DENSITY OF BOUNDED MAPS IN SOBOLEV SPACES INTO COMPLETE MANIFOLDS

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ABSTRACT. Given a complete noncompact Riemannian manifold $N^n$, we study the density of the set of bounded Sobolev maps on the cube $(W^{1,p}(Q^m; N^n))_{1 \leq p \leq m}$ in the Sobolev space $W^{1,p}(Q^m; N^n)$ for $1 \leq p \leq m$. The density always holds when $p$ is not integer. When $p$ is an integer, the density can fail, and we prove that a quantitative levelling property is equivalent with the density. This new condition is ensured by a uniform Lipschitz geometry or by bounds on the injectivity radius and on the curvature. As a byproduct, we give necessary and sufficient conditions for the density of the set of smooth maps $\mathcal{C}^\infty(Q^m; N^n)$ in the space $W^{1,p}(Q^m; N^n)$.

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1. INTRODUCTION

Bounded maps from the unit cube $Q^m \subset \mathbb{R}^m$ are dense in the class of Sobolev maps $W^{1,p}(Q^m; \mathbb{R}^n)$, and this follows from a straightforward truncation argument. In the setting of Sobolev maps with values into manifolds, this elementary approach is unable to handle additional constraints on the target. More precisely, we consider a complete Riemannian manifold $N^n$ isometrically imbedded in $\mathbb{R}^n$, and we define the class of Sobolev maps with values into $N^n$ as

$$W^{1,p}(Q^m; N^n) = \{ u \in W^{1,p}(Q^m; \mathbb{R}^n) : u \in N^n \text{ a.e. in } Q^m \}.$$ 

We first clarify what we mean by a bounded function in this framework. Indeed, for every ball $B(x; r) \subset \mathbb{R}^n$, no matter how small it is, the manifold $N^n$ can always be isometrically imbedded inside $B(x; r)$ [12 theorem 2; [13 theorem 3]. It is thus
preferable to use an intrinsic notion which does not depend on the imbedding of $N^n$ in $\mathbb{R}^r$. We then say that a measurable function $u : Q^n \to N^n$ is essentially bounded if the function $(x, y) \in Q^n \times Q^n \mapsto \text{dist}_{N^n}(u(x), u(y))$ is essentially bounded in $Q^n \times Q^n$, where $\text{dist}_{N^n}$ denotes the Riemannian distance in the manifold $N^n$.

When the manifold $N^n$ is imbedded as a closed submanifold of $\mathbb{R}^r$ [11], this property is equivalent with the usual notion of boundedness as a map into $\mathbb{R}^r$. The question addressed in this paper is the following: is the set $(W^{1,p} \cap L^\infty)(Q^n; N^n)$ dense in $W^{1,p}(Q^n; N^n)$ with respect to the $W^{1,p}$ distance?

One of our motivations to this problem comes from the following observation concerning the density of smooth maps in Sobolev spaces with values into complete manifolds: if $1 \leq p < m$, then every map in $(W^{1,p} \cap L^\infty)(Q^n; N^n)$ is the strong limit of a sequence of smooth maps in $C^\infty(Q^n; N^n)$ if and only if the homotopy group $\pi_{[p]}(N^n)$ is trivial; that is, every continuous map $f : S^{[p]} \to N^n$ on the $[p]$ dimensional sphere is homotopic to a constant map. This result can be deduced as in the case where the target is a compact manifold that was investigated by Schoen and Uhlenbeck [15] and by Bethuel [2], but whether such a conclusion holds for every Sobolev map, not necessarily bounded, is challenging and requires additional tools.

When $p > m$, Sobolev maps on the cube $Q^n$ are bounded, and even Hölder continuous, by the Morrey–Sobolev inequality. We can thus focus on the case $p \leq m$. In contrast with the setting of Euclidean targets, the answer to the question above depends on whether $p$ is an integer or not.

**Theorem 1.** For every $1 \leq p \leq m$ such that $p \notin \mathbb{N}$, the set $(W^{1,p} \cap L^\infty)(Q^n; N^n)$ is dense in $W^{1,p}(Q^n; N^n)$.

The case where $p$ is an integer is more subtle and the answer involves analytical properties of the manifold $N^n$. This surprising phenomenon arises even in the case $p = m$. In the related problem of density of smooth maps in $W^{1,m}(Q^n; N^n)$ when $N^n$ is a compact manifold, this critical case always has an affirmative answer, regardless of $\pi_m(N^n)$, and is a straightforward consequence of the fact that $W^{1,m}$ maps imbed into the class of vanishing mean oscillation (VMO) maps $[7,15]$. For complete but noncompact manifolds, this VMO property is not sufficient to imply density of bounded maps in $W^{1,m}$ even if $N^n$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$, and a counterexample is presented in Section 2 below. In fact, for integer exponents $p$ this density problem is equivalent to the following levelling property:

**Definition 1.1.** Given $p \in \mathbb{N}$, the manifold $N^n$ satisfies the **levelling property of dimension** $p$ if, for every map $f \in C^\infty(\partial Q^n; N^n)$ having a Sobolev extension $u \in W^{1,p}(Q^n; N^n)$, there exists a smooth extension $v \in C^\infty(\overline{Q^n}; N^n)$ such that $$\|Dv\|_{L^p(Q^n)} \leq C\|Du\|_{L^p(Q^n)},$$ for some constant $C > 0$ possibly depending on the manifold $N^n$.

This notion is satisfied by any manifold $N^n$ with uniformly Lipschitz geometry (in the sense of Definition 1.1 below), and in particular when $N^n$ is the covering space of a compact manifold. This case naturally arises in the study of Sobolev maps with values into a compact manifold, see e.g. [3,14]. We also observe that every complete manifold satisfies the levelling property of dimension 1: it suffices to take for $v$ a shortest geodesic connecting the points $f(-1)$ and $f(1)$. The answer to the density problem for integer exponents can now be stated as follows:
Theorem 2. For every $p \in \{1, \ldots, m\}$, the set $(W^{1,p} \cap L^\infty)(Q^m; N^n)$ is dense in $W^{1,p}(Q^m; N^n)$ if and only if $N^n$ satisfies the levelling property of dimension $p$.

As a consequence of Theorem 1 and Theorem 2, we characterize the class of complete manifolds $N^n$ for which smooth maps are dense in $W^{1,p}(Q^m; N^n)$. For non-integer values of the exponent $p$ we have:

Corollary 1.1. For every $1 \leq p \leq m$ such that $p \not\in \mathbb{N}$, the set $C^\infty(\overline{Q^m}; N^n)$ is dense in $W^{1,p}(Q^m; N^n)$ if and only if $\pi_p(N^n) \simeq \{0\}$.

For integer values of $p$, the characterization becomes:

Corollary 1.2. Case $p = 1$: The set $C^\infty(\overline{Q^m}; N^n)$ is dense in $W^{1,1}(Q^m; N^n)$ if and only if $\pi_1(N^n) \simeq \{0\}$.

Case $p \in \{2, \ldots, m-1\}$: The set $C^\infty(\overline{Q^m}; N^n)$ is dense in $W^{1,p}(Q^m; N^n)$ if and only if $\pi_p(N^n) \simeq \{0\}$ and $N^n$ satisfies the levelling property of dimension $p$.

Case $p = m$: The set $C^\infty(\overline{Q^m}; N^n)$ is dense in $W^{1,m}(Q^m; N^n)$ if and only if $N^n$ satisfies the levelling property of dimension $m$.

We now describe the plan of the paper. In the next section, we present a counterexample to the density of bounded maps in $W^{1,m}(Q^m; N^n)$ where $N^n$ is chosen as a suitable imbedding of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$. In Section 3 we have collected the main tools that will be used in the proofs of Theorem 1 and Theorem 2. Two of these tools, the zero degree homogenization and the adaptive smoothing, are standard in the context of Sobolev spaces with values into manifold. The third one, the opening technique, was introduced by Brezis and Li [6], and then pursued by the authors [5] in the framework of higher order Sobolev spaces $W^{k,p}(Q^m; N^n)$ for every $k \in \mathbb{N}$. Given a map $u \in W^{1,p}(Q^m; N^n)$ and an $\ell$ dimensional grid in $Q^m$, $0 \leq \ell \leq m-1$, one can use the opening technique by slightly modifying $u$ on a neighborhood of the grid to obtain a new map whose restriction to the grid belongs to $W^{1,p}$.

The interplay between the density of bounded maps and the geometry of the target manifold $N^n$ is described in Section 4 where we prove some characterizations of the levelling property given in Definition 1.1. We present the proofs of Theorem 1 and Theorem 2 in Section 5. Our strategy is based on the good and bad cubes method introduced by Bethuel [2] in the setting of a compact target manifold $N^n$. The idea is to approximate a map $u \in W^{1,p}(Q^m; N^n)$ in two different ways, depending on the oscillations of $u$. More precisely, one divides the domain $Q^m$ in a disjoint union of small cubes. Then, on a good cube where $u$ does not oscillate too much, $u$ is approximated by convolution with a smooth kernel.

On a bad cube instead, one uses the zero degree homogenization technique. When the target manifold is noncompact several difficulties arise compared to the compact case. In particular, oscillations of the map $u$ on a small cube cannot be estimated in terms of the $W^{1,p}$ norm only. The essential range of $u$ restricted to each of these small cubes must also be taken into account, before proceeding to the regularization by convolution. At this step, the levelling property is required to treat the case of integer exponents $p$. In the last section, we briefly show how the density of smooth maps in $W^{1,p}(Q^m; N^n)$ follows from the density of bounded maps (Theorem 1 and Theorem 2) and the density results in the setting of compact target manifolds.
2. Lack of Strong Density in $W^{1,m}(Q^n; N^n)$

In this section we give an example of a complete manifold for which $(W^{1,m} \cap L^\infty)(Q^n; N^n)$ is not strongly dense in $W^{1,m}(Q^n; N^n)$. For this purpose, we first observe that smooth maps with values into $N^n$ are always dense in $(W^{1,m} \cap L^\infty)(Q^n; N^n)$, and this essentially follows from the seminal work of Schoen and Uhlenbeck for compact manifolds [15].

For the convenience of the reader, we provide a proof based on their argument. Let us introduce some notation that will be used throughout the paper: we denote by $\Pi : O \to N^n$ the nearest point projection to $N^n$ from a neighborhood $O \supset N^n$ where $\Pi$ is smooth and $D\Pi \in L^\infty(O)$.

**Proposition 2.1.** For every $u \in (W^{1,m} \cap L^\infty)(Q^n; N^n)$, there exists a sequence in $C^\infty(Q^n; N^n)$ converging strongly to $u$ in $W^{1,m}$.

**Proof.** We extend the map $u$ by reflection on the larger cube $Q^n_2$ with radius 2. Hence, we may assume that $u \in (W^{1,m} \cap L^\infty)(Q^n_2; N^n)$. Let $K \subset N^n$ be a compact subset such that $u(x) \in K$ for almost every $x \in Q^n$. Given a family of mollifiers $(\varphi_{\varepsilon})_{\varepsilon > 0}$ of the form $\varphi_{\varepsilon}(z) = \frac{1}{\varepsilon^n} \varphi\left(\frac{z}{\varepsilon}\right)$, by the Poincaré-Wirtinger inequality we have, for every $x \in Q^n$ and for every $0 < \varepsilon < 1$,

$$\text{dist}_{R^n}(\varphi_{\varepsilon} * u(x), K)^m \leq \int_{B^n_R(x)} |\varphi_{\varepsilon} * u(y) - u(y)|^m \, dy \leq C_1 \int_{B^n_R(x)} |Du|^m,$$

for some constant $C_1 > 0$ independent of $\varepsilon$.

Taking $\underline{t}_K > 0$ such that $K + B^n_{\varepsilon K} \subset O$, we deduce from the estimate above that there exists $\varepsilon > 0$ such that, for every $0 < \varepsilon \leq \varepsilon$ and for every $x \in Q^n$, we have

$$\text{dist}_{R^n}(\varphi_{\varepsilon} * u(x), K) \leq \underline{t}_K.$$  

To conclude, it suffices to consider the sequence $(\Pi(\varphi_{\varepsilon} * u))_{\varepsilon \in \mathbb{N}}$ for some positive sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ converging to zero. □

We state the main result of this section as follows:

**Proposition 2.2.** Let $\nu \in \mathbb{N}_+$, $m, n \in \mathbb{N}_+$ be such that $m \geq n \geq 2$, and let $a \in S^n$. For every smooth imbedding $F : S^n \setminus \{a\} \to \mathbb{R}^\nu$ such that $\lim_{x \to a} |F(x)| = +\infty$ and $DF \in L^n(S^n \setminus \{a\})$, the manifold $N^n = F(S^n \setminus \{a\})$, equipped with the metric induced by the Euclidean distance in $\mathbb{R}^\nu$, is complete, but $(W^{1,n} \cap L^\infty)(Q^n; N^n)$ is not strongly dense in $W^{1,n}(Q^n; N^n)$.

For instance, one may take $F(x) = \lambda(x)x$, where $\lambda : S^n \setminus \{a\} \to \mathbb{R}$ is a positive smooth function such that $\lambda \in W^{1,n}(S^n; \mathbb{R})$ but $\lambda \not\in L^\infty(S^n; \mathbb{R})$. This is always possible in dimension $n \geq 2$, and an example is given by

$$\lambda(x) = \left(\log \frac{1}{\text{dist}_{S^n}(x, a)}\right)^\alpha$$

for $x$ in a neighborhood of $a$ and for any exponent $0 < \alpha < \frac{n-1}{m}$.

Let us also give an example of an algebraic complete manifold $N^n$ for which $(W^{1,n} \cap L^\infty)(Q^n; N^n)$ is not strongly dense in $W^{1,n}(Q^n; N^n)$:

$$N^n = \left\{y = (y', y_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+ : |y'|^2 = \frac{y_{n+1}^2}{(1 + y_{n+1})^{2\beta - 1}}\right\}$$

where $\beta > \frac{n}{n-1}$. The lack of density can be obtained using the map $u : Q^n \to N^n$.
defined for \( x \in Q^n \) by
\[
(2.1) \quad u(x) = \left( \frac{\sqrt{w(|x|)}}{1 + w(|x|)} \right)^{n-4} x^i w(|x|),
\]
where \( w : (0, \infty) \to \mathbb{R}_+ \) is a smooth function such that \( w(r) = \| \log r \|^\gamma \) for \( r \in (0, 1/3), w(r) = 0 \) for \( r \in (2/3, \infty) \) and \( w'/\sqrt{w} \) is bounded on \((1/3, 2/3)\). Then, \( u \) belongs to \( W^{1, n}(Q^n; N^n) \) provided that \( \frac{1}{n(n-1)} < \gamma < \frac{n-1}{n} \). A mere adaptation of the proof of Proposition 2.2 shows that there exists no sequence of maps in \( C^\infty(Q^n; N^n) \) converging strongly to \( u \) in \( W^{1, n}(Q^n; N^n) \).

Hajlasz and Schikorra [8, Section 3] have provided examples of noncompact manifolds \( N^n \) for which Lipschitz maps are not strongly dense in \( W^{1, n}(Q^n; N^n) \). Instead of taking an imbedding \( F \) that blows up at some point \( a \), they construct an imbedding that strongly oscillates in a neighborhood of \( a \).

**Proof of Proposition 2.2** We first assume that \( m = n \). Let \( f : \overline{Q^n} \to S^n \) be a diffeomorphism between \( \overline{Q^n} \) and a closed neighborhood of \( a \) in \( S^n \). We claim that the function \( u = F \circ f \) belongs to \( W^{1, n}(Q^n; N^n) \) but \( u \) cannot be approximated by bounded maps in \( W^{1, n}(Q^n; N^n) \). We assume by contradiction that the set \( (W^{1, n} \cap L^\infty)(Q^n; N^n) \) is dense in \( W^{1, n}(Q^n; N^n) \). Then, by density of \( C^\infty(Q^n; N^n) \) in the former space (Proposition 2.1), there exists a sequence of maps \((u_k)_{k \in \mathbb{N}}\) in \( C^\infty(Q^n; N^n) \) converging strongly to \( u \) in \( W^{1, n}(Q^n; N^n) \).

Without loss of generality, we may assume that \( f(0) = a \). Given a compact set \( K \subset N^n \) with \( H^n(K) > 0 \), since the imbedding \( F \) diverges at the point \( a \), there exists \( 0 < \delta < 1 \) such that
\[
K \cap u(Q^n_\delta) = \emptyset.
\]

By a Fubini-type argument, there exists a subsequence \((u_{k_j})_{j \in \mathbb{N}}\) such that, for almost every \( r \in (0, 1) \), \((u_{k_j}|_{\partial Q^n_r})_{j \in \mathbb{N}}\) converges to \( u|_{\partial Q^n_r} \) in \( W^{1, n}(\partial Q^n_r) \), whence also uniformly by the Morrey–Sobolev inequality. Since for each such \( r \leq \delta \) we have \( K \cap u(\partial Q^n_r) = \emptyset \), by uniform convergence of \((u_{k_j}|_{\partial Q^n_r})_{j \in \mathbb{N}}\) there exists \( J_r \in \mathbb{N} \) such that, for every \( j \geq J_r \),
\[
\|u_{k_j} - u\|_{L^\infty(\partial Q^n_r)} < \text{dist}(K, u(\partial Q^n_r)).
\]

In particular, \( K \cap u_{k_j}(Q^n_r) = \emptyset \).

We claim that
\[
(2.2) \quad K \subset u_{k_j}(Q^n_r).
\]
In order to prove this, let us introduce a homeomorphism $g : \overline{Q^m} \to \mathbb{S}^n \setminus f(Q^n_m)$ such that $g|_{\partial Q^m_n} = f|_{\partial Q^m_n}$. Since $K \cap (F \circ f(Q^n_m)) = \emptyset$, this implies that

$$F \circ g(Q^n_m) = F(S^n \setminus f(Q^n_m)) \supset K.$$ 

By continuity of the Brouwer degree with respect to uniform convergence, we have for every $y \in K$ and for every $j \geq J_r$,

$$\deg (u_{k_j}, Q^n_m, y) = \deg (F \circ g, Q^n_m, y) \neq 0.$$ 

This implies the claim \eqref{2.2}.

By monotonicity of the Hausdorff measure and by the area formula, we then have

$$\mathcal{H}^n(K) \leq \mathcal{H}^n(u_{k_j}(Q^n_m)) \leq \int_{Q^n_m} |\det Du_{k_j}|.$$

Using the pointwise inequality $|\det Du_{k_j}| \leq C_1 |Du_{k_j}|^n$, as $j$ tends to infinity we get

$$\mathcal{H}^n(K) \leq C_1 \int_{Q^n_m} |Du|^n.$$ 

Since the right-hand side tends to zero as $r$ tends to zero, we have a contradiction.

If $m > n$, we consider the map $u \circ \pi$, where $\pi : Q^m = Q^n \times Q^{m-n} \to Q^n$ is the canonical projection on the first component. Assume by contradiction that $(\nu_k)_{k \in \mathbb{N}}$ is a sequence of maps in $(W^{1,n} \cap L^\infty)(Q^m; N^n)$ that converges strongly to $u \circ \pi$ in $W^{1,n}(Q^m; N^n)$. Since, for every $k \in \mathbb{N}$, we have

$$\int_{Q^{m-n}} \left( \int_{Q^n} |Dv_k(\cdot, y) - Du(\cdot)|^n \right) dy = \int_{Q^m} |Dv_k - D(u \circ \pi)|^n,$$

and since the integral in the right-hand side converges to zero as $k$ tends to infinity, we deduce that there exists a sequence $(\nu_k)_{k \in \mathbb{N}}$ in $Q^{m-n}$ such that the sequence of maps $(v_k(\cdot, y_k))_{k \in \mathbb{N}}$ converges strongly to the map $u$ in $W^{1,n}(Q^n; N^n)$. In view of the first part of the proof, we obtain a contradiction. \hfill \qed

3. Main tools

In this section we explain the main tools used in the proofs of Theorem 1 and Theorem 2.

3.1. The opening technique. We recall the opening technique of maps introduced by Brezis and Li \cite{BrezisLi} and pursued in \cite{BrezisLi} Proposition 2.1. We first present this tool in the simplest situation. Given radii $0 < \rho < \bar{r}$ and a map $u \in W^{1,p}(Q^m_{\bar{r}}; \mathbb{R}^n)$, we want to construct a map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that $u \circ \Phi$ is constant on the smaller cube $Q^m_{\rho}$, $u \circ \Phi = u$ on a neighborhood of the boundary of the larger cube $\partial Q^m_{\bar{r}}$, and $u \circ \Phi \in W^{1,p}(Q^m_{\rho}; \mathbb{R}^n)$.

The opening construction goes as follows. We fix $\hat{\rho} < \frac{\bar{r} - \rho}{2}$ and a smooth map $\xi : \mathbb{R}^m \to \mathbb{R}^m$ such that $\xi = 0$ on $Q^m_{\hat{\rho} + \rho}$ and $\xi(x) = x$ for every $x \notin Q^m_{\hat{\rho} + \rho}$. For every $z \in Q^m_{\hat{\rho}}$, we then consider the map $\Phi_z(x) = \xi(x + z) - z$. We observe that the map $u \circ \Phi_z$ is constant on the smaller cube $Q^m_{\rho}$ and agrees with $u$ on the complement of the larger cube $\mathbb{R}^m \setminus Q^m_{\bar{r}}$. By further assuming that $\xi(Q^m_{\hat{\rho} - \rho}) \subset Q^m_{\rho - \rho}$, we also have...
\[ \Phi_z(x) = \xi(x + z) - z \in Q^m_\rho, \text{ for every } (x, z) \in Q^m_\rho \times Q^m_\rho. \] We now estimate the integral
\[ \int_{Q^m_\rho \times Q^m_\rho} \left( |u|^p + |Du|^p \right) (\xi(x + z) - z) \, dx \, dz. \]

By the change of variables \((x, z) \mapsto (x + z, \xi(x + z) - z)\) whose Jacobian is identically \(-1\), this quantity is seen to be not larger than
\[ |Q^m_\rho| \int_{Q^m_\rho} \left( |u|^p + |Du|^p \right). \]

Hence, by Fubini’s theorem, there exists \(z \in Q^m_\rho\) such that \(u \circ \Phi_z \in W^{1,p}(Q^m_\rho)\) and
\[ \|D(u \circ \Phi_z)\|_{L^p(Q^m_\rho)} \leq C_1 \|Du\|_{L^p(Q^m_\rho)}. \]

The map \(\Phi := \Phi_z\) then satisfies all the required properties.

More generally, one can open a map along the normals to a planar set, namely for every \(\ell \in \{0, \ldots, m - 1\}\). The following statement coincides with \([5, \text{Proposition 2.2}]\) and we omit the proof.

**Proposition 3.1.** Let \(\ell \in \{0, \ldots, m - 1\}\), \(\eta > 0\), \(0 < p < \bar{p}\) and \(A \subset \mathbb{R}^\ell\) be an open set. For every \(u \in W^{1,p}(A \times Q^{m-\ell}_\eta; \mathbb{R}^\ell)\), there exists a smooth map \(\zeta: \mathbb{R}^{m-\ell} \to \mathbb{R}^{m-\ell}\) such that
\begin{enumerate}[(i)]  
  \item \(\zeta\) is constant in \(Q^{m-\ell}_\eta\),  
  \item \(\{x \in \mathbb{R}^m : \zeta(x) \neq x\} \subset Q^{m-\ell}_\eta\) and \(\zeta(Q^{m-\ell}_\eta) \subset Q^{m-\ell}_\eta\),  
  \item if \(\Phi: \mathbb{R}^m \to \mathbb{R}^m\) is defined for every \(x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}\) by \(\Phi(x) = (x', \zeta(x''))\),
\end{enumerate}
then \(u \circ \Phi \in W^{1,p}(A \times Q^{m-\ell}_\eta; \mathbb{R}^\ell)\) and
\[ \|D(u \circ \Phi)\|_{L^p(A \times Q^{m-\ell}_\eta)} \leq C \|Du\|_{L^p(A \times Q^{m-\ell}_\eta)}, \]
for some constant \(C > 0\) depending on \(m, p, \rho\) and \(\bar{p}\).

Since the map \(u\) is only defined almost everywhere and \(\Phi\) is not one to one, the meaning of the composition \(u \circ \Phi\) is not obvious. This is clarified in \([5, \text{Lemma 2.3}]\). We only need to point out here that the essential range of \(u \circ \Phi\) is contained in the essential range of \(u\). We also observe that \(\Phi\) implies that \(\Phi\) is constant on the \(m - \ell\) dimensional cubes of radius \(\rho\) which are orthogonal to \(A\). The map \(u \circ \Phi\) thus only depends on \(\ell\) variables in a neighborhood of \(A\).

In order to present the opening technique in the framework of cubizations, we first need to introduce some vocabulary. First, given a set \(A \subset \mathbb{R}^m\) and \(\eta > 0\), a cubication of \(A\) of radius \(\eta > 0\) is a family of closed cubes \(S^m_\eta\) of radius \(\eta\) such that
\begin{enumerate}[(1)]  
  \item \(\bigcup_{\sigma \in S^m_\eta} \sigma^m = A\),  
  \item for every \(\sigma^1_\eta, \sigma^2_\eta \in S^m_\eta\) which are not disjoint, \(\sigma^1_\eta \cap \sigma^2_\eta\) is a common face of dimension \(i \in \{0, \ldots, m\}\).
\end{enumerate}

For \(\ell \in \{0, \ldots, m\}\), the set \(S^\ell\) of all \(\ell\) dimensional faces of all cubes in \(S^m_\eta\) is called the skeleton of dimension \(\ell\). We then denote by \(S^\ell\) the union of all elements of \(S^\ell\):
\[ S^\ell = \bigcup_{\sigma^\ell \in S^\ell} \sigma^\ell. \]
A subskeleton $E^\ell$ of $S^\ell$ is simply a subfamily of $S^\ell$. Accordingly, given an open set $A$ in $\mathbb{R}^m$ equipped with a cubication, a subskeleton of $A$ is a subfamily of the $\ell$ dimensional skeleton of the given cubication.

We proceed to state the main result of this section, which is essentially [5] Proposition 2.1.

**Proposition 3.2.** Let $p \geq 1$, $\ell \in \{0, \ldots, m-1\}$, $\eta > 0$, $0 < \rho < \frac{1}{2}$, and $E^\ell$ be a subskeleton of $\mathbb{R}^m$ of radius $\eta$. Then, for every $u \in W^{1,p}(E^\ell + Q^{m}_{2\rho \eta}; \mathbb{R}^\ell)$, there exists a smooth map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that

(i) for every $i \in \{0, \ldots, \ell\}$ and for every $i$ dimensional face $\sigma^i \in E^i$, $\Phi$ is constant on the $m-i$ dimensional cubes of radius $\rho \eta$ which are orthogonal to $\sigma^i$,

(ii) $\{x \in \mathbb{R}^m : \Phi(x) \neq x\} \subset E^\ell + Q^{m}_{2\rho \eta}$ and for every $\sigma^\ell \in E^\ell$, $\Phi(\sigma^\ell + Q^{m}_{2\rho \eta}) \subset \sigma^\ell + Q^{m}_{2\rho \eta}$,

(iii) $u \circ \Phi \in W^{1,p}(E^\ell + Q^{m}_{2\rho \eta}; \mathbb{R}^\ell)$, and

$$\|D(u \circ \Phi)\|_{L^p(E^\ell + Q^{m}_{2\rho \eta})} \leq C\|Du\|_{L^p(E^\ell + Q^{m}_{2\rho \eta})},$$

for some constant $C > 0$ depending on $m, p$ and $\rho$,

(iv) for every $\sigma^\ell \in E^\ell$,

$$\|D(u \circ \Phi)\|_{L^p(\sigma^\ell + Q^{m}_{2\rho \eta})} \leq C'\|Du\|_{L^p(\sigma^\ell + Q^{m}_{2\rho \eta})},$$

for some constant $C' > 0$ depending on $m, p$ and $\rho$.

For the convenience of the reader, and also because assertion (ii) in Proposition 3.2 is slightly more precise than the corresponding statement in [5], we sketch its proof.

**Proof of Proposition 3.2.** Let $\rho = \rho_0 < \cdots < \rho_i < \cdots < \rho_0 < 2\rho$. We define by induction on $i \in \{0, \ldots, \ell\}$ a map $\Phi^i : \mathbb{R}^m \to \mathbb{R}^m$ such that

(a) for every $r \in \{0, \ldots, i\}$ and every $\sigma^r \in E^r$, $\Phi^i$ is constant on the $m-r$ dimensional cubes of radius $\rho_i \eta$ which are orthogonal to $\sigma^r$,

(b) $\{x \in \mathbb{R}^m : \Phi^i(x) \neq x\} \subset E^\ell + Q^{m}_{2\rho \eta}$ and, for every $\sigma^i \in E^i$, $\Phi^i(\sigma^i + Q^{m}_{2\rho \eta}) \subset \sigma^i + Q^{m}_{2\rho \eta}$,

(c) $u \circ \Phi^i \in W^{1,p}(E^\ell + Q^{m}_{2\rho \eta}; \mathbb{R}^\ell)$,

(d) for every $\sigma^i \in E^i$,

$$\|D(u \circ \Phi^i)\|_{L^p(\sigma^i + Q^{m}_{2\rho \eta})} \leq C\|Du\|_{L^p(\sigma^i + Q^{m}_{2\rho \eta})},$$

for some constant $C > 0$ depending on $m, p$ and $\rho$.

The map $\Phi^\ell$ will satisfy the conclusion of the proposition.

For $i = 0$, $E^0$ is the set of vertices of cubes in $E^m$. The map $\Phi^0$ is obtained by applying Proposition 3.1 to $u$ around each $\sigma^0 \in E^0$ with parameters $\rho_0 < 2\rho$ and $\ell = 0$.

Assume that the maps $\Phi^0, \ldots, \Phi^{i-1}$ have been constructed. We then apply Proposition 3.1 to $u \circ \Phi^{i-1}$ around each $\sigma^i \in E^i$ with parameters $\rho_i < \rho_{i-1}$. This gives a smooth map $\Phi^i : \mathbb{R}^m \to \mathbb{R}^m$ which is constant on the $m-i$ dimensional cubes of radius $\rho_i \eta$ orthogonal to $\sigma^i$.

Let $\sigma^1, \sigma^2 \in E^\ell$ such that

$$(\sigma^1 + Q^{m}_{\rho_{i-1} \eta}) \cap (\sigma^2 + Q^{m}_{\rho_{i-1} \eta}) \neq \emptyset.$$  

We claim that for every $x$ in this set,

$$\Phi^{i-1}(\sigma^1(x)) = \Phi^{i-1}(\sigma^2(x)) = \Phi^{i-1}(x).$$  

(3.1)
Indeed, $\sigma_1^i$ and $\sigma_2^i$ are not disjoint and there exists $r \in \{0, \ldots, i - 1\}$ minimal and $\tau^r \in \mathcal{E}^i$ such that $\tau^r \subset \sigma_1^i \cap \sigma_2^i$ with $x \in \tau^r + Q_{\rho_1 - 1\eta}^{m}$. By the formula of $\Phi_{\sigma_2^i}$ given in Proposition 3.1 all the points $\Phi_{\sigma_2^i}(x), \Phi_{\sigma_2^i}(x)$ and $x$ belong to the same $m - r$ dimensional cube of radius $\rho_{i - 1}\eta$ which is orthogonal to $\tau^r$. By induction, $\Phi^{i - 1}$ is constant on the $m - r$ dimensional cubes of radius $\rho_{i - 1}\eta$ which are orthogonal to $\tau^r$. This proves claim (3.1).

We can thus define the map $\Phi^i : \mathbb{R}^m \rightarrow \mathbb{R}^m$ as follows:

$$
\Phi^i(x) = \begin{cases} 
  \Phi^{i - 1}(\Phi_{\sigma^i}(x)) & \text{if } x \in \sigma^i + Q_{\rho_{i - 1}\eta}^{m}, \\
  \Phi^{i - 1}(x) & \text{otherwise.}
\end{cases}
$$

Assertion (\textcircled{a}) follows from the above discussion which implies in particular that $\Phi^i = \Phi^{i - 1}$ on $\mathcal{E}^{i - 1} + Q_{\rho_{i - 1}\eta}^{m}$.

We proceed with the proof of assertion (\textcircled{b}). By definition of the map $\Phi^i$, we have $\Phi^i = \Phi^{i - 1}$ on $\mathbb{R}^m \setminus (\mathcal{E}^{i - 1} + Q_{\rho_{i - 1}\eta}^{m})$. By induction, $\Phi^{i - 1}$ agrees with the identity outside $\mathcal{E}^{i - 1} + Q_{\rho_{i - 1}\eta}^{m}$. Hence, $$\{x \in \mathbb{R}^m : \Phi^i(x) \neq x\} \subset \mathcal{E}^{i - 1} + Q_{\rho_{i - 1}\eta}^{m}.$$ Moreover, by induction, for every $\tau^i - 1 \in \mathcal{E}^{i - 1}$,

$$\Phi^{i - 1}(\tau^i - 1 + Q_{\rho_{i - 1}\eta}^{m}) \subset \tau^i - 1 + Q_{\rho_{i - 1}\eta}^{m}.$$ Thus for every $\sigma^i \in \mathcal{E}^{i}$, $\Phi^{i - 1}(\partial \sigma^i + Q_{\rho_{i - 1}\eta}^{m}) \subset \partial \sigma^i + Q_{\rho_{i - 1}\eta}^{m}$. Since $\Phi^{i - 1}(x) = x$ for $x \notin \mathcal{E}^{i - 1} + Q_{\rho_{i - 1}\eta}^{m}$ and $(\mathcal{E}^{i - 1} + Q_{\rho_{i - 1}\eta}^{m}) \cap (\sigma^i + Q_{\rho_{i - 1}\eta}^{m}) \subset \partial \sigma^i + Q_{\rho_{i - 1}\eta}^{m}$ it follows that

$$
(3.2) \\
\Phi^{i - 1}(\sigma^i + Q_{\rho_{i - 1}\eta}^{m}) \subset \sigma^i + Q_{\rho_{i - 1}\eta}^{m}.
$$

Now, let $x \in \sigma^i + Q_{\rho_{i - 1}\eta}^{m}$. If $x \notin \mathcal{E}^i + Q_{\rho_{i - 1}\eta}^{m}$ then $\Phi^i(x) = \Phi^{i - 1}(x)$. From (3.2), it follows that $\Phi^i(x) \in \sigma^i + Q_{\rho_{i - 1}\eta}^{m}$. We now assume that $x \in \mathcal{E}^i + Q_{\rho_{i - 1}\eta}^{m}$. Let $\tau^i \in \mathcal{E}^i$ be such that $x \in \tau^i + Q_{\rho_{i - 1}\eta}^{m}$. Then the cube $\tau^i := \tau^i \cap \sigma^i$ is not empty, $x \in \tau^i + Q_{\rho_{i - 1}\eta}^{m}$ and from the form of $\Phi_{\tau^i}$ we deduce that $\Phi_{\tau^i}(x) \in \tau^i + Q_{\rho_{i - 1}\eta}^{m}$.

In particular, $\Phi_{\tau^i}(x) \in \sigma^i + Q_{\rho_{i - 1}\eta}^{m}$ and thus by (3.2), $\Phi^i(x) = \Phi^{i - 1}(\Phi_{\tau^i}(x)) \in \sigma^i + Q_{\rho_{i - 1}\eta}^{m}$. This completes the proof of (\textcircled{b}).

The proof of (\textcircled{c}) and (\textcircled{d}) are the same as in (\textcircled{b}) and we omit them. \hfill \Box

Using the notation of Proposition 3.2 we have the following estimate:

Remark 3.1. If $\ell \leq p + 1$, then for every $\tau^\ell - 1 \in \mathcal{E}^{\ell - 1}$ and for every $Q_{\rho_{\ell - 1}\eta}^{m}(x) \subset \tau^{\ell - 1} + Q_{\rho_{\ell - 1}\eta}^{m}$,

$$
(3.3) \\
\frac{1}{\eta^{m - p}} \int_{Q_{\rho_{\ell - 1}\eta}^{m}(x)} |D(u \circ \Phi)|^p \leq \frac{C}{\eta^{m - p}} \int_{\tau^{\ell - 1} + Q_{\rho_{\ell - 1}\eta}^{m}} |D(u \circ \Phi)|^p.
$$

This estimate follows from the fact that $\Phi$ is constant on the $m - \ell + 1$ dimensional cubes of radius $\rho_{\ell - 1}\eta$ which are orthogonal to $\tau^{\ell - 1}$. Indeed, without loss of generality, we can assume that $\tau^{\ell - 1}$ has the form $[-\eta, \eta]^{\ell - 1} \times \{0\}^\ell$, where $0^\ell \in \mathbb{R}^{m - \ell + 1}$.

Accordingly, we write every $y \in \tau^{\ell - 1} + Q_{\rho_{\ell - 1}\eta}^{m}$ as $y = (y', y'\prime) \in \mathcal{E}^{\ell - 1} \times \mathbb{R}^{m - \ell + 1}$. By construction, for every $y \in \tau^{\ell - 1} + Q_{\rho_{\ell - 1}\eta}^{m}$, $(u \circ \Phi)(y') = (u \circ \Phi)(y', y'\prime)$ for some
$a'' \in \mathbb{R}^{m-\ell}$. This implies

\[
\int_{Q^\ell_m(x)} |D(u \circ \Phi)(y)|^p \, dy \leq C_2 r^{m-\ell+1} \int_{Q^{\ell-1}(x')} |D(u \circ \Phi)(y', a'')|^p \, dy' \\
\leq C_2 r^{m-\ell+1} \int_{Q^{\ell-1}(x') \times Q_m^{m-\ell+1}(0^m)} |D(u \circ \Phi)(y', y'')|^p \, dy' \, dy'' \\
\leq C_2 r^{m-\ell+1} \int_{\tau^{\ell-1} + Q_m^{m-\ell+1}} |D(u \circ \Phi)(y', y'')|^p \, dy' \, dy''.
\]

Hence,

\[
\frac{1}{r^{m-p}} \int_{Q^\ell_m(x)} |D(u \circ \Phi)|^p \leq \frac{C_5}{\eta^{m-p}} \left( \frac{r}{\eta} \right)^{p-\ell+1} \int_{\tau^{\ell-1} + Q_m^{m-\ell+1}} |D(u \circ \Phi)|^p,
\]

which proves estimate (3.3) since $r \leq \eta$ and $\ell \leq p + 1$.

\[\square\]

3.2. Adaptive smoothing. A second tool already used in [5] is adaptive smoothing, in which the function is smoothened by mollification at a variable scale.

**Proposition 3.3.** Let $\Omega$ be an open set in $\mathbb{R}^m$. Let $\varphi \in C^\infty_c(B^n)$ be a mollifier and let $\psi \in C^\infty(\Omega)$ be a nonnegative function such that $\|D\psi\|_{L^\infty(\Omega)} < 1$. For every $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ and for every open set $\omega \subset \{ x \in \Omega : \text{dist} (x, \partial \Omega) \geq \psi(x) \}$, we define the map $\varphi_\psi * u : \omega \to \mathbb{R}^n$ for each $x \in \omega$ by

\[
\varphi_\psi * u(x) = \frac{1}{\psi(x)m^m} \int \varphi\left( \frac{x - z}{\psi(x)} \right) \, dz.
\]

Then, $\varphi_\psi * u \in W^{1,p}(\omega; \mathbb{R}^n)$,

\[
\| \varphi_\psi * u - u \|_{L^p(\omega)} \leq \sup_{v \in B^m_t} \| \tau_\psi(v)(u) - u \|_{L^p(\omega)},
\]

\[
\|D(\varphi_\psi * u)\|_{L^p(\omega)} \leq \frac{C}{(1 - \|D\psi\|_{L^\infty(\omega)})^2} \|Du\|_{L^p(\Omega)},
\]

and

\[
\|D(\varphi_\psi * u) - Du\|_{L^p(\omega)} \leq \sup_{v \in B^m_t} \| \tau_\psi(v)(Du) - Du\|_{L^p(\omega)} + \frac{C'}{(1 - \|D\psi\|_{L^\infty(\omega)})^2} \|Du\|_{L^p(A)},
\]

for some constants $C > 0$ and $C' > 0$ depending on $m$ and $p$, where

\[
A = \bigcup_{x \in \omega \cap \text{supp } D\psi} B^m_{\psi(x)}(x).
\]

In this statement, $\tau_\psi(u) : \Omega + v \to \mathbb{R}^n$ denotes the translation of the map $u : \Omega \to \mathbb{R}^n$ with respect to the vector $v \in \mathbb{R}^m$ defined for each $x \in \Omega + v$ by $\tau_\psi(u)(x) = u(x - v)$.

3.3. Zero degree homogenization. We use this tool to extend a Sobolev map $u$ defined on the boundary of a star-shaped domain to the whole domain, by preserving the range of $u$. We first recall the notion of a Sobolev map on skeletons [7]:

**Definition 3.1.** Given $p \geq 1$, $\ell \in \{0, \ldots, m\}$, and an $\ell$ dimensional skeleton $S^\ell$, we say that a map $u$ belongs to $W^{1,p}(S^\ell; \mathbb{R}^n)$ if
where



the property satisfied by this construction is the following:



there exists $v$ in the sense of traces, and



Proposition 3.4. If $\ell \in \{1, \ldots, m\}$, $\eta > 0$ and $a \in \mathbb{R}^m$, we may consider the boundary of the cube $Q_\eta^i(a)$ as an $\ell - 1$ dimensional skeleton so that $W^{1, p}(\partial Q_\eta^i(a); \mathbb{R}^r)$ has a well-defined meaning in the sense of Definition 3.1.

Given $i \in \mathbb{N}$ and $\eta > 0$, the homogenization of degree 0 of a map $u : \partial Q_\eta^i(a) \to \mathbb{R}^r$ is the map $v : Q_\eta^i(a) \to \mathbb{R}^r$ defined for $x \in Q_\eta^i(a)$ by



(3.4)

$v(x) = u\left(\frac{x - a}{\|x - a\|_\infty}\right)$,

where $|y|_\infty = \max\{|y_1|, \ldots, |y_i|\}$ denotes the maximum norm in $\mathbb{R}^i$. The basic property satisfied by this construction is the following:

Proposition 3.4. If $1 \leq p < i$, then for every $u \in W^{1, p}(\partial Q_\eta^i(a); \mathbb{R}^r)$, the map $v : Q_\eta^i(a) \to \mathbb{R}^r$ defined in (3.4) belongs to $W^{1, p}(Q_\eta^i, \mathbb{R}^r)$ and



Iterating the zero degree homogenization above, we may extend Sobolev functions defined on lower dimensional subskeletons of $\mathbb{R}^m$ to an $m$ dimensional subskeleton. We apply this strategy to prove the following proposition that will be used in the proof of Theorems 1 and 2.

Proposition 3.5. Let $\ell \in \{0, \ldots, m - 1\}$, $\eta > 0$, $\mathcal{E}^m$ be a subskeleton of $\mathbb{R}^m$ of radius $\eta$ and $S^{m-1}$ be a subfamily of $\mathcal{E}^{m-1}$. If $p < \ell + 1$, then for every measurable function $u : E^\ell \cup S^{m-1} \to \mathbb{R}^r$ such that



(i) $u|_{E^\ell} \in W^{1, p}(E^\ell; \mathbb{R}^r)$,

(ii) for every $i \in \{\ell + 1, \ldots, m - 1\}$, $u|_{S^i} \in W^{1, p}(S^i; \mathbb{R}^r)$,

there exists $v \in W^{1, p}(E^m; \mathbb{R}^r)$ such that $v(E^m) \subset u(E^\ell \cup S^{m-1})$, $v = u$ on $S^{m-1}$ in the sense of traces, and



Proof. Let $v^i : E^\ell \to \mathbb{R}^r$ be defined by $v^i = u$ in $E^i$. We define by induction on $i \in \{\ell + 1, \ldots, m\}$ a map $v^i : E^i \to \mathbb{R}^r$ as follows:

(a) for every $\sigma^i \in E^i \setminus S^i$, we apply the zero degree homogenization on the face $\sigma^i$ to define $v^i$ on $\sigma^i$, so that $v^i = v^{i-1}$ on $\partial \sigma^i$ in the sense of traces and



(b) for every $\sigma^i \in S^i$, we take $v^i = u$ in $S^i$, whence





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We observe that with this definition we have \( v^i \in W^{1,p}(E^i; \mathbb{R}^p) \) since for any given \( \sigma_1, \sigma_2 \in \mathcal{E}^1 \) such that \( \sigma_1 \cap \sigma_2 \in \mathcal{E}^{i-1} \) we have \( v^i|_{\sigma_1} = v^i|_{\sigma_2} \) on \( \sigma_1 \cap \sigma_2 \) in the sense of traces.

From the estimates above we also deduce that
\[
\int_{E^i} |Dv^i|^p \leq C_2 \left( \eta \int_{E^{i-1}} |Dv^{i-1}|^p + \int_{S^i} |Du|^p \right).
\]
Iterating these estimates we get
\[
\int_{E^m} |Dv^m|^p \leq C_3 \left( \eta^{m-\ell} \int_{E^\ell} |Dv^\ell|^p + \sum_{i=\ell+1}^{m-1} \eta^{m-i} \int_{S^i} |Du|^p \right).
\]
From the construction of \( v^i \) we also have
\[
v^i(E^i) \subset v^{i-1}(E^{i-1}) \cup u(S^i).
\]
Iterating these inclusions we deduce that
\[
v^m(E^m) \subset v^i(E^i) \cup \bigcup_{i=\ell+1}^{m} u(S^i) \subset u(E^\ell \cup S^{m-1}).
\]
The function \( v^m \) satisfies the required properties. \( \square \)

4. Levelling property

The next proposition reformulates the levelling property (Definition 1.1) by replacing smooth maps with Sobolev maps.

**Proposition 4.1.** The manifold \( N^n \) satisfies the levelling property of dimension \( p \) if and only if, for every map \( u \in W^{1,p}(Q^p; N^n) \) such that \( u \in W^{1,p}(\partial Q^p; N^n) \), there exists \( v \in W^{1,p}(Q^p; N^n) \cap C^\infty(\overline{Q^p}; N^n) \) such that \( u = v \) on \( \partial Q^p \) in the sense of traces and
\[
\|Dv\|_{L^p(Q^p)} \leq C\|Du\|_{L^p(Q^p)}.
\]

**Proof.** We begin with the direct implication. For this purpose, let \( u \in W^{1,p}(Q^p; N^n) \) be such that \( u \in W^{1,p}(\partial Q^p; N^n) \). Given \( \frac{1}{2} \leq \lambda < 1 \), we set \( w : Q^p \rightarrow N^n \) to be the function defined for \( x \in Q^p \) by
\[
w(x) = \begin{cases} u(\frac{x}{|x|}) & \text{if } x \in Q^p_{\lambda}, \\ u(\frac{x}{r_1}) & \text{if } x \in Q^p \setminus Q^p_{\lambda}. \end{cases}
\]
Then \( w \in W^{1,p}(Q^p; N^n) \), \( w \) is continuous in \( Q^p \setminus Q^p_{\lambda} \) by the Morrey–Sobolev inequality and
\[
\int_{Q^p} |Dw|^p \leq C_1 \left( \int_{Q^p} |Du|^p + (1 - \lambda) \int_{\partial Q^p} |Du|^p \right).
\]
Take \( \lambda < r_1 < r_2 < 1 \) and \( 0 < \epsilon \leq \min \{ r_1 - \lambda, 1 - r_2 \} \). If \( (\varphi_\epsilon)_{\epsilon>0} \) is a family of smooth mollifiers, then the function \( \varphi_\epsilon * w \) is smooth on \( Q^p_{r_2} \setminus \overline{Q^p_{r_1}} \). Given \( \theta \in C^\infty_c(Q^p_{r_2} \setminus \overline{Q^p_{r_1}}) \), then for \( \epsilon > 0 \) small the function \( v : Q^p \rightarrow N^n \) defined for \( x \in Q^p \) by
\[
v(x) = \begin{cases} w(x) & \text{if } x \in Q^p \setminus (Q^p_{r_2} \setminus \overline{Q^p_{r_1}}), \\ \Pi((1 - \theta(x))w(x) + \theta(x)\varphi_\epsilon * w(x)) & \text{if } x \in Q^p_{r_2} \setminus \overline{Q^p_{r_1}}, \end{cases}
\]
is well-defined and belongs to $W^{1,p}(Q^p; N^n)$. Remember that $\Pi$ is the nearest point projection onto $N^n$ which is well defined and globally Lipschitz on a neighborhood $O$ of $N^n$. Here, we also use the fact that $w$ is continuous in $Q^p \setminus Q^p_1$. Moreover,

$$
\int_{Q^p} |Dv|^p \leq C_2 \left( \int_{Q^p} |Dw|^p + \|D\theta\|_{L^\infty(Q^p)} \int_{Q^p \setminus \overline{Q^p}} |w - \varphi_\varepsilon * w|^p \right) .
$$

Setting $\theta = 1$ on $\partial Q^p_1$ for some $r_1 < r < r_2$, then $v \in C^\infty(\partial Q^p_1; N^n)$. Applying the levelling property to the map $v$ on $Q^p_1$, there exists a map $\tilde{v} \in C^\infty(\overline{Q^p_1}; N^n)$ that coincides with $v$ on $\partial Q^p_1$ and such that

$$
\int_{Q^p_1} |D\tilde{v}|^p \leq C_3 \int_{Q^p_1} |Dv|^p .
$$

Extending $\tilde{v}$ as $v$ on $Q^p \setminus \overline{Q^p_1}$, we deduce from the estimates above that

$$
\int_{Q^p} |D\tilde{v}|^p \leq C_4 \left( \int_{Q^p} |Du|^p + (1 - \lambda) \int_{\partial Q^p} |Du|^p + \|D\theta\|_{L^\infty(Q^p)} \int_{Q^p_2 \setminus \overline{Q^p_1}} |w - \varphi_\varepsilon * w|^p \right) .
$$

To conclude the proof we may assume that $\int_{Q^p_1} |Du|^p > 0$. Choosing $\lambda$ close to 1 and then $\varepsilon > 0$ small, the second and third terms in the right-hand side can be controlled by $\int_{Q^p_1} |Du|^p$ and the direct implication follows.

In order to prove the converse implication, we take a map $f \in C^\infty(\partial Q^p_1; N^n)$ having an extension $u \in W^{1,p}(Q^p; N^n)$. By assumption, there exists a map $v \in W^{1,p}(Q^p; N^n) \cap C^0(\overline{Q^p}; N^n)$ such that $v|_{\partial Q^p} = f$ and

$$
\|Dv\|_{L^p(Q^p)} \leq C \|Du\|_{L^p(Q^p)} .
$$

Given $0 < \lambda < 1$, we fix a smooth extension $\tilde{f} \in C^\infty(\overline{Q^p} \setminus Q^p_1; N^n)$ of $f$. Given $0 < \lambda < r < \tau < 1$, we take $\theta \in C^\infty_c(\overline{Q^p} \setminus Q^p_1)$ such that $\theta = 1$ in $Q^p \setminus Q^p_1$. We note that, for $r$ close to 1 and for $\varepsilon > 0$ small, the function $\tilde{v} : Q^p \to N^n$ such that

$$
\tilde{v}(x) = \begin{cases} 
\tilde{f}(x) & \text{if } x \in Q^p_1 \\
\Pi((1 - \theta(x)) \varphi_\varepsilon \ast v(x) + \theta(x) \tilde{f}(x)) & \text{if } x \in Q^p \setminus Q^p_1, \\
\Pi(\varphi_\varepsilon \ast v(x)) & \text{if } x \in Q^p \setminus \overline{Q^p_1},
\end{cases}
$$

is well-defined and satisfies the estimate

$$
\int_{Q^p} |D\tilde{v}|^p \leq C_5 \left( \int_{Q^p} |Dv|^p + \int_{Q^p \setminus \overline{Q^p_1}} |D\tilde{f}|^p + \|D\theta\|_{L^\infty(Q^p)} \int_{Q^p \setminus \overline{Q^p_1}} |\varphi_\varepsilon \ast v - v|^p + \|D\theta\|_{L^\infty(Q^p)} \int_{Q^p \setminus \overline{Q^p_1}} |v - \tilde{f}|^p \right) .
$$

Since $v - \tilde{f} = 0$ on $\partial Q^p_1$, it follows from the Poincaré inequality that

$$
\int_{Q^p \setminus \overline{Q^p_1}} |v - \tilde{f}|^p \leq C_6 (1 - r)^p \int_{Q^p \setminus \overline{Q^p_1}} |D(v - \tilde{f})|^p .
$$

Taking $r < \tau < 1$ and $\theta$ such that $(1 - r)\|D\theta\|_{L^\infty(Q^p)} \leq C$, we get

$$
\int_{Q^p} |D\tilde{v}|^p \leq C_7 \left( \int_{Q^p} |Dv|^p + \int_{Q^p \setminus \overline{Q^p_1}} |D\tilde{f}|^p + (1 - r)^{-p} \int_{Q^p \setminus \overline{Q^p_1}} |\varphi_\varepsilon \ast v - v|^p \right) .
$$
We now assume that $\int_{Q^m} |Du|^p > 0$. The first integral in the right-hand side is by assumption estimated by $\int_{Q^m} |Du|^p$. Taking $r$ close to 1 and then $\epsilon > 0$ small the second and third integrals are also bounded by $\int_{Q^m} |Du|^p$, and the conclusion follows.

We prove the necessity part in Theorem 2.

**Proposition 4.2.** If $p \in \{2, \ldots, m\}$ and if the set $(W^{1,p} \cap L^\infty)(Q^m; N^n)$ is dense in $W^{1,p}(Q^m; N^n)$, then $N^n$ satisfies the levelling property of dimension $p$.

**Proof.** We first consider the case $p = m$. Let $u \in W^{1,m}(Q^m; N^n)$ be such that $u \in W^{1,m}(\partial Q^m; N^n)$. By the characterization given by Proposition 4.1, it suffices to prove that there exists $v \in W^{1,m}(Q^m; N^n) \cap C^0(\overline{Q^m}; N^n)$ such that $u = v$ on $\partial Q^m$ and $\|Dv\|_{L^\infty(Q^m)} \leq C\|Du\|_{L^\infty(Q^m)}$. Given $0 < \lambda < 1$, we introduce the same function $\theta \in C^\infty(\overline{Q^m}; N^n)$ converging strongly to $w$ in $W^{1,m}(Q^m; N^n)$. Then, for almost every $\lambda < r < 1$, there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $W^{1,m}(Q^m; N^n)$ converging to $u|_{\partial Q^m}$ in $W^{1,m}(\partial Q^m; N^n)$. By the Morrey–Sobolev imbedding, this convergence is also uniform on the set $\partial Q^m$. Since $w|_{\partial Q^m}$ is a compact subset of $N^n$, there exists $J \in \mathbb{N}$ such that for every $j \leq J$, for every $z \in \partial Q^m$ and every $t \in [0, 1]$, we have

$$tu_k(z) + (1-t)w(z) \in \Omega.$$ 

We also introduce a cut-off function $\theta \in C^\infty_c(Q^m)$ such that $0 \leq \theta \leq 1$ in $Q^m$ and $\theta = 1$ in $\overline{Q^m}$, with $(1-r\|D\theta\|_{L^\infty(Q^m)}) \leq C$. For every $j \geq J$, the map $v_j : \overline{Q^m} \to N^n$ defined by

$$v_j(x) = \begin{cases} \Pi(\theta(x)u_k_j(r \frac{x}{\|x\|} \| + (1 - \theta(x))w(r \frac{x}{\|x\|}) & \text{if } x \in Q^m \setminus \partial Q^m, \\ u_k_j(x) & \text{if } x \in \partial Q^m \end{cases}$$

is such that $v_j \in W^{1,m}(Q^m; N^n) \cap C^0(\overline{Q^m}; N^n)$, $v_j = u$ on $\partial Q^m$, and

$$\int_{Q^m} |Du_j|^m \leq C_1 \left( \int_{Q^m} |Du_k_j|^m + \|D\theta\|_{L^\infty(Q^m)}(1-r) \int_{\partial Q^m} |u_k_j - w|^m + (1-r) \int_{\partial Q^m} (|Du_k_j|^m + |Dw|^m) \right).$$

Without loss of generality, we can assume that $\int_{Q^m} |Du|^m > 0$. We take $j \geq J$ large enough so that the second term in the right-hand side is bounded from above by $\int_{Q^m} |Du|^m$. By convergence of the sequence $(u_k_j)_{j \in \mathbb{N}}$ we may also assume that

$$\int_{Q^m} |Du_k_j|^m \leq 2 \int_{Q^m} |Du|^m \quad \text{and} \quad \int_{\partial Q^m} |Du_k_j|^m \leq 2 \int_{\partial Q^m} |Dw|^m.$$ 

In view of the definition of $w$ in terms of $u$ we deduce from the estimates above that

$$\int_{Q^m} |Dv_j|^m \leq C_2 \left( \int_{Q^m} |Du|^m + (1-\lambda) \int_{\partial Q^m} |Du|^m \right).$$

To conclude the case $p = m$, it suffices to choose $\lambda$ sufficiently close to 1 so that the second term is bounded from above by $\int_{Q^m} |Du|^m$. 

We now consider the case where \( p < m \). Let \( u \in W^{1,p}(Q^p; N^n) \) be such that \( u|_{\partial Q^p} \in W^{1,p}(\partial Q^p; N^n) \). We then define the function \( v : Q'' \to N^n \) for \( x = (x', x'') \in Q'' \times (-1,1)^{m-1} \) by

\[
v(x) = u(x').
\]

By assumption, there exists a sequence \( (u_k)_{k \in \mathbb{N}} \) in \( (W^{1,p}(Q^m; N^n)) \) converging to \( v \) in \( W^{1,p}(Q^m; N^n) \). Hence, there exists a sequence \( (u_k(x))_{k \in \mathbb{N}} \) in the cube \((-1,1)^{m-1}\) such that \( (u_k(x), u_k)_{k \in \mathbb{N}} \) converges to \( u \) in \( W^{1,p}(Q^p; N^n) \). This proves that \( (W^{1,p}(Q^p; N^n)) \) is dense in \( W^{1,p}(Q^p; N^n) \). We are thus led to the first situation where \( p \) equals the dimension of the domain. We conclude that the manifold \( N^n \) satisfies the levelling property of dimension \( p \). □

**Definition 4.1.** A Riemannian manifold \( N^n \) has uniform Lipschitz geometry (or \( N^n \) is uniformly Lipschitz) if there exist \( \kappa > 0 \) and \( C > 0 \) such that, for every \( x \in N^n \), there exists a local chart \( \Psi : B_{N^n}(x; \kappa) \to \mathbb{R}^n \) such that \( B_{\mathbb{R}^n}(\Psi(x); \kappa) \subseteq \Psi(B_{N^n}(x; \kappa)) \), \(|\Psi|_{\text{Lip}} \leq C \) and \(|\Psi^{-1}|_{\text{Lip}} \leq C \).

Here, for \( x \in N^n \) and \( \kappa \geq 0 \), we have denoted by \( B_{N^n}(x; \kappa) \) the geodesic ball in \( N^n \) of center \( x \) and radius \( \kappa \). A natural candidate for \( \Psi \) is the inverse of the exponential map when the manifold \( N^n \) has a positive global injectivity radius and the exponential and its inverse are uniformly Lipschitz maps on balls of a fixed radius. If the injectivity radius of \( N^n \) is uniformly bounded from below and that the Riemannian curvature of \( N^n \) is uniformly bounded, then \( N^n \) has uniform Lipschitz geometry. By relying on harmonic coordinates instead of the normal coordinates given by the exponential maps, it can be proved that it is sufficient to bound the Ricci curvature instead of the Riemann curvature [1].

**Proposition 4.3.** If \( N^n \) is uniformly Lipschitz, then, for every \( p \in \mathbb{N}_* \), \( N^n \) satisfies the levelling property of dimension \( p \).

The proof of Proposition 4.3 is based on the following lemma which reduces the problem to a levelling property for maps with small energy.

**Lemma 4.4.** Let \( p \in \mathbb{N}_* \), and \( \alpha > 0 \). Assume that for every \( u \in W^{1,p}(Q^p; N^n) \) satisfying \( u|_{\partial Q^p} \in W^{1,p}(\partial Q^p; N^n) \) and \( \|Du\|_{L^p(\partial Q^p)} \leq \alpha \), there exists \( v \in (W^{1,p}(Q^p; N^n)) \) such that \( v = u \) on \( \partial Q^p \) and

\[
\|Dv\|_{L^p(\partial Q^p)} \leq C(\|Du\|_{L^p(\partial Q^p)} + \|Du\|_{L^p(\partial Q^p)})
\]

for some constant \( C > 0 \) independent of the map \( u \). Then, \( N^n \) satisfies the levelling property of dimension \( p \).

**Proof.** Let \( u \in W^{1,p}(Q^p; N^n) \) be such that \( u|_{\partial Q^p} \in W^{1,p}(\partial Q^p; N^n) \). For \( \frac{1}{2} < \lambda < 1 \), we introduce the same map \( w \) as in the proof of Proposition 4.3. Then, \( w|_{\partial Q^p} = u|_{\partial Q^p} \), \( w \) is bounded on \( Q^p \setminus Q^p_\lambda \) and

\[
\|Dw\|_{L^p(Q^p)} \leq C_1(\|Du\|_{L^p(Q^p)} + (1 - \lambda)\|Du\|_{L^p(\partial Q^p)}).
\]

Without loss of generality, we can assume that \( \|Du\|_{L^p(\partial Q^p)} > 0 \). We take \( \lambda > 0 \) such that

\[
(1 - \lambda)\|Du\|_{L^p(\partial Q^p)} \leq \|Du\|_{L^p(Q^p)}.
\]

This implies

\[
\|Du\|_{L^p(Q^p)} \leq C_2\|Du\|_{L^p(\partial Q^p)}.
\]

We fix \( 0 < \rho < \frac{1}{2} \). For every \( 0 < \mu < 1 \) sufficiently small, we consider a cubication \( K^p_\mu \) of side \( \mu \) such that

\[
Q^p_\lambda + 2\mu = K^p_\mu \subset K^p_\mu + Q^p_2 \subset Q^p.
\]
We open the map $w$ around $K_p^{\mu-1}$. More precisely, denoting by $\Phi^p : \mathbb{R}^m \to \mathbb{R}^m$ the smooth map given by Proposition 3.2 above, we consider

$$w^p = w \circ \Phi^p.$$  

In particular, $w^p \in W^{1,p}(Q^p; N^n)$, $w^p = w$ outside $K_p^{\mu-1} + Q_{2\mu}^p$ and, for every $\sigma^p \in K_p^\mu$, we have

$$\| Dw^p \|_{L^p(\partial \sigma^p + Q_{2\mu}^p)} \leq C_4 \| Dw \|_{L^p(\sigma^p + Q_{2\mu}^p)}.$$  

This implies that $w^p|_{\partial Q^p} = u|_{\partial Q^p}$ and

$$\| Dw^p \|_{L^p(Q^p)} \leq C_4 \| Dw \|_{L^p(Q^p)}.$$  

We also need the fact that the opening construction preserves the ranges of the maps. More precisely, for every $\sigma^{\mu-1} \in K_p^{\mu-1}$, we have

$$w^p(\sigma^{\mu-1} + Q_{2\mu}^p) \subset w(\sigma^{\mu-1} + Q_{2\mu}^p).$$  

We apply this remark to every $\sigma^{\mu-1} \in \partial K_p^\mu$ to get

$$w^p(\partial K_p^\mu + Q_{2\mu}^p) \subset w(\partial K_p^\mu + Q_{2\mu}^p).$$  

Together with the fact that $w$ is bounded on $Q^p \setminus Q_p^n \supset \partial K_p^\mu + Q_{2\mu}^p$, this proves that $w^p$ is bounded on $Q^p \setminus K_p^\mu$.

Since $w^p$ is $p - 1$ dimensional on $\partial \sigma^p + Q_{2\mu}^p$ for every $\sigma^p \in K_p^\mu$, we have

$$\| Dw^p \|_{L^p(\partial \sigma^p)} \leq \frac{C_5}{\mu^p} \| Dw^p \|_{L^p(\partial \sigma^p + Q_{2\mu}^p)} \leq \frac{C_5}{\mu^p} \| Dw \|_{L^p(\sigma^p + Q_{2\mu}^p)}.$$  

Taking $\mu > 0$ such that, for every $\sigma^p \in K_p^\mu$, we have

$$C_6 \| Dw \|_{L^p(\sigma^p + Q_{2\mu}^p)} \leq \alpha.$$  

Then,

$$\mu^{1-p} \| Dw^p \|_{L^p(\partial \sigma^p)} \leq \alpha.$$  

By the assumption applied to $w^p|_{\partial \sigma}$ for every $\sigma^p \in K_p^\mu$ and by a scaling argument, there exists a map $w_{\sigma^p} \in (W^{1,p} \cap L^\infty)(\sigma^p; N^n)$ which agrees with $w^p$ on $\partial \sigma^p$ and such that

$$\| Dw_{\sigma^p} \|_{L^p(\sigma^p)} \leq C_7 \left( \| Dw^p \|_{L^p(\sigma^p)} + \mu^{1-p} \| Dw^p \|_{L^p(\partial \sigma^p)} \right).$$  

We then define the map $\tilde{w}$ by

$$\tilde{w}(x) = w_{\sigma^p}(x) \text{ when } x \in \sigma^p \text{ and } \sigma^p \in K_p^\mu$$  

and we extend $\tilde{w}$ by $w^p$ outside $K_p^\mu$. Then, $\tilde{w} \in (W^{1,p} \cap L^\infty)(Q^p; N^n)$ and $\tilde{w}|_{\partial Q^p} = u|_{\partial Q^p}$. By additivity of the integral and by estimates (4.2) and (4.3), we also have

$$\| D\tilde{w} \|_{L^p(Q^p)}^p = \sum_{\sigma^p \in K_p^\mu} \| Dw_{\sigma^p} \|_{L^p(\sigma^p)}^p + \| Dw^p \|_{L^p(Q^p \setminus K_p^\mu)}^p \leq C_8 \sum_{\sigma^p \in K_p^\mu} \left( \| Dw_{\sigma^p} \|_{L^p(\sigma^p)}^p + \mu \| Dw_{\sigma^p} \|_{L^p(\partial \sigma^p)}^p \right) + \| Dw^p \|_{L^p(Q^p \setminus K_p^\mu)}^p \leq C_9 \| Dw \|_{L^p(Q^p)}^p.$$  

Applying estimate (4.1), we deduce that

$$\| D\tilde{w} \|_{L^p(Q^p)} \leq C_1 \| Dw \|_{L^p(Q^p)}.$$  

The map $\tilde{w}$ is continuous on a neighborhood of $Q^p \setminus (K_p^\mu + Q_{2\mu}^p)$ since it agrees with the map $w$ there. We introduce a cut-off function $\theta \in C_\infty^\infty(Q^p)$ such that $0 \leq \theta \leq 1$ in $Q^m$ and $\theta = 1$ on $K_p^\mu + Q_{2\mu}^p$. Given a family of mollifiers $(\varphi_\varepsilon)_{\varepsilon > 0}$, as in the proof of Proposition 2.1 there exists $\varepsilon > 0$ such that, for every $0 < \varepsilon \leq \varepsilon$, we
have \( \varphi_{\varepsilon} \ast \tilde{w}(x) \in O \) for almost every \( x \in Q^p \). Together with the continuity of \( \tilde{w} \) on a neighborhood of \( Q^p \setminus (K^p_\varepsilon + Q_{2\rho u}) \), this proves that for \( \varepsilon \) sufficiently small we can define

\[
v = \Pi[(\varphi_{\varepsilon} \ast \tilde{w}) + (1 - \theta)\tilde{w}].
\]

Such a map \( v \) is an extension of \( u \) in \( W^{1,p}(Q^p, N^n) \cap C^0(\overline{Q^p}; N^n) \). In view of Proposition 4.1 this completes the proof of the proposition.

\[\square\]

\textbf{Proof of Proposition 4.3} Let \( u \in W^{1,p}(Q^p, N^n) \) be such that \( u|_{\partial Q^p} \in W^{1,p}(\partial Q^p, N^n) \). Let \( \kappa > 0 \) be the radius of uniform Lipschitz geometry given by Definition 4.1. By the Morrey-Sobolev imbedding, there exists \( \alpha > 0 \) such that if \( \|Du\|_{L^p(\partial Q^p)} \leq \alpha \), then there exists \( x \in \partial Q^p \) such that, for almost every \( y \in \partial Q^p \), \( u(y) \in B_{N^n}(u(x); \kappa) \).

Given a local chart \( \Psi \) on \( B_{N^n}(u(x); \kappa) \) as in Definition 4.1 by the extension property of Sobolev functions, there exists \( w \in W^{1,p}(Q^p; \mathbb{R}^n) \) such that \( w|_{\partial Q^p} = \Psi \circ u|_{\partial Q^p} \), and

\[
\int_{Q^p} |Dw|^p \leq C_1 \int_{\partial Q^p} |D(\Psi \circ u)|^p \leq C_2 \int_{\partial Q^p} |Du|^p.
\]

Truncating \( w \) with a retraction on the ball \( B_{R_n}(\Psi(x); \kappa) \) if necessary, we may further assume that \( w(Q^p) \subset B_{R_n}(\Psi(x); \kappa) \). Defining the map \( v = \Psi^{-1} \circ w \), by composition of Sobolev maps with smooth functions it follows that \( v \in W^{1,p}(Q^p, N^n) \) and

\[
\int_{Q^p} |Dv|^p \leq C_3 \int_{Q^p} |Dw|^p \leq C_3C_2 \int_{\partial Q^p} |Du|^p.
\]

In view of Lemma 4.4 the proof is complete.

\[\square\]

\textbf{5. PROOFS OF THEOREM 11 AND THEOREM 22}

Let \( 1 \leq p \leq m \) and \( u \in W^{1,p}(Q^m, N^n) \). By using reflections across the boundary of \( Q^m \), we can extend \( u \) as a map in \( W^{1,p}(Q_{1+\gamma}^m; N^n) \) for some \( \gamma > 0 \). We also fix \( 0 < \rho < \frac{1}{2} \). Let \( K^m_\rho \) be a cubification of \( Q_{1+\gamma}^m \) of radius \( \rho \leq \gamma \) such that

\[
2\rho n \leq \gamma.
\]

For almost every \( x, y \in Q_{1+\gamma}^m \), the function \( t \mapsto u(tx + (1 - t)y) \) is an absolutely continuous path in \( N^n \) between \( u(x) \) and \( u(y) \). Hence, the geodesic distance \( \text{dist}_{N^n}(u(x), u(y)) \) between \( u(x) \) and \( u(y) \) can be estimated as follows:

\[
\text{dist}_{N^n}(u(x), u(y)) \leq \int_0^1 |Du(tx + (1 - t)y)(x - y)| \, dt.
\]

As in the proof of the Poincaré inequality, this implies that

\[
\int_{\partial Q_{1+\gamma}^m} |Du| \int_{Q_{1+\gamma}^m} \text{dist}_{N^n}(u(x), u(y)) \, dx \leq C_1 \int_{Q_{1+\gamma}^m} |Du|^p \, dx.
\]

Hence, for almost every \( y \in Q_{1+2\gamma}^m \),

\[
\int_{Q_{1+2\gamma}^m} \text{dist}_{N^n}(u(x), u(y)) \, dx < \infty.
\]

This implies that for every \( a \in \mathbb{N}^n \), the function \( x \mapsto \text{dist}_{N^n}(u(x), a) \) is summable on \( Q_{1+2\gamma}^m \).
We fix a point \( a \in \mathbb{N}^n \). For every \( R > 0 \) and \( \lambda > 0 \), we define the subskeleton \( G^m_\eta \) of \( K^m_\eta \) as the set of good cubes \( \sigma^m \in \mathcal{K}^m_\eta \) in the sense that
\[
\int_{\sigma^m + Q^m_{2\rho \eta}} \text{dist}_{\mathcal{N}^m}(u(x), a) \, dx \leq R \quad \text{and} \quad \frac{1}{\eta^m} \|Du\|_{L^p(\sigma^m + Q^m_{2\rho \eta})} \leq \lambda.
\]
We also introduce the subskeleton of bad cubes \( \mathcal{E}^m_\eta \) defined as the complement of \( G^m_\eta \) in \( K^m_\eta \). Thus, by definition of \( \mathcal{E}^m_\eta \), for every \( \sigma^m \in \mathcal{E}^m_\eta \) we have
\[
R < \int_{\sigma^m + Q^m_{2\rho \eta}} \text{dist}_{\mathcal{N}^m}(u(x), a) \, dx \quad \text{or} \quad \lambda < \frac{1}{\eta^m} \|Du\|_{L^p(\sigma^m + Q^m_{2\rho \eta})}.
\]
In the proof, we do not explicitly indicate the dependence of \( G^m_\eta \) and \( \mathcal{E}^m_\eta \) on the parameters \( R \) and \( \eta \).

The Lebesgue measure of the set \( E^m_\eta + Q^m_{2\rho \eta} \) can be made as small as we want by a suitable choice of parameters \( R \) and \( \eta \). This is a consequence of the following estimate:

**Claim 1.** The Lebesgue measure of the set \( E^m_\eta + Q^m_{2\rho \eta} \) satisfies
\[
|E^m_\eta + Q^m_{2\rho \eta}| \leq C \left( \frac{1}{R} \int_{Q^m_{1+2\gamma}} \text{dist}_{\mathcal{N}^m}(u(x), a) \, dx + \frac{\eta^m}{\lambda^p} \int_{Q^m_{1+2\gamma}} |Du|^p \right).
\]

**Proof of the claim.** By finite subadditivity of the Lebesgue measure, we have
\[
|E^m_\eta + Q^m_{2\rho \eta}| \leq \sum_{\sigma^m \in \mathcal{E}^m_\eta} |\sigma^m + Q^m_{2\rho \eta}| \leq C_2 \eta^m (\# \mathcal{E}^m_\eta).
\]
From the definition of \( \mathcal{E}^m_\eta \), we estimate the number \( \# \mathcal{E}^m_\eta \) of bad cubes as follows:
\[
\# \mathcal{E}^m_\eta \leq \sum_{\sigma^m \in \mathcal{E}^m_\eta} \frac{1}{|\sigma^m + Q^m_{2\rho \eta}|R} \int_{\sigma^m + Q^m_{2\rho \eta}} \text{dist}_{\mathcal{N}^m}(u(x), a) \, dx + \frac{\eta^m - p \eta^m}{\lambda^p} \int_{\sigma^m + Q^m_{2\rho \eta}} |Du|^p \, dx
\]
\[
\leq \frac{C_1}{\eta^m} \left( \frac{1}{R} \int_{Q^m_{1+2\gamma}} \text{dist}_{\mathcal{N}^m}(u(x), a) \, dx + \frac{\eta^m}{\lambda^p} \int_{Q^m_{1+2\gamma}} |Du|^p \, dx \right).
\]
Combining both estimates, we get the conclusion. \( \Box \)

Throughout the proof, we consider
\[
\ell = \begin{cases} 
[p] & \text{if } p < m, \\
m - 1 & \text{if } p = m,
\end{cases}
\]
where \([p]\) denotes the integer part of \( p \). We begin by opening the map \( u \) in a neighborhood of \( E' \). More precisely, if \( \Phi^\circ \) : \( \mathbb{R}^m \to \mathbb{R}^m \) is the smooth map given by Proposition 5.2 with the parameter \( \rho \), we consider the opened map \( u^\circ = u \circ \Phi^\circ \).

In particular, \( u^\circ \in W^{1,p}(Q^m_{1+2\gamma}; \mathbb{N}^n) \) and \( u^\circ = u \) in the complement of \( E_\eta + Q^m_{2\rho \eta} \). There exists \( C > 0 \) such that, for every \( \sigma^p \in \mathcal{E}^p_\eta \), we have
\[
\|Du^\circ\|_{L^p(\sigma^p + Q^m_{2\rho \eta})} \leq C \|Du\|_{L^p(\sigma^p + Q^m_{2\rho \eta})},
\]
and also
\begin{equation}
\|D u^\eta - Du\|_{L^p(Q^m_{1+2\gamma})} \leq C \|Du\|_{L^p(E^n + Q^{2m}_{\rho})}.
\end{equation}

Given $0 < \rho < \rho$, we consider a smooth function $\psi \in C^\infty(Q^m_{1+2\gamma})$ such that
(a) $0 \leq \psi \leq (\rho - \rho)\eta$,
(b) $\sup \psi \subset G^m_{\rho} + Q^{2m}_{\rho}$,
(c) $\|D\psi\|_{L^\infty(Q^m_{1+2\gamma})} < 1$.

The parameter $t$ is fixed throughout the proof and is independent of $\eta$, $R$ and $\lambda$. Condition (5) gives an upper bound on $t$, while condition (5) imposes $t$ to be typically smaller than $\rho - \rho$ and this can be achieved independently of the geometry of the cubication $G^m_{\eta}$.

Given a mollifier $\psi \in C^\infty(B^m_1)$, for every $x \in Q^m_{1+\gamma+\rho}$ let
\[ u^\eta_{\psi}(x) = (\varphi \ast u^\eta)(x) = \int_{B^m_1} u^\eta(x - y)\psi(y) \, dy. \]

Since $0 < \psi \leq \rho$, the smoothed map $u^\eta_{\psi} : Q^m_{1+\gamma+\rho} \rightarrow \mathbb{R}^\nu$ is well-defined.

**Claim 2.** The map $u^\eta_{\psi}$ satisfies the estimates
\begin{align}
\|u^\eta_{\psi} - u\|_{L^p(Q^m_{1+\gamma})} &\leq \sup_{v \in B^m_1} \|\tau_{\psi_{\psi}}(u) - u\|_{L^p(Q^m_{1+\gamma})} + C \|u - u^\eta_{\psi}\|_{L^p(Q^m_{1+\gamma})}, \\
\|Du^\eta_{\psi} - Du\|_{L^p(Q^m_{1+\gamma})} &\leq \sup_{v \in B^m_1} \|\tau_{\psi_{\psi}}(Du) - Du\|_{L^p(Q^m_{1+\gamma})} + C \|Du\|_{L^p(E^n + Q^{2m}_{\rho})}.
\end{align}

Proof of the claim. By Proposition 3.3 with $\omega = Q^m_{1+\gamma}$, we have
\[ \|u^\eta_{\psi} - u^\eta_{\psi}\|_{L^p(Q^m_{1+\gamma})} \leq \sup_{v \in B^m_1} \|\tau_{\psi_{\psi}}(u^\eta_{\psi}) - u^\eta_{\psi}\|_{L^p(Q^m_{1+\gamma})}. \]

We also observe that, for every $v \in B^m_1$, we have
\[ \|\tau_{\psi_{\psi}}(u^\eta_{\psi}) - u^\eta_{\psi}\|_{L^p(Q^m_{1+\gamma})} \leq \|\tau_{\psi_{\psi}}(u^\eta_{\psi}) - \tau_{\psi}(u)\|_{L^p(Q^m_{1+\gamma})} \]
\[ + \|\tau_{\psi_{\psi}}(u) - u\|_{L^p(Q^m_{1+\gamma})} + \|u - u^\eta_{\psi}\|_{L^p(Q^m_{1+\gamma})} \]
\[ \leq \|\tau_{\psi_{\psi}}(u) - u\|_{L^p(Q^m_{1+\gamma})} + C \|u - u^\eta_{\psi}\|_{L^p(Q^m_{1+\gamma})}, \]

and this proves (5.3).

We now consider the second estimate. Since $\|Du_{\psi}\|_{L^\infty(Q^m_{1+\gamma})} < 1$, it also follows from Proposition 3.3 that
\begin{equation}
\|Du_{\psi}^\eta - Du_{\psi}\|_{L^p(Q^m_{1+\gamma})} \leq \sup_{v \in B^m_1} \|\tau_{\psi_{\psi}}(Du_{\psi}^\eta) - Du_{\psi}\|_{L^p(Q^m_{1+\gamma})} + C \|Du\|_{L^p(E^n + Q^{2m}_{\rho})},
\end{equation}

where $A = \bigcup_{x \in Q^m_{1+\gamma}} \sup \psi_{\psi}(u)(x)$. From properties (5) and (6), we have
\[ \sup \psi_{\psi} \subset E^n_{\eta} + Q^{2m}_{\rho}. \]

and, since $\psi \leq \rho$, we deduce that $A \subset E^n_{\eta} + Q^{2m}_{\rho}$. By Proposition 3.2 we then get
\begin{equation}
\|Du_{\psi}^\eta\|_{L^p(A)} \leq C \|Du\|_{L^p(E^n + Q^{2m}_{\rho})}.
\end{equation}

As in the proof of the first estimate, for every $v \in B^m_1$ we also have
\begin{align}
\|\tau_{\psi_{\psi}}(Du_{\psi}^\eta) - Du_{\psi}\|_{L^p(Q^m_{1+\gamma})} &\leq \|\tau_{\psi_{\psi}}(Du) - Du\|_{L^p(Q^m_{1+\gamma})} \\
&\quad + C \|Du - Du_{\psi}\|_{L^p(Q^m_{1+\gamma})}.
\end{align}
Combining estimates (5.3)–(5.7), we complete the proof of (5.4).

**Claim 3.** There exists $\overline{R} > R$ such that, for every $\eta > 0$ and $\lambda > 0$, the directed Hausdorff distance to the geodesic ball $B_{N^m}(a; \overline{R})$ satisfies

$$\text{Dist}_{B_{N^m}(a; \overline{R})}(u^m_{\eta}(G^m_{\eta})) \leq \frac{C'}{\eta} \max_{\sigma_m \in G^m_{\eta}} \| Du \|_{L^p(\sigma_m + Q^{m}_{2\eta^m})},$$

for some constant $C' > 0$ depending on $m$, $p$ and $\nu$.

Here, the directed Hausdorff distance from a set $S \subset \mathbb{R}^\nu$ to the geodesic ball $B_{N^m}(a; \overline{R})$ is defined as

$$\text{Dist}_{B_{N^m}(a; \overline{R})}(S) = \sup \left\{ \text{dist}_{\mathbb{R}^\nu}(x, B_{N^m}(a; \overline{R})) : x \in S \right\},$$

where $\text{dist}_{\mathbb{R}^\nu}$ denotes the Euclidean distance in $\mathbb{R}^\nu$.

**Proof of the claim.** Given $\sigma^m \in G^m_{\eta}$ and $\overline{R} > 0$, we consider the sets

$$W^m_{\overline{R}} = \left\{ z \in \sigma^m + Q^{m}_{2\eta^m} : \text{dist}_{N^m}(u(z), a) < \overline{R} \right\},$$

$$Z^m_{\overline{R}} = \left\{ z \in \sigma^m + Q^{m}_{2\eta^m} : \text{dist}_{N^m}(u(z), a) \geq \overline{R} \right\},$$

and their counterparts for the map $u^{op}$ obtained by the opening construction,

$$W^{op}_{\overline{R}} = \left\{ z \in \sigma^m + Q^{m}_{2\eta^m} : \text{dist}_{N^m}(u^{op}(z), a) < \overline{R} \right\},$$

$$Z^{op}_{\overline{R}} = \left\{ z \in \sigma^m + Q^{m}_{2\eta^m} : \text{dist}_{N^m}(u^{op}(z), a) \geq \overline{R} \right\}.$$

Assuming that $|W^{op}_{\overline{R}}| > 0$, then for every $x \in \sigma^m$ we may estimate the distance from $u^m_{\eta}(x)$ to $B_{N^m}(a; \overline{R})$ in terms of an average integral as follows

$$\text{dist}_{\mathbb{R}^\nu}(u^m_{\eta}(x), B_{N^m}(a; \overline{R})) \leq \int_{W^{op}_{\overline{R}}} |u^m_{\eta}(x) - u^{op}_{\eta}(z)| \, dz.$$

We then have, for every $x \in \sigma^m$,

$$\text{dist}_{\mathbb{R}^\nu}(u^m_{\eta}(x), B_{N^m}(a; \overline{R})) \leq C_1 \int_{W^{op}_{\overline{R}}} \int_{Q^{m}_{\psi}(x)} |u^{op}_{\eta}(y) - u^{op}_{\eta}(z)| \, dy \, dz.$$

Since both sets $W^{op}_{\overline{R}}$ and $Q^{m}_{\psi}(x)$ are contained in $\sigma^m + Q^{m}_{2\eta^m}$ by the Poincaré–Wirtinger inequality we deduce that

$$\text{dist}_{\mathbb{R}^\nu}(u^m_{\eta}(x), B_{N^m}(a; \overline{R})) \leq \frac{C_{2\eta^m}}{|W^{op}_{\overline{R}}|} \frac{1}{\eta^m} \| Du^{op} \|_{L^p(\sigma^m + Q^{m}_{2\eta^m})}.$$

Since $\psi_{\eta} = t\eta$ on $G^m_{\eta}$, for every $x \in \sigma^m$ we have

$$|Q^{m}_{\psi}(x)| \geq C_{3}\eta^m.$$

We now estimate from below the quantity $|W^{op}_{\overline{R}}|$. Since $\sigma^m \in G^m_{\eta}$, then by definition of $G^m_{\eta}$ the average integral satisfies

$$\int_{\sigma^m + Q^{m}_{2\eta^m}} \text{dist}_{N^m}(u(x), a) \, dx \leq R,$$

whence by the Chebyshev inequality we have

$$\frac{|Z^m_{\overline{R}}|}{|\sigma^m + Q^{m}_{2\eta^m}|} \leq R.$$
We now proceed with the choice of $\overline{R}$. Taking any $\overline{R} > R$ such that
\begin{equation}
|\sigma^m + Q^m_{2\eta}| \overline{R} \leq \frac{[(\sigma^m + Q^m_{2\eta}) \setminus (\partial \sigma^m + Q^m_{2\eta})]}{2} \overline{R},
\end{equation}
we have
\begin{equation}
|Z_{\overline{R}}^m| \leq \frac{[(\sigma^m + Q^m_{2\eta}) \setminus (\partial \sigma^m + Q^m_{2\eta})]}{2}.
\end{equation}
Since $\sigma^m$ is a cube of radius $\eta$, by a scaling argument with respect to $\eta$ this choice of $\overline{R}$ is independent of $\eta$. Since the maps $\omega^m_{\eta}$ and $u$ coincide in $(\sigma^m + Q^m_{2\eta}) \setminus (\partial \sigma^m + Q^m_{2\eta})$, we have
\begin{equation}
Z^m_{\overline{R}} \subset Z \cup (\partial \sigma^m + Q^m_{2\eta}).
\end{equation}
By subadditivity of the Lebesgue measure and by the choice of $\overline{R}$ we get
\begin{equation}
|Z^m_{\overline{R}}| \leq \frac{[(\sigma^m + Q^m_{2\eta}) \setminus (\partial \sigma^m + Q^m_{2\eta})]}{2} + |\partial \sigma^m + Q^m_{2\eta}|,
\end{equation}
whence the measure of the complement set $W^m_{\overline{R}}$ satisfies
\begin{equation}
|W^m_{\overline{R}}| \geq \frac{[(\sigma^m + Q^m_{2\eta}) \setminus (\partial \sigma^m + Q^m_{2\eta})]}{2} = 2^{m-1}(\eta - 2\eta)^m = C_4 \eta^m.
\end{equation}
By estimate (5.8), for every $x \in \sigma^m$ we deduce that with the above choice of $\overline{R}$ we have
\begin{equation}
dist_{\nu} (u^m_{\eta}(x), B_{\overline{R}}(a, \overline{R})) \leq \frac{C_5}{\eta^{-p}} \|Du^m_{\eta}\|_{L^p(\sigma^m + Q^m_{2\eta})}.
\end{equation}
By subadditivity of the Lebesgue measure and by the properties of the opening construction, we have
\begin{equation}
\|Du^m_{\eta}\|_{L^p(\sigma^m + Q^m_{2\eta})} \leq \|Du^m_{\eta}\|_{L^p((\sigma^m + Q^m_{2\eta}) \setminus (E^\ell_{\eta} + Q^m_{2\eta}))} + \sum_{\sigma \in \mathcal{E}^\ell_{\eta}} \|Du^m_{\eta}\|_{L^p(\sigma^m + Q^m_{2\eta})}
\leq C_6 \|Du\|_{L^p(\sigma^m + Q^m_{2\eta})}.
\end{equation}
Together with (5.11), this implies the estimate we claimed.

We now study the behavior of the smoothened map $u^m_{\eta}$ on a part of the bad set $E^\ell_{\eta}$.

**Claim 4.** There exists $\overline{R} > R$ such that, for every $\eta > 0$ and $\lambda > 0$, the directed Hausdorff distance to the geodesic ball $B_{\lambda}(a, \overline{R})$ satisfies
\begin{equation}
\text{Dist}_{B_{\lambda}(a, \overline{R})}(u^m_{\eta}(E^\ell_{\lambda} \cap \text{supp } \psi_{\lambda})) \leq \frac{C^m}{\eta^{-p}} \max_{\sigma \in \mathcal{G}^\ell_{\eta}} \|Du\|_{L^p(\sigma^m + Q^m_{2\eta})},
\end{equation}
for some constant $C^m > 0$ depending on $m$, $p$ and $\nu$.

**Proof of the claim.** We first observe that
\begin{equation}
E^\ell_{\eta} \cap \text{supp } \psi_{\eta} \subset (E^\ell_{\eta} \cap G^\ell_{\eta}) \cup (E^\ell_{\eta} \cap G^\ell_{\eta}^{-1}) + Q^m_{2\eta}.
\end{equation}
By Claim 3 above it thus suffices to prove that, for every $\tau^\ell_{\eta} \in E^\ell_{\eta} \cap G^\ell_{\eta}^{-1}$, we have
\begin{equation}
\text{Dist}_{B_{\lambda}(a, \overline{R})}(u^m_{\eta}(\tau^\ell_{\eta} + Q^m_{2\eta})) \leq \frac{C_1}{\eta^{-p}} \max_{\sigma \in \mathcal{G}^\ell_{\eta}} \|Du\|_{L^p(\sigma^m + Q^m_{2\eta})}.
\end{equation}
For this purpose, we observe that there exists $R > R$ such that the map $u^{\text{op}}_\eta$ can be constructed with the following additional property: for every $\tau^{\ell-1} \in \mathcal{E}^{\ell-1}_\eta \cap \mathcal{G}^{\ell-1}_\eta$,

$$(5.13) \quad \text{Dist}_{B_{N^n}(a; R)} \left(u^{\text{op}}_\eta(\tau^{\ell-1} + Q^m_{\rho^m}) \right) \leq \frac{C_1}{\eta^{p/\ell}} \| Du^{\text{op}}_\eta \|_{L^p(\tau^{\ell-1} + Q^m_{\rho^m})}.$$  

Indeed, for every $\sigma^m \in \mathcal{G}^{\ell-1}_\eta$ and for every $R > R$ such that

$$(5.14) \quad |\sigma^m + Q^m_{\rho^m}| R \leq \frac{|Q^m_{\rho^m}|}{2},$$

we have, by (5.9),

$$|Z^m_{\eta^m}| \leq \frac{|Q^m_{\rho^m}|}{2}.$$  

Again by a scaling argument with respect to $\eta$, this choice of $R$ is independent of $\eta$. For each vertex $v$ of the cube $\sigma^m$ and for at least half of the points $x$ of the cube $Q^m_{\rho^m}(v)$, we thus have $u(x) \in B_{N^n}(a; R)$. Since the opening construction is based on a Fubini type argument (see the explanation preceding Proposition 3.1), we may thus assume that for every vertex $v$ of $\sigma^m \cap E_\eta$, the common value of $u^{\text{op}}_\eta$ in $Q^m_{\rho^m}(v)$ belongs to $B_{N^n}(a; R)$.  

Consider an $\ell-1$ dimensional face $\tau^{\ell-1} \in \mathcal{E}^{\ell-1}_\eta \cap \mathcal{G}^{\ell-1}_\eta$ of $\partial \sigma^m$. Since $p > \ell - 1$, by the Morrey--Sobolev inequality we have, for every $y, z \in \tau^{\ell-1}$,

$$\text{dist}_{\tau^{\ell-1}} \left( u^{\text{op}}_\eta(y), u^{\text{op}}_\eta(z) \right) \leq C_2 \eta^{1 - \frac{1}{\ell}} \| Du^{\text{op}}_\eta \|_{L^p(\tau^{\ell-1})}.$$  

On the other hand, since the map $u^{\text{op}}_\eta$ is, by construction, an $\ell - 1$ dimensional map in $\tau^{\ell-1} + Q^m_{\rho^m}$ we have $u^{\text{op}}_\eta(\tau^{\ell-1} + Q^m_{\rho^m}) = u^{\text{op}}_\eta(\tau^{\ell-1})$ and also

$$\| Du^{\text{op}}_\eta \|_{L^p(\tau^{\ell-1})} \leq \frac{C_3}{\eta^{p-\ell}} \| Du^{\text{op}}_\eta \|_{L^p(\tau^{\ell-1} + Q^m_{\rho^m})}.$$  

This implies that, for every $y, z \in \tau^{\ell-1} + Q^m_{\rho^m}$

$$\text{dist}_{\tau^{\ell-1} + Q^m_{\rho^m}} \left( u^{\text{op}}_\eta(y), u^{\text{op}}_\eta(z) \right) \leq \frac{C_4}{\eta^{p-\ell}} \| Du^{\text{op}}_\eta \|_{L^p(\tau^{\ell-1} + Q^m_{\rho^m})}.$$  

Taking as $z$ any vertex of $\tau^{\ell-1}$ in $\mathcal{G}^{\ell-1}_\eta$, we thus obtain estimate (5.13).

We now complete the proof of (5.12). Recall that the map $u^{m}_{\psi}$ is obtained from $u^{\text{op}}_\eta$ by convolution with parameter $\psi_\eta$. Hence, for every $\tau^{\ell-1} \in \mathcal{E}^{\ell-1}_\eta \cap \mathcal{G}^{\ell-1}_\eta$ and for every $x \in \tau^{\ell-1} + Q^m_{\rho^m}$ such that $\psi_\eta(x) \neq 0$, by the triangle inequality we have

$$\text{dist}_{\tau^{\ell-1} + Q^m_{\rho^m}} \left( u^{m}_{\psi}(x), B_{N^n}(a; R) \right) \leq C_5 \int \frac{\int \int |u^{\text{op}}_\eta(z) - u^{\text{op}}_\eta(y)| \, dy \, dz}{Q^m_{\psi}(x)}.$$  

Since $x \in \tau^{\ell-1} + Q^m_{\rho^m}$ and $\psi_\eta(x) < (\rho - \ell) \eta$, we have $Q^m_{\psi}(x) \subset \tau^{\ell-1} + Q^m_{\rho^m}$. Together with (5.13), this implies that, for every $y \in Q^m_{\psi}(x)$

$$\text{dist}_{\tau^{\ell-1} + Q^m_{\rho^m}} \left( u^{\text{op}}_\eta(y), B_{N^n}(a; R) \right) \leq \frac{C_6}{\eta^{p-\ell}} \| Du^{\text{op}}_\eta \|_{L^p(\tau^{\ell-1} + Q^m_{\rho^m})}.$$
By the Poincaré–Wirtinger inequality, we deduce that
\[
\text{dist}_{\mathbb{R}^n} (u_{\eta}^{sm}(x), \mathcal{B}_{N^n}(a, R)) \leq C_7 \left( \frac{1}{\eta^{m-p}} \| Du_{\eta}^{op} \|_{L^p(Q_{\eta}^m(x))} + \frac{1}{\eta^{-2}} \| Du_{\eta}^{op} \|_{L^p(\tau^{-1} + Q_{\eta}^m)} \right).
\]
By Remark 3.1 concerning the opening construction, for every \(Q_{\eta}^m(x) \subset \tau^{-1} + Q_{\eta}^m\) we have
\[
\frac{1}{\eta^{m-p}} \int_{Q_{\eta}^m(x)} |Du_{\eta}^{op}|^p \leq \frac{C_8}{\eta^{m-p}} \int_{\tau^{-1} + Q_{\eta}^m} |Du_{\eta}^{op}|^p.
\]
Combining these inequalities using \(r = \psi_\eta\), we get
\[
\text{dist}_{\mathbb{R}^n} (u_{\eta}^{sm}(x), \mathcal{B}_{N^n}(a, R)) \leq \frac{C_9}{\eta^{m-p}} \| Du_{\eta}^{op} \|_{L^p(\tau^{-1} + Q_{\eta}^m)}.
\]
In view of the estimates satisfied by the opening construction and the fact that \(\tau^{-1} \in G_{\eta}^{\tau^{-1}}\), for every \(x \in \tau^{-1} + Q_{\eta}^m\) such that \(\psi_\eta(x) \neq 0\) we have
\[
\text{dist}_{\mathbb{R}^n} (u_{\eta}^{sm}(x), \mathcal{B}_{N^n}(a, R)) \leq \frac{C_{10}}{\eta^{m-p}} \max_{\sigma_m \in Q_{\eta}^m} \| Du \|_{L^p(\sigma_m + Q_{\eta}^m)},
\]
from which (5.12) follows. If \(\psi_\eta(x) = 0\), then \(u_{\eta}^{sm}(x) = u_{\eta}^{op}(x)\), and the above inequality remains true by (5.13). This completes the proof of the claim.

Up to now, the parameters \(R, \lambda\) and \(\eta\) were arbitrary. In the following, they will be subject to some restrictions. For a given \(R > 0\), we take \(\overline{R} > R\) satisfying the conclusions of Claim 3 and Claim 4. For any such \(\overline{R}\), let \(\overline{\rho} > 0\) be such that
\[
\mathcal{B}_{N^n}(a; \overline{R}) + B_{\overline{\rho}}^{\overline{m}} \subset O.
\]
Remember that \(O\) is an open neighborhood of \(N^n\) in \(\mathbb{R}^n\) such that the nearest point projection \(\Pi : O \to N^n\) is well-defined and globally Lipschitz. We also take \(\lambda > 0\) depending on \(\overline{R} > 0\), whence also on \(R > 0\), such that
\[
(5.15) \quad \lambda \leq \frac{\overline{\tau}}{\max \{ C', C'' \}}
\]
where \(C'\) and \(C''\) are the constants given by Claim 3 and Claim 4 respectively. On the one hand, for every good cube \(\sigma_m \in G_{\eta}^m\) we have
\[
\frac{1}{\eta^{m-p}} \| Du \|_{L^p(\sigma_m + Q_{\eta}^m)} \leq \frac{\overline{\tau}}{\max \{ C', C'' \}}.
\]
By the estimate from Claim 3 this implies that
\[
u_{\eta}^{sm}(G_{\eta}^m) \subset \mathcal{B}_{N^n}(a; \overline{R}) + B_{\overline{\rho}}^{\overline{m}} \subset O.
\]
On the other hand, Claim 4 implies that
\[
u_{\eta}^{sm}(E_{\eta}^{pr} \cap \text{supp} \psi_{\eta}) \subset \mathcal{B}_{N^n}(a; \overline{R}) + B_{\overline{\rho}}^{\overline{m}} \subset O.
\]
On \(K_{\eta}^m \setminus \text{supp} \psi_{\eta}\), we have \(\nu_{\eta}^{sm} = u_{\eta}^{op}.\) In particular,
\[
u_{\eta}^{sm}(E_{\eta}^{pr} \setminus \text{supp} \psi_{\eta}) \subset N^n \subset O.
\]
This proves that \(\nu_{\eta}^{sm}(E_{\eta}^{pr}) \subset O.\)

We now define the projected map \(u_{\eta}^{pr} : G_{\eta}^m \cup E_{\eta}^{pr} \to N^n\) by setting
\[
u_{\eta}^{pr} = \Pi \circ \nu_{\eta}^{sm}.
\]
On $G^m_\eta$, $u^pr_\eta$ is smooth and we have:

**Claim 5.** The map $u^pr_\eta$ satisfies

$$
\|Du^pr_\eta - Du\|_{L^p(G^m_\eta)} 
\leq C \left( \|Du^sm_\eta - Du\|_{L^p(G^m_\eta)} + \|Du\| |DII(u^sm_\eta) - DII(u)| \right),
$$

for some constant $C > 0$ depending on the Lipschitz constant of $\Pi$ on $O$.

**Proof of the claim.** Since $u^pr_\eta = \Pi \circ u^sm_\eta$ and $u = \Pi \circ u$, by the triangle inequality we have

$$
\|Du^pr_\eta - Du\|_{L^p(G^m_\eta)} \leq \|D\Pi(u^sm_\eta)\|_{L^\infty(G^m_\eta)} \|Du^sm_\eta - Du\|_{L^p(G^m_\eta)}
+ \|D\Pi(u^sm_\eta) - D\Pi(u)\| \|Du\|_{L^p(G^m_\eta)}.
$$

We now consider the restriction of $u^pr_\eta$ to the set $E^\ell_\eta$.

**Claim 6.** The map $u^pr_\eta|_{E^\ell_\eta}$ belongs to $W^{1,p}(E^\ell_\eta; N^n)$ and, for every $\tau^\ell \in E^\ell_\eta$, we have

$$
\|Du^pr_\eta\|_{L^p(\tau^\ell)} \leq \frac{C}{\eta^{p-1}} \|Du\|_{L^p(\tau^\ell + Q^m_{\rho\eta})},
$$

for some constant $C > 0$ depending on $m$ and $p$.

**Proof of the claim.** Since $\Pi$ is globally Lipschitz on $O$ and $u^pr_\eta = \Pi \circ u^sm_\eta$, it is enough to prove the claim for $u^sm_\eta$ instead of $u^pr_\eta$. The map $u^op_\eta$ is $\ell - 1$ dimensional on $E^{\ell-1}_\eta + Q^m_{\rho\eta}$ and thus continuous by the Morrey-Sobolev imbedding. This implies that $u^sm_\eta$ is continuous on a neighborhood of $E^{\ell-1}_\eta$. Hence, we only need to prove that, for every $\tau^\ell \in E^\ell$, the restriction $u^sm_\eta|_{\tau^\ell}$ belongs to $W^{1,p}(\tau^\ell; \mathbb{R}^m)$ and satisfies the above estimate.

Without loss of generality, we can assume that

$$
\tau^\ell = (-\eta, \eta)^\ell \times \{0\}^{m-\ell},
$$

where $0^{\ell'} \in \mathbb{R}^{m-\ell}$. Accordingly, we write every vector $y \in \mathbb{R}^m$ as $y = (y^\ell, y^{m-\ell}) \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$. Since $D\psi_\eta$ is uniformly bounded with respect to $\eta$ and since there exists $c^{m-\ell} \in Q^m_{\rho\eta}$ such that for $y \in \tau^\ell + Q^m_{\rho\eta}$,

$$
u^{op}_\eta(y) = u^{op}_\eta(y^\ell, c^{m-\ell}),
$$

for every $x^\ell \in \tau^\ell$ we have

$$
|Du^{sm}_\eta(x^\ell, 0^{m-\ell})|^p \leq C_1 \int_{E^{m-\ell}_\eta} |Du^{op}_\eta(x^\ell - \psi_\eta(x^\ell)y^\ell, c^{m-\ell})|^p \, dy^\ell
\leq C_1 \int_{Q^{m-\ell}} \int_{Q^\ell} |Du^{op}_\eta(x^\ell - \psi_\eta(x^\ell)y^\ell, c^{m-\ell})|^p \, dy^\ell \, dy'
= C_1 2^{m-\ell} \int_{Q^\ell} |Du^{op}_\eta(x^\ell - \psi_\eta(x^\ell)y^\ell, c^{m-\ell})|^p \, dy' .
$$

Hence, by Fubini’s theorem,

$$
\int_{\tau^\ell} |Du^{sm}_\eta(x^\ell, 0^{m-\ell})|^p \, dx^\ell \leq C_1 2^{m-\ell} \int_{Q^\ell} \int_{\tau^\ell} |Du^{op}_\eta(x^\ell - \psi_\eta(x^\ell)y^\ell, c^{m-\ell})|^p \, dx^\ell \, dy' .
$$
By using the change of variables \( z' = x' - \psi_\eta(x')y' \) with respect to the variable \( x' \), we get

\[
\int_{\tau^\ell} |Du^{sm}_\eta(x',0')|^p \, dx' \leq \frac{C_2}{1 - \|D\psi_\eta\|_{L^\infty(K^m_{\eta})}} \int_{Q^\ell} dy' \int_{(-(1+\rho_\eta)(1+\rho_\eta))^\ell} |Du^{op}_\eta(z',c')|^p \, dz'.
\]

We observe that

\[
\int_{(-(1+\rho_\eta)(1+\rho_\eta))^\ell} |Du^{op}_\eta(z',c')|^p \, dz' \leq \frac{C_3}{\eta^{m-\ell}} \int_{\tau^\ell} |Du^{op}_\eta|^p.
\]

Combining both inequalities, we get

\[
\int_{\tau^\ell} |Du^{sm}_\eta|^p \leq \frac{C_5}{\eta^{m-\ell}} \int_{\tau^\ell} |Du^{op}_\eta|^p.
\]

In view of the estimate given by Proposition 3.2, which is satisfied by \( u^{op}_\eta \), the conclusion follows.

By construction, the map \( u^{op}_\eta \) is smooth on a neighborhood of \( G^i_\eta \cap E^i_\eta \) for every \( i \in \{\ell, \ldots, m-1\} \). In particular, \( u^{op}_\eta|_{G^i_\eta \cap E^i_\eta} \) belongs to \( W^{1,p}(G^i_\eta \cap E^i_\eta; \mathbb{R}^p) \), and we now estimate the \( L^p \) norm of \( Du^{op}_\eta|_{E^i_\eta \cap G^i_\eta} \):

**Claim 7.** For every \( i \in \{\ell, \ldots, m-1\} \), we have

\[
\|Du^{op}_\eta|_{L^p(G^i_\eta \cap E^i_\eta)} \leq \frac{C}{\eta^{m-\ell}} \|Du\|_{L^p(E^i_\eta \cap G^i_\eta; \mathbb{R}^p)},
\]

for some constant \( C > 0 \) depending on \( m \) and \( p \).

**Proof of the claim.** Once again, we only need to prove the estimate with \( u^{sm}_\eta \) instead of \( u^{op}_\eta \). Fix \( i \in \{\ell, \ldots, m-1\} \). For every \( x \in G^i_\eta \cap E^i_\eta \), we have \( \psi_\eta(x) = t_\eta \) and thus

\[
u^{sm}_\eta(x) = \int_{B^\eta_1} u^{op}_\eta(x - t_\eta y) \varphi(y) \, dy.
\]

Hence, by Jensen’s inequality and a change of variable,

\[
|Du^{sm}_\eta(x)|^p \leq C_1 \int_{B^\eta_1} |Du^{op}_\eta(x - t_\eta y)|^p \, dy = \frac{C_1}{(t_\eta)^m} \int_{B^\eta_1(x)} |Du^{op}_\eta|^p.
\]

Integrating both members with respect to the \( i \) dimensional Hausdorff measure over \( G^i_\eta \cap E^i_\eta \), by Fubini’s theorem and the co-area formula we get

\[
\int_{G^i_\eta \cap E^i_\eta} |Du^{sm}_\eta(x)|^p \, dx \leq \frac{C_2}{\eta^{m-1}} \int_{E^i_\eta \cap G^i_\eta} |Du^{op}_\eta|^p.
\]

By construction of \( u^{op}_\eta \), we also have

\[
\int_{E^i_\eta \cap G^i_\eta} |Du^{op}_\eta|^p \leq \int_{E^i_\eta \cap G^i_\eta} |Du|^p + \sum_{i \in F^i_\eta \cap G^i_\eta} \int_{\tau^\ell} |Du^{op}_\eta|^p
\]

\[
\leq C_3 \int_{E^i_\eta \cap G^i_\eta} |Du|^p.
\]
and the conclusion follows.

It follows from Claim 5 that if $\ell < p$, then the projected map $u^p_\eta$ is bounded on $E^\ell_q$. If $\ell = p$, then $u^p_\eta$ is bounded on the lower dimensional skeleton $E^{\ell-1}_q$. We now proceed to construct a bounded extension $u^b_\eta$ to $E^m_q$ of the map $u^p_\eta|_{G^m_q \cap E^m_q}$.

**Claim 8.** If $\ell < p$ or if $\ell = p$ and $N^m$ satisfies the leveling property of dimension $p$, then there exists a map $u^b_\eta \in (W^{1,p} \cap L^\infty)(E^m_q)$ such that

(i) $u^b_\eta = u^p_\eta$ on $G^m_q \cap E^m_q$,

(ii) for every $m^m \in E^m_q$,

$$\|Du^b_\eta\|_{L^p(E^m_q)} \leq C \left( \sum_{i=1}^{m-1} \eta_i \sum_{i=\ell+1}^{m} \eta_i \sum_{i=\ell+1}^{m} \eta_i \right),$$

for some constant $C > 0$ depending on $m$, $p$ and $N^m$.

**Proof of the claim.** We first define the extension $u^b_\eta$ on the subkeleton $E^\ell_q$. When $\ell < p$, the map $u^p_\eta$ is continuous on $E^\ell_q$, and we set $u^b_\eta = u^p_\eta$ on $E^\ell_q$. When $p \in \mathbb{N}$ and $\ell = p$, by Proposition 3.5 we may replace $u^p_\eta$ on each face $\tau^\ell \in E^\ell_q \setminus G^\ell_q$ without changing its trace on $E^\ell_q$ to get a continuous map $u^b_\eta : \tau^\ell \to N^m$ such that

$$\|Du^b_\eta\|_{L^p(\tau^\ell)} \leq C_1 \|Du^p_\eta\|_{L^p(\tau^\ell)}.$$

On every face $\tau^\ell \in E^\ell_q \setminus G^\ell_q$, the map $u^b_\eta$ is smooth, and we set $u^b_\eta = u^p_\eta$ in $\tau^\ell$. The map $u^b_\eta$ thus defined in $E^\ell_q$ is continuous and belongs to $W^{1,p}(E^\ell_q, N^m)$.

Let $S^{m-1}_q = G^m_q \cap E^{m-1}_q$, and we extend $u^b_\eta$ to $S^{m-1}_q$ as a continuous Sobolev map by $u^p_\eta$. This is possible since $u^p_\eta = u^b_\eta$ on $(S^{m-1}_q \cap E^m_q) \subset (G^m_q \cap E^m_q)$. We now apply Proposition 5.3 to the map $u^b_\eta : E^\ell_q \cup S^{m-1}_q \to N^m$. The resulting map, that we still denote by $u^b_\eta$, belongs to $W^{1,p}(E^m_q, N^m)$, agrees with $u^b_\eta$ on $G^m_q \cap E^m_q = S^{m-1}_q$ and

$$u^b_\eta(E^m_q) \subset u^b_\eta(E^\ell_q \cup S^{m-1}_q).$$

In particular, we have $u^b_\eta \in L^\infty(E^m_q)$. Finally, $u^b_\eta$ satisfies the estimate

$$\int_{E^\ell_q} |Du^b_\eta|^p \leq C_2 \left( \eta^{m-\ell} \int_{E^\ell_q} |Du^b_\eta|^p + \sum_{i=\ell+1}^{m} \eta^{m-i} \int_{S^i_q} |Du^b_\eta|^p \right).$$

Since $S^i_q \subset G^i_q \cap E^i_q$, the required estimate follows from the above inequality and (5.16).

We deduce from Claim 9 Claim 7 and Claim 8 that

$$\|Du^b_\eta\|_{L^p(E^m_q)} \leq C_1 \|Du\|_{L^p(E^m_q + Q^m_{2p^m})}.$$

We now complete the proof of the theorem. For this purpose, let $(R_i)_{i \in \mathbb{N}}$ be a sequence of positive numbers diverging to infinity. Accordingly, Claim 3 and Claim 4 yield a sequence $(R_i)_{i \in \mathbb{N}}$ from which we define a sequence of positive numbers $(\lambda R_i)_{i \in \mathbb{N}}$ satisfying (5.13). Finally, we take a sequence of positive numbers $(\eta_i)_{i \in \mathbb{N}}$ converging to zero such that

$$\lim_{i \to +\infty} \frac{\eta_i}{\lambda R_i} = 0.$$

By Claim 10 we have

$$\lim_{i \to +\infty} |E^m_q + Q^m_{2p^m}| = 0.$$
We proceed to prove that
\[ \lim_{i \to +\infty} \| Du^{m}_{n_{i}} - Du \|_{L^{p}(g^{m}_{n_{i}})} = 0. \]

Indeed, from estimate (5.3) in Claim 2 we have
\[ \| u^{m}_{n_{i}} - u \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})} \leq \sup_{v \in \bar{B}^{m}_{1}} \| \tau_{\psi_{n_{i}}, v}(u) - Du \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})} + C \| u - u^{op}_{n_{i}} \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})}. \]

Since \( u = u^{op}_{n_{i}} \) outside \( E^{m}_{n_{i}} + \bar{Q}^{m}_{2\rho_{n_{i}}}, \) by the Poincaré inequality for functions vanishing on a set of positive measure and by property 5.18 we have
\[ \| u^{op}_{n_{i}} - u \|_{L^{p}(\bar{Q}^{m}_{1+2\gamma})} \leq C_{2} \| Du - Du^{op}_{n_{i}} \|_{L^{p}(\bar{Q}^{m}_{1+2\gamma})} \leq C_{3} \| Du \|_{L^{p}(E^{m}_{n_{i}} + \bar{Q}^{m}_{2\rho_{n_{i}}})}. \]

Hence,
\[ \| u^{m}_{n_{i}} - u \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})} \leq \sup_{v \in \bar{B}^{m}_{1}} \| \tau_{\psi_{n_{i}}, v}(u) - Du \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})} + C_{4} \| Du \|_{L^{p}(E^{m}_{n_{i}} + \bar{Q}^{m}_{2\rho_{n_{i}}})}, \]

which proves that
\[ \lim_{i \to +\infty} \| u^{m}_{n_{i}} - u \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})} = 0. \]

By the dominated convergence theorem, we thus get
\[ \lim_{i \to +\infty} \| [DII(u^{m}_{n_{i}}) - DII(u)] \|_{L^{p}(g^{m}_{n_{i}})} = 0. \]

In view of estimate (5.3) in Claim 2 and of Claim 5 this implies (5.19).

Since \( u^{op}_{n_{i}} = u^{be}_{n_{i}} \) on \( G^{m}_{n_{i}} \cap E^{m}_{n_{i}}, \) the function obtained by juxtaposing \( u^{be}_{n_{i}} \) and \( u^{op}_{n_{i}} \) defined by
\[ u^{be}_{n_{i}}(x) = \begin{cases} u^{op}_{n_{i}}(x) & \text{if } x \in G^{m}_{n_{i}}, \\ u^{be}_{n_{i}}(x) & \text{if } x \in E^{m}_{n_{i}}, \end{cases} \]
belongs to \( W^{1,p}(\bar{Q}^{m}_{1+\gamma}; N^{n}) \) and is bounded. Moreover, by (5.17)–(5.19) we have
\[ \lim_{i \to +\infty} \| Du^{be}_{n_{i}} - Du \|_{L^{p}(\bar{Q}^{m}_{1+\gamma})} = 0. \]

This completes the proofs of Theorem 1 and Theorem 2. \( \square \)

6. Connection to density of smooth maps

By Theorem 1 and Theorem 2, every map in \( W^{1,p}(\bar{Q}^{m}; N^{n}) \) can be approximated by a sequence in \( (W^{1,p} \cap L^{\infty})(\bar{Q}^{m}; N^{n}) \) under the additional assumption that the manifold \( N^{n} \) satisfies the levelling property of dimension \( p \) when \( p \in \{2, \ldots, m\}. \) As we shall see, this reduces the problem of identifying the closure of \( C^{\infty}(\bar{Q}^{m}; N^{n}) \) to the case when the manifold \( N^{n} \) is compact. The first step of this reduction is given by the following lemma.

**Lemma 6.1.** Let \( N^{n}_{0} \) be a smooth compact submanifold of \( \mathbb{R}^{\nu} \) with boundary. Then for every compact set \( K \) in the relative interior of \( N^{n}_{0}, \) there exists a smooth compact submanifold \( L^{n} \) of \( \mathbb{R}^{\nu+1} \) without boundary such that

i. \( K \subset L^{n}, \)

ii. there exists a smooth map \( P : L^{n} \to N^{n}_{0} \) such that \( P(x) = x \) for every \( x \in K. \)

**Proof.** Let \( K \) be a compact subset in the interior of \( N^{n}_{0}. \) By the collar neighborhood theorem (see e.g. [10 Theorem 1.7.3]), there exists a relative open set \( U \subset N^{n}_{0} \) and a smooth diffeomorphism
\[ f : \partial N^{n}_{0} \times [0, 1] \to N^{n}_{0} \cap \overline{U} \]
such that \( f^{-1}(\partial N^{n}_{0}) = \partial N^{n}_{0} \times \{1\} \) and \( f^{-1}(\partial U \cap N^{n}_{0}) = \partial N^{n}_{0} \times \{0\}. \) By reducing \( U \) if necessary, we can assume that \( U \cap K = \emptyset. \)
Let \(\alpha, \beta : [0, 1] \to [0, 1]\) be two smooth functions such that
\[
\alpha(t) = \begin{cases} t & \text{if } t < 1/4, \\ 1 - t & \text{if } t > 3/4, \\ 1 & \text{if } t > 7/8, \end{cases}
\]
and \(\beta(t) = \begin{cases} 0 & \text{if } t < 1/8, \\ 1 & \text{if } t > 7/8. \end{cases}\)

We also require that \(\beta\) is nondecreasing and \(\beta' > 0\) on the set \([1/4, 3/4]\). We now define the set
\[
L^n = (\{N^0_0 \setminus U\} \times \{0, 1\}) \cup \{(f(\gamma, \alpha(t)), \beta(t)) : \gamma \in \partial N^0_n, t \in (0, 1)\}.
\]
We observe that
\[
(N^0_0 \times \{0\}) \cap \{(f(\gamma, \alpha(t)), \beta(t)) : \gamma \in \partial N^0_n, t \in (0, 1)\}
= \{(f(\gamma, t), 0) : \gamma \in \partial N^0_n, t \in (0, t_0)\}
\]
where \(t_0 = \max\{t : \beta(t) = 0\}\). Similarly,
\[
(N^0_0 \times \{1\}) \cap \{(f(\gamma, \alpha(t)), \beta(t)) : \gamma \in \partial N^0_n, t \in (0, 1)\}
= \{(f(\gamma, 1 - t), 1) : \gamma \in \partial N^0_n, t \in [t_1, 1)\}
\]
where \(t_1 = \min\{t : \beta(t) = 1\}\). This implies that \(L^n\) is a smooth submanifold of \(\mathbb{R}^{n+1}\). By construction, \(L^n\) is compact and has no boundary. Moreover, \(K\) (which is identified with \(K \times \{0\}\)) is contained in \(L^n\). Finally, we define \(P(x, s)\) for every \((x, s) \in L^n \subset N^0_0 \times [0, 1]\) by \(P(x, s) = x\). Then the map \(P\) satisfies all the required properties.

We can now apply the density results already obtained in the framework of compact manifolds. Given \(i \in \{0, \ldots, m - 1\}\), we denote by \(R_i(Q^m; N^n)\) the set of maps \(u : Q^m \to N^n\) which are smooth on \(Q^m \setminus T\), where \(T\) is a finite union of \(i\) dimensional planes, and such that, for every \(x \in Q^m \setminus T\), we have
\[
|Du(x)| \leq \frac{C}{\text{dist}(x, T)}.
\]
Here, the set \(T\) and the constant \(C\) depend on \(u\).

**Lemma 6.2.** Let \(1 \leq p < m\). The set \(R_{m-[p]-1} \cap L^\infty(Q^m; N^n)\) is dense in \((W^{1,p} \cap L^\infty)(Q^m; N^n)\). If moreover \(\pi_{[p]}(N^n) \simeq \{0\}\), then \(C^\infty(Q^m; N^n)\) is dense in \((W^{1,p} \cap L^\infty)(Q^m; N^n)\).

**Proof.** Let \(u \in (W^{1,p} \cap L^\infty)(Q^m; N^n)\). Then the essential range of \(u\) is contained in a compact set \(K\) of a smooth compact submanifold \(N^0_n \subset N^n\) with boundary. Let \(L^n\) be a compact smooth submanifold of \(\mathbb{R}^{n+1}\) satisfying the properties of Lemma 6.1. Then \(u\) belongs to \(W^{1,p}(Q^m; L^n)\). By [5] Theorem 2, there exists a sequence of maps \((u_j)_{j \in \mathbb{N}}\) in \(R_{m-[p]-1}(Q^m; L^n)\) which converges to \(u\) in \(W^{1,p}(Q^m; L^n)\). This implies that the sequence \((P(u_j))_{j \in \mathbb{N}}\) in \(R_{m-[p]-1}(Q^m; N^n)\) still converges to \(P(u) = u\) in \(W^{1,p}(Q^m; N^n)\). Since \(P(L^n) \subset N^0_n\), the sequence \((P(u_j))_{j \in \mathbb{N}}\) is also contained in \(L^\infty(Q^m; N^n)\). This completes the proof of the first part of the lemma.

If we further assume that \(\pi_{[p]}(N^n) \simeq \{0\}\), then we can approximate every map \(P(u_j) \in R_{m-[p]-1}(Q^m; N^n)\) by a sequence of smooth maps in \(C^\infty(Q^m; N^n)\). This fact was originally proved in the setting of a compact target manifold \(N^n\), see [2,9]. However, it does not require such an assumption, see the proof of the Claim in [5] Section 9 where the compactness of \(N^n\) is not used. By a diagonal argument, this implies that \(u\) itself belongs to the closure of \(C^\infty(Q^m; N^n)\).

We proceed with the
Proofs of Corollary 1.1 and Corollary 1.2. Let \( 1 \leq p \leq m \), and assume that the set \( C^\infty(Q^m; N^n) \) is dense in \( W^{1,p}(Q^m; N^n) \). When \( p < m \), this implies that \( \pi_{\lfloor p \rfloor}(N^n) \simeq \{0\} \) as in the case when \( N^n \) is compact \([4,15]\), with the same proof. When \( p \in \{2, \ldots, m\} \), the set \( (W^{1,p} \cap L^\infty)(Q^m; N^n) \) is then dense in \( W^{1,p}(Q^m; N^n) \), and it follows from Proposition 4.2 that \( N^n \) satisfies the levelling property of dimension \( p \).

Conversely, if \( 1 \leq p \leq m \) is not an integer and \( \pi_{\lfloor p \rfloor}(N^n) \simeq \{0\} \), then Lemma 6.2 implies that \( C^\infty(Q^m; N^n) \) is dense in \( (W^{1,p} \cap L^\infty)(Q^m; N^n) \). It also follows from Theorem 4.1 that \( (W^{1,p} \cap L^\infty)(Q^m; N^n) \) is dense in \( W^{1,p}(Q^m; N^n) \). Hence, the set \( C^\infty(Q^m; N^n) \) is dense in \( W^{1,p}(Q^m; N^n) \). This completes the proof of Corollary 1.1.

Finally, the sufficiency part of Corollary 1.2 follows from Theorem 2 and Lemma 6.2 when \( p \in \{1, \ldots, m-1\} \), and from Proposition 2.1 when \( p = m \). This completes the proof of Corollary 1.2. □

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