Dynamics for Galactic Archaeology

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Abstract

Our Galaxy is a complex machine in which several processes operate simultaneously: metal-poor gas is accreted, is chemically enriched by dying stars, and then drifts inwards, surrendering its angular momentum to stars; new stars are formed on nearly circular orbits in the equatorial plane and then diffuse through orbit space to eccentric and inclined orbits; the central stellar bar surrenders angular momentum to the surrounding disc and dark halo while acquiring angular momentum from inspiralling gas; the outer parts of the disc are constantly disturbed by satellite objects, both luminous and dark, as they sweep through pericentre. We review the conceptual tools required to bring these complex happenings into focus. Our first concern must be the construction of equilibrium models of the Galaxy, for upon these hang our hopes of determining the Galaxy’s mean gravitational field, which is required for every subsequent step. Ideally our equilibrium model should be formulated so that the secular evolution of the system can be modelled with perturbation theory. Such theory can be used to understand how stars diffuse through orbit space from either the thin gas disc in which we presume disc stars formed, or the debris of an accreted object, the presumed origin of many halo stars. Coupling this understanding to the still very uncertain predictions of the theory of stellar evolution and nucleosynthesis, we can finally extract a complete model of the chemodynamic evolution of our reasonably generic Galaxy. We discuss the relation of such a model to cosmological simulations of galaxy formation, which provide general guidance but cannot be relied on for quantitative detail.

Keywords:
1. Introduction

In this article I will focus on the aspects of Galactic dynamics which seem most relevant to Galactic archaeology. The latter is the endeavour to unravel the Galaxy’s history by measuring the positions, velocities, chemical compositions and if possible the ages of stars. From these data we can in principle determine the orbits of stars, so a major focus must be on how we should characterise orbits and what we can infer about a star’s history from its orbit. The latter raises the questions “how long has the star been on this orbit” and “what was its original Galactic orbit?”. It seems likely that most, perhaps all, disc stars were formed on nearly circular orbits in the plane. By contrast, many, perhaps most, halo stars were released onto eccentric and inclined orbits on the tidal disruption by the Galaxy of the system in which they were born. Therefore in addition to discussing how we can determine the current orbits of stars, we must be able to compute the probability that during the lifetime of a star that orbit can have evolved from either an in-plane circular orbit or a particular eccentric inclined orbit.

For a variety of reasons, mostly unconnected with our Galaxy, we believe that dark, unobserved matter plays a major role in generating the gravitational field in which stars move. A major issue is the extent to which we can use measurements of stars to constrain the spatial distribution of dark matter. Another important question is the extent to which the distribution of dark matter has been modified by gravitational interaction with gas and stars.

The central assumption of Galactic dynamics is that the galaxy is statistically in a steady state. In reality this assumption is false because the central rotating bar, spiral structure in the disc and infalling objects all cause the distribution of matter to evolve and thus the gravitational field to be time-dependent. However, our strategy must be to start from the assumption of a steady state and then to use perturbation theory to understand how time-dependent effects modify a steady-state model.

This strategy is possible because the gravitational force is a long-range one, so the force on a star is generally dominated by countless very distant stars rather than by its near neighbours\footnote{The force on a binary star is dominated by its companion, but in this case we consider the motion of the centre of mass of the binary rather than that of either component. Also during a rare close encounter of field stars the forces on each star can be dominated by}. Consequently, we can model the
Galaxy’s gravitational field by the field that would be generated by a system with the smooth density distribution $\rho(x)$ that would result from smearing the masses of stars and dark matter smoothly through interstellar space. When it is possible to model the gravitational field in this way, we say the system is collisionless.

2. Orbits

The simplest useful models of the Galaxy’s gravitational field are axisymmetric, so it is important to understand the nature of orbits in axisymmetric gravitational potentials $\Phi(R, z)$. Since the component of angular momentum $L_z$ about the potential’s symmetry axis $e_z$ is conserved, motion in an axisymmetric potential can be reduced to motion in the $(R, z)$ plane. This is governed by the Hamiltonian

$$H = \frac{1}{2}(p_R^2 + p_z^2) + \Phi_{\text{eff}}(R, z)$$

where $p_R = \dot{R}$, $p_z = \dot{z}$ and the effective potential is

$$\Phi_{\text{eff}}(R, z) = \Phi(R, z) + \frac{L_z^2}{2R^2}.$$  

Numerical integration of orbits in typical effective potentials reveals that the time series $R(t)$ and $z(t)$ are quasiperiodic, that is their Fourier transforms

$$\tilde{R}(\omega), \quad \tilde{z}(\omega)$$

contain only discrete frequencies $\omega_i$. Moreover, these frequencies can be expressed as integer linear combinations of two fundamental frequencies. In the generic non-resonant case, the fundamental frequencies can be taken to be the radial frequency $\Omega_r$, at which the star oscillates in and out, and the vertical frequency $\Omega_z$ at which it oscillated perpendicular to the potential’s symmetry plane. Thus in the generic case all the frequencies $\omega_i$ that occur in the Fourier transforms of $R(t)$ and $z(t)$ can be written

$$\omega_i = n_r(i)\Omega_r + n_z(i)\Omega_z,$$  

that from its nearest neighbour, but such encounters are so short-lived that long-term influence of the other star is small.
where $n_r(i)$ and $n_z(i)$ are both (possibly negative) integers.

When orbits in a Hamiltonian with $n$ degrees of freedom are quasi-periodic, one may show (Arnold, 1978) that there are $n$ integrals of motion – smooth functions on phase space $I_j(x, p)$ that are constant along each orbit: $dI_j/dt = 0$. Obviously one of these constants of motion can be taken to be the Hamiltonian $H(x, p)$. However, any function of integrals is itself an integral, so once we have more than one integral we have infinitely many integrals to choose from. It proves expedient to choose integrals that can be complemented by canonically conjugate variables, for if we use such coordinates only half a star’s coordinates evolve as the star moves because the momenta are constants of motion. Integrals that can be complemented with conjugate coordinates are called action integrals $J$ and the conjugate variables are called angle variables $\theta$. From Hamilton’s equations

$$\dot{J} = \frac{\partial H}{\partial \theta} = 0 \ ; \ \dot{\theta} = \frac{\partial H}{\partial J} = \Omega(J) \ (a \ constant),$$

it follows that the angle variables increase linearly in time, the fundamental frequencies playing the role of constants of proportionality. Thus, for example

$$\theta_r(t) = \theta_r(0) + \Omega_r t \ ; \ \theta_z(t) = \theta_z(0) + \Omega_z t.$$

Once we require integrals to be associated with conjugate variables and position in phase space to be $2\pi$-periodic in the $\theta_i$, our choice of integrals narrows dramatically. In fact, we are then confined to the action integrals $J_r$ and $J_z$ that quantify the extent of the star’s radial and vertical oscillations, and integer linear combinations of these basic actions. In practice the near uniqueness of action integrals is a significant advantage.

It turns out that the two-dimensional surfaces $J_r = \text{constant}, \ J_z = \text{constant}$ in the four-dimensional phase space $(R, p_R, z, p_z)$ can be mapped 1:1 and differentiably onto a 2-torus (doughnut); that is, a surface of constant $J_r, J_z$ has the topology of a 2-torus. When (as is generically the case) the fundamental frequencies are incommensurable (so $\Omega_R/\Omega_z \neq n_R/n_z$ for any integers $n_R, n_z$), the star eventually comes arbitrarily close to every point on the torus to which it is confined by its integrals $J_r, J_z$. Consequently, the orbit is essentially identical with the torus, and we identify quasi-periodic orbits with orbital tori.

Angle-action variables were introduced by Hamilton and Jacobi in the first half of the 19th century as tools for the study of the dynamics of the
solar system. The underlying Hamiltonian of that problem is the Kepler Hamiltonian, which governs the dynamics of a particle that moves in the gravitational field of a point mass. For that Hamiltonian it is possible to express ordinary polar coordinates \((r, p_r, \theta, p_\theta, \phi, p_\phi)\) as analytic functions of \(J\) and \(\theta\). Unfortunately, such formulae exist for very few Hamiltonians – essentially the \(n\)-dimensional harmonic oscillator and the spherical isochrone Hamiltonian, of which the Kepler Hamiltonian is a special case. None of these Hamiltonians is very close to that of a flattened Galaxy, so we must manage without explicit formulae for angle-action variables. Historically, this lack of formulae has discouraged people from using angle-action coordinates but, as I will detail below, there are now some effective work-arounds.

### 2.1. Integrable systems and resonances

A Hamiltonian that admits a global set of angle-action variables is said to be integrable. The Kepler Hamiltonian is integrable, as are the Hamiltonians associated with the gravitational potentials of certain flattened or triaxial massive ellipsoidal bodies - de Zeeuw (1985) discusses these Stäckel potentials in some detail. The Hamiltonians of real galaxies are almost certainly not integrable, but they are best understood as perturbations of integrable Hamiltonians, just as we consider the Hamiltonian of the solar system to be the Kepler Hamiltonian defined by the Sun perturbed by the gravitational fields of Jupiter, Saturn, etc., as well as relativistic corrections to Newtonian mechanics.

The impact on an orbit of adding a perturbation may not be a smooth function of the magnitude of the perturbation. This fact is of fundamental importance for perturbation theory because it implies that the usual procedure for doing perturbation theory – expanding all variables as power series in the “coupling constant” that specifies the strength of the perturbation – will in general fail because the perturbed motion is in fact not analytic in the coupling constant.

The reason the response to a perturbation can depend discontinuously on the magnitude of the perturbation is the quasi-periodic nature of the underlying motion: we should think of the unperturbed motion as oscillatory at the frequencies \(\omega_i\) that occur in the Fourier decompositions of the coordinates. If the perturbation is modulated at a frequency \(\omega\) that lies close to any of the \(\omega_i\), the frequency \(\omega - \omega_i\) at which the orbiting particle perceives the perturbation can become very small. Consequently, the particle can be pulled in the same sense for a long time \(\sim \pi/|\omega - \omega_i|\) and even a small perturbation
can have a large cumulative effect before it changes sign. In particular, for a critical magnitude of the perturbation the motion can become *resonantly trapped* by the perturbation, at which point the perturbation changes the motion qualitatively and not merely quantitatively. The classic paradigm for such trapping is the pendulum, which at a given energy will oscillate if gravity is strong, but circulate if gravity is sufficiently weak. In §8 we explain the mathematical theory of resonant trapping. If no resonant trapping occurs, the perturbation has simply transformed one integrable Hamiltonian into another. Once resonant trapping has occurred, the Hamiltonian is no longer integrable.

We can discover how important resonant trapping is for a given Hamiltonian, and thus how close to integrable that Hamiltonian is, by numerically integrating numbers of orbits and, if the motion is two-dimensional, inspecting a *surface of section*, or if the motion is three-dimensional, examining a *frequency map* — these tools are described in §§3.2.2 and 3.7.3(b) of Binney & Tremaine (2008). Surfaces of section reveal that realistic, axisymmetric models of the Galaxy’s potential are close to integrable. Surfaces of section for motion in planar rotating bars, or frequency maps for motion in realistic models of the Galaxy’s bar reveal that in the vicinity of the bar’s *corotation resonance* (where the circular frequency in the axisymmetrised Galaxy model coincides with the bar’s pattern speed $\Omega_p$) the Hamiltonian is far from integrable.

An integrable Hamiltonian admits a global set of angle-action coordinates. Orbits that have been trapped by a particular resonance form an orbit family that requires its own system of angle-action coordinates. Techniques for constructing such coordinates were discussed by Kaasalainen (1995a).

### 2.2. Chaos

When a single orbit in an integrable Hamiltonian is exposed simultaneously to significant perturbations at more than one nearly resonant frequency (“resonance overlap”), irregular behaviour can ensue (Chirikov, 1979). The standard interpretation of this *chaotic* behaviour is that the orbit becomes trapped first by one resonance and then by another, with the times of transition between resonantly trapped families varying erratically. In classic cases one can show that numerically integrated orbits do shift from one orbit family to another in an erratic way. Chaos is an important phenomenon for Galaxy dynamics in the region just inside the corotation resonance. By contrast chaos is unimportant in the immediate vicinity of the Sun.
2.3. What integrable Hamiltonian?

The account of orbits we have given hinges crucially on the concepts of resonance and perturbation amplitude. These concepts presuppose the existence of an integrable Hamiltonian $H_0$, which endows orbits with fundamental frequencies and defines the perturbing Hamiltonian as the difference $\delta H = H - H_0$ between the actual and integrable Hamiltonians.

In studies of planetary systems the Kepler Hamiltonian clearly stands out as $H_0$, and in plasma and accelerator physics the harmonic-oscillator Hamiltonian can plausibly serve as $H_0$. But for decades the integrable Hamiltonian remained a shadowy, unidentified feature of Galactic dynamics. A spherical Hamiltonian has been used (Saha, 1991; Weinberg, 1991), but this choice is highly unsatisfactory for the Galaxy because it implies that $\Omega_z$ is identically equal to the azimuthal frequency $\Omega_\phi$, whereas the majority of Galactic orbits $\Omega_z > \Omega_\phi$, and this inequality qualitatively changes the pattern of resonances. In studies of the solar neighborhood the harmonic oscillator has been used for $H_0$, but this choice is unsatisfactory because it requires $\Omega_R$ and $\Omega_z$ to be amplitude-independent whereas in the solar neighborhood $\Omega_z$ declines rapidly with amplitude.

A significant step forward was taken when de Zeeuw (1985) showed that Stäckel potentials could be generated by objects with mass distributions that are similar to those of real galaxies, so Stäckel Hamiltonians can be used for $H_0$. One drawback of Stäckel potentials is that they enforce a relationship between the radial density profile and the ellipticity of the isodensity surfaces of the generating body: the latter can be self-similar only if asymptotically the density declines as $\rho \sim r^{-4}$; in real galaxies we have $\rho \sim r^{-\alpha}$ with $3 < \alpha < 2$, and in this case the isodensity surfaces of a body that generates a Stäckel potential have to become steadily more spherical as $r \to \infty$. Another drawback is that for a Stäckel Hamiltonian the transformations between angle-action and spherical coordinates involve the numerical evaluation of one-dimensional integrals, so Stäckel Hamiltonians are not very convenient from a technical perspective.

Kaasalainen & Binney (1994b) introduced a general technique for constructing an integrable Hamiltonian that is close to any given galactic Hamiltonian. Kaasalainen (1994) showed that on account of the closeness of the constructed Hamiltonian to the given Hamiltonian, beautiful results for resonant orbits and chaotic regions could be obtained from (suitably modified) first-order perturbation theory.
Before explaining how integrable Hamiltonians can be constructed, we recall some facts about coordinate transformations. The transformation \((x, p) \leftrightarrow (\theta, J)\) between ordinary phase-space coordinates and angle-action coordinates is a \textbf{canonical transformation}. Every canonical transformation can be obtained from a generating function, and in our case we may choose to use a generating function of the form \(S(J, x)\). It then generates the transformation through
\[
\begin{align*}
  p &= \frac{\partial S}{\partial x} ; \\
  \theta &= \frac{\partial S}{\partial J}.
\end{align*}
\]
(7)

The classical way to find angle-action coordinates is to use the first of these equations to eliminate \(p\) from \(H(x, p) = E\) and to solve the resulting \textbf{Hamilton-Jacobi} equation as a non-linear partial differential equation for \(S\).

The Kaasalainen–Binney scheme for constructing an integrable Hamiltonian starts from angle-action coordinates \((\theta', J')\) obtained in this way for some \textbf{toy Hamiltonian} \(H_t\), which may be the isochrone Hamiltonian, or the harmonic oscillator Hamiltonian or a Stäckel Hamiltonian. Then a second generating function \(S(J, \theta')\) is numerically constructed that maps the tori \(J' = \text{constant of } H_t\) into the tori \(J = \text{constant of } H\), which is the actual or \textbf{target Hamiltonian}. The numerically-constructed generating function has the form
\[
S(J, \theta') = \theta' \cdot J + \sum_n s_n(J) e^{i n \cdot \theta'}.
\]
(8)

Here \(n\) is a vector with integer components and the \(s_n\), being functions of the orbit’s constant action \(J\), are numbers characteristic of the orbit. The construction consists in solving for these numbers. The true angle variables are
\[
\theta = \frac{\partial S}{\partial J} = \theta' + \sum_n \frac{\partial s_n}{\partial J} e^{i n \cdot \theta'}.
\]
(9)

Therefore we also have to solve for the derivatives of \(s_n\), which we do in a separate step (Binney & Kumar, 1993; Kaasalainen & Binney, 1994).

\[\text{A coordinate system } (Q, P) \text{ is canonical iff the Poisson bracket of two functions on phase space } f, g \text{ takes the form}
\]
\[
[f, g] = \sum_i \frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i} \frac{\partial g}{\partial Q_i}.
\]

A canonical transformation is simply one between canonical coordinate systems.
For any values of the \( s_n \) the mapping

\[
(\theta, J) \rightarrow (\theta', J') \rightarrow (x, p)
\]  

(10)

maps the torus \( J = \text{constant} \) into ordinary phase space, and a sufficient condition for determining the \( s_n \) is to require that the actual Hamiltonian \( H(x, p) \) is constant on this torus. In practice the Marquardt-Levenberg algorithm \([\text{Press et al.}, 2002]\) is used to minimise the variance of \( H \) over the image torus. It is not possible with a finite set of \( s_n \) to achieve complete constancy of the Hamiltonian, but one can come close to this goal.

By carrying out this fitting procedure at each point on a grid of actions \( J \), we foliate phase space with tori on which \( H \) is nearly constant. In principle further tori can be constructed by interpolating between the values \( s_n(J) \) taken by the \( s_n \) at the grid points. Thus we can completely fill phase space with tori on which \( H \) is approximately constant. The mean value of \( H \) on each of these tori defines an integrable Hamiltonian \( \overline{H}(J) \) and the difference \( \delta H(\theta, J) = H(\theta, J) - \overline{H}(J) \) is a well defined perturbation.

2.4. What technology to use?

Enough of general theory – how should we proceed in practice to interpret observational data in terms of angle-action variables? The first step is to decide whether one wants to proceed from \((x, p)\) to \((\theta, J)\) or in the opposite direction. Numerically constructed tori provide the natural way to go from \((\theta, J)\) to \((x, p)\) because the output from the torus machine is analytic expressions for \((x, p)\) given \((\theta, J)\). For example, a mock catalogue can be created from a model Galaxy by choosing actions \( J \) that sample the distribution function \( f(J) \) (discussed below) and then choosing the \( \theta_i \) uniformly within \((0, 2\pi)\). The numerically constructed torus then yields \((x, p)\) and a star drawn from the population described by the DF can be placed there. If you want to determine the velocity distribution at some point \( x \) in the model, for each sampled \( J \) you can determine the values of \( \theta \) (there are generally four or six) at which the star will pass through \( x \) and then compute the velocity at which each passage will be made.

Sometimes one will want to pass from \((x, p)\) to \((\theta, J)\). This can be done with numerically constructed tori (e.g. \textit{McMillan & Binney, 2008}) but doing so can be slow because it involves an iterative search for the torus that passes through the given \( x \) with the given \( p \). For disc stars \textit{Binney} \textit{(2010)} used the “adiabatic approximation” to estimate \( J \) from \((x, p)\) but this approach has
now been superseded by a faster and more accurate approximation based on Stäckel potentials \cite{Binney2012}. A related alternative is offered by \cite{Sanders2012}, who explicitly fits a Stäckel potential to the Galactic potential in the region covered by the given orbit.

These studies are restricted to axisymmetric models, which probably provide a good basis for understanding the extended solar neighbourhood and outer Galaxy, but will not deal with the Galaxy’s inner few kiloparsecs, which are dominated by the bar. Tori have been numerically fitted to non-rotating \cite{Kaasalainen1994} and rotating \cite{Kaasalainen1995} planar bars, but the extension to three-dimensional rotating bars has yet to be made. In principle this extension is straightforward.

### 3. Equilibrium models

Models in which the Galaxy is in a perfectly steady state play a fundamental role, not least because a good model of the Galaxy’s potential $\Phi(x)$ is fundamental to any investigation, and since $\Phi$ is substantially generated by dark matter that we cannot directly observe, we can determine $\Phi$ only to the extent that it is possible to argue that the phase-space distribution of stars is statistically in equilibrium.

Let $f(x, p) \, d^3x \, d^3p$ be the probability of finding a star of some particular type (specified by luminosity, colour, metalliticity, etc) in the phase-space volume $d^3x \, d^3p$ at $(x, p)$. Then $f$ is called the distribution function (DF) of that population of stars. By Jeans’ theorem \cite[§4.2]{Binney2008} a steady-state DF can depend only on the actions, so $f = f(J)$. Moreover, the Jacobian $\partial(x, p) / \partial(\theta, J)$ is unity and the angle variables all cover the range $(0, 2\pi)$ so the probability that a star of the given population lies in the volume $d^3J$ of action space around $J$ is

$$dP(J) = \left( \int_0^{2\pi} d^3\theta \right) d^3J \, f(J) = (2\pi)^3 d^3J \, f(J).$$

Hence, $(2\pi)^3 f(J)$ is the probability density of stars of the given population in three-dimensional action space. In other words, a steady-state model consists of a particular distribution of stars of each type in three-dimensional action space. It is in these terms that we should think of Galaxy models.

Conversely a steady-state model can be constructed by prescribing distributions of stars in action space and then calculating the spatial distribution of the stars and the kinematics that they have at every point. Crucially,
the spatial and kinematic properties of a population are firmly tied to one another by the potential $\Phi$ (which determines the structure of the orbital tori).

### 3.1. Disc DFs

Binney (2010, 2012b) showed that data for the solar neighbourhood can be largely accounted for by models synthesised from quasi-isothermal DFs. Specifically, we write

$$f(J_r, J_z, L_z) = f_{\sigma_r}(J_r, L_z)f_{\sigma_z}(J_z, L_z),$$

where $f_{\sigma_r}$ and $f_{\sigma_z}$ are defined to be

$$f_{\sigma_r}(J_r, L_z) \equiv \frac{\Omega \Sigma}{2\pi M \sigma^2 \kappa} [1 + \tanh(L_z/L_0)]e^{-\kappa J_r/\sigma^2_r},$$

and

$$f_{\sigma_z}(J_z, L_z) \equiv \frac{\nu}{2\pi \sigma^2_z} e^{-\nu J_z/\sigma^2_z}.$$ (14)

Here $\Omega(L_z)$, $\kappa(L_z)$ and $\nu(L_z)$ are, respectively, the circular, radial and vertical epicycle frequencies of the circular orbit with angular momentum $L_z$ and radius $R_c(L_z)$, while

$$\Sigma(L_z) = \Sigma_0 e^{-R_c/R_d}$$

governs the approximately exponential surface density of the disc and the disc’s mass is

$$M = 2\pi \Sigma_0 R^2_d.$$ (16)

The functions $\sigma_r(L_z)$ and $\sigma_z(L_z)$ control the radial and vertical velocity dispersions in the disc and are approximately equal to them at $R_c$. Given that the scale heights of galactic discs do not vary strongly with radius (van der Kruit & Searle, 1981), these quantities must increase inwards. The natural Ansatz to achieve this is

$$\sigma_r(L_z) = \sigma_{r0} e^{(R_0-R_c)/R_{\sigma r}}; \quad \sigma_z(L_z) = \sigma_{z0} e^{(R_0-R_c)/R_{\sigma z}},$$

which imply that the velocity dispersions decline outwards exponentially, with scale length $R_{\sigma i}$. Our expectation is that $R_{\sigma i} \sim 2R_d$ and in the simplest models one assumes that $R_{\sigma r} = R_{\sigma z}$.
The functions \( f_{\sigma_i} \) satisfy the normalisation conditions

\[
\int_0^\infty dJ_r f_{\sigma_r} = \frac{\Omega \Sigma}{2\pi M \kappa^2} [1 + \tanh(L_z/L_0)] ;
\int_0^\infty dJ_z f_{\sigma_z} = \frac{1}{2\pi}.
\]

Consequently, the number of stars per unit angular momentum

\[
g(L_z) \equiv \int dJ_r \int dJ_z f(J_r, J_z, L_z),
\]

decreases as \( \sim \Sigma(L_z)/\kappa(L_z) \), so roughly exponentially. Fig. 1 of Binney (2012b) shows that in a realistic potential \( \Phi(R, z) \) a quasi-isothermal DF with \( R_d/R_\sigma = 0.45 \) generates a disc in which, to a good approximation, the density declines exponentially with both \( R \) and \( |z| \) and the vertical velocity dispersion \( \sigma_z \) is essentially independent of \( z \) for \( |z| \lesssim 600 \) pc and rises very slowly at greater distances from the plane. Hence a quasi-isothermal DF provides a rigorous mathematical realisation of the isothermal mono-abundance components into which Bovy et al. (2012a) argue the disc should be decomposed as an alternative to the traditional divide into thin and thick discs.

Binney (2010, 2012b) represented the thick disc with a quasi-isothermal and the thin disc as a superposition of quasi-isothermals, one for the stars of each age \( \tau \). From the analysis of Hipparcos stars by Aumer & Binney (2009) he took the age dependence of the velocity-dispersion parameter of a coeval cohort’s quasi-isothermal, and then had

\[
\sigma_r(L_z, \tau) = \sigma_{r0} \left( \frac{\tau + \tau_1}{\tau_0 + \tau_1} \right)^\beta \exp\left(\frac{R_0 - R_c}{R_\sigma}\right),
\]

\[
\sigma_z(L_z, \tau) = \sigma_{z0} \left( \frac{\tau + \tau_1}{\tau_0 + \tau_1} \right)^\beta \exp\left(\frac{R_0 - R_c}{R_\sigma}\right).
\]

Here \( \sigma_{z0} \) is the approximate vertical velocity dispersion of local stars at age \( \tau_m \approx 10 \) Gyr, \( \tau_1 \approx 10 \) Myr sets velocity dispersion at birth, and \( \beta \approx 0.33 \) is an index that determines how the velocity dispersions grow with age. He further assumed that the star-formation rate in the thin disc has decreased exponentially with time, with characteristic timescale \( t_0 \), so the thin-disc DF is

\[
f_{\text{thn}}(J_r, J_z, L_z) = \int_0^{\tau_m} d\tau \frac{e^{\tau/t_0} f_{\sigma_r}(J_r, L_z) f_{\sigma_z}(J_z, L_z)}{t_0(e^{\tau/m} - 1)}.
\]

where \( \sigma_r \) and \( \sigma_z \) depend on \( L_z \) and \( \tau \) through equation (20). The normalising constant \( \Sigma_0 \) that appears in equation (15) he took to be the same for both
discs and used for the complete DF

\[ f(J_r, J_z, L_z) = (1 - \lambda)f_{\text{thn}}(J_r, J_z, L_z) + \lambda f_{\text{thk}}(J_r, J_z, L_z), \]

\[(22)\]

where \(\lambda\) is a parameter that controls the fraction of stars that belong to the thick disc.

In \cite{Binney2012b} the values of the three parameters \(\tau_1, \tau_m\) and \(\beta\) in the thin-disc DF were set from \cite{Aumer2009} and the remaining nine parameters in the overall DF, \(\sigma_{r0}, \sigma_{z0}, R_d\) and \(R_\sigma\) for each disc plus the thick-disc weighting \(\lambda\), were fitted to subsets of data for the solar cylinder using a plausible potential, generated by stellar and gas discs, a bulge and a dark halo that begins to dominate the radial force just inside \(R_0\). The principal results from this fitting exercise were:

- When the thick disc is omitted (\(\lambda = 0\)) and the thin-disc DF is fitted to the \(U, V\) and \(W\) distributions of stars in the Geneva-Copenhagen Survey \cite{Nordstrom2004, Holmberg2007} hereafter GCS, the density profile \(\rho(z)\) of the thin disc is in good agreement with that derived for \(|z| \lesssim 500\) pc from star counts. This finding provides support for the adopted potential.

- The \(U\) and \(V\) distributions are fitted by the thin-disc DF as well as can be expected given the presence in the solar neighbourhood of star streams that clearly lie beyond the scope of an axisymmetric equilibrium model. The wings of the observed \(W\) distribution are underpopulated by the DF.

- When a thick-disc component is added and used to improve the fits to the GCS velocity distributions while at the same time fitting the density profile \(\rho(z)\) of \cite{Gilmore1983}, the velocity and density constraints can be fitted simultaneously to good precision. Moreover, the resulting model does an excellent job predicting a preliminary estimate of \(\sigma_z(z)\) extracted from the RAdial Velocity Experiment \cite{Steinmetz2006} hereafter RAVE) by \cite{Burnett2010}. The model also does a reasonable job of predicting the distribution of \(V\) components at \(|z| \approx 1\) kpc published by \cite{Ivezic2008}.

- Since the wings of the GCS distributions of \(U\) and \(V\) had already been filled out by thin disc, the fitted value of the thick disc’s \(\sigma_{r0}\) parameter
(26 km s$^{-1}$) was much smaller than the fitted value of $\sigma_{z0}$ (45 km s$^{-1}$). That is, the thick disc that led to successful predictions was radially cooler than the thin disc (which had $\sigma_{r0} = 40$ km s$^{-1}$). To find a radially hot thick disc, the thin disc DF was set to values that did not fill out the wings of GCS $U, V$ distributions and then the thick disc was fitted as before. This resulted in $\sigma_{r0}$ being 30 km s$^{-1}$ for the thin disc and 40 km s$^{-1}$ for the thick disc. This model predicted values of $\sigma_{\phi}$ and $\langle v \rangle_{\phi}$ at $|z| \approx 1$ kpc that were, respectively, too large and too small. That is, a radially warm thick disc conflicts with the data by requiring too little rotation and too much random motion at large $|z|$.

Dividing the disc into components such as “thick” and “thin” only makes sense in so far as one can track the phase-space distributions of intrinsically distinguishable stars (i.e., stars that may be distinguished by [Fe/H] or [$\alpha$/Fe], or age – see Binney & Merrifield, 1998, §10.4.3). Since Binney (2012b) did not use data that distinguished stars by metallicity or age, the principal interest of his exercise is methodological – it shows how one can model data in a dynamically consistent way and derive testable predictions. Real progress will be made when the models are extended to include different DFs for several distinguishable groups of stars, such as stars younger than 1 Gyr, or with [Fe/H] in different ranges, etc. Setting up such models is straightforward; the complex step is fitting them to the data, for more sophisticated data are associated with strong selection effects.

### 3.2. DFs for the halos

In principle we should have DFs for both the stellar halo and the dark-matter halo. The only dynamically consistent halo DFs in the literature are functions $f(E, L_z)$ of energy and possibly azimuthal angular momentum. Such a DF implies that $\sigma_R = \sigma_z$ everywhere, which certainly does not hold for some classical denizens of the stellar halo, for example RR-Lyrae stars (Delhaye, 1965). A partial solution to this problem is to pretend the Galaxy’s potential is spherical, so the total angular momentum $L$ can be added to the list of isolating integrals (e.g. Deason et al. 2011; Gomez et al. 2010). Deason et al. (2011) have modelled the kinematics of blue horizontal-branch (BHB) stars using the DF

$$f = [1 + \chi \tanh(L_z/L_0)]L^{-2/3}E^{\alpha},$$

(23)
where $\chi$, $L_0$, $\beta$ and $s$ are all parameters. While a DF of this form permits the halo to display both systematic rotation and either radial or tangential bias in the velocity-dispersion tensor, it is implausible in key respects. The power-law dependence on $L$ causes the phase-space density to diverge as $L \to 0$ in the radially biased case ($\beta > 0$) and gives rise to a bimodal distribution of tangential velocities in the case of tangential bias. On account of these pathologies the DF cannot be expected to provide a good fit to observed velocity distributions, and when a model does not provide a good fit, the physical significance of the best-fitting parameters is doubtful.

Although useful qualitative insights into the halo’s dynamics can be obtained by approximating the Galaxy’s potential as spherical (e.g. Binney & Petrou, 1985), when fitting real data we should recognise that the Galaxy’s potential is not spherical, and indeed that there is much interest in determining its shape, and thus diagnosing the shape of the dark halo. In a non-spherical potential at least one of the integrals that appears in the DF has to be non-classical, and may as well be taken to be $J_z$. If the potential is non-axisymmetric, we must use $J_\phi$ in place of its limiting form for an axisymmetric system, $L_z$. Consequently, the only real choice we need to make is whether to use as the third integral $E$ or $J_r$. Using $J_r$ rather than $E$ has the following advantages

- In any potential the range of $J_r$ for bound orbits is $(0, \infty)$ whereas the range of $E$ is $(\Phi(0), 0)$ and thus depends on the depth of the adopted potential. Consequently it is impossible to make general statements about the physical implications a given DF $f(E, \ldots)$ in the way that one can about a DF $f(J)$.

- Whereas $f(J)$ is the density of stars in action space, $f(E, \ldots)$ is not the density of stars in $(E, \ldots)$ space because $d^3x d^2p = d^3\theta dJ_\phi dJ_z dE/\Omega_r$.

- Ultimately the Galaxy’s potential must be made consistent with the total density

$$\rho(x) = \sum_{\text{pops}} M_i \int d^3p f_i, \quad (24)$$

where $M_i$ is the mass of the $i$th population and the populations summed over include the dark matter. This requirement for self-consistency is easy to satisfy when we adopt $f_i(J)$: we take a trial potential, evaluate $\rho$ in this potential, solve for the corresponding potential, and re-determine $\rho$. This cycle converges after a handful of iterations. When
\( f_i(E, \ldots) \) is given, considerable cunning is required to devise a convergent scheme of this type (see Rowley, 1988; Binney & Tremaine, 2008, §4.4.2(b) for more detail).

- \( J \) is an adiabatic invariant whereas \( E \) is not. Consequently, when the potential changes slowly, through the growth of the stellar disc, for example, \( f(J) \) is constant while \( f(E, \ldots) \) is not. For this reason the classic work of Spitzer’s school on globular clusters (Spitzer, 1987) used \( J_r \) (usually denoted \( q \)) in numerical computations of cluster evolution even though the original DF was taken to be \( f(E, \ldots) \).

Pontzen & Governato (2013) estimated the actions of particles in some large simulations of cosmological dark-matter clustering using a spherically symmetrised potential and reached the interesting conclusion that in the portion of action space in which periods are less than the Hubble time, the DF is simply a product of exponentials of the actions. Consequently, it seems that there is a natural choice of functional form for the DF of the dark matter.

4. Cosmological simulations

As soon as electronic computers became widely available in the 1960s, they had a big impact on galactic dynamics. Henon & Heiles (1964) showed that quasiperiodic and chaotic motion can exist side by side, while Hockney & Hohl (1969) showed that cool, self-gravitating discs develop strong bars on essentially a dynamical timescale. A decade later N-body models showed that velocity anisotropy can flatten galaxies in the absence of rotation and that triaxial stellar systems are dynamically possible (Binney, 1976; Aarseth & Binney, 1978).

The structure of an N-body model is implicit in the initial conditions from which it is started, but these are typically far removed from the model’s final form, and the map from initial conditions to final equilibrium is not easily understood. In these circumstances a natural way to proceed is to imagine how a galaxy might have formed and to let this speculation suggest suitable initial conditions. For example Gott (1975) used initial conditions associated with a “top hat” over-density in the expanding Universe, while Binney (1976) adopted initial conditions inspired by Zeldovich’s picture of collapse to a pancake (Zel’dovich, 1970).

The advent of the Cold Dark Matter cosmology in the 1980s put the choice of initial conditions for the dominant dark-matter component on a
rigorous footing, and the next twenty years were taken up using ever more sophisticated software and faster hardware to understand the dynamical evolution of dark matter (DM) from the given initial conditions. Consequently, we now know what would happen in a universe free of baryons.

Unfortunately, the CDM cosmology does not make clean predictions for the initial conditions of galaxies’ stellar components because we are unable to compute with any confidence where and at what velocities gas turns into stars. The difficulty is that so long as matter remains gaseous it responds to pressure forces as well as gravity, and pressure depends on the temperature and density of gas. Hence to compute gas dynamics one needs to follow the heating and cooling of gas. The gas is heated by adiabatic compression and by shocks in which gas suffers an abrupt change of velocity. The velocities driving shocks may be gravitationally driven but they are often driven by outflows powered by supernovae and accreting objects. There is abundant evidence that outflows have a big impact on galaxy formation, but simulating them has proved extremely hard and can currently not be achieved with standard physics: some kind of ad-hoc suspension of the standard equations is required to generate significant outflows, such as shutting off cooling for a time or distributing by hand the energy of a supernovae through a kiloparsec-sized volume.

Ab initio simulations of galaxy formation are exceedingly hard because on the one hand the structure of the inner several kiloparsecs of the galaxy is strongly influenced by material that is brought to it by flows that have a characteristic length scale of megaparsecs, so to simulate the formation of a galaxy one has to follow the dynamics of a section of the Universe at least 10 Mpc across. On the other hand, to get the heating and cooling of gas right it is essential to resolve shocks and contact discontinuities that have sub-parsec scales. In short, simulations with enormous dynamic range are required. Nobody knows what the minimum dynamics range is, but it is certainly beyond the reach of current supercomputers.

Successive generations of stars synthesise heavy elements and the outflows driven by star formation drive a portion of these elements away from the sites of their synthesis. Heavy elements in the photospheres of low-mass, long-lived stars provide a fossil record that could tell us much about galaxy formation. Early attempts to understand the chemical evolution of galaxies modelled the annulus of gas within the Galaxy with \( R \sim R_0 \) as a box that initially contained just pristine gas but gradually contained more and more stars and more highly enriched gas (van den Bergh, 1962; Schmidt, 1963).
An early conclusion of these studies was that metal-poor gas continued to enter the box throughout the history of the Universe. We return to this crucial insight below.

Huge efforts have been made to understand the relative proportions of various nuclides each generation of stars produced (Thielemann et al., 2011; Nomoto, 2013) – for a single star these proportions depend on the star’s initial mass (which determines the nature of its death), its initial chemical composition (which determines the extent to which nuclides heavier than Fe can be built up by neutron capture), and potentially its speed of rotation (which affects convection within the star). Decades of work on this problem have made it clear that it is incredibly hard: not only does one require complete knowledge of the evolution of interacting binary stars, but one needs to be able to predict which parts of an exploding star will be ejected and which parts will fall back to form the remnant, and nothing about supernovae is straightforward – they are distinctly non-spherical, turbulent relativistic, and potentially highly magnetised non-equilibrium objects.

Given that it is so hard to predict the mix of heavy elements that a given generation of stars injects into the interstellar medium, a natural strategy is to determine this mix empirically by observing the chemistry of supernova remnants. This too is an extremely challenging programme because there are many different kinds of supernovae, each of which has to be studied independently from an appropriate supernova remnant, and it is uncertain which kind of supernova created each of the observable remnants. Moreover, the observationally accessible remnants probably do not cover all relevant supernova types.

Two islands of clarity emerge from this sea of difficulty and uncertainty. One is that deflagration supernovae, in which a C/O white dwarf suddenly burns to iron-peak elements, (i) are major contributors to the cosmic Fe production, and (ii) do not occur in large numbers until \( \sim 1 \) Gyr after a population of stars is formed because time is required for a star of initial mass \( < 8 M_\odot \) to evolve to a white dwarf and further time is required for its companion to dump significant mass on the white dwarf (Förster et al., 2006). Hence stars formed in the first \( \sim 1 \) Gyr of a galaxy’s life are Fe-poor (which is perversely usually designated \( \alpha \)-rich).

The other island of clarity is that stars form in clusters and the stars in a given cluster have a characteristic “fingerprint” of relative abundances that reflects the particular mix of massive stars that enriched the gas from which they formed (Freeman & Bland-Hawthorn, 2002). This fact suggests that we
could discover how stars move through phase space by dividing stars with measured phase-space coordinates into groups with characteristic relative abundances that probably formed in the same star-forming event.

We have seen that simulations of galaxy formation are formidably difficult, even when no attempt is made to follow chemistry, and we have seen how far the current theory of stellar evolution is from being able to predict reliably the chemical evolution of the interstellar medium. Given that to model current data one must simultaneously follow the dynamics and the chemistry of the interstellar medium (e.g. Brook et al., 2012), it is clear that definitive simulations of galaxy formation will not be available for many years, perhaps never. Nevertheless in the recent years significant progress has been made with simulations of galaxy formation and we now have a pretty good idea of how galaxies form (Mo et al., 2010).

Dark matter leads the way by collapsing on relatively small scales to form large numbers of cuspy dark halos. These tumble together to form larger halos. When the ratio of the masses of two halos is less than about 10, they merge rather completely through a combination of tidal disruption and dynamical friction. But when a large halo accretes much smaller halos, the latter are merely stripped of their outer layers and they live on as reduced systems orbiting through the outer halo of their massive host.

Gas falls into whatever dark halos are about, and loses energy by radiation. So it tends to accumulate in the inner regions of dark halos. Once gas dominates the local gravitational field, gravitational collapse driven by radiative cooling runs away at various locations and clusters of stars are formed. The most massive stars soon produce powerful outflows, which drive away most of the gas from which they formed. Some of this gas will fall into some other halo, other gas will return to the halo from it was ejected after a considerable delay. Thus the tumbling together of dark halos is punctuated by episodes of star formation that terminate when energy released by young stars temporarily drives away residual gas.

In any region of space there are a few massive halos and many much smaller ones, many of them orbiting within the massive halo. These orbiting halos have little opportunity to pick up gas from which to form stars because the ambient gas is moving in the dominant gravitational field of their host halo, and their own gravitational field is too weak to grab such gas at the relative velocity imposed by the host system. So the star-forming days of a halo come to an end soon after it starts to orbit within a much more massive halo.
Luminous galaxies are systems that have been the local centre of gravitational attraction for a long time in a region of high ambient gas density. In what were initially very over-dense regions, halos that are now part of a rich cluster of galaxies early on found themselves in collapsing regions of exceptional gas density. Consequently they accreted gas and formed stars fast. In these regions the scale of halo clustering exceeded the scale of an individual galaxy rather early on, and from this time even massive galaxies have been orbiting in cluster-sized halos so they have been unable to accrete gas and their star formation long ago faded. Hence in clusters we find luminous galaxies that are red and dead.

In lower density environments the scale of clustering increased more slowly and gas was accreted more slowly, so here we find luminous galaxies that are either still forming stars or only recently ceased to do so.

The atomic and molecular gas from which stars form has a natural velocity scale $7 - 10$ km s$^{-1}$: outflows from stars can sustain turbulence with this kind of velocity dispersion. If the cool gas in a halo forms a rotating disc, this disc will be thin if its orbital velocity is much greater than 10 km s$^{-1}$, otherwise it will be thick and lumpy. Hence spiral galaxies are associated with halos that have circular velocities in excess of $\sim 80$ km s$^{-1}$.

Ours is just such a galaxy. The visible Galaxy is the baryon-dominated core of a much larger halo within which orbit many much smaller halos. There are halos on orbits with periods much shorter than a Hubble time that have been through pericentre several times, and ones that are falling in for the first time. We expect halos to arrive in groups, and recent measurements of the proper motion of the Magellanic Clouds (Kallivayalil et al., 2006) suggest that the Clouds form just such an infalling group. The Sgr Dwarf, by contrast, fell in long ago and has already orbited the Galaxy several times.

As halos orbit through the Galaxy tides pull out of them leading and trailing streams, so the Galaxy’s halo is tracery of such streams. Since the outer parts of halos are DM-dominated, most stripped material is dark and stars are likely to enter the streams only after the halo has been orbiting for some time and has been stripped down to its core.

Several lines of argument imply that the outer Galaxy is filled with a corona of plasma at the virial temperature $\sim 2 \times 10^6$ K. Although it is too tenuous to be detected directly through its X-ray emission (in contrast to the analogous virial-temperature gas found in rich groups and clusters of galaxies), the corona manifests itself by confining denser cold gas that we can detect spectroscopically (Sembach et al., 2003), and by stripping cold
gas from the LMC/SMS system and forming it into a stream that stretches right across the southern Galactic hemisphere. Two arguments indicate that the Magellanic Stream is not a regular tidal stream: (i) it is not associated with stars, presumably because any stars that were gravitationally stripped alongside the gas were not subject to pressure forces and are consequently to be found at some distance from the gas; (ii) the leading arm of the Magellanic Stream is extremely short and does not in the least resemble the trailing arm, as it would if it were a stellar stream and generated by gravity alone.

The corona interacts with the gas disc in two ways: (i) hot plasma blown out of the star-forming disc must join the corona and enrich it with freshly synthesised heavy elements; (ii) clouds of cool (∼10^4 K) gas that are thrown 1 − 2 kpc off the disc by supernova bubbles interact strongly with the corona as they move at velocities ∼ 100 km s\(^{-1}\) through the coronal gas. In the first instance the flow of coronal plasma over the leading surfaces of clouds ablates the clouds. Downstream of the clouds cool ablated gas from the clouds mixes with coronal plasma in a wake. Depending on the density of the plasma, mixing in the wake may eliminate all ∼ 10^4 K gas. In this case the density of the corona steadily increases. Eventually the density will reach the regime in which the wake’s ablated gas cools the coronal gas and there is more gas at ∼ 10^4 K in the wake than has been stripped from the cloud. After ∼ 100 Myr the residual cloud and its bloated wake fall back to the disc and the net effect of the original ejection of the cloud has been to augment the supply of cool star-forming gas in the disc (Marasco et al., 2013).

Given the limitations of current simulations with respect to their inadequate resolution and uncertain foundations in stellar evolution, the picture we have just drawn is necessarily qualitative rather than quantitative. Moreover, an essential aspect of the picture is the stochastic nature of the formation process that is driven by infall of objects great and small. Hence we cannot even dream of one day having a precise simulation of the formation of the MW; the most we can hope for is an ensemble of simulations of which the MW might be a member, from a statistical perspective. That is, we have to focus on the statistics of simulations and develop tests that enable us to say that the MW is plausibly a member of this set of simulations even though it differs from all the set’s members.

What are the relevant statistics? Two might be the luminosity and mass functions of the Galaxy’s satellites. Another might be the amplitude and pitch angle of the spiral arms, another the mean and rms rates of gas accretion as functions of radius and time, and so on. In principle it should be
straightforward to determine such numbers from simulations, although huge computational resources will be required to determine these numbers from statistically significant samples of state-of-the-art simulations. Extracting such numbers from observations of the MW is by no means straightforward and will involve models that are refinements of the models we introduced in §3.

5. Fitting models to data

Astronomy is probably the science that is most affected by selection effects, and these effects are nowhere more important than in Galactic archaeology. We observe from a particular point in the plane, and which objects enter our catalogues is determined primarily by the ease of observation. Consequently, a major effort is required to extract from the catalogues the underlying frequencies of objects.

We inevitably select objects by spatial location – magnitude selection cannot be avoided, and the relationship between luminosity and apparent magnitude depends on distance and extinction, and is consequently a strong function of position. At a given location we can avoid selection by velocity, but often we do not because it is hard to resist the temptation to sift interesting targets from the generality by selecting on proper motion (e.g. Savres et al., 2012).

The traditional way of handling selection effects is correction of the data. For example, when determining the density profile $\rho(z)$ we multiply the number of stars in the catalogue at $z$ by the inverse of the fraction of the population’s luminosity function that are bright enough to pass the survey’s magnitude limit. Such corrections always rely on prior information – in this case the population’s luminosity function – and the required information is often precisely what we still need to determine. For example, spectroscopic surveys of the local disc tend to select targets by proper motion (Savres et al., 2012) or space velocity (Bensby et al., 2003), and to correct these data you need to know the complete velocity distribution of the underlying population. Consequently, you need a model of the population. Hence Galaxy models are absolutely fundamental to progress in Galactic archaeology.

The models traditionally used in the interpretation of spectroscopic surveys (e.g. Bensby et al., 2003; Bovy et al., 2012) are kinematic – they specify the velocity distribution without regard to Jeans theorem – and such models are open to two objections: (i) they have more degrees of freedom
than they should, and (ii) they are liable to specify physically unattainable velocity distributions. An excessive number of degrees of freedom is both a nuisance in that it decreases the strength of the inferences that can be drawn from a given survey and a huge missed opportunity for the following reason. If you measure the profiles $\rho(z)$ and $\sigma_z(z)$ of some population, you can probably find a potential $\Phi(z)$ for which they are reasonably consistent. If you now measure $\rho(z)$ or $\sigma_z(z)$ for any other population, one of these functions will immediately lead to a testable prediction of the other. Consequently, in a set of measured profiles $\rho(z)$, $\sigma_z(z)$ for a number of populations, there is considerable redundancy, and from the whole set it should be possible to determine $\Phi(z)$ with good precision even in the presence of significant observational errors in each data set. It follows that you cannot consistently hypothesise both $\rho(z)$ and $\sigma_z(z)$ for more than one population simultaneously.

Traditionally the positions and velocities of individual stars have been inferred from the data, which invariably consist of photometry, astrometry and sometimes spectroscopy. Then the stars are binned, either in phase space or some projection of it such as real space, and the resulting stellar densities are either compared with the predictions of models or analysed from the perspective of Jeans’ equations. The data from the Sloan Digital Sky Survey (SDSS) have been analysed in this way. Juric et al. (2008) showed that the spatial density of stars is very similar to that expected from a superposition of a thin and a thick disc. Ivezic et al. (2008) analysed the kinematics of these stars as a function of metallicity and showed that the sample divides naturally into a metal-poor halo and a metal-richer disc with distinct kinematics. Bovy et al. (2012a) determined the spatial distribution of stars as a function of their position in the ([Fe/H], [$\alpha$/Fe]) plane and concluded that each such population is consistent with forming a double-exponential disc. Bovy et al. (2012b) went on to argue that each such mono-abundance component is an isothermal in the sense that within it $\langle v_z^2 \rangle$ is almost independent of height.

There is, however, a strong case for not attempting to reconstruct the phase-space distribution directly from the data. The errors in the data are always appreciable for the majority of stars in any catalogue because most stars inevitably lie close to the catalogue’s magnitude limit, which will be close to the point at which measurements become too uncertain to be useful. Non-negligible errors need to be properly accounted for when making deductions from the data, and this is hard to do rigorously when directly inverting the data.

The distance $s$ to a star is a single number that affects not only the star’s
presumed location, but its presumed tangential velocity, luminosity, reddening and therefore intrinsic colours also. Distances are always significantly uncertain, and the error in \( s \) induces correlated errors in all these variables. A clean error analysis in the presence of correlated errors is hard. Still more worrying, the Gaia Catalogue will provide parallaxes \( \varpi \) for a billion stars, and many of these parallaxes will be negative. A negative parallax does not yield a meaningful distance, but it does constrain \( s \): to \( \gtrsim 1/\sigma_{\varpi} \). Moreover, many positive parallaxes will not be much larger than their errors, and in this case the probability distribution of \( s \) is very skew, with a long tail to large \( s \). We can overcome all these difficulties by comparing the data to the predictions of models in the space of observables \( (\alpha, \delta, \varpi, v_\parallel, \mu_{\alpha}, \mu_\delta, V, B - V, \ldots) \), for here the errors will be independent and, by the central limit theorem, to a good approximation Gaussian.

The downside of working in the space of observables is that physically significant information becomes non-local – an over-density of stars at some location in real space maps into less pronounced enhancements in the numbers of stars over a wide range of apparent magnitudes.

McMillan & Binney (2012) used orbital tori to investigate our ability to infer the DF of a population from various combinations of photometry, astrometry and measurements of line-of-sight velocities. They assumed throughout that the potential is known and that the stars are drawn from a very broad luminosity function, so the apparent magnitude of a star conveys very little information about \( s \) – the photometry of the SDSS constrains distances much more strongly, for example. Notwithstanding the breadth of their assumed luminosity function, McMillan & Binney (2012) found that photometry and proper motions with Gaia-like errors for 10 000 stars widely distributed over the sky are sufficient to constrain the DF of the population very narrowly. Adding parallaxes and line-of-sight velocities merely diminishes the already modest errors on the parameters of the DF.

The obvious way to constrain the gravitational potential \( \Phi \) is to choose the DF that maximises the likelihood of the data for a given potential, and then to choose the potential that gives the largest likelihood overall. When McMillan & Binney (2013) tried doing this with orbital tori they were thwarted by the extent of the Poisson noise in their likelihood values. They found that the more precise and complete the data in a catalogue are, the less strongly they constrain the potential because the Poisson noise increases as the data become more precise. This happens because as each star’s error ellipsoid shrinks, the number of tori that can contribute to the likeli-
hood of that star decreases. This is a generic problem for any approach to galaxy modelling that relies on a discrete orbit library. Such approaches include Schwarzschild modelling (Schwarzschild, 1979; Binney & Tremaine, 2008, §4.7.2), made-to-measure modelling (Syer & Tremaine, 1994; Dehnen, 2009) and straight N-body modelling. McMillan & Binney (2013) showed that the Poisson noise can be effectively eliminated from the problem if expressions that give $J(x, v)$ are available. (By contrast orbital tori yield inverse expressions $x(J, \theta)$ and $v(J, \theta)$.) With such expressions the scale height and radii of the disc that contributes to $\Phi$ can be constrained to better than 10% with a catalogue of just 10,000 stars.

6. Dynamics of the bulge/bar

The Galaxy’s bulge/bar is one of its three principal components (the other two being the disc and the dark halo). It is hard to observe, both because it is heavily obscured by dust and because we see it from within one of its principal planes. Its dynamics are complex and not fully understood. It has been modelled with Schwarzschild’s technique by Zhao et al. (1993) and Häfner et al. (2000) but the best current models have been obtained by either full N-body simulation (Martinez-Valpuesta & Gerhard, 2011, 2013) or the made-to-measure (M2M) technique (Bissantz et al., 2004; Long et al., 2013). These models have successfully reproduced both the kinematics of stars observed in regions of low extinction and the statistics of microlensing events.

On account of our unfavourable location for viewing the Galactic centre, there was no consensus that ours is a barred galaxy before 1991. Then Blitz & Spergel (1991) pointed out that near-IR photometry showed the signatures of a bar seen not far from end-on, signatures that subsequently became very clear in the photometry from the DIRBE experiment aboard the COBE satellite (Dwek et al., 1995; Binney et al., 1997). This evidence from surface photometry was reinforced by studies of the luminosity functions of stars seen to the left and right of the Galactic centre, which showed that stars of a given type seen to the left of the Galactic centre are systematically brighter than those seen to the right, presumably because they are nearer (Weinberg, 1992; Stanek et al., 1997).

Binney et al. (1991) made the case for a central bar on dynamical grounds by interpreting the longitude-velocity diagrams of CO and H I emission at longitudes $l \lesssim 12^\circ$ in terms of the two major families of closed orbits in the
equatorial plane of a rotating bar. These are the $x_1$ family, members of which are elongated parallel to the bar and become more elongated as one moves inwards, and the $x_2$ family, whose members exist at smaller radii and are elongated perpendicular to the bar (Contopoulos & Papayannopoulos, 1980; Binney & Tremaine, 2008, §3.3.2). In the Galaxy the $x_1$ orbits extend out to radii in excess of 2 kpc and gas clouds on these orbits contain mostly H$_1$. Clouds on the innermost of the populated $x_1$ orbits are moving towards us at $\sim 54$ km s$^{-1}$ as they pass in front of Sgr A*, and form the “expanding 3 kpc arm”. Dame & Thaddeus (2008) were eventually able to identify emission at positive line-of-sight velocities from clouds on these orbits as they pass behind Sgr A*, thus conclusively proving that the flow of gas in the inner few kiloparsecs is governed by the gravitational field of a rotating bar.

The $x_2$ orbits are populated by clouds of dense, mostly molecular gas and are sites of rapid star formation. The Galactic wind (Bland-Hawthorn & Cohen, 2003) is probably driven by this star formation. Gas on the outermost $x_2$ orbits at $R \approx 0.2$ kpc impacts gas on the innermost $x_1$ orbits, with the result that gas is constantly transferring from the innermost $x_1$ orbit to the outermost $x_2$ orbit. There the gas tends to accumulate, and in external galaxies a circum-nuclear ring of star formation is often seen (Buta & Combes, 1996), which is presumably the consequence of such gas accumulation.

Traditionally observations of bulge stars have been restricted to windows of low absorption that lie $\sim 4^\circ$ from the plane. In these windows bulge stars lie $\sim 0.5$ kpc from the plane, so they do not include more recently formed stars, and, despite the clear evidence of rapid star formation, the bulge is often considered old.

N-body simulations show that systems like the bulge/bar are the long-term products of the dynamical evolution of initially cool discs: a planar bar forms first, and when it has become strong enough and its particles are on highly eccentric orbits, the bar quite suddenly buckles out of the plane, the elongation decreases somewhat and a three-dimensional bar/bulge emerges (Combes & Sanders, 1981; Raha et al., 1991; Martinez-Valpuesta et al., 2006). When viewed from near its minor axis in the plane, such a bar/bulge has the characteristic peanut shape that is often seen in external galaxies, and is evident in the 2MASS image of our Galaxy (Carpenter et al., 2001). These systems have high pattern speeds in the sense that their corotation radii lie at about 1.2 times their major semi-axis lengths (Athanassoula, 2003). The pattern speed of the Galactic bar has been estimated to lie between 40 km s$^{-1}$ kpc$^{-1}$ (Long et al., 2013) and 53 km s$^{-1}$ kpc$^{-1}$ (Dehnen, 1999).
The bar/bulge must interact strongly with both the disc and the dark halo. Its coupling to the stellar disc was explored by Martinez-Valpuesta & Gerhard (2011), who showed that it creates over-densities in the surrounding disc at $R \simeq 4\, \text{kpc}$ and azimuthal locations that shift from leading the bar to trailing it and back again. This phenomenon reflects the fact that the dominant waves in a disc lie inside their corotation radius, so it is natural for the pattern of waves in the disc around the bar to rotate more slowly than the bar (Sparke & Sellwood, 1988). In fact Masset & Tagger (1997) present convincing evidence that a bar excites a spiral wave in the surrounding disc that is resonantly coupled to the bar in such a way that it rotates more slowly than the bar. Consequently, the density peaks in the disc are constantly overtaken by the ends of the bar, and we see alternation between leading and trailing morphology. This dynamical picture provides an elegant explanation for observation by Cabrera-Lavers et al. (2008) that clump stars in the disc delineate a “long thin bar” that leads the long axis of the “short-thick bar” by $\sim 15^\circ$.

Dehnen (1999) argued that the Hercules stream, which is a prominent feature in the velocity-space distribution of nearby stars, comprises stars that resonate with the bar’s pattern speed, and from this conjecture inferred that the pattern speed is $\Omega_p = (53 \pm 3)\, \text{km s}^{-1}\, \text{kpc}^{-1}$.

The bar must be constantly surrendering angular momentum to the dark halo (Tremaine & Weinberg, 1989; Sellwood, 2006) and to the stellar disc (Masset & Tagger, 1997). It must be acquiring angular momentum from gas which passes through the corotation resonance and moves on orbits of the $x_1$ family before transferring to an $x_2$ orbit (Combes, 2010). In simulations that lack gas, angular-momentum loss to the dark halo and disc causes the bar to strengthen and slow down (Athanassoula, 1992). If this process had continued for a Hubble time, the bar’s corotation radius would lie much further out than it does. Hence acquisition of angular momentum from gas must be an important process.

In the vicinity of the corotation resonance, phase space has significant chaotic zones. That is, many orbits in this region are not quasiperiodic and do not admit three isolating integrals. For the modeller this fact is a nuisance because we cannot use Jeans’ theorem to construct generic equilibrium models. But the implications go much deeper than that because in a chaotic phase space orbits diffuse. In §8 we will present a formalism that allows one to follow the evolution of a stellar population consequent on the diffusion of its stars through phase space. However, this apparatus is not immediately
applicable to dynamics near the ends of the bar, because it requires that orbits are essentially quasiperiodic. How to quantify diffusion through an intrinsically chaotic phase space is an open question.

7. The stellar and dark halos

We now discuss two components of the Galaxy that have rather similar dynamics but dramatically different properties in other respects: the dark and stellar halos.

7.1. The dark halo

The dark halo is generally assumed to have a structure similar to that of an NFW component (Navarro et al., 1997). Such halos form in cosmological simulations of the clustering of dark matter (DM) for reasons that are not properly understood. The most intriguing aspect of the formation of NFW components is that they arise independent of the power spectrum of the fluctuations in the original density of DM – if one suppresses the small-scale power in the fluctuation spectrum, low-mass halos are eliminated, but the massive halos that remain are still have central near power-law cusps (Moore et al., 1999). These cusps are a memory of the formally infinite initial phase-space density of cold dark-matter: as \( r \to 0 \), the density diverges like \( r^{-1} \), so the mass \( M(r) \) interior to radius \( r \) scales as \( r^2 \) and the velocity dispersion scales as \( \sigma \propto \sqrt{M/r} \propto r^{1/2} \) and thus the phase-space density diverges as \( \rho/\sigma^3 \propto r^{-5/2} \). Given that the velocity dispersion vanishes with \( r \), these cusps are fragile in the sense that particles within them can be liberated, and thus the cusp destroyed, by quite a weak scattering event. On the other hand their high spatial densities ensure that they are not easily tidally disrupted.

The statistics of microlensing events towards the Galactic centre tell us that inside \( r \simeq 3 \) kpc the Galaxy is baryon dominated (Bissantz et al., 2004). In fact we may be sure that baryons have been dominant wherever stars have formed in abundance since gas will not be unstable to gravitational collapse if the local gravitational field is not dominated by its self gravity. Dwarf spheroidal galaxies now appear to be dark-matter dominated, but this must be because at some point dying stars blew away most of the gas from which those stars formed.

What impact will a period of baryon domination have on the DM? Slow accumulation of baryons, through gas accretion for example, will adiabati-
cally compress the DM. In this compression the DF $f(J)$ of the DM will be invariant but the distribution of DM in $(x, v)$ space will change. Blumenthal et al. (1986) estimated how DM would respond to adiabatic compression by considering the trivially computed case in which all particles are on circular orbits. Sellwood & McGaugh (2005) treated the general case and showed that non-zero radial velocity dispersion diminishes the amplitude of compression.

It is far from clear that adiabatic compression is a useful approximation because star-forming gas is bound to fragment into massive, dense clumps, which will efficiently scatter DM particles onto more energetic orbits, thus decreasing the density of DM. The physics of this process is that of dynamical friction – a gas cloud of mass $m$ that is initially at radius $r$ will surrender its energy to background particles in $\sim M(r)/m$ dynamical times at $r$, where $M(r)$ is the mass of all material inside $r$. Note that in an NFW halo the dynamical time $r/\sigma \propto r^{1/2}$ tends to zero with $r$, so even if many dynamical times were required for modification of the DM distribution, this would be no problem. Thus once the gas has come to dominate the local mass density, only a few dynamical times are required for the DM profile to be profoundly modified by frictional heating if the gas fragments into a handful of massive clouds.

The density and velocity distribution of DM near the Sun is important for experiments that seek to detect DM with cryogenic underground detectors. Initially cold DM should be restricted to a nearly three-dimensional subset of six-dimensional phase space: that is, at each spatial point, all DM particles should have essentially the same velocity. As this “sheet” of DM moves through phase space under the influence of gravity, it is stretched and folded, so that long after virialisation, at any given spatial point many different folds of the sheet can be found, each with its own characteristic velocity. In fact the velocity distribution of the DM particles at a point must be made up of a finite number of contributions at particular velocities by distinct folds of the sheet. At some spatial points a fold of the sheet is tangent to a velocity direction $v_i$, so it has a significant velocity spread. By Poincaré’s invariant theorem (Binney & Tremaine, 2008, §D.4.2), the wide spread in velocity of this fold must be compensated by a narrow spread in the canonically conjugate spatial coordinate $x_i$, and the spatial density of this fold may be orders of magnitude larger than that of a generic fold. How irregular will the velocity distribution of the DM be on Earth, and can folds of exceptional density significantly enhance the rate of self-annihilation by DM? Vogelsberger & White (2011) used adapted cosmological N-body simulations to address these questions.
They concluded that very large numbers ($\sim 10^{14}$) of folds contribute to the velocity distribution at the Sun, with half of the DM being contributed by the $\sim 10^6$ most massive folds. They concluded that folds of anomalous spatial density do not dominate the total annihilation rate. Thus the naive model in which the DM has a smooth DF $f(x, v)$ should be able to account for all observable phenomena.

7.2. The stellar halo

In §3.2 we discussed the choice of DFs to represent the stellar halo. The idealisation of a smooth halo, which these DFs embody, is often useful even though we now know that the stellar halo is far from smooth — Bell et al. (2008) report that at least half of the halo’s luminosity is accounted for by lumps and streams. Thus the stellar halo is markedly less smooth than the dark halo is thought to be, presumably because by radiating binding energy baryons early on form more tightly bound clumps at each mass scale than (dissipationless) DM can.

A classical application of the concept of a smooth stellar halo is to the estimation of the escape speed $v_{\text{esc}}$ from the solar neighbourhood (Smith et al., 2009). Essentially one counts fast stars and following Leonard & Tremaine (1990) fits the number counts to a formula based on the assumption that the DF is of the form $f(E)$ and is such that $f \to 0$ as $E \to 0$. The weakness of this procedure is that as $E \to 0$ the periods of orbits tend to infinity. Consequently, in the limit $E \to 0$, the time for the system to phase mix tends to infinity, and we cannot invoke Jeans’ theorem. In fact, in this limit we must expect the DF to depend on the angle variables as well as on the integrals. Moreover, there will be a population of unbound, escaping stars, so the assumption that $f = 0$ for $E > 0$ is also unsafe. The determination of $v_{\text{esc}}$ involves extrapolation of the data, which is always dangerous. However, in so far as simulations of galaxy formation are realistic, it makes sense to estimate the parameters of the Galaxy’s potential, by fitting the observed velocity distribution of fast stars to simulations (Piffl et al., 2013).

7.3. Tidal streams

The SDSS revealed several long thin over-densities of stars (Grillmair, 2006; Belokurov et al., 2006). The most prominent of these is clearly the tidal debris of the Sagittarius dwarf spheroidal galaxy (Majewski et al., 2003; Fellhauer et al., 2007) and SDSS photometry of the vicinity of the globular cluster Palomar 5 allows us to study the process of disruption in detail.
(Newberg et al., 2002; Odenkirchen et al., 2009). Consequently, all linear over-densities are assumed to be generated by the tidal shredding of either a star cluster or a dwarf galaxy, even though some streams have no known progenitor.

When a stream’s progenitor passes through pericenter, the Galaxy’s tidal field detaches some stars from the point nearest the Galactic centre, and other stars from the furthest point. The stars detached from the near point have less angular momentum and shorter-period orbits than the progenitor, while the stars detached from the far point have more angular momentum and longer periods than the progenitor. When they are first stripped, all stars have similar phases, but over time the spread in the phases of stripped stars increases because the phases of stars detached from the near point advance more rapidly than do the phases of stars detached from the far point.

Mathematically, let \( \delta J \) be the difference between the actions of a detached star and the progenitor’s actions. Then the frequencies \( \Omega \) of the detached star differ from those of the progenitor by

\[
\delta \Omega = D \cdot \delta J,
\]

where

\[
D_{ij} = \frac{\partial^2 H}{\partial J_i \partial J_j} = \frac{\partial \Omega_j}{\partial J_i},
\]

is the Hessian matrix. \( D \) is a symmetric matrix, so it can be expanded as a sum over its orthogonal eigenvectors \( e_i \) and (real) eigenvalues \( \lambda_i \):

\[
D = \sum_i \lambda_i e_i \otimes e_i.
\]

Tremaine (1999) pointed out that for typical Galactic potentials \( D \) is a very anisotropic matrix in that one of its eigenvalues, \( \lambda_0 \), is much bigger than the others. This eigenvalue will dominate the sum (27), so to a good approximation we have

\[
\delta \Omega \simeq \lambda_0 e_0 (e_0 \cdot \delta J).
\]

A long time \( t \) after the star was detached from the progenitor, its displacement in angle space from the location of the progenitor is

\[
\delta \theta \simeq \delta \Omega t \simeq \lambda_0 t (e_0 \cdot \delta J) e_0,
\]

where we have neglected the small initial displacement. Thus in angle space the stream is drawn out into a straight line parallel to \( e_0 \) – the speed at which
a star moves along this line is proportional to $\delta \mathbf{J} \cdot \mathbf{e}_0$. This straight line in angle space maps into a curve in real space. Thus streams form because their stars are not all on the same orbit but on orbits that vary systematically along the stream, contrary to what was for many years assumed (Johnston et al., 2005; Binney, 2008; Eyre & Binney, 2009a,b). In angle space the progenitor’s orbit is the straight line parallel to $\Omega$ evaluated at the progenitor’s actions. Eyre & Binney (2010) showed that in the (spherical) isochrone potential the angle between $\mathbf{e}_0$ and $\Omega$ is typically $1 − 2$ degrees, and Sanders & Binney (2013a) show that in more realistic potentials the misalignment can exceed $10^\circ$. Moreover, even an angle difference $\sim 2^\circ$ can lead to large errors in the derived gravitational potential when a potential is sought that makes an orbit run along a stream.

Sanders & Binney (2013b) propose an alternative way to diagnose the potential with stream data and test its ability to infer correctly the flattening $q$ and circular speed $V_c$ of the flattened logarithmic potential in which a stream has been constructed by tidal disruption of a self-gravitating N-body model. For each trial potential one plots the stars of the stream in both angle space and frequency space. On account of the first equality of equation (29) the lines of regression through these two distributions should be parallel. One seeks potentials in which this condition is satisfied to within the errors. With data comparable to those that will be furnished by Gaia, the errors on the derived values of $V_c$ are $\sim 4\, \text{km} \, \text{s}^{-1}$ and those on $q \sim 6\%$.

8. Evolution of the potential

To this point we have been modelling the Galactic potential as a smooth, time-independent object. In reality the potential contains a significant fluctuating component caused by moving gas clouds, stellar systems and lumps of dark matter, and spiral structure. The potential also evolves secularly in response to the infall of gas, stars and dark matter, becoming ever deeper and probably flatter. Changes in the potential give rise to changes in stars’ orbits. We now discuss such changes.

A fundamental result is obtained by multiplying the equation of motion $\dot{\mathbf{v}} = -\nabla \Phi$ by $\mathbf{v}$ and rearranging the result to

$$\frac{dE}{dt} = \frac{\partial \Phi}{\partial t}. \tag{30}$$

Thus stars change their energies if and only if the potential is time-dependent. Fluctuations in the potential redistribute energy between stars. The overall
effect of this redistribution will be to increase the system’s entropy by making it hotter and more centrally concentrated (Binney & Tremaine, 2008, §4.10.1).

Changes in the potential that occur on a timescale that is significantly longer than the longest orbital timescale are easy to deal with because such changes leave the actions $\mathbf{J}$ invariant (Binney & Tremaine, 2008, §3.6). This fact greatly facilitates the process of determining the response of a stellar system to adiabatic distortion of its potential because all we have to do is to move each star from its orbit in the original potential to the orbit with the same actions in the distorted potential. In particular, the structure of the distorted system depends only on the initial and final configurations, and not on which configurations it passed through in between.

The issue of when a change is slow enough to leave $\mathbf{J}$ invariant is a tricky one because there are infinitely many timescales $t_n \equiv 2\pi/|\mathbf{n} \cdot \mathbf{\Omega}|$ associated with a given orbit, where $\mathbf{n}$ is any vector with integer components. For sufficiently large $|\mathbf{n}|$, $t_n$ can be arbitrarily long, and even a tiny perturbation can in principle lead to violation of adiabatic invariance through the mechanism of resonant trapping that we described qualitatively in §2.1. We now analyse this phenomenon mathematically.

8.1. Resonant trapping

Let $h(\theta, \mathbf{J})$ be the difference between the true $H$ and a nearby integrable Hamiltonian $H_0(\mathbf{J})$, which might be one constructed by the torus machine (§2.3). Hamilton’s equations for motion in $H$ read

$$\dot{\theta} = \frac{\partial H}{\partial \mathbf{J}} = \Omega_0 + \frac{\partial h}{\partial \mathbf{J}}, \quad \dot{\mathbf{J}} = -\frac{\partial H}{\partial \theta} = -\frac{\partial h}{\partial \theta},$$

(31)

where $\Omega_0 = \partial H_0/\partial \mathbf{J}$. The perturbing Hamiltonian, like any function on phase space, is a periodic function of the angles, so we can Fourier expand it:

$$h(\theta, \mathbf{J}) = \sum_n h_n(\mathbf{J}) \cos(\mathbf{n} \cdot \theta + \psi_n).$$

(32)

With this expansion, the equation of motion of $\mathbf{J}$ is

$$\dot{\mathbf{J}} = \sum_n \mathbf{n} h_n(\mathbf{J}) \sin(\mathbf{n} \cdot \theta + \psi_n) = \sum_n \mathbf{n} h_n(\mathbf{J}) \sin(\mathbf{n} \cdot \mathbf{\Omega} t + \psi'_n),$$

(33)

where $\psi'_n = \psi_n + \mathbf{n} \cdot \theta(0)$. So long as $\mathbf{n} \cdot \mathbf{\Omega} \neq 0$, the time-averaged value of $\dot{\mathbf{J}}$ vanishes and we expect $\mathbf{J}$ simply to oscillate slightly around its unperturbed
value. But if a \textbf{resonance condition} $N \cdot \Omega = 0$ is nearly satisfied, the argument of one or more of the sines will change very slowly, and the cumulative change in $J$ can be significant even for very small $h_N$. When we multiply the resonance condition by $t$, we obtain

$$N \cdot \theta(t) = \text{constant.} \quad (34)$$

This equation inspires us to make a canonical transformation to new angles and actions $(\theta', J')$ such that $N \cdot \theta$ becomes one of the new angles, say $\theta_1'$. It does not much matter what we adopt for $\theta_2'$ and $\theta_3'$; $\theta_2' = \theta_2$ and $\theta_3' = \theta_3$ works fine. Then the generating function of the required transformation is

$$S(\theta, J') = J'_1 N \cdot \theta + J'_2 \theta_2 + J'_3 \theta_3, \quad (35)$$

and the new angles are

$$\theta_1' = \frac{\partial S}{\partial J'_1} = N \cdot \theta, \quad \theta_2' = \frac{\partial S}{\partial J'_2} = \theta_2, \quad \theta_3' = \theta_3. \quad (36)$$

Since $\theta_1'$ does not evolve in time, the star explores only a two-dimensional set of its three-dimensional torus. While a star on a resonant torus does not come arbitrarily close to every point on its torus, the bigger the numbers $N_j$ are, the more effectively it samples its torus, and the less likely it is that the resonance condition will be dynamically important.

Mathematically, we use the new angle variables $\theta'$ defined by equation (36) and their conjugate actions $J'$, which follow from

$$J_1 = \frac{\partial S}{\partial \theta_1} = J'_1 N_1, \quad J_2 = \frac{\partial S}{\partial \theta_2} = J'_1 N_2 + J'_2, \quad J_3 = J'_1 N_3 + J'_3. \quad (37)$$

Thus

$$J'_1 = J_1/N_1 \quad ; \quad J'_2 = J_2 - J_1 N_2/N_1 \quad ; \quad J'_3 = J_3 - J_1 N_3/N_1. \quad (38)$$

Next we express $h$ as a Fourier series in the new angle variables and discard all terms that involve $\theta_2'$ or $\theta_3'$ on the grounds that they oscillate too rapidly to have a significant impact on the dynamics. Since our approximated Hamiltonian depends on neither $\theta_2'$ nor $\theta_3'$, to the level of our approximations the conjugate actions $J_2'$ and $J_3'$ will be constants of motion. The only non-trivial
equations of motion are now

\[ \dot{\theta}'_1 = \Omega'_{01}(J'_1) + \sum_n \frac{\partial h'_{(n,0,0)}}{\partial J'_1} \cos(n\theta'_1 + \psi'_{(n,0,0)}) \]

\[ \dot{J}'_1 = \sum_n nh'_{(n,0,0)}(J'_1) \sin(n\theta'_1 + \psi'_{(n,0,0)}), \]  \hspace{1cm} (39)

where

\[ \Omega'_{01} = \frac{\partial H_0(J')}{\partial J'_1} = \sum_i \Omega_{0i} \frac{\partial J_i}{\partial J'_1} = \Omega_{01}N_1 + \Omega_{02}N_2 + \Omega_{03}N_3 = \mathbf{N} \cdot \mathbf{\Omega}_0 \]  \hspace{1cm} (40)

and we have suppressed the dependence of \( \Omega'_{01} \) and \( h'_{(n,0,0)} \) on \( J'_2 \) and \( J'_3 \) because the latter are mere constants. We have reduced the particle’s motion in six-dimensional phase space to motion in the \((\theta'_1, J'_1)\) plane.

We differentiate the first of equations (39) with respect to time

\[ \ddot{\theta}'_1 = \frac{\partial \Omega'_{01}}{\partial J'_1} \dot{J}'_1 + \sum_n \left( \frac{\partial^2 h'_{(n,0,0)}}{\partial J'_1^2} \dot{J}'_1 \cos(n\theta'_1 + \psi'_{(n,0,0)}) - n\theta'_1 \frac{\partial h'_{(n,0,0)}}{\partial J'_1} \sin(n\theta'_1 + \psi'_{(n,0,0)}) \right). \]  \hspace{1cm} (41)

We can neglect the sum in this equation because each of its terms is the product of a derivative of \( h'_{(n,0,0)} \), which is small, and either \( \dot{J}'_1 \), which is of the same order, or \( \dot{\theta}'_1 \), which is also small because \( \Omega'_{01} \) vanishes at the resonance. Therefore we can dramatically simplify the \( \theta'_1 \) equation of motion to

\[ \ddot{\theta}'_1 = \frac{\partial \Omega'_{01}}{\partial J'_1} \dot{J}'_1 = \sum_n nh'_{(n,0,0)}(J'_1) \sin(n\theta'_1 + \psi'_{(n,0,0)}). \]  \hspace{1cm} (42)

If we approximate \( \frac{\partial \Omega'_{01}}{\partial J'_1} \) and \( h'_{(n,0,0)} \) by their values on resonance, and retain only the term for \( n = 1 \) in the sum, we are left with the equation of motion of a pendulum

\[ \ddot{\theta} = -\omega^2 \sin \theta, \]  \hspace{1cm} (43)

where

\[ \theta \equiv \theta'_1 + \psi'_{(1,0,0)} \quad \text{and} \quad \omega^2 \equiv -\frac{\partial \Omega'_{01}}{\partial J'_1} h'_{(1,0,0)}. \]  \hspace{1cm} (44)

A pendulum can move in two ways: at low energy its motion is oscillatory at an angular frequency that falls from \( \omega \) at the lowest energies to zero at the critical energy above which the pendulum circulates. Consequently,
Figure 1: The phase plane of a pendulum. Curves of constant energy $E$ (eq. 45) are plotted. The particle moves on these, from left to right in the upper half of the figure and from right to left in the lower half.

equation (42) predicts that close to the resonance ("low energy") $\theta_1'$ will oscillate. This is the regime of resonant trapping in which the particle librates around the underlying resonant orbit. At some critical distance from the resonance ("high energy") $\theta_1'$ will start to circulate. We obtain a useful pictorial representation of this behaviour by deriving the energy invariant of equation (43). We multiply both sides by $\dot{\theta}$ to obtain an equation which states that

$$E \equiv \frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta$$

(45)

is constant. Consequently, the particle moves in the $(\theta, \dot{\theta})$ plane along curves of constant $E$ like those shown in Fig. 1. The round contours near the centre of the figure show the motion of a particle that has been trapped by the resonance, while the contours at the top and bottom of the figure, which run all the way from left to right, show the motion of a particle that continues to circulate.

Near the threshold energy, the phase point moves close to one of the heavy curves in the figure, and the rate at which $\theta_1'$ advances in time is highly non-uniform, just as a pendulum that has only just enough energy to get over top dead centre slows markedly as it does so. As we move further and further
from the resonance and either up beyond the figure’s top boundary or down below its bottom boundary, the rate of advance of $\theta'_1$ becomes more and more uniform, and we gradually recover the unperturbed motion, in which the rate of advance of $\theta'_1$ is strictly uniform.

8.1.1. Levitation

Resonant trapping shows up clearly in a surface of section such as those shown in Fig. 2 for motion in two axisymmetric potentials $\Phi(R, z)$. Each orbit gives rise to a series of points that lie on the curve in which the orbit’s torus intersects the $(R, p_R)$ plane. All orbits contributing to a given panel of Fig. 2 have the same energy and value of $L_z$. Most of the curves move around a central point. The point itself is made by the shell orbit $J_r = 0$; in real space this orbit is a thin cylindrical shell that has a larger diameter at $z = 0$ than at its top or bottom edges. Each curve around this point in Fig. 2 is generated by a three-dimensional orbit that forms an annulus of finite thickness. In Fig. 2 the longer an orbit’s curve is, the thicker the real-space annulus and the smaller its vertical extent. The outermost curve in Fig. 2 is generated by the orbit $J_z = 0$, which is confined to the plane $z = 0$.

The lower panel in Fig. 2 is for motion in the potential of a flatter galaxy than the upper panel, and in this panel not all curves loop around the central point. Two resonant islands have appeared, formed by orbits that have been trapped by the resonance $\Omega_r = \Omega_z$.

Before the disc formed, when the Galaxy’s potential was nearly spherical, every orbit had a value of $\Omega_r$ that was bigger than either $\Omega_\phi$ or $\Omega_z$. Stars whose orbits are confined within a couple of kiloparsecs of the equatorial plane now have $\Omega_\phi < \Omega_z$. So at some point in the flattening process these stars satisfied the resonance condition $\Omega_r = \Omega_z$. In a flattened potential $\Omega_r/\Omega_z$ is smallest for orbits that are confined to the equatorial plane. Therefore the resonance condition was first satisfied by these orbits.

Fig. 2 shows surfaces of section for motion in a potential before and after the resonance condition $\Omega_r = \Omega_z$ is first satisfied: the islands visible in the lower panel are made up of orbits trapped by this resonance. Note that the areas of the curves that loop around the central point are

$$\int dR dp_R = 2\pi J_r,$$  \hspace{1cm} (46)

so these areas do not change as the potential flattens.
Figure 2: Surfaces of section for motion in flattened isochrone potentials: the upper panel is for the case of a mass distribution that has axis ratio $q = 0.7$, while the lower panel is for $q = 0.4$. In the lower panel we see resonant islands generated by the 1:1 resonance between the radial and vertical oscillations. No such island is evident in the upper panel.
The resonance condition is first satisfied by the orbit that is confined to the equatorial plane; in both panels of Fig. 2 the curve of this orbit lies on the outside. Hence the resonant islands first appeared just inside this curve. As the potential flattened more, $\Omega_r/\Omega_z$ dropped significantly below unity for the planar orbit, so the resonance condition was satisfied by orbits with non-zero $J_z$ and the islands moved inwards. Orbits whose curves lay in the path of a moving island did one of two things (Binney & Tremaine, 2008, §3.7.2): (a) they were trapped into the island, or (b) they abruptly increased their radial actions so that their curves went round the far side of the island. Which of these two outcomes happened in an individual case depended on the precise orbital phase of the star when the potential achieved a particular flattening, but it is most useful to average over phases and to consider the outcomes to occur with probabilities $P_a$ or $P_b$. The magnitudes of $P_a$ and $P_b$ depend on the relative speed with which the island increased its area and moved: if it simply grew, $P_a = 1$, and if it moved without growing $P_b = 1$. These results follow from Poincaré’s invariant theorem, which implies that the Hamiltonian flow shuffles particles around surfaces of section without changing the density of particles (for some subtleties in this see Binney et al., 1985).

Let’s imagine that after a period of stationary growth, the island moved inwards without growing, and then became stationary while it shrunk. In this case it would have swept up stars with large $J_r$ and small $J_z$ and released these stars into orbits with smaller $J_r$ and larger $J_z$. In other words, it will have turned radial motion into vertical motion. Sridhar & Touma (1996) have called this process “levitation”. Conversely, the moving island will have reduced the vertical motions and increased the radial motions of any stars it found in its path through action space. Thus resonances stir the contents of phase space. Levitation is a lovely idea but it’s not clear that it is of practical importance. In the next section we shall see that in a disc analogous scattering by resonances is very important.

8.1.2. Stellar migration

When the pattern speed $\Omega_p$ of a non-axisymmetric perturbation lies near the azimuthal frequency $\Omega_\phi$ of an orbit, the time $\pi/|\Omega_\phi - \Omega_p|$ during which the perturbation torques the orbit in the same sense is long. Consequently, in these circumstances even a weak perturbation can induce non-negligible changes in the orbit’s value of $L_z$. Since $L_z$ determines an orbit’s guiding-centre radius, changes in $L_z$ are associated with radial migration.

Fig. 3 shows the effect of a weak spiral perturbation on originally circular
orbits in a typical Galactic potential. Orbits of small radius (bottom of the figure) have $\Omega_\phi > \Omega_p$ and overtake the structure, while orbits with large radii have $\Omega_\phi < \Omega_p$ and are overtaken by the structure. Orbits for which $\Omega_\phi \simeq \Omega_p$ so they almost corotate with the structure, become resonantly trapped by it: on these **horse-shoe orbits** a star oscillates around the corotation radius $R_{\text{CR}}$ (where $\Omega_\phi = \Omega_p$) -- the ellipses at the centre of Fig. show these orbits. During the left half of a horse-shoe orbit, the perturbing force on the star is dominated by the potential trough that is marked by the left sloping dashed line, so the star is pulled to the left and its angular momentum diminishes. The star moves inwards in response. Conversely, on the right half of the orbit, the net perturbative force is to the right, $L_z$ increase, and the star moves outwards. The orbit librates around the trapping closed orbit that is marked by the dot.

Now suppose the perturbation grows in amplitude from zero, peaks and fades away again. As the strength of the perturbation grows, the range of orbits around corotation that become trapped increases, and the scale of the largest ellipse in Fig. grows. When the perturbation fades, stars are

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3The spatial location of the trapping orbit is one of the Lagrange points $L_4/L_5$. 

---
progressively released from entrapment, those on the largest ellipses first. The stars on a given ellipse will be a mix of stars that joined the ellipse on its upper half ($R > R_{CR}$) and ones that joined the ellipse on its lower half. During the lifetime of the perturbation stars circulate around the ellipse, so it’s unclear whether a star will be released from the top or the bottom of the ellipse. Roughly half the stars that became trapped when $R < R_{CR}$ are released at $R > R_{CR}$: the guiding-centre radii of these stars have been permanently increased. Conversely, roughly half the stars that were at $R > R_{CR}$ when they became trapped are released at $R < R_{CR}$. Thus a transient non-axisymmetric perturbation causes stars in a zone around the corotation radius to migrate radially either inwards or outwards.

Spiral structure is a natural source of transient non-axisymmetric perturbations. Sellwood & Binney (2002) showed that spiral features in N-body simulations cause a significant level of migration. Fig. 4 shows histograms of the final radii of stars that all started from the narrow radial bands marked in an N-body disc that was evolved for $\sim 5$ Gyr in the absence of any seeded spiral structure. Irregular and quite weak spiral structure emerged as the disc was evolved. Stars that started at the current radius of the Sun finished at radii that are often $1.2 - 2$ kpc from $R_0$. From the strength of spiral structure seen in NIR photometry by Rix & Zarisky (1995), Sellwood & Binney estimated that over the Hubble time stars will typically migrate $\sim 2$ kpc from their birth radii.

Although Fig. 3 is plotted for circular orbits, the mechanism of radial migration works also for stars on mildly eccentric orbits. Solway et al. (2012) have shown that the effectiveness of radial migration declines only slowly with increasing eccentricity of the initial orbits.

As we will discuss in §10, satellites, including completely dark ones, that are on eccentric orbits are a natural source of transient non-axisymmetric perturbations that complement spiral structure. At pericentre a satellite is moving faster than the local circular speed, so the corotation radius of its perturbation lies inside pericentre, and thus may lie in the body of the disc even if pericentre lies outside it.

Any non-axisymmetric perturbation that either does not last for ever, or changes its pattern speed, is likely to induce radial migration. Bars seem to be long-lived structures, but, as we saw in §6, their pattern speeds evolve in response to changes in the angular momentum of the population that supports the bar, and acquisition of angular momentum from gas, and loss to stars and DM particles, are likely to be constantly adjusting $\Omega_p$. We return
Figure 4: Each panel shows the distribution of the final guiding-centre radii of the stars in a disc simulation whose initial guiding-centre radii lay within the region between the dotted vertical lines. The disc had a flat rotation curve, $Q = 1.5$, and half of the radial force was provided by a fixed halo. The duration of the simulation was $\sim 4 \text{ Gyr}$. (From Sellwood & Binney 2002).
8.1.3. Heating versus migration

Let’s assume that a non-axisymmetric perturbation has a nearly constant pattern speed \( \Omega_p \). If we work in the frame of reference that rotates at frequency \( \Omega_p \), the motion of each star is governed by a time-independent Hamiltonian, the numerical value of which, the Jacobi constant, is an isolating integral. In terms of the energy \( E \) of motion in the non-rotating frame, the rotating-frame Hamiltonian is

\[
H = E - \Omega_p L_z. \tag{47}
\]

Since \( H \) is an integral, along an orbit, \( dH = 0 \) and changes in \( E \) and \( L_z \) caused by the perturbation satisfy

\[
dE = \Omega_p dL_z. \tag{48}
\]

Fig. 5 is a plot of \( E \) versus \( L_z \) for a planar axisymmetric potential. The physically accessible region is bounded below by the locus of circular orbits, which are the orbits with the largest value of \( L_z \) for each given \( E \), so there are no orbits in the shaded region below this boundary. The slope of the
boundary, \((\partial E/\partial L_z)_{J_r=0}\), is the circular frequency \(\Omega(L_z)\). Equation (48) states that a steadily rotating perturbation moves stars in this figure on lines of slope \(\Omega_p\) as indicated by arrows in Fig. 5. These lines are tangent to the bounding curve at the corotation resonance (CR), where \(\Omega(L_z) = \Omega_p\). Thus at CR the perturbation scatters stars along the boundary, from one circular orbit to another. Elsewhere, the perturbation scatters stars away from the boundary, to places where the energy exceeds that of the circular orbit of the given value of \(L_z\), and the additional energy will be invested in epicyclic motion. Inside the CR, the angular momenta of stars that are initially on circular orbits must be reduced, while outside the CR it must be increased. Thus perturbation of a cold disc must move angular momentum outwards.

In summary: away from CR in a cold disc stars have to migrate inwards at \(R < R_{CR}\) and outwards at \(R > R_{CR}\). In either case, migration will inevitably increase the stars’ in-plane velocity dispersion, that is, the perturbation will heat the disc by transferring angular momentum outwards – galactic discs are giant accretion discs. Near \(R_{CR}\) stars can move either in or out from circular orbit to circular orbit, so migration is reversible and occurs without heating.

### 8.2. Orbit-averaged Fokker-Planck equation

Our discussion of time-dependent perturbations has so far concentrated on approximately steadily rotating potentials. Significant effects are caused by perturbations that do not simply rotate rigidly, for example the gravitational fields of massive objects on non-circular orbits. To understand the impact of such perturbations we now develop a general framework for handling the impact of fluctuations. The general idea is that, by the strong Jeans theorem the galaxy’s distribution function is at all times a function \(f(J,t)\) of the actions. Fluctuations (and resonances) cause this function to evolve by causing innumerable small changes in the actions of individual stars. Let \(P(J,\Delta)d^3\Delta\delta t\) be the probability that in time \(\delta t\) a star with actions \(J\) is scattered to the action-space volume \(d^3\Delta\) centred on \(J + \Delta\). The number of stars in the action-space volume \(d^3J\) is \((2\pi)^3f(J,t)d^3J\), so the number of stars leaving this volume in \(\delta t\) is

\[
(2\pi)^3 f(J, t)d^3J \delta t \int d^3\Delta P(J, \Delta). \tag{49}
\]
Similarly, the number of stars that are scattered into this volume is

\[(2\pi)^3 d^3\Delta \int d^3\Delta f(J - \Delta, t)P(J - \Delta, \Delta). \quad (50)\]

Hence the rate of change of the distribution function is

\[\frac{\partial f}{\partial t} = \int d^3\Delta [f(J - \Delta, t)P(J - \Delta, \Delta) - f(J, t)P(J, \Delta)]. \quad (51)\]

Since scattering events change actions only slightly, \(P(J, \Delta)\) is appreciable only for \(|\Delta| \ll |J|\). So we can truncate after just a few terms the Taylor series expansion in \(J\) of the product \(f(J, t)P(J, \Delta)\):

\[f(J - \Delta, t)P(J - \Delta, \Delta) = f(J, t)P(J, \Delta) - \Delta_i \frac{\partial (fP)}{\partial J_i} + \frac{1}{2} \Delta_i \Delta_j \frac{\partial^2 (fP)}{\partial J_i \partial J_j} + \cdots \quad (52)\]

Substituting the first three terms on the right side of this expression into equation (51) and cancelling terms, we obtain

\[\frac{\partial f}{\partial t} = -\frac{\partial F_i}{\partial J_i}, \quad \text{where} \quad F_i \equiv f\overline{\Delta_i} - \frac{1}{2} \frac{\partial (f\overline{\Delta^2_{ij}})}{\partial J_j}, \quad (53)\]

\[\overline{\Delta_i}(J) \equiv \int d^3\Delta \Delta_i P(J, \Delta) \quad \text{and} \quad \overline{\Delta^2_{ij}}(J) \equiv \int d^3\Delta \Delta_i \Delta_j P(J, \Delta). \quad (54)\]

Equation (53) is the orbit-averaged Fokker-Planck equation. It states that the rate of change of the distribution function is minus the divergence of the flux \(F\) of stars in action space, and we have an expression for that flux in terms of the diffusion coefficients defined by equations (54). The latter are simply the expectation values and the variances of the probability distribution of changes in actions per unit time.

8.2.1. Diffusion coefficients

The diffusion coefficients reflect the physics of whatever is responsible for causing the fluctuations. In some circumstances, for example in a star cluster, the fluctuations will be approximately thermal in nature, with temperature \(T\). Then the principle of detailed balance requires that the stellar
flux vanish when the objects being scattered are in thermal equilibrium with the fluctuations. That is, \( \mathbf{F} = 0 \) for
\[
f(\mathbf{J}) = \text{constant} \times e^{-H/kT},
\]
where \( H(\mathbf{J}) \) is the Hamiltonian. When we substitute this form of the \( df \) into the equation \( \mathbf{F} = 0 \) and observe that for this \( df \)
\[
\frac{\partial f}{\partial J_i} = -f \frac{\Omega_i}{kT},
\]
we obtain an expression for the first-order diffusion coefficient in terms of the second-order coefficient (Binney & Lacey, 1988)
\[
\Delta_i = \frac{1}{2} \left( \frac{\partial \Delta_{ij}^2}{\partial J_j} - \frac{\Omega_j}{kT} \Delta_{ij}^2 \right).
\]
This expression is useful because it enables us to obtain the first-order diffusion coefficients \( \Delta_i \) from the second-order diffusion coefficients \( \Delta_{ij}^2 \), and, while \( \Delta_{ij}^2 \) can be obtained from first-order perturbation theory (see below), a direct calculation of \( \Delta_i \) requires second-order perturbation theory (see Appendix A of Binney & Lacey, 1988).

We calculate the second-order diffusion coefficients by expanding the potential in angle-action coordinates
\[
\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \Phi_1(\mathbf{x}, t) = \Phi_0(\mathbf{J}, t) + \sum_n \Phi_n(\mathbf{J}, t) \cos(n \cdot \mathbf{\theta} + \psi_n),
\]
where \( \Phi_0(\mathbf{x}) \) is the potential of the underlying Hamiltonian \( H_0(\mathbf{J}) \) and \( \Phi_1 \) is the fluctuating part of the potential. By integrating the equations of motion of the first-order change in the actions for a time \( T \) longer than the autocorrelation time of the fluctuations we may show (see Binney & Lacey, 1988; Binney, 2012c, for details) that
\[
\Delta_{ij}^2 = \frac{1}{2} \sum_n n_i n_j \tilde{c}_n(\mathbf{J}, \mathbf{n} \cdot \Omega_0),
\]
where \( \tilde{c}_n(\mathbf{J}, \omega) \) is the power spectrum of the fluctuations:
\[
\tilde{c}_n(\mathbf{J}, \omega) = \int_{-T}^{T} dv \Phi_n(\mathbf{J}, t) \Phi_n(\mathbf{J}, t - v) \cos(\omega v).
\]
The key implication of equation (59) is that the ability of a star to diffuse through phase space hinges on whether the fluctuations contain power at one of the star’s natural frequencies \( n \cdot \Omega_0 \). When the potential is constant in a steadily rotating frame, the expansion coefficients \( \Phi_n(J,t) \) will contain only multiples of \( \Omega_p \) in their temporal Fourier transforms. For example, if the potential is that of an \( m \)-armed spiral, the potential will be \( \propto \cos(m\Omega_p t + \psi) \), so the power spectrum of the potential \( \tilde{\mathcal{c}}_n(J,\omega) \) will be non-zero only when \( \omega = n \cdot \Omega_0 \) is equal to \( m\Omega_p \). In other words, the only stars that will be caused to diffuse by an \( m \)-armed spiral are those for which \( n \cdot \Omega_0 = m\Omega_p \) for some \( n \) (Barbanis & Woltjer, 1967; Lynden-Bell & Kalnajs, 1972). Besides the CR \( |n = (0,m,0)| \), the two most important resonances are the inner Lindblad resonance (ILR), where \( n = (-1,m,0) \), and the outer Lindblad resonance (OLR), where \( n = (1,m,0) \). At the ILR the Döppler-shifted frequency at which a star perceives the spiral is \( m(\Omega - \Omega_p) \) and this coincides with its radial frequency, \( \Omega_r \), while at the OLR the perceived frequency of the spiral is \( m(\Omega_p - \Omega) \) and this again coincides with \( \Omega_r \). We have shown that in a cold disc the spiral absorbs \( L_z \) at the ILR and emits it at the OLR, and that changes in \( L_z \) heat the disc at both locations.

Periodic fluctuations will depopulate narrow regions of phase space: stars for which \( n \cdot \Omega_0 \) is equal to any harmonic of the fluctuation will be scattered to new actions and then cease to be resonant because fundamental frequencies are functions of the actions. Sellwood & Kahn (1991) find evidence for such action-space “grooves” in numerical simulations of stellar discs and show that they can generate new spiral features, which in their turn generate other grooves.

### 8.2.2. Bar-driven migration?

Minchev & Famaey (2010) have argued that bars drive radial migration much faster than does the transient spiral structure as discussed in §8.1.2. The main point (Sellwood & Binney, 2002) were making was that resonant scattering at CR changes \( L_z \) without heating the disc, so we cannot infer from the coolness of the thin disc near the Sun that stars are still near their birth radii. The bar’s CR lies far interior to the Sun, where the disc is probably not so cool, so heating-free migration around the bar’s CR is not of observational interest, although it undoubtedly takes place. The Sun lies near the bar’s OLR (Dehnen, 1999) so changes in the angular momenta of local stars that are driven by the bar will be accompanied by heating and can be constrained by the coolness of the thin disc.
Minchev & Famaey (2010) rightly consider that the signature of scattering by the bar is enhancement of the changes in $L_z$ around $CR$ and $OLR$. However, they present evidence that the action of the bar is enhanced by transient spiral structure, with the implication that the bar is not solely responsible for changes in $L_z$. The role of spiral structure in the experiments presented by Minchev & Famaey (2010) and Minchev et al. (2011) is unclear. In Minchev & Famaey (2010) the bar and spiral structure have fixed pattern speeds and the disc consists of test particles. As we indicated above, a weak but steady pattern will not engender long-term orbital diffusion: rather it will restructure phase space as it is turned on and thereafter little will change. Spiral structure at a different frequency surely has the capacity to prevent the system settling down in the new order, by making the overall potential non-periodic. This is in fact the physics of resonance overlap discussed by Chirikov (1979). However, in this case one might expect the frequency of the spiral structure to play a bigger role than it seems to in the experiments of Minchev & Famaey (2010).

Minchev et al. (2011) present experiments with self-consistent N-body models of disc galaxies of low central concentration that appear to be strongly bar unstable. There is clear evidence of strong orbital diffusion driven by the bar, but no evidence that this diffusion is enhanced by spiral structure. Indeed, in their model of an Sa galaxy, which shows the strongest diffusion, the pattern speed of the bar decreases by $\sim 20\%$ over the short lifetime of the simulation, so the system is never in any danger of settling to a steady configuration in phase space.

8.2.3. Heating of the solar neighbourhood

It has been known since the 1950s that near the Sun younger stars tend to have smaller random velocities than older stars. Spitzer & Schwarzschild (1953) recognised that this phenomenon reflected stochastic acceleration of disc stars by the Galaxy’s fluctuating gravitational field. The Hipparcos data permitted beautiful quantification of the phenomenon, and it emerges that the velocity dispersion of a stellar cohort increases with age as $\sigma \sim t^{0.35}$ (Aumer & Binney, 2009).

It’s instructive to infer from this result how the diffusion coefficients must scale with $|J|$. We make two simplifying assumptions: (i) that the dominant scatterers are much more massive than stars, and (ii) that the velocity dis-
persions of groups of stars scale with the mean actions in the group as

\[ \sigma_r \propto \sqrt{\langle J_r \rangle} \quad \text{and} \quad \sigma_z \propto \sqrt{\langle J_z \rangle}. \] (61)

These relations are exact in the epicycle approximation, in which the radial and vertical oscillations of stars are harmonic, so for example \[ J_r = E_R / \kappa. \] Since scattering must be dominated by giant molecular clouds and spiral arms, the assumption of massive scatterers will be a good one. In thermal equilibrium with such massive bodies, stars would have velocity dispersions that are larger than those of the clouds and arms (\( \sim 7 \text{ km s}^{-1} \)) by the square root of the ratio of masses, so the stars’ velocity dispersion would be > 1000 km s\(^{-1}\). Consequently, we can use equation (57) in the limit of infinite temperature, when the action-space flux becomes

\[ F_i = -\frac{1}{2} \Delta_{ij} \frac{\partial f}{\partial J_j} \] (62)

so the Fokker-Planck equation simplifies to

\[ \frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J_i} \left( \Delta_{ij} \frac{\partial f}{\partial J_j} \right). \] (63)

Stars are born on orbits that have non-negligible angular momenta \( L_z \equiv J_\phi \) but small values of \( J_r \) and \( J_z \). Consequently, a young population is initially distributed in action space along the \( L_z \) axis, and diffusion of this population is predominantly away from this line, towards larger values of \( J_r \) and \( J_z \). For this reason we neglect derivatives with respect to \( J_\phi \) in equation (63).

On the line \( J_r = J_z = 0 \), the flux of stars \( F \) (eq. 62) must have vanishing \( r \) and \( z \) components to prevent stars diffusing out of the physical region \( J_r \geq 0, J_z \geq 0 \). As a consequence of this requirement, in the vicinity of the \( J_\phi \) axis the tensor \( \Delta^2 \) varies rapidly with \( J_r \) and \( J_z \). We neglect the much weaker dependence of \( \Delta^2 \) on \( J_\phi \) and consider \( \Delta^2(j) \) to be a function \( \Delta^2(j) \) of the two-vector

\[ j \equiv (J_r, J_z). \] (64)

\(^4\)Quite generally we have that \( \Omega_r J_r \) is equal to the time-averaged value of \( v_R^2 \) along any orbit.

\(^5\)See Appendix B of Binney & Lacey (1988) for a rigorous justification of this step.
In problems involving the ordinary diffusion equation, a key solution is the Green’s function \( \exp(-x^2/2t)/(2\pi t)^{1/2} \), which describes the spatial distribution at time \( t \) of particles injected at \( x = 0 \) at time \( t = 0 \). Analogously, we seek a Green’s function of the form

\[
f = t^{-2a} f_0(X) \quad \text{where} \quad X \equiv \frac{j}{t^a}.
\]

In the solution (65) the mean value of \( |j| \) will increase with time as \( t^a \), and the power of \( t \) multiplying \( f_0 \) ensures that the total number of stars \( \int dL_z \int d^2j f \) is conserved as stars diffuse from the axis. Suppose \( \Delta^2 \) scales such that \( \Delta^2(kj) = k^b \Delta^2(j) \). Then putting \( k = t^{-a} \) we have \( \Delta^2(X) = t^{-ab} \Delta^2(j) \).

Evaluating both sides of equation (63) with these assumptions yields

\[
-\frac{1}{t^{2a+1}} \left( 2af_0 + aX \cdot \frac{\partial f_0}{\partial X} \right) = \frac{1}{2} t^{a-4a} \frac{\partial}{\partial X_i} \left( \frac{\Delta^2(X)}{\partial X_j} \frac{\partial f_0}{\partial X_j} \right).
\]

This equation can be valid at all times only if \( 2a+1 = 4a - ab \), so \( b = 2 - 1/a \). Consequently, the empirical result \( \langle j_r \rangle \sim \sigma_r \sim t^{2/3} \) implies \( a = b = 1 \). A simple argument shows that it is implausible for the diffusion coefficients to grow so rapidly with \( |j| \).

In the epicycle approximation, \( J_r \) differs from the epicycle energy \( E_R \) only by the (constant) epicycle frequency, so \( \Delta_r \sim \Delta E_R = v \cdot \delta v \), where \( \delta v \) is the projection into the equatorial plane of the change in a star’s velocity as a result of a scattering event. Hence \( \langle \Delta^2 \rangle = |j| \) implies

\[
E_R \sim \langle (\Delta E_R)^2 \rangle \sim \langle (v \cdot \delta v)^2 \rangle \sim \langle E_R |\delta v|^2 \rangle.
\]

That is, \( \sigma_r \sim t^{1/2} \) implies that \( |\delta v| \) is independent of \( |v| \). However, gravitational scattering always causes the momentum change \( \delta v \) to decrease with increasing speed because the gravitational force is independent of speed and the time for which it acts decreases as \( 1/|v| \).

Can we derive \( \Delta^2(kj) \sim k^{1/2} \Delta^2(j) \) from physics? Binney & Lacey (1988) show that this scaling is predicted by the model of cloud-star scattering that was introduced by Spitzer & Schwarzschild (1953). In this model the clouds move at speed \( v_c \) on circular orbits and the stars are on orbits of non-zero eccentricity. The scattering of the star by the cloud is assumed to be essentially complete in a distance that is small compared to the epicycle
radius – thus the impact parameters of star-cloud collisions need to be small compared to epicycle radius. With these restrictions, the scattering can be analysed in the rotating frame in which the cloud is stationary. In this frame the Jacobi constant $\frac{1}{2}v^2 + \Phi_{\text{eff}}$ is invariant, so the speed $v$ with which the star recedes from the cloud is equal to its approach speed: in this frame the cloud merely deflects the star. Such deflections have two important effects. First, they can transfer energy between motion within the plane and motion perpendicular to the plane. Second, by increasing $v_\phi$ at the expense of $v_R$, or vice versa, they can change the magnitude of the epicycle energy $E_R$, which can be written (Binney & Tremaine, 2008, §3.2.3)

$$E_R = \frac{1}{2}v^2_R + \frac{1}{2}\gamma^2(v_\phi - v_c)^2,$$

(68)

where $\gamma \equiv 2\Omega_\phi/\kappa$ is the twice ratio of the circular and epicycle frequencies. Since $\gamma^2 > 1$, a deflection that decreases $v_R$ and increases $|v_\phi - v_c|$ increases $E_R$, and conversely for deflections that increase $v_R$. It turns out that deflections that decrease $v_R$ predominate, so overall deflections increase random velocity. It’s worth noting that the energy required to heat the disc comes not from the clouds but from the ample store of energy in the rotation of the stellar disc: the clouds catalyse an outward transfer of angular momentum by absorbing $L_z$ from stars that have guiding centres interior to their orbits and handing it out to stars with larger guiding centres.

Unfortunately, the demonstration by Binney & Lacey (1988) that the Spitzer-Schwarzschild model implies that $\Delta_{ij}^2(j) \propto |j|^{1/2}$ is defective in two respects: (i) it assumes that the relative velocity with which a star encounters a cloud is dominated by epicycle motion rather than differential rotation, and, more seriously, (ii) it assumes that stars are confined to the equatorial plane. In reality as a star ages it oscillates with increasing amplitude and period perpendicular to the plane, and these oscillations decrease its probability of being scattered by a cloud. Consequently, when this effect is taken into account, $\Delta_{ij}^2(j)$ increases with $|j|$ more slowly than as $|j|^{1/2}$.

Binney & Lacey (1988) show that three-dimensional scattering by molecular clouds generates a tensor of diffusion coefficients $\Delta_{ij}^2$ which is highly anisotropic. The consequence of this anisotropy is that we expect $\sigma_z/\sigma_r \sim 0.8$, which is significantly larger than the observed value, $\sim 0.6$. Sellwood (2008) argues that the discrepancy arises from the erroneous assumption of an isotropic distribution of encounters: as in two-body scattering, distant encounters are important, and since both stars and clouds lie within the disc,
distant encounters are dominated by the velocity components that lie within the plane and do not change $J_z$.

Thus it seems that scattering of stars by giant molecular clouds may set the ratio of the vertical and horizontal velocity dispersions of disc stars. While star-cloud scattering makes a significant contribution to the secular increase in the velocity dispersions of stars, it probably cannot account fully for the data because its effectiveness declines rapidly with increasing velocity dispersion and thus cannot account for the numbers of stars with radial dispersions $\gtrsim 30 \text{ km s}^{-1}$.

9. Chemodynamical evolution

It is more than half a century since it emerged from the work of Roman (1950) and others that when stars are divided by chemistry, the different groups have distinct kinematics. This phenomenon arises because age and chemistry are correlated through the metal-enrichment history of the ISM.

Abundances correlate with ages in two principal ways. Stars formed in the first $\sim \text{Gyr}$ have enhanced abundances of $\alpha$ elements (Mg, Ca, Si, etc.) relative to iron because deflagration supernovae (SN Ia) are major producers of Fe and, being the endpoints of binary-star evolution, they have a gestation periods $\gtrsim 1 \text{ Gyr}$ (e.g. Förster et al., 2006). In addition to a rapid decline in $[\alpha/\text{Fe}]$, we expect a longer-term increase in $[\text{Fe/H}]$. To the extent that all the stars of the thin and thick discs formed near the plane, where the star formation is certainly now concentrated, the number pair ($[\alpha/\text{Fe}], [\text{Fe/H}]$)
should encode the radius and time of a star’s formation because plots of \([\alpha/Fe]\) and \([Fe/H]\) as functions of \(t\) will have different forms at different radii (Fig. \(6\)), so from a star’s value of \([\alpha/Fe]\) we can argue that its time and place of birth, \((t_b, R_b)\) lie on a given curve in \((t, R)\) space. The star’s value of \([Fe/H]\) restricts \((t_b, R_b)\) to a different curve, so \((t_b, R_b)\) can be inferred from the intersection of these curves.

Unfortunately, this inference can be made only after the surfaces sketched in Fig. \(6\) have been computed with a model of chemical evolution, and, as we described in §4, such models are subject to many uncertainties. The general idea of a chemical-evolution model is that stars form on near-circular orbits at rates determined by the local ISM density. After birth the stars diffuse through action space. The more massive stars will at some point reach the ends of their lives and eject much or all of their mass as metal-enriched gas, and perhaps leave a degenerate remnant. A portion of the gaseous ejecta will eventually join the cold ISM and modify the chemical composition of subsequent generations of stars. To compute such a model one first modifies the Fokker-Planck equation (53) to include a source of stars on circular orbits. That is, one writes

\[
\frac{\partial f}{\partial t} + \frac{\partial F_i}{\partial J_i} = B(L_z, t)\delta(J_r)\delta(J_z),
\]

where \(B(L_z, t)\) specifies the rate at which stars form at the radius \(R_c(L_z)\) of the circular orbit of angular momentum \(L_z\). This rate is expected to be determined by the surface density of gas at radius \(R, \Sigma_g(R, t)\). One needs to solve equation (69) for a \(df\) that depends on chemical variables and age \(\tau\) in addition to \(J\), and from the developing solution and the theory of stellar evolution determine the radial distribution of ejection of H, He, Fe and \(\alpha\) elements at each time and use these data to update the chemistry of the ISM at each radius.

As we discussed in §4, it is incredibly hard to predict which mixture of heavy elements will over time be ejected from a cohort of stars because it hinges on details the life and death of massive single stars and less massive binary stars that are extremely complex and delicate. This fact is most unfortunate for these “yields” of heavy elements are key ingredients of any model of chemodynamical evolution.

Another important ingredient is the diffusion tensor \(\nabla^2\) that determines the action-space flux (eq. 62). The classical work on chemical evolution (Pagel & Patchett, 1974; Ferrini et al., 1992) restricted attention to the solar
cylinder, and later work (e.g., Chiappini et al., 2003) neglected diffusion in $L_z$. These restrictions greatly diminish the realism and predictive power of the models. Schönrich & Binney (2009) extended the classical work to include diffusion in orbit space, but did not use action integrals and used a very crude scheme for computing changes in inclination. This work needs to be revisited using a more rigorous and transparent approach to diffusion through action space.

The third and most uncertain ingredient is the dynamics of the ISM. A key conclusion of the earliest studies of chemical evolution was that the distribution in [Fe/H] of local long-lived stars is inconsistent with the solar neighbourhood having started with a fixed stock of gas that then turned into stars as a closed system (van den Bergh, 1962). The data then available could be explained either by a limited initial gas supply being constantly topped up by fresh, metal-poor gas, or by the original gas being pre-enriched by stars in the bulge and/or halo (Pagel & Patchett, 1974; Ostriker & Thuan, 1975). An indication that there has been accretion of metal-poor gas is that the star-formation rate near the Sun seems to have fallen by only a factor $\sim 3$ in the $\sim 10$ Gyr life of the thin disc (Aumer & Binney, 2009), and given that the star-formation rate must be a sensitive function of the density of the cold ISM it implies that the mass of cold ISM has decreased by no more than a factor 3 over that time. This being so, there simply wasn’t enough mass in the ISM 10 Gyr ago to form the present disc: much of the mass now in stars must have been accreted. This line of argument is strongly supported by measurements of the density of deuterium in the interstellar medium because, although deuterium is rapidly destroyed within any star, the abundance in the current ISM is lower than the primordial abundance by only a factor $\sim 1.4$ (Podanovic et al., 2010). This result clearly indicates that the ISM is constantly accreting material with a near-primordial deuterium abundance. Long-term accretion by halos is very much in line with the predictions of CDM Cosmology, but there is some confusion whether accretion is predominantly smooth or lumpy (i.e., associated with mergers). L’Huillier et al. (2012) find that 77% of accretion onto Milky-Way type halos is smooth and only 23% lumpy. A major issue is whether smooth accretion delivers baryons in a cold flow ($T \lesssim 10^4$ K) or via a galaxy’s virial-temperature corona. Marasco et al. (2012) suggest how gas may be accreted from the corona.

Bilitewski & Schönrich (2012) have investigated the distribution of accreted gas in $L_z$, which must control the radial distribution of infall. The
essential point is that the gas cannot all be accreted at large radii, outside
the optical disc, but must accrete to radii comparable to those at which most
star formation takes place.

Another key uncertainty for chemical evolution models is the extent to
which gas spirals inwards through the stellar disc. The spiralling of gas
in through most of the disc is an inevitable consequence of gas shocking
downstream of spiral arms, for then the surface density of gas has a phase
lag with respect to the density distribution of stars, and this phase lag causes
the gravitational field of the stars to drain angular momentum from the gas.
Most of the energy released by this accretion is radiated by the post-shock
gas.

When gas arrives just outside the corotation radius of the bulge/bar, it
can acquire angular momentum from the bar, so there is a tendency of gas to
pile up just outside the bar. Gas that crosses the bar’s corotation radius loses
most of its angular momentum in a few dynamical times and is then dumped
in the Central Molecular Zone, a disc of radius $\sim 200$ pc made up of dense,
largely molecular gas on $x_2$ orbits (Launhardt et al., 2002; Kim et al., 2011).
This is a region of intense star formation as is evidenced by the plethora of SN
remnants seen in radio continuum maps (Kassim et al., 1999) and the X-ray
emitting wind that emanates from this region (Bland-Hawthorn & Cohen,
2003).

One way of addressing these many uncertainties is to perform N-body
plus hydrodynamical simulations of the formation and chemical evolution of
a galaxy (e.g. Brook et al., 2012; Loebman et al., 2011). Ideally such simula-
tions would be embedded in a full cosmological context, but such simulations
are still not capable of resolving the detail that is crucial for studies of the
solar neighbourhood and the adjacent Galaxy ($\S$ 4).

10. Spiral structure

To this point we have largely ignored the self-consistency issue when
discussing perturbations: we have proceeded under the assumption that the
perturbing potential $\Phi_1$ is a given function of space-time. Ultimately it is
necessary to recognise that $\Phi_1$ is generated by the perturbation $\rho_1$ in
the mass density, but extending the mathematics to take this constraint into
account enormously increases the complexity of the problem. In particular,

it requires one to follow the dynamics of all the Galaxy’s many components,
not just that of the particular stars that you are observing.
The self-consistency constraint is of particular importance in connection with the dynamics of the thin disc. Fortunately, this is the area in which significant progress was made half a century ago through the work of Kalnajs, Lin, Lynden-Bell, Toomre and their collaborators. From their work it emerged that spiral disturbances, with their displacements either parallel to the plane (forming spiral arms) or perpendicular to the plane (bending waves) propagate radially. Bending waves simply propagate outwards with ever-increasing amplitude as the disc’s surface density declines. Eventually their energy is converted into heat in the form of a flared outer disc (Hunter & Toomre, 1969).

The situation regarding waves with displacements parallel to the plane is much more complex (Toomre, 1981). These waves are excluded in a zone around the CR, and outside this exclusion zone, two wavenumbers are permitted at a given frequency – there are short- and long-wave branches of the dispersion relation. Waves rattle between a Lindblad resonance and the edge of the forbidden zone around CR, steadily winding up as they travel: a short-leading wave becomes first a long-leading wave, then a long-trailing wave and finally a short-trailing wave before the wave finally thermalises its energy at a Lindblad resonance. As the wave transitions from leading to trailing form at a Lindblad resonance, the swing amplifier boosts its amplitude by a factor that can be large. Since in a steady state the leading waves in a disc have amplitudes that are smaller by this factor than those of trailing waves, observed spirals have trailing morphology overall. Moreover solutions to the normal-mode equations have their largest amplitudes inside CR (Toomre, 1981), and this is why spirals in the disc surrounding the bar are constantly overtaken by the bar (§ 6).

The results just described depend on the tight-winding approximation: that the radial wavelength $\lambda_r \ll R$. Unfortunately, this is a poor approximation on the long branches of the dispersion relation. In view of this problem, the logical next step is the construction of global normal modes by using angle-action coordinates to transform the normal-mode problem into a problem in linear algebra. This has been done for a few tractable discs (Kalnajs, 1977; Read & Evans, 1998) with the conclusion that the picture of a normal mode as a superposition of trapped spiral waves is fundamentally sound.

The pioneers of the theory of spiral structure hoped to be able to obtain a complete understanding of the dynamics of discs in terms of waves and normal modes (Lin & Shu, 1966). This hope has since been dashed by
numerical experiments, which show that discs that have no unstable normal modes, nonetheless demonstrate complex spiral instabilities that, given long enough, grow from Poisson noise to $O(1)$ amplitudes and ultimately lead to the formation of a strong central bar (Sellwood, 2012). The explanation of this phenomenon is probably as follows. In the case of an electrostatic plasma it is known that the set of ordinary normal modes is not complete (van Kampen, 1955). That is, it is not possible to express an arbitrary initial condition as a linear combination of normal modes. So one cannot argue that a small initial disturbance will stay small from the absence of growing modes. Gravitational plasmas presumably share this incompleteness property, so a knowledge of the stability of the ordinary normal modes can be used only to argue that the system will be unstable if it has an unstable normal mode, and not that it will be stable if all normal modes are stable.

The bottom line is that stellar discs are responsive dynamical systems because they support waves that can be amplified by self gravity as they move through the disc. The degree of amplification, and therefore the disc’s responsiveness, increases sharply as the velocity dispersion falls towards the critical value at which Toomre’s stability parameter (Toomre, 1964)

$$Q \equiv \frac{\sigma R \kappa}{3.36 G \Sigma}$$

falls to unity and the disc becomes unstable to radial fragmentation. Much of the energy carried by the waves is thermalised in the vicinity of a Lindblad resonance. Thus the waves heat the disc and render it less responsive.

10.1. Driving spiral structure

By counting faint stars in the outer reaches of both our Galaxy and the Andromeda nebula, M31, it has been shown that the outer parts of galaxies are a mass of stellar streams and full of faint satellite galaxies (McConnachie et al., 2009; Belokurov et al., 2006; Bell et al., 2008). From studies of the internal dynamics of satellite galaxies, we know that these systems are heavily dominated by dark matter, so we must anticipate that the dark-matter distribution that surrounds a galaxy like the Milky Way is lumpy. When a lump of dark matter sweeps through pericentre, its tidal field will launch a wave into the host galaxy’s disc, which we know to be a responsive system. The classic example of this process is M51, which has a satellite galaxy, NGC 5195, near the end of one of its exceptionally strong spiral arms. Few galaxies have such a luminous satellite so near to them,
so grand-design spirals like M51 are not prevalent. Most galaxies will be responding simultaneously to more than one much weaker stimulus, with the result that their spirals are both weaker and rather chaotic.

Masset & Tagger (1997) present convincing evidence that a bar excites a spiral wave in the surrounding disc that has ILR at the same radius as the bar’s CR – this is the real phenomenon of resonance overlap. Hence the Galactic bar must be important for the Galaxy’s spiral structure. In §6 we saw that the the “long thin bar” inferred by Cabrera-Lavers et al. (2008) is nicely explained by the more slowly-rotating spiral wave that the bar excites in the surrounding disc. Also the “molecular ring” that dominates the longitude-velocity plot of the Galaxy’s CO emission (Dame et al., 2001) quite likely consists of two giant spiral half-turns (Dobbs & Burkert, 2012) that the bar generates in the gas disc.

11. The warp

The early surveys of the Galaxy in neutral hydrogen revealed that the gas disc is warped (Burke et al., 1957; Kerr, 1957; Westerhout, 1957). Later it emerged that the gas discs of external spiral galaxies are routinely warped (Bosma, 1978; Briggs, 1990), so the phenomenon must be a generic outcome of disc dynamics. Warps in stellar discs are much harder to detect than warps in gas discs on account of the difficulty of obtaining precise measurements of the stellar velocity field, but it is clear that the Galaxy’s stellar disc is warped (Dehnen, 1998; Juric et al., 2008).

In the simplest approach to the dynamics of warps, we imagine that the disc is made up of a series of rigid, spinning rings. The centre of each ring is fixed, but can choose its own orientation in response to the torques exerted by the ambient gravitational field. The latter is in part generated by the rings themselves, and in part generated by the galaxy’s halo and bulge. Each ring intersects the galaxy’s equatorial plane in two nodes on opposite sides of the galaxy, and if the rings form a thin disc, the set of all nodes traces a smooth curve through the galactic centre that is called the line of nodes. Briggs (1990) reports that the line of nodes is reasonably straight out to $\sim 3R_d$ and then becomes an open leading spiral. A seductive picture of a warped galactic disc is one in which the shape of the disc is always the same, but the line of nodes steadily rotates because all the rings precess at the same rate. In the language of linear analysis, the disc then has an excited $m = 1$ normal mode.
It is helpful to consider the dynamics of the rings in two limiting special cases. Consider first the case in which the rings are massless so the only torques on them come from the gravitational field of the halo and bulge. Then each ring precesses independently of the other rings at a frequency that in general decreases outwards (being of order a fraction of the circular frequency at the ring’s radius) and increases with the flattening of the gravitational field. Since $\Omega_z > \Omega_\phi$, the precession is in the opposite sense to that in which stars circulate. Unless the flattening increases with radius in a very specific way, the precession frequency will vary with radius, with the consequence that the line of nodes will wind up into an ever tighter spiral. Hence a normal mode is not possible with massless rings.

Consider next the opposite case of very massive rings in which we can neglect the contributions of bulge and halo to the gravitational field. Then the rings form a system of coupled particles analogous to a chain of particles that are linked by springs. Hence bending waves can propagate through the disc, and we should ask whether normal modes can be constructed by waves of a suitable frequency propagating from centre to the edge and then back, returning to their point of departure with their original phase. The answer proves to be (Hunter & Toomre, 1969) that such standing-wave patterns are possible only if the disc’s outer edge is unrealistically sharp. In the absence of a sharp edge, the wavelength of an outwards propagating wave decreases, and its amplitude grows, in an ever more painful attempt to transmit a constant flux of mechanical energy outwards through a medium of ever decreasing density (the whiplash effect). When the wavelength become comparable with the typical amplitude of radial oscillations of the disc’s stars, the wave energy will be thermalised and serve to increase the disc’s thickness. Thus the outer boundary of a disc with a realistic density profile absorbs rather than reflects bending waves, and normal modes cannot arise.

While these considerations show that a steadily precessing warped disc can arise neither when the rings are very massive nor when they are very light, for some years it seemed that steady precession would be possible in the intermediate regime of moderately massive rings. To understand this case, imagine ramping up the mass of a system of initially massless rings. Before you ramp up the mass, the inner rings are precessing (backwards) faster than the outer rings, and on account of the phase lag that develops between inner and outer rings and an initially straight line of nodes will become a leading spiral. When you make the mass non-zero, a torque will act between the inner and outer rings. In the right circumstances this torque
can slow the precession of the inner rings and hasten that of the outer rings in just the way required to set all rings precessing together with the line of nodes remaining straight (Sparke & Casertano, 1988).

The problem with this model is that it treats the contribution of the bulge and halo to the galactic potential as fixed. In reality the halo/bulge experiences an equal and opposite force to that which it applies to the inner rings, and it will respond dynamically to this force. N-body experiments show that the response comes quite quickly, and tends to cause the bulge/halo to align at each radius with the disc rings of comparable radius. Consequently, if you set up a simulation of a system of rings embedded in an N-body halo in what would be a normal mode if the bulge/halo were unresponsive, the torques on rings from the bulge/halo quickly diverge from the values required by the mode, and the line of nodes winds up on a dynamical timescale (Binney et al., 1998).

Dubinski & Chakrabarty (2009) assumed that the disc would be aligned with the shortest principal axis of the inner halo (defined by \( r < r_s \) or \( r < 2r_s \) where \( r_s \) is the scale radius of the NFW halo) and determined the quadrupole component of the (interior) gravitational potential due to the outer halo. They analysed \( \sim 2000 \) halos in this way to define the statistics of the quadrupoles. Then they studied the response of an N-body disc to typical quadrupoles. They found that the disc precessed and warped. The precession of the N-body discs could be largely reproduced using rigid, spinning discs, so confirming that the self-gravity of the inner galaxy is strong enough to render it equivalent to a rigid body right out to the edge, where warping takes place. The amplitudes of the warps and their morphologies were entirely consistent with Briggs (1990). This work makes a very strong case that warps are caused by misalignments between the inner and outer parts of dark-matter halos. In doing so the study poses a serious problem for theories of modified gravity, for in these theories any quadrupole in the gravitational potential would be generated by the inner galaxy (where all mass would reside) and would consequently be aligned with the inner galaxy, so warps would not arise.

The misalignment of inner and outer halos is a natural consequence of evolution of the pattern of accretion over cosmic time: the inner halo formed long ago from an accretion pattern that reflected the structure of the cosmic web at that time, while the outer halo is forming now and reflects the current accretion pattern. Indeed, standard cosmology predicts that the direction of angular momentum vector of accreting material slews over cosmic
time through at least a radian (Quinn & Binney, 1992). In exceptional circumstances this slewing can create a disc in which roughly half the stars rotate one way and half the other, as is observed in the lenticular galaxy NGC 4550 (Rubin et al., 1992). This outcome is unusual, however, because when the angular momentum vector of accreting gas slews gradually, by exchanging a small amount of angular momentum with the stellar disc, gas will settle to the equatorial plane of the stellar disc and corotating with the stars. In response to the exchange of angular momentum with the gas, the spin axis of the stars shifts by a small amount in the sense of reducing the offset between the disc’s angular momentum vector and that of the recently accreted gas. Through this mechanism we expect the angular momentum vector of the stellar disc to remain roughly aligned with that of currently infalling gas, so the latter always ends up corotating with the stars. Nevertheless, freshly accreted material, baryonic and dark, will have an angular momentum vector that is not parallel to that of the inner Galaxy, and this offset naturally gives rise to a warp in the outer disc (Jiang & Binney, 1999).

In fact gas discs within cosmological simulations of galaxies frequently have warped outer zones.

12. Outlook

The study of our prototypical Galaxy is now recognised as a central task of contemporary astrophysics. The challenge is to synthesise a huge body of observational data from many different surveys at wavelengths from 21 cm to X-rays and beyond into a coherent picture of gas, stars and dark matter. Dynamical models are the key to this enterprise because they provide the mechanism by which we can combine constraints from disparate surveys, and map out the Galaxy’s gravitational field. The last task is both crucial and challenging: crucial because it provides our only realistic means to determine the distribution of dark matter, which provides both the Galaxy’s backbone and our surest connection to cosmology, and challenging because the dark-matter density is the divergence of the gravitational field, so we have to determine the latter with sufficient precision to differentiate it. A significant help in this enterprise should be the fact that the Galaxy comprises many chemically distinct populations, each with its own characteristic dynamics. Each such population probes the common gravitational field in a different way, so chemo-dynamical models should prove powerful diagnostics of the dark-matter distribution.
The Galaxy is a complex machine, which inhabits a noisy environment. We need to understand how it responds to that noise both because those responses will impact the observational data, and because they contribute to driving the Galaxy’s evolution. The standard tool for computing the response of a dynamical system is perturbation theory. To date perturbation theory has played a rather modest role in Galactic dynamics, and if we are to make good progress, this must change. In the eighteenth and nineteenth centuries the best minds in mathematical physics struggled with the perturbative dynamics of the solar system, and mastered it by inventing angle-action coordinates. Since these coordinates provide the key to progress in Galactic dynamics, they have been given prominence in this article.

Our first task is to learn how the Galaxy is currently structured and functions. Once that has been accomplished, it will be time to move on to the still more challenging task of understanding how it arrived at its present state from the cosmological initial conditions, which we believe we know. At this stage close interplay between perturbation theory and brute-force simulations is likely to be very productive. It is unrealistic to hope that we will ever have a cosmological simulation that yields a satisfactory representation of the current Galaxy. The role of simulations is rather to help us understand how different processes function and interact. If other branches of physics are useful guides, we will obtain this understanding by modelling simulations with perturbation theory, for perturbation theory has historically been the language of physics through which we develop understanding. In fact, perturbation theory will serve as intermediary between our actual Galaxy and many different simulations, none of which is quite the same as the Galaxy, but some of which resemble it in statistical measures that will be defined by perturbation theory.

With the launch of the Gaia satellite in late 2013 and data from complementary ground-based surveys such as the Gaia-ESO, APOGEE and Galah surveys, our empirical knowledge of our Galaxy will take a quantum leap. Much remains to be done before we can properly exploit the mines of data that these surveys will produce.

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