(Quasi-)linear time algorithm to compute LexDFS, LexUP and LexDown orderings

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Abstract
We consider the three graph search algorithm LexDFS, LexUP and LexDOWN. We show that LexUP orderings can be computed in linear time by an algorithm similar to the one which compute LexBFS. Furthermore, LexDOWN orderings and LexDFS orderings can be computed in time \( (n + m \log m) \) where \( n \) is the number of vertices and \( m \) the number of edges.

1 Introduction
A graph search is a mechanism for systematically visiting the vertices of a graph. Deep-First Search (DFS) and Breadth-First Search (BFS) have been studied for decades (see e.g. [CSRL01]). Those two graph searches can be computed in linear time. A particular kind of BFS, the Lexicographical BFS (LexBFS), has then been introduced in [RTL76]. And by similarity, the Lexicographical DFS (LexDFS) has been studied in [CK08]. And then LexUP and LexDOWN in [Dus14].

A LexBFS ordering of a graph \( G \) is a possible output of a LexBFS search applied to \( G \). While the LexBFS algorithm runs in time \( O(nm) \) where \( n \) is the number of vertices and \( m \) the number of edges, a LexBFS ordering can be computed in time \( O(n + m) \). LexDFS, LexUP and LexDOWN also run in time \( O(nm) \). We show that a LexUP ordering can be computed in linear time by an algorithm similar to the one which compute a LexBFS ordering. Furthermore, we prove that a LexDOWN ordering and a LexDFS ordering can be computed in time \( O(n + m \log m) \).

Definitions are given in Section 2. The four graph search algorithms considered in this paper are given in Section 3. An efficient algorithm to compute LexDFS and LexDOWN ordering are given in Section 4. Finally, efficient algorithm is given to compute a LexUP ordering in Section 5.

2 Definition
Definitions used in this paper are now introduced. Most of those definitions are standard. Let \( \mathbb{N} \) be the set of non-negative integer. For \( A \) a finite set, \( |A| \) denotes the cardinality of \( A \).

A word on \( \mathbb{N} \) is a sequence \( a_1 \ldots a_n \) with \( a_i \in \mathbb{N} \). The empty word is denoted \( \epsilon \). For \( \mathbf{a} = a_1 \ldots a_n \) and \( \mathbf{b} = b_1 \ldots b_m \) two words over \( A \), it is said that \( \mathbf{a} \) is (lexicographically) smaller than \( \mathbf{b} \) if there exists \( i \leq \min(n, m) \) such that, for all \( 1 \leq j \leq i \), \( a_j = b_j \), and (either \( i = n < m \) or \( a_{i+1} < b_{i+1} \)).

2.1 Graph
A (undirected) graph \( G \) with a source is a 3-tuple \((V,E,s)\) where \( V \) is a finite set, \( E \) is a set of subsets of \( V \) whose elements’s cardinality is 2 and \( s \in V \). The elements of \( V \) are called vertices.
The elements of $E$ are called edges. The vertex $s$ is called the source.

A vertex $v$ is said to be a neighbor of $w$ if $\{v, w\} \in E$. The neighborhood of a vertex $v$ is the set of neighbor of $w$, it is denoted $N(v)$. Formally, $N(v) = \{w \mid \{v, w\} \in E\}$. The degree of $v$, denoted $d(v)$, is the cardinality of its neighborhood. Formally, $d(v) = |N(v)|$.

Two vertices $v, w \in V$ are said to be connected if there exists a sequence $v = v_0, \ldots, v_p = w$ such that, for all $0 \leq i < p$, $\{v_i, v_{i+1}\} \in E$. A graph is said to be connex if all pair of distinct vertices are connected.

2.2 Data structures

In this section, we list the data structures used in this paper. We list the operation those data structures admit, and their time complexity. All of those notions are standard (see e.g. [CSRL01]).

In this paper, each type is represented as $\text{type}$, each variable is represented as $\text{var}$ and each function of parameter of an object $o$ is represented as $o.\text{param}$.

It is assumed through this paper that integers can be incremented and compared in constant time. During execution of the algorithm of this paper of a graph $(V, E)$, all integer variables are interpreted by a number whose absolute value is at most $\max(|V|, 2|E|)$. Hence, the constant time assumption is relatively safe. Assing a value $x$ to a variable $v$ is denoted $v := x$ and is assumed to take constant time.

**Arrays** It is assumed in this paper that arrays are created in time linear to their numbers of elements. The elements of an array $A$ with $n$ elements are numbered from 1 to $n$. The $i$-th element of $A$ is denoted $A[i]$, and can be read and assigned in constant time.

**Doubly linked lists** In this paper, all lists are assumed to be doubly-linked lists. A doubly-linked list of elements of type $t$ is a sequence of nodes, with direct access to its first and last nodes. Each node contains a value of type $t$. Each node has also a direct access to its list, to the preceding and following nodes. A doubly-linked list $l$ admits the following constant-time operations:

- Access to its first node: $l.\text{first}$.
- Access to its last node: $l.\text{last}$.
- Adding a node $c$ to the head of $l$: $l.\text{add-first}(c)$.
- Adding a node $c$ to the end of $l$: $l.\text{add-last}(c)$.

A list with $n$ nodes can be sorted in time $O(n \cdot \log(n))$, assuming that the comparison of two nodes of the list can be done in constant time: $l.\text{sort}$. The order will always be clear in the algorithms of this paper.

A node $e$ of a doubly-linked list $l$ admits the following operations:

- access to the preceding node: $e.\text{pred}$,
- access to the following node: $e.\text{next}$,
- access to the value at position $e$: $e.\text{value}$,
- inserting a value $i$ of type $t$ in a new node after $e$: $e.\text{add-after}(i)$ and
• inserting a value \( i \) of type \( t \) in a new node before \( e: e.\text{add-before}(i) \) and
• removing \( e: e.\text{remove} \).

Note that the first value of type \( t \) of a list \( l \) is \( l.\text{first}.\text{value} \) and not \( l.\text{first} \). Indeed, \( l.\text{first} \) is a node and not a value of type \( t \).

**Graphs** A graph \( G \) is represented as an array of size \( n \). The \( i \)-th element of the array contains the list of neighbors of \( v_i \). Formally, \( N(i) \) should be represented as \( G[i] \), however, \( N(i) \) is used in the algorithms of this paper for the sake of the readability.

### 3 Graph search algorithm

In this section, the four graph search algorithms considered in this paper are considered. A graph search algorithm is an algorithm as in [Algorithm 1]. Note that the standard definition of graph search algorithms is more general than the one used in this paper. The only difference between the four graph search algorithms considered in this paper appears in Line 12 of Algorithm 1.

In this paper, the label is always a list of integers. Each update always takes constant time and add exactly an integer to the label. The time complexity of this algorithm is now considered.

**Lemma 3.1.** Let \( G = (V, E, s) \) a graph with \( n \) vertices and \( m \) edges. Assuming the update consists in adding an integer in the front or in the rear of the list, the time complexity of [Algorithm 1] is \( O(nm) \).

**Proof.** Let us first consider the labels. The label of a vertex \( v \) contains at most \( |N(v)| \) elements. Hence the sum of the length of the label is at most \( 2m \).

Lines 2-4 are executed once and in constant time. Hence their time cost is \( O(1) \). Each execution of Line 10 may have to read the entire labels of each vertex. Finding the maximal label then cost \( O(m) \)-times. Since this line is executed \( n \) times, this Line costs \( O(nm) \)-times. Line 10 can be executed in constant time, and is executed \( n \) times, hence it costs \( O(n) \) time. Finally, each execution of line 14 takes constant time. And this line is executed once for each \( 1 \leq i \leq n \) and \( n \in N(v_i) \), hence it is executed \( 2m \)-times. Thus, it costs \( O(m) \) time.

Finally, the whole algorithm runs in time \( O(nm) \). □
For $A$ a graph search algorithm, an $A$ ordering of $G$ is a possible output of $A$ on $G$.

Those four algorithms are now defined, as in [Dus14].

### 3.1 Lexicographic Breadth-First Search

Let Lexicographic Breadth-First Search (LexBFS) be a graph search algorithm, as in Algorithm 1, where Line 12 is: “append $n - i$ to the neighbor’s label”.

Intuitively, at each step, the vertex $v$ is preferred to the vertex $v'$ if the first numbered neighbor of $v$ have been numbered earlier than the first numbered neighbor of $v'$. If their first neighbor are equal, then the same comparison is done on the second neighbor. And so on. If $v$ and $v'$ have $i$ and $i'$ numbered neighbors respectively, with $i' < i$, and furthermore if the $i'$ first numbered neighbors of $v$ are exactly the first $i$ numbered neighbors of $v'$ in the same order, then $v$ is also preferred.

Figure 1a shows examples of LexBFS ordering. Each arrow associates to a vertex $v$ its earliest numbered neighbor. Table 1 associates to each vertex $v$ its list of numbered neighbors when $v$ was numbered. This table also associates to $v$ its label.

| Vertex number | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|---|---|---|---|---|---|---|
| $v$'s label in Figure 1a | 8 | 8 | 76 | 7 | 6 | 54 | 53 | 21 |
| When $v$ is numbered, it’s numbered neighbors are | 1 | 1 | 23 | 2 | 3 | 45 | 46 | 78 |

Table 1: Label during the LexBFS search of Figure 1a

### 3.2 Lexicographic UP

Let Lexicographic UP (LexUp) be a graph search algorithm, as in Algorithm 1, where Line 12 is: “append $i$ to the neighbor’s label”.

Intuitively, at each step, the vertex $v$ is preferred to the vertex $v'$ if the first numbered neighbor of $v$ have been numbered later than the first numbered neighbor of $v'$. If their first neighbor are equal, then the same comparison is done on the second vertex. And so on. If $v$ and $v'$ have $i$ and $i'$ numbered neighbors respectively, with $i' < i$, and furthermore if the $i'$ first numbered neighbors of $v$ are exactly the first $i$ numbered neighbors of $v'$ in the same order, then $v$ is also preferred.

Note that this intuition is the same than for LexBFS, apart that the word “earlier” have been replaced by the word “later”. It is because, in both cases, integers are prepended to the label. But in
the former case, the sequence of prepended numbers decrease while in the second case it increases. Thus, the maximal numbers are added earlier in LexBFS and later in LexUP.

Figure 1b show an example of a LexBFS ordering. Each arrow associates to a vertex \( v \) its first numbered neighbor. Table 2 associates to each vertex \( v \) its label when \( v \) was numbered in each of those 4 examples respectively. Note that this list is also its list of numbered neighbors.

| Vertex number | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|---|---|---|---|---|---|---|
| \( v \)'s label in Figure 1b | 1 | 2 | 3 | 4 | 5 | 6 | 246 | 187 |

Table 2: Label during the LexUP search of Figure 1

### 3.3 Lexicographic Depth-First Search

Let Lexicographic Depth-First Search (LexDFS) be a graph search algorithm, as in Algorithm 1, where Line 12 is: “prepend \( i \) to the neighbor’s label”.

Intuitively, at each step, the vertex \( v \) is preferred to the vertex \( v' \) if the last numbered neighbor of \( v \) have been numbered later than the last numbered neighbor of \( v' \). If their last neighbor are equal, then the same comparison is done on the second last neighbor. And so on. If \( v \) and \( v' \) have \( i \) and \( i' \) numbered neighbors respectively, with \( i' < i \), and furthermore if the \( i' \) first numbered neighbors of \( v \) are exactly the last \( i \) numbered neighbors of \( v' \) in the same order, then \( v \) is also preferred.

Note that this intuition is the same than for LexUP, apart that the word first have been replaced by the word last. Indeed the same integers is added to the label in both cases. However, in the former case the integer is prepended while in the latter case the integer is appended. Hence, in both cases, neighbors with small number are prefered. But in LexUP they must be the earliest neighbors while in LexDFS they must be the latest neighbors.

Figure 1c show an example of a LexBFS ordering. Each arrow associates to a vertex \( v \) its last numbered neighbor. Table 3 associates to each vertex \( v \) its label when \( v \) was numbered. Note that this list is also its list of numbered neighbors.

| Vertex number | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|---|---|---|---|---|---|---|
| \( v \)'s label in Figure 1c | 1 | 2 | 31 | 4 | 53 | 6 | 73 | 82 |

Table 3: Label during the LexDFS search

### 3.4 LexDown

Let Lexicographic DOWN be a graph search algorithm, as in Algorithm 1, where Line 12 is: “prepend \( n - i \) to the label of neighbor”.

Intuitively, at each step, the vertex \( v \) is preferred to the vertex \( v' \) if the first numbered neighbor of \( v \) have been numbered later than the first numbered neighbor of \( v' \). If their first neighbor are equal, then the same comparison is done on the second neighbor. And so on. If \( v \) and \( v' \) have \( i \) and \( i' \) numbered neighbors respectively, with \( i' < i \), and furthermore if the \( i' \) first numbered neighbors of \( v \) are exactly the last \( i \) numbered neighbors of \( v' \) in the same order, then \( v \) is also preferred.
Note that this intuition is the same than for LexBFS (respectively, LexDFS), apart that the word earlier (respectively, first) have been replaced by the word later (respectively, last). The reason is similar to the previous explanations.

Figure 1d shows an examples of LexBFS ordering. Each arrow associates to a vertex $v$ its last numbered neighbor. Table 4 associates to each vertex $v$ its list of numbered neighbors when $v$ was numbered. This table also associate to $v$ its label. Note that when vertices 1 and 2 are fixed, this graph admits no other LexBFS ordering.

| Vertex number | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|---|---|---|---|---|---|---|
| v’s label in Figure 1d | 8 | 8 | 7 | 6 | 6 | 4 | 5 | 2 |
| When v is numbered, its numbered neighbors are | 1 | 1 | 2 | 3 | 2 | 3 | 54 | 7 | 865 |

Table 4: Labels during the LexDOWN search

4 LexDFS and LexDOWN

An algorithm is now given, which outputs a LexDOWN ordering in time $O(n + m \log(m))$. Note that $m \leq n^2$, hence $\log(m) \leq 2 \log n$, thus, this algorithm is more efficient than Algorithm 1.

**Theorem 4.1.** Let $G = (V, E, s)$ be a connex undirected graph with source $s$, with $n$ vertices and $m$ edges. A LexDFS ordering of $G$ can be computed in time $O(n + m \log(m))$.

An intuition of the algorithm is first given. Note that, in Algorithm 1 at each iteration of the loop of Line 10 all unnumbered states must be checked. At each iteration, this line runs in time $O(m)$. This time can be avoided if the list is already sorted. Since at the $i$-th iteration, at most $|N(v_{\sigma(i)})|$ labels change, it suffices to sort and move those $O(|N(v_{\sigma(i)})|)$ elements. The sorting can be done in time $O(|N(v_{\sigma(i)})| \log(|N(v_{\sigma(i)})|))$. Since all of those elements must be moved to the front of the list, a correct usage of pointers allow to move the $O(|N(v_{\sigma(i)})|)$ vertices in time $O(|N(v_{\sigma(i)})|)$. Summing over all $i$, the times taken by those operations is $O(\sum_{i=1}^{n} |N(v_{\sigma(i)})| \log(|N(v_{\sigma(i)})|)) = m \log(m))$.

A simplified version of the algorithm is given as Algorithm 2. In this simplified version, vertex is a type which contains an integer order and a label. Algorithm 3 furthermore shows exactly how to use pointers in order to obtain a quasi-linear time.

**Proof.** In Algorithm 3 a vertex is a data structure which contains 4 parameters

- **order** : an integer;
- **pos** : a node of a list of integers;
- **label** : a list of integers;
- **numbered** : a Boolean;

Let us first prove that Algorithm 3 returns a lexDFS ordering. For $1 \leq i \leq n$, let $\sigma_i$, unnumbered$_i$, vertices$_i$, and max$_i$ be the values of those variables when the iteration of the loop of Line 16 ends, with the variable $i$ interpreted by $j$. Finally, let $\sigma_0$, unnumbered$_0$, vertices$_0$ and max$_0$ be the values of those variables before the first iteration of this loop.

The loop invariants of this algorithm are:
Algorithm 2: Computing a LexDFS ordering-simplified

Input: $G = (V, E, s)$ an undirected graph with a source
Output: a LexDFS-Simple ordering $\sigma$ of the vertices of $G$

2 $\sigma$: array of $n$ integers;
4 vertices: array of $n$ elements of type vertex;
6 max:=0;
8 foreach $i$ from 1 to $n$ do /* Initialization */
10 vertices[$i$]:= {order:=-$\infty$; label:=[ ]};
12 vertices[$s$]:= {order:=0; label:=[ ]};
14 unnumbered:=[ ];
16 foreach $i$ from 1 to $n$ do
18 $\sigma$($i$):=unnumbered.first.value; /* Selecting the greatest value. */
20 sort the neighbors of $v_{\sigma(i)}$ in increasing order;
22 foreach neighb, unnumbered neighbor of $v_{\sigma(i)}$ in increasing order do
24 if neighb's label is empty then /* neighb must be removed from */
26 remove neighb from unnumbered; /* unnumbered if its was in it. */
28 prepend $i$ to neighb's label; /* neighb now has the greatest label */
30 add neighb to the front of unnumbered;
32 set max to max+1;
34 set neighb's order to max; /* and has the greatest order */
36 return $\sigma$

1. $\sigma[j][i]$ contains an element $k$ such that label$_i(v_k)$ is lexicographically maximal, for $0 < i \leq j$.
2. vertices$_j[i].label$ contains the label of $v_i$, as in the $j$-th step of LexDFS. Note that vertices$_j[i].label$ is not actually used in computation of the LexDFS ordering.
3. The variable unnumbered$_j$ contains the list of unnumbered vertices with a non-empty label. Those vertices appears in decreasing lexicographic order of their labels.
4. If $x$ appears before $y$ in unnumbered$_j$, then node$_j[x].order > node_j[y].order$.
5. If unnumbered$_j$ contains the vertex $v_i$, then vertices$_j[i].pos$ is the node of unnumbered$_j$ whose value is $v_i$. Otherwise, vertices$_j[i].pos$ is unspecified.
6. max$_j$ is greater than all finite vertices$_j[i].order$.
7. max$_j$ is less than the sum of the degree of the vertices $v$ which are numbered at the $j$-th step.

Let us show that, for $0 \leq j \leq n$, the 7 invariant are satisfied. Invariants 5 is satisfied at each step, because everytime an integer $i$ is added into unnumbered, vertices$[i].pos$ is modified accordingly. The proof for the other invariants is by induction on $j$. 

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Algorithm 3: Computing a LexDFS ordering

Input: $G = (V, E, s)$ an undirected graph with a source
Output: a LexDFS ordering $\sigma$ of the vertices of $G$

1. $\sigma$: array of $n$ integers initialized to $-1$;
2. vertices: array of $n$ elements of type vertex;
3. $\text{max} := 0$
4. foreach $i$ from 1 to $n$ do
   // Initialization
   5. $\text{vertices}[i] := \{\text{order} := -\infty; \text{label} := []; \text{numbered} := \text{false}\}$;
   6. $\text{unnumbered} := [s];$
   7. $\text{vertices}[s] := \{\text{order} := 0; \text{pos} := \text{unnumbered}.\text{last}; \text{label} := \{\infty\}; \text{numbered} := \text{false}\}$;
8. foreach $i$ from 1 to $n$ do
   // Selecting the greatest value.
   9. $\sigma(i) := \text{unnumbered}.\text{first}.\text{value}$;
   10. $\text{unnumbered}.\text{first}.\text{remove};$
   11. vertices[$\sigma(i)$].numbered := true;
   12. sort the neighbors of $v_{\sigma(i)}$ in increasing order;
   13. foreach $\text{neighb} := \text{unnumbered}$ neighbor of $v_{\sigma(i)}$ in increasing order do
      // neigh must be removed from
      14. if vertices[$\text{neighb}$].label $\neq []$ then /* neighb now has the greatest label */
      15. vertices[$\text{neighb}$].label.add-first($i$);
      16. vertices[$\text{neighb}$].pos := unnumbered.first;
      17. max := max + 1;
      18. vertices[$\text{neighb}$].order := max; /* and has the greatest order */
19. return $\sigma$

Let us show that, for $j = 0$, the $\text{[1]}$ invariant are satisfied.

Invariant $\text{[1]}$ holds, since there are no integer $0 < i \leq 0$.

By definition of LexDFS, all labels are empty at initialization, apart from the one of the source. It is the case in this program because of Lines $\text{[11]}$ and $\text{[14]}$. Hence invariant $\text{[2]}$ is satisfied.

Note that $v_s$ is the only labelled vertex and that no vertex is numbered. Furthermore $s$ is the element of unnumbered because of Line $\text{[14]}$. Hence invariant $\text{[3]}$ is satisfied.

Invariant $\text{[4]}$ is also trivially satisfied, since $s$ have the greatest order and the greatest label, and all other orders are equal and all other label are equals.

Invariant $\text{[5]}$ is trivially satisfied since for all $i$, vertices$_0[i].\text{order} = 0$.

Invariant $\text{[6]}$ is trivially satisfied since no vertices are numbered at the 0-th step.

Let $0 < j \leq n$. Let us now assume that the $\text{[1]}$ invariants holds at step $j - 1$, and let us prove that it holds for $j$.

Since Invariant $\text{[1]}$ holds at step $j - 1$, it clearly holds at step $j$ for all $i < j$. It remains to consider the case $i = j$. By invariant $\text{[3]}$, unnumbered$_j$ contains the list of unnumbered labelled vertices at step $j$, in decreasing lexicographic order of their labels. Hence Line $\text{[12]}$ correctly assigns to $\sigma[i]$ a vertex $w$ such that label$_i(w)$ has a maximal label. Thus, Invariant $\text{[1]}$ holds at step $j$.

Invariant $\text{[2]}$ clearly remains true since the updating of the label is exactly the one of the definition of the LexDFS algorithm.
At the $j$-th step, the list of unnumbered vertices with a non-empty label contains, in this order:

- The neighbors of $v_{\sigma(j)}$, which are unnumbered and have a non-empty label at step $j - 1$. The order, according to their labels, are in the same order in both lists.
- The vertices which are neither $v_{\sigma(j)}$ nor its neighbors, which are unnumbered and have a non-empty label at step $j - 1$. The order, according to their labels, are in the same order in both lists.
- The neighbors of $v_{\sigma(j)}$ which are unnumbered and have an empty label at step $j - 1$.

Thus, according to invariant 3, unnumbered$_j$ must contain, in the following order:

- the elements of unnumbered$_{j-1}$ which are neighbors of $v_{\sigma(j)}$, in the same order,
- the elements of unnumbered$_{j-1}$ which are neither neighbors of $v_{\sigma(j)}$ nor $j$, in the same order,
- the unlabelled neighbors of $v_{\sigma(j)}$, in an arbitrary order.

This is indeed the value of unnumbered$_j$, because of Lines 30 and 34. Hence invariant 3 holds at step $j$.

Since each time an element is moved to the front of unnumbered, its order is greater than any order presently assigned. Its order is greater than any previously assigned order, then Invariant 4 holds.

Invariant 6 clearly holds since the orders are assigned in increasing order, and since, each time an order is assigned, max is assigned to be its predecessor.

It is easy to see that $\text{max}_j \leq \text{max}_{j-1} + |N(\sigma(j))|$. Hence invariant 7 is true at step $j$.

Since the invariants are satisfied at each steps, by 1, at the end of the loop, $\sigma$ contains a LexDFS ordering of $G$. Hence the algorithm indeed returns a LexDFS ordering of $G$.

Let us now consider the computation time. The code of Lines 2, 4, 6, 14 and 12 are executed exactly once, and runs in time $O(n)$. Hence their cost is $O(n)$.

Lines 18, 20, 22, are executed $n$ times and runs in constant time. Hence their cost is $O(n)$. Line 24 is executed once for each vertex $v_i$. And for each vertex $v_i$, it runs in time $O(|N(v_i)| \log(|N(v_i)|))$. Hence the total cost of this line is $O(\sum_{i=1}^{n} |N(v_i)| \log(|N(v_i)|)) = O(m \log m)$.

Lines 32 to 34 are executed once by edge, and executed in constant time. Hence their cost is $O(m)$.

Finally, the total execution time is $O(n + m \log(m))$.

Note that the orders are either infinite, or integers between 0 and $2m$. Hence it is acceptable to assume that comparison of two order parameters can be done in constant time.

LexDOWN As stated in Section 3.4, LexDOWN is similar to LexDFS. It is now considered.

Theorem 4.2. Let $G = (V, E, s)$ be a connex undirected graph with source $s$, with $n$ vertices and $m$ edges. A LexDOWN ordering of $G$ can be computed in time $O(n + m \log(m))$.

Proof. The algorithm to compute a LexDOWN ordering is [Algorithm 3] with the three following changes:

- Line 32 is transformed into “vertices[neighbor].label.prepend.(n - i)”. 


• Line 34 is transformed into “\texttt{unnumbered.add-last.(neigh);}” and
• Line 38 is transformed into “\texttt{max:=max-1;}”.

Invariant 6 must be changed to “\texttt{max}_j\text{ is smaller than all finite vertices}_j[i].\texttt{order}”, and 7 must be changed to “\texttt{max}_j\text{ is less than the sum of the degree of the vertices which are numbered at the }j\text{-th step}”. Apart from those changes, the proof of this theorem is exactly the same than the proof of Theorem 4.1.

5 Efficient LexBFS and LexUP

In this section, it is shown that a LexUP ordering can be computed in linear time. The algorithm is very similar to the algorithm for efficiently computing LexBFS.

A simplified version of the linear time algorithm which computes a LexBFS ordering is recalled as Algorithm 4. This algorithm keeps a list, \texttt{unnumbered}, which contains all vertices, with a non-empty label, in decreasing order according to their label. More precisely, all (indices of) vertices with the same non-empty label belong to a set, and \texttt{unnumbered} is a list of sets. The sets are also encoded as lists. When a vertex \(v_i\) is numbered, the label of its neighbors increases. However, it does not increase enough to become greater than labels which used to be greater than it. Hence all neighbors belonging to the same set are moved to a new set placed before \(s\). As soon as a set is empty, it is removed from the list. Each vertex \(v\) is moved at most \(|\texttt{N}(v)|\) times in the list.

A correct usage of pointers, as shown in Algorithm 5, allows to move in dices from the previous set to the new set in constant time. Hence, the algorithm runs in time \(O(n + m)\). In this algorithm, a set is a data-structure with three parameters:

- \texttt{pos}: a node of a list of sets,
- \texttt{edited}: an integer and
- \texttt{elements}: a list of integers.

And a vertex is a data-structure with four parameters:

- \texttt{pos}: a node of a list of integers,
- \texttt{numbered}: a Boolean,
- \texttt{label}: a list of integers and
- \texttt{set}: a set.

Note that if a vertex \(v\) have an empty label, it has the lexicographically smallest label. Hence, when a first element is added to the label of \(v\), this vertex moves to the second least set (which may become the least set if there remains no more vertex with an empty label). Indeed, the first element of \(v\)'s label is \(n - i\). And \(n - i\) is smaller than the first element of the label of all other vertices \(w\) with non-empty label.

The preceding remark leads to the main difference between LexBFS and LexUP. In LexUP, the element added is \(i\), and not \(n - i\). Hence the first element of this vertex \(v\) is \(i\). Hence, it is greatest than all the first element of all other vertices \(w\) with a non-empty label. Hence, \(v\) moves to the greatest set. Therefore, to transform Algorithm 5 into an algorithm which computes a LexUP ordering, it suffices to do the following change:
Algorithm 4: Efficient computation of a LexBFS - simplified

**Input:** An undirected graph $G = (V, E)$

**Output:** an ordering $\sigma$ of the vertices of $G$

1. vertices: array of $n$ elements of type vertex;
2. **foreach** $i$ from 1 to $n$, distinct from $s$ do
   /* Initialization */
3. $v_i$'s label is set to $[]$;
4. set $s$ is set to $[s]$;
5. $s$'s label is set to $[\infty]$;
6. unnumbered := [set]$s$;
7. **foreach** $i$ from 1 to $n$ do
   8. greatest_set := the first element of unnumbered;
   9. $\sigma[i]$ := any element of greatest_set; /* Selecting a greatest vertex */
   10. Remove this element from greatest_set; /* and removing it from the list. */
   11. If greatest_set is empty, remove it from unnumbered;
   12. **foreach** neighb $\in N(v_{\sigma(i)})$, unnumbered do
      13. if neighb's label is not empty then
         14. if no vertices from neighb's set have been seen for this value of $i$ then
            15. new_set is set to the [];
            16. add new_set before neighb's set;
         17. else
            18. set new_set to the set preceding neighb's set;
            19. If neighb's set is a singleton, remove this set from unnumbered;
         20. else /* If neighb's label is empty */
            21. if No ununlabelled neighbor have been seen for this value of $i$ then
               22. Set new_set to a new set;
               23. Add new_set to the rear of unnumbered;
               24. Set new_set to the last set of unnumbered;
               25. Move neighb to new_set;
               26. append $n-i$ to neighb's label
      27. end if
   18. end if
   19. end if
   20. end if
   21. end if
   22. end if
23. return $\sigma$;

- Line 36 must be modified to “unnumbered.add-first(new_set);”.
- Line 44 must be changed to “vertices[neighb].label.add-last(i);”.

The proof that Algorithm 3 computes a LexDFS ordering is similar to the proof that Algorithm 5 computes a LexBFS ordering.

Note that, if unlabelled was not restricted to contain only labelled vertices, the algorithm would still be correct for LexBFS. Furthermore, the algorithm would be be shorter. However, the algorithm will not be correct anymore for LexUP.

6 Conclusion

In this paper, it has been proven that a LexUP ordering can be computed in linear time and that a LexDOWN ordering and a LexDFS ordering can be computed in time $O(n + m \log m)$.
The author thanks Michel Habib, who introduced this problem to him during his Graph Theory Lectures.

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Algorithm 5: Efficient computation of a LexUP

Input: An undirected graph $G = (V, E)$
Output: an ordering $\sigma$ of the vertices of $G$

1. vertices: array of $n$ elements of type vertex;
2. unlabelledEdited := 0;
3. foreach $i$ from 1 to $n$, distinct from $s$ do /* Initialization */
   4. vertices$[i]$ := {numbered := false, label := []};
   5. set$[s]$ := {pos := set$[s]$.elements.first, numbered := false, label := $\infty$; set := set$[s]$};
   6. unnumbered := [set$[s]$];
4. foreach $i$ from 1 to $n$ do
   5. greatest_set := unnumbered.first.value;
   6. $\sigma[i] := greatest_set$.elements.first.value; /* Selecting a greatest vertex */
   7. greatest_set.elements.first.remove; /* and removing it from the list. */
   8. vertices$[\sigma[i]]$.numbered := true;
   9. if greatest_set.elements$=[]$ then
      10. greatest_set.pos.remove;
   11. foreach neighb $\in N(v_{\sigma[i]})$, unnumbered do
      12. if vertices$[\text{neighb}]$.label$=[]$ then /* If the neighbor's label is not empty */
         13. if vertices$[\text{neighb}]$.set$.edited<i$ then /* no neighbors with the same label have been seen: a new set must be created before the current one. */
            14. vertices$[\text{neighb}]$.set$.edited := i$;
            15. new_set := {edited := i, elements := []};
            16. vertices$[\text{neighb}]$.set$.pos.add-before(new_set); 
            17. new_set$.pos := vertices$[\text{neighb}]$.set$.prec; 
         18. else /* A neighbor with the same label have already been seen */
            19. new_set := vertices$[\text{neighb}]$.set$.prec; 
            20. vertices$[\text{neighb}]$.pos.remove;
            21. if vertices$[\text{neighb}]$.set$.elements$=[]$ then
               22. vertices$[\text{neighb}]$.set$.remove;
            23. else /* If neighb's label is empty */
               24. if unlabelledEdited$<i$ then /* No unlabelled neighbors have been considered yet. */
                  25. unlabelledEdited := i;
                  26. new_set := {edited := i, elements := []};
                  27. unnumbered.add-last(new_set);
                  28. new_set$.pos := unnumbered.last;
               29. else
                  30. new_set := unnumbered.last.value;
                  31. new_set$.elements.add-last(neighb); /* Moving the neighbor */
                  32. vertices$[\text{neighb}]$.set$:=new_set$;
                  33. vertices$[\text{neighb}]$.pos$:=new_set$.elements$.last$;
                  34. vertices$[\text{neighb}]$.label$.add-last(n-i);$ /* Updating the label of neighb */
35. return $\sigma$;
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