Optimal Performance of Global Quantum Networks

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The development of a future, global quantum communication network (or quantum internet) will enable high rate private communication and entanglement distribution over very long distances. However, the large-scale performance of ground-based quantum networks (which employ photons as information carriers through optical-fibres) is fundamentally limited by the fibre quality and link length. While these fundamental limits are well established for arbitrary network architectures, the question of how to best design these global architectures remains open. In this work, we take a step forward in addressing this problem by modelling global quantum networks with weakly-regular architectures. Such networks are capable of idealising end-to-end performance whilst remaining sufficiently realistic. In this way, we may investigate the effectiveness of large-scale networks with consistent connective properties, and unveil the global conditions under which end-to-end rates remain analytically computable. Furthermore, by comparing the performance of ideal, ground-based quantum networks with satellite quantum communication protocols, we can establish conditions for which satellites can be used to outperform fibre-based quantum infrastructure.

Advancements in quantum information science will have a profound impact on society [1–4]. In particular, the overarching trajectory of quantum communication technologies is towards that of a global quantum communication network: A quantum internet [5–8]. This will facilitate high rate, provably secure communication and globally distributed quantum information processing with radical implications for science, technology and beyond.

The current, most promising point-to-point quantum communication protocols, where two parties are connected directly via a quantum channel, are based on Continuous Variable (CV) quantum systems [9–12] (such as bosonic modes). CV protocols achieve high performance and are compatible with current telecommunication infrastructure based upon optical-fibre connections, emphasising their near-term practical feasibility. However, the laws of quantum mechanics prohibit the ability to simultaneously achieve high rates and long distances, a fundamental law captured by the Pirandola-Laurenza-Ottaviani-Banchi (PLOB) bound [13]. This describes the absolute maximum rate that two parties may transmit qubits, distribute entanglement, or establish secret-keys over a bosonic lossy channel (optical-fibres) as $-\log_2(1 - \eta)$ bits per channel use, where $\eta$ is the channel transmissivity [13, 14].

Overcoming this point-to-point limitation requires the use of quantum repeaters/relays, or more generally the construction of quantum networks. Combining tools from classical networks [15–18] with the PLOB bound, ultimate limits have also been established for the end-to-end capacities of quantum networks [19]. These results confirm that the PLOB bound can be beaten via quantum networking, facilitating high rate communication at longer ranges. While these bounds are readily accessible for arbitrary networks, their assessment requires the specification of an architecture. Indeed, questions of network topology have been recently considered via the statistical study of complex, random quantum networks [20–23], which reveal critical and insightful phenomena associated with large-scale network properties. Yet, the ultimate limits of global quantum networks will not be realised by random constructions, but by specifically engineered architectures used to optimise performance. Furthermore, ground-based fibre channels are not the only conduits available for global quantum communications.

A rival infrastructure that may prove superior to fibre-based networks at global distances is Satellite Quantum Communication (SQC) [24–29]. SQC exploits ground-to-satellite communication channels to overcome the fundamental distance limitations offered by fibre/ground-based mechanisms. A satellite in orbit around the Earth may act as a dynamic repeater that physically passes over ground-based users and distributes very long-range entanglement/secret-keys. The ability to exploit a free-space connection with a satellite also carries the possibility of substantially more transmissive channels than optical-fibres, making it ideal for global communication protocols.

The following critical questions emerge: What is the best way to design a large-scale, quantum network to maximise end-to-end performance? And are fibre-based networks truly the best way to achieve long-distance quantum communication?

The goal of this work is to answer to these questions. Utilising ideas from quantum information theory, classical networks, and notions from graph theory [30], we design ideal architectures based on the property of weak-regularity (WR). Weakly-Regular Networks (WRNs) simultaneously idealise network connectivity whilst providing sufficient topological freedom to capture a broad class of realistic structures. Most importantly, we show that quantum WRNs which employ multi-path routing admit remarkably general and achievable upper-bounds.
on the network capacity. When employed in conjunction with suitable quantum network tools this allows for a characterisation of the ideal performance of a fibre-based quantum internet.

Furthermore, these exact, analytical results provide an immediate pathway to perform comparisons of SQC with global ground-based quantum communications. We study the average number of secret bits per day that can be distributed between two remote stations, using large-scale quantum networks or a single satellite repeater station. Our findings reveal the constraints associated with fibre-based networks, and the enormous resource demands required to overcome achievable rates offered by a single satellite in orbit. These results further motivate the study of SQC and its key role within future global quantum communications.

I. GLOBAL QUANTUM NETWORK DESIGN

A. Quantum Networks and Routing Strategies

A quantum network can be described as a finite, undirected graph $\mathcal{N} = (P, E)$ where $P = \{x_i\}$ collects all the nodes (points/vertices) on the graph, and $E = \{e_i\}$ collects valid connections between nodes (edges). A network node refers to either a user-node, such as a potential end-user pair Alice $a$ and Bob $b$, or a repeater/relay-node $r$. Each node $x_i$ possesses a local register of quantum systems which can be altered and exchanged with connected neighbours. Any two points $x, y \in P$ are connected via an undirected edge $e := (x, y) \in E$ if there exists a quantum channel $\mathcal{E}_{xy}$ through which they may communicate. Since each edge is undirected, this may be a forward or backward channel. Under the assistance of two-way classical communications (CCs) the optimal transmission of quantum information is connected with optimal entanglement distribution. It does not depend on the direction of communication but the Local Operations (LOs) at each point (and thus the direction of teleportation), referring to a logical flow of quantum information.

In a point-to-point communication setting, the logical flow of quantum information has a clear and obvious set of choices; Alice to Bob $a \rightarrow b$ or Bob to Alice $a \leftarrow b$. However, within a quantum network, a vast array of options emerge due to the various interconnections and possible paths that information may follow. To address this, users can devise an end-to-end routing strategy that constitutes a network protocol facilitating communication between end-users. The two key classes of strategy are single-path and multi-path routing.

Single-path routing is the simplest network communication method, which utilises point-to-point communications in a sequential manner. Quantum systems are exchanged from node-to-node followed by LOCC operations after each transmission until eventually communication has been established between the end-users. This kind of strategy is analogous to the use of a repeater-chain, and network performance is determined by the strength of each link along an optimal end-to-end route.

However, a more powerful strategy is multi-path routing, which properly exploits the multitude of possible end-to-end routes available in a quantum network. A user may exchange an initially multi-partite quantum state with a number of neighbouring receiver nodes, who may each then perform their own point-to-multi-point exchanges along its unused edges. The exchange of quantum systems can be interleaved with adaptive network LOCCs in order to distribute secret correlations, and this process continues until multi-point interaction is carried out with the end-user. A multi-path routing strategy in which all channels in the network are used once per end-to-end transmission is known as a flooding protocol. This is achieved via non-overlapping point-to-multi-point transmissions at each network node, such that receiving nodes only choose to transmit along unused edges for subsequent connections. This greatly enhances the end-to-end performance of quantum networks.

B. Regular Graphs

In an effort to provide a model for idealised large-scale networks, we may exploit properties of network regularity. Consider an undirected graph $\mathcal{N} = (P, E)$ of $n$-nodes, underlying a quantum network. The neighbourhood of any node $x \in P$ is the set of all nodes that are connected to $x$ via an edge, $N_x := \{y \mid (x, y) \in E\}$. A graph is $\lambda$-regular if all nodes in the graph possess the same degree $\lambda$, i.e. the neighbourhood of any node consists of strictly $\lambda$-elements, $|N_x| = \lambda$, $\forall x \in P$. We label the set of all $n$-node, $k$-regular graphs $\mathcal{R}_{n,k}$.

This is a very broad class of graphs and more detailed properties can be defined. There exist Strongly Regular (SR) graphs that satisfy stricter connective properties and are defined by four parameters $(n, k, \lambda, \mu)$ [31]. As before, $n, k$ indicate that there exist $n$-nodes, and that each node is $k$-regular. The parameter $\lambda$ is the adjacent commonality which counts the number of common neighbours shared by adjacent nodes,

$$\text{if } (x, y) \in E \implies \lambda(x, y) = |N_x \cap N_y| =: \lambda \in \mathbb{Z}^+. \quad (1)$$

Meanwhile, $\mu$ is the non-adjacent commonality which counts the number of common neighbours shared by non-adjacent nodes,

$$\text{if } (x, y) \notin E \implies \mu(x, y) = |N_x \cap N_y| =: \mu \in \mathbb{Z}^+. \quad (2)$$

Here, $\lambda(x, y)$ and $\mu(x, y)$ are counting functions for their respective property. The connectivity properties of such graphs are not independent, but follow the relation

$$\mu(n-k-1) = k(\lambda - \lambda - 1). \quad (3)$$

The set of all SR graphs is of course a subset of the larger set of all regular graphs, $S_{n,k}^{\lambda,\mu} \subset \mathcal{R}_{n,k}$. While these
graphs are desirably highly connected, their architectures are very strict, and the parameters \((k, \mu, \lambda)\) may inhibit the ability to use a large number of nodes rendering them impractical for network design.

Therefore a slightly more general classification describes Weakly-Regular (WR) graphs. Any regular graph that is not SR is technically WR, and can be characterised by a more general set of connectivity properties, \((n, k, \lambda, \mu)\). We invite greater generality by loosening the strict, single values of the adjacent/non-adjacent commonalities \(\lambda/\mu\) for all nodes and instead define vectors which contain potential values for the commonality properties, \(\lambda = \{\lambda_1, \ldots, \lambda_l\}\) and \(\mu = \{\mu_1, \ldots, \mu_m\}\). In this work, we consider graphs in which each node possesses an identical distribution of adjacent-commonalities with respect to each of its \(k\)-neighbours. That is, \(|\lambda| = k\) and there is a potentially unique \(\lambda_j \in \lambda\) for every node in a \(k\)-element neighbourhood. This means that all nodes \(x \in P\) satisfy
\[
\{\lambda(x, y) \mid y \in N_x\} = \text{perm}(\lambda),
\]
where \(\text{perm}(\lambda)\) is an unspecified permutation of \(\lambda\). This property imposes a useful level of symmetry, such that the adjacent common neighbour distribution around any node is always some permutation of \(\lambda\). We label the set of WR graphs that follow this construction \(W_{n,k}^{\lambda,\mu} \subseteq R_{n,k}\).

As a simple example, we depict a weakly-regular sub-graph from a larger network in Fig. 1. Clearly, the degree of the network is \(k = 6\), and the commonality properties are also illustrated by considering adjacent and non-adjacent nodes with respect to a root node. Provided that the regularity shown in Fig. 1 is consistent throughout the network, it can be shown that the adjacent commonality vector is \(\lambda = \{2\}^{56}\) [32], and the non-adjacent commonality vector is \(\mu = \{0, 1, 2\}\).

Weak-regularity is a desirable property for large-scale quantum networks, with useful qualities that can be exploited for analytical investigation. Indeed, the loosening of the \(\lambda, \mu\) commonality properties provides greater versatility for network construction. In the following section, we discuss how weak-regularity can be imposed on a quantum network.

C. Genuine and Internal Regularity

Let us first simplify the manner in which we refer to quantum Weakly-Regular Networks (WRNs). Since we are considering large-scale networks, we impose that the total number of nodes/repeaters greatly exceeds the degree and commonality parameters \(n \gg k, \lambda_j, \mu_j, \forall j\). This assumption is implicit throughout this paper and allows us to omit \(n\) from network characterisations.

We may also remove detailed reference to the non-adjacent-commonality \(\mu\). Typically, \(\mu\) is important for the characterisation of short-range connective structures since for large networks, the vast majority of non-adjacent nodes will simply share no neighbours, \(\mu_j = 0\).

We find that subsequent analyses do not require its explicit usage, and is dropped for purposes of clarity (unless otherwise specified). We henceforth study the general class of \((k, \lambda)\)-regular quantum networks \(N \in W_k^\lambda\), and consider two key formats in which regularity is satisfied: Genuine, and Internal regularity.

Genuine \((k, \lambda)\)-regularity refers to the original, rigorous definition. While this can in general always be satisfied, when defined on a two-dimensional plane this may lead to some undesirable characteristics, such as extremely long edges (channels) that undermine the integrity of the network. Yet, these conditions can be easily satisfied by considering closed networks embedded on a sphere, or other appropriate three-dimensional objects. Global quantum networks, in which we consider a network that spans the Earth (such as a future quantum internet) may therefore be appropriately and ideally modelled via genuinely-WR quantum networks (see Fig. 2).

Internal \((k, \lambda)\)-regularity provides a less rigorous (but nonetheless useful) model of network connectivity. As mentioned, defining regularity conditions on a two-dimensional plane can lead to unwanted features. However, it is possible to define a network that satisfies these connectivity properties within a network boundary. That is, one can identify a sequence of network nodes that form a boundary \(P_{\text{bound}} = \{p_1, \ldots, p_m, \ldots\}\), within which all other nodes \(P \setminus P_{\text{bound}}\) are \((k, \lambda)\)-regular. Importantly, we impose an implicit constraint that \(\exists \mu_j \geq 2 \in \mu\) for internally-WRNs, in order to eliminate the manifestation of graph structures with poor global connectivity [33].

We indicate a modification to the class of genuine-WRNs to include internally-WRNs via the relabelling \(W_k^\lambda \rightarrow U_k^\lambda\). Throughout this work we study networks that belong to this set, \(N \in U_k^\lambda\). Henceforth, we implicitly consider sufficiently large networks so that there exist pairs of remote nodes that do not share direct connections and assume deeply embedded end-users (distant from the boundary) for internally-WRNs.

Figure 1. A sub-graph from a \((k, \lambda, \mu) = (6, \{2\}^{56}, \{0, 1, 2\})\)-weakly regular network. Considering the yellow node as an end-user, the blue nodes thus represent the user neighbourhood, with a uniform adjacent commonality of \(\lambda = 2\). The non-adjacent commonality decreases as nodes increase in distance from the end-user.
Figure 2. A closed, genuinely weakly-regular quantum network defined on a global scale, which is \((k, \lambda) = (6, [2])^{\infty}\) regular. The sub-networks \(N_a\) and \(N_b\) represent the end-user neighbourhoods, with edge sets \(E_a\) and \(E_b\) respectively. The network-bulk \(N\) is the collections of nodes and edges that separate the neighbourhoods. The smallest cut-sets that can be collected on the network are \(C \in \{E_a, E_b\}\), while any cut on the network body \(C'\) is guaranteed to collect more edges.

II. ANALYTICAL BOUNDS FOR QUANTUM WEAKLY-REGULAR NETWORKS

A. Flooding Capacities

Consider a quantum network \(\mathcal{N} = (P, E)\) with edges described by channels \(\mathcal{E}_{xy}\) and a pair of remote users \(i = \{a, b\}\) contained within the network. Each channel is associated with a single-edge capacity \(C_{xy} := \mathcal{C}(\mathcal{E}_{xy})\) which describes the point-to-point communication quality between nodes. Hence, all networks have an associated capacity distribution \(\{C_{xy} | (x, y) \in E\}\) which informs the weights of the network graph.

The optimal end-to-end performance within \(\mathcal{N}\) is quantified by its multi-path or flooding capacity \(C^m(i, \mathcal{N})\), which describes the optimal number of target bits (such as secret-bits or entanglement-bits) that can be transmitted between end-users per use of a flooding protocol. The flooding capacity can be derived by solving the classical maximum-flow minimum-cut problem according to the capacity distribution of the network. Here we define a cut \(C\) as a means of disconnecting (or partitioning) the two end-users. Any cut generates an associated cut-set \(\tilde{C}\), a collection of edges within the network that perform this partition. The flooding capacity is found by locating the minimum-cut \(C_{\text{min}}\), which minimises the multi-edge capacity over all cut-sets \([19]\),

\[
C^m(i, \mathcal{N}) := \min_C \sum_{(x, y) \in \tilde{C}} C_{xy}. \tag{5}
\]

For general quantum networks, with arbitrary capacity distributions and network structures, this problem requires a numerical treatment for which there are a plethora of approaches \([34–36]\).

Yet, the imposition of network regularity invites the potential for analytical investigation. As a simple case, consider a WRN \(\mathcal{N} \in \mathcal{W}_k^\lambda\) with a uniform capacity distribution, i.e. all channels within the network are identical \(\mathcal{E}_{xy} = \mathcal{E}, \forall (x, y) \in P\). Then the max-flow min-cut problem simplifies to locating the network-cut \(C_{\text{min}}\) that generates the smallest possible cut-set \([\tilde{C}_{\text{min}}]\). For a WRN, this will simply be either end-user neighbourhood \(C_{\text{min}} = N_a = N_b\), such that the flooding capacity is just \(k\) times the single-edge capacity, \(C^m(i, \mathcal{N}) = kC(\mathcal{E})\).

Under general capacity distributions, this result does not hold. However, for highly connected networks, it is not unreasonable to assume that the minimum-cut-set will often be that with the smallest cardinality, even under more general capacity distributions. As one attempts to perform network-cuts that collect edges further from the end-users, the cut-set reliably becomes larger due to network regularity and it becomes decreasingly likely to find a minimum-cut. Hence, it is interesting to ask; under what network conditions does the minimum end-user neighbourhood remain the minimum-cut-set?

We find that this question is not only interesting but very useful for ideal network modelling. By assigning the minimum-cut as the min-neighbourhood cut, the network remains analytical, meaning that the flooding capacity between two-end users is always readily identified via the min-user neighbourhood. Furthermore, this imposition leads to surprisingly loose constraints on the network capacity distribution. As we will show, the min-neighbourhood cut captures the cheapest way to ensure high-performance flooding rates, such that it imposes weaker constraints than any other network-cut.

In the following sections, we answer this question to provide a valuable analytical tool that can be used to motivate the physical properties of large-scale quantum networks.

B. Minimum-Cut as Neighbourhood Isolation

Let us define some key network machinery in order to rigorously tackle this problem. Consider a quantum WRN, \(\mathcal{N} = (P, E) \in \mathcal{W}_k^\lambda\), and a pair of end-users denoted by the pair of network nodes \(i = \{a, b\}\). These two end-users Alice and Bob possess user-neighbourhood sub-networks \(\mathcal{N}_a := (P_a, E_a)\), and \(\mathcal{N}_b := (P_b, E_b)\), such that

\[
P_a := \{a\}, E_a := \{(a, p) | (a, p) \in E\}, \tag{6}
\]

\[
P_b := \{b\}, E_b := \{(b, p) | (b, p) \in E\}. \tag{7}
\]
Removing these neighbourhoods from $N$, the sub-network that remains is denoted by
\[ N' := (P \setminus \{a, b\}), E \setminus (E_a \cup E_b) =: (P', E'), \]
which we call the network-bulk, since $N'$ represents the “bulk” of the network that connects the end-users. In an internally-WR network, the user sub-networks are deeply embedded in $N'$; hence we can ignore edge-effects, i.e. variation in connectivity at network boundary. For closed network constructions this is implicitly satisfied, and we observe genuine-WR.

For subsequent methods, it is important to be able to differentiate between the capacity distributions of these sub-networks. To this end, we may define $\{C\}_{m,k}$ as the single-edge bulk capacities, and $\{C\}_{m,k} \mid (x, y) \in E'$ as the single-edge bulk capacities. These titles can be generalised to describe multi-edge quantities too. We define the min-neighbourhood capacity $C^m_{N_i}$ as the minimum single-edge capacity associated with the end-user neighbourhood cut-sets
\[
C^m_{N_i} = \min_{p \in E} \sum_{(x,y) \in E_p} |E_p| = k. \tag{9}
\]
Furthermore, we may define a multi-edge bulk-capacity $C^m_{N_i}$, as the capacity associated with a network-cut whose edges are strictly contained within the network-bulk. Given that $C'$ is a cut restricted to edges in $E'$, then the bulk-capacity takes the form
\[
C^m_{N_i}(C') = \sum_{(x,y) \in E'} C_{xy}. \tag{10}
\]

With these definitions in hand, it is possible to derive network conditions for which the flooding capacity is simply the min-neighbourhood capacity, $C^m(N) = C^m_{N_i}$ for some end-user pair $i = \{a, b\}$ leading to a key technical result of this paper.

**Theorem 1** Consider an arbitrary quantum WRN $N \in \mathcal{U}^r$. Select an end-user pair $i = \{a, b\}$, and demand they are sufficiently distant such that they do not share an edge or neighbour, and are deeply embedded in the network (if internally-WR). Then there exists a threshold single-edge bulk capacity $C^m_{\min}$ for which
\[
\text{if } C_{xy} \geq C^m_{\min}, \forall (x,y) \in E' \implies C^m(i, N) = C^m_{N_i}, \tag{11}
\]
where the threshold capacity is defined by the connectivity and neighbourhood properties of the network,
\[
C^m_{\min} := \frac{C^m_{N_i}}{\sum_{j=1}^k k - \lambda_j - 1} = \frac{\delta}{k} C^m_{N_i}, \tag{12}
\]
and we define $\delta$ as the cut-ratio,
\[
\delta := \frac{k}{\sum_{j=1}^k k - \lambda_j - 1}. \tag{13}
\]
Therefore, if all single-edge bulk capacities satisfy this minimum threshold, the flooding capacity is equal to the min-neighbourhood capacity.

A detailed proof can be found in Appendix A, but here we may outline the motivation. In a highly connected and regular network structure, as one moves further away from an end-user neighbourhood, the number of edges required to perform a valid cut increases exponentially. A visual representation of this phenomenon is portrayed in Fig. 2; while the users can be easily partitioned by collecting the $k$-edges in either of their neighbourhoods, a cut $C'$ performed on the network-bulk will inevitably collect more edges due to connectivity regularity. This point can be rigorously confirmed for our class of WRNs, such that when restricted to performing cuts on the network-bulk the smallest cut-set possesses the cardinality $C^m_{N'} := \sum_{j=1}^k k - \lambda_j - 1$. Hence, it is possible to impose a single-edge bulk capacity distribution for which the minimum-cut is always the minimum user-neighbourhood.

Furthermore, we can glean conditions as to when the previous result holds for a large class of capacity distributions:

**Corollary 1** As a consequence of Theorem 1, there exists a class of $(k, \lambda)$-regular networks for which the threshold single-edge bulk capacity is smaller than the maximum single-edge neighbourhood capacity. More precisely,
\[
\text{if } \sum_{j=1}^k \lambda_j \leq k(k-2) \implies C^m_{\min} \leq \max_{(x,y) \in E_i} C_{xy}. \tag{14}
\]

This is an important corollary (also see proof in Appendix A) to the previous theorem. This tells us that for networks where the regularity inequality in Eq. (14) holds, it follows that the threshold capacity throughout the network-bulk is smaller than that of the best single-edge neighbourhood capacity. Therefore, it is possible to permit poor quality channels in the network-bulk without compromising the minimum-cut. For networks with poor connectivity properties, this is often not the case, and the introduction of low quality channels elsewhere in the network would alter the minimum-cut.

These results also justify our previous remark about neighbourhood isolation as the cheapest network-cut, and motivates our analyses. By enforcing the minimum-cut to be an end-user neighbourhood, this generates the largest class of tolerable capacity distributions (and smallest threshold capacity $C^m_{\min}$). If we instead chose a cut $C^m$ as the minimum-cut, such that its cut-set satisfies $|C^m| > k$, then the single-edge capacities in all the smaller network-cuts must adopt higher capacity distributions. Thus, all other cuts place stricter constraints on the quantum network.

In other words, neighbourhood isolation describes the largest class of capacity distributions for quantum WRNs out of all possible minimum-cuts. Hence, we may study WRNs for which the flooding capacity will be the min-neighbourhood capacity for a potentially large range of
bulk capacity distributions, and allows for Theorem 1 to apply with strong generality. In the following section, we reveal the implications of these results for fibre-optic based quantum networks.

### C. Bosonic Lossy Quantum Networks

When considering fibre-based networks, point-to-point links are described by bosonic pure-loss (lossy) channels. A lossy channel $\mathcal{L}$ with transmissivity $\eta \in (0, 1)$ is a phase-insensitive Gaussian quantum channel, which transforms input quadratures $\hat{x} = (\hat{q}, \hat{p})^T$ according to $\hat{x} \mapsto \sqrt{\eta} \hat{x} + \sqrt{1 - \eta} \text{env}$ (where the environment is in a vacuum state) describing the interaction of bosonic mode with a zero-temperature bath [10].

For lossy quantum networks, the most important property is channel length, or from a network perspective, inter-nodal separation. For a given edge $(x, y) \in E$ connecting two users in a network, the inter-nodal separation is simply the distance $d_{xy}$ between them. All two-way capacities of the lossy channel are precisely known via the PLOB bound [13],

$$C_{\mathcal{L}}(d_{xy}) = -\log_2 \left( 1 - 10^{-\gamma d_{xy}} \right),$$

(15)

where the inter-nodal separation is related to the transmissivity via $\eta_{xy} = 10^{-\gamma d_{xy}}$. For current, state of the art fibre-optics the loss rate is $\gamma = 0.02$ per km [37]. Since these separations directly dictate the channel quality between nodes they must be precisely engineered and distributed in order to guarantee strong end-to-end performance.

Through the direct application of the two-way capacity $C_{\mathcal{L}}(d_{xy})$ within Theorem 1, the main result of this paper becomes clear. The minimum threshold capacity $C_{\min}^m$ translates into a max-bulk separation $d_{\max}^m$ which guarantees that the flooding capacity is equal to the min-neighbourhood capacity of the end-users. That is,

$$d_{xy} \leq d_{\max}^m, \forall (x, y) \in E' \implies C^m(i, N) = C_{\min}^m.$$  

(16)

This quantity describes the maximum distance that is permitted between any two nodes in the network-bulk, and is greatly informative for network topology and nodal density. A larger tolerable $d_{\max}^m$ leads to greater freedom in architectural design and lower resource demands for large-scale networks.

If the max-bulk separation condition is satisfied, then the minimum-cut is guaranteed to generate the minimum-neighbourhood separation. Furthermore, if this condition is not met then any alternative minimum-cut reliably generates a smaller flooding rate. This renders the min-neighbourhood capacity $C_{\min}^m$, an upper-bound on the optimal network performance. This is summarised in the following theorem.

**Theorem 2** Consider an arbitrary quantum WRN $N \in \mathcal{N}^m$ which is connected by bosonic lossy channels. Select an end-user pair $i = \{a, b\}$ within the network that do not share an edge or neighbour, and possess a minimum-neighbourhood capacity $C_{\min}^m$. Then, there exists a maximum inter-nodal separation within the network-bulk

$$d_{\max}^m = \frac{1}{\gamma} \log_{10} \left( 1 - 2^{-2C_{\min}^m} \right),$$

(17)

for which the flooding capacity is equal to the min-neighbourhood capacity,

$$C^m(i, N) = C_{\min}^m.$$  

(18)

Otherwise, if $\exists d_{xy} > d_{\max}^m, (x, y) \in E'$, then this becomes an upper-bound on the optimal network performance, $C^m(i, N) \leq C_{\min}^m$.

Therefore we have established an optimal performance bound on lossy quantum WRNs with respect to inter-nodal separation properties (see Appendix A for the proof). Figure 3 illustrates the relationship between flooding capacity and maximum inter-nodal separation in the network-bulk. The limiting separation $d_{\max}^m$ is inexorably linked with the regularity of the WRN. Networks with high connectivity possess a greater tolerance for longer distance channels since the enhanced multi-path capabilities of the network outweigh the effect of poor quality channels. This is clear from the examples shown in Fig. 3, where a WRN with degree $k = 16$ can tolerate channels of ~60 km longer than one with $k = 3$.

Theorem 2 applies to a pair of specified end-users $i = \{a, b\}$, and all other network nodes are assumed to be dedicated repeater-nodes. However, this result can be easily modified for any distribution of users and repeaters. One may define a set $\mathcal{I}$ of $N$-potential end-user pairs, i.e. valid end-to-end communicators

$$\mathcal{I} = \{i_1, i_2, \ldots, i_N\} = \{\{a_j, b_j\}\}_{j=1}^N.$$  

(19)
where \( \{ a_j, b_j \} \) are user-nodes within the \( j \)th end-user pair. It is then possible to derive a universal threshold condition that ensures the minimum-cut is the min-user neighbourhood for all pairs of users.

For a network in which all nodes are user-nodes, this theorem is readily translated. In this scenario, the distinction between a network-bulk and user-neighbourhoods vanishes since every node is part of a user-neighbourhood. Therefore, the derivation of a global threshold capacity translates into a maximum inter-nodal separation for all network edges \( d_{xy}^{\text{max}} \), which guarantees that the flooding capacity is always equal to the min-neighbourhood capacity. That is,

\[
d_{xy} \leq d_{N}^{\text{max}}, \forall (x, y) \in E \implies C^m(i, \mathcal{N}) = C^m_N(i), \forall i \in \mathcal{I}.
\]

We may then write the following corollary to Theorem 2 (see proof in Appendix B).

**Corollary 2** Consider an arbitrary quantum WRN \( \mathcal{N} \in \mathcal{U}_K \) which is connected by bosonic lossy channels, for which all network nodes are potential end-users. This generates a set of valid end-user pairs \( i \in \mathcal{I} \), and a distribution of min-neighbourhood capacities \( \{ C^m_N(i) \}_{i \in \mathcal{I}} \). Then there exists a maximum inter-nodal separation for all channels within the network,

\[
d_{N}^{\text{max}} = -\frac{1}{\gamma} \log_{10} \left( 1 - 2^{-\frac{2}{\gamma} \max_{i \in \mathcal{I}} C^m_N(i)} \right),
\]

for which the flooding capacity is equal to the min-neighbourhood capacity for all end-user pairs,

\[
C^m(i, \mathcal{N}) = C^m_N(i), \forall i \in \mathcal{I}.
\]

Otherwise, if \( \exists d_{xy} > d_{N}^{\text{max}}, (x, y) \in E \), then this remains an upper-bound on the optimal network performance, \( C^m(i, \mathcal{N}) \leq C^m_N(i), \forall i \in \mathcal{I} \).

This is an interesting consequence of the previous theorem which allows us to place a universal distance constraint on network edges. If this maximum inter-nodal separation is satisfied throughout the network, then optimal flooding performances are achievable for all potential end-user pairs. If not, the min-neighbourhood capacities will become potentially non-tight upper-bounds on the flooding rates of some end-user pairs, but may remain as much tighter bounds for many others. This is entirely dependent on where the network conditions are violated, and to what extent.

The high-resource cost of large quantum networks means that it is desirable to minimise the number of non-user nodes, and reduce the number of dedicated, static repeaters [22]. The insights offered by Theorem 2, Corollary 2 and variants thereof (discussed in the appendices) allow us to assess the ideal performance of global networks with respect to critical distance limitations.

### III. COMPARISON WITH SATELLITE QUANTUM COMMUNICATIONS

#### A. Satellite Quantum Communications

Here, we briefly review key results which facilitate a comparison of SQC with idealised, ground-based quantum networks. For more detailed derivations and discussions of these results, please refer to Refs. [24, 26].

Consider two users (Alice and Bob), who choose to communicate by means of an orbiting satellite (a dynamic repeater). Here we consider a ground station \( G \) at approximately sea-level, and a satellite \( S \) which is in orbit at an altitude \( h \geq 100 \) km and variable zenith angle \( \theta \). Given that the radius of the Earth is \( R_E \approx 6371 \) km, the slant distance between \( G \) and \( S \) is \( z(h, \theta) = \sqrt{h^2 + 2hR_E + R_E^2 \cos^2(\theta)} - R_E \cos(\theta) \), describing the true distance that an optical beam must travel from \( G \) to/from \( S \).

We may consider two unique configurations for information transmission: uplink, which refers to when \( G \) is the transmitter and \( S \) is the receiver, and downlink, where the converse is true. Both configurations will identically admit the effects of free-space diffraction (beam-spot widening) and atmospheric extinction (caused by molecular/aerosol absorption as the beam propagates). However additional loss/noise effects emerge with respect to uplink and downlink protocols, which invokes an asymmetry in their communication performance.

The effects of turbulence (caused by fluctuations in the atmospheric refractive index) and pointing errors (alignment of the optical signal with the receiver) are responsible for beam wandering, which instigates a fading process for the communication channels. For uplink protocols, turbulence is a significant factor for loss properties of the ground-satellite channel since it impacts the propagating beam immediately after transmission. However, pointing errors can be reduced thanks to the ability to easily access and optimise equipment at ground level. In downlink these effects are reversed; turbulence can be neglected while pointing errors are relevant due to limited onboard access and resources.

Considering each of these physical effects characterising the lossy free-space channel, it is possible to present an ultimate limit on the secret-key capacity \( K \) for SQC [26],

\[
K \leq -\Delta(\eta, \sigma) \log_2 (1 - \eta).
\]

Here \( \Delta(\eta, \sigma) \) is a correction factor to the PLOB bound, where \( \eta := \eta(h, \theta) \) is an effective transmissivity which is a function of geometric position, encompassing all the effects of diffraction, extinction, and optical imperfections/inefficiencies. Meanwhile, \( \sigma^2 = \sigma_{\text{ turb}}^2 + \sigma_{\text{ point}}^2 \) is the variance of the Gaussian random walk of the beam centroid caused by beam wandering, with contributions from turbulence and/or pointing-errors.

This bound can be further modified to account for the presence of thermal noise, which is highly dependent...
upon time of day (day/night-time) and weather conditions (cloudy/clear skies). For night-time communications, background noise is practically negligible, and the above bound requires little modification. However, for day-time operations this is generally not the case and the free-space lossy channels must be described as thermal-loss channels which account for additional noise.

While Eq. (23) represents an ultimate upper-bound on the capacity of a ground-to-satellite communication channel, it is important to provide an assessment of realistic and practical protocols which embody achievable lower-bounds for SQC. These lower-bounds will facilitate comparisons with global quantum networks, and help deduce the conditions for which we can expect satellite advantage for long-distance quantum communications. Here we summarise some achievable rates for different satellite configurations, utilising the pilot-guided protocol from Refs. [24, 26]. This is a coherent-state, Gaussian-modulated protocol in which signal-pulses are randomly interleaved with highly energetic pilot-pulses. These pilot-pulses are used to monitor the transmissivity of the communication channel and subsequently ascertain security.

We may consider the employment of such a protocol in conjunction with a near-polar sun-synchronous satellite used to communicate between two ground stations. This type of orbit ensures a consistent fly-over time for any point on the Earth’s surface, such that the satellite passes over any point at the same local mean solar time each day. This provides the possibility of stable conditions for satellite communications at around the same time each day. Let us assume that the stations lie along the orbital path such that the satellite crosses both of their zenith positions (which happens once per day). We further assume a worst-case scenario such that the stations only interact with the satellite when the zenith positions are crossed, and that both stations assume similar operational conditions.

It is possible to quantify the performance of satellite communications by considering a daily key rates, i.e. the number of secret-bits that may be shared per day. This allows us to utilise an average orbital rate $R_{\text{orb}}$ associated with up/downlink operations in day/night-time, representing an average secret-key rate per link usage. The number of secret-bits that can be shared in a zenith-crossing passage is then given by the effective transit time $t_Q(h)$ as a function of the altitude, and a typical clock frequency which we set as $\alpha = 10$ MHz. The average daily-rate in a given configuration is thus

$$R_{\text{day}}^{\text{rat}} \approx \alpha t_Q(h) R_{\text{orb}}^{i}, \quad (24)$$

for which $i$ labels the up/downlink and day/night-time. Importantly, thanks to the dynamic nature of SQC, these rates are constant with respect to ground-based end-to-end distances.

For downlink operations at altitude $h = 530$ km, initial beam-spot-size $\omega_0 = 40$ cm, receiver aperture $a_R = 1$ m, these setup parameters lead to the night-time/day-time rates [26],

$$R_{\text{down}}^{i} \approx \left\{ \begin{array}{ll} 3.066 \times 10^{-2} \text{\ bits/use (night)}, \\ 3.041 \times 10^{-2} \text{\ bits/use (day)}. \end{array} \right. \quad (25)$$

For uplink, we consider and altitude $h = 103$ km and similar setups (but now with a spot-size $\omega_0 = 60$ cm and wider aperture $a_R = 2$ m) leading to the rate,

$$R_{\text{up}}^{i} \approx \left\{ \begin{array}{ll} 4.244 \times 10^{-2} \text{\ bits/use (night)}, \\ 2.737 \times 10^{-2} \text{\ bits/use (day)}. \end{array} \right. \quad (26)$$

B. Comparison with Ground-Based Networks

The performance of ground-based fibre quantum networks is fundamentally limited by the length distribution of their constituent channels (inter-nodal separations) as derived in the previous sections. It is important to understand the limits of these networks for long-range communication and determine when SQC is superior. By idealising these networks as regular, highly connected structures, then we take a fundamental step in this study.

Let us consider a genuinely $(k, \lambda)$-regular quantum fibre-network $N \in U^k_\lambda$. As we have established in previous sections for large-scale WRNs (and similar to SQC) the end-to-end ground-distance is irrelevant. Network regularity successfully renders the flooding rate independent from this metric, which is instead critically dependent on the properties of connectivity and maximum inter-nodal separation. Here, we will consider global networks in which all nodes are potential user-nodes. This choice is for clarity, as it allows for the bounding of optimal performance with respect to universal network constraints. However, we remind the reader that our results apply to any distribution of user/repeater-nodes.

Hence, we proceed via the result of Corollary 2. For a network with end-user pairs $i \in I$ and a distribution of min-neighbourhood capacities $\{C_{N_{i}}^{m}\}_{i \in I}$, there exists a maximum inter-nodal separation $d_{\text{max}}^{N_{i}}$ for all network edges such that the flooding capacity is known exactly $C^{m}(i, N) = C_{N_{i}}^{m}$, for all $i \in I$. When this condition is violated, the min-neighbourhood capacities remain as upper-bounds on all flooding capacities. Furthermore, the threshold $d_{\text{max}}^{N_{i}}$ is intimately connected with the optimal min-neighbourhood capacity $C_{N_{i}}^{m}$ as per Eq. (21). Hence, given $t_{\text{day}} = 8.64 \times 10^4$ as the number of seconds in a day, and again assuming $\alpha = 10$ MHz, then the average number of secret-key bits per day satisfies [38]

$$R_{\text{day}}^{(k, \lambda)}(d_{\text{max}}^{N_{i}}) \lesssim \frac{\alpha t_{\text{day}} k}{\delta} \log_2(1 - 10^{-\gamma d_{\text{max}}^{N_{i}}}). \quad (27)$$

Repeater-chains can be considered in a similar manner. The repeater-chain capacity is equal to the single-edge
capacity associated with the longest inter-nodal separation in the chain. Hence, the average daily secret-key rate of a repeater-chain is [19]

\[ P_{\text{chain}}(d_N^{\text{max}}) \lesssim -\alpha \log_2(1 - 10^{-\gamma d_N^{\text{max}}}). \]  

(28)

In order to perform a quantitative comparison between satellite and ground-based quantum communications, we can compute the log-ratio between their daily-rates,

\[ \Delta K_{\text{day}} := 10 \log_{10} \left( \frac{R_{\text{day}}^{(k, \lambda)}}{P_{\text{max}}^{(\text{sat})}} \right), \]  

(29)

which determines a daily-rate advantage in decibels (dB). By studying the daily-rate advantage as a function of maximum inter-nodal separation \( \Delta K_{\text{day}}(d_N^{\text{max}}) \), we can then determine conditions for which SQC begin to outperform the global, ground-based networks. That is,

\[ \Delta K_{\text{day}} > 0 \implies \text{Fibre-Network Advantage}, \]
\[ \Delta K_{\text{day}} = 0 \implies \text{Equal Performance}, \]
\[ \Delta K_{\text{day}} < 0 \implies \text{Satellite Advantage}. \]

(30)

Hence, there exists some critical inter-nodal separation for which \( \Delta K_{\text{day}} = 0 \), beyond which a single, near-polar, sun-synchronous satellite is more beneficial than a global fibre-network,

\[ d_N^{\text{crit}} := \arg \min_{d_N^{\text{max}}} (|\Delta K_{\text{day}}|). \]  

(31)

Each architecture will possess a unique critical separation \( d_N^{\text{crit}} \), defining a limiting property of the network. Figure 4 illustrates results for the daily-rate advantage over SQC for a repeater-chain, and a number of quantum WRNs varying connectivity properties (symmetric network-cells shown in Fig. 3). We have considered a number of SQC setups and conditions, which are summarised in Table I in Fig. 4. In each plot, the critical inter-nodal separation \( d_N^{\text{crit}} \) has been identified as the \( d_N^{\text{max}} \) at which the single satellite in downlink is able to outperform each quantum WRN, under standard conditions.

The critical separation for a repeater-chain is 215 km, which offers a lower-bound on repeater-assisted, ground-based strategies. This can be extended by multi-path routing strategies and quantum networks, as WRNs are able to tolerate much longer lossy channels, at the expense of connectivity demands. Extending the critical separation by approximately 100 km requires a \( k = 16 \)-regular network, as shown above. To extend this critical separation \( d_N^{\text{crit}} \) even further to 400 km would require connectivity properties that satisfy \( k/\delta \approx 5000 \) [39].

It is important to appreciate that on a global scale, the resource demands for constructing a regular quantum network with a constant degree of even \( k = 16 \) are enormous, and is particularly true when the maximum inter-nodal separation places topological constraints. Furthermore, if such constraints are broken, a single satellite employing practical CVQKD can offer stronger performance over long distances. These results strongly suggest that a future quantum internet will significantly benefit from the use of SQC, and will be integral to the construction of global quantum communication networks.

IV. CONCLUSION

In this work, we have investigated the optimal performance of global, quantum communication networks to characterise the ultimate limits of a fibre-based quantum internet. This analysis is based on a underlying network architecture that exploits weak-regularity to construct powerful, highly-connected networks. In this way, we have been able to derive readily computable upper-bounds on the network flooding capacity of a very general class of ideal quantum network.

Crucially, these bounds allow us to benchmark the performance of a global quantum network versus that of a
single sun-synchronous satellite acting as a dynamic repeater. The result of this comparison emphasises the power of SQC, and vast network resources that are required to outperform a single satellite in orbit at global distances. These findings strongly motivate the utilisation of ground-satellite connections within large-scale quantum networks. It is clear that free-space ground-satellite links will be integral to long-range quantum communications, as their co-operation with ground-based infrastructure as global, dynamic repeaters will be invaluable.

This work introduces useful, analytical techniques for the study of ideal quantum networks which can be readily employed for future investigative paths. Indeed, the study of hybrid fibre/satellite networks is a topic of immediate interest; exploiting the power of SQC to enhance (rather than compete with) ground-based networks. Furthermore, the expansion of these methods to incorporate multiple satellites introduces the possibility of highly transmissive satellite-satellite channels at high altitudes.

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition, 10th ed. (Cambridge University Press, USA, 2011).
[2] J. Watrous, The Theory of Quantum Information (Cambridge University Press, 2018).
[3] A. S. Holevo, Quantum Systems, Channels, Information (De Gruyter, 2019).
[4] F. Arute et al., Nature 574, 505 (2019).
[5] H. J. Kimble, Nature 453, 1023 (2008).
[6] S. Pirandola and S. L. Braunstein, Nature 532, 169 (2016).
[7] M. Razzavi, An Introduction to Quantum Communications Networks, 2053-2571 (Morgan & Claypool Publishers, 2018).
[8] S. Pirandola et al., Advances in Optics and Photonics 12, 1012 (2020).
[9] A. Serafini, Quantum Continuous Variables: A Primer of Theoretical Methods (CRC Press, Taylor & Francis Group, 2017).
[10] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621 (2012).
[11] S. L. Braunstein and P. van Loock, Phys. Rev. A 61, 010303 (1999).
[12] S. Pirandola, R. Laurenza, C. Ottaviani, and L. Banchi, Nature Communications 8, 15043 (2017).
[13] S. Pirandola, R. García-Patrón, S. L. Braunstein, and S. Lloyd, Phys. Rev. Lett. 102, 050503 (2009).
[14] P. Slepian, Mathematical Foundations of Network Analysis (Springer-Verlag, New York, 1968).
[15] T. M. Cover and J. A. Thomas, Elements of Information Theory (Wiley, New Jersey, 2006).
[16] A. S. Tanenbaum and D. J. Wetherall, Computer Networks, 5th ed. (Pearson, 2010).
[17] A. El Gamal and Y.-H. Kim, Network Information Theory (Cambridge University Press, 2011).
[18] J. Biamonte, M. Faccin, and M. De Domenico, Communications Physics 2, 51 (2019).
[19] J. Biamonte, M. Faccin, and M. De Domenico, Communications Physics 2, 53 (2019).
[20] S. Brito, A. Canabarro, R. Chaves, and D. Cavalcanti, Phys. Rev. Lett. 124, 210501 (2020).
[21] Q. Zhuang and B. Zhang, arXiv:2011.07397 (2021).
[22] B. Zhang and Q. Zhuang, arXiv:2012.02241 (2020).
[23] S. Pirandola, Phys. Rev. Research 3, 013279 (2021).
[24] J. S. Sidhu et al., arXiv:2103.12749 (2021).
[25] S. Pirandola, arXiv:2012.01725 (2020).
[26] J.-G. Ren et al., Nature 549, 70–73 (2017).
[27] S.-K. Liao et al., Chinese Physics Letters 34, 090302 (2017).
[28] A. Villar et al., Optica 7, 734 (2020).
[29] R. Wilson, Introduction to Graph Theory (Longman, 1996).
[30] C. Bose, Pacific Journal of Mathematics 13, 389 (1963).
[31] Throughout this work we employ a superscript union notation to describe the repeated union of a single set, e.g. \( \{x\}^{\kappa} = \{x\} \cup \{x\} \cup \{x\} = \{x, x, x\} \). Without this constraint, one could propose an internally-WR, open-ended tree-graph with \( \lambda = 0 \) and \( \mu = \{0, 1\} \); this is an undesirable quantum network structure since there will exist only one unique path to any node.
[32] J. Edmonds and R. M. Karp, J. ACM 19, 248–264 (1972).
[33] J. B. Orlin, in Proceedings of the forty-fifth annual ACM symposium on theory of computing, STOC’13 (2013) pp. 765–774.
[34] This equates to a loss rate of 0.2 dB/km.
[35] This representation is easily obtained by rearranging Eq. (21) with respect to \( d_{\max} \).
[36] E.g. \( k \sim 70, \lambda \sim \{2\}^{\kappa} \). It is not obvious such an architecture is even possible.

Appendix A: Achievable Rates for Specific End-Users

1. Threshold Bulk Capacities

Lemma 1 Select two nodes on a genuinely-WR quantum network \( N = (P, E) \in \mathcal{W}_P^k \) that represent end-users, \( a, b \in P \), and demand they that they do not share an edge or neighbour. For any cut-set \( \tilde{C} \) that is restricted to edges in the network-bulk \( e \in E' \),

\[ \sum_{j=1}^{k} \lambda_j \leq k(k-2) \implies |\tilde{C}| \geq k. \]  

If \( \lambda_j = \lambda, \forall j \) then the condition holds if \( \lambda \leq k - 2 \).

Proof. For a genuine \((k, \lambda)\)-regular network the minimum-cut-set cardinality satisfies \( |C_{\min}| = k \). A cut-set limited to the network-bulk is unable to directly disconnect the neighbourhoods of \( a \) or \( b \) (\( N_a \) and \( N_b \) respectively). Then the next smallest cut-set will necessarily
cut the unique edges in the neighbourhoods of the \(a/b\)'s neighbouring nodes. That is, the cut-set will be either

\[
\tilde{C}_a = \bigcup_{x \in N_a} \{(x, y) \mid y \in N_x \setminus (N_a \cup N_b \cup \{a, b\})\}, \quad (A2)
\]
or

\[
\tilde{C}_b = \bigcup_{x \in N_b} \{(x, y) \mid y \in N_x \setminus (N_a \cup N_b \cup \{a, b\})\}. \quad (A3)
\]

Thanks to the network regularity, each of these neighbours possesses \((k - \lambda_j - 1)\)-edges (for \(j \in \{1, \ldots, k\}\)) that need to be cut in order to form a partition. The new cut-set cardinality is then

\[
|\tilde{C}_{a/b}| = \sum_{j=1}^k (k - \lambda_j - 1).
\]

as required. \(\blacksquare\)

We may provide a similar, but slightly weaker result for internally-WR networks:

**Lemma 2** Select two nodes on an internally-WR quantum network \(\mathcal{N} = (P, E, N)\) that represent end-users, \(a, b \in P\), and demand they that do not share an edge or neighbour. We further demand that these end-users are deeply embedded, such that we can ignore boundary effects. Then the result of Lemma 1 applies to \(\mathcal{N}\).

**Proof.** Open boundary edges \(E_{\text{bound}}\) add the complication of a potential cut \(C\) that utilises the boundary to find a smaller cut-set than that used in Lemma 1. However, if the end-user is deeply embedded within the network (there exist many links between the user and the boundary) then such cuts will always be larger than those restricted to the network body, \(E'\). This assumption allows us to apply the previous result directly. \(\blacksquare\)

With these lemmas in hand, we can now prove Theorem 1, and summarise the key mathematical tool used throughout this paper.

**Proof. (Theorem 1):**

Consider a network \(\mathcal{N} \in \mathcal{U}_n^k\) of \(n \gg k\) nodes. The network possesses a large set of valid cuts \(\mathcal{C}_{\mathcal{N}}\) which collect all the network cuts \(C\) that successfully partition end-users. Then, we may define an set of cut-set cardinalities, i.e. a set that collects the number of edges contained in the valid network-cuts,

\[
c_{\mathcal{N}} = \{|C_j| \mid j = 1, \ldots, |\mathcal{C}_{\mathcal{N}}|\}.
\]

The minimum-cut-set cardinality for the WRN \(\mathcal{N}\) is simply equal to the degree, i.e. \(c_{\mathcal{O}} = \min(c_{\mathcal{N}}) = k\), and can be achieved by isolating an end-user.

It follows that the minimum user-neighbourhood cut-set \(\tilde{C} = E_{\lambda}\) provides the multi-edge capacity \(C_{\lambda}^W\) defined in Eq. (9). Meanwhile, any cut \(C'\) and corresponding cut-set \(\tilde{C}'\) that strictly collects edges on the network-bulk \(\mathcal{N}'\) (cannot use end-user-neighbourhoods) provides a multi-edge capacity \(C_{\lambda}^W(C')\) defined in Eq. (10). In order to ensure \(C_{\lambda}^W\) is indeed the flooding capacity of the entire network, we must ensure that the minimum network-bulk based cut is *never* a minimum-cut, so that \(C_{\lambda}^W(C') \geq C_{\lambda}^W\), always.

When restricted to performing cuts only on the network-bulk, the cardinality distribution will be different from \(c_{\mathcal{N}}\), since now certain cuts are unavailable. Instead, we may define a new set of network-cuts \(\mathcal{C}_{\mathcal{N}'} \subset \mathcal{C}_{\mathcal{N}}\) which are restricted to the network-bulk. This generates an analogous set of cut-set cardinalities \(c_{\mathcal{N}'}\). Hence, we define the smallest network-bulk based cut on a \((k, \lambda)\)-regular network (with no boundary effects) \(C'\) as that which has cardinality,

\[
c_{\mathcal{N}'} = \min(c_{\mathcal{N}'}) = \sum_{j=1}^k (k - \lambda_j - 1).
\]

This corresponds to the minimum number of edges that must be cut from the neighbours of the *neighbours* of the minimum end-user (e.g. the green cut-set in Fig. 5). This generates \(\tilde{C}\) as the cut-set restricted to the network-bulk with minimum cardinality. Minimising \(C_{\lambda}^W(\tilde{C})\), we can impose

\[
c_{\mathcal{N}'} \cdot \min_{(x, y) \in \tilde{C}} C_{xy} \geq C_{\lambda}^W(\tilde{C}),
\]

and subsequently the minimum threshold network-bulk capacity satisfies,

\[
C_{\min} = \frac{C_{\lambda}^W(\tilde{C})}{\sum_{j=1}^k (k - \lambda_j - 1)} = \frac{\delta}{k} C_{\lambda}^W, \quad (A8)
\]

where we have used the definition of the cut-ratio in Eq. (13). Hence, provided \(C_{xy} \geq C_{min}, \forall (x, y) \in E'\)
then the minimum-cut is always the minimum user-neighbourhood $E_i$ as required. ■

Corollary 1 follows as a direct consequence of the previous theorem and lemmas.

Proof. (Corollary 1):
If the network regularity parameters satisfy
\[ \sum_{j=1}^{k} \lambda_j \leq k(k-2), \]  
(A9)
then via Lemma 1/2, $C_{N'}^i \geq k$. The user-neighbourhood multi-edge capacity can be upper-bounded using the maximum single-edge capacity in the minimum neighbour-

Therefore, given these connectivity properties, it follows that
\[ \frac{C_{\min}^i}{\max_{(x,y) \in E_i} C_{xy}} \leq \delta = \frac{k}{\sum_{j=1}^{k} k - \lambda_j - 1} \leq 1, \]
meaning that the minimum network-bulk threshold capacity is smaller than the maximum single-edge capacity in the minimum user-neighbourhood. ■

2. Bosonic Lossy Quantum WRNs

The direct application of Theorem 1 to bosonic lossy quantum networks allows us to translate the notion of threshold capacities into something more physical. Indeed, since the capacity of pure-loss channels is known exactly, it is possible to translate the threshold capacity into a maximum inter-nodal separation.

Proof. (Theorem 2):
Consider a valid pair of end-users $i = \{a, b\}$ embedded within a $(k, \lambda)$-regular quantum network $\mathcal{N} \in U_{k,\lambda}$. Then as before, there exists a threshold capacity $C_{\min}^i = \frac{\delta}{k} C_{N_i}^m$ that can be enforced to ensure the flooding capacity between these users is simply their min-neighbourhood capacity. Supplanting the PLOB bound into Theorem 1,
\[ C_{\mathcal{L}}(d_{xy}) \geq C_{\min}^i$, $\forall (x, y) \in E' \implies C_{\min}^m(i, \mathcal{N}) = C_{N_i}^m. \]
(A12)

With some algebra, this threshold condition readily translates to,
\[ d_{xy} \leq -\frac{1}{\gamma} \log_{10} \left(1 - 2^{-\frac{1}{2} \log_{10} C_{N_i}^m}\right), \forall (x, y) \in E'. \]  
(A13)
Therefore the threshold capacity becomes an upper-bound on the inter-nodal separation within the network-bulk. We can thus define a max-bulk separation,
\[ d_{N'}^{\max} = -\frac{1}{\gamma} \log_{10} \left(1 - 2^{-\frac{1}{2} \log_{10} C_{N_i}^m}\right), \]
(A14)
which when satisfied ensures that $C_{\min}^m(i, \mathcal{N}) = C_{N_i}^m$.

Now suppose that there exists channels within the network-bulk that violate this max-bulk separation, i.e. $\exists d_{xy} > d_{N'}^{\max}$ for $(x, y) \in E'$. This violates the threshold capacity condition from Theorem 1 meaning that the minimum-cut in the network is not guaranteed to be the min-neighbourhood cut. However, if the minimum-cut undergoes a transition due to the introduction of poor quality channels in the network-bulk, it cannot improve the network flooding capacity. It can only deteriorate network performance. Therefore the min-neighbourhood capacity remains an upper-bound on the optimal network performance, $C_{\min}^m(i, \mathcal{N}) \leq C_{N_i}^m$. ■

Appendix B: Achievable Rates for Multiple End-Users

It is possible to extend a threshold capacity that ensures that the flooding capacity for multiple different pairs of end-users will always be their min-neighbourhood capacity. That is, we can enforce a capacity distribution on the network-bulk that ensures the network is neighbour- 

Proof. (Corollary 2):
Any network-bulk, its set of nodes and edges are functions of the specifically chosen pair of end-users $i = \{a, b\}$,
\[ \mathcal{N}' = \mathcal{N}'(i), P' = P'(i), E' = E'(i). \]  
(B1)
Theorem 1 gives us a threshold condition for single-edge capacities on the network-bulk, relevant to specific end-users. It follows that the threshold capacity is of course a function of this end-user pair also
\[ C_{\min}^i = C_{\min}^m(i). \]  
(B2)

Suppose we want to derive a single threshold capacity which ensures that the minimum network-cut remains the min-neighbourhood cut for two different pairs of end-users $i_1 = \{a_1, b_1\}$ and $i_2 = \{a_2, b_2\}$. Without loss of generality we assume $i_1 \cap i_2 = \emptyset$ and that the user nodes do not share direct connections. Using Theorem 1 we can write their individual threshold capacities,
\[ C_{\min}^m(i_1) = \frac{\delta}{k} C_{N_{i_1}}^m, C_{\min}^m(i_2) = \frac{\delta}{k} C_{N_{i_2}}^m. \]
It is important to keep track of the network-bulks; $E'(i_1)$ will contain all edges $E \setminus (E_{a_1} \cup E_{b_1})$, while $E'(i_2)$ will contain all edges $E \setminus (E_{a_2} \cup E_{b_2})$. However, now the end-user neighbourhoods of $i_1$ will satisfy $E_{i_1} \subseteq E'(i_2)$ and similarly $E_{i_2} \subseteq E'(i_1)$, i.e. the neighbourhoods of an end-user pair will be found in the other pair’s network-bulk, and vice-versa.
By Theorem 1 this then leads to the following set of conditions. We can define a municipal network-bulk which describes all edges in the network that are not part of any end-user neighbourhood. In this case, the municipal network-bulk takes the form,

\[ E'(\mathcal{I}) = E \setminus (E_{a_1} \cup E_{a_2} \cup E_{b_1} \cup E_{b_2}). \] (B4)

Now all the edges in the municipal network-bulk must satisfy,

\[ C_{xy} \geq \max\{C'_{\min}(i_1), C'_{\min}(i_2)\}, \quad \forall (x, y) \in E'(i_1, i_2). \] (B5)

Meanwhile, all edges within the end-user neighbourhoods must satisfy the threshold set by the other end-user pair

\[ C_{xy} \geq C'_{\min}(i_1), \forall (x, y) \in (E_{a_2} \cup E_{b_2}), \]

\[ C_{xy} \geq C'_{\min}(i_2), \forall (x, y) \in (E_{a_1} \cup E_{b_1}). \] (B6)

It then follows that the flooding capacity for both pairs of end-users will simply be their min-neighbourhood capacity

\[ C^m(i_j, \mathcal{N}) = C^m_{\min}, \forall j \in \{1, 2\}. \] (B7)

This is the successful extension of Theorem 1 to 2 pairs of end-users. Extending to \( N \) possible pairs of end-users, we may collect these end-user pairs in the set \( \mathcal{I} = \{i_1, \ldots, i_N\} \). One can define a more general municipal network-bulk,

\[ E'(\mathcal{I}) = E \setminus \bigcup_{i \in \mathcal{I}} \bigcup_{p \in i} E_p, \] (B8)

which collects all edges that are not part of any end-user neighbourhood. There will now exist a universal condition that applies to a municipal network-bulk, captured by maximising over all threshold-capacities,

\[ C_{xy} \geq \max_{j \in [1, N]} C'_{\min}(i_j), \quad \forall (x, y) \in E'(\mathcal{I}), \] (B9)

and a condition for all edges within the end-user neighbourhoods,

\[ C_{xy} \geq \max_{j \neq i \in [1, N]} C'_{\min}(i_j), \quad \forall (x, y) \in \bigcup_{p \in i} E_p, \quad i \in [1, N], \] (B10)

such that the flooding capacity for all pairs of end-users \( i \in \mathcal{I} \) will simply be their min-neighbourhood capacity,

\[ C^m(i_j, \mathcal{N}) = C^m_{\min}, \forall j \in [1, N]. \] (B11)

These are the most general threshold conditions that we can offer. In essence, this method allows us to place unique restrictions on user-connected and non-user connected edges.

Finally, this can be extended to the case where all nodes in the network are considered as potential user nodes. In this case, \( \mathcal{I} \) is the set of all possible end-user pairs in the network. If we were to inspect the municipal network-bulk, \( E'(\mathcal{I}) = \emptyset \). Hence there will only exist one set of network capacity conditions described generally by Eq. (B10). However, this condition can be readily simplified since all nodes are user-nodes: Now, it is sufficient to ask that all single-edge capacities satisfy the threshold set by the maximum threshold capacity out of all possible end-user pairs anywhere in the network,

\[ C_{xy} \geq \max_{i \in \mathcal{I}} C'_{\min}(i), \forall (x, y) \in E. \] (B12)

If this condition is satisfied, the flooding capacity for all pairs of end-users will simply be their min-neighbourhood capacity,

\[ C^m(i, \mathcal{N}) = C^m_{\min}, \forall i \in \mathcal{I}. \] (B13)

Combining this result with Theorem 2 we arrive at the desired result. 

**Appendix C: Neighbour Sharing End-Users**

Throughout this work we consider end-user pairs that are not directly connected and do not share common neighbours. This is appropriate assumption since we are studying global quantum communications over very long distances; it is not interesting to consider short range users separated by single links. Furthermore, it allows for much clearer intuition surrounding increasing cut-set dimension with respect to cuts on the network-bulk as shown in Fig. 5. This assumption does not compromise the generality of our arguments, and only a minor detail is required to ensure that short-range end-user pairs can be studied.

Let us consider a \((k, \lambda)\)-regular network \( \mathcal{N} \in \cup \lambda^k \), and assume an end-user pair \( i = \{a, b\} \) which are not directly connected but share a neighbour. The number of common neighbours that these non-adjacent nodes share is \( \mu(a, b) > 0 \). The previous analyses from Appendix A do not directly apply, since cuts restricted to the network-bulk will not be able to partition the two users. This is true because there will exist clear paths along the edges connected to the common neighbours of \( a \) and \( b \). Hence, a valid network-cut of these end-users requires one to collect \( \mu(a, b) \) edges from a user-neighbourhood.

Hence, we must loosen our procedure, and locate a network-cut that uses the minimum number of user-connected edges possible. This cut \( C' \) still collects \( \sum_{j=1}^{k}(k - \lambda_j - 1) \) edges, but now \( \mu(a, b) \) of those edges are actually contained in one of the user-neighbourhoods. In a worst-case scenario, one may assume that these user-connected edges which are necessarily cut, possess the minimum single-edge capacity in the user-
neighbourhoods, defined as
\[
\begin{align*}
C_{\min}^N_i & := \min_{p \in i} \min_{(x,y) \in E_p} C_{xy}. \quad (C1)
\end{align*}
\]
Assuming that all edges in the network-bulk obey some threshold capacity \( C_{\min}'' \), the next smallest cut that collects the fewest number of user-neighbourhood edges will generate a multi-edge capacity,
\[
C_{\min}^N(C'') \geq \frac{k}{\delta} C_{\min}' + \mu(a, b)(C_{\min}^N - C_{\min}'). \quad (C2)
\]
In order for the min-cut to remain as the min-user neighbourhood, we must find that \( C_{\min}^N(C'') \geq C_{\min}^N \) always. It is clear that asserting \( C_{\min}^N = C_{\min}' \) is sufficient in order to retrieve Theorem 1.
Hence, if the single-edge user-neighbourhood capacities also satisfies the bulk threshold capacity then one can retrieve Theorem 1 for neighbour-sharing end-user pairs. Clearly Corollary 2 (when all network nodes are potential user-nodes) satisfies the above condition, meaning that one can consider neighbour-sharing end-user pairs. A similar analysis can be performed for edge-sharing end-user pairs, but as previously stated this is not overly interesting for long distance communications and need not be explicitly derived.