Entanglement entropy and Schmidt number as measures of delocalization of $\alpha$ clusters in one-dimensional nuclear systems

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We calculated the von Neumann entanglement entropy and the Schmidt number of one-dimensional (1D) cluster states and showed that these are useful measures to estimate entanglement caused by delocalization of clusters. We analyze system size dependence of these entanglement measures in the linear-chain $n\alpha$ states given by Tohsaki-Horiuchi-Schuck-Röpke wave functions for 1D cluster gas states. We show that the Schmidt number is an almost equivalent measure to the von Neumann entanglement entropy when the delocalization of clusters occurs in the entire system but it shows different behaviors in a partially delocalized state containing localized clusters and delocalized ones. It means that the Rényi-2 entanglement entropy, which relates to the Schmidt number, is found to be almost equivalent to the von Neumann entanglement entropy for the full delocalized cluster system but it is less sensitive to the partially delocalized cluster system than the von Neumann entanglement entropy. We also propose a new entanglement measure which has a generalized form of the Schmidt number. Sensitivity of these measures of entanglement to the delocalization of clusters in low-density regions was discussed.

I. INTRODUCTION

Nuclear many-body systems are self-bound systems of four species of Fermions, spin up and down protons and neutrons. There, a variety of phenomena arise originating in many-body correlations. One of the remarkable spatial correlations is “cluster” which is a subunit composed of strongly correlating nucleons. A typical cluster in nuclear systems is an $\alpha$ cluster, which is a composite particle consisting of four nucleons. If there is no correlation between nucleons in a nucleus, all nucleons behave as independent particles and the nucleus is an uncorrelated state with a clear Fermi surface written by a single Slater determinant wave function. However, in reality, correlations between nucleons are rather strong and $\alpha$ clusters are often formed at the surface in particular in light nuclei. Once $\alpha$ clusters are formed, delocalization of clusters occurs to gain the kinetic energy of center of mass motion (c.m.m.) of clusters in some situations. The delocalization of clusters involves many-body correlations beyond a Slater determinant.

In realistic nuclear systems, a degree of the (de)localization of clusters depends on the competition (balance) of the kinetic energy and potential energy of clusters and is strongly affected also by the Pauli blocking between nucleons in clusters and a core nucleus. The delocalization limit of $\alpha$ clusters is an $\alpha$ cluster gas state, where all $\alpha$ clusters move almost freely like a gas. Such a cluster gas has been predicted to appear in the second $0^+$ state of $^{12}$C [1,2]. To describe cluster states of delocalized $\alpha$ clusters, a new type of cluster wave function, the so-called “Tohsaki-Horiuchi-Schuck-Röpke” (THSR) wave function, has been introduced by Tohsaki et al. [2]. The THSR wave function is essentially based on $\alpha$ clusters in a common Gaussian orbit having a range of the system size, and suitable to describe general cluster gas states of $n\alpha$ clusters. Indeed, it has been shown that $^8$Be$(0^+_1)$ and $^{12}$C$(0^+_2)$ can be described well by the THSR wave functions of $2\alpha$ and $3\alpha$, respectively. Since the optimized THSR wave functions for these states have much larger ranges of the Gaussian orbit of clusters than the cluster size, $^8$Be$(0^+_1)$ and $^{12}$C$(0^+_2)$ are interpreted as gas-like cluster states of $2\alpha$ and $3\alpha$ [2,5].

The THSR wave function has been extended to apply to $^{20}$Ne, and found to be able to describe also $^{16}$O+$\alpha$ states in $^{20}$Ne [4,7]. Recently, Suhara et al. have proposed that this concept of the $\alpha$-cluster gas is applicable also to one dimension (1D) cluster motion in linear-chain $n\alpha$ structures [8]. They have proposed the 1D-THSR wave functions and shown that the 1D-THSR wave functions with the optimized Gaussian ranges can describe well the exact solutions of the linear-chain $3\alpha$ and $4\alpha$ states. Their result somewhat contradicts to the the conventional picture of the linear-chain $n\alpha$ structures that spatially localized $\alpha$ clusters are arranged in 1D with certain intervals [8]. Even though their model restricted in 1D is not enough to settle the problem of stability of the linear-chain states in realistic nuclear systems in 3D, their work provides a new picture of 1D cluster states and may lead to a understanding of cluster phenomena in nuclear many-body systems. Moreover, the 1D cluster state is an academically interesting problem of quantum many-Fermion systems.

To distinguish between localization and delocalization of composite particles (clusters) in microscopic wave functions of Fermion (nucleon) systems, one should carefully consider the antisymmetrization effect of nucleons between clusters. When clusters largely overlap with each other, motion of clusters is strongly affected by the Pauli blocking between nucleons in other clusters. As a result of Pauli blocking effect, when the system size is as small as or smaller than the cluster size, clusters can not move freely and the system becomes equivalent to a localized cluster state. In the case of a large system size, where the overlap between clusters is small, clusters can move rather freely. It means that the
The entanglement entropy and Schmidt number are the measures of entanglement in quantum many-body systems, which are defined by the density matrix. In this section, we describe these measures and also propose a new measure by extending the Schmidt number. We also define spatial distributions of these entanglement measures.

A. One-body density matrix

In the present paper, we use only the one-body density matrix, which has been often discussed in nuclear systems \cite{22}, though more general density matrices are used to define measures of entanglement. For a wave function $|\Psi^{(A)}\rangle$ of an $A$-particle state, the matrix element of the one-body density in an arbitrary basis is given as

$$
\rho_{pq} = \langle \Psi^{(A)} | c_q^\dagger c_p | \Psi^{(A)} \rangle,
$$

(1)

where $c_q^\dagger$ and $c_p$ are the creation and annihilation operators, respectively. The one-body density matrix is regarded as the matrix element of the one-body density operator $\hat{\rho}_\Psi$ for the wave function $\Psi^{(A)}$,

$$
\hat{\rho}_\Psi = \sum_{pq} |p\rangle \rho_{pq} \langle q|.
$$

(2)
The one-body density matrix can be diagonalized by a unitary transformation of single-particle basis
\[
(D^\dagger \rho D)_{ll'} = \rho_l \delta_{ll'},
\]
\[
a^\dagger_l = \sum_{l'} D^\dagger_{l'l} a^\dagger_{l'},
\]
where
\[
\rho_l = \langle \Psi^{(A)} | a^\dagger_l a_l | \Psi^{(A)} \rangle,
\]
\[
0 \leq \rho_l \leq 1
\]
is the eigen value of the one-body density matrix and means the occupation probability of the single-particle state \( l \) in the wave function \( \Psi^{(A)} \). \( \rho_l \) corresponds to so-called Schmidt coefficients in the Schmidt decomposition for the one-body density matrix. The trace of the one-body density matrix \( \rho \) equals to the particle number \( A \) as
\[
A = \text{Tr} \rho = \sum_l \rho_l.
\]

Note that the normalization of the one-body density matrix is not a unit but is the particle number \( A \). In the present paper, the entanglement entropy and the Schmidt number are defined by thus defined one-body density matrix normalized as \( \text{Tr} \rho = A \).

If a wave function \( | \Psi^{(A)} \rangle \) is a Slater determinant, \( \rho_l = 1 \) for occupied single-particle states and \( \rho_l = 0 \) for unoccupied single-particle states. It means that the one-body density operator \( \hat{\rho}_\Psi \) satisfies \( \hat{\rho}_\Psi^2 = \hat{\rho}_\Psi \) and is a projector in the single-particle Hilbert space if \( \Psi^{(A)} \) is a non-entangled state given by a Slater determinant wave function.

B. Measures of entanglement

1. Entanglement entropy

The von Neumann entanglement entropy has been introduced by Bennett et al. and proved to be an entanglement measure in quantum many-body systems (see, for instance, Refs. [15, 18] and references therein). The entanglement entropy is defined by the von Neumann entropy of one of the reduced density matrices and called “von Neumann entanglement entropy”, which we call the “entanglement entropy” unless otherwise noted. In the present paper, we consider the entanglement entropy only for the one-body density matrix of Fermion systems. The entanglement entropy that is defined by the one-body density matrix is given as,
\[
S = -\text{Tr} \rho \log \rho = -\sum_l \rho_l \log \rho_l.
\]
The entanglement entropy is zero if a wave function \( | \Psi^{(A)} \rangle \) is a Slater determinant, because \( \rho_l = 1 \) for occupied single-particle states and \( \rho_l = 0 \) for unoccupied single-particle states. It means that a system has non-zero positive value of the entanglement entropy only if the system contains many-body correlations beyond a Slater determinant, i.e., if the system is entangled.

2. Schmidt number

Another entanglement measure is the so-called Schmidt number, which has been introduced by Grobe et al. and used to measure many-body correlations in atomic and nuclear physics [13, 14]. The Schmidt number \( K \) is defined as,
\[
K = \frac{A}{\sum_l \rho_l^2} = \frac{A}{\text{Tr} \rho^2},
\]
which estimates the number of states involved in the Schmidt decomposition. \( K \) equals one if a wave function \( | \Psi^{(A)} \rangle \) is a Slater determinant, and \( K \) is greater than one for entangled states. In analogy to the entanglement entropy, which is generated by the entanglement, it is useful to consider the quantity \( K - 1 \),
\[
K - 1 = \frac{A}{\sum_l \rho_l^2} - 1 = \frac{\sum_l (\rho_l - \rho_l^2)}{\sum_l \rho_l^2}.
\]
It is clear that $K - 1 = 0$ only for non-entangled states because $\rho_l^2 = \rho_l$, i.e., $\hat{\rho}_q^2 = \hat{\rho}_q$ is satisfied only if a state is given by a Slater determinant. It means that a non-zero positive value of $K - 1$ is generated by entanglement, that is, many-body correlations beyond a Slater determinant. One of the merits of the $K$ number is that it is given by $\text{Tr} \rho^2$ which can be calculated by the matrix element $\rho_{pq}$ in an arbitrary basis without the diagonalization of the one-body density matrix.

We should comment that the logarithm of the Schmidt number is nothing but the Rényi entanglement entropy of order 2 (Rényi-2 entanglement entropy) for the one-body density matrix,

$$\log K = S_2^{\text{Rényi}},$$  
$$S_\xi^{\text{Rényi}} = \frac{1}{1 - \xi} \log \{ \frac{\text{Tr} \rho^\xi}{\rho} \}. $$

It is known that, in the $\xi \to 1$ limit, the Rényi entanglement entropy becomes equal to the von Neumann entanglement entropy. In this paper, we discuss the Schmidt number instead of Rényi-2 entropy though they are equivalent entanglement measures.

3. Extension of Schmidt number

In the present paper, we propose an entanglement measure which is regarded as a generalized version of the Schmidt number. As shown in Eq. (10), the origin of non-zero contributions in $K - 1$ is partially occupied single-particle state with $0 < \rho_l < 1$. Ignoring the total scaling factor $1/\sum_l \rho_l^2$, the contribution $W(\rho_l)$ (the weight function) of a single-particle state with the occupation probability $0 < \rho < 1$ in $K - 1$ has the $\rho_l$ dependence, $W(\rho_l) = \rho_l - \rho_l^2$, which is different from the weight function $W(\rho_l) = -\rho_l \log \rho_l$ in the entanglement entropy $S$. The former has relatively small weights for single-particle states with small occupation probability $\rho_l < 1/2$ compared with the latter although the occupied $\rho_l = 1$ and unoccupied $\rho_l = 0$ single-particle states have no weight in both cases (see the $\rho$ dependence of the weight functions in Fig. 1). We here introduce an alternative weight function, $W(\rho_l) = \rho_l^\gamma (1 - \rho_l)$, and define an entanglement measure $K_\gamma$ by extending the entanglement measure $K$ as follows,

$$K_\gamma = \frac{\sum_l \rho_l^\gamma}{\sum_l \rho_l^{(1+\gamma)}},$$

$$K_\gamma - 1 = \frac{\sum_l \rho_l^\gamma (1 - \rho_l)}{\sum_l \rho_l^{(1+\gamma)}},$$

where the parameter $\gamma$ is a positive constant. In the case of $\gamma = 1$, the $K_\gamma$ number becomes consistent with the Schmidt number $K$. Similarly to the known entanglement measures ($S$ and $K - 1$), $K_\gamma - 1$ equals to zero only for a Slater determinant, whereas a non-zero positive value of $K_\gamma - 1$ is generated by entanglement, that is, many-body correlations beyond a Slater determinant. Note that the diagonalization of the density matrix is required to obtain the $K_\gamma$ number for a non-integer $\gamma$ differently from the $K$ number. In this paper, we choose $\gamma = \log 2$ which gives the weight function $W(\rho_l) = \rho_l^{\log 2} (1 - \rho_l)$ having $\rho_l$ dependence similar to $W(\rho_l) = -\rho_l \log \rho_l$ for the entanglement entropy in $0.001 \leq \rho \leq 1$ (see Fig. 1).

It should be commented that $\log K_\gamma$ is given by the Rényi entanglement entropy of order $\gamma$ and $1 + \gamma$ as,

$$\log K_\gamma = (1 - \gamma)S_\gamma^{\text{Rényi}} + \gamma S_{1+\gamma}^{\text{Rényi}}.$$  

C. Spatial distributions of entanglement measures

To investigate the spatial distribution of the ”important single-particle states” that contribute to the non-zero entanglement entropy, we have defined the spatial distribution $s(r)$ of the entanglement entropy, which have been introduced in the previous paper as,

$$S = \sum_l (-\rho_l \log \rho_l) = \int s(r) dr,$$

$$S = \sum_l (-\rho_l \log \rho_l) \phi_l^*(r) \phi_l(r).$$
FIG. 1: Weight functions $W(\rho) = -\rho \log \rho, \rho(1-\rho), \rho^{\log 2}(1-\rho)$ for $S$, $K$, and $K_{\log 2}$, respectively.

Here the factor $-\rho \log \rho$ is contribution of the single-particle state $|l\rangle$ in $S$, and $\phi_l^*(r)\phi_l(r)$ means the density distribution in $|l\rangle$ and it is normalized as $\int \phi_l^*(r)\phi_l(r)dr = 1$. Therefore, the distribution $s(r)$ reflects spatial distributions of the important single-particle states $|l\rangle$ that contribute to the total entanglement entropy, whereas it is hardly affected by almost occupied single-particle states having $\rho_l \approx 1$. The expression of $S$ with $s(r)$ is analogous to that of the particle number with the density distribution,

$$ A = \sum_l \rho_l = \int \rho(r)dr, \quad (18) $$

$$ \rho(r) = \sum_l \rho_l \phi_l^*(r)\phi_l(r). \quad (19) $$

Note that the distribution $s(r)$ is not quantity determined only by local information at the position $r$. It is different from the density distribution $\rho(r)$ which is determined only by the local information.

We also define the distributions $\kappa(r)$ and $\kappa_\gamma(r)$ for $K - 1$ and $K_{\gamma} - 1$, respectively, to see spatial distributions of the "important single-particle states" in total amount of the measures,

$$ K - 1 = \frac{\sum_l (\rho_l - \rho_l^2)}{\sum_l \rho_l^2} = \int \kappa(r)dr, \quad (20) $$

$$ \kappa(r) = \sum_l \frac{1}{\sum_l \rho_l^2} (\rho_l - \rho_l^2) \phi_l^*(r)\phi_l(r) = \sum_l \frac{K}{A} (\rho_l - \rho_l^2) \phi_l^*(r)\phi_l(r), \quad (21) $$

and

$$ K_{\gamma} - 1 = \frac{\sum_l \rho_l^{1+\gamma} (1-\rho_l)}{\sum_l \rho_l^{1+\gamma}} = \int \kappa_\gamma(r)dr, \quad (22) $$

$$ \kappa_\gamma(r) = \sum_l \frac{1}{\sum_l \rho_l^{1+\gamma}} \rho_l^{1+\gamma} (1-\rho_l) \phi_l^*(r)\phi_l(r). \quad (23) $$
Similarly to s(r), these distributions, κ(r) and κs(r), show spatial distributions of the important single-particle states |l⟩ that contribute to the total amount of K − 1 and Kγ − 1, respectively. Note that these distributions are not local quantities.

III. S, K, AND Kγ OF IDEAL STATES IN A TOY MODEL

In this section, we discuss behaviors of S, K, and Kγ for correlated (entangled) states composed of delocalized clusters in a toy model. Here, γ in the Kγ number is assumed to be 0 < γ < 1.

Let us consider nf species of particles. For instance, nf = 2 for spin up and down Fermions, and nf = 4 for spin up and down protons and neutrons. We use the label σ for species of Fermions such as σ = p↑, p↓, n↑, n↓ for nuclear systems. We consider an A-body system containing the same number n = A/nf of σ particles. We assume that the system is symmetric for the exchange of species, that is, single-particle orbitals are occupied by all species of particles with an equal weight. Then the density matrix is diagonal with respect to σ, and all species of particles have the same occupation probability ρσl = ρl independent from σ. We consider the one-body density matrix and operator in the reduced space with the dimension n and define the entanglement measures, S, K, and Kγ, for a species of particles by using the reduced matrix of the one-body density. The total entanglement entropy Stotal and the total numbers Kγ, total and Kγ, total are given by the measures for each species as Stotal = nfS, Ktotal = K, and Kγ, total = Kγ. In this paper, we discuss S, K, and Kγ for a species of particles.

For simplicity, A particles are assumed to stay on sites in a space. The number of available sites (single-particle states) is m and we use the label kj for the jth site. Let us first consider a system of A = nf particles. If an A-body state is an ideal state of independent particles, the wave function can be written by a simple product of single-particle wave functions

\[ \Psi(1, 2, \ldots, n_f) = \psi(1)\psi(2)\cdots\psi(n_f), \]

\[ \psi(i) = \sum_{k=k_1,k_2,\ldots,k_m} c(k)\phi_k(i). \]

For instance, c(k) is constant as c(k) = 1/√m for a free gas state. For such an non-entangled state, S = 0, K − 1 = 0, and Kγ − 1 = 0 because the one-body density operator is given as \( \hat{\rho}_\psi = |\psi\rangle\langle\psi| \) and satisfies \( \hat{\rho}^2_\psi = \hat{\rho}_\psi \). Another example is a “localized cluster” system of a cluster, where all particles are localized at one site kj to form a composite particle (a cluster) at kj. The wave function is given as

\[ \Psi(1, 2, \ldots, n_f) = \prod_{i=1}^{n_f} \phi_{kj}(i). \]

This wave function for a localized clusters also has zero measures, S = 0, K − 1 = 0, and Kγ − 1 = 0, because the one-body density operator is given as \( \hat{\rho}_\psi = |\phi_{kj}\rangle\langle\phi_{kj}| \) and satisfies again \( \hat{\rho}^2_\psi = \hat{\rho}_\psi \). Namely, the localized cluster wave function is a non-entangled state.

We consider a state of a delocalized cluster in a strong correlation limit,

\[ \Psi(1, 2, \ldots, n_f) = \frac{1}{\sqrt{m}} \left\{ \prod_{i=1}^{n_f} \phi_{kj}(i) + \prod_{i=1}^{n_f} \phi_{kj}(i) + \cdots + \prod_{i=1}^{n_f} \phi_{km}(i) \right\}, \]

where the composite particle composed of A = nf particles moves freely in the entire space with an equal probability \( \frac{1}{m} \). This is a highly entangled (strongly correlated) state, where, if a particle is observed at a certain site, all other particles are always observed at the same site. This is a strong coupling limit and an example of a delocalized cluster. The one-body density operator,

\[ \hat{\rho}_\psi = \sum_{j=1}^{m} \frac{1}{m} |k_j\rangle\langle k_j|, \]

corresponds to the Schmidt decomposition with the common Schmidt coefficients, 1/m. We get S = log m, K = m, and Kγ = m. In general, if eigen values of \( \rho_l \) are constant \( \rho_l = 1/m_V \) (l = 1, ..., mV) for a given number \( m_V \) of states and \( \rho_l = 0 \) for \( l > m_V \), we get

\[ S = \log m_V, \quad e^S = m_V, \]

\[ K = m_V, \]

\[ K_\gamma = m_V. \]
It indicates that $e^S$, $K$, and $K_\gamma$ equal to the number $m_V$ of states involved in the Schmidt decomposition. $m_V$ is regarded as the effective volume size. Note that, for a Slater determinant, $e^S$, $K$, and $K_\gamma$ equal to 1 indicating that the effective volume size is one which can not be decomposed.

Next we consider an $A$-particle system of $n = A/n_{fV}$ clusters. Here $n$ is the number of clusters (composite particles) formed by $n_f$ constituent Fermions. For a state of $n$ localized clusters at $n$ sites $k = j_1,\ldots,k_n$, the wave function is given as,

$$
\Psi(1,2,\ldots,A) = (n!)^{-n_f/2}A\left\{\prod_{k=0}^{n-1} \phi_{k_j_1}(hn_f + 1)\cdots\phi_{k_j_1}(hn_f + n_f)\right\}
= (n!)^{-n_f/2}A\left\{\phi_{k_{j_1}}(1)\cdots\phi_{k_{j_1}}(n_f)\cdots\phi_{k_{j_n}}(A - n_f + 1)\cdots\phi_{k_{j_n}}(A)\right\}.
$$

Since the occupation probability is $\rho_l = 1$ for occupied single-particle states $k_{j_1},\ldots,k_{j_n}$ and $\rho_l = 0$ for unoccupied single-particle states, the system has the zero value of the measures, $S = 0$, $K - 1 = 0$, and $K_\gamma - 1 = 0$. Let us consider a state of $n$ clusters in the delocalized limit where all clusters are delocalized and move freely in a volume size $m_V$ like a gas. For this state of delocalized clusters, the occupation probability is $\rho_l = n/m_V$ for $l = 1,\ldots,m_V$. $m_V$ should not be less than $n$ because of the Pauli principle of $n$ Fermions. The measures of this delocalized cluster system are,

$$
S = n \log\frac{m_V}{n}, \quad e^{S/n} = \frac{m_V}{n},
$$

$$
K = \frac{m_V}{n},
$$

$$
K_\gamma = \frac{m_V}{n}.
$$

It indicates that $K$ and $K_\gamma$ are consistent with $e^{S/n}$, $ne^{S/n}$, $nK$, and $nK_\gamma$ equal to the number $m_V$ of the states involved in the Schmidt decomposition and they estimate the effective volume size of the delocalization of clusters. For the case of $m_V = n = A/n_{fV}$, all $m_V$ single-particle states are completely occupied by $A$ particles and clusters can not move at all. The state is equivalent to the localized cluster, and it has zero value, $S = 0$, $K - 1 = 0$, and $K_\gamma - 1 = 0$, of entanglement measures. In case $m_V$ is larger than $n$, the measures, $S$, $K - 1$, and $K_\gamma - 1$, become positive indicating that the delocalization of clusters occurs and the system becomes an entangled state. In the case that $m_V$ is much larger than $n$, the system corresponds to a low-density cluster gas and it is a highly entangled state.

Finally, we consider the case of a partial delocalization that $n - 1$ clusters are localized to form a core and only the last cluster is delocalized. This situation corresponds to an $\alpha$ cluster moving almost freely around a core nucleus. In this partially localized case of a delocalized cluster around a core, the occupation probability is $\rho_l = n/m_V$ for $l = 1,\ldots,m_V$ and $\rho_l = 1$ for the $n - 1$ single-particle states occupied by constituent particles of the core. Then, the measures of this partially delocalized system are

$$
S = \log m_V, \quad e^{S/n} = m_V^{1/n},
$$

$$
K = \frac{nm_V}{(n - 1)m_V + 1},
$$

$$
K_\gamma = \frac{(n - 1)m_\gamma + m_V}{(n - 1)m_V + 1}.
$$

In the low-density limit of the large $m_V$, we get

$$
K \rightarrow \frac{n}{n - 1},
$$

$$
K_\gamma \rightarrow m_V^{1 - \gamma}.
$$

Let us compare the fully delocalized cluster state and the partially delocalized cluster state in the large $m_V$ limit (the large volume size limit). The former state corresponds to a dilute cluster gas, where all clusters are delocalized moving freely like a gas, and is a highly entangled state. For the fully delocalized cluster state, all the entanglement measures are sensitive to $m_V$ as given in Eqs. 33, 34, and 35. That is, $e^{S/n}$, $K$, and $K_\gamma$ equal to $m_V/n$ and equivalently good indicators to measure the delocalization of clusters. However, for the latter case of the partial delocalization, which corresponds to a delocalized cluster around a core, $e^{S/n}$, $K$, and $K_\gamma$ are not equivalent but show different dependences on $m_V$. As clearly shown in Eqs. 36, 39, and 40, $e^{S/n}$ and $K_\gamma$ increases as the $m_V$ increases, but $K$ becomes constant and does not depend on $m_V$ in the large $m_V$ limit. It indicates that $S$ and
$K$, can be useful measures sensitive to the partial delocalization in subsystem, but $K$ is insensitive to the partial delocalization. The reason for the insensitivity of $K$ is the significant contribution from fully occupied single-particle states with $\rho_1 = 1$ in the denominator of the definition of the $K$ number, which makes the contribution from the delocalized part minor. From another point of view, it is found that $K$ can be a good probe to clarify whether the delocalization of clusters occurs in the entire system or not. $m_{1V}$ dependences of $e^{S_{1V}/n}$ and $K$, are the powers of $1/n$ and $1 - \gamma$, respectively. More generally, for the partially delocalized system that is composed of $n_g$ delocalized clusters and $n_0$ localized clusters, the $m_{1V}$ dependences of $e^{S_{1V}/n}$ and $K$, in the large $m_{1V}$ limit are the powers of $n_g/n$ and $1 - \gamma$, respectively. Here, $n_g$ and $n_0$ are the numbers of delocalized and localized clusters, respectively, and $n_0 + n_g = n$. This means that, in the case $n_g/n > 1 - \gamma$ of a small fraction of delocalized clusters, $K$, is more sensitive to the delocalization of clusters than $e^{S_{1V}/n}$, whereas, in the case of $n_g/n = 1 - \gamma$, the $K$, number has the $m_{1V}$ dependence similar to $e^{S_{1V}/n}$.

IV. APPLICATION TO 1D NUCLEAR SYSTEMS OF $\alpha$ CLUSTERS

In the present paper, we use the delocalized cluster wave functions in 1D for the linear-chain $n\alpha$ states which are investigated in the previous paper [21]. We also adopt the $\alpha + (2\alpha)$ wave functions for a state of an $\alpha$ cluster around a $2\alpha$ core. We analyze the entanglement measures, $S$, $K$, and $K_{log2}$, and also their spatial distributions, and discuss the system size dependence of these measures.

A. Localized and delocalized $\alpha$ cluster wave functions in 1D

We here briefly explain the adopted model wave functions for (de)localized cluster states in 1D. More details of the model wave functions are described in the previous paper [21]. We consider intrinsic wave functions of the linear-chain $n\alpha$- cluster states aligned to the $x$ axis. It means that the (de)localization of $\alpha$ clusters are defined for $\alpha$-cluster motion along the $x$ axis.

For a localized $n\alpha$-cluster wave function, we use the BB wave function [22] as,

$$
\Phi_{BB}^{n\alpha}(R_1, \ldots, R_n) = \frac{1}{\sqrt{A!}} A[\psi^a_{R_1} \ldots \psi^a_{R_n}], \quad (41)
$$

$$
\psi^a_{R_i} = \psi_0^{\alpha} \chi_{s0} \chi_{s1} \chi_{s2} \chi_{s3}, \quad (42)
$$

$$
\phi_0^{\alpha} = (\pi \rho_1^2)^{-3/4} \exp \left[ -\frac{1}{2b^2} (r - R_i)^2 \right]. \quad (43)
$$

$\psi^a_{R_i}$ is the four-nucleon wave function of the $i$th $\alpha$ cluster expressed by the (0$s$)$^4$ harmonic oscillator (ho) shell-model configuration localized around the spatial position $R_i$. $\chi$ is the spin-isospin part and $\phi_0^{\alpha}$ is the spatial part of the single-particle wave function. For 1D-cluster states, the position parameter $R_i$ is set to be $R_i = (R_i, 0, 0)$, and the 1D BB wave function is expressed as $\Phi_{BB}^{n\alpha}(R_1, \ldots, R_n)$. The parameter $b$ for the $\alpha$-cluster size is chosen to be $b = 1.376$ fm same as in Ref. [8]. Note that a single BB wave function is a localized cluster wave function written by a Slater determinant of single-particle wave functions and it has exactly zero values of the measures, $S = 0$, $K = 1$, and $K_{log2} = 1 = 0$. General wave functions for 1D $n\alpha$ systems can be written by linear combination of BB wave functions $\Phi_{BB}^{n\alpha}(R_1, \ldots, R_n)$.

For a delocalized cluster wave function, we use the 1D-THSR wave functions of $n\alpha$, which have been proposed by Suhara et al. to describe the linear-chain 3$\alpha$- and 4$\alpha$-cluster states in $^{12}$C and $^{16}$O systems [8]. The 1D-THSR wave functions are given by linear combination of BB wave functions with a Gaussian weight as,

$$
\Phi_{1D-THSR}^{n\alpha}(\beta) = \int dR_1 \cdots dR_n \exp \left\{ -\sum_{i=1}^{n} \frac{R_i^2}{\beta^2} \right\} \Phi_{BB}^{n\alpha}(R_1, \ldots, R_n). \quad (44)
$$

If the antisymmetrization is ignored, the $\Phi_{1D-THSR}^{n\alpha}(\beta)$ expresses the $n\alpha$ state where all $\alpha$ clusters are confined in the $y$ and $z$ directions while they move in the $x$ direction in the Gaussian orbit with the range parameter $\beta$. $\beta$ corresponds to the system size of the 1D $n\alpha$ state. In case of the system size $\beta$ is as small as or smaller than the $\alpha$-cluster size $b$, the 1D-THSR wave function is approximately equivalent to a localized cluster wave function given by a Slater determinant because of the antisymmetrization effect. As $\beta$ increases, the delocalization of $\alpha$ clusters occurs. When $\beta$ is large enough compared with the $\alpha$-cluster size $b$, the system goes to a dilute 1D $\alpha$-cluster gas where $n \alpha$ clusters move almost freely like a gas in the $x$ direction.
In the practical calculation, the $R_i$ integration is approximated by summation on mesh points in a finite box as done in the previous paper. For 2α, 3α, and 4α systems, we make a correction of the total c.m.m. to eliminate a possible artifact from β dependence in the total c.m.m. as described in the previous paper. It means that the 1D-THSR wave function of 1α without the c.m.m. correction expresses a system of an α cluster bound in an external field, which is not a realistic state of an isolate nucleus, whereas those of 2α, 3α, and 4α with the c.m.m. correction correspond to self-bound nα systems with linear-chain structures predicted in nuclear states such as excited states of $^{12}$C and $^{16}$O.

In this paper, we discuss the entanglement measures for 1α with no c.m.m. correction (nc) and those for 2α, 3α, and 4α with the c.m.m. correlation. We also show the entanglement entropy for 2α with no c.m.m. correction, just for comparison.

We also use the 1D-THSR wave function of $\alpha + (2\alpha)$ for the case of an α cluster around the 2α core,

$$\Phi^{\alpha(2\alpha)}_{1\text{D-THSR}}(\beta) = \int dR_1 \exp \left\{ -\frac{R_1^2}{\beta^2} \right\} \Phi^{3\alpha}_{BB}(R_1, R_2 = +\varepsilon, R_3 = -\varepsilon),$$

(45)

where the 2α core is located at the origin and an α cluster is distributed around the core with a Gaussian weight. When β is large enough compared with the α-cluster size $b$, the wave function describes a delocalized α cluster around the 2α core at the origin. This wave function is associated with the partially delocalized cluster wave function in the toy model discussed previously. We use a small value of $\varepsilon = 0.02$ fm to describe the 2α core almost equivalent to the h.o. (0s)$^4$(0p$_x$$^4$) configuration. This wave function has been originally introduced in previous paper to describe the $\alpha + ^{16}$O cluster states in $^{20}$Ne.

We calculate numerically the one-body density matrices for these 1D cluster wave functions, and obtain the measures, $S$, $K$, and $K_{\text{log2}}$ from the eigenvalues $\rho_i$ of the one-body density matrices. The detailed method of the practical calculation is described in the previous paper. Spatial distributions, $s(r)$, $\kappa(r)$, and $\kappa_\gamma(r)$ and the density distribution $\rho(r)$ are integrated out along $y$ and $z$ axes, and we get distributions, $s(x)$, $\kappa(x)$, $\kappa_\gamma(x)$, and $\rho(x)$, projected onto the $x$ axis.

B. nα cluster states

We analyze $S$, $K$, and $K_{\text{log2}}$ of the 1D nα states written by the 1D-THSR wave functions, $\Phi^{n\alpha}_{1\text{D-THSR}}(\beta)$. Figure 2 shows the system size dependence of the entanglement entropy of the 1α state with no c.m.m. correction ($1\alpha_{nc}$), and 2α, 3α, and 4α states with the c.m.m. correction, as well as that of a 2α state with no c.m.m. correction ($2\alpha_{nc}$), $S/n$, $e^{S/n}$, and $n(e^{S/n} - 1)$ are plotted as functions of the dimensionless system size $\beta/b$. In the $\beta/b = 0$ limit, the entanglement entropy equals zero, because the 1D-THSR wave functions are equivalent to localized nα cluster wave functions in this limit. As the $\beta/b$ increases, the delocalization of clusters occurs and the entanglement entropy is generated. $e^{S/n}$ for the $1\alpha_{nc}$ state increases almost linearly to $\beta/b$ as expected from a naive picture that a cluster moves freely in a finite volume, which is decomposed into $m_V \sim \beta/b$ states by the quantum decoherence in the one-body density matrix. Similarly to the $1\alpha_{nc}$ state, $S$ for 2α, 3α, and 4α states increases with the increase of the system size, indicating that the entanglement entropy is generated as the delocalization of clusters is enhanced. $e^{S/n}$ for 2α, 3α, and 4α states also shows almost linear dependence to $\beta/b$. The $n(e^{S/n} - 1)$ plots in Figs. 2(c) and 2(d) show a good correspondence of the $\beta$ dependence of the entanglement entropy between $1\alpha_{nc}$ and $2\alpha_{nc}$, and that between 2α, 3α, and 4α states. The values of $n(e^{S/n} - 1)$ for 2α, 3α, and 4α are relatively smaller compared with those for $1\alpha_{nc}$ and $2\alpha_{nc}$ because of the c.m.m. correction performed for the 2α, 3α, and 4α states.

Let us compare other measures, $K$ and $K_{\text{log2}}$, with the entanglement entropy. Figure 3 shows the system size dependence of $e^{S/n}$, $K$, and $K_{\text{log2}}$ of $1\alpha_{nc}$, 2α, 3α, and 4α states. $K$ and $K_{\text{log2}}$ shows the $\beta$ dependence quite similar to that of $e^{S/n}$ except for global normalization factors. This result indicates that the entanglement entropy, $K$, and $K_{\text{log2}}$, can be approximately equivalent measures for the cluster delocalization of nα states. It is consistent with the naive expectation from the analysis of the delocalized cluster states in the toy model discussed in the previous section. It means that $ne^{S/n}$, $nK$, and $nK_{\text{log2}}$ estimate the number of the states involved in the Schmidt decomposition as shown in Eqs. (34), (34), and (35).

We also compare the spatial distributions of these measures, $s(x)$, $\kappa(x)$, and $\kappa_{log2}(x)$ of $1\alpha_{nc}$ in Fig. 4. The density distribution is also shown. As described previously, $s(x)$, $\kappa(x)$, and $\kappa_{log2}(x)$ reflect spatial distributions of the important single-particle states that give non-zero contributions to total measures $S$, $K - 1$, $K_{\text{log2}} - 1$, respectively. Note that the shape, in particular, the spatial broadness of distributions is of importance, but their global scaling (normalization) is not so meaningful. It is found that $s(x)$ and $\kappa_{log2}(x)$ show quite similar distributions to each other. They are more broadly distributed than the density distribution indicating that $S$ and $K_{\text{log2}} - 1$ are generated in low-density regions but relatively suppressed in high-density regions. Differently from $s(x)$ and $\kappa_{log2}(x)$, the enhancement in low-density regions and the suppression in high-density regions are relatively weak in $\kappa(x)$. As a result, the spatial
extent of $\kappa(x)$ is not as remarkable as that of $s(x)$ and $\kappa_{\log 2}(x)$. This difference in the spatial distributions of $S$, $K-1$, $K_{\log 2}-1$ comes from the different weight functions $W(\rho)$, namely, $S$ and $K_{\log 2}-1$ are relatively sensitive to single-particle states with low occupation probability compared with $K-1$ as already shown in Fig. 1. The distributions $s(x)$, $\kappa(x)$, and $\kappa_{\log 2}(x)$ for the $2\alpha$, $3\alpha$, and $4\alpha$ states are shown in Fig. 4 as well as density distribution. Similarly to the $1\alpha_{\text{mc}}$ state, $s(x)$ and $\kappa_{\log 2}(x)$ are more broadly distributed than the density distribution and also slightly broader than $\kappa(x)$, indicating again that $S$ and $K_{\log 2}-1$ are generated in low-density regions but suppressed in high-density regions.

In the present result, we find that the $S$, $K$ and $K_{\log 2}$ can be useful measures to estimate the entanglement caused by the delocalization of clusters in the $1D\alpha$ states given by the 1D-THSR wave functions. As the system size increases, the delocalization of clusters develops and non-zero $S$, $K-1$, and $K_{\log 2}-1$ are generated. In the spatial distributions of these entanglement measures, significant contributions come from low-density regions than high-density regions. Quantitatively, the Schmidt number ($K$) is less sensitive to low-occupation probability single-particle states than the entanglement entropy ($S$) and the $K_{\log 2}$ number resulting in the less broad distribution of $\kappa(x)$ than $s(x)$ and $\kappa_{\log 2}(x)$, which are remarkably broader than the density distribution.

![Figure 2](image)

**FIG. 2:** System size $\beta/b$ dependence of the entanglement entropy $S$ and $e^{S/n}$ in the 1D-THSR wave functions of $n\alpha$. $n(e^{S/n}-1)$ is also shown. The 1D-THSR wave functions with the total c.m.m. correction are used for $2\alpha$, $3\alpha$, and $4\alpha$ states, and that with no c.m.m. correction is used for the $1\alpha$ system ($1\alpha_{\text{mc}}$). The results of the 1D-THSR wave function with no c.m.m. correction for $2\alpha$ system ($2\alpha_{\text{mc}}$) are also shown for comparison.

### C. an $\alpha$ cluster around a $2\alpha$ core

We investigate $S$, $K$, and $K_{\log 2}$ of $\alpha+(2\alpha)$ cluster wave functions. We use the 1D-THSR wave function $\Phi_{1D-\text{THSR}}^{\alpha-(2\alpha)}(\beta)$ in Eq. (45) for the $\alpha+(2\alpha)$ states associated with the partially delocalized cluster wave function. We also adopt the parity projected BB wave function with the fixed $\alpha-(2\alpha)$ distance $d$ as used in the previous paper as,

$$
\Phi_{BB}^{\alpha-(2\alpha),+}(d) = (1+\hat{P}_r)\Phi_{BB}^{3\alpha}(R_1 = d, R_2 = +\varepsilon, R_3 = -\varepsilon) = \Phi_{BB}^{3\alpha}(R_1 = d, R_2 = +\varepsilon, R_3 = -\varepsilon) + \Phi_{BB}^{2\alpha}(R_1 = -d, R_2 = +\varepsilon, R_3 = -\varepsilon),
$$

where $\hat{P}_r$ is the parity transformation operator. Note that $\Phi_{BB}^{\alpha-(2\alpha),+}(d)$ is given by the linear combination of two Slater determinants. This corresponds to the symmetry restored state where the parity symmetry is broken in the intrinsic state because of the cluster development.

Figure 8 shows $S$, $K$, and $K_{log 2}$ as well as $e^{S/n}$ for the 1D THSR wave function and the parity-projected BB wave function of the $\alpha+(2\alpha)$ system. Looking at the result of the parity-projected BB wave function, we find that, as $d$
FIG. 3: Comparison of system size $\beta/b$ dependences between $e^{S/n}$, $K$, and $K_{\log 2}$ in the 1D-THSR wave functions of $n\Omega$.

FIG. 4: Spatial distributions, $s(x)$, $\kappa(x)$, and $\kappa_{\log 2}(x)$, of $S$, $K-1$, and $K_{\log 2}-1$ in the 1D-THSR wave functions of $1\Omega_{\alpha c}$ with $\beta = 1$ fm, 3 fm, and 5 fm. The corresponding dimensionless system sizes are $\beta/b = 0.73$, 2.18, and 3.63. The density distribution $\rho(x)$ is also shown.

increases and the parity symmetry is broken in the intrinsic wave function, non-zero values of $S$, $K-1$, and $K_{\log 2}$ are generated in the projected state even though the intrinsic wave function before the parity projection is the localized cluster wave function. When $d$ is enough large, $P_\rho \Phi_{\Pi B}$ becomes independent to $\Phi_{\Pi B}$, and we get $S \to \log 2 = 0.693$, $K - 1 \to 1/\alpha = 0.2$, $K_{\log 2} - 1 \to 0.236$ from the eigen values of the one-body density matrix, $\rho_1 = \rho_2 = 1$ and $\rho_3 = \rho_4 = 1/2$. It means that non-zero values of $S$, $K-1$, and $K_{\log 2} - 1$ are generated by the symmetry breaking and restoration. Let us turn to the result of the 1D-THSR wave function. In the $\beta/b = 0$ limit, $S$, $K-1$, and $K_{\log 2} - 1$ are zero. In the $\beta/b \gtrsim 1$ region, $S$, $K$, and $K_{\log 2}$ increases rapidly with the increase of $\beta/b$ because of the symmetry breaking and restoration as seen in the parity-projected BB wave function. In the $\beta/b \lesssim 1$, $S$, $K$, and $K_{\log 2} - 1$ increases gradually as the system size increases indicating that the delocalization of cluster develops in this region.

As discussed in the previous section, the Schmidt number, $K$, should be less sensitive to the delocalization in the partially delocalized cluster states. To see the sensitivity of $e^{S/n}$, $K$, and $K_{\log 2}$ to the delocalization, we show in Fig. 3(1) the scaled measures, $(e^{S/n} - 1)/0.26$, $(K - 1)/0.2$, $K_{\log 2} - 1/0.236$, in the 1D-THSR wave function of $\alpha + (2\alpha)$, which are normalized to the values of the large $d$ limit of the parity-projected BB wave function. As expected from the analysis of the simple toy model, the result in Fig. 3(1) shows that $K$ is not so sensitive to the delocalization of cluster
in the \( \alpha + (2\alpha) \) state, whereas \( e^{S/n} \) and \( K_{\log 2} \) more strongly depend on the system size in the \( \beta/b \gtrsim 1 \) region than \( K \). The \( \beta/b \) dependence of \( K_{\log 2} \) is quite similar to that of \( e^{S/n} \), maybe, because of the accidental coincidence of the \( m^{1/n}_{\gamma} \) dependences, \( e^{S/n} \propto m^{1/n}_{\gamma} \) and \( K \propto m^{(1-\gamma)} \), for the partially delocalized cluster state discussed in the simple toy model as \( 1/n = 1/3 \) and \( 1 - \gamma = 1 - \log 2 = 0.31 \).

Figure 5 shows the distributions \( s(x) \), \( \kappa(x) \), and \( \kappa_{\log 2}(x) \) of \( S, K - 1 \), and \( K_{\gamma} - 1 \) as well as the density distribution in the 1D-THSR wave functions of \( \alpha + (2\alpha) \) for \( \beta = 2 \) fm and 5 fm. It is clear that distributions \( s(x) \), \( \kappa(x) \), and \( \kappa_{\log 2}(x) \) are strongly suppressed in the \( |x| \lesssim 2 \) fm because of the Pauli blocking effect from the \( 2\alpha \) core. In the \( \beta = 5 \)
of the parity-projected BB wave function.

Similarly to the distributions in $n\alpha$ states, the distribution $\kappa(x)$ for $K - 1$ is not so enhanced in low-density regions as $s(x)$ and $\kappa_{\log 2}(x)$ because the Schmidt number $K$ is relatively less sensitive to single-particle states with low occupation probability compared with the entanglement entropy and the $K_{\log 2}$ number.

$FIG. 7$: Same as Fig. [5] but spatial distribution, $\kappa_{\log 2}(x)$, for $K_{\log 2} = 1$.

$FIG. 8$: (a)(b)(c)(d) System size $\beta/b$ dependence of $S$, $e^{S/n}$, $K$, and $K_{\log 2}$ in the 1D-THSR wave function of $\alpha + (2\alpha)$, and (e)(f)(g)(h) $d/b$ dependence of $S$, $e^{S/n}$, $K$, and $K_{\log 2}$ in the parity-projected BB wave function of $\alpha + (2\alpha)$. (i) Scaled measures, $(e^{S/n} - 1)/0.26$, $(K - 1)/0.2$, $K_{\log 2}/0.236$, in the 1D-THSR wave function of $\alpha + (2\alpha)$, normalized to the large $d$ limit values of the parity-projected BB wave function.
localized clusters, the Schmidt number is not sensitive to the delocalization of the cluster around the core, whereas on the other hand, for the partially delocalized cluster state which contains a delocalized cluster and a core composed of localized clusters, the Schmidt number is not sensitive to the delocalization of the cluster around the core, whereas the entanglement entropy, the Schmidt number, and the equivalent entanglement measures to estimate the delocalization of a 1D gas of delocalized clusters. On the other hand, for the partially delocalized cluster state which contains a delocalized cluster and a core composed of localized clusters, the Schmidt number is not sensitive to the delocalization of the cluster around the core, whereas the entanglement entropy and the $K$ number can be good indicators to measure the partial delocalization. In other words, the Schmidt number can be a good probe to clarify whether the delocalization occurs for all clusters in the entire system or not, owing to the insensitivity to the partial delocalization. We should comment that the Schmidt number corresponds to the the Rényi-2 entanglement entropy; $K = e^{\frac{S}{n}}$. It means that the equivalence in the fully delocalized cluster states and difference in the partially delocalized cluster states between $e^{S/n}$ and $K$ are nothing but those between the von Neumann entanglement entropy and Rényi-2 entanglement entropy.

In the present analysis of the 1D $\alpha$-cluster states, the $K_{\log 2}$ number shows similar features to the entanglement entropy. It indicates that the $K_{\log 2}$ number can be an alternative measure to the entanglement entropy to estimate the delocalization in both cases of the partially and fully delocalized cluster states. When the delocalized part is minor compared with the core part, the $K_{\log 2}$ number is more sensitive to the partial delocalization than the entanglement entropy, and hence it is a promising measure.

We should point out that the calculation of the Schmidt number may be practically easier than those of the entanglement entropy and $K_\gamma$ with a non-integer $\gamma$ because the Schmidt number is given by $\text{Tr} \rho^2$ for which the diagonalization of the one-body density matrix is not required. This could be an advantage of the Schmidt number in large dimensional calculations because numerical errors might become a more serious problem in practical calculations of the entanglement entropy and the $K_\gamma$ number.

Recently, comparison between von Neumann and Rényi-2 entanglement entropies have been discussed in various fields such as field theories \[24\]. The equivalence and difference between von Neumann and Rényi-2 entanglement entropies shown in the present study are general features and may be found in various quantum systems. The present study of entanglement measures in cluster wave functions of nuclear systems may shed light on study of entanglement (correlation) in general quantum systems.

FIG. 9: Spatial distributions, $s(x)$, $\kappa(x)$, and $\kappa_{\log 2}(x)$, of $S$, $K$, and $K_{\log 2}$ in the 1D-THSR wave functions of $\alpha + (2\alpha)$ with $\beta = 2$ fm and 5 fm. The corresponding dimensionless system sizes are $\beta/b = 1.45$ and 3.63. The density distribution $\rho(x)$ is also shown.

V. SUMMARY

We calculated the entanglement entropy ($S$) and the Schmidt number ($K$) defined by the one-body density matrix in the 1D $\alpha$-cluster states to measure the entanglement caused by the delocalization of clusters in nuclear systems. We also propose a new entanglement measure $K_\gamma$ with a generalized form of the Schmidt number.

For the delocalized cluster states of $n\alpha$ given by the 1D-THSR wave functions, $e^{S/n}$, $K$, and $K_{\log 2}$ show good correspondence indicating that the entanglement entropy, the Schmidt number, and the $K_{\log 2}$ number are almost equivalent entanglement measures to estimate the delocalization of a 1D gas of delocalized $n\alpha$ clusters. On the other hand, for the partially delocalized cluster state which contains a delocalized cluster and a core composed of localized clusters, the Schmidt number is not sensitive to the delocalization of the cluster around the core, whereas the entanglement entropy and the $K_{\log 2}$ number can be good indicators to measure the partial delocalization. In other words, the Schmidt number can be a good probe to clarify whether the delocalization occurs for all clusters in the entire system or not, owing to the insensitivity to the partial delocalization. We should comment that the Schmidt number corresponds to the the Rényi-2 entanglement entropy; $K = e^{\frac{S}{n}}$. It means that the equivalence in the fully delocalized cluster states and difference in the partially delocalized cluster states between $e^{S/n}$ and $K$ are nothing but those between the von Neumann entanglement entropy and Rényi-2 entanglement entropy.

In the present analysis of the 1D $\alpha$-cluster states, the $K_{\log 2}$ number shows similar features to the entanglement entropy. It indicates that the $K_{\log 2}$ number can be an alternative measure to the entanglement entropy to estimate the delocalization in both cases of the partially and fully delocalized cluster states. When the delocalized part is minor compared with the core part, the $K_{\log 2}$ number is more sensitive to the partial delocalization than the entanglement entropy, and hence it is a promising measure.

We should point out that the calculation of the Schmidt number may be practically easier than those of the entanglement entropy and $K_\gamma$ with a non-integer $\gamma$ because the Schmidt number is given by $\text{Tr} \rho^2$ for which the diagonalization of the one-body density matrix is not required. This could be an advantage of the Schmidt number in large dimensional calculations because numerical errors might become a more serious problem in practical calculations of the entanglement entropy and the $K_\gamma$ number.

Recently, comparison between von Neumann and Rényi-2 entanglement entropies have been discussed in various fields such as field theories \[24\]. The equivalence and difference between von Neumann and Rényi-2 entanglement entropies shown in the present study are general features and may be found in various quantum systems. The present study of entanglement measures in cluster wave functions of nuclear systems may shed light on study of entanglement (correlation) in general quantum systems.
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\[ s_{(2\alpha)} \]

\[ \alpha_{(2\alpha)} \]

\[ BB \]

\[ \rho \]

\[ x \]

\[ (fm) \]

\[ d=3 \text{ fm} \]

\[ d=7 \text{ fm} \]