Higher dimensional dilaton black holes in the presence of exponential nonlinear electrodynamics

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We examine the higher dimensional action in which gravity is coupled to the exponential nonlinear electrodynamic and a scalar dilaton field. We construct a new class of \(n\)-dimensional static and spherically symmetric black hole solutions of this theory in the presence of the dilaton potential with two Liouville-type terms. In the presence of two Liouville-type dilaton potential, the asymptotic behavior of the obtained black holes are neither flat nor (A)dS. Due to the nonlinear nature of electrodynamic field, the electric field has finite value near the origin where \(r \to 0\) and goes to zero as \(r \to \infty\). Interestingly enough, we find that in the absence of the dilaton field, the electric field has a finite value at \(r = 0\), while as soon as the dilaton field is taken into account, the electric field diverges as \(r \to 0\). This implies that the presence of the dilaton field changes the behaviour of the electric field near the origin. In the limiting case where the nonlinear parameter \(\beta\) goes to infinity, our solutions reduce to dilaton black holes of Einstein-Maxwell-dilaton gravity in higher dimensions. We compute the conserved and thermodynamic quantities of the solutions and show that these quantities satisfy the first law of black holes thermodynamics on the horizon.

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I. INTRODUCTION

There are several motivations for studying black hole solutions in higher dimensional spacetimes. The first reason comes from string theory which contains gravity and requires more than four dimensions. In fact, the first successful statistical counting of black hole entropy in string theory was performed for a five dimensional black hole \([1]\). This example provides the best laboratory for the microscopic string theory of black holes. Another motivation originates from the AdS/CFT correspondence which relates the properties of an \(n\)-dimensional black hole with those of a quantum field theory in \((n - 1)\)-dimensions \([2]\). Besides, the production of higher-dimensional black holes in future colliders becomes a conceivable possibility in scenarios involving large extra dimensions and TeV-scale gravity \([3]\). In addition, as mathematical objects, black hole spacetimes are among the most important Lorentzian Ricci-flat manifolds in any dimension \([4]\).

In this paper, we turn to the investigation on higher dimensional black holes in the presence of nonlinear electrodynamics and dilaton field. The theory of nonlinear electrodynamics was first introduced in 1930’s by Born and Infeld to obtain a classical theory of charged particles with finite self-energy \([5]\). Born-Infeld (BI) theory has received renewed attentions since it turns out to play an important role in string theory. It arises naturally in open superstrings and D-branes \([6, 7]\). The low energy effective action for an open superstring in loop calculations lead to BI type actions. The BI action including a dilaton field and an axion field, appears in the coupling of an open superstring and an Abelian gauge field theory \([6]\). This action, describing a Born-Infeld-dilaton-axion system coupled to Einstein gravity, can be considered as a nonlinear extension in the Abelian field of Einstein-Maxwell-dilaton-axion gravity. Nonlinear BI theory in the context of dilaton gravity have been investigated by many authors. See for e.g \([8–17]\) and references therein. In the framework of Einstein-Maxwell-dilaton (EMd) gravity with a Liouville type dilaton potential non-degenerate and degenerate (extremal) Killing horizons of arbitrary geometry and topology were investigated \([18]\). Here we would like to consider another type of nonlinear electrodynamics, namely exponential form of the nonlinear electrodynamics in the setup of dilaton gravity. The lagrangian of the exponential nonlinear (EN) electrodynamics is given by \([19]\),

\[
L_{EN} = 4\beta^2 \left[ \exp \left( \frac{F^2}{4\beta^2} \right) - 1 \right],
\]

where \(\beta\) is called the nonlinear parameter with dimension of mass, \(F^2 = F_{\mu\nu} F^{\mu\nu}\), where \(F_{\mu\nu}\) is the electromagnetic field tensor. The advantages of the EN Lagrangian compared to BI nonlinear electrodynamics is that, it does not cancel, at least for some parameters, the divergency of the electric field at \(r = 0\), however, its singularity is much weaker than Einstein-Maxwell theory \([20]\). In our previous paper \([21]\), we studied dilaton black holes coupled to EN electrodynamics in four dimensional spacetime. In the present work, we would like to extend our study to all
higher dimensions and construct a new class of n-dimensional black holes in dilaton gravity which is coupled to nonlinear matter field. We shall also investigate the physical properties of the spacetime and obtain conserved and thermodynamic quantities of the solutions.

This paper is organized as follows. In the next section, we introduce the Lagrangian of EN electrodynamics in n-dimensions coupled to the dilaton field in Einstein gravity, and obtain the corresponding field equations by varying the action. In section III, we find a new class of static and spherically symmetric black hole solutions of this theory and investigate their properties. In section IV, we study thermodynamics of higher dimensional dilaton black holes in the presence of nonlinear electrodynamics. The last section is devoted to conclusion and discussion.

II. BASIC FIELD EQUATIONS

We consider the n-dimensional (n ≥ 4) action in which gravity is coupled to dilaton and nonlinear electrodynamic fields

\[ S = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-2} (\nabla \Phi)^2 - V(\Phi) + L(F, \Phi) \right), \quad (2) \]

where \( \mathcal{R} \) and \( \Phi \) are, respectively, the Ricci scalar curvature and the dilaton field, and \( V(\Phi) \) is a potential for \( \Phi \). We further assume the dilaton potential contain two Liouville terms,

\[ V(\Phi) = 2\Lambda_0 e^{2\zeta \Phi} + 2\Delta e^{2\zeta \Phi}, \quad (3) \]

where \( \Lambda_0 \), \( \Lambda \), \( \zeta_0 \) and \( \zeta \) are constants. This kind of potential was previously investigated in the context of BI-dilaton (BId) black holes [15, 16] as well as EMD gravity [22–27].

We choose the Lagrangian of the exponential nonlinear electrodynamics coupled to the dilaton field (ENd) in n-dimensions as

\[ L(F, \Phi) = 4\beta^2 e^{4\alpha \Phi/(n-2)} \left[ \exp \left( -\frac{e^{-8\alpha \Phi/(n-2)} F^2}{4\beta^2} \right) - 1 \right], \quad (4) \]

where \( \alpha \) is a constant determining the strength of coupling of the scalar and electromagnetic field. In order to justify such a choice for the Lagrangian of ENd field, let us invoke the BId Lagrangian in n-dimensions which is written as [15]

\[ L_{\text{BId}}(F, \Phi) = 4\beta^2 e^{4\alpha \Phi/(n-2)} \left( 1 - \sqrt{1 + \frac{e^{-8\alpha \Phi/(n-2)} F^2}{2\beta^2}} \right). \quad (5) \]

It is worthy to note that Lagrangian (5) originates from open string version of the BI action coupled to a dilaton field and only valid for the pure electric case [8]. Clearly, this version of the BId action does not enjoy electric-magnetic duality [12]. This form for the BId term have been investigated previously by a number of authors [8–17]. It is easy to check that the series expansion of both BId Lagrangian (5) and ENd Lagrangian (6), for large \( \beta \), have the same behavior

\[ L_{\text{BId}}(F, \Phi) = -e^{-4\alpha \Phi/(n-2)} F^2 + e^{-12\alpha \Phi/(n-2)} F^4 + e^{-20\alpha \Phi/(n-2)} F^6 + O \left( \frac{1}{\beta^6} \right), \quad (6) \]

\[ L(F, \Phi) = -e^{-4\alpha \Phi/(n-2)} F^2 + e^{-12\alpha \Phi/(n-2)} F^4 - e^{-20\alpha \Phi/(n-2)} F^6 + O \left( \frac{1}{\beta^6} \right). \quad (7) \]

This similarity implies that one can either consider (6) or (5) as the nonlinear electrodynamics Lagrangian coupled to the dilaton field. Here we would like to study the new ENd Lagrangian (6), and investigate the effects of this kind of nonlinear electrodynamics coupled to the dilaton field on the behavior of the solutions. In the absence of the dilaton field (\( \alpha = 0 \)) and in four dimensions where \( n = 4 \), \( L(F, \Phi) \) reduces to EN electrodynamics Lagrangian presented in [20]. On the other hand, in the limiting case where \( \beta \to \infty \), both \( L(F, \Phi) \) and \( L_{\text{BId}}(F, \Phi) \) recovers the standard linear Maxwell lagrangian coupled to the dilaton field in n-dimensions [22]

\[ L(F, \Phi) = L_{\text{BId}}(F, \Phi) = -e^{-4\alpha \Phi/(n-2)} F^2. \quad (8) \]
This is an expected result, since in this case the nonlinear electrodynamics reduces to the linear Maxwell electrodynamics. For latter convenience we rewrite

\[ L(F, \Phi) = 4\beta^2 e^{4\alpha \Phi/(n-2)} \mathcal{L}(Y), \]  

where

\[ \mathcal{L}(Y) = \exp(-Y) - 1, \]
\[ Y = \frac{e^{-8\alpha \Phi/(n-2)} F^2}{4\beta^2}. \]

By varying action (2) with respect to the gravitational field \( g_{\mu\nu} \), the dilaton field \( \Phi \) and the gauge field \( A_\mu \) we obtain the field equations as

\[ R_{\mu\nu} = \frac{4}{n-2} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) - 2e^{-4\alpha \Phi/(n-2)} \partial_\eta \mathcal{L}(Y) F_{\mu\eta} F^{\nu\eta} \]
\[ + \frac{4\beta^2}{n-2} e^{4\alpha \Phi/(n-2)} \left[ 2Y \partial_\eta \mathcal{L}(Y) - \mathcal{L}(Y) \right] g_{\mu\nu}, \]  

\[ \nabla^2 \Phi = \frac{n-2}{8} \frac{\partial V}{\partial \Phi} + 2\alpha \beta^2 \frac{e^{4\alpha \Phi/(n-2)}}{(n-2)} \left[ 2Y \partial_\eta \mathcal{L}(Y) - \mathcal{L}(Y) \right], \]
\[ \nabla_\mu \left( e^{-4\alpha \Phi/(n-2)} \partial_\eta \mathcal{L}(Y) F^{\mu\nu} \right) = 0. \]

When \( \beta \to \infty \), we have \( \mathcal{L}(Y) = -Y \), and the system of field equations (12)-(14) restore the well-known equations of EMD gravity [23–27].

### III. HIGHER DIMENSIONAL DILATON BLACK HOLE

In this section, we would like to find static and spherically symmetric solutions of the field equations (12)-(14). The metric of such a spacetime can be written

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r)d\Omega^2_{n-2}, \]

where \( d\Omega^2_{n-2} \) denotes the metric of an unit \((n-2)\)-sphere and \( f(r) \) and \( R(r) \) are functions of \( r \) which should be determined. Integrating the electromagnetic field equation (14) and assuming all components of \( F_{\mu\nu} \) are zero except \( F_{tr} \), we find

\[ F_{tr} = q e^{4\alpha \Phi/(n-2)} \frac{(rR)^{n-2}}{(rR)^{2n-4}} \exp \left[ -\frac{1}{2} L_W \left( \frac{q^2}{\beta^2 (rR)^{2n-4}} \right) \right], \]

where \( q \) is an integration constant which is related to the electric charge of the black hole. We recall that in BId theory the electric field can be written as [13]

\[ F_{tr} = q e^{4\alpha \Phi/(n-2)} \frac{(rR)^{n-2}}{(rR)^{2n-4}} \sqrt{1 + \frac{2q^2}{\beta^2 (n-2) (rR)^{2n-4}}}. \]

Using the Gauss’s law, \( Q = \frac{1}{4\pi} \int \exp[-4\alpha \Phi/(n-2)]^* Fd\Omega, \) we can calculate the flux of the electromagnetic field at infinity to obtain the electric charge of black hole as

\[ Q = \frac{q \omega_{n-2}}{4\pi}, \]

where \( \omega_{n-2} \) represents the volume of an unit \((n-2)\)-sphere. Note that in (16), \( L_W(x) = LambertW(x) \) is the Lambert function which satisfies the identity [28]

\[ L_W(x)e^{L_W(x)} = x, \]
and has the following series expansion
\[ L_W(x) = x - x^2 + \frac{3}{2} x^3 - \frac{8}{3} x^4 + \ldots \] (20)

Clearly, series (20) converges for \( |x| < 1 \). In the absence of the nonlinear dilaton field (\( \alpha = 0 \)) we have \( R(r) = 1 \), and Eq. (16) reduces to
\[ F_{tr} = \frac{q}{r^{n-2}} \exp \left[ -\frac{1}{2} L_W \left( \frac{q^2}{\beta^2 r^{2n-4}} \right) \right], \] (21)

while in the limiting case where \( \beta \to \infty \), it reduces to the electric field of \( n \)-dimensional EMd black holes [27]
\[ F_{tr} = \frac{g e^{4\alpha \phi/(n-2)}}{(r R)^{n-2}} + O \left( \frac{1}{\beta^2} \right). \] (22)

In order to solve the system of equations (12) and (13) for three unknown functions \( f(r) \), \( R(r) \) and \( \Phi(r) \), we make the ansatz [27]
\[ R(r) = e^{2\alpha \phi/(n-2)}. \] (23)

It is important to note that in the absence of the nontrivial dilaton field (\( \alpha = 0 \)), we have \( R(r) = 1 \), as one expected (see Eq. 15). Substituting (22), the electromagnetic field (16) and the metric (15) into the field equations (12) and (13), one can obtain the following solutions
\[ f(r) = -\frac{(n - 3)(\alpha^2 + 1)^2}{(\alpha^2 + n - 3)(\alpha^2 - 1)} b^{-\gamma r^{-\gamma}} - \frac{m}{r^{n-3-(n-2)\gamma/2}} + \frac{2(\alpha^2 + 1)^2(\Lambda + 2\beta^2) b^{-\gamma} r^{-\gamma}}{(n-2)(\alpha^2 - n + 1)} \]
\[ - \frac{4\beta q (\alpha^2 + 1) b^{(4-n)\gamma/2}}{(n-2)r^{n-3-(n-2)\gamma/2}} \int r^{-\gamma} \left( \sqrt{L_W(\eta)} - \frac{1}{\sqrt{L_W(\eta)}} \right) dr, \] (24)

\[ \Phi(r) = \frac{(n - 2)\alpha}{2(\alpha^2 + 1)} \ln \left( \frac{b}{r} \right), \] (25)

where \( b \) is an arbitrary constant, \( \gamma = 2\alpha^2/(1 + \alpha^2) \), and
\[ \eta \equiv \frac{q^2 b^{(2-n)\gamma}}{\beta^2 r^{(n-2)(2-\gamma)}}. \] (26)

In the above expression, \( m \) appears as an integration constant and is related to the mass of the black hole. The obtained solutions fully satisfy the system of equations (12) and (13) provided we take
\[ \zeta_0 = \frac{2}{\alpha(n-2)}, \quad \zeta = \frac{2\alpha}{n-2}, \quad \Lambda_0 = \frac{(n-2)(n-3)\alpha^2}{2b^2(\alpha^2 - 1)}. \] (27)

Notice that here \( \Lambda \) remains as a free parameter which plays the role of the cosmological constant. For later convenience, we redefine it as \( \Lambda = -(n-1)(n-2)/2l^2 \), where \( l \) is a constant with dimension of length. The integration of Eq. (21) can be performed using the Mathematica software. The resulting solution can be written
\[ f(r) = -\frac{(n - 3)(\alpha^2 + 1)^2}{(\alpha^2 - 1)(\alpha^2 + n - 3)} b^{-\gamma r^{-\gamma}} - \frac{m}{r^{n-3-(n-2)\gamma/2}} + \frac{2(\Lambda + 2\beta^2)(\alpha^2 + 1)^2 b^{-\gamma} r^{-\gamma}}{(n-2)(\alpha^2 - n + 1)} \]
\[ + \frac{2\beta q (\alpha^2 + 1) b^{(4-n)\gamma/2}}{(n-2)^2(\alpha^2 - 1)^2} \left( \frac{\beta^2 b^{(n-2)\gamma/2}}{q^2} \right)^{\frac{1}{(n-2)\gamma/2-n+3}} \int \frac{1 - \alpha^2}{2n-4} \left\{ - (n-2)^2(\gamma - 2)^2 \left[ \Gamma \left( \frac{\alpha^2 + 3n - 7}{2n-4}, \frac{1 - \alpha^2}{2n-4} L_W(\eta) \right) - \Gamma \left( \frac{\alpha^2 + 3n - 7}{2n-4} \right) \right] \right\}, \] (28)
where $\Gamma(a, z)$ and $\Gamma(a)$ are Gamma functions and they are related to each other via,

$$\Gamma(a, z) = \Gamma(a) - \frac{z^a}{a} F(a, 1 + a, -z).$$  \hspace{1cm} (29)

where $F(a, b, z)$ is hypergeometric function \cite{28}. Using (29), solution (28) can also be reexpressed in terms of hypergeometric function,

$$f(r) = -\frac{(n-3)(a^2 + 1)^2}{(a^2 + n - 3)(a^2 - 1)} b^{-\gamma} r^{-\gamma - 1} \frac{m}{r^{n-3-(n-2)\gamma/2}} + \frac{2\Delta(a^2 + 1)^2}{(n-2)(a^2 - n + 1)} b^{\gamma} r^{-2\gamma} +$$

$$+ \frac{4\beta q(a^2 + 1)^2 b^{(4-n)\gamma/2}}{(n-2) r^{n-3-(n-2)\gamma/2}} \left( \frac{a}{q^2} \right)^{\frac{4(n-2)\gamma}{(n-2)(n-3)\gamma}} L_W^{2n+1}(n)$$

$$\times \left\{ \frac{L_W^2(n)}{a^2 + 3n - 7} \left( \frac{\alpha^2 + 3n - 7}{2n - 4}, \frac{\alpha^2 + 5n - 11}{2n - 4}, \frac{\alpha^2 - 1}{2n - 4} L_W(n) \right) \right\}.$$  \hspace{1cm} (30)

Using the fact that $L_W(x)$ has a convergent series expansion for $|x| < 1$ as given in \cite{20}, we can expand (28) for large $\beta$. The result is

$$f(r) = -\frac{(n-3)(a^2 + 1)^2}{(a^2 + n - 3)(a^2 - 1)} b^{-\gamma} r^{-\gamma - 1} \frac{m}{r^{n-3-(n-2)\gamma/2}} + \frac{2\Delta(a^2 + 1)^2}{(n-2)(a^2 - n + 1)} b^{\gamma} r^{-2\gamma} +$$

$$+ \frac{2q^2(a^2 + 1)^2 b^{-(n-3)\gamma}}{(n-2)(a^2 + n - 3)r^{n-3-(n-3)\gamma} + 2n-6} - \frac{q^4(a^2 + 1)^2 b^{-(2n-5)\gamma}}{2\beta^2(n-2)(3n-7)r^{2n-6}} + \frac{1}{2\beta^2(n-2)(3n-7)r^{2n-6}} + \frac{q^4}{2\beta^2(n-2)(3n-7)r^{2n-6}} + O \left( \frac{1}{\beta^4} \right).$$  \hspace{1cm} (31)

When $\beta \to \infty$, solution (31) is exactly the one obtained for higher dimensional black holes in EMd gravity \cite{27}. This is an expected result, since as we discussed already in this limit the Lagrangian of ENd theory reduces to EMd gravity. In the absence of the dilaton field $(\alpha = 0 = \gamma)$, solution (31) can be further simplified as

$$f(r) = 1 - \frac{m}{r^{n-3}} + \frac{q^2}{r^{2n-6}} + \frac{2q^2}{(n-2)(n-3)r^{2n-6}} - \frac{1}{2\beta^2(n-2)(3n-7)r^{2n-6}} + O \left( \frac{1}{\beta^4} \right),$$  \hspace{1cm} (32)

which has the form of static spherically symmetric $n$-dimensional RN-AdS black holes in the limit $\beta \to \infty$. The last term in the right hand side of (32) is the leading nonlinear correction term to the RN-AdS black hole in the large $\beta$ limit.

**Physical properties of the solutions**

Now we back to the electric field obtained in (16). In order to study the behaviour of the electric field, we combine Eqs. (24) and (26) with (16). We find

$$F_{tr} = E(r) = \frac{q b^{(4-n)\gamma/2}}{r^{n-2-2\gamma-n/2}} \exp \left[ -\frac{1}{2} L_W \left( \frac{b^{(n-2)\gamma}}{r^{2(2\gamma-2)}(n-2)} \right) \right].$$  \hspace{1cm} (33)

Expanding for large $\beta$, we arrive at

$$E(r) = \frac{q b^{(4-n)\gamma/2}}{r^{n-2+(4-n)\gamma/2}} - \frac{q^3}{2\beta^2} b^{(4-3n/2)} - \frac{5q^5}{8\beta^4} b^{(6-5n/2)} + O \left( \frac{1}{\beta^6} \right).$$  \hspace{1cm} (34)

In order to analyze the behaviour of $F_{tr}$, we choose $b = 1$, $n = 6$ and $\alpha = \sqrt{2}$ $(\gamma = 4/3)$. For these values of the parameters we have

$$E(r) = \frac{q}{r^{3/3}} - \frac{q^3}{2\beta^2 r^{10/3}} + \frac{5q^5}{8\beta^4} + O \left( \frac{1}{\beta^6} \right).$$  \hspace{1cm} (35)

From Eq. (35) we see that in the presence of the dilaton field, the electric field diverges as $r \to 0$. For more details see Table A.
In order to have better understanding of the behavior of the electric field, we plot $E(r)$ versus $r$ for different values of the parameters in figures 1-4. From these figures and table A we see that the electric field goes to zero for large $r$ independent of the value of the other parameters. Figure 1 shows that for ENd black holes, and in the absence of the dilaton field ($\alpha = 0$), the electric field has a finite value near $r = 0$, while as soon as the dilaton field is taken into account ($\alpha > 0$), the electric field diverges as $r \to 0$. This implies that the presence of the dilaton field changes the behavior of the electric field near the origin. The behavior of the electric fields for ENd black holes in different dimensions are shown in figure 2. From this figure we see that in any dimension as $r \to 0$, the electric field goes to infinity. In figures 3 we have compared the behavior of $E(r)$ for BId, ENd and EMd black holes. From this figure we see that for ENd case the electric field has a finite value near the origin, while it diverges exactly at $r = 0$, however its singularity is weaker than EMd. This is in contrast to the BId electrodynamics which the electric field has finite value at $r = 0$. Finally, we have plotted in figure 4 the electric field of ENd black holes for different values of the nonlinear parameter $\beta$. From this figure we see that with increasing $\beta$, the electric field diverges as $r \to 0$. This is an expected result, since for large $\beta$ our theory reduces to the well-known EMd gravity [27].

| $r$ | $10^3$ | $10^2$ | $10^1$ | $10^{-1}$ | $10^{-10}$ | $10^{-1000}$ |
|-----|--------|--------|--------|-----------|------------|------------|
| $\alpha = 0.0$ | $10^{-11}$ | $10^{-7}$ | $10^{-4}$ | $10$ | $40$ | $400$ |
| $\alpha = 0.4$ | $10^{-10}$ | $10^{-7}$ | $10^{-4}$ | $20$ | $10^4$ | $10^{300}$ |
| $\alpha = 0.8$ | $10^{-9}$ | $10^{-6}$ | $10^{-3}$ | $50$ | $10^{10}$ | $10^{800}$ |

Table A: $E(r)$ for ENd versus $r$ for $\beta = 3$, $n = 6$, $q = 2$ and $b = 1$ and different values of $\alpha$. 
The next step is to investigate the causal structure of the solutions and check whether there is or not the curvature singularities and horizons. We find that Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges as $r \to 0$. This implies that our spacetime has an essential singularity located at $r = 0$.

In order to study the asymptotic behaviour of the solutions, we expand the metric function $f(r)$ for $r \to \infty$ limit. We find

$$
\lim_{r \to \infty} f(r) = -\frac{(n-3)(\alpha^2+1)^2}{(\alpha^2+n-3)(\alpha^2-1)} h^{-\gamma} r^\gamma + \frac{2\Lambda(\alpha^2+1)^2}{(n-2)(\alpha^2-n+1)} h^\gamma r^{2-\gamma}.
$$

Let us note that in the absence of the dilaton field ($\alpha = 0 = \gamma$), the metric function becomes

$$
\lim_{r \to \infty} f(r) = 1 - \frac{2\Lambda^2}{(n-1)(n-2)},
$$

which describes an asymptotically flat ($\Lambda = 0$), AdS ($\Lambda < 0$) or dS ($\Lambda > 0$) spacetimes. However, as one can see from Eq. (36), in the presence of the dilaton field the asymptotic behaviour is neither flat nor (A)dS. For example, taking $\alpha = \sqrt{2}$, $n = 6$ and $b = 1$, we have

$$
\lim_{r \to \infty} f(r) = -\frac{27}{5} r^{4/3} - \frac{3\Lambda}{2} r^{2/3}.
$$
Clearly, the metric function (38) is neither flat nor (A)dS. Indeed, it has been shown that no dilaton dS or AdS black hole solution exists with the presence of only one or two Liouville-type dilaton potential [22]. In the presence of one or two Liouville-type potential, black hole spacetimes which are neither asymptotically flat nor (A)dS have been explored by many authors (see e.g. [23–27]). It is important to note that this asymptotic behaviour is not due to the nonlinear nature of the electrodynamic field, since as \( r \to \infty \) the effects of the nonlinearity disappear. This is due to the fact that, \( r \to \infty \) limit corresponds to \( \beta^2 \to \infty \), and in this case \( F_r \) as well as the metric functions \( f(r) \) restore the result of EMD with unusual asymptotic [27].

Furthermore, from the dilaton field (25) we see that as \( r \to \infty \), the dilaton field does not vanishes, while in case of asymptotic flat or (A)dS we expect to have \( \lim_{r \to \infty} \Phi(r) = 0 \). Indeed, by solving the field equation (13) we find

\[
\Phi(r) = \left( \frac{n-2}{2(\alpha^2+1)} \right) \ln \left( \frac{a+b}{r} \right),
\]

however, the system of equation (12)–(14) will be fully satisfied provided we choose \( a = 0 \). From the above arguments we conclude that the asymptotic behaviour of the obtained solutions is neither flat nor (A)dS.

It is also worthwhile to note that in case of \( \alpha = \sqrt{n-1} \) and \( \alpha = 1 \) the solutions are ill-defined as one can see from Eq. (25). In order to obtain the location of the horizons of spacetime, we have to find the roots of \( f(r_+) = 0 \). However, due to the complexity of \( f(r) \) given in (25), it is not possible to find the roots of \( f(r+) = 0 \), analytically. Nevertheless, we can plot the function \( f(r) \) versus \( r \) for different model parameters as in figures 5 and 6. For simplicity, in these figures, we kept fixed the other parameters \( b = l = 1 \). Figure 5 shows that the obtained solutions may represent a black hole with two horizons, an extreme black hole or a naked singularity depending on the metric parameters. It also shows that for fixed value of the other parameters, the number of horizons decreases with increasing \( \alpha \). On the other hand from figure 6 we see that for fixed value of \( m \), \( \alpha \) and \( q \), there is a minimum (\( \beta_{\text{min}} \)) and extreme (\( \beta_{\text{ext}} \)) value for the nonlinear parameter for which we have black hole with a non-extreme horizon provided \( \beta \leq \beta_{\text{min}} \), black hole with two horizons for \( \beta_{\text{min}} < \beta < \beta_{\text{ext}} \), black hole with an extreme horizon for \( \beta = \beta_{\text{ext}} \) and naked singularity for \( \beta > \beta_{\text{ext}} \). Clearly, \( \beta_{\text{min}} \) and \( \beta_{\text{ext}} \) depend on the other parameters of the model. It is worth mentioning that \( \beta_{\text{ext}} \) is the value of \( \beta \) in which the two horizons meet and our black hole has only one horizon. In other words, \( f(r) = 0 \) has a degenerate solution. This correspond to green curve in figure 6. Besides in this figure, the red curve corresponds to \( \beta \leq \beta_{\text{min}} \), the blue curve shows the case \( \beta_{\text{min}} < \beta < \beta_{\text{ext}} \), and the purple curve indicates the case with \( \beta > \beta_{\text{ext}} \). On the other hand, since \( \beta_{\text{ext}} \) corresponds to the minimum value of \( f(r) \), thus in this case \( f'(r) \bigg|_{r=r_{\text{ext}}}=0 \), which implies that the surface gravity should be vanished.

The nature of the horizons can be further understood if we plot the mass parameter \( m \) as a function of the horizon radius \( r_h \) for different model parameters. Solving Eq. \( f(r_h) = 0 \), for the mass parameter yields

\[
m(r_h) = \frac{(n-3)(\alpha^2+1)^2}{(\alpha^2+n-3)(\alpha^2-1)} b^{-\gamma} r_h^{2\gamma + n - 3 - n\gamma/2} + \frac{4(\Lambda + 2\beta^2) (\alpha^2 + 1)^2 b^{2\gamma} r_h^{n - 1 - n\gamma/2}}{(2n - 4)(\alpha^2 - n + 1)} \frac{1}{n - 2} \frac{\beta^2 b^{n-2}\gamma}{q^2} \frac{a^2}{W^{\alpha-\gamma/2}} \frac{\alpha_{\text{ext}}+1}{L_W} (\eta_h)
\]
FIG. 6: $f(r)$ versus $r$ for $q = 2$, $m = 2.5$ and $n = 7$.

FIG. 7: The mass parameter $m$ versus $r_+$ for different $\alpha$ and $n = 8$ and $q = 1$.

FIG. 8: The mass parameter $m$ versus $r_+$ for different value of the charge parameter $q$. Here we have taken $\alpha = 0.5$, $n = 8$ and $\beta = 1$. 
FIG. 9: The mass parameter \( m \) versus \( r_h \) for \( \alpha = 0.6 \), \( n = 8 \), \( \beta = 2 \) and \( q = 1 \). \( m < m_{\text{ext}} \) (red line), \( m = m_{\text{ext}} \) (blue line), \( m > m_{\text{ext}} \) (purple line).

\[
\left\{ \frac{L_W^2(\eta_h)}{\alpha^2 + 3n - 7} \right\} \left( \frac{\alpha^2 + 3n - 7}{2n - 4}; \frac{\alpha^2 + 5n - 11}{2n - 4} L_W(\eta_h) \right) - \frac{1}{\alpha^2 - n + 1} \left( \frac{\alpha^2 - n + 1}{2n - 4}; \frac{\alpha^2 + n - 3}{2n - 4} L_W(\eta_h) \right) \right\},
\]

where \( \eta_h = \eta(r = r_h) \). Figures 7, 8 and 9 show that for fixed value of other parameters, the value of \( m \) determines the number of horizons. For simplicity in these figure we set \( l = b = 1 \). From figure 9 we see that if we solve Eq. \( m = \text{const} \) for \( r_h \), we can distinguish three cases depending on the value of \( m \). For \( m > m_{\text{ext}} \), there exist two value for \( r_h = r_{h+} \) and thus we have two horizons, for \( m = m_{\text{ext}} \) the two horizons meet. In this case we encounter an extremal black hole with zero temperature. As we will show in the next section the extremal black hole with mass \( m_{\text{ext}} \) and degenerate horizon has zero temperature. Besides, for \( m < m_{\text{ext}} \) there is no horizon. Furthermore, from figure 8 we see that in the limit \( r_h \to 0 \) we have a nonzero value for the mass parameter \( m \). This is in contrast to the Schwarzschild black holes in which the mass parameter goes to zero as \( r_h \to 0 \). This is due to the effect of the nonlinearity of the electrodynamic field and in case of \( q = 0 \), the mass parameter \( m \) goes to zero as \( r_h \to 0 \). Physically \( r_h \to 0 \) means that the radius of the horizon becomes very small. Since the horizon radius depends on the solutions parameters such as \( m \), thus we have plotted this behavior in figure 8. For the simple Schwarzschild black hole \( m = \frac{r_h}{2} \) and so as \( r_h \to 0 \) we have \( m \to 0 \).

**IV. THERMODYNAMICS OF DILATON BLACK HOLES**

The Hawking temperature of the black hole on the outer horizon \( r_+ \), may be obtained through the use of the definition of surface gravity [19]

\[
T_+ = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)},
\]

where \( \kappa \) is the surface gravity and \( \chi = \partial/\partial t \) is the null killing vector of the horizon. Taking \( \chi^\nu = (-1,0,0,...) \), we have \( \chi_\nu = (f(r_+),0,0,...) \) and hence \( (\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu) = -\frac{1}{2} [f'(r_+)]^2 \) which leads to

\[
\kappa = \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)} = \frac{1}{2} \left( \frac{df(r)}{dr} \right)_{r=r_+}.
\]

Thus, the temperature is obtained as

\[
T_+ = \frac{f'(r_+)}{4\pi} = \frac{\alpha^2 + 1}{4\pi} r_+^{1-\gamma} \left\{ \frac{(n-3)b^{-\gamma}b^{2\gamma-2}}{\alpha^2 - 1} + \frac{2(\Lambda + 2\beta^2)b^{\gamma}}{n-2} \right\},
\]

where \( \gamma = \frac{3}{2n-4} \).
of the model parameters, while for small values of \( r \) is degenerate. In this case we encounter a naked singularity. On the other hand, for an extremal black hole the temperature is zero and the horizon is degenerate. In this case \( r_{\text{ext}} \) is the positive root of the following equation:

\[
\frac{(n-3)b^{-\gamma}r_{\text{ext}}^{2}\gamma^{-2}}{\alpha^2 - 1} + \frac{2(\Lambda + 2\beta^2)b^\gamma}{n - 2} - \frac{4\beta q_{\text{ext}}}{n - 2} r_{\text{ext}}^{(n-2)(\gamma-2)/2}b^{(4-n)\gamma/2} \left( \frac{1}{\sqrt{L_W(\eta_{\text{ext}})}} - \sqrt{L_W(\eta_{\text{ext}})} \right) = 0.
\]

(44)

where

\[
\eta_{\text{ext}} \equiv \frac{q_{\text{ext}}}{b^{(n-2)\gamma} / 2^{(n-2)\gamma}}.
\]

(45)

From figures 10 and 11 we see that \( r_{\text{ext}} \) decreases as \( \alpha \) increases, while \( r_{\text{ext}} \) increases with increasing \( q \). Indeed, the metric of Eqs. (15) and (28) can describe a nonlinear dilaton black hole with inner and outer event horizons located at \( r_- \) and \( r_+ \), provided \( r > r_{\text{ext}} \), an extreme ENd black hole in the case of \( r = r_{\text{ext}} \), and a naked singularity if \( r < r_{\text{ext}} \).

Note that in the limiting case where \( \beta \to \infty \), expression (13) reduces to the temperature of higher dimensional EMd black holes [27],

\[
T_+ = -\frac{b^{-\gamma}(\alpha^2 + 1)(n-3)}{4\pi(\alpha^2 - 1)} r_+^{\gamma-1} - \frac{\Lambda(\alpha^2 + 1)b^\gamma}{2(n-2)\pi} r_+^{\gamma-1} - \frac{q^2 b^{-\gamma}(\alpha^2 + 1)}{2\pi(n-2)} r_+^{(\gamma_n+3\gamma-3n)/2} + O \left( \frac{1}{\beta^2} \right).
\]

(46)

The entropy of the ENd black hole still obeys the so called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area [29]. This near universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity [30]. It is easy to show

\[
S = \frac{b^{(n-2)\gamma/2} r_+^{(n-2)(1-\gamma/2)}}{4} \omega_{n-2}.
\]

(47)

The gauge potential \( A_t \) corresponding to the electromagnetic field (33) can be obtained through relation \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Since our solution is static, thus the gauge potential is only a function of \( r \). From \( F_{tr} = \partial_t A_r - \partial_r A_t \) with \( \partial_t A_r = 0 \) we have

\[
F_{tr} + \partial_r A_t(r) = 0,
\]

(48)

FIG. 10: \( T \) versus \( r_+ \) for different values of \( \alpha \) parameter. Here we take \( l = b = 1, \beta = 2, n = 5 \) and \( q = 1 \).
FIG. 11: $T$ versus $r_+$ for different values of charge parameter $q$. Here we take $l = b = 1$, $\alpha = 0.8$, $n = 6$ and $\beta = 2$.

and hence the gauge potential $A_t$ can be derived as

$$A_t(r) = -\int F_t dr = -\beta b^\gamma \int dr r^{-\gamma} \sqrt{L_W(\eta)}$$

where $\eta$ is defined in (26). Integrating yields

$$A_t = b^\gamma \beta (\alpha^2 + 1) \left( \frac{\beta^2 b^{(n-2)\gamma}}{q^2} \right)^{\frac{\alpha^2}{4n-4}} \left( \frac{\alpha^2 - 1}{4 - 2n} \right) \frac{\alpha^2 + n - 3}{2n - 4}$$

$$\times \left\{ -\frac{1}{2(n-2)} \Gamma \left( \frac{\alpha^2 + n - 3}{2n - 4}, \frac{1 - \alpha^2}{2n - 4} L_W(\eta) \right) + \frac{1}{\alpha^2 - 1} \left[ \Gamma \left( \frac{\alpha^2 + 3n - 7}{2n - 4}, \frac{1 - \alpha^2}{2n - 4} L_W(\eta) \right) - \frac{1}{2} \Gamma \left( \frac{\alpha^2 + n - 3}{2n - 4} \right) \right] \right\}.$$

(50)

The electric potential $U$, measured at infinity with respect to the horizon, is defined by

$$U = A_\mu \chi^\mu |_{r \to \infty} - A_\mu \chi^\mu |_{r = r_+},$$

(51)

where $\chi = \partial_t$ is the null generator of the horizon. It is a matter of calculation to show that

$$U = b^\gamma \beta (\alpha^2 + 1) \left( \frac{\beta^2 b^{(n-2)\gamma}}{q^2} \right)^{\frac{\alpha^2}{4n-4}} \left( \frac{\alpha^2 - 1}{4 - 2n} \right) \frac{\alpha^2 + n - 3}{2n - 4}$$

$$\times \left\{ -\frac{1}{2(n-2)} \Gamma \left( \frac{\alpha^2 + n - 3}{2n - 4}, \frac{1 - \alpha^2}{2n - 4} L_W(\eta) \right) + \frac{1}{\alpha^2 - 1} \left[ \Gamma \left( \frac{\alpha^2 + 3n - 7}{2n - 4}, \frac{1 - \alpha^2}{2n - 4} L_W(\eta) \right) - \frac{1}{2} \Gamma \left( \frac{\alpha^2 + n - 3}{2n - 4} \right) \right] \right\}.$$  

(52)

Expanding for large value of $\beta$, we get

$$U = \frac{q(\alpha^2 + 1)b^{(4-n)\gamma/2}}{(\alpha^2 + n - 3)^{\alpha^2/n - 3/2} + O\left(\frac{1}{\beta^2}\right)}.$$  

(53)

We have shown the the behavior of the electric potential $U$ as a function of horizon radius $r_+$ in figures 12 and 13 for $b = 1$. Due to the nature of the nonlinear electrodynamics, the electric potential can be finite as $r_+ \to 0$, depending on the model parameters, and goes to zero for large $r_+$ independent of the model parameters. From these figures we
find that for fixed value of other parameter, the divergency of $U$, for small $r_+$, increases with increasing $\alpha$ and $\beta$.

There are several ways for calculating the mass of the black holes. For example, for asymptotically AdS solution one can use the conterterm method inspired by (A)dS/CFT correspondence [31, 32]. Another way for calculating the mass is through the use of the substraction method of Brown and York [33]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. In our case, due to the presence of the non-trivial dilaton field, the asymptotic behaviour of the solutions are neither flat nor (A)dS, therefore we have used the reference background metric and calculate the mass. According to the substraction method of [33], if we write the metric of $n$-dimensional static and spherically symmetric spacetime in the form [23]

$$ds^2 = -W^2(r)dt^2 + \frac{dr^2}{V^2(r)} + r^2d\Omega^2_{n-2},$$

and the matter action contains no derivatives of the metric, then the quasilocal mass is given by [23]

$$M = \frac{n-2}{2} r^{n-3} W(r) \left( V_0(r) - V(r) \right).$$

(55)

Here $V_0(r)$ is an arbitrary function which determines the zero of the energy for a background spacetime and $r$ is the radius of the spacelike hypersurface boundary. It was argued that the $ADM$ mass $M$ is the $\mathcal{M}$ determined in (55) in the limit $r \to \infty$ [23]. Transforming metric (15) in the form (54), the mass of the solution is obtained as

$$M = \frac{b^{(n-2)\gamma/2} (n-2) \omega_{n-2}}{16\pi (\alpha^2 + 1)} m.$$ 

(56)
Next, we want to check the first law of thermodynamics for $n$-dimensional EN black holes. For this purpose, we first observe the mass $M$ as a function of extensive quantities $S$ and $Q$. Combining expressions for the charge, the mass and the entropy given in Eqs. (18), (56) and (47), and using the fact that $f(r_+) = 0$, we obtain a Smarr-type formula as

$$M(S, Q) = \frac{b^{-\alpha^2(n-2)(n-3)(\alpha^2+1)(4S)^{\frac{n^2+n-3}{n^2-n}}}}{16\pi(n\alpha^2+n-3)(n\alpha^2-1)} + \frac{(\alpha^2+1)\beta^2(1-\alpha^2)^{\frac{n^2+n-3}{n^2-n}}}{8\pi(n\alpha^2+n-1)}(4S)^{\frac{n^2+n-3}{n^2-n}}$$

$$+ \frac{b^{-\alpha^2(n-2)(n-3)(\alpha^2+1)(4S)^{\frac{n^2+n-3}{n^2-n}}}}{16\pi(n\alpha^2+n-3)(n\alpha^2-1)} + \frac{\beta^2(b^{-n-2}\gamma)}{16\pi^2Q^2} \left( \frac{\alpha^2+1}{2\alpha^2-1} \right)$$

$$\times \left\{ -\frac{(n-2)^2(\gamma-2)^2}{(\gamma-2)^2} \left[ \Gamma \left( \frac{\alpha^2+3n-7}{2\alpha^2-1} \right) - \Gamma \left( \frac{\alpha^2+3n-7}{2\alpha^2-1} \right) \right] \right\},$$

where $\zeta = \frac{\pi^2Q^2}{\alpha^2\beta}$. If we expand $M(S, Q)$ for large $\beta$, we arrive at

$$M(S, Q) = \frac{b^{-\alpha^2(n-2)(n-3)(\alpha^2+1)(4S)^{\frac{n^2+n-3}{n^2-n}}}}{16\pi(n\alpha^2+n-3)(n\alpha^2-1)} + \frac{\Lambda(\alpha^2+1)\beta^2}{8\pi(n\alpha^2+n-1)}(4S)^{\frac{n^2+n-3}{n^2-n}}$$

$$+ \frac{2\pi(\alpha^2+1)Q^2\beta^2}{\alpha^2+n-3}(4S)^{\frac{n^2+n-3}{n^2-n}} + O \left( \frac{1}{\beta^2} \right),$$

which is exactly the Smarr-type formula obtained for EMd black in the limit $\beta \to \infty$ [27]. Now, if we consider $S$ and $Q$ as a complete set of extensive parameters for the mass $M(S, Q)$, we can define the intensive parameters conjugate to $S$ and $Q$ as

$$T = \left( \frac{\partial M}{\partial S} \right)_Q, \quad U = \left( \frac{\partial M}{\partial Q} \right)_S.$$  

(57)

Numerical calculations show that the intensive quantities calculated by Eq. (59) coincide with Eqs. (48) and (52). Thus, these thermodynamic quantities satisfy the first law of black hole thermodynamics

$$dM = TdS + UdQ.$$  

(60)

The satisfaction of the first law of thermodynamics for the obtained conserved and thermodynamic quantities, together with the fact that these quantities in two limiting cases, namely in the absence of the dilaton field ($\alpha = 0 = \gamma$), and for large values of the nonlinear parameter ($\beta \to \infty$), reduce to the known results in the literature [20, 27], indicate that the conserved and thermodynamic quantities obtained in this paper are correct and in agreement with other method such as Euclidean action method [34].

V. CONCLUSION AND DISCUSSION

In this paper, we generalized the study on the EN electrodynamics by taking into account the dilaton scalar field in the action. We first proposed the suitable Lagrangian for EN electrodynamics coupled to the dilaton field and in the presence of two Liouville-type dilaton potential for the dilaton field in all higher dimensions. By varying the action we obtained the field equations of $n$-dimensional EN electrodynamics coupled to dilaton field in Einstein gravity. Then, we constructed a new class of higher dimensional static and spherically symmetric black hole solutions of this theory. When $\beta \to \infty$, our solutions reduce to higher dimensional EMd black hole solutions [27], while in the absence of the dilaton field, ($\alpha = 0 = \gamma$), they restore charged black holes coupled to EN electrodynamics. Although the behavior of the electric field near the origin depends on the model parameters, however for large $r$ the asymptotic behavior of electric field are exactly the same as linear Maxwell field. Interestingly enough, we found that the electric field of ENd black hole is finite near the origin and diverges exactly at $r = 0$ depending on the model parameters, however its divergency is much weaker than Maxwell field. Besides, in the absence of the dilaton field ($\alpha = 0$), the electric field has a finite value near $r = 0$, while as soon as the dilaton field is taken into account ($\alpha > 0$), the electric field
diverges as \( r \to 0 \). This implies that the presence of the dilaton field changes the behaviour of the electric field near the origin where \( r = 0 \).

We also found that the dilaton field changes the asymptotic behavior of the solutions to be neither flat nor (A)dS. This is consistent with the fact that no dilaton dS or AdS black hole solution exists with the presence of only one or two Liouville-type dilaton potential [22]. The obtained solutions can represent black holes with inner and outer horizons, an extreme black hole or naked singularity depending on the model parameters. For fixed value of \( m, \alpha \) and \( q \), we found that there is a minimum (extreme) value for nonlinear parameter \( \beta_{\text{min}} (\beta_{\text{ext}}) \), for which we have black hole with a non-extreme horizon provided \( \beta \leq \beta_{\text{min}} \), black hole with two horizons for \( \beta_{\text{min}} < \beta < \beta_{\text{ext}} \), black hole with an extreme horizon for \( \beta = \beta_{\text{ext}} \) and naked singularity for \( \beta > \beta_{\text{ext}} \). We computed conserved and thermodynamic quantities and obtained Smarr-type formula, \( M(S, Q) \). We checked that the conserved and thermodynamic quantities obtained for higher dimensional dilaton black holes satisfy the first law of black holes thermodynamics on the horizon.

Let us emphasize that the higher dimensional dilaton black holes coupled to EN electrodynamics constructed in this paper are static. Thus, it would be interesting to derive \( n \)-dimensional rotating black hole/brane solutions of these field equations. The study can also be applied for other type of nonlinear electrodynamics such as logarithmic nonlinear electrodynamics with suitable lagrangian in the presence of dilaton field. It is also intersecting to study stability of the solutions. These issues are now under investigations and the results will be appeared elsewhere.

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