CONTINUOUS DEPENDENCE AND OPTIMAL CONTROL OF A DYNAMIC ELASTIC-VISCOPLASTIC CONTACT PROBLEM WITH NON-MONOTONE BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we consider continuous dependence and optimal control of a dynamic elastic-viscoplastic contact model with Clarke subdifferential boundary conditions. Since the constitutive law of elastic-viscoplastic materials has an implicit expression of the stress field, the weak form of the model is an evolutionary hemivariational inequality coupled with an integral equation. By providing some equivalent weak formulations, we prove the continuous dependence of the solution on external forces and initial conditions in the weak topologies. Finally, the existence of optimal solutions to a boundary optimal control problem is established.

1. Introduction. The theory of hemivariational inequalities was introduced in the 1980s and is based on properties of the Clarke subdifferential. Compared with variational inequality, which is based on arguments of monotonicity and convexity, hemivariational inequality can deal with nonsmooth problems with a nonconvex structure ([6]). Hemivariational inequality has been applied to many subjects, especially to contact mechanics. Mathematical results for viscoelastic contact problems in the form of hemivariational inequality can be found in many works, see for example [5, 13, 22, 12, 14, 21].

Different from viscoelastic materials, elastic-viscoplastic materials like rubbers, metals, rocks and so on, can be described by constitutive law of the following form

\[
\sigma(t) = A \varepsilon(u'(t)) + B \varepsilon(u(t)) + \int_0^t G(\sigma(s) - A \varepsilon(u'(s)), \varepsilon(u(s))) ds.
\]  

(1)

Here \( u, \sigma, \varepsilon(u) \) denote the displacement field, the stress tensor and the linearized strain tensor, respectively. Operators \( A, B \) and \( G \) describe the purely viscous, the elastic and the viscoplastic properties of the material, respectively. For simplicity, the dependence on the spatial variable \( x \) is not indicated. We note that if the last term in (1) is dropped, then the constitutive law leads to the classical viscoelastic case. Elastic-isoplastic constitutive law (1) naturally contains the history-dependent term, and has an implicit expression of the stress field \( \sigma \). With a Clarke subdifferential term in the boundary conditions, the weak formulation of

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the elastic-viscoplastic contact model is governed by a history-dependent hemivariational inequality coupled with an integral equation. Thus, discussion here is more challenging.

We also note that if the elastic term $B$ in (1) is dropped, then the constitutive law leads to the viscoplastic case. For the viscoplastic contact in the form of hemivariational inequality, existence and uniqueness results are established in [16, 17], numerical analysis and simulations are derived in [4]. For the elastic-viscoplastic contact in the form of hemivariational inequality, existence and uniqueness results are established in [3, 20, 18], numerical analysis and simulations are derived in [30]. As far as we know, the continuous dependence result of viscoplastic or elastic-viscoplastic contact in the framework of hemivariational inequality has not been investigated.

We need to mention that, in the framework of variational inequality, the continuous dependence result of viscoplastic or elastic-viscoplastic contact can be found in [1, 2, 26, 27, 29] for quasistatic process, and [24] for dynamic process. It is proved in these papers that, the strong convergence of data leads to the strong convergence of solution. However, now we prove that the weak convergence of data leads to the weak convergence of solution. This conclusion is important since we will use it to prove the existence of optimal solutions. What’s more, the Clarke subdifferential term arising in hemivariational inequality is included in the model in this paper. Due to these differences, we use another approach than that in [24] to prove the continuous dependence result of a dynamic elastic-viscoplastic contact problem.

Optimal control for hemivariational inequalities has developed rapidly in the past few decades. It aims to minimize a given cost functional subject to a hemivariational inequality. Literature in this field for static contact models can be found in e.g. [31, 28, 15, 8, 25] with elastic or piezoelectric materials. For dynamic contact models, since the discussion is more challenging, publications are limited, see [11, 23, 19, 7] with viscoelastic or thermoviscoelastic materials. In this paper, we study the optimal control for a dynamic contact model with elastic-viscoplastic materials.

The paper is structured as follows. In Section 2, we present a dynamic elastic-viscoplastic contact problem in classical and weak forms, and recall the unique solvability. Then we derive some equivalent weak formulations. In Section 3, a continuous dependence result in weak topologies is proved. Finally, in Section 4, we consider the optimal control via external forces and initial conditions, and establish the existence of optimal solutions.

2. A elastic-viscoplastic contact model. We first recall some notation and introduce the elastic-viscoplastic contact model. Let $X$ be a Banach space. For a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the generalized directional derivative of $\varphi$ at $x \in X$ in the direction $v \in X$ is defined by

$$
\varphi^0(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.
$$

The generalized gradient of $\varphi$ at $x$ is a subset of a dual space $X^*$ given by $\partial \varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X \}$. The function $\varphi$ is called regular (in the sense of Clarke) at $x \in X$, if for all $v \in X$ the one-sided directional derivative $\varphi'(x; v)$ exists and satisfies $\varphi^0(x; v) = \varphi'(x; v)$ for all $v \in X$ ([6]).

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with Lipschitz continuous boundary $\Gamma$. The boundary $\Gamma$ is divided into three mutually disjoint measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$ such that $\text{meas}(\Gamma_1) > 0$. Since the unit outward normal vector exists a.e.
on $\Gamma$, we denote it by $\nu = (\nu_i) \in \mathbb{R}^d$. For a vector field $\mathbf{v}$, the normal and tangential components of $\mathbf{v}$ are $v_n = \mathbf{v} \cdot \nu$ and $v_t = \mathbf{v} - v_n \nu$, respectively. Similarly, for a tensor field $\sigma : \Omega \to \mathbb{S}^d$, the normal and tangential components are $\sigma_n = (\mathbf{\sigma} \nu) \cdot \nu$ and $\sigma_t = \mathbf{\sigma} \nu - \sigma_n \nu$, respectively.

We assume that the contact process is dynamic and we study it in the time interval $[0, T]$, with $0 < T < +\infty$. Let $Q = \Omega \times (0, T)$. The classical formulation of the contact problem is as follows.

**Problem.** Find a displacement field $\mathbf{u} : Q \to \mathbb{R}^d$ and a stress field $\mathbf{\sigma} : Q \to \mathbb{S}^d$ such that

\[
\mathbf{\sigma}(t) = A\varepsilon(\mathbf{u}'(t)) + B\varepsilon(\mathbf{u}(t)) + \int_0^t G(\mathbf{\sigma}(s) - A\varepsilon(\mathbf{u}'(s)), \varepsilon(\mathbf{u}(s)))ds \quad \text{in} \ Q, \tag{2}
\]

\[
\mathbf{u}''(t) - \text{Div} \mathbf{\sigma}(t) = f_0(t) \quad \text{in} \ Q, \tag{3}
\]

\[
\mathbf{u}(t) = 0 \quad \text{on} \ \Gamma_1 \times (0, T), \tag{4}
\]

\[
\mathbf{\sigma}(t)\nu = f_2(t) \quad \text{on} \ \Gamma_2 \times (0, T), \tag{5}
\]

\[
-\sigma_n(t) \in \partial j_p(u'_n(t)) \quad \text{on} \ \Gamma_3 \times (0, T), \tag{6}
\]

\[
\mathbf{\sigma}_t(t) = 0 \quad \text{on} \ \Gamma_3 \times (0, T), \tag{7}
\]

\[
\mathbf{u}(0) = \mathbf{u}_0, \ \mathbf{u}'(0) = \mathbf{w}_0 \quad \text{in} \ \Omega. \tag{8}
\]

Now we briefly comment on Problem $P$, and more details can be found in [20]. Equation (2) represents the elastic-viscoplastic constitutive law. Recall that the deformation operator $\varepsilon : H^1(\Omega; \mathbb{R}^d) \to L^2(\Omega; \mathbb{S}^d)$ is defined by:

\[
\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{ij} + u_{ji}),
\]

where index following comma indicates a partial derivative. Equation (3) is the normalized equilibrium equation for the dynamic process, with the divergence operator $\text{Div} \mathbf{\sigma} = (\sigma_{ij,j})$. Conditions (4) and (5) are displacement and traction boundary conditions, respectively. Boundary conditions (6)- (7) are used to model the frictionless contact with normal damped response. Initial conditions are given by (8).

In order to present the weak form of Problem $P$, we introduce the function spaces and list the hypotheses. Hilbert spaces $V$ and $H$ are defined with their inner products in the following:

\[
V = \{ v = (v_i) \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \Gamma_1 \}, \quad (\mathbf{u}, v)_V = \int_\Omega \varepsilon(\mathbf{u}) \cdot \varepsilon(v) \, dx,
\]

\[
H = \{ \tau = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) \}, \quad (\tau, \sigma)_H = \int_\Omega \tau \cdot \sigma \, dx.
\]

The associated norms in $V$ and $H$ are denoted respectively by $\| \cdot \|_V$ and $\| \cdot \|_H$. Completeness of the space $(V, \| \cdot \|_V)$ follows from the use of Korn’s inequality, which is allowed under the assumption $\text{meas}(\Gamma_1) > 0$. The duality pairing of $V$ and its dual $V^*$ is denoted by $(\cdot, \cdot)$.

Note that $V$ is a closed subspace of $H^1(\Omega; \mathbb{R}^d)$. Let $H = L^2(\Omega; \mathbb{R}^d)$ and $Z = H^p(\Omega; \mathbb{R}^d)$ with $p \in (1/2, 1)$. It is well known that

\[
V \subset Z \subset H \subset Z^* \subset V^*
\]
continuously and $V \subset Z$ compactly. Denoting by $i : V \to Z$ the embedding, and denoting by $c_e$ the embedding constant of $V$ into $Z$. Denoting by $\gamma_0 : Z \to L^2(\Gamma_3; \mathbb{R}^d)$ and $\gamma : V \to L^2(\Gamma_3; \mathbb{R}^d)$ the trace operators. Then $\gamma v = \gamma_0(i v)$ for all $v \in V$. Let $\|\gamma_0\|$ be the norm of the trace in $L(Z, L^2(\Gamma_3, \mathbb{R}^d))$. Let $\gamma^*$ be the adjoint operator of $\gamma$.

We introduce the Bochner-Lebesgue space $\mathcal{V} = L^2(0, T; V)$. It follows from Theorem 2.37 in [22] that $(L^2(0, T; V))^\ast \cong L^2(0, T; V^*)$, and here we denote the space simply by $\mathcal{V}^*$. In the following, when there is no confusion, we also use $\langle \cdot, \cdot \rangle$ to denote the duality pairing of $\mathcal{V}$ and its dual $\mathcal{V}^*$. Then we introduce the Bochner-Sobolev space $\mathcal{W} = \{v \in \mathcal{V} : v' \in \mathcal{V}^*\}$, where the time derivative is understood in the sense of vector valued distributions. Endowed with the norm $\|v\|_\mathcal{W} = \|v\|_\mathcal{V} + \|v'\|_\mathcal{V}^*$, the space $\mathcal{W}$ becomes a separable reflexive Banach space ([19]). Recall that the embeddings $\mathcal{W} \subset C(0, T; H)$ and $W^{1,2}(0, T; V) \subset C(0, T; V)$ are continuous ([11]).

We need the following hypotheses on the data of Problem P.

\[
\begin{align*}
H(A) & \quad A : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that:} \\
(i) & \quad A(\cdot, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d; \\
(ii) & \quad A(x, \cdot) \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } x \in \Omega; \\
(iii) & \quad (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}^2 \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\
& \quad \text{a.e. } x \in \Omega \text{ with } m_A > 0; \\
(iv) & \quad \|A(x, \varepsilon)\|_{\mathbb{S}^d} \leq \tau_0(x) + \tau_1 \|\varepsilon\|_{\mathbb{S}^d} \text{ for all } \varepsilon \in \mathbb{S}^d, \text{ a.e. } x \in \Omega \\
& \quad \text{with } \tau_0 \in L^2(\Omega), \tau_0 \geq 0 \text{ and } \tau_1 > 0; \\
(v) & \quad A(x, 0) = 0 \text{ for a.e. } x \in \Omega.
\end{align*}
\]

\[
\begin{align*}
H(B) & \quad B : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that:} \\
(i) & \quad B(\cdot, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d; \\
(ii) & \quad \|B(x, \varepsilon_1) - B(x, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_B \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega \\
& \quad \text{with } L_B > 0; \\
(iii) & \quad B(\cdot, 0) \in L^2(\Omega; \mathbb{S}^d).
\end{align*}
\]

\[
\begin{align*}
H(G) & \quad G : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that:} \\
(i) & \quad G(\cdot, \sigma, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \sigma, \varepsilon \in \mathbb{S}^d; \\
(ii) & \quad \|G(x, \sigma_1, \varepsilon_1) - G(x, \sigma_2, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_G (\|\sigma_1 - \sigma_2\|_{\mathbb{S}^d} + \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}) \\
& \quad \text{for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega \text{ with } L_G > 0; \\
(iii) & \quad G(\cdot, 0, 0) \in L^2(\Omega; \mathbb{S}^d).
\end{align*}
\]

\[
\begin{align*}
H(j_\nu) & \quad j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that:} \\
(i) & \quad j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and } j_\nu(\cdot, 0) \in L^1(\Gamma_3); \\
(ii) & \quad j_\nu(x, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } x \in \Gamma_3; \\
(iii) & \quad |\partial j_\nu(x, r)| \leq c_{j_\nu}(1 + |r|) \text{ for all } r \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \text{ with } c_{j_\nu} > 0; \\
(iv) & \quad j^0_\nu(x, r; -r) \leq c_{j_\nu}(1 + |r|) \text{ for all } r \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \text{ with } c_{j_\nu} > 0; \\
(v) & \quad (r^*_i - r^*_j)(r_i - r_j) \geq -m_j |r_i - r_j|^2 \text{ for all } r_i, r_j \in \mathbb{R}, \quad r^*_i \in \partial j_\nu(x, r_i), \\
& \quad i = 1, 2, \text{ a.e. } x \in \Gamma_3 \text{ with } m_j \geq 0; \\
(vi) & \quad \text{either } j_\nu(x, \cdot) \text{ or } -j_\nu(x, \cdot) \text{ is regular on } \mathbb{R} \text{ for a.e. } x \in \Gamma_3.
\end{align*}
\]
H(f): The densities of body forces and surface external forces satisfy:
\[ f_0 \in L^2(0, T; H), \quad f_2 \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^n)). \]

(H0): The initial values \( u_0 \in V, w_0 \in H \).

Note that the relaxed monotonicity condition \( H(j_\nu)(v) \) is equivalent to
\[ j_\nu^0(x, r_1; r_2 - r_1) + j_\nu^0(x, r_2; r_1 - r_2) \leq m_\nu |r_1 - r_2|^2 \]
for all \( r_1, r_2 \in \mathbb{R} \) and a.e. \( x \in \Omega \). For examples of functions satisfying \( H(j_\nu) \), we refer to [21]. Define the function \( f : (0, T) \to V^* \) by
\[ (f(t), v) = (f_0(t), v)_H + (f_2(t), v)_{L^2(\Gamma_2; \mathbb{R}^n)} \]
for all \( v \in V \) and a.e. \( t \in (0, T) \). Following [20], we have the weak formulation of Problem \( P \) and the unique solvability result.

**Problem.** \( P_V \) Find a displacement field \( u : [0, T] \to V \) and a stress field \( \sigma : [0, T] \to H \) such that
\[ \sigma(t) = A \epsilon(u'(t)) + B \epsilon(u(t)) \]
\[ + \int_0^t G(\sigma(s) - A \epsilon(u'(s)), \epsilon(u(s)))ds \quad \text{a.e. } t \in (0, T), \]
\[ (u''(t), v) + (\sigma(t), \epsilon(v))_H + \int_{\Gamma_3} j_\nu^0(u'_\nu(t); v_\nu) d\Gamma \geq (f(t), v) \quad \text{for all } v \in V, \text{ a.e. } t \in (0, T), \]
\[ u(0) = u_0, \quad u'(0) = w_0. \]

**Theorem 2.1.** Assume \( H(A), H(B), H(G), H(j_\nu), H(f), (H_0) \) and
\[ m_A > m_\nu c_\nu^2 \|\gamma_0\|^2. \]

Then Problem \( P_V \) has a unique solution which satisfies
\[ u \in W^{1,2}(0, T; V) \cap C^1(0, T; H), \quad u'' \in V^*, \]
\[ \sigma \in L^2(0, T; H), \quad \text{Div } \sigma \in V^*. \]

In this paper, we always assume that the hypotheses in Theorem 2.1 hold.

Finally, we derive some equivalent forms to Problem \( P_V \), which will be used in the next section. Different from other materials, the elastic-viscoplastic constitutive law leads to a coupled system. Hence, we need to introduce an auxiliary operator \( M u \), which maps from \([0, T]\) to \( H \). The following lemma is concluded from Lemma 4.5 in [20]. In fact, the operator \( M u \) is exactly the \( \eta^* \in L^2(0, T; H) \) in [20].

**Lemma 2.2.** For the \( u \) solving Problem \( P_V \), there exists a unique function \( M u \in L^2(0, T; H) \) such that
\[ (M u)(t) = B \epsilon(u(t)) + \int_0^t G((M u)(s), \epsilon(u(s)))ds \]
for a.e. \( t \in (0, T) \). Moreover, (9) and (10) in Problem \( P_V \) can be equivalently formulated as
\[ \sigma(t) = A \epsilon(u'(t)) + (M u)(t) \quad \text{a.e. } t \in (0, T), \]
\[ (u''(t), v) + (A \epsilon(u'(t)), \epsilon(v))_H + ((M u)(t), \epsilon(v))_H + \int_{\Gamma_3} j_\nu^0(u'_\nu(t); v_\nu) d\Gamma \] \[ \geq (f(t), v) \quad \text{for all } v \in V, \text{ a.e. } t \in (0, T). \]
We further define the operator \( A : V \to V^* \) by
\[
\langle Au, v \rangle = \langle A\varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \text{ for all } u, v \in V.
\]
For the sake of simplicity, \( A \) also denotes the Nemitsky operator from \( V \) to \( V^* \), given by
\[
\langle (Au)(t), v \rangle = \langle A\varepsilon(u(t)), \varepsilon(v) \rangle_{\mathcal{H}},
\]
for all \( u \in V, v \in V, \text{ a.e. } t \in (0, T). \) Define the operator \( M : V \to V^* \) by
\[
\langle (Mu)(t), v \rangle = \langle (\mathcal{M}u)(t), \varepsilon(v) \rangle_{\mathcal{H}},
\]
for all \( u \in V, v \in V, \text{ a.e. } t \in (0, T). \) Define the functional \( J : L^2(\Gamma_3; \mathbb{R}^d) \to \mathbb{R} \) by
\[
J(v) = \int_{\Gamma_3} j_\nu(v_\nu) d\Gamma
\]
for all \( v \in L^2(\Gamma_3; \mathbb{R}^d). \) The following useful properties can be found in Proposition 3.37 and Theorem 3.47 in [22].

**Lemma 2.3.** Assume \( j_\nu \) satisfies \( H(j_\nu) \) and \( J \) is defined by \( (18) \). Then

(i) \( J(\cdot) \) is locally Lipschitz for all \( u \in L^2(\Gamma_3; \mathbb{R}^d); \)

(ii) \( \|\xi\|_{L^2(\Gamma_3; \mathbb{R}^d)} \leq \bar{c}(1 + \|u\|_{L^2(\Gamma_3; \mathbb{R}^d)}) \) for all \( \xi \in \partial J(u), u \in L^2(\Gamma_3; \mathbb{R}^d) \)
with \( \bar{c} = \sqrt{2}c_{j_0} \max\{\sqrt{\text{meas}(\Gamma_3)}, 1\}; \)

(iii) \( J^0(u; v) = \int_{\Gamma_3} J^0(u_\nu; v_\nu) d\Gamma \) for all \( u, v \in L^2(\Gamma_3; \mathbb{R}^d); \)

(iv) \( \partial(J \circ \gamma)(u) = \gamma^* \partial J(\gamma(u)) \) for all \( u \in V. \)

Now we get the first equivalent form.

**Problem.** \( P_V^1. \) Find \( u : [0, T] \to V \) and \( \sigma : [0, T] \to \mathcal{H} \) such that

\[
\sigma(t) = A\varepsilon(u'(t)) + (Mu)(t) \quad \text{a.e. } t \in (0, T),
\]

\[
u''(t) + Au'(t) + (Mu)(t) + \gamma^* \partial J(\gamma u'(t)) \ni f(t) \quad \text{a.e. } t \in (0, T),
\]

\[
u(0) = u_0, \quad u'(0) = w_0.
\]

From Lemma 2.2 and Lemma 2.3(iii), every solution to Problem \( P_V^1 \) is a solution to Problem \( P_V. \) Conversely, let \( (u, \sigma) \) be the unique solution to Problem \( P_V. \) From Lemma 2.2 and Lemma 2.3(iii),

\[
\langle f(t) - \nu''(t) - Au'(t) - (Mu)(t), v \rangle \leq \int_{\Gamma_3} j_\nu^0(u_\nu'(t); v_\nu) = J^0(\gamma u'(t); \gamma v)
\]

holds for all \( v \in V \) and a.e. \( t \in (0, T). \) By Lemma 2.3(iv), we obtain

\[
f(t) - \nu''(t) - Au'(t) - (Mu)(t) \in \partial(J \circ \gamma)(u'(t)) = \gamma^* \partial J(\gamma u'(t)), \text{ a.e. } t \in (0, T),
\]

which implies (20). Thus, \( (u, \sigma) \) is also a solution to \( P_V^1. \)

Furthermore, denote the Nemitsky operator corresponding to \( \gamma \) by the same notation, and we have its adjoint \( \gamma^* : L^2(0, T; L^2(\Gamma_3; \mathbb{R}^d)) \to V^*. \) Then Problem \( P_V^1 \) can be equivalently written as the following Problem \( P_V^2. \)
Problem. $P^2_V$. Find $u \in V$ with $u' \in W$ and $\sigma \in L^2(0,T;H)$, such that there exists $\xi \in L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))$ satisfying
\[
\sigma = A\varepsilon(u') + Mu \quad \text{in } L^2(0,T;H),
\]
\[
u'' + Au' + Mu + \gamma^* \xi = f \quad \text{in } V^*,
\]
\[
\xi(t) \in \partial J(\gamma u'(t)) \quad \text{a.e. } t \in (0,T),
\]
\[
u(0) = u_0, \quad \nu'(0) = w_0.
\]

Remark 1. As stated before, $V^* = (L^2(0,T;V))^*$. Thus, (22) and (23) are understood in the following sense.
\[
\int_0^T (\sigma(t) - A\varepsilon(u'(t)) - (Mu)(t), \tau(t))_{H'} dt = 0 \quad \forall \tau \in L^2(0,T;H),
\]
\[
\int_0^T \langle u''(t) - Au'(t) - (Mu)(t) - \gamma^* \xi(t) - f(t), v(t) \rangle_{V^* \times V} dt = 0 \quad \forall v \in V.
\]

It is clear that every solution to Problem $P^1_V$ is a solution to Problem $P^2_V$, with the regularity of the solution presented in Theorem 2.1. For the boundedness of $\xi$ in $L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))$, since $u' \in V$, we can directly apply Lemma 2.3(ii) to get
\[
\|\xi\|_{L^2(0,T;L^2(\Gamma_3;\mathbb{R}^d))}^2 = \int_0^T \|\xi(x,t)\|_{L^2(\Gamma_3;\mathbb{R}^d)}^2 dt \\
\leq 2\tau^2 \int_0^T (1 + \|\gamma u'(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)}^2) dt \\
\leq 2\tau^2 T + 2\tau^2 \|\gamma\|_V^2 \|u'\|_V^2 \\
\leq c.
\]

Conversely, let $(u,\sigma)$ be the solution to Problem $P^2_V$. By the skills in [9], we introduce $\Phi(x,t) = \phi(t)v(x)$ with $\phi \in C_0^\infty(0,T)$ and $v \in V$. Multiply (23) with $\Phi$ and integrate on $(0,T)$, then
\[
\int_0^T (u''(t) + Au'(t) + (Mu)(t) + \gamma^* \xi(t) - f(t), v) dt = 0
\]
holds for all $\phi \in C_0^\infty(0,T), v \in V$. Thus,
\[
\langle u''(t) + Au'(t) + (Mu)(t) + \gamma^* \xi(t) - f(t), v \rangle = 0
\]
holds for all $v \in V$ and a.e. $t \in (0,T)$. It leads to
\[
u''(t) + Au'(t) + (Mu)(t) + \gamma^* \xi(t) = f(t), \quad \text{a.e. } t \in (0,T),
\]
which is (20). Similarly, from (22), we have
\[
\int_0^T \|\sigma(t) - A\varepsilon(u'(t)) - (Mu)(t)\|_H^2 dt = 0.
\]
It leads to
\[
\sigma(t) - A\varepsilon(u'(t)) - (Mu)(t) = 0 \quad \text{a.e. } t \in (0,T),
\]
which is (19). Hence, $(u,\sigma)$ is also a solution to $P^1_V$. 
3. Continuous dependence result. In this section, we provide the continuous dependence result of the solution to Problem $P_V$ on $f_2$, $u_0$ and $w_0$ in the weak topologies.

We first establish a boundedness result for the solution. Everywhere in the sequel, $c$ denotes a general positive constant, whose values may change in different inequalities. We also note that the condition ‘a.e. $t \in (0, T)$’ is not always indicated in order to simplify the notation.

**Lemma 3.1.** If $(u, \sigma)$ is the unique solution to Problem $P_V$, then there exists a constant $c > 0$ such that
\[
\|u\|_V + \|u'\|_W + \|\sigma\|_{L^2(0, T; \mathcal{H})} \leq c(1 + \|u_0\|_V + \|w_0\|_H + ||f||_V). \tag{27}
\]

**Proof.** Since $u(t) = \int_0^t u'(s)ds + u_0$, then
\[
\|u(t)\|_V \leq \int_0^t \|u'(s)\|_V ds + \|u_0\|_V, \tag{28}
\]
and
\[
\|u(t)\|_V^2 \leq c \int_0^t \|u'(s)\|_W^2 ds + c\|u_0\|_V^2. \tag{29}
\]

From (9), we have
\[
\|\sigma(t)\|_H \leq ||A\varepsilon(u'(t))||_H + ||B\varepsilon(u(t))||_H + \int_0^t ||G(\sigma(s) - A\varepsilon(u'(s)), \varepsilon(u(s)))||_H ds. \tag{30}
\]

From H(A),
\[
||A\varepsilon(u'(t))||_H = \left( \int_\Omega |A(x, \varepsilon(u'(x, t)))|^2 dx \right)^{1/2} \leq \left( \int_\Omega (|a_0(x)|^2 + 2|\pi_1\varepsilon(u'(x, t))|^2 dx \right)^{1/2} \leq \left( 2 \int_\Omega |a_0(x)|^2 dx + 2\pi_1 \|\varepsilon(u'(t))\|_{L^2}^2 \right)^{1/2} \leq c + c\|u'(t)\|_V.
\]

Similarly, from H(B) and (28),
\[
||B\varepsilon(u(t))||_H \leq ||B0||_H + L_B\|\varepsilon(u(t))\|_H \leq c + c\|u(t)\|_V \leq c + c\int_0^t \|u'(s)\|_V ds + c\|u_0\|_V.
\]

From H(G), H(A) and (28),
\[
\int_0^t ||G(\sigma(s) - A\varepsilon(u'(s)), \varepsilon(u(s)))||_H ds \leq \int_0^t ||G(0, 0)||_H ds + \int_0^t L_G \left( ||\sigma(s)||_H + ||A\varepsilon(u'(s))||_H + ||u(s)||_V \right) ds \leq c + c\int_0^t \left( ||\sigma(s)||_H + ||u'(s)||_V \right) ds + c\int_0^t \|u(s)\|_V ds \leq c + c\int_0^t \left( ||\sigma(s)||_H + ||u'(s)||_V \right) ds + c\|u_0\|_V.
\]
Hence, (30) leads to
\[ \|\sigma(t)\|_H \leq c + c\|u'(t)\|_V + c\|u_0\|_V + c \int_0^t (\|\sigma(s)\|_H + \|u'(s)\|_V)ds. \]

Taking the squares of both sides and integrating over \((0, \tau)\) with \(\tau \in (t, T)\), we obtain
\[ \int_0^\tau \|\sigma(t)\|^2_H dt \leq c + c \int_0^\tau \|u'(t)\|^2_V dt + c\|u_0\|^2_V + c \left( \int_0^\tau \|\sigma(s)\|^2_H ds + \int_0^t \int_0^t \|u'(s)\|^2_V dsdt \right). \]  (31)

Now we combine (9) and (10) with \(v = -u'(t)\) to get
\[ \langle u''(t), u'(t) \rangle + (A e(u'(t)), \varepsilon(u'(t)))_H \leq -(B e(u(t)), \varepsilon(u'(t)))_H - \left( \int_0^t \langle G(\sigma(s) - A e(u'(s)), \varepsilon(u(s)))ds, \varepsilon(u'(t)) \rangle_H \right) + \int_{\Gamma_3} j^0_\nu(u'_\nu(t); -u'_\nu(t))d\Gamma + \langle f(t), u'(t) \rangle. \]  (32)

From \(H(A)\),
\[ m_A \|u'(t)\|_V^2 \leq (A e(u'(t)), \varepsilon(u'(t)))_H. \]

From \(H(j_\nu)\),
\[ \int_{\Gamma_3} j^0_\nu(u'_\nu(t); -u'_\nu(t))d\Gamma \leq \int_{\Gamma_3} c_{j1} + c_{j1}|u'_\nu(t)|d\Gamma \leq c + c\|\gamma\|\|u'(t)\|_V. \]

Applying them to (32), we have
\[ \langle u''(t), u'(t) \rangle + m_A \|u'(t)\|_V^2 \leq \|B e(u(t))\|_H \|u'(t)\|_V + \int_0^t \|G(\sigma(s) - A e(u'(s)), \varepsilon(u(s)))\|_H ds \|u'(t)\|_V \\
+ c + c\|u'(t)\|_V + \|f(t)\|_V. \|u'(t)\|_V \leq c (1 + \int_0^t (\|\sigma(s)\|_H + \|u'(s)\|_V)ds + \|u_0\|_V + \|f(t)\|_V)\|u'(t)\|_V + c. \]

By Cauchy-Schwarz inequality, we can obtain
\[ \langle u''(t), u'(t) \rangle + \|u'(t)\|_V^2 \leq c + c\|f(t)\|_V^2 + c\|u_0\|_V^2 + c\int_0^t \|\sigma(s)\|_H^2 ds + \int_0^t \|u'(s)\|_V^2 ds. \]

Integrating the above inequality over \((0, \tau)\) with \(\tau \in (t, T)\), it leads to
\[ \|u'(\tau)\|_H^2 + \int_0^\tau \|u'(t)\|_V^2 dt \leq c + c\|f\|_V^2 + c\|u_0\|_H^2 + c\|u_0\|_V^2 + c \left( \int_0^\tau \int_0^t \|\sigma(s)\|_H^2 dsdt + \int_0^t \int_0^t \|u'(s)\|_V^2 dsdt \right). \]  (33)

Combining (31), (33) and using Gronwall argument, the routine manipulations yield
\[ \|u\|_{C(0,T;H)} + \|u\|_V + \|\sigma\|_{L^2(0,T;H)} \leq c + c\|f\|_V + c\|u_0\|_H + c\|u_0\|_V. \]  (34)
From (29) and (34), the boundedness for \( \| u \|_V \) can be obtained by
\[
\| u \|_V^2 = \int_0^T \| u(t) \|_V^2 dt \\
\leq c \int_0^T \left( \int_0^t \| u'(s) \|_V^2 ds + \| u_0 \|_V^2 \right) dt \\
\leq cT \| u' \|_V^2 + cT \| u_0 \|_V^2.
\]
(35)

What’s more, from (22), (23), (26) and (34), the boundedness for \( \| u'' \|_{V^*} \) can be obtained by
\[
\| u'' \|_{V^*} \leq \| A u' \|_{V^*} + \| M u \|_{V^*} + \| \gamma^* \xi \|_{V^*} + \| f \|_{V^*} \\
\leq c + c \| f \|_{V^*} + c \| w_0 \|_H + c \| u_0 \|_V.
\]
(36)

The proof is complete.

Next, we discuss the continuous dependence result in the weak topologies. Weakly-weakly continuous hypotheses are considered.

\( (H)_1 \)
\[
\begin{align*}
(i) \text{ if } u_n & \rightharpoonup u \text{ weakly in } V, \text{ then } Au_n \rightharpoonup Au \text{ weakly in } H; \\
(ii) \text{ if } u_n & \rightharpoonup u \text{ weakly in } V, \text{ then } Bu_n \rightharpoonup Bu \text{ weakly in } H; \\
(iii) \text{ if } \sigma_n & \rightharpoonup \sigma \text{ weakly in } H, \text{ then } G(\sigma_n, \tau_n) \rightharpoonup G(\sigma, \tau) \text{ weakly in } H.
\end{align*}
\]

We mention that the following classical example of the elastic-viscoplastic constitutive law satisfies hypotheses \( (H)_1 \). The viscosity tensor \( A \) and the elasticity tensor \( B \) are linear operators characterized by:
\[
(A \tau)_{ij} = \mu_1 (\tau_{11} + \tau_{22}) \delta_{ij} + \mu_2 \tau_{ij}, \quad 1 \leq i, j \leq 2,
\]
\[
(B \tau)_{ij} = \frac{E \kappa}{1 - \kappa^2} (\tau_{11} + \tau_{22}) \delta_{ij} + \frac{E}{1 + \kappa} \tau_{ij}, \quad 1 \leq i, j \leq 2.
\]

The coefficients \( \mu_1 \) and \( \mu_2 \) are viscosity constants, \( E \) and \( \kappa \) are Young’s modulus and Poisson’s ratio of the material, respectively, \( \delta_{ij} \) denotes the Kronecker symbol. The viscoplastic tensor \( G \) is of the Perzyna type:
\[
G(\sigma, \varepsilon) = -\frac{1}{2\lambda} A(\sigma - P_K \sigma),
\]
where \( \lambda > 0 \) is a constant and \( P_K \) is the orthogonal projection operator over the convex subset \( K \subset S^2 \). The subset \( K \) is given by
\[
K = \{ \tau \in S^2 : \| \tau \|_{S^2} \leq \sigma_Y \},
\]
where \( \sigma_Y \) is the uniaxial yield stress ([30]).

**Theorem 3.2.** Assume \( (H)_1 \) and
\[
\begin{align*}
\{ f_{2n} \} & \subset L^2(0, T; L^2(T_2; \mathbb{R}^d)), \quad f_{2n} \rightharpoonup f_2 \text{ weakly in } L^2(0, T; L^2(T_2; \mathbb{R}^d)), \quad (37) \\
\{ u_{0n} \} & \subset V, \quad u_{0n} \rightharpoonup u_0 \text{ weakly in } V, \quad (38) \\
\{ w_{0n} \} & \subset H, \quad w_{0n} \rightharpoonup w_0 \text{ weakly in } H. \quad (39)
\end{align*}
\]
Let \{ (u_n, \sigma_n) \}, (u, \sigma) be the unique solutions to Problem \( P_V \) corresponding to \{ (f_{2n}, u_{0n}, w_{0n}) \} and \( (f_2, u_0, w_0) \), respectively. Then

\[
\begin{align}
  u_n & \to u \quad \text{weakly in } V, \\
  u_n' & \to u' \quad \text{weakly in } W, \\
  \sigma_n & \to \sigma \quad \text{weakly in } L^2(0, T; \mathcal{H}).
\end{align}
\]

**Proof.** The unique solvability of Problem \( P_V \) follows from Theorem 2.1. Now we prove the continuous dependence result. Note that, for each \( n \in \mathbb{N} \), we have

\[
\begin{align}
  \sigma_n(t) &= \mathcal{A} (u_n'(t)) + \mathcal{B} (u_n(t)) \\
  & \quad + \int_0^t G(\sigma_n(s), \mathcal{A}(u_n'(s)), \mathcal{B}(u_n(s))) ds \quad \text{a.e. } t \in (0, T),
\end{align}
\]

where the function \( f_n : (0, T) \to V^* \) is defined by

\[
\langle f_n(t), v \rangle = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v ds \quad \text{for all } v \in V, \ t \in (0, T).
\]

By (37), it is clear that \( f_n \to f \) weakly in \( V^* \).

By Lemma 3.1, and the reflexivity of \( W \) and \( L^2(0, T; \mathcal{H}) \), we conclude that, by passing to a sequence if necessary,

\[
\begin{align}
  u_n & \to u \quad \text{weakly in } V, \\
  u_n' & \to u' \quad \text{weakly in } V, \\
  u_n'' & \to u'' \quad \text{weakly in } V^*, \\
  \sigma_n & \to \sigma \quad \text{weakly in } L^2(0, T; \mathcal{H}),
\end{align}
\]

as \( n \to +\infty \). Then we only need to prove that \( (u, \sigma) \) is the unique solution to Problem \( P_V \) corresponding to \( (f_2, u_0, w_0) \).

In fact, from (46)-(49), we have

\[
\begin{align}
  u_n(t) & \to u(t) \quad \text{weakly in } V, \\
  u_n'(t) & \to u'(t) \quad \text{weakly in } V, \\
  u_n''(t) & \to u''(t) \quad \text{weakly in } V^*, \\
  \sigma_n(t) & \to \sigma(t) \quad \text{weakly in } \mathcal{H}
\end{align}
\]

as \( n \to +\infty \), for a.e. \( t \in (0, T) \). What's more, by (47), we have

\[
\gamma u_n'(t) \to \gamma u'(t) \quad \text{in } L^2(\Gamma_3; \mathbb{R}^d), \quad \text{for a.e. } t \in (0, T),
\]

which directly leads to

\[
u_n'(t) \to u'(t) \quad \text{in } L^2(\Gamma_3), \quad \text{for a.e. } t \in (0, T).
\]

From [10], we obtain

\[
\limsup_{n \to \infty} j_0^0(u_n'(t); v_0) \leq j_0^0(u'(t); v_0).
\]
We take the weak limit of (43) and the upper limit of (44) to obtain
\[
\sigma(t) = A\varepsilon(u'(t)) + B\varepsilon(u(t)) \\
+ \int_0^t G(\sigma(s) - A\varepsilon(u'(s)), \varepsilon(u(s))) ds \quad \text{a.e.} \quad t \in (0,T),
\]
\[
\langle u''(t), v \rangle + \langle \sigma(t), \varepsilon(v) \rangle_H + \int_{\Gamma_3} f_\nu'(u'_n(t); v_\nu) d\Gamma \\
\geq \langle f(t), v \rangle \quad \text{for all} \quad v \in V, \text{a.e.} \quad t \in (0,T).
\]

Since we have the continuity of the embedding \( W \subset C(0,T;H) \) and the convergence \( u'_n \rightarrow u' \) weakly in \( W \), we obtain \( u'_n(t) \rightarrow u'(t) \) weakly in \( H \) for all \( t \in [0,T] \). Hence \( u'_n(0) \rightarrow u'(0) \) weakly in \( H \), which leads to \( u'(0) = w_0 \). Similarly, due to the continuity of the embedding \( W^{1,2}(0,T;V) \subset C(0,T;V) \) and the convergence \( u_n \rightarrow u \) weakly in \( W^{1,2}(0,T;V) \), we obtain \( u_n(t) \rightarrow u(t) \) weakly in \( V \) for all \( t \in [0,T] \). Hence \( u_n(0) \rightarrow u(0) \) weakly in \( V \), giving \( u(0) = w_0 \).

Thus, \( (u, \sigma) \) is the solution to Problem \( P_V \) corresponding to \( (f_2, u_0, w_0) \), which finishes the proof. \( \square \)

4. Optimal control problem. In this section, we study an optimal control problem via external forces and initial conditions for a system described by Problem \( P_V \).

Denote the control variable in Problem \( P_V \) by \( q = (f_2, u_0, w_0) \in Q \), where \( Q = L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)) \times V \times H \). For every \( q \in Q \) we introduce the solution set \( S(q) = \{ y \in S : y = y(q) \} \) is a solution of Problem \( P_V \), where \( y = y(q) = (u(q), u'(q), \sigma(q)) \) and \( S = V \times W \times L^2(0,T;H) \). It is known that, for every \( q \in Q \), Problem \( P_V \) admits a unique solution \( y = y(q) \in S \) under the hypotheses of Theorem 2.1.

Given a nonempty admissible set of controls \( Q_{ad} \subset Q \), and an objective functional \( F : Q \times S \rightarrow \mathbb{R} \), we formulate the optimal control problem, which is the main task of this section.

Find a control \( q^* \in Q_{ad} \) and a state \( y^* = y(q^*) \in S \) such that
\[
F(q^*, y^*) = \inf \{ F(q, y) \mid (q, y) \in Q_{ad} \times S(q) \}. \quad (50)
\]

We need the following hypotheses to obtain the existence result of optimal solutions.
\[
\begin{aligned}
H(Q_{ad}) & : Q_{ad} \text{ is a weakly compact subset of } Q ; \\
H(F) & : F \text{ is lower semicontinuous with respect to weak-} (Q \times S) \text{ topology.}
\end{aligned}
\]

Theorem 4.1. If \((H)_1, H(Q_{ad}) \text{ and } H(F) \) hold, then problem (50) has an optimal solution.

Proof. Let \( \{ (q_n, y_n) \} \) be a minimizing sequence for problem (50), that is, \( q_n \in Q_{ad}, y_n \in S(q_n) \) and
\[
\lim_{n \rightarrow \infty} F(q_n, y_n) = \inf \{ F(q, y) \mid (q, y) \in Q_{ad} \times S(q) \}. \quad (51)
\]

By assumption \( H(Q_{ad}) \), we may choose a subsequence \( \{ q_n \} \subset Q_{ad} \) such that
\[
q_n \rightarrow q^* \quad \text{weakly in } Q \text{ with } q^* \in Q_{ad}. \quad (52)
\]

Combined with Theorem 3.2, we have
\[
y_n \rightarrow y^* \quad \text{weakly in } S \text{ with } y_n \in S(q_n) \text{ and } y^* \in S(q^*). \quad (53)
\]
Example 7. Consider the objective functional $F(q^*, y^*)$, leads to
$$F(q^*, y^*) \leq \lim_{n \to \infty} F(q_n, y_n).$$
Thus, from (51), we get $F(q^*, y^*) = \inf \{ F(q, y) \mid (q, y) \in Q_{ad} \times S(q) \}$. It means that problem (50) has an optimal solution.

We conclude this section by providing some examples of the objective functional $F : Q \times S \to \mathbb{R}$ and the admissible set of controls $Q_{ad}$.

Example 6. Consider the objective functional $F : Q \times S \to \mathbb{R}$ of the form
$$F(f_2, u_0, w_0, u, u', \sigma) = \rho_1 \| \sigma - \sigma_d \|_{L^2(0, T; H)} + \rho_2 \| u_0 \|_{V} + \rho_3 \| w_0 \|_{H},$$
with $(f_2, u_0, w_0) \in Q$ and $(u, u', \sigma) \in S$. Here $\rho_i \geq 0$ for $i = 1, 2, 3$ are prescribed weights. With this choice, we look for the initial displacement $u_0 \in V$, the initial velocity $w_0 \in H$, and the external forces $f_2 \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d))$ such that the corresponding stress field $\sigma$ is as close as possible to the desired stress field $\sigma_d \in L^2(0, T; H)$. At the same time, the initial conditions $u_0$ and $w_0$ are chosen to be small.

Example 7. Consider the objective functional $F : Q \times S \to \mathbb{R}$ of the form
$$F(f_2, u_0, w_0, u, u', \sigma) = \rho_4 \int_{\Omega} \| u(x, T) - u_d(x) \|_{S^d} dx$$
$$+ \rho_5 \int_{\Omega} \| u'(x, T) - u'_d(x) \|_{S^d} dx + \rho_6 \| f_2 - f_2^* \|_{L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d))},$$
with $(f_2, u_0, w_0) \in Q$, $(u, u', \sigma) \in S$ and $\rho_i \geq 0$ for $i = 4, 5, 6$. Here we want the displacement at the terminal time $u(\cdot, T)$ and the velocity at the terminal time $v(\cdot, T)$ to be as close as possible to the desired displacement $u_d$ and velocity $u'_d$, respectively. Moreover, the possible external forces $f_2$ should be close to the given forces $f_2^* \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d))$.

Example 8. Given a bound $\bar{r}$. The admissible set of controls is given by
$$Q_{ad} = \{ f_2 \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d)) : \| f_2(t) \|_{L^2(\Gamma_2; \mathbb{R}^d)} \leq \bar{r} \ \text{a.e.} \ in \ [0, T] \} \times V \times H.$$ Consider the objective functional $F : Q \times S \to \mathbb{R}$ of the form
$$F(f_2, u_0, w_0, u, u', \sigma) = \int_{0}^{T} \int_{\Omega} \| \varepsilon(u(x, t)) \|_{S^d}^2 dx dt,$$
with $(f_2, u_0, w_0) \in Q_{ad}$ and $(u, u', \sigma) \in S$. In this case, with constrained external forces $f_2$, we aims to minimize the deformation in the body over the total time interval.

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