Recovery of a potential on a quantum star graph from Weyl’s matrix

Sergei A. Avdonin¹, Kira V. Khmelnytskaya², Vladislav V. Kravchenko³
¹ Department of Mathematics and Statistics, University of Alaska, Fairbanks, AK 99775, USA
² Faculty of Engineering, Autonomous University of Queretaro, Cerro de las Campanas s/n, col. Las Campanas Querétaro, Qro. C.P. 76010 México
³ Department of Mathematics, Cinvestav, Campus Querétaro, Libramiento Norponiente #2000, Fracc. Real de Juriquilla, Querétaro, Qro., 76230 México
e-mail: s.avdonin@alaska.edu, khmel@uaq.edu.mx, vkravchenko@math.cinvestav.edu.mx

October 28, 2022

Abstract

The problem of recovery of a potential on a quantum star graph from Weyl’s matrix given at a finite number of points is considered. A method for its approximate solution is proposed. It consists in reducing the problem to a two-spectra inverse Sturm-Liouville problem on each edge with its posterior solution. The overall approach is based on Neumann series of Bessel functions (NSBF) representations for solutions of Sturm-Liouville equations, and, in fact, the solution of the inverse problem on the quantum graph reduces to dealing with the NSBF coefficients.

The NSBF representations admit estimates for the series remainders which are independent of the real part of the square root of the spectral parameter. This feature makes them especially useful for solving direct and inverse problems requiring calculation of solutions on large intervals in the spectral parameter. Moreover, the first coefficient of the NSBF representation alone is sufficient for the recovery of the potential.

The knowledge of the Weyl matrix at a set of points allows one to calculate a number of the NSBF coefficients at the end point of each edge, which leads to approximation of characteristic functions of two Sturm-Liouville problems and allows one to compute the Dirichlet-Dirichlet and Neumann-Dirichlet spectra on each edge. In turn, for solving this two-spectra inverse Sturm-Liouville problem a system of linear algebraic equations is derived for computing the first NSBF coefficient and hence for recovering the potential. The proposed method leads to an efficient numerical algorithm that is illustrated by a number of numerical tests.

1 Introduction

The Weyl matrix of a quantum graph is in fact its Dirichlet-to-Neumann map. It plays essential role in all aspects of study of quantum graphs, including spectral theory and controllability
The problem addressed in the present work has a clear physical meaning. From a measured response at a number of frequencies of a system modelled by a quantum graph, recover the differential operator on each edge of the graph.

A number of surveys and collections of papers on quantum graphs appeared last years, and the first books on this topic by Berkolaiko and Kuchment [10], Mugnolo [25] and Kurasov [20] contain excellent lists of references. Inverse spectral theory of network-like structures is an important part of the rapidly developing area of applied mathematics — analysis on graphs. The known results in this direction concern almost exclusively trees, i.e. graphs without cycles, see e.g. [21, 31, 9, 6, 7, 4].

To date, there are few papers containing numerical results for inverse problems on graphs, all of them concern only very simple trees [9, 3]. It is known that the problems of space discretization of differential equations on metric graphs turn out to be very difficult, and even the forward boundary value problems on graphs contain a lot of numerical challenges (see, e.g. [2]).

In [5] we developed a new approach for solving inverse spectral problems on compact quantum star graphs which lends itself to efficient numerical algorithms. The approach is based on Neumann series of Bessel functions (NSBF) representations for solutions of Sturm-Liouville equations obtained in [16] (see also [14]). Here we extend this approach to the inverse problem of recovery the Schrödinger equation on a star graph from the values of the corresponding Weyl matrix at a finite set of points.

The NSBF representations (see Section 4 below) for solutions of Sturm-Liouville equations

\[-y'' + q(x)y = \rho^2 y,\]

possess two remarkable features which make them especially convenient for solving inverse problems. The remainders of the series admit \(\rho\)-independent bounds for \(\rho \in \mathbb{R}\), and the potential \(q(x)\) can be recovered from the very first coefficient of the series. The first feature allows us to work with the approximate solutions on very large intervals in \(\rho\), and the second implies that computationally satisfactory results require considering a reduced number of the terms of the series, which eventually results in a reduced number of linear algebraic equations which should be solved in each step.

In fact the overall developed approach reduces the solution of the inverse problem on a graph to operations with the NSBF coefficients. In the first step, using the given data, we compute the NSBF coefficients for solutions satisfying the homogeneous Dirichlet and Neumann conditions at the boundary vertices at the end point of each edge, which is associated with the common vertex of the star graph. This first step allows us to split the problem on the graph into separate problems on each edge. Second, the set of the coefficients for the series representation of the solution satisfying the homogeneous Dirichlet condition is used for computing the Dirichlet-Dirichlet eigenvalues of the potential on each edge, while the coefficients for the series representation of the solution satisfying the homogeneous Neumann condition are used for computing the Neumann-Dirichlet eigenvalues. Moreover, the first feature of the NSBF representations implies that, if necessary, hundreds of the eigenvalues can be computed with uniform accuracy, and for this, few coefficients of the series representations are sufficient.

Thus, the inverse problem on a graph is reduced to a two spectra inverse Sturm-Liouville problem on each edge. Results on the uniqueness and solvability of the two spectra problem are well known and can be found, e.g., in [12, 23, 28, 32]. For this problem we propose a method
which again involves the NSBF representations. It allows us to compute multiplier constants relating the Neumann-Dirichlet eigenfunctions associated with the same eigenvalues but normalized at the opposite endpoints of the interval. This leads to a system of linear algebraic equations for the coefficients of their NSBF representations already for interior points of the interval. Solving the system we find the very first coefficient, from which the potential is recovered. It is worth mentioning that the NSBF representations were first used for solving inverse Sturm-Liouville problems on a finite interval in [13]. Later on, the approach from [13] was improved in [14] and [18], [19]. In these papers the system of linear algebraic equations was obtained with the aid of the Gelfand-Levitan integral equation. In [15] another approach, based on the consideration of the eigenfunctions normalized at the opposite endpoints, was developed, and this idea was used in [3]. In the present work we adjust the same idea to the solution of the two-spectra inverse problem arising in the final step.

In Section 2 we recall the definition of the Weyl matrix and formulate the inverse problem. In Section 3 we write the system of equations in terms of the fundamental systems of solutions on each edge, which is obtained directly from the knowledge of the Weyl matrix. In Section 4 we recall the NSBF representations for solutions of the Sturm-Liouville equation and some of their relevant features. In Section 5 we explain the solution of the direct problem, i.e., the construction of the Weyl matrix when the potential on the graph is known. In Section 6 we give a detailed description of the proposed method for the solution of the inverse problem. In Section 7 we discuss the numerical implementation of the method. Finally, Section 8 contains some concluding remarks.

2 Problem setting

Let \( \Omega \) denote a compact star graph consisting of \( M \) edges \( e_1, \ldots, e_M \) connected at the vertex \( v \). Every edge \( e_j \) is identified with an interval \((0, L_j)\) of the real line in such a way that zero corresponds to the boundary vertex \( \gamma_j \), and the endpoint \( L_j \) corresponds to the vertex \( v \). By \( \Gamma \) we denote the set of the boundary vertices of \( \Omega \), \( \Gamma = \{ \gamma_1, \ldots, \gamma_M \} \). Let \( q \in L_2(\Omega) \) be real valued, and \( \lambda \) a complex number. A continuous function \( u \) defined on the graph \( \Omega \) is an \( M \)-tuple of functions \( u_j \in C[0, L_j] \) satisfying the continuity equalities at the vertex \( v \): \( u_i(v) = u_j(v) \) for all \( 1 \leq i, j \leq M \). Then \( u \in C(\Omega) \).

We say that a function \( u \) is a solution of the equation

\[
- u''(x) + q(x)u(x) = \lambda u(x)
\]

(2.1)
on the graph \( \Omega \) if besides (2.1), the following conditions are satisfied

\[
u \in C(\Omega)
\]

(2.2)

and

\[
\sum_{j=1}^{M} \partial u_j(v) = 0,
\]

(2.3)

where \( \partial u_j(v) \) denotes the derivative of \( u \) at the vertex \( v \) taken along the edge \( e_j \) in the direction outward the vertex. The sum in the equality (2.3), which is known as the Kirchhoff-Neumann condition, is taken over all the edges \( e_j, j = 1, \ldots, M \).
Let $\lambda \notin \mathbb{R}$ and $w_i$ be the so-called Weyl solution, that is, a solution of (2.1) on $\Omega$, satisfying the initial conditions at the boundary vertices

$$w_i(\gamma_i) = 1 \quad \text{and} \quad w_i(\gamma_j) = 0 \quad \text{for all } j \neq i.$$ 

**Definition 2.1** The $M \times M$ matrix-function $M(\lambda), \lambda \notin \mathbb{R}$, consisting of the elements $M_{ij}(\lambda) = \partial w_i(\gamma_j), \ i, j = 1, \ldots, M$ is called the Weyl matrix.

In fact, for a fixed value of $\lambda$, the Weyl matrix is the Dirichlet-to-Neumann map on the quantum graph because, if $u$ is a solution of (2.1) satisfying the Dirichlet condition at the boundary vertices $u(\gamma, \lambda) = f(\lambda)$, then $\partial u(\gamma, \lambda) = M(\lambda)f(\lambda), \lambda \notin \mathbb{R}$.

The problem we consider in the present paper can be formulated as follows.

**Problem 2.2** Given the Weyl matrix at a finite number of points $\lambda_k, k = 1, \ldots, m$, approximate the potential $q(x)$.

When the Weyl matrix or even its main diagonal is known everywhere, the potential $q(x)$ is determined uniquely (see, e.g., [31]). The knowledge of the Weyl matrix at a finite number of points allows one to recover the potential $q(x)$ only approximately.

### 3 Fundamental system of solutions and the Weyl matrix

In order to reformulate the inverse problem in terms suitable for its numerical solution, let us introduce for each edge $e_i$ a corresponding fundamental system of solutions. By $\varphi_i(\rho, x)$ and $S_i(\rho, x)$ we denote the solutions of the equation

$$-y''(x) + q_i(x)y(x) = \rho^2 y(x), \quad x \in (0, L_i)$$

satisfying the initial conditions

$$\varphi_i(\rho, 0) = 1, \quad \varphi_i'(\rho, 0) = 0,$$

$$S_i(\rho, 0) = 0, \quad S_i'(\rho, 0) = 1.$$ 

Here $q_i(x)$ is the potential $q(x)$ on the edge $e_i$ and $\rho = \sqrt{\lambda}, \text{Im } \rho \geq 0$. Then the Weyl solution $w_i$ has the form

$$w_i(\rho, x) = \varphi_i(\rho, x) + M_{ii}(\rho^2)S_i(\rho, x) \quad \text{on the edge } e_i$$

and

$$w_i(\rho, x) = M_{ij}(\rho^2)S_j(\rho, x) \quad \text{on every edge } e_j, \quad j \neq i.$$ 

Thus, the knowledge of the Weyl matrix at a point $\rho_k^2$ implies the knowledge of such constants $M_{ij}(\rho_k^2)$ that for all $i = 1, \ldots, M$, the equalities are valid

$$\varphi_i(\rho_k, L_i) + M_{ii}(\rho_k^2)S_i(\rho_k, L_i) = M_{ij}(\rho_k^2)S_j(\rho_k, L_j) \quad \text{for all } j \neq i$$

and

$$\varphi_i'(\rho_k, L_i) + \sum_{j=1}^M M_{ij}(\rho_k^2)S_j'(\rho_k, L_j) = 0.$$ 

The solutions $\varphi_i(\rho, x)$ and $S_i(\rho, x)$ admit very convenient series representations, which we discuss in the next section.
In analysis, according to [29, Chapter XVI], “Any series of the type
\[ \sum_{n=0}^{\infty} a_n J_{\nu+n}(z) \]
is called a Neumann series, although in fact Neumann considered only the special type of series for which \( \nu \) is an integer; the investigation of the more general series is due to Gegenbauer”. The papers by C. G. Neumann and L. B. Gegenbauer date from 1867 and 1877, respectively. Since then the Neumann series of Bessel functions (NSBF) were studied in numerous publications (see [29, 30] and the recent monograph on the subject [8] and references therein). In [16] it was shown that the solutions \( \varphi_i(\rho, x) \), \( S_i(\rho, x) \) and their derivatives with respect to \( x \) admit the NSBF representations which possess certain unique features which make them especially useful for solving direct and inverse spectral problems.

**Theorem 4.1 ([16])** The solutions \( \varphi_i(\rho, x) \) and \( S_i(\rho, x) \) of (3.1) and their derivatives with respect to \( x \) admit the following series representations

\[
\varphi_i(\rho, x) = \cos(\rho x) + \sum_{n=0}^{\infty} (-1)^n g_{i,n}(x) j_{2n}(\rho x),
\]

\[
S_i(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^n s_{i,n}(x) j_{2n+1}(\rho x),
\]

\[
\varphi'_i(\rho, x) = -\rho \sin(\rho x) + \frac{\cos(\rho x)}{2} \int_0^x q_i(t) \, dt + \sum_{n=0}^{\infty} (-1)^n \gamma_{i,n}(x) j_{2n}(\rho x),
\]

\[
S'_i(\rho, x) = \cos(\rho x) + \frac{\sin(\rho x)}{2\rho} \int_0^x q_i(t) \, dt + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^n \sigma_{i,n}(x) j_{2n+1}(\rho x),
\]

where \( j_k(z) \) stands for the spherical Bessel function of order \( k \), \( j_k(z) := \sqrt{\frac{\pi}{2z}} J_{k+\frac{1}{2}}(z) \) (see, e.g., [1]). The coefficients \( g_{i,n}(x) \), \( s_{i,n}(x) \), \( \gamma_{i,n}(x) \) and \( \sigma_{i,n}(x) \) can be calculated following a simple recurrent integration procedure (see [16] or [14, Sect. 9.4]), starting with

\[
g_{i,0}(x) = \varphi_i(0, x) - 1, \quad s_{i,0}(x) = 3 \left( \frac{S_i(0, x)}{x} - 1 \right),
\]

\[
\gamma_{i,0}(x) = g'_{i,0}(x) - \frac{1}{2} \int_0^x q_i(t) \, dt, \quad \sigma_{i,0}(x) = s_{i,0}(x) + s'_{i,0}(x) - \frac{3}{2} \int_0^x q_i(t) \, dt.
\]

For every \( \rho \in \mathbb{C} \) all the series converge pointwise. For every \( x \in [0, L_i] \) the series converge uniformly on any compact set of the complex plane of the variable \( \rho \), and the remainders of their partial sums admit estimates independent of \( \text{Re} \rho \).

This last feature of the series representations (the independence of \( \text{Re} \rho \) of the estimates for the remainders) is of crucial importance for what follows. In particular, it means that for
\[ S_{i,N}(\rho, x) := \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{N} (-1)^n s_{i,n}(x) j_{2n+1}(\rho x) \] (and analogously for \( \varphi_{i,N}(\rho, x) \)) the estimate holds

\[ |S_{i}(\rho, x) - S_{i,N}(\rho, x)| < \varepsilon_{i,N}(x) \] (4.6)

for all \( \rho \in \mathbb{R} \), where \( \varepsilon_{i,N}(x) \) is a positive function tending to zero as \( N \to \infty \). Roughly speaking, the approximate solution \( S_{i,N}(\rho, x) \) approximates the exact one equally well for small and for large values of \( \rho \in \mathbb{R} \). This is especially convenient when considering direct and inverse spectral problems that requires operating on a large range of the parameter \( \rho \). This unique feature of the series representations (4.1)-(4.4) is due to the fact that they originate from an exact Fourier-Legendre series representation of the integral kernel of the transmutation operator [16], [14, Sect. 9.4] (for the theory of transmutation operators we refer to [23], [24], [27], [32].

Moreover, for a fixed \( z \) the numbers \( j_k(z) \) rapidly decrease as \( k \to \infty \), see, e.g., [1, (9.1.62)].

Hence, the convergence rate of the series for any fixed \( \rho \) is, in fact, exponential. More detailed estimates for the series remainders depending on the regularity of the potential can be found in [16].

Note that formulas (4.5) indicate that the potential \( q_i(x) \) can be recovered from the first coefficients of the series (4.1) or (4.2). Indeed, we have

\[ q_i(x) = \frac{g''_{i,0}(x)}{g_{i,0}(x) + 1} \] (4.7)

and

\[ q_i(x) = \frac{(xs_{i,0}(x))''}{xs_{i,0}(x) + 3x}. \] (4.8)

### 5 Solution of the direct problem

Here we briefly explain how in terms of the solutions \( \varphi_{i}(\rho, x) \) and \( S_{i}(\rho, x) \) the construction of the Weyl matrix can be performed. Given \( q(x) \), fix \( \rho \). To find the first row of the matrix \( M(\rho^2) \), that is the entries \( M_{ij}(\rho^2) \), we consider the corresponding equalities at the common vertex \( v \):

\[
\varphi_1(\rho, L_1) + M_{11}(\rho^2)S_1(\rho, L_1) = M_{12}(\rho^2)S_2(\rho, L_2),
\]

\[
M_{12}(\rho^2)S_2(\rho, L_2) = M_{13}(\rho^2)S_3(\rho, L_3),
\]

\[
\ldots
\]

\[
M_{1M-1}(\rho^2)S_{M-1}(\rho, L_{M-1}) = M_{1M}(\rho^2)S_M(\rho, L_M),
\]

\[
\varphi'_1(\rho, L_1) + M_{11}(\rho^2)S'_1(\rho, L_1) + M_{12}(\rho^2)S'_2(\rho, L_2) + \ldots + M_{1M}(\rho^2)S'_M(\rho, L_M) = 0,
\]

which can be written in the form of a system of linear algebraic equations

\[
A(\rho)\tilde{M}_1(\rho^2) =
\]
\[
\begin{pmatrix}
S_1(L_1) & -S_2(L_2) & 0 & 0 & \cdots & 0 \\
0 & S_2(L_2) & -S_3(L_3) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & S_{M-1}(L_{M-1}) & -S_M(L_M) & \\
S_1'(L_1) & S_2'(L_2) & \cdots & S_{M-1}'(L_{M-1}) & S_M'(L_M)
\end{pmatrix}
\begin{pmatrix}
\mathbf{M}_{11} \\
\mathbf{M}_{12} \\
\vdots \\
\mathbf{M}_{1M-1} \\
\mathbf{M}_{1M}
\end{pmatrix} = \begin{pmatrix}
-\varphi_1(L_1) \\
0 \\
\vdots \\
0 \\
-\varphi_1'(L_1)
\end{pmatrix},
\]

where for the sake of space we omitted the dependence on \( \rho \).

It is easy to verify that all subsequent rows of the Weyl matrix can be computed by solving a system of linear algebraic equations with the same matrix \( \mathbf{A}(\rho) \)

\[
\mathbf{A}(\rho) = \begin{pmatrix}
S_1(\rho, L_1) & -S_2(\rho, L_2) & 0 & 0 & \cdots & 0 \\
0 & S_2(\rho, L_2) & -S_3(\rho, L_3) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & S_{M-1}(\rho, L_{M-1}) & -S_M(\rho, L_M) & \\
S_1'(\rho, L_1) & S_2'(\rho, L_2) & \cdots & S_{M-1}'(\rho, L_{M-1}) & S_M'(\rho, L_M)
\end{pmatrix},
\]

but with a different right-hand side. Namely,

\[
\mathbf{A}(\rho) \tilde{\mathbf{M}}_i(\rho^2) = \begin{pmatrix}
0 \\
\vdots \\
\varphi_i(\rho, L_i) \\
-\varphi_i(\rho, L_i) \\
0 \\
\vdots \\
-\varphi_i'(\rho, L_i)
\end{pmatrix}
\text{ for } i = 2, \ldots, M - 1,
\]

where the first \( \varphi_i(\rho, L_i) \) appears at \((i - 1)\)-th position, and for the last row we have

\[
\mathbf{A}(\rho) \tilde{\mathbf{M}}_M(\rho^2) = \begin{pmatrix}
0 \\
\vdots \\
\varphi_M(\rho, L_M) \\
0 \\
-\varphi_M'(\rho, L_M)
\end{pmatrix}.
\]

\section{Solution of the inverse problem}

Here we assume the Weyl matrix to be known at a finite number of points \( \rho_k^2 \notin \mathbb{R}, k = 1, \ldots, m \).

From this information we look to recover the potential \( q_i(x) \) on each edge of the star graph. The proposed method for solving this problem consists of several steps.

First, we compute a number of the constants \( \{g_{i,n}(L_i)\}_{n=0}^N \) and \( \{s_{i,n}(L_i)\}_{n=0}^N \) for every \( i = 1, \ldots, M \), that is, the values of the coefficients from (4.1) and (4.2) at the endpoint of the edge.
This first step allows us to split the problem on the graph and reduce it to the problems on the edges. Indeed, the knowledge of the coefficients \( \{g_{i,n}(L_i)\}_{n=0}^{N} \) and \( \{s_{i,n}(L_i)\}_{n=0}^{N} \) allows us in the second step to compute the Dirichlet-Dirichlet and Neumann-Dirichlet spectra for the potential \( q_i(x) \), \( x \in [0, L_i] \), thus obtaining a two-spectra inverse problem for each component of the potential \( q(x) \).

In the third step the two-spectra inverse problem is solved with the aid of the representation (4.1) and an analogous series representation for the solution \( T_i(x) \) of (3.1) satisfying the initial conditions at the endpoint \( L_i \):

\[
T_i(\rho, L_i) = 0, \quad T_i'(\rho, L_i) = 1. \tag{6.1}
\]

The two-spectra inverse problem is reduced to a system of linear algebraic equations, from which we obtain the coefficient \( g_{i,0}(x) \). Finally, the potential \( q_i(x) \) is calculated from (4.7).

### 6.1 Calculation of coefficients \( \{g_{i,n}(L_i)\}_{n=0}^{N} \) and \( \{s_{i,n}(L_i)\}_{n=0}^{N} \)

The knowledge of the Weyl matrix at a point \( \rho_k^2 \) means that we have equalities (5.1)-(5.3) valid for \( \rho = \rho_k \). In fact we will use only the equations which arise from the continuity condition (2.2) and not those which arise from the Kirchhoff-Neumann condition (2.3). Hence we work with the solutions \( \varphi_i(\rho, x) \) and \( S_i(\rho, x) \) and not with their derivatives. Thus, for every \( \rho_k \) from the set, using the representations (4.1) and (4.2) we have the equations

\[
\rho_k \sum_{n=0}^{\infty} (-1)^n g_{i,n}(L_i) j_{2n}(\rho_k L_i) + M_{ii}(\rho_k^2) \sum_{n=0}^{\infty} (-1)^n s_{i,n}(L_i) j_{2n+1}(\rho_k L_i)

- M_{ii+1}(\rho_k^2) \sum_{n=0}^{\infty} (-1)^n s_{i+1,n}(L_{i+1}) j_{2n+1}(\rho_k L_{i+1})

= M_{ii+1}(\rho_k^2) \sin(\rho_k L_{i+1}) - \rho_k \cos(\rho_k L_i) - M_{ii}(\rho_k^2) \sin(\rho_k L_i), \quad \text{for } i = 1, \ldots, M, \tag{6.2}
\]

where for \( i = M \) we replace \( i + 1 \) by 1 (the cyclic rule), as well as the equations

\[
M_{ij}(\rho_k^2) \sum_{n=0}^{\infty} (-1)^n s_{j,n}(L_j) j_{2n+1}(\rho_k L_j) - M_{ij+1}(\rho_k^2) \sum_{n=0}^{\infty} (-1)^n s_{j+1,n}(L_{j+1}) j_{2n+1}(\rho_k L_{j+1})

= M_{ij+1}(\rho_k^2) \sin(\rho_k L_{j+1}) - M_{ij}(\rho_k^2) \sin(\rho_k L_j), \tag{6.3}
\]

where \( j = 1, \ldots, M, j \neq i, j + 1 \neq i \), and again, for \( j = M \) we replace \( j + 1 \) by 1.

To compute the finite sets of the coefficients \( \{g_{i,n}(L_i)\}_{n=0}^{N} \) and \( \{s_{i,n}(L_i)\}_{n=0}^{N} \) for all \( i = 1, \ldots, M \) we chose the following strategy. Fix \( i \), and for each \( \rho_k^2 \) consider the continuity conditions corresponding to the Weyl solution \( w_i \) or, in other words, to the \( i \)-th row of the Weyl matrix. We have then one equation of the form (6.2) and \( M - 2 \) equations of the form (6.3). Considering these \( M - 1 \) equations for every \( \rho_k^2 \), we obtain \( m(M-1) \) equations for \( (M+1)(N+1) \) unknowns. Here the unknowns are \( M \) sets of the coefficients \( \{s_{j,n}(L_j)\}_{n=0}^{N} \) (for \( j = 1, \ldots, M \)) and one set of the coefficients \( \{g_{i,n}(L_i)\}_{n=0}^{N} \). Thus, we need at least \( m = \lceil \frac{(M+1)(N+1)}{M-1} \rceil \) points \( \rho_k^2 \) at which the Weyl matrix is known. Here \( \lceil x \rceil \) denotes the least integer greater than or equal to \( x \). In particular, the obtained system of equations allows us to compute \( \{g_{i,n}(L_i)\}_{n=0}^{N} \) and
\( \{ s_{i,n}(L_i) \}_{n=0}^{N} \) for the fixed \( i \). Thus in total, we consider \( M \) linear algebraic systems of this kind to compute all sets of the coefficients \( \{ g_{i,n}(L_i) \}_{n=0}^{N} \) and \( \{ s_{i,n}(L_i) \}_{n=0}^{N} \) for all \( i = 1, \ldots, M \).

An important observation however consists in the fact that in order to find the coefficients \( \{ g_{i,n}(L_i) \}_{n=0}^{N} \) and \( \{ s_{i,n}(L_i) \}_{n=0}^{N}, \ i = 1, \ldots, M \), there is no need to consider the equations of the form (6.3) or at least all such equations. In fact, it is enough to consider equation (6.2) alone, which means that to recover the potential on the edge \( e_i \), we need to know the main diagonal entry \( M_{ii}(\rho_k^2) \) as well as some \( M_{ij}(\rho_k^2) \) for one \( j \neq i \), which can be \( M_{ii+1}(\rho_k^2) \) (for \( i = M, i + 1 \) is replaced by 1). In this case, for each edge \( e_i \), from equations of the form (6.2), which we have for each \( \rho_k \), we compute \( \{ g_{i,n}(L_i) \}_{n=0}^{N}, \ \{ s_{i,n}(L_i) \}_{n=0}^{N} \) and \( \{ s_{i+1,n}(L_{i+1}) \}_{n=0}^{N} \), that is, \( 3(N + 1) \) unknowns. For this, the knowledge of the Weyl matrix (or of its main diagonal plus one more entry from each row) is required at least at \( 3(N + 1) \) points \( \rho_k \).

We show below that eventually the accuracy of the recovered potential is comparable when one uses all equations of the form (6.3), part of them or none. The number of equations of the form (6.3) used in computations we will denote by \( M_k \). Thus, if no such equation is used \( M_k = 0 \) (that is, two elements from each row of the Weyl matrix: \( M_{ii}(\rho_k^2) \) and \( M_{ii+1}(\rho_k^2) \) are used), while \( M_k = M - 2 \) means that all equations of the form (6.3) (and all elements of the Weyl matrix) are used.

### 6.2 Reduction to the two-spectra inverse problem on the edge

The first step, as described in the previous subsection, reduces the problem on the graph to \( M \) separate problems on the edges. Thus, consider an edge \( e_i \) for which at this stage we have computed the coefficients \( \{ g_{i,n}(L_i) \}_{n=0}^{N} \) and \( \{ s_{i,n}(L_i) \}_{n=0}^{N} \). Now we use them to compute the Dirichlet-Dirichlet and Neumann-Dirichlet spectra for the potential \( q_i(x), \ x \in [0, L_i] \). This is done with the aid of the approximate solutions evaluated at the end point

\[
\varphi_{i,N}(\rho, L_i) = \cos (\rho L_i) + \sum_{n=0}^{N} (-1)^n g_{i,n}(L_i) j_{2n}(\rho L_i)
\]

and

\[
S_{i,N}(\rho, L_i) = \frac{\sin (\rho L_i)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{N} (-1)^n s_{i,n}(L_i) j_{2n+1}(\rho L_i).
\]

Indeed, since \( S_i(\rho, x) \) (as well as \( S_{i,N}(\rho, x) \)) satisfies the Dirichlet condition at the origin, zeros of \( S_i(\rho, L_i) \) are precisely square roots of the Dirichlet-Dirichlet eigenvalues. That is, \( S_i(\rho, L_i) \) is the characteristic function of the Sturm-Liouville problem

\[
- y'' + q_i(x) y = \lambda y, \quad x \in (0, L_i), \tag{6.4}
\]

\[
y(0) = y(L_i) = 0, \tag{6.5}
\]

and its zeros coincide with the numbers \( \{ \mu_{i,k} \}_{k=1}^{\infty} \), such that \( \mu_{i,k}^2 \) are the eigenvalues of the problem (6.4), (6.5). In turn, zeros of the function \( S_{i,N}(\rho, L_i) \) approximate zeros of \( S_i(\rho, L_i) \) (see Theorem 6.1 below). Thus, the singular numbers \( \{ \mu_{i,k} \}_{k=1}^{\infty} \) are approximated by zeros of the function \( S_{i,N}(\rho, L_i) \).

9
The same reasoning is valid for the function $\varphi_{i,N}(\rho, L_i)$, whose zeros approximate the singular numbers $\{\nu_{i,k}\}_{k=1}^{\infty}$, which are the square roots of the eigenvalues of the Sturm-Liouville problem for (6.4) subject to the boundary conditions

$$y'(0) = y(L_i) = 0. \quad (6.6)$$

The following statement is valid.

**Theorem 6.1** For any $\varepsilon > 0$ there exists such $N \in \mathbb{N}$ that all zeros of the functions $S_i(\rho, L_i)$ and $\varphi_i(\rho, L_i)$ are approximated by corresponding zeros of the functions $S_{i,N}(\rho, L_i)$ and $\varphi_{i,N}(\rho, L_i)$, respectively, with errors uniformly bounded by $\varepsilon$. Moreover, $S_{i,N}(\rho, L_i)$ and $\varphi_{i,N}(\rho, L_i)$, have no other zeros.

**Proof.** The proof of this statement is completely analogous to the proof of Proposition 7.1 in [17] and consists in the use of properties of characteristic functions of regular Sturm-Liouville problems and application of the Rouché theorem. \[\Box\]

Below, in Section 7 we show that indeed, even for relatively small $N$, computing zeros of the functions $S_{i,N}(\rho, L_i)$ and $\varphi_{i,N}(\rho, L_i)$ one obtains hundreds of the Dirichlet-Dirichlet and Neumann-Dirichlet eigenvalues computed with remarkably uniform accuracy. Thus, on every edge $e_i$ we obtain the classical inverse problem of recovering the potential $q_i(x)$ from two spectra, which is considered in the next step.

### 6.3 Solution of two-spectra inverse problem

At this stage we dispose of two finite sequences of singular numbers $\{\mu_{i,k}\}_{k=1}^{K_D}$ and $\{\nu_{i,k}\}_{k=1}^{K_N}$ which are square roots of the eigenvalues of problems (6.4), (6.5) and (6.4), (6.6), respectively, as well as of two sequences of numbers $\{s_{i,n}(L_i)\}_{n=0}^{N}$ and $\{g_{i,n}(L_i)\}_{n=0}^{N}$, which are the values of the coefficients from (4.2) and (4.1) at the endpoint.

Let us consider the solution $T_i(\rho, x)$ of equation (6.4) satisfying the initial conditions at $L_i$:

$$T_i(\rho, L_i) = 0, \quad T_i'(\rho, L_i) = 1. \quad (6.4)$$

Analogously to the solution (4.2), the solution $T_i(\rho, x)$ admits the series representation

$$T_i(\rho, x) = \frac{\sin (\rho (x - L_i))}{\rho} + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^n t_{i,n}(x) j_{2n+1}(\rho (x - L_i)), \quad (6.7)$$

where $t_{i,n}(x)$ are corresponding coefficients, analogous to $s_{i,n}(x)$ from (4.2).

Note that for $\rho = \nu_{i,k}$ the solutions $\varphi_i(\nu_{i,k}, x)$ and $T_i(\nu_{i,k}, x)$ are linearly dependent because both are eigenfunctions of problem (6.4), (6.6). Hence there exist such real constants $\beta_{i,k} \neq 0$, that

$$\varphi_i(\nu_{i,k}, x) = \beta_{i,k} T_i(\nu_{i,k}, x). \quad (6.8)$$
Moreover, these multiplier constants can be easily calculated by recalling that $\varphi_i(\nu_{i,k}, 0) = 1$. Thus,

$$\frac{1}{\beta_{i,k}} = T_i(\nu_{i,k}, 0) \approx T_{i,N}(\nu_{i,k}, 0)$$

$$= -\frac{\sin (\nu_{i,k} L_i)}{\nu_{i,k}} - \frac{1}{\nu_{i,k}} \sum_{n=0}^{N} (-1)^n t_{i,n}(0) j_{2n+1}(\nu_{i,k} L_i),$$

(6.9)

where we took into account that the spherical Bessel functions of odd order are odd. The coefficients $\{t_{i,n}(0)\}_{n=0}^{N}$ are computed with the aid of the singular numbers $\{\mu_{i,k}\}_{k=1}^{K_D}$ as follows. Since the functions $T_i(\mu_{i,k}, x)$, $k = 1, 2, \ldots$ are eigenfunctions of the problem (6.4), (6.5), we have that $T_i(\mu_{i,k}, 0) = 0$ and hence

$$\sum_{n=0}^{\infty} (-1)^n t_{i,n}(0) j_{2n+1}(\mu_{i,k} L_i) = -\sin (\mu_{i,k} L_i), \quad k = 1, 2, \ldots.$$ 

This leads to a system of linear algebraic equations for computing the coefficients $\{t_{i,n}(0)\}_{n=0}^{N}$, which has the form

$$\sum_{n=0}^{N} (-1)^n t_{i,n}(0) j_{2n+1}(\mu_{i,k} L_i) = -\sin (\mu_{i,k} L_i), \quad k = 1, \ldots, K_D.$$

Now, having computed $\{t_{i,n}(0)\}_{n=0}^{N}$, we compute the multiplier constants $\{\beta_{i,k}\}_{k=1}^{K_N}$ from (6.9).

Next, we use equation (6.8) for constructing a system of linear algebraic equations for the coefficients $g_{i,n}(x)$ and $t_{i,n}(x)$. Indeed, equation (6.8) can be written in the form

$$\sum_{n=0}^{\infty} (-1)^n g_{i,n}(x) j_{2n}(\nu_{i,k} x) = \frac{\beta_{i,k}}{\nu_{i,k}} \sum_{n=0}^{\infty} (-1)^n t_{i,n}(x) j_{2n+1}(\nu_{i,k} (x - L_i))$$

$$= \frac{\beta_{i,k}}{\nu_{i,k}} \sin (\nu_{i,k} (x - L_i)) - \cos (\nu_{i,k} x).$$

We have as many of such equations as many Neumann-Dirichlet singular numbers $\nu_{i,k}$ are computed. For computational purposes we choose some natural number $N_c$ - the number of the coefficients $g_{i,n}(x)$ and $t_{i,n}(x)$ to be computed. More precisely, we choose a sufficiently dense set of points $x_m \in (0, L_i)$ and at every $x_m$ consider the equations

$$\sum_{n=0}^{N_c} (-1)^n g_{i,n}(x_m) j_{2n}(\nu_{i,k} x_m) = \frac{\beta_{i,k}}{\nu_{i,k}} \sum_{n=0}^{N_c} (-1)^n t_{i,n}(x_m) j_{2n+1}(\nu_{i,k} (x_m - L_i))$$

$$= \frac{\beta_{i,k}}{\nu_{i,k}} \sin (\nu_{i,k} (x_m - L_i)) - \cos (\nu_{i,k} x_m), \quad k = 1, \ldots, K_N.$$

Solving this system of equations we find $g_{i,0}(x_m)$ and consequently $g_{i,0}(x)$ at a sufficiently dense set of points of the interval $(0, L_i)$. Finally, with the aid of (1.7) we compute $q_i(x)$.

Schematically the proposed method for solving the inverse problem on a quantum star graph is presented in the following diagram.

$$M(\rho_k^2), \ k = 1, \ldots, m \xrightarrow{(1)} \{g_{i,n}(L_i), s_{i,n}(L_i)\}_{n=0}^{N} \xrightarrow{(2)} \text{two spectra } \{\mu_{i,k}\}_{k=1}^{K_D}, \ \{\nu_{i,k}\}_{k=1}^{K_N}$$
\[ \{\mu_{i,k}\}_{k=1}^{K} \stackrel{(3)}{\Rightarrow} \{t_{i,n}(0)\}_{n=0}^{N} \]
\[ \{t_{i,n}(0)\}_{n=0}^{N}, \{\nu_{i,k}\}_{k=1}^{K} \stackrel{(4)}{\Rightarrow} \{\beta_{i,k}\}_{k=1}^{K} \]
\[ \{\nu_{i,k}, \beta_{i,k}\}_{k=1}^{K} \stackrel{(5)}{\Rightarrow} g_{i,0}(x) \stackrel{(6)}{\Rightarrow} q_{i}(x). \]

Note that after step (1) the problem is reduced to separate problems on the edges. The two spectra problem arising after step (2) can be solved by different existing methods, nevertheless here we propose a method which uses the fact that the constants \(\{g_{i,n}(L_{i})\}_{n=0}^{N}\) and \(\{s_{i,n}(L_{i})\}_{n=0}^{N}\) are also known.

### 7 Numerical examples

**Example 1.** Let us consider a star graph of nine edges of lengths

\[
L_1 = \frac{e}{2}, \quad L_2 = 1, \quad L_3 = \frac{\pi}{2}, \quad L_4 = \frac{\pi}{3}, \quad L_5 = \frac{e^{2}}{4}, \quad L_6 = 1.1, \quad L_7 = 1.2, \quad L_8 = 1, \quad L_9 = 1.4 \quad (7.1)
\]

The corresponding nine components of the potential are defined as follows

\[ q_1(x) = |x - 1| + 1, \quad q_2(x) = e^{-(x-\frac{1}{2})^2}, \quad q_3(x) = \sin(8x) + \frac{2\pi}{3}, \quad q_4(x) = \cos(9x^2) + 2, \]
\[ q_5(x) = \frac{1}{x + 0.1}, \quad q_6(x) = \frac{1}{(x + 0.1)^2}, \quad q_7(x) = e^{x}, \]
\[ q_8(x) = \begin{cases} 
-35.2x^2 + 17.6x, & 0 \leq x < 0.25 \\
35.2x^2 - 35.2x + 8.8, & 0.25 \leq x < 0.75 \\
-35.2x^2 + 52.8x - 17.6, & 0.75 \leq x \leq 1, 
\end{cases}
\]
\[ q_9(x) = J_0(9x). \]

The potential \(q_8(x)\) (from [11], [26]) is referred to below as saddle potential.

In Fig. 1 we show the exact potentials (continuous line) together with the recovered ones (marked with asterisks), computed with the proposed algorithm. Here two elements from each row of the Weyl matrix (that is, \(M_k = 0\)) \(M_{ii}(\rho_k^2)\) and \(M_{i,i+1}(\rho_k^2)\) (for \(i = M, \ i+1\) was replaced by 1) were given at 190 points \(\rho_k\), and \(N\) was chosen as \(N = 9\) (ten coefficients \(g_{n}(L_{i})\) and ten coefficients \(s_{n}(L_{i})\)).

Here the maximum relative error was attained in the case of the potential \(q_6(x)\), and it resulted in approximately 0.047 at the endpoint \(x = 0\). All other potentials were computed more accurately.

It is interesting to track how accurately the two spectra were computed on each edge. For example, Table 1 presents some of the “exact” Dirichlet-Dirichlet eigenvalues on \(e_6\) computed with the aid of the Matslise package [22] (first column), the approximate eigenvalues, computed as described in Subsection 6.2 by calculating zeros of \(S_{6,0}(\rho, L_6)\) and the absolute error of each presented eigenvalue. Notice that both the absolute and relative errors remain small even for large indices.
Figure 1: The potential of the quantum star graph from Example 1 is recovered from two elements from each row of the Weyl matrix (that is, $M_k = 0$) $M_{ii}(\rho_k^2)$ and $M_{i+1,i}(\rho_k^2)$ (for $i = M$, $i + 1$ was replaced by 1) given at 190 points $\rho_k$ distributed uniformly on the segment $[1 + 0.1i, 100 + 0.1i]$. Here $N = 9$. 

\[ q_1(x) = |x-1|+1 \]
\[ q_2(x) = e^{-(x-1/2)^2} \]
\[ q_3(x) = \sin(8x)+2/3 \]
\[ q_4(x) = \cos(9x^2)+2 \]
\[ q_5(x) = 1/(x+0.1) \]
\[ q_6(x) = 1/(x+0.1)^2 \]
\[ q_7(x) = e^x \]
\[ q_8(x) = e^{-x^2} \]
\[ q_9(x) = j_0(9x) \]
Table 1: Dirichlet-Dirichlet eigenvalues of $q_6(x)$

| $n$ | $\lambda_n$   | $\tilde{\lambda}_n$ | $|\lambda_n - \tilde{\lambda}_n|$ |
|-----|---------------|----------------------|----------------------------------|
| 1   | 11.3620706   | 11.3620710           | 4.0 $\cdot 10^{-7}$              |
| 11  | 994.949643   | 994.949630           | 1.3 $\cdot 10^{-5}$              |
| 51  | 21223.885957 | 21223.885873         | 8.4 $\cdot 10^{-5}$              |
| 101 | 83214.803376 | 83214.803222         | 1.5 $\cdot 10^{-4}$              |

Similar results are obtained for the Neumann-Dirichlet spectrum, as shown in Table 2.

Table 2: Neumann-Dirichlet eigenvalues of $q_6(x)$

| $n$ | $\lambda_n$   | $\tilde{\lambda}_n$ | $|\lambda_n - \tilde{\lambda}_n|$ |
|-----|---------------|----------------------|----------------------------------|
| 1   | 10.21124706  | 10.21124734          | 2.8 $\cdot 10^{-7}$              |
| 11  | 908.123501   | 908.123578           | 7.7 $\cdot 10^{-5}$              |
| 51  | 20809.976547 | 20809.976583         | 3.6 $\cdot 10^{-5}$              |
| 101 | 82393.027033 | 82393.027162         | 1.3 $\cdot 10^{-4}$              |

The number and the distribution of the points $\rho_k$ influence the possibility of an accurate recovery of the potential. So much better results are obtained when a uniform distribution of the points $\rho_k$ is replaced, e.g., by their logarithmically uniform distribution. For example, the same accuracy as reported above was obtained for 90 points $\rho_k$ chosen according to the rule $\rho_k = 10^{\alpha_k} + 0.1i$ with $\alpha_k$ being uniformly distributed on $[0, 2]$. Such choice delivers a set of points which are more densely distributed near $\rho = 1 + 0.1i$ and more sparsely near $\rho = 100 + 0.1i$. A shift of the points $\rho_k$ to a relatively large distance from zero leads to a deterioration of the results. For example, when $\alpha_k$ are chosen in $[1, 2]$ the method with all the same parameters fails, while for $\alpha_k$ chosen in $[0.5, 2]$ it delivers accurate results. This deterioration of the results may happen due to the lack of information near the first eigenvalues of the Dirichlet spectrum of the graph.

Another interesting feature of the method is illustrated by Fig. 2. Namely, when the number of the points $\rho_k$ is small, the use of more equations of the form (6.3), i.e., of more elements of the Weyl matrix may help to improve the accuracy. Here we present the component of the potential $q_8(x)$. The whole potential on the graph was recovered with $N = 7$ from 30 points $\rho_k$ distributed logarithmically uniformly on the same segment as above ($\alpha_k \in [0, 2]$). The best accuracy corresponds to $M_k = 7$, that is when all the elements of the Weyl matrix were used, while the worst result corresponds to $M_k = 0$, that is when only two elements from each row were used.

It is worth noting that the whole computation takes few seconds performed in Matlab 2017 on a Laptop equipped with a Core i7 Intel processor.

The method copes equally well with inverse problems on star graphs with a larger number of edges, though to obtain a similar accuracy in this case more input data (points $\rho_k$) are required.

8 Conclusions

A new method for solving the inverse problem on quantum star graphs consisting in the recovery of the potential from the Weyl matrix is developed. The main role in the proposed approach
Figure 2: The component of the potential $q_k(x)$. The whole potential on the graph was recovered with $N = 7$ from 30 points $p_k$ distributed logarithmically uniformly on the same segment as above. The best accuracy corresponds to $M_k = 7$, that is when all the elements of the Weyl matrix were used, while the worst result corresponds to $M_k = 0$, that is when only two elements from each row were used.

is played by the coefficients of the Neumann series of Bessel functions expansion of solutions of the Sturm-Liouville equation. With their aid the given data lead to separate two-spectra inverse Sturm-Liouville problems on each edge. These two-spectra problems are solved by a direct method reducing each problem to a system of linear algebraic equations, and the crucial observation is that the potential is recovered from the first component of the solution vector.

The method is simple, direct and accurate. Its performance is illustrated by numerical examples. In subsequent works we plan to extend this method to other types of inverse spectral problems and to more general graphs.

**Funding** The research of Sergei Avdonin was supported in part by the National Science Foundation, grant DMS 1909869, and by Moscow Center for Fundamental and Applied Mathematics. The research of Vladislav Kravchenko was supported by CONACYT, Mexico via the project 284470 and partially performed at the Regional mathematical center of the Southern Federal University with the support of the Ministry of Science and Higher Education of Russia, agreement 075-02-2022-893.

**Data availability** The data that support the findings of this study are available upon reasonable request.

**Declarations**

**Conflict of interest** The authors declare no competing interests.

**References**

[1] Abramovitz M. and Stegun I. A. (1972), *Handbook of mathematical functions*, New York: Dover.
[2] Arioli M. and Benzi M. (2018), A finite element method for quantum graphs. IMA J. Numer. Anal., 38, no. 3, 1119–1163.

[3] Avdonin S., Belinskiy B., and Matthews J. (2011), Inverse problem on the semi-axis: local approach, Tamkang Journal of Mathematics, 42, no. 3, 1–19.

[4] Avdonin S. and Bell J. (2015), Determining physical parameters for a neuronal cable model defined on a tree graph, Journal of Inverse Problems and Imaging, 9, no. 3, 645-659.

[5] Avdonin S. and Kravchenko V. V. (2022) Method for solving inverse spectral problems on quantum star graphs. arXiv:2210.12500.

[6] Avdonin S. and Kurasov P. (2008), Inverse problems for quantum trees, Inverse Problems and Imaging, 2, no. 1, 1–21.

[7] Avdonin S., Leugering G., and Mikhaylov V. (2010), On an inverse problem for tree-like networks of elastic strings, Zeit. Angew. Math. Mech., 90, no. 2, 136–150.

[8] Baricz A., Jankov D., Pogány T. K. (2017) Series of Bessel and Kummer-type functions. Lecture Notes in Mathematics, 2207. Springer, Cham.

[9] Belishev M. and Vakulenko A. (2006), Inverse problems on graphs: Recovering the tree of strings by the BC-method, J. Inv. Ill-Posed Problems, 14, 29-46.

[10] Berkolaiko G. and Kuchment P. (2013), Introduction to Quantum Graphs, AMS, Providence, R.I.

[11] Brown B. M., Sanko V. S., Knowles I. W., Marletta M. (2003), Inverse spectral problem for the Sturm–Liouville equation, Inverse Probl. 19, 235–252.

[12] Chadan Kh., Colton D., Päivärinta L., Rundell W. (1997), An introduction to inverse scattering and inverse spectral problems. SIAM, Philadelphia.

[13] Kravchenko V. V. (2019), On a method for solving the inverse Sturm–Liouville problem, J. Inverse Ill-posed Probl. 27, 401–407.

[14] Kravchenko V. V. (2020), Direct and inverse Sturm-Liouville problems: A method of solution, Birkhäuser, Cham.

[15] Kravchenko V. V. (2022), Spectrum completion and inverse Sturm-Liouville problems. arXiv:2210.13460.

[16] Kravchenko V. V., Navarro L. J. and Torba S. M. (2017), Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions, Appl. Math. Comput. 314, 173–192.

[17] Kravchenko V. V. and Torba S. M. (2015), Analytic approximation of transmutation operators and applications to highly accurate solution of spectral problems, Journal of Computational and Applied Mathematics 275, 1-26.
[18] Kravchenko V. V. and Torba S. M. (2021), A direct method for solving inverse Sturm-Liouville problems, *Inverse Probl.* **37**, 015015 (32pp).

[19] Kravchenko V. V. and Torba S. M. (2021), A practical method for recovering Sturm-Liouville problems from the Weyl function, *Inverse Probl.* **37**, 065011 (26pp).

[20] Kurasov P. (2022) Quantum Graphs: Spectral Theory and Inverse Problems, Springer (to appear).

[21] Kurasov P. and Nowaczyk M. (2005), Inverse spectral problem for quantum graphs, *J. Phys. A.*, **38**, 4901-4915.

[22] Ledoux V., Daele M.V., Berghe G.V. (2005) MATSLISE: a MATLAB package for the numerical solution of Sturm–Liouville and Schrödinger equations, *ACM Trans. Math. Softw.* **31**, 532–554.

[23] Levitan B. M. (1987), *Inverse Sturm-Liouville problems*, VSP, Zeist.

[24] Marchenko V. A. (2011), *Sturm-Liouville operators and applications: revised edition*, AMS Chelsea Publishing.

[25] Mugnolo, D. (2014), *Semigroup Methods for Evolution Equations on Networks*, Understanding Complex Systems, Springer, Cham.

[26] Rundell W. and Sacks P. E. (1992) Reconstruction techniques for classical inverse Sturm–Liouville problems, *Math. Comput.* **58**, 161–183.

[27] Shishkina E. L. and Sitnik S. M. (2020), *Transmutations, singular and fractional differential equations with applications to mathematical physics*, Elsevier, Amsterdam.

[28] Savchuk A. M., Shkalikov A. A. (2005), Inverse problem for Sturm–Liouville operators with distribution potentials: reconstruction from two spectra, *Russ. J. Math. Phys.* **12**, 507–514.

[29] Watson G. N. (1996), *A Treatise on the theory of Bessel functions, 2nd ed., reprinted*, Cambridge University Press, Cambridge.

[30] Wilkins J. E. (1948) Neumann series of Bessel functions. *Trans. Amer. Math. Soc.* **64**, 359–385.

[31] Yurko V. A. (2005), Inverse Sturm-Liouville operator on graphs, *Inverse Problems*, **21**, 1075-1086.

[32] Yurko V. A. (2007), *Introduction to the theory of inverse spectral problems*, Fizmatlit, Moscow, (in Russian).