ATTRACTORS FOR WAVE EQUATIONS WITH NONLINEAR DAMPING ON TIME-DEPENDENT SPACE

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Abstract. In this paper, we consider the long time behavior of the solution for the following nonlinear damped wave equation

$$
\varepsilon(t)u_{tt} + g(u_t) - \Delta u + \varphi(u) = f
$$

with Dirichlet boundary condition, in which, the coefficient $\varepsilon$ depends explicitly on time, the damping $g$ is nonlinear and the nonlinearity $\varphi$ has a critical growth. Spirited by this concrete problem, we establish a sufficient and necessary condition for the existence of attractors on time-dependent spaces, which is equivalent to that provided by M. Conti et al.[10]. Furthermore, we give a technical method for verifying compactness of the process via contractive functions. Finally, by the new framework, we obtain the existence of the time-dependent attractors for the wave equations with nonlinear damping.

1. Introduction. In this paper, we consider the asymptotic properties of the dynamical system generated by the following wave equation with nonlinear damping on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial \Omega$,

$$
\begin{cases}
\varepsilon(t)u_{tt} + g(u_t) - \Delta u + \varphi(u) = f(x), & t > \tau, \tau \in \mathbb{R}, \\
u \mid_{\partial \Omega} = 0, \\
u(x, \tau) = u_0, & u_t(x, \tau) = u_1,
\end{cases}
$$

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where the unknown variable \( u = u(x, t) : \Omega \times [\tau, \infty) \to \mathbb{R} \) and \( u_0, u_1 : \Omega \to \mathbb{R} \) are assigned data, \( f \in L^2(\Omega). \)

Equation (1.1) arises as an evolutionary mathematical model in various systems for the relevant physical application. In particular, in the case that \( \varepsilon(t) \) is a positive constant function, equation (1.1) models a continuous Josephson junction with specific \( g, \varphi, f, \) etc. see [22, 32, 34] and the references therein.

It is well known that when \( \varepsilon(t) \) is a positive constant function, the system (1.1) becomes autonomous wave equation with nonlinear damping, and then one can use the solution operator to define a semigroup. Hence the long-time behavior of the solutions can be well characterized by using of the concept of global attractors in the framework of semigroup. The existence and the properties of the global attractor for (1.1) have been extensively studied in [5, 7, 8, 9, 15, 16, 17, 19, 21, 23, 27, 30, 33] etc.

When \( \varepsilon(t) \) is a positive constant function, meanwhile \( f \) depends on time, the system (1.1) is non-autonomous wave equation with nonlinear damping, and now the solution operator defines a process and constructs a skew product flow of the process, which made it possible to reduce the non-autonomous systems to autonomous ones in an extended phase space. Then the natural extension of the notion of global attractor to non-autonomous case is the concept of the so-called uniform attractor (see [3, 4]). However, one disadvantage of this uniform attractor is that it need not be invariant unlike the global attractor for autonomous systems. At the same time, the theory of pullback (or cocycle) attractors has been developed for both the non-autonomous and random dynamical systems (see [13, 14, 31], etc.). Consequently, the uniform attractor and pullback attractor are appropriate concepts, as we know, for the process in understanding the longtime behavior of the non-autonomous dynamical systems. In [34] and [32], the authors considered the existence of uniform attractor and pullback attractor for system (1.1) respectively.

On the other hand, when \( \varepsilon(t) \) depends on time (not a constant function), such as a positive decreasing function of time \( \varepsilon(t) \) vanishing at infinity, i.e. the coefficient of the differential operator depend on time explicitly, the classical theory (global attractor, uniform attractor, pullback attractor, etc.) generally fails to capture the dissipation mechanism, as mentioned in [10, 28].

To circumvent these issues, firstly, in [28], Di Plinio, Duane and Temam described the solution operators, which still be called a process, as a family of maps

\[
U(t, \tau) : X_\tau \to X_t, \ t \geq \tau \in \mathbb{R},
\]

acting on a time-dependent family of spaces \( X_t \), i.e. the norm of the space depends on the time explicitly. Then, the authors established a new theory of pullback flavor for dynamical systems by adopting a new point of view on pullback dissipativity. Furthermore, a suitable modification of the notion of pullback attractor, i.e. time-dependent global attractor which is a family of compact subsets satisfying invariance and pullback attracting w.r.t. \( X_t \) norm was presented.

Recently, in [10], combining with the perspective in [4], Conti, Pata and Temam recovered and improved the results in [28] by giving new insights on attractors on time-dependent space. They used the minimality to replace the invariance in the concept of the time-dependent global attractor. Furthermore, they showed that such an attractor (smallest family of compact subsets satisfying pullback attracting) is invariant whenever the process satisfies some kind of continuity. Meanwhile, within the new framework, the authors in [10] studied the longterm behavior of system
when \( g(u_t) = \alpha u_t \) (weak damping wave equation) and obtained the existence of the time-dependent global attractor, which converges in a suitable sense to the attractor of the parabolic equation \( \alpha u_t - \Delta u + f(u) = 0 \) (see [11]). Moreover, in [12], Conti, Pata considered a specific one-dimensional wave equation \( \varepsilon(t)u_{tt} - u_{xx} + [1 + \varepsilon f'(u)]u_t + f(u) = h \), they proved the existence of an invariant time-dependent attractor, which converges in a suitable sense to the classical Fourier equation.

In this paper, we will consider the system (1.1), where \( \varepsilon(t) \) depends on time explicitly, the damping \( g \) is nonlinear, and the nonlinearity \( \phi \) has a critical exponent.

Our basic assumptions about the coefficient \( \varepsilon(t) \), the nonlinear damping \( g \) and the nonlinearity \( \phi \) are as follows.

**Assumptions on \( \varepsilon \).** \( \varepsilon \in C^1(\mathbb{R}) \) is a decreasing bounded function and satisfies
\[
\lim_{t \to +\infty} \varepsilon(t) = 0. \tag{1.2}
\]
In particular, there exists \( L > 0 \) such that
\[
\sup_{t \in \mathbb{R}} [\varepsilon(t) + |\varepsilon'(t)|] \leq L. \tag{1.3}
\]

**Assumptions on \( g \).**
\[
g \in C^1(\mathbb{R}), \ g(0) = 0, \ g \text{ is strictly increasing,} \tag{1.4}
\]
\[
\lim_{|s| \to \infty} g'(s) > 0, \tag{1.5}
\]
\[
|g(s)| \leq C_0(1 + |s|^p), \tag{1.6}
\]
where \( 1 \leq p < 5 \).

**Assumptions on \( \phi \).** \( \phi \in C^1(\mathbb{R}) \) satisfies the growth condition
\[
|\phi'(s)| \leq C(1 + |s|^2), \tag{1.7}
\]
and dissipation condition
\[
\lim_{|s| \to \infty} \frac{\phi(s)}{s} > -\lambda_1, \tag{1.8}
\]
where \( \lambda_1 > 0 \) is the first eigenvalue of the strictly positive Dirichlet operator \( A = -\Delta \) with domain \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \) on \((L^2(\Omega), \langle \cdot, \cdot \rangle, \| \cdot \|)\).

In fact, the assumptions on the coefficient \( \varepsilon \) follow from [10], the assumptions (1.4)-(1.8) on \( g \) and \( \phi \) are the same as those in [32, 33, 34].

Whether the system is autonomous or non-autonomous, to establish the compactness of the system in some sense is the key point to prove the existence of attractors. In the case where \( \varepsilon(t) \) is a positive constant function, the difficult for obtaining compactness mainly comes from nonlinear damping and the nonlinear terms. In 2006, Sun etc.[33] firstly overcome this difficulty when system is autonomous (\( f \) did not depend on \( t \)), without assuming a large value for the damping parameter, and then they proved the existence of the global attractor for system (1.1) under the conditions that the growth order \( p \) of nonlinear damping term must be less than 5 and the growth order of nonlinear term can equal to 3, in which the exponent about \( \phi \) is called critical exponent since the nonlinearity \( \phi \) is not compact directly using Sobolev embedding. Later on, in [19], the author improved the result in [33] and obtained the existence of the global attractor in the case of \( p = 5 \), by estimating the boundedness of \( \int_0^T \int_{\Omega} g(u_t)u_t \), where the boundedness is independent of time. On the other hand, when \( f \) depends on \( t \), the case of \( p = 5 \), up to now, still
be open as we know, the main reason is that we can only get 
\( \int_0^T \int_\Omega g(u_t)u_t \leq C_T \),
where \( C_T \) depends on the time \( T \) explicitly.

When \( \varepsilon(t) \) is not a constant function, but rather a positive decreasing function of time \( \varepsilon(t) \) vanishing at infinity, even \( f \) does not depend on \( t \), this system is still non-autonomous, and then much more complex. In [10], the author proved the existence of the time-dependent global attractor for (1.1) where \( g(u_t) = \alpha u_t \), the strategy of verifying asymptotic compactness for the corresponding process consists in finding a suitable decomposition of the process in the sum of a decaying part and of a compact one. However, for the nonlinear damping, it appears difficult to apply the method of [10] to deal with the compactness of the system.

Motivated by the problem and inspired by [1, 24, 32, 34, 35], we first introduce a concept about compactness for the process on time-dependent space, which we term as pullback asymptotically compact. Then we give a criterion ensuring the existence of a time-dependent global attractor under the assumptions of pullback asymptotic compactness and the existence of a pullback absorbing family of sets, which seems to be more suitable to deal with the model (1.1). Moreover, we propose a technique method via contractive functions on time-dependent space to verify pullback asymptotic compactness for the process. The technique to prove some compactness of systems (in autonomous case) was initiated by I. Chueshov and I. Lasiecka [7, 8] in the context of wave equation with nonlinear damping, and subsequently, by A. Kh. Khanmamedov [20] in the context of Von Karman equations. Later on, Sun et.al [32, 34] extended this technique to non-autonomous systems. Our aim is to extend this technique to non-autonomous systems on time-dependent space. Finally, we show the existence of time-dependent global attractor for the system (1.1) by using the new framework.

The paper is organized as follows. In Section 2, we make some preparations for our consideration; in Section 3, we introduce the concept of pullback asymptotic compactness, and develop a criteria for the existence of time-dependent global attractor; in Section 4, a theoretical technique for verifying asymptotic compactness for the process is proposed; finally, in Section 5, the existence of time-dependent global attractor for the system (1.1) is proved by using the new framework.

Throughout the paper, \( C \) denotes any positive constant which may be different from line to line even in the same line (sometimes for special differentiation, we also denote the different positive constants by \( C_1, C_2, \cdots \)).

2. Preliminaries. In the following subsection 2.1 and 2.2, we review briefly the notations, some basic definitions and abstract results about processes on time-dependent spaces, which will be used in considering our problems, see [10, 28] for more details.

2.1. Notations. Let \( \{X_t\}_{t \in \mathbb{R}} \) be a family of normed spaces, we introduce the \( R \)-ball of \( X_t \),
\[
\mathbb{B}_t(R) = \{ z \in X_t : \| z \|_{X_t} \leq R \}.
\]
For any given \( \varepsilon > 0 \), the \( \varepsilon \)-neighborhood of a set \( B \subset X_t \) is defined as
\[
O^*_\varepsilon(B) = \bigcup_{x \in B} \{ y \in X_t : \| x - y \|_{X_t} < \varepsilon \} = \bigcup_{x \in B} \{ x + \mathbb{B}_t(\varepsilon) \}.
\]
We denote the Hausdorff semidistance of two (nonempty) sets \( B, C \subset X_t \) by
\[
\delta_t(B, C) = \sup_{x \in B} \inf_{y \in C} \| x - y \|_{X_t}.
\]
Given any set $B \subset X_t$, the symbol $\overline{B}$ stands for the closure of $B$ in $X_t$.

2.2. Concepts.

**Definition 2.1.** Let \( \{X_t\}_{t \in \mathbb{R}} \) be a family of normed spaces. A process is a two-parameter family of mappings \( \{U(t, \tau) : X_{\tau} \to X_t, \ t \geq \tau, \ \tau \in \mathbb{R} \} \) with properties

(i) \( U(\tau, \tau) = Id \) is the identity operator on \( X_{\tau}, \ \tau \in \mathbb{R} \);

(ii) \( U(t, s)U(s, \tau) = U(t, \tau), \ \forall t \geq s \geq \tau, \ \tau \in \mathbb{R} \).

For convenience, we sometimes abbreviate \( \{U(t, \tau) : X_{\tau} \to X_t, t \geq \tau, t, \tau \in \mathbb{R} \} \) to \( U(\cdot, \cdot) \).

**Definition 2.2.** A family \( \mathcal{C} = \{C_t\}_{t \in \mathbb{R}} \) of bounded sets \( C_t \subset X_t \) is called uniformly bounded if there exists a constant \( R > 0 \) such that \( C_t \subset B_R \), \( \forall t \in \mathbb{R} \).

**Definition 2.3.** A family \( \mathcal{B} = \{B_t\}_{t \in \mathbb{R}} \) is called pullback absorbing if it is uniformly bounded and for every \( R > 0 \), there exists a constant \( t_0 = t_0(R) \leq t \) such that

\[
\tau \leq t - t_0 \Rightarrow U(t, \tau)B(R) \subset B_t.
\]

The process \( U(t, \tau) \) is called dissipative whenever it admits a pullback absorbing family.

**Definition 2.4.** A time-dependent absorbing set for the process \( U(t, \tau) \) is a uniformly bounded family \( \mathcal{B} = \{B_t\}_{t \in \mathbb{R}} \) with the following property: for every \( R \geq 0 \) there exists a \( t_0 = t_0(R) \geq 0 \) such that

\[
\tau \leq t - t_0 \Rightarrow U(t, \tau)B(R) \subset B_t.
\]

**Remark 1.** It is obvious that the existence of a time-dependent absorbing set implies the dissipative of the corresponding process.

**Definition 2.5.** The time-dependent global attractor for \( U(t, \tau) \) is the smallest family \( \mathfrak{A} = \{A_t\}_{t \in \mathbb{R}} \) such that

(i) each \( A_t \) is compact in \( X_t \);

(ii) \( \mathfrak{A} \) is pullback attracting, i.e. it is uniformly bounded and the limit

\[
\lim_{\tau \to -\infty} \delta_t(U(t, \tau)C_{\tau}, A_t) = 0
\]

holds for every uniformly bounded family \( \mathcal{C} = \{C_t\}_{t \in \mathbb{R}} \) and every fixed \( t \in \mathbb{R} \).

**Remark 2.** The attracting property can be equivalently stated in terms of pullback absorbing: a (uniformly bounded) family \( \mathfrak{K} = \{K_t\}_{t \in \mathbb{R}} \) is called pullback attracting if for all \( \varepsilon > 0 \) the family \( \{O_t^\varepsilon(K_t)\}_{t \in \mathbb{R}} \) is pullback absorbing.

**Theorem 2.6.** The time-dependent global attractor \( \mathfrak{A} \) exists and it is unique if and only if the process is asymptotically compact, namely, the set

\[
\mathcal{K} = \{\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}} : K_t \subset X_t \text{ is compact}, \mathfrak{K} \text{ is pullback attracting}\}
\]

is not empty.

**Definition 2.7.** A process \( U(t, \tau) \) is \( \varepsilon \)-dissipative if for every \( t \in \mathbb{R} \) there exists a set \( F_t \subset X_t \) made of a finite number of points such that the family \( \{O_t^\varepsilon(F_t)\}_{t \in \mathbb{R}} \) is pullback absorbing. The process is called totally dissipative whenever it is \( \varepsilon \)-dissipative for every \( \varepsilon > 0 \).

**Theorem 2.8.** The time-dependent global attractor \( \mathfrak{A} \) exists and it is unique if and only if the process is totally dissipative.
Remark 3. It is apparent that an asymptotically compact process is totally dissipative, the corresponding relationship about semigroup is discussed in [4].

Definition 2.9. We say that \( \mathfrak{A} = \{ A_t \}_{t \in \mathbb{R}} \) is invariant if
\[
U(t, \tau) A_\tau = A_t, \quad \forall t \geq \tau.
\]

Remark 4. If the time-dependent global attractor \( \mathfrak{A} \) exists and the process \( U(t, \tau) \) is a strongly continuous process, then \( \mathfrak{A} \) is invariant, see more details in [10].

2.3. Some properties for nonlinear function \( g \) and \( \varphi \). In the following, we will state some properties of the nonlinear damping function \( g \) and the nonlinearity \( \varphi \) (see [10, 17, 20, 32, 34] etc. for more details), which will be used in Section 5.

Note that condition (1.6) implies that
\[
|g(s)|^{\frac{1}{p}} \leq C'_0 (1 + |s|),
\]
therefore, we have
\[
|g(s)|^{\frac{p+1}{p}} = |g(s)|^{\frac{1}{p}} \cdot |g(s)| \leq C'_0 (1 + |s|)|g(s)| \leq C'_0 + C'_0 g(s) \cdot s,
\]
feurthmore, we get
\[
|g(s)| \leq C + C(g(s)s)^{\frac{p}{p+1}}.
\]

Moreover, we review the following result.

Lemma 2.10. ([17, 20]) Let \( g \) satisfy (1.4) and (1.5). Then for any \( \gamma > 0 \), there exists \( C_\gamma > 0 \) depending on \( \gamma \) such that \( |u - v|^2 \leq \gamma + C_\gamma (g(u) - g(v))(u - v) \) for all \( u, v \in \mathbb{R} \).

Set
\[
\Phi(u) = \int_0^{u(x)} \varphi(y) dy,
\]
from (1.8) we can obtain that
\[
\int_\Omega \Phi(u) dx \geq -\frac{\lambda}{2} \|u\|^2 - C,
\]
and
\[
\langle \varphi(u), u \rangle \geq \int_\Omega \Phi(u) dx - \frac{\lambda}{2} \|u\|^2 - C,
\]
for some \( \lambda < \lambda_1 \), see [29].

3. Criterion for the existence of time-dependent global attractor. In this section, we firstly give the definition of pullback asymptotically compact in the time-dependent space case, and then using this definition, we provide a sufficient and necessary condition about the existence of attractor on time-dependent space.

Definition 3.1. We say that a process \( U(\cdot, \cdot) \) in a family of normed spaces \( \{ X_t \}_{t \in \mathbb{R}} \) is pullback asymptotically compact if and only if for any fixed \( t \in \mathbb{R} \), bounded sequence \( \{ x_n \}_{n=1}^\infty \subset X_{\tau_n} \) and any \( \{ \tau_n \}_{n=1}^\infty \subset \mathbb{R}^{-t} \) with \( \tau_n \to -\infty \) as \( n \to \infty \), sequence \( \{ U(t, \tau_n) x_n \}_{n=1}^\infty \) has a convergent subsequence, where \( \mathbb{R}^{-t} = \{ \tau : \tau \in \mathbb{R}, \tau \leq t \} \).
**Remark 5.** By Theorem 2.6 which is from [10], we know that the asymptotic compactness of the process will ensures the existence of time-dependent global attractor. However, for some concrete problem, the asymptotic compactness of the process is hard to be verified, especially, when it is hard to get higher regularity of the solutions. In [10], for the wave equation, the authors gave a very nice method to verify the necessary asymptotic compactness, they found a suitable decomposition of the process in the sum of a decaying part and a compact one to overcome the lack of regularity of the solutions, and then verified the asymptotic compactness of the process successfully. However, for our problem, due to the nonlinear damping, it seems to be difficult to apply the method of [10]. Hence, inspired by the concrete problem and the classical definition of asymptotic compactness (e.g. for the semi-group [24, 35], the semi-process [26] and the process [1, 34] etc.), we introduce a concept of the compactness for the process on time-dependent space, which we term as pullback asymptotically compact. Note that, pullback asymptotically compact is weaker than asymptotically compact.

**Lemma 3.2.** If \( U(\cdot, \cdot) \) is asymptotically compact, then it is pullback asymptotically compact.

**Proof.** If \( U(\cdot, \cdot) \) is asymptotically compact, i.e., there exists a pullback attracting family \( \mathcal{A} = \{ K_t \}_{t \in \mathbb{R}} \), where \( K_t \subset X_t \) is compact. According to Definition 2.6, 
\[
\lim_{\tau_n \to -\infty} \delta_t(U(t, \tau_n) x_n, K_t) = 0
\]
for any bounded sequences \( \{ x_n \}_{n=1}^\infty \subset X_{\tau_n} \), where \( \{ \tau_n \} \subset \mathbb{R}^- \) and \( \tau_n \to -\infty \) as \( n \to \infty \). Given \( k \in \mathbb{N} \), take the subsequence \( x_{n_k} \) of \( x_n \) such that \( \delta_t(U(t, \tau_{n_k}) x_{n_k}, K_t) < \frac{1}{k} \). Then, there exists \( y_k \in K_t \) with \( \| U(t, \tau_{n_k}) x_{n_k} - y_k \|_{X_t} < \frac{1}{k} \). Now, by the compactness of \( K_t \), there exists a subsequence (which we relabel) \( y_k \) such that \( \lim_{k \to \infty} y_k = y_0 \in K_t \). Furthermore, due to
\[
\| U(t, \tau_{n_k}) x_{n_k} - y_0 \|_{X_t} \leq \| U(t, \tau_{n_k}) x_{n_k} - y_k \|_{X_t} + \| y_k - y_0 \|_{X_t},
\]
then \( \{ U(t, \tau_{n_k}) x_{n_k} \}_{n=1}^\infty \) has a convergent subsequence, hence \( U(\cdot, \cdot) \) is pullback asymptotically compact.

Now, we provide the sufficient and necessary condition for existence of time-dependent global attractor.

**Theorem 3.3.** (Sufficient and necessary condition for existence of time-dependent global attractor)

Let \( U(\cdot, \cdot) \) be a process in a family of Banach spaces \( \{ X_t \}_{t \in \mathbb{R}} \). Then \( U(\cdot, \cdot) \) has a time-dependent global attractor \( \mathcal{A}^* = \{ A_t^* \}_{t \in \mathbb{R}} \) satisfying
\[
A_t^* = \bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau) B_r,
\]
if and only if
(i) \( U(\cdot, \cdot) \) has a pullback absorbing family \( \mathcal{B} = \{ B_t \}_{t \in \mathbb{R}} \);
(ii) \( U(\cdot, \cdot) \) is pullback asymptotically compact.

**Proof.** (Necessary.) If \( U(\cdot, \cdot) \) has a time-dependent global attractor \( \mathcal{A}^* = \{ A_t^* \}_{t \in \mathbb{R}} \), there exists a pullback attracting family \( \mathcal{A} = \{ A_t \}_{t \in \mathbb{R}} \) by Definition 2.5, where \( K_t \subset X_t \) is compact. Then according to Remark 2, the family \( \{ O_t(K_t) \}_{t \in \mathbb{R}} \) is pullback absorbing. Moreover, by Theorem 2.6, the process \( U(\cdot, \cdot) \) is asymptotically compact, which implies that the pullback asymptotic compactness of the process
immediately by Lemma 3.2.

(Sufficient.) We will verify \( \mathfrak{A}^* = \{A_t^*\}_{t \in \mathbb{R}} \) where

\[
A_t^* = \bigcap_{s \leq t \leq s} \overline{U(t, \tau)B_{\rho}}
\]

for any pullback absorbing set \( \mathfrak{B} = \{B_t\}_{t \in \mathbb{R}} \) is the time-dependent global attractor for the process \( U(\cdot, \cdot) \).

We will accomplish the proof by three steps.

**Step 1.** For any fixed \( t \in \mathbb{R}, A_t^* = \bigcap_{s \leq t \leq s} \overline{U(t, \tau)B_{\rho}} \) is nonempty and compact in \( X_t \).

For any fixed \( t \in \mathbb{R}, \) then for any \( \tau_n \in \mathbb{R}^{-t} \) and \( x_n \in B_{\tau_n}, \) by the definition of the pullback asymptotic compactness of the process we know that \( \{U(t, \tau_n)x_n\}_{n=1}^{\infty} \) has a convergent subsequence, without loss of generality, we assume that

\[
U(t, \tau_n)x_n \to y.
\]

Then by the construction of \( A_t^* \) we know that \( y \in A_t^* \), which implies that \( A_t^* \) is nonempty.

For any \( y_m \in A_t^*, m = 1, 2, \cdots, \) we will show that \( \{y_m\} \) has a convergent subsequence in \( X_t \). For each \( m \in \mathbb{N}, \) there exist \( \tau_m \in \mathbb{R}^{-t}, \tau_m \leq t \) and \( x_m \in B_t \) such that

\[
\rho(U(t, \tau_m)x_m, y_m) \leq \frac{1}{m},
\]

where \( \rho(\cdot, \cdot) \) is the metric on \( X_t \).

Therefore, by the assumption of the pullback asymptotic compactness of the process \( U(t, \tau) \), there exists a convergent subsequence of \( \{U(t, \tau_m)x_m\}_{m=1}^{\infty} \) in \( X_t \), without loss of generality, we assume that \( \{U(t, \tau_m)x_m\}_{m=1}^{\infty} \) is a Cauchy sequence in \( X_t \). Then

\[
\rho(y_n, y_m) \\
\leq \rho(y_n, U(t, \tau_n)x_n) + \rho(U(t, \tau_n)x_n, U(t, \tau_m)x_m) + \rho(U(t, \tau_m)x_m, y_m) \\
\leq \frac{1}{n} + \rho(U(t, \tau_n)x_n, U(t, \tau_m)x_m) + \frac{1}{m},
\]

we know that \( \{y_m\}_{m=1}^{\infty} \) is also a Cauchy sequence in \( X_t \). Moreover, from the construction we have that \( A_t^* \) is closed in \( X_t \).

Hence, \( A_t^* \) is compact in \( X_t \).

**Step 2.** The family \( \mathfrak{A}^* = \{A_t^*\}_{t \in \mathbb{R}} \) is pullback attracting, i.e. for every uniformly bounded family \( \mathfrak{C} = \{C_t\}_{t \in \mathbb{R}} \) and every \( t \in \mathbb{R}, \) the limit

\[
\lim_{\tau \to -\infty} \delta_t(U(t, \tau)C_t, A_t^*) = 0
\]

holds.

We argue by contradiction. Assume there exists a family of uniformly bounded sets \( \{C_t\}_{t \in \mathbb{R}} \) and some \( t_0 \in \mathbb{R} \) such that \( C_t \in X_t \), \( \delta_{t_0}(U(t_0, \tau)C_{t_0}, A_{t_0}^*) \) does not tend to 0 as \( \tau \to -\infty \). Thus there exists \( \varepsilon_0 > 0 \) and a sequence \( \tau_n \to -\infty \) such that

\[
\delta_{t_0}(U(t_0, \tau_n)C_{t_0}, A_{t_0}^*) \geq \varepsilon_0 > 0, \quad \forall n. \tag{3.1}
\]

For each fixed \( n \), there exists \( c_n \in C_{t_n} \) satisfying

\[
\delta_{t_0}(U(t_0, \tau_n)c_n, A_{t_0}^*) \geq \frac{\varepsilon_0}{2} > 0. \tag{3.2}
\]
Since $\mathcal{B}$ is pullback absorbing, $U(t, \tau_n)_{\tau_n} \subset B_t$, and hence $U(t, \tau_n)c_n \in B_t$, for $n$ sufficiently large (such that $t_1$ small enough and $\tau_n \leq t_1 \leq t$). The sequence $\{U(t, \tau_n)c_n\}$ is relatively compact, and hence $\{U(t, \tau_n)c_n\}$ has a convergent subsequence $\{U(t, \tau_n)c_{n_i}\}$ such that

$$\beta = \lim_{n_i \to \infty} U(t, \tau_n)c_{n_i} = \lim_{n_i \to \infty} U(t, t_1)U(t_1, \tau_n)c_{n_i}.$$ 

Since $U(t_1, \tau_n)c_{n_i} \in \mathcal{B}$, $\beta \in A^*_{r^*}$ and this contradicts (3.2).

**Step 3.** Minimality, i.e. if $K_t$ is compact and pullback attracts $C_t$ for every uniformly bounded family $\mathcal{C} = \{C_t\}_{t \in \mathbb{R}}$ and every $t \in \mathbb{R}$, then $A^*_{r^*} \subset K_t$.

Now we will show it. For $y \in A^*_{r^*}$, then there are $x_n \in B_{\tau_n}$ and $\tau_n \in \mathbb{R}^{-t}$ with $\tau_n \to -\infty$ such that $U(t, \tau_n)x_n \to y$. From the assumption $K_t$ pullback attracts $B_t$, obviously, we have

$$\delta_t(U(t, \tau_n)x_n, K_t) \to 0 \text{ as } n \to \infty.$$

At the same time, the compactness of $K_t$ implies $y \in K_t$. Hence, $A^*_{r^*} \subset K_t$. \qed

**Remark 6.** Note that, from Lemma 3.2, we know that pullback asymptotically compact is weaker than asymptotically compact, hence in order to provide sufficient and necessary conditions for the existence of time-dependent global attractor, we need some dissipativity of the system. In some concrete problems, some dissipativity is easier to be obtained than some compactness in certain sense. However, it is easily to conclude that, through comparing Theorem 2.6 in [10] with Theorem 3.3 in this paper, “the process $U$ is asymptotically compact” in [10] is equivalent to “the process $U$ is pullback asymptotically compact (see Definition 3.1) plus $U$ has a pullback absorbing family”.

### 4. A technique method for verifying asymptotically compactness

In this section, we present a technical method via contractive functions to verify the pullback asymptotic compactness on time-depend spaces. The technique to prove some compactness of autonomous system was initiated by I. Chueshov and I. Lasiecka [7, 8], and then was extended to non-autonomous systems by Sun et.al [32, 34]. Now, we extend this method to time-depend spaces case.

We start with a preliminary definition.

**Definition 4.1.** Let $\{X_t\}_{t \in \mathbb{R}}$ be a family of Banach spaces and $\mathcal{C} = \{C_t\}_{t \in \mathbb{R}}$ be a family of uniformly bounded subsets of $\{X_t\}_{t \in \mathbb{R}}$. We call a function $\phi^t_{\tau}(:, :)$, defined on $X_t \times X_t$, a contractive function on $C_\tau \times C_\tau$ if for any fixed $t \in \mathbb{R}$ and any sequence $\{x_n\}_{n=1}^{\infty} \subset C_\tau$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} \lim_{l \to \infty} \phi^t_{\tau}(x_{n_k}, x_{n_l}) = 0,$$

where $\tau \leq t$. We denote the set of all contractive functions on $C_\tau \times C_\tau$ by $\mathcal{C}(C_\tau)$.

**Theorem 4.2.** Let $U(:, :)$ be a process on $\{X_t\}_{t \in \mathbb{R}}$ and has a pullback absorbing family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$. Moreover, assume that for any $\varepsilon > 0$ there exist $T(\varepsilon) \leq t$, $\phi^t_{\tau} \in \mathcal{C}(B_T)$ such that

$$\|U(t, T)x - U(t, T)y\| \leq \varepsilon + \phi^t_{\tau}(x, y), \quad \forall x, y \in B_T,$$

for any fixed $t \in \mathbb{R}$. Then $U(:, :)$ is pullback asymptotically compact.
Proof. \( \mathcal{B} = \{B_t\}_{t \in \mathbb{R}} \) is pullback absorbing, we only need to show that
\[
U(t, \tau_n)x_n \text{ is precompact in } X_t
\]
for any \( \{x_n\}_{n=1}^\infty \subset B_{\tau_n} \) and \( \{\tau_n\} \) with \( \tau_n \to -\infty \) as \( n \to \infty \).

In the following, we will prove that \( \{U(t, \tau_n)x_n\}_{n=1}^\infty \) has a convergent subsequence via diagonal methods.

Taking \( \varepsilon_m > 0 \) with \( \varepsilon_m \to 0 \) as \( m \to \infty \).

At first, for \( \varepsilon_1 \), by the assumptions, there exist \( T_1 = T_1(\varepsilon_1) \leq t \) and \( \phi_{T_1}^t(\cdot, \cdot) \in \mathcal{E}(B_{T_1}) \) such that
\[
\|U(t, T_1)x - U(t, T_1)y\| \leq \varepsilon + \phi_{T_1}^t(x, y) \quad \forall x, y \in B_{T_1}
\]
for any fixed \( t \in \mathbb{R} \), where \( \phi_{T_1}^t \) depends on \( T_1 \).

Since \( \tau_n \to -\infty \), for such fixed \( T_1 \), without loss of generality, we assume that \( \tau_n \leq T_1 \), and for each \( n \in \mathbb{N} \) such that \( U(T_1, \tau_n)x_n \in B_{T_1} \). Set \( y_n = U(t, T_1)x_n \). Then from (4.1) we have
\[
\begin{align*}
    &\|U(t, \tau_n)x_n - U(t, \tau_m)x_m\| \\
    &= \|U(t, T_1)U(t, \tau_n)x_n - U(t, T_1)U(t, \tau_m)x_m\| \\
    &= \|U(t, T_1)y_n - U(t, T_1)y_m\| \leq \varepsilon_1 + \phi_{T_1}^t(y_n, y_m).
\end{align*}
\]
By the definition of \( \mathcal{E}(B_{T_1}) \) and \( \phi_{T_1}^t \in \mathcal{E}(B_{T_1}) \), we know that \( \{y_n\}_{n=1}^\infty \) has a subsequence \( \{y_{n_k}\}_{k=1}^\infty \) such that
\[
\lim_{k \to \infty} \lim_{l \to \infty} \phi_{T_1}^t(y_{n_k}, y_{n_k}) = \frac{\varepsilon_1}{2}.
\]
Similar to [20], we have
\[
\lim_{k \to \infty} \sup_{p \in \mathbb{N}} \|U(t, \tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)} - U(t, \tau_{n_k}^{(1)})x_{n_k}^{(1)}\| \\
\leq \lim_{k \to \infty} \sup_{p \in \mathbb{N}} \|U(t, \tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)} - U(t, \tau_{n_k}^{(1)})x_{n_k}^{(1)}\| \\
+ \lim_{k \to \infty} \sup_{p \in \mathbb{N}} \|U(t, \tau_{n_k}^{(1)})x_{n_k}^{(1)} - U(t, \tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)}\| \\
\leq \varepsilon_1 + \lim_{k \to \infty} \sup_{p \in \mathbb{N}} \phi_{T_1}^t(y_{n_{k+p}}, y_{n_{k+p}}) \\
+ \varepsilon_1 + \lim_{k \to \infty} \sup_{p \in \mathbb{N}} \phi_{T_1}^t(y_{n_k}, y_{n_k}),
\]
which, combining with (4.2) and (4.3), implies that
\[
\lim_{k \to \infty} \sup_{p \in \mathbb{N}} \|U(t, \tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)} - U(t, \tau_{n_k}^{(1)})x_{n_k}^{(1)}\| \leq 4\varepsilon_1.
\]
Therefore, there exists a \( K_1 \) such that
\[
\|U(t, \tau_{n_k}^{(1)})x_{n_k}^{(1)} - U(t, \tau_{n_l}^{(1)})x_{n_l}^{(1)}\| \leq 5\varepsilon_1, \text{ for all } k, l \geq K_1.
\]
By induction, we obtain that, for each \( m \geq 1 \), there is a subsequence \( \{U(t, \tau_{n_k}^{(m+1)})x_{n_k}^{(m+1)}\}_{k=1}^\infty \) of \( \{U(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^\infty \) and certain \( K_{m+1} \) such that
\[
\|U(t, \tau_{n_k}^{(m+1)})x_{n_k}^{(m+1)} - U(t, \tau_{n_l}^{(m+1)})x_{n_l}^{(m+1)}\| \leq 5\varepsilon_m, \text{ for all } k, l \geq K_{m+1}.
\]
Now, we consider the diagonal subsequence \( \{U(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^\infty \). Since for each \( m \in \mathbb{N} \), \( \{U(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^\infty \) is a subsequence of \( \{U(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^\infty \), then
\[
\|U(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)} - U(t, \tau_{n_l}^{(m)})x_{n_l}^{(m)}\| \leq 5\varepsilon_m, \text{ for all } k, l \geq \max\{m, K_m\},
\]
which combining with \(\varepsilon_m \rightarrow 0\) as \(m \rightarrow \infty\), implies that \(\{U(t, \tau_n^{(k)})x_n^{(k)}\}_{k=1}^{\infty}\) is a Cauchy sequence in \(X_t\). This shows that \(\{U(t, \tau_n)x_n\}_{n=1}^{\infty}\) is precompact in \(X_t\). Thus we complete the proof. 

\[\square\]

5. Time-dependent global attractor for non-autonomous wave equation.

In this section, we prove the existence of time-dependent global attractor for the wave equation with nonlinear damping by applying Theorem 3.3. First, the well-posedness of the problem (1.1) and then the corresponding process is established in 5.1; the dissipativity of the process is obtained by appropriate energy estimates in 5.2; then some a priori estimates are established which will be used to obtain the asymptotic compactness of the process in 5.3; in 5.4, the compactness of the process is verified by using the technique method presented in Section 4; the main result on the existence of the time-dependent global attractor is stated at the end of this section.

Denote the time-dependent space

\[
\mathcal{H}_t = H^1_0(\Omega) \times L^2(\Omega)
\]

equipped with norm

\[
\|\{u_0, u_1\}\|^2_{\mathcal{H}_t} = \|\nabla u_0\|^2 + \varepsilon(t)\|u_1\|^2.
\]

Note that the spaces \(\mathcal{H}_t\) are all the same as linear spaces and the norms \(\|\cdot\|^2_{\mathcal{H}_t}\) and \(\|\cdot\|^2_{\mathcal{H}_t}\) are equivalent for any fixed \(t, \tau \in \mathbb{R}\).

5.1. Well-posedness. In this subsection, we first state the results about the well-posedness of problem (1.1), which has been discussed in [6, 19, 33].

**Theorem 5.1.** Under the conditions (1.2) – (1.8), for any initial data \(z(\tau) = (u_0(\tau), u_1(\tau)) \in \mathcal{H}_\tau\), on any interval \([\tau, t]\) with \(t > \tau\), there exists a unique solution

\[u \in C([\tau, t], H^1_0(\Omega)), \quad u_t \in C([\tau, t], L^2(\Omega)),\]

which continuously depend on the initial data. That is, problem (1.1) generates a strongly continuous process \(U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t, \quad t \geq \tau \in \mathbb{R}\), where \(U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t\) acting as \(U(t, \tau)z(\tau) = (u(t), u_t(t))\).

As mentioned in [6, 19, 33], applying Faedo-Galerkin method (see [25]) one can show the existence of solution to (1.1). Rather than [6, 19, 33], \(\varepsilon(t)\) depends on time, which will bring some difficulties in energy estimates, see Theorem 5.3 below for detail.

Moreover, we state the continuous dependence estimate for \(U(t, \tau)\) on \(\mathcal{H}_\tau\), which can be to verify the uniqueness of the solution.

**Theorem 5.2.** Given any \(R > 0\), for every pair of initial data \(z_i(\tau) = (u_{0i}(\tau), u_{1i}(\tau)) \in \mathcal{H}_\tau\) such that \(\|z_i(\tau)\|_{\mathcal{H}_\tau} \leq R, i = 1, 2\), the difference of the corresponding solutions satisfies

\[
\|U(t, \tau)z_1(\tau) - U(t, \tau)z_2(\tau)\|_{\mathcal{H}_t} \leq e^{C(t-\tau)}\|z_1(\tau) - z_2(\tau)\|_{\mathcal{H}_\tau}, \quad \forall t \geq \tau, \tag{5.1}
\]

for some constant \(C = C(R) \geq 0\).

**Proof.** Given two different initial datum \(z_1(\tau), z_2(\tau) \in \mathcal{H}_\tau\) such that \(\|z_i(\tau)\|_{\mathcal{H}_\tau} \leq R, i = 1, 2\). By Theorem 5.3 below we know that

\[
\|U(t, \tau)z_i(\tau)\|_{\mathcal{H}_t} \leq C. \tag{5.2}
\]
Let \((u_t, \partial_t u_t(t)) = U(t, \tau)z_t(\tau)\), by (1.1) the difference \(\bar{z}(t) = (\bar{u}(t), \bar{u}_t(t)) = U(t, \tau)z_1(\tau) - U(t, \tau)z_2(\tau)\) satisfies
\[
\begin{cases}
\varepsilon \bar{u}_{tt} + g(u_{1t}(t)) - g(u_{2t}(t)) - \Delta \bar{u} + \varphi(u_1) - \varphi(u_2) = 0, & t > \tau, \\
\bar{u} \big|_{\partial \Omega} = 0, \\
(\bar{u}(x, \tau), \bar{u}_t(x, \tau)) = z_1 - z_2,
\end{cases}
\]

Multiplying the above equality by \(2\bar{u}_t\) and integrating over \(\Omega\), we have
\[
\frac{d}{dt}\|\bar{z}\|_{H_1}^2 + 2 \int_\Omega (g(u_{1t}(t)) - g(u_{2t}(t))\bar{u}_t \, dx - \varepsilon'\|\bar{u}_t\|^2 = -2\langle \varphi(u_1) - \varphi(u_2), \bar{u}_t \rangle.
\]
From (1.4), we have
\[
2 \int_\Omega (g(u_{1t}(t)) - g(u_{2t}(t))\bar{u}_t \, dx \geq 0.
\]
Combining with (1.7) and (5.2), we have
\[-2\langle \varphi(u_1) - \varphi(u_2), \bar{u}_t \rangle \leq C\|\nabla \bar{u}\|\|\bar{u}_t\| \leq C\|\nabla \bar{u}\|^2 + C\|\bar{u}_t\|^2,
\]
In addition, the following inequality is hold:
\[
\|\nabla \bar{u}\|^2 + \|\bar{u}_t\|^2 = \frac{1}{\varepsilon(t)}(\varepsilon(t)\|\nabla \bar{u}\|^2 + \varepsilon(t)\|\bar{u}_t\|^2)
\leq \frac{1}{\varepsilon(t)}(L\|\nabla \bar{u}\|^2 + \varepsilon(t)\|\bar{u}_t\|^2) \leq (L + 1)\frac{1}{\varepsilon(t)}\|\bar{z}(t)\|_{H_1}.
\]
Then we can obtain the differential inequality
\[
\frac{d}{dt}\|\bar{z}(t)\|_{H_1}^2 \leq C(L + 1)\frac{1}{\varepsilon(t)}\|\bar{z}(t)\|_{H_1}^2.
\]
Applying the Gronwall Lemma on \([\tau, t]\),
\[
\|\bar{z}(t)\|_{H_1}^2 \leq e^{C(L+1)\int_\tau^t \frac{1}{\varepsilon(s)} \, ds}\|\bar{z}(\tau)\|_{H_1}^2,
\]
the proof is completed.

5.2. Dissipativity. In this subsection, we are concerned with the dissipation properties of the process \(U(t, \tau)\) corresponding to (1.1). Inspired by [23] in the autonomous case, we will show the existence of a time-dependent absorbing set (hence pullback absorbing).

**Theorem 5.3.** Under assumptions (1.2)-(1.8), for any initial data \(z(\tau) = (u_0(\tau), u_1(\tau)) \in B_\varepsilon(R) \subseteq H_\varepsilon\), there exists \(R_0 > 0\) such that the family \(\mathcal{B} = \{B_\varepsilon(R_0)\}_{\varepsilon \in \mathbb{R}}\) is a time-dependent absorbing set for the process \(U(t, \tau)\) corresponding to (1.1).

**Proof.** Denote
\[
E(t) = \frac{1}{2} \int_\Omega (\bar{z}(t)|u_t(t)|^2 + |\nabla u(t)|^2) \, dx = \|U(t, \tau)z\|_{H_1}^2
\]
and
\[
E_0(t) = \frac{1}{2} \int_\Omega (\bar{z}(t)|u_t(t)|^2 + |\nabla u(t)|^2) \, dx + \int_\Omega \Phi(u) \, dx - \int_\Omega f \, dx.
\]
Multiplying (1.1) by \(u_t\) and integrating over \(\Omega\), we get
\[
\frac{d}{dt} E_0 + \int_\Omega g(u_t) u_t \, dx - \int_\Omega \frac{\varepsilon'}{2}|u_t|^2 \, dx = 0.
\]

(5.3)
Combining with (1.4) and the decreasing property of \( \varepsilon(t) \), we have
\[
\int_{\Omega} \left( g(u_t)u_t - \frac{\varepsilon'}{2} |u_t|^2 \right) dx \geq 0.
\]
Integrating (5.3) over \([\tau, t]\) we get
\[
E_0(t) \leq E_0(\tau), \quad \forall t \geq \tau.
\]
(5.4)
From (1.8) and Sobolev’s embeddings there exist some proper positive constant \( c_0, C_0 \) and the function \( C(s) \) such that
\[
c_0 E(t) - C_0 \leq E_0(t) \leq C(E(t)) \quad (5.5)
\]
On the other hand, combining with (2.2) and Hölder inequality, we have
\[
| \int_{\Omega} g(u_t)u_t dx | \leq C \int_{\Omega} |u| dx + C(\int_{\Omega} (g(u_t)u_t)^{\frac{4}{p+1}} dx)^{\frac{1}{4}} \| \nabla u \|
\]
\[
\leq C \int_{\Omega} |u| dx + C(\int_{\Omega} (g(u_t)u_t) dx)^{\frac{1}{4}} \| \nabla u \|
\]
\[
\leq C \int_{\Omega} |u| dx + C(1 + \int_{\Omega} |g(u_t)u_t| dx) \| \nabla u \|
\]
\[
\leq M_{g,\Omega} + \eta \| \nabla u \|^2 + C(E(\tau)) \int_{\Omega} g(u_t)u_t dx \quad (5.6)
\]
where \( \eta > 0 \) is small enough and will be determined later and \( M_{g,\Omega} \) is a generic constant different in various occurrences.

Multiplying (1.1) by \( u_t + \delta u \), and integrating over \( \Omega \), we infer
\[
\frac{d}{dt} E_1 = \int_{\Omega} \left( g(u_t)u_t - \frac{\varepsilon'}{2} |u_t|^2 - \delta |u_t|^2 \right) dx + \delta \int_{\Omega} |\nabla u|^2 dx
\]
\[
+ \delta \langle \varphi(u), u \rangle - \delta \langle f, u \rangle + \langle g(u), \delta u \rangle = \delta \varepsilon'(u_t, u), \quad (5.7)
\]
where
\[
E_1 = E_0 + \delta \varepsilon(u_t, u).
\]
In light of (1.3), Hölder inequality and Younger inequality, we infer
\[
|\varepsilon'(u_t, u)| \leq L_\Omega |u_t||u| \leq \frac{L^2}{4\eta \lambda_1} |u_t|^2 + \eta \| \nabla u \|^2,
\]
and
\[
\delta \varepsilon |u_t, u| \leq \frac{\varepsilon}{4} ||u_t||^2 + \frac{L \delta^2}{\lambda_1} \| \nabla u \|^2 \quad (5.8)
\]
Combining with (5.5) and (5.8) we can obtain that
\[
c_0 E(t) - C_0 \leq E_1(t) \leq C(E(t)). \quad (5.9)
\]
Combining with (5.6) and (5.7) we have
\[
\frac{d}{dt} E_1 + \delta E_1 + \Gamma \leq \delta M_{g,\Omega}, \quad (5.10)
\]
where
\[
\Gamma = \int_\Omega ((1 - \delta C(E(\tau)))g(u_t)u_t - \frac{\varepsilon'}{2}|u_t|^2 - \delta \left(\frac{3}{2}\varepsilon + \frac{L^2}{4\eta\lambda_1}\right)|u_t|^2)dx
+ \left(\frac{1}{2} - \frac{\lambda}{2\lambda_1} - 2\eta\right)\delta \int_\Omega |\nabla u|^2 dx - \delta^2 \varepsilon(u_t, u).
\]
By (5.8) we have
\[
\Gamma \geq \int_\Omega ((1 - \delta C(E(\tau)))g(u_t)u_t - \frac{\varepsilon'}{2}|u_t|^2 - \delta \left(\frac{5}{4}\varepsilon + \frac{L^2}{4\eta\lambda_1}\right)|u_t|^2)dx
+ \left(\frac{1}{2} - \frac{\lambda}{2\lambda_1} - 2\eta\right)\delta \int_\Omega |\nabla u|^2 dx.
\] (5.11)
By (1.5), there exist \(m > 0\) and \(R_g\) such that \(g' \geq m\) when \(|s| > R_g\). Combining with (1.4) we have
\[
\int_\Omega g(u_t)u_t dx \geq m \int_{\Omega \{u_t \geq R_g\}} |u_t|^2 dx.
\]
Then for \(\delta_1\) small enough, we have
\[
\int_\Omega g(u_t)u_t dx - \delta_1 \|u_t\|^2
\geq m \int_{\Omega \{u_t \geq R_g\}} |u_t|^2 dx - \delta_1 \int_{\Omega \{u_t \leq R_g\}} |u_t|^2 dx
\geq (m - \delta_1) \int_{\Omega \{u_t \geq R_g\}} |u_t|^2 dx + \delta_1 \int_{\Omega \{u_t \leq R_g\}} |u_t|^2 dx - 2\delta_1 R_g^2 |\Omega|. \quad (5.12)
\]
By setting \(\delta = \eta^2\) and choosing \(\eta\) small enough, such that \(1 - \delta C(E(\tau)) > \frac{1}{2}\), and inserting (5.12) into (5.11), we can obtain that \(\Gamma \geq -\delta M_{g,\Omega}\). From (5.10), we end up with
\[
\frac{d}{dt} E_1 + \delta E_1 \leq \delta M_{g,\Omega}.
\]
Using Gronwall lemma, we have
\[
E_1(t) \leq E_1(\tau)e^{-\delta (t-\tau)} + C.
\]
Together with (5.9), there exist a constant \(C_1 > 0\) and an increasing positive function \(Q\) such that
\[
\|U(t, \tau)z\|_{\mathcal{H}_t} \leq Q(\|z\|_{\mathcal{H}_t})e^{-\delta (t-\tau)} + C_1,
\]
For \(z \in \mathcal{B}_\tau(R)\) yields
\[
\|U(t, \tau)z\|_{\mathcal{H}_t} \leq Q(R)e^{-\delta (t-\tau)} + C_1 \leq 1 + 2C_1 = R_0,
\]
provides that \(t - \tau \geq t_0\), where
\[
t_0 = \max \{0, \delta^{-1} \ln \frac{Q(R)}{1 + C_1}\}.
\]
This concludes the proof of the existence of the time-dependent absorbing set. □
5.3. **A priori estimates.** The main purpose of this part is to establish (5.24)-(5.26), which will be used to obtain the asymptotic compactness of the process.

Let \((u_i(t), u_i(t))\) be the corresponding solution of (1.1) with initial datum \((u_0(\tau), v_0(\tau)) \in \{B_r\}_{\tau \in R}\). For convenience, we introduce notations

\[ g_i(t) = g(u_i(t)), \quad \varphi_i(t) = \varphi(u_i(t)), \quad i = 1, 2 \]

and

\[ w(t) = u_1(t) - u_2(t). \]

Then \(w(t)\) satisfies

\[
\begin{cases}
\varepsilon w_{tt} + g_1(t) - g_2(t) - \Delta w + \varphi_1(t) - \varphi_2(t) = 0, & t > T, \\
w|_{\partial\Omega} = 0, \\
w(x, T) = u_0^1(T) - u_0^2(T), \quad w_t(x, T) = v_0^1(T) - v_0^2(T).
\end{cases}
\]

(5.13)

Denote

\[ E_w(t) = \frac{1}{2} \int_{\Omega} (\varepsilon(t)|w_1(t)|^2 + |\nabla w(t)|^2) dx. \]

**Step 1.** Multiplying (5.13) by \(w_t(t)\), and integrating over \([s, t] \times \Omega\), we obtain

\[
E_w(t) - E_w(s) + \int_s^t \int_{\Omega} [(g_1(\xi) - g_2(\xi))w_1(\xi) - \frac{1}{2} \varepsilon'(t)|w_1|^2] d\xi dx
+ \int_s^t \int_{\Omega} (\varphi_1(\xi) - \varphi_2(\xi))w_1(\xi) dx d\xi = 0,
\]

(5.14)

where \(T \leq s \leq t\). Then

\[
\int_s^t \int_{\Omega} [(g_1(\xi) - g_2(\xi))w_1(\xi) - \frac{1}{2} \varepsilon'(\xi)|w_1|^2] d\xi dx
\leq E_w(s) - \int_s^t \int_{\Omega} (\varphi_1(\xi) - \varphi_2(\xi))w_1(\xi) dx d\xi.
\]

(5.15)

Note that \(\varepsilon\) is decreasing, hence \(\varepsilon'(t) < 0\), combining with Lemma 2.10, for any \(\delta > 0\), there exists \(C_\delta > 0\) such that

\[
\varepsilon(\xi)|w_1|^2 \leq L|w_1|^2 \leq L\delta + L\delta((g_1(\xi) - g_2(\xi))w_1(\xi) - \frac{1}{2} \varepsilon(\xi)|w_1|^2).
\]

(5.16)

Thus we have

\[
\int_s^t \int_{\Omega} \varepsilon(\xi)|w_1|^2 dx d\xi \leq L\delta mes(\Omega)(t - T)
+
L\delta E_w(s) - L\delta \int_s^t \int_{\Omega} (\varphi_1(\xi) - \varphi_2(\xi))w_1(\xi) dx d\xi.
\]

(5.17)

**Step 2.** Multiplying (5.13) by \(w(t)\), and integrating over \([T, t] \times \Omega\), we get

\[
\int_t^T \int_{\Omega} |\nabla w(s)|^2 dx ds + \int_\Omega \varepsilon(t)|w_1(t)|^2 dx
= \int_t^T \int_{\Omega} \varepsilon(\xi)|w_1|^2 dx d\xi + \int_T^t \int_{\Omega} \varepsilon'(\xi)w_1 w dx d\xi - \int_t^T \int_{\Omega} (g_1(\xi) - g_2(\xi))w dx d\xi
+ \int_\Omega \varepsilon(T)|w_1(T)|^2 dx - \int_T^t \int_{\Omega} (\varphi_1(\xi) - \varphi_2(\xi))w dx d\xi.
\]

(5.18)
Combining with (5.17) and (5.18), we have
\[
2 \int_T^t \mathcal{E}_w(\xi) d \xi
\]
\[
\leq 2L \delta \text{mes}(\Omega)(t - T) + 2LC_\delta \mathcal{E}_w(T) - 2LC_\delta \int_T^t \int_\Omega (\varphi_1(\xi) - \varphi_2(\xi)) w(t) \xi dx d \xi
\]
\[
- \int_\Omega \varepsilon(t) w(t) w(t) dx + \int_T^t \int_\Omega \varepsilon'(\xi) w(t) \xi dx d \xi - \int_T^t \int_\Omega (g_1(\xi) - g_2(\xi)) w(t) dx d \xi
\]
\[
+ \int_\Omega \varepsilon(T) w(T) w(T) dx - \int_T^t \int_\Omega (\varphi_1(\xi) - \varphi_2(\xi)) w(t) dx d \xi.
\]  

(5.19)  

**Step 3.** Integrating (5.14) over \([T, t]\) with respect to \(s\), we have that
\[
(t - T) \mathcal{E}_w(t) + \int_T^t \int_s^t \int_\Omega [(g_1(\xi) - g_2(\xi)) w(t) \xi - \frac{1}{2} \varepsilon'(\xi) |w(t)|^2] dx d \xi d s
\]
\[
= - \int_T^t \int_s^t \int_\Omega (\varphi_1(\xi) - \varphi_2(\xi)) w(t) \xi dx d \xi d s + \int_T^t \mathcal{E}_w(s) d s,
\]
(5.20)  

Note that \(\varepsilon(t)\) is decreasing and combining (1.4), we can get
\[
(g_1(\xi) - g_2(\xi)) w_1(\xi) - \frac{1}{2} \varepsilon'(\xi) |w_1|\quad \geq 0.
\]

Combining with (5.19) and (5.20), we have
\[
(t - T) \mathcal{E}_w(t)
\]
\[
\leq - \int_T^t \int_s^t \int_\Omega (\varphi_1(\xi) - \varphi_2(\xi)) w(t) \xi dx d \xi d s + L \delta \text{mes}(\Omega)(t - T)
\]
\[
+ LC_\delta \mathcal{E}_w(T) - LC_\delta \int_T^t \int_\Omega (\varphi_1(\xi) - \varphi_2(\xi)) w(t) \xi dx d \xi - \frac{1}{2} \int_\Omega \varepsilon(t) w(t) w(t) dx
\]
\[
+ \frac{1}{2} \int_T^t \int_\Omega L |w(t)|^2 dx - \frac{1}{2} \int_T^t \int_\Omega (g_1(\xi) - g_2(\xi)) w(t) dx d \xi
\]
\[
+ \frac{1}{2} \int_\Omega \varepsilon(T) w(T) w(T) dx - \frac{1}{2} \int_T^t \int_\Omega (\varphi_1(\xi) - \varphi_2(\xi)) w(t) dx d \xi.
\]
(5.21)  

**Step 4.** We will deal with \(\int_T^t \int_\Omega (g_1(\xi) - g_2(\xi)) w(t) dx d \xi\). Multiplying (1.1) by \(u_i\), and integrating over \(\Omega\) we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (\varepsilon(t) |u_i|^2) \xi dx + \int_\Omega g(u_i) u_i - \frac{1}{2} \int_\Omega \varepsilon'(t) |u_i|^2 dx
\]
\[
+ \int_\Omega \varphi(u_i) u_i dx = \int_\Omega f u_i dx,
\]
which combining with (5.5) and the existence of time-dependent absorbing set, implies that
\[ \int_{t}^{T} \int_{\Omega} g(u_{i})u_{i} \, dx \, \xi \leq C_{T}, \]
where the constant \( C_{T} \) depends on \( T \). Then, noticing (2.1), we obtain that
\[ \int_{t}^{T} \int_{\Omega} |g(u_{i})|^{\frac{p+1}{p}} \, dx \, \xi \leq C'_{T} + C'_{T}(t-T) \text{mes}(\Omega), \]
where \( C'_{T} = C'_{T}C_{T} \). Therefore, using Hölder inequality, from (5.22) we have
\[ | \int_{T}^{t} \int_{\Omega} g_{i}w \, dx \, \xi | \leq \left( \int_{T}^{t} \int_{\Omega} |g(u_{i})|^{\frac{p+1}{p}} \, dx \, \xi \right)^{\frac{p}{p+1}} \left( \int_{T}^{t} \int_{\Omega} |w|^{p+1} \, dx \, \xi \right)^{\frac{1}{p+1}} \]
\[ \leq (C'_{T}^{\frac{p}{p+1}} + (C'_{T}(t-T) \text{mes}(\Omega))^{\frac{p}{p+1}}) \left( \int_{T}^{t} \int_{\Omega} |w|^{p+1} \, dx \, \xi \right)^{\frac{1}{p+1}}, \]
which implies that
\[ | \int_{T}^{t} \int_{\Omega} (g_{1}(\xi) - g_{2}(\xi))w \, dx \, \xi | \]
\[ \leq 2 \left( C'_{T}^{\frac{p}{p+1}} + (C'_{T}(t-T) \text{mes}(\Omega))^{\frac{p}{p+1}} \right) \left( \int_{T}^{t} \int_{\Omega} |w|^{p+1} \, dx \, \xi \right)^{\frac{1}{p+1}}. \]
Combining with (5.21) and (5.23) we have
\[ (t-T)E_{w}(t) \]
\[ \leq L\delta \text{mes}(\Omega)(t-T) + L\delta E_{w}(T) + \frac{1}{2} \int_{T}^{t} \varepsilon(T)w_{i}(T)w(T) \, dx \]
\[ - \int_{T}^{t} \int_{s}^{t} \int_{\Omega} (\varphi_{1}(\xi) - \varphi_{2}(\xi))w_{i}(\xi) \, dx \, \xi \, ds \]
\[ - LC_{\delta} \int_{T}^{t} \int_{\Omega} (\varphi_{1}(\xi) - \varphi_{2}(\xi))w_{i}(\xi) \, dx \, \xi - \frac{1}{2} \int_{T}^{t} \varepsilon(t)w_{i}(t)w(t) \, dx \]
\[ + \frac{1}{2} \int_{T}^{t} \int_{\Omega} L|w_{i}|w \, dx \, \xi - \frac{1}{2} \int_{T}^{t} \int_{\Omega} (\varphi_{1}(\xi) - \varphi_{2}(\xi))w \, dx \, \xi \]
\[ + (C'_{T}^{\frac{p}{p+1}} + (C'_{T}(t-T) \text{mes}(\Omega))^{\frac{p}{p+1}}) \left( \int_{T}^{t} \int_{\Omega} |w|^{p+1} \, dx \, \xi \right)^{\frac{1}{p+1}}. \]
Set
\[ \phi_{T}^{i}((u_{0}^{i}(T), v_{0}^{i}(T)), (u_{0}^{i}(T), v_{0}^{i}(T))) \]
\[ = - \frac{1}{t-T} \int_{T}^{t} \int_{s}^{t} \int_{\Omega} (\varphi_{1}(\xi) - \varphi_{2}(\xi))w_{i}(\xi) \, dx \, \xi \, ds \]
\[ - \frac{LC_{\delta}}{t-T} \int_{T}^{t} \int_{\Omega} (\varphi_{1}(\xi) - \varphi_{2}(\xi))w_{i}(\xi) \, dx \, \xi - \frac{1}{2(t-T)} \int_{T}^{t} \varepsilon(t)w_{i}(t)w(t) \, dx \]
\[ + \frac{1}{2(t-T)} \int_{T}^{t} \int_{\Omega} L|w_{i}|w \, dx \, \xi - \frac{1}{2(t-T)} \int_{T}^{t} \int_{\Omega} (\varphi_{1}(\xi) - \varphi_{2}(\xi))w \, dx \, \xi \]
\[ + \frac{C'_{T}^{\frac{p}{p+1}}}{t-T} + (C'_{T}(t-T) \text{mes}(\Omega))^{\frac{p}{p+1}} \left( \int_{T}^{t} \int_{\Omega} |w|^{p+1} \, dx \, \xi \right)^{\frac{1}{p+1}}, \]
and
\[ C_M = L\delta \text{mes}(\Omega)(t - T) + LC_\delta \mathcal{E}_w(T) + \frac{1}{2} \int_\Omega \varepsilon(T)w_1(T)w(T)dx. \] (5.25)

Then we have
\[ \mathcal{E}_w \leq \frac{C_M}{T - t} + \phi_T((u_0^1(T), v_0^1(T)), (u_0^2(T), v_0^2(T))). \] (5.26)

5.4. **Asymptotically compact.** In this subsection, we will prove the process corresponding to problem (1.1) is pullback asymptotically compact. To this end, we will use the technique (via contractive functions) presented in Section 4.

**Theorem 5.4.** Under the assumptions (1.2)-(1.8), for any fixed \( t \in \mathbb{R} \), bounded sequence \( \{x_n\}_{n=1}^\infty \subset X_{\tau_n} \) and any \( \{\tau_n\}_{n=1}^\infty \subset \mathbb{R}^t \) with \( \tau_n \to -\infty \) as \( n \to \infty \), sequence \( \{U(t, \tau_n)x_n\}_{n=1}^\infty \) has a convergent subsequence.

**Proof.** For any fixed \( \epsilon > 0 \), we first choose some proper \( \delta \) such that \( L\delta \text{mes}(\Omega) \leq \frac{\epsilon}{4} \), for some fixed \( t \), let \( T < t \) such that \( t - T \) so large that
\[ \frac{C_M}{T - t} < \frac{\epsilon}{2}. \]

Hence, thanks to Theorem 4.2, we only need to verify that \( \phi_T \in \mathcal{E}(B_t) \) for each fixed \( T \).

Let \( (u_n, u_m) \) be the solution corresponding to initial data \( (u_n^n, v_n^n) \in B_T \) for the problem (1.1). From (5.3), \( \|\nabla u_n\|^2 + \varepsilon(\xi)\|u_m\|^2 \) is bounded, where the bound depends on the \( T \), furthermore, \( \|\nabla u_n\|^2 \) is bounded. Moreover, by (1.2) and (1.3), for fixed \( T \), \( \xi \in [T, t] \), \( \varepsilon(\xi) \) is bounded, hence \( \|u_m\|^2 \) is bounded. According to Alaoglu Theorem, without loss of generality (at most by passing to subsequence), we assume that
\[ u_n \to u \text{ weakly in } L^\infty(T, t; H_0^1(\Omega)), \] (5.27)
\[ u_n \to u \text{ weakly in } L^\infty(T, t; L^2(\Omega)), \] (5.28)
\[ u_n \to u \text{ in } L^{p+1}(T, t; L^{p+1}(\Omega)), \] (5.29)
\[ u_n(T) \to u(T) \text{ and } u_n(t) \to u(t) \text{ in } L^4(\Omega). \] (5.30)

Here we have used the compact embeddings \( H_0^1(\Omega) \hookrightarrow L^2(\Omega) \), \( H_0^1(\Omega) \hookrightarrow L^4(\Omega) \), \( H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega) \) (since \( p < 5 \)).

Now, we will deal with each term in (5.24) one by one.

Firstly, from Theorem 5.3, (5.27) and (5.28), (5.30) we get
\[ \lim_{n \to \infty} \lim_{m \to \infty} \int_\Omega \varepsilon(t)(u_n(t) - u_m(t))(u_n(t) - u_m(t))dx = 0, \] (5.31)
\[ \lim_{n \to \infty} \lim_{m \to \infty} \int_T^t \int_\Omega L(u_n(s) - u_m(s))(u_n(s) - u_m(s))dxds = 0, \] (5.32)
\[ \lim_{n \to \infty} \lim_{m \to \infty} \int_T^t \int_\Omega (\varphi(u_n(s)) - \varphi(u_m(s)))(u_n(s) - u_m(s))dxds = 0. \] (5.33)
Secondly, since

\[
\int_T^t \int_\Omega (u_n(s) - u_m(s))(\varphi(u_n(s)) - \varphi(u_m(s)))dx
ds
\]

\[
= \int_T^t \int_\Omega u_n \varphi(u_n(s))dx + \int_T^t \int_\Omega u_m \varphi(u_m(s))dx
- \int_T^t \int_\Omega u_n \varphi(u_m(s))dx - \int_T^t \int_\Omega u_m \varphi(u_n(s))dx
- \int_T^t \int_\Omega \varphi(u_n(s))dx - \int_T^t \int_\Omega \varphi(u_m(s))dx
\]

then, combining (5.27) with (5.28), (5.30) and (1.7), taking first \( m \to \infty \), then \( n \to \infty \), we can obtain that

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_T^t \int_\Omega (u_n(s) - u_m(s))(\varphi(u_n(s)) - \varphi(u_m(s)))dx
ds
= \int_\Omega \Phi(u(t))dx - \int_\Omega \Phi(u(T))dx + \int_\Omega \Phi(u(t))dx - \int_\Omega \Phi(u(T))dx
- \int_T^t \int_\Omega u_n \varphi(u(s))dx - \int_T^t \int_\Omega u_m \varphi(u(s))dx,
= 0. \quad \text{(5.34)}
\]

Similarly, we have

\[
\int_s^t \int_\Omega (u_n(\xi) - u_m(\xi))(\varphi(u_n(\xi)) - \varphi(u_m(\xi)))dxd\xi
\]

\[
= \int_\Omega \Phi(u_n(\xi))dx - \int_\Omega \Phi(u_m(\xi))dx + \int_\Omega \Phi(u_n(\xi))dx - \int_\Omega \Phi(u_m(\xi))dx
- \int_s^t \int_\Omega u_n \varphi(u_m(\xi))dxd\xi - \int_s^t \int_\Omega u_m \varphi(u_n(\xi))dxd\xi,
\]

At the same time, \( |\int_s^t \int_\Omega (u_n(\xi) - u_m(\xi))(\varphi(u_n(\xi)) - \varphi(u_m(\xi)))dxd\xi| \) is bounded for each fixed \( t \), then by Lebesgue dominated convergence theorem we have

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_T^t \int_s \int_\Omega (u_n(\xi) - u_m(\xi))(\varphi(u_n(\xi)) - \varphi(u_m(\xi)))dxd\xi ds
= \int_T^t \lim_{n \to \infty} \lim_{m \to \infty} \int_s \int_\Omega (u_n(\xi) - u_m(\xi))(\varphi(u_n(\xi)) - \varphi(u_m(\xi)))dxd\xi ds
= \int_T^t 0 ds = 0. \quad \text{(5.35)}
\]

Finally, by (5.29), we have

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_T^t \int_\Omega (|u_n(s) - u_m(s)|^{p+1}dx)^{\frac{1}{p+1}} ds = 0,
\]
where $T = \left( C' \frac{\tau}{\tau + 1} + (C_5 (t - \tau) \text{mes}(\Omega)) \frac{1}{\tau + 1} \right)^{\frac{1}{\tau + 1}}$

Hence, combining (5.31)-(5.35), we get that $\phi_T^1(\cdot, \cdot) \in \mathcal{C}(B_1)$, and this completes the proof of Theorem 5.4.

5.5. Existence of the time-dependent global attractor.

**Theorem 5.5.** Under the conditions (1.2)-(1.8), the process $U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t$ generated by problem (1.1) has a invariant time-dependent global attractor $\mathfrak{A} = \{ A_t \}_{t \in \mathbb{R}}$.

**Proof.** From Theorem 5.3, Theorem 5.4, then by Theorem 3.3, there exists a unique time-dependent global attractor $\mathfrak{A} = \{ A_t \}_{t \in \mathbb{R}}$. Furthermore, due to the strong continuity of the process stated in Theorem 5.1, we can obtain that $\mathfrak{A}$ is invariant by Remark 4.

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