Massive super Yang-Mills quantum mechanics: classification and the relation to supermembrane

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Abstract
We classify the supersymmetric mass deformations of all the super Yang-Mills quantum mechanics, which are obtained by dimensional reductions of minimal super Yang-Mills in space-time dimensions: ten, six, four, three and two. The resulting actions can be viewed as the matrix descriptions of supermembranes in nontrivial backgrounds of one higher dimensional supergravity theories. We also discuss the utmost generalization of the light-cone formulation of the Nambu-Goto action for a $p$-brane, including time dependent backgrounds.
1 Introduction and summary

Supersymmetry and gauge symmetry are two principal symmetries in the Lagrangian description of a p-brane in superstring or M-theory. Demanding both, the Lagrangian becomes considerably constrained. For instance, when the Lagrangian contains only the gauge multiplet, the action is uniquely given by the minimal super Yang-Mills theory. As is well known, the minimal super
Yang-Mills theories exist not in arbitrary spacetime dimensions. They do so only in ten, six, four, three and two dimensions, which coincide with the dimensions where the superstrings can be defined. The constraints are due to the requirement of the relevant Fierz identity in order to implement the non-Abelian gauge symmetry.

Dimensional reductions of the minimal super Yang-Mills field theories lead to one-dimensional gauge theories, i.e. super Yang-Mills quantum mechanics (SYMQM). In contrast to the rigidity of the formers, the latters allow mass deformations, without reducing the number of supersymmetries, as discovered for the \( \mathcal{N} = 16 \) [1], \( \mathcal{N} = 8 \) [2] and \( \mathcal{N} = 1 + 1 \) [3] cases, whose field theory origin can be traced in ten, six and two dimensions, respectively. Throughout the paper, \( \mathcal{N} \) denotes the number of real dynamical supersymmetries in super Yang-Mills quantum mechanics. Especially, the dimensional reduction of the minimal super Yang-Mills in two-dimensions doubles the number of dynamical supersymmetries [3], whence our notation ‘\( \mathcal{N} = 1 + 1 \)’.

In the present paper, we classify, in a systematic way, the mass deformations of all the super Yang-Mills quantum mechanics with \( \mathcal{N} = 16, 8, 4, 2, 1 + 1 \) supersymmetries. Undeformed SYMQM are given by the dimensional reductions of the minimal super Yang-Mills field theories. Utilizing the properties of the spinors and gamma matrices in each case, we first consider adding generic mass terms for fermions and then perform the supersymmetric completion. Our final results are summarized in Table 1 where ‘\( D_{\text{YM}} \)’ denotes the spacetime dimension of the corresponding minimal super Yang-Mills field theory, and \( \mu, \mu_1, \mu_2 \) are constant deformation parameters, while \( \Lambda(t), \rho(t) \) are arbitrary time dependent functions.

| massive SYM quantum mechanics | \( D_{\text{YM}} \) | splitting of \( \text{SO}(D_{\text{YM}} - 1) \) | superalgebra | deformation parameter |
|-----------------------------|-----------------|---------------------------------|-------------|-----------------|
| \( \mathcal{N} = 16 \)     | 10              | \( \text{SO}(6) \times \text{SO}(3) \) | \( \text{su}(2|4) \) | \( \mu \)        |
| \( \mathcal{N} = 8 \) type I | 6               | \( \text{SO}(3) \times \text{SO}(2) \) | \( \text{su}(2|2) \) | \( \mu \)        |
| \( \mathcal{N} = 8 \) type II | 6              | \( \text{SO}(4) \)               | \( \text{su}(2|1) \oplus \text{su}(2|1) \) | \( \mu \)        |
| \( \mathcal{N} = 4 \) type I | 4               | \( \text{SO}(3) \)               | \( \text{su}(2|1) \) | \( \mu_1, \mu_2 \) |
| \( \mathcal{N} = 4 \) type II | 4              | \( \text{SO}(2) \)               | \( \text{Clifford}_4(\mathbb{R}) \) | \( \mu \)        |
| \( \mathcal{N} = 2 \)       | 3               | \( \text{SO}(2) \)               | \( \text{Clifford}_2(\mathbb{R}) \) | \( \mu \)        |
| \( \mathcal{N} = 1 + 1 \)   | 2               | \( \text{SO}(1,2) \)             | \( \text{osp}(1|2, \mathbb{R}) \) | \( \Lambda(t), \rho(t) \) |

Table 1: Classification of massive super Yang-Mills quantum mechanics

For \( \mathcal{N} = 16 \) and \( \mathcal{N} = 2 \) cases the deformations are unique, given by a single mass parameter. Both for \( \mathcal{N} = 8 \) and \( \mathcal{N} = 4 \) cases, the deformations are two folds: type I and type II where only the former contains a Myers term [4]. In particular, the number of deformation parameters in \( \mathcal{N} = 4 \) type I case is two, while in other cases it is one. For \( \mathcal{N} = 1 + 1 \) case, the deformation
parameters are given by two arbitrary time dependent functions, and accordingly there are infinite number of deformation parameters [3].

Here we focus on the deformations which do not break any supersymmetry of the super Yang-Mills quantum mechanics. We refer to [5] for the analysis on more generic deformations of the $\mathcal{N} = 16$ SYMQM which break supersymmetries partially.

Given the diversity of the massive SYMQM we obtain, it is natural to ask the physical origin of them. So far, apart from our classification scheme, three different ways of predicting or constructing massive super Yang-Mills quantum mechanics are known: (i) the light-cone description of the superparticle or supermembrane, (ii) consistent truncations of the BMN matrix model caused by M5 branes, and (iii) the compactification of four-dimensional super Yang-Mills on $S^3$. In the following we briefly discuss how our results may fit into such physical insights.

As is well known, the light-cone formulation [6–9] of the superparticle or supermembrane in the eleven-dimensional flat background gives the undeformed $\mathcal{N} = 16$ SYMQM, which is conjectured to describe the eleven-dimensional $\mathcal{M}$-theory in the flat background [10–13]. In 2002, Berenstein, Maldacena and Nastase (BMN) derived a mass deformation of the $\mathcal{N} = 16$ SYMQM [1]. Their derivation was based on the light-cone formulation of the superparticle in the maximally supersymmetric eleven-dimensional pp-wave background [14–17]. An alternative derivation is also available from the supermembrane aspect [18]. The uniqueness of the mass deformation of the $\mathcal{N} = 16$ SYMQM we show in the present paper is consistent with the fact that the maximally supersymmetric backgrounds of the eleven-dimensional supergravity are exhausted by the pp-wave and $AdS_4 \times S^7$, $AdS_7 \times S^4$ [14].

Clearly one can expect the same phenomena occur in lower dimensional supergravity theories: seven, five, four and three. Namely, any supersymmetric pp-wave background therein should give rise to a novel massive SYMQM. Indeed, the known maximally supersymmetric plane wave solutions in five- and four-dimensional supergravity theories (see e.g. [19, 20]) correspond to our $\mathcal{N} = 4$ type II and $\mathcal{N} = 2$ massive SYMQM, respectively. In this spirit, our $\mathcal{N} = 8, \mathcal{N} = 4, \mathcal{N} = 2, \mathcal{N} = 1 + 1$ massive super Yang-Mills quantum mechanics can be identified as $\mathcal{M}$-theory matrix models in curved backgrounds of the noncritical dimensions: seven, five, four and three, respectively.\(^1\)

Supersymmetric embedding of the M2 and M5 branes into the eleven-dimensional pp-wave background has been analyzed in [26]. It turns out that there are two types of half BPS M5-branes, such that one preserves $SO(3) \times SO(2)$ isometry of the transverse five-dimensional geometry and the other preserves $SO(4)$, whilst the background pp-wave has the isometry $SO(3) \times SO(6)$. This

\(^1\)For the discussion of noncritical $\mathcal{M}$-theories see [21–25].
analysis consistently matches with our results: \( \mathcal{N} = 8 \) massive SYMQM type I and type II, respectively. Namely, truncating any longitudinal degrees of freedom (which should be described by a self-dual two-form tensor), our two types of \( \mathcal{N} = 8 \) massive super Yang-Mills quantum mechanics describe the transverse degrees of freedom of the M5-branes, treating them as ‘point-like’ particles (see [2, 27, 28] for further discussion).

Another way of obtaining massive SYMQM was proposed in [29], to compactify the maximally supersymmetric four-dimensional super Yang-Mills on \( S^3 \) utilizing the superconformal invariance of the four-dimensional gauge theory [30]. The harmonic analysis of the dimensional reduction on \( S^3 \) produces a quantum mechanical system with an infinite tower of massive modes, which can be arranged as irreducible representations of the \( \text{SO}(4) \equiv \text{SU}(2) \times \text{SU}(2) \) isometry of \( S^3 \). When one keeps ‘half’ of the lowest lying modes, the resulting quantum mechanical system is precisely the BMN matrix model [29]. The analysis was carefully revisited and generalized recently in [31]. It is obvious that one can apply this procedure to any other super Yang-Mills theory in four-dimensions with less supersymmetries and obtain a corresponding SYMQM. From the super Yang-Mills with eight supercharges one can reproduce our \( \mathcal{N} = 8 \) type I SYMQM, while from the super Yang-Mills with four supercharges, i.e. pure super QCD, one can partially derive our \( \mathcal{N} = 4 \) type I SYMQM with the restriction \( \mu_1 = 0 \). It is worth while to note that not all of the massive SYMQM in our classification can be identified in the manners discussed above.

The organization of the present paper is as follows:

In section 2 as a motivation to analyze the supersymmetric deformation of each super Yang-Mills quantum mechanics, we review the light-cone formulation of the Nambu-Goto action for a generic \( p \)-brane. In particular, we discuss the most general backgrounds which admit the light-cone formulation. The light-cone formulation converts, without any approximation, the relativistic square root action to a non-relativistic one where the kinetic term is simply velocity squared. Especially for a membrane, the resulting action resembles the Yang-Mills action.

In each of the following sections 3, 4, 5, 6, 7 we discuss the \( \mathcal{N} = 16, 8, 4, 2, 1 + 1 \) massive super Yang-Mills quantum mechanics separately. We first briefly set up our conventions for the gamma matrices as well as the spinors, such as \( \text{su}(2) \) Majorana-Weyl spinor in the \( \mathcal{N} = 8 \) case. We then present explicitly the massive super Yang-Mills quantum mechanical systems with the supersymmetry transformations of all the variables. We further identify the corresponding super Lie algebras and write down the maximally supersymmetric bosonic configurations.

Section 8 contains some comments. The appendix carries out our derivations of the most general mass deformations of all the super Yang-Mills quantum mechanics.
2 Light-cone formulation of a $p$-brane action revisited

In this section, we review, with the utmost generalization, the light-cone formulation of a $p$-brane action. We first consider a relativistic point particle, and then generalize the analysis to a generic $p$-brane, in view of applying to the $p = 2$ i.e. membrane case. We take Nambu-Goto action coupled to a $p + 1$ form field as a relativistic description of a $p$-brane. With an embedding of a $(p + 1)$-dimensional worldvolume into a $D$-dimensional target spacetime

$$x(\xi) : \xi^\mu (0 \leq \mu \leq p) \rightarrow x^M (0 \leq M \leq D - 1),$$

the action for a $p$-brane reads

$$S_{p-\text{brane}} = \int \! d^{p+1} \xi \, L_{p-\text{brane}}, \quad L_{p-\text{brane}} = L_{\text{N.G.}} + L_{C_{p+1}},$$

where the Lagrangian consists of two parts:

$$L_{\text{N.G.}} = -T \sqrt{-\det G_{\mu\nu}}, \quad L_{C_{p+1}} = -\frac{1}{(p+1)!} \epsilon^{\mu_1\mu_2\cdots\mu_{p+1}} C_{\mu_1\mu_2\cdots\mu_{p+1}}.$$ (2.3)

Here $T$ is the $p$-brane tension of mass dimension $p + 1$, while $G_{\mu\nu}$ and $C_{\mu_1\mu_2\cdots\mu_{p+1}}$ are respectively the induced metric and $p + 1$ form gauge field on the worldvolume

$$G_{\mu\nu}(\xi) = \partial_\mu x^M \partial_\nu x^N G_{MN}(x),$$

$$C_{\mu_1\mu_2\cdots\mu_{p+1}}(\xi) = \partial_{\mu_1} x^M \partial_{\mu_2} x^M \cdots \partial_{\mu_{p+1}} x^{M_{p+1}} C_{M_1 M_2 \cdots M_{p+1}}(x).$$ (2.4)

The Nambu-Goto Lagrangian $L_{\text{N.G.}}$ measures the volume of the $p$-brane in the target spacetime, and may be replaced by a Polyakov action:

$$L_{\text{Poly.}} = -\frac{1}{2} T \sqrt{-h} \left[ h^{-1\mu\nu} \partial_\mu x^L \partial_\nu x^M G_{LM}(x) + 1 - p \right].$$ (2.5)

Integrating out the auxiliary worldvolume metric $h_{\mu\nu}$, using its equation of motion

$$h_{\mu\nu} = \partial_\mu x^L \partial_\nu x^M G_{LM}(x) : \text{ for } p \neq 1,$$

$$h_{\mu\nu} \propto \partial_\mu x^L \partial_\nu x^M G_{LM}(x) : \text{ for } p = 1,$$ (2.6)

one recovers $L_{\text{N.G.}}$. Henceforth we focus on the Nambu-Goto Lagrangian.

We show in the following subsections that, for a certain class of backgrounds, any sector of a fixed ‘light-cone momentum’ can be exactly described by a ‘non-relativistic’ action where the kinetic term is simply velocity squared. An exact galilean invariance of the light-cone formalism
in a flat background has been known for the point particle by Susskind since 1967 [6, 7], and for the supermembrane by de Wit, Hoppe and Nicolai in 1988 [8] (for generic \(p\)-branes see [9]; also for a non-light-cone formalism see [32]). A new ingredient in the present review is an utmost generalization to nontrivial time-dependent backgrounds.

With the light-cone coordinates

\[
x^\pm = \frac{1}{\sqrt{2}} (\pm x^0 + x^{D-1}) ,
\]

we focus on a \(D\)-dimensional target spacetime background given by a metric \(G_{LM}\) and a \(p+2\) form field strength \(F_{(p+2)} = dC_{(p+1)}\). We require an isometry along the light-cone direction \(x^-\),

\[
\frac{\partial G_{LM}}{\partial x^-} = 0, \quad \frac{\partial F_{(p+2)}}{\partial x^-} = 0,
\]

and further set certain components of \(G_{LM}\), \(F_{(p+2)}\) to vanish,

\[
G_{--} = 0, \quad G_{-a} = 0, \quad F_{-M_1M_2\cdots M_{p+1}} = 0, \quad F_{a_1a_2\cdots a_{p+2}} = 0.
\]

Here we denote the target spacetime coordinates by \(x^M = (x^+, x^-, y^a)\) where \(a, b\) indices are for a \(d\)-dimensional Euclidean subspace, running from 1 to \(d = D - 2\). In summary, they are of the form:

\[
d s^2 = A(y, x^+) \left[ 2 dx^+ dx^- - 2 V(y, x^+) dx^+ dx^+ + 2 J_a(y, x^+) dx^+ dy^a + g_{ab}(y, x^+) dy^a dy^b \right],
\]

\[
F_{(p+2)} = \frac{1}{(p+1)!} F_{+a_1a_2\cdots a_{p+1}}(y, x^+) dx^+ \wedge dy^{a_1} \wedge \cdots \wedge dy^{a_{p+1}}.
\]

Especially, the only non-vanishing components of the field strength are \(F_{+a_1a_2\cdots a_{p+1}}(y, x^+)\). This implies, from \(dF = 0\) and the Poincaré lemma, that there exists a \(p\)-form field \(V_{(p)}(y, x^+)\) depending on \(y, x^+\) only such that

\[
F_{+a_1a_2\cdots a_{p+1}}(y, x^+) = \partial_{a_1} V_{a_2\cdots a_{p+1}} + (-1)^p \partial_{a_2} V_{a_3\cdots a_{p+1}a_1} + \cdots + (-1)^p \partial_{a_{p+1}} V_{a_1\cdots a_p}.
\]

These properties are crucial for the light-cone formulation we analyze below. All the known pp-wave type solutions assume this form (quite often \(A = 1\)). It is worth to note that unlike the metric \(G_{LM}(y, x^+)\) and the field strength \(F(y, x^+)\), the gauge field \(C_{(p+1)}(x)\) may depend on the light-cone coordinate \(x^-\).
### 2.1 Point particle aspect

In this subsection, we focus on a point particle \((p = 0)\) which propagates in the background \((2.10)\). After a gauge choice to identify the worldline with a target spacetime light-cone coordinate

\[
\tau = \xi^0 = x^+, \quad (2.12)
\]

the action \((2.3)\) reads

\[
S_{\text{particle}} = \int d\tau \left( L_{\text{N.G.}} - C_+(x) - \dot{x}^- C_-(x) - \dot{y}^a C_a(x) \right),
\]

\[
L_{\text{N.G.}} = -m \sqrt{\mathcal{A}(y,\tau)} \left( 2 \dot{x}^- - 2V(y,\tau) + 2J_a(y,\tau) \dot{y}^a + g_{ab}(y,\tau) \dot{y}^a \dot{y}^b \right). \quad (2.13)
\]

Here \(m\) is the mass of the particle, and dot denotes the derivative with respect to the worldline coordinate \(\tau\), such that the canonical momenta are

\[
p_- = \frac{mA}{\sqrt{\mathcal{A}(2\dot{x}^- - 2V + 2J_a \dot{y}^a + g_{ab} \dot{y}^a \dot{y}^b)}} - C_-, \quad p_a = \frac{mA \left(g_{ab} \dot{y}^b + J_a\right)}{\sqrt{\mathcal{A}(2\dot{x}^- - 2V + 2J_c \dot{y}^c + g_{cd} \dot{y}^c \dot{y}^d)}} - C_a = (p_- + C_-) \left(g_{ab} \dot{y}^b + J_a\right) - C_a. \quad (2.14)
\]

Inverting these, we express the velocities in terms of the phase space variables

\[
\dot{y}^a = \frac{\bar{g}^{ab} \mathcal{P}_b}{\mathcal{P}_-} - \bar{J}^a, \quad \dot{x}^- = V + \frac{1}{2} J_a \bar{J}^a - \frac{\bar{g}^{ab} \mathcal{P}_a \mathcal{P}_b + m^2 \mathcal{A}}{2 \mathcal{P}_-^2}, \quad (2.15)
\]

where we define ‘modified momenta’ \(\mathcal{P}_-, \mathcal{P}_a\) by

\[
\mathcal{P}_-(p, x) := p_- + C_-(x^-, y, \tau), \quad \mathcal{P}_a(p, x) := p_a + C_a(x^-, y, \tau), \quad (2.16)
\]

and set \(\bar{g}^{ab}\) to be the inverse of \(g_{cd}\) such that

\[
\bar{g}^{ab} g_{bc} = \delta^a_c, \quad \bar{J}^a(y, \tau) := \bar{g}^{ab}(y, \tau) J_b(y, \tau), \quad \bar{J}^2(y, \tau) := J_a(y, \tau) \bar{J}^a(y, \tau). \quad (2.17)
\]

Note that \(\mathcal{P}_-, \mathcal{P}_a\) reduce to the canonical momenta \(p_-, p_a\) in the absence of the one form field. In terms of the former, the Hamiltonian reads

\[
H = \frac{\bar{g}^{ab}(y,\tau) \mathcal{P}_a(p, x) \mathcal{P}_b(p, x) + m^2 \mathcal{A}(y,\tau)}{2 \mathcal{P}_-(p, x)} + C_+(x^-, y, \tau) - \mathcal{P}_a(p, x) \bar{J}^a(y, \tau)
\]

\[
+ \mathcal{P}_-(p, x) \left(V(y, \tau) + \frac{1}{2} \bar{J}^2(y, \tau)\right). \quad (2.18)
\]
The time evolution of the modified momenta $\mathcal{P}_-$, $\mathcal{P}_a$ are then, from the Hamiltonian dynamics,

\[
\frac{d\mathcal{P}_-}{d\tau} = F_{+-} + F_a \frac{\partial H}{\partial \mathcal{P}_a},
\]

\[
\frac{d\mathcal{P}_a}{d\tau} = F_{+a} + F_{-a} \frac{\partial H}{\partial \mathcal{P}_-} + F_{ba} \frac{\partial H}{\partial \mathcal{P}_b} - \hat{\partial} \frac{\partial}{\partial y^a} (H - C_+),
\]

where the derivative with a hat $\hat{\partial}$ denotes the partial derivative with respect to $(\mathcal{P}, x)$, rather than $(p, x)$. In fact, all the partial derivatives above can be replaced by $\hat{\partial}$, since in the Hamiltonian (2.18) the canonical momenta appear only through the modified momenta.

From our general assumption (2.10), we have $F_{-M} = F_{ab} = 0$, and there exists a function $V(y, \tau)$ satisfying $\partial_a V = F_{+a}$ (2.11). Thus, $\mathcal{P}_-$ is a constant of motion

\[
\frac{d\mathcal{P}_-}{d\tau} = 0,
\]

and for any sector with a fixed constant of motion $\mathcal{P}_-$, the dynamics of the rest variables, $y^a$, $1 \leq a \leq d$, can be described by the following Hamiltonian

\[
H^-(\mathcal{P}_a, y^b, \tau) = \frac{\bar{g}^{ab}(y, \tau) \mathcal{P}_a \mathcal{P}_b + m^2 A(y, \tau)}{2\mathcal{P}_-} - \mathcal{P}_a \bar{J}^a(y, \tau)
\]

\[
+ \mathcal{P}_- V(y, \tau) + \frac{1}{2} \mathcal{P}_- J^2(y, \tau) - V(y, \tau),
\]

such that

\[
\frac{dy^a}{d\tau} = \frac{\partial H^-}{\partial \mathcal{P}_a}, \quad \frac{d\mathcal{P}_a}{d\tau} = - \frac{\partial H^-}{\partial y^a}.
\]

Again, $\hat{\partial}$ denotes the partial derivative with respect to $(\mathcal{P}, x)$, rather than $(p, x)$. Clearly, this Hamiltonian dynamics can be derived from the following ‘non-relativistic’ Lagrangian

\[
\mathcal{L}^-(y, \tau) = \mathcal{P}_- \left[ \frac{1}{2} \bar{g}^{ab}(y, \tau) \dot{y}^a \dot{y}^b + J_a(y, \tau) \dot{y}^a - V(y, \tau) - \frac{1}{2} \hat{m}^2 A(y, \tau) \right] + V(y, \tau),
\]

where we set $\hat{m} := m\mathcal{P}_-^{-1}$, and $\mathcal{P}_-$ should be taken as a $c$-number rather than a dynamical variable. From (2.14), $\mathcal{P}_-$ is always positive.

$\mathcal{L}^-$ is a ‘non-relativistic’ Lagrangian in the sense that the kinetic term in (2.23) is velocity squared. Nevertheless, the resulting dynamics matches perfectly with the relativistic particle motion of a fixed constant of motion $\mathcal{P}_-$. 

8
Now we consider the dynamics of the D0-brane gas. We restrict on the flat subspace, $g_{ab} = \delta_{ab}$. In this case, the above non-relativistic Lagrangian for a single particle (2.23) has a natural\(^2\) generalization to the Yang-Mills quantum mechanics, \textit{i.e.} to the matrix model with $U(N)$ gauge symmetry for the description of $N$ D-particles,

$$L_{\text{YM}}^\text{−} = \mathcal{P} \text{tr} \left[ \frac{1}{2} D_t X^a D_t X_a + J_a (X, \tau) D_t X^a - V (X, \tau) - \frac{1}{2} \hat{m}^2 A (X, \tau) + \cdots \right] + \text{tr} \left[ V (X, \tau) \right].$$

(2.24)

Here $X^a$ is a Hermitian matrix of which the eigenvalues represent the positions of the D-particles. Moreover, in (2.24) the ordinary time derivative is replaced by the covariant time derivative involving a non-dynamical gauge field $A_0$,

$$D_t X = \dot{X} - i [A_0, X].$$

(2.25)

This allows for the gauge symmetry,

$$X \rightarrow U^{-1} X U, \quad A_0 \rightarrow U^{-1} A_0 U + i U^{-1} \partial_\tau U, \quad U \in U(N).$$

(2.26)

The equation of motion of the auxiliary gauge field $A_0$ is a secondary first-class constraint, and the physical states are in the gauge singlet sector.

The reason for the gauging is related to the identical property of the D-particles. The diagonalization of the $X$ is not unique: the Weyl group of $U(N)$ in (2.26) acts by permuting the eigenvalues of $X$. Physically, this reflects the fact that $D$-particles are identical particles [34]. In matrix models, gauging the $U(N)$ symmetry naturally takes care of the ambiguity in the diagonalization. Different diagonalizations correspond to the same gauge orbit and hence to the same physical state [35]. The gauging of the matrix model is also consistent with the gauge theory description of $D$-brane dynamics.

The abbreviated part in (2.24) corresponds to possible terms which vanish when $N = 1$ or the single particle case, such as a commutator of the matrices. Its explicit form can be uniquely determined by requiring both the maximal supersymmetry and the gauge symmetry simultaneously, as we analyze in detail later.

### 2.2 Membrane or $p$-brane aspect

Here we generalize the above analysis on the point particle to a generic $p$-brane of which the dynamics is given by the Nambu-Goto action coupled to a $p + 1$ form (2.3). Apparently the

\(^2\)This is somewhat in contrast to the ordering ambiguity appearing in non-Abelian generalizations of the relativistic action or DBI action. See [33] for some proposals to fix the ambiguity.
action is invariant under the \((p + 1)\)-dimensional worldvolume diffeomorphism \(\xi^\mu \rightarrow \xi'^\mu(\xi)\) provided \(x^M(\xi)\) transforms as a scalar

\[
x^M(\xi) \rightarrow x'^M(\xi) = x^M(\xi') .
\] (2.27)

This induces

\[
G_{\mu\nu}(\xi) \rightarrow G'_{\mu\nu}(\xi) = \frac{\partial \xi'^\kappa}{\partial \xi'^\mu} \frac{\partial \xi'^\lambda}{\partial \xi'^\nu} \ G_{\kappa\lambda}(\xi') .
\] (2.28)

Below we will break this gauge symmetry step by step via some gauge fixing conditions such that, at the end, only the \(p\)-dimensional static volume preserving diffeomorphism will survive.

With the decomposition of the worldvolume coordinates into the temporal and spatial parts

\[
\xi^0 = \tau , \quad \xi^i = \sigma^i \quad ( \, 1 \leq i \leq p ) ,
\] (2.29)

our first gauge fixing is to identify the worldvolume time with a light-cone coordinate in the target spacetime,

\[
\tau \equiv x^+ .
\] (2.30)

Now the unbroken gauge symmetry is

\[
\tau \rightarrow \tau' = \tau , \quad \sigma^i \rightarrow \sigma'^i = f^i(\tau, \sigma) ,
\] (2.31)

under which, in particular, \(G_{\tau i}(\xi)\) component transforms as

\[
G_{\tau i}(\xi) \rightarrow G'_{\tau i}(\xi) = \frac{\partial f^j}{\partial \sigma^i} G_{j k}(\xi') \left( \partial_\tau f^k(\tau, \sigma) + \hat{G}^{kl}(\xi') G_{\tau l}(\xi') \right) ,
\] (2.32)

where \(\hat{G}^{kl}\) is the inverse of the \(p \times p\) matrix \(G_{ij}\) i.e. \(G^{ij} G_{jk} = \delta^i_k\). With arbitrary functions \(f^k(0, \sigma), 1 \leq k \leq p\) at an initial time \(\tau = 0\), one can uniquely fix its time evolution by demanding

\[
\partial_\tau f^k(\tau, \sigma) = -\hat{G}^{kl}(\tau, f(\tau, \sigma)) G_{\tau l}(\tau, f(\tau, \sigma)) .
\] (2.33)

This recurrently determines all the higher order time derivatives of \(f^k(\tau, \sigma)\). Thus, it is possible to choose a gauge such that

\[
\forall \, \ 1 \leq i \leq p , \quad G_{\tau i} = 0 .
\] (2.34)

At this stage the unbroken gauge symmetry is the \(p\)-dimensional worldvolume ‘static’ diffeomorphism

\[
\tau \rightarrow \tau' = \tau , \quad \sigma^i \rightarrow \sigma'^i = f^i(\sigma) .
\] (2.35)
With the gauge choices above, \( \tau = x^+ \) and \( G_{\tau i} = 0 \), the Nambu-Goto Lagrangian and the \( p + 1 \) form Lagrangian in the target spacetime background read
\[
\mathcal{L}_{\text{N.G.}} = - TA^{\frac{n+1}{2}} \sqrt{\left(-2 \dot{x}^+ + 2V - 2J_a \dot{y}^a - g_{ab} \dot{y}^a \dot{y}^b \right) \det \left( \partial_i y^a \partial_j y^b g_{ab} \right)} ,
\]
\[
\mathcal{L}_{C_{p+1}} = - \frac{1}{p!} \epsilon^{\tau j_1 \cdots j_p} \partial_{j_1} x^{M_1} \cdots \partial_{j_p} x^{M_p} \left( C_{+M_1 \cdots M_p}(x) + \dot{x}^- C_{-M_1 \cdots M_p}(x) + \dot{y}^a C_{aM_1 \cdots M_p}(x) \right) ,
\]
where the determinant is for the \( p \times p \) matrix, \( \partial_i y^a \partial_j y^b g_{ab} \). The dynamical variables are \( x^- \) and \( y^a \).

Especially for the light-cone variable \( x^- \), we define a quantity \( P_- \) as
\[
P_- := \frac{\partial \mathcal{L}_{\text{N.G.}}}{\partial \dot{x}^-} = TA^{\frac{n+1}{2}} \sqrt{\frac{\det \left( \partial_i y^a \partial_j y^b g_{ab}(y, \tau) \right)}{-2 \dot{x}^- + 2V - 2J_a \dot{y}^a - g_{ab} \dot{y}^a \dot{y}^b}} .
\]
We note that \( P_- \) is a scalar density with weight one such that under the static diffeomorphism, it transforms as
\[
P_- (\tau, \sigma) \rightarrow P'_-(\tau, \sigma) = \left| \det \left( \frac{\partial \sigma'}{\partial \sigma} \right) \right| P_-(\tau, \sigma') .
\]
Hence, it is possible to choose a gauge such that at \( \tau = 0 \) the momentum has no spatial dependency,
\[
\forall \ 1 \leq i \leq p , \quad \frac{\partial P_-(0, \sigma)}{\partial \sigma^i} = 0 .
\]
This is the last of the gauge fixing conditions in our prescription. The remaining unbroken gauge symmetry is then the time-independent \( p \)-dimensional volume-preserving diffeomorphism.

Now we turn to the dynamics. The Euler-Lagrangian equations of the \( p \)-brane action are
\[
\partial_\mu \left( \frac{\partial \mathcal{L}_{\text{N.G.}}}{\partial \partial_\mu x^m} \right) - \frac{\partial \mathcal{L}_{\text{N.G.}}}{\partial x^m} + \partial_\mu \left( \frac{\partial \mathcal{L}_{C_{p+1}}}{\partial \partial_\mu x^m} \right) - \frac{\partial \mathcal{L}_{C_{p+1}}}{\partial x^m} = 0 ,
\]
where \( x^m \) is either \( x^- \) or \( y^a \). After straightforward manipulation one obtains
\[
\partial_\mu \left( \frac{\partial \mathcal{L}_{C_{p+1}}}{\partial \partial_\mu x^m} \right) - \frac{\partial \mathcal{L}_{C_{p+1}}}{\partial x^m} = \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1}} \partial_{\mu_1} x^{M_1} \cdots \partial_{\mu_{p+1}} x^{M_{p+1}} F_{M_1 \cdots M_{p+1}} .
\]
In particular, since \( \mathcal{L}_{\text{N.G.}} \) depends on neither \( x^- \) nor its spatial derivatives \( \partial_j x^- \), the equation of motion of \( x^- \) is
\[
\frac{\partial P_-}{\partial \tau} = \frac{1}{p!} \epsilon^{\tau j_1 \cdots j_p} \partial_{j_1} x^{M_1} \cdots \partial_{j_p} x^{M_p} \left( F_{-M_1 \cdots M_p} + \dot{y}^a F_{a-M_1 \cdots M_p} \right) .
\]
From our main assumption (2.10), the right hand side vanishes and hence, \( P_\tau = 0 \). Therefore along with the gauge choice (2.39), \( P_\tau \) becomes strictly a constant on-shell,
\[
\forall \ 0 \leq \mu \leq p, \quad \frac{\partial P_\tau}{\partial \xi^\mu} = 0.
\] (2.43)

Regarding the equations of motion of the remaining variables \( y^a \), which read with (2.10)
\[
\frac{\partial}{\partial \tau} \left( \frac{\partial L_{N,\text{G}}}{\partial \dot{y}^a} \right) + \frac{\partial}{\partial \sigma} \left( \frac{\partial L_{N,\text{G}}}{\partial \dot{y}^a} \right) - \frac{1}{p!} \epsilon^{\tau j_1 \cdots j_p} \partial_{j_1} y^{b_1} \cdots \partial_{j_p} y^{b_p} F_{+ab_1 \cdots b_p} = 0,
\] (2.44)
we have\(^3\) for a fixed \( c \)-number \( P_- \),
\[
\frac{\partial L_{N,\text{G}}}{\partial \dot{y}^a} = P_- \left( g_{ab} \dot{y}^b + J_a \right),
\]
\[
\frac{\partial L_{N,\text{G}}}{\partial \dot{y}^a} = -\left( \frac{T^2}{2P_-} \right) A^{p+1} \frac{\partial}{\partial \dot{y}^a} \det \left( \partial_{j_1} y^a \partial_{k} y^b g_{ab} (y, \tau) \right),
\]
\[
\frac{\partial L_{N,\text{G}}}{\partial y^a} = \frac{\partial}{\partial y^a} \left[ P_- \left( \frac{1}{2} g_{bc} \dot{y}^b \dot{y}^c + J_b \dot{y}^b - V \right) - \left( \frac{T^2}{2P_-} \right) A^{p+1} \det \left( \partial_{j_1} y^a \partial_{k} y^b g_{ab} \right) \right].
\] (2.45)

Furthermore, for the \( p \)-form \( V \) satisfying (2.11), if we set
\[
\mathcal{L}_V := + \frac{1}{p!} \epsilon^{\tau j_1 j_2 \cdots j_p} \partial_{j_1} y^{a_1} \partial_{j_2} y^{a_2} \cdots \partial_{j_p} y^{a_p} V_{a_1 a_2 \cdots a_p} (y, \tau),
\] (2.46)
then in a similar fashion to (2.41) we obtain
\[
\partial_i \left( \frac{\partial \mathcal{L}_V}{\partial \dot{y}^a} \right) - \frac{\partial \mathcal{L}_V}{\partial y^a} = -\frac{1}{p!} \epsilon^{\tau j_1 j_2 \cdots j_p} \partial_{j_1} y^{b_1} \cdots \partial_{j_p} y^{b_p} F_{+ab_1 \cdots b_p}.
\] (2.47)

Therefore, we conclude that for an arbitrary sector of the fixed constant of motion \( P_- \), the relativistic \( p \)-brane dynamics can be exactly described by the following ‘non-relativistic’ action:
\[
\mathcal{L}^- = P_- \left[ \frac{1}{2} g_{ab} (y, \tau) \dot{y}^a \dot{y}^b + J_a (y, \tau) \dot{y}^a - V (y, \tau) - \frac{1}{2} T^2 A (y, \tau)^{p+1} \det \left( \partial_{j_1} y^a \partial_{j_2} y^b \dot{y}^a \dot{y}^b \right) \right]
+ \frac{1}{p!} \epsilon^{\tau j_1 j_2 \cdots j_p} \partial_{j_1} y^{a_1} \cdots \partial_{j_p} y^{a_p} V_{a_1 \cdots a_p} (y, \tau),
\] (2.48)

\(^3\)Surely, the derivative of the determinant in (2.45) can be explicitly spelled out using the relation \( \delta \det M = M^{-1} \delta M \det M \). However, what is more illuminating expression in our analysis is the one in (2.46).
where we set \( T_- := T P_1 \), and from (2.37), \( P_- \) should be taken as a positive \( c \)-number, rather than a dynamical variable. The determinant in (2.38) is for the \( p \times p \) matrix. In the case \( p = 0 \) i.e. a point particle, by setting the determinant to be 1 the Lagrangian consistently reduces to the previous result (2.23). For \( p = 1 \) we obtain the usual string action in the light-cone gauge. After some scaling of the worldvolume coordinates we get

\[
L_{\text{string}} = \frac{1}{2} \left( \partial_{\tau} y^a \partial_{\tau} y^b - A(y, \tau)^2 \partial_{\sigma} y^a \partial_{\sigma} y^b \right) g_{ab}(y, \tau) + J_a(y, \tau) y^a - V(y, \tau) + T^{-1} \partial_{\sigma} y^a V_a .
\]

For generic values of \( p \), the determinant in the above action (2.48) can be expressed in terms of the Nambu bracket [36]

\[
\{ y_{a_1}, y_{a_2}, \ldots, y_{a_p} \}_{\text{N.B.}} := \epsilon^{j_1 j_2 \cdots j_p} \frac{\partial y^{a_1}}{\partial \sigma^{j_1}} \frac{\partial y^{a_2}}{\partial \sigma^{j_2}} \cdots \frac{\partial y^{a_p}}{\partial \sigma^{j_p}} ,
\]

\[
\det \left( \partial_i y^a \partial_j y^b g_{ab} \right) = \frac{1}{p!} \left\{ y_{a_1}, y_{a_2}, \ldots, y_{a_p} \right\}_{\text{N.B.}} \left\{ y_{b_1}, y_{b_2}, \ldots, y_{b_p} \right\}_{\text{N.B.}} g_{a_1 b_1} g_{a_2 b_2} \cdots g_{a_p b_p} .
\]

The matrix regularization is then to replace the dynamical fields \( y^a(\tau, \sigma) \) by \( N \times N \) Hermitian matrices \( X^a(\tau) \) depending on the time only; the ordinary time derivative by the covariant time derivative (2.25); and further the Nambu bracket by an anti-symmetrized matrix product,

\[
\left\{ y_{a_1}, y_{a_2}, \ldots, y_{a_p} \right\}_{\text{N.B.}} \Longleftrightarrow \left( \sqrt{-1} \right)^{\frac{1}{2} p(p-1)} \left[ X^{a_1}, X^{a_2}, \ldots, X^{a_p} \right] ,
\]

where we set

\[
\left[ M_1, M_2, \ldots, M_p \right] := \epsilon^{j_1 j_2 \cdots j_p} M_{j_1} M_{j_2} \cdots M_{j_p} .
\]

The numerical factor in (2.51) is chosen such that the right hand side is Hermitian provided \( X^a \)'s

\[ \text{4However, while the Nambu bracket satisfies the generalized Jacobi identity} \]

\[
\left\{ \left\{ f_1, f_2, \ldots, f_p \right\}_{\text{N.B.}}, g_2, \ldots, g_p \right\}_{\text{N.B.}} = \sum_{j=1}^{p} \left\{ f_1, \ldots, f_{j-1}, \left\{ f_j, g_2, \ldots, g_p \right\}_{\text{N.B.}}, f_{j+1}, \ldots, f_p \right\}_{\text{N.B.}} ,
\]

the anti-symmetrized matrix product (2.51) does not do so except \( p = 2 \) case. See [37,38] for related discussions.
are so. The resulting matrix model is then, for \( g_{ab} = \delta_{ab} \) (to avoid an ordering ambiguity),

\[
S_{\text{M.M.}} = \int d\tau ~ P_{-} L_{\text{M.M.}},
\]

\[
L_{\text{M.M.}} = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + J_a(X, \tau) D_t X^a - V(X, \tau) \right) \\
+ \text{tr} \left( -\frac{\kappa^2}{2p!} (-1)^{\frac{1}{2}p(p-1)} A(X, \tau)^p X^{a_1, a_2, \ldots, a_p} \right)^2 \\
+ \text{tr} \left( \frac{\lambda_p}{p!} (-\sqrt{-1})^{\frac{1}{2}p(p-1)} X^{b_1, b_2, \ldots, b_p} V_{b_1 b_2 \cdots b_p}(X, \tau) \right),
\]

where \( \kappa_p, \lambda_p \) are constants. It is noteworthy that the first line in the Lagrangian is universally present irrespective of \( p \).

When \( p = 1 \), \textit{i.e.} for the string, the matrix model essentially reduces to a \( D - 2 \) copies of harmonic oscillators subject to additional potentials \( V, J_a \) and \( V_a \); while for \( p = 2 \) \textit{i.e.} membrane, the Nambu bracket reduces to the Poisson bracket, and the matrix regularization can be justified in terms of the non-commutative geometry as follows. For a constant non-commutative deformation of a two-dimensional space

\[
[\sigma^1, \sigma^2] = i\theta,
\]

the non-commutative geometry can be realized either by Moyal-Weyl star product formalism on ordinary commutative space,

\[
f(\sigma) \star g(\sigma) = f(\sigma) e^{\frac{i}{\theta} \hat{\sigma} \hat{\sigma}} g(\sigma) \implies \sigma^1 \star \sigma^2 - \sigma^2 \star \sigma^1 = i\theta,
\]

or equivalently by a matrix formalism generated by a pair of \( \infty \times \infty \) matrices satisfying the matrix commutator relation, \( [\hat{\sigma}^1, \hat{\sigma}^2] = i\theta \). The equivalence between the two formalisms follows from the isomorphism\(^5\)

\[
O(f)O(g) = O(f \star g),
\]

where \( O(f) \) is a Weyl ordering map from an ordinary commutative function to a matrix, defined by

\[
O(\sigma^{j_1} \sigma^{j_2} \cdots \sigma^{j_n}) := \sum_{P=1}^{n!} \frac{1}{n!} \hat{\sigma}^{P_1} \hat{\sigma}^{P_2} \cdots \hat{\sigma}^{P_n}.
\]

\(^5\)For a proof of the isomorphism in the physics literature see \textit{e.g.} [39], and for the string theory aspect of the non-commutative geometry see [40]. The isomorphism guarantees the associativity of the star product, since the matrix product is associative.
Here \( P \) denotes the permutations of the \( n \) indices \( (j_1, j_2, \ldots, j_n) \).

The justification of the matrix regularization in the case of \( p = 2 \) then follows from an observation that the Poisson bracket corresponds to the leading order term in the start product commutator,

\[
[f, g]_\ast = i\theta \{ f, g \}_\text{PB} + O(\theta^2).
\] (2.58)

The matrix model for a membrane contains terms of matrix commutator squared, and this coincides with the potential in the Yang-Mills quantum mechanics. In the rest of the paper, we analyze the most general supersymmetric mass deformations of all the super Yang-Mills quantum mechanics, without breaking any supersymmetry. The resulting matrix models can be identified as the light-cone formulation of the relativistic superparticle or supermembrane actions.

### 3 \( \mathcal{N} = 16 \) super Yang-Mills quantum mechanics: BMN matrix model

In [14], Figueroa-O’Farrill and Papadopoulos classified the maximally supersymmetric solutions of the eleven-dimensional supergravity theory. Up to local isometry, \( \text{AdS}_4 \times S^4 \), \( \text{AdS}_7 \times S^7 \), pp-wave and the flat spacetime exhaust them. In particular, the pp-wave solution reads [14–17]

\[
ds^2 = 2dx^+dx^- - \frac{1}{36}\mu^2\left(x_1^2 + \cdots + x_6^2 + 4x_7^2 + 4x_8^2 + 4x_9^2\right)dx^+dx^- + \sum_{a=1}^9 dx^a dx^a,
\] (3.1)

\[
F_{789+} = \mu,
\]

where \( \mu \) is a characteristic mass parameter of the solution. When \( \mu = 0 \), the solution apparently reduces to the flat background.

In [1], Berenstein, Maldacena and Nastase (BMN) derived the \( \mathcal{M} \)-theory matrix model in the above maximally supersymmetric pp-wave background

\[
\mathcal{L}_{\text{BMN}}^{\mathcal{N}=16} = \text{tr} \left( \frac{1}{2} D_t X^a D_a X_a + \frac{1}{4} [X^a, X^b]^2 + i \frac{1}{2} \Psi \dagger D_t \Psi - \frac{1}{2} \Psi \dagger \Gamma^a [X_a, \Psi] \right)
\]

\[
+ i\mu \text{tr} \left( \frac{1}{8} \Psi \dagger \Gamma^{789} \Psi - X_7 [X_8, X_9] \right)
\] (3.2)

\[- \frac{1}{1728} \mu^2 \text{tr} \left( X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 + 4X_7^2 + 4X_8^2 + 4X_9^2 \right),
\]

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where with $a = 1, 2, \cdots, 9$, $\Gamma^a = (\Gamma^a)^\dagger$ are Euclidean nine-dimensional gamma matrices satisfying in general
\begin{equation}
(\Gamma^a)^T = (\Gamma^a)^* = C^{-1} \Gamma^a C, \quad C = C^T = (C^\dagger)^{-1},
\end{equation}
and $\Psi$ is a sixteen-component Majorana spinor $\Psi = C\Psi^*$. The matrix model possesses 16 real dynamical supersymmetries (and hence $N = 16$) :
\begin{align}
\delta A_0 &= i\Psi^\dagger \epsilon(t), \quad \delta X^a = i\Psi^\dagger \Gamma^a \epsilon(t), \\
\delta \Psi &= (D_t X^a \Gamma_a - i\frac{1}{2} [X^a, X^b] \Gamma_{ab} - \frac{1}{12} \mu X^7 \Gamma^{789} - \frac{1}{4} \mu \Gamma^{789} X) \epsilon(t),
\end{align}
where $X := X^a \Gamma_a$, and the supersymmetric parameter is time dependent
\begin{equation}
\epsilon(t) = e^{\frac{1}{12} \mu \Gamma^{789}} \epsilon(0).
\end{equation}
As we show in Appendix A.3 BMN matrix model is the unique mass deformation of the $N = 16$ super Yang-Mills quantum mechanics. The uniqueness is consistent with the fact that, among the maximally supersymmetric eleven-dimensional backgrounds, only the pp-wave background (3.1) can give the supersymmetric mass deformation of the $N = 16$ SYMQM via the light-cone quantization.

The dynamical supersymmetries form a super Lie algebra $\mathfrak{su}(2|4)$. The classification of its supermultiplets was achieved [41,42], and the corresponding classical configurations were analyzed in [43] using the ‘projection’ method [44]. For the perturbative analysis on the spectrum, see also [18, 45]. In particular, the maximally supersymmetric configuration preserving all the dynamical supersymmetries is given by a static fuzzy sphere spanning the 7, 8, 9 directions :
\begin{align}
[X_p, X_q] &= i\frac{1}{2} \epsilon_{pqr} X^r, \quad D_t X^p = 0, \quad X^1 = X^2 = X^3 = X^4 = X^5 = X^6 = 0, 
\end{align}
where $p, q, r = 7, 8, 9$ and $\epsilon_{789} = 1$. In addition, there are 16 real kinematical supersymmetries
\begin{align}
\delta A_0 &= \delta X^a = 0, \quad \delta \Psi = e^{-\frac{1}{4} \mu \Gamma^{789}} \epsilon'.
\end{align}

---

For simplicity one may take the real gamma matrices such that $C = 1$. 

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4 \( \mathcal{N} = 8 \) super Yang-Mills quantum mechanics

In this section we analyze the mass deformations of the \( \mathcal{N} = 8 \) super Yang-Mills quantum mechanics which originates from the six-dimensional minimal super Yang-Mills theory via dimensional reduction. We first review the six-dimensional super Yang-Mills in order to set up our notations, especially for the \( \text{su}(2) \) Majorana-Weyl spinor. We then present the most general mass deformations of the \( \mathcal{N} = 8 \) super Yang-Mills quantum mechanics. It turns out that there exist two distinct types of mass deformations: type I and type II, with the corresponding superalgebra \( \text{su}(2|2) \) and \( \text{su}(2|1) \oplus \text{su}(2|1) \) respectively. Only the former is compatible with the \( \text{su}(2) \) Majorana-Weyl condition, while the latter breaks the \( \text{su}(2) \) symmetry. The latter is a rederivation of an earlier work [2]. Here we simply present the results. The detailed derivation is carried out in Appendix A.1.

4.1 Minimal super Yang-Mills in six-dimensions

In six-dimensional Minkowskian spacetime of the metric \( \eta = \text{diag}(-++++) \), the \( 8 \times 8 \) gamma matrices satisfy with \( M = 0, 1, 2, 3, 4, 5 \) [46, 47]

\[
\Gamma^{\mathbf{M}} = \Gamma_{\mathbf{M}} = A \Gamma^{\mathbf{M}} A^{\dagger}, \quad A := \Gamma_{12345}^{12345} = A^{\dagger} = A^{-1}, \\
\Gamma^{\mathbf{MT}} = C \Gamma^{\mathbf{M}} C^{\dagger}, \quad C^{T} = -C, \quad C^{\dagger} = C^{-1}, \\
\Gamma^{\mathbf{M}^*} = B \Gamma^{\mathbf{M}} B^{\dagger}, \quad B = CA = -B^{T}, \quad B^{\dagger} = B^{-1}.
\]

The gamma “seven” is given by \( \Gamma^{(7)} = \Gamma_{012345}^{012345} \) which satisfies \( \Gamma^{(7)} = \Gamma^{(7)*} = \Gamma^{(7)}^{-1} \) and

\[
\Gamma^{\mathbf{LMN}} = \frac{1}{6} \epsilon^{\mathbf{LMNPQR}} \Gamma_{\mathbf{PQR}} \Gamma^{(7)},
\]

where \( \epsilon^{012345} = +1 \).

The \( \text{su}(2) \) Majorana-Weyl spinor \( \psi_{i}, i = 1, 2 \), satisfies then

\[
\Gamma^{(7)} \psi_{i} = + \psi_{i}, \quad \bar{\psi}^{i} \Gamma^{(7)} = - \bar{\psi}^{i} : \text{chirality} \\
\bar{\psi}^{i} = (\psi_{i})^{\dagger} A = \epsilon^{ij} (\psi_{j})^{T} C : \text{su}(2) \text{ Majorana},
\]

where \( \epsilon^{ij} \) is the usual \( 2 \times 2 \) skew-symmetric unimodular matrix. It is worth to note that \( \bar{\psi}^{i} \Gamma^{M_{1}M_{2} \ldots M_{2n+1}} \rho_{i} = 0 \) and

\[
\text{tr}(i \bar{\psi}^{i} \Gamma^{M_{1}M_{2} \ldots M_{2n+1}} \rho_{i}) = \left[ \text{tr}(i \bar{\psi}^{i} \Gamma^{M_{1}M_{2} \ldots M_{2n+1}} \rho_{i}) \right]^{\dagger} = -(-1)^{n} \text{tr}(i \bar{\rho}^{i} \Gamma^{M_{1}M_{2} \ldots M_{2n+1}} \psi_{i}),
\]

where \( \psi_{i}, \rho_{i} \) are two arbitrary Lie algebra valued \( \text{su}(2) \) Majorana-Weyl spinors.
The six-dimensional super Yang-Mills Lagrangian is
\[ \mathcal{L}_{6\text{DSYM}} = \text{tr} \left( -\frac{1}{4} F_{LM} F^{LM} - i \frac{1}{4} \bar{\psi} \Gamma^{L} D_{L} \psi \right), \] (4.5)
where all the fields are in the adjoint representation of the gauge group such that, with Hermitian Lie algebra valued gauge fields \( A_{M} \),
\[ D_{L} \psi_{i} = \partial_{L} \psi_{i} - i [A_{L}, \psi_{i}], \quad F_{LM} = \partial_{L} A_{M} - \partial_{M} A_{L} - i [A_{L}, A_{M}]. \] (4.6)
From (4.4) the action is real valued.

The supersymmetry transformations are, with a su(2) Majorana-Weyl supersymmetry parameter \( \varepsilon_{i} \),
\[ \delta A_{M} = + i \bar{\varepsilon}_{i} \Gamma_{M} \psi_{i} = - i \bar{\psi}_{i} \Gamma_{M} \varepsilon_{i}, \quad \delta \psi_{i} = - \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon_{i}. \] (4.7)
In particular, \( \delta \bar{\psi}^{i} = + \frac{1}{2} F_{MN} \bar{\varepsilon}^{i} \Gamma^{MN} \). There are eight real supersymmetries.

The crucial Fierz identity for the supersymmetry invariance is, with the chiral projection matrix \( P := \frac{1}{2} (1 + \Gamma^{(7)}) \),
\[ \left( \Gamma^{L} P \right)_{\alpha \beta} \left( \Gamma_{L} P \right)_{\gamma \delta} + \left( \Gamma^{L} P \right)_{\beta \gamma} \left( \Gamma_{L} P \right)_{\delta \alpha} = 0, \] (4.8)
which ensures the vanishing of the terms cubic in \( \psi_{i} \)
\[ \text{tr}(\bar{\psi}^{i} \Gamma^{L} [\delta A_{L}, \psi_{i}]) = \text{tr}(\bar{\psi}^{i} \Gamma^{L} [i \bar{\varepsilon}^{j} \Gamma_{L} \psi_{j}, \psi_{i}]) = 0. \] (4.9)

### 4.2 Deformation \( SO(5) \rightarrow SO(3) \times SO(2) \) : type I

After a dimensional reduction of (4.5) to the time \( t \), we obtain a \( \mathcal{N} = 8 \) super Yang-Mills quantum mechanics, containing five Hermitian matrices \( X^{a}, a = 1, 2, 3, 4, 5, \)
\[ \mathcal{L}_{0}^{\mathcal{N}=8} = \text{tr} \left( \frac{1}{2} D_{t} X^{a} D_{t} X_{a} + \frac{1}{4} [X^{a}, X^{b}]^{2} - i \frac{1}{2} \bar{\psi} \Gamma^{a} D_{t} \psi_{i} - \frac{1}{2} \bar{\psi} \Gamma^{a} [X_{a}, \psi_{i}] \right). \] (4.10)
The mass dimensions are 1 for the bosons and \( \frac{3}{2} \) for the fermions so that the Lagrangian has mass dimension 4.

We consider the most general mass deformations of the above matrix model which preserves all the supersymmetries. From the chirality of the spinors, the possible mass term for the fermion is of the form :
\[ \text{tr} \left[ \bar{\psi} \left( M_{L} \Gamma^{L} - i \frac{1}{3} M_{abc} \Gamma^{abc} \right) \psi \right], \] (4.11)
where $M_L, M_{abc}$ are real parameters, which \textit{a priori} may depend on time. However, as we show in Appendix (A.4), it turns out that in order to admit $\mathcal{N} = 8$ supersymmetries they must be constants, and furthermore it is required that either $M_L = 0$ or $M_{abc} = 0$. Namely there exist two distinct types of mass deformations: type I and type II. Only the former leads to a mass deformation which is still compatible with the $\text{su}(2)$ Majorana-Weyl condition.

With a constant mass parameter $\mu$, $\mathcal{N} = 8$ type I massive super Yang-Mills quantum mechanics reads

\[
\mathcal{L}^\mathcal{N}=8_{\text{type I}} = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b] \right)^2 - i \frac{1}{2} \bar{\psi}^i \Gamma^i D_t \psi_i - \frac{1}{2} \bar{\psi}^i \Gamma^a [X_a, \psi_i] \right) + \text{tr} \left( i \frac{1}{2} \mu \bar{\psi}^i \Gamma^{345} \psi_i - i \mu [X_3, X_4] X_5 - \frac{1}{72} \mu^2 \left( X_1^2 + X_2^2 + 4X_3^2 + 4X_4^2 \right) \right) \tag{4.12}
\]

The supersymmetry transformations are given by

\[
\delta A_0 = i \bar{\psi}^i \Gamma_t \epsilon_i(t), \quad \delta X_a = i \bar{\psi}^i \Gamma_a \epsilon_i(t), \quad \delta \psi_i = \left( \Gamma^{ta} D_t X_a - i \frac{1}{2} [X_a, X_b] \Gamma^{ab} - \frac{1}{12} \mu X \Gamma^{345} - \frac{1}{4} \mu \Gamma^{345} X \right) \epsilon_i(t), \tag{4.13}
\]

where

\[
X = \Gamma^a X_a, \quad \epsilon_i(t) = e^{\frac{1}{12} \mu t \Gamma^{345}} \epsilon_i(0). \tag{4.14}
\]

Note that the Lagrangian manifestly possesses a $\text{SO}(2) \times \text{SO}(3) \times \text{SU}(2)$ symmetry. The mass spectra are $\frac{1}{4} \mu$ for the fermions, $\frac{1}{72} \mu^2$ for the two scalars and $\frac{1}{4} \mu$ for the other three scalars.

### 4.3 $\text{su}(2|2)$ superalgebra : type I

Writing the Noether charge for the supersymmetry transformation (4.13) as

\[
\text{tr} \left( -i \bar{\psi}^i \Gamma^i \delta \psi_i \right) := i\bar{Q}^i \epsilon_i(t), \tag{4.15}
\]

the supercharge satisfies the $\text{su}(2)$ Majorana-Weyl condition with the opposite chirality to the fermions $\epsilon_i, \psi_i$,

\[
\Gamma^{(7)} Q_i = -Q_i, \quad \bar{Q}^i \Gamma^{(7)} = +\bar{Q}^i : \text{anti-chirality}, \quad \bar{Q}^i = (Q_i)^\dagger \Gamma^{(7)} A = \epsilon^{ij}(Q_j)^T C : \text{su}(2) \text{ Majorana}. \tag{4.16}
\]

This is consistent with the fact that the Noether charge is real, $i\bar{Q}^i \epsilon_i(t) = -i\bar{\epsilon}^i(t) Q_i = (i\bar{Q}^i \epsilon_i(t))^\dagger$. 19
The supersymmetry algebra of the $\mathcal{N} = 8$ type I massive SYMQM (4.12) reads explicitly

\[
[H, Q^i] = +i \frac{1}{12} \mu \Gamma_{12} Q^i, \quad [H, \bar{Q}^i] = -i \frac{1}{12} \mu \bar{Q}^i \Gamma_{12},
\]

\[
\{Q^i, \bar{Q}^j\} = 2 \delta_i^j \left( A(H - \frac{1}{6} \mu M_{12}) - \frac{1}{6} \mu \epsilon_{pqr} \Gamma^p M^{qr} \right) P_+ + i \frac{2}{7} \mu T^{i \dot{j}} \Gamma_{345} P_+ ,
\]

and, as usual,

\[
[M_{12}, Q^i] = +i \frac{1}{2} \Gamma_{12} Q^i, \quad [M_{12}, \bar{Q}^i] = -i \frac{1}{2} \bar{Q}^i \Gamma_{12},
\]

\[
[M_{pq}, Q^i] = +i \frac{1}{2} \Gamma_{pq} Q^i, \quad [M_{pq}, \bar{Q}^i] = -i \frac{1}{2} \bar{Q}^i \Gamma_{pq},
\]

\[
[M_{pq}, M_{rs}] = i (\delta_{pr} M_{qs} - \delta_{ps} M_{qr} - \delta_{qr} M_{ps} + \delta_{qs} M_{pr}) ,
\]

\[
[T^{i \dot{j}}, M_{12}] = 0, \quad [H, M_{pq}] = 0, \quad [M_{12}, M_{pq}] = 0, \quad [H, T^{i \dot{j}}] = 0, \quad [M_{ab}, T^{i \dot{j}}] = 0.
\]

Here $Q^i, H, M_{12}, M_{pq}, p, q = 3, 4, 5, T^{i \dot{j}} = (T^{i \dot{j}})^\dagger$ refer to the supercharges, Hamiltonian, the $\text{SO}(3) \times \text{SO}(2)$ generators, the $\text{su}(2)$ generator; and $P_+ = \frac{1}{2} \left( 1 + \Gamma^7 \right)$ is the chiral projector.

In particular, we note that $H - \frac{1}{6} \mu M_{12}$ is central\textsuperscript{7}

\[
[H - \frac{1}{6} \mu M_{12}, \text{anything}] = 0 ,
\]

and since \{\(Q^i, Q^i\dagger\)\} is non-negative, we have the unitary bound

\[
H \geq \frac{1}{6} \mu M_{12}.
\]

From the classification of the simple super Lie algebra by Kac [48, 49], we identify the corresponding superalgebra\textsuperscript{8} as a centrally extended $\text{su}(2|2)$.

---

\textsuperscript{7}By a field redefinition \eqref{A.7}, one can rewrite the Lagrangian such that the new Hamiltonian itself is central.

\textsuperscript{8}Any inclusion of a brane charge in the simple super Lie algebra (e.g. [50]) will inevitably lead to a noncentral extension of the super Lie algebra [51]. This can be seen easily from a Jacobi identity involving two supercharges $Q, \bar{Q}$ and a brane charge $Z$,

\[
\{\left\{Q, \bar{Q}\right\}, Z\} = \{Q, [\bar{Q}, Z]\} + \{\bar{Q}, [Q, Z]\}.
\]

The left hand side corresponds to an infinitesimal rotation of the brane charge. Since the brane charge is not singlet under the rotations in general, the left hand side does vanish. This show that the brane charge can not commute with supercharges [52]. In the present paper, we neglect the brane charges and identify only the non-extended simple super Lie algebras classified in [48, 49].

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4.4 $\mathcal{N} = 8$ supersymmetric configuration : type I

From the supersymmetry transformation (4.13), it is straightforward to obtain the following 8/8 BPS equations i.e. the conditions for the bosonic configuration to preserve all the eight supersymmetries,

\[ D_t X_1 = -\frac{1}{6} \mu X_2, \quad D_t X_2 = +\frac{1}{6} \mu X_1, \quad D_t X_p = 0, \]  

\[ [X_p, X_q] = i \frac{1}{6} \mu \epsilon_{pqr} X^r, \quad [X_i, X_p] = 0, \quad [X_1, X_2] = 0, \]  

where $i = 1, 2, \quad p, q, r = 3, 4, 5,$ and $\epsilon_{pqr}$ is a totally anti-symmetric tensor with $\epsilon_{345} = 1$. Note that the BPS equations imply all the Euler-Lagrangian equations of the $\mathcal{N} = 8$ super Yang-Mills quantum mechanics, including the Gauss constraint.\(^9\)

The most general irreducible solution is given by a fuzzy sphere spanning (4, 5, 6) directions and rotating on the (1, 2) plane

\[ X_1 = R \cos(\frac{1}{6} t \mu) 1, \quad X_2 = R \sin(\frac{1}{6} t \mu) 1, \quad X_p = \frac{1}{3} \mu J_p, \]  

where $R$ is the radius of the circular orbit on the (1, 2) plane, and $J_p, p = 4, 5, 6$ satisfy the standard so(3) commutator relations, $[J_p, J_q] = i \epsilon_{pqr} J^r$. The rotating fuzzy sphere saturates the unitary bound (4.20) as $H = \frac{1}{6} \mu M_{12} = (\frac{1}{6} \mu R)^2$.

4.5 Deformation $\text{SO}(5) \rightarrow \text{SO}(4), \quad \text{su}(2|1) \oplus \text{su}(2|1)$ superalgebra : type II

Type II mass deformation of the $\mathcal{N} = 8$ super Yang-Mills quantum mechanics breaks the su(2) symmetry of the su(2) Majorana-Weyl spinors. Hence, in this subsection we drop the su(2) index of the fermion

\[ \psi := \psi_1, \quad \bar{\psi} = \psi^\dagger A = \bar{\psi}_1. \]  

$\mathcal{N} = 8$ type II massive super Yang-Mills quantum mechanics then reads [2]

\[ \mathcal{L}_{\text{type II}}^{\mathcal{N}=8} = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4!} [X^a, X^b]^2 - i \bar{\psi} \Gamma^a D_t \psi - \bar{\psi} \Gamma^a [X_a, \psi] \right) \]  

\[ + \text{tr} \left( \frac{1}{4} \mu \bar{\psi} \Gamma^1 \psi - \frac{1}{12} \mu^2 \left( 4 X_1^2 + X_2^2 + X_3^2 + X_4^2 \right) \right). \]  

The supersymmetry transformations are given by

\[ \delta A_0 = \bar{\psi} \Gamma_0 \varepsilon(t) + \bar{\varepsilon}(t) \Gamma_0 \psi, \quad \delta X_a = \bar{\psi} \Gamma_a \varepsilon(t) + \bar{\varepsilon}(t) \Gamma_a \psi, \]  

\[ \delta \psi = \left( - i D_t X_a \Gamma^{ta} - \frac{1}{2} [X_a, X_b] \Gamma^{ab} + \frac{1}{4} \mu \Gamma^1 X + \frac{1}{12} \mu X T^1 \right) \varepsilon(t), \]  

\[ \delta \bar{\psi} = \left( - i D_t X_a \Gamma^{ta} - \frac{1}{2} [X_a, X_b] \Gamma^{ab} + \frac{1}{4} \mu \Gamma^1 X + \frac{1}{12} \mu X T^1 \right) \bar{\varepsilon}(t). \]

\(^9\)However, in general, the BPS equations for less supersymmetries do not necessarily imply the Gauss constraint [43].
where
\[ X = \Gamma^a X_a, \quad \epsilon(t) = e^{-i \frac{1}{12} \mu \Gamma^{11}} \epsilon(0). \] (4.26)

Note that the Lagrangian manifestly possesses a SO(4) symmetry. The mass spectra are $\frac{1}{4} \mu$ for the fermions, $\frac{1}{6} \mu$ for the four scalars and $\frac{1}{3} \mu$ for the one scalar.

With anti-chiral supercharges $Q = -\Gamma^{(7)} Q$, $\bar{Q} = Q^\dagger A = Q \Gamma^{(7)}$, the supersymmetry algebra corresponds to $\text{su}(2|1) \oplus \text{su}(2|1)$,
\[
\{Q, \bar{Q}\} = 2 \left( A H + i \frac{1}{12} \mu M_{mn} \Gamma^{mn} - \frac{1}{12} \mu T \Gamma^{11} \right) P_+, \\
\{Q, Q\} = 0, \quad [H, T] = 0, \\
[H, Q] = \frac{1}{12} \mu \Gamma^{11} Q, \quad [H, \bar{Q}] = \frac{1}{12} \mu \bar{Q} \Gamma^{11}, \\
[T, Q] = Q, \quad [T, \bar{Q}] = -\bar{Q},
\] (4.27)

where $M_{mn}$ corresponds to the $\text{so}(4) = \text{su}(2) \oplus \text{su}(2)$ generators for $(2, 3, 4, 5)$ directions, and $T$ is the $\text{u}(1)$ generator of the phase rotation of the fermion. Unlike type I, in this case there is no nontrivial $\mathcal{N} = 8$ supersymmetric configuration.

5 $\mathcal{N} = 4$ super Yang-Mills quantum mechanics

In this section we analyze the mass deformations of the $\mathcal{N} = 4$ super Yang-Mills quantum mechanics which originates from the four-dimensional minimal super Yang-Mills theory via dimensional reduction. After reviewing the four-dimensional super Yang-Mills, we present the most general mass deformations of the $\mathcal{N} = 4$ super Yang-Mills quantum mechanics. Like $\mathcal{N} = 8$ case, there exist two distinct types of mass deformations: type I and II. Type I is a two-parameter family of deformations, while type II contains only one parameter.
5.1 Minimal super Yang-Mills in four-dimensions

In four-dimensional Minkowskian spacetime of the metric $\eta = \text{diag}(- + + +)$, the $4 \times 4$ gamma matrices satisfy with $\mu = 0, 1, 2, 3$

$$\Gamma^\mu_\uparrow = \Gamma_\mu = -A \Gamma^\mu A^\dagger, \quad A = \Gamma^t = -A^\dagger,$$

$$\Gamma^\mu_\star = +B \Gamma^\mu B^\dagger, \quad B^T = B,$$  

$$B^\dagger = B^{-1},$$  

$$\Gamma^\mu T = -C \Gamma^\mu C^\dagger, \quad C = -C^T = B \Gamma^t,$$  

$$C^\dagger = C^{-1}. \quad (5.1)$$

The spinors are then taken to meet the Majorana condition

$$\bar{\psi} = \psi^\dagger \Gamma^t = \psi^T C \quad \iff \quad \psi^* = B \psi. \quad (5.2)$$

The four-dimensional super Yang-Mills Lagrangian is

$$L_{4D\text{SYM}} = \text{tr} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - i \frac{1}{2} \bar{\psi} \Gamma^\mu D_{\mu} \psi \right). \quad (5.3)$$

The supersymmetry transformations are

$$\delta A_\mu = i \bar{\epsilon} \Gamma_\mu \psi = -i \bar{\psi} \Gamma_\mu \epsilon, \quad \delta \psi = -\frac{1}{2} F_{\mu \nu} \Gamma^{\mu \nu} \epsilon. \quad (5.4)$$

There are four real supersymmetries.

The Fierz identity relevant to the supersymmetry invariance is

$$(C \Gamma^\mu)_{\alpha \beta} (C \Gamma_\mu)_{\gamma \delta} + (C \Gamma^\mu)_{\beta \gamma} (C \Gamma_\mu)_{\alpha \delta} + (C \Gamma^\mu)_{\gamma \alpha} (C \Gamma_\mu)_{\beta \delta} = 0. \quad (5.5)$$

5.2 Deformation SO(3) $\rightarrow$ SO(3), $\text{su}(2|1)$ superalgebra : type I

After a dimensional reduction, the four-dimensional super Yang-Mills gives a supersymmetric matrix model containing three Hermitian matrices, $X^a$, $a = 1, 2, 3$,

$$L_{N=4}^{\text{type I}} = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b]^2 + i \frac{1}{2} \bar{\psi} \Gamma^t D_t \psi - \frac{1}{2} \bar{\psi} \Gamma^a [X_a, \psi] \right). \quad (5.6)$$

As we show in Appendix A.2, there are two distinct ways of deforming this matrix model: type I and type II. The former corresponds to a two-parameter $\mu_1, \mu_2$ family of deformation

$$L_{N=4}^{\text{type I}} = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b]^2 + i \frac{1}{2} \bar{\psi} \Gamma^t D_t \psi - \frac{1}{2} \bar{\psi} \Gamma^a [X_a, \psi] \right)$$

$$+ \text{tr} \left( i \frac{1}{7} \mu_1 \bar{\psi} \psi + i \frac{1}{7} \mu_2 \bar{\psi} \Gamma^{123} \psi - i \mu_2 [X_1, X_2] X_3 - \frac{1}{108} (\mu_1^2 + \mu_2^2) X^a X_a \right). \quad (5.7)$$
Some characteristic features of the type I deformation are the unbroken SO(3) symmetry and the presence of the Myers term. The supersymmetry transformations are

$$
\delta A_0 = -i\bar{\psi}\Gamma_t \varepsilon(t), \quad \delta X_a = -i\bar{\psi}\Gamma_a \varepsilon(t),
$$

$$
\delta \psi = \left( -\Gamma^a D_t X_a + i\frac{1}{2} [X_a, X_b] \Gamma^{ab} - \frac{1}{3} \mu_1 X + \frac{1}{3} \mu_2 X \Gamma^{123} \right) \varepsilon(t),
$$

where

$$
\varepsilon(t) = e^{\frac{1}{2}t(\mu_1 \Gamma^t - \mu_2 \Gamma^{123})} \varepsilon(0).
$$

Diagonalizing the mass matrix $\mu_1 \Gamma^t + \mu_2 \Gamma^{123}$, we obtain the mass spectra, $\frac{1}{2} \sqrt{\mu_1^2 + \mu_2^2}$ for the fermions and $\frac{1}{3} \sqrt{\mu_1^2 + \mu_2^2}$ for the bosons.

With the Majorana supercharge $\bar{Q} = Q^T C = Q^\dagger \Gamma^t$, the corresponding supersymmetry algebra reads

$$
[H, Q] = -i\frac{1}{6} (\mu_1 \Gamma^t + \mu_2 \Gamma^{123}) Q, \quad \{Q, \bar{Q}\} = 2 \left( \Gamma^t H + \frac{1}{6} \mu_1 \Gamma^{ab} M_{ab} - \frac{1}{6} \mu_2 \epsilon_{abc} \Gamma^a M^{bc} \right).
$$

This can be identified as $su(2|1)$ super Lie algebra.

From (5.8), maximally supersymmetric configuration, preserving all the $\mathcal{N} = 4$ supersymmetries, exists only if $\mu_1 = 0$. It corresponds to the static fuzzy sphere

$$
D_t X_a = 0, \quad [X_a, X_b] = i\frac{1}{3} \mu_2 \epsilon_{abc} X^c.
$$

Physically, when $\mu_1 \neq 0$, the harmonic potential becomes too steep to support the fuzzy sphere favored by the Myers term. It is worth to note that the Lagrangian (5.7) can be rewritten such that the fuzzy sphere structure is manifest in the potential

$$
\mathcal{L}^{\mathcal{N} = 4}_{N = 4} = \text{tr} \left[ \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} \left( [X^a, X^b] - i\frac{1}{2} \mu_2 \epsilon_{abc} X^c \right)^2 - \frac{1}{18} \mu_1^2 X^a X_a \right] + \text{tr} \left[ -i\frac{1}{2} \bar{\psi}\Gamma^t D_t \psi - \frac{1}{2} \bar{\psi} \Gamma^{123} [X_a, \psi] + i\frac{1}{4} \bar{\psi} \left( \mu_1 + \mu_2 \Gamma^{123} \right) \psi \right].
$$

Finally we note an interesting property at $\mu_1 = 0$. In this case, a time dependent field redefinition of the fermion

$$
\psi \rightarrow e^{\frac{1}{2}t(\mu_3 - \mu_2) \Gamma^{123}} \psi,
$$

(5.13)
can change the mass of the fermion arbitrarily: $\frac{1}{7}\mu_2 \to \frac{1}{2}\mu_3$, without changing the mass of the bosons. The resulting matrix model reads

$$L_{N=4}^{\text{type I}|\mu_1=0} = \text{tr} \left[ \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} \left( [X^a, X^b] - i \frac{1}{4} \mu_2 \epsilon_{abc} X^c \right)^2 \right] + \text{tr} \left[ - i \frac{1}{2} \bar{\psi} \Gamma^a D_t \psi - \frac{1}{2} \bar{\psi} \Gamma^a [X_a, \psi] + i \frac{1}{7} \mu_3 \bar{\psi} \Gamma^{123} \psi \right].$$

(5.14)

The mass parameter $\mu_3$ is fictitious, since different values can be mapped to one another by the field redefinition. The supersymmetry transformations are

$$\delta A_0 = -i \bar{\psi} \Gamma_t \varepsilon(t), \quad \delta X_a = -i \bar{\psi} \Gamma_a \varepsilon(t),$$

$$\delta \psi = \left( - \Gamma^a D_t X_a + i \frac{1}{2} [X_a, X_b] \Gamma^{ab} + \frac{1}{7} \mu_2 X \Gamma^{123} \right) \varepsilon(t),$$

(5.15)

where

$$\varepsilon(t) = e^{\frac{1}{6} (2\mu_2 - 3\mu_3) \Gamma^{123} t} \varepsilon(0).$$

(5.16)

The corresponding supersymmetry algebra is now of the form

$$[H, Q] = i \frac{1}{7} (2\mu_2 - 3\mu_3) \Gamma^{123} Q, \quad [R, Q] = i \Gamma^{123} Q, \quad [H, R] = 0,$$

$$\{ Q, \bar{Q} \} = 2 \left( \Gamma^t (H + \frac{1}{2} (\mu_3 - \mu_2) R) - \frac{1}{6} \mu_2 \epsilon_{abc} \Gamma^a M^{bc} \right),$$

where, compared to (5.10), a new generator $R = R^t = i \frac{1}{2} \bar{\psi} \Gamma^{123} \psi$ appears which corresponds to the following Noether symmetry of the Lagrangian

$$\delta A_0 = 0, \quad \delta X_0 = 0, \quad \delta \psi = \Gamma^{123} \psi.$$

(5.17)

The superalgebra is then a central extension of $su(2|1)$.

### 5.3 Deformation $SO(3) \to SO(2)$, Clifford$_4$(R) superalgebra : type II

The other deformation, type II, of the $N = 4$ super Yang-Mills quantum mechanics is

$$L_{N=4}^{\text{type II}} = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b]^2 - i \frac{1}{2} \bar{\psi} \Gamma^t D_t \psi - \frac{1}{2} \bar{\psi} \Gamma^{12} [X_a, \psi] \right)$$

$$+ \text{tr} \left( i \frac{1}{8} \mu \bar{\psi} \Gamma^{12} \psi - \frac{1}{12} \mu^2 \left( X_1^2 + X_2^2 + 4X_3^2 \right) \right).$$

(5.19)

The supersymmetry transformations are

$$\delta A_0 = -i \bar{\psi} \Gamma_t \varepsilon(t), \quad \delta X_a = -i \bar{\psi} \Gamma_a \varepsilon(t),$$

$$\delta \psi = \left( - \Gamma^a D_t X_a + i \frac{1}{2} [X_a, X_b] \Gamma^{ab} - \frac{1}{4} \mu \Gamma^{12} X + \frac{1}{12} \mu X \Gamma^{12} \right) \varepsilon(t),$$

(5.20)
where
\[ \varepsilon(t) = e^{-\frac{1}{12}t\mu \Gamma^{12}} \varepsilon(0). \] (5.21)

The supersymmetry algebra reads
\[ [H, Q] = i \frac{1}{12} \mu \Gamma_{12} Q, \quad \{Q, \bar{Q}\} = 2 \Gamma^t (H - \frac{1}{6} \mu M_{12}). \] (5.22)

Similar to (4.19), (4.20), \( H - \frac{1}{6} \mu M_{12} \) is central and positive semi-definite. The corresponding superalgebra is then Clifford_{4}(R).

There exists a nontrivial \( \mathcal{N} = 4 \) supersymmetric configuration corresponding to a circular motion of the \( D \)-particles,

\[ X_1 = R \cos \left( \frac{1}{6} t \mu \right) 1, \quad X_2 = R \sin \left( \frac{1}{6} t \mu \right) 1, \quad X_3 = 0. \] (5.23)

This saturates the unitary bound \( H = \frac{1}{6} \mu M_{12} = \left( \frac{1}{6} \mu R \right)^2 \).

6 \( \mathcal{N} = 2 \) super Yang-Mills quantum mechanics

The three-dimensional Minkowskian spacetime of the metric \( \eta = \text{diag}(− + +) \) admits a Majorana spinor, and all the formulae in Section 5.1 for the four-dimensional super Yang-Mills can be freely adopted for the analysis on the three-dimensional super Yang-Mills. One only needs to note that the gamma matrices are now \( 2 \times 2 \) and \( \Gamma^{t12} = 1 \), such that there are two real supersymmetries.

After the dimensional reduction of the minimal super Yang-Mills in three-dimensions, we obtain a \( \mathcal{N} = 2 \) supersymmetric matrix model with two Hermitian matrices \( X^a, a = 1, 2 \),

\[ \mathcal{L}_0^{\mathcal{N}=2} = \text{tr} \left( \frac{1}{4} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b]^2 - i \frac{1}{2} \bar{\psi} \Gamma^t D_t \psi - \frac{1}{2} \bar{\psi} \Gamma^a [X_a, \psi] \right). \] (6.1)

The most general mass term we may add for the fermion, which is compatible with the Majorana condition, is

\[ i \frac{1}{8} \mu \text{tr} (\bar{\psi} \psi). \] (6.2)

On the other hand, there is no supersymmetric counter term for the bosons which are linear in \( \mu \), up to the field redefinition (A.7). Obviously the Myers term can not exist with two spatial
directions. As we show in Appendix A.3 the supersymmetric completion of the above mass term is unique. The resulting massive $\mathcal{N} = 2$ super Yang-Mills quantum mechanics reads

$$\mathcal{L}_{\text{Massive}}^{\mathcal{N}=2} = \text{tr} \left( \frac{1}{2} D_{t} X^{a} D_{t} X^{a} + \frac{1}{2} [X_{1}, X_{2}]^{2} - i \frac{1}{2} \bar{\psi} \Gamma_{t} D_{t} \psi - \frac{1}{2} \bar{\psi} \Gamma^{a} [X_{a}, \psi] \right)$$

$$+ \text{tr} \left( i \frac{1}{12} \mu \bar{\psi} \psi - \frac{1}{72} \mu^{2} (X_{1}^{2} + X_{2}^{2}) \right).$$

(6.3)

The supersymmetry transformations are

$$\delta A_{0} = -i \bar{\psi} \Gamma_{t} \varepsilon(t), \quad \delta X_{a} = -i \bar{\psi} \Gamma_{a} \varepsilon(t),$$

(6.4)

$$\delta \psi = \left( - \Gamma^{a} D_{t} X_{a} + i [X_{1}, X_{2}] \Gamma^{12} - \frac{1}{6} \mu \Gamma^{a} X_{a} \right) \varepsilon(t),$$

where

$$\varepsilon(t) = e^{\frac{1}{12} \mu t \Gamma_{t}} \varepsilon(0).$$

(6.5)

With the Majorana supercharge $Q = Q^{T} C = Q^{d} \Gamma^{d}$, the supersymmetry algebra of the $\mathcal{N} = 2$ massive super Yang-Mills quantum mechanics reads

$$[H, Q] = -i \frac{1}{12} \mu \Gamma^{d} Q, \quad \{Q, \bar{Q}\} = 2 \Gamma^{d} \left( H - \frac{1}{6} \mu M_{12} \right).$$

(6.6)

As before (4.19), (4.20), $H - \frac{1}{6} \mu M_{12}$ is central and positive semi-definite. The corresponding superalgebra is Clifford$_{2}(\mathbb{R})$.

The BPS state preserving all the two supersymmetries corresponds to a circular motion

$$X_{1} = R \cos \left( \frac{1}{12} \mu t \right) 1, \quad X_{2} = R \sin \left( \frac{1}{12} \mu t \right) 1,$$

(6.7)

which saturates the unitary bound $H = \frac{1}{6} \mu M_{12} = \left( \frac{1}{12} \mu R \right)^{2}$.

7 $\mathcal{N} = 1 + 1$ super Yang-Mills quantum mechanics

In two-dimensional Minkowskian spacetime, the Majorana-Weyl spinor has only one real component, $\psi = \psi^{t}$. Accordingly the minimal super Yang-Mills field theory has one real supersymmetry. However, it turns out that upon dimensional reduction the number of supersymmetries is doubled [3], and hence our notation $\mathcal{N} = 1 + 1$. The most general supersymmetric deformation was obtained in [3] and shown to allow for two arbitrary time dependent functions $\Lambda(t)$, $\rho(t)$,

$$\mathcal{L}_{\text{Massive}}^{\mathcal{N}=1+1} = \text{tr} \left[ \frac{1}{2} (D_{t} X)^{2} + i \frac{1}{2} \bar{\psi} D_{t} \psi + X \psi \bar{\psi} + \frac{1}{2} \Lambda(t) X^{2} + \rho(t) X \right].$$

(7.1)
Here our convention is not to use any gamma matrices so that $X$ is just a Hermitian matrix.\textsuperscript{10}

The two dynamical supersymmetries, which we distinguish by $+,-$ indices, are

\[ \delta_{\pm} A_0 = \delta_{\pm} X = i f_{\pm}(t) \psi_{\pm}, \quad \delta_{\pm} \psi = \left( f_{\pm}(t) D_t X - \dot{f}_{\pm}(t) X - \kappa_{\pm}(t) 1 \right) \varepsilon_{\pm}. \]  

(7.2)

Here $\varepsilon_{+}, \varepsilon_{-}$ are two real supersymmetry parameters; $f_{+}(t), f_{-}(t)$ are two different solutions of the following second order differential equation

\[ \ddot{f}_{\pm}(t) = f_{\pm}(t) \Lambda(t); \]  

(7.3)

and $\kappa_{+}(t), \kappa_{-}(t)$ are given by

\[ \kappa_{\pm}(t) := \int_{t_0}^{t} dt' \rho(t') f_{\pm}(t'). \]  

(7.4)

These two dynamical supersymmetries further reveal three bosonic symmetries which we denote by $\delta_{++}, \delta_{--}, \delta_{(+,-)}$, in order to indicate their origins as the anti-commutator of two supersymmetries:

\[ \delta_{++} A_0 = \delta_{++} X = f_{+} \left( f_{+} D_t X - \dot{f}_{+} X - \kappa_{+} 1 \right), \quad \delta_{++} \psi = 0, \]

\[ \delta_{--} A_0 = \delta_{--} X = f_{-} \left( f_{-} D_t X - \dot{f}_{-} X - \kappa_{-} 1 \right), \quad \delta_{--} \psi = 0, \]

\[ \delta_{(+,-)} A_0 = \delta_{(+,-)} X = 2 f_{+} f_{-} D_t X - \left( f_{+} \dot{f}_{-} + f_{-} \dot{f}_{+} \right) X - (f_{+} \kappa_{-} + f_{-} \kappa_{+}) 1, \quad \delta_{(+,-)} \psi = 0. \]  

(7.5)

These bosonic symmetries form $sp(2,R) \equiv so(1,2)$ Lie algebra, and with the two supersymmetries they form $osp(1|2,R)$ super Lie algebra.

From the supersymmetry transformations of the fermion, the BPS equations are

\[ f_{\pm}(t) D_t X = \dot{f}_{\pm}(t) X + \kappa_{\pm}(t) 1. \]  

(7.6)

A generic BPS configuration then decomposes into the traceless and $u(1)$ parts,

\[ X(t) = f_{+}(t) \mathcal{X} + h_{+}(t) 1 \quad \text{or} \quad X(t) = f_{-}(t) \mathcal{X} + h_{-}(t) 1. \]  

(7.7)

Here $\mathcal{X}$ is an arbitrary traceless constant matrix, and $h_{\pm}(t)$ are the solutions of the first order differential equation $f_{\pm} \dot{h}_{\pm} = \dot{f}_{\pm} h_{\pm} + \kappa_{\pm}$ corresponding to the center of mass $N^{-1} tr X(t) = h_{\pm}(t)$. Since $f_{+}(t) \neq f_{-}(t)$, the BPS state preserves only one supersymmetry.

\textsuperscript{10}Note that in the previous sections, our notation was “$X := \Gamma^a X_a$”. 

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8 Comments

Alternative to our approach \textit{i.e.} looking for the supersymmetric completions of the Yang-Mills quantum mechanics after adding mass terms for fermions, one can consider the light-cone formulation of the supersymmetric Nambu-Goto action in any supersymmetric background and try to obtain a corresponding supersymmetric matrix model, as was done for the eleven-dimensional background [8, 53].

Conversely, from our resulting massive super Yang-Mills quantum mechanics, utilizing the light-cone formulation (2.53) for the generic background (2.10), it is straightforward to deduce the corresponding supersymmetric background for each massive super Yang-Mills quantum mechanics. For example, while the background for the BMN matrix model (3.2) is the eleven-dimensional pp-wave background (3.1), for the $\mathcal{N} = 4$ type II massive super Yang-Mills quantum mechanics, the relevant background is

\begin{equation}
\frac{1}{36}\mu^2 \left(x_1^2 + x_2^2 + 4x_3^2\right)dx^+dx^- + \sum_{a=1}^3 dx^a dx^a.
\end{equation}

This background preserves 8 supersymmetries [19] which match the sum of the dynamical and kinematical supersymmetries in $\mathcal{N} = 4$ type II massive super Yang-Mills quantum mechanics.

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A Derivation of the mass deformations

A.1 $\mathcal{N} = 8$ super Yang-Mills quantum mechanics

After the dimensional reduction to time $t$, the six-dimensional super Yang-Mills gives a supersymmetric matrix model Lagrangian $\mathcal{L}_{N=8}^0$ containing five Hermitian matrices $X^a$, $a = 1, 2, 3, 4, 5$,

$$\mathcal{L}_{N=8}^0 = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b]^2 - i \frac{1}{2} \bar{\psi}^i \Gamma^t D_t \psi_i - \frac{1}{2} \bar{\psi}^i \Gamma^a [X_a, \psi_i] \right). \quad (A.1)$$

The mass dimensions are 1 for the bosons and $\frac{3}{2}$ for the fermions so that the Lagrangian has mass dimension 4.

Dropping the su(2) indices of the su(2) Majorana-Weyl spinor, i.e. $\psi \equiv \psi_1$, we can rewrite the Lagrangian as

$$\mathcal{L}_{N=8}^0 = \text{tr} \left( \frac{1}{2} D_t X^a D_t X_a + \frac{1}{4} [X^a, X^b]^2 - i \bar{\psi}^i \Gamma^t D_t \psi_i - \frac{1}{2} \bar{\psi}^i \Gamma^a [X_a, \psi_i] \right). \quad (A.2)$$

We look for the mass deformation of the above matrix model

$$\mathcal{L}_{N=8}^{\text{Massive}} = \mathcal{L}_{N=8}^0 + \mu \mathcal{L}_{N=8}^1 + \mu^2 \mathcal{L}_{N=8}^2 + \cdots, \quad (A.3)$$

where $\mu$ is a constant mass parameter we introduce. Accordingly, $\mathcal{L}_{N=8}^1$ has mass dimension three, and hence it should take the form:

$$\mathcal{L}_{N=8}^1 = \text{tr} \left( \bar{\psi} M \psi + \frac{1}{3!} S_{abc} X^a X^b X^c + J_{ab} X^a D_t X^b \right), \quad (A.4)$$

where $M_t, M_a, M_{abc} = M_{[abc]}$, $S_{abc}, J_{ab}$ are dimensionless and may depend on time. On the other hand, $\mathcal{L}_{N=8}^2$ has mass dimension two. Hence it should be purely bosonic and, in fact, quadratic in $X$

$$\mathcal{L}_{N=8}^2 = - \text{tr} \left( \frac{1}{2} S_{(ab)} X^a X^b \right). \quad (A.5)$$

It is clear that the expansion (A.3) terminates at the second order in $\mu$.

Without loss of generality we may set $J_{ab} = - J_{ba}$, since

$$\text{tr} \left( J_{ab} X^a D_t X^b \right) = \text{tr} \left( J_{[ab]} X^a D_t X^b \right) + \frac{d}{dt} \text{tr} \left( \frac{1}{2} J_{(ab)} X^a X^b \right) - \text{tr} \left( \frac{1}{2} \dot{J}_{(ab)} X^a X^b \right), \quad (A.6)$$

and the time derivative of $J_{ab}$ corresponds to $\mathcal{L}_{N=8}^2$. A time dependent SO(5) rotation of the dynamical variables $(X^a, \psi) \rightarrow (L^a X^b, \hat{L} \psi)$ with

$$L^T = L^{-1}, \quad \hat{L}^\dagger = \hat{L}^{-1}, \quad \hat{L} \Gamma_a \hat{L}^{-1} = \Gamma_a L^b, \quad \hat{L} \Gamma_t = \Gamma_t \hat{L}, \quad (A.7)$$

11In (A.4), the chirality of the fermion has been taken into account, so that $\Gamma^{iab}$ is absent in the expansion of $M$. 

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leaves the above generic form of the Lagrangian (A.3) invariant. Among others, it induces a shift of \( J_{ab} \),

\[
J_{ab} \rightarrow J_{ab} - 2 \left( L^T L \right)_{ab} .
\]  \( \text{(A.8)} \)

Hence, by solving \( \dot{L} = \frac{1}{2} LJ \) for a given \( J \), we can eliminate \( J_{ab} \) \cite{54,55}. Therefore without loss of generality, we may set \( J_{ab} = 0 \) in (A.4). Similarly by a time dependent U(1) phase transformation of the fermion we can set \( M_t = 0 \) as well, such that

\[
M = M_a \Gamma^a - i \frac{1}{\beta} M_{abc} \Gamma^{abc} .
\]  \( \text{(A.9)} \)

Now it is useful to note that

\[
\Gamma^t M \Gamma^t = M ,
\]

\[
\Gamma^a M \Gamma^b - \Gamma^b M \Gamma^a + M \Gamma^{ab} = 4iM^{abc} \Gamma_c - i \Gamma^{ab} M ,
\]

\[
[M, \Gamma^{12345}] = 0 ,
\]

\[
M^2 = M_a M^a + \frac{i}{\beta} M_{abc} \Gamma^{abc} - i M_{abc} M^c \Gamma^{ab} - \frac{i}{\beta} M_{abc} M_{cde} \Gamma^{abed} .
\]  \( \text{(A.10)} \)

The supersymmetry transformations for \( L_0^{N=8} \) should get modified. However, in order to benefit from the Fierz identity, \( \text{(A.8)} \), we keep the transformation of the bosons as before, but allow the supersymmetry parameter to be time dependent so that

\[
\delta A_0 = \bar{\psi} \Gamma_0 \varepsilon(t) + \bar{\varepsilon}(t) \Gamma_0 \psi ,
\]

\[
\delta X_a = \bar{\psi} \Gamma_a \varepsilon(t) + \bar{\varepsilon}(t) \Gamma_a \psi ,
\]

\[
\delta \psi = \left( -i D_t X_a \Gamma^{ta} - \frac{1}{2} [X_a, X_b] \Gamma^{ab} + \mu \Delta \right) \varepsilon(t) ,
\]  \( \text{(A.11)} \)

where \( \Delta \) is Lie algebra valued, depends on \( X^a \) and has mass dimension one. Its explicit form is to be determined. Although we allow \( \varepsilon(t) \) to be time dependent, it can not be Lie algebra valued.

Explicitly, with a constant and chiral supersymmetry parameter \( \varepsilon \), we set

\[
\varepsilon := G(t) \dot{\varepsilon} ,
\]

\[
\partial_t \varepsilon(t) = \mu \Pi(t) \varepsilon(t) ,
\]

\[
\mu \Pi(t) := \partial_t G(t) G(t)^{-1} .
\]  \( \text{(A.12)} \)

Accordingly under the above transformations we obtain

\[
\delta L_0^{N=8} \simeq \mu \text{tr} \left[ \bar{\psi} \Gamma^t \left( - i D_t \Delta + \Gamma^{ta} [X_a, \Delta] - D_t X_a \Gamma^{ta} \Pi + i \frac{1}{2} [X_a, X_b] \Gamma^{ab} \Pi - i \mu \Delta \Pi \right) \varepsilon \right] + \text{c.c} ,
\]

\[
\delta \text{tr} \left[ \bar{\psi} M \psi \right] = \text{tr} \left[ \bar{\psi} M \left( - i D_t X_a \Gamma^{ta} - \frac{1}{2} [X_a, X_b] \Gamma^{ab} + \mu \Delta \right) \varepsilon \right] + \text{c.c} ,
\]

\[
\delta \text{tr} \left[ \frac{1}{3!} S_{abc} X^a X^b X^c \right] = \text{tr} \left[ \frac{1}{3!} \bar{\psi} S_{abc} X^a X^b \Gamma^c \varepsilon \right] + \text{c.c} ,
\]  \( \text{(A.13)} \)

where c.c. denotes the complex conjugate containing \( \varepsilon \), and ‘\( \simeq \)’ for \( \delta L_0^{N=8} \) indicates that total derivative terms are neglected. From these, we can read off the expression for \( \delta L_{\text{Massive}}^{N=8} \). Especially the terms involving \( D_t X^a \) read

\[
\delta L_{\text{Massive}}^{N=8} \implies \mu \text{tr} \left[ \bar{\psi} D_t \left( - i \Gamma^t \Delta + X_a \Gamma^a \Pi - i M \Gamma^t X_a \Gamma^a \right) \varepsilon \right] + \text{c.c} ,
\]  \( \text{(A.14)} \)
which must vanish by itself. Hence we have

\[ \Delta \left( 1 + \Gamma^{(7)} \right) = (MX - iX\Gamma^4\Pi) \left( 1 + \Gamma^{(7)} \right), \]  

where we set \( X := X_a \Gamma^a \). Further, since \( \Delta \left( 1 - \Gamma^{(7)} \right) \) is irrelevant for the chiral spinor, we may simply set

\[ \Delta = MX - iX\Gamma^4\Pi. \]  

Now using (A.10), it is straightforward to see

\[ \delta \left[ L^N = 8 \right] = \mu \text{tr} \left( \bar{\psi} \left( M_a \Gamma^a - i \frac{1}{3!} M_{abc} \Gamma^{abc} \right) \psi + i \frac{4}{3} M_{abc} X^a X^b X^c \right) \]

which shows

\[ \Pi = -i \frac{1}{3} \Gamma^t M, \]  

and fixes \( L^N = 8 \) as

\[ S_{abc} = i8 M_{abc}. \]  

Provided these, (A.17) further reduces to

\[ \delta \left[ L^N = 8 \right] + \mu \text{tr} \left( \bar{\psi} \left( M_a \Gamma^a - i \frac{1}{3!} M_{abc} \Gamma^{abc} \right) \psi + i \frac{4}{3} M_{abc} X^a X^b X^c \right) \]

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which shows

\[ \Pi = -i \frac{1}{3} \Gamma^t M, \]  

and fixes \( L^N = 8 \) as

\[ S_{abc} = i8 M_{abc}. \]  

Provided these, (A.17) further reduces to

\[ \delta \left[ L^N = 8 \right] + \mu \text{tr} \left( \bar{\psi} \left( M_a \Gamma^a - i \frac{1}{3!} M_{abc} \Gamma^{abc} \right) \psi + i \frac{4}{3} M_{abc} X^a X^b X^c \right) \]

which shows

\[ \Pi = -i \frac{1}{3} \Gamma^t M, \]  

and fixes \( L^N = 8 \) as

\[ S_{abc} = i8 M_{abc}. \]  

Provided these, (A.17) further reduces to

\[ \delta \left[ L^N = 8 \right] + \mu \text{tr} \left( \bar{\psi} \left( M_a \Gamma^a - i \frac{1}{3!} M_{abc} \Gamma^{abc} \right) \psi + i \frac{4}{3} M_{abc} X^a X^b X^c \right) \]

which shows

\[ \Pi = -i \frac{1}{3} \Gamma^t M, \]  

and fixes \( L^N = 8 \) as

\[ S_{abc} = i8 M_{abc}. \]  

Provided these, (A.17) further reduces to

\[ \delta \left[ L^N = 8 \right] + \mu \text{tr} \left( \bar{\psi} \left( M_a \Gamma^a - i \frac{1}{3!} M_{abc} \Gamma^{abc} \right) \psi + i \frac{4}{3} M_{abc} X^a X^b X^c \right) \]

which shows

\[ \Pi = -i \frac{1}{3} \Gamma^t M, \]  

and fixes \( L^N = 8 \) as

\[ S_{abc} = i8 M_{abc}. \]  

Provided these, (A.17) further reduces to

\[ \delta \left[ L^N = 8 \right] + \mu \text{tr} \left( \bar{\psi} \left( M_a \Gamma^a - i \frac{1}{3!} M_{abc} \Gamma^{abc} \right) \psi + i \frac{4}{3} M_{abc} X^a X^b X^c \right) \]

which shows

\[ \Pi = -i \frac{1}{3} \Gamma^t M, \]  

and fixes \( L^N = 8 \) as

\[ S_{abc} = i8 M_{abc}. \]
Contracting this with $\Gamma^a$ from the left and from the right separately, we get
\[
5M^2 - \Gamma^a M^2 \Gamma_a = -4i M_{abc} M^{c} \Gamma^{ab} - 8 \frac{i}{4} M_{abc} M_{cd} \epsilon^{abcd} = 0,
\]
\[
5\dot{M} + \Gamma^a \dot{M} \Gamma_a = 2 \dot{M}_a \Gamma^a - i \dot{M}_{abc} \Gamma^{abc} = 0.
\]
Thus, $M$ must be time independent $\dot{M} = 0$ and satisfy
\[
M_{abc} M_{cd} \epsilon^{abcd} = 0, \quad M_{abc} M^c = 0.
\]
To solve these two constraints we set
\[
M_{abc} := \frac{1}{2} \epsilon_{abcde} M^{de},
\]
and take the canonical form for $M^{ab}$ utilizing the SO(5) rotation:
\[
M^{ab} = c_1 \begin{pmatrix}
0 & \cos \theta & 0 & 0 & 0 \\
-\cos \theta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sin \theta & 0 \\
0 & 0 & -\sin \theta & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}^{ab},
\]
where $c_1$ and $\theta$ are some constants. The first constraint in (A.25) implies $\epsilon_{abcde} M^{bc} M^{de} = 0$ so that
\[
c_1 \cos \theta \sin \theta = 0.
\]
Hence, without loss of generality, we get $\theta = 0$. The second constraint then shows $M_3 = M_4 = M_5 = 0$. Using the SO(2) rotation we further set $M_a = c_2 \delta^1_a$, and hence all together
\[
M = c_2 \Gamma^1 - ic_1 \Gamma^{345}.
\]
Substituting this into (A.22) we obtain the final constraint
\[
c_1 c_2 = 0.
\]
This implies that there are two distinct mass deformations of the $\mathcal{N} = 8$ super Yang-Mills quantum mechanics: type I and type II. The case $c_1 \neq 0$, $c_2 = 0$ corresponds to a deformation compatible with the su(2) Majorana-Weyl spinors (type I), while the other deformation by $c_1 = 0$, $c_2 \neq 0$ is not so (type II). The final results on the mass deformations of the $\mathcal{N} = 8$ super Yang-Mills quantum mechanics are spelled out in (4.12) for type I and (4.24) for type II.
A.2 \( \mathcal{N} = 4 \) super Yang-Mills quantum mechanics

After the dimensional reduction to the time, the four-dimensional super Yang-Mills gives a supersymmetric matrix model \( \mathcal{L}_0^{N=4} \) containing three Hermitian matrices \( X^a, a = 1, 2, 3 \),

\[
\mathcal{L}_0^{N=4} = \text{tr} \left( \frac{i}{2} \partial_t X^a \partial_t X_a + \frac{i}{4} [X^a, X^b]^2 - i \frac{i}{2} \bar{\psi} \Gamma^t \partial_t \psi - \frac{1}{2} \bar{\psi} \Gamma^a [X_a, \psi] \right). \quad (A.31)
\]

Most general mass terms for fermions which are compatible with the Majorana condition are

\[
\mu \mathcal{L}_\psi^{N=4} = -i \mu \text{tr} \left[ \bar{\psi} \left( c \Gamma^{123} + \Gamma^t H + r \cos \theta + r \sin \theta \Gamma^{t123} \right) \psi \right], \quad (A.32)
\]

where \( H = \frac{i}{2} \Gamma^{ab} H_{ab} \) and all the coefficients, \( c, r \) are real with \( 0 \leq r, 0 \leq \theta < 2\pi \). Under the chiral transformation

\[
\left( \psi, \bar{\psi} \right) \rightarrow \left( e^{i \Gamma^{t123}} \psi, \bar{\psi} e^{i \Gamma^{t123}} \right), \quad 0 \leq \phi < 2\pi, \quad (A.33)
\]

\( \mathcal{L}_0^{N=4} \) is invariant, while the above mass terms for the fermions transform as \( \theta \rightarrow \theta + 2\phi \). Thus, without loss of generality, we fix \( \theta = 0 \) henceforth.

For the supersymmetric counter terms for the bosons which are linear in \( \mu \), Myers term is the unique candidate up to field redefinitions, as discussed in Section A.1,

\[
\mu \mathcal{L}_\text{Myers} = \mu \text{tr} \left( 4i [X_1, X_2] X_3 \right). \quad (A.34)
\]

Now letting the modified supersymmetry transformations be

\[
\delta A_0 = -i \bar{\psi} \Gamma_t \varepsilon(t), \quad \delta X_a = -i \bar{\psi} \Gamma_a \varepsilon(t),
\]

\[
\delta \psi = \left( - \Gamma^a \partial_t X_a + i \frac{1}{2} [X_a, X_b] \Gamma^{ab} + \mu \Delta \right) \varepsilon(t), \quad (A.35)
\]

\[
\partial_t \varepsilon(t) = \mu \Pi \varepsilon(t),
\]

we consider the terms linear and quadratic in \( \mu \) appearing in \( \delta \left( \mathcal{L}_0^{N=4} + \mu \mathcal{L}_\psi^{N=4} + e \mu \mathcal{L}_\text{Myers} \right) \), where a real coefficient \( e \) is introduced in order to allow the case of the Myers term being absent. Without loss of generality we can take \( e = 0 \) or \( e = 1 \). The linear terms containing \( \partial_t \) read

\[
- i \mu \text{tr} \left[ \bar{\psi} \left( \partial_t \Pi + \Gamma^t \partial_t \Delta - 2 \left( c \Gamma^{123} + \Gamma^t H + r \right) \Gamma^t \partial_t X \right) \varepsilon \right]. \quad (A.36)
\]

Requiring this to vanish, we get

\[
\Delta = 2 \left( r + \Gamma^t H - c \Gamma^{123} \right) X + \Gamma^t \Pi X. \quad (A.37)
\]
Consequently, with \( F = \frac{1}{2} \Gamma^{ab} F_{ab} \) and using
\[
\Gamma^a D_a X = 2F, \quad \Gamma^a H D_a X = F_{ab} H^{ab} 1,
\]
the remaining linear terms become
\[
- i \mu \text{tr} \left[ \bar{\psi} F \left( -3 \Gamma^t \Pi + 2r + 2 \Gamma^t H + 2(2e - 3c) \Gamma^{123} \right) \varepsilon \right].
\]
Hence, to make this vanish for arbitrary \( \psi \) and \( \varepsilon \), we should set
\[
\Pi = \frac{1}{3} \left( -2r \Gamma^t + (6c - 4e) \Gamma^{123} + 2H \right).
\]
Substituting this into (A.37), we also obtain
\[
\Delta = \frac{4}{3} \left( r - e \Gamma^{123} \right) X + 2 \Gamma^t H X + \frac{2}{3} \Gamma^t H X H.
\]
The quadratic terms are now
\[
- i \mu^2 \text{tr} \left[ \bar{\psi} \left( \Gamma^t \Delta \Pi + 2(r + e \Gamma^{123} + \Gamma^t H) \Delta \right) \varepsilon \right],
\]
where explicitly,
\[
\Gamma^t \Delta \Pi + 2(r + e \Gamma^{123} + \Gamma^t H) \Delta
= \frac{16}{9} \Gamma^t \left( XH + 3HX \right) \left( r - e \Gamma^{123} \right) - \frac{8}{3} HXH - 4H^2 X - \frac{4}{9} XH^2
\]
\[
+ \frac{16}{9} X \left[ e^2 + r^2 + 3r(e - 3c) \Gamma^{123} \right].
\]
For the supersymmetry invariance, this quadratic terms must be cancelled by the variation of \( \mathcal{L}_{\Sigma}^{\chi=4} \) which is purely bosonic. This implies that in the expansion of (A.43) by the gamma matrix products \( \{1, \Gamma^\mu, \Gamma_{\mu\nu}, \Gamma_{\lambda\mu\nu}, \Gamma_{\mu123} \} \), only the linear order gamma matrices, \( \Gamma_a, a = 1, 2, 3 \) must appear. In particular, \( \Gamma_t \) is not also allowed, as the explicit appearance of the gauge field \( A_0 \), would break the gauge symmetry. From
\[
XH + 3HX = 2\epsilon_{abc} X^a H^{bc} \Gamma^{123} + 2 H_{ab} X^b \Gamma^a,
\]
the term linear in \( \Gamma^t \) is \( \frac{32}{9} \epsilon \Gamma^t \epsilon_{abc} X^a H^{bc} \). Since this should vanish for arbitrary \( X^a \), we must require
\[
H = 0 : \text{type I,} \quad \text{or} \quad e = 0 : \text{type II}.
\]
If \( H = 0 \) (type I), Eq. (A.43) gets simplified and we note \( c = e \) or \( r = 0 \). However, when \( r = 0 \), a field redefinition of the fermions which is given by a time dependent chiral rotation 
\[
\psi \to \exp(-2\mu\delta c \Gamma^{t123} t) \psi,
\]
shifts the constant \( c \) only as \( c \to c + \delta c \). Thus, the \( r = 0 \) case can be treated as a special case of \( c = e \).

On the other hand, if \( e = 0 \) and \( H \neq 0 \) (type II), we observe \( r \) should vanish, and hence, as explained above, we can safely set \( c = 0 \). A canonical choice \( H = \frac{1}{2} \Gamma_{12} \) gives
\[
\delta \left[ \mathcal{L}^{N=4}_0 + \mu \mathcal{L}^{N=4}_\psi - \frac{2}{3} \mu^2 \left( X^2_1 + X^2_2 + 4X^2_3 \right) \right] = \text{total derivative}.
\]  
(A.46)

This completes our analysis. The final results on two different mass deformations of \( N = 4 \) super Yang-Mills quantum mechanics are spelled out in (5.7) for type I and (5.19) for type II.

### A.3 \( N = 2 \) super Yang-Mills quantum mechanics

The three-dimensional Minkowskian spacetime of the metric \( \eta = \text{diag}(-+++) \), admits a Majorana spinor, and all the formulae in Section 5.1 for the four-dimensional super Yang-Mills can be freely adopted, with the understanding that the gamma matrices are now \( 2 \times 2 \), and \( \Gamma^{t12} = 1 \). There are two real supersymmetries.

After the dimensional reduction to the time, the super Yang-Mills in three-dimensions gives a supersymmetric matrix model \( \mathcal{L}^{N=2}_N \) which is essentially of the same form as (5.6) but contains two Hermitian matrices \( X^a, a = 1, 2 \).

The most general mass term for fermions which is compatible with the Majorana condition reads
\[
\mu \mathcal{L}^{N=2}_\psi = -i\frac{1}{2} \mu \text{tr} \left( \bar{\psi} \psi \right).
\]  
(A.47)

On the other hand, there is no supersymmetric counter term for the bosons which is linear in \( \mu \), up to the field redefinition (A.7). Clearly the Myers term does not exist with two spatial directions.

Now we take the modified supersymmetry transformations to be as before (A.35), and consider the terms linear and quadratic in \( \mu \) appearing in \( \delta \left( \mathcal{L}^{N=4}_0 + \mu \mathcal{L}^{N=2}_\psi \right) \). The linear terms containing \( D_t \) read
\[
- i \mu \text{tr} \left[ \bar{\psi} \left( D_t X \Pi + \Gamma^t D_t \Delta - \Gamma^t D_t \chi \right) \right].
\]  
(A.48)

Requiring this to vanish, we get
\[
\Delta = X + \Gamma^t X \Pi.
\]  
(A.49)

Consequently, with \( F = \frac{1}{2} \Gamma^{ab} F_{ab} \) and using \( \Gamma^a D_a X = 2F \) the remaining linear terms become
\[
- i \mu \text{tr} \left[ \bar{\psi} F \left( 1 - 3 \Gamma^t \Pi \right) \right].
\]  
(A.50)
Hence
\[ \Pi = -\frac{1}{3} \Gamma'. \] (A.51)

Substituting this into (A.37), we also obtain
\[ \Delta = \frac{2}{3} X. \] (A.52)

This completes our analysis. The final result on the mass deformation of the $\mathcal{N} = 2$ super Yang-Mills quantum mechanics is spelled out in (6.3).

### A.4 $\mathcal{N} = 1 + 1$ super Yang-Mills quantum mechanics

The dimensional reduction of the minimal super Yang-Mills in two-dimensions leads to the following supersymmetric matrix model
\[ \mathcal{L}_0 = \text{tr} \left[ \frac{i}{2} D_t X D_t X + i \frac{1}{2} \psi D_t \psi + X \psi \psi \right]. \] (A.53)

The supersymmetry transformation $\delta_{\text{YM}}$ descending from the two-dimensional super Yang-Mills is, with a constant supersymmetry parameter $\varepsilon$,
\[ \delta_{\text{YM}} A_0 = \delta_{\text{YM}} X = i \psi \varepsilon, \quad \delta_{\text{YM}} \psi = D_t X \varepsilon. \] (A.54)

Now, following [3], we look for the generalization of the above Lagrangian as well as the supersymmetry transformations. First of all, we note from
\[ \text{tr} \left[ i \frac{1}{2} \psi D_t \psi + X \psi \psi \right] = \text{tr} \left[ i \frac{1}{2} \psi \partial_t \psi + (X - A_0) \psi \psi \right], \] (A.55)

that, in order to cancel the possible cubic term of $\psi$ which may arise from the transformation of $(X - A_0)$, it is inevitable to impose $\delta A_0 = \delta X$. Hence, introducing a time dependent function $f(t)$, we set the generalized supersymmetry transformation to be
\[ \delta A_0 = \delta X = if(t) \psi \varepsilon, \quad \delta \psi = \left( f(t) D_t X + \Delta \right) \varepsilon, \] (A.56)

where $\Delta$ is a bosonic quantity having mass dimension 2, whose explicit form is to be determined shortly. After some straightforward manipulation, we obtain
\[ \delta \mathcal{L}_0 = \text{tr} \left[ i \psi \varepsilon \left( D_t \left( f X + \Delta \right) - f X + i [X, \Delta] \right) \right] + \partial_t \mathcal{K}, \] (A.57)

where the total derivative term is given by
\[ \mathcal{K} = \text{tr} \left( D_t X \delta X - i \frac{1}{2} \psi \delta \psi \right). \] (A.58)
Of course, the simplest case where \( f(t) = 1 \) and \( \Delta = 0 \) reduces to the supersymmetry of the original two-dimensional super Yang-Mills, (A.54). For the generic cases, we are obliged to set

\[
\Delta = -\dot{f}X - \kappa 1, 
\]

and obtain the following supersymmetry invariance

\[
\delta \left[ L_0 + \text{tr} \left( \frac{1}{2} (\dot{f}/f) X^2 + (\dot{\kappa}/f) X \right) \right] = \partial_t K.
\]  

This essentially leads to the final result (7.1) with the supersymmetry enhancement: \( \mathcal{N} = 1 \rightarrow \mathcal{N} = 1 + 1 \). This kind of supersymmetry enhancement after the dimensional reduction can be noted elsewhere e.g. [54, 55]. A physical reason for the enhancement is that the D-brane which the higher dimensional field theory describes preserves only a fraction of the supersymmetries of the corresponding \( \mathcal{M} \)-theory.

### A.5 \( \mathcal{N} = 16 \) super Yang-Mills quantum mechanics

Here we show that for \( \mathcal{N} = 16 \) super Yang-Mills quantum mechanics, the BMN matrix model (3.2) is the unique mass deformation. For simplicity we employ the real and symmetric nine-dimensional gamma matrices, \( \Gamma^a = \Gamma^{a*} = \Gamma^a \), \( a = 1, 2, 3, \ldots, 9 \) such that \( \Gamma^{123456789} = 1 \) and the sixteen component Majorana spinor is real. The undeformed super Yang-Mills quantum mechanics follows from the dimensional reduction of the ten-dimensional super Yang-Mills,

\[
L_{0}^{\mathcal{N}=16} = \text{tr} \left( \frac{1}{2} D^a X^b D_b X^a + \frac{1}{4} [X^a, X^b]^2 + i \frac{1}{2} \Psi^T D_t \Psi - \frac{1}{2} \Psi^T \Gamma^a [X_a, \Psi] \right).
\]

In a similar fashion to the previous analysis, henceforth we look for the mass deformation of the above matrix model:

\[
L_{\text{Massive}}^{\mathcal{N}=16} = L_{0}^{\mathcal{N}=16} + \mu L_{1}^{\mathcal{N}=16} + \mu^2 L_{2}^{\mathcal{N}=16}.
\]  

In particular, \( L_{1}^{\mathcal{N}=16} \) has mass dimension three, and hence it should take the form

\[
L_{1}^{\mathcal{N}=16} = \text{tr} \left( \frac{1}{2} i \Psi^T M \Psi + \frac{1}{4!} S_{abc} X^a X^b X^c + J_{ab} X^a D_t X^b \right),
\]

where \( M \) is a real and anti-symmetric matrix, so that it can be expressed as

\[
M = \frac{1}{2} M_{ab} \Gamma^{ab} + \frac{1}{3!} M_{abc} \Gamma^{abc}.
\]  

As discussed before through \( A.6, A.7, A.8 \), without loss of generality we may set \( J_{ab} = 0 \), and consider the following supersymmetry transformation

\[
\delta A_0 = i \Psi^T \epsilon(t), \quad \delta X^a = i \Psi^T \Gamma^a \epsilon(t), \quad \delta \Psi = (D_t X + F + \mu \Delta) \epsilon(t),
\]

where we set \( X := X^a \Gamma_a, F := -i \frac{1}{2} [X^a, X^b] \Gamma_{ab} \). We further write \( \partial_t \epsilon(t) = \mu \Pi \epsilon(t) \).
The transformation induces
\[ \delta \mathcal{L}_{\text{N}=16}^0 = i \mu \text{tr} \left[ \Psi^T \left( D_t \Delta + D_t X \Pi + F \Pi + i \Gamma^a [X_a, \Delta] + \mu \Delta \Pi \right) \right] \varepsilon(t) + \text{total derivative}, \]
\[ \delta \mathcal{L}_{\text{I}}^{\text{N}=16} = i \text{tr} \left[ \Psi^T \left( M (D_t X + F + \mu \Delta) + \frac{1}{2} S_{abc} X^a X^b \Gamma^c \right) \right] \varepsilon(t). \]

(A.66)

\[ \text{Requiring } \delta \mathcal{L}_{\text{N}=16}^0 \text{ to vanish, in analogy to (A.16), (A.18), (A.19), we obtain se quently} \]
\[ \Delta = -X \Pi - M X, \]
\[ \Pi = \frac{1}{3} M, \]
\[ S_{abc} = -\frac{8}{9} i M_{abc}. \]

(A.67)

Hence, we have
\[ \delta (\mathcal{L}_{\text{N}=16}^0 + \mu \mathcal{L}_{\text{I}}^{\text{N}=16}) = i \mu \text{tr} \left[ \Psi^T F_{ab} M_{cd} \Gamma^{abcd} \right] \varepsilon(t) + O(\mu^2) + \text{total derivative}. \]

(A.68)

Thus for the supersymmetry invariance it is required to set \( M_{ab} = 0 \), which is essentially due to our gauge choice \( J_{ab} = 0 \). The remaining terms are second order in \( \mu \) as
\[ \delta (\mathcal{L}_{\text{N}=16}^0 + \mu \mathcal{L}_{\text{I}}^{\text{N}=16}) = -i \mu^2 \text{tr} \left[ \Psi^T X_a \mathcal{M}^a \varepsilon(t) \right] + \text{total derivative}, \]

(A.69)

where we set
\[ \mathcal{M}^a := \mu^{-1} \left( \dot{M} \Gamma^a + \frac{1}{3} \Gamma^a \dot{M} \right) + M^2 \Gamma^a + \frac{2}{3} M \Gamma^a M + \frac{1}{9} \Gamma^a M^2 \]
\[ = -\frac{2}{27} M_{bcd} \Gamma^{bde} + \frac{1}{9} \mu^{-1} M_{abc} \Gamma^{bde} - \frac{1}{9} M_{abc} M_{def} \Gamma^{bde} \]

(A.70)

This must be linear in gamma matrices, and hence from the second order term we see \( M \) must be time independent \( \dot{M}_{abc} = 0 \). Other higher order terms give the following two constraints
\[ M^{ab} [c M_{de}]b = 0, \quad M^a [bc M_{def}] = 0. \]

(A.71)

Now we look for the most general solution of these constraints, up to the SO(9) rotation. Without loss of generality we set \( M_{[ab} M_{9]}^{ab} \neq 0 \). Using the SO(9) rotation, we further let among the components \( M_{9ij}, 1 \leq i, j \leq 8 \), only \( M_{129}, M_{349}, M_{569}, M_{789} \) may be non-vanishing, while \( M_{789} \) is strictly nonzero. Now, considering the case \( a = d = 9, b = 7, c = 8 \) for the latter constraint in (A.71), we see that all the components \( M_{9ij} \) are vanishing except \( M_{789} \). Moreover, the case \( a = 9 \) for the constraint shows that in fact all the components \( M_{abc} \) are vanishing except \( M_{789} \). In this case, the first constraint in (A.71) is automatically satisfied and the resulting mass deformation of the \( N = 16 \) super Yang-Mills quantum mechanics is uniquely given by the BMN matrix model (3.2). This completes our proof.
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