FLAT CONNECTIONS AND QUANTUM GROUPS

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Abstract. We review the Kohno–Drinfeld theorem and a conjectural analogue relating quantum Weyl groups to the monodromy of a flat connection $\nabla_C$ on the Cartan subalgebra of a complex, semi-simple Lie algebra $\mathfrak{g}$ with poles on the root hyperplanes and values in any $\mathfrak{g}$-module $V$. We sketch our proof of this conjecture when $\mathfrak{g} = \mathfrak{sl}_n$ and when $\mathfrak{g}$ is arbitrary and $V$ is a vector, spin or adjoint representation. We also establish a precise link between the connection $\nabla_C$ and Cherednik’s generalisation of the Knizhnik–Zamolodchikov connection to finite reflection groups.

1. Introduction

The aim of this paper is to discuss a principle which first arose in the work of Kohno and Drinfeld and states, roughly speaking, that quantum groups are natural receptacles for the monodromy of certain integrable, first order PDE’s. Quite how general this principle is I do not know, but, as I will try to show, it does extend beyond its original formulation.

The following diagram gives an overview of the paper

Here’s how it should be read. To any complex, semi-simple Lie group $G$ with Lie algebra $\mathfrak{g}$, we may canonically attach two finite groups. The first,

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the symmetric group $\mathfrak{S}_n$, is an external symmetry group and acts on the $n$–fold tensor product of any finite–dimensional $G$–module $V$. This action admits two remarkable deformations through representations of Artin’s braid group $B_n$. The first is the monodromy of the Knizhnik–Zamolodchikov (KZ) equations, and is analytic in the deformation parameter. The second is the $R$–matrix representation of the Drinfeld–Jimbo quantum group $U_\hbar \mathfrak{g}$ associated to $\mathfrak{g}$, and is formal. The remarkable theorem of Kohno and Drinfeld alluded to above states that these two seemingly very different deformations are in fact equivalent.

The second finite group attached to $G$ is its Weyl group $W$. It is an internal symmetry group and it is tempting to think that it acts on any finite–dimensional $G$–module $U$. This isn’t quite the case, but, as Tits showed \cite{33}, $W$ possesses a canonical abelian extension $\tilde{W}$

$$1 \to \mathbb{Z}_2^r \to \tilde{W} \to W \to 1$$

by the sign group $\mathbb{Z}_2^r$, with $r$ the rank of $G$, which does act on $U$. This action is canonical only up to conjugation by a fixed maximal torus $T$ of $G$, but since this has little effect on the constructions I will discuss, I will overlook this point and abusively speak of the action of $\tilde{W}$ on $U$.

Returning to the main story, this action possesses a formal deformation through representations of the generalised braid group $B_{\mathfrak{g}}$ of type $\mathfrak{g}$ known as the quantum Weyl group action, which is constructed via the quantum group $U_\hbar \mathfrak{g}$. It is natural to ask whether it also possesses an analytic deformation obtained as the monodromy of a suitable flat connection. As I will explain, the answer turns out to be affirmative and is given by a new connection $\nabla_C$, which I will call the Casimir connection. The latter was discovered by De Concini around 1995 (unpublished), and independently by J. Millson and myself \cite{24, 32}, see also \cite{11}. The conjectural relation between these two deformations, due to De Concini and myself will also be discussed.

Here’s a brief overview of the paper. In section 2 we describe a general method for constructing flat vector bundles on hyperplane complements. This is applied in sections 3 and 5 to obtain the KZ and Casimir connections respectively. Along the way, we describe in §4 Cherednik’s generalisation of the KZ connection to other root systems since it is closely related to the KZ and Casimir connections. In section 6, we briefly review the definition of the quantum group $U_\hbar \mathfrak{g}$ and of the associated $R$–matrix and $q$Weyl group representations. Sections 7 and 8 describe the Kohno–Drinfeld theorem and its conjectural extension relating the monodromy of the Casimir connection to $q$Weyl group actions. We also sketch a proof of this conjecture for the case $\mathfrak{g} = sl_n$, referring to \cite{32} for more details. In section 9 we study the relation of the Casimir and Cherednik connections.
2. Flat connections on hyperplane complements

Artin’s braid groups and their generalised counterparts are, up to the action of the corresponding finite Coxeter groups, fundamental groups of hyperplane complements. This topological incarnation leads to the analytic deformations mentioned in the Introduction by taking the monodromy representations of suitable flat vector bundles on these spaces. We begin by describing a general method for constructing such bundles.

Recall that a hyperplane complement $X$ is defined by the following data

- the base $\mathcal{B}$, a finite–dimensional complex vector space
- the arrangement $\mathcal{A} = \{ \mathcal{H}_i \}_{i \in I}$, a finite collection of linear hyperplanes

and by setting $X = \mathcal{B} \setminus \mathcal{A}$. To describe flat vector bundles over $X$, we need two additional pieces of data

- the fibre $\mathcal{F}$, another finite–dimensional complex vector space
- the residues $r_i \in \text{End}(\mathcal{F})$, labelled by the hyperplanes in $\mathcal{A}$

With this at hand, we consider the following meromorphic connection on the trivial vector bundle $\mathcal{V} = X \times \mathcal{F}$ over $X$

$$\nabla = d - \sum_{i \in I} \frac{d\phi_i}{\phi_i} \cdot r_i$$

where $\phi_i \in \mathcal{B}^*$, $i \in I$, are linear equations for the hyperplanes, so that $\mathcal{H}_i = \text{Ker}(\phi_i)$. The following useful criterion of Kohno [19] tells us when such a connection is flat

**Lemma 2.1.** The above connection is flat iff, for any subcollection of linear forms $\{ \phi_j \}_{j \in J}$ which is maximal for the property that their span in $\mathcal{B}^*$ is two–dimensional, one has

$$[r_j, \sum_{j' \in J} r_{j'}] = 0$$

for any $j \in J$.

The Lie theoretic nature of the above equations prompts one to make, following Chen and Sullivan, the following

**Definition 2.2.** The holonomy Lie algebra $\mathfrak{a}(\mathcal{A})$ of the arrangement $\mathcal{A}$ is the quotient of the free Lie algebra generated by symbols $r_i$, $i \in I$, by the relations of lemma 2.1.

Thus, if we decide to regard the $r_i$ as abstract generators of $\mathfrak{a}(\mathcal{A})$ rather than endomorphisms of $\mathcal{F}$, we may rephrase Kohno’s lemma by saying that any linear representation

$$\pi : \mathfrak{a}(\mathcal{A}) \longrightarrow \text{End}(\mathcal{F})$$
of \( \mathfrak{a}(\mathcal{A}) \) gives rise to a flat connection on \( X \times \mathcal{F} \). In fact, since the relations satisfied by the \( r_i \) are homogeneous, \( \pi \) gives rise to a one–parameter family of flat connections labelled by \( h \in \mathbb{C} \), namely
\[
\nabla = d - h \sum_{i \in I} \frac{d\phi_i}{\phi_i} \cdot r_i
\]
and therefore to a one–parameter family of monodromy representations of the fundamental group \( \pi_1(X) \) of \( X \). These analytically deform the trivial representation of \( \pi_1(X) \) on \( \mathcal{F} \) which is obtained by setting \( h = 0 \). Thinking of this as a process of exponentiation, we shall denote them by \( e_h^\pi \) and think of \( \mathfrak{a}(\mathcal{A}) \) as the Lie algebra of \( \pi_1(X) \). Odd as this may sound, this point of view is vindicated by the following

**Proposition 2.3.** The monodromy representation \( e_h^\pi : \pi_1(X) \to GL(\mathcal{F}) \) is generically irreducible iff the infinitesimal representation \( \pi : \mathfrak{a}(\mathcal{A}) \to \text{End}(\mathcal{F}) \) is irreducible.

see, e.g., [24]. Here generically irreducible means irreducible for all values of \( h \) outside the zero set of some holomorphic function \( f \neq 0 \).

### 3. The Knizhnik–Zamolodchikov equations

Let \( X_n \) be the configuration space of \( n \) ordered points in \( \mathbb{C} \). Thus
\[
X_n = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \Delta_{ij}
\]
where \( \Delta_{ij} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n|z_i = z_j\} \) so that \( X_n \) is a hyperplane complement. To construct the Knizhnik–Zamolodchikov (KZ) connection on \( X_n \), we fix a complex, semi–simple Lie algebra \( g \), one of its finite–dimensional representations \( V \) and set \( \mathcal{F} = V^\otimes n \). The residue matrices \( r_{ij} \) are usually denoted by \( \Omega_{ij} \) and are given by [13]
\[
\Omega_{ij} = \sum_a \pi_i(X_a)\pi_j(X^a)
\]
where \( \{X_a\}, \{X^a\} \) are dual basis of \( g \) with respect to the basic inner product i.e., the multiple \( \langle \cdot, \cdot \rangle \) of the Killing form such that the highest root of \( g \) has squared length 2, and \( \pi_k(X) \) denotes the action of \( X \in g \) on the 4th tensor factor in \( V^\otimes n \). A simple application of Kohno’s lemma then shows that
\[
\nabla_{\text{KZ}} = d - h \sum_{1 \leq i < j \leq n} \frac{d(z_i - z_j)}{z_i - z_j} \cdot \Omega_{ij}
\]
is a flat connection on \( X_n \times V^\otimes n \) for any \( h \in \mathbb{C} \). Its monodromy yields a representation of Artin’s pure braid group on \( n \) strands
\[
P_n = \pi_1(\mathbb{C}^n \setminus \{z_i = z_j\}) \to GL(V^\otimes n)
\]
which deforms the trivial representation of \( P_n \) on \( V^\otimes n \). We can however do a little better by noticing that the symmetric group \( \mathcal{S}_n \) acts on \( V^\otimes n \) and \( X_n \). \( \nabla_{\text{KZ}} \) is readily seen to be equivariant for the combination of these
two actions and therefore descends to a flat connection on the quotient
bundle \((X_n \times V^\otimes n)/\mathfrak{S}_n\) over \(\tilde{X}_n = X_n/\mathfrak{S}_n\) i.e., the configuration space of
\(n\) unordered points in \(\mathbb{C}\). Taking its monodromy, we obtain a one–parameter
family of representations of Artin’s braid group on \(n\) strands
\(\rho_h : B_n = \pi_1(\mathbb{C}^n \setminus \{z_i = z_j\}/\mathfrak{S}_n) \rightarrow GL(V^\otimes n)\)

\(\rho_h\) depends analytically in \(h\) and deforms the natural action of \(\mathfrak{S}_n\) on \(V^\otimes n\)
since \(\rho_0\) factors through this action.

Recall that \(B_n\) is presented on elements \(T_i, 1 \leq i \leq n - 1\) subject to Artin’s
braid relations

\[T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1 \ldots n - 1\]

\[T_i T_j = T_j T_i, \quad |i - j| \geq 2\]

Each \(T_i\) may be realised as a small loop in \(\tilde{X}_n\) around the image of the
hyperplane \(\{z_i = z_{i+1}\}\). In particular, \(\rho_h(T_i)\) is generically conjugate to
\((i i + 1) \cdot \exp^{\pi\sqrt{-1}h}\Omega_{i+1}\).

**Example 3.1.** Take \(g = \mathfrak{gl}_m\) with vector representation \(V = \mathbb{C}^m\) and basic
inner product \(\langle X, Y \rangle = tr_V(XY)\). If \(e_1, \ldots, e_n\) is the standard basis of \(V\)
and \(E_{ij} e_k = \delta_{jk} e_i\) the corresponding elementary matrices then, on \(V^\otimes 2\),

\[\Omega_{12} e_k \otimes e_l = \sum_{1 \leq i, j \leq m} E_{ij} \otimes E_{ji} e_k \otimes e_l = e_l \otimes e_k\]

so that \(\Omega_{ij}\) acts on \(V^\otimes n\) as the transposition \((i j)\) and its eigenvalues are
therefore \(\pm 1\). The corresponding monodromy representation

\[\rho_h : B_n \rightarrow GL(V^\otimes n)\]

therefore factors through the Iwahori–Hecke algebra \(H_{\mathfrak{S}_n}(q)\), i.e., the quo-
tient of the group algebra \(\mathbb{C}[B_n]\) by the relations

\[(T_i - q)(T_i + q^{-1}) = 0\]

where \(q = e^{i\pi h}\).

**Example 3.2.** Choose now an orthogonal vector space \(V \cong \mathbb{C}^n\), \(g = \mathfrak{so}(V)\)
and let \(e_1, \ldots, e_n\) be an orthonormal basis of \(V\). Since the basic inner
product on \( g \) is \( \langle X, Y \rangle = \frac{1}{2} \text{tr}_V(XY) \), we find

\[
\Omega_{12} = \sum_{1 \leq i < j \leq n} (E_{ij} - E_{ji}) \otimes (E_{ji} - E_{ij}) = \sum_{1 \leq i,j \leq n} E_{ij} \otimes E_{ji} - \sum_{1 \leq i,j \leq n} E_{ij} \otimes E_{ij} = (1\,2) - n \, p_0
\]

where \( p_0 \) is the orthogonal projection onto the \( g \)-fixed line spanned by

\[
v_0 = \sum_{i=1}^{n} e_i \otimes e_i \in S^2 V
\]

If, on the other hand, \( V \cong \mathbb{C}^{2n} \) is a symplectic vector space with symplectic form \( \omega \) and \( g = \text{sp}(V) \), a similar computation in a basis \( e_{\pm 1}, \ldots, e_{\pm n} \) of \( V \) satisfying \( \omega(e_i, e_j) = \text{sign}(i) \delta_{i+j,0} \) shows that

\[
\Omega_{12} = (1\,2) - 2n \, q_0
\]

where \( q_0 \) is now the orthogonal projection onto the \( g \)-fixed line spanned by

\[
v_0 = \sum_{i=-n}^{n} \text{sign}(i) \, e_i \otimes e_{-i} \in \Lambda^2 V
\]

Thus, in either case, each generator of monodromy \( T_i \) only has the three eigenvalues \( q, -q^{-1}, r^{-1} \), where \( q = e^{i\pi h} \) and

\[
r = \varepsilon e^{i\pi(h(\dim(V)) - \varepsilon)} \quad \text{with} \quad \varepsilon = \begin{cases} +1 & \text{if } V \text{ is orthogonal} \\ -1 & \text{if } V \text{ is symplectic} \end{cases}
\]

With a little more work, one can show that the monodromy of \( \nabla_{KZ} \) factors in this case through the Birman–Wenzl–Murakami algebra \( \mathcal{BMW}_n(q, r) \) \([3, 25]\) defined as the quotient of \( \mathbb{C}[B_n] \) by the relations

\[
(T_i - q)(T_i + q^{-1})(T_i - r^{-1}) = 0
\]

\[
E_i T_i^{\pm 1} E_i = r^{\pm 1} E_i
\]

where \( E_i = 1 - (T_i - T_i^{-1})(q - q^{-1}) \) is a multiple of the spectral projection of \( T_i \) corresponding to the eigenvalue \( r^{-1} \).

4. The Coxeter–KZ Connection

The connection described below was introduced by Cherednik \([3]\), to whom the results of this section are due, and is usually referred to as the KZ connection. In order to distinguish it from the one introduced in the previous section, we shall use the term Coxeter–KZ (CKZ) connection instead. Let \( W \) be a Weyl group, or more generally a finite reflection group, with complexified reflection representation \( \mathfrak{h} \cong \mathbb{C}^r \) and root system \( R = \{ \alpha \} \subset \mathfrak{h}^* \). The base space and arrangement are now \( \mathcal{B} = \mathfrak{h} \) and

\[
\mathcal{A} = \bigcup_{\alpha \in R} \text{Ker}(\alpha)
\]
so that $X = B \setminus A$ is the space $\mathfrak{h}_{\text{reg}}$ of regular elements in $\mathfrak{h}$. Set $\mathcal{F} = U$ where $U$ is a finite-dimensional $W$–module and let the residue $r_\alpha$ be given by the reflection $s_\alpha \in W$.

**Theorem 4.1.** For any choice of weights $k_\alpha \in \mathbb{C}$ satisfying $k_{w\alpha} = k_\alpha$, $\forall w \in W$, the connection

$$\nabla_{\text{CKZ}} = d - \sum_{\alpha > 0} k_\alpha \frac{d\alpha}{\alpha} \cdot s_\alpha$$

is a $W$–equivariant, flat connection on $\mathfrak{h}_{\text{reg}} \times U$.

The monodromy of $\nabla_{\text{CKZ}}$ yields a family of representations of the *generalised pure braid group* $P_W$ of type $W$

$$\rho_h : P_W = \pi_1(\mathfrak{h}_{\text{reg}}) \to GL(U)$$

deforming the trivial representation of $P_W$ on $U$. Each $W$–orbit in $R$ carries a deformation parameter $k_\alpha$. As for the KZ connection however, one can do a little better and use the action of $W$ on $\mathfrak{h}_{\text{reg}}$ and $U$ to push $\nabla_{\text{CKZ}}$ down to the quotient $\mathfrak{h}_{\text{reg}}/W$. This yields a a representation of the *generalised braid group* of type $W$

$$\rho_h : B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W) \to GL(U)$$

which, for $k_\alpha = 0$, factors through the action of $W$ on $U$.

By Brieskorn’s theorem [4], $B_W$ is presented on generators $S_1, \ldots, S_r$ labelled by a choice of simple reflections $s_1, \ldots, s_r$ in $W$ with relations

$$S_i S_j \cdots = S_j S_i \cdots$$

for any $1 \leq i < j \leq r$ where the number $m_{ij}$ of factors on each side is equal to the order of $s_i s_j$ in $W$. Each $S_i$ may be obtained as a small loop in $\mathfrak{h}_{\text{reg}}/W$ around the reflecting hyperplane $\text{Ker}(\alpha_i)$ of $s_i$ so that $\rho_h(S_i)$ is generically conjugate to $s_i \exp^{\pi \sqrt{-1} k_\alpha_i s_i}$. Since each simple reflection $s_\alpha$ has at most two eigenvalues in $U$, the monodromy of $\nabla_{\text{CKZ}}$

therefore factors through the (unequal length) Hecke algebra $\mathcal{H}_W(q_i)$ of $W$ i.e., the quotient of $\mathbb{C}[B_W]$ by the relations

$$(S_i - q_i)(S_i + q_i^{-1}) = 0$$

where $q_i = e^{\pi \sqrt{-1} k_\alpha_i}$. Choosing $U$ to be the direct sum of the irreducible representations of $W$, so that $\text{End}(U) \cong \mathbb{C}[W]$, and the weights $k_\alpha$ to be generic, the monodromy does in fact yield an algebra isomorphism of $\mathcal{H}_W(q_i)$
Example 4.2. When $W = \mathfrak{S}_n$, the Coxeter–KZ connection is a particular instance of the KZ connection. Indeed, we already noted in Example 3.1. that, for $\mathfrak{g} = \mathfrak{gl}_n$ acting on the $n$-fold tensor product $V^\otimes n$ of its vector representation $V \cong \mathbb{C}^m$, the KZ operator $\Omega_{ij}$ is given by the transposition $(i \ j)$. Thus,

Proposition 4.3. The KZ connection for $\mathfrak{g} = \mathfrak{gl}_m$ with values in $V^\otimes n$ coincides with the Coxeter–KZ connection for $W = \mathfrak{S}_n$ with values in $V^\otimes n$ and weights given by $k_\alpha = h$.

A finer version of this statement may of course be obtained using Schur–Weyl duality. If $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ is a Young diagram with at most $m$ rows and such that $|\lambda| = \sum \lambda_i$ is equal to $n$, the irreducible representation of $\mathfrak{gl}_n$ with highest weight $\lambda$ is a summand in $V^\otimes n$. The corresponding multiplicity space $M^n_\lambda$ is an irreducible representation of $\mathfrak{S}_n$ and the KZ and CKZ connections with values in $M^n_\lambda$ coincide.

5. The Casimir connection

We shall now use the Lie algebra $\mathfrak{g}$ in a rather different way. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $R = \{\alpha\} \subset \mathfrak{h}^*$ be the corresponding root system. The base space and arrangement are the same as those of the Coxeter–KZ connection for the Weyl group $W$ of $\mathfrak{g}$, so that

$$X = \mathfrak{h} \setminus \bigcup_{\alpha \in R} \text{Ker}(\alpha) = \mathfrak{h}_{\text{reg}}$$

The fibre $\mathcal{F}$ of the vector bundle is now a finite-dimensional $\mathfrak{g}$-module $U$. To describe the residue matrices $r_\alpha$, recall that for any root $\alpha$, there is a corresponding subalgebra $\mathfrak{sl}_\alpha^2 \subset \mathfrak{g}$ spanned by the triple $e_\alpha, f_\alpha, h_\alpha$, where $h_\alpha = \alpha^\vee \in \mathfrak{h}$ is the corresponding coroot and $e_\alpha, f_\alpha$ are a choice of root vectors normalised by $[e_\alpha, f_\alpha] = h_\alpha$. The restriction of the basic inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ to $\mathfrak{sl}_\alpha^2$ determines a canonical Casimir element

$$C_\alpha = \frac{\langle \alpha, \alpha \rangle}{2} \left( e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2} h^2_\alpha \right) \in U \mathfrak{sl}_2^0 \subset U \mathfrak{g}$$

which we shall use as the residue on the hyperplane Ker($\alpha$). The following result was discovered by De Concini around 1995 (unpublished), and independently by J. Millson and myself [24, 32], see also [11].

Theorem 5.1. For any $h \in \mathbb{C}$, the Casimir connection

$$\nabla_C = d - h \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \cdot C_\alpha$$

is a flat connection on $\mathfrak{h}_{\text{reg}} \times U$ which is reducible with respect to the weight space decomposition of $U$. 
Proof. Kohno’s flatness criterion translates into the statement that $\nabla_C$ is flat iff for any rank 2 root system $R_2 \subseteq R$ determined by the intersection of $R$ with a two–dimensional plane in $\mathfrak{h}^*$, the following holds for any positive root $\alpha \in R_2$,

$$[C_\alpha, \sum_{\beta \in R_2, \beta \succ 0} C_\beta] = 0$$

Our original proof of this statement was a cumbersome case–by–case check for the root systems $R_2 = A_1 \times A_1, A_2, B_2, G_2$. This was immediately made obsolete by A. Knutson’s elegant observation that the second term in the commutator above is, modulo terms in $\mathfrak{h}$, the Casimir operator of the sub–algebra $\mathfrak{g}_2 \subseteq \mathfrak{g}$ with root system $R_2$ and therefore commutes with $C_\alpha$. The reducibility of $\nabla_C$ follows from the fact that the $C_\alpha$ commute with $\mathfrak{h}$ □

We now wish to push the Casimir connection down to the quotient $\mathfrak{h}_{reg}/W$ to get a monodromy representation of the generalised braid group $B_{\mathfrak{g}} = B_{\mathfrak{w}}$. This requires a little work because the Weyl group $W$ does not act on $U$ and its Tits extension $\tilde{W}$, while acting on $U$, does not act freely on $\mathfrak{h}_{reg}$. To circumvent this difficulty, we pull–back the Casimir connection $\nabla_C$ to the universal cover $\tilde{\mathfrak{h}}_{reg} \overrightarrow{\pi} \mathfrak{h}_{reg}$. Since $\tilde{W}$ is a quotient of $B_{\mathfrak{g}}$, the latter acts on $U$ and, freely, on $\tilde{\mathfrak{h}}_{reg}$. The desired one–parameter family $\rho$ of representations is obtained by taking the monodromy of the flat vector bundle $(\tilde{\mathfrak{h}}_{reg} \times U, p^*\nabla_C)/B_{\mathfrak{g}}$. It factors through the action of $\tilde{W}$ on $U$ for $h = 0$.

Example 5.2. Let $V = \mathfrak{g}$ be the adjoint representation of $\mathfrak{g}$ so that the zero weight space $V[0]$ of $V$ is the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. $V[0]$ is acted upon by the Casimirs $C_\alpha$ as well as the Weyl group $W$ of $\mathfrak{g}$ and, if $t \in \mathfrak{h}$

$$C_\alpha \ t = \langle \alpha, \alpha \rangle \ \text{ad}(e_\alpha) \ \text{ad}(f_\alpha) \ t$$

$$= \langle \alpha, \alpha \rangle \langle \alpha, t \rangle h_\alpha$$

$$= \langle \alpha, \alpha \rangle (1 - s_\alpha) t$$

From this we conclude that the Casimir connection with values in $\mathfrak{h} = V[0]$ coincides with the Coxeter–KZ connection with values in the reflection representation of $W$, provided the weights $k_\alpha$ are given by $-h(\alpha, \alpha)$ and we tensor the CKZ connection with the character of $\pi_1(\mathfrak{h}_{reg}/W)$ given by the multi–valuedness of the function

$$f = \prod_{\alpha > 0} \alpha^{h(\alpha, \alpha)}$$

One cannot expect a similar coincidence to arise on the zero weight space of any $\mathfrak{g}$–module $V$ because the monodromy of the CKZ connection with values in $V[0]$ always factors through the Hecke algebra of $W$ while simple calculations show that that of the Casimir connection hardly ever does. We shall however return to this point in section 9.
Remark 5.3. Using the rigidity of the Hecke algebra of $W$ i.e., the fact that its representations are uniquely determined by their specialisation at $q_i = 1$ it is easy to see that the monodromy representation of $B \mathfrak{g}$ on $\mathfrak{h} = \mathfrak{g}[0]$ is equivalent to the reduced Burau representation of $B_n = B \mathfrak{g}$ when $\mathfrak{g} = \mathfrak{sl}_n$ and to the Squier representation of $B \mathfrak{g}$ when $\mathfrak{g}$ is simply–laced.

Remark 5.4. It is tempting to think that, since the Casimir operators $C_\alpha$ are self–adjoint in any finite–dimensional $\mathfrak{g}$–module, the connection $\nabla_C$ is unitary whenever $\mathfrak{h}$ is purely imaginary. I am grateful to P. Boalch for slapping my fingers on this point and pointing out that this isn’t (of course) so. Determining the values of $\mathfrak{h}$ for which the Casimir connection is unitary seems a very interesting problem.

6. Formal Deformations via Quantum Groups

We turn now to formal deformations. These will be obtained via the Drinfeld–Jimbo quantum group $U_{\hbar} \mathfrak{g}$. Recall that the latter is a deformation of the enveloping algebra $U \mathfrak{g}$ of $\mathfrak{g}$, i.e., a Hopf algebra over the ring $\mathbb{C}[h]$ of formal power series in the variable $h$, which is topologically free as $\mathbb{C}[h]$–module and endowed with an isomorphism $U_{\hbar} \mathfrak{g} / \hbar U_{\hbar} \mathfrak{g} \cong U \mathfrak{g}$ of Hopf algebras.

The simplest of these quantum groups corresponds to $\mathfrak{g} = \mathfrak{sl}_2$, where the standard generators $e, f, h$ of $\mathfrak{g}$ given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

together with the relations

$$[h, e] = 2e, \quad [h, f] = -2f$$

$$[e, f] = h$$

which they satisfy are replaced by the generators $E, F, H$ of $U_{\hbar} \mathfrak{sl}_2$ subject to

$$[H, E] = 2E, \quad [H, F] = -2F$$

and

$$[E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}$$

At first sight, the representation theory of $U_{\hbar} \mathfrak{g}$ offers few new features. This is so because any finite–dimensional representation $\mathcal{V}$ of $U_{\hbar} \mathfrak{g}$, i.e., one which is finitely generated and topologically free as $\mathbb{C}[h]$–module, is uniquely determined by the $\mathfrak{g}$–module $\mathcal{V} = \mathcal{V} / h \mathcal{V}$. Indeed, since $H^2(\mathfrak{g}, U \mathfrak{g}) = 0$, the multiplication in $U \mathfrak{g}$ does not possess non–trivial deformations and $U_{\hbar} \mathfrak{g}$ is isomorphic as $\mathbb{C}[h]$–algebra to

$$U \mathfrak{g}[h] = \{ \sum_{n \geq 0} x_n h^n \mid x_n \in U \mathfrak{g} \}$$
Using this to let $U\mathfrak{g}$ act on $\mathcal{V}$, we may regard the latter as a deformation of $V$. Since $H^1(\mathfrak{g}, \text{End}(\mathcal{V})) = 0$ however, $\mathcal{V}$ is isomorphic, as $U\mathfrak{g}$ and therefore as $U_h\mathfrak{g}$–module, to the trivial deformation $V[\hbar]$ of $V$.

The first novelty arises when one considers the action of the symmetric group $\mathfrak{S}_n$ on tensor products of $\mathfrak{g}$–modules. When implemented on the $n$–fold tensor product $\mathcal{V}^\otimes n$ of a finite–dimensional $U_h\mathfrak{g}$–module $\mathcal{V}$, the latter does not commute with the action of $U_h\mathfrak{g}$. The following result shows however that this problem may be corrected by deforming the action of $\mathfrak{S}_n$.

**Theorem 6.1** (Faddeev–Reshetikhin–Takhtajan, Drinfeld, Jimbo). There exists a universal $R$–matrix $R \in U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ such that the elements $R_i^\vee \in GL(\mathcal{V}^\otimes n)$, $i = 1 \ldots n − 1$, given by

$$R_i^\vee = (i \ i + 1) \cdot 1 \otimes \cdots \otimes 1 \otimes R \otimes 1 \otimes \cdots \otimes 1$$

commute with $U_h\mathfrak{g}$ and satisfy

i. the braid relations :

$$R_i^\vee R_{i+1}^\vee R_i^\vee = R_{i+1}^\vee R_i^\vee R_{i+1}^\vee \quad i = 1 \ldots n − 1$$

$$R_i^\vee R_j^\vee = R_j^\vee R_i^\vee \quad |i − j| \geq 2$$

ii. the deformation property :

$$R_i^\vee = (i \ i + 1) + o(h)$$

Thus, if one is prepared to replace $\mathfrak{S}_n$ by the braid group $B_n$, there is an interesting, $U_h\mathfrak{g}$–equivariant, ‘permutation’ action of $B_n$ on $\mathcal{V}^\otimes n$ which, when reduced mod $\hbar$, factors through the natural action of $\mathfrak{S}_n$. Moreover, this action is local in the sense that $i$th generator of $B_n$ acts on the $i$ and $i + 1$ tensor copies in $\mathcal{V}^\otimes n$ only, as does the transposition $(i \ i + 1)$. What is lost in this replacement is the fact that the $R_i^\vee$ do not square to 1 and do not therefore give an action of $\mathfrak{S}_n$.

A similar phenomenon occurs for the action of the Tits extension $\tilde{W}$ on finite–dimensional $\mathfrak{g}$–modules. There is no known, canonical way to implement it on $U_h\mathfrak{g}$–modules, but one may define an action of the braid group $B_\mathfrak{g}$ on these, known as the quantum Weyl group action, which deforms that of $\tilde{W}$. Before stating the precise result, recall that the latter action arises by mapping $\tilde{W}$ to the completion $\hat{U}\mathfrak{g}$ of $U\mathfrak{g}$ with respect to its finite–dimensional representations via

$$s_i \rightarrow \exp(e_i) \exp(-f_i) \exp(e_i)$$

Let $q_i = e^{\hbar(\alpha_i, \alpha_i)/2}$ and consider the triple $q$–exponentials [14, 28]

$$S_i = \exp_{q_i^{-1}}(q_i^{-1}E_i q_i^{-H_i}) \exp_{q_i^{-1}}(-F_i) \exp_{q_i^{-1}}(q_i E_i q_i H_i)$$
where $E_i, F_i, H_i$ are the generators of the subalgebra $U_h\mathfrak{sl}_2 \subseteq U_h\mathfrak{g}$ corresponding to the simple root $\alpha_i$,

$$\exp_q(x) = \sum_{n \geq 0} q^{n(n-1)/2} x^n \left[ \frac{n}{[n]_q} \right]$$

and

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}) \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

are the usual $q$–numbers and factorials. Viewing the $S_i$ as lying in the completion $\hat{U}_h\mathfrak{g}$ of $U_h\mathfrak{g}$ with respect to its finite–dimensional representations, we have the following

**Theorem 6.2** (Lusztig, Kirillov–Reshetikhin, Soibelman). The elements $S_1, \ldots, S_r$ satisfy

i. the braid relations:

$$S_i S_j S_i \cdots = S_j S_i S_j \cdots \quad \text{with } m_{ij} = 1$$

ii. the deformation property:

$$S_i = s_i + o(h)$$

The quantum Weyl group action is given by the $S_i$. Just as the operators $R_i^\vee$, each $S_i$ is local in that it lies in the completion $\hat{U}_h\mathfrak{sl}_2$ of $U_h\mathfrak{sl}_2$, and does not square to 1.

7. Monodromy theorems for Artin’s braid groups

Let us summarise what we have found so far for Artin’s braid group $B_n$. We let as usual $\mathfrak{g}$ be a complex, semi–simple Lie algebra, $V$ a finite–dimensional representation of the quantum group $U_h\mathfrak{g}$ and $V$ the $\mathfrak{g}$–module $V/\hbar V$.

$$\nabla_{\text{KZ}} : GL(V \otimes \cdots \otimes V[h]) \to B_n$$

On the one hand, $B_n$ acts on $V^\otimes n$ via the monodromy of the KZ equations. The latter depends analytically on the deformation parameter $h \in \mathbb{C}$ and can therefore be regarded as an action of $B_n$ on $V^\otimes n \{h\}$. Forgetting about convergence, we regard $h$ as a formal variable, which we rename $h/2\pi i$, and consider the monodromy of $\nabla_{\text{KZ}}$ as an action of $B_n$ on $V^\otimes n [h]$. On the other hand, $B_n$ acts on $V^\otimes n$ via the $R$–matrix representation of $U_h\mathfrak{g}$. One now has the following beautiful

**Theorem 7.1** (Kohno, Drinfeld). The monodromy representation of the KZ equations on $V^\otimes n [h]$ is equivalent to the $R$–matrix action of $B_n$ on $V^\otimes n$. 
One may wonder whether the stated equivalence could be promoted to an equality and proved by a direct calculation. There are several reasons why this cannot be so.

- The monodromy representation depends upon a number of choices, most notably that of a base point in the configuration space $\tilde{X}_n$. Thus, the upper row is an equivalence class of representations rather than a single one.
- This, in a sense, is also true of the $R$–matrix representation. Indeed, to implement the latter on $V \otimes^n \mathbb{C}[\hbar]$ rather than on $V \otimes^n$, one has to choose an algebra isomorphism $\phi : \mathcal{U}_\hbar \mathfrak{g} \to \mathcal{U}_\hbar \mathfrak{g}[\hbar]$ to make $\mathcal{U}_\hbar \mathfrak{g}$ act on $V[\hbar]$. As mentioned, such a $\phi$ exists, but is only unique up to conjugation by an element $a \in \mathcal{U}_\hbar \mathfrak{g}[\hbar]$ of the form $1 + o(\hbar)$.
- This last objection partially disappears if one works modulo $\hbar^2$ since in that case there is a preferred algebra isomorphism

$$\mathcal{U}_\hbar \mathfrak{g}/\hbar^2 \mathcal{U}_\hbar \mathfrak{g} \to \mathcal{U}_\hbar \mathfrak{g} \otimes \mathbb{C}[\hbar]/(\hbar^2)$$

obtained by lifting the given isomorphism $\mathcal{U}_\hbar \mathfrak{g}/\hbar \mathcal{U}_\hbar \mathfrak{g} \cong \mathcal{U}_\hbar \mathfrak{g}$. Even then however one finds that the monodromy representation, when computed mod $\hbar^2$ in a basis of horizontal sections of $\nabla_{KZ}$ is not local, contrary to the $R$–matrix action.

The stated equivalence is in fact given by a rather explicit, albeit cohomological expression (Drinfeld’s twist) which is not $\mathfrak{g}$–equivariant [8, 9, 10]. Thus, the monodromy and $R$–matrix pictures are complementary. The first is obtained from the representation theory of $\mathfrak{g}$, and is non–local, the second is obtained from the representation theory of $\mathcal{U}_\hbar \mathfrak{g}$ and is local.

Finally, we remark that when read from top to bottom, the Kohno–Drinfeld theorem gives a concise description of the monodromy of the KZ equations while, when read from bottom to top, it is a sort of Riemann–Hilbert theorem since it asserts that the $R$–matrix representation of $B_n$ is the monodromy of a flat connection on the trivial bundle over the configuration space $X_n$.

Let us summarise the previous theorem as the following

**Kohno–Drinfeld Principle.** If $\nabla$ is a flat connection depending on a deformation parameter, there exists a quantum group describing its monodromy.
8. Monodromy theorems for generalised braid groups

Turning now to the generalised braid group $B_\mathfrak{g}$, we have a similar diagram

$$
\begin{array}{ccc}
\n & \nabla_c & GL(V[h]) \\
\downarrow & & \downarrow \\
B_\mathfrak{g} & \xrightarrow{qW} & GL(V) \\
\end{array}
$$

where the top row is the monodromy of the Casimir connection, regarded as depending formally on the deformation parameter $h$, here renamed $\hbar/2\pi i$, while the bottom one is the quantum Weyl group action of $B_\mathfrak{g}$ on the $U_{\hbar\mathfrak{g}}$–module $V$. In the light of the Kohno–Drinfeld principle, it seems natural to make the following

**Monodromy Conjecture.** The monodromy of the Casimir connection with values in $V[h]$ is equivalent to the quantum Weyl group action of $B_\mathfrak{g}$ on $V$.

This conjecture was formulated by De Concini in unpublished work around 1995 and independently by myself in [31, 32]. The difficulties in promoting its statement to a conjectural equality are the same as for the Kohno–Drinfeld theorem. In this case, the lack of locality of the monodromy representation means that, even when computing mod $\hbar^2$, the image of a small loop around the hyperplane $\text{Ker}(\alpha_i)$ does not lie in the completion $\hat{U}_{\mathfrak{s}\mathfrak{l}_2}$ of the $\mathfrak{s}\mathfrak{l}_2$–subalgebra corresponding to the simple root $\alpha_i$.

A number of things can be proved in support of the above conjecture, namely

- It is true for all representations of $\mathfrak{g} = \mathfrak{s}\mathfrak{l}_2$ where $B_\mathfrak{g} \cong \mathbb{Z}$.
- The spectra of the generators of $B_\mathfrak{g}$ agree in both representations.
- It is true mod $\hbar^2$.

Moreover, one has the following

**Theorem 8.1** ([31]). The monodromy conjecture holds for the following pairs $(\mathfrak{g}, V)$

- All fundamental representations of $\mathfrak{g} = \mathfrak{s}\mathfrak{l}_n$.
- Vector representation of $\mathfrak{g} = \mathfrak{s}\mathfrak{o}_n, \mathfrak{s}\mathfrak{p}_n$.
- Spin representation(s) of $\mathfrak{g} = \mathfrak{s}\mathfrak{o}_n$.
- Minuscule representations of $\mathfrak{g} = \mathfrak{e}_6, \mathfrak{e}_7$.
- The 7–dimensional representation of $\mathfrak{g} = \mathfrak{g}_2$.
- Adjoint representation of any $\mathfrak{g}$.

**Proof (sketch).** All listed representations, except for the adjoint one, have the property that their weight spaces are one–dimensional. This makes it possible to compute the monodromy representation explicitly, since, when restricted to the pure braid group $P_\mathfrak{g}$, it is just a sum of one–dimensional
characters. One the other hand, it is easy to deform these same \( V \) to representations of \( U_q \) explicitly \([27]\), and therefore to compute the corresponding quantum Weyl group action using the triple \( q \)-exponentials that define it. One finds in this case that the two representations are conjugate by a diagonal matrix. The adjoint representation of \( \mathfrak{g} \) requires a little more work. We first break \( \mathfrak{g} \) up as \( \mathfrak{n} \oplus \mathfrak{h} \) where \( \mathfrak{n} = \mathfrak{n}_- \oplus \mathfrak{n}_+ \) is the direct sum of the upper and lower nilpotent subalgebras, and \( \mathfrak{h} \) is the Cartan algebra. Since both \( \mathfrak{h} \) and \( \mathfrak{n} \) are preserved by the two actions, it suffices to prove the monodromy conjecture for each piece. Since the weight spaces of \( \mathfrak{n} \) are one-dimensional, the corresponding monodromy representation of \( B_q \) is readily computed. For the \( q \)-Weyl group action, one uses Lusztig’s explicit deformation of the adjoint representation \([22]\). The equivalence on \( \mathfrak{n} \) is readily obtained from this. For \( \mathfrak{h} \) we use the fact that both representations factor through the Hecke algebra \( H_W(q_i) \). This was shown in Example 5.2, for the monodromy representation and is a simple, and old observation of Lusztig and Levendorskii–Soibelman for the \( q \)-Weyl group action. The equivalence is then obtained from the rigidity of the Hecke algebra and the fact that both representations are deformations of the reflection action of \( W \) on \( \mathfrak{h} \).

**Remark 8.2.** The above list of representations contains, for any simple \( \mathfrak{g} \), at least one generator of the representation ring of \( \mathfrak{g} \), i.e., a \( V \) such that any finite-dimensional irreducible \( \mathfrak{g} \)-module is contained in a tensor power \( V \otimes^n \) of \( V \). The monodromy conjecture would therefore be proved if one could show that it holds for \( V_1 \otimes V_2 \) whenever it holds for each of the tensor factors. This seems difficult.

For the case of \( \mathfrak{g} = \mathfrak{sl}_n \), we can say more.

**Theorem 8.3 (32).** The monodromy conjecture holds for all representations of \( \mathfrak{g} = \mathfrak{sl}_n \).

**Proof (sketch).** The basic idea, summarised in the diagram below, is to use the duality between \( \mathfrak{gl}_k \) and \( \mathfrak{gl}_n \) obtained from their joint action on \( k \times n \) matrices to reduce the monodromy conjecture for \( \mathfrak{sl}_n \) to the Kohno–Drinfeld
Let then $A = \mathbb{C}[x_{11}, \ldots, x_{kn}]$ be the algebra of polynomial functions on the space of $k \times n$ matrices. $A$ is well–known to be multiplicity–free, see e.g., [34, §132]. Specifically, if $A^d \subset A$ is the subspace of homogeneous polynomials of degree $d \in \mathbb{N}$, one has

$$A^d = \bigoplus_{\lambda \in \mathcal{Y}_p \min(k,n), \lambda \cdot d} V^{(k)}_\lambda \otimes V^{(n)}_\lambda$$

where $\mathcal{Y}_p$ is the set of Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{N}^p$ with at most $p$ rows, $|\lambda| = \sum_i \lambda_i$ and $V^{(p)}_\lambda$ is the simple $\mathfrak{gl}_p$–module with highest weight $\lambda$. If $k \geq n$, which we henceforth assume, this allows one to identify the $\mathfrak{gl}_n$–weight space $V^{(n)}_\lambda[\mu]$ corresponding to a weight $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ to the space $M^\mu_\lambda$ of highest weight vectors of weight $\lambda$ for the diagonal $\mathfrak{gl}_k$–action on

$${\mathbb{C}^{\mu_1}[x_{11}, \ldots, x_{k1}] \otimes \cdots \otimes \mathbb{C}^{\mu_n}[x_{1n}, \ldots, x_{kn}]}$$

where $\mathbb{C}^{\mu_j}[x_{1j}, \ldots, x_{kj}]$ is the space of polynomials in $x_{1j}, \ldots, x_{kj}$ which are homogeneous of degree $\mu_j$. An explicit computation then proves the following

**Proposition 8.4.** Under this identification, the Casimir connection $\nabla_C$ for $\mathfrak{g} = \mathfrak{sl}_n$ with values in $V^{(n)}_\lambda[\mu]$ coincides with the KZ connection $\nabla_{KZ}$ for $\mathfrak{g}' = \mathfrak{sl}_k$ with values in $M^\mu_\lambda$.

Thus, the identification

$$\bigoplus_{\nu \in \mathfrak{S}_n[\mu]} V^{(n)}_\lambda[\nu] \longrightarrow \bigoplus_{\nu \in \mathfrak{S}_n[\mu]} M^\nu_\lambda$$

is equivariant for the monodromy actions of $B_n$ given by the Casimir and KZ connections respectively.
Turning now to the $q$–setting, the algebra $A$ possesses a non–commutative, graded deformation $A_\hbar$ over $\mathbb{C}[\hbar]$ on which both $U_\hbar\mathfrak{gl}_k$ and $U_\hbar\mathfrak{gl}_n$ act, and which is multiplicity free. This allows as before to identify a weight space $V_\lambda^{(n)}[\mu]$ for $U_\hbar\mathfrak{gl}_n$ with a corresponding space $M_\lambda^\mu$ of singular vectors for $U_\hbar\mathfrak{gl}_n$. An explicit, but a little more involved computation shows that

**Proposition 8.5.** Under the identification

$$\bigoplus_{\nu \in \mathfrak{S}_n,\mu} V_\lambda^{(n)}[\mu] \rightarrow \bigoplus_{\nu \in \mathfrak{S}_n,\mu} M_\lambda^\mu$$

the $(U_\hbar\mathfrak{gl}_n)$–$q$ Weyl group action of $B_n$ on the left–hand side coincides with the $R$–matrix action for $U_\hbar\mathfrak{gl}_k$ on the right–hand side.

Proposition 8.5 is the $q$–analogue of the simple fact that the action of $\mathfrak{S}_n$ on $\mathbb{C}[x_{11}, \ldots, x_{kn}] = \mathbb{C}[x_{11}, \ldots, x_{k\lambda}] \otimes \cdots \otimes \mathbb{C}[x_{1\lambda}, \ldots, x_{kn}]$ obtained by permuting the columns of a $k \times n$ matrix is equal to the one obtained by right multiplying the matrix by a permutation matrix in $GL(n)$. The former action is the classical limit of the $R$–matrix action of $U_\hbar\mathfrak{gl}_k$, the latter that of the quantum Weyl group action of $U_\hbar\mathfrak{gl}_n$.

Putting together the above two propositions together with the Kohno–Drinfeld theorem for $\mathfrak{sl}_k$, one obtains the monodromy conjecture for $\mathfrak{sl}_n$.

### 9. The Casimir and Coxeter–KZ connection (encore)

In this section, we pursue the study of the relations between the Casimir connection for a Lie algebra $g$ and the Coxeter–KZ connection for its Weyl group $W$. The calculation of Example 5.2. for the adjoint representation of $g$ generalises as follows.

**Proposition 9.1.** For any finite–dimensional $g$–module $V$, define

$$V[0] = \{ v \in V[0] | e_\alpha^2 v = 0, \forall \alpha > 0 \}$$

Then,

i. $V[0]$ is invariant under $W$ and the $C_\alpha$.

ii. On $V[0]$, one has

$$C_\alpha = \langle \alpha, \alpha \rangle (1 - s_\alpha)$$

so that the Casimir connection for $g$ with values in $V[0]$ coincides with the Coxeter–KZ connection for $W$ with values in $V[0]$ and weights given by $k_\alpha = -h(\alpha, \alpha)$.

**Proof.** The $W$–invariance of $V[0]$ is clear. Its $C_\alpha$–invariance follows from (ii). Any $v \in V[0]$ may be written as $v_0^0 + v_0^2$ where $v_0^1$ lies in the zero weight space of the irreducible $\mathfrak{sl}_2$–module $V_i$ of dimension $i + 1$. (ii) then follows from the fact that $s_\alpha$ and $2/(\alpha, \alpha) C_\alpha$ act as multiplication by $(-1)^{i/2}$ and
Note that, if $V$ is a small representation in the sense of Broer and Reeder, i.e., such that $2\alpha$ is not a weight of $V$ for any root $\alpha$, then $V[0] = V[0]$. This is the case of the adjoint representation for example. In general however, $V[0]$ can be a proper, non-zero subspace of $V[0]$.

Proposition 9.1 raises the question of whether every irreducible representation $U$ of $W$ may be realised inside some $V[0]$. On the positive side, we have the following.

Proposition 9.2.

i. If $U_\lambda$ is the simple $\mathfrak{S}_n$–module corresponding, via the Schur–Weyl parametrisation, to the Young diagram $\lambda$, then

\[ U_\lambda \cong V_{\lambda'}[0] = V_{\lambda'}[0] \]

where $V_{\lambda'}$ is the irreducible representation of $\mathfrak{sl}_n$ with highest weight given by the tranposed Young diagram $\lambda'$.

ii. For any $\mathfrak{g}$, the equality $\mathfrak{h} = \mathfrak{g}[0]$ induces an inclusion of $W$–modules

\[ \Lambda^i \mathfrak{h} \hookrightarrow \Lambda^i \mathfrak{g}[0] \]

Proof. (i) Let $V \cong \mathbb{C}^n$ be the vector representation of $\mathfrak{sl}_n$ so that the $V_{\lambda'}$ span all irreducible summands of $V^{\otimes n}$ as $\lambda$ varies over all partitions of $n$. A simple inspection shows that $V^{\otimes n}$ is a small representation so that $V^{\otimes n}[0] = V^{\otimes n}[0]$. The isomorphism $U_\lambda \cong V_{\lambda'}[0]$ is a simple corollary of Schur–Weyl duality due to Kostant and Gutkin. (ii) follows from an easy calculation.

On the negative side however, one has the following

Proposition 9.3. There exist irreducible representations of the orthogonal Weyl groups $B_n = W(\mathfrak{so}_{2n+1}), n \geq 2$ and $D_n = W(\mathfrak{so}_{2n}), n \geq 4$ which are not contained in any $V[0]$.

It seems an interesting problem to determine, for any $\mathfrak{g}$, the Springer parameters of the irreducible representations of $W$ which arise inside some $V[0]$. A further motivation for this question comes from the following simple corollary of proposition 9.1. Let $\{U_i\}_{i \in I}$ be the isomorphism classes of irreducible representations of $W$ which may be realised inside some $V[0]$, and let $P_0 \in \mathbb{C}[W]$ be the corresponding central projection onto $\bigoplus_{i \in I} \text{End}(U_i)$. Let $C_\mathfrak{g}$ be the Casimir algebra of $\mathfrak{g}$, i.e., the subalgebra

\[ C_\mathfrak{g} = \langle C_\alpha \rangle_{\alpha > 0} \subset U_\mathfrak{g} \]

of the enveloping algebra of $\mathfrak{g}$ generated by the $C_\alpha$. Then,

Proposition 9.4. The assignement

\[ C_\alpha \to \langle \alpha, \alpha \rangle P_0 (1 - s_\alpha) P_0 \]
extends uniquely to a surjective, \( W \)-equivariant algebra homomorphism of the Casimir algebra \( C_g \) of \( g \) onto the subalgebra

\[
P_0C[W]P_0 = \bigoplus_{i \in I} U_i \otimes U_i^*
\]

of \( \mathbb{C}[W] \).

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