UNCONDITIONAL WELL-POSEDNESS FOR WAVE MAPS

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ABSTRACT. We prove uniqueness of solutions to the wave map equation in the natural class, namely \((u, \partial_t u) \in C([0, T); \dot{H}^{d/2}) \times C^1([0, T); \dot{H}^{d/2-1})\) in dimensions \(d \geq 4\). This is achieved through estimating the difference of two solutions at a lower regularity level. In order to reduce to the Coulomb gauge, one has to localize the gauge change in suitable cones as well as estimate the difference between the frames and connections associated to each solutions and take advantage of the assumption that the target manifold has bounded curvature.

1. INTRODUCTION

Let \((N, g)\) be a complete riemannian manifold of dimension \(k\) without boundary. We denote \((x^\alpha)\), \(0 \leq \alpha \leq d\) the canonical coordinate system of \(\mathbb{R} \times \mathbb{R}^d\) where \(t = x^0\) denotes the time variable. Moreover, we denote \(\partial_\alpha = \partial/\partial x^\alpha\) and use the Minkowski metric on \(\mathbb{R} \times \mathbb{R}^d\) to raise and lower indices. In particular, \(\partial^0 = -\partial_0\) and \(\partial^\alpha = \partial_\alpha\) for \(1 \leq \alpha \leq d\). The wave map equation, from \(\mathbb{R} \times \mathbb{R}^d\) into \(N\), reads

\[
\begin{cases}
D_\alpha \partial^\alpha u &= 0, \\
u(x, 0) &= u_0(x), \\
\partial_t u(x, 0) &= u_1(x) & x \in \mathbb{R}^d, \ t \geq 0,
\end{cases}
\]

where \(D_\alpha\) is the pull-back of the covariant derivative on the target Riemannian manifold \(N\).

1.1. Statement of the Result. Low regularity strong (e.g. unique) solutions to semilinear wave equations like (1) are usually constructed via fixed point methods. Hence, while one is ultimately seeking solutions which are continuous evolutions of the data, that is \((u, \partial_t u) \in C([0, T); \dot{H}^s) \times C([0, T); \dot{H}^{s-1})\), the necessary requirements to set up a fixed point lead to a smaller Banach space. For example, this translates into additional space-time integrability conditions, like \(u \in L^p_t(L^q)\) for suitable \(p, q\). The resulting well-posedness result is often deemed conditional (to these additional requirements). Our aim is to remove these
assumptions which are incorporated in the uniqueness and existence statement given by, say, Picard’s theorem, and prove unconditional well-posedness (sometimes called unconditional uniqueness in the literature), that is uniqueness in the natural class, where the flow is continuous. Note that in the wave map situation, one does not construct a solution directly by iteration, at least when working at the critical regularity. Nevertheless, in order to obtain a priori estimates, one is led to add similar requirements ($\partial u \in L^2_t(L^2_{x_1})$ for example in [15, 11]).

From now on, we generically denote $(\nabla u, \partial_t u)$ as $\partial u$, so that any statement regarding $u$ and $\partial_t u$ can be summarized into one, like $\partial u \in C_t(\dot{H}^{s-1})$. Also, for any Banach space $X$, $C(X)$ will denote the space $C([0,T);X)$, $L^p(X)$ will denote the space $L^p(0,T;X)$ and $L^p(L^q)$ will denote the space $L^p(0,T;L^q(\mathbb{R}^d))$.

**Theorem 1.1.** Let $u$ be a solution to (1) on $[0,T^*)$, with $d \geq 4$. Then $u$ is the unique solution of (1) in the class $\partial u \in C(\dot{H}^{\frac{d}{2}-1})$.

**Remark 1.2.** The same result should hold for $d = 3$ if one is willing to consider $\partial u \in L^\infty(\dot{H}^{\frac{d}{2}+\epsilon})$, $\epsilon > 0$. In fact, both schemes of proof from [15, 11] work in that framework, modulo technicalities related to the low regularity (one has to take into account the null form structure in the elliptic equation).

Recently, there were many works proving unconditional well-posedness for several hyperbolic systems (see [12, 10] for the critical wave equation, [3, 2] for the nonlinear Schrödinger equation, [9] for the Zakharov system, [8] for the Maxwell-Dirac system). There is now a huge literature about unconditional well-posedness for parabolic equations such as the Navier-Stokes system [1, 6, 7].

A desirable goal would be to prove that strong solutions to the wave map equation do coincide with weak solutions in (spatial) dimension $d = 2$, at least before possible blow-up; recall that when $d = 2$ the scale-invariant space is the natural energy space. Such a goal appeared totally out of reach when the present work was started. However, a great deal of progress was made in recent years on the strong Cauchy theory for $d = 2$, eventually leading to global well-posedness for large finite energy data ([17, 16] for negatively curved compact targets, [5] for the hyperbolic space $\mathbb{H}^2$ as target, and [18] and references therein again for the hyperbolic space). We hope our present high dimensional result provides a clear view of uniqueness issues in a relatively straightforward functional setting, but that the strategy itself will be of interest in the more intricate lower dimensional setting.
2. Existence of solutions: From large data to small data

We recall that results on global well-posedness at the critical level for small data can actually be extended to local well-posedness for large data. All these results rely on a gauge change, which requires a certain connection (associated to the map) to be small. In high dimension, one may choose the Coulomb gauge, and in the large data case, there is no reason for this connection to be small, as the elliptic system linking the connection to the map needs not have a unique solution. Fortunately, one can take advantage of a fundamental property of the wave equation, namely the finite speed of propagation. The equation (11) can be seen as a semilinear wave equation, and the nonlinearity is a local one, since it can be written as a product of \( \partial u \) and (function of) \( u \). Remark that after performing the gauge transform alluded to above, this local character is lost, the new nonlinearity involves pseudodifferential operators.

Given an arbitrary initial data \((u_0, u_1)\), let us explain how we can construct a local solution \( u \) using the known results for small data. We can choose \( r > 0 \) small enough such that

\[
\sup_{x \in \mathbb{R}^d} \| (\partial u_0, u_1) \|_{\dot{H}^{d/4-1}(B(x,r))} \leq \epsilon_0
\]

where \( \epsilon_0 \) is a small parameter which will be chosen later. For each ball \( B(x,r) \), the initial data \((u_0, u_1)\) can be extended to the whole space by \((\tilde{u}_0, \tilde{u}_1)\) in such a way that

\[
\| (\partial \tilde{u}_0, \tilde{u}_1) \|_{\dot{H}^{d/4-1}((\mathbb{R}^d)^+)} \leq C\epsilon_0
\]

for some constant \( C \) which only depends on \( N \) and \( r \). For each \( x \in \mathbb{R}^d \), we can use the results of [15] to construct a global solution \( u_x \) which is unique in the class \( \partial u \in C(\dot{H}^{d/4-1}(\mathbb{R}^{d+1})) \cap L^2(L^2_x) \). For each \( t < \frac{r}{2} \), we can define \( u \) by

\[
u(t, y) = u_x(t, y) \quad \text{if} \quad (t, y) \in C(r, x) \]

where \( C(r, x) \) denotes the backward light cone of vertex \((r, x)\)

\[
C(r, x) = \{(t, y) \mid |y - x| \leq r - t \}.\]

We have only to make sure that if \( |x - x'| < 2r \) then for all \((t, y) \in C(r, x) \cap C(r, x')\), \( u_{x'}(t, y) = u_x(t, y) \). Let \( x_m = \frac{x + x'}{2} \) and \( r_m = r - |x - x_m| \) then \( C(r_m, x_m) = C(r, x) \cap C(r, x') \). Writing the energy estimate on \( w = u_x - u_{x'} \) in the cone \( C(r_m, x_m) \) and using the same computation as in the uniqueness result of [15] (which uses the smallness condition as well as the extra bound \( L^2(L^2_x) \)), we infer that \( u_x = u_{x'} \) in \( C(r_m, x_m) \).
3. Proof of theorem \[11\]

In order to avoid distracting dependencies on the dimension, we shall restrict to the case \(d = 4\). The proof proceeds through several reductions. We start with two solutions \(u\) and \(v\) of \[1\] with the same initial data \((u_0, u_1)\) defined on some time interval \([0, T)\) and such that \(\partial u, \partial v \in C([0, T); \dot{H}^1])\). Without loss of generality, we can assume that \(u\) is the solution which was obtained in the previous section (see also Shatah and Struwe \[13\]). This solution \(u\) is known to satisfy some extra estimates which will be useful in the proof.

Next, notice that to prove uniqueness, it is enough to prove that \(u\) and \(v\) coincide on some small time interval. Indeed, if we can prove that there exists \(\tau, 0 < \tau < T\) such that, \(\forall t, 0 < t < \tau\), we have \(u(t) = v(t)\) then by continuity, we deduce that \(u(\tau) = v(\tau)\) and we can iterate the argument to prove that \(\forall t, 0 < t < T\), we have \(u(t) = v(t)\).

Finally, to prove uniqueness it is sufficient to prove that \(u\) and \(v\) are equal on each backward light cone with a vertex \((t, x)\) such that \(0 < t < r\).

3.1. Gauge transform. Given a (now small) data \(\partial u_0\) and the wave map

\[ D_\alpha \partial^\alpha u = 0, \]

one can choose to work within the Coulomb gauge and take advantage of carefully chosen frames to obtain a new system of the form

\[
\begin{align*}
\Box q &= A \cdot \partial q + q \partial \cdot A + A^2 q + q (R(u)q^2) \\
\Delta A &= \nabla (A^2) + \nabla (R(u)q^2),
\end{align*}
\]

where we simplified the system to a model case where \(q\) is scalar, \(A\) is a vector and powers of \(A\) are to be understood as bilinear forms of its coefficients. To get a sense of perspective, one should see \(q \approx \partial u\), or more accurately any of the components of the 1-form \(du\), and \(R\) should be seen as the curvature of the target manifold, which we assume bounded along with all its derivative (target with “bounded geometry”).

For targets which are symmetric spaces, the coefficient \(R(u)\) just disappears. This new system can then be solved using an iteration scheme, using Strichartz estimates up to the end-point (thus, the restriction on \(n \geq 4\)), as is done in \[11\].

If one takes two small data which are the same, \(\partial u_0 = \partial v_0\), the reduced system for \(u\) and \(v\) will be the same, and in particular their respective data coincide. Hence, all there is to do is to actually prove uniqueness for the system \[2, 3\].
3.2. A model case. As we just saw, if the target happens to be a symmetric space, the renormalized wave map system reduces to the following simpler system:

\[
\begin{align*}
\Box q &= A \cdot \nabla q + q \nabla \cdot A + A^2 q + q^3 \\
\Delta A &= \nabla (A^2) + \nabla (q^2).
\end{align*}
\]

Essentially, the curvature term has disappeared in the elliptic equation (which we refer to as (RWMe) while the wave equation part will be (RWMh)), and we are left with a system involving only \(q\) and \(A\). Moreover, if we make a smallness assumption, \(A\) is entirely determined by \(q\), and in the present situation where (RWM) is derived from (I), we are under such an assumption.

**Theorem 3.1.** The system (RWM) has a unique small solution in the class \(\partial q \in L^\infty(L^2)\).

**Proof.** Recall that one can perform a fixed point in the class \(E = C(\dot{H}^1) \cap L^2(\dot{B}_6^{1/6,2})\) for \(q\), and \(F = C(\dot{H}^1) \cap L^1(\dot{B}_4^{1,1})\) for \(A\) ([11]). We therefore can prove uniqueness by comparing any solution \(v\) such that \(\partial v \in C(L^2)\) with the reference solution \(u \in E\) (and its associated \(A \in F\)).

**Remark 3.2.** Note that \(\tilde{F} = L^\infty(L^4) \cap L^1(L^\infty)\) is enough to solve (and this is what happens, mutatis mutandis, in [15]), but the additional regularity information we carry in \(F\) will be useful later.

The idea to obtain uniqueness is to write an estimate at a lower regularity level. This idea is recurrent when proving uniqueness for hyperbolic systems, since taking differences yields a loss of derivative. In fact, in [15], uniqueness for \(u \in C(\dot{H}^2) \cap L^2(L^8)\) is established in this way, writing a difference estimate in \(\dot{H}^1\). At the level of \(q\), this translate to uniqueness for \(q \in C(\dot{H}^1) \cap L^2(\dot{W}^{-1}_8)\) (though writing directly the estimate at the \(q\) level is most likely more involved than directly on the true system).

Consider \(\delta = q - q'\) the difference between two solutions, and \(\alpha = A - A'\) the difference between the vectors, and set \((q, A)\) to be the fixed point solution, namely the solution in \(E \times F\). The equation for \((\delta, \alpha)\) will be

\[
\begin{align*}
\Box \delta &\equiv A \cdot \nabla \delta + \alpha \nabla (q - \delta) + \delta \nabla A + (q - \delta) \nabla \alpha \\
&\quad + q \alpha (2A - \alpha) + \delta (A - \alpha)^2 + \delta (q^2 + q^3 + \delta^2) \\
\Delta \alpha &\equiv \nabla (2A \alpha - \alpha^2) + \nabla (2q \delta - \delta^2).
\end{align*}
\]
Remark 3.3. We are in a situation where $q \in L^\infty(\dot{H}^1)$, small, say $\lesssim \varepsilon_0$. From

$$A = |\nabla|^{-1}(A^2 + q^2),$$

we know that we can solve this elliptic equation:

$$\|A\|_{\dot{H}^1} \lesssim \|A\|_{L^4}^2 + \varepsilon_0^2,$$

hence

$$\|A\|_{L^4} \lesssim \|A\|_{L^4}^2 + \varepsilon_0^2,$$

which gives both $\|A\|_{L^4}$ and $\|A\|_{\dot{H}^1}$ small, and the same is true also for $q', A'$. We also have space-time estimates on $A$ from what we know on $q$: say $q \in L^2(\dot{B}_{1/6}^{1/3}, 2^{1/6})$, then $q^2 \in L^1(\dot{B}_{4}^{0,1})$, and we can write

$$\|A\|_{L^1(\dot{B}_{4}^{1/3})} \lesssim \|A\|_{L^1(\dot{B}_{4}^{1/3})} + \|q^2\|_{L^1(\dot{B}_{4}^{0,1})}.$$

Such an estimate will prove useful later, note that this immediately gives $A \in L^1L^\infty$.

We will write an estimate for $\delta$ in the following Strichartz space

$$X = C(\dot{H}^{1/4}) \cap L^2(\dot{B}_{6}^{-\frac{5}{2},2}).$$

For $\alpha$, one may think that, heuristically, $\alpha = |\nabla|^{-1}(\delta^2)$, and this leads to (using one factor $\delta$ in $C(\dot{H}^1)$ and the other factor $\delta$ in $L^2(\dot{B}_{6}^{-\frac{5}{2},2})$)

$$\alpha \in Z = L^2(\dot{B}_{12/7}^{1,2}) \hookrightarrow L^2(L^3).$$

Remark 3.4. Let us motivate the choice of $X$ and $Z$: if it was not for the derivatives, the model equation for $q$ would be $\Box q = q^3$. In [12], uniqueness for this equation is established for $\dot{H}^1$ data; this relies on a contraction estimate, with $\delta \in X_{-\frac{1}{4}}$ where $X_s = L^2(\dot{B}_{6}^{-s,2})$. In fact, one has some freedom in the choice of $X$, and any $X_s$ with $-1 < s \leq -\frac{1}{3}$ would do. However, from the embedding $\dot{H}^1 \hookrightarrow \dot{B}_{6}^{-\frac{3}{2},2}$ the choice $X_{-\frac{1}{4}}$ seems straightforward. In our setting, however, the source term is more like $|\nabla|^{-1}(\delta^2)|\nabla\delta$. Using our knowledge $\nabla\delta \in L^2$, the requirement on $|\nabla|^{-1}(\delta^2)$ becomes $L^2(\dot{B}_{6}^{\frac{3}{2},2})$; Part of the product in the source involves low frequencies of this term, producing the worst possible situation. Since we cannot afford to use Sobolev embedding, this requires $s + \frac{2}{3} \leq 0$. Thus one is led naturally to pick $s = -\frac{2}{3}$. This in turn requires to check that $\delta$ belongs to the chosen $X$ space, which we do below.

Lemma 3.5. If $u, v$ are two solutions of [RWM] such that $\partial u, \partial v \in C(L^2)$, then $\delta \in X$ and $\alpha \in Z$. 

Proof. Certainly $\delta, \alpha, q, A \in \dot{H}^1 \hookrightarrow L^4 \hookrightarrow \dot{B}^{-1/3,2}_6$. Using the equation and looking only at the worst possible term,

$$|\nabla|^{-1}(ab)\nabla c \in L^4 \hookrightarrow \dot{H}^{-1},$$

where $a, b, c$ are $\delta, \alpha, q, A$, and $ab \in (L^4)^2 = L^2$, hence $|\nabla|^{-1}(ab) \in \dot{H}^1 \hookrightarrow L^4$ and $\nabla c \in L^2$.

By Strichartz estimates, we deduce that $\delta \in C(\dot{H}^0) \cap L^2(\dot{B}^{-5/6,2})$ and by interpolation (recall we are local in time, $L^2_t$ is controlled by $L^\infty_t$) we obtain $\delta \in X$.

Remark 3.6. In all the remaining part of the paper, we will have to perform various product estimates, and rely heavily on the first part of the Appendix to do so. We refer to the Appendix for precise definitions of the LF/MF/HF interactions, in connection with paraproduct decomposition. We simply recall here that LF (resp. MF, HF) is meant for low frequencies (resp. medium, high) interactions between frequencies of factors in a product.

Now we check that similarly $\alpha \in Z$: using the elliptic equation again, one has to check that $\delta q \in L^2(\dot{B}^{-1/6,1/2}_6)$ and $q \in L^\infty(\dot{B}^{1/6}_2)$, therefore by (12) this interaction will be in $L^2(\dot{B}^{1/6,1/2}_2) \subset L^2(\dot{B}^{1/6,1/2}_2)$; the HF-LF interaction is just as fine since $\delta \in L^\infty(\dot{B}^{1/6}_2)$ and $q \in L^2(\dot{B}^{1/6}_2) \subset L^2(\dot{B}^{-1/6,1/2}_6)$ which results in a $L^2(\dot{B}^{1/6,1/2}_2)$ term from (12).

We have

$$\|\alpha\|z \lesssim \varepsilon_0\|\alpha\|z + \varepsilon_0\|x\| + \|\delta\|q\|L^2(\dot{B}^{-1/6}_6),$$

and the $\delta^2$ is just as easy, HF in $L^\infty(\dot{H}^1)$ and MF in $L^2(\dot{B}^{-2/3,2}_6)$. This ends the proof of the lemma.

We now aim at closing an estimate in $X$. We will prove

Proposition 3.7. Let $\delta = u - v$ be the difference of two solutions with the same initial data. Then

$$\|\delta\|X \lesssim \varepsilon_0\|x\|,$$

from which we can infer $\delta = 0$.

Proof. The source term in the equation on $\delta$ should be anywhere (in term of interpolation) between $L^2(L^{6/5})$ and $L^1(\dot{H}^{-5/6})$, which are then pulled back to $X$ by $\Box^{-1}$. Indeed, let us denote $X' = L^2(L^{6/5}) + L^1(\dot{H}^{-5/6})$ and recall the following end-point Strichartz estimate: let
\[ \Box \delta = F \text{ and } (\delta, \partial_t \delta)(t = 0) = 0, \text{ then} \]

\[ \|\delta\|_X \lesssim \|F\|_{X'} \tag{5} \]

Let us go through each term appearing in the right-hand side of (\Delta RWM).

- The easy terms: \( \alpha \nabla (q - \delta), (\nabla \alpha)(q - \delta) \). These can all be dealt with by Sobolev embedding and Hölder inequality.

As \( \alpha \in L^2(L^3) \) and \( \nabla \delta, \nabla q \in L^\infty L^2 \) we have the corresponding term in \( L^2(L^{6/5}) \). Similarly, \( \alpha \in L^2(\dot{B}^{1,2}_{12/7}) \) and \( q - \delta \in L^\infty(L^4) \) give \( L^2(L^{6/5}) \). Hence, we have

\[ \|\alpha \nabla (q - \delta) + (q - \delta) \nabla \alpha\|_{X'} \lesssim \|q - \delta\|_{E} \|\alpha\|_{Z} \lesssim \epsilon_0 \|\delta\|_{X}. \tag{6} \]

- The term \( \delta^3 \): interpolation between \( \delta \in C(\dot{H}^{1}) \) and \( \delta \in L^2(\dot{B}^{-2/3}_6) \) yields \( \delta \in L^4(\dot{B}^{1/2}_{12/7}) \hookrightarrow L^4(L^{24/7}) \), hence \( \delta^2 \in L^2(L^{12/7}) \). Then using that \( \delta \in L^\infty(L^4) \), one gets that \( \delta^3 \in L^2(L^{6/5}) \), namely

\[ \|\delta^3\|_{X'} \lesssim \|\delta\|_{E}^2 \|\delta\|_{X}. \]

- The two terms \( A \nabla \delta \) and \( \delta \nabla A \): here, one has to perform a paraproduct decomposition and deal with the different frequency interactions in a suitable way. We follow the conventions set up in the Appendix for the different interactions in the paraproduct decomposition of a product.

- LF-HF interaction: \( \nabla \delta \in C(\dot{H}^{-5/6}) \), hence we are forced to have \( A \in L^1(L^\infty) \), which is fortunately true since \( A \) is the good connection from local Cauchy theory (recall actually \( A \in L^1(\dot{B}^{1,1}_1) \)).

- HF-MF interaction: same information on \( \delta \), but using \( A \in L^1(\dot{B}^{1,1}_1) \) (so that \( 1 + 5/6 > 0 \)) and embedding. Notice how we need the regularity on \( A \) \( (A \in L^1(L^\infty) \) would be too weak).

We thus have

\[ \|A \nabla \delta\|_{X'} \lesssim \|A\|_{F} \|\delta\|_{X}. \]

For \( (\nabla A)\delta \), we proceed similarly:

- LF-HF interaction: \( \delta \in C(\dot{H}^{1/6}) \) and \( \nabla A \in L^1(\dot{B}^{-1,1}_\infty) \) yields a term in \( L^1(\dot{H}^{-5/6}) \).

- HF-LF interaction: similarly, \( \nabla A \in L^1(L^4), \delta \in C(L^{24/11}) \) yields a term in \( L^1(\dot{H}^{-5/6}) \) after embedding.

- HF-HF interaction: again, \( \nabla A \in L^1(L^4) \) and \( \delta \in C(\dot{H}^{1/6}) \) yields a term in \( L^1(\dot{H}^{-5/6}) \) after embedding.
Thus, we obtain
\[ \| \delta \nabla A \|_{X'} \lesssim \| A \|_F \| \delta \|_X. \]

- The terms \( \delta^2 q \) and \( \delta q^2 \); we only proceed with the details of \( \delta q^2 \), \( \delta^2 q \) being easier: \( q^2 \) is just like \( \nabla A \), that is \( q^2 \in L^1(L^4) \), hence we do exactly as the previous one \( \delta \nabla A \).

Summing all the above estimates, we get
\[ \| \delta \|_X \lesssim \| \text{R.H.S} \|_{X'} \lesssim \varepsilon_0 \| \delta \|_X, \]
where R.H.S. denotes the source term in \((\Delta \text{RWM})\), which ends the proof.

3.3. The general case. We start with two solutions \( u \) and \( v \) such that \( \partial u \) and \( \partial v \) are in \( C((\dot{H}^1)) \). Without loss of generality, we can assume that \( u \) is the solution constructed by Shatah and Struwe in [15]. The argument given in the last section was based on some smallness condition. Using the finite speed of propagation for the wave equation we will reduce our problem to the small case. We choose \( r > 0 \) small enough that
\[ \sup_{x \in \mathbb{R}^d} \| \partial u(0) \|_{\dot{H}^1(B(x,r))} \leq \varepsilon_0, \]
where \( \varepsilon_0 \) is a small parameter which will be chosen later.

Next, using the continuity of \( u \) and \( v \) with respect to time, we can choose \( \tau, 0 < \tau \leq \frac{r}{2} \) such that
\[ \forall t, 0 \leq t \leq \tau, \| \partial u(t) - \partial u(0) \|_{\dot{H}^1} + \| \partial v(t) - \partial u(0) \|_{\dot{H}^1} \leq \varepsilon_0. \]

To prove that \( u \) and \( v \) coincide for all \( 0 \leq t < \tau \), it is sufficient to prove that they coincide on each truncated backward light cone of the form
\[ C^+_0(x_0, r) = \{(t, x) / \| x - x_0 \| \leq r - t , 0 \leq t \leq \tau \}. \]

In the sequel, we restrict ourselves to the backward light cone of center \( x_0 = 0 \). The proof will be reduced to the proof given in the model case. There are only two extra difficulties we have to handle.

The first difficulty lies in the choice of some coordinate system where we can write our equation in a form similar to \((RWM)\). As in [15], we have to choose a frame \( e \) and a connection \( A \) satisfying the Coulomb gauge and express \( \partial u \) in that frame using the coordinates \( q \). Moreover, to get good estimates for \( \alpha = A - A' \) and \( e - e' \) in terms of \( \delta = q - q' \) we have to construct local frames. The second difficulty comes from the fact that the curvature tensor is no longer constant and an extra term \( R(u) \) will appear in the equation \((RWM)\) and hence we have to estimate \( R(u) - R(v) \) in terms of \( \delta \).
Let us start by constructing the the local frame \( e \) and \( e' \) associated respectively to \( u \) and \( v \).

3.3.1. Construction of the frame. Without loss of generality, we can assume that \( T N \) is parallelizable; hence, we can find smooth vector fields \( \bar{e}_1, ..., \bar{e}_k \) such that at each \( p \in N \) the family \( \{\bar{e}_1, ..., \bar{e}_k\} \) is an orthonormal basis of \( T_p N \) (see for instance [4]). Given the map \( u \) or \( v \) from \( R^{d+1} \) into \( N \), the family \( \{\bar{e}_1 \circ u, ..., \bar{e}_k \circ u\} \) is a smooth orthonormal frame of the pull-back bundle \( u^* T N \). Moreover, we may freely rotate this frame at any point \((t, x) \in R^{d+1}\) with a matrix \( (R_{ba}^a(z)) \in SO(k) \), thus obtaining the frame

\[
e_a = R^b_a \bar{e}_b \circ u, \quad 1 \leq a \leq k.
\]

For our uniqueness proof, it will be important that \( R_{ba}^a \) only depend on the solution in the cone \( C^\tau_0(x_0, r) \). We choose \( R_{ba}^a \) in the truncated cone \( C^\tau \) by minimizing for each time \( 0 \leq t < \tau \) the following functional

\[
F(R) = \int_{B(0, r-t)} \sum_{i=1}^{d} \sum_{a,b=1}^{k} \left( \frac{\partial e_a}{\partial x^i}, e_b \right)^2 \, dx
\]

The existence of a minimizer can be proved following the same proof as in [4]. Moreover, denoting \( A_{b,a}^a = \left( \frac{\partial e_a}{\partial x^i}, e_b \right) \) for \( 1 \leq \alpha \leq d \) and \( 1 \leq a, b \leq k \), we get the following Euler-Lagrange equation

\[
\begin{align*}
\partial_\alpha A_{b,a}^a &= 0 \quad \text{in} \quad B(0, r-t) \\
A_{b,a}^a n_\alpha &= 0 \quad \text{on} \quad \partial B(0, r-t)
\end{align*}
\]

where \( n \) denotes the normal to the ball \( B(0, a-t) \). We need an extra equation to determine \( A \), which we can get from the curvature of the pull-back covariant derivative \( D = (D_\alpha)_{0 \leq \alpha \leq d} \). Indeed, using that the Lie bracket between \( D_\alpha \) and \( D_\beta \) vanishes, we get that for all \( 1 \leq a, b \leq k \)

\[
\partial_\alpha A_{b,\beta}^a - \partial_\beta A_{b,\alpha}^a + [A_\alpha, A_\beta]^a_b = R(u)(\partial_\alpha u, \partial_\beta u).
\]

Expressing \( du \) in the frame \( e \), we get \( \partial_\alpha u = q_\alpha^a e_a \). On the other hand, written in the \( q \) coordinate system, the wave map equation yields

\[
\square q_\beta \equiv 2A^\alpha \partial_\alpha q_\beta + (\partial^\alpha A_\alpha) q_\beta + A^\alpha A_\alpha q_\beta + F_{\alpha}^\beta q_\alpha.
\]

Notice that the system of equations we obtained, namely (11), (9) and (10) is very similar to the model problem we studied in the previous subsection.

The construction of the frame \( e \), the connection \( A \) and the components \( q \) of \( du \) can be carried out also for the solution \( v \). We denote \( e' \),
$A'$ and $q'$ the resulting frame, connection and components of $dv$. In the sequel, we denote $\delta = q - q'$, $\alpha = A - A'$. Using the different equations we have at hand, we will estimate $du - dv$ and $\alpha$ in terms of $\delta$ and then prove a closed estimate for $\delta$ from which we deduce that $\delta$ should vanish as well as $du - dv$ and $\alpha$.

3.3.2. Proof of Theorem 1.1. In the generic situation, one has (as a model) the following two equations: the first one is the wave equation (11) holding inside a space-time cone, and the second is an elliptic equation. It is a short-hand for what is really an elliptic div-curl system ((9) and (10)) holding on fixed time slices with appropriate boundary conditions. The elliptic theory yields the same regularity estimates in that case as in our simplified model with a Laplacian.

\begin{align*}
\delta & = q - q' \\
\alpha & = A - A'.
\end{align*}

Using the different equations we have at hand, we will estimate $du - dv$ and $\alpha$ in terms of $\delta$ and then prove a closed estimate for $\delta$ from which we deduce that $\delta$ should vanish as well as $du - dv$ and $\alpha$.

Provided we seek an estimate on $R(u) - R(u')$ such that we only use $R' \in L_\infty^{1,}\infty$, we are left with the other factor, namely $(u - u')$. However, we can control $\partial(u - u')$ by $\delta$, and it turns out to be sufficient to close the estimates. The new system for $(\delta, \alpha)$ is (denoting by $w = u - u'$)

$R(u) - R(u') = (u - u') \int_0^1 R'(\theta u + (1 - \theta) u') d\theta,$

Thus the modification appears in the elliptic equation on the connection. From the computation in the previous section, one infers that $2q\delta - \delta^2 \in L^2(L^{12})$, and combined with $R \in L^\infty(L^{\infty})$ we dispose of the first term with $R$ as we did in the model case. The next term is the real novelty here. Assuming that $\partial w \equiv \delta$, we get that

$w \in L^2(\dot{B}^{1/2}_{\infty})) \hookrightarrow L^2(L^{12}),$

and using $q^2 \in L^\infty(L^2)$, we get the desired $L^2(L^{12})$ estimate for the source term.
All other terms may be estimated like in the model case, and we can therefore close an estimate on $\delta$ as we did in the previous section, up to localization to the interior of the light cone for space-time estimates and localization to balls for elliptic estimates. Fortunately, all the required estimates may easily be transposed in such a situation, as explained in Appendix B. This ends the proof.

Acknowledgments

The first author was partially supported by NFS grant DMS-0703145. The second author wishes to thank the Courant institute of mathematical science for its kind hospitality during a visit where this work was initiated; he was partially supported by A.N.R. grant SWAP.

Appendices

Appendix A. Product estimates

In this appendix we describe various product estimates in Besov spaces, which are used throughout the rest of the paper. We do not claim novelty here, but we do however emphasize that thinking about the product in terms of different frequency interactions is crucial in our situation.

Proposition A.1. Let $f \in \dot{B}_{p_1}^{s_1,q_1} = B_1$ and $g \in \dot{B}_{p_2}^{s_2,q_2} = B_2$. Assume $s_i - \frac{d}{p_i} < 0$, define $r_i$ such that $s_i - \frac{d}{p_i} = -\frac{d}{r_i}$ (Sobolev embedding exponent if $s_i > 0$), and assume moreover that $\frac{1}{r_1} + \frac{1}{r_2} < 1$.

1. Suppose $s_1 > 0$ and $s_2 < 0$, and $r_1 \geq q_1$. Then, $fg = \pi_1 + \pi_2$ where $\pi_1 \in \dot{B}_{p_1}^{s_1 + s_2, q}$, $\pi_2 \in \dot{B}_{p_2}^{s_2, q_2}$, with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{P_2} = \frac{1}{p_2} + \frac{1}{r_1},$$

and

$$\|\pi_1\|_{\dot{B}_{p_1}^{s_1 + s_2, q}} + \|\pi_2\|_{\dot{B}_{p_2}^{s_2, q_2}} \lesssim \|f\|_{B_1} \|g\|_{B_2}. \tag{12}$$

We call $\pi_1$ the high-medium frequencies interaction term, the high frequencies referring to the $f$ factor and the medium to the $g$ factor. We abbreviate it to $HF - MF$ (or $MF - HF$ if $f$ and $g$ are switched). Similarly, $\pi_2$ is the low-high frequencies interaction term, or $LF - HF$ for short.
Suppose $s_1, s_2 > 0$ and $r_i \geq q_i$, then $fg = \pi_1 + \pi_2 + \pi_3$ where
\[
\pi_3 \in \dot{B}^{s_1+s_2,q}_{p,P_1}, \quad \pi_1 \in \dot{B}^{s_1,q}_{P_1}, \quad \pi_2 \in \dot{B}^{s_2,q}_{P_2},
\]
with $p, q, P_2$ as above, $\frac{1}{P_1} = \frac{1}{p_1} + \frac{1}{r_2}$ and
\[
\|\pi_3\|_{\dot{B}^{s_1+s_2,q}_{p}} + \|\pi_1\|_{\dot{B}^{s_1,q}_{p_1}} + \|\pi_2\|_{\dot{B}^{s_2,q}_{P_2}} \lesssim \|f\|_{B_1}\|g\|_{B_2}.
\]
We refer to $\pi_1$ as the $HF - LF$ term, $\pi_2$ as the $LF - HF$ term and $\pi_3$ as the $HF - HF$ term, similarly to the previous case.

Such product estimates are classical, see e.g. [13]. Consider the first case: we decompose $fg$ as
\[
fg = \pi_1 + \pi_2 = \sum_j S_{j+2}g\Delta_jf + \sum_j S_{j-1}f\Delta_jg.
\]
The term $\pi_2$ is a sum of frequency localized pieces, meaning that for a finite number of $k$ close to $j$,
\[
\Delta_j\pi_2 = \sum_{k \approx j} \Delta_j(S_{k-1}f\Delta_kg).
\]
For convenience we only deal with the $k = j$ term. For the low frequencies $S_{j-2}f$, we use Sobolev embedding and the fact that $r_1 \geq q_1$ to get
\[
\|S_{j-2}f\|_{r_1} \lesssim \|f\|_{B_1}.
\]
For the high frequencies $\Delta_jg$,
\[
\|\Delta_jg\|_{p_2} \lesssim 2^{s_j} \varepsilon_j \|g\|_{B_2},
\]
where $\varepsilon_j \in l^{q_2}$. The result follows by Hölder.

The other term $\pi_1$ is a sum of dyadic terms localized in balls of radius $2^j$. We estimate
\[
\Delta_j\pi_1 = \sum_{j \leq k} \Delta_j(\Delta_kfS_{k+2}g),
\]
and, since $s_2 < 0$ and recalling $S_j = \sum_{l < j} \Delta_l$,
\[
\|S_{j+2}g\|_{p_2} \lesssim 2^{-s_2} \mu_j \|g\|_{B_2},
\]
with $\mu_j \in l^{q_2}$. Thus
\[
\|\Delta_j\pi_1\|_p \lesssim \sum_{j \leq k} 2^{-(s_1+s_2)k} \mu_k \eta_k \|g\|_{B_2} \|f\|_{B_1} = 2^{-(s_1+s_2)j} \lambda_j \|g\|_{B_2} \|f\|_{B_1},
\]
with $\lambda_j \in l^t$, and we are done.

The other case proceeds similarly, except we use the full paraproduct decomposition, namely
\[
fg = \pi_1 + \pi_2 + \pi_3 = \sum_j S_{j-1}g\Delta_jf + \sum_j S_{j-1}f\Delta_jg + \sum_{|k-k'| \leq 2} \Delta_kf\Delta_{k'}g.
\]
The first two terms are treated like the term $\pi_2$, and the term $\pi_3$ is treated as the term $\pi_1$.

**Appendix B. Localizing Besov spaces**

In our context and in order to take advantage of the finite speed of propagation, one wishes to localize all the usual estimates to a backward light cone. Such a procedure is well-known in the context of the critical semilinear wave equation: see for example [14], where an explicit extension procedure is given to achieve this goal.

In our setting we do not need to worry about space-time Besov spaces and the geometry of cones: we are always interested in estimates in a truncated backward cone (avoiding the tip of the cone). Denote by $(x_0, t_0)$ the vertex of such a backward cone $C(x_0, t_0) \ (t_0 > 0)$, we consider a slab $C_0^a(x_0, t_0) = C(x_0, t_0) \cap \mathbb{R}^d \times [0, a]$. For each $t \in [0, a]$, denote by $B_t$ the corresponding time slice of $C_0^a(x_0, t_0)$: this (space) ball or radius $t$ is a smooth domain of $\mathbb{R}^d$.

**Definition B.1** ([19]). A function $f(x) \in \mathcal{D}'(B_t)$ belongs to the (spatial) Besov space $\dot{B}^{s,a}_p(B_t)$ iff there exists $g \in \dot{B}^{s,a}_p(\mathbb{R}^d)$ such that $f$ is the restriction of $g$ to $B_t$ (as distributions). The norm of $f$ is then the minimum over all possible extensions $g$ of their Besov norm in $\mathbb{R}^d$.

The main property we need is the existence of an extension operator: if we call $R$ the restriction operator, i.e. $f = Rg$, then there exists an operator $E$ from $\dot{B}^{s,a}_p(B_t)$ to $\dot{B}^{s,a}_p(\mathbb{R}^d)$ such that $f = Ef$. Moreover, this extension operator can be chosen to be the same whenever $(p, q, s)$ are in a bounded domain (which is always the case for us). We refer again to [19] for a detailed presentation.

Next, for a given space-time function $u(x, t)$, we can define $u \in L_T^p(\dot{B}^{s,a}_p(B_t))$ by

$$\int_0^T \|u(\cdot, t)\|_{\dot{B}^{s,a}_p(B_t)}^p \, dt < +\infty.$$  

After these preliminaries, we can localize estimates in this way:

- **Product estimates.**
  Let $f_1 \in L_T^r \dot{B}^{s_1,q_1}_p(B_t)$, $f_2 \in L_T^r \dot{B}^{s_2,q_2}_p(B_t)$. Then $f_1 f_2 \in L_T^r \dot{B}^{s,a}_p(B_t)$ and
  $$\|f_1 f_2\|_{L_T^r \dot{B}^{s,a}_p(B_t)} \lesssim \|f_1\|_{L_T^r \dot{B}^{s_1,q_1}_p(B_t)} \|f_2\|_{L_T^r \dot{B}^{s_2,q_2}_p(B_t)},$$
  where $r, s, p, q$ are the same as in $\mathbb{R}^d$. In fact, we have $g_1$ and $g_2$ the extensions of $f_1$ and $f_2$, we perform the product $g_1 g_2$ and then we have $f_1 f_2 = R(g_1 g_2)$, and the inequality between the two norms.
• Linear estimates for the wave equation.

Consider first the inhomogeneous wave equation, \( \square u = F \), with zero Cauchy data at time \( t = 0 \). By finite speed of propagation, \( u \) inside the backward cone \( C(t_0, x_0) \) depends only on \( F \) inside the same region. Moreover (causality), \( u \) at \( t = a \) depends only on \( F \) on \( C(a, x_0) \). Given \( F \in L^p_{\nu}(\dot{B}^{s_2}_{\nu}(\mathbb{R}^d)) \) where \((\nu', \nu')\) is a dual admissible Strichartz pair, we can extend it to an \( \tilde{F} \in L^{p'}_{\nu'}(\dot{B}^{s_2}_{\nu'}(\mathbb{R}^d)) \), apply Strichartz estimates, recover \( \tilde{u} = \square^{-1}\tilde{F} \) such that \( \tilde{u} \in L^\lambda_{\mu}(\dot{B}^{s_2}_{\mu}(\mathbb{R}^d)) \) where \((\lambda, \mu)\) is any admissible Strichartz pair and \( u = R\tilde{u} \). Therefore,

\[
\|u\|_{L^\lambda_{\mu}(\dot{B}^{s_2}_{\mu}(\mathbb{R}^d))} \lesssim \|F\|_{L^{p'}_{\nu'}(\dot{B}^{s_2}_{\nu'}(\mathbb{R}^d))}.
\]

One can proceed similarly for the data to obtain the full range of estimates for the Cauchy problem.

Combining these two observations, we can localize all the estimates from Subsection 3.2 without modification, whenever we are facing a product of functions or an estimate on a solution to the wave equation (through the use of the Duhamel formula).

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