Conditional Markov Chains Revisited
Part I: Construction and properties.

Tomasz R. Bielecki\textsuperscript{1} \hspace{0.5cm} Jacek Jakubowski\textsuperscript{2,3} \hspace{0.5cm} Mariusz Niewęgłowski\textsuperscript{3}

\texttt{bielecki@iit.edu} \hspace{1cm} \texttt{jakub@imuw.edu.pl} \hspace{1cm} \texttt{m.nieweglowski@mini.pw.edu.pl}

\textsuperscript{1}Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA
\textsuperscript{2}Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
\textsuperscript{3}Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warszawa, Poland

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Abstract

In this paper we continue the study of conditional Markov chains (CMCs) with finite state spaces, that we initiated in Bielecki, Jakubowski and Niewęgłowski (2014a) in an effort to enrich the theory of CMCs that was originated in Bielecki and Rutkowski (2004). We provide an alternative definition of a CMC and an alternative construction of a CMC via a change of probability measure. It turns out that our construction produces CMCs that are also doubly stochastic Markov chains (DSMCs), which allows for study of several properties of CMCs using tools available for DSMCs.

Keywords: conditional Markov chain; doubly stochastic Markov chain; compensator of a random measure; change of probability measure.

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1 Introduction

In this paper we continue the study of conditional Markov chains (CMCs) with finite state spaces, that we initiated in Bielecki, Jakubowski and Niewęgłowski [5] in an effort to enrich the theory of CMCs that was originated in Bielecki and Rutkowski [8].

CMCs were conceptualized in the context of credit risk, where they have been found to provide a useful tool for modeling credit migrations. In many ways, a CMC is an important generalization of the concept of a default time with stochastic compensator, a key concept in the models of financial markets allowing for default of parties of a financial contract. Such a default time is really just a special example of a CMC: it is a CMC taking values in a state space consisting of only two states, say 0 and 1, where 0 is the transient state and 1 is the absorbing state.

In [5] we proposed a modified definition of the conditional Markov property, which was less general than Definition 11.3.1 used in Chapter 11.3 in [8]. The reason for this was that the definition of conditional Markov property proposed in [5] was aimed at providing a suitable framework for study of Markov consistency properties for conditional Markov chains and study of Markov copulae for conditional Markov chains, a feat that can’t be achieved within the framework of the CMC framework proposed in [8]. Still, the definition of the conditional Markov property, and the related construction of a CMC as presented in [5] were not general enough, as they did not allow for study of conditional Markov families. This is because in [5] we only dealt with processes starting from a fixed, non-random, initial state. Here, we generalize the definition of a conditional Markov property and construction of a CMC that allow for the initial state of the chain to have a nondegenerate conditional initial distribution, and, consequently, allow for study of conditional Markov families. Such study will be conducted elsewhere.
Classical conditional Markov chains, that is, the ones defined originally in [8], have already proven to play important role in applications in finance and in insurance, for example (cf. Bielecki and Rutkowski [6], [7], [8], Jakubowski and Niewęgłowski [19], Eberlein and Özkan [12], Eberlein and Grbac [11], Biagini, Groll and Widenmann [1]). The main advantage of these processes is that, via appropriate conditioning, their primary Markov properties are mixed with dependence of their infinitesimal characteristics on relevant random factors, that do not have to be Markovian. The CMCs studied in this paper may lead to many more applications since, as already has been mentioned above, the present modified definition allows for study dependence properties between CMCs, which are crucial in applications to credit and counterparty risk, among other applications. In fact, the present paper is a companion paper to Bielecki, Jakubowski and Niewęgłowski [4], where we complement the study done here by investigating the issues of modeling dependence between CMCs, and we propose some specific applications.

An important family of jump processes, so called doubly stochastic Markov chains (DSMC), was introduced in Jakubowski and Niewęgłowski [18]. The conditional Markov chains constructed in the present paper turn out to be doubly stochastic Markov chains. Thus, the benefit from the construction provided here is two-fold:

- The constructed CMCs enjoy the conditional Markov property, which has unquestionable practical appeal, and
- The constructed CMCs enjoy the doubly stochastic Markov property, which has critical theoretical implications allowing for applying important tools from stochastic analysis to studying CMCs.

The paper is organized as follows: In Section 2 we introduce the concept of CMC, which underlies the present study. In this section we also introduce and discuss the relevant concept of stochastic generator (or an intensity matrix) of a CMC. In addition, we give there two examples of $(\mathcal{F}, \mathcal{G})$-CMC, one which does not have the intensity, and one with the intensity. Section 3 is devoted to presentation of a specific method for constructing a CMC. In Section 4 we relate conditional Markov chains to doubly stochastic Markov chains. In particular, we show that any conditional Markov chain constructed using the change of measure technique used in Section 3 is also a doubly stochastic Markov chain. Finally, in the last section we collect all needed technical results used throughout the paper.

2 Conditional Markov Chain and Its Intensity

Let $T > 0$ be a fixed finite time horizon. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an underlying complete probability space, which is endowed with two filtrations, $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ and $\mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]}$, that are assumed to satisfy the usual conditions. For the future reference we also define

$$\hat{\mathcal{G}}_t := \mathcal{F}_T \lor \mathcal{G}_t, \quad t \in [0,T],$$

(2.1)
as well as the corresponding filtration $\hat{G} := (\hat{G}_t)_{t \in [0,T]}$.

Typically, processes considered in this paper are defined on $(\Omega, \mathcal{A}, \mathbb{P})$, and are restricted to the time interval $[0,T]$. Moreover, for any process $U$ we denote by $\mathbb{F}^U$ the completed right-continuous filtration generated by this process. In addition, we fix a finite set $S$, and we denote by $d$ the cardinality of $S$. Without loss of generality we take $S = \{1,2,3,\ldots,d\}$.

**Definition 2.1.** An $S$-valued, $\mathbb{G}$-adapted càdlàg process $X$ is called an $(\mathbb{F}, \mathbb{G})$-conditional Markov chain if for every $x_1,\ldots,x_k \in S$ and for every $0 \leq t \leq t_1 \leq \ldots \leq t_k \leq T$ it satisfies the following property

$$
\mathbb{P}(X_{t_1} = x_1,\ldots,X_{t_k} = x_k | \mathcal{F}_t \vee \mathcal{G}_t) = \mathbb{P}(X_{t_1} = x_1,\ldots,X_{t_k} = x_k | \mathcal{F}_t \vee \sigma(X_t)).
$$

(iii) It needs to be stressed that an $(\mathbb{F}, \mathbb{G})$-conditional Markov chain may not be a classical Markov chain (in any filtration). However, if $\mathbb{G}$ is independent of $\mathbb{F}$, then the above definition reduces to the case of a classical Markov chain with respect to filtration $\mathbb{G}$, or $\mathbb{G}$–Markov chain. In other words, a classical $\mathbb{G}$-Markov chain is an $(\mathbb{F}, \mathbb{G})$-conditional Markov chain for the reference filtration independent of the base filtration.

In what follows we shall write $(\mathbb{F}, \mathbb{G})$-CMC, for short, in place of $(\mathbb{F}, \mathbb{G})$-conditional Markov chain.

### 2.1 Intensity of an $(\mathbb{F}, \mathbb{G})$-CMC

Let $X$ be an $(\mathbb{F}, \mathbb{G})$-CMC. For each $x \in S$ we define the corresponding state indicator process of $X$,

$$
H_t^x := 1_{\{X_t = x\}}, \quad t \in [0,T].
$$

Accordingly, we define a column vector $H_t = (H_t^x, x \in S)^\top$, where $\top$ denotes transposition. Similarly, for $x, y \in S$, $x \neq y$, we define process $H_t^{xy}$ that counts the number of transitions from $x$ to $y$,

$$
H_t^{xy} := \# \{ u \leq t : X_{u-} = x \text{ and } X_u = y \} = \int_{[0,t]} H_u^x dH_u^y, \quad t \in [0,T].
$$

The following definition generalizes the concept of the generator matrix (or intensity matrix) of a Markov chain.

**Definition 2.3.** We say that an $\mathbb{F}$-adapted (matrix valued) process $\Lambda_t = [\lambda_t^{xy}]_{x,y \in S}$ such that

$$
\lambda_t^{xy} \geq 0, \quad \forall x, y \in S, x \neq y, \quad \text{and} \quad \sum_{y \in S} \lambda_t^{xy} = 0, \quad \forall x \in S,
$$

is an $\mathbb{F}$-stochastic generator or an $\mathbb{F}$-intensity matrix process for $X$, if the process $(M_t^x, x \in S)^\top$ defined as

$$
M_t = H_t - \int_0^t \Lambda_u \top H_u du, \quad t \in [0,T],
$$

is an $\mathbb{F} \vee \mathbb{G}$–local martingale (with values in $\mathbb{R}^d$).
Remark 2.4. We remark that even though the above definition is stated for an \((\mathcal{F},\mathcal{G})\)-CMC process \(X\), it applies to \(S\)-valued semimartingales.

We will now discuss the question of uniqueness of \(\mathcal{F}\)-intensity.

Definition 2.5. We say that two processes \(\Lambda\) and \(\hat{\Lambda}\) are equivalent relative to \(X\) if
\[
\int_0^t (\Lambda_u - \hat{\Lambda}_u)^\top H_u du = 0, \quad \forall t \in [0,T]. \tag{2.7}
\]

Proposition 2.6. Let \(X\) be an \((\mathcal{F},\mathcal{G})\)-CMC.

i) If \(\Lambda\) and \(\hat{\Lambda}\) are \(\mathcal{F}\)-intensities of \(X\), then they are equivalent relative to \(X\). In particular \(\mathcal{F}\)-intensity of \(X\) is unique up to equivalence relative to \(X\).

ii) Let \(\Lambda\) be an \(\mathcal{F}\)-intensity of \(X\). If \(\hat{\Lambda}\) is an \(\mathcal{F}\)-adapted process equivalent to \(\Lambda\) relative to \(X\), then \(\hat{\Lambda}\) is \(\mathcal{F}\)-intensity of \(X\).

Proof. i) By assumption, \(M\) given by (2.6) and \(\hat{M}\) defined as
\[
\hat{M}_t = H_t - \int_0^t \hat{\Lambda}_u^\top H_u du, \quad t \in [0,T],
\]
have \(\mathcal{F} \cup \mathcal{G}\)-local martingales. We have that
\[
\hat{M}_t - M_t = \int_0^t (\Lambda_u - \hat{\Lambda}_u)^\top H_u du.
\]
Thus \(\hat{M} - M\) is a continuous finite variation \(\mathcal{F} \cup \mathcal{G}\)-martingale starting from 0, and hence it is a constant null process. Thus (2.7) holds.

ii) Note that (2.7) implies that for \(\mathcal{F} \cup \mathcal{G}\) martingale \(M\) given by (2.6) it holds
\[
M_t = H_t - \int_0^t \Lambda_u^\top H_u du + \int_0^t (\Lambda_u - \hat{\Lambda}_u)^\top H_u du = H_t - \int_0^t \hat{\Lambda}_u^\top H_u du, \quad t \in [0,T].
\]
Thus \(\hat{\Lambda}\) is an \(\mathcal{F}\)-intensity of \(X\). \(\square\)

In [4, Example 3.9] we exhibit an \((\mathcal{F},\mathcal{G})\)-CMC \(X\), which admits two different intensities \(\Gamma\) and \(\Lambda\) that are equivalent relative to \(X\).

In the case of classical Markov chains with finite state space, intensity matrix may not exist if the matrix of transition probabilities is not differentiable (e.g. when \(X\) is not quasi left continuous). In the case of \((\mathcal{F},\mathcal{G})\)-CMC the situation is similar. That is, there exist \((\mathcal{F},\mathcal{G})\)-CMCs that do not admit \(\mathcal{F}\)-intensities. We illustrate this possibility by means of the following example (see [5] for details):

Example 2.7. Suppose that \((\Omega,\mathcal{A},\mathbb{P})\) supports a real valued standard Brownian motion \(W\), and a random variable \(E\) with unit exponential distribution\(^1\) and independent from \(W\). Define a nonnegative process \(\gamma\), by formula
\[
\gamma_t := \sup_{u \in [0,t]} W_u, \quad t \geq 0.
\]
\(^1\)That is, \(E\) is exponentially distributed with mean 1.
By definition, \( \gamma \) is an increasing and continuous process. It is well known (cf. Section 1.7 in Itô and McKean [15]) that trajectories of \( \gamma \) are not absolutely continuous with respect to the Lebesgue measure on real line. It is shown in [5] that the process \( X \) defined by

\[ X_t := 1_{\{\tau \leq t\}}, \quad t \geq 0, \]

where

\[ \tau := \inf\{t > 0 : \gamma_t > E\} \]

is an \((\mathbb{F}^W, \mathbb{F}^X)\)-CMC which does not admit an \( \mathbb{F}^W \)-intensity matrix.

Theorem 2.8 below provides more insight into the issue of existence of \( \mathbb{F} \)-intensity for an \((\mathbb{F}, \mathbb{G})\)-CMC.

**2.1.1 Intensity of an \((\mathbb{F}, \mathbb{G})\)-CMC and \( \mathbb{F} \vee \mathbb{G} \)-compensators of counting processes \( H^{xy} \)**

The \( \mathbb{F} \)-intensity matrix of an \((\mathbb{F}, \mathbb{G})\)-CMC \( X \) is related to the \( \mathbb{F} \vee \mathbb{G} \)-compensators of processes \( H^{xy} \), \( x, y \in S \), \( x \neq y \). In fact, we have the following result, which is a special case of [18, Lemma 4.3], which deals with general jump semimartingales, and thus its proof is omitted.

**Theorem 2.8.** Let \( X \) be an \((\mathbb{F}, \mathbb{G})\)-CMC.

1) Suppose that \( X \) admits an \( \mathbb{F} \)-intensity matrix process \( \Lambda \). Then for every \( x, y \in S \), \( x \neq y \), the process \( H^{xy} \) admits an absolutely continuous \( \mathbb{F} \vee \mathbb{G} \)-compensator given as

\[ \int_0^t H^{xy}_u \lambda^{xy}_u du, \quad t \in [0, T], \]

(2.8)

is an \( \mathbb{F} \vee \mathbb{G} \)-local martingale.

2) Suppose that we are given a family of nonnegative \( \mathbb{F} \)-progressively measurable processes \( \lambda^{xy} \), \( x, y \in S \), \( x \neq y \), such that for every \( x, y \in S \), \( x \neq y \), the process \( K^{xy} \) given in (2.8) is an \( \mathbb{F} \vee \mathbb{G} \)-local martingale. Then, the matrix valued process \( \Lambda_t = [\lambda^{xy}]_{x,y \in S} \), with diagonal elements defined as

\[ \lambda^{xx} = -\sum_{y \in S, y \neq x} \lambda^{xy}, \quad x \in S, \]

is an \( \mathbb{F} \)-intensity matrix of \( X \).

We see that the \( \mathbb{F} \)-intensity may not exist since \( \mathbb{F} \vee \mathbb{G} \)-compensators of \( H^{xy} \) may not be absolutely continuous with respect to Lebesgue measure. On the other hand, absolute continuity of \( \mathbb{F} \vee \mathbb{G} \)-compensators of all processes \( H^{xy} \), for \( x \in S \), \( x \neq y \), is not sufficient for existence of an \( \mathbb{F} \)-intensity. This is due to the fact that the density of \( \mathbb{F} \vee \mathbb{G} \) compensator is, in general, \( \mathbb{F} \vee \mathbb{G} \)-adapted, whereas the \( \mathbb{F} \)-intensity is only \( \mathbb{F} \)-adapted.

In order to focus our study, we now introduce the following restriction:

In the rest of this paper we restrict ourselves to CMCs, which admit \( \mathbb{F} \)-intensity. CMCs that do not admit intensities will be studied in a follow-up paper.
2.2 \((\mathbb{F}, \mathbb{G})\)-CMC as a pure jump semimartingale

It is important to note that an \((\mathbb{F}, \mathbb{G})\)-CMC \(X\) admitting \(\mathbb{F}\)-intensity process \(\Lambda\) can be viewed as a pure jump semimartingale,\(^2\) with values in \(S\), whose corresponding random jump measure \(\mu\) defined by (cf. Jacod \(\cite{16}\))

\[
\mu(\omega, dt, dz) = \sum_{n \geq 1} \delta_{(T_n(\omega), X_{T_n(\omega)}(\omega))}(dt, dz) \mathbb{1}_{\{T_n(\omega) < T\}},
\]

where

\[
T_n := \inf \{t : T_{n-1} < t \leq T, X_t \neq X_{T_{n-1}}\} \land T, \quad T_0 = 0,
\]

has the \(\mathbb{F} \lor \mathbb{G}\) predictable projection under \(\mathbb{P}\) (the \((\mathbb{F} \lor \mathbb{G}, \mathbb{P})\)-compensator) given as

\[
\nu(dt, dz) = \sum_{y \in S} \delta_y(dz) \left( \sum_{x \in S \setminus \{y\}} H^x_t \lambda^{xy}_t \right) dt = \sum_{y \in S} \delta_y(dz) \left( \sum_{x \in S \setminus \{y\}} \mathbb{1}_{\{X_t=x\}} \lambda^{xy}_t \right) dt. \quad (2.9)
\]

So the problem of construction of an \((\mathbb{F}, \mathbb{G})\)-CMC with an \(\mathbb{F}\)-intensity (matrix) process \(\Lambda\) is equivalent to the problem of construction of any \(\mathbb{G}\)-adapted, \(S\)-valued pure jump semimartingale with the \((\mathbb{F} \lor \mathbb{G}, \mathbb{P})\)-compensator \(\nu\) given by (2.9), and additionally satisfying condition (2.2).

**Remark 2.9.** With a slight abuse of terminology, we shall refer to a \(\mathbb{G}\)-adapted, \(S\)-valued pure jump semimartingale \(X\) with the \(\mathbb{F} \lor \mathbb{G}\) compensator \(\nu\) given by (2.9), as to a \(\mathbb{G}\)-adapted, \(S\)-valued pure jump semimartingale admitting the \(\mathbb{F}\)-intensity process \(\Lambda\). In particular, this also means that the process \(M\) corresponding to \(X\) as in (2.6) (see Remark 2.4) is an \(\mathbb{F} \lor \mathbb{G}\)-local martingale and, even though \(X\) is not necessarily \((\mathbb{F}, \mathbb{G})\)-CMC, the conclusions 1) and 2) of Theorem 2.8 hold.

Theorem 2.11 below shows that a \(\mathbb{G}\)-adapted, \(S\)-valued pure jump semimartingale admitting \(\mathbb{F}\)-intensity process \(\Lambda\) is, under some additional conditions, an \((\mathbb{F}, \mathbb{G})\)-CMC with the same \(\mathbb{F}\)-intensity process \(\Lambda\). Before stating the theorem, we recall the notion of immersion between two filtrations.

**Definition 2.10.** We say that a filtration \(\mathbb{F}\) is \(\mathbb{P}\)-immersed in a filtration \(\mathbb{H}\) if \(\mathbb{F} \subset \mathbb{H}\) and if every \((\mathbb{P}, \mathbb{F})\)-local martingale is a \((\mathbb{P}, \mathbb{H})\)-local martingale.

We now have,

**Theorem 2.11.** Assume that

\[
\mathbb{F} \text{ is } \mathbb{P}\text{-immersed in } \mathbb{F} \lor \mathbb{G}. \quad (2.10)
\]

Let \(X\) be a \(\mathbb{G}\)-adapted, \(S\)-valued pure jump semimartingale admitting the \(\mathbb{F}\)-intensity process \(\Lambda\). Moreover suppose that

all real valued \(\mathbb{F}\) – local martingales are orthogonal to components \(M^x, x \in S, \quad (2.11)\)

of process \(M\) given by (2.6).

\(^2\)We adhere to the standard convention that semimartingale processes (taking values in finite dimensional spaces) are càdlàg.
Then $X$ is an $(\mathbb{F}, \mathbb{G})$-CMC with the $\mathbb{F}$-intensity process $\Lambda$.\(^3\)

**Proof.** Let us fix $0 = t_0 \leq t_1 \leq \ldots \leq t_k \leq T$, and $x_1, \ldots, x_k \in S$. It is enough to show that the martingale $N$, given as

$$N_t = \mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_k} = x_k | \mathcal{F}_t \vee \mathcal{G}_t), \quad t \in [0, T],$$

is such that $N_u$ is $\mathcal{F}_u \vee \sigma(X_u)$ measurable for any $u \in [0, t_1]$. Indeed, this implies that $\mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_k} = x_k | \mathcal{F}_u \vee \mathcal{G}_u) = \mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_k} = x_k | \mathcal{F}_u \vee \sigma(X_u)), \quad u \in [0, t_1],$

which is the $(\mathbb{F}, \mathbb{G})$-CMC property. To this end, for each $n = 1, \ldots, k$, we define a process $V^n_t$ by

$$V^n_t := \prod_{l=1}^{n-1} \mathbb{1}_{\{X_{t_l} = x_l\}} H_t^\top \mathbb{E}\left( Z_t Y_{t_n} e_{x_n} \prod_{m=n}^{k-1} e_{x_m}^\top Z_{t_m} Y_{t_{m+1}} e_{x_{m+1}} | \mathcal{F}_t \right), \quad t \in [0, T],$$

where $e_x$ denotes a column vector in $\mathbb{R}^d$ with 1 at the coordinate corresponding to state $x$ and with zeros otherwise, and $Z, Y$ are solutions of the random ODE’s\(^4\)

$$dZ_t = -\Lambda_t Z_t dt, \quad Z_0 = I, \quad t \in [0, T],$$

$$dY_t = Y_t \Lambda_t dt, \quad Y_0 = 1, \quad t \in [0, T].$$

We will show, that

$$V^n_t = N_t, \quad t \in [t_{n-1}, t_n], \quad n = 1, 2, \ldots, k,$$  \hspace{1cm} (2.12)

which, in particular, implies that for every $t \in [0, t_1]$ the random variable $N_t = V^1_t$ is measurable with respect to $\mathcal{F}_t \vee \sigma(X_t)$.

We first note that, in view of Lemma 5.5 in Appendix B, the process $V^n$ is an $\mathbb{F} \vee \mathbb{G}$ martingale on $[t_{n-1}, t_n]$. Moreover, we have that

$$V^n_{t_n} = V^{n+1}_{t_n}. \hspace{1cm} (2.13)$$

Indeed,

$$V^{n+1}_{t_n} = \prod_{l=1}^{n} \mathbb{1}_{\{X_{t_{l+1}} = x_{l+1}\}} H_{t_n}^\top \mathbb{E}\left( Z_{t_n} Y_{t_{n+1}} e_{x_{n+1}} \prod_{m=n+1}^{k-1} e_{x_m}^\top Z_{t_m} Y_{t_{m+1}} e_{x_{m+1}} | \mathcal{F}_{t_n} \right)$$

$$= \prod_{l=1}^{n} \mathbb{1}_{\{X_{t_{l+1}} = x_{l+1}\}} H_{t_n}^\top e_{x_n} H_{t_n}^\top \mathbb{E}\left( Z_{t_n} Y_{t_{n+1}} e_{x_{n+1}} \prod_{m=n+1}^{k-1} e_{x_m}^\top Z_{t_m} Y_{t_{m+1}} e_{x_{m+1}} | \mathcal{F}_{t_n} \right)$$

$$= \prod_{l=1}^{n-1} \mathbb{1}_{\{X_{t_{l+1}} = x_{l+1}\}} H_{t_n}^\top \mathbb{E}\left( Z_{t_n} Y_{t_{n+1}} e_{x_n} e_{x}^\top e_{x_n} Z_{t_{n+1}} Y_{t_{n+2}} e_{x_{n+2}} \prod_{m=n+1}^{k-1} e_{x_m}^\top Z_{t_m} Y_{t_{m+1}} e_{x_{m+1}} | \mathcal{F}_{t_n} \right)$$

$$= \prod_{l=1}^{n-1} \mathbb{1}_{\{X_{t_{l+1}} = x_{l+1}\}} H_{t_n}^\top \mathbb{E}\left( Z_{t_n} Y_{t_{n+1}} e_{x_n} e_{x_n} Z_{t_{n+1}} Y_{t_{n+2}} e_{x_{n+2}} \prod_{m=n+1}^{k-1} e_{x_m}^\top Z_{t_m} Y_{t_{m+1}} e_{x_{m+1}} | \mathcal{F}_{t_n} \right) = V^n_{t_n},$$

\(^3\)We refer to He, Wang and Yan [14, Definition 7.33], for notion of orthogonality of local martingales.

\(^4\)The symbol "1" used below is a generic symbol for the identity matrix, whose dimension may vary depending on the context.
where the third equality follows from Lemma 5.4 formula (5.7), and from the fact that

$$H_{t_n}^e x_n H_{t_n}^e = H_{t_n}^e x_n e_{x_n}^T.$$  

We will finish the proof by demonstrating (2.12) with use of backward induction. Towards this end, we start from the last interval, i.e. $n = k$. Observing that

$$V_{t_k}^k = \prod_{l=1}^{k-1} \mathbf{1}_{\{X_{t_l} = x_l\}} H_{t_k}^e \mathbb{E}\left(Z_{t_k} Y_{t_k} e_{x_k} | \mathcal{F}_{t_k}\right) = \prod_{l=1}^{k-1} \mathbf{1}_{\{X_{t_l} = x_l\}} H_{t_k}^e e_{x_k} = \prod_{l=1}^{k} \mathbf{1}_{\{X_{t_l} = x_l\}},$$

and using the martingale property of $V^k$ on $[t_{k-1}, t_k]$, we conclude that for $t \in [t_{k-1}, t_k]$

$$V_t^k = \mathbb{E}(V_{t_k}^k | \mathcal{F}_t \lor \mathcal{G}_t) = \mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_k} = x_k | \mathcal{F}_t \lor \mathcal{G}_t) = N_t.$$  

Now, suppose that for some $n = 2, \ldots, k - 1$, the process $V^n$ coincides with $N$ on $[t_{n-1}, t_n]$. This, together with (2.13), yields that

$$N_{t_{n-1}} = V_{t_{n-1}}^n = V_{t_{n-1}}^{n-1}.$$  

Thus, by the martingale property of $V^{n-1}$ on the interval $[t_{n-2}, t_{n-1}]$, we obtain that

$$V_t^{n-1} = \mathbb{E}(N_{t_{n-1}} | \mathcal{F}_t \lor \mathcal{G}_t) = N_t, \quad t \in [t_{n-2}, t_{n-1}].$$

So the (backward) induction principle completes the proof. 

□

Remark 2.12. A sufficient condition for orthogonality of real valued $\mathbb{F}$–local martingales and components of process $M$ is that $\mathbb{F}$–local martingales and the process $M$ do not have common jumps or, equivalently, that $\mathbb{F}$–local martingales and the process $X$ do not have common jumps. Indeed, let $Z$ be an $(\mathbb{F}, \mathbb{P})$–local martingale. Since $M^x$ is a local martingale of finite variation we have that

$$[Z, M^x]_t = \sum_{0 < u \leq t} \Delta Z_u \Delta M^x_u = \sum_{0 < u \leq t} \Delta Z_u \Delta H^x_u, \quad t \in [0, T].$$

Now, note that $X$ jumps iff one of the processes $H^x, x \in S$, jumps. Thus if $X$ and $Z$ do not have common jumps then $[Z, M^x]$ is the null process, hence it is a local martingale. Consequently $Z$ and $M^x$ are orthogonal local martingales.

We complete this section with the following proposition, which furnishes an interesting example of filtrations $\mathbb{F}$ and $\mathbb{G}$ that satisfy conditions (2.10) and (2.11) of Theorem 2.11.

Proposition 2.13. Let $X$ be an $S$-valued pure jump semimartingale in its own filtration. Let $W$ be a Brownian motion in filtration $\mathbb{F}^W \lor \mathbb{F}^X$. Then (2.10) and (2.11) of Theorem 2.11 are satisfied with $\mathbb{G} = \mathbb{F}^X$ and $\mathbb{F} = \mathbb{F}^W$. 

Proof. Note that $W$ is also $\mathbb{F}^W$ Brownian motion, thus any square integrable $\mathbb{F}^W$-martingale $N$ can be represented as

$$N_t = N_0 + \int_0^t \phi_u dW_u, \quad t \in [0, T],$$

for some $\mathbb{F}$-predictable process $\phi$. The assumption that $W$ is a Brownian motion in $\mathbb{F}^W \lor \mathbb{F}^X$, implies that $N$ is also an $\mathbb{F}^W \lor \mathbb{F}^X$-martingale. This proves that $\mathbb{F}^W$ is immersed in $\mathbb{F}^W \lor \mathbb{F}^X$. So (2.10) holds. Condition (2.11) is satisfied, since all $\mathbb{F}^W$ martingales are continuous.

Remark 2.14. The assumption that $W$ is a Brownian motion in the filtration $\mathbb{F}^W \lor \mathbb{F}^X$ is in fact equivalent to immersion of $\mathbb{F}^W$ in $\mathbb{F}^W \lor \mathbb{F}^X$.

3 Construction $(\mathbb{F}, \mathbb{G})$–CMC via change of measure

The construction of CMC given in this section generalizes the construction provided in [5]. In [5] the authors constructed CMCs that are starting from a given state with probability one. Here, we construct a process $(X_t)_{t \in [0, T]}$ such that $X$ is an $(\mathbb{F}, \mathbb{G})$–CMC with the $\mathbb{F}$-intensity matrix process $\Lambda$, and with $X_0$ satisfying

$$\mathbb{P}(X_0 = x|\mathcal{F}_T) = \mathbb{P}(X_0 = x|\mathcal{F}_0), \quad x \in S. \quad (3.1)$$

Even though in case of ordinary Markov chains a construction of a chain starting from a given state with probability one directly leads to construction of a chain with arbitrary initial distribution, this is not the case any more when one deals with CMCs. In fact, some non-trivial modifications of the construction argument used in [5] will need to be introduced below.

3.1 Preliminaries

In our construction we start from some underlying probability space, say $(\Omega, \mathcal{A}, \mathbb{Q})$, on which we are given:

(I1) A (reference) filtration $\mathbb{F}$.

(I2) An $S$-valued random variable $\xi$, such that for any $x \in S$ we have that

$$\mathbb{Q}(\xi = x|\mathcal{F}_T) = \mu_x, \quad (3.2)$$

for some $\mathcal{F}_0$–measurable random variable $\mu_x$ taking values in $[0, 1]$.

(I3) A family $\mathcal{N} = (N^{xy})_{x, y \in S}^{y \neq x}$ of mutually independent Poisson processes, that are independent of $\mathcal{F}_T \lor \sigma(\xi)$ and with non-negative intensities $(a^{xy})_{x, y \in S}$ (of course for $a^{xy} = 0$ we put $N^{xy} = 0$).
Remark 3.1. We observe that condition (3.2) is satisfied iff $\xi = \xi' + \xi''$, where $\xi'$ is $\mathcal{F}_0$-measurable and $\xi''$ is orthogonal to $\mathcal{F}_T$.

In what follows we take
\begin{equation}
\mathcal{G}_t = \left( \bigvee_{x,y \in S \atop y \neq x} \mathcal{F}_t^{N_{xy}} \right) \vee \sigma(\xi),
\end{equation}
and we recall that
\begin{equation}
\hat{\mathcal{G}}_t = \mathcal{F}_t \vee \mathcal{G}_t, \quad t \in [0,T].
\end{equation}
Next, we will construct $\hat{\mathcal{G}}$-Markov chain, say $X$, as a solution of an appropriate stochastic differential equation. This is an intermediate step in our goal of constructing an $\left(\mathbb{F}, \hat{\mathcal{G}}\right)$-CMC with the $\mathbb{F}$-intensity matrix process $\Lambda$, and with $X_0$ satisfying
\begin{equation}
\mathbb{P}(X_0 = x|\mathcal{F}_T) = \mathbb{P}(X_0 = x|\mathcal{F}_0), \quad x \in S,
\end{equation}
for a measure $\mathbb{P}$ to be constructed later.

**Proposition 3.2.** Let $A = [a_{xy}]_{x,y \in S}$, where the diagonal elements of $A$ are defined as $a_{xx} := -\sum_{y \in S \atop y \neq x} a_{xy}$. Assume that $\xi$ is an $S$-valued random variable and $N = (N_{x,y})$ are Poisson processes satisfying (I3). Then the unique strong solution of the following SDE
\begin{equation}
dX_t = \sum_{x,y \in S \atop x \neq y} (y - x)1_{\{x\}}(X_t-)dN_{xy}^t, \quad t \in [0,T], \quad X_0 = \xi,
\end{equation}
is a $\hat{\mathcal{G}}$-Markov chain with the infinitesimal generator $A$. Moreover, $A$ is an $\mathbb{F}$-intensity of $X$ under $\mathbb{Q}$.

**Proof.** In view of (I3), the processes $N_{xy}$ and $N_{xy}'$, $y \neq y'$, do not jump together. Thus, the process $H_{xy}$ defined for $x,y \in S$, $x \neq y$ by
\begin{equation}
H_{xy}^t = \int_0^t H_{x-}^u dN_{xy}^u, \quad t \in [0,T],
\end{equation}
(cf. (2.3) for definition of $H^x$) counts number of transitions of $X$ from state $x$ to state $y$. Independence of $N_{xy}$ from $\mathcal{F}_T \vee \sigma(\xi)$ implies that $N_{xy}$ is also a $\hat{\mathcal{G}}$-Poisson processes with intensity $a_{xy}$. Thus, by boundedness and $\hat{\mathcal{G}}$-predictability of $(H_{x-}^t)_{t \in [0,T]}$, the process $L_{xy}^t$ given as
\begin{equation}
L_{xy}^t = \int_0^t H_{x-}^u (dN_{xy}^u - a_{xy}^u du) = H_{xy}^t - \int_0^t H_{x-}^u a_{xy}^u du = H_{xy}^t - \int_0^t H_{x}^u a_{xy}^u du, \quad t \in [0,T],
\end{equation}
is a $\hat{\mathcal{G}}$-martingale. Consequently, application of relevant characterization theorem [18, Thm. 4.1] yields that $X$ is a $\hat{\mathcal{G}}$-Markov chain with the infinitesimal generator $A$.

A random variable $\xi''$ and sigma field $\mathcal{F}_T$ are said to be orthogonal if $E_{\mathbb{Q}}(\xi'' \eta) = 0$, for every $\eta \in L^\infty(\mathcal{F}_T)$. 

\footnote{A random variable $\xi''$ and sigma field $\mathcal{F}_T$ are said to be orthogonal if $E_{\mathbb{Q}}(\xi'' \eta) = 0$, for every $\eta \in L^\infty(\mathcal{F}_T)$.}
To finish the proof we observe that since $X$ given by (3.6) is a pure jump process with finite variation, it is a semimartingale. The $(\hat{\mathcal{G}}, \mathbb{Q})$-compensator of the jump measure of $X$, that is, the jump characteristic of $X$ relative to $(\hat{\mathcal{G}}, \mathbb{Q})$, is given in terms of matrix $A$ (cf. (3.7)). Moreover, since $X$ is adapted to filtration $\mathbb{F} \lor \mathbb{G} \subseteq \hat{\mathcal{G}}$, then we see that $X$ is a semimartingale with the $(\mathbb{F} \lor \mathbb{G}, \mathbb{Q})$-compensator of its jump measure given in terms of matrix $A$. Now, $A$ is $\mathbb{F}$-adapted (since it is deterministic), so, in view of the terminology introduced earlier (cf. Definition 2.3), $A$ is an $\mathbb{F}$-intensity of $X$ under $\mathbb{Q}$. □

The fact that $X$ is a Markov chain in filtration $\hat{\mathcal{G}}$ will be critically important below.

3.2 Canonical conditions

Let $\Lambda_t = [\lambda_t^{xy}]_{x,y \in S}$ be matrix valued process satisfying the following conditions:

(C1) $\Lambda$ is an $\mathbb{F}$-progressively measurable and it fulfills (2.5).

(C2) The processes $\lambda_t^{xy}, x, y \in S, x \neq y$, have countably many jumps $\mathbb{Q}$-a.s.

Definition 3.3. A process $\Lambda$ satisfying conditions (C1) and (C2) is said to satisfy canonical conditions relative to the pair $(S, \mathbb{F})$.

Any $\mathbb{F}$-adapted càdlàg process $\Lambda_t = [\lambda_t^{xy}]_{x,y \in S}$, for which (2.5) holds, satisfies canonical conditions.

We are now ready to proceed with construction of a CMC via change of measure.

3.3 Construction of a CMC

In this section we provide a construction of a probability measure $\mathbb{P}$, under which the process $X$ following the dynamics (3.6) is an $(\mathbb{F}, \mathbb{G})$-CMC with a given $\mathbb{F}$-intensity matrix $\Lambda$ and with $\mathcal{F}_T$-conditional initial distribution satisfying (3.5).

Theorem 3.4. Let $\Lambda$ satisfy canonical conditions relative to the pair $(S, \mathbb{F})$ and assume that $\xi$ satisfies (I2). Suppose that $a^{xy}$, introduced in (I3), is strictly positive for all $x, y \in S, x \neq y$. Moreover, let $X$ be the unique solution of SDE (3.6). For each pair $x, y \in S, x \neq y$, define the processes $\kappa_t^{xy}$ as

$$\kappa_t^{xy} = \frac{\lambda_t^{xy}}{a_t^{xy}} - 1, \quad t \in [0, T],$$

and assume that the random variable $\vartheta$ given as

$$\vartheta = \prod_{x, y \in S : x \neq y} \exp \left( - \int_0^T H_u^{xy} a_u^{xy} \kappa_u^{xy} du \right) \prod_{0 < u \leq T} \left( 1 + \kappa_u^{xy} \Delta H_u^{xy} \right),$$

satisfies $\mathbb{E}_{\mathbb{Q}}[\vartheta] = 1$. 6 Finally, define on $(\Omega, \hat{\mathcal{G}}_T)$ the probability $\mathbb{P}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} |_{\hat{\mathcal{G}}_T} = \vartheta,$$

6There exist many different sufficient conditions ensuring that $\mathbb{E}_{\mathbb{Q}}[\vartheta] = 1$. For example uniform boundedness of $\Lambda$ is such a condition.
Then

\[(i) \quad \mathbb{P}|_{\mathcal{F}_T} = \mathbb{Q}|_{\mathcal{F}_T}, \quad (3.9)\]

\[(ii) \quad X \text{ is an } (\mathbb{F}, \mathbb{G})-\text{CMC under } \mathbb{P} \text{ with the } \mathbb{F}\text{-intensity matrix process } \Lambda, \text{ and with the initial distribution satisfying} \]

\[\mathbb{P}(X_0 = x|\mathcal{F}_T) = \mathbb{Q}(X_0 = x|\mathcal{F}_T), \quad x \in S. \quad (3.10)\]

**Proof.** In view of Theorem 2.11, in order to prove (ii) it suffices to prove that:

(a) under measure \(\mathbb{P}\) process \(X\) has an \(\mathbb{F}\)-intensity \(\Lambda\),

(b) \(\mathbb{F}\) is \(\mathbb{P}\)-immersed in \(\mathbb{F} \vee \mathbb{G}\),

(c) all real valued \((\mathbb{F}, \mathbb{P})\)-martingales are orthogonal (under \(\mathbb{P}\)) to martingales \(M^x, x \in S\),

(d) \((3.10)\) holds.

We will prove these claims in separate steps. In the process, we will also demonstrate (i).

**Step 1:** Here we will show that \(\Lambda\) is an \(\mathbb{F}\)-intensity of \(X\) under \(\mathbb{P}\). Towards this end, we consider a \(\hat{G}\)-adapted process \(\eta\) given as

\[
\eta_t = \prod_{x,y \in S: x \neq y} \exp \left( - \int_0^t H^x_{u-} a^{xy} \kappa_{u}^{xy} du \right) \prod_{0 < u \leq t} (1 + \kappa_{u}^{xy} \Delta H^x_{u}), \quad t \in [0, T],
\]

so that

\[d\eta_t = \eta_{t-} \left( \sum_{x,y \in S: x \neq y} \kappa_{t}^{xy} dL^x_{t} \right), \quad \eta_0 = 1,
\]

where \(L^x_{\cdot}\) is a \((\hat{G}, \mathbb{Q})\)-martingale given by \((3.7)\). Consequently, process \(\eta\) is a \((\hat{G}, \mathbb{Q})\)-local martingale. Now, note that \(\eta_T = \vartheta\), and thus \(\mathbb{E}_{\mathbb{Q}} \eta_T = 1 = \eta_0\). Thus \(\eta\) is \((\hat{G}, \mathbb{Q})\)-martingale (on \([0, T]\)).

Since \(\kappa^{xy}\) is a left-continuous and \(\mathbb{F}\)-adapted process, and since \(\mathbb{F} \subset \hat{G}\), we conclude that \(\kappa^{xy}\) is \(\hat{G}\)-predictable. Thus, by the Girsanov theorem (see Brémaud [10, Thm. VI.T3]), we conclude that the \((\hat{G}, \mathbb{P})\) compensator of \(H^{xy}\) has density with respect to the Lebesgue measure given as\(^7\)

\[\mathbb{1}_{\{x\}}(X_{t-}) a^{xy}(1 + \kappa_{t}^{xy}) = \mathbb{1}_{\{x\}}(X_{t-}) a^{xy} \left( 1 + \frac{\lambda_{t}^{xy}}{a^{xy}} - 1 \right) = \mathbb{1}_{\{x\}}(X_{t-}) \lambda_{t-}^{xy}, \quad t \in [0, T].\]

So, for any \(x \neq y\), the process \(\widehat{K}^{xy}_{\cdot}\) defined as

\[\widehat{K}^{xy}_{t} := H^{xy}_{t} \mathbb{1}_{\{x\}}(X_{t-}) \lambda_{t-}^{xy} du,
\]

\(^7\)We use the usual convention that \(U_{0-} := 0\) for any real valued process \(U\).
is a $\hat{G}$–local martingale under $\mathbb{P}$. Since $X$ is a càdlàg process and since $\lambda^{xy}$ satisfies condition (C2) we see that
\[ K_t^{xy} = H_t^{xy} - \int_0^t H_u^{xy} du, \quad t \in [0, T], \tag{3.11} \]
is a $\hat{G}$–local martingale under $\mathbb{P}$. Since $\mathbb{F} \vee \mathbb{G} \subset \hat{G}$, and the process $\hat{K}^{xy}$ is $\mathbb{F} \vee \mathbb{G}$-adapted, we conclude that $\hat{K}^{xy}$ is also a $\mathbb{F} \vee \mathbb{G}$–local martingale. Thus according to Remark 2.9 we can use Theorem 2.8 to conclude that $\Lambda$ is an $\mathbb{F}$-intensity of $X$ under $\mathbb{P}$.

Step 2: In this step we prove (3.9). By definition of $\mathbb{P}$ and by the tower property of conditional expectations we conclude that for an arbitrary $\psi \in L^\infty(\mathcal{F}_T)$ we have
\[ E_{\mathbb{P}}(\psi) = E_{\mathbb{Q}}(\psi_{\mathbb{F}T}) = E_{\mathbb{Q}}(E_{\mathbb{Q}}(\psi_{\mathbb{F}T} | \hat{G}_0)) = E_{\mathbb{Q}}(\psi_{\mathbb{F}T} | \hat{G}_0)) = E_{\mathbb{Q}}(\psi). \]

Step 3: Next, we show that $\mathbb{F}$ is $\mathbb{P}$-immersed in $\mathbb{F} \vee \mathbb{G}$. In view of Proposition 5.9.1.1 in Jeanblanc, Yor and Chesney [20] it suffices to show that for any $\psi \in L^\infty(\mathcal{F}_T)$ and any $t \in [0, T]$ it holds that
\[ E_{\mathbb{F}}(\psi | \mathcal{F}_t \vee \mathcal{G}_t) = E_{\mathbb{P}}(\psi | \mathcal{F}_t), \quad \mathbb{P} - a.s. \tag{3.12} \]

Now, observe that
\[ \mathbb{P}(\eta_t > 0) = E_{\mathbb{Q}}(1_{\{\eta_t > 0\} \eta_T}) \geq E_{\mathbb{Q}}(1_{\{\eta_T > 0\} \eta_T}) = E_{\mathbb{Q}}(\eta_T) = 1, \]
so that $\mathbb{P}(\eta_t > 0) = 1$. Moreover, $\eta_t$ is $\mathcal{F}_t \vee \mathcal{G}_t$ measurable by (I3), (3.7) and (C1). Thus we have
\[ E_{\mathbb{P}}(\psi | \mathcal{F}_t \vee \mathcal{G}_t) = \frac{E_{\mathbb{Q}}(\psi \eta_T | \mathcal{F}_t \vee \mathcal{G}_t)}{E_{\mathbb{Q}}(\eta_T | \mathcal{F}_t \vee \mathcal{G}_t)} = \frac{E_{\mathbb{Q}}(\psi \mathbb{Q} \mathbb{E}_{\mathbb{Q}}(\eta_T | \hat{G}_t) | \mathcal{F}_t \vee \mathcal{G}_t)}{\eta_t} = \frac{E_{\mathbb{Q}}(\psi \eta_T | \mathcal{F}_t \vee \mathcal{G}_t)}{\eta_t} = E_{\mathbb{Q}}(\psi | \mathcal{F}_t \vee \mathcal{G}_t) = E_{\mathbb{Q}}(\psi | \mathcal{F}_t), \quad \mathbb{P} - a.s., \]
where the third equality holds in view of the fact that $\eta$ is $(\hat{G}, \mathbb{Q})$-martingale, and where the last equality holds since $\mathbb{F}$ is $\mathbb{Q}$-immersed in $\mathbb{F} \vee \mathbb{G}$ (see Appendix A, Corollary 5.2). Hence, using (3.9) we conclude
\[ E_{\mathbb{P}}(\psi | \mathcal{F}_t \vee \mathcal{G}_t) = E_{\mathbb{P}}(\psi | \mathcal{F}_t), \quad \mathbb{P} - a.s. \]

Consequently, (3.12) holds.

Step 4: Now we show the required orthogonality, that is we prove claim (c). Towards this end it suffices to prove that all real valued $(\mathbb{F}, \mathbb{P})$-martingales do not have common jumps with $X$ under $\mathbb{P}$ (see Remark 2.12). Let us take $Z$ to be an arbitrary real valued $(\mathbb{F}, \mathbb{P})$-martingale. Then, in view of (3.9) $Z$ is an $(\mathbb{F}, \mathbb{Q})$-martingale. By (I3), we have that $(\mathbb{F}, \mathbb{Q})$-martingales and Poisson processes in $\mathcal{N}$ are independent under $\mathbb{Q}$. Thus, by Lemma 5.3 in the Appendix A, the $\mathbb{Q}$ probability that process $Z$ has common jumps with any process from family $\mathcal{N}$ is zero. Consequently, in view of (3.6), the $(\mathbb{F}, \mathbb{Q})$-martingale $Z$ does not jump together with $X$, $\mathbb{Q}$-a.s. Therefore, by absolute continuity of $\mathbb{P}$ with respect to $\mathbb{Q}$, $\mathbb{P}$ probability that $Z$ jumps at the same time as $X$ is zero.
Step 5: Finally, we will show that (3.10) holds. Towards this end, let us take an arbitrary real valued function \( h \) on \( S \). The abstract Bayes rule yields

\[
\mathbb{E}_P(h(X_0)|\mathcal{F}_T) = \frac{\mathbb{E}_Q(h(X_0)\eta_T|\mathcal{F}_T)}{\mathbb{E}_Q(\eta_T|\mathcal{F}_T)} = \frac{\mathbb{E}_Q(h(X_0)\eta_T|\hat{G}_0)|\mathcal{F}_T)}{\mathbb{E}_Q(\eta_T|\hat{G}_0)|\mathcal{F}_T)}
\]

\[
= \mathbb{E}_Q(h(X_0)\mathbb{E}_Q(\eta_T|\hat{G}_0)|\mathcal{F}_T) = \mathbb{E}_Q(h(X_0)|\mathcal{F}_T) = \mathbb{E}_Q(h(X_0)|\mathcal{F}_0),
\]

where the last equality follows from the fact that by assumption (3.2) the initial condition of the process \( X \) satisfies

\[
\mathcal{Q}(X_0 = x|\mathcal{F}_T) = \mathcal{Q}(X_0 = x|\mathcal{F}_0), \quad x \in S.
\]  

(3.13)

Consequently,

\[
\mathbb{E}_P(h(X_0)|\mathcal{F}_0) = \mathbb{E}_P(\mathbb{E}_P(h(X_0)|\mathcal{F}_T)|\mathcal{F}_0) = \mathbb{E}_P(\mathbb{E}_Q(h(X_0)|\mathcal{F}_0)|\mathcal{F}_0) = \mathbb{E}_Q(h(X_0)|\mathcal{F}_0)
\]

\[
= \mathbb{E}_P(h(X_0)|\mathcal{F}_T).
\]

This completes the proof of (3.10), and the proof of the theorem. \( \square \)

4 \((\mathbb{F}, \mathbb{G})\)-CMC vs \((\mathbb{F}, \mathbb{G})\)-DSMC

In this section we first re-visit the concept of the doubly stochastic Markov chain. Then, we study relationships between conditional Markov chains and doubly stochastic Markov chains. These relationships are crucial for the theory of consistency of CMCs and for the theory of CMC copulæ, that are put forth in the companion paper [4].

4.1 \((\mathbb{F}, \mathbb{G})\)-DSMC

We start with introducing the concept of \((\mathbb{F}, \mathbb{G})\)-doubly stochastic Markov chain (\((\mathbb{F}, \mathbb{G})\)-DSMC for brevity), which generalizes the notion of \(\mathbb{F}\)-doubly stochastic Markov chain (cf. [18]), as well as the notion of continuous time \(\mathbb{G}\)-Markov chain.

**Definition 4.1.** A \(\mathbb{G}\)-adapted càdlàg process \( X = (X_t)_{t \in [0, T]} \) is called an \((\mathbb{F}, \mathbb{G})\)-doubly stochastic Markov chain with state space \( S \) if for any \( 0 \leq s \leq t \leq T \) and for every \( y \in S \) we it holds that

\[
\mathbb{P}(X_t = y | F_T \vee G_s) = \mathbb{P}(X_t = y | F_t \vee \sigma(X_s)).
\]  

(4.1)

We refer to [18] for examples of processes, which are \((\mathbb{F}, \mathbb{F}^X)\)-DSMCs. We remark that in [18] it was assumed that the chain \( X \) starts from some point \( x \in S \) with probability one, whereas here, we allow for the initial state \( X_0 \) to be a non-constant random variable.

With any \( X \), which is an \((\mathbb{F}, \mathbb{G})\)-DSMC, we associate a matrix valued random field

\[
P = (P(s, t), \ 0 \leq s \leq t \leq T), \text{ where } P(s, t) = (p_{xy}(s, t))_{x,y \in S}
\]

is defined by

\[
p_{xy}(s, t) = \frac{\mathbb{P}(X_t = y, X_s = x | F_t)}{\mathbb{P}(X_s = x | F_t)} \mathbb{1}_{\{P(X_s = x | F_t) > 0\}} + \mathbb{1}_{\{x = y\}} \mathbb{1}_{\{P(X_s = x | F_t) = 0\}}.
\]  

(4.2)

The following result provides a characterization of \((\mathbb{F}, \mathbb{G})\)-DSMC.
Proposition 4.2. A process \( X \) is an \((\mathbb{F}, \mathcal{G})\)-DSMC iff there exists a stochastic matrix valued random field \( \tilde{P}(s, t) = (\tilde{p}_{xy}(s, t))_{x,y \in S}, \ 0 \leq s \leq t \leq T \), such that:

1) for every \( s \in [0, T] \), the process \( \tilde{P}(s, \cdot) \) is \( \mathcal{F} \)-adapted on \([s, T]\),

2) for any \( 0 \leq s \leq t \leq T \) and for every \( x, y \in S \) we have

\[
\mathbb{I}_{\{X_s=x\}} \mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \mathcal{G}_s) = \mathbb{I}_{\{X_s=x\}} \tilde{p}_{xy}(s, t) \tag{4.3}
\]

or, equivalently, for every \( y \in S \) we have

\[
\mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \mathcal{G}_s) = \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} \tilde{p}_{xy}(s, t). \tag{4.4}
\]

Proof. We first prove the sufficiency. Using (4.3) we have that

\[
\mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \mathcal{G}_s) = \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} \tilde{p}_{xy}(s, t). \tag{4.5}
\]

So, taking conditional expectations with respect to \( \mathcal{F}_t \vee \sigma(X_s) \) on both sides of (4.5), observing that \( \mathcal{F}_t \vee \sigma(X_s) \subset \mathcal{F}_T \vee \mathcal{G}_s \), and using the tower property of conditional expectations, we obtain

\[
\mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \sigma(X_s)) = \mathbb{E}\left( \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} \tilde{p}_{xy}(s, t) \mid \mathcal{F}_t \vee \sigma(X_s) \right) = \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} \tilde{p}_{xy}(s, t),
\]

where the last equality follows from measurability of \( \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} \tilde{p}_{xy}(s, t) \) with respect to \( \mathcal{F}_t \vee \sigma(X_s) \). This and (4.5) imply

\[
\mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \mathcal{G}_s) = \mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \sigma(X_s)),
\]

which is (4.1).

Now we prove the necessity. First we observe that, using similar arguments as in Jakubowski and Niewegłowski [17, Lemma 3] (see also Bielecki, Crépey, Jeanblanc and Rutkowski [2, Lemma 2.1]), we have that for \( t \geq s \)

\[
\mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \sigma(X_s)) \tag{4.6}
\]

\[
= \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} \left( \frac{\mathbb{P}(X_t = y, X_s = x \mid \mathcal{F}_t)}{\mathbb{P}(X_s = x \mid \mathcal{F}_t)} \mathbb{I}_{\{\mathbb{P}(X_s=x|\mathcal{F}_t)>0\}} + \mathbb{I}_{\{y=x\}} \mathbb{I}_{\{\mathbb{P}(X_s=x|\mathcal{F}_t)=0\}} \right) \ \mathbb{P} - \text{a.s.}
\]

Consequently, in view of (4.2) we have

\[
\mathbb{P}(X_t = y \mid \mathcal{F}_t \vee \sigma(X_s)) = \sum_{x \in S} \mathbb{I}_{\{X_s=x\}} p_{x,y}(s, t).
\]

It is enough now to let \( \tilde{p}_{x,y}(s, t) = p_{x,y}(s, t) \), for \( x, y \in S, 0 \leq s \leq t \leq T \). \( \square \)

As we saw in the proof of Proposition 4.2 we can take \( \tilde{P} = P \), where \( P \) is given by (4.2).
Remark 4.3. Observe that, in view of the results in Rao [22], we have that for every $s \in [0, T]$ and almost every $\omega \in \Omega$ the function $P(s, \cdot)$ is measurable on $[s, T]$, and that for every $t \in [0, T]$ and almost every $\omega \in \Omega$, the function $P(\cdot, t)$ is measurable on $[0, t]$.

This, and (4.3) justify the following definition

Definition 4.4. The matrix valued random field $P = (P(s, t), 0 \leq s \leq t \leq T)$, defined by (4.2) is called the conditional transition probability matrix field (c-transition field for short) of $X$.

Remark 4.5. For the future reference, we note that (4.4) in the definition of an $(\mathcal{F}, \mathcal{G})$-DSMC, can be written in the following form (recall that we take $\tilde{P} = P$):

$$
\mathbb{E}(H^y_t | \mathcal{F}_T \vee \mathcal{G}_s) = \sum_{x \in S} H^x_t p_{xy}(s, t) \quad \text{for every } y \in S,
$$

which is equivalent to

$$
\mathbb{E}(H_t | \mathcal{F}_T \vee \mathcal{G}_s) = P(s, t)^\top H_s. \quad (4.7)
$$

We know that in the case of classical Markov chains the transition semigroup and the initial distribution of the chain characterize the finite dimensional distributions of the chain, and thus they characterize the law of the chain. The next proposition shows that, in case of an $(\mathcal{F}, \mathcal{G})$–DSMC $X$, the c-transition field $P$ of $X$ and the conditional law of $X_0$ given $\mathcal{F}_T$ characterize conditional law of $X$ given $\mathcal{F}_T$.

Proposition 4.6. If $X$ is an $(\mathcal{F}, \mathcal{G})$–DSMC with c-transition field $P$, then for arbitrary $0 \leq t_1 \leq \ldots \leq t_n \leq t \leq T$ and $(x_1, \ldots, x_n) \in S^n$ it holds that

$$
\mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_n} = x_n | \mathcal{F}_T) = \sum_{x_0 \in S} \mathbb{P}(X_0 = x_0 | \mathcal{F}_T) p_{x_0, x_1}(0, t_1) \prod_{k=0}^{n-1} p_{x_k, x_{k+1}}(t_k, t_{k+1}).
$$

Moreover, if

$$
\mathbb{P}(X_0 = x_0 | \mathcal{F}_T) = \mathbb{P}(X_0 = x_0 | \mathcal{F}_0) \quad \text{for every } x_0 \in S, \quad (4.9)
$$

then for arbitrary $0 \leq t_1 \leq \ldots \leq t_n \leq t \leq T$ and $(x_1, \ldots, x_n) \in S^n$

$$
\mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_n} = x_n | \mathcal{F}_T) = \mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_n} = x_n | \mathcal{F}_T). \quad (4.10)
$$

Proof. Let us fix arbitrary $x_1, \ldots, x_k \in S$ and $0 \leq t_1 \leq \ldots \leq t_k \leq t \leq T$, and let us define a set $A$ by

$$
A = \{X_{t_1} = x_1, \ldots, X_{t_k} = x_k\}.
$$

Note that by Lemma 3.1 in [18] we have

$$
\mathbb{P}(A | \mathcal{F}_T \vee \mathcal{G}_0) \mathbb{1}_{\{X_0 = x_0\}} = \mathbb{1}_{\{X_0 = x_0\}} p_{x_0, x_1}(0, t_1) \prod_{n=1}^{k-1} p_{x_n, x_{n+1}}(t_n, t_{n+1}).
$$
Consequently,
\[
\mathbb{P}(A \mid \mathcal{F}_T) = \sum_{x_0 \in S} \mathbb{P}(X_0 = x_0 \mid \mathcal{F}_T)p_{x_0,x_1}(0,t_1) \prod_{k=0}^{n-1} p_{x_k,x_{k+1}}(t_k,t_{k+1})
\]
which proves (4.8). Thus, in view of (4.9), the following equality is satisfied
\[
\mathbb{P}(A \mid \mathcal{F}_T) = \sum_{x_0 \in S} \mathbb{P}(X_0 = x_0 \mid \mathcal{F}_0)p_{x_0,x_1}(0,t_1) \prod_{k=0}^{n-1} p_{x_k,x_{k+1}}(t_k,t_{k+1}).
\]
Since \( P \) is a \( c \)-transition field we obtain that \( \mathbb{P}(A \mid \mathcal{F}_T) \) is \( \mathcal{F}_t \)-measurable as a product of \( \mathcal{F}_t \)-measurable random variables. Thus, the tower property of conditional expectations yields (4.10).

\[\square\]

**Corollary 4.7.** Let \( X \) be an \((\mathcal{F},\mathcal{G})\)-DSMC with \( X_0 \) satisfying (4.9). Then \( \mathbb{F} \) is \( \mathbb{P} \)-immersed in \( \mathbb{F} \vee \mathbb{F}^X \).

**Proof.** In view of Proposition 4.6 process \( X \) satisfies (4.10). This, by [17, Lemma 2], is equivalent to \( \mathbb{P} \)-immersion of \( \mathbb{F} \) in \( \mathbb{F} \vee \mathbb{F}^X \). \[\square\]

In analogy to the concept of \( \mathbb{F} \)-intensity for \((\mathcal{F},\mathcal{G})\)-CMCs, one considers the concept of intensity with regard to \((\mathcal{F},\mathcal{G})\)-DSMCs. Definition 4.8 introduces a concept of such intensity. This definition is stated in the form, which is consistent with the way the original definition of intensity for DSMCs was introduced in [18]. Later on, we will show that this definition can be equivalently stated in the form analogous to Definition 2.3.

**Definition 4.8.** We say that an \( \mathcal{F} \)-adapted matrix-valued process \( \Gamma = (\Gamma_s)_{s \geq 0} = ([\gamma_{xy}^s]_{x,y \in S})_{s \geq 0} \) is an intensity of \((\mathcal{F},\mathcal{G})\)-DSMC \( X \) if the following conditions are satisfied:

1) \[
\int_{[0,T]} \sum_{x \in S} |\gamma_{xx}^s| \, ds < \infty. \tag{4.11}
\]

2) \[
\gamma_{xy}^s \geq 0 \quad \forall x, y \in S, x \neq y, \quad \gamma_{xx}^s = -\sum_{y \in S; y \neq x} \gamma_{xy}^s \quad \forall x \in S. \tag{4.12}
\]

3) The Kolmogorov backward equation holds: for all \( v \leq t \),
\[
P(v,t) - I = \int_v^t \Gamma_u P(u,t) \, du. \tag{4.13}
\]

4) The Kolmogorov forward equation holds: for all \( v \leq t \),
\[
P(v,t) - I = \int_v^t P(v,u) \Gamma_u \, du. \tag{4.14}
\]
Remark 4.9. The above Kolmogorov equations admit unique solution provided that $\Gamma$ satisfies (4.11). The unique solution of Kolmogorov equation (4.13) is given by the formula which is known as Peano-Baker series

$$P(v,t) = I + \sum_{n=1}^{\infty} \int_v^t \int_v^{v_1} \cdots \int_v^{v_{n-1}} \Gamma_{v_1} \cdots \Gamma_{v_n} dv_n \cdots dv_1,$$

and the solution of (4.14) is given by

$$P(v,t) = I + \sum_{n=1}^{\infty} \int_v^t \int_v^{v_1} \cdots \int_v^{v_{n-1}} \Gamma_{v_1} \cdots \Gamma_{v_n} dv_n \cdots dv_1.$$

There is also a different useful representation of the solution of Kolmogorov equations. It is given in terms of a matrix exponential, and it is called the Magnus expansion:

$$P(v,t) = \exp(\Phi(v,t)),$$

where $\Phi(v,t)$ is the Magnus series

$$\Phi(v,t) = \sum_{k=1}^{\infty} \Phi_k(v,t).$$

We refer to Blanes, Casas, Oteo and Ros [9] for a detailed statement of the Magnus expansion in deterministic case. The formulae found in [9] are adequate in our case, as here we use the Magnus expansion of $P(v,t)$ for every $\omega \in \Omega$.

It is easily seen from the Magnus expansion, that $P(v,t)$ has inverse $Q(v,t) = \exp(-\Phi(v,t))$. For an alternative proof of invertibility of $P(v,t)$ we refer to [18, Proposition 3.11.iii)].

4.1.1 Martingale characterizations of $(F,G)$-DSMC

It turns out that the $(F,G)$-DSMC property of process $X$ is fully characterized by the martingale property (with respect to the filtration $\hat{G}$ given by (2.1)) of some processes related to $X$. These characterizations are given in the next theorem.

Theorem 4.10. Let $(X_t)_{t \in [0,T]}$ be an $S$-valued stochastic process and $(\Gamma_t)_{t \in [0,T]}$ be an $F$-adapted matrix valued process satisfying (4.11) and (4.12). The following conditions are equivalent:

i) The process $X$ is an $(F,G)$-DSMC with the intensity process $\Gamma$.

ii) The processes $\hat{M}^x$ defined by

$$\hat{M}^x_t := H^x_t - \int_{[0,t]} \gamma^{X_u,x} du, \quad x \in S,$$

are $\hat{G}$-local martingales.

iii) The processes $K^{xy}$ defined by

$$K^{xy}_t := H^{xy}_t - \int_{[0,t]} H^x_s \gamma^{xy}_s ds, \quad x, y \in S, \ x \neq y,$$

are $\hat{G}$-martingales.
where

\[ H_{xy}^t := \int_{[0,t]} H_{u_y}^x dH_{u_x}^y, \quad (4.17) \]

are $\hat{G}$-local martingales.

iv) The process $L$ defined by

\[ L_t := Z_t^\top H_t, \quad (4.18) \]

where $Z$ is a unique solution to the random integral equation

\[ dZ_t = -\Gamma_t Z_t dt, \quad Z_0 = I, \quad (4.19) \]

is a $\hat{G}$-local martingale.

v) For any $t \in [0,T]$, the process $N^t$ defined as

\[ N_t^s := P(s,t)^\top H_s \quad \text{for} \ 0 \leq s \leq t. \quad (4.20) \]

is a $\hat{G}$ martingale, where

\[ P(s,t) := Z_s Y_t \]

with

\[ dY_t = Y_t \Gamma_t dt, \quad Y_0 = I, \quad t \in [0,T]. \]

Proof. The proof of equivalence of (i)--(iv) goes along the lines of the proof of [18, Theorem 4.1]; only minor and straightforward modifications are needed, and therefore the proof is omitted. Equivalence of (iv) and (v) follows from formula

\[ N_t^s = Y_t^\top L_s \quad \text{for} \ 0 \leq s \leq t \]

and the fact that $Y_t$ is uniformly bounded $\hat{G}_0$ measurable invertible matrix (Lemma 5.4).

The following result is direct counterpart of Proposition 2.6 and therefore we omit its proof.

**Proposition 4.11.** Let $X$ be an $(\mathcal{F}, \mathcal{G})$-DSMC.

i) If $\Gamma$ and $\hat{\Gamma}$ are intensities of $X$, then they are equivalent relative to $X$. In particular intensity of $X$ is unique up to equivalence relative to $X$.

ii) Let $\Gamma$ be an intensity of $X$. If $\hat{\Gamma}$ is an $\mathcal{F}$-adapted process equivalent to $\Gamma$ relative to $X$, then $\hat{\Gamma}$ is intensity of $X$.

We will not discuss here the question of existence of an $(\mathcal{F}, \mathcal{G})$-DSMC with intensity $(\Gamma_t)_{t \in [0,T]}$. This question will be addressed in some generality in Bielecki, Jakubowski and Niewęgłowski [3]. Instead, in the next section, we will show that any $(\mathcal{F}, \mathcal{G})$-CMC process $X$ constructed in Theorem 3.4 is also an $(\mathcal{F}, \mathcal{G})$-DSMC.

Since an $(\mathcal{F}, \mathcal{G})$-DSMC $X$ is a $S$-valued càdlàg process, then it is a pure jump semimartingale. This observation sheds a new light on the intensity of $X$ as the following corollary shows.
Corollary 4.12. Intensity of an \((\mathcal{F}, \mathcal{G})\)-DSMC \(X\) is an \(\mathcal{F}\)-intensity of \(X\) in the sense of Definition 2.3.

Proof. The process \(\hat{M}\) is a \(\hat{\mathcal{G}}\)-local martingale by Theorem 4.10.ii). But \(\hat{M}\) is also \(\mathcal{F} \vee \mathcal{G}\)-adapted. Hence \(\hat{M}\) is an \(\mathcal{F} \vee \mathcal{G}\)-local martingale, which implies that the \(\mathcal{F}\)-adapted process \(\Gamma\) is an \(\mathcal{F}\)-intensity of \(X\).

4.2 Relation between CMC and DSMC

In this section we present some aspects of relationship between the classes of \((\mathcal{F}, \mathcal{G})\)-CMCs and \((\mathcal{F}, \mathcal{G})\)-DSMCs.

4.2.1 DSMCs that are CMCs

Proposition 4.13. Assume that \(\mathcal{F}\) and \(\mathcal{G}\) satisfy the immersion property (2.10), and that \(X\) is an \((\mathcal{F}, \mathcal{G})\)-DSMC. Then \(X\) is an \((\mathcal{F}, \mathcal{G})\)-CMC. In addition if \(X\) considered as an \((\mathcal{F}, \mathcal{G})\)-DSMC admits intensity \(\Gamma\), then \(X\) considered as an \((\mathcal{F}, \mathcal{G})\)-CMC admits \(\mathcal{F}\)-intensity \(\Lambda = \Gamma\).

Proof. Let us fix arbitrary \(x_1, \ldots, x_k \in S\) and \(0 \leq t \leq t_1 \leq \ldots \leq t_k \leq T\), and let us define a set \(A\) by

\[
A = \left\{ X_{t_1} = x_1, \ldots, X_{t_k} = x_k \right\}.
\]

We need to show that

\[
P(A|\mathcal{F}_T \vee \mathcal{G}_t) = P(A|\mathcal{F}_t \vee \sigma(X_t)).
\]

Towards this end we first note that by Lemma 3.1 in [18] we have

\[
P(A|\mathcal{F}_T \vee \mathcal{G}_t) \mathbb{I}_{\{X_{t_1} = x\}} = \mathbb{I}_{\{X_{t_1} = x\}} p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}).
\]

The tower property of conditional expectation and (4.22) imply

\[
P(A|\mathcal{F}_t \vee \mathcal{G}_t) = \mathbb{E} \left( \sum_{x \in S} \mathbb{E}(\mathbb{I}_{A|\mathcal{F}_T \vee \mathcal{G}_t} \mathbb{I}_{\{X_{t_1} = x\}} | \mathcal{F}_t \vee \mathcal{G}_t) \right) = \mathbb{E} \left( \sum_{x \in S} \mathbb{I}_{\{X_{t_1} = x\}} p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}) | \mathcal{F}_t \vee \mathcal{G}_t \right) = \sum_{x \in S} \mathbb{I}_{\{X_{t_1} = x\}} \mathbb{E} \left( p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}) | \mathcal{F}_t \vee \mathcal{G}_t \right).
\]

Thus using the assumed immersion property of \(\mathcal{F}\) in \(\mathcal{F} \vee \mathcal{G}\) we obtain

\[
P(A|\mathcal{F}_t \vee \mathcal{G}_t) = \sum_{x \in S} \mathbb{I}_{\{X_{t_1} = x\}} \mathbb{E} \left( p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}) | \mathcal{F}_t \right),
\]

which implies the CMC property.

The second claim of the theorem follows immediately from Corollary 4.12. \qed
The following example illustrates the use of Proposition 4.13.

**Example 4.14.** (Time changed discrete Markov chain) Consider process $\bar{C}$, which is a discrete time Markov chain with values in $S = \{1, \ldots, K\}$ and with transition probability matrix $P$. In addition consider process $N$, which is a Cox process with càdlàg $\mathbb{F}$-intensity process $\tilde{\lambda}$. From [17, Theorem 7 and 9] we know that under assumption that the processes $(\bar{C}_k)_{k \geq 0}$ and $(N_t)_{t \in [0,T]}$ are independent and conditionally independent given $\mathcal{F}_T$, the process

$$C_t := \bar{C}_{N_t}$$

is an $(\mathbb{F}, \mathbb{F}_C)$-DSMC. Moreover $C$ admits intensity process $\Gamma = [\gamma^{xy}]$ given as

$$\gamma^{xy}_t = (P - I)_{x,y} \tilde{\lambda}_t.$$ 

Thus, by Corollary 4.7 and Proposition 4.13, the process $C$ is an $(\mathbb{F}, \mathbb{F}_C)$-CMC with $\mathbb{F}$-intensity $\Lambda = \Gamma$.

### 4.2.2 CMCs that are DSMCs

**Theorem 4.15.** Suppose that $X$ is an $(\mathbb{F}, \mathbb{G})$-CMC admitting an $\mathbb{F}$-intensity $\Lambda$. In addition, suppose that $X$ is also an $(\mathbb{F}, \mathbb{G})$-DSMC with an intensity $\Gamma$. Then $\Gamma$ is an $\mathbb{F}$-intensity of $X$ and $\Lambda$ is an intensity of $X$.

**Proof.** It follows from Corollary 4.12 that $\Gamma$ is an $\mathbb{F}$-intensity. Thus by Proposition 2.6 $\Lambda$ and $\Gamma$ are equivalent relative to $X$. Consequently, by Proposition 4.11 process $\Lambda$ is an intensity of $X$. 

This and Proposition 4.6 imply

**Corollary 4.16.** If $X$ is an $(\mathbb{F}, \mathbb{G})$-CMC with $\mathbb{F}$-intensity and also an $(\mathbb{F}, \mathbb{G})$-DSMC with intensity, then $\mathbb{F}$-intensity (or, equivalently, intensity) and $\mathcal{F}_T$-conditional distribution of $X_0$ determine the $\mathcal{F}_T$-conditional distribution of $X$.

In case of process $X$ constructed in Theorem 3.4 the result of Theorem 4.15 can be strengthen as follows.

**Proposition 4.17.** Let $X$ be a process constructed in Theorem 3.4, so that $X$ is an $(\mathbb{F}, \mathbb{G})$-CMC process with an $\mathbb{F}$-intensity process $\Lambda$. Then $X$ is also an $(\mathbb{F}, \mathbb{G})$-DSMC with an intensity process $\Gamma = \Lambda$.

**Proof.** In Step 1 of the proof of Theorem 3.4 we showed that the processes $\hat{K}^{xy}$, $x, y \in S$, $x \neq y$, given by (3.11), are $\mathbb{G}$–local martingales. Thus, by Theorem 4.10, $X$ is an $(\mathbb{F}, \mathbb{G})$-DSMC with intensity $\Lambda$. 

□
4.2.3 Pure jump semimartingales that are both CMCs and DSMCs

**Theorem 4.18.** Let \( \mathbb{F}, \mathbb{G} \) satisfy the immersion property (2.10). Assume that \( S \)-valued \( \mathbb{G} \)-adapted pure jump semimartingale \( X \) admits an \( \mathbb{F} \)-intensity \( \Lambda \). Moreover suppose that the orthogonality property (2.11) is fulfilled. Then \( X \) is an \( (\mathbb{F}, \mathbb{G}) \)-CMC and an \( (\mathbb{F}, \mathbb{G}) \)-DSMC with intensity \( \Lambda \).

**Proof.** In Theorem 2.11 we showed that \( X \) is an \( (\mathbb{F}, \mathbb{G}) \)-CMC. In order to prove that \( X \) is an \( (\mathbb{F}, \mathbb{G}) \)-DSMC it suffices to show that for for every \( A \in \mathcal{F}_T, B \in \mathcal{G}_t, t \leq u \) and \( y \in S \) it holds that

\[
\mathbb{E}(1_A 1_B 1_{\{X_u=y\}}) = \mathbb{E}(1_A 1_B H_t^\top Z_t Y_u e_y),
\]

where \( Y \) and \( Z \) are defined by (5.5) and (5.4), respectively. Indeed, by the monotone class theorem, the above yields

\[
\mathbb{P}(X_u = y | \mathcal{F}_T \vee \mathcal{G}_t) = H_t^\top Z_t Y_u e_y.
\]

Consequently, since the right hand side of (4.24) is measurable with respect to \( \mathcal{F}_t \vee \sigma(X_t) \) we obtain the desired \( (\mathbb{F}, \mathbb{G}) \)-DSMC property of \( X \).

It remains to prove (4.23). Since \( Z_t Y_u e_y 1_A \in L^1(\mathcal{F}_T) \) (see Lemma 5.4), the following formula

\[
V_s := 1_B H_s^\top \mathbb{E}(Z_s Y_u e_y 1_A | \mathcal{F}_s), \quad s \in [t, u],
\]

well defines a process \( V \) on \([t, u]\). Now, let a Doob martingale \( D \) be defined on \([0, T]\) by

\[
D_s = \mathbb{E}(1_A 1_B 1_{\{X_u=y\}} | \mathcal{F}_s \vee \mathcal{G}_s), \quad s \in [0, T].
\]

The immersion property (2.10) leads to

\[
V_t = 1_B H_t^\top \mathbb{E}(Z_t Y_u e_y 1_A | \mathcal{F}_t \vee \mathcal{G}_t) = \mathbb{E}(1_A 1_B H_t^\top Z_t Y_u e_y | \mathcal{F}_t \vee \mathcal{G}_t).
\]

Next, we will show that \( V_t = D_t \), which in turn will imply that

\[
\mathbb{E}(1_A 1_B 1_{\{X_u=y\}}) = \mathbb{E} V_t = \mathbb{E}(1_A 1_B H_t^\top Z_t Y_u e_y),
\]

which is (4.23).

In order to show that \( D_t = V_t \) we will demonstrate a stronger result, namely that \( V = D \) on \([t, u]\). To this end, note that by Lemma 5.5 \( V \) is an \( \mathbb{F} \vee \mathbb{G} \)-martingale on the interval \([t, u]\).

Thus, to show that \( V = D \) on \([t, u]\) it suffices to show that \( V_u = D_u \). For this purpose, let us define on \([u, T]\) the process \( W \) by

\[
W_s := 1_B 1_{\{X_u=y\}} \mathbb{E}(1_A | \mathcal{F}_s).
\]

Next we observe that for \( s \in [u, T] \)

\[
D_s = 1_B 1_{\{X_u=y\}} \mathbb{E}(1_A | \mathcal{F}_s \vee \mathcal{G}_s) = 1_B 1_{\{X_u=y\}} \mathbb{E}(1_A | \mathcal{F}_s) = W_s,
\]
where the penultimate equality follows from immersion of $F$ in $F \vee G$. Hence, using the fact that $Z_u Y_u = I$ (see Lemma 5.4), we have

$$D_u = W_u = \mathbb{1}_B \mathbb{1}_{X_u = y} \mathbb{E}(\mathbb{1}_A|F_u) = \mathbb{1}_B H_u^\top e_y \mathbb{E}(\mathbb{1}_A|F_u) = \mathbb{1}_B H_u^\top \mathbb{E}(Z_u Y_u e_y \mathbb{1}_A|F_u) = V_u.$$  

Thus, by the martingale property of $D$ and $V$ (on $[t, u]$), we conclude that $D = V$ on $[t, u]$. This completes the proof of (4.23) and, consequently, demonstrates that $X$ is an $(\mathbb{F}, \mathbb{G})$-DSMC.

In order to verify that $X$ admits intensity $\Lambda$ we first note that the random field $P$ defined as

$$P(t, u) := Z_t Y_u$$  

solves the Kolmogorov equations (4.13) and (4.14). Next we observe that (4.24) implies the martingale property of $N^u$ given as in (4.20), with $P(t, u)$ as in (4.26). Thus, by Theorem 4.10, $\Lambda$ is an intensity of $X$. The proof of the theorem is now complete. 

$\square$

5 Appendixes

Appendix A

In this appendix we provide technical results needed for derivations done in Section 3.

**Lemma 5.1.** Let $\xi$ be an $S$–valued random variable defined on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{H}, \mathbb{P})$ with $\mathbb{H} = \{\mathcal{H}_t\}_{t \in [0, T]}$. Suppose that

$$\mathbb{E}_\mathbb{P}(h(\xi)|\mathcal{H}_T) = \mathbb{E}_\mathbb{P}(h(\xi)|\mathcal{H}_0)$$  

for every real valued function $h$ on $S$. Then $\mathbb{H}$ is $\mathbb{P}$–immersed in $\mathbb{H} \vee \sigma(\xi)$.

$\text{Proof.}$ It is sufficient to prove (c.f. [8, Lemma 6.1.1]) that for every $\psi \in L^\infty(\mathcal{H}_T)$ it holds that

$$\mathbb{E}_\mathbb{P}(\psi|\mathcal{H}_t \vee \sigma(\xi)) = \mathbb{E}_\mathbb{P}(\psi|\mathcal{H}_t), \quad \forall t \in [0, T].$$  

Let us fix $t \in [0, T]$ and $\psi \in L^\infty(\mathcal{H}_T)$. By the standard $\pi - \lambda$ system arguments it is enough to show that

$$\mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A \mathbb{1}_B(\xi)) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi|\mathcal{H}_t) \mathbb{1}_A \mathbb{1}_B(\xi)), \quad \forall A \in \mathcal{H}_t, B \subseteq S, \quad (5.3)$$

where

$$\mathbb{1}_B(\xi) = \begin{cases} 1, & \xi \in B, \\ 0, & \xi \notin B. \end{cases}$$

Towards this end we first derive another representation of the right hand side in (5.3),

$$\mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi|\mathcal{H}_t) \mathbb{1}_A \mathbb{1}_B(\xi)) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A|\mathcal{H}_t) \mathbb{1}_B(\xi)) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A|\mathcal{H}_t) \mathbb{1}_B(\xi)|\mathcal{H}_T))$$

$$= \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A|\mathcal{H}_t) \mathbb{1}_B(\xi)|\mathcal{H}_T)) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A|\mathcal{H}_t) \mathbb{1}_B(\xi)|\mathcal{H}_0))$$

$$= \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A|\mathcal{H}_0) \mathbb{1}_B(\xi)|\mathcal{H}_T)) = \mathbb{E}_\mathbb{P}(\psi \mathbb{1}_A \mathbb{1}_B(\xi)|\mathcal{H}_0),$$
where the fourth equality follows from (5.1). The left hand side of (5.3) can be rewritten as

$$\mathbb{E}_{\tilde{P}}(\psi 1_A 1_B(\xi)) = \mathbb{E}_{\tilde{P}}(\mathbb{E}_{\tilde{P}}(\psi 1_A 1_B(\xi) | \mathcal{H}_T)) = \mathbb{E}_{\tilde{P}}(\psi 1_A \mathbb{E}_{\tilde{P}}(1_B(\xi) | \mathcal{H}_0)),$$

where the last equality follows from (5.1). This proves (5.3) and thus concludes the proof of the lemma. □

**Corollary 5.2.** Let $\mathbb{K}$ be a filtration on $(\Omega, \mathcal{A}, \tilde{P})$, such that it is independent of $\mathbb{H} \vee \sigma(\xi)$. Suppose that $\xi$ satisfies (5.1). Then $\mathbb{H}$ is $\tilde{P}$-immersed in $\mathbb{H} \vee \mathbb{K} \vee \sigma(\xi)$.

**Proof.** The result follows from Lemma 5.1 and from the fact that if $\mathbb{H}^1$ and $\mathbb{H}^2$ are two independent filtrations on $(\Omega, \mathcal{A}, \tilde{P})$, then $\mathbb{H}^1$ is $\tilde{P}$-immersed in $\mathbb{H}^1 \vee \mathbb{H}^2$. □

In the next lemma we use the same probabilistic setup as in Section 2.

**Lemma 5.3.** Let $X$ be an $\mathbb{F}$ adapted càdlàg process, and let $N$ be a Poisson process. Suppose that $N$ and $\mathbb{F}$ are independent. Then

$$\mathbb{P}(\{\omega \in \Omega : \exists t \in [0,T] \text{ s.t. } \Delta X_t(\omega) \Delta N_t(\omega) \neq 0\}) = 0.$$

**Proof.** First note that both $X$ and $N$ have countable number of jumps on $[0,T]$, and let denote their jump times as $(T_n)_{n \geq 1}$ and $(S_n)_{n \geq 1}$, respectively. Independence of $N$ and $\mathbb{F}$ implies that $(T_n)_{n \geq 1}$ and $(S_n)_{n \geq 1}$ are independent. Since each random variable $S_n$ is Gamma distributed and thus has density, then for any $n, k \geq 1$ it holds that

$$\mathbb{P}(T_n = S_k) = 0.$$

Since

$$A := \{\omega : \exists t \in [0,T] \text{ s.t. } \Delta X_t(\omega) \Delta N_t(\omega) \neq 0\} = \bigcup_{n,k \geq 1} \{\omega : T_n(\omega) = S_k(\omega)\}$$

we have

$$\mathbb{P}(A) \leq \sum_{n,k \geq 1} \mathbb{P}(T_n = S_k) = 0.$$

□

**Appendix B**

In this appendix we derive some technical results that are used in Section 2 and Section 4.

**Lemma 5.4.** Let $Z$ and $Y$ be solutions of the random ODE’s

$$dZ_t = -\Psi_t Z_t dt, \quad Z_0 = I, \quad t \in [0,T], \quad (5.4)$$

$$dY_t = Y_t \Psi_t dt, \quad Y_0 = I, \quad t \in [0,T], \quad (5.5)$$
where $\Psi$ is an appropriately measurable matrix valued process satisfying (2.5)\(^8\) and such that
\[
\sum_{x \in S} \int_0^T |\psi_{x}^{xy}| \, du < \infty. \tag{5.6}
\]

Then, the matrix valued random processes $(Y_t)_{0 \leq t \leq T}$ and $(Z_t Y_v)_{0 \leq t \leq v}$, $v \in [0,T]$, have elements that are nonnegative and bounded by 1. Moreover
\[
Z_t Y_t = I \quad \text{for } t \in [0,T]. \tag{5.7}
\]

**Proof.** Using Remark 4.9 one can verify that for each $t$, the functions $Y_t(\cdot)$ and $Z_t(\cdot)$ are measurable, so that $Y$ and $Z$ are matrix valued random processes. Since $\Psi$ satisfies (2.5), then for every $\omega$, $Y(\omega)$ is a solution of matrix forward Kolmogorov equation, and so its elements belong to the interval $[0,1]$ (since they give conditional probabilities, see e.g. Gill and Johansen [13, Thm. 12 and Thm. 13]).

Next, observe that, letting $Z(t,v) = Z_t Y_v$ we have that
\[
d_t Z(t,v) = (dZ_t) Y_v = -\Psi_t Z_t Y_v dt = -\Psi_t Z(t,v) dt, \quad 0 \leq t \leq v.
\]
Moreover, it is easy to verify that $Z(v,v) = Z_v Y_v = Z_0 Y_0 = I$. We thus see that for every $\omega$, $Z(\cdot,v)(\omega)$ satisfies the Kolmogorov backward equation,
\[
d_t Z(t,v) = -\Psi_t Z(t,v) dt, \quad 0 \leq t \leq v, \quad Z(v,v) = I,
\]
and so, it has non-negative elements bounded by 1. \hfill \Box

The following lemma is used in the proof of Theorem 2.11.

**Lemma 5.5.** Suppose that assumptions of Theorem 2.11 are satisfied. Let $U$ be an $\mathbb{R}^d$-valued bounded random variable, and let $Z$ and $Y$ be solutions of the random ODE’s (5.4) and (5.5), respectively. Fix $u$ and $v$ satisfying $0 \leq u < v \leq T$, and fix set $A \in \mathcal{F}_u \lor \mathcal{G}_u$.

Then, process $V$ given by
\[
V_t = 1_A H_t^{\top} Z_t E(Y_v U | \mathcal{F}_t), \quad t \in [0,T],
\]
is an $\mathbb{F} \lor \mathbb{G}$ martingale on the interval $[u,v]$.

**Proof.** It suffices to prove that the process $\widehat{V}$ given as
\[
\widehat{V}_t = H_t^{\top} Z_t E(Y_v U | \mathcal{F}_t), \quad t \in [0,T],
\]
is an $\mathbb{F} \lor \mathbb{G}$ martingale on $[0,v]$. Furthermore, since all components of $H_t$ and $Z_t Y_v$ are non-negative and bounded by 1 (for the latter see Lemma 5.4), and since random variable $U$ is bounded, then it suffices to show that $\widehat{V}$ is an $\mathbb{F} \lor \mathbb{G}$ local martingale.

\(^8\)For any $\omega$ for which $\Psi$ does satisfy (2.5), we set $\Psi_t(\omega) = 0$ for all $t \in [0,T]$. 

Towards this end we first verify that vector valued process \( L = (L^x, x \in S)^\top \) defined by \( L_t := H_t^\top Z_t, \ t \in [0, T] \), is an \( \mathbb{F} \lor \mathbb{G} \)-local martingale with the following representation

\[
L_t = H_0^\top + \int_0^t dM_u^\top Z_u, \ t \in [0, T].
\] (5.8)

Indeed, since \( \Lambda \) is an \( \mathbb{F} \)-intensity, integration by parts yields that

\[
dL_t = d(H_t^\top Z_t) = H_t^\top dZ_t + dH_t^\top \cdot Z_t = -H_t^\top \Lambda_t Z_t dt + dH_t^\top \cdot Z_t = dM_t^\top Z_t.
\]

Next, we observe that the vector valued process \( U(\cdot, v) = (U^x(\cdot, v), x \in S)^\top \) defined by

\[
U^x(t, v) = \sum_{y \in S} \mathbb{E}(Y_{v}^{xy}U^y|\mathcal{F}_t), \ t \in [0, T],
\]

is an \( \mathbb{F} \)-martingale.

Thus, by assumptions (2.10) and (2.11) in Theorem 2.11, its components are orthogonal to components of \( M \). Hence the square bracket processes \( [M^y, U^x(\cdot, v)], x, y \in S \), are \( \mathbb{F} \lor \mathbb{G} \)-local martingales. By properties of square brackets (cf. Protter [21, Thm. II.6.28]) we obtain

\[
[L^x, U^x(\cdot, v)]_t = \sum_{y \in S} \int_0^t Z_{u}^{xy} d[M^y, U^x(\cdot, v)]_u.
\]

Thus, by predictability and local boundedness of \( Z \), and by [21, Thm. IV.2.29], we conclude that process \( [L^x, U^x(\cdot, v)] \) is a local martingale, and consequently that local martingales \( L^x \) and \( U^x(\cdot, v) \) are orthogonal. Since,

\[
\hat{V}_t = L_t U(t, v) = \sum_{x \in S} L_t^x U^x(t, v), \ t \in [0, T],
\]

we conclude that \( \hat{V} \) is an \( \mathbb{F} \lor \mathbb{G} \)-local martingale as a sum of local martingales. \( \square \)

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