Optimisation of hybrid high-modulus/high-strength carbon fiber reinforced plastic composite drive

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Abstract

This study deals with the optimisation of hybrid composite drive shafts operating at subcritical or supercritical speeds, using a genetic algorithm. A formulation for the flexural vibrations of a composite drive shaft mounted on viscoelastic supports including shear effects is developed. In particular, an analytic stability criterion is developed to ensure the integrity of the system in the supercritical regime. Then it is shown that the torsional strength can be computed with the maximum stress criterion. A shell method is developed for computing drive shaft torsional buckling. The optimisation of a helicopter tail rotor driveline is then performed. In particular, original hybrid shafts consisting of high-modulus and high-strength carbon fibre reinforced epoxy plies were studied. The solutions obtained using the method presented here made it possible to greatly decrease the number of shafts and the weight of the driveline under subcritical conditions, and even more under supercritical conditions. This study yielded some general rules for designing an optimum composite shaft without any need for optimisation algorithms.

Keywords: Drive shaft, Optimisation, Hybrid carbon fibre reinforced plastic

1. Introduction

Since the 1970s, composite materials have been regarded as potential candidates for manufacturing drive shafts because of their high specific stiffness and strength [1]. Previous studies on this topic have dealt mainly with composite shaft design in the subcritical case, that is when the first critical speed is never exceeded. However, when a long driveline is required (in the case of helicopters, tilt-rotors, tail-less aircraft with twin turboprops, etc.), an additional means of increasing the drive shaft length consists in operating above the first critical speed, in the so-called supercritical regime. The main advantage of long shafts is that they reduce the number of bearing supports required, and thus greatly decrease the maintenance costs and the weight of the driveline. The design process is more complex, however, because the shaft has to cross a critical speed, and dynamic instabilities due to rotating damping can occur in this regime. Aeronautical applications lend themselves well to operating in the supercritical regime because the driveline always rotates at the nominal speed during flight, since they undergo acceleration and deceleration processes on the ground. The aim of this paper is to optimise a supercritical drive shaft in this practical case.

Many different numerical methods have been used to design optimised composite drive shafts in order to reduce their weight, for example. Traditional methods based on the gradients of continuous functions have been used for this purpose by several authors [2–4]. These methods are unsuitable in the case of composite laminates, however, because many of the variables which have to be optimised are discrete variables (such as the number of plies and the ply angle in prepreg lay-up processes). It is therefore necessary to assume these variables to be continuous in order to be able to compute the gradients required. The optimisation techniques available for solving problems involving discrete variables are known as metaheuristic methods. For example, Gubran and Gupta [5] have used simulated annealing techniques based on a neighbourhood approach. A review of the literature shows that genetic algorithms (GA) [6, 7] are well adapted to designing laminate structures. GA were recently used to optimise a flexible matrix composite drive shaft in [8]. Here it is proposed to use a GA with penalisation methods to account for the constraint functions. In addition, in order to reduce the CPU time, all the design aspects are handled without requiring the use of finite element methods.

In drive shaft applications, the choice of composite material is of great importance. Several authors have recommended the use of hybrid composites in the production of drive shafts. Xu et al. [9], Gubran [10], and Badie et al. [11] recently studied the advantages of a mixture of glass and carbon fibres in a modified epoxy matrix. Lee et al. [12], Gubran [10], Mutasher [13] and Abu Talib et al. [14] recently have studied the design and manufacture of hybrid metallic/composite drive shafts. Here it is proposed to study the use of...
2. Design aspects

2.1. Flexural vibration analysis

When designing supercritical shafts, the external damping has to be maximised in order to reduce the flexural imbalance responses and increase the stability in the supercritical regime. Rolling-element bearings provide insufficient damping. Dissipative materials such as elastomers have recently been used as bearing supports as a passive means of enhancing the non-rotating damping [17, 18]. A low cost configuration consisting of an axisymmetric composite shaft and increase the stability in the supercritical regime. Rolling-element bearings provide insufficient damping. Dissipative materials such as elastomers was studied here (Fig. 1).
Various approaches based on beam and shell theories have been used to compute the critical speeds of composite shafts, most of which were placed on infinitely rigid supports. The simplest of these theories is the Equivalent Modulus Beam Theory (EMBT) [1]. Based on this approach, it is proposed to investigate a rotating beam with Timoshenko’s assumptions [22], replacing the isotropic properties of the material by the homogenised properties of the composite. These equations are also adapted to account for the motion of the supports and the internal damping terms. Lastly, the three complex governing equations and boundary conditions used can be written in the following form:

\[
\ddot{u} - \frac{I_y}{S} \left( 1 + \frac{E}{kG} \right) \frac{\dot{u}}{S} + \frac{E}{\rho S} \left( \frac{\dot{u}'''}{S} + \frac{e_i}{\rho S} (\dot{u}_s - i \Omega u_s) \right) = 0,
\]

(1)

\[
\int_0^l \rho S \ddot{u} \ddot{x} + 2m_s \ddot{u}_s + 2c_e \ddot{u}_b + 2k_e \ddot{u}_b = 0,
\]

(2)

\[
\int_0^l \rho S \left( x - \frac{1}{2} \right) \ddot{u} \ddot{x} + 2m_b \ddot{u}_b + 2c_b \ddot{u}_b + 2k_b \ddot{u}_b = 0,
\]

(3)

\[
\ddot{u}'(0, t) = \ddot{u}''(l, t) = 0, \quad u_s(0, t) = u_s(l, t) = 0
\]

(4)

where \( \ddot{u} \) is the deflection of the shaft (see the list of parameters for the nomenclature). Using the method presented in [18], the above equations yield the four critical speeds for the nth harmonic:

\[
\omega_{cnF\pm} = \pm \frac{1}{\sqrt{2} \Delta_n} \sqrt{\omega_{sn}^2 + \Lambda_n - \omega_{bn}^2}
\]

(5)

\[
\omega_{cnB\pm} = \pm \frac{1}{\sqrt{2} \Delta_n} \sqrt{\omega_{sn}^2 + \Lambda_n + \omega_{bn}^2}
\]

(6)

where \( \omega_{sn} = n^2 \pi^4 EI_y = \frac{k_n}{m_s}, \omega_{bn} = \frac{k_e}{m_b + \frac{m_s}{2n^2 c_i + 4n^2 s_i}}, \Delta_n = \frac{n^2 \pi^2 l_t^2}{S l^2} \),

\[
\Pi_n = 1 + \frac{n^2 \pi^2 l_t^2}{S l^2} \left( 1 + \frac{E}{kG} \right), \quad \Phi_n = \frac{m_e}{m_b + \frac{m_s}{2n^2 c_i + 4n^2 s_i}},
\]

\[
\Psi_n = \frac{\Delta_n}{n^2 \pi^2 l_t^2} \left( \frac{1}{\rho S} \right), \quad \Lambda_{n\pm} = \Psi_n \pm \Gamma_n, \quad \Lambda_{n\pm} = \Pi_n \pm \Gamma_n
\]

(7)

In addition, in the case of a composite shaft consisting of a symmetric laminate, the homogenised properties can be computed with the following equations: \( E = 1/\alpha_{11} \), \( G = 1/\alpha_{66} \), and \( v = -\alpha_{12}/\alpha_{11} \) where \( \alpha = A^{-1} \) [23]. It is also assumed that \( \kappa = 2(1 + v)/(4 + 3v) \).

In the field of rotordynamics, internal damping, which is also referred to as rotating damping, is known to cause whirl instability in the supercritical regime. In the literature, the internal damping resulting from dissipation in the shaft material and dry friction between the assembled components has been usually approached using the viscous damping model. However, most materials, such as carbon/epoxy materials in particular, show vibratory damping, which resembles hysteretic damping much more than viscous damping [24, 29]. Using the classical equivalence between viscous and hysteretic damping [18], the analytical instability criterion suitable for shaft optimisation purposes, can be written in the following form:

\[
\pm \left\{ \eta_n k_n \Phi_n (\Pi_n \omega_{bn}^2 - 2 \omega_{sn}^2) - \eta_k k_n (\omega_{bn}^2 - \omega_{sn}^2) \right\}
\]

\begin{align*}
&< 0 \implies \omega_{bn} \neq \omega_{sn} \\
&> 0 \implies \text{stable}
\end{align*}

(8)

where

\[
\omega_{bn} = \frac{1}{\sqrt{2 \Pi_n}} \sqrt{\omega_{sn}^2 + \Pi_n \omega_{bn}^2}
\]

\[
\pm \sqrt{\omega_{sn}^4 + 2(\Pi_n - 2 \Psi_n) \omega_{sn}^2 \omega_{bn}^2 + \Pi_n^2 \omega_{bn}^4}
\]

(9)

and where the equivalent longitudinal loss factor denoted \( \eta_l \) is computed with Adams, Bacon and Ni’s theory [26, 27] using complex
2.2. Torsional vibration analysis

Torsional vibrations are computed using classical methods with the relations presented by Lim and Darlow [3]:

\[ \omega_n = \frac{v_n}{T} \sqrt{\frac{G}{\rho}} \]  

with

\[ \eta_1 = 0.11\% \text{, } \eta_2 = 0.70\% \text{ and } \eta_6 = 1.10\% \].

It then suffices to compute the lowest threshold speed in order to determine the spin speed limit of the shaft.

A dynamic shaft test rig corresponding to the case of Fig. 1 has been developed to validate the theoretical results (Fig. 2). Basically, the test rig consists in a shaft that is powered by an electric motor via a belt and pulley system and is capable of a maximum test velocity of 12 000 rpm. Two non-contact laser-optical displacement sensors are able to measure the cross-section displacement in real-time. The detailed characteristics of the test rig are given in [25-28]. In the case of a long aluminium shaft (\( E = 69 \text{ GPa} \), \( \rho = 2700 \text{ kg.m}^{-1} \), \( l = 1.80 \text{ m} \), \( r_m = 23.99 \text{ mm} \), \( t = 2.02 \text{ mm} \)) supported on viscoelastic supports (\( m_b = 2.817 \text{ kg} \), \( k_e = 5.64 \times 10^5 \text{ N.m}^{-1} \)), the critical speeds \( \omega_{c1} \) and \( \omega_{c1} \) were measured to be 251 rad.s\(^{-1} \) and 446 rad.s\(^{-1} \), respectively. The results obtained with the above model are 250 rad.s\(^{-1} \) and 460 rad.s\(^{-1} \) which is very close to the experiment.

The experimental investigation of the instabilities can initiate catastrophic risks for the dynamic test rig. For this reason, it was proposed to study the instabilities using PVC material. Another advantage of the PVC material is its low stiffness and high damping (\( E = 2.2 \text{ GPa} \), \( \rho = 1350 \text{ kg.m}^{-1} \) and \( \eta_l = 0.025\% \)) which decrease the critical and threshold speeds. Several shafts with four different lengths were tested in the supercritical domain (\( r_m = 23.25 \text{ mm} \), \( t = 2.55 \text{ mm} \) with \( m_b = 2.608 \text{ kg} \), \( k_e = 2.58 \times 10^5 \text{ N.m}^{-1} \) and \( \eta_e = 0.07\% \)). A high level of acceleration was required in order to run over the first critical speed. Only the shafts of length 0.8 m and 0.9 m were found to be unstable in the supercritical regime. The theoretical uncoupled natural frequencies (\( \omega_{n1} \) and \( \omega_{n2} \)) forward critical speeds (\( \omega_{n1} \)) and threshold speeds (\( \omega_{n1}, \omega_{n2} \)) vs. experimental data in the case of tubes in PVC material of various length.

\[ v_1 \approx \sqrt{\frac{I_G J_s + J_T J_s + J_s^2}{J_G J_s + J_T J_s + 2 J_G J_T}}, \]

\[ v_{n\pm 1} \approx \frac{(n - 1)\pi}{2} + \frac{(n - 1)^2 \pi^2}{4} + \frac{J_s}{J_T} \]

where \( J_G \), \( J_T \) and \( J_s \) are the mass moment of inertia of the main gear, the tail rotor and the shaft, respectively.

2.3. Failure strength analysis

Only the torsional resistance of the shaft is taken into account here. The transmitted torque causes only in-plane shear, which can be computed with classical laminate theory [23]. Contrary to what occurs with an unsymmetrical free-edge laminate plate, the tubular structure blocks the coupling effects in the case of small displacements. This can be modelled simply by assuming the classical coupling matrix \( B \) to be null before performing the inversion procedure required to compute the strain state.

A conservative approach often used in the case of helicopter drive shafts consists in computing only the fracture of the first ply. The Tsai-Wu criterion [29] can be used in this case to account for the differences between the tensile and compressive strengths, which can be of great importance in the case of HM carbon/epoxy (see Table 1):

\[ \frac{1}{X^2} \sigma_{11}^2 + \frac{1}{Y^2} \sigma_{22}^2 + \frac{F_{12}}{\sqrt{X^2 Y^2}} \sigma_{11} \sigma_{22} + \frac{1}{Y^2} \sigma_{11}^2 \leq 1 \]

\[ + \frac{1}{X^2} \sigma_{22}^2 + \frac{1}{X^2} \sigma_{11}^2 + \frac{1}{Y^2} \sigma_{11} + \frac{1}{X^2} \sigma_{22} \leq 1 \]  

(11)

where \( F_{12} \) is an interaction parameter which is taken to be equal to 0.5. It should be noted that the Tsai-Wu criterion includes the transverse fracture mechanism. In the HS carbon/epoxy material, the transverse failure strain is approximately equal to 0.6 %, while the longitudinal one is equal to 1.8 %. This type of fracture generally has no direct effects on the fracture of the laminate, and this approach therefore seems to be too conservative. Assuming that the structure will be safe up to the occurrence of the first longitudinal or shear failure, a more realistic torque resistance limit can be obtained [15-20].

Figure 2: The dynamic shaft test rig: the zoom image (left) corresponds to the non-contact laser-optical displacement sensors (with the scheme of the laser beams in red), the detail (right) corresponds to the bearing mounted on the six viscoelastic supports (the left plate was removed to take the snapshot).

Figure 3: The theoretical uncoupled natural frequencies (\( \omega_{n1}, \omega_{n2} \)), forward critical speeds (\( \omega_{n1} \)) and threshold speeds (\( \omega_{n1}, \omega_{n2} \)) vs. experimental data in the case of tubes in PVC material of various length.
This limit can be computed quite simply with a maximum stress criterion:

\[-X' \leq \sigma_{11} \leq X ; \quad |\sigma_{12}| \leq s\]  \hspace{1cm} (12)

The comparisons made in Table 2 between the results obtained with these criteria and the experimental data confirm the validity of this approach. In the case of tubes Nos. 1 and 2, the values obtained with the maximum stress criterion showed better agreement with the experimental data than those obtained using the Tsai-Wu criterion. The assumption involving the presence of a null coupling mechanism in the case of unsymmetrical laminates was also found to be true. The Tsai-Wu values could be improved by taking a greater transverse tensile strength \(Y\).

2.4. Torsional buckling analysis

Finite element methods are those most frequently employed to predict torsional buckling. However, an alternative method is presented here, which requires less computing time, and consists in solving the buckling shell problem in the case of orthotropic circular cylinders, using Flügge's buckling shell theory [32]. The laminate theory is included in the shell equations, as established in [33]. This gives the Eqs. (A.1-A.3) (see Appendix A). Since we are dealing here with very long shafts, it is possible to neglect the boundary condition effects. In this case, a simplified displacement field presented by Flügge can be used:

\[
\begin{align*}
  u &= a \sin(hp + \frac{px}{l}) \\
  v &= b \sin(hp + \frac{px}{l}) \\
  w &= c \cos(hp + \frac{px}{l})
\end{align*}
\]  \hspace{1cm} (13)

where \((u, v, w)\) is the displacement field of the middle-surface of the cylinder, \(h\) is the number of half-waves along the cylinder's circumference and \(p\) is the number of half-waves along the axis of the cylinder with a fictive length \(l\). When this displacement field is applied to the shell equations, a classical eigenvalue problem is obtained:

\[
\mathbf{K} \mathbf{U} = 0 \quad \text{with} \quad \mathbf{U} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
\]  \hspace{1cm} (14)

where \(\mathbf{K}\) is the stiffness matrix (the elements of \(\mathbf{K}\) are given in Appendix A). A non-trivial solution exists when the determinant of \(\mathbf{K}\) is null.

The numerical method used here consists in finding the minimum value of the buckling torque \(T_{\text{buck}}\) which cancels the determinant among all the values of \(h \in \mathbb{N}^n\) and \(p \in \mathbb{R}^+\). The computing time was reduced as follows. First, we have observed that the minimum value of the buckling torque is always obtained at \(h = 2\). Second, instead of searching for the value of \(p\) between 0 and \(+\infty\), this unknown can be found by searching around the value of \(l(48r^2/12r^2)^{1/4}/\pi r\) obtained by Flügge [22] in the case of isotropic material. Thirdly, the search for the buckling torque is conducted around the value of \(T_{\text{buck}}\), which can be obtained with an analytic criterion such as Hayashi's criterion [31].

\[
T_{\text{buck}} = 11 \sqrt{7} \left( A_{11} - \frac{A_{12}^2}{A_{22}} \right)^{1/4} D^{3/4} \frac{1}{22}
\]  \hspace{1cm} (15)

It is worth noting that this criterion, like other classical criteria, does not account for the coupling mechanism involved in unsymmetrical laminates.

The shell method was first tested on off-axis stacking sequences. The buckling torque obtained are presented in Table 3 and compared with those obtained with the finite element method, the above criterion and by Bert and Kim [33]. The results obtained with the finite element method using ABAQUS (s4 elements) [34], which were previously validated in [25] based on experimental results obtained by Bauchau et al. [35] in the case of short tube, are taken as reference values. Bert and Kim buckling theory is based on the Sanders shell theory and take the boundary conditions into account. In the table, the results obtained by Bert and Kim correlate perfectly well with the finite element computations. Those obtained with the method presented here show higher differences which is explain by the too low length-to-diameter ratio \((l/(2r_m) = 20)\). The results show the highest differences at the extremum cases \((0°\) and \(90°)\) which yet are symmetric, however, the method is conservative. The error obtained with the Hayashi criterion can reach 82%.

The shell method was then tested on unsymmetrical stacking sequences for higher length-to-diameter ratio \((l/(2r_m) = 50)\) in Table 4. All the tubes presented in the table are of the same size and the laminates all have the same thickness. Buckling torque was computed in the positive and then in the negative direction. The table shows that the Hayashi criterion overestimates the buckling torque, especially in the largest unsymmetrical laminates (Nos. 9-12) by up to 40%. The results obtained with shell theory show good agreement with the finite element calculations, giving a conservative estimate on the whole. The largest errors amounted to only 8% and the mean error was 4%.

2.5. Driveline mass

The driveline is composed of shafts and intermediate supports, which include bearings, fittings, and supports. The intermediate support mass \(m_b\) can be computed with an empirical equation from Lim and Darlow [36]:

\[
m_b = 17.1288 \left( \frac{P_{\text{dv}}}{\Omega_{\text{nom}}} \right)^{0.69}
\]  \hspace{1cm} (16)

where \(P_{\text{dv}}\) is the power transmitted with the driveline (in W) and \(\Omega_{\text{nom}}\) is the nominal spin speed (in rev / min). The driveline's mass can then be computed using the following expression:

\[
m_{\text{dv}} = N_s \times m_s + N_b \times m_b \quad \text{with} \quad m_s = \rho SI
\]  \hspace{1cm} (17)

where \(N_s\) is the number of shafts and \(N_b\) is the number of intermediate supports.

3. Shaft optimisation using a genetic algorithm

The principles underlying the GA algorithm are the same as those on which Darwin's theory of evolution was based. At the beginning, a randomly created population is evaluated with a fitness function. The result gives the fitness of each individual. Starting with this information, the new generation of the population can be deduced using selection, crossover and mutation operators. This process is iterated up to convergence.

The main risk with this stochastic method is that of not obtaining the optimum solution. In particular, GA may tend to converge on local optima and may not be able to cross these attracting points. Another weakness of the method is the large amount of fitness function calculations required. This means that the evaluation procedure must not be too time-consuming.

3.1. Individual

An individual in this driveline optimisation procedure consists of the medium diameter \(r_m\) (which can be fixed or otherwise), the bearing stiffness \(k_e\) (fixed or not), the nominal spin speed \(\Omega_{\text{nom}}\) and
the stacking sequence with various materials, symmetric or otherwise, as in the following example: \([a_1^{\text{mat}_1} \times n_1, ..., a_j^{\text{mat}_j} \times n_j, ..., a_q^{\text{mat}_q} \times n_q]\) where \(a_j, n_j\) and \(\text{mat}_j\) are the orientation, the number of plies and the material of which the ply \(j\) is made, respectively. Under supercritical conditions, the stiffness of the bearings is a necessary optimisation variable because it appears in both the rigid mode frequencies Eq. (1) and the stability criterion Eq. (2). The shaft length corresponds to the driveline length divided by the number of shafts.

There are several possible ways of modelling the chromosomes of individuals. It is proposed here to fix the number of possible orientations, the number of plies, and the number of shafts. This sets the size of the chromosome in the case of a particular optimisation process, which simplifies the crossover operations. Individuals are classically represented by an array of binary numbers. The orientation \(\alpha_j\) can be written with 2 or 3 bits, standing for the sets \([-45, 0, 45, 90]\) or \([-67.5,-45,0,45,67.5,90]\) (in degree units), respectively, which correspond to realistic prepreg hand lay-up orientations. The quantity \(n_j\) is written with 2 or 3 bits corresponding to the sets \([1,2,3,4]\) and \([1.2,3,4,5,6,7,8]\), respectively. The material \(\text{mat}_j\) is written with one bit to take advantage of both HM and HS carbon/epoxy, or metal and HM carbon/epoxy, for example. Lastly, \(K_B\) and \(r_m\) are bounded and generally encoded with 3 bits. For example, a shaft with the following stacking sequence \([45^{\text{mat}_1}, 0^{\text{mat}_2}, 45^{\text{mat}_3}, 0^{\text{mat}_4} \times q]\) (i.e. \(q \geq 2\), with \(r_m = 52 \text{ mm}\) and \(\Omega_{\text{nom}} = 4000 \text{ rev} / \text{min}\), with the bearing stiffness fixed and where \(a_j\) and \(n_j\) are encoded with 2 bits, \(r_m\) with 4 bits, \(\Omega_{\text{nom}}\) with 3 bits and \(\text{mat}_j\) with 1 bit, is defined by the following chromosome:

\[
\begin{array}{cccccccccc}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 \times 45^\circ & n_1 = 2 & \text{mat}_1 & a_2 = 0^\circ & n_2 = 3 & \text{mat}_2 & \ldots
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\ldots & 52 \times 30,60 & \Omega_{\text{nom}} = 4000c(3700,4400)
\end{array}
\]

The string length is therefore simply \((\text{bit}_{\alpha} + \text{bit}_n + \text{bit}_{\text{mat}}) \times q + \text{bit}_{K_B} + \text{bit}_{r_m} + \text{bit}_{\Omega_{\text{nom}}}\).

3.2. Constraints and fitness

The mass is the optimised value generally used in driveline problems. In this paper, the fitness function is equal to the inverse of the mass of one shaft. The other part depends on the strength, buckling and dynamic constraints previously investigated.

The stacking sequence (from inner to outer radius) is written with 2 or 3 bits corresponding to the sets \([1,2,3,4]\) or \([1.2,3,4,5,6,7,8]\).

### Table 2: Comparison between torque resistance calculations on various short BE tubes

| Nos. | 1 | 2 | 3 |
|------|---|---|---|
| Stacking sequence (from inner to outer radius) | ° | [90, 45, -45, 90] | [90, 45, 0, 90] | [90, 0, 90] |
| Outer radius × length | mm | 25.4 × 50.8 | 63.5 × 305 | 25.4 × 50.8 |
| Experimental | N.m | 581 | 4689a | 132 |
| Tsai-Wu criterion | N.m | 2407 (71%) | 1905 (66%) | 130 (3%) |
| Tsai-Wu criterion with \(B = 0\) | N.m | 313 (-40%) | 2613 (-44%) | 130 (3%) |
| Maximum stress in fibre and shear directions | N.m | 517 (-10%) | 1610 (-66%) | 130 (3%) |
| Maximum stress in fibre and shear directions with \(B = 0\) | N.m | 585 (2%) | 4880 (4%) | 130 (3%) |
| Buckling torque computed with Hayashi [31] criterion | N.m | 1049 | 13016 | 2835 (-14) |

2. Mean value of two specimen tests.

### Table 3: Comparison between buckling torque calculations on off-axis BE drive shafts

| Ply orientation angle | ° | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
|----------------------|---|---|----|----|----|----|----|----|
| Abaqus               | N.m (%) | 966 (-35) | 735 (-22) | 979 (-13) | 1647 (-7) | 2445 (-5) | 2957 (-6) | 2835 (-14) |
| Bert & Kim           | N.m (%) | 1587 (7) | 974 (0) | 1126 (0) | 1790 (1) | 2617 (1) | 3156 (1) | 3016 (-8) |
| Hayashi criterion    | N.m (%) | 1887 (27) | 1776 (82) | 1607 (43) | 1648 (-7) | 2216 (-14) | 3925 (-25) | 3653 (3) |
|                     | N.m | 585 (2%) | 4880 (4%) | 130 (3%) | 130 (3%) | 130 (3%) | 130 (3%) | 130 (3%) |
|                     | N.m | 1489 | 974 | 1121 | 1769 | 2587 | 3131 | 3278 |

Maximum stress in fibre and shear directions:

- \(g_1 = K_{\text{str}} T_{\text{str}} / T_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{str}} \leq 1 \) (18)
- \(g_2 = K_{\text{buck}} / T_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{buck}} \leq 1 \) (19)
- \(g_3 = \Omega_{\text{nom}} / T_{\text{nom}} \geq 1 \Rightarrow \text{with } \Omega_{\text{nom}} \leq T_{\text{nom}} \) (20)
- \(g_4 = 1 - K_{\text{inf}} / \Omega_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{inf}} \leq 1 \) (21)
- \(g_5 = 1 - K_{\text{sup}} / \Omega_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{sup}} \leq 1 \) (22)

and in the subcritical case:

- \(g_6 = 1 - K_{\text{inf}} / \Omega_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{inf}} \leq 1 \) (23)
- \(g_7 = 1 - K_{\text{inf}} / \Omega_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{inf}} \leq 1 \) (24)
- \(g_8 = 1 - K_{\text{sup}} / \Omega_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{sup}} \leq 1 \) (25)
- \(g_9 = K_{\text{inf}} / \Omega_{\text{nom}} \geq 1 \Rightarrow \text{with } K_{\text{inf}} \leq 1 \) (26)

where \(g_j\) and \(K_j\) are constraint functions and reserve factors, respectively. Eq. (18) corresponds to the torsional strength constraint, which requires that the torque computed with the strength criterion multiplied by the reserve factor is smaller than the torque required. Eq. (19) is the same equation but for the subcritical buckling. The other equations are those giving the dynamic constraints. Eqs. (21) correspond to the positioning of the nominal spin speed between torsional modal frequencies. As regards the bending modes, the constraints depend on whether the subcritical or supercritical case applies. In the first case, Eq. (23) corresponds to the subcritical assumption, i.e. the nominal spin speed multiplied by the reserve factor must be smaller than the first critical speed. In the supercritical case, Eqs. (25) correspond to the positioning of the nominal spin speed between the flexural critical speeds, and Eq. (26) corresponds to the stability constraint.
Table 4: Comparison between buckling torque calculations on unsymmetrical CFRP drive shafts (l = 4 m, r_m = 40 mm, E_11 = 134 GPa, E_22 = 8.5 GPa, E_66 = E_{55} = 4.6 GPa, E_{44} = 4.0 GPa, E_{12} = 0.29 and t_s = 1.067 mm)

| Laminate | ABAQUS | Present work | Hayashi criterion |
|----------|---------|--------------|-------------------|
| Nos.     | Mesh^a  | Nm | % | Nm | % | Nm | % |
| 1 [-15,-15]_4 | 60-150 | 210 | -8 | 222 | 6 |
| 2 [-15,15]_4 | 60-150 | 214 | -8 | 222 | 4 |
| 3 [30,-30]_4 | 60-150 | 263 | -4 | 283 | 8 |
| 4 [-30,30]_4 | 60-150 | 268 | -3 | 283 | 6 |
| 5 [45,-45]_4 | 60-150 | 385 | -1 | 419 | 9 |
| 6 [-45,45]_4 | 60-150 | 385 | -1 | 419 | 9 |
| 7 [0_2,45,-45,-45,0_2] | 60-150 | 389 | -4 | 493 | 27 |
| 8 [0_2,-45,45,-45,0_2] | 60-150 | 389 | -4 | 493 | 27 |
| 9 [0_2,45,0,-45,0,45] | 30-100 | 358 | -4 | 420 | 17 |
| 10 [0_2,-45,0,45,-45] | 30-100 | 329 | -4 | 420 | 28 |
| 11 [0_2,45,0,-45,0,45] | 30-100 | 355 | -4 | 420 | 28 |
| 12 [0_2,-45,0,45,-45] | 30-100 | 313 | -4 | 420 | 28 |
| 13 [-45,-15,15,45,15,-15,-45,45] | 60-150 | 389 | -4 | 493 | 27 |
| 14 [45,15,-15,15,45,15,-15,-45] | 60-150 | 439 | 2 | 493 | 12 |
| 15 [15,-15,15,-15,15,-15,-15] | 60-150 | 219 | 206 | 6 | 265 | 21 |
| 16 [-15,15,-15,-15,15,-15,-15] | 60-150 | 241 | 226 | 6 | 265 | 10 |

a Number of circumferential elements - number of lengthwise elements.

lem can be overcome by using a penalisation method consisting in deteriorating the quality of an individual that violates one or more constraints, by decreasing the fitness function. The fitness function used for mass minisisation purposes can be written in the following general form:

\[ f = \frac{1}{m_s} + \sum_j \gamma_j \min(0, g_j) \]  

where \( \gamma_j \) are the penalisation factors. The reserve factors and penalisation factors are given in Table 5.

3.3. The genetic algorithm method
3.3.1. Initialisation
The algorithm is initialised by randomly generating a population of 300-600 individuals. The number depends on the size of the problem.

3.3.2. Elitism
After evaluating the population with the fitness function, the two fittest individuals, which are also called the elites, are selected and kept for the next generation.

3.3.3. Scaling, selection and crossover
With the progression of the GA, the fitness of all the individuals tends to converge on that of the fittest ones. This slows down the progress of the algorithm. A windowing method [6] is used here, whereby the fitness of the lowest ranking individual is subtracted from the fitness of each individual. Two parents are then selected, based on their scaled fitness values and a multi-point crossover operation is performed. The cutting point is selected randomly. This operation gives two children, forming the next generation.

3.3.4. Mutation
The mutation consists in randomly modifying the bits of the chromosomes. The probability of occurrence of the mutation must be very high to obtain a highly diverse population. But if the mutation process is too strong, the algorithm may not converge on the optimum fitness. Note that elites are not subject to mutations. After the mutation, the process is restarted at the elitism stage until the maximum fitness function is reached. The search parameters in the GA are given in Table 6.

4. Case study
The helicopter tail rotor driveline presented by Zinberg and Symonds [1] is investigated with the GA. The original driveline, having a total length of 7.41 m, which is assumed to transmit a power of 447.4 kW, is composed of five subcritical aluminium alloy tubes and four intermediate supports. Zinberg and Symonds suggested replacing the conventional driveline by three subcritical composite shafts consisting of BE material. The properties of this composite shaft are compared with those of the aluminium one in Table 7. Note that the Zinberg shaft was obtained only on the basis of physical considerations.

In line with Lim and Darlow [3], who studied the same driveline case, the mass moment of inertia of the main gearing and tail rotor are assumed to be equal to 0.94 kg m^2 and 3.76 kg m^2, respectively. To take the difference between the connections in the metallic and composite shafts into account, a weight penalty of 1.5 kg per composite shaft is added here.

4.1. Subcritical shaft optimisation
Subcritical shaft optimisation was first studied with the GA. The case of two single-carbon fibre/epoxy composites (BE and HM) and

Table 6: Search parameters of the genetic algorithm

| Parameter             | Value          |
|-----------------------|----------------|
| Population size       | 300-600        |
| Chromosome length     | 24-34          |
| Crossover probability | 90%            |
| Mutation probability  | 10%            |
| Number of generation  | 150-40000      |
The stacking sequence obtained (\([90^\circ, 22.5^\circ, 45^\circ, -45^\circ, 90^\circ]\)) was selected as the best individual. The optimisation of the HM carbon/epoxy was then carried out \(\alpha = 3\) i.e. \(\{\pm 67.5^\circ, 45^\circ, -45^\circ, 0^\circ, 90^\circ\}\). Only two \(0^\circ\) plies were replaced here by a \(90^\circ\) ply. The decrease in the shaft thickness and shaft radius explain the slight increase in weight saving from 29\% to 30\% obtained in comparison with the conventional aluminium shaft (see Table 7). The computing time required for one evolution was approximately equal to one hour using MATLAB \([36]\] on a Xeon E5540.

The second material tested was HM carbon/epoxy (Fig. 4b). Convergence was reached after 200 generations, but five different solutions were obtained with the same fitness. Only the order between \(0^\circ\), \(45^\circ\) and \(-45^\circ\) plies and the operating speed were different. The optimum shaft stacking sequence maximising the strength and buckling margins was \([90^\circ, 0^\circ, 45^\circ, 45^\circ, -45^\circ]\). This gives the minimum thickness authorized (1 mm). Due to the high level of stiffness in comparison with BE, only one \(90^\circ\) ply was necessary to prevent buckling and three \(0^\circ\) plies were required to avoid reaching the first critical speed. The number of \(\pm 45^\circ\) plies increased two-fold due to the low strength of HM carbon/epoxy. The weight saving increased considerably in comparison with the previous example, reaching 41\% due to several combined effects: the decrease in the density, the mean tube radius, the thickness and the weight of the supports (due to the increase in the operating speed, see Eq. (16)).

The optimisation of the HM carbon/epoxy was then carried out with \(b_{in} = 3\) i.e. \(a \in \{-67.5^\circ, -45^\circ, -22.5^\circ, 0^\circ, 45^\circ, 67.5^\circ, 90^\circ\}\) (Fig 4c). The chromosome length increased from 24 to 30. This considerably increased the search-space, and hence the number of generations required to obtain convergence and the computing time (approximately 3h/evolution). Convergence was obtained after approximately 6000 generations. Four different optimum individuals were obtained with the same fitness. All of them contained (+ and

### Table 5: Reserve factors and penalisation factors

| Material                  | Conv. aluminium | BE | BE | HM | HM\(^b\) | HS/HS |
|---------------------------|-----------------|----|----|----|---------|-------|
| Operating speed \(\Omega_{in}\) rev / min | 5 540 | 4 320 | 3 800 | 4 800 | 4 600 | 4 400 |
| 1st critical speed \(\omega_1\) rev / min | 8 887 | 5 697 | 4 606 | 5 800 | 5 695 | 5 344 |
| 1st torsion mode \(\alpha_1\) rev / min | 2 058 | 1 292 | 1 065 | 1 534 | 1 254 | 635 |
| 2nd torsion mode \(\alpha_2\) rev / min | 65 370 | 35 318 | 36 428 | 64 965 | 59 599 | 34 510 |

### Table 7: Optimised composite tail rotor driveline under subcritical conditions in comparison with the conventional aluminium driveline

| Material                  | Conv. aluminium | BE | BE | HM | HM\(^b\) | HS/HS |
|---------------------------|-----------------|----|----|----|---------|-------|
| Number of tubes           | -               | 5  | 3  |    |         |       |
| String length             | -               | bit | -  | 24 | 24      | 30    |
| Stacking sequence (from inner to outer radius) | [90°, 0°, 45°, -45°, 90°] | [90°, 0°, 90°, 90°, -22.5°, -45°, 22.5°, 22.5°, 0°, 45°, 90°] |
| Operating speed \(\Omega_{in}\) rev / min | 5 540 | 4 320 | 3 800 | 4 800 | 4 600 | 4 400 |
| 1st critical speed \(\omega_1\) rev / min | 8 887 | 5 697 | 4 606 | 5 800 | 5 695 | 5 344 |
| 1st torsion mode \(\alpha_1\) rev / min | 2 058 | 1 292 | 1 065 | 1 534 | 1 254 | 635 |
| 2nd torsion mode \(\alpha_2\) rev / min | 65 370 | 35 318 | 36 428 | 64 965 | 59 599 | 34 510 |
| Nominal torque \(T_{nom}\) N m | 771 | 989 | 1 124 | 891 | 929 | 971 |
| Strength torque \(T_{str}\) N m | 4 925 | 4 880 | 3 149 | 2 268 | 2 267 | 3 349 |
| Buckling torque \(T_{buck}\) N m | 3 090 | 2 671 | 2 645 | 2 108 | 2 105 | 2 206 |
| Tube length \(l\) m | 1.482 | 2.470 |
| Mean tube radius \(r_m\) mm | 56.3 | 62.84 | 56 | 54 | 50 | 46 |
| Tube thickness \(t_s\) mm | 1.65 | 1.321 | 1.19 | 1.00 | 1.00 | 1.00 |
| Tubes weight \(N_r m_r\) kg (%) | 13.38 | 8.16 (61) | 6.09 (46) | 4.26 (32) | 3.96 (30) | 3.57 (27) |
| Supports weight \(N_h m_h\) kg | 15.42 | 7.71 | 9.68 | 8.24 | 8.48 | 8.75 |
| Weight penalty kg | - | 4.5 | 4.5 | 4.5 | 4.5 | 4.5 |
| Total weight \(m_{dv}\) kg | 28.80 | 20.37 | 20.27 | 17 | 16.95 | 16.82 |
| Weight saving kg (%) | - | 8.4 (29) | 8.5 (30) | 11.8 (41) | 11.9 (41) | 12.0 (42) |

\(^a\) Value computed with presented methods. \(^b\) with \(b_{in} = 3\)
(a) BE (bit\(\alpha = 2\), bit\(\text{mat} = 0\), \(r_m\) \(\in\) [0.05, 0.064] mm)

(b) HM (bit\(\alpha = 2\), bit\(\text{mat} = 0\), \(r_m\) \(\in\) [0.05, 0.064] mm)

(c) HM (bit\(\alpha = 3\), bit\(\text{mat} = 0\), \(r_m\) \(\in\) [0.04, 0.054] mm)

(d) HM/HS (bit\(\alpha = 2\), bit\(\text{mat} = 1\), \(r_m\) \(\in\) [0.04, 0.054] mm)

Figure 4: Evolution of the best individual fitness of several shaft populations in the case of various materials, subcritical conditions and three tubes forming the Zinberg and Symonds tail rotor driveline (there are 300 individuals in each evolution with bit\(n = 1\), bit\(r_m = 3\), bit\(\Omega_{\text{nom}} = 3\), \(q = 6\), \(\Omega_{\text{nom}} \in\) [3800, 5200] rev/min, \(l_{\text{min}} = 1\) mm and \(l = 2.470\) m)

-22.5° plies, and most of them contained (+ and/or -)67.5° plies. The optimum shaft selected from four solutions was \([90°, -23°, 23°, -23°, 23°, -68°]\). This shaft did not contain 0° plies. The weight saving increased slightly in comparison with the previous case, mainly due to the decrease in the mean tube radius.

The last case tested was that of the hybrid HM/HS carbon/epoxy (Fig. 4d). Convergence was again obtained after approximately 6000 generations, despite the fact that only three evolutions yielded the optimum individual. The optimum stacking sequence obtained was \([90°_{\text{HM}}, 45°_{\text{HR}}, 0°_{\text{HM}}, 45°_{\text{HR}}, 90°_{\text{HM}}]\). This result requires some simple comments. The 90° and 0° plies consisted of HM carbon/epoxy because these plies determine the stiffness problems (the dynamics and buckling). The ±45° plies consisted of HS fibres because these plies determine the strength problem. The weight reduction obtained in comparison with the HM case was lower than expected. In fact, the decrease in the weight of the shaft was practically balanced by the increase in the weight of the supports.

The optimisation procedure was also carried out in the case of HS carbon/epoxy material (results not presented here). In the configuration studied here, the HS carbon/epoxy material gave a fitness score in between that obtained with BE and HM composite materials, due to its low density.

In addition, it is worth noting that the number of generations required to converge on the global optimum depended on the size of the search-space, as well as on the basin of attraction of the local and global optima. For example in the case of two materials with the same chromosome length, the number of generations required to reach convergence increased from 250 to 2000 (Figs. 4a-4b).

4.2. Supercritical shaft optimisation

The subcritical condition was then removed. The optimisation was performed in the case of one single-carbon fibre/epoxy composite (HM) and one hybrid composite (HM/HS). The results obtained here show that the five conventional shafts can be replaced...
Table 8: Optimised composite tail rotor driveline under supercritical conditions in comparison with the conventional aluminium driveline

| Material          | Conv. aluminium | CE_L [3] | HM | HM/HS | HM/HS |
|-------------------|-----------------|----------|----|-------|-------|
| Number of tubes   | -               | 5        | 1  | 2     | 2     | 1     |
| String length     | bit             | -        | -  | 27    | 33    | 34    |
| Stacking sequence | (from inner to outer radius) | -46%, -90/64, -45/33, -45/45, -45/45 | -90/23, 90°/45, -90/45, 90°/45, 90°/45 |
| Operating speed   | Ω_nom rev / min | 5.540    | 6000 | 5400 | 4800 | 7000 |
| 1st critical speed| ω_1 rev / min   | 8.887    | 490  | 2696 | 2647 | 1018 |
| 2nd critical speed| ω_2 rev / min   | -        | 2636 | 10589| 10859| 4072 |
| 3rd critical speed| ω_3 rev / min   | -        | 42264| 29324| 29824| 9161 |
| 4th critical speed| ω_4 rev / min   | -        | 43136| 43355| 41355| 16287|
| 1st torsion mode  | ω_1 rev / min   | -        | 3268 | 23064| 23142| 8000 |
| 2nd torsion mode  | ω_2 rev / min   | -        | 34594| 43326| 43326| 8300 |
| Nominal torque    | T_nom N m      | 771      | 712  | 791   | 891   | 610   |
| Strength torque   | T_str N m      | 4925     | 1492 | 2439  | 2096  | 4352  |
| Buckling torque   | T_buck N m     | 3090     | 1460 | 1963  | 2137  | 1657  |
| Tube length       | l m            | 4.182    | 7.41 | 3.705 | 3.705 | 7.41  |
| Mean tube radius  | r_m mm         | 56.3     | 47.7 | 56.0  | 50.0  | 62.0  |
| Tube thickness    | t_s mm         | 1.65     | 1.69 | 1.0   | 1.0   | 1.375 |
| Support stiffness | k_e kN m^-1    | -        | 2864 | 2864  | 1437  | -     |
| Tubes weight      | N_t m_t kg (%) | 13.38    | 6.08 | 4.43  | 3.60  | 6.62  |
| Supports weight   | N_b m_b kg     | 15.42    | 3.80 | 4.12  | 0     | 0     |
| Weight penalty    | - kg           | 1.5      | 3.0  | 3.0   | 1.5   | -     |
| Total weight      | m_dv kg        | 28.80    | 7.58 | 11.23 | 10.72 | 8.15  |
| Weight saving     | kg (%)         | 21.22    | 17.6 | 18.1  | 20.7  | 22.7  |

(a) HM (bit_mat = 0), r_m ∈ [0.046, 0.06] mm, Ω_nom ∈ [4800, 6200] rev / min
(b) HM/HS (bit_mat = 1), r_m ∈ [0.046, 0.06] mm, Ω_nom ∈ [4800, 6200] rev / min

Figure 5: Evolution of the best individual fitness of several shaft populations in the case of various materials, supercritical conditions and two tubes forming the Zinberg and Symonds tail rotor driveline (there are 300 individuals in each evolution with bit_α = 2, bit_β = 1, bit_r = 3, bit_k = 3, bit_Ω = 3, q = 6, k_e ∈ [10^4, 10^7] N m^-1, t_min = 1 mm, η_e = 0.1 and l = 3.705 m)

by either one or two supercritical shafts (see Table 8 and Figs. 5, 6). Contrary to the subcritical optimisation, it was necessary here to take the first four critical speeds and the threshold speed into account. The support stiffness was used as a supplementary optimisation variable to maximise the dynamic stability margin. Lim and Darlow [3] suggested optimising one shaft case with a carbon/epo-
are 600 individuals in each evolution with bit mutations in the case of HM/HS composite material, supercritical conditions.

Figure 6: Evolution of the best individual fitness of several shaft populations in the case of HM/HS composite material, supercritical conditions and one tube forming the Zinberg and Symonds tail rotor drive line (there are 600 individuals in each evolution with bit \( b_1 = 2, b_{max} = 1, b_{crit} = 2, b_{fmax} = 3, b_{fmin} = 3, q = 5, \tau_m \in [0.052, 0.068] \) mm, \( \Omega_{nom} \in \{5600, 7000\} \) rev/min). The latter authors used a generalised reduced gradient method involving continuous variables such as the fraction and the orientation of the laminate plies. Among the stacking sequences tested, [0°, +/−45°] plies made of HM carbon/epoxy, two 90° plies made of HM carbon/epoxy (Fig. 5b). Convergence was reached with five populations after approx. 10,000 generations. All the stacking sequences consisted of five 0° plies made of HM carbon/epoxy, two 90° plies made of HM carbon/epoxy, and one (+ or −)45° ply made of HS carbon/epoxy. The optimum one was [0°, +/−45°] plies made of HS carbon/epoxy plies should be used far from the shaft middle surface in order to maximise the torsional buckling torque; 3. 90° HM carbon/epoxy plies should be used far from the shaft middle surface in order to maximise the torsional buckling torque; 4. the laminate does not generally have to be symmetrical.

**Appendix A. Torsional buckling equations**

Equilibrium equations used to solve the torsional buckling problem in the case of a circular cylinder with orthotropic properties:

\[
\begin{align*}
\sum_{i=1}^{3} A_{11} + \frac{B_{11}}{r} &\ddot{u}'' + \frac{2A_{13} - \frac{T}{\pi r^2}}{r^2} \dot{u}' \\
+ A_{33} - \frac{B_{33}}{r} &\ddot{w}'' + \frac{2A_{13} + \frac{D_{13}}{r^2}}{r^2} \dot{u}' \\
+ A_{12} + \frac{B_{12}}{r} &\ddot{v}' + A_{32} \ddot{v}' - \frac{B_{11}}{r} + \frac{D_{11}}{r^2} \dot{w}'' \\
+ A_{12} u' + \frac{B_{23}}{r} &\ddot{w}'' + \frac{D_{23}}{r^2} \dot{w}'' - \frac{2A_{13} + D_{13}}{r^2} \dot{w}'' \quad \text{(A.1)}
\end{align*}
\]
\[
\begin{align*}
&A_{13} + \frac{2B_{13}}{r} + \frac{D_{13}}{r^2} \frac{u''}{r} + \left( A_{12} + \frac{B_{12}}{r} + A_{33} + \frac{B_{33}}{r} + \frac{D_{33}}{r^2} \right) \frac{\dot{u}}{r} \\
+ &\left( A_{23} + \frac{3D_{23}}{2r^2} \frac{u''}{r} + \frac{A_{33} + \frac{3B_{33}}{r} + \frac{D_{33}}{2r^2}}{r^2} \right) \frac{\dot{u}}{r} \\
+ &\left( 2A_{23} + \frac{4B_{23}}{r} + \frac{2D_{23}}{r^2} - \frac{T}{\pi r^2} \right) u' + \left( A_{22} + \frac{B_{22}}{r} \right) \frac{\dot{u}}{r} + A_{22} \frac{\dot{u}}{r} \\
- &\left( \frac{B_{13}}{r} - \frac{D_{12}}{r} - \frac{2B_{33}}{r} - \frac{3D_{33}}{r^2} \right) \frac{u''}{r} + \left( \frac{3B_{33}}{r} + \frac{D_{33}}{r^2} \right) \frac{\dot{u}}{r} = 0 \quad (A.2)
\end{align*}
\]

\[
\begin{align*}
&\left( \frac{B_{11}}{r} + \frac{D_{11}}{r^2} \right) \frac{u''}{r} + \left( 3 \frac{B_{13}}{r} + \frac{D_{13}}{r^2} \right) \frac{\dot{u}}{r} \\
+ &\left( 2 \frac{B_{33}}{r} + \frac{D_{33}}{r^2} + \frac{B_{12}}{r} + \frac{D_{12}}{r^2} \right) \frac{u''}{r} + \left( \frac{3B_{23}}{r} + \frac{2D_{23}}{r^2} \right) \frac{\dot{u}}{r} \\
+ &\left( 2 \frac{B_{33}}{r} + \frac{3D_{33}}{r^2} + \frac{B_{12}}{r} + \frac{D_{12}}{r^2} \right) \frac{u''}{r} + \left( \frac{3B_{33}}{r} + \frac{2D_{23}}{r^2} \right) \frac{\dot{u}}{r} \\
+ &\left( B_{22} \frac{u''}{r} + \left( -A_{23} + \frac{B_{33}}{r} - \frac{D_{33}}{r^2} \right) \frac{\dot{u}}{r} - \frac{T}{\pi r^2} \right) u'' \\
- &\frac{4D_{13}}{r} \frac{u''}{r} + \left( -A_{23} + \frac{B_{33}}{r} - \frac{D_{33}}{r^2} \right) \frac{\dot{u}}{r} - \frac{D_{11}}{r^2} \frac{u''}{r} \\
- &\frac{D_{22} \frac{u''}{r} + \frac{3B_{12}}{r} \frac{\dot{u}}{r}}{\frac{D_{23}}{r^2} + \frac{2D_{23}}{r^2} - \frac{T}{\pi r^2}} \frac{\dot{u}}{r} \\
+ &\left( \frac{2B_{22}}{r} - \frac{2D_{22}}{r^2} \right) \frac{u''}{r} + \left( -A_{22} + \frac{B_{22}}{r} \right) \frac{\dot{u}}{r} = 0 \quad (A.3)
\end{align*}
\]

where \( r' = \frac{\partial}{\partial x} \) and \( \varphi = \frac{\partial}{\partial \varphi} \).

Elements of the stiffness matrix in the torsional buckling problem in the case of a very long circular cylinder with orthotropic properties:

\[
K(1,1) = -\left( A_{11} + \frac{B_{11}}{r} \right) \lambda^2 - \left( 2A_{13} - \frac{T}{\pi r^2} \right) h \lambda
\]

\[
K(1,2) = -\left( A_{13} - \frac{B_{33}}{r} + \frac{D_{33}}{r^2} \right) \lambda^2
\]

\[
K(1,3) = -\left( A_{12} + \frac{B_{12}}{r} + A_{33} + \frac{B_{33}}{r} \right) \lambda h - A_{23} \lambda^2
\]

\[
K(2,1) = -\left( A_{11} + \frac{2B_{11}}{r} + \frac{D_{11}}{r^2} \right) \lambda^2
\]

\[
K(2,2) = -\left( A_{22} + \frac{2B_{22}}{r} + \frac{D_{22}}{r^2} \right) \lambda^2
\]

\[
K(2,3) = -\left( \frac{B_{13}}{r} + \frac{2D_{13}}{r^2} \right) \lambda^3 + \left( A_{23} + \frac{B_{23}}{r} + \frac{T}{\pi r^2} \right) \lambda
\]

\[
K(3,1) = -\left( B_{11} + \frac{D_{11}}{r^2} \right) \lambda^3 - \left( \frac{3B_{13}}{r} + \frac{D_{13}}{r^2} \right) h \lambda^2
\]

\[
K(3,2) = -\left( A_{13} - \frac{B_{33}}{r} + \frac{D_{33}}{r^2} \right) \lambda^3
\]

\[
K(3,3) = -\frac{D_{11}}{r^2} \lambda^4 - \frac{D_{12}}{r^2} \lambda h \lambda^3 - \left( \frac{D_{23}}{r} + \frac{2D_{23}}{r^2} \right) h \lambda^2
\]

\[
K(3,4) = -\frac{D_{22} \lambda}{r^2} - \frac{D_{22} h}{r^2} \lambda - \frac{D_{23} \lambda h}{r^2} + \left( A_{22} + \frac{B_{22}}{r} \right) \frac{\dot{u}}{r} = 0
\]

where \( \lambda = \frac{\pi n r}{l} \).

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