Path-Dependent Optimal Stochastic Control and Viscosity Solution of Associated Bellman Equations*

Shanjian Tang and Fu Zhang†

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Abstract

In this paper we study the optimal stochastic control problem for a path-
dependent stochastic system under a recursive path-dependent cost functional,
whose associated Bellman equation from dynamic programming principle is a
path-dependent fully nonlinear partial differential equation of second order. A
novel notion of viscosity solutions is introduced by restricting the semi-jets on
an $\alpha$-Hölder space $C^\alpha$ for $\alpha \in (0, \frac{1}{2})$. Using Dupire’s functional Itô calculus, we
characterize the value functional of the optimal stochastic control problem as the
unique viscosity solution to the associated path-dependent Bellman equation.

Keyword: path-dependent optimal stochastic control, path-dependent Bellman equa-
tion, viscosity solution, backward SDE, dynamic programming.

1 Introduction

Since the initial investigation of Krylov [26], it has been the subject of many studies
to verify that the value function of an optimal stochastic control problem should be
the unique solution of the associated Bellman equation from the dynamic programming
principle for the optimal stochastic control problem. Nowadays, the notion of viscosity

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†Institute of Mathematical Finance and Department of Finance and Control Sciences, School of
Mathematical Sciences, Fudan University, Shanghai 200433, China. Email: sjtang@fudan.edu.cn
(Shanjian Tang), 09110180028@fudan.edu.cn (Fu Zhang)
solution invented in 1983 by Crandall and Lions \cite{10} has become a universal tool to study such a broad fundamental subject. For detailed exposition of such a tool and the related general dynamic programming theory on optimal stochastic control, see among others the survey paper of Crandall, Ishii & Lions \cite{9} and the monographs of Fleming & Soner \cite{20} and Yong & Zhou \cite{51}.

Such a theme has also been developing in terms of backward stochastic differential equations (BSDEs). For instance, a BSDE depending on a Markovian diffusion in a Markovian way via the generator and the terminal condition, is associated to a second-order partial differential equation (PDE), and a fully coupled forward and backward stochastic differential equation (FBSDE) is associated to a quasi-linear PDE. For relevant details, see Pardoux & Peng \cite{37}, Ma, Proter & Yong \cite{35}, and Pardoux & Tang \cite{38}. These developments appear quite natural in view of the close relation between BSDEs and minimax problems as exposed by Tang \cite{48}. Furthermore, the second-order BSDE (2BSDE) is associated to a fully nonlinear PDE, and for such a relation see Cheritdito, Soner, Touzi & Victoir \cite{6} and Soner, Touzi & Zhang \cite{45}. More generally, Peng \cite{39} shows that an optimal stochastic control problem—where the coefficients of both the system and the cost functional depend on the history path of the underlying Brownian motion—should be associated to a fully nonlinear backward stochastic PDE as the underlying Bellman equation.

Recently Dupire \cite{14} introduced horizontal and vertical derivatives in the path space of a path functional, non-trivially generalized the classical Itô formula to a functional Itô formula (see Cont & Fournie \cite{7, 8} for a more general and systematic research), and provided a functional extension of the classical Feynman-Kac formula. His insightful work is becoming a foundation to stochastic analysis of path functionals, and has stimulated extensions of some above-mentioned developments to the functional case. In fact, he has shown that a path functional in $C^{1,2}(\Lambda)$ solves a linear path-dependent PDE (PPDE) if its composition with a Brownian motion generates a martingale. In the plenary lecture at the International Congress of Mathematicians of the year 2010, Peng \cite{41} pointed out that a non-Markovian BSDE is a PPDE. In view of Dupire’s functional Itô formula, it is very natural to associate a BSDE to a “semi-linear” PPDE, and a stochastic optimal control problem of BSDE to a path-dependent Bellman equation. Peng & Wang \cite{42} studied the former relation and give some sufficient conditions for a semi-linear PPDE to admit a classical solution. However, a PPDE, even for the simplest heat equation, rarely has a classical solution, and a path-dependent Bellman equation, even in the simpler state-dependent case, also appeals to a generalized solution in many occasions. Therefore, generalized solution of general PPDEs are demanding, and it has to be developed.

This paper incorporates Dupire’s functional Itô calculus to discuss the optimal
stochastic control problem for a path-dependent stochastic system under a recursive path-dependent cost functional. The associated Bellman equation from the dynamic programming principle in this general setting is a path-dependent fully nonlinear partial differential equation of second order, and we are concerned with its viscosity solution. The familiar optimal control of SDEs with delay can be addressed within our current framework.

In the classical theory of viscosity solution to PPDEs (see Crandall, Ishii & Lions \cite{9} and Fleming & Soner \cite{20}), local compactness of the state space and smoothness of the norm play a crucial role. The main difficulty for our path-dependent case lies in both facts that the path space \( \Lambda \) is an infinite dimensional Banach space and lacks of a local compactness, and that the maximal norm \( \| \cdot \|_0 \) is not smooth. The arguments for the case of Hilbert space introduced by Lions (see \cite{31,29}) contain a limiting procedure based on the structure of Hilbert space, and are difficult to be adapted to our path-dependent case where we have to work in a subspace \( C^\alpha \) (which has a local compactness, but whose \( \alpha \)-Hölder norm is not smooth).

In the generalization of Kim’s \( i \)-smooth theory \cite{25}, Luyakonov \cite{34} developed a theory of viscosity solution to fully non-linear path-dependent (also called functional in literature) Hamilton-Jacobi (HJ) equations of first order—which include conventionally called Bellman and Isaacs equations arising from deterministic optimal control problems and differential games for time-delayed ordinary differential equations. He used the so-called co-invariant derivatives (it is Clio-derivative in Kim’s terminology), which coincide with the restriction of Dupire’s derivatives on continuous paths though their definitions appear different. A deterministic functional differential system, subject to some proper conditions, starting at a uniformly Lipschitz continuous path, actually evolves forever in the locally compact space \( C^{0,1} \) of all uniformly Lipschitz continuous paths. This property of deterministic dynamical systems allows Luyakonov \cite{34} to define the jet functionals and to localize the application of dynamical programming all in \( C^{0,1} \) (in fact in a sequence of expanding compact subsets) of paths. Due to these conveniences, his proof of existence and uniqueness of viscosity solutions appears fairly straightforward. In our stochastic case, the situation changes in a dramatic manner. Even starting at a uniformly Lipschitz continuous path, due to the essential diffusion nature, our dynamic system could not live in \( C^{0,1} \) anymore, and it is impossible in general to enclose under proper conditions our dynamical system within any given compact space. We have to choose an \( \alpha \)-Hölder continuous paths space \( C^\alpha \) (\( \forall \alpha \in (0, \frac{1}{2}) \)) to substitute \( C^{0,1} \), and also define our jet functional on a family of expanding compact subsets \( \{ C^\alpha_{\mu} : \mu \text{ is sufficiently large} \} \) in \( C^\alpha \). We give an example to illustrate the following phenomenon in our stochastic dynamic system (see Remark \[5.2\] for details): starting at the boundary of \( C^\alpha_{\mu} \), our stochastic system might leave away from the set
with probability one within an arbitrary small time. This essential nature prevents us from starting the dynamic programming at the boundary of \( C^\alpha_\mu \) to show the Bellman equation holds there for the value functional. More precisely, to show that the value functional is a viscosity solution, we could not follow the conventional way to start the dynamic programming directly at the minimum/maximum path of the difference between the value functional and a jet functional since the extremum path might be at the boundary. To get around the difficulty, we construct a specific perturbation \( \gamma^\epsilon_t \in C^\alpha_\mu \) (around the extremum path \( \gamma_t \)) where we can start the dynamic programming and use the exit time \( \hat{\tau}^\epsilon \) from \( C^\alpha_\mu \) (see the definition of \( \hat{\tau}^\epsilon \) in the existence proof of Section 5; the probability for our dynamical system starting from \( \gamma^\epsilon_t \) at time \( t \) to stay within \( C^\alpha_\mu \) up to time \( t + \delta \) is shown to converge to one, uniformly with respect to sufficiently large \( \mu \), as \( \delta \to 0^+ \)) to localize our dynamic programming within the compact subset \( C^\alpha_\mu \). It seems to be the first here for us to define \( \hat{\tau}^\epsilon \) in association to the \( \alpha \)-Hölder modulus of the system’s history path. We finally prove that the value functional is a viscosity solution in Section 5. In the proof of the uniqueness, we adapt the smoothing (and viscosity vanishing for the degenerate case) methodology of Lions [30] to our path-dependent case, and also use the natural approximating arguments of parameterized state-dependent PDEs. In the passage to the limit, our a priori maximal and Hölder estimates on the second-order derivatives of the solutions to the approximate PPDEs play a crucial role. Our methodology is expected to be used to study the path-dependent Isaacs equation arising from stochastic differential games (see [19]) and other related fully nonlinear PPDEs. We note that using Gateaux or Frechet derivatives, Goldys & Gozzi [22] and Fuhrman, Masiero & Tessitore [21] considered time-delay stochastic optimal control problems with the diffusion being independent of the control variable, and studied the mild solutions to the associated semi-linear functional Bellman equations on Hilbert space and Banach space, respectively.

Defining the “semi-jets” on non compact subsets in terms of a nonlinear expectation, Ekren et al. [15] studied in a quite different way viscosity solution to second order path-dependent PDE. They prove the existence and uniqueness of viscosity solutions (in their sense) for a semi-linear PPDE by Peron’s approach. In the subsequent works, Ekren, Touzi, and Zhang (see for details [16, 17, 18]) use their previous notion of viscosity solution to study the fully nonlinear PPDE, Pham and Zhang (see [43]) discuss path-dependent Bellman-Isaacs equations. However, their relevant results on the path-dependent Bellman equation require stronger conditions: Their Assumption 2.8 requires, as they have noted, the diffusion coefficient \( \sigma \) to be path-invariant (see [18] page 8 for details), and, in the degenerate case, their Assumption 7.1 (i) requires further approximating structures (see [18] page 29 for details). Our Theorems 4.6 (on the non-degenerate case) and 4.7 (on the degenerate case) give more general uniqueness...
results on path-dependent Bellman equation. Furthermore, Ekren, Touzi, and Zhang [16,17,18] directly work with an abstract fully nonlinear PDE, and use a more complicated definition of super- and sub-jets in their notion of viscosity solution, in particular their definitions involve the unnatural and advanced notion of nonlinear expectation.

Backward stochastic PDE is another tool to study non-Markovian optimal control problems and FBSDEs. Peng [39,40] established the non-Markovian stochastic dynamic programming principle where he derived the backward stochastic Bellman equations in a heuristic way. Ma & Yong [36] gave the relationship between FBSDEs and a class of semi-linear BSPDEs, and further developed the stochastic Feynman-Kac formula. For Sobolev and classical solution of BSPDE, we refer to Zhou [52], Tang [47], Du & Meng [11], Du & Tang [12], Du, Tang & Zhang [13] and Qiu & Tang [44]. For viscosity solution of BSPDE or SPDE, we refer to Lions [32,33], Buckdahn & Ma [2,3,4,5] and Boufoussi et al. [1]. In Example 4.11, the relationship is exposed between path-dependent Bellman equations and backward stochastic Bellman equations.

The rest of the paper is organized as follows. In Section 2, we introduce the calculus for path functionals of [7,8] and [14], and preliminary results on BSDEs. In Section 3, we formulate the path-dependent stochastic optimal control problem and discuss the dynamic programming principle, which is crucial in the proof of the existence of a viscosity solution. In Section 4, we define classical and viscosity solutions to our path-dependent Bellman equation, state our main results, and prove a verification theorem in the context of classical solutions. In Section 5, we prove that the value functional is a viscosity solution, which implies our existence of a viscosity solution to the path-dependent Bellman equations. Finally in Section 6 we prove the uniqueness of viscosity solutions for the path-dependent Bellman equations.

2 Preliminaries

2.1 Calculus of path functionals

2.1.1 Space of cadlag paths

Let $n$ be a positive integer and $T$ be a fixed positive number. For each $t \in [0,T]$, define $\Lambda_t(\mathbb{R}^n) := D([0,t],\mathbb{R}^n)$ as the set of all cadlag (right continuous with left limit) $\mathbb{R}^n$-valued functions on $[0,t]$. For $\gamma \in \Lambda_T(\mathbb{R}^n)$, $\gamma(s)$ is the value of $\gamma$ at time $s \in [0,T]$, and for some $t \in [0,T]$ we denote the part of $\gamma$ up to time $t$ by $\gamma_t := \{\gamma(s), s \in [0,t]\} \in \Lambda_t(\mathbb{R}^n)$. Define $\Lambda(\mathbb{R}^n) := \bigcup_{t \in [0,T]} \Lambda_t(\mathbb{R}^n)$. Write $\Lambda$ for $\Lambda(\mathbb{R}^n)$ if there is no confusion.
For convenience, define for \(0 \leq t < \bar{t} \leq T\), and \(\bar{\gamma}_t, \gamma_t \in \hat{\Lambda}\),
\[
\gamma_t^x(s) := \gamma_t(s)\chi_{[0,t]}(s) + (\gamma_t(t) + x)\chi_{[t,\bar{t}]}(s), \quad s \in [0,t];
\]
\[
\gamma_{t,i}(s) := \gamma_t(s)\chi_{[0,t]}(s) + (\gamma_t(t) + x)\chi_{[t,\bar{t}]}(s), \quad s \in [0,t];
\]
\[
\gamma_{t,t} := \bar{\gamma}_t^0; \quad \bar{\gamma}_t^x(s) := \gamma_t(s)\chi_{[0,t]}(s) + (\bar{\gamma}_t(t) + \gamma_t(t))\chi_{[t,\bar{t}]}(s), \quad s \in [0,\bar{t}].
\]

We define the quasi-norm and metric in \(\hat{\Lambda}\) as follows: for each \(0 \leq t \leq \bar{t} \leq T\) and \(\gamma_t, \bar{\gamma}_t \in \hat{\Lambda}_T(\mathbb{R}^n)\),
\[
\|\gamma_t\|_0 := \sup_{0 \leq s \leq t} |\gamma_t(s)|,
\]
\[
d_p(\gamma_t, \bar{\gamma}_t) := \sqrt{|t - \bar{t}|} + \sup_{0 \leq s \leq \bar{t}} |\gamma_t(s) - \bar{\gamma}_t(s)|. \tag{2.1}
\]

Here \(|\cdot|\) is the standard metric of the Euclid space, \(d_p\) is called parabolic metric. It is easy to verify that \((\hat{\Lambda}_T(\mathbb{R}^n), \|\cdot\|_0)\) is a Banach space, \((\hat{\Lambda}, d_p)\) is a complete metric spaces.

**Definition 2.1.** (Continuity). Let \(E\) be a Banach space. A map \(v : \hat{\Lambda} \to E\) is said to be continuous at \(\gamma_t\), if for any \(\varepsilon > 0\) there exits \(\delta > 0\) such that for each \(\bar{\gamma}_t \in \hat{\Lambda}\) such that \(d_p(\gamma_t, \bar{\gamma}_t) < \delta\), we have \(|v(\gamma_t) - v(\bar{\gamma}_t)| < \varepsilon\). \(v\) is said to be continuous on \(\hat{\Lambda}\) and is denoted by \(v \in \mathcal{C}(\hat{\Lambda}, E)\) if \(v\) is continuous at each \(\gamma_t \in \hat{\Lambda}\). Moreover, we write \(v \in \mathcal{C}_b(\hat{\Lambda}, E)\) if \(v\) is bounded, and if \(v(\gamma_t) \leq C(1 + \|\gamma_t\|_0)\) for all \(\gamma_t \in \hat{\Lambda}\) holds for some constant \(C\), we write \(v \in \mathcal{C}_l(\hat{\Lambda}, E)\) (the subscript indicates linear growth).

(Uniform continuity). A continuous map \(v : \hat{\Lambda} \to E\) is said to be uniformly continuous, and is denoted by \(v \in \mathcal{C}_u(\hat{\Lambda}, E)\), if for any \(\varepsilon > 0\), there exists a \(\delta > 0\) such that for each \(\gamma_t, \bar{\gamma}_t \in \hat{\Lambda}\) satisfying \(d_p(\gamma_t, \bar{\gamma}_t) < \delta\), we have \(|v(\gamma_t) - v(\bar{\gamma}_t)| \leq \varepsilon\).

We write \(\mathcal{C}(\hat{\Lambda}), \mathcal{C}_b(\hat{\Lambda}), \mathcal{C}_l(\hat{\Lambda})\) and \(v \in \mathcal{C}_u(\hat{\Lambda})\) for \(\mathcal{C}(\hat{\Lambda}, \mathbb{R}), \mathcal{C}_b(\hat{\Lambda}, \mathbb{R}), \mathcal{C}_l(\hat{\Lambda}, \mathbb{R})\), and \(v \in \mathcal{C}_u(\hat{\Lambda}, \mathbb{R})\), respectively.

Now we define the vertical and horizontal derivatives of Dupire [14].

**Definition 2.2.** (Vertical derivative). Consider functional \(v : \hat{\Lambda}(\mathbb{R}^n) \to \mathbb{R}\) and \(\gamma_t \in \hat{\Lambda}(\mathbb{R}^n)\). The vertical (space) derivative of \(v\) at \(\gamma_t\) is defined as
\[
D_xv(\gamma_t) := (D_1v(\gamma_t), \ldots, D_nv(\gamma_t))
\]
where
\[
D_i\bar{v}(\gamma_t) := \lim_{h \to 0} \frac{1}{h}[v(\gamma_{t,i}) - v(\gamma_t)], \quad i = 1, \ldots, n. \tag{2.2}
\]
if all the limits exist, with \( e_i, i = 1, \cdots, n \), being coordinate unit vectors of \( \mathbb{R}^n \). If (2.2) is well-defined for all \( \gamma_t \), the map \( D_x v := (D_1 v, \cdots, D_n v) : \hat{\Lambda}(\mathbb{R}^n) \to \mathbb{R}^n \) is called the vertical derivative of \( v \). We define the Hessian \( D_{xx} v(\gamma_t) \) in an obvious way. Then \( D_{xx} v \) is an \( \mathbb{S}(n) \)-valued functional defined on \( \hat{\Lambda}(\mathbb{R}^n) \), where \( \mathbb{S}(n) \) is the space of all \( n \times n \) symmetric matrices.

(Horizontal derivative). The horizontal derivative at \( \gamma_t \in \hat{\Lambda} \) of a functional \( v : \hat{\Lambda} \to \mathbb{R} \) is defined as

\[
D_t v(\gamma_t) := \lim_{h \to 0, h > 0} \frac{1}{h} [v(\gamma_{t,t+h}) - v(\gamma_t)], \tag{2.3}
\]

if the limit exists. If (2.3) is well-defined for all \( \gamma_t \in \hat{\Lambda} \), the functional \( D_t v : \hat{\Lambda} \to \mathbb{R} \) is called the horizontal derivative of \( v \). Note that it is a right derivative.

**Definition 2.3.** Define \( \mathcal{C}^{j,k}(\hat{\Lambda}) \) as the set of functionals \( v : \hat{\Lambda} \to \mathbb{R} \) which are \( j \) times horizontally and \( k \) times vertically differentiable in \( \hat{\Lambda} \) such that all these derivatives are continuous. Moreover, we write \( v \in \mathcal{C}^{j,k}_b(\hat{\Lambda}) \) if \( v \) together with all its derivatives are bounded, and \( v \in \mathcal{C}^{j,k}_b(\hat{\Lambda}) \) if \( v \in \mathcal{C}^{j,k}(\hat{\Lambda}) \) and \( v \) grows in a linear way.

**Remark 2.4.** For \( v(\gamma_t) = f(t, \gamma_t(t)) \) with \( f \in \mathcal{C}^{1,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \), we have

\[
D_t v(\gamma_t) = \partial_t f(t, \gamma_t(t)), \quad D_x v(\gamma_t) = \partial_x f(t, \gamma_t(t)),
\]

which shows the coincidence of Dupire’s derivatives with the classical ones.

### 2.1.2 Space of continuous paths

Let \( \Lambda_t(\mathbb{R}^n) := \mathcal{C}_0([0, t], \mathbb{R}^n) \) be the set of all continuous \( \mathbb{R}^n \)-valued functions defined over \([0, t]\) which vanish at time zero, and \( \Lambda(\mathbb{R}^n) := \bigcup_{t \in [0, T]} \Lambda_t(\mathbb{R}^n) \). In the sequel, for notational simplicity, we use \( 0 \) to denote \( \gamma_0 \) or vectors and matrices whose components are all zero. Clearly, \( \Lambda(\mathbb{R}^n) \subset \hat{\Lambda}(\mathbb{R}^n) \). \( \Lambda_t(\mathbb{R}^n), \| \cdot \|_0 \) is a Banach space, and \( (\Lambda(\mathbb{R}^n), d_\rho) \) is a complete metric space. We write \( \Lambda \) for \( \Lambda(\mathbb{R}^n) \) if there is no confusion.

Let \( \mathbb{E} \) be a Banach space. \( \hat{v} : \hat{\Lambda} \to \mathbb{E} \) and \( v : \Lambda \to \mathbb{E} \) are called consistent on \( \Lambda \) if \( v \) is the restriction of \( \hat{v} \) on \( \Lambda \).

**Definition 2.5.** Consider a map \( v : \Lambda \to \mathbb{E} \).

(i) We write \( v \in \mathcal{C}(\Lambda) \) if \( v \) is continuous at every path \( \gamma_t \in \Lambda \) under \( d_\rho \). We write \( v \in \mathcal{C}_b(\Lambda) \) (resp. \( v \in \mathcal{C}_l(\Lambda), v \in \mathcal{C}_u(\Lambda) \)) if \( v \in \mathcal{C}(\Lambda) \) and \( v \) is bounded (resp. linearly growth, uniformly continuous).

(ii) We write \( v \in \mathcal{C}^{j,k}(\Lambda) \) if there exists \( \hat{v} \in \mathcal{C}^{j,k}(\hat{\Lambda}) \) which is consistent with \( v \) on \( \Lambda \), we shall define

\[
D_t^i v := D_t^i \hat{v}, \quad D_x^j v := D_x^j \hat{v}, \quad \text{on } \Lambda, \tag{2.4}
\]
where \(0 \leq i \leq j\) and multi index \(\beta = (\beta_1, \cdots, \beta_n)\) with the non-negative integers \(\alpha_1, \cdots, \alpha_n\) satisfying \(\beta_1 + \cdots + \beta_n \leq k\). Similarly, we define the spaces \(\mathcal{C}^{j,k}_b(\Lambda)\), \(\mathcal{C}^{j,k,i}(\Lambda)\) and \(\mathcal{C}^{j,k}_u(\Lambda)\) in an obvious way.

Remark 2.6. By [14] and [8], the derivatives of \(v\) in (2.4) is independent of the choice of \(\hat{v}\), i.e., if \(\hat{v}' \in \mathcal{C}^{j,k}(\hat{\Lambda})\) is another functional consistent with \(v\), then \(D_x^i \hat{v} = D_x^i \hat{v}'\) and \(D_x^\alpha \hat{v} = D_x^\alpha \hat{v}'\) on \(\Lambda\). Therefore, Definition 2.5 (ii) is well defined.

Definition 2.7. (Hölder continuity). For \(\alpha \in (0,1]\), we say that a functional \(v\) defined on \(Q \subset \Lambda\) is Hölder continuous on \(Q\) with exponent \(\alpha\) if the quantity

\[
[v]_{\alpha;Q} := \sup_{\gamma, \gamma' \in Q, \gamma \neq \gamma'} \frac{|v(\gamma_t) - v(\gamma'_t)|}{d^\alpha_p(\gamma_t, \gamma'_t)}
\]

is finite. Let \(v \in \mathcal{C}^{1,2}(\Lambda)\), define

\[
|v|_{\alpha;Q} := \sup_{\gamma \in Q} |v(\gamma_t)| + [\gamma]_{\alpha;Q},
\]

\[
|v|_{2,\alpha;Q} := |v|_{\alpha;Q} + |D_t v|_{\alpha;Q} + \sum_{1 \leq |\beta| \leq 2} |D_x^\beta v|_{\alpha;Q}.
\]

If (2.5) is finite, we write \(v \in \mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q)\).

2.1.3 Filtration and localization

Now we introduce the filtration of \(\Lambda_T\). Let \(\mathcal{G}_T := \mathcal{B}(\Lambda_T)\), the smallest Borel \(\sigma\)-field generated by metric space \((\Lambda_T, \| \cdot \|_0)\). For any \(t \in [0,T]\), define \(\mathcal{G}_t := \theta_t^{-1}(\mathcal{G}_T) = \sigma(\theta_t^{-1}(\mathcal{G}_T))\), where \(\theta_t : \Lambda_T \to \Lambda_T\) is the mapping

\[
(\theta_t \gamma)(s) = \gamma(t \wedge s), \quad 0 \leq s \leq T, \text{ for any } \gamma \in \Lambda_T,
\]

and \(\sigma\) means the smallest \(\sigma\)-field generated by the underlying class of subsets. \(\mathcal{G} := \{\mathcal{G}_t, t \in [0, T]\}\) is a filtration. \(\mathcal{G}_t\) is just the smallest \(\sigma\)-algebra generated by the collection of finite-dimensional cylinder sets of the form

\[
\{\gamma \in \Lambda_T; (\gamma(t_1), \cdots, \gamma(t_k)) \in A\}; \quad k \geq 1, A \in \mathcal{B}(\mathbb{R}^k),
\]

where, for all \(i = 1, \cdots, k\), \(t_i \in [0, t]\). For more details, see the monograph of Stroock and Varadhan [46, Section 1.3].

Define \(\pi_t : \Lambda_T \to \Lambda_t\) as follows:

\[
(\pi_t \gamma)(s) = \gamma_t(s), \quad (s, \gamma) \in [0, t] \times \Lambda_T.
\]
It is easy to observe that \( \pi_t^{-1}(\mathcal{G}_T) = \sigma(\pi_t^{-1}(\mathcal{G}_T)) = \mathcal{B}(\Lambda_t) \), the smallest Borel \( \sigma \)-field generated by metric space \((\Lambda_t, \| \cdot \|_0)\).

A map \( H : [0, T] \times \Lambda_T \rightarrow \mathbb{E} \) is called a functional process. Moreover, we say a process \( H \) is adapted to the filtration \( \mathcal{G} \), if \( H(t, \cdot) \) is \( \mathcal{G}_t \)-measurable for any \( t \in [0, T] \). Obviously an adapted process \( H \) has the property that, for any \( \gamma^1, \gamma^2 \in \Lambda_T \) satisfying \( \gamma^1(s) = \gamma^2(s) \) for all \( s \in [0, t] \), \( H(t, \gamma^1) = H(t, \gamma^2) \). Hence \( H(t, \gamma_T) \) can be view as \( H(\gamma_t) \), that is to say a functional adapted process equals a path functional defined on \( \Lambda \). Let \( B = \{B(t), t \in [0, T]\} \) be the canonical process on \( \Lambda \), i.e. \( B(t, \gamma) = \gamma(t) \). Define \( B_t := \{B(s), s \in [0, t]\} \), and it is \( \mathcal{G}_t \)-measurable. For any continuous map \( v : \Lambda \rightarrow \mathbb{E} \), we know that \( \{v(B_t), t \in [0, T]\} \) is an adapted functional process.

Let \( P_0 \) be the Wiener measure of space \((\Lambda_T, \mathcal{G})\), under which the canonical process \( B \) (i.e. \( B(t, \gamma) = \gamma(t) \)) is a standard Brownian motion. For any integrable \( \mathcal{G}_T \)-measurable variable \( \xi \), denote \( P_0[\xi] \) the integration of \( \xi \) under measure \( P_0 \).

Let \( X := \{X(t), t \in [0, T]\} \) be an \( n \)-dimensional adapted continuous stochastic process on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\). For \( t \in [0, T] \), \( X(t) \) is the value of \( X \) at time \( t \) and \( X_t \) is the path of \( X \) up to time \( t \), i.e., \( X_t := \{X(r), r \in [0, t]\} \). \( X \) can be viewed as a map from \( \Omega \) to \( \Lambda_T \), the continuity and adaption of \( X \) imply that \( X_t \) is \( \mathcal{F}_t/\mathcal{G}_t \)-measurable for any \( t \in [0, T] \). For any \( v \in \mathcal{C}(\Lambda) \), \( \{v(X_t), t \in [0, T]\} \) is an \( \mathcal{F} \)-adapted stochastic process.

The following functional Itô formula was initiated by Dupire [14] and later extended to a more general context by Cont & Fournie [8].

**Theorem 2.8 (Functional Itô formula).** Suppose \( X \) is a continuous semi-martingale and \( v \in \mathcal{C}^{1,2}(\Lambda) \). Then for any \( t \in [0, T] \):

\[
v(X_t(\omega)) - v(X_0(\omega)) = \int_0^t D_s v(X_s(\omega)) ds + \frac{1}{2} \int_0^t D_{xx} v(X_s(\omega)) d\langle X \rangle(s, \omega) + \int_0^t D_x v(X_s(\omega)) dX(s, \omega), \quad a.s. \quad \omega.
\]

(2.7)

A time functional \( \tau : \Lambda_T \rightarrow [0, T] \) (resp. \( \tau : \Lambda_T \rightarrow [0, T] \)) is said to be a \( \mathcal{G} \)-stopping time, if \( \{\gamma_t : \gamma \in \Lambda_T, \tau(\gamma) \leq t\} \in \mathcal{G}_t \) for every \( t \in [0, T] \). Let \( \mathcal{T} \) be the set of all \( \mathcal{G} \)-stopping times. Define \( \mathcal{T}_+ := \{\tau \in \mathcal{T} : \tau > 0\} \).

Let \( u \in \mathcal{C}(\Lambda) \). Obviously, for any constant \( \lambda > 0 \), \( \tau(\gamma) := \inf \{t : u(\gamma_t) \geq \lambda\} \land T \) is a stopping time.

### 2.1.4 Space shift

For any fixed \( \gamma_t \in \hat{\Lambda} \) and \( s \in [t, T] \), we introduce the shifted spaces of cadlag and continuous paths.
Let \( \hat{\Lambda}^n := \{ \bar{\gamma}_s \in \hat{\Lambda}_s; \bar{\gamma}_s(r) = \gamma_t(r) \) for any \( r \in [0, t] \} \) be the space of cadlag paths originating at \( \gamma_t \), and \( \hat{\Lambda}^n := \cup_{t \leq s \leq T} \hat{\Lambda}^n_s \). Let \( \Lambda^n_s := \{ \bar{\gamma}_s \in \hat{\Lambda}^n_s; \bar{\gamma}_s \) is continuous on \( [t, s] \} \) be the spaces with continuous paths originating at \( \gamma_t \) and \( \Lambda^n_s := \cup_{t \leq s \leq T} \Lambda^n_s \). It is easy to verify that \( (\hat{\Lambda}^n, \| \cdot \|_0) \) and \( (\Lambda^n, \| \cdot \|_0) \) are Banach spaces, and \( (\hat{\Lambda}^n, d_p) \) and \( (\Lambda^n, d_p) \) are complete metric spaces.

For any map \( v : \hat{\Lambda}^n \to \mathbb{R} \), define the derivatives in the spirit of Definition 2.2, and define the spaces \( \mathcal{C}(\hat{\Lambda}^n), \mathcal{C}_b(\hat{\Lambda}^n), \mathcal{C}_u^{1, k}(\hat{\Lambda}^n) \) in the spirit of Definitions 2.1 and 2.3. We define \( \mathcal{C}(\Lambda^n), \mathcal{C}_b(\Lambda^n), \mathcal{C}_u^{1, k}(\Lambda^n) \) in the spirit of Definition 2.5.

Let \( \mathcal{G}^n_t := \mathcal{B}(\Lambda^n_t) \), the smallest Borel \( \sigma \)-field generated by metric space \( (\Lambda^n_t, \| \cdot \|) \). For \( s \in [t, T] \), define \( \mathcal{G}^n_s := \theta^{-1}_s(\mathcal{G}^n_t) = \sigma(\theta^{-1}_s(\mathcal{G}^n_t)) \), with \( \theta \) being defined by (2.6). \( \mathcal{G}^n := \{ \mathcal{G}^n_s, s \in [t, T] \} \) is said to be \( \mathcal{G}^n \)-filtration. A time functional \( \tau : \hat{\Lambda}^n_t \to [t, T] \) is called \( \mathcal{G}^n \)-stopping time, if \( \{ \gamma_s : \gamma \in \Lambda^n_t, \tau(\gamma) \leq s \} \in \mathcal{G}^n_t \) for any \( s \in [t, T] \). The set of all \( \mathcal{G}^n \)-stopping is denoted \( \mathcal{T}^n \). Denote \( \mathcal{T}^n_+ := \{ \tau \in \mathcal{T}^n : \tau > t \} \).

Let \( \{ H_t, t \in [0, T] \} \) be a \( \mathcal{G} \)-progressively measurable functional process, and \( \xi \) be a \( \mathcal{G}_T \) measurable functional variable, let \( \gamma_t \in \hat{\Lambda} \). Define the process \( H^n \) on \( [0, T] \times \Lambda^n_T \) and the variable \( \xi^n \) on \( \Lambda^n_T \), as the restriction on \( \Lambda^n \) of \( H \) and \( \xi \), respectively; that is,

\[
H^n := H|_{[0, T] \times \Lambda^n_T}, \quad \xi^n := \xi|_{\Lambda^n_T}.
\]

Then, \( \{ H^n, s \in [t, T] \} \) is a \( \mathcal{G}^n \)-progressively measurable functional process, and \( \xi^n \) is a \( \mathcal{G}^n_T \)-measurable functional variable.

### 2.1.5 Space of \( \alpha \)-Hölder continuous paths

For any \( \alpha \in (0, 1] \), we say that \( \gamma \in \Lambda(\mathbb{R}^n) \) is \( \alpha \)-Hölder continuous if

\[
\| \gamma_t \|_\alpha := \sup_{0 \leq s < r \leq t} \frac{|\gamma_t(s) - \gamma_t(r)|}{|s - r|^\alpha} < \infty.
\]

We call \( \| \gamma_t \|_\alpha \) the \( \alpha \)-Hölder modulus of \( \gamma_t \). Define the \( \alpha \)-Hölder space:

\[
C^\alpha(\mathbb{R}^n) := \{ \gamma_t \in \Lambda : \| \gamma_t \|_\alpha < \infty \}.
\]

Clearly, \( C^\alpha(\mathbb{R}^n) \subset \Lambda(\mathbb{R}^n) \). We write \( C^\alpha \) for \( C^\alpha(\mathbb{R}^n) \) if there is no confusion.

For any \( \alpha \in (0, 1] \) and \( \mu > 0 \), denote

\[
C^\alpha_\mu := \{ \gamma_t \in \Lambda : \| \gamma_t \|_\alpha \leq \mu \}. \tag{2.8}
\]

Analogous to the proof of the Arzela-Ascoli theorem, we show the following compact property.
Proposition 2.9. For \( \alpha \in (0, 1] \), \( C^\alpha \) is a compact subset of \( (\Lambda, d_p) \).

Proof. Since \( (\Lambda, d_p) \) is a metric space, it suffices to prove that every sequence \( \{\gamma_{t_k}^k\}_{k=1}^\infty \subset C^\alpha \) has a convergence subsequence and the limit lies in \( C^\alpha \).

For \( \{t_k\}_{k=1}^\infty \subset [0, T] \), it has a convergence subsequence, still denoted by \( \{\gamma_{t_k}^k\}_{k=1}^\infty \).

Without loss of generality, we suppose that \( t_k \) converges increasingly to \( t \in [0, t] \) as \( k \to \infty \). From (2.9), we have

\[
|\gamma_{t_k}^k(r)| \leq |\gamma_{t_k}^k(0)| + \mu r^\alpha \leq \mu T^\alpha, \quad \forall r \in [0, t_k], \quad k = 1, 2 \cdots .
\]

Let \( Q \) be the set of rational numbers, and \( \{r_1, r_2, \cdots \} \) be an enumeration of \( Q \cap [0, t) \). By (2.9), we can choose a subsequence \( \{\gamma_{t_k}^{(1)k}\}_{k=1}^\infty \) of \( \{\gamma_{t_k}^k\}_{k=1}^\infty \) such that \( r_1 \in [0, t^{(1)}_{k}] \) for all \( k = 1, 2, \cdots \) and \( \{\gamma_{t_k}^{(1)k}(r_1)\}_{k=1}^\infty \) converges to a limit, denoted by \( \gamma(t_1) \). From \( \{\gamma_{t_k}^{(1)k}\}_{k=1}^\infty \), choose a further subsequence \( \{\gamma_{t_k}^{(2)k}\}_{k=1}^\infty \) such that \( r_2 \in [0, t^{(2)}_{k}] \) for all \( k = 1, 2, \cdots \) and \( \{\gamma_{t_k}^{(2)k}(r_2)\}_{k=1}^\infty \) converges to a limit \( \gamma(t_2) \). Continue this process, and then let \( \{\gamma_{t_k}^k\}_{k=1}^\infty = \{\gamma_{t_k}^{(k)k}\}_{k=1}^\infty \) be the “diagonal sequence”. We have \( \{\gamma_{t_k}^k(r) : r \in [0, t_k]\}_k \) has a unique accumulation point \( \gamma_t(r) \) for any \( r \in Q \cap [0, t) \).

For any \( r, s \in Q \cap [0, t) \),

\[
|\gamma_{t_k}^k(r) - \gamma_{t_k}^k(s)| \leq \mu |r - s|^\alpha, \quad \forall k \geq K,
\]

where \( K \) is a sufficiently large integer such that \( r, s \in [t, t_k] \) for all \( k \geq K \). Setting \( n \to \infty \), we have

\[
|\gamma_t(r) - \gamma_t(s)| \leq \mu |r - s|^\alpha, \quad \forall s, r \in Q \cap [0, t).
\]

Hence \( \gamma_t \) has a continuous extension on \([0, t]\), still denoted by \( \gamma_t \), and it lies in \( C^\alpha \).

It remains to show the limit \( \lim_{k \to \infty} d_p(\gamma_{t_k}^k, \gamma_t) = 0 \). In fact, for any \( \varepsilon > 0 \), define

\[
r_j = j\left(\frac{\varepsilon}{\mu}\right)^{\frac{1}{\alpha}}, \quad j = 1, 2, \cdots, \left[ t\left(\frac{\varepsilon}{\mu}\right)^{\frac{1}{\alpha}} \right],
\]

where \( \left\lfloor s \right\rfloor \) denotes the greatest integer less than or equal to \( s \). Then, for \( s \in [0, t] \), there is some \( r_j \) such that \( |r_j - s| \leq \left(\frac{\varepsilon}{\mu}\right)^{\frac{1}{\alpha}} \). For a sufficiently large \( K \), we have \( r_k \in [0, t_k] \) and \( |\gamma_{t_k}^k(r_j) - \gamma_t(r_j)| < \varepsilon \), for all \( j = 1, 2 \cdots, \left[ t\left(\frac{\varepsilon}{\mu}\right)^{\frac{1}{\alpha}} \right] \) and \( k > K \). Consequently,

\[
|\gamma_{t_k}^k(s) - \gamma_t(s)| \leq |\gamma_{t_k}^k(s) - \gamma_{t_k}^k(r_j)| + |\gamma_{t_k}^k(r_j) - \gamma_t(r_j)| + |\gamma_t(r_j) - \gamma_t(s)| \leq 3\varepsilon, \quad \forall s \in [0, t], \quad k > K.
\]
Furthermore,
\[ d_p(\gamma^k_{t_k}, \gamma_t) = \max_{0 \leq s \leq t} d_p(\gamma^k_{t_k,t}, \gamma_t(s)) + \sqrt{|t_k - t|} \leq \varepsilon + \sqrt{|t_k - t|}, \]
which leads to \( \lim_{k \to \infty} d_p(\gamma^k_{t_k}, \gamma_t) = 0. \)

Define the following (random) time for the path to oscillate beyond a given \( \alpha \)-Hölder modulus:
\[ \tau^\alpha_\mu(\gamma) := \inf\{ t > 0 : \|\gamma(t)\|_\alpha > \mu\}, \quad \gamma \in \Lambda_T. \]

It is a \( \mathcal{G} \)-stopping time due to the following
\[ \{\tau^\alpha_\mu \leq t\} = \{\gamma \in \Lambda_T : \|\gamma\|_\alpha \leq \mu\} = \left\{ \gamma \in \Lambda_T : \sup_{s, r \in Q \cap [0, t], s \neq r} \frac{|\gamma(s) - \gamma(t)|}{|s - r|^\alpha} \leq \mu \right\} = \bigcap_{s, r \in Q \cap [0, t], s \neq r} \{\gamma \in \Lambda_T : |\gamma(s) - \gamma(t)| \leq \mu|s - r|^\alpha\} \in \mathcal{G}_t. \]

This kind of exit time will play a crucial role in the subsequent proof of the existence of a viscosity solution.

### 2.2 Backward stochastic differential equations

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\) be a probability space with the usual condition (see Karatzas and Shreve [23]), and \(\{W(t), t \in [0, T]\}\) be a \(d\)-dimensional standard Brownian motion. Let \(\mathcal{N} \subset \mathcal{P}\) be the collection of all \(P\)-null sets in \(\Omega\), for any \(0 \leq t \leq r \leq T\), \(\mathcal{F}_r^t\) denotes the completion of \(\sigma(W(s) - W(t); t \leq s \leq r)\), i.e., \(\mathcal{F}_r^t := \sigma(W(s) - W(t); t \leq s \leq r) \cup \mathcal{N}\). We also write \(\mathcal{F}^t\) for \(\{\mathcal{F}_s, s \in [t, T]\}\).

For any \(t \in [0, T]\), denote by \(\mathcal{M}^2(t, T)\) the space of all \(\mathcal{F}^t\)-adapted, \(\mathbb{R}^d\)-valued processes \(\{Y(s), s \in [t, T]\}\) such that \(E[\int_t^T |Y(s)|^2 ds] < \infty\) and by \(\mathcal{I}^2(0, T)\) the space of all \(\mathcal{F}^t\)-adapted, \(\mathbb{R}\)-valued continuous processes \(\{Y(s), s \in [t, T]\}\) such that \(E[\sup_{s \in [t, T]} |Y(s)|^2] < \infty\).

**Lemma 2.10.** Consider \(f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) such that \(\{f(t, y, z), t \in [0, T]\}\) is progressively measurable for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), and the following two conditions are satisfied:

(i) \(f\) is uniformly Lipschitz continuous about \((y, z) \in \mathbb{R} \times \mathbb{R}^d\);  
(ii) \(f(\cdot, 0, 0) \in \mathcal{M}^2(0, T)\). For any \(\xi \in L^2(\Omega, \mathcal{F}_T, P)\), the BSDE
\[ Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dW_s, \quad 0 \leq t \leq T, \tag{2.10} \]
has a unique adapted solution \((Y, Z) \in \mathcal{I}^2(0, T) \times \mathcal{M}^2(0, T)\).
Lemma 2.11. Let two BSDEs of data \((\xi_1, f_1)\) and \((\xi_2, f_2)\), satisfy all the assumptions of Lemma 2.10. Denote by \((Y^1, Z^1)\) and \((Y^2, Z^2)\) their respective adapted solutions. Then we have:

1. (Monotonicity). If \(\xi_1 \geq \xi_2\) and \(\tilde{f}_1 \geq \tilde{f}_2\), a.s., then \(Y^1(t) \geq Y^2(t)\), a.s., for all \(t \in [0, T]\).

2. (Strict monotonicity). If, in addition to (1), we also \(P\{\xi_1 > \xi_2\} > 0\), then \(P\{Y^1(t) > Y^2(t)\} > 0\) for any \(t \in [0, T]\), and in particular, \(Y^1(0) > Y^2(0)\).

3 Formulation of the path-dependent optimal stochastic control problem and dynamic programming principle

Let the set of admissible control processes \(U\) be the set of all \(\mathcal{F}\)-progressively measurable process valued in some compact metric space \(U\). For any \(t \in [0, T]\), \(L^p(\Omega, \mathcal{F}_t; \Lambda_t, \mathcal{G}_t)\) is the set of all \(\mathcal{F}/\mathcal{G}\)-measurable maps \(\Gamma_t : \Omega \rightarrow \Lambda_t\) satisfying \(E\|\Gamma_t\|^p \leq \infty\).

Consider the following functionals \(b : \Lambda \times U \rightarrow \mathbb{R}^n\), \(\sigma : \Lambda \times U \rightarrow \mathbb{R}^{n \times d}\), \(g : \Lambda_T \rightarrow \mathbb{R}\) and \(f : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times U \rightarrow \mathbb{R}\). We make the following assumption.

(H1) There exists a constant \(C > 0\) such that, for all \((t, \gamma_T, y, z, u), (t', \gamma_T', y', z', u') \in [0, T] \times \Lambda_T \times \mathbb{R} \times \mathbb{R}^d \times U\),

\[
\begin{align*}
|b(\gamma_t, u) - b(\gamma_{t'}, u')| & \leq C(d_p(\gamma_t, \gamma_{t'}) + |u - u'|), \\
|\sigma(\gamma_t, u) - \sigma(\gamma_{t'}, u')| & \leq C(dp(\gamma_t, \gamma_{t'}) + |u - u'|), \\
|f(\gamma_t, y, z, u) - f(\gamma_{t'}, y', z', u')| & \leq C(dp(\gamma_t, \gamma_{t'}) + |y - y'| + |z - z'| + |u - u'|), \\
|g(\gamma_T) - g(\gamma_{T}')| & \leq C\|\gamma_T - \gamma_{T}'\|_0.
\end{align*}
\]

For given \(t \in [0, T]\), \(\mathcal{F}_t/\mathcal{G}_t\)-measurable map \(\Gamma_t : \Omega \rightarrow \Lambda_t\) and admissible control \(u \in U\), consider the following SDE:

\[
\begin{cases}
X_{t}^{\Gamma_t, u}(s) = \Gamma_t(s), & \text{all } \omega, s \in [0, t]; \\
X_{t}^{\Gamma_t, u}(s) = \Gamma_t(t) + \int_{t}^{s} b(X_{r}^{\Gamma_t, u}, u(r)) \, dr \\
+ \int_{t}^{s} \sigma(X_{r}^{\Gamma_t, u}, u(r)) \, dW(r), & \text{a.s.}, \omega, s \in [t, T].
\end{cases}
\]
**Lemma 3.1.** Take $p \geq 2$. Let Assumption (H1) hold. For

$$t \in [0, T], \Gamma_t \in \mathcal{L}^p(\Omega, F_t; \Lambda_t, \mathcal{B}(\Lambda_t)), \text{ and } u \in \mathcal{U},$$

SDE (3.1) admits a unique strong solution $X^{\Gamma_t,u} : \Omega \to \Lambda_T$ such that $X^{\Gamma_t,u}_s : \Omega \to \Lambda_s$ is $\mathcal{F}_s/\mathcal{B}(\Lambda_s)$ measurable for all $s \in [t, T]$, and $E[\|X^{\Gamma_t,u}_T\|^p] < \infty$. Moreover, there is a positive constant $C_p$ such that for any $t \in [0, T]$, $u, u' \in \mathcal{U}$, and $\Gamma_t, \Gamma'_t \in \mathcal{L}^p(\Omega, \Lambda_t; \mathcal{F}_t/\mathcal{B}(\Lambda_t))$, we have, $P$-a.s.,

$$E[\|X^{\Gamma_t,u}_t - X^{\Gamma'_t,u'}_t\|^p] \leq C_p (\|\Gamma_t - \Gamma'_t\|^p_\infty + E[\int_t^T |u(r) - u'(r)|^p dr |\mathcal{F}_t]),$$

$$E[\|X^{\Gamma_t,u}_T\|^p] \leq C_p (1 + \|\Gamma_t\|^p_0),$$

$$E[\|X^{\Gamma_t,u}_r - \Gamma_t, r\|^p] \leq C_p (1 + \|\Gamma_t\|^p_0) (r - t)^{\frac{p}{2}}, \quad r \in [t, T].$$

The constant $C_p$ only depends on the Lipschitz constant of $b$ and $\sigma$ in $(\gamma_t, t)$.

Combing Lemmas 2.10 and 3.1 we have

**Lemma 3.2.** Take $p \geq 2$. Let (H1) hold. For any $t \in [0, T]$, $\Gamma_t \in \mathcal{L}^p(\Omega, F_t; \Lambda_t, \mathcal{B}(\Lambda_t))$ and $u \in \mathcal{U}$, $X^{\Gamma_t,u}$ is the solution of the stochastic equation (3.1). Then BSDE

$$Y^{\Gamma_t,u}(s) = g(X^{\Gamma_t,u}_T) + \int_s^T f(X^{\Gamma_t,u}_r, Y^{\Gamma_t,u}(r), Z^{\Gamma_t,u}(r), u(r)) dr$$

$$- \int_s^T Z^{\Gamma_t,u}(r) dB(r), \quad a.s.-\omega, \text{ all } s \in [t, T], \quad (3.2)$$

has a unique solution $(Y^{\Gamma_t,u}, Z^{\Gamma_t,u}) \in \mathcal{S}^2(t, T) \times \mathcal{M}^2(t, T)$. Furthermore, there is a constant $C_p$ such that for any $t \in [0, T]$, $\Gamma_t, \Gamma'_t \in \mathcal{L}^p(\Omega, F_t; \Lambda_t, \mathcal{B}(\Lambda_t))$, and $u \in \mathcal{U}$, $P$-a.s.,

$$E\left[\sup_{t \leq s \leq T} |Y^{\Gamma_t,u}(s) - Y^{\Gamma'_t,u'}(s)|^p |\mathcal{F}_t\right] \leq C_p (\|\Gamma_t - \Gamma'_t\|^p_\infty + E[\int_t^T |u(r) - u'(r)|^p dr |\mathcal{F}_t]),$$

$$E\left[\sup_{t \leq s \leq T} |Y^{\Gamma_t,u}(s)|^p |\mathcal{F}_t\right] \leq C_p (1 + \|\Gamma_t\|^p_0), \quad (3.3)$$

$$E\left[\sup_{t \leq s \leq T} |Y^{\Gamma_t,u}(s)|^p |\mathcal{F}_t\right] \leq C_p (1 + \|\Gamma_t\|^p_0) (r - t)^{\frac{p}{2}}, \quad (3.4)$$

$$E\left[\sup_{t \leq s \leq T} |X^{\Gamma_t,u}(s) - Y^{\Gamma_t,u}(t)|^p |\mathcal{F}_t\right] \leq C_p (1 + \|\Gamma_t\|^p_0) (r - t)^{\frac{p}{2}}, \quad (3.5)$$

14
For the particular case of a deterministic $\Gamma_t$, i.e. $\Gamma_t = \gamma_t \in \Lambda_t$:

$$
\begin{align*}
X^{\gamma_t,u}(s) &= \gamma_t(s), \quad \text{all } \omega, s \in [0, t); \\
X^{\gamma_t,u}(s) &= \gamma_t(t) + \int_t^s b(X^{\gamma_t,u}_r, u(r)) \, dr + \int_t^s \sigma(X^{\gamma_t,u}_r, u(r)) \, dW(r), \quad \text{a.s.-}\omega, s \in [t, T].
\end{align*}
$$

(3.6)

$$
Y^{\gamma_t,u}(s) = g(X^{\gamma_t,u}_T) + \int_s^T f(X^{\gamma_t,u}_r, Y^{\gamma_t,u}_r, Z^{\gamma_t,u}_r, u(r)) \, dr \\
- \int_s^T Z^{\gamma_t,u}_r \, dW(r), \quad \text{a.s.-}\omega, \text{ all } s \in [t, T].
$$

(3.7)

Given the control process $u \in U$, we introduce the following cost functional:

$$
J(\gamma_t, u) := Y^{\gamma_t,u}(t), \quad \text{P-a.s. } \forall \gamma_t \in \Lambda.
$$

The value functional of the optimal control is defined by

$$
\tilde{v}(\gamma_t) := \text{ess sup}_{u \in U} Y^{\gamma_t,u}(t), \quad \forall \gamma_t \in \Lambda.
$$

(3.8)

We easily prove that for $t \in [0, T]$ and $\Gamma_t \in L^2(\Omega, \Lambda_t; \mathcal{F}_t/\mathcal{B}(\Lambda_t))$,

$$
J(\Gamma_t, u) = Y^{\Gamma_t,u}(t), \quad \text{P-a.s.}
$$

Remark 3.3. The above essential supremum should be understood as one with respect to indexed families of random variables (see Karatzas and Shreve [24, Appendix A] for details). For the convenience of reader we recall the notion of esssup of random variables. Given a family of real-valued random variables $\eta_\alpha$, $\alpha \in I$, a random variable $\eta$ is said to be $\text{ess sup}_{\alpha \in I} \eta_\alpha$, if

1. $\eta \leq \eta_\alpha$, $\text{P-a.s.}$, for any $\alpha \in I$;

2. if there is another random variable $\xi$ such that $\xi \leq \eta_\alpha$, $\text{P-a.s.}$, for any $\alpha \in I$, then $\xi \leq \eta$, $\text{P-a.s.}$.

The existence of $\text{ess sup}_{\alpha \in I} \eta_\alpha$ is well known.

Under Assumption (H1), the random variable $\tilde{v}(\gamma_t) \in L^p(\Omega)$ is $\mathcal{F}_t$-measurable. We have

Proposition 3.4. The value functional $\tilde{v}$ is deterministic.
Proof. Firstly we show that there exist \( \{u_n\}_{n=1}^{\infty} \subset \mathcal{U} \) such that
\[
\bar{v}(\gamma_t) = \text{ess sup}_{u(\cdot) \in \mathcal{U}} Y^{\gamma, u_n}(t) = \lim_{n \to \infty} \wedge Y^{\gamma, u_n}(t). \tag{3.9}
\]
In view of [24, Theorem A.3], it is sufficient to prove that, for any \( u_1, u_2 \in \mathcal{U} \), we have
\[
Y^{\gamma, u_1}(t) \vee Y^{\gamma, u_2}(t) = Y^{\gamma, u}(t), \quad P\text{-a.e.} \tag{3.10}
\]
for \( u \in \mathcal{U} \) such that \( u(s) := u_1(s)\chi_{A_1} + u_2(s)\chi_{A_2}, s \in [t, T] \), where \( A_1 := \{ Y^{\gamma, u_1}(t) > Y^{\gamma, u_2}(t) \} \) and \( A_2 := \{ Y^{\gamma, u_1}(t) \leq Y^{\gamma, u_2}(t) \} \). Since \( \sum_{i=1,2} \varphi(x_i)\chi_{A_i} = \sum_{i=1,2} \varphi(x_i\chi_{A_i}) \), we have
\[
\sum_{i=1,2} \chi_{A_i}X^{\gamma, u_i}(s) = \gamma(t) + \int_t^s b(\sum_{i=1,2} \chi_{A_i}X_r^{\gamma, u_i}, \sum_{i=1,2} \chi_{A_i}u_i(r)) \, dr \tag{3.11}
\]
\[
+ \int_t^s \sigma(\sum_{i=1,2} \chi_{A_i}X_r^{\gamma, u_i}, \sum_{i=1,2} \chi_{A_i}u_i(r)) \, dW(r), \quad s \in [t, T]
\]
and
\[
\sum_{i=1,2} \chi_{A_i}Y^{\gamma, u_i}(s) = \sum_{i=1,2} \chi_{A_i}g(X_T^{\gamma, u_i}) \tag{3.12}
\]
\[
+ \int_T^s f(\sum_{i=1,2} \chi_{A_i}X_r^{\gamma, u_i}, \sum_{i=1,2} \chi_{A_i}Y_r^{\gamma, u_i}(r), \sum_{i=1,2} \chi_{A_i}Z_r^{\gamma, u_i}(r), \sum_{i=1,2} \chi_{A_i}u_i(r)) \, dr \tag{3.11}
\]
\[
- \int_s^T \sum_{i=1,2} \chi_{A_i}Z_r^{\gamma, u_i}(r) \, dW(r), \quad s \in [t, T].
\]
By the uniqueness of solution of BSDE, we have
\[
Y^{\gamma, u} = \sum_{i=1,2} \chi_{A_i}Y^{\gamma, u_i}, \quad P\text{-a.e.}. \tag{3.13}
\]
From Lemma 2.11 we have \( \sum_{i=1,2} \chi_{A_i}Y_r^{\gamma, u_i}(t) = Y^{\gamma, u_1}(t) \vee Y^{\gamma, u_2}(t) \), which yields (3.10).

Suppose that \( \{u_i(\cdot)\}_{i=1}^{\infty} \subset \mathcal{U} \) satisfy (3.9). Since \( Y^{\gamma, u} \) is continuous in \( u \in \mathcal{U} \), we suppose without lost of generality that \( u_i(\cdot) \) takes the following form:
\[
u_i(s) = \sum_{j=1}^{N} \chi_{A_{i,j}}u_{ij}(s), \quad s \in [t, T].
\]
Here, \( u_{ij}(s) \in \mathcal{F}_s^t, t \leq s \leq T \) and \( \{A_{ij}\}_{j=1}^N \subset \mathcal{F}_t \) is a partition of \((\Omega, \mathcal{F}_t)\), i.e., \( \cup_{i=1}^N A_i = \Omega \) and \( A_i \cap A_j = \emptyset, i \neq j \). Like \((3.11)\) and \((3.12)\), we know that

\[
J(\gamma_t, u_t(\cdot)) = \sum_{j=1}^N \chi_{A_{ij}} J(\gamma_t, u_{ij}(\cdot)).
\]

It is easy to prove that \( J(\gamma_t, u_{ij}(\cdot)) \) is deterministic. Without lost of generality, we suppose

\[
J(\gamma_t, u_{ij}(\cdot)) \leq J(\gamma_t, u_{i1}(\cdot)).
\]

Immediately, we have \( J(\gamma_t, u_t(\cdot)) \leq J(\gamma_t, u_{i1}(\cdot)) \). Combining \((3.9)\), we have

\[
\lim_{i \to \infty} J(\gamma_t, u_{i1}(\cdot)) = \tilde{v}(\gamma_t).
\]

Therefore, \( \tilde{v}(\gamma_t) \) is deterministic. \( \square \)

From \((3.3)\) and \((3.4)\), we have the following estimates on functional \( \tilde{v} \).

**Lemma 3.5.** There exists a constant \( C > 0 \) such that, for all \( 0 \leq t \leq T \), \( \gamma_t, \gamma_t' \in \Lambda_t \),

1. \( |\tilde{v}(\gamma_t) - \tilde{v}(\gamma_t')| \leq C \|\gamma_t - \gamma_t'\|_0; \)
2. \( |\tilde{v}(\gamma_t)| \leq C(1 + \|\gamma_t\|_0). \)

To formulate the DPP for the optimal control problem \((3.6), (3.7) \) and \((3.8)\), we define the family of backward semi-groups generated by BSDE \((3.7)\) in the spirit of of Peng \([40]\). Given the initial path \( \gamma_t \in \Lambda \), an \( \mathcal{F} \)-stopping time \( \hat{\tau} \geq t \), an admissible control process \( u \in \mathcal{U} \), and a real-valued random variable \( \eta \in L^2(\Omega, \mathcal{F}_\hat{\tau}, P; \mathbb{R}) \), we put

\[
\mathbb{G}^{\gamma_t,u}_{s,\hat{\tau}}[\eta] := \tilde{Y}^{\gamma_t,u}(s), \quad s \in [t, \hat{\tau}],
\]

where the pair \((\tilde{Y}^{\gamma_t,u}, \tilde{Z}^{\gamma_t,u})\) solves the following BSDE of the terminal time \( \hat{\tau} \):

\[
\tilde{Y}^{\gamma_t,u}(s) = \eta + \int_s^{\hat{\tau}} f(X^{\gamma_t,u}_r, \tilde{Y}^{\gamma_t,u}(r), \tilde{Z}^{\gamma_t,u}(r), u(r)) \, dr
- \int_s^{\hat{\tau}} \tilde{Z}^{\gamma_t,u}(r) \, dW(r), \quad \text{a.s.-}\omega, \quad \text{all } s \in [t, \hat{\tau}],
\]

with \( X^{\gamma_t,u} \) being the solution to SDE \((3.3)\). Then, obviously, for the solution \((Y^{\gamma_t,u}, Z^{\gamma_t,u})\) of BSDE \((3.7)\), the uniqueness of the BSDE yields

\[
J(\gamma_t, u) = \mathbb{G}^{\gamma_t,u}_{t,\hat{\tau}}[g(X^{\gamma_t,u}_\hat{\tau})] = \mathbb{G}^{\gamma_t,u}_{t,\hat{\tau}}[Y^{\gamma_t,u}(\hat{\tau})]
= \mathbb{G}^{\gamma_t,u}_{t,\hat{\tau}}[Y^{\gamma_t,u}(\hat{\tau})] = \mathbb{G}^{\gamma_t,u}_{t,\hat{\tau}}[J(X^{\gamma_t,u}, u)].
\]

17
The following dynamic programming principle (DPP) is adapted from the Markovian case, by mimicking the method of Peng \[39, 40\].

**Theorem 3.6.** Let Assumption (H1) be satisfied. Then for any $\delta \in (0, T-t)$, the value functional $\bar{v}$ obeys the following:

$$
\bar{v}(\gamma_t) = \text{ess sup}_{u \in U} \mathbb{G}^{\gamma_t,u}_{t,t+\delta} \left[ \bar{v}(X_{t+\delta}^{\gamma_t,u}) \right], \quad \gamma_t \in \Lambda. \tag{3.15}
$$

Our proof requires the following lemma

**Lemma 3.7.** Let $t \in [0, T]$, $\Gamma_t \in L^2(\Omega, \mathcal{F}_t; \Lambda_t, \mathcal{B}(\Lambda_t))$. For $u \in U$, we have

$$
\bar{v}(\Gamma_t) \geq Y^{\Gamma_t,u}(t). \tag{3.16}
$$

For any $\varepsilon > 0$, there is $u \in U$ such that

$$
\bar{v}(\Gamma_t) \leq Y^{\Gamma_t,u}(t) + \varepsilon. \tag{3.17}
$$

**Proof.** Since $\bar{v}$ is continuous in $\gamma_t \in \Lambda_t$ and $Y^{\gamma_t,u}$ is continuous in $(\gamma_t, u) \in \Lambda_t \times U$, it is sufficient to prove \eqref{3.16} for the following class of $\Gamma_t$ and $u$:

$$
\Gamma_t = \sum_{i=1}^{N} \chi_{A_i} \gamma_t^i
$$

and

$$
u(s) = \sum_{i=1}^{N} \chi_{A_i} u^i(s), \quad t \leq s \leq T.
$$

Here, $N$ is a positive integer, $u^i$ is $\mathcal{F}_t$-adapted, and $\{A_i\}_{i=1}^{N} \subset \mathcal{P}_t$ is a partition of $(\Omega, \mathcal{F}_t)$, that is, $\cup_{i=1}^{N} A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. We have

$$
Y^{\Gamma_t,u}(t) = \sum_{i=1}^{N} \chi_{A_i} Y^{\gamma_t^i,u^i}(t) \leq \sum_{i=1}^{N} \chi_{A_i} \bar{v}(\gamma_t^i) = \bar{v}(\Gamma_t).
$$

We obtain the first assertion \eqref{3.16}.

In a similar way, we prove \eqref{3.17}. Obviously there exists $\Gamma_t \in L^2(\Omega, \mathcal{F}_t; \Lambda_t, \mathcal{B}(\Lambda_t))$ of the form

$$
\bar{\Gamma}_t = \sum_{i=1}^{\infty} \chi_{A_i} \gamma_t^i.
$$
such that
\[ \|\Gamma_t - \bar{\Gamma}_t\| \leq \frac{1}{3} C^{-1} \varepsilon, \]
where \( C \) is the constant in Lemmas 3.1 and 3.2. \( \gamma_t^i \in \Lambda_t \) and \( \{A_i\}_{i=1}^\infty \subset \mathcal{F}_t \) satisfies
\[ \cup_{i=1}^\infty A_i = \Omega \) and \( A_i \cap A_j = \emptyset, i \neq j. \) By Lemmas 3.1 and 3.2 we have for any \( u \in \mathcal{U}, P\text{-a.e.}, \)
\[ |Y^{\Gamma_t, u}(t) - Y^{\bar{\Gamma}_t, u}(t)| \leq \frac{1}{3} \varepsilon, \quad (3.18) \]
\[ |\tilde{v}(\Gamma_t) - \tilde{v}(\bar{\Gamma}_t)| \leq \frac{1}{3} \varepsilon. \]

Then for any \( \gamma_t^i, \) we can choose an \( \mathcal{F}_t\)-adapted admissible control \( u^i \in \mathcal{U} \) such that
\[ \tilde{v}(\gamma_t^i) \leq Y^{\gamma_t^i, u^i}(t) - \frac{1}{3} \varepsilon. \]

Define
\[ u(\cdot) = \sum_{i=1}^\infty \chi_{A_i} u^i(\cdot). \]

Combining (3.18), we have
\[ Y^{\Gamma_t, u}(t) \geq -|Y^{\Gamma_t, u}(t) - Y^{\bar{\Gamma}_t, u}(t)| + Y^{\bar{\Gamma}_t, u}(t) \geq -\frac{1}{3} \varepsilon + \sum_{i=1}^\infty \chi_{A_i} Y^{\gamma_t^i, u^i}(t) \]
\[ \geq -\frac{1}{3} \varepsilon + \sum_{i=1}^\infty \chi_{A_i} (\tilde{v}(\gamma_t^i) - \frac{1}{3} \varepsilon) = -\frac{2}{3} \varepsilon + \tilde{v}(\bar{\Gamma}_t) \]
\[ \geq -\varepsilon + \tilde{v}(\Gamma_t). \]

The proof is complete.

**Proof of Theorem 3.6.** On one hand,
\[ \tilde{v}(\gamma_t) = \sup_{u \in \mathcal{U}} G^{\gamma_t, u}_{t, T} [g(X_T^{\gamma_t, u})] = \sup_{u \in \mathcal{U}} G^{\gamma_t, u}_{t, t+\delta} \left[ Y^{X_t^{\gamma_t, u}, u}(t + \delta) \right]. \]

From (3.16) and the definition of \( \tilde{v}, \) we have
\[ \tilde{v}(\gamma_t) \leq \sup_{u \in \mathcal{U}} G^{\gamma_t, u}_{t, t+\delta} \left[ \tilde{v}(X_t^{\gamma_t, u}) \right]. \]
On the other hand, by (3.17) for any \( \varepsilon > 0 \), there exists \( \bar{u} \in \mathcal{U} \) such that, a.e. \( P \)

\[
\tilde{v}(X_{t+\delta}^{\gamma;u}) \leq Y_{t+\delta}^{\gamma;u} \bar{u}(t+\delta) + \varepsilon.
\]

From (2.11) we have

\[
\tilde{v}(\gamma_t) \geq \sup_{u \in \mathcal{U}} G_{t,t+\delta}^{\gamma;u} \left[ \tilde{v}(X_{t+\delta}^{\gamma;u}) - \varepsilon \right]
\geq \sup_{u \in \mathcal{U}} G_{t,t+\delta}^{\gamma;u} \left[ \tilde{v}(X_{t+\delta}^{\gamma;u}) \right] - C\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we have (3.15). \( \square \)

In Lemma 3.5, the value functional \( \tilde{v} \) is Lipschitz continuous in \( \Lambda_t \), uniformly in \( t \). Theorem 3.6 implies the following continuity in \( t \).

**Lemma 3.8.** Let Assumption (H1) be satisfied. There is a constant \( C \) such that for every \( \gamma_T \in \Lambda_T \), \( t, t' \in [0, T] \),

\[
|\tilde{v}(\gamma_t) - \tilde{v}(\gamma_{t'})| \leq C(1 + \|\gamma_{t,t'}\|_0)(|t - t'|)^{\frac{1}{2}}.
\]  

(3.19)

**Proof.** Suppose that \( t \leq t' \). From Theorem 3.6 we see that for any \( \varepsilon > 0 \), there is \( u \in \mathcal{U} \) such that

\[
|\tilde{v}(\gamma_t) - G_{t,t'}^{\gamma;u} [\tilde{v}(X_{t'}^{\gamma;u})]| \leq \varepsilon.
\]

Hence

\[
|\tilde{v}(\gamma_t) - \tilde{v}(\gamma_{t'})| \leq \text{Part1} + \text{Part2} + \varepsilon,
\]

where

\[
\begin{align*}
\text{Part1} & := E \left| G_{t,t'}^{\gamma;u} [\tilde{v}(X_{t'}^{\gamma;u})] - G_{t,t'}^{\gamma;u} [\tilde{v}(\gamma_{t'})] \right|, \\
\text{Part2} & := E \left| G_{t,t'}^{\gamma;u} [\tilde{v}(\gamma_{t'})] - \tilde{v}(\gamma_{t'}) \right|.
\end{align*}
\]

Since \( \tilde{v} \) is uniformly continuous in \( \gamma_t \), we have from Lemmas 3.1 and 3.2

\[
\text{Part1} \leq C(E|\tilde{v}(X_{t'}^{\gamma;u}) - \tilde{v}(\gamma_{t'})|^2)^{\frac{1}{2}} \leq C(E\|X_{t'}^{\gamma;u} - \gamma_{t'}\|_0^2)^{\frac{1}{2}} \leq C(1 + \|\gamma_{t'}\|_0)(t' - t)^{\frac{1}{2}}
\]

20
for a positive constant $C$ being independent of $u$. By the definition of $G_{t,t'}^{\gamma,u}$, we have
\[
\text{Part 2} = \left| E \left[ \tilde{v}(\gamma') + \int_t^{t'} f(X^\gamma_{r,u}, Y^\gamma_{r,u}(r), Z^\gamma_{r,u}(r), u(r)) \, dr \right. \\
+ \left. \int_t^{t'} Z^\gamma_{r,u}(r) \, dW(r) \right] - \tilde{v}(\gamma') \right|
\leq C \left( 1 + \|\gamma'_t\|_0 \right) \left( t' - t \right)^{1/2}.
\]

Since $\varepsilon$ is arbitrary, we have (3.19). \qed

From Lemmas 3.5 and 3.8, we have the regularity for the value functional $\tilde{v}$.

**Theorem 3.9.** Let Assumption (H1) be satisfied. There is a constant $C > 0$ such that for any $0 \leq t \leq t' \leq T$ and $\gamma_t, \gamma'_t \in \Lambda$, we have
\[
|\tilde{v}(\gamma_t) - \tilde{v}(\gamma'_t)| \leq C \left( \|\gamma_t\|_0 + (1 + \|\gamma'_t\|_0 + \|\gamma_t\|_0) (t' - t)^{1/2} \right).
\] (3.20)

From (3.20), we have the stronger version of Theorem 3.6:

**Theorem 3.10.** Let Assumption (H1) be satisfied. For any $\mathcal{F}$-stopping time $\hat{\tau} \geq t$, a.s., the value functional $\hat{v}$ obeys the following:
\[
\hat{v}(\gamma_t) = \text{ess sup}_{u \in U} G_{t,\hat{\tau}}^{\gamma,u} \left[ \tilde{v}(X^\gamma_{\hat{\tau}}) \right], \quad \gamma_t \in \Lambda.
\] (3.21)

4 Associated path-dependent Bellman equation

4.1 Path-dependent Bellman equation and viscosity solution

Define the Hamiltonian $H : \Lambda \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times U \to \mathbb{R}$ by
\[
H(\gamma_t, r, p, A, u) := \frac{1}{2} \text{Tr} \left( \sigma \sigma^T(\gamma_t, u) A \right) + \langle b(\gamma_t, u), p \rangle + f(\gamma_t, r, \sigma^T(\gamma_t, u)p, u)
\] (4.1)
for $(\gamma_t, r, p, A, u) \in \Lambda \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times U$. For each $(\gamma_t, u) \in \Lambda \times U$, define the differential operator $\mathcal{L}(\gamma_t, u) : \mathcal{C}^{1,2}(\Lambda) \to \mathbb{R}$ by
\[
(\mathcal{L}\psi)(\gamma_t, u) := D_t\psi(\gamma_t) + H(\gamma_t, \psi(\gamma_t), D_x\psi(\gamma_t), D_{xx}\psi(\gamma_t), u), \quad \psi \in \mathcal{C}^{1,2}(\Lambda).
\] (4.2)
Consider the following path-dependent Bellman equation:

\[- D_t v(\gamma_t) - \sup_{u \in U} \mathcal{H}(\gamma_t, v(\gamma_t), D_x v(\gamma_t), D_{xx} v(\gamma_t), u) = 0, \quad \gamma_t \in \Lambda, t < T, \quad (4.3)\]

with the terminal condition

\[v(\gamma_T) = g(\gamma_T), \quad \gamma_T \in \Lambda_T. \quad (4.4)\]

**Definition 4.1.** (Classical solution). A functional \(v \in \mathcal{C}^{1.2}(\Lambda)\) is called a classical solution to the path-dependent Bellman equation \((4.3)\) if it satisfies the path-dependent Bellman equation \((4.3)\) point-wisely in the sense of Definition 2.2.

For \((\kappa, \iota) \in (0, \infty) \times (0, T)\) and \(\gamma_t \in \Lambda\) with \(t \in [0, T - \iota)\), define the cylinder

\[Q_{\kappa, \iota}(\gamma_t) := \{\gamma_{t'} \in \Lambda : t \leq t' \leq t + \iota, \|\gamma_{t,t'}\|_0 < \kappa\}. \quad (4.5)\]

Write \(Q_{\kappa, \iota}\) for \(Q_{\kappa, \iota}(0)\), i.e.,

\[Q_{\kappa, \iota} := \{\gamma_{t'} \in \Lambda : 0 \leq t' \leq t + \iota, \|\gamma_{t'}\|_0 < \kappa\}. \quad (4.6)\]

Throughout the rest of this paper, we fix \(\alpha \in (0, \frac{1}{2})\) and \(\beta \in (0, 1)\).

We now generalize the classical notions of semi-jets (see Crandall, Ishii, and Lions [9]). For \(\gamma_t \in \mathcal{C}^\alpha\) with \(t \in [0, T)\), \((\mu, \kappa) \in (\|\gamma_t\|_0, \infty) \times (0, T - t)\), and \(v \in \mathcal{C}(\Lambda)\), define the super-jet of \(v\) at \(\gamma_t\) sliced by the double index of Hölder modulus \((\mu, \kappa)\):

\[J_{\mu, \kappa}^+(\gamma_t, v) \quad (4.7)\]

\[:= \{\psi \in \mathcal{C}^{1,2}(\Lambda) : |\psi|_{2, \beta; Q_{\kappa, \iota}(\gamma_t) \cap C^\alpha_{\mu}} \leq \kappa^{-1}, \quad 0 = \psi(\gamma_t) - v(\gamma_t) \leq \psi(\gamma_{t'}) - v(\gamma_{t'}), \quad \forall \gamma_{t'} \in Q_{\kappa, \iota}(\gamma_t) \cap C^\alpha_{\mu}\}\]

and the sub-jet of \(v\) at \(\gamma_t\) sliced by the double index of Hölder modulus \((\mu, \kappa)\):

\[J_{\mu, \kappa}^-(\gamma_t, v) \quad (4.8)\]

\[:= \{\psi \in \mathcal{C}^{1,2}(\Lambda) : |\psi|_{2, \beta; Q_{\kappa, \iota}(\gamma_t) \cap C^\alpha_{\mu}} \leq \kappa^{-1}, \quad 0 = \psi(\gamma_t) - v(\gamma_t) \geq \psi(\gamma_{t'}) - v(\gamma_{t'}), \quad \forall \gamma_{t'} \in Q_{\kappa, \iota}(\gamma_t) \cap C^\alpha_{\mu}\}\]

**Remark 4.2.** Both \(J_{\mu, \kappa}^+(\gamma_t, v)\) and \(J_{\mu, \kappa}^-(\gamma_t, v)\) may be empty.

Our notion of viscosity solutions is defined as follows.
Definition 4.3. (i) We call \( v \in \mathcal{C}(\Lambda) \) a viscosity sub-solution to the path-dependent Bellman equation (4.3), if for any \((M_0, \kappa) \in (0, \infty) \times (0, T)\), we have
\[
\lim_{\mu \to \infty} \sup_{\gamma_t \in Q_{M_0, T - \kappa} \cap C_{\mu}^\alpha} \left\{ -D_t \psi - \sup_{u \in U} \mathcal{H}(\cdot, \psi, D_x \psi, D_{xx} \psi, u) \right\}(\gamma_t) \leq 0. \tag{4.9}
\]

(ii) We call \( v \in \mathcal{C}(\Lambda) \) a viscosity super-solution to the path-dependent Bellman equation (4.3), if for any \((M_0, \kappa) \in (0, \infty) \times (0, T)\), we have
\[
\lim_{\mu \to \infty} \inf_{\gamma_t \in Q_{M_0, T - \kappa} \cap C_{\mu}^\alpha} \left\{ -D_t \psi - \sup_{u \in U} \mathcal{H}(\cdot, \psi, D_x \psi, D_{xx} \psi, u) \right\}(\gamma_t) \geq 0. \tag{4.10}
\]

(iii) We call \( v \in \mathcal{C}(\Lambda) \) a viscosity solution to the path-dependent Bellman equation (4.3) if it is both the viscosity sub- and super-solution.

Note that in (4.9) and (4.10): \( \sup \emptyset := -\infty \), \( \inf \emptyset := +\infty \).

Remark 4.4. (1) A viscosity solution of the path-dependent Bellman equation \( u \) is a classical solution if it furthermore lies in \( \mathcal{C}^{1,2}(\Lambda) \).

(2) In the classical uniqueness proof of viscosity solution to state-dependent PDEs in an unbounded domain (which is locally compact), a conventional technique is to construct an auxiliary smooth function decaying outside a compact domain. In our path-dependent case, we find it difficult to construct such smooth functionals. For the sake of the uniqueness proof, our new notion of jets is enlarged to be defined only on \( C_{\mu}^\alpha \), which is compact in \( \Lambda \). However, at a cost, our modification leads to additional difficulty in the existence proof.

(3) Assume that all the coefficients of Bellman equation (4.3) and terminal condition (4.4) are state-dependent. Let state-dependent function \( u \) be a viscosity solution to (4.3) as a path-dependent functional. Then \( u \) is also a classical viscosity solution as a function of time and state.

4.2 Main results

Our main results on the existence and the uniqueness of the viscosity solution to the path-dependent Bellman equation (4.3) are formulated below.

Theorem 4.5. Let assumption (H1) be satisfied. Then \( \tilde{v} \) defined by (3.8) is a viscosity solution to the path-dependent Bellman equation (4.3).

23
The uniqueness is given on both non-degenerate and degenerate cases. We first address the non-degenerate case.

We make the following assumption, extending our previous assumption (H1) to the larger path space \( \hat{\Lambda} \).

(H2) The functionals \( b : \hat{\Lambda} \times U \to \mathbb{R}^n, \sigma : \hat{\Lambda} \times U \to \mathbb{R}^{n \times d}, g : \hat{\Lambda}_T \to \mathbb{R}, \) and \( f : \hat{\Lambda} \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R} \) are all bounded. There is a constant \( C > 0 \) such that for all \((t, \gamma_T, y, z, u), (t', \gamma'_T, y', z', u') \in [0, T] \times \hat{\Lambda}_T \times \mathbb{R} \times \mathbb{R}^d \times U,\)

\[
|b(\gamma_t, u) - b(\gamma'_t, u')| \leq C(d_p(\gamma_t, \gamma'_t) + |u - u'|),
|\sigma(\gamma_t, u) - \sigma(\gamma'_t, u')| \leq C(d_p(\gamma_t, \gamma'_t) + |u - u'|),
|f(\gamma_t, y, z, u) - f(\gamma'_t, y', z', u')| \leq C(d_p(\gamma_t, \gamma'_t) + |y - y'| + |z - z'| + |u - u'|),
\]

(4.11)

\[
|g(\gamma_T) - g(\gamma'_T)| \leq C\|\gamma_T - \gamma'_T\|_0.
\]

Moreover, \( \sigma \) satisfies the non-degenerate condition

\[
\sigma^T > \frac{1}{C} I_n.
\]

The uniqueness of viscosity solutions of (4.3) is an immediate consequence of the following representation theorem.

**Theorem 4.6.** Suppose that (H2) holds. Let \( v \in C_b(\Lambda) \cap C_u(\Lambda) \). If \( v \) is a viscosity solution to the path-dependent Bellman equation (4.3), and \( v = g \) on \( \Lambda_T \), then \( v \) is the value functional \( \tilde{v} \) defined by (3.8).

In the degenerate case of \( \sigma \sigma^T \geq 0 \), we have the following extra smooth conditions on the coefficients.

(H3) Functionals \( b : \hat{\Lambda} \times U \to \mathbb{R}^n, \sigma : \hat{\Lambda} \times U \to \mathbb{R}^{n \times d}, g : \hat{\Lambda}_T \to \mathbb{R}, \) and \( f : \hat{\Lambda} \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R} \) satisfy (4.11). Furthermore, for any \( u \in U \), the functionals \( \sigma(\cdot, u), b(\cdot, u) \in C^{1,2}_b(\hat{\Lambda}), g \in C^{1,2}_b(\Lambda_T), \) and \( f(\cdot, \cdot, \cdot, u) \in C^{1,2,2}_b(\hat{\Lambda} \times \mathbb{R} \times \mathbb{R}^n) \) and all their differentials are bounded uniformly w.r.t. \( u \in U \).

We have the following representation theorem.

**Theorem 4.7.** Suppose that (H3) holds. Let \( v \in C_b(\Lambda) \cap C_u(\Lambda) \) be a viscosity solution to the path-dependent Bellman equation (4.3), and \( v = g \) on \( \Lambda_T \). Then \( v = \tilde{v} \), where \( \tilde{v} \) is defined by (3.8).
Remark 4.8. Analogous to the proof of Lemma 3.5 and Theorem 3.9 from the bounded and Lipschitz assumption on the coefficients in (H2) or (H3), the value functional $\tilde{v}$ can be shown to be bounded and to satisfy

$$|v(\gamma_t) - v(\gamma'_t)| \leq C d_p(\gamma_t, \gamma'_t), \quad \forall \gamma_t, \gamma'_t \in \Lambda,$$

which implies $\tilde{v} \in C_b(\Lambda) \cap C^u(\Lambda)$. Note that the non-degeneracy condition (4.12) is not needed here.

In view of the comparison theorem of BSDEs, we immediately have the following comparison theorem.

Corollary 4.9. Suppose that either (H2) or (H3) holds. Let $v_1, v_2 \in C_b(\Lambda)$ satisfy (4.13), and $g_1, g_2 \in C_b(\Lambda_T)$ satisfy $g_1 \leq g_2$. Furthermore, let $v_1$ and $v_2$ be viscosity solutions to the path-dependent Bellman equation (4.3) with the terminal conditions:

$$v_1(\gamma_T) = g_1(\gamma_T), \quad v_2(\gamma_T) = g_2(\gamma_T), \quad \forall \gamma_T \in \Lambda_T.$$

Then $v_1 \leq v_2$.

For an initial path $\gamma_t \in \Lambda$, define

$$\mathcal{U}_t := \{u \in [t, T] \times \Omega \to U | u \text{ is } \mathcal{F}^t\text{-progressive measurable}\}.$$

We have

Theorem 4.10 (Verification Theorem). Let $v$ be a classical solution to (4.3) and (4.4). Then we have the following two assertions:

(i) $v(\gamma_t) \geq J(\gamma_t, u), \quad \forall (u, \gamma_t) \in \mathcal{U}_t \times \Lambda.$

(ii) If the following holds for an admissible control $u^* \in \mathcal{U}_t$: for every $\gamma_t \in \Lambda$,

$$0 = (\mathcal{L} v)(X_{s}^{\gamma_t, u^*}, u^*(s)) = \text{ess max}_{\beta \in U} \{(\mathcal{L} v)(X_{s}^{\gamma_t, u^*}, \beta)\}, \quad \text{a.s.-} \omega, \text{ a.e.s \in } [t, T],$$

then $v(\gamma_t) = J(\gamma_t)$ for any $\gamma_t \in \Lambda$.

Proof. For $u \in \mathcal{U}_t$, since $v$ is a classical solution, we have

$$(\mathcal{L} u)(X_{s}^{\gamma_t, u}, u(s)) \leq 0.$$
Applying Itô formula to compute $v(X^{\gamma_t, u}_s)$, we have

$$v(X^{\gamma_t, u}_s) = g(X^{\gamma_t, u}_T)$$

$$+ \int_s^T \left( f(s, v, \sigma^T(s, u(r)) \cdot D_x v, u(r)) - (\mathcal{L} v)(s, u(r)) \right) (X^{\gamma_t, u}_r) dr$$

$$- \int_s^T \sigma^T(X^{\gamma_t, u}_r, u(r)) D_x v(X^{\gamma_t, u}_r) dW(r), \quad s \in [t, T].$$

Define for $s \in [t, T]$,

$$Y^1(s) := v(X^{\gamma_t, u}_s) - Y^{\gamma_t, u}(s), \quad Z^1(s) := \sigma^T(X^{\gamma_t, u}_s, u(s)) D_x v(X^{\gamma_t, u}_s) - Z^{\gamma_t, u}(s).$$

In view of (3.7), we have

$$Y^1(s) = \hat{T}^T s (A(r) Y^1(r) + \langle \bar{A}, Z^1 \rangle(r) - (\mathcal{L} v)(X^{\gamma_t, u}_r, u(r))) dr - Z^1(r) dW(r).$$

Denote by $\Gamma^t(\cdot)$ the unique solution of the linear SDE

$$d\Gamma^t(s) = \Gamma^t(s) \left( A(s) ds + \bar{A}(s) dW(s) \right), \quad s \in [t, T]; \quad \Gamma^t(t) = 1.$$

From [19, Proposition 2.2], we have

$$E[Y^1(t)] = -E \left[ \int_t^T \Gamma^t(r) (\mathcal{L} v)(X^{\gamma_t, u}_r, u(r)) dr \right] \geq 0,$$

and the equality holds for $u = u^*$. This proves Assertions (i) and (ii). The proof is complete.

**Example 4.11.** In what follows, we show that the conventional non-Markovian optimal stochastic control problem is included as a particular case of our problem (3.6), (3.7) and (3.8). Under some suitable smooth conditions, the corresponding path-dependent Bellman equation is associated to a backward stochastic Bellman equation via Dupire’s functional calculus.
Let \( \{B_t, 0 \leq t \leq T\} \) be a \( d \)-dimensional Winner process on the probability space \( (\Omega := \Lambda_T(\mathbb{R}^d), P_0) \). Consider functionals \( \tilde{b} : \Lambda(\mathbb{R}^d) \times \mathbb{R}^n \times U \to \mathbb{R}^n \), \( \tilde{\sigma} : \Lambda(\mathbb{R}^d) \times \mathbb{R}^n \times U \to \mathbb{R} \), \( \tilde{f} : \Lambda(\mathbb{R}^d) \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R} \), and \( \tilde{g} : \Lambda_T(\mathbb{R}^d) \times \mathbb{R}^n \to \mathbb{R} \). The non-Markovian stochastic optimal control problem is formulated as follows. For any differential systems:

\[
\begin{align*}
\int_{0}^{T} & \int_{\mathbb{R}^{d+n}} \tilde{b}(B_s, X^t,x,u(s), u(s))ds + \tilde{\sigma}(B_s, X^t,x,u(s), u(s))dB(s), \quad s \in [t, T]; \\
X^t,x,u(t) & = x,
\end{align*}
\]

and

\[
\begin{align*}
\int_{0}^{T} & \int_{\mathbb{R}^{d+n}} -\tilde{f}(B_s, X^t,x,u(s), Z^t,x,u(s), u(s))ds - Z^t,x,u(s)dB(s), \quad s \in [t, T]; \\
Y^t,x,u(T) & = \tilde{g}(B_T, X^t,x,u(T)).
\end{align*}
\]

The optimal value field \( \bar{v} : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R} \) is given by

\[
\bar{v}(t, x) := \text{ess sup}_{u \in \mathcal{U}} \bar{Y}^t,x,u(t).
\]

This problem depends on the Brownian path \( B_t \) and the state \( X(t) \). Now we translate this problem into the path-dependent case. For any \( (\gamma_t, \xi_t) \in \Lambda(\mathbb{R}^d) \times \Lambda(\mathbb{R}^n) \) and \( u \in U \), define \( b : \Lambda(\mathbb{R}^{d+n}) \times U \to \mathbb{R}^{d+n} \), \( \sigma : \Lambda(\mathbb{R}^{d+n}) \times U \to \mathbb{R}^{(d+n)\times d} \), \( f : \Lambda(\mathbb{R}^{d+n}) \times \mathbb{R} \times \mathbb{R}^{d+n} \times U \to \mathbb{R} \), and \( g : \Lambda_T(\mathbb{R}^{d+n}) \to \mathbb{R} \) as follows:

\[
\begin{align*}
b((\gamma_t, \xi_t), u) & := \begin{pmatrix} 0 \\
\tilde{b}(\gamma_t, \xi_t(t), u) \\
\end{pmatrix}, \\
\sigma((\gamma_t, \xi_t), u) & := \begin{pmatrix} I_d \\
\tilde{\sigma}(\gamma_t, \xi_t(t), u) \\
\end{pmatrix}, \\
f(\gamma_t, \xi_t, y, z, u) & := \tilde{f}(\gamma_t, \xi_t(t), y, z, u), \\
g(\gamma_T, \xi_T) & := \tilde{g}(\gamma_T, \xi(T)).
\end{align*}
\]

Following (3.1), (3.7) and (3.8), for any \( (\gamma_t, \xi_t) \in \Lambda(\mathbb{R}^d) \times \Lambda(\mathbb{R}^n) \) and \( u \in U \), we define \( X^{(\gamma_t, \xi_t), u}, Y^{(\gamma_t, \xi_t), u}, \) and \( \bar{v}(\gamma_t, \xi_t) := \text{ess sup}_{u \in \mathcal{U}} Y^{(\gamma_t, \xi_t), u}(t) \). Note that \( \bar{v}(\gamma_t, \xi_t) \) only depends on the state \( x = \xi_t(t) \) of the path \( \xi_t \) at time \( t \)—instead of its whole history up to time \( t \), and thus we can rewrite \( X^{(\gamma_t, \xi_t), u}, Y^{(\gamma_t, \xi_t), u}, \) and \( \bar{v}(\gamma_t, \xi_t) \) into \( X^{t, x, u}, Y^{t, x, u}, \) and \( \bar{v}(t, x) \), respectively. The uniqueness of solution to the FBSDE implies that for any \( (t, x) \in [0, T] \times \mathbb{R}^n \),

\[
\begin{align*}
X^{t, x, u}(s) & = X^{B_t, x, u}(s), \quad \text{P-a.s.}, \quad t \leq s \leq T, \\
Y^{t, x, u}(s) & = Y^{B_t, x, u}(s), \quad \text{P-a.s.}, \quad t \leq s \leq T, \\
\bar{v}(t, x) & = \bar{v}(B_t, x), \quad \text{P-a.s.}, \quad t \leq s \leq T.
\end{align*}
\]
Furthermore, in view of Theorem 4.5, $\tilde{v}(\gamma_t, x)$ is a solution to the PDE:

$$-D_t \tilde{v} - \sup_{u \in U} \left[ \frac{1}{2} \text{Tr} (\bar{\sigma} \bar{\sigma}^T (\gamma_t, x, u) \partial_{xx} \tilde{v}) + \langle \bar{b}(\gamma_t, x, u), \partial_x \tilde{v} \rangle + \frac{1}{2} \text{Tr} D_{\gamma\gamma} \tilde{v} \right] + \bar{\sigma}^T (\gamma_t, x, u) D_{x\gamma} \tilde{v} + \tilde{f}(\gamma_t, x, u, D_{\gamma} \tilde{v} + \bar{\sigma}^T (\gamma_t, x, u) \partial_x \tilde{v}, u) = 0, \quad (\gamma_t, x) \in \Lambda(\mathbb{R}^d) \times \mathbb{R}^n. \quad (4.14)$$

Here, $D_\gamma$ and $D_{\gamma\gamma}$ are the path vertical derivatives in $\gamma_t \in \Lambda(\mathbb{R}^d)$, and $\partial_x$ and $\partial_{xx}$ are the classical partial derivatives in the state variable $x$.

If $\tilde{v}(\gamma_t, x)$ is smooth enough, applying Itô formula to $\tilde{v}(B_t, x)$, we have

$$d\tilde{v}(B_t, x) = (D_t \tilde{v}(B_t, x) + \frac{1}{2} \text{Tr} D_{\gamma\gamma} \tilde{v}(B_t, x)) dt + D_{\gamma} \tilde{v}(B_t, x) dB(t).$$

In view of (4.14), we have

$$d\bar{v}(B_t, x) = -\sup_{u \in U} \left[ \frac{1}{2} \text{Tr} (\bar{\sigma} \bar{\sigma}^T (B_t, x, u) \partial_{xx} \bar{v}) + \langle \bar{b}(B_t, x, u), \partial_x \bar{v} \rangle \right] + \bar{\sigma}^T (B_t, x, u) D_{x\gamma} \bar{v} + \bar{f}(B_t, x, u, D_{\gamma} \bar{v} + \bar{\sigma}^T (B_t, x, u) \partial_x \bar{v}, u) \right] dt + D_{\gamma} \bar{v}(B_t, x) dB(t).$$

Define the pair of $\mathcal{F}_t$-adapted processes $(\bar{v}(t, x), p(t, x)) := (\bar{v}(B_t, x), D_{\gamma} \bar{v}(B_t, x))$. Then we have

$$d\bar{v}(t, x) = -\sup_{u \in U} \left\{ \frac{1}{2} \text{Tr} (\bar{\sigma} \bar{\sigma}^T (B_t, x, u) \partial_{xx} \bar{v}) + \langle \bar{b}(B_t, x, u), \partial_x \bar{v} \rangle \right\} + \bar{\sigma}^T (B_t, x, u) \partial_x p + \bar{f}(B_t, x, u, p + \bar{\sigma}^T (B_t, x, u) \partial_x \bar{v}, u) \right\} dt + p dB(t),$$

with the terminal condition

$$\bar{v}(T, x) = \bar{g}(B_T, x). \quad (4.16)$$

The fully nonlinear BSPDE (4.15) and (4.16) with $\bar{f}$ being invariant in the third and fourth arguments $(y, z)$, is the so-called stochastic Bellman equation, introduced by Peng [39, 40].

28
5 Existence of viscosity solutions

In this section we give the solution of the path-dependent Bellman equation (4.3) with the help of FBSDEs (3.6) and (3.7).

First, let us perturb a path \( \gamma_t \in C^\alpha_\mu \). For \( \mu > 0, \varepsilon \in (0, \mu) \) and \( \gamma_t \in C^\alpha_\mu \), define a perturbation of \( \gamma_t \) in the following manner:

\[
\gamma^\varepsilon_t(s) := \begin{cases} 
\gamma_t(s), & |\gamma_t(s) - \gamma_t(t)| \leq (\mu - \varepsilon)|s - t|^\alpha; \\
\gamma_t(t) + (\mu - \varepsilon)(t - s)^\alpha \frac{\gamma_t(s) - \gamma_t(t)}{|\gamma_t(s) - \gamma_t(t)|}, & |\gamma_t(s) - \gamma_t(t)| > (\mu - \varepsilon)|s - t|^\alpha.
\end{cases}
\] (5.1)

We have

Lemma 5.1. Let \( \mu > 0, M_0 > 0 \). Assume that \( \|\gamma_t\|_\alpha \leq \mu, \|\gamma_t\|_0 \leq M_0 \), and \( \varepsilon \leq \frac{1}{2} \mu \).

We have

(i) \( \|\gamma^\varepsilon_t - \gamma_t\|_0 \leq 2M_0\varepsilon(\mu - \varepsilon)^{-1} \leq 4M_0\varepsilon^{-1} \);

(ii) \( \|\gamma^\varepsilon_t\|_\alpha \leq \mu \);

(iii) there is a constant \( C \), independent of \( \mu \) and \( u \in U \), such that for some \( p, p(\frac{1}{2} - \alpha) > 1 \) and for all \( \delta < T - t \),

\[
P\{\|X^\varepsilon_{t+\delta}\|_\alpha > \mu\} \leq C\delta^{p(\frac{1}{2} - \alpha)}\varepsilon^{-p}.
\]

Proof. Assertion (i) is obvious. Now we prove Assertion (ii).

Since \( |\gamma^\varepsilon_t(s) - \gamma^\varepsilon_t(t)| = |\gamma^\varepsilon_t(s) - \gamma_t(t)| \leq \mu|s - t|^\alpha \) for \( s \in [0, t] \), it is sufficient to show that for any \( s_1, s_2 \in [0, t] \) such that \( s_1 > s_2 \), we have

\[
|\gamma^\varepsilon_t(s_1) - \gamma^\varepsilon_t(s_2)| \leq \mu|s_1 - s_2|^\alpha.
\] (5.2)

Define

\[
r_1 := |\gamma^\varepsilon_t(s_1) - \gamma_t(t)|, \quad r_2 := |\gamma^\varepsilon_t(s_2) - \gamma_t(t)|,
\]

\[
x_1 := |\gamma_t(s_1) - \gamma_t(t)| - r_1, \quad x_2 := |\gamma_t(s_2) - \gamma_t(t)| - r_2,
\]

\[
C^\varepsilon := |\gamma^\varepsilon_t(s_1) - \gamma^\varepsilon_t(s_2)|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos \theta,
\]

\[
C^2 := |\gamma_t(s_1) - \gamma_t(s_2)|^2 = (r_1 + x_1)^2 + (r_2 + x_2)^2 - 2(r_1 + x_1)(r_2 + x_2)\cos \theta.
\]

Here, \( \theta \) is the angle between both vectors \( \gamma_t(s_1) - \gamma_t(t) \) and \( \gamma_t(s_2) - \gamma_t(t) \) and it is equal to the angle between both vectors \( \gamma^\varepsilon_t(s_1) - \gamma_t(t) \) and \( \gamma^\varepsilon_t(s_2) - \gamma_t(t) \). We have

\[
0 \leq r_1 \leq (\mu - \varepsilon)(t - s_1)^\alpha, \quad 0 \leq r_2 \leq (\mu - \varepsilon)(t - s_2)^\alpha, \quad x_1 \geq 0, \quad x_2 \geq 0.
\] (5.3)

and

\[
C^2 - C^\varepsilon = x_1^2 + x_2^2 + 2r_1x_1 + 2r_2x_2 - 2(x_1x_2 + r_1x_2 + r_2x_1)\cos \theta.
\] (5.4)

29
We assert that \( C^2 \leq C^2 \), which implies (5.2) immediately. It is obvious if \( x_2 = 0 \). If \( x_2 > 0 \), we have
\[
r_2 = (\mu - \varepsilon)|t - s_2|^\alpha > (\mu - \varepsilon)|t - s_1|^\alpha \geq r_1
\]
by the definition (5.1), which together with equality (5.5) gives \( C^2 - C^2 \geq 0 \).

The proof of inequality (5.2) is divided into the following three cases.

**The case of \( x_1 = 0 \).** We have
\[
C^2 - C^2 = x_2^2 + 2r_2x_2 - 2r_1x_2 \cos \theta = x_2^2 + 2x_2(r_2 - r_1 \cos \theta).
\]
(5.5)
We assert that \( C^2 \leq C^2 \), which implies (5.2) immediately. It is obvious if \( x_2 = 0 \). If \( x_2 > 0 \), we have
\[
r_2 = (\mu - \varepsilon)|t - s_2|^\alpha > (\mu - \varepsilon)|t - s_1|^\alpha \geq r_1
\]
by the definition (5.1), which together with equality (5.5) gives \( C^2 - C^2 \geq 0 \).

**The case of \( x_2 = 0 \) and \( r_2 \leq r_1 \).** We have
\[
r_2 \leq r_1 = (\mu - \varepsilon)|t - s_1|^\alpha < (\mu - \varepsilon)|t - s_2|^\alpha.
\]
Therefore, we have \( x_2 = 0 \) from the definition (5.1), and thus
\[
C^2 - C^2 = x_1^2 + 2r_1x_1 - 2r_2x_1 \cos \theta = x_1^2 + 2x_1(r_1 - r_2 \cos \theta) \geq 0.
\]

**The case of \( x_1 > 0 \) and \( r_2 > r_1 \).** We have
\[
r_2 \leq (\mu - \varepsilon)|t - s_2|^\alpha, \quad r_1 = (\mu - \varepsilon)|t - s_1|^\alpha.
\]
If \( C^2 \geq C^2 \), the proof is complete. If \( C^2 < C^2 \), we have
\[
\cos \theta > \frac{x_1^2 + x_2^2 + 2r_1x_1 + 2r_2x_2}{2(x_1^2 + x_2^2 + r_1x_1 + r_2x_2)} \geq \frac{r_1}{r_2}.
\]
Then
\[
C^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \theta \leq r_1^2 + r_2^2 - 2r_1^2 = r_2^2 - r_1^2
\]
\[
\leq (\mu - \varepsilon)^2|t - s_2|^{2\alpha} - (\mu - \varepsilon)^2|t - s_1|^{2\alpha} \leq (\mu - \varepsilon)^2|s_1 - s_2|^{2\alpha},
\]
and thus (5.2) holds. The last inequality is deduced from the following fact: if \( 2\alpha \in [0, 1] \), then \( a^{2\alpha} + b^{2\alpha} \geq (a + b)^{2\alpha} \) for all \( a > 0, b > 0 \).

It remains to show Assertion (iii). For any \( \delta < T - t \) and \( \tilde{\gamma}_{t+\delta} \in \Lambda \) such that
\[
\sup_{t \leq s_1 < s_2 \leq t + \delta} \frac{|\tilde{\gamma}_{t+\delta}(s_1) - \tilde{\gamma}_{t+\delta}(s_2)|}{|s_1 - s_2|^{\alpha}} \leq \varepsilon,
\]
in view of (5.1) and Assertion (ii), we have \( \|\tilde{\gamma}_{t+\delta}\|_\alpha \leq \mu \). Therefore, we have
\[
\left\{ \left[ X_{t+\delta}^{\tilde{\gamma}_{t+\delta}} \right]_\alpha > \mu \right\} \subset \left\{ \sup_{t \leq s_1 < s_2 \leq t + \delta} \frac{|X_{t+\delta}^{\tilde{\gamma}_{t+\delta}}(s_1) - X_{t+\delta}^{\tilde{\gamma}_{t+\delta}}(s_2)|}{|s_1 - s_2|^{\alpha}} > \varepsilon \right\}.
\]
Assertion (iii) then follows from Proposition 7.1 in the Appendix. □
Proof of Theorem 4.5. Firstly, we show that \( \tilde{v} \) is a viscosity sub-solution. Let \( M_0 > 0, \mu > 0, \) and \( \kappa \in (0, T) \). For \( \gamma_t \in Q_{M_0, T-k} \cap C^a_\mu \) and \( \psi \in \mathcal{J}^{+}_{\mu, \kappa}(\gamma_t, \tilde{v}) \). Note that the cylinder \( Q_{M_0, T-k}(\gamma_t) \) is defined by (4.6).

For any \( \mu > 1 \) and \( \varepsilon < \frac{1}{2} \wedge \left( \frac{1}{16} \kappa \mu M_0^{-1} \right) \), from Assertion (ii) of Lemma 5.1, we have

\[
\| \gamma_t^\varepsilon - \gamma_t \|_0 \leq 4M_0\varepsilon\mu^{-1} < \frac{1}{2}\kappa.
\]

For any \( \mu > 1 \) and \( u \in \mathcal{U} \), we define an \( F \)-stopping time

\[
\hat{\tau}^\varepsilon := \inf \{ s > t : \| X^\gamma_{s:t} u \|_\alpha > \mu \} \wedge \inf \{ s > t : \| X^\gamma_{s:t} u - \gamma_{s:t} \|_0 > \kappa \} \wedge (t + \kappa).
\]

Obviously, \( X^\gamma_{\hat{\tau}^\varepsilon} u \in Q_{\kappa, \kappa}(\gamma_t) \cap C^a_\mu \), and for any \( \delta < \kappa \),

\[
\{ \hat{\tau}^\varepsilon \geq t + \delta \} \supset \left\{ \| X^\gamma_{t+\delta} u \|_\alpha \leq \mu \right\} \cap \left\{ \| X^\gamma_{t+\delta} - \gamma_{t+\delta} \|_0 \leq \frac{1}{2}\kappa \right\}.
\]

Therefore,

\[
P \{ \hat{\tau}^\varepsilon \geq t + \delta \} \geq P \left\{ \| X^\gamma_{t+\delta} u \|_\alpha \leq \mu \right\} - P \left\{ \| X^\gamma_{t+\delta} u - \gamma_{t+\delta} \|_0 > \frac{1}{2}\kappa \right\}.
\]

From Lemma 3.1 and Assertion (iii) of Lemma 5.1, we have

\[
P \left\{ \| X^\gamma_{t+\delta} u \|_\alpha \leq \mu \right\} \geq 1 - C\delta^{p(\frac{1}{2} - \alpha)} \varepsilon^{-p},
\]

\[
P \left\{ \| X^\gamma_{t+\delta} - \gamma_{t+\delta} \|_0 > \frac{1}{2}\kappa \right\} \leq C\delta\kappa^{-2}.
\]

Note that \( \hat{\tau}^\varepsilon \) depends on \( u \) and \( \mu \), while the R.H.S. of both inequalities are independent of the pair \( (u, \mu) \). Hence, uniformly with respect to \( (u, \mu) \),

\[
P \{ \hat{\tau}^\varepsilon \geq t + \delta \} \geq 1 - C\delta^{p(\frac{1}{2} - \alpha)} \varepsilon^{-p} - C\delta\kappa^{-2} \wedge 1 \quad \text{as} \quad \delta \to 0.
\]

where \( p(\frac{1}{2} - \alpha) > 1 \). In particular, there is a positive constant \( \delta_1(\varepsilon, \kappa, p) < \kappa \) such that,

\[
P \{ \hat{\tau}^\varepsilon \geq t + \delta \} \geq \frac{1}{2}, \quad \forall \delta \in (0, \delta_1(\varepsilon, \kappa, p)). \quad (5.6)
\]

Define

\[
\hat{\tau}^{\varepsilon, \delta} := \hat{\tau}^\varepsilon \wedge (t + \delta). \quad (5.7)
\]
Comparing (5.8) and (5.9), we have for $r$ where

$$\psi(\gamma_t) = \psi(X_r^{\gamma, u}) \in \mathbb{F}_t$$

where $L$ is defined as (4.2). Let $(Y, Z, \tilde{\gamma})$ be the solution of the following BSDE

$$
\begin{aligned}
-dY(r) &= f(X_r^{\gamma, u}, Y(r), Z(r), u(r)) dr - Z(r) dW(r), \quad r \in [t, \tilde{\gamma}]; \\
Y(\tilde{\gamma}) &= \bar{v}(X_{\tilde{\gamma}}^{\gamma, u}).
\end{aligned}
$$

Comparing (5.8) and (5.9), we have for $r \in [t, \tilde{\gamma}]$, $P$-a.s.,

$$
\begin{aligned}
-dY^{2, \varepsilon, \delta, u}(r) &= -\mathcal{L}Y^{2, \varepsilon, \delta, u}(r) dr + f(X_r^{\gamma, u}, \psi(X_r^{\gamma, u}), \sigma^T(X_r^{\gamma, u}, u(r)) D_x \psi(X_r^{\gamma, u}), u(r)) dr \\
&\quad - f(X_r^{\gamma, u}, Y^{1, \varepsilon, \delta, u}(r), Z^{1, \varepsilon, \delta, u}(r), u(r)) dr \\
&\quad + Z^{2, \varepsilon, \delta, u}(r) dW(r)
\end{aligned}
$$

where $|A|, |\tilde{A}| \leq C$ ($C$ depends on Lipschitz constant of $f$, and is independent of the triplet $(u, \varepsilon, \delta)$. Therefore, we have (see [19] Proposition 2.2])

$$
\begin{aligned}
Y^{2, \varepsilon, \delta, u}(t) &= E \left[ Y^{2, \varepsilon, \delta, u}(\tilde{\gamma}) \Gamma^t(\tilde{\gamma}, \delta) - \int_t^{\tilde{\gamma}} \Gamma^t(r) (\mathcal{L}Y)(X_r^{\gamma, u}, u(r)) dr \big| \mathcal{F}_t \right],
\end{aligned}
$$

where $\Gamma^t(\cdot)$ solves the linear SDE

$$
d\Gamma^t(s) = \Gamma^t(s) \left( A(s) ds + \tilde{A}(s) dW(s) \right), \quad s \in [t, \tilde{\gamma}]; \quad \Gamma^t(t) = 1.
$$
Obviously, $\Gamma^t \geq 0$. Since $\psi \in \mathcal{J}_{\mu,\kappa}^+(\gamma_t, \bar{v})$, $\psi - \bar{v}$ is minimized at $\gamma_t$ over $Q_{\kappa,\mu}(\gamma_t) \cap C_\mu^\alpha$, and in view of Theorem 3.10 and Proposition 3.4, we have

$$Y^{2,\epsilon,\delta}(\bar{\tau}^{\epsilon,\delta}) \geq Y^{2,\epsilon,\delta,u}(t) = 0,$$

$$\inf_{u \in U} EY^{2,\epsilon,\delta,u}(t) = \text{ess inf} \inf_{u \in U} Y^{2,\epsilon,\delta,u}(\bar{\tau}^{\epsilon,\delta}) = \psi(\gamma_t^\epsilon) - \bar{v}(\gamma_t^\epsilon).$$

From equation (5.12), we have

$$\psi(\gamma_t^\epsilon) - \bar{v}(\gamma_t^\epsilon) \geq \inf_{u \in U} E[Y^{2,\epsilon,\delta,u}(t)]$$

$$= \inf_{u \in U} E \left[ Y^{2,\epsilon,\delta,u}(\bar{\tau}^{\epsilon,\delta}) \Gamma^t(\bar{\tau}^{\epsilon,\delta}) - \int_t^{\tau^{\epsilon,\delta}} \Gamma^t(r)(\mathcal{L}\psi)(X^{\gamma_t^\epsilon,u} r, u(r))dr \right]$$

$$\geq - \sup_{u \in U} E \left( \int_t^{\tau^{\epsilon,\delta}} (\mathcal{L}\psi)(\gamma_t^\epsilon, u(r))dr \right)$$

$$= - \sup_{u \in U} E \left( \int_t^{\tau^{\epsilon,\delta}} \left[ (\mathcal{L}\psi)(X^{\gamma_t^\epsilon,u} r, u(r)) - (\mathcal{L}\psi)(\gamma_t^\epsilon, u(r)) \right]dr \right)$$

$$- \sup_{u \in U} E \left( \int_t^{\tau^{\epsilon,\delta}} (\Gamma^t(r) - 1)(\mathcal{L}\psi)(X^{\gamma_t^\epsilon,u} r, u(r))dr \right)$$

$$:= - \sup_{u \in U} \text{Part1} - \sup_{u \in U} \text{Part2} - \sup_{u \in U} \text{Part3}.$$

Since the coefficients in $\mathcal{L}$ are Lipschitz continuous, combining the regularity of $\psi$ (see (4.11)), we have for any $\gamma_{t_1}^1, \gamma_{t_2}^2 \in Q_{\kappa,\mu}(\gamma_t) \cap C_\mu^\alpha$ and $\bar{u} \in U$,

$$|\psi(\gamma_{t_1}^1) - \psi(\gamma_{t_2}^2)| \leq d_p^\beta(\gamma_{t_1}^1, \gamma_{t_2}^2),$$

$$|\mathcal{L}\psi(\gamma_{t_1}^1, \bar{u}) - \mathcal{L}\psi(\gamma_{t_2}^2, \bar{u})| \leq C d_p^\beta(\gamma_{t_1}^1, \gamma_{t_2}^2).$$

Thus we have

$$\psi(\gamma_t^\epsilon) - \bar{v}(\gamma_t^\epsilon) = (\psi - \bar{v})(\gamma_t^\epsilon) + \psi(\gamma_t^\epsilon) - \psi(\gamma_t) + \bar{v}(\gamma_t^\epsilon) - \bar{v}(\gamma_t)$$

$$\leq C \left( \|\gamma_t^\epsilon - \gamma_t\|_0^\beta + \|\gamma_t^\epsilon - \gamma_t\|_0 \right) \leq C \left( 4M_0 \epsilon^{-1} \right)^\beta.$$
and

\[
\sup_{u \in U} \text{Part1} \leq \sup_{u \in U} E \left[ (\tau_{\varepsilon, \delta} - t) \sup_{\bar{u} \in U} \mathcal{L} \psi(\gamma_t^\varepsilon, \bar{u}) \right] \leq \sup_{u \in U} \mathcal{L} \psi(\gamma_t^\varepsilon, \bar{u}) \sup_{u \in U} E[(\tau_{\varepsilon, \delta} - t)] \\
\leq \left( \sup_{\bar{u} \in U} \mathcal{L} \psi(\gamma_t, \bar{u}) + C (4M_0\varepsilon\mu^{-1})^\beta \right) \sup_{u \in U} E[(\tau_{\varepsilon, \delta} - t)].
\]

Now we estimate higher order terms Part2 and Part3. In view of Lemma 3.1 and (5.16), we have

\[
E \left[ \sup_{t \leq r \leq \tau_{\varepsilon, \delta}} |\mathcal{L} \psi(X_{r, u}^\gamma, \gamma_t^\varepsilon) - \mathcal{L} \psi(\gamma_t^\varepsilon, u(r))| \right] \leq C E \delta^2.
\]

Hence

\[
|\text{Part2}| \leq E \left[ (\tau_{\varepsilon, \delta} - t) \sup_{t \leq r \leq \tau_{\varepsilon, \delta}} |\mathcal{L} \psi(X_{r, u}^\gamma, \gamma_t^\varepsilon) - \mathcal{L} \psi(\gamma_t^\varepsilon, u(r))| \right] \\
\leq \delta E \left[ \sup_{t \leq r \leq \tau_{\varepsilon, \delta}} |\mathcal{L} \psi(X_{r, u}^\gamma, \gamma_t^\varepsilon) - \mathcal{L} \psi(\gamma_t^\varepsilon, u(r))| \right] \\
= C \delta^{1+\frac{\beta}{2}} \quad (5.19)
\]

and

\[
|\text{Part3}| \leq C E \int_{\tau_{\varepsilon, \delta}}^{\tau_{\varepsilon, \delta}} \left| \Gamma^t(r) - 1 \right| dr \leq C E \left[ (\tau_{\varepsilon, \delta} - t) \sup_{t \leq r \leq \tau_{\varepsilon, \delta}} \left| \Gamma^t(r) - 1 \right| \right] \\
\leq C \delta \left[ \sup_{t \leq r \leq \tau_{\varepsilon, \delta}} \left| \Gamma^t(r) - 1 \right| \right] \leq C \delta^{\frac{3}{2}}. \\
\leq C \delta \left[ \sup_{t \leq r \leq \tau_{\varepsilon, \delta}} \left| \Gamma^t(r) - 1 \right| \right] \leq C \delta^{\frac{3}{2}}. \\
\quad (5.20)
\]

Substituting (5.17) - (5.20) into (5.15), we have

\[
- C (4M_0\varepsilon\mu^{-1})^\beta \\
\leq \left( \sup_{\bar{u} \in U} \mathcal{L} \psi(\gamma_t, \bar{u}) + C (4M_0\varepsilon\mu^{-1})^\beta \right) \sup_{u \in U} E[(\tau_{\varepsilon, \delta} - t)] + C \delta^{1+\frac{\beta}{2}}.
\]

34
In view of (5.6), then for all \( \delta \in (0, \delta_1(\kappa, \varepsilon, p)) \), uniformly for every \( u \in \mathcal{U} \), we have

\[
E[\hat{\tau}^{\varepsilon, \delta} - t] \geq E \left[ \chi_{\{\hat{\tau}^{\varepsilon, \delta} \geq t+\delta\}}(\hat{\tau}^{\varepsilon, \delta} - t) \right] \geq \frac{1}{2}\delta.
\]

(5.22)

Therefore,

\[
- \sup_{u \in \mathcal{U}} \mathcal{L} \psi (\gamma_t, u) \leq C \left( 4M_0 \varepsilon \mu^{-1} \right) \delta (2\delta^{-1} + 1) + 2C\delta^\frac{3}{2}.
\]

(5.23)

Note that the constants \( C \) throughout this proof only depend on \( M_0, \kappa \) and the Lipschitz constants of the coefficients in \( \mathcal{L} \), and they do not depend on \( \psi \in \mathcal{J}_{\mu, \kappa}^+(\gamma_t, \tilde{v}), \gamma_t \in \mathcal{Q}_{M_0, T-\kappa} \cap \mathcal{C}_\mu^\alpha \), and \( \delta \in (0, \delta_1(\varepsilon, \kappa)) \). Taking the supremum on both sides over \( \psi \in \mathcal{J}_{\mu, \kappa}^+(\gamma_t, \tilde{v}) \) and \( \gamma_t \in \mathcal{Q}_{M_0, T-\kappa} \cap \mathcal{C}_\mu^\alpha \), setting \( \delta := \mu^{-\frac{3}{2}} \), and then sending \( \mu \) to \( \infty \), we have (4.9). This shows that \( \tilde{v} \) is a viscosity sub-solution to the path-dependent Bellman equation (4.3).

In a symmetric (also easier) way, we show that \( \tilde{v} \) is a super-solution to the path-dependent Bellman equation (4.3). The proof is complete.

\[\square\]

**Remark 5.2.** (1) Our existence proof is more complicated than the classical counterpart (for the state-dependent case). The complication arises from the fact that we start the dynamic programming at the perturbation \( \gamma_t^\varepsilon \) instead of directly at the minimum path \( \gamma_t \) of \( \psi - \tilde{v} \) like the conventional arguments. Since our jets are defined on some compact subset \( \mathcal{C}_\mu^\alpha \) of \( \Lambda \), the minimum path \( \gamma_t \) might happen to be at the boundary of \( \mathcal{C}_\mu^\alpha \), i.e. \( \|\gamma_t\|_\alpha = \mu \). If we started at \( \gamma_t \), BSDE (5.11) would be trivial and nothing from the localized dynamic programming principle could be derived if

\[
P\{\|X_s^\gamma\|_\alpha \leq \mu, \, \exists s > t\} = 0.
\]

(5.24)

The following example illustrates that (5.24) might happen, and therefore explains why we have to start the dynamic programming at the perturbation \( \gamma_t^\varepsilon \).

Let \( W \) be a one-dimensional standard Brownian Motion and \( \gamma_t \in \Lambda(\mathbb{R}) \) such that for some \( t_1 \in [0, t) \)

\[
\gamma_t(t) - \gamma_t(t_1) = \mu|t - t_1|^{\alpha}.
\]

Define

\[
W^\gamma_t(s) := \gamma_t(s)\chi_{[0,t]}(s) + (W(s) - W(t) + \gamma_t(t))\chi_{[t,T]}(s), \quad s \in [0, T].
\]

Then

\[
\begin{align*}
\{ \exists \delta > 0, \text{ s.t. } & \|W^\gamma_{t+\delta}\|_\alpha \leq \mu \} \\
\subset & \{ \exists \delta > 0, \text{ s.t. } W^\gamma_t(s) - \gamma_t(t_1) \leq \mu|s - t_1|^{\alpha}, \forall s \in (t, t+\delta) \} \\
= & \{ \exists \delta > 0, \text{ s.t. } W^\gamma_t(s) - \gamma_t(t) + \gamma_t(t) - \gamma_t(t_1) \leq \mu|s - t_1|^{\alpha}, \forall s \in (t, t+\delta) \} \\
= & \{ \exists \delta > 0, \text{ s.t. } W^\gamma_t(s) - \gamma_t(t) \leq \mu(|s - t_1|^{\alpha} - |t - t_1|^{\alpha}), \forall s \in (t, t+\delta) \}.
\end{align*}
\]
Since the function \( \mu(|\cdot-t_1|^{\alpha} - |t-t_1|^{\alpha}) \in C^1[t,t+\delta] \), by the law of iterated logarithm (see [23, Theorem 9.23, Chapter 2]), we have
\[
P \{ \exists \delta > 0, \text{ s.t. } W_{\gamma}^{\gamma_{t+\delta}}_{t} \leq \mu \} = 0.
\]
This example enlightens us to perturb the left \( \mu \)-Hölder modulus of \( \gamma_t \) at time \( t \) in (5.1).

(2) The introduction of \( Q_{M_0,T-\kappa} \) in Definition 4.3 plays a crucial role in the proof of Theorem 4.5. Otherwise, we only have the following too rough estimate on our perturbation:
\[
\|\gamma_t^\epsilon - \gamma_t\| \leq C \epsilon, \quad \text{from which and (5.23) only results the following inequality}
\]
\[- \sup_{u \in U} L \psi (\gamma_t, u) \leq C \epsilon^\beta (2\delta^{-1} + 1) + 2C\delta^{\frac{1}{2}}.\]
It does not help us, for the relation of \( \delta^{\frac{1}{2} - \alpha} = o(\epsilon) \) is required in the estimate (5.22) by Proposition 7.1 and implies that \( \epsilon^\beta \delta^{-1} \) increases to \( \infty \) as \( \delta \) is decreasing to zero.

However, with the restriction of \( \gamma_t \in Q_{M_0,T-\kappa} \), in (5.23) we could fix \( \epsilon \), while sending \( \delta \rightarrow 0 \) and \( (4M_0^\epsilon \mu^{-1})^\delta \delta^{-1} \rightarrow 0 \) simultaneously.

(3) In the above proof, both parameters \( \mu \) and \( M_0 \) in our definition of viscosity sub-solutions play a key role, while the parameter \( \kappa \) is fixed such that the following associated family of path functionals
\[
\{ \psi, D_t \psi, D_x \psi, D_{xx} \psi : \Omega_{\kappa,\alpha} (\gamma_t) \cap C^\alpha_\mu \rightarrow \mathbb{R}; \gamma_t \in Q_{M_0,T-\kappa} \cap C^\alpha_\mu, \psi \in J^+_{\mu,\kappa} (\gamma_t, u), \mu \geq \mu_0 \}
\]
for some sufficiently large \( \mu_0 \), share a common Hölder modulus, which implies the so-called equi-continuous but with the underlying functionals being considered on varying domains.

6 Uniqueness of viscosity solution

6.1 Non-degenerate case

We assume without loss of generality that, there exists a constant \( K > 0 \), such that, for all \( (\gamma_t, p, A, u) \in \Lambda \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times U \) and \( r_1, r_2 \in \mathbb{R} \) such that \( r_1 < r_2 \),
\[
\mathcal{H}(\gamma_t, r_1, p, A, u) - \mathcal{H}(\gamma_t, r_2, p, A, u) \geq K (r_2 - r_1), \tag{6.1}
\]
Otherwise, define \( \bar{v}(\gamma_t) = e^{-\lambda T} v(\gamma_t) \) for \( \lambda > 0 \). Then \( v \) is a viscosity solution of PHJB equation (4.3) if and only if \( \bar{v} \) is a viscosity solution of the following PPDE
\[
\begin{cases}
-D_t \bar{v} - \sup_{u \in U} \mathcal{H}(\gamma_t, \bar{v}, D_x \bar{v}, D_{xx} \bar{v}, u) = 0, & \gamma_t \in \Lambda; \\
\bar{v}(\gamma_T) = e^{-\lambda T} g(\gamma_T), & \gamma_T \in \Lambda_T,
\end{cases}
\]

36
where
\[ \tilde{H}(\gamma_t, r, p, A, u) := -\lambda r + e^{-\lambda t}H(\gamma_t, e^{\lambda t}r, e^{\lambda t}p, e^{\lambda t}A, u). \]

Obviously, \( \tilde{H} \) satisfies (6.1) for sufficiently large \( \lambda \).

### 6.1.1 State-dependent smooth approximations

First, we construct the state-dependent approximations of the path functional \( \tilde{v} \) defined by (3.8). Let \( m \) be a positive integer, and \( t_i := \frac{i}{m}T, i = 0, 1, \ldots, m \), which divide the time interval \([0, T]\) into \( m \) equal parts. Now for all \( t \in [0, T] \), we define the truncating operator \( P^m : \Lambda_t \to \hat{\Lambda}_t \) by

\[
(P^m \gamma_t)(r) = \sum_{i=0}^{k-2} \gamma_t(t_i) \chi_{[t_i, t_{i+1})}(r) + \gamma_t(t_{k-1}) \chi_{[t_{k-1}, t)}(r) + \gamma_t(t) \chi_t(r)
\]

where \( \gamma_t(t_i) := \gamma_t(t_i) - \gamma_t(t_{i-1}) \), and \( k \) is the positive integer such that \( t \in (t_{k-1}, t_k) \).

We define functions \( b^m : \Lambda \times U \to \mathbb{R}^n \), \( \sigma^m : \Lambda \times U \to \mathbb{R}^{n \times d} \), \( f^m : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R} \), \( g^m : \Lambda_T \to \mathbb{R} \), and \( H^m : \Lambda \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times U \to \mathbb{R} \) as follows:

\[
\begin{align*}
    b^m(\gamma_t, u) &:= b(P^m \gamma_t, u), \\
    \sigma^m(\gamma_t, u) &:= \sigma(P^m \gamma_t, u), \\
    f^m(\gamma_t, y, z, u) &:= f(P^m \gamma_t, y, z, u), \\
    g^m(\gamma_T) &:= g(P^m \gamma_T), \\
    H^m(\gamma_t, r, p, A, u) &:= H(P^m \gamma_t, r, p, A, u).
\end{align*}
\]

Assumption (H2) implies the following estimates

\[
\begin{align*}
    |b^m(\gamma_t, u) - b(\gamma_t, u)| &\leq C\|P^m \gamma_t - \gamma_t\|_0, \\
    |\sigma^m(\gamma_t, u) - \sigma(\gamma_t, u)| &\leq C\|P^m \gamma_t - \gamma_t\|_0, \\
    |f^m(\gamma_t, y, z, u) - f(\gamma_t, y, z, u)| &\leq C\|P^m \gamma_t - \gamma_t\|_0, \\
    |g^m(\gamma_t) - g(\gamma_t)| &\leq C\|P^m \gamma_t - \gamma_t\|_0, \\
    |H^m(\gamma_t, r, p, A, u) - H(\gamma_t, r, p, A, u)| &\leq C(1 + |p| + |A|)\|P^m \gamma_t - \gamma_t\|_0.
\end{align*}
\]
Consider the following FBSDE: for any \( \gamma_t \in \Lambda, \ t < T, \) and \( u \in \mathcal{U}, \)
\[
\begin{cases}
X^{m, \gamma_t, u}(s) = (P^m \gamma_t)(s), & \text{all } \omega, \ s \in [0, t], \\
X^{m, \gamma_t, u}(s) = (P^m \gamma_t)(t) + \int_t^s b^m(X^{m, \gamma_t, u}_r, u(r))dr \\
+ \int_t^s \sigma^m(X^{m, \gamma_t, u}_r, u(r))dW(r), & \text{a.s.-} \omega, \ s \in [t, T];
\end{cases}
\]
(6.3)

\[
Y^{m, \gamma_t, u}(s) = g^m(X^{m, \gamma_t, u}_T) + \int_s^T f^m(X^{m, \gamma_t, u}_r, Y^{m, \gamma_t, u}_r, Z^{m, \gamma_t, u}_r, u(r))dr \\
- \int_s^T Z^{m, \gamma_t, u}(r)dW(s), \ s \in [t, T].
\]

Define the first approximating value functional

\[
v^m(\gamma_t) := \text{ess sup}_{u \in \mathcal{U}} Y^{m, \gamma_t, u}(t), \ \gamma_t \in \Lambda.
\]

Proposition 6.1. For \((u, \gamma_t) \in (\mathcal{U} \times \Lambda), \) and \( p > 2, \) we have

\[
E\left[ \|X^{m, \gamma_t, u}_T - X^{\gamma_t, u}_T\|_0^p \right] \leq C \left( \text{Osc}(\gamma_t, m^{-1})^p + m^{-\frac{p}{2}} \right),
\]
(6.4)

\[
E\left[ \sup_{s \in [t, T]} \|Y^{m, \gamma_t, u}(s) - Y^{\gamma_t, u}(s)\|_0^p \right] \leq C \left( \text{Osc}(\gamma_t, m^{-1})^p + m^{-\frac{p}{2}} \right),
\]
(6.5)

where

\[
\text{Osc}(\gamma_t, m^{-1}) := \max_{0 < s < s + \delta < t, 0 < \delta < m^{-1}} |\gamma(s + \delta) - \gamma(s)|
\]
is the oscillating amplitude with time \( m^{-1} \) of \( \gamma_t \) in the interval \((0, t)\).

Proof. From (6.2), we have

\[
|b^m(X^{m, \gamma_t, u}_s) - b(X^{\gamma_t, u}_s)| \leq C\|P^m X^{m, \gamma_t, u}_s - X^{\gamma_t, u}_s\|_0
\]
\[
\leq C(\|P^m X^{m, \gamma_t, u}_s - P^m X^{\gamma_t, u}_s\|_0 + \|P^m X^{\gamma_t, u}_s - X^{\gamma_t, u}_s\|_0)
\]
\[
\leq C(\|X^{m, \gamma_t, u}_s - X^{\gamma_t, u}_s\|_0 + \|P^m X^{\gamma_t, u}_s - X^{\gamma_t, u}_s\|_0).
\]

38
In a similar way, we have
\[ |\sigma^m(X^{\gamma,t,u}_s) - \sigma(X^{\gamma,t,u}_s)| \leq C(\|X^{m,\gamma,t,u}_s - X^{\gamma,t,u}_s\|_0 + \|P^m X^{\gamma,t,u}_s - X^{\gamma,t,u}_s\|_0), \]
\[ |f^m(X^{m,\gamma,t,u}_s, y', z', u) - f(X^{\gamma,t,u}_s, y, z, u)| \leq C(\|X^{m,\gamma,t,u}_s - X^{\gamma,t,u}_s\|_0 + \|P^m X^{\gamma,t,u}_s - X^{\gamma,t,u}_s\|_0 + |y - y'| + |z - z'|, \]
\[ |g^m(X^{m,\gamma,t,u}_T) - g(X^{\gamma,t,u}_T)| \leq C (\|X^{m,\gamma,t,u}_s - X^{\gamma,t,u}_s\|_0 + \|P^m X^{\gamma,t,u}_s - X^{\gamma,t,u}_s\|_0). \]

Applying Itô formula, BDG and Gronwall inequality, using standard arguments, we have
\[ E\|X^{m,\gamma,t,u}_T - X^{\gamma,t,u}_T\|^p \leq C E \|P^m X^{\gamma,t,u}_T - X^{\gamma,t,u}_T\|^p \]
\[ = C E \left[ \max_{1 \leq i \leq m} \max_{t_{i-1} \leq s \leq t_i} |X^{\gamma,t,u}(s) - X^{\gamma,t,u}(t_i)| \right] \]
\[ \leq C \left( \text{Osc}(\gamma_t, m^{-1})^p + m^{-\frac{p}{2}} \right). \]

\[ E\|Y^{m,\gamma,t,u}(T) - Y^{\gamma,t,u}(T)\|^p \leq C \left( E\|X^{m,\gamma,t,u}_T - X^{\gamma,t,u}_T\|^p + E\|P^m X^{\gamma,t,u}_T - X^{\gamma,t,u}_T\|^p \right) \]
\[ \leq C \left( \text{Osc}(\gamma_t, m^{-1})^p + m^{-\frac{p}{2}} \right). \]

\[ \Box \]

Obviously, (6.5) yields, for any \( \gamma_t \in \Lambda \) and positive integer \( m \),
\[ |\tilde{v}(\gamma_t) - v^m(\gamma_t)| \leq C \left( \text{Osc}(\gamma_t, m^{-1}) + m^{-\frac{1}{2}} \right). \] (6.6)

Similar to the state-dependent optimal stochastic control problem, \( v^m \) has a PDE interpretation. For each \( m \) and \( i = 1, \ldots, m \), define functions \( B^{m,i} : (t_{i-1}, t_i] \times \mathbb{R}^{i \times n} \times U \to \mathbb{R}^n \), \( \Sigma^{m,i} : (t_{i-1}, t_i] \times \mathbb{R}^{i \times n} \times U \to \mathbb{R}^{n \times d} \), \( F^{m,i} : (t_{i-1}, t_i] \times \mathbb{R}^{i \times n} \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R} \), \( G^m : \mathbb{R}^{m \times n} \to \mathbb{R} \), and \( H^{m,i} : (t_{i-1}, t_i] \times \mathbb{R}^{i \times n} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{i \times n} \times U \to \mathbb{R} \) as follows (with
\( \overrightarrow{x}_i = (x_1, \cdots, x_i) \in \mathbb{R}^{1 \times n} \):

\[
B^{m,i}(t, \overrightarrow{x}_i, u) := b \left( \sum_{j=1}^{i-1} x_j \chi_{[t_j, t]} + x_i \chi_{\{t\}}, u \right),
\]

\[
\Sigma^{m,i}(t, \overrightarrow{x}_i, u) := \sigma \left( \sum_{j=1}^{i-1} x_j \chi_{[t_j, t]} + x_i \chi_{\{t\}}, u \right),
\]

\[
F^{m,i}(t, \overrightarrow{x}_i, y, z, u) := f \left( \sum_{j=1}^{i-1} x_j \chi_{[t_j, t]} + x_i \chi_{\{t\}}, y, z, u \right),
\]

\[
G^m(x_1, \cdots, x_m) := g \left( \sum_{j=1}^{m-1} x_j \chi_{[t_j, T]} + x_m \chi_{\{T\}} \right),
\]

\[
H^{m,i}(t, \overrightarrow{x}_i, r, p, A, u) := \frac{1}{2} \text{Tr} \left( \Sigma^{m,i} \right) \left( \Sigma^{m,i} \right)^T(t, \overrightarrow{x}_i, u) + \left( B^{m,i}(t, \overrightarrow{x}_i, u), p \right) + F^{m,i}(t, \overrightarrow{x}_i, r, \Sigma^{m,i}) \left( t, \overrightarrow{x}_i, u \right) p, u \right).
\]

Obviously, for any \( \gamma \in \Lambda_T, u \in U \) and \( t \in (t_{i-1}, t_i] \),

\[
b^m(\gamma(t), u) = B^{m,i}(t, \gamma(t(t_1), \gamma(t_1) t_2, \cdots, \gamma(t_{i-2}, \gamma(t_{i-1}, u)),
\]

\[
\sigma^m(\gamma(t), u) = \Sigma^{m,i}(t, \gamma(t_1), \gamma(t_1) t_2, \cdots, \gamma(t_{i-2}, \gamma(t_{i-1}), u)),
\]

\[
f^m(\gamma(t), y, z, u) = F^{m,i}(t, \gamma(t_1), \gamma(t_1) t_2, \cdots, \gamma(t_{i-2}, \gamma(t_{i-1}, y, z, u)),
\]

\[
g^m(\gamma, u) = G^m(t, \gamma(t_1), \gamma(t_1) t_2, \cdots, \gamma(T)),
\]

\[
H^m(\gamma(t), r, p, A, u) = H^{m,i}(t, \gamma(t), \gamma(t_1) t_2, \cdots, \gamma(t_{i-2}, \gamma(t_{i-1}, r, p, A, u)),
\]

Here \( \gamma(t) := \gamma(t) - \gamma(s) \). Furthermore, from Assumption (H2), \( B^{m,i}, \sigma^{m,i}, F^{m,i} \), and \( G^m \) are uniformly Lipschitz continuous in \( (t, \overrightarrow{x}_i, y, z, u) \), and for \( i < m \),

\[
B^{m,i}(t_i, \overrightarrow{x}_i, u) = B^{m,i+1}(t_i, +, \overrightarrow{x}_i, 0, x_i, u),
\]

\[
\Sigma^{m,i}(t_i, \overrightarrow{x}_i, u) = \Sigma^{m,i+1}(t_i, +, \overrightarrow{x}_i, 0, x_i, u),
\]

\[
F^{m,i}(t_i, \overrightarrow{x}_i, y, z, u) = F^{m,i+1}(t_i, +, \overrightarrow{x}_i, 0, x_i, y, z, u),
\]

\[
H^{m,i}(t_i, \overrightarrow{x}_i, r, p, A, u) = H^{m,i+1}(t_i, +, \overrightarrow{x}_i, 0, x_i, r, p, A, u).
\]

Here \( f(t_i) \) is the right limit of the function \( f \) at time \( t_i \).

Let \( V^{m,i}, i = 1, \cdots m \) be the unique viscosity solutions of second order parametrized
nonlinear parabolic equations

\[
\begin{cases}
-\partial_t V^{m,i}(t, \vec{x}_i) - \sup_{u \in U} H^{m,i}(t, \vec{x}_i, (1, \partial_{x_i}, \partial_{\gamma, x_i}) V^{m,i}, u) = 0, \\
V^{m,i}(t_i, \vec{x}_i) = V^{m,i+1}(t_{i+1}, \vec{x}_{i+1}, 0, x_i), \\
V^{m,m}(T, \vec{x}_m) = G^m(\vec{x}_m),
\end{cases}
\]

\( (t, \vec{x}_i) \in (t_{i-1}, t_i) \times \mathbb{R}^{i \times n}, \ i = 1, \cdots, m; \) \( (6.7) \)

According to the relationship between viscosity solution of Bellman equations and the optimal control problems, we have, for any \( \gamma_t \in \Lambda, \)

\[
v^m(\gamma_t) = \sum_{i=1}^{m} \chi_{[t_{i-1}, t_i]}(t) V^{m,i}(t, \gamma_t(t_1), \gamma_t|_{t_1}, \cdots, \gamma_t|_{t_{i-1}}, \gamma_t|_{t_{i-1}}).
\]

Second, we construct the smooth approximations of \( v^m. \) For this purpose, we mollify \( \sup_u H^{m,i}. \) Consider the following mollifier \( \varphi_i : \mathbb{R}^i \rightarrow \mathbb{R}, \ i = 1, 2, \cdots, \)

\[
\varphi_i(x) := \begin{cases} C_i \exp \left( -\left( 1 - \frac{|x|^2}{2} \right)^{-1} \right), & |x|^2 < 1; \\
0, & \text{else},
\end{cases}
\]

where \( C_i \) is the constant such that \( \int \varphi_i = 1. \) Let

\[
\varphi_\varepsilon(t, x, r, p, A) := \varepsilon^{-(\alpha^2 + 2n + 2)} \varphi_1\left( \frac{t + 1/\varepsilon}{\varepsilon} \right) \varphi_n\left( \frac{x}{\varepsilon} \right) \varphi_1\left( \frac{r}{\varepsilon} \right) \varphi_n\left( \frac{p}{\varepsilon} \right) \varphi_n^2\left( \frac{A}{\varepsilon} \right).
\]

Now we extend \( H^{m,i}, \ i < m, \) on the interval \( t \in [t_i, t_{i+1}] \) by

\[
H^{m,i}(t, \vec{x}_i, r, p, A, u) := H^{m,i+1}(t, \vec{x}_{i+1}, 0, x_i, r, p, A, u),
\]

and \( H^{m,m} \) on the interval \( t \in [T, T + 1/m] \) by

\[
H^{m,m}(t, \vec{x}_m, r, p, A, u) := H^{m,m}(T, \vec{x}_m, r, p, A, u),
\]

and mollify \( \sup_u H^{m,i} \) on interval \( (t_{i-1}, t_i) \) as

\[
\hat{H}^{m,i; \varepsilon_1}(\cdot, \vec{x}_{i-1}, \cdot, \cdots, \cdot) := \left( \sup_u H^{m,i}(\cdot, \vec{x}_{i-1}, \cdot, \cdots, \cdot, u) \right) \ast \varphi_\varepsilon(\cdot), \quad (6.8)
\]

where \( \ast \) is the convolution operator in \( (t, x_i, r, p, A) \) and \( \varepsilon_1 < m^{-1}. \) Obviously, \( \hat{H}^{m,i; \varepsilon_1} \)

is differentiable in \( (t, x_i, r, p, A) \) and Lipschitz continuous in \( \vec{x}_{i-1}. \) Noting the structure condition, we obtain that

\[
|\hat{H}^{m,i; \varepsilon_1}(t, \vec{x}_i, r, p, A) - \sup_u H^{m,i}(t, \vec{x}_i, r, p, A, u)| \leq C(1 + |p| + |A|)\varepsilon_1, \quad (6.9)
\]

41
Similarly, we define $\hat{H}^{m, i; \varepsilon}$ satisfy the following structure conditions:

$$|\partial_t \hat{H}^{m, i; \varepsilon}| + |\partial_x \hat{H}^{m, i; \varepsilon}| \leq C(1 + |p| + |A|),$$

$$|\partial_p \hat{H}^{m, i; \varepsilon}| + |\hat{H}^{m, i; \varepsilon}(t, \overrightarrow{x}_i, r, p, 0)| \leq C,$$

$$C^{-1} I_n \leq \partial_t \hat{H}^{m, i; \varepsilon} \leq CI_n, \quad \partial_t \hat{H}^{m, i; \varepsilon} \leq -C.$$  \hspace{5cm} (6.10)

Define $G_0 := \sup_{m, x_m} |H_0 := \sup_{m, i, t, \overrightarrow{x}_i}| G^{m}_{\varepsilon}(x_m) \varepsilon |, H_0 := \sup_{m, i, t, \overrightarrow{x}_i} |\hat{H}^{m, i; \varepsilon}(t, \overrightarrow{x}_i, 0, 0, 0)|.$

We have the following key lemma.

**Lemma 6.2.** Assume (H2). Then the system (6.7) has unique viscosity solutions \{V^{m, i; \varepsilon}_{\varepsilon}\}_{i=1}^m. Moreover, there is some positive constants $C$ which are independent of $m$, $i$, and $\varepsilon$ $(\varepsilon < 1 < m^{-1})$, such that:

1. $V^{m, i; \varepsilon}_{\varepsilon}(\cdot, \overrightarrow{x}_i) \in C^{1,2}(\overrightarrow{t}_{i-1}, \overrightarrow{t}_i) \times \mathbb{R}^n$, and for any $t \in \overrightarrow{t}_{i-1}, \overrightarrow{t}_i$ and $\overrightarrow{x}_i \in \mathbb{R}^{i \times n}$,

$$|V^{m, i; \varepsilon}_{\varepsilon}(t, \overrightarrow{x}_i)| \leq G_0 e^{-K(T-t)} + (1 - e^{-K(T-t)}) H_0 K^{-1},$$

(6.11)

where $K$ is the constant in (6.1).

2. Hölder continuity: for any $t \in \overrightarrow{t}_{i-1}, \overrightarrow{t}_i$, $\overrightarrow{x}_i, \overrightarrow{y}_i \in \mathbb{R}^{i \times n}$, and $s \in \overrightarrow{t}_{j-1}, \overrightarrow{t}_j$, $i \leq j \leq m$,

$$|\partial_t V^{m, i; \varepsilon}_{\varepsilon}(t, \overrightarrow{x}_i)| + |\partial_x V^{m, i; \varepsilon}_{\varepsilon}(t, \overrightarrow{x}_i)| + |\partial_{x_i} V^{m, i; \varepsilon}_{\varepsilon}(t, \overrightarrow{x}_i)| \leq C(\varepsilon),$$

(6.13)

$$|(1, \partial_{x_i}, \partial_{x_i}, \partial_{t}) V^{m, i; \varepsilon}_{\varepsilon}(t, \overrightarrow{x}_i)| \leq C(\varepsilon) \max_{1 \leq k \leq l} |(x_1 - y_1) + \cdots + (x_k - y_k)|^\beta,$$

(6.14)

$$|\partial_t V^{m, i; \varepsilon}_{\varepsilon}(t, \overrightarrow{x}_i) - (1, \partial_{x_i}, \partial_{x_i}, \partial_{t}) V^{m, i; \varepsilon}_{\varepsilon}(s, \overrightarrow{x}_i, 0, \cdots, 0, \overrightarrow{x}_i)| \leq C(\varepsilon)|s - t|^\frac{\beta}{2},$$

(6.15)

$$\leq C(\varepsilon)|s - t|^\frac{\beta}{2},$$

$$\leq C(\varepsilon)|s - t|^\frac{\beta}{2},$$

42
(3) smoothly approximating rate:

\[ |V^{m,i;\varepsilon}_\varepsilon - V^{m,i}| \leq C(\varepsilon_1 + \varepsilon). \tag{6.16} \]

**Proof.** Firstly we prove the existence of viscosity solution. Define

\[ H_T := \sup_{\tilde{x}_m \in \mathbb{R}^{m \times n}} \hat{H}^{m,m;\varepsilon}_\varepsilon(T, \tilde{x}_m, G^m_\varepsilon, \partial_{x_m} G^m_\varepsilon, \partial_{x_m x_m} G^m_\varepsilon). \]

It is easy to verify that

\[ G^m_\varepsilon + (1 - e^{-K(T-t)})H_TK^{-1}, t \in [0,T] \text{ and } G^m_\varepsilon - (1 - e^{-K(T-t)})H_TK^{-1}, t \in [0,T] \]

are respectively viscosity super- and sub-solutions of system \( (6.11) \) on interval \([t_{m-1}, T]\), where \( K \) is the constant in \( (6.1) \). By Perron’s method and comparison principle (see Crandall, Ishii and Lions [9]), system \( (6.11) \) has unique viscosity solutions \( V^{m,m;\varepsilon}_\varepsilon \) on \([t_{m-1},T]\). If assertion (1) holds on \([t_{m-1},T]\), then \( V^{m,m;\varepsilon}_\varepsilon(t_m, x_{m-2}, 0, x_{m-1}) \), the terminal value of \( V^{m,m-1;\varepsilon}_\varepsilon \), is bounded and twice differentiable in \( x_{m-1} \in \mathbb{R}^n \). Similarly we have the existence of the unique viscosity solution \( V^{m,i;\varepsilon}_\varepsilon \) recursively.

In view of Wang [50, Theorems 1.1 and 1.3], using the interior \( C^{1,\alpha} \) and \( C^{2,\alpha} \) estimates for the equations of the structure conditions \( (6.10) \) (see Lieberman [28, Chapter 14, Sections 2-4]), we have that \( V^{m,i;\varepsilon}_\varepsilon(t_{i-1}, t_i) \in C^{1,2}([t_{i-1}, t_i] \times \mathbb{R}^n) \). Since

\[ G_0 e^{-K(T-t)} + (1 - e^{-K(T-t)})H_0K^{-1}, t \in [0,T], \]

and

\[ -G_0 e^{-K(T-t)} - (1 - e^{-K(T-t)})H_0K^{-1}, t \in [0,T], \]

are viscosity super- and sub-solutions of system \( (6.11) \), respectively, we have \( (6.12) \). Then combining the interior \( C^{2,\alpha} \) estimates and interpolation inequality, we have \( (6.13) \). Assertion (1) is proved.

Noting that \( \hat{V} := V^{m,i;\varepsilon}_\varepsilon(\cdot, \overrightarrow{x_{i-1}}, \cdot) - V^{m,i;\varepsilon}_\varepsilon(\cdot, \overrightarrow{y_{i-1}}, \cdot) \) is the solution of

\[ -\partial_t \hat{V} - \text{Tr}(a \partial_{x_i x_i} \hat{V}) - \langle b, \partial_{x_i} \hat{V} \rangle - c \hat{V} - h_0 = 0, \]
where

\[
a := \int_0^1 \partial_x \hat{H}^{m, \epsilon \varepsilon_1}(t, \overrightarrow{x_{i-1}}, x_i, (1, \partial_x) V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{x_{i-1}}, x_i),
\]

\[
\partial_{x_i} V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i) + \theta \partial_{x_i} V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i)\bigg|_{\overrightarrow{y_i = \overrightarrow{x_i}}}) d\theta,
\]

\[
b := \int_0^1 \partial_p \hat{H}^{m, \epsilon \varepsilon_1}(t, \overrightarrow{x_{i-1}}, x_i, V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{x_{i-1}}, x_i), \partial_x V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i)
\]

\[
\quad + \theta \partial_x V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i)\bigg|_{\overrightarrow{y_i = \overrightarrow{x_i}}}) d\theta,
\]

\[
c := \int_0^1 \partial_t \hat{H}^{m, \epsilon \varepsilon_1}(t, \overrightarrow{x_{i-1}}, x_i, V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i) + \theta V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i)\bigg|_{\overrightarrow{y_i = \overrightarrow{x_i}}})
\]

\[
\quad \left(\partial_{x_i}, \partial_{x_{i-1}}\right) V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i)\bigg|_{\overrightarrow{y_i = \overrightarrow{x_i}}}) d\theta,
\]

\[
h_0 := \hat{H}^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i, (1, \partial_x) V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i))\bigg|_{\overrightarrow{y_i = \overrightarrow{x_i}}}
\]

In view of Assertion (1), we know \(|a| + |b| + |c| < C\), and

\[
|h_0| \leq C \max_{1 \leq k \leq i-1} |(x_1 - y_1) + \cdots + (x_k - y_k)|.
\]

Similar to recursive method in Assertion (1), we have

\[
|V(t, x_i)| = |V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{x_i}) - V^{m, \epsilon \varepsilon_1}(t, \overrightarrow{y_{i-1}}, x_i)|
\]

\[
\leq e^{-K(T-t)} G_{\overrightarrow{x_{i-1}}, \overrightarrow{y_{i-1}}} + (1 - e^{-K(T-t)}) K^{-1} L_{\overrightarrow{x_{i-1}}, \overrightarrow{y_{i-1}}}
\]

\[
\leq C \left(1 \wedge \max_{1 \leq k \leq i-1} |(x_1 - y_1) + \cdots + (x_k - y_k)|\right),
\]

where

\[
G_{\overrightarrow{x_{i-1}}, \overrightarrow{y_{i-1}}} := \sup_{x_i, \cdots, x_m} |G^{m}_\epsilon (z, x_i, \cdots, x_m)|_{z = \overrightarrow{x_{i-1}}} ^{z = \overrightarrow{y_{i-1}}}
\]

\[
\leq C \left(1 \wedge \max_{1 \leq k \leq i-1} |(x_1 - y_1) + \cdots + (x_k - y_k)|\right)
\]

and

\[
L_{\overrightarrow{x_{i-1}}, \overrightarrow{y_{i-1}}} := \sup_{k \geq i, x_{i+1}, \cdots, x_m} \left|\hat{H}^{m, \epsilon \varepsilon_1}(t, z, x_i, \cdots, x_k, 0, 0, 0)\bigg|_{z = \overrightarrow{x_{i-1}}} ^{z = \overrightarrow{y_{i-1}}}|\right.
\]

\[
\leq C(\epsilon) \left(1 \wedge \max_{1 \leq k \leq i-1} |(x_1 - y_1) + \cdots + (x_k - y_k)|\right).
\]
In view of the interior Schauder estimate for linear parabolic equation (see Lieberman [28, Theorem 4.9]), we have
\[
\left|(1, \partial_{x_i}, \partial_{x_ix_i}, \partial_t)V^m_{\varepsilon \xi}(t, x_i)\right| \leq C(\varepsilon) \left(1 + \max_{1 \leq k \leq i-1} |(x_1 - y_1) + \cdots + (x_k - y_k)|\right).
\]

Incorporating the following interior $C^{2,\alpha}$ estimate of $V^m_{\varepsilon \xi}:
\[
\left|(1, \partial_{x_i}, \partial_{x_ix_i}, \partial_t)V^m_{\varepsilon \xi}(t, \bar{x}_{i-1}, \cdot)\right|_{x_i} \leq C(\varepsilon)|x_i - y_i|^{\beta},
\]

we have (6.14).

For any $t \in [t_{i-1}, T]$ and $\bar{x}_i \in \mathbb{R}^{i \times n}$, define
\[
\hat{V}(t, \bar{x}_i) := \sum_{k=i}^{m} \chi_{[t_{k-1}, t_k]}(t)V^m_{k \xi}(t, \bar{x}_{i-1}, \cdot)_{x_i},
\]
\[
\hat{H}(t, \bar{x}_i, r, p, A) := \sum_{k=i}^{m} \chi_{[t_{k-1}, t_k]}(t)\hat{H}^m_{k \xi}(t, \bar{x}_{i-1}, \cdot)_{x_i, r, p, A}.
\]

Obviously, $\hat{H}$ is smooth and satisfies structure condition (6.10), and $\hat{V}$ is the classical solution of
\[
\begin{aligned}
\partial_t \hat{V} + \hat{H}(T, \bar{x}_i, \hat{V}, \partial_{x_i} \hat{V}, \partial_{x_i x_i} \hat{V}) &= 0, \quad t \in [t_{i-1}, T); \\
\hat{V}(T, \bar{x}_i) &= G^m_{\varepsilon \xi}(\bar{x}_{i-1}, \cdot)_{x_i}.
\end{aligned}
\]

By the Schauder interior estimate, we have for any $t \in [t_{i-1}, t_i)$, $s \in [t_{j-1}, t_j)$, $i \leq j \leq m$, and $\bar{x}_i \in \mathbb{R}^{i \times n}$,
\[
\left|(1, \partial_{x_i}, \partial_{x_ix_i}, \partial_t)V^m_{\varepsilon \xi}(t, \bar{x}_i) - (1, \partial_{x_i}, \partial_{x_ix_i}, \partial_t)V^m_{\varepsilon \xi}(s, \bar{x}_{j-1}, \cdot)_{x_i} \right|
\]
\[
= \left|\hat{V}(t, \bar{x}_i) - \hat{V}(s, \bar{x}_i)\right| \leq C(\varepsilon)|t - s|^\frac{\alpha}{2}.
\]

We proved (6.15).

It remains to show Assertion (3). Form (6.13) and (6.9), we see that, when $C$ is sufficiently large, $V^m_{\varepsilon \xi} + C(\varepsilon_1 + \varepsilon)$ is a viscosity super-solution and $V^m_{\varepsilon \xi} - C(\varepsilon_1 + \varepsilon)$ a viscosity sub-solution of equation (6.7), which imply (6.16) by the comparison principle.
Remark 6.3. Note that the constant in estimate (6.12) does not depend on \( m \), which allows us to conclude that those constants in the estimates (6.13) - (6.15) do not depend on \( m \).

Define the smooth approximating functional

\[
v^m_{\varepsilon;1}(\gamma_t) := \sum_{i=1}^{m} \chi_{[t_{i-1},t_i)}(t) V^m_{\varepsilon;1,i}(t, \gamma_t(t_1), \gamma_t|_{t_1}, \cdots, \gamma_t|_{t_{i-2}}, \gamma_t|_{t_{i-1}}), \quad \gamma_t \in \Lambda. \tag{6.17}
\]

Then, \( v^m_{\varepsilon;1} \in C^{1,2}(\Lambda) \). In fact, (6.17) is well-defined as well for \( \gamma \in \hat{\Lambda} \). It is obvious that \( v^m_{\varepsilon;1} \in C^{1,2}(\hat{\Lambda}) \), which implies by definition that the restriction of \( v^m_{\varepsilon;1} \) on \( \Lambda \) lies in \( C^{1,2}(\Lambda) \).

Define

\[
\hat{H}^m_{\varepsilon;1}(\gamma_t, r, p, A) := \sum_{i=1}^{m} \chi_{[t_{i-1},t_i)}(t) \hat{H}^m_{\varepsilon;1,i}(t, \gamma_t(t_1), \gamma_t|_{t_1}, \cdots, \gamma_t|_{t_{i-2}}, \gamma_t|_{t_{i-1}}; r, p, A).
\]

Obviously, \( v^m_{\varepsilon;1} \) is the classical solution of the following path-dependent PDE

\[
\begin{cases}
-D_t v^m_{\varepsilon;1} - \hat{H}^m_{\varepsilon;1}(\gamma_t, v^m_{\varepsilon;1}, D_x v^m_{\varepsilon;1}, D_{xx} v^m_{\varepsilon;1}) = 0, & \forall \gamma_t \in \Lambda, 0 < t < T; \\
v^m_{\varepsilon;1}(\gamma_T) = G^m_{\varepsilon}(\gamma_T), & \forall \gamma_T \in \Lambda_T.
\end{cases}
\tag{6.18}
\]

Moreover, we have the path version of Lemma 6.2.

Proposition 6.4. Let (H2) hold. There are some positive constants \( C(\varepsilon) \) (independent of \( m \) and \( \varepsilon_1 < m^{-1} \)), such that for all \( \gamma_t, \gamma_{\tilde{t}} \in \Lambda, t, \tilde{t} < T \), we have:

1. \( C^{1+\frac{p}{2}, 2+\beta} \) boundedness:

\[
|v^m_{\varepsilon;1}(\gamma_t)|_{2,\beta;\Lambda} \leq C(\varepsilon). \tag{6.19}
\]

2. smoothly approximating rate:

\[
|v^m_{\varepsilon;1}(\gamma_t) - \tilde{v}(\gamma_t)| \leq C(Osc(\gamma_t, m^{-1}) + m^{-\frac{1}{2}} + \varepsilon + \varepsilon_1). \tag{6.20}
\]

Proof. Assertion (1) is immediate consequence of Assertions (1) and (2) of Lemma 6.2. Assertion (2) follows from (6.6) and (6.16). \( \square \)

At the end of this subsection, we introduce the following auxiliary path functional

\[
\tilde{v}_0(\gamma_t) := E[\|W_{\gamma_t}^m\|_0], \quad \gamma_t \in \Lambda,
\]

46
and the smooth approximating functional

\[ v^m_{0,\varepsilon}(\gamma_t) := E[g^m_{0,\varepsilon}(W^\gamma_t)], \quad \gamma_t \in \Lambda. \] (6.21)

Here,

\[ W^\gamma_t(s) := \gamma_t(s)\chi_{[0,t]}(s) + (W(s) - W(t) + \gamma_t(t))\chi_{[t,T]}(s), \quad s \in [0, T], \]

and

\[ g^m_{0,\varepsilon}(\gamma_t) := G^m_{0,\varepsilon} \left( \gamma_T(t_1), \gamma_T|_{t_1}^{t_2}, \cdots, \gamma_T|_{t_{m-1}}^{t_m} \right), \]

\[ G^m_{0,\varepsilon}(\vec{x}_{m-1}, \cdot) := \left( \left( \max_{1 \leq k \leq m-1} |x_1 + \cdots + x_k| \right) \vee |x_1 + \cdots + x_{m-1} + \cdot | \right) \ast \varphi_{n,\varepsilon}, \]

\[ \varphi_{n,\varepsilon} \] is a mollifier in \( \mathbb{R}^n \). Obviously

\[ V^{m,i}_{0,\varepsilon}(t, \vec{x}_i) := v^m_{0,\varepsilon}\left( \sum_{j=1}^{i-1} x_j \chi_{[t_j,t_i]} + x_i \chi_{(t_i)}, t, \vec{x}_i \right), \quad (t, \vec{x}_i) \in [t_{i-1}, t_i] \times \mathbb{R}^{i \times n}, i = 1, \cdots, m, \]

are the classical solutions of

\[
\begin{aligned}
&\partial_t V^{m,i}_{0,\varepsilon}(t, \vec{x}_i) + \frac{1}{2} \Delta x_i V^{m,i}_{0,\varepsilon}(t, \vec{x}_i) = 0, \quad (t, \vec{x}_i) \in (t_{i-1}, t_i) \times \mathbb{R}^{i \times n}; \\
&V^{m,i}_{0,\varepsilon}(t_i, \vec{x}_i) = V^{m,i+1}_{0,\varepsilon}(t_{i+1}, \vec{x}_{i+1}, 0, x_{i}), \quad \vec{x}_{i} \in \mathbb{R}^{i \times n}, i = 1, \cdots, m - 1; \\
&V^{m,i}_{0,\varepsilon}(T, \vec{x}_m) = G^m_{0,\varepsilon}(\vec{x}_m), \\
&\vec{x}_{m} \in \mathbb{R}^{m \times n}.
\end{aligned}
\]

Similarly as in Proposition 6.4, \( v^m_{0,\varepsilon} \in C^{1,2}(\Lambda) \) satisfies the following estimates:

\[ |v^m_{0,\varepsilon}|_{2,\beta, Q_{M_0, T}} \leq C(\varepsilon)(1 + M_0), \] (6.22)

\[ |v^m_{0,\varepsilon}(\gamma_t) - \tilde{v}_0(\gamma_t)| \leq C \left( \text{Osc}(\gamma_t, m^{-1}) + m^{-\frac{3}{2}} + \varepsilon \right). \] (6.23)

### 6.1.2 Proof of Theorem 4.6

Let \( v \in \mathcal{C}_b(\Lambda) \cap \mathcal{C}_u(\Lambda) \) satisfying (4.13) be a viscosity solution to the path-dependent Bellman equation (4.3), and \( \tilde{v} \) be defined by (3.8). From Remark 4.8 we know \( \tilde{v} \in \mathcal{C}_b(\Lambda) \cap \mathcal{C}_u(\Lambda) \). It is sufficient to show \( \tilde{v} \geq v \) since the inverse inequality can be proved in a similar way.

Otherwise, we have

\[ \inf_{\gamma_t \in \Lambda} \left( \tilde{v}(\gamma_t) - v(\gamma_t) \right) := -r_0 < 0. \]
For sufficiently large $\lambda > 1$, we have
\[
\inf\left(\bar{v} + e^{-\lambda(t+1)}\tilde{v}_0 - v\right) < -\frac{7}{8}r_0.
\]

We fix $\lambda$. Since $\cup_{\mu > 0} C^a_\mu$ is dense in $\Lambda$, then for sufficiently large number $\mu$,
\[
\inf_{\gamma_t \in C^a_\mu} \left[\bar{v}(\gamma_t) + e^{-\lambda(t+1)}\tilde{v}_0(\gamma_t) - v(\gamma_t)\right] < -\frac{3}{4}r_0.
\]

Besides, from the definition of $\tilde{v}_0$ we know $\tilde{v}_0(\gamma_t) \geq \|\gamma_t\|_0$, and noting that $v$, $\bar{v} \in C_b(\Lambda)$, thus there is $M_0 = M_0(\lambda) > 0$ such that
\[
\tilde{v} + e^{-\lambda(t+1)}\tilde{v}_0 - v > 0, \quad \text{on } \Lambda \setminus Q_{M_0,T}.
\] (6.24)

Now, we fix $\mu$ and $M_0$ firstly. From (6.20) and (6.23), we have for all $\varepsilon < \frac{1}{32}r_0C$, $m > m := (32C\mu_0^{-1})^{\frac{1}{2}} \lor [(32C)^2r_0^{-2}]$ and $\varepsilon_1 < \varepsilon_1 := (\frac{1}{32}r_0C^{-1}) \land m^{-1}$, for any $\gamma_t \in C^a_\mu$,
\[
|(v^m_{\varepsilon_1} - \tilde{v})(\gamma_t)| \leq C(\text{Osc}(\gamma_t, m^{-1}) + m^{-\frac{1}{2}} + \varepsilon_1 + \varepsilon) \leq C(\mu m^{-\alpha} + m^{-\frac{1}{2}} + \varepsilon_1 + \varepsilon) < \frac{1}{8}r_0,
\]
\[
|(v^m_{\varepsilon_1} - \tilde{v}_0)(\gamma_t)| \leq C(\text{Osc}(\gamma_t, m^{-1}) + m^{-\frac{1}{2}} + \varepsilon) \leq C(\mu m^{-\alpha} + m^{-\frac{1}{2}} + \varepsilon) < \frac{1}{8}r_0.
\]

Hence
\[
\inf_{\gamma_t \in C^a_\mu} \left(v^m_{\varepsilon_1} - e^{-\lambda(t+1)}v^m_{0,\varepsilon} - v\right)(\gamma_t) < -\frac{1}{2}r_0,
\] (6.25)
\[
(v^m_{\varepsilon_1} + e^{-\lambda(t+1)}v^m_{0,\varepsilon} - v)(\gamma_t) > -\frac{1}{4}r_0, \quad \forall \gamma_t \in C^a_\mu \setminus Q_{M_0,T},
\] (6.26)
and
\[
(v^m_{\varepsilon_1} + e^{-\lambda(T+1)}v^m_{0,\varepsilon} - v)(\gamma_{T_t}) > (\bar{v} + e^{-\lambda(T+1)}\tilde{v}_0 - v)(\gamma_{T}) - \frac{1}{4}r_0 > -\frac{1}{4}r_0, \quad \forall \gamma_{T_t} \in C^a_\mu.
\] (6.27)

Since $v \in C_u(\Lambda)$, $v^m_{\varepsilon_1}$ satisfies (6.19) and $v^m_{0,\varepsilon}$ satisfies (6.22) uniformly w.r.t. all $m$ and $\varepsilon_1$, therefore, there is a constant $\kappa_1 = \kappa_1(\varepsilon, \lambda, M_0) \in (0, T)$ such that for any $\gamma_t \in C^a_\mu$, $t > T - \kappa_1$,
\[
|(v^m_{\varepsilon_1} + e^{-\lambda(t+1)}v^m_{0,\varepsilon} - v)(\gamma_t) - (v^m_{\varepsilon_1} + e^{-\lambda(T+1)}v^m_{0,\varepsilon} - v)(\gamma_{T_t})| \leq \frac{1}{4}r_0.
\] (6.28)
Combining (6.27), we have
\[ |(v^{m,\varepsilon}_\varepsilon + e^{-\lambda(t+1)}v^{m}_{0,\varepsilon} - v)(\gamma_t)| \leq \frac{1}{2} \varepsilon_0, \quad \forall \gamma_t \in C_\mu, t > T - \kappa_1. \] (6.29)

This together with (6.25) and (6.26) yield that there is \( \gamma_t \in C^\alpha_\mu \cap Q_{M_0, T-1} \), where the functional \( v^{m,\varepsilon}_\varepsilon + e^{-\lambda(t+1)}v^{m}_{0,\varepsilon} - v \) is minimized over \( C^\alpha_\mu \).

Define
\[
\psi(\gamma_t) := v^{m,\varepsilon}_\varepsilon(\gamma_t) + e^{-\lambda(t+1)}v^{m}_{0,\varepsilon}(\gamma_t) - (v^{m,\varepsilon}_\varepsilon + e^{-\lambda(t+1)}v^{m}_{0,\varepsilon})(\gamma_t), \quad \gamma_t \in \Lambda. \] (6.30)

By (6.19) and (6.22), there is \( \kappa = \kappa(M_0, \varepsilon, \lambda) < \kappa_1 \) such that \( \psi \in \mathcal{F}_{\mu, \kappa}(\bar{\gamma}_t, v) \) for all \( m > \bar{m} \) and small \( \varepsilon_1 > \bar{\varepsilon}_1 \). Consider the following estimates:

\[
-D_t \psi(\bar{\gamma}_t) - \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, \psi(\bar{\gamma}_t), D_x \psi(\bar{\gamma}_t), D_{xx} \psi(\bar{\gamma}_t), u) \]
\[
= -D_t(v^{m,\varepsilon}_\varepsilon + e^{-\lambda(t+1)}v^{m}_{0,\varepsilon})(\bar{\gamma}_t)
- \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, v(\bar{\gamma}_t), (D_x, D_{xx})(v^{m,\varepsilon}_\varepsilon + e^{-\lambda(t+1)}v^{m}_{0,\varepsilon})(\bar{\gamma}_t), u)
\geq e^{-\lambda(t+1)} \left( \lambda v^{m}_{0,\varepsilon} - D_t v^{m}_{0,\varepsilon} - \sup_u |\sigma|^2 |D_{xx}v^{m}_{0,\varepsilon}| - \sup_u (|b| + |\sigma|)|D_xv^{m}_{0,\varepsilon}| \right) (\bar{\gamma}_t)
- D_t v^{m,\varepsilon}_\varepsilon(\bar{\gamma}_t) - \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, v(\bar{\gamma}_t), (D_x, D_{xx})v^{m,\varepsilon}_\varepsilon(\bar{\gamma}_t), u)
\geq e^{-\lambda(t+1)} \left( \lambda v^{m}_{0,\varepsilon} - C(|D_t v^{m}_{0,\varepsilon}| + |D_{xx}v^{m}_{0,\varepsilon}| + |D_xv^{m}_{0,\varepsilon}|) \right) (\bar{\gamma}_t)
+ \left[ -D_t v^{m,\varepsilon}_\varepsilon - \mathcal{H}^{m,\varepsilon}_\varepsilon(\cdot, (1, D_x, D_{xx})v^{m,\varepsilon}_\varepsilon) \right] (\bar{\gamma}_t)
+ \left[ \sup_{u \in U} \mathcal{H}_\varepsilon(\cdot, (1, D_x, D_{xx})v^{m,\varepsilon}_\varepsilon, u) - \sup_{u \in U} \mathcal{H}(\cdot, (1, D_x, D_{xx})v^{m,\varepsilon}_\varepsilon, u) \right] (\bar{\gamma}_t)
+ \left[ \sup_{u \in U} \mathcal{H}(\cdot, v^{m,\varepsilon}_\varepsilon, (D_x, D_{xx})v^{m,\varepsilon}_\varepsilon, u) - \sup_{u \in U} \mathcal{H}(\cdot, v, (D_x, D_{xx})v^{m,\varepsilon}_\varepsilon, u) \right] (\bar{\gamma}_t)
:= \text{Part 1} + \text{Part 2} + \text{Part 3} + \text{Part 4} + \text{Part 5}.

From estimates (6.22)-(6.23), if \( \lambda \) is sufficiently large,

\[
\text{Part 1} \leq -e^{-\lambda} \left( \lambda(\bar{v}_0 - \text{Osc}(\bar{\gamma}_t, m^{-1}) - m^{-\frac{1}{2}} - \varepsilon) - C(|D_t v^{m}_{0,\varepsilon}| + |D_{xx}v^{m}_{0,\varepsilon}| + |D_xv^{m}_{0,\varepsilon}|) \right) (\bar{\gamma}_t)
\leq -e^{-\lambda} \left( \lambda(\|\bar{\gamma}_t\|_0 - \mu m^{-\alpha} - m^{-\frac{1}{2}} - \varepsilon) - C(1 + \|\bar{\gamma}_t\|_0) \right) \quad \text{(here choosing} \lambda > C)\]
\leq - e^{-\lambda}(\mu m^{-\alpha} + m^{-\frac{1}{2}} + \varepsilon + C).

49
Since \( v_{\varepsilon}^{m,\varepsilon_1} \) is a classical solution to PPDE (6.18), we have

\[
\text{Part}2 = 0.
\]

From (6.9), we have

\[
|\mathcal{H}^{m,\varepsilon_1} - \sup_u \mathcal{H}^m| \leq C(1 + |p| + |A|)\varepsilon_1,
\]

which together with (6.19) gives

\[
|\text{Part}3| \leq C\varepsilon_1.
\]

From the estimates (6.2) and (6.19), noting \( \bar{\gamma}_t \in C_\mu^\alpha \), we have

\[
|\text{Part}4| \leq C\|P^{m,\bar{\gamma}_t} - \bar{\gamma}_t\|_0 \leq C\mu m^{-\alpha}.
\]

Both (6.1) and (6.25) imply

\[
\text{Part}5 \geq -C(v_{\varepsilon}^{m,\varepsilon_1}(\bar{\gamma}_t) - v(\bar{\gamma}_t)) \geq \frac{1}{2}Cr_0.
\]

Note that the constants \( C \) in this proof all do not depend on \( m, \varepsilon_1, \mu \) and \( \lambda \). Setting \( m \to \infty, \varepsilon \to 0 \), and then considering the upper-limit as \( \mu \to \infty \) on both sides of (6.31), we have from Definition (4.10) the following inequality:

\[
0 \geq -e^{-\lambda}(\varepsilon + C) + \frac{1}{2}Cr_0,
\]

which is a contradiction when \( \lambda \) tends to \( \infty \). The proof is complete.

\textit{Remark} 6.5. Assertions of Theorems 4.6 and 4.7 are still true if the coefficients in Assumptions (H2) and (H3) are relaxed to grow in a linear way.

### 6.2 Degenerate case

In this subsection, we prove Theorem 4.7 using the vanishing viscosity method (see [30]).

\textit{Proof of Theorem 4.7} Similarly as in the non-degenerate case, we assume that \( \mathcal{H} \) strictly decreases in \( y \in \mathbb{R} \) without loss of generality, i.e., (6.1) holds. We only prove \( \bar{v} \geq v \), and the reverse inequality can be proved in a symmetric (also easier) way.
First we construct an approximation of \( \tilde{v} \). For any \( \theta > 0 \), \( \gamma_t \in \Lambda \) and \( u \in \mathcal{U} \), let \( X_{\gamma_t,u;\theta}^{\gamma_t,u;\theta} \) and \( Y_{\gamma_t,u;\theta}^{\gamma_t,u;\theta} \) solve following stochastic equations

\[
\begin{align*}
X_{\gamma_t,u;\theta}^{\gamma_t,u;\theta}(s) &= \gamma_t(s), \quad \text{all } \omega, s \in [0, t); \\
X_{\gamma_t,u;\theta}^{\gamma_t,u;\theta}(s) &= \gamma_t(t) + \int_t^s b(X_{\gamma_t,u;\theta}^{\gamma_t,u;\theta}(r), u(r))dr \\
&+ \int_t^s \sigma(X_{\gamma_t,u;\theta}^{\gamma_t,u;\theta}(r), u(r))dW(r) + \theta(\hat{W}(t) - \hat{W}(s)), \quad \text{a.s. } \omega, s \in [t, T];
\end{align*}
\]

\[
Y_{\gamma_t,u;\theta}^{\gamma_t,u;\theta}(s) = g(X_T^{\gamma_t,u;\theta}) + \int_s^t f(X_r^{\gamma_t,u;\theta}, Y_r^{\gamma_t,u;\theta}, Z_r^{\gamma_t,u;\theta}(r), u(r))dr \\
- \int_s^t Z_r^{\gamma_t,u;\theta}(r)dW(r) - \int_s^t \tilde{Z}_r^{\gamma_t,u;\theta}(r)d\hat{W}(s), \quad s \in [t, T].
\]

where \( \{\hat{W}_t, 0 \leq t \leq T\} \) is an \( n \)-dimensional Brownian motion, independent of \( W \). Define

\[
\tilde{v}^{\theta}(\gamma_t) := \text{ess sup}_{u \in \mathcal{U}} Y_{\gamma_t,u;\theta}^{\gamma_t,u;\theta}(t).
\]

Obviously, we have

\[
|\tilde{v}^{\theta}(\gamma_t) - \tilde{v}(\gamma_t)| \leq C\theta.
\]

(6.33)

For any positive integer \( m, \varepsilon > 0 \), and \( \varepsilon_1, (\varepsilon_1 < m^{-1}) \), let \( V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta} : [t_{i-1}, t_i] \times \mathbb{R}^{n} \rightarrow \mathbb{R}, i = 1, \cdots, m \) be the viscosity solutions to the following state-dependent PDEs

\[
\begin{align*}
-\partial_t V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}(t, \overrightarrow{x}_i) &= \frac{1}{2}\theta^2 \Delta_{x_i,x_i} V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}(t, \overrightarrow{x}_i) \\
&- H_{m,i;\varepsilon_1}(t, \overrightarrow{x}_i, (1, \partial_{x_i}, \partial_{x_i}, x_i)) V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}(t, \overrightarrow{x}_i)) = 0, \\
(t, \overrightarrow{x}_i) &\in (t_{i-1}, t_i) \times \mathbb{R}^{n}, i = 1, \cdots, m; \\
V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}(t_i, \overrightarrow{x}_i) &= V_{\varepsilon,m,i+1;\varepsilon_1}^{m,i+1;\theta}(t_i + \overrightarrow{x}_i, 0, x_i), \quad i < m, \overrightarrow{x}_i \in \mathbb{R}^{n}; \\
V_{\varepsilon,m,m;\varepsilon_1}(T, \overrightarrow{x}_m) &= G_{\varepsilon}^{m}(\overrightarrow{x}_m), \quad \overrightarrow{x}_m \in \mathbb{R}^{m}.
\end{align*}
\]

(6.34)

Like in Subsection 6.1.1, we know \( V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta} : (t_{i-1}, t_i) \in \mathcal{C}^{1,2}([t_{i-1}, t_i] \times \mathbb{R}^{n}) \) and, there are some constants \( C \), independent of \( m, i, \) and \( \varepsilon_1 \), such that

\[
|V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}| + |\partial_{x_i} V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}| + |\partial_{x,x_i} V_{\varepsilon,m,i;\varepsilon_1}^{m,i;\theta}| \leq C(\theta),
\]

(6.35)

51
and for all \((t, \overrightarrow{x_i}) \in [t_{i-1}, t_i] \times \mathbb{R}^{i \times n}, (s, \overrightarrow{y_j}) \in [t_{j-1}, t_j] \times \mathbb{R}^{j \times n}, i \leq j \leq m,\)

\[
\left| (1, \partial_{x_i}, \partial_{x_i}, \partial_t) V^m_i(t, \cdot) \right|^{\overrightarrow{x_i}}_{/y_i} \leq C(\theta) \max_{1 \leq k \leq i} |(x_1 - y_1) + \cdots + (x_k - y_k)|^\beta, \tag{6.36}
\]

\[
\left| (1, \partial_{x_i}, \partial_{x_i}, \partial_t) V^m_i(t, \overrightarrow{x_i}) - (1, \partial_{x_i}, \partial_{x_i}, \partial_t) V^m_j(s, \overrightarrow{x_i}; 0, \ldots, 0, x_i) \right| \leq C(\theta) |s - t|^{\overrightarrow{\beta}}, \tag{6.37}
\]

Since the coefficients in \(H^m_i\) are twice differentiable, in view of the method in Krylov \[27, \text{Lemma 1, Section 7.1} \], we have the following lower bound estimate

\[
\partial_{x_i} V^m_i(t, \cdot) \geq -C. \tag{6.38}
\]

Hence

\[
u^m_i(\gamma_t) := \sum_{i=1}^m \chi_{[t_{i-1}, t_i]}(t) V^m_i(t, \gamma(t_1), \gamma(t_2), \ldots, \gamma(t_{i-2}, t_i))
\]

is the classical solution of PPDE

\[
\begin{cases}
-D_t v^m_i(\gamma_t) - \frac{1}{2} \theta^2 \Delta v^m_i(\gamma_t, 1, D_x, D_{xx}) \varepsilon_1 = 0, & \gamma_t \in \Lambda, t < T; \\
\hat{H} v^m_i(\gamma_T) = g(\gamma_T),
\end{cases}
\tag{6.39}
\]

and satisfies the following estimates from \((6.35) - (6.38),\)

\[
|v^m_i|_{2, \beta; \Lambda} \leq C(\theta), \tag{6.40}
\]

\[
D_{xx} v^m_i \geq -C. \tag{6.41}
\]

Besides, we easily have the following approximating rate

\[
|v^m_i(\gamma_t) - \tilde{v}(\gamma_t)| < C(\text{Osc}(\gamma_t, m^{-1}) + m^{-\frac{1}{4}} + \theta + \varepsilon_1 + \varepsilon).
\]

Now we assert that \(\tilde{v} \geq v\). Otherwise, we have

\[
\inf_{\gamma_t \in \Lambda} \{ \tilde{v}(\gamma_t) - v(\gamma_t) \} := -r_0 < 0.
\]

52
As in the proof of Theorem 4.6, for any sufficiently large $\lambda > 1$, there are $M_0 = M_0(\lambda) > 0$ and $\kappa_1 = \kappa_1(\lambda, M_0) \in (0, T)$, such that for all sufficiently large number $\mu$, $\theta = \theta(r_0), \varepsilon = \varepsilon(r_0), \varepsilon_1 < \varepsilon_1(\mu, r_0)$ and $m > m(\mu, r_0)$, we have

\[
(v_{\varepsilon}^{m, \theta, \varepsilon_1} + e^{-\lambda(t+1)}v_{0, \varepsilon}^m - \psi)\bar{\gamma}_t = \inf_{\gamma_t \in C^m_{\mu}} (v_{\varepsilon}^{m, \theta} + e^{-\lambda(t+1)}v_{0, \varepsilon}^m - \psi)\bar{\gamma}_t > -\frac{1}{2} r_0,
\]

here $\bar{\gamma}_t \in C^m_\mu \cap Q_{M_0, T-\kappa_1}$. Noting (6.22), (6.40), there is a constant $\kappa = \kappa(M_0, \theta, \lambda, \varepsilon) < \kappa_1$ such that

\[
\psi := v_{\varepsilon}^{m, \theta, \varepsilon_1} + e^{-\lambda(t+1)}v_{0, \varepsilon}^m - (v_{\varepsilon}^{m, \theta, \varepsilon_1} + e^{-\lambda(t+1)}v_{0, \varepsilon}^m - \psi)\bar{\gamma}_t
\]

lies in $\mathcal{J}_{\mu, \kappa}(\bar{\gamma}_t, v)$. Since $v$ is a viscosity sub-solution to the path-dependent Bellman equation (4.3), we consider the following formula:

\[
- D_t \psi(\bar{\gamma}_t) - \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, \psi(\bar{\gamma}_t), D_x \psi(\bar{\gamma}_t), D_{xx} \psi(\bar{\gamma}_t), u) \\
= - D_t (v_{\varepsilon}^{m, \theta, \varepsilon_1} + e^{-\lambda(t+1)}v_{0, \varepsilon}^m)(\bar{\gamma}_t) \\
- \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, v(\bar{\gamma}_t), (D_x, D_{xx})(v_{\varepsilon}^{m, \theta, \varepsilon_1} + e^{-\lambda(t+1)}v_{0, \varepsilon}^m)(\bar{\gamma}_t), u) \\
\geq e^{-\lambda(t+1)} \left( \lambda v_{\varepsilon}^{0, \varepsilon} - D_t v_{\varepsilon}^{m, \varepsilon_1} - \sup_{u \in U} \left| \sigma \right|^2 |D_{xx} v_{\varepsilon}^{m, \varepsilon_1} - \sup_{u \in U} \left| \sigma \right| |D_x v_{\varepsilon}^{m, \varepsilon_1}| \right)(\bar{\gamma}_t) \\
- D_t v_{\varepsilon}^{m, \theta, \varepsilon_1}(\bar{\gamma}_t) - \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, v(\bar{\gamma}_t), (D_x, D_{xx})v_{\varepsilon}^{m, \theta, \varepsilon_1}(\bar{\gamma}_t), u) \\
\geq e^{-\lambda(t+1)} \left( \lambda v_{\varepsilon}^{0, \varepsilon} - C(|D_t v_{\varepsilon}^{m, \varepsilon_1}| + |D_{xx} v_{\varepsilon}^{m, \varepsilon_1} + |D_x v_{\varepsilon}^{m, \varepsilon_1}|) \right)(\bar{\gamma}_t) + \frac{1}{2} \theta^2 \Delta v_{\varepsilon}^{m, \theta, \varepsilon_1}(\bar{\gamma}_t) \\
+ \left[ - D_t v_{\varepsilon}^{m, \theta, \varepsilon_1} - \frac{1}{2} \theta^2 \Delta v_{\varepsilon}^{m, \theta, \varepsilon_1} - \mathcal{H}^{m, \varepsilon_1}(\bar{\gamma}_t, (1, D_x, D_{xx})v_{\varepsilon}^{m, \theta, \varepsilon_1}) \right](\bar{\gamma}_t) \\
+ \left[ \sup_{u \in U} \mathcal{H}^{m, \varepsilon_1}(\bar{\gamma}_t, (1, D_x, D_{xx})v_{\varepsilon}^{m, \theta, \varepsilon_1}, u) - \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, (1, D_x, D_{xx})v_{\varepsilon}^{m, \theta, \varepsilon_1}, u) \right](\bar{\gamma}_t) \\
+ \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, v_{\varepsilon}^{m, \theta, \varepsilon_1}, (D_x, D_{xx})v_{\varepsilon}^{m, \theta, \varepsilon_1}, u) - \sup_{u \in U} \mathcal{H}(\bar{\gamma}_t, v, (D_x, D_{xx})v_{\varepsilon}^{m, \theta, \varepsilon_1}, u) \right](\bar{\gamma}_t) \\
:= e^{-\lambda(t+1)} \left( \lambda v_{\varepsilon}^{0, \varepsilon} - C(|D_t v_{\varepsilon}^{m, \varepsilon_1}| + |D_{xx} v_{\varepsilon}^{m, \varepsilon_1} + |D_x v_{\varepsilon}^{m, \varepsilon_1}|) \right)(\bar{\gamma}_t) + \frac{1}{2} \theta^2 \Delta v_{\varepsilon}^{m, \theta, \varepsilon_1}(\bar{\gamma}_t) \\
+ \text{Part1} + \text{Part2} + \text{Part3} + \text{Part4}. \]
From Assumption (H3) and the corresponding estimates, similar to the proof of Theorem 4.6, we have
\[
e^{-\lambda(\bar{t}+1)} \left( \lambda v_{m,0,\xi} - C(|D_tv_{0,\xi}^m| + |D_xxv_{0,\xi}^m| + |D_xzv_{0,\xi}^m|) \right) (\gamma_t) \\
geq -e^{-\lambda}(\mu m^{-\alpha} + m^{-\frac{1}{2}} + \varepsilon + C), \quad \text{if } \lambda \text{ sufficiently large,}
\]
\[
\theta^2 \Delta v_{\varepsilon;m,\theta,\varepsilon}^{m,\theta,\varepsilon}(\bar{\gamma}_t) \geq -\theta^2 C,
\]
and
\[
\text{Part1} = 0, \quad \text{Part2} \geq -C(\theta)\varepsilon_1, \quad \text{Part3} \geq -C(\theta)\mu m^{-\alpha}, \quad \text{Part4} \geq \frac{1}{2}Cr_0.
\]
Since the constants $C$ and $C(\theta)$ do not depend on $\mu, m, \varepsilon_1$ and $\lambda$, first setting $m \to \infty, \varepsilon_1 \to 0$ and then considering the upper-limit as $\mu \to \infty$ on both sides of (6.42), we have from the definition (4.9) the following inequality
\[
0 > -e^{-\lambda}(C + \varepsilon) - \frac{1}{2}\theta^2 C + \frac{1}{2}Cr_0,
\]
which yields a contradiction when sending $\lambda$ to $\infty$ and $\theta$ to 0. The proof is complete. \qed

7 Appendix

In this Appendix, we prove the $\alpha$-Hölder continuity of the path for the following SDE:
\[
X(t) = \int_0^t b(X_r)dr + \int_0^t \sigma(X_r)dW(r), \tag{7.1}
\]
where $b : \Lambda \to \mathbb{R}^n$ and $\sigma : \Lambda \to \mathbb{R}^{n \times d}$ are uniformly Lipschitz continuous.

**Proposition 7.1.** Let any $\alpha \in (0, \frac{1}{2})$, $\mu > 0$, and $X$ be the unique strong solution of (7.1). For any $p$ satisfying $(\frac{1}{2} - \alpha)p > 1$, there is a constant $C = C(p, \alpha)$ such that
\[
P\{ \|X_T\|_\alpha \geq \mu \} \leq CT^{(\frac{1}{2} - \alpha)p} \mu^{-p}. \tag{7.2}
\]

Now we give an auxiliary result.

**Lemma 7.2.** Fix $\delta \in (0, T]$, $\mu > 0$, $\alpha \in (0, \frac{1}{2})$ and $p > 1$. Then we have
\[
P \left\{ \max_{0 \leq s < t \leq \delta} \left| X(s) - X(t) \right| > \mu \delta^\alpha \right\} \leq 2 \cdot 3^p C \delta^{p(\frac{1}{2} - \alpha) - 1} \mu^{-p}.
\]
Proof. Let \( m = m(\delta) \geq 2 \) be the unique integer satisfying \( T/m < \delta \leq T/(m-1) \). Suppose that \( |X(t) - X(s)| > \mu \delta \alpha \) for some \( s \) and \( t \) such that \( 0 \leq s < t \leq T \) and \( |s - t| \leq \delta \). Then there is a unique integer \( q \in [0, m-1] \) such that \( s \in [q \delta, (q+1)\delta) \). There are two possibilities for \( t \). One is \( t \in [q \delta, (q+1)\delta) \), where we have either of both inequalities:

\[
|X(q\delta) - X(s)| > \frac{1}{3} \mu \delta \alpha \quad \text{and} \quad |X(q\delta) - X(t)| > \frac{1}{3} \mu \delta \alpha .
\]

The other is \( t \in [(q+1)\delta, (q+2)\delta) \) (with \( q \leq m + 2 \)), where we have one of the three inequalities:

\[
|X(q\delta) - X(s)| > \frac{1}{3} \mu \delta \alpha, \quad |X(q\delta) - X((q+1)\delta)| > \frac{1}{3} \mu \delta \alpha , \quad \text{and} \quad |X((q+1)\delta) - X(t)| > \frac{1}{3} \mu \delta \alpha .
\]

In conclusion, we have

\[
\left\{ \max_{0 \leq s < t \leq T \atop |s-t| \leq \delta} |X(s) - X(t)| > \mu \delta \alpha \right\} \subseteq \bigcup_{q=0}^{m-1} \left\{ \max_{q \delta \leq s \leq (q+1)\delta} |X(s \wedge T) - X(q\delta)| > \frac{1}{3} \mu \delta \alpha \right\}.
\]

In view of Chebyshev’s inequality and Lemma 3.1, we have

\[
P \left\{ \max_{0 \leq s < t \leq T \atop |s-t| \leq \delta} |X(s) - X(t)| > \mu \delta \alpha \right\} \leq \sum_{q=0}^{m-1} P \left\{ \max_{q \delta \leq s \leq (q+1)\delta} |X(s \wedge T) - X(q\delta)| > \frac{1}{3} \mu \delta \alpha \right\}
\]

\[
\leq \sum_{q=0}^{m-1} E \left[ \max_{q \delta \leq s \leq (q+1)\delta} |X(s \wedge T) - X(q\delta)|^p \right] 3^p (\mu \delta \alpha)^{-p}
\]

\[
\leq 3^p (T \delta^{-1} + 1) C_p \delta^2 \mu^p (\mu \delta \alpha)^{-p}
\]

\[
\leq 2 \cdot 3^p C_p T \delta \mu^p (\frac{1}{\delta^{1/2} \Delta})^{-p} \mu^{-p} .
\]

\( \square \)

Proof of Proposition 7.1. For any \( s, t \in [0, T] \) such that \( s < t \), there is a unique integer \( q \geq 0 \) such that \( 2^{-(q+1)} T < t - s \leq 2^{-q} T \). Obviously,

\[
\left\{ |X(s) - X(t)| > \mu |s-t|^{\alpha} \right\} \subset \left\{ |X(s) - X(t)| > \mu T^{\alpha} 2^{-\alpha(q+1)} \right\}
\]

\[
\subset \left\{ \max_{0 \leq s < t \leq T \atop |s-t| \leq 2^{-q} T} |X(s) - X(t)| > 2^{-\alpha} \mu (2^{-q} T)^{\alpha} \right\}.
\]

55
Thus, applying Lemma 7.2,

\[
P \left\{ \max_{0 \leq s < t \leq T} \frac{|X(s) - X(t)|}{|s - t|^{\alpha}} > \mu \right\} \leq \sum_{q=0}^{\infty} P \left\{ \max_{0 \leq s < t \leq T \mid |s-t| \leq 2^{-q}T} |X(s) - X(t)| > 2^{-\alpha} \mu (2^{-q}T)^{\alpha} \right\}
\]

\[
\leq \sum_{q=0}^{\infty} 2 \cdot 3^{p} C_{p} T (2^{-q}T)^{p(\frac{1}{2} - \alpha) - 1} (2^{-\alpha} \mu)^{-p} = CT^{(\frac{1}{2} - \alpha)p} \mu^{-p}.
\]

This completes the proof.

\[\square\]

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