ENTIRE FUNCTIONS WITH JULIA SETS OF POSITIVE MEASURE

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Abstract. Let $f$ be a transcendental entire function for which the set of critical and asymptotic values is bounded. The Denjoy-Carleman-Ahlfors theorem implies that if the set of all $z$ for which $|f(z)| > R$ has $N$ components for some $R > 0$, then the order of $f$ is at least $N/2$. More precisely, we have $\log \log M(r, f) \geq \frac{1}{2} N \log r - O(1)$, where $M(r, f)$ denotes the maximum modulus of $f$. We show that if $f$ does not grow much faster than this, then the escaping set and the Julia set of $f$ have positive Lebesgue measure. However, as soon as the order of $f$ exceeds $N/2$, this need not be true. The proof requires a sharpened form of an estimate of Tsuji related to the Denjoy-Carleman-Ahlfors theorem.

1. Introduction and results

The Julia set $J(f)$ of an entire function is defined as the set of all points in $\mathbb{C}$ where the iterates $f^n$ of $f$ do not form a normal family; see [3] for an introduction to transcendental dynamics.

McMullen [16] proved that $J(\sin(\alpha z + \beta))$ has positive Lebesgue measure and that $J(\lambda e^z)$ has Hausdorff dimension 2, for $\alpha, \beta, \lambda \in \mathbb{C}$, $\alpha, \lambda \neq 0$. The result on the Hausdorff dimension of $J(\lambda e^z)$ has been extended to large classes of functions; see [1, 5, 21, 25]. It is the purpose of this paper to exhibit a class of functions whose Julia sets have positive measure. However, we begin by briefly describing the results on Hausdorff dimension.

We first recall that the Eremenko-Lyubich class $B$ consists of all entire functions for which the set of finite asymptotic values and critical values is bounded. Eremenko and Lyubich [10] proved that if $f \in B$, then the escaping set $I(f)$ consisting of all points $z \in \mathbb{C}$ for which $f^n(z) \to \infty$ is contained in $J(f)$. In fact, it follows that $J(f) = I(f)$ for $f \in B$. It is easily seen that $\sin(\alpha z + \beta) \in B$ and $\lambda e^z \in B$. McMullen actually proved that $I(\sin(\alpha z + \beta))$ has positive measure and $I(\lambda e^z)$ has Hausdorff dimension 2.

Next we note that the order $\varrho(f)$ of an entire function $f$ is defined by

$$\varrho(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r},$$

where

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

Thus $\varrho(f)$ is the infimum of the set of all $\mu$ such that $|f(z)| \leq \exp(|z|^\mu)$ for large $|z|$, with $\varrho(f) = \infty$ if no such $\mu$ exists. We note that $\varrho(\lambda e^z) = \varrho(\sin(\alpha z + \beta)) = 1$.

McMullen’s result on the Hausdorff dimension of $J(\lambda e^z)$ was substantially generalized by Barański [1] and, independently, Schubert [21]. They proved that if $f \in B$ and

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\( \varrho(f) < \infty \), then \( J(f) \) has Hausdorff dimension 2. The special case where \( f \) has the form

\[
(1.1) \quad f(z) = \int_0^z P(t)e^{Q(t)}dt + c,
\]

with polynomials \( P \) and \( Q \) and with \( c \in \mathbb{C} \) had been treated before by Taniguchi \[25\]. These functions are in \( B \) and we have \( \varrho(f) = \deg P \).

A generalisation of the result of Barański and Schubert was given in \[5\] where it is shown that if \( f \in B \) and

\[
q = \limsup_{r \to \infty} \frac{\log \log \log M(r, f)}{\log \log r} < \infty,
\]

then \( I(f) \) and hence \( J(f) \) have Hausdorff dimension at least \( (q + 1)/q \). For further results on the Hausdorff dimension of Julia sets of entire functions we refer to \[2, 6\] and, in particular, the survey \[24\].

While McMullen’s result on the Hausdorff dimension of \( J(\lambda e^z) \) thus has prompted a lot of further research, there seem to be no papers whose main intention is to extend McMullen’s result that \( J(\sin(\alpha z + \beta)) \) has positive measure to more general classes of functions. However, there are some papers devoted to ergodic properties of transcendental entire and meromorphic functions and their results in particular imply that the Julia sets of certain functions have positive measure. We mention the work of Bock \[7\] whose results imply that if \( f \in B \) and if there exist \( \alpha > 0 \) and \( R > 0 \) such that the set of all \( z \) for which \( |z| > R \) and \( |f(z)| < e^t \) is contained in finitely many domains of the form \( \{ z : \arg z - s \leq t/(\log |z|)^\alpha \} \) for all large \( t \), then \( I(f) \) has positive measure. For example, this result applies to \( f(z) = R(e^z) \), where \( R \) is a rational function with \( R(0) = R(\infty) = 0 \), or to \( f(z) = \sin P(z) \), where \( P \) is a polynomial. Skorulski \[22\] considered functions of the form

\[
f(z) = \frac{a \exp(z^p) + b \exp(-z^p)}{c \exp(z^p) + d \exp(-z^p)},
\]

where \( p \in \mathbb{N} \) and \( a, b, c, d \in \mathbb{C} \), and Hemke \[13\] studied a class which contains all functions of the form \((1.1)\). Both Skorulski and Hemke proved that \( J(f) \) has positive measure for the functions considered, if the singularities of the inverse have a certain behavior under iteration. For a more detailed description of the above and other results on the measure of Julia and escaping sets we refer to the survey by Kotus und Urbański \[14\] section 7.

We shall exhibit a condition which depends only on the growth of \( f \) and which, for \( f \in B \), implies that \( I(f) \) and \( J(f) \) have positive measure. Before stating this condition we recall that (one version of) the Denjoy–Carleman–Ahlfor s-Theorem (see \[11\] p. 173], \[12\] section 8.3) or \[17\] p. 309)\) says that if \( f \) is entire, \( R > 0 \) and \( N \) denotes the number of components of

\[
A_R = \{ z \in \mathbb{C} : |f(z)| > R \},
\]

then \( N \leq \max\{1, 2\varrho(f)\} \). As we shall see below, we have \( \varrho(f) \geq \frac{1}{2} \) for \( f \in B \). (This seems to have been observed first in \[4, 15\]; see also \[20\] Lemma 3.5\). Thus \( N \leq 2\varrho(f) \) in this case. More precisely, we even have (see \[12\] Theorem 8.9 or \[17\] p. 312)

\[
(1.2) \quad \log \log M(r, f) \geq \frac{N}{2} \log r - O(1)
\]
as \( r \to \infty \). We shall show that if \( f \in B \) does not grow much faster than guaranteed by \((1.2)\), then \( I(f) \) and \( J(f) \) have positive measure. In particular, we shall see that this
is the case if we have equality in (1.2) or, more generally, if
\[
\log \log M(r, f) \leq \left( \frac{N}{2} + \frac{1}{\log^m r} \right) \log r
\]
for large \( r \), where \( \log^m \) denotes the \( m \)-th iterate of the logarithm.

To formulate a more precise condition we fix \( \beta \in (0, 1/e) \) and note that the function \( E_\beta(x) = e^{\beta x} \) has a repelling fixed point \( \xi > e \) with multiplier
\[
\mu = E_\beta'(\xi) = \beta E_\beta(\xi) = \beta \xi > 1.
\]
Now Schröder’s functional equation
\[
(1.3) \quad \Phi(E_\beta(z)) = \mu \Phi(z)
\]
has a unique solution \( \Phi \) holomorphic in a neighborhood of \( \xi \) and satisfying \( \Phi(\xi) = 0 \) and \( \Phi'(\xi) = 1 \). It is not difficult to see that \( \Phi \) is real on the real axis and that \( \Phi \) has a continuation \( \Phi : [\xi, \infty) \to [0, \infty) \) so that (1.3) is satisfied for \( \xi \leq z \leq \infty \). Moreover, \( \Phi \) is increasing on the interval \( [\xi, \infty) \) and we have \( \lim_{x \to \infty} \Phi(x) = \infty \) while
\[
\lim_{x \to \infty} \frac{\Phi(x)}{\log^m x} = 0
\]
for all \( m \in \mathbb{N} \). Thus \( \Phi \) tends to \( \infty \), but slower than any iterate of the logarithm. Hence the function
\[
(1.4) \quad \varepsilon : (\xi, \infty) \to (0, \infty), \quad \varepsilon(x) = \frac{1}{\Phi(x)}
\]
is decreasing and tends to 0 as \( x \to \infty \), but it tends to 0 slower than any of the functions \( 1/\log^m x \). We mention that the function \( \Phi \) also appears in recent work of Peter [18] on Hausdorff measure of exponential Julia sets.

**Theorem 1.1.** Let \( f \in B \) and suppose that \( A_R \) has \( N \) components for some \( R > 0 \). Let \( \varepsilon(x) \) be defined by (1.4) and suppose that
\[
\log \log M(r, f) \leq \left( \frac{N}{2} + \varepsilon(r) \right) \log r
\]
for large \( r \). Then \( I(f) \) and \( J(f) \) have positive Lebesgue measure.

In section 4 we will give an example which shows that the function \( \varepsilon(r) \) in Theorem 1.1 cannot be replaced by a positive constant \( \varepsilon \).

The proof of Theorem 1.1 will use some ideas connected to the Denjoy-Carleman-Ahlfors-Theorem. One way to prove the latter theorem is based on an estimate of Tsuji [26, p. 116]. To formulate this result, let \( U \) be a component of \( A_R \), let
\[
(1.5) \quad \theta(r) = \operatorname{meas} \left( \{ t \in [0, 2\pi] : r e^{it} \in U \} \right),
\]
put \( \theta^*(r) = \theta(r) \) if \( \{ z \in \mathbb{C} : |z| = r \} \not\subset U \) and put \( \theta^*(r) = \infty \) and thus \( 1/\theta^*(r) = 0 \) otherwise. Tsuji’s result says that for \( 0 < \kappa < 1 \) there exist constants \( C \) and \( r_0 \) such that
\[
(1.6) \quad \log \log M(r, f) \geq \pi \int_{r_0}^{rr} \frac{dt}{t\theta^*(t)} - C
\]
for \( r > r_0/\kappa \).

Eremenko and Lyubich [10] proved that if \( f \in B \) and \( R \) is chosen such that all critical and finite asymptotic values have modulus less than \( R \), then all components of \( A_R \) are
simply connected and unbounded. For large \( r \) we thus have \( \theta^*(r) = \theta(r) \leq 2\pi \) and hence (1.6) yields
\[
\log \log M(r, f) \geq \frac{1}{2} \int_{r_0}^{r} \frac{dt}{t} - C = \frac{1}{2} \log r - C - \log \frac{\kappa}{r_0}
\]
for large \( r_0 \) and \( r > r_0/\kappa \). In particular, it follows that \( \varrho(f) \geq \frac{1}{2} \) for \( f \in B \), as mentioned above.

To prove Theorem 1.1 we shall need a refinement of (1.6) in the case that (1.7)
\[
\{ z \in \mathbb{C} : |z| = r \} \not\subset U
\]
and hence \( \theta^*(r) = \theta(r) \) for large \( r \).

**Theorem 1.2.** Let \( f \) be entire, \( R > 0 \) and let \( U \) be a component of \( A_R \) such that (1.7) holds for all large \( r \). Let \( 0 < \beta < \frac{1}{2} \) and put
\[
V = \{ z \in U : |f(z)| \geq \exp(|z|^\beta) \}
\]
and \( \psi(r) = \text{meas}\{ t \in [0, 2\pi] : re^{it} \in V \} \). Then for \( 0 < \kappa < 1 \) there exist constants \( C \) and \( r_0 \) such
\[
\log \log M(r, f) \geq \pi \int_{r_0}^{r} \frac{dt}{t \psi(t)} - C
\]
for \( r \geq r_0/\kappa \).

We shall prove Theorem 1.2 in section 2 and Theorem 1.1 in section 3.

**2. Proof of Theorem 1.2**

In this section we prove Theorem 1.2 following an argument by Tsuji [26, section III.17]; see also [11, section 5.1] and, for a slightly different approach, [12, section 8.1]. The original result is stated for subharmonic functions and the main difference here is that our function \( \log |f(z)| - |z|^\beta \) is not subharmonic.

The following lemma [26, p. 112] is known as Wirtinger’s inequality.

**Lemma 2.1.** Let \( f \) and \( f' \) be continuous in \([a, b]\) and \( f(a) = f(b) = 0 \). Then
\[
\int_{a}^{b} f'(x)^2 dx \geq \frac{\pi^2}{(b-a)^2} \int_{a}^{b} f(x)^2 dx.
\]

Let \( v(z) = \log |f(z)| - |z|^\beta \) and let
\[
V = \{ z \in U : v(z) \geq 0 \} \subset U,
\]
where \( U \) is a component of \( A_R \). Define \( V_r = \{ \theta \in [t_r, 2\pi + t_r] : re^{i\theta} \in V \} \) where \( t_r \) is chosen such that \( re^{it_r} \not\in U \) and put
\[
m(r) = m_v(r) = \frac{1}{2\pi} \int_{V_r} v(re^{it})^2 dt.
\]
Hence \( \text{meas}(V_r) = \psi(r) \). Recall that \( \theta(r) \) is defined by (1.5).

Next we prove the following.

**Lemma 2.2.** There exist positive constants \( c \) and \( r_0 \) such that \( m(r) \geq cr \) for \( r \geq r_0 \).
Proof. Let $u(z) = \log |f(z)| - \log R$ for $z \in U$ and $u(z) = 0$ outside $U$. Then $u \geq 0$ and $u$ is subharmonic. Let

$$m_u(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})^2 d\theta.$$ 

Note that $\log M(r, f) \geq \sqrt{m_u(r)}$. Inequality (1.6) is actually a corollary of a more general result [12, Theorem 8.2] which says that

$$\log \sqrt{m_u(r)} \geq \pi \int_{r_0}^{r} \frac{dt}{t\theta(t)} - C,$$

for $r > r_0/\kappa$. Since $\theta(t) \leq 2\pi$ we obtain

$$\log m_u(r) \geq 2\pi \int_{r_0}^{r} \frac{dt}{t\theta(t)} - C \geq \int_{r_0}^{r} \frac{dt}{t} - C \geq \log r - O(1),$$

and we conclude that $m_u(r) \geq c' r$ for some $c' > 0$, if $r > r_0/\kappa$.

To obtain a similar estimate for $m(r)$, first write

$$\int_{V_r} v(re^{i\theta})^2 d\theta = \int_{V_r} u(re^{i\theta})^2 d\theta - 2 \int_{V_r} u(re^{i\theta}) r^{\beta} d\theta + \int_{V_r} r^{2\beta} d\theta.$$

By the Cauchy-Schwarz inequality,

$$\int_{V_r} u(re^{i\theta}) r^{\beta} d\theta \leq \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} \sqrt{\int_{V_r} r^{2\beta} d\theta} \leq \sqrt{2\pi r^\beta} \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta}.$$

Hence

$$\int_{V_r} v(re^{i\theta})^2 d\theta \geq \int_{V_r} u(re^{i\theta})^2 d\theta - \sqrt{8\pi r^\beta} \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta}$$

(2.1)

$$= \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} \left( \sqrt{\int_{V_r} u(re^{i\theta})^2 d\theta} - \sqrt{8\pi r^\beta} \right).$$

We have

$$c' r \leq m_u(r)$$

$$= \frac{1}{2\pi} \int_{V_r} u(re^{i\theta})^2 d\theta + \frac{1}{2\pi} \int_{\{\theta: 0 \leq u(re^{i\theta}) < r^\beta\}} u(re^{i\theta})^2 d\theta$$

$$\leq \frac{1}{2\pi} \int_{V_r} u(re^{i\theta})^2 d\theta + r^{2\beta}.$$

Hence

$$\int_{V_r} u(re^{i\theta})^2 d\theta \geq 2\pi c' r - 2\pi r^{2\beta} \geq c'' r$$

for some $c'' > 0$ and $r \geq r_0$, provided $r_0$ is large enough.

Hence (2.1) yields

$$m(r) = \frac{1}{2\pi} \int_{V_r} v(re^{i\theta})^2 d\theta \geq \frac{1}{2\pi} \sqrt{c'' r} (\sqrt{c'' r} - \sqrt{8\pi r^\beta}) \geq cr,$$

for some $c > 0$ and $r \geq r_0$, if $r_0$ is sufficiently large. □

Next we prove (following Tsuji [26]) that $m(r)$ is a convex function of $\log r$ for large $r$. 

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Lemma 2.3. There exist \( r_0 > 0 \) such that
\[
\frac{d^2 m(r)}{d (\log r)^2} \geq 0,
\]
for all \( r \geq r_0 \).

Proof. The Laplacian in polar coordinates is given by
\[
(2.2) \quad \frac{1}{r^2} \left( \frac{\partial^2 v(re^{i\theta})}{\partial (\log r)^2} + \frac{\partial^2 v(re^{i\theta})}{\partial \theta^2} \right) = \Delta v(re^{i\theta}) = \Delta \left( \log |f(re^{i\theta})| - r^\beta \right) = -\beta^2 r^{\beta - 2}.
\]

The set \( V_r \) consists of finitely many intervals \([\alpha_j(r), \beta_j(r)]\). Then
\[
\frac{dm(r)}{d \log r} = \frac{1}{2\pi} \sum_j \frac{d}{d \log r} \int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta
\]
\[
= \frac{1}{2\pi} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} \frac{\partial v(re^{i\theta})}{\partial \log r} \frac{\partial v(re^{i\theta})}{\partial \theta} + v(re^{i\beta_j(r)})^2 \frac{d \beta_j(r)}{d \log r} - v(re^{i\alpha_j(r)})^2 \frac{d \alpha_j(r)}{d \log r}
\]
\[
= \frac{1}{\pi} \int_{V_r} v(re^{i\theta}) \frac{\partial v(re^{i\theta})}{\partial \log r} d\theta,
\]
since \( v(re^{i\alpha_j(r)}) = v(re^{i\beta_j(r)}) = 0 \).

Also,
\[
(2.3) \quad \frac{d^2 m(r)}{d (\log r)^2} = \frac{1}{\pi} \int_{V_r} \left( \frac{\partial v(re^{i\theta})}{\partial \log r} \right)^2 + v(re^{i\theta}) \frac{\partial^2 v(re^{i\theta})}{\partial (\log r)^2} d\theta.
\]

Now
\[
\frac{\partial^2}{\partial \theta^2} (v(re^{i\theta})^2) = \frac{\partial}{\partial \theta} \left( 2v(re^{i\theta}) \frac{\partial v(re^{i\theta})}{\partial \theta} \right) = 2v(re^{i\theta}) \frac{\partial^2 v(re^{i\theta})}{\partial \theta^2} + 2 \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2,
\]
and since \( v(re^{i\alpha_j(r)}) = v(re^{i\beta_j(r)}) = 0 \) we have
\[
\int_{\alpha_j(r)}^{\beta_j(r)} \frac{\partial^2}{\partial \theta^2} (v(re^{i\theta})^2) d\theta = \left[ 2v(re^{i\theta}) \frac{\partial v(re^{i\theta})}{\partial \theta} \right]_{\alpha_j(r)}^{\beta_j(r)} = 0
\]
for all \( j \). Thus
\[
\int_{V_r} v(re^{i\theta}) \frac{\partial^2 v(re^{i\theta})}{\partial \theta^2} d\theta = -\int_{V_r} \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2 d\theta
\]
and using (2.2) and (2.3) we obtain
\[
\frac{d^2 m(r)}{d (\log r)^2} = \frac{1}{\pi} \int_{V_r} \left( \frac{\partial v(re^{i\theta})}{\partial \log r} \right)^2 + \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2 - v(re^{i\theta}) \beta^2 r^{\beta - 2} d\theta.
\]
Let us write
\[ J_1 = \frac{1}{\pi} \int_{V_r} \left( \frac{\partial v(re^{i\theta})}{\partial \log r} \right)^2 d\theta, \]
\[ J_2 = \frac{1}{\pi} \int_{V_r} \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2 d\theta, \]
\[ J_3 = \frac{1}{\pi} \int_{V_r} v(re^{i\theta}) \beta^2 r^{\beta} d\theta. \]

To estimate \( J_1 \) we use the Cauchy-Schwarz inequality to obtain
\[ \left( \frac{dm(r)}{d\log r} \right)^2 = \left( \frac{1}{\pi} \int_{V_r} v(re^{i\theta}) \frac{\partial v(re^{i\theta})}{\partial \log r} d\theta \right)^2 \leq \frac{1}{\pi^2} \int_{V_r} v(re^{i\theta})^2 d\theta \int_{V_r} \left( \frac{\partial v(re^{i\theta})}{\partial \log r} \right)^2 d\theta \leq 2m(r)J_1. \]

Hence
\[ J_1 \geq \frac{1}{2m(r)} \left( \frac{dm(r)}{d\log r} \right)^2. \]

Recall that \( V_r \) is a union of intervals \([\alpha_j(r), \beta_j(r)]\) so that \( \psi(r) = \sum_j (\beta_j(r) - \alpha_j(r)) \).

Using Wirtinger’s inequality on each of these intervals we get
\[ \frac{1}{\pi} \int_{\alpha_j(r)}^{\beta_j(r)} \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2 d\theta \geq \frac{\pi}{(\beta_j(r) - \alpha_j(r))^2} \int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta \geq \frac{\pi}{\psi(r)^2} \int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta. \]

Summing over all \( j \) yields
\[ J_2 = \frac{1}{\pi} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} \left( \frac{\partial v(re^{i\theta})}{\partial \theta} \right)^2 d\theta \geq \frac{\pi}{\psi(r)^2} \sum_j \int_{\alpha_j(r)}^{\beta_j(r)} v(re^{i\theta})^2 d\theta \geq \frac{\pi}{\psi(r)^2} \int_{V_r} v(re^{i\theta})^2 d\theta = \frac{2\pi^2}{\psi(r)^2} m(r). \]

To estimate \( J_3 \) we use the Cauchy-Schwarz inequality again to obtain
\[ J_3 = \frac{\beta^2 r^{\beta}}{\pi} \int_{V_r} v(re^{i\theta}) d\theta \leq \frac{\beta^2 r^{\beta}}{\pi} \left( \int_{V_r} v(re^{i\theta})^2 d\theta \right)^{1/2} \left( \int_{V_r} 1 d\theta \right)^{1/2} \leq 2\beta^2 r^{\beta} \sqrt{m(r)} \]
\[ = 2\beta^2 r^{\beta - 1/2} r^{1/2} \sqrt{m(r)}. \]
Using Lemma 2.2 and putting $\gamma = 2\beta^2/\sqrt{c}$ we obtain

$$J_3 \leq \gamma r^{\beta - 1/2} m(r).$$

Hence

$$\frac{d^2 m(r)}{d(\log r)^2} \geq J_1 + J_2 - J_3 \quad (2.4)$$

$$= \frac{1}{2m(r)} \left( \frac{dm(r)}{d\log r} \right)^2 + m(r) \left( \frac{2\pi}{\psi(r)} \right)^2 - \frac{\gamma r^{\beta - 1/2} m(r)}{2}.$$

Since $\psi(r) \leq 2\pi$ and $\beta < \frac{1}{2}$ there is some $r_0 > 0$ such that

$$\frac{2\pi}{\psi(r)} \geq 2 \gamma r^{\beta - 1/2} \geq 0 \quad \text{for all } r \geq r_0.\quad (2.5)$$

Hence

$$\frac{d^2 m(r)}{d(\log r)^2} \geq 0 \quad \text{for all } r \geq r_0.$$

By (2.5) we may define $\alpha$ and $\tilde{\alpha}$ by

$$\tilde{\alpha}(r) = \alpha(\log r) = \sqrt{\left( \frac{2\pi}{\psi(r)} \right)^2 - 2 \gamma r^{\beta - 1/2}}$$

for $r \geq r_0$. Put $\mu(t) = m(e^t)$. We use the change of variables $r = e^t$, so $\alpha(t) = \tilde{\alpha}(r)$ and $\mu(t) = m(r)$. Now inequality (2.4) becomes

$$\mu''(t) \geq \frac{\mu'(t)^2}{2\mu(t)} + \frac{1}{2} \alpha(t)^2 \mu(t).$$

From this we deduce (see Tsuji \[26, pp. 114-115\]) that

$$\left( \frac{\mu''(t)}{\mu'(t)} \right)^2 \geq \alpha(t)^2.\quad (2.6)$$

We now argue that in fact also $\mu'(t) \geq 0$ for large enough $t$. Lemma 2.2 implies that $\mu(t) = m(e^t) \geq ce^t$ for all $t \geq \log r_0$. This means that $\mu'(t) = rm'(r) > 0$ for some $t$ because otherwise $\mu$ would be bounded. Since also $\mu''(t) = d^2 m(r)/d(\log r)^2 \geq 0$ for large $t$ this implies that actually $\mu'(t) > 0$ for all large $t$, say $t \geq \log r_0$.

Hence from (2.6) we get

$$\frac{\mu''(t)}{\mu'(t)} \geq \alpha(t) \quad \text{for all } t \geq \log r_0.$$  

To conclude the proof of Theorem 1.2 let $\tau > t_0 = \log r_0$ and note that

$$\log \mu'(\tau) - \log \mu'(t_0) = \int_{t_0}^{\tau} \frac{\mu''(\rho)}{\mu'(\rho)} d\rho \geq \int_{t_0}^{\tau} \alpha(\rho) d\rho,$$

so

$$\mu'(\tau) \geq \mu'(t_0) \exp \left\{ \int_{t_0}^{\tau} \alpha(\rho) d\rho \right\}.$$
With \( t = \log r > t_0 \) we have, since \( \mu(t) \) is increasing for \( t \geq \log r_0 \),
\[
\mu(t) \geq \mu(t_0) = \int_{t_0}^{t} \mu'(\tau) d\tau \geq \mu'(t_0) \int_{t_0}^{t} \exp \left\{ \int_{t_0}^{\tau} \alpha(\rho) d\rho \right\} d\tau.
\]

With \( \rho = \log s \) and \( \tau = \log \sigma \) we get
\[
\mu(t) \geq \mu'(t_0) \int_{r_0}^{r} \exp \left\{ \int_{r_0}^{\sigma} \frac{\tilde{\alpha}(s)}{s} ds \right\} \frac{d\sigma}{\sigma}.
\]

For \( r \geq r_0/\kappa \), with \( 0 < \kappa < 1 \), we thus have
\[
\mu(t) \geq \mu'(t_0) \int_{r_0}^{\kappa r} \exp \left\{ \int_{r_0}^{\sigma} \frac{\tilde{\alpha}(s)}{s} ds \right\} \frac{d\sigma}{\sigma} \geq \mu'(t_0)(1-\kappa) \exp \left\{ \int_{r_0}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds \right\}.
\]

With \( c_0 = \mu'(t_0) \) thus
\[
\mu(t) \geq c_0(1-\kappa) \exp \left\{ \int_{r_0}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds \right\}.
\]

We want to estimate the integral on the right side. We have
\[
\tilde{\alpha}(s) = \frac{2\pi}{\psi(s)} \sqrt{1 - \frac{\psi(s)^2}{2\pi^2} \gamma s^{\beta-1/2}} \geq \frac{2\pi}{\psi(s)} \left( 1 - \frac{\psi(s)^2}{2\pi^2} \gamma s^{\beta-1/2} \right) = \frac{2\pi}{\psi(s)} - \frac{\gamma}{\pi} \psi(s) s^{\beta-1/2},
\]
for \( s \geq r_0 \), where we used that \( \sqrt{x} \geq x \) for \( x \in [0, 1] \). Therefore,
\[
\int_{r_0}^{\kappa r} \frac{\tilde{\alpha}(s)}{s} ds \geq \frac{2\pi}{\psi(s)} \int_{r_0}^{\kappa r} \frac{ds}{s} - \frac{\gamma}{\pi} \int_{r_0}^{\kappa r} \psi(s) s^{\beta-3/2} ds
\]
But, since \( \beta < 1/2 \) and \( \psi(s) \leq 2\pi \),
\[
\frac{\gamma}{\pi} \int_{r_0}^{\kappa r} \psi(s) s^{\beta-3/2} ds \leq c_1
\]
for some constant \( c_1 > 0 \). Hence (2.8) yields
\[
\mu(t) \geq c_0(1-\kappa)e^{-c_1} \exp \left\{ 2\pi \int_{r_0}^{\kappa r} \frac{ds}{s\psi(s)} \right\}.
\]

Recalling that \( t = \log r \) and \( m(r) = \mu(t) \), we get
\[
m(r) \geq c_2 \exp \left\{ 2\pi \int_{r_0}^{\kappa r} \frac{ds}{s\psi(s)} \right\},
\]
where \( c_2 = c_0(1-\kappa)e^{c_1} \).

From this and the fact that
\[
\log \log M(r,f) \geq \log \max_{|z|=r} v(z) \geq \log \sqrt{m(r)} = \frac{1}{2} \log m(r),
\]
Theorem 1.2 follows.

\textit{Remark 2.1.} With some more effort (see again [26]), one can show that
\[
m(\rho) \leq \frac{2e^{c_1+1}m(r)}{1-\kappa} \exp \left\{ -2\pi \int_{\rho}^{\kappa r} \tilde{\alpha}(s) ds \right\}
\]
for \( r_0 \leq \rho < \kappa r \). Using this it follows that the constant \( C \) in Theorem 1.2 only depends on \( \kappa \), \( r_0 \) and \( \beta \).
3. Proof of Theorem 1.1

Before we begin with the proof of Theorem 1.1 we need some auxiliary results. We begin by describing the logarithmic change of variable which was the main tool used by Eremenko and Lyubich to study the dynamics of a function \( f \in B \). We choose \( R > |f(0)| \) such that \( \Delta_R = \{ z \in \mathbb{C} : |z| > R \} \) contains no critical values and no asymptotic values of \( f \). As already mentioned in the introduction, Eremenko and Lyubich showed that every component \( U \) of \( A_R = f^{-1}(\Delta_R) \) is simply connected. The map \( f : U \rightarrow \Delta_R \) is thus a universal covering. With \( H = \{ z \in \mathbb{C} : \Re z > \log R \} \) the map \( \exp : H \rightarrow \Delta_R \) is also a universal covering and so there exists a biholomorphic map \( G : U \rightarrow H \) such that \( f = \exp \circ G \). This construction can be done for every component \( U \) of \( A_R \) and putting \( W = \exp^{-1}(A_R) \) we can thus define \( F : W \rightarrow H, F(z) = G(e^z) \). Thus \( \exp F(z) = f(e^z) \) and \( F \) maps every component of \( W \) univalently onto \( H \). We say that \( F \) is the function obtained from \( f \) by a logarithmic change of variable.

If \( \phi \) is a branch of the inverse function of \( F \) and if \( w \in H \), then \( \phi \) is defined in particular in the disk of radius \( \Re w - \log R \) around \( w \). Thus Koebe’s one quarter theorem implies that \( \phi(H) \) contains a disk of radius \( \frac{1}{4}|\phi'(w)|(\Re w - \log R) \) around \( \phi(w) \). Since \( W \) and hence \( \phi(H) \) do not contain vertical segments of length greater than \( 2\pi \), and thus in particular no disc of radius greater than \( \pi \), it follows that

\[
|\phi'(w)| \leq \frac{4\pi}{\Re w - \log R}.
\]

In terms of \( F \) this inequality takes the form

\[
|F'(z)| \geq \frac{\Re F(z) - \log R}{4\pi}
\]

for \( z \in W \).

Another tool we shall use is the Besicovitch covering lemma [8, Theorem 3.2.1]. Here we denote the ball of radius \( r \) around a point \( x \in \mathbb{R}^n \) by \( B(x, r) \).

**Lemma 3.1.** Let \( K \subset \mathbb{R}^n \) be bounded and \( r : K \rightarrow (0, \infty) \). Then there exists an at most countable subset \( L \) of \( K \) satisfying

\[
K \subset \bigcup_{x \in L} B(x, r(x))
\]

such that no point in \( \mathbb{R}^n \) is contained in more than \( 4^{2^n} \) of the balls \( B(x, r(x)) \), \( x \in L \).

We now begin with the proof of Theorem 1.1. Let \( U_1, U_2, \ldots, U_N \) be the components of \( \{ z \in \mathbb{C} : |f(z)| > R \} \). We may assume that \( R \) is so large that \( E^\beta_n(x) \rightarrow \infty \) as \( n \rightarrow \infty \) for \( x > \log R \). For \( j = 1, \ldots , N \) we put

\[
V_j = \{ z \in U_j : |f(z)| \geq \exp \left( |z|^\beta \right) \}
\]

and denote by \( \psi_j(r) \) the measure of the set of all \( t \in [0, 2\pi] \) such that \( re^{it} \in V_j \). It follows from Theorem 1.2 that for \( 0 < \kappa < 1 \) there exist constants \( r_0 \) and \( C \) such that

\[
\log \log M(r, f) \geq \pi \int_{r_0}^{kr} \frac{dt}{t \psi_j(t)} - C
\]

for \( r > r_0/\kappa \). Hence

\[
\log \log M(r, f) \geq \pi \int_{r_0}^{kr} \left( \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\psi_j(t)} \right) \frac{dt}{t} - C.
\]
By the Cauchy-Schwarz inequality we have

\[ N^2 = \left( \sum_{j=1}^{N} \sqrt{\psi_j(t)} \right)^2 \leq \left( \sum_{j=1}^{N} \frac{1}{\psi_j(t)} \right) \cdot \left( \sum_{j=1}^{N} \psi_j(t) \right). \]

With

\[ \psi(t) = \sum_{j=1}^{N} \psi_j(t) \]

we deduce that

\[ \sum_{j=1}^{N} \frac{1}{\psi_j(t)} \geq \frac{N^2}{\psi(t)} \]

and hence that

\[ \log \log M(r, f) \geq N\pi \int_{r_0}^{kr} \frac{1}{\psi(t)} \frac{dt}{t} - C. \]

By hypothesis we have

\[ \log \log M(r, f) \leq \frac{N}{2} \log r + \varepsilon(r) \log r = \frac{N}{2} \left( \int_{r_0}^{kr} \frac{dt}{t} + \log \frac{r_0}{\kappa} \right) + \varepsilon(r) \log r. \]

It follows from the last two inequalities that

\[ \varepsilon(r) \log r + C + \frac{N}{2} \log \frac{r_0}{\kappa} \geq \frac{N}{2} \left( \frac{1}{\psi(t)} - \frac{1}{2\pi} \right) \frac{dt}{t} \]

\[ = \frac{N\pi}{2} \int_{r_0}^{kr} \frac{2\pi - \psi(t)}{2\pi \psi(t)} \frac{dt}{t} \]

\[ \geq \frac{N}{4\pi} \int_{r_0}^{kr} (2\pi - \psi(t)) \frac{dt}{t}. \]

Since \( \varepsilon(r) \) is decreasing and \( \varepsilon(r) \log r \to \infty \) as \( r \to \infty \) we obtain

\[ (3.3) \quad \int_{r_0}^{r} (2\pi - \psi(t)) \frac{dt}{t} \leq \frac{4\pi}{N} \left( \varepsilon \left( \frac{r}{\kappa} \right) \log \frac{r}{\kappa} + C + \frac{N}{2} \log \frac{r_0}{\kappa} \right) \leq \frac{5\pi}{N} \varepsilon(r) \log r \]

for large \( r \).

Let now \( F \) be the function obtained from \( f \) by the logarithmic change of variable. With \( W = \bigcup_{j=1}^{N} \exp^{-1}(U_j) \) and \( H = \{ z \in \mathbb{C} : \text{Re} z > \log R \} \) we thus have

\[ F : W \to H, \quad F(z) = \log f(e^z), \]

for some branch of the logarithm. Moreover, \( F \) maps every component of \( W \) bijectively onto \( H \).

The real part of \( F \) is large on the set \( L = \bigcup_{j=1}^{N} \exp^{-1}(V_j) \). In fact,

\[ L = \{ z \in W : \text{Re} F(z) \geq \exp(\beta \text{Re} z) \}. \]

We put

\[ T = \{ z \in L : F^n(z) \in L \text{ for all } n \in \mathbb{N} \}. \]

For \( z \in T \) we then have

\[ \text{Re} F^n(z) \geq E^n_\beta(\text{Re} z) \]

and thus \( \text{Re} F^n(z) \to \infty \) as \( n \to \infty \). It follows that

\[ |f^n(e^z)| = \exp(\text{Re} F^n(z)) \to \infty \]

so that \( \exp(T) \subset I(f) \).
We shall show that area($T$) > 0. This then implies that area($I(f)$) > 0. In order to prove that area($T$) > 0 we consider for $n \geq 0$ the set

$$T_n = \{ z \in L : F^k(z) \in L \text{ for } 0 \leq k \leq n \}$$

so that $T_0 = L$. Then

$$T = \bigcap_{n=1}^{\infty} T_n.$$ 

Let $S = \mathbb{C} \setminus L$ and put

$$\Psi(x) = \text{meas} \{ y \in [0, 2\pi] : x + iy \in S \}$$

for $x > \log R$. Since for $x + iy \in L$ we have $e^x e^{iy} \in \bigcup_{j=1}^{N} V_j$ it follows that

$$\Psi(x) = 2\pi - \psi(e^x).$$

From (3.3) we deduce that

$$\int_{x}^{e^x} \Psi(s) ds = \int_{r_0}^{e^x} \Psi(\log t) \frac{dt}{t} = \int_{r_0}^{e^x} (2\pi - \psi(t)) \frac{dt}{t} \leq \frac{5\pi}{N} \epsilon(e^x).$$

We put $\delta(x) = \epsilon(e^x)$. It follows that if $x \geq \log r_0$, then

$$\int_{x_0}^{x} \Psi(s) ds \leq \frac{5\pi}{N} \delta(x).$$

For $z = x + iy \in \mathbb{C}$ with $x > 2\log R$ we denote by $Q(z)$ the square of sidelength $x$ centered at $z$. Thus

$$Q(z) = \left\{ \zeta \in \mathbb{C} : |\text{Re } \zeta - x| \leq \frac{1}{2} x, |\text{Im } \zeta - y| \leq \frac{1}{2} x \right\}.$$ 

Now

$$\text{area} \left( \{ z \in S : \{ x_1 \leq \text{Re } z \leq x_2, y_0 \leq \text{Im } z \leq y_0 + 2\pi \} \right) = \int_{x_1}^{x_2} \Psi(s) ds$$

for $\log R < x_1 < x_2$ and $y_0 \in \mathbb{R}$. Since $Q(z)$ can be covered by $[\frac{x}{2\pi} + 1]$ horizontal strips of width $2\pi$ we obtain

$$\text{area}(Q(z) \cap S) \leq \left( \frac{x}{2\pi} + 1 \right) \int_{\frac{x}{2\pi}}^{\frac{x}{2\pi}} \Psi(s) ds \leq \left( \frac{x}{2\pi} + 1 \right) \frac{5\pi}{N} \delta \left( \frac{3}{2} x \right) \frac{3}{2} x \leq \frac{4}{N} \delta(x) x^2$$

for large $x$. Recall that for measurable sets $A, B \subset \mathbb{C}$ the density of $A$ in $B$ is defined by

$$\text{dens}(A, B) = \frac{\text{area}(A \cap B)}{\text{area}(B)}.$$ 

With this notation (3.4) takes the form

$$\text{dens}(S, Q(z)) \leq \frac{4}{N} \delta(x).$$

We now fix $n \in \mathbb{N}$ and consider $u \in T_{n-1} \setminus T_n$ with $\text{Re } u > x_0$ for some large number $x_0$ to be determined later. Put $v = F^n(u)$ and $x_n = E^n_\beta(x_0)$, where $E^n_\beta(x) = e^{\beta x}$. Then

$$\text{Re } v \geq E^n_\beta(\text{Re } u) \geq x_n.$$ 

A standard argument (cf. Remark 3.1 at the end of this section) using Koebe’s distortion theorem shows that for large $u$ and $v$ the branch $\phi_n$ of the inverse function of
$F^n$ which satisfies $\phi_n(v) = u$ extends to a univalent map on $B(v, \frac{3}{4} \text{Re } v)$ and thus has bounded distortion on $Q(v)$. It follows that there exists a constant $K$ such that

$$\text{dens}(\phi_n(Q(v) \cap S), \phi_n(Q(v))) \leq K \text{dens}(S, Q(v)) \leq \frac{4K}{N} \delta(x_n).$$

Moreover, Koebe’s theorem yields that there exist positive constants $\sigma, \tau$ such that if $r_n(u) = |\phi'_n(v)| \cdot \text{Re } v = \frac{\text{Re } F^n(u)}{|(F^n)'(u)|}$, then

$$B(u, \sigma r_n(u)) \subset \phi_n(Q(v)) \subset B(u, \tau r_n(u)).$$

It can be deduced from (3.2) and the chain rule that

$$r_n(u) \leq 5\pi$$

if $x_0$ is sufficiently large. From (3.6) and (3.7) we can deduce that

$$\text{dens}(F^{-n}(S), B(u, \tau r_n(u))) \leq \frac{4K}{N} \left(\frac{T}{\sigma}\right)^2 \delta(x_n).$$

We now fix $w_0$ with $\text{Re } w_0 > 2x_0$ and consider the square $P = Q(w_0)$. Suppose that $n \in \mathbb{N}$ and

$$\text{dens}(T_{n-1}, P) \geq \frac{1}{2}.$$

By Lemma 3.1 we can find an at most countable subset $A$ of $T_{n-1} \cap P$ such that the disks $B(u, \tau r_n(u)), u \in A$, cover $T_{n-1} \cap P$, with no point being covered more than $4^4$ times. With

$$P' = \left\{ z \in \mathbb{C} : |\text{Re } (z - w_0)| < \frac{1}{2} \text{Re } w_0 + 5\pi \tau, |\text{Im } (z - w_0)| < \frac{1}{2} \text{Re } w_0 + 5\pi \tau \right\}$$

we have $B(u, \tau r_n(u)) \subset P'$ for all $u \in A$ by (3.8), and for large $x_0$ we also have $\text{area}(P') \leq 2 \text{area}(P)$.

We now deduce from (3.9) and (3.10) that

$$\text{area}(F^{-n}(S) \cap T_{n-1} \cap P) \leq \text{area} \left( F^{-n}(S) \cap \bigcup_{u \in A} B(u, \tau r_n(u)) \right)$$

$$\leq \sum_{u \in A} \text{area}(F^{-n}(S) \cap B(u, \tau r_n(u)))$$

$$\leq \frac{4K}{N} \left(\frac{T}{\sigma}\right)^2 \delta(x_n) \sum_{u \in A} \text{area}(B(u, \tau r_n(u)))$$

$$\leq \frac{4K}{N} \left(\frac{T}{\sigma}\right)^2 4^4 \delta(x_n) \text{area}(P')$$

$$\leq \frac{8K}{N} \left(\frac{T}{\sigma}\right)^2 4^4 \delta(x_n) \text{area}(P)$$

$$\leq \frac{16K}{N} \left(\frac{T}{\sigma}\right)^2 4^4 \delta(x_n) \text{area}(T_{n-1} \cap P).$$

With

$$\eta = \frac{16K}{N} \left(\frac{T}{\sigma}\right)^2 4^4$$
we thus have
\[ \text{dens}(F^{-n}(S), T_{n-1} \cap P) \leq \eta \delta(x_n). \]
Since \( F^{-n}(S) \cap T_{n-1} = T_{n-1} \setminus T_n \) we obtain
\[ \text{dens}(T_{n-1} \setminus T_n, T_{n-1} \cap P) \leq \eta \delta(x_n) \]
and thus
\[ \text{dens}(T_n, T_{n-1} \cap P) \geq 1 - \eta \delta(x_n). \]
Induction shows that
\[ \text{dens}(T_n, T_0 \cap P) \geq \prod_{k=1}^{n} (1 - \eta \delta(x_k)), \]
as long as
\[ \text{dens}(T_k, P) \geq \frac{1}{2} \quad \text{for } k \leq n - 1. \]
Now
\[ \delta(x_n) = \delta(E_{\beta}^n(x_0)) = \varepsilon(\exp(E_{\beta}^n(x_0)) \leq \varepsilon(E_{\beta}^{n+1}(x_0)) = \frac{1}{\Phi(E_{\beta}^{n+1}(x_0))} = \frac{1}{\mu_{n+1} \Phi(x_0)}. \]
We conclude that the infinite product \( \prod_{k=1}^{\infty} (1 - \eta \delta(x_k)) \) converges and by choosing \( x_0 \)
large we may achieve that
\[ \prod_{k=1}^{\infty} (1 - \eta \delta(x_k)) \geq \frac{3}{4}. \]
From (3.12) we deduce that
\[ \text{dens}(T_0, P) = 1 - \text{dens}(L, P) \geq \frac{2}{3} \]
for large \( w_0. \)
Suppose now that (3.12) and hence (3.11) holds for some \( n \in \mathbb{N}. \) Then, since \( T_n \subset T_0, \)
\[ \text{dens}(T_n, P) = \text{dens}(T_n, P \cap T_0) \cdot \text{dens}(T_0, P) \geq \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2} \]
by (3.13) and (3.14).
Thus (3.12) and hence (3.11) hold with \( n - 1 \) replaced by \( n. \) Induction thus shows
that (3.11) holds for all \( n \in \mathbb{N}. \) It follows that
\[ \text{dens}(T, P \cap T_0) \geq \prod_{k=1}^{\infty} (1 - \eta \delta(x_k)) \geq \frac{3}{4}. \]
In particular, \( \text{area}(T) > 0. \)

Remark 3.1. We used in the proof that the branch \( \phi_n \) of the inverse function of \( F^n \) which
maps \( v = F^n(u) \) to \( u \) extends to a univalent map on \( B(v, \frac{3}{4} \text{Re } v). \) In order to see this
we note that if \( \phi \) is the branch of \( F^{-1} \) which maps \( v \) to \( F^{-1}(u) \), then \( \phi \) is univalent
in \( H \) and it follows from (3.1) that
\[ \text{diam } \phi \left( B(v, \frac{3}{4} \text{Re } v) \right) \leq \frac{3}{2} \text{Re } v \max_{w \in B(v, \frac{3}{4} \text{Re } v)} |\phi'(w)| \leq \frac{3}{2} \text{Re } v \frac{4\pi}{\frac{4}{3} \text{Re } v - \log R} \leq 48\pi \]
if \( \text{Re } v < 8 \log R. \) We conclude that if \( u \) and hence \( F^{-1}(u) \) are large enough, then
\[ \phi \left( B(v, \frac{3}{4} \text{Re } v) \right) \subset B \left( F^{-1}(u), \frac{3}{4} F^{-1}(u) \right). \]
The above claim now follows by induction.

Essentially the same argument can be found, e.g., in [1, 5]. The argument gets much simpler if the postsingular set

\[ P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1})) \]

is bounded. (Here \( \text{sing}(f^{-1}) \) denotes the set of singularities of the inverse function of \( f \).) We note that if \( f \in B \) and \( f_\lambda(z) = \lambda f(z) \), then \( P(f_\lambda) \) is bounded for small \( \lambda \). A theorem of Rempe [19] implies that there exists \( R_\lambda > 0 \) such that \( f \) and \( f_\lambda \) are quasiconformally conjugate on the set \( \{ z : |f^n(z)| \geq R_\lambda \text{ for all } n \geq 0 \} \). Since quasiconformal mappings map sets of positive area to sets of positive area, the conclusion for \( f \) follows from that for \( f_\lambda \). Thus it actually suffices to consider the special case that \( P(f_\lambda) \) is bounded.

**Remark 3.2.** Let \( f \) be a function meromorphic in the plane which has \( N \) logarithmic singularities over infinity. Denote by \( U_1, U_2, \ldots, U_N \) the corresponding logarithmic tracts. With

\[ U = \bigcup_{j=1}^{N} U_j \quad \text{and} \quad M_U(r, f) = \max_{|z|=r, z \in U} |f(z)| \]

we deduce from the proof that the conclusion of Theorem 1.1 holds if

\[ \log \log M_U(r, f) \leq \left( \frac{N}{2} + \varepsilon(r) \right) \log r \]

for large \( r \). It follows from standard estimates of Nevanlinna theory [11, Theorem 7.1] that \( \log M_U(r, f) \leq 3m(2r, f) \). Using this it is not difficult to see that the conclusion of Theorem 1.1 holds if

\[ \log m(r, f) \leq \left( \frac{N}{2} + \varepsilon(r) \right) \log r \]

and thus, in particular, if

\[ \log T(r, f) \leq \left( \frac{N}{2} + \varepsilon(r) \right) \log r. \]

The dynamics of meromorphic functions with logarithmic singularities are studied for example in [2, 6].

4. An example

We consider Mittag-Leffler’s function

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \]

for a parameter \( \alpha \in (0, 2) \). It satisfies the following conditions:

(i) \( g(E_\alpha) = \frac{1}{\alpha} \)
(ii) \( E_\alpha \) is bounded in the sector \( \{ re^{it} : r > 0, |t - \pi| \leq (1 - \frac{1}{2}\alpha) \pi \} \)
(iii) \( E_\alpha \in B \)

Properties (i) and (ii) are well-known; see, e.g., [11, pp. 83-86]. Since we could not find a proof of (iii) in the literature, we indicate an argument to prove (iii) below.

It follows from (ii) and (iii) and a theorem of Eremenko and Lyubich [10, Theorem 7] that \( \text{area}(I(E_\alpha)) = 0 \). Moreover, the arguments yield (cf. [10, Theorem 8]) that if \( \lambda > 0 \)
is chosen so small that the Fatou set of \( \lambda E_\alpha \) consists of a single, completely invariant attracting basin, then area\((J(\lambda E_\alpha))\) = 0.

We see that there exist functions \( f \in B \) whose order is arbitrarily close to \( \frac{1}{2} \) such that area\((I(f))\) = area\((J(f))\) = 0. Considering \( f(z) = E_\alpha(z^N) \) we obtain functions where \( A_R \) has \( N \) components and where \( \theta(f) \) is close to \( \frac{1}{2}N \). Thus the function \( \varepsilon(r) \) in Theorem 1.1 cannot be replaced by a positive constant \( \varepsilon \).

Let us now prove property (iii). From [11, pp. 84-85] we get the following representation for \( E_\alpha \), where \( \theta = 1/\alpha \):

\[
E_\alpha(z) = w_1(z) \quad \text{for} \quad \frac{1}{2}\alpha \pi < |\arg(z)| \leq \pi, \\
E_\alpha(z) = w_2(z) + \theta \exp(z^\theta) \quad \text{for} \quad |\arg(z)| \leq \frac{1}{2}\alpha \pi + \delta,
\]

where \( 0 < \delta \leq \max \{ \frac{1}{2}\alpha \pi, (1 - \frac{1}{2}\alpha) \pi \} \) and \( w_i(z) = O(1/|z|) \) as \( |z| \to \infty \), for \( i = 1, 2 \).

Note that Properties (i) and (ii) follow immediately from (4.1) and (4.2).

To prove Property (iii), put \( S_\delta = \{ z : |\arg(z)| \leq \frac{1}{2}\alpha \pi + \delta \} \). For \( z \in S_{\delta/2} \) we have \( B(z, |z| \sin(\delta/2)) \subset S_\delta \) and Cauchy’s formula yields

\[
|E'_\alpha(z) - \theta^2 z^{\theta - 1} \exp(z^\theta)| = |w'_2(z)| = \frac{1}{2\pi i} \int_{\partial B(z, |z| \sin(\frac{\delta}{2}))} \frac{w_2(z)}{z - \zeta^2} d\zeta = O\left( \frac{1}{|z|^2} \right)
\]
as \( |z| \to \infty \), uniformly in \( z \in S_{\delta/2} \). For \( z \in \mathbb{C} \setminus S_0 \) we have \( B(z, |z| \sin(\delta/2)) \subset \mathbb{C} \setminus S_0 \) and in the same way Cauchy’s formula yields

\[
|E'_\alpha(z)| = |w'_1(z)| = O\left( \frac{1}{|z|^2} \right)
\]
as \( |z| \to \infty \), uniformly in \( \mathbb{C} \setminus S_{\delta/2} \).

We now show that the set of critical values of \( E_\alpha \) is bounded. Since \( E_\alpha \) is bounded in \( \mathbb{C} \setminus S_0 \) we have to consider only the critical points in \( S_0 \). So let \( \xi \in S_0 \) be a critical point of \( E_\alpha \); that is, \( E'_\alpha(\xi) = 0 \). Then

\[
\theta^2 |\xi|^\theta - 1 |\exp(\xi^\theta)| \leq \frac{C_1}{|\xi|^2}
\]
for some constant \( C_1 \) by (4.3) and thus

\[
|E_\alpha(\xi)| \leq \theta |\exp(\xi^\theta)| + \frac{C_2}{|\xi|} \leq \frac{C_1}{\theta |\xi|^\theta + 1} + \frac{C_2}{|\xi|^2}
\]
for some constant \( C_2 \) by (4.2). It follows that the set of critical values of \( E_\alpha \) is bounded. Since \( E_\alpha \) has only finitely many asymptotic values by the Denjoy-Carleman-Ahlfors-Theorem, it follows that \( f \in B \). (Actually the only asymptotic value of \( E_\alpha \) is 0. This can be deduced from (4.1) and (4.2).

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