Geometric action for extended Bondi-Metzner-Sachs group in four dimensions

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ABSTRACT: The constrained Hamiltonian analysis of geometric actions is worked out before applying the construction to the extended Bondi-Metzner-Sachs group in four dimensions. For any Hamiltonian associated with an extended BMS\textsubscript{4} generator, this action provides a field theory in two plus one spacetime dimensions whose Poisson bracket algebra of Noether charges realizes the extended BMS\textsubscript{4} Lie algebra. The Poisson structure of the model includes the classical version of the operator product expansions that have appeared in the context of celestial holography. Furthermore, the model reproduces the evolution equations of non-radiative asymptotically flat spacetimes at null infinity.

KEYWORDS: Classical Theories of Gravity, Models of Quantum Gravity, Sigma Models, Space-Time Symmetries

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1 Introduction

Whereas sigma models on symmetric spaces $G/H$ involve a Killing form for the construction of a $G$-invariant spacetime action principle, geometric actions yield $G$-invariant first order worldline action principles for $G/H_{b_0}$ that do not rely on an invariant metric but rather on the coadjoint representation and the choice of a fixed coadjoint vector $b_0$ with $H_{b_0}$ the stabilizer subgroup of $b_0$. Geometric actions appear in the context of the orbit method [1–4] when constructing group characters through path integral quantization [5]. For infinite-dimensional groups such as Kac-Moody or Virasoro groups [6, 7] (see also e.g. [8–10]), they appear as Hamiltonian gauge field theories with spatial sections that are circles. In the same context, a geometric action for the (centrally extended) BMS group in three spacetime dimensions [11, 12] has recently been constructed in [13].

The main objective of the current paper is to apply this construction to the BMS group in four spacetime dimensions [14–17], or more precisely the extended version [18, 19] that appears in the celestial holography program [20–24], by combining the ingredients of the construction in three dimensions with the detailed understanding of the coadjoint representation in four dimensions [25].
In the course of the construction, the question whether one may consider the coadjoint vectors as time-dependent, dynamical variables in addition to the group elements has come up. In order to clarify this issue, after a brief review of geometric actions, we work out the constrained Hamiltonian analysis for geometric actions (see also [26]), before applying the construction to the group of interest.

In the final two sections, general comments about the relevance of the models to non-radiative asymptotically flat gravity at null infinity are provided. In particular, the proper choice of a Hamiltonian reproduces the time-dependence of the asymptotic symmetry vectors and gravitational flux-balance relations, without the fluxes.

2 Generalities

We briefly summarize here the construction of geometric actions following the conventions of [13], up to an overall sign in the definition of the Hamiltonian and the Noether charges. Furthermore, the Lie algebra bracket for the diffeomorphism group in 1 dimension is taken here to be minus the Lie bracket of vector fields, as it should if all signs related to the passage from the group to the algebra are to be correct. As a consequence, some formulas will have the opposite signs to those that appear commonly in the literature.

Let \( g \in G \) be a group element and \( b_0 \in \mathfrak{g}^* \) a fixed coadjoint vector. The coadjoint orbit \( O_{b_0} \) is the set of coadjoint vectors \( b = \text{Ad}_{g^{-1}}b_0 \in \mathfrak{g}^* \) that can be reached from \( b_0 \) through the coadjoint action \( \text{Ad}^* \). This orbit is isomorphic to \( G/H_{b_0} \) with \( H_{b_0} \) the isotropy sub-group of \( b_0 \), i.e., the elements \( h \in G \) such that \( \text{Ad}_h b_0 = b_0 \).

Consider left/right translations by elements \( g \),

\[
L_g : h \rightarrow gh \quad / \quad R_g : h \rightarrow hg,
\]

and let \( \theta/\kappa \) be the left-invariant/right-invariant Maurer-Cartan forms satisfying

\[
d\theta = -\frac{1}{2}\text{ad}_\theta \theta \quad / \quad d\kappa = \frac{1}{2}\text{ad}_\kappa \kappa.
\]

For \( X \in \mathfrak{g} = \frac{d}{ds}g(s)|_{s=0} \), let

\[
v^R_X = \frac{d}{ds}(hg(s))|_{s=0} \quad / \quad v^L_X = \frac{d}{ds}(g(s)h)|_{s=0},
\]

the vector fields which generate right/left translations. These vector fields are the left/right invariant vector fields that reduce to \( X \) at the identity, with

\[
i_{v^R_X} \theta = X \quad / \quad i_{v^L_X} \kappa = X,
\]

and

\[
i_{v^L_X} \theta = \text{Ad}_{g^{-1}}X \quad / \quad i_{v^R_X} \kappa = \text{Ad}_g X.
\]

On the level of the generators, the left/right invariance of the Maurer-Cartan forms translates into

\[
\mathcal{L}_{v^L_X} \theta = 0 \quad / \quad \mathcal{L}_{v^R_X} \kappa = 0.
\]
The presymplectic potential and two-form are
\[ a = \langle b, \theta \rangle = \langle b_0, \kappa \rangle, \quad \Omega = da, \tag{2.7} \]
with \( \langle \cdot, \cdot \rangle \) the pairing between \( \mathfrak{g}^* \) and \( \mathfrak{g} \). Furthermore, let \( X_0 \in \mathfrak{g} \) be a fixed Lie algebra element and
\[ H_{X_0} = \langle b, X_0 \rangle. \tag{2.8} \]
In particular, it follows from the second of (2.5) that
\[ dH_{X_0} = -i_{v_{X_0}^R} \Omega. \tag{2.9} \]
The geometric actions we will be considering below are of the form
\[ I_G[g; b_0, X_0] = \int_\gamma [a - H_{X_0} dt] = \int dt L_H, \quad L_H = i_V a - H_{X_0}, \tag{2.10} \]
where \( \gamma : t \to g(t) \) represents a path on \( G \) with tangent vector \( V = \dot{g} \). Models with different \( b_0 \)'s that belong to the same coadjoint orbit are equivalent in the sense that they are described by geometric actions that are related by “field redefinitions”, that is to say invertible reparametrizations of the configuration space variables that correspond to left translations,
\[ I_G[g; Ad_{h^{-1}}b_0, X_0] = I_G[g'; b_0, X_0], \quad g' = hg, \tag{2.11} \]
for constant \( h \in G \).
In other words, in order to cover all inequivalent models of this type, it is enough to study them for the different representatives \( b_0 \) of the partition of \( \mathfrak{g}^* \) into coadjoint orbits.
For a family \( g(t, \lambda) \) of such paths, and their infinitesimal variation characterized by \( W = \frac{\partial g}{\partial \lambda} \), the associated variation of the action is
\[ \delta S = \delta \lambda \int dt \left[ i_W (-i_V \Omega - dH_{X_0}) + \frac{d}{dt}(i_W a) \right], \tag{2.12} \]
so that extremal paths satisfy the equations of motion
\[ i_V \Omega + dH_{X_0} = 0 \iff \langle b_0, [i_V \kappa - Ad_g X_0, \kappa] \rangle = 0. \tag{2.13} \]
For a time dependent Lie algebra element \( X = X(t) \in \mathfrak{g} \), let
\[ Q_X = \langle b, X(t) \rangle. \tag{2.14} \]
Under an infinitesimal right translation generated by \( v_{X_0}^R \), the variation of the Lagrangian density is
\[ \frac{d}{dt} Q_X - i_V dQ_X - Q_{[X, X_0]} = Q_X - Q_{[X, X_0]}. \tag{2.15} \]
It follows that right translations define global symmetries if the time dependence of \( X(t) \) is fixed through
\[ \dot{X} = -\text{ad}_{X_0} X = [X, X_0]. \tag{2.16} \]
The associated Noether charges are $Q_X$. When acting with a global symmetry, they satisfy
\[ \mathcal{L}_{v_{X_1}^L} Q_{X_2} = Q_{[X_1,X_2]} \quad (2.17) \]

The little algebra $\mathfrak{h}_{b_0}$ is the subalgebra defined by elements $\epsilon \in \mathfrak{g}$ such that $\text{ad}_\epsilon^* b_0 = 0$. As will be explicitly shown below, the zero eigenvectors of $\Omega$ are exhausted by the vector fields $v_L^\epsilon$.
\[ \mathcal{L}_{v_L^\epsilon} Q_X = 0 \quad (2.18) \]

When $\epsilon = \epsilon(t)$, these transformations are gauge invariances of the action. Indeed, the variation of the Lagrangian density is $\frac{d}{dt} \langle b_0, \epsilon(t) \rangle$ so that the variation of the action vanishes for all $\epsilon(t)$ that vanish at the end points of the path $\gamma$. In these terms, (2.18) means that the Noether charges for the global symmetries, including the Hamiltonian, are gauge invariant.

3 Constrained Hamiltonian analysis of geometric actions

Even though geometric actions are already in first order form, the Hamiltonian analysis is not complete because of the degeneracies of the pre-symplectic two form. In order to have explicit expressions for Poisson and Dirac brackets, required in the context of operator quantization and extended formulations of the theory with both group elements and coadjoint vectors as dynamical variables, it is instructive to perform a complete constrained Hamiltonian analysis. Conversely, such world-line actions are prime examples where Dirac’s theory comes into its own in the case of completely tractable mechanical systems as opposed to field theories. We refer to the reviews [27] on Lie groups and [28, 29] on constrained Hamiltonian systems for more details and proofs. There is of course no claim of originality as all results are known in one form or the other in the (mathematical) literature.

3.1 Lie groups and algebras in local coordinates

We find it convenient to perform the analysis by using explicit (arbitrary) local coordinates $g^i$ on $G$. At the same time, even though not necessary for our purpose here, we provide in parenthesis the simplified expressions for the objects of section 2 for the case of (subgroups) of $\text{GL}(n)$, where the inner product $\langle \cdot , \cdot \rangle$ is the matrix trace.

In local coordinates, the left/right translations $L_g/R_g$ are encoded in the multiplication table $L^i(g^j,h^k) / R^i(g^j,h^k)$, while their differentials $L'_g/R'_g$ needed to push-forward vector fields from $h$ to $gh/hg$ are characterized by
\[ \frac{\partial L^i(g^j,h^k)}{\partial h^j} / \frac{\partial R^i(g^j,h^k)}{\partial h^j}. \quad (3.1) \]

The matrices
\[ L^i_j(g^l) = \frac{\partial L^i(g^j,h^k)}{\partial h^j} \bigg|_{h=e} / \quad R^i_j(g^l) = \frac{\partial R^i(g^j,h^k)}{\partial h^j} \bigg|_{h=e}, \quad (3.2) \]

are invertible, reduce to $\delta^i_j$ at the identity $e$, and commute, $L^i_j R^j_k = R^i_j L^j_k$. 


Denoting by $e_i = \frac{\partial}{\partial q^i}$, the coordinate basis for tangent vector fields at the identity, bases for the generators of right/left translations which are the left/right invariant vector fields $v_{e_i}^R(=ge_i)/v_{e_i}^L(=e_ig)$ that reduce to $e_i$ at the identity $e$ are given by
\[
v_{e_i}^R = L^j_i \frac{\partial}{\partial g^j} \quad / \quad v_{e_i}^L = R^j_i \frac{\partial}{\partial g^j}. \tag{3.3}\]
Left/right invariance of these vector fields translates into
\[
\frac{\partial L^i_j(g,h)}{\partial g^l} L^l_j(h) = L^i_j(gh) \quad / \quad \frac{\partial R^i_j(g,h)}{\partial g^l} R^l_j(h) = R^i_j(hg). \tag{3.4}\]
These bases are mutually commuting,
\[
[v_{e_i}^R, v_{e_j}^L] = 0, \tag{3.5}\]
and their rotation coefficients are determined by the Lie algebra structure constants,
\[
[v_{e_i}^R, v_{e_j}^R] = f_{ijk} v_{e_k}^R \quad / \quad [v_{e_i}^L, v_{e_j}^L] = -f_{ijk} v_{e_k}^L. \tag{3.6}\]
The left/right invariant Maurer-Cartan forms $\theta(=g^{-1}dg), \kappa(=dgg^{-1})$ are given by
\[
\theta = e_i(L^{-1})^j_i dg^j \quad / \quad \kappa = e_i(R^{-1})^i_j dg^j, \tag{3.7}\]
and satisfy
\[
d\theta + \frac{1}{2}[\theta, \theta] = 0 \quad / \quad d\kappa - \frac{1}{2}[\kappa, \kappa] = 0, \tag{3.8}\]
with $[e_i, e_j] = f_{ijk} e_k$. The adjoint representation is determined by
\[
\text{Ad}_g e_i = e_j(R^{-1}L)^j_i, \tag{3.9}\]
with $(R^{-1}L)^j_i = (R^{-1})^k_j L^k_i$. In the following, we will use
\[
v_{e_i}^L(R^{-1}L)^j_i = f_{ijk}^j(R^{-1}L)^h_k, \tag{3.10}\]
which holds on account of (3.5) and (3.6).

In order to explicitly show in local coordinates that replacing $b_{0i}$ by $b'_0 = b_{0i}(R^{-1}L)^i_j$ amounts to replacing $g^i$ by $g^h = L^i(h, g)$, one uses left invariance in the form of (3.4) and the matrix expression for $\text{Ad}_g \text{Ad}_g = \text{Ad}_g$.  

### 3.2 Legendre transform, primary constraints and canonical generators

In terms of local coordinates, the geometric action (2.10) becomes
\[
I_G[g; b_0, X_0] = \int dt \left[ b_{0i}(R^{-1})^i_j \dot{g}^j - b_{0i}(R^{-1}L)^i_j X_0^j \right]. \tag{3.11}\]
Denoting by $p_j$ the canonical momenta, $\{g^i, p_j\} = \delta^i_j$, $\{g^i, g^j\} = 0 = \{p_i, p_j\}$, the primary constraints and the canonical Hamiltonian are
\[
\dot{\phi}^j_{b_0} = p_j - b_{0i}(R^{-1})^i_j \approx 0, \quad H_C = p_i L^i_j X_0^j \approx b_{0i}(R^{-1}L)^i_j X_0^j. \tag{3.12}\]
By construction, geometric actions are then equivalent to

\[ I^H_G[g^i, p_j, \tilde{u}^m; b_0, X_0] = \int dt \left[ p_j \dot{g}^j - H_G - \tilde{u}^i \phi_i^{b_0} \right]. \tag{3.13} \]

Models with different $b_0$’s that belong to the same coadjoint orbit are described by equivalent actions,

\[ I^H_G[g, \pi, u^m; \text{Ad}^{-1}_h b_0, X_0] = I^H_G[g^i, p_j, \tilde{u}^m; b_0, X_0] \]

that are related through the canonical transformations,

\[ g^i' = L_i^k(h^k, g^i), \quad p_j' = L^m_i(g)(L^{-1})^l_j(h) p_m, \tag{3.14} \]

together with

\[ \tilde{u}'^m = L^m_j(h)(L^{-1})^l_n(g) \tilde{u}^n. \tag{3.15} \]

In the following, it turns out to be convenient not to use Darboux coordinates $(g^i, p_j)$, but rather to change coordinates on phase space to $g^i$ and

\[ \pi_j = R^k_j p_k. \tag{3.16} \]

The fundamental Poisson brackets in terms of these coordinates are

\[ \{ g^i, g^j \} = 0, \quad \{ g^i, \pi_j \} = R^i_j, \quad \{ \pi_i, \pi_j \} = f^k_{ij} \pi_k. \tag{3.17} \]

In particular, $\{ \pi_i, \cdot \} = \pi_k f^k_{ij} \frac{\partial}{\partial \pi_j} - u^L_j \pi_i$. Under the canonical transformation designed to compensate a change of the orbit representative $b_0$, these variables transform as

\[ \pi'_i = \pi_i (R^{-1} L)^i_j (h). \tag{3.18} \]

In the mathematical literature, when considering $\pi_i$ as coordinates on $g^*$, $\pi = \pi_i e^{*i}$, the above Poisson brackets for the $\pi_i$’s are referred to as the Lie-Poisson bracket or Kirillov-Kostant-Souriau bracket on $g^*$.

The primary constraints are equivalent to

\[ \phi_i^{b_0} = \pi_i - b_0 \approx 0, \tag{3.19} \]

while the canonical Hamiltonian may be chosen as

\[ H^*_X_0 = \pi_i (R^{-1} L)^i_j X_0^j. \tag{3.20} \]

By construction, the theory defined by the geometric action $I[g; b_0, X^0]$ is equivalent to the one defined by

\[ I^H_G[g, \pi, u; b_0, X^0] = \int dt \left[ \pi_i (R^{-1})^i_j \dot{g}^j - H^*_X_0 - u^i \phi_i^{b_0} \right], \tag{3.21} \]

where $u^i$ are Lagrange multipliers that may be considered as elements of $g$, $u = u^i e_i$, that transform in the adjoint representation,

\[ u'^i = (R^{-1} L)^i_j (h) u^j. \tag{3.22} \]
When using the second of (3.8), variations with respect to the dynamical variables gives,

\[
\delta I_G^H = \int dt \left[ \delta \pi_i ((R^{-1})^i_j \dot{g}^j - (R^{-1}L)^i_j X_0^j - u^i) - \delta u^i \phi^{b_0} \right. \\
+ \left. [\pi_i (R^{-1})^i_j \dot{g}^j - \pi_i \partial_j (R^{-1})^i_m \dot{g}^m - \pi_i \partial_j (R^{-1}L)^i_m X_0^m] \delta g^j \right].
\]

(3.23)

The Euler-Lagrange equations with respect to \( g^j \) may then be simplified using (3.10), and the associated dynamics is provided by the primary constraints (3.19), together with the Hamiltonian evolution equations

\[
\dot{g}^i = \{g^i, H_{\pi} X_0 + u^j \phi^{b_0} \} = L^i_j X_0^j + R^i_j u^j, \\
\dot{\pi}_i = \{\pi_i, H_{\pi} X_0 + u^j \phi^{b_0} \} = \pi_k f^k_{ij} u^j.
\]

(3.24)

In the Hamiltonian formalism, the Noether charges

\[
Q_{\pi}^X = \pi_i (R^{-1}L)^i_j X^j,
\]

(3.25)

canonically generate right translations and do not act on the \( \pi_i \),

\[
\delta_X g^i = v^R (g^i) = L^i_j X^j = \{g^i, Q_{\pi}^X \}, \quad \delta_X \pi_i = \{\pi_i, Q_{\pi}^X \} = 0.
\]

(3.26)

They form a Poisson bracket realization of \( g^i \),

\[
\{Q_{\pi X_1}^X, Q_{\pi X_2}^X \} = Q_{[X_1, X_2]}^X.
\]

(3.27)

### 3.3 Dirac algorithm: first and second class constraints

The preservation in time of the primary constraints, \( \{\phi_i, H_{\pi} X_0 + u^j \phi^{b_0} \} \approx 0 \) leads to

\[
b_{0k} f^k_{ij} u^j = 0.
\]

(3.28)

To solve this equation, one considers vectors \( e^i_a \) that constitute a complete set of zero eigenvectors of the matrix \( C_{ij} = b_{0k} f^k_{ij} \),

\[
b_{0k} f^k_{ij} v^j = 0 \iff v^j = v^a e^i_a,
\]

(3.29)

and introduces an associated change of basis in \( g \) and \( g^\ast \) defined through constant matrices \( e^i_a, e^a_i, e^A_i, e^a_i \) such that

\[
e^a_i e^b_i = \delta^b_a, \quad e^A_i e^B_i = \delta^B_A, \quad e^a_i e^A_j + e^A_i e^a_j = \delta^j_i.
\]

(3.30)

In the following, we will use these constant vielbeins and their inverse to transform quantities with greek indices into the same quantities with small and capital Latin indices. In terms of the new basis, the little algebra \( b_0 \) is determined by vectors such that \( v^A = 0 \). Furthermore,

\[
f^C_{ab} = 0, \quad C_{ab} = 0, \quad C_{aB} = 0
\]

(3.31)

while the matrix

\[
C_{AB} = b_{0c} f^C_{AB} + b_{0C} f^C_{AB}
\]

(3.32)
is invertible, with inverse denoted by \((C^{-1})^{AB}C_{BC} = \delta^A_B\). Equation (3.28) leaves the Lagrange multipliers \(u^a\) undetermined and sets to zero the remaining Lagrange multipliers \(u^A = 0\). There are thus no secondary constraints, while the primary constraints split into first and second class constraints

\[
\phi^b_a = \pi_a - b_0 a \approx 0, \quad \phi^b_A = \pi_A - b_0 A \approx 0, \tag{3.33}
\]

with

\[
\{\phi^b_a, \phi^b_b\} = f_{ab}^c \phi^b_c, \quad \{\phi^b_a, \phi^b_B\} = f_{ab}^C \phi^b_C + f_{AB}^C \phi^b_C + C_{AB}. \tag{3.34}
\]

The gauge transformations are generated by the first class constraints,

\[
\delta_c g^i = \{g^i, \phi^b_a e^a\} = R^i_a e^a, \quad \delta_c \pi_i = \{\pi_i, \phi^b_a e^a\} \approx 0, \tag{3.35}
\]

with \(e^a = e^a(t)\). The Hamiltonian \(H^X_0\) is first class while the total Hamiltonian

\[
H^C_{X_0} = H^X_0 + u^a \phi_a, \tag{3.36}
\]

is also the extended Hamiltonian since there are no secondary constraints. In terms adapted to the classification of the constraints, one has

\[
I^H_C [g^i, \pi_b, \pi_B, u^a, u^C; b_0, X^0] = \int dt \left[ a_i^H g^i - H^X_0 - u^a \phi_a - u^A \phi_A \right],
\]

\[
a_i^H = \pi_a (R^{-1})^a_i + \pi_A (R^{-1})^A_i. \tag{3.37}
\]

### 3.4 Reduced theory and Dirac brackets

At this stage, one may solve the second class constraints in the action and eliminate the \(\pi_A\) in favor of \(b_0 A\). The reduced theory becomes

\[
I^R_C [g^i, \pi_a, u^a; b_0, X_0] = \int dt \left[ a_i^R g^i - H^R_{X_0} - u^a \phi^b_a \right], \tag{3.38}
\]

\[
a_i^R = \pi_a (R^{-1})^a_i + b_0 A (R^{-1})^A_i, \quad H^R_{X_0} = \pi_a (R^{-1})^0_j X^j_0 + b_0 A (R^{-1})^A_j X^j_0,
\]

with associated reduced brackets defined by the inverse of the symplectic form \(\Omega^R = da^R\), with \(a_R = a^R dg^i\). More explicitly,

\[
\Omega^R = d\pi_a (R^{-1})^a_i dg^i + \frac{1}{2} C_{AB} (R^{-1})^A_i (R^{-1})^B_j dg^idg^j. \tag{3.39}
\]

and

\[
\{g^i, g^j\}^R = R^i_A (C^{-1})^{AB} R^j_B, \quad \{g^i, \pi_a\}^R = R^i_a, \quad \{\pi_a, \pi_b\}^R = 0. \tag{3.40}
\]

When using (3.10), it follows that the dynamics of the reduced theory is

\[
\phi^b_a = 0, \quad \phi^b_A = 0, \quad g^i = \{g^i, H^R_{X_0}\}^R + u^b \{g^i, \phi^b_a\}^R \approx L^i_j X^j_0 + R^i_j u^b, \tag{3.41}
\]

\[
\pi_a = \{\pi_a, H^R_{X_0}\}^R + u^b \{\pi_a, \phi^b_a\}^R \approx 0.
\]

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Note that if one also eliminates the first class constraints by solving them in the action, one recovers $I_G[g; b_0, X_0]$ in (2.10) with degenerate two-form $\Omega$.

Alternatively, one may work with Dirac brackets and keep the variables $\pi_A, u^A$ together with the second-class constraints $\phi^{b_0}_A \approx 0$ for reasons of Lie algebra covariance. In this case, the Dirac brackets are given by

$$
\{g^i, g^j\}^* = R^i_A(C^{-1})^{AB} R^j_B, \\
\{g^i, \pi_a\}^* = R^i_a - R^i_A(C^{-1})^{AB}(f_{Ba}^{c} \phi^{b_0}_c + f_{Ba}^{C} \phi^{b_0}_C), \\
\{\pi_a, \pi_b\}^* = f_{aB}^{c} \phi^{b_0}_c - (f_{aA}^{C} \phi^{b_0}_A + f_{aA}^{b_0}_C)(C^{-1})^{AB}(f_{Bb}^{d} \phi^{b_0}_d + f_{Bb}^{D} \phi^{b_0}_D),
$$

which agree with the reduced brackets on the constraint surface, $\{\cdot, \cdot\}^* \approx \{\cdot, \cdot\}^R$, while the additional Dirac brackets all vanish,

$$
\{g^i, \pi_A\}^* = 0, \quad \{\pi_a, \pi_B\}^* = 0, \quad \{\pi_A, \pi_B\}^* = 0.
$$

A point on the coadjoint orbit of a given covector $b_0$ can be parametrized by the group element needed to reach it,

$$
b_0^g = \text{Ad}_{g^{-1}}^* b_0.
$$

The time dependence of such a covector is given by

$$
\frac{db_0^g}{dt} = -\text{Ad}_{g^{-1}}^*(\text{ad}_{V^\kappa} b_0).
$$

When using the equations of motion (3.41) together with the fact that $u^b e_b$ belongs to the little algebra of $b_0$, it follows that

$$
\frac{db_0^g}{dt} = -\text{ad}_{X^0}^* b.
$$

### 3.5 Unconstrained model

One may decide to use the Lie algebra covariant form (3.21) of the model,

$$
I_G^H[g, \pi, u; b_0, X_0] = \int_\gamma [\langle \pi, \kappa \rangle - dt(H_{X_0}^\pi + \langle \phi^{b_0}, u \rangle)],
$$

where

$$
H_{X_0}^\pi = \langle \text{Ad}^*_g \pi, X_0 \rangle, \quad \phi^{b_0} = \pi - b_0,
$$

without explicitly splitting into first and second class constraints. As seen above, the latter can always be done once $b_0$ is fixed and the little algebra has been worked out.

One may also go a step further and drop the constraints $\phi^{b_0}$ to study the unconstrained model

$$
I_G^U[g, \pi; X_0] = \int_\gamma [\langle \pi, \kappa \rangle - dtH_{X_0}^\pi],
$$

with Poisson brackets given in (3.17) and Hamiltonian evolution equations that simplify to

$$
\dot{g}^i = \{g^i, H_{X_0}^\pi\} = L^i_j X_0^j \iff i_V \kappa = \text{Ad}_g X_0, \quad \dot{\pi}_i = \{\pi_i, H_{X_0}^\pi\} = 0.
$$

It follows that, besides the $Q^\pi_X$, the $\pi$’s themselves are constants of the motion,

$$
\pi_i = b_0^i,
$$

with $b_0^i$ constant. On these level sets, one can study Hamiltonian reduction. This amounts to performing the analysis in the previous section.
4 Geometric action for extended BMS\(_4\) group

4.1 Group and algebra

The extended BMS\(_4\) group is a semi-direct product group of the form

\[ S_\sigma = G \ltimes_\sigma A, \]  

(4.1)

with \(A\) and abelian ideal,

\[ (g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1 \cdot g_2, \alpha_1 + \sigma_g \alpha_2). \]  

(4.2)

The non-abelian factor \(G\) corresponds to conformal coordinate transformations on the complex plane minus the origin, \(g = (f, \bar{f})\),

\[ z'(z) = f(z), \quad \bar{z}'(\bar{z}) = \bar{f}(\bar{z}), \]  

(4.3)

with group law on the level of \(f, \bar{f}\) defined by composition.

The abelian ideal \(A\) consists of real fields \(T\) of conformal dimensions \((-\frac{1}{2}, -\frac{1}{2})\), with

\[ (\sigma_g T)(z', \bar{z}') = \left(\frac{\partial z}{\partial z'}\right)^{-\frac{1}{2}} \left(\frac{\partial \bar{z}}{\partial \bar{z'}}\right)^{-\frac{1}{2}} T(z, \bar{z}). \]  

(4.4)

More generally \(G\) acts on conformal fields of dimensions \((h, \bar{h})\) as

\[ (\sigma_g \phi^h) (z', \bar{z}') = \left(\frac{\partial z}{\partial z'}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{z'}}\right)^{\bar{h}} \phi^{h,\bar{h}}(z, \bar{z}). \]  

(4.5)

while \(\Sigma_X\) acts as

\[ (Y, \bar{Y}) \cdot \phi^{h,\bar{h}} = - [Y \partial + \bar{Y} \bar{\partial} + h\partial Y + \bar{h}\partial \bar{Y}] \phi^{h,\bar{h}}. \]  

(4.6)

The associated Lie algebra \(\mathfrak{bms}_4^E\) is of the form \(\mathfrak{g} \oplus_\Sigma A\),

\[ [(X_1, \alpha_1), (X_2, \alpha_2)] = ([X, Y], \Sigma_X, \alpha_2 - \Sigma_X \alpha_1), \]  

(4.7)

where \(\Sigma_X\) is the differential of \(\sigma_g\) and we identify the Lie algebra elements of \(A\) with elements of \(A\) itself. The Lie algebra of \(G\) is described by chiral fields \(Y, \bar{Y}\) of conformal dimensions \((-1, 0)\) and \((0, -1)\),

\[ \partial Y = 0 = \partial \bar{Y}, \]  

(4.8)

while the \(\mathfrak{bms}_4^E\) Lie bracket is explicitly given by

\[ [(Y_1, \bar{Y}_1, T_1), (Y_2, \bar{Y}_2, T_2)] = -(\hat{Y}, \hat{\bar{Y}}, \hat{T}), \]

\[ \hat{Y} = Y_1 \partial Y_2 - (1 \leftrightarrow 2), \quad \hat{\bar{Y}} = \bar{Y}_1 \partial \bar{Y}_2 - (1 \leftrightarrow 2), \]

\[ \hat{T} = Y_1 \partial T_2 - \frac{1}{2} \partial Y_1 T_2 + \text{c.c.} - (1 \leftrightarrow 2). \]  

(4.9)
4.2 Adjoint and coadjoint representation

The adjoint action is of the form

$$\text{Ad}_{(g,\alpha)}(X,\beta) = (\text{Ad}_g X, \sigma_g \beta - \Sigma_{\text{Ad}_g} \chi \alpha),$$

(4.10)

where $\text{Ad}_g X$ is given by $(g \cdot (Y, \tilde{Y}))(x') = (Y'(x'), \tilde{Y}(x'))$ with

$$Y'(z') = \left(\frac{\partial z}{\partial z'}\right)^{-1} Y(z), \quad \tilde{Y}'(z') = \left(\frac{\partial \tilde{z}}{\partial z'}\right)^{-1} \tilde{Y}(z).$$

(4.11)

If $\alpha = T_1, \beta = T_2, \sigma_g \beta - \Sigma_{\text{Ad}_g} \chi \alpha$ is given by

$$T'(x') = \left(\frac{\partial z}{\partial z'}\right)^{-\frac{1}{2}} \left(\frac{\partial \tilde{z}}{\partial z'}\right)^{-\frac{1}{2}} \left(T_2 + \left(Y \partial T_1 - \frac{1}{2} T_1 \partial Y + \text{c.c.}\right)\right)(x).$$

(4.12)

The dual space to the Lie algebra is of the form $\mathfrak{g}^* \oplus A^*$, with non-degenerate pairing denoted by

$$\langle (j, p), (X, \alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle.$$

(4.13)

In terms of

$$\times : A \oplus A^* \to \mathfrak{g}^*, \quad \langle \alpha \times p, X \rangle = \langle p, \Sigma_X \alpha \rangle,$$

(4.14)

and $\sigma^*$, the dual realization associated with $\sigma, \sigma^* : G \times A^* \to A^*, \langle \sigma^* p, \alpha \rangle = \langle p, \sigma_{g^{-1}} \alpha \rangle$, the coadjoint actions of the group and algebra are of the form

$$\text{Ad}_{(g,\alpha)}^*(j, p) = (\text{Ad}_g^* j + \alpha \times \sigma^*_g p, \sigma^*_g p),$$

$$\text{ad}^*_{(X,\alpha)}(j, p) = (\text{ad}_X^* j + \alpha \times p, \Sigma^*_X p).$$

(4.15)

In the case of $(\mathfrak{sl}_2^{\mathbb{F}})^*$, elements are denoted by $([[J], [\bar{J}], P])$. Here $J, \bar{J}$ have conformal dimensions $(1, 2)$ and $(2, 1)$, while $P$ has dimensions $\left(\frac{3}{2}, \frac{3}{2}\right)$, with pairing given by

$$\langle ([J], [\bar{J}], P), (Q, \bar{Q}, T) \rangle = \int dx [\bar{J}Y + J\bar{Y} + PT].$$

(4.16)

As discussed in more details in [25], if we assume that conformal fields may be expanded in terms of suitable series in $z, \bar{z}$,

$$\phi_{h,\bar{h}}(z, \bar{z}) = \sum_{k,l} a_{k,l} z^{-k} \bar{z}^{-\bar{h}^{-1}},$$

(4.17)

the integral corresponds to taking residues in $z, \bar{z}$: $\int di(x) = \int dz d\bar{z}$ with

$$\int dz \phi_{h,\bar{h}}(z, \bar{z}) = \text{Res}_z \phi_{h,\bar{h}}(z, \bar{z}) = \sum_l a_{1-h,1} z^{-l},$$

$$\int d\bar{z} \phi_{h,\bar{h}}(z, \bar{z}) = \text{Res}_{\bar{z}} \phi_{h,\bar{h}}(z, \bar{z}) = \sum_k a_{k,1} \bar{z}^{-\bar{h}-1}.$$
Because $Y, \bar{Y}$ are chiral fields, one has to consider equivalence classes, $J \sim J + \partial L$, $\bar{J} \sim \bar{J} + \partial \bar{L}$ with $L, \bar{L}$ of dimensions $(0, 2)$ and $(2, 0)$. In these terms, the coadjoint representation is given by

$$J'(x') = \left( \frac{\partial z}{\partial \bar{z}} \right)^2 \left( \frac{\partial \bar{z}}{\partial \bar{z}} \right)^2 \left( J - \left( \frac{1}{2} T \partial P + \frac{3}{2} \partial TP \right) \right)(x),$$

$$\bar{J}'(x') = \left( \frac{\partial \bar{z}}{\partial \bar{z}} \right)^2 \left( \frac{\partial z}{\partial \bar{z}} \right)^2 \left( \bar{J} - \left( \frac{1}{2} T \partial P + \frac{3}{2} \partial TP \right) \right)(x),$$

$$P'(x') = \left( \frac{\partial z}{\partial \bar{z}} \right)^2 \left( \frac{\partial \bar{z}}{\partial \bar{z}} \right)^2 P(x).$$

Representatives for the equivalence classes $[\bar{J}], [J]$ are given by

$$\bar{J}(z, \bar{z}) = \bar{J}(z) \delta(\bar{z}, 0), \quad J(z, \bar{z}) = \delta(z, 0) J(\bar{z}),$$

where $\bar{J}(z) = \int d\bar{z} \bar{J}(z, \bar{z})$ is of conformal dimension $h = 2$, while $J(\bar{z}) = \int dz J(z, \bar{z})$ is of conformal dimension $\bar{h} = 2$. This follows from the fact that the only term in a series in $z, \bar{z}$ that cannot be written as a $\partial/\partial \bar{z}$ derivative is $\bar{z}^{-1}/z^{-1}$ and from the series expansion of the delta function: if

$$\bar{J}(z, \bar{z}) = \sum_{k,l} \bar{J}_{k,l} z^{-2-k} \bar{z}^{-l-1}, \quad J(z, \bar{z}) = \sum_{k} J_{k,l} z^{-1-k} \bar{z}^{-2-l},$$

we have

$$\bar{J}(z, \bar{z}) = \sum_{k} \bar{J}_{k,0} z^{-2-k} \bar{z}^{-1} + \partial \bar{L}, \quad J(z, \bar{z}) = \sum_{l} J_{0,l} z^{-1-2-l} + \partial L,$$

with

$$\bar{L} = - \sum_{k,l \neq 0} \bar{J}_{k,l} z^{-2-k} \bar{z}^{-l}, \quad L = - \sum_{k \neq 0, l} J_{k,l} z^{-1-k} \bar{z}^{-2-l}.$$  \hspace{1cm} (4.23)

and furthermore,

$$\delta(z, w) = \sum_{k} z^{k-1} w^{-k}, \quad \delta(\bar{z}, \bar{w}) = \sum_{k} \bar{z}^{k-1} \bar{w}^{-k}.  \hspace{1cm} (4.24)$$

4.3 Unconstrained model for extended BMS$_4$

For a semi-direct product group of the form (4.1), the left/right invariant Maurer-Cartan forms $\theta_{g,\alpha}/\kappa_{g,\alpha}$ are given by

$$\theta_{g,\alpha} = (\theta_{g,\sigma_{g-1}d\alpha}) / \kappa_{g,\alpha} = (\kappa_{g, d\alpha - \Sigma_{\kappa_{g}}} \alpha),$$

where $\theta_{g}/\kappa_{g}$ denote the left/right invariant Maurer-Cartan forms of the non-abelian group $G$. It then follows from (4.14) that the kinetic term of the unconstrained model (3.49) is

$$\langle \pi, \kappa \rangle = \langle j - \alpha \times p, \kappa_{g} \rangle + \langle p, d\alpha \rangle,  \hspace{1cm} (4.26)$$

while the Hamiltonian $\langle \pi, \text{Ad}_{g} X_{0} \rangle$ is

$$H(\delta_{0}) = \langle j - \alpha \times p, \text{Ad}_{g} X_{0} \rangle + \langle p, \sigma_{g} \delta_{0} \rangle.  \hspace{1cm} (4.27)$$
Furthermore, the identity $\langle \pi, \kappa \rangle = \langle \text{Ad}^*_{g^{-1}}, \pi \rangle$ becomes
\[
\langle j - \alpha \times p, \kappa_g \rangle + \langle p, \text{d} \alpha \rangle = \langle \text{Ad}^*_{g^{-1} - \sigma_g^{-1} \alpha}(j, p), (\theta_g, \sigma_g^{-1} \text{d} \alpha) \rangle = \langle \text{Ad}^*_{g^{-1} - \sigma_g^{-1} \alpha}(j - \sigma_g^{-1} \alpha \times \sigma_g^{-1} p, \theta_g) + \langle \sigma_g^{-1} p, \sigma_g^{-1} \text{d} \alpha \rangle, \quad (4.28)
\]
while the Hamiltonian may also be written as $\langle \text{Ad}^*_{g^{-1}} j, X_0 \rangle$.

For conformal coordinate transformations, $z \mapsto z' = f(z)$, $\bar{z} \mapsto \bar{z}' = \bar{f}(\bar{z})$ the left/right invariant Maurer-Cartan forms are
\[
\theta_g = \left( \frac{1}{f} \text{d} f \frac{\partial}{\partial z}, \text{c.c.} \right) \quad / \quad \kappa_g = \left( \text{d} f \circ f^{-1} \frac{\partial}{\partial \bar{z}}, \text{c.c.} \right). \quad (4.29)
\]

As a consequence, the unconstrained model (3.49) for the extended BMS$_4$ group may be written either as
\[
I_{\text{BMS}_4}^U[f, \bar{f}, T, J, \tilde{J}, P; Y_0, \bar{Y}_0, T_0] = \int dt dz \bar{d} z \left( \left[ \tilde{J} + \left( \frac{1}{2} T \partial P + \frac{3}{2} \bar{\partial} T \bar{P} \right) \right] \left[ (f - (f') Y_0) \circ f^{-1} \right] + \text{c.c.} \right.
\]
\[
+ P \bar{T} - P \left( \left[ (f' \bar{f}) \frac{1}{2} T_0 \right] \circ (f^{-1}, \bar{f}^{-1}) \right) \left( \left[ (f' \bar{f}) \frac{1}{2} T_0 \right] \circ (f^{-1}, \bar{f}^{-1}) \right), \quad (4.30)
\]
or, in a chiral boson like form, as
\[
I_{\text{BMS}_4}^U[f, \bar{f}, T, J, \tilde{J}, P; Y_0, \bar{Y}_0, T_0] = \int dt dz \bar{d} z \left( \left[ \tilde{J} + \left( \frac{1}{2} T \partial P + \frac{3}{2} \bar{\partial} T \bar{P} \right) \right] \circ (f, \bar{f}) \right) \left[ (f' \bar{f}) \tilde{j} - (f')^2 \bar{f} \bar{f} Y_0 \right] + \text{c.c.} \right.
\]
\[
+ P \bar{T} - [P \circ (f, \bar{f})](f' \bar{f}) \frac{1}{2} T_0 \right). \quad (4.31)
\]
That these forms of the model are equivalent may also be shown by replacing in the integral in (4.30) of the relevant terms the dummy variables $z, \bar{z}$ by $z', \bar{z}'$ and then performing the change of coordinates $z' = f(z)$, $\bar{z}' = \bar{f}(\bar{z})$.

By construction, the model is invariant under infinitesimal right BMS$_4$ transformations,
\[
\delta_R f = f' Y_R, \quad \delta_R \bar{f} = \bar{f}' \bar{Y}_R, \quad \delta_R T = [(f' \bar{f}) \frac{1}{2} T_R] \circ (f^{-1}, \bar{f}^{-1}), \quad (4.32)
\]
provided that
\[
\bar{Y}_R = Y_0 \partial Y_R - Y_R \partial Y_0, \quad \bar{Y}_R = Y_0 \partial \bar{Y}_R - \bar{Y}_R \partial \bar{Y}_0,
\]
\[
\bar{T}_R = Y_0 \partial T_R - \frac{1}{2} \partial Y_0 T_R - Y_R \partial T_0 + \frac{1}{2} \partial Y_R T_0 + \text{c.c.}. \quad (4.33)
\]
The direct check of invariance on the form (4.30) uses $\delta_R f^{-1} = -Y_R \circ f^{-1}$ and the associated complex conjugate relation, as well as spatial integrations by parts.

The associated equations of motion are explicitly given by
\[
\bar{f} = f' Y_0, \quad \bar{f} = f' \bar{Y}_0, \quad \bar{T} = [(f' \bar{f}) \frac{1}{2} T_0] \circ (f^{-1}, \bar{f}^{-1}), \quad (4.34)
\]
\[
\dot{\bar{f}} = 0, \quad \dot{\bar{f}}(z) = 0, \quad \dot{\bar{f}}(\bar{z}) = 0.
\]
while the Poisson brackets \( \{ \pi_i, \pi_j \} = f^k_{ij} \pi_k \) read explicitly

\[
\{ \tilde{J}(z), P(w, \bar{w}) \} = \frac{3}{2} \partial_w \delta(z, w) + \delta(z, w) \partial_w \] P(w, \bar{w}),
\]
\[
\{ \tilde{J}(z), P(w, \bar{w}) \} = \frac{3}{2} \partial_w \delta(\bar{z}, \bar{w}) + \delta(\bar{z}, \bar{w}) \partial_w \] P(w, \bar{w}),
\]
\[
\{ \tilde{J}(z), \tilde{J}(w) \} = [2 \partial_w \delta(z, w) + \delta(z, w) \partial_w] \tilde{J}(w),
\]
\[
\{ J(z), J(w) \} = [2 \partial_{\bar{w}} \delta(\bar{z}, \bar{w}) + \delta(\bar{z}, \bar{w}) \partial_{\bar{w}}] J(\bar{w}),
\]
\[
\{ J(z), \tilde{J}(w) \} = 0,
\]
\[
\{ P(z, \bar{z}), P(w, \bar{w}) \} = 0.
\]

This can be shown from \( \{ \pi_i X^1_i, \pi_j X^1_j \} = \pi_k [X_1, X_2]^k \). An manifestly skew-symmetric form of the brackets may be obtained by using the relations

\[
\partial_z \delta(z, w) = -\partial_w \delta(z, w), \quad F(w) \partial_z \delta(z, w) = F(z) \partial_z \delta(z, w) + \partial_z F(z) \delta(z, w).
\]

5 Relation to asymptotically flat gravity at null infinity and celestial holography

The Lie-Poisson or Kirillov-Kostant-Souriau brackets (4.35) are the classical analogs of the operator product expansions that have recently appeared in the context of celestial holography [30–33].

In the spirit of effective field theories, since the symmetry group of asymptotically flat spacetimes at null infinity is the BMS_4 group, the current algebra and conserved charges of the model are expected to reproduce the behavior of the currents and charges of these spacetimes [34–36]. In particular, when choosing the Hamiltonian associated to (retarded) time-translations \( t = u, \) \( X_0 = (0, 0, 1), \) \( Y_0 = 0 = \bar{Y}_0, \) \( T_0 = 1, \)

\[
H_{(0,0,1)} = \int dudzd\bar{z} P \left[ (f' \bar{f'} \right) \frac{1}{2} \circ (f^{-1}, \bar{f}^{-1}) \right] = \int dudzd\bar{z} [P \circ (f, \bar{f})](f' \bar{f'} \right) \frac{1}{2},
\]

the time dependence of the symmetry generators of right translations is determined by

\[
\dot{Y}_R = 0 = \dot{\bar{Y}}_R, \quad \dot{T}_R = \frac{1}{2} (\partial Y_R + \partial \bar{Y}_R).
\]

This is consistent with the time dependence of the leading parts of the asymptotic symmetries generators in the gravitational computation at future null infinity. The equations for the group elements in (4.34) simplify to

\[
\dot{f} = 0 = \dot{\bar{f}}, \quad \dot{T} = [f' \bar{f'} \right] \frac{1}{2} \circ (f^{-1}, \bar{f}^{-1}),
\]

so that \( T = T(z, \bar{z}, 0) + u[f' \bar{f'} \right] \frac{1}{2} \circ (f^{-1}, \bar{f}^{-1}), \) while those for the coadjoint vectors are unchanged. Furthermore, in the constraint model with points on the coadjoint orbits
described by \( b_0^g = \text{Ad}_{g^{-1}} b_0 \),

\[
J_0^g = \left[ J_0 + \left( \frac{1}{2} T \partial P_0 + \frac{3}{2} \partial T P_0 \right) \right] \circ (f, \bar f)(f')^2 f',
\]

\[
\bar J_0^g = \left[ \bar J_0 + \left( \frac{1}{2} T \partial P_0 + \frac{3}{2} \partial T P_0 \right) \right] \circ (f, \bar f)(f')^2 \bar f',
\]

\[
P_0^g = P_0 \circ (f, \bar f)(f' \bar f')^2,
\]

it follows from (3.46) or by direct computation that

\[
\frac{dP_0^g}{du} = 0, \quad \frac{df_0^g}{du} = \frac{1}{2} \partial P_0^g, \quad \frac{d\bar f_0^g}{du} = \frac{1}{2} \partial \bar P_0^g.
\]  

(5.5)

If \( \sigma^0, \Psi_0^0, \Psi_0^1, \Psi_1^0 \) are the leading components of shear and of suitable components of the Weyl tensor in the Newman-Penrose description of asymptotically flat spacetimes at null infinity and \( f = T + \frac{1}{2} u (\partial Y + \bar \partial Y) \), the former transform under \( \mathfrak{bms}_4^E \) as

\[
- \delta \sigma^0 = \left( f \partial_u + Y \partial + \bar Y \partial + \frac{3}{2} \partial Y - \frac{1}{2} \partial \bar Y \right) \sigma^0 - \bar \partial^2 f,
\]

\[
- \delta \Psi_0^1 = \left( f \partial_u + Y \partial + \bar Y \partial + \frac{5}{2} \partial Y + \frac{1}{2} \partial \bar Y \right) \Psi_0^0,
\]

\[
- \delta \Psi_0^3 = \left( f \partial_u + Y \partial + \bar Y \partial + 2 \partial Y + \bar \partial Y \right) \Psi_0^0 \partial f + \Psi_0^1 \partial f,
\]

\[
- \delta \Psi_0^2 = \left( f \partial_u + Y \partial + \bar Y \partial + \frac{3}{2} \partial Y + \frac{3}{2} \partial \bar Y \right) \Psi_0^0 + 2 \Psi_0^1 \partial f,
\]

(5.6)

\[
- \delta \Psi_1^1 = \left( f \partial_u + Y \partial + \bar Y \partial + \partial Y + 2 \bar \partial Y \right) \Psi_1^0 + 3 \Psi_0^1 \partial f.
\]

Furthermore, they are related through

\[
\Psi_0^1 = - \partial_\sigma^0, \quad \Psi_0^3 = - \bar \partial \sigma^0, \quad \Psi_0^2 = \partial \sigma^0 - \bar \partial \bar \sigma^0 + \sigma^0 \partial \sigma^0 - \bar \sigma^0 \partial \bar \sigma^0,
\]

(5.7)

and satisfy the evolution equations,

\[
\partial_u \Psi_0^3 = \bar \partial \Psi_0^1, \quad \partial_u \Psi_0^2 = \bar \partial \Psi_0^3 + \sigma^0 \Psi_0^1, \quad \partial_u \Psi_1^1 = \bar \partial \Psi_1^2 + 2 \sigma^0 \Psi_3^0.
\]

(5.8)

We define here non-radiative spacetimes by the conditions

\[
\Psi_1^0 = 0 = \Psi_3^0,
\]

(5.9)

and their complex conjugates. Furthermore, we require \( \Psi_0^0 \) to be real,

\[
\Psi_0^0 = \bar \Psi_0^0.
\]

(5.10)

On account of the relations (5.7), these are constraints on the asymptotic part of the shear and the news that are somewhat weaker than \( \partial_u \sigma^0 = 0 \) and its complex conjugate, together with \( \partial^2 \sigma^0 = \bar \partial \bar \sigma^0 \), i.e., the requirement that the news vanishes together with the analog of the electric condition. The reason is that the latter would require \( \bar \partial^3 \bar Y = 0 = \)
\[ \partial^3 Y \] and eliminate superrotations, while the former are invariant under extended BMS\textsuperscript{4} transformations. With these conditions, the map

\[
m \left( -\frac{1}{G} \Psi_0^0 \right) = P_0^g, \quad m \left( -\frac{1}{2G} \Psi_1^0 \right) = J_0^g, \quad m \left( -\frac{1}{2G} \bar{\Psi}_1^0 \right) = J_0^g
\]  

(5.11)
is compatible with the transformations laws, while the remaining non-trivial evolution equations (5.8) are compatible with (5.5).

6 Discussion

The Chern-Simons to chiral Wess-Zumino-Witten [37–39] to Liouville theory [40, 41] reductions have been used in the context of three-dimensional gravity [42–44] (see also [45–47]). They provide a direct approach to constructing holographic action principles that may be compared to the group theoretic constructions [13, 48].

Due to the absence of a pure Chern-Simons formulation, this avenue is less straightforward in four dimensions, see however [49–51] for such constructions. From this point of view, our models provide consistent targets for holographically dual theories that may be compared to the group theoretic constructions [13, 48].

Due to the absence of a pure Chern-Simons formulation, this avenue is less straightforward in four dimensions, see however [49–51] for such constructions. From this point of view, our models provide consistent targets for holographically dual theories that may be compared to the group theoretic constructions [13, 48].

More generally, considerations on the coadjoint orbits of closely related groups that appear in gravitational theories may be found in [57, 58], while prescriptions for gravitational charges in terms of the left hand sides of (5.11) have recently been discussed in [59–62]. Note that the reality condition imposed in (6.1) explicitly excludes magnetic mass, which has played a prominent role in a number of recent studies [63–68].

In the case of the global BMS\textsuperscript{4} group, the Maurer-Cartan forms for the non-abelian part SL(2, \mathbb{C}) can be obtained directly from the matrix representations

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{-E/2} & -\bar{A}e^{E/2} \\ -Be^{-E/2} & (1 + AB)e^{E/2} \end{pmatrix},
\]  

(6.1)

with \( a, b, c, d, E, A, B \in \mathbb{C} \) and \( ad - bc = 1 \). The second parametrization corresponds to a composition of Lorentz rotations of type \( II \circ I \circ II \) in the terminology of [69, 70]. Explicit formulas for the associated geometric action will be given elsewhere.

Finally, since BMS groups are conformal Carroll groups [71], the BMS\textsuperscript{3} invariant models of [13, 72] and the BMS\textsuperscript{4} invariant models constructed here are explicit examples of conformal Carroll field theories in \( 1 + 1 \) and \( 2 + 1 \) dimensions, respectively.
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