Commuting quantum traces for quadratic algebras

Zoltán Nagy\textsuperscript{a,1}, Jean Avan\textsuperscript{a,2}, Anastasia Doikou\textsuperscript{b,3}, Geneviève Rollet\textsuperscript{a,4}

\textsuperscript{a}Laboratoire de Physique Théorique et Modélisation
Université de Cergy-Pontoise (CNRS UMR 8089), 5 mail Gay-Lussac,
Neuville-sur-Oise,
F-95031 Cergy-Pontoise Cedex, France

\textsuperscript{b} Laboratoire d’Annecy-Le-Vieux de Physique Théorique,
LAPTH (CNRS UMR 5108), B.P. 110, Annecy-Le-Vieux, F-74941, France

Abstract

Consistent tensor products on auxiliary spaces, hereafter denoted “fusion procedures”, and commuting transfer matrices are defined for general quadratic algebras, non-dynamical and dynamical, inspired by results on reflection algebras. Applications of these procedures then yield integer-indexed families of commuting Hamiltonians.

\textsuperscript{1}e-mail: nagy@ptm.u-cergy.fr
\textsuperscript{2}e-mail: avan@ptm.u-cergy.fr
\textsuperscript{3}e-mail: doikou@lapp.in2p3.fr
\textsuperscript{4}e-mail: rollet@ptm.u-cergy.fr
1 Introduction

A procedure to construct commuting quantum traces for a particular form of quadratic exchange algebras, known as reflection algebra [1], was recently developed in [2], building on the pioneering work in [3]. We recall that it entails three different steps: construction of the quadratic exchange algebra itself, and its so-called “dual” (this notion will be clarified soon); construction of realizations of the exchange algebra and its dual on consistent tensor products of the initial auxiliary space (which we will denote here as “fusion” procedure) while keeping a single “quantum” Hilbert space on which all operators are assumed to act; combination of these realizations into traces over the tensorized auxiliary spaces, yielding commuting operators acting on the original quantum space, labeled by the integer set of tensorial powers of the auxiliary space.

We immediately insist that this procedure is distinct of, and in a sense complements, the familiar construction of transfer matrices by tensoring over distinct quantum spaces (using an appropriate comodule structure of the quantum algebra) while keeping a single common auxiliary space; the trace is then taken over the auxiliary space to yield a generating functional of commuting operators [4]. In the case when there exists a universal formulation of the algebra as a bialgebra with a coproduct structure, both constructions stem from two separate applications of this coproduct. However, the resulting operators are quite distinct: the trace of the monodromy matrix yields commuting operators acting on a tensor product of Hilbert spaces (as in e.g. the case of spin chains); the trace of the fused auxiliary matrix yields operators acting on one single Hilbert space. These can be shown in some particular cases to realize the quantum analogue of the classical Poisson-commuting traces of powers of the classical Lax-matrix $Tr(L^n)$ (see [2, 5, 6]). This is the reason for our phrasing of “quantum traces” actually borrowed from [7]. In addition it must be emphasized that the procedure itself, combining a construction of a “dual” algebra and the establishing of exact fusion formulas, yields very interesting results on the quadratic exchange algebra itself, and its possible identification as a coalgebra (e.g. Hopf or quasi-Hopf). As we will later comment, it also plays a central role in the (similarly named) Mezincescu-Nepomechie fusion constructions for spin chains [8, 9].

A word of caution is in order. Throughout the paper, we use the term “fusion” in a restrictive sense, insofar as we only consider the possibility of acting on auxiliary spaces. The general fusion procedure itself has been
applied also to the quantum spaces, yielding e.g. higher spin interactions \cite{10} or multiparticle bound states \(S\)-matrices.

Our purpose here is to fully describe the quantum trace procedure for three types of general quadratic algebras. The first one is the quantum non-dynamical quadratic exchange algebra introduced in \cite{3}. The second one was formulated in \cite{11} as a dynamical version of the quadratic exchange algebras in \cite{3} with particular zero-weight conditions. It will be denoted “semi-dynamical” here, for reasons to be explicited later. The third one (similarly denoted here as “fully dynamical”) was first built in \cite{12} for the \(\mathfrak{sl}(2)\) case, and extended to the \(\mathfrak{sl}_n\) case in \cite{13}, albeit with particular restrictions on the coefficient matrices. The zero-weight conditions are different; the algebra structure itself mimicks the reflection algebra introduced by Cherednik et Sklyanin in \cite{1}; a comodule structure was identified and a universal structure was proposed in \cite{14}. We will here briefly comment on the differences between the quantum traces built in both dynamical cases.

## 2 Non-dynamical quadratic algebras

These algebras were recognized \cite{1,7} as generalizations of the usual \(R\)-matrix and quantum group structure, leading to non skew symmetrical \(r\)-matrices in the quasiclassical limit.

They are characterised by the following exchange relations.

\[
A_{12} T_1 B_{12} T_2 = T_2 C_{12} T_1 D_{12} \tag{1}
\]

where, as usual, the quantum generators sit in the matrix entries of \(T\). Let us recall some examples of this structure.

- The Yangian and quantum group structures where \(A = D, B = C = 1\)

- Donin-Kulish-Mudrov (DKM) reflection algebra without spectral parameters \cite{15}: \(A = C, B = D = A^\pi\), \(\pi\) denotes the permutation of auxiliary spaces: \((A^\pi)_{12} = A_{21}\).

- Kulish-Sklyanin type reflection algebra containing spectral parameters \cite{2,16}: \(A = R_{12}^-, B = R_{21}^+, C = R_{12}^+, D = R_{21}^-\) (\(\pm\) signs refer to the relative signs of spectral parameters in the \(R\)-matrix).
In [3, 17] consistency relations involving the structure matrices were derived and it was found that they had the form of cubic relations on the matrices $A, B, C, D$.

\[
\begin{align*}
A_{12} A_{13} A_{23} &= A_{23} A_{13} A_{12} \\ 
A_{12} C_{13} C_{23} &= C_{23} C_{13} A_{12} \\ 
D_{12} D_{13} D_{23} &= D_{23} D_{13} D_{12} \\ 
D_{12} B_{13} B_{23} &= B_{23} B_{13} D_{12}
\end{align*}
\] (2-5)

We can see that $A$ and $D$ obey the usual YB-equations whereas $C$ and $B$ are their respective representations.

Furthermore, generalized unitarity conditions can be derived from self-consistency of (1) under exchange of spaces 1 and 2 which imposes:

\[
A_{12} = \alpha A_{21}^{-1} \quad D_{12} = \beta D_{21}^{-1} \quad B_{12} = \gamma C_{21} \quad (\alpha, \beta, \gamma \in \mathbb{C})
\] (6)

The constants of proportionality have to obey an additional constraint: $\alpha \gamma = \beta \gamma^{-1}$. In the sequel, we will restrict ourselves to the simplest choice of $\alpha = \beta = \gamma = 1$.

Let us also note that although $B_{12} = C_{21}$, for aesthetical and mnemotechnical reasons we continue to use $C$ whenever it allows for the more familiar and significant $(12, 13, 23)$ display of indices.

In [3] the authors had already introduced an algebra which they called “dual” to (1). This “dual” structure is characterised by the following exchange relation.

\[
(A_{12}^{-1})^{t_1 t_2} K_1 \left( (B_{12}^{t_1})^{-1} \right)^{t_2} K_2 = K_2 \left( (C_{12}^{t_2})^{-1} \right)^{t_1} K_1 \left( D_{12}^{t_1 t_2} \right)^{-1}
\] (7)

Two respective representations of (1) and (7) (assumed to act on different quantum spaces) can be combined by means of a trace on the common auxiliary space to generate commuting quantum operators. It is with respect to this trace that equation (7) can be characterized as the dual of equation (1). We formulate the conjecture that this is the trace of a $*$-algebra structure on some underlying universal algebra. Some freedom remains as to the actual form of the trace and in the sequel we will stick to the choice of $H$ as $Tr_V(K^t T)$. Here the superscript $t$ stands for any antimorphism on the auxiliary space $V$, which satisfies also the trace invariance property $Tr(K T) = Tr(K^t T^t)$, for all matrices $K$ and $T$. The actual
antimorphism may differ from the usual transposition (e.g. by additional conjugation, crossing operation) since the proof of commutation uses only (see theorem 5,6 and 14) the antimorphism and trace invariance properties (see e.g. the super-transposition in superalgebras, or the crossing operation in $R$-matrices). Let us also remark here that it is possible to choose a trace formula where the antimorphism acts on the quantum space, as it is the case in [2], but we prefer not to do so here. Our particular choice is motivated by the fact that transposition on the auxiliary space is always defined whereas on the quantum space it is not necessarily straightforward and could require a supplementary hypothesis on this quantum representation which may not be easily implemented.

The quantum trace formulation for such a non-dynamical algebra stems from the results in [2, 3]; it is however interesting to give a rather detailed derivation of it in the general case, since both dynamical algebras will present similar features, albeit with crucial modifications in the fusion and trace formulas induced by the dynamical dependence.

We will describe two fusions (consistent tensor product of auxiliary spaces) of equation (1) respectively inspired by [2] (itself relying on [1]) and [15]. While the fusion of the structure matrices is uniquely defined in each case, the solutions of the fused exchange relations are not. In particular, they can be dressed, i.e. multiplied by suitable “coupling” factors. This dressing procedure turns out to be crucial: indeed, when the simplest solutions of the fused exchange relation are combined in a quantum trace, they decouple, giving rise to products of lower order hamiltonians. To obtain nontrivial commuting quantities these fused $T$-matrices must be dressed.

We will finally show that the two fusion procedures identified in [2, 15] are related by a coupling matrix $L_M$ and that they generate the same commuting quantities.

2.1 First fusion procedure

Let us first start by introducing some convenient notations (see [2]) for fused matrices.
\[ A_{MN'} = \prod_{i \in M} \prod_{j \in N'} A_{ij} = A_{11'} A_{12'} \ldots A_{1n'} A_{21'} A_{22'} \ldots A_{2n'} \ldots A_{m1'} \ldots A_{mn'} \] (8)

where \( M = \langle 1, 2, \ldots, m \rangle \) and \( N' = \langle 1', 2', \ldots, n' \rangle \) are ordered sets of labels. The same sets with reversed ordering are denoted by \( \bar{M} \) and \( \bar{N}' \). A set \( M \) deprived of its lowest (highest) element is denoted by \( M_0 \) (\( M^0 \)).

**Remark.** In many explicit examples we would have to deal only with one single exchange formula (1) with two isomorphic auxiliary spaces. However our derivation also applies to a situation where more general coupled sets of exchange relations would occur as \( A_{ij}T_iB_{ij}T_j = T_jC_{ij}T_iD_{ij} \) with \( \{i, j\} \subset \{1, \ldots, m_0 < \infty\} \) and generically \( V_i \not\approx V_j \). Such situations will occur whenever a universal structure is identifiable and the auxiliary spaces \( V_i \) carry different representations of the algebra, as in e.g. [15]. It is therefore crucial that the order in the index set be stipulated.

Similar notations are used for the fusion of the other structure matrices. The next lemma states that that the structure matrices in (1) can be fused in a way that respects the YB-equations (2)-(5).

**Lemma 1.** Let \( A, B, C, D \) be solutions of the Yang-Baxter equations (2)-(5). Then the following fused Yang-Baxter equations hold:

\[ A_{MN'} A_{ML'} A_{N'L'} = A_{N'L'} A_{ML'} A_{MN'} \] (9)
\[ A_{MN'} C_{ML'} C_{N'L'} = C_{N'L'} C_{ML'} A_{MN'} \] (10)
\[ D_{MN'} D_{ML'} D_{N'L'} = D_{N'L'} D_{ML'} D_{MN'} \] (11)
\[ D_{MN'} B_{ML'} B_{N'L'} = B_{N'L'} B_{ML'} D_{MN'} \] (12)

**Proof.** Simple induction on \( \#M + \#N' \). \[ \blacksquare \]

We now describe a fusion procedure for the algebra characterized by (1), generalizing the one introduced in [2].

**Theorem 1.** If \( T \) is a solution of

\[ A_{12} T_1 B_{12} T_2 = T_2 C_{12} T_1 D_{12} \] (13)
then

\[ T_M = \prod_{i \in M} \left( T_i \left( \prod_{i < j} B_{ij} \right) \right) \]  

(14)

verifies the following fused equation:

\[ A_{M'N'} T_M B_{MN'} T_{N'} = T_{N'} C_{MN'} T_M D_{M'N'} \]  

(15)

**Proof.** Induction on the cardinality \( n \) of the index sets: \( n = \#M + \#N' \) which repeats and generalizes the steps in [2].

The solution \( T_M \) obtained above can be dressed, i.e. can be multiplied from the left and the right by suitable factors.

**Proposition 1.** Let \( T_M \) be a solution of the fused exchange relation. Then \( Q_M T_M S_M \) is also a solution of the fused exchange relation provided \( Q_M \) and \( S_M \) verify:

\[
\begin{align*}
[Q_M, A_{MN'}] &= [Q_{N'}, A_{MN'}] = [Q_{N'}, B_{MN'}] = [Q_M, C_{MN'}] = 0 \\
[S_M, D_{M'N'}] &= [S_{N'}, D_{M'N'}] = [S_{N'}, C_{MN'}] = [S_M, B_{MN'}] = 0
\end{align*}
\]

(16)

A particular solution of these constraints is provided by:

\[
\begin{align*}
Q_M &= \hat{A}_{12} \hat{A}_{23} \cdots \hat{A}_{m-1,m} \\
S_M &= \hat{D}_{12} \hat{D}_{23} \cdots \hat{D}_{m-1,m}
\end{align*}
\]

(17)

where \( \hat{A}_{12} = P_{12} A_{12}, \ldots, P_{12} \) being the permutation exchanging two auxiliary spaces.

**Proof.** again by induction on the cardinality of the index sets. In the induction step we use the decomposition: \( Q_{N'} B_{MN'} = \hat{A}_{12} \cdots \hat{A}_{n'-1,n'} B_{M,N'00} B_{M,n'-1} B_{M,n'} \), for example.

The fusion procedure can be repeated for the dual exchange relation as follows.
Theorem 2. If $K$ is a solution of the dual exchange relation:

$$(A_{12}^{-1})^{t_1 t_2} K_1 \left( (B_{12}^{t_1})^{-1} \right)^{t_2} K_2 = K_2 \left( (C_{12}^{t_2})^{-1} \right)^{t_1} K_1 \left( D_{12}^{t_1 t_2} \right)^{-1}$$

then

$$K_M = \prod_{i \in M} K_i \left( \prod_{i < j} \left( (B_{ij}^{t_{ij}})^{-1} \right)^{t_2} \right)$$

is a solution of the dual fused equation

$$(A_{MN'}^{-1})^{t_M t_{N'}} K_M \left( (B_{MN'}^{t_M})^{-1} \right)^{t_{N'}} K_{N'} = K_{N'} \left( (C_{MN'}^{t_{N'}})^{-1} \right)^{t_M} K_M \left( D_{MN'}^{t_M t_{N'}} \right)^{-1}$$

**Proof.** similar to that of Theorem 1. Note that the dual structure matrices obey a set of appropriate YB-equations, isomorphic to (9)-(12), for instance

$$(A_{MN'}^{-1})^{t_M t_{N'}} \left( (B_{MN'}^{t_M})^{-1} \right)^{t_{N'}} \left( (D_{MN'}^{t_M})^{-1} \right)^{t_M} K_M = K_M \left( (D_{MN'}^{t_M})^{-1} \right)^{t_M} K_M$$

$$= K_M \left( (D_{MN'}^{t_M})^{-1} \right)^{t_M} K_M$$

A similar dual dressing procedure exists: Any dressing of a solution of (18) should obey the commutativity constraints

$$[Q_M', (A_{MN'}^{-1})^{t_M t_{N'}}] = [Q_{N'}', (A_{MN'}^{-1})^{t_M t_{N'}}] = [Q_{N'}', ((B_{MN'}^{t_M})^{-1})^{-1}] = (22)$$

$$[Q_M', ((C_{MN'}^{t_{N'}})^{-1})^{-1}] = [S_M', ((D_{MN'}^{t_M})^{-1})^{-1}] = (23)$$

$$[S_{N'}', ((C_{MN'}^{t_{N'}})^{-1})^{-1}] = [S_M', ((D_{MN'}^{t_M})^{-1})^{-1}] = 0$$

involving fused dual structure matrices. It is easy to check that if $Q_M$ and $S_M$ dress solutions of (15) then $Q'_M = Q'^{t_M}_M$ and $S'_M = S'^{t_M}_M$ dress solutions of (18).

### 2.2 Second fusion procedure

Results in [15] hint that relation (11) admits another fusion procedure. We will explicitly link the fusion described in the preceding section to the one inspired by ref. [15].
The DKM type fusion is characterized by the following fused exchange relation for fused matrices \( \tilde{T} \) to be described in the following:

\[
A_{MN'} \, \tilde{T}_M \, B_{MN'} \, \tilde{T}_{N'} = \tilde{T}_{N'} \, C_{MN'} \, \tilde{T}_M \, D_{MN'} \tag{23}
\]

This equation can actually be obtained from a multiplication of the KS exchange relation (15) by suitable factors reversing the ordering of indices where it is needed. The next lemma specifies this statement.

**Lemma 2.** Let \( T_M \) be a solution of the fused exchange relation (15). If \( L_M \) verifies the following commutation rules

\[
L_M A_{MN'} = A_{MN'} L_M \\
L_{N'} B_{MN'} = B_{MN'} L_{N'} \\
L_M C_{MN'} = C_{MN'} L_M
\]

then \( \tilde{T}_M = L_M T_M \) is a solution of the exchange relation

\[
A_{MN'} \, \tilde{T}_M \, B_{MN'} \, \tilde{T}_{N'} = \tilde{T}_{N'} \, C_{MN'} \, \tilde{T}_M \, D_{MN'} \tag{25}
\]

An example of such an \( L_M \) is given by:

\[
L_M = A_{12} \ldots A_{1m} A_{23} \ldots A_{2m} \ldots A_{m-1,m} = \prod_{1 \leq i < j \leq m} A_{ij} \tag{26}
\]

**Proof.** The first part is straightforward. Example (26) is verified by induction using \( L_M = A_{1M_0} L_{M_0} \). For instance, the first relation of (24) is proved as:

\[
L_M A_{MN'} = A_{1M_0} L_{M_0} A_{1N'} A_{M_0N'} = A_{1M_0} A_{1N'} L_{M_0} A_{M_0N'} = A_{1M_0} A_{1N'} A_{M_0} A_{M_0N'} = A_{MN'} L_M
\]

where fused YB-equations are used. \( \blacksquare \)

Combined with Theorem 1, this lemma leads to

**Theorem 3.** If \( \tilde{T} \) is a solution of

\[
A_{12} \tilde{T}_1 B_{12} \tilde{T}_2 = \tilde{T}_2 C_{12} \tilde{T}_1 D_{12} \tag{27}
\]

then

\[
\tilde{T}_M = \prod_{i \in M} \left( \prod_{j > i} A_{ij} \tilde{T}_i \prod_{j > i} B_{ij} \right) \tag{28}
\]
is a solution of
\[
A_{MN'} \tilde{T}_M B_{M\tilde{N}'} \tilde{T}_{N'} = \tilde{T}_{N'} C_{M\tilde{N}'} \tilde{T}_M D_{M\tilde{N}'}
\] (29)

Proof. The only property left to check is that the solution \(\tilde{T}_M\) in (28) is obtained from \(T_M\) in (14) by a multiplication by \(L_M\) in (26). It is enough to show that
\[
\tilde{T}_M = A_{1M_0} T_1 B_{1M_0} \tilde{T}_{M_0}
\] We only develop the induction step.

\[
L_M T_M = A_{1M_0} L_{M_0} T_1 B_{1M_0} T_{M_0} = A_{1M_0} T_1 L_{M_0} B_{1M_0} T_{M_0} = (30)
\]

The next proposition describes the dressing of the solutions.

Proposition 2. Let \(\tilde{T}_M\) be a solution of the DKM-type fused exchange relations. Then \(\tilde{Q}_M \tilde{T}_M \tilde{S}_M\) is also a solution provided \(\tilde{Q}_M\) and \(\tilde{S}_M\) verify
\[
[\tilde{Q}_M, A_{MN'}] = [\tilde{Q}_{N'}, A_{MN'}] = [\tilde{Q}_{N'}, B_{M\tilde{N}'}] = [\tilde{Q}_M, C_{M\tilde{N}'}] = 0 \quad (31)
\]
\[
[\tilde{S}_M, D_{M\tilde{N}'}] = [\tilde{S}_{N'}, D_{M\tilde{N}'}] = [\tilde{S}_{N'}, C_{MN'}] = [\tilde{S}_M, B_{M\tilde{N}'}] = 0
\]

These equations are solved by
\[
\tilde{Q}_M = L_M Q_M L_M^{-1} \quad \tilde{S}_M = S_M
\]
where \(Q_M\) and \(S_M\) dress the solutions of the fused exchange relation (15) and \(L_M\) is a solution of (24).

Proof. Straightforward.

We saw that \(T_M\) and \(\tilde{T}_M\) were linked by a factor \(L_M\). The question arises whether there is a similar relation between the corresponding dual exchange algebras and their solutions. The relation is established in

Theorem 4. Let \(K_M\) be a solution of the first fused exchange relation (15) and \(L_M\) be a solution of (24). Then \(\tilde{K}_M = (L_M^{-1})^{-1} K_M\) is a solution of the KDM-type dual fused exchange relation:
\[
(A_{MN'}^{-1})^{tM_tN'} \tilde{K}_M ((B_{MN'}^{tM_tN'})^{-1})^{tN'} \tilde{K}_{N'} = \tilde{K}_{N'} ((C_{MN'}^{tM_tN'})^{-1})^{tM} \tilde{K}_M (D_{MN'}^{tM_tN'})^{-1} \quad (32)
\]
Proof. We first see that (32) is indeed the dual exchange relation associated with (25). The next step is to check that $(L^t_M)^{-1}$ obeys the appropriate commutation relations that enable it to transform the fused dual AD type algebra (18) into the fused dual DKM-type one (32). It is obvious since these equations are the inverse-transposed of (24).

Dressings of these dual fused solutions are obtained from dressings of (25) by the same operation as for the AD type fusion, i.e. by transposing.

2.3 Commuting traces

In the preceding sections we have derived two distinct fusion procedures both of which allow for building commuting quantities. In this section we will describe this construction, and show the two different quantum traces are identified once the dressing is used.

We first establish:

**Theorem 5.** Let $T_M$ be a solution of the fused AD-type exchange relation (15). $T_M$ acts on the tensor product of the auxiliary spaces labeled by $M$ and on the quantum space $V_q$.

Let $K_M$ be a solution of the dual fused AD-type exchange relation (18). $K_M$ acts on the tensor product of the auxiliary spaces labeled by $M$ and on the quantum space $V_q$.

The following operators

$$H_M = Tr_M (K_M^t M T_M)$$

constitute a family of mutually commuting quantum operators acting on $V_q \otimes V_q'$:

$$[H_M, H_{N'}] = 0$$

**Proof.** It repeats the steps of [2, 16] The proof is independent of the particular fusion procedure so it remains valid for the DKM case too. Thus we have

**Theorem 6.** Let $\tilde{T}_M$ be a solution of the fused DKM-type exchange relation (25). $\tilde{T}_M$ acts on the tensor product of the auxiliary spaces labeled by $M$ and on the quantum space $V_q$. 


Let $\tilde{K}_M$ be a solution of the dual fused DKM-type exchange relation (22). $\tilde{K}_M$ acts on the tensor product of the auxiliary spaces labeled by $M$ and on the quantum space $V_q$. The following operators

$$\tilde{H}_M = \text{Tr}_M \left( \tilde{K}^t_M \tilde{T}_M \right)$$

(constitute a family of mutually commuting quantum operators acting on $V_q \otimes V_q'$:

$$\left[ \tilde{H}_M, \tilde{H}_{M'} \right] = 0$$

(36)

So far we have two seemingly different sets of commuting quantities obtained from the same defining relations (1) via two distinct fusion procedures. However we will show that the operation consisting in dressing and taking the trace smears out this difference and one is left with only one set of commuting hamiltonians. This is summarized in:

**Proposition 3.** The quantum commuting Hamiltonians obtained from any set of solutions $T_M$, $K_M$ of (14), (19) are identified with the quantum commuting Hamiltonians obtained from a suitable set of solutions $\tilde{T}_M$, $\tilde{K}_M$ of (25), (32). This identification is implemented by a coupling matrix $L_M$.

**Proof.** Let $T_M$ be the solution (14) and $K_M$ the corresponding dual solution (19). The results of the multiplication by $L_M$ and $(L^{-1}_M)$ are denoted by $\tilde{T}_M$ and $\tilde{K}_M$. We calculate the tilded hamiltonians after dressing and we find that they are equal to the dressed untilded ones.

$$\text{Tr}_M(\tilde{K}^t_M \tilde{Q} \tilde{T} \tilde{S}) = \text{Tr}_M(\tilde{K}^t_M \tilde{T} \tilde{S})$$

(37)

The following propositions justifies the technical relevance of dressings.

**Proposition 4.** Operators built from the solution (14) decouple as $H_N = \text{Tr}_N(k^t N T_N) = \text{Tr}(k^t T)^N$. 

11
Proof. By induction using the property $T_N = T_1 B_{1N_0} T_{N_0}$. Let us detail the induction step.

\[
H_N = Tr_N(K^{t_N}_N T_N) = Tr((K_1(B^{t_1}_{1N_0})^{-1} t_{N_0} K_{N_0})^{t_N} T_1 B_{1N_0} T_{N_0}) =
\]

\[
Tr((K_1(B^{t_1}_{1N_0})^{-1} t_{N_0} K_{N_0})^{t_N} (T_1 B_{1N_0} T_{N_0})^{t_1}) =
\]

\[
Tr(K_1 K^{t_N}_{N_0} (B^{t_1}_{1N_0})^{-1} B^{t_1}_{1N_0} T^{t_1} T_{N_0}) =
\]

\[
Tr(K_1 T^{t_1}_1) Tr(K^{t_N}_{N_0} T_{N_0})
\]

\[\blacksquare\]

Note that the result in Proposition 3 implies that the same goes for the operators built using the second fusion. Three important remarks are in order here.

The use of dressed quantum traces

Dressed quantum traces yield a priori independent operators. Indeed, the classical limit of a quantum trace computed with the particular dressing (17) in Proposition 1 will yield $Tr T^n$ instead of $(Tr T)^n$ (since $A, B, C, D \to 1 \otimes 1$ but $P_{12} \to P_{12}^{-1}$). Quantum traces are directly, in this particular case, (as was already known in the context of quantum group structures [6]) quantum analogues of the classical Poisson-commuting power traces $Tr T^n$.

The use of undressed quantum traces

It must on the other hand be emphasized that the decoupling of the undressed fused quantities plays an essential role in the formulation of the analytical Bethe ansatz solution of $\mathfrak{sl}(n)$ spin chains (as is seen in [9]) and more generally in the formulation of a generalized Mezincescu-Nepomechie procedure for fusion of transfer matrices [8], in that it gives a natural construction of products of monodromy matrices such as are required by this formulation.

Explicit computation of the dressings

From a more theoretical point of view, it must be noticed that eqn. (24), as already discussed for the particular example treated in [2], would appear as a condition obeyed by coproducts of the central elements of a (hypothetical) universal algebra, thereby promoting the dressing matrices $Q$ and $S$ from
“technical auxiliaries” to get non-trivial traces, to representations of Casimir elements of the algebra itself\(^1\).

A second more technical remark is required here regarding the actual computation of the quantum traces with the particular explicit dressing determined in Proposition 1. Difficulties in applying (35) with the explicit dressings (17) may occur when the auxiliary space \(V\) is a loop space \(V^{(n)} \otimes \mathbb{C}(z)\) \((n=\text{finite dimension of the vector space})\). Indeed, the permutation of spectral parameters required in formula (17) is only achieved at a formal level by the singular distribution \(\delta(z_i/z_j)\) (see [2] for discussions). Hence the actual explicit computations of such quantum traces may entail delicate regularization procedures. However, if one only focuses on the practical purpose of the quantum trace procedure, which is to build a set of commuting operators, use of higher-power fused objects as in (14) and (19) is mostly required when no spectral parameter is present in the represented exchange algebra (1). Otherwise one needs to consider only the first order trace \(\text{Tr}_1 \tilde{K}_1(z_1) \text{Tr}_1(z_1)\) and expand it in formal series in \(z_1\). If no spectral parameter is available, one can then use (14), (19), (17) and (35) to build explicitly without difficulties a priori independent commuting quantum operators. (For an application to a different algebraic structure see [5]).

2.4 Further example: “Soliton non-preserving” boundary conditions: Twisted Yangians.

We have mentioned in the Introduction several examples of non-dynamical quadratic exchange algebras. Another interesting example to which we plan to apply this scheme is related to the so-called “soliton non-preserving” boundary conditions in integrable lattice models (see [18]). To characterize it we will focus on the \(\mathfrak{su}(n)\) invariant \(R\) matrix given by

\[
R_{12}(\lambda) = \lambda I + i\mathcal{P}_{12}
\]

where \(\mathcal{P}\) is the permutation operator on the tensor product \(V_1 \otimes V_2\). The \(R\) matrix is a solution of the Yang–Baxter equation \([19, 20, 21, 22]\) and also satisfies:

(i) Unitarity

\[
R_{12}(\lambda) R_{21}(-\lambda) = \zeta(\lambda)
\]

\(^1\)this was pointed out to us by Daniel Arnaudon
where $R_{21}(\lambda) = \mathcal{P}_{12} R_{12}(\lambda) \mathcal{P}_{12} = R_{12}^{t}\lambda$ and $\mathcal{P}$ is the permutation operator.

(ii) Crossing-unitarity

\[
R_{12}^{t}\lambda M_{1} R_{12}^{t}\lambda - 2i\rho M_{1}^{-1} = \zeta'(\lambda + i\rho)
\]

$M = V^{t} V$, $(M = 1$ for the $\mathfrak{su}(n)$ case $) \rho = \frac{n}{2}$ and also

\[
[M_{1}M_{2}, R_{12}(\lambda)] = 0,
\]

\[
\zeta(\lambda) = (\lambda + i)(-\lambda + i), \quad \zeta'(\lambda) = (-\lambda + i\rho)(\lambda + i\rho).
\]

It is interpreted as the scattering matrix [23, 22, 24] describing the interaction between two solitons –objects that correspond to the fundamental representation of $\mathfrak{su}(n)$.

One may also derive the scattering matrix that describes the interaction between a soliton and an anti-soliton, which corresponds to the conjugate representation of $\mathfrak{su}(n)$. It reads:

\[
R_{12}(\lambda) = R_{12}(\lambda) = \bar{R}_{12}(\lambda) = U_{1} R_{12}^{t}\lambda - \lambda - i\rho U_{1},
\]

and it can also be written as

\[
\bar{R}_{12}(\lambda) = (-\lambda - i\rho)I + iQ
\]

where $Q$ is a projector onto a one dimensional space, and where $U$ is a matrix of square 1. Note that for the $\mathfrak{su}(2)$ case

\[
R_{12}(\lambda) = R_{12}(\lambda),
\]

which is expected because $\mathfrak{su}(2)$ is self conjugate. The $\bar{R}$ matrix also satisfies the Yang–Baxter equation and

(i) Unitarity

\[
\bar{R}_{12}(\lambda) \bar{R}_{21}(-\lambda) = \zeta'(\lambda)
\]

(ii) Crossing-unitarity

\[
R_{12}^{t}\lambda M_{1} R_{12}^{t}\lambda - 2i\rho M_{1}^{-1} = \zeta(\lambda).
\]
The reflection equation

The usual reflection equation \([1]\) describes physically the reflection of a soliton (fundamental representation of \(\mathfrak{su}(n)\)) as a soliton. The associated quadratic algebra was considered e.g. in \([2]\).

\[
R_{12}(\lambda_1 - \lambda_2) T_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) T_2(\lambda_2) = T_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) T_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2).
\]

(48)

Considering now the reflection of a soliton as anti-soliton one is similarly lead to the formulation of another quadratic algebra:

\[
R_{12}(\lambda_1 - \lambda_2) T_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) T_2(\lambda_2) = T_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) T_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2).
\]

(49)

More specifically equation (49) is the definition of the so–called twisted Yangian. Its dual reflection equation is obtained essentially by taking its formal transposition:

\[
R_{12}(-\lambda_1 + \lambda_2) K_1^{1t}(\lambda_1) M_1^{-1} \bar{R}_{21}(-\lambda_1 - \lambda_2 - 2i\rho) M_1 K_2^{1t}(\lambda_2) = K_2^{1t}(\lambda_2) M_1 R_{12}(-\lambda_1 - \lambda_2 - 2i\rho) M_1^{-1} K_1^{1t}(\lambda_1) R_{21}(-\lambda_1 + \lambda_2).
\]

(50)

This indeed realizes the general quadratic exchange relation \([11], [15]\) with the following identifications (using unitarity and crossing symmetries of the \(R\)-matrix)

\[
A_{12} = R_{12}(\lambda_1 - \lambda_2), \quad B_{12} = \bar{R}_{21}(\lambda_1 + \lambda_2), \quad C_{12} = \bar{R}_{12}(\lambda_1 + \lambda_2), \quad D_{12} = R_{21}(\lambda_1 - \lambda_2)
\]

\[
(A_{12}^{-1})^{t_{12}} = R_{12}(-\lambda_1 + \lambda_2), \quad ((B_{12}^{-1})^{-1})^{t_{12}} = M_1 R_{12}(-\lambda_1 - \lambda_2 - 2i\rho) M_1^{-1}, \quad ((C_{12}^{-1})^{-1})^{t_1} = M_1^{-1} \bar{R}_{21}(-\lambda_1 - \lambda_2 - 2i\rho) M_1, \quad (D_{12}^{-1})^{t_{12}} = R_{21}(-\lambda_1 + \lambda_2)
\]

Explicit application of the quantum trace procedure to this particular algebra will be left for further studies.

3 Quantum traces for semi-dynamical quadratic algebras

The second type of quadratic exchange relations considered here consists of the dynamical quadratic algebras generically described and studied in \([11]\)
which were first exemplified in the context of scalar Ruijsenaars-Schneider models in [26]. Fusion procedures and commuting traces can be built up for these dynamical quadratic algebras following the same overall procedure as in the non-dynamical case, albeit with crucial, non-trivial differences.

### 3.1 The semi-dynamical quadratic algebra

Let us recall here the basic definitions. Our starting point is the dynamical quadratic exchange relation:

$$A_{12}(\lambda)T_1(\lambda)B_{12}(\lambda)T_2(\lambda + \gamma h_1) = T_2(\lambda)C_{12}(\lambda)T_1(\lambda + \gamma h_2)D_{12}$$  \hspace{1cm} (51)

This describes an algebra generated by the matrix entries of $T$. $A, B, C, D$ are matrices in $\text{End}(V \otimes V)$ depending on $\lambda \in \mathfrak{h}^*$ where $\mathfrak{h}$ is a commutative Lie algebra, of dimension $n$, making $V$ a diagonalizable $\mathfrak{h}$-module. Introducing coordinates $\lambda_i$ on $\mathfrak{h}^*$ and the dual base $h_i$ on $\mathfrak{h}$ the shift $\lambda + \gamma h$ can be defined in the following way. For any differentiable function $f(\lambda) = f(\{\lambda_i\})$:

$$f(\lambda + \gamma h) = e^{\gamma D} f(\lambda) e^{-\gamma D},$$  \hspace{1cm} (52)

where

$$D = \sum_i h_i \partial_{\lambda_i}$$  \hspace{1cm} (53)

In the forthcoming calculations $\gamma$ is set to 1 for simplification. Zero weight conditions are imposed on the first space of $B$ and the second one of $C$; $D$ is of total weight zero.

$$[B_{12}, h \otimes 1] = [C_{12}, 1 \otimes h] = [D_{12}, h \otimes 1 + 1 \otimes h] = 0 \hspace{1cm} (h \in \mathfrak{h})$$  \hspace{1cm} (54)

These particular conditions, together with the absence of dynamical shift in two out of four $T$ matrices in (51), lead us to denote this structure as “semi-dynamical”. We will restrict ourselves from now on to the case where $V$ is of dimension $n$: the basis of $V$ and the generators of $\mathfrak{h}$, can then be chosen so that one identifies: $h_i = E_{ii}$ (diagonal basis elements of $\mathfrak{gl}(n)$, see e.g. [25] for introduction of this condition). These conditions mean in particular that $B$ and $C$ are diagonal on the corresponding spaces, respectively $V_1$ and $V_2$. In addition, $D$ has components on basis elements $E_{ij} \otimes E_{kl}$ of $\mathfrak{gl}(n) \otimes \mathfrak{gl}(n)$ only when the sets $\{i, k\}$ and $\{j, l\}$ are equal (property ZW). In other words
non-zero elements have identical *unordered* multiplets of line and column indices.

For the consistency of the exchange relations the following set of coupled “dynamical” YB-equations is imposed.

\[
A_{12} A_{13} A_{23} = A_{23} A_{13} A_{12} \quad (55)
\]

\[
D_{12}(\lambda + \gamma h_3) D_{13} D_{23}(\lambda + \gamma h_1) = D_{23} D_{13}(\lambda + \gamma h_2) D_{12} \quad (56)
\]

\[
D_{12} B_{13} B_{23}(\lambda + \gamma h_1) = B_{23} B_{13}(\lambda + \gamma h_2) D_{12} \quad (57)
\]

\[
A_{12} C_{13} C_{23} = C_{23} C_{13} A_{12}(\lambda + \gamma h_3) \quad (58)
\]

The simplest example of this algebra is related to the elliptic scalar \( \mathfrak{gl}(n) \) Ruijsenaars-Schneider model and was first written in [26]. We only write down its rational limit here.

\[
A(\lambda) = 1 + \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} (E_{ii} - E_{ij}) \otimes (E_{jj} - E_{ji}) \quad (59)
\]

\[
B(\lambda) = C(\lambda) = 1 + \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} E_{ij} \otimes (E_{ii} - E_{ij}) \quad (60)
\]

\[
D(\lambda) = 1 - \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} E_{ii} \otimes E_{jj} + \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} E_{ij} \otimes E_{ji} \quad (61)
\]

where \( E_{ij} \) is the elementary matrix whose entries are \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \) and \( \lambda_{ij} = \lambda_i - \lambda_j \). These matrices verify the consistency conditions (55)-(58).

A scalar representation of the exchange algebra defined with these structure matrices is then provided by:

\[
T(\lambda) = \sum_{ij} \frac{\prod_{a \neq i} (\lambda_{aj} + \tilde{\gamma})}{\prod_{a \neq j} \lambda_{aj}} E_{ij} \otimes 1 \quad (62)
\]

The word “scalar” is used here in the sense that \( T(\lambda) \) acts on a one-dimensional (trivial) quantum space. The exchange relation (51) is just a \( c \)-number equality. Representation of (51) on non-trivial quantum spaces is provided in this context by the comodule structure in [11].

Let us note here that the condition \( AB = CD \) found in [26] means in this context that the identity matrix is also a solution of (51). This is not a trivial statement; in fact it does not hold in general, and is not preserved by fusion procedures.
3.2 Fusion procedures and the “dual” algebra

Let $A, B, C, D$ be solutions of the dynamical exchange relation. We will define their fusion by induction as follows. We omit the dependence on $\lambda$ and simplify the notations of the shifts as $(h_{i,j})$; otherwise we use the notations introduced in section 2.1, defining the multiple-index matrices by induction as:

$$A_{MN'} = A_{1N'}A_{M0N'} = A_{M0'}A_{M0N'}$$
$$B_{MN'} = B_{M1'}B_{M0N'} = B_{1N'}[B_{M0N'}(h_1)]$$
$$C_{MN'} = C_{1N'}C_{M0N'} = C_{M1'}[C_{MN0}(h_1)]$$
$$D_{MN'} = D_{1N'}[D_{M0N'}(h_1)] = [D_{M0'}(h_{1',n'-1})] D_{MN0}$$

where $h_{i,j} := \sum_{k=i}^j h_k$. These fused structure matrices verify the fused dynamical YB-equations which are gathered together in the next proposition.

Proposition 5. Let $A, B, C, D$ be solutions of the dynamical Yang-Baxter equations (55)-(58). Then the following fused dynamical Yang-Baxter equations hold:

$$A_{MN'}A_{ML''}A_{N'L''} = A_{N'L''}A_{ML''}A_{MN'}$$
$$A_{MN'}C_{ML''}C_{N'L''} = C_{N'L''}C_{ML''}A_{MN'}(h_{L''})$$
$$D_{MN'}(h_{L''})D_{ML''}D_{N'L''}(h_M) = D_{N'L''}D_{ML''}(h_{N'})D_{MN'}$$
$$D_{MN'}B_{ML''}B_{N'L''}(h_M) = B_{N'L''}B_{ML''}(h_{N'})D_{MN'}$$

Proof. by induction, using at crucial stages the zero weight properties. The fusion procedure respects the property ZW for $D$ and the diagonality of $B$. It is also clear from the fusion procedure that the fused shift matrix $h_M$ is identified with $h_{(1,m)}$. ■

Theorem 7. Let $T$ be a solution of the dynamical quadratic exchange relation

$$A_{12}T_1B_{12}T_2(h_1) = T_2C_{12}T_1(h_2)D_{12}$$

then

$$T_M = \prod_{i \in M} \left( T_i \sum_{\substack{k<i \atop k \in M}} h_k \right) \left( \prod_{\substack{j>i \atop j \in M}} B_{ij} \right)$$
verifies the fused dynamical exchange relation

\[ A_{M N'} T_M B_{M N'} T_N' (h_M) = T_N' C_{M N'} T_M (h_{N'}) D_{M N'} \]  

(65)

**Proof.** Similar to that of Theorem 1 but the induction step uses the fact that \( T_M = T_1 B_{1M_0} T_{M_0} (h_1) \) and uses the fused dynamical YB-equations. ■

The dual exchange relation and an associated fusion procedure are described in the next theorem.

**Theorem 8.** Let \( K \) be a solution of the dynamical quadratic exchange relation

\[ (A_{12}^{-1})_{t_1 t_2} K_1 (B_{12}^{-1})_{t_1} K_2 (h_1) = K_2 (C_{12}^{-1})_{t_1} K_1 (h_2) (D_{12})_{t_1 t_2} \]  

(66)

then

\[ K_M = \prod_{i \in M} (K_i (\sum_{k<i} h_k) \left( \prod_{j>i} (B_{ij}^{-1}) \right)) \]  

(67)

verifies the fused dynamical exchange relation

\[ (A_{M N'}^{-1})_{t_M t_N} K_M (B_{M N'}^{-1})_{t_M} K_{N'} (h_M) = K_{N'} (C_{M N'}^{-1})_{t_M} K_M (h_{N'}) (D_{M N'})_{t_M} \]  

(68)

**Proof.** Similar to the nondynamical case. ■

Note that the structure matrices of this dual relation are related to original ones in the same way as in the nondynamical case once we take into account the partial zero weight property of \( B \) and \( C \) which implies diagonality on the corresponding spaces, respectively \( V_1 \) and \( V_2 \).

### 3.3 Second fusion

As in the nondynamical case, one can define another KDM-type fusion with the appropriate shifts. This fusion is characterized by the following exchange relation

\[ A_{M N'} T_M B_{M N'} T_N' (h_M) = T_N' C_{M N'} T_M (h_{N'}) D_{M N'} \]  

(69)

The analogy with the nondynamical case can be pushed further i.e. there exists an object \( L_M \) linking the fusions in Theorem 7 and 9. This allows us to use directly the proofs of Theorem 3 and 4.
Lemma 3. Let $T_M$ be a solution of the fused equation (65). If $L_M$ verifies the following commutation rules

\begin{align}
L_M A_{MN'} &= A_{MN'} L_M \\
L_{N'} A_{MN'} &= A_{MN'} L_{N'} \\
L_{N'} B_{MN'} &= B_{MN'} L_{N'}(h_M) \\
L_M C_{MN'} &= C_{MN'} L_M(h_{N'})
\end{align}

then $L_M T_M$ is a solution of the exchange relation

\begin{align}
A_{MN'} T_M B_{MN'} T_{N'}(h_M) &= T_{N'} C_{MN'} T_M(h_{N'}) D_{MN'}
\end{align}

An example of such an $L_M$ is given by

\begin{align}
L_M &= A_{12} \ldots A_{1m} A_{23} \ldots A_{2m} \ldots A_{m-1,m} = \prod_{1 \leq i < j \leq m} A_{ij} 
\end{align}

Proof. Straightforward, using the dynamical YB-equations (55)-(57).

Now we state the dynamical versions of Theorem 3 and 4.

Theorem 9. Let $T$ be a solution of the dynamical quadratic exchange relation

\begin{align}
A_{12} T_1 B_{12} T_2(h_1) &= T_2 C_{12} T_1(h_2) D_{12}
\end{align}

then

\begin{align}
T_M &= \prod_{i \in M} \left( \prod_{j > i} A_{ij} T_i \left( \sum_{k < i} h_k \right) \prod_{j > i} B_{ij} \right)
\end{align}

verifies the fused dynamical exchange relation

\begin{align}
A_{MN'} T_M B_{MN'} T_{N'}(h_M) &= T_{N'} C_{MN'} T_M(h_{N'}) D_{MN'}
\end{align}

Proof. Reproduces the proof of Theorem 3 with suitable dynamical shifts.

The dual exchange relation and an associated fusion procedure are described in the next theorem.
Theorem 10. Let $K_M$ be a solution of the first fused exchange relation (66) and $L_M$ be a solution of (70). Then $\tilde{K}_M = (L_M^t)^{-1}K_M$ is a solution of the KDM-type dual fused exchange relation:

$$(A_{MN'}^{-1})^{t_Mt_{N'}} \tilde{K}_M((B_{MN'}^{-1})^{-1})^{t_M'}\tilde{K}_{N'}(h_M) = \tilde{K}_{N'}((C_{MN'}^{-1})^{-1})^{t_M}\tilde{K}_M(h_{N'})((D_{MN'}^{-1})^{-1})^{t_{N'}}(76)$$

Proof. Reproduces the proof of Theorem 4 with suitable dynamical shifts. ■

3.4 Dressing.

Solutions $T_M$ of the fused dynamical exchange relations also admit dressing procedures. However, because of the dynamical nature of the exchange relations some of the equations that the dressings $Q_M$ and $S_M$ obey exhibit shifts, too. Specifically we have

Proposition 6. Let $T_M$ be a solution of the fused dynamical exchange relation. Then $Q_M T_M S_M$ is also a solution of the fused exchange relation provided $Q_M$ and $S_M$ verify:

$$[Q_M, A_{MN'}] = [Q_{N'}, A_{MN'}] = 0 \quad (77)$$

$$Q_{N'} B_{MN'} = B_{MN'} Q_{N'}(h_M) \quad Q_M C_{MN'} = C_{MN'} Q_M(h_{N'})$$

$$[S_{N'}, C_{MN'}] = [S_M, B_{MN'}] = 0 \quad (78)$$

$$S_M(h_{N'}) D_{MN'} = D_{MN'} S_M \quad S_{N'} D_{MN'} = D_{MN'} S_{N'}(h_M)$$

A particular solution of these constraints is given by:

$$Q_M = \tilde{A}_{12} \tilde{A}_{23} \cdots \tilde{A}_{m-1,m}$$

$$S_M = \tilde{D}_{12} \tilde{D}_{23}(h_1) \cdots \tilde{D}_{m-1,m}(h_{(1,m-2)})$$

Proof. By induction, similar to the non-dynamical dressings. ■

An interesting comparison can be drawn between this formula for $S_M$ and the formula used in [5] to dress the quantum traces for dynamical quantum groups. The formula for $S_M$ is exactly the “mirror image” of the formula:

$$S_M^{A_{BB}} = \tilde{R}_{12}(h_{(3,m)}) \cdots \tilde{R}_{m,m-1}.$$
3.5 Three lemmas: dynamical and cyclic properties of $D$.

Three easy technical lemmas are required to proceed with the construction.

**Lemma 4 (Dynamical transposition).** Let $R(q)$ and $S(q)$ be two matrices with mutually commuting entries depending on a set of commuting coordinates $\{q_k\}_{k=1}^n$. We then have:

$$
(R(q)e^D S(q))^t = [S^{SL}(q)]^t e^D [R^{SC}(q)]^t
$$

(79)

where $S^{SL}(q)_{ij} = e^{\partial_i} S(q)_{ij} e^{-\partial_i} = S_{ij}((q_1, \ldots, q_{(i)} + 1, \ldots, q_n))$ (shift on line index) and $R^{SC}(q)_{ij} = e^{-\partial_j} R(q)_{ij} e^{\partial_j}$ (shift on column index).

**Proof.** We compare the $ij$-th entry on both sides using the fact that entries of $S^{SL}$ and $R^{SC}$ do not contain explicit shift quantities $e^{\partial}$ and therefore commute with each other. If (in the case of $k$-tensor products) $"i"$ denotes a $k$-uple of indices $(i_1, \ldots, i_k)$, the notation $q_{(i)} + 1$ must be interpreted as $q_{i_1 + 1}, \ldots, q_{i_k + 1}$. ■

**Remark.** Later we will use this lemma in the special case when $R(q)$ is diagonal. This implies $R^{SC}(q) = e^{-D} R(q) e^D$.

**Lemma 5 (Matrix dynamical shift).** Let $D(q)$ be a matrix obeying the zero weight condition:

$$
[D_{12}, h \otimes 1 + 1 \otimes h] = 0 \quad (h \in \mathfrak{h})
$$

(80)

Then the exponentials can be “pushed through” $D$, that is we have

$$
e^{-D_{12} - D_2} D_{12} = \bar{D}_{12} e^{-D_{12} - D_2}
$$

(81)

where $\bar{D}_{12} = D_{12}^{SL}$.

**Proof.** What this lemma means is that one can write $e^{-D_{12} - D_2} D_{12} e^{D_{12} + D_2}$ in a matrix form where the exponentials of derivatives cancel out. The proof is straightforward because the zero weight condition implies the identification of incoming and outgoing indices of $D$. One then verifies easily the equality of the two sides. ■
Lemma 6. Let $D(q)$ be a matrix obeying the zero weight condition:

$$[D_{12}, h \otimes 1 + 1 \otimes h] = 0 \quad (h \in h) \quad (82)$$

Then $D$ is cyclic with respect to the trace operation over $V_1 \otimes V_2$ as follows:

$$Tr_{12} (D_{12} X_{12} D^{-1}_{12} e^{D_1} e^{D_2}) = Tr_{12} (X_{12} e^{D_1} e^{D_2}) \quad (83)$$

where $X$ is an arbitrary matrix the entries of which commute with the entries of $D$.

Proof. Consequence of the ZW property of $D$, which allows to reinterpret the matrix indices of $e^{D_1 + D_2}$ as line instead of column indices of $D^{-1}_{12}$, allowing then to independently sum over the now decoupled column indices of $D^{-1}_{12}$ with line indices of $D_{12}$ to altogether eliminate the matrix $D$ from the trace. Labels 1 and 2 formally denote here tensored auxiliary spaces. ■

3.6 Commuting hamiltonians.

We can now state the fundamental result of this section.

Theorem 11. Let $T_M$ be a solution of the fused dynamical exchange relations (65). $T_M$ acts on the tensor product of the auxiliary spaces labeled by $M$ and on the quantum space $V_q$.

Let $K_M$ be a solution of the dual fused dynamical exchange relation (68). $K_M$ acts on the tensor product of the auxiliary spaces labeled by $M$ and on the quantum space $V_q$.

The following operators

$$H_M = Tr_M (T_M e^{D_M} (K_M^{SC})^t M) \quad (84)$$

constitute a family of mutually commuting quantum operators acting on $V_q \otimes V_q'$

$$[H_M, H_{N'}] = 0 \quad (85)$$

Proof. Similar to the preceding one, but extra care must be taken because of the shift operators that enter the expression. Using the dynamical transposition lemma for $K_{N'}$ one has:

$$H_M H_{N'} = Tr \left( T_M e^{D_M} (K_M^{SC})^t M T_{N'} e^{D_{N'}} (K_{N'}^{SC})^t N' \right) = \quad (86)$$

$$Tr \left( T_M e^{D_M} (K_M^{SC})^t M T_{N'}^{N'} K_{N'} e^{D_{N'}} \right)$$
since the invariance of the trace with respect to transposition is preserved in the dynamical case. One then writes:

\[ \text{Tr} \left( T_M T_{N'}^{tN'}(h_M) e^{D_M (K_M^{SC})^tM} K_{N'} e^{D_{N'}} \right) = \]
\[ \text{Tr} \left( T_M T_{N'}^{tN'}(h_M) B_{MN'}^{tN'}(B_{MN'}^{tN'})^{-1} e^{D_M (K_M^{SC})^tM} K_{N'} e^{D_{N'}} \right) = \]
\[ \text{Tr} \left( (T_M B_{MN'} T_{N'}(h_M))^{tM}_{tN'} ((B_{MN'}^{tN'})^{-1} e^{D_M (K_M^{SC})^tM})^{tM} K_{N'} e^{D_{N'}} \right) \]

In the last equality the identification \( T_{N'}^{tN'}(h_M) B_{MN'}^{tN'} = (B_{MN'} T_{N'}(h_M))^{tN'} \) uses the zero-weight condition \( [B_{MN'}, h_M] = 0 \). Using once again the dynamical transposition lemma and the zero-weight condition on \( B \) which guarantees \( e^{D_M ((B_{MN'}^{tN'})^{-1})^{SCtM}} = (B_{MN'}^{tN'})^{-1} e^{D_M} \) as commented above, one gets:

\[ \text{Tr} \left( (T_M B_{MN'} T_{N'}(h_M))^{tM}_{tN'} A_{MN'}^{tM}_{tN'} (A_{MN'}^{tM}_{tN'})^{-1} K_M (B_{MN'}^{tN'})^{-1} e^{D_M} K_{N'} e^{D_{N'}} \right) = \]
\[ \text{Tr} \left( (A_{MN'} T_M B_{MN'} T_{N'}(h_M))^{tM}_{tN'} (A_{MN'}^{tM}_{tN'})^{-1} K_M (B_{MN'}^{tN'})^{-1} K_{N'}(h_M) e^{D_M} e^{D_{N'}} \right) = \]

One here identifies the direct and dual exchange relation. to yield:

\[ \text{Tr} \left( (T_{N'} C_{MN'} T_M (h_{N'}))^{tM}_{tN'} K_{N'} (C_{MN'}^{tM}_{tN'})^{-1} K_M (h_{N'}) (D_{MN'}^{tM}_{tN'})^{-1} e^{D_M} e^{D_{N'}} \right) = \]
\[ \text{Tr} \left( (D_{MN'}^{tM}_{tN'}) (T_{N'} C_{MN'} T_M (h_{N'}))^{tM}_{tN'} K_{N'} (C_{MN'}^{tM}_{tN'})^{-1} K_M (h_{N'}) (D_{MN'}^{tM}_{tN'})^{-1} e^{D_M} e^{D_{N'}} \right) \]

Here Lemma 6 is at work.

\[ \text{Tr} \left( (T_{N'} C_{MN'} T_M (h_{N'}))^{tM}_{tN'} K_{N'} (C_{MN'}^{tM}_{tN'})^{-1} K_M (h_{N'}) e^{D_M} e^{D_{N'}} \right) = \]
\[ \text{Tr} \left( T_{N'} (C_{MN'} T_M (h_{N'}))^{tM}_{tN'} (K_{N'} (C_{MN'}^{tM}_{tN'})^{-1} e^{D_{N'}})^{tN'} K_{M} e^{D_M} \right) = \]
\[ \text{Tr} \left( T_{N'} T_{M}^{tM}_{tN'} (h_{N'}) C_{MN'}^{tM}_{tN'} (C_{MN'}^{tM}_{tN'})^{-1} e^{D_{N'}} K_{N'}^{SCtN'} K_{M} e^{D_M} \right) \]

Once again we have used the dynamical transposition lemma and the partial weight zero property of \( C_{MN'}^{tN'} \)

\[ \text{Tr} \left( T_{N'} T_{M}^{tM}(h_{N'}) e^{D_{N'}} (K_{N'}^{SC})^{tN'} K_{M} e^{D_M} \right) = \]
\[ \text{Tr} \left( T_{N'} e^{D_{N'}} (K_{N'}^{SC})^{tN'} T_{M} K_{M} e^{D_M} \right) = \text{Tr} \left( T_{N'} e^{D_{N'}} K_{N'} T_{M} e^{D_M} (K_{M}^{SC})^{tM} \right) \]
Without the dressing described by Proposition 6 the traces constructed in (64) decouple just as in the nondynamical case. Indeed we have

**Proposition 7.** Operators built from the solution (64) decouple as

\[
\text{Tr}_M(T_M e^{D_M} (K_M^{SC}(t_M)) = \text{Tr}(T e^D (K^{SC})^t)_{#M}
\]

**Proof.** We will prove the proposition for \( M \) with two elements. The statement remains valid for higher powers by induction. We also need to put the trace under a more amenable form. In fact, \( \text{Tr}(T e^D ((B_1^{12})^{-1}K_1^{SC})^t)_{#M} \) by virtue of Lemma 4

\[
\text{Tr}([T_1 B_1 T_2 (h_1)]^{l_1 t_2} K_1^{-1}K_2 e^{D_1 + D_2}) = \text{Tr}(T_1 [B_1 T_2 (h_1)]^{l_2} [K_1 e^{D_1 ((B_1^{12})^{-1}SC_1)^t} K_2 e^{D_2}) =
\]

where \(( )^{SC_1}\) means \(( )^{SC}\) operation applied on the first space.

\[
\text{Tr}(T_1 T_2 (h_1) B_1^{12} (B_1^{12})^{-1} e^{D_1 K_1^{SC_1} K_2 e^{D_2}}) = \text{Tr}(T_1 e^{D_1} T_2^t K_1^{SC_1} K_2 e^{D_2}) = \text{Tr}(T e^D K^{SC})^2
\]

Of course, the three comments made after Proposition 4 in the nondynamical case remain valid, although we do not know yet of explicit examples for Mezincescu-Nepomechie procedure in a dynamical context.

### 4 The fully dynamical algebra

The third type of quadratic algebra considered here is the extension to general structure matrices \( A, B, C, D \) of the “boundary dynamical algebra” (BDA) considered in [13, 27]. Fusion and trace formulas were defined in [13] for the particular case of BDA where \( A = D = R(u_1 - u_2), B = C = R(u_1 + u_2), R \) being the IRF \( \mathbb{Z}_n \) \( R \)-matrix. The most general “fully dynamical” (denomination to be justified presently) exchange algebra reads:

\[
A_{12}(\lambda) T_1(l + \gamma h_2)B_{12}(l)T_2(l + \gamma h_1) = T_2(l + \gamma h_1)C_{12}(l)T_1(l + \gamma h_2)D_{12}(l)(87)
\]

Once again we assume \( \text{dim } \mathfrak{h} = \text{dim } V \) [25]. The following conditions are imposed on the structure matrices (\( R = A, B, C \) or \( D \)) Unitarity
\( R_{12}(u_1, u_2; l) R_{21}(u_2, u_1; l) = 1 \) \hfill (88)

Zero weight property

\[ [h \otimes 1 + 1 \otimes h, R_{12}(u_1, u_2; l)] = 0 \quad (h \in h) \] \hfill (89)

\( R \) then verifies the same ZW property as in the semi-dynamical case. By contrast with the previous case all four matrices in (87) exhibit a dynamical shift and all four structure matrices have \((1 + 2)\)-zero weight, hence the denomination “fully dynamical”. In some specific examples [27, 28] the structure matrices also obey the dynamical zero weight property:

\[ [D \otimes 1 + 1 \otimes D, R_{12}(u_1, u_2; l)] = 0 \] \hfill (90)

Structure matrices all obey Gervais-Neveu-Felder type equations.

\[
\begin{align*}
A_{12}(l)A_{13}(l + \gamma h_2)A_{23}(l) &= A_{23}(l + \gamma h_1)A_{13}(l)A_{12}(l + \gamma h_3) \\
A_{12}(l)C_{13}(l + \gamma h_2)C_{23}(l) &= C_{23}(l + \gamma h_1)C_{13}(l)A_{12}(l + \gamma h_3) \\
D_{12}(l + \gamma h_3)D_{13}(l)D_{23}(l + \gamma h_1) &= D_{23}(l)D_{13}(l + \gamma h_2)D_{12}(l) \\
D_{12}(l + \gamma h_3)B_{13}(l)B_{23}(l + \gamma h_1) &= B_{23}(l)B_{13}(l + \gamma h_2)D_{12}(l)
\end{align*}
\] \hfill (91)

If the dynamical zero weight property is verified then all equations can be rewritten under the more familiar ‘alternating shift’ form

\[
\begin{align*}
R_{12}(l - \gamma h_3)R_{13}(l + \gamma h_2)R_{23}(l - \gamma h_1) &= R_{23}(l + \gamma h_1)R_{13}(l - \gamma h_2)R_{12}(l + \gamma h_3)
\end{align*}
\]

As in the previous situation, these equations ensure the compatibility of the algebra in the following sense. Let us take the left hand side of exchange relation (87), embed it in a triple tensor product and shift it on the third space. Then let us multiply it with \( B_{13}(u_1, u_3; l)B_{23}(u_2, u_3; l + \gamma h_1)T_3(u_3; l + \gamma h_1 + \gamma h_2) \). One can reverse the order of the \( T \)’s in two different ways which yield the same result if equations (91) are obeyed.

4.1 Fusion procedure and the “dual” algebra

The fusion of the structure matrices is again defined by induction as follows:
Proposition 8. Let $A, B, C, D$ be solutions of the dynamical Yang-Baxter equations. Then the following fused dynamical Yang-Baxter equations hold:

$$A_{MN'} = A_{N'}(h_{(2,m)})A_{M0N'} = A_{Mn'}A_{MN_0}(h_{n'})$$

$$B_{MN'} = B_{1N'}B_{M0N'}(h_1) = B_{MV}(h_{(2',n')})B_{MN_0}$$

$$C_{MN'} = C_{1N'}(h_{(2,m)})C_{M0N'} = C_{MV}C_{MN_0}(h_1)$$

$$D_{MN'} = D_{1N'}D_{M0N'}(h_1) = D_{Mn'}(h_{(r',n'-1)})D_{MN_0}$$

These fused matrices verify the corresponding fused YB-equations and the ZW property.

**Proposition 8.** Let $A, B, C, D$ be solutions of the dynamical Yang-Baxter equations. Then the following fused dynamical Yang-Baxter equations hold:

$$A_{MN'}A_{ML'}(h_{n'})A_{N'L'} = A_{N'L'}(h_{M})A_{ML'}A_{MN'}(h_{l''})$$

$$A_{MN'}C_{ML'}(h_{n'})C_{N'L'} = C_{N'L'}(h_{M})C_{ML'}A_{MN'}(h_{l''})$$

$$D_{MN'}(h_{l''})D_{ML'}D_{N'L'}(h_{M}) = D_{N'L'}D_{ML'}(h_{M})D_{MN'}$$

$$D_{MN'}(h_{l''})B_{ML'}B_{N'L'}(h_{M}) = B_{N'L'}B_{ML'}(h_{M})D_{MN'}$$

**Proof.** Straightforward by induction.  

Note that the dynamical zero weight property does not survive fusion, but algebraic zero weight does. In this sense this dynamical zero weight property is not relevant for the construction of commuting traces, and is not (generically) a feature of the universal algebra. We will from now on disregard it. In addition, we will concentrate here on the most relevant features of quantum trace building, ignoring for instance the possibility of a “second fusion”.

**Theorem 12.** Let $T$ be a solution of the dynamical quadratic exchange relation

$$A_{12}T_1(h_2)B_{12}T_2(h_1) = T_2C_{12}T_1(h_2)D_{12}$$

then

$$T_M = \prod_{i \in M} \left( T_i \left( \sum_{k \neq i} h_k \right) \left( \prod_{j > i} B_{ij}(h_j + \sum_{k < j} h_k) \right) \right)$$

verifies the fused dynamical exchange relation

$$A_{MN'}T_M(h_{n'})B_{MN'}T_{N'}(h_{M}) = T_{N'}(h_{M})C_{MN}T_M(h_{n'})D_{MN'}$$

27
Proof. Similar to that of Theorem 11 but the induction step uses the fact that $T_M = T_1(h_{M_0})B_{1M_0}T_{M_0}(h_1)$ and uses the fused dynamical YB-equations. ■

The dual exchange relation and the associated fusion procedure are described in the next theorem.

Theorem 13. Let $K$ be a solution of the dynamical quadratic exchange relation

$$A_{12}^d(l)K_1(l + \gamma h_2)B_{12}^d(l)K_2(l + \gamma h_1) = K_2(l + \gamma h_1)C_{12}^d(l)K_1(l + \gamma h_2)D_{12}^d(l)$$

where

$$A_{12}^d = ((A_{12}^{-SL_{12}})^{-1})^{-SC_{12}t_{12}} B_{12}^d = (((B_{12}^{-SL_{12}})^{-S_{C_{12}t_{12}}}^{-1})^{SL_{12}} t_{12}$$

$$C_{12}^d = (((C_{12}^{-SL_{12}})^{-SC_{12}t_{12}}}^{-1})^{SL_{12}} D_{12}^d = ((D_{12}^{-SL_{12}})^{-1})^{SL_{12}} t_{12}$$

then

$$K_M = \prod_{i \in M} \left( K_i \left( \prod_{k \neq i, j \in M} \left( \prod_{k \leq l \in M} B_{ij}^d \left( \sum_{k \leq l} h_k + \sum_{k > l} h_k \right) \right) \right) \right)$$

verifies the fused dynamical exchange relation

$$A_{MN'}^d K_M(h_{N'})B_{MN'}^d K_N(h_M) = K_{N'}(h_M)C_{MN'}^d K_M(h_{N'})D_{MN'}^d$$

Proof. Straightforward once one has established that the fused dual structure matrix is equal to the dual of the fused structure matrix and that the YB-equations obeyed by the dual structure matrices derive from the equations (91). ■

4.2 Dressing

Proposition 9. Let $T_M$ be a solution of the fused fully dynamical exchange relation. Then $Q_M T_M S_M$ is also a solution of the fused exchange relation provided $Q_M$ and $S_M$ verify:

$$Q_M A_{MN'} = A_{MN'} Q_M(h_{N'}) \quad Q_M(h_M) A_{MN'} = A_{MN'} Q_M(h_{N'})$$

$$Q_{N'} B_{MN'} = B_{MN'} Q_{N'}(h_M) \quad Q_M C_{MN'} = C_{MN'} Q_M(h_{N'})$$

$$S_{N'}(h_M) C_{MN'} = C_{MN'} S_{N'}(h_M) \quad S_M(h_{N'}) B_{MN'} = B_{MN'} S_M(h_{N'})$$

$$S_M(h_{N'}) D_{MN'} = D_{MN'} S_M \quad S_{N'} D_{MN'} = D_{MN'} S_{N'}(h_M)$$

28
A particular solution of these constraints is given by:

\[ Q_M = \tilde{A}_{12}(h_{(3,m)})\tilde{A}_{23}(h_{(4,m)}) \ldots \tilde{A}_{m-1,m} \]

\[ S_M = \tilde{D}_{12}\tilde{D}_{23}(h_1) \ldots \tilde{D}_{m-1,m}(h_{(1,m-2)}) \]

**Proof.** By induction. ■

### 4.3 Commuting traces

We use the following properties inferred from lemma 5.

\[ e^{-D_1 - D_2}A_{12} = A_{12}^{-SL}e^{-D_1 - D_2} = \tilde{A}_{12}e^{-D_1 - D_2} \]

\[ e^{-D_2}A_{12}e^{D_1} = e^{D_1}\tilde{A}_{12}e^{-D_2} \]

and their transposed variants:

\[ e^{D_1}(\tilde{A}_{12}^{-SLt_2})^{-1}e^{D_2} = e^{D_2}(A_{12}^{-SLt_2})^{-1}e^{D_1} \]

and so on. Since these relations are immediately derived from the ZW property on the structure matrices, they remain valid for fused structure matrices, too, since the fusion respects the zero weight property as opposed to the dynamical zero weight property (cf. remark above). In this case labels 1 and 2 formally denote tensored auxiliary spaces.

**Theorem 14.** Let \( T_M \) be a solution of the fused dynamical exchange relations (65). \( T_M \) acts on the tensor product of the auxiliary spaces labeled by \( M \) and on the quantum space \( V_q \).

Let \( K_M \) be a solution of the dual fused dynamical exchange relation (69). \( K_M \) acts on the tensor product of the auxiliary spaces labeled by \( M \) and on the quantum space \( V_q' \).

The following operators

\[ H_M = Tr_M e^{-D_M}T_M e^{D_M}K_{M}^{SCt_M} \]

constitute a family of commuting operators acting on \( V_q \otimes V_{q'} \).

\[ [H_M, H_N] = 0 \]
Proof. It is worth to give a detailed description of the proof as in theorem \[\text{11}\] since the occurrence of derivative objects \(\sim e^{D_M}\) considerably modifies it in comparison to the standard Sklyanin-type proof for non-dynamical algebras. Once again the dynamical transposition lemma plays an essential role.

\[H_M H_{N'} = Tr \ e^{-D_M} T_M e^{D_M} K^{SCMtM} \ e^{-D_{N'}} T_{N'} e^{D_{N'}} K^{\bar{SC}_{N'tN'}} =\]
\[Tr \ e^{-D_M} T_M e^{D_M} K^{SCMtM} \ (e^{-D_{N'}} T_{N'})^{t_{N'}} \left( e^{D_{N'}} K^{SC_{N'tN'}} \right)^{t_{N'}} =\]
\[Tr \ e^{-D_M} T_M e^{D_M} K^{SCMtM} T_{N'}^{-SL_{N'tN'}} e^{-D_{N'}} K_{N'} e^{D_{N'}} =\]
\[Tr \ e^{-D_M} T_M e^{D_M} T_{N'}^{-SL_{N'tN'}} K^{SCMtM} e^{-D_{N'}} K_{N'} e^{D_{N'}} =\]
\[Tr \ e^{-D_M} \left[ e^{-D_{N'}} A^{-1}_{MN',T_{N'}(h_M) C_{MN'} T_{M}(h_{N'}) D_{MN'} e^{D_M}} \right]^{t_{N'}} \times\]
\[\left( B_{MN'}^{-SC_{N'tN'}} \right)^{-1} e^{D_{N'}} K^{SCMtM} e^{-D_{N'}} K_{N'} e^{D_{N'}} =\]
\[Tr \ [ e^{-D_M - D_{N'}} A^{-1}_{MN',T_{N'}(h_M) C_{MN'} T_{M}(h_{N'}) D_{MN'}} ]^{t_{MN'}} \times\]
\[e^{D_M} ( B_{MN'}^{-SC_{N'tN'}} )^{-1} e^{D_{N'}} K^{SCMtM} \left[ e^{-D_{N'}} K_{N'} e^{D_{N'}} \right] =\]

Pushing exponentials through \(B\).

\[Tr \left[ e^{-D_M - D_{N'}} A^{-1}_{MN',T_{N'}(h_M) C_{MN'} T_{M}(h_{N'}) D_{MN'}} \right]^{t_{MN'}} e^{D_{N'}} \times\]
\[\left[ ( B_{MN'}^{-SL_{N'tN'}} )^{-1} e^{D_M} K^{SCMtM} \right]^{t_{MN'}} e^{-D_{N'}} K_{N'} e^{D_{N'}} =\]
\[Tr \left[ A^{-1}_{MN'} e^{-D_M - D_{N'}} T_{N'}(h_M) C_{MN'} T_{M}(h_{N'}) D_{MN'} \right]^{t_{MN'}} e^{D_{N'}} \times\]
\[K_{N'} \left[ e^{D_M} \left( ( B_{MN'}^{-SL_{N'tN'}} )^{-1} \right) SC_{MtM} \right] e^{-D_{N'}} K_{N'} e^{D_{N'}} \]
Using zero weight of $A$ and $B$ transposed.

$$
\begin{align*}
Tr \left[ T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} \right]^{-SL_{MN't}M_{N'}} e^{-D_{MN't}N'} & \times \\
(\bar{A}_{MN'})^{-SC_{MN't}M_{N'}} e^{B_{MN't}N'}K_M e^{-D_{MN't}D_{N'}} & \times \\
Tr \left[ T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} \right]^{-SL_{MN't}M_{N'}} & \times \\
\{(\bar{A}_{MN'})^{-SC_{MN't}M_{N'}}K_M(h_N')((B_{MN't}N')^{-1})SL_{t}M_kN'_N(h_M)\} & e^{D_{MN't}D_{N'}} = \\
Tr \left[ T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} \right]^{-SL_{MN't}M_{N'}} & \times \\
K_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't}^{-1})SL_{N't}N'_K(h_{N'})((\bar{D}_{MN't}N')SL_{MN't}M_{N'}) & e^{D_{MN't}D_{N'}} = \\
Tr \left[ e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} \right] & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}C_{MN't})^{-1})SL_{N't}N'_K(h_{N'})] & e^{D_{MN't}D_{N'}} = \\
Tr e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't})^{-1}) & e^{D_{MN't}D_{N'}} = \\
Tr e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't})^{-1}) & e^{D_{MN't}D_{N'}} = \\
Tr e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't})^{-1}) & e^{D_{MN't}D_{N'}} = \\
Tr e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't})^{-1}) & e^{D_{MN't}D_{N'}} = \\
Tr e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't})^{-1}) & e^{D_{MN't}D_{N'}} = \\
\end{align*}
$$

Using zero weight of $C$.

$$
\begin{align*}
Tr e^{-D_{MN't}D_{N'}}T_{N'}(h_M)C_{MN'T}M(h_{N'})D_{MN'} & \times \\
[\bar{K}_{N'}(h_M)((\bar{C}_{MN't}t)C_{MN't})^{-1}) & e^{D_{MN't}D_{N'}} = \\
\end{align*}
$$

5 Conclusion

We have now defined fusion and trace procedures in view of obtaining commuting hamiltonians of “quantum trace type”, for the non-dynamical general quadratic algebra (I), for the semi-dynamical quadratic algebra (II).
and for the fully dynamical quadratic algebra (87). Our immediate interest is now to apply this procedure to some particularly interesting examples of such quadratic algebras, the most relevant being at this time the scalar Ruijsenaars-Schneider quantum Lax formulation (semi-dynamical type) [29].

Note in this respect that previous application of an order-one trace formulation (i.e. without auxiliary space tensor products) to the specific case of “boundary dynamical $\mathfrak{sl}(2)$ algebras” considered in [12] yielded models described in [27] as generalizations of the Gaudin models. Positions of the sites were associated with values of the spectral parameters (in a spin-chain type construction), not with the dynamical variable itself whose interpretation is unclear.

As already emphasized, our elucidation of tensor product structure for quadratic algebras is also very important in formulating generalizations of the Mezincescu-Nepomechie fusion procedure in general open spin chains [9].

Our constructions moreover also shed light on some characteristic properties of the quadratic algebra. The building of commuting traces requires first of all the introduction of a dual exchange relation. It seems possible that this notion reflects the existence of anti-automorphisms of the underlying hypothetical algebra structure, of which the transposition and crossing-relations used in the non-dynamical cases (see [2]) would be realizations.

The explicit formulation of consistent fusion relations should also help in understanding the meaning of quantum algebra (QA) structures and characterizing in particular their coalgebra properties. As pointed out, the DKM-type fusions do stem in at least one case from a universal structure [15], and so does the fusion for boundary dynamical algebra (case when $A, B, C, D$ stem from one single dynamical $R$-matrix [14]). Regarding the semi-dynamical QA it was already known [11] that one could extend the quantum space on which entries of $T$ act, by auxiliary spaces of $A$ and $B$ or $C$ and $D$ matrices, thereby obtaining spin-chain like construction of a monodromy matrix (comodule structure). We have now defined the complementary procedure, extending the auxiliary space by a “fusion” procedure. This yields the full “coproduct” or rather comodule structure of the DQA (51).

Acknowledgments: AD is supported by the TMR Network “EUCLID”; “Integrable models and applications: from strings to condensed matter”, contract number HPRN-CT-2002-00325.
References

[1] I. Cherednik, Theor. Math. Phys. 61 (1984), 977; E. K. Sklyanin, J. Phys. A21 (1988), 2375; D. Fioravanti, M. Rossi, J. Phys A34 (2001), 567; M. Mintchev, E. Ragoucy, P. Sorba, hep-th/0303187; A. Kundu, Mod. Phys. Lett. A10 (1995), 2955; L. Hlavaty, Journ. Math. Phys. 35 (1994), 2560; S. Majid, Journ. Math. Phys 32 (1991), 3246.

[2] J. Avan, A. Doikou, Commuting quantum traces: the case of reflection algebras, J. Phys A 36 (2003), p. 1; math.QA/0305424

[3] L. Freidel, J. M. Maillet, Quadratic algebras and integrable systems, Phys. Lett B 262 (1991), p. 268.

[4] L. Faddeev, Integrable models in (1+1)-dimensional quantum field theory, in: Les Houches 1982 ed.: J. B. Zuber and R. Stora pp. 561-608

[5] J. Avan, O. Babelon, E. Billey, The Gervais-Neveu-Felder equation and the quantum Calogero-Moser systems, Comm. Math. Phys. 178 (1996), p. 281; hep-th/9505091

[6] J.M. Maillet, Lax equations and quantum groups, Phys. Lett. B 245 (1990), p. 480.

[7] J. M. Maillet, Phys. Lett B 162 (1985), p. 137.

[8] L. Mezincescu, R. I. Nepomechie, Fusion procedure for open spin chains, J. Phys. A 25 (1992) p. 2533.

[9] D. Arnaudon, J. Avan , N. Crampé, A. Doikou, L. Frappat, E. Ragoucy, General boundary conditions for the $sl(N)$ and super $sl(M/N)$ open spin chains, J. Stat. Mech.: Theor. Exp. (JSTAT) P08 (2004) P08005, math-ph/0406021

[10] A. N. Kirillov, N. Yu. Reshetikhin, Exact solution of the integrable XXZ Heisenberg model, J.Phys. A 20 (1987) p. 1565; P. P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. 5 (1981) p. 393.

[11] Z. Nagy, J. Avan and G. Rollet, Construction of dynamical quadratic algebras, Lett. Math. Phys. 67 (2004) p. 1; math.QA/0307026
[12] Heng Fan, Bo-Yu Hou, Kang-Jie Shi, Representation of the boundary elliptic quantum group $BE_{r,q}(sl_2)$ and the Bethe ansatz, Nucl. Phys. B 496 (1997) p. 551-570;

[13] Heng Fan, Bo-You Hou, Guang-Liang Li, Kang-Jie Shi, Integrable $A_n^{(1)}$ IRF model with reflecting boundary conditions Mod. Phys. Lett. A 26 (1997) pp. 1929-1942.

[14] P. P. Kulish, A. I. Mudrov: Dynamical reflection equation, math.QA/0405556

[15] J. Donin, A. I. Mudrov, Reflection equation, twist and equivariant quantization, Isr. J. Math. 136 (2003), p. 11., math.QA/0204295; J. Donin, P. P. Kulish and A. I. Mudrov, On universal solution to reflection equation, Lett.Math.Phys., 63 (2003) p.179; math.QA/0210242

[16] P. P. Kulish, E. K. Sklyanin: Algebraic structure related to the reflection equation, J. Phys. A 25 1992) p. 5963.

[17] L. Freidel, J. M. Maillet On classical and quantum integrable field theories associated to Kac-Moody current algebras, Phys. Lett B 263 (1991), p. 403.

[18] A. Doikou, J. Phys. A33 (2000) 8797.

[19] J.B. McGuire, J. Math. Phys. 5 (1964) 622.

[20] C.N. Yang, Rev. Lett. 19 (1967) 1312.

[21] R.J. Baxter, Ann. Phys. 70 (1972) 193; J. Stat. Phys. 8 (1973) 25; Exactly solved models in statistical mechanics (Academic Press, 1982)

[22] V.E. Korepin, Theor. Math. Phys. 76 (1980) 165; V.E. Korepin, G. Izergin and N.M. Bogoliubov, Quantum inverse scattering method, correlation functions and algebraic Bethe Ansatz (Cambridge University Press, 1993).

[23] A.B. Zamolodchikov, Al.B. Zamolodchikov, Ann.Phys. 120 (1979) 253.

[24] L.D. Faddeev and L.A. Takhtajan, J.Sov.Math. 24 (1984) 241; L.D. Faddeev and L.A. Takhtajan, Phys.Lett. 85A (1981) 375.
[25] P. Etingof, A. Varchenko, Solution of the quantum dynamical Yang-Baxter equation and dynamical quantum groups, Comm. Math. Phys. 196 (1998) p. 591, q-alg/9708015

[26] G.E.Arutyunov, L.O. Chekhov, S.A. Frolov, R-matrix quantization of the elliptic Ruijsenaars-Schneider model, Comm. Math. Phys. 192 (1998), pp. 405-432, q-alg/9612032 G. E. Arutyunov, S. A. Frolov, Quantum dynamical R-matrices and quantum Frobenius group Comm. Math. Phys. 191 (1998), pp. 15-29, q-alg/9610009

[27] Mark D. Gould, Yao-Zhong Zhang, Shao-You Zao, Elliptic Gaudin models and elliptic KZ equations, Nucl.Phys. B 630 (2002) p.492-508, nlin.SI/0110038;

[28] G.Felder , Proc. ICM Zürich hep-th/9407154 (1994), 1247; Proc. ICMP Paris (1994), 211.

[29] Z. Nagy, J. Avan: Spin chains from dynamical quadratic algebras, in preparation.