Renormalization of Tamm-Dancoff Integral Equations

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Abstract

During the last few years, interest has arisen in using light-front Tamm-Dancoff field theory to describe relativistic bound states for theories such as QCD. Unfortunately, difficult renormalization problems stand in the way. We introduce a general, non-perturbative approach to renormalization that is well suited for the ultraviolet and, presumably, the infrared divergences found in these systems. We reexpress the renormalization problem in terms of a set of coupled inhomogeneous integral equations, the “counterterm equation.”

The solution of this equation provides a kernel for the Tamm-Dancoff integral equations which generates states that are independent of any cutoffs. We also introduce a Rayleigh-Ritz approach to numerical solution of the counterterm equation. Using our approach to renormalization, we examine several ultraviolet divergent models. Finally, we use the Rayleigh-Ritz approach to find the counterterms in terms of allowed operators of a theory.
Introduction

First attempted in the 1950’s by Tamm and Dancoff [1, 2], the idea of describing relativistic bound states in terms of a small number of particles (Fock space truncation) ran into considerable difficulties. There were severe divergences associated with connected Green’s functions as well as divergences associated with particle creation and annihilation from the vacuum and the approach was soon abandoned. In 1966, Weinberg noticed that, in the infinite momentum frame, creation and annihilation of particles from the vacuum is forbidden and the divergences associated with the vacuum itself are removed [3]. This is similarly true for light front quantized field theories. During the last few years, interest has arisen in combining these two ideas, Tamm-Dancoff Fock space truncation and light front quantization, in an attempt to describe relativistic bound states [4, 5]. Although divergences associated with the vacuum are removed, very difficult renormalization problems remain.

In perturbation theory, divergences fall into two categories: ultraviolet (UV) divergences and infrared (IR) divergences. One generally associates UV divergences with particles having large energy. In the context of light-front physics, large energies arise from particles having large transverse or small longitudinal momenta, and thus UV divergences and some IR divergences have a common origin. New non-perturbative divergences arise because the Tamm-Dancoff truncation does not include all diagrams for any given order in perturbation theory. To date, these divergences in light-front Tamm-Dancoff calculations have only been renormalized within the context of perturbation theory [6]. What we propose here is a general, nonperturbative approach to renormalization that naturally handles the UV and, we believe, some of the IR divergences found in light-front Tamm-Dancoff field theory.

In this approach, it is shown that the renormalization problem can be rewritten as a set of coupled integral equations, much in the same way that the eigenvalue problem is written as a set of coupled integral equations. One can then solve this set of coupled integral equations numerically using the same techniques that are used in solving the eigenvalue problem itself. Although the context of light-front field theory is the motivation for this work, our approach will not depend on any of its special properties. Presumably, our approach to renormalization could be used in other, unrelated contexts.

In section 1, we will introduce the renormalization problem using a simple one
dimensional UV divergent model that has been discussed by other authors. We will introduce the so-called “high-low analysis” and show how it can be used to renormalize the model. In section 2 we will use the high-low analysis to examine the general case and derive the “counterterm equation,” a set of coupled integral equations which relate the bare and renormalized Hamiltonians. In section 3, we will show how renormalization group ideas are expressed in the context of our renormalization procedure. Next, in section 4, we will discuss a variational, or Rayleigh-Ritz, approach to solving the counterterm equation. In sections 5 and 6, we will apply our renormalization scheme to some simple examples. Finally, in section 7, we use the Rayleigh-Ritz approach to find the counterterm in terms of the allowed operators of a theory.

1. Model A

Let us start by looking at model A, a simple toy model that has been studied by a number of authors [6, 7, 8, 9]. Consider the homogeneous integral equation

\[(p - E) \phi(p) + g \int_0^\Lambda dp' \phi(p') = 0\]  (1.1)

with eigenvalue \(E\) and eigenvector \(\phi(p)\). This is a model for a single particle of momentum \(p\) with Hamiltonian \(H(p, p') = p \delta(p - p') + g\). We will focus on the \(E < 0\) solution:

\[\phi(p) = \text{const} \frac{p}{p - E}, \quad E = \frac{\Lambda}{1 - e^{-1/g}}.\]  (1.2)

Note that the eigenvalue diverges in the limit \(\Lambda \to \infty\). Proper renormalization involves modifying the system to make \(E\) and \(\phi(p)\) independent of \(\Lambda\) in the limit \(\Lambda \to \infty\). Toward this end we add a counterterm \(C_\Lambda\) to the Hamiltonian. Invoking the high-low analysis [10], we divide the interval \(0 < p < \Lambda\) into two subintervals: \(0 < p < L\), a “low-momentum region,” and \(L < p < \Lambda\), a “high-momentum region,” where the momentum scales characterized by \(E\), \(L\), and \(\Lambda\) are assumed to be widely separated. The idea is that the eigenvalue and eigenvector should be independent of the behavior of the system in the high-momentum region. The eigenvalue equation can be written as two coupled equations
Integrating (1.3b) in the limit $L, \Lambda \gg E$,
\[ \int_L^\Lambda dp \phi(p) = -\frac{(g + C_\Lambda) \ln \frac{\Lambda}{L}}{1 + (g + C_\Lambda) \ln \frac{\Lambda}{L}} \int_0^L dp' \phi(p') , \] (1.4)
and substituting this expression into (1.3a), we obtain an eigenvalue equation with the high momentum region integrated out:
\[ p \epsilon [0, L] \quad (p - E) \phi(p) + \frac{(g + C_\Lambda)}{1 + (g + C_\Lambda) \ln \frac{\Lambda}{L}} \int_0^L dp' \phi(p') = 0 . \] (1.5)

If we demand that this expression be independent of $\Lambda$,
\[ \frac{d}{d\Lambda} \left( \frac{(g + C_\Lambda)}{1 + (g + C_\Lambda) \ln \frac{\Lambda}{L}} \right) = 0 , \] (1.6)
we obtain a differential equation for $C_\Lambda$
\[ \frac{dC_\Lambda}{d\Lambda} = \frac{(g + C_\Lambda)^2}{\Lambda} . \] (1.7)
Solving this equation, we are free to insert an arbitrary constant $-1/A_\mu - \ln \mu$
\[ g + C_\Lambda = \frac{A_\mu}{1 - A_\mu \ln \frac{\Lambda}{\mu}} . \] (1.8)
Substituting this result back into (1.5),
\[ p \epsilon [0, L] \quad (p - E) \phi(p) + \frac{A_\mu}{1 - A_\mu \ln \frac{\Lambda}{\mu}} \int_0^L dp' \phi(p') = 0 \] (1.9)
we see that $\Lambda$ has been removed from the equation entirely. Using (1.8) in the original eigenvalue equation
\[(p - E) \phi(p) + \frac{A_{\mu}}{1 - A_{\mu} \ln \frac{\Lambda}{\mu}} \int_0^{\Lambda} dp' \phi(p') = 0 \quad (1.10)\]

gives the same equation as (1.9) with \(L\) replaced by \(\Lambda\). The eigenvalue is now,

\[E = \frac{\Lambda}{1 - \frac{\Lambda}{\mu} e^{-1/A_{\mu}}} \quad \lim_{\Lambda \to \infty} E = -\mu e^{1/A_{\mu}}. \quad (1.11)\]

Although the eigenvalue is still a function of the cutoff for finite \(\Lambda\), the eigenvalue does become independent of the cutoff in the limit \(\Lambda \to \infty\), and the system is properly renormalized.

We can think of \(A_{\mu}\) as the renormalized coupling constant and \(\mu\) as the renormalization scale. In that case, the eigenvalue should depend on the choice of \(A_{\mu}\) for a given \(\mu\) but be independent of \(\mu\) itself. Suppose, for equation (1.11), we want to change \(\mu\) to a new value, say \(\mu'\). In order that the eigenvalue remain the same, we must also change the coupling constant from \(A_{\mu}\) to \(A_{\mu'}\)

\[\mu e^{1/A_{\mu}} = \mu' e^{1/A_{\mu'}}. \quad (1.12)\]

In the same manner, one can write down a \(\beta\) function for \(A_{\mu}\)

\[\mu \frac{d}{d\mu} A_{\mu} = A_{\mu}^2. \quad (1.13)\]

### 2. The Counterterm Equation

Using the ideas introduced in the previous section we can examine the general case. Throughout, we will be working with operators projected onto some Tamm-Dancoff subspace (finite particle number) of the full Fock space. In addition, we will regulate the system by demanding that each component of momentum of each particle lies within some finite interval. We define the “cutoff” \(\Lambda\) to be an operator which projects onto this subspace of finite particle number and finite momenta. Thus, for any operator \(O\), \(O \equiv \Lambda O \Lambda\). Let us introduce the Hamiltonian

\[H = H_0 + V + C_{\Lambda} \quad (2.1)\]
where, in the standard momentum space basis, $H_0$ is the diagonal part of the Hamiltonian, $V$ is the interaction term, and $C_\Lambda$ is the counterterm which is to be determined and is a function of the cutoff. Each term of the Hamiltonian is Hermitian and compact. Schrödinger’s equation can be written

$$(H_0 - E) \phi + (V + C_\Lambda) \phi = 0 \quad (2.2)$$

with energy eigenvalue $E$ and eigenvector $\phi$. Our goal is to choose $C_\Lambda$ such that $E$ and $\phi$ are independent of $\Lambda$ in the limit of large cutoff.

Now we will make an important assumption: the physics that we are interested in, characterized by energy scale $E$, is independent of physics near the boundary of the space spanned by $\Lambda$. Thus, we define two projection operators, $\mathcal{H}$ and $\mathcal{L}$, where $\Lambda = \mathcal{H} + \mathcal{L}$, $\mathcal{H}\mathcal{L} = \mathcal{L}\mathcal{H} = 0$, and $\mathcal{H}$ and $\mathcal{L}$ commute with $H_0$. $\mathcal{H}$ projects onto a “high-momentum region” which contains energy scales we do not care about, and $\mathcal{L}$ projects onto a “low-momentum region” which contains energy scales characterized by $E$. Schrödinger’s equation (2.2) can be rewritten as two coupled equations:

$$(H_0 - E) \mathcal{L}\phi + \mathcal{L}(V + C_\Lambda) \mathcal{L}\phi + \mathcal{L}(V + C_\Lambda) \mathcal{H}\phi = 0 \quad (2.3a)$$

and

$$(H_0 - E) \mathcal{H}\phi + \mathcal{H}(V + C_\Lambda) \mathcal{H}\phi + \mathcal{H}(V + C_\Lambda) \mathcal{L}\phi = 0 . \quad (2.3b)$$

Using equation (2.3b), we can formally solve for $\mathcal{H}\phi$ in terms of $\mathcal{L}\phi$

$$\mathcal{H}\phi = \frac{1}{\mathcal{H}(E - H)\mathcal{H}}(V + C_\Lambda) \mathcal{L}\phi . \quad (2.4)$$

The term with the denominator is understood to be defined in terms of its series expansion in $V$. We can substitute this result back into (2.3a)

$$(H_0 - E) \mathcal{L}\phi + \mathcal{L}(V + C_\Lambda) \mathcal{L}\phi + \mathcal{L}(V + C_\Lambda) \frac{1}{\mathcal{H}(E - H)\mathcal{H}}(V + C_\Lambda) \mathcal{L}\phi = 0 . \quad (2.5)$$

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In order to properly renormalize the system, we could choose $C_\Lambda$ such that (2.5) is independent of one’s choice of $\Lambda$ for a fixed $\mathcal{L}$ in the limit of large cutoffs. However, we will make a stronger demand: that (2.5) should be equal to (2.2) with the cutoff $\Lambda$ replaced by $\mathcal{L}$. Consider the following ansatz for $C_\Lambda$:

$$C_\Lambda = a_0 V + a_1 V \mathcal{F} V + a_2 V \mathcal{F} V \mathcal{F} V + \ldots$$  \hspace{1cm} (2.6)

where $F$ commutes with $\mathcal{H}$ and $\mathcal{L}$ and the coefficients $a_0, a_1, \ldots$ are to be determined. We will discuss some specific choices for $F$ below. Substituting (2.6) into (2.2) with $\Lambda$ replaced by $\mathcal{L}$ everywhere and setting it equal to (2.5), order by order in $V$,

$$
(1 + a_0) V + a_1 V \mathcal{L} F V + a_2 V \mathcal{L} F V \mathcal{L} F V + \ldots \\
= (1 + a_0) V + a_1 V F V + a_2 V F V F V \\
+ (1 + a_0)^2 V \frac{\mathcal{H}}{E - H_0} V + a_1 (1 + a_0) \left( V F V \frac{\mathcal{H}}{E - H_0} V + V \frac{\mathcal{H}}{E - H_0} V F V \right) \\
+ (1 + a_0)^3 V \frac{\mathcal{H}}{E - H_0} V \frac{\mathcal{H}}{E - H_0} V + \ldots .
$$  \hspace{1cm} (2.7)

Using $\Lambda = \mathcal{L} + \mathcal{H}$, we find that $a_0$ is arbitrary and the rest of the coefficients are uniquely determined

$$a_i = (-1)^i (1 + a_0)^i , \quad i = 1, 2, \ldots$$  \hspace{1cm} (2.8)

provided that we can make the approximation

$$V \frac{\mathcal{H}}{E - H_0} V \approx V \mathcal{H} F V .$$  \hspace{1cm} (2.9)

This is what we will call the “renormalizability condition.” A system is properly renormalized if, as we increase the cutoffs $\Lambda$ and $\mathcal{L}$, (2.9) becomes an increasingly good approximation. Since the choice of $a_0$ rescales $V$, we absorb the factor of $1 + a_0$ into the definition of $V$. The arbitrary scale $1 + a_0$ corresponds to a particular solution of (2.7) and therefore depends on $F$, where $F$ may involve an arbitrary energy scale $\mu$. Thus $1 + a_0$ is an implicit function of $\mu$. In simple models where $V$ is proportional to a coupling constant, this $\mu$-dependent rescaling is equivalent to introducing a running
coupling \( g(\mu) \) that runs with the arbitrary scale in \( F \). In both models A and B discussed below the coupling will therefore run with \( \mu \). Summing the series (2.6), we can express \( C_A \) as the solution of an operator equation, the “counterterm equation,”

\[
C_A = -VFV - VFC_A
\]

(2.10)

or, defining \( V_A = V + C_A \),

\[
V_A = V - VFC_A.
\]

(2.11)

This is our central result. In the standard momentum space basis, this becomes a set of coupled inhomogeneous integral equations. Such equations generally have a unique solution, allowing us to renormalize systems without having to resort to perturbation theory. This includes cases where the perturbative expansion diverges or converges slowly.

There are many possible choices for \( F \) that satisfy the renormalizability condition. For instance, one might argue that we want \( F \) to resemble \( 1/(E - H_0) \) as much as possible and choose

\[
F = \frac{1}{\mu - H_0}
\]

(2.12)

where the arbitrary constant \( \mu \) is chosen to be reasonably close to \( E \). In this case, one might be able to use a smaller cutoff in numerical calculations.

One might argue that physics above some energy scale \( \mu \) is simpler and that it is numerically too difficult to include the complications of the physics at energy scale \( E \) in the solution of the counterterm equation. Thus one could choose

\[
F = -\frac{\theta(H_0 - \mu)}{H_0}
\]

(2.13)

where the arbitrary constant \( \mu \) is chosen to be somewhat larger than \( E \) but smaller than the energy scale associated with the cutoff. The \( \theta \) function is assumed to act on each diagonal element in the standard momentum space basis. The difficulty with this renormalization scheme is that it involves three different energy scales, \( E, \mu \) and
the cutoff which might make the numerical problem more difficult.

3. Renormalization-Group Analysis

We can relate our approach to conventional renormalization-group concepts. In renormalization-group language, \( V_\Lambda \) is the bare interaction term and \( V \) is the renormalized interaction term. In both of the renormalization schemes introduced above, we introduced an arbitrary energy scale \( \mu \); this is the renormalization scale. Now, physics (the energy eigenvalues and eigenvectors) should not depend on this parameter or on the renormalization scheme itself, for that matter. How does one move from one renormalization scheme to another? Consider a particular choice of renormalized interaction term \( V \) associated with a renormalization scheme which uses \( F \) in the counterterm equation. We can use the counterterm equation to find the bare coupling \( V_\Lambda \) in terms of \( V \). Now, to find the renormalized interaction term \( V' \) associated with a different renormalization scheme using a different operator \( F' \) in the counterterm equation, we simply use the counterterm equation with \( V_\Lambda \) as given and solve for \( V' \)

\[
V' = V_\Lambda + V_\Lambda F' V'.
\]

Expanding this procedure order by order in \( V \) and summing the result, we can obtain an operator equation relating the two renormalized interaction terms directly:

\[
V' = V + V (F' - F) V'.
\]

The renormalizability condition ensures that this expression will be independent of the cutoff in the limit of large cutoff.

For the two particular renormalization schemes mentioned above, (2.12) and (2.13), we can regard the renormalized interaction term \( V \) as an implicit function of \( \mu \). We can see how the renormalized interaction term changes with \( \mu \) in case (2.12):

\[
\mu \frac{d}{d\mu} V = -V \frac{\mu}{(H_0 - \mu)^2} V
\]

and in the case (2.13):
\[ \mu \frac{d}{d\mu} V = V \delta (H_0 - \mu) V. \quad (3.4) \]

This is a generalization of the \( \beta \) function.

4. The Rayleigh-Ritz Method

It is necessary to find efficient numerical techniques for solving the counterterm equation. The approach we discuss here is a simple extension of a standard technique used in solving Fredholm-type integral equations \[11\]. Suppose we have some operator \( K \); how do we measure, given \( V \) and \( F \), whether \( K \) is a good approximation to \( V_\Lambda \)? Let us define a functional \( J[K] \) such that \( J[K] \) is stationary at \( K = V_\Lambda \). The expression

\[ \text{Tr} \ (K - V_\Lambda) (F + FVF) (K - V_\Lambda) \quad (4.1) \]

is stationary at \( K = V_\Lambda \). Using the counterterm equation (2.10), \( (F + FVF)V_\Lambda = FV \), and (4.1) can be rewritten as

\[ \text{Tr} \ K (F + FVF) K - \text{Tr} \ KFV - \text{Tr} \ VFK + \text{term independent of } K. \quad (4.2) \]

Thus, the functional

\[ J[K] = \text{Tr} \ K (F + FVF) K - \text{Tr} \ K (FV + VF) \quad (4.3) \]

is stationary at \( K = V_\Lambda \) \[12\]. Let us choose a linearly independent set of Hermitian operators \( \{O_i\} \) and define

\[ K = \sum_i x_i O_i. \quad (4.4) \]

We want to find the vector \( x \) such that \( J[K] \) is stationary; this will give us the best approximation to \( V_\Lambda \) from the subspace spanned by \( \{O_i\} \). Defining the Hermitian matrix \( B \) and the real-valued vector \( c \)

\[ B_{i,j} = \text{Tr} \ O_i (F + FVF) O_j \quad c_i = \text{Tr} \ O_i (FV + VF), \quad (4.5a,b) \]

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the functional is

\[ J[K] = \mathbf{x}^\dagger B \mathbf{x} - \mathbf{x} \cdot \mathbf{c} \]  

(4.6)

with stationary point given by

\[ 2B \mathbf{x} = \mathbf{c} . \]  

(4.7)

Unless the number of operators is limited, this approach is not very useful. If we can approximate the space spanned by \( \Lambda \) using \( n \) vectors, then it will generally take \( n^2 \) operators to span the truncated Fock space; finding \( \mathbf{x} \) involves solving an \( n^2 \times n^2 \) linear system, too large for practical applications.

Alternatively, we can express \( K \) as an outer product of \( n \) linearly independent vectors \( \{ \chi_i \} \)

\[ K = \sum_{i,j=1}^{n} X_{i,j} \chi_i \chi_j^\dagger . \]  

(4.8)

Define the Hermitian matrices \( A, E, \) and \( C \):

\[ A_{i,j} = \chi_i^\dagger (F + FV) \chi_j \]
\[ E_{i,j} = \chi_i^\dagger \chi_j \]
\[ C_{i,j} = \chi_i^\dagger (FV + VF) \chi_j \]  

(4.9a, b, c)

Note that \( E \) is positive definite. The functional is now written as

\[ J[K] = \text{Tr} \ XEXA - \text{Tr} \ CX ; \]  

(4.10)

its stationary point is given by the matrix equation

\[ C = AXE + EXA . \]  

(4.11)

One can solve for \( X \) using standard numerical techniques; the solution is basically equivalent to solving an \( n \times n \) generalized eigenvalue problem \([14,15]\).
5. Model A Revisited

Model A is an example of what could be called a “separable system.” Separable systems have interaction terms which can be written as an outer product of a Fock space vector: \( V = \eta \eta^\dagger \). Then, the counterterm equation is

\[
V_\Lambda = \eta \eta^\dagger - \eta \eta^\dagger F V_\Lambda
\]  
(5.1)

which has the solution

\[
V_\Lambda = \frac{\eta \eta^\dagger}{1 + \eta^\dagger F \eta}.
\]  
(5.2)

So, in this case, the counterterm equation produces an ordinary coupling constant renormalization. For model A in the standard momentum space basis, we have \( \eta(p) = \sqrt{g} \) where \( g = g(\mu) \), and the bare interaction term is

\[
V_\Lambda(p, p') = \frac{g}{1 + g \int_0^\Lambda dq \, dq' \, F(q, q')}.  
\]  
(5.3)

For the choice \( F(q, q') = \delta(q - q') / (\mu - q) \) the eigenvalue is, using a principal-value prescription,

\[
E = \frac{\Lambda}{1 - e^{-1/g} (\Lambda - \mu)}  
\]  
(5.4)

which, in the limit \( \Lambda \to \infty \) becomes independent of \( \Lambda \), giving an expression consistent with our previous result,

\[
\lim_{\Lambda \to \infty} E = -|\mu| e^{1/g}. 
\]  
(5.5)

We can easily verify the renormalizability condition for this model. For the \( \mathcal{H} \) region, we restrict \( L < p < \Lambda \). The renormalizability condition is,

\[
g \ln \left( \frac{\Lambda - E}{L - E} \right) \approx g \ln \left( \frac{\Lambda - \mu}{L - \mu} \right)
\]  
(5.6)
which becomes an equality in the limit $L, \Lambda \to \infty$. Finally, we can use equation (3.3) to calculate the $\beta$ function:

$$\mu \frac{d}{d\mu} g(\mu) = g(\mu)^2 \left( 1 + \frac{\mu}{\Lambda - \mu} \right) \lim_{\Lambda \to \infty} \mu \frac{d}{d\mu} g(\mu) = g(\mu)^2. \quad (5.7)$$

which is the same result we found in (1.13).

6. Model B

Let us consider an extension to model A that has a more complicated interaction term. In the momentum space basis, we define the Hamiltonian of model B to be

$$H_0(p, p') = p \delta(p - p') \quad V(p, p') = g \frac{p + p' + a}{p + p' + b}. \quad (6.1)$$

The interaction term becomes constant in the limit of large $p$ or $p'$, and the unrenormalized eigenvalue is linear in $\Lambda$.

Applying the Rayleigh-Ritz procedure, we can solve for the bare interaction term $V_\Lambda(p, p')$ in terms of $V(p, p')$. We will use (4.8) with a lattice in momentum space as our linearly independent set of $n$ vectors

$$\chi_i(p) = \begin{cases} 1 & \frac{i-1}{n}\Lambda < p < \frac{i}{n}\Lambda \\ 0 & \text{otherwise} \end{cases} \quad (6.2)$$

and $F(p, p') = \delta(p - p') / (\mu - p)$. Thus, the relevant matrices, $A$, $E$, and $C$, are:

$$A_{i,j} = \frac{\Lambda}{n} \frac{\delta_{i,j}}{p_i - \mu} - \left( \frac{\Lambda}{n} \right)^2 V(p_i, p_j) \frac{1}{(p_i - \mu)(p_j - \mu)}$$

$$E_{i,j} = \frac{\Lambda}{n} \delta_{i,j}$$

$$C_{i,j} = \left( \frac{\Lambda}{n} \right)^2 V(p_i, p_j) \left[ \frac{1}{p_i - \mu} + \frac{1}{p_j - \mu} \right] \quad (6.3a, b, c)$$

where $p_i = \frac{i-1/2}{n}\Lambda$. Typical numerical results for $V$ and $V_\Lambda$ are shown in Figures 1 and 2 for $n = 100$, $\Lambda = 40$, $\mu = -8.5$, $a = 9$, $b = 3$, and $g = -2.85$. Note that $V_\Lambda(p, p')$ has a slightly different functional form in the regions where only one momentum is large and is almost zero in the region where both momenta are large.
Using $V_\Lambda$, we can solve Schrödinger’s equation and find the renormalized eigenvalues and eigenvectors. If we plot the negative eigenvalue and eigenvector as a function of $\Lambda$ as shown in Figures 3 and 4, we can see that the system is properly renormalized.

7. Local Counterterms

An operator that is an arbitrary function of momenta in the momentum-space basis is generally nonlocal in the coordinate basis. In light-front physics, proper renormalization requires that one abandon local operators, at least for the longitudinal coordinates. However, when our renormalization procedure is applied to systems without such infrared divergences, nonlocal counterterms may still be produced. This is unacceptable. One possible solution to this problem is to choose only functions that have the desired locality when choosing a basis of linearly independent operators in the Rayleigh-Ritz procedure. Such a basis does not span the space of all operators, but the Rayleigh-Ritz procedure will, if possible, choose the coefficients so that the divergences are still canceled. While not conceptually pleasing, this will properly renormalize a system using, by construction, counterterms that are sufficiently local.

Let us see how this works in the context of model B. Since the interaction term becomes constant in the limit of large $p$, one can renormalize model B using only a local counterterm. For simplicity, we will apply the Rayleigh-Ritz procedure to the counterterm equation for $C_\Lambda$ (2.10) using $\{O_1\}$ as our set of linearly independent operators where $O_1$ is defined to be 1 in the momentum space basis. In this case, we want to find the stationary point of the functional

$$ J [K] = \text{Tr} K (F + FVF) K + \text{Tr} K (FVVF + VFVF) $$

(7.1)

where $K = x_1 O_1$; this will give us the best possible approximation to $C_\Lambda$ from the subspace spanned by $\{O_1\}$. Defining

$$ B_{1,1} = \text{Tr} O_1 (F + FVF) O_1 \quad \tilde{c}_1 = -\text{Tr} O_1 (FVVF + VFVF) $$

(7.2a, b)

the stationary point of (7.1) is given by

$$ 2B_{1,1} x_1 = \tilde{c}_1. $$

(7.3)
In the momentum space basis, we will use $F(p, p') = \delta(p - p')/\Lambda - p$ and,

\[
B_{1,1} = -\Lambda \ln \frac{\Lambda}{\mu} + \Lambda \int_{\mu}^{\Lambda} dp \, dp' \frac{V(p, p')}{p, p'}
\]

\[
\tilde{c}_1 = -2 \int_{\mu}^{\Lambda} dp \, dp' \, dq \frac{V(p, p') V(p', q)}{p, p'}.
\]

(7.4a, b)

To leading order in $\Lambda$,

\[
B_{1,1} = g\Lambda \left( \ln \frac{\Lambda}{|\mu|} \right)^2 - \Lambda \ln \frac{\Lambda}{|\mu|} - \frac{\Lambda g(a - b)}{b + 2\mu} \left\{ \frac{\pi^2}{6} + 2 \text{Li}_2 \left( \frac{b + \mu}{\mu} \right) \right\}
\]

\[
\tilde{c}_1 = -2g^2\Lambda \left( \ln \frac{\Lambda}{|\mu|} \right)^2 + \frac{\Lambda g^2(a - b)}{b + 2\mu} \left\{ \frac{\pi^2}{6} + 2 \text{Li}_2 \left( \frac{b + \mu}{\mu} \right) \right\}
\]

(7.5a, b)

where $\text{Li}_2(x)$ is the diluarithm function, and to leading order in $\ln \frac{\Lambda}{|\mu|}$,

\[
x_1 = -g(\mu) + \frac{g(\mu)}{1 - g(\mu)} \ln \frac{\Lambda}{|\mu|}.
\]

(7.6)

in agreement with references [6] and [7]. Numerically calculating the eigenvalue as a function of $\Lambda$ shows that this choice of counterterm does correctly renormalize the system as shown in Figures 5 and 6 where $n = 100$, $\Lambda = 60$, $\mu = 5$, $a = 8$, $b = 2.2$, and $g = -0.65$.

For a more realistic example, one would start with $V$ written as a linear combination of local operators and use the Rayleigh-Ritz procedure to find $V_\Lambda$ in terms of those same operators. We have not yet established general conditions for the validity of this procedure.

Conclusion

We have introduced an approach to renormalization that is nonperturbative, general, and successfully handles the UV divergences found in light-front Tamm-Dancoff field theories. In this approach counterterms that remove all cutoff dependence are obtained from the solution of the counterterm equation. We have shown that the Rayleigh-Ritz procedure offers a practical way of solving the counterterm equation,
and we have illustrated how the renormalization procedure works for some simple examples.

In general, the Hamiltonian that one uses in field theory is not the one that is obtained by canonical quantization. In principle one must include all operators allowed by power counting. We have shown that the Rayleigh-Ritz method can be used to find the combination of allowed operators that provides the best solution of the counterterm equation. A simple example of this procedure is given.

Although we have only examined ultraviolet divergences, the discussion that led to the counterterm equation should also be valid for some of the infrared divergences found in light-front quantized field theory. In light-front coordinates, the Hamiltonian for a free particle with mass $m$, longitudinal momentum $p^+$, and transverse momenta $p_\perp$ is

$$H_f = \frac{m^2 + p_\perp^2}{2p^+}.$$ 

Thus, if we associate the $\mathcal{H}$ region with small values of $p^+$, the renormalizability condition can be fulfilled since $H_f \gg E$ for sufficiently small $p^+$. Future work will examine the issue of infrared divergences in detail.

Clearly, there is a lot of work left to be done before we can successfully describe relativistic bound states using the light-front Tamm-Dancoff approach. However, we hope that the approach to renormalization outlined in this paper will provide a useful framework for addressing the remaining problems.

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[12] More generally, $J [M, K] = \text{Tr} \; MK (F + FVF) K - \text{Tr} \; K (FVM + MVF)$ is stationary at $K = V_\Lambda$. The choice $M = F$ is useful in some cases.
[13] In the context of quantum chemistry calculations, the choosing of such bases has become a refined art. See K. Wilson, in *Lattice '89*, Proceedings of the International Symposium, Capri, Italy, 1989, edited by R. Petronzio *et al.* [Nucl. Phys. B (Proc. Suppl.) **17** (1989)].

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[15] The matrix equation is solved by finding a nonsingular matrix $U$ such that $U^\dagger AU = \text{diag}(\lambda_1, \lambda_2, \ldots)$ is diagonal and $U^\dagger EU$ is the identity. The solution is $X = U \tilde{X} U^\dagger$ where $\tilde{X}_{i,j} = \left(U^\dagger CU\right)_{i,j} / (\lambda_i + \lambda_j)$. 
Figure Captions

Figure 1. Plot of the renormalized interaction term $-V(p, p')$ as a function of $p$ and $p'$ for model B.

Figure 2. Plot of the bare interaction term $-V_{\Lambda}(p, p')$ as a function of $p$ and $p'$.

Figure 3. Eigenvalue as a function of $\Lambda$ along with a fit to $x+y/\Lambda$ and the asymptotic value $x$. The fit parameters are $x = -6.55$ and $y = 5.34$.

Figure 4. Eigenfunction $p \phi(p)$ as a function of $p$ for various $\Lambda$. The wavefunctions are not normalized.

Figure 5. Eigenvalue as a function of $\Lambda$ along with a fit to $x+y/\Lambda$ and the asymptotic value $x$. The fit parameters are $x = -6.84$ and $y = 30.52$.

Figure 6. Plot of the bare interaction term $-V_{\Lambda}(p, p')$ as a function of $p$ and $p'$.