EULERIAN POLYNOMIALS FOR SUBARRANGEMENTS OF WEYL ARRANGEMENTS

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ABSTRACT. Let $A$ be a Weyl arrangement. We introduce and study the notion of $A$-Eulerian polynomial producing an Eulerian-like polynomial for any subarrangement of $A$. This polynomial together with shift operator describe how the characteristic quasi-polynomial of a certain class of subarrangements of $A$ can be expressed in terms of the Ehrhart quasi-polynomial of the fundamental alcove. The method can also be extended to define two types of deformed Weyl subarrangements containing the families of the extended Shi, Catalan, Linial arrangements and to compute their characteristic quasi-polynomials. We obtain several known results in the literature as specializations, including the formula of the characteristic polynomial of $A$ via Ehrhart theory due to Athanasiadis (1996), Blass-Sagan (1998), Suter (1998) and Kamiya-Takemura-Terao (2010); and the formula relating the number of coweight lattice points in the fundamental parallelepiped with the Lam-Postnikov’s Eulerian polynomial due to the third author.

1. INTRODUCTION

Motivation. One of typical problems in enumerative combinatorics is to count the sizes of sets depending upon a positive integer $q$. This often gives rise to polynomials, for instance, the chromatic polynomial of an undirected graph, going back to Birkhoff and Whitney, encodes the number of ways of coloring the vertices with $q$ colors so that adjacent vertices get different colors. However, it may happen that enumerating the cardinalities of sets leads to quasi-polynomials. Generally speaking, a quasi-polynomial is a refinement of polynomials, of which the coefficients may not come from a ring but instead are periodic functions with integral period. Thus a quasi-polynomial is made of a bunch of polynomials, the constituents of the quasi-polynomial. One of the most classical examples in the theory is
that the number of integral points in the $q$-fold dilation of a rational polytope agrees with a quasi-polynomial in $q$, broadly known as the Ehrhart quasi-polynomial.

We are interested in the connection between the counting problems and the arrangement theory. A finite list (multiset) $\mathcal{A}$ of vectors in $\mathbb{Z}^d$ determines an arrangement $\mathcal{A}(\mathbb{R})$ of hyperplanes in the vector space $\mathbb{R}^d$, an arrangement $\mathcal{A}(S^1)$ of subtori in the torus $(S^1)^d$, and especially an arrangement $\mathcal{A}(\mathbb{Z}_q)$ of subgroups in the finite abelian group $\mathbb{Z}_q^d$. Enumerating the cardinality of the complement of $\mathcal{A}(\mathbb{Z}_q)$ produces a quasi-polynomial, the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ of $\mathcal{A}$ [KTT08]. This single quasi-polynomial encodes a number of combinatorial and topological information of several types of arrangements and has generated increasing interest recently (e.g., [CW12, BM14, Yos18a, Yos18b, TY19, Tra19]). Among the others, $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ has the first constituent identical with the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{R})}(t)$ of $\mathcal{A}(\mathbb{R})$ which justified its name (e.g., [Ath96, KTT08]), and the last constituent identical with the characteristic polynomial $\chi_{\mathcal{A}(S^1)}(t)$ of $\mathcal{A}(S^1)$ [LTY, TY19]. One of the methods used in [KTT08] for showing that $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ is indeed a quasi-polynomial is to express it as a sum of the Ehrhart quasi-polynomials of rational polytopes, or in the sense of [BZ06], as the Ehrhart quasi-polynomial of an “inside-out” polytope. Such an expression is certainly interesting as it reveals the connection between two seemingly unrelated quasi-polynomials, one would hope for a more explicit expression if the list $\mathcal{A}$ was chosen to be a more special vector configuration.

Objective. A particularly well-behaved class of the hyperplane arrangements is that of Weyl arrangements. More precisely, if $\mathcal{A} = \Phi^+$ is a positive system of an irreducible root system $\Phi$, then $\mathcal{A}(\mathbb{R})$ is called the Weyl arrangement of $\mathcal{A}$. It is proved that $\chi_{\Phi^+}^{\text{quasi}}(q)$ is expressed in terms of the Ehrhart quasi-polynomial $L_{A^\circ}(q)$ of the fundamental alcove $A^\circ$, the Weyl group, and the index of connection of $\Phi$ (e.g., [Ath96, BS98, Sut98, KTT10]). Thus we arrive at a natural and essential problem that for which subset $\Psi \subseteq \Phi^+$, $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ can be computed by using the fundamental invariants of $\Phi$, and more importantly, by means of the Ehrhart quasi-polynomials. Some partial results are known. If the root system $\Phi$ is of classical type and $\Psi$ is an ideal of $\Phi^+$, then $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ can be computed from information of the signed graph associated with $\Psi$ [Tra19]. A result due to [Yos18b] applied to any root system, asserts that $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ (or simply $q^\ell$) can be written in terms of the Lam-Postnikov’s Eulerian polynomial [LP18], shift operator, and $L_{A^\circ}(q)$.

Results. Inspired by the works of [LP18] and [Yos18b], we introduce the notion of $\mathcal{A}$-Eulerian polynomial $E_\mathcal{A}(t)$ - an arrangement theoretical generalization of the classical Eulerian polynomial, and the notion of compatible
subsets of $\Phi^+$ with respect to the Worpitzky partition. We prove that when $\Psi \subseteq \Phi^+$ is compatible, $\chi^\text{quasi}_\Psi(q)$ can be expressed in terms of $E_\Psi(t)$, shift operator, and $L_{A^\circ}(q)$. The formula specializes correctly to the two formulas in the extreme cases ($\Psi = \Phi^+$ and $\Psi = \emptyset$) mentioned above. Using the similar methods, we further define two types of deformed Weyl subarrangements containing the families of the extended Shi, Catalan, Linial arrangements and compute their characteristic quasi-polynomials.

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2, we recall definitions and basic facts of the irreducible root systems, their (affine) Weyl groups and the Worpitzky partition. In Section 3, we recall the definitions of the characteristic and Ehrhart quasi-polynomials and specify the choices of lattices for these quasi-polynomials (Remarks 3.4 and 3.6). We also recall the formula between the quasi-polynomials in the extreme case $\Psi = \Phi^+$ (Theorem 3.8), and derive a generalization of it (Proposition 3.11). In Section 4, we introduce the notion of $A$-Eulerian polynomial (Definition 4.2), of which the main specialization is the Lam-Postnikov’s Eulerian polynomial (Remark 4.3). We then define the notion of compatible sets (Definition 4.8) that interpolates between the two extreme cases $\Psi = \Phi^+$ and $\Psi = \emptyset$. We prove that the $A$-Eulerian polynomial is an essential tool to compute $\chi^\text{quasi}_\Psi(q)$ for any compatible subset $\Psi \subseteq \Phi^+$ (Theorem 4.11) and its generating function (Theorem 4.14). In Section 5, we define two types of the deformed Weyl subarrangements (Definition 5.1), which the main examples are the truncated affine Weyl and deleted Shi arrangements (Remark 5.2). Then we compute the characteristic quasi-polynomials of these deformed arrangements according to two special choices of intervals (Theorem 5.6 and 5.8).

2. **ROOT SYSTEMS AND WORPITZKY PARTITION**

Our standard references for root systems and their Weyl groups is [Hum90]. Assume that $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot, \cdot)$. Let $\Phi$ be an irreducible (crystallographic) root system in $V$ with the Coxeter number $h$ and the Weyl group $W$. Fix a positive system $\Phi^+ \subseteq \Phi$ and let $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple roots (base) of $\Phi$ associated with $\Phi^+$. The highest root, denoted by $\tilde{\alpha} \in \Phi^+$, can be expressed uniquely as a linear combination $\tilde{\alpha} = \sum_{i=1}^\ell c_i \alpha_i \ (c_i \in \mathbb{Z}_{\geq 0})$. Set $\alpha_0 := -\tilde{\alpha}$, $c_0 := 1$, and $\tilde{\Delta} := \Delta \cup \{\alpha_0\}$. Then we have the linear relation

$$c_0 \alpha_0 + c_1 \alpha_1 + \cdots + c_\ell \alpha_\ell = 0.$$  

The coefficients $c_i$ are important in our study and will appear frequently throughout the paper.
For \( \alpha \in V \setminus \{0\} \), denote \( \alpha' := \tfrac{2\alpha}{(\alpha,\alpha)} \). The root lattice \( Q(\Phi) \), coroot lattice \( Q(\Phi') \), weight lattice \( Z(\Phi) \), and coweight lattice \( Z(\Phi') \) are defined as follows:

\[
Q(\Phi) := \bigoplus_{i=1}^{\ell} \mathbb{Z} \alpha_i,
\]

\[
Q(\Phi') := \bigoplus_{i=1}^{\ell} \mathbb{Z} \alpha_i',
\]

\[
Z(\Phi) := \left\{ x \in V \mid (\alpha', x) \in \mathbb{Z} (1 \leq i \leq \ell) \right\},
\]

\[
Z(\Phi') := \left\{ x \in V \mid (\alpha_i, x) \in \mathbb{Z} (1 \leq i \leq \ell) \right\}.
\]

Then \( Q(\Phi) \) is a subgroup of \( Z(\Phi) \) of finite index \( f \), and similarly \( Q(\Phi') \) is a subgroup of \( Z(\Phi') \) of index \( f \). The number \( f \) is called the index of connection. Let \( \{ \alpha_1', \ldots, \alpha_\ell' \} \subseteq Z(\Phi') \) be the dual basis of the base \( \Delta \), namely, \( (\alpha_i, \alpha_j') = \delta_{ij} \). Then \( Z(\Phi') = \bigoplus_{i=1}^{\ell} \mathbb{Z} \alpha_i' \) and \( c_i = (\alpha_i', \alpha) \).

For \( m \in \mathbb{Z} \) and \( \alpha \in \Phi \), the affine hyperplane \( H_{\alpha,m} \) is defined by

\[
H_{\alpha,m} := \left\{ x \in V \mid (\alpha, x) = m \right\}.
\]

A connected component of \( V \setminus \bigcup_{\alpha \in \Phi^+, m \in \mathbb{Z}} H_{\alpha,m} \) is called an alcove. The fundamental alcove \( A^0 \) is defined by

\[
A^0 := \left\{ x \in V \mid (\alpha_i, x) > 0 (1 \leq i \leq \ell), (\alpha_0, x) > -1 \right\}.
\]

The closure \( \overline{A^0} = \left\{ x \in V \mid (\alpha_i, x) \geq 0 (1 \leq i \leq \ell), (\alpha_0, x) \geq -1 \right\} \) is a simplex, which is the convex hull of \( 0, \overline{\alpha}_1', \ldots, \overline{\alpha}_\ell' \). The supporting hyperplanes of the facets of \( \overline{A^0} \) are \( H_{\alpha_1,0}, \ldots, H_{\alpha_\ell,0}, H_{\alpha_0,-1} \). The affine Weyl group \( W_{\text{aff}} := W \ltimes Q(\Phi') \) acts simply transitively on the set of alcoves and admits \( \overline{A^0} \) as a fundamental domain for its action on \( V \).

The fundamental domain \( P^\diamond \) of the coweight lattice \( Z(\Phi') \), called the fundamental parallelepiped, is defined by

\[
P^\diamond := \sum_{i=1}^{\ell} (0, 1] \overline{\alpha}_i' = \left\{ x \in V \mid 0 < (\alpha_i, x) \leq 1 (1 \leq i \leq \ell) \right\}.
\]

Let \( N := \#W / \ell \), and denote \( [N] := \{1, 2, \ldots, N\} \). Then the cardinality of the set \( \Sigma \) of all alcoves contained in \( P^\diamond \) equals \( N \) (see, e.g., [Hum90, Theorem 4.9]). Let us write

\[
\Sigma = \left\{ A_i^0 \subseteq P^\diamond \mid i \in [N] \right\},
\]
where each alcove \( A^\circ_i \) is written uniquely as
\[
A^\circ_i = \left\{ x \in V \left| \begin{array}{l}
(\alpha, x) > k_\alpha (\alpha \in I), \\
(\beta, x) < k_\beta (\beta \in J)
\end{array} \right. \right\},
\]
where \( k_\alpha \in \mathbb{Z}_{\geq 0}, k_\beta \in \mathbb{Z}_{> 0} \) and the sets \( I, J \subseteq \Phi^+ \) with \( \#(I \cup J) = \ell + 1 \) indicate the constraints on \( x \in V \) according to the inequality symbols \( >, \leq \), respectively.

**Definition 2.1.** For each \( A^\circ_i \in \Sigma \), the partial closure \( A^\Diamond_i \) is defined by
\[
A^\Diamond_i := \left\{ x \in V \left| \begin{array}{l}
(\alpha, x) > k_\alpha (\alpha \in I), \\
(\beta, x) \leq k_\beta (\beta \in J)
\end{array} \right. \right\}.
\]

**Theorem 2.2 (Worpitzky partition).**
\[
P^\Diamond = \bigsqcup_{i \in [N]} A^\Diamond_i.
\]

*Proof.* See, e.g., [Yos18b, Proposition 2.5], [Hum90, Exercise 4.3]. \( \Box \)

3. Characteristic and Ehrhart quasi-polynomials

A function \( g : \mathbb{Z} \to \mathbb{C} \) is called a quasi-polynomial if there exist \( \rho \in \mathbb{Z}_{>0} \) and polynomials \( f^k(t) \in \mathbb{Z}[t] \) \( (1 \leq k \leq \rho) \) such that for any \( q \in \mathbb{Z}_{>0} \) with \( q \equiv k \mod \rho \),
\[
g(q) = f^k(q).
\]
The number \( \rho \) is called a period and the polynomial \( f^k(t) \) is called the \( k \)-constituent of the quasi-polynomial \( g \).

Let \( \Gamma := \bigoplus_{i=1}^\ell \mathbb{Z} \beta_i \simeq \mathbb{Z}^\ell \) be a lattice. Let \( \mathcal{L} \) be a finite list (multiset) of elements in \( \Gamma \). Let \( q \in \mathbb{Z}_{>0} \) and denote \( \mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z} \). For \( \alpha = \sum_{i=1}^\ell a_i \beta_i \in \mathcal{L} \) and \( m_\alpha \in \mathbb{Z} \), define a subset \( H_{\alpha,m_\alpha,z_q} \) of \( \mathbb{Z}^\ell \) by
\[
H_{\alpha,m_\alpha,z_q} := \left\{ z \in \mathbb{Z}^\ell \left| \sum_{i=1}^\ell a_iz_i \equiv m_\alpha \mod q \right. \right\}.
\]

Given a vector \( m = (m_\alpha)_{\alpha \in \mathcal{L}} \in \mathbb{Z}^\mathcal{L} \), we can define the \( \mathbb{Z}_q \)-plexification (or \( q \)-reduced) arrangement of \( (\mathcal{L}, m) \) by
\[
(\mathcal{L}, m)(\mathbb{Z}_q) := \{ H_{\alpha,m_\alpha,z_q} \mid \alpha \in \mathcal{L} \}.
\]
The complement of \( (\mathcal{L}, m)(\mathbb{Z}_q) \) is defined by
\[
\mathcal{M}( (\mathcal{L}, m); \Gamma; \mathbb{Z}_q ) := \mathbb{Z}^\ell_q \setminus \bigcup_{\alpha \in \mathcal{L}} H_{\alpha,m_\alpha,z_q}.
\]
For each \( S \subseteq \mathcal{L} \), write \( \Gamma/(S) \simeq \bigoplus_{i=1}^n \mathbb{Z} d_{S,i} \oplus \mathbb{Z}^{n-\ell} \) where \( n_S \geq 0 \) and \( 1 < d_{S,i} | d_{S,i+1} \). The LCM-period \( \rho_{\mathcal{L}} \) of \( \mathcal{L} \) is defined by
\[
\rho_{\mathcal{L}} := \text{lcm}(d_{S,n_S} \mid S \subseteq \mathcal{L}).
\]
Theorem 3.1. \#\(M((\mathcal{L}, m); \Gamma, \mathbb{Z}_q)\) is a monic quasi-polynomial in \(q\) for which \(\rho_\mathcal{L}\) is a period. The quasi-polynomial is called the characteristic quasi-polynomial of \((\mathcal{L}, m)\) with respect to the lattice \(\Gamma\), and denoted by \(\chi_{\mathcal{L},m}^{\text{quasi}}(q)\).

Proof. See [KTT08, Theorem 2.4] and [KTT11, Theorem 3.1].

We can also define the \(\mathbb{R}\)-plexification (in fact, the real hyperplane arrangement) of \((\mathcal{L}, m)\) as follows:

\((\mathcal{L}, m)(\mathbb{R}) := \{H_\alpha, m \mid \alpha \in \mathcal{L}\}\),

where \(H_\alpha, m := \{x \in \mathbb{R}^\ell \mid \sum_{i=1}^\ell a_i x_i = m_\alpha\}\). For a real hyperplane arrangement \(A\), denote by \(\chi_A(t)\) the characteristic polynomial (e.g., [OT92, Definition 2.52]) of \(A\).

Theorem 3.2. The first constituent of \(\chi_{\mathcal{L},m}^{\text{quasi}}(q)\) coincides with the characteristic polynomial of \((\mathcal{L}, m)(\mathbb{R})\), i.e.,

\(f_{\mathcal{L},m}^1(t) = \chi_{(\mathcal{L}, m)(\mathbb{R})}(t)\).

Proof. See, e.g., [KTT08, Theorem 2.5] and [KTT11, Remark 3.3].

Convention: When \(m = (0)\) the zero vector, we simply write \(\mathcal{L}, H_\alpha, \mathbb{Z}_q, H_\alpha, \mathbb{R}\) instead of \((\mathcal{L}, m)\), \(H_\alpha, 0, \mathbb{Z}_q, H_\alpha, 0, \mathbb{R}\), respectively.

Remark 3.3. \(\mathcal{L}(\mathbb{Z}_q)\) and \(\mathcal{L}(\mathbb{R})\) are examples of a more abstract concept, the \(G\)-plexifications (\(G\) is an abelian group) introduced in [LTY] (see also [TY19] for more information on their combinatorial properties) by viewing \(G = \mathbb{Z}_q, \mathbb{R}\), respectively. In addition, the characteristic quasi-polynomial can be generalized to the chromatic quasi-polynomial by replacing the lattice \(\Gamma\) by a finitely generated abelian group, see, e.g., [Tra18] for more details.

Remark 3.4. Throughout the paper, for every \(\Psi \subseteq \Phi^+ \subseteq Q(\Phi)\), the characteristic quasi-polynomial \(\chi_{\Psi}^{\text{quasi}}(q)\) is always defined with respect to the root lattice \(\Gamma = Q(\Phi)\).

For each \(\Psi \subseteq \Phi^+\), define the Weyl arrangement of \(\Psi\) by \(A_\Psi := \{H_\alpha \mid \alpha \in \Psi\}\), where \(H_\alpha = \{x \in V \mid (\alpha, x) = 0\}\) is the hyperplane orthogonal to \(\alpha\). It is not hard to see that \(H_\alpha \simeq H_{\alpha, \mathbb{R}}\) (as vector spaces), thus we can view \(A_\Psi\) as the \(\mathbb{R}\)-plexification of \(\Psi\), i.e., \(A_\Psi = \Psi(\mathbb{R})\). In standard terminology, \(A_{\Phi^+}\) is known with the name Weyl (or Coxeter) arrangement, and clearly \(A_\Psi\) is a (Weyl) subarrangement of \(A_{\Phi^+}\).

Remark 3.5. Sometimes, when we say “the” characteristic quasi-polynomial of \((\mathcal{L}, m)(\mathbb{Z}_q)\) or of \((\mathcal{L}, m)(\mathbb{R})\), we literally mean \(\chi_{\mathcal{L},m}^{\text{quasi}}(q)\). In particular,
χ_ψ^{quasi}(q) will be referred to the characteristic quasi-polynomial χ_ψ^{quasi}(A_ψ) of A_ψ. We will use this notation later in Section 5 when we deal with deformed Weyl arrangements of Ψ.

Let Γ be a lattice. For a polytope P with vertices in the rational vector space generated by Γ, the Ehrhart quasi-polynomial L_P(q) of P with respect to Γ is defined by

\[ L_P(q) := \#(qP \cap \Gamma) \]

We denote by P^\circ the relative interior of P. Similarly, we can define

\[ L_{P^\circ}(q) := \#(qP^\circ \cap \Gamma) \]

For q > 0, the following reciprocity law holds:

\[ L_P(-q) = (-1)^{\dim P} L_{P^\circ}(q) \]

Remark 3.6. Throughout the paper, the Ehrhart quasi-polynomials L_{A^\circ}(q), L_{A^\circ}(q) are defined with respect to the coweight lattice Γ = Z(Φ^∨).

Let F_0 := H_{α_0,-q}, F_i := H_{α_i,0} (1 ≤ i ≤ ℓ) denote the supporting hyperplanes of the facets of qA^∞. Then the number of coweight lattice points in qA^∞ after removing some facets can be computed by L_{A^\circ} with the scale factor of dilation being reduced.

Proposition 3.7. Let \{i_1, \ldots, i_k\} ⊆ \{0, 1, \ldots, ℓ\}. Suppose that q > c_{i_1} + \cdots + c_{i_k}. Then

\[ \#(qA^∞ \cap Z(Φ^∨) \setminus (F_{i_1} \cup \cdots \cup F_{i_k})) = L_{A^\circ}(q - (c_{i_1} + \cdots + c_{i_k})) \]

In particular, for q ∈ ℤ,

\[ L_{A^\circ}(q) = L_{A^\circ}(q - h) \]

Proof. See [Yos18b, Corollaries 3.4 and 3.5].

In general, it is not easy to find explicit formulas which involve both characteristic and Ehrhart quasi-polynomials. With regards to root systems, there is an interesting relation between these quasi-polynomials.

Theorem 3.8.

χ_ψ^{quasi}(q) = \#W/f L_{A^\circ}(q).

Proof. See, e.g., [KTT10], [Yos18b, Proposition 3.7].

Theorem 3.9. The minimum period of χ_ψ^{quasi}(q) is equal to lcm(c_1, \ldots, c_ℓ). Furthermore, by a case-by-case argument, we can verify that the minimum period coincides with the LCM-period ρ_ψ^+.

Proof. See [KTT10, Corollary 3.2 and Remark 3.3].
Corollary 3.10. \( \chi_{\Phi^+}^{\text{quasi}} (q) > 0 \) (equivalently, \( L_{A^v}(q) > 0 \)) if and only if \( q \geq h \).

Proof. See, e.g., \([KTT10, \text{Corollary 3.4}]\). \( \square \)

We can generalize the result above to have a formula for \( \chi_{\Psi}^{\text{quasi}} (q) \) for any \( \Psi \subseteq \Phi^+ \) in terms of lattice point counting functions. It will also explain why the choice of lattices for the characteristic and Ehrhart quasi-polynomials is important. In the proposition below, we view \( \Psi \) as a list with possible repetitions of elements.

Proposition 3.11. Let \( m = (m_\alpha)_{\alpha \in \Psi} \) be a vector in \( \mathbb{Z}^\Psi \). Set
\[
X_{(\Psi, m)} (q) := qP^\circ \cap Z(\Phi^\vee) \setminus \bigcup_{\alpha \in \Psi, k \in \mathbb{Z}} H_{\alpha, k q + m_\alpha},
\]
\[
Y_{(\Psi, m)} (q) := \{ \pi \in Z(\Phi^\vee)/qZ(\Phi^\vee) \mid (\alpha, x) \not\equiv m_\alpha \mod q, \forall \alpha \in \Psi \}.
\]
We have bijections between sets
\[
X_{(\Psi, m)} (q) \simeq Y_{(\Psi, m)} (q) \simeq M((\Psi, m); \mathbb{Z}^\ell, \mathbb{Z}_q).
\]
As a result,
\[
\chi_{(\Psi, m)}^{\text{quasi}} (q) = \#X_{(\Psi, m)} (q) = \#Y_{(\Psi, m)} (q).
\]

Proof. The bijection \( X_{(\Phi^+, m)} (q) \simeq Y_{(\Phi^+, m)} (q) \) is proved in \([Yos18b, \text{§3.3}]\). We can use exactly the same argument applied to any \( \Psi \). The proof of \( Y_{\Psi} (q) \simeq M(\Psi; \mathbb{Z}^\ell, \mathbb{Z}_q) \) for an arbitrary \( \Psi \subseteq \Phi^+ \) runs as follows:
\[
Y_{(\Psi, m)} (q) = \{ \pi = \sum_{i=1}^\ell z_i \omega_i^\vee \mid (\alpha, x) \not\equiv m_\alpha \mod q, \forall \alpha \in \Psi \}
\]
\[
= \{ \pi = \sum_{i=1}^\ell z_i \omega_i^\vee \mid (\sum_{i=1}^\ell S_{ij} \alpha_i, \sum_{i=1}^\ell z_i \omega_i^\vee) \not\equiv m_\alpha \mod q, (1 \leq j \leq \#\Psi) \}
\]
\[
\simeq \{ z = (z_1, \ldots, z_\ell) \in \mathbb{Z}_q^\ell \mid \sum_{i=1}^\ell z_i S_{ij} \not\equiv m_\alpha \mod q, (1 \leq j \leq \#\Psi) \}
\]
\[
= M((\Psi, m); \mathbb{Z}^\ell, \mathbb{Z}_q).
\]

Remark 3.12. The bijection \( X_{\Phi^+} (q) \simeq M(\Phi^+; \mathbb{Z}^\ell, \mathbb{Z}_q) \) appeared (without proof) in \([KTT10, \text{Proof of Theorem 3.1}]\). Theorem 3.8 is a special case of Proposition 3.11 because \( \chi_{\Phi^+}^{\text{quasi}} (q) = \#X_{\Phi^+} (q) = \#M_{\Phi^+} (q) \) [Yos18b, \text{§3.3}].
4. Eulerian polynomials for Weyl subarrangements

Let $\Psi \subseteq \Phi +$, and set $\Psi^c := \Phi + \setminus \Psi$.

**Definition 4.1.** The descent $dsc_\Psi$ with respect to $\Psi$ is the function $dsc_\Psi : W \to \mathbb{Z}_{\geq 0}$ defined by

$$dsc_\Psi(w) := \sum_{0 \leq i \leq \ell, w(\alpha_i) \notin -\Psi^c} c_i.$$

**Definition 4.2.** The (arrangement theoretical Eulerian or) $A$-Eulerian polynomial of $\Psi$ is defined by

$$E_\Psi(t) := \frac{1}{f} \sum_{w \in W} t^{-dsc_\Psi(w)}.$$

**Remark 4.3.**

(a) If $\Psi = \Phi +$, then $dsc_{\Phi +}(w) = 0$ for all $w \in W$, and $E_{\Phi +}(t) = \frac{#W}{f} t^h$.

(b) If $\Psi = \emptyset$, then $dsc = cdes$, the descent statistic (see, e.g., [LP18, Definition 6.2], [Yos18b, Definition 4.1]). Then $E_{\emptyset}(t) = R_{\emptyset}(t)$, the generalized Eulerian polynomial (e.g., [Yos18b, Definition 4.4]). Note that if $\Phi$ is of type $A_\ell$, then $R_{\Phi}(t) = A_\ell(t)$, the classical $\ell$-th Eulerian polynomial [LP18, Theorem 10.1].

**Lemma 4.4.** For all $w \in W$, $0 \leq dsc_\Psi(w) < h$. In particular, $E_\Psi(0) = 0$.

**Proof.** If $\Psi = \Phi +$, then the statements are clearly true by Remark 4.3(a). Assume that $\Psi \neq \Phi +$. If $w(\alpha_i) \notin -\Psi^c$ for some $1 \leq i \leq \ell$, then $dsc_\Psi(w) < h$. Otherwise, we have $w(\alpha_0) = -\sum_{i=1}^{\ell} c_i w(\alpha_i) \in \Phi +$. Thus $w(\alpha_0) \notin -\Psi^c$, and hence $dsc_\Psi(w) < h$. □

**Lemma 4.5.**

(i) Let $w \in W$. Suppose that $w$ induces a permutation on $\tilde{\Delta}$. If $w(\alpha_i) = \alpha_p$, then $c_i = c_p$.

(ii) Let $w_1, w_2 \in W$. If there exists $\gamma \in V$ such that $w_1(A^0) = w_2(A^0) + \gamma$, then $dsc_\Psi(w_1) = dsc_\Psi(w_2)$.

**Proof.** (i) is exactly [Yos18b, Lemma 4.3(1)], and (ii) is similar to [Yos18b, Lemma 4.3(2)]. □

Let $A'$ be an arbitrary alcove. We can write $A' = w(A^0) + \gamma$ for some $w \in W$ and $\gamma \in Q(\Phi^\vee)$. By Lemma 4.5, we can extend $dsc_\Psi$ to a function on the set of all alcoves (in particular, on the set $\Sigma$ of alcoves contained in $P^\Phi$) as follows:

**Definition 4.6.**

$$dsc_\Psi(A') := dsc_\Psi(w).$$
Theorem 4.7.
\[ E_\psi(t) = \sum_{i \in [N]} t^{h - \text{dsc}_\psi(A_i^\diamondsuit)}. \]

Proof. Similar to [Yos18b, Theorem 4.7]. \qed

In what follows, we shall sometimes abuse terminology and call a face of \( \overline{A_i^\diamondsuit} \) (resp., \( \overline{P^\diamondsuit} \)) that is contained in the partial closure \( A_i^\diamondsuit \) (resp., \( P^\diamondsuit \)), a face of \( A_i^\diamondsuit \) (resp., \( P^\diamondsuit \)).

Definition 4.8. A subset \( \Psi \subseteq \Phi^+ \) is said to be compatible (with the Wor-pitzky partition) if for each \( A_i^\diamondsuit \subseteq P^\diamondsuit \) the following condition holds: if \( A_i^\diamondsuit \cap H_{\alpha,m_\alpha} \neq \emptyset \) for each \( \alpha \in \Psi, m_\alpha \in \mathbb{Z} \), then there exist \( \beta \in \Psi, m_\beta \in \mathbb{Z} \) such that \( A_i^\diamondsuit \cap H_{\alpha,m_\alpha} \subseteq A_i^\diamondsuit \cap H_{\beta,m_\beta} \) and \( A_i^\diamondsuit \cap H_{\beta,m_\beta} \) is a facet of \( A_i^\diamondsuit \).

Example 4.9. Clearly, \( \Psi \) is compatible if \( \Psi = \emptyset \), or \( \Psi = \Phi^+ \). If \( \Psi = \{ \tilde{\alpha} \} \), where \( \tilde{\alpha} \) is the highest root, then \( \Psi \) is not compatible because \( H_{\tilde{\alpha},\text{ht}(\tilde{\alpha})} \) intersects with \( P^\diamondsuit \) only at \( x = \sum_{i=1}^\ell \omega_i^\gamma \).

Definition 4.10. Let \( f : \mathbb{Z} \to \mathbb{C} \) be a function and let \( P(S) = \sum_{k=0}^n a_k S^k \) be a polynomial in \( S \). The shift operator via \( P(S) \) acting on \( f \) is defined by
\[ (P(S)f)(t) := \sum_{k=0}^n a_k f(t - k). \]

Theorem 4.11. If \( \Psi \) is compatible, then
\[ \chi_{\Psi}^{\text{quasi}}(q) = (E_\psi(S) L_{\overline{A_i^\gamma}})(q). \]

Proof. The proof is similar in spirit to [Yos18b, Proof of Theorem 4.8]. Since both sides are quasi-polynomials, it is sufficient to prove the formula for \( q \gg 0 \) (actually, \( q > h \) is sufficient). For \( i \in [N] \), we have
\[
\begin{aligned}
\# \left( qA_i^\diamondsuit \cap Z(\Phi^\vee) \right) &\setminus \mathbf{U}_{\mu \in \Psi,k \in \mathbb{Z}} H_{\mu,kq} \\
= &\# \left\{ x \in Z(\Phi^\vee) \bigg| \begin{array}{l}
(\alpha, x) > qk_\alpha (\alpha \in I) \\
(\beta, x) < qk_\beta (\beta \in J \cap \Psi) \\
(\delta, x) \leq qk_\delta (\delta \in J \cap \Psi^c)
\end{array} \right\} \\
= &L_{\overline{A_i^\gamma}}(q - h + \text{dsc}_\psi(A_i^\diamondsuit)).
\end{aligned}
\]

Note that we used Definition 2.1 of \( A_i^\diamondsuit \) in the first equality. We applied Proposition 3.7 and Definition 4.6 in the second equality. More precisely, if \( A_i^\diamondsuit = w(A^\diamondsuit) + \gamma \) for some \( w \in W \) and \( \gamma \in Q(\Phi^\vee) \), then \( qA_i^\diamondsuit \) can be written as
\[
qA_i^\diamondsuit = \left\{ x \in V \bigg| \begin{array}{l}
(-w(\alpha_0), x) < q((-w(\alpha_0), \gamma) + 1), \\
(-w(\alpha_i), x) < q(-w(\alpha_i), \gamma), (1 \leq i \leq \ell)
\end{array} \right\}.
\]
Thus the half-spaces defined by \((\delta, x) \leq qk_\delta (\delta \in J \cap \Psi^c)\) correspond exactly to the roots \(\alpha_i \in \tilde{\Delta}\) satisfying \(w(\alpha_i) \in -\Psi^c\). The compatibility of \(\Psi\) is needed when we apply Proposition 3.7 because it ensures that if lattice points in \(qA_i^\diamond\) are being removed, then these lattice points must belong to a facet of \(qA_i^\diamond\) in which the lattice points are also being removed.

By the Worpitzky partition (Theorem 2.2) and Proposition 3.11, we have

\[
\chi_{\Psi}^\text{quasi}(q) = \sum_{i \in [N]} \# \left( qA_i^\diamond \cap Z(\Phi^\vee) \setminus \bigcup_{\mu \in \Psi, k \in \mathbb{Z}} H_{\mu,kq} \right)
= \sum_{i \in [N]} L_{A_i^\diamond}(q - h + dsc_\Psi(A_i^\diamond)).
\]

Now using Theorem 4.7 together with the shift operator (Definition 4.10), we conclude that

\[
\chi_{\Psi}^\text{quasi}(q) = (E_\Psi(S)L_{A_i^\diamond})(q).
\]

\[\square\]

**Proposition 4.12.**

\[
\sum_{q \geq 1} L_{A_i^\diamond}(q) t^q = \frac{t^h}{\prod_{i=0}^t (1 - e_i^\Psi)}.
\]

**Proof.** See, e.g., [KTT10, Proof of Theorem 3.1]. \[\square\]

**Theorem 4.13.**

(i) \[
\sum_{q \geq 1} \chi_{\Phi^+}^\text{quasi}(q) t^q = \frac{\#W t^h}{\prod_{i=0}^t (1 - e_i^\Psi)}.
\]

(ii) Let \(R_\Phi(t)\) be the generalized Eulerian polynomial (see Remark 4.3(b)). Then

\[
\sum_{q \geq 1} q^\ell t^q = \frac{R_\Phi(t)}{\prod_{i=0}^t (1 - e_i^\Psi)}.
\]

**Proof.** For a proof of (i), see, e.g., [Ath96], [BS98, Theorem 4.1], [KTT10, Theorem 3.1]. (ii) follows from [LP18, Theorem 10.1]. \[\square\]

**Theorem 4.14.** If \(\Psi\) is compatible, then

\[
\sum_{q \geq 1} \chi_{\Psi}^\text{quasi}(q) t^q = \frac{E_\Psi(t)}{\prod_{i=0}^t (1 - e_i^\Psi)}.
\]
Proof. By Theorem 4.11, Corollary 3.10 and Proposition 4.12, we have
\[
\sum_{q \geq 1} \chi^\text{quasi}_\Psi(q) t^q = \sum_{q \geq 1} \sum_{i \in [N]} \mathcal{L}_{A^\circ}(q + \text{dsc}_\Psi(A^\circ_i)) t^q
\]
\[
= \sum_{i \in [N]} t^{-\text{dsc}_\Psi(A^\circ_i)} \sum_{q \geq 1} \mathcal{L}_{A^\circ}(q + \text{dsc}_\Psi(A^\circ_i)) t^{q + \text{dsc}_\Psi(A^\circ_i)}
\]
\[
= \sum_{i \in [N]} t^{-\text{dsc}_\Psi(A^\circ_i)} \sum_{n \geq 0} \mathcal{L}_{A^\circ}(n) t^n
\]
\[
= \frac{\sum_{i \in [N]} t^{-\text{dsc}_\Psi(A^\circ_i)}}{\prod_{i=0}^t (1 - t^{c_i})} = \frac{E_\Psi(t)}{\prod_{i=0}^t (1 - t^{c_i})}.
\]
\[
\square
\]

Remark 4.15. By Remark 4.3, Theorems 4.11 and 4.14, if
(a) \( \Psi = \Phi^+ \), then we recover Theorems 3.8 and 4.13(ii).
(b) \( \Psi = \emptyset \), then we recover [Yos18b, Theorem 4.8] and Theorem 4.13(ii).

5. DEFORMATIONS OF WEYL SUBARRANGEMENTS

Let \( \Psi \subseteq \Phi^+ \), and recall the notation \( \Psi^c = \Phi^+ \setminus \Psi \). Let \( a \leq b \) be integers, and denote \( [a, b] := \{ m \in \mathbb{Z} | a \leq m \leq b \} \). Also, if \( b \geq 1 \), then write \( [b] \) instead of \( [1, b] \).

Definition 5.1. Let \( a \leq b, c \leq d \) be integers. Define two types of the deformed Weyl arrangements of \( \Psi \) as follows:

(Type I) \( \mathcal{I}_\Psi^{[a,b]} := \{ H_{\alpha, m} | \alpha \in \Psi, m \in [a, b] \} \).

(Type II) \( \mathcal{I}_\Psi^{[a,b],[c,d]} := \mathcal{I}_\Psi^{[a,b]} \cup \mathcal{I}_\Psi^{[c,d]} \).

Remark 5.2.

(a) There is an obvious duality \( \mathcal{I}_\Psi^{[a,b],[c,d]} = \mathcal{I}_\Psi^{[c,d],[a,b]} \). We can list some specializations: \( \mathcal{I}_\emptyset^{[a,b]} = \emptyset \) the empty arrangement, \( \mathcal{I}_\Phi^{[a,b],[c,d]} = \mathcal{A}_{\Phi^+}^{[c,d]} \) the truncated affine Weyl arrangement, including the extended Shi, Catalan, Linial arrangements, see, e.g., [SP00, §9]. In addition, \( \mathcal{I}_\Phi^{[a,b]} = \mathcal{I}_{\Psi, \Phi^+}^{[a,b]} \) the deformed Weyl arrangements of an arbitrary \( \Psi \) are less well-known.

(b) The deformed Weyl arrangements of an arbitrary \( \Psi \) are less well-known. When \( \Phi \) is of type \( A \), the deleted (or graphical) Shi arrangement, see, e.g., [Ath99, §3] or [AR12], is the product [OT92, Definition 2.13] of the one dimensional empty arrangement and \( \mathcal{I}_{\Psi, \Phi^+}^{[0,1],[0,0]} \).
Definition 5.3. For \( w \in W \), define
\[
\overline{dsc}_\Psi(w) := \sum_{0 \leq i \leq \ell, w(\alpha_i) \in -\Psi} c_i,
\]
\[
asc_\Phi(w) := \sum_{0 \leq i \leq \ell, w(\alpha_i) \in \Psi^c} c_i.
\]
\[
asc_\Psi(w) := \sum_{0 \leq i \leq \ell, w(\alpha_i) \in \Psi} c_i.
\]

Obviously, \( asc_\Psi(w) + asc_\Phi(w) + dsc_\Phi(w) + dsc_\Psi(w) = h \) for all \( w \in W \).

Similar to Lemma 4.4, each function defined above takes values in \([0, h-1]\).

Furthermore, \( asc_\emptyset(w) = asc_\Psi(w) + asc_\Phi(w) = asc_\Psi(w) + asc_\Psi(w) \).

\[\overline{dsc}_\emptyset(w) = \overline{dsc}_\Phi(w) + dsc_\Phi(w) = dsc_\Psi(w) + dsc_\Psi(w).\]

Similar to Definition 4.6, we can extend the functions above to functions on the set of all alcoves.

Now let us formulate a deformed version of Proposition 3.7. Set
\[
F_i^{[a_i, b_i]} := \bigcup_{m \in [a_i, b_i]} H_{\alpha, m} \quad (1 \leq i \leq \ell),
\]
\[
F_0^{[a_0, b_0]} := \bigcup_{m \in [a_0, b_0]} H_{\alpha_0, -q+m}.
\]

Proposition 5.4. Let \( \{i_1, \ldots, i_k\} \subseteq [0, \ell] \), and let \( b_{i_j} \geq 0 \) for all \( 1 \leq j \leq k \). Suppose that \( q > \sum_{j \in [k]} (b_{i_j} + 1)c_{i_j} \). Then
\[
\# \left( qA^\circ \cap Z(\Phi^\vee) \setminus \bigcup_{j \in [k]} F_i^{[0, b_{i_j}]} \right) = \overline{L}_{\Phi^\circ}(q - \sum_{j \in [k]} (b_{i_j} + 1)c_{i_j}).
\]

Proof. The formula was implicitly used in [Yos18b, §5] and its proof is very similar to the non-deformed case. Note that if \( i \in [\ell] \), then
\[
\# \left( qA^\circ \cap Z(\Phi^\vee) \setminus F_i^{[a_i, b_i]} \right) = \# \left\{ x \in Z(\Phi^\vee) \bigg| \begin{array}{l} (\alpha_i, x) \geq b_i + 1 \\
(\alpha_j, x) \geq 0 \quad (j \in [\ell] \setminus \{i\}) \\
(\alpha_0, x) \geq -q \end{array} \right\}
\]
\[
= \overline{L}_{\Phi^\circ}(q - (b_i + 1)c_i).
\]

Here the last equality follows from the bijection \( x \mapsto x + (b_i + 1)\omega_i^\vee \). The proof for \( i = 0 \) is similar. Then apply the formula above repeatedly. \( \square \)
Remark 5.5. If we replace the interval \([0, b_i]\) in Proposition 5.4 by \([a, b_i]\) with \(a \geq 1\), there might be a large change in the right-hand side of the formula above. For example, if \(1 \leq a_1 \leq b_1\), then

\[
\# \left( qA^\circ \cap Z(\Phi^\vee) \setminus F_{1^{[a_1, b_1]}} \right) = L_{\mathcal{A} \circ}(q-(b_1+1)c_1)+L_{\mathcal{A} \circ}(q-a_1c_1). 
\]

Theorem 5.6. Let \(\Psi\) be a compatible subset of \(\Phi^+\).

(i) If \(a, b \geq 0\), then

\[
\chi^{\text{quasi}}_{I^{[a,b]}}(q) = \sum_{i \in [N]} L_{\mathcal{A} \circ}(q-(b+1)\overline{\text{asc}}_{\Psi}(A_i^\circ) - \text{asc}_{\Psi}(A_i^\circ) - (a+1)\overline{\text{dsc}}_{\Psi}(A_i^\circ)).
\]

(ii) If \(b \geq 1\), then

\[
\chi^{\text{quasi}}_{I^{[-a,b]}}(q) = \sum_{i \in [N]} L_{\mathcal{A} \circ}(q-(b+1)\overline{\text{asc}}_{\Psi}(A_i^\circ) - \text{asc}_{\Psi}(A_i^\circ)).
\]

Proof. Proofs of (i) and (ii) are similar, and both are similar in spirit to the proof of [Yos18b, Theorem 5.1]. See also Proof of Theorem 4.11 in this paper. First, we give a proof for (i). Since both sides are quasi-polynomials, it is sufficient to prove the equality for \(q \gg 0\) (actually, \(q > (a+b+3)h\) is sufficient). By Proposition 5.4, for \(i \in [N]\),

\[
\# \left( qA_i^\circ \cap Z(\Phi^\vee) \setminus \bigcup_{\mu \in \Psi, k \in \mathbb{Z}, m \in [-a,b]} H_{\mu, kq+m} \right)
= \# \left\{ x \in Z(\Phi^\vee) \left| \begin{array}{l}
(\alpha, x) \geq qk_\alpha + b + 1 (\alpha \in I \cap \Psi) \\
(\eta, x) > qk_\eta (\eta \in I \cap \Psi^c) \\
(\beta, x) \leq qk_\beta - a - 1 (\beta \in J \cap \Psi) \\
(\delta, x) \leq qk_\delta (\delta \in J \cap \Psi^c)
\end{array} \right. \right\}
= L_{\mathcal{A} \circ}(q-(b+1)\overline{\text{asc}}_{\Psi}(A_i^\circ) - \text{asc}_{\Psi}(A_i^\circ) - (a+1)\overline{\text{dsc}}_{\Psi}(A_i^\circ)).
\]

By Proposition 3.11, we have

\[
\chi^{\text{quasi}}_{I^{[-a,b]}}(q) = \sum_{i \in [N]} L_{\mathcal{A} \circ}(q-(b+1)\overline{\text{asc}}_{\Psi}(A_i^\circ) - \text{asc}_{\Psi}(A_i^\circ) - (a+1)\overline{\text{dsc}}_{\Psi}(A_i^\circ)).
\]
For (ii), note that for $i \in [N]$,

$$\# \left( qA_i^\circ \cap Z(\Phi^\vee) \setminus \bigcup_{\mu \in \Psi, k \in \mathbb{Z}, m \in [1, b]} H_{\mu, kq+m} \right)$$

$$= \# \left\{ x \in Z(\Phi^\vee) \left| \begin{array}{l}
(\alpha, x) \geq qk+ b + 1 (\alpha \in I \cap \Psi) \\
(\eta, x) > qk (\eta \in I \cap \Psi^c) \\
(\beta, x) \leq qk (\beta \in J \cap \Psi) \\
(\delta, x) \leq qk (\delta \in J \cap \Psi^c)
\end{array} \right. \right\}$$

$$= L_{A^\circ}(q - (b + 1)\text{asc}_\Psi(A_i^\circ) - \text{asc}_\Psi(A_i^\circ)).$$

\[\square\]

**Remark 5.7.** Theorem 5.6 is a generalization of several known results.

(a) When $\Psi = \Phi^+$, we obtain [Ath04, Theorem 1.2] ($a = b \geq 0$), and Theorem 5.1 ($a = b - 1 \geq 0$), Theorem 5.2 ($b \geq 1$), Theorem 5.3 ($b = n + k$, $a = k - 1$, $n, k \geq 1$) in [Yos18b]. When $\Psi = \emptyset$, we obtain [Yos18b, Theorem 4.8].

(b) When $\Psi \subseteq \Phi^+$ is compatible and $a = b = 0$, we obtain Theorem 4.11.

By using the same method, we have the following result for the arrangements of type II.

**Theorem 5.8.** Let $\Psi$ be a compatible subset of $\Phi^+$.

(i) If $a, b, c, d \geq 0$, then

$$\chi_{\text{quasi}}^{[a,b],[c,d]}(q) = \sum_{i \in [N]} L_{A^\circ}(q - (b + 1)\text{asc}_\Psi(A_i^\circ) - (d + 1)\text{asc}_\Psi(A_i^\circ) - (a + 1)\text{dsc}_\Psi(A_i^\circ) - (c + 1)\text{dsc}_\Psi(A_i^\circ)).$$

(ii) If $a, b \geq 0, d \geq 1$, then

$$\chi_{\text{quasi}}^{[a,b],[1,d]}(q) = \sum_{i \in [N]} L_{A^\circ}(q - (b + 1)\text{asc}_\Psi(A_i^\circ) - (d + 1)\text{asc}_\Psi(A_i^\circ) - (a + 1)\text{dsc}_\Psi(A_i^\circ)).$$

(iii) If $b, d \geq 1$, then

$$\chi_{\text{quasi}}^{[1,b],[1,d]}(q) = \sum_{i \in [N]} L_{A^\circ}(q - (b + 1)\text{asc}_\Psi(A_i^\circ) - (d + 1)\text{asc}_\Psi(A_i^\circ)).$$

**Remark 5.9.**

(a) One can work with other intervals $[a, b]$ but the computation may become more complicated (see Remark 5.5).
(b) One can define and study the arrangement $\bigsqcup_{k=1}^{n} I_{\Psi_k}^{[a_k, b_k]}$ where $\Phi^+ = \bigsqcup_{k=1}^{n} \Psi_k$ with $n \geq 3$. See, e.g., [Ath96, Theorem 3.11] for an example when $n = 3$. We choose not to develop this direction here.

It is interesting to compare the following result with [AR12, Theorem 3.2] and [Ath96, Theorem 3.9].

**Corollary 5.10.** Define $M_{\Psi}(t) := \frac{1}{t} \sum_{w \in W} t^{h + \max \Psi(w)}$. If $\Psi$ is compatible, then

$$
\chi_{\Pi_{\Psi,\Psi}}^{\text{quasi}}(q) = (M_{\Psi}(S) \mathcal{L}_{\mathcal{V}})(q).
$$

**Acknowledgements.** The first author was partially supported by Mitacs Canada Globalink Research Award to visit Japan and carry out the collaboration. The second author is partially supported by JSPS Research Fellowship for Young Scientists Grant Number 19J12024. The third author is partially supported by JSPS KAKENHI Grant Number JP18H01115. The second author would like to thank Akiyoshi Tsuchiya for pointing out an error in the proof of the main result of the manuscript (Theorem 4.11) in a previous version.

**REFERENCES**

[AR12] D. Armstrong and B. Rhoades. The Shi arrangement and the Ish arrangement. *Trans. Amer. Math. Soc.*, 364(3):1509–1528, 2012.

[Ath96] C. A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields. *Adv. Math.*, 122(2):193–233, 1996.

[Ath99] C. A. Athanasiadis. Extended Linial hyperplane arrangements for root systems and a conjecture of Postnikov and Stanley. *J. Algebr. Comb.*, 10:207–225, 1999.

[Ath04] C. A. Athanasiadis. Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. *Bull. Lond. Math. Soc.*, 36:294–302, 2004.

[BM14] P. Brändén and L. Moci. The multivariate arithmetic Tutte polynomial. *Trans. Amer. Math. Soc.*, 366(10):5523–5540, 2014.

[BS98] A. Blass and B. Sagan. Characteristic and Ehrhart polynomials. *J. Algebr. Comb.*, 7(2):115–126, 1998.

[BZ06] M. Beck and T. Zaslavsky. Inside-out polytopes. *Adv. Math.*, 205(1):134–162, 2006.

[CW12] B. Chen and S. Wang. Comparison on the coefficients of characteristic quasi-polynomials of integral arrangements. *J. Combin. Theory Ser. A.*, 119:271–281, 2012.

[Hum90] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[KTT08] H. Kamiya, A. Takemura, and H. Terao. Periodicity of hyperplane arrangements with integral coefficients modulo positive integers. *J. Algebr. Comb.*, 27(3):317–330, 2008.
H. Kamiya, A. Takemura, and H. Terao. The characteristic quasi-polynomials of the arrangements of root systems and mid-hyperplane arrangements. *Arrangements, local systems and singularities*, pages 177–190, Progr. Math., 283, Birkhäuser Verlag, Basel, 2010.

H. Kamiya, A. Takemura, and H. Terao. Periodicity of non-central integral arrangements modulo positive integers. *Ann. Comb.*, 15(3):449–464, 2011.

T. Lam and A. Postnikov. Alcoved polytopes II. In: Kac V., Popov V. (eds) *Lie Groups, Geometry, and Representation Theory*, pages 253–272, Progr. Math., 326, Birkhäuser Verlag, Cham, 2018.

Y. Liu, T. N. Tran, and M. Yoshinaga. G-Tutte polynomials and abelian Lie group arrangements. *Int. Math. Res. Not. IMRN*, to appear.

P. Orlik and H. Terao. *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 1992.

R.P. Stanley and A. Postnikov. Deformations of Coxeter hyperplane arrangements. *J. Combin. Theory Ser. A*, 91:544–597, 2000.

R. Suter. The number of lattice points in alcoves and the exponents of the finite Weyl groups. *Math. Comp.*, 67(222):751–758, 1998.

T. N. Tran. An equivalent formulation of chromatic quasi-polynomials. arXiv preprint, 2018. [https://arxiv.org/abs/1803.08649](https://arxiv.org/abs/1803.08649).

T. N. Tran. Characteristic quasi-polynomials of ideals and signed graphs of classical root systems. *European J. Combin.*, 79:179–192, 2019.

T. N. Tran and M. Yoshinaga. Combinatorics of certain abelian Lie group arrangements and chromatic quasi-polynomials. *J. Combin. Theory Ser. A*, 165:258–272, 2019.

M. Yoshinaga. Characteristic polynomials of Linial arrangements for exceptional root systems. *J. Combin. Theory Ser. A*, 157:267–286, 2018.

M. Yoshinaga. Worpitzky partitions for root systems and characteristic quasi-polynomials. *Tohoku Math. J.*, 70(1):39–63, 2018.