GREEN FUNCTION OF A RANDOM WALK IN A CONE.

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Abstract. This paper studies the asymptotic behavior of the Green function of a multidimensional random walk killed when leaving a convex cone with smooth boundary. Our results imply uniqueness, up to a multiplicative factor, of the positive harmonic function for the killed random walk.

1. Introduction, main results and discussion

Consider a random walk \( \{S(n), n \geq 1\} \) on \( \mathbb{R}^d \), \( d \geq 1 \), where
\[
S(n) = X(1) + \cdots + X(n)
\]
and \( \{X(n), n \geq 1\} \) is a family of independent copies of a random variable \( X = (X_1, X_2, \ldots, X_d) \). Denote by \( S^{d-1} \) the unit sphere of \( \mathbb{R}^d \) and \( \Sigma \) an open and connected subset of \( S^{d-1} \). Let \( K \) be the cone generated by the rays emanating from the origin and passing through \( \Sigma \), i.e. \( \Sigma = K \cap S^{d-1} \).

Let \( \tau_x \) be the exit time from \( K \) of the random walk with starting point \( x \in K \), that is,
\[
\tau_x = \inf \{ n \geq 1 : x + S(n) \notin K \}.
\]
Denisov and Wachtel \([4, 5]\) have constructed a positive harmonic function \( V(x) \) for the random walk \( \{S(n)\} \) killed at leaving \( K \). That is,
\[
V(x) = \mathbb{E}[V(x + X); \tau_x > 1], \quad x \in K.
\]
They have also proved standard and local limit theorems for random walks conditioned to stay in the cone \( K \).

In the present paper we determine the asymptotic behavior of the Green function for \( \{S(n)\} \) killed at leaving \( K \) and prove by using the Martin compactification the uniqueness of the positive harmonic function for such processes.

Next introduce the assumptions on the cone \( K \) and on the random walk \( \{S(n) : n \geq 1\} \). Let \( u(x) \) be the unique strictly positive on \( K \) solution of the following boundary problem:
\[
\Delta u(x) = 0, \quad x \in K \quad \text{with boundary condition } u|_{\partial K} = 0.
\]
Let \( L_{S^{d-1}} \) be the Laplace-Beltrami operator on \( S^{d-1} \) and assume that \( \Sigma \) is regular with respect to \( L_{S^{d-1}} \). With this assumption, there exists (see, for example, \([1]\)) a complete set of orthonormal eigenfunctions \( m_j \) and corresponding eigenvalues \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) satisfying
\[
L_{S^{d-1}} m_j(x) = -\lambda_j m_j(x), \quad x \in \Sigma
\]
\[
m_j(x) = 0, \quad x \in \partial \Sigma.
\]

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Define
\[ p := \sqrt{\lambda_1 + (d/2 - 1)^2} - (d/2 - 1) > 0. \]
The function \( u(x) \) is given by
\[ u(x) = |x|^p m_1 \left( \frac{x}{|x|} \right), \quad x \in K. \] (2)

As in [5], we shall impose the following condition on the cone \( K \):

- \( K \) is either convex or starlike and \( C^2 \).

We impose the following assumptions on the increments of the random walk:

- Normalisation assumption: We assume that \( \mathbb{E}X_j = 0, \mathbb{E}X_j^2 = 1, j = 1, \ldots, d \). In addition we assume that \( \text{cov}(X_i, X_j) = 0 \).
- Moment assumption: We assume that \( \mathbb{E}|X|^\alpha < \infty \) with \( \alpha = p \) if \( p > 2 \) and some \( \alpha > 2 \) if \( p \leq 2 \).
- Lattice assumption: \( X \) takes values on a lattice \( R \) which is a non-degenerate linear transformation of \( \mathbb{Z}^d \).

Our first result describes the asymptotic behavior of the Green function for endpoints \( y \) which lie deep inside the cone \( K \).

**Theorem 1.** Set \( r_1(p) = p + d - 2 + (2 - p)^+ \) and assume that \( \mathbb{E}|X|^{r_1(p)} \) is finite. If \( |y| \to \infty \) and \( \text{dist}(y, \partial K) \geq \alpha |y| \) for some positive \( \alpha \), then
\[ G_K(x, y) := \sum_{n=0}^{\infty} \mathbb{P}(x + S(n) = y, \tau_x > n) \sim c V(x) \frac{u(y)}{|y|^{2p+d-2}}. \] (3)

Moreover, this relation remains valid if one replaces the moment condition \( \mathbb{E}|X|^{r_1(p)} < \infty \) by the following restriction on the local structure of \( X_1 \):
\[ \mathbb{P}(X = x) \leq |x|^{-p-d+1} f(|x|) \] (4)
for some decreasing function \( f \) such that \( u^{(3-p)/2} f(u) \to 0 \) as \( u \to \infty \).

Uchiyama [22] has shown, see Theorem 2 there, that if \( d \geq 5 \) and \( \mathbb{E}|X|^{d-2} < \infty \) then
\[ G_{R^d}(0, z) \sim \frac{c}{|z|^{d-2}}, \quad |z| \to \infty. \]
If \( d = 4 \) or \( d = 3 \) then the same is valid provided that respectively \( \mathbb{E}|X|^2 \log |X| < \infty \) or \( \mathbb{E}|X|^2 < \infty \).

Uchiyama mentions also that this moment condition is optimal: for every \( \varepsilon > 0 \) there exists a random walk satisfying \( \mathbb{E}|X|^{d-2-\varepsilon} < \infty \) such that
\[ \limsup_{|z| \to \infty} |z|^{d-2} G_{R^d}(0, z) = \infty. \]
He has considered the dimensions 4 and 5 only, but it is quite simple to show that this statement holds in every dimension \( d \geq 5 \). We now give an example in our setting of a random walk which shows the optimality of Uchiyama’s condition and of the moment condition in Theorem [3]. Our example is just a multidimensional variation of the classical Williamson example, see [24].

**Example 2.** Let \( d \) be greater than 4 and consider \( X \) with the following distribution. For every \( n \geq 1 \) and for every basis vector \( e_k \) put
\[ \mathbb{P}(X = \pm 2^n e_k) = \frac{q_n}{2d}, \]
where the sequence $q_n$ is such that

$$
\sum_{n=1}^{\infty} q_n = 1 \quad \text{and} \quad q_n \sim \frac{e \log n}{2^n (d-2)}.
$$

Clearly,

$$
E|X|^{d-2} = \infty \quad \text{and} \quad E \frac{|X|^{d-2}}{\log^{1+\varepsilon}|X|} < \infty.
$$

Using now the obvious inequality $G_{R^d}(0,x) \geq P(X = x)$, we conclude that, for every $j = 1, 2, \ldots, d$,

$$
\lim_{n \to \infty} 2^{(d-2)n} G_{R^d}(0, \pm 2^n e_j) = \infty.
$$

If we have a cone $K$ such that $p \geq 2$ and $e_j \in \Sigma$ for some $j$, then, choosing $q_n \sim \frac{c \log n}{2^{n (p+2-d)}},$ we also have

$$
\lim_{n \to \infty} 2^{(p+2-d)n} G_K(e_j, (1 + 2^n)e_j) = \infty.
$$

Therefore, the finiteness of $E|X(1)|^{r_1(p)}$ can not be replaced by a weaker moment assumption.

But Uchiyama shows that the moment assumption $E|X|^{d-2}$ is not necessary, as it can be replaced by $P(X = x) = o(|x|^{-d-2}),$ which implies the existence of the second moment only. In Theorem 1 we have a similar situation: the moment condition $E|X|^{r_1(p)} < \infty$ is not necessary and can be replaced by the assumption (5), which yields the finiteness of $E|X|^{p/2}$ only. It has been shown in [4], if $p > 2$ then the condition $E|X|^p < \infty$ is an optimal moment condition for the existence of the harmonic function $V(x)$.

We now turn to the asymptotic behavior of the Green function along the boundary of the cone.

**Theorem 3.** Assume that $K$ is convex and $C^2$. Assume also that $E|X|^{r_1(p)+1} < \infty$. If $|y|/|x|$ converges to $\sigma \in \partial \Sigma$ as $|y| \to \infty$ then there exists a strictly positive function $v_\sigma(y)$ such that

$$
G_K(x,y) \sim \frac{V(x)v_\sigma(\operatorname{dist}(y, \partial K))}{|y|^{p+d-1}}.
$$

(5)

The function $v_\sigma$ is asymptotically linear, that is,

$$
v_\sigma(t) \sim c_\sigma t \quad \text{as} \quad t \to \infty.
$$

Moreover, the same relation for $G_K$ holds if one replaces the moment condition $E|X|^{r_1(p)+1} < \infty$ by the following restriction on the local structure of $X_1$:

$$
P(X = x) \leq |x|^{-p-d+1} f(|x|)
$$

(6)

for some decreasing function $f$ such that $\log(u)u^{(3-p)/2}f(u) \to 0$ as $u \to \infty$.

Clearly, one can adapt the random walk from Example 2 to show that the moment assumption in Theorem 3 is minimal as well. Indeed, it suffices to take $q_n \sim \frac{c \log n}{2^{n (p+2-d)}}$ and to assume that one of the vectors $\pm e_j$ belongs to the boundary of the cone $K$.

Theorems 1 and 3 describe the asymptotic behavior of $G_K(x,y)$ along all possible directions inside the cone $K$. Combining these two results, we conclude that, for all $x, x' \in K$,

$$
\frac{G(x,y)}{G(x',y)} \to \frac{V(x)}{V(x')} \quad \text{as} \quad |y| \to \infty.
$$
As a result we have the following

**Corollary 4.** Assume that the assumptions of Theorem 3 are valid. Then the function $V(x)$ is the unique, up to multiplicative factor, positive harmonic function for $\{S(n)\}$ killed at leaving $K$.

Doney [8] has shown that the harmonic function for any one-dimensional oscillating random walk killed at leaving the positive half-axis is unique without any additional moment assumption.

For multidimensional cones much less is known. Raschel [19] has shown the uniqueness of the positive harmonic function for random walks with small steps killed at leaving the positive quadrant $\mathbb{Z}_+^2$, and some particular cases of such random walks have been studied by the same author in [17] and [18]. The approach in [19] is based on a functional equation which is satisfied by all harmonic functions. It should be also mentioned that [17] and [18] describe actually the asymptotic behavior of the Green function and the uniqueness of the harmonic function is just a consequence of the results on the Green function.

Uchiyama [23] derives asymptotics for the Green function for random walks in $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, see Theorem 5 in [23]. He assumes that $E|X|^d$ is finite. This is slightly weaker than the moment assumption in Theorem 3, which reduces in the case of a half-space to $E|X|^{d+1} < \infty$. On the other hand, Theorem 3 can be applied to any half-space, and Uchiyama’s proof uses in a crucial way that the half-space is precisely $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, i.e. aligned with the lattice of the random walk.

Bouaziz, Mustapha and Sifi [2] have shown uniqueness for a wide class of random walks with finite number of steps killed at leaving the orthant $\mathbb{Z}_+^d$, $d \geq 2$.

Ignatiouk-Robert [15] has studied the properties of harmonic function for random walks on semigroups by introducing a special renewal structure. Applying this to a random walk in a cone, she has shown that, if the cone $K$ satisfies the assumptions from [4] with some $p \leq 2$ and $E|X|^\alpha$ is finite for some $\alpha > 2$, then the harmonic function is unique. From our results on the Green function one can deduce uniqueness if it is assumed that $E|X|^{d+1} < \infty$. Therefore, for cones with $p \leq 2$ the result in [15] holds under much weaker restrictions on the random walk than Corollary 4. This does not provide any information on the asymptotic behavior of the Green function.

Raschel and Tarrago [20] have derived an implicit asymptotic representation for the Green function of random walks in cones under the assumptions $p \geq 1$ and $E|X|^n (p) \rho < \infty$. This is the main difference between their assumptions and ours: we need less moments but we impose the condition that the cone is $C^2$. The latter excludes, for example, Weyl chambers, which appear often in models from physics.

We conclude this section by describing our approach. Using the local limit theorem for $S(n)$ killed at leaving $K$ one can easily determine the asymptotic behaviour of the sum

$$\sum_{n \geq \varepsilon |y|^2} P(x + S(n) = y, \tau_x > n),$$

for every fixed $\varepsilon > 0$. Thus, the main problem in studying $G_K(x, y)$ consists in estimating the sum of local probabilities over $n \leq \varepsilon |y|^2$. The local limit theorem is useless in this domain, since $y$ belongs to the region of large deviations. For unconditioned one-dimensional transient random walks Caravenna and Doney [3] estimate every probability separately to obtain asymptotics for the Green function.
Our strategy is completely different: we consider the sum of probabilities as the expectation of the number of visits to \( y \) up to time \( \varepsilon |y|^2 \). In the proof of Theorem \[4\] we derive an upper bound for this expectation in terms of the Green function \( G_{\mathbb{R}^d} \) and apply the functional limit theorems from \[10\]. It turns out that if \( y \) goes to infinity along the boundary then the estimate via \( G_{\mathbb{R}^d} \) becomes too rough. For that reason we first consider the special case when \( K \) is a half-space. In this situation we derive the asymptotic for \( G_K \) via appropriate estimates for local probabilities of large deviations. This part is similar to the approach in \[3\]. Given the well-known asymptotic of Green functions for half-spaces we follow the same strategy as in the proof of Theorem \[4\] to estimate the expected number of visits to \( y \) in terms of the Green function for the half-space whose normal is perpendicular to the direction of convergence for the Green function. For that estimate we need to assume that \( K \) is convex and \( C^2 \).

2. Asymptotic behavior inside the cone.

**Proof of Theorem \[4\].** Fix some \( \varepsilon > 0 \) and split \( G_K(x, y) \) into two parts:

\[
G_K(x, y) = \sum_{n<|y|^2} P(x + S(n) = y, \tau_x > n) + \sum_{n\geq|y|^2} P(x + S(n) = y, \tau_x > n)
= : S_1(x, y, \varepsilon) + S_2(x, y, \varepsilon).
\]

By Theorem 5 in \[4\],

\[
n^{p/2+d/2} P(x + S(n) = y, \tau_x > n) = \varkappa H_0 V(x) u \left( \frac{y}{\sqrt{n}} \right) e^{-|y|^2/2n} + o(1)
\]

uniformly in \( y \in K \). Consequently, as \( |y| \to \infty \),

\[
S_2(x, y, \varepsilon) = \varkappa H_0 V(x) \sum_{n\geq|y|^2} \frac{1}{n^{p/2+d/2}} u \left( \frac{y}{\sqrt{n}} \right) e^{-|y|^2/2n} + o \left( \sum_{n\geq|y|^2} \frac{1}{n^{p/2+d/2}} \right)
= \varkappa H_0 V(x) u(y) \sum_{n\geq|y|^2} \frac{1}{n^{p+d/2}} e^{-|y|^2/2n} + o \left( |y|^{-p-d+2} \right)
= \varkappa H_0 V(x) u(y) |y|^{-2p-d+2} \int_\varepsilon^{\infty} z^{-p-d/2} e^{-1/(2z)} dz + o \left( |y|^{-p-d+2} \right).
\]

Letting here \( \varepsilon \to 0 \) and recalling that \( u(y) \geq c(\alpha) |y|^p \) for \( \text{dist}(y, \partial K) \geq \alpha |y| \), we obtain

\[
\lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^{2p+d-2}}{u(y)} S_2(x, y, \varepsilon) = \varkappa H_0 V(x) \int_0^\infty z^{-p-d/2} e^{-1/(2z)} dz. \quad (7)
\]

Therefore, it remains to show that

\[
\lim_{\varepsilon \to 0} \limsup_{|y| \to \infty} \frac{|y|^{2p+d-2}}{u(y)} S_1(x, y, \varepsilon) = 0. \quad (8)
\]

Fix additionally some small \( \delta > 0 \) and define

\[
\theta_y := \inf \{ n \geq 1 : x + S(n) \in B_{\delta, y} \},
\]

where \( B_{\delta, y} \) denotes the ball of radius \( \delta |y| \) around point \( y \).
Then we have
\[ S_1(x, y, \varepsilon) = \sum_{n \leq |y|^2} P(x + S(n) = y, \tau_x > n \geq \theta_y) \]
\[ = \sum_{n \leq |y|^2} \sum_{k=1}^{n} \sum_{z \in B_{k,y}} P(x + S(k) = z, \tau_x > k = \theta_y) P(z + S(n-k) = y, \tau_z > n-k) \]
\[ \leq \sum_{k \leq |y|^2} P(x + S(k) = z, \tau_x > k = \theta_y) \sum_{j \leq |y|^2-k} P(z + S(j) = y) \]
\[ \leq \mathbb{E} \left[ G(|y|^2)(y - x - S(\theta_y)); \tau_x > \theta_y, \theta_y \leq \varepsilon|y|^2 \right], \quad (9) \]
where
\[ G^{(t)}(z) := \sum_{n < t} P(S(n) = z). \]

We focus first on the case \( d \geq 3 \). Then, according to Theorem 2 in Uchiyama [22],
\[ G(z) := G^{(\infty)}(z) \leq \frac{C}{1+|z|^{d-2}}, \quad z \in \mathbb{Z}^d, \quad (10) \]
provided that \( \mathbb{E}|X_1|^\alpha < \infty \), where \( s_d = 2 + \varepsilon \) for \( d = 3, 4 \) and \( s_d = d - 2 \) for \( d \geq 5 \). Since \( r_1(p) > s_d \), (10) yields
\[ S_1(x, y, \varepsilon) \]
\[ \leq CE \left[ \frac{1}{1+|y-x-S(\theta_y)|^{d-2}}; \tau_x > \theta_y, \theta_y \leq \varepsilon|y|^2 \right] \quad (11) \]
\[ \leq CP(|y-x-S(\theta_y)| \leq \delta^2|y|; \tau_x > \theta_y, \theta_y \leq \varepsilon|y|^2) + \frac{C(\delta)}{|y|^{d-2}} P(\tau_x > \theta_y, \theta_y \leq \varepsilon|y|^2). \]

Noting now that \( |y-x-S(\theta_y)| \leq \delta^2|y| \) yields \( |X(\theta_y)| > \delta(1-\delta)|y| \) and using our moment assumption, we conclude that
\[ P(|y-x-S(\theta_y)| \leq \delta^2|y|, \tau_x > \theta_y, \theta_y < \varepsilon|y|^2) \]
\[ \leq \sum_{k \leq |y|^2} P(|X(k)| > \delta(1-\delta)|y|, \tau_x > k = \theta_y) \]
\[ \leq P(|X| > \delta(1-\delta)|y|) \sum_{k \leq |y|^2} P(\tau_x > k - 1) \]
\[ = o \left( |y|^{-r_1(p)} \mathbb{E}[\tau_x; \tau_x < |y|^2] \right) = o \left( |y|^{-d-p+2} \right). \quad (12) \]

Recalling that \( V \) is harmonic for \( S(n) \) killed at leaving \( K \), we obtain
\[ P(\tau_x > \theta_y, \theta_y < \varepsilon|y|^2) \]
\[ = \sum_{k \leq |y|^2} \sum_{|z-y| \leq \delta|y|} P(\tau_x > k, \theta_y = k, x + S(k) = z) \]
\[ = \sum_{k \leq |y|^2} \sum_{|z-y| \leq \delta|y|} \frac{V(x)}{V(z)} P^{(V)}(\theta_y = k, x + S(k) = z) \]
\[ \leq \frac{V(x)}{\min_{|z-y| \leq \delta|y|} V(z)} \frac{V^{(V)}(\theta_y < \varepsilon|y|^2)}{V(z)}. \]
It follows from the assumption $\text{dist}(y, \partial K) \geq \alpha |y|$ and Lemma 13 in [4], that

$$\min \{ z : |z-y| \leq \delta |y| \} V(z) \geq C|y|^p$$

for all $\delta$ sufficiently small. As a result,

$$|y|^p P(\tau_x > \theta_y, \theta_y < \varepsilon |y|^2) \leq C(\delta)^{-} \sum_{k=1}^m P(\tau_x > k-1) \mathbb{E}[|\tau_x - |z' - y||] .$$

Applying now the functional limit theorem for $S(n)$ under $P(V)$, see Theorem 2 and Corollary 3 in [10], we conclude that

$$\lim_{\varepsilon \to 0} \limsup_{|y| \to \infty} |y|^p P(\tau_x > \theta_y, \theta_y < \varepsilon |y|^2) = 0. \quad (13)$$

Note that the functional limit theorem from [10] only requires $p \vee (2 + \varepsilon)$-moments.

Combining (11)–(13), we infer that (8) is valid under the assumption $\mathbb{E}|X_1|^p < \infty$ in all dimensions $d \geq 3$.

Assume now that (4) holds. It is clear that this restriction implies that $\mathbb{E}|X_1|^p < \infty$. Therefore, Theorem 5 in [4] is still applicable and (7) remains valid for all random walks satisfying (4). In order to show that (8) remains valid as well, we notice that

$$S_1(x, y, \varepsilon) = \mathbb{E} \left[ 1 + |y - x - S(\theta_y)|^{d-2} : |y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right]$$

In view of (13), we have to estimate the first term on the right hand side only. For every $z$ such that $|z - y| \leq \delta^2 |y|$ we have

$$\mathbb{P}(x + S(\theta_y) = z, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2) \leq \sum_{k=1}^m \sum_{z' \in K \setminus \partial K} \mathbb{P}(x + S(k-1) = z', \tau_x > k-1) \mathbb{P}(X_k = z - z').$$

Since $|z - z'| \geq \delta(1 - \delta)|y|$, we infer from (14) that

$$\mathbb{P}(x + S(\theta_y) = z, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2) \leq C(\delta) |y|^{-p+2} f(\delta(1 - \delta) |y|) \sum_{k=1}^m \mathbb{P}(\tau_x > k-1)$$

Here and in the following we use that $\mathbb{E}[\tau_x ; \tau_x < |y|^2] \sim C|y|^{-p+2}$ if $p \leq 2$. 


For every natural $m$ there are $O(m^{d-1})$ lattice points $z$ such that $|z - y| \in (m,m + 1]$. Then, using \((14)\), we obtain
\[
\begin{align*}
\mathbb{E} \left[ \frac{1}{1 + |y - x - S(\theta_y)|^{d-2}} |y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] \\
\leq C(\delta) |y|^{-p-d+1} f(\delta(1-\delta)/|y|) \mathbb{E}[\tau_x; \tau_x < |y|^2] \sum_{m=1}^{\infty} \frac{m^{d-1}}{1 + m^{d-2}} \\
\leq C(\delta) |y|^{-p-d+3} f(\delta(1-\delta)/|y|) \mathbb{E}[\tau_x; \tau_x < |y|^2].
\end{align*}
\]
Recalling that $u^{(3-p)\nu_1} f(u) \to 0$, we conclude that
\[
\mathbb{E} \left[ \frac{1}{1 + |y - x - S(\theta_y)|^{d-2}} |y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] = o(|y|^{-p-d+2}).
\]
This completes the proof of the theorem for $d \geq 3$.

We now focus on $d = 2$. In this case we can not use the full Green function. We will obtain bounds for $G(t)(x)$ directly from the local limit theorem for unrestricted walks. More precisely, we shall use Propositions 9 and 10 from Chapter 2 in Spitzer’s book \(21\), which say that
\[
P(S_n = z) = \frac{1}{2\pi n} e^{-|z|^2/2n} + \rho(n,z) \left| z \right| \nu n, \quad \text{as } n \to \infty
\]
where
\[
\sup_{z \in \mathbb{Z}^2} \rho(n,z) \to 0, \quad \text{as } n \to \infty.
\]
This asymptotic representation implies that
\[
\sup_{z \in \mathbb{Z}^2} G(t)(z) \leq C \log t, \quad t \geq 2.
\]
Furthermore, for $|z| \to \infty$ and $t \leq a |z|^2$ one has
\[
G(t)(z) \leq \sum_{n=1}^{a |z|^2} \frac{1}{2\pi n} e^{-|z|^2/2n} + o(1) = \frac{1}{2\pi} \int_0^{a |z|^2} \frac{1}{v} e^{-v/2} dv + o(1).
\]
As a result,
\[
\sup_{z \in \mathbb{Z}^2} G(a |z|^2)(z) \leq C(a) < \infty.
\]
Using \((16)\) and \((17)\), we obtain
\[
S_1(x,y,\varepsilon) \leq C \log |y| \mathbb{P}(|y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2) \\
+ C(\varepsilon) \mathbb{P}(\tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2)
\]
According to \((12)\),
\[
\mathbb{P}(|y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2) = o(|y|^{-r(\nu)} \mathbb{E}[\tau_x; \tau_x \leq |y|^2])
\]
\[
= o(|y|^{-p}/\log |y|).
\]
Combining this with \((13)\), we conclude that \(\text{(8)}\) holds for $d = 2$. \(\Box\)
3. Random walks in a half-space

In this section we shall consider a particular cone

\[ K = \{ x \in \mathbb{R}^d : x_d > 0 \} . \]

Since the rotations of the space do not affect our moment assumptions, the results of this section remain valid for any half-space in \( \mathbb{R}^d \).

For this very particular cone we have

- \( u(x) = x_d \);
- \( \tau_x = \inf \{ n \geq 1 : x_d + S_d(n) \leq 0 \} ; \)
- \( V(x) \) depends on \( x_d \) only and is proportional to the renewal function of ladder heights of the random walk \( \{ S_d(n) \} \).

In other words, the exit problem from \( K \) is actually a one-dimensional problem. This allows the use of exiting results for one-dimensional walks. As a result we obtain the asymptotic behavior of the Green function.

**Theorem 5.** Assume that \( \mathbb{E}|X|^{d+1} < \infty \). Assume also that \( x = (0, \ldots, 0, x_d) \) with \( x_d = o(|y|) \). Then

\[ G_K(x, y) \sim \frac{V(x)V'(y)}{|y|^d} . \]

Here, \( V' \) is the harmonic function for the killed reversed random walk \( \{-S(n)\} \).

The proof of this result is based on the following simple generalization of known results for cones.

**Lemma 6.** Assume that \( \mathbb{E}|X|^{2+\delta} \) is finite. Then, uniformly in \( x \in K \) with \( x_d = o(\sqrt{n}) \)

1. \( \mathbb{P}(\tau_x > n) \sim \mathcal{N} V(x)n^{-1/2} ; \)
2. \( \{ \frac{2 \mathbb{E}[|S(n)|]}{\sqrt{n}} \} \), \( t \in [0, 1] \) conditioned on \( \{ \tau_x > n \} \) converges weakly to the Brownian meander in \( K \);
3. \( \sup_{y \in K} \left| n^{1/2+d/2} \mathbb{P}(x + S(n) = y; \tau_x > n) - cV(x)\frac{y_d}{\sqrt{n}}e^{-|y-x|^2/2n} \right| \to 0. \)

**Proof.** The first statement is the well-known result for one-dimensional random walks, see Corollary 3 in Doney [9]. The second and the third statements for fixed starting points \( x \) have been proved in [10] and in [4] respectively. To consider the case of growing \( x_d \) one has to make only one change: Lemma 24 from [4] should be replaced by the estimate

\[ \lim_{n \to \infty} \frac{1}{V(x)} \mathbb{E} \left[ |x + S(\nu_n)|; \tau_x > \nu_n, |x + S(\nu_n)| > \theta_n\sqrt{n}, \nu_n \leq n^{1-\epsilon} \right] = 0 \]

uniformly in \( x_d \leq \theta_n\sqrt{n}/2 \). If \( x_d \geq n^{1/2-\epsilon} \) then \( \nu_n = 0 \) and the expectation equals zero. If \( x_d \leq n^{1/2-\epsilon} \) then one repeats the proof of Lemma 24 in [4] with \( p \) replaced by 1 and uses the part (a) of the lemma to obtain a uniform in \( x_d \) estimate for the sum \( \sum_{j \leq n^{1-\epsilon}} \mathbb{P}(\tau_x > j - 1) \). (In [4], the Markov inequality has been used, since one does not have the statement (a) in general cones.)

**Lemma 7.** Uniformly in \( y \) with \( y_d = o(\sqrt{n}) \),

\[ \mathbb{P}(x + S(n) = y, \tau_x > n) \sim c \frac{V(x)V'(y)}{n^{1+d/2}} e^{-|y|^2/2n} . \]
Applying part (c) of Lemma 6 to the random walk $\{M_k\}$, where

It follows from part (b) of the previous lemma that $y - y'$ is Brownian meander. Combining these observations with

This completes the proof. □
Proof of Theorem 3. If \( y \) is such that \( y_d \geq \alpha |y| \) for some \( \alpha > 0 \) then it suffices to repeat the proof of Theorem 1.

We consider then the 'boundary case' \( y_d = o(|y|) \).

Using Lemma 8 one obtains easily

\[
\lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^d}{V(x)V'(y)} S_2(x, y, \varepsilon) = c.
\]

Namely, it follows that

\[
\lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^d}{V(x)V'(y)} S_2(x, y, \varepsilon) = c \lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \sum_{n \geq |y|^2} |y|^d n^{-1} - \frac{d}{2} e^{-\frac{|y|^2}{2n}}
\]

\[
= c \int_0^\infty \left( \frac{1}{v} \right)^{1+\frac{d}{2}} e^{-\frac{v}{2}} dv
\]

and the last integral is finite. It follows that the theorem will be proven if we show that

\[
\lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^d}{V(x)V'(y)} S_1(x, y, \varepsilon) = 0. \tag{18}
\]

Using an appropriate rotation we can reduce everything to the case \( y_k = o(|y|) \) for every \( k = 2, \ldots, d - 1 \) and \( y_1 \sim |y| \). This also implies \( y_d = o(|y|) \).

We first split the probability \( P(x+S(n) = y, \tau_x > n) \) into two parts:

\[
P(x+S(n) = y, \tau_x > n) = P(x+S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y_1) + P(x+S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| > \gamma y_1),
\]

where \( \gamma \in (0, 1) \). Introduce the stopping time

\[
\sigma_\gamma := \inf \{ k \geq 1 : |X_1(k)| > \gamma y_1 \}.
\]

Then, by the Markov property,

\[
P(x+S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| > \gamma y_1)
\]

\[
= \sum_{k=1}^n P(x+S(n) = y, \tau_x > n, \sigma_\gamma = k)
\]

\[
\leq \sum_{k=1}^n P(\tau_x > k-1)P(|X_1| > \gamma y_1) \max_z P(S(n-k) = z).
\]

Using now the bounds \( P(\tau_x > k) \leq CV(x)k^{-1/2} \) and \( \max_z P(S(k) = z) \leq Ck^{-d/2} \), we obtain

\[
P(x+S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| > \gamma y_1)
\]

\[
\leq CV(x)P(|X_1| > \gamma y_1) \sum_{k=1}^n \frac{1}{\sqrt{k} (n-k+1)^{d/2}}
\]

\[
\leq CV(x)P(|X_1| > \gamma y_1) \frac{(\log n)^{1(d=2)}}{\sqrt{n}}.
\]

Here, in the last step we have splitted the sum \( \sum_{k=1}^n \frac{1}{\sqrt{k} (n-k+1)^{d/2}} \) into \( \sum_{k=1}^\frac{n}{d} \) and \( \sum_{k=\frac{n}{d}} \) and used elementary inequalities.
This implies that
\[
\sum_{n=1}^{\varepsilon|y|^2} P(x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| > \gamma y_1) 
\leq C \sqrt{V(x)} P(|X_1| > \gamma y_1) |y| (\log |y|^2)^{1(d=2)}.
\]
As a result, for all random walks satisfying
\[
E \left[ |X|^{d+1} (\log |X|)^{1(d=2)} \right] < \infty,
\]
we have
\[
\sum_{n=1}^{\varepsilon|y|^2} P(x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| > \gamma y_1) = o\left( \frac{V(x)}{|y|^d} \right). \tag{19}
\]
In order to estimate \( P(x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y_1) \) we shall perform the following change of measure:
\[
P(X(k) \in dz) = \frac{e^{hz_1}}{\varphi(h)} P(X(k) \in dz; |X_1(k)| \leq \gamma y_1),
\]
where
\[
\varphi(h) = E \left[ e^{hX_1}; |X_1| \leq \gamma y_1 \right].
\]
Therefore,
\[
P(x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y_1) 
= e^{-hy_1} \varphi^n(h) P(x + S(n) = y, \tau_x > n). \tag{20}
\]
According to (21) in Fuk and Nagaev [14],
\[
e^{-hy_1} \varphi^n(h) 
\leq \exp \left\{ -hy_1 + h \gamma y_1 n E[X_1; |X_1| \leq \gamma y_1] + \frac{e^{h\gamma y_1} - 1 - h\gamma y_1}{\gamma^2 y_1^2} \frac{1}{n} E[X_1^2; |X_1| \leq \gamma y_1] \right\}.
\]
Choosing
\[
h = \frac{1}{\gamma y_1} \log \left( 1 + \frac{\gamma y_1^2}{n E[X_1^2; |X_1| \leq \gamma y_1]} \right) \tag{21}
\]
and noting that
\[
|E[X_1; |X_1| \leq \gamma y_1]| = |E[X_1; |X_1| > \gamma y_1]| \leq \frac{1}{\gamma y_1} E[X_1^2] = \frac{1}{\gamma y_1},
\]
we conclude that uniformly for \( n \leq \gamma |y|^2 \) it holds
\[
e^{-hy_1} \varphi^n(h) \leq \left( \frac{e^{n}}{\gamma y_1^2} \right)^{1/\gamma}.
\]
Plugging this into (20), we obtain uniformly for \( n \leq \gamma |y|^2 \)
\[
P \left( x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y_1 \right) 
\leq C(\gamma) \left( \frac{n}{|y|^2} \right)^{1/\gamma} \overline{P}(x + S(n) = y, \tau_x > n). \tag{22}
\]
According to Theorem 6.2 in Esseen [13], there exists an absolute constant $C$ such that
\[
\sup_z \mathbb{P}(S(n) = z) \leq \frac{C}{n^{d/2}} x^{-d/2},
\]
where
\[
\chi := \sup_{u \geq 1} \frac{1}{u^2} \inf_{|t| = 1} \mathbb{E}[(t, X(1) - X(2)); |X(1) - X(2)| \leq u].
\]

Since $h$ defined in (21) converges to zero as $|y| \to \infty$ uniformly in $n \leq \gamma |y|^2$,
\[
\mathbb{E}[(t, X(1) - X(2)); |X(1) - X(2)| \leq u] \to \mathbb{E}[(t, X(1) - X(2)); |X(1) - X(2)| \leq u]
\]
for every fixed $u$. Since $S(n)$ is truly $d$-dimensional under the original measure, \(\inf_{|y|=1} \mathbb{E}[(t, X(1) - X(2)); |X(1) - X(2)| \leq u] > 0\) for all large values $u$. As a result, there exists $\chi_0 > 0$ such that $\chi \geq \chi_0$ for all $|y|$ large enough and all $n \leq \gamma |y|^2$. Consequently,
\[
\sup_z \mathbb{P}(S(n) = z) \leq \frac{C\chi_0^{-d/2}}{n^{d/2}}.
\]

Combining this bound with (22), we obtain for all $r \in (0, 1)$, $\gamma < 2/d$
\[
\sum_{n=1}^{y^{2-r}} \mathbb{P} \left( x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y_1 \right)
\]
\[
\leq C(\gamma)x_0^{-d/2}\gamma^{-2/\gamma} \sum_{n=1}^{y^{2-r}} n^{1/\gamma - d/2} \leq C(\gamma)x_0^{-d/2}\gamma^{-2/\gamma} |y|^{(2-r)(1/\gamma-d/2+1)},
\]
for all $n \leq \gamma |y|^2$. If we choose $\gamma$ so small that $r(1/\gamma - d/2 + 1) > 2$, then
\[
\sum_{n=1}^{y^{2-r}} \mathbb{P} \left( x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y_1 \right) = o \left( \frac{1}{|y|^2} \right).
\]

In the case $n \geq |y|^{2-r}$ we can not ignore the condition $\tau_x > n$. By the Markov property at times $n/3$ and $2n/3$ and by (23),
\[
\mathbb{P}(x + S(n) = y, \tau_x > n)
\]
\[
\leq \sum_{z, z'} \mathbb{P}(x + S(n/3) = z, \tau_x > n/3) \mathbb{P}(z + S(n/3) = z', \tau_{z'} > n/3)
\]
\[
= \sum_{z, z'} \mathbb{P}(x + S(n/3) = z, \tau_x > n/3) \mathbb{P}(z + S(n/3) = z', \tau_y > n/3)
\]
\[
\leq \frac{C}{n^{d/2}} \mathbb{P}(\tau_x > n/3) \mathbb{P}(\tau_y > n/3).
\]

Therefore, it remains to show that, uniformly in $n \in [|y|^{2-r}, |y|^2]$,
\[
\mathbb{P}(\tau_x > n/3) \leq C \frac{1 + x_d}{\sqrt{n}}.
\]

Indeed, from this estimate and from the corresponding estimate for the inversed walk we get
\[
\mathbb{P}(x + S(n) = y, \tau_x > n) \leq C \frac{(x_d + 1)(y_d + 1)}{n^{d/2+1}}.
\]
This implies with help of (22) that
\[
\sum_{n=|y|^{2-r}}^{e|y|^2} P \left( x + S(n) = y, \tau_x > n, \max_{k \leq n} |X_1(k)| \leq \gamma y \right)
\leq C \varepsilon^{1/2(d-2)} (x_d + 1)(y_d + 1) |y|^{-d}.
\]
Combining this with (19) and with (24), we obtain (18).

To derive (25) we first estimate some moments of the random walk \(S_d(n)\) under \(F\). By the definition of this probability measure,
\[
E[X_x] = \frac{1}{\varphi(h)} E \left[ X_d e^{hX_1}; |X_1| \leq \gamma y \right].
\]
For the expectation on the right hand side we have the representation
\[
E \left[ X_d e^{hX_1}; |X_1| \leq \gamma y \right]
= E \left[ X_d; |X_1| \leq \gamma y \right] + h E \left[ X_d X_1; |X_1| \leq \gamma y \right]
+ E \left[ X_d(e^{hX_1} - 1 - hX_1); |X_1| \leq \gamma y \right]
= -E \left[ X_d; |X_1| > \gamma y \right] - h E \left[ X_d X_1; |X_1| > \gamma y \right]
+ E \left[ X_d(e^{hX_1} - 1 - hX_1); |X_1| \leq \gamma y \right].
\]
In the last step we have used the equalities
\[
E[X_x] = E[X_d X_1] = 0. \tag{26}
\]
then, by the Markov inequality,
\[
E \left[ X_d; |X_1| > \gamma y \right] + h E \left[ X_d X_1; |X_1| > \gamma y \right] = o(y_1^{-2}) = o(n^{-1}).
\]
Therefore,
\[
E \left[ X_d e^{hX_1}; |X_1| \leq \gamma y \right] = o(n^{-1}) + E \left[ X_d(e^{hX_1} - 1 - hX_1); |X_1| \leq \gamma y \right].
\]
It is obvious that \(|e^x - 1 - x| \leq \frac{x^2}{2} e^{|x|}\). Therefore,
\[
\left| E \left[ X_d(e^{hX_1} - 1 - hX_1); |X_1| \leq \gamma y \right] \right|
\leq \frac{h^2}{2} E \left[ |X_d| |X_1|^2 e^{h|X_1|}; |X_1| \leq \gamma y \right]
\leq \frac{h^2}{2} E \left[ |X_d| |X_1|^2 + h^2 e^{h\gamma y}; E \left[ |X_d| |X_1| > \frac{1}{n} \right] \right]
\leq \frac{c}{2} h^2 E \left[ |X_d| |X_1|^2 + h^{2+\delta} e^{h\gamma y}; E \left[ |X_d| |X_1| \right]^{2+\delta} \right]
\leq \frac{c}{2} h^2 E \left[ |X|^3 + h^{2+\delta} e^{h\gamma y}; E \left[ |X| \right]^{3+\delta} \right].
\]
Here in the last step we have used Hölder’s inequality. It is immediate from the definition of \(h\) that \(h^2 \leq cn^{-1}\). Furthermore, if \(n \geq |y|^{2-r}\) with some \(r < \frac{1}{2}\), then \(h^{2+\delta} e^{h\gamma y} = o(n^{-1})\). From these estimates and from assumption (20), we obtain
\[
\left| E \left[ X_d e^{hX_1}; |X_1| \leq \gamma y \right] \right| \leq \frac{c}{n} \tag{27}
\]
uniformly in \(n \in [|y|^{2-r}, |y|^2]\).
By the same arguments,
\[ \varphi(h) = E \left[ e^{hX_1}; |X_1| \leq \gamma y \right] \]
\[ = P(|X_1| \leq \gamma y) + hE[X_1; |X_1| \leq \gamma y] + E \left[ e^{hX_1} - 1 - hX_1; |X_1| \leq \gamma y \right] \]
\[ = 1 - P(|X_1| > \gamma y) - hE[X_1; |X_1| > \gamma y] + E \left[ e^{hX_1} - 1 - hX_1; |X_1| \leq \gamma y \right] \]
\[ = 1 + o(n^{-1}). \tag{28} \]
Combining this with (27), we finally obtain
\[ |EX_d| \leq \frac{c_1}{n}. \tag{29} \]
We now turn to the second and the third moments of \( X_d \) under \( \mathbf{P} \). Using (28) and the moment assumption we have
\[ EX_d^2 = \frac{1}{\varphi(h)} E[X_d^2 e^{hX_1}; |X_1| \leq \gamma y] = (1 + o(1)) E[X_d^2 e^{hX_1}; |X_1| \leq \gamma y] \]
\[ = E(X_d^2; |X_1| \leq \gamma y) + o(1) + O \left( E \left[ X_d^2 (e^{hX_1} - 1); |X_1| \leq \gamma y \right] \right) \]
\[ = 1 + o(1) + O \left( h e^{h\gamma y} \right). \]
Noting that \( h e^{h\gamma y} = o(1) \) for all \( n \geq |y|^{2-r} \), we get
\[ EX_d^2 = 1 + o(1). \tag{30} \]
Similarly,
\[ EX_d^3 = (1 + o(1)) E[|X_d|^3 e^{hX_1}; |X_1| \leq \gamma y] \]
\[ \leq c \left( E[|X_d|^3; |X_1| \leq 1/h] + e^{h\gamma y} E[|X_d|^3; |X_d| > 1/h] \right) \]
\[ \leq c \left( E[|X_d|^3] + h^\delta e^{h\gamma y} E[|X_d|^3] \right) \]
Using once again the fact that \( h^\delta e^{h\gamma y} = o(1) \) for \( n \geq |y|^{2-r} \), we arrive at
\[ E|X_d|^3 \leq c_3. \tag{31} \]

Now we can derive (25). First, it follows from (29) that
\[ \mathbf{P}(\tau_x > n/3) \leq \mathbf{P}(\tau_x^0 > n/3), \]
where
\[ \tau_x^0 := \inf\{ k \geq 1 : y + S_d^0(k) \leq 0 \} \quad \text{and} \quad S_d^0(k) = S_d(k) - kEX_d. \]
Applying Lemma 25 in [1] to the random walk \( S_d^0 \), we have
\[ \mathbf{P}(\tau_y^0 > k) \leq \frac{E[y + S_d^0(k); \tau_y^0 > k]}{E(y + S_d^0(k))^+} \]
Relations (30) and (31) allow the application of the central limit theorem to the walk \( S_d^0(k) \), which gives \( E(y + S_d^0(k))^+ \geq c \sqrt{k} \). Consequently,
\[ \mathbf{P}(\tau_y^0 > k) \leq \frac{C}{\sqrt{k}} E[y + S_d^0(k); \tau_y^0 > k]. \]
Further, by the optional stopping theorem,
\[ E[y + S_d^0(k); \tau_y^0 > k] = y - E[y + S_d^0(\tau_y^0); \tau_y^0 \leq k] \]
\[ \leq y - E[y + S_d(\tau_y^0)]. \]
We now use inequality (7) in [16] which states that there exists an absolute constant $A$ such that

$$-\mathbb{E}[y + S_d^0(\tau_y^0)] \leq A \frac{\mathbb{E}|X_d|^3}{\mathbb{E}X_d^2}.$$ 

Combining this with (30) and (31), we finally get

$$\mathbb{P}(\tau_y^0 > k) \leq \frac{C(y + 1)}{\sqrt{k}},$$

which implies (25). \hfill \Box

4. Asymptotics close to the boundary

4.1. Limit theorems for random walks starting far from the origin but close to the boundary. Let $|y| \to \infty$ in such a way that $\text{dist}(y, \partial K) = o(|y|)$. Let $y_\perp \in \partial K$ be defined by the relation $\text{dist}(y, \partial K) = |y - y_\perp|$. Set $\sigma(y) := y_\perp/|y| \in \partial \Sigma$ and assume that $\sigma(\cdot)$ converges as $|y| \to \infty$ to some $\sigma \in \partial \Sigma$. Let $H_y$ denote a tangent hyperplane at point $y_\perp$. Let $P_n$ be the distribution of the linear interpolation of $t \to (y + S(nt))/\sqrt{n}$ conditioned to stay in the half-space $K_y$ containing the cone K and having boundary $H_y$. Then $P_n \to P$ weakly on $C[0,1]$.

Denote

$$A_n := \{f \in C[0,1] : f(k/n) \in K \text{ for all } 1 \leq k \leq n\}.$$ 

Then

$$\liminf A_n \supseteq \{f \in C[0,1] : f(t) \in K \text{ for all } t \in (0,1]\}$$

and

$$\limsup \overline{A_n} \subseteq \{f \in C[0,1] : f(t) \in \overline{K} \text{ for all } t \in (0,1]\},$$

where $\overline{A}$ denote the closure of $A$.

Denote for every fixed $n$ by $[0,1] \ni t \mapsto S(nt)$ the linear interpolation of $S(k), k \leq n$. The conditions to apply Theorem 2.3 from Durrett [11] are given. This leads to an invariance principle: $[0,1] \ni r \mapsto \frac{y + S(nt)}{\sqrt{n}}$ converges weakly as $\frac{n}{|y|^2} \to t$ to the Brownian meander $\{B_r, r \leq 1\}$ inside the cone $K$ started at $\frac{y}{\sqrt{2}}$. In particular it holds with $T_y := \inf\{n \geq 1 : y + S(n) \notin K_y\}$

$$\mathbb{P}\left(\frac{y + S(n)}{\sqrt{n}} \in B \mid \tau_y > n\right) \sim Q_{\sigma,t}(B) = \int_B q_{\sigma,t}(z)dz, \quad \frac{n}{|y|^2} \to t$$

(32)

where $q_{\sigma,t}(z)$ is the density of the Brownian meander in $K$, started at $\frac{y}{\sqrt{2}}$ and evaluated at time 1. Theorem 2.3 in [11] also leads to

$$P(\tau_y > n \mid T_y > n) \to c_{\sigma,t}.$$ 

(33)

where

$$T_y := \inf\{n \geq 1 : y + S(n) \notin K_y\}.$$ 

Limiting relations (32) and (33) imply that

$$V(y) \geq c|y|^{p-1}(1 + \text{dist}(y, \partial K)).$$ 

(34)

Indeed, by the harmonicity of $V$

$$V(y) = \mathbb{E}[V(y + S(n)); \tau_y > n], \quad n \geq 1.$$
Fix now some $\epsilon > 0$ and note that choosing $n = \lceil |y|^2 \rceil$ it follows that $V(z) \sim u(z)$ uniformly as $z \to \infty$ as long as distance to $\partial K$ of $z$ is at least $\epsilon |z|$ (Lemma 13 in [4]). We obtain, as $|y| \to \infty$, and $\epsilon \to 0$

\[
V(y) \geq P(T_y > |y|^2) c_{1,1} |y|^p \int_K u(z) q_{1,1}(z) dz.
\]

Due to results for the one-dimensional random walk we arrive at

\[
P(T_y > |y|^2) \geq c \frac{1 + \text{dist}(y, \partial K)}{|y|}.
\]

This establishes (34).

Before proving Theorem 3 we record an auxiliary estimate needed in its proof.

**Lemma 8.** Define

\[
\phi_\sigma(t) = c_{\sigma, t} \int_K u(z) e^{-\frac{|z|^2}{2}} q_{\sigma, t}(z) dz.
\]

It holds $\phi_\sigma(t) = o(e^{-c/t})$ as $t \to 0$ for some $c > 0$.

**Proof.** First we record that due to the invariance principle for the halfspace it holds

\[
c_{\sigma, t} = P_\sigma(\tau^{me} > t) = P_{\sigma/\sqrt{t}}(\tau^{me} > 1),
\]

where $\tau^{me} := \inf\{t > 0 : M^\sigma(t) \notin K_y\}$. Here $M^\sigma(t)$ is a Brownian meander in $K_y$ whereas we will denote the Brownian meander in $K$ by $M^\sigma_K(t)$. Since $|\sigma| = 1$ and $K$ is contained in $K_y$ it is clear then that $c_{\sigma, t} \to 1$ as $t \to 0$.

It follows

\[
\phi_\sigma(t) \leq C E_{\sigma/\sqrt{t}}[u(M^\sigma_K(1)) e^{-\frac{1}{2} (\frac{|M^\sigma_K(1)|^2}{t})}] \leq C E_{\sigma/\sqrt{t}}[u(M^\sigma(1)) e^{-\frac{1}{2} (\frac{|M^\sigma(1)|^2}{t})}].
\]

The second inequality can be easily justified using the invariance principles for meanders in $K$ and $K_y$ as well as the fact that $c_{\sigma, t} \to 1$ is bounded away from zero.

It follows

\[
\phi_\sigma(t) \leq C E_{\sigma/\sqrt{t}}[e^{-\frac{1}{2} (\frac{|M^\sigma(1)|^2}{t})}].
\]

Due to rotation invariance of Brownian motion the expectation doesn’t depend on $\sigma$ so that we can choose $\sigma = (1, 0, \ldots, 0)$ and $K_y = \mathbb{R}^{d-1} \times \mathbb{R}_+$. The first $d - 1$ coordinates become independent Brownian motions whereas the last one is a 1-dimensional Brownian meander (see [12] for its density). This finishes the proof.

4.2. **Proof of Theorem 3.** To estimate the contribution coming from large values of $n$ one does not need the limit theorems from the previous paragraph, quite rough estimates turn out to be sufficient.

Set $m = \lceil n/2 \rceil$. Then, applying the Markov property at time $m$ and inverting the time in the second part of the path, we obtain

\[
P(x + S(n) = y, \tau_x > n)
\]

\[
= \sum_{z \in K} P(x + S(m) = z, \tau_x > m) P(y + S'(n - m) = z, \tau'_y > n - m)
\]

\[
\leq \max_{z \in K} P(x + S(m) = z, \tau_x > m) P(\tau'_y > n - m).
\]
By Theorem 5 in [4],
\[ \max_{z \in K} P(x + S(m) = z, \tau_x > m) \leq C \frac{V(x)}{m^{p/2+d/2}}. \]

Furthermore, due to results for the one-dimensional random walk (see, for example Lemma 3 in [5])
\[ P(\tau_y' > n - m) \leq P(T'_y > n - m) \leq C \frac{1 + \text{dist}(y, \partial K)}{\sqrt{n - m}}. \] (35)

Combining these estimates, we obtain
\[ P(x + S(n) = y) \leq CV(x)(1 + \text{dist}(y, \partial K))n^{-(p+d+1)/2}. \]

Consequently, for \( A \geq 2 \) and \( |y| \geq 1 \),
\[ \sum_{n \geq A|y|^2} P(x + S(n) = y) \leq CV(x)(1 + \text{dist}(y, \partial K)) \sum_{n \geq A|y|^2} n^{-(p+d+1)/2} \]
\[ \leq CV(x)A^{-(p+d-1)/2} \frac{1 + \text{dist}(y, \partial K)}{|y|^{p+d-1}}. \] (36)

We turn now to the ‘middle’ part: \( n \in (\varepsilon |y|^2, A|y|^2) \). Using again the Markov property at time \( m = \lfloor n/2 \rfloor \) and applying Theorem 5 in [4], we obtain
\[ P(x + S(n) = y, \tau_x > n) \]
\[ = \sum_{z \in K} P(x + S(m) = z, \tau_x > m)P(y + S'(n - m) = z; \tau'_y > n - m) \]
\[ = \frac{\varkappa H_0 V(x)}{m^{p/2+d/2}} \sum_{z \in K} \left( u \left( \frac{z}{\sqrt{m}} \right) e^{-i \frac{z^2}{2m}} + o(1) \right) P(y + S'(n - m) = z; \tau'_y > n - m) \]
\[ = \frac{\varkappa H_0 V(x)}{m^{p/2+d/2}} E \left[ u \left( \frac{S'(n - m)}{\sqrt{m}} \right) e^{-i \frac{|S'(n - m)|^2}{2m}} ; \tau'_y > n - m \right] + o \left( \frac{1 + \text{dist}(y, \partial K)}{n^{(p+d+1)/2}} \right). \]

Taking into account (35), we have
\[ P(x + S(n) = y, \tau_x > n) \]
\[ = \frac{\varkappa H_0 V(x)}{m^{p/2+d/2}} E \left[ u \left( \frac{S'(n - m)}{\sqrt{m}} \right) e^{-i \frac{|S'(n - m)|^2}{2m}} ; \tau'_y > n - m \right] + o \left( \frac{1 + \text{dist}(y, \partial K)}{n^{(p+d+1)/2}} \right). \]

Next, it follows from (32) and (33) that if \( \frac{m}{|y|^2} \sim t \) then
\[ E \left[ u \left( \frac{S'(n - m)}{\sqrt{m}} \right) e^{-i \frac{|S'(n - m)|^2}{2m}} ; \tau'_y > n - m \right] \sim P(T'_y > n - m)\phi_\sigma(t/2). \]

Since \( T'_y \) is an exit time from a half space,
\[ P(T'_y > k) \sim v'(y)k^{-1/2}, \]
where \( v'(y) \) is the positive harmonic function for \( S' \) killed at leaving the half-space \( K_\sigma \). As a result,
\[ P(x + S(n) = y, \tau_x > n) = C_0 \frac{V(x)v'(y)}{n^{(p+d+1)/2}} \phi_\sigma \left( \frac{n}{|y|^2} \right) + o \left( \frac{1 + \text{dist}(y, \partial K)}{n^{(p+d+1)/2}} \right), \]
where
\[ C_0 := \varkappa H_0 2^{(p+d+1)/2}. \]
where

Applying Theorem 5 and (17) to the random walk

The first term has been estimated in (12):

Therefore,

This representation implies that

for random walks having finite moments of order

of Theorem 1, but instead of the Green function for the whole space we shall use

the Green function for the half-space

Combining this with (36) and letting \( A \to \infty \), one can easily obtain

From Lemma \( \ref{lemma1} \) it follows

It remains to estimate \( S_1(x, y, \varepsilon) \). We shall use the same strategy as in the proof of Theorem \( \ref{thm1} \) but instead of the Green function for the whole space we shall use the Green function for the half-space \( K_y \). More precisely,

\[
S_1(x, y, \varepsilon) = \sum_{n < \varepsilon |y|^2} \mathbb{P}(x + S(n) = y, \tau_x > n \geq \theta_y)
\]

\[
= \sum_{n < \varepsilon |y|^2} \sum_{k=1}^{n} \mathbb{P}(x + S(n) = z, \tau_x > k = \theta_y) \mathbb{P}(z + S(n-k) = y, \tau_x > n-k)
\]

\[
= \sum_{k<\varepsilon |y|^2} \mathbb{P}(x + S(n) = z, \tau_x > k = \theta_y) \sum_{j<\varepsilon |y|^2-k} \mathbb{P}(z + S(j) = y, \tau_x > j)
\]

\[
\leq \sum_{k<\varepsilon |y|^2} \mathbb{P}(x + S(n) = z, \tau_x > k = \theta_y) \sum_{j<\varepsilon |y|^2} \mathbb{P}(y + S'(j) = z, T'_y > j)
\]

\[
= \mathbb{E} [G_{\varepsilon,y}(x + S(\theta_y)); \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2],
\]

where

\[
G_{\varepsilon,y}(z) = \sum_{j<\varepsilon |y|^2} \mathbb{P}(y + S'(j) = z, T'_y > j).
\]

Applying Theorem \( \ref{thm5} \) and \( \ref{thm17} \) to the random walk \( S'(n) \), we obtain

\[
G_{\varepsilon,y}(z) \leq C \frac{\psi'(y)(1 + \text{dist}(z, H_y))}{1 + |z - y|^d} \wedge 1.
\]

Therefore,

\[
S_1(x, y, \varepsilon) \leq C \mathbb{P}(|y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2)
\]

\[+ C(\delta) \frac{\psi'(y)}{|y|^d} \mathbb{E} [(1 + \text{dist}(x + S(\theta_y), H_y)); \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2]. \quad (38)
\]

The first term has been estimated in (12):

\[
\mathbb{P}(|y - x - S(\theta_y)| \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2) = o(|y|^{-p-1}) \quad (39)
\]

for random walks having finite moments of order \( r_2(p) := p + d - 1 + (2 - p)^+ \).
In order to estimate the second term in (38), we shall perform again the change of measure with the harmonic function $V$:

$$
E \left[ 1 + \text{dist}(x + S(\theta_y), H_y); \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] = V(x)E^{(V)} \left[ \frac{1 + \text{dist}(x + S(\theta_y), H_y)}{V(x + S(\theta_y))}; \theta_y \leq \varepsilon |y|^2 \right].
$$

Applying now (34), we obtain

$$
E \left[ 1 + \text{dist}(x + S(\theta_y), H_y); \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] \leq CV(x)|y|^{-p+1}P^{(V)}(\theta_y \leq \varepsilon |y|^2).
$$

From this estimate and (13) we conclude that

$$
\lim_{|y| \to \infty} \lim_{\varepsilon \to 0} |y|^{p-1}E \left[ 1 + \text{dist}(x + S(\theta_y), H_y); \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] = 0.
$$

Combining this estimate with (38) and (39) as well as Lemma 13 in [1] we get

$$
\lim_{|y| \to \infty} \lim_{\varepsilon \to 0} |y|^{p+d-1}S_1(x, y, \varepsilon) = 0.
$$

(40)

Since $v'(y)$ is bounded from below by a positive number, (40) and (37) yield the desired result for the case $E|X|^r(\sigma) < \infty$ due to classical results for the one-dimensional random walk.

Assume now that (3) holds. It is easy to see that the above proof that

$$
\lim_{|y| \to \infty} \lim_{\varepsilon \to 0} \frac{|y|^{p+d-1}}{V(x)v'(y)}S_2(x, y, \varepsilon) = C_0 \int_0^\infty \phi_x(t)t^{-(p+d+1)/2}dt,
$$

(41)

goes through again word for word. Therefore we focus on the asymptotic of $S_1(x, y, \varepsilon)$ in the following. With similar steps as above it holds

$$
S_1(x, y, \varepsilon) \leq C(\delta)v'(y)E \left[ 1 + \text{dist}(x + S(\theta_y), H_y) \right] \frac{|y - x - S(\theta_y)|}{1 + |x + S(\theta_y) - y|^d}, |y - x - S(\theta_y)| \leq \delta^2|y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] + C(\delta)v'(y)\frac{|y|^d}{|y|^d}E \left[ 1 + \text{dist}(x + S(\theta_y), H_y); \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right].
$$

The second summand can be treated just as above with help of (34) so that we need to show

$$
E \left[ 1 + \text{dist}(x + S(\theta_y), H_y) \right] \frac{|y - x - S(\theta_y)|}{1 + |x + S(\theta_y) - y|^d}, |y - x - S(\theta_y)| \leq \delta^2|y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] = O(|y|^{-p-d+1}).
$$

It holds

$$
1 + \text{dist}(x + S(\theta_y), H_y) \leq 1 + |S(\theta_y) - y| + |y - y_{\perp}| = o(|y|) + |S(\theta_y) - y|.
$$

To complete the proof we now show for $r = d - 1, d$

$$
S_{2,r}(x, y, \varepsilon) := E \left[ 1 + \frac{1}{|x + S(\theta_y) - y|^r}, |y - x - S(\theta_y)| \leq \delta^2|y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right] = o(|y|^{-p-d+1}).
$$
With a similar calculation as in the proof of Theorem 1 (using (13)) we obtain
\[
\mathbb{E} \left[ \frac{1}{1 + |y - x - S(\theta_y)|^{d-1}} \right] y - x - S(\theta_y) \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right]
\leq C(\delta) |y|^{-p-d+1} f(\delta(1 - \delta)|y|) \mathbb{E}[\tau_x; \tau_x < |y|^2] \sum_{m=1}^{\frac{\varepsilon}{\delta^2}} \frac{m^{d-1}}{m^d}
\leq C(\delta) |y|^{-p-d+2} f(\delta(1 - \delta)|y|)|y|^{(2-p)^+}.
\]

Finally,
\[
\mathbb{E} \left[ \frac{1}{1 + |y - x - S(\theta_y)|^d} \right] y - x - S(\theta_y) \leq \delta^2 |y|, \tau_x > \theta_y, \theta_y \leq \varepsilon |y|^2 \right]
\leq C(\delta) |y|^{-p-d+1} f(\delta(1 - \delta)|y|) \mathbb{E}[\tau_x; \tau_x < |y|^2] \sum_{m=1}^{\frac{\varepsilon}{\delta^2}} \frac{m^{d-1}}{m^d}
\leq C(\delta) \log((|y|)|y|^{-p-d+2} f(\delta(1 - \delta)|y|)|y|^{(2-p)^+}.
\]

This finishes the proof of Theorem 3.

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