NUMERICAL METHODS FOR SOLUTION OF SINGULAR INTEGRAL EQUATIONS

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Abstract ................................................................. 4
Preface ................................................................. 4
Introduction ............................................................ 10
1. Classes of Functions .............................................. 10
2. Designations and Auxiliary Statements ....................... 12
  2.1. Designations of Optimal Algorithms ....................... 12
  2.2. Elements of Functional Analysis ............................ 14
  2.3. Elements of Approximation Theory ......................... 19
  2.4. Inverse Theorems .............................................. 20
Chapter 1 .............................................................. 22
Approximate Solution of Singular Integral Equations ...... 22
  1. An Smoothness of Solutions of Singular Integral Equations 22
    1.1. The Integral Operators on the Smooth Functions ....... 23
    1.1.1. Fundamental Statements ................................ 23
    1.2. On Smoothness of Solutions of Singular Integral Equa-
         tions on Closed Contours .................................. 24
      1.2.1. Fundamental Statements ............................... 24
  2. Approximate Solution of Linear Singular Integral Equations
      on the Closed Circuits of Integration (Basis in Holder Spaces) . 24
    2.1. Methods of Collocations and Mechanical Quadratures .. 26
9.1.2. Approximation Solution of the Boundary Value Problem (9.3), (9.2) ................................................................. 122
9.3. Approximate Solution of Nonlinear Singular Integro-Differential Equations on Closed Contours of Integration ............... 126
9.4. Approximate Solution of Linear Singular Integro-Differential Equations with Discontinuous Coefficients and on Open Contours of Integration ...................................................... 130
9.5. Approximate Solution of Nonlinear Singular Integro-Differential Equations on the Open Contour of Integration ............... 136

Chapter 2 ................................................................. 138
Approximate Solution of Multi-Dimensional Singular Integral Equations ................................................................. 8
1. Bisingular Integral Equations ................................................. 139
2. Riemann Boundary Value Task ............................................. 144
3. Approximate Solution of Multi-Dimensional Singular Integral Equations ................................................................. 147
3.1. Approximate Solution of Multi-Dimensional Singular Integral Equations on Holder Classes of Functions .................. 148
3.2. Approximate Solution of Linear Multi-Dimensional Singular Integral Equations on Sobolev Classes of Functions ....... 153
3.3. Parallel Method for Solution of Multi-Dimensional Singular Integral Equations ................................................................. 156

Appendix 1 ................................................................. 157
1. Stability of Solutions of Numerical Schemes ......................... 157
1.1. Stability of Solutions of Bisingular Integral Equation ..... 158
1.2. Stability of Solutions of Multi-Dimensional Singular Integral Equations ................................................................. 161
1.3. Stability of Solutions of Nonlinear Singular Integral Equations ................................................................. 162

References ................................................................. 165
Abstract
This paper is devoted to an overview of the authors own works for numerical solution of singular integral equations (SIE), polysingular integral equations and multi-dimensional singular integral equations of the second kind. Considered iterative - projective methods and parallel methods for solution of singular integral equations, polysingular integral equations and multi-dimensional singular integral equations. The paper is the second part of the overview of the authors own works devoted to numerical methods for calculation singular and hypersingular integrals and to approximate methods for solution of singular integral equations.

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Preface

Hilbert and Poincare created the theory of singular integral equations early in the 20th century, which has undergone an intense growth during the last 100 years. The theory is associated with numerous applications of singular integral equations, as well as with Riemann boundary value problem. The Riemann boundary value problem and singular integral equations are widely used as basic techniques of mathematical modeling in physics (quantum field theory, theory of close and long-distance interaction, soliton theory, theory of elasticity and thermoelasticity, aerodynamics, and electrodynamics, etc.

A closed-form solution of singular integral equations is only possible in exceptional cases. A comprehensive presentation and an extensive literature survey associated with all methods of solution of singular integral equations of the first and second kinds are found in 35, 54, 55, 56, 57, 58, 68, 71, 83, 100, 105, 117, 126, 127, 132, 137.
Various numerical methods for solution of singular integral equations have been the topic of a great many of papers, most of which can be divided in two fields. The first field is devoted to numerical methods for solution of singular integral equations of the first kind. The second field is devoted to numerical methods for solution of singular integral equations of the second kind.

The early works belong to the first field. Singular integral equations are the main tool for simulation aerodynamics tasks. M.A. Lavrent’ev paper an theory of aerofoil [98] was the first work devoted to numerical methods for solution of the singular integral equations of the first kind. Later numerical methods for solution of the first kind singular integral equations was devoted many papers based on different methods. A. I. Kalandia proposed the method of mechanical quadrature [89], F. Erdogan and G.D. Gupta proposed the Gauss-Chybyshhev method [61], N.I. Ioakimidis and P.S. Theocaris proposed the Lobatto-Chebyshev method [81], S. Krenk proposed Gauss-Jacobi, Lobatto-Jacobi methods [96]. Discrete vortexes method for solution of the singular integral equations of the first kind was investigated by S.M. Belotserkovsky and I.K. Lifanov [6], I.K. Lifanov [100], N.F. Vorob’ev [136]. A.V. Dzhishkariani was printed the detailed analyses of numerical methods for solution of the singular integral equations of the first kind [54], [55].

The numerical methods for solution of the singular integral equations of the second kind we can divide on two parts: direct and indirect methods. For different cases of singular integral equations Du Jinyuan [85], [86], [87], A. Gerasolus [67], J.G. Graham [74], N.L. Gori [73], D. Elliot [57], [58], [59], [60], B.I. Musaev [108], J.G. Sanikidze [122], I. Ioakimidis [80], [81], M.A. Sheshko [124], R.P. Srivastav and E Jen [127], G. Tsamasphyras and P.S. Theocaris [132] constructed various numerical schemes of indirect methods and proved its convergence.
We shall not touch the part which are devoted to indirect methods for solution of singular integral equations. One of these methods consists in transformation of singular integral equations

\[ a(t)x(t) + \frac{b(t)}{\pi i} \int_{-1}^{1} \frac{x(\tau)}{\tau - t} d\tau + \int_{-1}^{1} h(t, \tau)x(\tau)d\tau = f(t) \]

to the equivalent Fredholm integral equations

\[ x(t) + \int_{-1}^{1} h^*(t, \tau)x(\tau)d\tau = f^*(t). \]

Numerical methods are applied to the last equations.

One can become acquainted with this method by papers G. Tsamasphyras and P.S. Theocaris [132], M.A. Sheshko [123].

For the first time direct methods for solution of SIE as

\[ a(t)x(t) + \frac{b(t)}{\pi i} \int_{\gamma} \frac{x(\tau)}{\tau - t} d\tau + \int_{\gamma} h(t, \tau)x(\tau)d\tau = f(t), \quad (1.1) \]

where \( \gamma = \{z, z \in C, |z| = 1\} \), was considered by V.V. Ivanov [83], [82]. He proved the convergence of collocation method, moments method, Galerkin-Petrov method for the equation (1.1).

For solution of SIE as

\[ a(t)x(t) + b(t) \int_{-1}^{1} \frac{x(\tau)d\tau}{\tau - t} + \int_{-1}^{1} h(t, \tau)x(\tau)d\tau = f(t) \quad (1.2) \]

V.V. Ivanov proposed a method for transformation the equation (1.2) to the equation (1.1).

For solution of the equation (1.1) V.V. Ivanov used the method which is based on connection between singular integral equations and Riemann boundary value problem. According this method the singular integral equation is transformed to the Riemann
boundary value problem. Simultaneously, numerical scheme for solution of the singular integral equation is transformed to numerical scheme for approximate solution of Riemann boundary value problem. Using the Kantorovich common theory of approximate methods of analysis [90] V.V. Ivanov proved existence and unique of approximate solution of Riemann boundary value problem and, as corollary, existence and unique of approximate solution of a singular integral equation. In this way he investigated collocation method, moments method, Bubnov - Galerkin method, least square method for one-dimensional singular integral equations. Convergence of these methods are given in the spaces $W$ and $L_2$. Historical this method was the first common method for solution of singular integral equations as (1.1).

Development of this method make up the first direction in numerical methods for solution of second kind singular integral equations.

In the frame of this direction was received following results.

B.G. Gabdulhaev [62] have proved convergence of the mechanical quadrature method for the equation (1.1) in the space $H_\beta$.

I.V. Boykov [11], [15], [17], [35] offered modifications of collocation and mechanical quadrature methods for solution of linear singular integral equations

$$a(t)x(t) + \frac{1}{\pi i} \int_{\gamma} \frac{h(t, \tau)x(\tau)}{\tau - t} d\tau = f(t) \quad (1.3)$$

and nonlinear singular integral equations

$$a(t, x(t)) + \frac{1}{\pi i} \int_{\gamma} \frac{h(t, \tau, x(\tau))}{\tau - t} d\tau = f(t). \quad (1.4)$$

Convergence of these methods are given in $H_\beta$ and $L_2$.

I.V. Boykov and I.I. Zhechev [42], [43], [44] proved convergence of numerical methods for solution of singular integro - differential
integral equations
\[
\sum_{k=0}^{m} \left[ a_k(t)x^{(k)}(t) + \frac{b_k(t)}{\pi i} \int_{\gamma} \frac{x^{(k)}(\tau)}{\tau-t} d\tau + \frac{1}{2\pi i} \int_{\gamma} h_k(t,\tau)x^{(k)}(\tau)d\tau \right] = f(t)
\]
with boundary dates
\[
\int_{\gamma} x(\tau)\tau^{-k-1}d\tau = 0, \quad k = 0, 1, \ldots, m - 1.
\]
Approximate solution of nonlinear singular integro-differential equations as
\[
a(t, x(t), x'(t), \ldots, x^{(m)}(t)) + \\
\frac{1}{\pi i} \int_{\gamma} \frac{h(t, \tau, x(\tau), x'(\tau), \ldots, x^{(m)}(\tau))}{\tau-t} d\tau = f(t)
\]
under boundary dates (1.6) with method of mechanical quadrature was investigated by I.V. Boykov and I.I. Zhechev [45], [46].

Collocation method and method of mechanical quadrature for system of linear and nonlinear singular integral equations was investigated by I.I. Zhechev [137].

Projective methods for solution of singular integral equations as (1.2) in weight spaces was considered by I.V. Boykov [18], [24].

Some results, received for solution of one-dimensional singular integral equations, was diffused by I.V. Boykov [26], [34], [35] to bisingular integral equations as
\[
a(t_1, t_2)x(t_1, t_2) + \frac{b(t_1, t_2)}{\pi i} \int_{\gamma_1} \frac{x(\tau_1, \tau_2)}{\tau_1-t_1} d\tau_1 + \\
+ \frac{c(t_1, t_2)}{\pi i} \int_{\gamma_2} \frac{x(t_1, \tau_2)}{\tau_2-t_2} d\tau_2 + \frac{d(t_1, t_2)}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \frac{x(\tau_1, \tau_2)}{(\tau_1-t_1)(\tau_2-t_2)} d\tau_1 d\tau_2 +
\]
\[ + \int_{\gamma_1} \int_{\gamma_2} h(t_1, t_2, \tau_1, \tau_2) x(\tau_1, \tau_2) d\tau_1 d\tau_2 = f(t_1, t_2), \]

where \( \gamma_i = \{ z_i, z_i \in C, |z_i| = 1, i = 1, 2 \} \).

Some results, which are received in the first direction, are printed in the books [26], [35].

The second direction in approximate methods for solution of singular integral equation is connected with common projective methods for solution of integral equations with convolution.

I.Tz. Gohberg and I.A. Feldman [68] offered common method for investigation of integral equations with convolution. The method is based on the theory of rings.

Later this method was diffused to singular integral equations of non normal type and polysingular integral equations. Basic results in this direction was printed in the books [114], [105], [117].

The third direction in approximate methods for solution of singular integral equations is spline-collocation methods.

Apparently, the first works in this direction was the papers of S. Pressdorf [115], S. Pressdorf and G. Shmidt [116]. In these papers verification of spline-projective method was based on study of the spectrum of the linear system of algebraic equations, which are approximated the given singular integral equation.

During the last twenty five years spline-collocation methods was diffused on many types of singular and polysingular integral equations.

Basic results of this direction was printed in the works [73], [74], [105], [117].

For solution of aerodynamics problems discrete vortex method was offered by S.M. Belotserkovsky [6]. This method was diffused to many types of singular and polysingular integral equations which are used as basic techniques of mathematical modeling in aerodynamics [101].
Recently the wavelet collocation method and moment method, based on wavelet functions, begin the rapid development [7], [8].

Apparently approximate methods for solution of singular and polysingular integral equations, based on theory of wavelets, will form the fifth direction in numerical methods for solution of singular integral equations.

For solution of the equation (1.2) was offered some special methods which we can not associate to concrete direction. So direct methods for solution of SIE as (1.2) given by D. Elliot in the seria of important papers [57] - [60].

In this paper we do not give review of all directions. We only give the short review of some results in the first direction of numerical methods for solution of singular integral equations, polysingular integral equations and multi-dimensional singular integral equations of the second kind which was received author and his disciples.

Note. The author’s name in various publications was translated from Russian to English as Boikov or Boykov.

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Introduction

1. Classes of Functions

In this section we will list several classes of functions, which will be used later.

Let γ be the unit circle: γ = \{z : |z| = 1\}.

To measure the continuity of a function \( f \in C[a, b] \) we proceed as follows [102]. We consider the first difference with step \( h \)

\[
\Delta_h(f(x)) = f(x + h) - f(x)
\]
of the function $f$ and put
\[ \omega(f, \delta) = \omega(\delta) = \max_{x, h(|h| \leq \delta)} |f(x + h) - f(x)|. \]

The function $\omega(\delta)$, called the modulus of continuity of $f$, is defined for $0 \leq \delta \leq b - a$.

**Definition 1.1.** A function $f$, defined on $\Delta = [a, b]$ or $\Delta = \gamma$, satisfies a Lipschitz condition with constant $M$ and exponent $\alpha$, or belongs to the class $H_\alpha(M)$, $M \geq 0$, $0 < \alpha \leq 1$, if
\[ |f(x') - f(x'')| \leq M|x' - x''|\alpha, \quad x', x'' \in \Delta. \]

**Definition 1.2.** A function $f$, defined on $\Delta = [a, b]$ or $\Delta = \gamma$, satisfies a Zigmund condition with constant $M$, or belongs to the class $Z(M)$, $M \geq 0$, if
\[ |f(x') - f(x'')| \leq M|x' - x''||\ln |x' - x''||, \quad x', x'' \in \Delta. \]

**Definition 1.3.** The class $W^r(M, \Delta)$, $r = 1, 2, \ldots$, $\Delta = [a, b]$ or $\Delta = \gamma$, consists of all functions $f \in C(\Delta)$, which have an absolutely continuous derivative $f^{(r-1)}(x)$ and piecewise derivative $f^{(r)}(x)$ with $|f^{(r)}(x)| \leq M$.

**Definition 1.4** [102]. Let $r = 0, 1, \ldots, M_i \geq 0, i = 0, 1, \ldots, r + 1$, let $\omega$ be a modulus of continuity and let $\Delta = [a, b]$ or $\Delta = \gamma$. Then $W^r_\omega = W^r(M_0, \ldots, M_{r+1}; \Delta)$ is the set of all functions $f \in C(\Delta)$, which have continuous derivatives $f, f', \ldots, f^{(r)}$ on $\Delta$, satisfying
\[ |f^{(i)}(x)| \leq M_i, \quad x \in \Delta, \quad i = 0, 1, \ldots, r, \quad \omega(f^{(r)}, \delta) \leq M_{r+1}\omega(\delta). \]

They write $W^r H_\alpha$ if $\omega(\delta) = \delta^\alpha$, $0 < \alpha \leq 1$.

Let us consider functions $f(x_1, \ldots, x_l)$ of $l$ variables on $\Delta$, where $\Delta$ is either an $l$-dimensional parallelepiped (that is, the product of $l$ intervals $a_k \leq x_k \leq b_k$, $k = 1, 2, \ldots, l$) or an $l$-dimensional torus $\gamma^{(l)}$, the product of $l$ circles $\gamma$. The modulus of continuity $\omega(f, \delta)$ are defined as the maximum of
\[ |f(y_1, \ldots, y_l) - f(x_1, \ldots, x_l)| \text{ for } |y_k - x_k| \leq \delta, k = 1, 2, \ldots, l. \]
Sometimes the partial modulus of continuity $\omega^{(k)}(f, \delta)$ of $f$ are used. The $k$th modulus among them is the maximum of (1.1) when the increment is only with respect to the $k$th coordinate: $x_i = y_i$, $i \neq k$, $|y_k - x_k| \leq \delta$.

In [102] was defined classes $W_{l\omega}^r = W_{l\omega}(M_0, \ldots, M_{r+1}; \Delta)$ of functions $f(x_1, \ldots, x_l)$ of $l$ variables on $l$-dimensional set $\Delta$, which is either a parallelepiped or a torus.

**Definition 1.5** [102]. A function $f$ belongs to $W_{l\omega}^r$ if and only if all its partial derivatives $D^j f$ of order $j = 0, 1, \ldots, r$ exist and continuous and satisfy the following conditions: For each partial derivative $D^j f$ of order $j$, $\|D^j f\| \leq M_j$, $j = 0, 1, \ldots, r$, and in addition for each derivative of order $r$, $\omega(D^r f, \delta) \leq M_{r+1} \omega(\delta)$.

They write $W_{l\omega}^r H_\alpha$ if $\omega(\delta) = \delta^\alpha$, $0 < \alpha \leq 1$.

If coefficients $A$ and $M$ are not essential we use designations $H_\alpha$, $W^r H_\alpha$, $W^{rr} H_{\alpha\alpha}$ instead of $H_\alpha(A)$, $W^r H_\alpha(A, M)$, $W^{rr} H_{\alpha\alpha}(A, M)$ respectively.

2. Designations and Auxiliary Statements

2.1. Designations of Optimal Algorithms

In this paper we will use definitions of optimal algorithms for solution of problems of mathematical physics, given N.S. Bakhvalov [3]. These definitions we will use in the study of optimal algorithms for solving singular integral equations.

Let $H = \{h\}$ be a class of smooth functions. Let $F = \{f\}$ be a class of smooth functions. Let $C = \{c\}$ be a class of smooth functions.

Let $\Psi$ be a class of vector-functionals, defined on $H$. Let $\Psi^*$ be a class of vector-functionals, defined on $C$. Let $\Psi^{**}$ be a class of vector-functionals, defined on $F$.

Let $M$ be a set of Markov algorithms. Let $R = R(K, h, c, f, A, \{\psi_v\}^N_1, \{\psi_v^*\}^N_1, \{\psi_v^{**}\}^N_1, t)$ be a result of
numerical solution of the equation

\[ Kx \equiv c(t)x(t) + \frac{1}{\pi i} \int_{\gamma} \frac{h(t, \tau)x(\tau)d\tau}{\tau - t} = f(t) \quad (2.1) \]

with a algorithm \( A \in M \).

Algorithm \( A \) uses \( N^2 \) functionals \( \psi_v(h), \nu = 1, 2, \ldots, N^2, \psi_v \in \Psi, N \) functionals \( \psi_v^*(f), \nu = 1, 2, \ldots, N, \psi_v^* \in \Psi^*, \) and \( N \) functionals \( \psi_v^*(c), \nu = 1, 2, \ldots, N, \psi_v^* \in \Psi^* \).

Let us introduce the following designations:

\[ v(K, h, c, f, A, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N) = \rho(x^*, R), \]

\[ v(K, H, C, F, A, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N) = \]

\[ = \sup_{c \in C, f \in F, h \in H} v(K, h, c, f, A, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N), \]

\[ v(K, H, C, F, M, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N) = \]

\[ = \inf_{A \in M} v(K, H, C, F, A, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N), \]

\[ v_N(K, H, C, F, M, \Psi, \Psi^*, \Psi^{**}) = \]

\[ = \inf_{\{\psi_v\}_1^{N^2} \in \Psi^*, \{\psi_v^*\}_1^N \in \Psi^*, \{\psi_v^{**}\}_1^N \in \Psi^{**}} v(K, H, C, F, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N), \]

\[ v_N(K, H, C, F) = \]

\[ = \inf_{\{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N} v(K, H, C, F, \{\psi_v\}_1^{N^2}, \{\psi_v^*\}_1^N, \{\psi_v^{**}\}_1^N). \]

Here \( \rho(x^*, R) \) is the distance between exact solution \( x^* \) of the equation \( Kx = f \) and approximate solution \( R \) of the equation (2.1).
Functional $v_N(K, H, C, F, \Psi, \Psi^*, \Psi^{**})$ is infimum of values

$v_N(K, H, C, F, \{\psi_v(h)\}_{1}^{N^2}, \{\psi_v^*(c)\}_{1}^{N}, \{\psi_v^{**(f)}\}_{1}^{N})$ on functionals

$\Psi, \Psi^*, \Psi^{**}$.

Functional $v_N(K, H, C, F)$ is infimum of values

$v_N(K, H, C, F, \Psi, \Psi^*, \Psi^{**})$

using all kind of functional classes $\Psi, \Psi^*, \Psi^{**}$.

Algorithm $A$, using functionals $\overline{\psi_v} 
_{1}^{N^2} \in \Psi, \{\overline{\psi_v^*}\}_{1}^{N} \in \Psi^*, \{\overline{\psi_v^{**}}\} \in \Psi^{**},$ called optimal, asymptotically optimal, optimal with respect to order if

$$v(K, H, C, F, A, \overline{\psi_v} 
_{1}^{N^2}, \overline{\psi_v^*} 
_{1}^{N}, \overline{\psi_v^{**}} 
_{1}^{N}) = 1, \sim 1, \asymp 1.$$

2.2. Elements of Functional Analysis

Let us remind some statements of functional analysis.

Let $X$ is a normed space. Let $K$ is a linear bounded operator from a normed space $X$ to a normed space $Y$. This fact we will write as $K \in [X, Y]$.

**Banach Theorem** [103]. Let $B$ be a Banach space. Let $A \in [B, B]$ with a norm $\|A\| = q < 1$. In this case the operator $K = I + A$ has the linear inverse operator $K^{-1}$ with the norm $\|K^{-1}\| \leq 1/(1 - q)$.

**Generalized Banach Theorem** [103]. Let $A$ and $B$ are linear bounded operators from Banach space $X$ to Banach space $Y$. Let the operator $A$ has the inverse operator $A^{-1} \in [Y, X]$. Let $q = \|A - B\|$. If $q\|A^{-1}\| < 1$, then the operator $B$ has the linear bounded operator $B^{-1} \in [Y, X]$ and the inequality

$$\|A^{-1} - B^{-1}\| \leq \frac{q\|A^{-1}\|^2}{1 - \|A^{-1}\|q}$$

is valid.

**Note.** From the proof of the Generalized Banach Theorem follows the inequality

$$\|B^{-1}\| \leq \|A^{-1}\|/(1 - q).$$
We will need in Kantorovich theory for approximate methods of analysis [90].

Let $X$ be a Banach space and let $X_n$ be a closed subspace of $X$.

Let us consider two equations:

$$Kx \equiv x + Hx = f, \quad K \in [X, X] \quad (2.2)$$

and

$$K_n x_n \equiv x_n + H_n x_n = f_n, \quad K_n \in [X_n, X_n]. \quad (2.3)$$

The equation (2.2) is a given equation. The equation (2.3) is an approximate equation.

Let the operator $K$ has the inverse bounded operator $K^{-1}$. Let $P_n$ is a projector from $X$ on $X_n$. Let the operators $K$ and $K_n$ are connected by following conditions:

I. For each $x \in X$ exists such $x_n \in X_n$, that

$$\|Hx - x_n\| \leq \eta_1(n)\|x\|;$$

II. For each $x_n \in X_n$

$$\|P_n Hx - H_n x_n\| \leq \eta_2(n)\|x_n\|;$$

III. For each $x \in X$ exists such $x_n \in X_n$ such that

$$\|x - x_n\| \leq \eta_3(n, x)\|x\|.$$

Note. In the conditions I and II constants $\eta_1, \eta_2$ are independent from $x$ or $x_n$.

Under conditions I-III the following statements is valid.

**Theorem 2.1** [90]. Let the operator $K$ is inverse. Let the conditions I and II are valid. If

$$q = [\eta_2(n) + \|I - P_n\|\eta_2(n)]\|K^{-1}\| < 1,$$

then the operator $K_n$ has the inverse operator $K_n^{-1}$ with the norm

$$\|K_n^{-1}\| \leq \|K^{-1}\|/(1 - q).$$
Theorem 2.2 [90]. Let the conditions of the Theorem 1 are fulfilled. Let the condition III is valid. The equation (2.3) has a unique solution $x^*_n$ and have place the inequality
\[ \|x^* - x^*_n\| \leq 2\eta_2(n)\|K_n^{-1}\| + (\eta_1(n) + \eta_3(x^*))\|K\|(1 + \|K_n^{-1}P_nK\|), \]
where $x^*$ is a unique solution of the equation (2.2).

During the paper we will use the following well known Hadamard Theorem.

Hadamard Theorem [65]. Let $|c_{jj}| > \sum_{k=0,k\neq j}^{n} |c_{jk}|$ for $j = 1, 2, \ldots, n$. Then the system $Cx = b$, where $C = \{c_{ij}\}, i, j = 1, 2, \ldots, n$, $x = (x_1, \ldots, x_n)^T$, $b = (b_1, \ldots, b_n)^T$, has a unique solution.

The convergence of numerical methods for solution nonlinear singular integral equations follows from convergence theorems for Newton – Kantorovich’s method in Banach spaces. Let us recall convergence criteria for the Newton – Kantorovich method.

Let $X, Y$ be Banach spaces. Consider the equation
\[ Kx = 0, \quad (2.4) \]
where $K$ is a nonlinear operator from $X$ into $Y$. Let the operator $K$ has the Frechet derivative in a neighborhood of initial point $x_0$. Assume that $[K'(x)]^{-1}$, or in a general case, the right inverse operator $[K'(x_0)]^{-1}$ exists.

We will seek for a solution of the equation (2.4) in the form of the following iteration processes – the basic
\[ x_{n+1} = x_n - [K'(x_n)]^{-1}K(x_n) \quad (2.5) \]
and the modified
\[ x_{n+1} = x_n - [K'(x_0)]^{-1}K(x_n). \quad (2.6) \]

If an operator $Kx$ has the second order Frechet derivative, the following assertion is used to justify the iteration processes (2.5) and (2.6).
Theorem 2.3 [90]. Let the operator $K$ is defined in $\Omega$ ($\|x - x_0\| < R$) and has continuous second order derivative in $\Omega_0$ ($\|x - x_0\| \leq r, r < R$). Let 
i) a countinuous linear operator $\Gamma_0 = [K'(x_0)]^{-1}$ exists; 
ii) $\|\Gamma_0 K(x_0)\| \leq \eta$; 
iii) $\|\Gamma_0 K''(x)\| \leq b_0$ ($x \in \Omega_0$).

Then, if $h = b_0 \eta \leq 1/2$ and $r = r_0 = (1 - \sqrt{1 - 2h})\eta/h$, the Newton methods (basic and modified) converge to a solution $x^*$ of the equation (2.4). Moreover, $\|x^* - x_0\| \leq r_0$.

If for $h < 1/2 \quad r < r_1 = (1 + \sqrt{1 - 2h})\eta/h$, and for $h = 1/2 \quad r \leq r_1$, then a solution $x^*$ is unique in the ball $\Omega_0$.

The convergence speed for the basic method is characterized by
$$\|x^* - x_n\| \leq \frac{1}{2^n} (2h)^{2^n} \frac{\eta}{h} \quad (n = 0, 1, ...),$$

the convergence speed for the modified method for $h < 1/2$ is characterized by
$$\|x^* - x'_n\| \leq \frac{\eta}{h} (1 - \sqrt{1 - 2h})^{n+1} \quad (n = 0, 1, ...).$$

If the Frechet derivative of $Kx$ satisfies the Lipschitz condition, convergence of the modified Newton - Kantorovich method follows from the following

Theorem 2.4 [95]. Let an operator $K$ is defined and Frechet differentiable on a ball $\Omega(x_0, R)$ ($\|x - x_0\| < R$), and its derivative $K'(x)$ satisfies on $\Omega(x_0, R)$ the Lipschitz condition $\|K'(x) - K'(y)\| \leq L\|x - y\|$. Let a linear bounded operator $\Gamma_0 = [K'(x_0)]^{-1}$ exists, and $\|\Gamma_0\| \leq b_0$, $\|\Gamma_0 K(x_0)\| \leq \eta_0$. Let $h_0 = b_0 L\eta_0 < 1/2, r_0 = (1 - \sqrt{1 - 2h_0})\eta_0/h_0 \leq R$. Then successive approximations (2.5) converge to a solution $x^* \in \Omega$ of the equation $K(x) = 0$.

In the more common case we have the following statement.

Theorem 2.5 [35]. Let $X$ and $Y$ be Banach spaces. Suppose the following conditions are fulfilled:
i) \[ \| K(x_0) \| \equiv \eta_0; \]

ii) The operator \( K \) has the Frechet derivative in neighbourhood of initial point \( x_0 \);

iii) There exists a right reverse operator \([K'(x_0)]^{-1}_r\) with the norm
\[ \| [K'(x_0)]^{-1}_r \| = B_0; \quad (2.7) \]

iii)
\[ \| K'(x_1) - K'(x_2) \| \leq q/(B_0(1 + q)) \quad (2.8) \]
in the sphere \( S\{ x : \| x - x_0 \| \leq \frac{B_0\eta_0}{1 - q} \} \) \( (q < 1) \).

Then, equation (2.4) has a solution \( x^* \) in \( S \). A sequence
\[ x_{n+1} = x_n - [K'(x_n)]^{-1}_r K x_n \]
converges to \( x^* \). The estimate \[ \| x^* - x_n \| \leq q^n \eta_0 B_0/(1 - q) \] is valid.

**Note.** Let a solution \( x^* \) of (2.4) enters the domain \( S_0 = S \cap \Delta R(x) \). Here \( \Delta R(x) \) is a domain in which \( \| [K'(x)]^{-1}_r \| \leq B_0 \), and \( \| K'(x_1) - K'(x_2) \| \leq q/(B_0(1 + q)) \), \( x_1, x_2 \in \Delta R(x) \) are valid.

Then, \( x^* \) is a unique solution in \( S_0 \).

Let us present the proof of the Theorem 2.6. First recall the Rakovchik Lemma.

**Theorem 2.6** [35]. Let \( X \) and \( Y \) be Banach spaces. Suppose the following conditions are fulfilled:

i) \[ \| K(x_0) \| \equiv \eta_0; \]

ii) The operator \( K \) has the Frechet derivative in a neighborhood of a initial point \( x_0 \), and there exists the right inverse operator \([K'(x_0)]^{-1}_r\) with the norm \( \| [K'(x_0)]^{-1}_r \| = B_0; \)

iii) The condition \[ \| K'(x_1) - K'(x_2) \| \leq q/B_0 \] is fulfilled in the sphere \( S\{ x : \| x - x_0 \| \leq \frac{B_0\eta_0}{1 - q} \} \) \( (q < 1) \).

Then, equation (2.4) has a solution \( x^* \) in \( S \). A sequence \( x_{n+1} = x_n - [K'(x_n)]^{-1}_r K x_n \) converges to \( x^* \). The estimate \[ \| x^* - x_n \| \leq q^n \eta_0 B_0/(1 - q) \] is valid. A solution \( x^* \) is unique in \( S \cap (\Delta R(x)) \).

Let us present the proof of the Theorem 2.6. First recall the Rakovchik Lemma.
Lemma 2.1 [120]. Let $X$ and $Y$ be Banach spaces, $A$ and $B$ be linear bounded operators mapping $X$ into $Y$. If $A$ has a bounded right inverse operator $A^{-1}_r$ and if an operator $B$ satisfies $\|A - B\|_r A^{-1}_r < 1$, it has a bounded right inverse operator $B^{-1}_r, \|B^{-1}_r\| \leq \|A^{-1}_r\|/(1 - \|A - B\|_r A^{-1}_r)$.

Proof of the Theorem 2.6. First we prove that a uniformly bounded right inverse operator $[K(x_n)]^{-1}_r$ exists in the domain $S$. Indeed, from Rakovchik’s lemma it follows

$$\|[K'(x_n)]^{-1}_r\|_Y \leq (1 + q)B_0, \quad (2.9)$$

Show that all the approximations obtained by the iterative process (2.6) are in $S$.

Clearly, $\|x_1 - x_0\|_X = \|[K'(x_0)]^{-1}_r K(x_0)\|_X \leq B_0 \eta_0 < B_0 \eta_0/(1 - q)$. So, $x_1 \in S$.

Let $x_m \in S$ for $m \leq n$ have already been proved. Since $x_n - x_{n-1} = [K'(x_{n-1})]^{-1}_r K(x_{n-1})$, then by Lemma we have

$$\|x_{n+1} - x_n\|_X \leq
\leq \|[K'(x_n)]^{-1}_r [K(x_n) - K(x_{n-1}) - K'(x_{n-1})(x_n - x_{n-1})]\|_X \leq
\leq q\|x_n - x_{n-1}\|_X. \quad (2.10)$$

Therefore $\|x_{n+1} - x_0\|_X \leq \sum_{k=0}^n q^k B_0 \eta_0$, i.e. $x_{n+1} \in S$.

From (2.10) it follows that $\{x_n\}$ is a fundamental sequence. Hence $x^* = \lim x_n$ exists. Since $K(x_n) = -K'(x_n)(x_{n+1} - x_n)$, then $K(x^*) = 0$. It follows from (2.10) that

$$\|x^* - x_n\|_X \leq \sum_{k=n}^{\infty} q^k B_0 \eta_0 \leq \frac{q^n B_0 \eta_0}{1 - q}.$$ 

The theorem is proved.

2.3. Elements of Approximation Theory

We give some well known results from theory of approximation functions of real variable.
Theorem 2.7 \cite{109}. If a trigonometric polynomial $T_n(x)$ of order $n$ satisfies on $[0, 2\pi]$ the inequality $|T_n(x)| \leq M$, then

$$|T'_n(x)| \leq nM, \quad x \in [0, 2\pi]. \quad (2.11)$$

Let $P_n(x), -1 \leq x \leq 1$, is a algebraic polynomial of degree $n$.

Theorem 2.8 \cite{109}. Let $P_n(x) \leq M$ for $-1 \leq x \leq 1$. Then

$$|P'_n(x)| \leq \frac{Mn}{\sqrt{1 - x^2}}, \quad -1 < x < 1. \quad (2.12)$$

Theorem 2.9 (A.A. Markov) \cite{109}. Let $P_n(x), -1 \leq x \leq 1$. If $P_n(x) \leq M$ on $[a, b]$, then

$$|P'_n(x)| \leq \frac{2Mn^2}{b - a}, \quad a \leq x \leq b.$$

Theorem 2.10 (S.M. Nikolskii) \cite{111}. Let $T_n(x)$ is $n$-order trigonometric polynomial on $[0, 2\pi]$. Then

$$\|T_n(x)\|_C \leq n^{1/p}\|T_n\|_{L_p} \quad (1 \leq p \leq \infty).$$

The $n$th degree of the best approximation of a function $f \in \tilde{C}[0, 2\pi]$ by trigonometrical polynomial $T_n$ of degree $n$ is defined by

$$\tilde{E}_n(f) = \min_{T_n} \max_x |f(x) - T_n(x)|.$$

In this section we will give estimations of $\tilde{E}_n(f)$.

Theorem 2.11 \cite{109}, \cite{102}. There exists a constant $M$ such that, for each $f \in \tilde{C}[0, 2\pi]$

$$\tilde{E}_n(f) \leq M \omega(f; \frac{1}{n}), \quad n = 1, 2, \ldots.$$ \(\text{Note.}\) N.P. Korneichuk \cite{92} proved that $M = 1$.

Theorem 2.12 \cite{109}. For each $r = 1, 2, \ldots$ there is a constant $M_r$ with the property that if $f \in \tilde{C}^r[0, 2\pi]$ has a continuous derivative $f^{(r)}(x)$, then

$$\tilde{E}_n(f) \leq \frac{M_r}{n^r} \omega(f^{(r)}; \frac{1}{n}), \quad n = 1, 2, \ldots.$$
Theorem 2.13 [130]. If the function \( f(x) \) has continuous derivatives \( f'(x), \ldots, f^{(r)}(x) \) on \([-1, 1]\), then is a sequence of polynomials \( P_n(x) \) for which

\[
|f(x) - P_n(x)| \leq M_r \left[ \frac{1}{n} \left( \sqrt{1 - x^2} + \frac{1}{n} \right) \right]^r \omega \left( f^{(r)}; \frac{1}{n} \left( \sqrt{1 - x^2} + \frac{1}{n} \right) \right),
\]

where the constant \( M_r \) depend only upon \( r \).

2.4. Inverse Theorems

In this section we will give some statements that are derived smoothness properties of a function \( f \in \tilde{C}[0, 2\pi] \) from the hypothesis that the numbers \( \tilde{E}_n(f) \) approach zero with a given rapidity. Theorems of this kind were first obtained by Bernstein. His results have been developed further by Zigmund, Gaier, Stechkin and other. (For detail see [109], [102].)

Theorem 2.14 [109], [102]. A function \( f \in \tilde{C}[0, 2\pi] \) belongs to the class \( H_\alpha \) \( (0 < \alpha < 1) \), if and only if

\[
\tilde{E}_n(f) = O(n^{-\alpha}).
\]

Theorem 2.15 [109], [102]. A function \( f \in \tilde{C}[0, 2\pi] \) belongs to the class \( Z \) if and only if

\[
\tilde{E}_n(f) = O(n^{-1}).
\]

Theorem 2.16 [109], [102]. The \( 2\pi \)-periodic continuous function \( f \) belongs to the class \( W^r H_\alpha \) \( (0 < \alpha < 1) \) if and only if

\[
\tilde{E}_n(f) = O \left( \frac{1}{n^{r+\alpha}} \right).
\]

Here we give the designations which are used in the paper.

Let \( f(x) \in \tilde{C}[0, 2\pi] \) be a periodic function with \( 2\pi \) period. Let \( s_k = 2k\pi/(2n + 1), k = 0, 1, \ldots, 2n \). Then interpolating polynomial \( P_n(x) \) can be written in the form

\[
P_n(x) = \sum_{k=0}^{2n} f(s_k) \psi_k(x),
\]
where
\[ \psi_k(x) = \frac{1}{2n+1} \frac{\sin \frac{2n+1}{2}(x-s_k)}{\sin \frac{1}{2}(x-s_k)}. \]

Let \( \gamma = \{ z : |z| = 1 \} \). For a function \( f(t), t \in \gamma \), the interpolating polynomial can be written in the form
\[ P_n(f) = \sum_{k=0}^{2n} f(t_k) \psi_k(s), \]
where \( t_k = e^{is_k}, s_k = 2k\pi/(2n+1), k = 0, 1, \ldots, 2n, s \in [0, 2\pi] \).

In this paper we will use the Holder space of functions \( x(t) \in H_\beta (0 < \beta \leq 1) \) with the norm
\[ \|x(t)\| = M(x) + H(x, \beta) = \max_{t \in \gamma} |x(t)| + \sup_{t_1, t_2 \in \gamma, t_1 \neq t_2} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\beta} \]
and its subspace \( X_n \), consisting of polynomials \( x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k \).

Chapter 1

Approximate Solution of Singular Integral Equations

1. An Smoothness of Solutions of Singular Integral Equations

Consider the connection between a smoothness of coefficients of singular integral equations and a smoothness of their solutions.
1.1. The Integral Operators on the Smooth Functions

1.1.1. Fundamental Statements

**Lemma 1.1** [26], [35]. If $h(t, \tau) \in H_{\alpha\alpha}, 0 < \alpha \leq 1$, then the operator

$$Kx \int_0^{2\pi} (h(t, \tau) - h(t, t)) | \cot \frac{\tau - t}{2} |^\eta x(\tau)d\tau,$$

$0 \leq \eta < 1$, transforms every function $x(t) \in C[0, 2\pi]$ to the function belonging to the class $H_{\alpha}(0 < \alpha \leq 1)$.

**Lemma 1.2** [26], [35]. Let $x \in H_{\alpha}(0 < \alpha \leq 1)$. Then the function $v(t)$,

$$v(t) = \int_0^{2\pi} x(\tau) | \cot \frac{\tau - t}{2} |^\eta d\tau,$$

belongs to the class $Z$ if $\alpha = \eta$ or belongs to the class $H_{\gamma}$ if $\alpha \neq \eta$, moreover $\gamma = 1$ if $\alpha > \eta$ and $\gamma = \alpha + 1 - \eta$ if $\alpha < \eta$.

**Lemma 1.3** [26], [35]. Let the integral equation

$$Hx \equiv x(t) + \int_0^{2\pi} h(t, \tau) | \cot \frac{\tau - t}{2} |^\eta x(\tau)d\tau = f(t), \quad 0 \leq \eta < 1,$$

where $h(t, \tau) \in H_{\alpha\alpha}, f(t) \in H_{\alpha}(0 < \alpha \leq 1)$, has a unique solution $x^*(t)$. Then $x^*(t) \in H_{\alpha}$.

**Lemma 1.4** [26], [35]. Let the integral equation $Hx = f$, where $h(t, \tau) \in W^{rr}H_{\alpha\alpha}, f(t) \in W^rH_{\alpha}$, has a unique solution $x^*(t)$. Then $x^*(t) \in W^rH_{\alpha}$. 
1.2. On Smoothness of Solutions of the Singular Integral Equations on Closed Contours

1.2.1. Fundamental Statements
Let us consider the singular integral equation

\[ Kx \equiv a(t)x(t) + b(t)(S_\gamma x)(t) + U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau)) = f(t) \]

(1.1)

where \( \gamma = \{ z : |z| = 1 \} \),

\[ S_\gamma(x) \equiv \frac{1}{\pi i} \int_\gamma \frac{x(\tau)d\tau}{\tau - t}, \quad U_\gamma(hx) = \frac{1}{2\pi i} \int_\gamma h(t, \tau)x(\tau)d\tau, \quad 0 \leq \eta < 1. \]

Let us put the characteristic operator \( K^0x = ax + bS_\gamma x \) in accordance to the operator \( K \).

**Theorem 1.1** [26], [35]. Let \( a(t), b(t), f(t) \in H_\alpha, h(h, \tau) \in H_{\alpha\alpha} (0 < \alpha < 1) \). Suppose that the index of the operator \( K^0 \) is equal to zero, and the operator \( K \) is continuously invertible in the space \( H_\beta (0 < \beta \leq \alpha) \). Then a unique solution \( x^*(t) \) of the equation (1.1) belongs to the class \( H_\alpha \).

**Note.** If the conditions of Theorem 1.1 are fulfilled, then the operator \( K \) operating from \( H_\alpha \) into \( H_\alpha \) is continuously invertible.

**Theorem 1.2** [26], [35]. Let \( a(t), b(t), f(t) \in W^r H_\alpha, h(t, \tau) \in W^{rr\alpha} H_{\alpha\alpha}, 0 < \alpha < 1 \). Suppose that the index of the operator \( K^0 \) is equal to zero, and the operator \( K \) is continuously invertible in the space \( H_\beta (0 < \beta \leq \alpha) \). Then the solution \( x^*(t) \) of the equation (1.1) belongs to the space \( W^r H_\alpha \).

2. Approximate Solution of Linear Singular Integral Equations on Closed Contours of Integration
    (Basis in Holder Space)

In this section we investigate approximate methods for solution of singular integral equations

\[ Kx \equiv a(t)x(t) + b(t)S_\gamma(x) + U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau)) = \]
\[ Lx \equiv a(t)x(t) + S_\gamma(h(t,\tau)x(\tau)) = f(t). \] (2.2)

Here \( a(t), b(t), f(t) \in H_\alpha, h(t, \tau) \in H_{\alpha,\alpha}, 0 < \alpha \leq 1. \)

General form of one-dimensional singular integral equations is

\[ a(t)x(t) + \int_\gamma \frac{k(\tau,t)}{\tau - t} d\tau = f(t). \]

Let us put

\[ b(t) = k(t,t), \quad \frac{h(t,\tau)}{|\tau - t|^{\eta}} = \frac{k(t,\tau) - k(t,t)}{\tau - t}. \]

In result we receive the singular integral equation in the form (2.1).

Here we use the following designations

\[ S_\gamma x = \frac{1}{\pi i} \int_\gamma \frac{x(\tau)d\tau}{\tau - t}, \quad U_\gamma(h(t,\tau)x(\tau)) = \frac{1}{2\pi i} \int_\gamma h(t,\tau)x(\tau)d\tau, \]

where \( \gamma \) is the unit circle with the center in origin of coordinates: \( \gamma = \{z : |z| = 1\} \).

In this section we will consider collocation method and method of mechanical quadrature.

Proofs of these methods we give in Holder space \( X \) of functions \( x(t) \in H_\beta (0 < \beta \leq \alpha < 1) \) with the norm

\[ \|x(t)\| = \max_{t \in \gamma} |x(t)| + \sup_{t_1,t_2 \in \gamma, t_1 \neq t_2} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\beta} \]

and its subspace \( X_n \), consisting of \( n \)-order polynomials \( x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k. \)

Also we will need in the space \( W(\gamma) \) introduced in the book [83].

Let us associate with a function \( \varphi(z) \) functions \( \varphi^+(t) \) and \( \varphi^-(t) \), which are analytical inside and outside of the contour
\( \gamma \) respectively and connected with \( \varphi(t) \) by Sohotzky - Plemel formulas
\[
\varphi^+(t) - \varphi^-(t) = \varphi(t), \quad \varphi^+(t) - \varphi^-(t) = S_\gamma \varphi(t).
\]
Well known [64] that Sohotzky - Plemel formulas have place if the function \( \varphi(t) \) belongs to common enough classes of functions.

The space \( W(\gamma) \) consists of functions \( \varphi(t), t \in \gamma \), for which functions \( \varphi^+(t) \) and \( \varphi^-(t) \) are continuous. The norm in the space \( W(\gamma) \) is introduced with the formula
\[
\| \varphi \|_{W(\gamma)} = \max_{t \in \gamma} |\varphi^+(t)| + \max_{t \in \gamma} |\varphi^-(t)|.
\]

2.1. Methods of Collocations and Mechanical Quadratures

2.1.1. Fundamental Statements

Many works was devoted to numerical solution of the equations (2.1) with \( \eta = 0 \). First of them was the V.V. Ivanov’s paper [82], in which the collocation method was used for solution of the equation (2.1) with \( \eta = 0 \).

An approximate solution of equation (2.1) is sought in the form of the polynomial
\[
x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k.
\]

According the collocation method, coefficients \( \alpha_k, k = -n, n \), are defined from the system of linear algebraic equations, that in the operator form is written as
\[
P_n[a(t)x_n(t) + b(t)S_\gamma(x_n) + U_\gamma(h(t, \tau)x_n(\tau)) = P_n[f], \quad (2.4)
\]
where operator \( P_n \) is the projector of interpolation onto the set of polynomials, constructed on the knots \( t_k = \exp(is_k), s_k = 2k\pi/(2n + 1), k = 0, 1, \ldots, 2n \).

**Theorem 2.1** [82], [83]. Let the functions \( a, b, f \in H_\alpha \), \( h(t, \tau) \in H_{\alpha\alpha} \) \((0 < \alpha \leq 1)\), \( \eta = 0 \) and the operator \( K \) is
continuously invertible in the space $W(\gamma)$. Then for large enough $n$ the system (2.4) has a unique solution $x^*_n(t)$ and the estimate

$$\|x^*(t) - x^*_n(t)\|_{W(\gamma)} \leq A \ln n \|x^*(t) - T_n(x^*)\|_{W(\gamma)}$$

is valid. Here $x^*(t)$ is a solution of the equation (2.1), $T_n(t) = \sum_{k=-n}^{n} \beta_k t^k$ is the polynomial of the best approximation to $x^*(t)$ in the metric of the space $W(\gamma)$.

**Note.** The similar results is valid when the index of the operator $K$ do not equal to zero.

Later the similar results was received in the Holder space.

**Theorem 2.2 [62].** Let the functions $a(t), b(t), f(t) \in H_\alpha$, $h(t, \tau) \in H_{\alpha\alpha}, \eta = 0$ and the operator $K$ is continuously invertible in the Holder space $X = H_\beta (0 < \beta < \alpha).$ Then for $n$ such that $q = An^{-\alpha+\beta} \ln n < 1$, the system (2.4) has a unique solution $x^*_n(t)$ and the estimation $\|x^* - x^*_n\|_{\beta} \leq An^{-(\alpha-\beta)} \ln n$, where $x^*(t)$ is a solution of the equation (2.1), is valid.

Apparently the first works devoted to numerical methods for solution of the equations (2.1) (for $0 < \eta < 1$) and (2.2) was papers [11], [15]. In these papers was investigated the collocation method and mechanical quadrature method for solution of the equations (2.1) and (2.2).

Let the index of the operator $K$ is equal to zero.

An approximate solution of the equation (2.1) we look for in the form of the polynomial (2.3).

The coefficients $\alpha_k$, $k = -n, \ldots, -1, 0, 1, \ldots, n$, are defined from the system of linear algebraic equations

$$K_n x_n \equiv P_n[a(t)x_n(t)+b(t)S_\gamma(x_n)+U_\gamma(P_n^* h(t, \tau)d(t, \tau)x_n(\tau))] = P_n[f(t)],$$

where

$$d(t, \tau) = \begin{cases} \begin{array}{l} |\tau - t|^{\eta}, \quad |\sigma - s| \geq 2\pi/(2n + 1), \\ |e^{i2\pi/(2n+1)} - 1|^{\eta}, \quad |\sigma - s| < 2\pi/(2n + 1), \end{array} \end{cases}$$
\[ \tau = e^{i\sigma}, t = e^{is}. \]

**Theorem 2.3** [11], [15], [17]. Let the functions \( a, b, f \in H_\alpha \), \( h \in H_{\alpha,\alpha} \) \((0 < \alpha \leq 1)\) and the operator \( K \) is continuously invertible in the Holder space \( X = H_\beta(0 < \beta < \min(\alpha, 1 - \eta)) \). Then for \( n \) such that \( q = An^{-\xi}\ln n < 1 \) \((\xi = \min(\alpha - \beta, 1 - \eta - \beta, \beta))\), the system of equations (2.5) has a unique solution \( x_n^* \) and the estimation \( \|x^* - x_n^*\|_\beta < An^{-\xi}\ln n \) is valid. Here \( x^* \) is a unique solution of the equation (2.1).

The estimate of error obtained in preceding Theorem depends on the constant \( \eta \). Let us build a calculating scheme the estimation of the error of which depends only on a smoothness of the functions \( a, b, h, f \).

For simplicity we will take \( G(t) = (a(t) - b(t))/(a(t) + b(t)) \), \( a(t) + b(t) \equiv 1 \).

With each function \( x(t) \in H_\beta \) we associate functions \( x^+(t) \) and \( x^-(t) \) which are analytical inside and outside \( \gamma \) respectively and connected with \( x(t) \) by Sohotzky - Plemel formulas \( x^+(t) - x^-(t) = x(t), S_\gamma x = x^+(t) + x^-(t) \).

The approximate solution of the equation (2.1) we will look for in the form of the polynomial (2.3), the coefficients of which are defined from the system of equations, that representable in the operator form by the expression

\[
K_n x_n = P_n[a(t)x_n(t) + b(t)S_\gamma(x_n(\tau))] + \sum_{k=0}^{2n} h(t, t_k)x_n(t_k) \int_{t_k'}^{t_{k+1}'} |\tau - t|^{-\eta} d\tau = P_n[f(t)], \quad (2.6)
\]

where \( t_{k+1}' = e^{is_{k+1}}, s_{k+1}' = (2k + 1)\pi/(2n + 1), k = 0, 1, \ldots, 2n \).

**Theorem 2.4** [11], [26], [35]. Let the conditions of Theorem 2.3 are fulfilled. Then among all possible approximate methods for solution of the equation (2.1), using \( n \) values of the functions
a, b, f and \( n^2 \) values of the function \( h \), the method, being described by the calculating scheme (2.6), is optimal with respect to order on the class \( H_\alpha \). The error of this method is equal to 
\[
\|x^* - x^*_n\|_\beta = 0((n^{-\alpha+\beta} + n^{-(1-\gamma)+\beta}) \ln n),
\]
where \( x^* \) and \( x^*_n \) are solutions of the equations (2.1) and (2.6) respectively.

Now assume that \( a(t), b(t), f(t) \in W^r \), \( h(t, \tau) \in W^{r, r} \).

The approximate solution of the equation (2.1) we will seek in the form of the polynomial (2.3), the coefficients \( \alpha_k \) of which are defined from the system of linear algebraic equations

\[
K^{(1)}_n x_n \equiv P_n[a(t)x_n(t) + b(t)S_\gamma(x_n(\tau)) + U_\gamma[P^\tau_n[h(t, \tau)]x_n(\tau)|\tau - t|^{-\eta}] = P_n[f(t)]. \tag{2.7}
\]

The next Theorem is a generalization of some optimal with respect to order algorithms for solution of singular integral equations, printed in [11], [26], [35].

**Theorem 2.5.** Let the operator \( K \) is continuously invertible in the Holder spaces \( X = H_\beta(0 < \beta < \alpha, 0 < \alpha < 1) \). Let

\[
a, b, f \in W^r H_\alpha(M), h \in W^{r, r} H_{\alpha, \alpha}(M), r = 0, 1, \ldots \tag{2.8}
\]

Among all possible algorithms for solution of the equation (2.1), using \( n \) values of the functions \( a, b, f \) and \( n^2 \) values of the function \( h \), the calculating scheme (2.3), (2.7) is optimal with respect to order. The error of this calculating scheme is equal to 
\[
\|x^* - \tilde{x}^*_n\|_\beta \leq An^{-(r+\alpha-\beta)} \ln n,
\]
where \( x^* \) and \( x^*_n \) are solutions of the equations (2.1), (2.7) respectively.

The approximate solution of the equation (2.2) is sought in the form of the polynomial (2.3), the coefficients \( \alpha_k \) of which are defined from the system of linear algebraic equations

\[
L_n x_n \equiv P_n[a(t)x_n(t) + S_\gamma(P^\tau_n[h(t, \tau)]x_n(\tau))] = P_n[f(t)]. \tag{2.9}
\]

**Theorem 2.6** [26], [35]. Let the operator \( L \) is continuously invertible in the Holder spaces \( X = H_\beta(0 < \beta < \alpha \leq 1) \). Then
for $n$ such that $q = A(n^\beta \ln^2 n(E_n(a) + E_n(b) + E_n(\psi) + E_n^t(h) + E_n^\tau(h)) < 1$, the system of the equations (2.9) has a unique solution $x_n^*$ and the inequality $\|x^* - x_n^*\| \leq A(n^\beta \ln^2 n(E_n(a) + E_n(b) + E_n(\psi) + E_n^t(h) + E_n^\tau(h)) + n^\beta E_n(f))$ is valid, where $x^*$ is a solution of the equation (2.2),

$$
\psi(z) = \exp \left\{ \frac{1}{2\pi} \int_\gamma (\ln G(\tau))(\tau - z)^{-1} d\tau \right\},
$$

$$
G(t) = (a(t) - b(t))/(a(t) + b(t)), b(t) = h(t, t).
$$

Let $\overline{P}_n$ is the projector for interpolation onto the set of trigonometrical polynomials, constructed on the knots $\overline{t}_k = e^{i\overline{s}_k}$, $\overline{s}_k = 2k\pi/(2n + 1), k = 0, 1, \ldots, 2n$.

Let an approximate solution of the equation (2.2) is sought in the form of the polynomial (2.3), the coefficients $\alpha_k (k = -n, \ldots, -1, 0, 1, \ldots, n)$ of which are defined from the system of the linear algebraic equations

$$
\tilde{L}_n x_n \equiv \overline{P}_n \left[ a(t)x_n(t) + \frac{1}{\pi i} \int_\gamma P_n^\tau \left[ \frac{h(t, \tau) x_n(\tau)}{\tau - t} \right] d\tau \right] = \overline{P}_n [f(t)].
$$

**Theorem 2.7** [15], [26], [35]. Let the operator $L$ is continuously invertible. Then for $n$ such that $q = A(E_n(a) + E_n(\psi) + E_n^t(h) + E_n^\tau(h)) \ln^2 n < 1$, the system of the equations (2.10) has a unique solution $x_n^*$ and the inequality $\|x^* - x_n^*\|_\beta \leq A(q + E_n(f))$ holds, where $x^*$ is a solution of the equation (2.2),

$$
\psi(z) = \exp \left\{ \frac{1}{2\pi} \int_\gamma (\ln G(\tau))(\tau - t)^{-1} d\tau \right\},
$$

$$
G(t) = (a(t) - b(t))/(a(t) + b(t)), b(t) = h(t, t).
$$
Let us build the optimal with respect to order calculating scheme for approximate solution of the singular integral equations (2.2). We shall seek an approximate solution in the form of the polynomial (2.3), the coefficients of which $\alpha_k (k = -n, \ldots, n)$ are defined from the system of linear algebraic equations

$$\overline{L}_nx_n \equiv \overline{P}_n \left[ a(t)x_n(t) + \frac{h(t,t)}{\pi i} \int_{\gamma} \frac{x_n(\tau)}{\tau - t} d\tau + \right.$$

$$+ \frac{1}{\pi i} \int_{\gamma} P^n T \left[ \frac{h(t,\tau) - h(t,t)}{\tau - t} x_n(\tau) \right] d\tau \left. \right] = \overline{P}_n[f(t)]. \quad (2.11)$$

**Theorem 2.8** [15], [17], [35]. Let the operator $L$ is continuously invertible. Among all possible algorithms for approximate solution of the singular integral equation (2.2), using $n$ values of the functions $a(t), f(t)$ and $n^2$ values of the function $h(t, \tau)$, the calculating scheme (2.3), (2.11) is optimal with respect to order and has the error

$$\|x^* - x_n^*\|_C \leq A(E_n(a) + E_n(\psi) + E^t_n(h(t,\tau) + E^t_n(h(t,\tau)) + E_n(f)) \ln n,$$

where $x^*$ and $x_n^*$ are solutions of the equations (2.2) and (2.9) respectively, $\psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\gamma} (\ln G(\tau))(\tau - z)^{-1} d\tau \right\}$,

$$G(t) = (a(t) - b(t))/(a(t) + b(t)), b(t) = h(t,t).$$

### 2.1.2. Proofs of Theorems

We drop proofs of the Theorem 2.1 and the Theorem 2.2, because its are special cases of the Theorem 2.3.

**Proof of Theorem 2.3.** Here we give the full proof of the Theorem 2.3, because it had not been printed before.

At first let us investigate the collocation method

$$K_nx_n \equiv P_n[a(t)x_n(t) + b(t)S_\gamma x_n + U_\gamma(h(t,\tau)| \tau - t |^{-\eta}x_n(\tau)] =$$
\[= P_n[f(t)]. \quad (2.12)\]

Fix function \(x(t) \in H_\beta\) and associate with it the functions \(x^+(t)\) and \(x^-(t)\), which are analytical inside and outside of \(\gamma\) respectively and connected with \(x(t)\) by Sohotzky - Plemel formulas \(x^+(t) - x^-(t) = x(t), S_\gamma x = x^+(t) + x^-(t)\). Since the function \(G(t)\) is represented \([64]\) as

\[G(t) = \frac{\psi^+(t)}{\psi^-(t)}, \psi^\pm(z) = \exp(\Theta^\pm(z)), \Theta(z) = \frac{1}{2\pi i} \int_\gamma \ln G(\tau) d\tau,\]

then the equations (2.1), (2.5) and (2.12) are equivalent to the following equations respectively

\[F\psi \equiv \psi^- - \psi^+ \psi^- + \psi^- U_\gamma(h(t, \tau)| \tau - t |^{-\eta}x(\tau)) = \psi^- f;\]
\[\tilde{F}_n\psi \equiv P_n[\psi^- x^+_n - \psi^+ x^-_n + \psi^- U_\gamma[h(t, \tau)d(t, \tau)x_n(\tau)]] = P_n[\psi^- f];\]
\[F_n x = P_n[\psi^- x^+_n - \psi^+ x^-_n + \psi^- U_\gamma[h(t, \tau)| \tau - t |^{-\eta}x_n(\tau))]] = P_n[\psi^- f].\]

The operator \(K\) is continuously invertible. So the operator \(F\) make one-to-one mapping of the space \(X\) onto inself. It is follows from Banach Theorem that operator \(F\) has linear inverse operator in the space \(X\) also.

Let us introduce the operator

\[
V\psi \equiv \psi^- x^+ - \psi^+ x^- + T_n[\psi^- U_\gamma(h(t, \tau)| \tau - t |^{-\eta}x(\tau))],
\]

where \(\psi_n = \psi^+_n - \psi^-_n, \psi^+_n(\psi^-_n)\) is the polynomial of best uniform approximation of degree \(n\) for the function \(\psi^+_n(\psi^-_n): \psi^\pm = T_n\psi^\pm.\) Here \(T_n\) is the projector, which put to according each continuous function \(f(x)\) its polynomial of best uniform approximation. It was marked in the first section of this chapter, that the operator \(U_\gamma(h(t, \tau)| \tau - t |^{-\eta}x(\tau))\) maps any continuous function into function belonging to the Holder class \(H_\delta(\delta = \min(\alpha, 1 - \eta)).\)
Therefore $\|Fx - Vx\|_\beta \leq An^{-(\delta-\beta)}\|x\|_\beta$;
$\|P_n[Fx - Vx]\|_\beta \leq \|P_n\|_\beta \|Fx - Vx\|_\beta \leq An^{-(\delta-\beta)}\ln n\|x\|_\beta$.

In obtaining the last relation was used the known [109], [110] inequality $\|P_n\|_C \leq 8 + \frac{4}{\pi}\ln n$. It follows from the equality $P_nVx_n = Vx_n$, that $\|Fx_n - F_nx_n\|_\beta = \|Fx_n - Vx_n + P_nVx_n - F_nx_n\|_\beta \leq \|Fx_n - Vx_n\|_\beta + \|P_nVx_n - F_nx_n\|_\beta \leq An^{-(\delta-\beta)}\ln n\|x_n\|_\beta$.

It is easy to see that $\|F_nx_n\| = \|Fx_n - (Fx_n - F_nx_n)\| \geq \|Fx_n\| - \|(Fx_n - F_nx_n)\| \geq \frac{1}{\|F^{-1}\|}\|x_n\| - An^{-(\delta-\beta)}\ln n\|x_n\| \geq A\|x_n\|$.

From this inequality and the Theorem about left inverse operator [90], p. 449, it follows that, for $n$ such that $An^{-(\alpha-\beta)}\ln n < 1$, the operator $F_n$ has left inverse operator. Since $F_n$ is finite-dimensional operator, then the operator $F_n$ has continuously invertable operator $F_n^{-1}$.

Let solutions of the equations (2.1) and (2.12) are denoted by $x^*$ and $x_n^*$ respectively. Then, as follows from the general theory of the approximate methods of the analysis, the estimate $\|x^* - x_n^*\|_\beta \leq An^{-(\delta-\beta)}\ln n$ is valed. The collocation method is justified.

Introduce the operator

$V_nx_n \equiv P_n[\psi^-x_n^+ - \psi^+x_n^- + \psi^-U_\gamma(h(t, \tau)d(t, \tau)x_n(\tau))]$.

Let us estimate the $\|F_nx_n - V_nx_n\|_\beta$.

It is easy to see that $\|F_nx_n - V_nx_n\|_\beta = \|P_n[\psi^-U_\gamma(h(t, \tau)(|\tau - t|^{-\eta} - d(t, \tau))x_n(\tau))]\|_\beta$.

In the space $\tilde{C}[0, 2\pi]$

$\|P_n[\psi^-U_\gamma(h(t, \tau)(|\tau - t|^{-\eta} - d(t, \tau))x_n(\tau))]\|_C \leq \|P_n\|\|\psi^-U_\gamma(h(t, \tau)(|\tau - t|^{-\eta} - d(t, \tau))x_n(\tau))\|_C \leq$
\[
\leq A \ln n \| \psi^+ U_{\gamma}(h(t, \tau)(| \tau - t |^{-\eta} - d(t, \tau)) x_n(\tau)) \|_C \leq An^{-(1-\eta)} \ln n \| x_n \|_C \leq An^{-(1-\eta)} \ln n \| x_n \|_\beta
\]

and hence
\[
\| F_n x_n - V_n x_n \|_\beta \leq An^{-(1-\eta-\beta)} \ln n \| x_n \|_\beta. \quad (2.13)
\]

Let us estimate the norm of the difference
\[
\| V_n x_n - \tilde{F}_n x_n \|_\beta = \| P_n[\psi^+ U_{\gamma}(R_n^{\tau}[h(t, \tau)d(t, \tau)x_n(t)])] \| \leq
\]
\[
\leq P_n \left[ \frac{\psi^{-*}(s)}{2\pi} \int_0^{2\pi} h^*(s, \sigma) d^*(s, \sigma) e^{i\sigma} (x_n(e^{i\sigma}) - \overline{x}_n(e^{i\sigma})) d\sigma \right] +
\]
\[
+ P_n \left[ \frac{\psi^{-*}(s)}{2\pi} \sum_{k=0}^{2n} \int_{s_k}^{s_{k+1}} h^*(s, \sigma) d^*(s, \sigma) e^{i\sigma} e^{i\sigma} - h^*(s, s_k) d^*(s, s_k) e^{i\sigma} \right] \times
\]
\[
\times \overline{x}_n(e^{i\sigma}) d\sigma \right] \| = \| I_1(s) \|_\beta + \| I_2(s) \|_\beta,
\]

where \( \overline{x}_n(e^{i\sigma}) \) is the step-function equal to \( x_n(e^{i\sigma}) \) on the interval \([s_k, s_{k+1})\), \( \psi^{-*}(s) = \psi^-(e^{is}) \), \( h^*(s, \sigma) = h(e^{is}, e^{i\sigma}) \), \( d^*(s, \sigma) = d(e^{is}, e^{i\sigma}) \), \( R_n^{\tau} = I - P_n^{\tau} \).

Since \( x_n(e^{i\sigma}) \in H_\beta \) and \( \overline{x}_n(e^{i\sigma}) \) is a step-function coinciding with \( x_n(e^{i\sigma}) \) at the points \( s_k \), then
\[
\| I_1(s) \|_\beta \leq A \ln n \max | x_n(e^{i\sigma}) - \overline{x}_n(e^{i\sigma}) | \leq A \ln n \| x_n \|_\beta / n^\beta.
\]

Using the fact that \( P_n f = \sum_{k=0}^{2n} f(s_k) \psi_k(s) \), where \( \psi_k(s) \) are fundamental polynomials constructed on the points \( s_k \), and that
\[
\max | P_n f | \leq (A + B \ln n) \max | f |,
\]
one has
\[
\max_s | I_2(s) | \leq
\]
\[
\leq A \ln n \max_{0 \leq j \leq 2n} \left\{ \frac{\psi^{-*}(s_j)}{2\pi} \sum_{k=0}^{2n} \int_{s_k}^{s_{k+1}} h^*(s_j, \sigma) d^*(s_j, \sigma) e^{i\sigma} -
\]

34
\[- h^*(s_j, s_k) d^*(s_j, s_k) e^{is_k} \bar{x}_n(e^{i\sigma}) d\sigma \right\} | \leq \\
\leq A \ln n \max_{0 \leq j \leq 2n} \left\{ \left| \frac{\psi^* - (s_j)}{2\pi} \sum_{k=0}^{2n} s_k^{s_k+1} \int h^*(s_j, \sigma) e^{i\sigma} - \\
- h^*(s_j, s_k) e^{is_k} d^*(s_j, \sigma) \bar{x}_n(e^{i\sigma}) d\sigma \right| + \\
+ \left| \frac{\psi^* - (s_j)}{2\pi} \sum_{k=0}^{2n} s_k^{s_k+1} \int h^*(s_j, s_k) e^{is_k} [d^*(s_j, \sigma) - d^*(s_j, s_k)] \bar{x}_n(e^{i\sigma}) \right| \right\} = \\
= A \ln n \max_{0 \leq j \leq 2\pi} (| I_3(s_j) | + | I_4(s_j) |). \]

It is easy to see that

\[ \max_j | I_3(s_j) | \leq A \| x_n \| \beta / n^\alpha, \]

\[ \max_j | I_4(s_j) | \leq A \| x_n \| C \max_j \sum_{k=0}^{2n} s_k^{s_k+1} \int d^*(s_j, \sigma) - d^*(s_j, s_k) | d\sigma. \]

Below the prime in the summation indicate that \( k \neq j - 1, j \). Without loss of generality we assume \( j = 0 \). Then

\[ \sum_{k=0}^{2n} s_k^{s_k+1} \int (\ast) d\sigma = \sum_{k=1}^{n-1} s_k^{s_k+1} \int (\ast) d\sigma + \int (\ast) d\sigma + \sum_{k=n+1}^{2n} s_k^{s_k+1} \int (\ast) d\sigma = \\
= I_5 + I_6 + I_7, \]

where \( (\ast) = | d^*(0, \sigma) - d^*(0, s_k) | \).

For values \( \sigma \), belonging to the segment \([2\pi/(2n + 1), 2\pi - 2\pi/(2n + 1)]\), the formula \( d^*(0, \sigma) = d(0, \sigma) = \frac{1}{2} \csc \frac{\sigma}{2} | \eta \) is valid.

Therefore

\[ I_5 = \frac{1}{2} \sum_{k=1}^{n-1} s_k^{s_k+1} \int (| \csc \frac{\sigma}{2} | \eta - | \csc \frac{s_k}{2} | \eta \}) d\sigma \leq \\
\leq
\[
\leq A \sum_{k=1}^{n-1} \int_{s_k}^{s_{k+1}} (s_k + \theta(\sigma - s_k))^{-1-\eta}(\sigma - s_k) d\sigma \leq \]

\[
\leq A \sum_{k=1}^{n-1} \left( \frac{2n+1}{2k\pi} \right)^{1+\eta} \int_{s_k}^{s_{k+1}} (\sigma - s_k) d\sigma \leq An^{-(1-\eta)}. \]

The sum \( I_7 \) can be estimated similarly. It is not difficult to see that

\[
I_6 \leq An^{-(1-\eta)}. \]

It follows from the estimations for \( I_3 - I_7 \) that \( \max |I_2(s_j)| \leq An^{-\delta} \ln n \| x_n \|_\beta \). Since \( I_2(s) \) is a polynomial of degree \( n \), then \( \| I_2 \|_\beta \leq An^{-(\delta-\beta)} \ln n \| \tilde{x}_n \|_\beta \). From the last inequality and values of the norm \( \| I_1 \|_\beta \) we have \( \| \tilde{F}_n x_n - V_n x_n \|_\beta \leq An^{-\xi} \ln n \| x_n \|_\beta \). Hence, it follows from the last inequality and the inequality (2.13) and Kantorovich theory of approximate methods of analyses, that for \( n \) such that \( q = An^{-\xi} \ln n < 1 \), the operator \( \tilde{F}_n \) is continuous invertible and the estimation \( \| \tilde{x}_n^* - x_n^* \|_\beta \leq An^{-\xi} \ln n \) is valid.

Theorem is proved.

**Proof of Theorem 2.4.**

The equation (2.1) is equivalent to the equation

\[
K_1 x \equiv x^+(t) - G(t)x^-(t) + U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau)) = f(t). 
\]

(2.14)

The last equation is uniquely solvable for any right-hand side. Using well known Banach Theorem [103], we see that the operator \( K_1 \) is continuously invertible and \( \| K_1^{-1} \| = C \).

The equation (2.14) is equivalent to the equation

\[
K_2 x \equiv \psi^-(t)x^+(t) - \psi^+(t)x^-(t) + \psi^-(t)U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(t)) = \psi^-(t)f(t),
\]

(2.15)
where
\[
\psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{\ln G(\tau)}{\tau - t} d\tau \right\}.
\]

Repeated the previous arguments we see that the operator \( K_2 \) is continuously invertible and \( \|K_2^{-1}\| = C \).

Let \( \psi^+_n(\psi^-_n) \) is the polynomial of best uniform approximation of degree \( n \) for the function \( \psi^+(\psi^-) \); \( T_n \) is projector onto the polynomial of best uniform approximation of degree \( n \).

Let us consider the equation
\[
K_3x \equiv \psi^-_n(t)x^+(t) - \psi^+_n(t)x^-(t) + T_n[\psi^-(t)U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau))],
\]

(2.16)

From conditions \( a(t), b(t) \in H_\alpha \) it follows that \( G(t) \in H_\alpha \). As \( G(t) \neq 0 \) then \( \ln G(t) \in H_\alpha, t \in \gamma \). Using the Sohotzky-Plemelj formulas, we have
\[
\psi^+(t) = \exp \left\{ \frac{1}{2} \ln G(t) + \frac{1}{2\pi i} \int_{\gamma} \frac{\ln G(\tau)}{\tau - t} d\tau \right\},
\]
\[
\psi^-(t) = \exp \left\{ -\frac{1}{2} \ln G(t) + \frac{1}{2\pi i} \int_{\gamma} \frac{\ln G(\tau)}{\tau - t} d\tau \right\}.
\]

So, using the Privalov Theorem, we have \( \psi^\pm(t) \in H_\alpha \).

Let us estimate the norm
\[
\|K_2x - K_3x\| \leq \|\psi^- - \psi^-_n\|x^+\| + \|\psi^+ - \psi^+_n\|x^-\| +
\]
\[
+\|\left(I - T_n\right)[\psi^-(t)U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau))]\| = J_1 + J_2 + J_3.
\]

(2.17)

It is easy to see that
\[
\|\psi^- - \psi^-_n\|x^+\| \leq \|\psi^- - \psi^-_n\||x^+\|.
\]
\[ \| (\psi^+ - \psi_n^+)x^- \| \leq \| (\psi^+ - \psi_n^+) \| \| x^- \|; \]
\[ \| \psi^\pm - \psi_n^\pm \|_C(\gamma) \leq An^{-\alpha}. \]

Repeating the proof of Bernstein inverse theorem, one can see that
\[ \| \psi^\pm - \psi_n^\pm \| \leq An^{-\alpha+\beta}. \]

From Sohotzky – Plemel formulas we receive the inequality
\[ \| x^\pm \| = \| \pm \frac{1}{2}x + \frac{1}{2}S(x) \| \leq \frac{1}{2}\| x \| + \frac{1}{2}\| S(x) \| \leq A\| x \|. \]

From two last inequalities we have
\[ J_1 + J_2 \leq An^{-(\alpha-\beta)}\| x \|. \quad (2.18) \]

Let \( x(t) \in H_\beta, \ 0 < \beta < \alpha \leq 1. \) According the Lemma 1.2 the function
\[ v(t) = \int_0^{2\pi} x(\tau) \left| \cot \frac{\tau - t}{2} \right|^\eta d\tau \]

belongs to Zygmund class of functions if \( \beta = \eta, \) or to Holder class \( H_1 \) if \( \beta > \eta, \) or to Holder class \( \beta + 1 - \eta \) if \( \beta < \eta. \)

It is easy to see that
\[ \int_\gamma \frac{x(\tau)d\tau}{|\tau - t|^\eta} = \int_0^{2\pi} \frac{e^{i\sigma}ix(e^{i\sigma})d\sigma}{|e^{i\sigma} - e^{is}|^\eta} = i \int_0^{2\pi} \frac{e^{i\sigma}x(e^{i\sigma})d\sigma}{|e^{i\sigma} - e^{-i\sigma}|^\eta} = \]
\[ = i \int_0^{2\pi} \frac{e^{i\sigma}x(e^{i\sigma})d\sigma}{|e^{i\sigma} - e^{-i\sigma}|^\eta} = i \int_0^{2\pi} \frac{e^{i\sigma}x(e^{i\sigma})d\sigma}{|2\sin \frac{\sigma}{2}|^\eta}. \]

Repeating the proof of the Lemma 1.2, we see that the function \( g(t) = \int_\gamma \frac{x(\tau)d\tau}{|\tau - t|^\eta} \) belongs to Zygmund class \( Z \) if \( \beta = \eta \) or Holder class \( H_\mu, \) where \( \mu = 1 \) if \( \beta > \eta \) and \( \mu = \beta + 1 - \eta \) if \( \beta < \eta. \)

Let us put \( \beta < \eta. \)
In this case function \( g_1(t) = \psi^-(t)U_\gamma(h(t, \tau) | \tau - t|^{-\eta}x(\tau)) \) belongs to the Holder class \( H_\chi, \chi = \min(\alpha, \beta + 1 - \eta) \).

Using the smoothness of the function \( g_1(t) \), we have

\[
\| (I - T_n)g_1(t) \|_{C(\gamma)} \leq A \frac{1}{n^{\chi}} \| x \|.
\]

Repeating the proof of Bernstein inverse theorem, one can see that

\[
\| (I - T_n)g_1(t) \| \leq A \frac{1}{n^{\chi - \beta}} \| x \|.
\]

Let us put \( \beta = \eta \).

In this case function \( g_1(t) \) belongs to the Zygmund class \( Z \).
From this it follows that \( g(t) \in H_\alpha \) for \( \beta < \alpha < 1 \) and

\[
\| (I - T_n)g_1(t) \| \leq A \frac{1}{n^{\alpha - \beta}} \| x \|.
\]

Let us put \( \beta > \eta \).

In this case function \( g_1(t) \in H_\alpha \) and

\[
\| (I - T_n)g_1(t) \| \leq A \frac{1}{n^{\alpha - \beta}} \| x \|.
\]

The cases \( \beta < \eta, \beta = \eta, \beta > \eta \) are considered identically. For simplicity we consider the case when \( \eta < \beta < \min(\alpha, 1 - \eta) \).

In this case

\[
J_3 \leq A \frac{1}{n^{\alpha - \beta}} \| x \|. \tag{2.19}
\]

Collecting the inequalities (2.16) - (2.19), we receive the estimate

\[
\| K_2x - K_3x \| \leq A \frac{1}{n^{\alpha - \beta}} \| x \|.
\]

From this inequality and Banach Theorem follows that, for \( n \) such that \( q = A \| K_2^{-1} \| n^{-(\alpha - \beta)} < 1 \), the operator \( K_3 \) is continuously invertible and the estimate \( \| K_3^{-1} \| \leq \| K_2^{-1} \|/(1 - q) \) is valid.
The operator $K_3$ has a left inverse operator on subspace $X_n$ and

$$\|K_3x_n\| \geq \frac{1}{\|K_3^{-1}\|}\|x_n\|, \quad x_n \in X_n.$$ 

Let us consider the equation

$$K_{3,n} \equiv P_n[\psi_n^- x_n^+ + \psi_n^+ x_n^- + l(t)U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau))],$$

$$= P_n[l(t)f(t)], \quad l(t) = \psi^-(t)/(a(t) + b(t)). \quad (2.20)$$

It is easy to see that

$$P_n[\psi_n^- x_n^+ + \psi_n^+ x_n^-] \equiv \psi_n^- x_n^+ + \psi_n^+ x_n^-$$

and

$$\|P_n[D_n[l(t)U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x_n(\tau))]] \leq An^{-(\alpha-\beta)}\ln n\|x_n\|.$$ 

So,

$$\|K_3x_n - K_{3,n}x_n\| \leq An^{-(\alpha-\beta)}\ln n\|x_n\|$$

and

$$\|K_{3,n}, x_n\| = \|K_3x_n - (K_3x_n - K_{3,n}x_n)\| \geq \left(\frac{1}{\|K_3^{-1}\|} - An^{-(\alpha-\beta)}\ln n\right)\|x_n\| \geq C\|x_n\|.$$ 

From the Theorem about left inverse operator [90] follows that, for $n$ such $q_1 = A\|K_3^{-1}\|n^{-(\alpha-\beta)}\ln n < 1$, the operator $K_{3,n}$ has left inverse operator. As the $X_n$ is $n$-dimensional space, the operator $K_{3,n}$ has continuously inverse operator $K_{3,n}^{-1}$.

Now we introduce the equation

$$K_{4,n}x_n \equiv P_n[\psi^- x_n^+ + \psi^+ x_n^- + l(t)U(h(t, \tau)|\tau - t|^{-\eta}x_n(\tau))],$$

$$= P_n[l(t)f(t)]$$

and estimate the norm

$$\|K_{3,n}x_n - K_{4,n}x_n\| \leq \|P_n[(\psi^- - \psi_n^-)x_n^+]\| + \|P_n[(\psi^+ - \psi_n^+)x_n^-]\| = J_4 + J_5.$$
It is easy to see that

$$\|P_n[(\psi^- - \psi_n^-)x_n^+]\|_{C(\gamma)} \leq An^{-\alpha} \ln n \|x_n\|,$$

$$\|P_n[(\psi^+ - \psi_n^+)x_n^-]\| \leq An^{-\alpha} \ln n \|x_n\|.$$  

So,

$$\|P_n[(\psi^\mp - \psi_n^\mp)x_n^\pm]\| \leq An^{-\alpha+\beta} \ln n \|x_n\|$$

and

$$\|K_{3,n}x_n - K_{4,n}x_n\| \leq An^{-\alpha+\beta} \ln n \|x_n\|.$$  

From this inequality and Banach Theorem follows that, for \(n\) such \(q_2 = A\|K_{3,n}^{-1}\|n^{-(\alpha-\beta)} \ln n < 1\), the operator \(K_{4,n}\) has continuously invertable operator \(K_{4,n}^{-1}\) and the estimate \(\|K_{4,n}^{-1}\| \leq \|K_{3,n}^{-1}\|/(1 - q_2)\) is valid.

Let us consider the equations

$$K_{5,n}x_n \equiv P_n[x_n^+(t)+G(t)x_n^-(t)+l(t)U(h(t, \tau)|\tau-t|^{-\eta}x(\tau))] = f_n(t),$$

$$K_{6,n}x_n \equiv P_n[a(t)x_n(t) + b(t)S_\gamma(x_n(\tau)) + U_\gamma(h(t, \tau)|\tau-t|^{-\eta}x_n(\tau))] = f_n(t). \quad (2.21)$$

Here \(f_n(t) = P_n[f(t)]\).

It is obviously that the equation \(K_{5,n}x_n = f_n\) is equivalent to the equation \(K_{4,n}x_n = P_n[l(t)f]\) and the equation \(K_{6,n}x_n = f_n\) is equivalent to the equation \(K_{5,n}x_n = f_n\). So, the operator \(K_{6,n}\) has the continuously invertable operator \(K_{6,n}^{-1}\) and \(\|K_{6,n}^{-1}\| \leq A\). Here \(f_n = P_n[f]\).

The equation (2.21) is the operator form of collocation method for solution of the equation (2.1). Using methods of functional analysis we receive the estimate \(\|x^* - \bar{x}_n^*\| \leq An^{-(\alpha-\beta)} \ln n\), where \(x^*\) and \(\bar{x}_n^*\) are solutions of equations (2.1) and (2.21) respectively.

Let us estimate the norm

$$\|K_nx_n - K_{6,n}x_n\| =$$
It is easy to see that

\[ \|K_n x_n - K_{6,n} x_n\| \leq \]

\[ \leq P_n \left[ \sum_{k=0}^{2n} \int_{t_k'}^{t_{k+1}'} \frac{(h(t, \tau) - h(t, t_k)) x_n(\tau)}{|\tau - t|^\eta} d\tau \right] + \]

\[ + A \ln n \left[ \sum_{k=0}^{2n} \int_{t_k'}^{t_{k+1}'} \frac{h(t, t_k)(x_n(\tau) - x_n(t_k))}{|\tau - t|^\eta} d\tau \right] = J_6 + J_7. \quad (2.23) \]

Obviously,

\[ \left\| P_n \left[ \sum_{k=0}^{2n} \int_{t_k'}^{t_{k+1}'} \frac{h(t, \tau) - h(t, t_k)}{|\tau - t|^\eta} x(\tau) d\tau \right] \right\|_{C(\gamma)} \leq A \frac{\ln n}{n^\alpha} \|x_n\|. \quad (2.24) \]

Then

\[ J_6 \leq A \frac{\ln n}{n^{\alpha-\beta}} \|x_n\|. \quad (2.25) \]

Let

\[ g_2(t) = \sum_{k=0}^{2n} \int_{t_k'}^{t_{k+1}'} \frac{h(t, t_k)(x_n(\tau) - x_n(t_k))}{|\tau - t|^\eta} d\tau. \]

For simplicity we will restrict youself with the function

\[ g_2^*(s) = \left( x_n(e^{i\sigma}) - \bar{x}_n(e^{i\sigma}) \right) i e^{i\sigma} \]

where \( \bar{x}_n(e^{i\sigma}) = x_n(e^{i\sigma_k}), \sigma \in [s_k', s_k'), k = 0, 1, \ldots, 2n. \)
It is easy to see that
\[ \|g_2^*(s)\|_{\tilde{C}[0,2\pi]} \leq A \frac{1}{n^\beta} \|x_n\|. \]

The difference \( g_2^*(s + \delta) - g_2^*(s) \) is estimated more hard:
\[
|g_2^*(s + \delta) - g_2^*(s)| = \left| \int_0^{2\pi} (x_n(e^{i\sigma}) - \bar{x}_n(e^{i\sigma}))i e^{i\sigma} \left( \frac{1}{|\sin \sigma - s - \delta|^{\eta}} - \frac{1}{|\sin \sigma - s|^{\eta}} \right) d\sigma \right| \\
\leq \sum_{j=1}^{4} \int_{\Delta_j} |\psi(\sigma)| = J_{7,1} + \cdots + J_{7,4},
\]
where \( \Delta_1 = [s - 2\delta, s + 2\delta], \Delta_2 = [s + 2\delta, s + \pi - 2\delta], \Delta_3 = [s + \pi - 2\delta, s + \pi + 2\delta], \Delta_4 = [s + \pi + 2\delta, s + 2\pi - 2\delta]. \)

One can see that
\[
|J_{7,1}| \leq A \frac{1}{n^\beta} \delta^{1-\eta} \|x_n\|,
\]
\[
|J_{7,2}| \leq A \frac{1}{n^\beta} \delta^{1-\eta} \|x_n\|,
\]
\[
|J_{7,3}| \leq A \frac{1}{n^\beta} \delta^{1-\eta} \|x_n\|,
\]
\[
|J_{7,4}| \leq A \frac{1}{n^\beta} \delta^{1-\eta} \|x_n\|,
\]
So,
\[
|g_2^*(s + \delta) - g_2^*(s)| \leq A \frac{1}{n^\beta} \delta^{1-\eta} \|x_n\|
\]
and
\[
J_7 \leq A \frac{\ln n}{n^\beta} \|x_n\|. \quad (2.26)
\]

From (2.22) - (2.26) it follows that
\[
\|K_n x_n - K_{6,n} x_n\| \leq A \left( \frac{1}{n^\beta} + \frac{1}{n^{\alpha - \beta}} \right) \ln n \|x_n\|.
\]
So, the operator $K_n$ has continuously invertible operator $K_n^{-1}$ with the norm $\|K_n^{-1}\| \leq A$.

Let $x^*(t)$ is a unique solution of the equation (2.1). So,

$$P_n[a(t)x^*(t) + b(t)S_\gamma(x^*(\tau)) + U_\gamma(h(t, \tau)|\tau-t|^{-\eta}x^*(\tau))] = P_n[f(t)].$$

(2.27)

The identity (2.27) we can rewrite as

$$P_n[a(t)x^* + b(t)S_\gamma(x^*(\tau)) + \sum_{k=0}^{2n} h(t, t_k)x^*(t_k) \int_{t'_k}^{t'_{k+1}} |\tau - t|^{-\eta}d\tau] =$$

$$= P_n[f(t) - U_\gamma(h(t, \tau)|\tau-t|^{-\eta}x^*(\tau)) +$$

$$+ \sum_{k=0}^{2n} h(t, t_k)x^*(t_k) \int_{t'_k}^{t'_{k+1}} |\tau - t|^{-\eta}d\tau].$$

(2.28)

Let $x^*_n(t)$ is a unique solution of the equation (2.6). Subtracted the identity

$$P_n[a(t)x^*_n(t) + b(t)S_\gamma(x^*_n(\tau)) + \sum_{k=0}^{2n} h(t, t_k)x^*_n(t_k) \int_{t'_k}^{t'_{k+1}} |\tau - t|^{-\eta}d\tau] =$$

$$= P_n[f(t)]$$

from the identity (2.28) we have

$$P_n[a(t)(x^*(t) - x^*_n(t)) + b(t)S_\gamma(x^*(\tau) - x^*_n(\tau)) +$$

$$+ \sum_{k=0}^{2n} (h(t, t_k)(x^*(t_k) - x^*_n(t_k)) \int_{t'_k}^{t'_{k+1}} |\tau - t|^{-\eta}d\tau] =$$

$$= P_n \left[ \int_{\gamma} \frac{h(t, \tau)x^*(\tau)}{|\tau-t|^{-\eta}} d\tau - \sum_{k=0}^{2n} h(t, t_k)x^*(t_k) \int_{t'_k}^{t'_{k+1}} |\tau - t|^{-\eta}d\tau \right].$$

(2.29)
From the Theorem 1.1 it follows that a unique solution \( x^*(t) \) of equation (2.1) belongs to the Holder class \( H_\alpha \).

In this case
\[
\left\| P_n \left[ \int_\gamma \frac{h(t, \tau)x^*(\tau)}{|\tau - t|^{\eta}} d\tau - \sum_{k=0}^{2n} h(t, t_k)x^*(t_k) \int_{t_k'}^{t_{k+1}'} |\tau - t|^{-\eta} d\tau \right] \right\| \leq \frac{\ln n}{n^{\alpha-\beta}}.
\]

Applied the inverse operator \( K_n^{-1} \) to both side of the equation (2.29) we have
\[
\|x^* - x_n^*\| \leq An^{-(\alpha-\beta)} \ln n. \tag{2.30}
\]

Let us prove that this estimate can not be improved. For this aim we consider the equation
\[
Kx \equiv \frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau) \cot \frac{\sigma - s}{2} d\tau = f(s). \tag{2.31}
\]

Well known \([64]\) that the solution of this equation is
\[
K^{-1}f \equiv x(t) = \frac{1}{2\pi} \int_0^{2\pi} f(\sigma) \cot \frac{\sigma - s}{2} d\sigma.
\]

Let \( n \) be a integer. Let us solve the equation (2.31) with collocation method, using arbitrary knots \( t_k, k = 1, 2, \ldots, n \). Without loss of generality we can put \( t_1 = 0 \). Let us introduce the function \( f^*(s) = \min_k |s - t_k|^\alpha, 0 \leq s \leq 2\pi \). It easy to see that the approximate solution \( x_n^*(s) \) of the equation (2.31) received with collocation method based on knots \( t_k, k = 1, 2, \ldots, n, \) is equal to zero \( (x_n^*(s) \equiv 0) \).

From the Theorem of inverse operator \([90]\) follows that
\[
\|x\| \geq \frac{1}{m} \|f\|, \text{ where } m = \|K\|. \text{ So } \|x^*(s)\| \geq m^{-1} \|f^*(s)\|, \text{ where } x^*(s) \text{ is a solution of the equation (2.31).}
\]
It is easy to see that the function \( f^*(s), 0 \leq \sigma \leq 2\pi \), belongs to the Holder class \( H_\alpha \) and \( \|f^*(s)\| \geq An^{-\alpha+\beta} \).

So,

\[
\|x^*(s) - x^*_n(s)\| = \|x^*(s)\| \geq An^{-\alpha+\beta}. \tag{2.32}
\]

Correctness of Theorem follows from the comparison of the inequalities (2.30) and (2.32).

**Proof of Theorem 2.5.** In proving of Theorem 2.4 it was noticed that the equations (2.1) and (2.14) are equivalent. The equation (2.14) is equivalent to the Riemann boundary value problem

\[
K_7x \equiv \psi^-(t)x^+(t) - \psi^+(t)x^-(t) + \psi^-(t)U_\gamma(h(t, \tau)| \tau - t |^{-\eta}x(\tau)) = \psi^-(t)f(t), \tag{2.33}
\]

where

\[
\psi(z) = \exp \left( \frac{1}{2\pi i} \int_\gamma [\ln G(\tau)](\tau - z)^{-1}d\tau \right).
\]

Hence, the operator \( K_7 \) has the continuous inverse operator \( K_7^{-1} \) with the norm \( \|K_7^{-1}\| \leq A \).

Let us approximate functions \( \psi^+(t) \) and \( \psi^-(t) \) by interpolation polynomials \( \psi_n^+(t) \) and \( \psi_n^-(t) \), constructed on the knots \( t_k = \exp\{is_k\}, s_k = 2k\pi/(2n+1), k = 0, 1, \ldots, n \). It is easy to see that the functions \( \psi_n^+(t) \) and \( \psi_n^-(t) \) can be represented in the forms \( \psi_n^+(t) = \sum_{k=0}^{n} \gamma_k t^k \) and \( \psi_n^-(t) = \sum_{k=-n}^{0} \gamma_k t^k \).

Let us introduce the equation

\[
K_8x \equiv \psi_n^-(t)x^+(t) - \psi_n^+(t)x^-(t) + \psi^-(t)U_\gamma[P_n^-[h(t, \tau)]| \tau - t |^{-\eta}x(\tau)] = \psi^-(t)f(t). \tag{2.34}
\]

Easy to see that

\[
\|K_7x - K_8x\| \leq \|\psi^- - \psi_n^-\|\|x^+\| + \|\psi^+ - \psi_n^+\|\|x^-\| \leq A \frac{1}{n^{r+\alpha-\beta}}\|x\|.
\]
From this inequality and Banach Theorem it follows that, for 
\( n \) such that \( q = A\|K_7^{-1}\|n^{-(r+\alpha-\beta)} < 1 \), the operator \( K_8 \) has 
continuous invertible operator \( K_8^{-1} \) with the norm \( \|K_8^{-1}\| \leq \|K_7^{-1}\|/(1-q) \).

Method of mechanical quadrature for solution of the equation
(2.34) can be written as

\[
K_{8,n}x_n \equiv P_n[\psi_n^-(t)x_n^+(t) - \psi_n^+(t)x_n^-(t) + \psi^-(t)U_\gamma[P_n^r[h(t,\tau)]| \tau - t |^{-\eta}x_n(\tau)] = P_n[\psi^-(t)f(t)].
\] (2.35)

Using Kantorovich theory of approximate methods of analysis,
one can prove that, under condition \( q = An^{-(r+\alpha-\beta)} \ln n < 1 \),
the equation (2.35) has a unique solution \( x_{4,n}^* \) and the estimate
\( \|x^* - x_{8,n}^*\| \leq An^{-(r+\alpha-\beta)} \ln n \) is valid.

Let us consider the sequence of equations

\[
K_{9,n}x_n \equiv P_n[\psi_n^-(t)x_n^+(t) - \psi_n^+(t)x_n^-(t) + \psi^-(t)U_\gamma[P_n^r[h(t,\tau)]| \tau - t |^{-\eta}x_n(\tau)] = P_n[\psi^-(t)f(t)];
\] (2.36)

\[
K_{10,n}x_n \equiv P_n[x_n^+(t)G(t)x_n^-(t) + U_\gamma[P_n^r[h(t,\tau)]| \tau - t |^{-\eta}x_n(\tau)] = P_n[f(t)];
\] (2.37)

\[
K_{11,n}x_n \equiv P_n[a(t)x_n(t) + b(t)S_\gamma(x_n(\tau) + U_\gamma[P_n^r[h(t,\tau)]| \tau - t |^{-\eta}x_n(\tau)] = P_n[f(t)].
\] (2.38)

Using the arguments repeatedly cited above, we prove the
unique solvability of the equations (2.36) - (2.38).

So, the equation \( K_{11,n}x_n = f_n, \ f_n = P_n[f(t)] \), has a unique
solution \( x_{11,n}^* \) and the estimate

\[
\|x^* - x_{11,n}^*\| \leq An^{-(r+\alpha-\beta)} \ln n
\] (2.39)
is valid.

The equation \( K_{11,n}x_n = f_n \) is the equation (2.7).

Theorem is proved.
Now we will prove that this estimation can not be improved. Let \( n \) is a integer. Let \( t_k, k = 1, 2, \ldots, n, \) are arbitrary knots, \( 0 = t_1 < t_2 < \cdots < t_n < 2\pi. \) Let us introduce the function \( f^*(s) \) which is equal to \( \frac{((s-t_k)(t_{k+1}-s))^{r+\alpha}}{h_k^{r+\alpha}}, \) \( h_k = t_{k+1} - t_k, \) on each segment \([t_k, t_{k+1}], k = 1, 2, \ldots, n - 1, \) where \( t_{n+1} = 2\pi. \) It is easy to see that \( f^*(s) \in W^rH_\alpha(M) \) and \( \|f^*(s)\|_\beta = An^{-(r+\alpha-\beta)}. \)

Let us consider the equation

\[
\frac{1}{2\pi} \int_0^{2\pi} x(\tau) \cot \frac{\sigma - s}{2} d\sigma = f^*(s). \tag{2.40}
\]

Well known that a solution of this equation is

\[
x^*(s) = \frac{1}{2\pi} \int_0^{2\pi} f^*(\sigma) \cot \frac{\sigma - s}{2} d\sigma.
\]

Solving the equation (2.40) by collocation methods with knots \( t_k, k = 1, 2, \ldots, n, \) we receive approximate solution \( x^*_n(t) \equiv 0. \) Repeating arguments given behind, we see that \( \|x^* - x^*_n\|_\beta = \|x^*\|_\beta \geq An^{-(r+\alpha-\beta)} \ln n. \)

Comparing this estimation with (2.39) we finish the proof of the Theorem.

Theorem is proved.

Proof of Theorem 2.6. Let us consider the collocation method for the equation (2.2). In the operator form the collocation method is written by the expression

\[
L^1_n x_n \equiv P_n[a(t)x_n(t) + S_\gamma(h(t, \tau)x_n(\tau))] = P_n[f(t)], \tag{2.41}
\]

where \( x_n(t) \) is the polynomial defined in (2.3).

The equations (2.2) and (2.41) are equivalent to the Riemann boundary value problems

\[
L^{(2)} x \equiv Vx + Wx = y
\]
and
\[ L_n^{(2)} x_n \equiv \tilde{V}_n x_n + \tilde{W}_n \tilde{x}_n = y_n, \]
where
\[ V x = \psi^- x^+ - \psi^+ x^-, \quad W x = \psi^- S_\gamma((h(t, \tau) - h(t, t))x(\tau)), \]
\[ \tilde{V}_n x_n = P_n [V x_n], \quad \tilde{W}_n x_n = P_n [W x_n], \quad y = \psi^- f, \quad y_n = P_n [y], \]
\[ G(t) = (a(t) - b(t))/(a(t) + b(t)), \quad b(t) = h(t, t), \]
\[ \psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\gamma} (\ln G(\tau))(\tau - z)^{-1} d\tau \right\}. \]

Let us introduce the polynomial
\[ \varphi_n(t) = V_n x_n + T^t_{[n/3]}[\psi^- (t)] S_\gamma((\tilde{h}(t, \tau) - \tilde{h}(t, t))x_n(\tau)), \]
where
\[ V_n x_n = \psi^- x^+_n - \psi^+ x^-_n, \quad x_n = T_n(\psi), \quad \tilde{h}(t, \tau) = T^t_{[n/3]} T^\tau_{[n/3]} [h(t, \tau)]. \]

It is obvious, that
\[ ||V_n x_n - V x_n||_\beta \leq A ||x_n||_\beta \max(E_n(a), E_n(\psi), E_n(b)) n^\beta. \]

Since
\[ ||h(t, \tau) - \tilde{h}(t, \tau)||_C \leq A \ln n \max(E^t_n[h(t, \tau)], E^\tau_n[h(t, \tau)]), \]
then the function \( \eta(t, \tau) = h(t, \tau) - \tilde{h}(t, \tau) \) belongs to the Holder class with respect to \( \tau \) with index \( 1/\ln n \) and with the coefficient \( A \ln n E^*_n(h) \), where \( E^*_n(h) = \ln n \max(E^t_n(h(t, \tau)), E^\tau_n(h(t, \tau))) \).

Let us prove this statement. Consider the function \( \eta(t, \tau) = h(t, \tau) - \tilde{h}(t, \tau) \), where \( t \) is fixed variable. It is easy to see that
\[ E_m(\eta(t, \tau)) \leq A \ln n \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)]), \]
for \( 1 \leq m \leq n \) and
\[ E_m(\eta(t, \tau)) \leq A \ln m \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)]). \]
for $n < m < \infty$.

These inequalities we can rewrite as

$$E_m(\eta(t, \tau)) \leq A \ln m \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)])$$

for $n < m < \infty$ and

$$E_m(\eta(t, \tau)) \leq \frac{A}{m^\gamma} (m^\gamma \ln n \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)])) \leq \frac{A}{m^\gamma} (n^\gamma \ln n \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)]))$$

for $1 \leq m \leq n$.

Here $\gamma$ is an arbitrary number, $0 < \gamma < 1$.

Repeating the proof of Bernstein inverse theorem we see that

$$\omega(\eta(t, \tau), \delta) \leq \left(\frac{2^{1-\gamma}}{2^{1-\gamma} - 1} (1 + 2^{\gamma}) + \frac{1 + 2^{\gamma}}{1 - 2^{-\gamma}}\right) \times$$

$$\times An^\gamma \ln n \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)]) \delta^\gamma.$$

Let us put $\gamma = 1/\ln n$. In result we have

$$\omega(\eta(t, \tau), \delta) \leq An^\gamma \ln^2 n \max(E^t_m[h(t, \tau)], E^\tau_m[h(t, \tau)]) \delta^{1/\ln n}.$$

Therefore

$$\|S_\gamma(h(t, \tau) - \tilde{h}(t, \tau) - (h(t, t) - \tilde{h}(t, t)))x_n(\tau)\|_\beta \leq \|x_n\|_\beta n^\beta \ln^2 n E^*_n(h).$$

It follows from this inequality and received above estimation of the norm $\|V_n x_n - V x_n\|$ that the operator $L_n^{(2)}$ is continuously invertible for $n$ such that

$$q_1 = A \ln^3 n^\beta \max(E_n(a), E_n(b), E_n(\psi), E^t_n(h(t, \tau)), E^\tau_n(h(t, \tau))) < 1.$$
As marked above the equations $L_n^{(2)} x_n = y_n$ and $L_n^{(1)} x_n = y_n$ are equivalent. So, the operator $L_n^{(1)}$ has the continuous invertible operator $(L_n^{(1)})^{-1}$ and $\|(L_n^{(1)})^{-1}\| \leq A$.

It is easy to see that

$$\|L_n^{(1)} x_n - L_n x_n\| \leq AE_n^*[h(t, \tau)] \ln^2 n \|x_n\|.$$ 

From this inequality and Banach Theorem follows that the operator $L_n$ has the continuous invertible operator $L_n^{-1}$ with the norm $\|(L_n^{(1)})^{-1}\| \leq A$ and the following estimate $\|x_{1,n}^* - x_n^*\| \leq AE_n^*[h(t, \tau)] \ln^2 n$ is valid. Here $x_n^*$ is a unique solution of the equation (2.7).

Using the estimations for $\|x^* - x_{1n}^*\|_\beta$ and for $\|x_{1n}^* - x_n^*\|_\beta$, we verify in correctness of Theorem.

**Proof of Theorem 2.7.**

It was shown in proving of preceding Theorem that the operator $L_n^{(2)}$ is continuously invertible and $\|([L_n^{(2)}]^{-1})\| \leq A$ for $n$ such that $q_1 < 1$. Repeating arguments of proof of the Theorem 2.5, one can see that the operator

$$L_n^{(3)} x_n \equiv \bar{P}_n \left[ a(t) x_n(t) + \frac{1}{\pi i} \int_{\gamma} h(t, \tau) \frac{x_n(\tau)}{\tau - t} d\tau \right]$$

is continuously invertible and $\|L_n^{(3)}\|^{-1} \leq A$ for $n$ such that $q_1 < 1$.

The correctness of the identity

$$\overline{P}_n \left[ \frac{1}{\pi i} \int_{\gamma} P_n^\tau \left[ \frac{h(t, \tau) x_n(\tau)}{\tau - t} \right] d\tau \right] \equiv$$

$$\equiv \overline{P}_n \left[ \frac{1}{\pi i} \int_{\gamma} P_n^\tau [h(t, \tau) x_n(\tau)] d\tau \right] \equiv$$

$$\equiv \overline{P}_n \left[ \frac{1}{\pi i} \int_{\gamma} P_n^\tau [h(t, \tau)] x_n(\tau) d\tau \right]$$

(2.42)
follows from the results on the quadrature rules of the highest algebraic exactness.

Using this identity we can show that
\[\|L_n^{(3)} - L_n\|_\beta \leq An^2 \ln n (E_n^t(h(t, \tau)) + E_n^\tau(h(t, \tau))).\]

Therefore the operator \(L_n\) is continuously invertible for \(n\) such that \(q = An^2 \ln n (E_n(a(t)) + E_n(h(t, t)) + E_n(\psi) + E_n^t(h(t, \tau)) + E_n^\tau(h(t, \tau))) < 1\). The estimation of error \(\|x^* - x_n^*\|_\beta\) is evaluated just as in preceding Theorems. Theorem is proved.

**Proof of Theorem 2.8.** Joined proofs of the Theorem 2.7 and the Theorem 2.5 we are verified in the correctness of the Theorem 2.8.

3. Approximate Solution of Singular Integral Equations on Closed Paths of Integration (Basis in Space \(L_2\))

Let us extend investigation of approximate methods for solution of singular integral equations as

\[Kx \equiv a(t)x(t) + b(t)S_\gamma(x) + U_\gamma(h(t, \tau)|\tau - t|^{-n}x(\tau)) = f(t)\]  \hspace{1cm} (3.1)

and

\[Lx \equiv a(t)x(t) + S_\gamma(h(t, \tau)x(\tau)) = f(t).\]  \hspace{1cm} (3.2)

According to accepted beyond designations

\[S_\gamma x = \frac{1}{\pi i} \int_{\gamma} \frac{x(\tau)d\tau}{\tau - t}, \quad U_\gamma(h(t, \tau)x(\tau)) = \frac{1}{2\pi i} \int_{\gamma} h(t, \tau)x(\tau)d\tau,\]

where \(\gamma\) is a unit circle with the center in origin of coordinates.

The verification of the suggested below calculating schemes is carried out in the functions space \(X = L_2(\gamma)\) with scalar product

\[(f_1, f_2) = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{is})\overline{f_2(e^{is})}ds\]
and its subspace $X_n$ consisting of the polynomials

$$x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k. \quad (3.3)$$

In this paragraph we take all designations, which are introduced in the previous paragraph.

### 3.1. Basis Statements

The approximate solution of the equations (3.1) we will seek in the form of the polynomial (3.3), the coefficients $\alpha_k$ of which are determined from the system of linear algebraic equations written in the operator form as

$$K_n x_n \equiv P_n[a(t)x_n(t) + b(t)S_\gamma(x_n) + U_\gamma(P_n^r[h(t, \tau)d(t, \tau)x_n(\tau)]))] = P_n[f(t)], \quad (3.4)$$

where

$$d(t, \tau) = \begin{cases} |\tau - t|^{-\eta} & \text{for } |\sigma - s| \geq 2\pi/(2n + 1), \\ |e^{i2\pi/(2n+1)} - 1|^{-\eta} & \text{for } |\sigma - s| < 2\pi/(2n + 1), \end{cases}$$

$\tau = e^{is}, t = e^{is}; P_n$ is the projector of interpolation onto the set of trigonometrical interpolated polynomials constructed on the knots $t_k = e^{is_k}, s_k = 2k\pi/(2n + 1), k = 0, 1, \ldots, 2n.$

**Theorem 3.1** [14], [16], [26], [35]. Let the operator $K \in [X, X]$ has the linear invertible operator and functions $a, b \in C[0, 2\pi], h \in C([0, 2\pi]^2).$ Then for $n$ such that $q = A[\omega(a; n^{-1/2}) + \omega(b; n^{-1/2}) + n^{-1/2+\varepsilon} + [\omega(h; n^{-1})]^{(1-\eta)/(1+\eta)}] < 1,$ the equation (3.4) is uniquely solvable for any right-hand side and the estimate $\|x^* - x_n^*\| \leq A[q + \omega(f; n^{-1})]$ is valid, where $x^*$ and $x_n^*$ are solutions of equations (3.1) and (3.4), $\varepsilon$ is arbitrary small number ($\varepsilon > 0$).

The approximate solution of the equation (3.2) we will seek in the form of the polynomial (3.3), coefficients $\alpha_k$ of which are
determined from the system of linear algebraic equations
\[ L_n x_n \equiv \overline{P}_n \left[ a(t)x_n(t) + \frac{1}{\pi i} \int_\gamma P_n^\tau \left[ \frac{h(t, \tau)x_n(\tau)}{\tau - t} \right] d\tau \right] = \overline{P}_n[f(t)]. \]

(3.5)

**Theorem 3.2** [16], [17], [26], [35]. Let the operator \( L \) has a linear inverse one in the space \( X \). Then for \( n \) such that \( q = A(E_n(a) + E_n(b) + n^{-1/2+\varepsilon} + E^t_n(h(t, \tau)) + E^r_n(h(t, \tau))) \ln^3 n < 1 \), the system of equations (3.5) is uniquely solvable for any right-hand side and the estimate \( \|x^* - x_n^*\| \leq A(q + E_n(f)) \) is valid, where \( x^* \) and \( x_n^* \) are solutions of the equations (3.2) and (3.5), \( \varepsilon \) is arbitrary small number (\( \varepsilon > 0 \)).

The approximate solution of the equation (3.2) is sought in the form of the polynomial (3.3), coefficients \( \alpha_k \) of which are determined from the system of equations
\[ \overline{L}_n x_n \equiv \overline{P}_n[a(t)x_n(t) + S_\gamma P_n^\tau[h(t, \tau)x_n(\tau)]] = \overline{P}_n[f(t)]. \]

(3.6)

**Theorem 3.3** [26], [35]. Let the operator \( L \) is continuously invertible and the functions \( a, f \in W^r, h \in W^{r,r}(r = 1, 2, \ldots) \) Then for \( n \), such that \( q = An^{-r} \ln^3 n < 1 \), the system of equations (3.6) is uniquely solvable for any right-hand side and the estimate \( \|x^* - x_n^*\| \leq An^{-r} \ln n \) is valid, where \( x^* \) and \( x_n^* \) are solutions of equations (3.2) and (3.6).

Note. These results are diffused to the space \( L_p(\gamma) \), \( 1 \leq p < \infty \), in the papers I.V. Boykov and Zhechev [42] - [46].

### 3.2. Proofs of Theorems

**Proof of Theorem 3.1.** Let us introduce the equation
\[ K_1 x \equiv \tilde{a}_m(t)x(t) + \tilde{b}_mS_\gamma(x) + U_\gamma(h(t, \tau)d^*(t, \tau)x(\tau)) = f(t), \]
\[ \text{where } a_m(t) = T_m[a(t)], b_m(t) = T_m[b(t)], d^*(t, \tau) = |\tau - t|^{-\eta} \text{ for } |\tau - t| \geq \rho, d^*(t, \tau) = \rho^{-\eta} \text{ for } |\tau - t| < \rho, \rho \text{ is positive number, } \rho \geq 2/\pi(2n+1). \] The numbers \( \rho \) and \( m \) are fixed below.
It follows from Banach Theorem that for \( m, \rho \) such that \( q_1 = A(\rho^{1-\eta} + \omega(a, m^{-1}) + \omega(b, m^{-1})) < 1 \), the operator \( K_1 \) has inverse one with the norm \( \| K_1^{-1} \| \leq \| K^{-1} \|/(1 - q_1) \).

The method of mechanics quadratures in the operator form for the equation (3.7) is

\[
K_{1,n}x_n \equiv P_n[\tilde{a}_m(t)x_n(t) + \tilde{b}_m(t)S_\gamma(x_n) + \\
U_\gamma(P_n^*[h(t, \tau)d^*(t, \tau)x_n(\tau)])] = P_n[f(t)].
\] (3.8)

The equations (3.7) and (3.8) are equivalent to the Riemann boundary value problems \( K_2x = Vx + W^*x = y \) and \( K_{2,n}x_n = V_nx_n + \tilde{W}_n^*x_n = y_n \), where

\[
Vx = \psi^-x^+ - \psi^+x^-, \quad \tilde{V}_n x_n = P_n[Vx_n],
\]

\[
W^*x = \psi^-h(t, \tau)d^*(t, \tau)x(\tau),
\]

\[
\tilde{W}_n^*x_n = P_n[\psi^-h(t, \tau)d^*(t, \tau)x_n(\tau)],
\]

\[
\psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_\gamma \frac{\ln((a_m(\tau) - b_m(\tau))/(a_m(\tau) + b_m(\tau)))}{\tau - z} d\tau \right\},
\]

\[
y = \psi^-f, \quad y_n = P_n[\psi^-f].
\]

Input the polynomial

\[
\tilde{\phi}_n(t) = \psi^-_n(t)x_n^+(t) - \psi^+_n(t)x_n^-(t) + T_n[U_\gamma(\psi^-_n(t)h(t, \tau)d^*(t, \tau)x_n(\tau))],
\]

where \( \psi_n = T_n\psi \).

It is easy to see that

\[
\| K_2x_n - \tilde{\phi}_n \| + \| P_nK_2x_n - \tilde{\phi}_n \| = A \left( \frac{m^{1-\varepsilon}}{n^{1-\varepsilon}} + \frac{\omega(h; n^{-1})}{\rho^{2\eta}} \right) \| x_n \|,
\] (3.9)

where \( \varepsilon \) is an arbitrary small number \((0 < \varepsilon < 1)\). Necessity of introduction the \( \varepsilon \) follows from Privalov Theorem [64], as \( \tilde{a}_m, \tilde{b}_m \) are belong to the class \( H_1 \).
It is evident,
\[
\|K_{2,n}x_n - P_nK_{2,n}x_n\| \leq A\omega(h; n^{-1})\|x_n\|/\rho^{2\eta}. \quad (3.10)
\]

It follows from (3.9)-(3.10) (assuming \(\rho = (\omega(h, n^{-1}))^{1/(1+\eta)}\), \(\varepsilon = 1/\ln n\), \(m = n^{1/2}\)) and Kantorovich theory of approximate methods of analysis, that, for \(n\) such that
\[
q_2 = \frac{A[\omega(a; n^{-1/2}) + \omega(b, n^{-1/2}) + E_n(\psi^\pm) + [\omega(h; 1)]^{(1-\eta)/(1+\eta)} + \omega(f; n^{-1})]}{1 - \eta/(1+\eta)} < 1,
\]
the linear operator \(K_{2,n}^{-1}\) with the norm \(\|K_{2,n}^{-1}\| \leq \|K_2^{-1}\|/(1 - q_2)\) exists.

In addition
\[
\|x^* - \tilde{x}^*\| \leq A(\omega(a, 1/n^{1/2}) + \omega(b; 1/n^{1/2}) + E_n(\psi^\pm) + [\omega(h; 1)]^{(1-\eta)/(1+\eta)} + \omega(f; n^{-1})]
\]
where \(x^*\) and \(x_n^*\) are solutions of equations (3.7) and \(K_{2,n}x_n = y_n\).

Existence of the linear operator \(K_{1,n}^{-1}\) follows from equivalence of equations \(K_{2,n}x_n = y_n\) and (3.8).

Let us estimate its norm. Since
\[
\|x_n^*\| \leq \|K_{2,n}^{-1}\| \|\psi^-\| \|P_n[f]\|;
\]
then \(\|K_{1,n}^{-1}\| \leq \|K_{2,n}^{-1}\| \max_t |\psi^-(t)|\).

It is easy to see that
\[
\|K_nx_n - K_{1,n}x_n\| \leq A[\|a - \tilde{a}_m\| + \|b - \tilde{b}_m\|]|x_n| + I_1], \quad (3.11)
\]
where
\[
I_1 = \|P_n[\frac{1}{2\pi i} \int_\gamma P_n^\tau[h(t, \tau)x_n(\tau)b(t, \tau)]d\tau]\|
\]
\[
b(t, \tau) = |d^*(t, \tau) - d(t, \tau)|.
\]

\(I_1\) may be estimated as
\[
I_1 \leq \max_s \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P_n^\sigma[h(s, \sigma)\text{sgn}[b(s, \sigma)] - b(s, \sigma)]^{1/2} \right|^2 d\sigma \right\}^{1/2} \times
\]

56
\[ \times \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left| P_{n}^{\sigma} [\tilde{x}_{n}(\sigma)e^{i\sigma} P_{n}^{s}[b(s,\sigma) \, |^{1/2}]]^{2} \right| d\tau \right\}^{1/2} \right\} = \]

\[ = I_{2}I_{3}, \quad (3.12) \]

where \( h(s, \sigma) = h(e^{is}, e^{i\sigma}), b(s, \sigma) = b(e^{is}, e^{i\sigma}), x_{n}(\sigma) = x_{n}(e^{i\sigma}). \)

Having designated the value \([\rho(2n+1)/2\pi]+1\) by \( v \) we estimate \( I_{2} : \)

\[ I_{2}^{2} \leq A \frac{1}{2n+1} \sum_{k=1}^{v} \left( \frac{1}{s_{k}} \right)^{\eta} \leq A \left[ \frac{1}{(2n+1)^{1-\eta}} + \rho^{1-\eta} \right]. \quad (3.13) \]

It is easy to see that

\[ I_{3}^{2} = \frac{1}{(2n+1)^{2}} \sum_{k=0}^{2n} \sum_{i=0}^{2n} |x_{n}(s_{i})|^{2} |b(s_{k}, s_{i})| \leq \]

\[ \leq \frac{1}{(2n+1)^{2}} \sum_{i=0}^{2n} |x_{n}(s_{i})|^{2} \sum_{k=0}^{2n} |b(s_{k}, s_{i})| \leq A \|x_{n}\|_{2} \rho^{1-\eta}. \quad (3.14) \]

The validity of Theorem follows from the estimations (3.11) - (3.14) for reduced beyond values \( m, \rho \) and Banach Theorem.

**Proof of Theorem 3.2.**

The collocation method for equation (3.2) may be written in the form of the following expression

\[ L_{1,n}x_{n} \equiv \mathcal{P}_{n}[a(t)x_{n}(t) + \]

\[ + b(t)S_{\gamma}(x_{n}(\tau)) + S_{\gamma}((h(t, \tau) - h(t, t))x_{n}(\tau)) = \]

\[ = \mathcal{P}_{n}[f(t)], \quad (3.15) \]

where \( b(t) = h(t, t). \)

Let us represent the equations (3.2) and (3.15) in the form of Riemann boundary value problems \( L_{2}x \equiv Vx + Wx = y \) and \( L_{2,n}x_{n} \equiv \tilde{V}_{n}x_{n} + \tilde{W}_{n}x_{n} = y_{n} \), the operators \( V, W, \tilde{V}_{n}, \tilde{W}_{n} \) and the
elements $y$ and $y_n$ of which are defined beyond in proving of the Theorem 3.1.

Introduce the polynomial

$$
\varphi_n(t) = \psi_n^+ x_n^+ - \psi_n^- x_n^- + T_{n/3}[\psi^-(t)] S_\gamma((\tilde{h}(t, \tau) - \tilde{h}(t, t)) x_n(\tau)),
$$

where $\tilde{h}(t, \tau) = T_{n/3} t T_{n/3}^\tau h(t, \tau)$.

It is obvious

$$
\|V x_n - V_n x_n\| \leq A \|x_n\| (E_n(a) + E_n(b) + E_n(\psi)). \quad (3.16)
$$

Since

$$
|h(t, \tau) - \tilde{h}(t, \tau)| < A E_n^*(h) = A \ln n (E_n^t(h) + E_n^\tau(h)),
$$

then the function $\eta(t, \tau) = h(t, \tau) - \tilde{h}(t, \tau)$ belongs to the H"older class with respect to variable value $\tau$ with the exponent $1/\ln n$ and with the coefficient $A \ln n E_n^*(h)$.

Therefore

$$
\|S_\gamma((h(t, \tau) - \tilde{h}(t, \tau)) - (h(t, t) - \tilde{h}(t, t)) x_n(\tau))\| \leq \leq A \|x_n\| \ln n (E_n(b) + E_n^*(h)).
$$

An analogous estimate is valid for $\|P_n [L_2 x_n - \varphi_n]\|$. Indeed

$$
\|P_n [(V_n - V) x_n]\| \leq A \|x_n\| (E_n(a) + E_n(b) + E_n(\psi)).
$$

It remains to carry out the following proof

$$
\|P_n [S_\gamma((h(t, \tau) - \tilde{h}(t, \tau)) - (h(t, t) - \tilde{h}(t, t)) x_n(\tau))}\| \leq 
\leq \|P_n [S_\gamma(v(\tau, \tau)(x_n(\tau) - x_n(t)))]\| + \|P_n [x_n(t) S_\gamma(v(\tau, \tau))]\| + 
+ \|P_n [S_\gamma(v(t, \tau) - v(\tau, \tau)) x_n(\tau)]\| + 
+ \|P_n [(h(t, t) - \tilde{h}(t, t)) S_\gamma(x_n(\tau)))]\| \leq 
\leq A \|x_n\| E_n^*(h) \ln^2 n,
$$

where $v(t, \tau) = P_n [h(t, \tau) - \tilde{h}(t, \tau)]$.

Since the operator $L_2$ has the linear inverse one, then, for $n$ such that $q_1 = A \ln^2 n (E_n(a) + E_n(b) + E_n(\psi) + E_n^*(h)) < 1$, the
operator $L_{2,n}$ has the linear inverse operator $L_{2,n}^{-1}$ with the norm 
$\|L_{2,n}^{-1}\| \leq \|L_{2,n}^{-1}\|/(1 - q_1)$.

Since the equation $L_{2,n}x_n = y_n$ is equivalent to (3.15), then the linear operator $L_{1,n}^{-1}$ exists. Let us estimate its norm. It is evident

$$\|\tilde{x}_1^*\| \leq \|L_{2,n}^{-1}\|\|\tilde{y}\| = \|\tilde{K}_{2,n}^{-1}\|\|P_n[\psi_f]\| \leq \|L_{2,n}^{-1}\| \max_t |\psi(t)\| \|P_n[f]\|,$$

i.e. $\|L_{1,n}^{-1}\| \leq A$.

In addition $\|x^* - x_{1,n}^*\| \leq A \ln^2(n)[E_n^*(h) + E_n(a) + E_n(b) + E_n(\psi)]$, where $x^*$ and $x_{1,n}^*$ are solutions of equations (3.2) and (3.15).

Let us now show that the linear operator $L_{n}^{-1}$ exists. To this end let us estimate

$$\|L_n x_n - L_{1,n} x_n\| \leq \frac{1}{\pi i} \int_{\gamma} \frac{\omega(t, \tau) - \omega(\tau, \tau)}{\tau - t} x_n(\tau) d\tau +$$

$$+ \|P_n \left[ \frac{1}{\pi i} \int_{\gamma} \frac{\omega(\tau, \tau)x_n(\tau)}{\tau - t} d\tau \right]\|,$$

where $w(t, \tau) = P_n^t [h(t, \tau) - \tilde{h}^*(t, \tau)], \tilde{h}^*(t, \tau) = P_n^\tau [h(t, \tau)]$.

Since the function $w(t, \tau) - w(\tau, \tau)$ belongs to the Holder class with respect to variable $t$ with index $1/\ln n$ and with coefficient $A \ln^2 n E_n^\tau(h)$, then, by repeating preceding arguments, we make sure of correctness of the estimate

$$\|L_n x - L_{1,n} x_n\| \leq A \ln^2 n E_n^\tau(h) \|x_n\|.$$ 

So, for $n$ such that $q_2 = \max(q_1, A E_n^\tau(h) \ln^2 n) < 1$, the operator $L_{n}^{-1}$ exists and $\|L_{n}^{-1}\| \leq \|L_{n}^{-1}\|/(1 - q_2)$, i.e. $\|L_{n}^{-1}\| \leq A$.

To this end $\|x_n^* - x_{1,n}^*\| \leq A \ln^2 n E_n^\tau(h)$, where $x_n^*$ is a solution of the equation (3.5). The Theorem is proved.
Proof of the Theorem 3.3 is similar to the proof of the Theorem 3.2.

4. An Approximate Solution of Singular Integral Equations with Discontinuous Coefficients and on Open Contours of Integration

Let us investigate the projective method for solution of equation

\[ Kx \equiv a(t)x(t) + b(t)S_\gamma(x(\tau)) + U_\gamma(h(t, \tau)| t - \tau |^{-\eta}x(\tau)) = f(t), \] (4.1)

\[ Lx \equiv e(t)x(t) + S_L(k(t, \tau)x(\tau)) = g(t). \] (4.2)

Here \( \gamma \) is a unit circle with the center at origin, \( L = (c_1, c_2) \) is a segment of \( \gamma \). We will consider that the functions \( a(t), b(t), f(t) \in H_\alpha, h(t, \tau) \in H_{\alpha,\alpha} \) \((0 < \alpha \leq 1)\) everywhere on the circle \( \gamma \) except the point \( t = 1 \), where the functions \( a(t), b(t) \) have discontinuity of the first kind. Let \( e(t), g(t) \in H_\alpha, k(t, \tau) \in H_{\alpha,\alpha} \) \((0 < \alpha \leq 1)\). In the complex plane the cut from the origin across the point \( c \ (t = 1) \) to infinity is carried out. On the complex plane with thus slit the functions \((t - 1)^\delta \) and \( t^\delta \), which are used below, are analytical.

4.1. Fundamental Statements

The singular integral equation (4.2) on the open contours of integration are reduced to the singular integral equations with discontinuous coefficients. Because we will pay main attention to singular integral equations (4.1). It follows from theory of singular integral equations [64], [107] that the singularities of a solution \( x^*(t) \) of equation (4.1) coincides with singularities of canonical function of characteristic operator

\[ K^0x \equiv a(t)x(t) + b(t)S_\gamma(x(\tau)). \]
Let the solution $x^*(t)$ in the vicinity of the point $c$ has the form $(t - c)^\delta \varphi(t)$, where

$$
\delta = \frac{1}{2\pi i} \ln(G(c - 0)/G(C + 0)) = \xi + i\zeta, -1 < \xi < 1,
$$

$\varphi \in H_\alpha, G(t) = d(t)s(t), d(t) = a(t) - b(t), s(t) = (a(t) + b(t))^{-1}.$

Depending on the value $\xi$ (it is assumed that $-\eta + \xi > -1$) it is necessary to distinguish two cases: a) $0 < \xi < 1$, b) $-1 < \xi \leq 0$. In each case it is necessary to consider a separate calculating scheme.

At first we turn our attention to the case when $0 < \xi < 1$.

The approximate solution of equation (4.1) is sought as the polynomial

$$
x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k,
$$

which coefficients are defined from the system of equations

$$
K_n x_n \equiv \overline{P}_n [a(t)x_n(t) + b(t)S_\gamma(x_n(\tau))] +
+ U_\gamma (\overline{P}_n [h(t, \tau)d(t, \tau)x_n(\tau)])] = \overline{P}_n [f(t)].
$$

The justification of this calculating scheme is carried out in space $X = H_\beta$ ($\beta < \lambda_1 = \min(\alpha, \xi, 1 - \eta)$) and in its subspace $X_n$ consisting of $n$-order polynomials.

**Theorem 4.1** [24], [26], [35]. Let the following conditions are satisfied: the functions $a, b, f \in H_\alpha, h \in H_{\alpha\alpha}(0 < \alpha \leq 1)$ everywhere except the point $t = 1$; in the point $t = 1$ functions $a(t)$ and $b(t)$ have the discontinuity of the first kind; the Riemann boundary value problem $\psi^+(t) = G(t)\psi^-(t)$ has a solution of the form $\psi = (t - 1)^\delta \varphi(t), \delta = \xi + i\zeta; \xi > 0$; the operator $K \in [X, X]$ is continuously invertible. Then for $n$ such that $q = A(n^{-(\lambda_1 - \beta)} + n^{-\beta}) \ln n < 1$, the system (4.4) is uniquely solvable and the estimate $\|x^* - x^*_n\|_\beta \leq A(n^{-(\lambda_1 - \beta)} + n^{-\beta}) \ln n$ is valid. Here $x^*$ and $x^*_n$ are solutions of equations (4.1) and (4.4).
Let us consider another numerical scheme.

The approximate solution of equation (4.1) is sought in the form of the polynomial (4.3), which coefficients are defined from the system

$$K_n x_n \equiv \overline{P}_n [a(t)x_n(t) + b(t)S_\gamma(x_n(\tau)) + $$

$$+ \frac{1}{2\pi i} \sum_{k=0}^{2n} h(t, t_k) x_n(t_k) \int_{t_{k-1}}^{t_k} |\tau - t|^{-\eta} d\tau = \overline{P}_n[f(t)], \quad (4.5)$$

where $t_k' = e^{is_k}, \ s_k' = (2k + 1)\pi/(2n + 1), \ k = 0, 1, \ldots, 2n.$

**Theorem 4.2** [24], [26], [35]. Let all hypothesis of preceding Theorem are realized and besides a solution $x^*(t)$ of equation (4.1) has the form $x^*(t) = (t - 1)^\delta \varphi^*(t)$, where $\varphi^*(t) \in H_\alpha$. Then for $n$ such that $q = A(n^{-(\alpha - \beta)} + n^{-(\xi - \beta)}) \ln n < 1$, the system (4.5) is uniquely solvable for any right-hand side and the estimate

$$\|x^* - x^*_n\|_\beta \leq A(n^{-(\alpha - \beta)} + n^{-(\xi - \beta)}) \ln n$$

is valid, where $x^*$ and $x^*_n$ are solutions of equations (4.1) and (4.5).

Let us get over to the case when $-1 < \xi \leq 0$.

We denote by $X^*$ the space of functions $x(t) = (t - 1)^\delta \varphi(t), \ \varphi \in H_\beta$, with the norm

$$\|x(t)\| = \max_{t \in \gamma} |\varphi(t)| + \sup_{t_1 \neq t_2, t \notin (t_1, t_2)} |\varphi(t_1) - \varphi(t_2)| / |t_1 - t_2|^\beta,$$

where $\beta < \lambda_2 = \min(\alpha, 1 - \eta - \xi)$. We denote by $Y$ the space of functions satisfying the Holder condition $H_\beta$ with the norm

$$\|y\| = \max_{t \in \gamma} |y(t)| + \sup_{t_1 \neq t_2, t \notin (t_1, t_2)} |y(t_1) - y(t_2)| / |t_1 - t_2|^\beta.$$

We denote by $Y_n$ subspace of space $Y$ consisting of polynomials $\sum_{k=-n}^{n} \alpha_k t^k$.

The approximate solution of the equation (4.1) is sought in the space $X^*_n$, consisted from functions $x_n(t) = x^*_n(t) + x^-_n(t)$, where

$$x^*_n(t) = (t - 1)^\delta \varphi^*_n(t) = (t - 1)^\delta \sum_{k=0}^{n} \alpha_k t^k,$$

$$x^-_n(t) = \left(\frac{t - 1}{t}\right)^\delta \varphi^-_n(t) = \left(\frac{t - 1}{t}\right)^\delta \sum_{k=-n}^{-1} \alpha_k t^k, \quad (4.6)$$

62
coefficients \{\alpha_k\} of which are defined from the system
\[\overrightarrow{K}_n x_n \equiv \overrightarrow{P}_n [x_n^+(t) - G(t) x_n^-(t) + s(t) U_\gamma(\overrightarrow{P}_n[h(t, \tau) d(t, \tau) x_n(\tau)])] = \overrightarrow{P}_n [s(t) f(t)].\] (4.7)

**Theorem 4.3** [26], [35]. Let the following conditions are realized:
1) the functions \(a, b, f \in H_\alpha, h \in H_{\alpha \alpha} (0 < \alpha < 1)\) everywhere except the point \(t = 1\), in which functions \(a(t)\) and \(b(t)\) have a discontinuity of the first kind;
2) the operator \(K\) acting from the space \(X^*\) into the space \(Y\) has continuously invertible one;
3) the Riemann boundary value problem \(\psi^+(t) = G(t) \psi^-(t)\) has a solution of the form \((t - 1)^\delta \varphi(t), \delta = \xi + i\zeta, -1 < \xi < 0.\)

Then for \(n\) such that \(q = A(n^{- (\lambda_2 - \beta)} + n^{- \beta}) \ln n < 1, \lambda_2 = \min(\alpha, 1 - \eta + \xi, 1 + \xi)\), the system of equations (4.7) has a unique solution \(x_n^*\) and the estimate \(\|x^* - x_n^*\| \leq A(n^{- (\lambda_2 - \beta)} + n^{- \beta}) \ln n\) is valid. Here \(x^*\) is a solution of the equation (4.1).

Let us note the changes which appear in constructing numerical scheme if one supposes that a free term of equation (4.1) has a singularity as \((t - 1)^v\) at the point \(c = 1\). Let us input the space \(Y^*\) of the functions \(y(t) = (t - 1)^v \varphi(t), \varphi \in H_\beta\), with the norm
\[\|y\| = \max_{t \in \gamma} |\varphi(t)| + \sup_{t_1 \neq t_2, 1 \notin (t_1, t_2)} |\varphi(t_1) - \varphi(t_2)| / |t_1 - t_2|^\beta\]
and its subspace \(Y_{n, \gamma}^* \in Y^*\) consisting of functions of form \(y_n^* = (t - 1)^v \sum_{k=-n}^n \alpha_k t^k.\) We denote by \(\overrightarrow{P}_n \in [Y^*, Y_{n, \gamma}^*]\) the projector \(\overrightarrow{P}_n y(t) = \overrightarrow{P}_n[(t - 1)^v \varphi(t)] = (t - 1)^v \overrightarrow{P}_n [\varphi(t)].\)

We propose that the operator \(K\) acts from \(X^*\) to \(Y^*\) and has the continuous inverse one. An approximate solution of equation (4.1) we will seek in the form of a function \(x_n(t)\), defined by the expression (4.6), which coefficients are defined from the system of algebraic equations:
\[K_n^* x_n \equiv \]
\[
\left( P_n x^\gamma \right)_\eta \bigl( t, \tau \bigr) \bigl[ h(t, \tau) d(t, \tau) x_n(\tau) \bigr] d\tau = P_n^* [s(t) f(t)]. \] (4.8)

**Theorem 4.4** [24], [26], [35]. Let the hypotheses of Theorem 4.3 are realized. Then for \( n \) such that \( q = A(n^{-(\lambda_2-\beta)} + n^{-\beta}) \ln n < 1 \), the system (4.8) is uniquely solvable for any right-hand side and the estimate \( \| x^* - x_n^* \|_{X^*} \leq A(n^{-(\lambda_2-\beta)} + n^{-\beta}) \ln n \) is valid, where \( x^* \) and \( x_n^* \) are solutions of equations (4.1) and (4.8).

In a certain case it appears to consider the equations (4.1) and (4.2) is more preferable as operator equations in the space \( L_p(1 < p < \infty) \) and to lead the proof of calculating schemes in spaces \( L_p \) and their subspaces \( L_{n,p} \) consisting of polynomials as (4.3).

**Theorem 4.5** [24], [26], [35]. Let the operator \( K \) is continuously invertible in the space \( L_2 \), the coefficients \( a(t), b(t), f(t) \in H_\alpha \), \( h(t, \tau) \in H_{\alpha\alpha} \) are continuous everywhere over \( \gamma \) except the point \( t = 1 \), in which functions \( a(t), b(t) \) have a discontinuity of the first kind. Then for \( n \) such that \( q = A(n^{-\alpha} n^\theta + n^{-\eta(1-\eta)/(1+\eta)}) < 1 \), the system (4.8) has a unique solution \( x_n^* \) and an estimate \( \| x^* - x_n^* \|_{L^2} \leq A(n^{-\lambda_2-\beta} + n^{-\beta}) \ln n \) is valid, where \( x^* \) is a solution of equation (4.1). Here \( \Theta = -(1 - | \xi |) \) for \( \xi \leq 0, \Theta = -\xi \) for \( \xi > 0 \); the function \( (t - 1)^\delta \varphi_0(t) \), where \( \delta = \xi + i\zeta, \varphi_0 \in H_\alpha \) is a solution of the Riemann boundary value problem \( \varphi^+(t) = G(t)\varphi^-(t) \).

**Theorem 4.6** [24], [26], [35]. Let the operator \( K \) is continuously invertible in the space \( L_p \) (1 \( < \) \( p \) \( \leq \) 2), functions \( a, b, h, f \) everywhere, except the point \( t = 1 \), satisfy the Holder conditions with the exponent \( \alpha \) \( (0 < \alpha < 1) \), in the point \( t = 1 \) functions \( a, b \) have a discontinuity of the first kind, and \( (t - 1)^\delta \varphi_0(t) \) \( (\delta = \xi + i\zeta) \) is a solution of Riemann boundary value problem \( \varphi^+(t) = G(t)\varphi^-(t) \). Then for \( n \) such that \( q = A(n^{-\alpha} + n^{-\Theta} + n^{-\eta(1-\eta)/(1+\eta)}) < 1 \) \( (\Theta = \xi \) for \( \xi > 0, \Theta = 1 - | \xi | \)
for $\xi \leq 0$ the system of equations (4.8) has a unique solution $x_n^*$
and the estimate $|x^* - x_n^*|_{L_p} \leq A(n^{-\alpha} + n^{-\Theta} + n^{-\eta(1-\eta)/(1+\eta)})$ is
valid. Here $x^*$ is a solution of the equation (4.1).

5. Singular Integral Equations with Constant
Coefficients

Let us consider the singular integral equation

$$Kx \equiv ax(t) + \frac{b}{\pi} \int_{-1}^{1} \frac{x(\tau)}{\tau - t} d\tau + \frac{1}{\pi} \int_{-1}^{1} h(t, \tau)x(\tau)d\tau \equiv$$

$$\equiv ax(t) + bSx + Hx = f(t) \quad (5.1)$$

with constant coefficients $a$ and $b$.

The index of the equation (5.1) is defined as $\xi = -(\alpha + \beta)$, where

$$\alpha = \frac{1}{2\pi i} \ln \left( \frac{a - ib}{a + ib} \right) + N,$$

$$\beta = -\frac{1}{2\pi i} \ln \left( \frac{a - ib}{a + ib} \right) + M,$$

where $N$ and $M$ are integers, which we choose as follows:

1) $\xi = 1, -1 < \alpha, \beta < 0$;
2) $\xi = -1, 0 < \alpha, \beta < 1$;
3) $\xi = 0, \alpha = -\beta, 0 < |\alpha| < 1$.

These cases cover the well-known problems of mechanics.

Under these indexes a solution of the equation (5.1) has the form $x(t) = \omega_\xi(t)z(t)$, where

$$\omega_\xi(t) = (1 - t)^\alpha(1 + t)^\beta, 0 < |\alpha|, |\beta| < 1, \xi = -(\alpha + \beta),$$

$z(t)$ is a smooth function.

Also, the number $\alpha$ can be determined by the formula

$$a + b \cot \pi \alpha = 0.$$
L. Gori and E. Santi [73] used a projective-splines methods for solution of the equation (5.1).

A.V. Dzhishkariani are devoted papers [54], [55] to numerical methods for solution of singular integral equations (5.1).

His results are summed in the review [56].

Here, following papers [48], [49], we will give reviews of some numerical algorithms for solution the equation based on other approach.

We will consider the particle case of the equation (5.1) - the equation

\[
Kx \equiv \frac{1}{\pi} \int_{-1}^{1} \frac{x(\tau)}{\tau - t} d\tau \equiv Sx = f(t).
\]  

(5.2)

This equation is important in the aerodynamics.

Results, which will be given for the equation (5.2), one can easily diffuse to the equation

\[
Kx \equiv \frac{b}{\pi} \int_{-1}^{1} \frac{x(\tau)}{\tau - t} d\tau + \frac{1}{\pi} \int_{-1}^{1} h(t, \tau)x(\tau)d\tau \equiv
\]

\[
\equiv bSx + Hx = f(t),
\]

where \(Hx\) is a compact operator.

It is known, that the index of the equation (5.2) can take three values:

\[
\xi = \begin{cases} 
1, & \alpha = \beta = -1/2 \\
-1, & \alpha = \beta = 1/2 \\
0, & \alpha = -1/2, \beta = 1/2 \text{ (or } \alpha = 1/2, \beta = -1/2) 
\end{cases}
\]

**The first case. Index \(\xi = 0\).**

Let \(\xi = 0, \alpha = -\beta, |\alpha| = |\beta| = 1/2\).

There are two possibilities:

1) \(\alpha = -1/2, \beta = 1/2\);
2) \(\alpha = 1/2, \beta = 1/2\).
Consider the first case. The second case is similar.
The solution of equation (5.2) has the form
\[ x(t) = (1 - t)^{-1/2}(1 + t)^{1/2}\varphi(t). \]

We introduce the function space \( L_{2,\rho_1}[-1, 1] \) with the weight \( \rho_1 = [(1 - t)(1 + t)]^{1/2} \).

The integral operator \( H x \) is completely continuous in the space \( L_{2,\rho_1}[-1, 1] \).

Known [?], that the system of functions
\[ z_k(t) = c_k(1 - t)^{-1/2}(1 + t)^{1/2}P_k^{(-1/2,1/2)}(t), \]
where \( P_k^{(-1/2,1/2)}(t) \) is the Jacobi polynomial, \( k = 0, 1, \ldots, \)
\[ c_0 = \pi; c_k = (h_k^{(-1/2,1/2)})^{-1/2}, k = 1, 2, \ldots, \]
\[ h_k^{(-1/2,1/2)} = h_k^{(1/2,-1/2)} = \left( \frac{2\Gamma(k + 1/2)\Gamma(k + 3/2)}{(2k + 1)(k!)^2} \right); \]
complete and orthonormal in the space \( L_{2,\rho_1}[-1, 1] \). Besides,
\[ S z_k = c_k P_k^{(1/2,-1/2)}(t). \]

Let \( t_k(k = 0, 1, \ldots, n) \) are the knots of the polynomial \( P_n^{(1/2,-1/2)}(t) \). Let
\[ \gamma_k^{(1/2,-1/2)} = \sum_{l=0}^{n} (P_l^{(1/2,-1/2)}(t_k))^2, k = 0, 1, \ldots, n. \]

The polynomial
\[ f_n(t) = \sum_{k=0}^{n} \frac{1}{\gamma_k^{(1/2,-1/2)}} \sum_{l=0}^{n} P_l^{(1/2,-1/2)}(t_k)P_l^{(1/2,-1/2)}(t)f(t_k) \]
interpolate a function \( f(t) \) on the knots \( t_k, k = 0, 1, \ldots, n. \)

An approximate solution of equation (5.1) will be sought in the form of function
\[ x_n(t) = \sum_{k=0}^{n} \frac{1}{\gamma_k^{(1/2,-1/2)}} \sum_{l=0}^{n} P_l^{(1/2,-1/2)}(t_k)P_l^{(-1/2,1/2)}(t)x_k. \]
The right-hand part $f(x)$ of the equation (5.1) is approximated by a polynomial

$$f_n(t) = \sum_{k=0}^{n} \frac{1}{\gamma_k} \sum_{l=0}^{n} P^{(1/2,-1/2)}(t_k)P_l^{(1/2,-1/2)}(t) f_k,$$

where $f_k = f(t_k)$.

Substituting in the equation (5.2) $x_n(t)$ instead of $x(t)$ and $f_n(t)$ instead of $f(t)$ and setting the left and right side of this expression in the points $t_i, i = 0, 1, ..., n$, we have $x_i = f_i, i = 0, 1, ..., n$.

Thus, an approximate solution of equation (5.2) has the form

$$x^*(t) = \sum_{k=0}^{n} \frac{1}{\gamma_k} \sum_{l=0}^{n} P^{(1/2,-1/2)}(t_k)P_l^{(-1/2,1/2)}(t) f_k.$$

Since $f_n(t)$ is the interpolation polynomial for a function $f(t)$ on the nodes of the Jacobi polynomial $P_{n+1}^{(1/2,-1/2)}(t)$ then

$$||f(t) - f_n(t)||_{C[-1,1]} \leq (1 + \lambda_{n+1})E_n(f),$$

where $E_n(f)$ is the best uniform approximation of $f(t)$ by $n$-order polynomials, $\lambda_{n+1}$ is the Lebesgue constant over the nodes of the polynomial $P_{n+1}^{(1/2,-1/2)}(t)$.

Since the operator $K$ is continuously invertible in the space $L_{2,\rho_1}[-1,1]$ [54], then

$$||x^*(t) - x_n^*(t)||_{[-1,1]} \leq C(1 + \lambda_{n+1})E_n(f),$$

where $x^*(t)$ is a solution of the equation (5.1).

**The second case. Index $\xi = 1$.**

Let $\xi = 1, \alpha = \beta = -1/2$. In this case, the equation (5.1) has a solution with the singularities at both ends: $x(t) = (1 - t^2)^{-1/2} \varphi(t)$, where $\varphi(t)$ is a smooth function.

Consider the Chebyshev polynomials of the first kind $T_n(t) = \cos n \arccos t, n = 0, 1, \ldots |t| \leq 1$.

The polynomials $T_n(t)$ are orthogonal in the space $L_2[-1,1]$ with weight $(1 - t^2)^{-1/2}$. 68
Let $U_n(t)$ is the Chebyshev polynomial of the second kind:

$$U_n(t) = \frac{1}{(1 - t^2)^{1/2}} \sin(n + 1) \arccos t, \quad n = 0, 1, \ldots, |t| \leq 1.$$ 

It is known that

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_n(\tau)d\tau}{\sqrt{1 - \tau^2(\tau - t)}} = U_{n-1}(t), \quad n \geq 1;$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{d\tau}{\sqrt{1 - \tau^2(\tau - t)}} = 0.$$ 

Let $t_k$, $k = 0, 1, \ldots, n - 1$, are nodes of the Chebyshev polynomial of the second kind $U_n(t)$.

On nodes $t_k$, $k = 0, 1, \ldots, n - 1$, we construct the polynomial $f_{n-1}(t)$, which interpolate the function $f$:

$$f_{n-1}(t) = \sum_{k=0}^{n-1} \frac{1}{\gamma_k} \sum_{l=0}^{n-1} U_l(t_k)U_l(t)f(t_k),$$

$$\gamma_k = \sum_{l=0}^{n-1} (U_l(t_k))^2.$$ 

An approximate solution of equation (5.2) is sought in the form of a polynomial

$$x_n(t) = \frac{1}{\sqrt{1 - t^2}} \sum_{k=0}^{n-1} \frac{1}{\gamma_k} \sum_{l=0}^{n-1} U_l(t_k)T_{l+1}(t)x_{l+1} + \frac{x_0}{\sqrt{1 - t^2}}.$$ 

Substituting the functions $x_n(t)$ and $f_n(t)$ in the equation (5.2) instead of functions $x(t)$ and $f(t)$, we have

$$\sum_{k=0}^{n-1} \frac{1}{\gamma_k} \sum_{l=0}^{n-1} U_l(t_k)U_l(t)x_{l+1} = \sum_{k=0}^{n-1} \frac{1}{\gamma_k} \sum_{l=0}^{n-1} U_l(t_k)U_l(t)f(t_l).$$

Equating both sides of this equality in knots $t_k = 0, 1, \ldots, n - 1$, we have $x_{l+1} = f_l, \ l = 0, 1, \ldots, n - 1.$
So,
\[ x_n(t) = \frac{1}{\sqrt{1 - t^2}} \sum_{k=0}^{n-1} \frac{1}{\gamma_k} \sum_{l=0}^{n-1} U_l(t_k) T_{l+1}(t) f_l + \frac{x_0}{\sqrt{1 - t^2}}. \]

To find a unknown value \( x_0 \) we need an additional condition. In problems of mechanics are usually given an additional condition
\[ \int_{-1}^{1} x(\tau) d\tau = p, \quad p = \text{const.} \quad (5.3) \]

Substituting the function \( x_n(t) \) in the last equation, we find \( x_0 \).
Thus, we received a unique solution of the equation (5.2) under the presence of additional condition (5.3).

6. Approximate Methods for Solution of Nonlinear Singular Integral Equations

Iterative methods for solution of nonlinear singular integral equations with Hilbert and Cauchy kernels are devoted many papers. In these papers are considered simple methods of iteration, Newton-Kantorovich method, functional correction method and other. Reviews of these methods and rich bibliography are given in [77].

Obviously, the first works devoted to projective methods for solution of nonlinear singular integral equations was the paper [12].

6.1. Projective Methods for Solution of Nonlinear Equations on Closed Paths of Integration

Let us consider the nonlinear singular integral equation
\[ Kx \equiv a(t, x(t)) + \frac{1}{\pi i} \oint_{\gamma} \frac{h(t, \tau, x(\tau))}{\tau - t} d\tau = f(t), \quad (6.1) \]
where $\gamma$ is the unit circle with the center at origin.

**Basis Statements.**

**Calculating Scheme 1.** An approximate solution of equation (6.1) is sought in the form of polynomial

$$x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k,$$

(6.2)

the coefficients of which are determined from the system

$$a(t_j, x_n(t_j)) + \frac{1}{2n + 1} \sum_{k=0}^{2n} h(t_j, t_k, x_n(t_k))(1 - i \cot \frac{s_k - s_j}{2}) -$$

$$- \frac{2i}{2n + 1} h'_u(t_j, t_j, x_n(t_j)) x'_{ns}(e^{is_j}) = f(t_j), j = 0, 1, \ldots, 2n,$$

(6.3)

where $t_j = \exp(is_j), s_j = 2\pi j/(2n + 1)$, the prime in the summation indicate that $k \neq j$, $x'_{ns}$ means differentiation of the function $x_n(s)$ with respect to $s$ and $h'_u(t, \tau, u)$ means differentiation of the function $h(t, \tau, u)$ with respect to $u$.

**Theorem 6.1** [12], [13], [26], [35]. Let the equation (6.1) has a unique solution $x^*$ in certain sphere $S$, it exists the linear operator $[K'(x^*)]^{-1}$ and the conditions $x^*(t), f(t) \in H_\alpha$, $a(t, u), a'_u(t, u), a''_u(t, u) \in H_\alpha_1$, $h(t, \tau, u), h'_u(t, \tau, u), h''_u(t, \tau, u) \in H_{\alpha_1}$, where $0 < \alpha \leq 1, |u| < \infty$, are fulfilled. Then for $n$ such that $q = An^{-(\alpha/2-\beta)} \ln^5 n < 1$, the system (6.3) has a solution $x^*_n$ and in the metric of the space $X = H_\beta$ the inequality $\|x^* - x^*_n\| \leq An^{-(\alpha/2-\beta)} \ln^2 n$ is valid.

**Calculating Scheme 2.** An approximate solution of equation (6.1) is sought in the form of polynomial

$$x_n(s) = \sum_{k=0}^{2n} \alpha_k \psi_k(s),$$

(6.4)
the coefficients of which are defined from the system of equations
\[ a(t_j, x_n(t_j)) + \frac{1}{2n+1} \sum_{k=0}^{2n} h(t_j, t_k, \alpha_k)(1 - i \cot \frac{2(k - j)\pi - \pi}{4n + 2}) = f(t_j), \]
where
\[ t_j = \exp(is_j), \psi_k(s) = \frac{1}{2n+1} \sin \frac{2n+1}{2}(s - s_k) \cdot \frac{1}{\sin \frac{s - s_k}{2}}, s_k = \frac{2k\pi}{2n + 1}, \]
\[ \bar{t}_k = \exp(i\bar{s}_k), \bar{s}_k = (2k + 1)\pi/(2n + 1). \]

Let us plan of obtaining the system (6.5). The equation (6.1) may be written as
\[ a(s, x(s)) - \frac{i}{2\pi} \int_0^{2\pi} h(s, \sigma, x(\sigma)) \cot \frac{\sigma - s}{2} d\sigma - \]
\[ -\frac{i}{2\pi} \int_0^{2\pi} h(s, \sigma, x(\sigma))d\sigma = f(s), \]  
where \( a(s, x(s)) = a(e^{is}, x(e^{is})), h(s, \sigma, x(\sigma)) = h(e^{is}, e^{i\sigma}, x(e^{i\sigma})), \) \( f(s) = f(e^{is}). \)

Having subtracted the polynomial \( x_n(s) \) into equation (6.6) for \( x(s) \), setting equal left-hand sides of this equation to their right-hand sides at the points \( \bar{s}_j = (2\pi j + \pi)/(2n+1) \) \( (j = 0, 1, \ldots, 2n) \) and taking as quadrature rule for both integrals the quadrature rule of left rectangle on the point \( s_j \) we come to the following algebraic scheme
\[ a(\bar{s}_j, x_n(\bar{s}_j)) - \frac{1}{2n+1} \sum_{k=0}^{2n} h(\bar{s}_j, s_k, \alpha_k) \cot \frac{s_k - \bar{s}_j}{2} + \]
\[ + \frac{1}{2n+1} \sum_{k=0}^{2n} h(\bar{s}_j, s_k, \alpha_k) = f(\bar{s}_j), \]
\( j = 0, 1, \ldots, 2n. \)
Changing \( a(x_j, x_n(x_j)), h(x_j, s_k, \alpha_k), f(x_j) \) into \( a(s_j, x_n(s_j)), h(s_j, s_k, \alpha_k), f(s_j) \) we arrive at the system (6.5).

**Theorem 6.2** [12], [13], [26], [35]. Let the equation (6.1) has a unique solution \( x^*(t) \) in certain sphere \( S \), it exists a right bounded inverse operator \( [K'(x)]_r^{-1}(x \in S) \) and the conditions \( a(t), f(t), x^*(t) \in H_{\alpha}, h(t, \tau, u), h_u(t, \tau, u) \in H_{\alpha,\alpha,1} \) \( (0 < \alpha \leq 1, |u| < \infty) \) are fulfilled. Then for \( n \) such that \( q = An^{2} \ln^5 n < 1 \) the system (6.5) has a unique solutions \( x_n^* \) and in the metric of the space \( X = H_{\beta}(0 < \beta < \alpha^2/4) \) the estimate \( \|x^* - x_n^*\| \leq An^{-(\alpha/2-\beta)} \ln^2 n \) is valid.

**Calculating Scheme 3.** An approximate solution of equation (6.1) is sought in the form of the polynomial (6.4), the coefficients \( \alpha_k \) of which are defined from the equation

\[
K_n x_n \equiv P_n \left[ a(t, x_n(t)) + \frac{1}{\pi i} \int P_\gamma^n \left[ \frac{h(t, \tau, x_n(\tau))}{\tau - t} \right] d\tau \right] = P_n[f(t)]. \tag{6.7}
\]

**Theorem 6.3** [12], [13], [26], [35]. Let the equation (6.1) has a unique solution \( x^*(t) \) in the certain sphere \( S \), exists a right bounded inverse operator \( [K'(x)]_r^{-1}(x \in S) \) and it is carried out one of the following conditions:

a) \( x^*(t) \in H_{\alpha}, a(t, u) \in H_{\alpha,1}^{r,r+1}, h(t, \tau, u) \in H_{\alpha,\alpha,1}^{r,r,r+1} \),

b) the functions \( x^*(t), a(t, x^*(t)), h(t, \tau, x^*(t)), h(\tau, t, x^*(t)) \) are analytical that in the domains \( R_1 < |t| < R_2, R_1 < |\tau| < R_2 \), where \( R_1 < 1, R_2 > 1 \).

If the condition "a" was fulfilled, then, for \( n \) such that \( q = An^{-r-\alpha+2\beta} \ln^6 n < 1 \), the system (6.7) has a solution \( x_n^* \) and in the metric of the space \( X = H_{\beta}(0 < \beta < (r+\alpha)/2) \) \( \|x^* - x_n^*\| \leq An^{-r-\alpha+\beta} \ln^2 n \). If the condition "b" was fulfilled, then, for \( n \) such that \( q = A[R_1^{n+1} + R_2^{-n-1}]n^{2\beta} \ln^6 n < 1 \), the equations system (6.7) has a solution \( x_n^* \) and the estimate \( \|x^* - x_n^*\| < \)
\[ A[R_1^{n+1} + R_2^{-n-1}]n^\beta \ln^2 n \] is valid.

In series of cases the condition to nonlinear singular integral equations in the space \( L_2 \) is more preferable. We are looking for approximate solution of the equation (6.1) by means of the iterative process

\[ x_n^{m+1} = x_n^m - \frac{\|K_n x_n^m\|^2}{\|K'_n(x_0^0)K_n x_n^m\|^2} [K'_n(x_0^0)]^* K_n x_n^m, \] (6.8)

where

\[ K_n x_n = \overline{P}_n[a(s, x_n(s)) - \frac{i}{2\pi} \int_0^{2\pi} P_n^\sigma[h(s, \sigma, x_n(\sigma)) \cot \frac{\sigma - s}{2}] d\sigma + \]

\[ \frac{1}{2\pi} \int_0^{2\pi} P_n^\sigma[h(s, \sigma, x_n(\sigma))] d\sigma - f(s), \]

\[ x_0^0(s) = P_n[x_0^0(s)], \]

\( x_0^0(s) \) is well enough approximation for the solution \( x^* \) of equation (6.1), \( x_n(s) \) is trigonometric polynomial of degree not higher than \( n, a(s) = a(e^{i\sigma}), h(s, \sigma) = h(e^{i\sigma}, e^{i\sigma}), x_n(s) = x_n(e^{i\sigma}). \)

Let the following condition are fulfilled:

a) functions \( a(t, x_0^0(t)), f(t), x_0^0(t), h(t, \tau, x_0^0(\tau)) \) (with respect to each variable) either are analytical inside the ring \( R_1 \leq |t| \leq R_2, R_1 < 1, R_2 > 1 \) or belong to the class \( W^r H_\alpha; \)

b) in some sphere defined below

\[ \max_{t, \tau \in \gamma} |h'_u(t, \tau, u_1) - h'_u(t, \tau, u_2)|, |h^*(t, \tau, u_1) - h^*(t, \tau, u_2)| \leq F, \]

where \((u_1, u_2 \in S),\)

\[ h^*(t, \tau, u) = [h'_u(t, \tau, u) - h'_u(\tau, \tau, u)]/ |\tau - t|^\beta \exp(i\theta_1) \}, \]

\( \beta \) is an arbitrary value \( 0 < \beta < \alpha, \theta_1 = \theta_1(\tau, t) = \arg |\tau - t| \).

Let us introduce the designations: \( \delta_0 = \|K x_0 - f\|, \delta_0 = \|K_n x_n^0\|, B_0 = \|K'_n(x_0^0)\|^{-1}, B_* = \max\{\|K'(x_0)\|, \|K'_n(x_0^0)\|\}. \)

The existence of constants \( B_0 \) and \( \|K'(x_0^0)\| \) and their relation to the values \( \|K'(x_0^0)\|^{-1} \) and \( \|K'(x_0^0)\| \) follow from results of section 3.
Theorem 6.4 \cite{15}, \cite{26}, \cite{35}. Let in the sphere $S[x; \|x - x^0\| \leq r], r = B_*B_0^2\delta_0/(1 - q) + A\ln nE_n(x^0), q = AF_1B_*B_0^2 + \sqrt{(1 - B_*^2B_0^{-2})} < 1$, the condition "b" are carried out and the operator $K'(x^0)$ has a linear inverse (it is enough left inverse). Then for $n$ such that $p = A\ln^2 n\max[E_n(a), E_n(\psi), E_n(x^0), E_n(h'_u(t, t, x^0_n(t))), E_n^{t, \tau}(h'_u(t, \tau, x^0_n(\tau)))] < 1$, the equation $K_nx_n = 0$ has in $S$ a unique solution $x^*_n$ and to which the iterative process (6.8) converges with the rate $\|x^*_n - x^m_n\| \leq q^m\tilde{\delta}_0/B_*(1 - q)$. The distance between $x^*_n$ and solution $x^*$ of equation (6.1) is estimated by inequalities

$$\|x^* - x^*_n\| \leq \ln^2 n\max(E_n(a), E_n(\psi), E_n(t), E_n^{t, \tau}(h(t, \tau, x^*_n(\tau))),$$

$$\|x^* - x^*_n\| \leq B_*\tilde{\delta}_0/(1 - q) + E_n(x_0).$$

Proof of Theorem 6.1. An approximate solution is sought in the subspace $X_n \subset X$, consisted of polynomials of the form (6.2). Under the conditions of the theorem the operator $K$ has Frechet derivative

$$K'(x)z \equiv a'_u(t, x(t))z(t) + \frac{1}{\pi i} \int_\gamma \frac{h'_u(t, \tau, x(\tau))z(\tau)}{\tau - t}d\tau, \quad (6.14)$$

satisfying in a ball $S(x^*, r)$ with an some radius $r$ the Lipschitz condition $\|K'(x_1) - K'(x_2)\| \leq A(r)\|x_1 - x_2\|$. The system of equations (6.3) in the operator form is written in the form of expression

$$K_n(x_n) \equiv P_n[a(s, x_n(s)) + \frac{1}{2\pi} \sum_{k=0}^{2n} h(s, s_k, x_n(s_k))\psi_k(\sigma)d\sigma -$$

$$- \frac{i}{2\pi} \int_0^{2\pi} P_n \left[ h(s, s, x_n(\sigma)) - h(s, s, x_n(s)) \right] \cot \frac{\sigma - s}{2} d\sigma -$$
Here $a(s, x_n(s)) = a(e^{is}, x_n(e^{is}))$. Functions $h(s, s_k, x_n(s_k))$, $f(s)$ are defined similarly. Here $\psi_k(\sigma)$ are basic trigonometric polynomials, constructed on knots $s_k = \frac{2k\pi}{2n+1}$, $k = 0, 1, \ldots, 2n$.

It is easy to see that the Frechet derivative of the operator $K_n$ has the form

$$K_n'(x_n)z_n \equiv \sum P_n\left\{ \int^0_0 \sum' \{ h_u'(s, s_k, x_n(s_k)) - h_u'(s, s, x_n(s_k)) \} \cot \frac{s_k - s}{2} \psi_k(\sigma) d\sigma \right\},$$

$$I_1(x_n) = \frac{i}{2\pi} P_n \left\{ \int^0_0 \sum' \{ h_u'(s, s_k, x_n(s_k)) - h_u'(s, s, x_n(s_k)) \} \cot \frac{s_k - s}{2} \psi_k(\sigma) d\sigma \right\};$$

$$I_2(x_n) = \frac{i}{2\pi} P_n \left\{ \int^0_0 \sum' \{ h_u'(s, s, x_n(s)) z_n(s) \} \cot \frac{s - s}{2} \psi_k(\sigma) d\sigma \right\};$$

$$I_3(x_n) = \frac{1}{2\pi} P_n \left\{ \int^0_0 \sum' \{ h_u'(s, s_k, x_n(s_k)) z_n(s_k) \} \psi_k(\sigma) d\sigma \right\}. $$

We will show that the Frechet derivative of $K_n$ belongs to Lipschitz class of functions:

$$\|K_n'(x_n') - K_n'(x_n'')\| \leq An^{\beta} \ln^2 n \|x_n' - x_n''\|. \quad (6.15)$$

We will give a proof only for $I_2$ (for $I_1$ and $I_3$ the proof is more simply than for $I_2$).
It was proved \cite{109, 110} that \( \sum_{k=0}^{2n} |\psi_k(s)| \leq A \ln n \).

Therefore
\[
|I_2(x_n') - I_2(x_n'')| \leq A \ln n \left\{ \max_{0 \leq j \leq 2n} \frac{2\pi}{2n+1} \left| \sum_{k=0}^{2n} \bigl\{ [h_u'(s_j, s_j, x_n'(s_k)) - h_u'(s_j, s_j, x_n''(s_k))] z_n(s_k) - [h_u'(s_j, s_j, x_n'(s_j)) - h_u'(s_j, s_j, x_n''(s_j))] z_n(s_j) \bigr\} \cot \frac{s_k - s_j}{2} + \frac{4}{2n+1} \left| [h_u''(s_j, s_j, x_n'(s_j)) x_n'(s_j) - h_u''(s_j, s_j, x_n''(s_j)) x_n''(s_j)] z_n(s_j) \right| + \right. \\
\left. \frac{4}{2n+1} \left| [h_u'(s_j, s_j, x_n'(s_j)) - h_u'(s_j, s_j, x_n''(s_j))] z_n'(s_j) \right| \right\} \leq A \ln n [I_4 + I_5 + I_6].
\]

Let us estimate \(|I_4|\):

\[
|I_4| \leq A |x_n' - x_n''| |z_n| \sum_{k=1}^{n} k^{-1} \leq A |x_n' - x_n''||z_n| \ln n.
\]

It is known the following inequality of Riesz \cite{121}. If \( f(s) \) is a trigonometric polynomial of degree \( n \), then \( |f'(s)| \leq n \max_{s} |f(s + \pi/2n) - f(s)|/2 \). Using this inequality, we obtain the estimates
\[
|I_5| \leq A |x_n' - x_n''||z_n| n^{-\beta}, \quad |I_6| \leq A |x_n' - x_n''||z_n| n^{-\beta}.
\]

As \((I_2(x_n') - I_2(x_n''))\) is the trigonometric polynomial of degree \( n \), then \(|(I_2(x_n') - I_2(x_n''))| \leq An^{\beta} \ln n |x_n' - x_n''||z_n|\). Now it is easy to verify the validity of the estimate (6.15).

According the conditions of the Theorem, \( x^*(t) \in H_\alpha \). So, it exists such polynomial \( x_0(t) \in X_\alpha \), that \( ||x^* - x_0|| \leq An^{-(\alpha - \beta)} \).

According the conditions of the Theorem the operator \([K'(x^*)]^{-1}\) exists. So, the linear operator \([K_n'(x_n^0)]^{-1}\) exists too. Indeed, let \(||[K'(x^*)]^{-1}|| = B_0\). From Banach Theorem follows that, for \( n \)
such \( q = An^{-(\alpha - \beta)} < 1 \), the operator \([K'(x_n^0)]^{-1}\) exists with the norm \( \|K'(x_n^0)\|^{-1} \leq B_0/(1 - q) \).

In §2 was shown that, for \( n \) such that \( q = A\ln n/n^{\alpha - \beta} < 1 \), the operator \( K_n'(x_n^0) \) has linear inverse operator \([K_n'(x_n^0)]^{-1}\) with the norm \( \|[K_n'(x_n^0)]^{-1}\| \leq A\ln n \). Here

\[
K_n'(x_n^0)z_n \equiv P_n[a'_u(t, x_n^0(t))z_n(t) + S_N[h'_u(t, \tau, x_n^0(\tau))z_n(\tau)].
\]

Let us estimate the norm \( \|[K_n'(x_n^0) - K_n'(x_n^0)]\| \).

At first we estimate the norm

\[
\|J\| = \|P_n \left[ \int_0^{2\pi} R_n^\sigma [h'_u(s, s, x_n^0(\sigma))z_n(\sigma) - h'_u(s, s, x_n^0(s))z_n(s)] \cot \frac{\sigma - s}{2} d\sigma \right] \|
\]

where \( R_n = I - P_n \).

Let us approximate the function \( h'_u(s, s, x_n^0(\sigma)) \) (with respect to variable \( \sigma \)) with interpolation polynomial, which is constructed on knots \( s_k = 2k\pi(2n + 1)^{-1} \), \( k = 0, 1, \ldots, 2n \). This polynomial will be denoted by \( h_n(s, s, \sigma) \). It is easy to see that function \([h_n(s, s, \sigma)z_n(\sigma) - h_n(s, s, s)z_n(s)] \cot \frac{\sigma - s}{2} \) is a polynomial of \( 2n \) degree with respect to variable \( \sigma \).

It is easy to see that \( \|J\| \leq \|J_1\| + \|J_2\| \), where

\[
J_1 = P_n \left[ \int_0^{2\pi} \left[ h'_u(s, s, x_n^0(\sigma)) - h_n(s, s, \sigma) \right] z_n(\sigma) \cot \frac{\sigma - s}{2} d\sigma \right]
\]

\[
J_2 = \sum_{k=0}^{2n} S_k(s) \psi_k(s),
\]

\[
S_k(s) = \int_0^{2\pi} P_n^\sigma \left[ [h'_u(s_k, s_k, x_n^0(\sigma)) - h_n(s_k, s_k, \sigma)] z_n(\sigma) -
\right]
\]
\[-[h_u'(s_k, s_k, x_n^0(s_k)) - h_n(s_k, s_k, s_k)]z_n(s_k)] \cot \frac{\sigma - s_k}{2} d\sigma.\]

Obviously, \(\| J_1 \| \leq A \ln^2 n \| z_n \| / n^{\alpha - \beta},\)

\[|S_k(s)| \leq \left\{ \left\| J_2 \right\| \leq \frac{4\pi}{2n + 1} \left( |J_3| + |J_4| \right) = \frac{4\pi}{2n + 1} |J_3|, \right\}\]

since \(J_4 = 0\) by the definition of the polynomial \(h_n(s, s, \sigma)\).

Let us estimate \(|J_3|\). The function \(h_u'(s, s, x_n^0(\sigma))\) has the first derivative with respect to variable \(\sigma\). This derivative is belong to the Holder class \(H_\alpha\) with the coefficient \(A_n\). It follows from conditions of the Theorem and the Riesz inequality. It is easy to see that \(|J_3| \leq A n^{1-\alpha} \| z_n \| \ln n.\)

From two last inequalities follows \(|S_k(s)| \leq A \| z_n \| n^{-\alpha} \ln n.\) So \(\| J_2 \| \leq A \ln^2 n \| z_n \| n^{-(\alpha - \beta)}.\) From this and estimate for \(\| J_1 \|\) we have \(\| J \| \leq A \ln^2 n \| z_n \| n^{-(\alpha - \beta)}.\)

Let us estimate the following integral

\[\| J_5 \| =\]

\[= \left\{ \left\| \frac{i}{2\pi} P_n \int_0^{2\pi} \left[ h_u'(s, \sigma, x_n^0(\sigma)) - h_u'(s, s, x_n^0(\sigma)) \right] z_n(\sigma) \cot \frac{\sigma - s}{2} d\sigma \right\| -\right\}

\[-I_1(x_n^0) \leq A \ln n \| \int_0^{2\pi} \left[ h_u'(s, \sigma, x_n(\sigma)) - h_u'(s, s, x_n(\sigma)) \right] \right\| -\]

\[-[h_u'(s, s, x_n^0(\sigma)) - h_n^*(s, s, x_n^0(\sigma))]z_n(\sigma) \cot \frac{\sigma - s}{2} d\sigma \right\| +\]

\[+ \left\{ \left\| P_n \left[ \int_0^{2n} \sum_{k=0}^{2n} \left[ h_u'(s, s_k, x_n^0(s_k)) - h_n^*(s, s_k, x_n^0(s_k)) \right] \right\] \right\} -\]

79
$- [h'_u(s, s, x^0_n(s_k)) - h^*_n(s, s, x^0_n(s_k))] z_n(s_k) \psi_k(\sigma) \cot \frac{s_k - s}{2} d\sigma] +$

$+ \left\{ \left[ \sum_{k=0}^{2n} \frac{2i}{2n + 1} [h^*_n(s_k, \sigma, x^0_n(s_k))]'_{\sigma = s_k} z_n(s_k) \psi_k(\sigma) \right] \right\} = \|J_6\| + \|J_7\|,$

where $h^*_n(s, \sigma, x^0_n(\nu))$ is the interpolation trigonometric polynomial of $[n/2]$ degree (with respect to variable $\sigma$) for $P^{\nu}_{[n/2]}[h(s, \sigma, x^0_n(\nu))]$.

Polynomial is constructed on knots $s_k(k = 0, 1, \ldots, 2[n/2]).$

So, $h^*(s, \sigma, x^0_n(\nu)) = P^\sigma_{[n/2]} P^{\nu}_{[n/2]}[h(s, \sigma, x^0_n(\nu))].$

It is easy to see that $\|J_6\| \leq A \ln^2 n \|z_n\| n^{-(\alpha-\beta)}$.

Using the Riesz inequality we have $\|J_7\| \leq A \ln^2 n \|z_n\| n^{-(\alpha-\beta)}$.

From estimates for $\|J_6\|$ and $\|J_7\|$ it follows that

$\|J_5\| \leq A \ln^2 n \|z_n\| n^{-(\alpha-\beta)}.$

It is easy to receive the estimate

$\| P_n \left[ \int_0^{2\pi} h'_u(s, \sigma, x^0_n(\sigma)) z_n(\sigma) d\sigma \right] - I_3(x^0_n) \| \leq$

$\leq A \ln n \|z_n\| n^{-(\alpha-\beta)}.$ \hspace{1cm} (6.16)

From estimates for $\|J\|$, $\|J_5\|$ and from the inequality (6.16) it follows that

$\|K'_1(x^0_n) - K'_n(x^0_n)\| \leq A \ln^2 n/n^{\alpha-\beta}.$

We have proved that the operator $K'_1(x^0_n)$ has continuously invertible operator. Using the Banach theorem we see that, for $n$ such that $q = A n^{-\alpha + \beta} \ln^2 n < 1$, the operator $K'_n(x^0_n)$ has continuously invertible operator with the norm $\|(K'_n(x^0_n))^{-1}\| \leq A \ln n.$

Let us note that $\|K_n x^0_n\| \leq \|K_n x^0_n - K x^0_n\| + \|K x^0_n - K x^*\| \leq A n^{-(\alpha-\beta)} \ln n$. In the ball $S[x : \|x - x^0_n\| \leq A \ln^2 n/n^{\alpha-\beta}]$, for $n$ such that $q = \frac{A \ln n}{n^{\alpha-\beta}} < 1$, all conditions of the Theorem 2.6
from the Chapter 1 is valid. Using this Theorem, we prove the existence of a unique solution \( x_n^*(t) \) of the equation (6.3) and the correctness of the estimate \( \| x^* - x_n^* \| \leq A \ln^2 n / n^{\alpha-\beta} \).

The Theorem is proved.

**Proof of the Theorem 6.2.**

Justification for computing algorithm 2 is performed in space \( X = H_\beta \quad (\beta < \alpha/4) \) and its subspace \( X_n \), which consists of polynomials (6.2).

Repeating the calculations made in the proof of the previous theorem, we can show that the operator \( K \) has Frechet derivative

\[
K'(x)z \equiv a(t)z(t) + \frac{1}{\pi i} \int_{\gamma} h'_u(t, \tau, x(\tau))z(\tau)(\tau - t)^{-1}d\tau,
\]
satisfying in the sphere \( \| x - x^0 \| \leq r \) the Holder condition \( \| K'(x_1) - K'(x_2) \| \leq A\| x_1 - x_2 \|^{\alpha-\beta} \), where \( x^0 \) is a element of the space \( X \), and \( A \) is a constant, which is dependent only from \( r \) and \( x_0 \).

The system of equations (6.7) in the operator form is written in the form of expression

\[
K_n(x_n) \equiv \tilde{P}_n[\tilde{a}(s)x_n(s)] - \frac{i}{2\pi} \int_0^{2\pi} P_n^\sigma[\tilde{h}(s, \sigma, x_n(\sigma))] \cot \frac{\sigma - s}{2} d\sigma + \\
\quad + \frac{1}{2\pi} \int_0^{2\pi} P_n^\sigma[\tilde{h}(s, \sigma, x_n(\sigma))]d\sigma = \tilde{P}_n[\tilde{f}],
\]
where

\[
\tilde{a}(s) = a \left( s - \frac{\pi}{2n+1} \right), \quad \tilde{h}(s, \sigma, x_n(\sigma)) = h \left( s - \frac{\pi}{2n+1}, \sigma, x_n(\sigma) \right), \quad \tilde{f}(s) = f \left( s - \frac{\pi}{2n+1} \right).
\]
The Frechet derivative of the operator \( K_n(x_n) \) is

\[
K_n'(x_n)z_n \equiv \bar{P}_n[\tilde{a}(s)z_n(s)-i \frac{2 \pi}{\sigma - s} d\sigma + \frac{1}{2 \pi} \int_0^{2 \pi} P_n^\sigma [\tilde{h}'_u(s, \sigma, x_n(\sigma))z_n(\sigma)]d\sigma]
\]

where \( \tilde{h}'_u(s, \sigma, x_n(\sigma)) = h'_u(s - \pi/(2n + 1), \sigma, x_n(\sigma)) \).

The arguments that differ little from the proof of Theorem 6.1, suggest that

\[
\|K_n'(x'_n) - K_n(x''_n)\| \leq An^\beta \ln^2 n \|x'_n - x''_n\|^\alpha.
\]

Let \( x^0_n \) is the polynomial of best approximation of the degree \( n \) for function \( x^*(s) \).

Let the operator \( K'(x) \) has bounded right inverse operator \( [K'(x)]^{-1} (x \in S_1) \) with the norm \( \|[K'(x)]^{-1}\| = B_0 \).

Let us prove the existence of the bounded right inverse operator \( [K'(x^0_n)]^{-1} (x \in S_1) \).

The existence of a right inverse operator \( [K'(x^0_n)]^{-1} \) with such \( n \) that \( q = An^{-(\alpha - \beta)} < 1 \), follows from the results of the Section 2 of Introduction. Note that \( \|[K'(x^0_n)]^{-1}\| = B_0/(1 - q) \).

As in the §2, we can see that, for such \( n \) that \( q = An^{-(\alpha - \beta)} \ln^2 n < 1 \), the operator \( K_{1n}'(x^0_n) \), where

\[
K_{1n}'(x^0_n)z_n \equiv \bar{P}_n[a(t)z_n(t)-i \frac{2 \pi}{\sigma - s} d\sigma + \frac{1}{2 \pi} \int_0^{2 \pi} P_n^\sigma [h'_u(s, \sigma, x^0_n(\sigma))z_n(\sigma)]d\sigma],
\]
has the linear invertible operator \( [K'_{1n}(x_n^0)]^{-1} \) with the norm \( \| [K'_{1n}(x_n^0)]^{-1} \| \leq A \ln n \).

We will show that if in some neighborhood \( S_1 \) of the point \( x^* \) exists a bounded right inverse operator \( [K'(x)]_{r_1}^{-1}(x \in S_1) \) with the norm \( \| [K'(x)]_{r_1}^{-1} \| = B_0 \), that there exists a linear operator \( [K'_{n}(x_n^0)]_{r}^{-1} \) for sufficiently large \( n \).

To prove the existence of a linear operator \( [K'_{n}(x_n^0)]^{-1} \) we need in assessment of the norm

\[
\| K'_{1n}(x_n^0)z_n - K'_{n}(x_n^0)z_n \| \leq \| \tilde{P}_n[(a(t) - \tilde{a}(t)]z_n(t)\| + \\
+ \frac{1}{2\pi} \tilde{P}_n \int_0^{2\pi} P_n(\| h'(s, \sigma, x_n^0(\sigma)) - \tilde{h}'(s, \sigma, x_n^0(\sigma))\|z_n(\sigma)\cot \frac{\sigma - \bar{s}}{2}\|d\sigma\| + \\
+ \frac{1}{2\pi} \tilde{P}_n \int_0^{2\pi} P_n(\| h'(s, \sigma, x_n^0(\sigma)) - \tilde{h}'(s, \sigma, x_n^0(\sigma))\|z_n(\sigma)\|d\sigma\| = \\
= \| J_1 \| + \| J_2 \| + \| J_3 \|.
\]

Easy to see that

\[
\| J_1 \| + \| J_3 \| \leq A \ln^2 n \| z_n \| n^{-(a-\beta)},
\]

\[
| I_2 | \leq A \ln n \max_{0 \leq k \leq 2n} \left| \int_0^{2\pi} P_n(\| h'(s_k, \sigma, x_n^0(\sigma)) - \tilde{h}_u(s_k, \sigma, x_n^0(\sigma))\|z_n(\sigma)\cot \frac{\sigma - \bar{s}_k}{2}\|d\sigma\| \right| \leq A \ln^2 n \| z_n \| n^{-\alpha}.
\]

Since \( I_2 \) is the trigonometrical polynomial of the order \( n \), that

\[
\| I_2 \| \leq A \ln^2 n \| z_n \| n^{-(a-\beta)}.
\]

From estimates for \( \| J_1 \|, \| J_2 \|, \| J_3 \| \) and Banach Theorem it follows, that, for such \( n \) that \( q = An^{-(a-\beta)} \ln^2 n < 1 \), the linear operator \( [K'_n(x_n^0)]^{-1} \) exists with the norm \( \| [K'_n(x_n^0)]^{-1} \| = B_2 \leq B_1/(1 - q) \).
Was proved (see the proof of the Theorem 6.1) that \( \|K'_n(x^0_n)\| \leq An^{-(\alpha-\beta)} \ln n \).

Equation (6.7) satisfies all conditions of Theorem 2.6, given in the introduction.

So, we proved the existence of a unique solution \( x^*_n \) of the equation (6.7). The estimate of norm \( \|x^* - x^*_n\| \) follows from Theorem 2.6, given in the introduction.

**Proof of the Theorem 6.3.** The proof of the Theorem 6.3 is similar to the proof of the Theorem 6.2.

**Proof of the Theorem 6.4.**

We will show that under the conditions of "b" the operator \( K \) has the Frechet derivative in the space \( L_2 : \)

\[
K'(x)z_n \equiv a'(t,x(t))z_n(t) + S_\gamma(h'_u(t,\tau,x(\tau))z_n(\tau)),
\]

and inequality

\[
\|K'(x_1) - K'(x_2)\| \leq AF \tag{6.17}
\]

is valid.

To prove the existence of the derivative \( K'(x)z_n \), let us estimate the difference

\[
\|I_1\|/\|z_n\| = \|K(x + z_n) - K(x) - K'(x)z_n\|/\|z_n\|.
\]

We restrict ourselves to estimate the integral term in the expression for \( K'(x)z_n \).

Obviously,

\[
|I_1| = \left| \frac{1}{\pi i} \int \left\{ \int_0^1 (1 - \nu)h''_u(t,\tau,x(\tau) + \nu z_n(\tau))d\nu \right\} \frac{z^2_n(\tau)d\tau}{\tau - t} \right|.
\]

Represent \( h''_u(t,\tau,u) \) in the form \( h''_u(t,\tau,u) = [h''_u(t,\tau,u) - h''_u(\tau,\tau,u)] + h''_u(\tau,\tau,u) \).

It is easy to see that \( \|I_1\| \leq A\|z_n\| \max |z_n| \) and \( \|I_1\|/\|z_n\| \leq An^{1/2}\|z_n\| \).
From this estimate follows that \( \lim_{\|z_n\| \to 0} \|I_1\|/\|z_n\| = 0 \). The inequality (6.17) is proved.

It is easy to see that the Frechet derivative of the operator \( K_n \) is

\[
K'_n(x_n)z_n \equiv \bar{P}^s_n [a'_u(s, x_n(s))z_n(s) + \frac{1}{2\pi} \int_0^{2\pi} P^\sigma_n [h'_u(s, \sigma, x_n(\sigma))z_n(\sigma)]d\sigma] - \frac{i}{2\pi} \int_0^{2\pi} P^\sigma_n [h'_u(s, \sigma, x_n(\sigma))z_n(\sigma)\cot\frac{\sigma - s}{2}]d\sigma.
\]

Let us proved justice of the inequality

\[
\|K'_n(x'_n) - K'_n(x''_n)\| \leq AF. \quad (6.18)
\]

Indeed, \( \|K'_n(x'_n)z_n - K'_n(x''_n)z_n\| \leq I_2 + I_3 \), where

\[
I_2 = \|\bar{P}_n [S_\gamma (P^\tau_n [h'_u(\tau, \tau, x'_n(\tau)) - h'_u(\tau, \tau, x''_n(\tau))] z_n(\tau))]\|, 
\]

\[
I_3 = \|\bar{P}_n [\frac{1}{\pi i} \int_\gamma P^\tau_n [\frac{h^*(t, \tau, x'_n(\tau)) - h^*(t, \tau, x''_n(\tau))}{|\tau - t|^{1-\beta}}]z_n(\tau)]\| = \|\bar{P}_n [\frac{1}{\pi} \int_0^{2\pi} P^\sigma_n [(h^*(s, \sigma, x'_n(\sigma)) - h^*(s, \sigma, x''_n(\sigma))] z_n(\sigma)e^{is}p(s, \sigma)]d\sigma]\|.
\]

Here \( p(s, \sigma) = |e^{i\sigma} - e^{is}|^{-1} \) for \( |\sigma - s| \geq \pi/(2n + 1) \) and \( p(s, \sigma) = |e^{i\pi/(2n+1)} - 1|^{-1} \) for \( |\sigma - s| < \pi/(2n + 1) \), \( h^*(t, \tau, x(\tau)) = (h(t, \tau, x(\tau)) - h(\tau, \tau, x(\tau))(\cot\frac{\tau - t}{2})|\tau - t|^{1-\beta} \).

It is easy to see that

\[
I_2 \leq A\|P[[h'_u(\tau, \tau, x'_n(\tau)) - h'_u(\tau, \tau, x''_n(\tau))] z_n(\tau)]\| \leq A \max \|h'_u(t, t, x'_n(t)) - h'_u(t, t, x''_n(t))\|z_n\|.
\]
We proceed to the evaluation of the expression $I_3$. If integrand member in $I_3$ is a real function then $I_3 \leq I_4 I_5$, where

$$I_4 = \max \left\{ \frac{1}{\pi} \int_0^{2\pi} P_n^\sigma \left\{ h^*(s, \sigma, x'_n(\sigma)) - h^*(s, \sigma, x''_n(\sigma)) \right\} \left\{ p(s, \sigma) \right\}^{1/2} \right\}^2 \, d\tau \right\}^{1/2},$$

$$I_5 = \left\{ \frac{1}{2\pi} \int_0^{2\pi} ds \right\} \left\{ \int_0^{2\pi} P_n^\sigma \left\{ z_n(\sigma) e^{i\sigma} P_n^s \left\{ p(s, \sigma) \right\}^{1/2} \right\} \right\}^2 \, d\sigma \right\}^{1/2}.$$

One can see that

$$I_4 \leq A \max \left\{ \left\| h^*(s, s, x'_n(s)) - h^*(s, s, x''_n(s)) \right\| \right\}.$$

Similar,

$$I_5 = \left\{ \frac{2}{(2n + 1)^2} \sum_{i=0}^{2n} \sum_{k=0}^{2n} \left\| z_n(s_k) \left\| p(s_i, s_k) \right\| \right\}^{1/2} =

\left\{ \frac{2}{(2n + 1)^2} \sum_{i=0}^{2n} \sum_{k=0}^{2n} \left\| z_n(s_k) \left\| \sum_{i=0}^{2n} p(s_i, s_k) \right\| \right\}^{1/2} \leq A \left\| z_n \right\|.$$

From these calculations, it follows that $I_3 \leq A F \left\| z_n \right\|$. From estimates of $I_2$ and $I_3$ the inequality (6.18) is followed.

Now we can prove the existence of the linear operator $[K'(x^0_n)]^{-1}$ with the norm $\left\| [K'(x^0_n)]^{-1} \right\| \leq \left\| [K'(x^0_n)]^{-1} \right\|/(1 - q_1)$.

It follows, at such $n$ that $q_1 = A \ln n E_n(x_0) < 1$, from existence of the linear operator $[K'(x_0)]^{-1}$ and Banach Theorem.

In the section 3 was proved, at such $n$ that

$q_2 = A \ln^2 n \max \left[ E_n(h'_u(t, t, x^0_n(t))), E^t_{n, \tau}(h'_u(t, \tau, x^0_n(\tau))), E_n(\psi(t)) \right] < 1$, the existence of the linear operator $[K'_n(x^0_n)]^{-1}$ with norm $B_0$.

The validity of the theorem follows from the results of the item 2 of Introduction.

6.2. Projection Methods for Solving Nonlinear Singular Integral Equations on Open Contours of Integration
Let us consider nonlinear singular integral equation

\[ G(x) \equiv a(t, x(t)) + \frac{1}{\pi i} \int_{L} \frac{h(t, \tau, x(\tau))}{\tau - t} d\tau = f(t), \quad (6.19) \]

where \( L = (c_1, c_2) \) is a segment of the unit circle \( \gamma \), centered at the origin.

Approximate solution of the equation (6.19) we will search as polynomial

\[ x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k. \]

Coefficients \( \{\alpha_k\} \) are determined from the system

\[
\begin{align*}
G_n(x_n) & \equiv \bar{P}_n \left[ a^*(t, x_n(t)) + \frac{1}{\pi i} \int_{L} P_n^\tau \left[ \frac{h^*(t, \tau, x_n(\tau))}{\tau - t} \right] d\tau \right] = \\
& = \bar{P}_n[f^*(t)], \quad (6.20)
\end{align*}
\]

where

\[
\begin{align*}
a^*(t, x_n(t)) & = \begin{cases} 
 a(t, x(t)) & \text{if } t \in L, \\
 x(t) & \text{if } t \notin L;
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
h^*(t, \tau, x_n(\tau)) & = \begin{cases} 
 h(t, \tau, x_n(\tau)) & \text{if } t \in L, \\
 0 & \text{if } t \notin L;
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
f^*(t) & = \begin{cases} 
 f(t) & \text{if } t \in L, \\
 0 & \text{if } t \notin L.
\end{cases}
\end{align*}
\]

Justification of the computational scheme (6.20) is carried out along the same lines as the study of numerical schemes, cited in the preceding paragraph.

The difference is that, when communication between reversibility operators \( G \) and \( G_n \) is researched, we need to use results of the section §4 instead of the results of sections §2 and §3.
Omitting the intermediate calculations, we formulate statements which are similar to statements of the preceding paragraph.

**Theorem 6.5** [27], [35]. Let in the ball $S(x^*, \tau)$, where

$$r = B_2 B_0^2 / (1 - q), \quad B_2 = \|R(x^*)\| + O(n^{-\alpha} \ln n),$$

$q = O(F_1 B_2 B_0^2) + \sqrt{1 - 1/(B_2^4 B_0^4)} < 1,$

$$R(x^*)z \equiv a'_u(t, x^*(t))z(t) + \frac{1}{\pi i} \int_L h'_u(t, \tau, x^*(\tau))z(\tau)(\tau - t)^{-1}d\tau,$$

the following conditions are fulfilled:

1) $a'_u(t, u) \in H_{\alpha\alpha}, h'_u(t, \tau, u) \in H_{\alpha\alpha\alpha}, f(t) \in H_\alpha (0 < \alpha < 1);$  
2) $\max_{t, \tau \in L}\{|h'_u(t, \tau, u_1) - h'_u(t, \tau, u_2)|, |h^*(t, \tau, u_1) - h^*(t, \tau, u_2)|\} \leq F_1;$  
3) the Frechet derivative $R(x^*)z$ of the operator $G(x^*)$ is invertible in the space $L_2$ and $||[R(x^*)]^{-1}|| \leq B_0;$  
4) characteristic equation

$$a'_u(t, x^*(t))z(t) + \frac{h'_u(t, t, x^*(t))}{\pi i} \int_L \frac{z(\tau)d\tau}{\tau - t} = 0$$

has a solution $z(t) = (t - c_1)^{\delta_1}(t - c_2)^{\delta_2}\varphi(t)$, where $\delta_1 = \zeta_1 + i\xi_1, \delta_2 = \zeta_2 + i\xi_2, \varphi(t) \in H_\alpha$.

Then, for such $n$ that $O((n^{-\alpha} + n^{-\theta}) \ln n) < 1 (\theta = 1$ for $\zeta > 0, \theta = 1 - \zeta$ for $\xi \leq 0, \zeta = \min(\zeta_1, \zeta_2)$), the system (6.20) has a unique solution $x^*_n$ and the estimate $||x^* - x^*_n|| = O((n^{-\alpha} + n^{-\theta}) \ln n)$ is valid. Here $x^*$ is a solution of the equation (6.19).

**7. Spline-Collocation Method**

**7.1. Linear Singular Integral Equations**

Let us consider the singular integral equation

$$Kx \equiv a(t)x(t) + b(t) \frac{1}{\tau - t} \int_{-1}^1 \frac{x(\tau)d\tau}{\tau - t} + \int_{-1}^1 h(t, \tau)x(\tau)d\tau = f(t), \quad (7.1)$$
where \( a, b, f \in W^r(1), h(t, \tau) \in W^{r,r}(1), b(t) \neq 0, a^2(t) + b^2(t) \geq C > 0 \).

Methods are discussed below may be extended to singular integral equations on piecewise continuous loops. It works in exceptional cases too.

Introduce points \( t_k = -1 + 2k/n, \ k = 0, 1, \ldots, n, \) and \( t_{kj} = t_k + jh/(r + 1), \ j = 1, 2, \ldots, r, \ k = 0, 1, \ldots, n - 1, \ h = 2/n. \) Let \( \Delta_k = [t_k, t_{k+1}], \ k = 0, 1, \ldots, n - 1. \)

Let \( f(t) \in C[-1, 1]. \) Let us construct on each segment \( \Delta_k \) \((k = 0, 1, \ldots, n-1)\) interpolated polynomial \( L_r(f, \Delta_k). \) The polynomial \( L_r(f, \Delta_k) \) interpolate the function \( f(t) \) on the segment \( \Delta_k \) on knots \( t_{kj}, \ j = 1, 2, \ldots, r, \ k = 0, 1, \ldots, n - 1. \)

The polynomial \( L_r(f, \Delta_k) \) has the form

\[
L_r(f, \Delta_k) = \sum_{j=1}^{r} f(t_{kj}) \psi_{kj}(t),
\]

where \( \psi_{kj}(t) \) are fundamental polynomials on knots \( t_{kj}, \ j = 1, 2, \ldots, r, \ k = 0, 1, \ldots, n - 1. \)

Spline \( f_n(t), -1 \leq t \leq 1, \) consists of polynomials \( L_r(f, \Delta_k), \ k = 0, 1, \ldots, n - 1. \)

To each knot \( t_{kj} \) we put in correspondence the segment \( \Delta_{kj} = [t_{kj} - qh^*, t_{kj} + h^*], \) where \( h^*(0 < h^* < h/(r + 1)) \) and \( q \) are parameters, values of which will be determined below.

Approximate solution of the equation (7.1) we will seek in the form of the spline \( x_n(t), \) which is composed from polynomials \( L_r(x, \Delta_k). \) Values \( x_{kj} = x(t_{kj}), \ j = 1, 2, \ldots, r, \ k = 0, 1, \ldots, n - 1, \) of these polynomials are determined from the system

\[
a(t_{kl})x_{kl} + b(t_{kl}) \int_{\Delta_{kl}} \frac{x_n(\tau)}{\tau - t_{kl}} d\tau + \sum_{i=0}^{n-1} b'(t_{kl}) \int_{\Delta_i} \frac{x_n(\tau)}{\tau - t_{kl}} d\tau +
\]
\[ + \sum_{i=0}^{n-1} \int_{\Delta_i} h(t_{kl}, \tau) x_n(\tau) d\tau = f(t_{kl}), \quad (7.2) \]

\( k = 0, 1, \ldots, n - 1, \ l = 1, 2, \ldots, r, \) where the prime in the summation indicates that \( i \neq k - 1, k, k + 1. \)

Let us prove that the parameter \( h^* \) can be chosen so way, that the system (7.2) has a unique solution. This proof based on Hadamard theorem about invertibility of matrix.

Let us write the system (7.2) as the matrix equation

\[ CX = F, \quad (7.3) \]

where \( X = (x_1, \ldots, x_N), \ F = (f_1, \ldots, f_N), \ C = \{c_{ij}\}, \ i, j = 1, N, \ N = nr. \)

Here \( x_l = x_{ij}, \ f_l = f_{ij}, \) where \( l = ri + j, \ i = 0, 1, \ldots, n - 1, \ j = 1, 2, \ldots, r, \ x_{ij} = x(t_{ij}), \ f_{ij} = f(t_{ij}). \)

Let \( l = ri + j, \ i = 0, 1, \ldots, n - 1, \ j = 1, 2, \ldots, r, \ k = rv + w, \ v = 0, 1, \ldots, n - 1, \ w = 1, 2, \ldots, r. \) Then elements \( c_{kl} \) of matrix \( C \) have the form

\[ c_{ll} = a(t_{ij}) + b(t_{ij}) \int_{\Delta_i} \frac{\psi_{ij}(\tau)d\tau}{\tau - t_{ij}} + \int_{\Delta_i} \psi_{ij}(\tau)h(t_{ij}, \tau)d\tau, \ l = 1, 2, \ldots, N, \]

\[ c_{lk} = b(t_{ij}) \int_{\Delta_i} \frac{\psi_{vw}(\tau)d\tau}{\tau - t_{ij}} + \int_{\Delta_i} \psi_{vw}(\tau)h(t_{ij}, \tau)d\tau, \]

if \( t_{vw} \in \Delta_i \) and

\[ c_{lk} = \int_{\Delta_v} \psi_{vw}(\tau)h(t_{ij}, \tau)d\tau, \]

if \( t_{vw} \in \Delta_i. \)
Let us estimate from below the diagonal coefficients $c_{ll}$.
If $b(t_{ij}) \neq 0$, then

$$|c_{ll}| \geq |b(t_{ij})| \left| \int_{\Delta_{ij}} \frac{\psi_{ij}(\tau)d\tau}{\tau - t_{ij}} \right| - |a(t_{ij})| - \left| \int_{\Delta_i} \psi_{ij}(\tau)h(t_{ij}, \tau)d\tau \right|.$$  \hspace{1cm} (7.4)

Easy to see that $|a(t_{ij})| \leq A$, $\left| \int_{\Delta_i} \psi_{ij}(\tau)h(t_{ij}, \tau)d\tau \right| = O(n^{-1})$,
where $A$ is a constant, which is not depend from indexes $i, j, i = 0, 1, \ldots, n - 1, j = 1, 2, \ldots, r$.

Well known that $\psi_{ij}(t_{ij}) = 1$. Let us show, that, for big enough number $M$, exist parameters $h^*$ and $q$, under which the inequalities

$$\int_{\Delta_{ij}} \frac{\psi_{ij}(\tau)d\tau}{\tau - t_{ij}} \geq M, \quad i = 0, 1, \ldots, n - 1, \quad j = 1, 2, \ldots, r, \quad (7.5)$$

is valid.
Indeed

$$\left| \int_{\Delta_{ij}} \frac{\psi_{ij}(\tau)d\tau}{\tau - t_{ij}} \right| \geq \left| \int_{\Delta_{ij}} \frac{d\tau}{\tau - t_{ij}} \right| -$$
$$- \left| \int_{\Delta_{ij}} \frac{\psi_{ij}(\tau) - 1}{\tau - t_{ij}}d\tau \right| \geq |\ln q| - \left| \int_{\Delta_{ij}} \frac{\psi_{ij}(\tau) - 1}{\tau - t_{ij}}d\tau \right|. \quad (7.6)$$

We can chose the parameter $q$ so way that $|\ln q| \geq M + 1 + a$.
Also we can chose the parameter $h^*$ so way that

$$\left| \int_{\Delta_{ij}} \frac{\psi_{ij}(\tau) - 1}{\tau - t_{ij}}d\tau \right| \leq \varepsilon, \quad (7.7)$$

where $\varepsilon (\varepsilon > 0)$ is a arbitrary shall number.
So, we can chose parameters $q$ and $h^*$ so way, that $|c_{il}| \geq M, \ l = 1, 2, \ldots, N$, where $M$ is an arbitrary big number.

Let us estimate coefficients $|c_{kl}|$ for $k \neq l, l, k = 1, 2, \ldots, N$.

The function $\psi_{vw}(t_{ij}) = 0$ for $v \neq i$ and $w \neq j$. Using this fact one can see that

$$\left| \int_{\Delta_{ij}} \frac{\psi_{vw}(\tau)d\tau}{\tau - t_{ij}} \right| = O\left(\frac{1}{n}\right) \quad (7.8)$$

and

$$\left| \int_{\Delta_i} \psi_{vw}(\tau)h(t_{ij}, \tau)d\tau \right| = o\left(\frac{1}{n}\right). \quad (7.9)$$

Collecting inequalities (7.4) – (7.9) we see, that exist such parameters $h^*$ and $q$, that conditions of Hadamard theorem on invertibility of matrices are valid. From Hadamard theorem it follows, that the system (7.2) has a unique solution. Obviously, the system (7.3) has a unique solution too.

Easy to see that, in the space $R_N$ of vectors $X_N = (x_1, \ldots, x_N)$ with the norm $\|X_N\| = \max_{1 \leq k \leq N} |x_k|$, the norm of the matrix $C^{-1}$ is evaluated as $\|C^{-1}\| \leq A/M$.

Let $P_N$ is the projector from space $C[-1, 1]$ onto set of vector-functions: $P_N f = f(t_{ij}), i = 0, 1, \ldots, n - 1, j = 1, 2, \ldots, r$. This is defined as $P_N[f] = f_N(t)$.

In the operator form the system (7.2) can be written as

$$K_N x_N \equiv$$

$$\equiv P_N \left[ a(t)x_n t) + b(t) \int_{-1}^{1} e(t, \tau) \frac{x_n(\tau)}{\tau - t} d\tau + \int_{-1}^{0} h(t, \tau)x(\tau)d\tau \right] = P_N[f(t)].$$
The function $e(t, \tau)$ is defined by formula
\[
e(t_{kl}, \tau) = \begin{cases} 
1 & \text{if } \tau \in \Delta_{kl} \text{ or } \tau \in [-1, 1] \setminus (\Delta_{k-1} \cup \Delta_k \cup \Delta_{k+1}), \\
0 & \text{if } \tau \in (\Delta_{k-1} \cup \Delta_k \cup \Delta_{k+1}) \setminus \Delta_{kl}.
\end{cases}
\]

The estimate $\| K_N^{-1} \| \leq A/M$ is valid.

Let us estimate the error of the offered method.

Let $x^*(t)$ be a solution of the equation (7.1). Equating left and right sides of the equation (7.1) in points $t_{kl}, k = 0, 1, \ldots, n - 1, l = 1, 2, \ldots, r$, we have
\[
a(t_{kl})x^*_{kl} + \sum_{i=0}^{n-1} b(t_{kl}) \int_{\Delta_i} \frac{x^*(\tau)}{\tau - t_{kl}} d\tau + \\
+ \sum_{i=0}^{n-1} \int_{\Delta_i} h(t_{kl}, \tau) x^*(\tau) d\tau = f(t_{kl}),
\]
(7.10)
k = 0, 1, \ldots, n - 1, l = 1, 2, \ldots, r, where $x^*_{kl} = x^*(t_{kl}).$

Subtracting from the system (7.2) the system (7.10) we have
\[
a(t_{kl})(x_{kl} - \tilde{x}^*_{kl}) + b(t_{kl}) \int_{\Delta_k} \frac{x_n(\tau) - \tilde{x}^*_n(\tau)}{\tau - t_{kl}} d\tau + \\
+ \sum_{i=0}^{n-1} b(t_{kl}) \int_{\Delta_i} \frac{x^*(\tau) - \tilde{x}^*_n(\tau)}{\tau - t_{kl}} d\tau + \sum_{i=0}^{n-1} \int_{\Delta_i} h(t_{kl}, \tau)(x_n(\tau) - \tilde{x}^*_n(\tau)) d\tau = \\
= b(t_{kl}) \int_{\Delta_k} \frac{x^*(\tau) - \tilde{x}^*_n(\tau)}{\tau - t_{kl}} d\tau + b(t_{kl}) \int_{\Delta_k} \frac{x^*(\tau)}{\tau - t_{kl}} d\tau + \\
+ \sum_{i=0}^{n-1} b(t_{kl}) \int_{\Delta_i} \frac{x^*(\tau) - \tilde{x}^*_n(\tau)}{\tau - t_{kl}} d\tau + \sum_{i=0}^{n-1} \int_{\Delta_i} h(t_{kl}, \tau)(x^*(\tau) - \tilde{x}^*_n(\tau)) d\tau,
\]
(7.11)
k = 0, 1, \ldots, n - 1, l = 1, 2, \ldots, r, where $\Delta_k^* = \Delta_{k-1} \cup (\Delta_k \setminus \Delta_{kl}) \cup \Delta_{k+1}$, $x^*$ and $x^*_n$ are solutions of equations (7.1) and (7.2).
Here $\tilde{x}^*_n(\tau)$ is the local spline, which approximate the solution $x^*(\tau)$ on the set of knots $t_{k,l}$, $k = 0, \ldots, n - 1, l = 1, 2, \ldots, r$.

Let us estimate the right side of the expression (7.11).

Easy to see that

$$r_1 = \left| b(t_{kl}) \int_{\Delta_{kl}} \frac{x^*(\tau) - \tilde{x}^*_n(\tau)}{\tau - t_{kl}} d\tau \right| \leq \left| b(t_{kl}) \int_{\Delta_{kl}} \frac{x^*(\tau) - \tilde{x}^*_n(\tau) - (x^*(t_{kl}) - \tilde{x}^*_n(t_{kl}))}{\tau - t_{kl}} d\tau \right| \leq An^{-r};$$

$$r_3 = \sum_{i=0}^{n-1} 'b(t_{kl}) \int_{\Delta_i} \frac{x^*(\tau) - \tilde{x}^*_n(\tau)}{\tau - t_{kl}} d\tau \leq An^{-r} \ln n,$$

$$r_4 = \sum_{i=0}^{n-1} \int_{\Delta_i} h(t_{kl}, \tau)(x^*(\tau) - \tilde{x}^*_n(\tau)) d\tau \leq An^{-r}.$$

For evaluating the integral

$$r_2 = \left| b(t_{kl}) \int_{\Delta^*_{kl}} \frac{x^*(\tau)}{\tau - t_{kl}} d\tau \right|$$

we introduce a function $\psi(\tau)$, which satisfy the following conditions:

1) $\int_{\Delta^*_{kl}} \frac{\psi(\tau)}{\tau - t_{kl}} d\tau = 0$;

2) in the domain $\Delta^*_{kl}$ the function $\psi(t)$ realize the best approximation for the function $x^*(t)$ in the metric of the space $C$. 

94
Introduce the following designations:

\[ E_{kl}^*(x^*) = \left| \int_{\Delta_k^*} \frac{x^*(\tau) - \psi(\tau)}{\tau - t_{kl}} d\tau \right|, \]

\[ E^*(x^*) = \max_{k,l} E_{kl}^*(x^*). \]

Here \( \psi(t) \) satisfy to conditions 1) and 2).

Obviously, \( r_2 \leq B E^*(x^*) \), where \( B = \|b(t)\|_C \).

Now it is easy to see that

\[ \max |x^*(t_{kl}) - x_n(t_{kl})| \leq A(n^{-r} \ln n + E^*(x^*)). \]

**Theorem 7.1** [35, 38, 39]. Let \( a, b, f \in W^r, \ h \in W^{rr}, \ |b(t)| \geq C > 0 \) on the segment \([-1, 1]\). Let the equation (7.1) has a unique solution \( x^*(t) \). Then exist such parameters \( h^* \) and \( q \), that the system of equations (7.2) has a unique solution \( x^*_n \) and estimate \( \|x^*(t_{kl}) - x^*_n(t_{kl})\|_C \leq A(n^{-r} \ln n + E^*(x^*)) \) is valid.

### 7.2. Nonlinear Equations

Let us consider the nonlinear singular integral equation

\[ Kx \equiv a(s)x(s) + S(b(s, \sigma, x(\sigma))) + T(h(s, \sigma, x(\sigma))) \equiv \]

\[ \equiv a(s)x(s) + \frac{1}{2\pi} \int_0^{2\pi} b(s, \sigma, x(\sigma)) \cot \frac{\sigma - s}{2} d\sigma + \int_0^{2\pi} h(s, \sigma, x(\sigma)) d\sigma = \]

\[ = f(s). \quad (7.12) \]

Let us assume that \( a(s), f(s) \in H_\alpha, \ b'_3(s, \sigma, u) \in H_{\alpha, \alpha, \alpha}, \) \( 0 < \alpha \leq 1, \) where \( b'_3(s, \sigma, u) \equiv \partial b(s, \sigma, u)/\partial u. \)

Easy to see, that Frechet derivative of the nonlinear operator \( K(x) \) in the space \( X = H_\beta \) \( (0 < \beta < \alpha) \) has the form

\[ K'(x_0)z \equiv a(s)z(s) + \frac{1}{2\pi} \int_0^{2\pi} b'_3(s, s, x_0(s)) z(\sigma) \cot \frac{\sigma - s}{2} d\sigma + \]

\[ + \int_0^{2\pi} h(s, \sigma, x_0(\sigma)) z(\sigma) d\sigma. \]
\[
+ \frac{1}{2\pi} \int_0^{2\pi} h_3'(s, \sigma, x_0(\sigma))z(\sigma)d\sigma.
\]

We assume, that the function \(a^2(s) - (b_3'(s, s, x_0(s)))^2\) can be equal to zero on sets of points with measure not bigger than zero. For construction numerical scheme for solution of the equation (7.12), we choose the sets of knots

\[
s_k = \frac{\pi k}{n}, \quad s_k^* = s_k + h, \quad 0 < h \leq \frac{\pi}{2n}, \quad k = 0, \ldots, 2n.
\]

The value of the parameter \(h\) will be defined later.

Approximate solution of the equation (7.12) is defined from the following system of nonlinear equations

\[
a(s_j)x(s_j^*) + \frac{1}{2\pi} \sum_{k=0, k \neq j-1, j+1}^{2n-1} b(s_j^*, s_k^*, x(s_k^*)) \int_{s_k}^{s_{k+1}} \cot \frac{\sigma - s_j^*}{2}d\sigma +
\]

\[
+ \frac{\pi}{n} \sum_{k=0}^{2n-1} h(s_j^*, s_k^*, x(s_k^*)) = f(s_j^*), \quad j = 0, \ldots, 2n - 1. \quad (7.13)
\]

For solution of the system (7.13) we use the Newton - Kantorovich method.

After found solution \(x(s_j^*), j = 0, 1, \ldots, 2n - 1,\) of nonlinear system of equations (7.13), we restore the approximate solution of the equation (7.12) in the form of trigonometric polynomial

\[
x_n(s) = \sum_{k=0}^{2n-1} x(s_k^*)\psi_k(s), \quad (7.14)
\]

where

\[
\psi_k(s) = \frac{1}{2n + 1} \frac{\sin \frac{2n+1}{2}(s - s_k^*)}{\sin \frac{s - s_k^*}{2}}.
\]

One can restore the approximate solution of the equation (7.13) in the form of polygons, constructed on knots \(s_k^*, x(s_k^*), k = 0, \ldots, 2n - 1.\)
Let $P_n$ is the projector from the space $X$ onto the space $X_n \subset X$. The space $X_n$ consist of trigonometric polynomial of $n$ order. The projector $P_n$ is introduced by the formula

$$P_n x = \sum_{k=0}^{2n-1} x(s^*_k)\psi_k(s).$$

In the space $X_n$ the system (7.13) can be written as operator equation $K_n x_n = f_n$.

The Frechet derivative $K'_n(x_0)z_n, z_n \in X_n$, of the operator $K_n$ on a element $x_0$ can be written as the vector

$$a(s^*_j)z_n(s^*_j) +$$

$$+ \frac{1}{2\pi} \sum_{k=0, k \neq j-1, j+1}^{2n-1} b'_3(s^*_j, s^*_k, x_0(s^*_k))z_n(s^*_k) \left( \int_{s_k}^{s_{k+1}} \cot \frac{\sigma - s^*_j}{2} d\sigma + \frac{\pi}{n} \sum_{k=0}^{2n-1} h'_3(s^*_j, s^*_k, x_0(s^*_k))z_n(s^*_k), j = 0, \ldots, 2n - 1.\right)$$

We will show that the parameter $h$ can be chosen so way, that the Frechet derivative $K'_n(x_0)$ is invertible. Under this condition the approximate solution of the equation (7.12) we will seek by Newton-Kantorovich modified method

$$\tilde{x}_{m+1} = \tilde{x}_m - [K'_n(x_0)]^{-1}K_n(\tilde{x}_m), m = 0, 1, 2, \ldots \quad (7.15)$$

Here $K'_n(x_0)$ is the Frechet derivative of the operator $K_n$ in the space $X_n$ on a initial element $\tilde{x}_0 = P_n(x_0)$.

Let us transform the iterative method (7.15) to the following form

$$K'_n(x_0)\tilde{x}_{m+1} = K'_n(x_0)\tilde{x}_m - K_n(\tilde{x}_m). \quad (7.16)$$

On the each step of the iterative method (7.16) we must decide the following system of linear algebraic equations

$$Lx = g_n, \quad (7.17)$$
where \( g_n = K_n'(x_0)\tilde{x}_m - K_n\tilde{x}_m, \ L = K_n'(x_0). \)

Let us prove that the system (7.17) has a unique solution on each step of the iterative process (7.15). For this aim we must prove that the operator \( K_n'(x_0) \) is invertible. Let be \( a_j = a(s_j^*), \ b_{jk} = b'_3(s_j^*, s_k^*, x_0(s_k^*)), \ h_{jk} = h'_3(s_j^*, s_k^*, x_0(s_k^*)), \ j, k = 0, \ldots, 2n - 1. \) Assume that derivatives \( b'_3(s_j^*, s_k^*, x_0(s_k^*)), \ j, k = 0, \ldots, 2n - 1, \) are not equal to zero on elements \( x_0(s) \) in neighborhood of the solution \( x^*(s) \) of the equation (7.12). Diagonal elements of the matrix \( L \) of the system (7.17) have the form

\[
|l_{jj}| = \left| a_j + \frac{b_{jj}}{\pi} \ln \frac{\sin \left( \frac{s_{j+1} - s_j^*}{2} \right)}{\sin \frac{s_j - s_j^*}{2}} \right| + \frac{\pi h_{jj}}{n} = \\
= \left| a_j + \frac{b_{jj}}{\pi} \ln \left( \frac{\sin \left( \frac{\pi}{2n} - \frac{h}{2} \right)}{\sin \frac{h}{2}} \right) \right| + \frac{\pi h_{jj}}{n},
\]

\( j = 0, 1, \ldots, 2n - 1. \)

From conditions \( b_{jj} \neq 0, \ j = 0, \ldots, 2n - 1, \) follow that for sufficient small \( h \)|\( l_{jj} | > A + B \ln n. \) Let us estimate the sum

\[
\sum_{k=0, k \neq j}^{2n-1} |l_{jk}| \leq \frac{1}{2\pi} \sum_{k=0, k \neq j, j-1, j+1}^{2n-1} |b_{jk}| \int_{s_k}^{s_{k+1}} \frac{\sigma - s_j^*}{\sin \frac{h}{2}} d\sigma + \\
+ \frac{\pi}{n} \sum_{k=0, k \neq j}^{2n-1} h_{jk} \leq A + B \ln n.
\]

Using the Hadamard theorem about invertibility of matrix, we see that the system (7.17) has a unique solution.

Let us prove the convergence of the iterative process (7.15) to the exact solution of the equation (7.12).

The proof of the convergence we will give in the space \( R_{2n} \) of vectors \( v = (v_1, \ldots, v_{2n}) \) with the norm \( \|v\| = \max_{1 \leq i \leq 2n} |v_i|. \)
One can write the matrix of the operator $K'_n(x_0)$ in the form

$$K'_n(x_0) = D + E,$$

with elements

$$d_{jk} = \begin{cases} 
0, & j \neq k; \\
\frac{a_j}{2\pi} + \frac{b_{jj}}{2\pi} \int_{s_j}^{s_{j+1}} \cot \frac{\sigma - s^*_j}{2} d\sigma + \frac{\pi h_{jj}}{n}, & j = k; 
\end{cases}$$

and

$$e_{jk} = \begin{cases} 
0, & j = k; \\
\frac{\pi h_{jk}}{n}, & j = k - 1, k + 1; \\
\frac{b_{jk}}{2\pi} \int_{s_k}^{s_{k+1}} \cot \frac{\sigma - s^*_j}{2} d\sigma + \frac{\pi h_{jk}}{n}, & \text{in other cases.}
\end{cases}$$

Easy to see that

$$\|D^{-1}\| \leq \left( C \ln \left| \sin \left( \frac{\pi}{2n} - \frac{h}{2} \right) \right| + G \right)^{-1}, \quad \|E\| \leq A + B \ln n,$$

$$\|D^{-1}\| \|E\| \leq (A + B \ln n) \left| G + C \ln \left| \sin \left( \frac{\pi}{2n} - \frac{h}{2} \right) \right| \right|^{-1} \leq q < 1,$$

where $C$ and $G$ are constants.

From the Banach theorem follows that

$$\|[[K'_n(x_0)]^{-1}]\| \leq \left( G + C \ln \left| \sin \left( \frac{\pi}{2n} - \frac{h}{2} \right) \right| - A - B \ln n \right)^{-1}.$$
Assume that in some ball $S(x_0, r)$ the inequality
\[
||[K'_n(x_0)]^{-1}||||K'_n(u_1) - K'_n(u_2)|| \leq q < 1, u_1, u_2 \in S(x_0, r)
\] (7.18)
is valid.

Using statements for convergence of Newton-Kantorovich method, which are given in the section 2 of the Introduction, we prove that the equation (7.13) has a unique solution $x^*_n(s)$ in ball $S(x_0, r)$ and iterative process (7.15) converges to this solution.

Combining proof of convergence of Newton-Kantorovich method, given in the section 2 of the Introduction, and statements of previous items, we see that
\[
||x^* - x^*_n||_C \leq An^{-\alpha} \ln n, ||x^* - x^*_n||_X \leq An^{-\alpha+\beta} \ln n.
\]

**Theorem 7.2** [35]. Let the equation (7.12) has a unique solution $x^*(s) \in H_\alpha$. Let $a(s), f(s) \in H_\alpha$, $b'_3(s, \sigma, u), h'_3(s, \sigma, u) \in H_{\alpha,\alpha,\alpha}$, $0 < \alpha \leq 1$. Then exist values of the parameter $h$, for which conditions (7.18) is valid and the system of equations (7.17) has a unique solution on each step of the iterative process (7.15). Under these values of the parameter $h$ and under some additional conditions estimates
\[
||x^*(s) - x^*_n(s)||_C \leq An^{-\alpha} \ln n, ||x^*(s) - x^*_n(s)||_X \leq An^{-\alpha+\beta} \ln n
\] is valid. Here $x^*_n(s)$ is a solution of the system (7.13).

**8. Singular Integral Equations in Exceptional Cases**

Numerical methods for solution of singular integral equations in exceptional cases are investigated since 70 years of the past century. As a rule, they was investigated in cases, when conditions of normality of singular operators was broken at separate points. The results in this direction was printed in the books [105], [114], [117].
There are many physical and technical problems, where conditions of normality are violated on whole segments or on all domains of definition of singular operators. M.M. Lavrentyev formulated [99] some problems for investigation of the existence and stability of solution of singular integral equations of the kinds

\[
x(t) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{x(\tau)d\tau}{\tau - t} = f(t),
\]

\[
\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{x(\tau_1, t_2)d\tau_1}{\tau_1 - t_1} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{x(t_1, \tau_2)d\tau_2}{\tau_2 - t_2} = f(t_1, t_2),
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_1, \tau_2)\left(\frac{1}{\tau_1 - t_1} - \frac{1}{\tau_2 - t_2}\right)d\tau_1d\tau_2 = f(t_1, t_2).
\]

These problems was decided in [22].

We need in numerical methods for solution of the singular integral equations of the types (8.1) - (8.3). In this section we describe some of these methods.

8.1. Equations on Closed Circles

Let us consider singular integral equation

\[
a(t)x(t) + \frac{b(t)}{\pi} \int_{\gamma} \frac{x(\tau)d\tau}{\tau - t} + \int_{\gamma} h(t, \tau)x(\tau)d\tau = f(t),
\]

where \(\gamma = \{\gamma : |z| = 1\}\), \(b(t) \neq 0\), \(a, b, f \in H_\alpha\), \(h(t, \tau) \in H_{\alpha\alpha}\), \(0 < \alpha \leq 1\).

Assume that the function \(a^2(t) - b^2(t)\) can be equal to zero on arbitrary sets, which are belong to \(\gamma\), or \(a^2(t) - b^2(t) \equiv 0\) on whole circle \(\gamma\).

Using Hilbert transform we pass to the equation

\[
a(e^{is})x(e^{is}) - \frac{ib(e^{is})}{2\pi} \int_{0}^{2\pi} x(e^{i\sigma})\text{ctg}\frac{\sigma - s}{2}d\sigma +
\]
\[ + i \int_0^{2\pi} h(e^{is}, e^{i\sigma})x(e^{i\sigma})e^{i\sigma} d\sigma + \frac{b(e^{is})}{2\pi} \int_0^{2\pi} x(e^{i\sigma})d\sigma = f(e^{is}), \quad (8.5) \]

where \(0 \leq s < 2\pi\).

For simplicity, instead of the equation (8.5), we will consider the equation

\[ a(s)x(s) + \frac{b(s)}{2\pi} \int_0^{2\pi} x(\sigma)\cotg \frac{\sigma - s}{2} d\sigma + \frac{1}{2\pi} \int_0^{2\pi} h(s, \sigma)x(\sigma)d\sigma = f(s). \quad (8.6) \]

Let us introduce knots

\[ s_k = \frac{\pi k}{n}, \quad s_k^* = \frac{\pi k}{n} + h, \quad 0 < h \leq \frac{\pi}{2n}, \quad k = 0, \ldots, 2n, \]

where parameter \(h\) will be defined later.

Solution of the equation (8.6) we will find in the form

\[ x_n(s) = \sum_{k=0}^{2n-1} a_k \psi_k(s), \]

where

\[ \psi_k(s) = \frac{1}{2n+1} (\sin \frac{2n+1}{2}(s - s_k^*))/(\sin \frac{s - s_k^*}{2}). \]

Unknown coefficients \(a_k\) we will find from the following system of linear algebraic equations

\[ a(s_j^*)x(s_j^*) + \frac{b(s_j^*)}{2\pi} \sum_{k=0, k \neq j-1, j+1}^{2n-1} x(s_k^*) \int_{s_k}^{s_{k+1}} \cotg \frac{\sigma - s_j^*}{2} d\sigma + \]

\[ + \frac{\pi}{n} \sum_{k=0}^{2n-1} h(s_j^*, s_k^*)x(s_k^*) = f(s_j^*), \quad j = 0, \ldots, 2n - 1. \quad (8.7) \]

Under some conditions for \(h\), we will prove invertibility of the system (8.7), using the well-known Hadamard Theorem.
The system (8.7) can be rewritten as the algebraic equations system

\[ CX = F, \]

where \( C = \{c_{kl}\}, \ k, l = 1, 2, \ldots, 2n + 1, \ X = (x_1, \ldots, x_{2n+1}), \ F = (f_1, \ldots, f_{2n+1}). \)

Let be \( a_j = a(s_j^*) \), \( b_j = b(s_j^*) \), \( h_{jk} = h(s_j^*, s_k^*) \). It is easy to see that

\[ |c_{jj}| = |a_j + \frac{b_j}{\pi} \ln \left| \frac{\sin \left( \frac{\pi}{2n} - \frac{h}{2} \right)}{\sin \frac{h}{2}} \right| \]

and

\[
\sum_{k=0, k \neq j}^{2n-1} |c_{jk}| \leq \frac{\pi}{n} \sum_{k=0, k \neq j}^{2n-1} |h_{jk}| + \frac{|b_j|}{2\pi} \sum_{k=0, k \neq j, j+1}^{2n-1} \left| \int_{s_k}^{s_{k+1}} \frac{\sigma - s_j^*}{2} d\sigma \right| \leq A + B \ln n.
\]

From two last inequality follows, that parameter \( h \) can be chosen so way, that \( |c_{jj}| > \sum_{k=0, k \neq j}^{2n-1} |c_{jk}| \). So from Hadamard Theorem follows, that the system (8.7) has a unique solution.

For each knot \( s_j^* \), \( j = 0, 1, \ldots, 2n - 1 \), exists a function \( \psi(s) \) for which

\[ \psi(s_j^*) = x(s_j^*) \]

and

\[
\int_{s_{j-1}}^{s_j} \psi(\sigma) \cot \frac{\sigma - s_j^*}{2} d\sigma + \int_{s_{j+1}}^{s_{j+2}} \psi(\sigma) \cot \frac{\sigma - s_j^*}{2} d\sigma = 0. \tag{8.8}
\]

As a example we can to conside the straight line

\[ \psi(\sigma) = x(s_j^*) + k_j(\sigma - s_j^*), \tag{8.9} \]

\[ k_j = -\frac{nx(s_j^*)}{2\pi} \ln \left( \frac{\sin \frac{h}{2}}{\sin \left( \frac{\pi}{n} - \frac{h}{2} \right) \sin \left( \frac{\pi}{n} + \frac{h}{2} \right)} \right). \]
Assume that the equation (8.6) with right side \( f(s) \) has a unique solution \( x^* \in H_\alpha \). We proved that the system (8.7), under corresponding parameter \( h \), has a unique solution \( x^*_n(t) \). Let \( P_n \) be the projector from space \( X = H_\beta \) \((0 < \beta < \alpha)\) onto interpolated polynomials on knots \( s^*_k, k = 0, 1 \ldots, 2n - 1 \). We can write the system (8.7) in operator form as \( K_n x_n = f_n \). Then

\[
x^*_n - P_n x^* = K_n^{-1}(K_n(x^*_n - P_n x^*)) =
\]

\[
= K_n^{-1}(P_n f - K_n P_n x^*) = K_n^{-1}(P_n K x^* - K_n P_n x^*) =
\]

\[
= K_n^{-1}(P_n K x^* - P_n K_n x^*) + K_n^{-1}(P_n K_n x^* - P_n K_n P_n x^*).
\]

Let us estimate in the metric \( C(\gamma) \) the difference \( P_n K x^* - P_n K_n x^* \).

One can see that

\[
\left| \sum_{l=0, l \neq j-1, j+1}^{2n-1} \int_{s_l}^{s_{l+1}} [(x^*(\sigma) - (s^*_l)) \ctg \frac{\sigma - s^*_j}{2} d\sigma \right| +
\]

\[
\left| \int_{s_{j-1}}^{s_j} (x^*(\sigma) - \psi(\sigma)) \ctg \frac{\sigma - s^*_j}{2} d\sigma \right| +
\]

\[
\left| \int_{s_{j+1}}^{s_{j+2}} (x^*(\sigma) - \psi(\sigma)) \ctg \frac{\sigma - s^*_j}{2} d\sigma \right| \leq \left| \sum_{k=0, k \neq j, j-1, j+1}^{2n-1} \int_{s_k}^{s_{k+1}} (x^*(\sigma) - x^*(s^*_j)) \ctg \frac{\sigma - s^*_j}{2} d\sigma \right| +
\]
\[
\begin{align*}
&+ \int_{s_{j-1}}^{s_j} (x^*(\sigma) - x^*(s_j^*)) \cotg \frac{\sigma - s_j^*}{2} d\sigma + \\
&+ \int_{s_{j-1}}^{s_j} (\psi_j(s_j^*) - \psi_j(\sigma)) \cotg \frac{\sigma - s_j^*}{2} d\sigma + \\
&+ \int_{s_{j+1}}^{s_{j+2}} (x^*(\sigma) - x(s_j^*)) \cotg \frac{\sigma - s_j^*}{2} d\sigma + \\
&+ \int_{s_{j+1}}^{s_{j+2}} (\psi(\sigma) - \psi(s_j^*)) \cotg \frac{\sigma - s_j^*}{2} d\sigma = I_1 \div I_6.
\end{align*}
\]

One can see that: \(I_1 \leq An^{-\alpha}; I_2 \leq An^{-\alpha} \ln n; I_3 + I_4 \leq An^{-\alpha}; I_5 + I_6 \leq Akn^{-1},\) where \(k = \max_{1 \leq j \leq 2n} |k_j|\).

If coefficients \(k_j\) in the equations (8.9) satisfy the inequality \(k = \max_{1 \leq j \leq 2n} |k_j| \leq n^{1-\alpha}\), then \(R_n \leq An^{-\alpha} \ln n\).

In the metric of space \(R_{2n+1}\) we have

\[
\|x^* - P_n x^*\| \leq A \frac{\ln^3 n}{n^\alpha}
\]

So,

\[
\|x^* - x_n^*\|_C \leq \|x^* - P_n x^*\|_C + \|P_n x^* - x_n^*\|_C \leq A \frac{\ln^3 n}{n^\alpha}.
\]

**Theorem 8.1** [35], [38]. Let the equation (8.6) has a unique solution \(x^*(t) \in H_\alpha, 0 < \alpha \leq 1.\) Then, for sufficient small value \(h,\) the system (8.7) has a unique solution \(x_n^*(t)\) and for \(h\) so, that the coefficient \(k\) of straight line (8.9) is smaller than \(n^{1-\alpha},\) the estimate \(\|x^* - x_n^*\|_C \leq A \frac{\ln^3 n}{n^\alpha}\) is valid.
8.2. Equations on Segments

Let us consider a singular integral equation

\[ a(t)x(t) + \frac{b(t)}{\pi} \int_{-1}^{1} \frac{x(\tau)}{\tau - t} d\tau + \int_{-1}^{1} h(t, \tau) d\tau = f(t), \quad -1 < t < 1, \]  

where \( f(t), a(t), b(t) \in H_\alpha, h(t, \tau) \in H_{\alpha, \alpha}, 0 < \alpha \leq 1. \)

We assume that \( b(t) \neq 0, \) but \( a^2(t) - b^2(t) \) can be equal to zero on sets with measure bigger than zero.

We will use knots \( t_k = -1 + \frac{k}{n}, \) \( k = 0, \ldots, 2n, \) \( t_j^* = t_j + h, \) \( j = 1, 2, \ldots, n-1, \) \( t_j^* = t_{j+1} - h, \) \( j = n, n+1, \ldots, 2n-2, \) \( 0 < h < \frac{1}{2n}. \)

The parameter \( h \) will be chosen later.

The integral

\[ I_2x = \int_{-1}^{1} \frac{x(\tau)}{\tau - t} d\tau \]  

we approximate by the following quadrature rule.

This quadrature rule for \( t \in [t_j, t_{j+1}), j = 1, 2, \ldots, 2n - 2, \) is

\[
I_2x = R_N + \left\{ \begin{array}{c}
\int_{t_0}^{t_2} \frac{x(t_j^*)}{\tau - t} d\tau + \sum_{k=2, k \neq j-1, j+1}^{2n-3} \int_{t_k}^{t_{k+1}} \frac{x(t_j^*)}{\tau - t} d\tau + \\
+ \int_{t_2}^{t_{2n-2}} \frac{x(t_{2n-1})}{\tau - t} d\tau, \quad j \neq 1, 2n - 2; \\
\int_{t_1}^{t_2} \frac{x(t_j^*)}{\tau - t} d\tau + \sum_{k=3}^{2n-3} \int_{t_k}^{t_{k+1}} \frac{x(t_j^*)}{\tau - t} d\tau + \\
+ \int_{t_1}^{t_{2n-2}} \frac{x(t_{2n-1})}{\tau - t} d\tau, \quad j = 1; \\
\int_{t_0}^{t_2} \frac{x(t_j^*)}{\tau - t} d\tau + \sum_{k=2}^{2n-4} \int_{t_k}^{t_{k+1}} \frac{x(t_j^*)}{\tau - t} d\tau + \\
+ \int_{t_0}^{t_{2n-2}} \frac{x(t_{2n-1})}{\tau - t} d\tau, \quad j = 2n - 2; \\
\end{array} \right. \]  

(8.12)
Using this quadrature rule we receive the system of algebraic equations for approximate solution of the singular integral equation (8.10):

\[ a(t^*_j)x(t^*_j) + \frac{b(t^*_j)}{\pi} \left( \int_{t_0}^{t_2} \frac{x(t^*_1)}{\tau - t^*_j} d\tau + \sum_{k=2}^{2n-2} \int_{t_k}^{t_{k+1}} \frac{x(t^*_k)}{\tau - t^*_j} d\tau + \sum_{k=2, k \neq j-1, j+1}^{2n-2} \int_{t_k}^{t_{k+1}} \frac{x(t^*_k)}{\tau - t^*_j} d\tau \right) + \frac{1}{t_{2n-1}} \int_{t_{2n-1}}^{1} \frac{x(t^*_{2n-1})}{\tau - t^*_j} d\tau \right) = f(t^*_j), \quad j \neq 1, 2n - 2; \]

\[ a(t^*_j)x(t^*_j) + \frac{b(t^*_j)}{\pi} \left( \int_{t_1}^{t_2} \frac{x(t^*_1)}{\tau - t^*_j} d\tau + \sum_{k=2}^{2n-3} \int_{t_k}^{t_{k+1}} \frac{x(t^*_k)}{\tau - t^*_j} d\tau + \sum_{k=2}^{2n-3} \int_{t_k}^{t_{k+1}} \frac{x(t^*_k)}{\tau - t^*_j} d\tau \right) + \frac{1}{t_{2n-2}} \int_{t_{2n-2}}^{1} \frac{x(t^*_{2n-2})}{\tau - t^*_j} d\tau \right) = f(t^*_j), \quad j = 1; \]

\[ a(t^*_j)x(t^*_j) + \frac{b(t^*_j)}{\pi} \left( \int_{t_0}^{t_2} \frac{x(t^*_1)}{\tau - t^*_j} d\tau + \sum_{k=2}^{2n-4} \int_{t_k}^{t_{k+1}} \frac{x(t^*_k)}{\tau - t^*_j} d\tau + \sum_{k=2}^{2n-4} \int_{t_k}^{t_{k+1}} \frac{x(t^*_k)}{\tau - t^*_j} d\tau \right) + \frac{1}{t_{2n-2}} \int_{t_{2n-2}}^{1} \frac{x(t^*_{2n-1})}{\tau - t^*_j} d\tau \right) = f(t^*_j), \quad j = 2n - 2. \] (8.13)

Using Hadamard Theorem, we prove, that the system (8.13) has a unique solution.

We use following designations: \( a_j = a(t^*_j), b_j = b(t^*_j), h_{jk} = h(t^*_j, t^*_k) \). The system (8.13) can be written as \( Cx = b \), where
$C$ is the matrix with elements $\{c_{ij}\}$, $i, j = 1, 2, \ldots, 2n - 2$, $x = (x_1, \ldots, x_{2n-2})$, $f = (f_1, \ldots, f_{2n-2})$.

Easy to see that

$$|c_{jj}| = \left| a_j + \frac{b_j}{\pi} \int_{t_j}^{t_{j+1}} \frac{d\tau}{\tau - t_j} + \frac{h_{jj}}{2n} \right| \geq \left| a_j + \frac{b_j}{\pi} \ln \frac{t_{j+1} - t_j^*}{t_j - t_j^*} \right| - \left( \frac{h_{jj}}{n} \right) =$$

$$= \left| a_j + \frac{b_j}{\pi} \ln \frac{1}{n - h} \right| - \left( \frac{h_{jj}}{n} \right).$$

We assumed that $b_j \neq 0, j = 1 \ldots, 2n - 2$. So we can select the parameter $h$ such way, that $|c_{jj}|$ will become bigger than any arbitrary integer $M$.

From the second hand, for $j \neq 1, 2n - 2$, we have

$$\sum_{k=1, k \neq j}^{2n-1} |c_{jk}| \leq A \left( \left| \int_{t_2}^{t_0} \frac{d\tau}{\tau - t_j^*} \right| + \sum_{k=2, k \neq j}^{2n-2} \left| \int_{t_k}^{t_{k+1}} \frac{d\tau}{\tau - t_j^*} \right| + \right) +$$

$$+ \left| \int_{t_{2n-2}}^{t_{2n-1}} \frac{d\tau}{\tau - t_j^*} \right| + \frac{\pi}{n} \sum_{k=0, k \neq j}^{2n-1} |h_{jk}| \leq$$

$$\leq A \left( \ln \frac{t_{j+1} - t_0}{t_j^* - t_{j-1}} + \ln \frac{t_{2n-2} - t_j^*}{t_{j+2} - t_j^*} \right) + D \leq E \ln n + D.$$

For other cases we have similar estimations. From Hadamard Theorem follows that system (8.13) has a unique solution.

Let us estimate the error of offered method.

For each point $t_j^*$, $j = 1, \ldots, 2n - 2$, we find a function $\psi(t)$ with properties

$$\psi(t_j^*) = x(t_j^*), \int_{t_j-1}^{t_j} \psi(\tau) \frac{1}{\tau - t_j^*} d\tau + \int_{t_j}^{t_{j+1}} \psi(\tau) \frac{1}{\tau - t_j^*} d\tau = 0. (8.14)$$
As example of such function we can take the straight line  
\[ \psi_j(t) = x(t_j^*) + k(t - t_j^*), \]
where
\[ k = -\frac{x(t_j)n}{2} \ln \frac{h(2/n - h)}{(1/n + h)(1/n - h)}. \]  

(8.15)

For case when \( j \neq 1, 2n - 2 \), we have the following estimate of error of the quadrature formula (8.12)

\[ |R_n| \leq \left| \int_{-1}^{t_2} \frac{x(\tau) - x(t_1^*)}{\tau - t_j^*} d\tau \right| + \left| \int_{t_{2n-2}}^{1} \frac{x(\tau) - x(t_{2n-1}^*)}{\tau - t_j^*} d\tau \right| + \\
+ \sum_{k=2, k \neq j-1, j+1}^{2n-2} \left| \int_{t_k}^{t_{k+1}} \frac{x(\tau) - x(t_k^*)}{\tau - t_j^*} d\tau \right| + \\
+ \left| \int_{t_{j-1}}^{t_j} \frac{x(\tau) - \psi(\tau)}{\tau - t_j^*} d\tau \right| + \left| \int_{t_{j+1}}^{t_{j+2}} \frac{x(\tau) - \psi(\tau)}{\tau - t_j^*} d\tau \right| + \\
+ \left| \int_{t_j}^{t_{j+1}} \frac{x(\tau) - x(t_j^*)}{\tau - t_j^*} d\tau \right| + \\
+ \sum_{k=3, k \neq j, j-1, j+1}^{2n-1} \left| \int_{t_k}^{t_{k+1}} \frac{x(\tau) - x(t_k^*)}{\tau - t_j^*} d\tau \right| + \\
+ \left| \int_{t_{j-1}}^{t_j} \frac{x(\tau) - \psi(\tau)}{\tau - t_j^*} d\tau \right| + \left| \int_{t_{j+1}}^{t_{j+2}} \frac{x(\tau) - \psi(\tau)}{\tau - t_j^*} d\tau \right| = r_1 + \cdots r_6. \]
Estimating each sum, we see that
\[ R_N \leq An^{-\alpha} + Akn^{-1} + \frac{A}{n^\alpha} \ln \left( \frac{(t_{2n} - t^*_j) \cdot (t^*_j - t_0)}{n^2} \right). \]

Similar estimations we have in other cases. Repeated arguments of previous item, we formulate the following statement.

**Theorem 8.2** [35], [38]. Let \( a(t), b(t), f(t) \in H_\alpha \), \( h(t, \tau) \in H_{\alpha, \alpha} \), \( 0 < \alpha \leq 1 \); \( b(t) \neq 0 \) for \( t \in [-1, 1] \). Let the equation (8.10) has a unique solution \( x^*(t) \in H_\alpha(0 < \alpha \leq 1) \). Then it exists a small parameter \( h \) such that the system (8.13) has a unique solution \( x^*_n(t) \). If coefficients \( k_j \) in the equations \( \psi_j(t) = x(t^*_j) + k_j(t - t^*_j) \) satisfy the inequality \( \max |k_j| \leq n^{1-\alpha} \), then the estimate \( \|x^* - x^*_n\|_{H_\beta} \leq An^{-(\alpha-\beta)} \ln^3 n \) is valid.

### 8.3. Approximate Solution of the Equation (8.1)

For approximate solution of the equation (8.1) we use two sets of knots: \( t_k = -A + kA/N \), \( k = 0, 1, \ldots, 2N \), \( t^*_k = t_k + h \), \( k = 1, \ldots, N - 1 \), \( t^*_k = t_{k+1} - h \), \( k = N, \ldots, 2N - 2 \), \( 0 < h \leq A/2N \). We assume that \( A \) is a sufficient large constant and \( h \) is a small constant.

Repeating arguments of previous item we can construct the numerical scheme for solution of the equation (8.1) and formulate conditions of solvability.

### 8.4. Approximate Solution of the Equation (8.2)

Let us consider the approximate method for solution of the equation (8.2). We will use two sets of knots: \( t_k = -A + kA/N \), \( k = 0, 1, \ldots, 2N \) and \( t^*_k = t_k + h \), \( k = 1, \ldots, N - 1 \), \( t^*_k = t_k - h \), \( k = N + 1, \ldots, 2N - 2 \), where \( A \) and \( h \) are constants. We assume that \( A \) is a sufficient large constant and \( h \) is a small constant, \( 0 < h < A/N \).
We assume that the right side of the equation (8.2) has the form of function

\[ f(t_1, t_2) = \frac{g(t_1, t_2)}{(1 + t_1^2 + t_2^2)^\lambda}, \]

\( \lambda > 3/2, \) where \( g(t_1, t_2) \) is a function which is belongs to the Holder class of functions.

An approximate solution of equation (8.2) we will find in the form of

\[ x_N(t_1, t_2) = \sum_{k=1}^{2N-2} \sum_{l=1}^{2N-2} a_{kl} \bar{\psi}_{kl}(t_1, t_2), \]

where

\[ \bar{\psi}_{kl}(t_1, t_2) = \begin{cases} 
\psi_{kl}(t_1, t_2), & 2 \leq k, l \leq 2N - 2, \\
\psi_{00}(t_1, t_2) \cup \psi_{01}(t_1, t_2) \cup \psi_{10}(t_1, t_2) \cup \psi_{11}(t_1, t_2), & k = l = 1; \\
\psi_{2n-1,0}(t_1, t_2) \cup \psi_{2n-2,0}(t_1, t_2) \cup \psi_{2n-1,1}(t_1, t_2) \cup \psi_{2n-2,1}(t_1, t_2), & k = 2n - 2, l = 1, \\
\psi_{2n-1,2n-1}(t_1, t_2) \cup \psi_{2n-1,2n-2}(t_1, t_2) \cup \psi_{2n-2,2n-1}(t_1, t_2) \cup \\
\psi_{2n-2,2n-2}(t_1, t_2), & k = l = 2n - 2, \\
\psi_{0,2n-1}(t_1, t_2) \cup \psi_{1,2n-1}(t_1, t_2) \cup \psi_{0,2n-2}(t_1, t_2) \cup \psi_{1,2n-2}(t_1, t_2), & k = 1, l = 2n - 2, \\
\psi_{0,l}(t_1, t_2) \cup \psi_{1,l}(t_1, t_2), & 1 \leq k, l \leq 2n - 2, \\
\psi_{2n-1,l}(t_1, t_2) \cup \psi_{2n-2,l}(t_1, t_2), & k = 2n - 2, 1 \leq l \leq 2n - 2; \\
\psi_{k,0}(t_1, t_2) \cup \psi_{k,1}(t_1, t_2), & 1 \leq k \leq 2n - 2, l = 1, \\
\psi_{k,2n-2}(t_1, t_2) \cup \psi_{k,2n-1}(t_1, t_2), & 1 \leq k \leq 2n - 2, l = 2n - 2; \\
\end{cases} \]

\[ \psi_{kl}(t_1, t_2) = \begin{cases} 
1, & (t_1, t_2) \in \Delta_{kl}, \\
0, & (t_1, t_2) \notin \Delta_{kl}; \\
\end{cases} \]

\[ \Delta_{kl} = [t_k, t_{k+1}] \times [t_l, t_{l+1}). \]

Unknown coefficients \( \{a_{k,l}\}, k, l = 1, \ldots, 2N - 2, \) are defined
from the system

\[
\frac{A}{\pi N i} \sum_{k=1}^{2N-2} t_k - t_{k_1}^* x_N(t_k^*, t_{l_1}^*) + \frac{A}{\pi N i} \sum_{l=1}^{2N-2} t_l - t_{l_1}^* x_N(t_{k_1}^*, t_l^*) = f(t_{k_1}^*, t_{l_1}^*),
\]

(8.16)

\(k_1, l_1 = 1, \ldots, 2N - 2\), where one prime in the first sum indicates that \(k \neq k_1 - 1, k_1 + 1\), two primes in the second sum indicate that \(l \neq l_1 - 1, l_1 + 1\).

If the constant \(h\) is a sufficient small number, the system (8.16) has a unique solution \(x^*\).

Error of numerical scheme (8.16) is estimated as in previous items.

8.5. Approximate Solution of the Equation (8.3)

Let us consider the approximate method for solution of the equation (8.3). We will use two sets of knots: \(t_k = -A + kA/N, k = 0, 1, \ldots, 2N\) and \(t_k^* = t_k + h_1, k = 1, \ldots, N - 1, t_k^{**} = t_k - h_2, k = N + 1, N + 2, \ldots, 2N - 2\), where \(A\) and \(h_1, h_2\) are constants. We assume that \(A\) is a sufficient large constant and \(h_i\) are small constants, \(0 < h_i \leq A/N, i = 1, 2\).

The approximate solution of equation (8.3) we will find in the form of the function

\[
x_N(t_1, t_2) = \sum_{k=1}^{2N-2} \sum_{l=1}^{2N-2} a_{kl} \tilde{\psi}_{kl}(t_1, t_2),
\]

where \(\tilde{\psi}_{kl}(t_1, t_2)\) was defined in the previous item.

Unknown coefficients \(\{a_{kl}\}, k, l = 1, \ldots, 2N - 2\), are defined from the system

\[
\frac{A}{N} \sum_{k=1}^{2N-2} \sum_{l=1}^{2N-2} \left(\frac{1}{t_k - t_{k_1}^*} - \frac{1}{t_l - t_{l_1}^*}\right) x_N(t_k, t_l) = f(t_{k_1}^*, t_{l_1}^*),
\]

(8.17)
\[ k_1, l_1 = 1, \ldots, 2N - 2, \text{ where the prime in the sum indicates} \]
\[ (k, l) \neq (k_1 - 1, l_1 - 1), (k_1 - 1, l_1), (k_1 - 1, l_1 + 1), (k_1, l_1 - 1), (k_1, l_1 + 1), (k_1 + 1, l_1 - 1), (k_1 + 1, l_1), (k_1 + 1; l_1 + 1). \]

Constants \( h_1 \) and \( h_2 \) can be chosen so way that the system (8.17) has a unique solution.

9. Approximate Solution of Singular Integro-Differential Equations

9.1. Linear Equations

Introduction. In the section 9.1.1 proposed a calculation scheme of the mechanical quadratures method for solution of singular integro-differential equations (SIDE)

\[ Kx \equiv \sum_{k=0}^{m} \left[ a_k(t)x^{(k)}(t) + \frac{b_k(t)}{\pi i} \int_{L} \frac{x^{(k)}(\tau)}{\tau - t} d\tau + \right. \]
\[ \left. + \frac{1}{2\pi i} \int_{L} \frac{h_k(t, \tau)x^{(k)}(\tau)}{|\tau - t|^\gamma} d\tau \right] = f(t) \quad (9.1) \]

under conditions

\[ \int_{L} x(t)t^{-t-1}dt = 0, k = 0, 1, \ldots, m - 1, \quad (9.2) \]

where \( a_k(t), b_k(t), h_k(t, \tau), f(t) \in C_{2\pi}, L \) is the unit circle centered at the origin, \( 0 \leq \gamma < 1 \), and given the justification of this scheme.

In the section 9.1.2 proposed calculation scheme of mechanical quadrature method for approximate solution of SIDE

\[ Fx \equiv \sum_{k=0}^{m} \left[ a_k(t)x^{(k)}(t) + \frac{1}{\pi i} \int_{L} \frac{h_k(t, \tau)x^{(k)}(\tau)}{\tau - t} d\tau \right] = f(t) \quad (9.3) \]
under conditions (9.2), where functions \( f(t) \in H_{\alpha}, a_k(t) \in H_{\alpha}, k = 0, m, h_k(t, \tau) \in H_{\alpha,\alpha}, k = 0, m, (0 < \alpha < 1) \), and provided its justification.

### 9.1.1. Approximate Solution of the Boundary Value Problem (9.1), (9.2)

1°. Computational scheme.

An approximate solution of the boundary value problem (9.1), (9.2) is sought in the form of a polynomial

\[
\tilde{x}_n(t) = \sum_{k=0}^{n} \alpha_k t^{k+m} + \sum_{k=-n}^{-1} \alpha_k t^k
\]

(9.4)

which coefficients \( \{\alpha_k\} \) are determined from the system of equations

\[
\tilde{K}_n \tilde{x}_n = P_n \left[ \sum_{k=0}^{m} \left[ a_k(t) \tilde{x}^{(k)}(t) + \frac{b_k(t)}{\pi i} \int_{L} \frac{x_n^{(k)}(\tau)d\tau}{\tau-t} + \frac{1}{2\pi i} \int_{L} P_n^\tau \left[ h_k(t, \tau) \tilde{x}^{(k)}(\tau)d(t, \tau) \right] d\tau \right] \right] = P_n [f(t)],
\]

(9.5)

where \( P_n \) is the projector onto the set of interpolating trigonometric polynomials of degree \( n \) built over the knots \( t_k = e^{i s_k}, s_k = 2k\pi/(2n+1) (k = 0, 2n), d(t, \tau) = |t-\tau|^{-\gamma} \) for \(|\sigma-s| \geq 2\pi/(2n+1)\), \( d(t, \tau) = |e^{is_1} - 1|^{-\gamma} \) for \(|\sigma-s| \leq 2\pi/(2n+1)\), \( \tau = e^{i\sigma}, t = e^{is} \).

**Justification of the method.** First of all, the boundary value problem (9.1), (9.2) and the equation (9.5) are reduced to the equivalent Riemann boundary value problems. For this we introduce the function \( \Phi(z) = \frac{1}{2\pi i} \int_{L} \frac{x(\tau)d\tau}{\tau-z} \)

We will need in Sohotzky - Plemel formulas

\( \Phi^+(t) - \Phi^-(t) = x(t), \Phi^+(t) + \Phi^-(t) = \frac{1}{\pi i} \int_{L} \frac{x(\tau)d\tau}{\tau-t}, \ldots \),

114
\[
\Phi^{(m)}(t) - \Phi^{(m)}(t) = x^{(m)}(t), \quad \Phi^{(m)}(t) + \Phi^{(m)}(t) = \frac{1}{i\pi} \int_{L} \frac{x^{(m)}(\tau) \, d\tau}{\tau - t}.
\]

Substituting the previous formulas in equations (9.1) and (9.5), we arrive to the following Riemann boundary value problems:

\[
Kx \equiv \sum_{k=0}^{m} \left[ (a_k(t) + b_k(t)) \Phi^{(k)}(t) - (a_k(t) - b_k(t)) \Phi^{(k)}(t) + \right. \\
+ \frac{1}{2\pi i} \int_{L} \frac{h_k(t, \tau) \left( \Phi^{(k)}(\tau) - \Phi^{(k)}(\tau) \right)}{|\tau - t|^{\gamma}} \, d\tau \left. \right] = f(t) \quad (9.6)
\]

under conditions (9.2) and

\[
\tilde{K}_n \tilde{x}_n \equiv \quad \\
\equiv P_n \left[ \sum_{k=0}^{m} \left[ (a_k(t) + b_k(t)) \tilde{\Phi}_n^{(k)}(t) - (a_k(t) - b_k(t)) \tilde{\Phi}_n^{(k)}(t) + \right. \\
+ \frac{1}{2\pi i} \int_{L} P_n^{\tau} \left( h_k(t, \tau) d(t, \tau) \left( \tilde{\Phi}_n^{(k)}(\tau) - \tilde{\Phi}_n^{(k)}(\tau) \right) \right) \, d\tau \left. \right] = \\
= P_n [f(t)]. \quad (9.7)
\]

Easy to see that \( \tilde{\Phi}_n(t) = \sum_{k=0}^{n} \alpha_k t^{k+m}, \tilde{\Phi}_n(t) = \sum_{k=-n}^{-1} \alpha_k t^k. \)

We reduce the problem (9.6), (9.2) and (9.7) to the equivalent singular integral equations. For this we use the well known integral representation of Krikunov [97]. Functions \( \frac{d^m \Phi^+(z)}{dz^m} \) and \( \frac{d^m \Phi^-(z)}{dz^m} \) are represented into the integral of Cauchy type with the same density

\[
\frac{d^m \Phi^+(z)}{dz^m} = \frac{1}{2\pi i} \int_{L} \frac{v(\tau) \, d\tau}{\tau - z}, \quad \frac{d^m \Phi^-(z)}{dz^m} = \frac{z^{-m}}{2\pi i} \int_{L} \frac{v(\tau) \, d\tau}{\tau - z}. \quad (9.8)
\]

To transform the approximate system (9.7) for the Riemann boundary value problem (9.6) into the equivalent approximate
system for singular integral equation, we will use the integral representation
\[
\frac{d^m \tilde{\Phi}_n^+(z)}{dz^m} = \frac{1}{2\pi i} \int_L \tilde{v}_n(\tau) d\tau, \quad \frac{d^m \tilde{\Phi}_n^-(z)}{dz^m} = \frac{z^{-m}}{2\pi i} \int_L \tilde{v}_n(\tau) d\tau, \tag{9.9}
\]
where \( \tilde{v}_n(t) = \sum_{k=0}^n \frac{(m+k)!}{k!} \alpha_k t^k - \sum_{k=1}^n (-1)^m \alpha_{-k} \frac{(m+k-1)!}{(k-1)!} t^{-k} \).

Repeating arguments given in the work [97], and using integral representations (9.8) and (9.9), we reduce the boundary value problems (9.6), (9.2) and (9.7) to equivalent singular integral equations
\[
K_1 v \equiv a^*_1 v(t) + \frac{b^*_1(t)}{\pi i} \int_L \frac{v(\tau) d\tau}{\tau - t} + \frac{1}{(2\pi i)^2} \int_L \frac{h_0(t, \tau)}{|\tau - t|^{\gamma}} \left( \int_L k_0(\tau, \sigma) \ln(\tau - \sigma) v(\sigma) d\sigma \right) d\tau + \ldots + \frac{1}{2\pi i} \int_L \frac{h_m(t, \tau)}{|\tau - t|^{\gamma}} \left( \frac{1}{2\pi i} \int_L \frac{v(\sigma) + \tau^{-m} v(\sigma)}{\sigma - \tau} d\sigma \right) d\tau = f_1(t), \tag{9.10}
\]
and
\[
\tilde{K}_{1,n} \check{v}_n = P_n \left[ a^*_1 \check{v}(t) + \frac{b^*_1(t)}{\pi i} \int_L \frac{\check{v}_n(\tau) d\tau}{\tau - t} \right. + \frac{1}{(2\pi i)^2} \int_L P_n^\tau \left[ h_0(t, \tau) d(t, \tau) \left[ \int_L k_0(\tau, \sigma) \ln(\tau - \sigma) \check{v}(\sigma) d\sigma \right] \right] d\tau + \ldots + \frac{1}{(2\pi i)^2} \int_L P_n^\tau \left[ h_m(t, \tau) d(t, \tau) \left[ \int_L \frac{\check{v}(\sigma) + \tau^{-m} \check{v}(\sigma)}{\sigma - \tau} d\sigma \right] d\tau \right] = P[f_1(t)], \tag{9.11}
\]
where \( k_0(t, \tau), \ldots, k_{m-1}(t, \tau) \) are Fredholm kernels; \( a_1^*(t), b_1^*(t), f_1^*(t) \in C_{2\pi} \); explicit form of these functions can be discharged on the basis of presentation Yu. M. Krikunov [97].

Let \( X \) is the space of square-integrable functions with scalar product

\[
(g_1, g_2) = \frac{1}{2\pi} \int_0^{2\pi} g_1(t) g_2(t) ds, \quad t = e^{is}.
\]

Approximate solution \( \tilde{v}_n(t) \) we will seek in the subspace \( X_n \) of the space \( X \). Subspace \( X_n \) consists of \( n \)-order polynomials

\[
x_n = \left\{ \sum_{k=-n}^{n} \alpha_k t^k \right\}.
\]

Obviously , \( \tilde{K}_{1,n} \in [X_n \to X_n] \).

We assume that the operator \( K_1 \) has the linear inverse. (So, the boundary value problem (9.1), (9.2) has a unique solution for any \( f \)).

As

\[
\left\| \int_L k_0(\tau, \sigma) \ln(\tau - \sigma) v(\sigma) d\sigma \right\| \leq A\|v\|, \left\| \int_L \frac{v(\sigma)}{(\sigma - \tau)} d\sigma \right\| \leq A\|v\|,
\]

where (as everywhere else) \( A \) are well-defined constants independent from \( v \) and \( n \), instead of equations (9.10), (9.11) we can restrict ourself by the equations

\[
K_1 v = a(t) v(t) + b(t) \int \frac{v(\tau) d\tau}{\tau - t} + \frac{1}{2\pi i} \int \frac{h(t, \tau) v(\tau)}{|\tau - t|^\gamma} d\tau = f(t),
\]

(9.12)

\[
K_{1,n} \tilde{v}_n = P_n \left[ a(t) \tilde{v}_n(t) + \frac{b(t)}{\pi i} \int \frac{\tilde{v}_n(\tau) d\tau}{\tau - t} + \right.
\]

\[
+ \frac{1}{2\pi i} \int L P_n^\tau [h(t, \tau) d(t, \tau) \tilde{v}_n(\tau)] d\tau \right] = P_n[f(t)].
\]

(9.13)

Consider the equation

\[
K_2 v = \tilde{a}(t) v(t) + \tilde{b}(t) S(v) + H(v) = f(t),
\]

(9.14)

\(^1\)The explicit form of these functions is not issued, since below we use only their characteristics.
where \( S(v) = \frac{1}{\pi i} \int_{L} \frac{v(\tau)d\tau}{\tau - t}, \) \( H(v) = \frac{1}{2\pi i} \int_{L} \frac{h(t,\tau)v(\tau)d\tau}{\tau - t}; \) \( \tilde{a}(t), \tilde{b}(t) \) are polynomials of best uniform approximation degree at most \( m \) for functions \( a(t), b(t), t \in \gamma, \) respectively. The value of \( m \) will be fixed below.

From Banach theorem it implies that, for such \( n \) that \( q_1 = A [\omega(a; m^{-1}) + \omega(b; m^{-1})] < 1, \) operator \( K_2 \) has a linear inverse operator with the norm \( \|K_2^{-1}\| \leq \|K_1^{-1}\|/(1 - q_1). \)

Collocation method for the equation (9.14) is written as follows

\[
K_{2,n} \tilde{\nu}_n \equiv P_n[\tilde{a}(t)\tilde{\nu}_n(t) + \tilde{b}(t)S(\tilde{\nu}_n) + H(\tilde{\nu}_n)] = P_n[f]. \tag{9.15}
\]

Following the method proposed in the paragraph 3, equations (9.14) and (9.15) can be written in the equivalent form:

\[
K_3 v \equiv \Psi v + W v = y, K_3 \in [X \to X] \tag{9.16}
\]

and

\[
K_{3,n} \tilde{\nu}_n \equiv \tilde{\Psi}_n \tilde{\nu}_n + \tilde{W}_n \tilde{\nu}_n = \tilde{y}_n, K_{3,n} \in [\tilde{X}_n \to \tilde{X}_n], \tag{9.17}
\]

where \( \Psi v = \psi^{-}v^{+} - \psi^{+}v^{-}, \) \( W v = lH(v), l = \psi^{-}/(\tilde{a} + \tilde{b}), \)

\[
\psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_{L} \ln \left( \frac{\ln \left( \frac{\tilde{a}(\tau) - \tilde{b}(\tau)}{\tilde{a}(\tau) + \tilde{b}(\tau)} \right)}{\tau - z} \right) d\tau \right\},
\]

\[
\tilde{\Psi}_n \tilde{\nu}_n = P_n[\Psi \tilde{\nu}_n], \tilde{W}_n \tilde{\nu}_n = P_n[lH(\tilde{\nu}_n)], y = lf, \tilde{y}_n = P_n[y].
\]

Let us introduce the polynomial

\[
\tilde{\varphi}(t) = \Psi_n \tilde{\nu}_n + \left( T^{[n/2]}[l] \right) \frac{1}{2\pi i} \int_{L} T_i^{[n/2]} [h(t,\tau)d^*(t,\tau)] \tilde{\nu}_n(\tau)d\tau,
\]

where \( \Psi_n \tilde{\nu}_n = \psi^{-}_n \tilde{\nu}^+_n - \psi^+_n \tilde{\nu}^-_n, \) \([n/2]\) is antje \( n/2; \) \( T^{[n/2]}[f] \) and \( \psi_n \) are polynomials of the best uniform approximation degree to \([n/2]\) and \( n \) for functions \( f \) and \( \psi, \) respectively; \( d^*(t,\tau) = |\tau - t|^{-\gamma} \) for \( |\tau - t| > \rho, \) \( d^*(t,\tau) = \rho^{-\gamma} \) for \( |\tau - t| \leq \rho; \) \( \rho \) is a fixed positive number, \( \rho \geq |e^{is_1} - 1|. \)
Let us estimate
\[ \|K_3\tilde{v}_n - \tilde{\varphi}\| \leq \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\|, \tag{9.18} \]
where
\[ I_1 = \Psi_n\tilde{v}_n - \Psi\tilde{v}_n, \quad I_2 = D^{[n/2]}[l]H(\tilde{v}_n), \]
\[ I_3 = (T^{[n/2]}[l]) \frac{1}{\pi i} \int_L h(t, \tau)\tilde{v}_n(\tau) \left[ |\tau - t|^{-\gamma} - d^*(t, \tau) \right] d\tau, \]
\[ I_4 = (T^{[n/2]}[l]) \frac{1}{\pi i} \int_L D^{[n/2]}[h(t, \tau)d^*(t, \tau)]\tilde{v}_n(\tau) d\tau, \]
\[ D = E - T, \]  
\[ E \text{ is the identity operator.} \]

Since \( \tilde{a}, \tilde{b} \) are included in the Holder class \( H_1 \) with the factor \( Am \), then \( \psi \in H_\delta, \delta = 1 - \epsilon \), where \( \epsilon \) is an arbitrary number, \( 0 < \epsilon < 1 \). It follows from Privalov theorem \[64\].

So
\[ ||I_1|| \leq Am\|\tilde{v}_n\|/n^{1-\epsilon} \tag{9.19} \]

Easy to see that
\[ ||I_2|| \leq A\frac{m}{n^{1-\epsilon}}\|\tilde{v}_n\|, \]
\[ ||I_3|| \leq A\|\tilde{v}_n\|\rho^{1-\gamma}, \]
\[ ||I_4|| \leq A\|\tilde{v}_n\|\frac{\omega(h; n^{-1})}{\rho^{2\gamma}}. \tag{9.20} \]

Let us estimate
\[ \|P_nK_3\tilde{v}_n - \tilde{\varphi}\| \leq \|I_5\| + \|I_6\| + \|I_7\| + \|I_8\|, \tag{9.21} \]
where \( I_5 = P_nI_1, I_6 = P_nI_2, I_7 = P_nI_3, I_8 = P_nI_4. \)

Obviously,
\[ ||I_5|| = \left[ \frac{2\pi}{2n+1} \sum_{k=0}^{2n} |\psi_n(t_k) - \psi(t_k)|^2 |\tilde{v}_n(t_k)|^2 \right]^{1/2} \leq \]
\[ \leq Am\|\tilde{v}_n\|/n^{1-\epsilon}. \tag{9.22} \]

We proceed to the estimation of \( ||I_7|| \). For this we represent \( I_7 \) as follows:
\[ I_7 = I_9 + I_{10}, \tag{9.23} \]
where

\[ I_9 = P_n \left[ \frac{1}{\pi i} \int_{L} \frac{c(t, \tau)\tilde{v}_n(\tau)}{\tau - t} d\tau \right] = P_n \left[ \frac{1}{\pi i} \int_{L} \frac{P_n^t[c(t, \tau)]\tilde{v}_n(\tau)}{\tau - t} d\tau \right] = \]

\[ = \frac{1}{\pi i} \int_{L} \frac{c(t, \tau) - c(\tau, \tau)}{\tau - t} \tilde{v}_n(\tau) d\tau + \frac{1}{\pi i} \int_{L} \frac{c(\tau, \tau)[\tilde{v}_n(\tau) - \tilde{v}_n(t)]}{\tau - t} d\tau + \]

\[ + P_n \left[ \frac{\tilde{v}_n(t)}{\pi i} \int_{L} \frac{c(\tau, \tau) d\tau}{\tau - t} \right] = I_{11} + I_{12} + I_{13}, \quad (9.24) \]

\[ I_{10} = P_n \left[ \frac{1}{\pi i} \left( T^{n/2}[l] \right) \int_{L} h(t, \tau)\tilde{v}_n(\tau)[d(t, \tau) - d^*(t, \tau)] d\tau \right], \]

\[ c(t, \tau) = \left( T^{n/2}[l] \right) h(t, \tau)(1 - |\tau - t|^\gamma d(t, \tau))|\tau - t|^{1-\gamma} e^{i\theta}, \quad \theta = \theta(\tau, t) = \arg(\tau - t), \quad c_1(t, \tau) = P_n^t[c(t, \tau)]. \]

From the definition of function \( d(t, \tau) \) follows that \(|c_1(t, \tau)| \leq An^\gamma - 1\). Since \( c(t, \tau) \) with respect to the variable \( t \) is a trigonometric polynomial of degree not higher than the \( n \), it is possible to show that \( c_1(t, \tau) \) belongs to the class Holder with degree \( 1/\ln n \) and with the factor \( A \left( n^{\gamma - 1}n^{1/\ln n} \right) = An^\gamma - 1 \).

Therefore

\[ \|I_{11}\| \leq An^\gamma - 1 \ln n \|\tilde{v}_n\|. \quad (9.25) \]

For \( I_{12}, I_{13} \) we have:

\[ \|I_{12}\| + \|I_{13}\| \leq An^\gamma - 1 \ln n \|\tilde{v}_n\|. \quad (9.26) \]

Now we will estimate \( \|I_9\| \). Let us introduce the notations

\[ b(t, \tau) = d(t, \tau) - d^*(t, \tau), \quad g(t, \tau) = \left( T^{[n/2][l]} \right) h(t, \tau). \]

Obviously

\[ \|I_9\| = \]
\[
\mathbf{I} = \left[ \frac{2\pi}{2n+1} \sum_{k=0}^{2n} \left[ \frac{1}{\pi i} \int_{L} g(t_k, \tau) \tilde{v}_n(\tau) [b(t_k, \tau)]^{\frac{1}{2}} P_n^t \left[ b(t, \tau) \right]^{\frac{1}{2}} d\tau \right] \right]^{2} \leq \]

\[
\leq \left[ \frac{2\pi}{2n+1} \sum_{k=0}^{2n} \frac{1}{\pi i} \left[ \int_{L} \left| g(t_k, \tau) \left[ b(t, \tau) \right]^{\frac{1}{2}} \right|^{2} d\tau \right] \right]^{\frac{1}{2}} \times \]

\[
\times \left[ \int_{L} \left| P_n^t \left[ b(t, \tau) \right]^{\frac{1}{2}} \tilde{v}_n(\tau) \right|^{2} d\tau \right]^{\frac{1}{2}} \leq \]

\[
\leq \left[ \max_{t} \int_{L} \left| g(t, \tau) \left[ b(t, \tau) \right]^{\frac{1}{2}} \right|^{2} d\tau \right]^{\frac{1}{2}} \times \]

\[
\times \left[ \int_{0}^{2\pi} ds \frac{1}{\pi} \int_{0}^{2\pi} \left| P_n^s \left[ b(e^{i\sigma}, e^{i\sigma}) \right]^{\frac{1}{2}} \tilde{v}_n(e^{i\sigma}) \right|^{2} e^{i\sigma} d\sigma \right]^{\frac{1}{2}} \leq \]

\[
\leq A \rho^{1-\gamma} \| \tilde{v}_n \|. \quad (9.27)
\]

Repeating the previous discussion, we obtain

\[
\| I_6 \| \leq A m \ln n \| \tilde{v}_n \| / n^{1-\epsilon}. \quad (9.28)
\]

Norm of \( \| I_8 \| \) is estimated in the same way as the \( \| I_4 \| \):

\[
\| I_8 \| \leq A \| \tilde{v} \| \omega(h; n^{-1}) / \rho^{2\gamma}. \quad (9.29)
\]

From the estimates (9.18) - (9.29) it follows that

\[
\| (K_3 \tilde{v}_n - \tilde{\varphi}) + (P_n \tilde{\varphi} - P_n K_3 \tilde{v}_n) \| \leq \]

\[
\leq A \left[ \frac{m}{n^{1-\epsilon}} + \frac{\omega(h; n^{-1})}{\rho^{2\gamma}} + \rho^{1-\epsilon} + n^{\gamma-1} \right] \| \tilde{v}_n \|.
\]

Note. It is assumed that \( \gamma > 0 \). Otherwise it is possible to put \( \rho = 0, m/n^{1-\epsilon} < \min[w(a; n^{-1}), \omega(b; n^{-1})] \).
Assume $\varepsilon = \gamma/2$, $m = n^{\gamma/2}$, $\rho = [\omega(h; n^{-1})^{1/(1+\gamma)}]$. Using the results of the paragraph 3, we obtain, for such $n$ that

$$q_2 = A [\omega(h; n^{-1})]^{1-\gamma/2} < 1,$$

that the equation (9.17), and hence (9.15), is uniquely solvable for any right-hand side, and the estimate $\|\tilde{K}_3^{-1}\| \leq A$ is valid.

Let us estimate $\|K_2^{-1}\|$. Let $\tilde{v}_1^*$ is the solution of the equation (9.14), and hence (9.17).

Then

$$\|\tilde{v}_1^*\| \leq \|\tilde{K}_{3,n}\| \|\tilde{y}\| \leq \|\tilde{K}_{3,n}\| \|P_n[lP[f]]\| \leq \|\tilde{K}_{3,n}\|\|l\|\|P_n[f]\|.$$

Hence $\|\tilde{K}_2^{-1}\| \leq A$.

Repeating the above discussion, we see that

$$\|\tilde{K}_{2,n} - \tilde{K}_{1,n}\| \leq A \left[ [\omega(h; n^{-1})]^{1-\gamma} + \omega(a; n^{\gamma-1}) + \omega(b; n^{\gamma-1}) \right].$$

Hence, from the Banach theorem we have

**Theorem 9.1** [42], [44]. Let the problem (9.1), (9.2) has a unique solution for any right-hand side, and let the functions $a_i$, $b_i$, $h_i$, $f \in C_{2\pi}$, $i = 0, m$. Then for $n$ such that

$$q = A \sum_{k=0}^{m} \left[ \omega(a_k; n^{\gamma-1}) + \omega(b_k; n^{\gamma-1}) + [\omega(h_k; n^{-1})]^{1-\gamma/2} \right] < 1,$$

the system of equations (9.5) has a unique solution $\tilde{x}_n^*$ and the estimate $\|x^* - \tilde{x}_n^*\| \leq A[q + \omega(f; n^{-1})]$ is valid. Here $x^*$ is the solution of the boundary value problem (9.1), (9.2).

**9.1.2. Approximation Solution of the Boundary Value Problem (9.3), (9.2)**

**Computational scheme.**

An approximate solution of the boundary value problem (9.3), (9.2) is sought in the form of the polynomial (9.4) with coefficients $\{\alpha_k\}$, which are determined from the system of algebraic equations

$$\tilde{F}_n \tilde{x}_n \equiv$$
\[ \equiv P_n \left[ \sum_{k=0}^{m} \left[ a_k(t) \tilde{x}_n^{(k)}(t) + \frac{1}{\pi} \int_{L} \left[ P_n^{\tau} [h_{k}(t, \tau) \tilde{x}_n^{(k)}(\tau)] \frac{\tau - t}{\tau - t} \right] d\tau \right] \right] = \]
\[ = P[f], \quad (9.30) \]

where the operator \( P_n \) is the projection operator onto the set of interpolating polynomials of degree at most \( n \), constructed on nodes \( \tilde{t}_k = e^{i\pi k}, \tilde{s}_k = (2k\pi + \pi)/(2n + 1), k = 0, 1, \ldots, 2n. \)

**Justification of the method.** As in the previous item boundary value problem (9.3), (9.2) and the equation (9.30) is reduced to an equivalent singular integral equations.

To do this, we will use the identity

\[ \equiv P_n \left[ \int_{0}^{2\pi} P_n^{\sigma} [h(s, \sigma) \tilde{g}(\sigma) \cot \frac{\sigma - s}{2}] d\sigma \right] \equiv \]
\[ \equiv P_n \left[ \int_{0}^{2\pi} P_n^{\sigma} [h(s, \sigma) \tilde{g}(\sigma)] \frac{\sigma - s}{2} d\sigma \right] \equiv \]
\[ \equiv P_n \left[ \int_{0}^{2\pi} P_n^{\sigma} [h(s, \sigma)] \tilde{g}(\sigma) \cot \frac{\sigma - s}{2} d\sigma \right], (\tilde{g} \in \tilde{X}_n). \]

(9.31)

The validity of this identity follows from the fact, that if \( \tilde{g}(s) \) is a polynomial of degree \( n \), then \( [\tilde{g}(\sigma) - \tilde{g}(s)] \cot \frac{\sigma - s}{2} \) is a polynomial of degree \( n \), and \( \sum_{k=0}^{2n} \cot \frac{2k\pi + \pi}{4n + 2} = 0. \)

Using the identity (9.31), the equations (9.3) (9.30) can be represented as follows:

\[ Fx \equiv \sum_{k=0}^{m} \left[ a_k(t) x^{(k)}(t) + \frac{b_k(t)}{\pi i} \int_{L} \frac{x^{(k)}(\tau)}{\tau - t} d\tau + \right. \]
\[ \left. + \frac{1}{2\pi i} \int_{L} \frac{h_k(t, \tau) - h_k(t, t)}{\tau - t} x^{(k)}(\tau) d\tau \right] = f(t) \quad (9.32) \]
\[
\tilde{F}_n \tilde{x}_n \equiv \overline{P}_n \left[ \sum_{k=0}^{m} \left[ a_k(t) \tilde{x}^{(k)}_n(t) + \frac{b_k(t)}{\pi i} \int_{L} \frac{\tilde{x}^{(k)}_n(\tau)}{\tau - t} d\tau \right] + \frac{1}{2\pi i} \int_{L} \frac{P^\tau_n \left[ h_k(t, \tau) - h_k(t,t) \right] \tilde{x}^{(k)}_n(\tau)}{\tau - t} d\tau \right] \right] = \overline{P}_n[f], \quad (9.33)
\]

where \( b_k(t) = h_k(t,t), n \geq m. \)

Repeating the arguments of the previous section, the boundary value problem (9.32) (9.2) and the equation (9.33) can be reduced to equivalent singular integral equations.

Therefore, the justification of the computational scheme (9.30) for the boundary value problem (9.3), (9.2) can be reduced to the justification of the computing scheme

\[
\tilde{F}_{1,n} \tilde{v}_n \equiv \overline{P}_n \left[ a(t) \tilde{v}_n(t) + \frac{b(t)}{\pi i} \int_{L} \frac{\tilde{v}_n(\tau)d\tau}{\tau - t} + \frac{1}{2\pi i} \int_{L} \frac{P^\tau_n \left[ h(t, \tau) - h(t,t) \right] \tilde{v}_n(\tau)}{\tau - t} d\tau \right] = \overline{P}_n[f] \quad (9.34)
\]

for the singular integral equation

\[
F_1 v \equiv a(t)v(t) + \frac{b(t)}{\pi i} \int_{L} \frac{v(\tau)d\tau}{\tau - t} + \frac{1}{2\pi i} \int_{L} \frac{h(t, \tau) - h(t,t)}{\tau - t} v(\tau)d\tau = f(t), \quad (9.35)
\]

where
\[
a(t), b(t), f(t) \in H_\alpha, h(t, \tau) \in H_{\alpha,\alpha}, (0 < \alpha < 1). \quad (9.36)
\]

Justification held in the spaces \( X \) and \( \tilde{X}_n \), introduced in the previous section (\( v \in X, \tilde{v}_n \in \tilde{X}_n \)).
As in the previous section, equations (9.35) and (9.34) can be written in the following equivalent forms:

\[ F_2v \equiv \Psi v + Wy = y, \quad F_2 \in [X \to X] \quad (9.37) \]

and

\[ \tilde{F}_{2,n}\tilde{v}_n \equiv \tilde{\Psi}_n\tilde{v}_n + \tilde{W}_n\tilde{v}_n = \tilde{y}_n, \quad \tilde{F}_{2,n} \in [\tilde{X}_n \to \tilde{X}_n], \quad (9.38) \]

where

\[ Wv = \frac{l}{2\pi i} \int L h(t, \tau) - h(t, t) v(\tau) d\tau, \]

\[ \tilde{W}_n\tilde{v}_n = \overline{P}_n \left[ \frac{l}{2\pi i} \int L \frac{P_n^\tau \left[ [h(t, \tau) - h(t, t)] \tilde{\nu}_n(\tau) \right]}{\tau - t} d\tau \right], \]

\[ \tilde{\Psi} = \overline{P}_n[\Psi], \quad \tilde{y}_n = \overline{P}_n[y]. \]

The functions \( \Psi, l, y \) are introduced in the section 9.1.1.

First of all, we justify the method of collocation for the equation (9.37). The collocation method for this equation can be written as

\[ \tilde{F}_{3,n}\tilde{v}_n \equiv \overline{P}_n[\Psi \tilde{v}_n + W\tilde{v}_n] = \tilde{y}_n, \quad \tilde{F}_{3,n} \in [\tilde{X}_n \to \tilde{X}_n]. \quad (9.38) \]

To justify the collocation method we introduce the polynomial

\[ \tilde{\varphi}_n(t) = \Psi_n\tilde{v}_n + \frac{1}{2\pi i} \int L \frac{[\tilde{h}(t, \tau) - \tilde{h}(t, t)] \tilde{v}_n(\tau)}{\tau - t} d\tau, \]

where

\[ \tilde{h}(t, \tau) = T_{[n/2]}^t T_{[n/2]}^\tau [h(t, \tau)l(t)]. \]

Symbols \( \Psi_n, T, [n] \) was introduced in the section 9.1.1.

It is easy to see that

\[ |l(t)h(t, \tau) - \tilde{h}(t, \tau)| \leq A \ln n [E_n^t[h(t, \tau)l(t)] + E_n^\tau[h(t, \tau)l(t)]], \]

where \( E_n[f] \) is the the best approximation of \( f \) by trigonometric polynomials of degree at most \( n \).
Hence it follows that \( \{ l(t)h(t, \tau) - \tilde{h}(t, \tau) \} \) belongs to the class Holder with exponent \( 1 / \ln n \) and coefficient \( A \ln n [E_n^t(h(t, \tau)l(t)) + E_n^\tau(h(t, \tau)l(t))] \).

Then \( \| F_2 \tilde{v} - \varphi \| \leq A \ln^2 n[E_n(\psi) + E_n^t(hl) + E_n^\tau(hl)] \).

From the previous arguments follow that a similar estimate is valid for \( \| \overline{F}_n F_2 \tilde{v}_n - \tilde{\varphi}_n \| \). Therefore the operator \( \tilde{F}_{3,n} \) has a linear inverse with the norm \( \| \tilde{F}_{3,n}^{-1} \| \leq \| F_{2,n}^{-1} \| / (1 - q_1) \), when \( n \) such that \( q_1 = A_4 \ln^2 n [E_n(\psi) + E_n^t(hl) + E_n^\tau(hl)] < 1 \). Using the equality (9.31) we obtain the estimate \( \| \tilde{F}_{3,n} - \tilde{F}_{2,n} \| \leq A_5 \ln^2 n [E_n^t(hl) + E_n^\tau(hl)] \). From this estimate and Banach theorem implies the following theorem.

**Theorem 9.2** [42], [44]. Let the boundary value problem (9.3), (9.2) has a unique solution for any right-hand side. Then for such \( n \) that \( q = An^{-\alpha} \ln^2 n < 1 \), the system of equations (9.30) has a unique solution \( \tilde{x}_n^* \) and the estimate \( \| x_n^* - \tilde{x}_n^* \| \leq A[q + \ln^2 n E_n(f)] \) is valid, where \( x_n^* \) is a unique solution of the boundary value problem problem (9.3), (9.2).

**9.3. Approximate Solution of Nonlinear Singular Integro-differential Equations on Closed Contours of Integration**

In the items 9.1.1 and 9.1.2 we investigated the numerical methods for solution of linear singular integro-differential equations on closed contours of integration.

Now we examine the application of these methods to nonlinear singular integro-differential equations of the form

\[
Kx \equiv a(t, x(t), \ldots, x^{(m)}(t)) + S_\gamma(h(t, \tau, x(\tau), \ldots, x^{(m)}(\tau))) = f(t)
\]

under conditions

\[
\int_\gamma x(t)t^{-k-1}dt = 0, \ k = 0, 1, \ldots, m - 1.
\]
Here $\gamma$ is the unit circle centered at the origin, 

$$S_\gamma(x(\tau)) = \frac{1}{\pi i} \int_{\gamma} \frac{x(\tau)}{\tau - t} d\tau,$$

$$a'_u(t, u_0, u_1, \ldots, u_m) \in H_{\alpha, \ldots, \alpha},$$

$$h'_u(t, \tau, u_0, u_1, \ldots, u_m) \in H_{\alpha, \ldots, \alpha}; f(t) \in H_{\alpha}.$$ (9.41)

**Computational scheme.** An approximate solution of the boundary value problem (9.39) (9.40) is sought in the form of the polynomial

$$\tilde{x}_n(t) = \sum_{k=0}^{n} \alpha_k t^{k+m} + \sum_{k=-n}^{-1} \alpha_k t^k,$$ (9.42)

the coefficients of which are determined from the system of nonlinear algebraic equations

$$\tilde{K}_n \tilde{x}_n \equiv$$

$$\equiv \tilde{P}_n t[a(t, \tilde{x}_n(t), \ldots, \tilde{x}_n^{(m)}(t)) + S_\gamma(P_n^\tau[h(t, \tau, \tilde{x}_n(\tau), \ldots, \tilde{x}_n^{(m)}(\tau))])] =$$

$$= \tilde{P}_n t[f(t)],$$ (9.43)

where $P_n(\tilde{P}_n)$ is the projector of interpolation onto the set of $n$ order trigonometric polynomials on knots

$$s_k = 2k\pi/(2n + 1) \ (\tilde{s}_k = (2k\pi + \pi)/(2n + 1)), \ k = 0, 1, \ldots, 2n.$$

**Justification of the method.** We transform the boundary value problem (9.39), (9.40) and approximating its equation (9.43) to the equivalent nonlinear equations.

Let us introduce the function $\Phi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{x(\tau)}{\tau - z} d\tau.$ Using the Sohotzky - Plemel formulas, we receive the equations

$$K x \equiv a(t, \Phi^+(t) - \Phi^-(t), \ldots, \Phi^{(m)+}(t) - \Phi^{(m)-}(t)) +$$

$$+ S_\gamma(h(t, \tau, \Phi^+(\tau) - \Phi^-(\tau), \ldots, \Phi^{(m)+}(\tau) - \Phi^{(m)-}(\tau))) =$$

127
\[ \tilde{K}_n \tilde{x}_n \equiv \tilde{P}_n^t[a(t, \tilde{\Phi}_n^+(t) - \tilde{\Phi}_n^-(t), \ldots, \tilde{\Phi}_n^{(m)+}(t) - \tilde{\Phi}_n^{(m)-}(t)] + \\
+ S_\gamma(P_n^\tau[h(t, \tau, \tilde{\Phi}_n^+(\tau) - \tilde{\Phi}_n^-(\tau), \ldots, \tilde{\Phi}_n^{(m)+}(\tau) - \tilde{\Phi}_n^{(m)-}(\tau)])] = \\
= \tilde{P}_n^t[f(t)], \quad (9.45) \]

where \( \tilde{\Phi}_n(z) = \frac{1}{2\pi i} \int_{\gamma} \tilde{x}_n(\tau) \frac{d\tau}{\tau - z} \).

Using the integral representation of Y. M. Krikunov, the equation (9.44) under the conditions (9.40) and the equation (9.45) can be transformed to nonlinear singular integral equations

\[
K_1 x \equiv a(t, \eta_0(v(t)), \ldots, \eta_m(v(t))) + \\
+ S_\gamma(h(t, \tau, \eta_0(v(\tau)), \ldots, \eta_m(v(\tau)))) = f_1(t), \quad (9.46)
\]

\[
\tilde{K}_{1,n} \tilde{x}_n \equiv \tilde{P}_n^t[a(t, \eta_0(\tilde{v}_n(t))), \ldots, \eta_m(\tilde{v}_n(t)))] + \\
+ S_\gamma(P_n^\tau[h(t, \tau, \eta_0(\tilde{v}_n(\tau)), \ldots, \eta_m(\tilde{v}_n(\tau)))] = \tilde{P}_n[f_{1,n}(t)], \quad (9.47)
\]

where

\[
\eta_0(v(t)) = \frac{1}{2\pi i} \int_{\gamma} k_0(t, \tau)v(\tau)d\tau, \\
\eta_{m-1}(v(t)) = \frac{1}{2\pi i} \int_{\gamma} k_{m-1}(t, \tau)v(\tau)d\tau, \\
\eta_m(v(t)) = \frac{1}{2} v(t)(1 + t^{-m}) + S_\gamma(v(\tau) - t^{-m}S_\gamma(v(\tau))),
\]

\( k_0(t, \tau), \ldots, k_m(t, \tau) \) are Fredholm kernels (explicit form of these functions discharged in \[97\]), \( v \) is the integral density of Y.M. Krikunov presentation, \( \tilde{v}_n(t) = \sum_{k=-n}^n \beta_k t^k \), \( \beta_k \) are constants, uniquely expressed in terms of \( \alpha_k \).

Since \( \eta_i(v) \) \((i = 0, 1, \ldots, m)\) are linear operators, then for simplicity of calculations, instead of the equations (9.46) and (9.47), we can restrict ourself with equations

\[
K_2(v) \equiv a(t, v(t)) + S_\gamma(h(t, \tau, v(\tau))) = f(t) \quad (9.48)
\]
and 
\[ \tilde{K}_{2,n}(\tilde{v}) \equiv \tilde{P}_{n}[a(t, \tilde{v}_n(t)) + S_{\gamma}(P_{n}^r[h(t, \tau, \tilde{v}_n(\tau))])] = \]
\[ = \tilde{P}_{n}[f(t)], \quad (9.49) \]

where

\[ a'_u(t, u) \in H_{\alpha, \alpha}, \quad h'_u(t, \tau, u) \in H_{\alpha, \ldots, \alpha}; \quad f(t) \in H_{\alpha} \quad (0 < \alpha < 1). \]

We can choose the space, in which we will justificate the collocation method (9.49) for the equation (9.48). At first, we will work in the space \( L_2 \) with the scalar product

\[ (v_1, v_2) = \frac{1}{2\pi} \int_0^{2\pi} v_1(t)v_2(t)dt, \quad t = e^{is}. \]

The approximate solution is sought in the subspace \( \tilde{L}_{2,n} \subset L_2 \), consisting of \( n \) order polynomials. In these conditions, method collocation (9.49) for the equation (9.48) was justified in the item 3. Using the results given there, we receive the following statement.

**Theorem 9.2** [42], [46]. Suppose that in a ball \( S \) boundary problem (9.39) (9.40) has a unique solution \( x^* \in W^r H_{\alpha}^{(m)} \) and the equation \( K'(x^*)z = f \) for arbitrary \( f \) has a unique solution that satisfies (9.40). Then, for such \( n \) that \( q = A \ln^2 n/n^{\alpha} < 1 \), the system of equations (9.43) has a unique solution \( \tilde{x}^*_n \), and the estimation

\[ \| (x^*)^{(m)} - (\tilde{x}^*_n)^{(m)} \| \leq A \ln n/n^\alpha \]

is valid.

Let us now take as the space, in which the collocation method (9.49) for the equation (9.48) will be studied, the space \( X = H_{\beta} \) \((0 < \beta < \alpha)\) with the norm

\[ \| x \| = \max |x(t)| + \sup_{t_2 \neq t_1} |x(t_1) - x(t_2)|/|t_1 - t_2|^\beta. \]
Approximate solution of the equation (9.48) we seek in the space $\tilde{X}_n \subset X$ consisting of $n$ order polynomials. The justification of collocation method for the equation (9.48) is given in the paragraph 2.

Using the results given there, we receive the following statement.

**Theorem 9.3** [46] Suppose that in a ball $S$ the boundary problem (9.39) (9.40) has a unique solution $x^* \in H^{(m)}_{\alpha}$, and the equation $K'(x^*)z = f$ with an arbitrary right side has a unique solution satisfying the conditions (9.40). Then, for $n$ such that $q = A \ln^6 n/n^\alpha < 1$, the system of equations (9.43) has a unique solution $\tilde{x}^*_n(t)$, and the estimate

$$\|(x^*)^{(m)} - (\tilde{x}^*_n)^{(m)}\| \leq A \ln^2 n/n^\alpha$$

is valid.

### 9.4. Approximate Solution of Linear Singular Integro-Differential Equations with Discontinuous Coefficients and on Open Contours of Integration

In this and the following items are studied directly (without regularization) methods for approximate solution of singular integro-differential equations with discontinuous coefficients and on open contours of integration. For the first time, approximate methods for solving singular integral equations with discontinuous coefficients have been studied in monograph [82], in which an efficient algorithm for transformation singular integral equations with discontinuous coefficients to equivalent singular integral equations with continuous coefficients was proposed.

Let us consider the singular integro-differential equation

$$Kx \equiv \sum_{k=0}^{m} \left[ a_k(t)x^{(k)}(t) + b_k(t)S_\gamma(x^{(k)}(\tau)) + U_\gamma \left( \frac{h_k(t, \tau)}{|\tau - t|^{\eta}} x^{(k)}(\tau) \right) \right] = f(t)$$

(9.50)
under conditions
\[ \int_\gamma x(t) t^{-k-1} dt = 0, \quad k = 0, 1, \ldots, m - 1, \quad (9.51) \]

where \( a_k(t), b_k(t), h_k(t, \tau) \ (k = 0, m), f(t) \) are functions, which are continuous in the metric \( C \) everywhere on \( \gamma \), except of the point \( t = 1 \), in which \( a_k(t) \) and \( b_k(t) \) \((k = 0, m)\) have discontinuity of the first kind.

An approximate solution of the boundary value problem (9.50), (9.51) is sought in the form of polynomial (9.42), whose coefficients determined from the system of algebraic equations

\[
\tilde{K}_n \tilde{x}_n \equiv P_n \left[ \sum_{k=0}^{m} \left[ a_k(t) \tilde{x}_n^{(k)}(t) + b_k(t) S_\gamma(\tilde{x}_n^{(k)}(\tau)) + U_\gamma(P_n^\tau[h_k(t, \tau)d(t, \tau)\tilde{x}_n^{(k)}(\tau)]) \right] \right] = P_n[f(t)], \quad (9.52)
\]

where \( d(t, \tau) = |\tau - t|^{-\eta} \) for \(|\sigma - s| \geq 2\pi/(2n + 1)\), \( d(t, \tau) = |e^{i2\pi/(2n+1)} - 1|^{-\eta} \) for \(|\sigma - s| \leq 2\pi/(2n + 1)\), \( t = e^{is}, \tau = e^{i\sigma} \).

Just as in the case of singular integro-differential equations with continuous coefficients, boundary value problem (9.50), (9.51) and the system (9.52) are reduced to equivalent singular integral equation and approximating its algebraic system. They are written in the item 9.1 (formulas (9.6) and (9.7)). Instead of considering these equations, we confine ourselves to equations

\[ Kx \equiv a(t)x(t) + b(t)S_\gamma(x(\tau)) + U_\gamma \left( \frac{h(t, \tau)}{|\tau - t|^\eta}x(\tau) \right) = f(t) \quad (9.53) \]

and

\[
\tilde{K}_n \tilde{x}_n \equiv P_n[a(t)\tilde{x}_n(t) + b(t)S_\gamma(\tilde{x}_n(\tau)) + U_\gamma(P_n^\tau[h(t, \tau)d(t, \tau)\tilde{x}_n(\tau)])] = P_n[f(t)], \quad (9.54)
\]
where $a(t)$ and $b(t)$ have discontinuity of the first kind in the point $t = 1$.

Justification of the computational scheme (9.54) will be carried out in the space $L_2$.

Equations (9.53) and (9.54) are transformed to the equivalent Riemann boundary value problems

$$K_1 x \equiv x^+(t) + G(t)x^-(t) + D(t)U_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau)) =$$

$$= D(t)f(t), \quad (9.55)$$

$$\tilde{K}_{1,n}\tilde{x}_n \equiv$$

$$\equiv P_n[x^+_n(t) + G(t)x^-_n(t) + D(t)U_\gamma(P_n^\tau[h(t, \tau)d(t, \tau)x_n(\tau)])] =$$

$$= P_n[D(t)f(t)], \quad (9.56)$$

where $G(t) = S(t)D(t)$, $S(t) = a(t) - b(t)$, $D(t) = (a(t) + b(t))^{-1}$.

Let a solution of the Riemann boundary value problem $\varphi^+(t) = G(t)\varphi^-(t)$ is

$$\varphi(t) = (t - 1)^\delta\varphi_0(t), \text{ where } \delta = \zeta + i\xi, \ \varphi_0 \in H_\alpha, \ 0 < \alpha < 1.$$  

As in the case of singular integral equations with continuous coefficients, equations (9.55) and (9.56) can be represented in the following equivalent forms:

$$K_2 x \equiv Vx + Wx = y, \quad (9.57)$$

$$\tilde{K}_{2,n}\tilde{x}_n \equiv \tilde{V}_n\tilde{x}_n + \tilde{W}_n\tilde{x}_n = \tilde{y}_n, \quad (9.58)$$

where $Vx = \psi^-x^+ - \psi^+x^-$, $Wx = lU_\gamma(h(t, \tau)|\tau - t|^{-\eta}x(\tau))$, $l = \psi^-/(a + b)$, $y = lf$, $\tilde{y} = P[y]$, $\tilde{V}_n\tilde{x}_n = P_n[V\tilde{x}_n]$, $\tilde{W}_n\tilde{x}_n = P_n[lU_\gamma(P_n^\tau[h(t, \tau)d(t, \tau)x_n(\tau)])],$

$$\psi(z) = \begin{cases} (z - 1)\exp\left\{\frac{1}{2\pi i}\int_{\gamma}\frac{\ln|S(\tau)D(\tau)|}{\tau - z}d\tau\right\} \text{ for } \zeta \leq 0, \\ \exp\left\{\frac{1}{2\pi i}\int_{\gamma}\frac{\ln|S(\tau)D(\tau)|}{\tau - z}d\tau\right\} \text{ for } \zeta > 0. \end{cases}$$
To justify the proposed computational scheme we introduce a polynomial
\[ \tilde{\varphi}_n(t) = V_n \tilde{x}_n + U_\gamma \left( T_t^{(n)}[l(t)h(t, \tau)d^*(t, \tau)]\tilde{x}_n(\tau) \right), \]
where \( V_n \tilde{x}_n = \psi_n^- \tilde{x}^+ - \psi_n^+ \tilde{x}^-, \psi_n = T^{(n)}(\psi), T_t^{(n)} \) is projector of interpolation with respect to variable \( t \) onto the set of the \( n \) order trigonometric polynomials of the best uniform approximation; \( d^*(t, \tau) = |\tau - t|^{-\eta} \) for \( |\tau - t| \geq \rho \), \( d^*(t, \tau) = \rho^{-\eta} \) for \( |\tau - t| \leq \rho \), a constant \( \rho \) is fixed below. It can be shown that
\[ \| K_{2,n} \tilde{x}_n - \tilde{\varphi}_n \| \leq A[w(a; n^{-1}) + w(b; n^{-1}) + w(h; n^{-1})\rho^{1-\eta} + \rho^{-2\eta}n^{-\eta} + n^\theta]\| \tilde{x}_n \|, \quad (9.59) \]
\[ \| PK_{2,n} \tilde{x}_n - \tilde{\varphi}_n \| \leq A[w(a; n^{-1}) + w(b; n^{-1}) + w(h; n^{-1})\rho^{1-\eta} + \rho^{-2\eta}n^{-\eta} + n^\theta]\| \tilde{x}_n \|, \quad (9.60) \]

where \( \theta = -(1 - |\zeta|) \) for \( \zeta \leq 0 \) and \( \theta = -\zeta \) for \( \zeta > 0 \).

Let us estimate \( \| PK_{2,n} \tilde{x}_n - \tilde{K}_{2,n} \tilde{x}_n \| \). Repeating the arguments, given in the paragraph 3, we obtain the estimate
\[ \| PK_{2,n} \tilde{x}_n - \tilde{K}_{2,n} \tilde{x}_n \| \leq A[w(h; n^{-1}) + \rho^{-2\eta}n^{-\eta}]\| \tilde{x}_n \|. \quad (9.61) \]

Let \( \rho = n^{-\eta/(1+\eta)} \). Collecting the estimates (9.59) - (9.61), we receive the following statement.

\textbf{Уклоняясь 9.4 [46].} Let the operator \( K \) has the linear inverse operator \( K^{-1} \) in the space \( L_2 \); functions \( a(t), b(t), h(t, \tau), f(t) \in C \) everywhere, except the point \( t = 1 \), where the functions \( a(t), b(t) \) have a first-order gap. Then, for \( n \) such that
\[ q = A[w(a; n^{-1}) + w(b; n^{-1}) + w(h; n^{-1}) + n^\theta + n^{-\eta(1-\eta)/(1+\eta)}] < 1, \]
the system of equation (9.54) has a unique solution \( \tilde{x}^* \). The estimate \( \| x^* - \tilde{x}^* \| \leq A[q + w(f; n^{-1})] \), where \( x^* \) is a solution of the equation (9.53), is valid. Here \( \theta = -(1 - |\zeta|) \) for \( \zeta \leq 0 \),

\[ ^2 \text{modulo continuity of functions } a(e^{ix}) \text{ and } b(e^{ix}) \text{ are determined in open interval } 0 < s < 2\pi. \]
\( \theta = -\zeta \) for \( \zeta > 0 \), \( \delta = \zeta + i\xi \), \( (t-1)\delta \varphi_0(t) \) is a solution of Riemann boundary task \( \psi^+(t) = [(a(t) - b(t))/(a(t) + b(t))]\psi^-(t) \).

A corollary of the Theorem 9.4 is the following statement.

**Theorem 9.4'.** Let the value problem (9.50), (9.51) has a unique solution for any right-hand side, and conditions \( a_k(t), b_k(t), h_k(t, \tau) \) \( (k = 0, m) \), \( f(t) \in C \) are performed everywhere, except the point \( t = 1 \), in which the functions \( a_k(t) \) i \( b_k(t) \) have a first-order gap. Then for such \( n \) that

\[
q = A \max_{0 \leq k \leq m} \left[ w(a_k; \frac{1}{n}) + w(b_k; \frac{1}{n}) + w(h_k; \frac{1}{n}) + n^\theta + \frac{1}{n^{\eta(1-n)/(1+\eta)}} \right] < 1,
\]

the system of equations (9.52) has a unique solution \( \tilde{x}_n^* \) and the estimate \( ||x^* - \tilde{x}_n^*|| \leq A[q + w(f; n^{-1})] \) is valid. Here \( x^* \) is a solution of the boundary value problem (9.50), (9.51); \( \theta = -(1-|\zeta|) \) for \( \zeta \leq 0 \), \( \theta = -\zeta \) for \( \zeta > 0 \); \( \delta = \zeta + i\xi \), \( (t-1)\delta \varphi_0(t) \) is a solution of Riemann boundary value problem \( \Phi^+(t) = [(a_m(t) - b_m(t))/(a_m(t) + b_m(t))]\Phi^-(t) \).

Consider the singular integro-differential equation

\[ Fx \equiv \sum_{k=0}^{m} [a_k(t)x^{(k)}(t) + S_L(h_k(t, \tau)x^{(k)}(\tau))] = f(t), \quad t \in [c_1, c_2], \tag{9.62} \]

with conditions

\[ x(1) = x'(1) = \cdots = x^{(m-1)}(1) = 0, \tag{9.63} \]

where \( L \) is a segment of the counter \( \gamma \), wherein one end of segment \( L = (c_1, c_2) \) (say, \( c_1 \)) coincides with the point \( t = 1 \).

An approximate solution of the boundary value problem (9.62), (9.63) is sought in the form of the polynomial

\[ \tilde{x}_n(t) = (t - 1)^{m-1} \sum_{k=-n}^{n} \alpha_k t^k, \tag{9.64} \]

coefficients \( \{\alpha_k\} \) of which are determined from the system of
\[ \tilde{F}_n \tilde{x}_n \equiv \sum_{k=0}^{m} \left[ \bar{P}_n[\bar{a}_k(t)\tilde{x}_n^{(k)}(t) + S_\gamma(P_\gamma^* \bar{h}_k(t, \tau)\tilde{x}_n^{(k)}(\tau))] \right] = \bar{P}_n[\bar{f}(t)], \]  

(9.65)

where \( \bar{a}_k(t) = a_k(t) \) (\( k = 0, \ldots, m \)) for \( t \in L \), \( \bar{a}_k(t) = 0 \) (\( k = 0, \ldots, m-1 \)), \( \bar{a}_m(t) = 1 \) for \( t \notin L \), \( \bar{h}_k(t, \tau) = h_k(t, \tau) \) for \( t \in L \) and \( \tau \in L \), \( \bar{h}_k(t, \tau) = 0 \) for \( t \notin L \) and \( \tau \in \gamma \) or \( t \in \gamma \) and \( \tau \notin L \) (\( k = 0, \ldots, m \)), \( \bar{f}(t) = f(t) \) for \( t \in L \), \( \bar{f}(t) = 0 \) for \( t \notin L \).

Equation (9.62) can be written in the following equivalent forms:

\[ Fx \equiv \]
\[ \equiv \sum_{k=0}^{m} \left[ a_k(t)x^{(k)}(t) + b_k(t)S_L(x^{(k)}(\tau)) + S_L((h_k(t, \tau) - b_k(t))x^{(k)}(\tau)) \right] = \]
\[ = f(t), \quad t \in L, \]
\[ Fx \equiv \]
\[ \equiv \sum_{k=0}^{m} \left[ \bar{a}_k(t)x^{(k)}(t) + \bar{b}_k(t)S_\gamma(x^{(k)}(\tau)) + U_\gamma(g_k(t, \tau)x^{(k)}(\tau)) \right] = \]
\[ + \bar{f}(t), \quad t \in \gamma, \]  

(9.66)

where \( b_k(t) = h_k(t, t) \), \( g_k(t, \tau) = (h_k(t, \tau) - h_k(t, t))/(\tau - t) \) for \( t \in L \) and \( \tau \in L \), \( g_k(t, \tau) = 0 \) for \( t \in L \) and \( \tau \in \gamma \) or \( t \in \gamma \) and \( \tau \notin L \).

Taking advantage of the following identity

\[ \bar{P}_n \left[ \int_\gamma P_\gamma^*[h(s, \sigma)\tilde{\phi}(\sigma) \cot \frac{\sigma - s}{2}] \, d\sigma \right] = \]
\[ = \bar{P}_n \left[ \int_\gamma P_\gamma^*[h(s, \sigma)]\tilde{\phi}(\sigma) \cot \frac{\sigma - s}{2} \, d\sigma \right], \]  

135
true if $\tilde{\phi}(\sigma)$ is a polynomial of degree not exceeding $n$, the system (9.66) reduces to

$$
\tilde{F}_n \tilde{x}_n \equiv P_n \left[ \sum_{k=0}^{m} [\tilde{a}_k(t)\tilde{x}_n^{(k)}(t) + \tilde{h}_k(t, t) S_\gamma(\tilde{x}_n^{(k)}(\tau))] + U_\gamma(P_n^\tau [g_k(t, \tau)\tilde{x}_n^{(k)}(\tau)])] \right] = \tilde{P}_n[\tilde{f}(t)]. \quad (9.67)
$$

The relationship between the equations (9.66) and (9.67) is studied above and formulated in the Theorem 9.4. Using this theorem and the results of the item 9.1, we receive the following statement.

**Theorem 9.5** [46]. Let the problem (9.62), (9.63) has a unique solution for any right-hand side, and the conditions $h_k(t, \tau) \in H_{\alpha,\alpha}, k = 0, m$, for $t$ and $\tau \in L$; $a_k(t) \in H_{\alpha}, f(t) \in H_{\alpha}$, for $t \in L$, are implemented. Then for $n$ such that $q = A \ln^2 n (n^{-\alpha} + n^{-\theta})$, the system of equations (9.65) has a unique solution $\tilde{x}_n^*$, and the estimate $\|x^{(m)} - \tilde{x}_n^{(m)}\| \leq Aq$ is valid. Here $x^*$ is a solution of the boundary value task (9.62) (9.63); $\theta = \min\{\theta_1, \theta_2\}$, $\theta_1 = 1 - |\zeta_1|$ for $\zeta_1 \leq 0$, $\theta_1 = \zeta_1$ for $\zeta_1 > 0$, $\theta_2 = 1 - |\zeta_2|$ for $\zeta_2 \leq 0$, $\theta_2 = \zeta_2$ for $\zeta_2 > 0$; $(t - c_1)^{\delta_1}(t - c_2)^{\delta_2} \varphi_0$ ($\varphi_0 \in H$) is a solution of Riemann boundary value task $\Phi^+(t) = [(a(t) - b(t))/(a(t) + b(t))]\Phi^-(t)$, $\delta_1 = \zeta_1 + i\xi_1$, $\delta_2 = \zeta_2 + i\xi_2$.

**9.5. Approximate Solution of Nonlinear Singular Integro-Differential Equations on the Open Contour of Integration**

Consider the nonlinear singular integro-differential equation

$$
Gx \equiv a(t, x(t), \ldots, x^{(m)}(t)) + S_L(h(t, \tau, x(\tau), \ldots, x^{(m)}(\tau))) = f(t) \quad (9.68)
$$

with conditions

$$
x(1) = x'(1) = \cdots = x^{(m-1)}(1) = 0. \quad (9.69)
$$
An approximate solution of the boundary value problem (9.68), (9.69) is sought in the form of the polynomial (9.64), the coefficients \( \{\alpha_k\} \) of which are determined from the system of nonlinear algebraic equations

\[
\tilde{G}_n \tilde{x}_n \equiv \\
\equiv \tilde{P}_n[\tilde{a}(t, \tilde{x}_n(t), \ldots, \tilde{x}_n^{(m)}(t)) + \tilde{S}_\gamma(P_n^\tau[\tilde{h}(t, \tau, \tilde{x}_n(\tau), \ldots, \tilde{x}_n^{(m)}(\tau))])] = \\
= \tilde{P}_n[\tilde{f}(t)], 
\]

(9.70)

where

\[
\tilde{a}(t, \tilde{x}_n(t), \ldots, \tilde{x}_n^{(m)}(t)) = a(t, \tilde{x}_n(t), \ldots, \tilde{x}_n^{(m)}(t)) \\
\text{for } t \in L,
\]

\[
\tilde{a}(t, \tilde{x}_n(t), \ldots, \tilde{x}_n^{(m)}(t)) = \tilde{x}_n^{(m)}(t) \\
\text{for } t \notin L,
\]

\[
\tilde{h}(t, \tau, \tilde{x}_n(\tau), \ldots, \tilde{x}_n^{(m)}(\tau)) = h(t, \tau, \tilde{x}_n(\tau), \ldots, \tilde{x}_n^{(m)}(\tau)) \\
\text{for } t \in L \text{ and } \tau \in L,
\]

\[
\tilde{h}(t, \tau, \tilde{x}(\tau), \ldots, \tilde{x}^{(m)}(\tau)) = 0 \\
\text{for } t \notin L, \tau \in \gamma \text{ or } t \in \gamma, \tau \notin L,
\]

\[
\tilde{f}(t) = f(t) \\
\text{for } t \in L,
\]

\[
\tilde{f}(t) = 0 \\
\text{for } t \notin L.
\]

Justification of the computational scheme (9.64) (9.70) for approximate solution of the boundary value problem (9.68), (9.69) is conducted considering the results of the preceding item, as well as the results of the item 9.1.

**Theorem 9.6** [43], [46]. Let the boundary value problem (9.68), (9.69) has a unique solution \( x^* \) in a ball \( S = B(x^*, r) \) \( r > 0; \) the conditions \( h_{u_i}'(t, \tau, u_0, u_1, \ldots, u_m) \in H_{\alpha, \ldots, \alpha} \) (i =
$a'_{u_i}(t, u_0, u_1, \ldots, u_m) \in H_{\alpha, \ldots, \alpha}$, $f(t) \in H_\alpha$ are fulfilled on $L$, and equation $G'(x^*) z = f$ has a unique solution, that satisfies (9.69). Then, for such $n$ that $q = A \ln^5 n(n^{-\alpha} + n^{-\theta}) < 1$, the system of equations (9.70) has in a ball $S_1 \subset S$ a unique solution $\tilde{x}_n^*$ and the estimate

$$
\|x^{(m)} - \tilde{x}_n^{(m)}\|_{L^2} \leq A \ln^2 n(n^{-\alpha} + n^{-\theta})
$$
is valid. Here $\theta = \min\{\theta_1, \theta_2\}$, $\theta_1 = 1 - |\zeta_1|$ for $\zeta_1 \leq 0$, $\theta_1 = \zeta_1$ for $\zeta_1 > 0$, $\theta_2 = 1 - |\zeta_2|$ for $\zeta_2 \leq 0$, $\theta_2 = \zeta_2$ for $\zeta_2 > 0$; $(t - c_1)^{\delta_1}(t - c_2)^{\delta_2}\varphi_0$ is a solution of the Riemann boundary value problem

$$
\Phi^+(t) = a'_{um}(t, x(t), \ldots, x^{(m)}(t)) - h'_{um}(t, t, x(t), \ldots, x^{(m)}(t)) \Phi^-(t),
$$

$\delta_1 = \zeta_1 + i\xi_1$, $\delta_2 = \zeta_2 + i\xi_2$.

**Note 1.** The above we have studied the computing schemes under the assumption that the solutions of the corresponding boundary value problems owned to $L_2$. If we assume that the solutions of boundary value problems belong to the class $L_p (1 < p \leq 2)$, then, after similar in nature, but more cumbersome calculations, we will see that the results, which are set forth above, are valid in the space $L_p (1 < p \leq 2)$.

**Note 2.** Justification of applicability of the method Newton-Kantorovich to the approximate solution of nonlinear algebraic systems, approximating non-linear singular integral-differential equations, carried out in the same manner as in the case of non-linear singular integral equations.

**Chapter 2**

**Approximate Solution of Multi-Dimensional Singular Integral Equations**

The chapter consists of the three sections. In the first section we will consider polysingular integral equations. In the second
section we will consider multi-dimensional Riemann boundary value problems. The third section is devoted to numerical methods for solution of multi-dimensional singular integral equations with Zygmund-Calderon type of kernels. Also in this section we will investigate parallel iterative-projection methods for solution of multi-dimensional singular integral equations.

The main attention we will spare to collocation method, because the proofs of convergence for moment and the Galerkin methods are similar to collocation method.

There are three principle different methods for proof of the convergence of projector methods for polysingular integral equations. Historical the first method is based on transform the polysingular integral equations into multi-dimensional Riemann boundary value problems. The second method is based on Simonenko local principle [125], [53]. The third method is based on the theory of commutative rings. The main results of the second direction and the rich bibliography are given in [114]. In the introduction to the work it was noticed, that we will not touch two last directions.

1. Bisingular integral equations

Let us consider bisingular integral equations of the following kind

\[ Kx \equiv a(t_1, t_2)x(t_1, t_2) + b(t_1, t_2)S_1(x(\tau_1, t_2)) + \\
+ c(t_1, t_2)S_2(x(t_1, \tau_2)) + d(t_1, t_2)S_{12}(x(\tau_1, \tau_2)) + \\
+ U_{12}[h(t_1, t_2, \tau_1, \tau_2)x(\tau_1, \tau_2)] = f(t_1, t_2), \tag{1.1} \]

where

\[ S_1(x(\tau_1, t_2)) = \frac{1}{\pi i} \int_{\gamma_1} \frac{x(\tau_1, t_2)}{\tau_1 - t_1} d\tau_1, \]

\[ S_2(x(t_1, \tau_2)) = \frac{1}{\pi i} \int_{\gamma_2} \frac{x(t_1, \tau_2)}{\tau_2 - t_2} d\tau_2, \]
\[ S_{12}(x(\tau_1, \tau_2)) = -\frac{1}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \frac{x(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)} d\tau_1 d\tau_2, \]

\[ U_{12}[h(t_1, t_2, \tau_1, \tau_2)x(\tau_1, \tau_2)] = \int_{\gamma_1} \int_{\gamma_2} h(t_1, t_2, \tau_1, \tau_2)x(\tau_1, \tau_2)d\tau_1 d\tau_2, \]

\[ \gamma_i = \{ z_i : |z_i| = 1, \ i = 1, 2 \}, \ \gamma_{12} = \gamma_1 \times \gamma_2. \]

The equation (1.1) we will consider under conditions \( a^2(t_1, t_2) - b^2(t_1, t_2) \neq 0. \)

Numerical methods for solution of the equation (1.1) was devoted many works.

Convergence of one common projective method was proved by A.V. Kozak and I.B. Simonenko [94].

In cases when \( b = c = 0 \) or \( a(t_1, t_2) = a(t_1)a(t_2), \ b(t_1, t_2) = b(t_1)b(t_2), \ c(t_1, t_2) = c(t_1)c(t_2), \ d(t_1, t_2) = d(t_1)d(t_2) \) convergence of the method of mechanical quadrature was proved by I.V. Boykov [23], [26], [35].

I.V. Boykov proved convergence of projector method in common case when functions \( a(t_1, t_2), \ b(t_1, t_2), \ c(t_1, t_2), \ d(t_1, t_2) \) are belong to Holder class and factorable.

I.K. Lifanov [100] used the discrete vortex method for solution of the equation (1.1) under conditions \( a(t_1, t_2) \equiv b(t_1, t_2) \equiv c(t_1, t_2) \equiv h(t_1, t_2, \tau_1, \tau_2) \equiv 0. \)

At first we consider the equation

\[ Kx \equiv a(t_1, t_2)x(t_1, t_2) + d(t_1, t_2)S_{12}(x(\tau_1, \tau_2)) = f(t_1, t_2). \ (1.2) \]

Let us introduce the function

\[ X(z_1, z_2) = -\frac{1}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \frac{x(\tau_1, \tau_2)}{(\tau_1 - z_1)(\tau_2 - z_2)} d\tau_1 d\tau_2. \]

Let \( D_1^+(D_1^-), \ i = 1, 2, \) be the set of points \( z_i, \ i = 1, 2, \) on the complex plane \( z_i, \ i = 1, 2, \) which satisfy the following conditions: \( |z_i| < 1 \ (|z_i| > 1), \ i = 1, 2. \) Let \( D^{++} = D_1^+D_2^+, \ D^{+-} = D_1^+D_2^-, \)
\[ D^{++} = D_1^- D_2^+, \; D^{--} = D_1^- D_2^-. \] If a function \( f(z_1, z_2) \) is analytical in the domain \( D^{++} \), we write \( f^{++}(z_1, z_2) \). If the function \( f(z_1, z_2) \) is analytical in the domain \( D^{++}(D^{--} \text{ or } D^{--}) \), we write \( f^{+-}(z_1, z_2) (f^{-+}(z_1, z_2) \text{ or } f^{--}(z_1, z_2)) \).

Let \((z_1, z_2) \in D^{++} \). Let \( t_i \in \gamma_i, i = 1, 2 \). We will write \( f^{++}(t_1, t_2) = \lim_{(z_1, z_2) \to (t_1, t_2)} f^{++}(z_1, z_2) \). Limiting values \( f^{+-}(t_1, t_2), \; f^{-+}(t_1, t_2), \; f^{--}(t_1, t_2) \) of the functions \( f^{+-}(z_1, z_2), \; f^{-+}(z_1, z_2), \; f^{--}(z_1, z_2) \) are defined by the similar way.

We need in Sohotsky - Plemel formulas \[ 64 \]

\[ X^{++}(t_1, t_2) + X^{+-}(t_1, t_2) + X^{-+}(t_1, t_2) + X^{--}(t_1, t_2) = S_{12}x, \]
\[ X^{++}(t_1, t_2) - X^{+-}(t_1, t_2) - X^{-+}(t_1, t_2) + X^{--}(t_1, t_2) = x. \]

Using Sohotsky - Plemel formulas (1.3), we can write the equation (1.2) as

\[
(a(t_1,t_2)+b(t_1,t_2))X^{++}(t_1,t_2)-(a(t_1,t_2)-b(t_1,t_2))X^{-+}(t_1,t_2)-
-(a(t_1,t_2)-b(t_1,t_2))X^{+-}(t_1,t_2)+(a(t_1,t_2)+b(t_1,t_2))X^{--}(t_1,t_2) =
= f(t_1,t_2).
\]

Dividing the equation (1.4) on \( a + b \), we receive the equation

\[
X^{++}(t_1,t_2) - G(t_1,t_2)(X^{+-}(t_1,t_2) + X^{-+}(t_1,t_2)) + X^{--}(t_1,t_2) =
= f_1(t_1,t_2),
\]

where \( G(t_1,t_2) = ((a(t_1,t_2) - d(t_1,t_2))/(a(t_1,t_2) + d(t_1,t_2)), \)
\( f_1(t_1,t_2) = f(t_1,t_2))/(a(t_1,t_2) + d(t_1,t_2)) \).

Let \( a(t_1,t_2) \pm b(t_1,t_2) \) be not equal to zero, \( a(t_1,t_2),b(t_1,t_2) \in H_{\alpha\alpha}(0 < \alpha < 1) \). Let the partial indexes \[ 88 \] of the operator \( K \) are equal to zero.

Well known \[ 88 \] the representation

\[
G(t_1,t_2) = \frac{\psi^{++}(t_1,t_2)\psi^{--}(t_1,t_2)}{\psi^{+-}(t_1,t_2)\psi^{-+}(t_1,t_2)},
\]

141
where
\[ \psi^\pm(t_1, t_2) = \exp\{P^{\pm} \ln G(t_1, t_2)\}, \]
\[ P^{\pm} = P^\pm P^\pm, P^\pm = \frac{1}{2} (I \pm S), Sf = \frac{1}{\pi i} \int_\gamma \frac{f(\tau)d\tau}{\tau - t}, \]
\[ \gamma = \{ z : |z| = 1 \}. \]

An approximate solution of the equation (1.5) we will look for in the form
\[
\begin{align*}
X_{n}^{++}(t_1, t_2) &= \psi^{++}(t_1, t_2) \sum_{k=0}^{n} \sum_{l=0}^{n} \alpha_{kl} t_1^k t_2^l, \\
X_{n}^{-+}(t_1, t_2) &= \psi^{-+}(t_1, t_2) \sum_{k=0}^{n} \sum_{l=0}^{-1} \alpha_{kl} t_1^k t_2^l, \\
X_{n}^{--}(t_1, t_2) &= \psi^{--}(t_1, t_2) \sum_{k=-1}^{n} \sum_{l=-1}^{-1} \alpha_{kl} t_1^k t_2^l, \\
X_{n}^{+-}(t_1, t_2) &= \psi^{+-}(t_1, t_2) \sum_{k=-1}^{-1} \sum_{l=-1}^{n} \alpha_{kl} t_1^k t_2^l.
\end{align*}
\]

So, approximate solution of the equation (1.2) we will seek in the form
\[ X_n(t_1, t_2) = X_{n}^{++}(t_1, t_2) - X_{n}^{-+}(t_1, t_2) - X_{n}^{--}(t_1, t_2) + X_{n}^{+-}(t_1, t_2). \]

The coefficients \( d_{kl}, k, l = -n, \ldots, -1, 0, 1, \ldots, n, \) are defined from the system of linear algebraic equations
\[
K_n X_n \equiv P_{nn} [ X_{n}^{++}(t_1, t_2) - G(t_1, t_2)( X_{n}^{--}(t_1, t_2) + X_{n}^{+-}(t_1, t_2)) + X_{n}^{-+}(t_1, t_2) ] = P_{nn} [ f_1(t_1, t_2) ].
\]

The convergence of collocation method (1.7) we investigate in the Banach space \( E \) of functions \( f(t_1, t_2) \), which are belong to the Holder class \( H_{\alpha, \alpha}, 0 < \alpha < 1 \), with the norm
\[
\| f(t_1, t_2) \| = \\
= \max_{(t_1, t_2) \in \gamma_{12}} \| f(t_1, t_2) \| + \max_{t_2} \sup_{t_1' \neq t_1''} \frac{|f(t_1', t_2) - f(t_1'', t_2)|}{|t_1' - t_1''|^\beta} + \]
\[ + \max_{t_1} \sup_{t_2' \neq t_2''} \frac{|f(t_1, t_2') - f(t_1, t_2'')|}{|t_2' - t_2''|^\beta}, \]

\[ 0 < \beta < \alpha. \]

**Theorem 1.1** [23], [26], [35]. Let the following conditions are satisfied:
1) the operator \( K \) is continuously invertible in the space \( E \);
2) \( a \pm b \neq 0 \);
3) the partial indexes of the operator \( K \) are equal to zero;
4) \( a, b, f \in H_{\alpha, \alpha}, 0 < \alpha < 1 \).

Then, for \( n \) such that \( q = An^{\alpha+\beta} \ln^2 n < 1 \), the system (1.7) has a unique solution \( x^*(t_1, t_2) \) and the estimate \( \|x^* - x_n^*\|_E \leq An^{\alpha+\beta} \ln^2 n \) is valid, where \( x^*(t_1, t_2) \) is a unique solution of the equation (1.1).

**Theorem 1.2** [26], [35]. Let the following conditions are satisfied:
1) the operator \( K \) is continuously invertible in the space \( L_2(\gamma_{12}) \);
2) \( a \pm b \neq 0 \);
3) the partial indexes of the operator \( K \) is equal to zero;
4) \( a, b, f \in H_{\alpha, \alpha}, 0 < \alpha < 1 \).

Then, for \( n \) such that \( q = A \max(\omega(a; n^{-1}), n^{-\alpha} \ln n, \omega(b; n^{-1})) < 1 \), the system (1.7) has a unique solution \( x^*_n(t_1, t_2) \) and the estimate \( \|x^* - x^*_n\| \leq A \max(\omega(a; n^{-1}), n^{-\alpha} \ln n, \omega(b; n^{-1}), \omega(f; n^{-1})) \) is valid in the space \( L_2[\gamma_{12}] \). Here \( x^* \) is a unique solution of the equation (1.1).

The similar statements is valid for singular integral equations

\[ a(t_1, t_2)x(t_1, t_2) - \frac{1}{\pi^2} \int \int_{\gamma_1 \gamma_2} \frac{x(\tau_1, \tau_2)d\tau_1d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)} + \]

\[ + \int \int_{\gamma_1 \gamma_2} h(t_1, t_2, \tau_1, \tau_2)x(\tau_1, \tau_2)d\tau_1d\tau_2 = f(t_1, t_2) \]

and, under special conditions [26], [35] for coefficients \( a(t_1, t_2), \)
for singular integral equations

\[ a(t_1, t_2)x(t_1, t_2) + \frac{b(t_1, t_2)}{\pi} \int_{\gamma_1} \frac{x(\tau_1, t_2)}{\tau_1 - t_1} d\tau_1 + \]

\[ + \frac{c(t_1, t_2)}{\pi} \int_{\gamma_2} \frac{x(t_1, \tau_2)}{\tau_2 - t_2} d\tau_2 + \frac{d(t_1, t_2)}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \frac{x(\tau_1, \tau_2) d\tau_1 d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)} + \]

\[ + \int_{\gamma_1} \int_{\gamma_2} h(t_1, t_2, \tau_1, \tau_2) x(\tau_1, \tau_2) d\tau_1 d\tau_2 = f(t_1, t_2). \]

2. Riemann Boundary Value Problem

Let us consider iterative methods for solution of characteristic bisingular equations

\[ a(t_1, t_2)\varphi(t_1, t_2) + b(t_1, t_2)S_{12}(\varphi(\tau_1, \tau_2)) = f(t_1, t_2). \quad (2.1) \]

Conditions on functions \( a, b, f \) will be imposed below. We now assume that they are continuous and \( a \pm b \neq 0 \) on \( \gamma_{12} \). In the previous section was shown that the preceding equation reduces to the Riemann boundary value problem

\[ \varphi^{++} - \frac{a - b}{a + b} \varphi^{--} - \frac{a - b}{a + b} \varphi^{-+} - \varphi^{--} = \frac{1}{a + b} f \quad (2.2) \]

Let us suppose that for any points \( t_2 \in \gamma_2 \) the values of the function

\[ G(t_1, t_2) = \frac{a(t_1, t_2) - b(t_1, t_2)}{a(t_1, t_2) + b(t_1, t_2)} \quad (G(t_1, t_2) \neq 0) \]

lie inside of an angle \( \Gamma_1 \), which has the vertex in origin of the plane of complex variable \( Z_1 \). Also we suppose that values of the angle \( \Gamma_1 \) are not exceed \( \pi - \delta_1 (\delta_1 > 0) \). Then one can find such constant \( \alpha_1 \) that \( \max_{(t_1, t_2) \in \gamma_{12}} | \alpha_1 G(t_1, t_2) - 1 | \leq q_1 < 1. \)

Let for any fixed points \( t_1 \in \gamma_1 \) values of functions \( G(t_1, t_2) \) lie inside of an angle \( \Gamma_2 \), which has the center at origin. Also we suppose that the values of angle are not exceed \( \pi - \delta_2 (\delta_2 > 0) \). Then
it exists a constant $\alpha_2$ such that $\max_{(t_1,t_2) \in \gamma_{12}} | \alpha_2 G_1(t_1, t_2) - 1 | \leq q_2 < 1$.

Let be $q = \min(q_1, q_2)$. For certainty we put $q = q_1$ and $\alpha = \alpha_1$. We will change the functions $\varphi^{\pm\pm}$ using the formulas

$$\psi^{\pm\pm} = \varphi^{\pm\pm}; \psi^{+-} = \alpha^{-1} \varphi^{+-}; \psi^{-+} = \alpha^{-1} \varphi^{-+}; \psi^{--} = \varphi^{--}$$

and receive the equation

$$\psi^{++} - \alpha G(t_1, t_2) \psi^{+-} - \alpha G(t_1, t_2) \psi^{-+} + \psi^{--} = (a + b)^{-1} f.$$

By adding the function $\psi$ to the both sides of the preceding equation and by using the Sohotzky – Plemel formulas, we obtain the equation

$$\psi(t_1, t_2) = (\alpha G(t_1, t_2) - 1) \psi^{+-}(t_1, t_2) + (\alpha G(t_1, t_2) - 1) \psi^{-+}(t_1, t_2) + (a(t_1, t_2) + b(t_1, t_2))^{-1} f(t_1, t_2).$$

For solving this equation the following iterative process may be used

$$\psi_{n+1} = (\alpha G - 1) \psi^{+-}_n + (\alpha G - 1) \psi^{-+}_n + (a + b)^{-1} f. \quad (2.3)$$

We will investigate its convergence in the space $L_2(\gamma_{12})$. From Sohotzky – Plemel formulas follows that

$$(\alpha G - 1) \psi^{+-} + (\alpha G - 1) \psi^{-+} = (\alpha G - 1)(-\frac{1}{2} \psi + \frac{1}{2} S_{12} \psi).$$

Since $\|S_{12}\|_{L_2} = 1$, then

$$\|(\alpha G - 1) \psi^{+-} + (\alpha G - 1) \psi^{-+}\|_{L_2} \leq$$

$$\leq \frac{1}{2} \max_{(t_1,t_2) \in \gamma_{12}} | \alpha G(t_1, t_2) - 1 | \left( \| \psi \| + \| S_{12} \psi \| \right) \leq$$

$$\leq \max_{(t_1,t_2) \in \gamma_{12}} | \alpha G(t_1, t_2) - 1 | \| \psi \| \leq q \| \psi \|.$$

The iterative process (2.3) thus converges. The estimate $\| \varphi^* - \varphi_n^* \| \leq A q^n$ is valid, where $\varphi^*$ is a solution of the equation (2.1),
(2.3) \( \varphi_\ast^n = \psi_\ast^n \), \( \varphi_\ast ^{++} = \alpha \psi_\ast ^{++} \), \( \varphi_\ast ^{--} = \alpha \psi_\ast ^{--} \), \( \varphi_\ast ^{--} = \psi_\ast ^{--} \); \( \psi_\ast \) is result of \( n \)-iteration which is obtained by the formula (2.3).

**Theorem 2.1** [26], [35]. Let the following conditions are fulfilled:
1) \( a(t_1, t_2) \pm b(t_1, t_2) \neq 0 \) for \( (t_1, t_2) \in \gamma_{12} \);
2) \( a(t_1, t_2), b(t_1, t_2) \in C[\gamma_{12}] \);
3) for any fixed \( t_1 \in \gamma_1 (t_2 \in \gamma_2) \) the values of function \( G(t_1, t_2) \) lie inside of an angle of the solution \( \pi - \alpha_1 (\pi - \alpha_2) \), \( \alpha_i > 0 (i = 1, 2) \) with the center in origin.

Then the equation (2.1) has a unique solution \( \varphi^* \) to which the iterative process (2.3) (after returning from the function \( \psi \) to the function \( \varphi \)) converges with the rate of geometric progression.

It is not difficult to see (using Sohotzky – Plemel formulas), that an iterative processes may be built analogously for equations

\[
\begin{align*}
a(t_1, t_2)\varphi(t_1, t_2) + b(t_1, t_2)S_1\varphi(\tau_1, t_2) &= f(t_1, t_2), \\
a(t_1, t_2)\varphi(t_1, t_2) + b(t_1, t_2)S_2\varphi(t_1, \tau_2) &= f(t_1, t_2), \\
a(t_1, t_2)S_1\varphi(\tau_1, t_2) + b(t_1, t_2)S_2\varphi(t_1, \tau_2) &= f(t_1, t_2), \\
a(t_1, t_2)S_1\varphi(\tau_1, t_2) + b(t_1, t_2)S_{12}\varphi(\tau_1, \tau_2) &= f(t_1, t_2), \\
a(t_1, t_2)S_2\varphi(t_1, \tau_2) + b(t_1, t_2)S_{12}\varphi(\tau_1, \tau_2) &= f(t_1, t_2).
\end{align*}
\]

We turn our attention to one moment.

The equation

\[
\begin{align*}
a(t_1, t_2)x(t_1, t_2) + b(t_1, t_2)S_{12}(x(\tau_1, \tau_2)) + \\
+U_{12}(h(t_1, t_2, \tau_1, \tau_2)x(\tau_1, \tau_2)) &= f(t_1, t_2),
\end{align*}
\]

(2.4)

where \( U_{12} \) is totally continuous operator, can be studied as the equation (2.1) and as above mentioned equations.

Let us now consider the more general singular integral equation

\[
\begin{align*}
a(t_1, t_2)x(t_1, t_2) + b(t_1, t_2)S_1(x(\tau_1, t_2)) + \\
+c(t_1, t_2)S_2(x(t, \tau_2)) + d(t_1, t_2)S_{12}(x(\tau_1, \tau_2)) &= f(t_1, t_2).
\end{align*}
\]

(2.5)
This equation is equivalent to the Riemann boundary value problem

\[ a_1(t_1, t_2)\psi^{++}(t_1, t_2) + b_1(t_1, t_2)\psi^{+-}(t_1, t_2) + \\
+c_1(t_1, t_2)\psi^{-+}(t_1, t_2) + d_1(t_1, t_2)\psi^{--}(t_1, t_2) = f(t_1, t_2), \]

in which the coefficients \(a_1, b_1, c_1, d_1\) are determined, using Sobotzky – Plemel formulas for functions \(x(t_1, t_2), S_1(x(\tau_1, t_2)), S_2(x(t_1, \tau_2)), S_{12}(x(\tau_1, \tau_2))\).

Assume that the conditions, described in this section above, are fulfilled. These conditions guarantee the existence of constants \(\alpha, \beta, \gamma, \delta\) such that

\[ |\alpha a_1(t_1, t_2) + 1| < q_1, \quad |\beta b_1(t_1, t_2) - 1| < q_2, \]
\[ |\gamma c_1(t_1, t_2) - 1| < q_3, \quad |\delta d_1(t_1, t_2) + 1| < q_4. \]

Then, provided \(q_1 + q_2 + q_3 + q_4 < 1\), the iterative process

\[ v_{n+1} = (\alpha a_1(t_1, t_2) + 1)v^{++}_n + (\beta b_1(t_1, t_2) - 1)v^{+-}_n + \\
+(\gamma c_1(t_1, t_2) - 1)v^{-+}_n + (\delta d_1(t_1, t_2) + 1)v^{--}_n + f(t_1, t_2), \]

where \(v^{++} = \alpha^{-1}\psi^{++}, v^{+-} = \beta^{-1}\psi^{+-}, v^{-+} = \gamma^{-1}\psi^{-+}, v^{--} = \psi^{---}\), converges to a solution of equation (2.5) with the rate of geometric progression, of course, after replacing the function \(v\) on the function \(\psi\).

The proof of this statement is conducted in the same way, as in case of simplest bisingular integral equation (2.1).

### 3. Approximate Solution of Multi-Dimensional Singular Integral Equations

In this section we will consider numerical methods for solution of multi-dimensional singular integral equation

\[ Kx \equiv a(t)x(t) + b(t) \int_{E_m} \frac{\varphi(t, \Theta)x(\tau)}{(r(t, \tau))^m} d\tau + \]

\[
\int_{E_m} h(t, \tau) x(\tau) d\tau = f(t),
\]
(3.1)

where \( t = (t_1, \ldots, t_m), \tau = (\tau_1, \ldots, \tau_m), m = 2, 3, \ldots, r(t, \tau) = (\sum_{k=1}^{m} (t_k - \tau_k)^2)^{1/2}, \Theta = ((t - \tau) / r(t, \tau)), a(t), b(t), f(t) \) are smooth functions of \( m \) variables, \( h(t, \tau) \) is a smooth function of \( 2m \) variables.

These equations have many applications in the elasticity theory, in the oscillating theory and other. In spite of this there are only a few works devoted to approximate methods for solution of multi-dimensional singular integral equations. At first we notice the papers [106], [119], in which the method of moments and the method of least squares are used for solution of the equation \( Kx = f \).

Several iterative methods for solution of equations (3.1) was offered in [108], [113].

In this section we give a review of works [40], [41] and [35].

3.1. Approximate Solution of Multi-Dimensional Singular Integral Equations on Holder Classes of Functions

Let us consider the equation

\[
Kx \equiv a(t)x(t) + b(t) \int_{G} \frac{\varphi(\Theta)x(\tau)}{(r(t, \tau))^2} d\tau = f(t),
\]
(3.2)

where \( G \) is a simply connected domain on \( E_2, t = (t_1, t_2), \tau = (\tau_1, \tau_2), \Theta = ((\tau - t) / r(t, \tau)), r(t, \tau) = ((t_1 - \tau_1)^2 + (t_2 - \tau_2)^2)^{1/2} \).

Assume that functions \( a(t), b(t), f(t), \varphi(\Theta) \in H_{\alpha\alpha}(1), \alpha (0 < \alpha \leq 1) \).

All results of this item can be diffused on singular integral
equation
\[ a(t)x(t) + \int_G \frac{\varphi(t, \Theta)x(\tau)}{(r(t, \tau))^2} d\tau = f(t), \]
where \( G = [-A, A]^l, l = 2, 3, \ldots, t = (t_1, \ldots, t_l), \tau = (\tau_1, \ldots, \tau_l), \)
\[ \Theta = ((t - \tau)/r(t, \tau)), r(t, \tau) = (t_1 - \tau_1)^2 + \ldots + (t_l - \tau_l)^2)^{1/2}. \]

For simplicity we consider the case when \( G = [-A, A; -A, A]. \)

Let us cover the domain \( G \) by squares \( \Delta_{kl} = [t_k, t_{k+1}; t_l, t_{l+1}], \)
\( k, l = 0, N + 1, \) where \( t_k = -A + 2Ak/(N + 2), k = 0, N + 2. \)

Together with squares \( \Delta_{kl}, k, l = 0, N + 1, \) we will use rectangles \( \tilde{\Delta}_{kl}, k, l = 1, N, \) which are defined by following way:
\( \tilde{\Delta}_{kl} = \Delta_{kl} \) for \( k = 2, N - 1 \) and \( l = 2, N - 1; \)
\[ \tilde{\Delta}_{11} = \Delta_{00} \cup \Delta_{01} \cup \Delta_{10} \cup \Delta_{11}; \]
\[ \tilde{\Delta}_{1,N} = \Delta_{0,N} \cup \Delta_{1,N} \cup \Delta_{0,N+1} \cup \Delta_{1,N+1}; \]
\[ \tilde{\Delta}_{N,1} = \Delta_{N,0} \cup \Delta_{N+1,0} \cup \Delta_{N,1} \cup \Delta_{N+1,1}; \]
\[ \tilde{\Delta}_{N,N} = \Delta_{N,N} \cup \Delta_{N,N+1} \cup \Delta_{N+1,N} \cup \Delta_{N+1,N+1}; \]
\[ \tilde{\Delta}_{1,l} = \Delta_{0,l} \cup \Delta_{1,l} \text{ for } l = 2, N - 1; \]
\[ \tilde{\Delta}_{N,l} = \Delta_{N,l} \cup \Delta_{N+1,l} \text{ for } l = 2, N - 1; \]
\[ \tilde{\Delta}_{k,1} = \Delta_{k,0} \cup \Delta_{k,1} \text{ for } k = 2, N - 1; \]
\[ \tilde{\Delta}_{k,N} = \Delta_{k,N} \cup \Delta_{k,N+1} \text{ for } k = 2, N - 1. \]

Approximate solution of the equation (3.1) we will seek as piecewise constant function \( x_N(t_1, t_2). \) The function \( x_N(t_1, t_2) \) is equal to unknown constant \( x_{kl} \) on each rectangular \( \tilde{\Delta}_{kl}, k, l = 1, 2, \ldots, N. \)

Unknown values \( x_{kl}, k, l = 1, 2, \ldots, N, \) are defined from the system of linear algebraic equations
\[ a(\bar{t}_{kl})x_{kl} + b(\bar{t}_{kl}) \sum_{i=1}^N \sum_{j=1}^N 'd_{ij}(\bar{t}_{kl})x_{ij} + x_{kl}d_{kl}(\bar{t}_{kl}) = f(\bar{t}_{kl}), \quad (3.3) \]
k, l = 1, 2, \ldots, N,
where \( \sum \sum' \) denote that the sum is taken over rectangles \( \tilde{\Delta}_{ij}, \) which intersections with squares \( \Delta_{kl} \) are empty, \( t_{kl} = (t_k, t_l), \)
\[ \tau_{kl} = (\tau_k, \tau_l), \quad \bar{t}_{kl} = (t_k + \bar{h}_1, t_l + \bar{h}_2), \]

\[
d_{ij}(\bar{t}_{kl}) = \int_{\Delta_{ij}} \varphi \left( \frac{\bar{t}_{kl} - \tau}{r(\bar{t}_{kl}, \tau)} \right) \frac{d\tau}{r^2(\bar{t}_{kl}, \tau)},
\]

\[
\bar{d}_{ij}(\bar{t}_{kl}) = \int_{\Delta_{ij}} \varphi \left( \frac{\bar{t}_{kl} - \tau}{r(\bar{t}_{kl}, \tau)} \right) \frac{d\tau}{r^2(\bar{t}_{kl}, \tau)},
\]

values of \( \bar{h}_1 \) and \( \bar{h}_2 \) will be given later.

Let us estimate the expression

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \bar{d}_{ij}(\bar{t}_{kl}) \leq Bh^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(t_k - t_j)^2 + (t_l - t_j)^2} \leq
\]

\[
\leq 8Bh^2 \sum_{i=1}^{\lfloor N/2 \rfloor + 1} \sum_{j=1}^{\lfloor N/2 \rfloor + 1} \frac{N^2}{(i^2 + j^2)4A^2} =
\]

\[
= 8B \sum_{i=1}^{\lfloor N/2 \rfloor + 1} \sum_{j=1}^{\lfloor N/2 \rfloor + 1} \frac{1}{i^2 + j^2} \leq D \ln N, \quad (3.4)
\]

where \( h = 2A/N \).

Let us denote the matrix of the system (3.3) as \( C = \{c_{ij}\}, i, j = 1, 2, \ldots, N^*, N^* = N^2 \). So, for arbitrary \( j, 1 \leq j \leq N^2 \), we have

\[
\sum_{k=1, k \neq j}^{N^*} |c_{jk}| \leq D \ln N. \quad (3.5)
\]

Let us show that we can choose parameters \( \bar{h}_1, \bar{h}_2 \), so way, that the matrix \( C \) will be invertible.

Well known [104], that the necessary and sufficient condition of the existence of multi-dimensional singular integrals
\[ \int \int_{E_2} \frac{\varphi(\Theta) x(\tau)}{r^2(t, \tau)} d\tau, \quad t = \bar{t}_{kl}, \] is the equality \[ \int_{S} \varphi \left( \frac{\bar{t}_{kl} - \tau}{r(t_{kl}, \tau)} \right) ds = 0, \] where \( S \) is the unit circle with the center in the point \( \bar{t}_{kl} \).

From this condition follows existence at least of two rays from point \( \bar{t}_{kl} \), where the function \( \varphi(\Theta) = 0 \). So, varying values \( \bar{h}_1 \) and \( \bar{h}_2 \), we can make the integral

\[ \int \int_{\Delta_{kl}} \varphi \left( \frac{\tau_1 - \bar{t}_k}{r(\tau, \bar{t}_{kl})}, \frac{\tau_2 - \bar{t}_l}{r(\tau, \bar{t}_{kl})} \right) \frac{d\tau}{r^2(\tau, \bar{t}_{kl})} \]

bigger than any fixed number \( M \). We can select \( M \) bigger than \( D \ln N \). This inequality is the sign, that conditions of Hadamard Theorem about invertibility of matrix is fulfilled. So, we proved the existence of a unique solution \( x^*_N(t_1, t_2) \) of the system (3.3).

We will estimate the error of approximation of exact solution \( x^*(t_1, t_2) \) of the equation (3.2) with respect to solution \( x^*_N(t_1, t_2) \) of the equation (3.3).

Let \( x^*(t_1, t_2) \) be a solution of the equation (3.2). Equating left and right sides of the equation (3.3) in points \( \bar{t}_{kl} \), we have:

\[ a(\bar{t}_{kl})x^*(\bar{t}_{kl}) + \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\Delta_{ij}} \varphi \left( \frac{\tau_1 - \bar{t}_k}{r(\tau, \bar{t}_{kl})}, \frac{\tau_2 - \bar{t}_l}{r(\tau, \bar{t}_{kl})} \right) x^*(\tau) \frac{d\tau}{r^2(\tau, \bar{t}_{kl})} = \]

\[ = f(\bar{t}_{kl}), \quad k, l = 1, N. \quad (3.6) \]

Let \( P_N \) is the projector from space \( X = C(G) \) onto the set of piece-constant functions. This projector is defined as

\[ P_N f(t) = \begin{cases} 
 f(\bar{t}_k, \bar{t}_l), & \text{if } t \in \bar{\Delta}_{kl}, \\
 0, & \text{if } t \in G \setminus \bar{\Delta}_{kl}.
\end{cases} \]

The system (3.3) in operator form can be written as

\[ K_N x_N \equiv P_N \left[ a(t_1, t_2) x_N(t_1, t_2) + 
 b(t) \int_{G} P(t, \tau) \left[ \varphi \left( \frac{\tau_1 - t_1}{r(\tau, t)}, \frac{\tau_2 - t_2}{r(\tau, t)} \right) \frac{x_N(\tau)}{r^2(\tau, t)} \right] d\tau \right] = \]

151
\[ = P_N[f(t)], \]

where
\[
P(\bar{t}_{kl}, \tau) = \begin{cases} 
1, & \text{if } \tau \in G \setminus g_{kl}, \\
0, & \text{if } \tau \in g_{kl}, 
\end{cases}
\]

\[ g_{kl} = [t_{k-1}, t_{k+2}; t_{l-1}, t_{l+2}] \setminus \Delta_{kl}. \]

Then
\[
x_N^* - P_N x^* = K_N^{-1}(K_N(x_N^* - P_N x^*)) = \]
\[
= K_N^{-1}(P_N f - K_N P_N x^*) = K_N^{-1}(P_N K x^* - K_N P_N x^*) = 
= K_N^{-1}(P_N K x^* - P_N K P_N x^*) + K_N^{-1}(P_N K P_N x^* - P_N K_N P_N x^*). \]

The system (3.3) we can consider in the space \( R_N^* \) with the norm \( ||u|| = \max_{1 \leq k \leq N^*} |u_k| \), where \( u = (u_1, \ldots, u_{N^*}) \). In the space \( R_N^* \) \( \max_{1 \leq k, l \leq N-1} |x_{k,l}| \leq B \max_{1 \leq k, l \leq N-1} |f_{k,l}| \), where \( f_{kl} = f(\bar{t}_{k,l}) \). The solution \( x_N^*(t_1, t_2) \) is a piece-constant function with the norm \( ||x_N^*(t_1, t_2)||_C = \max_{1 \leq k, l \leq N-1} |x_{k,l}| \). Easy to see that \( ||P_N f(t_1, t_2)||_C = \max_{1 \leq k, l \leq N-1} |f_{k,l}| \). From these equalities follow that \( ||K_N^{-1}|| \leq B \).

In the space \( C \) we have
\[
||x_N^* - P_N x^*|| \leq B N^{-\alpha} + B ||P_N[KP_N x^* - K_N P_N x^*]||. \quad (3.7)
\]

Well known, that \( ||P_N|| = 1 \). So, we need to estimate the integral
\[
I_1 = \left| \int_{g_{kl}} \varphi \left( \frac{\tau_1 - \bar{t}_k}{r(\tau, \bar{t}_{kl})}, \frac{\tau_2 - \bar{t}_l}{r(\tau, \bar{t}_{kl})} \right) \frac{1}{r^2(\tau, \bar{t}_{kl})} P_N[x^*]d\tau \right|.
\]

Let us construct a function \( \psi(x_1, x_2) \in H_{\alpha\alpha} (0 < \alpha \leq 1) \), which has properties: 1) \( \psi(\bar{t}_{kl}) = (P_N x^*)(\bar{t}_{kl}) \) and
\[
2) \int_{g_{kl}} \varphi \left( \frac{\tau_1 - \bar{t}_k}{r(\tau, \bar{t}_{kl})}, \frac{\tau_2 - \bar{t}_l}{r(\tau, \bar{t}_{kl})} \right) \frac{\psi(\tau_1, \tau_2)}{r^2(\tau, \bar{t}_{kl})} d\tau_1 d\tau_2 = 0. \quad (3.8)
\]

Existence of such function follows from the fact, that function \( \varphi(\Theta) \) is equal to zero at least on two rays outbound from point \( \bar{t}_{kl} \).
Using (3.8), we can prove that
\[ \left| \int_{g_{kl}} \varphi \left( \frac{\tau_1 - \bar{t}_k}{r(\tau, \bar{t}_{kl})}, \frac{\tau_2 - \bar{t}_l}{r(\tau, \bar{t}_{kl})} \right) \frac{1}{r^2(\tau, \bar{t}_{kl})} (P_Nx^* - \psi(\tau_1, \tau_2)) d\tau_1 d\tau_2 \right| \leq \]
\[ \leq \int_{g_{kl}} \left| \varphi \left( \frac{\tau_1 - \bar{t}_k}{r(\tau, \bar{t}_{kl})}, \frac{\tau_2 - \bar{t}_l}{r(\tau, \bar{t}_{kl})} \right) \right| \frac{1}{r^2(\tau, \bar{t}_{kl})} \left( |(P_Nx^*)(\tau_1, \tau_2) - (P_Nx^*)(\bar{t}_{kl})| + |\psi(\tau_1, \tau_2) - \psi(\bar{t}_{kl})| \right) d\tau_1 d\tau_2 \leq \]
\[ \leq AB \int_{g_{kl}} \frac{d\tau_1 d\tau_2}{r^{2-\alpha/2}(\tau, \bar{t}_{kl})} \leq \frac{AB}{N^{\alpha/2}}. \] (3.9)

So, from (3.7) — (3.9) we have
\[ \|x^* - x_N^*\| \leq \|x^* - P_Nx^*\| + \|P_Nx^* - x_N^*\| \leq AN^{-\alpha/2}. \]

**Theorem 3.1** [40], [41], [35]. Let the equation (3.1) has a unique solution \( x^*(t) \). Let \( \varphi(\Theta), f(t) \) are belong to Holder class of functions with exponent \( \alpha \). Then exist parameters \( \bar{h}_1, \bar{h}_2 \), such, that the system (3.3) has a unique solution \( x_N^*(t) \) and the estimate \( \|x^* - x_N^*\|_{C(G)} \leq AN^{-\alpha/2} \) is valid.

### 3.2. Approximate Solution of Linear Multi-Dimensional Singular Integral Equations on Sobolev Classes of Functions

Let us consider the two-dimensional singular integral equation
\[ a(t)x(t) + b(t) \int_{G} \frac{\varphi(\Theta)x(\tau)}{r^2(t, \tau)} d\tau = f(t), \] (3.10)

where \( G = [-A, A]^2 \). \( t = (t_1, t_2), \tau = (\tau_1, \tau_2), r(t, \tau) = ((t_1 - \tau_1)^2 + (t_2 - \tau_2)^2)^{1/2}, \Theta = ((t - \tau)/r(t, \tau)); \) functions \( a, b, f, \varphi \) have derivatives up to \( r \) order.
We consider two-dimensional singular integral equations for simplicity. All received below results can be diffused to multi-dimensional singular integral equations.

Let us construct the numerical method for solution of the equation (3.1).

Let us cover the domain $G$ by squares

$$\Delta_{kl} = [t_k, t_{k+1}; t_l, t_{l+1}], \quad k, l = 0, N + 1, \quad t_k = -A + 2Ak/(N + 2),$$

$k = 0, N + 2$.

Together with squares $\Delta_{kl}, \quad k, l = 0, N + 1$, we need rectangles $\bar{\Delta}_{kl}$, which are defined by following way:

- $\bar{\Delta}_{kl} = \Delta_{kl}$ for $k = 2, N - 1$ and $l = 2, N - 1$;
- $\bar{\Delta}_{1,l} = \Delta_{0,l} \cup \Delta_{1,l}$ for $l = 2, N - 1$;
- $\bar{\Delta}_{N,l} = \Delta_{N,l} \cup \Delta_{N+1,l}$ for $l = 2, N - 1$;
- $\bar{\Delta}_{k,1} = \Delta_{k,0} \cup \Delta_{k,1}$ for $k = 2, N - 1$;
- $\bar{\Delta}_{k,N} = \Delta_{k,N} \cup \Delta_{k,N+1}$ for $k = 2, N - 1$.

Approximate solution of the equation (3.1) we will seek in the form of local spline $x_{nn}(t_1, t_2)$.

Let us describe the construction of the spline $x_{nn}(t_1, t_2)$.

Let us consider a function $f(x_1, x_2)$, which is given on the square $G$. In the each rectangular $\bar{\Delta}_{kl}$ the function $f(x_1, x_2)$ is approximated by interpolation polynomial, constructed by following way. Let $\Delta_{kl} = [t_k, t_{k+1}; t_l, t_{l+1}]$. Let us introduce knots $(t^i_k, t^j_l)$, where $t^i_k = t_k + (t_{k+1} - t_k)i/(r + 1)$, $i = 1, 2, \ldots, r$, $k, l = 1, \ldots, N$.

The polynomial $L_r(f, \Delta_{kl})$ is introduced by formula

$$L_r(f, \Delta_{kl}) = \sum_{i=1}^{r} \sum_{j=1}^{r} f(t^i_k, t^j_l)\psi_{ki}(t_1)\psi_{lj}(t_2), \quad (3.11)$$

where $\psi_{ki}(t_1)$ and $\psi_{lj}(t_2)$ are fundamental polynomials constructed
on knots \( t^i_k \) and \( t^j_l \).

The interpolated polynomial (3.11) is diffused to domains \( \bar{\Delta}_{kl} \), \( k, l = 1, 2, \ldots, N \), in following way. If \( k, l = 2, \ldots, N - 1 \), then \( L_r(f, \bar{\Delta}_{kl}) = L_r(f, \Delta_{kl}) \). Let us consider the square \( \bar{\Delta}_{11} \). We will construct the polynomial \( L_r(f, \Delta_{11}) \) and diffuse this polynomial on the square \( \bar{\Delta}_{11} \). In result we have the polynomial \( L_r(f, \bar{\Delta}_{11}) \).

Polynomials \( L_r(f, \bar{\Delta}_{kl}) \), \( k, l = 1, N \), we will receive by similar way. Let us consider the square \( \bar{\Delta}_{11} \). We will construct the polynomial \( L_r(f, \Delta_{11}) \) and diffuse this polynomial on the square \( \bar{\Delta}_{11} \). In result we have the polynomial \( L_r(f, \bar{\Delta}_{11}) \).

Polynomials \( L_r(f, \bar{\Delta}_{kl}) \), \( k, l = 1, N \), we will receive by similar way. Let us consider the rectangular \( \bar{\Delta}_{1,l} \), \( l = 2, \ldots, N - 1 \). In this case we construct the polynomial \( L_r(f, \Delta_{1l}) \) and diffuse this polynomial on domain \( \bar{\Delta}_{1l} \). In result we have polynomial \( L_r(f, \bar{\Delta}_{1l}) \), \( l = 2, \ldots, N - 1 \).

Spline \( f_{nn}(t_1, t_2) \) consists of polynomials \( L_r(f, \bar{\Delta}_{kl}) \), where \( k, l = 1, \ldots, N \).

Approximate solution of the equation (3.1) we seek as local spline \( x_{nn}(t_1, t_2) \) with unknown values \( x_{nn}(t^i_k, t^j_l) \), \( k, l = 1, \ldots, N \), \( i, j = 1, \ldots, r \).

To each knots \( t^i_{kl}, \ (t^j_{kl} = (t^i_k, t^j_l)) \), \( k, l = 1, 2, \ldots, N \), \( i, j = 1, 2, \ldots, r \), we put in correspondence the rectangle \( \Delta_{ij} = [t^i_k - q_1 h, t^i_k + h; t^j_l - q_2 h, t^j_l + h] \), where \( h \leq 2A/(rN) \), \( q_1, q_2 \) are parameters, values of which will be defined later.

Values \( x^i_{kl} = x_{nn}(t^i_k, t^j_l) \), \( k, l = 1, 2, \ldots, N \), \( i, j = 1, 2, \ldots, r \), we define from the system of linear algebraic equations

\[
\begin{align*}
a_{kl} x^i_{kl} + b_{kl} \int_{\Delta_{ij}} \varphi \left( \frac{\tau_1 - t^i_k}{r(\tau, M_{ij}^{kl})}, \frac{\tau_2 - t^j_l}{r(\tau, M_{ij}^{kl})} \right) \frac{x_{nn}(\tau) d\tau}{r^2(\tau, M_{ij}^{kl})} + \\
b_{kl} \sum_{k_1=1}^{N} \sum_{l_1=1}^{N} \int_{\Delta_{kl}} \varphi \left( \frac{\tau_1 - t^i_k}{r(\tau, M_{ij}^{kl})}, \frac{\tau_2 - t^j_l}{r(\tau, M_{ij}^{kl})} \right) \frac{x_{nn}(\tau) d\tau}{r^2(\tau, M_{ij}^{kl})} = f^i_{kl},
\end{align*}
\]  

(3.12)
\(i, j = 1, 2, \ldots, r, k, l = 1, 2, \ldots, N,\)

where \(M_{kl}^{ij} = (t_k^i, t_l^j), a_{kl}^{ij} = a(M_{kl}^{ij}), b_{kl}^{ij} = b(M_{kl}^{ij}), f_{kl}^{ij} = f(M_{kl}^{ij}),\)

\(i, j = 1, 2, \ldots, r, k, l = 1, 2, \ldots, N,\) the prime in the summation indicate, that \((k_1, l_1) \neq (k + v, l + w), v, w = -1, 0, 1.\)

From necessary and sufficient conditions for the existence of singular integral follow, that the function \(\varphi(\Theta), \Theta = \left(\frac{\tau_1}{r(0, \tau)}, \frac{\tau_2}{r(0, \tau)}\right)\)

must be equal to zero on rays which are radiated from the origin. So, one can select parameters \(q_1, q_2, h\) so way that the integral

\[
\int \int_{\Delta_{kl}^{ij}} \varphi \left(\frac{\tau_1 - t_k^i}{r(\tau, M_{kl}^{ij})}, \frac{\tau_2 - t_l^j}{r(\tau, M_{kl}^{ij})}\right) \frac{\psi_{ki}(\tau_1)\psi_{ij}(\tau_2)}{r^2(\tau, M_{kl}^{ij})} d\tau_1 d\tau_2
\]

will be arbitrarily large.

Repeating arguments given in the previous item, one can see that conditions of the Hadamard theorem are fulfilled. So, the system (3.12) has a unique solution. The convergence of solution \(x_{nn}(t_1, t_2)\) of the system (3.12) to exact solution of the equation (3.1) is proved by the method similar to the method, which we have used for polysingular integral equations.

In result we can formulate the following statement.

**Theorem 3.2** [40], [41], [35]. Let \(a(t), b(t), \varphi(t) \in W^{r,r}(M), b(t) \neq 0.\) Let the equation (3.1) has a unique solution \(x^*_1(t).\)

Then exist such parameters \(h, q_1, q_2,\) that the system (3.12) has a unique solution \(x_{nn}^*\) and under some additional conditions the estimate \(\|x^* - x_{nn}^*\|_C \leq An^{-r} \ln^2 n\) is valid.

### 3.3. Parallel Method for Solution of Multi-Dimensional Singular Integral Equations

In this section we investigate numerical methods for solution of multi-dimensional singular integral equations on parallel computers with \(P\) processors. For determination we assume that offered method will be realized on computer with MIMD architecture.
(Multiple Instruction stream/ Multiple Date stream) in M.J. Flynn classification.

For simplicity we will describe this method only for the equation (3.2).

Approximate solution of the equation (3.2) we seek as piece-
constant function $x_N(t_1, t_2)$, unknown values $x_{kl}, k, l = 1, 2, \ldots, N,$ of which we define from the system (3.3). The system (3.3) can be written as matrix equation

$$CX = F,$$  \hspace{1cm} (3.13)

where $C = \{c_{kl}\}, k, l = 1, 2, \ldots, M, M = N^2, X = (x_1, \ldots, x_M)^T, F = (f_1, \ldots, f_M)^T.$

It is easy to see that for the system (3.13) conditions of the Hadamard theorem is valid.

Let $M = PL$. Let us decompose the matrix $C$ of the system (3.13) on $P^2$ blocks, vectors $X$ and $F$ on $P$ blocks. In this case the system (3.13) can be written as

$$\bar{C}\bar{X} = \bar{F},$$  \hspace{1cm} (3.14)

where $\bar{C} = \{B_{ij}\}, i, j = 1, 2, \ldots, P, \bar{X} = (\bar{x}_1, \ldots, \bar{x}_P), \bar{F} = (\bar{f}_1, \ldots, \bar{f}_P),$

$$B_{ij} = \begin{pmatrix} c_{(i-1)L+1,(j-1)L+1}, \ldots, c_{(i-1)L+1,(j-1)L+L} \\ \vdots \\ c_{(i-1)L+L,(j-1)L+1}, \ldots, c_{(i-1)L+L,(j-1)L+L} \end{pmatrix},$$

$i, j = 1, 2, \ldots, P,$

$$\bar{x}_i = (x_{(i-1)L+1}, \ldots, x_{(i-1)L+L}), \quad i = 1, 2, \ldots, P,$$

$$\bar{f}_i = (f_{(i-1)L+1}, \ldots, f_{(i-1)L+L}), \quad i = 1, 2, \ldots, P.$$

For solving the system (3.14) we use block-iterative method

$$B_{kk}\tilde{x}_{k}^{n+1} = f_k - \sum_{l=1, l \neq k}^{P} B_{kl}\tilde{x}_{l}^{n},$$  \hspace{1cm} (3.15)
\( k = 1, 2, \ldots, P. \)

From conditions, imposed on the matrix \( C \), one can see that matrixes \( B_{kk}, k = 1, 2, \ldots, P, \) are invertible and norms \( \|B_{kk}^{-1}\|, k = 1, 2, \ldots, P, \) can be made as small as it is necessary for the convergence of the iterative process (3.15).

### Appendix 1

1. Stability of numerical schemes

In this section we will investigate a criteria of stability of numerical schemes for solution of singular integral equations.

We investigate stability of the three types of singular integral equations. One can notice that stability of solutions of numerical schemes for other types of singular integral equations which was considered in this paper can be investigated by similar way.

#### 1.1. Stability of Solutions of Bisingular Integral Equation

Let us consider the bisingular integral equation

\[
d(t_1, t_2)x(t_1, t_2) + \frac{b(t_1, t_2)}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \frac{x(\tau_1, \tau_2)d\tau_1d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)} = f(t_1, t_2).
\]

(1.1)

Approximate method for solution of the equation (1.1) was given in the section 1 of the chapter 2. Below we use definitions introduced in the section 1 of the chapter 2. It was shown that one can transform the equation (1.1) into Riemann value boundary problem

\[
L \varphi =
\]

\[
\equiv \psi^{++}(t_1, t_2) - G(t_1, t_2)(\psi^{+}(t_1, t_2) + \psi^{-}(t_1, t_2)) + \psi^{--}(t_1, t_2) =
\]

\[
= f^*(t_1, t_2), \quad f^*(t_1, t_2) = \frac{f(t_1, t_2)}{a(t_1, t_2) + b(t_1, t_2)}.
\]

(1.2)
Approximate solution of the equation (1.2) we seek in the form of the functions

\[
\varphi_{nn}^{++}(t_1, t_2) = \psi^{++}(t_1, t_2) \sum_{k=0}^{n} \sum_{l=0}^{n} \alpha_{kl} t_1^k t_2^l,
\]

\[
\varphi_{nn}^{+-}(t_1, t_2) = \psi^{+-}(t_1, t_2) \sum_{k=0}^{n} \sum_{l=-n}^{-1} \alpha_{kl} t_1^k t_2^l,
\]

\[
\varphi_{nn}^{-+}(t_1, t_2) = \psi^{-+}(t_1, t_2) \sum_{k=-n}^{-1} \sum_{l=0}^{n} \alpha_{kl} t_1^k t_2^l,
\]

\[
\varphi_{nn}^{--}(t_1, t_2) = \psi^{--}(t_1, t_2) \sum_{k=-n}^{-1} \sum_{l=-n}^{-1} \alpha_{kl} t_1^k t_2^l. \quad (1.3)
\]

Coefficients \(\{\alpha_{kl}\}, k, l = -n, \ldots, -1, 0, 1, \ldots, n\), are defined from the system of linear algebraic equations

\[
L_n \varphi_{nn} \equiv P_{nn}[L \varphi_{nn}] = P_{nn}[f^*(t_1, t_2)], \quad (1.4)
\]

where \(P_{nn}\) is the operator from the space \(C[\gamma_1 \times \gamma_2]\) onto the set of interpolation polynomials, constructed on knots \(v_{kl} = (v_k, v_l), k, l = 0, 1, \ldots, 2n, v_k = \exp\{is_k\}, s_k = 2k\pi/(2n + 1), k, l = 0, 1, \ldots, 2n\).

Let the operator \(L\) is invertible in Holder space \(H_\beta\) and the inverse operator \(\|L^{-1}\|_\beta\) has the norm \(\|L^{-1}\|_\beta = B\).

In the chapter 2 was proved that the operator \(L_n\) is invertible under condition \(q = A n^\beta (E_{nn}(\psi^{++}) + E_{nn}(\psi^{+-}) + E_{nn}(\psi^{-+}) + E_{nn}(\psi^{--})) \ln^2 n < 1\).

Let the values \(a(v_i, v_j), d(v_i, v_j)\) are determined with error \(\varepsilon\). Let \(\min_{i,j} |a(v_i, v_j) + d(v_i, v_j)| \geq d\). Then the error of determination of values \(G(t_i, t_j)\) is not bigger than \(\varepsilon_1 = 2\varepsilon/(d - \varepsilon) \approx 2\varepsilon/d\).

Let us denote by \(\tilde{a}(v_i, v_j), \tilde{d}(t_i, t_j), \tilde{f}(v_i, v_j)\) perturbed values of \(a(v_i, v_j), d(v_i, v_j), f(v_i, v_j)\): \(|\tilde{a}(v_i, v_j) - a(v_i, v_j)| \leq \varepsilon, |\tilde{d}(v_i, v_j) - d(v_i, v_j)| \leq \varepsilon, |\tilde{f}(v_i, v_j) - f(v_i, v_j)| \leq \varepsilon, i, j = 0, 1, \ldots, 2n\).
With perturbed values of \( a(t_i, t_j), d(v_i, v_j), f(v_i, v_j) \) collocation method for equation (1.2) has the view

\[
\tilde{L}_n \varphi_{nn} \equiv P_{nn}[\varphi_{nn}^+(t_1, t_2) - \tilde{G}(t_1, t_2)(\varphi_{nn}^-(t_1, t_2) + \varphi_{nn}^-(t_1, t_2)) + \\
+ \varphi_{nn}^-(t_1, t_2)] = P_{nn}[\tilde{f}^*(t_1, t_2)],
\]

where \( \tilde{G}(v_i, v_j) = (\tilde{a}(v_i, v_j) - \tilde{b}(v_i, v_j))/(\tilde{a}(v_i, v_j) + \tilde{b}(v_i, v_j)) \),
\( \tilde{f}^*(v_i, v_j) = \tilde{f}(v_i, v_j)/(\tilde{a}(v_i, v_j) + \tilde{b}(v_i, v_j)) \), \( i, j = 0, 1, \ldots, 2n \).

It is easy to see that

\[
|G(v_i, v_j) - \tilde{G}(v_i, v_j)| \leq \varepsilon_1 = 2\varepsilon/(d - \varepsilon),
\]
\[
|f(v_i, v_j) - \tilde{f}(v_i, v_j)| \leq \varepsilon_2 = \varepsilon/(d - \varepsilon).
\]

Using inverse theorems of approximation theory, one can see that

\[
\|P_{nn}[(G(t_1, t_2) - \tilde{G}(t_1, t_2))\varphi_{nn}^\pm]\|_\beta \leq A\varepsilon_1 n^\beta \ln^2 n \|\varphi_{nn}\|_\beta;
\]
\[
\|P_{nn}[f^*(t_1, t_2) - \tilde{f}^*(t_1, t_2)]\|_\beta \leq An^\beta \varepsilon_2 \ln^2 n.
\]

Here \( \tilde{G}(t_1, t_2) \) and \( \tilde{f}^*(t_1, t_2) \) are Holder functions, which are equal to \( \tilde{G}(v_i, v_j) \) and \( \tilde{f}^*(v_i, v_j) \) in points \( v_i, v_j, i, j = 0, 1, \ldots, 2n \).

Using Banach Theorem for inverse operator, we see that for such \( \varepsilon_1 \) and \( \varepsilon_2 \) that \( A\varepsilon_1 n^\beta \ln^2 n < 1, A\varepsilon_2 n^\beta \ln^2 n < 1 \), the operator \( \tilde{L}_n \) is continuously invertible and the equation (1.5) has a unique solution.

So, under conditions \( \varepsilon_1 \leq An^{-\beta}(\ln^2 n)^{-1}, \varepsilon_2 \leq An^{-\beta}(\ln^2 n)^{-1} \), the system (1.5) has a unique solution and collocation method (1.2) is stable.

Under conditions \( \|a(t) - \tilde{a}(t)\|_C \leq \varepsilon, \|b(t) - \tilde{b}(t)\|_C \leq \varepsilon, \|f(t) - \tilde{f}(t)\|_C \leq \varepsilon \), where \( \varepsilon \leq An^{-\beta}(\ln n)^{-1} \), similar statements we have for collocation method for equation

\[
a(t)x(t) + \frac{b(t)}{\pi i} \int_\gamma \frac{x(\tau)d\tau}{\tau - t} = f(t).
\]
The collocation method for singular integral equations was investigated in the item 2 of the chapter 1.

*Note.* We drop compact operator

\[ Tx \equiv \int_{\gamma} h(t, \tau)x(\tau)d\tau, \]

because stability of collocation method and method of mechanical quadrature for Fredholm integral equations of the second kind is well known. By this reason we have dropped compact operator in the equation (1.1) too.

### 1.2. Stability of Solutions of Multi-Dimensional Singular Integral Equations

Let us consider multi-dimensional singular integral equation

\[ Kx \equiv a(t)x(t) + b(t) \int_{G} \frac{\varphi(\Theta)x(\tau)}{(r(t, \tau))^2}d\tau = f(t). \quad (1.6) \]

Mechanical quadrature method for solution of the equation (1.6) was investigated in the item 3 of the chapter 2. This method in operator form can be written in the form of the equation (3.3) from the item 3 of the chapter 2. Below we will use the definitions introduced in the item 3 of the chapter 2. In the base of the proof of the solvability the equation (3.3) from the chapter 2 was put the Hadamard Theorem for invertibility of matrix. Using this theorem one can make the following conclusion.

Let elements of the system (3.3) are given with error \( \varepsilon \):

\[
|a(\bar{t}_{kl}) - \tilde{a}(\bar{t}_{kl})| \leq \varepsilon, \quad |f(\bar{t}_{kl}) - \tilde{f}(\bar{t}_{kl})| \leq \varepsilon,
\]

\[
k, l = 1, 2, \ldots, N;
\]

\[
|d_{ij}(\bar{t}_{kl}) - \tilde{d}_{ij}(\bar{t}_{kl})| \leq \varepsilon, \quad |d_{ij}(\bar{t}_{kl}) - \tilde{d}_{ij}(t_{kl})| \leq \varepsilon,
\]

\[
k, l = 1, 2, \ldots, N, \; i, j = 1, 2, \ldots, N.
\]
With values of ε such, that

\[ |(a(t_{kl}) \pm \varepsilon) + (d_{kl}(t_{kl}) \pm \varepsilon)| > \sum_{i=1}^{N} \sum_{j=1}^{N} |\tilde{d}_{ij}(t_{kl}) \pm \varepsilon|, \]

the system

\[
\tilde{a}(t_{kl})x_{kl} + \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{d}_{ij}(t_{kl})x_{ij} + x_{kl}\tilde{d}_{kl}(t_{kl}) = \tilde{f}(t_{kl}), \quad (1.7)
\]

\[ k, l = 1, 2, \ldots, N, \]

has a unique solution \{\tilde{x}_{kl}^*\} and for \( k, l = 1, 2, \ldots, N \), estimations \( \|\tilde{x}_{kl}^* - x_{kl}^*\|_{R_n} \leq A\varepsilon \) are valid, where \{\( x_{kl}^* \)\} is a solution of the system \( (3.3) \) from of the chapter 2. Here \( \sum \sum^* \) denote that the \( \sum \) is taken by \((i,j) \neq (k,l)\).

**1.3. Stability of Solutions of Nonlinear Singular Integral Equations**

Let us consider stability of solutions of nonlinear singular integral equation

\[
Kx \equiv a(t, x(t)) + \frac{1}{\pi i} \int_{\gamma} \frac{h(t, \tau, x(\tau))d\tau}{\tau - t} = f(t), \quad (1.8)
\]

where \( f(t) \in W^r(M), a(t, u) \in W^{rr}(M), a(t, \tau, u) \in W^{rrr}(M). \)

Collocation method and method of mechanical quadrature for the equation (1.8) was investigated in the item 6 of the Chapter 1. Below we will use designations which was introduced in that item.

Approximate solution of the equation (1.8) we look for in the form of the polynomial

\[
x_n(t) = \sum_{k=-n}^{n} \alpha_k t^k \quad (1.9)
\]
coefficients \( \{\alpha_k\} \) of which are defined from the system

\[
K_n x_n \equiv \tilde{P}_n^t \left[ a(t, x_n(t)) + \frac{1}{\pi i} \int P_n^\tau \left[ \frac{h(t, \tau, x_n(\tau))}{\tau - t} \right] d\tau \right] = \tilde{P}_n[f(t)]. \tag{1.10}
\]

In the Theorem 2.6 of the Introduction were given conditions for convergence of numerical scheme (1.10).

We will investigate stability of solution of the system (1.10) under conditions, that functions \( a(t, \tau), h(t, \tau, u), f(t) \) were given with error: \( |a(t, \tau) - \tilde{a}(t, \tau)| \leq \varepsilon, |h(t, \tau, u) - \tilde{h}(t, \tau, u)| \leq \varepsilon, |f(t) - \tilde{f}(t)| \leq \varepsilon \).

So, instead of the system (1.10), we have the system

\[
\tilde{K}_n x_n \equiv \tilde{P}_n^t \left[ \tilde{a}(t, x_n(t)) + \frac{1}{\pi i} \int P_n^\tau \left[ \tilde{h}(t, \tau, x_n(\tau)) \right] d\tau \right] = \tilde{P}_n^t[\tilde{f}(t)]. \tag{1.11}
\]

Let us introduce the functions

\[
a_\varepsilon(t, x_n(t)) = \tilde{P}_n^t[\tilde{a}(t, x_n(t)) - a(t, x_n(t))],
\]

\[
h_\varepsilon(t, \tau, x_n(\tau)) = \tilde{P}_n^t P_n^\tau[\tilde{h}(t, \tau, x_n(\tau)) - h(t, \tau, x_n(\tau))],
\]

\[
f_\varepsilon(t) = \tilde{P}_n^t[\tilde{f}(t) - f(t)].
\]

From Bernstein inequality and from Bernstein inverse theorems one can see, that

\[
\|a_\varepsilon(t, x_n(t))\|_\beta \leq A\varepsilon n^\beta \ln n,
\]

\[
\|h_\varepsilon(t, \tau, x_n(\tau))\|_\beta \leq A\varepsilon n^\beta \ln^2 n,
\]

\[
\|f_\varepsilon(t)\|_\beta \leq A\varepsilon n^\beta \ln n,
\]

\[
\|a_\varepsilon'(t, x_n(t))\|_\beta \leq A\varepsilon n^{1+\beta} \ln n,
\]

\[
\|a_\varepsilon'_{2}(t, x_n(t))\|_\beta \leq A\varepsilon n^{1+\beta} \ln n,
\]

163
\[ \| h'_{\varepsilon,3}(t, \tau, x_n(\tau)) \|_\beta \leq A \varepsilon n^{1+\beta} \ln^2 n. \]

Frechet derivatives of operators \( K_n \) and \( \tilde{K}_n \) have the forms

\[
K'_n(x^0_n)x_n \equiv \\
\equiv \tilde{P}_n \left[ a'_2(t, x^0_n(t))x_n(t) + \frac{1}{\pi i} \int P_n \left[ \frac{h'_3(t, \tau, x^0_n(\tau))x_n(\tau)}{\tau - t} \right] d\tau \right] = \tilde{P}_n[f(t)] \tag{1.12}
\]

and

\[
\tilde{K}'_n(x^0_n)x_n \equiv \\
\equiv \tilde{P}_n \left[ \tilde{a}'_2(t, x^0_n(t))x_n(t) + \frac{1}{\pi i} \int P_n \left[ \frac{\tilde{h}'_3(t, \tau, x^0_n(\tau))x_n(\tau)}{\tau - t} \right] d\tau \right] = \tilde{P}_n[\tilde{f}(t)] \tag{1.13}
\]

Repeating the arguments from the item 6 of the Chapter 1 one can see, that

\[ q_1 = \| \tilde{K}'_n(x^0_n)x_n - K'_n(x^0_n)x_n \|_\beta \leq A \varepsilon n^{1+\beta} \ln n \| x_n \|_\beta. \]

In the item 6 of the Chapter 1 was proved, that, for such \( n \) that

\[ q = A \| [K'(x_0)]^{-1} \|_\beta n^{-r+\beta} \ln n < 1, \]

the operator \( K'_n(x_0) \) is continuously invertible in the space \( H_\beta \) and \( \| [K'_n(x_0)]^{-1} \|_\beta \leq \| K'(x_0) \|_\beta/(1 - q) \).

From this and Banach Theorem of invertible operators follows, that, if \( \| [\tilde{K}'_n(x_0)]^{-1} \|_\beta q_1 < 1 \), the operator \( \tilde{K}'_n(x_0) \) is invertible and the estimate

\[ \| [\tilde{K}'_n(x_0)]^{-1} \|_\beta \leq \| [K'_n(x_0)]^{-1} \|_\beta/(1 - q_1) \]

is valid.
So, the equation (1.11) has a unique solution, which we design as $\tilde{x}_n^*$. Let us estimate the norm $\|x_n^*(t) - \tilde{x}_n^*(t)\|_\beta$, where $x_n^*(t)$ is a unique solution of the equation (1.10). For this aim we use the Newton – Kantorovich method for solution of the equation (1.11):

$$
\tilde{x}_n^{m+1}(t) = \tilde{x}_n^m(t) - [\tilde{K}'_n(\tilde{x}_n^0)]^{-1}(\tilde{K}_n(\tilde{x}_n^m) - \tilde{f}_n(t)), \quad m = 0, 1, \ldots, \quad (1.14)
$$

where $\tilde{f}_n(t) = \tilde{P}_n[\tilde{f}(t)]$, $\tilde{x}_n^0(t) = x_n^*(t)$.

It is easy to see that

$$
\|\tilde{x}_n^1(t) - \tilde{x}_n^0\|_\beta \leq
\leq \|[\tilde{K}'(\tilde{x}_n^0)]^{-1}\|_\beta \|\tilde{K}_n(\tilde{x}_n^0) - K_n(\tilde{x}_n^0)\|_\beta - (\tilde{f}_n(t) - f_n(t))\|_\beta \leq
\leq A\varepsilon n^\beta \ln \ln n.
$$

So, $\eta_0 = A\varepsilon n^\beta \ln n$ and, repeating arguments from the item 6 of the Chapter 1, we see that, under conditions which was described in that item, the estimate $\|\tilde{x}_n^*(t) - x_n^*(t)\|_\beta \leq A\varepsilon n^\beta \ln^2 n$ is valid.

From given above arguments we can make conclusion that, for so small $\varepsilon$ that $A\varepsilon n^\beta \ln^2 n < 1$, the method of mechanical quadrature is stable in the space $H_\beta$.

References

[1] D.N. Arnold and W.L. Wendland, The Convergence of Spline Collocation for Strongly Elliptic Equations on Curves, Numer. Math. 47, (1985), 317-341.

[2] K.I. Babenko, Some Problems of Approximation Theory and Numerical Analysis, Uspechi Matematicheskix Nauk, Vol. 40, No 1(241), 1985, 3-27 [In Russian]; also in: Russian Math. Surveys, Vol. 40, No 1, 1985, 1-30.
[3] N.S. Bakhvalov, The Properties of Optimal Methods of Solving Problems in Mathematical Physics, Zh. Vychisl. Mat. i Mat. Fiz., Vol. 10, No 3, 1970, 555-588 [In Russian]; also in: Computational Mathematics and Mathematical Physics, Vol. 10, No. 3, 1970, 1-19.

[4] N.S. Bakhvalov, On the Optimality of Linear Methods for Operator Approximation in Convex Classes of Functions, Zh. Vychisl. Mat. i Mat. Fiz., Vol. 11, No 4, 1971, 1014-1018 [In Russian]; also in: Computational Mathematics and Mathematical Physics, Vol. 11, No 4, 1971, 244-249.

[5] S.M. Belotserkovsky, Thin carrying surface in subsonic stream of gas, Moscow, Nauka, 1965 [In Russian].

[6] S.M. Belotserkovsky and I.K. Lifanov, Method of Discrete Vortices, CRC Press, 1993.

[7] G. Beylkin and R. Cramer, A multiresolution approach to regularization of singular integral equations, SIAM J. Sci. Comp., Vol. 24, No 1, 2002, 81 - 87.

[8] I.A. Blatov and N.V. Bubnova, On estimations of matrix units in wavelet-Galerkin methods for singular integral equations, Vestnik of Samara State University, Natural science series, Special issue, 2004, 68 - 79 [in Russian].

[9] N.N. Bogolubov, V.A. Mesheryakov, A.N. Tavkhelidze, An application of Muskhelishvili’s methods in the theory of elementary particles, Proceedings of Symposium on Solid Mechanics and Related Problems of Analysis, Metsniereba, Tbilisi, Vol. 1, 1971, 5-11 [in Russian].

[10] I.V. Boykov, The approximate solution of Fredholm integral equations with Cauchy-Hadamard kernels, in collection of works: Functional'nii analiz i teoria funkthii, Kazan State University, No 7, 1970, 3-23 [in Russian].
[11] I.V. Boykov, An Approximate Solution of Some Types of Integral Equations with Singularity, in the book: *Sbornik aspirantskich rabot. Tochnue nauki*, Kazan, Kazan State University, 1970, 73-82 [in Russian].

[12] I.V. Boykov, An Approximate Solution of Nonlinear Singular Integral Equation by Quadrature Method, in the book: *Sbornik aspirantskich rabot. Tochnue nauki*, Kazan, Kazan State University, 1970, 61-72 [in Russian].

[13] I.V. Boykov, An Application of Quadrature Method to Approximate Solution of Nonlinear Singular Integral Equations, in the book: *Funkthionalnii analiz i teoria funkthii*, Kazan, Kazan State University, 1971, 3-12 [in Russian].

[14] I.V. Boykov, Quadrature Methods for Solving Singular Integral Equations with Continuous Coefficients, in the book: *Sbornik aspirantskikh rabot. Tochnue nauki*, Kazan, Kazan State University, 1971, 140-147 [in Russian].

[15] I.V. Boykov, An Approximate Solution of Singular Integral Equations, *Dokl. Akad. Nauk SSSR*, Vol. 203, No 3, 1972, 511 - 514 [in Russian]; also in: *Soviet Math. Dokl.*, Vol. 13, 1972, 400 - 404.

[16] I.V. Boykov, To Approximate Solution of Singular Integral Equations, *Matematicheskie zametki*, Vol. 12, No 2, 1972, 177-186 [in Russian]; also in: *Mathematical Notes*, Vol. 14, No 2, 1972, 541 - 546.

[17] I.V. Boykov, An One Direct Method of Solution of Singular Integral Equations, *Zh. Vychisl. Mat. i Mat. Fis.* Vol. 12, No 6, 1972, 1381-1390 [in Russian]; also in: *Computational Mathematics and Mathematical Physics*, Vol. 12, No 3, 1972, 1-19.
[18] I.V. Boykov, Approximate Solution of Integro-Differential Equations with Hadamard Integrals, in the book: *Uche- nie zapiski Penzenskogo politichnicheskogo instituta*, Penza, Vol. 4, 1973, 42-61 [in Russian].

[19] I.V. Boykov, An Approximate Solution of Nonlinear Singular Integral Equations by Iteration Methods, in the book: *Razlogenie po ortogonalnum mnogochlenam*, Kazan, Kazan State University, 1973, 10-13 [in Russian].

[20] I.V. Boykov, The Approximate Solution of Singular Integral Equations, *Dokl. Akad. Nauk SSSR*, Vol. 16, No 5, 1975, 1367-1371 [in Russian].

[21] I.V. Boykov, An Approximate Methods for Solution of Singular Integral Equations of Gravitation Measurement, in the book: *Issledovania po dinamicheskoi gravimetrii*, Moscow, Institut Fiziki Zemli, 1977, 118-152 [in Russian].

[22] I.V. Boykov, Approximate Solution of Singular Integral Equations in Exceptional Cases, in the book: *Metodi izmerenii i obrabotki nabludenii v morskoi gravimetrii*, Moscow, Institut Fiziki Zemli, 1980, 108-124 [in Russian].

[23] I.V. Boykov, Approximate Solution of Multi-dimensional Singular Integral Equations and Its Applications, in the book: *Primenenie vichislitelnikh metodov v nauchno-tekhnicheskikh issledovanijakh*, Penza, Penza polytechnical institute, 1980, 3-18 [in Russian].

[24] I.V. Boykov, *Optimal Methods for Approximate Calculation of Integrals and Approximate Solution of Integral Equations*, Penza, Penza polytechnical institute, 1981, 105 pp. [in Russian].

[25] I.V. Boykov, Projection Methods for Solution of Singular Integral Equation of the Elasticity Theory, in the
book: Primenenie vichislitel’nikh metodov v nauchno-
tekhnicheskih issledovaniyakh, Penza, Penza polytechnical
institute, 1981, 3-21 [in Russian].

[26] I.V. Boykov, *Optimal Methods of Calculation in Auto-
matic Control Problems*, Penza, Penza Politechnical Insti-
tute Press, 1983. 96 pp. [in Russian].

[27] I.V. Boykov, *Optimal with Respect to Accuracy Algorithms
for Approximate Calculation of Singular Integrals*, Saratov,
Saratov State University Press, 1983. 210 pp. [in Russian].

[28] I.V. Boykov, At One Exceptional Case in Singular Integral
Equations, in the book: Primenenie vichislitel’nikh
metodov v nauchno-tekhnichestvikh issledovaniyakh, Penza,
Penza Polytechnical Institute, 1984, 3-11 [in Russian].

[29] I.V. Boykov, *Optimal with Respect to Accuracy Algorithms
for Calculation of Singular Integrals and Solution of Sin-
gular Integral Equations*, Dissertation of Doctor of Sci-
ences, Novosibirsk, Vichislitel’niy Center of Siberian branch
of Academy Science of USSR, 1991, 474 p. [In Russian].

[30] I.V. Boykov, *Passive and Adaptive Algorithms for Approx-
imate Calculation of Singular Integrals*, Ch. 1, Publishing
Penza Technical State Univ., 1995, 214 p. [in Russian].

[31] I.V. Boykov, *Passive and Adaptive Algorithms for Approx-
imate Calculation of Singular Integrals*, Ch. 2, Publishing
Penza Technical State Univ., 1995, 128 p. [in Russian].

[32] I.V. Boykov, The Optimal Algorithms for Calculation of
Singular Integrals, Decision of Singular Integral Equations
and Its Applications, *15 th IMACS World Congress on Sci-
entific Computation, Modelling and Applied Mathematics,
Berlin, Vol. 1*, Computational Mathematics, Wissenschaft
Technik Verlag, Berlin, 1997, 587-592.
[33] I.V. Boykov, Numerical methods of computation of singular and hypersingular integrals, *International Journal of Mathematics and Mathematical Sciences*, Vol. 28, No 3, 2001, 127-179.

[34] I.V. Boykov, Optimal methods for calculation of polysingular integrals and solution polysingular integral equations, *Trudi Srednevolgskogo matemativcheskogo obshestva*, Vol. 5, No 1, 2003, 109-118 [in Russian].

[35] I.V. Boykov, *Approximate methods for solution of singular integral equations*, Penza, Publishing Penza State Univ., 2004, 316 p. [in Russian].

[36] I.V. Boykov, Parallel calculations in singular integral equations, *International Conference on Computational Mathematics*, Part two, Novosibirsk, 2004, 806-812 [in Russian].

[37] I.V. Boykov, N.F. Dobrunina and L.N. Domnin, *Approximate Methods for Calculation of Hadamard Integrals and Solution of Hypersingular Integral Equations*, Penza, Publishing Penza Technical State Univ., 1996, 188 p. [in Russian].

[38] I.V. Boykov and N.Ju. Kudryashova, Approximate methods for singular integral equations in exceptional cases, *Differentialnie uravnenija*, Vol. 36, No 9, 2000, 1230-1237 [in Russian]; also in: *Differential equations*, Vol. 36, No 9, 2000, 1360-1369.

[39] I.V. Boykov and N.Ju. Kudryashova, An approximate metod for solution of nonlinear singular integral equations in exceptional cases, *Trudi Mezdunarodnogo simposiuma Nadeznost i kachestvo*, Penza, 2002, 187-189 [in Russian].

[40] I.V. Boykov and Ju. F. Zakharova, An approximate methods for solution of multi-dimensional singular integral
equations, *Trudi Mezdunarodnogo simposiuma Nadeznost i kachestvo*, Penza, 2002, 185-187 [in Russian].

[41] I.V. Boykov and Ju. F. Zakharova, An approximate methods for solution of multi-dimensional singular integral equations, *Voprosi matematicheskogo analiza*, Krasnoyarsk, Publishing Krasnoyarsk Tech. Univ., Vol. 6, 2003, 30-50 [in Russian].

[42] I.V. Boykov and I.I. Zhechev, Approximate Solution of Singular Integro-Differential Equations, in the book: *Shornik aspirantskich rabot. Tochnue nauki*, Kazan, Kazan State University, 1972, 169-172 [in Russian].

[43] I.V. Boykov and I.I. Zhechev, Approximate Solution of Singular Integro-Differential Equations on closed contours of integration, in the book: *Issledovaniay po prikladnoi matematike*, Kazan, Kazan State University, Vol. 2, 1974, 3-17 [in Russian]; also in: *Journal of Soviet Mathematics*, Vol. 41, No 3, 1988, 1003 - 1013.

[44] I.V. Boykov and I.I. Zhechev, To Approximate Solution of Singular Integro-Differential Equations (Linear Equations), *Differentialnie uravneniya*, Vol. 9, No 8, 1973, 1493-1502 [in Russian].

[45] I.V. Boykov and I.I. Zhechev, Approximate Solution of Singular Integro-Differential Equations on Open Circles, in the book: *Prilogenie funktsionalnogo analiza k pribligenim vichsleniaym*, Kazan, Kazan State University, 1974, 21-28 [in Russian].

[46] I.V. Boykov and I.I. Zhechev, To Approximate Solution of Singular Integro-Differential Equations (Nonlinear Equations), *Differentialnie uravneniya*, Vol. 11, No 3, 1975, 562-571 [in Russian].
[47] A.I. Boykova, On One Class of Interpolated Polynomials, in the book: Optimalnie Methodi Vychislenii i ikh Prime- nenie, Penza, Publishing Penza Technical State University, Vol. 12, 1996, 141-149 [in Russian].

[48] A.I. Boykova, Methods for approximate calculation of the Hadamard integral and solution of integral equations with Hadamard integrals, C. Constanda, J. Saranen and S. Siekkala (editors), in the book: Integral Methods in Science and Engineering, Vol. two: Approximate methods, Addison Wesley Longman Limited, 1997, 59-63.

[49] A.I. Boykova, On One Approximate Method for Solution of Differential and Integral Equations, in the book: Matematicheskii analiz, Krasnoyarsk, 1997, 93-102 [in Russian].

[50] A.I. Boykova, An Oder One Approximate Method of Solution of Differential and Integral Equations, B. Bertram, C. Constanda, S. Struchers (editors), in the book: Integral Methods in Science and Engineering, Research Notes in Mathematics, Vol. 418, CRC PRESS, London, 2000, 73-78.

[51] D.D. Brown and A.D. Jackson, The Nucleon-Nucleon Interaction, North-Holland, Amsterdam, 1976.

[52] L.A. Chikin, Special cases of the Riemann boundary value problems and singular integral equations, Scientific Notes of the Kazan State University, 113, 53-105 [in Russian].

[53] R.V. Duducava, On Bisingular Integral Operators with Discontinuous Coefficients, Mathematics of the USSR - Sbornik, Vol. 30, No 4, 1976, 515 - 540 [in Russian].

[54] A.V. Dzhishkariani, To solution of singular integral equations by approximate projective methods, Zh. Vychisl. Mat. i Mat. Fiz., Vol. 19, No 5, 1979, 1149-1151 [in Russian]; also
in: *Computational Mathematics and Mathematical Physics*, Vol. 19, No 5, 1979, 61-74.

[55] A.V. Dzhishkariani, To Solution of Singular Integral Equations by Collocation Methods, *Zh. Vychisl. Mat. i Mat. Fiz.* Vol. 21, No 2, 1981, 355-362 [in Russian]; Also in: *Computational Mathematics and Mathematical Physics*, Vol. 21, No 2, 1981, 99-107.

[56] A. Dzhishkariani, *Approximate solution of one class of singular integral equations by means of projective and projective-iterative methods*, Memoirs of Differential Equations and Mathematical Physics, Vol. 34, 2005, 1-76.

[57] D. Elliott, The classical collocation method for singular integral equations, *SIAM J Numer Anal*, Vol. 19, 1982, 816-832.

[58] D. Elliott, Rates of convergence for the method of classical collocation for solving singular integral equations, *SIAM J Numer Anal*, Vol. 21, 1984, 136-148.

[59] D. Elliott, The numerical treatment of singular integral equations-a review, In: C. T. H. Baker and G. F. Miller, eds. *Treatment of Integral Equations by Numerical Methods*, New York,: Acad. Press, 1982.

[60] D. Elliott, Projection methods for singular integral equations, *J. of Integral Equations and Applications*, Vol. 2, 1989, 95-106.

[61] F. Erdogan and G.D. Gupta, On the numerical solution of singular integral equations, *Quart. Appl. Math.*, Vol. 24, 1974, 525 - 534.

[62] B.G. Gabdulkhaev, Approximate solution of singular integral equations with method of mechanical quadrature, *Dok-*
ladi Akademii Nauk SSSR, Vol. 179, No 2, 1968, 260-264 [in Russian].

[63] B.G. Gabdulkhaev, Direct methods for solution of some operator equations, Izvestia Vishich Uchebnich Zavedeniy, Matematika, No 4, 1972, 32-43 [in Russian].

[64] F.D. Gakhov, Boundary value problems, Dover Publication, USA, 1990, 561 p.

[65] F.R. Gantmacher, The Theory of Matrices, Vol. 1, AMS Chelsea Publishing, Providence, R.I., 1977, 277 p.

[66] F.R. Gantmacher, The Theory of Matrices, Vol. 2, AMS Chelsea Publishing, Providence, R.I., 1989, 281 p.

[67] A. Gerasoulis, On the existence of approximate solutions for singular integral equations of Cauchy type discretised by Gauss-Chebyshev quadrature formulae, BIT, 21, 1981, 377-380.

[68] I.C. Gohberg and I.A. Fel’dman, Convolution Equation and Projection Methods for Their Solution, Nauka, Moscow, 1971 [in Russian]; also in: Transl. Math. Monographs, Vol. 41, Amer. Math. Soc., Providence R.I., 1974.

[69] M.A. Golberg, The convergence of several algorithms for solving integral equations with finite-part integrals, 1993. I, J. Integral Equations, Vol. 5, No 4, 329-340.

[70] M.A. Golberg, The convergence of several algorithms for solving integral equations with finite-part integrals, II, J. Integral Equations, 1997. Vol. 9, No 3, 267-275.

[71] M.A. Golberg, Introduction to the numerical solution of Cauchy singular integral equations, Math. Concepts and Methods in Science and Appl., Vol. 42, Plenum Press, New York, 1990, 83-107.
[72] V.L. Goncharov, *The theory of interpolation and approximation of functions*, Moscow, 1954 [in Russian].

[73] L. Gori and E. Santi, On the numerical solution of Cauchy singular integral equations. 2. A projector-splines method for solution, *Advanced Mathematical Tools in Metrology. II* Edited by P. Ciarlini, M. G. Cox, F. Pavese, D. Richter, 1996, World Scientific Publishing Company, 70-80.

[74] I.G. Graham, Collocation methods for two dimensional weakly singular integral equations, *Journal of the Australian Math. Society*, Seria B, Vol. 22, 1981, 456-473.

[75] I.G. Graham and Y. Yan, Piecewise-constant collocation for first-kind boundary integral equations, *Journal of the Australian Math. Society*, Seria B, Vol. 33, No 1, 1991, 39-64.

[76] N.M. Gunter, *Potential Theory and its Applications to Basic Problems of Mathematical Physics*. Ungar. New-York. 1967.

[77] A.I. Guseinov and X.Sh. Muchtarov, *Introduction to Theory of Nonlinear Singular Integral Equations*, Moscow, Nauka, 1980 [in Russian].

[78] L. Faddeev and L. Takhtajan, *Hamiltonian Approach to Soliton Theory*, Springer, 1986.

[79] F. Hartmann and E.P. Stephan, Rates of convergence collocation with Jacobi polynomials for the airfoil equation, *Journal of Computational and Applied Mathematics*, 1991, Vol. 51, 179-191.

[80] N. I. Ioakimidis, On the nature interpolation formula for Cauchy type singular integral equations of the first kind, *Computing*, Vol. 16, 1981, 73-77.
[81] N. I. Ioakimidis and P. S. Theocaris, On convergence of two direct methods for solution of Cauchy type singular integral equations of the first kind, *BIT*, Vol. 20, 1980, 83-87.

[82] V.V. Ivanov, An using the moment method and the mixed method for solution of singular integral equations, *Soviet Math. Dokl.* Vol. 114, No 5, 1957, 945-948.

[83] V.V. Ivanov, *The Theory of Approximate Methods and their Application to the Numerical Solution of Singular Integral Equations*, Kiev, Naukova dumka, 1968 [in Russian]; also in: *The Theory of Approximate Methods and their Application to the Numerical Solution of Singular Integral Equations*, Noordhoff International Publishing, Leiden, The Netherlands, 1976, 330 p.

[84] Lu Jianke, *Boundary Value Problems for Analytic Functions*, World Scientific, 1993.

[85] Du Jinyuan, On methods for numerical solutions for singular integral equations (I), *Acta Math Sci*, Vol. 7, No 2, 1987, 169-189.

[86] Du Jinyuan, On methods for numerical solutions for singular integral equations (II), *Acta Math Sci*, Vol. 8, No 1, 1988, 33-45.

[87] Du Jinyuan, Singular integral operators and singular quadrature operators associated with singular integral equations of the first kind and their applications, *Acta Math. Sci.*, Vol. 15, No 2, 1995, 219-234.

[88] V.A. Kakichev, *Methods of solution of some boundary task for two-dimensional analytical functions*, Tumen, Tumen State University, 1978 [in Russian].
[89] A.I. Kalandia, *The mathematical task of two-dimensional theory elasticity*, Moscow, Nauka, 1972 [in Russian].

[90] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Moscow, Nauka, 1977 [in Russian]; also in *Functional Analysis*, Elsevier, 2014, 604 p.

[91] A.C. Kaya and F.Erdogan, On The Solution of Integral Equations with Strongly Singular Kernels, *Quart. Appl. Math.*, Vol. 95, 1987, 105-122.

[92] N.P. Kornechuk, *Extreme tasks of approximation theory*, Moscow, Nauka, 1976, 320 p. [in Russian].

[93] N.P. Korneichuk, *Splines in the Theory of the Approximation*, Moscow, Nauka, 1984 [in Russian].

[94] A.V. Kozak and I.B. Simonenko, An projective methods for solution of two dimensional singular integral equations on a toruc, *Functional analysis and its applications*, 1978, Vol. 12, No 1, 74-75.

[95] M.A. Krasnoselskii, G.M.Vainikko, P.P.Zabreyko, Y.B.Ruticki and V.Y.Stetsenko, *Approximate Solution of Operator Equations*, Moscow, Nauka, 1969, 456 p.[in Russian]; also in: *Approximate solution of Operator Equations*, Wolters-Noordhoff Publ., Groninger, 1972.

[96] S. Krenk, Numerical quadrature of periodic singular integral equations, *J. Inst. Math. Appl.*, 1978, Vol. 21, 181-187.

[97] Ju. M. Krikunov, A general boundary Riemann problem and linear singular integro-differential equation, *The Scientific Notes of the Kazan University*, Vol. 116, No 4, 1956, 3-30 [in Russian].
[98] M.A. Lavrent’ev, An construction of stream which flowing the given arc, *Trudi TzAGI*, 118, 1932, 3 - 56 [in Russian].

[99] M.M. Lavrentyev, An One Class of Singular Integral Equations, *Siberian Math. J.*, Vol. 21, No 3, 1980, 225-228 [in Russian].

[100] I.K. Lifanov, *Singular Integral Equations and Discrete Vortices*, VSP, Utrecht, The Netherlands, 1996, 475 p.

[101] I.K. Lifanov, L.N. Poltavskii and G.M. Vainikko, *Hypersingular integral equations and their applications*, CRC, 2003. 408 p.

[102] G.G. Lorentz, *Approximation of functions*, Chelsia Publishing Company, N.Y., 1986, 190 p.

[103] L.A. Lusternik and V.I. Sobolev, *Elements of functional analysis*, Moscow, Nauka, 1965, 510 p. [in Russian].

[104] S.G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Moscow, Fizmatgiz,1962, 256 p. [in Russian]; also in: International Series of Monographs in Pure and Applied Mathematics, Vol. 83, 1965, 259 p.

[105] S.G. Mikhlin and S. Prossdorf, *Singulare Integraloperatoren*, Berlin, Acad.- Verl., 1980, 514 p.

[106] S.G. Mikhlin and P.C. Radeva, An approximate solution for singular integral equations, *Izvestia Vischikch Uchebnich Zavedenii, Mathematika*, Vol. 5, 1974, 158-162 [in Russian].

[107] N.I. Muskhelishvili, *Singular Integral Equations*, Groningen, Nordhoff, 1963.

[108] B.I. Musaev, *Constructive methods in the theory of singular integral equations*, Dissertation of Doctor of Sciences, Tbilisi, Tbilisi State University, 1988 [in Russian].
[109] I.P. Natanson, *Constructive Function Theory. Volume I. Uniform Approximation*, Frederick Undar Publishing Co., N.Y., 1965.

[110] I.P. Natanson, *Constructive Function Theory. Volume II. Approximation in Mean*, Frederick Undar Publishing Co., N.Y., 1965.

[111] S.M. Nikolskii, *Approximation of functions of multivariables and theorems of embedding*, Moscow, Nauka, 1977, 456 p. [in Russian].

[112] L.I. Oblomskaya, Methods of successive approximation for the linear equations in Banah spaces, *Zh. Vychisl. Mat. i Mat. Fiz.*, Vol. 8, No 2, 1968, 417-426 [In Russian]; also in: *USSR Computational Mathematics and Mathematical Physics*, Vol. 8, No 2, 1968, 239-253.

[113] V.Z. Parton and P.I. Perlin, *Integral equations in elasticity*, Moscow, Nauka, 1977, [in Russian]; also in: Moscow, Mir, 1982, 303 p.

[114] S. Prossdorf, *Einige Klassen sungularer Gleichungen*, Akademia-Verlag-Berlin, 1974 [in Germany]; also in: North Holland Amsterdam, 1978; also in: Moscow, Mir., 1979.

[115] S. Prossdorf, *Approximation Methods for Solving Singular Integral Equations*, Preprint - P. - Math. -12/81, Berlin, 1981, 31 p.

[116] S. Prossdorf and G. Shmidt, A Finite Element Collocation Method for Singular Integral Equations, *Math. Nachr.*, Vol. 100, 1981, 33-60.

[117] S. Prossdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations*, Berlin, Acad. Verl., 1991, 544 p.
[118] A.P. Prudnikov, I.A. Bruchkov and O.I. Marichev, *Integrals and Series*, Moscow, Nauka, 1981 [in Russian].

[119] P.C. Radeva, Approximate solution for linear equations with not unique solution, *Izvestia Vyschikh Uchebnich Zavedenii, Matematika*, 1978, Vol. 2, 76-80 [in Russian].

[120] L. S. Rakovshchik, On the Newton - Kantorovich method, *Zh. Vychisl. Mat. i Mat. Fiz.*, Vol. 8, No 6, 1968, 1208-1217 [in Russian]; also in: *USSR Computational Mathematics and Mathematical Physics*, Vol. 8, No 6, 1968, 31 -43.

[121] M. Riesz, Eine trigonometrische Interpolationsformel and einige Ungleichungen fur polynome, *Deutsche Mat. Ver.*, Vol. 23, 1914, 354 - 368.

[122] D.G. Sanikidze, *Numerical processes for Singular Integrals with Cauchy kernel and some their applications*, Dissertation of Doctor of Sciences, Moscow, Moscow Fiziko-technicheskii institute, 1986 [in Russian].

[123] M.A. Sheshko, On the convergence of quadrature process for singular integral, *Sov. Math.*, Vol. 20, 1976, 86-92.

[124] M.A. Sheshko, *Singular Integral Equations with Cauchy and Hilbert Kernels and Their Approximated Solutions*, Catholic University of Lublin, Lublin, 2003.

[125] I.B. Simonenko, On the question of the solvability of bisingular and polysingular equations, *Functional Analysis and Its Applications*, Vol. 5, No 1, 1971, 81-83 [in Russian].

[126] V.S. Sizikov, A.V. Smirnov and A.V. Fedotov, Numerical solution of the singular Abel integral equation by the generalized quadrature methods, *Izv. Vysh. Uchebn. Zaved., Matematika*, No. 8, 2004, 62-70 [in Russian]; also in: *Rus-
sian Mathematics (Izvestiya VUZ., Mathematika), No 8, 2004, 59 - 66.

[127] R.P. Srivastav and E. Jen, On the polynomials interpolating approximate solutions of singular integral equations, *Appl. Anal.*, Vol. 14, 1983, 275-285.

[128] G. Szego, *Orthogonal Polynomials*, NY, 1959.

[129] *Theoretical Bases and Construction of Numerical Algorithms for the Tasks of Mathematical Physics*, (Ed. K.I. Babenko), Moscow, Nauka, 1979, 296 p. [in Russian].

[130] A.F. Timan, *Theory of approximation functions of real variable*, Moskow. Fizmatlit. 1960. 624 p. [in Russian].

[131] A.N. Tikhonov and V.J. Arsenin, *Solutions of Ill-Posed Problems*, Wiley, New-York, 1977.

[132] S. Tsamasphyras and P.S. Theocaris, Equivalence and Convergence of Direct and Indirect Methods for the Numerical Solution of Singular Integral Equations, *Computing*, Vol. 27, 1981, 71-80.

[133] I.F. Traub and H. Wozniakowski, *A General Theory of Optimal Algorithms*, N. Y., Academic Press, 1980.

[134] E. Venturino, Recent developments in the numerical solution of singular integral equations, *J of Math Anal Appl*, Vol. 115, 1986, 239-277.

[135] V.V. Voevodin and V.V. Voevodin, *Parallel Calculations*, S.-Peterburg, BHV-Petersburg, 2002, 608 p. [in Russian].

[136] N.F. Vorob’ev, *Carrying surfaces in stationary stream*. Novosibirsk, Nauka, 1985, 240 p. [in Russian].
[137] I.I. Zhechev, Approximate Solution of System of Nonlinear singular Integro-differential Equations on Closed Curves, *Nature, Plovdiv*, Vol. 6, No. 1, 1973, 19-25.