Constacyclic Codes over $F_p + vF_p$

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Abstract

In this paper, we study constacyclic codes over $F_p + vF_p$, where $p$ is an odd prime and $v^2 = v$. The polynomial generators of all constacyclic codes over $F_p + vF_p$ are characterized and their dual codes are also determined.

Keywords: Constacyclic code, polynomial generator, generating set in standard form.

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1 Introduction

Since the discovery that many good non-linear codes over finite fields are actually closely related to linear codes over $\mathbb{Z}_4$ via the Gray map (see [7]), codes over finite rings have received a great deal of attention (e.g. see [1] - [6], [10]).

In these studies, most of them are concentrated on the case that the ground rings associated with codes are finite chain rings. However, it turns out that optimal codes over non-chain rings exist. In [9], Yildiz and Karadeniz considered linear codes over the ring $R_1 = F_2 + uF_2 + vF_2 + uvF_2$ with $u^2 = v^2 = 0$ and $uv = vu$; some good binary codes were obtained as the images of cyclic codes over $R_1$ under two Gray maps. In [12], Zhu, Wang and Shi studied the structure and properties of cyclic codes over $F_2 + vF_2$, where $v^2 = v$; the authors showed that cyclic codes over the ring are principally generated. In the subsequent paper [11], Zhu and Wang investigated a class of constacyclic codes over $F_p + vF_p$ with $p$ being an odd prime and $v^2 = v$. It was proved that the image of a $(1 - 2v)$-constacyclic code of length $n$ over $F_p + vF_p$ under the Gray map is a distance-invariant cyclic code of length $2n$ over $F_p$ and $(1 - 2v)$-constacyclic codes over the ring are principally generated. These rings in the mentioned papers are not finite chain rings.

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In this paper, we generalize the results from [11] to all constacyclic codes over $R = F_p + vF_p$, where $v^2 = v$ and $p$ is an odd prime. We characterize the polynomial generators of all constacyclic codes over $F_p + vF_p$, and show that constacyclic codes of any length are principally generated. The dual codes of the constacyclic codes over $R$ are also discussed. The rest sections of this paper are organized as follows. In Section 2, the preliminary concepts and results are provided. In Section 3, the polynomial generators of all constacyclic codes over $F_p + vF_p$ are characterized and their dual codes are also determined.

2 Preliminaries

Let $F_p$ be the finite field of order $p$ and $F_p^*$ the multiplicative group of $F_p$, where $p$ is an odd prime. It is known that $F_p[x]/(x^n - \lambda)$ is a principal ideal ring for any $\lambda \in F_p^*$. If $p(x) + \langle x^n - \lambda \rangle \in F_p[x]/(x^n - \lambda)$, then the ideal generated by $p(x) + \langle x^n - \lambda \rangle$, denoted by $\langle p(x) \rangle$, is the smallest ideal in $F_p[x]/(x^n - \lambda)$ containing $p(x) + \langle x^n - \lambda \rangle$. In addition, we adopt the notation $[g(x)]$ to denote the ideal in $F_p[x]/(x^n - \lambda)$ generated by $g(x) + \langle x^n - \lambda \rangle$ with $g(x)$ being a monic divisor of $x^n - \lambda$; in that case, $g(x)$ is called a generator polynomial.

Throughout this paper, $R$ denotes the commutative ring $F_p + vF_p = \{a + vb | a, b \in F_p\}$ with $v^2 = v$. It turns out that $R$ is a principal ideal ring and has only two non-trivial ideals, namely $\langle v \rangle = \{va | a \in F_p\}$ and $\langle 1 - v \rangle = \{(1 - v)b | b \in F_p\}$. One can easily check that $\langle v \rangle$ and $\langle 1 - v \rangle$ are maximal ideals in $R$, hence $R$ is not a chain ring. Let $R^n$ be the $R$-module of $n$-tuples over $R$. A linear code $C$ of length $n$ over $R$ is an $R$-submodule of $R^n$. For any linear code $C$ of length $n$ over $R$, the dual $C^\perp$ is defined as $\langle C^\perp \rangle = \{u \in R^n | u \cdot w = 0, \text{for any } w \in C\}$, where $u \cdot w$ denotes the standard Euclidean inner product of $u$ and $w$ in $R^n$. Note that $C^\perp$ is linear whether or not $C$ is linear. The Gray map $\phi$ from $R$ to $F_p \oplus F_p$ given by $\phi(c) = (a, a + b)$, is a ring isomorphism, which means that $R$ is isomorphic to the ring $F_p \oplus F_p$. Therefore $R$ is a finite Frobenius ring. If $C$ is linear, then $|C||C^\perp| = |R|^n$ (See [3]).

Let $\theta$ be a unit in $R$. A linear code $C$ of length $n$ over $R$ is called $\theta$-constacyclic if for every $(c_0, c_1, \cdots, c_{n-1}) \in C$, we have $(\theta c_{n-1}, c_0, c_1, \cdots, c_{n-2}) \in C$. It is well known that a $\theta$-constacyclic code of length $n$ over $R$ can be identified with an ideal in the quotient ring $R[x]/\langle x^n - \theta \rangle$ via the $R$-module isomorphism as follows:

$$
R^n \rightarrow R[x]/\langle x^n - \theta \rangle
$$

$$(c_0, c_1, \cdots, c_{n-1}) \mapsto c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \pmod{x^n - \theta}).$$

If $\theta = 1$, $\theta$-constacyclic codes are just cyclic codes and while $\theta = -1$, $\theta$-constacyclic codes are known as negacyclic codes.

Let $A, B$ be codes over $R$. We denote $A \oplus B = \{a + b | a \in A, b \in B\}$. Note that any element $c$ of $R^n$ can be expressed as $c = a + vb = v(a + b) + (1 - v)a$, where $a, b \in F_p^n$. Let $C$ be a linear code of length $n$ over $R$. Define

$$
C_v = \{b \in F_p^n | va + (1 - v)b \in C, \text{for some } a \in F_p^n\},
$$

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and
\[ C_{1-v} = \{ a \in F_p^n \mid va + (1-v)b \in C, \text{for some } b \in F_p^n \}. \]
Obviously, \( C_v \) and \( C_{1-v} \) are linear codes over \( F_p \). By the definition of \( C_v \) and \( C_{1-v} \), we have that \( C \) can be uniquely expressed as \( C = vC_{1-v} \oplus (1-v)C_v \).

It can be routine to check that for any elements \( a \in C_{1-v} \) and \( b \in C_v \), we get \( va + (1-v)b \in C \); in particular, \( |C| = |C_v||C_{1-v}| \).

3 Constacyclic codes over \( R = F_p + vF_p \)

In this section, we let \( R_n = R[x]/\langle x^n - \theta \rangle \) with \( \theta = \lambda + v\mu \) being a unit in \( R \), where \( \lambda \) and \( \mu \) are elements in \( F_p \). As usual, we identify \( R_n \) with the set of all polynomials over \( R \) of degree less than \( n \). Let \( f_1(x), f_2(x), \cdots, f_s(x) \in R_n \). The ideal generated by \( f_1(x), f_2(x), \cdots, f_s(x) \) will be denoted by
\[ \langle f_1(x), f_2(x), \cdots, f_s(x) \rangle. \]

The following lemma characterizes the units in \( R \).

**Lemma 3.1.** Let \( \theta = \lambda + v\mu \) be an element in \( R \), where \( \lambda \) and \( \mu \) are elements in \( F_p \). Then \( \theta = \lambda + v\mu \) is a unit of \( R \) if and only if \( \lambda \neq 0 \) and \( \lambda + \mu \neq 0 \).

**Proof.** (\( \Rightarrow \)) Suppose that \( \theta = \lambda + v\mu \) is a unit of \( R \). Then there exists elements \( a, b \in F_p \) and \( \theta' = a + vb \in R \) such that \( \theta \theta' = 1 \), that is, \( (\lambda + v\mu)(a + vb) = \lambda a + v(\lambda b + \mu a + \mu b) = 1 \). So we have that \( \lambda a = 1 \) and \( (\lambda + \mu)b + \mu a = 0 \), which implies that \( \lambda \neq 0 \) and \( \lambda + \mu \neq 0 \).

(\( \Leftarrow \)) Let \( \theta = \lambda + v\mu \in R \), where \( \lambda \neq 0 \) and \( \lambda + \mu \neq 0 \). Setting \( \theta' = \lambda^{-1} + v[-(\lambda + \mu)^{-1}\mu\lambda^{-1}] \). Then
\[
\theta\theta' = \lambda + v\mu\left[\lambda^{-1} + v\left(-\frac{1}{\lambda + \mu}\mu\lambda^{-1}\right)\right]
= 1 + v[\mu\lambda^{-1} - \mu(\lambda + \mu)^{-1} - \mu(\lambda + \mu)^{-1}\cdot\mu\lambda^{-1}]
= 1 + v[\mu\lambda^{-1} - \mu(\lambda + \mu)^{-1}(1 + \mu\lambda^{-1})]
= 1 + v[\mu\lambda^{-1} - \mu(\lambda + \mu)^{-1}(\mu\lambda^{-1} + \mu\lambda^{-1})]
= 1 + v[\mu\lambda^{-1} - \mu(\lambda + \mu)^{-1}(\lambda + \mu)\lambda^{-1}]
= 1.
\]
Hence \( \theta = \lambda + v\mu \) is a unit of \( R \). \( \square \)

**Theorem 3.2.** Let \( C = vC_{1-v} \oplus (1-v)C_v \) be a linear code of length \( n \) over \( R \). Then \( C \) is a \( \theta \)-constacyclic code of length \( n \) over \( R \) if and only if \( C_v \) and \( C_{1-v} \) are \( \lambda \)-constacyclic and \( (\lambda + \mu) \)-constacyclic codes of length \( n \) over \( F_p \), respectively.

**Proof.** (\( \Rightarrow \)) Let \( (r_0, r_1, \cdots, r_{n-1}) \) be an arbitrary element in \( C_{1-v} \), and let \( (q_0, q_1, \cdots, q_{n-1}) \) be an arbitrary element in \( C_v \). We assume that \( c_i = vr_i + \)}
(1−v)q_i, i = 0, 1, · · · , n−1; hence we get that (c_0, c_1, · · · , c_{n−1}) ∈ C. Since C is a θ-constacyclic code of length n over R, then (θc_{n−1}, c_0, · · · , c_{n−2}) ∈ C. Note that

θc_{n−1} = (λ + vμ)[vr_{n−1} + (1−v)q_{n−1}]
= v[(λ + μ)r_{n−1}] + (1−v)(λq_{n−1}),

then

(θc_{n−1}, c_0, · · · , c_{n−2}) = v[(λ + μ)r_{n−1}, r_0, · · · , r_{n−2}]
+ (1−v)(λq_{n−1}, q_0, · · · , q_{n−2}) ∈ C,

hence ((λ + μ)r_{n−1}, r_0, · · · , r_{n−2}) ∈ C_{1−v} and (λq_{n−1}, q_0, · · · , q_{n−2}) ∈ C_v, which implies that C_v and C_{1−v} are λ-constacyclic and (λ + μ)-constacyclic codes of length n over F_p, respectively.

(⇐) Suppose that C_v and C_{1−v} are λ-constacyclic and (λ + μ)-constacyclic codes of length n over F_p, respectively. Let (c_0, c_1, · · · , c_{n−1}) ∈ C, where c_i = vr_i + (1−v)q_i, i = 0, 1, · · · , n−1. It follows that (r_0, r_1, · · · , r_{n−1}) ∈ C_{1−v} and (q_0, q_1, · · · , q_{n−1}) ∈ C_v. Note that

(θc_{n−1}, c_0, · · · , c_{n−2}) = v[(λ + μ)r_{n−1}, r_0, · · · , r_{n−2}]
+ (1−v)(λq_{n−1}, q_0, · · · , q_{n−2})
∈ vC_{1−v} ⊕ (1−v)C_v,

hence C is a θ-constacyclic code of length n over R.

**Theorem 3.3.** Let C = vC_{1−v} ⊕ (1−v)C_v be a θ-constacyclic code of length n over R. Then C = ⟨vg_{1−v}(x), (1−v)g_v(x)⟩, where g_{1−v}(x) and g_v(x) are the generator polynomials of C_{1−v} and C_v, respectively.

**Proof.** Since C_v and C_{1−v} are λ-constacyclic and (λ + μ)-constacyclic codes of length n over F_p, respectively, we may assume that the generator polynomials of C_v and C_{1−v} are g_v(x) and g_{1−v}(x), respectively. Then vg_{1−v}(x) ∈ vC_{1−v} ⊆ C and (1−v)g_v(x) ∈ (1−v)C_v ⊆ C, hence ⟨vg_{1−v}(x), (1−v)g_v(x)⟩ ⊆ C.

Let f(x) ∈ C. Since C = vC_{1−v} ⊕ (1−v)C_v, then there are s'(x) = g_{1−v}(x)s(x) ∈ C_{1−v} and u'(x) = u(x)g_v(x) ∈ C_v such that f(x) = vs'(x) + (1−v)u'(x) = vg_{1−v}(x)s(x) + (1−v)g_v(x)u(x), where s(x), u(x) ∈ F_p[x] ⊆ R[x]. Hence f(x) ∈ ⟨vg_{1−v}(x), (1−v)g_v(x)⟩. Therefore C ⊆ ⟨vg_{1−v}(x), (1−v)g_v(x)⟩. This gives that C = ⟨vg_{1−v}(x), (1−v)g_v(x)⟩.

**Proposition 3.4.** Let C = vC_{1−v} ⊕ (1−v)C_v be a θ-constacyclic code of length n over R and g_{1−v}(x), g_v(x) are the generator polynomials of C_{1−v} and C_v, respectively. Then |C| = p^{2n−deg(g_{1−v}(x))−deg(g_v(x))}.

**Proof.** Since |C| = |C_v||C_{1−v}|, then |C| = p^{2n−deg(g_{1−v}(x))−deg(g_v(x))}. 

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For the proof of the following theorem we introduce some notations. Since the ring $R$ has two maximal ideals $\langle v \rangle$ and $(1 - v)$ with the same residue field $F_p$, thus we have two canonical projections defined as follows:

$$\sigma : R = F_p + vF_p \longrightarrow F_p$$

$$va + (1 - v)b \longmapsto a;$$

and

$$\tau : R = F_p + vF_p \longrightarrow F_p$$

$$va + (1 - v)b \longmapsto b.$$  

Denote by $r^\sigma$ and $r^\tau$ the images of an element $r \in R$ under these two projections, respectively. These two projections can be extended naturally from $R^n$ to $F_p^n$ and from $R[x]$ to $F_p[x]$.

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$, where $a_i \in R$ for $0 \leq i \leq n - 1$, and we denote

$$f(x)^\sigma = a_0^\sigma + a_1^\sigma x + \cdots + a_{n-1}^\sigma x^{n-1}; f(x)^\tau = a_0^\tau + a_1^\tau x + \cdots + a_{n-1}^\tau x^{n-1}.$$  

Hence $f(x)$ has a unique expression as $f(x) = vf(x)^\sigma + (1 - v)f(x)^\tau$.

For a code $C$ of length $n$ over $R$, $a \in R$. The submodule quotient is a linear code of length $n$ over $R$, defined as follows:

$$(C : a) = \{ r \in R^n | ar \in C \}.$$  

**Theorem 3.5.** Let $C = vC_{1-v} \oplus (1 - v)C_v$ be a $\theta$-constacyclic code of length $n$ over $R$. If $C = \langle vh_1(x), (1 - v)h_2(x) \rangle$, where $h_1(x)$ and $h_2(x) \in F_p[x]$ are monic with $h_1(x) | (x^n - (\lambda + \mu))$ and $h_2(x) | (x^n - \lambda)$, then $C_{1-v} = [h_1(x)]$ and $C_v = [h_2(x)]$, that is, $h_1(x)$ and $h_2(x)$ are the generator polynomials of constacyclic codes $C_{1-v}$ and $C_v$, respectively.

**Proof.** We shall prove the theorem by carrying out the following steps.

Step 1. If $C = vC_{1-v} \oplus (1 - v)C_v$, then $(C : v)^\sigma = C_{1-v}$ and $(C : (1 - v))^\tau = C_v$.

Let $a \in (C : v)$, then $va \in C$. Setting $a = va^\sigma + (1 - v)b$, where $b \in F_p^n$. Hence $va^\sigma = v[va^\sigma + (1 - v)b] = va \in C$. Therefore $a^\sigma \in C_{1-v}$, which implies that $(C : v)^\sigma \subseteq C_{1-v}$. Let $y \in C_{1-v}$, then there exists $z \in F_p^n$ such that $vy + (1 - v)z \in C$. Note that $vy = v[vy + (1 - v)z] \in vC \subseteq C$ and $y = vy + (1 - v)y$, so $y \in (C : v)$ and $y = y^\sigma$, then $y \in (C : v)^\sigma$. Hence $C_{1-v} \subseteq (C : v)^\sigma$. Therefore $(C : v)^\sigma = C_{1-v}$.

Let $c \in (C : (1 - v))$, then $(1 - v)c \in C$. Setting $c = va^\tau + (1 - v)c^\tau$, where $a^\tau \in F_p^n$. Hence $(1 - v)c^\tau = (1 - v)[va^\tau + (1 - v)c^\tau] = (1 - v)c \in C$. Therefore $c^\tau \in C_v$, which implies that $(C : (1 - v))^\tau \subseteq C_v$. Let $y \in C_v$, then there exists $z \in F_p^n$ such that $vz + (1 - v)y \in C$. Note that $(1 - v)y = (1 - v)[vz + (1 - v)y] \in (1 - v)C \subseteq C$ and $y = vz + (1 - v)y$, so $y \in (C : (1 - v))$ and $y = y^\tau$. Hence $C_v \subseteq (C : (1 - v))^\tau$. Therefore $(C : (1 - v))^\tau = C_v$. 

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Step 2. If \( C = \langle vh_1(x), (1 - v)h_2(x) \rangle \), then \((C : v)^0 = [h_1(x)]\) and \((C : (1 - v))^\tau = [h_2(x)]\).

Let \( f(x) \in (C : v) \), then \( vf(x) \in C \). Since \( C = \langle vh_1(x), (1 - v)h_2(x) \rangle \), we have that \( vf(x) = vh_1(x)s(x) + (1 - v)h_2(x)t(x) \), for some \( s(x), t(x) \in R_n \).

Write
\[
f(x) = vf(x)^\sigma + (1 - v)f(x)^\tau, s(x) = vs(x)^\sigma + (1 - v)s(x)^\tau \text{ and } t(x) = vt(x)^\sigma + (1 - v)t(x)^\tau,
\]
where \( f(x)^\sigma, s(x)^\sigma, t(x)^\sigma, t(x)^\tau \in F_p[x] \). Hence
\[
v[f(x)^\sigma + (1 - v)f(x)^\tau]^\tau = vh_1(x)[vs(x)^\sigma + (1 - v)s(x)^\tau]
+ (1 - v)h_2(x)[vt(x)^\sigma + (1 - v)t(x)^\tau].
\]
Thus \( v[f(x)^\sigma + (1 - v)f(x)^\tau]^\tau \), which forces that \( f(x)^\sigma = h_1(x)s(x)^\sigma \).

This shows that \( f(x)^\sigma \in [h_1(x)] \). Therefore \( (C : v)^0 \subseteq [h_1(x)] \). Conversely, if \( f(x) \in [h_1(x)] \), then \( f(x) = h_1(x)u(x) \), for some \( u(x) \in F_p[x] \). Hence \( vf(x) = vh_1(x)u(x) \in \langle vh_1(x), (1 - v)h_2(x) \rangle = C \), which shows that \( f(x) \in (C : v) \); note that \( f(x) = v[f(x) + (1 - v)f(x)] \), so \( f(x) = f(x)^\sigma \). Hence \( f(x) \in (C : v)^0 \). We obtain that \([h_1(x)] \subseteq (C : v)^\tau \). Then we have that \((C : v)^\tau = [h_1(x)] \).

Now we prove the second equality in this step. Let \( f(x) \in (C : (1 - v)) \), then \((1 - v)f(x) \in C \). So we have that \((1 - v)f(x) = vh_1(x)d(x) + (1 - v)h_2(x)q(x) \), for some \( d(x), q(x) \in R_n \). Write
\[
f(x) = vf(x)^\sigma + (1 - v)f(x)^\tau, d(x) = vd(x)^\sigma + (1 - v)d(x)^\tau, q(x) = vq(x)^\sigma + (1 - v)q(x)^\tau,
\]
where \( f(x)^\sigma, d(x)^\sigma, d(x)^\tau, q(x)^\sigma, q(x)^\tau \in F_p[x] \). Thus \((1 - v)f(x)^\tau = vh_1(x)d(x)^\sigma + (1 - v)h_2(x)q(x)^\tau \), which forces that \( f(x)^\tau = h_2(x)q(x)^\tau \). This shows that \( f(x)^\tau \in [h_2(x)] \). Therefore \((C : (1 - v))^\tau \subseteq [h_2(x)] \). Conversely, if \( f(x) \in [h_2(x)] \), then \( f(x) = h_2(x)w(x) \), for some \( w(x) \in F_p[x] \). Hence \((1 - v)f(x) = (1 - v)h_2(x)w(x) \in \langle vh_1(x), (1 - v)h_2(x) \rangle = C \), which shows that \( f(x) \in (C : (1 - v)) \); note that \( f(x) = f(x)^\tau \), hence \( f(x) \in (C : (1 - v))^\tau \). Therefore we obtain that \([h_2(x)] \subseteq (C : (1 - v))^\tau \), and thus \((C : (1 - v))^\tau = [h_2(x)] \).

By the above two steps, we can obtain our desired results. Specially, \( h_1(x) \) and \( h_2(x) \) are the generator polynomials of constacyclic codes \( C_{1-v} \) and \( C_v \), respectively.

**Definition 3.6.** Let \( C = vC_{1-v} \oplus (1 - v)C_v \) be a \( \theta \)-constacyclic code of length \( n \) over \( R \). We say that the set \( S = \{vg_1(x), (1 - v)g_2(x)\} \) is a generating set in standard form for the \( \theta \)-constacyclic code \( C = \langle S \rangle \) if both the following two conditions are satisfied:

1. For each \( i \in \{1, 2\} \), \( g_i(x) \) is either monic in \( F_p[x] \) or equals to 0;
2. If \( g_1(x) \neq 0 \), then \( g_1(x)|(x^n - (\lambda + \mu)) \); if \( g_2(x) \neq 0 \), then \( g_2(x)|(x^n - \lambda) \).

Now combining Theorem 3.3 and 3.5, the following result is obtained.

**Theorem 3.7.** Any nonzero constacyclic code \( C \) over \( R \) has a unique generating set in standard form.

**Corollary 3.8.** Let \( C \) be an ideal in \( R_n \), then there exists a unique polynomial \( g(x) = vg(x)^\sigma + (1 - v)g(x)^\tau \in C \) such that \( C = \langle g(x) \rangle \) with \( g(x)^\sigma \) and \( g(x)^\tau \) being monic in \( F_p[x] \); furthermore, \( g(x) \) is a divisor of \( x^n - \theta \). In particular, \( R_n \) is a principal ideal ring.
Proof. According to Theorem 3.7, we have $C = \langle vg_1 - v(x), (1 - v)g_v(x) \rangle$, where $\{vg_1 - v(x), (1 - v)g_v(x)\}$ is a generating set in standard form for $C$. Let $g(x) = vg_1 - v(x) + (1 - v)g_v(x)$. Note that
\[ vg_1 - v(x) = v[vg_1 - v(x) + (1 - v)g_v(x)], \]
and
\[ (1 - v)g_v(x) = (1 - v)[vg_1 - v(x) + (1 - v)g_v(x)], \]
which imply that $C = \langle g(x) \rangle$. Note that there exist polynomials $r_1 - v(x)$ and $r_v(x)$ in $F_p[x]$ such that
\[ x^n - (\lambda + \mu) = g_1 - v(x) r_1 - v(x), \]
\[ x^n - \lambda = g_v(x) r_v(x). \]
Then we get
\[ x^n - \theta = g(x)[vr_1 - v(x) + (1 - v)r_v(x)]. \]

Finally, we prove the uniqueness of such a polynomial. Suppose that $C = \langle h(x) \rangle$. Write $h(x) = vh(x)^\sigma + (1 - v)h(x)^\tau$, where $h(x)^\sigma$ and $h(x)^\tau$ are monic in $F_p[x]$. In the following we shall prove that $h(x)^\sigma = g_1 - v(x)$ and $h(x)^\tau = g_v(x)$. Since $C = \langle h(x) \rangle$ and $vh(x) \in C$, so $h(x) \in (C : v)$, that is, $h(x)^\sigma \in (C : v)^\sigma = C_1 - v$. Then $g_1 - v(x) | h(x)^\sigma$; similarly, we have that $g_v(x) | h(x)^\tau$. On the other hand, there exists some polynomial $s(x) \in R_n$ such that
\[ vg_1 - v(x) + (1 - v)g_v(x) = [vs(x)^\sigma + (1 - v)s(x)^\tau][vh(x)^\sigma + (1 - v)h(x)^\tau], \]
it follows that $s(x)^\sigma h(x)^\sigma = g_1 - v(x)$ and $s(x)^\tau h(x)^\tau = g_v(x)$. Hence $h(x)^\sigma | g_1 - v(x)$ and $h(x)^\tau | g_v(x)$. Therefore we obtain that $h(x)^\sigma = g_1 - v(x)$ and $h(x)^\tau = g_v(x)$, which is the required results. \( \square \)

Remark 3.9. We mention that, the approach used in the above results also valid when $p = 2$. In particular, Corollary 3.8 yields a generalization of a main result in [12], which showed that cyclic codes over $F_2 + \langle 2 \rangle$ are principally generated. In the following, we shall see that the hypothesis with $p$ being an odd prime is necessary on the discussion of $(1 - 2v)$ or $(-1 + 2v)$-constacyclic codes over $R$.

Now we give the definition of polynomial Gray map over $R_n$. Let $f(x) \in R_n$ with degree less than $n$, then $f(x)$ can be expressed as $f(x) = r(x) + vq(x)$, where $r(x), q(x) \in F_p[x]$ and their degrees are both less than $n$. Let $\theta = \lambda + v\mu \in R^*$. Define the polynomial Gray map as follows:
\[ \phi_\theta : R_n \rightarrow F_p[x]/(x^{2n} - 1) \]
\[ f(x) = r(x) + vq(x) \rightarrow \lambda(\lambda + \mu)q(x) + x^n[-\mu r(x) - (\lambda + \mu)q(x)]. \]
Obviously the above polynomial Gray map $\phi_\theta$ is well-defined. If $\mu \neq 0$, then the map $\phi_\theta$ is bijective.
Hence

\[ \phi(C) \subseteq \langle g_{1-v}(x)g_v(x) \rangle. \]

Proof. Since \( g_{1-v}(x) \) divides \( x^n - (\lambda + \mu) \), and \( g_v(x) \) divides \( x^n - \lambda \), there exists \( q_1(x), q_2(x) \in F_p[x] \) such that

\[ x^n - (\lambda + \mu) = g_{1-v}(x)q_1(x) \quad \text{and} \quad x^n - \lambda = g_v(x)q_2(x). \]

By the proof of Corollary 3.8, we have that \( C = \langle v \rangle \) and \( C = \langle 1 - v \rangle \). Let \( f(x) \) be any element in \( C \). Then \( f(x) = [v \cdot g_{1-v}(x) + (1 - v)g_v(x)]h(x) \), for some \( h(x) \in R_n \). Since \( h(x) \) can be written as \( h(x) = vh(x) + (1 - v)h(x) \), where \( h(x) \), \( h(x) \) is in \( F_p[x] \), it follows that \( f(x) = g_{1-v}(x)h(x) + v[1 - v]g_v(x)h(x) - g_v(x)h(x) \). Then we have that

\[
\phi(f(x)) = \lambda(\lambda + \mu)[g_{1-v}(x)h(x)g_{1-v}(x)h(x)] + x^n[-\mu g_v(x)h(x) - (\lambda + \mu)(g_{1-v}(x)h(x) - g_v(x)h(x))]
= \lambda g_v(x)h(x)g_{1-v}(x)q_1(x) - (\lambda + \mu)g_{1-v}(x)h(x)g_v(x)q_2(x)
= \lambda g_v(x)h(x)g_{1-v}(x)[\lambda h(x)q_1(x) - (\lambda + \mu)h(x)g_v(x)q_2(x)].
\]

Hence \( \phi(C) \subseteq \langle g_{1-v}(x)g_v(x) \rangle \). \( \square \)

**Theorem 3.10.** Let \( C \) be a \( \theta \)-constacyclic code of length \( n \) over \( R \) with a generating set in standard form \( \{v_{g_{1-v}(x)}, (1 - v)g_v(x)\} \). Then

\[ \phi(C) \subseteq \langle g_{1-v}(x)g_v(x) \rangle. \]

**Corollary 3.11.** Let \( \theta = 1 - 2v \) or \( -1 + 2v \) and let \( C \) be a \( \theta \)-constacyclic code of length \( n \) over \( R \) with a generating set in standard form \( \{v_{g_{1-v}(x)}, (1 - v)g_v(x)\} \). Then \( \phi(C) \subseteq \langle g_{1-v}(x)g_v(x) \rangle \).

**Proof.** Note that \( g_{1-v}(x) \) divides \( x^n - (\lambda + \mu) \), and \( g_v(x) \) divides \( x^n - \lambda \), where \( \lambda + \nu = 1 - 2v \) or \( -1 + 2v \), then \( x^n - (\lambda + \mu)(x^n - \lambda) = x^{2n} - 1 \), hence \( g_{1-v}(x)\) divides \( x^{2n} - 1 \), which means that \( g_{1-v}(x) \) divides \( g_{1-v}(x)g_v(x) \) is the generator polynomial for cyclic code \( \langle g_{1-v}(x)g_v(x) \rangle \), that is, \( \langle g_{1-v}(x)g_v(x) \rangle = \langle g_{1-v}(x)g_v(x) \rangle \).

By Theorem 3.10, we have that \( \phi(C) \subseteq \langle g_{1-v}(x)g_v(x) \rangle \). On the other hand, \( \|C\| = \|\phi(C)\| = p^{2n - \deg(g_{1-v}(x)) - \deg(g_v(x))} \) and

\[ ||g_{1-v}(x)g_v(x)|| = p^{2n - \deg(g_{1-v}(x)) - \deg(g_v(x))}. \]

Hence, \( \phi(C) = \langle g_{1-v}(x)g_v(x) \rangle \). \( \square \)

The above Corollary 3.11 shows that the Gray image of a \( \theta \)-constacyclic code over \( R \) under the Gray map \( \phi_\theta \) is a cyclic code of length \( 2n \) over \( F_p \). In order to study the converse part, we now give the corresponding Gray map on \( R^n \). Let \( \theta = \lambda + \nu \mu \),

\[ \phi_\theta : R^n \longrightarrow F_p^{2n} \]

\[ (c_0, c_1, \cdots, c_{n-1}) \longmapsto (\lambda(\lambda + \mu)c_0, \lambda(\lambda + \mu)c_1, \cdots, \lambda(\lambda + \mu)c_{n-1}, \cdots, \lambda(\lambda + \mu)c_{n-1}). \]

8
The dual code

\[-\mu r_0 - (\lambda + \mu)q_0, -\mu r_1 - (\lambda + \mu)q_1, \ldots, -\mu r_{n-1} - (\lambda + \mu)q_{n-1}, \]

where \( c_i = r_i + vq_i, 0 \leq i \leq n - 1. \)

**Lemma 3.12.** Let \( \vartheta = 1 - 2v \) or \( -1 + 2v \) and \( \phi_\theta \) be the Gray map of \( R^n \) into \( F_p^{2n} \). Let \( \alpha \) be the \( \vartheta \)-constacyclic shift of \( R^n \) and \( \beta \) the cyclic shift of \( F_p^{2n} \). Then \( \phi_\theta \alpha = \beta \phi_\theta. \)

**Proof.** We first consider the case when \( \vartheta = -1 + 2v \). Let \( c = (c_0, c_1, \ldots, c_{n-1}) \in R^n \), where \( c_i = r_i + vq_i, 0 \leq i \leq n - 1. \) By the definition of \( \phi_\theta \), we have that

\[ \phi_\theta(c) = (-q_0, -q_1, \ldots, -q_{n-1}, -2r_0 - q_0, -2r_1 - q_1, \ldots, -2r_{n-1} - q_{n-1}). \]

Hence

\[ \beta(\phi_\theta(c)) = (-2r_{n-1} - q_{n-1}, -q_0, \ldots, -q_{n-1}, -2r_0 - q_0, \ldots, -2r_{n-2} - q_{n-2}). \]

On the other hand,

\[
\alpha(c) = (\vartheta c_{n-1}, c_0, \ldots, c_{n-2})
\]

\[
= (-r_{n-1} + v(2r_{n-1} + q_{n-1}), r_0 + vq_0, \ldots, r_{n-2} + vq_{n-2}).
\]

So we obtain

\[ \phi_\theta(\alpha(c)) = (-2r_{n-1} - q_{n-1}, -q_0, \ldots, -q_{n-1}, -2r_0 - q_0, \ldots, -2r_{n-2} - q_{n-2}). \]

Therefore, \( \phi_\theta \alpha = \beta \phi_\theta. \)

The case when \( \vartheta = 1 - 2v \) is [11], Proposition 3.3. \( \square \)

**Theorem 3.13.** Let \( \vartheta = 1 - 2v \) or \( -1 + 2v \). A linear code \( C \) of length \( n \) over \( R \) is a \( \vartheta \)-constacyclic code if and only if \( \phi_\vartheta(C) \) is a cyclic code of length \( 2n \) over \( R. \)

**Proof.** It is obtained by Lemma 3.12. \( \square \)

The following result is routine to check or see the proof of [24, Proposition 2.4].

**Proposition 3.14.** Let \( C \) be a \( \theta \)-constacyclic code of length \( n \) over \( R. \) Then the dual code \( C^\perp \) for \( C \) is a \( \theta^{-1} \)-constacyclic code of length \( n \) over \( R. \)

Let \( g_{1-v}(x)h_{1-v}(x) = x^n - (\lambda + \mu), g_v(x)h_v(x) = x^n - \lambda. \) Let \( \tilde{h}_{1-v}(x) = x^{\deg(h_{1-v}(x))}h_{1-v}(\frac{1}{x}) \) and \( \tilde{h}_v(x) = x^{\deg(h_v(x))}h_v(\frac{1}{x}) \) be the reciprocal polynomials of \( h_{1-v}(x) \) and \( h_v(x) \), respectively. We write \( h_{1-v}^*(x) = \frac{1}{h_{1-v}(0)}\tilde{h}_{1-v}(x) \) and \( h_v^*(x) = \frac{1}{h_v(0)}\tilde{h}_v(x) \).

**Theorem 3.15.** Let \( C = vC_{1-v} \oplus (1-v)C_v \) be a \( \vartheta \)-constacyclic code of length \( n \) over \( R. \) Then \( C^\perp = vC_{1-v}^\perp \oplus (1-v)C_v^\perp. \)
Proof. From Theorem 3.2, $C_{1-v}$ and $C_v$ are constacyclic codes over $F_p$, then $C_{1-v}^\perp$ and $C_v^\perp$ are also constacyclic codes over $F_p$. Let $g_{1-v}(x)$ and $g_v(x)$ are generator polynomials for $C_{1-v}$ and $C_v$, respectively. Then $C_{1-v}^\perp = [h_{1-v}^*(x)]$ and $C_v^\perp = [h_v^*(x)]$. Thus we have that $|C_{1-v}^\perp| = p^{\deg(g_{1-v}(x))}$ and $|C_v^\perp| = p^{\deg(g_v(x))}$.

For any $a \in C_{1-v}^\perp$, $b \in C_v^\perp$ and $c = vr + (1-v)q \in C$, where $r \in C_{1-v}$, $q \in C_v$, we have that
\[
C \cdot (va + (1-v)b) = (vr + (1-v)q) \cdot (va + (1-v)b) = v(r \cdot a) + (1-v)(q \cdot b) = 0,
\]
and hence $vC_{1-v}^\perp \oplus (1-v)C_v^\perp \subseteq C^\perp$.

Furthermore, suppose that $va + (1-v)b = va' + (1-v)b'$, where $a, a' \in C_{1-v}^\perp$ and $b, b' \in C_v^\perp$, then $v(a - a') = (1-v)(b' - b)$, so
\[
v(a - a') = v[v(a - a')] = v[(1-v)(b - b')] = 0.
\]
Hence $a = a'$, which forces $b = b'$. Thus every element $c$ of $vC_{1-v}^\perp \oplus (1-v)C_v^\perp$ has a unique expression as $c = vr + (1-v)q$, where $r \in C_{1-v}^\perp$, $q \in C_v^\perp$. This shows that
\[
|vC_{1-v}^\perp \oplus (1-v)C_v^\perp| = |C_{1-v}^\perp||C_v^\perp| = p^{\deg(g_{1-v}(x)) + \deg(g_v(x))}.
\]

Finally, by Proposition 3.3, $|C| = p^{2n - \deg(g_{1-v}(x)) - \deg(g_v(x))}$. Since $R$ is a Frobenius ring, $|C||C^\perp| = |R|^n$, so
\[
|C^\perp| = \frac{|R|^n}{|C|} = p^{2n - \deg(g_{1-v}(x)) - \deg(g_v(x))} = p^{\deg(g_{1-v}(x)) + \deg(g_v(x))} = |vC_{1-v}^\perp \oplus (1-v)C_v^\perp|.
\]
Note that $vC_{1-v}^\perp \oplus (1-v)C_v^\perp \subseteq C^\perp$ as above, we have that $C^\perp = vC_{1-v}^\perp \oplus (1-v)C_v^\perp$, as required. 

According to the above results and their proofs, we can carry over the results regarding constacyclic codes corresponding to their dual codes.

**Theorem 3.16.** With notations as above. Let $C$ be a $\theta$-constacyclic code of length $n$ over $R$ with a generating set in standard form $\{v\eta_{1-v}(x), (1-v)\eta_v(x)\}$. Then

1. $C^\perp = \langle vh_{1-v}^*(x), (1-v)h_v^*(x) \rangle$ and $|C^\perp| = p^{\deg(g_{1-v}(x)) + \deg(g_v(x))}$;
2. $C^\perp = \langle vh_{1-v}^*(x) + (1-v)h_v^*(x) \rangle$;
3. $\phi_\theta(C^\perp) \subseteq \langle h_{1-v}^*(x)h_v^*(x) \rangle$.
Proof. (1) By Proposition 3.14, \( C^\perp \) is a \( \theta^{-1} \)-constacyclic code over \( R \); by Theorem 3.15, we have that \( C^\perp = vC_{1-v}^\perp \oplus (1-v)C_1^\perp \), where according to Theorem 3.2, \( C_{1-v}^\perp \) and \( C_1^\perp \) are two constacyclic codes over \( F_p \). Since \( h^{*}_{1-v}(x) \) and \( h^{*}_{v}(x) \) are generator polynomials for \( C_{1-v}^\perp \) and \( C_1^\perp \), respectively, we have that \( \{vh^{*}_{1-v}(x), (1-v)h^{*}_{v}(x)\} \) is the generating set in standard form for \( C^\perp \). So \( C^\perp = \langle vh^{*}_{1-v}(x), (1-v)h^{*}_{v}(x) \rangle \). In addition, \( |C^\perp| = |C_{1-v}^\perp||C_1^\perp| = p^{|\deg(g_{1-v}(x))|} \).

(2) Since \( \{vh^{*}_{1-v}(x), (1-v)h^{*}_{v}(x)\} \) is the generating set in standard form for \( C^\perp \), according to the proof of Corollary 3.8, we have that \( C^\perp = \langle vh^{*}_{1-v}(x) + (1-v)h^{*}_{v}(x) \rangle \).

(3) Similar to the proof of Theorem 3.10. 

\[ \square \]

**Theorem 3.17.** Let \( \theta = 1-2v \) or \(-1+2v\) and let \( C \) be a \( \theta \)-constacyclic code of length \( n \) over \( R \) with a generating set in standard form \( \{vg_{1-v}(x), (1-v)g_{v}(x)\} \). Then

1. \( \phi_{\theta}(C^\perp) = [h^{*}_{1-v}(x)h^{*}_{v}(x)] \);
2. \( \phi_{\theta}(C^\perp) = (\phi_{\theta}(C))^\perp \).

**Proof.** (1) According to the proof of Corollary 3.11, we can obtain the result.

(2) Note that the facts that \( \phi_{\theta}(C) = [g_{1-v}(x)g_{v}(x)] \) and \( \phi_{\theta}(C^\perp) = [h^{*}_{1-v}(x)h^{*}_{v}(x)] \), we have

\[
\phi_{\theta}(C^\perp) = [g_{1-v}(x)g_{v}(x)]^\perp \\
= [h^{*}_{1-v}(x)h^{*}_{v}(x)] \\
= \phi_{\theta}(C^\perp),
\]

which is the required result. 

\[ \square \]

**Example 3.18.** In \( F_3[x] \),

\[
x^{10} + 1 = (x^2 + 1)(x^4 + x^3 - x + 1)(x^4 - x^3 + x + 1);
\]

\[
x^{10} - 1 = (x - 1)(x + 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1).
\]

Let \( C \) be the \((-1+2v)\)-constacyclic code of length 10 over \( F_3 + vF_3 \) with generating polynomial

\[
g(x) = v(x^4 + x^3 - x + 1) + (1-v)(x^4 - x^3 + x^2 - x + 1) \\
= x^4 + (2v - 1)x^3 + (1-v)x^2 - x + 1.
\]

The Gray image \( \phi_{\theta}(C) \) is a [20,12,4] cyclic code over \( F_3 \) with generator polynomial \((x^4 + x^3 - x + 1)(x^4 - x^3 + x^2 - x + 1)\).

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