PRODUCTS OF UNBOUNDED NORMAL OPERATORS

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Abstract. The present paper partly constitutes an "unbounded" follow-up of a paper by I. Kaplansky dealing with bounded products of normal operators. Results on the normality of unbounded products are also included.

1. Introduction

In this paper we are concerned with the normality of the products $AB$ and $BA$ for two operators, where $B$ is unbounded. The problems of this sort lie among the most fundamental questions in the Hilbert space theory and were explored by many authors, especially in the bounded case (see [7, 8, 19, 20]). See also the paper [3] and the references therein. Nonetheless, only a few shy attempts where made in the unbounded case. See the very recent papers [6] and [?].

We also cite the reference [11] where the following result (among others) has been obtained

**Theorem 1.**

(1) Assume that $B$ is a unitary operator. Let $A$ be an unbounded normal operator. If $B$ and $A$ commute (i.e. $BA \subset AB$), then $BA$ is normal.

(2) Assume that $A$ is a unitary operator. Let $B$ be an unbounded normal operator. If $A$ and $B$ commute (i.e. $AB \subset BA$), then $BA$ is normal.

For results involving normality and self-adjointness of unbounded operator products, see [9, 10, 11, 6]. For similar papers on the sum of two normal operators, see [12] and [14].

One purpose of this paper is to try to get an analog for unbounded operators of the following result

**Theorem 2.** [Kaplansky, 7] Let $A$ and $B$ be two bounded operators on a Hilbert space such that $AB$ and $A$ are normal. Then $B$ commutes with $AA^*$ iff $BA$ is normal.

In order to do that and to allow a broader audience to read the present paper, we recall basic definitions and results on unbounded operators. Some important references are [1, 4, 5, 17].

All operators are assumed to be densely defined (i.e. having a dense domain) together with any operation involving them or their adjoints. Bounded operators are assumed to be defined on the whole Hilbert space. If $A$ and $B$ are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then $B$ is called an extension of $A$, and we write $A \subset B$, if $D(A) \subset D(B)$ and if $A$ and $B$ coincide on $D(A)$. We

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write $A \subseteq B$ if $A \subset B$ or $A = B$ (meaning that $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$).

If $A \subset B$, then $B^* \subset A^*$. An unbounded operator $A$ is said to be closed if its graph is closed; self-adjoint if $A = A^*$ (hence from known facts self-adjoint operators are automatically closed); normal if it is closed and $AA^* = A^*A$ (this implies that $D(AA^*) = D(A^*A)$).

A densely defined operator $A$ is said to be hyponormal if $D(A) \subset D(A^*)$ and $\|A^*x\| \leq \|Ax\|$ for $x \in D(A)$. We say that a densely defined operator $S$ in a Hilbert space $H$ is subnormal if there is another Hilbert space $L \supset H$ and a normal operator $N$ in $L$ such that $S \subset N$. It is wellknown that each subnormal operator is hyponormal and that each hyponormal operator is closable.

The Fuglede-Putnam theorem (see [2] and [16]) is important to prove our results so we recall it here.

**Theorem 3.** Let $A$ be a bounded operator. Let $N$ and $M$ be two unbounded normal operators. If $AN \subseteq MA$, then $AN^* \subseteq M^*A$.

We digress a little bit to say that a new version of this famous theorem has been obtained by the author where all the operators involved are unbounded. See [13].

It is also convenient to recall the following theorem which appeared in [18], but we state it in the form we need.

**Theorem 4.** If $T$ is a closed subnormal (resp. closed hyponormal) operator and $S$ is a closed hyponormal (resp. closed subnormal) operator verifying $XT^* \subset SX$ where $X$ is a bounded operator, then both $S$ and $T^*$ are normal once $\ker X = \ker X^* = \{0\}$.

2. **Main Results**

We start by giving a counterexample that shows that the same assumptions, as in Theorem 2, would not yield the same results if $B$ is an unbounded operator, let alone the case where both operators are unbounded.

What we want is a normal bounded operator $A$ and an unbounded (and closed) operator $B$ such that $BA$ is normal, $A^*AB \subset BA^*A$ but $AB$ is not normal.

**Example 1.** Let

$$Bf(x) = e^{x^2}f(x) \text{ and } Af(x) = e^{-x^2}f(x)$$

on their respective domains

$$D(B) = \{f \in L^2(\mathbb{R}) : e^{x^2}f \in L^2(\mathbb{R})\} \text{ and } D(A) = L^2(\mathbb{R}).$$

Then $A$ is bounded and self-adjoint (hence normal). $B$ is self-adjoint (hence closed).

Now $AB$ is not normal for it is not closed as $AB \subset I$. $BA$ is normal as $BA = I$ (on $L^2(\mathbb{R})$). Hence $AB \subset BA$ which implies that

$$AAB \subset ABA \Rightarrow AAB \subset ABA \subset BAA.$$ 

Now, we state and prove the generalization of Theorem 2. We have

**Theorem 5.** Let $B$ be an unbounded closed operator and $A$ a bounded one such that $AB$ (resp. $BA$) and $A$ are normal. Then

$$BA \text{ normal (resp. } AB) \Rightarrow A^*AB \subset BA^*A.$$
Proof. Since $AB$ and $BA$ are normal, the equation
$$A(BA) = (AB)A$$
implies that
$$A(BA)^* = (AB)^*A$$
by the Fuglede-Putnam theorem. Hence
$$AA^*B^* \subset B^*A^*A \text{ or } A^*AB \subset BA^*A.$$ 

□

We already observed in Example [1] that the converse in the previous theorem does not hold. An extra hypothesis combined with a result by Stochel [18] yield the following

**Theorem 6.** If $B$ is an unbounded closed operator and and if $A$ is a bounded one such that $AB$ and $A$ are normal, and if further $BA$ is hyponormal (resp. subnormal), then

$$BA \text{ normal } \iff A^*AB \subset BA^*A.$$ 

Proof. The idea of proof is similar in core to Kaplansky’s ([7]). Let $A = UR$ be the polar decomposition of $A$, where $U$ is unitary and $R$ is positive (remember that they also commute and that $R = \sqrt{A^*A}$), then one may write
$$U^*ABU = U^*URBU = RBU \subset BRU = BA$$
or
$$U^*AB = U^*\sqrt{AB} = U^*((AB)^*)^* \subset BAU^*$$
(by the closedness of $AB$). Since $(AB)^*$ is normal, it is closed and subnormal. Since $B$ is closed and $A$ is bounded, $BA$ is closed. Since it is hyponormal, Theorem [4] applies and yields the normality of $BA$ as $U$ is invertible.

The proof is very much alike in the case of subnormality. □

Now, if we assume that $A$ is unitary, then we have the following interesting result (cf Theorem [1]) that bypasses the commutativity of operators. Besides, this constitutes generalization of Theorem [2] with the assumption $A$ unitary.

**Theorem 7.** If $A$ is unitary and $B$ is an unbounded normal operator, then

$$BA \text{ is normal } \iff AB \text{ is normal}.$$ 

Proof. First, recall that self-adjoint operators are maximally symmetric, that is, if $T$ is self-adjoint and $S$ is symmetric, then $T \subset S \Rightarrow T = S$ (we shall call this the MS-property). Second, if $T$ is closed, then $TT$ and $TT^*$ are both self-adjoint (see [17] for both results).

Now, assume that $BA$ is normal. To show that $AB$ is normal observe that it is first closed (which is essential) thanks to the invertibility of $A$ and the closedness of $B$. We then have
$$(AB)^*AB = B^*A^*AB = B^*B$$
and
$$AB(AB)^* = ABB^*A^*$$
and we must show the equality of the two quantities. We have
$$A^*B^*BA \subset (BA)^*BA \text{ and } BB^* \subset BA(AB)^*.$$
By the closedness of $B$ and that of $BA$, and the MS-property

$$BB^* = BA(BA)^*.$$  

By the normality of $BA$, we obtain

$$A^*B^*BA \subset BB^* \text{ or } A^*B^*B \subset BB^*A^*.$$  

Hence

$$AB(AB)^* = ABB^*A^* \supset AA^*B^*B = B^*B = (AB)^*AB.$$  

Therefore, and by the MS-property again, we must have

$$AB(AB)^* = (AB)^*AB.$$  

To prove the converse, we may argue similarly with some minor changes. Suppose that $AB$ is normal and let us show that $BA$ is normal too. 
Firstly, note that $BA$ is closed, but this time, since $B$ is closed and $A$ is bounded.  
Secondly, and by the normality of $AB$, we have

$$(AB)^*AB = B^*B = AB(AB)^* = ABB^*A^*$$  

and hence

$$ABB^* = B^*BA.$$  

Accordingly, we have

$$(BA)^*BA \supset A^*B^*BA = A^*ABB^* = BB^*$$  

and

$$BA(BA)^* \supset BB^*.$$  

By the MS-property, we must have

$$(BA)^*BA = BB^* \text{ and } BA(BA)^* = BB^*,$$  

proving the normality of $BA$. The proof is complete.  

\[ \square \]

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