1. Introduction

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces and let \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) be the space of bounded linear operators \( A : \mathcal{H}_1 \to \mathcal{H}_2 \). An operator \( A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) is said to be Fredholm if first, the kernel of \( A \) is finite-dimensional, and second the image of \( A \) is closed and has finite codimension. An application of the open mapping theorem shows that the closedness requirement on the image is redundant. A well-known example of Fredholm operators (F. Riesz): if \( C \) is a compact operator then \( 1 - C \) is Fredholm. It is easy to see that the Fredholm property is equivalent to invertibility modulo finite-rank operators or compact operators.

For a Fredholm operator \( A \) its index is defined by

\[
\text{ind}(A) = \dim_{\mathbb{C}}(\ker A) - \dim_{\mathbb{C}}(\text{coker} A).
\]

The set of all Fredholm operators in \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) is an open set in the norm topology of \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) and the index is locally constant in this set. This means that the index is stable under perturbations that are small with respect to the operator norm. This stability suggests that it might be possible to calculate the index in some concrete analytic situation.

The main example of such a situation is given by elliptic differential operators acting in sections of vector bundles over compact manifolds. Choosing \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) to be appropriate Sobolev spaces of these sections, we find that elliptic operators are Fredholm. Probably the most important result concerning these elliptic operators is the Atiyah-Singer index theorem, which gives the index of the operator in terms of some characteristic classes involving its principal symbol, \( [BGV] \).

Let \( M \) be a noncompact manifold (possibly with boundary) and \( A \) an elliptic differential operator on \( M \). Then \( A \) is not necessarily Fredholm. That is, the kernel and/or cokernel of \( A \) may be infinite-dimensional and/or the image of \( A \) may not be closed. In particular, the index as defined above may not be well-defined, but there are notions generalizing the Fredholm property and the index. In this paper we will use one of these generalized Fredholm properties that makes sense when there is a free action of a unimodular Lie group \( G \) on \( M \) with quotient \( X = M/G \), a compact manifold. Making appropriate choices of metric on \( M \) and in the vector bundles over \( M \) and using a Haar measure on \( G \), we obtain
Hilbert spaces of sections on which the $G$-action is unitary. This action allows us to define an trace $\text{tr}_G$ in the algebra of operators commuting with the action of $G$. Restricting this trace to orthogonal projections $P_L$ onto $G$-invariant subspaces $L$ provides a dimension function $\dim_G$ given by

$$\dim_G(L) = \text{tr}_G(P_L).$$

Generalizing the previous definition, a $G$-invariant operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be $G$-Fredholm if $\dim_G \ker A < \infty$ and if there exists a closed, invariant subspace $Q \subset \text{im}(A)$ so that $\dim_G(\mathcal{H}_2 \ominus Q) < \infty$. With this definition, we prove the following:

**Theorem 1.1.** Let $M$ be a complex manifold with boundary which is strongly pseudoconvex. Let $G$ be a unimodular Lie group acting freely by holomorphic transformations on $M$ so that $M/G$ is compact. Then, for $q > 0$, the Kohn Laplacian $\Box$ in $L^2(M, \Lambda^{p,q})$ is $G$-Fredholm.

**Corollary 1.2.** If $M$ is as before and $q > 0$, then the reduced Dolbeault cohomologies $H^{p,q}(M)$ have finite $G$-dimension.

**Corollary 1.3.** If $M$ is as before, let $L \subset (\ker \bar{\partial})^\perp$ be closed and $G$-invariant. Then $\bar{\partial}|_L : L \to \bar{\partial}L$ is $G$-Fredholm.

**Remark 1.4.** Examples of manifolds satisfying the hypotheses of the theorem are Grauert tubes of unimodular Lie groups. The unimodularity of $G$ is necessary for the definition of the $G$-Fredholm property.

The $\bar{\partial}$-Neumann problem was proposed by Spencer in the 1950s as a method of obtaining existence theorems for holomorphic functions. Morrey in [Mo] introduced the key basic estimate and the problem was solved by Kohn in [K]. We use variants of the techniques in [FK] in this work.

This generalized Fredholm property was first introduced in an abstract setting by M. Breuer [B]. In an analytical context, it was first used by L. Coburn, R. Moyer and I.M. Singer [CMS] to define and calculate the real-valued index of elliptic almost-periodic pseudodifferential operators in $\mathbb{R}^n$. Similarly, M. Atiyah [A] defined and computed the real-valued index of elliptic operators on covering spaces of compact manifolds. B. Fedosov and M. Shubin [FS] working analytically (without Breuer’s theory) defined and calculated the index of random elliptic operators in $\mathbb{R}^n$. A. Connes and H. Moscovici [CM] proved an $L^2$-index theorem for homogeneous spaces of noncompact Lie groups. Also, in [S] M. Shubin used similar techniques to obtain an $L^2$-Riemann-Roch theorem for elliptic operators. In all this work, an important part of the analysis consists of showing that the operators under consideration have the property stressed previously: their images contain closed, invariant subspaces with finite codimension in an appropriate sense. In [GHS], $\Gamma$ is taken to be a discrete group and it is shown that the Kohn Laplacian...
\[ \Box \text{ is } \Gamma\text{-Fredholm. Note that the natural boundary value problem for } \Box \text{ (called the } \partial\text{-Neumann problem) is not elliptic, but only subelliptic. In the present paper we extend this result from [GHS] to the situation in which } G \text{ is a unimodular Lie group. When } G \text{ has a discrete cocompact subgroup } \Gamma \subset G \text{ the } \Gamma\text{-Fredholm property easily implies the } G\text{-Fredholm property. Generically, however, it is not the case that a unimodular Lie group have such a subgroup, cf. [M]. Using different methods, questions posed in [GHS] have been answered and some results there strengthened in [Br] and [TCM].}

In section 2 we will introduce the } G\text{-trace for invariant operators in Hilbert } G\text{-modules. Section 3 contains a description of abstract } G\text{-Fredholm operators and several useful properties. Section 4 treats the relevant results from the theory of the } \partial\text{-Neumann problem. In section 5 we discuss Hodge theory which links analytic results we obtain for } \Box \text{ to the reduced } L^2 \text{ Dolbeault cohomology for } q > 0. \text{ We also explore some easy consequences of the main theorem regarding the operator } \partial \text{ on functions.}

2. Preliminaries

A Hilbert } G\text{-module is a Hilbert space with a (left) strongly continuous unitary action of } G. \text{ A free Hilbert } G\text{-module is a Hilbert } G\text{-module which is unitarily and } G\text{-equivariantly isomorphic to the Hilbert space tensor product } L^2(G) \otimes \mathcal{H}, \text{ where } \mathcal{H} \text{ is a Hilbert space and the Hilbert module structure is given by the action of } G \text{ given by } s \mapsto R_s \otimes 1_\mathcal{H} \text{, where } R_s : L^2(G) \to L^2(G) \text{ is induced by the right translation } t \mapsto ts \text{ on } G. \text{ A projective Hilbert } G\text{-module is a Hilbert } G\text{-module that can be embedded isometrically and } G\text{-equivariantly into a free Hilbert } G\text{-module. Later on we will denote by } R_s \text{ the operator of the action of } s \in G \text{ on arbitrary Hilbert } G\text{-modules. We will only need projective } G\text{-modules in this work, so from now on projective Hilbert } G\text{-modules will be called simply Hilbert } G\text{-modules.}

If there is an action of a group } G \text{ in a Hilbert space } \mathcal{H}, \text{ denote the space of } G\text{-equivariant bounded linear operators in } \mathcal{H} \text{ by } \mathcal{B}(\mathcal{H})^G. \text{ In other words } P \in \mathcal{B}(\mathcal{H})^G \text{ if } P \in \mathcal{B}(\mathcal{H}) \text{ and } R_s P = PR_s \text{ for every } s \in G. \text{ An example of a projective Hilbert } G\text{-module is the image of a projection } P \in \mathcal{B}(L^2(M))^G.

We will describe the Hilbert } G\text{-modules important to our discussion later, but first we restrict our attention to invariant operators on the group. Then we will define the } G\text{-invariant trace we actually need in the invariant operators on } L^2(M). \text{ For any } s \in G \text{ define left and right translations } L_s, R_s : L^2(G) \to L^2(G) \text{ by } (L_s u)(t) = u(s^{-1}t), (R_s u)(t) = u(ts). \text{ For } f \in L^1(G) \text{ and } u \in L^2(G), \text{ let}

\[ (L_f u)(t) = \int_G f(s)(L_s u)(t) ds = \int_G f(s)u(s^{-1}t) ds. \]

The set } \{L_f \mid f \in L^1(G)\} \text{ forms an associative algebra of bounded operators in } L^2(G) \text{ which are right-invariant (i.e. commute with right translations). Define}
\( \mathcal{L}_G \subset \mathcal{B}(L^2(G)) \) to be the weak closure of this algebra. Then \( \mathcal{L}_G \) is a von Neumann algebra. We will also need to consider operators \( L_f \) for \( f \in L^2(G) \). These are defined on \( C_\infty(G) \) and we may try to extend them by continuity to \( L^2(G) \). This is not always possible, but we will be concerned only with those \( L_f \) which are bounded, or, equivalently, can be extended to bounded linear operators in \( L^2(G) \). The extended operator will be still denoted \( L_f \) and it is then right-invariant and belongs to \( \mathcal{L}_G \). It follows from the Schwartz kernel theorem that any bounded right-invariant operator in \( L^2(G) \) can be presented in the form \( L_f \) for a distribution \( f \) on \( G \).

We will need the following fact from about group von Neumann algebras (cf. [P], sections 5.1 and 7.2). There is a unique trace \( \text{tr}_G \) on \( \mathcal{L}_G \subset \mathcal{B}(L^2(G)) \) agreeing with

\[ \text{tr}_G(L_f^*L_f) = \int_G |f(s)|^2 ds, \]

whenever \( L_f \in \mathcal{B}(L^2(G)) \) and \( f \in L^2(G) \). Furthermore, \( \text{tr}_G(A^*A) < \infty \) if and only if there is an \( f \in L^2(G) \) for which \( A = L_f \in \mathcal{B}(L^2(G)) \). If we define \( \tilde{f}(t) = \overline{f}(t^{-1}) \), and if \( f_k, g_k \in L^2(G) \), \( k = 1, \ldots, N \), then the operator \( \sum L_{f_k}^*L_{g_k} \) is in \( \text{Dom}(\text{tr}_G) \). Furthermore, \( h \) is continuous and \( \text{tr}_G(L_h) = h(e) \).

**Remark 2.1.** The unimodularity of the group is necessary for the trace property of \( \text{tr}_G \).

Now we bring our results on the group up to the manifold. Let \( G \) be a Lie group and \( G \to M \xrightarrow{p} X \) be a principal \( G \)-bundle with compact base \( X \). In particular, this means that we have a free right action of \( G \) on \( M \) with quotient space \( X \), and \( p : M \to X \) is the canonical projection. Having a smooth free action of \( G \) on a manifold \( M \) with a \( G \)-invariant measure \( dx \), and fixing a Haar measure \( dt \) on \( G \), we obtain a natural quotient measure \( dx \) on \( X = M/G \) which allows us to present the Hilbert \( G \)-module \( L^2(M) \) in the form

\[ L^2(M) \cong L^2(G) \otimes L^2(X), \]

which makes it a free Hilbert \( G \)-module. It follows that we have a decomposition of the von Neumann algebra of bounded invariant operators

\[ \mathcal{B}(L^2(M))^G \cong \mathcal{B}(L^2(G))^G \otimes \mathcal{B}(L^2(X)) \cong \mathcal{L}_G \otimes \mathcal{B}(L^2(X)), \]

where we have made the identification \( \mathcal{L}_G \cong \mathcal{B}(L^2(G))^G \). In order to measure the invariant subspaces of \( L^2(M) \), we need a trace on \( \mathcal{L}_G \otimes \mathcal{B}(L^2(X)) \). It occurs that there exists a natural normal, faithful, semifinite trace on this algebra. It is denoted \( \text{Tr}_G \) and formally presented in the form

\[ \text{Tr}_G = \text{tr}_G \otimes \text{Tr}, \]
where $\text{Tr}$ is the usual trace on $\mathcal{B}(L^2(X))$. We describe the trace $\text{Tr}_G$ in more detail. Let $(\psi_l)_{l \in \mathbb{N}}$ be an orthonormal basis for $L^2(X)$. Then

$$L^2(M) \cong L^2(G) \otimes L^2(X) \cong \bigoplus_{l \in \mathbb{N}} L^2(G) \otimes \psi_l.$$ 

Denoting by $P_m$ the projection onto the $m^{th}$ summand, we obtain a matrix representation of $A \in \mathcal{B}(L^2(M))$ with elements $A_{lm} = P_l A P_m \in \mathcal{B}(L^2(G))$. If $A \in \mathcal{B}(L^2(M))^G$, then these matrix elements are invariant operators in $L^2(G)$, and there exist distributions $h_{lm}$ on $G$ so that $A \in \mathcal{B}(L^2(M))^G$ has a matrix representation

\begin{equation}
A \leftrightarrow [A_{lm}]_{lm} = [L_{h_{lm}}]_{lm}.
\end{equation}

**Definition 2.2.** For positive $A \in \mathcal{B}(L^2(M))^G$ define

$$\text{Tr}_G(A) = \sum_{l \in \mathbb{N}} \text{tr}_G(A_{ll}).$$

The functional $\text{Tr}_G$ is a normal, faithful, and semifinite trace and is independent of the basis $(\psi_l)_{l}$ used in its construction, cf. Section V.2 of [T]. We define the $G$-Hilbert-Schmidt operators

\begin{equation}
\text{Dom}_{1/2}(\text{Tr}_G) = \{ A \in \mathcal{B}(L^2(M))^G \mid \text{Tr}_G(A^* A) < \infty \}.
\end{equation}

Also, define the $G$-trace-class, $\text{Dom}(\text{Tr}_G)$, to be the vector space of finite linear combinations of the form $A^* B$, where $A, B \in \text{Dom}_{1/2}(\text{Tr}_G)$.

**Remark 2.3.** If $L$ is an arbitrary (projective) Hilbert $G$-module, then $L$ is the image of a $G$-invariant orthogonal projection $P$ in $L^2(G) \otimes \mathcal{H}$. Thus the trace $\text{Tr}_G$ on $L^2(G) \otimes \mathcal{H}$ restricts to one on $L$ defined by $A \mapsto \text{Tr}_G(PAP)$.

We will have to describe smoothness of functions, forms, and sections of vector bundles using $G$-invariant Sobolev spaces which we describe here. The $G$ action induces an invariant Riemannian metric on $M$ so that with respect to this structure $M$ has bounded geometry. As in [Gro] and [S1] we may construct appropriate partitions of unity and, with local geodesic coordinates, assemble $G$-invariant integer Sobolev spaces $H^s(M)$. If $E$ is a vector $G$-bundle over $M$, then we may introduce a $G$-invariant inner product structure on $E$. Together with the $G$-invariant measure on $M$ that we have described previously, we define the Hilbert spaces of sections of $E$ which we denote $H^s(M, E)$, for $s = 0, 1, 2, \ldots$. Because $X = M/G$ is compact, the spaces $H^s(M, E)$ do not depend on the choices of invariant metric on $M$ or of invariant inner product on $E$. Note that, in particular, spaces of sections in natural tensor bundles on a $G$-manifold have natural, invariant Sobolev structures.
3. G-Fredholm Operators

We will explain and modify a generalized notion of the Fredholm property as introduced in [B] in the setting of bounded operators in arbitrary von Neumann algebras. By using the graph norm on the domain of the operator, it is easy to extend the results in [B] to closed, densely defined operators as in [S]. There, the von Neumann algebras in question were of invariant operators on Hilbert Γ-modules with Γ a discrete group. Here we make the trivial extension to von Neumann algebras of invariant operators acting in Hilbert $G$-modules where $G$ is a unimodular Lie group rather than a discrete group. Also we will describe and utilize the property called Γ-density which was introduced and exploited in [S]. A lemma regarding restrictions of Fredholm operators is proven here.

Lemma 3.1. [GHS] (2.1) Let $L$ be a Hilbert $G$-module and $L_1, L_2$ two Hilbert submodules of $L$ such that $\dim_G L_1 > \text{codim}_G L_2$ where the codimension means the dimension of the orthogonal complement of $L_2$ in $L$. Then $L_1 \cap L_2 \neq \{0\}$ and $\dim_G L_1 \cap L_2 \geq \dim_G L_1 - \text{codim}_G L_2$.

Definition 3.2. Let $L_0$, $L_1$ be Hilbert $G$-modules, $A : L_0 \to L_1$ a closed densely-defined linear operator commuting with the action of $G$. Such an operator is called $G$-Fredholm if the following conditions are satisfied:

• $\dim_G \ker A < \infty$
• There exists a $G$-invariant closed subspace $Q \subset L_1$ so that $Q \subset \text{im} A$ and $\text{codim}_G Q = \dim_G (L_1 \cap Q^\perp) < \infty$.

Remark 3.3. Henceforth we will also use another notation: $L \ominus Q \overset{\text{def}}{=} L \cap Q^\perp$.

Definition 3.4. Let $L$ be a Hilbert $G$-module and $Q \subset L$ a $G$-invariant subspace, not necessarily closed. Then

• If for every $\epsilon > 0$ there is a $G$-invariant subspace $Q_\epsilon \subset Q$ such that $Q_\epsilon$ is closed in $L$ and $\text{codim}_G Q_\epsilon < \epsilon$ in $L$, then $Q$ is called $G$-dense in $L$.
• $Q$ is called almost closed if $Q$ is $G$-dense in its closure $\overline{Q}$.

Remark 3.5. It could happen that a $G$-invariant dense subspace $M \subset L$ in a Hilbert $G$-module $L$ not be $G$-dense. For example, if $G$ is countable, then the space $Q$ of all functions on $G$ with finite support is not $G$-dense in $L^2(G)$. Indeed, a closed subspace in $Q$ is necessarily finite-dimensional in the usual sense while any nontrivial closed invariant subspace in $L^2(G)$ must be infinite-dimensional.

Lemma 3.6. (Lemma 1.15 of [S]) If $A : L_0 \to L_1$ is a $G$-Fredholm operator, then its image is almost closed. That is $\text{im}(A)$ is $G$-dense in $\overline{\text{im}(A)}$.

Corollary 3.7. (GHS lemma 2.6) Let $A : L_0 \to L_1$ be a $G$-Fredholm operator and $L$ a $G$-submodule of $L_1$ such that $L \subset \text{im}(A)$. Then $L \cap \text{im}(A)$ is $G$-dense in $L$. 
Lemma 3.8. ([S], lemma 1.17) Let $L$ be a Hilbert $G$-module, $L_1 \subset L$, and $Q \subset L$ be $G$-invariant subspaces in $L$ so that $L_1$ is closed and $Q$ is $G$-dense in $L$. Then $Q \cap L_1$ is $G$-dense in $L_1$. More generally, if $Q$ is almost closed, then $Q \cap L_1$ is almost closed with closure equal $\overline{Q \cap L_1}$.

Lemma 3.9. If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is $G$-Fredholm and $L \hookrightarrow \mathcal{H}_1$ is closed and $G$-invariant, then $A|_L : L \to A(L)$ is $G$-Fredholm.

Proof. This follows immediately from Lemma 3.7 and Lemma 3.8. \(\square\)

4. The $\bar{\partial}$-Neumann Problem

The principal references for this section are [E, FK, GHS]. Let $M$ be a complex manifold with nonempty, smooth, strongly pseudoconvex boundary $bM$, $\tilde{M} = M \cup bM$, so that $M$ is the interior of $\tilde{M}$, and $\text{dim}_\mathbb{C}(M) = n$. For simplicity, let us also assume that $\tilde{M} \subset \tilde{M}$, where $\tilde{M}$ is a complex neighborhood of $\tilde{M}$ of the same dimension, such that $bM$ is in the interior of $\tilde{M}$. Let us choose a smooth function $\rho : \tilde{M} \to \mathbb{R}$ so that $M = \{z \mid \rho(z) < 0\}$, $bM = \{z \mid \rho(z) = 0\}$, and for all $z \in bM$, we have $d\rho(z) \neq 0$.

We describe the construction of $\square$ and its relevance to the solution of the $\bar{\partial}$-Neumann problem. We seek a solution $u \in L^2(M)$ to the equation $\bar{\partial} u = \phi$ with $\phi \in L^2(M, \Lambda^{0,1})$, $\bar{\partial} \phi = 0$. Note that solutions will only be determined modulo the kernel of $\bar{\partial}$ consisting of all square-integrable holomorphic functions on $M$. It is preferable to deal with self-adjoint operators, so since the Hilbert space adjoint $\overline{\partial}$ of $\partial$ satisfies $\text{im} \partial = (\ker \bar{\partial})^\perp$, it is natural to seek $u$ of the form $u = \bar{\partial}^* v$, so that

\begin{equation}
\partial \overline{\partial}^* v = \phi.
\end{equation}

Note that $\partial \overline{\partial}^*$ is a self-adjoint operator. In order to do away with the compatibility condition on $\phi$, let us add a term $\bar{\partial} \bar{\partial} v$, thus obtaining

\begin{equation}
(\partial \overline{\partial}^* + \overline{\partial} \bar{\partial}) v = \phi,
\end{equation}

where $\phi$ need not satisfy $\bar{\partial} \phi = 0$. Notice that when $\bar{\partial} \phi = 0$, \(\square\) reduces to \(\square\) because applying $\bar{\partial}$ to \(\square\) gives $\partial \overline{\partial} \bar{\partial} v = 0$, which in turn implies

\[ 0 = \langle \partial \overline{\partial}^* \bar{\partial} v, \bar{\partial} v \rangle = \|\bar{\partial}^* \bar{\partial} v\|^2_{L^2(M)}. \]

Thus the new term in \(\square\) vanishes when the compatibility condition holds. Let us consider $\bar{\partial}$ as the maximal operator in $L^2(M)$ and let $\overline{\partial}$ be the Hilbert space
adjoint operator. We will also use the corresponding Laplacian
\[ \Box = \Box_{p,q} = \overline{\partial \partial^*} + \overline{\partial^* \partial} \text{ on } L^2(M, \Lambda^{p,q}). \]

We will denote the domain of any operator \( A \) by \( \text{Dom}(A) \). The following lemma gives a description of the operators \( \overline{\partial} \), \( \Box \) as well as their domains \( \text{Dom}(\overline{\partial}) \), \( \text{Dom}(\Box) \). Let \( \partial \) be the formal adjoint operator to \( \overline{\partial} \), and let \( \sigma = \sigma(\partial, \cdot) \) be its principal symbol.

**Lemma 4.1.** [GHS] Let us assume that \( M \) is strongly pseudoconvex.
(i) The operator \( \overline{\partial} \) can be obtained as the closure of \( \partial \) from the initial domain
\[ \text{Dom}_0(\overline{\partial}) = \{ \omega, \omega \in C^\infty_c(M, \Lambda^*) \}, \quad \sigma(\partial, d\rho)\omega = 0 \text{ on } bM. \]
(ii) The space \( \text{Dom}_0(\overline{\partial}^*) \) is dense in \( \text{Dom}(\overline{\partial}^*) \cap \text{Dom}(\overline{\partial}) \) in the norm
\[ (\|\omega\|^2_0 + \|\overline{\partial}\omega\|^2_0 + \|\overline{\partial^*}\omega\|^2_{1/2}, \omega \in \text{Dom}(\overline{\partial}^*) \cap \text{Dom}(\overline{\partial}). \]
(iii) The operator \( \Box = \Box_{p,q} \) can be obtained as the closure of the operator \( \overline{\partial} \partial + \partial \overline{\partial} \) from the initial domain
\[ \text{Dom}_0(\Box) = \{ \omega, \omega \in C^\infty_c(M, \Lambda^*) \cap L^2(M), \quad \sigma(\partial, d\rho)\omega = 0, \]
\[ \sigma(\partial, d\rho)\overline{\partial}\omega = 0 \text{ on } bM \}. \]

For any \( \omega \in \text{Dom}(\Box) \) defined as \( \{ \omega \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*) : \overline{\partial}\omega \in \text{Dom}(\overline{\partial}), \overline{\partial}^*\omega \in \text{Dom}(\overline{\partial}) \} \) the following integral identity holds
\[ (\Box \omega, \omega) = \|\overline{\partial}\omega\|^2_0 + \|\overline{\partial}^*\omega\|^2_0. \]

The boundary conditions on \( \omega \) are called the \( \overline{\partial} \)-Neumann conditions.

We describe the Friedrichs construction here for completeness [FK]. Suppose \( \mathcal{H} \) is a Hilbert space and \( Q \) is a Hermitian form defined on a dense subspace \( \mathcal{D} \subset \mathcal{H} \) so that \( Q(\phi, \phi) \geq \|\phi\|^2, \phi \in \mathcal{D} \). Suppose further that \( \mathcal{D} \) is a Hilbert space under the inner product \( Q \). Then there is a self-adjoint operator \( F \) on \( \mathcal{H} \) associated with \( Q \): For each \( \alpha \in \mathcal{H}, \psi \mapsto \langle \alpha, \psi \rangle \) is a \( Q \)-bounded functional of \( \psi \in \mathcal{D} \) since
\[ |\langle \alpha, \psi \rangle| \leq \|\alpha\||\psi\| \leq \|\alpha\|\sqrt{Q(\psi, \psi)}. \]
By Riesz, we have a unique representative \( \phi \in \mathcal{D} \) so that for all \( \psi \in \mathcal{D}, Q(\phi, \psi) = \langle \alpha, \psi \rangle \). Now define \( T : \mathcal{H} \rightarrow \mathcal{D} \subset \mathcal{H} \) by \( T\alpha = \phi. \) Then \( \|T\alpha\|^2 \leq Q(T\alpha, T\alpha) = \langle T\alpha, T\alpha \rangle \leq \|\alpha\||T\alpha\| \) so \( T \) is a bounded operator. Further, \( T\alpha = 0 \) implies that \( \forall \psi \in \mathcal{D}, Q(T\alpha, \psi) = \langle \alpha, \psi \rangle = 0, \) hence \( \alpha = 0 \) since \( \mathcal{D} \) is assumed dense. So \( T \) is injective. Now, \( \langle T\alpha, \beta \rangle = \langle \beta, T\alpha \rangle = Q(T\beta, T\alpha) = Q(T\alpha, T\beta) = \langle \alpha, T\beta \rangle \). Therefore \( T \) is self-adjoint. Put \( F = T^{-1} \).

The Friedrichs Extension theorem says that \( F \) is the unique self-adjoint operator with \( \text{Dom}(F) \subset \mathcal{D} \) satisfying \( Q(\phi, \psi) = \langle F\phi, \psi \rangle \) for all \( \phi \in \text{Dom}(F) \) and \( \psi \in \mathcal{D} \).

In our case we will put \( Q(\phi, \psi) = \langle \overline{\partial}\phi, \overline{\partial}\psi \rangle + \langle \partial\phi, \partial\psi \rangle + \langle \phi, \psi \rangle \) on the smooth forms satisfying the \( \overline{\partial} \)-Neumann boundary conditions. Thus \( F = \Box + 1 \).

The following is a regularity result for \( F \) and is the crux of the problem.
Theorem 4.2. Let $M$ be strongly pseudoconvex, $U$ an open subset of $\bar{M}$ with compact closure, and $\zeta, \zeta_1 \in C_0^\infty(U)$ for which $\zeta_1|_{\text{supp}(\zeta)} = 1$. If $q > 0$ and $\alpha|_U \in H^q(U, \Lambda^{p,q})$, then $\zeta(\Box + 1)^{-1}\alpha \in H^{q+1}(\bar{M}, \Lambda^{p,q})$ and there exist constants $C_s > 0$ so that

\[ \|\zeta(\Box + 1)^{-1}\alpha\|_{s+1}^2 \leq C_s(\|\zeta_1\alpha\|_s^2 + \|\alpha\|_0^2). \]

Proof. This is Prop. 3.1.1 from [FK] extended to the noncompact case in [E]. \qed

Corollary 4.3. Let $q > 0$ and $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of the Laplacian in $L^2(M, \Lambda^{p,q})$. If $\delta > 0$ and $P = \int_0^\delta dE_\lambda$ then $\text{im}(P) \subset C^\infty(\bar{M}, \Lambda^{p,q})$.

Proof. We show that $\text{im}(P) \subset H^s_{\text{loc}}(\bar{M}, \Lambda^{p,q})$ for all $s$. Let $U, \zeta, \zeta_1$ be as in the previous theorem. Since $\text{im}(\Box + 1)^{-1} = \text{Dom}(\Box)$ we have the following. For every $u \in \text{Dom}(\Box)$ with $\Box u + u \in H^s_{\text{loc}}(M)$, we have $u \in H^{s+1}_{\text{loc}}(M)$ and

\[ \|\zeta u\|_{s+1}^2 \leq C_s(\|\zeta_1(\Box + 1) u\|_s^2 + \|(\Box + 1) u\|_0^2). \]

Let $u \in \text{im}(P)$. Applying the theorem with $s = 0$, we have $\text{im}(P) \subset H^1_{\text{loc}}(M, \Lambda^{p,q})$. Now assume $u \in \text{im}(P) \subset H^{-1}_{\text{loc}}(M, \Lambda^{p,q})$. Then $\Box + 1)u = (\Box + 1)Pu = P(\Box + 1)u \in H^{s-1}_{\text{loc}}(\bar{M}, \Lambda^{p,q})$. We conclude that $u \in H^s_{\text{loc}}(\bar{M}, \Lambda^{p,q})$ and so $\text{im}(P) \subset H^s_{\text{loc}}(M, \Lambda^{p,q})$. \qed

Here we repeat Theorem 2.2.9 of [FK], which gives interior Sobolev regularity of the Laplacian. Notice that it is local.

Lemma 4.4. Let $U, V$ be regions with $V \subset \bar{V} \subset U \subset \bar{U} \subset M$, and let $\zeta_1$ be a real $C^\infty$ function supported in $U$ with $\zeta_1 = 1$ on $V$. If $\phi \in \text{Dom}(F)$ and $\zeta_1 F \phi \in H^s(M, \Lambda^{p,q})$ for some $s \geq 0$, then $\zeta \phi \in H^{s+2}(M, \Lambda^{p,q})$ for any real $\zeta \in \Lambda^{0,0}_0(V)$.

Remark 4.5. We have that the images of the spectral projections of $\Box$ corresponding to bounded intervals consist of forms that are smooth to the boundary, but we need that these forms belong to Sobolev spaces as in the interior as well. We cannot glue local estimates together as in [GHS] because the last term in

\[ \|\zeta u\|_{s+1}^2 \leq C_s(\|\zeta_1(\Box + 1) u\|_s^2 + \|(\Box + 1) u\|_0^2) \]

is not cut off and the proof uses crucially the compact support of the cutoff functions. To adjust for this, we will need to modify a number of claims from Sections 2.3 and 2.4 of [FK]. As the proof is long and computationally detailed we will relegate it to an appendix and give the result here below, cf. Prop. 3.1.11 of [FK].

Theorem 4.6. For every smooth $u \in \text{Dom}(\Box) \cap \Lambda^{p,q}$, we have

\[ \|u\|_{s+1}^2 \lesssim \|\Box u\|_s^2 + \|u\|_0^2 \]

for each positive integer $s$. 

Proof. From Lemma 7.9 and again as in [Gro] and [SI] we may construct appropriate partitions of unity and glue together the local a priori estimates
\[ \|\zeta u\|_{s+1}^2 \lesssim \|\zeta_0 Fu\|_s^2 + \|\zeta_0 u\|_s^2 \quad (u \in \text{Dom}(\square) \cap C^\infty) \]
to obtain the global estimate. \[ \square \]

Corollary 4.7. Let \( q > 0 \) and \( \square = \int_0^\infty \lambda dE_\lambda \) be the spectral decomposition of the Laplacian in \( L^2(M, \Lambda^{p,q}) \). If \( \delta > 0 \) and \( P = \int_0^\delta dE_\lambda \) then \( \text{im}(P) \subset H^\infty(M, \Lambda^{p,q}) \).

We need the following fact about Sobolev spaces on manifolds with boundary.

Definition 4.8. For \( s > 0 \), denote by \( H^{-s}(\bar{M}) \) the dual space of \( H^s(\bar{M}) \). I.e. \( H^{-s}(\bar{M}) = (H^s(\bar{M}))' \).

Lemma 4.9. Let \( M \) be a manifold with boundary and \( s > 0 \). Then \( H^{-s}(\bar{M}) \) consists of elements of \( H^{-s}(\bar{M}) \) whose support is in \( \bar{M} \).

Proof. See Remark 12.5 of [LM]. \[ \square \]

Corollary 4.10. Let \( q > 0 \) and \( \square = \int_0^\infty \lambda dE_\lambda \) be the spectral decomposition of the Laplacian in \( L^2(M, \Lambda^{p,q}) \). If \( \delta > 0 \) and \( P = \int_0^\delta dE_\lambda \) then \( P : H^{-s}(\bar{M}, \Lambda^{p,q}) \to H^s(\bar{M}, \Lambda^{p,q}) \) for any positive integer \( s \).

Proof. In Lemma 4.7 we established that spectral projections \( P \) of \( \square \) take \( L^2(M) \) to \( H^s(M) \) for all \( s > 0 \). It follows that \( P : H^{-s}(\bar{M}) \to L^2(M) \). Since \( P^2 = P \) on \( H^\infty(M) \subset L^2(M) \), a dense subspace of all the \( H^s(\bar{M}) \), \( (s \in \mathbb{R}) \) we conclude that \( P : H^{-s}(\bar{M}) \to H^s(M) \) for all \( s > 0 \). \[ \square \]

5. Dolbeault-Hodge-Kodaira

Let us describe the reduced \( L^2 \) Dolbeault cohomology spaces on a complex (generally non-compact) manifold \( M \) with a given hermitian metric. Denote the Hilbert space of all (measurable) square-integrable \((p,q)\)-forms on \( M \) by \( L^2(M, \Lambda^{p,q}) \). The operator
\[ \overline{\partial} : L^2(M, \Lambda^{p,q}) \longrightarrow L^2(M, \Lambda^{p,q+1}) \]
is defined as the maximal operator, i.e. its domain \( D^{p,q} = D^{p,q}(\overline{\partial};M) \) is the set of all \( \omega \in L^2(M, \Lambda^{p,q}) \) such that \( \overline{\partial}\omega \in L^2(M, \Lambda^{p,q+1}) \) where \( \overline{\partial} \) is applied in the sense of distributions. Obviously \( \overline{\partial}^2 = 0 \) on \( D^{p,q} \) and we can form a complex
\[ L^2(M, \Lambda^{p,*}) : \quad 0 \longrightarrow D^{p,0} \longrightarrow D^{p,1} \longrightarrow \ldots \longrightarrow D^{p,n} \longrightarrow 0. \]
The reduced \( L^2 \)-Dolbeault cohomology spaces of \( M \) are defined by:
\[ L^2H^{p,q}(M) = \ker(\overline{\partial} : D^{p,q} \to D^{p,q+1}) / \text{im} (\overline{\partial} : D^{p,q-1} \to D^{p,q}). \]
Since \( \ker \overline{\partial} \) is a closed subspace in \( L^2 \), the reduced cohomology space \( L^2H^{p,q}(M) \) is a Hilbert space. Note that the space \( L^2H^{0,0}(M) \) coincides with the space \( L^2\mathcal{O}(M) \) of all square-integrable holomorphic functions on \( M \).
Lemma 5.1. The following orthogonal decompositions hold:

\[ L^2(M, \Lambda^*) = \text{im} \partial \oplus \ker \Box \oplus \text{im} \overline{\partial} \]

In particular, we have an isomorphism of Hilbert \( G \)-modules

\[ L^2 \overline{H}^{p,q}(M) = \ker \Box_{p,q}. \]

Corollary 5.2. \( \text{im} \overline{\partial} \subset \text{im} \Box \).

6. The \( G \)-Fredholm Property of \( \Box \)

We will need a description of \( G \)-operators in terms of their Schwartz kernels, cf. [2]. If \( P \in \mathcal{B}(L^2(M))^G \), its kernel \( K_P \) satisfies

\[ K_P(x, y) = K_P(x, yt), \quad t \in G. \]

Thus \( K_P \) descends to a distribution on the quotient \( \frac{M \times M}{G} \). The measure taken on \( \frac{M \times M}{G} \) is simply the quotient measure.

Lemma 6.1. If \( P : L^2(M) \to H^\infty(M) \) is a self-adjoint projection, then its Schwartz kernel \( K_P \) is smooth.

Proof. Since \( y \mapsto \delta_y \) is a smooth function on \( M \) with values in \( H^{-\infty}(\overline{M}) \), the composition

\[ (x, y) \mapsto (P\delta_y)(x) = \int_M K_P(x, z)\delta_y(z)dz = K_P(x, y) \]

is jointly smooth. \( \square \)

Lemma 6.2. If \( P \in \mathcal{B}(L^2(M))^G \) is a self-adjoint, invariant projection so that \( \text{im}(P) \subset C^\infty(M) \), then \( K_P \in L^2\left( \frac{M \times M}{G} \right) \).

Proof. Fix \( x \in M \). If \( P : L^2(M) \to C^\infty(M) \), the closed graph theorem applied to \( P \) implies \( u \in L^2(M) \mapsto (Pu)(x) \in \mathbb{C} \) is a bounded linear functional. The Riesz representation theorem then gives that there exists a function \( h_x \in L^2(M) \) so that

\[ (Pu)(x) = \langle h_x, u \rangle \quad u \in L^2(M). \]

Since \( (Pu)(x) = \int_M K_P(x, y)u(y)dy \), and agrees with \( \langle h_x, u \rangle \) when \( u \) has compact support, \( h_x = K_P(x, \cdot) \) almost everywhere. We conclude that for any \( x \in M \), \( \int_M |K_P(x, y)|^2dy \) is finite.

Now consider \( \phi(x) = \int_M |K_P(x, y)|^2dy \). The function \( \phi \) is constant on orbits since the measure on \( M \) is invariant;

\[ \phi(xt) = \int_M |K_P(xt, y)|^2dy = \int_M |K_P(x, yt^{-1})|^2dy = \int_M |K_P(x, y)|^2dy = \phi(x). \]
Thus $\phi$ descends to a function on $M/G = X$. Since the map from $M$ to $C_c^{-\infty}(M)$ defined by $y \mapsto \delta_y$ is continuous, the composition

$$y \mapsto P\delta_y = K_P(\cdot, y)$$

is a continuous function $M \to L^2(M)$. We may conclude that $\phi : X \to \mathbb{R}_+$ is continuous. Denote by $\frac{dx}{dt}$ the quotient measure on $X$. The compactness of $X$ together with continuity of $\phi$ imply that $\int_X \phi(x) \frac{dx}{dt} < \infty$. Thus we have that $K_P \in L^2(M \times M_G)$.

Choosing a measurable global section $x$ in $M$ and representing points $x \in M$, $x \to (t, x) \in G \times X$, we obtain an isomorphism of measure spaces $(M, dx) \sim (G \times X, dt \otimes dx)$. Whenever $P \in \mathcal{B}(L^2(M))^G$ and $K_P \in L^2_{loc}(M \times M)$, this isomorphism and the criterion for invariance allow a representation

$$K_P(x, y) \longrightarrow K_P(t, x; s, y) \overset{\text{def}}{=} \kappa(ts^{-1}; x, y), \quad s, t \in G, \, x, y \in X$$

with $\kappa \in L^2_{loc}(G \times X \times X)$.

**Lemma 6.3.** Let $P \in \mathcal{B}(L^2(M))^G$. Then $\text{Tr}_G(P^*P) = \int_{M \times M} |K_P|^2$.

**Proof.** Let $(\psi_k)_k$ be an orthonormal basis for $L^2(X)$. In the decomposition $L^2(M) \cong \bigoplus_k L^2(G) \otimes \psi_k$, the invariant operator $P$ has a matrix representation $P \to [L_{h_{kl}}]_{kl}$. In terms of this, we compute

$$\text{Tr}_G(P^*P) = \sum_l \text{tr}_G((P^*P)_l) = \sum_l \text{tr}_G \left( \sum_k (P^*)_lk P_{kl} \right)$$

$$= \sum_l \text{tr}_G \left( \sum_k P_{kl}^* P_{kl} \right) = \sum_{kl} \text{tr}_G(L_{h_{kl}}^* L_{h_{kl}}) = \sum_{kl} \|h_{kl}\|^2_{L^2(G)}$$

by normality of $\text{tr}_G$.

Now, except on a set of measure zero, we have a description of $P$

$$(Pu)(x) = \int_M K_P(x, y)u(y)dy = (Pu)(t, x) = \int_{G \times X} dsdy \, \kappa(s; x, y)u(st, y).$$

Now, the distributional kernels $h_{ij}$ can be recovered from $\kappa$ by projecting into the summands in $L^2(M) \cong \bigoplus_i (L^2(G) \otimes \psi_i)$,

$$h_{ij} = \int_{X \times X} dxdy \, \kappa(\cdot; x, y)\psi_j(y)\overline{\psi}_i(x).$$

Let us compute the norm of $\kappa$ in $L^2(G \times X \times X)$. Since $(\psi_j)_j$ is an orthonormal basis for $L^2(X)$, the set $(\overline{\psi}_i \otimes \psi_j)_{ij}$ forms an orthonormal basis for $L^2(X \times X)$. 
By construction, $h_{ij}$ is equal to the $ij$th Fourier coefficient of $\kappa$ with respect to the decomposition $L^2(G \times X \times X) \cong \bigoplus_{ij} (L^2(G) \otimes \psi_i \otimes \psi_j)$. Hence

$$\sum_{ij} \|h_{ij}\|^2_{L^2(G)} = \|\kappa\|^2_{L^2(G \times X \times X)}.$$  

Thus $\text{Tr}_G(P^*P) = \|\kappa\|^2_{L^2(G \times X \times X)} = \int_{M \times M} |K_P(x,y)|^2 \frac{dxdy}{d\gamma}$. □

**Corollary 6.4.** If $P \in \mathcal{B}(L^2(M))^G$ is an invariant self-adjoint projection such that $\text{im}(P) \subset H^\infty(M)$, then $\text{Tr}_G(P) < \infty$.

**Remark 6.5.** All the previous results extend trivially to operators acting in bundles.

**Theorem 6.6.** For $q > 0$, the operator $\Box$ on $M$ is $G$-Fredholm.

**Proof.** Let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of $\Box$ and for $\delta > 0$, $P = \int_0^\delta dE_\lambda$. Thus $\text{im}(1 - P) \subset \text{im}(\Box)$. Further, $\text{im}(P) \subset L^2(M, \Lambda^{p,q})$ is closed, invariant and, by Corollary 6.4, $\text{im}(P) \subset C^\infty(M, \Lambda^{p,q})$. Corollary 6.4 implies that $\text{codim}(\text{im}(\Box)) < \infty$. The requirement on the kernel of $\Box$ is verified noting that $\ker(\Box) \subset \text{im}(P)$ in the above. □

**Remark 6.7.** By Theorem 5.4.9 of [FK] and the discussion immediately following, one can deduce the same results for the boundary Laplacian $\Box_b$.

**Corollary 6.8.** If $q > 0$, $\dim_G L^2(\bar{H}^{p,q}(M)) < \infty$.

**Proof.** By Lemma 5.1 $L^2(\bar{H}^{p,q}(M)) = \ker(\Box_{p,q}) = \text{im}(E_0)$ which has finite $G$-dimension. □

**Corollary 6.9.** For the operator $\bar{\partial} : L^2(M, \Lambda^{0,0}) \to L^2(M, \Lambda^{0,1})$ we have that $\text{im}(\bar{\partial})$ is $G$-dense in $\overline{\text{im}(\bar{\partial})}$. Consequently, $\bar{\partial} : L^2(M, \Lambda^{0,0})$ restricted to $(\ker \bar{\partial})^\perp$ is $G$-Fredholm.

**Proof.** By Lemma 3.8 we have that $\text{im}(\Box) \cap \overline{\text{im}(\bar{\partial})}$ is $G$-dense in $\overline{\text{im}(\bar{\partial})}$. The decomposition of $L^2(\bar{H}^{p,q}(M))$ implies that $\text{im}(\Box) \cap \overline{\text{im}(\bar{\partial})} \subset \overline{\text{im}(\bar{\partial})}$. Thus $\overline{\text{im}(\bar{\partial})}$ is almost closed. □

**Corollary 6.10.** If $L$ is a closed and invariant subspace of $(\ker \bar{\partial}_{0,0})^\perp$, then $\bar{\partial}|_L : L \to \overline{\partial L}$ is $G$-Fredholm.

**Proof.** Apply Lemma 3.9 to Corollary 6.9. □

**Corollary 6.11.** For any closed, invariant $L \subset L^2(M, \Lambda^{0,0})$, we have that $\bar{\partial}L$ is almost closed.

**Proof.** Consider $L \cap (\ker \bar{\partial}_{0,0})^\perp$. Then $\bar{\partial}(L \cap (\ker \bar{\partial}_{0,0})^\perp)$ is $G$-dense in $\overline{\partial L}$. □
7. Appendix

Here we derive an a priori estimate for the Laplacian \( \Box \) by modifying some lemmata from [FK]. To that end, we repeat some of their definitions.

**Definition 7.1.** Denote by \( D^{p,q} \) the domain of the formal adjoint \( \vartheta \) of \( \bar{\partial} \) in \( C_c^\infty(\bar{M}, \Lambda^{p,q}) \).

**Definition 7.2.** A special boundary chart \( U \) is a chart intersecting \( bM \) having the following properties:

1. With \( \rho \) the function defining \( bM \), the functions \( t_1, \ldots, t_{2n-1}, \rho \) form a coordinate system on \( U \).
2. The coordinates \( \{t_1, \ldots, t_{2n-1}\}_\rho = 0 \) form a coordinate system on \( bM \cap U \).
3. Having chosen a Riemannian structure in the cotangent bundle, we choose a local orthonormal basis \( \omega_1, \ldots, \omega_n \) for \( \Lambda^{1,0}(\bar{M}) \) such that \( \omega_n = \sqrt{2} \partial \rho \) on \( U \).

With the tangential Fourier transform in a special boundary chart

\[
\hat{u}(\tau, \rho) = \frac{1}{(2\pi)^{(2n-1)/2}} \int_{\mathbb{R}^{2n-1}} e^{-i\langle t, \tau \rangle} u(t, \rho) dt,
\]

define for \( s \in \mathbb{R} \), the operators

\[
\Lambda^s u(t, \rho) = \frac{1}{(2\pi)^{(2n-1)/2}} \int_{\mathbb{R}^{2n-1}} e^{i(t, \tau)} (1 + |\tau|^2)^{s/2} \hat{u}(\tau, \rho) d\rho d\tau
\]

(t means tangential) and define the tangential Sobolev norms by

\[
|||u|||^2_s = \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^0 (1 + |\tau|^2)^s |\hat{u}(\tau, \rho)|^2 d\rho d\tau.
\]

With \( D^j = D^j_t = \frac{1}{t} \frac{\partial}{\partial t_j} \) for \( j = 1, \ldots, 2n-1 \) the derivatives in tangential directions and \( D^{2n} = D_\rho \), define the norms

\[
|||D u|||^2_s = \sum_{j=1}^{2n} |||D^j u|||^2_s + \sum_{j=1}^{2n} |||D^j u|||^2_s + \sum_{j=1}^{2n} \sum_{k=1}^{2n} |||D^j u|||^2_s + \sum_{j=1}^{2n} \sum_{k=1}^{2n} \sum_{l=1}^{2n} |||D^j u|||^2_s.
\]

In order to state the basic estimate, we need the quantity

\[
E(u)^2 = \sum_{j,k} |||\partial_\rho u_j|||^2 + \int_{bM} |u|^2 + |||u|||^2.
\]

**Definition 7.3.** That the basic estimate is satisfied means that there exists a \( C > 0 \) such that \( E(u)^2 \leq CQ(u, u) \) uniformly for \( u \) in \( D^{0,1} \). We will abbreviate this and similar estimates.
If $M$ is strongly pseudoconvex, then the basic estimate holds in $D^{0,1}$ (Prop 2.1.4, [FK]) and in fact in all $D^{p,q}$ for which $q > 0$ (Corollary 3.2.12, [FK]).

We will systematically label sequences of real-valued, cutoff functions $(\zeta_k)_k \subset C_0^\infty(M)$ such that $\zeta_k|_{\text{supp}(\zeta_{k+1})} = 1$ for $k = 0, 1, 2, \ldots$.

**Lemma 7.4.** Let $U$ be a special boundary chart and let $\zeta, \zeta_0, \zeta_1$ be real-valued functions in $C_0^\infty(U)$ with $\zeta_1 = 1$ on $\text{supp}(\zeta)$ and $\zeta_0 = 1$ on $\text{supp}(\zeta_1)$. Then for $A = \zeta_1 \Lambda \zeta$ and for $A'$ the formal adjoint of $A$ with respect to the inner product on $L^2(M)$,

$$Q(Au, Au) - \text{Re} Q(u, \zeta_0 A' Au) = \mathcal{O}(|||Du|||_{2-1}^{2})$$

$$Q(\zeta u, \zeta u) - \text{Re} Q(u, \zeta_0 \zeta_0^2 u) = \mathcal{O}(||\zeta_0 u||^2),$$

uniformly for $u \in D^{p,q} \cap \Lambda^{p,q}_0(U \cap \bar{M})$.

**Proof.** These are simple consequences of the fact that the domain $D^{p,q}$ of $\vartheta$ is preserved under the application of a cutoff function (cf. 2.3.2 of [FK]) and lemmata 2.4.2 and 2.4.3 of [FK] applied to $\zeta_0 u$. \hfill $\square$

**Remark 7.5.** If we assume further that $u \in \text{Dom}(F)$, (cf. [FK], Prop. 1.3.5) we may write

$$Q(Au, Au) - \text{Re} \langle \zeta_0 Fu, A' Au \rangle = \mathcal{O}(|||D\zeta_0 u|||_{2-1}^{2})$$

$$Q(\zeta u, \zeta u) - \text{Re} \langle \zeta_0 Fu, \zeta \zeta_0^2 u \rangle = \mathcal{O}(||\zeta_0 u||^2).$$

It is in this localized form that 2.4.2, 2.4.3 of [FK] will be useful in our Lemma 7.7, a substantial modification of Lemma 2.4.6 of [FK]. We will need the following theorem (2.4.4 from [FK]) unchanged.

**Lemma 7.6.** For every $p \in bM$ there is a (small) special boundary chart $V$ containing $p$ such that $|||Du|||_{-1/2}^{2} \lesssim E(u)^2$ uniformly for $u \in \Lambda^{p,q}_0(V \cap \bar{M})$.

The following is our local replacement of Lemma 2.4.6 of [FK].

**Lemma 7.7.** Suppose the basic estimate holds in $D^{p,q}$. Let $V$ be a special boundary chart in which the conclusions of Lemma 7.6 hold, and let $(\zeta_k)_k$ be a sequence of real functions in $\Lambda^{0,0}_0(V \cap \bar{M})$ such that $\zeta_k = 1$ on $\text{supp} \zeta_{k+1}$. Then for each positive integer $k$,

$$|||D\zeta_k u|||_{(k-2)/2}^{2} \lesssim ||\zeta_0 Fu|||_{(k-2)/2}^{2} + ||\zeta_0 u||^2,$$

uniformly for $u \in \text{Dom}(F) \cap D^{p,q}$.
Proof. Assuming the basic estimate, using Lemma 7.6 and noting that multiplication by $\zeta$ preserves $\mathcal{D}^{p,q}$, we have

$$|||D\zeta_1u|||_{-1/2}^2 \lesssim Q(u, \zeta_1u), \quad u \in \mathcal{D}^{p,q} \cap \Lambda^0_{p,q}(V \cap \tilde{M}).$$

If we insert a real-valued cutoff function $\zeta_0$ equal 1 on the support of $\zeta_1$ and apply Lemma (7.4), to the form $\zeta_0u$ we have

$$|||D\zeta_1u|||_{-1/2}^2 \lesssim \Re Q(u, \zeta_0\zeta_1^2u) + \mathcal{O}(|||\zeta_0u|||^2).$$

Substituting (11) into (10) gives

$$\zeta_0 \langle Fu, \zeta_1^2u \rangle + \mathcal{O}(|||\zeta_0u|||^2) = \Re \langle \zeta_1Fu, \zeta_1u \rangle + \mathcal{O}(|||\zeta_0u|||^2).$$

Now, by the generalized Schwartz inequality, we have

$$|||D\zeta_1u|||_{-1/2}^2 \lesssim \Re \langle \zeta_1Fu, \zeta_1u \rangle + \mathcal{O}(|||\zeta_0u|||^2) \lesssim |||\zeta_1Fu|||_{-1/2}|||\zeta_1u|||_{1/2} + \mathcal{O}(|||\zeta_0u|||^2).$$

But for any $c > 0$ there exists a $C > 0$ sufficiently large so that

$$|||D\zeta_1u|||_{-1/2}^2 \lesssim C|||\zeta_1Fu|||_{-1/2}^2 + c|||\zeta_1u|||_{1/2}^2 + \mathcal{O}(|||\zeta_0u|||^2).$$

By the equivalence in (7), $|||\zeta_1u|||_{1/2} \leq |||D\zeta_1u|||_{-1/2}$, so

$$|||D\zeta_1u|||_{-1/2}^2 \lesssim |||\zeta_0Fu|||_{-1/2}^2 + |||\zeta_0u|||^2,$$

and we have shown that the lemma is true for $k = 1$. Assume the lemma true for $k - 1$ i.e.

$$(9) \quad |||D\zeta_{k-1}u|||_{(k-3)/2}^2 \lesssim |||\zeta_0Fu|||_{(k-3)/2}^2 + |||\zeta_0u|||^2.$$

We follow the proof of [FK] 2.4.6, citing intermediate results. Abbreviating $A_{-1/2}^{k-1/2} = \Lambda$ and $A = \zeta_1\Lambda\zeta_k$,

$$(10) \quad |||D\zeta_ku|||_{(k-2)/2}^2 \lesssim |||D\zeta_1\Lambda\zeta_ku|||_{-1/2}^2 + |||D\zeta_{k-1}u|||_{(k-3)/2}^2.$$

Substituting (11) into (10) gives

$$|||D\zeta_ku|||_{(k-2)/2}^2 \lesssim |||\zeta_1Fu|||_{(k-2)/2}^2 + |||D\zeta_{k-1}u|||_{(k-3)/2}^2.$$

Using the inductive hypothesis (9) yields

$$|||D\zeta_ku|||_{(k-2)/2}^2 \lesssim |||\zeta_1Fu|||_{(k-2)/2}^2 + |||\zeta_0Fu|||_{(k-3)/2}^2 + |||\zeta_0u|||^2.$$

Because of the support properties of the $\zeta_k$,

$$|||D\zeta_ku|||_{(k-2)/2}^2 \lesssim |||\zeta_0Fu|||_{(k-2)/2}^2 + |||\zeta_0u|||^2.$$

This implies

$$|||D\zeta_ku|||_{(k-2)/2}^2 \lesssim |||\zeta_0Fu|||_{(k-2)/2}^2 + |||\zeta_0u|||^2$$

for the following two reasons: First,

$$|||\zeta_0Fu|||_{(k-2)/2}^2 \lesssim |||\zeta_0u|||^2$$

since the latter differentiates in the normal direction and the former does not. Second, $|||\zeta_0Fu|||_{(k-3)/2}^2 \lesssim |||\zeta_0u|||^2$ obviously. \qed
Remark 7.8. Lemma 2.4.6 needs real modification if we are to obtain a local statement; cutting off naïvely:

\[ |||Dζu|||^2_{(k-2)/2} \lesssim |||ζ_1 Fu|||^2_{(k-2)/2} + ||ζ_1 Fu||^2 \]

is false! To see this, let \( g \) be a function with small support near the origin and choose \( ζ_1 \) so that \( ζ_1 g = 0 \). Furthermore, let \( u ∈ \ker(□)⊥ \) solve \( Fu = g \). Then the right-hand side of the inequality is zero while the left is not.

The following lemma corresponds to [FK] (2.4.8).

Lemma 7.9. Suppose the basic estimate holds in \( D^{p,q} \). Let \( V \) be a special boundary chart on which the conclusions of Lemma 7.6 hold. Let \( U ⊂ U_0 ⊂ V \), and choose a real \( ζ_1 ∈ \Lambda^0_0(V ∩ M) \) with \( ζ_1 = 1 \) on \( U \). Then for each real \( ζ ∈ \Lambda^0_0(V ∩ M) \), and each positive integer \( s \).

\[ |||ζu|||^2_{s+1} \lesssim ||ζ_0 Fu||^2_s + ||ζ_0 u||^2 \]

uniformly for \( u ∈ \text{Dom}(F) ∩ D^{p,q} \).

Proof. Induction on \( s \): For \( s = 0 \), set \( ζ = ζ_2 \) and apply the previous lemma with \( k = 2 \) and \( 0 = ζ_3 = ζ_4 = \ldots \)

\[ ||ζu||^2_1 \lesssim ||Dζ_1 u||^2_1 \lesssim ||ζ_0 Fu||^2 + ||ζ_0 u||^2. \]

Now assume the claim true for \( s - 1 \). Then

\[ ||ζu||^2_{s+1} \lesssim \sum_{|β|=s+1} ||D^βζu||^2 + ||ζ_0 u||^2 \]

\[ ||ζu||^2_{s+1} \lesssim \sum_{|β|=s+1} ||D^βζu||^2 + ||ζ_1 Fu||^2_{s-1} + ||ζ_0 u||^2 \]

(12)

\[ ||ζu||^2_{s+1} \lesssim \sum_{|β|=s+1} ||D^βζu||^2 + ||ζ_1 Fu||^2_{s} + ||ζ_0 u||^2, \]

so estimate \( ||D^βζu||^2 \) for \( |β| = s + 1 \). Construct a sequence of cutoffs \( \{ζ_k\}_{2s+1}^2 \) so that \( ζ = ζ_{2s+2} \) and \( ζ_k = 1 \) on \( \text{supp } ζ_{k+1} \). Then apply Lemma 7.7 with \( k = 2s + 2 \) and \( ζ_j = 0 \) for \( j > 2s + 2 \). Then

(13)

\[ ||D^βζu||^2 \lesssim |||Dζu||^2_s \lesssim ||ζ_1 Fu||^2_s + ||ζ_0 u||^2. \]

Thus we got part of the first term on the right of (12) estimated by the latter terms. For \( |β| = s \) we have

(14)

\[ ||D^βDζu||^2 \lesssim |||Dζu||^2_s \lesssim ||ζ_1 Fu||^2_s + ||ζ_0 u||^2. \]

It remains to estimate \( D^βDζu \) with \( |β| + m = s + 1 \), \( m ≥ 2 \). Follow FK back to p 34, equation (2.3.5). Here \( F \) is written in terms of differentiation with respect
to the coordinates of the special boundary chart:

\[ Fu = A_0 D^2_{\rho} u + \sum_{j=1}^{2n-1} A_j D^j_{\xi} D^0_{\rho} u + \sum_{j,k=1}^{2n-1} A_{j,k} D^j_{\xi} D^k_{\xi} u + B_0 D^0_{\rho} u + \sum_{j=1}^{2n-1} B_j D^j_{\xi} u + Cu. \]

Since \( F \) is an elliptic operator, the matrices \( A \) are invertible. Thus we may solve

\[ D^2_{\rho} u = -A^{-1}_0 \left[ -Fu + \sum_{j=1}^{2n-1} A_j D^j_{\xi} D^0_{\rho} u + \sum_{j,k=1}^{2n-1} A_{j,k} D^j_{\xi} D^k_{\xi} u + B_0 D^0_{\rho} u + \sum_{j=1}^{2n-1} B_j D^j_{\xi} u + Cu \right]. \]

Applying \( \zeta D^\beta D^m_{\rho} \) with \(|\beta| + m = s + 1, m \geq 2\) and inserting a cutoff \( \zeta_1 \), we obtain

\[ \zeta D^\beta D^m_{\rho} u = -\zeta D^\beta D^m_{\rho} A^{-1}_0 \left[ -\zeta_1 Fu + \sum_{j=1}^{2n-1} A_j D^j_{\xi} D^0_{\rho} u + \sum_{j,k=1}^{2n-1} A_{j,k} D^j_{\xi} D^k_{\xi} u + B_0 D^0_{\rho} u + \sum_{j=1}^{2n-1} B_j D^j_{\xi} u + Cu \right]. \]

As in Folland and Kohn, at this point an induction on \( m \) (commuting the \( \zeta \) through) gives that \( \zeta D^\beta D^m_{\rho} u \) is expressed in terms of derivatives of \( \zeta_1 Fu \) of order \( s - 1 \) (\( = |\beta| + m - 2 \)) and derivatives of \( \zeta u \) which have been previously estimated in \((13)\) and \((14)\).

\[ \square \]

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References

[A] Atiyah, M.F.: Elliptic Operators, Discrete Groups, and von Neumann Algebras, *Soc. Math. de France, Astérisque* 32-3 (1976) 43–72

[B] Breuer M.: Fredholm Theories in von Neumann Algebras I, II, *Math Annalen* 178, 1968, 243–254 & 180, 1969, 313–325

[BGV] Berline, N., Getzler, E., Vergne, M.: Heat kernels and Dirac operators, Grundlehren der Mathematischen Wissenschaften, vol. 298, Springer-Verlag, Berlin, 1992.

[Br] Brudnyi, A.: On Holomorphic \( L^2 \) functions on Coverings of Strongly Pseudoconvex Manifolds, arxiv.org/pdf/math.CV/0508237

[CMS] Coburn, L.A.; Moyer, R.D.; Singer, I.M.: \( C^* \)-Algebras of Almost-Periodic Pseudo-differential Operators, *Acta Math.* 130 (1973) 279–307

[CM] Connes A. & Moscovici H.: The \( L^2 \)-Index Theorem for Homogeneous Spaces of Lie Groups *Ann. of Math.*, 115, (1982), no. 2, 291–330

[E] Engliš, M.: Pseudolocal Estimates for \( \mathcal{J} \) on General Pseudoconvex Domains, *Indiana Univ. Math. J.*, 50, (2001) no 4. 1593–1607, and Erratum, to appear in *Indiana Univ. Math. J.*
[FK] Folland, G. B. & Kohn J. J.: The Neumann Problem for the Cauchy-Riemann Complex, 
*Ann. Math. Studies*, No. 75. Princeton University Press, Princeton, N.J. 1972

[FS] Fedosov, B. & Shubin, M.A.: The Index of Random Elliptic Operators I & II, *Mat. Sb. (N.S.)* 106(148) (1978) no. 1, 108–140, 144. & 106(148) (1978) no. 3, 455–483, 496.

[Gra] Grauert, H.: On Levi's Problem and the Imbedding of Real-Analytic Manifolds, *Ann. of Math.*, 68, (1958), 460–472

[GHS] Gromov, M., Henkin, G. & Shubin, M.: Holomorphic $L^2$ Functions on Coverings of Pseudoconvex Manifolds, *Geom. Funct. Anal.*, v. 8, no. 3, (1998), 552–585

[Gro] Gromov, M.: Curvature, Diameter, and Betti Numbers, *Comment. Math. Helv.*, 56, (1981), no. 2, 179–195

[K] Kohn, J. J.: Harmonic Integrals on Strongly Pseudoconvex Manifolds, I & II, *Ann. of Math.*, 78 (1963) 112–148 & 79 (1964) 450–472

[LM] Lions, J.L., Magenes, E.: *Non-Homogeneous Boundary Value Problems and Applications*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 181 Springer-Verlag, Berlin, 1972

[M] Margulis, G.A.: *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 17 Springer-Verlag, Berlin, 1991

[Mo] Morrey, C.B.: The Analytic Embedding of Abstract Real-Analytic Manifolds, *Ann of Math.*, 68 (1968) 159–201

[P] Pedersen, G.K.: *C*-Algebras and their Automorphism Groups*, London Mathematical Society Monographs 14, Academic Press, Inc., London-New York, 1979

[R] Roe, J.: An index theorem on open manifolds. I. II. *J. Differential Geom.* 27 (1988), no. 1, 87–113, 115–136.

[S] Shubin, M.A.: $L^2$ Riemann-Roch Theorem for Elliptic Operators. *Geom. Funct. Anal.*, 5 (1995) no. 2, 482–527

[S1] Shubin, M.A.: Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, 207:5, 35-108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).

[S2] Shubin, M.A.: *Von Neumann Algebras and $L^2$ Techniques in Geometry and Topology*, preprint

[T] Takesaki, M.: *Theory of Operator Algebras* vol I, Springer-Verlag, Berlin, 1979

[TCM] Todor, R., Chiose, I., Marinescu, G.: $L^2$-Riemann-Roch Inequalities for Covering Manifolds, arxiv.org/pdf/math.AG/0002049