 3.3. Let \( X, Y \in \mathcal{V}(n,c) \) be two stable ADHM data such that \( [X] = [Y] \). Then \( \ker \Phi_X \cong \ker \Phi_Y \).

Proof. Suppose that \( [X] = [(B_0, \ldots, B_{n-1}, I)] = [Y] = [(A_0, \ldots, A_{n-1}, J)] \), then there exists an element \( g \in GL(V) \) such that \( A_i = gB_ig^{-1} \), for all \( 0 \leq i \leq n-1 \) and \( J = gI \). Now, for any polynomial \( f \in \mathbb{C}[Z_0, \ldots, Z_{n-1}] \) one has \( f(A_0, \ldots, A_{n-1}) = gf(B_0, \ldots, B_{n-1})g^{-1} \), hence

\[
f(A_0, \ldots, A_{n-1})J(1) = gf(B_0, \ldots, B_{n-1})g^{-1}(gI(1)) = gf(B_0, \ldots, B_{n-1})I(1),
\]

in other words, \( \Phi_Y = g\Phi_X \). Since \( g \) is invertible, it then follows that \( \ker \Phi_X \cong \ker \Phi_Y \).

\[ \square \]

Lemma 3.4. The ADHM datum \( X = (B_0, \ldots, B_{n-1}, I) \in \mathcal{V}(n,c) \) is stable if and only if the set \( \{B_0^{i_0} \cdot B_1^{i_1} \cdots B_{n-1}^{i_{n-1}} I(1) \in V \mid i_k = 0, \ldots, c - 1 \} \) spans \( V \) as a complex vector space.

Proof. The result follows from item (2) of Proposition 2.3 and the fact that the set \( \{B_0^{i_0} \cdot B_1^{i_1} \cdots B_{n-1}^{i_{n-1}} I(1) \in V \mid i_k = 0, \ldots, c - 1 \} \) spans \( \text{Im} \ \mathcal{R}_n(X) \).

\[ \square \]

We are finally in position to complete the Proof of Theorem 3.1.

Proof of Theorem 3.1. We will consider the map

\[
\Psi : \mathcal{M}(n,c) \rightarrow \text{Hilb}^c(\mathbb{C}^n)
\]

\[
[X] \mapsto \ker \Phi_X
\].
which associates the ideal \( \ker \Phi_X \) to the class \([X] = [(B_0, \ldots, B_{n-1}, I)]\) of a stable ADHM datum \( X = (B_0, \ldots, B_{n-1}, I) \in \mathcal{V}(n, c)^{st} \). By Lemma 3.3, the map \( \Psi \) is well-defined and it is clear from lemma 3.2 that \( \ker \Phi_X \) belong to \( \text{Hilb}^{[c]}(\mathbb{C}^n) \).

Inversely, we define the map

\[
\Psi': \text{Hilb}^{[c]}(\mathbb{C}^n) \rightarrow \mathcal{M}(n, c) \\
J \mapsto [(B_0, \ldots, B_{n-1}, I)]
\]

from \( \text{Hilb}^{[c]} \) to \( \mathcal{M}(n, c) \) as the following:

Given an ideal \( J \in \text{Hilb}^{[c]}(\mathbb{C}^n) \) we denote by \( V = \mathbb{C} [Z_0, \ldots, Z_{n-1}] / J \) the vector space associated to it. The multiplication by \( Z_i \) mod \( J \) define endomorphisms \( B_i \in \text{End}(V) \), for \( 0 \leq i \leq n-1 \), in the following way

\[
B_i: [p(Z_0, \ldots, Z_{n-1})] \mapsto [Z_i p(Z_0, \ldots, Z_{n-1})]
\]

One can also define \( I \in \text{Hom}(W, V) \) as the linear mapping which associates to the unit vector \( 1 \in W \) the class \( 1 \) mod \( J \in V \). Since all \( B_i's \) commute, then \( (B_0^{(i)}, \ldots, B_{n-1}^{(i)}) \in \mathcal{V}(n, c) \). Moreover, the set

\[
\begin{align*}
\{ B_0^{(i)} \cdots B_i^{(i)} \cdots B_{n-1}^{(i)} | & \quad i_k = 0, \ldots, c-1 \}
\end{align*}
\]

spans \( V \) as complex vector space. Therefore, by Lemma 3.3 the ADHM datum \( X \) is stable.

To complete the proof, we only have to show that \( \Psi' \circ \Psi = 1_{\mathcal{M}(n, c)} \) and \( \Psi \circ \Psi' = 1_{\text{Hilb}^{[c]}} \), i.e., the maps \( \Psi \) and \( \Psi' \) are inverse to each other.

Indeed, to each class \([X] \in \mathcal{M}(n, c)\), one associates the ideal \( \Psi([X]) = \ker \Phi_X \) in \( \text{Hilb}^{[c]}(\mathbb{C}^n) \). Moreover, one associates to the later ideal, \( \ker \Phi_X \), the ADHM datum class \( \Psi(\ker \Phi_X) = [\tilde{X}] \). Then one has \([X] = [\tilde{X}] \) if and only if there exists an element \( g \in \text{GL}(V) \) such that \( \tilde{X} = g \cdot X \).

Let \( pr: \mathbb{C}[Z_0, \ldots, Z_{n-1}] \rightarrow \mathbb{C}[Z_0, \ldots, Z_{n-1}] / \ker \Phi_X \) be the natural projection. From Lemma 3.2 it is clear that the diagram

\[
\begin{array}{ccc}
\mathbb{C}[Z_0, \ldots, Z_{n-1}] & \xrightarrow{id} & \mathbb{C}[Z_0, \ldots, Z_{n-1}] \\
pr \downarrow & & \downarrow \Phi_X \\
\mathbb{C}[Z_0, \ldots, Z_{n-1}] / \ker \Phi_X & \xrightarrow{g} & V
\end{array}
\]

commutes. Hence \( pr = g^{-1} \circ \Phi_X \), since \( g \) is an isomorphism. On the other hand, from the \( Z_i \) multiplication one has \( pr \circ Z_i = B_i \circ pr : \mathbb{C}[Z_0, \ldots, Z_{n-1}] \rightarrow [Z_0, \ldots, Z_{n-1}] / \ker \Phi_X \) and \( \Phi_X \circ Z_i = B_i \circ \Phi_X : \mathbb{C}[Z_0, \ldots, Z_{n-1}] \rightarrow V \). That is, one has the following diagram
Moreover, one has

\[ \text{in which all faces commute. Then, one has } g \circ B_i \circ g = B_i \text{, for all } i \in \{0, \ldots, n-1\}. \]

Moreover, \( g \circ \tilde{I}(1) = g(1 \text{ mod } J) = \Phi_X(1) = I(1) \), i.e., \( g \circ \tilde{I} = I \). Therefore, \( [X] = [\tilde{X}] \in \mathcal{M}(n, c) \), in other words, we have just shown that \( \Psi' \circ \Psi = 1_{\mathcal{M}(n, c)} \).

To prove that \( \Psi' \circ \Psi = 1_{\text{Hilb}^c_{\mathbb{C}^n}} \), we only need to show that for a given \( J \in \text{Hilb}^c_{\mathbb{C}^n} \), one has \( J = \ker \Phi_X \), where \( X \) is a ADHM datum in \( \mathcal{V}(n, c)^{st} \) such that \( \Psi'(J) = [X] \in \mathcal{M}(n, c) \). For a polynomial

\[ p(Z_0, \ldots, Z_{n-1}) = \sum_\alpha a_\alpha Z_0^{\alpha_0} \cdots Z_{n-1}^{\alpha_{n-1}} \in \mathbb{C}[Z_0, \ldots, Z_{n-1}] \]

we have

\[ \Phi_X(p(Z_0, \ldots, Z_{n-1})) = \sum_\alpha a_\alpha B_0^{\alpha_0} \cdots B_{n-1}^{\alpha_{n-1}} I(1) \]

where \( I(1) \) is the class \([1 \text{ mod } J] = [1] \). Moreover, since \( B_i \circ \Phi_X = \Phi_X \circ Z_i \) then

\[ \sum_\alpha a_\alpha B_0^{\alpha_0} \cdots B_{n-1}^{\alpha_{n-1}} I(1) = \left[ \sum_\alpha a_\alpha Z_0^{\alpha_0} \cdots Z_{n-1}^{\alpha_{n-1}} \right] = [p(Z_0, \ldots, Z_{n-1})] . \]

Thus, if the polynomial \( p(Z_0, \ldots, Z_{n-1}) \) belongs to the ideal \( J \), then

\[ \sum_\alpha a_\alpha B_0^{\alpha_0} \cdots B_{n-1}^{\alpha_{n-1}} I(1) = 0, \]

and therefore \( p(Z_0, \ldots, Z_{n-1}) \in \ker \Phi_X \).

On the other hand, suppose that \( p(Z_0, \ldots, Z_{n-1}) \in \ker \Phi_X \). Then

\[ \Phi_X(p(Z_0, \ldots, Z_{n-1})) = \sum_\alpha a_\alpha B_0^{\alpha_0} \cdots B_{n-1}^{\alpha_{n-1}} I(1) = 0. \]

Again, one has

\[ \sum_\alpha a_\alpha B_0^{\alpha_0} \cdots B_{n-1}^{\alpha_{n-1}} I(1) = \left[ \sum_\alpha a_\alpha Z_0^{\alpha_0} \cdots Z_{n-1}^{\alpha_{n-1}} \right] = [p(Z_0, \ldots, Z_{n-1})] . \]

hence \( [p(Z_0, \ldots, Z_{n-1})] = 0 \in \mathbb{C}[Z_0, \ldots, Z_{n-1}]/J \), that is, \( p(Z_0, \ldots, Z_{n-1}) \in J \). Thus \( J = \ker \Phi_X \). This finishes our proof.
4. Extended monads and perfect extended monads

In this section we shall generalize the concept of monads, introduced by Horrocks (the reader may consult [18] for definitions and properties), in order to describe ideal sheaves for zero-dimensional subschemes of $\mathbb{C}^n$ and $\mathbb{P}^n$, $n \geq 2$.

Let $X$ be a smooth projective algebraic variety of dimension $n$ over the field of complex numbers $\mathbb{C}$, and let $\mathcal{O}_X(1)$ be a polarization on it.

4.1. $l$–extended monads. The objects we now wish to introduce are defined as follows.

**Definition 4.1.** An $l$–extended monad over $X$ is a complex

\[
C^\bullet : C^{-l-1} \xrightarrow{\alpha_{-l-1}} C^{-l} \xrightarrow{\alpha_{-l}} \cdots \xrightarrow{\alpha_{-2}} C^{-1} \xrightarrow{\alpha_{-1}} C^0 \xrightarrow{\alpha_0} C^1
\]

of locally free sheaves over $X$ which is exact at all but the $0$–th position, i.e. $\mathcal{H}^i(C^\bullet) = 0$ for $i \neq 0$. The coherent sheaf $E := \mathcal{H}^0(C^\bullet) = \ker \alpha_0 / \text{Im } \alpha_{-1}$ will be called a the cohomology of $C^\bullet$.

Note that a monad on $X$, in the usual sense, is just a 0–extended monad.

Moreover, one can associate to any $l$–extended monad $C^\bullet$ a display of exact sequences as the following

\[
\begin{array}{ccccccccccccc}
0 & \to & 0 & \to & C^{-l-1} & \xrightarrow{\alpha_{-l-1}} & C^{-l} & \xrightarrow{\alpha_{-l}} & \cdots & \xrightarrow{\alpha_{-2}} & C^{-1} & \xrightarrow{\alpha_{-1}} & C^0 & \xrightarrow{\alpha_0} & C^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & K & \xrightarrow{\phi_{-1}} & C^0 & \xrightarrow{\alpha_0} & C^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & F & \xrightarrow{\phi_{-1}} & Q & \xrightarrow{\phi_0} & C^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
\]

where $K := \ker \alpha_0$ and $Q := \text{coker } \alpha_{-1}$

A morphism $\phi : C^\bullet_1 \to C^\bullet_2$ of two $l$–extended monads $C^\bullet_1$ and $C^\bullet_2$ is an $(l + 3)$–tuple of morphisms such that the following diagram commutes:

\[
\begin{array}{ccccccccccccc}
C^\bullet_1 : & C^{-l-1} & \to & C^{-l} & \to & \cdots & \to & C^{-2} & \to & C^{-1} & \to & C^0 & \to & C^1 \\
\phi & \phi_{-l-1} & \phi_{-l} & \cdots & \phi_{-2} & \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{-1} & \phi_0 & \phi_1 & \phi_0 & \phi_1 \\
C^\bullet_2 : & C^{-l-1} & \to & C^{-l} & \to & \cdots & \to & C^{-2} & \to & C^{-1} & \to & C^0 & \to & C^1
\end{array}
\]
With these definitions, the category of \( l \)-extended monads form a full subcategory of the category \( \text{Kom}^b(X) \) of bounded complexes of coherent sheaves on \( X \).

\( l \)-extended monads have already appeared in the literature. The most important example of a locally-free sheaf that can be obtained as the cohomology of a 2-extended monad on \( \mathbb{P}^4 \) is the dual of the Horrocks–Mumford bundle; indeed, Fløystad shows in [4, Introduction: example b.] that the Horrocks–Mumford bundle is given by the cohomology at degree zero, where the grading is given by the twist, of a complex of the form

\[
C^5_{\mathbb{P}^4}(-1) \rightarrow C^5_{\mathbb{P}^4} \rightarrow C^5_{\mathbb{P}^4}(1) \rightarrow C^5_{\mathbb{P}^4}(2).
\]

Dualizing such complex we get a 2-extended monad on \( \mathbb{P}^4 \) whose cohomology is the dual of the Horrocks–Mumford bundle.

Moreover, object very closely related to 2-extended monads on \( \mathbb{P}^3 \) have also appeared in the mathematical physics literature, see [3, Section 4].

An \( l \)-extended monad can be broken into the following two complexes: first,

\[
N^\bullet: \quad 0 \longrightarrow C^{l-1} \longrightarrow C^{l-2} \longrightarrow \cdots \longrightarrow C^{-2} \longrightarrow C^{-1} \longrightarrow I_{l-1} \xrightarrow{\alpha_{l-1}} G \xrightarrow{\phi} 0
\]

which is exact, and a locally free resolution of the sheaf \( G = \text{coker} \alpha_{l-2} \), and

\[
M^\bullet: \quad G \xrightarrow{I_{l-1}} C^0 \xrightarrow{\alpha_0} C^1
\]

where \( I_{l-1} \circ J_{l-1} = \alpha_{l-1} \). \( M^\bullet \) is a monad-like complex in which the coherent sheaf \( G \) might not be locally free; indeed, \( G \) is not locally free for the extended monads describing ideal sheaves of 0-dimensional subschemes, the situation most relevant to the present paper.

For a given \( l \)-extended monad, we refer to the complexes \( M^\bullet \) and \( N^\bullet \) as the associated resolution and the associated monad, respectively. Therefore, the morphism \( \phi: C_1^\bullet \rightarrow C_2^\bullet \) can be thought of as a pair of morphisms \((\phi_N: N_1^\bullet \rightarrow N_2^\bullet, \phi_M: M_1^\bullet \rightarrow M_2^\bullet) \in \text{Hom}(N_1^\bullet, N_2^\bullet) \times \text{Hom}(M_1^\bullet, M_2^\bullet)\).

Remark that as long as we have \( \phi_0(\text{Im} \, \alpha_1^{-1}) \subseteq \text{Im} \, \alpha_2^{-1} \) and \( \phi_0(\text{ker} \, \alpha_0^1) \subseteq \text{ker} \, \alpha_0^2 \), then \( \phi \) is determined by only \( \phi_0 \); indeed, the conditions

\[
\phi_0(\text{Im} \, \alpha_1^{-1}) \subseteq \text{Im} \, \alpha_2^{-1} \quad \text{and} \quad \phi_0(\text{Im} \, J_1^{-1}) \subseteq \text{Im} \, J_2^{-1}
\]

are equivalent (here we considered the morphism of the associated monads). Hence \( \phi_0 \) determines the morphism \( \phi_M \), and consequently it also determines the whole morphism \( \phi: C_1^\bullet \rightarrow C_2^\bullet \). This is because \( N_1^\bullet \) and \( N_2^\bullet \) are locally free resolutions and hence projective resolutions, so that giving a morphism \( \phi_G: G \rightarrow G \) determines all the morphisms \( \phi_i: C_i^{-1} \rightarrow C_i^0 \) for \( i \leq -1 \).

Since taking cohomology is a functorial operation, a morphism \( \phi: C_1^\bullet \rightarrow C_2^\bullet \) of two \( l \)-extended monads \( C_1^\bullet \) and \( C_2^\bullet \), induces a morphism between their respective cohomologies

\[
H(\phi): H^0(C_1^\bullet) \rightarrow H^0(C_2^\bullet).
\]

Of course, isomorphic complexes induce isomorphic cohomologies. It follows that there is natural notion of equivalence for \( l \)-extended monads with the same terms \( C^i \) provided by the action of the automorphism group \( \text{Aut}(C^i) = \text{Aut}(C^{l-1}) \times \text{Aut}(C^{l-2}) \times \cdots \times \text{Aut}(C^0) \times \text{Aut}(C^1) \).

Our goal now is to study families of ideal sheaves of zero-cycles in \( \mathbb{P}^n \). It turns out that such ideal sheaves are given by cohomologies of a special kind of \( l \)-extended monads. However, before proving this claim, which will be done only in Section
below, we tackle a more general question, namely under which conditions a ho-
morphism $H^0(C_1^*) \to H^0(C_2^*)$ lifts to a homomorphism $C_1^* \to C_2^*$ between the

Our next result provides a sufficient condition, by showing when the cohomology

Proposition 4.2. Let

$$C_1^* : \quad C_1^{-l} \ar[rr]^{\alpha_{-l-1}^1} \quad \cdots \quad \ar[rr]^{\alpha_{-2}^1} C_1^{-1} \ar[rr]^{\alpha_{-1}^0} & & C_1^0 \ar[rr]^{\alpha_0^1} & & C_1^1$$

and

$$C_2^* : \quad C_2^{-l} \ar[rr]^{\alpha_{-l-1}^2} \quad \cdots \quad \ar[rr]^{\alpha_{-2}^2} C_2^{-1} \ar[rr]^{\alpha_{-1}^0} & & C_2^0 \ar[rr]^{\alpha_0^2} & & C_2^1$$

be two $l$-extended monads, and let us denote by $F_1$ and $F_2$ their respective coho-
mologies. Then

$$H : \text{Hom}(C_1^*, C_2^*) \to \text{Hom}(F_1, F_2)$$

is surjective if

$$\text{Ext}^1(C_1^0, C_2^0) = 0,$$

$$\text{Ext}^k(C_1^0, C_2^{-k}) = 0 \text{ for } k \geq 1, \quad \text{Ext}^k(C_1^1, C_2^{-k+1}) = 0 \text{ for } k \geq 2.$$

Moreover if

$$\text{Hom}(C_1^1, C_2^0) = 0,$$

$$\text{Ext}^k(C_1^0, C_2^{-k+1}) = 0 \text{ for } k \geq 1, \quad \text{Ext}^k(C_1^0, C_2^{-k-1}) = 0 \text{ for all } k \geq 0,$$

then $H$ is an isomorphism.

Proof. Let $G_1 = \text{Im } \alpha_{-1}^1$ and $G_2 = \text{coker } \alpha_{-1}^2$. The associated resolution $N_2^*$ can be broken into sequences

$$0 \to G_2^{-i} \to C_2^{-i} \to G_2^{-i+1} \to 0, \quad 1 \leq i \leq l + 1$$

where we put $G_2^{-l-1} = C_2^{-l}$ and $G_2^0 = G_2$. Then, by applying either $\text{Hom}(C_1^0, \bullet)$ or $\text{Hom}(C_1^1, \bullet)$ on the above sequences and incorporating the conditions given in the proposition, it follows that $H$ is surjective if

$$\text{Ext}^1(C_1^0, C_2^0) = \text{Ext}^1(C_1^1, G_2) = \text{Ext}^2(C_1^1, G_2) = 0,$$

and it is an isomorphism if

$$\text{Hom}(C_1^1, C_2^0) = \text{Hom}(C_1^0, G_2) = \text{Ext}^1(C_1^0, G_2) = 0.$$

To finish the proof, it suffice to apply [13 Lemma 4.1.3] to the associated monad

$M_2^*$ of the $l$-extended monad $C_2^*$.

Corollary 4.3. Let

$$C^* : \quad C^{-l} \ar[rr]^{\alpha_{-l-1}^1} \quad \cdots \quad \ar[rr]^{\alpha_{-2}^1} C^{-1} \ar[rr]^{\alpha_{-1}^0} & & C^0 \ar[rr]^{\alpha_0^1} & & C^1$$

and

$$C'^* : \quad C^{-l} \ar[rr]^{\alpha_{-l-1}^1} \quad \cdots \quad \ar[rr]^{\alpha_{-2}^1} C^{-1} \ar[rr]^{\alpha_{-1}^0} & & C'^0 \ar[rr]^{\alpha_0^1} & & C'^1$$

be $l$-extended monads, and let $F$ and $F'$ be their cohomologies, respectively. Suppose that

$$\text{Ext}^1(C^1, C^0) = 0,$$

$$\text{Ext}^k(C^0, C^{-k}) = 0 \text{ for } k \geq 1, \quad \text{Ext}^k(C^1, C^{-k+1}) = 0 \text{ for } k \geq 2.$$
Then $F$ and $F'$ are isomorphic if and only if $C^\bullet$ and $C'^\bullet$ are isomorphic (as $l$-extended monads).

4.2. Perfect extended monads. We now introduce the class of $l$--extended monads which is relevant to the description of the Hilbert scheme of points.

Definition 4.4. An $l$--extended monad $C^\bullet$ is called pure if $C^{-i} = \mathcal{L}_{-i}^{a_{-i}}$, for all $-1 \leq i \leq l-1$, where $\mathcal{L}_{-i}$ are invertible sheaves, and it is called linear if all maps $\alpha_{-i}$ are given by matrices of linear forms.

Before our next definition, recall that $\mathcal{O}_X(1)$ is the chosen polarization on the $n$--dimensional projective algebraic variety $X$.

Definition 4.5. A perfect extended monad on a $n$-dimensional projective variety $X$ is a linear $(n-2)$--extended monad $P^\bullet$ on $X$ of the following form

$$\mathcal{O}_X(1-n)^{\oplus a_{1-n}} \xrightarrow{\alpha_{1-n}} \mathcal{O}_X(2-n)^{\oplus a_{2-n}} \xrightarrow{\alpha_{2-n}} \cdots \xrightarrow{\alpha_{n-1}} \mathcal{O}_X^{\oplus a_0} \xrightarrow{\alpha_0} \mathcal{O}_X \xrightarrow{a_1} \mathcal{O}_X(1)^{\oplus a_1}.$$  

We recall to the reader that a projective scheme $X$ is arithmetically Cohen-Macaulay, or simply ACM, if its homogeneous coordinate ring is Cohen-Macaulay ring. Moreover let us denote by $\mathfrak{Per}$ the full subcategory of $\text{Kom}^\bullet(X)$ consisting of perfect extended monads.

Corollary 4.6. If $X$ is an $n$--dimensional ACM variety, then the cohomology functor

$$H : \mathfrak{Per}(X) \to \text{Coh}(X)$$

is full and faithful.

Proof. This follows easily from Proposition 4.2 since $X$ is ACM, we have that

$$\text{Hom}(C^1_1, C^0_2) = H^0(X, \mathcal{O}_X(-1)) = 0$$ and

$$\text{Ext}^i(\mathcal{O}_X(a), \mathcal{O}_X(b)) = H^i(X, \mathcal{O}_X(b-a)) = 0, \text{ for } 1 \leq i \leq n-1.$$  

It follows from the Corollary above that automorphism group of a perfect extended monad on an ACM variety is just $GL_{a_{1-n}}(\mathbb{C}) \times GL_{a_{-n}}(\mathbb{C}) \times \cdots \times GL_{a_1}(\mathbb{C})$.

We finish this section by describing the cohomology of sheaves which arise as cohomologies of perfect extended monads on $\mathbb{P}^n$, $n \geq 2$.

Proposition 4.7. If $F$ is the cohomology of a perfect extended monad on $\mathbb{P}^n$ ($n \geq 2$) then:

(i) $H^0(F(k)) = 0$ for $k < 0$;
(ii) $H^0(F(k)) = 0$ for $k > -n-1$;
(iii) $H^i(F(k)) = 0 \forall k, 2 \leq i \leq n-1$, when $n \geq 3$. 

□
Proof. We twist the middle column of the display (3) by $\mathcal{O}_{\mathbb{P}^n}(k)$, then break it into short exact sequences

\begin{align*}
(7) & \quad 0 \to \mathcal{O}_{\mathbb{P}^n}(k + 1 - n)^{\oplus a_{1-n}} \to \mathcal{O}_{\mathbb{P}^n}(k + 2 - n)^{\oplus a_{2-n}} \to Q_{2-n}(k) \to 0 \\
& \quad 0 \to Q_{2-n}(k) \to \mathcal{O}_{\mathbb{P}^n}(k + 3 - n)^{\oplus a_{2-n}} \to Q_{3-n}(k) \to 0 \\
& \quad \vdots \\
& \quad 0 \to Q_{-p-1}(k) \to \mathcal{O}_{\mathbb{P}^n}(k - p)^{\oplus a_{-p}} \to Q_{-p}(k) \to 0 \\
& \quad \vdots \\
& \quad 0 \to Q_{-2}(k) \to \mathcal{O}_{\mathbb{P}^n}(k - 1)^{\oplus a_{-1}} \to Q_{-1}(k) \to 0 \\
& \quad 0 \to Q_{-1}(k) \to \mathcal{O}_{\mathbb{P}^n}(k)^{\oplus a_0} \to Q_0(k) \to 0
\end{align*}

where $Q_0 := Q = \text{coker} \alpha_{-1}$.

**Step 1:** From the long sequences in cohomology of the first row above, we have

$$H^i(\mathcal{O}_{\mathbb{P}^n}(k + 2 - n))^{\oplus a_{2-n}} \to H^i(Q_{2-n}(k)) \to H^{i+1}(\mathcal{O}_{\mathbb{P}^n}(k + 1 - n))^{\oplus a_{1-n}} \to \cdots$$

Then, from the vanishing properties of line bundles on $\mathbb{P}^n$, it follows that

(i) $H^0(Q_{2-n}(k)) = 0$ for $k < n - 2$;
(ii) $H^n(Q_{2-n}(k)) = 0$ for $k > -1$;
(iii) $H^i(Q_{2-n}(k)) = 0 \forall k, 1 \leq i \leq n - 1$.

**Step 2:** Using induction on the remaining rows in (7) it follows that, for $p > 2$,

(i) $H^0(Q_{q-p-n}(k)) = 0$ for $k < n - p$;
(ii) $H^n(Q_{q-p-n}(k)) = 0$ for $k > -p - 1$;
(iii) $H^i(Q_{q-p-n}(k)) = 0 \forall k, 1 \leq i \leq n - 1$.

**Step 3:** From the long exact sequence in cohomology of the lower row in (3) twisted by $\mathcal{O}_{\mathbb{P}^n}(k)$ one has

$$H^{i-1}(\mathcal{O}_{\mathbb{P}^n}(k + 1))^{\oplus a_1} \to H^i(\mathcal{F}(k)) \to H^i(Q(k)) \to \cdots$$

Using the vanishing obtained in Step 2 for $Q_0 = Q$, the claims of items (i), (ii), (iii) and (iv) follow.

The last item is obtained by dualizing the lower row of (3).

Now let us denote by $\Omega_{\mathbb{P}^n}^p$ the bundle of holomorphic $(-p)$-forms on $\mathbb{P}^n$, where $p \leq 0$ in our convention.

**Proposition 4.8.** If a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ ($n \geq 2$) satisfies:

(i) $H^0(\mathcal{E}(-1)) = H^n(\mathbb{P}^n, \mathcal{E}(-n)) = 0$;
(ii) $H^q(\mathbb{P}^n, \mathcal{E}(k)) = 0 \forall k, 2 \leq q \leq n - 1$ when $n \geq 3$;
(iii) $H^1(\mathbb{P}^n, \mathcal{E} \otimes \Omega_{\mathbb{P}^n}^p(-p - 1)) \neq 0$ for $-n \leq p \leq 0$;

then $\mathcal{E}$ is the cohomology of a perfect extended monad.
Proof. Applying Beilinson’s theorem [18, Theorem 3.1.4] to the sheaf $E(-1)$, one gets a spectral sequence with $E_1$-term given by

$$E_1^{p,q} = H^q(E \otimes \Omega_{P^n}^{-p}(-p-1)) \otimes O_{P^n}(p)$$

which converges to the graded sheaf associated to a filtration of $E(-1)$ itself.

Twist the Euler sequence for the sheaves of differential forms

$$0 \to \Omega^p(p) \to O_{P^n}^{N(p)} \to \Omega^{p-1}(p) \to 0, \quad N = \left(\begin{array}{c} n+1 \\ p \end{array}\right)$$

by $E(k-p)$ and use hypotheses (i) and (ii) above to conclude, after long but straightforward calculations with the associated long exact sequences of cohomology, that $E_1^{p,q} = 0$ for $q \neq 1$.

It follows immediately that the Beilinson spectral sequence degenerates already at the $E_2$-term, i.e. $E_2 = E_\infty$. Beilinson’s theorem then implies that the complex $E_n^{p,1}$ given by

$$V_n \otimes O_{P^n}(-n) \to \cdots \to V_1 \otimes O_{P^n}(-1) \to V_0 \otimes O_{P^n},$$

with $V_p := H^1(E \otimes \Omega_{P^n}^{-p}(-p-1)), -n \leq p \leq 0$, is exact everywhere except at position $p = -1$, and its cohomology at this position is precisely $E(-1)$.

The third hypothesis implies that none of the vector spaces $V_p$ vanishes. So twisting the complex (8) by $O_{P^n}(1)$, we obtain a perfect extended monad whose cohomology is exactly $E$, as desired. \qed

5. Ideal sheaves of zero-dimensional subschemes of $P^n$

We now consider sheaves $E$ of rank $r$ on $P^n$ fitting in the following short exact sequence

$$0 \to E \to O_{P^n}^{\oplus r} \to Q \to 0,$$

where $Q$ is a pure torsion sheaf of length $c$ supported on a 0-dimensional subscheme $Z \subset P^n$.

Note that the Chern character of $E$ is given by $ch(E) = r - cH^n$, and that $E$ is necessarily torsion free. Such sheaves can also be regarded as points in the Quot scheme $Quot^P_{c}(O_{P^n}^{\oplus r})$.

In the case $r = 1$, it is clear that $E$ is the sheaf of ideals in $O_{P^n}$ associated to the zero-dimensional subscheme $Z$, i.e. $Q = O_Z$; in this case, we will then denote $E$ by $I_Z$.

Proposition 5.1. Every sheaf $E$ on $P^n$ given by sequence (9) is the cohomology of a perfect extended monad $P^\bullet$ with terms of the form $P^{-i} := V_i \otimes O_{P^n}(i), i = 1 - n, \ldots, 0, 1$, where

$$V_i := H^1(E \otimes \Omega_{P^n}^{1-i}(-i)) \cong H^0(Q \otimes \Omega_{P^n}^{1-i}(-i)).$$

Furthermore, we the following isomorphisms:

$$V_i \cong \begin{cases} V_i^{\oplus n} \oplus \mathbb{C}^r & \text{for } i = 0 \\ V_i^{\oplus (n_1)} & \text{for } i < 0 \end{cases}$$

(11)

(12)
In particular, we conclude that
\[
\dim V_i = \begin{cases} 
  c & i = 1 \\
  nc + r & i = 0 \\
  (\frac{n}{i})c & i < 0 
\end{cases}
\]

**Proof.** Conditions (i) and (ii) in Proposition 4.8 follow easily from twisting sequence (9) by \(\mathcal{O}_{\mathbb{P}^n}(k)\) and using that fact that \(\mathcal{Q}\) is supported in dimension zero. Next, twist sequence (9) by \(\Omega_{\mathbb{P}^n}^{-p}(-p-1)\) and use Bott’s formula to obtain the isomorphisms in (10).

The isomorphisms (11) and (12) can be proved as follows. First, we have for \(i = 1\)
\[
V_1 := H^1(\mathcal{E}(-1)) \cong H^0(\mathcal{Q}(-i)) \cong H^0(\mathcal{Q})
\]
since \(\mathcal{Q}\) is supported in dimension zero.

The space \(V_0\) fits in the sequence
\[
0 \to H^0(\mathcal{Q} \otimes \Omega_{\mathbb{P}^n}^1) \to H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^1) \to H^1(\Omega_{\mathbb{P}^n}^{1-q}) \to 0
\]
which is obtained from sequence (13) twisted by \(\Omega_{\mathbb{P}^n}^1\). On the other hand, we know from the Euler sequence that \(H^1(\Omega_{\mathbb{P}^n}^1) \cong H^0(\mathcal{O}_{\mathbb{P}^n})\). Moreover, since
\[
H^0(\mathcal{Q} \otimes \Omega_{\mathbb{P}^n}^1) \cong H^0(\mathcal{Q}) \cong V_1^{\oplus n},
\]
it follows that \(H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{1-i}) \cong V_1^{\oplus n} \oplus \mathbb{C}^r\).

Finally, note that
\[
V_{-i} = H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{1-i}(-i)) \cong H^0(\mathcal{Q}^{\oplus n}) = V_1^{\oplus n}.
\]

\(\square\)

In particular, for the case \(r = 1\), we have the following Corollary.

**Corollary 5.2.** For every zero dimensional subscheme \(Z \subset \mathbb{P}^n\), there exists a perfect extended monad \(P^*\) of the form
\[
V_{1-n} \otimes \mathcal{O}_{\mathbb{P}^n}(1-n) \xrightarrow{\alpha_{1-n}} V_{2-n} \otimes \mathcal{O}_{\mathbb{P}^n}(2-n) \xrightarrow{\cdots} V_0 \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\alpha_o} V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(1)
\]
where \(V_1 := H^0(\mathcal{O}_Z)\) and
\[
V_{-i} \cong \begin{cases} 
  V_1^{\oplus n} \oplus \mathbb{C} & \text{for } i = 0 \\
  V_1^{\oplus n} & \text{for } i < 0
\end{cases}
\]
whose cohomology is the ideal sheaf \(\mathcal{I}_Z\).

6. **The \(\mathbb{P}^3\) Case**

In this section, we fix a hyperplane \(\varphi \subset \mathbb{P}^n\). We shall describe how to get linear algebraic data out of the perfect extended monad corresponding to a 0–dimensional subscheme \(Z \subset \mathbb{P}^3 \setminus \varphi\), as in Corollary 5.2.

Let us start by fixing notation; we choose homogeneous coordinates \([z_0 \, z_1 \, z_2 \, z_3]\) on \(\mathbb{P}^3\) in such a way that the hyperplane \(\varphi\) is given by the equation \(z_3 = 0\). We also regard such coordinates as a basis for the space of global sections \(H^0(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1))\).

By Corollary 5.2 there is a perfect extended monad \(P^*\) with cohomology equal to the ideal sheaf \(\mathcal{I}_Z\). It is given by
\[
V_1 \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha_{-2}} V_1^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_{-1}} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha} V_1 \otimes \mathcal{O}_{\mathbb{P}^3}(1)
\]
where \( \alpha_{-2} \in \text{Hom}(V_1, V_1^{\oplus 3}) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)), \ \alpha_{-1} \in \text{Hom}(V_1^{\oplus 3}, V_1 \otimes \mathcal{O}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))) \) and \( \alpha_0 \in \text{Hom}(V_1^{\oplus 3} \oplus W, V_1) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \). Then we can write the \( \alpha \)'s as:

\[
\begin{align*}
\alpha_{-2} &= \alpha_{-2} z_0 + \alpha_{-2}^1 z_1 + \alpha_{-2}^2 z_2 + \alpha_{-2}^3 z_3; \\
\alpha_{-1} &= \alpha_{-1}^0 z_0 + \alpha_{-1}^1 z_1 + \alpha_{-1}^2 z_2 + \alpha_{-1}^3 z_3; \\
\alpha_0 &= \alpha_0^0 z_0 + \alpha_0^1 z_1 + \alpha_0^2 z_2 + \alpha_0^3 z_3,
\end{align*}
\]

The conditions \( \alpha_{-1} \circ \alpha_{-2} = 0 \) and \( \alpha_0 \circ \alpha_{-1} = 0 \), which guarantee that (13) is a complex, are equivalent to

\[
\begin{align*}
\alpha_{-1}^{k-i} \circ \alpha_{-1}^k &= 0 \quad \forall k, i \quad \text{and} \quad \alpha_{-1}^{k-i} \circ \alpha_{-1}^l + \alpha_{-1}^l \circ \alpha_{-1}^k &= 0 \quad \forall i, k \neq l.
\end{align*}
\]

We also have to impose the condition \( \ker \alpha_{-1} = \text{Im} \alpha_{-2} \), since \( H^{-1}(P^*) = 0 \).

Restricting \( P^* \) to the plane \( \wp \cong \mathbb{P}^2 \) we get the following 1-extended monad on \( \wp \):

\[
V_1 \otimes \mathcal{O}(\wp)(-2) \xrightarrow{\alpha_{-2}(\wp)} V_1^{\oplus 3} \otimes \mathcal{O}(\wp)(-1) \xrightarrow{\alpha_{-1}(\wp)} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}(\wp) \xrightarrow{\alpha_0(\wp)} V_1 \otimes \mathcal{O}(\wp)(1)
\]

and the maps of this complex are just given by

\[
\begin{align*}
\alpha_{-2}(\wp) &= \alpha_{-2}^0 z_0 + \alpha_{-2}^1 z_1 + \alpha_{-2}^2 z_2; \\
\alpha_{-1}(\wp) &= \alpha_{-1}^0 z_0 + \alpha_{-1}^1 z_1 + \alpha_{-1}^2 z_2; \\
\alpha_0(\wp) &= \alpha_0^0 z_0 + \alpha_0^1 z_1 + \alpha_0^2 z_2.
\end{align*}
\]

The resolution and the monad associated to the perfect extended monad \( P^* \) are given by, respectively,

\[
\begin{align*}
0 &\to V_1 \otimes \mathcal{O}(\wp)(-2) \xrightarrow{\alpha_{-2}(\wp)} V_1^{\oplus 3} \otimes \mathcal{O}(\wp)(-1) \xrightarrow{\alpha_{-1}(\wp)} \mathcal{G}(\wp) \otimes \mathcal{O}(\wp) \xrightarrow{\alpha_0(\wp)} V_1 \otimes \mathcal{O}(\wp)(1). \\
0 &\to \mathcal{G}(\wp) \otimes \mathcal{O}(\wp) \xrightarrow{\alpha_{-1}(\wp)} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}(\wp) \xrightarrow{\alpha_0(\wp)} V_1 \otimes \mathcal{O}(\wp)(1).
\end{align*}
\]

**Lemma 6.1.** The sheaf \( \mathcal{G}(\wp) \) is locally free and satisfies

(i) \( H^0(\wp, \mathcal{G}(\wp)) = H^1(\wp, \mathcal{G}(\wp)) = H^2(\wp, \mathcal{G}(\wp)) = 0 \);

(ii) \( H^1(\wp, \mathcal{G}(\wp)^*) = H^2(\wp, \mathcal{G}(\wp)^*) = 0 \), and \( h^0(\wp, \mathcal{G}(\wp)^*) = 3c \).

**Proof.** Taking the restriction of the display of the perfect monad to the plane \( \wp \) one has \( \mathcal{I}(\wp) = \mathcal{O}(\wp) \), since \( \text{supp}(Z) \cap \wp = \emptyset \). Moreover, from the lowest row of the restricted display, namely

\[
0 \to \mathcal{O}(\wp) \to Q(\wp) \to V_1 \otimes \mathcal{O}(\wp) \to 0,
\]

it follows that \( Q(\wp) \) is a locally free sheaf. Furthermore, from the middle column of the restricted display, namely

\[
0 \to \mathcal{G}(\wp) \to (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}(\wp) \to Q(\wp) \to 0,
\]

it also follows the sheaf \( \mathcal{G}(\wp) \) is locally free.

The first item follows from the long exact sequence in cohomology of the associated resolution (13) and the fact that \( H^i(\wp, \mathcal{O}(\wp)(k)) = 0 \), for \( i = 0, 1, 2 \) and \( k = -1, -2 \).

For the second item, dualize the exact sequence (16) and apply the global sections functor \( \Gamma \) to obtain the exact sequence

\[
0 \to H^0(\wp, \mathcal{G}(\wp)^*) \to (V_1^{\oplus 3} \oplus H^0(\wp, \mathcal{O}(\wp)(1))) \to V_1 \otimes H^0(\wp, \mathcal{O}(\wp)(2)) \to H^1(\wp, \mathcal{G}(\wp)^*) \to 0.
\]
and \( H^2(\wp, \mathcal{G}^*_0) = 0 \), since \( H^{1,2}(\wp, \mathcal{O}|_{\wp}(1)) = H^{1,2}(\wp, \mathcal{O}|_{\wp}(1)) = 0 \). On the other hand, from the dual display of the associated monad \( (17) \) one has the exact sequence

\[
0 \rightarrow Q|_{\wp} \rightarrow (V_1^* + W^*) \otimes \mathcal{O}|_{\wp} \rightarrow \mathcal{G}^*_0 \rightarrow 0
\]

where \( Q := \text{coker} \alpha_1 \). Moreover \( Q|_{\wp}^* = V_1^* \otimes \mathcal{O}|_{\wp}(1) \otimes \mathcal{O}|_{\wp} \), since \( Q|_{\wp} \in \text{Ext}^1(V_1 \otimes \mathcal{O}|_{\wp}(1), \mathcal{O}|_{\wp}) = V_1^* \otimes H^1(\wp, \mathcal{O}|_{\wp}(-1)) = 0 \). Then, from the long exact sequence in cohomology associated to \( (19) \), it follows that \( H^0(\wp, \mathcal{G}^*_0) \) fits in the exact sequence

\[
0 \rightarrow C \rightarrow (V_1^*)^{\oplus 3} \oplus W \rightarrow H^0(\wp, \mathcal{G}^*_0) \rightarrow 0,
\]

hence \( h^0(\wp, \mathcal{G}^*_0) = 3c \) and from \( (18) \) it follows that \( h^1(\wp, \mathcal{G}^*_0) = 0 \). \( \square \)

Remark that the sequence \( (18) \) becomes just

\[
0 \rightarrow H^0(\wp, \mathcal{G}^*_1) \rightarrow (V_1^*)^{\oplus 3} \oplus (V_1^*)^{\oplus 3} \oplus (V_1^*)^{\oplus 3} \rightarrow (V_1^*)^{\oplus 3} \oplus (V_1^*)^{\oplus 3} \rightarrow 0,
\]

since \( H^0(\wp, \mathcal{O}|_{\wp}(1)) \simeq \mathbb{C}^3 \), and \( H^0(\wp, \mathcal{O}|_{\wp}(2)) \simeq \mathbb{C}^6 \). So one can identify \( H^0(\wp, \mathcal{G}^*_1) \) with \( (V_1^*)^{\oplus 3} \). Furthermore, by \( (20) \), one can identify \( W = H^0(\wp, \mathcal{I}_Z|_{\wp}) \simeq \mathbb{C} \), since \( Z \cap \wp = \emptyset \).

Combining sequences \( (21) \) and \( (20) \), and dualizing the resulting sequence one gets

\[
0 \rightarrow V_1^{\oplus 3} \oplus V_1^{\oplus 3} \rightarrow V_1^{\oplus 3} \oplus V_1^{\oplus 3} \oplus V_1^{\oplus 3} \oplus V_1^{\oplus 3} \rightarrow W \rightarrow C \rightarrow 0,
\]

The maps \( i \) and \( j \) are just \( H^0(\alpha_{-2}) \) and \( H^0(\alpha_{-1}) \), respectively. Thus we have

\[
\ker H^0(\alpha_{-2}) = \ker \alpha_{-2} \cap \ker \alpha_{-2} \cap \ker \alpha_{-2} = \{0\},
\]

and

\[
\ker H^0(t\alpha_{-1}) = t\ker \alpha_{-1} \cap \ker \alpha_{-1} \cap \ker t\alpha_{-1} = \mathbb{C}.
\]

The subscript \( t \), in the last equation, stands for transposition. Remark also that the sequence \( (22) \) reflects the fact the complex \( (18) \) is exact at degree \(-1\), i.e., \( \alpha_{-1} \circ \alpha_{-2} = 0 \).

We can then choose the maps \( \alpha_{-1}^t \) in the following way. First,

\[
\alpha_{-2}^0, \quad \alpha_{-2}^1, \quad \alpha_{-2}^2 : V_1 \rightarrow V_1 \oplus V_1 \oplus V_1,
\]

with:

\[
\alpha_{-2}^0 = \begin{pmatrix} 0 \\ 0 \\ \mathbb{I}_{V_1} \end{pmatrix}, \quad \alpha_{-2}^1 = \begin{pmatrix} 0 \\ -\mathbb{I}_{V_1} \\ 0 \end{pmatrix}, \quad \alpha_{-2}^2 = \begin{pmatrix} \mathbb{I}_{V_1} \\ 0 \\ 0 \end{pmatrix},
\]

and where \( \mathbb{I}_{V_1} \) denotes the identity in \( \text{End}(V_1) \).

One also has

\[
\alpha_{-1}^0, \quad \alpha_{-1}^1, \quad \alpha_{-1}^2 : V_1 \oplus V_1 \oplus V_1 \rightarrow V_1 \oplus V_1 \oplus V_1 \oplus \mathbb{C}
\]
given by

\[
\alpha_{-1}^0 = \begin{pmatrix}
0 & 0 & 0 \\
\mathbb{I}_V & 0 & 0 \\
0 & \mathbb{I}_V & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\alpha_{-1}^1 = \begin{pmatrix}
-\mathbb{I}_V & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mathbb{I}_V \\
0 & 0 & 0 \\
\end{pmatrix}
\]

(24)

\[
\alpha_{-1}^2 = \begin{pmatrix}
0 & -\mathbb{I}_V & 0 \\
0 & 0 & -\mathbb{I}_V \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]

Finally, for

\[
\alpha_0^0, \alpha_0^1, \alpha_0^2 : V_1 \oplus V_1 \oplus V_1 \oplus \mathbb{C} \to V_1
\]

one has

\[
\alpha_0^0 = \begin{pmatrix}
-\mathbb{I}_V & 0 & 0 & 0 \\
\end{pmatrix},
\alpha_0^1 = \begin{pmatrix}
0 & -\mathbb{I}_V & 0 & 0 \\
\end{pmatrix},
\alpha_0^2 = \begin{pmatrix}
0 & 0 & -\mathbb{I}_V & 0 \\
\end{pmatrix}.
\]

(25)

Now, to complete our construction, we have to add the maps \(\alpha_{-2}^3, \alpha_{-1}^3\) and \(\alpha_0^3\)
such that conditions (14) are satisfied. By putting

\[
\alpha_{-2}^3 = \begin{pmatrix}
-B_2 \\
B_1 \\
-B_0
\end{pmatrix};
\alpha_{-1}^3 = \begin{pmatrix}
B_1 & B_2 & 0 \\
-B_0 & 0 & -B_0 \\
0 & B_2 & -B_1 \\
0 & 0 & 0
\end{pmatrix};
\alpha_0^3 = \begin{pmatrix}
B_0 & B_1 & B_2 & I
\end{pmatrix},
\]

where \(B_i \in \text{End}(V_1)\) and \(I \in \text{Hom}(\mathbb{C}, V_1)\), then all the equations are satisfied, since

\[
\alpha_{-1}^3 \circ \alpha_{-2}^3 = 0 \text{ and } \alpha_0^3 \circ \alpha_{-1}^3 = 0 \text{ are equivalent to}
\]

(27)

\[ [B_0, B_1] = 0; \quad [B_0, B_2] = 0; \quad [B_1, B_2] = 0, \]

Summing up what we have done so far, for a given 0-dimensional subscheme

\[
\mathbb{Z} \subset \mathbb{P}^3 \setminus \emptyset \text{ we have constructed a perfect extended monad } P^* \text{ of the form}
\]

(28)

\[ V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \xrightarrow{\alpha_{-2}^2} V_1^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha_{-1}^1} (V_1^{\oplus 2} \otimes W) \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\alpha_0^0} V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(1) \]

where the maps maps \(\alpha_{-2}, \alpha_{-1}\) and \(\alpha_0\) are given by

(29)

\[
\alpha_{-2} = \begin{pmatrix}
-B_2 z_3 + z_2 \\
B_1 z_3 - z_1 \\
-B_0 z_3 + z_0
\end{pmatrix};
\alpha_{-1} = \begin{pmatrix}
B_1 z_3 - z_1 & B_2 z_3 - z_2 & 0 \\
-B_0 z_3 + z_0 & 0 & -B_1 z_3 + z_1 \\
0 & -B_0 z_3 + z_0 & -B_2 z_3 - z_2
\end{pmatrix};
\alpha_0 = \begin{pmatrix}
B_0 z_3 - z_0 & B_1 z_3 - z_1 & B_2 z_3 - z_2 & I z_3
\end{pmatrix}.
\]

It only remains for us to show the the ADHM datum

\[ (B_0, B_1, B_2, I) \in \text{End}(V_1)^{\oplus 3} \oplus \text{Hom}(\mathbb{C}, V_1) \]

obtained from the above construction is indeed stable. Such claim will follow from the following observation.

\[ \begin{enumerate}
\item We omit writing the identity in front of the coordinates so \( z_i \mathbb{I}_{V_1} \) will just be written \( z_i \)
\end{enumerate}
Lemma 6.2. The map $\alpha_0$ given above is surjective if and only if the ADHM datum $(B_0, B_1, B_2, I)$ is stable.

Proof. Recall that a map of sheaves is surjective if and only if it is surjective at every fiber.

So if $\alpha_0$ is not surjective, then there is a point $z = [z_0; z_1; z_2; z_3] \in \mathbb{P}^3$ such that $\alpha_0(z)$ is not surjective, while its dual map $\alpha_0^*$ is not injective. Hence there exists a vector $\bar{v} \in V^*$ such that $(B_i^*z_3 - z_i)\bar{v} = 0$, where $i = 0, 1, 2$, and $I^*\bar{v} = 0$. Then the subspace $\bar{S} \subseteq V^*$ generated by all such vectors is $B_i^*$-invariant, for $i = 0, 1, 2$, while the restriction $I^*|_{\bar{S}}$ of $I^*$ to $\bar{S}$ is zero.

Now, consider the following subspace of $V$:

$$S = \{ v \in V | \bar{v}(v) = 0, \forall \bar{v} \in \bar{S} \}.$$ 

It follows that $S$ is $B_i$-invariant, for $i = 0, 1, 2$, since $B_i^*\bar{v}(v) = \bar{v}(B_i v) = 0$ for $\bar{v} \in \bar{S}$ and $v \in S$. Moreover $I(1) \in S$ since $I^*|_{\bar{S}} = 0$. Thus $(B_0, B_1, B_2, I)$ is not stable.

Conversely, suppose that $(B_0, B_1, B_2, I)$ is not stable. Then there exists a $B_i$-invariant subspace $S \subseteq V$, for $i = 0, 1, 2$, such that $\text{Im } I \subseteq S$. Set

$$\bar{S} = \{ v \in V^* | \bar{v}(v) = 0, \forall \bar{v} \in S \}.$$ 

Then $\bar{S}$ is $B_i^*$-invariant and $\bar{S} \subset \ker I^*$. Since the $B_i$'s commutes, there exists a vector $\bar{v} \in V^*$ such that $B_i^*\bar{v} = \lambda_i \bar{v}$ for some $\lambda_i \in \mathbb{C}$, for $i = 0, 1, 2$. Hence the map $\alpha_0^*(\lambda_0; \lambda_1; \lambda_2; 1)$ is not injective, and equivalently, $\alpha_0$ is not surjective.

\[\square\]

Theorem 6.3 (Inverse construction). To a stable ADHM datum $X = (B_0, B_1, B_2, I) \in \mathcal{V}(3, c)^{st}$ one can associate the perfect extended monad $\mathcal{B}$ with maps $\alpha_{-2}, \alpha_{-1}, \alpha_0$ given as in $\mathcal{B}$, such that its cohomology is an ideal sheaf whose restriction to $\mathbb{C}^3 = \mathbb{P}^3 \setminus \rho$ is isomorphic to the one given by Theorem 5.7.

Proof. Given the stable ADHM datum $(B_0, B_1, B_2, I)$, one can put together the maps $\alpha_{-2}, \alpha_{-1}$ and $\alpha_0$ and a perfect extended monad like $\mathcal{B}$. Restricting the obtained perfect monad to $\mathbb{C}^3$ one has the complex

$$\begin{array}{c}
\mathcal{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\
\oplus \\
\mathcal{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\
\oplus \\
\mathcal{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\
\oplus \\
\mathcal{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\
\oplus \\
\mathcal{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\
\end{array} \overset{a_{-2} \otimes \mathcal{O}_{\mathbb{C}^3}}{\longrightarrow} \overset{a_{-1} \otimes \mathcal{O}_{\mathbb{C}^3}}{\longrightarrow} \overset{a_0 \otimes \mathcal{O}_{\mathbb{C}^3}}{\longrightarrow} \mathcal{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \longrightarrow 0. $$



Moreover, by projecting on the fourth summand in the degree 0 term, one has the injection $\ker a_0/\text{Im } a_{-1} \hookrightarrow \mathcal{O}_{\mathbb{C}^3}$. We denote its image by $J$. Then, it is clear that $J$ is an ideal of $c$ points in $\mathbb{C}^3$.

To prove that $J$ is isomorphic to $\ker \Phi_X = \{ f \in \mathcal{O}_{\mathbb{C}^3} | f(B_0, B_1, B_2) = 0 \}$, let us suppose, first, that $f \in J$. Then, there exists three vectors $u_0(z), u_1(z), u_2(z) \in V \otimes \mathcal{O}_{\mathbb{C}^3}$ such that $f(z)I(1) = (B_0 - z_0)u_0(z) + (B_1 - z_1)u_1(z) + (B_2 - z_2)u_2(z)$, since $f$ represents an element in $\ker a_0$. Hence $f(B_0, B_1, B_2)I(1) = 0$. But $(B_0, B_1, B_2, I)$ is stable and $B_0^0, B_1^1, B_2^2$ span all $V$, for $l_0, l_1, l_2 \geq 0$. Thus $f(B_0, B_1, B_2) = 0$, i.e., $f \in \ker \Phi_X$. 

On the other hand, let $f \in \mathcal{O}_{\mathbb{P}^3}$ such that $f(B_0, B_1, B_2)$. One has (unless otherwise specified, all sums are taken with respect to the indices $l_0, l_1, l_2 \geq 0$):

$$f(z_0, z_1, z_2) = \sum_{i=0}^{l_0} a_{i_0 i_1 i_2} z_0^{i_0} z_1^{i_1} z_2^{i_2} f_v$$

$$= \sum_{i=0}^{l_0} a_{i_0 i_1 i_2} (z_0 - B_0 + B_0)^{l_0} (z_1 - B_1 + B_1)^{l_1} (z_2 - B_2 + B_2)^{l_2}$$

$$= \sum_{i=0}^{l_0} a_{i_0 i_1 i_2} \left\{ \sum_{i=0}^{l_0} \alpha_i (z_0 - B_0)^{i_0} B_0^{l_0-i} \right\} \cdot \left\{ \sum_{j=0}^{l_0} \beta_j (z_1 - B_1)^{i_1} B_1^{l_1-j} \right\}$$

$$\left\{ \sum_{k=0}^{l_0} \gamma_k (z_2 - B_2)^{i_2} B_2^{l_2-k} \right\}$$

where we used the expansion $(a + b)^n = \sum_{i=0}^{n} \alpha_i a^i b^{n-i}, \alpha_i = \binom{n}{i}$ in the third line. Expanding again

$$\left\{ \sum_{i=0}^{l_0} \alpha_i (z_q - B_q)^{i_0} B_q^{l_0-i} \right\} = B_q^{l_0} + (z_q - B_q)^{l_0} + \left\{ \sum_{i=1}^{l_0-1} \alpha_i (z_q - B_q)^{i_0} B_q^{l_0-i} \right\}$$

with $q = 0, 1, 2$, thus one obtains

$$f(z_0, z_1, z_2) = \sum_{i=0}^{l_0} a_{i_0 i_1 i_2} B_0^{i_0} B_1^{i_1} B_2^{i_2} + \sum_{q=0}^{2} (z_q - B_q) A_q(z)$$

$$= f(B_0, B_1, B_2) + \sum_{q=0}^{2} (z_q - B_q) A_q(z)$$

for some vectors $A_q \in \text{End}(V) \otimes \mathcal{O}_{\mathbb{P}^3}$, $q = 0, 1, 2$. But $f(B_0, B_1, B_2) = 0$ by hypothesis. If we put $u_q = A_q I(1), q = 0, 1, 2$, then

$$f(z_0, z_1, z_2) I(1) = (z_0 - B_0) u_0(z) + (z_1 - B_1) u_1(z) + (z_2 - B_2) u_2(z).$$

Hence, $f \in \mathcal{J}$. □

The automorphisms of $P^* \mathcal{M}$ are clearly given by the action of the group $GL(V)$. Since, by Corollary 4.16, the cohomology functor is fully faithfull, we recover the correspondence, given in Section 2, between equivalence classes of ideal sheaves $\mathcal{I}_Z$ and the space $\mathcal{M}(3, c)$ defined as the quotient $\mathcal{V}(3, c)^{st}/GL(V_1)$, in the $3$–dimensional case.

We complete this section by writing down the maps $\alpha_0$ and $\alpha_{-1}$ in the more general $n$–dimensional case. Starting with a hyperplane $\wp \subset \mathbb{P}^n$ and a $0$-dimensional subscheme $Z \subset \mathbb{P}^n \setminus \wp$, the maps $\alpha_{-1}$ in the corresponding perfect extended monad can also be constructed as done above for the $3$–dimensional case:

$$\alpha_0 = \left( \begin{array}{cccc} B_0 z_n - z_0 & B_1 z_n - z_1 & \cdots & B_{n-1} z_n - z_{n-1} & I z_n \end{array} \right).$$

$$\alpha_{-1} = \left( \begin{array}{cccc} A_0 & A_1 & \cdots & A_{n-2} \\ 0 & 0 & \cdots & 0 \end{array} \right).$$

(31)
where each block $A_i$, $0 \leq i \leq n-2$ is an $[(n-i) \cdot c \times n \cdot c]$-matrix of the form

$$
A_i = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -B_i z_n + z_i & 0 & \cdots & 0 \\
-B_i z_n + z_i & B_{i+1} z_n - z_{i+1} & B_{i+2} z_n - z_{i+2} & \cdots & B_{n-1} z_n - z_{n-1}
\end{pmatrix}
$$

One can similarly show that $\alpha_0 \circ \alpha_{i-1} = 0 \iff [B_i, B_j] = 0$, for all $0 \leq i, j \leq n-1$, and that the map $\alpha_0$ is surjective if and only if the ADHM datum $(B_0, \ldots, B_{n-1}, I)$ is stable.

Once again, this reflects the set theoretic bijection between the Hilbert scheme of length $c$ zero-dimensional subschemes of $\mathbb{C}^n \simeq \mathbb{P}^n \setminus \varnothing$ and the quotient space $\mathcal{M}(n, c) := \mathcal{V}(n, c)^n/GL(V_i)$.

7. Representability of the Hilbert Functor of Points

Let us start this Section by introducing notation; for every two sheaves, $\mathcal{F}$ on $\mathbb{P}^n$ and $\mathcal{G}$ on a scheme $S$, we put $\mathcal{F} \boxtimes \mathcal{G} := p^* \mathcal{F} \otimes q^* \mathcal{G}$, where $p : \mathbb{P}^n \times S \rightarrow \mathbb{P}^n$ is the projection on the first factor and $q$ is the projection $\mathbb{P}^n \times S \rightarrow S$ on the second one. We also denote by $k(s)$ the residue field of a closed point $s \in S$.

Using the ingredients developed in the previous sections, we now proceed to prove that $\mathcal{M}(n, c)$ represents the Hilbert functor

$$
\mathcal{Hilb}_{\mathbb{C}^n}^{[c]} : \mathcal{S} \mathcal{h} \rightarrow \mathcal{S} \mathcal{e} \mathcal{t}
$$

from the category of schemes $\mathcal{S} \mathcal{h}$ to the category of sets $\mathcal{S} \mathcal{e} \mathcal{t}$, which associates to every scheme $S$ the set

$$
\mathcal{Hilb}_{\mathbb{C}^n}^{[c]}(S) := \left\{ Z \subset \mathbb{C}^n \times S \mid \begin{array}{l}
Z \text{ is a closed subscheme,} \\
\pi \downarrow \quad \downarrow q \\
S \simeq S \\
\chi(\mathcal{O}_{\pi^{-1}(s)} \otimes \mathcal{O}_{\mathbb{C}^n}(m)) = c, \quad \forall m \in \mathbb{Z}
\end{array} \right\}
$$

of flat families of 0-dimensional subschemes of $\mathbb{C}^n$.

For any noetherian scheme $S$ of finite type over the field of complex numbers $\mathbb{C}$, consider the following diagram:

$$
\begin{array}{ccc}
\mathbb{P}^n \times \mathbb{P}^n \times S & \xrightarrow{pr_{13}} & \mathbb{P}^n \times S \\
pr_{23} \downarrow & & \downarrow q \\
\mathbb{P}^n \times S & \xrightarrow{q} & S
\end{array}
$$

and the relative Euler sequence:

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^n \times S}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n \times S}^{\oplus (n+1)} \rightarrow TP^n(-1) \boxtimes \mathcal{O}_S \rightarrow 0
$$
Theorem 7.1 (Relative Beilinson’s Theorem). For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n \times S$ there is a spectral sequence $E_1^{i,j}$ with $E_1$-term

$$E_1^{i,j} = \mathcal{O}_{\mathbb{P}^n}(i) \boxtimes \mathcal{R}^j q_* (\mathcal{F} \otimes \Omega_{\mathbb{P}^n \times S/S}^{-i}(-i))$$

which converges to

$$E_\infty^{i,j} = \begin{cases} \mathcal{F} & i + j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is similar, to that of [10, Theorem 4.2].

Let $\mathcal{J}$ be an $S$-flat family of ideal sheaves of 0-dimensional subschemes of $\mathbb{P}^n$ of length $c$, for a noetherian scheme $S$ of finite type.

Theorem 7.2. There exists an 1-extended monad given by

$$\mathcal{O}_{\mathbb{P}^n}(1-n) \boxtimes \mathcal{R}^1 q_* (\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^{-1}(n-1)) \to \mathcal{O}_{\mathbb{P}^n}(2-n) \boxtimes \mathcal{R}^1 q_* (\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^{-1}(n-2)) \to \cdots$$

such that it’s cohomology is exactly the family $\mathcal{J}$.

Proof. By the relative Beilinson theorem, we only need the $S$-flatness of $\mathcal{J}$ and the fact that at point $s \in S$ one has

$$\mathcal{R}^1 q_* (\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^{-1}(1-i)) \otimes k(s) \simeq H^1(\mathbb{P}^n, \mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^{-1}(1-i),$$

where $Z(s)$ is the 0-dimensional subscheme of $\mathbb{P}^n$ corresponding to the point $s \in S$.

The rest of the proof follows from the vanishing properties of Lemma 5.4.

Therefore, on every point $s \in S$, one has a perfect extended monad $P^*(s)$ given by

$$H^1(\mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^{-1}(n-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1-n) \to H^1(\mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^{-1}(n-2)) \otimes \mathcal{O}_{\mathbb{P}^n}(2-n) \to \cdots$$

Moreover, in the case of the space $\mathcal{V}(n,c)^{ext}$ defined by one have the universal extended monad

$$\mathcal{O}_{\mathbb{P}^n}(1-n) \boxtimes (V_1 \otimes \mathcal{O}_{\mathcal{V}(n,c)^{ext}}) \to \mathcal{O}_{\mathbb{P}^n}(2-n) \boxtimes (V_1^{\oplus(n+n)} \otimes \mathcal{O}_{\mathcal{V}(n,c)^{ext}}) \to \cdots$$

Finally we have the following

Theorem 7.3. The scheme $\mathcal{M}(n,c)$ is a fine moduli space for the Hilbert functor $\mathcal{Hilb}_C$, of $c$ points on $\mathbb{C}^n$.

Proof. The proof is similar, mutatis mutandis, to that of [10, Theorem 4.2].

It follows by universality of the Hilbert scheme that

Corollary 7.4. $\mathcal{Hilb}_C(\mathbb{C}^n) \simeq \mathcal{M}(n,c)$ as schemes.
8. The Hilbert–Chow map

Let $S^c$ denote the group of permutations on $c$ elements, and consider the symmetric product of $c$ copies of $\mathbb{C}^n$:

$$S^{(c)}\mathbb{C}^n := (\mathbb{C}^n \times \mathbb{C}^n \times \ldots \times \mathbb{C}^n)/S^c.$$ 

In this section, we show how one can describe the Hilbert–Chow morphism

$$HC : \text{Hilb}^{[c]}(\mathbb{C}^n) \to S^{(c)}\mathbb{C}^n$$

in terms of the linear data $[(B_0, B_1, \ldots, B_{n-1}, I)]$.

Recall that a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$, with $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_k \geq 0$, of $c$ of length $k$, gives a stratum

$$Z^{(c)}_\nu = \{ \sum_{i=1}^k \nu_i[p_i] \in S^{(c)}\mathbb{C}^n | p_i \neq p_j \text{ for } j \neq j \}$$

of the symmetric product $S^{(c)}\mathbb{C}^n$ corresponding to $k$ ordered points $p_1, p_2, \ldots, p_k$, in $\mathbb{C}^n$ with multiplicities $\nu_1, \nu_2, \ldots, \nu_k$, respectively. There is a set theoretic stratification

$$\text{Hilb}^{[c]}(\mathbb{C}^n) = \bigsqcup_{\nu} U^{[c]}_{\nu},$$

where each stratum $U^{[c]}_{\nu}$ is given by the inverse image $HC^{-1}(Z^{(c)}_\nu)$.

Now let $[(B_0, B_1, \ldots, B_{n-1}, I)]$ be a datum class in $\text{Hilb}^{[c]}(\mathbb{C}^n)$; the endomorphisms $B_i$, $i = \{0, 1, \ldots, n-1\}$ can be put simultaneously into upper-triangular form, since the $B_i$'s commutes two by two. That is one can write the $B_i$'s as

$$B_1 = \begin{pmatrix}
\lambda_1^1 & * & * & \ldots & * \\
0 & \lambda_2^1 & * & \ldots & * \\
0 & 0 & \lambda_3^1 & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_k^1
\end{pmatrix}, \ldots, B_n = \begin{pmatrix}
\lambda_1^c & * & * & \ldots & * \\
0 & \lambda_2^c & * & \ldots & * \\
0 & 0 & \lambda_3^c & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_k^c
\end{pmatrix},$$

where the diagonal elements $\lambda_1^1, \lambda_2^1, \ldots, \lambda_k^1$ are the, not necessarily all distinct, eigenvalues of the endomorphism $B_i$.

The Hilbert-Chow map can be represented by:

$$HC : \text{Hilb}^{[c]}(\mathbb{C}^n) \longrightarrow S^{(c)}\mathbb{C}^n
\xrightarrow{\nu_i} \sum_i^k \nu_i[p_i].$$

where $\nu_i$ counts the number of repeating $n$-tuples $(\lambda_1^l, \lambda_2^l, \ldots, \lambda_{n-1}^l)$, for $l = 1, \cdots, c$ and $\{p_l = (\lambda_1^l, \lambda_2^l, \ldots, \lambda_{n-1}^l)\}_{l=1,\cdots,k}$ is a set of $k$ points in $\mathbb{C}^n$, each with multiplicity $\nu_i$. Indeed, $\sum_i^k \nu_i = c$ and $\sum_i^k \nu_i[p_i]$ is the topological support of the zero dimensional subscheme corresponding to the datum $[Z]$.

Of course, when all the eigenvalues are distinct, the endomorphisms $B_i$ are all, simultaneously diagonalizable and the multiplicity $\nu_i = 1$, for all $l = 1, \cdots, c$. Hence $\sum_i^c [p_i]$ is a point in the smooth stratum $S^{(c)}\mathbb{C}^n \setminus \Delta$, where $\Delta \subset S^{(c)}\mathbb{C}^n$ is the big diagonal.
9. The Hilbert scheme of points on affine varieties

In this section we realise a scheme-theoretic bijection between ideals of zero dimensional subschemes, with constant Hilbert polynomial $c$, on affine varieties $Y$ and points of a subscheme of $\text{Hilb}^{[c]}(\mathbb{C}^n)$ which is defined out of the ideal associated to $Y$.

Let us consider the following data: Let $Y = Z(Y) \subset \mathbb{C}^n$ be an affine variety, given by the zero locus of the ideal $Z_Y \subseteq \mathbb{C}[x_1, \cdots, x_n]$. We denote by $A(Y)$ the affine coordinate ring of the variety $Y$, i.e., $A(Y) = \frac{\mathbb{C}[x_1, \cdots, x_n]}{Z_Y}$.

To each stable datum $X = (\bar{B}, I)$, as defined in Section 2 one can associate a unique ideal $J \subset \mathbb{C}[x_1, \cdots, x_n]$, up to a $GL(V)$ action, such that the quotient $\mathbb{C}[x_1, \cdots, x_n] / J$ is a vector space of dimension $c$. In other words, $J$ is an ideal corresponding to a zero dimensional subschemes, of length $c$, of the affine space $\mathbb{C}^n$.

Theorem 3.1. Then one has the following exact sequence

$$0 \to J \xrightarrow{\phi_X} \mathbb{C}[x_1, \cdots, x_n] \xrightarrow{\Phi_X} V \to 0,$$

where $\phi_X$ is defined by $\phi_X(p(x_1, \cdots, x_n)) := p(B_1, \cdots, B_n)I(1)$ for any polynomial $p \in \mathbb{C}[x_1, \cdots, x_n]$.

Furthermore, suppose that $X = (\bar{B}, I)$ satisfies the condition $f(\bar{B}) = 0$, $\forall f \in Z_Y$. Then $\phi_X(f) = 0$ for all $f \in Z_Y$, and we have an injective map $Z_Y \xhookrightarrow{i} J$. Moreover there exists an injective map $J' = J \xrightarrow{i'} \mathbb{C}[x_1, \cdots, x_n]$, since $J \xrightarrow{\pi'} J'$ is surjective then for any $g' \in J'$ there exists an element $g \in J$ such that $g' = \pi'(g)$. Taking the image of $g$ under the composition $J \xrightarrow{j'} \mathbb{C}[x_1, \cdots, x_n] \xrightarrow{\pi} A(Y)$ one has the well defined map $j'$, since any difference in the choice of $g$ lies in the ideal $Z_Y$. One can resume the above situation in the following commutative diagram

$$
\begin{array}{cccccc}
0 & \to & J & \xrightarrow{j} & \mathbb{C}[x_1, \cdots, x_n] & \xrightarrow{\phi_X} V & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & J' & \xrightarrow{j'} & A(Y) & \xrightarrow{\phi_X'} V & \to 0 \\
\end{array}
$$

Moreover, we have $\phi_X = \phi_X' \circ \pi$ and the correspondence $X \leftrightarrow J := \ker \Phi_X$ is unique up to isomorphism (Theorem 3.1). Since $\pi$ is fixed for the given variety $Y$ and its kernel is obviously given by $f \in Z_Y$, then the map that associates $J' := \ker \Phi_X$ to $X = (\bar{B}, I) \in \mathbb{V}(n, c)^{ab}$, such that $f(\bar{B}) = 0$ for all $f \in Z_Y$, is also unique up to isomorphism.

Now, assume we have an ideal $J' \subset A(Y)$ such that $V := \frac{A(Y)}{J'}$ is a $\mathbb{C}$-vector space of dimension $c$, that is,

$$0 \to J' \xrightarrow{j'} A(Y) \xrightarrow{\phi_X'} V \to 0.$$
Then one has commutative diagram as above, where $\Phi : \mathbb{C}[x_1, \ldots, x_n] \to V$ is given by the composition $\mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\pi} A(Y) \xrightarrow{\Phi'} V$. Define $J := \ker \Phi$. Then, by Theorem 3.1 one can define a datum $X = (\tilde{B}, I) \in \mathcal{V}(n, c)^{st}$ up to a $GL(V)$ action. Furthermore, there is a map $\pi' : J \to J'$ with $\ker \pi' = Z_Y$. Thus, $\Phi(f) := f(\tilde{B})I(1) = 0$, and by the stability of $X = (\tilde{B}, I)$ we have $f(\tilde{B}) = 0$, for all $f \in Z_Y$.

Now we put $\mathcal{V}_Y(c)^{st} : \{X = (\tilde{B}, I) \in \mathcal{V}(n, c)^{st}/f(\tilde{B}) = 0, \ \forall f \in Z_Y\}$. Hence we have the following:

**Theorem 9.1.** There exists a set-theoretical bijection between the quotient space $\mathcal{M}_Y(c) := \mathcal{V}_Y(c)^{st}/GL(V)$ and the Hilbert scheme $\text{Hilb}_Y^c$ of $c$ points in $Y$.

### 9.1. Scheme structure on $\mathcal{M}_Y(c)$.

The scheme structure of $\mathcal{M}_Y(c)$ is given as the following: For a given datum $X = (B_1, \cdots, B_n, I) \in \mathcal{V}(n, c)^{st}$, the map $\phi_X'$ is given by

$$
\phi_X' : A(Y) \to V, \quad [p \mod Z_Y] \mapsto \phi'([p \mod (Z_Y)]) := [p(\tilde{B}) \mod (Z_Y)]I(1).
$$

By stability, one has $\ker \phi_X' = \{f \in A(Y) = \mathbb{C}[x_1, \ldots, x_n]/f(B_1, \cdots, B_n) = 0\}$. In particular $f(B_1, \cdots, B_n) = 0$ for all $f \in Z_Y$. Conversely, one can define an ideal $\tilde{Z}_Y$ in the ring of regular functions $\Gamma(\mathcal{V}(n, c)^{st})$, on $\mathcal{V}(n, c)^{st}$, which is defined by $\tilde{Z}_Y = \{f(B_1, \cdots, B_n) = 0 \in \text{End}(V) \text{ for } f \in Z_Y\}$. Then $\mathcal{V}_Y(c)^{st}$ is the subscheme of $\mathcal{V}(n, c)^{st}$ given by $\tilde{Z}_Y \subset \Gamma(\mathcal{V}(n, c)^{st})$. Of course, by choosing a basis for $V$, one has $\text{End}(V) \cong \text{Mat}_{c \times c}(\mathbb{C})$ and every polynomial equation $f \in Z_Y$ gives rise to $c$ polynomial equations in $\tilde{Z}_Y$ given the rows of $f(B_1, \cdots, B_n)$. Thus, if $Z_Y$ is generated by $k$ element in $\mathbb{C}[x_1, \cdots, x_n]$, then $\tilde{Z}_Y$ will be generated by, at most, $c \times k$ polynomials in the entries $b_{ab}^{i}$ of $B_i$, for $i = 1, \ldots, n$, and $a, b = 1, \cdots, c$.

Now, consider $O_{\text{Hilb}_Y^c} := \bigoplus_{i \geq 0} \Gamma((\mathcal{V}(n, c))^{G \cdot X^i})$, where $\Gamma((\mathcal{V}(n, c))^{G \cdot X^i})$ is the ring equivariant regular functions on $\mathcal{V}(n, c)$, of weight $i$ with respect to the character $\chi$ corresponding to the $GL(V)$—action, as defined in Section 2. One can form the space

$$
\mathcal{M}_Y(c) := \text{Proj} \left( \bigoplus_{i \geq 0} \left( \frac{\Gamma(\mathcal{V}(n, c))}{\tilde{Z}_Y} \right)^{G \cdot X^i} \right) \hookrightarrow \text{Hilb}_Y^c(\mathbb{C}^n) := \text{Proj} \left( \bigoplus_{i \geq 0} \Gamma((\mathcal{V}(n, c))^{G \cdot X^i}) \right).
$$

Hence, $\mathcal{M}_Y(c)$ is a closed subscheme of the Hilbert scheme of points $\text{Hilb}_Y^c(\mathbb{C}^n)$, and represents, of course a subfunctor, $\mathcal{M}_Y(c)$ of the Hilbert functor $\text{Hilb}_Y^c(\cdot)$.
9.2. The schematic isomorphism $\mathcal{M}_Y(c) \cong \text{Hilb}_Y^c$. Recall that the Hilbert scheme of points $\text{Hilb}_Y^c$ represents the functor $\mathcal{H}ilb_Y^c(\cdot) : \text{Sch} \rightarrow \text{Set}$ which associates to any noetherian scheme of finite type $S$ the set

$$
\text{Hilb}_Y^c(S) := \begin{cases}
Z \subset Y \times S & \text{if } Z \text{ is a closed subscheme,} \\
\cap C^n \times S & \pi \downarrow \downarrow q \text{ with } \pi \text{ flat, and } \\
S & S \simeq S \\
\chi(O_{\pi^{-1}(s)} \otimes O_Y(m)) = c \quad \forall m \in \mathbb{Z}
\end{cases}
$$

of flat families of 0-dimensional subschemes of $Y \xrightarrow{\cdot} C^n$. In particular there exists a universal flat family $\mathcal{X} \rightarrow Y \times \text{Hilb}_Y^c$ of 0-dimensional subschemes of length $c$ on $Y$. Moreover, $\mathcal{X}$ is a flat subfamily of the universal family $\tilde{\mathcal{X}} \subset C^n \times \text{Hilb}_Y^c$, of 0-dimensional subschemes of length $c$ in $C^n$. This follows from the fact that $\text{Hilb}_Y^c(\cdot) \xrightarrow{h^*} \text{Hilb}_{C^n}^c(\cdot)$ is a subfunctor and $\mathcal{X}$ is just the restriction $h^*\tilde{\mathcal{X}}$ of $\tilde{\mathcal{X}}$, for the morphism $h := i \times h : Y \times \text{Hilb}_Y^c \rightarrow C^n \times \text{Hilb}_{C^n}^c$. Furthermore, since the scheme $\mathcal{M}_Y(c)$ also parameterizes 0-dimensional subschemes of length $c$ in $Y \rightarrow C^n$, then any flat family $\mathcal{Z} \subset Y \times \mathcal{M}_Y(c)$ is the pull-back of the family $\mathcal{X}$ under a morphism $Y \times \mathcal{M}_Y(c) \xrightarrow{id_Y \times f} Y \times \text{Hilb}_Y^c$. One can resume the above situation in the following diagram:

\[
\begin{array}{c}
\xymatrix{ 
\mathcal{X} \ar[d]_{\tilde{\beta}} & \tilde{\mathcal{X}} \ar[l] \ar[d] & Y \times \text{Hilb}_Y^c \ar[dl]_{\tilde{f}} \ar[r]_{\beta} \ar[d]_{\alpha} & Y \times \mathcal{M}_Y(c) \ar[d]_{\alpha} \ar[l]_{f} \\
\mathcal{X} \ar[d]_{\beta} & \mathcal{X} \ar[l] \ar[d] & C^n \times \text{Hilb}_{C^n}^c \ar[l]_{\alpha} & Y \times \mathcal{M}_Y(c) \ar[l]_{\beta} & \mathcal{X} \ar[l] \ar[d] \ar[l] \ar[l] \\
\text{Hilb}_Y^c & \text{Hilb}_{C^n}^c & \mathcal{X} & \mathcal{X} & \mathcal{X}
\end{array}
\]

where $\tilde{\alpha} := i \times \beta$, $\tilde{\beta} := i \times \beta$ and $\tilde{f} := id_Y \times f$. On the other hand, $\mathcal{M}_Y(c)$ represent a closed subfunctor of the Hilbert functor $\text{Hilb}_Y^c(\cdot)$ such that, for any closed point $\text{spec}(k)$ such that the following diagram commutes:

\[
\begin{array}{c}
\xymatrix{ 
\text{Hilb}_Y^c \ar[r]_{\beta} \ar[d]_{\alpha} & \mathcal{M}_Y(c) \ar[d]_{\alpha} \\
\text{Hilb}_{C^n}^c & \text{spec}(k) \ar[l]_{\rho} \ar[l]_{\rho}
\end{array}
\]

where $\rho := \rho$.
One has $\text{Hom}(\text{spec}(k), \mathcal{M}_Y(c)) \cong \text{Hilb}_Y^{[c]}(\text{spec}(k))$, since $f$ is isomorphic on points, as it follows from Theorem 9.1.

On the other hand, let us consider the restriction of the universal monad \( \mathcal{Y} \) to $Y \times \mathcal{V}_Y(c)^{st}$. This gives the following extended monad

$$
\mathcal{O}_Y(1 - n) \otimes (V_1 \otimes \mathcal{O}_{\mathcal{V}_Y(c)^{st}}) \to \cdots \to \mathcal{O}_Y(2 - n) \otimes (V_1^{\otimes (\cdot - 1)} \otimes \mathcal{O}_{\mathcal{V}_Y(c)^{st}}) \to \cdots
$$

(35) \( \cdots \to \mathcal{O}_Y \otimes ((V_1^{\otimes n} \otimes W) \otimes \mathcal{O}_{\mathcal{V}_Y(c)^{st}}) \to \mathcal{O}_Y(1) \otimes (V_1 \otimes \mathcal{O}_{\mathcal{V}_Y(c)^{st}}) \)

We denote by $\mathfrak{z}$ the family of ideals which arises as its cohomology. The restriction at any point $\text{spec}(k) \to \mathcal{V}_Y(c)^{st}$ one has an $(n - 2)$—extended monad whose cohomology is an ideal of $c$ points on $Y$ represented by the stable datum $(B_1, \cdots, B_n, I)$ which satisfies $f(B_1, \cdots, B_n) = 0$, for all $f \in \mathcal{Z}_Y$.

Now for any noetherian scheme of finite type $S$, parameterizing zero-dimensional schemes of $Y$, suppose there exists a flat family of ideals $\eta$ on $Y \times S$. We consider an open cover $\{S_i\} \subset S$ of $Y$. Then, for a fixed $i \in J$, $\eta_i := \eta(Y \times S_i)$ is the cohomology of the following $(n - 2)$—extended monad on $S_i$

$$
\mathcal{O}_Y(1 - n) \otimes (V_1 \otimes \mathcal{O}_{S_i}) \to \cdots \to \mathcal{O}_Y(2 - n) \otimes (V_1^{\otimes (\cdot - 1)} \otimes \mathcal{O}_{S_i}) \to \cdots
$$

(36) \( \cdots \to \mathcal{O}_Y \otimes ((V_1^{\otimes n} \otimes W) \otimes \mathcal{O}_{S_i}) \to \mathcal{O}_Y(1) \otimes (V_1 \otimes \mathcal{O}_{S_i}) \)

At each point $s \in S_i$, this gives an $(n - 2)$—extended monad whose cohomology is an ideal of zero dimensional subscheme of $Y$. In particular, there exists a datum in $\mathcal{V}_Y(c)^{st}$ representing it. Since $S_i$ is an open subset, one obtains a morphism $g_{\eta_i} : S_i \to \mathcal{V}_Y(c)^{st}$, such that on the overlaps of the form $S_i \cap S_j$ one has

$$
g_{\eta_i}(s) \sim_{\text{GL}(V_1)} g_{\eta_j}(s), \quad \forall s \in S_i \cap S_j.
$$

Thus giving rise to a well defined morphism $g_\eta : S \to \mathcal{M}_Y(c)$ such that $\eta = (id_Y \times g_\eta)^* \mathfrak{z}$. In particular, there exists a morphism $h_X : \text{Hilb}_Y^{[c]} \to \mathcal{M}_Y(c)$, such that $X = (id_Y \times h_X)^* \mathfrak{z}$ and $\text{Hilb}_Y^{[c]}(\text{spec}(k)) \cong \text{Hom}(\text{spec}(k), \mathcal{M}_Y(c))$, that is, $h = f^{-1}$ as a set-theoretic maps.

Moreover, for the inclusions $\tilde{\alpha} : Y \times \mathcal{M}_Y(c) \to \mathbb{C}^n \times \text{Hilb}_Y^{[c]}$ and $\tilde{\beta} : Y \times \text{Hilb}_Y^{[c]} \to \mathbb{C}^n \times \text{Hilb}_Y^{[c]}$, one has $\tilde{\alpha}_*(\mathcal{Z}(U)) := \tilde{\beta}_*(\mathcal{X}(U)) := \mathcal{X}(U \cap \mathbb{C}^n \times \text{Hilb}_Y^{[c]})$, on every open $U \subset \mathbb{C}^n \times \text{Hilb}_Y^{[c]}$. Furthermore, for a point $x = \text{spec}(k)$ as in (34), we have an isomorphism of stalks

$$
\lim_{V \ni x} (\tilde{\alpha}_*(\mathcal{Z}(V)) \cong \lim_{V \ni x} \tilde{\beta}_*(\mathcal{X}(V)),
$$

for all $V \subseteq U$, in some directed system. Finally, the families $\mathfrak{z}$ and $\mathfrak{x}$ are both restrictions, of the universal family $\mathcal{F} \to \mathbb{C}^n \times \text{Hilb}_Y^{[c]}$, to subschemes, with the same topological support, and such that $\mathfrak{z} = (id_Y \times h_X)^* \mathfrak{z} = [id_Y \times (h_X \circ f)]^* \mathfrak{x}$ and $\mathfrak{z} = (id_Y \times f)^* \mathfrak{z} = [id_Y \times (f \circ h_X)]^* \mathfrak{x}$. Hence $f$ is lifted to an (unique) isomorphism of schemes $\text{Hilb}_Y^{[c]} \cong \mathcal{M}_Y(c)$, with inverse $h_X$. 

10. **Irreducible components of the Hilbert scheme of points**

The variety $\mathcal{C}(n, c)$ of $n$ commuting $c \times c$ matrices have been much studied by various authors since a 1961 paper by Gerstenhaber [7]. The results concerning the irreducibility of $\mathcal{C}(n, c)$ can be summarized as follows:

- $\mathcal{C}(2, c)$ is irreducible for every $c$ (originally proved by Motzkin and Taussky [14], see also [7]);
- $\mathcal{C}(3, c)$ is irreducible for $c \leq 10$ and reducible for $c \geq 30$, see [11] [21] and the references therein;
- for $n \geq 4$, $\mathcal{C}(n, c)$ is irreducible if and only if $c \leq 3$ [7].

In particular, determining the highest possible value of $c$ for which $\mathcal{C}(3, c)$ is irreducible is an important open problem.

On the other hand, much less is known about the irreducibility of the Hilbert scheme $\text{Hilb}^{[c]}(\mathbb{C}^n)$ of $c$ points on $\mathbb{C}^n$, see for instance [2] Section 7 and the references therein.

- $\text{Hilb}^{[c]}(\mathbb{C}^2)$ is irreducible for every $c$, see [3];
- $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible for $c \leq 8$ [2] Theorem 1.1, while it is reducible for $c \geq 78$, cf. [13];
- for $n \geq 4$, $\text{Hilb}^{[c]}(\mathbb{C}^n)$ is irreducible if and only if for $c \leq 7$ [2] Theorem 1.1.

Similarly, determining the highest possible value of $c$ for which $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible is also an important open question.

In this section, we connect the two problems through the following result.

**Proposition 10.1.** The number of irreducible components of $\text{Hilb}^{[c]}(\mathbb{C}^n)$ is smaller than, or equal to, the number of irreducible components of $\mathcal{C}(n, c)$. In particular, if $\mathcal{C}(n, c)$ is irreducible, then $\text{Hilb}^{[c]}(\mathbb{C}^n)$ is also irreducible.

**Proof.** Clearly, the number of irreducible components of $\mathcal{C}(n, c)$ is the same as the number of irreducible components of $\mathcal{V}(n, c) := \mathcal{C}(n, c) \times \text{Hom}(W, V)$. Let $\mathcal{V}_1(n, c), \ldots, \mathcal{V}_p(n, c)$ denote the irreducible components of $\mathcal{V}(n, c)$, and set $\mathcal{V}_l(n, c)^{st} := \mathcal{V}_l(n, c) \cap \mathcal{V}(n, c)^{st}$, with $l = 1, \ldots, p$.

It is possible that some components of $\mathcal{V}(n, c)$ contain no stable points; one can then order the irreducible components of $\mathcal{V}(n, c)$ in such a way that $\mathcal{V}_l(n, c)^{st} \neq \emptyset$ for $l = 1, \ldots, q$ and $\mathcal{V}_l(n, c)^{st} = \emptyset$ for $l = q + 1, \ldots, p$.

Since the group $G := GL(V)$ is irreducible, it is easy to see that if $x \in \mathcal{V}_l(n, c)^{st}$ then its orbit $G \cdot x \subset \mathcal{V}_l(n, c)^{st}$. Note also that

$$\mathcal{V}_l(n, c) / /_x G = \mathcal{V}_l(n, c)^{st} / G$$

is irreducible, for each $l = 1, \ldots, q$.

Since the GIT quotient $\mathcal{M}(n, c)$ coincides, by Proposition 2.7 with the set of stable $G$-orbits, we have that

$$\mathcal{M}(n, c) = \left( \mathcal{V}_1(n, c)^{st} / G \right) \cup \cdots \cup \left( \mathcal{V}_q(n, c)^{st} / G \right)$$

and the desired conclusion follows from Corollary 7.3.

As an immediate consequence of [21] Theorems 26 & 32, we obtain the following new result on the irreducibility of the Hilbert scheme of points in dimension 3.

**Corollary 10.2.** $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible for $c \leq 10$. 
As a final comment, we remark that determining which components of $V(n,c)$ admit stable solutions seems to be a very interesting problem both from the linear algebra and the algebraic geometry points of view. More precisely, given an $n$-tuple of commuting matrices $(B_1, \ldots, B_n)$, when is it possible to find a deformation $(B'_1, \ldots, B'_n)$ and a vector $I$ such that the datum $(B'_1, \ldots, B'_n, I)$ is stable?

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