Relative Chow–Künneh decompositions for morphisms of threefolds

Dedicated to Professor Jacob Murre

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Abstract. We show that any nonconstant morphism of a threefold admits a relative Chow–Künneh decomposition. As a corollary we get sufficient conditions for threefolds to admit an absolute Chow–Künneh decomposition. In case the image of the morphism is a surface, this implies another proof of a theorem on the absolute Chow–Künneh decomposition for threefolds satisfying a certain condition, which was obtained by the first author with P. L. del Angel. In case the image is a curve, this improves in the threefold case a theorem obtained by the second author where the singularity of the morphism was assumed isolated and the condition on the general fiber was stronger.

Introduction

Let \( f : X \to S \) be a surjective projective morphism of complex algebraic varieties with \( X \) smooth connected and \( S \) reduced. By the decomposition theorem of Beilinson, Bernstein and Deligne [3], there are noncanonical and canonical isomorphisms respectively in \( D^b_{\text{c}}(S, \mathbb{Q}) \) and \( \text{Perv}(S, \mathbb{Q}) \):

\[
\mathbb{R}f_*\mathbb{Q}_X[\dim X] \cong \bigoplus_i p^Rf_* (\mathbb{Q}_X[\dim X])[-i],
\]

\[
(0.1) \quad p^Rf_* (\mathbb{Q}_X[\dim X]) = \bigoplus_Z \text{IC}_Z L^i_{Z^o},
\]

where \( Z \) runs over closed irreducible subvarieties of \( S \). Here \( p^Rf_* = p^R\mathcal{H}^i p^*_s \) with \( p^*\mathcal{H} \) the perverse cohomology functor, \( \text{Perv}(S, \mathbb{Q}) \) is the category of perverse sheaves on \( S \), and \( \text{IC}_Z L^i_{Z^o} \) denotes the intersection complex associated with a \( \mathbb{Q} \)-local system \( L^i_{Z^o} \), defined on a Zariski-open smooth subvariety \( Z^o \) of \( Z \), see [3]. Moreover, (0.1) holds in the (derived) category of mixed Hodge modules, and the local system \( L^i_{Z^o} \) naturally underlies a polarizable variation of Hodge structure on \( Z^o \) whose weight is \( \dim X - \dim Z + i \), see [17], [18]. Recall that the level of a Hodge structure \( H \) is the difference between the maximal and minimal numbers \( p \) such that \( \text{Gr}^p_H \mathbb{C} \neq 0 \), see [7]. We have the following.
Proposition 1. Let \( n = \dim X, m = \dim S \). Then \( L^i_\mathcal{Z}_0 = 0 \) unless \( |i| \leq n - m \) with \( Z = S \) or \( |i| \leq n - \dim Z - 2 \) with \( Z \neq S \). Moreover, the level of the Hodge structure on each stalk of \( L^i_\mathcal{Z} \) is at most \( n - m - |i| \) if \( Z = S \), and is at most \( n - \dim Z - |i| - 2 \) if \( Z \neq S \).

This is a special case of a more general theorem on the direct images of arbitrary pure Hodge modules [4]. Using Proposition 1, we can show the following.

Theorem 1. Let \( f : X \to S \) be a nonconstant projective morphism of quasi-projective varieties over \( \mathbb{C} \). Assume \( X \) is smooth and 3-dimensional. Then \( f \) admits a relative Chow–Künneth decomposition.

This means that there are mutually orthogonal projectors

\[
\pi^f_{i,\mathcal{Z}} \in \text{Cor}^0_S(X, X) := \text{CH}_{\dim X}(X \times_S X)_\mathbb{Q}
\]

such that their sum is the diagonal, and their action on the perverse cohomology sheaf \( \mathop{\mathcal{R}\mathcal{H}\pi}^j_\mathcal{Z}(\mathcal{Q}_X[\dim X]) \) is the projection to \( \text{IC}_\mathcal{Z}L^j_\mathcal{Z}_0 \) for \( j = i \), and vanishes otherwise. (For the relative Chow–Künneth decomposition, see also [5], [11], [16], [21], etc.) Note that Theorem 1 in the case \( \dim X \leq 2 \) is rather easy, and is known to specialists. For instance, the assertion for \( \dim X = 2 \) follows from (1.5.1) and Proposition 1.8 below, see also [21]. In the case \( \dim S = 1 \), a similar assertion on the relative Chow–Künneth decomposition for \( f \) was proved in [21] under the hypothesis that the level of the cohomology of the generic fibers are at most one and \( f \) has at most isolated singularities. In this paper we simplify some arguments by using the isomorphisms in (1.5.1) below, see also [16], Remark 1.9.

For the moment it is very difficult to prove the Chow–Künneth decomposition for general threefolds \( X \). From Theorem 1 we deduce some sufficient conditions for threefolds to admit an absolute Chow–Künneth decomposition as below. We will denote by \( H^2_\mathcal{U}(X, \mathbb{Q}) \) the quotient of \( H^2(X, \mathbb{Q}) \) by the subgroup generated by the divisor classes (called the transcendental part).

Theorem 2. Set \( m = \dim S \). Let \( X \) be a smooth complex projective variety of dimension 3 satisfying the following conditions.

(a) In case \( m = 2 \), the transcendental part \( H^2_\mathcal{U}(X, \mathbb{Q}) \) is generated by the images of \( f^*\mathcal{I}H^2(S, \mathbb{Q}) \) and \( H^1(X, \mathbb{Q}) \cap f^*\mathcal{I}H^1(S, \mathbb{Q}) \).

(b) In case \( m = 1 \), the restriction morphism \( H^2_\mathcal{U}(X, \mathbb{Q}) \to H^2_\mathcal{U}(X_s, \mathbb{Q}) \) vanishes for general \( s \in S^0 \) where \( X_s := f^{-1}(s) \).

Then \( X \) admits an absolute Chow–Künneth decomposition. If moreover \( S \) is normal and \( f \) has connected fibers (replacing \( S \) with the Stein factorization if necessary), then the absolute projectors \( \pi^f_i \) can be obtained by decomposing the relative projectors \( \pi^f_{i,\mathcal{Z}} \) in Theorem 1.

Here \( \mathcal{I}H^*(S, \mathbb{Q}) \) denotes the intersection cohomology (see [13] and also [3]). Since \( X \) is smooth, \( \mathcal{I}H^*(S, \mathbb{Q}) \) in the hypothesis (a) is canonically a direct factor of \( \mathcal{H}^*(X, \mathbb{Q}) \) by the decomposition theorem (0.1) together with Proposition 1, see Remarks 1.14(i) below. Note that \( S \) can be replaced by the Stein factorization \( S' \) in Theorem 2, see Remark 1.14(iii). In the case (a) the assertion was shown by the first author with P. L. del Angel ([1], [2])
using another method (by taking a blow-up of $X$ having a morphism onto a smooth $S$).

Set $r := \dim X - \dim S = 3 - m$. Then $L^0_{S^0} = R^jf_*\mathbb{Q}_X|_{S^0}$, and there is a canonical decomposition compatible with the variation of Hodge structure

\[(0.2)\]

\[L^0_{S^0} = (L^0_{S^0})^c \oplus (L^0_{S^0})^{nc},\]

such that $(L^0_{S^0})^c$ is constant and $(L^0_{S^0})^{nc}$ has no nontrivial global section. By the global invariant cycle theorem [7], each stalk of $(L^0_{S^0})^c$ coincides with the image of the restriction morphism by $X_s \hookrightarrow X$ for $s \in S^0$. Assume $f$ has connected fibers (replacing $S$ if necessary). The hypothesis (a) then means that $H^1(S, IC_S(L^0_{S^0})^{nc})$ has type $(1,1)$, i.e.

\[(0.3)\]

\[F^2H^1(S, IC_S(L^0_{S^0}, \mathbb{C})^{nc}) = 0,\]

where $(L^0_{S^0, \mathbb{C}})^{nc} := (L^0_{S^0})^{nc} \otimes \mathbb{C}$, see Remark 1.14(iv) below. Condition (b) means that each stalk of $(L^0_{S^0})^c$ has type $(1,1)$. This improves a theorem in [21] for $\dim X = 3$. Note that $H^2(X, \mathbb{Q})$ is not assumed to be of type $(1,1)$ in our paper. Indeed, $H^2(X, \mathbb{Q})$ contains $H^1(X, \mathbb{Q})^{inv} \otimes IH^1(S, \mathbb{Q})$ in both cases and also $IH^2(S, \mathbb{Q})$ in the case (a), where no conditions are imposed on these Hodge structures. Here $H^1(X, \mathbb{Q})^{inv}$ denotes the monodromy invariant part.

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**Convention.** In this paper a variety means a quasi-projective variety over $\mathbb{C}$.

### 1. Preliminaries

For the convenience of the reader we give here a short proof of Proposition 1 using the nearby and vanishing cycle functors [8].

**1.1. Proof of Proposition 1.** The assertion can be proved by using the fact that the direct factors $IC_ZL^i_{Z^0}$ with $Z \subset S$ are subquotients of

\[pR^if_*(\phi_{h,1}\mathbb{Q}_X[\dim X]) = \phi_{h,1}pR^if_*(\mathbb{Q}_X[\dim X]),\]

where $h = gf$ with $g$ a function locally defined on $S$ such that $f$ is smooth over $\{g \neq 0\}$. Here $pR^if_* = p\mathcal{H}^i\mathcal{H}_f$ with $p\mathcal{H}_f$ the perverse cohomology functor (see [3]), and $\phi_{h,1}$ is the unipotent monodromy part of the vanishing cycle functor [8] which is shifted by $-1$ so that it preserves perverse sheaves. (Note that $\phi_{h,1}M = M$ if $\text{supp } M \subset g^{-1}(0)$. ) We may assume that $h^{-1}(0)$ is a divisor with normal crossings applying the decomposition theorem to a resolution of singularities. Then the $Gr^W_k \phi_{h,1}\mathbb{Q}_X[\dim X]$ are direct sums of the con-
stant sheaves supported on intersections of irreducible components of $h^{-1}(0)$ where $W$ is the monodromy filtration up to a shift, and moreover the codimensions of the strata are at least 2 in $X$. Indeed, $\varphi_{h,1}^1_{\mathbb{Q}X}[\dim X]$ is identified with the image of the logarithm $N$ of the monodromy $T$ on the nearby cycles $\psi_{h,1}^1_{\mathbb{Q}X}[\dim X]$ (see [17], Lemma 5.1.4), and the assertion for the nearby cycles is rather well-known, see e.g. [18], Theorem 3.3, and also [22]. Then Proposition 1 follows by induction on $\dim X$.

1.2. Relative correspondences. Let $X$, $Y$ be smooth complex quasi-projective varieties with projective morphisms $f : X \to S$, $g : Y \to S$ where $S$ is a reduced complex quasi-projective variety. The group of relative correspondences is defined by

$$\text{Cor}^i_S(X, Y) = \text{CH}_{\dim Y - i}(X \times_S Y)_{\mathbb{Q}},$$

if $Y$ is equidimensional. In general we take the direct sum over the connected components of $Y$. The composition of relative correspondences is defined by using the pull-back associated to the cartesian diagram (see [10])

$$\begin{array}{ccc}
X \times_S Y \times_S Z & \longrightarrow & (X \times_S Y) \times (Y \times_S Z) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \times Y,
\end{array}$$

together with the pushforward by $X \times_S Y \times_S Z \to X \times_S Z$, see [5] for details.

We have a canonical morphism

$$(1.2.1) \quad \text{Cor}^i_S(X, Y) \to \text{Cor}^i(X, Y) := \text{Cor}^i_C(X, Y),$$

and this is compatible with composition. So we get a forgetful functor from the category of relative Chow motives over $S$ to the category of Chow motives over $\mathbb{C}$, see loc. cit.

We have moreover the action of correspondences

$$(1.2.2) \quad \text{Cor}^i_S(X, Y) \to \text{Hom}\left(\mathbb{R}f_*\mathbb{Q}_X, \mathbb{R}g_*\mathbb{Q}_Y(i)[2i]\right)$$

$$\to \bigoplus_j \text{Hom}\left(\mathbb{R}^j f_*\mathbb{Q}_X, \mathbb{R}^{j+2i} g_*\mathbb{Q}_Y(i)\right),$$

where the decomposition (0.1) is used for the second morphism. This action is compatible with the composition of correspondences, see [5], Lemma 2.21. The isomorphisms in (0.1) can be lifted respectively in $D^b_{\text{MHM}}(S)$ and $\text{MHM}(S)$ where $\text{MHM}(S)$ is the category of mixed Hodge modules, see [18]. So we can take the first Hom of (1.2.2) either in $D^b_{\text{MHM}}(S, \mathbb{Q})$ or $D^b_{\text{MHM}}(S)$, and the second Hom either in $\text{Perv}(S, \mathbb{Q})$ or $\text{MHM}(S)$. Note that the second morphism of (1.2.2) is not an isomorphism as in (1.5.1) below since the information of morphisms belonging to higher extension groups $\text{Ext}^i (i > 0)$ is lost.

1.3. Relative Chow motives. For a projector $\pi \in \text{Cor}^0_S(X, X)$, the relative Chow motive defined by $\pi$ is denoted by $(X/S, \pi)$. More precisely, it is an abbreviation of $(X/S, \pi, 0)$.
implies that $f$ and (1.3.3) are satisfied, since (1.4.1) becomes (1.3.2) under condition $p$ with $p$. Here we assume $p$.

Remark 1.9) $f$ and (1.3.3) and moreover (1.3.2) are satisfied, since (1.4.1) becomes (1.3.2) under condition $p$.

The last isomorphism follows by lifting the first isomorphism of (0.1) in $D^b\text{MHM}(S)$ as explained in (1.2). Applying (1.5.1) to the case $f = g$, the decomposition theorem (0.1) implies that $f$ admits a relative Chow–Künneth decomposition if $f$ is flat with relative dimension at most 1.

As for the proof of (1.5.1), let $\mathbb{D}$ denote the functor associating the dual, and set $Z = X \times_S Y$ with canonical morphisms $f' : Z \to Y$, $g' : Z \to X$, $h : Z \to S$. We have

$$\mathbb{D}Q_Y = Q_Y(\text{dim } Y)[2\text{ dim } Y],$$

since $Y$ is smooth. Consider first the case where the relative dimension is 1. We have canonical isomorphisms (see [5], Lemma 2.21)
there is a projective morphism
an isomorphism as relative Chow motives
such that
perverse cohomology sheaves are orthogonal). Set
\[ M = \text{Hom}(\mathbb{R}f^!\mathbb{Q}_X, \mathbb{R}g_*\mathbb{Q}_Y) = \text{Hom}(g^!\mathbb{R}f_*\mathbb{Q}_X, \mathbb{Q}_Y) = \text{Hom}(\mathbb{R}f^!g_*\mathbb{Q}_X, \mathbb{Q}_Y) = \text{Hom}(\mathbb{Q}_Z, (\mathbb{D}\mathbb{Q}_Z)(1 - \dim Z)[2 - 2 \dim Z]). \]

So (1.5.1) follows from Proposition (1.15) below. The argument is similar in case the relative dimension is 0.

1.6. Orthogonal decomposition of projectors. Recall that a projector \( \pi' \) is called a direct factor (or a refinement) of a projector \( \pi \) if

\[ \pi' = \pi \circ \pi' \circ \pi, \quad \text{or equivalently} \quad \pi' = \pi \circ \pi' = \pi' \circ \pi. \]

In this case \( \pi'' := \pi - \pi' \) is also a projector and is called the orthogonal complement of \( \pi' \) in \( \pi \). We say that \( \pi = \sum \pi_i \) is an orthogonal decomposition of a projector \( \pi \) if the \( \pi_i \) are mutually orthogonal projectors. In this case \( \pi_i \) is a direct factor of \( \pi \) in the above sense.

1.7. Good projectors. Let \( f : X \to S \) be a surjective projective morphism where \( X \) is smooth connected. We say that a relative projector \( \pi \in \text{Cor}_S^0(X, X) \) is a good projector if there is a projective morphism \( g : Y \to S \) together with a projector \( \pi' \in \text{Cor}_Y^0(Y, Y) \) and an isomorphism as relative Chow motives

\[ (X/S, \pi) \cong (Y/S, \pi')(-i) \quad \text{for some} \ i \in \mathbb{Z}, \]

such that \( \pi' \) is a direct factor of a relative Chow–Künneth projector \( \pi_{0, Z}^\theta \) with \( Z := g(Y) \), and moreover

\[ \text{End}(Y/S, \pi') \cong \text{End}_{\text{MHM}(S)}(\mathcal{A}) \subset \text{End}_{\text{MHM}(S)}(\theta R^0 g_*(\mathbb{Q}_Y[\dim Y])), \]

where \( \mathcal{A} := \text{Im} \pi' \subset \theta R^0 g_*(\mathbb{Q}_Y[\dim Y]) \in \text{MHM}(S) \).

Note that (1.7.1) is satisfied if \( Y \) is flat with relative dimension 1 over \( Z \) by 1.5 or if \( Y \) is purely 2-dimensional and is generically finite over \( Z \) at each generic point of \( Y \) by Proposition 1.8 below.

Let \( \gamma_i \in \text{Cor}_X^0(X, X) \) be mutually orthogonal projectors, and \( \pi \in \text{Cor}_S^0(X, X) \) be a good projector. Assume \( \pi \) is cohomologically orthogonal to the \( \gamma_i \) (i.e. their actions on the perverse cohomology sheaves are orthogonal). Set

\[ \tilde{\pi} = \tilde{\gamma} \circ \pi \circ \tilde{\gamma} \quad \text{with} \quad \tilde{\gamma} := \prod_i (1 - \gamma_i). \]

Then \( \tilde{\pi} \) is a projector which is orthogonal to the \( \gamma_i \) using (1.7.1), see also [21].

1.8. Proposition. Let \( \pi : \tilde{S} \to S \) be a surjective projective morphism of purely 2-dimensional varieties such that \( \tilde{S} \) is smooth and every irreducible component of \( \tilde{S} \) is dominant over \( S \). Let \( j : U \to S \) be the largest open subset such that the restriction \( \pi_U : \tilde{S}_U := \pi^{-1}(U) \to U \) is finite etale. Set \( L_U = (\pi_U)_*\mathbb{Q}_{\tilde{S}_U} \). Let \( s_i \) be the points of \( S \) such that \( D_i := \pi^{-1}(s_i) \) has positive dimension. Let \( D_{i,k} \) be the 1-dimensional irreducible compo-
nents of \( D_i \). Then \( \mathbb{R} \pi_! Q_S[2] \) is a perverse sheaf naturally underlying a mixed Hodge module on \( S \), and there are canonical isomorphisms

\begin{align*}
\mathbb{R} \pi_! Q_S[2] &= IC_S L_U \oplus \left( \bigoplus_{i,k} Q[D_{i,k} ]_{s_i} \right) \quad \text{in Perv}(S) \text{ and also in } \text{MHM}(S), \\
\text{CH}_2(\tilde{S} \times_S \tilde{S}) &\cong \text{End}(\mathbb{R} \pi_! Q_S[2]) \\
&= \text{End}(IC_S L_U) \oplus \left( \bigoplus_i \text{End}\left( \bigoplus_k Q[D_{i,k} ]_{s_i} \right) \right).
\end{align*}

Here \( Q[D_{i,k} ]_{s_i} \) is a sheaf supported on \( \{ s_i \} \) and is generated by \( [D_{i,k} ]_{s_i} \) over \( Q \), and \( \text{End} \) in \( (1.8.2) \) can be taken in both \( \text{Perv}(S) \) and \( \text{MHM}(S) \).

Proof. Using the base change theorem for the direct image by a proper morphism, we can calculate the stalks of the higher direct image sheaf \( (\mathbb{R}^2 \pi_! Q_S)_{s} \) for \( s \in S \). It is non-zero if and only if \( s = s_i \) for some \( i \), and we have

\((\mathbb{R}^2 \pi_! Q_S)_{s_i} = \bigoplus_k Q[D_{i,k} ]_{s_i} \).

Since \( IC_S L_U = f_! (L_U[2]) \), we have the vanishing of \( \mathcal{H}^0 IC_S L_U \). Then we get \( (1.8.1) \) using the decomposition theorem \( (0.1) \). This implies the last isomorphism of \( (1.8.2) \) since there are no nontrivial morphisms between intersection complexes with different supports and \( L_U \) underlies a variation of Hodge structure of type \( (0,0) \) on \( U \).

For the proof of the first isomorphism of \( (1.8.2) \), we have

\[ \text{CH}_2(\tilde{S} \times_S \tilde{S}) = \text{CH}^0(\tilde{S}_U \times_U \tilde{S}_U) \oplus \left( \bigoplus_i \bigoplus_{j,k} \text{CH}^0 (D_{i,j} \times D_{i,k}) \right), \]

since \( \dim \tilde{S} = \dim S = 2 \). Then, using the canonical morphism from the associated short exact sequence

\[ 0 \rightarrow \bigoplus_i \left( \bigoplus_{j,k} \text{CH}^0 (D_{i,j} \times D_{i,k}) \right) \rightarrow \text{CH}_2(\tilde{S} \times_S \tilde{S}) \rightarrow \text{CH}^0 (\tilde{S}_U \times_U \tilde{S}_U) \rightarrow 0, \]

to the corresponding short exact sequence

\[ 0 \rightarrow \bigoplus_i \text{End} \left( \bigoplus_k Q[D_{i,k} ]_{s_i} \right) \rightarrow \text{End}(\mathbb{R} \pi_! Q_S[2]) \rightarrow \text{End}(IC_S L_U) \rightarrow 0, \]

the assertion follows.

1.9. Notation. The projector corresponding to \( (id,0) \) by Proposition 1.8 will be denoted by

\[ \pi_{\tilde{S}_U/U} \in \text{Cor}_S^2(\tilde{S}, \tilde{S}) = \text{CH}_2(\tilde{S} \times_S \tilde{S})_Q. \]

By definition its action is the identity on \( IC_S L_U \), and vanishes on the other direct factors.
1.10. Proposition. Assume $S$ projective. With the above notation, let $\pi_j$ denote the absolute Chow–Künneth projectors for $S$ constructed in [14]. Set $\tilde{\pi}_j = \pi_{S_U/V} \circ \pi_j \circ \pi_{S_U/U}$ as an absolute projector. Then the $\tilde{\pi}_j$ are mutually orthogonal projectors and give a decomposition of $\pi_{S_U/U}$ as an absolute projector.

Proof. It is enough to show that the $\tilde{\pi}_j$ for $j \neq 2$ are mutually orthogonal projectors. So the assertion is reduced to

$$\pi_i \circ \pi_{S_U/U} \circ \pi_j = \delta_{i,j} \pi_j \quad \text{for } i, j \neq 2. \tag{1.10.1}$$

Set $\tilde{\pi}' = \Delta_S - \pi_{S_U/U}$ as an absolute projector. By the direct sum decomposition (1.8.1) its action on the cohomology $H^j(S, \mathbb{Q})$ vanishes for $j \neq 2$, and is the projection to the subspace generated by the classes of $D_{i,k}$ for $j = 2$. By the construction in [14], the $\pi_j$ for $j \neq 2$ are good projectors over the base space $S = \text{Spec } \mathbb{C}$, see 1.7. So (1.10.1) for $i = j$ follows since the action of $\pi_{S_U/U}$ on $H^j(S, \mathbb{Q})$ is the identity for $j \neq 2$. For $i \neq j$, we have to show

$$\pi_i \circ \tilde{\pi}' \circ \pi_j = 0 \quad \text{if } i, j \neq 2, i \neq j. \tag{1.10.2}$$

We have isomorphisms of absolute Chow motives

$$(\tilde{S}, \pi_j) \cong (Y_j, \eta_j)(-k_j) \quad (\tilde{S}, \tilde{\pi}') = (Y', \eta'')(-1),$$

where $\dim Y_j = 0, 1, 1, 0$ and $k_j = 0, 0, 1, 2$ for $j = 0, 1, 3, 4$ respectively, and $\dim Y' = 0$. So it is enough to show the vanishing of the composition of morphisms of Chow motives

$$\begin{array}{c}
(Y_i, \eta_i)(-k_i) \xrightarrow{\xi} (Y', \eta'')(-1) \xrightarrow{\xi'} (Y_j, \eta_j)(-k_j), \tag{1.10.3}
\end{array}$$

assuming that the action of $\xi$ and $\xi'$ on the cohomology vanishes (since $i, j \neq 2$). By (1.5.1) for $S = \text{Spec } \mathbb{C}$, we can calculate the composition in the derived category of mixed Hodge structures since $\dim Y' \leq 1$ and $\dim Y' = 0$. Then we get a composition of elements of $\text{Ext}^1$ by the hypothesis that the action on the cohomology vanishes. So the composition (1.10.3) vanishes since the higher extension groups $\text{Ext}^i$ for $i > 1$ vanish in the category of mixed Hodge structures. This finishes the proof of Proposition 1.10.

1.11. Constant part of a relative projector. Let $f : X \rightarrow S$ be a surjective projective morphism where $X$ is smooth connected and $S$ is reduced. Let $\pi$ be a relative projector of $X/S$. Assume $\pi$ has pure relative degree $j$, i.e. the action of $\pi$ on $\mathcal{H}^k f_*(\mathbb{Q}_X [\dim X])$ vanishes for $k \neq j$. We say that $\pi^c$ is the constant part of $\pi$ if $\pi^c$ is a direct factor of $\pi$, the image of the action of $\pi^c$ on the shifted local system $\mathcal{H}^1 f_*(\mathbb{Q}_X [\dim X])|_{S^0}$ is its constant part, and there is an isomorphism as relative Chow motives for some integer $k$:

$$(X/S, \pi^c) \cong (C_S/S, \pi_S^c)(-k).$$

Here $C_S := C \times S$ with $C$ an equidimensional smooth projective variety (which is not assumed to be connected), and $\pi_S^c := \pi'' \times [S] \in \text{Coh}^0(C_S, C_S)$ is the pull-back of a direct factor $\pi''$ of the middle Chow–Künneth projector $\pi^c_{\dim C}$ of $C$. In this case we define
\( \pi_{nc} := \pi - \pi^c \), see (1.6). This is called the nonconstant part of \( \pi \). We say that the constant part has relative level \( \leq i \) if we can take \( C \) as above with \( \dim C \leq i \).

The direct factor \( (X/S, \pi^c) \) is well-defined as a relative Chow motive, if \( \pi^c \) has relative level at most 1. This follows from (1.5.1). For the well-definedness of \( \pi^c \) as a direct factor of \( \pi \), we have to assume, for example, \( f \) is flat with relative dimension \( \leq 1 \) and use (1.5.1).

Let \( f : X \to S \) be a surjective projective morphism where \( X \) is smooth connected and \( S \) is reduced. We say that a relative projector \( \pi \) of \( X/S \) has generically relative level at most 1, if there is a dense smooth open subvariety \( U \) of \( S \) together with a surjective smooth projective morphism \( g : Y \to U \) and a relative projector \( \pi' \) of \( Y/U \) such that \( Y \) is equidimensional with relative dimension \( r := \dim Y - \dim U \leq 1 \), \( \pi' \) is a direct factor of a relative Chow–Kühneth projector \( \pi_{0,S}^d \) (in particular, the action of \( \pi' \) on \( \mathcal{R}^d g_*(\mathcal{Q}_Y | \dim Y) \) vanishes for \( j \neq 0 \)), and there is an isomorphism as relative Chow motives over \( U \)

\[
(X/S, \pi)|_U \cong (Y/U, \pi').
\]

Here \( X|_U := f^{-1}(U) \) may be assumed smooth over \( U \) shrinking \( U \) if necessary. In the above definition we always assume that \( \pi_{0,S}^d \) is induced by (0.1) and (1.5.1) so that we get in the case of relative dimension \( r \leq 1 \)

\[
(1.11.1) \quad \text{End}((X/S, \pi)|_U) \cong \text{End}(Y/U, \pi') \cong \text{End}_{VHS(U)}(M),
\]

where \( M \) is the image of the action of \( \pi' \) on \( L := \mathcal{R}^d g_*(\mathcal{Q}_Y) \) in the category of variations of Hodge structures \( VHS(U) \).

For the convenience of the reader we show that \( \pi^c \) exists if \( \pi \) has generically relative level at most 1 by using (1.5.1). For an argument using the theory of abelian schemes (which is valid also in the positive characteristic case), see [21].

1.12. Proposition. Let \( f : X \to S \) be a surjective projective morphism where \( X \) is smooth connected and \( S \) is reduced. In the notation of 1.11, let \( \pi \) be a relative projector of \( X/S \) which has generically relative level at most 1. Then the constant part \( \pi^c \) of \( \pi \) exists.

Proof. We first consider the case where \( g \) in 1.11 has relative dimension 1. Set

\[
L := \mathcal{R}^1 g_* \mathcal{Q}_Y \quad \text{in } VHS(U).
\]

Let \( L^c \) be the constant part of \( L \), and \( L^nc \) the orthogonal complement of \( L^c \) under a polarization of \( L \) so that

\[
L = L^c \oplus L^nc \quad \text{in } VHS(U).
\]

By the semisimplicity of \( L \), there is no nontrivial morphism between \( L^c \) and \( L^nc \) in \( VHS(U) \). Hence the decomposition is compatible with the action of \( \pi' \). In the notation of (1.11.1) we get then

\[
M = M^c \oplus M^{nc} \quad \text{in } VHS(U),
\]
where \( M^c = M \cap L^c, M^{nc} = M \cap L^{nc} \). This is also compatible with the action of \( \pi' \). So we get by (1.11.1) a canonical orthogonal decomposition of projectors

\[
\pi' = \pi^c + \pi^{nc},
\]
corresponding to the above decomposition. By (1.11.1) there is a smooth projective curve \( C \) together with a projector \( \pi'' \) which is a direct factor of a Chow–Küneth projector \( \pi'_1 \) of \( C \) and such that

\[
(\gamma / U, \pi^c) \cong (C_U / U, \pi''),
\]
where \( C_U := C \times U \) and \( \pi''_U := \pi'' \times [U] \). Here \( C \) can be a general complete intersection in a general fiber of \( g \) using (1.11.1). Moreover, the above decomposition together with the isomorphism \( (X / S, \pi)|_U \cong (\gamma / U, \pi') \) implies the orthogonal decomposition of projectors

\[
\pi|_U = (\pi|_U)^c + (\pi|_U)^{nc},
\]
such that

\[
(X|_U / U, (\pi|_U)^c) \cong (\gamma / U, \pi^c), \quad (X|_U / U, (\pi|_U)^{nc}) \cong (\gamma / U, \pi^{nc}).
\]
So we get

\[
(X|_U / U, (\pi|_U)^c) \cong (C_U / U, \pi''_U).
\]
By assumption \( \pi \) has pure relative degree, say \( i \). Then there are

\[
\zeta \in \text{Cor}_U^i(X|_U, C_U), \quad \zeta' \in \text{Cor}_U^i(C_U, X|_U),
\]
inducing the above isomorphism, i.e.

\[
\zeta' \circ \zeta = (\pi|_U)^c, \quad \zeta \circ \zeta' = \pi'',
\]

\[
\zeta = (\pi''_U) \circ \zeta = \zeta \circ (\pi|_U)^c, \quad \zeta' = (\pi|_U)^c \circ \zeta' = \zeta' \circ (\pi''_U).
\]
Since \( (\pi|_U)^c = (\pi|_U) \circ (\pi|_U)^c = (\pi|_U)^c \circ (\pi|_U) \), the last equalities imply

\[
\zeta = \zeta \circ (\pi|_U), \quad \zeta' = (\pi|_U) \circ \zeta'.
\]
Take any extensions

\[
\tilde{\zeta} \in \text{Cor}_S^i(X, C_S), \quad \tilde{\zeta}' \in \text{Cor}_S^i(C_S, X),
\]
of \( \zeta \) and \( \zeta' \) respectively. Replacing \( \tilde{\zeta} \) and \( \tilde{\zeta}' \) respectively with

\[
\pi'' \circ \tilde{\zeta} \circ \pi \quad \text{and} \quad \pi \circ \tilde{\zeta}' \circ \pi''_S,
\]
if necessary, we may assume
\[ \tilde{\zeta} = \pi'' S \circ \tilde{\xi} = \zeta \circ \pi, \quad \tilde{\zeta}' = \pi \circ \tilde{\xi}' = \tilde{\xi}' \circ \pi'', \]
since the composition of relative correspondences is compatible with the restriction over \( U \).
Using the injection in (1.11.1), we get
\[ \pi'' S = \tilde{\zeta} \circ \tilde{\zeta}', \]
since this hold by restricting over \( U \). Define
\[ \pi^c = \tilde{\zeta}' \circ \tilde{\zeta}. \]
Then \( \pi^c \) is a projector and
\[ \pi^c = \pi \circ \pi^c = \pi^c \circ \pi. \]
So the assertion follows. The argument is similar in case the relative dimension of \( g \) is 0. This finishes the proof of Proposition 1.12.

1.13. Decomposition of a constant relative correspondence. With the notation of (1.11), let \( \pi^c \) be the constant part of \( \pi \) so that
\[
(1.13.1) \quad (X/S, \pi^c) \cong (C_S, \pi'' S) (-k).
\]
By definition (1.3.1) this isomorphism is induced by
\[ \zeta \in \text{Cor}_{S}^{-k}(X, C_S), \quad \zeta' \in \text{Cor}_{S}^{k}(C_S, X), \]
satisfying the conditions
\[ \zeta' \circ \zeta = \pi^c, \quad \zeta \circ \zeta' = \pi'' S, \]
together with
\[ \zeta = \pi'' S \circ \zeta = \zeta \circ \pi^c, \quad \zeta' = \pi^c \circ \zeta' = \zeta' \circ \pi'' S. \]
If \( S \) admits an absolute Chow–Künneth decomposition with projectors \( \pi^S_j \), then we have an orthogonal decomposition as absolute correspondences
\[
(1.13.2) \quad \pi^c = \sum_{j} \pi^S_j \quad \text{with} \quad \pi^S_j = \zeta' \circ (\pi'' \times \pi^S_{j-2k-\dim C}) \circ \zeta,
\]
where \( k \) is as in (1.13.1), and the action of \( \pi^S_j \) on \( H^i(X, \mathcal{Q}) \) vanishes unless \( i = j \).

1.14. Remarks. (i) The first isomorphism of (0.1) is not canonical although the second is. Set \( r = \dim X - \dim S \). In the notation of (0.1), we have
\[ L_{S^0} = \mathcal{H}^0 f_*, \mathcal{Q}_X |_{S^0}, \]
by restricting $f$ over $S^\circ$. By the decomposition theorem (0.1), $\text{IC}_{S}L_{S^\circ}^{c}[r]$ is a direct factor of $\mathbb{R}f_{*}\mathbb{Q}_{X}[\dim X]$, and we have an inclusion morphism

$$\text{IC}_{S}L_{S^\circ}^{c}[r] \hookrightarrow \mathbb{R}f_{*}\mathbb{Q}_{X}[\dim X],$$

which splits and induces an injection

$$\text{IH}^{j}(S, L_{S^\circ}^{c}) \hookrightarrow H^{j}(X, \mathbb{Q}).$$

These inclusion morphisms are canonical if $\mathcal{P}R^{i}f_{*}(\mathbb{Q}_{X}[\dim X]) = 0$ for $i < -r$, since the negative extension groups vanish, see also the second isomorphism of (1.5.1).

(ii) More generally, let $L_{k}$ denote the filtration on $\mathbb{R}f_{*}\mathbb{Q}_{X}[\dim X]$ defined by the truncation $\mathcal{P}r_{\leq k}$ (see [3]) so that

$$\text{Gr}_{k}^{f}(\mathbb{R}f_{*}\mathbb{Q}_{X}[\dim X]) = \mathcal{P}R^{k}f_{*}(\mathbb{Q}_{X}[\dim X])[-k].$$

Let $L$ denote also the induced filtration on $H^{j}(X, \mathbb{Q})$. There are canonical isomorphisms

$$\text{Gr}_{k}^{f}H^{j}(X, \mathbb{Q}) = H^{j-\dim X-k}(S, \mathcal{P}R^{k}f_{*}(\mathbb{Q}_{X}[\dim X])).$$

This means that the injective morphism

$$H^{j-\dim X-k}(S, \mathcal{P}R^{k}f_{*}(\mathbb{Q}_{X}[\dim X])) \hookrightarrow H^{j}(X, \mathbb{Q})$$

is canonical modulo $L_{k-1}H^{j}(X, \mathbb{Q})$. This is a generalization of Remark (i) above.

(iii) Let $f : X \to S$ be as in (0.1). Let $f' : X \to S'$ be the Stein factorization of $f$ with canonical finite morphism $\pi : S' \to S$ such that $f = \pi \circ f'$. We have the decomposition theorem (0.1) for $f$ and $f'$. Since the intersection complexes are stable by the direct image by the finite morphism $\pi$, the direct image by $\pi$ of the decomposition (0.1) for $f'$ gives the decomposition (0.1) for $f$. This implies that, if the hypothesis (a) or (b) of Theorem 2 is satisfied, then the same hypothesis holds with $S$ replaced by $S'$, since $\text{IC}_{S}^{0}\mathbb{Q}$ is a canonical direct factor of $\pi_{*}\text{IC}_{S'}^{0}\mathbb{Q}$ and $f^{-1}(s)$ is the disjoint union of $f'^{-1}(s')$ for $s' \in \pi^{-1}(s)$.

(iv) Let $f : X \to S$ be as in Theorem 2. Assume $\dim S = 2$ and $f$ has connected fibers. The decomposition theorem (0.1) together with Proposition 1 implies

$$H^{1}(X, \mathbb{Q}) = \text{IH}^{1}(S, \mathbb{Q}) \oplus H^{1}(X, \mathbb{Q})^{\text{inv}},$$

where $H^{1}(X, \mathbb{Q})^{\text{inv}}$ denotes the monodromy invariant part which is identified with the stalk of the constant part of $(L_{S^\circ}^{c})$ at $s \in S^\circ$. We have

$$\text{IH}^{1}(S, (L_{S^\circ}^{c})) = H^{1}(X, \mathbb{Q})^{\text{inv}} \otimes \text{IH}^{1}(S, \mathbb{Q})$$

$$= H^{1}(X, \mathbb{Q}) \cup \text{IH}^{1}(S, \mathbb{Q}) \mod L_{-1}H^{2}(X, \mathbb{Q}).$$
Indeed, the first isomorphism is clear. For the second isomorphism, note that the cup product is induced by
\[ Rf_* \mathbb{Q}_X[3] \otimes Rf_* \mathbb{Q}_X[3] \to (Rf_* \mathbb{Q}_X[3])[3],\]
where the source contains as a direct factor
\[ \text{IC}_S L_{S_0}^{-1}[1] \otimes \text{IC}_S L_{S_0}^{-1}[1].\]

This is a perverse sheaf shifted by 4 since
\[ \dim \supp H^{-1}(\text{IC}_S L_{S_0}^{-1}) = 0 \text{ and } H^0(\text{IC}_S L_{S_0}^{-1}) = 0.\]

Then its image by the morphism to \((Rf_* \mathbb{Q}_X[3])[3]\) is contained in \(L_{-1}(Rf_* \mathbb{Q}_X[3])[3]\). So (1.14.1) follows. Note that we have in the notation of the decomposition (0.1)
\[ \text{Gr}^I_k H^2(X, \mathbb{Q}) = \begin{cases} 
\text{IH}^2(S, \mathbb{Q}) \oplus \left( \bigoplus_{i \geq 0} L_{(Z_i)}^{-1} \right) & \text{if } k = -1, \\
\text{IH}^1(S, L_{S_0}^0) \oplus \left( \bigoplus_{\dim Z = 1} H^0(Z^0, L_{Z^0}^0) \right) & \text{if } k = 0, \\
\text{IH}^0(S, \mathbb{Q})(-1) & \text{if } k = 1.
\end{cases}\]

Here the following Hodge structures have type \((1, 1)\) (see Proposition 1):
\[ \text{IH}^0(S, \mathbb{Q})(-1), \quad L_{(Z_i)}^{-1}, \quad H^0(Z^0, L_{Z^0}^0) \quad (\dim Z = 1).\]

(v) In (0.2) and (0.3) in the introduction, it might be possible to replace \((L_{S_0})^c\) and \((L_{S_0})^{nc}\) respectively with \((L_{S_0})^{pc}\) and \((L_{S_0})^{npc}\) where \(pc\) and \(npc\) respectively stand for potentially constant and non-potentially constant. The former is defined by the condition that the monodromy group of the local system is finite (or the local system becomes trivial by taking the pull-back under a finite étale covering \(r: S_0 \to S_0\)). The latter is the sum of simple local subsystems which are not potentially constant. Here we have to use a relative correspondence for \(S'/S\) in order to capture \(L_{pc}\) since \(\rho_* \rho^* L_{pc}\) is too big. Moreover, we have to study the relation with the Chow–Künneth projector of \(S'\) and the argument is not so simple. Note that we essentially replace \(S\) with the Stein factorization. In this case it is rather rare that we have \(L_{pc} \neq L_{pc}\). Note also that the hypothesis (a) cannot be stated as in the form in Theorem 2 if we replace \((L_{S_0}, \mathbb{C})^{nc}\) in (0.3) with \((L_{S_0}, \mathbb{C})^{npc}\).

The following is an improvement of [16], Proposition 1.10, which is needed for the proof of (1.5.1). Some argument is similar to 3.5 in later versions (or 2.9 or 2.10 in some earlier versions) of an unpublished preprint [20].

1.15. Proposition. Let \(X\) be a complex algebraic variety of dimension at most \(d\). Then we have the bijectivity of the cycle map
\[ (1.15.1) \quad \text{CH}_{d-1}(X, \mathbb{Q}) \cong \text{Hom}_{D^b \text{MHM}(X)}(\mathbb{Q}_X, (\mathbb{D}\mathbb{Q}_X)(1-d)[2-2d]).\]
Proof. Set $D = \text{Sing} X$ and $U = X \setminus D$. Let $\pi : X' \to X$ be the normalization. Set $D' = \pi^{-1}(D)$ with $\pi' : D' \to D$ the restriction of $\pi$. The Chow group $\text{CH}_{d-1}(X)$ does not change by deleting a closed subvariety of dimension at most $d - 2$. This is the same for the right-hand side of (1.15.1) since $\mathcal{H}^i \mathbb{D}\mathbb{Q}_X = 0$ for $i < -2 \dim X$ (see also the proof of [16], Proposition 1.10). Thus we may assume that $X'$, $D$, $D'$ are smooth, $D$ is purely $(d - 1)$-dimensional, and $\pi'$ is étale, shrinking $X$ if necessary.

Let $D_i$ be the connected components of $D$, and $D'_{i,j}$ be the connected components of $\pi^{-1}(D_i)$ with $d_{i,j}$ the degree over $D_i$. Define

$$E := \bigoplus_i E_i \text{ with } E_i := \text{Ker} \left( \bigoplus_j d_{i,j} : \bigoplus_j \mathbb{Z}[D_{i,j}] \to \mathbb{Z}[D_i] \right),$$

where $\mathbb{Z}[D_{i,j}]$ is a free $\mathbb{Z}$-module with (formal) generator $[D_{i,j}]$ (similarly for $\mathbb{Z}[D_i]$), and the morphism $d_{i,j} : \mathbb{Z}[D_{i,j}] \to \mathbb{Z}[D_i]$ is the multiplication by $d_{i,j}$, which is identified with the trace morphism for $D'_{i,j} \to D_i$.

Let $\mathcal{I}_i(X)$ be the set of integral (i.e. irreducible and reduced) closed subvarieties of $X$ with dimension $i$. We have the following commutative diagram of exact sequences (which is part of the diagram of the snake lemma):

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \bigoplus_{Y' \in \mathcal{I}_i(X')} \mathbb{C}(Y')^* & \longrightarrow & \bigoplus_{Y \in \mathcal{I}_i(X)} \mathbb{C}(Y)^* \\
\downarrow & & \downarrow & & \downarrow \\
E & \hookrightarrow & \bigoplus_{D' \in \mathcal{I}_{d-1}(X')} \mathbb{Z}[D'] & \longrightarrow & \bigoplus_{D \in \mathcal{I}_{d-1}(X)} \mathbb{Z}[D] \\
\downarrow & & \downarrow & & \downarrow \\
E & \longrightarrow & \text{CH}_{d-1}(X') & \longrightarrow & \text{CH}_{d-1}(X) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

(1.15.2)

where $\mathbb{Z}[D]$ is a free $\mathbb{Z}$-module with (formal) generator $[D]$, and the vertical morphism from $\mathbb{C}(Y)^*$ is the divisor map which associates the multiplicity of a rational function along each divisor $D$ (and similarly for $Y'$, $D'$).

On the other hand, we have a distinguished triangle in $\mathbb{D}\text{MHM}(X)$:

$$\mathbb{Q}_X \to \pi_* \mathbb{Q}_{X'} \to \bigoplus_i \left( \bigoplus_j \pi'_{D'_i,j} \mathbb{Q}_{D_i,j} \right) / \mathbb{Q}_{D_i} \to .$$

Applying the dual $\mathbb{D}$ and $\text{Hom}(\mathbb{Q}_X(d - 1)[2d - 2], *)$, and then using an isomorphism as in (1.5.2) together with the adjunction for $\pi^*$, $\pi_*$, we get an exact sequence

(1.15.3) $E_{\mathbb{Q}} \to \text{Hom}(\mathbb{Q}_{X'}, \mathbb{Q}_{X'}(1)[2]) \to \text{Hom}(\mathbb{Q}_X, \mathbb{D}\mathbb{Q}_X(1-d)[2-2d]) \to 0,$
where the Hom are taken in $D^b\text{MHM}(X')$ or $D^b\text{MHM}(X)$. Indeed, we have

$$E_i, Q = \text{Hom} \left( \mathcal{O}_{D_i}, \text{Ker} \left( \text{Tr} : \bigoplus_j \pi'_s \mathcal{O}_{D_{i,j}} \to \mathcal{O}_{D_i} \right) \right),$$

and the surjectivity of the last morphism of (1.15.3) follows from

$$\text{Ext}^1 \left( \mathcal{O}_{D_{i,j}}, \text{Ker} \left( \text{Tr} : \bigoplus_j \pi'_s \mathcal{O}_{D_{i,j}} \to \mathcal{O}_{D_i} \right) \right) = 0,$$

which is a consequence of the semisimplicity of pure Hodge modules.

The first morphism of (1.15.3) is induced by the cycle map for the cycles $[D_{i,j}]$ in $X'$. Indeed, the latter is induced by the Gysin morphism

$$\mathcal{O}_{D_{i,j}} \to \mathcal{O}_{X'}(1)[2],$$

which is the dual of the restriction morphism $\mathcal{O}_{X'} \to \mathcal{O}_{D_{i,j}}$. Then, comparing (1.15.3) with the third row of (1.15.2) tensored by $Q$, the assertion for $X$ is reduced to that for $X'$, and follows from [21], Proposition 3.4. This finishes the proof of Proposition 1.15.

### 2. Proof of the main theorems

#### 2.1. Proposition. Let $f : X \to S$ be a nonconstant surjective projective morphism of complex quasi-projective varieties where $X$ is smooth, connected, and 3-dimensional. Let $Z$ be a closed irreducible subvariety of $S$ and $i$ be an integer such that $(Z, i) \neq (S, 0)$. Then there is a good projector $\pi'_i|Z \in \text{Cor}^0(X, X)$ in the sense of 1.7 such that its action on the perverse cohomology sheaf $\mathcal{H}^jR^i f_* (\mathcal{O}_X, \dim X))$ is the projection to the direct factor $\text{IC}_Z L^j_{Z^0}$ for $j = i$, and vanishes for $j \neq i$.

**Proof.** By assumption we have $0 < \dim S \leq \dim X = 3$. We may assume $L^j_{Z^0} \neq 0$ since $\pi'_i|Z = 0$ otherwise. In case $Z = S$, we have $|i| \leq \dim X - \dim S$, and we may assume $\dim S \leq 2$ since $(Z, i) \neq (S, 0)$. In case $Z \neq S$, we have $|i| \leq 1 - \dim Z$ by Proposition 1, and in particular, $\dim Z \leq 1$.

Let $\tilde{X}_Z$ be a desingularization of $X_Z := f^{-1}(Z)$. It is denoted by $\tilde{X}_s$ if $Z = \{s\}$.

**Case 1 (dim $Z = 0$).** Here $\dim S$ can be arbitrary as in Case 2. We have $Z = \{s\}$ for some $s \in S$. Let $\tilde{X}'_s \subset \tilde{X}_s$ be the union of the 2-dimensional irreducible components of $\tilde{X}_s$. For $|i| \leq 1$ we have the injectivity of the composition of morphisms

$$L^i_{|s|} \to \text{Gr}^W_{i+3} H^{i+3}(X_s, \mathcal{O}) \to H^{i+3}(\tilde{X}_s, \mathcal{O}) \to H^{i+3}(\tilde{X}'_s, \mathcal{O}).$$

Here the first morphism is induced by the decomposition theorem (0.1), and is injective. The second morphism is also injective by the construction of the weight spectral sequence using a simplicial resolution, see [9]. So the composition (2.1.1) is injective for $i = 0, 1$, since $\tilde{X}_s \setminus \tilde{X}'_s$ is 1-dimensional. Then the injectivity for $i = -1$ is reduced to the case $i = 1$ using the relative hard Lefschetz theorem for the direct image $\mathbb{R}f_* \mathcal{O}_X, \dim X$ (see [3]) since the
latter implies an isomorphism \( \eta : L_{(s)}^{-1} \sim L_{(s)}^1 \) where \( \eta \) is the cohomology class of a relative \( \mathcal{f} \)-ample line bundle.

We first consider the case \( i = 0 \). We take a smooth projective curve \( C_s \) over \( s \) (which is not necessarily connected) together with a correspondence

\[
\xi \in \text{Cor}_S^{-1}(X_s', C_s) = \text{Cor}^{-1}(X_s', C_s) = \text{CH}^1(X'_s \times C_s)_\mathbb{Q},
\]

such that the composition below is injective:

\[
L_{(s)}^0 \xrightarrow{\mathcal{f}} H^3(X_s', \mathbb{Q}) \xrightarrow{\xi} H^1(C_s, \mathbb{Q})(-1),
\]

where the first morphism is given by (2.1.1).

Let \( t'_s : X'_s \to X \) denote the canonical morphism. Then (2.1.1) is induced by the morphism of perverse sheaves

\[
(t'_s)^* : pR^0f_*(\mathbb{Q}[3]) \to H^3(X'_s, \mathbb{Q})_{(s)}.
\]

This is induced by (1.2.2), and preserves the decomposition by the support of intersection complexes. Here \( M_{(s)} \) for an abelian group \( M \) in general denotes the sheaf supported on \( \{s\} \) and whose stalk is \( M \). Thus the injective morphism (2.1.2) is induced by

\[
\zeta := \xi \circ (t'_s)^* \in \text{Cor}_S^{-1}(X, C_s),
\]

using (1.2.2) and the compatibility with composition. Let \( \pi^C_{(s)} \) be an absolute Chow–Künneth projector for \( C_s \). Replacing \( \zeta \) with \( \pi^C_{(s)} \circ \zeta \) if necessary, we may assume

\[
\zeta = \pi^C_{(s)} \circ \zeta.
\]

The dual of (2.1.2) is surjective, and is induced by \( \mathcal{t}' \zeta \in \text{Cor}_S^1(C_s, X) \). So there is

\[
\gamma_{0,s} \in \text{Cor}_S^0(C_s, C_s) = \text{Cor}^0(C_s, C_s),
\]

such that \( \gamma_{0,s} = \pi^C_{(s)} \circ \gamma_{0,s} \circ \pi^C_{(s)} \) and moreover, setting

\[
\pi^f_{0,(s)} := \zeta' \circ \zeta \in \text{Cor}_S^0(X, X) \quad \text{with} \quad \zeta' := \mathcal{t}' \zeta \circ \gamma_{0,s},
\]

the action of \( \pi^f_{0,(s)} \) on \( L_{(s)}^0 \) is the identity. (Note that the action on \( L_{(s')}^0 \) for \( s' \neq s \) vanishes by considering the support.) The last condition on \( \gamma_{0,s} \) depends only on its action on \( H^1(C_s, \mathbb{Q}) \). By the first condition on \( \gamma_{0,s} \) we get

\[
\zeta' = \zeta' \circ \pi_{(s)}^C.
\]

Let \( H_s \subset H^1(C_s, \mathbb{Q}) \) be the image of (2.1.2) (up to a Tate twist), and \( H'_s \) be its orthogonal complement giving the orthogonal decomposition

\[
H^1(C_s, \mathbb{Q}) = H_s \oplus H'_s.
\]
We may assume that the action of $\gamma_{0,s}$ is compatible with this decomposition and moreover its restriction to $H^i_s$ vanishes.

Set

$$\pi'_{(s)} := \zeta \circ \zeta' = \zeta \circ \tau \circ \gamma_{0,s} \in \text{Cor}^0_{S}(C_s, C_s).$$

The action of $\pi'_{(s)}$ on $H^1(C_s, \mathbb{Q})$ is the projection to $H^i_s$ associated to the orthogonal decomposition using the above hypothesis on the action of $\gamma_{0,s}$. Since $\pi'_{(s)}$ is a direct factor of $\pi'_1$ (i.e. $\pi'_{(s)} = \pi'_1 \circ \pi'_{(s)} \circ \pi'_1$) by the above argument, it implies that $\pi'_{(s)}$ is a projector. Then $\pi'_{(s)}$ is also a projector using the above hypothesis on $\gamma_{0,s}$ since

$$(\pi'_{0,(s)})^2 = \tau \circ \pi'_1 \circ \gamma_{0,s} \circ \pi'_{(s)} \circ \pi'_1 \circ \zeta.$$ 

It is a good projector since (1.7.1) for $\pi'_{(s)}$ is clear. Then, replacing $\zeta$ and $\zeta'$ respectively with $\pi'_{(s)} \circ \zeta \circ \pi'_{0,(s)}$ and $\pi'_{0,(s)} \circ \zeta' \circ \pi'_{(s)}$ if necessary, the assertion follows.

The argument is similar for $|i| = 1$ where $C$ is replaced by a disjoint union of a finite number of points.

**Case 2** ($\dim Z = 1$, $Z \not= S$). Let $\tilde{X}'_Z \subset \tilde{X}_Z$ be the subvariety consisting of the irreducible components whose image in $S$ is $Z$. Let $\tilde{Z}$ be the Stein factorization of $\tilde{X}'_Z \rightarrow Z$. Note that $\tilde{Z}$ is smooth since $\tilde{X}'_Z$ is smooth and $\dim Z = 1$. Let $i'_Z : \tilde{X}'_Z \rightarrow X$, $p_Z : \tilde{X}'_Z \rightarrow \tilde{Z}$, and $q_Z : \tilde{Z} \rightarrow Z$ denote the canonical morphisms. Set

$$\zeta' := (p_Z)_* \circ (i'_Z)^*.$$ 

Its action induces an injection

$$L^0_{Z} \hookrightarrow (q_Z)_* \mathbb{Q}_Z(-1).$$ 

This is shown by taking a smooth curve intersecting $Z$ transversally at a sufficiently general point of $Z$ and using the base change over it. Note that the restriction to the other direct factor of $HR^0_{s_i}(\mathbb{Q}_X[\dim X])$ vanishes by the property of the strict support decomposition. Then the projector $\pi'_{0,Z}$ is defined by

$$\pi'_{0,Z} := \zeta' \circ \zeta \in \text{Cor}^0_{S}(X, X) \text{ with } \zeta' := \tau \circ \gamma_{0,Z},$$

where $\gamma_{0,Z} \in \text{Cor}^0_{S}(\tilde{Z}, \tilde{Z}) = \text{CH}^0(\tilde{Z} \times Z, Z)_{\mathbb{Q}}$ is chosen so that $\pi'_{0,Z}$ is a projector. As in Case 1 we assume that the action of $\gamma_{0,Z}$ is compatible with the orthogonal decomposition associated with the image of the above injective morphism and moreover the action of $\gamma_{0,Z}$ on the orthogonal complement vanishes.

Set

$$\pi'_Z := \zeta \circ \zeta' \in \text{Cor}^0_{S}(\tilde{Z}, \tilde{Z}).$$ 

This is also a projector. Then, replacing $\zeta$ and $\zeta'$ respectively with $\pi'_Z \circ \zeta \circ \pi'_Z$ and $\pi'_{0,Z} \circ \zeta' \circ \pi'_Z$ if necessary, the assertion follows.
Case 3 ($Z = S$, $\dim S = 1$). If $i = -1$, let $Y$ be a sufficiently general hyperplane section of $X$ which is smooth, 2-dimensional, and flat over $S$. Let $i_Y: Y \hookrightarrow X$ denote the inclusion. Then $\zeta$ is defined by $i_Y^* \zeta$, and the remaining argument is similar to the above cases since $Y$ is flat with relative dimension 1. The argument is similar for $i = 1$ (taking the transpose).

If $i = \pm 2$, we replace $Y$ with a complete intersection so that $Y$ is smooth, 1-dimensional and flat over $S$. Then the argument is similar.

Case 4 ($Z = S$, $\dim S = 2$). Let $S' \to S$ be the Stein factorization of $f$, and $\tilde{S} \to S'$ be a resolution of singularities. Let $\zeta \in CH_2(X)\mathbb{Q}$ such that $f_*\zeta = [S]$. Taking the pull-back of $\zeta$ by $X_{\tilde{S}} := X \times_S \tilde{S} \to X$, we get

$$\tilde{\zeta}_S \in Cor^0_S(X, \tilde{S}) = CH_2(X_{\tilde{S}})\mathbb{Q},$$

such that $(f_{\tilde{S}})_*\tilde{\zeta}_S = [\tilde{S}]$ where $f_{\tilde{S}}: X_{\tilde{S}} \to \tilde{S}$ is the base change of $f$. We have

$$[X_{\tilde{S}}] \in Cor^0_S(\tilde{S}, X) = CH_3(X_{\tilde{S}})\mathbb{Q}.$$ 

By Proposition 1.8 and with the notation of 1.9 we have

$$\pi_{\tilde{S}/U} \circ \tilde{\zeta}_S \circ [X_{\tilde{S}}] \circ \pi_{\tilde{S}/U} = \pi_{\tilde{S}/U},$$

using the action on the perverse sheaves. So the projector $\pi^f_{i,1,S}$ is defined by

$$(2.1.3) \quad \pi^f_{i,1,S} = [X_{\tilde{S}}] \circ \pi_{\tilde{S}/U} \circ \tilde{\zeta}_S.$$ 

and the assertion follows, see Remark 1.4. The argument is similar for $\pi^f_{i,S}$ (taking the transpose). This finishes the proof of Proposition 2.1.

2.2. Proof of Theorem 1. The relative projectors $\pi^f_{i,Z}$ for $(Z, i) \neq (S, 0)$ are constructed in Proposition 2.1. These can be modified so that they are orthogonal to each other by 1.7. Then $\pi^f_{0,S}$ is defined to be the remainder so that $\sum_{i,Z} \pi^f_{i,Z}$ is the diagonal $\Delta_X$, and the assertion follows.

2.3. Proof of Theorem 2. If the hypothesis (a) or (b) is satisfied, then it is also satisfied by replacing $S$ with the Stein factorization of $f$, see Remark 1.14(iii). So we may assume that $S$ is normal and $f$ has connected fibers. We will decompose every projector $\pi^f_{i,S}$ viewed as an absolute projector by (1.2.1) into a direct sum of mutually orthogonal projectors $(\pi^f_{i,S})_j$ in the sense of 1.6 so that the action of $(\pi^f_{i,S})_j$ on $H^k(X, \mathbb{Q})$ vanishes for $k \neq j$.

In the case $Z = \{s_j\}$, the projectors $\pi^f_{i,s_j}$ are essentially absolute projectors (over $\{s_j\}$), and we do not have to decompose them further. So we may assume $\dim Z > 0$. We first consider the case (a) which is more difficult.

Case (a.1). $\dim Z = 1$. We have $i = 0$ by Proposition 1, and Proposition 1.12 implies the orthogonal decomposition

$$\pi^f_{0,Z} = (\pi^f_{0,Z})^c + (\pi^f_{0,Z})^{nc},$$
since \((X/S, \pi^f_{0,Z}) \cong (\tilde{Z}/S, \pi^f_Z)\) in the notation of Case 2 in 2.1. By 1.13 we have the decomposition

\[(\pi^f_{0,Z})^c = \sum_{j=0}^2 (\pi^f_{0,Z})^c_j.\]

Note that \((\pi^f_{0,Z})^{nc} = (\pi^f_{0,Z})^{nc}_S\), i.e. \((\pi^f_{0,Z})^{nc}_j = 0\) for \(j \neq 3\).

Case (a.2). \(Z = S, |i| = 1\). Here Propositions 1.12 and 1.13 are not sufficient. Let \(\tilde{S} \to S\) be a resolution of singularities, and \(\pi^S_j \in \text{Cor}^0(\tilde{S}, S)_Q\) be the Chow–Künneth projectors for \(\tilde{S}\), see [14]. Here we may assume that \(\pi^S_j\) for \(j \neq 2\) are good projectors in the sense of 1.7 over \(S = pt\) since only curves are used in the construction. By (2.1.3) we have

\[
\pi^f_{-1,S} = \zeta^f_{-1,S} \circ \zeta_{-1,S} \quad \text{with} \quad \zeta^f_{-1,S} = \gamma^f_{-1,S} \circ [X_{\tilde{S}}] \circ \pi_{S_{\tilde{S}}/U}, \quad \zeta_{-1,S} = \pi_{S_{\tilde{S}}/U} \circ \xi_S \circ \gamma_{-1,S}.
\]

Here \(\gamma_{-1,S}\) and \(\gamma^f_{-1,S}\) are added in order to get mutually orthogonal projectors as in (1.7.2), see the proof of Theorem 1 in 2.2. By Proposition 1.10 we have absolute projectors

\[(\pi^f_{-1,S})_j := \zeta^f_{-1,S} \circ \pi^S_j \circ \zeta_{-1,S} \in \text{Cor}^0(X, X),\]

giving the orthogonal decomposition

\[\pi^f_{-1,S} = \sum_{j=0}^4 (\pi^f_{-1,S})_j \quad \text{in} \ \text{Cor}^0(X, X).\]

Note that \(f^*\text{IH}^2(S, Q) \subset H^2(X, Q)\) coincides with the image of the action of \((\pi^f_{-1,S})_2\) on \(H^2(X, Q)\) by the decomposition (1.8.1). Similarly we get \((\pi^f_{1,S})_{j+2}\) giving the orthogonal decomposition

\[\pi^f_{1,S} = \sum_{j=0}^4 (\pi^f_{1,S})_{j+2} \quad \text{in} \ \text{Cor}^0(X, X).\]

Case (a.3). \(Z = S, i = 0\). Proposition (1.12) implies the orthogonal decomposition

\[\pi^f_{0,S} = (\pi^f_{0,S})^c + (\pi^f_{0,S})^{nc},\]

and we have by 1.13 the decomposition

\[(\pi^f_{0,S})^c = \sum_{j=0}^4 (\pi^f_{0,S})^c_{j+1}.
\]

We need the assumption (a) to construct the decomposition

\[(\pi^f_{0,S})^{nc} = \sum_{j=1}^3 (\pi^f_{0,S})^{nc}_{j+1},\]
Here \((\pi^f_{0,S})_{i+j+1} = 0\) for \(j = 0, 4\), since \(H^j(S, IC_S(L^0_{0,S})) = 0\) for \(|j| = 2\) (using the vanishing of \(\Gamma(S^o, (L^0_{0,S})^{nc})\)). By condition (a) the image of the action of \((\pi^f_{0,S})^{nc}\) has type \((1, 1)\), see Remark 1.14(iii). So there is an absolute projector \(\gamma\) such that the action of \(\gamma\) on \(H^2(X, \mathbb{Q})\) coincides with that of \((\pi^f_{0,S})^{nc}\) and \((X, \gamma)\) is isomorphic to a finite direct sum of copies of \((pt, id)(-1))\). Let \(\zeta\) and \(\zeta'\) be algebraic cycles on a disjoint union of copies of \(X\) inducing the last isomorphism so that

\[
\gamma = \zeta' \circ \zeta, \quad \zeta \circ \zeta' = id.
\]

Then we can set

\[
(\pi^f_{0,S})^{nc}_{2} = (\pi^f_{0,S})^{nc}_{2} \circ \gamma \circ (\pi^f_{0,S})^{nc}_{2},
\]

since \(\zeta \circ (\pi^f_{0,S})^{nc}_{2} \circ \zeta' = id\) and this implies that \((\pi^f_{0,S})^{nc}_{2}\) is an idempotent.

The argument is similar for \((\pi^f_{0,S})^{nc}_{4}\) where \(H^2(X, \mathbb{Q})\) and \((pt, id)(-1)\) are respectively replaced by \(H^4(X, \mathbb{Q})\) and \((pt, id)(-2)\). The orthogonality of \((\pi^f_{0,S})^{nc}_{2}\) and \((\pi^f_{0,S})^{nc}_{4}\) follows from the fact that the projectors factor through the direct sums of copies of \((pt, id)(-1)\) or \((pt, id)(-2)\). Then \((\pi^f_{0,S})^{nc}_{2}\) is defined by the remaining so that the above decomposition holds.

Thus we get an orthogonal decomposition of every \(\pi^f_{i,Z}\) as explained at the beginning of this subsection, and Theorem 2 is proved in the Case (a).

**Case (b).** We may assume \(Z = S\) by the argument at the beginning of this subsection. By 1.12 we have orthogonal decompositions as relative projectors

\[
\pi^f_{i,S} = (\pi^f_{i,S})^c + (\pi^f_{i,S})^{nc},
\]

such that the image of the action of \((\pi^f_{i,S})^c\) on the local systems is the constant part of \(L^i_{S^o}\), see 1.6. Here we use condition (b) in the case \(i = 0\), since it implies that the invariant part of \(H^2(X, \mathbb{Q})\) has type \((1, 1)\) and is generated by divisor classes. By 1.13 we then get an orthogonal decomposition as absolute projectors

\[
(\pi^f_{i,S})^c = \sum_{j=0}^{2} (\pi^f_{i,S})^c_{i+j+2}.
\]

On the other hand, we do not have to decompose further \((\pi^f_{i,S})^{nc}\), i.e.

\[
(\pi^f_{i,S})^{nc} = (\pi^f_{i,S})^{nc}_{i+3}, \quad (\pi^f_{i,S})^{nc}_{i+j+2} = 0 \quad (j \neq 1),
\]

since \(H^0(S^o, (L^0_{S^o})^{nc}) = 0\) and \(\dim S = 1\).

Thus we get an orthogonal decomposition of every \(\pi^f_{i,Z}\) as explained at the beginning of this subsection, and Theorem 2 is proved also in the Case (b).
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