AVERAGE-VALUE TVERBERG PARTITIONS VIA FINITE FOURIER ANALYSIS

STEVEN SIMON

Abstract. The topological Tverberg conjecture claimed, for any continuous map from the boundary of a $N(q,d) := (q - 1)(d + 1)$-simplex to $d$-dimensional Euclidian space, the existence of $q$ pairwise disjoint faces whose images have non-empty $q$-fold intersection. The affine cases, true for all $q$, constitute Tverberg’s famous generalization of Radon’s theorem on partitioning point collections into disjoint sets with overlapping convex hulls. Although the conjecture was established for all prime powers in 1987 by Özaydin, counterexamples for all non-prime-powers were shown to exist in 2015 by Frick. Reformulating this conjecture in terms of finite harmonic analysis and considering maps below the tight dimension $N(q,d)$, we show that one can nonetheless guarantee collections of $q$ pairwise disjoint faces – including when $q$ is not a prime power – which satisfy a variety of “average value” coincidences arising from the vanishing of Fourier transforms.

1. Introduction and Statement of Results

1.1. A brief history of the topological Tverberg conjecture. The celebrated Tverberg theorem [21] of 1966, recovering the classical Radon’s Theorem of 1921 [18] when $q = 2$, states that any $(q - 1)(d + 1) + 1$ (or more) points in $\mathbb{R}^d$ can be partitioned into $q$ pairwise disjoint sets whose convex hulls have non-empty $q$-fold intersection. A famous conjecture of topological combinatorics claimed a continuous extension, with Tverberg’s theorem recovered as the affine cases:

Conjecture 1. Let $\partial \Delta^n$ denote the boundary of the $n$-simplex. Any continuous map $f : \partial \Delta^{(q - 1)(d + 1)} \to \mathbb{R}^d$ admits a topological Tverberg partition, i.e., $q$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_q) \neq \emptyset$.

The algebraic argument of [21] in the affine cases showed that $N := (q - 1)(d + 1)$ is minimal for given $d$ and $q$. Conjecture 1 was established for $q = 2$ in 1979 by Bagmóczy and Bárány [3], for odd primes by Bárány, Schlosman, and Szücs [4] in 1981, and for all prime powers by Özaydin in 1987 [17] (as well as by Volovinkov [22] and Sarkaria [19]):

Theorem 1.1. If $q$ is a prime power, then any continuous map $f : \partial \Delta^{(q - 1)(d + 1)} \to \mathbb{R}^d$ admits a topological Tverberg partition.

Although true for arbitrary $q$ when $d = 1$, the conjecture remained completely open for all non-prime-powers in all dimensions $d \geq 2$ until 2015, when the existence of counterexamples was proven by Frick [11]:

Theorem 1.2. For any non-prime-power $q$ and any $k \geq 3$, there exists a continuous map $f : \partial \Delta^{(q - 1)(qk + 2)} \to \mathbb{R}^{qk + 1}$ which does not admit a topological Tverberg partition.

1.2. Average-Value Tverberg Partitions. This note considers continuous maps $f : \partial \Delta^n \to \mathbb{R}^d$ when $n$ is smaller than the tight dimension $N := (q - 1)(d + 1)$. Although
full Tverberg partitions can fail to exist, we show that one is nonetheless guaranteed collections of \( q \) pairwise disjoint faces which satisfy other symmetric coincidences, including when \( q \) is not a prime power. In essence, our main such result guarantees a collection of \( q \) points, one from each of \( q \) pairwise disjoint faces, which for varying \( m \geq 1 \) contains a relatively large number of order \( m \) subsets \( \{x_1, \ldots, x_m\} \) whose average values
\[
(1.1) \quad \text{Avg}(f; x_1, \ldots, x_m) = \frac{f(x_1) + \ldots + f(x_m)}{m}
\]
amre fixed. More precisely:

**Theorem 1.3.** Let \( q = p^k r \), \( p \) prime; \( G = \mathbb{Z}_p^k \oplus \mathbb{Z}_r \); and for any \( 0 \leq a \leq k \), let \( h = \mathbb{Z}_p^a \) and \( \bar{G} = \mathbb{Z}_p^{k-a} \oplus \mathbb{Z}_r \). If \( n = (d+1)(q-1) - d[(k-a)(p-1) + r-1] \), then for any continuous map \( f : \partial \Delta^n \rightarrow \mathbb{R}^d \) there exist \( q \) points \( \{x_g \mid x_g \in \sigma_g \}_{g \in G} \) from \( q \) pairwise disjoint faces such that
(i) \( f(x_{h+\bar{g}}) = f(x_\bar{g}) \) for all \( h \in H \) and all \( \bar{g} \in \bar{G} \) (so \( \cap_{h \in H} f(\sigma_{h+\bar{g}}) \neq \emptyset \) for all \( \bar{g} \in \bar{G} \)),
(ii) for any \( \ell \leq \min\{r, p\} \) and each of the \( \binom{n}{\ell} \) subsets \( S = \sigma_{a+1} \times \ldots \times \sigma_{k+1} \subset G \) with \( |S| = \ell \) for all \( a + 1 \leq j \leq k + 1 \), there exist \( \ell \)-cycles \( \{x_g \}_{g \in G} \) distinct subsets \( \{x_g \}_{g \in G} \) with fixed average value: If \( \Phi \) is the set of all permutations \( \phi = (\phi_{a+1}, \ldots, \phi_{k+1}) \) of \( S \) such that each \( \phi_j \) is a \( \ell \)-cycle of \( S_j \), then
\[
(1.2) \quad \text{Avg}(f; x_s, x_{\phi(s)}, \ldots, x_{\phi^{(r-1)}(s)}) = c(S, \Phi)
\]
is constant for all \( s \in S \) and all \( \phi \in \Phi \).

**Remark 1.** From the viewpoint of the topological Tverberg problem, the “worst” possible outcome of Theorem 1.3 occurs when the \( f(x_{h+\bar{g}}) \in \cap_{h \in H} f(\sigma_{h+\bar{g}}) \) are all distinct (otherwise, one has the intersection of even more of the \( f(\sigma_g) \) than guaranteed by (i)). In that case, the average-value condition (ii) has the simple geometric meaning that for each \( S \subset G \) as above, the \( (\ell)! \)-distinct point sets \( \{f(x_s), \ldots, f(x_{\phi^{(r-1)}(s)})\} \) have identical barycenters.

When \( a = 0 \) and \( r = p \) is prime, the average values \( \text{Avg}(f; x_m, x_{\phi(m)}, \ldots, x_{\phi^{(r-1)}(m)}) \) are constant for all \( m \in \mathbb{Z}_p^{k+1} \) and all \( (k+1) \)-tuples \( \phi \) of \( p \)-cycles. For instance, if \( G = \mathbb{Z}_2^3 \) and \( f : \partial \Delta^{4d+8} \rightarrow \mathbb{R}^d \), this gives 9 points \( x_0, \ldots, x_8 \), one each from 9 disjoint faces, for which \( \text{Avg}(f; x_0, x_4, x_8) = \text{Avg}(f; x_0, x_5, x_7) = \text{Avg}(f; x_1, x_3, x_6) = \text{Avg}(f; x_2, x_3, x_7) = \text{Avg}(f; x_2, x_4, x_5) \). This can be compared to the \( N = 8d + 8 \) necessary to guarantee a full Tverberg partition. For an example in the non-prime-power cases, considering \( G = \mathbb{Z}_2^3 \oplus \mathbb{Z}_2 \), \( a = 1 \), and \( f : \partial \Delta^{14d+17} \rightarrow \mathbb{R}^d \) gives 18 points \( x_0, \ldots, x_{17} \), one from each of 18 disjoint faces, for which (i) \( f(x_j) = f(x_{j+6}) = f(x_{j+12}) \) for each \( 0 \leq j \leq 5 \) and (ii) \( f(x_j) + f(x_{j+1}) = f(x_{j+3}) + f(x_{j+4}) \) for all \( 0 \leq j \leq 2 \) (modulo 6). Again, a full Tverberg partition would require \( N = 17d + 17 \), with counterexamples for all \( d = 17k + 1 \) when \( k \geq 3 \).

We give one other average-value generalization of Theorem 1.1, the latter again recovered when \( r = 1 \):

**Theorem 1.4.** Let \( q = p^k r \), \( p \) prime. For any continuous map \( f : \partial \Delta^{(d+r)(p^k-1)+r-1} \rightarrow \mathbb{R}^d \), there exist \( q \) points \( x_1, \ldots, x_q \), one from each of \( q \) pairwise disjoint faces, such that
\[
(1.3) \quad \text{Avg}(f; x_1, \ldots, x_r) = \text{Avg}(f; x_{r+1}, \ldots, x_{2r}) = \ldots = \text{Avg}(f; x_{(p^k-1)r+1}, \ldots, x_q)
\]
As with Theorem 1.3, distinct \( f(x_j) \) here means that the image of the \( q \) points can be partitioned into \( p^k \) sets of \( r \) points each whose barycenters are all equal. By letting \( p = 2 \) and \( k = 1 \), Theorem 1.4 gives an alternating-sum generalization of the topological Radon theorem [3] which, in the planar cases, forces all but at most 2 of the disjoint faces to be vertices:

**Corollary 1.5.** For any continuous map \( f : \partial \Delta^{2r+1} \rightarrow \mathbb{R}^2 \), there exist either (a) \( 2r - 1 \) vertices \( x_1, \ldots, x_{2r-1} \) and a point \( x_{2r} \) from the remaining disjoint 2-dimensional face, or (b) \( 2r - 2 \) vertices \( x_1, \ldots, x_{2r-2} \) and points \( x_{2r} \) and \( x_{2r+1} \), one from each of the remaining two disjoint edges, for which

\[
\sum_{j=1}^{2r} (-1)^j f(x_j) = 0
\]

1.3. **Methodology.** As discussed in Section 2, the above results have as their starting point a reformulation of the topological Tverberg problem in terms of harmonic analysis on finite groups, with the average-value partitions obtained as the vanishing of prescribed Fourier transforms. The existence of this vanishing is established via the standard equivariant cohomological techniques in combinatorial geometry, described in Section 3. In this case, one is reduced to the computation of characteristic classes of group representations arising from the transforms considered, thereby resulting in the non-zero polynomial conditions \((2.3)\) and \((2.4)\) of Lemma 2.1. Thus new partition types can be obtained by carefully selecting the transforms to be annihilated, including those for which the number of faces is not a prime power. A similar Fourier approach was used in [20] to produce a variety of measure equipartitions, another central topic of the field.

2. A Finite Fourier Approach

A finite harmonic analysis interpretation of the topological Tverberg problem can be given as follows. Let \( f = (f_1, \ldots, f_d) : \partial \Delta^n \rightarrow \mathbb{C}^d \), and let \( Q \) be any collection of \( q \) points of \( \partial \Delta^n \) with pairwise disjoint support (i.e., one from each of \( q \) pairwise disjoint faces). Indexing \( Q = \{x_g\}_{g \in G} \) by a finite group \( G \) of order \( q \) gives maps \( F_1, \ldots, F_d : G \rightarrow \mathbb{C}, g \mapsto f_i(x_g) \), each of which has a Fourier expansion (see, e.g., [7]). Choosing \( G = \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_k} \) to be abelian, this expansion takes the particularly simple form

\[
F_i(g) = \sum_{\epsilon \in G} c_{i,\epsilon} \chi_{\epsilon}(g),
\]

where the

\[
c_{i,\epsilon} = \frac{1}{|G|} \sum_{u \in \mathbb{C}} f_i(x_u) \chi_{\epsilon}^{-1}(u) \in \mathbb{C}
\]

are the Fourier transforms of the 1-dimensional unitary representations \( \chi_{\epsilon} : G \rightarrow U(1) \), given explicitly by \( \chi_{\epsilon}(g) = \prod_{j=1}^k \zeta_{q_j}^{\epsilon_j b_j} \) for each \( \epsilon = (\epsilon_1, \ldots, \epsilon_k) \) and \( g = (b_1, \ldots, b_k) \in G \), \( \zeta_{q_j} = \exp(2\pi i/q_j) \). Note that \( c_{i,\epsilon} = c_{i,-\epsilon} \) if \( f_i \) is real-valued, and that \( \chi_{\epsilon} \) is real-valued valued iff \( \epsilon \) is of order 2.

In particular, a topological Tverberg partition for a given map \( f : \partial \Delta^n \rightarrow \mathbb{R}^d \) is equivalent to a collection for which all coefficients except \( c_{1,0}, \ldots, c_{d,0} \) (i.e., those arising from the trivial representation) are zero. One has the following general condition for the annihilation of prescribed transforms:
Lemma 2.1. Let \( q = q_1 \cdots q_k \) and \( \epsilon_1, \ldots, \epsilon_m \in \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_k}, \) \( \epsilon_j = (\epsilon_{j,1}, \ldots, \epsilon_{j,k}) \).

(a) If

\[
(2.3) \quad h(y_1, \ldots, y_k) = \Pi_{j=1}^{m}(\epsilon_{j,1} y_1 + \cdots + \epsilon_{j,k} y_k)^d
\]

is non-zero in \( \mathbb{Z}[y_1, \ldots, y_k]/(q_1 y_1, \ldots, q_k y_k) \), then for any continuous map \( f : \partial \Delta^{2dm+q-1} \to \mathbb{C}^d \) there exist \( q \) points of \( \partial \Delta^n \) with pairwise disjoint support such that \( c_i,\epsilon_j = 0 \) for each \( 1 \leq j \leq m \) and each \( 1 \leq i \leq d \) in the Fourier expansion (2.1).

(b) Suppose that \( d \) is odd, \( q_1 = 2r_1, \ldots, q_k = 2r_k \) are even, and that \( \epsilon_1, \ldots, \epsilon_{m'} \) are the elements of order 2. If

\[
(2.4) \quad h(x_1, y_1, \ldots, x_{k'}, y_{k'}) = \Pi_{j=1}^{m'}(\epsilon_{j,1} x_1 + \cdots + \epsilon_{j,k'} x_{k'})^d \Pi_{j=m'+1}^{m}(\epsilon_{j,1} y_1 + \cdots + \epsilon_{j,k'} y_{k'})^d
\]

is non-zero in \( \mathbb{Z}_2[x_1, \ldots, x_{k'}, y_{k'}]/(x_1^2 - r_1 y_1, \ldots, x_{k'}^2 - r_k y_{k'}) \), then for any continuous map \( f : \partial \Delta^{d(2m-m')+q-1} \to \mathbb{R}^d \) there exist \( q \) points of \( \partial \Delta^n \) with pairwise disjoint support such that \( c_i,\epsilon_j = c_i,-\epsilon_j = 0 \) for each \( 1 \leq j \leq m \) and each \( 1 \leq i \leq d \) in (2.1).

2.1. Proof of Theorems 1.3 and 1.4. We defer the proof of this inherently topological lemma to Section 3. Note that for \( G = \mathbb{Z}_p^a \oplus \mathbb{Z}_r \), \( p \) prime, the polynomials (2.3) and (2.4) are non-zero provided each \( \epsilon_j \notin 0 \oplus \mathbb{Z}_r \): (2.3) is non-zero in \( \mathbb{Z}_p[y_1, \ldots, y_k] \) even after quotienting by \( (y_{k+1}) \), and likewise (2.4) is non-zero in \( \mathbb{Z}_2[x_1, \ldots, x_k] \) even after quotienting by the ideal \( (x_{k+1}, y_{k+1}) \).

Theorems 1.3 and 1.4 follow easily:

\[
(2.5) \quad F_i(\tilde{g}) = c_{i,0} + \sum_{u=1}^{p-1} c_{i,u} \zeta_p^{u_{b_{k+1}}} \zeta_p^{u_{b_{k+1}}} + \ldots + \sum_{u=1}^{r-1} c_{i,u} \zeta_r^{u_{b_{k+1}}},
\]

\[ \tilde{g} = (b_{a+1}, \ldots, b_{k+1}) \in \tilde{G} \]. Thus (i) \( F_i(h + \tilde{g}) = F_i(\tilde{g}) \) for all \( h \in H \) and all \( \tilde{g} \in \tilde{G} \), and (ii) if \( \ell \leq \min(p, r) \) and \( S = S_{a+1} \times \ldots \times S_{a+k} \subset \tilde{G} \) with \( |S_j| = \ell \) for all \( a+1 \leq j \leq k+1 \), it follows that the sums \( F_i(s) = F_i(\phi(s)) + \ldots + F_i(\phi^{\ell-1}(s)) \) are constant for all \( s \in S \) and all \( k \)-tuples \( \phi = (\phi_{a+1}, \ldots, \phi_{k+1}) \) of \( \ell \)-cycles of the \( S_j \).

For Theorem 1.4, again let \( G = \mathbb{Z}_p^a \oplus \mathbb{Z}_r \), but choose each \( c_{i,\epsilon} \) with \( \epsilon \in (\mathbb{Z}_p^a - 0) \oplus 0 \). The same arguments as before show that both the polynomial and dimension conditions of Lemma 2.1 are satisfied. The expansion is \( F_i(h, b_{k+1}) = c_{i,0} + \sum_{\epsilon_{k+1} \neq 0} c_{i,\epsilon} \chi_\epsilon(h, 0) \zeta_r^{\epsilon_{k+1} b_{k+1}}, \) \( h \in \mathbb{Z}_p \), so \( F_i(h, 0) + \ldots + F_i(h, r-1) = r c_{i,0} \). \( \square \)
3. Equivariant Topological Underpinnings

Our proof of Lemma 2.1 follows the usual configuration-space/test-map scheme, the standard method for the reduction of problems in discrete and combinatorial geometry to corresponding ones of algebraic topology. See, e.g., [15, 23] for introductions.

As in [4, 17, 22], all collections of q points of \( \partial \Delta^n \) with pairwise disjoint support can be parametrized by the 2-fold deleted product

\[
X := (\partial \Delta^n)_{(2)} = \{ x = (x_1, \ldots, x_q) \in \sigma_1 \times \cdots \times \sigma_q \mid \sigma_i \cap \sigma_j = \emptyset \forall i \neq j \}
\]

The symmetric group \( S_q \) acts freely on \( X \) by permutations, though (as in [22], and similarly in [19]) we shall restrict to the free \( G \)-action induced from left multiplication after indexing \( \{1, \ldots, q\} \). Evaluating the various Fourier transforms yields continuous maps \( F : X \to \mathbb{C}^{dm} \) in part (a) and \( F : X \to \mathbb{R}^{dm} \oplus \mathbb{C}^{d(m-m')} = \mathbb{R}^{d(2m-m')} \) in part (b),

\[
F : x \mapsto \frac{1}{|G|} \sum_{g \in G} f_i(x_g) \chi_{c_j}^{-1}(g),
\]

1 \( \leq i \leq d \) and 1 \( \leq j \leq m \), where \( G \) acts linearly on the respective target spaces via the representation \( \rho = \oplus_{j=1}^m \chi_{c_j} \).

The existence of a collection \( Q \) with prescribed vanishing coefficients is equivalent to a zero of the map \( F \). Crucially, this map is equivariant with respect to the given actions, so that a zero can be guaranteed by standard equivariant cohomological methods, most commonly via Fadell-Husseini index theory [10] or the theory of vector bundles and characteristic classes (see, e.g., [13, 16]). Opting for the latter, for simplicity we let \( \ell = dm \) if \( F = \mathbb{C} \) and \( \ell = d(2m-m') \) if \( F = \mathbb{R} \) and \( d \) is odd. Quotienting \( X \times \mathbb{F}^\ell \) by the diagonal \( G \)-action produces a vector bundle \( \mathbb{F}^\ell \to E := X \times_G \mathbb{F}^\ell \to X := X/G, \) and the map \( x \mapsto (x, F(x)) \) induces a section \( s : X \to E \) of this bundle, a zero of which is the same as that for \( F \). In particular, the desired \( Q \) exists if it can be shown more generally that \( E \) never admits a non-vanishing section, and the latter follows from an identification of the bundle’s top Chern class in part (a) and Stiefel-Whitney class in part (b) with the polynomials (2.3) and (2.4), respectively.

**Proposition 3.1.** Let \( E = X \times_G \mathbb{F}^\ell \) be the vector bundle above.

(a) If \( F = \mathbb{C} \), \( \ell = dm \), and \( n = 2\ell + q - 1 \), then \( E \) admits a non-vanishing section iff the polynomial (2.3) is zero.

(b) If \( F = \mathbb{R} \), \( d \) is odd, \( \ell = d(2m-m') \), and \( n = \ell + q - 1 \), then \( E \) does not admit a non-vanishing section if the polynomial (2.4) is non-zero. \( E \) is orientable iff \( \epsilon_1, i + \ldots + \epsilon_{m'+1} = 0 \) for each 1 \( \leq i \leq k' \).

As the vector bundle of part (b) is often non-orientable, one sees that the mod 2 computations of Lemma 2.1(b) are necessary.

**Proof.** A non-vanishing section of \( E \) implies a vanishing top Chern class \( c_\ell(E) \in H^{2\ell}(X; \mathbb{Z}) \) or top Stiefel-Whitney class \( w_\ell(E) \in H^{\ell}(X; \mathbb{Z}_2) \), so we show as usual that these are non-zero when their respective polynomials (2.3) and (2.4) are non-zero. On the other hand, a non-vanishing section exists iff \( c_\ell(E) = 0 \) in part (a), since in this case \( \dim(X) = 2\ell \) (see below) and hence \( c_\ell(E) \) (as the Euler class of the underlying oriented real bundle) is the primary and only obstruction class (see, e.g., [8]).

As in [19], computations can be reduced to those of group cohomology via the key technical fact, first shown in [4], that \( (\partial \Delta^n)_{(2)} \) is a \( (n-q) \)-connected, \( (n-q+1) \)-dimensional CW complex when \( n > q \). As the \( G \)-action on each \( (\partial \Delta^n)_{(2)} \) is free, \( EG = \cup_n (\partial \Delta^n)_{(2)} \) is a model for the total space of the universal bundle \( G \hookrightarrow EG \to BG = EG/G \) of the group.
G (see, e.g., [13]). Concretely, each \((\partial \Delta^n)_2 = E_{n-q+1}G\) is the \((n-q+1)\)-skeleton of \(EG\), so each quotient \((\partial \Delta^n)_q / B = B_{n-q+1}G\) is the \((n-q+1)\)-skeleton of the classifying space \(BG = \bigcup_n B_{n-q+1}G\). We recall that classifying spaces \(BG\) are unique up to homotopy.

With this viewpoint, \(E\) is the pullback under the inclusion \(i : \overline{X} \to BG\) of the bundle \(\mathbb{G}^\ell \to E_\sigma := EG \times_G \mathbb{G}^\ell \to BG\), whose Chern and Stiefel-Whitney classes are denoted (as in [2]) by \(c_i(\sigma)\) and \(w_i(\sigma)\). By naturality, the total Chern class of \(E\) is \(c(E) = i^*(c(\rho))\), and likewise the total Stiefel Whitney class is \(w(E) = i^*(w(\rho))\). As cellular cohomology shows that \(i^* : H^*(BG; \mathbb{R}) \to H^*(\overline{X}; \mathbb{R})\) is injective in all dimensions \(d' \leq n - q + 1\) for any choice of coefficient ring, it is enough to show that \(c_i(\rho) \in H^*(BG; \mathbb{Z})\) and \(w_j(\rho) \in H^*(BG; \mathbb{Z}_2)\) are precisely the polynomials (2.3) and (2.4), respectively. These identifications are essentially classical, so we provide only a sketch in each case:

For \((\alpha, \beta)\), it is a basic fact that \(H^*(BG; \mathbb{Z}) \cong \mathbb{Z}^\ell \oplus \mathbb{Z}^\ell \oplus \mathbb{Z}^\ell \oplus \ldots\) where \(\ell = c_1(\chi_1)\) is the first Chern class of the standard representation \(\chi_1 : \mathbb{Z}_q \to \mathbb{C}\). As the evaluation \(c_1 : \text{Vect}_B^1(B) \to H^2(B; \mathbb{Z})\) of first Chern classes is an isomorphism for any paracompact space \(B\) (the space \(\text{Vect}_B^1(B)\) of all \(F\)-line bundles over \(B\) being a group under tensor product), the general Künneth formula (see, e.g., [12]) applied to \(BG = B\mathbb{Z}_q \times \ldots \times B\mathbb{Z}_q\) implies that \(c_i(\chi_j) = \varepsilon_1.1 y_1 + \cdots + \varepsilon_j.k.y_k \in \mathbb{Z}[y_1, \ldots, y_k]/(qy_1, \ldots, qy_k)\) and each \(\chi_1 : G \to \mathbb{C}\). As \(c(\rho) = \Pi_{\ell=1}^n c(\chi_{\ell})\) by the Whitney sum formula, \(c_i(\rho) = \Pi_{\ell=1}^n (\varepsilon_1.1 y_1 + \cdots + \varepsilon_{j,k}y_k)^d = h(y_1, \ldots, y_k)\).

As for \((\beta, \gamma)\), a cellular cochain argument applied to lens spaces as in [12] shows that \(H^*(B\mathbb{Z}_q; \mathbb{Z}_2) \cong \mathbb{Z}_2\) if \(q\) is odd. For \(q = 2\) even, \(H^*(B\mathbb{Z}_q; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/(x^2 - ry)\) (see, e.g., [1]), and by considering the projection \(B\mathbb{Z}_2 \to \mathbb{Z}_2 \cong \mathbb{Z}^\infty\) and the latter’s mod 2 cohomology it follows easily that \(x = w_1(\chi_1), \chi_1 : \mathbb{Z}_2 \to \mathbb{R}\), and that \(y = w_2(\chi_1)\) is the Chern class \(c_1(\chi_1)\) reduced mod 2. Hence \(H^*(BG; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y_1, \ldots, y_k, y_{k'}]/(x^2 - r_1 y_1, \ldots, x^2 - r_{k'} y_{k'})\) by the Künneth formula with field coefficients. For \(1 \leq j \leq m'\), the isomorphism \(w_1 : \text{Vect}_B^1(B) \to H^1(B; \mathbb{Z}_2)\) again implies that \(w_1(\chi_{\ell}) = \varepsilon_1.1 x_1 + \cdots + \varepsilon_{j,k}.x_{k'}\). On the other hand, each \(E_{\ell}\) is complex when \(m' < j \leq m\), so \(w_1(\chi_{\ell}) = 0\) and \(w_2(\chi_{\ell}) = \varepsilon_1.1 y_1 + \cdots + \varepsilon_{j,k}.y_{k'}\) is the mod 2 reduction of \(c_i(\chi_{\ell})\). Again, \(w(\rho) = \Pi_{\ell=1}^n w(\chi_{\ell})\) by the Whitney sum formula, from which it follows that \(w_i(\rho) = \Pi_{\ell=1}^n (\varepsilon_1.1 x_1 + \cdots + \varepsilon_{j,k}.x_{k'}) \Pi_{\ell=m'+1}^n (\varepsilon_1.1 y_1 + \cdots + \varepsilon_{j,k}.y_{k'})^d = h(x_1, y_1, \ldots, x_{k'}, y_{k'})\), as claimed. Finally, any real bundle is orientable iff its first Stiefel-Whitney class is zero, and since \(w_1(\rho) = d \sum_{j=1}^{m'} w_1(\chi_{\ell})\) and \(d\) is odd, \(E_\rho\) (and hence \(E\)) is orientable iff \(\varepsilon_1 + \cdots + \varepsilon_{m'} = 0\) for each \(1 \leq i \leq k'\).

We conclude with some remarks on Proposition 3.1, Lemma 2.1, and the recent developments on the topological Tverberg conjecture. First, note that the dimension and polynomial conditions of Lemma 2.1(a) are optimal, at least in a certain homotopic sense: the desired partition follows from the existence of a zero for any \(G\)-equivariant map \((\partial \Delta^n)_2 \to \mathbb{C}^d\), guaranteed by precluding the existence of a non-vanishing section \(\overline{s} : \overline{X} \to E\), but such a section always exists when \(n < 2dm + q - 1\) (since then \(\dim(\overline{X}) < 2\ell\), so the primary obstruction class automatically vanishes), and exists \(\overline{s} = h = 0\) when \(n = 2dm + q - 1\).

This optimality of (2.3) should be compared to that of Özaydin’s proof [17] of Theorem 1.1, in which the full symmetric group action on the 2-fold deleted product was considered. Full Tverberg partitions would follow from the existence, when \(n = (q - 1)(d + 1)\), of a zero for any \(S_q\)-equivariant continuous map \((\partial \Delta^n)_2 \to \mathbb{R}^d\), where \(W = \{w_1, \ldots, w_q\} \in \mathbb{R}^q\) and \(\sum_{j=1}^q w_j = 0\) is the standard representation of \(S_q\) restricted to the orthogonal complement of the diagonal subspace of \(\mathbb{R}^q\). That all such maps admit a zero iff \(q\) is a prime power was the essential reason Conjecture 1 remained open for so long. It was not until (a) Mabillard and Wagner’s theorem [14] on the equivalence, for a \((q - 1)k\)-dimensional simplicial complex
$K$, $k \geq 3$, of (i) the existence of a $S_q$-equivariant map from the 2-fold deleted product $K^q_{(2)}$ to $W^{qk}$ without zeros and (ii) the existence of a continuous map $f : K \to \mathbb{R}^{qk}$ with $q$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_q) = \emptyset$, combined with (b) the combinatorial reduction argument of Frick [11] (or, as in [5], with the “constrained” topological Tverberg methods of [6]) that counterexamples to Conjecture 1 for non-prime-powers were shown to exist.

4. ACKNOWLEDGEMENTS

The author thanks Pavle Blagojević and Alfredo Hubard for helpful comments.

REFERENCES

[1] A. Adem and J. Milgram. Cohomology of Finite Groups, Grundlehren der mathematischen Wisen-
chaften, Vol. 309, Springer (2004).
[2] M.F. Atiyah. Characters and Cohomology of Finite Groups, Inst. Hautes Études Sci. Publ. Math., Vol. 9 (1961) 23–64.
[3] E. G. Bagjmočzy and I. Bárány. On a common generalization of Borsuk’s and Radon’s theorem, Acta math. Acad. Sci. Hung., Vol. 34 (1979) 347–350.
[4] I. Bárány, S.B. Schlosman, and A. Szücs. On a topological generalization of a Theorem of Tverberg, J. London Math. Soc., Vol. 23, No. 2 (1981) 158–164.
[5] P. Blagojević, F. Frick, and G.M. Ziegler. Counterexamples to the topological Tverberg conjecture and other applications of the constraint method, in preparation.
[6] P. Blagojević, F. Frick, and G.M. Ziegler. Tverberg plus constraints, Bull. London Math. Soc., Vol. 46, No. 5 (2014) 953–967.
[7] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli. Harmonic Analysis on Finite Groups: Representa-
tion Theory, Gelfand Pairs, and Markov Chains, Cambridge studies in advanced mathematics Vol. 108, Cambridge University Press (2008).
[8] J.F. Douglas and P. Kirk. Lecture Notes in Algebraic Topology, Graduate Studies in Mathematics Vol. 35, American Mathematical Society (2001).
[9] T. tom Dieck. Transformation Groups, de Gruyter Studies in Math, Vol 8., de Gruyter, Berlin (1987).
[10] E. Fadell and S. Husseini. An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems. Ergodic Theory and Dynamical Systems, Vol. 8 (1988) 73–85.
[11] F. Frick. Counterexamples to the topological Tverberg conjecture. [arXiv:1502.00947 [math.CO]].
[12] A. Hatcher. Algebraic Topology, Cambridge University Press (2002).
[13] D. Husemoller. Fiber Bundles, Springer-Verlag (1994).
[14] I. Mabillard and U. Wagner. Eliminating Tverberg points, I. An analogue of the Whitney trick, Pro-
cedings of the Thirtieth Annual Symposium on Computational Geometry (New York, NY, USA), SOCG’14, ACM, 2014 (171–180).
[15] J. Matoušek. Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and
Geometry, Springer-Verlag (2008).
[16] J.W. Milnor and J.D. Stasheff. Characteristic Classes, Princeton University Press (1974).
[17] M. Özaydin. Equivariant maps for the symmetric group, Preprint, 17 pages, 1987.
[18] J. Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, Math. Annal., Vol. 83 (1921) 113–115.
[19] K.S. Sarkaria. Tverberg partitions and Borsuk–Ulam theorems, Pacific J. Math., Vol. 196, No. 1 (2000) 231–241.
[20] S. Simon. Measure partitions via harmonic analysis on finite and compact groups. [arXiv:1403.7094 [math.MG]]
[21] H. Tverberg. A generalization of Radon’s Theorem, J. London Math. Soc., Vol. 41 (1966) 123–128.
[22] A. Yu. Volokov. On a topological generalization of the Tverberg theorem, Math. Notes, Vol. 59, No. 3 (1996) 324–326.
[23] R.T. Živaljević. Topological Methods. Chapter 14 in Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O’Rourke, eds, Chapman & Hall/CRC (2004) 305–330.