Anomalous scaling in homogeneous isotropic turbulence

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Abstract

The anomalous scaling exponents $\zeta_n$ of the longitudinal structure functions $S_n$ for homogeneous isotropic turbulence are derived from the Navier-Stokes equations by using field theoretic methods to develop a low energy approximation in which the Kolmogorov theory is shown to act effectively as a mean field theory. The corrections to the Kolmogorov exponents are expressed in terms of the anomalous dimensions of the composite operators which occur in the definition of $S_n$. These are calculated from the anomalous scaling of the appropriate class of nonlinear Green’s function, using an $uv$ fixed point of the renormalisation group, which thereby establishes the connection with the dynamics of the turbulence. The main result is an algebraic expression for $\zeta_n$, which contains no adjustable constants. It is valid at orders $n$ below $g_*^{-1}$, where $g_*$ is the fixed point coupling constant. This expression is used to calculate $\zeta_n$ for orders in the range $n = 2$ to 10, and the results are shown to be in good agreement with experimental data, key examples being $\zeta_2 = 0.7$, $\zeta_3 = 1$ and $\zeta_6 = 1.8$.

I. INTRODUCTION

The study of homogeneous isotropic turbulence has as its aim the derivation of the statistical features of small scale velocity fluctuations at high Reynolds numbers, based on the assumption that they exhibit universal characteristics independent of the form of the large scale flow structures [1-3]. A key quantity of interest is the longitudinal velocity increment, $v_+ - v_-$, where $v_\pm = v_1(x \pm r/2, y, z, t)$, the velocity component $v_1$ and the separation distance $r$ both being in the same direction, here the $x$-axis. An empirical fact is that its $n$th order moment, the longitudinal structure function $S_n(r)$, defined by

$$S_n(r) = \langle (v_+ - v_-)^n \rangle ,$$

(1)
exhibits multiscaling. That is, the exponent $\zeta_n$, defined by the scaling relation

$$S_n(r) \sim r^{\zeta_n},$$

is a nonlinear function of the order $n$. This behaviour is not explained by the classical turbulence theory of Kolmogorov [4] which yields a linear dependence

$$\zeta_n^{Kol} = \frac{n}{3}.$$ 

Moreover, the amount by which $\zeta_n$ differs from $\zeta_n^{Kol}$, called the anomaly, has proved stubbornly resistant to attempts at quantitative explanation [1-3,5,6]. The obstacle to progress with the theory is the strong nonlinearity of the governing Navier-Stokes (NS) equations. In this paper, our aim is to show how modern statistical field theory can be used to overcome this difficulty and provide theoretical predictions for $\zeta_n$, which agree well with turbulent flow data.

The idea that statistical field theory can be brought to bear on the problem of turbulence is not itself new. Indeed, interest in describing turbulence in terms of the underlying functional probability distribution of the velocity field, together with its corresponding generating functional $W$, has a long history [5,6]. But such work has suffered from the weakness of relying on conventional perturbation theory to effect closure of the statistical hierarchy, whereas it is widely believed that a non-perturbative treatment is necessary, because the NS equations lack a small parameter. Consequently, progress with this approach has been disappointing.

The question is whether we can find a middle course, which avoids the limitations of conventional perturbation theory, while not demanding an intractable non-perturbative approach. Here we explore the possibility of formulating a more efficient perturbation theory by developing a zero-order solution which already accounts for the dominant nonlinear interactions, in an attempt, as it were, to deplete the effect of the nonlinearity. We shall do this by adopting a more general quadratic form in $W$ in place of the viscous form which arises naturally. The modified quadratic form is determined self-consistently from the NS nonlinearity using the linear response function and the energy equation. In the inertial range, it
leads directly to the Kolmogorov distribution, after allowing for the kinematic effect of the sweeping of the smaller scales by the larger ones. The difference between these quadratic forms then appears as a perturbation, which, as we shall see, is not critical, provided that the force spectrum function is non-zero only at small wavenumbers and yields a finite input power.

Having incorporated the dominant nonlinearities which are responsible for the turbulence energy cascade into the zero-order solution, what is then lacking is the effect of the fluctuating dissipation rate, which is the well-known defect of the Kolmogorov theory [5]. In this approach, the perturbation theory is then, in effect, only required to accommodate the residual coupling associated with these fluctuations, which are directly responsible for the anomalies. The fact that the anomalies are small, and associated with a weak residual coupling, provides good reason to expect that a small expansion parameter might emerge, thereby rendering the problem accessible to perturbation theory, essentially by means of a standard loop-expansion of the generating functional.

Although the use of the modified quadratic form as an initial approximation would appear to be an attractive option, providing a sound physical basis for the approximate evaluation of the generating functional, it poses severe technical problems, the most significant being the occurrence of divergences at higher orders in perturbation theory, due to the incomplete representation of the large scale flow. These divergences are of two types: power divergences (including power × logarithmic), which are associated with the sweeping and pure logarithmic divergences, which describe the cascade process. On the other hand, statistical field theory [7], provides the mathematical techniques needed to compensate for such divergences, in the form of the well-known processes of resummation and renormalisation. In particular, renormalisation [8] provides a procedure whereby the scale invariance in (2) can be recovered from a divergent theory, yielding the exponents in terms of the anomalous dimensions of the composite operators appearing in (1), which we can calculate from the appropriate nonlinear Green’s functions. The modified quadratic form itself follows uniquely from the requirement for the absence of non-renormalisable terms, after renormalising the basic parameters of $W$.
and allowing for sweeping.

Fortunately, sweeping effects do not pose an insuperable obstacle, notwithstanding that
the initial formulation is Eulerian. Indeed, we show that the power divergences associated
with sweeping can be removed by introducing a single sweeping interaction term, which
can be derived from the generating functional itself using a random Galilean transforma-
tion of the velocity field, having an $rms$ convection speed which is calculated from the NS
nonlinearity. The application of this transformation does not, of course, affect the values
of $S_n(r)$ and, thus, enables the straining interactions which determine the spectrum to be
separated from the background of sweeping convection, yielding, in effect, quasi-Lagrangian
forms. In this way, as we shall show below, it proves possible to demonstrate multiscaling
and calculate the anomalies of the structure functions accurately.

II. THEORETICAL FOUNDATIONS

Our starting point is the NS equations, describing flow in an incompressible fluid of unit
density, velocity $v$, kinematic viscosity $\nu$ and pressure $p$, and driven by a random solenoidal
stirring force $f$, which are

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \nu \nabla^2 v + f,$$

and

$$\text{div } v = 0.$$  

Suppose that

$$v(\hat{x}) = V(\hat{x} | f)$$

is the solution of (4) at the space-time point $\hat{x} = (x, t)$, corresponding to a force $f(\hat{x})$, which
has a Gaussian probability distribution $P(f)$. Then the generating functional $W$ for the
correlation functions of the velocity field can be written as the functional integral

$$W = \int Df \exp \left\{ -\frac{1}{2} \int \left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] \right\}.$$
\[ W = \int \exp(S) \mathcal{P}(f) \mathcal{D}f, \]  

(7)

where the source term is given by

\[ S(J) = \int J(\hat{x}) \cdot \mathbf{V}(\hat{x}|f) \, d\hat{x}, \]  

(8)

and the correlators follow by functional differentiation with respect to the source field \( J(\hat{x}) \).

Given that we cannot obtain an explicit expression for the solution (6), the crux of the problem is how to approximate (7) with the accuracy required to calculate the \( \zeta_n \). In our approach, as indicated above, we prove that the Kolmogorov theory can be used effectively as a mean field theory in a saddle-point evaluation of (7), and that this leads to an expansion which has a genuinely small coupling constant.

Within the context of a field theoretic interpretation of (7), each term of the binomial expansion of (1) must be regarded as an operator product of the usual Wilson type, (see eg [7,8]). Correspondingly, the powers of \( v_\pm \) must be treated as composite operators, which, in accordance with standard procedures [7], must be generated from \( W \) by independent sources. Here, our aim is to limit the composite operators that need to be allowed for to those which appear explicitly in the definition of \( S_n(r) \), as given in (1). To this end, we define a set of longitudinal composite operators \( O_s(\hat{x}) \), for \( s = 2, 3, 4, \ldots \), by

\[ O_s(\hat{x}) = v_1(\hat{x})^s/s!, \]  

(9)

which we generate from \( W \) by adding to the source term (8), the additional term

\[ - \sum_s \int t_s(\hat{x})O_s(\hat{x})d\hat{x}. \]  

(10)

We also need to include in the definition of \( W \) a means of establishing the vital link between the time-independent definition of \( S_n(r) \) and the dynamics of the turbulence. This requires the introduction of a dynamic response operator, which we define to be the functional differentiation operator

\[ \mathcal{F}_\alpha(\hat{x}) = \frac{\delta}{i\delta f_\alpha(\hat{x})}, \]  

(11)
Its inclusion in the definition of $W$ adds a final source term to $S$, given by

$$\int J_\alpha(\hat{x})F_\alpha(\hat{x})d\hat{x}, \quad (12)$$

where summation over repeated vector indices is implied here and below.

The terms (8), (10) and (12) together constitute the full source term for (7) which becomes, therefore,

$$S(J, \tilde{J}, t_s) = \int \{J_\alpha(\hat{x})V_\alpha(\hat{x}) + \tilde{J}_\alpha(\hat{x})F_\alpha(\hat{x}) - \sum_s t_s(\hat{x})O_s(\hat{x})\}d\hat{x}, \quad (13)$$

and this completes the definition of $W$. Thus, (9) and (13) provide the foundation of our approach to the calculation of $\zeta_n$. However, before we proceed with this calculation, we need to cast $W$ into a conventional field theory form, and introduce the modified quadratic form.

A straightforward method of transforming (7) into a conventional field theory form is to replace $P(f)$ by its functional Fourier transform and then integrate over $f$. This is the stage at which we make explicit use of the NS equations. Essentially, to effect the transformation, we change our perspective by replacing the velocity field $V(f)$ generated by the force $f$, by the force $F(v)$ needed to excite a particular realisation $v$ of the flow field. The operator (11) is then replaced by an equivalent conjugate vector field $\tilde{v}$.

To carry out this transformation, we work in the Fourier domain setting

$$v(\hat{x}) = \int \exp(i\hat{k} \cdot \hat{x}) v(\hat{k})D\hat{k},$$

where $\hat{k}$ denotes $(k,\omega)$, so that $\hat{k} \cdot \hat{x} = \omega t - k \cdot x$, while $D\hat{k} = d\omega dk/(2\pi)^4$. Then, from (4) and (3), we have

$$F_\alpha(\hat{k}, v) = G_0(\hat{k})^{-1}v_\alpha(\hat{k}) - \frac{i}{2}(2\pi)^4 P_{\alpha\beta\gamma}(k) \int v_\beta(\hat{p})v_\gamma(\hat{q})\delta(\hat{p} + \hat{q} - \hat{k})D\hat{p}D\hat{q}. \quad (14)$$

The notation here is the following. $G_0(\hat{k})$ is the zero-order approximation to the response function $G(\hat{k})$ defined below in (64) and (65), namely

$$G_0(\hat{k}) = \frac{1}{i\omega + \tau_\nu(\hat{k})^{-1}}, \quad (15)$$
where
\[ \tau_{\nu}(k) = \nu k^2. \]

\( P_{\alpha\beta\gamma}(k) \) is the NS vertex defined by
\[ P_{\alpha\beta\gamma}(k) = k_\beta P_{\alpha\gamma}(k) + k_\gamma P_{\alpha\beta}(k). \]

where
\[ P_{\alpha\beta}(k) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2. \]

Next we write the Gaussian distribution of \( f \) in the form
\[ \mathcal{P}(f) = N \exp \left\{ -\frac{1}{2} \int f_\alpha(\hat{k}) h(k)^{-1} P_{\alpha\beta}(k) f_\beta(\hat{k}) D\hat{k} \right\}, \quad (16) \]

for which the corresponding force covariance is
\[ \left\langle f_\alpha(\hat{k}) f_\beta(\hat{l}) \right\rangle = (2\pi)^4 \delta(\hat{k} + \hat{l}) h(k) P_{\alpha\beta}(k), \]

where the force spectrum function \( h(k) \) is an arbitrary function which is assumed to be peaked near the origin so that the power input \( \int h(k) d\mathbf{k} \) is finite. We now change the functional integration over \( f \) in (7) to an integration over \( v \) by means of the transformation
\[ v(\hat{k}) = V(\hat{k}|f), \]

and substitute the representation
\[ \mathcal{P}(f) = N \int \exp \left\{ -\frac{1}{2} \int \tilde{v}_\alpha(\hat{k}) h(k) P_{\alpha\beta}(k) \tilde{v}_\beta(\hat{k}) D\hat{k} + i \int \tilde{v}_\alpha(\hat{k}) f_\alpha(\hat{k}) D\hat{k} \right\} \mathcal{D}\tilde{v}, \quad (17) \]

Since the Jacobian only contributes an unimportant constant, we get
\[ W(J, \tilde{J}, t_s) = \int \exp \left[ -L(v, \tilde{v}) + S(J, \tilde{J}, t_s) \right] \mathcal{D}v \mathcal{D}\tilde{v}, \quad (18) \]

where
\[ L(v, \tilde{v}) = \frac{1}{2} \int \tilde{v}_\alpha(\hat{k}) h(k) P_{\alpha\beta}(k) \tilde{v}_\beta(\hat{k}) D\hat{k} - i \int \tilde{v}_\alpha(\hat{k}) F_\alpha(\hat{k}, v) D\hat{k}, \quad (19) \]

while the source term \( \mathcal{B} \) becomes
\[ S(J, \tilde{J}, t_s) = \int \left\{ J_\alpha(-\hat{k})v_\alpha(\hat{k}) + \tilde{J}_\alpha(-\hat{k})\tilde{v}_\alpha(\hat{k}) - \sum_s t_s(\hat{k})O_s(\hat{k}) \right\} D\hat{k}, \]  

(20)

The expression (18) casts \( W \) into the form of an MSR type functional integral [9].

Now the quadratic form appearing in (19) does not provide a good initial approximation for inertial range scaling because, of course, it merely describes the viscous decay of an externally driven random flow, with no account taken of the nonlinear interactions. It is thus essential in developing an expansion theorem for (18) to introduce a more appropriate quadratic form. Now the general theory of quadratic forms in a Hilbert space indicates that we can introduce at most two functions. These can be taken as an apparent force spectrum \( D_0(k) \) and an effective micro timescale \( \tau_0(k) \), which are related to the energy in wavemode \( k, Q(k) \), by

\[ Q(k) = \tau_0(k)D_0(k). \]  

(21)

The modified quadratic form in \( L(v, \tilde{v}) \) is then obtained, firstly, by replacing \( h(k) \) with \( D_0(k) \) and, secondly, by replacing the viscous timescale \( \tau_\nu(k) \) by the effective timescale \( \tau_0(k) \), so that the viscous propagator (15) in (14) is replaced by

\[ G_0(k) = \frac{1}{i\omega + \tau_0(k)D_0(k)}. \]

Thus, we now have in place of (19)

\[ L(v, \tilde{v}) = \frac{1}{2} \int \tilde{v}_\alpha(-\hat{k})D_0(k)P_{\alpha\beta}(k)\tilde{v}_\beta(\hat{k})D\hat{k} - i \int \tilde{v}_\alpha(-\hat{k})F_\alpha(\hat{k}, v)D\hat{k}, \]  

(22)

in which \( D_0(k) \) and \( \tau_0(k) \) are, as yet, unknown functions to be determined in an appropriate way from the energy equation and the linear response function. The idea that one should replace the viscous quadratic form by a modified form was suggested originally in [10], where it was used in conjunction with a variational principle based on an entropy functional, but recent work [11] has shown that this approach contains an arbitrary element. However, we shall not need to invoke any additional principle, because we shall be able to deduce the modified quadratic form in a self-consistent way from the 1-loop expansion, as we have already indicated.
The introduction of the modified quadratic form as a basis for an expansion theorem for \( (18) \) requires the inclusion of the difference terms as perturbations, which contributes an additional term to \( L \) given by

\[
\Delta L_0 = \frac{1}{2} \int \tilde{v}_\alpha(-\hat{k}) \{ h(k) - D_0(k) \} P_{\alpha\beta}(k) \tilde{v}_\beta(\hat{k}) D \hat{k} - i \int \tilde{v}_\alpha(-\hat{k}) \{ \tau_\nu(k)^{-1} - \tau_0(k)^{-1} \} v_\alpha(\hat{k}) D \hat{k}.
\]

(23)

These terms have the same form as the counterterms introduced below in (27) to accommodate the pure logarithmic divergences but their role, as we shall see, is not critical as regards calculating the inertial range exponents.

The derivation of the functions \( D_0(k) \) and \( \tau_0(k) \) occurring in the modified quadratic form entails a detailed discussion of sweeping convection, the structure of the Feynman diagrams associated with the loop expansion of \( W \) and the establishment of the condition for the absence from the linear response function of non-renormalisable terms. We shall defer detailed discussion of these topics until Sections VII and VIII and, meanwhile, proceed with the calculation of the anomalous exponents by anticipating their forms, which, in the inertial range, are

\[
D_0(k) = D_0 k^{-3}, \quad (24)
\]

and

\[
\tau_0(k)^{-1} = \nu_0 k^{2/3}. \quad (25)
\]

Clearly, these forms imply that the zero order solution behaves in the inertial range as if the fluid were stirred by a random force with a \( k^{-3} \) force correlation spectrum and responds to it with a Lagrangian time scale \( \propto k^{-2/3} \). Thus, they lead to the Kolmogorov distribution. We shall explain how this result follows from the generating functional in Section VIII. The advantage of this approximation is that it achieves a prime requirement of any efficient perturbation theory, which is a zero-order approximation that already closely approximates the desired solution.
On the other hand, as we have indicated, the resulting perturbation theory yields divergences at higher orders. But these divergences can be handled by standard renormalisation procedures. Fortunately, as regards the calculation of \( \zeta \), we need consider only logarithmic divergences. As discussed above, this is because the power divergences represent the kinematic effect of the sweeping of small scales by larger scales. Indeed, as we shall show in Section VII, such terms are precisely those which can be generated by applying a random Galilean transformation of the velocity field to \( W \). Consequently, they can be cancelled by introducing the appropriate vertex into \( W \), yielding quasi-Lagrangian approximations. Hence, from a purely practical calculational point of view, the effect of sweeping can be removed from the calculation of \( \zeta \), simply by discarding power divergences. We are then left with the logarithmic divergences, which we can sum by renormalisation group methods.

Thus, an important implication of using the modified quadratic form as an initial approximation for the calculation of \( \zeta \) is that renormalisation becomes a necessary preliminary. So we need to identify the counterterms which arise in \( W \) under renormalisation and obtain the transformation rule which connects the bare and renormalised generating functionals. Renormalisation is applied to the viscosity and force constants appearing in (24) and (25) in the usual way by introducing renormalisation constants \( Z_\nu \) and \( Z_D \), which relate their bare values \( \nu_0 \) and \( D_0 \) to their renormalised replacements \( \nu \) and \( D \) by

\[
\nu_0 = \nu Z_\nu \quad \text{and} \quad D_0 = D Z_D. \tag{26}
\]

This generates counterterms in (19) for the elementary fields (\( \mathbf{v} \) and \( \tilde{\mathbf{v}} \)) given by

\[
\Delta L_{ef} = -\Delta Z_\nu i \int \tilde{v}_\alpha(-\hat{k}) \tau(k)^{-1} v_\alpha(\hat{k}) \hat{k}
\]

\[
+ \Delta Z_D \frac{1}{2} \int \tilde{v}_\alpha(-\hat{k}) D(k) P_{\alpha\beta}(k) \tilde{v}_\alpha(\hat{k}) \hat{k},
\]

where we have defined renormalisation constant increments by

\[
\Delta Z_{\nu,D} = Z_{\nu,D} - 1.
\]

The additional renormalisation which must be applied to the composite operators (9) also takes the standard form.
\[(O_s)_B = Z_s(O_s)_R. \quad (28)\]

The corresponding counterterm is obtained by substituting (28) in (20) to get

\[\Delta L_{\text{co}} = \sum_s \Delta Z_s \int t_s(-\hat{k})O_s(\hat{k})D\hat{k},\]

where

\[\Delta Z_s = Z_s - 1.\]

We conclude this section by giving the transformation which relates the generating functional of the bare correlation functions \(W_B\) to its corresponding renormalised form \(W_R\). To provide a convenient means of handling the dependence of the correlation and response functions on the dimensional parameters \(\nu_0\) and \(D_0\), we rescale \(V\) and \(f\) by introducing bare fields defined by

\[V(\hat{k}) = \left(\frac{D_0}{\nu_0^3}\right)^{1/2} V_B(k, \omega_B),\]

and

\[f(\hat{k}) = \left(\frac{D_0}{\nu_0^3}\right)^{1/2} f_B(k, \omega_B),\]

with bare frequency

\[\omega_B = \frac{\omega}{\nu_0}.\]

These bare fields preserve the form of the NS equations, apart from explicitly introducing the non-dimensional coupling constant, defined by

\[g_0 = \frac{D_0}{6\pi^2 \nu_0^3}; \quad (29)\]

in which the appropriateness of the numerical factor will appear later from the loop-expansion of \(W\).

Under the renormalisation (26), the bare fields are replaced by renormalised fields, to which they are related by
\[ V_B(k, \omega_B) = \left( \frac{Z_{\nu}}{Z_D} \right)^{1/2} V_R(k, \omega_R) \]

and

\[ f_B(k, \omega_B) = \left( \frac{Z_{\nu}}{Z_D} \right)^{1/2} f_R(k, \omega_R), \]

where

\[ \omega_R = \frac{\omega}{\nu}. \]

These relations follow from two requirements. First, the form of the NS equations (14) must again be preserved, with the new constants \( \nu \) and \( D \) resulting in a renormalised coupling constant

\[ g = \frac{D}{6\pi^2 \nu^2}. \tag{30} \]

Second, we have to satisfy the crucial requirement that \( \mathcal{P}(f) \), as given in (14), remains invariant under renormalisation. Indeed, satisfaction of these conditions implies the desired relation between \( W_B \) and \( W_R \), which from (18) and (20), is readily found to be

\[ W_R(J, \bar{J}, t_s) = W_B \left( \frac{1}{Z_{\nu}} \left( \frac{Z_D}{Z_{\nu}} \right)^{1/2}, \frac{1}{Z_{\nu}} \left( \frac{Z_D}{Z_{\nu}} \right)^{1/2}, \frac{Z_s}{Z_{\nu}} \left( \frac{Z_D}{Z_{\nu}} \right)^{s/2}, t_s \right). \tag{31} \]

The foregoing provides the basis of our calculation of \( \zeta_n \), which involves the following four stages. First, we use (31) and the binomial expansion of (1) to develop a short distance expansion for \( S_n(r) \), by substituting an operator product expansion (OPE) \([7,8]\) for each term, based on the operators (9). As shown in Section \([13]\), this yields the scaling of \( S_n(r) \) in terms of \( uv \) fixed point values of standard RG functions. The second stage of the calculation is to demonstrate that the required \( uv \) fixed point of the RG actually exists, and then to deduce the corresponding fixed point coupling constant \( g_* \). This is done in Section \([14]\) by considering the renormalisation of the linear response function, using the renormalised functional in the form obtained from (15). The third stage is to calculate the specific fixed point RG parameters which give the anomalous component of \( \zeta_n \). To do this, we have to consider
the renormalisation of appropriate nonlinear Green’s functions involving the composite operators defined in (I). These are identified and evaluated in Section V. Having calculated the anomalous scaling exponent $-\tau_{np}$ of the $p$th term in the binomial expansion of $S_n(r)$, in the fourth and final stage of the calculation, we derive a simple algebraic expression for $\zeta_n$ by maximising $\tau_{np}$, with respect to integer values of $p$, and subtracting this maximum from the Kolmogorov value (3). The results obtained for $\zeta_n$ are presented in Section VI, where they are shown to be in good agreement with experimental measurements at all orders for which reliable data exists. Finally, the mathematical proofs, deferred during the calculation of the exponents, are presented in Sections VII-X, and comprise: (a) the demonstration that sweeping effects can be eliminated by means of a random Galilean transformation of the velocity field; (b) the derivation of the modified quadratic form from the 1-loop expansion; and (c) the derivation of the dominant terms of the OPEs.

III. THE STRUCTURE FUNCTION EXPANSIONS

In applying the OPE technique to (I) the first point to appreciate is that the orders $n = 2$, $n = 3$ and $n \geq 4$ require separate treatment. The factor distinguishing $S_2$ and $S_3$ from the higher order $S_n$ is that the latter involve composite operator products, whereas $S_2$ and $S_3$ do not. Also, $S_3$ is exceptional in representing a transition at which corrections to the Kolmogorov exponents (3) change from positive at $n = 2$ to negative at $n \geq 4$, with no correction occurring at $n = 3$ in accordance with the known exact scaling law, which is verified, within the present framework, in Section VIII. This sign change is caused precisely because composite operator products appear in $S_n$ when $n \geq 4$.

We begin, therefore, with the relatively straightforward case of $S_2$. According to (I), we have

$$S_2(r) = 2 \left( \langle v^2 \rangle - \langle v_+ v_- \rangle \right),$$

which shows that the scaling of $S_2$ is determined by the behaviour of the operator product $v_+ v_-$ as $r \to 0$. The form of its OPE is established in Section IX after the necessary
mathematical apparatus has been set up. Its proof is given there to the accuracy of the calculation, ie up to and including terms of order $g^2$. We shall show that the operators which appear in its OPE are: (a) the unit operator $I$, with constant coefficient $E/3$, where $E$ is the turbulence energy; (b) the dominant longitudinal quadratic composite operator $O_2(\hat{x})$, which gives the leading scaling behaviour; and (c) subdominant operators including all transverse operators and the longitudinal higher order composite operators $O_s(\hat{x})$. However, we shall only be concerned with the dominant operators and so we write the expansion as

$$v_+v_- = \frac{1}{3}EI + C_2(r)O_2(\hat{x}) + \ldots,$$

(33)

where the dots indicate the additional subdominant terms. The scaling behaviour of this operator product can be found in the usual way from the RG equation satisfied by the leading Wilson coefficient $C_2(r)$ [12].

We start by considering an arbitrary equal time correlation function of order $l$, given by

$$H_{\alpha_1\ldots\alpha_l}(\hat{x}_1,\ldots,\hat{x}_l) = \langle v_{\alpha_1}(\hat{x}_1)\ldots v_{\alpha_l}(\hat{x}_l) \rangle.$$

(34)

If we insert (33) into this correlation function, we get

$$H_{\alpha_1\ldots\alpha_l}(\hat{x}_1,\ldots,\hat{x}_l, \hat{x} + \frac{r}{2}\hat{i}, \hat{x} - \frac{r}{2}\hat{i}) = \frac{E}{3}H_{\alpha_1\ldots\alpha_l}(\hat{x}_1,\ldots,\hat{x}_l) + C_2(r)Q_{\alpha_1\ldots\alpha_l}(\hat{x}_1,\ldots,\hat{x}_l, \hat{x}) + \ldots,$$

(35)

where, in general, $Q_{\alpha_1\ldots\alpha_l}^{(s)}$ is the inserted correlation function defined by

$$Q_{\alpha_1\ldots\alpha_l}(\hat{x}_1,\ldots,\hat{x}_l, \hat{x}) = \langle v_{\alpha_1}(\hat{x}_1)\ldots v_{\alpha_l}(\hat{x}_l)O_s(\hat{x}) \rangle,$$

(36)

and $\hat{i}$ is a unit vector along the $x$-axis.

We can deduce the RG equation satisfied by the Wilson coefficient $C_2$ in (33) from (35), given the RG equations satisfied by $H_{\alpha_1\ldots\alpha_l}$ and $Q_{\alpha_1\ldots\alpha_l}^{(2)}$. To obtain the latter, we need the transformation rule for the equal time generator of these correlation functions, which we denote by $W^{(e)}(J,t_s)$. This follows in a straightforward manner by taking time independent sources in (31), and integrating with respect to $\omega_B$ and $\omega_R$, with the $\tilde{J}$ dependence, which
is irrelevant here, suppressed. To simplify the result, we shall anticipate the fact, which we demonstrate in Section IV, that

\[ Z_D = Z_\nu. \]  

(37)

We then get

\[ W^{(e)}_R(J, t_s) = W^{(e)}_B(J, Z_s t_s). \]  

(38)

According to this relation, the bare and renormalised forms of \( H_{\alpha_1...\alpha_l} \) are equal. Hence, when we change the renormalisation scale, which we denote by \( \mu \), the Fourier transform of \( H_{\alpha_1...\alpha_l} \) changes according to the RG equation

\[ \mathcal{D} H_{\alpha_1...\alpha_l} = 0, \]  

(39)

where \( \mathcal{D} \) is the standard RG operator defined by

\[ \mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}, \]  

(40)

with

\[ \beta(g) = \mu \frac{dg}{d\mu}. \]  

(41)

In the case of \( Q_{\alpha_1...\alpha_l}^{(s)} \), we obtain from (36) and (38) the relation

\[ (Q_{\alpha_1...\alpha_l}^{(s)})_R = Z_s (Q_{\alpha_1...\alpha_l}^{(s)})_B, \]

which leads to the RG equation

\[ \mathcal{D} Q_{\alpha_1...\alpha_l}^{(s)} = \gamma_s Q_{\alpha_1...\alpha_l}^{(s)}, \]  

(42)

where \( \gamma_s \) is the anomalous dimension of \( O_s \) given by

\[ \gamma_s = \mu \frac{d}{d\mu} \log Z_s. \]  

(43)

For ease of notation, we have dropped the suffix \( R \) in the RG equations (39) and (42), since we shall always be dealing with relations between renormalised functions.
We now apply the RG operator (40) to the Fourier transform of (35), and make use of (39) and (42), to get

\[ 0 = (\mathcal{D}C_2 + \gamma_2 C_2)Q_{\alpha_1...\alpha_l}^{(2)} + \ldots. \]  

\[ (44) \]

As this equation holds for arbitrary \( Q_{\alpha_1...\alpha_l}^{(2)} \), it follows that

\[ \mathcal{D}C_2 = -\gamma_2 C_2. \]  

\[ (45) \]

which is the RG equation satisfied by the Wilson coefficients in (33).

The standard solution of this equation, corresponding to an \( uv \) fixed point [12], now gives for the leading term of (33) the scaling behaviour

\[ C_2(r) \sim r^{2/3 - \gamma_2^*}, \]  

\[ (46) \]

where the star denotes the fixed point value of (43). This result, in conjunction with (32) and (33), yields the scaling exponent for \( S_2(r) \), namely

\[ \zeta_2 = \frac{2}{3} + \Delta_2, \]  

\[ (47) \]

where

\[ \Delta_2 = -\gamma_2^*. \]  

\[ (48) \]

We shall calculate \( \Delta_2 \) in Section V.

Consider now the general case for even orders \( n = 2m > 2 \). Introducing the general composite operator product

\[ \Lambda_{ss'}(\hat{x}, r) = O_s(\hat{x} + \frac{r}{2}\hat{i})O_{s'}(\hat{x} - \frac{r}{2}\hat{i}), \]  

\[ (49) \]

and taking advantage of the isotropic symmetry, we can write the binomial expansion of (1) as

\[ S_n(r) = n! \left[ 2 \sum_{p=0}^{m-1} (-)^p \Lambda_{n-p,p}(\hat{x}, r) + (-)^m \Lambda_{m,m}(\hat{x}, r) \right]. \]  

\[ (50) \]
We can identify the dominant term of the OPE of $\Lambda_{n-p,p}$ by factoring out the product $(v_+v_-)^p$ and using the fact that, by [33], its expansion begins with the unit operator. We will justify this process in Section IX. This implies that the OPE of $\Lambda_{n-p,p}$ itself takes the form

$$\Lambda_{n-p,p}(\hat{x}, r) = C_{p,m-p}(r)O_{2(m-p)}(\hat{x}) + \ldots,$$

where again the dots indicate subdominant terms. Substituting (51) in (50), we get

$$S_n(r) = n! \left\{ 2 \sum_{p=0}^{m-1} (-)^p C_{p,m-p}(r) \langle O_{2(m-p)}(\hat{x}) \rangle + (-)^m C_{m,m}(r) \right\} + \ldots,$$

the averages of the composite operators being independent of $\hat{x}$ for homogeneous isotropic turbulence.

To find $\zeta_n$ from this expansion, we have to determine which term or terms on the right hand side yield the negative correction of maximum magnitude to $\zeta_n^{Kol}$. As before, this is deduced from the RG equation for the Wilson coefficient $C_{p,s}$, which we derive next.

We begin by inserting (49) into the general correlation function (34) to obtain the general inserted correlation function

$$R_{\alpha_1 \ldots \alpha_l}^{(ss')} (\hat{x}_1, \ldots, \hat{x}_l, \hat{x} + \frac{1}{2} r \hat{\imath}, \hat{x} - \frac{1}{2} r \hat{\imath}) = \langle v_{\alpha_1}(\hat{x}_1) \ldots v_{\alpha_l}(\hat{x}_l) \Lambda_{ss'}(\hat{x}, r) \rangle.$$

According to (53) its bare and renormalised forms are connected by the relation

$$\left( R_{\alpha_1 \ldots \alpha_l}^{(ss')} \right)_R = Z_s Z_{s'} \left( R_{\alpha_1 \ldots \alpha_l}^{(ss')} \right)_B,$$

from which it follows that $R_{\alpha_1 \ldots \alpha_l}^{(ss')}$ satisfies the RG equation

$$\mathcal{D} R_{\alpha_1 \ldots \alpha_l}^{(ss')} = (\gamma_s + \gamma_{s'}) R_{\alpha_1 \ldots \alpha_l}^{(ss')}.$$  

Next, we insert the expansion (51) into the general correlation function (34), and use the definitions (36) and (53), to get

$$R_{\alpha_1 \ldots \alpha_l}^{(n-p,p)} (\hat{x}_1, \ldots, \hat{x}_l, \hat{x} + \frac{r}{2} \hat{\imath}, \hat{x} - \frac{r}{2} \hat{\imath}) = C_{p,m-p}(r) Q_{\alpha_1 \ldots \alpha_l}^{(2s)} (\hat{x}_1, \ldots, \hat{x}_l, \hat{x}).$$
We then apply the RG operator (40) to the Fourier transform of this equation, and substitute (42) and (54) to obtain

\[ Q_2^{m-p} \{ DC_{p,m-p} + (\gamma_2(m-p) - \gamma_p - \gamma_{n-p}) C_{p,m-p} \} + \ldots = 0, \]

from which it follows that

\[ DC_{p,m-p} = - (\gamma_2(m-p) - \gamma_p - \gamma_{n-p}) C_{p,m-p}. \]  (55)

We now invoke the standard solution of (55), applicable at the \( uv \) fixed point [12], to obtain the scaling relation

\[ C_{p,m-p}(r) \sim r^{n/3 - \tau_{np}}, \]  (56)

where

\[ \tau_{np} = \gamma_2^*(m-p) - \gamma_p^* - \gamma_{n-p}^*. \]  (57)

Upon substituting (56) in (52), it is immediately evident that the scaling exponent of \( S_n(r) \) is given by

\[ \zeta_n = n/3 - \tau_n, \]  (58)

where

\[ \tau_n = \max_p \tau_{np}, \quad \text{for} \quad n = 2m > 2. \]  (59)

Once \( \gamma_2^* \) has been evaluated from (43), at the fixed point, which we do in Section V, it is a simple matter to evaluate \( \tau_n \), as we show in Section VI.

Odd orders with \( n = 2m + 1 > 3 \) may be treated similarly with minor adjustments to allow for the fact that the expansions involve odd powers. In this case, however, it is immediately evident that the dominant scaling must arise from the Wilson coefficient of the unit operator corresponding to \( p = m \), because averaging wipes out other terms by virtue of the fact that \( \langle O_{2s+1} \rangle = 0 \). Hence, we obtain

\[ \tau_n = - (\gamma_{m}^* + \gamma_{m+1}^*) \quad \text{for} \quad n = 2m + 1 > 3. \]  (60)

Again, the justification of the relevant expansions is given in Section IX.
IV. THE LINEAR RESPONSE

In order to evaluate $\tau_n$, we have to establish that an $uv$ fixed point exists, which entails showing that the RG $\beta$ function (41) possesses a zero

$$\beta(g_*) = 0,$$  \hspace{1cm} (61)

at which

$$\frac{d\beta}{dg} < 0.$$  \hspace{1cm} (62)

To do this we must first determine the dependence of the renormalisation constants $Z_\nu$ and $Z_D$ on the renormalisation scale $\mu$. We will then verify that (37) holds and use this fact to calculate $g_*$ from $Z_\nu$.

Consider $Z_\nu$. According to the general theory of renormalisation [14], we have an expansion of the form

$$Z_\nu = 1 + g a_{1\nu} \log \left( \frac{\mu}{\kappa} \right) + g^2 \left\{ \frac{a_{1\nu}^2}{2} \log^2 \left( \frac{\mu}{\kappa} \right) + a_{2\nu} \log \left( \frac{\mu}{\kappa} \right) \right\} + \ldots .$$  \hspace{1cm} (63)

Here $\kappa$ is the wavenumber cut-off which provides the intermediate regulation of the divergent integrals. This is an ir wavenumber of the order of $L^{-1}$, where $L$ is the typical length scale of the large scale flow. Divergences arise in the limit $\kappa \to 0$, corresponding to the inertial range limit $r/L \to 0$. The constants $a_{1\nu}$ and $a_{2\nu}$ will be calculated by eliminating the logarithmic divergences, at 1 and 2-loop orders respectively, from the 1PI Green’s function $\Gamma_{\alpha\beta}(\hat{k}, \hat{l})$, which is the inverse of the Fourier transform $G_{\alpha\beta}(\hat{x}, \hat{x}')$ of the linear response function

$$G_{\alpha\beta}(\hat{x}, \hat{x}') = \left\langle \frac{\delta v_{\alpha}(\hat{x})}{\delta f_{\beta}(\hat{x}')} \right\rangle .$$  \hspace{1cm} (64)

$\Gamma_{\alpha\beta}$, and the other 1PI functions that we shall require, are generated from the functional $K$, which is obtained in the usual way by performing a Legendre transformation on $W_c = \log W$, with respect to the sources of the elementary fields, $J$ and $\tilde{J}$, while holding the composite operator sources $t_s$ fixed [7,13]. The new source fields for $K$ are therefore given by
\[ u(\hat{k}) = (2\pi)^4 \frac{\delta W_c}{i\delta J(-\hat{k})}, \]

and

\[ \tilde{u}(\hat{k}) = (2\pi)^4 \frac{\delta W_c}{i\delta \tilde{J}(-\hat{k})}, \]

with \( K \) itself given in terms of its source fields by

\[ K(u, \tilde{u}, t_s) = -W_c + i \int \left\{ J(-\hat{k}) \cdot u(\hat{k}) + \tilde{J}(-\hat{k}) \cdot \tilde{u}(\hat{k}) \right\} D\hat{k}. \]

It follows, therefore, that

\[ \Gamma_{\alpha\beta}(\hat{k}, \hat{l}) = (2\pi)^8 \frac{\delta^2 K}{i\delta \tilde{u}_\alpha(\hat{k})\delta u_\beta(\hat{l})}. \]

Introduction of the reduced forms

\[ G_{\alpha\beta}(\hat{k}, \hat{l}) = (2\pi)^4 \delta(\hat{k} + \hat{l})P_{\alpha\beta}(k)G(\hat{k}), \quad (65) \]

and

\[ \Gamma_{\alpha\beta}(\hat{k}, \hat{l}) = (2\pi)^4 \delta(\hat{k} + \hat{l})P_{\alpha\beta}(k)\Gamma(\hat{k}), \]

then leads to the standard relation

\[ \Gamma(\hat{k}) = G(\hat{k})^{-1}. \quad (66) \]

We can now use (31) to show that the connection between the bare and renormalised forms is

\[ \Gamma_R = Z_\nu \Gamma_B, \quad (67) \]

which demonstrates the suitability of \( \Gamma(\hat{k}) \) as a basis for determining \( Z_\nu \).

In carrying out the renormalisation of \( \Gamma(\hat{k}) \) to obtain the coefficients in (X3), we choose the normalisation point to be \( \hat{k} = \hat{m} \), where

\[ \hat{m} = (m, \omega_m = 0). \]
Here \( \mathbf{m} \) is a fixed vector of magnitude
\[
|\mathbf{m}| = \mu,
\]
the direction of which need not be specified, because the geometrical factor is contained in \( P_{\alpha\beta}(\mathbf{m}) \) which cancels off. The expansion (63) is used in conjunction with a normalisation condition that sets \( \Gamma(\hat{m}) \) equal to its tree level value. Thus, from (15) and (66), we have
\[
\Gamma(\hat{m}) = \Gamma_0(\hat{m}) = G_0(\hat{m})^{-1} = \tau(\mu)^{-1},
\]
and so the 1-loop term satisfies the normalisation condition
\[
\Gamma_1(\hat{m}) = 0.
\]

The Feynman diagram giving the 1-loop term of \( \Gamma_{\alpha\beta}(\hat{m}) \) is shown in Fig.3(i). The standard rules apply to such diagrams with the following assignments, which are shown in Fig.1:

1. External lines represent functional differentiation with respect to \( u(\hat{k}) \) when continuous, and \( \tilde{u}(\hat{k}) \), when dotted. The diagram is divided by a factor of \( i \) for each differentiation with respect to \( \tilde{u} \).

2. A continuous line linking two vertices denotes the reduced velocity correlation function defined through
\[
\langle v_\alpha(\hat{k})v_\beta(\hat{l}) \rangle = (2\pi)^4 \delta(\hat{k} + \hat{l}) Q_{\alpha\beta}(\hat{k}),
\]
and given by
\[
Q_{\alpha\beta}(\hat{k}) = D(\hat{k}) \left| G(\hat{k}) \right|^2 P_{\alpha\beta}(\hat{k}).
\]

For ease of notation, we omit zero-order labels in writing down mathematical expressions for the diagrams.
3. A half dotted/half continuous line connecting two vertices represents $i$ times the zero-order response function

$$G_{\alpha\beta}(\hat{k}) = G(\hat{k})P_{\alpha\beta}(k).$$

4. The NS vertex with one dotted and two continuous lines represents $P_{\alpha\beta\gamma}(k)$, the argument of which is associated with the dotted leg, with $k$ directed away from the node.

Returning now to the 1-loop diagram for $\Gamma_{\alpha\beta}(\hat{m})$, we note that it has a symmetry factor of 1. Hence, it yields a contribution to $\Gamma_1(\hat{m})$ given by

$$P_{\alpha\beta}(m)\Gamma'_1(\hat{m}) = \int D\hat{p}\, P_{\alpha\beta\delta}(m)P_{\lambda\nu\beta}(m - p)G_{\gamma\lambda}(\hat{m} - \hat{p})Q_{\delta\nu}(\hat{p}).$$

We can extract the logarithmic divergence from this integral by expanding its integrand in powers of $p/m$. This is possible because the divergence emanates from the region $p \sim \kappa$, while $\kappa \ll \mu$. A simple calculation leads to

$$\Gamma'_1(\hat{m}) = \frac{3}{2}g\tau(\mu)^{-1}I_0(\varepsilon),$$

where

$$I_0(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{dx}{x^2(x + 1)},$$

in which the lower limit of integration is

$$\varepsilon = \frac{\tau(\mu)}{\tau(\kappa)}.$$  

Extracting the logarithmic singularity from this integral gives

$$\Gamma'_1(\hat{m}) = -\tau(\mu)^{-1}g \log \left( \frac{\mu}{\kappa} \right).$$

To this we have to add the term arising from the counterterm vertex shown in Fig.2(i). This contributes the term $P_{\alpha\beta}(m)\Gamma''_1(\hat{m})$ where
\[ \Gamma''(\hat{m}) = \Delta Z_\nu \tau(\mu)^{-1} = a_{1\nu} \tau(\mu)^{-1} g \log \left( \frac{\mu}{\kappa} \right). \] (73)

But, from the normalisation condition (69), we have

\[ \Gamma'(\hat{m}) + \Gamma''(\hat{m}) = 0, \]

which, upon substituting (72) and (73), yields

\[ a_{1\nu} = 1. \]

We next carry out the analogous calculation for \( Z_D \) and show that its corresponding coefficient \( a_{1D} \) also equals 1, thereby verifying that the condition (37) is satisfied at 1-loop order. Here the relevant 1PI function is the correlation function given by

\[ \Pi_{\alpha\beta}(\hat{k}, \hat{l}) = (2\pi)^8 \frac{\delta^2 K}{i\delta \hat{u}_\alpha i\delta \hat{u}_\beta}, \]

which is readily shown to be related to the velocity correlation function \( Q_{\alpha\beta}(\hat{k}, \hat{l}) \) by [13]

\[ \Pi_{\alpha\beta}(\hat{k}, \hat{l}) = \int \Gamma_{\alpha\lambda}(\hat{k}, \hat{p}) \Gamma_{\beta\mu}(\hat{l}, \hat{q}) Q_{\lambda\mu}(\hat{p}, \hat{q}) d\hat{p}d\hat{q}. \]

Substituting the reduced forms

\[ \Pi_{\alpha\beta}(\hat{k}, \hat{l}) = (2\pi)^4 \delta(\hat{k} + \hat{l}) P_{\alpha\beta}(k) \Pi(\hat{k}), \]

and

\[ Q_{\alpha\beta}(\hat{k}, \hat{l}) = (2\pi)^4 \delta(\hat{k} + \hat{l}) P_{\alpha\beta}(k) Q(\hat{k}), \]

we get

\[ \Pi(\hat{k}) = \left| \Gamma(\hat{k}) \right|^2 Q(\hat{k}). \]

From this result and (31) and (66), we find that the bare and renormalised forms of \( \Pi \) are related by

\[ \Pi_R = Z_D \Pi_B, \]
which confirms that $\Pi(\hat{k})$ is the appropriate 1PI function to use for calculating $Z_D$.

The normalisation condition is again chosen to be consistent with the tree level approximation. That is, we set

$$\Pi(\hat{m}) = \Pi_0(\hat{m}) = D(\mu),$$

so that the 1-loop term satisfies the normalisation condition

$$\Pi_1(\hat{m}) = 0.$$  \hspace{1cm} (74)

The 1-loop Feynman diagram for $\Pi_{\alpha\beta}(\hat{m})$ is shown in Fig.3(ii). It has a symmetry factor of 1/2, and makes a contribution to $\Pi_1(\hat{m})$ which is given by

$$P_{\alpha\beta}(m)\Pi_1'(\hat{m}) = \frac{1}{2} \int D\hat{p} P_{\alpha\gamma\delta}(m) P_{\beta\lambda\nu}(m) Q_{\gamma\lambda}(\hat{p}) Q_{\delta\nu}(\hat{m} - \hat{p}).$$

In extracting the logarithmic singularity from this integral, we must take into account the fact that the symmetry of the integrand results in singularities of equal strength at both $p \sim \kappa$, and $|p - m| \sim \kappa$, the effect of which compensates for the symmetry factor. Consequently, we get

$$\Pi_1'(\hat{m}) = -D(\mu) g \log \left(\frac{\mu}{\kappa}\right).$$  \hspace{1cm} (75)

The $Z_D$ counterterm, which is shown in Fig.2(ii), contributes a term to $\Pi_1(\hat{m})$ given by

$$\Pi_1''(\hat{m}) = \Delta Z_D D(\mu) = a_{1D} D(\mu) g \log \left(\frac{\mu}{\kappa}\right).$$  \hspace{1cm} (76)

But, from the normalisation condition (74), we have

$$\Pi_1'(\hat{m}) + \Pi_1''(\hat{m}) = 0,$$

and substitution of (75) and (76) leads to

$$a_{1D} = 1.$$

We shall take the equality of the 1-loop coefficients of $Z_\nu$ and $Z_D$ as establishing that (37) holds. This allows us to calculate the $uv$ fixed point from the linear response function alone as follows. We use the standard result [14]
where \( Z_g \) is the renormalisation constant associated with the coupling constant. From the definition \( Z_g = g_0/g \) and (26), (28), (30), and (17), we get

\[
Z_g = Z^\nu_{-2}.
\]

Inserting this result in (77) and substituting the expansion (63), leads to

\[
\beta(g) = 2g^2(1 + a_{2\nu}g).
\]

This yields an \( uv \) fixed point

\[
g_* = -\frac{1}{a_{2\nu}},
\]

which satisfies (61) and (62) provided that \( a_{2\nu} < 0 \). It remains, then, to calculate \( a_{2\nu} \).

The constant \( a_{2\nu} \) is obtained from the 2-loop term of \( \Gamma_{\alpha\beta}(\hat{k}) \), namely \( P_{\alpha\beta}(m)\Gamma^\nu_2(\hat{k}) \). At the normalisation point it must satisfy the condition

\[
\Gamma^\nu_2(\hat{m}) = 0,
\]

by virtue of (49). Only two Feynman diagrams yield logarithmic divergences. They are shown in Figs. 3(iii) and (iv). They contribute the terms

\[
P_{\alpha\beta}(m)\Gamma^\nu_2(\hat{m}) = -\int D\hat{p} D\hat{q} Q_{\delta\nu}(\hat{p}) Q_{\lambda\mu}(\hat{q})
\]
\[\times G_{\kappa\gamma}(\hat{m} - \hat{p}) G_{\nu\sigma}(\hat{m} - \hat{p} - \hat{q}) G_{\tau\mu}(\hat{m} - \hat{p})
\]
\[\times P_{\alpha\gamma\beta}(m) P_{\mu\beta\epsilon}(m - p) P_{\kappa\lambda\nu}(m - p) P_{\rho\sigma\tau}(m - p - q),
\]

and

\[
P_{\alpha\beta}(m)\Gamma^\nu_2(\hat{m}) = -\int D\hat{p} D\hat{q} Q_{\delta\sigma}(\hat{p}) Q_{\lambda\mu}(\hat{q})
\]
\[\times G_{\kappa\gamma}(\hat{m} - \hat{p}) G_{\omega\nu}(\hat{m} - \hat{p} - \hat{q}) G_{\epsilon\tau}(\hat{m} - \hat{q})
\]
\[\times P_{\alpha\gamma\beta}(m) P_{\mu\beta\epsilon}(m - q) P_{\kappa\lambda\nu}(m - p) P_{\rho\sigma\tau}(m - p - q).
\]
To extract the logarithmic singularities from these integrals, we expand their integrands in powers of both $p/m$ and $q/m$. We do this in two steps. First, we integrate over frequencies and solid angles to get

$$\Gamma'_{2}(\hat{m}) = -\frac{9}{4}g^2\tau(\mu)^{-1}I_1(\varepsilon),$$  \hfill (81)$$

and

$$\Gamma''_{2}(\hat{m}) = -\frac{9}{4}g^2\tau(\mu)^{-1}I_2(\varepsilon),$$  \hfill (82)$$

where

$$I_1(\varepsilon) = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{dxdy}{x^2y^2(1+x)^2(1+x+y)},$$  \hfill (83)$$

and

$$I_2(\varepsilon) = \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{dxdy}{x^2y^2(1+x)(1+y)(1+x+y)}.$$  \hfill (84)$$

Secondly, we expand these double integrals for small $\varepsilon$ to obtain

$$\Gamma'_{2}(\hat{m}) = 8g^2\tau(\mu)^{-1}\log\left(\frac{\mu}{\kappa}\right),$$  \hfill (85)$$

and

$$\Gamma''_{2}(\hat{m}) = \frac{21}{2}g^2\tau(\mu)^{-1}\log\left(\frac{\mu}{\kappa}\right).$$  \hfill (86)$$

To these two contributions to $\Gamma_{2}(\hat{m})$, we must add the counterterm, which, by analogy with $\Gamma_{2}(\hat{m})$, takes the form

$$\Gamma''''_{2}(\hat{m}) = a_{2\nu}g^2\tau(\mu)^{-1}\log\left(\frac{\mu}{\kappa}\right).$$  \hfill (87)$$

Thus, the normalisation condition $\Gamma_{2}(\hat{m})$ becomes

$$\Gamma'_{2}(\hat{m}) + \Gamma''_{2}(\hat{m}) + \Gamma''''_{2}(\hat{m}) = 0,$$

which, by (85)-(87), yields
Therefore, from (78), we obtain the fixed point coupling constant

$$g_* = \frac{2}{37},$$

which verifies that the residual coupling can be treated as weak.

Finally, we explain why the 2-loop topologies, which have been discarded in calculating $\Gamma$, do not contribute to $a_{2\nu}$. As we shall explain further in Section VII, divergences arise in these diagrams when it is possible for one or more soft wavevectors (ie values of $p$ and/or $q \ll m$) to flow through a correlator. However, if this entails the flow of some or all of these wavevectors through the active (ie dotted) leg of the NS vertex, then the logarithmic divergence will be suppressed by the extra powers of $p$ and/or $q$. In the case of the 2-loop diagrams which we have just calculated, the external hard wavevector $\hat{m}$ flows through the active legs of all vertices, so no suppression occurs. However, in the case of the remaining topologies at least one soft wavevector flowing through a correlator must also flow through the active leg of a NS vertex. In the case of the four remaining two loop topologies containing vertex corrections, the logarithmic divergence is suppressed individually for each diagram, after integration over the solid angles. In the case of the three remaining 2-loop diagrams containing insertions of the 1-loop diagrams (i) and (iv) of Fig.3, suppression results after integrating over the solid angles and summing over the diagrams, the overall cancellation being related to the fact that the coefficients $a_{1\nu}$ and $a_{1D}$ associated with the two types of insertion are equal. Likewise the four 1-loop diagrams containing the counterterm vertices yield no net contribution to $a_{2\nu}$. The treatment of the power and power×logarithmic divergences arising in integrals like (71),(83) and (84) is given in Section VII. For the moment we discard them because they are not directly relevant to the actual calculation of the scaling exponents for reasons already given.
V. THE NONLINEAR RESPONSE

Having established that an $uv$ fixed point exists, we can proceed with the calculation of the anomalous dimension $\gamma_s$ of the general operator $O_s(\hat{x})$, which is required for the evaluation of the anomaly $\tau_n$. To do this in the simplest possible way, we must identify a 1PI response function which can be renormalised by means of $Z_s$. Elimination of the logarithmic divergences from such a function will then enable us to determine the constants in the expansion

$$Z_s = 1 + g_s a_1^{(s)} \log \left( \frac{\mu}{\kappa} \right) + g^2 \left\{ \frac{1}{2} \left( a_1^{(s)} \right)^2 \log^2 \left( \frac{\mu}{\kappa} \right) + a_2^{(s)} \log \left( \frac{\mu}{\kappa} \right) \right\} + \ldots \quad (89)$$

so that we can calculate $\gamma_s$ using (43).

Consider first the case $s = 2$. Obviously, the required function must involve $O_2(\hat{x})$, which is the composite operator associated with the longitudinal turbulence energy. In addition, it must involve the dynamic response operator (11) in order to relate the anomaly $\tau_2$ to the dynamics of the turbulence. This suggests that we should consider how the turbulence energy responds on average to a change in the forcing. Clearly, we can characterise the response of the turbulence energy at a point $\hat{x}$ to a change in the forcing at two points $\hat{x}'$ and $\hat{x}''$ by means of the nonlinear Green’s function

$$G^{(2)}_{\alpha\beta}(\hat{x}', \hat{x}'', \hat{x}) = \left\langle \frac{\delta^2}{\delta f_\alpha(\hat{x}') \delta f_\beta(\hat{x}'')} \left( \nu_1(\hat{x})^2 \right) \right\rangle. \quad (90)$$

But the complexity of this object is such that its logarithmic divergences cannot be summed using the renormalisation group in terms of the $Z_2$ and $Z_\nu$ counterterms alone. On the other hand, its average $\overline{G}^{(2)}_{\alpha\beta}(\hat{x})$ taken over the forcing separation $\hat{x}' - \hat{x}'''$, which gives a mean response to forcing at the centroid of the excitation points, can be, as we shall show shortly. Hence, its corresponding 1PI function provides a direct means of obtaining the expansion (89) and so it provides an adequate basis for the calculation of $\gamma_2$.

This 1PI function is obtained as follows. We start with the Fourier transform of (90), the reduced form of which is given by
\[ G_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}, \hat{p}) = (2\pi)^4 \delta(\hat{k} + \hat{l} + \hat{p}) G_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}), \]  
(91)

where

\[ G_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}) = P_{\alpha_1}(k) P_{\beta_1}(l) G_{2}(\hat{k}, \hat{l}). \]  
(92)

Its corresponding 1PI response function follows from

\[ \Theta_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}, \hat{p}) = (2\pi)^{12} \frac{\delta^3 K}{\delta u_{\alpha}(\hat{k}) \delta u_{\beta}(\hat{l}) \delta t_2(\hat{p})}, \]  
(93)

with a reduced form given by

\[ \Theta_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}, \hat{p}) = (2\pi)^4 \delta(\hat{k} + \hat{l} + \hat{p}) P_{\alpha_1}(k) P_{\beta_1}(l) \Theta_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}), \]  
(94)

A standard calculation shows that it is related to \( G_{\alpha\beta}^{(2)} \) by

\[ \Theta_{\alpha\beta}^{(2)}(\hat{k}, \hat{l}, \hat{p}) = -\int \Gamma_{\lambda\alpha}(\hat{q}, \hat{k}) \Gamma_{\mu\beta}(\hat{q}', \hat{l}) G_{\lambda\mu}^{(2)}(\hat{q}, \hat{q}', \hat{p}) \, d\hat{q} d\hat{q}', \]  
(95)

from which, on making use of (91)-(94), we obtain

\[ \Theta^{(2)}(\hat{k}, \hat{l}) = -\Gamma(\hat{k}) \Gamma(\hat{l}) G_{2}(\hat{k}, \hat{l}). \]  
(96)

Next, we average \( G_{\alpha\beta}^{(2)} \) over the forcing separation to get

\[ \overline{G}_{\alpha\beta}(\hat{x}) = 2 \int G_{\alpha\beta}^{(2)}(\hat{k}, \hat{k}) \exp(2i\hat{k} \cdot \hat{x}) D\hat{k}. \]  
(97)

This integral shows that the Fourier transform of \( \overline{G}_{\alpha\beta}^{(2)}(\hat{x}) \) depends only on the diagonal components of the reduced function (92). It follows, therefore, from (96) and (97), that the 1PI object which we need to consider, in order to determine \( Z_2 \), is

\[ \Theta^{(2)}(\hat{k}, \hat{k}) = -\Gamma(\hat{k})^2 G_{2}(\hat{k}, \hat{k}). \]  

Indeed, an application of (31), together with (67), shows that its bare and renormalised forms are connected by

\[ \Theta_R^{(2)}(\hat{k}, \hat{k}) = Z_2 \Theta_B^{(2)}(\hat{k}, \hat{k}). \]
In this way, as we have indicated, we arrive at a function which can be renormalised using the $Z_2$ counterterm alone.

The normalisation condition for $\Theta^{(2)}(\hat{k}, \hat{k})$ is again applied at the point $\hat{k} = \hat{m}$, and chosen to be consistent with the tree level approximation, which gives

$$\Theta^{(2)}(\hat{m}, \hat{m}) = \Theta^{(2)}_0(\hat{m}, \hat{m}) = -1,$$

so that the 1 and 2-loop terms satisfy the normalisation conditions

$$\Theta^{(2)}_1(\hat{m}, \hat{m}) = \Theta^{(2)}_2(\hat{m}, \hat{m}) = 0. \tag{99}$$

The diagrams giving $\Theta^{(2)}(\hat{m}, \hat{m})$ to 2-loop order are shown in Fig.4. Their new feature is the appearance of the heavy dot vertex. This represents the $O_2$ composite operator vertex, which is shown in Fig.1(iv) for the general case of $O_s$. We can understand how these diagrams arise from the loop expansion of $K$ by using the general procedure described in [15]. This depends on the fact that (80) is a special case of the 4th order correlation function of elementary fields defined by

$$B^{(4)}_{\alpha\beta\gamma\delta}(\hat{x}', \hat{x}'', \hat{x}, \hat{z}) = \frac{i^2}{2} \left\langle \bar{v}_\alpha(\hat{x}') v_\beta(\hat{x}'') v_\gamma(\hat{x}) v_\delta(\hat{z}) \right\rangle,$$

in which the arguments $\hat{x}$ and $\hat{z}$ coalesce. Hence, their Fourier transforms are related. In particular, the connection between their respective 1PI functions is

$$\Theta^{(2)}_{\alpha\beta}(\hat{k}, \hat{l}, \hat{m}) = \frac{1}{2} \int \Phi_{\alpha\beta\lambda\mu}(\hat{k}, \hat{l}, \hat{m} - \hat{q}) G_1(\hat{m} - \hat{q}) G_{\mu 1}(\hat{q}) \, D\hat{q},$$

where $\Phi$ is the 1PI form corresponding to $B^{(4)}$, which is generated by

$$\Phi_{\alpha\beta\gamma\delta}(\hat{k}, \hat{l}, \hat{p}, \hat{q}) = (2\pi)^{16} \frac{\delta^4 K}{\delta u_\alpha(\hat{k}) \delta u_\beta(\hat{l}) i\delta \bar{u}_\gamma(\hat{p}) i\delta \bar{u}_\delta(\hat{q})}.$$  

This implies that the diagrams for $\Theta^{(2)}(\hat{m}, \hat{m})$ are constructed from the diagrams for $\Phi$ by tying the two dotted external legs of the latter to form the $O_2$ vertex.

The 1-loop diagram for $\Theta^{(2)}(\hat{m}, \hat{m})$ shown in Fig.4(iv) is constructed from the tree level diagram for $\Phi$, which is shown opposite to it in Fig.4(i). Similarly, the two 2-loop diagrams
for $\Theta^{(2)}(\hat{m}, \hat{m})$, shown in Figs.4(v) and (vi), are constructed from the 1-loop diagrams for $\Phi$, again shown opposite to them in Figs.4(ii) and (iii). The other possible 2-loop diagrams for $\Theta^{(2)}(\hat{m}, \hat{m})$, which arise from the two remaining 1-loop diagrams for $\Phi$, are discarded because the logarithmic divergences disappear, after integration over the solid angles. In addition, diagrams which produce longitudinal terms obviously make no contribution to $Z_2$ and can also be discarded.

The 1-loop diagram of Fig.4(iv) contributes to $\Theta^{(2)}_1$ the term

$$P_{\alpha_1}(m)P_{\beta_1}(m)\Theta^{(2)\prime}_1(m, \hat{m}) = \int D\hat{p}P_{\lambda\gamma}(m - p)P_{\nu\delta}(m + p)G_{\lambda\nu}(\hat{m} + \hat{p})G_{\gamma\delta}(\hat{m} - \hat{p})Q_{\alpha\beta}(\hat{p}),$$

which yields a logarithmic divergence

$$\Theta^{(2)\prime}_1(m, \hat{m}) = -g \log \left(\frac{\mu}{\kappa}\right) .$$

The contractions implied in (98) again permit us to discard the longitudinal part of the above integral. The counterterm vertex shown in Fig.2(iii) adds a contribution

$$-P_{\alpha_1}(m)P_{\beta_1}(m)\Theta^{(2)\prime\prime}_1(m, \hat{m}) = -P_{\alpha_1}(m)P_{\beta_1}(m)\Delta Z_2,$$

so that by (99) its contribution to $\Theta^{(2)}_1$ is

$$\Theta^{(2)\prime\prime}_1(m, \hat{m}) = -a^{(2)}_1 g \log \left(\frac{\mu}{\kappa}\right) .$$

But, from the normalisation condition (99), we have

$$\Theta^{(2)}_1(m, \hat{m}) = \Theta^{(2)\prime}_1(m, \hat{m}) + \Theta^{(2)\prime\prime}_1(m, \hat{m}) = 0,$$

which, upon substituting (100) and (101), gives

$$a^{(2)}_1 = -1 .$$

At 2-loop order the diagrams in Figs.4(v) and (vi) contribute the terms

$$\Theta^{(2)\prime}_2(m, \hat{m}) = \frac{9}{4} g^2 I_3 ,$$

$$\Theta^{(2)\prime\prime}_2(m, \hat{m}) = \frac{9}{4} g^2 I_3 ,$$

$$\Theta^{(2)\prime\prime\prime}_2(m, \hat{m}) = \frac{9}{4} g^2 I_3 ,$$

$$\Theta^{(2)\prime\prime\prime\prime}_2(m, \hat{m}) = \frac{9}{4} g^2 I_3 .$$
and

$$\Theta^{(2)\mu}_{2}(\hat{m}, \hat{m}) = \frac{9}{4}g^2 I_4,$$

where

$$I_3 = -\int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{2 + x + y}{x^2y^2(1 + x)(2 + y)(1 + x + y)} dxdy,$$

and

$$I_4 = -\int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{(2 + x)(2 + x + y)(1 + 3y + y^2) - (1 + x)y(2 + y)(3 + x + y)}{x^2y^2(1 + x)(2 + x)(1 + y)^2(2 + y)(1 + x + y)} dxdy.$$

The latter yield logarithmic divergences

$$\Theta^{(2)\nu}_{2}(\hat{m}, \hat{m}) = \frac{9}{4}g^2 \left( \frac{5}{3} - \frac{1}{6} \log 2 \right) \log \left( \frac{\mu}{\kappa} \right),$$

(103)

and

$$\Theta^{(2)\mu}_{2}(\hat{m}, \hat{m}) = \frac{9}{4}g^2 \left( \frac{4}{3} + \frac{1}{3} \log 2 \right) \log \left( \frac{\mu}{\kappa} \right).$$

(104)

To these we must add the 2-loop counterterm corresponding to (101), namely

$$\Theta^{(2)\mu}_{2}(\hat{m}, \hat{m}) = -a^{(2)}_{2} g^2 \log \left( \frac{\mu}{\kappa} \right).$$

(105)

But the normalisation condition (99) gives

$$\Theta^{(2)\mu}_{1}(\hat{m}, \hat{m}) + \Theta^{(2)\nu}_{1}(\hat{m}, \hat{m}) + \Theta^{(2)\mu}_{1}(\hat{m}, \hat{m}) = 0,$$

which, after substituting (103)-(105), yields

$$a^{(2)}_{2} = 7.0.$$

(106)

The foregoing can be generalised to arbitrary $s$. In place of (93), we now consider the general 1PI response function

$$\Theta_{\alpha_1...\alpha_s}(\hat{k}_1, ..., \hat{k}_s, \hat{p}) = (2\pi)^{4(s+1)} \frac{\delta^{s+1}K}{\delta u_{\alpha_1}(k_1)\ldots\delta u_{\alpha_s}(k_s)\delta t_s(-\hat{p})},$$
with a reduced form defined by
\[
\Theta^{(s)}_{\alpha_1...\alpha_s}(\hat{k}_1, ..., \hat{k}_s, \hat{p}) = (2\pi)^4 \delta(\hat{k}_1 + ... + \hat{k}_s + \hat{p}) P_{\alpha_1}(k_1)...P_{\alpha_s}(k_s) \Theta^{(s)}(\hat{k}_1, ..., \hat{k}_s).
\]
Then \(Z_s\) can be found by eliminating the logarithmic divergences from the diagonal component \(\Theta^{(s)}(\hat{m}, ..., \hat{m})\) as above. The relevant diagrams are again those shown in Figs.4(iv)-(vi), except that the heavy dot now symbolises the \(O_s\) vertex of Fig.1(iv), so the \(s-2\) external legs of \(O_s\) are not shown explicitly. Each diagram has a symmetry factor \(s(s-1)/2\). As this is the only respect in which these diagrams differ from those just considered, we have the relation
\[
a^{(s)}_{1,2} = \frac{s(s-1)}{2} a^{(2)}_{1,2}.
\]
However, this is an approximate result, because it is not valid for diagrams containing more than 2-loops. But, as we discuss further below, it suffices for the calculation of low order exponents. Thus, we have now calculated all the numerical constants that we require for the evaluation of \(\zeta_n\).

VI. THE SCALING EXPONENTS

For \(n = 2\), we have, from (107),
\[
\zeta_2 = \frac{2}{3} + \Delta_2,
\]
where, from (13),(48) and (89).
\[
\Delta_2 = -g_*(a_1^{(2)} + a_2^{(2)} g_*).
\]
Substituting the numerical values calculated above, as given in (88),(102) and (106), we get
\[
\Delta_2 = \frac{46}{372} = 0.0336,
\]
which yields

34
\[ \zeta_2 = 0.70. \]

For \( n = 3 \), we shall verify in Section \textbf{VIII} that the known exact result

\[ \zeta_3 = 1, \]

holds.

In the general case, for \( n > 3 \), we have from (58)

\[ \zeta_n = \frac{n}{3} - \tau_n. \]

For even orders \( n = 2m \), the anomaly is given by (59),

\[ \tau_n = \max_p \tau_{np}. \quad (109) \]

But, from (13),(57),(89),(107) and (108), we have

\[ \tau_{np} = \{p(p-1) + (n-p)(n-p-1) - 2(m-p)[(2(m-p)-1)]\} \frac{\Delta_2}{2}. \]

A simple calculation shows that the maximum value of this expression is attained by the two terms in the series (52) with (a) \( p = m \) and (b) \( p = m - 1 \); which gives for (109)

\[ \tau_n = m(m-1)\Delta_2. \quad (110) \]

For odd orders, \( n = 2m + 1 \), the anomaly is given directly by (60), which yields

\[ \tau_n = m^2\Delta_2, \]

where we have again used (13),(59),(107) and (108).

The above results have been used to calculate \( \zeta_n \) up to \( n = 10 \). The results are shown in Fig.5, together with the experimental data taken from [16-20]. It can be seen that the agreement is good up to about \( n = 7 \) and fair beyond, if we allow for the uncertainties in the experimental data which begin to arise. In particular, it may be noted that the key values \( \zeta_2 = 0.70 \) and \( \zeta_6 = 1.8 \) are in good agreement with experimental data, the respective data sets from [16-20] giving for \( \zeta_2 \) the values (0.71, 0.70, 0.71, 0.70, 0.71) and for \( \zeta_6 \) the
values (1.78, 1.8, 1.8, 1.71, 1.71). The divergence of the experimental data at higher orders reflects the fact that the experimental determination of ζₙ is not yet fully satisfactory for the reasons given in [20]. Hence, the good agreement between our calculations at higher values of n with the particular data sets from [16-18] must be treated with caution, particularly as the expression we have derived above is not applicable at large orders. This limitation stems from the fact that the mean nonlinear response function, being an average over the forcing configuration, does not represent the effect of multiple correlations with sufficient accuracy at large n. In addition, the approximation (107), as we have noted, only holds up to 2-loop order. Indeed, it is evident from the foregoing that the overall approximation must fail when ngₙ ∼ 1. However, this occurs at roughly n = 20, which is well above the current limit of reliable experimental data. Equally, the divergence of our theoretical values at higher values of n from the other two data sets [19,20] could indicate that the accuracy of our low order approximation is already beginning to deteriorate at around n ∼ 10.

VII. ELIMINATION OF SWEEPING

We now return to the question of the power and power×logarithmic divergences which, up to this point, we have simply discarded. The fact that power divergences arise when field-theoretic methods are applied to turbulence, using an Eulerian approach, was noticed originally in [21]. Their origin was subsequently identified as being due to the kinematic effect of the sweeping of small eddies by large eddies, having an almost uniform velocity [22,23]. The remedy was to change from an Eulerian to a Lagrangian description, but this greatly complicates the subsequent analysis [24]. However, it has been shown that the elimination of sweeping can be accomplished more simply by transforming to a frame moving with the local velocity of the large scale eddies at some chosen reference point, [25,26]. We shall show that a similar approach can be used to eliminate the power and power×logarithmic divergences within the present framework. In this way, we shall demonstrate that, although we have started out from an Eulerian formulation, we ultimately obtain quasi-Lagrangian
approximations for the renormalised functions.

The problem, therefore, is to find a sweeping interaction term, $\Delta L_s$, say, which can be used to eliminate the effect of sweeping convection. To this end, we introduce a uniform convection $U$ into $W$ and average over its probability distribution, which we assume to be a Gaussian distribution $\propto \exp(-U^2/2U_0^2)$. This adds to $L$ an additional interaction term given by

$$
\Delta L_U = -\frac{U^2}{2} \int \mathbf{l} \cdot \mathbf{m} \tilde{\mathbf{v}}(\hat{m}) \cdot \mathbf{v}(\hat{m}) \tilde{\mathbf{v}}(\hat{l}) \cdot \mathbf{v}(\hat{l}) D\hat{l}D\hat{m},
$$

which represents the effect of a random Galilean transformation of the velocity field. We have not distinguished between $\mathbf{v}$ before and after the transformation for consistency with the earlier expressions, such as (18), and bearing in mind that the transformation does not affect the statistical averages required for the structure functions.

To represent diagrammatically the additional terms which arise in the loop expansion of $W$ after the inclusion of the sweeping interaction term we need to introduce a new 4-leg ‘sweeping’ vertex of the type shown in Fig.6(i). The two wavevectors $\hat{l}$ and $\hat{m}$ in (111) enter this vertex along its continuous legs and leave along the dotted legs. A pair of legs carrying a particular wavevector must also carry the same vector index to represent the scalar product. Free wavevectors in a diagram containing one or more of these sweeping vertices are identified, as previously, by overall wavenumber conservation, together with conservation at any NS vertex. Each sweeping vertex then contributes a factor $U_0^2 \mathbf{l} \cdot \mathbf{m}$, where $\mathbf{l}$ and $\mathbf{m}$ are the two wavevectors which enter the vertex along its two continuous legs. In all other respects the diagrams are to be interpreted in accordance with the rules given in Section IV.

Consider now the set of diagrams, containing only NS vertices, which are associated with a particular Green’s function or velocity correlator, $\mathcal{G}$, say. Let $C_{NS}$ denote any such diagram contributing to $\mathcal{G}$. We shall show that it is possible to generate all power and power×logarithmic divergences of any $C_{NS}$ from a single sweeping interaction of the form (111). Let $C_U$ denote any diagram containing at least one sweeping vertex. If $C_U$ contains
no NS vertices at all, then it will only generate power divergences. But if it contains at least one NS vertex, it will also generate power \times logarithmic divergences. The following topological argument demonstrates that the power divergences of $C_{NS}$ can be put into 1-1 correspondence with the $C_U$ diagrams relating to $g$.

Each factor $\tau(\kappa)$ (or, equivalently, $\varepsilon^{-1}$) in a power divergence of $C_{NS}$ arises because it is possible for a soft wavevector $q$ to flow through a particular velocity correlator without flowing through the active legs of the two NS vertices which it connects, as already discussed in Section [V]. This situation can be represented diagrammatically by contracting the correlator into a 4-leg vertex formed by merging the two NS vertices which it links, whilst leaving the hard lines intact. This can be demonstrated as follows. First, the new vertex must consist of two in-coming full lines which carry hard wavevectors, $l$ and $m$ (say), and two outgoing dotted lines along which they leave. This is because two full legs disappear from the merged NS vertices and wavevectors leave NS vertices along the dotted leg. Furthermore, after integrating over the directions of the soft wavevector $q$, the two merged NS vertices generate, through contraction of the projectors, a factor proportional to the scalar product of the in-coming hard lines, $l \cdot m$, while the two legs of a pair carrying the same wavevector acquire the same vector index. The final integration over the wavenumbers then produces the constant

$$
\frac{1}{6\pi^2} \int_{\kappa}^{\infty} q^2 D(q) \tau(q) dq = \frac{3}{2} g \tau(\kappa) \nu^3.
$$

So, such a vertex must, in fact, be of the sweeping convection type (111), with a coefficient given by

$$
U_0^2 = \frac{3}{2} g \tau(\kappa) \nu^3,
$$

which relates the rms velocity of the sweeping eddies to the strength of the nonlinear interaction. Note that $U_0$ is scale dependent, as it depends on the renormalisation scale $\mu$ through $g$ and $\nu$, and, hence, it differs according to the fluctuation scale on which the RG focuses. In physical terms, this reflects the fact that the rms velocity of the sweeping eddies depends
Clearly, if a subset of correlators of $C_{NS}$, each of which carries a soft wavenumber, is contracted into such vertices, in a manner which allows hard wavevectors to flow through $C_{NS}$, then the result is a diagram which is identical to one of the $C_U$ diagrams. Moreover, it is clear that there are always exactly as many ways to contract the correlators in $C_{NS}$ as there are different $C_U$ diagrams and that their symmetry factors must match. This argument demonstrates, therefore, the important point that the power and power×logarithmic divergences generated by the NS vertex must arise on account of the background of kinematic sweeping effects. Moreover, we also see that, in order to eliminate them, it is only necessary to introduce a sweeping interaction term into $W$ of opposite sign to the one from which they can be generated, which, according to (111) and (112) yields the sweeping interaction term

$$\Delta L_s = \frac{3}{4} \frac{g \nu^3}{\tau(k)} \int l \cdot m \tilde{v}(\hat{m}) \cdot v(-\hat{m}) \tilde{v}(\hat{l}) \cdot v(-\hat{l}) D\hat{l} D\hat{m}. \quad (113)$$

Thus, the sweeping vertex shown in Fig.6(i) is taken to represent the algebraic factor

$$\text{Vertex } 6(i) = -\frac{3}{2} g \tau(k) \nu^3 l \cdot m.$$

Having inserted (113) into $W$ one is then left with only the pure logarithmic divergences generated by the NS vertex, which, as we have shown, can be summed using the RG. This justifies our procedure whereby power and power×logarithmic divergences are discarded when calculating anomalous exponents.

We now illustrate the cancellation of power and power×logarithmic divergences in concrete terms by eliminating them to 2-loop order from $\Gamma(\hat{k})$. This will demonstrate how the various symmetry factors match up. Consider first the 1-loop diagram for $\Gamma_1(\hat{k})$ arising from the NS vertex. From our previous result (100), we find that its power divergence is given, at the normalisation point, by

$$\Gamma_1(\text{diagram } 3(i)) = \frac{3}{2} g \frac{\tau(k)}{\tau(\mu)^2}. \quad (114)$$

Here the Feynman rules applied to the sweeping vertex yield the single diagram of Fig.6(ii),
as we anticipate from the fact that the NS vertices in Fig.3(i) can be merged in only one way. In this case, a trivial calculation yields

$$\Gamma_1(\text{diagram 6(ii)}) = -\frac{3}{2}g^2 \frac{\tau(\kappa)}{\tau(\mu)^2},$$

which cancels (114), as required.

Explicit verification that there are no power or power×logarithmic divergences in $\Gamma(\hat{k})$ at 2-loop order is less trivial. Consider first diagram (iv) of Fig.3. The power divergences arising from this diagram follow from (82) which gives

$$\Gamma_2(\text{diagram 3(iv)}) = -\frac{9}{4}g^2 \frac{\tau(\mu)}{\tau(\mu)} \left\{ \frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \frac{4}{\epsilon} \log \epsilon \right\}. \quad (115)$$

For this diagram the corresponding sweeping diagrams are diagrams (i)-(iii) of Fig.7. This follows from the Feynman rules and can be checked from diagram (iv) of Fig.3 by first contracting its correlators individually and then together. By applying the Feynman rules to diagram (i) of Fig.7 we obtain, at the normalisation point,

$$P_{\alpha\beta}(m)\Gamma_2(\text{diagram 7(i)}) = \frac{3}{2}g^2 \frac{\tau(\kappa)\nu^3}{\tau(\mu)} \int \mathbf{p} \cdot (\mathbf{k} - \mathbf{p}) P_{\lambda\mu}(m) P_{\tau\rho}(m - \mathbf{p})$$

$$\times Q_{\rho\nu}(\hat{p}) G_{\alpha\lambda}(\hat{m}) G_{\sigma\mu}(\hat{m} - \hat{p}) G_{\mu\tau}(\hat{m} - \hat{p}).$$

We can evaluate this integral using the method described in Section IV. This gives

$$\Gamma_2(\text{diagram 7(i)}) = \frac{9}{4}g^2 \frac{1}{\tau(\mu)} \int_{\epsilon}^{\infty} \frac{dx}{x^2(x + 1)^2}$$

$$= \frac{9}{4}g^2 \frac{1}{\tau(\mu)} \left\{ \frac{1}{\epsilon^2} + \frac{2}{\epsilon} \log \epsilon + \frac{1}{\epsilon} \right\}.$$

Diagram (ii) of Fig.7 yields the same value

$$\Gamma_2(\text{diagram 7(ii)}) = \Gamma_2(\text{diagram 7(i)}).$$

Finally, evaluation of the diagram (iii) of Fig.7 is trivial and yields

$$\Gamma_2(\text{diagram 7(iii)}) = -\frac{9}{4}g^2 \frac{1}{\tau(\mu)} \frac{1}{\epsilon^2}. $$
Evidently, the sum of these three diagrams cancels (115) exactly.

Similarly, we can show that the sweeping vertex eliminates the power divergences arising from the second 2-loop diagram, shown in Fig.3(iii). From (81), these are given by
\[
\Gamma_2(\text{diagram 3(iii)}) = -\frac{9}{4} \frac{g^2}{\tau(\mu)} \left\{ \frac{1}{\varepsilon^2} + \frac{5}{2} \frac{1}{\varepsilon} + \frac{4}{\varepsilon} \log \varepsilon \right\}. \tag{116}
\]
In this case, the corresponding diagrams generated by the sweeping vertex are diagrams (iv)-(vi) of Fig.7 which contribute the terms
\[
\Gamma_2(\text{diagram 7(iv)}) = \frac{9}{4} \frac{g^2}{\tau(\mu)} \left\{ \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \log \varepsilon \right\},
\]
\[
\Gamma_2(\text{diagram 7(v)}) = -\frac{9}{4} \frac{g^2}{\tau(\mu)} \frac{1}{\varepsilon^2},
\]
and
\[
\Gamma_2(\text{diagram 7(vi)}) = \frac{9}{4} \frac{g^2}{\tau(\mu)} \left\{ \frac{1}{\varepsilon^2} + \frac{3}{\varepsilon} \log \varepsilon + \frac{5}{2} \frac{1}{\varepsilon} \right\}.
\]
Again, their sum exactly cancels (116). We have thereby verified to 2-loop order that the sweeping interaction eliminates power divergences from the linear response function.

VIII. THE KOLMOGOROV APPROXIMATION

The fact that it has been possible to calculate the anomalies successfully by means of perturbation theory stems, in part, from the incorporation of the Kolmogorov theory into the zero order approximation. As we have seen, this has been done by replacing the actual viscous quadratic form in $W$, arising from the NS equations, by a modified quadratic form, characterised by an effective random stirring force spectrum $D(k)$ and the effective timescale $\tau(k)$. We now demonstrate that these two functions can be deduced self-consistently as part of the calculation and confirm that that they do have the inertial range forms given in (24) and (25).

To determine these functions, we need two conditions. As in [10], one condition is supplied by evaluating the energy equation to 1-loop order, which gives the convergent DIA
form, corresponding to the so-called line renormalisation [24]. In the inertial range, it reduces to the condition that the energy flux across wavenumbers $\Pi_E(k)$ is independent of $k$ and equal to the mean dissipation rate $\epsilon$:

$$\Pi_E(k) = \epsilon.$$ 

Thus, evaluation of $\Pi_E(k)$ to 1-loop order gives the well-known result [24]

$$\Pi_E(k) = \int_k^\infty T(p)dp,$$

where

$$T(p) = 8\pi^2 \int \int_{\Delta}dqdr \frac{p^3qr}{\tau(p)^{-1} + \tau(q)^{-1} + \tau(r)^{-1}} \times \{b(p, q, r)Q(r)(Q(q) - Q(p)) + b(p, r, q)Q(q)(Q(r) - Q(p))\}.$$  \hspace{1cm} (117)

Here $\Delta$ indicates integration over the region of the $p, q$ plane in which $p, q, r$ can form a triangle and

$$b(p, q, r) = \frac{(p^2 + q^2 - r^2)^3}{8p^2q^2} + \frac{r^4 - (p^2 - q^2)^2}{4p^2r^2}.$$ 

The second condition must be deduced from the linear response function. This is where difficulties have arisen with this approach in the past, when using an Eulerian framework, because of the $iv$ divergences arising from sweeping. On the other hand, it is known that no divergence problems arise from sweeping convection in the case of the energy equation [24]. However, we have just shown how these power divergences can be systematically removed from the response function (and, indeed, all such functions) by means of a random Galilean transformation of the velocity field. This leaves the logarithmic divergences which, as we have seen, are to be eliminated from $\Gamma(\hat{k})$ using the $Z_\nu$ counterterm. Recall that to fix the finite part of $\Gamma(\hat{k})$, after this renormalisation, we imposed the normalisation condition (68) which specifies that its tree level term should be exact at the normalisation scale $\mu$. Thus, after eliminating sweeping convection, as described in Section [VII], and using the 1-loop
normalisation condition (68) to eliminate the logarithmic divergences, we obtain, at an arbitrary wavevector \( k \) (with \( \omega = 0 \)), the renormalised linear response function

\[
\Gamma(k, 0) = \tau(k)^{-1} + \frac{\mu^2 \tau(\mu)^3 - k^2 \tau(k)^3}{6\pi^2 \tau(k) \tau(\mu)} \int_0^\infty \frac{p^2 \tau(p)^2 D(p) dp}{(\tau(k) + \tau(p))(\tau(\mu) + \tau(p))} \\
+ \frac{\mu^2 \tau(\mu)^2 - k^2 \tau(k)^2}{6\pi^2} \int_0^\infty \frac{p^2 \tau(p) D(p) dp}{(\tau(k) + \tau(p))(\tau(\mu) + \tau(p))}.
\]

(118)

It is precisely the condition that this expression should, indeed, yield a finite renormalised value which provides the required second relation, as we now explain.

In the inertial range limit, we seek scaling solutions with \( \tau(k) \propto k^{-a} \) and \( Q(k) \propto k^b \), in which case \( D(k) = \tau(k)^{-1} Q(k) \propto k^{a+b} \). Now standard dimensional analysis shows that for (117) to hold in these circumstances, we must have \( a + 2b = -8 \), [24]. Furthermore, if this scaling solution were to produce a non-renormalisable divergence in the response function, it would arise in the second term of (118), since we can assume that \( a > 0 \). To prevent this from occurring, the coefficient of the integral must be zero, which requires

\[
\frac{\tau(k)}{\tau(\mu)} = \left( \frac{\mu}{k} \right)^{2/3},
\]

giving \( a = 2/3 \), and, hence, \( b = -11/3 \), so that \( a + b = -3 \). Thus, these relations do, in fact, yield the solution (24) and (25), which we may conveniently re-write as

\[
\tau(k)^{-1} = \beta \epsilon^{1/3} k^{2/3}
\]

(119)

and

\[
D(k) = \frac{\alpha}{2\pi} \epsilon^{2/3} k^{-3}.
\]

(120)

Therefore, the energy spectrum function

\[
E(k) = 4\pi k^2 Q(k) = 4\pi k^2 D(k) \tau(k)
\]

takes the Kolmogorov inertial range form

\[
E(k) = \alpha \epsilon^{2/3} k^{-5/3},
\]
and the integral in the third term of (118) is, indeed, finite and yields

\[ \Gamma(k, 0) = \tau(k)^{-1} \{ 1 - g \log \left( \frac{k}{\mu} \right) \}. \]

For present purposes, explicit evaluation of the two constants is unnecessary, since they ultimately disappear from the calculation of the exponents, because they only occur through the coupling constant which, as we have seen, is eventually evaluated in terms of its fixed point value.

Next, we comment briefly on the effect of allowing for the perturbation terms (23) which give the difference between the modified quadratic form and the original viscous form. As in [10], we treat these terms as being of nominal order \( g \). Their effect is, firstly, to reintroduce into \( \Gamma(\hat{k}) \) the viscous timescale \( \tau_{\nu}(k) \) which was replaced by \( \tau_0(k) \). Secondly, and more significantly, new divergences appear. However, it is not difficult to show that the divergent terms which are independent of \( h(k) \) and \( \nu \) sum exactly to the amount cancelled by the counterterms, as would be expected. In the inertial range limit \( \nu \to 0 \), this leaves the term arising from \( h(k) \), which is given by

\[ \Delta \Gamma = -\frac{k^2}{\tau(k)} \int_0^\infty \frac{p^2 h(p) dp}{\tau(k)^{-1} + \tau(p)^{-1}}. \]

Given that the actual stirring force spectrum function \( h(k) \) has remained arbitrary, subject only to the condition that it yields a finite input power given by

\[ 4\pi \int_0^\infty p^2 h(p) dp = \epsilon, \]

it is clear that the above integral for \( \Delta \Gamma \) must be finite.

Thus, the role of these perturbation terms is not critical as regards calculating the anomalous exponents, provided that the spectrum of the stirring forces is non-zero only at small \( k \), as it should be. However, what we find is that, although forced at large scales, the above solution behaves in the inertial range as if the fluid were stirred with a force spectral function \( \propto k^{-3} \). In this context, it is interesting to note that, in a study of the randomly forced NS equations by a stochastic force with zero mean and variance
\(\propto k^{-3} [27],\) evidence of multiscaling of the structure functions has been found. In particular, the results obtained for the ratios \(\zeta_n/\zeta_2\) with the \(k^{-3}\) spectrum have been shown to agree with the values computed from the NS equations forced at large scales. This, of course, is exactly what one might expect from the above approximation. The present results are also consistent with the numerical calculations in [28], which suggest the scaling \(\tau_L(k) \propto k^{-2/3},\) as in [19], for the Lagrangian micro timescale, as opposed to the scaling \(\tau_E(k) \propto k^{-1}\) for the Eulerian micro timescale, evidence for which has also been presented in [29]. As we have seen, the reason why the Lagrangian timescale applies in the present calculation is because we have eliminated sweeping by referring the velocity field to a frame moving with the local velocity of the large scale eddies which prevail at any chosen scale. This extracts the straining interactions, which shape the spectrum, from the background of convection, to yield quasi-Lagrangian approximations.

In a sense, this derivation of the Kolmogorov quadratic form is analogous to a multiple timescale expansion in nonlinear wave theory, where part of the nonlinear behaviour is incorporated into the linear approximation, e.g. via a slowly changing wave amplitude, the variation of which is then determined from the nonlinear interaction by requiring the absence of secular terms in the higher order approximation. Here the requirement is similar in that it demands the absence of non-renormalisable terms in order to determine the nonlinear behaviour of the modified quadratic form.

An integral part of the Kolmogorov theory is the exact result that in the inertial range limit

\[
S_3(r) = -\frac{4}{5} \epsilon r, \tag{121}
\]

[5]. So we conclude this section by verifying that this result follows from the present treatment.

Using standard symmetry relations, we can express \(S_3(r)\) in terms of the longitudinal component of the equal time triple velocity correlator

\[
B_{\alpha\beta\gamma}(x) = \langle v_\alpha(0) v_\beta(0) v_\gamma(x) \rangle,
\]
giving

\[ S_3(r) = 6B_{111}(r, 0, 0). \]  (122)

Now the general form of the Fourier transform of \( B_{\alpha\beta\gamma} \) must be

\[ B_{\alpha\beta\gamma}(k) = i F(k) P_{\gamma\alpha\beta}(k), \]

and so \( F(k) \) can be expressed in terms of the transfer spectrum \( T(k) \) by

\[ F(k) = \frac{\pi^2}{k^4} T(k), \]

while \( T(k) \) is given to 1-loop order by (117). Substituting these results in (122) gives

\[ S_3(r) = 12i\pi \int \frac{T(k)}{k^4} k_1 \left( 1 - \frac{k_1^2}{k^2} \right) \exp(ik_1r) Dk. \]

This integral can be expanded in powers of \( r \) the lowest order term giving

\[ S_3(r) = -12\pi^2 r \int \frac{T(k)}{k^4} k_1^2 \left( 1 - \frac{k_1^2}{k^2} \right) Dk. \]

After integrating over the solid angle, we get

\[ S_3(r) = -\frac{4}{5} \int_{\kappa}^{\infty} T(k) dk. \]

This latter integral is, of course, the transport power \( \Pi_E(\kappa) \), which is a finite quantity at 1-loop order and equal to the mean dissipation rate, as indicated in above, and, hence, we recover (121).

The correlation function \( B_{111}(x) \) also has an important role in the derivation of the OPEs required for the structure functions with higher odd orders, as we shall see shortly.

**IX. DERIVATION OF THE OPEs**

We give finally the derivation of the dominant terms of the OPEs which we have used in Section [III] to obtain the structure function expansions. We deal first with the expansions
required for the higher order structure functions with orders \( n > 3 \). These can be obtained using the technique described in [30]. We defer discussion of the particular case \( n = 2 \) until last, because it requires a different approach for the reasons given in Section [III].

We begin by considering the OPE of the general product, defined in (49), as it appears in the expansion (50) for \( S_n(r) \), taking first the case of even orders \( n = 2m \), with \( p = 0, 1, \ldots, m \), namely

\[
\Lambda_{n-p,p}(\hat{x}, r) = \frac{v_+^{n-p}v_-^p}{p!(n-p)!},
\]

where, as previously, \( v_\pm = v_1(x \pm r/2, y, z, t) \), and we have used the definition (3). Let us consider the effect of inserting \( \Lambda_{n-p,p} \) into a correlation function containing an arbitrary set of elementary fields \( v_{\alpha_1}(\hat{x}_1), \ldots, v_{\alpha_l}(\hat{x}_l) \), as in (34). Then, following the approach of [30], we can derive the dominant terms which we have used in Section [III] by considering how many of the \( v_+ \) fields can be paired with a \( v_- \) field to form products of lower order correlation functions.

Consider the case \( p = m \), ie

\[
\langle v_{\alpha_1}(\hat{x}_1) \ldots v_{\alpha_l}(\hat{x}_l)\Lambda_{m,m}(\hat{x}, r) \rangle.
\]

Here each \( v_+ \) can be paired with a \( v_- \) to yield a product term

\[
\langle (v_+ v_-)^{m} \rangle \langle v_{\alpha_1}(\hat{x}_1) \ldots v_{\alpha_l}(\hat{x}_l) \rangle,
\]

which corresponds to the presence of a unit operator term in the OPE, [30]. If, instead, we only select \( m - 1 \) pairs of \( v_+ v_- \) products, we obtain a term of the type

\[
2 \langle (v_+ v_-)^{m-1} \rangle \langle v_{\alpha_1}(\hat{x}_1) \ldots v_{\alpha_l}(\hat{x}_l) \left( \frac{v_+^2}{2} \right) \rangle.
\]

Now, in the limit as \( r \to 0 \), \( v_+^2/2 \) behaves like an insertion of \( O_2(\hat{x}) \) into the correlation function of elementary fields [30]. Hence, this product tends to

\[
2 \langle (v_+ v_-)^{m-1} \rangle \langle v_{\alpha_1}(\hat{x}_1) \ldots v_{\alpha_l}(\hat{x}_l)O_2(\hat{x}) \rangle.
\]
But the averages of powers of $v_+v_-$ simply yield non-stochastic functions of $r$, which we shall denote generically by $C_0(r), C_2(r), \ldots$, as appropriate. Thus, from (123) and (124), we obtain, in the limit as $r \to 0$,

$$\langle v_{\alpha_1}(\hat{x}_1) \cdots v_{\alpha_l}(\hat{x}_l) \Lambda_{m,m}(\hat{x}, r) \rangle = \langle v_{\alpha_1}(\hat{x}_1) \cdots v_{\alpha_l}(\hat{x}_l) [C_0(r) + C_2(r)O_2(\hat{x}) + \ldots] \rangle .$$

Since the elementary fields are arbitrary, it follows that we have an OPE of the form

$$\Lambda_{m,m}(\hat{x}, r) = C_0(r)I + C_2(r)O_2(\hat{x}) + \ldots .$$

The point about expansions of this type is that the operators of increasing complexity do, indeed, produce subdominant terms in the expansion of $S_n(r)$. Here, for example, the unit operator term, as we have shown, produces the dominant scaling with anomalous exponent given by (110), whereas the quadratic term can be readily shown to give the smaller exponent $\tau_n = [m(m - 1) - 1] \Delta_2$, and, hence, is subdominant, while further terms in the expansion would produce even greater reductions.

A similar argument applies when $p = m - 1$. In this case, however, we cannot pair every $v_+$ with a $v_-$. Therefore, the unit operator term cannot appear in the OPE for $\Lambda_{m+1,m-1}$. If, however, we pair every $v_-$ with a $v_+$ then the remaining $v_+^2$ pairs with the elementary fields and, in the limit as $r \to 0$, again appears as an $O_2(\hat{x})$ insertion. In this case, therefore, the OPE starts with $O_2(\hat{x})$ to give

$$\Lambda_{m+1,m-1}(\hat{x}, r) = C_2(r)O_2(\hat{x}) + \ldots .$$

By continuing with this argument, we see that the dominant term of the OPE for the general case of $\Lambda_{n-p,p}$ must take the form given in (51).

Consider next odd orders, $n = 2m + 1$. When $p = m$, we have a term of the form

$$\langle v_+^2v_- \rangle \langle (v_+v_-)^m \rangle \langle v_{\alpha_1}(\hat{x}_1) \cdots v_{\alpha_l}(\hat{x}_l) \rangle ,$$

which, again, corresponds to the presence of a unit operator term, which is, thus, the dominant term of the OPE, giving
\[ \Lambda_{m+1,m}(\hat{x}, r) = C_0(r)I + \ldots. \]

When \( p = m - 1 \), by pairing each \( v_- \) with a \( v_+ \), we obtain a term of the form

\[ \langle (v_+ v_-)^{m-1} \rangle v_{\alpha_1}(\hat{x}_1) \ldots v_{\alpha_l}(\hat{x}_l) v_+^3 \].

In the limit as \( r \to 0 \), \( v_+^3 \) appears as an insertion of the cubic operator \( O_3(\hat{x}) \), so that here the OPE takes the form

\[ \Lambda_{m+2,m-1}(\hat{x}, r) = C_3(r)O_3(\hat{x}) + \ldots. \]

Continuing this process, we get for the next OPE

\[ \Lambda_{m+3,m-2}(\hat{x}, r) = C_5(r)O_5(\hat{x}) + \ldots, \]

and so on. But, in fact, the only term which contributes to \( S_n(r) \) for odd \( n \) is the unit operator term of \( \Lambda_{m+1,m} \) because \( \langle O_{2s+1}(\hat{x}) \rangle = 0 \) for any integer \( s \), in the case of homogeneous isotropic turbulence.

In the particular case of \( v_+ v_- \), we can establish the form of its OPE by using an expansion in the Fourier domain, in which the wavenumber \( q \), corresponding to the separation \( r \), tends to infinity, as described for instance, in [7,8]. To this end, we start by considering the general correlation function

\[ H_{\alpha\beta\lambda\mu}(\hat{x}_1, \hat{x}_2 | \hat{x}', \hat{x}'') = \langle v_{\alpha}(\hat{x}_1) v_{\beta}(\hat{x}_2) v_{\lambda}(\hat{x}') v_{\mu}(\hat{x}'') \rangle, \]

for the case in which \( \hat{x}' \) and \( \hat{x}'' \) tend to a common point \( \hat{x} \), well separated from \( \hat{x}_1 \) and \( \hat{x}_2 \). For simplicity of presentation here, we have included only two arbitrary fields \( v_{\alpha}(\hat{x}_1) \) and \( v_{\beta}(\hat{x}_2) \). Denote its Fourier transform by

\[ H_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' | \hat{k}, \hat{k}') = \left\langle v_{\alpha}(\hat{p}) v_{\beta}(\hat{p}') v_{\lambda}(\hat{k}) v_{\mu}(\hat{k}') \right\rangle = (2\pi)^4 \delta(\hat{p} + \hat{p}' + \hat{k} + \hat{k}') \tilde{H}_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' | \hat{k}, \hat{k}'). \]

Then, in terms of the reduced correlation function, we can write
\[ H_{\alpha\beta\lambda\mu}(\hat{x}_1, \hat{x}_2 \mid \hat{x}', \hat{x}'') = \int D\hat{p}D\hat{p}'D\hat{q} \tilde{H}_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q} - \frac{\hat{p} + \hat{p}'}{2}, -\hat{q} - \frac{\hat{p} + \hat{p}'}{2}) \times \exp \left\{ i\hat{p} \cdot \left( \hat{x}_1 - \frac{\hat{x}' + \hat{x}''}{2} \right) + i\hat{p}' \cdot \left( \hat{x}_2 - \frac{\hat{x}' + \hat{x}''}{2} \right) + i\hat{q} \cdot (\hat{x}' - \hat{x}'') \right\} . \] (126)

When the arguments in (123) coalesce to the common point \( \hat{x} \), we obtain the correlation function

\[ Q_{\alpha\beta\lambda\mu}(\hat{x}_1, \hat{x}_2 \mid \hat{x}) = \langle v_\alpha(\hat{x}_1) v_\beta(\hat{x}_2) v_\lambda(\hat{x}) v_\mu(\hat{x}) \rangle, \]

with Fourier transform

\[ Q_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q}) = \langle v_\alpha(\hat{p}) v_\beta(\hat{p}') (v_\lambda v_\mu)(\hat{q}) \rangle = (2\pi)^4 \delta(\hat{p} + \hat{p}' + \hat{q}) \hat{Q}_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q}). \]

Thus, corresponding to (126), we have

\[ Q_{\alpha\beta\lambda\mu}(\hat{x}_1, \hat{x}_2 \mid \hat{x}) = \int D\hat{p}D\hat{p}'Q_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid -\hat{p} - \hat{p}') \exp \left\{ i\hat{p} \cdot \left( \hat{x}_1 - \hat{x} \right) + i\hat{p}' \cdot \left( \hat{x}_2 - \hat{x} \right) \right\}. \]

Let \( \Psi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{k}, \hat{k}') \) and \( \Xi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q}) \) be the 1PI functions associated with the connected parts of \( \tilde{H}_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{k}, \hat{k}') \) and \( \hat{Q}(\hat{p}, \hat{p}' \mid \hat{q}) \). Denoting the connected part by superscript \( c \), we have, as in Section V,

\[ \tilde{H}^{(c)}_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{k}, \hat{k}') = -G_{\alpha\alpha'}(\hat{p}) G_{\beta\beta'}(\hat{p}') G_{\lambda\lambda'}(\hat{k}) G_{\mu\mu'}(\hat{k}') \psi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{k}, \hat{k}'), \] (127)

and

\[ \hat{Q}^{(c)}_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q}) = -G_{\alpha\alpha'}(\hat{p}) G_{\beta\beta'}(\hat{p}') \xi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q}). \] (128)

According to the standard procedure [7,8], the behaviour of the correlation function \( H_{\alpha\beta\lambda\mu}(\hat{x}_1, \hat{x}_2 \mid \hat{x}', \hat{x}'') \) as a function of \( \hat{x}' - \hat{x}'' \), when \( \hat{x}' \) and \( \hat{x}'' \) both tend to a common value \( \hat{x} \), can be deduced from the behaviour of \( \Psi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \hat{q} - (\hat{p} + \hat{p}')/2, -\hat{q} - (\hat{p} + \hat{p}')/2) \), in the limit as \( \hat{q} \to \infty \), which is, indeed, apparent from (126) and (127). Now, the diagrams which contribute to this 1PI correlation function are diagram (i) of Fig.8, together with its permutation \( (\hat{p}, \alpha) \leftrightarrow (\hat{p}', \beta) \), and diagram (ii). However, it is easy to see from these
diagrams that, as $\hat{q} \to \infty$, diagram (ii) yields a contribution which is smaller than that from diagram (i) by a factor $Q_{\sigma\nu}(\hat{q}) \sim q^{-11/3}$. So to derive the dominant term, we need to focus on diagram (i) and its permutation. The corresponding diagrams for $\Xi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid -\hat{p} - \hat{p}')$ are diagram (iii) of Fig.8 plus its permutation $\lambda \leftrightarrow \mu$.

Evaluation of these diagrams using the Feynman rules is straightforward and yields

$$\Psi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \frac{\hat{q} + \hat{p}'}{2}, -\hat{q} - \frac{\hat{p} + \hat{p}'}{2}) = \int P_{\lambda\xi\rho}(\hat{q} - \frac{\hat{p} + \hat{p}'}{2})P_{\mu\tau\eta}(\hat{q} - \frac{\hat{p} + \hat{p}'}{2})P_{\alpha\gamma\sigma}(\hat{p})P_{\beta\nu\delta}(\hat{p}')$$

$$\times Q_{\gamma\delta}(\hat{s})Q_{\eta\sigma}(\hat{p} - \hat{s})Q_{\xi\nu}(\hat{p}' + \hat{s})Q_{\rho\tau}(\hat{q} + \frac{\hat{p}' - \hat{p}}{2} + \hat{s})D\hat{s}$$

$$+ (\hat{p}, \alpha) + (\hat{p}', \beta),$$

and

$$\Xi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid -\hat{p} - \hat{p}') = -\int P_{\alpha\gamma}(\hat{p})P_{\beta\nu}(\hat{p}')Q_{\gamma\delta}(\hat{s})Q_{\sigma\mu}(\hat{p} - \hat{s})Q_{\lambda\nu}(\hat{p}' + \hat{s})D\hat{s}$$

$$+ (\lambda \leftrightarrow \mu).$$

Hence, for large $\hat{q}$, we obtain from the last two equations the relation

$$\Psi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid \frac{\hat{q} + \hat{p}'}{2}, -\hat{q} - \frac{\hat{p} + \hat{p}'}{2}) = P_{\lambda\xi\rho}(\hat{q})P_{\mu\tau\eta}(\hat{q})\Xi_{\alpha\beta\lambda\mu}(\hat{p}, \hat{p}' \mid -\hat{p} - \hat{p}').$$

Combining this with (127) and (128) yields the approximation

$$H_{\alpha\beta\lambda\mu}^{(c)}(\hat{p}, \hat{p}' \mid \frac{\hat{q} + \hat{p}'}{2}, -\hat{q} - \frac{\hat{p} + \hat{p}'}{2}) = C_{\lambda\mu\xi\eta}(\hat{q})Q_{\lambda\mu\xi\eta}^{(c)}(\hat{p}, \hat{p}' \mid -\hat{p} - \hat{p}'),$$

where, to this order,

$$C_{\lambda\mu\xi\eta}(\hat{q}) = P_{\lambda\xi\rho}(\hat{q})P_{\mu\tau\eta}(\hat{q})|G(\hat{q})|^2 Q_{\rho\tau}(\hat{q}).$$

To obtain the required expansion for $v_+ v_-$, we must take the inverse Fourier transform of (129) for the particular case $\lambda = \mu = 1$ with

$$\hat{x}' = (x + \frac{r}{2}, y, z, t) \quad \text{and} \quad \hat{x}'' = (x - \frac{r}{2}, y, z, t).$$

The coefficient $C_{11\xi\eta}(\hat{x}' - \hat{x}'')$ then depends only upon $r$ and, according to (130), it must have the form
\[ C_{11\xi\eta}(r) = \int \frac{q_\xi q_\eta (q_2^2 + q_3^2)}{q^4} F(q) \exp(-iq_1 r) Dq, \]

where \( F(q) \) is a function only of the wavenumber \( q \). It is clear from this integral that \( C_{11\xi\eta} \)

must be diagonal in the indices \( \xi, \eta \), and have equal transverse components: \( C_{1122} = C_{1133} \).

We now define \( Q^{(L)}_{\alpha\beta}(\hat{x}_1, \hat{x}_2 | \hat{x}) \) to be the connected correlation function formed from the

elementary fields \( v_\alpha(\hat{x}_1) \) and \( v_\beta(\hat{x}_2) \), with the insertion of the longitudinal energy operator

\( O_2(\hat{x}) \), ie it is the particular case of (36) with \( s = 2 \) and \( l = 2 \). Then

\[ Q^{(c)}_{\alpha\beta11}(\hat{x}_1, \hat{x}_2 | \hat{x}) = 2Q^{(L)}_{\alpha\beta}(\hat{x}_1, \hat{x}_2 | \hat{x}). \]

Similarly, we define \( Q^{(T)}_{\alpha\beta}(\hat{x}_1, \hat{x}_2 | \hat{x}) \) to be the correlation function with \( v_\alpha(\hat{x}_1) \) and \( v_\beta(\hat{x}_2) \),

and the insertion of the transverse energy operator

\[ O_2^{(T)}(\hat{x}) = \frac{1}{2} (v_2^2 + v_3^2). \]

Thus, we have

\[ Q^{(c)}_{\alpha\beta22}(\hat{x}_1, \hat{x}_2 | \hat{x}) + Q^{(c)}_{\alpha\beta33}(\hat{x}_1, \hat{x}_2 | \hat{x}) = 2Q^{(T)}_{\alpha\beta}(\hat{x}_1, \hat{x}_2 | \hat{x}). \]

Finally, we define longitudinal and transverse coefficients by writing

\[ C_2(r) = 2C_{1111}(r), \]

and

\[ C_2(r) = 2C_{1122}(r) = 2C_{1133}(r). \]

Using these definitions, and taking into account the diagonality of \( C_{11\xi\eta} \), enables us to express

the inverse Fourier transform of (129), for the case \( \lambda = \mu = 1 \), as

\[ H^{(c)}_{\alpha\beta11}(\hat{x}_1, \hat{x}_2 | \hat{x}, r) = C_2(r)Q^{(L)}_{\alpha\beta}(\hat{x}_1, \hat{x}_2 | \hat{x}) + C'_2(r)Q^{(T)}_{\alpha\beta}(\hat{x}_1, \hat{x}_2 | \hat{x}), \]

which, in the limit as \( r \to 0 \), leads to

\[ \langle v_\alpha(\hat{x}_1)v_\beta(\hat{x}_2)v_+v_- \rangle = \left\langle v_\alpha(\hat{x}_1)v_\beta(\hat{x}_2) \left[ \frac{E}{3} + C_2(r)O_2(\hat{x}) + C_2(r)O_2^{(T)}(\hat{x}) + \ldots \right] \right\rangle. \]
Since the fields $v_\alpha(\hat{x}_1)$ and $v_\beta(\hat{x}_2)$ are arbitrary, we may conclude that
\[
v_+ v_- = \frac{E}{3} I + C_2(r)O_2(\hat{x}) + \ldots.
\]
Note that we have discarded the transverse operator because it is subdominant. This follows immediately from the analysis of Section V. For example, in the case of $O_2^{(T)}$, when we calculate the corresponding value of the constant $a_1^{(2)}$, as defined in (39), we get twice the value given in (102) for the longitudinal operator $O_2(\hat{x})$, because, by isotropy, each of the two transverse components of $O_2^{(T)}(\hat{x})$ contributes an amount equal to the value obtained for $O_2(\hat{x})$ and, hence, the right hand side of (108) then yields an anomalous exponent of $2\Delta_2$, indicating that $O_2^{(T)}(\hat{x})$ makes a subdominant contribution to $S_2(r)$. Thus, we have shown, to within the order $g^2$ of the calculation, that the dominant term of the OPE for $v_+ v_-$ has the form given in (33).

X. SUMMARY AND DISCUSSION

The fact that it has been possible to demonstrate multiscaling and calculate anomalous exponents successfully from the generating functional by means of perturbation theory, notwithstanding the strong nonlinearity of the NS equations, is attributable to several factors. These include: (1) the use of a modified quadratic form, which is derived self-consistently from the NS nonlinearity; (2) the incorporation in the generating functional of the composite operators which appear in the definition of the general structure function; (3) the application of OPEs to derive corrections to the Kolmogorov exponents in terms of the anomalous dimensions of these operators; (4) the identification of a class of irreducible Green’s functions containing insertions of these operators, which facilitate the calculation of their anomalous dimensions; (5) the elimination of sweeping convection effects using a random Galilean transformation of the velocity field; and, finally, (6) the deduction of the inertial range scaling using an $uv$ fixed point of the RG to achieve the required small wavenumber limit. Let us now consider how each of these factors contributes to overcoming the obstacles encountered in previous applications of the RG.
The use of the modified quadratic form is an important element in the success of our calculation, because it provides an accurate initial approximation, which yields the Kolmogorov distribution in the inertial range limit. By contrast, in the early work which employed a field theoretic RG [31], and in subsequent developments of it [32-34], including equivalent formulations based on [35], reviewed recently in [36], the zero order approximation is based solely on the linear terms of the NS equations, as in a conventional field theory calculation. Because this is a poor approximation for turbulence, it does not result in a genuine weak expansion parameter. For example, in the previous applications of RG techniques based on an expansion in the force spectrum exponent (i.e. the $\epsilon$-expansion), in which the expansion about $\epsilon = 0$ is extrapolated to $\epsilon = 4$, the value of the coupling constant is not small, at the $ir$ fixed point which is used. Therefore, the accuracy of the expansion is uncontrolled. Indeed, according to [37], it may even be uncontrolled when $\epsilon \ll 1$, and there are problems in establishing its radius of convergence and the value of $\epsilon$ at which long range driving becomes technically irrelevant [38].

However, our expansion is of a different nature. First, we do not use an $\epsilon$-expansion. Actually, there is no force power spectrum in our calculation as such. As we showed, the force spectrum $h(k)$ remains in the calculation as an arbitrary function, subject only to the requirement that it yields a finite input power. What the modified quadratic form provides, however, is an apparent force power spectrum $D(k)$, but its exponent is fixed by the solution (120), and, thus, cannot be varied. Second, we do not use an $ir$ fixed point, because we are interested in taking the short wavelength limit, for which purpose we require an $uv$ fixed point. Together, these differences result in a genuinely small coupling constant $g$, which is about 1/20 at the fixed point, as shown in Section IV. Hence, our expansion is inherently more accurate than the $\epsilon$-expansion. In fact, given that our calculation is carried out to 2-loop order, its errors are controlled at $g^3 \sim 10^{-4}$. Another significant consequence of using the modified quadratic form is that no convergence problems are encountered in the $uv$ region. This, together with the fact that we do not use an $\epsilon$-expansion or an $ir$ fixed point, means that none of the ingredients which cause marginality by power counting in previous
applications of the RG [37], are present in our approach.

On the other hand, there is a similar problem to be faced in the present calculation. Any fully renormalised theory of turbulence must contain an infinite number of renormalised functions because it must be equivalent to the hierarchy of equations for the cumulants. This equivalence has been demonstrated recently [39]. In fact, each cumulant will have a representation as a expansion in terms of irreducible renormalised functions. Thus, one has an infinite set of vertex functions to contend with. Now, when any one of these irreducible functions is calculated in perturbation theory using the modified quadratic form, the overall logarithmic divergence will remain, after sweeping divergences have been eliminated. So the problem in the present approach amounts to the resummation of these logarithms. However, we showed in Section V that this difficulty could be overcome, in relation to multiscaling, by identifying the infinite sub-class of functions which yields the desired information relating to anomalous exponents while being, at the same time, amenable to resummation using the RG. The irreducible inserted nonlinear Green’s functions defined in Section V satisfy both requirements. Being fully irreducible they give full \( n \)-point correlations. However, as we have seen, to render them tractable, it was expedient to obtain a mean response to forcing at the centroid of the excitation points. This averaging thus constitutes a closure approximation. Although this type of closure approximation permits considerable progress to be made with the calculation of the exponents, the averaging process limits its applicability to relatively low orders, \( n \lesssim 10 \), because the multiple correlations between the apparent forcing at different space-time points are not then approximated accurately enough at higher orders. Thus, a different approximation would be required to obtain the asymptotic scaling at large orders and it remains for future work to discover a suitable approach.

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CAPTIONS TO FIGURES

1. FIG.1: Components of the diagrams: (i) velocity correlator; (ii) linear response function; (iii) Navier-Stokes vertex; (iv) composite operator $O_s$ vertex.

2. FIG.2: Counterterm vertices associated with the renormalization of the elementary fields and the composite operators.

3. FIG.3: The 1PI Feynman diagrams for the linear response evaluated in Section [Ⅳ].

4. FIG.4: The 1PI diagrams for the nonlinear response functions evaluated in Section [Ⅴ].

5. FIG.5: Comparison of the theoretical expression for $\zeta_n$ (full line) with experimental data.

6. FIG.6: (i) The ‘sweeping’ vertex; (ii) the 1-loop ‘sweeping’ diagram for the linear response function evaluated in Section [Ⅶ].

7. FIG.7: The 2-loop ‘sweeping’ diagrams for the linear response function evaluated in Section [Ⅶ].

8. FIG.8: The 1-loop diagrams for the correlation functions evaluated in Section [Ⅸ] in connection with the OPEs.
\[ a \xrightarrow{k} b = Q_{ab} H k L \]
\[ a \xrightarrow{k} b = i G_{ab} H k L \]
\[ = P_{abg} H k L \]
\[ = P_{a_1 H k_1 L} \]
\[ i = 1 \]
\[ H_{ii} \quad \Xi_{ab}^k = iDZ_n P_{ab}^{HkL} tHkL \]

\[ H_{iii} \quad \Xi_{ab}^k = -DZ_P P_{ab}^{HkL} DkL \]

\[ s \quad = -DZ_s \quad P_{a_i}^{1Hk_iL} \quad i=1 \]
