SMALL SCALE QUANTUM ERGODICITY IN CAT MAPS. II.
QUASIMODES THAT ARE NOT EQUIDISTRIBUTED AT THE
LOGARITHMICAL SCALES

XIAOULONG HAN

Abstract. In this series, we investigate quantum ergodicity at small scales for linear hyperbolic maps of the torus (“cat maps”). In Part II of the series, we construct quasimodes that are quantum ergodic but are not equidistributed at the logarithmical scales.

1. Introduction

We recall the setup of cat maps briefly and refer to Part I [Han] of the series for more background. A classical cat map on the torus $T^2$ is defined by a matrix $M \in SL(2, \mathbb{Z})$ with $|\text{Tr} M| > 2$. Its iterations $M^t (t \in \mathbb{Z})$ induce a discrete hyperbolic (i.e., chaotic) dynamical system.

In the quantum cat system, the phase space is the 2-dim torus $T^2 = \{(q, p) : q, p \in \mathbb{T}^1\}$, in which $q$ and $p$ denote the position and momentum variables, respectively. Therefore, a quantum state can be represented by a distribution $\psi(q)$ on $\mathbb{R}$ such that $\psi$ and its Fourier transform are both periodic. One can then determine that the spaces of such quantum states are $N$-dim Hilbert spaces with $N \in \mathbb{N}$, by which we denote $\mathcal{H}_N$. Finally, the quantum cat map $\hat{M}$ is a unitary operator acting on $\mathcal{H}_N$ and $\hbar = 1/(2\pi N) \to 0$ plays the role of the Planck parameter.

One of the main problems in Quantum Chaos for cat maps is concerned with the density distribution of eigenstates (or more generally, approximate eigenstates, i.e., quasimodes) of $\hat{M}$ as $\hbar \to 0$.

The density distribution of a quantum state $\psi$ in the physical space can be studied via $\int_\Omega |\psi(q)|^2 dq$, in which $\Omega \subset \mathbb{T}^1$. In particular, we say that a sequence of normalized states $\{\psi_j\}_{j=1}^\infty$ with $\psi_j \in \mathcal{H}_{N_j}$ tend equidistributed at the macroscopic scale in the physical space $\mathbb{T}^1$ if for any open subset $\Omega \subset \mathbb{T}^1$,

$$\int_\Omega |\psi_j(q)|^2 dq \to \text{Vol}(\Omega) \text{ as } j \to \infty.$$ 

We say that $\{\psi_j\}_{j=1}^\infty$ tend equidistributed at a small scale $r = r(N) \to 0$ as $N \to \infty$ in the physical space if

$$\int_{B_1(q_0, r_j)} |\psi_j(q)|^2 dq = \text{Vol}(B_1(q_0, r_j)) + o(r_j) \text{ as } j \to \infty,$$

uniformly for all $q_0 \in \mathbb{T}^1$. Here, $r_j = r(N_j)$ and $B_d(x, r)$ is the geodesic ball in $\mathbb{T}^d$ with radius $r$ and center $x$.

The density distribution of a quantum state $\psi$ in the phase space $T^2$ can be studied via $\langle \psi | \hat{f} | \psi \rangle$, in which $f \in C^\infty(T^2)$ and $\hat{f}$ is its quantization. In particular, we say that a

2010 Mathematics Subject Classification. 35P20, 58G25, 81Q50, 37D20, 11F25.
Key words and phrases. Hyperbolic linear maps of the torus, quantum ergodicity, small scale.
sequence of normalized states \( \{ \psi_j \}_{j=1}^\infty \) with \( \psi_j \in \mathcal{H}_{N_j} \) is quantum ergodic (i.e., they tend equidistributed at the macroscopic scale in the phase space \( T^2 \)) if for all \( f \in C^\infty(T^2) \),

\[
\langle \psi_j | \hat{f} | \psi_j \rangle \to \int_{T^2} f(x) \, dx \quad \text{as } j \to \infty.
\]

Here, \( dx \) is the Lebesgue measure on \( T^2 \). It is obvious that a quantum ergodic sequence is also equidistributed in the physical space.

We say that \( \{ \psi_j \}_{j=1}^\infty \) is quantum ergodic at a small scale \( r = r(N) \to 0 \) as \( N \to \infty \) if

\[
\langle \psi_j | \hat{b}_{x,r}^\pm | \psi_j \rangle = \text{Vol}(B_1(x,r_j)) + o(r_j^2) \quad \text{as } j \to \infty,
\]

uniformly for all \( x \in T^2 \). Here, \( b_{x,r}^\pm \) are the appropriate smooth functions that approximate the indicator function of \( B_2(x,r) \subset T^2 \). See Subsection 3.4 for more details.

Quantum ergodicity at small scales characterize refined density distribution properties of the states than the one at the macroscopic scale. Establishing these small scale results in various dynamical systems, as well as proving optimal scales for such properties, have attracted a lot of attention. See Part I [Han] of the series for the recent history.

In the context of cat maps, the Quantum Ergodicity theorem [BouDB, Ze] asserts that a full density subsequence of any eigenbasis is quantum ergodic at logarithmical scale \( s \) (log-ergodicity) at some logarithmical scales. In particular, we proved that a full density subsequence of any eigenbasis is quantum ergodic at logarithmical scales \( (\log N)^{-\alpha} \) for some \( \alpha > 0 \). In addition, the scale of quantum ergodicity can be greatly improved to polynomial scales \( N^{-\beta} \) for some \( \beta > 0 \), under certain conditions such as the Hecke symmetry requirement.

In Part II, we investigate the potential failure of quantum ergodicity at certain small scales. In particular, we construct approximate eigenstates (i.e., quasimodes) which are quantum ergodic at the macroscopic scale but fail quantum ergodicity at some logarithmical scales. Here, we say that a sequence of states \( \{ \psi_j \}_{j=1}^\infty \) are quasimodes of order \( R(N) \to 0 \) as \( N \to \infty \) if \( \psi_j \in \mathcal{H}_{N_j} \) and there are \( \{ \phi_j \}_{j=1}^\infty \subset \mathbb{R} \) such that

\[
\left\| \left( \hat{M} - e^{i\phi_j} \right) \psi_j \right\|_{L^2(T^1)} = R(N_j) \left\| \psi_j \right\|_{L^2(T^1)}.
\]

Note that since \( \hat{M} \) is unitary, the eigenvalues (or energy) have module one. Therefore, \( e^{i\phi_j} \) is the “quasi-energy” of the quasimode \( \psi_j \). Setting the reminder \( R = 0 \) reduces the quasimodes to exact eigenstates.

Our main theorem states that

**Theorem 1.1.** Let \( M \) be a cat map on \( T^2 \). Then there exist a quantum ergodic sequence of normalized quasimodes \( \{ \psi_j \}_{j=1}^\infty \) of order \( O((\log N)^{-1/2}) \) such that \( \psi_j \in \mathcal{H}_{N_j} \) satisfies the following non-equidistribution conditions.

For any \( \varepsilon > 0 \), there are constants \( c_0 = c_0(\varepsilon, M) > 0 \) and \( j_0 = j_0(\varepsilon, M) \in \mathbb{N} \) such that for all \( j \geq j_0 \),

(i). at the scale \( r_j = c_0(\log N_j)^{-1} \) in the physical space,

\[
\sup_{q \in T^1} \left\{ \frac{\int_{B_1(q,r_j)} |\psi_j|^2}{\text{Vol}(B_1(q,r_j))} \right\} \geq \varepsilon^{-1} \quad \text{and} \quad \inf_{q \in T^1} \left\{ \frac{\int_{B_1(q,r_j)} |\psi_j|^2}{\text{Vol}(B_1(q,r_j))} \right\} \leq \varepsilon,
\]


(ii). at the scale $r_j = c_0 (\log N_j)^{-1/2}$ in the phase space,

$$\sup_{x \in \mathbb{T}^2} \left\{ \frac{\langle \psi_j | \hat{b}_{x,r_j}^\pm | \psi_j \rangle}{\text{Vol}(B_2(x,r_j))} \right\} \geq \varepsilon^{-1} \quad \text{and} \quad \inf_{x \in \mathbb{T}^2} \left\{ \frac{\langle \psi_j | \hat{b}_{x,r}^\pm | \psi_j \rangle}{\text{Vol}(B_2(x,r_j))} \right\} \leq \varepsilon.
$$

Theorem 1.1 does not apply to exact eigenstates and we leave it for the future study. In the case of quasimodes, we in fact prove a much more general result than Theorem 1.1. To state the result, we say that a measure $\mu$ on $\mathbb{T}^2$ is a semiclassical measure induced by a sequence of states $\{\psi_j\}_{j=1}^\infty$ if for all $f \in C^\infty(\mathbb{T}^2)$

$$\langle \psi_j | \hat{f} | \psi_j \rangle \rightarrow \int_{\mathbb{T}^2} f(x) \, d\mu \quad \text{as} \quad j \rightarrow \infty. \quad (1.2)$$

So the semiclassical measure $\mu$ characterizes the density distribution of the states $\{\psi_j\}_{j=1}^\infty$ at the macroscopic scale. For example, if $\mu = \mu_\gamma$ as the delta measure on some closed orbit of the classical cat map $M$ on $\mathbb{T}^2$, then the states concentrate near the orbit asymptotically and are said to be “scarring”. (See (3.1) for the delta measure $\mu_\gamma$ defined on a closed orbit $\gamma$.) On the other hand, the states are quantum ergodic at the macroscopic scale if and only if the corresponding semiclassical measure coincides with the Lebesgue measure.

Our next theorem states that

**Theorem 1.2.** Let $M$ be a cat map on $\mathbb{T}^2$ and $\mu$ be an invariant probability measure of $M$. Then there exists a sequence of normalized quasimodes $\{\psi_j\}_{j=1}^\infty$ of order $O((\log N)^{-1/2})$ with corresponding semiclassical measure $\mu$ such that the non-equidistribution conditions (i) and (ii) in Theorem 1.1 hold.

Since the Lebesgue measure is an invariant probability measure of $M$, Theorem 1.1 follows directly from Theorem 1.2.

**Remark.** Suppose that $\{\psi_j\}_{j=1}^\infty$ are normalized quasimodes of order $O((\log N)^{-1/2})$ as considered in Theorems 1.1 and 1.2. Then it is well known that the corresponding semiclassical measures are probability measures which are invariant under $M$. (In fact, the result remains valid for quasimodes of order $o(1)$. See Zworski [Zw, Chapter 5] and also Subsection 3.2 for a short proof in the context of cat maps.)

From this point of view, Theorem 1.2 provides a reverse statement that any invariant probability measure of $M$ must arise as a semiclassical measure that is induced by quasimodes of order $O((\log N)^{-1/2})$, moreover, these quasimodes are not equidistributed at the same logarithmical scales as in Theorem 1.1.

We shall also point out that Theorem 1.2 is invalid for exact eigenstates. Indeed, the set of semiclassical measures corresponding to eigenstates is smaller than the one of invariant probability measures, see [BH, BonDB, FN, R]. For example, the delta measure on a closed prime orbit can not be the semiclassical measure induced by eigenstates. Therefore, Theorem 1.2 demonstrates the sharp difference between the exact eigenstates and quasimodes of logarithmical order.

**Outline of the plan.** The construction of the quasimodes in Theorems 1.1 and 1.2 is inspired by Faure-Nonnenmacher-De Bièvre [FND]. Let $\phi \in \mathbb{R}$ and $\gamma = \{x_t\}_{t=0}^{T-1}$ be a closed prime orbit of the cat map $M$ on $\mathbb{T}^2$. Construct the quantum state $\Psi_\phi^\gamma \in \mathcal{H}_N$ by

$$|\Psi_\phi^\gamma\rangle = \sum_{t=0}^{T-1} e^{-i\phi t} \hat{M}^t |x_0, \tilde{c}_0, \theta\rangle.$$
Here, \(|x_0, \tilde{c}_0, \theta| \in \mathcal{H}_N\) is a coherent state centered at \(x_0 \in \mathbb{T}^2\) and localized in a region with width \(\sim h^2\). (The precise localization of a quantum state is analyzed via its Husimi function. See Subsection 4.1.)

Under the quantum evolution with \(t > 0\), \(\hat{M}^t |x_0, \tilde{c}_0, \theta)\) becomes less localized. Assume that \(M\) has Lyapunov exponent \(\lambda > 0\) (so the eigenvalues of \(M\) are \(e^{\pm \lambda}\)). Then \(\hat{M}^t |x_0, \tilde{c}_0, \theta)\) has center at \(M^t x_0 = x_t\) and localization with width \(\sim h^2 e^{\lambda T}\). We therefore introduce the Ehrenfest time

\[
T_E = \frac{|\log h|}{\lambda}.
\]

Thus, \(\hat{M}^t |x_0, \tilde{c}_0, \theta)\) remains well localized near \(x_t\) if \(t \leq \delta T_E\) for \(\delta < 1/2\). From the basic property of cat maps, the points on a closed prime orbit \(\{x_t\}_{t=0}^{T-1}\) are separated by distance at least \(e^{-\lambda T}\) in \(\mathbb{T}^2\). Combining these two estimates,

\[
\hbar^2 e^{\lambda T} \ll e^{-\lambda T} \quad \text{if} \quad T \leq \delta T_E \quad \text{for} \quad \delta < \frac{1}{4}.
\]

That is, within this time frame, \(\hat{M}^t |x_0, \tilde{c}_0, \theta)\) \((t = 0, \ldots, T - 1)\) are well localized in disjoint regions so the state \(\Psi^\gamma_\phi\) is approximately a direct sum of them.

Suppose that \(\mu\) is an invariant probability measure of \(M\) on \(\mathbb{T}^2\). Then by Sigmund \(\mathcal{S}\), there is a sequence of closed prime orbits \(\{\gamma_j\}_{j=1}^\infty\) such that the delta measures \(\mu_{\gamma_j} \to \mu\) weakly.

If the lengths \(|\gamma_j|\) of \(\gamma_j\) are bounded, then \(\mu\) must be itself a delta measure on some closed prime orbit. This is because the orbits of \(M\) are enumerable by length. In this case, the construction of quasimodes with corresponding semiclassical measure \(\mu\) was done by Faure-Nonnenmacher-De Bièvre \(\mathcal{FNDB}\). (Indeed, the quasimodes are the localized parts of the ones constructed in \(\mathcal{FNDB}\). See related literature below for the difference between the approach in \(\mathcal{FNDB}\) and the one in this paper.) These quasimodes are not quantum ergodic at the macroscopic scale therefore satisfy the non-equidistribution conditions as in (i) and (ii) of Theorem 1.1 at any small scale.

In this paper, we discuss the case when the lengths \(|\gamma_j| \to \infty\). Fix \(0 < \delta < 1/4\). Assign \(\hbar_j \sim e^{N|\gamma_j|/\delta}\) so \(|\gamma_j| \sim \delta |\log \hbar_j|/\lambda\). Construct the quantum state \(\Psi^\gamma_\phi \in \mathcal{H}_N\) as above. For notational simplicity, we drop the subscription for now. We have seen that \(\Psi^\gamma_\phi\) localizes near \(\gamma = \{x_t\}_{t=0}^{T-1}\), on which the points are also well separated. It then follows immediately that \(\|\Psi^\gamma_\phi\|_{L^2} \sim \sqrt{T}\). Observe that

\[
\left\| \left( \hat{M} - e^{i \phi \hat{I}} \right) \sum_{t=0}^{T-1} e^{-i \phi t} \hat{M}^t \right\| = \left\| e^{-i \phi T} \hat{M}^T - \hat{I} \right\| \leq 2.
\]

The states \(\Psi^\gamma_\phi\) therefore are quasimodes of order \(O(1/\sqrt{T}) = O(|\log \hbar|^{-1/2})\). In addition, \(\Psi^\gamma_\phi\) localizes near the classical orbit \(\gamma\), on which the delta measure tends to \(\mu\) as \(T \to \infty\). Then the semiclassical measure induced by \(\Psi^\gamma_\phi\) is also \(\mu\). The rigorous analysis requires the detailed study of their Husimi functions.

To observe the non-equidistribution phenomenon, we first note that a closed prime orbit \(\gamma\) of length \(T\) in \(\mathbb{T}^2\) can not be equidistributed at any scale \(r\) if \(r \ll T^{-1/2} \sim |\log \hbar|^{-1/2}\). Indeed, one can find \(\sim r^{-2} \gg T\) disjoint balls in \(\mathbb{T}^2\). From the pigeon-hole principle, there are balls that do not intersect \(\gamma\). This readily shows the non-equidistribution of \(\gamma\) in \(\mathbb{T}^2\). Since the quasimode \(\Psi^\gamma_\phi\) is localized near the orbit \(\gamma\), it must also display non-equidistribution at the same scale in \(\mathbb{T}^2\). The analysis here again needs the study of their Husimi functions. The
non-equidistribution of $\Psi^\gamma_{\phi}$ at scales $r \ll T^{-1} \sim |\log \hbar|^{-1}$ in the physical space $\mathbb{T}^1$ can be argued similarly.

Related literature. The cat maps are the simplest examples of hyperbolic dynamical systems. We expect that the Quantum Chaos study in this series would motivate a more general approach for other hyperbolic systems, such as the geodesic flow on compact manifolds with negative curvature. The eigenstates in the corresponding quantum system can be described by Laplacian eigenfunctions on the manifold. The density equidistribution of eigenfunctions as well as quasimodes have been extensively studied. See Part I [Han] for the discussion on these results.

The study of non-equidistribution of quasimodes of logarithmical order have been studied by Brooks [B2] on surfaces of constant curvature, Eswarathasan-Nonnenmacher [EN] on surfaces of variable curvature, and Eswarathasan-Silberman [ES] on higher dimensional manifolds of constant curvature. In these various settings, they construct quasimodes that concentrate near a fixed closed orbit and therefore fail equidistribution at the macroscopic scale. These arguments are similar in spirit to Faure-Nonnenmacher-De Bièvre [FNDB] for cat maps.

In our construction however, there are a family of closed orbits with lengths tending infinity. The quasimodes are associated with this family, instead of one fixed orbit. When the delta measures on the orbits of this family tend to a measure, the quasimodes are then designed to recover the same measure in the semiclassical limit. It is interesting to see if such construction can also be carried out on manifolds.

Organization of the paper. In Sections 2 and 3, we recall the necessary tools of classical and quantum dynamics to construct the quasimodes (4.2). In Section 4, we describe the Husimi functions of these quasimodes and prove the properties in Theorem 1.2.

2. Classical dynamics and quantum dynamics on the plane

In this section, we introduce the classical linear hyperbolic systems on the phase space $\mathbb{R}^2$ and their quantum systems. We follow the setup in Faure-Nonnenmacher-De Bièvre [FNDB]. In particular, we mention several interpretations of the classical system, define the corresponding quantum system, and analyze the quantum evolution of coherent states.

2.1. Classical dynamics on the plane. Consider the quadratic Hamiltonian on the phase space $\mathbb{R}^2$ that is given by

$$H(q, p) = \frac{1}{2}\alpha q^2 + \frac{1}{2}\beta p^2 + \gamma qp.$$  \hspace{1cm} (2.1)

It generates the Hamiltonian flow $M(t) : x(0) = (q(0), p(0)) \rightarrow x(t) = (q(t), p(t))$, in which

$$M(t) = \exp \left\{ t \left( \begin{array}{cc} \gamma & \beta \\ -\alpha & -\gamma \end{array} \right) \right\}.$$  \hspace{1cm} (2.2)

Define

$$M = M(1) = \exp \left\{ \left( \begin{array}{cc} \gamma & \beta \\ -\alpha & -\gamma \end{array} \right) \right\} = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{SL}(2, \mathbb{R}).$$

Set $\lambda = \sqrt{\gamma^2 - \alpha \beta}$. Then

$$\begin{cases}
A = \cosh \lambda + \frac{\gamma}{\lambda} \sinh \lambda, & B = \frac{\beta}{\lambda} \sinh \lambda, \\
C = -\frac{\alpha}{\lambda} \sinh \lambda, & D = \cosh \lambda - \frac{\gamma}{\lambda} \sinh \lambda.
\end{cases}$$
**Definition** (Classical linear dynamical systems on $\mathbb{R}^2$).

- If $\gamma^2 > \alpha\beta$, then $M(t)$ is a hyperbolic flow with Lyapunov exponent $\lambda = \sqrt{\gamma^2 - \alpha\beta}$ and $M$ is a hyperbolic map with eigenvalues $e^{\pm\lambda}$ and two eigenaxes that correspond to the unstable and stable directions for the dynamics. They have slopes $s_+ = \tan \psi_+$ and $s_- = \tan \psi_-$.

- If $\gamma^2 < \alpha\beta$, then $M$ is an elliptic map.

Any hyperbolic map $M \in \text{SL}(2, \mathbb{R})$ with $\text{Tr} M > 2$ is of the above form. (If $\text{Tr} M < -2$, then consider the map $-M$.) Throughout the paper, we use $M$ to denote both the map and the matrix that defines it.

**Remark.** We remark that $M \in \text{SL}(2, \mathbb{R})$ preserves the symplectic product on $\mathbb{R}^2$:

$$Mu \wedge Mv = u \wedge v.$$  

Here,  

$$u \wedge v = u_2v_1 - u_1v_2 \text{ for } u = (u_1, u_2) \text{ and } v = (v_1, v_2) \in \mathbb{R}^2.$$  

We now rewrite the hyperbolic flow $M(t)$ and the hyperbolic map $M$ in complex coordinates. Let $z = (q + ip)/\sqrt{2}$. Then the Hamiltonian in (2.1) is

$$H(z, \overline{z}) = \frac{c}{2} z^2 + \frac{c}{2} \overline{z}^2 + b z \overline{z}, \text{ in which } b = \frac{\alpha + \beta}{2} \in \mathbb{R} \text{ and } c = \frac{\alpha - \beta}{2} - i\gamma \in \mathbb{C}.$$  

Since $\gamma^2 - \alpha\beta = |c|^2 - b^2$, $M = M_{(c,b)}$ is hyperbolic if $|c|^2 > b^2$ and is elliptic if $|c|^2 < b^2$.

**Definition.** Let $\mu \in (0, \infty)$.

- Define

$$D(\mu) = M_{(c = -i\mu, b = 0)} = \begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu} \end{pmatrix},$$

which is hyperbolic with $q$-axis and $p$-axis as the unstable and stable axes, respectively.

- Define

$$B(\mu) = M_{(c = -\mu, b = 0)} = \begin{pmatrix} \cosh \mu & \sinh \mu \\ \sinh \mu & \cosh \mu \end{pmatrix},$$

which is hyperbolic with the unstable and stable axes forming $\psi_+ = \pi/4$ and $\psi_- = -\pi/4$ with the $q$-axis.

- Define

$$R(\mu) = M_{(c = 0, b = -\mu)} = \begin{pmatrix} \cos \mu & -\sin \mu \\ \sin \mu & \cos \mu \end{pmatrix},$$

which is a rotation of angle $\mu$ and is therefore elliptic.

Then any hyperbolic map $M_{(c,b)}$ can be decomposed as follows: There are $b_1 \in [-1/2, 1/2]$ and $b_2 \in \mathbb{R}$ such that

$$M_{(c,b)} = QD(\lambda)Q^{-1}, \text{ in which } Q = R(b_1)B(b_2). \tag{2.3}$$

That is, $M_{(c,b)}$ is obtained from the special case $D(\lambda)$ ($\lambda = \sqrt{|c|^2 - b^2} > 0$) by a change of coordinates. Notice that $D(\lambda)$ has unstable and stable axes given by vectors $e_q$ and $e_p$ in the $q$-axis and $p$-axis, respectively. The map $Q$ transforms from this $(q,p)$-frame to the unstable/stable-frame given by vectors $v_+ = Qe_q$ and $v_- = Qe_p$. 
2.2. **Quantum dynamics on the plane.** Let $h$ be the Planck constant and we are interested in the semiclassical limit as $h \to 0$ in this paper. Denote $\hbar = h/(2\pi)$. The states in the quantum system are represented by functions in $L^2(\mathbb{R})$; the quantum observables are operators acting on $L^2(\mathbb{R})$ which are quantization of the classical observables in $C^\infty(\mathbb{R}^2)$.

We first define the quantization of the position and momentum observables as the self-adjoint operators

\[
\hat{q}\psi(q) = q\psi(q) \quad \text{and} \quad \hat{p}\psi = \frac{h}{i} \frac{d\psi}{dq} \quad \text{for} \quad \psi \in C_0^\infty(\mathbb{R}).
\]

So we have that

\[
[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar \hat{I}.
\]

Here, $\hat{I}$ is the identity map that $\hat{I}\psi = \psi$.

The Weyl quantization of the Hamiltonian in (2.1) is the self-adjoint operator

\[
\hat{H} = \frac{1}{2}\alpha\hat{q}^2 + \frac{1}{2}\beta\hat{p}^2 + \gamma (\hat{q}\hat{p} + \hat{p}\hat{q}).
\]

It generates the Schrödinger flow $\psi(0) \to \psi(t)$ such that

\[
\psi(t) = e^{-it\hat{H}/\hbar}\psi(0)
\]

solves the Schrödinger equation

\[
i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t).
\]

**Definition** (Quantum maps). The quantum map (or quantum evolution operator) corresponding to a hyperbolic map $M$ is defined as $\hat{M} = e^{-i\hat{H}/\hbar}$.

**Definition** (Quantum translation operators). Let $v = (v_1, v_2) \in \mathbb{R}^2$ and the translation $T_v(x) = x + v$ for $x \in \mathbb{R}^2$. Define the quantum translation operator as

\[
\hat{T}_v = \exp \left( -\frac{i}{\hbar} (v_1\hat{p} - v_2\hat{q}) \right).
\]

**Proposition 2.1.** Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$. Then

(1). the adjoint operator

\[
\hat{T}_v^* = \hat{T}_{-v},
\]

(2). the conjugation

\[
\hat{M}\hat{T}_v\hat{M}^{-1} = \hat{T}_{Mv},
\]

(3). the composition

\[
\hat{T}_u\hat{T}_v = e^{-iu\cdot v}/(2\pi) \hat{T}_{u+v}.
\]

2.3. **Coherent states and their evolution on the plane.** The standard coherent state $|0\rangle$ at the origin is the ground state of the quantum harmonic oscillator $\hat{q}^2 + \hat{p}^2$. The standard coherent state at $x = (q, p) \in \mathbb{R}^2$ is then $|x\rangle = \hat{T}_x |0\rangle$. In the $L^2(\mathbb{R})$ representation, $|x\rangle$ is a Gaussian wave packet that is given by

\[
|x\rangle \langle q'| = \frac{1}{(\pi \hbar)^{\frac{1}{2}}} e^{\frac{iq^*q}{\hbar}} e^{-\frac{1}{2\hbar} |q'-q|^2},
\]

which is localized at $x = (q, p)$ with width $\sim \sqrt{\hbar}$. 

Let $M$ be a hyperbolic map on $\mathbb{R}^2$ and $\hat{M}$ be its quantum map. The evolution of the standard coherent state $|x\rangle$ under $\hat{M}$ is not straightforward. This is partly due to the fact that the unstable/stable-frame of $M$ is in general different from the $(q,p)$-frame. To remedy this issue, we introduce the squeezed coherent states.

**Definition** (Squeezed coherent states). Let $\tilde{c} \in \mathbb{C}$. Define the squeezed coherent states

$$|\tilde{c}\rangle = \hat{M}_{(\tilde{c},0)} |0\rangle \quad \text{and} \quad |x,\tilde{c}\rangle = \hat{T}_x |\tilde{c}\rangle.$$  

We use the notations $|\tilde{c}\rangle$ and $|x,\tilde{c}\rangle$ with tildes to indicate the squeezed coherent states. In particular, choosing $\tilde{c} = 0$ reduces $|\tilde{c}\rangle$ and $|x,\tilde{c}\rangle$ to the standard coherent states $|0\rangle$ and $|x\rangle$. The properties of the coherent states $|\tilde{c}\rangle$ are analyzed via the Bargmann and Husimi functions.

**Definition** (Bargmann and Husimi functions on the plane). Let $\tilde{w} \in \mathbb{C}$ and $\psi \in L^2(\mathbb{R})$. Define

- the Bargmann function of $\psi$ as
  $$B_{\tilde{w},\psi}(x) = \langle x, \tilde{w} | \psi \rangle,$$

- the Husimi function of $\psi$ as
  $$H_{\tilde{w},\psi}(x) = \frac{|\langle x, \tilde{w} | \psi \rangle|^2}{2\pi \hbar},$$
  which satisfies
  $$\frac{1}{2\pi \hbar} \int_{\mathbb{R}^2} |x, \tilde{w}\rangle \langle x, \tilde{w}| \, dx = \hat{I}. \tag{2.6}$$

That is, taking the inner product of $\psi$ with the coherent state $|x, \tilde{w}\rangle$, $B_{\tilde{w},\psi}(x)$ and $H_{\tilde{w},\psi}(x)$ measure the localization of the state $\psi$ at the point $x$ in the phase space $\mathbb{R}^2$. We use the squeezed coherent states $|x, \tilde{w}\rangle = \hat{T}_x |\tilde{w}\rangle$ here to allow full generality so $B_{\tilde{w},\psi}(x)$ and $H_{\tilde{w},\psi}(x)$ depend on $\tilde{w}$. The Bargmann and Husimi functions of various states $\psi$ can be simplified by making appropriate choices of $\tilde{w}$ and the frame in $\mathbb{R}^2$.

**Example** (Bargmann function of the squeezed coherent states). Let $|\tilde{c}\rangle = \hat{M}_{(\tilde{c},0)} |0\rangle$ be a squeezed coherent state. Choose $\tilde{w} = 0$ and the unstable/stable-frame $x = (\tilde{q}, \tilde{p})$ of $M_{(\tilde{c},0)}$. Then we have that

$$B_{0,\tilde{c}}(x) = \langle x, 0 | \tilde{c} \rangle = \frac{1}{\sqrt{\cosh |\tilde{c}|}} \exp\left(-\frac{i}{2\pi \hbar} \frac{2h}{\cosh \mu} \frac{\tilde{c}}{\tilde{q} \tilde{p}}\right) \exp\left(-\frac{1}{2} \left(\frac{\tilde{q}^2}{\Delta q^2} + \frac{\tilde{p}^2}{\Delta p^2}\right)\right),$$

in which

$$\Delta q^2 = \frac{2h}{1 - \tanh |\tilde{c}|} \quad \text{and} \quad \Delta p^2 = \frac{2h}{1 + \tanh |\tilde{c}|}.$$  

In particular, if $\tilde{c} = -i\mu$ for $\mu > 0$, then $M_{(\tilde{c},0)} = D(\mu)$ and the unstable/stable-frame coincides with the $(q,p)$-frame. In this case, we have that

$$\langle x, 0 | \tilde{c} \rangle = \langle x | \hat{D}(\mu) |0\rangle = \frac{1}{\sqrt{\cosh \mu}} \exp\left(-\frac{i}{2\pi \hbar} \frac{2h}{\cosh \mu} \frac{\mu}{q p}\right) \exp\left(-\frac{1}{2} \left(\frac{q^2}{\Delta q^2} + \frac{p^2}{\Delta p^2}\right)\right). \tag{2.7}$$

We next describe the quantum evolution of coherent states under $\hat{M} = \hat{M}_{(\tilde{c},b)}$. Based on the decomposition of $M = M_{(c,b)}$ in (2.3), the evolution can be described explicitly for properly chosen squeezed coherent states. Recall that $M_{(c,b)} = QD(\lambda)Q^{-1}$, in which $Q = R(b_1)B(b_2)$ with $b_1 \in [\frac{b}{2}, \frac{b}{2}]$ and $b_2 \in \mathbb{R}$. Put

$$\tilde{c}_0 = -b_2 e^{-2ib_1}. \tag{2.8}$$
Since \( M(\tilde{c}_0,0) = R(b_1)B(b_2)R(-b_1) \), \( \tilde{M}(\tilde{c}_0,0) = \tilde{R}(b_1)\tilde{B}(b_2)\tilde{R}(-b_1) = \tilde{Q}\tilde{R}(-b_1) \), which implies that
\[
|\tilde{c}_0\rangle = M(\tilde{c}_0,0) |0\rangle = \tilde{Q}\tilde{R}(-b_1) |0\rangle = e^{-\frac{ib}{\hbar}}\tilde{Q} |0\rangle.
\]
It thus follows from (2.7) that
\[
\tilde{M}^t |\tilde{c}_0\rangle = e^{-\frac{ib}{\hbar}}\tilde{Q}\tilde{D}(\lambda t) |0\rangle \quad \text{and} \quad \langle \tilde{c}_0|\tilde{M}^t|\tilde{c}_0\rangle = \langle 0|\tilde{D}(\lambda t)|0\rangle = \frac{1}{\sqrt{\cosh(\lambda t)}}.
\]
Denote the quantum evolution of the coherent state \( |\tilde{c}_0\rangle \) in (1) by
\[
|t;\tilde{c}_0\rangle = \tilde{M}^t |\tilde{c}_0\rangle.
\]
Then the Husimi function (2.6) of \( |t;\tilde{c}_0\rangle \) is explicit by choosing \( \tilde{w} = \tilde{c}_0 \) and the unstable/stable frame of \( M \).

**Proposition 2.2.** Let \( x = (q',p') = Q^{-1}(q,p) \). Then for \( t \geq 0 \),
\[
\mathcal{H}_{\tilde{c}_0,t}(x) = \frac{|\langle x,\tilde{c}_0|t;\tilde{c}_0\rangle|^2}{2\pi\hbar} = \frac{1}{2\pi\hbar\cosh(\lambda t)} \exp \left( - \frac{q'^2}{\Delta q'^2} - \frac{p'^2}{\Delta p'^2} \right),
\]
in which
\[
\begin{align*}
\Delta q'^2 &= \frac{2\hbar}{1-\tanh(\lambda t)} \sim \hbar e^{2\lambda t} \quad \text{as} \ t \to \infty, \\
\Delta p'^2 &= \frac{2\hbar}{1+\tanh(\lambda t)} = e^{-2\lambda t} \Delta q'^2 \to \hbar \quad \text{as} \ t \to \infty.
\end{align*}
\]

**Remark.**

- The Husimi function \( \mathcal{H}_{\tilde{c}_0,t}(x) \) of \( |t;\tilde{c}_0\rangle \) spreads in the unstable direction of the map \( M \) by a rate of \( \sqrt{\hbar}e^{\lambda t} \) while stays in the \( \sqrt{\hbar} \) neighborhood of the stable direction.
- \( \mathcal{H}_{\tilde{c}_0,t}(x) \) is concentrated in the elliptic region around the origin with two axes \( \Delta q' \) and \( \Delta p' \) (in the unstable/stable-frame of \( M \)).
- The concentration region of \( \mathcal{H}_{\tilde{c}_0,t}(x) \) has area \( \sim \Delta q'\Delta p' \sim \hbar e^{\lambda t} \). Hence, \( \mathcal{H}_{\tilde{c}_0,t}(x) \sim \sqrt{\hbar}e^{\lambda t/2} \) in this region due to conservation of the \( L^2 \) norm in (2.9).
- The Husimi function of the evolution \( |t;\tilde{c}_0\rangle \) in negative times \( t < 0 \) can be described similarly as above. In this case, the concentration region of \( \mathcal{H}_{\tilde{c}_0,t}(x) \) spreads in the stable direction of the map \( M \) by a rate of \( \sqrt{\hbar}e^{\lambda t} \) while stays in the \( \sqrt{\hbar} \) neighborhood of the unstable direction.
- Because of the explicit Husimi function of the evolution \( |t;\tilde{c}_0\rangle \), we exclusively use the squeezed coherent states \( |\tilde{c}_0\rangle \) (which depends on \( M \)) in our construction of quasimodes. See Subsection 3.3.

3. Classical dynamics and quantum dynamics on the torus

In this section, we introduce the classical linear hyperbolic systems on the phase space \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \) and their quantum systems, which are referred as classical and quantum cat maps, respectively. We again follow the setup in Faure-Nonnenmacher-De Bièvre [FNDB].

3.1. Classical dynamics on the torus.

**Definition** (Classical cat maps). Let \( M \in SL(2, \mathbb{R}) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a hyperbolic map. Suppose further that \( M \in SL(2, \mathbb{Z}) \), i.e., \( A, B, C, D \in \mathbb{Z} \) in (2.2). Since
\[
M(x+n) = Mx + Mn = Mx \mod 1 \quad \text{for} \ x \in \mathbb{R}^2 \text{and} \ n \in \mathbb{Z}^2,
\]
\( M \) induces a map on \( \mathbb{T}^2 \) that is hyperbolic, by which we refer as a classical cat map.
**Example** (Arnold cat map). The Arnold cat map is defined by
\[ M_{\text{Arnold}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \]
The eigenvalues of \( M_{\text{Arnold}} \) are \((3 \pm \sqrt{5})/2\) with the Lyapunov exponent \( \log((3 + \sqrt{5})/2) \).

Consider the classical cat map \( M \) with Lyapunov exponent \( \lambda > 0 \) so the eigenvalues are \( e^{\pm \lambda} \).

We recall some standard facts about the discrete hyperbolic dynamical system \( M^t : \mathbb{T}^2 \to \mathbb{T}^2 \) \((t \in \mathbb{Z})\) that are useful later. See Katok-Hasselblatt [KH] for more details.

**Definition** (Periodic points and closed orbits).
- We say that \( x \in \mathbb{T}^2 \) is periodic if \( x \) is a fixed point of \( M^t \) for some \( t \geq 1 \), i.e., \( M^t x = x \).
- Then \( \gamma = \{ M^s x \}_{s=0}^{t-1} \) form a closed orbit of \( M \) and we denote the length of \( \gamma \) by \( |\gamma| = t \).
- The period of a periodic point \( x \in \mathbb{T}^2 \) is defined as \( p(x) = \min \{ T : T \geq 1 \text{ and } M^T x = x \} \).

Then \( \gamma = \{ M^t x \}^{p(x)-1} \) is called a prime closed orbit of \( M \). In this case, \( p(x) = |\gamma| \) for all \( x \in \gamma \).

Denote \( \mathcal{F} = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \) as a fundamental domain of \( \mathbb{T}^2 \). Then each periodic point \( x = (q, p) \in \mathcal{F} \) satisfies that
\[ M^t x = M^t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} j \\ k \end{pmatrix} \quad \text{for some } j, k \in \mathbb{Z}. \]

Since \( M^t \in \text{SL}(2, \mathbb{Z}) \),
\[ \begin{pmatrix} q \\ p \end{pmatrix} = (M^t - \text{Id})^{-1} \begin{pmatrix} j \\ k \end{pmatrix} \]
has rational coordinates. Here, \( \text{Id} \) is the identity matrix. Moreover, if \( x \in \mathcal{F} \) is periodic with period \( p(x) \), then its coordinates can be written as rational numbers with denominator that is not larger than \( \det(M^{p(x)} - \text{Id}) \). To summarize,

**Proposition 3.1.**

(i) Let \( x \in \mathcal{F} \) be a periodic point with period \( p(x) \). Then
\[ x \in \mathcal{L}_l, \quad \text{in which } l = \det \left( M^{p(x)} - \text{Id} \right) = e^{\lambda p(x)} + e^{-\lambda p(x)} - 2. \]
Here,
\[ \mathcal{L}_l = \left\{ \begin{pmatrix} j \\ k \end{pmatrix} : \frac{-l}{2} \leq j, k < \frac{l}{2} \right\} \subset \mathcal{F} \]
is the lattice of rational points with denominator \( l \in \mathbb{N} \) (which are not necessarily in the simplest form).

(ii) Let \( x \in \mathcal{L}_l \). Then \( x \) is periodic with period \( p(x) \leq l^2 \).

**Definition** (Delta measures on closed orbits). Given a closed orbit \( \gamma = \{ x_t \}_{t=0}^{T-1} \), define the delta measure on \( \gamma \)
\[ \mu_\gamma = \frac{1}{T} \sum_{t=0}^{T-1} \delta_{x_t}, \quad (3.1) \]
in which $\delta_x$ is the delta measure at $x$. That is, for any $f \in C(\mathbb{T}^2)$,
\[
\int_{\mathbb{T}^2} f \, d\mu_\gamma = \frac{1}{T} \sum_{t=0}^{T-1} f(x_t).
\]

It is clear that $\mu_\gamma$ is an invariant probability measure of the cat map $M$ on $\mathbb{T}^2$. Moreover, by Sigmund [S],

**Theorem 3.2.** For any invariant probability measure $\mu$ of a cat map $M$ on the torus $\mathbb{T}^2$, there is a sequence of closed prime orbits $\{\gamma_j\}_{j=1}^{\infty}$ such that the delta measures $\mu_{\gamma_j} \to \mu$ weakly, that is, for any $f \in C(\mathbb{T}^2)$,
\[
\int_{\mathbb{T}^2} f \, d\mu_{\gamma_j} \to \int_{\mathbb{T}^2} f \, d\mu, \quad \text{as } j \to \infty.
\]

**Remark.**
- In particular, since the Lebesgue measure $dx$ is invariant on $\mathbb{T}^2$, there is a sequence of closed prime orbits $\{\gamma_j\}_{j=1}^{\infty}$ such that the delta measures $\mu_{\gamma_j}$ converge to the Lebesgue measure weakly.
- Let $\{\gamma_j\}_{j=1}^{\infty}$ be a sequence of closed prime orbits such that $\mu_{\gamma_j} \to \mu$. Suppose that $|\gamma_j| \leq C$ for some uniform constant $C > 0$. Since the closed prime orbits are enumerable by their lengths, there are only finitely many orbits with length bounded by $C$. Therefore, $\mu$ is itself a delta measure on some closed prime orbit.

### 3.2. Quantum dynamics on the torus

We first need to describe the space of states in the quantum system of a cat map with phase space $\mathbb{T}^2$. Each state is represented by $\psi \in L^2(\mathbb{R})$ that is periodic in position and in momentum. This means that $\psi$ is invariant under the phase translations $\hat{T}_n$ for $n = (n_1, n_2) \in \mathbb{Z}^2$. In particular,
\[
\hat{T}_{(1,0)} |\psi\rangle = e^{i\theta_1} |\psi\rangle \quad \text{and} \quad \hat{T}_{(0,1)} |\psi\rangle = e^{i\theta_2} |\psi\rangle.
\]

Here, we allow the phase shifts $e^{i\theta_1}$ and $e^{i\theta_2}$ for some angle $\theta = (\theta_1, \theta_2) \in [0, 2\pi) \times [0, 2\pi)$, because under such phase shifts the function defines the same quantum state. It then follows from such periodicity that
\[
\hat{T}_{(1,0)} \hat{T}_{(0,1)} = \hat{T}_{(0,1)} \hat{T}_{(1,0)}
\]
restricted to the space of quantum states. But in the view of (2.5), since $(1,0) \wedge (0,1) = -1$, it requires that $e^{i/\hbar} = 1$. Hence,
\[
N = \frac{1}{2\pi \hbar} \in \mathbb{N}. \tag{3.3}
\]

Under the conditions (3.2) and (3.3), the space of quantum states $\mathcal{H}_{N,\theta}$ is an $N$-dim Hilbert space. Moreover, $L^2(\mathbb{R})$ can then be decomposed as
\[
L^2(\mathbb{R}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \mathcal{H}_{N,\theta} \, d\theta.
\]

**Definition (Projector).** The projector $\hat{P}_\theta : S'(\mathbb{R}) \to \mathcal{H}_{N,\theta}$ is defined as
\[
\hat{P}_\theta = \sum_{n=(n_1,n_2) \in \mathbb{Z}^2} e^{-in_1\theta_1-in_2\theta_2} \hat{T}_{(1,0)}^{n_1} \hat{T}_{(0,1)}^{n_2} = \sum_{n=(n_1,n_2) \in \mathbb{Z}^2} e^{-i\theta n + i\delta_n} \hat{T}_n, \tag{3.4}
\]
in which $\delta_n = -n_1n_2N\pi$ by (2.5).
Let $M \in \text{SL}(2, \mathbb{Z})$ be a hyperbolic map on $\mathbb{R}^2$. Then by Section 2, $\hat{M}$ defines a quantum map that acts on $L^2(\mathbb{R})$. From (2.4), we have that
\[
\hat{M} \hat{P}_\theta \hat{M}^{-1} = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} e^{-i\theta \cdot n + i\delta_n} \hat{T}_{Mn} = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} e^{-i\theta \cdot M^{-1}n + i\delta_{M^{-1}n}} \hat{T}_n.
\]
Hence,
\[
\hat{M} \hat{P}_\theta = \hat{P}_{\theta'} \hat{M}, \quad \text{in which } \theta' = M^{-1}\theta + N\pi \left( \frac{CD}{AB} \right).
\]
If $\theta' = \theta \mod (2\pi)$, then $\hat{M}$ commutes with $\hat{P}_\theta$ and therefore defines an endomorphism on $\mathcal{H}_{N,\theta}$. For each $N \in \mathbb{N}$, such choices of $\theta$ are always possible, for example, $\theta = (0, 0)$ if $N$ is even and $\theta = (\pi, \pi)$ if $N$ is odd.

**Definition** (Quantum cat maps). Let $M \in \text{SL}(2, \mathbb{Z})$ be a classical cat map. Then for any $N \in \mathbb{N}$, there exists $\theta \in [0, 2\pi) \times [0, 2\pi)$ such that $\hat{M} : \mathcal{H}_{N,\theta} \to \mathcal{H}_{N,\theta}$. We fix such choice of $\theta$ that depends on $M$ and $N$. The operator $\hat{M}$ restricted on $\mathcal{H}_{N,\theta}$ is called the quantum cat map. (If there is no confusion, then we simply write $\mathcal{H}_N$.)

Any quantum translation operator $\hat{T}_v$ acts on $\mathcal{H}_N$ only if $\hat{T}_v$ commutes with $\hat{T}_n$ for all $n \in \mathbb{Z}^2$. Applying (2.5) again, $e^{i(v \cdot n)/\hbar} = 1$ for all $n \in \mathbb{Z}^2$. So $v \in \mathbb{Z}^2/N$. For notational convenience, we write
\[
\hat{T}_N(n) = \hat{T}_{n/N}.
\]

**Definition** (Quantization). Let $a \in C^\infty(\mathbb{T}^2)$. Define its Weyl quantization as an operator on $\mathcal{H}_N$:
\[
\hat{a}^w = \sum_{n \in \mathbb{Z}^2} \hat{a}(n) \hat{T}_N(n).
\]
Here, $a$ is called the symbol and $\hat{a}(n)$ is the Fourier coefficients of $a$ that
\[
a(x) = \sum_{n \in \mathbb{Z}^2} \hat{a}(n) e^{2\pi i(n \cdot x)}.
\]

**Remark** ($L^2$ boundedness). We have that $\hat{a}^w$ is bounded on $L^2(\mathbb{T}^1)$, that is,
\[
\langle \psi | \hat{a}^w | \psi \rangle \leq C \langle \psi | \psi \rangle,
\]
in which $C$ depends on finite number of derivatives of $a$. See Zworski [Zw] Setion 4.5.

Since the cat maps are linear, we have the following exact Egorov’s theorem. See Part I [Han] for a shoot proof.

**Theorem 3.3** (Egorov’s theorem). Let $a \in C^\infty(\mathbb{T}^2)$. Then
\[
\hat{M}^{-t} \hat{a}^w \hat{M}^t = a \circ \hat{M}^t \quad \text{for all } t \in \mathbb{Z}.
\]
As an immediate consequence, we have that

**Proposition 3.4** (Invariance of semiclassical measures). Let $\{\psi_j\}_{j=1}^\infty$ be a sequence of quasi-modes of order $o(1)$ in $L^2$. Suppose that $\mu$ is the corresponding semiclassical measure. Then $\mu$ is invariant under $M$. 

Proof. It suffices to prove that for any \( f \in C^\infty(\mathbb{T}^2) \),
\[
\int_{\mathbb{T}^2} f \ d\mu = \int_{\mathbb{T}^2} f \circ M \ d\mu.
\]
Since \( \{ \psi_j \}_{j=1}^\infty \) is a sequence of quasimodes of order \( o(1) \), we have that
\[
\hat{M} |\psi_j\rangle = e^{i\phi_j} |\psi_j\rangle + o_L(1) \quad \text{for some } \phi_j \in \mathbb{R}.
\]
Then by the \( L^2 \) boundedness of \( \hat{f}_w \) and the Egorov’s theorem above,
\[
\langle \psi_j | \hat{f} \circ M^w | \psi_j \rangle = \langle \psi_j | \hat{M}^{-1} \hat{f} \hat{M} | \psi_j \rangle = \langle \psi_j | \hat{f}_w | \psi_j \rangle + o_f(1).
\]
The proposition follows by taking limits of both sides as \( j \to \infty \). \( \square \)

3.3. Coherent states and their evolution on the torus. Let \( M \) be a classical cat map on the torus \( \mathbb{T}^2 \) and \( \hat{M} \) be its quantization on \( \mathcal{H}_{N,\theta} \) with \( N = 1/(2\pi \hbar) \in \mathbb{N} \).

In this section, we investigate the coherent states on the torus and their evolution under the quantum cat map \( \hat{M} \). For technical convenience, we begin from the squeezed coherent state \( |\tilde{c}_0\rangle \) on the plane (3.8). Its evolution \( |t; \tilde{c}_0\rangle = \hat{M}_t |\tilde{c}_0\rangle \) on the plane (3.9) has an explicit Husimi function which is given in Proposition 2.2.

The squeezed coherent states \( |\tilde{c}_0, \theta\rangle \) and \( |x, \tilde{c}_0, \theta\rangle \) on the torus are defined via the projector \( \hat{P}_\theta \) in (3.1):
\[
|\tilde{c}_0, \theta\rangle = \hat{P}_\theta |\tilde{c}_0\rangle \quad \text{and} \quad |x, \tilde{c}_0, \theta\rangle = \hat{P}_\theta |x, \tilde{c}_0\rangle.
\]
Write the evolution of \( |\tilde{c}_0, \theta\rangle \) under \( \hat{M} \) as \( |t; \tilde{c}_0, \theta\rangle = \hat{M}_t |\tilde{c}_0, \theta\rangle \in \mathcal{H}_{N,\theta} \) for \( t \in \mathbb{Z} \). We use the Husimi function on the torus to analytize \( |t; \tilde{c}_0, \theta\rangle \).

For any quantum state \( \psi \) on the torus, define the Husimi function of \( \psi \) as
\[
\mathcal{H}_{\tilde{c}_0, \psi, \theta}(x) = N |\langle x, \tilde{c}_0, \theta | \psi \rangle|^2, \quad \text{which satisfies} \quad \int_{\mathbb{T}^2} N |x, \tilde{c}_0, \theta \rangle \langle x, \tilde{c}_0, \theta \rangle \ dx = \hat{I}.
\](3.6)
Then the Husimi function of \( |t; \tilde{c}_0, \theta\rangle \)
\[
\mathcal{H}_{\tilde{c}_0, t, \theta}(x) = N |\langle x, \tilde{c}_0, \theta | t; \tilde{c}_0, \theta \rangle|^2
\]
\[
= N |\langle x, \tilde{c}_0 | \hat{P}_\theta | t; \tilde{c}_0 \rangle|^2
\]
\[
= N \left| \sum_{n \in \mathbb{Z}^2} e^{-i \theta \cdot n + i \Delta n_1} \langle x, \tilde{c}_0 | \hat{T}_n | t; \tilde{c}_0 \rangle \right|^2
\]
\[
= N \left| \sum_{n \in \mathbb{Z}^2} e^{-i \theta \cdot n + i \Delta n_1} \langle x + n, \tilde{c}_0 | t; \tilde{c}_0 \rangle \right|^2.
\]
That is, \( \langle x, \tilde{c}_0 | \hat{P}_\theta | t; \tilde{c}_0 \rangle \) is the sum (up to some phases) of the translates of \( \langle x, \tilde{c}_0 | t; \tilde{c}_0 \rangle \) in different phase space cells of size 1. Use \( x = (q', p') \) in the unstable/stable-frame of \( M \) and recall that \( \mathcal{F} \) is a fundamental cell of \( \mathbb{T}^2 \). By Proposition 2.2, the Husimi function
\[
\mathcal{H}_{\tilde{c}_0, t}(x) = \frac{|\langle x, \tilde{c}_0 | t; \tilde{c}_0 \rangle|^2}{2\pi \hbar} = \frac{N}{\cosh(\lambda t)} \exp \left( -\left( \frac{q'^2}{\Delta q'^2} + \frac{p'^2}{\Delta p'^2} \right) \right),
\]
in which
\[
\Delta q'^2 = \frac{2\hbar}{1 - \tanh(\lambda t)} \sim \hbar e^{2\lambda t} \quad \text{and} \quad \Delta p' = \frac{2\hbar}{1 + \tanh(\lambda t)} \sim \hbar.
\]
Recall that the Ehrenfest time $T_E = |\log \hbar|/\lambda$ in (1.3). Suppose that $0 \leq t \leq \delta T_E$ for some $0 \leq \delta < 1/2$. Then $\mathcal{H}_{\bar{c}_0, t}$ is concentrated in the region

$$\left\{|q'| \lesssim \sqrt{\hbar e^\lambda}, |p'| \lesssim \sqrt{\hbar} \right\} \subset B_2 \left(o, Ch^{1/2} e^\lambda \right) \subset \mathcal{F},$$

in which $C > 0$ is an absolute constant. Here, $B_2(o, r) \subset \mathbb{T}^2$ is the geodesic ball centered at the origin $o$ and with radius $r$. Therefore, within such a time frame, all the terms but the one when $n = (0, 0)$ are negligible in the sum of $\mathcal{H}_{\bar{c}_0, t, \theta}$. In particular,

$$\mathcal{H}_{\bar{c}_0, t, \theta}(x) = \begin{cases} \frac{N}{\cosh(\lambda t)} \exp \left( -\left( \frac{q'^2}{\Delta q^2} + \frac{p'^2}{\Delta p^2} \right) \right) + O \left( e^{-\frac{1}{ct}} \right) & \text{if } x \in B_2 \left(o, Ch^{1/2} e^\lambda \right), \\ O \left( e^{-\frac{1}{ct}} \right) & \text{if } x \in \mathcal{F} \setminus B_2 \left(o, Ch^{1/2} e^\lambda \right). \end{cases} \quad (3.7)$$

Taking $x = o$ and $t = 0$, we have that $\mathcal{H}_{\bar{c}_0, 0, \theta}(o) = N + O(e^{-1/c})$. This means that the squeezed coherent state $|\bar{c}_0, \theta\rangle$ is asymptotically normalized in $L^2(\mathbb{T}^2)$:

$$\langle \bar{c}_0, \theta | \bar{c}_0, \theta \rangle = \frac{\mathcal{H}_{\bar{c}_0, 0, \theta}(o)}{N} = 1 + O \left( e^{-\frac{1}{ct}} \right).$$

Since $\hat{M}$ preserves the $L^2$ norm, the above estimate remains valid for $|t; \bar{c}_0, \theta\rangle$:

$$\langle t; \bar{c}_0, \theta | t; \bar{c}_0, \theta \rangle = 1 + O \left( e^{-\frac{1}{ct}} \right). \quad (3.8)$$

Moreover, by (3.6) and (3.7), the $L^2$ norm of $|t; \bar{c}_0, \theta\rangle$ can be recovered by the integral of the Husimi function in its concentration region modulo an exponential error.

**Corollary 3.5.** Let $0 \leq t \leq \delta T_E$ for some $0 \leq \delta < 1/2$. Then

$$\langle t; \bar{c}_0, \theta | t; \bar{c}_0, \theta \rangle = \int_{B_2 \left(o, Ch^{1/2} e^\lambda \right)} \mathcal{H}_{\bar{c}_0, t, \theta}(x) \, dx + O \left( e^{-\frac{1}{ct}} \right).$$

**Remark.** If $t \geq T_E/2$, then $\Delta q' \gtrsim \hbar e^\lambda$ reaches the size of the fundamental cell $\mathcal{F}$. That is, the concentration region of $\mathcal{H}_{\bar{c}_0, t}$ spreads from $\mathcal{F}$ to other cells. Hence, the terms when $n \neq (0, 0)$ may contribute in the sum of $\mathcal{H}_{\bar{c}_0, 0, \theta}$. This phenomenon of “interference effects” has been extensively studied in Faure-Nonnenmacher-De Bi`evre [FNDB]. In this paper, we restrict the evolution below $T_E/2$ so the interference effects are negligible.

### 3.4. Anti-Wick quantization.

In the previous subsection, we discussed the coherent states and their evolution under the quantum cat map. The localization of these quantum states are described by their Husimi functions (3.6). It is therefore most convenient to use the anti-Wick quantization to study the density distribution of the states.

**Definition** (Anti-Wick quantization). Let $a \in L^\infty(\mathbb{T}^2)$. Define its anti-Wick quantization as an operator on $\mathcal{H}_N$:

$$\hat{a}^{aw} = N \int_{\mathbb{T}^2} a(x) \langle x, \bar{c}_0, \theta \rangle \langle x, \bar{c}_0, \theta \rangle \, dx. \quad (3.9)$$

Then we immediately have that

$$\langle \psi | \hat{a}^{aw} | \psi \rangle = N \int_{\mathbb{T}^2} a(x) \langle \langle x, \bar{c}_0, \theta | \psi \rangle \rangle^2 \, dx = \int_{\mathbb{T}^2} a(x) \mathcal{H}_{\bar{c}_0, \psi, \theta}(x) \, dx. \quad (3.10)$$

In particular, for any $\Omega \subset \mathbb{T}^1$,

$$\int_{\Omega} \langle \psi(q) \rangle^2 \, dq = \langle \psi | \hat{1}_\Omega^{aw} | \psi \rangle = \int_{\Omega} \int_{\mathbb{T}^1} \mathcal{H}_{\bar{c}_0, \psi, \theta}(q, p) \, dp \, dq. \quad (3.11)$$
To accommodate the discussion of density distribution of states at small scales in the phase space $\mathbb{T}^2$ in Theorems 1.1 and 1.2 we allow the classical symbols in quantizations (3.5) and (3.9) to depend on the semiclassical parameter $\hbar$:

**Definition** (Small scale symbols). Let $\rho \in [0, 1/2)$ and $\hbar_0 \in (0, 1)$. We say that $a(x; \hbar) \in S_{\rho}(\mathbb{T}^2)$ if $a \in C^\infty(\mathbb{T}^2 \times (0, \hbar_0))$ and for each multiindex $\alpha$, there is a constant $C_\alpha > 0$ such that

$$|\partial^\alpha_x a(x; \hbar)| \leq C_\alpha \hbar^{-\rho|\alpha|},$$

for all $x \in \mathbb{T}^2$ and $\hbar \in (0, \hbar_0)$.

The Weyl and anti-Wick quantizations are asymptotically equivalent for symbols in $S_{\rho}(\mathbb{T}^2)$ with $\rho \in [0, 1/2)$. See for example Bouzouina-De Bièvre [BouDB] for a proof.

**Lemma 3.6.** Let $\rho \in [0, 1/2)$ and $a(x; \hbar) \in S_{\rho}(\mathbb{T}^2)$. Then

$$\hat{a}^w - \hat{a}^{aw} = O_{L^2 \to L^2} \left( \hbar^{1-2\rho} \right).$$

Hence,

$$\langle \psi | \hat{a}^w \psi \rangle - \langle \psi | \hat{a}^{aw} \psi \rangle = O \left( \hbar^{1-2\rho} \right),$$

for all normalized states $\psi$.

A direct consequence is that the semiclassical measure defined in (1.2) are independent of the quantization.

**Remark.** In Theorems 1.1 and 1.2 we are concerned with the density distribution at small scales in the physical space and in the phase space.

(i). Let $B_1(q_0, r) \subset \mathbb{T}^1$ with $r = r(\hbar) \geq \hbar^{\rho}$ for $\rho \in [0, 1/2)$. From (3.11),

$$\int_{B_1(q_0, r)} \psi(q)^2 dq = \langle \psi | \hat{\chi}^{aw}_{B_1(q_0, r)} \psi \rangle = \int_{\Omega} \int_{B_1(q_0, r)} \mathcal{H}_{\hat{\chi}_{q, \rho}(q, p)} dp dq. \quad (3.12)$$

(ii). Let $B_2(x, r) \subset \mathbb{T}^2$ with $r = r(\hbar) \geq \hbar^{\rho}$ for $\rho \in [0, 1/2)$. Then as in Part I [Han, Lemma 3.1], there are $b^\pm_{x_0, r} \in C^\infty(\mathbb{T}^1 \times (0, \hbar))$ such that

$$b^-_{x_0, r} \leq \chi_{B_2(x_0, r)} \leq b^+_{x_0, r} \quad \text{and} \quad \int_{\mathbb{T}^2} b^\pm_{x_0, r}(x) dx = \text{Vol}(B(x_0, r)) + o \left( \hbar^2 \right). \quad (3.13)$$

In addition, $b^\pm_{x_0, r} \in S_{\rho}(\mathbb{T}^2)$. For later use in Subsection 4.2 we also require that $b^-_{x_0, r} = 1$ in $B_2(x_0, 2r/3)$ and $b^-_{x_0, r} = 1$ in $B_2(x_0, 3r/2)$. The symbols $b^\pm_{x_0, r}$ are therefore the appropriate approximation of the indicator function $\chi_{B_2(x_0, r)}$.

By Lemma 3.6 we have that for any normalized state $\psi$,

$$\langle \psi | \hat{b}^{aw}_{x_0, r} \psi \rangle - \langle \psi | \hat{b}^{aw}_{x_0, r} \psi \rangle = O \left( \hbar^{1-\rho} \right).$$

With the choice of $r(\hbar) = O(|\log \hbar|^{-1/2})$ in Theorems 1.1 and 1.2 the symbols $b^\pm_{x_0, r} \in S_{\rho}(\mathbb{T}^2)$ for all $\rho \in (0, 1/2)$. Hence, Statement (ii) there applies to both Weyl and anti-Wick quantizations.
4. CONSTRUCTION OF THE QUASIMODES

Throughout this section, we fix $M : \mathbb{T}^2 \to \mathbb{T}^2$ as a classical cat map with Lyapunov $\lambda$. Denote $\hat{M} : \mathcal{H}_N \to \mathcal{H}_N$ its quantum cat map. Recall that $N = 1/(2\pi \hbar) \in \mathbb{N}$. In this section, we construct the quasimodes in $\mathcal{H}_N$ that satisfy the conditions in Theorem 1.2. Their localization properties are described by the Husimi functions (3.6), while the density distribution properties are studied via the anti-Wick quantization (3.10).

Let $\gamma = \{x_t\}_{t=0}^{T-1} \subset \mathbb{T}^2$ be a closed prime orbit of $M$ with length $|\gamma| = T$. Here, we allow $T$ to depend on $\hbar$ and require that

$$T \leq \delta T_E = \frac{\delta |\log \hbar|}{\lambda},$$

in which $T_E$ is the Ehrenfest time (1.3) and $\delta > 0$ is to be determined later. Suppose that $\phi \in \mathbb{R}$. Construct the quantum state $\Psi_\phi^\gamma$ associated with $\gamma$ by

$$|\Psi_\phi^\gamma\rangle = \sum_{t=0}^{T-1} e^{-i\phi t} \hat{M}^t |x_0, \bar{c}_0, \theta\rangle = \sum_{t=0}^{T-1} e^{-i\phi t} \hat{M}^t \hat{P}_\theta \hat{T}_{x_0} |\bar{c}_0\rangle \in \mathcal{H}_N.$$

We first describe the Husimi function of $\Psi_\phi^\gamma$ in Subsection 4.1. Then we use this description to establish the distribution of $\Psi_\phi^\gamma$ at various scales in Subsections 4.2 and 4.3.

4.1. Description of the Husimi function. For notational simplicity, we omit the scripts in $\Psi_\phi^\gamma$ and write $\Psi$. Compute the Husimi function (3.6) of $\Psi$:

$$\mathcal{H}_{\bar{c}_0, \psi, \theta}(x) = N \left| \langle x, \bar{c}_0, \theta | \Psi \rangle \right|^2 = N \left| \sum_{t=0}^{T-1} e^{i\phi t} \langle \bar{c}_0 | \hat{T}_{-x} \hat{P}_\theta \hat{M}^t \hat{T}_{x_0} | \bar{c}_0 \rangle \right|^2$$

We know from the Egorov’s theorem in Theorem 3.3 that $\hat{M}^t \hat{T}_{x_0} = \hat{T}_{M^t x_0} \hat{M}^t = \hat{T}_{x_t} \hat{M}^t$. Thus,

$$N \left| \langle \bar{c}_0 | \hat{T}_{-x} \hat{P}_\theta \hat{M}^t \hat{T}_{x_0} | \bar{c}_0 \rangle \right|^2 = N \left| \langle \bar{c}_0 | \hat{T}_{-x} \hat{T}_{x_t} \hat{P}_\theta \hat{M}^t | \bar{c}_0 \rangle \right|^2 = N \left| \langle x - x_t, \bar{c}_0 | t; \bar{c}_0, \theta \rangle \right|^2 = \mathcal{H}_{\bar{c}_0, \psi, \theta}(x - x_t).$$

Now if $\delta < 1/2$ in (4.1), then by (3.7), $\mathcal{H}_{\bar{c}_0, \psi, \theta}(x - x_t)$ is exponentially small unless $x - x_t \in B_2(o, \sqrt{\hbar} e^{\lambda T})$, that is, $x \in B_2(x_t, \sqrt{\hbar} e^{\lambda T}) \subset B_2(x_t, \sqrt{\hbar} e^{\lambda T})$.

Next we are concerned about the separation of the balls $B_2(x_t, \sqrt{\hbar} e^{\lambda T}) \subset \mathbb{T}^2$ for $t = 0, \ldots, T-1$. By Proposition 3.1, the prime closed orbit $\gamma$ of length $T$ lives on the lattice $\mathcal{L}_l$ of rational points with denominator

$$l = e^{\lambda T} + e^{-\lambda T} - 2 \leq e^{\lambda T}.$$

Since $\gamma = \{x_t\}_{t=0}^{T-1}$ is prime,

$$|x_t - x_s| \geq \frac{1}{l} \geq e^{-\lambda T}, \quad \text{if } t \neq s.$$

Thus, if $\delta < 1/4$ in (4.1), then for all $T \leq \delta T_E$,

$$|x_t - x_s| \geq e^{-\lambda T} \geq 2 \sqrt{\hbar} e^{\lambda T}.$$

This means that

$$B_2(x_t, \sqrt{\hbar} e^{\lambda T}) \cap B_2(x_s, \sqrt{\hbar} e^{\lambda T}) = \emptyset, \quad \text{if } t \neq s.$$
Combining with (3.8) and Corollary 3.5, we summarize the description of the Husimi function of $\Psi$ as follows.

**Theorem 4.1.** Let $0 < \delta < 1/4$. Then there is a constant $C_0 > 0$ depending only on $M$ and $\delta$ such that the following statement holds.

Suppose that $\phi \in \mathbb{R}$ and $\gamma = \{x_t\}_{t=0}^{T-1}$ is a closed prime orbit with length $|\gamma| = T \leq \delta T_E$. Construct $\Psi = \Psi_0^\gamma$ as in (4.2). Then

$$U_\Psi = \bigcup_{t=0}^{T-1} B_2 \left( x_t, C_0 \hbar^2 e^{\lambda T} \right)$$

is a disjoint union.

Moreover,

(i). If $x \in U_\Psi$, then there is a unique $t \in \{0, \ldots, T - 1\}$ such that

$$\mathcal{H}_{\tilde{c}_0, \Psi, \theta}(x) = \mathcal{H}_{\tilde{c}_0, t, \theta}(x - x_t) + O \left( e^{-\frac{1}{C_0 \hbar}} \right),$$

in which

$$\int_{\mathbb{T}^2} \mathcal{H}_{\tilde{c}_0, t, \theta}(x - x_t) \, dx = \int_{B_2 \left( \xi, \hbar^2 e^\gamma T \right)} \mathcal{H}_{\tilde{c}_0, t, \theta}(x) \, dx + O \left( e^{-\frac{1}{C_0 \hbar}} \right) = 1 + O \left( e^{-\frac{1}{C_0 \hbar}} \right),$$

(ii). If $x \notin U_\Psi$, then

$$\mathcal{H}_{\tilde{c}_0, \Psi, \theta}(x) = O \left( e^{-\frac{1}{C_0 \hbar}} \right).$$

The reminder estimates in (i) and (ii) are uniform for all $\phi \in \mathbb{R}$, closed prime orbits $\gamma$ with $|\gamma| \leq \delta T_E$, and $x \in \mathbb{T}^2$.

The proposition asserts that for all $T \leq \delta T_E$ such that $\delta < 1/4$, the Husimi function $\mathcal{H}_{\tilde{c}_0, \Psi, \theta}(x)$ of $\Psi$ is a direct sum of $\mathcal{H}_{\tilde{c}_0, t, \theta}(x - x_t)$ ($t = 0, \ldots, T - 1$) with disjoint essential supports in $B_2 \left( x_t, C_0 \hbar^2 e^\gamma T \right)$. The analysis in the rest of this section are based upon this assertion. For example, we immediately have the $L^2$ norm estimate of $\Psi$.

**Proposition 4.2** ($L^2$ norm estimate). Let $0 < \delta < 1/4$ and $T \leq \delta T_E$. Then there is a constant $C > 0$ depending only on $M$ and $\delta$ such that

$$\|\Psi\|_{L^2(T^1)}^2 = \langle \Psi | \Psi \rangle = T + O \left( e^{-\frac{1}{C_0 \hbar}} \right).$$

**Proof.** By (3.6) and Theorem 4.1 we have that

$$\langle \Psi | \Psi \rangle = \int_{\mathbb{T}^2} \mathcal{H}_{\tilde{c}_0, \Psi, \theta}(x) \, dx$$

$$= \sum_{t=0}^{T-1} \int_{B_2 \left( x_t, C_0 \hbar^2 e^\gamma T \right)} \mathcal{H}_{\tilde{c}_0, \Psi, \theta}(x) \, dx + O \left( e^{-\frac{1}{C_0 \hbar}} \right)$$

$$= \sum_{t=0}^{T-1} \int_{B_2 \left( x_t, C_0 \hbar^2 e^\gamma T \right)} \mathcal{H}_{\tilde{c}_0, t, \theta}(x - x_t) \, dx + O \left( e^{-\frac{1}{C_0 \hbar}} \right)$$

$$= T \int_{B_2 \left( \xi, \hbar^2 e^\gamma T \right)} \mathcal{H}_{\tilde{c}_0, t, \theta}(x) \, dx + O \left( e^{-\frac{1}{C_0 \hbar}} \right)$$

$$= T + O \left( e^{-\frac{1}{C_0 \hbar}} \right). \square$$
From the above $L^2$ norm estimate, we now see that

**Corollary 4.3.** Let $0 \leq \delta < 1/4$ and $T \leq \delta T_E$. Then the normalized state $|\Psi\rangle_n = |\Psi\rangle / \sqrt{\langle\Psi,\Psi\rangle}$ are quasimodes of order $O(1/\sqrt{T})$ with quasi-energy $e^{i\phi}$.

**Proof.** In the view of (1.4), we have that

$$\left\| \left( \hat{M} - e^{i\phi} \right) |\Psi\rangle \right\|_{L^2(T^2)} \leq 2 \left\| |\Psi\rangle \right\|_{L^2(T^2)} = O \left( \frac{1}{\sqrt{T}} \right),$$

that is,

$$\left\| \left( \hat{M} - e^{i\phi} \right) |\Psi\rangle \right\|_{L^2(T^2)} \leq \frac{2}{\sqrt{T} + O \left( e^{-\frac{1}{c_T}} \right)} = O \left( \frac{1}{\sqrt{T}} \right),$$

by the $L^2$ norm estimate of $\Psi$ in Proposition 4.2. \qed

### 4.2. Non-equidistribution of the quasimodes at small scales

Let $\Psi = \Psi_{\Phi}^{\gamma}$ be the quasimode that is associated with a closed prime orbit $\gamma = \{x_i\}_{t=0}^{T-1}$ of length $T = |\gamma|$. In this section, we show that $\Psi$ must display non-equidistribution at certain scales depending on $T$.

The next lemma is a direct consequence of the description of the Husimi function in Proposition 4.1.

**Lemma 4.4.** Let $0 < \delta < 1/4$ and $T \leq \delta T_E$. Suppose that $r = r(\hbar) \geq 2C_0\hbar^{\frac{1}{2}}e^{\lambda T}$ in Theorem 4.1. Then there is a constant $C > 0$ depending only on $M$ and $\delta$ such that

(i). \[
\left\{ \begin{array}{ll}
\langle \Psi | \hat{X}_{B_1(q,3)}^{\lambda} |\Psi\rangle \geq 1 + O \left( e^{-\frac{1}{c_T}} \right) & \text{if } B_1(q,r) \cap P_q(\gamma) \neq \emptyset, \\
\langle \Psi | \hat{X}_{B_1(q,r)}^{\lambda} |\Psi\rangle = O \left( e^{-\frac{1}{c_T}} \right) & \text{if } B_1(q,r) \cap P_q(\gamma) = \emptyset,
\end{array} \right.
\]

(ii). \[
\left\{ \begin{array}{ll}
\langle \Psi \hat{b}_{x,r}^{\lambda} |\Psi\rangle \geq 1 + O \left( e^{-\frac{1}{c_T}} \right) & \text{if } B_2(x,r) \cap \gamma \neq \emptyset, \\
\langle \Psi \hat{b}_{x,r}^{\lambda} |\Psi\rangle = O \left( e^{-\frac{1}{c_T}} \right) & \text{if } B_2(x,r) \cap \gamma = \emptyset,
\end{array} \right.
\]

Here, $b_{x,r}^{\pm}$ are given in (3.13) and $P_q(x) = q$ for $x = (q,p) \in T^2$ is the projection onto space of the position variable $q$.

**Proof.** (i). If $B_1(q,r) \cap P_q(\gamma) \neq \emptyset$, then there is $x_t = (q_t,p_t) \in \gamma$ such that $q_t \in B_1(q,r)$. Hence, $B_2(x_t, C_0\hbar^{\frac{1}{2}}e^{\lambda T}) \subset P_q^{-1}(B_1(q,3r))$ since $r \geq 2C_0\hbar^{\frac{1}{2}}e^{\lambda T}$. Therefore, by (3.12) and Theorem 4.1

$$\langle \Psi | \hat{X}_{B_1(q,3r)}^{\lambda} |\Psi\rangle = \int_{B_1(q,3r)} \int_{T^1} \mathcal{H}_{\tilde{c}_0,\Phi,\psi}(q',p') \, dp' \, dq'$$

$$\geq \int_{B_2(x_t, C_0\hbar^{\frac{1}{2}}e^{\lambda T})} \mathcal{H}_{\tilde{c}_0,\Phi,\psi}(q',p') \, dp' \, dq'$$

$$= 1 + O \left( e^{-\frac{1}{c_T}} \right).$$

On the other hand, if $B_1(q,r) \cap P_q(\gamma) = \emptyset$, then $B_2(x_t, C_0\hbar^{\frac{1}{2}}e^{\lambda T}) \cap P_q^{-1}(B_1(q,r/3)) = \emptyset$ for all $t = 0, ..., T - 1$. Hence, $U_{\psi} \cap P_q^{-1}(B_1(q,r/3)) = \emptyset$. By (3.12) and Theorem 4.1 again,

$$\langle \Psi | \hat{X}_{B_1(q,r/3)}^{\lambda} |\Psi\rangle = O \left( e^{-\frac{1}{c_T}} \right).$$
(ii). If \( B_2(x,r) \cap \gamma \neq \emptyset \), then there is \( x_t \in \gamma \) such that \( x_t \in B_2(x,r) \) since \( r \geq 2C_0\hbar^\sharp e^{\lambda T} \). Hence, \( B_2(x_t, C_0\hbar^\sharp e^{\lambda T}) \subset B_2(x,2r) \). Therefore, by \((3.13)\) and Theorem 4.1

\[
\langle \Psi | \hat{\beta}_{x,3r}^{-\text{aw}} | \Psi \rangle \geq \int_{B_2(x,2r)} \mathcal{H}_{\bar{c}_0, \Psi, \theta}(x') \, dx' \\
\geq \int_{B_2(x_t, C_0\hbar^\sharp e^{\lambda T})} \mathcal{H}_{\bar{c}_0, \Psi, \theta}(x') \, dx' \\
= 1 + O\left( e^{-\frac{1}{3\kappa}} \right).
\]

On the other hand, if \( B_1(q, r) \cap \gamma = \emptyset \), then \( B_2(x_t, C_0\hbar^\sharp e^{\lambda T}) \cap B_1(q, r/2) = \emptyset \) for all \( t = 0, \ldots, T - 1 \). Hence, \( U_\Psi \cap B_1(x, r/2) = \emptyset \). By \((3.13)\) and Theorem 4.1 again,

\[
\langle \Psi | \hat{\beta}_{x,r/3}^{+,\text{aw}} | \Psi \rangle \leq \int_{B_2(x,r/2)} \mathcal{H}_{\bar{c}_0, \Psi, \theta}(x') \, dx' = O\left( e^{-\frac{1}{3\kappa}} \right).
\]

Next we establish the non-equidistribution of \( \Psi \) at certain scales.

**Theorem 4.5.** Let \( 0 < \delta < 1/4 \) and \( T \leq \delta T_E = \delta |\log \hbar|/\lambda \). Then there are constant \( c_1, C > 0 \) depending only on \( M \) and \( \delta \) such that for any closed prime orbit \( \gamma \) with length \( T \), the associated quasimode \( \Psi \) in \((1.2)\) satisfies the following non-equidistribution conditions.

(i). For any \( r \in [2C_0\hbar^\sharp e^{\lambda T}, c_1T^{-1}] \), there are \( q_1, q_2 \in \mathbb{T}^1 \) such that

\[
\langle \Psi | \hat{\chi}_{B_1(q_1,r)}^{\text{aw}} | \Psi \rangle \geq 1 + O\left( e^{-\frac{1}{3\kappa}} \right) \quad \text{and} \quad \langle \Psi | \hat{\chi}_{B_1(q_2,r)}^{\text{aw}} | \Psi \rangle = O\left( e^{-\frac{1}{3\kappa}} \right).
\]

(ii). For any \( r \in [2C_0\hbar^\sharp e^{\lambda T}, c_1T^{-\frac{1}{2}}] \), there are \( x_1, x_2 \in \mathbb{T}^2 \) such that

\[
\langle \Psi | \hat{\beta}_{x_1,r}^{\text{aw}} | \Psi \rangle \geq 1 + O\left( e^{-\frac{1}{3\kappa}} \right) \quad \text{and} \quad \langle \Psi | \hat{\beta}_{x_2,r}^{\text{aw}} | \Psi \rangle = O\left( e^{-\frac{1}{3\kappa}} \right).
\]

**Proof.** (i). Given any closed prime orbit \( \gamma \) of length \( T \) and \( x_1 = (q_1, p_1) \in \gamma \),

\[
\langle \Psi | \hat{\chi}_{B_1(q,r)}^{\text{aw}} | \Psi \rangle \geq \int_{B_2(x_1, C_0\hbar^\sharp e^{\lambda T})} \mathcal{H}_{\bar{c}_0, \Psi, \theta}(x) \geq 1 + O\left( e^{-\frac{1}{3\kappa}} \right),
\]

using Theorem 4.1.

On the other hand, select a maximal family of disjoint balls \( \{B_1(q_k, 3r)\}_{k=1}^K \subset \mathbb{T}^1 \). Then we have that \( K \geq cr^{-1} \) for some absolute constant \( c > 0 \). Suppose that \( r \leq c/(2T) \) so \( K \geq cr^{-1} \geq 2T > T \). However, there are only \( T \) points on the orbit \( \gamma \). By the pigeon-hole principle, there is \( 1 \leq k \leq K \) such that \( B_1(q_k, 3r) \cap P_q(\gamma) = \emptyset \). In this case, by the previous lemma,

\[
\langle \Psi | \hat{\chi}_{B_1(q_k,r)}^{\text{aw}} | \Psi \rangle = O\left( e^{-\frac{1}{3\kappa}} \right).
\]

That is, (i) is proved by choosing \( c_1 = c/2 \).

(ii) can be proved in the same fashion so we omit the details. \( \square \)

4.3. **Semiclassical measures of the quasimodes.** Let \( \mu \) be any invariant probability measure of the cat map \( M \) on the torus \( \mathbb{T}^2 \). In this subsection, we construct a sequence of quasimodes for which the corresponding semiclassical measure is \( \mu \), and in addition, they satisfy the non-equidistribution conditions as in Theorem 1.2.
By Theorem 3.2, there is a sequence of closed prime orbits \( \{ \gamma_j \}_{j=1}^{\infty} \) such that the delta measures \( \mu_{\gamma_j} \to \mu \) weakly. Denote \( T_j = |\gamma_j| \). Then for all \( f \in C(\mathbb{T}^2) \),

\[
\int_{\mathbb{T}^2} f \, d\mu_{\gamma_j} = \frac{1}{T_j} \sum_{t=0}^{T_j-1} f(x_t^j) \to \int_{\mathbb{T}^2} f \, d\mu \quad \text{as} \quad j \to \infty, \tag{4.3}
\]

in which the closed prime orbit \( \gamma_j = \{x_t^j\}_{t=0}^{T_j-1} \).

If \( \{T_j\}_{j=1}^{\infty} \) is bounded, then \( \mu = \mu_{\gamma} \) is itself a delta measure on some closed prime orbit \( \gamma \). As mentioned in the introduction, this case has been treated in Faure-Nonnenmacher-De Bièvre [FNDH].

We therefore assume that \( \{T_j\}_{j=1}^{\infty} \) is unbounded. Without loss of generality, we assume that \( T_j \to \infty \) is increasing. (If not, then choose a subsequence of \( \{\gamma_j\}_{j=1}^{\infty} \) such that the lengths are increasing.)

Fix \( 0 < \delta < 1/4 \). Let

\[
N_j = \frac{1}{2\pi h_j} = \left[ \frac{e^{\frac{\lambda T_j}{\sqrt{\delta}}} \pi}{2\pi} \right].
\]

Then

\[
T_j \leq \frac{\delta |\log(2\pi N_j)|}{\lambda} = \frac{\delta |\log h_j|}{\lambda} \quad \text{and} \quad T_j = \frac{\delta |\log h_j|}{\lambda} + O(h_j). \tag{4.4}
\]

For any \( \phi_j \in \mathbb{R} \), construct the quantum states \( \Psi_{\phi_j}^{\gamma_j} \in \mathcal{H}_{N_j} \) associated with the prime closed orbit \( \gamma_j \) as in (4.2). In Subsection 4.1, we know that the normalized states \( |\Psi_{\phi_j}^{\gamma_j}\rangle_n \) are quasimodes of order \( O(T_j^{-1/2}) = O(|\log h_j|^{-1/2}) \).

Next, we show that the semiclassical measure induced by the normalized quasimodes

\[
|\psi_j\rangle = |\Psi_{\phi_j}^{\gamma_j}\rangle_n = \frac{|\Psi_{\phi_j}^{\gamma_j}\rangle}{\sqrt{\langle\Psi_{\phi_j}^{\gamma_j}|\Psi_{\phi_j}^{\gamma_j}\rangle}}
\]

coincides with the probability measure \( \mu \) on \( \mathbb{T}^2 \).

By Theorem 4.1 the Husimi function \( \mathcal{H}_{c_0,\Psi_{\phi_j}^{\gamma_j}}(x) \) of \( \Psi_{\phi_j}^{\gamma_j} \) is a direct sum of \( \mathcal{H}_{c_0,t,\theta_j}(x-x_t^j) \) \((t = 0, \ldots, T - 1)\) with disjoint essential support in \( B_2(x_t^j, C_0 h_j^{\frac{1}{2}} e^{\lambda T_j}) \), modulo exponential errors. Together with the \( L^2 \) norm estimate of \( \Psi_{\phi_j}^{\gamma_j} \) in Proposition 4.2, we have that

\[
\langle \psi_j | \hat{f}^{aw} | \psi_j \rangle = \frac{1}{\langle \Psi_{\phi_j}^{\gamma_j}|\Psi_{\phi_j}^{\gamma_j}\rangle} \langle \Psi_{\phi_j}^{\gamma_j}|\hat{f}^{aw}|\Psi_{\phi_j}^{\gamma_j}\rangle
\]

\[
= \frac{1}{T_j + O\left(e^{-\frac{1}{c_0 h_j}}\right)} \left( \int_{\mathbb{T}^2} \mathcal{H}_{c_0,\Psi_{\phi_j}^{\gamma_j}}(x) f(x) \, dx + O_f \left(h_j^{\frac{1}{2}}\right) \right)
\]

\[
= \frac{1}{T_j} \sum_{t=0}^{T_j-1} \int_{\mathbb{T}^2} \mathcal{H}_{c_0,t,\theta_j}(x-x_t^j) f(x) \, dx + O_f \left(h_j^{\frac{1}{2}}\right)
\]

\[
= \frac{1}{T_j} \sum_{t=0}^{T_j-1} \int_{B_2(x_t^j, C_0 h_j^{\frac{1}{2}} e^{\lambda T_j})} \mathcal{H}_{c_0,t,\theta_j}(x-x_t^j) f(x) \, dx + O_f \left(h_j^{\frac{1}{2}}\right)
\]
\[
\begin{align*}
\frac{1}{T_j} \sum_{t=0}^{T_j-1} \left( f(x_i^j) \int_{B_2} (\mathcal{H}_{c_0, t, \theta_j} (x - x_i^j) \, dx + O_f \left( \frac{1}{\epsilon} e^{\lambda T_j} \right)) + O_f \left( \frac{1}{\epsilon} \right) \right) \\
\frac{1}{T_j} \sum_{t=0}^{T_j-1} f(x_i^j) \int_{B_2} (\mathcal{H}_{c_0, t, \theta_j} (x) \, dx + O_f \left( \frac{1}{\epsilon} e^{\lambda T_j} \right)) \\
\frac{1}{T_j} \sum_{t=0}^{T_j-1} f(x_i^j) + O_f \left( \frac{1}{\epsilon} \right) \\
\to \int_{\mathbb{T}^2} f \, d\mu \quad \text{as } j \to \infty,
\end{align*}
\]

in which the last step follows from (4.3).

Finally, we use Theorem 4.5 to establish the non-equidistribution conditions in Theorem 1.2 for \( \{ \psi_j \}_{j=1}^{\infty} \). Let \( \varepsilon > 0 \).

For the non-equidistribution in the physical space, observe that if \( r_j = c_0 (\log N_j)^{-1} \) for \( c_0 \) small enough, then \( r_j \in [2C_0 \frac{1}{\epsilon} e^{\lambda T_j}, c_1 T_j^{-1}] \) so Theorem 4.5 applies. Note that

\[
\langle \Psi_{\phi_j}^\psi | \Psi_{\phi_j}^\psi \rangle = T_j + O \left( e^{-\frac{1}{C \phi_j^2}} \right)
\]

by the \( L^2 \) norm estimate of \( \Psi_{\phi_j}^\psi \) in Proposition 4.2. Then by Theorem 4.5 there are \( q_1, q_2 \in \mathbb{T}^1 \) such that

\[
\int_{B_1(q_1, r_j)} |\psi_j|^2 = \langle \psi_j | \tilde{X}_{B_1(q_1, r_j)} | \psi_j \rangle \geq \frac{1}{T_j} + O \left( e^{-\frac{1}{C \phi_j^2}} \right) \geq \frac{c}{\log N_j},
\]

where \( c > 0 \) depends only on \( M \) and \( \delta \) in the view of (4.3), such that

\[
\int_{B_1(q_2, r_j)} |\psi_j|^2 = \langle \psi_j | \tilde{X}_{B_1(q_2, r)} | \psi_j \rangle = O \left( e^{-\frac{1}{C \phi_j^2}} \right).
\]

However,

\[
\text{Vol}(B_1(q, r_j)) = 2r_j = \frac{2c_0}{\log N_j},
\]

Hence, there are \( c_0 > 0 \) and \( j_0 \in \mathbb{N} \) such that

\[
\frac{\int_{B_1(q_1, r_j)} |\psi_j|^2}{\text{Vol}(B_1(q, r_j))} \geq \frac{c}{2c_0} \geq \varepsilon^{-1} \quad \text{and} \quad \frac{\int_{B_1(q_2, r_j)} |\psi_j|^2}{\text{Vol}(B_1(q, r_j))} = O \left( e^{-\frac{1}{C \phi_j^2}} \right) \leq \varepsilon.
\]

The non-equidistribution of \( \{ \psi_j \}_{j=1}^{\infty} \) at small scale \( c_0 (\log N_j)^{-1/2} \) in the phase space \( \mathbb{T}^2 \) can be argued similarly so we omit the details.

**References**

[B1] S. Brooks, *On the entropy of quantum limits for 2-dimensional cat maps.* Comm. Math. Phys. 293 (2010), no. 1, 231–255.

[B2] S. Brooks, *Logarithmic-scale quasimodes that do not equidistribute.* Int. Math. Res. Not. IMRN 2015, no. 22, 11934–11960.

[BouDB] A. Bouzouina and S. De Bièvre, *Equipartition of the eigenfunctions of quantized ergodic maps on the torus,* Comm. Math. Phys. 178 (1996), no. 1, 83–105.

[BonDB] F. Bonechi and S. De Bièvre, *Controlling strong scarring for quantized ergodic toral automorphisms.* Duke Math. J. 117 (2003), no. 3, 571–587.

[FN] F. Faure and S. Nonnenmacher, *On the maximal scarring for quantum cat map eigenstates.* Comm. Math. Phys. 245 (2004), no. 1, 201–214.
S. Eswarathasan and S. Nonnenmacher, *Strong scarring of logarithmic quasimodes*. Ann. Inst. Fourier (Grenoble) 67 (2017), no. 6, 2307–2347.

S. Eswarathasan and L. Silberman, *Scarring of quasimodes on hyperbolic manifolds*. Nonlinearity 31 (2018), no. 1, 1–29.

F. Faure, S. Nonnenmacher, and S. De Bièvre, *Scarred eigenstates for quantum cat maps of minimal periods*, Comm. Math. Phys. 239 (2003), no. 3, 449–492.

X. Han, *Small scale quantum ergodicity in cat maps. I*. arXiv:1810.11949

A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*. Cambridge University Press, Cambridge, 1995.

G. Riviére, *Entropy of semiclassical measures for symplectic linear maps of the multidimensional torus*. Int. Math. Res. Not. IMRN 2011, no. 11, 2396–2443.

K. Sigmund, *Generic properties of invariant measures for Axiom A diffeomorphisms*. Invent. Math. 11 (1970), 99–109.

S. Zelditch, *Index and dynamics of quantized contact transformations*. Ann. Inst. Fourier (Grenoble) 47 (1997), no. 1, 305–363.

M. Zworski, *Semiclassical analysis*. American Mathematical Society, Providence, 2012.

E-mail address: xiaolong.han@csun.edu

Department of Mathematics, California State University, Northridge, CA 91330, USA