Bounded weak and strong time periodic solutions to a three-dimensional chemotaxis-Stokes model with porous medium diffusion

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Abstract

In this paper, we study the time periodic problem to a three-dimensional chemotaxis-Stokes model with porous medium diffusion $\nabla^m n$ and inhomogeneous mixed boundary conditions. By using a double-level approximation method and some iterative techniques, we obtain the existence and time-space uniform boundedness of weak time periodic solutions for any $m > 1$. Moreover, we improve the regularity for $m \leq \frac{4}{3}$ and show that the obtained periodic solutions are in fact strong periodic solutions.

Key words: chemotaxis-Stokes system; porous medium diffusion; mixed boundary; time periodic solution
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1 Introduction and main result

In this paper, we consider the following coupled chemotaxis-Stokes model

\begin{equation}
\begin{aligned}
\rho_{11} + u \cdot \nabla \rho &= \Delta \rho^m - \chi \nabla \cdot (n \nabla c) + \mu n(a(x, t) - n) + g(x, t), \\
c_t + u \cdot \nabla c &= \Delta c - cn,
\end{aligned}
\end{equation}

where $m > 1$, $(x, t) \in Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. This model describes the motion of oxygen-driven bacteria living in a water drop containing oxygen. $n, c$ denote the bacterial density, the oxygen concentration respectively, $n \nabla c$ is the chemotactic flux, $\chi > 0$ is the sensitivity coefficient of aggregation induced by the concentration changes of oxygen, $\mu > 0$ is a parameter, $\mu n(a(x, t) - n)$ reflects the proliferation and death of bacteria

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in a logistic law, \( g(x,t) \geq 0 \) is source term, \( cn \) is the consumption term of oxygen, \( u, \pi \) are the fluid velocity and the associated pressure, \( \nabla \varphi \) is the gravitational potential. Here, we assume that \( a, g \) and \( \nabla \varphi \) are time periodic functions with period \( T \).

The chemotaxis-fluid model originated from the experimental observation \([17]\) in concentrated suspensions of swimming bacteria of the species \textit{Bacillus subtilis}, and then some biologists and mathematicians began to work on the qualitative description of the pattern formation process in this experiment \([10, 11]\). Based on the experimental observations, in 2005, Tuval et al. \([33]\) introduced the following chemotaxis-fluid model \((m = 1)\).

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\
    c_t + u \cdot \nabla c &= \Delta c - n f(c), \\
    u_t + \tau u \cdot \nabla u &= \Delta u - \nabla \pi + n \nabla \varphi, \\
    \nabla \cdot u &= 0,
\end{align*}
\]  

(1.2)

which describes the motion of oxygen-driven swimming bacteria in incompressible fluid, that is, \textit{Bacillus subtilis} suspending in a drop of water will swim up an oxygen gradient, and when the upper bacteria-rich boundary layer is too dense, it becomes unstable and an overturning instability develops, leading to the formation of falling bacterial plumes. In this model, the fluid motion is governed by the Navier-Stokes equations \((\tau = 1)\). However, the viscous force plays a leading role in slow viscous flows, and the inertial force is far less than the viscous force. Thus, for which, the Navier-Stokes equations can be approximated using Stokes equations by ignoring the convective term \( u \cdot \nabla u \) (see \([18]\)).

The employed linear diffusion in (1.2) is the normal type of diffusion associated with Brownian processes. However, there is evidence that at least some morphogens may not freely diffuse \([2]\) and it is needed to develop some nonlinear diffusion models in biology. Considering that the finite size of bacteria causes the nonlinear enhancement of random movement of cells at large densities and the diffusion of cells is more like movement in a porous medium in a viscous fluid, Di Francesco et al. \([6]\) proposed the following chemotaxis-fluid system by modifying the linear diffusion \( \Delta n \) in (1.2) by the nonlinear diffusion \( \Delta n^m \):

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n\chi(c)\nabla c), \\
    c_t + u \cdot \nabla c &= \Delta c - n f(c), \\
    u_t + \tau u \cdot \nabla u &= \Delta u - \nabla \pi + n \nabla \varphi, \\
    \nabla \cdot u &= 0.
\end{align*}
\]  

(1.3)

Before going into our mathematical analysis, let us briefly recall some important progresses on system (1.3) and its variants. For the two dimensional case of (1.3), that is, \( \tau = 1 \), the global solvability and boundedness of weak solutions are established completely for any \( m > 1 \) in \([31]\). While in three dimensional space, the study of (1.3) with \( \tau = 0 \) is rather tortuous. In 2011, a global very weak solution \((n \ln n \in L^1)\) is obtained for \( m = \frac{4}{3} \) by Liu and Lorz \([23]\) in dimension 3. Subsequently, Duan and Xiang improved this results to any \( m > 1 \) \([7]\). However,
this kind of weak solutions may be unbounded, and it is impossible to identify the singularity of the solution. In 2010, Di Francesco et al. [6] obtained the existence of a global bounded weak solution for \( m \in (m^*, 2) \) \((m^* \approx 1.81)\); a locally bounded global weak solution was then obtained for \( m > \frac{8}{5} \) in 2013 [32]; the uniform boundedness of solutions was subsequently supplemented for \( m > \frac{4}{3} \) [37]; further extension was made by Winkler for \( m > \frac{9}{8} \) in [38] to a convex domain; recently, Jin [16] obtained the existence of bounded weak solutions for \( m > 1 \) and bounded strong solutions for \( m < \frac{4}{3} \). If the logistic growth term \( \mu n(1 - n) \) is added to this model, Lankeit [20] established the global weak solution for the linear diffusion case \( m = 1 \), and proved that after some waiting time the weak solution becomes smooth and finally converge to the semi-trivial steady state \((1, 0, 0)\); for the nonlinear diffusion case, Jin [13] established the existence of global bounded weak solutions for any \( m > 1 \) to the fluid-free case.

For the time periodic problem of the chemotaxis-fluid models, there are few works concerned. In 2019, Jin [15] considered the chemotaxis-fluid model (1.1) without source term for the linear diffusion case \((m = 1)\), and proved the existence of bounded strong (and classical) time periodic solutions in dimension 3. Recently, for the following chemotaxis-Stokes model with porous medium diffusion in dimension 3,

\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n^m - \chi \nabla \cdot (n \nabla c) + \mu n(1 - n) + g(x, t), \\
    c_t + u \cdot \nabla c &= \Delta c - c + n, \\
    u_t &= \Delta u - \nabla \pi + n \nabla \varphi, \\
    \nabla \cdot u &= 0, \\
    \frac{\partial n^m}{\partial \nu} |_{\partial \Omega} &= \frac{\partial c}{\partial \nu} |_{\partial \Omega} = u |_{\partial \Omega} = 0,
\end{aligned}
\]

Huang and Jin [12] established the existence of uniformly bounded time periodic solution for any \( m \geq \frac{6}{5} \).

It is worth noting that most of the results have been carried out for the closed system, that is, there is no flux of oxygen and cells through the fluid-air interface. However, in fact, the experiment is to place the well mixed suspension of Bacillus subtilis in an open chamber. On the surface of the water layer, oxygen is allowed to exchange with the outside air. Therefore, for the oxygen in the model, Dirichlet boundary condition in [5, 24] or Robin boundary condition in [3, 39] are more realistic. In [5, 24], some numerical results are given, the global existence of small strong solutions around a equilibrium state is established by Peng and Xiang [29] in dimension 3, global classical solutions [3, 4] and time periodic solutions [15] are established respectively in dimension 2 or 3. Inspired by the above works, we assume that the water drop is surrounded by air, oxygen exchange will take place on the boundary of \( \Omega \), that is, the solved oxygen in the water drop may leave, and the free oxygen in the air may diffuse into the drop. The behaviour of the oxygen exchange can be described by Raoult’s law, which connects the rate of incoming oxygen to the partial vapour pressure of the oxygen in the surroundings. We assume that the vapour pressure of the free oxygen is given, and thereby, the incoming rate of
oxygen is known. The leaving rate of the oxygen molecules is proportional to the total number of molecules on the surface. Therefore, we have the following Robin boundary condition

\[
\frac{\partial c}{\partial \nu}|_{\partial \Omega} = -a_1(x,t)c(x,t) + a_2(x,t),
\]

where \(a_1, a_2 \in C^\infty(\partial \Omega \times [0, \infty))\), \(a_1 > 0\) is the leaving rate of the oxygen molecules, \(a_2 \geq 0\) with \(a_2 \neq 0\) is the incoming oxygen and depends on the known vapour pressure of the free oxygen. By [3, 22], there exist

\[
g_1, g_2 \in C^\infty_T(\overline{\Omega} \times [0, \infty))
\]

such that

\[
\frac{\partial g_1(x,t)}{\partial \nu} = -a_1(x,t) < 0, \quad g_2(x,t) = \frac{a_2}{a_1} \geq 0, \quad \frac{\partial g_2(x,t)}{\partial \nu} = 0
\]

for \((x,t) \in \partial \Omega \times [0, \infty)\). So the boundary condition of \(c\) can be rewritten as

\[
\frac{\partial c}{\partial \nu}|_{\partial \Omega} = \frac{\partial g_1(x,t)}{\partial \nu}(c(x,t) - g_2(x,t)).
\]

For \(u\), we still consider the no-slip boundary condition, namely,

\[
u|_{\partial \Omega} = 0.
\]

Inside the suspension, chemotaxis is ultimately responsible for the maintenance of the fluid convection, and thus, for the shape of plumes at large times [5, 24]. To propose the the boundary condition for \(n\) naturally, we need to consider the effect of chemotaxis on the boundary of suspension. It’s now widely recognized that chemotaxis allows microbial cells to colonize surfaces or interfaces and grow on them, in the form of multicellular aggregates embedded in matrices commonly referred to as biofilms, which provide the cells with strength in numbers to cope with environmental stresses [30]. *Bacillus subtilis* has long served as a robust model organism to examine the molecular mechanisms of biofilm formation. Due to the aerotaxis of the cells, *Bacillus subtilis* (less domesticated strains) preferably produce biofilms at the air-liquid interface rather than on the surface of a solid phase in a liquid [27, 30]. For the pictures of *Bacillus subtilis* biofilms in liquid medium, please refer to Fig.1 in [27] and Fig.2 in [35]. Attachment is initially reversible, and the suspended cell comes and goes until it sticks to the interface and commits to a sessile existence. However, biofilms are not static entities and cells can be released from the biofilms through an active process of dispersal due to resource limitation and waste product accumulation, see [25, 35].

Taking account of the fact that inducing biofilm formation is an important selective advantage of chemotaxis and biofilm formation is a nearly universal bacterial trait [30, 35], in the present paper, we assume that aerobic bacteria (such as *Bacillus subtilis*) may form biofilms at the boundary due to chemotaxis. We also assume that bacteria may “cross the boundary”, that is, the cells escape from the system when they are attached at the interface and become
nonmotile, and the cells enter the system when they are released from biofilms. Therefore, we propose Neumann boundary condition for $n$:

$$\frac{\partial n}{\partial \nu}|_{\partial \Omega} = 0. \tag{1.8}$$

In view of the cell flux $-\nabla n + nu + \chi n \nabla c$, the boundary conditions (1.7)–(1.8) imply that there may be three cases occurring on the boundary:

1. If no exchange of oxygen takes place at $\Gamma_1$ on the boundary, that is, $\frac{\partial c}{\partial \nu}|_{\Gamma_1} = 0$ with $\Gamma_1 \subset \partial \Omega$, then no bacteria “cross” $\Gamma_1$. This case exists at the bottom (and the sides) of the liquid medium and implies that biofilms can not be formed at these regions. In particular, this situation includes the no-flux boundary conditions for $c$ and $n$ by letting $\Gamma_1 = \partial \Omega$, which can be considered as an extension of early works;

2. If the incoming oxygen molecules are more than leaving oxygen molecules at some region $\Gamma_2 \subset \partial \Omega$, that is, $\frac{\partial c}{\partial \nu}|_{\Gamma_2} > 0$, then more bacteria escape from the system at $\Gamma_2$ to produce biofilms;

3. If the leaving oxygen molecules are more than incoming oxygen molecules at some region $\Gamma_3 \subset \partial \Omega$, that is, $\frac{\partial c}{\partial \nu}|_{\Gamma_3} < 0$, then either more bacteria are released from the biofilms and enter the system, or there is no bacteria at the region $\Gamma_3$.

Thus, we have the following inhomogeneous mixed boundary conditions

$$\frac{\partial n}{\partial \nu}|_{\partial \Omega} = 0, \quad \frac{\partial c}{\partial \nu}|_{\partial \Omega} = \frac{\partial g_1(x,t)}{\partial \nu}(c(x,t) - g_2(x,t)), \quad u|_{\partial \Omega} = 0. \tag{1.9}$$

The purpose of this paper is to establish the existence of bounded weak and strong time periodic solutions for the problem (1.1) and (1.9) in dimension 3. Since the boundary condition of oxygen concentration $c$ is inhomogeneous, it is necessary to make a transformation to apply the standard Neumann heat semigroup argument and integration by parts to (1.1)–(1.9).

Let

$$\tilde{c} = e^{-g_1}(c - g_2). \tag{1.10}$$

Then we have

$$\frac{\partial \tilde{c}}{\partial \nu}|_{\partial \Omega} = -e^{-g_1}\frac{\partial g_2}{\partial \nu}|_{\partial \Omega} = 0.$$

And the problem (1.1) and (1.9) is transformed into

$$\begin{cases}
   n_t + u \cdot \nabla n = \Delta n^n - \chi \nabla \cdot (e^{g_1} n \nabla \tilde{c} + e^{g_1} n \nabla g_1 + n \nabla g_2) + \mu n(a - n) + g, \\
   \tilde{c}_t - \Delta \tilde{c} + (u - 2 \nabla g_1) \cdot \nabla \tilde{c} = (|\nabla g_1|^2 + \Delta g_1 - n - u \nabla g_1 - g_2)\tilde{c} + (\Delta g_2 - u \nabla g_2 - n g_2 - g_2) e^{-g_1}, \\
   u_t = \Delta u - \nabla \pi + n \nabla \varphi, \\
   \nabla \cdot u = 0, \\
   \frac{\partial n}{\partial \nu}|_{\partial \Omega} = \frac{\partial \tilde{c}}{\partial \nu}|_{\partial \Omega} = u|_{\partial \Omega} = 0.
\end{cases} \tag{1.11}$$
By using a double-level approximation method \[12\] and some iterative techniques, we obtain the existence and time-space uniform boundedness of weak time periodic solutions of (1.11) for any \(m > 1\). Moreover, by deriving the following estimate

\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m-4} \left| \nabla n_\varepsilon \right|^4 \, dx \, ds \leq C,
\]

where \(C\) is independent of \(\varepsilon\), we improve the regularity for \(m \leq \frac{4}{3}\) and show that the obtained periodic solutions are in fact strong periodic solutions.

It is worth mentioning that there exist some essential difficulties to establish the prior estimates of time periodic solutions for the problem (1.11).

Firstly, considering that (1.11) may be degenerate due to \(m > 1\), and in general does not allow for classical solvability as the well-known porous medium equations \[34\], there may not be enough compactness to get the existences of weak and strong periodic solutions. In this paper, we use a fourth order regularized problem (see (3.1) below) to approach the original problem (1.11). However, different from the second order parabolic system, there is no positivity for the fourth order regularized system. The most basic and natural \(L^1\)-norm estimate of \(n\) is no longer valid, which brings great difficulties to the later proof. So we By using a double-level approximation method and introducing three terms \(\varepsilon |n|^s, A|n|\) and \(An\) in the regularized equation (3.1), we solve the difficulties caused by the lack of positivity. For more details, please see the proof of Lemma 3.3 and 3.4.

Secondly, the Neumann boundary condition for \(n\) may produce boundary integral at \(\partial \Omega\). Indeed, integrating (1.11) over \(\Omega \times (t_0, t)\) formally, we see that the following term in the resulting equation

\[
- \chi \int_{t_0}^t \int_\Omega \nabla \cdot (e^{g_1} n \hat{c} \nabla g_1) \, dx \, ds = - \chi \int_{t_0}^t \int_{\partial \Omega} e^{g_1} n \hat{c} \frac{\partial g_1}{\partial \nu} \, d\Gamma \, ds \neq 0,
\]

(1.12)
cannot be eliminated or estimated directly, which also prevent us from deriving the positivity, \(L^1\)-norm and \(L^2\)-norm estimates of \(n\). By properly choosing the form of chemotactic flux in the regularized equation (3.1), that is,

\[
e^{g_1} n_+ \nabla \hat{c} + e^{g_1} n_+ \hat{c} \nabla g_1 + n_+ \nabla g_2,
\]

and making some more accurate estimates on (1.12) by virtue of boundary trace imbedding lemma, we are able to overcome these difficulties. For more details, please refer to Section 4.

Now we are the position to give the first result in this paper.

**Theorem 1.1** Let \(m > 1\). Assume that (1.5) and (1.6) hold, \(a, g, \nabla \varphi \in L^\infty_T (Q)\) and \(g \geq 0\). Then the problem (1.11) admits a bounded weak time periodic solution \((n, c, u)\) with \(n, c \geq 0\),
such that
\[ n \in X_1, \ c \in X_2, \ u \in X_3, \]

where
\[ X_1 = \{ \mathcal{D}_1 \}, \]
\[ X_2 = \{ \mathcal{D}_2 \}, \]
\[ X_3 = \{ \mathcal{D}_3 \}, \]

such that
\[ \sup_t (\| n(t) \|_{L^\infty} + \| c(t) \|_{W^{1,\infty}} + \| u(t) \|_{V^{1,\infty}} + \| \nabla n(t) \|_{L^2}) \leq C, \]
\[ \int_0^T (\| \nabla n^m \|_{L^2}^2 + \| (n^{m+1})_t \|_{L^2}^2) \, dt \leq C, \]
\[ \int_0^T (\| u_t \|_{L^p}^p + \| u \|_{W^{2,p}}^p + \| c_t \|_{L^p}^p + \| c \|_{W^{2,p}}^p) \, dt \leq C \]
\[ \text{for any } p > 1, \]

where \( C \) only depends on \( m, \chi, \mu, \Omega, T, a, g, \varphi, p \).

The second result of this paper is concerned with the existence of strong time periodic solutions.

**Theorem 1.2** Let \( 1 < m \leq \frac{1}{4} \). Assume that \([1.5] \) and \([1.6] \) hold, \( a, g, \nabla \varphi \in L^\infty(Q) \) and \( g \geq 0 \). Then the problem \([1.11] \) admits a bounded strong time periodic solution \((n, c, u)\) with \( n, c \geq 0 \), \( n \in \mathcal{D}_1, \ c \in \mathcal{D}_2, \ u \in \mathcal{D}_3, \)

such that
\[ \sup_t (\| n(t) \|_{L^\infty} + \| c(t) \|_{W^{1,\infty}} + \| u(t) \|_{V^{1,\infty}} + \| \nabla \sqrt{n}(t) \|_{L^2}) \leq C, \]
\[ \int_0^T (\| \Delta n^m \|_{L^2}^2 + \| \nabla \sqrt{n} \|_{L^2}^2 + \| n_t \|_{L^2}^2) \, dt \leq C, \]
\[ \int_0^T (\| u_t \|_{L^p}^p + \| u \|_{W^{2,p}}^p + \| c_t \|_{L^p}^p + \| c \|_{W^{2,p}}^p) \, dt \leq C \]
\[ \text{for any } p > 1, \]

where \( C \) only depends on \( m, \chi, \mu, \Omega, T, a, g, \varphi, p \).

The remainder of this paper is organized as follows. In Section 2, we recall some auxiliary lemmas which will be used in this paper. In Section 3, we prove the existence of time periodic solutions for a fourth order regularized problem which approaches the original problem \([1.11] \).

In section 4, by using a double-level approximation method and some iterative techniques, we obtain the existence and time-space uniform boundedness of weak time periodic solutions for any \( m > 1 \). Finally, we show that the obtained time periodic solutions are in fact strong periodic solutions for \( m \leq \frac{1}{2} \) in the last section.
2 Some auxiliary lemmas

Notations.

- \( Q = \Omega \times \mathbb{R}^+ \) and \( Q_T = \Omega \times (0, T) \).
- \( f \in L^p_T(\mathbb{R}^+; X) \iff f \) is a time periodic function with period \( T \), and \( f \in L^p(0, T; X) \). For simplicity, we denote \( L^p_T(\mathbb{R}^+; L^p(\Omega)) \) by \( L^p_T(Q) \).
- The outward unit normal to \( \partial \Omega \) is denoted by \( \nu \).
- \( C^\infty_0,\sigma(\Omega) \) denotes the set of all \( C^\infty_0,\sigma(\Omega) \)-real functions \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \) with compact support in \( \Omega \), such that \( \nabla \cdot \varphi = 0 \). The closure of \( C^\infty_0,\sigma(\Omega) \) with respect to norm \( L^r \) is denoted by \( L^r_\sigma(\Omega) \).
- \( C \) stands for a generic positive constant which may vary from line to line.

By [9], each \( u \in L^r(\Omega) \) has a unique decomposition

\[
u = v + \nabla p, \quad v \in L^r_\sigma, \quad \nabla p \in G^r
\]

with \( G^r = \{ \nabla p; \nabla p \in L^r, p \in L^r(\Omega) \} \), and the projection \( P : L^r(\Omega) \to L^r_\sigma(\Omega) \) is called Helmholtz projection. Let \( A_\omega := -P \Delta \omega \), then \( A \) generates a bounded analytic semigroup \( \{ e^{-tA}; t \geq 0 \} \) on \( L^r_\sigma(\Omega) \), and the time periodic solution \( u \) of (1.1) can be expressed as

\[
u(t) = \int_{-\infty}^{t} e^{-t-s} A P(\nabla \varphi) \, ds.
\]

(2.1)

For more details, please refer to [8, 19].

By [15], we have the following two lemmas.

Lemma 2.1 Let \( T > 0, a > 0, \sigma \geq 0 \), and suppose that \( f : \mathbb{R}^+ \to [0, \infty) \) is absolutely continuous, \( f, h \) are time periodic functions with period \( T \), and \( f \) satisfies

\[
u(t) - f(t_0) + a \int_{t_0}^{t} f^{1+\sigma}(s) \, ds \leq \int_{t_0}^{t} h(s) \, ds, \quad 0 \leq t_0 < t,
\]

where \( 0 \leq f, h \in L^1_T(\mathbb{R}^+) \) and \( \int_{0}^{T} h(s) \, ds \leq \beta \). Then we have

\[
u \left( \sup_{t \in (0, T)} f(t) + a \int_{0}^{T} f(t) \, dt \right) \leq \left( \frac{\beta}{aT} \right)^{1/(1+\sigma)} + 2\beta.
\]

Lemma 2.2 Let \( T > 0, a > 0, \sigma > 0 \), and suppose that \( f : \mathbb{R}^+ \to [0, \infty) \) is absolutely continuous, \( f, g, h \) are time periodic functions with period \( T \), and satisfy

\[
u(t) - f(t_0) + a \int_{t_0}^{t} f^{1+\sigma}(s) \, ds \leq \int_{t_0}^{t} g(s) f(s) \, ds + \int_{t_0}^{t} h(s) \, ds, \quad 0 \leq t_0 < t,
\]

For more details, please refer to [8, 19].
where $0 \leq g, h \in L^1_1(\mathbb{R}^+)$, $\int_0^T g(s) \, ds \leq \alpha$ and $\int_0^T h(s) \, ds \leq \beta$. Then we have
\[
\sup_{t \in (0, T)} f(t) + a \int_0^T f^{1+\tau}(t) \, dt \leq C,
\]
where $C$ is a constant depending only on $a, \alpha, \beta, T$. While, if $a = 0$ and $\int_0^T f(s) \, ds \leq \gamma$. Then we also have
\[
\sup_{t \in (0, T)} f(t) \leq C,
\]
where $C$ is a constant depending only on $\gamma, \alpha, \beta, T$.

By [14, 28], we also have the following lemma.

**Lemma 2.3** Assume that $f \in L^p_1(\mathbb{R}^+; L^p(\Omega))$ with $p > 1$. Then the following problem
\[
\begin{aligned}
&u_t - \Delta u + u = f(x, t), \\
&\frac{\partial u}{\partial \nu} |_{\partial \Omega} = 0
\end{aligned}
\]
admits a unique strong time periodic solution $u \in W^{2,1}_2(Q_T)$, and
\[
\|u\|_{W^{2,1}_2(Q_T)}^p \leq C\|f\|_{L^p(Q_T)}^p,
\]
where $C$ is a positive constant.

By Gagliardo-Nirenberg interpolation inequality [21], we see that

**Lemma 2.4** Let $m, k$ be non-negative integers, $1 \leq p, q, r, s \leq \infty$ and $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Suppose that $\frac{1}{q}$ is between $\frac{1}{p} - \frac{m-k}{N}$ and $\frac{1}{mp} + \frac{m-k}{mr}$, and when $\frac{1}{q} = \frac{1}{p} - \frac{m-k}{N}$, $m - k - \frac{N}{p}$ is not non-negative integer. Then if $u \in L^r(\Omega) \cap L^s(\Omega)$ and $\partial^m u \in L^p(\Omega)$, we have $\partial^k u \in L^q(\Omega)$ and
\[
\|\partial^k u\|_{L^q(\Omega)} \leq C\|u\|_{L^r(\Omega)}^\theta \|\partial^m u\|_{L^p(\Omega)}^{1-\theta} + C\|u\|_{L^s(\Omega)},
\]
where $C > 0$ depends only on $N, m, k, p, q, r, s, \Omega$, and $\theta \in [0, \frac{m-k}{m}]$ satisfying
\[
\frac{1}{q} = \theta\left(\frac{1}{r} + \frac{k}{N}\right) + (1-\theta)\left(\frac{1}{p} - \frac{m-k}{N}\right).
\]
Especially, if $u|_{\partial \Omega} = 0$, then (2.3) is reduced to
\[
\|\partial^k u\|_{L^q(\Omega)} \leq C\|u\|_{L^r(\Omega)}^\theta \|\partial^m u\|_{L^p(\Omega)}^{1-\theta},
\]
(2.4)

The following boundary trace imbedding lemma follows from [1].

**Lemma 2.5** Let $m$ be a non-negative integer, $1 \leq p, q < \infty$ and $\Omega$ be a domain in $\mathbb{R}^N$ satisfying the uniform $C^m$-regularity condition, and suppose there exists a simple $(m, p)$-extension operator $E$ for $\Omega$. Also suppose that $mp < N$ and $p \leq q \leq \frac{N-1}{N-mp}$. Then
\[
W^{m,p}(\Omega) \hookrightarrow L^q(\partial \Omega).
\]
(2.5)

If $mp = N$, then imbedding (2.5) holds for $1 \leq p \leq q < \infty$. 

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Lemma 2.6 ([26]) Assume that $\Omega$ is bounded and $\omega \in C^2(\Omega)$ satisfying $\frac{\partial \omega}{\partial \nu}|_{\partial \Omega} = 0$. Then we get that

$$\frac{\partial |\nabla \omega|^2}{\partial \nu} \leq 2\kappa |\nabla \omega|^2 \text{ on } \partial \Omega,$$

where $\kappa > 0$ is an upper bound for the curvature of $\Omega$.

### 3 Time periodic solutions for a fourth-order regularized problem

To obtain the compactness of the operator, we use a fourth order regularized system as follows to approach the original system.

$$\begin{cases}
    n_t - m \nabla \cdot ((|n| + \varepsilon)^{m-1} \nabla n) + \delta \Delta^2 n + \varepsilon |n|^s n + u \cdot \nabla n + An \\
    = -\chi \nabla \cdot (e^{g_1} n + \varepsilon |n|^s n + u \cdot \nabla g_1 + n + \nabla g_2) + \mu |n|(a - n) + A|n| + g,
\end{cases}$$

$$\begin{align*}
    \dot{c}_t - \Delta \dot{c} + (u - 2 \nabla g_1) \cdot \nabla \dot{c} &= (|\nabla g_1|^2 + \Delta g_1 - n + u \nabla g_1 - g_{1t}) \dot{c} \\
    &+ (\Delta g_2 - u \nabla g_2 - n g_2 - g_{2t}) e^{-g_1},
\end{align*}$$

$$\begin{cases}
    u_t = \Delta u - \nabla \pi + n \nabla \varphi, \\
    \nabla \cdot u = 0, \\
    \frac{\partial u}{\partial \nu}|_{\partial \Omega} = \frac{\partial \Delta u}{\partial \nu}|_{\partial \Omega} = \frac{\partial \varphi}{\partial \nu}|_{\partial \Omega} = u|_{\partial \Omega} = 0,
\end{cases}$$

where $\max\{2(m-1), 2\} < s \leq 5m - 1$, $\delta, \varepsilon > 0$, $m > 1$, $A > 0$. The terms $\varepsilon |n|^s n$, $An$ and $A|n|$ are introduced in (3.1) to solve the difficulties caused by the lack of positivity of the fourth order regularized problem.

In this section, we prove the existence of time periodic solutions for the fourth-order problem (3.1). For this purpose, we linearize this problem. Consider the following problem

$$\begin{cases}
    u_t - \Delta u + \nabla \pi = \eta \dot{\varphi}, \\
    \nabla \cdot u = 0, \\
    u|_{\partial \Omega} = 0,
\end{cases}$$

where $\eta \in [0, 1]$ is a constant. By [14], we have

**Lemma 3.1** Assume $\nabla \varphi \in L^\infty_t(\Omega)$, $\dot{u}, \dot{\varphi} \in L^4_t(\mathbb{R}^+; L^4_\Omega)$ and $\dot{\varphi} \in L^2_t(\mathbb{R}^+; L^2(\Omega))$. Then (3.2) admits a unique strong time periodic solution $u$ with $u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $u_t \in L^2(0, T; H^2(\Omega))$.

For the above solution $u$, we consider the following problem for any $\eta \in [0, 1]$.

$$\begin{cases}
    c_t - \Delta c + u \cdot \nabla c + (1 - \eta)c = -\dot{n}_x c, \\
    \frac{\partial c}{\partial \nu}|_{\partial \Omega} = \eta \frac{\partial \varphi}{\partial \nu}(x, t) (c(x, t) - g(x, t)).
\end{cases}$$

(3.3)
Let $\hat{c} = e^{-\eta g_1}(c - g_2)$. Then (3.2) is equivalent to

$$
\begin{cases}
\hat{c}_t - \Delta \hat{c} + (u - 2\eta \nabla g_1) \cdot \nabla \hat{c} + (1 - \eta + \hat{n}_+)\hat{c} = \eta(\eta|\nabla g_1|^2 + \Delta g_1 - u\nabla g_1 - g_1)\hat{c} \\
+ (\Delta g_2 - u\nabla g_2 - \hat{n}_+ g_2 - g_2t - (1 - \eta)g_2)e^{-\eta g_1},
\end{cases}
(3.4)
$$

By [12], we have

**Lemma 3.2** Assume $\hat{n} \in L^2_T(\mathbb{R}^+; H^1(\Omega))$. Let $u$ be the time periodic solution of the problem (3.2). Then (3.3) (or 3.3) admits a unique strong time periodic solution $\hat{c}$ with $c \geq 0$, $\hat{c} \in L^\infty_T(\Omega) \cap L^\infty_T(\mathbb{R}^+; H^2(\Omega)) \cap L^2_T(\mathbb{R}^+; H^3(\Omega))$ and $\hat{c}_t \in L^\infty_T(\mathbb{R}^+; L^2(\Omega))$.

For the above obtained solutions $u, \hat{c}$, we consider the following problem.

$$
\begin{cases}
\begin{align*}
&n_t - m \nabla \cdot ((|\hat{n}| + \varepsilon)^{m-1} \nabla n) + \delta \Delta^2 n + \varepsilon |\hat{n}|^\alpha n + An + u \cdot \nabla n \\
&= -\eta \chi \nabla \cdot (e^{g_1(n_+ \nabla \hat{c} + e^{g_1} n_+ \nabla g_1 + n_+ \nabla g_2)} + \eta(\mu a + A)|\hat{n}| - |\hat{n}| n + \eta g,
\end{align*}
\end{cases}
(3.5)
$$

$$
\frac{\partial n}{\partial \tau}|_{\partial \Omega} = 0.
$$

For the above linear parabolic problem, when $A$ is sufficiently large, the existence of time periodic solutions can be easily obtained by a fixed point method. That is, define a Poincaré map from $n(x, 0)$ to $n(x, T)$, the time-periodic solution is then identified as a fixed point of this Poincaré map (see also [12]). We only give the regularity estimates.

For simplicity, in what follows, we may assume that the solution $n$ is sufficiently smooth, otherwise, we can approximate $u, \hat{c}, n$ with a sequence of sufficiently smooth functions $u_k, \hat{c}_k, n_k$ such that the corresponding solutions $n$ are sufficiently smooth, and the following energy estimates can be obtained through an approximate process.

**Lemma 3.3** Assume $a, g \in L^\infty_T(Q)$, $\hat{n} \in L^\infty_T(Q) \cap L^\infty_T(\mathbb{R}^+; H^1(\Omega)) \cap L^2_T(\mathbb{R}^+; H^3(\Omega))$ and $\frac{\partial n}{\partial \tau}|_{\partial \Omega} = 0$. Let $u, \hat{c}, n$ be the time periodic solutions of the problem (3.2), (3.3) and (3.5), respectively. If $A > 1$ is a sufficiently large constant, then $n \in L^\infty_T(\mathbb{R}^+; H^2(\Omega)) \cap L^2_T(\mathbb{R}^+; H^4(\Omega))$ and $n_t \in L^2_T(\mathbb{R}^+; L^2(\Omega))$.

**Proof.** Multiplying (3.5) by $n$, integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t \leq t_0 + T$, and
using Lemma 2.4, 2.5 and 3.2, we see that for sufficiently large \( A \),

\[
\frac{1}{2} \int_{\Omega} \left( |n(x, t)|^2 - |n(x, t_0)|^2 \right) dx + \int_{t_0}^{t} \int_{\Omega} (\delta |\Delta n|^2 + \varepsilon |\dot{n}|^2 n^2 + \mu |\dot{n}| n + \mu_n) dx ds \\
\leq \eta \chi \int_{t_0}^{t} \int_{\Omega} \left( e^{g_n} + e^{g_n} n + \nabla g_n \right) \nabla n dx ds - \eta \chi \int_{t_0}^{t} \int_{\partial \Omega} e^{g_n} n \frac{\partial g}{\partial \nu} d\Gamma ds \\
+ \int_{t_0}^{t} \int_{\Omega} \left( \eta (\mu a + A) |\dot{n}| - \mu |\dot{n}| n + \eta g \right) n dx ds \\
\leq C \int_{t_0}^{t} \|n\|_{L^2} \|\nabla \tilde{e}\|_{L^2} \|\nabla n\|_{L^2} ds + C \int_{t_0}^{t} \|n\|_{L^2} \|\nabla n\|_{L^2} ds + C \int_{t_0}^{t} \|n\|_{L^2}^2 ds \\
+ \int_{t_0}^{t} \left( \eta (\mu a + A) |\dot{n}| - \mu |\dot{n}| n + \eta g \right) \|L^2\|_{L^2} ds \\
\leq C \int_{t_0}^{t} \|\nabla n\|_{L^2}^2 ds + \frac{A}{2} \int_{t_0}^{t} \|n\|_{L^2}^2 ds + C,
\]

which implies

\[
\int_{\Omega} \left( |n(x, t)|^2 - |n(x, t_0)|^2 \right) dx + \int_{t_0}^{t} \int_{\Omega} (2\delta |\Delta n|^2 + \mu_n) dx ds \leq C \int_{t_0}^{t} \|\nabla n\|_{L^2}^2 ds + C.
\]

By Lemma 2.1 it follows that

\[
\sup_t \|n(t)\|_{L^2}^2 + \int_{0}^{T} \|n(s)\|_{L^2}^2 ds \leq C \int_{0}^{T} \|\nabla n\|_{L^2}^2 ds + C,
\]

where the constant \( C > 0 \) is independent of \( A \). Multiplying (3.5) by \((-\Delta)n\) and integrating
it over $\Omega \times (t_0, t)$, and using Lemma 2.4 and 3.2, we have

$$
\frac{1}{2} \int_\Omega (|\nabla n(x, t)|^2 - |\nabla n(x, t_0)|^2) \, dx + \int_{t_0}^t \int_\Omega (\delta |\nabla \Delta n|^2 + A|\nabla n|^2) \, dx \, ds
$$

\[
\begin{align*}
&= \int_{t_0}^t \int_\Omega \left( m(\hat{n}) + \varepsilon \right)^{m-1} \nabla n - \eta \chi e^{\gamma_1 t} + \nabla \nabla \Delta n \, dx \, ds \\
&\quad + \eta \chi \int_{t_0}^t \int_\Omega \nabla \cdot (e^{\gamma_1 t} \nabla \nabla \Delta n) \, dx \, ds + \int_{t_0}^t \int_\Omega u \cdot \nabla n \, dx \, ds \\
&\quad + \int_{t_0}^t \int_\Omega \varepsilon \hat{n}^n n + \mu |\hat{n}| n - \eta (\mu a + A) |\hat{n}| - \eta g \Delta n \, dx \, ds \\
&\leq C \int_{t_0}^t (\|\nabla n\|_{L^2} + \|n\|_{L^\infty} \|\nabla \nabla \Delta n\|_{L^2} + \|n\|_{L^2} \|\nabla \Delta n\|_{L^2}) \, ds \\
&\quad + \eta \chi \int_{t_0}^t \int_\Omega \|\nabla \cdot (e^{\gamma_1 t} \nabla \nabla \Delta n)\|_{L^2} \|\Delta n\|_{L^2} \, ds + \int_{t_0}^t \|u\|_{L^6} \|\nabla n\|_{L^6} \|\Delta n\|_{L^6} \, ds \\
&\quad + \int_{t_0}^t \|\varepsilon \hat{n}^n n + \mu |\hat{n}| n - \eta (\mu a + A) |\hat{n}| - \eta g\|_{L^2} \|\Delta n\|_{L^2} \, ds \\
&\leq C \int_{t_0}^t (\|n\|_{H^1} + \|n\|_{H^2} \|\nabla \nabla \Delta n\|_{L^2}) \|\nabla \Delta n\|_{L^2} \, ds + C \int_{t_0}^t \|n\|_{H^1} \|\Delta n\|_{L^2} (1 + \|\hat{n}\|_{H^1}) \, ds \\
&\quad + C \int_{t_0}^t \|u\|_{H^1} \|\nabla n\|_{H^1} \|\Delta n\|_{L^2} \, ds + C \int_{t_0}^t (\|n\|_{L^2} + 1) \|\Delta n\|_{L^2} \, ds \\
&\leq \frac{\delta}{2} \int_{t_0}^t \|\nabla \Delta n\|_{L^2}^2 \, ds + \frac{A}{2} \int_{t_0}^t \|\nabla n\|_{L^2}^2 \, ds + C \int_{t_0}^t (\|n\|_{L^2}^2 + \|\Delta n\|_{L^2}^2) \, ds + C.
\end{align*}
\]

Then,

$$
\int_\Omega (|\nabla n(x, t)|^2 - |\nabla n(x, t_0)|^2) \, dx + \int_{t_0}^t \int_\Omega (\delta |\nabla \Delta n|^2 + A|\nabla n|^2) \, dx \, ds \\
\leq C \int_{t_0}^t (\|n\|_{L^2}^2 + \|\Delta n\|_{L^2}^2) \, ds + C.
$$

If $A$ is large enough, combining Lemma 2.1 with inequality 3.6, we obtain

$$
\sup_t \|\nabla n(t)\|_{L^2}^2 + \int_0^T \|n(s)\|_{H^1}^2 \, ds \leq C,
$$

where the constant $C$ depends on $\delta$ and $\varepsilon$. Multiplying (3.5) by $\Delta^2 n$ and integrating it over
Combining with Lemma 2.1, we finally get
\[ C \] where the constant
\[ F \] Summing up, we complete the proof.
\[ \text{depends on} \] and using the estimate \( \delta \), Lemma 2.4 and 3.2, we also have
\[ C \] and integrating it, we can obtain
\[ \sup_t \| n(t) \|_{L^2}^2 + \int_0^T \| n(s) \|_{H^2}^2 \, ds \leq C, \quad (3.8) \]
where the constant \( C \) depends on \( \delta \) and \( \varepsilon \). Similarly, multiplying \( \{5.5\}_1 \) by \( n_t \) and integrating it, we can obtain
\[ \int_0^T \| n_t(s) \|_{L^2}^2 \, ds \leq C. \quad (3.9) \]
Summing up, we complete the proof.

Next, we show the existence of time periodic solutions in dimension 3 by Leray-Schauder’s fixed point theorem. Define an operator \( \mathcal{F} : \mathcal{E} \times [0, 1] \to \mathcal{E} \) as follows:
\[ \mathcal{F}(\tilde{n}, \eta) = n, \]
where
\[ \mathcal{E} = \{ n; n \in L_T^\infty(Q) \cap L_T^\infty(\mathbb{R}^+; H^1(\Omega)) \cap L_T^2(\mathbb{R}^+; H^2(\Omega)) \}, \]
endowed with the norm
\[ \| n \|_{\mathcal{E}} = \sup_t (\| u(\cdot, t) \|_{L^\infty} + \| n(\cdot, t) \|_{H^1}) + \int_0^T \| n(\cdot, t) \|_{H^2}^2 \, dt \]
and \( u, \hat{c}, n \) are the time periodic solutions of the problem \((3.2), (3.4)\) and \((3.5)\), respectively.
Lemma 3.4 Assume \( s > \max\{2m - 2, 2\} \), \( a, g, \nabla \varphi \in L^\infty_T(Q) \) and let \( F(n, \eta) = n \) with \( \eta \in (0, 1] \). Then there exists \( R > 0 \) such that
\[
\|n\|_E \leq R,
\]
where \( R \) depends on \( \delta, \varepsilon \), and is independent of \( A \).

**Proof.** Take \( \hat{n} = n \) in \((3.10)\). Multiplying \((3.10)\) by \( n \), integrating it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \), and using Lemma 2.1, 2.5 and 3.2, we see that
\[
\begin{align*}
\frac{1}{2} \int_\Omega (|n(x, t)|^2 - |n(x, t_0)|^2) \, dx + \int_{t_0}^t \int_\Omega m(|n| + \varepsilon)^{m-1} |\nabla n|^2 \, dx \, ds \\leq \frac{\eta \chi}{2} \int_{t_0}^t \int_\Omega (\varepsilon n \cdot \Delta \hat{c} + \Delta g_1) n^2 \, dx \, ds - \eta \int_{t_0}^t \int_\Omega \hat{c} \cdot \hat{c} dx \, ds - \eta \int_{t_0}^t \int_\Omega \nabla \cdot (\varepsilon n \cdot \nabla \varphi) n \, dx \, ds \\leq C \eta \int_{t_0}^t \left( \|\nabla \hat{c}\|_{H^1}^2 + \|\hat{n}\|_{L^2}^2 + \frac{m \varepsilon^{m-1}}{2} \|\nabla n\|_{L^2}^2 \right) \, ds + C \eta,
\end{align*}
\]
which implies
\[
\begin{align*}
\int_\Omega (|n(x, t)|^2 - |n(x, t_0)|^2) \, dx + \int_{t_0}^t \int_\Omega m(|n| + \varepsilon)^{m-1} |\nabla n|^2 \, dx \, ds \\leq C \eta \int_{t_0}^t \left( \|\nabla \hat{c}\|_{H^1}^2 + \|\hat{n}\|_{L^2}^2 \right) \, ds + C \eta.
\end{align*}
\]
Taking \( \hat{n} = n \) in \((3.2)\) and multiplying \((3.2)\) by \( u \) and \( u_t \), respectively, then combining the two resulting inequalities and applying Lemma 2.1 we have
\[
\sup_t \|u(t)\|_{H^1}^2 + \int_0^T \left( \|u(s)\|_{H^1}^2 + \|u_t(s)\|_{L^2}^2 \right) \, ds \, ds \leq C \int_0^T \|n(s)\|_{L^2}^2 \, ds. \tag{3.11}
\]
Noticing that
\[-\Delta u + \nabla \pi = -u_t + \eta n \nabla \varphi, \quad u|_{\partial \Omega} = 0,\]
using \( L^2 \)-theory of Stokes operator, we have
\[
\|u\|_{H^2}^2 \leq C(\|u_t\|_{L^2}^2 + \|n\|_{L^2}^2),
\]
that is
\[
\int_0^T \|u(s)\|_{H^2}^2 \, ds \leq C \int_0^T \left( \|u_t(s)\|_{L^2}^2 + \|n(s)\|_{L^2}^2 \right) ds.
\]

Combining with (3.11) yields

\[ \sup_t \|u(t)\|^2_{H^1} + \int_0^T \left( \|u(s)\|^2_{H^2} + \|u_t(s)\|^2_{L^2} \right) ds \leq C \int_0^T \|n\|^2_{L^2} ds. \]  

(3.12)

Take \( \tilde{n} = n \) in (3.4). We first multiply (3.4) by \( \tilde{c} \), integrate it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \), and apply Lemma 2.1 to the resulting inequality. It is easy to see that

\[ \sup_t \|\tilde{c}(t)\|^2_{L^2} + \int_0^T \|\nabla \tilde{c}(s)\|^2_{L^2} ds \leq C \int_0^T \|n(s)\|^2_{L^2} ds. \]  

(3.13)

Then we multiply (3.4) by \( -(\Delta)\tilde{c} \) and integrate it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \). By Lemma 2.4, we have

\[
\frac{1}{2} \int_0^T \left( \|\nabla \tilde{c}(x, t)\|^2 - \|\nabla \tilde{c}(x, t_0)\|^2 \right) dx + \int_0^T \int_\Omega (|\Delta \tilde{c}|^2 + (1 - \eta + n_+) |\nabla \tilde{c}|^2) \, dx \, ds \\
= \int_0^t \int_\Omega n_+ \cdot (\tilde{c} \nabla \tilde{c}) \, dx \, ds + \int_0^t \int_\Omega u \cdot \nabla \tilde{c} \Delta \tilde{c} \, dx \, ds - 2\eta \int_0^t \int_\Omega \Delta \tilde{c} \cdot \nabla \tilde{c} \Delta \tilde{c} \, dx \, ds \\
- \int_0^t \int_\Omega (\Delta \tilde{g}_2 - u \nabla g_2 - n_+ g_2 - g_2 e^{-\eta g_1} \Delta \tilde{c}) \, dx \, ds \\
\leq \int_0^t \|n\|_{L^2} (\|\nabla \tilde{c}\|^2_{L^2} + \|\tilde{c}\|_{L^\infty} \|\Delta \tilde{c}\|_{L^2}) \, ds + \int_0^t \|\nabla u\|_{L^2} \|\nabla \tilde{c}\|^2_{L^2} \, ds \\
+ C \int_0^t \|\nabla \tilde{c}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \, ds + \int_0^t (\|n\|_{L^2} + \|u\|_{L^2} + C) \|\Delta \tilde{c}\|_{L^2} \, ds \\
\leq C \int_0^t (\|n\|_{L^2} + \|\nabla u\|_{L^2}) (\|\tilde{c}\|_{L^\infty} \|\Delta \tilde{c}\|_{L^2} + \|\tilde{c}\|_{L^2}) \, ds \\
+ \frac{1}{4} \int_0^t \|\Delta \tilde{c}\|^2_{L^2} \, ds + C \int_0^t (\|n\|^4_{L^4} + \|u\|^2_{L^2} + \|\nabla \tilde{c}\|^2_{L^2}) \, ds \\
\leq \frac{1}{2} \int_0^t \|\Delta \tilde{c}\|^2_{L^2} \, ds + C \int_0^t (\|n\|^4_{L^4} + \|u\|^2_{H^1} + \|\tilde{c}\|^2_{H^1}) \, ds.
\]

Then, combining (3.4) with (3.11) with Lemma 2.1, we obtain

\[ \sup_t \|\nabla \tilde{c}(t)\|^2_{L^2} + \int_0^T \|\Delta \tilde{c}(s)\|^2_{L^2} ds \leq C \int_0^T \|n\|^2_{L^2} ds. \]  

(3.14)

Combining (3.10) with (3.14) and noting \( s > 2 \), we have

\[
\sup_t \int_\Omega |n(x, t)|^2 \, dx + \int_0^T \int_\Omega (2\delta |\Delta n|^2 + \varepsilon |n|^{s+2} + \mu |n|^3) \, dx \, ds \\
\leq C \int_0^T (\|n\|^4_{L^4} + \|n\|^2_{L^2}) \, ds + C \\
\leq \frac{\varepsilon}{2} \int_0^T \|n\|^{s+2} \, dx \, ds + C.
\]
which implies

\[
\sup_t \|n(t)\|_{L^2}^2 + \int_{t_0}^T \int_\Omega m(|n| + \varepsilon)^{m-1} \|\nabla n\|^2 \, dx \, ds \\
+ \int_{\Omega}^T \left( \delta \|\Delta n\|_{L^2}^2 + \varepsilon \|n\|_{L^{p+2}}^2 + \mu \|n\|_{L^\infty}^3 \right) \, ds \leq C, \quad (3.15)
\]

where \( C \) is independent of \( \delta \) and \( A \), but depends on \( \varepsilon \). Recalling (3.12) and Lemma (3.2) we also have

\[
\sup_t (\|u(t)\|_{H^1}^2 + \|\tilde{c}(t)\|_{H^2}^2) + \int_{t_0}^T (\|\nabla u(s)\|_{H^1}^2 + \|\nabla \Delta \tilde{c}(s)\|_{L^2}^2) \, ds \leq C, \quad (3.16)
\]

where \( C \) is also independent of \( \delta \) and \( A \), but depends on \( \varepsilon \). Take \( \hat{n} = n \) in (3.5). Multiplying (3.5) by \((-\Delta)n\), integrating it over \( \Omega \times (t_0, t) \) for any \( t_0 < t \leq t_0 + T \), and using (3.15) - (3.16) and \( s > 2(m - 1) \), we see that

\[
\frac{1}{2} \int_{t_0}^T (|\nabla n(x, t)|^2 - |\nabla n(x, t_0)|^2) \, dx + \int_{t_0}^T (\delta \|\nabla \Delta n\|^2 + \varepsilon(s + 1)|n|^4|\nabla n|^2 + 2\mu |n| \|\nabla n\| + A |\nabla n|^2) \, dx \\
= \int_{t_0}^T m(|n| + \varepsilon)^{m-1} \nabla n \cdot \nabla \Delta n \, dx + \int_{t_0}^T \int_{\Omega} u \cdot \nabla \Delta n \, dx ds \\
- \eta \chi \int_{t_0}^T \int_{\Omega} (e^{g_1} n + \nabla \tilde{c} + n + \nabla g_2) \cdot \nabla \Delta n \, dx ds + \eta \chi \int_{t_0}^T \int_{\Omega} \nabla \cdot (e^{g_1} n + \nabla g_1) \Delta n \, dx ds \\
- \int_{t_0}^T \int_{\Omega} \eta \mu |n| + A |n| + g) \Delta n \, dx ds \\
\leq \frac{m^2}{\delta} \int_{t_0}^T \int_{\Omega} \eta \mu |n| + A |n| + g) \Delta n \, dx ds \\
+ \frac{\varepsilon}{2} \int_{t_0}^T \int_{\Omega} |n|^4 |\nabla n|^2 \, dx ds + C \int_{t_0}^T \|\nabla u\|_{L^\infty} \|\nabla n\|_{L^2} \|\Delta n\|_{L^2} \, dx ds \\
+ \left( \int_{t_0}^T \int_{\Omega} |n| \nabla \tilde{c}\|^2 \, dx ds + \int_{t_0}^T \|\nabla n\|_{L^2}^2 \, dx ds + \eta A \int_{t_0}^T \|\nabla n\|_{L^2}^2 \, dx ds \\
+ C \int_{t_0}^T \|u\|_{H^1} \|\nabla n\|_{H^1} \|\Delta n\|_{L^2} \, dx ds + C \int_{t_0}^T \|n\|_{H^2}^2 \|\nabla \tilde{c}\|_{L^4}^2 \|\nabla n\|_{L^2} \, dx ds \right).
\]

Since

\[
\int_{t_0}^T \|u\|_{H^1} \|\nabla n\|_{H^1} \|\Delta n\|_{L^2} \, dx \leq \sup_t \|u(t)\|_{H^1}^2 \int_{t_0}^T (\|\nabla n\|_{H^1}^2 + \|\Delta n\|_{L^2}^2) \, ds,
\]

\[
\int_{t_0}^T |n| \nabla \tilde{c}\|^2 \, dx ds \leq C \int_{t_0}^T \|n\|_{L^2}^2 \|\nabla \tilde{c}\|_{L^4}^2 \, dx ds \leq C \sup_t \|\tilde{c}(t)\|_{H^2}^2 \int_{t_0}^T \|n\|_{H^1}^2 \, ds,
\]

we have

\[
\int_{\Omega} (|\nabla n(x, t)|^2 - |\nabla n(x, t_0)|^2) \, dx \\
+ \int_{t_0}^T (\delta |\nabla \Delta n|^2 + \varepsilon(s + 1)|n|^4 |\nabla n|^2 + \mu |n| |\nabla n|^2) \, dx ds \leq C \int_{t_0}^T \|n\|_{H^2}^2 \, ds.
\]
By Lemma 3.2 and (3.15), we obtain
\[
\sup_t \|\nabla n(t)\|_{L^2}^2 + \int_0^t \int_{\Omega} (|\nabla n|^2 + \varepsilon(s + 1)|n|^\gamma|\nabla n|^2 + \mu|n||\nabla n|^2) \, dx \, ds \leq C,
\]  
(3.17)
where \( C \) depends on \( \varepsilon \) and \( \delta \), and is independent of \( A \). By (3.15) and (3.17), we complete the proof of Lemma 3.4.

By Lemma 3.3 we see that \( \mathcal{F} \) is a compact operator. Furthermore, by (3.10), it is easy to see that
\[
\mathcal{F}(n, 0) = 0.
\]
Combining this equality with Lemma 3.4 and applying Leray-Schauder's fixed point theorem, we see that the operator \( \mathcal{F}(\cdot, 1) \) has a fixed point in \( \mathcal{E} \), that is, the following problem admits a solution \((n, \tilde{c}, u)\),
\[
\begin{cases}
  n_t - n \nabla \cdot ((|n| + \varepsilon)^{m-1} \nabla n) + \delta \Delta n + \varepsilon|n|^\gamma n + u \cdot \nabla n + An = -\chi \nabla \cdot (e^{\eta} n_n \nabla \tilde{c} + n_+ \tilde{c} e^{\eta_1} + n_+ \nabla g_2) + (\mu a + A)|n| - \mu |n| n + g, \\
  \tilde{c}_t - \Delta \tilde{c} + (u - 2 \nabla g_1) \cdot \nabla \tilde{c} = (|\nabla g_1|^2 + \Delta g_1 - n_+ - u \nabla g_1 - g_1) \tilde{c} \\
  + (\Delta g_2 - u \nabla g_2 - n_+ g_2 - g_2) e^{-g_1}, \\
  u_t = \Delta u - \nabla \pi + n \nabla \varphi, \\
  \nabla \cdot u = 0, \\
  \frac{\partial n}{\partial \nu}|_{\partial \Omega} = \frac{\partial \tilde{c}}{\partial \nu}|_{\partial \Omega} = \frac{\partial \varphi}{\partial \nu}|_{\partial \Omega} = u|_{\partial \Omega} = 0,
\end{cases}
\]  
(3.18)
where \( s > \max\{2(m - 1), 2\} \), \( \delta, \varepsilon > 0 \). Taking advantage of Lemma 3.3 we have the following proposition.

**Proposition 3.1** Assume \( s > \max\{2(m - 1), 2\} \), and \( a, g, \nabla \varphi \in L^\infty(Q) \). Then the problem (3.18) admits a strong time periodic solution \((n, \tilde{c}, u)\) with
\[
\begin{align*}
  u &\in L^\infty(\mathbb{R}^+, H^1_\sigma(\Omega)) \cap L^2_T(\mathbb{R}^+, H^2_\sigma(\Omega)), & u_t &\in L^2_T(\mathbb{R}^+, L^2_\sigma(\Omega)), \\
  \tilde{c} &\in L^\infty(\mathbb{R}^+, H^2(\Omega)) \cap L^2_T(\mathbb{R}^+, H^3(\Omega)), & \tilde{c}_t &\in L^2_T(\mathbb{R}^+, L^2(\Omega)), \\
  n &\in L^\infty(\mathbb{R}^+, H^2(\Omega)) \cap L^2_T(\mathbb{R}^+, H^3(\Omega)), & n_t &\in L^2_T(\mathbb{R}^+, L^2(\Omega)).
\end{align*}
\]

## 4 Weak time periodic solutions

In this section, we prove the existence of weak time periodic solutions for the problem (1.11). As in [12], the proof is based on two level approximation schemes (corresponding to \( \delta \) and \( \varepsilon \), respectively). We first consider the approximation corresponding to that \( \delta \to 0 \).
Lemma 4.1 Assume \( m > 1, s > \max\{2(m - 1), 2\} \), and \( a, g, \nabla \varphi \in L^\infty(Q) \). Let \((n, \hat{c}, u)\) be the strong time periodic solution of (3.18) obtained in Proposition 3.1. Then we have

\[
\sup_t \int_\Omega (\delta |\nabla n|^2 + (|n| + \varepsilon)^{m+1}) dx + \int_{t_0}^t \int_\Omega |\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n|^2 dx ds \\
+ \int_{t_0}^t \int_\Omega (|n| + \varepsilon)^m (|n|^{s+1} + \mu n^2) dx ds \leq C,
\]

(4.1)

where \( C \) is independent of \( \delta \), and depends only on \( \varepsilon, \Omega, T, a, \nabla \varphi \). Recalling (3.18), if \( s \leq 5m - 1 \), we also have \( n_t \in L^4(0, T; H^{-1}(\Omega)) \).

Proof. Multiplying (3.18) by \(-\delta \Delta n + (|n| + \varepsilon)^{m} \text{sgn} n\) and using (3.15) and (3.16), we obtain

\[
\frac{\delta}{2} \int_\Omega \left((\nabla n(x, t))^2 - |\nabla n(x, t_0)|^2\right) dx + \frac{1}{m + 1} \int_\Omega \left( (|n(x, t)| + \varepsilon)^{m+1} - (|n(x, t_0)| + \varepsilon)^{m+1} \right) dx \\
+ \int_{t_0}^t \int_\Omega |\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n|^2 dx ds + \int_{t_0}^t \int_\Omega (\varepsilon \delta (s + 1)|n|^s + 2\mu |n| + A\delta) |\nabla n|^2 dx ds \\
+ \int_{t_0}^t \int_\Omega (|n|^{s+1} + \mu n^2 + A|n|)(|n| + \varepsilon)^m dx ds \\
= \delta \int_{t_0}^t \int_\Omega u \cdot \nabla n \Delta n dx ds - \chi \int_{t_0}^t \int_\Omega \left( (\partial_t n_+ \nabla \hat{c} + n_+ \nabla g_1)(\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n| \right) dx ds \\
+ \chi \int_{t_0}^t \int_\Omega \nabla \cdot \left( (\partial_t n_+ \nabla \hat{c} + n_+ \nabla g_1)(\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n| \right) dx ds \\
- \int_{t_0}^t \int_\Omega (\partial_t n_+ \nabla \hat{c} + n_+ \nabla g_1)(\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n| dx ds \\
\leq \delta \int_{t_0}^t \int_\Omega \|u\|_{L^2} \|\nabla n\|_{L^2} \|\Delta n\|_{L^2} ds + \frac{1}{2} \int_{t_0}^t \int_\Omega |\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n|^2 dx ds \\
+ C \int_{t_0}^t \left( |n|^3_{L^3} + \|\nabla \hat{c}\|_{H^1} + \|n_+\|_{H^1} \right) ds + 2\delta \int_{t_0}^t \|\Delta n\|_{L^2}^2 ds + C \int_{t_0}^t \left( |n| + \varepsilon \right)^{m} dx ds \\
+ \delta A \int_{t_0}^t \|\nabla n\|_{L^2}^2 ds + A \int_{t_0}^t \int_\Omega |n|(|n| + \varepsilon)^m dx ds + \int_{t_0}^t \int_\Omega (|n| + \varepsilon)^m dx ds \\
\leq \delta \int_{t_0}^t \int_\Omega \|\nabla u\|_{L^2} \|\nabla n\|_{L^2} \|\Delta n\|_{L^2} \|\nabla n\|_{L^2} ds + \frac{1}{2} \int_{t_0}^t \int_\Omega |\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n|^2 dx ds \\
+ C \int_{t_0}^t \left( |n|^3_{L^3} + \|\nabla \hat{c}\|_{H^1} + \|n_+\|_{H^1} \right) ds + 2\delta \int_{t_0}^t \|\Delta n\|_{L^2}^2 ds + C \int_{t_0}^t \left( |n| + \varepsilon \right)^{2m} dx ds \\
+ \delta A \int_{t_0}^t \|\nabla n\|_{L^2}^2 ds + A \int_{t_0}^t \int_\Omega |n|(|n| + \varepsilon)^m dx ds + \int_{t_0}^t \int_\Omega (|n| + \varepsilon)^m dx ds \\
\leq 3\delta \int_{t_0}^t \|\Delta n\|_{L^2}^2 ds + \frac{1}{2} \int_{t_0}^t \int_\Omega |\delta \nabla \Delta n - m(|n| + \varepsilon)^{m-1} |\nabla n|^2 dx ds \\
+ C \int_{t_0}^t \left( |\nabla \hat{c}|_{H^1} + \|n_+\|_{H^1} \right) ds + \frac{\varepsilon}{2} \int_{t_0}^t \int_\Omega |n|^{m+1} dx ds \\
+ \delta A \int_{t_0}^t \|\nabla n\|_{L^2}^2 ds + A \int_{t_0}^t \int_\Omega |n|(|n| + \varepsilon)^m dx ds + C.
Then we have

\[
\frac{\delta}{2} \int_{\Omega} \left( |\nabla n(x,t)|^2 - |\nabla n(x,t_0)|^2 \right) dx + \frac{1}{m+1} \int_{\Omega} \left( (|n(x,t)| + \varepsilon)^m - (|n(x,t_0)| + \varepsilon)^m \right) dx \\
+ \frac{1}{2} \int_{t_0}^t \int_{\Omega} |\Delta n - m(|n| + \varepsilon)^{m-1} \nabla n|^2 dx ds + \int_{t_0}^t \int_{\Omega} (\varepsilon \delta(s+1)|n|^s + 2\mu|n|)|\nabla n|^2 dx ds \\
+ \frac{1}{2} \int_{t_0}^t \int_{\Omega} (\varepsilon|n|^{s+1} + \mu n^2)(|n| + \varepsilon)^m dx ds \\
\leq 3\delta \int_{t_0}^t \|\Delta n\|^2_{L^2} ds + C \int_{t_0}^t (\|\nabla c\|^2_{H^1} + \|n\|^2_{L^3} + \|n\|^2_{H^1} + \|u\|^2_{H^1}) ds + C.
\]

Combining the above inequality with (3.15)–(3.16) and applying Lemma 2.2, we obtain (4.1).

By (4.1), it is easy to see that \(\delta\) and (4.1). Taking \(\varepsilon\) energy estimates for the problem (4.2) independent of \(\mu, g, \varphi \in L^2(Q)\).

Let \((u_{\epsilon, \delta}, \tilde{c}_{\epsilon, \delta}, n_{\epsilon, \delta})\) be a time periodic solution of the problem (3.18) satisfying (3.15), (3.16) and (4.1). Taking \(\delta \to 0\), we obtain that (if necessary, we may choose a subsequence)

\[
\begin{align*}
  u_{\epsilon, \delta} & \to u_\epsilon, & & \text{in } L^0(Q_T), \\
  \tilde{c}_{\epsilon, \delta} & \to \tilde{c}_\epsilon, & & \text{in } C(\overline{Q_T}), \\
  u_{\epsilon, \delta} & \to u_\epsilon, & & \text{in } W^{2,1}_2(Q_T), \\
  n_{\epsilon, \delta} & \to n_\epsilon, & & \text{in } L^p(Q_T) \text{ for any } p < s + m + 1, \\
  \nabla n_{\epsilon, \delta} & \to \nabla n_\epsilon, & & \text{in } L^2(Q_T), \\
  n_{\epsilon, \delta} & \to n_\epsilon, & & \text{in } L^2(\partial \Omega \times (0, T)).
\end{align*}
\]

Then \((u_\epsilon, \tilde{c}_\epsilon, n_\epsilon)\) is the solution of the following problem

\[
\begin{align*}
  n_t - m \nabla \cdot \left((|n| + \varepsilon)^{m-1} \nabla n \right) + \varepsilon |n|^s n + u \cdot \nabla n + An \\
  = -\chi \nabla \cdot \left(e^{a_1}n_+ \nabla \tilde{c} + n_+ \nabla c \right) + n_+ \nabla g_2) + (\mu a + A)|n| - \mu |n| n + g, \\
  \tilde{c}_t - \Delta \tilde{c} + (u - 2\nabla g_1) \cdot \nabla \tilde{c} = (|\nabla g_1|^2 + \Delta g_1 - n_+ - u \nabla g_1 - g_1 \tilde{c}) + (\Delta g_2 - u \nabla g_2 - n_+ g_2 - g_{21}) e^{-\varphi_1}, \\
  u_t = \Delta u - \nabla \pi + n \nabla \varphi, \\
  \nabla \cdot u = 0, \\
  \frac{\partial u}{\partial \nu}|_{\partial \Omega} = \frac{\partial \tilde{c}}{\partial \nu}|_{\partial \Omega} = u|_{\partial \Omega} = 0,
\end{align*}
\]

such that \((u_\epsilon, \tilde{c}_\epsilon, n_\epsilon)\) satisfies (3.15) and (3.16).

Next, we consider the second level approximation. For this purpose, we need the following energy estimates for the problem (4.2) independent of \(\varepsilon\).

**Proposition 4.1** Assume \(m > 1\), \(\max\{2(m - 1), 2\} < s \leq 5m - 1\), \(g \geq 0\) and \(a, g, \nabla \varphi \in L^2(\Omega)\). If \(A > 0\) is sufficiently large, the problem (4.2) admits a time periodic solution.
\[ (n, c, \tilde{c}, u) \text{ such that } n, c, \tilde{c} \geq 0 \text{ and } \]
\[ \sup_t (\|u(t)\|_{W^{1,\infty}} + \|n(t)\|_{L^\infty} + \|\tilde{c}(t)\|_{W^{1,\infty}}) \leq C, \]
\[ \sup_t (\|u(t)\|_{H^1}^2 + \|\tilde{c}(t)\|_{H^1}^2) + \int_0^T (\|u\|_{H^2}^2 + \|\tilde{c}\|_{H^2}^2 + \|u_{tt}\|_{L^2}^2 + \|\tilde{c}_{tt}\|_{L^2}^2) \, ds \leq C, \]
\[ \sup_t (\|n\|_{L^1} + \|n \ln n\|_{L^1}) + \int_0^T (n + \varepsilon)^m - TV \|n\| dx - \int_0^T \|n\|_{L^2}^2 \, ds \leq C, \]
where \( C \) is independent of \( \varepsilon \).

The proof of Proposition 4.1 is given by the following lemmas. For simplicity, we will denote the limit \((u, \tilde{c}, n)\) by \((u, \tilde{c}, n)\).

**Lemma 4.2** Assume \( m > 1, \max\{2(m-1), 2\} < s \leq 5m - 1 \), \( g \geq 0 \) and \( a, g, \nabla \varphi \in L^\infty_t(Q) \). Let \((n, \tilde{c}, u)\) be a periodic solution of (4.2). Then for sufficiently large \( A \), we have \( n, c, \tilde{c} \geq 0 \) and
\[ \sup_t \|\tilde{c}(t)\|_{L^\infty} \leq C, \quad (4.3) \]
\[ \sup_t (\|u(t)\|_{H^1}^2 + \|\tilde{c}(t)\|_{H^1}^2) + \int_0^T (\|u\|_{H^2}^2 + \|\tilde{c}\|_{H^2}^2 + \|u_{tt}\|_{L^2}^2 + \|\tilde{c}_{tt}\|_{L^2}^2) \, ds \leq C, \quad (4.4) \]
\[ \sup_t (\|n\|_{L^1} + \|n \ln n\|_{L^1}) + \int_0^T (n + \varepsilon)^m - TV \|n\| dx - \mu \int_0^T \|n\|_{L^2}^2 \, ds \leq C, \quad (4.5) \]
where \( C \) is independent of \( \varepsilon \).

**Proof.** By Lemma 3.2, we have
\[ 0 \leq c \leq |g_2|_{L^\infty}. \]

Next, we show \( n \geq 0 \) by examining the set \( J(t) = \{ x \in \Omega; n(x, t) < 0 \} \). Assume that \( J(t) \) is a differentiable submanifold and \( \frac{\partial n}{\partial \nu} \) denote the outward normal derivative of \( n \) on \( J(t) \). It’s easy to see that \( n = 0 \) and \( \frac{\partial n}{\partial \nu} \geq 0 \) on \( \partial\{J(t)\} \setminus \partial \Omega \), and \( n_+ = 0 \) and \( \frac{\partial n}{\partial \nu} = 0 \) on \( \partial\{J(t)\} \cap \partial \Omega \). A direct integration of \( (4.2)_1 \) on \( J(t) \times (0, T) \) gives
\[ 0 \geq -m \int_0^T \int_{\partial\{J(t)\}} (|n| + \varepsilon)^{m-1} \frac{\partial n}{\partial \nu} \, d\Gamma ds + \int_0^T \int_{J(t)} (|n|^{\mu} + A) n dx ds \]
\[ = \int_0^T \int_{J(t)} (\mu a + A)|n| dx ds + \int_0^T \int_{J(t)} (g - \mu |n|) n dx ds \geq 0, \]
where \( A > 0 \) is sufficiently large. It implies that
\[ \int_0^T \int_{J(t)} n dx ds = 0, \]
21
namely, \( n \geq 0 \). While if \( J(t) \) is not a regular submanifold, we can construct a sufficiently smooth approximating sequence \((n_k, \tilde{c}_k, u_k)\) of \((n, c, u)\) such that the corresponding approximating solutions \( n_k \) satisfying that \( n_k(\cdot, t) \) are continuously differentiable. Thus, the sets \( J_k(t) \) are measurable and \( \partial J_k(t) \) are differentiable submanifolds. Then the above result can be obtained by letting \( k \to 0 \).

Similarly, in what follows, we still assume that the solution \( n \) is sufficiently smooth. Otherwise, the following estimates can be obtained by an approximating process. Take \( n_+ = |n| = n \) in the problem (4.2). Combining (4.2) with (3.12)–(3.14), we have

\[
\partial J \text{ measurable and } \partial J_k(t) \text{ are differentiable submanifolds. Then the above result can be obtained by letting } k \to 0.
\]

\[
\text{Similarly, in what follows, we still assume that the solution } n \text{ is sufficiently smooth. Otherwise, the following estimates can be obtained by an approximating process. Take } n_+ = |n| = n \text{ in the problem (4.2). Combining (4.2) with (3.12)–(3.14), we have}
\]

\[
sup_t (\|u(t)\|_{H^1}^2 + \|\tilde{c}(t)\|_{H^1}^2) + \int_0^T (\|u\|_{H^2}^2 + \|\tilde{c}\|_{H^2}^2 + \|u_1\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2) \, ds \leq C \int_0^T \|n\|_{L^2}^2 \, ds + C,
\]

where \( C \) is independent of \( \varepsilon \). In order to estimate the term \( \int_0^T \|n\|_{L^2}^2 \, ds \), we integrate the equation (4.2) over \( \Omega \times (t_0, t_0 + t) \) with \( t_0 < t \leq t_0 + T \). It is easy to obtain that

\[
\left\{ beginning \text{ of the equation (4.2)} \right\}
\]

\[
\text{Combining with Lemma 2.2, we have}
\]

\[
\sup_t \|n\|_{L^1} + \mu \int_0^T \|n\|_{L^2}^2 \, ds \leq C \int_0^T \int_{\partial \Omega} n \, d\Gamma \, ds + C.
\]

\[\text{(4.7)}\]

Obviously, the estimate of the term \( \int_0^T \int_{\partial \Omega} n \, d\Gamma \, ds \) is a key point to prove (4.4) and (4.5).

Multiplying (4.2) by \( 1 + \ln n \), and integrating it over \( \Omega \times (t_0, t) \) with \( t_0 < t \leq t_0 + T \), we have

\[
\int_\Omega (n(x, t) \ln n(x, t) - n(x, t_0) \ln n(x, t_0)) \, dx + 4m \int_{t_0}^t \int_\Omega (n + \varepsilon)^{m-1} |\nabla \sqrt{n}|^2 \, dx \, ds
\leq \chi \int_{t_0}^t \int_\Omega (e^{g_1} \nabla \tilde{c} + e^{g_1} \tilde{c} \nabla g_1 + \nabla g_2) \nabla n \, dx \, ds
\leq \chi \int_{t_0}^t \int_{\partial \Omega} e^{g_1} \tilde{c} \frac{\partial g_1}{\partial \nu} n(1 + \ln n) \, d\Gamma \, ds
\leq C \int_{t_0}^t (\|\nabla \tilde{c}\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2) \, ds + C \int_{t_0}^t \int_{\partial \Omega} n(1 + \ln n) \, d\Gamma \, ds + C,
\]

where we use the fact that

\[
1 + \ln n \leq \frac{1}{\sigma} e^{(\sigma - 1) \cdot n^\sigma}
\]

\[\text{(4.9)}\]
for any $n > 0$ and $\sigma > 0$. Applying Lemma 2.1 and combining with (4.6) and (4.7), we obtain

$$\sup_t \int_\Omega n(x, t) \ln n(x, t) \, dx + 4m \int_0^T \int_\Omega (n + \varepsilon)^{m-1} |\nabla \sqrt{n}|^2 \, dx \, ds \leq C \mu \int_0^T \int_\Omega n \, d\Gamma \, ds + C \mu \int_0^T \int_\Omega (1 + \ln n)_+ \, d\Gamma \, ds + C.$$  \hfill (4.10)

Taking $\sigma \in (0, 1)$, adding (4.7) and (4.10) together and recalling the embedding relation $W^{1,1}(\Omega) \hookrightarrow L^1(\partial \Omega)$ by Lemma 2.3, yield

$$\sup_t (\|n\|_{L^1} + \|n \ln n\|_{L^1}) + \mu \int_0^T \int_\Omega \|n\|^2_{L^2} \, ds + 4m \int_0^T \int_\Omega (n + \varepsilon)^{m-1} |\nabla \sqrt{n}|^2 \, dx \, ds \leq C \frac{1 + \mu}{\mu} \int_0^t \int_\Omega n \, d\Gamma \, ds + C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} n(1 + \ln n)_+ \, d\Gamma \, ds + C$$

$$= C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} n(2 + \ln n - (1 + \ln n)) \, d\Gamma \, ds + C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} n(1 + \ln n)_+ \, d\Gamma \, ds + C$$

$$\leq C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} n(2 + \ln n)_+ \, d\Gamma \, ds - C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} n(1 + \ln n)_+ \, d\Gamma \, ds$$

$$+ C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} (1 + \ln n)_+ \, d\Gamma \, ds + C$$

$$\leq C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} n(2 + \ln n)_+ \, d\Gamma \, ds + C \frac{1 + \mu}{\mu} \int_0^t \int_{\partial \Omega} |\nabla(n(2 + \ln n)_+)| \, d\Gamma \, ds + C$$

$$\leq C \int_0^t \int_{\partial \Omega} n^{1+\sigma} \, dx \, ds + C \int_0^t \int_{\partial \Omega} |\nabla n|(2 + \ln n)_+ \, dx \, ds$$

$$+ C \int_0^t \int_{\partial \Omega \setminus \text{supp } (2 + \ln n)_+} |\nabla n| \, dx \, ds + C$$

$$\leq C \int_0^t \int_{\partial \Omega} n^{1+\sigma} \, dx \, ds + C \int_0^t \int_{\partial \Omega} |\nabla n|(2 + \ln n)_+ \, dx \, ds$$

$$+ C \int_0^t \int_{\partial \Omega \setminus \text{supp } (2 + \ln n)_+} |\nabla n| \, dx \, ds + C$$

$$\leq C \int_0^t \int_{\partial \Omega} n^{1+\sigma} \, dx \, ds + C \int_0^t \int_{\partial \Omega} |\nabla n| n^{\sigma-1} \, dx \, ds + C$$

$$\leq \frac{\mu}{2} \int_0^T \|n\|^2_{L^2} \, ds + m \int_0^T \int_{\partial \Omega} |\nabla n|^2 n^{m-2} \, dx \, ds + C,$$

which implies that

$$\sup_t (\|n\|_{L^1} + \|n \ln n\|_{L^1}) + \mu \int_0^T \|n\|^2_{L^2} \, ds + \int_0^T \int_{\partial \Omega} (n + \varepsilon)^{m-1} |\nabla \sqrt{n}|^2 \, dx \, ds \leq C.$$  \hfill (4.11)

where $C$ depends on $\mu$ and is independent of $\varepsilon$. Combining (4.6) with (4.11), we obtain (4.3) and (4.5). The proof of Lemma 4.2 is completed. \hfill \Box
Since that \(n \geq 0\), the problem (4.12) can be reduced to the following form.

\[
\begin{align*}
n_t - \Delta (n + \varepsilon)^m + \varepsilon n^{s+1} + u \cdot \nabla n &= -\chi \nabla \cdot (e^{g_1} n \nabla \tilde{c} + u \nabla e^{g_1} + n \nabla g_2) + \mu n (a - n) + g, \\
\tilde{c}_t - \Delta \tilde{c} + (u - 2 \nabla g_1) \cdot \nabla \tilde{c} &= (\nabla g_1)^2 + \Delta g_1 - n - u \nabla g_1 - g_{tt} \tilde{c} \\
&+ (\Delta g_2 - u \nabla g_2 - n g_2 - g_{tt}) e^{-g_1}, \\
u_t &= \Delta u - \nabla \pi + n \nabla \varphi, \\
\nabla \cdot u &= 0, \\
\frac{\partial u}{\partial \nu}\big|_{\partial \Omega} &= \frac{\partial \varphi}{\partial \nu}\big|_{\partial \Omega} = u|_{\partial \Omega} = 0.
\end{align*}
\]

(4.12)

**Lemma 4.3** Assume \(m > 1\), \(\max\{2(m-1), 2\} = \frac{5m-1}{2}\), \(g \geq 0\) and \(a, g, \nabla \varphi \in L_{\infty}^\varepsilon(Q)\). Let \((\tilde{c}, u, n)\) be the periodic solution of (4.12). Then for any \(q \geq 2(m-1)\), we have

\[
\sup_t \int_\Omega n^{1+q} \, dx + \int_0^T \int_\Omega (n^{m+q-2} |\nabla n|^2 + n^{2+q}) \, dx \, ds \leq C(q),
\]

(4.13)

where \(C(q)\) is independent of \(\varepsilon\), but depends on \(q\).

**Proof.** Multiplying (4.12) by \(n^q\) with \(q \geq 2(m-1)\), and integrating it over \(\Omega \times (t_0, t)\) with \(t_0 < t \leq t_0 + T\), we have

\[
\begin{align*}
&\frac{1}{1 + q} \int_{t_0}^t \int_\Omega (n^{1+q}(x, t) - n^{1+q}(x, t_0)) \, dx + \int_{t_0}^t \int_\Omega (qmn^{m+q-2} |\nabla n|^2 + \mu n^{2+q}) \, dx \, ds \\
&\leq q\chi \int_{t_0}^t \int_\Omega n^q (e^{g_1} \nabla \tilde{c} + e^{g_1} \nabla g_1 + \nabla g_2) \cdot \nabla n \, dx \, ds - \chi \int_{t_0}^t \int_{\partial \Omega} e^{g_1} \frac{\partial g_1}{\partial \nu} n^{1+q} \, d\Gamma \, ds \\
&\quad + \int_{t_0}^t \int_{\Omega} (\mu n + g) n^q \, dx \, ds \\
&\leq q \int_{t_0}^t \int_{\Omega} n^{q+m-2} |\nabla n|^2 \, dx \, ds + C \int_{t_0}^t \int_{\Omega} n^{m-q+2} |\nabla \tilde{c}|^2 \, dx \, ds \\
&\quad + C \int_{t_0}^t \int_{\Omega} (n^q + n^{q+1} + n^{q-m-2}) \, dx \, ds + C \int_{t_0}^t \int_{\partial \Omega} n^{1+q} \, d\Gamma \, ds \\
&\leq q \int_{t_0}^t \int_{\Omega} n^{q+m-2} |\nabla n|^2 \, dx \, ds + \frac{\mu}{4} \int_{t_0}^t \int_{\Omega} n^{q+2} \, dx \, ds + C \int_{t_0}^t \int_{\partial \Omega} |\nabla \tilde{c}|^{\frac{q+m}{m-2}} \, d\Gamma \, ds \\
&\quad + C \int_{t_0}^t \int_{\partial \Omega} n^{1+q} \, d\Gamma \, ds + C.
\end{align*}
\]

Recalling the embedding relation \(W^{1,1}(\Omega) \hookrightarrow L^1(\partial \Omega)\) by Lemma 2.4 yields

\[
\begin{align*}
&\frac{C}{1 + q} \int_{t_0}^t \int_{\partial \Omega} n^{1+q} \, d\Gamma \, ds \\
&\leq C \int_{t_0}^t \int_{\Omega} n^{1+q} \, dx \, ds + C q \int_{t_0}^t \int_{\Omega} n^q |\nabla n| \, dx \, ds \\
&\leq \frac{\mu}{4} \int_{t_0}^t \int_{\Omega} n^{2+q} \, dx \, ds + C \int_{t_0}^t \int_{\Omega} n^q \, dx \, ds \\
&\quad + \frac{q}{4} \int_{t_0}^t \int_{\Omega} n^{m-q+2} |\nabla n|^2 \, dx \, ds + C q \int_{t_0}^t \int_{\Omega} n^{q-m+2} \, dx \, ds.
\end{align*}
\]

(4.15)
Substituting (4.14) into (4.12) and combining with Lemma 2.1 give

\[
\sup_t \int_\Omega n^{1+q}(x,t) \, dx + \int_0^T \int_\Omega \left( qmn^{m+q-2}|\nabla n|^2 + \mu n^{2+q} \right) \, dx \, ds \\
\leq C \int_0^T \int_\Omega |\nabla \tilde{c}|^{\frac{2(q+2)}{m}} \, dx \, ds + C
\]

\[
\leq C \sup_t \|\tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \int_0^T \|\Delta \tilde{c}\|_{L^{\frac{m+2}{m}}} \, ds + C \sup_t \|\tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{2(q+2)}{m}} + C
\]

\[
\leq C \int_0^T \|\Delta \tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds + C,
\]

(4.16)

where \(C\) is independent of \(\varepsilon\). By Lemma 2.3 and the equality \(4.12\), we have

\[
\int_0^T \|\Delta \tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds \\
\leq C \int_0^T \int_\Omega \left( |u - 2\nabla g_1| \cdot \nabla \tilde{c}|_{\frac{2(q+2)}{m}} \, dx \right) \, ds + C \int_0^T \int_\Omega \left( |nc|^{\frac{2(q+2)}{m}} + |\tilde{c}|^{\frac{2(q+2)}{m}} \right) \, dx \, ds \\
+ C \int_0^T \int_\Omega \left( |\nabla g_1|^2 + \Delta g_1 - u \nabla g_1 - g_1 \right) \tilde{c}|_{\frac{2(q+2)}{m}} \, dx \, ds \\
+ C \int_0^T \int_\Omega \left( |\Delta g_2 - u \nabla g_2 - n g_2 - g_2| e^{-g_1} \tilde{c}|_{\frac{2(q+2)}{m}} \right) \, dx \, ds \\
\leq C \int_0^T \left( ||u||_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + 1 ||\nabla c||_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + 1 \right) \, ds + C \int_0^T \left( ||u||_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + ||n||_{L^{\frac{m+2}{m}}}^{\frac{2(q+2)}{m}} \right) \, ds \\
\leq C \int_0^T \|\nabla \tilde{c}\|_{L^{\frac{2(q+2)}{m}}}^{\frac{q+2}{m}} \, ds + C \int_0^T \left( ||u||_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + ||u||_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + ||n||_{L^{\frac{m+2}{m}}}^{\frac{2(q+2)}{m}} \right) \, ds.
\]

(4.17)

The following proof is divided into three steps.

Step 1. We assume that \(q \in [2m - 2, 3m - 2]\) such that \(\frac{2(q+2)}{m} \in [4, 6]\).

By Lemma 2.3 it is easy to see that for any \(\lambda > 0\),

\[
\|u\|_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + \|u\|_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} \leq C(||u||_{L^{\frac{2(q+2)}{m}}}^{\frac{2(q+2)}{m}} + \|n\|_{L^{\frac{m+2}{m}}}^{\frac{2(q+2)}{m}}).
\]

(4.18)

\[
\|\nabla \tilde{c}\|_{L^{\frac{2(q+2)}{m}}}^{\frac{q+2}{m}} \leq C\|\tilde{c}\|_{L^{\frac{2(q+2)}{m}}}^{\frac{q+2}{m}} + C\|\tilde{c}\|_{L^{\frac{2(q+2)}{m}}}^{\frac{q+2}{m}} \leq \lambda ||\Delta \tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} + C\lambda\|\tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}}.
\]

(4.19)

Taking sufficiently small \(\lambda\) and recalling Lemma 4.2 we reduces (4.17) to that

\[
\int_0^T \|\Delta \tilde{c}\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds \leq C \int_0^T \|n\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds + C.
\]

(4.20)

Substituting (4.20) into (4.16), we obtain

\[
\sup_t \int_\Omega n^{1+q} \, dx + \int_0^T \int_\Omega \left( n^{m+q-2}|\nabla n|^2 + \mu n^{2+q} \right) \, dx \, ds \leq C \int_0^T \|n\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds + C,
\]

(4.21)

which implies that

\[
\int_0^T \|n\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds \leq C \int_0^T \|n\|_{L^{\frac{m+2}{m}}}^{\frac{q+2}{m}} \, ds + C, \quad m > 1,
\]

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where \( q + 2 \in [2m, 3m] \) and \( C \) is independent of \( \varepsilon \), but depends on \( m, q, \mu, \chi, \Omega \) and \( T \). After finite iterations, we will finally derive that

\[
\sup_t \int_\Omega n^{1+q} \, dx + \int_0^T \int_\Omega n^{m+q-2} \nabla n^2 \, dx \, ds + \int_0^T \int_\Omega n^{1+q} \, dx \, ds \\
\leq C \int_0^T \|n\|_{L^2}^2 \, ds + C \leq C.
\] (4.22)

Taking \( q = 3m - 2 \) in (4.22) gives

\[
\sup_t \int_\Omega n^{3m-1} \, dx + \int_0^T \int_\Omega n^{4m-4} \nabla n^2 \, dx \, ds + \int_0^T \int_\Omega n^{3m} \, dx \, ds \leq C.
\] (4.23)

**Step 2.** Based on (4.23), we prove the time-space uniform boundedness of \( u \).

Note that \(-\frac{3}{2(3m-1)} > -1\). Recalling (2.1) and according to standard smoothing properties of the Stokes semigroup, there exists \( \kappa > 0 \) such that

\[
\|u(t)\|_{L^\infty} = \int_{-\infty}^t \|e^{-(t-s)A}P(n(s)\nabla \varphi(s))\|_{L^\infty} \, ds \\
\leq C \int_{-\infty}^t e^{-\kappa(t-s)}(t-s)^{-\frac{3}{2(3m-1)}} \|n(s)\nabla \varphi(s)\|_{L^{3m-1}} \, ds \\
\leq C \int_{-\infty}^t e^{-\kappa(t-s)}(t-s)^{-\frac{3}{2(3m-1)}} \|n(s)\|_{L^{3m-1}} \|\nabla \varphi(s)\|_{L^\infty} \, ds \\
\leq C \sup_t (\|n(t)\|_{L^3} \|\nabla \varphi(t)\|_{L^\infty}) \int_0^\infty e^{-\kappa s} s^{-\frac{3}{2(3m-1)}} \, ds \\
\leq C,
\]

which implies that

\[
\sup_t \|u(t)\|_{L^\infty} \leq C.
\] (4.24)

**Step 3.** Finally, we assume that \( q \in (3m - 2, \infty) \).

Combining (4.17) with (4.19) and (4.23), and taking sufficiently small \( \lambda \), we have

\[
\int_0^T \|\Delta \tilde{c}\|_{L^\infty} \, ds \leq C \int_0^T \|n\|_{L^\infty} \, ds + C.
\]

Then we can reduce (4.16) to that

\[
\sup_t \int_\Omega n^{1+q} \, dx + \int_0^T \int_\Omega (n^{m+q-2}|\nabla n|^2 + \mu n^{2+q}) \, dx \, ds \leq C \int_0^T \|n\|_{L^\frac{2+q}{2+m}}^\frac{2+q}{2+m} \, ds + C.
\]

Following the same procedure as the first step, we completes the proof. \( \square \)

Next, we prove the time-space uniform boundedness of \( \nabla \tilde{c} \) and \( n \). By Lemma 4.3 it is easy to see that

\[
\sup_t \left(\|n(t)\|_{L^4} + \|n(t)\|_{L^{2m+4}}\right) \leq C,
\] (4.25)

where \( C \) is independent of \( \varepsilon \).
Lemma 4.4 Assume $m > 1$, $\max\{2(m-1), 2\} < s \leq 5m-1$ and $a, g, \nabla \varphi \in L^\infty_T(Q)$. Let $(n, \tilde{c}, u)$ be a periodic solution of (4.12). Then we have

$$\sup_t (\|\nabla \tilde{c}(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} + \|n(t)\|_{L^\infty}) \leq C,$$  \hspace{1cm} (4.26)

where $C$ is independent of $\varepsilon$.

Proof. Recalling (4.12), we see that

$$\dot{\tilde{c}} - \Delta \tilde{c} + \tilde{c} = F(n, \tilde{c}, u),$$

where

$$F(n, \tilde{c}, u) = -(u - 2\nabla g_1) \cdot \nabla \tilde{c} + (|\nabla g_1|^2 + \Delta g_1 + 1 - n - u \nabla g_1 - g_{1t}) \tilde{c}$$

$$+ (\Delta g_2 - u \nabla g_2 - ng_2 - g_{2t}) e^{-g_1}.$$  

Notice that the time periodic solution $\tilde{c}$ of (4.12) can be expressed as follows

$$\tilde{c} = \int_{-\infty}^t e^{-(t-s)} e^{(t-s)\Delta} F \, ds,$$

where \{e^{t\Delta}\}_{t \geq 0} is the Neumann heat semigroup in $\Omega$, for more properties of Neumann heat semigroup, please refer to [36]. By lemmas 4.2 and 4.3, we obtain that

$$\|\nabla \tilde{c}(t)\|_{L^\infty} = \int_{-\infty}^t e^{-(t-s)} \|\nabla e^{(t-s)\Delta} F(s)\|_{L^\infty} \, ds$$

$$\leq C \sup_t \|F(t)\|_{L^4} \int_{-\infty}^t e^{-(t-s)} (t-s)^{-\frac{7}{8}} \, ds$$

$$\leq C \sup_t \left( (\|u\|_{L^\infty} + 1)(\|\nabla \tilde{c}\|_{L^4} + 1) + (\|n\|_{L^4} + 1)\|\tilde{c}\|_{L^\infty} \right) \int_0^\infty e^{-s}s^{-\frac{7}{8}} \, ds$$

$$\leq C \sup_t \left( \|n\|_{L^4} + \|\nabla \tilde{c}\|_{L^\infty}^\frac{1}{4} \left\| \nabla \tilde{c} \right\|_{L^2}^\frac{3}{4} \right) + C$$

$$\leq C + C \sup_t \|\nabla \tilde{c}\|_{L^\infty}^\frac{1}{4},$$

which implies that

$$\sup_t \|\nabla \tilde{c}(t)\|_{L^\infty} \leq C.$$  \hspace{1cm} (4.27)

Similarly, we can also obtain

$$\sup_t \|\nabla u(t)\|_{L^\infty} \leq C.$$  \hspace{1cm} (4.28)

Next, we prove the time-space uniform boundedness of $n$. Multiplying (4.12) by $n^q$ with
For any $q > 3m$, and integrating it over $\Omega \times (t_0, t)$ with $t_0 < t \leq t_0 + T$, we have

$$
\frac{1}{1 + q} \int_{t_0}^t \int_{\Omega} (n^{1+q}(x,t) - n^{1+q}(x,t_0)) \, dx + \int_{t_0}^t \int_{\Omega} (qmn^{m+q-2} + n q^2 + n^{1+q}) \, dx

\leq q \chi \int_{t_0}^t \int_{\Omega} n^q (e^{\varphi_1} \nabla \tilde{\varphi} + e^{\varphi_1} \nabla \varphi_1 + \nabla \varphi_2) \cdot \nabla n \, dx - \chi \int_{t_0}^t \int_{\partial \Omega} e^{\varphi_1} \frac{\partial \varphi_1}{\partial \nu} n^{1+q} \, d\Gamma ds

+ \int_{t_0}^t \int_{\Omega} ((mn + g)n^q + n^{1+q}) \, dxds

\leq \frac{q}{4} \int_{t_0}^t \int_{\Omega} n^{m+q-2} |\nabla n|^2 \, dxds + Cq \int_{t_0}^t \int_{\Omega} n^{q-m+2} \, dxds + C \int_{t_0}^t \int_{\partial \Omega} n^{1+q} \, d\Gamma ds

+ \frac{\mu}{4} \int_{t_0}^t \int_{\Omega} n^{2+q} \, dxds + C \int_{t_0}^t \int_{\Omega} n^q \, dxds

\leq \frac{q}{2} \int_{t_0}^t \int_{\Omega} n^{m+q-2} |\nabla n|^2 \, dxds + Cq \int_{t_0}^t \int_{\Omega} n^{q-m+2} \, dxds

+ \frac{\mu}{2} \int_{t_0}^t \int_{\Omega} n^{2+q} \, dxds + C \int_{t_0}^t \int_{\Omega} n^q \, dxds,

$$

which implies that

$$
\int_{\Omega} (n^{1+q}(x,t) - n^{1+q}(x,t_0)) \, dx + \frac{4mh(1 + q)}{(m + q)^2} \int_{t_0}^t \int_{\Omega} |\nabla n^{m+q}|^2 \, dxds + \int_{t_0}^t \int_{\Omega} n^{1+q} \, dxds

\leq Cq^2 \int_{t_0}^t \int_{\Omega} n^{q-m+2} \, dxds + Cq \int_{t_0}^t \int_{\Omega} n^q \, dxds.

(4.29)

By Lemma 2.4 and Young inequality, for any $\lambda \in (0, 1)$, we derive that

$$
Cq^2 \int_{\Omega} n^{q-m+2} \, dx = Cq^2 \| n^{m+q} \|_{L^{2(1+q)}} \frac{2(1+q)}{2(q-m+2)}

\leq Cq^2 \| n^{m+q} \|_{L^{2(1+q)}} \frac{2(1+q)}{2(q-m+2)}

\leq \lambda \| \nabla n^{m+q} \|_{L^2} + Cq^2 \| n^{q-m+2} \|_{L^{2(1+q)}}

(4.30)

and

$$
Cq \int_{\Omega} n^q \, dx = Cq \| n^{m+q} \|_{L^{2(1+q)}} \frac{2q}{2(1+q)}

\leq Cq \| n^{m+q} \|_{L^{2(1+q)}} \frac{2q}{2(1+q)}

\leq \lambda \| \nabla n^{m+q} \|_{L^2}^2 + Cq^2 \| n^{q-m+2} \|_{L^{2(1+q)}}

(4.31)

$$

for some $\kappa \geq 2$, where $\theta_1 = \frac{4(1+q)(q+2m-1)}{(7q+9m-2)(q-m+2)}$, $\theta_2 = \frac{4(1+q)(2q+3m)}{q(7q+9m-2)}$, and the constants $C, \kappa$ are independent of $q$ and $\varepsilon$. 

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Substituting (4.30)–(4.31) into (4.32) and letting \( \lambda \) be sufficiently small, we obtain
\[
\int_\Omega (n^{1+q}(x, t) - n^{1+q}(x, t_0)) \, dx + \int_{t_0}^t \int_\Omega n^{1+q} \, dx \, ds \\
\leq C q^\kappa \int_{t_0}^t \|n\|_{L^{2(1+q)}_x}^{q \theta_2(m+q)} \, ds + C q^\kappa \int_{t_0}^t \|n\|_{L^{2(1+q)}_x}^q \, ds \\
+ C q^2 \int_{t_0}^t \|n\|_{L^{2(1+q)}_x}^{q-m+2} \, ds + C q \int_{t_0}^t \|n\|_{L^{2(1+q)}_x}^{q} \, ds.
\]
(4.32)

Since \( q \geq 3m > 3 \), it is easy to check that
\[
\frac{2(1+q)}{3} \leq \frac{\theta_1(m+q)(q-m+2)}{2(m-1) + \theta_1(q + 2 - m)} \frac{q \theta_2(m+q)}{m + q \theta_2}, \quad q - m + 2, q \leq 1 + q
\]

Then applying Lemma 2.1 to (4.32), we have
\[
\sup_{t} \|n\|_{L^{1+q}}^{1+q} \leq C q^\kappa (\sup_{t} \|n\|_{L^{2(1+q)}}^{2(1+q)}) + \sup_{t} \|n\|_{L^{1+q}}^{1+q},
\]
(4.33)

where \( C \) and \( \kappa \) are independent of \( q \) and \( \varepsilon \). Define a monotonically increasing sequence
\[
\{r_j\}_{j=0}^{\infty}, \quad r_{j+1} = \frac{3}{2} r_j, \quad r_0 = 2m + \frac{2}{3}.
\]
(4.34)

Take \( q = r_{j+1} - 1 \) in (4.33) and \( M_j = \max\{1, \sup_t \|n\|_{L^{r_j}} \} \) with \( j = 0, 1, 2, \cdots \). Then we have
\[
M_j \leq C^{\frac{1}{r_j}} (r_j - 1)^{\frac{1}{r_j}} (M_j^{r_j-1} + M_j^{r_j-1})^{\frac{1}{r_j}} \\
\leq (2C)^{\frac{1}{r_j}} (r_j - 1)^{\frac{1}{r_j}} M_j - 1 \\
\leq (2C)^{\sum_{i=1}^j \frac{1}{r_i}} M_1 (r_i - 1)^{\frac{1}{r_i}} M_0, \quad j = 1, 2, \cdots.
\]

Notice that \( \sum_{i=1}^j \frac{1}{r_i} \) and \( \Pi_i (r_i - 1)^{\frac{1}{r_i}} \) converge as \( j \to \infty \). Letting \( j \to \infty \) gives
\[
\sup_{t} \|n(t)\|_{L^\infty} \leq C + C \sup_{t} \|n(t)\|_{L^{2m+4}} \leq C,
\]
(4.35)

where \( C \) is independent of \( \varepsilon \). The proof is completed. \( \square \)

By virtue of (4.28) and Lemma 4.4, we give the following estimate.

**Lemma 4.5** Assume \( m > 1 \), \( \max\{2(m-1), 2\} < s \leq 5m-1 \), \( g \geq 0 \) and \( \nabla \varphi \in L_T^\infty (Q) \). Let \( (n, \tilde{c}, u) \) be a periodic solution of (1.12). Then we have
\[
\int_0^T (\|u_t\|_{L^p} + \|u\|_{W^{2,p}} + \|\tilde{c}_t\|_{L^p} + \|\tilde{c}\|_{W^{2,p}}) \, ds \leq C \text{ for any } p > 1,
\]
(4.36)
\[
\sup_t \int_\Omega |\nabla (n + \varepsilon)|^m \, dx + \int_0^T \int_\Omega (n + \varepsilon)^{m-1} \left| \frac{\partial n}{\partial t} \right|^2 \, dx \, ds \leq C,
\]
(4.37)

where \( C \) is independent of \( \varepsilon \).
Combining with Lemma 2.1, 4.2, and 4.4, we derive that
which implies that

Proof. Applying Lemma 2.3–2.4 and the uniform boundedness of \( u, \nabla \hat{c} \) and \( n \) to (4.12), we can obtain (4.36). Multiplying (4.12) by \( \frac{1}{m} \frac{\partial (n + \varepsilon)^m}{\partial t} \), and integrating it over \( \Omega \times (t_0, t) \) with \( t_0 < t \leq t_0 + T \), we have

\[
\frac{m}{2} \int_\Omega \left( (n + \varepsilon)^{2(m-1)}|\nabla n|^2(x, t) - (n + \varepsilon)^{2(m-1)}|\nabla n|^2(x, t_0) \right) \, dx
\]

\[
+ \int_{t_0}^{t} \int_\Omega (n + \varepsilon)^{m-1} \left| \frac{\partial n}{\partial t} \right|^2 \, dx \, ds
\]

\[
= \int_{t_0}^{t} \int_\Omega (-u \cdot \nabla n + \mu n - \mu n^2 + g)(n + \varepsilon)^{m-1} \frac{\partial n}{\partial t} \, dx \, ds
\]

\[
- \chi \int_{t_0}^{t} \int_\Omega \nabla \cdot (e^{\beta_1 n} \nabla \hat{c} + \varepsilon \beta_1 \nabla g_1 + n \nabla g_2)(n + \varepsilon)^{m-1} \frac{\partial n}{\partial t} \, dx \, ds
\]

\[
\leq \frac{1}{2} \int_{t_0}^{t} \int_\Omega (n + \varepsilon)^{m-1} \left| \frac{\partial n}{\partial t} \right|^2 \, dx \, ds + C \int_{t_0}^{t} \int_\Omega (n + \varepsilon)^{m-1} |\nabla n|^2 \, dx \, ds
\]

\[
+ C \int_{t_0}^{t} \int_\Omega |\Delta \hat{c}|^2 \, dx \, ds + C,
\]

which implies that

\[
\int_\Omega \left( (n + \varepsilon)^{2(m-1)}|\nabla n|^2(x, t) - (n + \varepsilon)^{2(m-1)}|\nabla n|^2(x, t_0) \right) \, dx
\]

\[
+ \int_{t_0}^{t} \int_\Omega (n + \varepsilon)^{m-1} \left| \frac{\partial n}{\partial t} \right|^2 \, dx \, ds
\]

\[
\leq C \int_{t_0}^{t} \int_\Omega (n + \varepsilon)^{m-1} |\nabla n|^2 \, dx \, ds + C \int_{t_0}^{t} \int_\Omega |\Delta \hat{c}|^2 \, dx \, ds + C.
\]

Combining with Lemma 2.1, 4.2, and 4.4, we derive that

\[
\sup_t \int_\Omega (n + \varepsilon)^{2(m-1)}|\nabla n|^2(x, t) \, dx + \int_0^T \int_\Omega (n + \varepsilon)^{m-1} \left| \frac{\partial n}{\partial t} \right|^2 \, dx \, ds
\]

\[
\leq C \int_0^T \int_\Omega (n + \varepsilon)^{m-1} |\nabla n|^2 \, dx \, ds + C \int_0^T \int_\Omega |\Delta \hat{c}|^2 \, dx \, ds + C
\]

\[
\leq C \int_0^T \int_\Omega (n + \varepsilon)^{m-1} |\nabla n|^2 \, dx \, ds + C
\]

\[
\leq C,
\]

where \( C \) is independent of \( \varepsilon \). The proof is completed. \( \square \)

The proof of Proposition 4.1 is a consequence of Lemma 4.2–4.5.

Proof of Theorem 1.1 Let \((u_\varepsilon, \hat{c}_\varepsilon, n_\varepsilon)\) be a time periodic solution of the problem (4.12)
satisfying Proposition 4.3. Then we have

\[ - \iint_{Q_T} n_\varepsilon \phi_1 \, dx \, ds + \iint_{Q_T} \nabla(n_\varepsilon + \varphi)^m \nabla \phi_1 \, dx \, ds + \iint_{Q_T} (\varepsilon n_\varepsilon^{s+1} + u_\varepsilon \cdot \nabla n_\varepsilon) \phi_1 \, dx \, ds \\
= \chi \iint_{Q_T} (\varepsilon g_1 n_\varepsilon \nabla \tilde{c}_\varepsilon + n_\varepsilon \tilde{c}_\varepsilon \nabla g_1 + n_\varepsilon \nabla g_2) \cdot \nabla \phi_1 \, dx \, ds - \chi \int_0^T \int_{\partial \Omega} n_\varepsilon \tilde{c}_\varepsilon \phi_1 \frac{\partial g_1}{\partial \nu} \, d\Gamma \, ds \\
+ \iint_{Q_T} (\mu_\varepsilon (a - n_\varepsilon) + g) \phi_1 \, dx \, ds, \quad (4.38) \]

\[ - \iint_{Q_T} \tilde{c}_\varepsilon \phi_2 \, dx \, ds + \iint_{Q_T} \nabla \tilde{c}_\varepsilon \nabla \phi_2 \, dx \, ds + \iint_{Q_T} ((u_\varepsilon - 2\nabla g_1) \cdot \nabla \tilde{c}_\varepsilon) \phi_2 \, dx \, ds \\
= \iint_{Q_T} (|\nabla g_1|^2 + \Delta g_1 - n_\varepsilon - u_\varepsilon \nabla g_1 - g_1 \tilde{c}_\varepsilon) \phi_2 \, dx \, ds \\
+ \iint_{Q_T} (\Delta g_2 - u_\varepsilon \nabla g_2 - n_\varepsilon g_2 - g_2)e^{-g_1} \phi_2 \, dx \, ds, \quad (4.39) \]

\[ - \iint_{Q_T} u_\varepsilon \phi_3 \, dx \, ds + \iint_{Q_T} \nabla u_\varepsilon \nabla \phi_3 \, dx \, ds = \iint_{Q_T} n_\varepsilon \nabla \phi_3 \, dx \, ds, \quad (4.40) \]

for any \( \phi_1, \phi_2, \phi_3 \in H^1_p(Q) \) with \( \frac{\partial \phi_3}{\partial \nu} |_{\partial \Omega} = 0 \), \( \phi_3 |_{\partial \Omega} = 0 \) and \( \nabla \cdot \phi_3 = 0 \). Using Sobolev imbedding theorem and taking \( \varepsilon \to 0 \), we have (if necessary, we may choose a subsequence)

\[ u_\varepsilon \to u, \quad \tilde{c}_\varepsilon \to \tilde{c}, \quad \text{uniformly}, \]

\[ u_\varepsilon \to u, \quad \tilde{c}_\varepsilon \to \tilde{c}, \quad \text{in } W^{2,1}_p(Q_T) \text{ for any } p > 1, \]

\[ n_\varepsilon, \quad n_\varepsilon + \varepsilon \to n, \quad \text{in } L^p(Q_T) \text{ for any } p > 1, \]

\[ \varepsilon n_\varepsilon^{s+1} \to 0, \quad n_\varepsilon \to n, \quad \text{in } L^\infty(Q_T), \]

\[ \nabla(n_\varepsilon + \varepsilon)^m \to \nabla n^m, \quad \text{in } L^2(Q_T). \]

Then \((u, \tilde{c}, n)\) is a time periodic solution of the problem 1.1.11 satisfies 1.1.13 - 1.1.15. \( \square \)

5 \ Strong time periodic solutions

In this section, we improve the regularity for \( m \in (1, \frac{4}{3}] \) and prove that the obtained time periodic solution is strong solution. Let \((n_\varepsilon, \tilde{c}_\varepsilon, u_\varepsilon)\) be a time periodic solution of 1.1.12. We still assume that \((n_\varepsilon, \tilde{c}_\varepsilon, u_\varepsilon)\) is sufficiently smooth. Otherwise, the following estimates can be obtained by an approximating process.

**Lemma 5.1** Assume \( 1 < m \leq \frac{4}{3}, \) \( \max\{2(m - 1), 2\} \leq s \leq 5m - 1, \) \( g \geq 0 \) and \( a, \nabla \varphi \in L^\infty_T(Q). \)

Let \((n_\varepsilon, \tilde{c}_\varepsilon, u_\varepsilon)\) be a time periodic solution of 1.1.12. Then we have

\[ \sup_\iint |\nabla \sqrt{n_\varepsilon + \varepsilon}|^2 \, dx + \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m-4} |\nabla n_\varepsilon|^4 \, dx \, ds \leq C, \quad (5.1) \]

where \( C \) is independent of \( \varepsilon \).
Proof. The proof is the similar with that of Lemma 3.5 in [16]. Actually, the proof is more concise in our case by applying the estimate \((4.36)\) and the uniform boundedness of \(u_\varepsilon, \tilde{c}_\varepsilon\) and \(n_\varepsilon\). So we omit it here.

\[\Box\]

Lemma 5.2 Assume \(1 < m \leq \frac{s}{2}, \max\{2(m-1), 2\}\) \(< s \leq 5m-1, g \geq 0\) and \(a, \nabla \varphi \in L_t^\infty(Q)\). Let \((n_\varepsilon, \tilde{c}_\varepsilon, u_\varepsilon)\) be a time periodic solution of \((4.12)\). Then we have

\[
\int_0^T \left(\|n_\varepsilon\|^2_{L^2} + \|\Delta(n_\varepsilon + \varepsilon)^m\|^2_{L^2}\right) \, ds \leq C,
\]

where \(C\) is independent of \(\varepsilon\).

Proof. Multiplying \((4.12)\) by \(n_\varepsilon t\), integrating it over \(\Omega \times (t_0, t)\) with \(t_0 < t \leq t_0 + T\), and applying \((4.36), (5.1)\) and the uniform boundedness of \(u_\varepsilon, \tilde{c}_\varepsilon\) and \(n_\varepsilon\), we have

\[
\begin{align*}
\frac{m}{2} & \int_\Omega \left( (n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon|^2(x, t) - (n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon|^2(x, t_0) \right) \, dx + \int_{t_0}^t \int_\Omega |n_\varepsilon t|^2 \, dx \, ds \\
& = -\chi \int_{t_0}^t \int_\Omega \nabla \cdot (e^{\beta n_\varepsilon \nabla \tilde{c}_\varepsilon} + e^{\beta n_\varepsilon \tilde{c}_\varepsilon} \nabla g_1 + n_\varepsilon \nabla g_2) n_\varepsilon t \, dx ds \\
& + \int_{t_0}^t \int_\Omega (-\varepsilon n_\varepsilon^{s+1} - u_\varepsilon \nabla n_\varepsilon + \mu n_\varepsilon - \mu n_\varepsilon^2 + g) n_\varepsilon t \, dx ds \\
& + \frac{m(m-1)}{2} \int_{t_0}^t \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon t |\nabla n_\varepsilon|^2 \, dxds \\
& \leq \frac{1}{2} \int_{t_0}^t \int_\Omega |n_\varepsilon t|^2 \, dxds + C \int_{t_0}^t \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^4 \, dxds + C \\
& \leq \frac{1}{2} \int_{t_0}^t \int_\Omega |n_\varepsilon t|^2 \, dxds + C \int_{t_0}^t \int_\Omega (n_\varepsilon + \varepsilon)^{m-4} |\nabla n_\varepsilon|^4 \, dxds + C \\
& \leq \frac{1}{2} \int_{t_0}^t \int_\Omega |n_\varepsilon t|^2 \, dxds + C.
\end{align*}
\]

Combining with Lemma 2.1, we derive that

\[
\int_0^T \int_\Omega |n_\varepsilon t|^2 \, dxds \leq C,
\]

where \(C\) is independent of \(\varepsilon\). Similarly, multiplying \((4.12)\) by \(-\Delta(n_\varepsilon + \varepsilon)^m\), integrating it over \(\Omega \times (t_0, t)\) with \(t_0 < t \leq t_0 + T\), and applying \((4.36), (5.1), (5.3)\) and the uniform boundedness of \(u_\varepsilon, \tilde{c}_\varepsilon\) and \(n_\varepsilon\), we also obtain that

\[
\int_0^T \int_\Omega |\Delta(n_\varepsilon + \varepsilon)^m|^2 \, dxds \leq C,
\]

where \(C\) is independent of \(\varepsilon\). The proof is completed.

\[\Box\]

Next, we prove Theorem 1.2
Proof of Theorem 1.2 Let $(u_\varepsilon, \tilde{c}_\varepsilon, n_\varepsilon)$ be a weak time periodic solution of the problem (4.12) satisfying Proposition 4.1. Using Lemma 5.2 and Sobolev imbedding theorem and taking $\varepsilon \to 0$, we obtain that (if necessary, we may choose a subsequence)

\[ u_\varepsilon \to u, \quad \tilde{c}_\varepsilon \to \tilde{c}, \quad \text{uniformly,} \]
\[ u_\varepsilon \to u, \quad \tilde{c}_\varepsilon \to \tilde{c}, \quad \text{in } L^p(Q_T) \text{ for any } p > 1, \]
\[ \tilde{c}_\varepsilon \to \tilde{c}_t, \quad \nabla \tilde{c}_\varepsilon \to \nabla \tilde{c}, \quad \Delta \tilde{c}_\varepsilon \to \Delta \tilde{c}, \quad \text{in } L^p(Q_T) \text{ for any } p > 1, \]
\[ u_\varepsilon \to u_t, \quad \nabla u_\varepsilon \to \nabla u, \quad \Delta u_\varepsilon \to \Delta u, \quad \text{in } L^p(Q_T) \text{ for any } p > 1, \]
\[ n_\varepsilon, n_\varepsilon + \varepsilon \to n, \quad \text{in } L^p(Q_T) \text{ for any } p > 1, \]
\[ \varepsilon n_\varepsilon^{k+1} \to 0, \quad n_\varepsilon \to n, \quad \text{in } L^\infty(Q_T), \]
\[ \nabla n_\varepsilon \to \nabla n, \quad \Delta (n_\varepsilon + \varepsilon)n_\varepsilon \to \Delta n^m, \quad \text{in } L^p(Q_T) \text{ for any } p \in (1, 6), \]
\[ n_\varepsilon \to n_t, \quad \Delta(n_\varepsilon + \varepsilon)n_\varepsilon \to \Delta n^m, \quad \text{in } L^2(Q_T). \]

Applying integration by parts to (4.38)–(4.40) and letting $\varepsilon \to 0$, we derive that

\[ \int_{Q_T} (n_t - \Delta n^m + u \cdot \nabla n) \phi_1 \, dx \, ds - \int_{Q_T} (\mu n \varepsilon (a - n \varepsilon) + g) \phi_1 \, dx \, ds \]
\[ = -\chi \int_{Q_T} \nabla \cdot (e^{g_1} n \varepsilon \nabla \varepsilon + n \varepsilon \varepsilon \varepsilon \varepsilon e^{g_1} + n \varepsilon \nabla g_2) \phi_1 \, dx \, ds, \quad (5.5) \]
\[ \int_{Q_T} (\tilde{c}_t - \Delta \tilde{c} + (u - 2\nabla g_1) \cdot \nabla \tilde{c}) \phi_2 \, dx \, ds \]
\[ = \int_{Q_T} (|\nabla g_1|^2 + \Delta g_1 - n - u \nabla g_1 - g_1) \tilde{c} \phi_2 \, dx \, ds \]
\[ + \int_{Q_T} (\Delta g_2 - u \nabla g_2 - n g_2 - g_2) e^{-g_1} \phi_2 \, dx \, ds, \quad (5.6) \]
\[ \int_{Q_T} (u_t - \Delta u + \nabla \pi - n \nabla \phi) \phi_3 \, dx \, ds = 0, \quad (5.7) \]

for any $\phi_1, \phi_2, \phi_3 \in H^1_T(Q)$ with $\frac{\partial \phi}{\partial \nu}|_{\partial \Omega} = 0$, $\phi_3|_{\partial \Omega} = 0$ and $\nabla \cdot \phi_3 = 0$. Then $(u, \tilde{c}, n)$ is a strong time periodic solution of the problem (1.11) satisfying Proposition 4.1 and Lemma 5.2. The proof is completed. \[ \square \]

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References

[1] R. A. Adams, J. J. F. Fournier, Sobolev spaces. Second edition. Elsevier/Academic Press, Amsterdam, 2003.

[2] P. W. Atkins, J. D. Paula, Physical Chemistry, 8th edn. Oxford University Press, 2006.

[3] M. Braukhoff, Global (weak) solution of the chemotaxis-Navier-Stokes equations with nonhomogeneous boundary conditions and logistic growth, Ann. Inst. H. Poincaré Anal. Non Linéaire, 34 (2017), 1013–1039.

[4] M. Braukhoff, B. Tang, Global solutions for chemotaxis-Navier-Stokes system with Robin boundary conditions, J. Differential Equations, 269(2020), 10630–10669.

[5] A. Chertock, K. Fellner, A. Kurganov, A. Lorz, P. A. Markowich, Sinking, merging and stationary plumes in a coupled chemotaxis-fluid model: a high-resolution numerical approach, J. Fluid Mech., 694 (2012), 155–190.

[6] M. Di Francesco, A. Lorz, P. Markowich, Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior. Discrete Contin. Dyn. Syst., 28 (2010), 1437–1453.

[7] R. Duan, Z. Xiang, A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion, Int. Math. Res. Not. IMRN, 2014 (2014), 1833–1852.

[8] R. Farwig, T. Okabe, Periodic solutions of the Navier-Stokes equations with inhomogeneous boundary conditions, Ann Univ Ferrara, 56 (2010), 249–281.

[9] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Springer-Verlag, New York, 1994. (pp. 169–174)

[10] A. J. Hillesdon, T. J. Pedley, J. O. Kessler, The development of concentration gradients in a suspension of chemotactic bacteria, Bull. Math. Biol, 57(1995), 299–344.

[11] A. J. Hillesdon, T. J. Pedley, Bioconvection in suspensions of oxytactic bacteria : linear theory, J.Fluid. Mech. 324(1996), 223–259.

[12] J. Huang, C. Jin, Time periodic solution to a coupled chemotaxis-fluid model with porous medium diffusion, Discrete Contin. Dyn. Syst., 40 (2020), 5415–5439.

[13] C. Jin, Boundedness and global solvability to a chemotaxis model with nonlinear diffusion, J. Differential Equations, 263 (2017), 5759–5772.

[14] C. Jin, Large time periodic solutions to coupled chemotaxis-fluid models, Z. Angew. Math. Phys., 68 (2017), 24pp.
[15] C. Jin, Periodic pattern formation in the coupled chemotaxis-(Navier-)Stokes system with mixed nonhomogeneous boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A, 150 (2020) 3121–3152.

[16] C. Jin, Global bounded solution in three-dimensional chemotaxis-Stokes model with arbitrary porous medium slow diffusion, arXiv:2101.11235v1, 2021.

[17] J. O. Kessler, Path and pattern - The mutual dynamics of swimming cells and their environment, Comments theor. Biol., 1 (1989), 85–108.

[18] M. Kohr, I. Pop, Viscous incompressible flow for low reynolds numbers. Advances in boundary elements, 16, 427 pp, Southampton: WIT Press, 2004.

[19] H. Kozono, T. Yanagisawa, Leray’s problem on the stationary Navier-Stokes equations with inhomogeneous boundary data, Math. Z., 262 (2009), 27–39.

[20] J. Lankeit, Long-term behaviour in a chemotaxis-fluid system with logistic source, Math. Mod. Methods Appl. Sci., 26 (2016), 2071–2109.

[21] G. Leoni, A First Course in Sobolev Spaces: Second Edition, 181, American Mathematical Society, Providence, Rhode Island, 2017.

[22] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol. 1, Dunod, 1968.

[23] J. Liu, A. Lorz, A coupled chemotaxis-fluid model: global existence, Ann. I. H. Poincaré-AN, 28 (2011), 643–652.

[24] H. Lee, J. Kim, Numerical investigation of falling bacterial plumes caused by bioconvection in a three-dimensional chamber, European Journal of Mechanics B/Fluids, 52 (2015), 120–130.

[25] M. T. Madigan, K. S. Bender, D. H. Buckley, W. M. Sattley, D. A. Stahl, Brock Biology of Microorganisms, 15th edition, Pearson Education, 2019.

[26] N. Mizoguchi, P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), 851–875.

[27] M. Morikawa, Beneficial biofilm formation by industrial bacteria Bacillus subtilis and related species, Journal of Bioscience and Bioengineering, 101 (2006), 1–8.

[28] M. Nakao, R. Koyanagi, Existence of classical periodic solutions of semilinear parabolic equations with the Neumann boundary condition, Funkcial. Ekvac., 28 (1985) 213–219.
[29] Y. Peng, Z. Xiang, Global existence and convergence rates to achemotaxis-fluids system with mixed boundary conditions, *J. Differential Equations*, 267 (2019), 1277–1321.

[30] M. Swanson, G. Reguera, M. Schaechter, F. Neidhardt, Microbe, Second edition. Washington, DC: ASM Press, 2016.

[31] Y. Tao, M. Winkler, Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion, *Discrete Contin. Dyn. Syst.*, 32 (2012), 1901–1914.

[32] Y. Tao, M. Winkler, Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion, *Ann. I. H. Poincaré AN*, 30 (2013), 157–178.

[33] I. Tuval, L. Cisneros, C. Dombrowski, C. Wolgemuth, J. Kessler, R. Goldstein, Bacterial swimming and oxygen transport near contact lines, *Proc. Natl. Acad. Sci. USA*, 102 (2005), 2277–2282.

[34] J. L. Vázquez, The Porous Medium Equations: Mathematical Theory. Oxford Mathematical Monographs. Oxford: Oxford University Press, 2006.

[35] H. Vlamakis, Y. Chai, P. Beauregard, R. Losick, R. Kolter, Sticking together: building a biofilm the Bacillus subtilis way, *Nature Reviews Microbiology*, 11 (2013), 157–168.

[36] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, *Communications in Partial Differential Equations*, 37 (2012), 319–351.

[37] M. Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity, *Calc. Var. PDE*, 54 (2015), 3789–3828.

[38] M. Winkler, Global existence and stabilization in a degenerate chemotaxis-Stokes system with mildly strong diffusion enhancement, *J. Differential Equations*, 264 (2018), 6109–6151.

[39] C. Wu, Z. Xiang, Asymptotic dynamics on a chemotaxis-Navier-Stokes system with nonlinear diffusion and inhomogeneous boundary conditions, *Math. Models Methods Appl. Sci.*, 30 (2020), 1325–1374.