ANY SASAKIAN STRUCTURE IS APPROXIMATED BY EMBEDDINGS INTO SPHERES

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ABSTRACT. We show that, for any given \( q \geq 0 \), any Sasakian structure on a closed manifold \( M \) is approximated in the \( C^q \)-norm by structures induced by CR embeddings into weighted Sasakian spheres. In order to obtain this result, we also strengthen the approximation of an orbifold Kähler form by projectively induced ones given in [21] in the \( C^0 \)-norm to a \( C^q \)-approximation.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

Sasakian geometry is an established field of research in geometry and physics which has received ever growing interest in the past decades. The focus on Sasakian geometry is partly justified by its connection to the AdS/CFT theory from the physical point of view. From a geometric standpoint the interests is also justified by the abundance of structures underlying a Sasakian manifold and the connection to Kähler geometry. In particular, as Kähler geometry lies in the intersection of complex, symplectic and Riemannian geometry, a Sasakian structure requires the compatibility of CR, contact and Riemannian structures. This allows several different approaches to the study of Sasakian geometry. In fact, a Sasakian manifold comes equipped with a characteristic vector field (the Reeb vector field) and a transverse Kähler geometry. When the leaves of the Reeb vector field are compact, its orbit space is a Kähler orbifold. A fruitful approach is then to study the Kähler base to obtain informations on the Sasakian manifold itself, see for instance [1, 3, 11, 18]. Other techniques involve the study of the Kähler cone over a Sasakian manifold. These methods do not require the compactness of the orbits of the Reeb vector field and have proved to be equally effective, see e.g. [4, 6, 13]. In this paper we take advantage of both approaches and their interplay.

Given their close relation, many interesting problems in Sasakian geometry originate by analogy from its older sister, Kähler geometry. This is the case in our paper. Namely, we are motivated by a classical result of Tian [25] on \( C^2 \)-approximations of polarized Kähler metrics, later improved by Ruan and Zelditch [22, 26] to a smooth approximation, and its extension due to Ross and Thomas [21] to the orbifold case. In particular, in [21] it is proven that any Kähler form on a polarized Kähler orbifold with cyclic quotient singularities can be continuously approximated by embeddings into weighted projective spaces. It is known that not every polarized Kähler manifold admits an isometric embedding into a complex projective space \( \mathbb{C}P^N \).

In this paper we are interested in the analogous problem in Sasakian geometry. Namely, is there a model space such that all Sasakian structures are approximations of structures induced by embeddings in such a model space? The natural candidate for such a space is a (weighted) Sasakian sphere as it is the Sasakian manifold whose Kähler base is a (weighted) complex projective space.

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Such Sasakian structures on spheres are obtained as simple deformations of the standard Sasakian structure, cf. Example 7. Our main theorem is the following $C^q$-approximation theorem.

**Theorem 1.** Let $(M, \eta, g)$ be a compact Sasakian manifold and fix $q \in \mathbb{N}$. Then there exist a sequence of Sasakian structures induced by CR embeddings $\varphi_k : M \rightarrow S^{2N_k+1}$ into weighted Sasakian spheres which converge to $(\eta, g)$ in the $C^q$-norm.

Let us briefly comment this result. In analogy with the Kähler setting, the dimension $2N_k + 1$ of the model space tends to infinity as $k \rightarrow \infty$. The embeddings of Theorem 1 do not preserve the Reeb vector field. In fact, the Reeb vector field is the conjugate to the radial direction on the Kähler cone over $M$. This is determined by the hermitian metric on the cone (see Equation (3)) so it is not preserved unless the embedding is in fact a Sasakian embedding. Notice that there is no assumption on the regularity of the Sasakian structure in Theorem 1. In fact, there is no assumption other than compactness of the Sasakian manifold.

The main ingredients in the proof of Theorem 1 are:

1. $C^\infty$-approximation of irregular structures by quasi-regular ones due to Rukimbira [23].
2. $C^q$-approximation of a quasi-regular Sasakian structure by Sasakian structures induced by embeddings into weighted Sasakian spheres.

What allows us to get the convergence is the fact that the Kähler cone does not vary with the structures in point (1). In fact, Theorem 1 can be rephrased as a statement on embeddings of the Kähler cone of $M$ into $\mathbb{C}^{N_k+1} \setminus \{0\}$. Moreover, the structures induced via embedding are simple deformations (see Definition 6 below) of the original one. All these deformations do not change the CR structure so we get a sequence of CR embeddings into weighted Sasakian spheres that approximate a given structure. This is in analogy with the Kähler setting where the embeddings are holomorphic, since the CR structure can be regarded as the transverse holomorphic structure.

When the structure $(M, \eta, g)$ is quasi-regular, the space of leaves is a polarized Kähler orbifold $X$. This allows us to work on the Kähler geometry of the base $X$ and approximate it by embeddings into projective spaces. Such approximations given by the asymptotic of the Bergman kernel for the Kodaira-Baily embedding were studied by Dia, Liu and Ma in [5]. Unfortunately, this does not suit our needs as the Sasakian structures so obtained would be regular. Thus we require an orbifold embedding (cf. [21, Section 2.1]) in a weighted projective space. This was done by Ross and Thomas [21] in the $C^2$-norm. The main technical difficulty of part (2) is to extend their result to a $C^q$-convergence. This is done in Theorem 2.

When the Sasakian structure on $M$ is regular, that is, the Reeb action is free, it is natural to ask whether one can get a similar result under the requirement that the model space is a standard Sasakian sphere. We address this problem, both in the compact and noncompact setting, in a related work [19].

**Related Works.** Recently, a closely related problem was considered by Herrmann, Hsiao, Marinescu, and Shen in [10]. There, using different methods, the authors provide several results on CR embeddings into spheres with the weighted CR structure. Remarkably, they obtain smooth approximation of CR structures. In particular, one should compare our Theorem 1 with [10, Theorem 6.5] One should compare Theorem 1 also with the work of Ornea and Verbitsky [14, 15, 16, 17] dealing with CR embeddings of Sasakian manifolds into spheres. In particular, our result can be regarded as a strengthening of [17, Theorem 11.21]. More generally, CR embeddings $\Phi_m : M \rightarrow \mathbb{C}^{N_m}$ of any CR manifold $M$ were produced in [8, 9] for all $m$ sufficiently large. Moreover, the authors studied the asymptotic of the Szegő kernel. However, this cannot be applied to the Sasakian case.
in a straightforward manner as the Szegő kernel considered in [8, 9] relates to the pullback of the standard metric of $\mathbb{C}^{N_m}$ rather than the cone metric inducing the Sasakian structure on the weighted sphere.

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2 we review Sasakian geometry and define the basic objects needed in the remainder of the article. In Section 3 we recall some notions on weighted Bergman kernels and their asymptotic expansions from [21] and use them to prove Theorem 2 in Section 4. Finally Section 5 is dedicated to the proof of Theorem 1.

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2. Sasaki manifolds and Kähler cones

Sasakian geometry can be presented in terms of contact metric geometry as in the monograph of Boyer and Galicki [2] or via the associated Kähler cone, see e.g. [7, 13, 24] for easily readable references. We will present both formulations because a combination of the two is most suitable to our purpose. In the following all manifolds and orbifolds are assumed to be connected, closed and oriented.

A K-contact structure $(\eta, \Phi, R, g)$ on a manifold $M$ consists of a contact form $\eta$ and an endomorphism $\Phi$ of the tangent bundle $TM$ satisfying the following properties:

- $\Phi^2 = -\text{Id} + R \otimes \eta$ where $R$ is the Reeb vector field of $\eta$,
- $\Phi|_D$ is an almost complex structure compatible with the symplectic form $d\eta$ on $D = \ker \eta$,
- the Reeb vector field $R$ is Killing with respect to the metric $g = \frac{1}{2}d\eta \circ \text{Id} \otimes \Phi + \eta \otimes \eta$.

Given such a structure one can consider the almost complex structure $J$ on the Riemannian cone $(M \times \mathbb{R}^+, t^2g + dt^2)$ given by

- $J = \Phi$ on $D = \ker \eta$, and
- $R = J(t\partial_t)|_{\{t=1\}}$.

A Sasakian structure is a K-contact structure $(\eta, \Phi, R, g)$ such that the associated almost complex structure $J$ is integrable, and therefore $(M \times \mathbb{R}^+, t^2g + dt^2, J)$ is Kähler. A Sasakian manifold is a manifold $M$ equipped with a Sasakian structure $(\eta, \Phi, R, g)$.

Equivalently, one can define Sasakian manifolds in terms of Kähler cones. Namely, a Sasakian structure on a compact smooth manifold $M$ is defined to be a Kähler cone structure on $Y := M \times \mathbb{R}^+$. That is, A Kähler structure $(g_Y, J)$ on $Y$ of the form $g_Y = t^2g + dt^2$ where $t$ is the coordinate on $\mathbb{R}^+$ and $g$ a metric on $M$. Then $(Y, g_Y, J)$ is called the Kähler cone of $M$ which, in turn, is identified with the submanifold $\{t = 1\}$. The Kähler form on $Y$ is then given by

$$\Omega_Y = \frac{i}{2} \partial \bar{\partial} t^2.$$  

The Reeb vector field on $Y$ is defined as

$$R = J(t\partial_t).$$

This defines a holomorphic Killing vector field on $Y$ with metric dual 1-form

$$\eta = g_Y(R, \cdot) = \frac{g_Y(R, \cdot)}{t^2} = d^c \log t = i(\partial - \bar{\partial}) \log t$$

where $d^c = i(\partial - \bar{\partial})$. With a slight abuse of notation we write $R$ and $\eta$ to indicate both the objects on $Y$ and their restrictions to $M$. Notice that $J$ induces an endomorphism $\Phi$ of $TM$ by setting
\[ \Phi = J \text{ on } \mathcal{D} = \ker \eta_{\text{TM}}, \text{ and} \]
\[ \Phi(R) = 0. \]

Equivalently, the endomorphism \( \Phi \) is determined by \( g \) and \( \eta \) by setting
\[ g(X, Z) = \frac{1}{2} \langle \eta(X, \Phi Z) \rangle \text{ for } X, Z \in \mathcal{D}. \]

It is easy to see that, when restricted to \( M = \{ t = 1 \} \), \( (\eta, \Phi, R, g) \) is a Sasakian structure in the contact metric sense whose Kähler cone is \( \omega(1) \).

In particular, we have
\[ \omega^T = \frac{1}{2} \partial \eta, \quad J^T = \Phi_{|D}, \text{ and } g^T(X, Z) = \frac{1}{2} \partial \eta(X, J^TZ) = g|_\mathcal{D}. \]

In particular, we have
\begin{equation}
\omega^T = \frac{1}{2} \partial \eta = \frac{i}{2} \partial(\overline{\partial} - \partial) \log t = \frac{i}{2} \overline{\partial} \partial \log t^2.
\end{equation}

Regular and quasi-regular Sasakian manifold are fairly well understood due to the following result

**Theorem** ([2]). *Let \((M, \eta, \Phi, R, g)\) be a quasi-regular compact Sasakian manifold. Then the space of leaves of the Reeb foliation \((X, \omega)\) is a compact Kähler cyclic orbifold with integral Kähler form \( \frac{1}{2} \omega \) so that the projection \( \pi : M \rightarrow X \) is a Riemannian submersion. Moreover, \( X \) is a smooth manifold if and only if the Sasakian structure on \( M \) is regular.*

*Viceversa, any principal \( S^1 \)-orbibundle \( M \) with Euler class \(-\frac{1}{\pi} [\omega] \in H^2_{\text{orb}}(X, \mathbb{Z})\) over a compact Kähler cyclic orbifold \((X, \omega)\) admits a Sasakian structure.*

We refer to [2] for the basics of Kähler orbifolds too. This result allows us to reformulate the geometry of a quasi-regular Sasakian manifold \( M \) in terms of the Kähler geometry of the space of leaves \( X \). We will illustrate in detail this correspondence for its importance in the remainder of the paper. In order to do so, let us first introduce the concept of \( \mathcal{D} \)-homothetic transformation of a Sasakian structure.

**Definition 1** (\( \mathcal{D} \)-homothety or transverse homothety [2, Section 7.3]). *Let \((M, \eta, \Phi, R, g)\) be a Sasakian manifold and \( a \in \mathbb{R} \) a positive number. One can define the Sasakian structure \((\eta_a, \Phi_a, R_a, g_a)\) from \((\eta, \Phi, R, g)\) as
\[ \eta_a = a \eta, \quad \Phi_a = \Phi, \quad R_a = \frac{R}{a}, \quad g_a = ag + (a^2 - a) \eta \otimes \eta = ag^T + \eta_a \otimes \eta_a. \]

Equivalently, the Sasakian structure \((\eta_a, \Phi_a, R_a, g_a)\) on \( M \) can be obtained from the Kähler cone by setting the new coordinate \( t = t^a \).

Now let the quasi-regular Sasakian manifold \((M, \eta, \Phi, R, g)\) be given and consider the projection \( \pi : (M, g) \rightarrow (X, \omega) \) given above. Notice that \( \pi \) locally identifies the contact distribution \( \mathcal{D} \) with the tangent space of \( X \). Therefore, up to \( \mathcal{D} \)-homothety, we have that \( \pi^*(\omega) = \frac{1}{2} \partial \eta \). Moreover,
the endomorphism $\Phi$ determines the complex structure on $X$ and $g$ induces the Kähler metric $g_\omega$ compatible with $\omega$, i.e. $g^T = \pi^* g_\omega$.

In this case the class $\frac{1}{\pi} [\omega] \in H^2_{orb}(X, \mathbb{Z})$ defines an orbisample line bundle over $X$ in the sense of [21, Definition 2.7]. Moreover, the cone $Y = M \times \mathbb{R}^+$ is identified with $L^*$ without the zero section in the following way. Let $h$ be a hermitian metric on $L$ such that

$$\omega = -\frac{i}{2} \partial \overline{\partial} \log h. \tag{2}$$

Then its dual $h^*$ on $L^*$ defines the second coordinate $(p, t) \in M \times \mathbb{R}^+ = L^* \setminus \{0\}$ by

$$t : L^* \setminus \{0\} \to \mathbb{R}^+
(p, v) \mapsto |v|_{h^*} \tag{3}$$

where $v$ is a vector of $L^*$ in the fiber over $p$. With this notation the Kähler form on the Kähler cone $(M \times \mathbb{R}^+, t^2 g + dt^2, J)$ is given by

$$\Omega = \frac{i}{2} \partial \overline{\partial} t^2. \tag{4}$$

The Sasakian structure can be read from this data as

$$\omega^T = -\frac{i}{2} \partial \overline{\partial} \log h, \quad \eta = i(\overline{\partial} - \partial) \log t. \tag{5}$$

Therefore, the choice of a hermitian metric $h$ on an orbisample line bundle $L$ over a compact Kähler orbifold $X$ completely determines a Sasakian structure on the $U(1)$-oribundle associated to $L^*$. Further, this is a smooth manifold if the local uniformizing groups of the orbifold inject into $U(1)$. The Sasakian manifold so obtained is called a Boothby-Wang bundle over $(X, \omega)$ (cf. for instance [2, Chapter 7]).

The most basic example is the standard Sasakian structure on $S^{2n+1}$, that is, the Boothby-Wang bundle determined by the Fubini-Study metric $h = h_{FS}$ on $O(1)$ over $\mathbb{C}P^n$. We give the details of this construction to further illustrate the formulation above.

**Example 2 (Standard Sasakian sphere).** Let $h = h_{FS}$ be the Fubini-Study hermitian metric on the holomorphic line bundle $O(1)$ over $\mathbb{C}P^n$. Recall that its dual metric $h^*$ on $O(-1) \setminus \{0\} = \mathbb{C}^{n+1} \setminus \{0\}$ is given by the euclidean norm. This defines a coordinate $t$ on the Kähler cone $O(-1) \setminus \{0\} = \mathbb{C}^{n+1} \setminus \{0\} = S^{2n+1} \times \mathbb{R}^+$. Namely, for coordinates $z = (z_0, z_1, \ldots, z_n)$ on $\mathbb{C}^{n+1}$ we have

$$t : \mathbb{C}^{n+1} \to \mathbb{R}^+
(z) \mapsto |z| = \sqrt{\sum_{i=0}^{n} z_i \overline{z_i}} \tag{7}$$

Now the Kähler metric on the cone is nothing but the flat metric

$$\Omega_{flat} = \frac{i}{2} \partial \overline{\partial} t^2 = \frac{i}{2} \sum dz_i \wedge d\overline{z}_i. \tag{8}$$

The Reeb vector field $R_0$ and the contact form $\eta_0$ read

$$R_0 = J(t \partial_t) = i \sum z_i \partial_{z_i} - \overline{z}_i \partial_{\overline{z}_i}, \quad \eta_0 = i(\overline{\partial} - \partial) \log t = \frac{i}{2t^2} \sum z_i d\overline{z}_i - \overline{z}_i dz_i. \tag{9}$$
It is clear that, when restricted to $S^{2n+1}$, $\eta_0$ and $R_0$, together with the round metric $g_0$ and the restriction $\Phi_0$ of $J$ to $\ker \eta_0$ give a Sasakian structure on $S^{2n+1}$. This corresponds exactly to the Hopf bundle $S^{2n+1} \rightarrow \mathbb{CP}^n$. Moreover, we have
\[
\pi^* \omega_{FS} = \omega^T = \frac{1}{2} d\eta_0 = \frac{i}{2|z|^2} \sum_i |z_i|^2 d\bar{z}_i \wedge dz_i - \sum_{i,j} z_i z_j dz_i \wedge d\bar{z}_j
\]
where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$ is the standard projection and the Fubini-Study form is normalized to give $\text{vol}(\mathbb{CP}^n) = \pi^n$.

2.1. Some Sasakian deformations and weighted Sasakian spheres. We begin this section by recalling some well known classes of deformations of Sasakian structures. Let us start with transverse Kähler transformations. Namely, given a Kähler cone $Y = M \times \mathbb{R}^+$ we consider all Kähler metrics on $(Y, J)$ that are compatible with the Reeb field $R$. In other terms, these are potentials $t^2$ such that $t \partial_i = t \partial_i$. This means that the corresponding Kähler and contact forms satisfy
\[
\tilde{\Omega} = \Omega + i \partial \bar{\partial} e^{2f}, \quad \tilde{\eta} = \eta + df
\]
for a function $f$ invariant under $\partial_i$ and $R$. Such functions are called basic functions. We still need to identify the manifolds $\{\tilde{t} = 1\}$ and $\{t = 1\}$. This is done by means of the diffeomorphism
\[
F : Y \rightarrow Y \quad (p,t) \mapsto (p,te^{-f(p)})
\]
which maps $\{t = 1\}$ to $\{t = e^{-f(p)}\} = \{\tilde{t} = 1\}$. It is elementary to check that $\eta$, $R$ and $df$ are invariant under $F$ so that $\tilde{\eta} = \eta + df$ holds on $M$. Furthermore, the transverse Kähler forms are related by $\tilde{\omega}^T = \omega^T + i \partial \bar{\partial} f$. Notice that when the Sasaki structure is quasi-regular basic functions correspond to function on the base orbifold $X$. Thus, if $t$ comes from a hermitian metric $h^*$ on $L^*$, such a transformation is given by replacing $h^*$ with $e^f h^*$ for a function $f : X \rightarrow \mathbb{C}$ such that $\omega + i \partial \bar{\partial} f > 0$. This is equivalent to picking a different Kähler form $\tilde{\omega}$ in the same class as $\omega$. We summarize the above discussion in the following definition (see e.g. [24, Proposition 1.4]).

**Definition 3** (Transverse Kähler deformations). Let $(M, \eta, R, g, \Phi)$ be a Sasakian manifold with Kähler cone $(Y, J)$ and Kähler potential $t^2$. A transverse Kähler transformation is given by replacing $t$ with $\tilde{t} = e^f t$ for a basic function $f$ and leaving $(Y, J, R)$ unchanged. When the Sasaki structure is quasi-regular and given as in (5), a transverse Kähler transformation is given by replacing $h^*$ with $e^f h^*$.

We now focus on deformations induced by a modification of the Reeb vector field rather than the the Kähler potential. Such deformations are called Type-I deformations. Notice that a transformation of the Sasakian structure $(M, \eta, R, g, \Phi)$ naturally induces an automorphism of the Kähler cone $(Y, J)$. Therefore, the group $\text{Aut}(M, \eta, R, g, \Phi)$ of diffeomorphisms of $M$ preserving the Sasaki structure is a subgroup of $\text{Aut}(Y, J)$. Fix a maximal torus $\mathbb{T}$ in $\text{Aut}(M, \eta, R, g, \Phi)$ and pick an element $R'$ of its Lee algebra $\mathfrak{t}$ such that $\eta(R') > 0$. Then there exists a $\mathbb{T}$-invariant coordinate on the Kähler cone $Y$ inducing a $\mathbb{T}$-invariant Kähler cone metric on $(Y, J)$ with Reeb vector field $R'$, see e.g. [7, Lemma 2.2]. We give the deformed Sasakian structure in the following

**Definition 4** (Type-I deformations). Let $(M, \eta, R, g, \Phi)$ be a Sasakian manifold. A type-I deformation is given by a choice of a maximal torus $\mathbb{T}$ in $\text{Aut}(M, \eta, R, g, \Phi)$ and an element $R'$ of its...
Lee algebra \( \mathfrak{t} \) such that \( \eta(R') > 0 \). The deformed structure is then given by

\[
\eta' = \frac{\eta}{\eta(R')}, \quad \Phi' = \Phi - \Phi R' \otimes \eta', \quad g = \eta' \otimes \eta' + \frac{1}{2}d\eta'(\text{Id} \otimes \Phi').
\]

Notice that if a Sasakian manifold is irregular, then \( R \) is an irrational element in the Lie algebra \( \mathfrak{t} \) of a maximal torus \( \mathbb{T} \) in which it is contained. This allows us to reformulate a classical result of Rukimbira [23] in terms of the Kähler cone. A proof of the following proposition can be found, for instance, in [4, Corollary 2.8].

**Proposition 5.** Let \( (M, \eta, R, g, \Phi) \) be a Sasakian manifold such that its Reeb vector field is irrational in a maximal torus \( \mathbb{T} \) in \( \text{Aut}(M, \eta, R, g, \Phi) \) (i.e. an irregular Sasakian manifold). Then there exists a sequence of Reeb fields \( R_j \in \mathfrak{t} \) such that the corresponding type-I deformations \( (M, \eta_j, R_j, g_j, \Phi_j) \) are quasi-regular and converge smoothly to \( (M, \eta, R, g, \Phi) \).

We now conclude the section with the definition of weighted Sasaki sphere.

**Definition 6** (Simple deformation). Let \( (M, \eta, R, g, \Phi) \) be a Sasakian manifold with Kähler cone \((Y, J)\) and fix a maximal torus \( \mathbb{T} \) in \( \text{Aut}(M, \eta, R, g, \Phi) \). A simple deformation of \( (M, \eta, R, g, \Phi) \) is a Sasakian structure on \( M \) induced by a Kähler cone metric on \((Y, J)\) with Reeb vector field \( R' \in \mathfrak{t} \) such that \( \eta(R') > 0 \).

Notice that a simple deformation amounts to a type-I deformation followed by a transverse Kähler deformation. Since neither of the two changes the CR structures, simple deformations preserve the underlying CR structure.

**Example 7** (Weighted Sasaki sphere). Let \( (S^{2n+1}, \eta_0, R_0, g_0, \Phi_0) \) be the standard structure coming from the flat Kähler metric on \( \mathbb{C}^{n+1} \setminus \{0\} \). Then \( \text{Aut}(S^{2n+1}, \eta_0, R_0, g_0, \Phi_0) = U(n + 1) \) and we fix the maximal torus \( \mathbb{T}^{n+1} \subset U(n + 1) \) acting by diagonal elements. The Lie algebra \( \mathfrak{t} \) of \( \mathbb{T} \) is generated by the elements \( R_j = i (z_j \partial_{z_j} - \bar{z}_j \partial_{\bar{z}_j}) \). Therefore an element \( R_w \in \mathfrak{t} \) satisfies the positivity condition \( \eta(R_w) > 0 \) if and only if \( R_w = \sum_j w_j R_j \) where \( w = (w_0, w_1, \ldots, w_n) \) is a positive vector in \( \mathbb{R}^{n+1} \).

A simple Sasakian structure on \( S^{2n+1} \) is a Sasakian structure that can be obtained as simple deformation of the standard Sasakian sphere with Reeb field \( R_w \) as above. The sphere \( S^{2n+1} \) endowed with a simple Sasakian structure is called a weighted Sasaki sphere. Notice that when the weights \( w_j \) are rational the Sasakian structure is quasi-regular and the base orbifold is the quotient of \( \mathbb{C}^{n+1} \setminus \{0\} \) by the \( \mathbb{C}^* \)-action

\[
\lambda \cdot (z_0, z_1, \ldots, z_n) = (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n).
\]

This is a weighted projective space \( \mathbb{CP}(w) \) and the weighted Sasaki structure corresponds to a hermitian metric \( h \) on \( \mathcal{O}(1) \) such that its curvature pulls back to \( 2d\eta_w \) on \( S^{2n+1} \). Notice that up to transverse homothety one can assume that the weights \( w_j \) are positive integers. Moreover, one gets (a \( D \)-homothety of) the standard Sasakian structure if and only if all weights are equal.

This realizes \( S^{2n+1} \) as the \( U(1) \) bundle associated to the line bundle \( \mathcal{O}(-1) \) over the weighted projective space \( \mathbb{CP}(w) \), that is, the quotient of \( \mathbb{C}^{n+1} \setminus \{0\} \) by the weighted \( \mathbb{C}^* \) action above. The weighted Sasakian structure is then determined by a metric \( h \) on \( \mathcal{O}(1) \) such that its curvature is a Kähler form on \( \mathbb{CP}(w) \). Notice that there is more than one natural choice for \( h \), see [21, Section 3].

In analogy with the unweighted case, one could wish to restrict to the case where \( h \) is the Fubini-Study hermitian metric whose curvature is the Fubini-Study Kähler metric on \( \mathbb{CP}(w) \). This
is what is done, for instance, in [2, Example 7.1.12] while our definition allows different choices for $h$. Nevertheless, the metric used in our Theorem 1 coincides exactly with the one in [2]. In fact, it was noted in [24, Theorem 1.6] that $\frac{1}{2}t^2_w$ is the Hamiltonian function for the Reeb vector field $R_w$ (where $t_w$ is the coordinate on $\mathbb{C}^{n+1} \setminus \{0\}$ coming from the type-I deformation). This implies that Kähler orbifold $(\mathbb{C}P(w), \omega)$ of the weighted Sasakian sphere is the one obtained as Kähler reduction of the Kähler cone $\mathbb{C}^{n+1} \setminus \{0\}$ under the $U(1)$-action of the Reeb field $R_w$ (cf. Definition 8 below).

3. Asymptotic Expansion of the Weighted Bergman Kernel

3.1. Kodaira embedding for Kähler orbifolds. In this section we recall some results from [20, 21] for clarity and future reference. In order to improve comprehensibility we follow their notation closely.

Let $(X, L)$ be a polarized cyclic orbifold and let $h$ be a hermitian metric on $L$ with curvature $2\pi\omega$ (cf. Remark 12 below). Consider the weighted vector space

$$V = \bigoplus_i H^0(L^{k+i})^*$$

where $i$ ranges from 1 to the order $m = \text{ord}(X)$ of the orbifold $X$ and the $i$-th summand has weight $k+i$. Then, for large enough $k \gg 0$ we have an embedding $\phi_k : X \rightarrow \mathbb{P}(V)$ such that $\phi_k^*\mathcal{O}(1) = L$ ([20]) given by

$$\phi_k(x) = \left[\oplus \text{ev}_{k+i}^x \right]$$

where $\text{ev}_{k+i}^x$ is the evaluation element at $x$ of $H^0(L^{k+i})^*$ which sends a section $s$ of $L^{k+i}$ to $s(x) \in L^{k+i}$ identified with $\mathbb{C}$ via a local trivialization.

We now explain the relation between the hermitian metric $h$ on $L$ and the Fubini-Study metric on $\mathbb{P}(V)$. Let $| \cdot |_V$ be a hermitian metric on $V$ such that the summands $H^0(L^{k+i})^*$ and $H^0(L^{k+j})^*$ are orthogonal for $i \neq j$, that is, a collection of hermitian metrics $| \cdot |_{k+i}$ on each summand of $V$. The group $U(1)$ acts on each $H^0(L^{k+i})^*$ with weight $k+i$. For an element $v = \oplus_i v_{k+i} \in V$ the moment map of the $U(1)$ action is given by

$$\mu_{U(1)}(v) = \frac{1}{2} \left( \sum_i (k+i)|v_{k+i}|^2 - c \right) \quad \text{with} \quad c := \sum_i (k+i)c_i h_0(L^{k+i})$$

where $c_i$ are arbitrary positive real constants.

**Definition 8.** The Fubini-Study metric $\omega_{FS}$ on $\mathbb{P}(V)$ is defined as $\frac{1}{c}$ times the metric induced by $| \cdot |_V$ on $\mathbb{P}(V) = \mu_{U(1)}^{-1}(0) / U(1)$ by Kähler reduction.

Moreover, the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ in [21] is defined so that the vectors of $\mu_{U(1)}^{-1}(0)$ are unitary. Namely, we have the following

**Definition 9.** The Fubini-Study hermitian metric $h_{FS}$ on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is defined as $|v|_{h_{FS}} := \lambda(v)^{-1}$ where $\lambda(v)$ is the unique positive real number such that $\lambda(v) \cdot v \in \mu_{U(1)}^{-1}(0)$. In other terms, $\lambda(v)$ is the unique positive real solution to $\sum_i (k+i)\lambda(v)^{2(k+i)}|v_{k+i}|^2 = c$.

By abuse of notation we use $h_{FS}$ for the induced metric on $\mathcal{O}_{\mathbb{P}(V)}(i)$ as well. Unfortunately the curvature $2\pi\omega_{h_{FS}} := i\partial\bar{\partial}\log h_{FS}$ of $h_{FS}$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$ is not given by $2\pi\omega_{FS}$. In fact, their difference is described in the next
Lemma 10. [21, Lemma 3.6] In the notation above we have
\[ \omega_{FS} = \omega_{h_{FS}} + \frac{i}{2c} \bar{\partial} \partial f \]
where \( f = \sum_{i} \sum_{\alpha} |t_{i\alpha}^{a}|^2 \) and \( \{t_{i\alpha}^{a}\} \) is an orthonormal basis of \((V^*, |\cdot|_V)\).

Remark 11. In particular this implies that there exists a hermitian metric \( h_{FS} = e^{\pi} h_{FS} \) on \( O_{\mathbb{P}(V)}(1) \) whose curvature is exactly \( 2\pi \omega_{FS} \).

Remark 12. Notice that the convention \( 2\pi \omega_h = i \partial \bar{\partial} \log h \) used here for the curvature form \( \omega_h \) of a hermitian metric \( h \) on a line bundle \( L \) differs from Equation (2) by a factor of \( -\pi \). This is of no consequence for the convergence results we are aiming for. Thus we follow the convention of [21] throughout this section and Section 4.

Notice that there is a correspondence between pairs \((h, \omega)\), consisting of a hermitian metric \( h \) on \( L \) and an orbifold Kähler metric \( \omega \) on \( X \) such that \( 2\pi \omega \) is the curvature of some hermitian metric on \( L \), and metrics \(|\cdot|_V\) on \( V = \bigoplus_i H^0(L^{k+i})^* \). Namely, given a pair \((h, \omega)\) as above, one can define a metric on \( V \) (which following [21] we denote by \( \text{Hilb}(h, \omega) \)) by setting
\[ |\phi|^2_{\text{Hilb}(h, \omega)} = \frac{1}{c_1 \text{vol}} \int_X |s|^2 \omega^n \]
where \( s \) is a section of \( L^{k+1} \) and
\[ \text{vol} := \int_X \frac{c_1(L)^n}{n!} = \int_X \frac{\omega^n}{n!}. \]

Conversely, a metric \(|\cdot|_V\) on \( V \) gives a pair of metrics \((h_{FS}, \omega_{FS})\) on \( O_{\mathbb{P}(V)}(1) \) and \( \mathbb{P}(V) \) as constructed above. In turn they determine a pair \((h, \omega) = (\phi_{h_{FS}}^*, \phi_{\omega_{FS}}^*) \).

If the pair \((h, \omega)\) is fixed and is inducing the metric \(|\cdot|_V = \text{Hilb}(h, \omega)\), then we denote the pair \((\phi_{h_{FS}}^*, \phi_{\omega_{FS}}^*)\) by \((h_{FS,k}, \omega_{FS,k})\). From now on we will assume that this is the case.

3.2. The weighted Bergman kernel. Let \((X, L)\) be a polarized orbifold, \( h \) a hermitian metric on \( L \) and \( \omega \) a Kähler metric on \( X \). As in the smooth case one can give the following

Definition 13. The Bergman kernel of \( L^{k+i} \) is defined to be the function \( B_{k+i}(h, \omega) = \sum_{\alpha} |t_{i\alpha}|^2 \) where \( \{t_{i\alpha}\} \) is an orthonormal basis of \( H^0(L^{k+i}) \) with respect to \( \text{Hilb}(h, \omega) \).

The functions \( B_{k+i} \) satisfy a crucial property that will allow us the control we need to prove Theorem 2. This is proven in [20, Corollary 4.10] and, alternatively, can be deduced from [12, Theorem 5.4.11].

Lemma 14. If \( D \) be a differential operator of order \( p \), then
\[ DB_k = O(k^{n+p}) \]
uniformly on \( X \) for large \( k \).

This is not the only Bergman kernel we are interested in, though. In fact, we consider the weighted Bergman kernel, that is, the version of it related to orbifold embeddings. In order to define the weighted Bergman kernel, let us pick a basis \( \{s_{i\alpha}^j\} \) of \( \bigoplus_i H^0(L^{k+i}) \) orthonormal with respect to the \( L^2 \)-metric and a basis \( \{t_{i\alpha}\} \) orthonormal for \( \text{Hilb}(h, \omega) \). For instance, one could take \( t_{i\alpha} = \sqrt{c_v \text{vol}} s_{i\alpha}^j \).
Definition 15. The weighted Bergman kernel is defined to be the function

\[ B_k = B_k(h, \omega) = \text{vol} \sum_i c_i(k + i) \sum_{\alpha} |s^i_{\alpha}|_h^2 \]

or, equivalently,

\[ B_k = \sum_i (k + i) \sum_{\alpha} |t^i_{\alpha}|_h^2 = \sum_i (k + i)B_{k+i} \]

where \( i \) runs from 1 to \( m = \text{ord}(X) \).

A crucial element for the proof of Theorem 2 is the asymptotic expansion of the weighted Bergman kernel \( B_k = \sum_i (k + i)B_{k+i} \) proven in [20, Theorem 1.7 and Remark 4.13]. Namely, for coefficients \( c_i \) given by

\[ \sum_i c_it^i := (t^{n-1} + t^{m-2} + \cdots + 1)^p \]

there is an asymptotic expansion

\[ B_k = b_0k^{n+1} + b_1k^n + \cdots \text{ for } k \to \infty \]

for some smooth functions \( b_i \). Moreover, for a given \( q \in \mathbb{N} \) we can pick the coefficients \( c_i \) given as in Equation (10) with \( p \gg r + q \). This will ensure that the expansions holds in \( C^q \) up to order \( O(k^{n+1-r}) \). Assume from now on that \( q \in \mathbb{N} \) is fixed. We will always assume that \( p \) is chosen large enough so that the expansions holds in \( C^q \).

In our case \( 2\pi \omega \) is the curvature of \( h \) so that [20, (1.11) and Corollary 4.12] give

\[ b_0 = \text{vol} \sum_i c_i, \quad b_1 = \text{vol} \sum_i c_i \left( (n+1)i + \frac{1}{2}\text{Scal}(\omega) \right) \]

where \( \text{Scal}(\omega) \) is the scalar curvature of \( \omega \). In particular, \( b_0 \) is constant over \( X \). Moreover, integrating over \( X \) one sees that \( c = \sum_i c_i(k + i)h^0(L^{k+i}) \) has an expansion

\[ c = \text{vol} \sum_i c_i \left[ k^{n+1} + \left( (n+1)i + \frac{3}{2}\overline{\text{S}} \right) k^n \right] + O(k^{n-1}) \]

where \( \overline{\text{S}} \) is the average of the scalar curvature of \( \omega \).

We conclude this section with a definition that will be useful for the estimates in the proof of Theorem 2.

Definition 16 ([21, p. 137]). A sequence of real-valued functions \( f_k \) on \( X \) is of order \( \Omega(k^p) \) if there exists a constant \( \delta > 0 \) such that \( f_k \geq \delta k^p \) uniformly on \( X \) for all \( p \gg 0 \).

4. \( C^\infty \)-approximation of Hermitian metrics

As we anticipated in the Introduction, in this section we prove the approximation results in the Kähler setting needed in the proof of Theorem 1. More precisely, given a cyclic Kähler orbifold \((X, \omega)\) polarized by \((L, h)\) with \( 2\pi \omega = i\partial\bar{\partial} \log h \), we get a \( C^q \) approximation of \( h \) (and consequently a \( C^{q-2} \) approximation of \( \omega \)) by pulling back the Fubini-Study metric \( h'_{FS} \) of Remark 11
via embeddings into a weighted projective space (cf. Proposition 17 below). Nevertheless, in order to highlight the connection to [21], we first prove the strengthening of their Theorem 4.6 to the $C^q$-norm.

**Theorem 2.** Let $(X, L)$ be a polarized cyclic orbifold with $h$ a hermitian metric on $L$ whose curvature is $2\pi\omega$. Denote with $\text{Scal}(\omega)$ the scalar curvature and with $\overline{S}$ its average. Then for any given $q > 1$ the pair $(h_{FS,k}, \omega_{FS,k})$ $C^{q-2}$-converges to $(h, \omega)$ when $k$ tends to infinity. Namely, we have

$$
\frac{h_{FS,k}}{h} = 1 + \frac{\overline{S} - \text{Scal}(\omega)}{2} k^{-2} + O(k^{-3})
$$
in the $C^q$-norm, and

$$
\omega = \omega_{FS,k} + O(k^{-2})
$$
in the $C^{q-2}$-norm.

**Proof of (14).** Fix a $q \geq 2$ and pick the coefficients $c_i$ in (10) so that the expansion (11) holds in $C^q$.

We want to show that

$$
\alpha_k := \frac{h_{FS,k}}{h} = 1 + \frac{\overline{S} - \text{Scal}(\omega)}{2} k^{-2} + O(k^{-3})
$$
in $C^q$.

Notice that $\alpha_k$ satisfies the equation

$$
\sum_i (k+i) \alpha_k^{k+i} B_{k+i} = c
$$
where $c$ and $B_{k+i}$ are as above (cf. [21, (4.10)]).

It was proven in [21, page 138] that $\alpha_k = 1 + O(k^{-2})$ and there exist positive constants $C_1$ and $C_2$ such that

$$
C_1 \leq \alpha_k^j \text{ for all } \frac{k}{2} \leq j \leq k;
$$

$$
\alpha_k^j \leq C_2 \text{ for all } 0 \leq j \leq k.
$$

Now define

$$
\beta_k := 1 + \frac{\overline{S} - \text{Scal}(\omega)}{2} k^{-2} + \sum_{j=1}^q \tau_j k^{-2-j}
$$
for smooth functions $\tau_j$ independent of $k$. Analogously to [21, page 139], we get the equality

$$
\sum_i (k+i) \beta_k^{k+i} B_{k+i} = \sum_i (k+i) \left(1 + \frac{\overline{S} - \text{Scal}(\omega)}{2k} + O(k^{-2})\right) B_{k+i}
$$

$$
= \text{vol} \sum_i c_i \left[ k^{n+1} + \left( \frac{\overline{S} - \text{Scal}(\omega)}{2} + (n+1)i + \text{Scal}(\omega) \right) k^n \right] + O(k^{n-1})
$$

$$
= \text{vol} \sum_i c_i \left[ k^{n+1} + \left( \frac{\overline{S}}{2} + (n+1)i \right) k^n \right] + O(k^{n-1})
$$

$$
= c + O(k^{n-1})
$$
(21)
in $C^0$ as $B_{k+i} = O(k^n)$ in $C^0$ by (7). We now pick the functions $\tau_j$ in (20) so that the terms of order $n-j$ in (21) vanish for all $1 \leq j \leq q$. In order to do so, notice that the coefficient of $k^{n-j}$ in (21) equals $b_0\tau_j + f_j$ where $f_j$ is independent of $k$ and of the functions $\tau_l$ for all $l \geq j$. Thus we can set $\tau_j = -\frac{f_j}{b_0}$ so that the terms of order $n-j$ in $k$ vanish for $1 \leq j \leq q$. Hence we get

$$
\sum_i (k+i)\beta_k^{i+j}B_{k+i} = c + O(k^{n+p-q-1}) \text{ in } C^p \text{ for } p = 0, 1, \ldots, q,
$$

where we have used (7). Subtracting (22) from (17) yields

$$
(\alpha_k - \beta_k)\sum_i (k+i)\gamma_k B_{k+i} = O(k^{n+p-q-1}) \text{ in } C^p \text{ for } p = 0, 1, \ldots, q,
$$

where

$$
\gamma_k = \sum_{j=1}^{k+i} \alpha_k^{j-1}\beta_k^{i+j}.
$$

Notice that $\gamma_k = \Omega(k)$ (cf. Definition 16) since it is the sum of $k+i$ terms which are uniformly bounded from below. Thus the sum on the left hand side of (23) is of order $\Omega(k^{n+2})$. In particular, for $p = 0$ we get

$$
\alpha_k - \beta_k = O(k^{-q-3}) \text{ in } C^0.
$$

In fact, we claim that $D^p(\alpha_k - \beta_k) = O(k^{p-q-3})$ in $C^p$ for all $p = 1, \ldots, q$, which yields (16) for $p = q$.

Our strategy is to prove the following facts by induction on $p = 1, 2, \ldots, q$, from which the claim follows

(i) $D^p\alpha_k = O(k^{p-1})$;
(ii) $D^p\gamma_k = O(k^{p+1})$;
(iii) $D^p(\gamma_k B_{k+i}) = O(k^{n+p+1})$.
(iv) $D^p(\alpha_k - \beta_k) = O(k^{p-q-3})$ in $C^p$.

**Case** $p = 1$: this is proved in [21, page 140], but we recall the argument here for completeness. Differentiating (17) yields

$$
D\alpha_k \sum_i (k+i)^2\alpha_k^{i+j}B_{k+i} = -\sum_i (k+i)\alpha_k^{i+j}DB_{k+i}.
$$

The sum on the left hand side is of order $\Omega(k^{n+2})$ while the sum on the right hand side is $O(k^{n+2})$ because $DB_{k+i} = O(k^{n+1})$ by (7). Thus $D\alpha_k = O(1)$ which proves (i).

Now for (ii) let us first notice that for $u, v \geq 0$ with $u + v = k + i - 1$ we have

$$
D(\alpha_k^u \beta_k^v) = u\alpha^{u-1}\beta_k^v D\alpha_k + v\alpha_k^u \beta_k^{v-1} D\beta_k.
$$

Therefore $D(\alpha_k^u \beta_k^v) = O(k)$ because $D(\alpha_k) = O(1)$, $D(\beta_k) = O(k^{-2})$ and both $\alpha_k$ and $\beta_k$ are uniformly bounded above. Now $D\gamma_k = O(k^2)$ follows from the fact that

$$
D\gamma_k = \sum_{j=1}^{k+i} D\left(\alpha_k^{j-1}\beta_k^{i+j}\right)
$$

is a sum of $k+i$ terms of order $O(k)$, and this proves (ii).
To show (iii) we simply consider
\[ D(\gamma_k B_{k+i}) = D(\gamma_k)B_{k+i} + \gamma_k D(B_{k+i}). \]
Since \( \gamma_k = O(k) \) and \( D(B_{k+i}) = O(k^{n+1}) \), the claim follows from (ii).

Now differentiating (23) yields
\[ D(\alpha_k - \beta_k) \Omega(k^{n+2}) = -(\alpha_k - \beta_k) \sum_i (k+i)D(\gamma_k B_{k+i}) + O(k^{n-q}) \text{ in } C^1. \]

Part (iv) then follows from (25) and part (iii).

**Inductive step:** Suppose that parts (i)-(iv) hold for all natural numbers between 1 and \( p - 1 \). In particular, since part (iv) holds, we get
\[ D^j(\alpha_k) = D^j(\beta_k) + O(k^{j-q-3}) = O(k^{-2}) \text{ for all } 1 \leq j \leq p - 1. \]

Let us now prove (i). In order to do so, consider (17) differentiated \( p - 1 \) times, that is,
\[ D^{p-1}(\alpha_k) \sum_i (k+i)^{2}\alpha_k^{k+i-1}B_{k+i} = O(k^{n+p}). \]

The right hand side is a sum of terms of order at most \( O(k^{n+p}) \). Each of these summands is a product involving some (or all) of the following factors

- (I) \( D^u B_{k+i} \) for \( u \leq p - 1 \);
- (II) powers of \( D^v \alpha_k \) for \( v \leq p - 2 \);
- (III) powers of \( \alpha_k \);
- (IV) constants possibly involving powers of \( k \).

Now differentiating (27) we obtain the equation
\[
D^p(\alpha_k) \Omega(k^{n+2}) = -D^{p-1}(\alpha_k) \sum_i (k+i)^2(k+i-1)\alpha_k^{k+i-2}D\alpha_k B_{k+i} - D(\text{RHS})
\]
\[ = -D^{p-1}(\alpha_k) \sum_i (k+i)^2\alpha_k^{k+i-1}DB_{k+i} + D(\text{RHS}). \]

where \( D(\text{RHS}) \) is obtained by differentiating the right hand side of (27). Thus each term in \( D(\text{RHS}) \) is obtained by differentiating one of the factors (I)-(III) in the summands of the right hand side of (27). Notice that differentiating (I) raises the order of the summand by 1 (again by (7)), differentiating (II) does not change the order of the summand and differentiating (III) lowers the order of the summand by 1. We conclude that \( D(\text{RHS}) \) has order at most \( O(k^{n+p+1}) \). Since the order of the first two sums is \( O(k^{n-1}) \) and \( O(k^{n+1}) \) respectively, we have proven (i).

Now for (ii) notice first that for \( u, v \geq 0 \) with \( u + v = k + i - 1 \) we have
\[ D^p(\alpha_k^u \beta_k^v) = u\alpha^{u-1} \beta_k^v D^p \alpha_k + O(k^{-2}) = O(k^p) \]
where the first equality follows from (26) and the second equality comes from part (i). Thus (ii) follows from the fact that
\[ D^{k+i} \gamma_k = \sum_{j=1}^{k+i} D^j \left( \alpha_k^{j-1} \beta_k^{k+i-j} \right) \]
is a sum of \( k + i \) terms of order \( O(k^p) \).

---

1 In fact, the term of highest order in \( k \) is \( \sum_i (k+i)\alpha_k^{k+i}D^p B_{k+i} = O(k^{n+p+1}) \).
Now also (iii) follows from (ii), (7) and the inductive hypothesis because
\[ D^p(\gamma_k B_{k+i}) = \sum_{j=0}^{p} \binom{p}{j} D^{p-j}(\gamma_k) D^j(B_{k+i}). \]

In order to show (iv) and conclude the proof, differentiate (23) to get
\[ D^p(\alpha_k - \beta_k) \Omega(k^{n+1}) = -\sum_{j=0}^{p-1} \binom{p}{j} D^j(\alpha_k - \beta_k) \sum_{i} (k+i) D^{p-j}(\gamma_k B_{k+i}) + O(k^{n+p-q-1}) \text{ in } C^p. \]

Notice that \( \Phi(\gamma_k B_{k+i}) = O(k^{n+p-j+1}) \) and \( D^j(\alpha_k - \beta_k) = O(k^{j-3}) \) so that we obtain (iv) as wanted. \( \square \)

**Proof of (15).** This proof is carried analogously to [21] but we write it here for completeness.

Applying \( \partial \bar{\partial} \log \) to (14) we get
\[ \omega_{h_{FS,k}} = \omega + O(k^{-2}) = \omega + O(k^{-2}) \text{ in } C^q-2. \]
Together with Lemma 10 this yields
\[ \omega_{FS,k} = \omega_{h_{FS,k}} + \frac{i}{2c} \partial \bar{\partial} f_k = \omega + \frac{i}{2c} \partial \bar{\partial} f_k + O(k^{-2}) \text{ in } C^q-2. \]
Now, since \( c \) is of order \( \Omega(k^{n+1}) \) by (13), it suffices to show that \( f_k \) is constant up to terms of order \( O(k^{n-1}) \) in the \( C^q \)-norm. From the definition of the metric \( h_{FS,k} \) it follows
\[ f_k(x) = \text{vol} \sum_i c_i \sum_{\alpha} |s^i_\alpha(x)|^2_{h_{FS,k}} = \text{vol} \sum_i c_i \frac{h_{FS,k}^{k+i}}{\tilde{h}_{FS,k}^{k+i}} \sum_{\alpha} |s^i_\alpha(x)|^2_\tilde{h}. \]
Now by (14) we get
\[ f_k(x) = \text{vol} \sum_i c_i \left( 1 + \frac{\text{Scal}(\omega) - \bar{S}}{2k} + O(k^{-2}) \right) \sum_{\alpha} |s^i_\alpha(x)|^2_\tilde{h}. \]
Finally by [20, Theorem 1.7] we have the following \( C^q \) expansion:
\[ \text{vol} \sum_i c_i \sum_{\alpha} |s^i_\alpha(x)|^2_\tilde{h} = b_0 k^n + b'_1 k^{n-1} + \ldots \]
where \( b_0 = \text{vol} \sum_i c_i \) is a constant. Therefore we get
\[ f_k(x) = b_0 k^n + O(k^{n-1}) \text{ in } C^q \]
which is what we were aiming for. \( \square \)

Notice that Theorem 2 is not the result we need for our purposes as \( h_{FS} \) is not the hermitian metric whose curvature is proportional to the Fubini-Study form \( \omega_{FS} \) on the weighted projective space. In order to apply this study to weighted Sasakian spheres we need to prove the convergence of the metrics \( h_{FS,k} \) obtained as pullback to \( L \) of the metric \( h_{FS} \) defined in Remark 11.

**Proposition 17.** Let \((X, L)\) be a polarized cyclic orbifold with \( h \) a hermitian metric on \( L \) whose curvature is \( 2\pi \omega \). Then for a given \( q > 1 \) the metrics \( h_{FS,k} \) converge to \( h \) when \( k \) tends to infinity. Namely, we have
\[ \frac{h_{FS,k}}{h} = 1 + O(k^{-1}) \]

(29)
in the $C^q$-norm.

Proof. Fix $q > 1$ so that the coefficients $c_i$ in (10) define a hermitian metric $h_{FS}$ as in Definition 9.

By definition

$$h_{FS}' = e^f h_{FS}$$

where $f = \sum_{i} \sum_{\alpha} |t_i^\alpha|^2 h_{FS}$ and $\{t_i^\alpha\}$ is an orthonormal basis of $(V^*, |\cdot|_V)$.

Therefore

$$h_{FS,k}' = e^{f_k} h_{FS,k}$$

where $f_k(x) = b_0 k^n + O(k^{n-1})$ in $C^q$ as above. Moreover, since $c$ is of order $\Omega(k^{n+1})$ by (13), we have that $\frac{f_k}{2c}$ is a $O(k^{-1})$ in $C^q$. Therefore,

$$\frac{h_{FS,k}'}{h_{FS,k}} = e^{\frac{f_k}{2c}} = 1 + O(k^{-1}) \text{ in } C^q.$$ 

Finally, making use of (14), we have

$$\frac{h_{FS,k}'}{h} = \frac{h_{FS,k}'}{h_{FS,k}} \frac{h_{FS,k}}{h} = (1 + O(k^{-1}))(1 + O(k^{-2})) = 1 + O(k^{-1}) \text{ in } C^q.$$ 

The convergence of $\omega_{FS,k}$ to $\omega$ was already proved in Theorem 2. □

5. PROOF OF THEOREM 1

The line of argument to prove Theorem 1 is the following. A Sasakian embedding of a Sasakian manifold $(M, \eta, R, g, \Phi)$ into a weighted Sasakian sphere determines an isometric Kähler embedding of its Kähler cone $(Y, J)$ into $\mathbb{C}^n+1 \setminus \{0\}$. Our strategy is to define a sequence of holomorphic embeddings of the Kähler cone $(Y, J)$ into $\mathbb{C}^n+1 \setminus \{0\}$ such that the induced Kähler cone metrics converge to the original one in the $C^q$-norm for a given $q$. This is done firstly for quasi-regular manifolds where we can exploit the fact that $Y$ is in fact a holomorphic line bundle over a Kähler orbifold and make use of Theorem 2.

Let $(M, \eta, R, g, \Phi)$ be a compact quasi-regular Sasakian manifold. By the discussion following Theorem 2, possibly after performing a transverse homothety, the Kähler cone of $M$ is the complement of the zero section of the dual $L^*$ of a orbibample line bundle $L$ over a compact Kähler orbifold $(X, \omega)$. In particular, the Sasakian structure on $M$ is determined by a hermitian metric $h$ on $L$ whose curvature is $2\pi\omega$ and $M$ identifies with the $U(1)$ orbibundle determined by $L^*$. As in Section 2 denote with $\pi : M \longrightarrow X$ the Riemannian submersion given by the restriction to $M$ of the bundle projection $p : L^* \longrightarrow X$.

By Theorem 2 and Proposition 17 this, together with the choice of $q + 2 \in \mathbb{N}$, determines a sequence of embeddings $\phi_k$ into weighted projective spaces $(\mathbb{P}(V), \omega_{FS})$ such that

$$\phi_k^* (\mathcal{O}_{\mathbb{P}(V)}(1)) = L$$

and $\phi_k^*(h_{FS}', \omega_{FS}) = (h_{FS,k}', \omega_{FS,k})$ converges to $(h, \omega)$ in $C^q$ as $k \longrightarrow \infty$. Visually we have
where \( \tilde{\phi}_k \), i.e. the lift of the embedding \( \phi_k \) to \( L \), is an isometric Kähler embedding of Kähler cones. Moreover, the vertical maps restricted to the \( U(1) \)-orbibundles are Boothby-Wang fibrations. Now the hermitian metric \( h'_{FS,k} \) defines a Sasakian structure \( (\eta_k, R_k, g_k, \Phi_k) \) on \( M \) as in (5). Furthermore, these structures converge to \( (\eta, g) \) in the \( C^q \)-norm because the metrics \( h'_{FS,k} \) converge to \( h \) in \( C^{q+2} \) and so do the associated coordinates \( t_k \) on the Kähler cone \( Y = L^* \setminus \{0\} \). Notice that we could pull back the Kähler cone structure on \( \mathcal{O}_{\mathbb{P}(V)}(-1) = V \setminus \{0\} \) after performing a \( D \)-homothetic transformation so that the induced Sasakian structure do converge to the original metric on \( M \). This concludes the proof in the case where the Sasakian structure \( (M, \eta, g) \) is quasi-regular.

Suppose now that the structure \( (M, \eta, R, g, \Phi) \) is irregular. By Proposition 5, any irregular structure can be \( C^\infty \)-approximated by a sequence of quasi-regular structures \( (M, \eta_j, R_j, g_j, \Phi_j) \) obtained by type-I deformation. Moreover, by the previous part each of these admits a \( C^q \)-approximation by structures \( (M, \eta_{j,k}, R_{j,k}, g_{j,k}, \Phi_{j,k}) \) induced by embeddings \( \varphi_{j,k} \) into a weighted Sasakian sphere. Therefore, the Sasakian structures induced on \( M \) by the embeddings \( \varphi_k := \varphi_{k,k} \) into weighted spheres converge \( C^q \) to the structure \( (\eta, g) \). This concludes the proof of Theorem 1.

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