LORENTZ-MORREY REGULARITY FOR NONLINEAR ELLIPTIC PROBLEMS WITH IRREGULAR OBSTACLES OVER REIFENBERG FLAT DOMAINS

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Abstract. A global Calderón-Zygmund estimate type estimate in Weighted Lorentz spaces and Lorentz-Morrey spaces is obtained for weak solutions to elliptic obstacle problems of $p$-Laplacian type with discontinuous coefficients over Reifenberg flat domains.

1. Introduction. An obstacle problem arises when one studies the shape of a soap film holding a fixed boundary condition and constrained by an obstacle, see Plateau’s problem, and also it arises when one finds the optimal stopping time of a stochastic problem with a payoff function in control theory. Roughly speaking, an obstacle problem treats a partial differential equation in a given constrained situation, the so-called obstacle. Areas of its numerous applications include computer science, economics, biology, engineering, etc., see [3, 11, 17, 24, 27].

In this paper, we investigate the solvability to nonlinear elliptic obstacle problems in both weighted Lorentz spaces and Lorentz-Morrey spaces by obtaining global Calderón-Zygmund type estimates in those spaces. Lorentz spaces are generalization of familiar Lebesgue spaces. We refer to [2, 12, 18, 30] for an introduction of Lorentz spaces and their applications. They play an important role in a mathematical analysis to provide tighter control for both the height and spread of the graph of a function versus the Lebesgue spaces. There have been rich research activities in those spaces, see [1, 13, 16, 20, 21, 25] and references therein.

Our work is motivated by the recent papers [20, 21]. In [20] the global regularity for an unconstrained nonlinear elliptic equation was established when the $p$-Laplacian type nonlinearity is given by $a(\xi, x) = (A(x)\xi \cdot \xi)^{\frac{p-2}{2}}A(x)\xi$ in weighted

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Lorentz spaces and Lorentz-Morrey spaces on a nonsmooth domain. In [21] quasi-linear Riccati type equations with distributional data were investigated for the existence and regularity estimates in the framework of Morrey or Lorentz-Morrey spaces. We here consider nonvariational problems having general nonlinearities of $p$-Laplacian type and involving irregular obstacles. In a given obstacle problem, the regularity of a solution is deeply related to that of the obstacle and the nonhomogeneous term. Indeed it was proved that the gradient of a solution to an elliptic obstacle problem of the type considered in this work is as globally integrable as that of the obstacle and the nonhomogeneous term in Lebesgue spaces, see [8]. We refer to [1, 4, 9, 10, 28, 29] for a further discussion on the regularity of obstacle problems in the literature.

For the global Calderón-Zygmund type estimates, we need as necessary make various additional assumptions on the nonlinearity and a suitable geometric assumption on the boundary of the underlying domain. The paper [8] suggests a smallness assumption in BMO on the nonlinearity and a sufficiently flatness on the boundary for a nonlinear elliptic problem with irregular obstacle in the framework of Lebesgue spaces. This result was extended to the setting of weighted Lebesgue spaces and Morrey spaces in [7]. As its natural continuation, this work argues that similar results can be obtained within the more general framework of weighted Lorentz spaces and Lorentz-Morrey spaces, respectively. Our results ensure the existence of solutions to elliptic obstacle problems in general spaces. In the weighted Lorentz space $L^{q,t}_{w}(\Omega)$, see (11)-(12), we need to make a very careful analysis of each computation by considering the parameters $q$, $t$ and the weight $w$ lying in a suitable Muckenhoupt class. For the estimate in the Lorentz-Morrey space $L^{q,t;\theta}(\Omega)$, see (16), we will use the global estimate in the weighted Lorentz space by taking a special weight function.

The paper is organized as follows. In the next section, we introduce our problem and main assumptions on the nonlinearity and the boundary, and then review preliminaries about weighted Lorentz spaces and Lorentz-Morrey spaces to state our main results. In the last section, we present the main analytic and geometric tools to give a complete derivation of the required estimates.

2. Preliminaries and main results. Let $p \in (1, \infty)$ be a fixed real number and $\Omega \subset \mathbb{R}^n$ a bounded domain with nonsmooth boundary $\partial \Omega$, $n \geq 2$. For a given obstacle function $\psi \in W^{1,p}(\Omega)$ with $\psi \leq 0$ a.e. on $\partial \Omega$, we consider the convex admissible set

$$
\mathcal{C} = \left\{ v \in W^{1,p}_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega \right\}.
$$

(1)

Then we are interested in investigating a solution $u \in \mathcal{C}$ to the following variational inequality

$$
\int_{\Omega} a(Du, x) \cdot D(v - u) \, dx \geq \int_{\Omega} |F|^{p-2} F \cdot D(v - u) \, dx \quad \text{for all } v \in \mathcal{C}.
$$

(2)

Here the nonhomogeneous term $F$ is a vector valued function in $L^p(\Omega, \mathbb{R}^n)$ and the nonlinearity term $a(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in $\xi$ for almost every $x \in \mathbb{R}^n$, and is measurable in $x$ for all $\xi \in \mathbb{R}^n$. Additionally, we assume that there are constants $0 < \gamma_1 \leq 1 \leq \gamma_2$ so that $a(\xi, x)$ has the following growth and ellipticity conditions:

$$
|a(\xi, x)| + |\xi| |D_\xi a(\xi, x)| \leq \gamma_2 |\xi|^{p-1}
$$

(3)
and
\[ D_\xi a(\xi, x) \eta \cdot \eta \geq \gamma_1 |\xi|^{p-2} |\eta|^2 \]
for almost every \( x \in \mathbb{R}^n \) and for all \( \xi, \eta \in \mathbb{R}^n \).

With the basic structure conditions (3) and (4), it is well known that there exists a unique solution \( u \in C \) to the variational inequality (2) with the estimate
\[ \|Du\|_{L^p(\Omega)} \leq c \left( \|F\|_{L^1(\Omega)} + \|D\psi\|_{L^1(\Omega)} \right), \]
the constant \( c \) depending only on \( n, p, \gamma_1 \) and \( \gamma_2 \), see [4, 8, 14].

As announced earlier in Introduction, the purpose of this paper is to show the well-posedness in both weighted Lorentz spaces \( L_{q,t}^{r,\epsilon}(\Omega) \) and Lorentz-Morry spaces \( L_{q,t}^{r,\epsilon}(\Omega) \) by essentially replacing \( L^1(\Omega) \) by both \( L_{q,1}^{1,\epsilon} \) and \( L_{q,\epsilon}^{1,1} \). We clearly point out that \( L_{q,1}^{1,\epsilon} \) is strictly contained in \( L^1 \) and so is \( L_{q,\epsilon}^{1,1} \). We will return to make a brief description about these function spaces later in this section.

According to the regularity results in [15, 22], we know that such well-posedness in general can not hold true for every \( q \in (1, \infty) \), even if \( q = t \) and \( w \equiv 1 \). One needs some extra regularity assumptions on both \( a \) and \( \partial \Omega \). We report here that they are sufficient smallness of \( a \) in \( BMO \) and \( \delta \)-Reifenberg flatness of \( \partial \Omega \), as we now introduce.

For a ball \( B_r(y) \) with center \( y \in \mathbb{R}^n \) and radius \( r > 0 \), we consider a function on \( B_r(y) \) defined by
\[ \beta(a, B_r(y))(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| \frac{a(\xi, x) - \bar{a}_{B_r(y)}(\xi)}{|\xi|^{p-1}} \right|, \]
where
\[ \bar{a}_{B_r(y)}(\xi) := \int_{B_r(y)} a(\xi, x) \, dx = \frac{1}{|B_r(y)|} \int_{B_r(y)} a(\xi, x) \, dx \]
is the integral average of \( a(\xi, \cdot) \) over \( B_r(y) \).

**Definition 2.1.** We say that \( (a, \Omega) \) is \( (\delta, R) \)-vanishing, if there exist positive constants \( \delta \) and \( R \) such that \( a \) satisfies
\[ \sup_{0 < r \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_r(y)} \left| \beta(a, B_r(y))(x) \right| \, dx \leq \delta, \]
and for every \( x \in \partial \Omega \) and every \( r \in (0, R] \) there is a coordinate system \( \{z_1, \cdots, z_n\} \), which may depend on \( x \) and \( r \), with origin at \( x \), so that
\[ B_r(0) \cap \{z: z_n > \delta r\} \subset B_r(0) \cap \partial \Omega \subset B_r(0) \cap \{z: z_n > -\delta r\}. \]

**Remark 1.** We remark that \( R \) can be any positive number by a scaling, even we will use a specific number for the sake of convenience. On the other hand \( \delta \in (0, \frac{1}{8}) \) is to be selected so small that our main results hold true.

This \( (\delta, R) \)-vanishing assumption has been considered to be a minimal regularity condition for global Calderón-Zygmund type estimates of the gradient of solutions to divergence form problems of \( p \)-Laplacian type, as it follows from the papers [5, 19, 20, 32, 33] and references therein. Roughly speaking, this assumption is that \( \frac{a(\xi, x)}{|\xi|^{p-1}} \) has a small deviation from being its integral average with respect to \( x \), uniformly in \( \xi \), while the boundary can be locally trapped between two hyperplanes and includes even rough fractal boundaries such as Van Koch snowflake, see [26, 31].
Another point in this paper is that we are dealing with Muckenhoupt weights. We recall here that the *Muckenhoupt class* \( A_s \), \( 1 < s < \infty \), consists of all positive functions \( w \in L^1_{loc}(\mathbb{R}^n) \) such that the quantity

\[
[w]_s = \sup_{y \in \mathbb{R}^n} \sup_{r > 0} \left( \frac{\int_{B_r(y)} w(x) \, dx}{\int_{B_r(y)} (\frac{1}{r^s})^s \, dx} \right)^{s-1}
\]

(8)
is finite. We denote \( w(D) \) for any measurable set \( D \subset \mathbb{R}^n \), to mean that

\[
w(D) = \int_D w(x) \, dx.
\]

(9)

As a typical example for the Muckenhoupt class \( A_s \), we look at the weight defined by \( w(x) = |x|^{-\tau}, -n < \tau < n(s - 1) \). This Muckenhoupt class \( A_s \) is monotone increasing in \( s \). Moreover it satisfies the open-end property which says that for any \( w \in A_s, 1 < s < \infty \), there exists \( \tau = \tau(n, s, [w]_s) > 0 \) such that \( s - \tau > 1 \) and \( w \in A_{s-\tau} \). In other words, we have

\[
A_s = \bigcup_{s' \in (1,s)} A_{s'},
\]

(10)

see for instance \([12, \text{Corollary 9.2.6}]\).

We now recall the so called doubling property on the Muckenhoupt class \( A_s \).

**Lemma 2.2. ([12])** Let \( w \) be an \( A_s \) weight for some \( 1 < s < \infty \). Then there exist positive constants \( \nu \) and \( \alpha \), depending only on \([w]_s\) and \( n \), such that for any measurable subset \( E \) of a ball \( B \)

\[
\frac{1}{\alpha} \left( \frac{|E|}{|B|} \right)^s \leq \frac{w(E)}{w(B)} \leq \alpha \left( \frac{|E|}{|B|} \right) \nu.
\]

Thanks to this property, one can transfer the density results with respect to Lebesgue measures into those of weighted measures, see (26).

We will use the following lemma later in this paper of Theorem 2.5.

**Lemma 2.3. ([20])** Let \( w \in A_s \) for \( 1 < s < \infty, k > 0 \) and \( z \in \mathbb{R}^n \). Then

1. \( w_1(x) = w(x - z) \) is an \( A_s \) weight with the same \([w]_s\) constant.
2. \( w_2(x) = \min\{w, k\} \) is an \( A_s \) weight and satisfies

\[
[w_2]_s \leq c_s[w]_s,
\]

where \( c_s = \max\{2, 2^{s-1}\} \).

We now return to weighted Lorentz spaces and Lorentz-Morrey spaces which are the function spaces under consideration. We start with weighted Lorentz spaces. Fix \( 1 < q < \infty, 0 < t \leq \infty \) and \( w \in A_s \) for some \( 1 < s < \infty \). Then the weighted Lorentz space \( L_{w}^{q,t}(\Omega) \) is the collection of all measurable functions \( f \) defined on \( \Omega \) with

\[
\|f\|_{L_{w}^{q,t}(\Omega)} := \left( q \int_0^{\infty} \left( \lambda^{q} \left[ w(\{x \in \Omega : |f(x)| > \lambda\}) \right]^t \frac{d\lambda}{\lambda} \right)^{\frac{1}{t}} < \infty.
\]

(11)

For \( t = \infty \), the space \( L_{w}^{q,\infty}(\Omega) \) is known as Marcinkiewicz space with

\[
\|f\|_{L_{w}^{q,\infty}(\Omega)} := \sup_{\lambda > 0} \left[ \lambda^q w(\{x \in \Omega : |f(x)| > \lambda\}) \right]^\frac{1}{q} < \infty.
\]

(12)

In particular, the space \( L_{w}^{q,q}(\Omega) \) is the weighted Lebesgue space \( L_{w}^q(\Omega) \).
Remark 2. Let $\Omega \subset \mathbb{R}^n$ be bounded and let $w \in A_q$ for some $1 < q < \infty$. Then according to (10), $w \in A_{q-\tau}$ for some $\tau > 0$. Then we find that for all $0 < t \leq \infty$, $L^{q,t}_w(\Omega) \subset L^{q-\tau}_w(\Omega)$ by making use of standard H"{o}lder inequality in Marcinkiewicz spaces, see [23, Lemma 2.8]. We further observe that $L^{q,t}_w(\Omega) \subset L^1(\Omega)$ for any $w \in A_1$ with $1 < s < \infty$. We then see that for all $0 < t \leq \infty$,

$$L^{q,t}_w(\Omega) \subset L^1(\Omega)$$

(13)

with the estimate

$$\|f\|_{L^1(\Omega)} \leq c\|f\|_{L^{q,t}_w(\Omega)},$$

(14)

where $c$ is a constant independent from $f$.

Now we state one of the main results of the paper.

Theorem 2.4. Let $1 < q < \infty$ and $0 < t \leq \infty$. We assume that a weight $w \in A_q$, $|F|^p \in L^{q,t}_w(\Omega)$ and $|D\psi|^p \in L^{q,t}_w(\Omega)$. Then there exists a small constant $\delta = \delta(n,p,q,t,\gamma_1,\gamma_2, [w]_q) > 0$ such that if $(a, \Omega)$ is $(\delta, R)$-vanishing, then the gradient $Du$ of the weak solution $u$ to the variational inequality (2) satisfies $|Du|^p \in L^{q,t}_w(\Omega)$ and we have the estimate

$$\|Du\|^p_{L^{q,t}_w(\Omega)} \leq c \left( \|F|^p\|_{L^{q,t}_w(\Omega)} + \|D\psi|^p\|_{L^{q,t}_w(\Omega)} \right)$$

(15)

for some constant $c$ depending only on $n$, $p$, $q$, $t$, $\gamma_1$, $\gamma_2$, $[w]_q$ and $\Omega$.

Remark 3. Under the assumptions of Theorem 2.4 we ensure the existence of the unique weak solution to (2) from the estimate (5) and Remark 2. The case that $q = t$ and $w \equiv 1$ was studied in [8] while the case that $q = t$ and $w \in A_q$ was done in [7]. Interior results for Theorem 2.4 were obtained in the recent paper [1]. Thus the present work is a natural growth of the previous works [1, 8, 7].

We next recall the Lorentz-Morrey spaces. Fix $1 < q < \infty$, $0 < t \leq \infty$ and $0 < \theta < n$. Then a function $f$ belongs to $L^{q,t,\theta}(\Omega)$, if

$$\|f\|_{L^{q,t,\theta}(\Omega)} := \sup_{0 < r \leq \text{diam } \Omega} r^{-\frac{n}{\theta}} \|f\|_{L^{q,t}(B_r(z) \cap \Omega)} < \infty.$$  

(16)

When $q = t$ the space $L^{q,q,\theta}(\Omega)$ is the usual Morrey space.

We then state our second regularity result for the variational inequality (2) in the setting of Lorentz-Morrey spaces.

Theorem 2.5. Let $1 < q < \infty$, $0 < t \leq \infty$ and $0 < \theta < n$. Assume that $|F|^p \in L^{q,t,\theta}(\Omega)$ and $|D\psi|^p \in L^{q,t,\theta}(\Omega)$. Then there exists a small positive constant $\delta = \delta(n,p,q,\theta,t,\gamma_1,\gamma_2)$ such that if $(a, \Omega)$ is $(\delta, R)$-vanishing, then the gradient $Du$ of the weak solution $u$ to (2) satisfies $|Du|^p \in L^{q,t,\theta}(\Omega)$ with the estimate

$$\|Du\|^p_{L^{q,t,\theta}(\Omega)} \leq c \left( \|F|^p\|_{L^{q,t,\theta}(\Omega)} + \|D\psi|^p\|_{L^{q,t,\theta}(\Omega)} \right)$$

(17)

for some constant $c$ depending only on $n$, $p$, $q$, $\theta$, $t$, $\gamma_1$, $\gamma_2$ and $\Omega$.

Remark 4. From the definition of the Lorentz-Morrey space $L^{q,t,\theta}(\Omega)$ and Remark 2, we see that $L^{q,t,\theta}(\Omega) \subset L^{q,t}(\Omega) \subset L^1(\Omega)$ for $1 < q < \infty, 0 < t \leq \infty$ and $0 < \theta < n$. Then under the assumptions of Theorem 2.5, there is a unique weak solution $u$ to (2). We would like to point out that the case $q = t$ was investigated in [7].
3. Global gradient estimates in weighted Lorentz spaces and Lorentz-Morrey spaces. In this section we shall give the global gradient estimates in weighted Lorentz spaces and Lorentz-Morrey spaces announced in the previous section. Our approach is based on the Hardy-Littlewood maximal function operator, standard measure theory and a Vitali type covering lemma.

Recall that for any function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), the Hardy-Littlewood maximal function \( Mf \) is given by
\[
(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy.
\]
If \( f \) is defined only on a bounded domain \( D \), we define
\[
Mf = M(f\chi_D)
\]
where \( \chi_D \) is a standard characteristic function on \( D \).

We point out that from the inclusion (13) the maximal function operator is well defined on the spaces \( L^{q,t}_{w}(\mathbb{R}^n) \) with respect to \( w \in A_q \) for \( 1 < q < \infty \) and \( 0 < t \leq \infty \).

**Lemma 3.1.** Let \( 1 < q < \infty \) and \( 0 < t \leq \infty \). Assume \( w \in A_q \). Then there is a constant \( c = c(n,q,[w]_q) > 0 \) such that
\[
1/c \| f \|_{L^{q,t}_{w}(\mathbb{R}^n)} \leq \| Mf \|_{L^{q,t}_{w}(\mathbb{R}^n)} \leq c \| f \|_{L^{q,t}_{w}(\mathbb{R}^n)} \quad \text{for each } f \in L^{q,t}_{w}(\mathbb{R}^n).
\]
Furthermore we have the following weak 1-1 estimate
\[
\left| \{ x \in \mathbb{R}^n : (Mf)(x) > \lambda \} \right| \leq \frac{c}{\lambda} \int |f(x)| \, dx \quad \text{for every } \lambda > 0
\]
with a constant \( c = c(n) \).

We will use the following standard measure theory in the weighted Lorentz spaces.

**Lemma 3.2.** Assume that \( f \) is a nonnegative measurable function on a bounded domain \( D \subset \mathbb{R}^n \). Let \( \sigma > 0 \) and \( m > 1 \) be constants and \( w \in A_q \) with \( 0 < q < \infty \). Then for \( 0 < t < \infty \),
\[
f \in L^{q,t}_{w}(D) \iff S := \sum_{k \geq 1} m^{tk} \left[ w(\{ x \in D : f(x) > \sigma m^k \}) \right]^\frac{1}{t} < \infty
\]
and
\[
\frac{1}{c} S \leq \| f \|_{L^{q,t}_{w}(D)} \leq c \left( [w(D)]^\frac{1}{q} + S \right),
\]
where \( c > 0 \) is a constant depending only on \( \theta \), \( m \) and \( t \).

On the other hand, for \( t = \infty \),
\[
f \in L^{q,\infty}_{w}(D) \iff T := \sup_{k \geq 1} m^k \left[ w(\{ x \in D : f(x) > \sigma m^k \}) \right]^\frac{1}{t} < \infty
\]
and
\[
\frac{1}{c} T \leq \| f \|_{L^{q,\infty}_{w}(D)} \leq c \left( [w(D)]^\frac{1}{q} + T \right),
\]
where \( c > 0 \) is a constant depending only on \( \theta \) and \( m \).

We also use the following Vitali type covering Lemma whose proof can be found in [6] or [19].
Lemma 3.3. Suppose that \( \Omega \) is \((\delta, 1)\)-Reifenberg flat and that \( w \in A_s \) for some \( s \in (1, \infty) \). Let \( E \) and \( G \) be measurable sets with \( E \subset G \subset \Omega \). We further assume that there is a constant \( c \in (0, 1) \) such that

\[
\forall z \in \Omega, \quad w(E) < cw(B_1(z)),
\]
and for each \( z \in \Omega \) and for each \( \rho \in (0, 1) \),

\[
B_{\rho}(z) \cap \Omega \subset G \quad \text{whenever} \quad w(E \cap B_{\rho}(z)) \geq cw(B_{\rho}(z)).
\]

Then

\[
w(E) \leq c^* w(G)
\]
with a constant \( c^* = c^*(n, s, [w]_s) > 0 \).

Using a perturbation lemma based on the existence of Lipschitz interior and boundary estimates for the limiting equations \( \text{div } a(Dv) = 0 \) in small balls and small half balls, one can derive the following density lemma for the upper-level set for \( \mathcal{M}(|Du|^p) \) when \( u \) is a weak solution to the variational inequality (2).

Lemma 3.4. Let \( w \in A_2 \) for some \( 1 < q < \infty \). Suppose that \( u \in W^{1,p}_0(\Omega) \) is the weak solution to (2). Then there exists a constant \( N_0 = N_0(\gamma_1, \gamma_2, n, p) > 1 \) so that for any fixed \( 0 < \epsilon < 1 \) there exists a small \( \delta = \delta(\epsilon, \gamma_1, \gamma_2, n, p, [w]_q) > 0 \) such that if \((a, \Omega)\) is \((\delta, 42)\)-vanishing, and a ball \( B_r(z) \) with \( z \in \Omega \) and \( r \in (0, 1) \) satisfies

\[
w(\{ x \in \Omega : \mathcal{M}(|Du|^p) > N_0^p \} \cap B_r(z)) \geq c w(B_r(z)),
\]
then we have

\[
B_r(z) \cap \Omega \subset \{ x \in \Omega : \mathcal{M}(|Du|^p) > 1 \}
\]
\[
\cup \{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p \} \cup \{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p \}.
\]

Proof. For the proof we refer to Lemma 3.4 in [7] and Lemma 4.8 in [8]. Here we add one thing to clarify the proof of [8, Lemma 4.4] (page 3130) which was used for [8, Lemma 4.8]. It follows from the condition (2.4) in [8] that for any \( B_r(y) \) and for any \( t \geq 1 \),

\[
\int_{B_r(y)} \beta^t \, dx \leq (2\Lambda)^{t-1} \int_{B_r(y)} \beta \, dx.
\]

Then we see that

\[
\left( \int_{B_4} \beta^{\frac{p(q+\alpha)}{p\alpha+\gamma_0}} \, dx \right)^{\frac{\alpha}{p\alpha+\gamma_0}} \leq c \delta^\frac{\alpha}{p+\gamma_0},
\]
where \( c \) and \( \epsilon_0 \) depend only on \( n, p \) and the structural constants on \( a \).

We are now all set to prove the main results of the paper. Hereafter we denote the letter \( c \) for any universal constants depending only on \( n, p, t, [w]_q, \gamma_1, \gamma_2, \) and \( \Omega \), and hence it may be different from line to line.

Proof of Theorem 2.4. We first note from Remark 2 that we have the unique weak solution \( u \in W_0^{1,p}(\Omega) \) to (2) with the estimate (5). We then take \( N_0 \) and the corresponding \( \delta > 0 \) for a fixed \( \epsilon > 0 \) from Lemma 3.4. Our strategy is to show that

\[
|||F|||_{L^{q,s'}(\Omega)} + |||D\psi|||_{L^{q,s'}(\Omega)} \leq \delta^p \implies |||Du|||_{L^{q,s'}(\Omega)} \leq c.
\]

In fact, the estimate (15) is then deduced from the following way. Consider for any \( \sigma > 0 \),

\[
F_\sigma = \frac{F}{\lambda_\sigma}, \quad \psi_\sigma = \frac{\psi}{\lambda_\sigma}, \quad \text{and} \quad u_\sigma = \frac{u}{\lambda_\sigma}.
\]
where \( \lambda_\sigma = \frac{1}{\sigma} \left( \| F \|^p_{L^q(\Omega)} + \| D\psi \|^p_{L^q(\Omega)} + \sigma \right) \). Then \( u_\sigma \) solves the obstacle problem (2) with the datum \( F_\sigma \), the obstacle \( \psi_\sigma \) and the nonlinearity \( \alpha(x) = \frac{\alpha(\lambda_\sigma \xi, x)}{\lambda_\sigma} \) of the same structural conditions as \( \alpha \). We thus apply (24) to \( u_\sigma, F_\sigma \) and \( \psi_\sigma \), and then let \( \sigma \to 0 \), to derive the required estimate

\[
\|Du\|^p_{L^q(\Omega)} \leq c\|F\|^p_{L^q(\Omega)} + \|D\psi\|^p_{L^q(\Omega)},
\]

Under the assumption \( \|F\|^p_{L^q(\Omega)} + \|D\psi\|^p_{L^q(\Omega)} \leq \delta^p \), we know from (14) that

\[
\|F\|^p_{L^1(\Omega)} \leq c\|F\|^p_{L^q(\Omega)} \quad \text{and} \quad \|D\psi\|^p_{L^1(\Omega)} \leq c\|D\psi\|^p_{L^q(\Omega)}.
\]

Hence

\[
\|F\|^p_{L^1(\Omega)} + \|D\psi\|^p_{L^1(\Omega)} \leq c\delta^p. \tag{25}
\]

We apply Lemma 3.3. To do this, set

\[
E = \{ x \in \Omega : \mathcal{M}(|Du|^p) > N_0^p \}
\]

and

\[
G = \{ x \in \Omega : \mathcal{M}(|Du|^p) > 1 \} \cup \{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p \} \cup \{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p \}.
\]

We now check the hypotheses of Lemma 3.3. Select a sufficiently large ball \( B \) so that

\[
\Omega \subset \bigcup_{z \in \Omega} B_1(z) \subset B.
\]

By the estimates (19), (5) and (25), we find that for any \( z \in \Omega \),

\[
|E| \leq c \int_\Omega |Du|^p \, dx \leq c \left( \|F\|^p_{L^q(\Omega)} + \|D\psi\|^p_{L^q(\Omega)} \right) \leq c\delta^p < \left( \frac{\epsilon}{\alpha^2 \left( \frac{|B_1(z)|}{|B|} \right)^q} \right) \frac{1}{|B|},
\]

by taking \( \delta \) so that the last inequality holds true. Here, we note that \( \delta \) is independent of the point \( z \). Then in light of Lemma 2.2, we discover that

\[
w(E) \leq \alpha \left( \frac{|E|}{|B|} \right) w(B) < \frac{\epsilon}{\alpha} \left( \frac{|B_1(z)|}{|B|} \right)^q w(B) \leq \epsilon w(B_1(z)), \tag{26}
\]

which is the first hypothesis (20) of Lemma 3.3. On the other hand, the second condition (21) directly follows from Lemma 3.4. Thus we are in the hypotheses of Lemma 3.3, which implies

\[
w(\{ \mathcal{M}(|Du|^p) > N_0^p \}) \leq c^* \epsilon \left[ w(\{ \mathcal{M}(|Du|^p) > 1 \}) + w(\{ \mathcal{M}(|D\psi|^p) > \delta^p \}) \right] + w(\{ \mathcal{M}(|F|^p) > \delta^p \}),
\]

for the constant \( c^* \) given as in Lemma 3.3.

We first look at the case that \( 0 < t < \infty \). In this case, we have that

\[
[w(\{ \mathcal{M}(|Du|^p) > N_0^p \})]^{\frac{1}{q}} \leq \epsilon_1 [w(\{ \mathcal{M}(|Du|^p) > 1 \})]^{\frac{1}{q}} + \epsilon_1 [w(\{ \mathcal{M}(|D\psi|^p) > \delta^p \})]^{\frac{1}{q}} + \epsilon_1 [w(\{ \mathcal{M}(|F|^p) > \delta^p \})]^{\frac{1}{q}}, \tag{27}
\]

where \( \epsilon_1 = c^*(1 + 4^{\frac{q}{q-1}} - 1) \epsilon. \)
Using a natural normalization argument, we iterate the estimate (27) to discover that for \( k \geq 1, \)
\[
\left[ w(\{ \mathcal{M}(|Du|^p) > N_0^{P_k} \}) \right]^\frac{1}{q} \leq c_k \left[ w(\{ \mathcal{M}(|Du|^p) > 1 \}) \right]^\frac{1}{q} \\
+ \sum_{i=1}^{k} c_i \left[ w(\{ \mathcal{M}(|F|^p) > \delta^p N_0^{P(k-i)} \}) \right]^\frac{1}{q} \\
+ \sum_{i=1}^{k} c_i \left[ w(\{ \mathcal{M}(|D\psi|^p) > \delta^p N_0^{P(k-i)} \}) \right]^\frac{1}{q}.
\]
(28)

We then use this power decay estimate (28) to compute as follows:
\[
\sum_{k=1}^{\infty} N_0^{p k} \left[ w(\{ \mathcal{M}(|Du|^p) > N_0^{P_k} \}) \right]^\frac{1}{q} \\
\leq \sum_{k=1}^{\infty} N_0^{p k} c_k \left[ w(\{ \mathcal{M}(|Du|^p) > 1 \}) \right]^\frac{1}{q} \\
+ \sum_{k=1}^{\infty} N_0^{p k} \sum_{i=1}^{k} c_i \left[ w(\{ \mathcal{M}(|F|^p) > \delta^p N_0^{P(k-i)} \}) \right]^\frac{1}{q} \\
+ \sum_{k=1}^{\infty} N_0^{p k} \sum_{i=1}^{k} c_i \left[ w(\{ \mathcal{M}(|D\psi|^p) > \delta^p N_0^{P(k-i)} \}) \right]^\frac{1}{q}.
\]

We then apply Lemma 3.2 when both \( f = \mathcal{M}(\delta^{-p}|F|^p) \) and \( f = \mathcal{M}(\delta^{-p}|D\psi|^p), \)
\( \sigma = 1 \) and \( m = N_0^p, \) and then recall Lemma 3.1 and use (24), to discover that
\[
\sum_{k=1}^{\infty} N_0^{p k} \left[ w(\{ \mathcal{M}(|Du|^p) > N_0^{P_k} \}) \right]^\frac{1}{q} \\
\leq c \left( \left[ w(\Omega) \right]^\frac{1}{q} + \| \mathcal{M}(\delta^{-p}|F|^p) \|_{L_{q}^{\gamma'}(\Omega)} + \| \mathcal{M}(\delta^{-p}|D\psi|^p) \|_{L_{q}^{\gamma'}(\Omega)} \right) \sum_{k=1}^{\infty} (N_0^{p k} c_k) \\
\leq c \left( \left[ w(\Omega) \right]^\frac{1}{q} + \delta^{-p t} \| |F|^p \|_{L_{q}^{\gamma'}(\Omega)} + \delta^{-p t} \| |D\psi|^p \|_{L_{q}^{\gamma'}(\Omega)} \right) \sum_{k=1}^{\infty} (N_0^{p k} c_k) \\
\leq c \left( \left[ w(\Omega) \right]^\frac{1}{q} + 1 \right) \sum_{k=1}^{\infty} (N_0^{p k} c_k).
\]

Now choosing \( \epsilon \) small so that \( N_0^{p k} c_1 < 1, \) we obtain
\[
S := \sum_{k=1}^{\infty} N_0^{p k} \left[ w(\{ x \in \Omega : \mathcal{M}(|Du|^p) > N_0^{P_k} \}) \right]^\frac{1}{q} \leq c.
\]
We again apply Lemma 3.2 when \( f = \mathcal{M}(|Du|^p) \), \( \sigma = 1 \) and \( m = N_0^p \), to conclude that
\[
\|\mathcal{M}(|Du|^p)\|_{L^t_w(\Omega)}^t \leq c \left( |w(\Omega)|^{\frac{4}{3}} + S \right) \leq c.
\]
This estimate and Lemma 3.1 finally imply
\[
|||Du|||_{L^t_w(\Omega)} \leq c,
\]
for some positive constant \( c \) depending only on \( n, p, q, r, \gamma_1, \gamma_2, [w]_q \) and \( \Omega \).

For \( t = \infty \), we use the estimate (28) with \( \frac{4}{3} \) replaced by \( \frac{1}{q} \), to calculate as follows:
\[
\sup_{k \geq 1} N_0^{p_k} \left[ w\left( \{ |\mathcal{M}(Du|^p) > N_0^{p_k} \} \right) \right]^{\frac{1}{q}} \\
\leq \sup_{k \geq 1} N_0^{p_k} \varepsilon_1^k \left[ w\left( \{ |\mathcal{M}(Du|^p) > 1 \} \right) \right]^{\frac{1}{q}} \\
+ \sup_{k \geq 1} N_0^{p_k} \sum_{i=1}^{k} \varepsilon_1^i \left[ w\left( \{ |\mathcal{M}(|F|^p) > \delta^p N_0^{p(k-i)} \} \right) \right]^{\frac{1}{q}} \\
+ \sup_{k \geq 1} N_0^{p_k} \sum_{i=1}^{k} \varepsilon_1^i \left[ w\left( \{ |\mathcal{M}(|D\psi|^p) > \delta^p N_0^{p(k-i)} \} \right) \right]^{\frac{1}{q}} \\
\leq 3 \sum_{i=1}^{\infty} (N_0^{p_1})^i \left[ w(\Omega) \right]^{\frac{1}{q}} \\
+ \sum_{i=1}^{\infty} (N_0^{p_1})^i \sup_{j \geq 1} N_0^{p_j} \left[ w\left( \{ |\mathcal{M}(\delta^{-p}|F|^p) > N_0^{p_j} \} \right) \right]^{\frac{1}{q}} \\
+ \sum_{i=1}^{\infty} (N_0^{p_1})^i \sup_{j \geq 1} N_0^{p_j} \left[ w\left( \{ |\mathcal{M}(\delta^{-p}|D\psi|^p) > N_0^{p_j} \} \right) \right]^{\frac{1}{q}}.
\]
Therefore, by the same reasoning as in the case that \( 0 < t < \infty \), we conclude that
\[
|||Du|||_{L^\infty_w(\Omega)} \leq c
\]
for some positive constant \( c \) depending only on \( n, p, q, \gamma_1, \gamma_2, [w]_q \) and \( \Omega \). This completes the proof. \( \square \)

We next shall prove the Theorem 2.5.

**Proof of Theorem 2.5.** We start with the case that \( 0 < t < \infty \). Fix any \( z \in \Omega \) and any \( 0 < r \leq \text{diam}(\Omega) \). Choose a number \( \rho \in (0, n - \theta) \) and consider a function \( w \) on \( \mathbb{R}^n \) by
\[
w(x) = \min\{|x - z|^{-\theta - \rho}, r^{-\theta - \rho}\}.
\]
Note that \( -n < -\theta - \rho < n(q - 1) \), and so by Lemma 2.3, \( w \) is an \( A_q \) weight. Since \( w = r^{-\theta - \rho} \) in \( B_r(z) \), we find that for some constant \( c = c(n, p, q, t, \gamma_1, \gamma_2, \theta, \Omega) \),
\[
|||Du|||_{L^t_w(B_r(z) \cap \Omega)} = r^{(\theta + \rho)q} \|Du\|^t_{L^t_w(B_r(z) \cap \Omega)} \\
\leq cr^{(\theta + \rho)q} \left( \|F\|^{t}_{L^t_w(\Omega)} + |||Du|||_{L^t_w(\Omega)} \right),
\]
where we have used the estimate (15) in Theorem 2.4 for the last inequality.

To proceed further, we consider the upper-level set of \(|F|^p\) denoted by
\[
Q_\lambda := \{ x \in \Omega : |F(x)|^p > \lambda \}, \quad \lambda > 0.
\]
Then we have
\[
\|F\|^p_{L^q(\Omega)} = q \int_0^\infty \lambda^t \left( \int_{Q_\lambda} w(x) \, dx \right)^{\frac{q}{t}} \frac{d\lambda}{\lambda}
\]
\[
= q \int_0^\infty \lambda^t \left( \int_0^\infty \left| \{ x \in Q_\lambda : w(x) > s \} \right| \, ds \right)^{\frac{q}{t}} \frac{d\lambda}{\lambda}
\]
\[
\leq q \int_0^\infty \lambda^t \left( \int_0^{r^{-\theta-\rho}} \lambda^q \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \, ds \right)^{\frac{q}{t}} \frac{d\lambda}{\lambda},
\]
since \( w \leq r^{-\theta-\rho} \) and \( \{ x \in Q_\lambda : w(x) > s \} \subset Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \).

If \( q < t < \infty \), then we apply Minkowski’s integral inequality to the last term in (31), (11) for \( w \equiv 1 \), and then (16), to discover that
\[
\|F\|^p_{L^q(\Omega)} \leq \left\{ \int_0^{r^{-\theta-\rho}} \left( q \int_0^\infty \lambda^t \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \frac{d\lambda}{\lambda} \right)^{\frac{q}{t}} \frac{d\lambda}{\lambda} \right\}^{\frac{1}{q}}
\]
\[
\leq \left[ \int_0^{r^{-\theta-\rho}} \left\{ \left( \int_0^\infty \lambda^t \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \frac{d\lambda}{\lambda} \right)^{\frac{q}{t}} \right\}^{\frac{1}{q}} \frac{d\lambda}{\lambda} \right]^{\frac{1}{q}}
\]
\[
\leq \frac{1}{r^{-\frac{\theta}{q}}} \|F\|^p_{L^q, t, \theta, (\Omega)} \int_0^{r^{-\theta-\rho}} \frac{1}{s} \frac{d\lambda}{\lambda},
\]
for some positive constant \( c = c(n, \theta, q, t) \).

Likewise, we have
\[
\|D\psi\|^p_{L^q(\Omega)} \leq c \|D\psi\|^p_{L^q, t, \theta, (\Omega)} r^{-\frac{\theta}{q}}.
\]
(33)

Combining (30), (32) and (33), we obtain that for \( q < t < \infty \),
\[
r^{-\frac{\theta}{q}} \|Du\|^p_{L^q(\Omega)} \leq c \|F\|^p_{L^q, t, \theta, (\Omega)} + \|D\psi\|^p_{L^q, t, \theta, (\Omega)}.
\]
(34)

If \( 0 < t \leq q \), we then return to the integrand in the right-hand side of the last inequality in (31). We have
\[
\left( \int_0^{r^{-\theta-\rho}} \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \, ds \right)^{\frac{1}{q}} = \left( \sum_{i=1}^{\infty} \int_{2^{-i-1}r^{-\theta-\rho}}^{2^{-i}r^{-\theta-\rho}} \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \, ds \right)^{\frac{1}{q}}
\]
\[
\leq \left( \sum_{i=1}^{\infty} 2^{-i} r^{-\theta-\rho} \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \right)^{\frac{1}{q}}
\]
We note that \( \left( \sum_{i=1}^{\infty} a_i \right)^{\sigma} \leq \sum_{i=1}^{\infty} a_i^{\sigma} \) for \( a_i \geq 0 \) and for \( 0 < \sigma \leq 1 \). Thus
\[
\left( \int_0^{r^{-\theta-\rho}} \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \, ds \right)^{\frac{1}{q}} \leq \sum_{i=1}^{\infty} \left( 2^{-i} r^{-\theta-\rho} \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \right)^{\frac{1}{q}}
\]
\[
\leq 2 \int_0^{r^{-\theta-\rho}} \left( \frac{1}{s} \left| Q_\lambda \cap B_{\frac{s}{r^{-\theta-\rho}}} (z) \right| \right)^{\frac{1}{q}} \frac{ds}{s}
\]
(35)
But then (31) and (35) imply
\[
|||F|||_{L^{q,\gamma}(\Omega)} \leq 2q \int_0^\infty \lambda^t \int_0^\infty \left( \frac{1}{s^{\frac{1}{q}-\rho}} \left( \frac{s}{\lambda} \right)^{\frac{1}{q}} \frac{d\lambda}{\lambda} \right) \frac{ds}{s}
\]
\[
= 2q \int_0^\infty \frac{1}{s^{\frac{1}{q}-\rho}} \left( \int_0^\infty \lambda^t \left( \frac{s}{\lambda} \right)^{\frac{1}{q}} \frac{d\lambda}{\lambda} \right) \frac{ds}{s}
\]
\[
\leq 2q \int_0^\infty \frac{1}{s^{\frac{1}{q}-\rho}} \left( \int_0^\infty \lambda^t \left( \frac{s}{\lambda} \right)^{\frac{1}{q}} \frac{d\lambda}{\lambda} \right) \frac{ds}{s}
\]
\[
\leq c |||F|||_{L^{n,q,\gamma}(\Omega)}^t \int_0^\infty \frac{1}{s^{\frac{1}{q}-\rho}} \frac{ds}{s},
\]
for some positive constant \(c = c(n, \theta, q, t)\).

Similarly, we have
\[
|||D\psi|||_{L^{q,\gamma}(\Omega)} \leq c |||D\psi|||_{L^{n,q,\gamma}(\Omega)}^t \frac{c^t}{s}.\]

Then this estimate, (36) and (30) imply that for \(0 < t \leq q\),
\[
r^{-\frac{\alpha}{q}} |||Du|||_{L^{q,\gamma}(B_r(\gamma) \cap \Omega)} \leq c \left( |||F|||_{L^{n,q,\gamma}(\Omega)} + |||D\psi|||_{L^{q,\gamma}(\Omega)} \right)
\]
(37)

If \(t = \infty\), we then use the weight \(\omega\) in (29) and the estimate (15) in Theorem 2.4, to discover
\[
|||D\omega|||_{L^{q,\infty}(B_r(\gamma) \cap \Omega)} = r^{q+\rho} |||D\omega|||_{L^{q,\infty}(B_r(\gamma) \cap \Omega)} \leq c r^{q+\rho} \left( |||F|||_{L^{q,\infty}(\Omega)} + |||D\omega|||_{L^{q,\infty}(\Omega)} \right),
\]
for some positive constant \(c = c(n, p, q, \gamma_1, \gamma_2, \theta, \Omega)\).

Recalling the upper-level set \(Q_{\lambda}\) of \(|F|^r\), we deduce that for any \(\lambda > 0\) and for some \(c = c(n, \theta) > 0\),
\[
\lambda^q w(Q_{\lambda}) = \int_{Q_{\lambda}} \lambda^q w(x) \, dx
\]
\[
= \int_0^\infty \lambda^q \left( \int_0^\infty \lambda^{\frac{1}{q}} \left( \frac{s}{\lambda} \right)^{\frac{1}{q}} \frac{d\lambda}{\lambda} \right) \frac{ds}{s}
\]
\[
\leq \int_0^\infty \lambda^q \left( \int_0^\infty \lambda^{\frac{1}{q}} \left( \frac{s}{\lambda} \right)^{\frac{1}{q}} \frac{d\lambda}{\lambda} \right) \frac{ds}{s}
\]
\[
\leq \int_0^\infty \lambda^q \left( \frac{1}{s^{\frac{1}{q}-\rho}} \right) \frac{ds}{s}
\]
\[
\leq \frac{\\sqrt{\lambda}}{s^{\frac{1}{q}-\rho}} \int_0^\infty \lambda^q \left( \frac{1}{s^{\frac{1}{q}-\rho}} \right) \frac{ds}{s}
\]
\[
\leq c |||F|||_{L^{q,\infty}(\Omega)}^t \int_0^\infty \frac{ds}{s^{\frac{1}{q}-\rho}},
\]
where we have used the fact that \(\{x \in Q_{\lambda} : w(x) > s\} \subset Q_{\lambda} \cap B_s x \frac{1}{\rho} (z)\) and that \(w \leq r^{-\theta-\rho}\), in order to have (39). So in view of (16) and (40), we find
\[
|||F|||_{L^{q,\infty}(\Omega)}^t = \sup_{\lambda > 0} \lambda^q w(Q_{\lambda}) \leq c |||F|||_{L^{n,q,\gamma}(\Omega)}^t r^{-\rho}.
\]
We similarly have
\[ \| |D\psi|^p\|_{L^{p,\infty}(\Omega)}^q \leq c \| |D\psi|^p\|_{L^{q,\infty}(\Omega)}^q r^{-\rho}. \] (42)

Then in light of (38), (41) and (42), we discover that for \( t = \infty \),
\[ r^{-\frac{p}{q}} \| |D\psi|^p\|_{L^{p,\infty}(B_r(z)\cap\Omega)} \leq c \left( \| |F|^p\|_{L^{q,\infty}(\Omega)} + \| |D\psi|^p\|_{L^{q,\infty}(\Omega)} \right). \] (43)

Making use of (34), (37) and (43) and recalling (16), we at last obtain the required estimate (17). This completes the proof.

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