STRINGS IN A WORMHOLE BACKGROUND

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Abstract

Exact solutions of the string equations of motion in a specific Lorentzian wormhole background are obtained. These include both closed and open string configurations. Perturbations about some of these configurations are investigated using the manifestly covariant formalism of Larsen and Frolov. Finally, the generalized Raychaudhuri equations for the corresponding string worldsheet deformations are written down and analysed briefly.

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I. INTRODUCTION

The study of the string equations of motion and constraints in a curved background spacetime has been a topic of active research over the last few years. Since the equations are nonlinear it is often quite difficult to obtain exact solutions. However, presently, there exists several exact solutions in a variety of curved backgrounds. These include solutions in the spacetime around a cosmic string [1], De Sitter space [2], black hole geometries [3], cosmological backgrounds [4], and gravitational wave backgrounds [5]. More interestingly, several multi-string solutions have been found in De Sitter space [6]. The aim of this paper is to analyse the string equations of motion and constraints in the background of a Lorentzian wormhole geometry, obtain some exact string configurations and study their properties related to perturbations.

If the background spacetime is static and we consider only stationary strings there is a nice simplification of the equations of motion. It turns out that one essentially has to solve for the geodesics in a certain unphysical three dimensional Riemannian space in order to obtain specific string configurations. This fact was first put to use by Frolov et. al. [3] to obtain string configurations in a Kerr–Newman background and we shall also exploit it in our investigations here. Our analysis, as we shall see, will reveal interesting open as well as closed string configurations. The case of the closed string is particularly important because in black hole backgrounds such solutions are not possible [7]. Furthermore, following a recent analysis due to Larsen and Frolov [8] we shall investigate the perturbations about these string configurations. Lastly, we shall write down the generalized Raychaudhuri equations [9] for worldsheet deformations and comment briefly on its solutions and the issue of worldsheet focussing.

Since a wormhole geometry will be used throughout as the background spacetime it is probably necessary to say a few words on wormholes in general. Lorentzian signature wormholes have been a topic of great interest in recent times primarily because of two reasons. Firstly, these geometries require the violation of the Energy conditions for the
matter that is required to support them [10]. One does not know till today whether such matter is possible although quantum stress energy seems to be natural choice. Secondly, such wormholes can very easily be converted into a time machine by performing a relative motion of the wormhole mouths [11]. This makes the possibility of going back to ones past a seemingly simple notion and a large number of physicists have spent a lot of time in trying to understand the consequences of such backward time travel within the framework of simple models [12].

The organisation of the paper is as follows. Section II contains a discussion of the string equations of motion and the constraints and derives the various possible solutions representing string configurations. In Section III we derive the perturbation equations and solve them for some cases. The Raychaudhuri equations are written down and analysed in Section IV. Section V is a conclusion with comments on possible future directions. The Appendix to the paper lists the various Affine Connections and Riemann tensors used extensively in the calculations.

Units and sign conventions in this paper are those due to Misner, Thorne and Wheeler [13]. $(2\pi \alpha)^{-1}$, where $\alpha$ is the inverse of the string tension is set to one.

II. STRING EQUATIONS OF MOTION, CONSTRAINTS AND THEIR SOLUTIONS

As mentioned earlier, the background spacetime for all the analyses in this paper will be that of a Lorentzian wormhole. The metric for such a geometry is represented as:

$$ds^2 = -\chi^2(l)dt^2 + dl^2 + r^2(l)(d\theta^2 + \sin^2\theta d\phi^2)$$ (1)

where $r(l)$ and $\chi(l)$ are two functions which characterize the nature of the geometry. The metric has the features of a wormhole if $r(l)$ and $\chi(l)$ satisfy the following requirements:

(i) $r(l) \sim l \quad \text{as} \quad l \to \pm\infty \quad (\text{Asymptotic flatness})$  \quad (2)

(ii) $r(l=0) = b_0 \quad (\text{Existence of a throat})$  \quad (3)
(iii) \( \chi(l) \) finite everywhere (Nonexistence of Horizons) \hspace{1cm} (4)

Spacelike sections when embedded in a higher dimensional Euclidean space resemble two asymptotically flat regions connected by a bridge.

The wormhole metric can also be written in an alternative form by using the radial coordinate \( r \) instead of the proper radial distance \( l \). This is given as:

\[
ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 \left( d\theta^2 + \sin^2\theta d\phi^2 \right)
\]  

(5)

where \( \frac{b(r)}{r} \leq 1 \), \( b(r = b_0) = b_0 \), \( \frac{b(r)}{r} \to 0 \) as \( r \to \infty \) and \( e^{2\Phi(r)} \) is always finite.

For the special case of an Ellis geometry [14] we have \( \Phi(r) = 1 \) and \( b(r) = \frac{b^2_0}{r} \) or \( r(l) = \sqrt{b^2_0 + l^2} \) and \( \chi(l) = 1 \).

Some of the interesting features of the Ellis geometry are:

(i) The matter required to support it violates all versions of the Energy Conditions [15]. In fact, \( \rho(l) \) i.e. the energy density is negative everywhere.

(ii) Exact solutions of the massless scalar wave equation can be obtained for both the 2 + 1 and 3 + 1 dimensional versions of the metric. These involve the Modified Mathieu and the Radial Oblate Spheroidal Functions in 2 + 1 and 3 + 1 dimensions respectively [16].

(iii) the \( t = \text{constant} \) \( \theta = \frac{\pi}{2} \) sections when embedded in \( R^3 \) represent a catenoid which is a minimal surface \( (Tr(K_{ij}) = 0) \).

We now move on to the analysis of string configurations in the background of the Ellis geometry.

The bosonic string equations of motion and constraints in a general curved background are obtained by extremizing the Nambu–Goto action. These are given as:

\[
\ddot{x}^\mu - x^{\mu\nu} + \Gamma^\mu_{\rho\sigma} \left( \dot{x}^\rho \dot{x}^\sigma - x^{\rho\sigma} \right) = 0
\]  

(6)

\[
g_{\mu\nu} \ddot{x}^\mu . \ddot{x}^\nu = g_{\mu\nu} \dot{x}^\mu . \dot{x}^\nu + g_{\mu\nu} \dot{x}^{\mu\nu} = 0
\]  

(7)

where the primes denote derivatives with respect to \( \sigma \) and dots denote derivatives with respect to \( \tau \) (\( \sigma, \tau \) being the worldsheet coordinates).
For the case of a stationary string in a static background we assume:

\[ t = \tau \quad x^i = x^i(\sigma) \] (8)

The string equations and constraints reduce to the following equations:

\[ x^{i''} + \Gamma^i_{jk}x^j x^{k'} - \Gamma^i_{00} = 0 \] (9)

\[ g_{00} + g_{ij}x^{i''}x^{j'} = 0 \] (10)

where \( \Gamma^i_{jk} \) are the Affine Connections for the background metric.

One can also write (9) as a geodesic equation in a Riemannian space endowed with a metric

\[ ds^2 = H_{ij}dx^idx^j \] (11)

where \( H_{ij}x^i x^j = 1 \). The \( H_{ij} \) are related to the background metric by the following:

\[ g_{00} = -\chi^2 \quad g_{ij} = \frac{H_{ij}}{\chi^2} \text{ and } g_{0i} = 0. \]

We shall mostly be concerned with the special case of Ellis geometry except while discussing closed strings. Note that we can reduce the problem of obtaining string configurations to that of finding the geodesics in a fictitious Riemannian space. However we have to make sure that the constraint equations are satisfied.

For our metric, i.e Ellis geometry the corresponding components of \( H_{ij} \) are fairly simple.

\[ H_{11} = 1 \quad H_{22} = r^2(l) = b_0^2 + l^2 \]

\[ H_{33} = r^2(l) \sin^2 \theta = (b_0^2 + l^2) \sin^2 \theta \quad H_{ij} = 0 \forall i \neq j \] (13)

The equations of motion now reduce to the following geodesic equations:

\[ l'' - r\ddot{r}\theta'^2 - r\ddot{r}\sin^2 \theta \dot{\phi}' = 0 \] (14)

\[ \theta'' + 2\frac{\ddot{r}}{r}\theta' l' - \sin \theta \cos \theta \dot{\phi}' = 0 \] (15)

\[ \phi'' + 2\frac{\ddot{r}}{r}\phi' l' + 2\cot \theta \theta' \phi' = 0 \] (16)
In the above the $\sim$ sign denotes differentiation w.r.t the variable $l$.

The constraint equation leads to the following:

$$-1 + l'^2 + r^2 \left( \theta'^2 + \sin^2 \theta \phi'^2 \right) = 0 \quad (17)$$

We now specialize by choosing $\theta = \frac{\pi}{2}$. The second of the geodesic equations is solved straightaway and therefore we need to look for solutions of only the $l$ and $\phi$ equations. The $\phi$ equation can be integrated once to give:

$$\phi' = \frac{C_1}{r^2} \quad (18)$$

where $C_1$ is an integration constant. Integrating the $l$ equation once by making use of the above expression for $\phi'$ one gets:

$$-C_2 + l'^2 + \frac{C_1^2}{r^2} = 0 \quad (19)$$

The above equation matches with the constraint equation only if we choose $C_2 = 1$. We can now obtain the various string configurations by solving the two equations for $\phi'$ and $l'$.

A. General Solution for $\theta = \frac{\pi}{2}$

The solutions of the equations (18) and (19) are as follows:

$$\sigma - \sigma_0 = b_0 F(\beta, q) - b_0 E(\beta, q) + l \sqrt{\frac{b_0^2 + l^2}{a_0^2 + l^2}} \quad (20)$$

where $\sigma_0$ is an integration constant, $a_0^2 = b_0^2 - l^2$ and

$$\beta = \arctan \frac{b_0}{l}, \quad q = \frac{C_1}{b_0} \quad (21)$$

$F(\beta, q)$ and $E(\beta, q)$ are the Elliptic Functions of the first and second kinds given by:

$$F(\beta, k) = \int_0^\beta \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} \quad (22)$$

$$E(\beta, k) = \int_0^\beta d\alpha \sqrt{1 - k^2 \sin^2 \alpha} \quad (23)$$
with $k^2$ less than one.

The solution for $\phi$ as a function of $l$ is also given in terms of Elliptic functions.

$$\phi - \phi_0 = \frac{C_1}{b_0} F(\alpha, q)$$

where

$$\alpha = \arctan \frac{l}{a_0}, \quad q = \frac{C_1}{b_0} \quad (25)$$

Much simpler functional forms can be obtained by considering special cases of the above expressions. These are discussed below.

**B. Straight Strings**

From the original string equations of motion and the constraints one can easily see that the following simple configuration represents a solution.

$$l = \sigma \quad \theta = \frac{\pi}{2} \quad \phi = \phi_0 \quad (26)$$

This represents a straight string in the sense that the proper radial distance $l$ is identified with the worldsheet coordinate $\sigma$ and $\phi$ does not vary throughout.

**C. Strings With $\phi \neq \text{Constant}$**

A special case of the general solution described in the Case 1 can be obtained by setting $a_0 = 0$ which implies $b_0 = C_1$. This yields the following expressions for $\sigma(l)$ and $\phi(l)$.

$$\sigma - \sigma_0 = r(l) + \frac{1}{2} \ln \frac{r(l) - b_0}{r(l) + b_0} \quad \forall \ l \neq 0$$

$$= 0 \quad \text{for} \ l = 0 \quad (27)$$

where $r(l) = \sqrt{b_0^2 + l^2}$.

Also,
\[ \phi - \phi_0 = \frac{1}{2} \ln \frac{r(l) - b_0}{r(l) + b_0} \quad \forall l \neq 0 \]

\[ = 0 \quad \text{for} \quad l = 0 \quad (28) \]

One can also obtain an expression for \( l(\phi) \) for \( l \neq 0 \). This is given as :

\[ l = \pm b_0 (\sinh(\phi_0 - \phi))^{-1} \quad (29) \]

In this configuration the value of \( \phi \) changes as we move along the wormhole by varying \( l \). At \( l = 0 \) i.e at the throat \( \phi = \phi_0, \sigma = \sigma_0 \). As \( l \to \pm \infty \) also \( \phi \to \phi_0 \) but \( \sigma \to \pm \infty \). Thus the string may lie in the wormhole by spiralling around the surface if one chooses \( \phi_0 \) appropriately.

**D. Closed Strings**

A closed, stationary string configuration can be defined in the following way :

\[ t = \tau, \quad l = l_0, \quad \theta = \frac{\pi}{2}, \quad \phi = C_1 \sigma \quad (30) \]

where \( l_0 \) and \( C_1 \) are constants. A very simple closed string solution in the background of a wormhole can be defined as

\[ t = \tau, \quad l = 0, \quad \theta = \frac{\pi}{2}, \quad \phi = C_1 \sigma \quad (31) \]

If \( b_0 \) is the throat radius of the wormhole then one can easily check that one needs \( b_0 = \frac{1}{C_1} \) in order to satisfy the constraint equation. This, at least in the case of the Ellis geometry is a possible string configuration.

One recalls from [7] that closed strings are not possible in a stationary black hole background, such as the Schwarzschild. The obvious question that arises is — How does a wormhole background help in having closed strings? In order to understand this we have to put back the \( -\chi^2(l) \) factor in front of the coefficient of \( dt^2 \) in the background wormhole metric. This will illustrate the role of the horizon in forbidding the existence of closed strings.
The string equations of motion for the coordinates $\theta$ and $\phi$ remain the same as in (15) and (16) while that for the coordinate $l$ becomes:

$$l'' - \chi'\theta - \tilde{r}^2 - \tilde{r}\sin^2 \theta\phi^2 = 0$$  \hfill (32)

The constraint equation now changes to:

$$-\chi^2 + l'^2 + r^2\theta'^2 + r^2\sin^2 \theta\phi^2 = 0$$  \hfill (33)

Now for any wormhole geometry we have $\tilde{r} = 0$ at the throat (i.e. at $l = 0$). Therefore, for $\theta = \frac{\pi}{2}$ one gets the following requirements on $\chi(l)$.

$$\left(\frac{d}{dl}[\chi^2(l)]\right)_{l=0} = 0 , \quad \chi^2(l = 0) = b_0^2 C_1^2$$  \hfill (34)

Note that if $\chi(l = 0) = 0$ –which is the situation in the Schwarzschild black hole at the horizon one does not have a closed string solution. In general, however, for any wormhole (which is a spacetime free of horizons and singularities) one is bound to have at least one closed string configuration residing at the throat.

III. PERTURBATIONS AROUND SPECIFIC STRING CONFIGURATIONS

In this section, we shall concentrate on analysing the nature of perturbations around two of the specific string configurations derived in the previous section. A manifestly covariant formalism for the study of these perturbations has been recently set up by Larsen and Frolov [8]. The variations are defined as

$$\delta x^\mu = \delta x^\mu_i n_i^\mu + \delta x^\mu_a x^\mu_a$$  \hfill (35)

where $n_i^\mu$ are two vectors normal to the worldsheet described by the solution. Infact $\{x^\mu_a, n_i^\mu\}$ forms a basis at every point on the worldsheet ($a = \sigma, \tau; i = 1, 2$). These vectors obviously satisfy

$$g_{\mu\nu} n_i^\mu n_j^\nu = \delta_{ij} , \quad g_{\mu\nu} x^\mu_a x^\nu_a = 0$$  \hfill (36)
We will only consider normal variations because the tangential ones leave the action invariant – a consequence of the reparametrization invariance of the worldsheet. Details and the exact expressions for the second variation of the Nambu–Goto action can be found in the paper by Larsen and Frolov [8]. Since we consider only stationary strings our equations will be much simpler than those for the most general case. It turns out that the perturbations are governed by a set of equations written in compact form as:

\[
\left( \partial_{\sigma^2} - \partial_{\tau^2} \right) \delta x_i = U_{ij} \delta x^j
\]  

(37)

where \( U_{ij} \) is given by

\[
U_{ij} = V \delta_{ij} + F^2 V_{ij}
\]  

(38)

and

\[
V = \frac{1}{4} \left( F''^2 - 2F F'' \right)
\]  

(39)

\[
V_{ij} = x^{p\mu} x^{q\nu} \tilde{R}_{pqrs} n_i^p n_j^r
\]  

(40)

with \( d\sigma_c = \frac{d\sigma}{\chi^2} \), \( F \equiv \chi^2 \), and \((n^p_i, x'^i)\) form an orthonormal basis in the space with metric \( H_{ij} \).

Thus, to obtain information regarding the nature of perturbations one needs to solve the coupled differential equations involving the \( \delta x^i \). We now concentrate on doing so for the specific string configurations discussed in Section II.

**A. Straight String**

The normal vectors for this case can be chosen as:

\[
n_1^\mu \equiv (0, 0, \frac{1}{r}, 0) \quad ; \quad n_2^\mu \equiv (0, 0, \frac{1}{r})
\]  

(41)

and the perturbations in terms of \( \delta x_1, \delta x_2 \) are
\[ \delta l = 0 \quad , \quad \delta \theta = \delta x_1/r \quad , \quad \delta \phi = \delta x_2/r \]  

(42)

Using the expression for \( V_{ij} \) we find that:

\[ V_{11} = V_{22} = \frac{b_0^2}{(b_0^2 + l^2)^2} \quad ; \quad V_{12} = V_{21} = 0 \]  

(43)

Since \( F = \chi^2 = 1 \) we have \( V = 0 \).

Hence we need to solve only one differential equation in order to obtain \( \delta x_1 \) and \( \delta x_2 \).

This is given as

\[ \left( \partial^2_{\sigma} - \partial^2_{r} \right) \delta x_{1,2} = \frac{b_0^2}{(b_0^2 + l^2)^2} \delta x_{1,2} \]  

(44)

Now we perform an usual Fourier expansion of \( \delta x_{1,2} \). This yields the equation

\[ \frac{d^2 D_\omega}{d\sigma^2} + \left( \omega^2 - \frac{b_0^2}{(b_0^2 + l^2)^2} \right) D_\omega = 0 \]  

(45)

where

\[ \delta x_{1,2} = \int e^{-i\omega t} D_\omega(\sigma) d\omega \]  

(46)

This differential equation for \( D_\omega \) can be exactly solved. Infact it appears in the study of massless scalar waves in the background of the Ellis geometry and has been analysed in detail in [16]. Redefining \( D_\omega = \sqrt{b_0^2 + \sigma^2} F_\omega \) and with \( \xi = \frac{\sigma}{b_0} \) we get

\[ (1 + \xi^2) \frac{d^2 F_\omega}{d\xi^2} + 2\xi \frac{dF_\omega}{d\xi} + [\omega^2 b_0^2 (1 + \xi^2)] F_\omega = 0 \]  

(47)

The differential equation for Radial Oblate Spheroidal Functions is

\[ (1 + \xi^2) \frac{d^2 V_{mn}}{d\xi^2} + 2\xi \frac{dV_{mn}}{d\xi} + [-\lambda_{mn} + k^2 \xi^2 - \frac{m^2}{1 + \xi^2}] V_{mn} = 0 \]  

(48)

If we take \( m = 0 \) \( \lambda_{0n} = -k^2 = -\omega^2 b_0^2 \) then we get our equation governing the perturbations of a straight string as a special case of the Radial Oblate Spheroidal Equation. However we only get solutions for specific values of \( \lambda_{0n} \) (i.e \( \lambda_{0n} \) negative). Consulting [17] we notice that this is possible only for \( n = 0, 1 \). The solutions to the equations are finite at infinity – they behave like simple sine/cosine waves. The scattering problem for the
Schroedinger–like equation has been analysed numerically in [18] and one obtains a smooth transmittivity curve rising from almost zero at small values of $\omega$ and increasing to one as the energy becomes comparable or larger than the barrier height. Explicit expressions for the solutions of the equations governing the perturbations can be written only for $n = 0, 1$ and these involve a series of products of Bessel Functions (for details see [17]).

**B. Closed String**

The equations governing the perturbations of the closed string are even more simple. The normals can be chosen as:

$$n_1^\mu \equiv (0, 0, \frac{1}{r}, 0) \ ; \ n_2^\mu \equiv (0, 1, 0, 0)$$

and the $\delta x^\mu$ turn out to be

$$\delta l = \delta x^2 \ , \ \delta \theta = \delta x^1 / r \ , \ \delta \phi = 0$$

The potentials are either constant or zero (by virtue of the fact that $l = 0$).

$$V_{11} = -V_{22} = C_1^2 \ ; \ V_{12} = V_{21} = 0$$

Performing a Fourier expansion similar to the case discussed earlier one arrives at the following expressions for $D_1^\omega$ and $D_2^\omega$. These are:

$$D_1^\omega(\sigma) = A_1 \exp(\pm in\sigma)$$

where $\omega_n^2 = n^2 - C_1^2$ and

$$D_2^\omega(\sigma) = A_2 \exp(\pm in\sigma)$$

where $\omega_n^2 = n^2 + C_1^2$

The $\tau$ parts of the $\delta x^{1,2}$ differ in the argument of the exponential. For $\delta x^1$ it is $\exp(\pm i\sqrt{n^2 - C_1^2}\tau)$ whereas for $\delta x^2$ it is $\exp(\pm i\sqrt{n^2 + C_1^2}\tau)$. Notice that for $\delta x^1$ the $\tau$ part may be exponentially damped or growing if $C_1^2 \geq n^2$ whereas the solution for $\delta x_2$ is always oscillatory.
IV. THE GENERALIZED RAYCHAUDHURI EQUATIONS

The Raychaudhuri Equations in GR [19] essentially deal with the evolution of timelike or null geodesic congruences. The assumption of an Energy Condition leads to the focussing theorem in GR which states that if matter satisfies certain restrictions then initially converging sets of geodesics will always tend to focus at a point within a finite value of the affine parameter that characterises each geodesic. Recently Capovilla and Guven [9] have come up with a generalisation of the Raychaudhuri Equations for $D$ dimensional timelike worldsheets embedded in an $N$ dimensional curved background. We shall be concerned with the $D = 2, N = 4$ case of these generalised Raychaudhuri Equations.

We now very briefly recall the ingredients of the generalised Raychaudhuri equations (for details the reader is referred to Capovilla and Guven[9]).The equation which we shall deal with is given by (this is a special case of the more general equation quoted in [9]):

$$\Delta \gamma + \frac{1}{2} \partial_a \gamma \partial^a \gamma + (M^2)^i_i = 0 \quad (54)$$

where $\nabla_a$ is the worldsheet covariant derivative, and $\partial_a \gamma = \theta_a \gamma$. $\theta_a \gamma$ is related to a quantity $J_{a}^{ij} = \frac{1}{2} \delta_{ij} \theta_a$ where the $J_a^{ij}$ are defined as quantities related to the normal gradients of the orthonormal spacetime basis $(E_{a}^{\mu}, n_{i}^{\mu})$ defined at each point on the worldsheet through the analogs of the Gauss–Weingarten equations:

$$D_i E_a = J_{aij} n^j + S_{abi} E^b \quad (55)$$

$$D_i n^j = -J_{aij} E^a + \gamma^k_{ij} n^k \quad (56)$$

$S_{abi}$ and $\gamma^k_{ij}$ are defined as:

$$S_{ab}^i = g_{\mu\nu} n^{\alpha i} (D_{a} E_{b}^{\mu \nu}) E_{\nu}^{\alpha} \quad (57)$$

$$\gamma^k_{ij} = g_{\mu\nu} n^{\alpha i} (D_{a} n_{j}^{\mu}) n_{k}^{\nu} \quad (58)$$

The quantity $(M^2)^{ij}$ is given as (see [20]):

$$(M^2)^{ij} = K_{ab}^{i} K^{abj} + R_{\mu \nu \rho \sigma} E_{a}^{\mu \nu} n^{\rho i} E_{\rho a} n^{\sigma j} \quad (59)$$
Notice that the quantity \((M^2)^{ij}\) contains contributions from the extrinsic curvatures
\[ K^i_{ab} = -g_{\mu\nu}(D_a E^\mu_b)n^\nu \] of the worldsheet as well as the Riemann tensor components of the
background spacetime. For geodesic curves one can arrive at the usual Raychaudhuri equation
by noting that \(K^i_{00} = 0\), the \(J_{aij}\) are related to their spacetime counterparts \(J_{\mu\nu a}\) through
the equation \(J_{\mu\nu a} = n_i^\mu n_i^\nu J_{aij}\), and the \(\theta\) is defined by contracting with the projection tensor
\(h_{\mu\nu}\).

The \(\theta\) or \(\gamma\) basically tell us how the spacetime basis vectors change along the normal
directions as we move along the worldsheet. If \(\theta\) diverges somewhere, it induces a divergence
in \(J_{aij}\), which, in turn means that the gradients of the spacetime basis along the normals
have a discontinuity. Thus the family of worldsheets meet along a curve and a cusp/kink
is formed. This can be called as a worldsheet focussing effect in analogy with geodesic
focussing in GR.

We emphasize that the simplified form of the generalised Raychaudhuri equation we
are using is valid only when the quantity \((M^2)^{ij}\) does not contain any off
diagonal terms. Quantities analogous to the shear and rotation present in the case of geodesic congruences
are also present here but we have put them to zero in order to simplify the analysis.

We now move on to the discussion of the special cases.

**A. Straight String**

The induced metric \(\gamma_{ab} = g_{\mu\nu}x^\mu_a x^\nu_b\) is equal to \(\eta_{ab}\) here—thus the \(K^i_{ab}\) vanish for all \(a, b\).
The normal vectors can be chosen as before. The quantity \((M^2)^i\) turns out to be

\[
(M^2)^i = -\frac{2\beta^2}{(b_0^2 + \sigma^2)^2}
\]  

Thus the Raychaudhuri equation for \(\gamma = 2\ln \beta\) turns out to be:

\[
-\frac{\partial^2 \beta}{\partial \tau^2} + \frac{\partial^2 \beta}{\partial \sigma^2} - \frac{2\beta^2}{(b_0^2 + \sigma^2)^2} \beta = 0
\]

Separating variables \(\beta = T(\tau)\Sigma(\sigma)\) we have
\[
\frac{d^2 T}{d\tau^2} + \omega^2 T = 0
\]  \hspace{1cm} (62)

and
\[
\frac{d^2 \Sigma}{d\sigma^2} + \left( \omega^2 - \frac{2b_0^2}{(b_0^2 + \sigma^2)^2} \right) \Sigma = 0
\]  \hspace{1cm} (63)

Since \( \gamma = 2 \ln \beta = 2 \ln T + 2 \ln \Sigma \) we have:
\[
\theta_\tau = 2 \frac{\dot{T}}{T} \hspace{1cm} ; \hspace{1cm} \theta_\sigma = 2 \frac{\Sigma'}{\Sigma} \hspace{1cm} (64)
\]

Now, from several theorems in the theory of differential equations [21] one can conclude that as long the quantity \( H(x) \) in the differential equation \( \frac{d^2 F}{dx^2} + H(x)F = 0 \) is positive and continuous everywhere in the domain of \( x \) one can say that there exists zeros in the solutions. If we use this fact as an input in the two differential equations for \( T \) and \( \Sigma \) we find that both \( \theta_\tau \) and \( \theta_\sigma \) must necessarily diverge somewhere. For \( T(\tau) \) we can locate the points whereas for \( \Sigma \) the only statement we can make is \( \omega^2 \geq \frac{2}{b_0^2} \) (this guarantees the positivity of \( H(\sigma) \))

**B. Closed String**

Similarly for the case of closed strings the generalised Raychaudhuri Equations can be written down. We find
\[
(M^2)^i_i = 0
\]  \hspace{1cm} (65)

and
\[
-\frac{\partial^2 \beta}{\partial \tau^2} + \frac{\partial^2 \beta}{\partial \sigma^2} = 0
\]  \hspace{1cm} (66)

Using analysis similar to the previous case one obtains the following solutions for \( T(\tau) \) and \( \Sigma(\sigma) \):
\[
T(\tau) = \sin n\tau \hspace{1cm} ; \hspace{1cm} \cos n\tau
\]  \hspace{1cm} (67)

\[
\Sigma(\sigma) = \sin n\sigma \hspace{1cm} ; \hspace{1cm} \cos n\sigma
\]  \hspace{1cm} (68)

The conclusions on focussing are similar to the case for Straight strings.
C. Strings with $\phi \neq \text{constant}$

Finally we write down the corresponding Raychaudhuri equations for the string solutions with a nonconstant $\phi$. These, after the usual separation of variables turn out to be the following two ordinary differential equations for $\Sigma$ and $T$.

$$\frac{d^2 \Sigma}{d\sigma^2} + \left(\omega^2 - \frac{2l^2(\sigma) h_0^2}{r_0^2(\sigma)}\right) \Sigma = 0$$  \hspace{1cm} (69)

$$\frac{d^2 T}{d\tau^2} + \omega^2 T = 0$$  \hspace{1cm} (70)

The second equation is the same as for the previous two cases. For the first, one has to check the positivity of the quantity $H(\sigma)$. This can be done by locating the maximum value of the potential function which occurs here at $l = \frac{h_0}{2\sqrt{2}}$. Thus $\omega^2 \geq \frac{128}{729h_0^2}$ is the criterion for the positivity of $H(\sigma)$. Thus a focussing effect is possible in the sense that there exist zeros in the solutions of the differential equations and $\theta_\alpha$ diverges at certain points.

V. CONCLUDING REMARKS

Let us now summarize the results we have obtained.

(i) Specific string configurations in a wormhole background have been obtained. These include both closed and open strings. The open ones can be straight or spiral while the closed one is possible only at the throat of the wormhole. All geodesics on the two-sphere residing at the throat of the wormhole are closed string configurations. In fact it should be emphasized that for all wormholes there exists at least one closed string configuration at the throat. It is important to contrast this with the case of the black hole where no such closed string configurations are possible. This fact can be attributed to the presence of the black hole horizon. We have also discussed a general solution given in terms of elliptic functions.

(ii) Perturbations about two of the string configurations have been analysed using the manifestly covariant formalism of Larsen and Frolov [8]. For the straight string the
perturbation equations can be solved exactly in terms of the Radial Oblate Spheroidal Functions. For closed strings the solutions are even simpler – they are only sines and cosines.

(iii) The Raychaudhuri Equations for deformations of the string worldsheets obtained in Section I have also been written down and analysed for the various configurations. We have been able to arrive at the concept of worldsheets focussing in a manner similar to geodesic focussing in GR. In the general case of a string moving in any curved background one can see very easily that the generalised Raychaudhuri equation (with quantities analogous to shear and rotation put to zero) takes the following form:

\[-\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma^2} + \Omega^2(\sigma, \tau) (M^2 i_1(\sigma, \tau) F = 0 \quad (71)\]

This is possible because the worldsheet is two–dimensional and any 2D metric can be written in a conformally flat form by choosing coordinates appropriately. The operator \(\Delta\) therefore reduces to \(\frac{1}{M^2} \Delta_M\) where \(\Delta_M\) is the D’Alembertian in flat Minkowski space. and \(\Omega^2\) is the conformal factor. Therefore, to derive a focussing theorem for string worldsheets one has to understand the nature of the solutions of the above equation with emphasis on the zeros.

(iv) Since a closed string exists only at the throat and the wormhole requires matter that violates the Weak Energy Condition one can ask – Can quantum strings provide the source for WEC violating matter? If this is true then one has a solution to the problem of WEC violation for wormhole material.

Finally, as a continuation of the search for exact solutions of the string equations and constraints in curved spacetimes one can look into string motion in other wormhole backgrounds. An interesting case could be the evolving wormholes, where the analysis used here would not apply but that of string configurations in Friedman–Robertson–Walker cosmologies certainly would [21].

Work along these lines is in progress and will be communicated in due course.
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In this appendix we list all the relevant affine connections and Riemann tensors used in the calculations in this paper.

The metric is assumed as:

\[ ds^2 = -\chi^2(l) dt^2 + dl^2 + r^2(l) \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \]  \hspace{1cm} (A1)

the nonzero Affine Connections (Christoffel symbols) are given as

\[ \Gamma_{tt} = \frac{\chi'}{\chi} ; \quad \Gamma_{tl} = \chi' ; \quad \Gamma_{t\theta} = -rr' \]  \hspace{1cm} (A2)

\[ \Gamma_{\phi\phi} = -rr' \sin^2 \theta ; \quad \Gamma_{\theta l} = \frac{r'}{r} ; \quad \Gamma_{\phi\theta} = -\sin \theta \cos \theta \]  \hspace{1cm} (A3)

\[ \Gamma_{l\phi} = \frac{r'}{r} ; \quad \Gamma_{\phi\theta} = \cot \theta \]  \hspace{1cm} (A4)

From these one can evaluate the Riemann tensor components which are:

\[ R_{t\theta l} = R_{\phi l\phi} = -rr'' ; \quad R_{\phi\theta\phi} = \sin^2 \theta (1 - r'^2) \]  \hspace{1cm} (A5)

In the above expressions the prime denotes derivative w.r.t \( l \). With the help of these one can easily evaluate the relevant expressions for Ellis geometry (\( \chi = 1 \) and \( r(l) = \sqrt{b_0^2 + l^2} \)). These are

\[ \Gamma_{\theta l} = \Gamma_{\phi l} = -l ; \quad \Gamma_{\phi l} = \Gamma_{l\phi} = \frac{l}{b_0^2 + l^2} \]  \hspace{1cm} (A6)

\[ \Gamma_{\phi\phi} \text{ and } \Gamma_{\phi\theta} \text{ are the same as quoted above.} \]

\[ R_{\phi l\phi} = R_{\phi l\phi} = -\frac{b_0^2}{b_0^2 + l^2} ; \quad R_{\phi\theta\phi} = \frac{b_0^2 \sin^2 \theta}{b_0^2 + l^2} \]  \hspace{1cm} (A7)