We review (and elaborate on) the ‘dual graviton problem’ in the context of duality covariant formulations of M-theory (exceptional field field theories). These theories require fields that are interpreted as components of the dual graviton in order to build complete multiplets under the exceptional groups $E_{d(d)}$, $d = 2, \ldots, 9$. Circumventing no-go arguments, consistent theories for such fields have been constructed by incorporating additional covariant compensator fields. The latter fields are needed for gauge invariance but also for on-shell equivalence with $D = 11$ and type IIB supergravity. Moreover, these fields play a non-trivial role in generalized Scherk-Schwarz compactifications, as for instance in consistent Kaluza-Klein truncation on $AdS_4 \times S^7$.

1. Introduction

In string theory and supergravity it is often convenient or even necessary to pass from certain field variables to their ‘Poincaré duals’. For differential $p$-forms with suitable couplings this can be done straightforwardly by introducing a master action that treats the $(p+1)$-form field strength $F_{p+1}$ as an independent field whose Bianchi identity $dF_{p+1} = 0$ is imposed by a Lagrange multiplier field, viewed as a differential form $A_{p+1}$. Upon integrating out $F_{p+1}$ one obtains an on-shell equivalent action for the dual $(D - p - 2)$-form. Such duality transformations are instrumental, first, in order to describe the world-volume actions of certain branes and, second, to realize the U-duality symmetries of M-theory (and its low-energy actions) arising in toroidal compactifications in a manifestly local and covariant formulation.

The natural question arises whether theories of more complicated tensor fields have such ‘dual’ formulations. Gravity linearized about flat space, i.e., the free massless spin-2 theory for a symmetric second-rank tensor $h_{\mu\nu}$ on $D$-dimensional Minkowski space, permits a dual formulation in terms of a mixed Young tableau field $C_{\mu_1, \ldots, \mu_D, \ldots, \nu}$ — the ‘dual graviton’.[1,2] This formulation can be obtained by essentially the same procedure as for $p$-forms, passing to a master action and then integrating out auxiliary fields. The existence of this dual formulation of linearized gravity has led to numerous speculations that the dual graviton (and more general mixed Young tableaux fields) may play an important role in the search for the elusive fundamental formulation of string/M-theory.[3-4] Specifically, it has been suggested that, among other reasons, the dual graviton may be needed for a proper description of Kaluza-Klein monopole solutions of string theory[5] and to realize enhanced U-duality-type symmetries. There are, however, strong no-go theorems excluding the existence of a manifestly covariant and local formulation of mixed Young tableaux fields beyond linear order.[6,7]

In order to understand the significance of these no-go theorems, it is worthwhile to pause here for a moment and to reflect about what exactly the issue is. The issue is not to find a new physical theory for a dual graviton field, because (non-linear) general relativity in physical or light-cone gauge is indistinguishable from a hypothetical theory of the dual graviton in light-cone gauge. Both would be formulated in terms of a symmetric tensor $\gamma_{ij}$ under the little group $SO(D - 2)$[8] — here one uses that the dual graviton $C_{i_1, \ldots, i_{D-3}}$ can be replaced by $\gamma_{ij}$ by means of the $SO(D - 2)$ epsilon symbol. Indeed, graviton and dual graviton are supposed to encode the same physical content. The real issue is rather one of a suitable formulation, namely one that is non-linear, manifestly local and covariant, with a gauge symmetry so that i) linearizing about flat space one recovers the free action of the dual graviton; and ii) in light-cone gauge it is equivalent to general relativity. It is the existence of such a formulation that is excluded by the no-go theorems of [6,7].

More generally, as far as we can tell, there is no sharp physical problem whose solution would require a theory of dual gravity as defined above. Nevertheless, it is reasonable to ask whether there are reformulations of (super-)gravity that feature a dual graviton-type field and that are useful for particular applications. It is indeed possible (in a surprisingly trivial fashion) to formulate general relativity so that it contains the dual graviton together with the usual graviton and a compensator gauge field.[9-11] This

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1 The name ‘dual graviton’ is hence somewhat of a misnomer as it agrees with the ‘graviton’ as usually defined: a massless spin-2 state whose interactions are governed by Einstein gravity (plus possible higher order corrections) no matter what field variables are used.

2 More precisely, the no-go theorems of [6,7] exclude the existence of a non-linear theory that is manifestly local and covariant and satisfies requirement i), irrespective of whether such a theory would be equivalent to general relativity and hence also satisfy requirement ii).

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formulation is such that linearization about flat space fields, depending on a gauge choice, either standard linearized gravity or dual gravity, but this still begs the question what such a formulation is good for. There is one (reasonably sharp) mathematical problem that has a bearing on the dual graviton issue, namely the problem of finding a formulation of, say, $D = 11$ supergravity that is duality covariant under U-duality groups such as $E_{8(8)}$. In the following we outline how this problem arises and how it is resolved in exceptional field theory.

The $E_{8(8)}$ U-duality symmetry arises upon torus compactification of $D = 11$ or type IIB supergravity to three dimensions. It is a non-linearly realized global symmetry, with the physical bosonic degrees of freedom being organized in a symmetric matrix $M_{MN}$ parametrizing the coset space $E_{8(8)}/SO(16)$. Upon decomposing $E_{8(8)}$ w.r.t. $GL(8)$ one may parametrize $M_{MN}$ in terms of (scalar) components, which include in particular fields $\varphi_m$, $m = 1, \ldots, 8$, that are the on-shell duals of the Kaluza-Klein vector fields $A^m_{\mu}$ originating from the metric in $D = 11$. As such, one may think of $\varphi_m$ as originating from the dual graviton, but as long as we are in three dimensions there is no dual graviton problem. The problem arises if one attempts to formulate $D = 11$ supergravity prior to compactification in an $E_{8(8)}$ covariant way, as is done in exceptional field theory (ExFT). Here one decomposes all tensor fields and their indices as in Kaluza-Klein compactifications, but without truncating coordinate dependence. The goal is then to reorganize the fields into duality covariant objects such as ‘scalars’ $M_{MN}$, vectors $A_m^\mu$ and, more generally, higher tensors. The vector fields $A_m^\mu$ generally contain the Kaluza-Klein vectors $A^m_{\mu}$ and so for the $E_{8(8)}$ theory the question arises of how the eight components $\varphi_m$ of $M_{MN}$ should be interpreted, in particular how such a theory should be matched with $D = 11$ supergravity, which does not contain a dual graviton. (One could introduce a dual graviton, using the formulation of [10,11], but it should definitely be possible to match $D = 11$ supergravity in the standard formulation.)

The resolution of the apparent conflict hinges on additional gauge fields and their associated gauge symmetries, which guarantee the correct counting of degrees of freedom and which are precisely as needed for the match with $D = 11$ supergravity. In order to explain this we recall that in ExFT the fields depend on ‘external’ coordinates $x^\nu$ and extended ‘internal’ coordinates $Y^M$, subjects to ‘section constraints’ for their dual derivatives of the form

$$\mathbb{P}_{MN}^{KL} \partial_K \otimes \partial_L = 0. \tag{1.1}$$

Here, $\mathbb{P}$ is the projector onto suitable sub-representations in the tensor product of the fundamental representation (labelled by indices $M, N, \ldots$) with itself. This constraint should be interpreted in the sense that for any fields $A, B$ we set $\mathbb{P}_{MN}^{KL} \partial_K \partial_L A = 0$ and $\mathbb{P}_{MN}^{KL} \partial_K A \partial_L B = 0$. There are also generalized diffeomorphisms of the internal and external coordinates, which are consistent (obeying closure relations) thanks to the constraints (1.1). For the $E_{8(8)}$ theory their gauge parameters are $\Lambda^M$, $\Sigma_m$, but importantly the latter parameter needs to be covariantly constrained in the sense that it satisfies constraints of the same type as the derivatives,

$$\mathbb{P}_{MN}^{KL} \partial_K \otimes \Sigma_L = 0, \quad \text{etc.} \tag{1.2}$$

This constraint implies that the $\Sigma_m$ feature significantly fewer components than 248, with the precise non-vanishing field components depending on the ‘dual’ choice of non-trivial coordinates among the $Y^M$. The vector fields are gauge fields for the internal (generalized) diffeomorphisms, and so we have a doubled set of gauge vectors $A^m_{\mu}$, $B_m^\lambda$, with the latter satisfying similar constraints as (1.2). Upon solving the section constraint, say as appropriate for $D = 11$ supergravity, the $\Sigma_m$ gauge symmetries reduce to eight St"uckelberg symmetries with parameters $\Sigma_m$, which are precisely sufficient in order to render the dual graviton components $\varphi_m$ pure gauge, thereby restoring the proper counting of degrees of freedom. (The same holds true for the type IIB solution of the constraint.)

In the remainder of this review we explain this resolution of the dual graviton problem in more technical detail for different ExFTs and how it is useful for applications such as using generalized Scherk-Schwarz compactifications for the consistency proofs of non-toroidal Kaluza-Klein truncations. Specifically, in Section 2 we review the dualization of linearized gravity and explain the compensator formulation of [10,11] that describes full, non-linear (super-)gravity. Moreover, we discuss the dimensional reduction of the dual graviton, which sets the stage for our subsequent discussion of ExFT, where certain dimensionally reduced components of the dual graviton are visible. In Section 3 we review ExFT, with a particular emphasis on how the components of the dual graviton enter the ExFT p-forms and induce additional compensating fields with their associated gauge transformations. We illustrate this in detail for $E_7(7)$ and $E_{8(8)}$ ExFT. Finally, in Section 4 we discuss the fate of the dual graviton in consistent Kaluza-Klein truncations on non-trivial backgrounds. Within ExFT these truncations are described as generalized Scherk-Schwarz reductions whose consistency requires the dual graviton and its compensating gauge field to be included. We close with a summary and outlook in Section 5.

2. Linearized Dual Gravity and its Dimensional Reduction

In this section we review the dualization of linearized gravity and discuss a reformulation of general relativity in terms of a dual graviton, the original graviton and a compensator gauge field. In the second subsection we briefly discuss the dimensional reduction of the dual graviton.

2.1. Dual Gravity in Linearized and Compensator Form

We begin with the frame-formulation of the Einstein-Hilbert action, in a form that is quadratic in first derivatives. It is written in terms of the coefficients of anholonomy

$$\Omega_{ab}^c = e_a^\rho e_b^\sigma (\partial_{\rho} e_c^\sigma - \partial_{\sigma} e_c^\rho), \tag{2.1}$$

where $e_a^\mu$ is the $D$-dimensional vielbein, as

$$S_{EH} = - \int d^D x e \left( \right.$$

$$\left. \Omega^{abc} \Omega_{abc} + 2 \Omega^{abc} \Omega_{a[b} \Omega_{c]d} - 4 \Omega^{ab} \Omega^{cd} \right). \tag{2.2}$$
We next pass to a first-order formulation by introducing an auxiliary field $Y_{ab}$, that is antisymmetric in its first two indices but otherwise lives in a reducible representation of the Lorentz group.\(^2\)

$$S[Y, e] = -2 \int d^D x \epsilon \left( Y_{ab}^{\mu} \partial^{\mu} 
abla_{ab} - \frac{1}{2} Y_{ab}^{\mu} Y_{ac}^{\nu} b + \frac{1}{4(D-2)} Y_{ab}^{\mu} Y_{ac}^{\nu} c \right).$$

(2.3)

To show the equivalence to (2.2) we use that the field equation of $Y$ can be used to solve for $Y$ in terms of $\Omega$.

$$Y_{ab} = \Omega_{ab} - 2 \Omega_{[a|b]} + 4 \eta_{[a} \Omega_{b]}.$$  

(2.4)

Upon back-substitution into (2.3) one recovers the Einstein-Hilbert action (2.2). In order to obtain the dual formulation it is convenient to first rewrite the action in terms of the Hodge dual of $Y_{ab}^{\mu}$.

$$Y_{ab}^{\mu} = \frac{1}{(D-2)!} \epsilon^{ab}_{\mu \nu \rho \cdots} Y_{1 \cdot \cdot \cdot 2}^\nu,$$

(2.5)

which yields

$$S = -2 \int \frac{d^D x}{(D-2)!} \epsilon \left( \epsilon^{ab}_{\mu \nu \rho \cdots} Y_{1 \cdot \cdot \cdot 2}^\nu \partial^{\mu} Y_{2 \cdot \cdot \cdot 1}^\nu - \frac{1}{2} \epsilon^{ab}_{\mu \nu \rho \cdots} Y_{ab}^{\mu} \delta_{abc} \right),$$

(2.6)

Second, one may check by an explicit computation that the action $S[Y]$ with Lagrangian (2.9) is invariant under the following St"uckelberg symmetry\(^3\)

$$\delta \Lambda C_{\mu \nu \rho \cdots} = \partial_{[\mu} \Lambda_{\rho] \nu \cdots}.$$  

(2.11)

with completely antisymmetric shift parameter. From the point of view of the master action (2.6) (that is equivalent to the Einstein-Hilbert action) this is simply a consequence of the local Lorentz symmetry that to first order acts on the fluctuation as $\delta \Lambda h_{\mu \nu} \sim \Lambda_{\nu \mu}$. Consequently, the totally antisymmetric part of $C_{\mu \nu \rho \cdots}$ can be gauge-fixed to zero inside $S[C]$, in the same way that the antisymmetric part of $h_{\mu \nu}$ can be gauged away in standard linearized gravity. (It should be emphasized, however, that in the master action (2.6) we cannot gauge away the antisymmetric parts of $h$ [13]).

The field $D_{\mu \nu \rho \cdots}$ obtained by gauging away the totally antisymmetric part of $C$ carries a specific Young-diagram symmetry. The characteristics of such mixed Young tableaux fields have been studied in [12,14], where it has been shown that they transform under two types of gauge transformations as follows

$$\delta D_{\mu \nu \rho \cdots} = \partial_{[\mu} \alpha_{\nu \rho \cdots]} + \partial_{[\mu} \beta_{\nu \rho \cdots]} - (D-3)! \partial_{[\mu} \theta_{\nu \rho \cdots]} \sim 0,$$

(2.12)

where $\alpha$ lives in the $(D-4, 1)$ Young tableau, and $\beta$ is completely antisymmetric. These transformations originate after gauge fixing from (2.10) and compensating local Lorentz transformations (2.11). The field strength (2.8), with the field replaced by the $D$ field, is invariant under $\alpha$-transformations, which therefore are a manifest invariance of the Curtright action. In contrast, the $\beta$ transformations are a non-manifest invariance. Indeed, the relative coefficients can be fixed by requiring gauge invariance under $\beta$-transformations.

After having discussed the dualization of linearized gravity, let us now return to full non-linear Einstein gravity. We first note that the above dualization procedure cannot be applied to the non-linear theory, because the field equations for $h_{\mu \nu}$ no longer imply that the curl of $Y$ vanishes, c.f. (2.7). Rather, one obtains an equation of the schematic form $\partial Y \sim Y^2$, which does not imply that $Y$ can be written as the curl of a dual graviton field. This obstacle for extending the dualization to the non-linear level is in perfect agreement with the no-go theorems of [6,7] that prohibit the existence of manifestly covariant and local dual formulation. Instead, we will discuss now a non-linear formulation that features both the graviton and dual graviton together with a compensating gauge field.\(^9,10\) This theory is a straightforward reinterpretation of the master action (2.6) that, however, turns out to be quite prescient for the exceptional field theory formulations to be discussed below.

This theory is obtained by starting from the quadratic Curtright action (2.9) and coupling it to dynamical gravity described...
by a vielbein field $e_{\mu}^{\ a}$. This yields the ‘covariantized’ Curtright Lagrangian

$$\mathcal{L}_C(e, \tilde{F}) = \frac{D-3}{2D-2} e^{\mu_1 \cdots \mu_{D-1}} \Lambda_{\mu_1 \cdots \mu_{D-2}}^a \mathcal{F}_{\mu_1 \cdots \mu_{D-2}}^a$$

$$- \frac{D-2}{2} e^{\mu} e^{\rho} \mathcal{F}^{\mu_1 \cdots \mu_{D-1}}_{\rho} \Lambda_{\mu_1 \cdots \mu_{D-3}}^a \mathcal{F}_{\mu_1 \cdots \mu_{D-3}}^a$$

$$+ \frac{1}{2} e^{\mu} e^{\nu} e^{\rho} \mathcal{F}^{\mu_1 \cdots \mu_{D-3}}_{\rho} \Lambda_{\mu_1 \cdots \mu_{D-4}}^a \mathcal{F}_{\mu_1 \cdots \mu_{D-4}}^a, \quad (2.13)$$

which is now fully diffeomorphism invariant. Here we interpret the dual graviton $\mathcal{C}_{\mu_1 \cdots \mu_{D-3}}^a$ as a $(D - 3)$-form in the vector representation of the Lorentz group, whose field strength $\mathcal{F}_{[\mu_1 \cdots \mu_{D-2}]a}$ is defined as in (2.8). The action then admits a (still abelian) dual diffeomorphism symmetry that acts on $\mathcal{C}_{\mu_1 \cdots \mu_{D-3}}^a$ as an ordinary $p$-form gauge symmetry (with a $(D - 4)$-form gauge parameter in the vector representation of the Lorentz group). However, the (dual) local Lorentz transformations (2.11) are no longer an invariance of the action, and hence (2.13) is not fully consistent. In order to repair this, we have to modify the field strength by adding a compensating gauge field in the form of a Stückelberg coupling.

$$\tilde{F}_{\mu_1 \cdots \mu_{D-2}}^a = \partial_{\mu_1} C_{\mu_2 \cdots \mu_{D-2}}^a + Y_{\mu_1 \cdots \mu_{D-2}}^a. \quad (2.14)$$

The field $Y$ is now interpreted as a compensating gauge field for Stückelberg gauge symmetries that act as

$$\delta Y_{\mu_1 \cdots \mu_{D-2}}^a = \partial_{[\mu_1} \Sigma_{\mu_2 \cdots \mu_{D-2}]^a,$$

$$\delta C_{\mu_1 \cdots \mu_{D-3}}^a = - \Sigma_{\mu_1 \cdots \mu_{D-3}}^a, \quad (2.15)$$

and hence leave the field strength $\tilde{F}$ invariant.

Since we coupled the free Curtright action to the dynamical gravity field $e_{\mu}^{\ a}$, it would be natural to add a kinetic Einstein-Hilbert term to (2.13). However, this would lead to a doubling of the gravity degrees of freedom, instead of a reformulation of Einstein gravity, and it would also not restore the local Lorentz invariance. The correct procedure is instead to add a topological term that couples $Y_{\mu_1 \cdots \mu_{D-2}}^a$ to the compensating shift gauge field $Y$. We thus consider the total action

$$S[e, Y] = \int d^D x \left[ \mathcal{L}_C(e, \tilde{F}) + 2\kappa^{-1} e^{\mu_1 \cdots \mu_{D-3}} \partial_{\mu_1 Y_{\mu_2 \cdots \mu_{D-2}}^a} \partial_{\mu_2} e_{\mu_3}^a \right]. \quad (2.16)$$

where we restored Newton’s constant $\kappa$. Let us verify that the theory defined by this action is equivalent to Einstein gravity. To this end we note that the Stückelberg shift symmetry (2.15) can be gauge fixed by setting $C = 0$ so that $\tilde{F} = Y$, in which case (2.16) reduces to the original master action (2.6). As the latter leads to the Einstein-Hilbert action upon integrating out $Y$, we have shown that (2.16) is equivalent to Einstein’s general relativity. On the other hand, we may linearize about flat space before gauge fixing and/or integrating out fields. In this case, the field equation of $e_{\mu}^{\ a}$ reduces to $dY = 0$, which in turn implies that the Stückelberg symmetry (2.15) can be gauge fixed by setting $Y = 0$. The action (2.16) then reduces to the Curtright action (2.9). The action (2.16) thus provides a universal formulation of gravity that features both the graviton and dual graviton, together with a compensating gauge field. Although this formulation is a minor extension of the original master action (2.6), it turns out that its basic mechanism of compensating gauge fields is realized, in a more subtle and duality covariant version, in exceptional field theory.

It is instructive to investigate the field equations following from the above action, which take the form of first-order duality relations. Varying with respect to the gauge field $Y$ one finds

$$e^{-1} g_{\mu_1 \cdots \mu_{D-2}} \omega_{\rho}^{\ a} = - \frac{D-3}{D-2} \tilde{F}_{\mu_1 \cdots \mu_{D-2}}^a$$

$$+ (-1)^{D-1} (D-2) e_{\rho}^{\ a} e_{\mu_1}^{\ a} \tilde{F}^{\mu_2 \cdots \mu_{D-2}}_{\rho} \tilde{F}^{\mu_1 \cdots \mu_{D-3}}_{\rho} b_{\rho}, \quad (2.17)$$

while the field equation for $e_{\mu}^{\ a}$ reads

$$e^{-1} g_{\mu_1 \cdots \mu_{D-3}} \partial_{\mu_1 Y_{\mu_2 \cdots \mu_{D-2}}^a} = \frac{1}{2} e^{-1} \frac{\delta \mathcal{L}_C(e, \tilde{F})}{\delta e_{\mu}^a}. \quad (2.18)$$

These combined first-order equations imply the full non-linear Einstein equations, which is of course guaranteed from the definition of the master action. To this end one has to take suitable derivatives of (2.17) and use on the right-hand side the Bianchi identity of $\tilde{F}$, which reads schematically $d\tilde{F} = dY$. One can then use the second duality relation (2.18) in order to eliminate $dY$. Alternatively, one may solve (2.17) for $Y$ in terms of $e$ and $C$ and then insert into (2.18) upon which $C$ drops out and the Einstein equations are obtained. In particular, the first equation (2.17) by itself has no physical content (in contrast to the linear duality relation without compensating gauge field) in that it can be viewed as a mere definition of $Y$, but the point is that $Y$ in turn satisfies an equation, Equation (2.18), so that the combined system implies the dynamical second order Einstein equations. This mechanism of ‘hierarchical’ duality relations is very natural for the tensor hierarchy structure in gauged supergravity and ExFT and will recur in several places below. Finally, we note that here arbitrary matter couplings could be introduced by adding the matter action to (2.16), without modifying $\mathcal{L}_C$. This leaves the first duality relation unchanged, but adds to the second duality relation (2.18) the standard energy-momentum tensor $T_{\mu_1 \cdots \mu_{D-2}} \sim \delta \mathcal{L}_M / \delta e_{\mu}^a$, which in turn re-appears in the Einstein equation in the usual way.

### 2.2. Dimensional Reduction of Dual Gravity

We will now discuss some aspects of the dimensional or Kaluza-Klein reduction of theories involving the dual graviton. In principle, we could work out the reduction of the full non-linear master action (2.16) including all fields, but here we content ourselves with a more schematic discussion of the type of fields appearing in lower dimensions. This sets the stage for our subsequent discussion of related fields in exceptional field theory. It is then sufficient to inspect the linearized theory, and here it is convenient to work with the master action in the form (2.3). In order to distinguish between world indices in different dimensions we will temporarily change notation and denote full $D$-dimensional spacetime indices by $\hat{\mu}, \hat{\nu}, \ldots = 0, \ldots, D - 1$,
so that the (linearized) master action reads

\[ S[Y, e] = -2 \int d^DX \epsilon \left( \gamma^\rho_1^\beta_1 \Omega_{\beta_1^\rho_1^\alpha} - \frac{1}{2} \gamma_{\beta_1^\rho_1^\alpha} \gamma^{\beta_1^\rho_1^\alpha} + \frac{1}{2 \Omega_{\rho_1}^\beta} \gamma_{\rho_1}^\beta \right), \]

(2.19)

where

\[ \Omega_{\beta_1^\rho_1^\alpha} = \partial_\beta h_{\rho_1^\alpha} - \partial_\rho h_{\beta_1^\alpha}, \]

(2.20)

are the linearized coefficients of anholonomy. This action is invariant under (linearized) local Lorentz transformations, with \( Y \) transforming as

\[ \delta \gamma_{\beta_1^\rho_1^\alpha} = -2 \partial_\beta A_{\rho_1^\alpha} - \partial_\rho A_{\beta_1^\alpha}. \]

(2.21)

We now perform the dimensional reduction of (2.19) by decomposing the spacetime indices as \( \mu = (\mu, \nu) \), where \( \mu = 0, \ldots, n-1 \) are the external indices and \( m = 1, \ldots, d = D - n \) are the internal indices, and assuming that all fields are independent of the internal coordinates, thus setting \( \partial_m = 0 \). For the (linearized) frame field we then write

\[ h_{\beta_1^\rho_1^\alpha} = \left( h_{\mu\nu} A_{nm} \right). \]

(2.22)

Note that usually one fixes the Lorentz gauge (partially) by setting \( h_{\mu\nu} = 0 \), but in the present context it is important to keep all fields so that we do not lose field equations. We will confirm momentarily, however, that \( h_{\mu\nu} \) is non-propagating. Moreover, we recall that \( h_{\beta_1^\rho_1^\alpha} \) and \( \phi_{m,n} \) carry symmetric and antisymmetric parts. The surviving gauge symmetries are given in terms of these components by

\[ \delta h_{\mu\nu} = \partial_\mu \xi_\nu - \Lambda_{\mu\nu}, \]

\[ \delta A_{\mu m} = \partial_\mu \xi_m - \Lambda_{\mu m}, \]

\[ \delta \phi_{m,n} = - \Lambda_{m,n}, \]

\[ \delta h_{\mu\nu} = \Lambda_{\mu\nu}. \]

(2.23)

The non-vanishing components of the coefficients of anholonomy (2.20) are given by

\[ \Omega_{\mu\nu} = 2 \partial_\mu h_{\nu}, \]

\[ \Omega_{\mu m} = F_{\mu m} \equiv \partial_\mu A_{m} - \partial_m A_{\mu}, \]

\[ \Omega_{\mu n} = \partial_\mu \phi_{m,n}, \]

\[ \Omega_{\rho m} = \partial_\rho h_{m}. \]

(2.24)

Integrating out \( Y \) from (2.19) naturally yields the free kinetic terms for the graviton \( h_{\mu\nu} \), the Kaluza-Klein vector \( A_{\mu m} \) and the Kaluza-Klein scalars \( \phi_{m,n} \), while the unphysical \( h_{\mu\nu} \) drops out. (More precisely, for the vectors only the shift-invariant combination \( A_{\mu m} + h_{\mu\nu} \) enters the action, which can therefore be identified with the Kaluza-Klein vectors.) In order to obtain the dual theory, we vary w.r.t. the original Kaluza-Klein fields in (2.22). To this end, we need the dimensional reduction of the \( Y \Omega \) term in (2.19):

\[ \gamma^{\rho_1^\beta_1} \Omega_{\beta_1^\rho_1^\alpha} = 2 \gamma^{\mu\nu} \partial_\mu h_{\nu} + \gamma^{\mu\nu} F_{\nu m} + 2 \gamma^{\mu\nu} \partial_\mu \phi_{m,n}, \]

(2.25)

which contain the only couplings to the original Kaluza-Klein fields (2.22). Thus, varying the action w.r.t. the physical fields \( h \), \( A \) and \( \phi \) respectively, yields

\[ \partial_\mu \gamma^{\mu\nu} = 0 \Rightarrow \gamma^{\mu\nu} = \partial_\nu C^{\mu \nu}; \]

\[ \partial_\mu \gamma^{\mu m} = 0 \Rightarrow \gamma^{\mu m} = \partial_\mu B^{\mu m}; \]

\[ \partial_\mu \gamma^{\mu n} = 0 \Rightarrow \gamma^{\mu n} = \partial_\mu E^{\mu n}. \]

(2.26)

where we used the Poincaré lemma. (Here we use the convention that like-wise indices that are not separated by a bar are assumed to be totally antisymmetric.) Varying w.r.t. the unphysical field \( h_{\mu\nu} \) yields

\[ \partial_\mu \gamma^{\mu m} = 0 \Rightarrow \gamma^{\mu m} = \partial_\mu K^{\mu m}. \]

(2.27)

Upon reinserting into the action, all terms coming from the expansion of \( Y \Omega \) then reduce to total derivatives, while the \( Y^2 \) terms yield the proper kinetic terms for the dual (generally propagating) fields \( C, A \) and \( E \).

The fields thus obtained can be written a little more suggestively as

\[ C_{\mu \nu \ldots \gamma} = \epsilon_{\mu \nu \ldots \gamma} C^{\mu \nu \ldots \gamma}, \]

\[ B_{\mu \nu \ldots \gamma} = \epsilon_{\mu \nu \ldots \gamma} B^{\mu \nu \ldots \gamma}, \]

\[ E_{\mu \nu \ldots \gamma} = \epsilon_{\mu \nu \ldots \gamma} E^{\mu \nu \ldots \gamma}. \]

(2.28)

and are thus naturally identified with components of the dimensionally reduced dual graviton \( C_{\ldots 3} \). Eliminating the \( Y \) fields determined by (2.26) inside (2.19) then yields the second-order actions for the dual fields that one would also obtain by dimensionally reducing the Curtright action directly. Note that the \((n-3)\)-forms \( B_\nu \) and the \((n-2)\)-forms \( E_{m,n} \) are the standard duals of vectors and scalars, respectively. More precisely, a suitable combination of the \((n-3)\)-forms determined in the second line of (2.26) and of the \((n-3)\)-form sub-representations contained in the \( K \) fields in (2.27) play the role of the duals to the vectors. The remaining sub-representations of the \( K \)-field are either non-propagating or pure gauge (noting that with (2.21) the local Lorentz transformations imply \( \delta K^{\mu \nu \ldots \gamma} = -4 \eta_{\mu \nu \ldots \gamma} \bar{A}_{\nu \ldots \gamma} \), so that the trace part is pure gauge). Finally, the components of the \( Y \) fields that have not yet been determined, which are

\[ Y_{\mu m} (\xi), \quad Y_{\mu m | \nu}, \quad Y_{\mu m | j}, \]

(2.29)

then enter the action purely quadratically and hence are non-propagating and can be eliminated algebraically. Relatedly, we note that the naïve decomposition of the dual graviton \( C_{\ldots 3} \) under the Kaluza-Klein split yields more component fields than are contained in (2.28), it follows from the above analysis that these fields are non-propagating and can thus directly be eliminated from the action.
Let us summarize the dual graviton components relevant in each dimension. In the exceptional field theory formulations to be discussed in the next section, the external graviton degrees of freedom will always be described conventionally. In the above formulation this corresponds to integrating out the external components $Y_{\mu
u|m}$, which leads to the standard Einstein-Hilbert action. The dual graviton components needed in each dimension are those dual to the Kaluza-Klein vectors and hence given for $D = 3, 4, 5$ by

$$
D = 3: \quad B_m \rightarrow D_m B_m = \partial_m B_{\hat{\mu}} + \cdots + \hat{Y}_{\mu|m},$$

$$
D = 4: \quad B_{\mu m} \rightarrow F_{\mu\nu|m} = 2 \partial_\mu B_{\nu|m} + \cdots + \hat{Y}_{\mu\nu|m}, \quad (2.30)
$$

$$
D = 5: \quad B_{\mu
u|m} \rightarrow H_{\mu\nu\rho|m} = 3 \partial_\mu B_{\nu\rho|m} + \cdots + \hat{Y}_{\mu\nu\rho|m}.
$$

Here we also indicated the compensating gauge fields entering through Stückelberg-type couplings in the compensator formulation that exists at the full non-linear level. These fields correspond, in dimensions $D = 3, 4, 5$, to extra vectors, two-forms and three-forms, respectively, and are a re-interpretation (and dualization) of the $Y$ fields.

In addition, also the dual graviton components dual to the Kaluza-Klein scalars will typically be visible below, which in dimensions $D = 3, 4, 5$ are given by $E_{\mu m,n}, E_{\mu\nu m,n}$ and $E_{\mu\nu\rho m,n}$, respectively. Let us finally note that there is an intriguing interplay between these dual graviton components and those originating from the Kaluza-Klein vectors, which is already visible at linearized level provided one keeps the full coordinate dependence as in exceptional field theory. To illustrate this, let us return to the master action (2.19) and again perform the Kaluza-Klein split (2.22), but now without truncating the coordinate dependence. This leads to further terms in (2.25) involving the internal derivative $\partial_\mu$, and hence modifies the field equations of the Kaluza-Klein fields accordingly. In particular, the field equations for the Kaluza-Klein vectors are now solved by

$$
Y_{\mu\nu|m} = \partial_\mu B_{\mu\nu|m} + \partial_\nu E_{\mu\nu|m}. \quad (2.31)
$$

(This equation can of course be obtained directly by solving the field equation for the full $h_{\hat{\nu}\hat{\rho}}$ as $Y_{\nu\rho|\mu} = \partial_\mu C_{\nu\rho|m}$ and reading off this component.) Dualizing now as in (2.28), say in four external dimensions, this naturally leads to the field strength

$$
F_{\mu\nu|m}(B) = 2 \partial_\nu B_{\mu|m} + \cdots + \partial_\mu E_{\mu\nu|m} + \hat{Y}_{\mu\nu|m}, \quad (2.32)
$$

where we also included the compensating two-form, while the ellipsis again denotes terms that would arise in the full non-linear theory. The terms displayed in this field strength follow naturally by Kaluza-Klein decomposing (2.14) (where one has to recall that, as in (2.26), redefinitions with the external and internal epsilon symbols are needed), while the non-linear terms not displayed are generated by Kaluza-Klein redefinitions, c.f. eqs. (4.30) in [16]. The above field strength is invariant under gauge transformations with one-form parameters $\Sigma_{\mu,n}$:

$$
\delta B_{\mu,m} = -\partial_\mu \Sigma_{\mu,n}, \quad \delta E_{\mu\nu|m} = 2 \partial_\mu \Sigma_{\mu\nu|m}.
$$

which is a direct consequence of the dual diffeomorphism symmetry, c.f. (2.10). Below we will repeatedly encounter this general structure of ‘hierarchical’ gauge symmetries and their invariant field strengths.

### 3. The Dual Graviton in Exceptional Field Theory

In this section, we review the structure of $p$-forms in exceptional field theories and discuss explicitly the structure and couplings of $(8 - d)$- and $(9 - d)$-forms. Among the latter feature the covariantly constrained compensator fields that are required for the construction of covariant field strengths and closure of the gauge algebra. Upon solving the section constraints and recovering the 11-dimensional field equations, these forms carry components from the 11-dimensional dual graviton and compensator field, respectively. We first discuss the generic structure and then illustrate the results for $E_{(7)}$ and $E_{(8)}$ ExFT.

#### 3.1. Exceptional Field Theory and the Tensor hierarchy

Exceptional field theory (ExFT) is a framework that embeds all $D = 11$ and $D = 10$ supergravities in a way manifestly covariant under the $E_{(d)}$ group that becomes a global symmetry after dimensional reduction. More precisely, in ExFT fields fall into representations of $E_{(d)}$ and coordinates split into $(11 - d)$ external coordinates $\{x^\mu\}$ and internal coordinates $\{y^\mu\}$ of which the latter are embedded into the fundamental representation of $E_{(d)}$ with the coordinate dependence restricted by the section constraints

$$
Y^M_{NL} \partial_M \otimes \partial_N = 0. \quad (3.1)
$$

Here, $\partial_M$ define derivatives w.r.t. coordinates $\{Y^M\}$ transforming in the fundamental representation of $E_{(d)}$, and $Y^M_{NL}$ is a constant $E_{(d)\text{-invariant}}$ tensor, see [19]. Any solution to (3.1) breaks $E_{(d)}$ by restricting the coordinate dependence of all fields to $d$ or $(d - 1)$ coordinates, whereupon the ExFT field equations reproduce the $D = 11$ and IIB field equations, respectively.

The ExFT field equations are manifestly invariant under generalized diffeomorphisms, acting as

$$
L^{(i)}_\Lambda V^M = \Lambda^K \delta_K V^M + \kappa \{P^M_{N|L} \partial_N \Lambda^K \partial_L V^N + \lambda \partial_N \Lambda^K V^M, \quad (3.2)
$$

on a vector field $V^M$ of internal weight $\lambda$. Here, $\kappa$ is a constant (fixed by closure of the algebra), and $P^M_{N|L}$ denotes the projector onto the adjoint representation of $E_{(d)}$, explicitly given by

$$
\kappa \{P^M_{N|L} = \kappa \{t_0\}_N \partial_0 (t^K)_L \quad (3.3)
$$

Upon redefining $E_{\mu\nu|m} \rightarrow E_{\mu\nu|m} + \alpha \partial_\mu E_{\nu|m} + \delta_\mu \delta_n^n$, we could change the structure of this term to resemble more closely some of the formulas below.
in terms of the $E_{d[\ell]}$ generators $(t_\ell)_N$, and related to the tensor $Y^{MK}_{NL}$ defining the section constraint (3.1), with $\lambda_d \equiv \frac{1}{d-1}$. We refer to the three contributions to (3.2) as the transport term, the rotation term, and the weight term, respectively.

In ExFT, the generalized diffeomorphisms (3.2) are a local symmetry w.r.t. the external coordinates, implemented by a gauge connection $A_\mu^M$ and standard covariant external derivatives

$$D_\mu = \partial_\mu - \mathcal{L}_{A_\mu}.$$  

(3.4)

The remaining ExFT fields organize into $E_{d[\ell]}$ representations that are scalars and $p$-forms w.r.t. the external coordinates. Accordingly, the latter come with gauge transformations with gauge parameters of rank $(p-1)$, defining a non-abelian tensor hierarchy on top of the generalized diffeomorphisms (3.2). In particular, the components $B_m$ and $E_{m,n}$ (2.28) from the higher-dimensional dual graviton sit within $(8-d)$-forms $B_M$ and $(9-d)$-forms $C_m$, in the fundamental and the adjoint representation of $E_{d[\ell]}$, respectively.

As a characteristic feature of non-abelian tensor hierarchies,[89] the covariant field strength associated to a $p$-form gauge potential $C^{[p]}$ is of the schematic form

$$\mathcal{F}^{[p+1]} = DC^{[p]} + \cdots + DC^{[p+1]}.$$  

(3.5)

Here, $D$ is the covariant external derivative (3.4), the ellipsis represents possible Chern-Simons-like contributions polynomial in lower-rank $p$-forms, and $D$ is a differential operator in the internal derivatives acting on a $(p+1)$-form gauge potential $C^{[p+1]}$. In standard non-abelian field theories, $D$ is typically given by an algebraic operator and describes the St"uckelberg-type coupling of a higher-rank gauge potential.

Accordingly, the $p$-form potentials are subject to gauge transformations of the type

$$\delta_\lambda C^{[p]} = D\Lambda^{[p-1]} + \cdots + D\Lambda^{[p]}.$$  

(3.6)

in order to leave the field strength (3.5) invariant. Consistency (in particular closure of the full gauge algebra) requires that the internal derivative operator $D$ defines a tensor under generalized diffeomorphisms (in analogy to the exterior derivative $D$ in the external sector). As observed in [21], this holds true for $p \leq 7-d$ but fails at $p = 8-d$, where the components of the dual graviton first enter the ExFT fields. Let us make this explicit. The $(8-d)$-forms $B_M$ in ExFT transform in the (dual) fundamental representation of $E_{d[\ell]}$, and the gauge transformations (3.6) take the explicit form

$$\delta_\lambda B_M = DA_M + \cdots - (t^{\mu})^N_\mu \partial_N A_\mu,$$  

(3.7)

with the $(8-d)$-form gauge parameter $A_\mu$ in the adjoint representation of $E_{d[\ell]}$, and the operator $D$ explicitly realized in terms of the $E_{d[\ell]}$ generators $(t^{\mu})^N_\mu$. The notation $\delta$ indicates that (3.7) is not the final answer. Using (3.2) and (3.3) it is a straightforward exercise that given a tensor $A_\mu$ transforming as

$$\delta_\lambda A_\mu = C^{[1]}_\lambda A_\mu \equiv \Lambda^K \partial_K A_\mu + \kappa \int_{\mu\nu} (t^{\nu})_L^K \partial_L A_\nu,$$  

(3.8)

with weight $\lambda$ under generalized diffeomorphisms, the particular combination

$$T_M \equiv (t^{\mu})^N_\mu \partial_N A_\mu,$$  

(3.9)

of internal derivatives, does not transform tensorially under generalized diffeomorphisms, but rather as

$$\delta_\lambda T_M = L^K_\lambda \partial_K T_M + (t^{\nu})^N_\nu (\lambda - 1) \partial_N \partial_K \Lambda^L \Lambda_\nu,$$  

(3.10)

provided $Y^{MN}_{KL} (t_\ell)_L^p \partial_{\mu} \partial_\ell = 0$ which holds for all $E_{d[\ell]}$, $d < 8$, as a consequence of the section constraint. In particular, for $\lambda = 1 = (9-d) \lambda_d$, which is the correct ExFT weight for $(9-d)$-forms, the non-covariant transformation of $T_M$ reduces to the last term of (3.10) and can be absorbed by introducing a covariantly constrained object $\Xi_M$ to which one assigns the transformation law

$$\delta_\lambda \Xi_M = C^{[1]}_\lambda \Xi_M - (t^{\nu})^N_\nu \partial_N \Lambda_\nu - (t^{\nu})^N_\nu \partial_\nu \Lambda^L \Lambda_\nu,$$  

(3.11)

under generalized diffeomorphisms. ‘Covariantly constrained’ indicates that this object satisfies the same section constraint (3.1) as the internal derivative operators, i.e.

$$Y^{MN}_{KL} \Xi_M \partial_\nu = 0 = Y^{MK}_{NL} \Xi_M \Xi_K.$$  

(3.12)

The resulting complete gauge transformation for $B_M$ then extends (3.7) to

$$\delta_\lambda \Xi_B_M = DA_M + \cdots - (t^{\mu})^N_\mu \partial_N A_\mu - \Xi_M,$$  

(3.13)

with an $(8-d)$-form covariantly constrained gauge parameter $\Xi_M$. Its associated $(9-d)$-form gauge field $C_M$ then appears in the full covariant field strength as

$$\mathcal{F}_M = DB_M + \cdots + (t^{\mu})^N_\mu \partial_N C_\mu + C_M.$$  

(3.14)

We will give the explicit expressions for $E_{(7)}$ and $E_{(8)}$ ExFT in the next subsections. W.r.t. the higher-dimensional origin of the ExFT fields, the presence of $C_M$ in the field strength $\mathcal{F}_M$ is precisely the remnant of the required presence of the compensating field $Y$ in the full non-linear field strength of the dual graviton, c.f. (2.30).

Let us briefly describe the generic structure of the dynamics of the field $C_M$. The Bianchi identity for the field strength (3.14) takes the form

$$D \mathcal{F}_M = \bullet (f_{\mu\nu})_M + (t^{\mu})^N_\mu \partial_N \mathcal{F}_\nu + \mathcal{G}_M,$$  

(3.15)

with the Chern-Simons contributions $\bullet (f_{\mu\nu})_M$ resulting from the ellipsis in (3.14) and $\mathcal{F}_\nu$ and $\mathcal{G}_M$ denoting the non-abelian field strengths of $C_\nu$ and $C_M$, respectively. The dynamics of the $(8-d)$-forms is encoded in a first-order duality equation

$$\mathcal{F}_M = \mathcal{M}_{MN} \bullet \mathcal{F}_N,$$  

(3.16)
where $M_{MN}$ denotes the scalar dependent $E_{(d)}$ matrix parametrizing the scalar target space, and $F^N$ represents the non-abelian field strength for the ExFT vector fields in the fundamental representation of $E_{(d)}$. The second order field equations for the latter are derived from the ExFT Lagrangian and take the generic form

$$D(M_{MN} \star F^M) = (J_{CS})_M + (t^a)_M \partial_N J_a + I_M.$$  

(3.17)

Here, $(J_{CS})_a$ comes from variation of the topological terms of the Lagrangian, while $J_a$ and $I_M$ derive from variation of the vector fields within connections (3.4), with $J_a$ carrying the contributions from the rotation term and $I_M$ carrying the contributions from the transport and the weight term. In particular, $I_M$ is covariantly constrained according to the notion of (3.12).

Combining exterior derivative of the duality equation (3.16) with the Bianchi identity (3.15) and the Yang-Mills equations (3.17), shows that the CS terms cancel as they do in the dimensionally reduced theory\cite{23} (when $\partial M = 0$), such that we are left with the duality equations

$$F_a = \star J_a,$$  

(3.18)

$$G_M = \star I_M,$$  

(3.19)

describing the dynamics of the fields $C_a$ and $C_M$ descending from the higher-dimensional dual graviton and the compensating field $Y$, respectively. They encode some of the components of the higher-dimensional duality equation (2.17) and (2.18), respectively.

Finally, we would like to point out that $F_a$ and $G_M$ (and likewise $\Lambda_a$ and $\Xi_M$) naturally form an irreducible object w.r.t. an underlying Lie algebra from which the algebra of generalized diffeomorphisms can be derived by means of an ‘embedding tensor’. Specifically, starting from the Lie algebra $g$ of (conventional) diffeomorphisms and local U-duality transformations, spanned by parameters $(\lambda^M, \sigma^M)$, the generalized diffeomorphisms of ExFT can be defined in terms of $g$-representations and the embedding tensor $\vartheta : R \to g$, where $R$ denotes the representation labelled by indices $M, N, \ldots$ as

$$\vartheta(\Lambda) = (\Lambda^M, -\kappa(t^a)_M \partial_N \Lambda^M).$$  

(3.20)

The objects $F_a$ and $G_M$ form now the irreducible coadjoint representation of $g$ in that the transformations (3.8) and (3.11) can be rewritten in terms of the coadjoint action as

$$\delta_a F_a = \text{ad}_{\vartheta(\Lambda)}(F_a) = (\Lambda^M \star F_{aM}),$$  

(3.21)

as one may quickly verify with Equation (A.9) in [23].

### 3.2. E_{17} General Structure and Solving the Section Constraint

In $E_{17}$ ExFT, the fields with origin in the 11-dimensional dual graviton show up among the vector and the two-form fields. More precisely, the fields (2.28) give rise to 7 vectors $B_{\mu \nu}$ and 49 two-forms $E_{\mu \nu \lambda}$, together with the $D = 4$ dual graviton $C_{\mu \nu \lambda \sigma}$. In this case, the field strength (3.14) in which the constrained compensator field first appears is the field strength $F^M = \Omega^{MN} F_{N}$ of the vector fields itself which is explicitly given by

$$F_{\mu \nu}^M = 2 \partial_{[\mu} A_{\nu]}^M - 2 A_{[\mu}^K \partial_{\nu]} A_{\nu]}^M$$

$$- \frac{1}{2} \Omega^{MP} (24 (t^a)_P (t^a)_Q - \delta^P_Q \delta^M_N) \Omega^{QN} A_{[\mu}^N \partial_{\nu]} A_{\nu]}^L,$$

$$- 12 \Omega^{MK} (t^a)_K \partial_{\nu]} B_{\mu \nu K} = \frac{1}{2} \Omega^{MK} B_{\mu \nu K},$$

(3.22)

in terms of the $E_{17}$, generators $(t^a)_M^N$ and the symplectic matrix $\Omega_{MN}$. The last two terms in (3.22) correspond to the couplings introduced in (3.14) with two-form gauge potentials $B_{\mu \nu a}$ and $B_{\mu \nu M}$ in the adjoint and the fundamental representation, respectively.

In $D = 4$ ExFT (and supergravity), the generic duality equation (3.16) is replaced by the twisted self-duality equations\cite{22}

$$F_{\mu \nu}^M = \frac{1}{2 \sqrt{|g|}} \epsilon_{\mu \nu \rho \sigma} \Omega^{MN} M_{NK} F_{\rho \sigma}^K,$$

(3.23)

It is accompanied by the duality equations between two-forms and scalar fields (3.1), (3.19), which here take the explicit form

$$H_{\mu \nu \alpha a} = - \sqrt{|g|} \epsilon_{\mu \nu \rho \sigma} (t^a)_K \partial_{(a, k)} \left(D^a M + \mathcal{M}_{L P} \right),$$

$$H_{\mu \nu \alpha a} = \sqrt{|g|} \epsilon_{\mu \nu \rho \sigma} \left(\mathcal{M}_M - \frac{1}{12} D^a M_{KL} \partial_{(a)} M_{KL} \right),$$

(3.24)

see [18] for details.

The ExFT section constraint is solved by decomposing the adjoint representation of $E_{17}$ under its maximal GL$(7)$ subgroup and restricting internal derivatives according to

$$\partial_M = (\partial_0, \ldots, 0).$$

(3.25)

The same decomposition applies to vector and two-forms and implies

$$A_{\mu}^M \rightarrow (A_{\mu}^M, A_{\mu m 0}, A_{\mu m 0}, B_{\mu a}),$$

$$B_{\mu \nu a} \rightarrow (B_{\mu \nu m}, B_{\mu \nu m}, C_{\mu \nu m} \ldots),$$

(3.26)

$$B_{\mu \nu M} \rightarrow (C_{\mu \nu m}, 0, 0, 0),$$

where we have restricted to those fields actually appearing in (3.23) and made explicit the fields $B_{\mu m}$, $E_{\mu \nu m}$, $C_{\mu \nu m}$, descending from the $D = 11$ dual graviton $(B_{\mu m}, E_{\mu \nu m})$ and the compensator field $(C_{\mu \nu m})$, respectively. Zooming in on their field equations, the relevant component of the field strength (3.22) gives the full completion of (2.32) as

$$F_{\mu \nu m} = 2 \partial_{[\mu} B_{\nu]} m - 2 A_{[\mu}^k B_{\nu]} k - 2 \partial_{[\mu} A_{\nu]}^k B_{\nu]} k$$

$$- 2 A_{[\mu}^k B_{\nu]} k + 2 \partial_{[\mu} A_{\nu]}^k A_{\nu]} k - 3 A_{[\mu}^k \partial_{\nu]} A_{\nu]} k$$

$$+ \partial_{[\mu} E_{\nu]} ^{m \nu} - \partial_{\nu] m} E_{\mu \nu} - C_{\mu \nu m},$$

(3.27)

By virtue of (3.23), this field strength is expressed as the dual of the remaining field strengths, reproducing the relevant components of the duality equations (2.17). In turn, the second equation of (3.24) reproduces the corresponding components of the
eleven-dimensional equation (2.18) for the compensator field. The combination of (3.23) and (3.24) thus reproduces the correct eleven-dimensional dynamics. Comparing the explicit form of the field strength (3.27) to its higher-dimensional ancestor (2.14), the first line of (3.27) descends from the first term of (2.14) upon expanding the objects in the standard Kaluza-Klein basis (see e.g. [16] for a detailed discussion). The $A^I$ terms in the second line of (3.27) indicate the translation of the relevant component $\mathcal{V}_{\mu
u}$ of the higher-dimensional compensator field into the component $C_{\nu\rho
u}$ of the duality covariant ExFT object $\mathcal{B}_{\nu\rho
u}$ (3.26).

This illustrates how the $E_7(7)$ covariant ExFT field equations (3.23), (3.24) require the inclusion of certain components of the eleven-dimensional equations (2.17), (2.18) for the dual graviton and the compensator field, respectively, which in turn become part of the duality covariant ExFT objects as in (3.26).

### 3.3. $E_8(8)$ General Structure and Solving the Section Constraint

We now spell out the details of the general structures introduced above for the $E_8(8)$ case. The adjoint representation of $E_8(8)$ is 248-dimensional, and we denote its indices by $M, N = 1, \ldots, 248$ and the structure constants by $f^{MNK}$. The tensor product $248 \otimes 248$ decomposes as

$$248 \otimes 248 \rightarrow 1 \oplus 248 \oplus 3875 \oplus 27000 \oplus 30380,$$

and the section constraints project out the sub-representation $1 \oplus 248 \oplus 3875$, i.e.,

$$\eta^{MN} \partial_M \otimes \partial_N = 0, \quad f^{MNK} \partial_N \otimes \partial_K = 0, \quad \eta^{MN} \partial_N \otimes \partial_K = 0, \quad \eta^{MN} \partial_K \otimes \partial_N = 0.$$

The explicit form of the projectors can be found in [24]. In $E_8(8)$ ExFT, the fields on which additional gauge transformations have to be introduced as in (3.13) are the scalar fields. As a result, these transformations modify the generalized diffeomorphisms (3.2) themselves which become

$$L^{(2)}_{(\lambda, \Sigma)} V^M = \Lambda^N \partial_N V^M + f^{MNK} R^K V^K + \partial_N \Lambda^N V^M,$$

where

$$R^K = f^{MNK} \partial_N \Lambda^K + \Sigma^K.$$

This expression is the $E_8(8)$ implementation of the general structure (3.13) discussed above, combining the gauge parameter of $(8 - d)$-forms (i.e., scalars for the $E_8(8)$ case) with a ‘covariantly constrained’ parameter in the adjoint representation (i.e., here in the fundamental representation of $E_8(8)$). As shown recently in [23,25], the space of these extended gauge parameters, which we group as $Y = (\Lambda^N, \Sigma)$, carries the structure of a Leibniz-Loday algebra with product

$$\mathcal{T}_1 \circ \mathcal{T}_2 \equiv \left( L^{(2)}_{\mathcal{T}_1, \mathcal{T}_2} \right), \quad \mathcal{T}_{2M} \Sigma_{2M} + \Lambda^N \partial_M R_N (\mathcal{T}_1),$$

which satisfies the Leibniz rule

$$\mathcal{T}_1 \circ (\mathcal{T}_2 \circ \mathcal{T}_3) = (\mathcal{T}_1 \circ \mathcal{T}_2) \circ \mathcal{T}_3 + \mathcal{T}_2 \circ (\mathcal{T}_1 \circ \mathcal{T}_3).$$

Thanks to this algebraic structure, the $E_8(8)$ ExFT can be formulated efficiently in terms of ‘doubled’ parameters and fields, which makes manifest that the consistency of the theory hinges on the precise interplay of the naive (generalized diffeomorphism) parameter $\Lambda^M$ and the new companion parameter $\Sigma$ required for consistency of the dual graviton couplings, as we will explain in more detail in the following.

In order to explain the resolution of the dual graviton problem for the $E_8(8)$ case more explicitly, we first review the bosonic field content. It is given by the external dreibein $e^a \mu$, carrying curved and flat 3D indices, the internal 248-bein $V^M$, being a matrix in the adjoint of $E_8(8)$, and gauge vectors $A_{\mu} = (A_{\mu}^M, B_{\mu M})$ taking values in the Leibniz-Loday algebra. All fields depend on $3 + 248$ coordinates $(x^m, Y^M)$, modulo the section constraints. The action takes the schematic form

$$S = \int d^3 x d^{248} Y \sqrt{|g|} \left( \hat{R} + \frac{1}{2} \nabla^2 V \right) + \frac{1}{240} g^{a\mu} D_a M^{MN} D_\nu M_{MN} - V(M, g).$$

The first term is the 3D Einstein-Hilbert term for $e^a \mu$, suitably covariantized w.r.t. the gauge connection $A_{\mu}$. The second term is a Chern-Simons term for $A_{\mu}$ based on the Leibniz-Loday algebra structure (3.33), taking the third term is the kinetic kinetic for the $E_8(8)/SO(16)$ coset scalar matrix $M \equiv \nabla^M$. The final term is the ‘potential’ (in the sense that it involves only external derivatives and hence reduces upon Kaluza-Klein reduction to a genuine scalar potential),

$$V(M, g) = -\frac{1}{240} g^{a\mu} D_a M^{MN} D_\nu M_{MN} + \frac{1}{2} \nabla^M g^{a\mu} \nabla_M D_\nu M_{MN} + \frac{1}{7200} f^{a\mu} f^{b\nu} f^{c\rho} g^{a\mu} g^{b\nu} g^{c\rho} + (\partial g) terms,$$

where we suppressed terms carrying derivatives of the external metric $\partial_M g_{a\nu}$, see [24] for the full expression. Upon solving the section constraint by decomposing the adjoint representation of $E_8(8)$ under its maximal $GL(8)$ subgroup and writing for the internal derivatives

$$D_\mu = (\partial_\mu, 0, \ldots, 0),$$

where $m = 1, \ldots, 8$ is the fundamental $GL(8)$ index, the above action is fully equivalent to $D = 11$ supergravity in $3 + 8$ split.

Let us now discuss this match with $D = 11$ supergravity schematically, focusing on the role of dual fields such as the dual graviton. Decomposing the ExFT fields under the maximal $GL(8)$ subgroup that survives as a manifest symmetry, one obtains the following fields: The dreibein $e^a \mu$, being an $E_8(8)$ singlet, does not

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4 This doubled structure can also be understood as a special case of the general realization in terms of the coadjoint action of a Lie algebra in (3.21), using that here the adjoint and $R$ representations coincide.
decompose and originates directly from the corresponding 3D component of the vielbein in $D = 11$ under the $3 + 8$ split. In particular, there is no ‘external dual graviton’. The $E_{8(8)}/SO(16)$ coset degrees of freedom can be parameterized in terms of the following $GL(8)$ covariant scalar fields:

$$\psi_{\alpha} = \phi_{\alpha m} \in GL(8)/SO(8), \quad c_{m_{1}...m_{8}}, \quad \epsilon_{n_{1}...n_{8}} = \epsilon_{n_{1}...n_{8}} \phi_{m_{1}...m_{8}}. \quad (3.37)$$

Here $\phi_{\alpha m}$ is symmetric and originates directly as ‘Kaluza-Klein scalars’ from the $D = 11$ frame field, as in $(2.22)$. The fields denoted by $c$ are completely antisymmetric and thus naturally originate as internal components from a 3-form and 6-form in $D = 11$. Finally, the field $\phi_{m}$ is equivalent, as indicated here, to an $[8,1]$ field, thus suggesting an interpretation as a dual graviton component. Next, the vector fields decompose into $GL(8)$ covariant fields as follows:

$$A_{\mu} = A_{\mu}^{m_{1}}A_{\mu}^{m_{2}}A_{\mu}^{n_{1}n_{2}}A_{\mu}^{m_{1}m_{2}}, \cdots$$

$$B_{\mu \nu} = B_{\mu \nu}, \cdots. \quad (3.38)$$

Here we have left out a number of vector fields, indicated by the ellipsis, that drop out in the final action following the section constraint. In particular, among the covariantly constrained vector fields $B_{\mu \nu}$ only eight components survive, c.f. $(4.3),(4.4)$ in [24].

We are now ready to account for the $D = 11$ fields under the $3 + 8$ split. The $D = 11$ frame field yields a 3D frame field and an internal $GL(8)/SO(8)$ coset matrix, both of which were already identified among the above fields. It also gives rise to eight Kaluza-Klein vectors $A_{\mu}^{m}$ that are among the components of $A_{\mu}^{m}$ in $(3.38)$. However, here we seem to encounter the dual graviton problem, because naively the scalars $\psi_{m}$ in $(3.37)$ also encode the degrees of freedom of the Kaluza-Klein vectors. Indeed, in dimensional reduction it is necessary to dualize the Kaluza-Klein vectors $A_{\mu}^{m}$ into $3D$ scalars $\phi_{m}$ in order to complete the $E_{8(8)}/SO(16)$ coset matrix. Since in ExFT we have both $A_{\mu}^{m}$ and $\phi_{m}$, then it seems to be an over-counting of fields. In order to see that, on the contrary, the degrees of freedom properly match we have to take into account the compensating gauge vectors $B_{\mu \nu}$. The Chern-Simons and scalar kinetic terms then take the schematic form

$$L_{\text{scalar-vector}} = \frac{1}{2} \epsilon_{\mu \nu \rho} B_{\mu \nu}^{\rho} F^{\mu \nu} = \frac{1}{2} \phi_{\mu}^{\rho} D^{\rho} \phi_{\mu} - L_{A_{\mu}}, \quad (3.39)$$

where $F^{\mu \nu}$ is the field strength for the Kaluza-Klein vectors and

$$D_{\mu} \phi_{m} = \partial_{\mu} \phi_{m} = L_{A_{\mu}} \phi_{m} + B_{\mu \nu}. \quad (3.40)$$

We observe here Stückelberg couplings, in agreement with the residual gauge invariance originating from the $\Sigma_{M}$ transformations, $\delta \phi_{m} = -\Sigma_{m}, \delta B_{\mu \nu} = D_{\mu} \Sigma_{m}$. Consequently, upon integrating out $B_{\mu \nu}$ from $(3.39)$ the scalar components $\phi_{m}$ drop out, and the action reduces to the Yang-Mills term for the vectors $A_{\mu}^{m}$, in precise agreement with the $3 + 8$ split of $D = 11$ supergravity. Moreover, working out the explicit form of the covariant derivatives $(3.4)$ with $(3.30)$ shows that the vector components $A_{\mu}^{m}$ (which are the three-dimensional version of the $(n - 2)$-forms $E_{m,n}$ discussed in Section 2.2) only enter in the combination $\delta_{\mu} A_{m}^{\mu} + B_{\mu \nu}$, so that integrating out $B_{\mu \nu}$ also eliminates $A_{m}^{\mu}$ from the Lagrangian. In addition, in the scalar potential $(3.35)$ the components $\phi_{m}$ drop out after solving the section constraint, as it should be in view of the Stückelberg invariance. Thus, there is no conflict with the presence of the ‘dual graviton’ field $\phi_{m}$ in ExFT and the fact it can be matched with $D = 11$ supergravity without a dual graviton field.

Finally, let us identify the degrees of freedom corresponding to the 3-form in $D = 11$, which gives rise to scalars $c_{m_{1}m_{2}}$, vectors $A_{m_{1}}$, 2-forms $B_{m_{1}m_{2}}$ and a 3-form $C_{m_{1}m_{2}m_{3}}$. The scalars and vectors are already contained in $(3.37)$ and $(3.38)$, respectively. The 2-forms are not present explicitly in ExFT but rather encoded in the scalars $c_{m_{1}m_{2}}$ in $(3.37)$ via the duality relation

$$H_{\mu \nu \rho} = \varepsilon_{\mu \nu \rho} F_{m_{1}...m_{8}} = \varepsilon_{\mu \nu \rho} \rho_{m_{1}...m_{8}} F_{m_{1}...m_{8}}. \quad (3.41)$$

where $H_{\mu \nu \rho}$ is the field strength of the 2-form, and $F_{m_{1}...m_{8}} = \delta_{m_{1}...m_{8}} + 140 \epsilon_{m_{1}m_{2}m_{3}m_{4}} \delta_{m_{1}m_{2}m_{3}m_{4}}$. The 4-form curvature of the 3-form vanishes identically in 3D and can be eliminated from the action (using techniques similar to those in [16]). Thus, all fields originating from the 3-form in $D = 11$ are accounted for. Finally, the vectors $A_{m}^{\mu}$ in $(3.38)$ are defined in terms of the scalars $c_{m_{1}m_{2}}$, which were introduced in $(3.41)$, via

$$F_{\mu \nu \rho \sigma} = \varepsilon_{\mu \nu \rho \sigma} \varepsilon_{m_{1}m_{2}} \varepsilon_{m_{3}m_{4}} \varepsilon_{m_{5}m_{6}} \varepsilon_{m_{7}m_{8}}. \quad (3.42)$$

with the current $J_{\mu}^{m_{1}m_{2}} = \varepsilon_{m_{1}m_{2}m_{3}m_{4}} \varepsilon_{m_{5}m_{6}m_{7}m_{8}} \varepsilon_{m_{9}m_{10}} \varepsilon_{m_{11}m_{12}}$, with the ellipsis indicating the connection term.

4. Application: Reductions and Consistent Truncation on $AdS_{4} \times S^{7}$

Exceptional field theory provides a powerful tool to study consistent truncations of maximal supergravity by virtue of a generalized Scherk-Schwarz reduction. These are truncations to lower-dimensional theories, such that every solution to the lower-dimensional field equations lifts to a solution of the higher-dimensional field equations. In particular, this requires that all dependence on the internal coordinates fact out from the higher-dimensional field equations. In this section, we work out the reduction formulas for the dual graviton and the compensator field for the prominent example of the truncation of $D = 11$ supergravity on $AdS_{4} \times S^{7}$[26]. In turn, this illustrates the necessity of the ExFT compensator field in order to consistently reproduce part of the higher-dimensional equations for the dual graviton around the seven-sphere.

The reduction formulas for the $p$-form fields in a consistent Scherk-Schwarz type reduction of $E_{7(7)}$ ExFT read[27]

$$A_{\mu}^{m}(x, Y) = \rho^{-1}(Y) \left( U^{-1} N^{m}(Y) A_{\mu}^{N}(x) \right),$$

$$B_{\mu \nu \alpha}(x, Y) = \rho^{-2}(Y) U_{\alpha}^{\beta}(Y) B_{\mu \nu \beta}(x)$$

$$B_{\mu \nu \alpha \beta}(x, Y) = -2 \rho^{-2}(Y) \left( U^{-1} S^{\alpha}(Y) \partial_{\mu} U_{\nu}^{\beta}(Y) \right) B_{\mu \nu \beta}(x), \quad (4.1)$$
in terms of an E\textsubscript{7(7)} valued twist matrix \(U\) in the fundamental representation, together with a scaling factor \(\rho\). For the \(S^2\) reduction, the twist matrix \(U\) lives in the subgroup \(\text{SL}(8) \subset E_{7(7)}\) and is of the explicit form \(U^a_m = \{U^a_n, U^a_n\}\) with

\[
U^a_n = \omega^{-1/4} Y^a - 6 \omega^{-1/4} \xi^n \partial_n Y^a,
\]

\[
U^a_m = \omega^{-1/4} \partial_m Y^a,
\]

in terms of sphere harmonics \(Y^a Y^a = 1\) and \(\partial_n \xi^n = \omega \equiv \sqrt{|g^{27|28|}|}\). The scale factor \(\rho\) is given by \(\rho = \omega^{-1/2}\).

For the different components (3.26) of the ExFT vector fields, the reduction formula (4.1) then implies the explicit reduction formulas

\[
A^a_m = K^a_m A^a_m, \quad A^a_m = (\omega K^m_{ab} + 12 \xi^m K^a_{n}) A^a_m, \quad A^a_m = K^a_{mn} A^a_{nm}, \quad B^m_m = (\omega K^a_{ab} + 6 \xi^a K^m_{ab}) A^a_{nm},
\]

(4.3)

with \((A^a_m, A^a_{nm}) = (A^a M)\) denoting the 28 electric and 28 magnetic vector fields of \(D = 4\) supergravity, and the \(S^2\) Killing vectors and tensors defined by

\[
K^a_{mn} \equiv Y^a \partial_m Y^n, \quad K^a_{mn} \equiv \partial_m K^a_{n}.
\]

This reproduces the formulas from [26,29,30]. In particular, the last formula of (4.3) provides the reduction formula for the components \(B_{nm}\) from the 11-dimensional dual graviton (2.28).

The other components \(C_{\mu\nu,m}^n\) of the 11-dimensional dual graviton are identified within the ExFT two-forms according to (3.26). The relevant reduction formula (4.1) then implies its reduction as

\[
C_{\mu\nu,m}^n = \left(6 \partial_m Y^n \partial_n Y^\mu + 6 \partial_m Y^n \partial_n Y^\nu + 6 \partial_m Y^n \partial_n Y^\mu + 6 \partial_m Y^n \partial_n Y^\nu \right) B_{\mu\nu,a}^b,
\]

(4.5)

in terms of 63 of the 4-dimensional two-forms \(\{B_{\mu\nu,a}^b\} \subset \{B_{\mu\nu,a}\}\). Finally, the components \(C_{\mu\nu,m}^n\) from the \(D = 11\) compensator field sit within the constrained ExFT field \(B_{\mu\nu,a}\) according to (3.26) such that the last equation of (4.1) implies the following reduction formula

\[
C_{\mu\nu,m}^n = -\left(\omega \partial^n Y^\mu \partial_m \partial_n Y^\alpha + \omega \partial^n Y^\nu \partial_m \partial_n Y^\alpha + \partial_m \omega \partial^n Y^\mu \partial_n \partial^\alpha Y^\nu + \partial_m \omega \partial^n Y^\nu \partial_n \partial^\alpha Y^\mu \right) B_{\mu\nu,a}^b.
\]

(4.6)

As discussed above, the eleven-dimensional duality equation for the dual graviton features the non-abelian field strength (3.27). Consistency thus crucially requires that the full non-linear expression (3.27) is compatible with the reduction formulas (4.3)–(4.6), such that all dependence on the sphere coordinates factors out precisely as in the first term

\[
\partial_{\mu} B_{\nu}^a = (\omega K^a_{ab} + 6 \xi^a K^m_{ab}) \partial_{\mu} A^a_{\nu b}.
\]

(4.7)

Collecting all the \(AB\) and \(A^2\) terms from (3.27), it is lengthy but straightforward to show that the various reduction formulas from (3.26) indeed combine into

\[
AB + AA \rightarrow (\omega K^a_{ab} + 6 \xi^a K^m_{ab}) A^a_{\mu c} A^a_{\nu c},
\]

(4.8)

i.e. complete (4.7) into the non-abelian Yang-Mills field strength \(F_{\mu\nu,a}\) for the magnetic vector fields.

It remains to analyze the two-form contributions in (3.27). The dual graviton contributions from \(\partial_{\mu} C_{\nu,m}^n\) are computed from (4.5) as

\[
\partial_{\mu} C_{\nu,m}^n - \partial_{\mu} C_{\nu,m}^n = \left(12 K^a_{mn} \xi^n - 6 \partial_m Y^\mu \partial_n \xi^n + \omega \partial_m \partial_n Y^\mu \partial^n \partial^\nu Y_b + \partial_m \omega \partial^n \partial^\nu Y^\mu \partial_n \partial^\alpha Y^\nu \right) B_{\mu\nu,a}^b.
\]

(4.9)

i.e. it is not consistent with the factorized form of (4.7), (4.8), unless the combination (4.6) from the compensator field is taken into account. Upon combining all contributions from (4.3)–(4.6), the final result is

\[
F_{\mu\nu,a} = (\omega K^a_{ab} + 6 \xi^a K^m_{ab}) (F_{\mu\nu,a} + 2 B_{\mu\nu,a}^b).
\]

(4.10)

The field strength (3.27) thus reduces in factorized form with the combination \(F_{\mu\nu,a} + 2 B_{\mu\nu,a}^b\) precisely capturing the Stickelberg corrected magnetic field strengths of maximal \(D = 4\) gauged supergravity.\[31\] As a result, the eleven-dimensional duality equation (2.17) for the dual graviton also allows for a consistent truncation on the seven-sphere.

5. Summary and Outlook

We reviewed the status of the so-called ‘dual graviton’, with a particular emphasis on its role within duality covariant formulations of the low-energy actions of string/M-theory. There is no sharp physical problem that would require a field theory of a dual graviton, since one may always choose physical or light-cone gauge for which there is no difference between ‘graviton’ and ‘dual graviton’. Nevertheless, in order to make certain features (such as duality symmetries) manifest, it is necessary to work with field variables that include dual graviton components. The main take-home message of this review should then be this: to the extent that duality-covariant formulations are the dual graviton’s purpose in life there is a fully satisfactory formulation.

The formulation of ‘covariantly constrained’ compensator gauge fields is the key ingredient that allowed us to circumvent the no-go theorems for the dual graviton. Here we summarize several results and observations that independently confirm that the seemingly bizarre notion of covariantly constrained fields and symmetries is self-consistent and necessary:

(1) The covariantly constrained fields and symmetries arise for any ExFT with duality group \(E_{7(7)}\), at the appropriate level of the tensor hierarchy of \(p\)-form potentials, to render the dual graviton pure gauge. In particular, for the recently discussed generalized diffeomorphisms for the affine \(E_{7(7)}\), such fields are present already among the scalar fields.\[14\]

(2) The dynamical equations for the covariantly constrained fields are necessary in order to recover the correct eleven-dimensional dynamics from the duality covariant ExFT field equations.
(3) In supersymmetric versions of ExFT, the covariantly constrained fields receive their own independent supersymmetry variations,\cite{15,30} which are indispensable for closure of the supersymmetry algebra.

(4) The consistency of generalized Scherk-Schwarz compactifications requires the inclusion of the compensator fields with an appropriate Scherk-Schwarz ansatz (that is manifestly compatible with the constraints on the new fields). These fields emerge already in consistency proofs of fully geometric compactifications such as $D = 11$ supergravity on $AdS_5 \times S^5$, as reviewed here.

(5) The doubled structure of gauge symmetries and vectors for the $E_{6(6)}$ theory has recently been shown to have a deeper mathematical significance in that, say, the doubled vectors $\mu \equiv (A^\mu, B^\mu_M)$ can be seen as gauge vectors for so-called Leibniz-Loday algebras with a corresponding Chern-Simons formulation of the action.\cite{21,25}

Finally, we point out that the above of course does not exclude the possibility that some future applications may require a formulation containing dual graviton-type fields going beyond those discussed here. For instance, one may imagine that eventually a formulation is called for in which all mixed-Young tableaux fields are encoded in representations of the full duality group, most likely along the lines already investigated for double field theory at the linearized level.\cite{32,33}

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Conflict of Interest

The authors have declared no conflict of interest.

Keywords

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