Smooth and singular multisoliton solutions of a modified Camassa–Holm equation with cubic nonlinearity and linear dispersion

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Abstract
We develop a direct method for solving a modified Camassa–Holm equation with cubic nonlinearity and linear dispersion under the rapidly decreasing boundary condition. We obtain a compact parametric representation for the multisoliton solutions and investigate their properties. We show that the introduction of a linear dispersive term exhibits various new features in the structure of solutions. In particular, we find the smooth solitons whose characteristics are different from those of the Camassa–Holm equation, as well as the novel types of singular solitons. A remarkable feature of the soliton solutions is that the underlying structure of the associated tau-functions is the same as that of a model equation for shallow-water waves introduced by Ablowitz et al (1974 Stud. Appl. Math. 53 249–315). Finally, we demonstrate that the short-wave limit of the soliton solutions recovers the soliton solutions of the short pulse equation which describes the propagation of ultra-short optical pulses in nonlinear media.

Keywords: modified Camassa–Holm equation, smooth soliton, peakon
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1. Introduction
In this paper, we consider the following modified Camassa–Holm (mCH) equation with cubic nonlinearity and linear dispersion

\[ m_t + 2\kappa^2 u_t + \left[ m(u^2 - u_x^2) \right]_x = 0, \quad m = u - u_{xx}. \]
where \( u = u(x, t) \) is a real-valued function of time \( t \) and a spatial variable \( x \), and the subscripts \( x \) and \( t \) appended to \( m \) and \( u \) denote partial differentiation. The positive parameter \( \kappa \) characterizes the magnitude of the linear dispersion. The mCH equation was introduced independently by several researchers using a novel procedure that generates new integrable systems from known integrable bi-Hamiltonian systems [1–3]. Actually, the method yields the CH equation [4] when applied to the Korteweg–de Vries (KdV) equation whereas it yields the mCH equation when applied to the modified KdV equation. In a physical context, it was derived from the two-dimensional Euler equation by using a singular perturbation method in which the variable \( u \) represents the velocity of fluid [5]. It also arises from an intrinsic invariant planar curve flow in Euclidean geometry [6]. The mCH equation admits a Lax pair representation and hence, in principle, it may be solved by means of the inverse scattering transform (IST) method [7].

The dispersionless version of the mCH equation (equation (1.1) with \( \kappa = 0 \)) has attracted much attention and has been studied extensively in recent years. There exists a variety of solutions depending on the boundary conditions. Specifically, under the vanishing boundary condition, it exhibits the usual peakons [6] whereas under the nonvanishing boundary condition, it supports smooth bright soliton solutions [8, 9]. See also [10] for the Lie algebraic approach for constructing solutions. A stability analysis reveals that the single peakon is stable for small perturbations in the energy space [11]. If, on the other hand, one includes the linear dispersion as shown in equation (1.1), then various new features appear in the structure of solutions. In particular, it will admit smooth soliton solutions which vanish at infinity, unlike the dispersionless mCH equation for which the nonexistence result for smooth traveling-wave solutions has been established under the vanishing boundary condition [5, 12]. Some qualitative results for the Cauchy problem of equation (1.1) were also reported in a later work; these are mainly concerned with the local well-posedness for the strong solutions and the blow-up phenomena [12]. The scattering problem for equation (1.1) was investigated recently by means of the IST and the time evolution of the scattering data was derived [13]. However, the resolution of the inverse problem remains open.

Another interesting aspect of the mCH equation is that it reduces to the short pulse (SP) equation

\[
v_{tx} = 2\kappa^2 v + \tfrac{1}{3} (v^3)_{xx}, \quad v = v(x, t),
\]

(1.2)
in the short-wave limit [6]. Equation (1.2) describes the propagation of ultra-short optical pulses in nonlinear media [14]. Soliton and periodic solutions to the equation are known and their properties have been explored in detail [15–17].

The main purpose of this paper is to construct soliton solutions of the mCH equation which decay rapidly at infinity and investigate their properties. We will show that it admits smooth soliton solutions like the bright solitons and breathers as well as the singular solitons. A direct method is employed to obtain solutions which mimics the construction of the soliton solutions of the generalized sine-Gordon (sG) equation [18, 19], the dispersionless mCH equation [9] and the Novikov equation [20].

This paper is organized as follows. In section 2, we transform the mCH equation to a system of nonlinear partial differential equations (PDEs) by a reciprocal transformation. We then apply a dependent variable transformation to reduce this system to a system of bilinear equations for the tau-functions. Subsequently, we analyze the latter system by means of the bilinear transformation method and present a compact parametric representation for the \( N \)-soliton solution of the mCH equation, where \( N \) is an arbitrary positive integer. Remarkably, we find that the underlying structure of the tau-functions constituting the \( N \)-soliton solution is essentially the same as that of the \( N \)-soliton solution of a model equation for shallow-water waves introduced by Ablowitz et al [21]. We remark that the same statement is true for
the tau-functions of the $N$-soliton solutions of the CH [22–25] and dispersionless mCH [9] equations. In section 3, we investigate the properties of the solutions focusing on the one- and two-soliton solutions. First, we show that the smooth soliton exists if the amplitude parameter of the soliton satisfies a certain condition. Furthermore, we obtain two types of the singular solitons. One has a symmetric profile and the other an antisymmetric profile. We analyze the limiting profiles of these singular solitons when the dispersion parameter $\kappa$ tends to zero and find that they do not recover the peakons constructed in [6]. Subsequently, we perform the asymptotic analysis of the smooth two-soliton solution and confirm its solitonic behavior. We find that the formula for the phase shift coincides precisely with that of the two-soliton solution of the CH equation. We also describe briefly the interaction of a smooth soliton and a singular symmetric soliton. We then construct the breather solution by specifying the complex conjugate pair for the amplitude parameters characterizing the smooth two-soliton solution. Finally, we address the general $N$-soliton solution and present the formula for the phase shift.

In section 4, we demonstrate that the $N$-soliton solution reduces to the $N$-soliton solution of the SP equation (1.2) under an appropriate limiting procedure. Section 5 is devoted to some concluding remarks. In the appendix, we prove the bilinear identities for the tau-functions associated with the $N$-soliton solution by means of mathematical induction.

2. Exact method of solution

In this section, we develop a systematic procedure for solving the mCH equation by means of the bilinear transformation method [26, 27]. Specifically, we seek soliton solutions which decay rapidly at infinity. We show that the system of bilinear equations deduced from the mCH equation is closely related to that of the generalized sG equation [18, 19]. This fact helps us to construct soliton solutions of the mCH equation.

2.1. Reciprocal transformation

First, we rewrite the mCH in the form of the conservation law

$$ r_t + \left[r(u^2 - u_x^2)\right]_x = 0, \quad r = \sqrt{m^2 + \kappa^2}. $$ (2.1)

This enables us to introduce the coordinate transformation $(x, t) \to (y, \tau)$ by

$$ dy = r \, dx - r(u^2 - u_x^2) \, dt, \quad d\tau = dt. $$ (2.2a)

The $x$ and $t$ derivatives transform as

$$ \frac{\partial}{\partial x} = r \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - r(u^2 - u_x^2) \frac{\partial}{\partial y}. $$ (2.2b)

Applying the transformation (2.2) to equation (2.1), we can recast it into the form

$$ r_\tau + 2r^2m u_y = 0. $$ (2.3)

It then follows from (2.2) that the variable $x = x(y, \tau)$ obeys a system of linear PDEs

$$ x_y = \frac{1}{r}, \quad x_\tau = u^2 - r^2 u_x^2. $$ (2.4a)

The system of equations (2.4) is integrable since its compatibility condition $x_{\tau y} = x_{y \tau}$ is assured by virtue of equation (2.3).

If we define the new angular variable $\phi = \phi(y, \tau)$ by

$$ m = \kappa \tan \phi, $$ (2.5)
then \( r \) from (2.1) can be represented in terms of \( \phi \) as
\[
    r = \frac{\kappa}{\cos \phi},
\]
with \(-\pi/2 < \phi < \pi/2\). Substitution of (2.5) and (2.6) into equation (2.3) gives
\[
    u_y + \frac{1}{2\kappa^2} (\sin \phi)_\tau = 0.
\]

Next, we express \( u \) in the form \( u = m + r^2 u_y + rr_y \) and rewrite this expression in terms of \( u \) and \( \phi \) with the aid of (2.5)–(2.7) and obtain the equation
\[
    \phi_{\tau_y} + 2u \cos \phi - 2\kappa \sin \phi = 0.
\]
Using the relation \( ru_y = -\phi_{\tau}/(2\kappa) \) which follows from (2.6) and (2.7), the linear system (2.4) can be put into the form
\[
    \begin{align*}
    x_y &= \frac{1}{\kappa} \cos \phi, \\
    x_\tau &= u^2 - \frac{1}{4\kappa^2} \phi^2; 
    \end{align*}
\]
Note that the \( x \) derivative of \( u \) is expressed simply as
\[
    u_x = -\frac{1}{2\kappa} \phi_y. 
\]

The system of nonlinear PDEs (2.7) and (2.8) for \( u \) and \( \phi \) is the starting point in the following analysis. Actually, the procedure for constructing solutions consists of two steps. First, solve this system under the boundary conditions \( u \to 0 \) and \( \phi \to 0 \) as \( |y| \to \infty \) to obtain \( u \) and \( \phi \) as functions of \( y \) and \( \tau \). Subsequently, integrate equation (2.9a) to give the mapping from \( y \) to \( x \)
\[
    x = \frac{y}{\kappa} + \frac{1}{\kappa} \int_{-\infty}^{y} (\cos \phi - 1) \, dy + d,
\]
where \( d \) is an integration constant depending generally on \( \tau \). If we differentiate (2.11) by \( \tau \) and use (2.9b) as well as (2.7) and (2.8), we confirm that \( d'(\tau) = 0 \) and so this constant is indeed independent of \( \tau \). Last, performing the integration with respect to \( y \) in (2.11), we obtain a parametric representation for the solution of the form \( u = u(y, \tau), x = x(y, \tau) \).

### 2.2. Bilinearization

Here, we perform the procedure for constructing soliton solutions as described in section 2.1. First, we bilinearize the system of PDEs (2.7) and (2.8) by introducing a dependent variable transformation. To this end, we seek solutions of the form
\[
    \begin{align*}
    u &= \frac{1}{2i\kappa} \left( \ln \frac{F}{\tilde{F}} \right)_\tau, \quad F = F(y, \tau), \quad \tilde{F} = \tilde{F}(y, \tau), \\
    \phi &= i \ln \frac{G}{\tilde{G}}, \quad G = G(y, \tau), \quad \tilde{G} = \tilde{G}(y, \tau),
    \end{align*}
\]
subjected to the boundary conditions \( F, \tilde{F}, G \) and \( \tilde{G} \to 1 \) as \( y \to -\infty \), where \( F, \tilde{F}, G \) and \( \tilde{G} \) are tau-functions. Substituting (2.12) into (2.7) and integrating the resultant expression by \( \tau \) under the boundary conditions specified above, we obtain
\[
    \frac{1}{2i\kappa} \left( \ln \frac{F}{\tilde{F}} \right)_y + \frac{1}{4ik^2} \left( \frac{G}{\tilde{G}} - \frac{\tilde{G}}{G} \right) = 0.
\]
If we impose an auxiliary condition
\[ \tilde{F} F = \tilde{G} G, \]  
(2.14)
for the tau-functions, then we can transform (2.13) into the bilinear equation
\[ D_y \tilde{F} \cdot F + \frac{1}{2\kappa} (G^2 - \tilde{G}^2) = 0, \]  
(2.15)
where the bilinear operators are defined by
\[ D^m D^n F \cdot G = (\partial_x - \partial_y)^m (\partial_x - \partial_y)^n F(y, \tau) G(y', \tau') |_{y=y', \tau=\tau}, \quad (m, n = 0, 1, 2, \ldots). \]  
(2.16)

To solve equation (2.8), we use the following relations which stem from (2.12) and the definition of the bilinear operators:
\[ \phi_{ty} = -\frac{i}{2G^2} D_x D_y G \cdot G + \frac{i}{2G^2} D_x D_y \tilde{G} \cdot \tilde{G}, \]  
(2.17a)
\[ u \cos \phi = \frac{1}{4\kappa} \frac{\left(G^2 + \tilde{G}^2\right) D_x \tilde{F} \cdot F}{\tilde{F} F \tilde{G} G}, \]  
(2.17b)
\[ \kappa \sin \phi = \frac{\kappa}{2i} \frac{G^2 - \tilde{G}^2}{G \tilde{G}}. \]  
(2.17c)
Substituting (2.17) into (2.8) and using (2.14), we can recast (2.8) into the form
\[ \tilde{G}^2 \left(D_x D_y G \cdot G + \frac{1}{\kappa} D_x \tilde{F} \cdot F + 2\kappa \tilde{G} G\right) = G^2 \left(D_x D_y \tilde{G} \cdot \tilde{G} + \frac{1}{\kappa} D_x F \cdot F + 2\kappa G \tilde{G}\right). \]  
(2.18)

We decouple the above equation as
\[ D_x D_x G \cdot G + \frac{1}{\kappa} D_x \tilde{F} \cdot F + 2\kappa \tilde{G} G = \mu G^2, \]  
(2.19a)
\[ D_x D_x \tilde{G} \cdot \tilde{G} + \frac{1}{\kappa} D_x F \cdot F + 2\kappa G \tilde{G} = \mu \tilde{G}^2, \]  
(2.19b)
by introducing a parameter \( \mu \) which depends generally on \( y \) and \( \tau \). To determine \( \mu \), we divide (2.19a) by \( G^2 \), take the limit \( y \rightarrow -\infty \) and use the boundary conditions for \( F, \tilde{F}, G \) and \( \tilde{G} \). We then find that \( \mu = 2\kappa \) which, substituted in (2.19), gives
\[ D_x D_x G \cdot G + \frac{1}{\kappa} D_x \tilde{F} \cdot F + 2\kappa (G - \tilde{G}) G = 0, \]  
(2.20a)
\[ D_x D_x \tilde{G} \cdot \tilde{G} + \frac{1}{\kappa} D_x F \cdot F + 2\kappa (G - \tilde{G}) \tilde{G} = 0. \]  
(2.20b)

Thus, the problem under consideration has been reduced to solving the system of bilinear equations (2.15) and (2.20) for \( F, \tilde{F}, G \) and \( \tilde{G} \) subject to condition (2.14). Fortunately, we have encountered a similar problem in the analysis of the generalized sG equation. Specifically, we recall that the bilinear equation (2.15) is essentially the same as (2.23a) of [18] if the asterisk is replaced by the tilde. Bearing this in mind, we put
\[ F = fg, \quad \tilde{F} = \tilde{f} \tilde{g}, \quad G = f \tilde{g}, \quad \tilde{G} = \tilde{f} g, \]  
(2.21)
where \( f, \tilde{f}, g \) and \( \tilde{g} \) are new tau-functions, and then impose the bilinear equations among these tau-functions
\[ D_x f \cdot \tilde{g} - \frac{1}{2\kappa} (f \tilde{g} - \tilde{f} g) = 0, \]  
(2.22a)
\[ D_x \tilde{f} \cdot g - \frac{1}{2\kappa} (\tilde{f} g - f \tilde{g}) = 0. \]  
(2.22b)
Obviously, the tau-functions $F, \tilde{F}, G$ and $\tilde{G}$ specified in (2.21) satisfy (2.14). If we substitute (2.21) into (2.15) and use (2.22), we can show that the bilinear equation (2.15) is satisfied automatically. Under these settings, the following proposition holds.

**Proposition 2.1.** Assume the relations (2.21) and (2.22). Then, the bilinear equations (2.20) reduce to the bilinear equations for $f, \tilde{f}, g$ and $\tilde{g}$

\[
D_\tau \frac{\partial f \cdot \tilde{g}}{\partial y} - \frac{1}{2\kappa} D_\tau f \cdot \tilde{g} - \frac{1}{2\kappa} D_\tau \tilde{f} \cdot g - \kappa (f \tilde{g} - \tilde{f} g) = 0, \tag{2.23a}
\]

\[
D_\tau \frac{\partial \tilde{f} \cdot g}{\partial y} - \frac{1}{2\kappa} D_\tau \tilde{f} \cdot g - \frac{1}{2\kappa} D_\tau f \cdot \tilde{g} - \kappa (\tilde{f} g - f \tilde{g}) = 0. \tag{2.23b}
\]

**Proof.** We first use (2.21) to derive the following identities which are verified easily by direct computation:

\[
D_\tau \tilde{F} \cdot F = f \tilde{g} D_\tau f \cdot \tilde{g} - \tilde{f} g D_\tau \tilde{f} \cdot g,
\]

\[
D_\tau D_y G \cdot G = 2 f \tilde{g} D_\tau D_y f \cdot \tilde{g} - 2 (D_\tau f \cdot \tilde{g} - D_\tau \tilde{f} \cdot g)(D_\tau f \cdot \tilde{g}).
\]

If we substitute these into (2.20a), use (2.22a) to eliminate a term $D_\tau f \cdot \tilde{g}$ and then divide the resultant expression by $2G$, we arrive at the bilinear equation (2.23a). The derivation of (2.23b) can be done in the same way. \qed

The system of bilinear equations (2.22) and (2.23) is more tractable than the original system (2.15) and (2.20) since the former system has no constraint on the tau-functions such as (2.14).

### 2.3. Parametric representation for the N-soliton solution

Here, we provide an explicit parametric representation for the $N$-soliton solution of the mCH equation. The following two theorems refer to the main results.

**Theorem 2.1.** The mCH equation (1.1) admits the parametric representation

\[
u(y, \tau) = \frac{1}{2\kappa \tau} \left( \ln \frac{\tilde{f} \tilde{g}}{fg} \right), \tag{2.24a}
\]

\[
\lambda (y, \tau) = \frac{y}{\kappa} + \ln \frac{\tilde{g}}{\tilde{f}}, \tag{2.24b}
\]

where the tau-functions $f, \tilde{f}, g$ and $\tilde{g}$ solve the system of bilinear equations (2.22) and (2.23) and $d$ is an arbitrary constant.

**Proof.** The expression (2.24a) is a consequence of (2.12a) and (2.21). To derive (2.24b), we deduce from (2.12b), (2.21) and (2.22) that

\[
\cos \phi = 1 + \kappa \left( \ln \frac{\tilde{g} \tilde{g}}{\tilde{f} f} \right) \tau.
\]

We substitute this relation into (2.11) and perform the integral with respect to $y$ under the boundary conditions $f, \tilde{f}, g, \tilde{g} \to 1$ as $y \to -\infty$ which are consistent with the boundary conditions for $F, \tilde{F}, G$ and $\tilde{G}$. Then, the expression (2.24b) follows immediately. The constancy of $d$ has already been demonstrated. \qed
Theorem 2.2. The tau-functions \( f, \tilde{f}, g \) and \( \tilde{g} \) constituting the \( N \)-soliton solution are given respectively by the expressions

\[
\begin{align*}
f &= \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j + \psi_j + \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k y_{jk} \right], \quad (2.25a) \\
\tilde{f} &= \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j - \psi_j - \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k y_{jk} \right], \quad (2.25b) \\
g &= \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j - \psi_j + \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k y_{jk} \right], \quad (2.25c) \\
\tilde{g} &= \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j - \psi_j - \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k y_{jk} \right]. \quad (2.25d)
\end{align*}
\]

where

\[
\begin{align*}
\xi_j &= k_j \left( y - \frac{2\gamma^3}{1 - (\kappa k_j)^2} \tau - y_{j0} \right), \quad (j = 1, 2, \ldots, N), \quad (2.25e) \\
e^{\nu_{ji}} &= \left( \frac{k_j - k_i}{k_j + k_i} \right)^2, \quad (j, i = 1, 2, \ldots, N; j \neq i), \quad (2.25f) \\
e^{-\psi_i} &= \sqrt{\frac{1 - \kappa k_j}{1 + \kappa k_j}}, \quad (j = 1, 2, \ldots, N). \quad (2.25g)
\end{align*}
\]

Here, \( k_j \) and \( y_{j0} \) are arbitrary complex parameters satisfying the conditions \( k_j \neq k_i \) for \( j \neq i \), \( \text{Re} k_j > 0 \) for all \( j \), and \( N \) is an arbitrary positive integer. The notation \( \sum_{\mu=0,1} \) implies the summation over all possible combinations of \( \mu_1 = 0, 1, \mu_2 = 0, 1, \ldots, \mu_N = 0, 1 \).

In the appendix, we show by means of mathematical induction that \( f, \tilde{f}, g \) and \( \tilde{g} \) from (2.25) solve the system of bilinear equations (2.22) and (2.23). The \( N \)-soliton solution given by (2.24) with (2.25) is characterized by the 2\( N \) complex parameters \( k_j \) and \( y_{j0} \) \( (j = 1, 2, \ldots, N) \). The parameters \( k_j \) determine the amplitude and the velocity of the solitons, whereas the parameters \( y_{j0} \) determine the position (or phase) of the solitons. We have imposed the conditions \( \text{Re} k_j > 0 \) \( (j = 1, 2, \ldots, N) \) to satisfy the boundary conditions for the tau-functions \( f, \tilde{f}, g \) and \( \tilde{g} \). In the following analysis, we consider the real solutions which are realized simply by imposing the conditions \( \tilde{f} = f^* \) and \( \tilde{g} = g^* \), where the asterisk denotes complex conjugate.

The parametric solution (2.24) would become a multi-valued (or singular) function of \( x \), as was the case for the SP [15] and generalized sG equations [18, 19], unless we impose certain conditions on the parameters \( k_j \) \( (j = 1, 2, \ldots, N) \). To establish a criterion for obtaining single-valued (or smooth) functions, we require that the mapping (2.2) is one-to-one which demands \( x_\gamma > 0 \). It follows from (2.9a), (2.12b) and (2.21) with \( f = f^*, \tilde{g} = g^* \) that this condition yields the inequality

\[
|\text{Re} f g^*| > |\text{Im} f g^*|. \quad (2.26)
\]

Although it is difficult in general to extract the condition for the parameters \( k_j \) from (2.26) for the \( N \)-soliton solution, we will give it explicitly in the case of the one-soliton solution.
Remark 2.1. The parametric representation for the N-soliton solution of the mCH equation has the structure similar to that of the generalized sG equation

\[ u_{xx} = (1 - \partial_x^2) \sin u, \quad u = u(x, t). \]  

(2.27)

Actually, it can be written in the form [18]

\[ u(y, \tau) = i \ln \frac{\tilde{g}}{f}, \quad \tau = \tau(y), \]  

(2.28a)

\[ x(y, \tau) = y + \tau + \ln \frac{\tilde{g}}{f} + y_0. \]  

(2.28b)

Here, the tau-functions \( f, \tilde{f}, g \) and \( \tilde{g} \) follow from (2.25) with the identification \( \kappa = 1, k_j = p_j \), and by replacing the \( \tau \) dependence as \( \tau/p_j^2 \) for \( j = 1, 2, \ldots, N \) so that \( \xi_j = p_j(y + (1/p_j^2)\tau - y_0) \), where \( p_j \) are complex parameters. See expressions (2.30), (2.35) and (2.36) as well as remark 2.5 in [18]. The implication of this interesting observation will be considered in a separate context.

Remark 2.2. By using an elementary theory of determinants, we can show that the tau-functions \( f, \tilde{f}, g \) and \( \tilde{g} \) from (2.25) solve the system of bilinear equations (2.22) and (2.23). To this end, we first shift the phase variables \( \xi_j \) as \( \xi_j \rightarrow \xi_j - \psi_j \) \( (j = 1, 2, \ldots, N) \) and then express the \( \tau \) and \( y \) derivatives of \( f \) and \( \tilde{f} \) as well as those of \( g \) and \( \tilde{g} \) in terms of the bordered determinants following the procedure developed for the \( N \)-soliton solution of the CH and generalized sG equations [18, 25]. Then, the proof of (2.22) follows from the result given in the appendix of [18]. Now, we differentiate (2.22a) (2.22b) by \( \tau \) and add the resultant expression to (2.23a) (2.23b) to obtain the alternative bilinear equations

\( f_{\tau y} - \frac{1}{2k} f_{\tau} - \kappa^2 \tilde{f}_{\tau} \tilde{g} - (f_{\tau} - \kappa^2 \tilde{f}) \tilde{g}_y + \frac{1}{2k} \tilde{f}_\tau \tilde{g} = 0, \)  

(2.29a)

\( \tilde{f}_{\tau y} - \frac{1}{2k} \tilde{f}_{\tau} - \kappa^2 \tilde{f}_{\tau} \tilde{g} - (\tilde{f}_{\tau} - \kappa^2 \tilde{f}) \tilde{g}_y + \frac{1}{2k} \tilde{f}_\tau \tilde{g} = 0. \)  

(2.29b)

With the aid of Jacobi’s formula as well as a few basic formulas for determinants, we can verify that equations (2.29) are satisfied with the tau-functions (2.25). The computation is performed straightforwardly but it is too lengthy to reproduce here.

Remark 2.3. The tau-functions \( f, \tilde{f}, g \) and \( \tilde{g} \) from (2.25) have the same structure as that of the \( N \)-soliton solution of a model equation for shallow-water waves [21]

\[ q_t + 2\kappa^2 q_y + 4\kappa^2 q_q - 2\kappa^2 q_q \int_0^\infty q_t dy - \kappa^2 q_{tyy} = 0, \quad q = q(y, \tau). \]  

(2.30)

To see this, we shift the phase variables \( \xi_j \) as \( \xi_j \rightarrow \xi_j - \psi_j - \pi i/2 \) \( (j = 1, 2, \ldots, N) \) in (2.25a), and then introduce the dependent variable transformation \( q = -2(\ln f)/\gamma \). It turns out that \( q \) solves equation (2.30) [21, 28]. We point out that the tau-function \( f \) thus obtained is the basic constituent for the \( N \)-soliton solutions of the CH [22–25] and dispersionless mCH [9] equations. Thus, at the level of the tau-functions, the \( N \)-soliton solutions for these equations have a common structure.

3. Properties of soliton solutions

In this section, we describe the properties of soliton solutions. We show that a variety of solutions arise from the parametric representation (2.24) with (2.25) in accordance with the values of the soliton parameters. First, we deal with the one-soliton solutions which include the
smooth soliton, symmetric singular soliton and antisymmetric singular soliton. Subsequently, we address the two-soliton solutions such as the smooth two-soliton and a smooth soliton and symmetric singular soliton pair, as well as a breather which stems from the smooth two-soliton solution as a degenerate case. Finally, we provide the formula for the phase shift of the \( N \)-soliton solution.

### 3.1. One-soliton solution

#### 3.1.1. Smooth soliton

The tau-functions \( f \) and \( g \) corresponding to the one-soliton solution are given by (2.25) with \( N = 1 \). Explicitly,

\[
f = 1 + i e^{\xi + \psi}, \tag{3.1a}
g = 1 + i e^{\xi - \psi}, \tag{3.1b}
\]

with

\[
\xi = k \left( y - \frac{2k^3}{1 - (\kappa k)^2} \tau - y_0 \right), \tag{3.1c}
\]

\[
e^{-\psi} = \sqrt{1 - \kappa k} \left( 1 + \kappa k \right), \tag{3.1d}
\]

where we have put \( \xi = \xi_1, \psi = \psi_1, k = k_1 \) and \( y_0 = y_{10} \) for simplicity. The boundary conditions for \( f \) and \( g \), i.e., \( f, g \to 1 \) as \( y \to -\infty \) require that the real part of \( k \) is positive. Since we are concerned with the real one-soliton solutions, we assume that all the parameters are real. The complex parameters will be introduced for constructing the breather solutions.

The parametric representation of the one-soliton solution follows by introducing (3.1) into (2.24). We write it in the form

\[
u = \frac{4\kappa^2 k}{\left( 1 - (\kappa k)^2 \right)^{3/2}} \cosh \xi \cosh 2\xi + \frac{1 + (\kappa k)^2}{1 - (\kappa k)^2}, \tag{3.2a}
\]

\[
X \equiv x - ct - x_0 = \frac{\xi}{\kappa k} + \ln \frac{1 - \kappa k \tanh \xi}{1 + \kappa k \tanh \xi}, \tag{3.2b}
\]

with

\[
c = \frac{2\kappa^2}{1 - (\kappa k)^2}, \tag{3.2c}
\]

where \( c \) is the velocity of the soliton in the \((x, t)\) coordinate system, \( x_0 = y_{10}/\kappa \) and the constant \( d \) in (2.24b) has been chosen appropriately such that \( \xi = 0 \) corresponds to \( X = 0 \).

The \( X \) derivative of \( u \) can be computed by using the relation \( u_X = u_\xi / X_\xi \), which gives

\[
u_X = -\frac{4\kappa^2 k^2}{\left( 1 - (\kappa k)^2 \right)^{3/2}} \sinh \xi \cosh 2\xi + \frac{1 + (\kappa k)^2}{1 - (\kappa k)^2} \tag{3.3}
\]

It can be checked by direct substitution that the parametric solution (3.2) indeed satisfies equation (1.1).

The smooth soliton solution is obtainable if one imposes a certain condition on the parameter \( k \) which can be derived from (2.26) and (3.1). Alternatively, we compute the quantity \( x_y \), directly from (3.2b) and obtain

\[
x_y = \frac{1}{\kappa} \left[ 1 - \frac{4(\kappa k)^2}{1 - (\kappa k)^2} \cosh 2\xi + \frac{1 + (\kappa k)^2}{1 - (\kappa k)^2} \right]. \tag{3.4}
\]
The condition $x_y > 0$ must hold for arbitrary value of $\xi$ to assure the smoothness of the solution. This leads to the inequality

$$0 < \kappa k < \frac{1}{\sqrt{2}}.$$  \hspace{1cm} (3.5)

The smooth one-soliton solution represents a bright soliton whose center position $x_c$ locates at $x_c = ct + x_0$ and has the amplitude $A$ given by

$$A = \sqrt{2(c - 2\kappa^2)}.$$  \hspace{1cm} (3.6)

This amplitude–velocity relation follows immediately by eliminating the parameter $k$ from the amplitude $A = u_{|\xi=0} = 2\kappa^2 k/(1 - (\kappa k)^2)^{1/2}$ and the velocity $c$ given by (3.2c). The inequality (3.5) restricts allowable values of $c$ and $A$. To be more specific, $2\kappa^2 < c < 4\kappa^2$, $0 < A < 2\kappa$.

Figure 1 depicts the profile of smooth solitons against the stationary variable $X$ defined by (3.2b) for three distinct values of $\kappa k$ with $\kappa = 1$. As the value of the parameter $\kappa k$ increases, the amplitude grows and the width narrows. When it tends to the upper limit $\kappa k = 1/\sqrt{2}$ of the inequality (3.5), then the smoothness of the solution is lost at the crest of the soliton. To see this in more detail, we expand $u$ and $X$ near the crest $X = 0$. Specifically, when $\kappa k = 1/\sqrt{2}$, the leading terms of the expansions read

$$u = 2\kappa \left[ 1 - \frac{1}{8} \xi^4 + O(\xi^6) \right],$$ \hspace{1cm} (3.7a)

$$X = \frac{1}{3\sqrt{2}} \xi^3 + O(\xi^5).$$ \hspace{1cm} (3.7b)

By eliminating the variable $\xi$ from (3.7), we find that the profile $u$ of the soliton near the crest $X = 0$ is approximated by

$$u = 2\kappa \left[ 1 - \frac{(3\sqrt{2})^{2/3}}{8} X^{4/3} + O(X^2) \right].$$ \hspace{1cm} (3.8)

We can see from (3.8) that the $n$th derivative of $u$ with respect to $X$ does not exist for $n \geq 2$. This novel feature of the solution is striking contrast to the usual peakon which has a discontinuous first derivative at the crest. However, the solitonic nature of the peaked solution presented here must be justified after its stability has been established.

A similar structure to this solution has been observed in the analysis of the smooth soliton solution of the dispersionless mCH equation where a constant background field plays the role of the parameter $\kappa$ [9].
3.1.2. Symmetric singular soliton. The singular solitons with a symmetric profile exist in the range of the parameter $1/\sqrt{2} < \kappa k < 1$. In figure 2, the profile of symmetric singular solitons is depicted for three distinct values of $\kappa k$ with $\kappa = 1$. They exhibit two crests and become three-valued functions of $X$ in the range $-X_0 < X < 0$, $0 < X < X_0$, where $X_0$ will be specified below. At the origin $X = 0$, $u$ takes two values $u_1$ and $u_2$ ($0 < u_1 < u_2$). We can show that if $1/\sqrt{2} < \kappa k < 1$, then the coordinate $X = X(\xi)$ from (3.2b) has three zeros $\xi = 0, \pm \xi_1$ where the value of $\xi_1$ can be computed numerically. Consequently, $u_1 = u|_{\xi = \pm \xi_1}$ and $u_2 = u$ with $A$ being given by (3.6). The maximum value $u_{\text{max}}$ of the amplitude is attained at $X = \pm X_0$ ($\xi = \mp \xi_0$), where $\xi_0 = \tanh^{-1}[\sqrt{2}(\kappa k)^2 - 1]/\kappa k]$ and

$$X_0 = -\frac{1}{2\kappa k} \left[ \frac{\kappa k + \sqrt{2(\kappa k)^2 - 1}}{\kappa k} + \frac{1 + \sqrt{2(\kappa k)^2 - 1}}{1 - \sqrt{2(\kappa k)^2 - 1}} \right].$$

Then, $u_{\text{max}} = c/2k$. The slope $u_X$ at $X = \pm X_0$ is evaluated simply by putting $\xi = \mp \xi_0$ in (3.3), which gives $\pm \sqrt{(2(\kappa k)^2 - 1)/2(1 - (\kappa k)^2)}$.

Finally, it is instructive to take the small dispersion limit $\kappa \to 0$ with $c$ being fixed. This limiting procedure is called the peakon limit. It has been used successfully to produce the peakons from the smooth solitons of the CH [30, 31] and Degasperis–Procesi (DP) [32] equations. In view of (3.2e), we must take the limit $\kappa k \to 1$ simultaneously. It turns out that $u_{\text{max}} \to \infty$, $u_X \to \pm \infty$ ($X_0 \to \pm \infty$), showing that the two crests located at $X = \pm X_0$ tend to $\pm \infty$ and their profile evolves into cusp. Note that, in this limit, $u_1 \to \sqrt{c/2}$ and $u_2 \to \sqrt{2c}$. We recall that a similar singular solution has been obtained for the dispersionless mCH equation [9]. Thus, unlike the W-shaped singular soliton of the Novikov equation which reduces to a peakon in an appropriate limit (see figure 4 of [20]), the symmetric singular soliton under consideration does not recover the peakon obtained in [6].

3.1.3. Antisymmetric singular soliton. A novel type of singular soliton appears in the parameter range $\kappa k > 1$. Such singular solitons can be constructed from the smooth solitons if one shifts the phase variable $\xi$ by $\xi \to \xi \pm \pi i/2$, or equivalently replaces the phase constants $x_0$ and $y_0$ by $x_0 - \pi i/(2\kappa k)$ and $y_0 - \pi i/(2\kappa k)$, respectively. In this setting, $\cosh \xi \to i \sinh \xi$, $\cosh 2\xi \to -\cosh 2\xi$ and $\tanh \xi \to \coth \xi$, giving rise to the parametric representation of the singular soliton solution

$$u = \frac{4\kappa^2 k}{\{((\kappa k)^2 - 1)^{3/2} \cosh 2\xi + \frac{(\kappa k)^2 + 1}{(\kappa k)^2 - 1}\}} \sinh \xi$$

(3.9a)

![Figure 2. The profile of symmetric singular solitons with $\kappa = 1$: $\kappa k = 0.86$ (dashed curve), $\kappa k = 0.90$ (dotted curve), $\kappa k = 0.94$ (solid curve).](image-url)
Figure 3. The profile of antisymmetric singular solitons with $\kappa = 1$: $\kappa k = 1.1$ (dashed curve), $\kappa k = 1.2$ (dotted curve), $\kappa k = 1.5$ (solid curve).

\[ X \equiv x - ct - x_0 = \frac{\xi}{\kappa k} + \ln \frac{\kappa k - \tanh \frac{\xi}{\kappa k} + \tanh \frac{\xi}{\kappa k}}{\kappa k + \tanh \frac{\xi}{\kappa k}}. \]  

(3.9b)

The expression (3.9) becomes an antisymmetric function of $X$ as evidenced from the relations $u(-\xi) = -u(\xi)$ and $X(-\xi) = -X(\xi)$. It follows from (3.9) that

\[ u X = -\frac{4\kappa^3 k^2}{(\kappa k)^2 - 1} \left( \frac{1}{2} \cosh \frac{\xi}{\kappa k} \cosh \frac{2\xi}{\kappa k} + \frac{(\kappa k)^2 + 1}{(\kappa k)^2 - 1} \right). \]  

(3.10)

Thus, $u_X$ takes only negative and finite values, showing that contrary to the symmetric singular soliton, the profile always exhibits negative slope.

Figure 3 depicts the profile of antisymmetric singular solitons for three distinct values of $\kappa k$ with $\kappa = 1$. We can see from (3.9a) that $u$ attains the maximum (minimum) value $\kappa/\{(\kappa k)^2 - 1\}^{1/2}$ at $\xi = -\xi_2$ ($\xi = \xi_2$), where $\xi_2 = \tanh^{-1}(\kappa k/\sqrt{2(\kappa k)^2 - 1})$. The value of $X$ corresponding to $\xi_2$, which is denoted by $X_2$, is found from (3.9b) as

\[ X_2 = -\frac{1}{2k} \ln \frac{\sqrt{2(\kappa k)^2 - 1} + \kappa k}{\sqrt{2(\kappa k)^2 - 1} - \kappa k} + \ln \frac{\sqrt{2(\kappa k)^2 - 1} + 1}{\sqrt{2(\kappa k)^2 - 1} - 1}. \]  

(3.11)

In the interval $-X_2 < X < X_2$, $u$ becomes a three-valued function of $X$. In the limit of $\kappa k \to 1$, the positions $\pm X_2$ of the two crests move to infinity and their amplitudes grow indefinitely.

We recall that the parametric solution (3.9) has been obtained recently in classifying the traveling-wave solutions of the mCH equation [29]. However, the detailed analysis of the solution is presented here for the first time.

### 3.2. Two-soliton solution

The tau-functions (2.25) with $N = 2$ for the two-soliton solutions can be written in the form

\[ f = 1 + i (e^{i\xi_1 + \psi_1} + e^{i\xi_2 + \psi_2}) - \frac{k_1 - k_2}{k_1 + k_2} e^{i(\xi_1 + \xi_2 + \psi_1 + \psi_2)}, \]  

(3.12a)

\[ g = 1 + i (e^{i\xi_1 - \psi_1} + e^{i\xi_2 - \psi_2}) - \frac{k_1 - k_2}{k_1 + k_2} e^{i(\xi_1 + \xi_2 - \psi_1 - \psi_2)}. \]  

(3.12b)

The parametric solution (2.24) with (3.12) exhibits a variety of solutions describing the interaction of two solitons. Here, we consider the three types of solutions which are composed
3.2.1. Smooth soliton–smooth soliton. The smooth two-soliton solution is obtained if one chooses the real parameters $\kappa_j$ subjected to the conditions $0 < \kappa_j < 1/\sqrt{2}$, ($j = 1, 2$). As already pointed out in remark 2.1, the structure of the tau-functions (3.12) is the same as that of the two-soliton solutions of the generalized sG equation except for the $\tau$ dependence. The asymptotic analysis of the solution mimics that of the generalized sG equation. Hence, we omit it and outline the results (see section 3.2 of [18] for details).

Figure 4 illustrates the interaction of two smooth solitons for four distinct values of $t$. This figure shows clearly the solitonic nature of the solution. The asymptotic state of the solution for large time is represented by a superposition of two smooth solitons, each of which has the profile given by (3.2). The net effect of the interaction is the phase shift. Let $\Delta_1$ and $\Delta_2$ be the phase shifts of the large and small solitons, respectively. It then follows from the expressions (3.6) of [18] with $p_1 = \kappa k_1$ and $p_2 = \kappa k_2$ ($0 < \kappa k_2 < \kappa k_1 < 1/\sqrt{2}$) that

$$
\Delta_1 = -\frac{1}{\kappa k_1} \ln \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2 - \ln \left( \frac{1 + \kappa k_2}{1 - \kappa k_2} \right)^2,
$$

(3.13a)

$$
\Delta_2 = \frac{1}{\kappa k_2} \ln \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2 + \ln \left( \frac{1 + \kappa k_1}{1 - \kappa k_1} \right)^2.
$$

(3.13b)

In the illustrated example, the amplitude of the large soliton is 1.96 whereas that of the small soliton is 1.16. The phase shifts are evaluated by the formulas (3.13), giving $\Delta_1 = 2.92$ and $\Delta_2 = -3.70$.

The first terms of (3.13) coincide with the corresponding formulas for the KdV and sG equations whereas the second terms originate from the coordinate transformation (2.2). Recall that the above formulas for the phase shifts are exactly the same as those for the two-soliton solution of the CH equation (see, for example, [25]). We first summarize the features of the phase shift of the CH two-soliton solution and then proceed to the mCH case.

In the CH case, the allowable values of the parameters are restricted by the inequality $0 < \kappa k_2 < \kappa k_1 < 1$. The detailed analysis reveals that the large soliton is always shifted forwards ($\Delta_1 > 0$) whereas the sign of $\Delta_2$ depends on the values of $\kappa k_1$ and $\kappa k_2$.
There arise three cases for the allowable values of the phase shifts: (i) $\Delta_2 < 0 \leq \Delta_1$, (ii) $0 < \Delta_2 \leq \Delta_1$, (iii) $0 < \Delta_1 \leq \Delta_2$. We can show that if $\kappa k_1 < \kappa k_{1c}$, then $\Delta_2$ always takes a negative value (case (i) above), where $\kappa k_{1c} = 0.8336$ is a root of the transcendental equation $-4/(\kappa k_1) + \ln[(1 + \kappa k_1)^2/(1 - \kappa k_1)^2] = 0$ which follows from (3.13b) by taking the limit $\kappa k_2 \rightarrow 0$. The cases (ii) and (iii) exhibit quite peculiar characteristics which have never been observed in the interaction process of the KdV and sG solitons.

Figure 5 plots the critical curves in the $(p, q)$ plane with $p = \kappa k_1$ and $q = \kappa k_2$ which separate the above three cases. Note that the allowable values of $p$ and $q$ are restricted by the inequality $0 < q < p < 1$. For any pair $(p, q)$ lying in the left region of the curve $C$, the phase shifts satisfy the inequality indicated in case (i). Notice that the curve $C$ starts from the point $(\kappa k_{1c}, 0)$, increases monotonically and ends at the point $(1, 1)$. The phase shift $\Delta_2$ for the small soliton becomes zero along this curve. The narrow region surrounded by the curve $C$ and the curve separating the white and gray regions corresponds to case (ii) whereas both the gray and black regions correspond to case (iii). In particular, on the boundary separating the gray and black regions, the relation $\Delta_2 - \Delta_1 = 1$ holds. See also figure 2 of [32] which depicts an analogous diagram for the phase shift of the two-soliton solution of the DP equation.

Now, in the mCH case considered here, since $\kappa k_1 < 1/\sqrt{2} < 0.8336$, we see from figure 5 that $\Delta_2 < 0$, implying that the small soliton is always shifted backward after the interaction of solitons. Thus, the behavior of smooth solitons is similar to that of the KdV and sG solitons despite the different structure of the formulas for the phase shifts.

3.2.2. Smooth soliton–symmetric singular soliton. The two-soliton solution composed of a smooth soliton and a symmetric singular soliton is obtained if one sets the parameters so that the inequalities $1/\sqrt{2} < \kappa k_1 < 1$ and $0 < \kappa k_2 < 1/\sqrt{2}$ are satisfied. Figure 6 illustrates the interaction process for four distinct values of $t$. The solitonic feature of the solution is apparent from the figure. Actually, we can see that the singular soliton overtakes and emerges ahead of the smooth soliton. After the interaction, both solitons appear without changing their profiles and suffer only the phase shifts, which can be evaluated by making use of (3.13). The characteristic of the interaction process differs from that of the smooth two-soliton case.
Figure 6. The interaction between a smooth soliton \((k_2 = 0.6, y_{20} = 0)\) and a symmetric singular soliton \((k_1 = 0.9, y_{10} = 0)\). The parameter \(\kappa\) is set to 1 for both solitons.

Indeed, if \(\kappa k_1 < \kappa k_{1c}\), then \(\Delta_2 < 0\) whereas if \(\kappa k_{1c} < \kappa k_1\), then the allowable value of \(\Delta_2\) is classified into either case (ii) or case (iii) mentioned above in accordance the value of \(\kappa k_2\). In the present example, \(\Delta_1 = 0.804\) and \(\Delta_2 = 0.524\) (case (ii)).

3.2.3. Breather. The breather solution has a localized structure which oscillates with time and decays exponentially in space. In the sG model, it can be interpreted as the bound state of a kink and an antikink. We show that in the mCH equation, the corresponding breather solution is produced from the two-soliton solution by specifying the complex conjugate pair for the parameters.

Now, we put

\[
k_1 = a + ib, \quad k_2 = a - ib (= k_1^*), \quad a > 0, \quad (3.14a)
\]

\[
y_{10} = \eta + i\delta, \quad y_{20} = \eta - i\delta (= y_{10}^*). \quad (3.14b)
\]

Then, the tau-functions \(f\) and \(g\) from (3.12) reduce to

\[
f = 1 + i(e^{\xi_1 + \psi_1} + e^{\xi_1^* + \psi_1^*}) + \left(\frac{b}{a}\right)^2 e^{\xi_1 + \xi_1^* + \psi_1 + \psi_1^*}, \quad (3.15a)
\]

\[
g = 1 + i(e^{\xi_2 - \psi_1} + e^{\xi_2^* - \psi_1^*}) + \left(\frac{b}{a}\right)^2 e^{\xi_2 + \xi_2^* - \psi_1 - \psi_1^*}, \quad (3.15b)
\]

where

\[
\xi_1 = \theta + i\chi, \quad (3.16a)
\]

\[
\theta = a(y - v_1 \tau) - a\eta + b\delta, \quad \psi_1 = \frac{2\kappa^3[1 - \kappa^2(a^2 + b^2)]}{(1 - \kappa^2(a^2 - b^2))^2 + 4\kappa^4(ab)^2}, \quad (3.16b)
\]

\[
\chi = b(y - v_2 \tau) - b\eta - a\delta, \quad \psi_2 = \frac{2\kappa^3[1 + \kappa^2(a^2 + b^2)]}{(1 - \kappa^2(a^2 - b^2))^2 + 4\kappa^4(ab)^2}, \quad (3.16c)
\]
Figure 7. The time evolution of a breather solution with the parameters $\kappa = 1$, $a = 0.2$, $b = 0.5$ and $\eta = \delta = 0$.

$$e^{-\psi_1} = \sqrt{\frac{1 - \kappa^2(a^2 - b^2) + 2i\kappa^2ab}{(1 + \kappa a)^2 + (\kappa b)^2}} = \alpha e^{-i\vartheta}. \quad (3.16d)$$

In terms of the new variables defined by (3.16), the tau-functions $f$ and $g$ can be rewritten as

$$f = 1 + 2i\alpha^{-1} e^{i\vartheta} \cos(\chi + \beta) + \alpha^{-2} \left(\frac{b}{a}\right)^2 e^{2\vartheta}, \quad (3.17a)$$

$$g = 1 + 2i\alpha e^{i\vartheta} \cos(\chi - \beta) + \alpha^2 \left(\frac{b}{a}\right)^2 e^{2\vartheta}. \quad (3.17b)$$

The smooth breather solutions are produced if we choose the parameters $a$ and $b$ such that condition (2.26) is satisfied. As in the case of the corresponding problem for the generalized sG equation, it is not easy to find the allowable values of $a$ and $b$ analytically. However, an inspection reveals that if the ratio $a/|b|$ is sufficiently small compared to 1, then the regularity of the solution would be assured. Figure 7 depicts the time evolution of a smooth breather solution for four distinct values of $t$. The breather propagates to the right while changing its profile and whose characteristic is similar to that of the breather solution of the generalized sG equation [18].

### 3.3. N-soliton solution

The general multisoliton solutions are classified in accordance values of the parameters $\kappa k_j$ ($j = 1, 2, \ldots, N$). Actually, the constituents of the solutions are composed of the smooth solitons, symmetric singular solitons and antisymmetric singular solitons as well as the breathers. The asymptotic analysis of the general $N$-soliton solution will not be performed here since the similar analysis has been done for the $N$-soliton solution of the generalized sG equation [18]. Here, we provide the formula for the phase shift. To this end, let us order the magnitude of the parameters $\kappa k_j$ as $0 < \kappa k_N < \kappa k_{N-1} < \cdots < \kappa k_1$ so that the velocity of each soliton satisfies the condition $0 < c_N < c_{N-1} < \cdots < c_1$ by (3.2c), where $c_j = 2\kappa^2/[1 - (\kappa k_j)^2]$ ($j = 1, 2, \ldots, N$). Then, the phase shift of the $n$th soliton is given by the formula
\[ \Delta_n = \frac{1}{k_k} \left\{ \sum_{j=1}^{n-1} \ln \left( \frac{k_n - k_j}{k_n + k_j} \right)^2 - \sum_{j=n+1}^{N} \ln \left( \frac{k_n - k_j}{k_n + k_j} \right)^2 \right\} \\
\quad + \sum_{j=1}^{n-1} \ln \left( \frac{1 + kk_j}{1 - kk_j} \right)^2 - \sum_{j=n+1}^{N} \ln \left( \frac{1 + kk_j}{1 - kk_j} \right)^2, \quad (n = 1, 2, \ldots, N). \tag{3.18} \]

See formula (3.26) of [18] with the identification \( p_j = kk_j \) \( (j = 1, 2, \ldots, N) \). If we restrict the largest parameter \( k k_1 \) as \( kk_1 < 1/\sqrt{2} \), then the above formula gives the phase shift for \( N \) interacting smooth solitons. If \( kk_{N-m+1} < 1/\sqrt{2} < kk_{N-m} < kk_{N-m-1} < \cdots < kk_1 < 1 \), for example, then the asymptotic state of the solution for large time is composed of \( m \) smooth solitons and \( N - m \) symmetric singular solitons. In this specific case, formula (3.18) gives the phase shift of the smooth solitons for \( 1 \leq n \leq m \) and that of the singular symmetric solitons for \( m + 1 \leq n \leq N \), respectively. It is also possible to construct the pure multibreather solutions as well as the multisoliton–multibreather solutions following the recipe described in section 3.2.3. See section 3.3 of [18] for details.

4. Reduction to the short pulse equation

The SP equation (1.2) was obtained for the first time in an attempt to construct integrable differential equations associated with pseudospherical surfaces [33]. Later, it was proposed as an alternative model to the cubic nonlinear Schrödinger (NLS) equation [14]. In the context of self-focusing of ultra-short pulses in nonlinear media, its validity would be beyond the scope of applicability of the NLS equation which has been derived on the assumption of a slowly varying envelope approximation. See the recent review articles [17, 34] for the SP equation and related topics. Here, we demonstrate that the SP equation, its \( N \)-soliton solution and formula for the phase shift are all recovered from the mCH equation under an appropriate scaling limit, or the short-wave limit. Note that the similar limiting procedure has been performed in [18, 19] for the generalized sG equation, leading to the same results.

4.1. Short-wave limit of the mCH equation

The mCH equation is reducible to the SP equation in the short-wave limit [6]. Here, we demonstrate it for completeness. We recall that the similar limiting procedure has been undertaken for the CH and DP equations [35].

First, we introduce the scaling variables in accordance with the relations
\[
\begin{align*}
  u &= \epsilon^2 \bar{u}, \quad x = \epsilon \bar{x}, \quad y = \epsilon \bar{y}, \quad t = \frac{\bar{t}}{\epsilon}, \quad \tau = \frac{\bar{\tau}}{\epsilon}, \\
  k_j &= \frac{\bar{k}_j}{\epsilon}, \quad y_j = \epsilon \bar{y}_j \quad (j = 1, 2, \ldots, N), \quad d = \epsilon \bar{d}.
\end{align*} \tag{4.1} \]

where \( \epsilon \) is a small parameter. Rewriting the derivatives in terms of the new variables, the mCH equation (1.1) is recast to
\[
\epsilon (\epsilon^2 \bar{u} - \bar{u}_{i\bar{x}})_t + 2\kappa^2 \epsilon \bar{u}_{i\bar{x}} + \frac{1}{\epsilon} [ (\epsilon^2 \bar{u} - \bar{u}_{i\bar{x}}) (\epsilon^4 \bar{u}^2 - \bar{u}_{i\bar{x}}^2) ]_t = 0. \tag{4.2} \]

If we expand \( \bar{u} \) in powers of \( \epsilon \) as \( \bar{u} = \bar{u}_0 + \epsilon \bar{u}_1 + \cdots \) and substitute it into (4.2), we obtain, at the the leading order of the expansion, the equation for \( \bar{u}_0 
\]
\[
- \bar{u}_{0,i\bar{x}} + 2\kappa^2 \epsilon \bar{u}_{0,i\bar{x}} + [ \bar{u}_{0,\bar{x}i} \bar{u}_{0,i\bar{x}} ]_t = 0. \tag{4.3} \]

If we put \( \bar{v} = \bar{u}_0, \bar{v}_i \) in (4.3), we arrive, after dropping the bar attached to the variables \( t, x \) and \( v \), at the SP equation (1.2).
4.2. Short-wave limit of the N-soliton solution

First, shift the phase variables $\xi_j$ as $\xi_j \rightarrow \xi_j - \psi_j$ ($j = 1, 2, \ldots, N$) and then take the limit $\epsilon \rightarrow 0$. The tau-functions $f$ and $\tilde{f}$ from (2.25a) and (2.25b), respectively have the limiting forms, which are given by

$$ f \rightarrow \tilde{f} = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j + \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k \overline{\gamma}_{jk} \right], $$

(4.4a)

$$ \tilde{f} \rightarrow \tilde{\tilde{f}} = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j - \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k \overline{\gamma}_{jk} \right], $$

(4.4b)

with

$$ \xi_j = \overline{k}_j \left( \overline{\tilde{\tilde{f}}} + \frac{2\kappa}{k_j^2} \overline{\tilde{\tilde{f}}} \right), \quad (j = 1, 2, \ldots, N), $$

(4.4c)

$$ e^{\tilde{\tilde{y}}_{il}} = \left( \frac{\overline{k}_j - \overline{k}_l}{\overline{k}_j + \overline{k}_l} \right)^2, \quad (j, l = 1, 2, \ldots, N; j \neq l). $$

(4.4d)

To perform the limiting procedure for the tau-function $g$, we need to retain terms up to order $\epsilon$. Using the expansion

$$ \exp \left( -2 \sum_{j=1}^{N} \mu_j \psi_j \right) = \prod_{j=1}^{N} \left( \frac{1 - \kappa k_j}{1 + \kappa k_j} \right)^{\mu_j}, $$

$$ = \exp \left( -\pi i \sum_{j=1}^{N} \mu_j \right) \left( 1 - 2\epsilon \sum_{j=1}^{N} \frac{\kappa k_j}{\kappa k_j} \right) + O(\epsilon^2), $$

(4.5)

the tau-function $g$ from (2.25c) can be developed in powers of $\epsilon$ as

$$ g = \sum_{\mu=0,1} \left( 1 - 2\epsilon \sum_{j=1}^{N} \frac{\kappa k_j}{k_j} \right) \exp \left[ \sum_{j=1}^{N} \mu_j \left( \xi_j - \frac{\pi}{2} i \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k \overline{\gamma}_{jk} \right] + O(\epsilon^2), $$

$$ = \tilde{f} - \frac{\epsilon}{\kappa^2} \tilde{f} + O(\epsilon^2). $$

(4.6a)

The corresponding expansion of the tau-function $\tilde{g}$ from (2.25d) reads

$$ \tilde{g} = \tilde{\tilde{f}} - \frac{\epsilon}{\kappa^2} \tilde{f} + O(\epsilon^2). $$

(4.6b)

Now, the relation (2.10) has the leading order expansion

$$ \tilde{v} \equiv \tilde{u}_{0,1} \equiv -\tilde{\phi} / (2\kappa), $$

(4.7)

where we have put $\phi = \tilde{\phi}$. The scaling variable $\tilde{\phi}$ has a limiting form $\tilde{\phi} = 2\ln (\tilde{f} / \tilde{\tilde{f}})$ by virtue of (2.12b), (2.21), (4.4) and (4.6) which, substituted in (4.7), gives the expression of $\tilde{v}$ in terms of the tau-functions $f$ and $\tilde{f}$:

$$ \tilde{v} = -\frac{i}{\kappa} \left( \ln \frac{\tilde{f}}{f} \right) \tilde{f}. $$

(4.8)
Similarly, it follows from (4.4) and (4.6) that
\[
\ln \frac{\tilde{g}}{f f} = \ln \left[ \frac{\left( \tilde{f} - \frac{\kappa}{\epsilon} \ddot{f} + O(\epsilon^2) \right) \left( \tilde{f} - \frac{\kappa}{\epsilon} \ddot{f} + O(\epsilon^2) \right)}{f f} \right] = -\frac{\epsilon}{\kappa^2} (\ln \tilde{f}) f + O(\epsilon^2).
\]

Introducing the scaling variables \(x, \tilde{y}\) and \(\tilde{d}\) from (4.1) as well as (4.9) into (2.24b), we obtain the limiting form of \(x\):
\[
\tilde{x} = \frac{\tilde{y}}{\kappa} - \frac{1}{\kappa^2} (\ln f) \tilde{f} + \tilde{d}.
\]
(4.10)

The expressions (4.8) and (4.10) coincide with the parametric representation for the \(N\)-soliton solution of the SP equation [15].

**Remark 4.1.** Performing the short-wave limit to the bilinear equations (2.22) and (2.23) with use of (4.4) and (4.6), they reduce to the following system of bilinear equations for \(\tilde{f}\) and \(\ddot{f}\):
\[
\begin{align*}
D_1 D_2 \dddot{f} &= \kappa (\tilde{f}^2 - \ddot{f}^2), \\
D_1 D_2 \dddot{\tilde{f}} &= \kappa (\ddot{f}^2 - \dddot{f}^2).
\end{align*}
\]
(4.11a, 4.11b)

Recall that the system of equations (4.11) is a bilinear form of the sG equation. Actually, the sG equation \(\ddot{u}_x = \sin \ddot{u}\) can be transformed to the bilinear equations (4.11) through the dependent variable transformation \(\dddot{u} = 2i \ln(\ddot{f}/\dddot{f})\).

### 4.3. Short-wave limit of the phase shift

The short-wave limit of formula (3.18) for the phase shift can be performed simply. Indeed, the scaling \(\Delta_n = \epsilon \Delta_n\) of the phase shift and that of the parameters \(k_j\) given by (4.1) lead, after taking the limit \(\epsilon \to 0\), to the phase shift of the \(n\)th soliton
\[
\Delta_n = \frac{1}{\kappa k_n} \left\{ \sum_{j=1}^{n-1} \ln \left( \frac{k_n - k_j}{k_n + k_j} \right)^2 - \sum_{j=n+1}^{N} \ln \left( \frac{k_n - k_j}{k_n + k_j} \right)^2 \right\}
+ \frac{4}{\kappa k_j} - \frac{4}{\kappa k_j}, \quad (n = 1, 2, \ldots, N).
\]
(4.12)

This formula recovers formula for the phase shift of the \(N\)-soliton solution of the SP equation presented in [15].

**Remark 4.2.** Under the scaling transformations
\[
\begin{align*}
u &= \epsilon \dddot{u}, & x - t &= \epsilon \ddot{x}, & y &= \epsilon \dddot{y}, & t &= \dddot{t}, & \tau &= \dddot{\tau}, \\
k_j &= \frac{\dddot{k}_j}{\epsilon}, & y_{j0} &= \epsilon \dddot{y}_{j0} (j = 1, 2, \ldots, N), & d &= \dddot{d},
\end{align*}
\]
(4.13)

the generalized sG equation (2.27) reduces to the SP equation (1.2) in the limit of \(\epsilon \to 0\). Indeed, rewriting (2.27) in terms of the new scaling variables introduced in (4.13) and using the Taylor series expansion of the function \(\sin \epsilon \dddot{u}\), we can develop (2.27) to
\[
\epsilon \left( \ddot{u}_{ij} - \frac{1}{\epsilon^2} \dddot{u}_{ij} \right) = \epsilon \dddot{u} - \frac{\epsilon^3}{6} \dddot{u}^3 + \cdots + \frac{1}{\epsilon^2} \left( \epsilon \dddot{u} - \frac{\epsilon^3}{6} \dddot{u}^3 + \frac{\epsilon^5}{120} \dddot{u}^5 \right) \dddot{u}_{ij}.
\]

The leading terms of order \(\epsilon\) in the above expansion yield the SP equation (1.2). Note that the terms of order \(\epsilon^{-1}\) are canceled each other. The \(N\)-soliton solution (2.28) recovers the
parametric representation (4.8) and (4.10) with (4.4) for the \(N\)-soliton solution of the SP equation whereas formula (3.18) for the phase shift reduces to formula (4.12).

5. Concluding remarks

In this paper, a systematic method has been developed for solving the mCH equation under the rapidly decreasing boundary condition. We have obtained a variety of solutions which include the smooth and singular solitons and breathers, and investigated their properties. The existence of the smooth soliton solutions of the mCH equation was found to be restricted to a certain range of the amplitude parameter \(\kappa\). The same situation has been encountered in the analysis of the smooth soliton solutions of the dispersionless mCH equation subjected to the nonvanishing boundary condition \([9]\). To be more specific, for the former case, the allowable range of the parameter is \(0 < \kappa k < 1/\sqrt{2}\) (see (3.5)) whereas for the latter case, this is given by \(0 < u_0 k < \sqrt{3}/2\), where \(u_0\) is a constant background field such that \(u \to u_0\) as \(|x| \to 0\). Thus, the peakon limit, i.e., \(\kappa k \to 1\) (\(u_0 k \to 1\)) with the velocity of the soliton being fixed, is not relevant for these smooth solitons. This is in striking contrast to the peakon limit of the smooth solitons of the CH and DP equations \([30–32]\) as well as that of the Novikov equation \([20]\), for which the limiting procedure recovered peakons. On the other hand, the peakon solution of the form \(u = \sqrt{3}/2 e^{x \text{exp}(−|x − ct − x_0|)}\) has been shown to exist for the dispersionless mCH equation \((1.1)\) with \(\kappa = 0\) \([6]\). A possible way to recover the peakon is to take the peakon limit for the singular soliton. Indeed, this procedure has been applied to the symmetric singular soliton of the Novikov equation \([20]\). Unfortunately, a similar procedure has not succeeded for the current problem, as already shown here for the symmetric singular soliton (section 3.1.2). We will postpone this interesting issue for a future study.

The exact method of solution developed here will be applied to construct soliton solutions of a variant of the mCH equation

\[m_t + 2\kappa^2 u_t + \alpha_1 \left[m (u^2 - u_x^2)\right]_x + \alpha_2 (2mu_x + m_xu) = 0, \quad m = u - u_{xx}, \quad u = u(x, t), \quad (5.1)\]

where \(\alpha_1\) and \(\alpha_2\) are arbitrary constants. This equation is a linear combination of the CH and mCH equations. The particular cases \(\alpha_1 = 0\) and \(\alpha_2 = 0\) reduce to the CH and mCH equations, respectively. It is an integrable generalization of the Gardner equation which is a linear combination of the KdV and modified KdV equations \([1–3]\). Actually, the integrability of equation (5.1) was established recently by constructing the Lax pair \([36]\). While the smooth and singular single soliton solutions of traveling-wave type were obtained in \([36]\) for equation (5.1), the multisoliton solutions are not yet available. The various problems mentioned above will be dealt with in subsequent papers.

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Appendix. Proof of (2.22) and (2.23)

In this appendix, we show that the tau-functions (2.25) solve the bilinear equations (2.22) and (2.23). We use a mathematical induction similar to that used for the \(N\)-soliton solution of the DP and Novikov equations \([20, 37]\). We first prove (2.22a) and then proceed to (2.23a). The proof of (2.22b) and (2.23b) can be done in the same way and hence it will be omitted.
A.1. Proof of (2.22a)

Substituting the tau-functions $f$, $\tilde{f}$, $g$ and $\tilde{g}$ from (2.25) into the bilinear equation (2.22a) and using the formula

$$D^m f^p \exp \left[ \sum_{i=1}^{N} \mu_i \xi_i \right] \cdot \exp \left[ \sum_{i=1}^{N} \nu_i \xi_i \right] = \left\{ - \sum_{i=1}^{N} (\mu_i - \nu_i) \kappa_i \right\} \left\{ \sum_{i=1}^{N} (\mu_i - \nu_i) \kappa_i \right\}^n$$

$$\times \exp \left[ \sum_{i=1}^{N} (\mu_i + \nu_i) \xi_i \right], \quad (m, n = 0, 1, 2, \ldots),$$

where $\xi_i = 2\kappa^3 / (1 - (\kappa k)^2)$, the identity to be proved becomes

$$\sum_{\mu, \nu = 0, 1} \left\{ \sum_{i=1}^{N} (\mu_i - \nu_i) \kappa_i - \frac{1}{2\kappa} \right\} \exp \left[ \frac{\pi i}{2} \sum_{i=1}^{N} (\mu_i - \nu_i) \right] + \frac{1}{2\kappa} \exp \left[ -\frac{\pi i}{2} \sum_{i=1}^{N} (\mu_i - \nu_i) \right]$$

$$\times \exp \left[ \sum_{i=1}^{N} (\mu_i + \nu_i) \xi_i + \sum_{i=1}^{N} (\mu_i - \nu_i) \psi_i + \sum_{1 \leq i < j \leq N} (\mu_i \mu_j + \nu_i \nu_j) \gamma_{ij} \right] = 0.$$  

(A.1)

Let $P_{m,n}$ be the coefficient of the factor $\exp[\sum_{i=1}^{n} \xi_i + \sum_{i=n+1}^{m+1} 2\xi_i](1 \leq n < m \leq N)$ on the left-hand side of (A.1). Correspondingly, the summation with respect to $\mu_i$ and $\nu_i$ must be performed under the conditions

$$\mu_i + \nu_i = 1 \quad (i = 1, 2, \ldots, n), \quad \mu_i = \nu_i = 1 \quad (i = n + 1, n + 2, \ldots, m),$$

$$\mu_i = \nu_i = 0 \quad (i = m + 1, m + 2, \ldots, N).$$  

(A.2)

To proceed, it is crucial to introduce the new summation indices $\gamma_i$ by the relations $\mu_i = (1 + \gamma_i)/2$, $\nu_i = (1 - \gamma_i)/2$ for $i = 1, 2, \ldots, n$, where $\gamma_i$ takes either the value +1 or −1. It turns out that $\mu_i \mu_j + \nu_i \nu_j = (1 + \gamma_i \gamma_j)/2$.

Now, under conditions (A.2), we deduce that

$$\sum_{1 \leq i < j \leq N} (\mu_i \mu_j + \nu_i \nu_j) \gamma_{ij} = \frac{1}{2} \sum_{1 \leq i < j \leq N} (1 + \gamma_i \gamma_j) \gamma_{ij} + \sum_{m = n+1}^{m} \sum_{j \neq i} \gamma_{ij}.$$  

(A.3)

Using (A.3), $P_{m,n}$ can be written in the form

$$P_{m,n} = \sum_{\sigma = \pm 1} \left\{ \sum_{i=1}^{n} \sigma_i \kappa_i - \frac{1}{2\kappa} \right\} \exp \left[ \frac{\pi i}{2} \sum_{i=1}^{n} \sigma_i \right] + \frac{1}{2\kappa} \exp \left[ -\frac{\pi i}{2} \sum_{i=1}^{n} \sigma_i \right]$$

$$\times \exp \left[ \sum_{i=1}^{n} \sigma_i \psi_i + \frac{1}{2} \sum_{1 \leq i < j \leq N} (1 + \gamma_i \gamma_j) \gamma_{ij} + \sum_{m = n+1}^{m} \sum_{j \neq i} \gamma_{ij} \right].$$  

(A.4)

The following relations stem from (2.25f), (2.25g) and the definition of $\gamma_i$:

$$\exp \left[ \frac{1}{2} \sum_{1 \leq i < j \leq n} (1 + \gamma_i \gamma_j) \gamma_{ij} \right] = \prod_{1 \leq i < j \leq n} \frac{(\sigma_i \kappa_i - \sigma_j \kappa_j)^2}{(k_i + k_j)^2},$$

$$\exp \left[ \sum_{i=1}^{n} \sigma_i \psi_i \right] = \prod_{i=1}^{n} \frac{1 + \kappa \sigma_i \kappa_i}{(1 - (\kappa \kappa_i)^2)^{1/2}}, \quad \exp \left[ \frac{\pi i}{2} \sum_{i=1}^{n} \sigma_i \right] = i^n \prod_{i=1}^{n} \sigma_i.$$  

(A.5)
If we insert (A.5) into (A.4) and drop a multiplicative factor independent of the summation indices \( n \), the identity to be proved reduces to

\[
\begin{align*}
P_n(1, k_2, \ldots, k_n) &\equiv \sum_{\sigma_{\pm 1}} \left[ \sum_{i=1}^{n} \sigma_{\pm k_i} - \frac{1}{2\kappa} + \frac{(-1)^n}{2\kappa} \right] \prod_{i=1}^{n} \sigma_{\pm} \prod_{i=1}^{n} (1 + \kappa \sigma_{k_i}) \\
&\times \prod_{1 \leq i < j \leq n} (\sigma_{k_i} - \sigma_{k_j})^2 = 0, \quad (n = 1, 2, \ldots, N). \quad (A.6)
\end{align*}
\]

Obviously, \( P_n \) is a symmetric polynomial of \( k_i \) (\( i = 1, 2, \ldots, n \)) by virtue of the summation indices \( \sigma_i \) (\( i = 1, 2, \ldots, n \)) and is odd with respect to each \( k_i \) due to the factor \( \prod_{i=1}^{n} \sigma_i \).

The proof proceeds by mathematical induction. The identity (A.6) can be proved easily for \( n = 1, 2 \). Assume that \( P_{n-2} = 0 \). First, we note that the relation

\[
P_{n \mid k_1 = 0} = \sum_{\sigma_{\pm 1}} \sigma_{\pm} \times \text{(terms independent of } \sigma_1) = 0,
\]

holds because of the summation with respect to \( \sigma_1 \), showing that \( k_1 = 0 \) is a single zero of \( P_n \). Then,

\[
P_{n \mid k_1 = k_2} = -8k_1^2(1 - (\kappa k_1)^2) \prod_{i=3}^{n} (k_i^2 - k_1^2)^2 P_{n-2}(k_3, k_4, \ldots, k_n) = 0,
\]

by the assumption of induction. On the other hand,

\[
\frac{\partial P_n}{\partial k_1} = \sum_{\sigma_{\pm 1} = 1}^{-2} \prod_{i=2}^{n} \sigma_i \prod_{i=1}^{n} (1 + \kappa \sigma_{k_i}) \prod_{1 \leq i < j \leq n} (\sigma_{k_i} - \sigma_{k_j})^2 \\
+ \sum_{\sigma_{\pm 1}} \kappa \left[ \sum_{i=1}^{n} \sigma_{\pm k_i} - \frac{1}{2\kappa} + \frac{(-1)^n}{2\kappa} \right] \prod_{i=1}^{n} \sigma_{\pm} \prod_{i=1}^{n} (1 + \kappa \sigma_{k_i}) \prod_{1 \leq i < j \leq n} (\sigma_{k_i} - \sigma_{k_j})^2 \\
+ \sum_{\sigma_{\pm 1}} \left[ \sum_{i=1}^{n} \sigma_{\pm k_i} - \frac{1}{2\kappa} + \frac{(-1)^n}{2\kappa} \right] \prod_{i=1}^{n} \sigma_{\pm} \prod_{i=1}^{n} (1 + \kappa \sigma_{k_i}) \frac{\partial}{\partial k_1} \left[ \prod_{1 \leq i < j \leq n} (\sigma_{k_i} - \sigma_{k_j})^2 \right]
\]

\[
= P_{n1} + P_{n2} + P_{n3}.
\]

We evaluate \( P_{n1} \) at \( k_1 = k_2 \) to obtain

\[
P_{n1 \mid k_1 = k_2} = -4k_1^2(1 - (\kappa k_1)^2) \prod_{i=3}^{n} (k_i^2 - k_1^2)^2 \\
\times \sum_{\sigma_{\pm 1} = 1}^{-2} \sigma_{\pm} \sum_{\sigma_{\pm 1} = 1}^{-2} \sigma_{\pm} \prod_{i=1}^{n} (1 + \kappa \sigma_{k_i}) \prod_{3 \leq i < j \leq n} (\sigma_{k_i} - \sigma_{k_j})^2,
\]

where the notation \( \sum_{\sigma_{\pm 1} = 1}^{-2} \) implies the exclusion of \( \sigma_1 \) and \( \sigma_2 \) from the set of the indices \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \). Performing the summation with respect to \( \sigma_1 \), we eventually arrive at

\[
P_{n1 \mid k_1 = k_2} = 0. \quad (A.10)
\]

Similarly,

\[
P_{n2 \mid k_1 = k_2} = -4\kappa k_1^2 \prod_{i=3}^{n} (k_i^2 - k_1^2)^2 \sum_{\sigma_{\pm 1} = 1}^{-2} \sigma_{\pm} (1 - \kappa \sigma_1) P_{n-2}(k_3, k_4, \ldots, k_n) = 0.
\]

(A.11)
Last, by a straightforward computation, we find that

\[
P_{n3|k_1=k_2} = -4k_1(1 - (\kappa k_1)^2) \left[ \prod_{i=3}^{n} (k_i^2 - k_j^2)^2 + 2k_1^2 \sum_{j=3}^{n} (k_i^2 - k_j^2) \prod_{j=3}^{n} (k_i^2 - k_j^2)^2 \right]
\]

\[
\times P_{n-2}(k_3, k_4, \ldots, k_n) - 8k_1^2(1 - (\kappa k_1)^2)
\]

\[
\times \sum_{\sigma_1=\pm 1}^{\sigma_2=\pm 1} \sum_{\sigma_1=\pm 1}^{\sigma_2=\pm 1} \left[ \sum_{i=3}^{n} \sigma_i k_i - \frac{1}{2\kappa} + (-1)^{\sigma_i} \right] \prod_{i=3}^{n} \sigma_i \prod_{i=3}^{n} (1 + \kappa \sigma i k_i)
\]

\[
\times \sum_{i=3}^{n} (k_i^2 - k_j^2) \sigma_i k_i \prod_{j=3}^{n} (k_i^2 - k_j^2)^2 \prod_{3 \leq i < j \leq n} (\sigma_i k_i - \sigma_j k_j)^2.
\] (A.12)

The first term on the right-hand side of (A.12) vanishes by the assumption of induction whereas the second term becomes zero due to the summation with respect to \(\sigma_1\) and hence \(P_{n3|k_1=k_2} = 0\). Thus, \(\partial P_n/\partial k_1 = 0\) at \(k_1 = k_2\). By the similar argument, we confirm that the equalities \(P_n = \partial P_n/\partial k_1 = 0\) hold at \(k_1 = -k_2\) as well. It turns out that \(k_1 = \pm k_2\) are double zeros of \(P_n\). When coupled with (A.7), we see that \(P_n\) has a factor \((k_1 - k_2)^2(k_1 + k_2)^2\). Taking into account the symmetry of \(P_n\) in \(k_i (i = 1, 2, \ldots, n)\), the above result reveals that \(P_n\) can be factored by a polynomial

\[
\prod_{i=1}^{n} k_i \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,
\]

of \(k_i (i = 1, 2, \ldots, n)\) of degree \(2n^2 - n\). On the other hand, the degree of \(P_n\) from (A.6) is \(n^2 + 1\) at most, which is impossible for \(n \geq 2\) except \(P_n \equiv 0\). This completes the proof of (2.22a).

\[\square\]

A.2. Proof of (2.23a)

The proof of (2.23a) parallels that for (2.22a). Hence, we omit the detail and outline the result. The expression corresponding to (A.6), which is denoted by \(Q_n\), takes the form

\[
Q_n(k_1, k_2, \ldots, k_n) \equiv \sum_{\sigma_1=\pm 1}^{\sigma_2=\pm 1} \kappa \left\{ -2\kappa \sum_{i=1}^{n} \sigma_i k_i + 1 - (-1)^{\sigma_i} \right\} \prod_{j=1}^{n} \sigma_i k_i \prod_{j=1}^{n} (1 - (\kappa k_j)^2)
\]

\[
- \{1 - (-1)^{\sigma_i} \} \prod_{i=1}^{n} (1 - (\kappa k_j)^2) \prod_{i=1}^{n} \sigma_i \prod_{i=1}^{n} (1 + \kappa \sigma_j k_i)
\]

\[
\times \prod_{1 \leq i < j \leq n} (\sigma_i k_i - \sigma_j k_j)^2 = 0, \quad (n = 1, 2, \ldots, N). \] (A.13)

Now, the identity (A.13) holds for \(n = 1, 2, 3\), as checked easily by direct computation. Assume that \(Q_{n-2} = 0\). Then, we can show that

\[
Q_n|_{k_1=0} = 0, \quad Q_n|_{k_1=\pm k_2} = 0, \quad \frac{\partial Q_n}{\partial k_1} |_{k_1=\pm k_2} = 0. \] (A.14)
The symmetry of $Q_n$ in $k_i (i = 1, 2, \ldots, n)$ as well as (A.14) ensures that $Q_n$ has a factor
\[
\prod_{i=1}^{n} k_i \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,
\]
whose degree in $k_i (i = 1, 2, \ldots, n)$ is $2n^2 - n$. On the other hand, the degree of $Q_n$ from (A.12) is $n^2 + 2n$ at most. This is impossible for $n \geq 4$ except $Q_n = 0$. Since (A.13) holds up to $n = 3$, we conclude that $Q_n = 0$ for all $n$, completing the proof of (2.23a). \[\square\]

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