ε-constants and Arakelov Euler characteristics

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1 Introduction

Let $X$ be a regular scheme projective and flat over Spec$(\mathbb{Z})$, equidimensional of relative dimension $d$. Consider the Hasse-Weil zeta function of $X$, $\zeta(X,s) = \prod_x (1 - N(x)^{-s})^{-1}$ where $x$ ranges over the closed points of $X$ and $N(x)$ is the order of the residue field of $x$. Denote by $L(X,s)$ the zeta function with Γ-factors $L(X,s) = \zeta(X,s)\Gamma(X,s)$. The $L$-function conjecturally satisfies a functional equation

$$L(X,s) = \epsilon(X)A(X)^{-s}L(X,d + 1 - s)$$

where $\epsilon(X)$ and $A(X)$ are real numbers defined independently of any conjectures (the “ε-constant” and the “conductor”). In fact, the unconditional definition of $\epsilon(X)$ and $A(X)$ involve choices of auxiliary primes $l$ with embeddings $\mathbb{Q}_l \subset \mathbb{C}$ (see [De]). In this note, we will suppress any notation regarding these choices; this should not cause any confusion.

The purpose of this note is to explain a way to obtain the absolute value $|\epsilon(X)|$ as an “arithmetic” Euler de Rham characteristic in the framework of the higher dimensional Arakelov theory of Gillet and Soulé. Choose a hermitian metric on the tangent bundle of $X(\mathbb{C})$ which is Kähler; it gives a hermitian metric on $\Omega^1_{X(\mathbb{C})}$. Recall the definition of the arithmetic Grothendieck group $\mathcal{K}_0(X)$ of hermitian vector bundles of Gillet and Soulé ([GS1, II, §6]; all hermitian metrics are smooth and invariant under the complex conjugation on $X(\mathbb{C})$). There is an arithmetic Euler characteristic homomorphism

$$\chi_Q : \mathcal{K}_0(X) \rightarrow \mathbb{R}$$

such that if $(\mathcal{F}, h)$ is a vector bundle on $X$ with a hermitian metric on $\mathcal{F}_{\mathbb{C}}$, then $\chi_Q((\mathcal{F}, h))$ is the Arakelov degree of the hermitian line bundle on Spec$(\mathbb{Z})$ formed by the determinant of the cohomology of $\mathcal{F}$ with its Quillen metric. The arithmetic Grothendieck group $\mathcal{K}_0(X)$ is a λ-ring with $\lambda^i$-operations defined in loc. cit. §7: If $(\mathcal{F}, h)$ is the class of a vector bundle with a hermitian metric on $\mathcal{F}_{\mathbb{C}}$ then $\lambda^i((\mathcal{F}, h))$ is

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the class of the vector bundle \( \wedge^i F \) with the exterior power metric on \( \wedge^i F \) induced from \( h \). Now consider the sheaf of differentials \( \Omega^1_{X/Z} \); this is a “hermitian coherent sheaf” in the terminology of [GS3, 2.5]. Since \( X \) is regular, by loc. cit. 2.5.2 \( \Omega^1_{X/Z} \) defines an element \( \Omega \) in \( \tilde{K}_0(X) \) as follows: Each embedding of \( X \) into projective space over \( \text{Spec}(\mathbb{Z}) \)

\[
E : 0 \to N \to P \to \Omega^1_{X/Z} \to 0
\]

with \( P \) and \( N \) vector bundles on \( X \) (here \( P \) is the restriction of the relative differentials of the projective space to \( X \) and \( N \) is the conormal bundle of the embedding). Pick hermitian metrics \( h^P \) and \( h^N \) on \( P \) and \( N \) respectively and denote by \( \widetilde{\text{ch}}(E) \) the secondary Bott-Chern characteristic class of the exact sequence of hermitian vector bundles \( E \) (as defined in [GS1]; there is a difference of a sign between this definition and the definition in [GS3, 2.5.2]). Then

\[
\Omega = ((P, h^P), 0) - ((N, h^N), 0) + ((0, 0), \widetilde{\text{ch}}(E)) \in \tilde{K}_0(X)
\]

depends only on the original choice of Kähler metric.

For each \( i \geq 0 \) we can consider now the element \( \lambda^i(\Omega) \) in \( \tilde{K}_0(X) \). Motivated by the “higher dimensional Fröhlich conjecture” of [CEPT], we conjecture that

\[
- \log |\epsilon(X)| = \sum_{i=0}^{d} (-1)^i \chi_Q(\lambda^i(\Omega)).
\]

(1.1)

Denote by \( X_S \) the disjoint union of the singular fibers of \( f : X \to \text{Spec}(\mathbb{Z}) \). In [B], S. Bloch conjectures that the conductor \( A(X) \) is given by

\[
A(X) = \text{ord}((-1)^d c_{d+1}^{X_S}(\Omega^1_{X/Z}))
\]

where \( c_{d+1}^{X_S}(\Omega^1_{X/Z}) := c_{d+1}^X(\Omega^1_{X/Z}) \cap [X] \) is the localized \( d+1 \)-st Chern class in \( \text{CH}_0(X_S) \) described in loc. cit. Here for a zero cycle \( \sum_i n_i x_i \), \( \text{ord}(\sum_i n_i x_i) = \prod_i (\# k(x_i))^{n_i} \), with \( k(x_i) \) the residue field of \( x_i \). In this paper we show:

**Theorem 1.2** The equality (1.1) is equivalent to Bloch’s conjecture.

The main ingredients in the proof are the Arithmetic Riemann-Roch theorem of Gillet and Soulé and the fact (Proposition 3.1) that Bloch’s localized Chern class agrees with the corresponding “arithmetic” Chern class of Gillet-Soulé.

Since Bloch has proven in [B] his conjecture for an arithmetic surface \( (d = 1) \) we see that (1.1) holds in this case. In this note we also show:

**Theorem 1.3** Bloch’s conjecture, and therefore equality (1.1), holds when for all primes \( p \), the fiber of \( X \to \text{Spec}(\mathbb{Z}) \) over \( p \) is a divisor with strict normal crossings with multiplicities relative prime to \( p \).
In fact, under the hypothesis of the above theorem, we can show \( \epsilon \) directly by replacing the use of the arithmetic Riemann-Roch theorem by Serre duality and the fact, due to Ray and Singer ([RS], Theorem 3.1), that the analytic torsion of the de Rham complex is trivial. We are grateful to C. Soulé for pointing this out to us; this approach is explained in detail in [CPT2]. Also, since we have \( \epsilon(X)^2 = A(X)^{d+1} \), we could have expressed \( \epsilon \) using the conductor \( A(X) \). However, it seems that \( \epsilon \) is more canonical and it could generalize in a motivic framework (for example to varieties with a group action). Indeed, the inspiration for \( \epsilon \) comes from [CEPT], see also [CPT1], where we observed a close connection between an equivariant version of an Euler de Rham characteristic as above and \( \epsilon \)-constants. Viewed this way, Theorem 1.2 also provides some indirect positive evidence for the general higher dimensional Fröhlich conjecture of [CEPT]. In [CPT2], we use the results of this note to obtain the actual \( \epsilon \)-constant (not just its absolute value) of the Artin motive obtained from the pair \((X, V)\) of an arithmetic variety \(X\) with an action of a finite group \(G\) and a symplectic character \(V\) of \(G\).

We would like to express our thanks to C. Soulé; this note would not have existed without his advice. We would also like to thank T. Saito for useful conversations and B. Erez for pointing out the reference [A]. After a preliminary version of this note was completed we have learned that K. Kato and T. Saito have announced a proof of a stronger version of Theorem 1.3 in which the assumption on the multiplicities is dropped; their proof is significantly more involved than the proof of the tame case that we consider here. T. Saito informed us that a similar argument to ours for the proof of the tame case is given by K. Arai in his thesis, which is currently in preparation.

2 Arithmetic Riemann-Roch

The formulae of [De] imply that \( \epsilon(X)^2 = A(X)^{d+1} \) (we can see that this also follows directly from the conjectural functional equation). Therefore, \( \epsilon \) translates to

\[
\frac{d+1}{2} \cdot \log A(X) = - \sum_{i=0}^{d} (-1)^i \chi_Q(\lambda^i(\Omega)). \tag{2.1}
\]

Denote by \( \overset{\cdot}{\text{CH}}(X) \), \( \overset{\cdot}{\text{CH}}(X) \) the arithmetic Chow groups of Gillet and Soulé ([GS1-2]), graded by codimension and dimension of cycles respectively. Since \( X_S \) has empty generic fiber, there is a natural homomorphism

\[ z_S : \text{CH}_0(X_S) \to \overset{\cdot}{\text{CH}}_0(X) = \overset{\cdot}{\text{CH}}^{d+1}(X). \]

The direct image homomorphism

\[ f_* : \overset{\cdot}{\text{CH}}^{d+1}(X) \to \overset{\cdot}{\text{CH}}^1(\text{Spec}(\mathbb{Z})) = \mathbb{R} \]

satisfies \( f_*(z_S(a)) = \log(\text{ord}(a)) \) for \( a \in \text{CH}_0(X_S) \). Therefore, Theorem 1.2 will follow if we show:
Theorem 2.2 \[ \sum_{i=0}^{d} (-1)^i \chi Q(\lambda^i(\Omega)) = (-1)^{d+1} \frac{d+1}{2} f_*(z_S(c_{d+1}^X(\Omega^1_{X/Z}))). \]

In what follows we will use heavily the notations and results of [GS1], [GS2] and [GS3].

First observe that from the definition of $\Omega$, we obtain $\text{ch}(\Omega) = c_h(\Omega^1_{X(C)})$, where $\text{ch}$ denotes the Chern character form (its domain can be extended to $\hat{\mathbb{K}}_0(X)$ as in [GS1]). By [GS1, Lemma 7.3.3], we have $\text{ch}(\Omega^i_{X(C)}) = \lambda^i(\text{ch}(\Omega^1_{X(C)}))$; here $\Omega^i_{X(C)}$, $0 \leq i \leq d$, has the exterior power metric and the $\lambda$-ring structure on differential forms is given by the grading as in loc. cit. We obtain that $\text{ch}(\lambda^i(\Omega)) = \lambda^i(\text{ch}(\Omega)) = \lambda^i(\text{ch}(\Omega^1_{X(C)}))$ where the first equality follows from the fact that $\text{ch} = \omega \cdot \hat{\text{ch}} : \hat{\mathbb{K}}_0(X) \to A(X_R)$ is a $\lambda$-ring homomorphism (see loc. cit.).

From the Arithmetic Riemann Roch theorem of Gillet and Soulé ([GS3], Theorem 7, see also 4.1.5 loc. cit.) we now have

\[ \frac{1}{2} \int_{X(C)} \text{ch}(\sum_{i=0}^{d} (-1)^i \Omega^i_{X(C)} \cdot \text{Td}(T_X(C)) \cdot R(T_X(C))) = 0. \]

Proof. (Shown to us by C. Soulé.) By the classical identity applied on the level of Chern forms we obtain

\[ \text{ch}(\lambda_{-1}(\Omega^1_{X(C)})) \cdot \text{Td}(T_X(C)) = c_d(T_X(C)) \]

(see [R, 6.19]). Therefore the integral is equal to:

\[ \int_{X(C)} c_d(T_X(C)) \cdot R(T_X(C)). \]

But $R(T_X(C))$ is non-zero in positive degrees only; therefore the degree of the form $c_d(T_X(C)) \cdot R(T_X(C))$ is at least $d + 1$ and the integral vanishes.

It remains to deal with the first term of the right hand side of 2.2. We will show:

Proposition 2.4 \[ (\hat{\text{ch}}(\sum_{i=0}^{d} (-1)^i \lambda^i(\Omega)) \cdot \text{Td}(X))^{(d+1)} = (-1)^{d+1} \frac{d+1}{2} \hat{c}_{d+1}(\Omega). \]
Proof. Recall the definition of $\hat{\text{Td}}(X)$ from [GS3]; we have an exact sequence

$$E^\ast_C : 0 \to T_X C = (\Omega^1_X)^\ast \to P^\ast_C \to N^\ast_C \to 0.$$ 

We set

$$\hat{\text{Td}}(X) := \hat{\text{Td}}(\bar{P}^\ast)\hat{\text{Td}}^{-1}(\bar{N}^\ast) + a(\hat{\text{Td}}(E^\ast_C)\hat{\text{Td}}(\bar{N}^\ast)^{-1})$$

where $\hat{\text{Td}}(E^\ast_C)$ is the Todd-Bott-Chern secondary form attached to the sequence $E^\ast_C$ (see [GS3], p. 503) and $\text{Td}$ is the usual Todd form. We are just interested in the terms of degree 0 and 1 of $\hat{\text{Td}}(X)$. If $\bar{E}$ is a hermitian vector bundle, we have

$$\hat{\text{Td}}(\bar{E}^\ast) = 1 + \frac{\hat{c}_1(\bar{E}^\ast)}{2} + \cdots,$$

$$\hat{\text{Td}}^{-1}(\bar{E}^\ast) = 1 - \frac{\hat{c}_1(\bar{E}^\ast)}{2} + \cdots.$$

The $(0,0)$ component of $\text{Td}(\bar{N}^\ast_C)^{-1}$ is 1. We can also see that the $(0,0)$ component of the secondary form $\hat{\text{Td}}(E^\ast_C)$ is given by

$$\hat{\text{Td}}(E^\ast_C)^{0,0} = \frac{\hat{c}_1(E^\ast_C)}{2}$$

where $\hat{c}_1(E^\ast_C)$ the “secondary” first Bott-Chern form associated to $E^\ast_C$. This gives

$$\hat{\text{Td}}(X) = 1 + \frac{\hat{c}_1(\bar{P}^\ast) - \hat{c}_1(\bar{N}^\ast)}{2} + a(\frac{\hat{c}_1(E^\ast_C)}{2}) + \cdots = 1 + \frac{\hat{c}_1(\Omega^\ast)}{2} + \cdots$$

and therefore

$$\hat{\text{Td}}(X) = \hat{\text{Td}}(\Omega^\ast) \mod \text{CH}^{\geq 2}(X)_Q. \quad (2.5)$$

Now consider the $\gamma$ operations on the $\lambda$-ring $\hat{K}_0(X)$ with augmentation $\epsilon : \hat{K}_0(X) \to \mathbb{Z}$ given by $\epsilon((\bar{E}, \eta)) = \text{rk}(E)$ (see [R, §4]). If $\epsilon(x) = d$, then (as in [CPT1] §1) we have:

$$(-1)^d \gamma^d(x - \epsilon(x)) = \sum_{i=0}^{d} (-1)^i \lambda^i(x). \quad (2.6)$$

Therefore $\hat{\text{ch}}(\sum_{i=0}^{d} (-1)^i \lambda^i(x))$ is concentrated in degrees $d$ and $d + 1$ only and so by (2.5)

$$\hat{\text{ch}}(\sum_{i=0}^{d} (-1)^i \lambda^i(\Omega)) \cdot \hat{\text{Td}}(X) = \hat{\text{ch}}(\sum_{i=0}^{d} (-1)^i \lambda^i(\Omega)) \cdot \hat{\text{Td}}(\Omega^\ast).$$

By the above and (2.6) it is enough to show that for $x \in \hat{K}_0(X)$ we have

$$\hat{\text{ch}}(\gamma^d(x - \epsilon(x)) \cdot \hat{\text{Td}}(x^\ast))^{(d+1)} = -\frac{d + 1}{2} \hat{c}_{d+1}(x).$$

Let $a_1, \ldots, a_{d+1}$ be the “arithmetic Chern roots” of $x$. By definition, these are formal symbols such that the arithmetic Chern classes of $x$ are the elementary symmetric
functions of $a_i$; we can perform our calculation using these symbols. The Chern roots of the dual $x^*$ are $-a_1, \ldots, -a_{d+1}$. A standard argument using [GS1, Theorem 4.1] shows that we have

$$\widehat{\text{ch}}(\gamma^d(x - \epsilon(x))) = \sum_{i=0}^{d+1} \prod_{j \neq i} (e^{a_j} - 1),$$

while by definition

$$\widehat{\text{Td}}(x^*) = \prod_{i=0}^{d+1} \frac{-a_i}{1 - e^{-(a_i)}} = \prod_{i=0}^{d+1} \frac{a_i}{e^{a_i} - 1}.$$

The product is equal to

$$\sum_{j=1}^{d+1} \frac{a_1 a_2 \cdots a_{d+1}}{e^{a_j} - 1} = \sum_{j=1}^{d+1} (a_1 \cdots \hat{a}_j \cdots a_{d+1} - \frac{a_1 \cdots a_{d+1}}{2}) + \cdots$$

$$= \hat{c}_d(x) - \frac{d+1}{2} \hat{c}_{d+1}(x) + \cdots$$

which gives the desired result.

3 Localized Chern classes.

We continue with the same assumptions and notations. Recall the homomorphism

$$z_S : \text{CH}_0(X_S)_{\mathbb{Q}} \to \widehat{\text{CH}}_0(X)_{\mathbb{Q}} = \widehat{\text{CH}}^{d+1}(X)_{\mathbb{Q}}.$$ 

Proposition 3.1 $z_S(c_{d+1}^{X_S}(\Omega^1_{X/\mathbb{Z}})) = \hat{c}_{d+1}(\Omega)$.

Theorem 2.2 follows from Propositions 3.1, 2.3, 2.4 and equation 2.2.

Proof of Proposition 3.1: We review the construction of the localized Chern class via the Grassmannian graph construction (as described in [B] §1, or in [GS3] §1) applied to the complex $0 \to N \to P$ with cokernel $\Omega^1_{X/\mathbb{Z}}$. Set $U = X - X_S$. Let $p$ be the projection $X \times \mathbb{P}^1 \to X$. Set $M := p^*N(1) \oplus p^*P$ where (1) denotes the Serre twist (which we view as tensoring with the pull-back of $\mathcal{O}_{\mathbb{P}^1}(\infty)$ under $X \times \mathbb{P}^1 \to \mathbb{P}^1$). Let us consider the Grassmannian $\text{Gr}(r, M)$ over $X \times \mathbb{P}^1$ of rank $r = \text{rk}(N)$ local direct summands of $M$. Denote by $\pi_0 : \text{Gr}(r, M) \to X \times \mathbb{P}^1$ the natural projection morphism. The diagonal embedding $p^*N \subset p^*N(1) \oplus p^*P$ gives a section $s$ of $\pi_0$ over the subscheme $(X \times \mathbb{A}^1) \cup (U \times \mathbb{P}^1)$. In fact, over $X \times \mathbb{A}^1$ the image of $p^*N$ can be identified with the graph of $\delta$. Denote by $W$ the Zariski closure of the image

$$s((X \times \mathbb{A}^1) \cup (U \times \mathbb{P}^1)) \subset \text{Gr}(r, M);$$
this is an integral subscheme of $\text{Gr}(r,M)$ which is called the Grassmannian graph of $N \to P$. The morphism $\pi := \pi_0|_W$ is projective and gives an isomorphism on the generic fibers. Let $W_\infty$ be the effective Cartier divisor on $W$ given by the inverse image of $X \times \{\infty\}$ under $\pi$. Also let $\tilde{X}$ be the Zariski closure in $W_\infty$ of the restriction of the section $s$ to $U \times \{\infty\}$. Then $\pi|_{\tilde{X}} : \tilde{X} \to X$ is birational (an isomorphism over $U$). As in [GS3], we see that the cycle $Z = [W_\infty] - [\tilde{X}]$

is supported in the inverse image of $X_S$. Looking at supports, we have $|W_\infty| = |\tilde{X}| \cup |Z|$. Denote by $\xi_1$ the universal subbundle of rank $r$ on $\text{Gr}(r,M)$ and by $\xi_0$ the “constant” bundle which is the base change of $P$ under the (smooth) morphism $\text{Gr}(r,M) \to X$. The section $s$ gives $s_C : X_C \times \mathbf{P}^1 = W_C \to \text{Gr}(r,M)_C$. The pull-back of $\xi_0$ under $s_C$ is $p^*\xi_C$; the pull-back of $\xi_1$ under $s_C$ is $p^*N_C$. Denote the restrictions $\xi_0|_W$, $\xi_1|_W$ by $\xi_0$, $\xi_1$. Equip $\xi_0|_C$, $\xi_1|_C$ with the hermitian metrics which correspond to the hermitian metrics on $p^*\xi_C$, $p^*N_C$ obtained via base change from the metrics on $P_C$, $N_C$. We will denote by $\tilde{\xi}_1$, $\tilde{\xi}_0$ the vector bundles $\xi_1$, $\xi_0$ on $W$ endowed with the above hermitian metrics on $W_C$. Set $\zeta = (\tilde{\xi}_0, 0) - (\tilde{\xi}_1, 0) \in \tilde{K}_0(W)$.

There is a natural morphism $\xi_1 \to \xi_0$ obtained by the natural inclusion $\xi_1 \subset \pi_0^*M$ followed by the projection $\pi_0^*M \to \xi_0 = \pi_0^*p^*P$. After restricting to $W_C$ this corresponds to the composition $p^*N_C \to p^*P_C$.

Over $X_C$ we have the exact sequence $\mathcal{E}_C : 0 \to N_C \to P_C \to \Omega^1_{X_C} \to 0$. This gives an exact sequence over $X_C \times \mathbf{P}^1 = W_C$: $p^*\mathcal{E}_C : 0 \to p^*N_C \to p^*P_C \to p^*\Omega^1_{X_C} \to 0$.

Consider $A = (0, \tilde{\chi}(p^*\mathcal{E}_C))$ in $\tilde{K}_0(W)$. Let us now define the elements $b = \tilde{c}_{d+1}(\zeta + A) \in \tilde{\text{CH}}^{d+1}(W)_\mathbf{Q}$, $\mu = \pi_*(b) \in \text{CH}^{d+1}(X \times \mathbf{P}^1)_\mathbf{Q}$.

**Lemma 3.2** *The restrictions of $\mu$ to $X \times \{0\}$ and $X \times \{\infty\}$ are equal.*

**Proof.** By [GS2, Theorem 4.4.6] the restrictions are well defined and their difference is given by $a(\int_{\mathbf{P}^1(C)} \omega(\mu) \log |z|^2)$ where $\omega$ and $a$ are defined in [GS2, 3.3.4]; $\omega(\mu)$ is a $(d+1,d+1)$-form on $(X \times \mathbf{P}^1)(C)$ and the integral in the parenthesis gives a $(d,d)$-form on $X(C)$. Since $\pi$ is an isomorphism on
the generic fibers, by the definition of $\tilde{\zeta}$ and $A$, we can see that the form $\omega(\mu)$ is obtained by pulling back via the projection $p_C : X(C) \times P^1(C) \to X(C)$ a $(d + 1, d + 1)$-form on $X(C)$. It follows that
\[ \int_{P^1(C)} \omega(\mu) \log |z|^2 = 0 \]
(the integral changes sign when $z$ is replaced by $1/z$).

Recall that the morphism $\pi : W \to X \times P^1$ restricts to give a projective morphism $\pi^{|Z|} : |Z| \to X_S \times \infty = X_S$. Here $|Z|$ is the (reduced) support of $Z$. Set $\xi = \xi_0 - \xi_1 \in K_0(\text{Gr}(r, M))$ and denote by $[Z]$ the fundamental cycle of $Z$ in $\text{CH}_{d+1}(|Z|)$.

**Lemma 3.3**  a) The restriction of $\mu$ to $X \times \{0\}$ is equal to $\check{c}_{d+1}(\Omega)$;

b) The restriction of the class $\mu$ to $X \times \{\infty\}$ is equal to the image of $\pi^{|Z|}_*(c_{d+1}(\xi||Z|) \cap [Z]) \in \text{CH}_0(X_S)Q$ under $z_S$.

Before we continue with the proof, let us point out that since by definition $c^X_{d+1}(\Omega^1_{X/\mathbb{C}}) = c_{d+1, X_S}(\Omega^1_{X/S} \otimes \mathcal{O}_S/z) \cap |X| = \pi^{|Z|}_*(c_{d+1}(\xi||Z|) \cap [Z])$, Lemmas 3.2 and 3.3 together imply the proof of Proposition 3.1.

**Proof.** Part (a) is straightforward; indeed $\check{\xi}_0$ restricts to give $\check{P}$, $\check{\xi}_1$ gives $\check{N}$ and $A$ gives $(0, \check{\mu}(\mathcal{E}_C))$.

Let us show part (b). Recall $W$ is integral of dimension $d + 2$, $W_\infty$ is an effective Cartier divisor in $W$ and we have $|W_\infty| = Z + [\check{X}]$. Denote by $|W_\infty|$ the reduced support of $W_\infty$ in $W$. Since $\pi^{|W_\infty|} : |W_\infty| \to X \times \{\infty\} = X$ is a projective morphism which is an isomorphism on the generic fiber,

$$\pi^{|W_\infty|}_* : \text{CH}^{d+1}(|W_\infty|)Q \to \text{CH}_0^{d+1}(X)Q$$

is well-defined. Also, since $i : W_\infty \to W$ is the inclusion of an effective Cartier divisor with smooth generic fiber, the pull-back $i^*(b)$ makes sense in $\text{CH}^{d+1}(W_\infty)Q = \text{CH}^{d+1}(|W_\infty|)Q = \text{CH}_0(|W_\infty|)Q$ and we have

$$\mu|_{X \times \{\infty\}} = \pi^{|W_\infty|}_*(b)|_{X \times \{\infty\}} = \pi^{|W_\infty|}_*(i^*(b)).$$

(see for example $[\text{GS3}, 2.2.7]$).

In what follows, we will calculate $i^*(b)$. For simplicity set $G = \text{Gr}(r, M)$. Equip the bundles $\xi_1$, $\xi_0$ on $G$ with hermitian metrics and set $\check{\xi} = (\xi_0, 0) - (\xi_1, 0) \in K_0(G)$. Consider $B = \check{c}_{d+1}(\check{\xi})$ in $\text{CH}^{d+1}(G)$ and $B|_W = \check{c}_{d+1}(\check{\xi}|_W)$ in $\text{CH}^{d+1}(W)$. Note that $\check{\xi}|_W \in \text{K}_0(W)$ need not agree with $\check{\zeta}$ because the metrics might not agree. In any case, we can write

$$B|_W - \check{c}_{d+1}(\check{\zeta} + A) = a(\eta) \quad (3.4)$$

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with \( \eta \) a \((d,d)\)-form on \( W(\mathbb{C}) \). The pull-back \( i^*(B|_W) \) is the \( d+1 \)-st arithmetic Chern class of the restriction of the bundle \( \xi_0 - \xi_1 \) to \( W_\infty \). We have

\[
i^*(b) = i^*(B|_W) - a(i_C^*(\eta)). \tag{3.5}\]

By [GS3, Theorem 4 (1)], \( i^*(B|_W) = B \cdot j[W_\infty] \) in \( \widehat{CH}^{d+1}(|W_\infty|)_\mathbb{Q} = \widehat{CH}_0(|W_\infty|)_\mathbb{Q} \); here \( j : |W_\infty| \to G \) is the natural embedding and \( |W_\infty| \in \widehat{CH}_{d+1}(|W_\infty|)_\mathbb{Q} \) is the fundamental cycle of \( W_\infty \) (the notations are as in loc.cit.). We may also consider \([\tilde{X}] \in \widehat{CH}_{d+1}(|W_\infty|)_\mathbb{Q} \) so that we have \( [W_\infty] = Z + [\tilde{X}] \) in \( \widehat{CH}_{d+1}(|W_\infty|)_\mathbb{Q} \). We obtain

\[
i^*(B|_W) = B \cdot j[W_\infty] = B \cdot jZ + B \cdot j[\tilde{X}]. \tag{3.6}\]

Denote by \( \phi : |Z| \to G \) and \( \psi : \tilde{X} \to G \) the natural immersions. By [GS3, Theorem 3 (4)] the elements \( B \cdot \phi Z \) and \( B \cdot [\tilde{X}] \) are the images of the elements \( B \cdot \phi Z \) and \( B \cdot \psi [\tilde{X}] \) of \( CH_0(|Z|)_\mathbb{Q} \) and \( CH_0(\tilde{X})_\mathbb{Q} \) under the maps \( CH_0(|Z|)_\mathbb{Q} \to \widehat{CH}_0(|W_\infty|)_\mathbb{Q} \) and \( CH_0(\tilde{X})_\mathbb{Q} \to \widehat{CH}_0(|W_\infty|)_\mathbb{Q} \) respectively. We have

\[
B \cdot \phi Z = c_{d+1}(\xi_{||Z|}) \cap [Z]
\]

and by [GS1, Theorem 4 (1)],

\[
B \cdot \psi [\tilde{X}] = c_{d+1}(\xi_{||\tilde{X}|}) \cap [\tilde{X}] = \hat{c}_{d+1}(\xi_{||\tilde{X}|})
\]

in \( \widehat{CH}_0(\tilde{X})_\mathbb{Q} = \widehat{CH}^{d+1}(\tilde{X})_\mathbb{Q} \) (recall \( \tilde{X} \) is integral of dimension \( d+1 \)).

Now subtract \( a(i_C^*(\eta)) \) from both sides of \( 3.4 \). Using \( 3.3 \) and the above, we obtain that \( i^*(b) \) can be written as a sum of the image of the class \( \hat{c}_{d+1}(\xi_{||X|}) - a(i_C^*(\eta)) \) under the map \( \widehat{CH}_0(\tilde{X})_\mathbb{Q} \to \widehat{CH}_0(|W_\infty|)_\mathbb{Q} \) plus the image of \( c_{d+1}(\xi_{||Z|}) \cap [Z] \) under \( CH_0(|Z|)_\mathbb{Q} \to \widehat{CH}_0(|W_\infty|)_\mathbb{Q} \). Since \( W_\infty \) and \( \tilde{X} \) have the same generic fiber we can see from \( 3.4 \) that

\[
\hat{c}_{d+1}(\xi_{||X|}) - a(i_C^*(\eta)) = \hat{c}_{d+1}(\xi_{||X|} - A_{||X|})
\]

Hence, part (b) will follow if we show that \( \hat{c}_{d+1}(\xi_{||X|} - A_{||X|}) = 0 \).

Over \( \tilde{X} \), there is an exact sequence of vector bundles

\[
0 \to \xi_{1|\tilde{X}} \to \xi_{0|\tilde{X}} \to Q \to 0
\]

with \( Q \) of rank \( d \). We have \( \tilde{X}_C = X_C \) and, as we have seen before, there is an isomorphism \( Q_C \simeq \Omega_{X_C} \) which can be used to identify the above exact sequence with \( E_C \). This implies that

\[
((\xi_0)_{||\tilde{X}|}, 0) - ((\xi_1)_{||\tilde{X}|}, 0) + (0, \tilde{\chi}(E_C)) = (\tilde{Q}, 0)
\]

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in $\hat{K}_0(\hat{X})$. Since $A|_{\hat{X}} = A|_{X \times \{\infty\}} = (0, \hat{c}_d(E_C))$, this translates to $(\hat{\zeta} + A)|_{\hat{X}} = (\hat{Q}, 0)$ in $\hat{K}_0(\hat{X})$. Since $d + 1 > \text{rk}(Q) = d$, by [GS1, 4.9, p. 198], $\hat{c}_{d+1}((\hat{Q}, 0)) = 0$. Therefore, we obtain

$$\hat{c}_{d+1}((\hat{\zeta} + A)|_{\hat{X}}) = 0.$$ 

This completes the proof of Lemma 3.3 and therefore also of Proposition 3.1.

**Remark:** Let $\bar{F}$ be a hermitian coherent sheaf on $X$. Suppose that $Y \subset X$ is a fibral closed subscheme and assume that $F$ is locally free of rank $m$ on the complement $X - Y$. Let $z_{i,Y} : \text{CH}_{d+1-i}(Y) \to \hat{\text{CH}}_{d+1-i}(X)$ be the natural homomorphism. The same argument as in the proof above can be used to show that for $i > m$,

$$z_{i,Y}(c^X_{i,Y}(F) \cap [X]) = \hat{c}_i(F),$$

where $c^X_{i,Y}(F)$ is the localized Chern class of $[B, \S 1]$.

### 4 Tame reduction

Here we show Theorem 1.3. Write $I$ for an index set for the irreducible components of the singular fibers of $X \to \text{Spec}(\mathbb{Z})$. If $i \in I$, we denote by $T_i$ the corresponding irreducible component and by $m_i$ its multiplicity in the divisor of the corresponding special fiber. For a non-empty subset $J$ of $I$, set

$$T_J = \cap_{i \in J} T_i$$

(scheme-theoretic intersection). Under our assumptions, $T_J$ is either empty or a smooth projective scheme of dimension $d + 1 - |J|$ over a finite field. The union $\bigcup_{J \neq \emptyset} T_J$ is a divisor with strict normal crossings on $T_J$. We start with the following proposition:

**Proposition 4.1** With the assumptions of Theorem 1.3, we can consider the sheaf of relative logarithmic differentials $\Omega^1_{X/Z}(\log X_{S/\log S})$ (see below); it is locally free of rank $d$ on $X$. There is a morphism

$$\omega : \Omega^1_{X/Z} \to \Omega^1_{X/Z}(\log X_{S/\log S})$$

whose kernel and cokernel are isomorphic to the kernel and cokernel of the morphism

$$a : \oplus_{p \in S} \mathcal{O}_{X/p} \to \oplus_{i \in I} \mathcal{O}_{T_i}.$$ 

**Proof:** The statement is local on the base, and so to simplify notation we will assume there is only one prime in $S$. We will use the logarithmic differentials $\Omega^1_{X/Z}(\log X_{p/\log S})$ defined in [K] §2. By definition,

$$\Omega^1_{X/Z}(\log X_{p}) := (\Omega^1_{X/Z} \oplus (OX \otimes j_* \mathcal{O}^*_{X/\hat{p}}))/\mathcal{F}$$

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where \( j \) is the open immersion \( j : X[\frac{1}{p}] \to X \) and \( F \) is the \( \mathcal{O}_X \)-subsheaf generated by elements of the form \((da, 0) - (0, a \otimes a)\) for \( a \in \mathcal{O}_X \cap j_*\mathcal{O}_{X[\frac{1}{p}]}^*\). We will write the element \( a \otimes b \) as \( a \cdot d \log(b) \). Notice that \( j_*\mathcal{O}_{X[\frac{1}{p}]}^* \) is the sheaf of elements of the function field of \( X \) whose divisor has support contained in the special fiber. By definition, \( \Omega_{X/Z}^1(\log X_S^{\text{red}} / \log S) \) is the quotient of \( \Omega_{X/Z}^1(\log X_p^{\text{red}}) \) by the \( \mathcal{O}_X \)-subsheaf generated by \( d \log(p) \). There is an exact sequence

\[
\mathcal{O}_X/p\mathcal{O}_X \xrightarrow{\phi} \Omega_{X/Z}^1(\log X_p^{\text{red}}) \xrightarrow{\omega} \Omega_{X/Z}^1(\log X_S^{\text{red}} / \log S) \to 0
\]

where the homomorphism \( \phi \) maps \( f \) to \( f \cdot d \log(p) \). There is also a natural exact sequence

\[
0 \to \Omega_{X/Z}^1 \xrightarrow{\omega_2} \Omega_{X/Z}^1(\log X_p^{\text{red}}) \xrightarrow{\oplus \text{Res}_i} \oplus \mathcal{O}_{T_i} \to 0. \tag{4.2}
\]

Here the right hand homomorphism is given by taking residues along \( T_i \). The homomorphism \( \omega \) is equal to the composition \( \omega_1 \cdot \omega_2 \).

Under our assumptions, the scheme \( X \) is locally étale isomorphic to

\[
Y = \text{Spec}(\mathcal{O}[t_1, \ldots, t_d]/(t_1^{m_1} \cdots t_d^{m_d} - p))
\]

with all \( m_i \) prime to \( p \). The above constructions of logarithmic differentials etc. make sense for the scheme \( Y \); we can see by an explicit calculation that \( \phi_Y \) is injective and that the analogue of the sequence \( 4.2 \) for \( Y \) is exact. It follows from the fact that taking (logarithmic) differentials commutes with étale base change that \( \phi \) is injective and that the sequence \( 4.2 \) is exact. On \( Y \) we have \( t_1^{m_1} \cdots t_d^{m_d} = p \) and so

\[
d \log(p) = m_1 \frac{dt_1}{t_1} + \cdots + m_d \frac{dt_d}{t_d}.
\]

This shows that for \( f \in \mathcal{O}_Y/p\mathcal{O}_Y \), \( \phi_Y(f) \) gives an element in the kernel of \( \omega \) if and only if \( f \in (t_1 \cdots t_d) \); this translates to \( a(f) = 0 \). Furthermore, \( f \cdot d \log(p) = 0 \) if and only if \( f = 0 \) in \( \mathcal{O}_Y/p\mathcal{O}_Y \). This shows the statement about the kernels for \( X \). Let us now discuss the cokernels: Let \( \beta : \oplus \mathcal{O}_{T_i} \to \oplus \mathcal{O}_{T_i} \) be the automorphism defined by \( \beta((f_i)_i) = (m_if_i)_i \) (recall that all the \( m_i \) are prime to \( p \)). The above calculation on \( Y \) implies that the composition

\[
\mathcal{O}_X/p\mathcal{O}_X \xrightarrow{\phi} \Omega_{X/Z}^1(\log X_p^{\text{red}}) \xrightarrow{\oplus \text{Res}_i} \oplus \mathcal{O}_{T_i}
\]

coincides with \( f \mapsto (m_1f, \ldots, m_df) \). The residue homomorphism \( \text{Res} = \oplus \text{Res}_i \) now gives a surjection:

\[
\Omega_{X/Z}^1(\log X_S^{\text{red}} / \log S) \xrightarrow{\beta^{-1} \text{Res}} \text{coker}(a) \to 0
\]

and we have \( \ker(\beta^{-1} \cdot \text{Res}) = \ker(\text{Res}) = \omega(\Omega_{X/Z}^1) \). This implies \( \text{coker}(\omega) \simeq \text{coker}(a) \).
Let $K_0^X(X)$ be the Grothendieck group of complexes of locally free $\mathcal{O}_X$-sheaves which are exact off $X_S$; since $X$ is regular, $K_0^X(X)$ can be identified with $K_0'(X_S)$. Set $q = \prod_{p \in S} p$. Consider the following complexes of locally free $\mathcal{O}_X$-sheaves which are exact off $X_S$:

\[ \mathcal{E}_1 : N \overset{\delta}{\rightarrow} P \rightarrow \Omega_X^1(\log X_S^{\text{red}} / \log S) \]

\[ \mathcal{E}_2 : \mathcal{O}_X \overset{(q^{-g})}{\rightarrow} \mathcal{O}_X \oplus (\oplus_i \mathcal{O}_X(-T_i)) \rightarrow \oplus_i \mathcal{O}_X \]

concentrated in degrees $-1, 0, 1$. The second homomorphism of $\mathcal{E}_1$ is the composition of $P \rightarrow \Omega_X^1(\log X_S^{\text{red}} / \log S)$ with $\omega$; the second homomorphism of $\mathcal{E}_2$ is given by $(g, (h_i)_i) \mapsto (g + h_i)_i$.

Proposition 4.1 implies that $[\mathcal{E}_1] = [\mathcal{E}_2]$ in $K_0^X(X)$. Consider also the complex

\[ \mathcal{E}_3 : N \overset{\delta}{\rightarrow} P \oplus \Omega_X^1(\log X_S^{\text{red}} / \log S) \overset{(0,1\text{id})}{\rightarrow} \Omega_X^1(\log X_S^{\text{red}} / \log S) \]

concentrated in degrees $-1, 0, 1$. The complex $\mathcal{E}_3$ is quasi-isomorphic to the complex $N \overset{\delta}{\rightarrow} P$ (in degrees $-1$ and $0$). There is an exact sequence of complexes

\[ 0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_3 \overset{pr}{\rightarrow} \Omega_X^1(\log X_S^{\text{red}} / \log S) \rightarrow 0 \]

where on the right end, $\Omega_X^1(\log X_S^{\text{red}} / \log S)$ is considered as a complex supported on degree $0$. Therefore, the main result of [A] (see loc. cit. Proposition 1.4 also [B] Prop. 1.1) implies that

\[ c^X_{d+1}(\Omega_X^1 / \mathbb{Z}) = \sum_{k+l = d+1} c_k(\Omega_X^1(\log X_S^{\text{red}} / \log S)) \cdot c^X_l([\mathcal{E}_1]). \] (4.3)

In fact, since $[\mathcal{E}_1] = [\mathcal{E}_2]$ we can replace $c^X_l([\mathcal{E}_1])$ by $c^X_l([\mathcal{E}_2])$ in this equality. We have

\[ [\mathcal{E}_2] = [\mathcal{O}_X / q \mathcal{O}_X] - \sum_i [\mathcal{O}_{T_i}] \]

(here we identify $K_0^X(X)$ with $K_0'(X_S)$) and so

\[ c^X_l([\mathcal{E}_2]) = c^X_l([\mathcal{O}_X / q \mathcal{O}_X] + \sum_i (-[\mathcal{O}_{T_i}])). \] (4.4)

We have $c^X_l([\mathcal{O}_X / q \mathcal{O}_X]) = \sum_i m_i[T_i]$, $c^X_l([\mathcal{O}_X / q \mathcal{O}_X]) = 0$ for $l > 1$. Similarly, $c^X_l([-\mathcal{O}_{T_i}]) = -[T_i]$, $c^X_l([-\mathcal{O}_{T_i}]) = 0$, for $l > 1$. Combining these with (4.4) we obtain from the usual Chern class identities

\[ c^X_l([\mathcal{E}_2]) = \sum_{J \subset I, |J| = l} (-1)^{|J|}[T_J] + (\sum_{i \in I} m_i[T_i])( \sum_{J' \subset I, |J'| = l-1} (-1)^{|J'|}[T_{J'}]). \] (4.5)

Now since $\sum_i m_i[T_i]$ is a principal divisor in $X$ we get for $l \geq 2$

\[ (\sum_{i \in I} m_i[T_i])( \sum_{J' \subset I, |J'| = l-1} (-1)^{|J'|}[T_{J'}]) = 0 \in \text{CH}_*(X_S). \]
Combining this with \[4.3\] and \[4.5\] we get
\[
c_{d+1}(\Omega_{X/Z}^1) = \sum_{i \in I} (m_i - 1)c_d(\Omega_{X/Z}^1(\log X_S^\text{red} / \log S)) \cdot [T_i] + \]
+ \sum_{J \subset I, |J| \geq 2} (-1)^{|J|}c_{d+1-|J|}(\Omega_{X/Z}^1(\log X_S^\text{red} / \log S)) \cdot [T_J].
\]
Therefore
\[
c_{d+1}(\Omega_{X/Z}^1) = \sum_{i \in I} (m_i - 1)c_d(\Omega_{X/Z}^1(\log X_S^\text{red} / \log S)|_{T_i}) + \]
+ \sum_{J \subset I, |J| \geq 2} (-1)^{|J|}c_{d+1-|J|}(\Omega_{X/Z}^1(\log X_S^\text{red} / \log S)|_{T_J}).
\]

**Proposition 4.8** For a non-empty subset \(J\) of \(I\), set \(T_J^* = T_J - \bigcup_{J \not\subset J'} T_J'\). We have
\[
\deg(c_{d+1-|J|}(\Omega_{X/Z}^1(\log X_S^\text{red} / \log S)|_{T_J})) = (-1)^{d+1-|J|} \chi_c(T_J^*)
\]
where \(\chi_c(T_J^*)\) is the l-adic \((l \not\in S)\) Euler characteristic with compact supports of \(T_J^*\).

**Proof.** Denote by \(X_p^\text{red}|_{T_J}\) the logarithmic structure on \(T_J\) obtained by restricting the logarithmic structure given by \((X, X_p^\text{red})\) to \(T_J\). This is isomorphic to the logarithmic structure defined on \(T_J\) by its divisor with strict normal crossings \(\bigcup_{J \not\subset J'} T_J'\). We will show that
\[
[\Omega_{X/Z}^1(\log X_S^\text{red} / \log S)|_{T_J}] = [\Omega_{T_J/k}^1(\log X_p^\text{red}|_{T_J})] + (|J| - 1)[O_{T_J}]
\]
in \(K_0(T_J)\). The proposition will follow from \[4.9\] and the well-known fact (see for example [S], p. 402) that
\[
\deg(c_{d+1-|J|}(\Omega_{T_J/k}^1(\log X_p^\text{red}|_{T_J}))) = (-1)^{d+1-|J|} \chi_c(T_J^*)
\]
From the proof of Proposition \[4.4\] there is an exact sequence
\[
0 \to O_{T_i} \to \Omega_{X/Z}^1(\log X_p^\text{red}|_{T_i}) \to \Omega_{X/Z}^1(\log X_S^\text{red} / \log S)|_{T_i} \to 0.
\]
By [K] \$2\$ (see also [S], p. 404) there are also exact sequences
\[
0 \to \Omega_{T_i/F_p}(\log X_p^\text{red}|_{T_i}) \to \Omega_{X/Z}^1(\log X_p^\text{red})|_{T_i} \to O_{T_i} \to 0,
\]
and for \(|J'| = |J| + 1\),
\[
0 \to \Omega_{T_J/F_p}(\log X_p^\text{red}|_{T_J}) \to \Omega_{T_J/F_p}(\log X_p^\text{red}|_{T_J})|_{T_J'} \to O_{T_J'} \to 0.
\]
We can now see that \[4.9\] follows by induction on the cardinality of \(J\).
Proposition 4.8 and 4.7 give for $p \in S$:

$$\deg((-1)^dX^S \cdot (\Omega^1_{\mathbb{X}/\mathbb{Z}})|_{X_p}) = -\sum_{i \in I_p} (m_i - 1) \chi^*_c(T_i) + \sum_{J \subset I_p, |J| \geq 2} \chi^*_c(T_J) = -\sum_{i \in I_p} m_i \chi^*_c(T_i) + \chi(X_p) \quad (4.12)$$

where $I_p$ is the subset of $I$ that corresponds to components over $p$.

Under our assumption, the ramification is tame (there is no Swan term in the conductor) and for each $p \in S$,

$$\chi(X_p) = \sum_{i \in I_p} m_i \chi^*_c(T_i)$$

(see for example [S], Cor. 2, p. 407). Therefore,

$$A(X) = \prod_{p \in S} p^{\chi(X_p) - \chi(X_p)} = \prod_{p \in S} p^{\sum_{i} m_i \chi^*_c(T_i) - \chi(X_p)}.$$

This together with (4.12) completes the proof of (1.3).

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