On the Derivation of Conserved Quantities in Classical Mechanics

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Abstract. Using a theorem of partial differential equations, we present a general way of deriving the conserved quantities associated with a given classical point mechanical system, denoted by its Hamiltonian. Some simple examples are given to demonstrate the validity of the formulation.

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1. Introduction.

It has been well-known for many years of the history of physics that symmetry principles and conservation laws play a crucial role in understanding physical phenomena. Many physical problems can be solved significantly easier by taking into account symmetry consideration, and most importantly, one can gain deep insight of the nature of the phenomena. Many fields of study in modern physics developed rapidly through the understanding of symmetry principles, for instance, the gauge symmetries in quantum field theory and elementary particle physics.

The existence of symmetries in a physical system means that there exists one or more transformations that leave the physical system unaltered. Since physical systems can be completely described by their Lagrangians / Hamiltonians, it is therefore natural to expect the symmetry transformations to be canonical ones. The number of symmetry transformations that can be generated from the system depends on the number of conserved quantities of the associated system. It has been well-known that the conserved quantities of a physical system relate closely to the generator of the associated symmetry transformations.

In this paper, we shall present a derivation of all possible conserved quantities associated with a given physical system directly from its associated conservation laws. We shall limit our task on classical point mechanics, although it may be possible to extend the derivation techniques to quantum mechanics and field theories, once the connection between classical and quantum mechanics is established via canonical
realizations \[\text{\textsuperscript{11\textsuperscript{8}}}\]. Some simple examples associated with time-independent and time-dependent systems will be considered.

The paper is organized as follows: a brief overview of classical mechanics is given in Section \[\text{\textsuperscript{2}}\] in which we shall use Lagrangian and Hamiltonian approaches. This includes a brief review of canonical transformations, especially a subset of the transformations called infinitesimal canonical transformations. The relation between symmetry canonical transformations and conservation laws, as well as the use of a theorem of partial differential equations to obtain the conserved quantities are given in Section \[\text{\textsuperscript{3}}\]. In Section \[\text{\textsuperscript{4}}\] we shall discuss the derivation of conserved quantities associated with time-independent systems. A similar discussion for time-dependent cases is given in Section \[\text{\textsuperscript{5}}\]. Some simple examples are used to demonstrate the validity of the formulation. Section \[\text{\textsuperscript{6}}\] presents summary and conclusion of the discussion, as well as a brief remark of the extension possibility of the techniques to quantum mechanics and field theories.

2. Classical Mechanics: Canonical Formalism.

A classical system can always be represented by an associated Lagrangian \(L(q_i, \dot{q}_i, t)\) that satisfies
\[
\delta A \equiv \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0 \quad (\text{II.1})
\]
and leads to Euler-Lagrange equations of motions
\[
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad (\text{II.2})
\]
where \(i = 1...N, \ N = \text{number of degrees of freedom}\).

Defining the conjugate momenta \(p_i\) associated with the canonical coordinates \(q_i\) as
\[
p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad (\text{II.3})
\]
we may construct a new function \(H(q_i, p_i, t)\), the Hamiltonian, that can equivalently represent the physical system as \(L(q_i, \dot{q}_i, t)\) via a Legendre transformation \[\text{\textsuperscript{7}}\]:
\[
H(q_i, p_i, t) \equiv \sum_j p_j \dot{q}_j - L(q_i, \dot{q}_i, t). \quad (\text{II.4})
\]
\(H(q_i, p_i, t)\) is known as Hamiltonian. Using Eqs. (\text{\textsuperscript{II.2}}) and (\text{\textsuperscript{II.4}}), a set of equations of motion equivalent to Eq. (\text{\textsuperscript{II.2}}), the so-called Hamilton’s equations of motion, can be obtained as follows:
\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (\text{II.5})
\]
A set of invertible phase space transformations
\[
(q_i, p_i, t) \Longleftrightarrow (Q_i, P_i, t) \quad (\text{II.6})
\]
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which preserves Eq. (II.5) is called canonical transformations. It is easy to show that the set of canonical transformations forms a non-abelian group \([1]\), and that the transformations can be generated by one of the four types of generating functions \(F_1(q_i, Q_i, t), F_2(q_i, P_i, t), F_3(p_i, Q_i, t), F_4(p_i, P_i, t)\) \([7]\). The old Hamiltonian \(H(q_i, p_i, t)\) and the new counterpart \(K(Q_i, P_i, t)\) are related by

\[
K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_\alpha}{\partial t}, \quad \alpha = 1, ..., 4.
\]

(II.7)

In the case of a time-independent system, the value of the old and new Hamiltonian are the same, even if they have different functional forms. In general, \(K(Q_i, P_i, t)\) and \(H(q_i, p_i, t)\) have different functional forms, leading to different physical interpretations.

It is more convenient to use the Poisson bracket formulation, defined as follows for \(A(q_i, p_i, t)\) and \(B(q_i, p_i, t)\):

\[
\{A, B\} \equiv \sum_{i=1}^{N} \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).
\]

(II.8)

The first two equations in Eq. (II.5) can be written as

\[
\dot{q}_i = \{q_i, H\}
\]

\[
\dot{p}_i = \{p_i, H\}.
\]

(II.9)

In the case of canonical transformations, it is easy to show that the necessary and sufficient condition for the transformation (II.6) to be canonical is

\[
\{Q_i, P_j\} = \{q_i, p_j\} = \delta_{ij}.
\]

(II.10)

We are now focusing our attention to a subset of canonical transformations that lies in the neighborhood of the identity transformation, namely the infinitesimal canonical transformations. The transformation can be generated by the following generating function \([7]\)

\[
F_2(q_i, P_i, t) = \sum_j (q_j P_j + \epsilon_j f_j(q_i, p_i, t)),
\]

(II.11)

where \(f_j(q_i, p_i, t)\) are the generators of infinitesimal canonical transformations, and \(\epsilon_j\) are infinitesimal parameters. The transformation generated by (II.11) can be written as

\[
Q_i = q_i + \sum_j \epsilon_j \{q_i, f_j\},
\]

\[
P_i = p_i + \sum_j \epsilon_j \{p_i, f_j\}.
\]

(II.12)

It can be shown that the set of infinitesimal canonical transformations forms an abelian group, and hence it can be used to form a linear vector space and an algebra \([9]\). The algebra of infinitesimal canonical transformation satisfies the following Poisson bracket relations \([1, 8]\):

\[
\{f_i, f_j\} = \sum_k C_{ij}^k f_k + d_{ij}
\]

(II.13)

where \(C_{ij}^k\) and \(d_{ij}\) are constants depending on the nature of the transformation. In some transformations, the constant \(d_{ij}\) may be removed by a suitable substitution \([1, 8]\). The relations (II.13) are closely related to the commutation relations in Lie algebras \([10]\).
3. Symmetries, Conservation Laws and General Conserved Quantities.

The infinitesimal canonical transformation (III.12) that leaves the Hamiltonian of a given physical system invariant is called the symmetry canonical transformation of the associated system. The term *invariant* means that the nature of the physical system is unaltered by the transformation, which means that the old and transformed Hamiltonians have to be of the same *functional forms*. As a consequence, Eq. (II.7) can be written in the form of

\[ H(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_2}{\partial t}, \]  

(III.14)

Inserting Eqs. (II.11) and (II.12) into Eq. (III.14) and applying the Taylor expansion on \( H(Q_i, P_i, t) \) to the first order of \( \epsilon_i \) together with Eqs. (II.8) and (II.9), we get

\[ \frac{df_j(q_i, p_i, t)}{dt} = \{ f_j, H \} + \frac{\partial f_j}{\partial t} = 0, \]  

(III.15)

which means that the generators \( f_j(q_i, p_i, t) \) are the conserved quantities of the given physical system. Hence there is a closed relation between the existence of a conserved quantity and its associated symmetry canonical transformation. Since the generators \( f_j(q_i, p_i, t) \) are independent functions due to independent parameters \( \epsilon_j \), the conserved quantities obtained must be independent to each other.

We may obtain the conserved quantities \( f_j(q_i, p_i, t) \) associated with a given physical system by solving Eq. (III.15), which lead to the constructions of its associated symmetry canonical transformations according to Eq. (II.12). Since Eq. (III.15) is a linear partial differential equation of the first order, the following theorem of partial differential equations provides the general solutions of the equation (III.15):

**THEOREM 1** Consider a linear partial differential equation of the first order in the form of

\[ \sum_{i=1}^{n} P_i(x_1, ..., x_n) \frac{\partial z}{\partial x_i} = R(x_1, ..., x_n, z). \]  

(III.16)

If \( u_i(x_1, ..., x_n, z) = c_i, i = 1 ... n \) are independent solutions of the subsidiary equation

\[ \frac{dx_1}{P_1} = \frac{dx_2}{P_2} = ... = \frac{dx_n}{P_n} = \frac{dz}{R}, \]  

(III.17)

then the relation \( \Phi(u_1, ..., u_n) = 0 \) (where \( \Phi \) is arbitrary function) is a general solution of Eq. (III.16).

A rigorous proof of Theorem 1 is given in Ref. [12]. It should be noted that there are a maximum of \( n \) independent solutions satisfying Eq. (III.17).

As the consequences of Theorem 1 the general solution of Eq. (III.15) may be written as

\[ f_j(q_i, p_i, t) = \Phi(u_1, ..., u_{2n}), \]  

(III.18)
where $n$ is the number of configuration dimension, $\Phi$ is an arbitrary function, and $u_k(q_i, p_i, t), \ k = 1, 2, ..., 2n$ are the independent solutions of

$$
\frac{dq_1}{\partial p_1} = \ldots = \frac{dq_n}{\partial p_n} = - \frac{dp_1}{\partial q_1} = \ldots = - \frac{dp_n}{\partial q_n} = dt. \quad (\text{III.19})
$$

The form of the solution (\text{III.18}) indicates that there could be infinitely many solutions satisfying Eq. (\text{III.15}).

Inserting (\text{III.18}) into Eq. (\text{III.15}), we have

$$
\frac{du_k}{dt}(q_i, p_i, t) = \{u_k, H\} + \frac{\partial u_k}{\partial t} = 0, \quad k = 1, 2, ..., 2n. \quad (\text{III.20})
$$

This means that each independent solution of the subsidiary equation (\text{III.19}) corresponds to a conserved quantity of the associated physical system. The maximum number of independent solutions may be interpreted as the maximum number of all independent conserved quantities one can possibly find in the system, which depends on the number of configuration degrees of freedom.

4. Time-independent Conserved Quantities.

For the time-independent case, Eq. (\text{III.15}) reduces to

$$
\frac{df}{dt}(q_i, p_i) = \{f, H\} = 0. \quad (\text{IV.21})
$$

The general solution of (\text{IV.21}) may then be written in the form of

$$
f(q_i, p_i) = \Phi(u_1, ..., u_{2n-1}), \quad (\text{IV.22})
$$

where $u_i, \ i = 1 \ldots (2n - 1)$ are the independent solution of the subsidiary equation

$$
\frac{dq_1}{\partial p_1} = \ldots = \frac{dq_n}{\partial p_n} = - \frac{dp_1}{\partial q_1} = \ldots = - \frac{dp_n}{\partial q_n}. \quad (\text{IV.23})
$$

It is clear from Eq. (\text{IV.23}) that it is not possible to obtain a time-independent conserved quantity from a time-dependent physical system, since the $\frac{\partial H}{\partial p_i}$'s and $\frac{\partial H}{\partial q_i}$'s will contain explicit time-dependence. The maximum number of possible independent conserved quantities is $2n - 1$. It should be noted that Eq. (\text{IV.21}) admits a trivial solution $f(q_i, p_i) = H(q_i, p_i)$, and hence $f(q_i, p_i) = \Phi(H(q_i, p_i))$. This kind of solution will be automatically satisfied in any time-independent case.

For the purpose of demonstration, we shall use Eqs. (\text{IV.22}) and (\text{IV.23}) to re-derive time-independent conserved quantities of some well-known cases in one- and many-particle systems.

4.1. One-Particle Systems : Free Particles.

The simplest one-particle system is the free particle system, whose Hamiltonian in 3-dimensional case is given by

$$
H(q, p) = \frac{p^2_x + p^2_y + p^2_z}{2m}. \quad (\text{IV.24})
$$
The correspondent subsidiary equation can be written as
\[
\begin{align*}
\frac{dq_x}{p_x} &= \frac{dq_y}{p_y} = \frac{dq_z}{p_z} = \frac{-dp_x}{0} = \frac{-dp_y}{0} = \frac{-dp_z}{0},
\end{align*}
\] (IV.25)
and its independent solutions are
\[
\begin{align*}
p_x &= C_1, \\
p_y &= C_2, \\
p_z &= C_3, \\
L_x &\equiv q_y p_z - q_z p_y = C_4, \\
L_y &\equiv q_z p_x - q_x p_z = C_5.
\end{align*}
\] (IV.26)
The general solution of Eq. (IV.21) of the system is then
\[
f(q, p) = \Phi(p_x, p_y, p_z, L_x, L_y).
\] (IV.27)
From Eq. (IV.26), we have 5 independent solutions correspond to momenta in the \(x\), \(y\) and \(z\) directions, as well as angular momenta in the \(x\) and \(y\) directions. The angular momentum in the \(z\) direction \(L_z \equiv q_x p_y - q_y p_x\) can be expressed as the function of 5 independent conserved quantities as given in Eq. (IV.27).

4.2. One-Particle Systems: Central Forces.

The Hamiltonian of a central force system is given by
\[
H(q, p) = \frac{p^2}{2m} + V(r),
\] (IV.28)
where \(V(r)\) is the central force potential. For the 3-dimensional case, it is defined as
\[
V(r) = V(q_x^2 + q_y^2 + q_z^2). \tag{IV.29}
\]
The associated subsidiary equation associated is given by
\[
\begin{align*}
\frac{dq_x}{\frac{p_x}{m}} &= \frac{dq_y}{\frac{p_y}{m}} = \frac{dq_z}{\frac{p_z}{m}} = -\frac{dp_x}{\left(\frac{\partial V}{\partial q_x}\right)} = -\frac{dp_y}{\left(\frac{\partial V}{\partial q_y}\right)} = -\frac{dp_z}{\left(2q_x \frac{\partial V}{\partial u}\right)} \\
&\quad \iff \quad \frac{dq_x}{\frac{p_x}{m}} = \frac{dq_y}{\frac{p_y}{m}} = \frac{dq_z}{\frac{p_z}{m}} = -\frac{dp_x}{\left(\frac{\partial V}{\partial q_x}\right)} = -\frac{dp_y}{\left(\frac{\partial V}{\partial q_y}\right)} = -\frac{dp_z}{\left(2q_x \frac{\partial V}{\partial u}\right)},
\end{align*}
\] (IV.30)
where \(u = q_x^2 + q_y^2 + q_z^2\). The independent solutions are
\[
\begin{align*}
H(q, p) &= \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(q_x^2 + q_y^2 + q_z^2) = C_1, \\
L_x &= q_y p_z - q_z p_y = C_2, \\
L_y &= q_z p_x - q_x p_z = C_3, \\
L_z &= q_x p_y - q_y p_x = C_4.
\end{align*}
\] (IV.31)
There are 4 independent solutions associated with an arbitrary 3-dimensional central force system, i.e. the trivial solution associated with the total energy of the system, and 3 non-trivial solutions associated with the angular momentum in $x$, $y$ and $z$ directions. However, there may be 5 independent solutions obtained in some 3-dimensional systems, for instance, the 3-dimensional harmonic oscillators. The general solution of Eq. (IV.21) for the 3-dimensional central force system is then

$$f(q, p) = \Phi(H(q, p), L_x, L_y, L_z).$$

(IV.32)

4.3. Many-Particle Systems.

We now consider the simplest case in many-particle systems. Suppose there are $N$ objects with the same masses $m$. All the objects lie on a straight line, and each object is connected with its closest neighbourhoods by springs with constant $k$. For the simplicity, let us consider only vibrations along the straight line formed by the objects. The Hamiltonian of the system above is given by

$$H(q_i, p_i) = \frac{p_i^2}{2m} + \sum_{i=2}^{N} \left[ \frac{p_i^2}{2m} + \frac{(q_i - q_{i-1})^2}{2} \right],$$

(IV.33)

where $q_i$ and $p_i$ are conjugate coordinate and momentum of the $i$-th object. The associated subsidiary equation is

$$\left. \frac{dq_1}{(p_1/m)} \right|_{p_1} = \ldots = \left. \frac{dq_i}{(p_i/m)} \right|_{p_i} = \ldots = \left. \frac{dq_N}{(p_N/m)} \right|_{p_N} = \frac{dp_1}{k(q_2 - q_1)} = \frac{dp_2}{k(q_3 - 2q_2 + q_1)} = \ldots = \frac{dp_{N-1}}{k(q_N - 2q_{N-1} + q_{N-2})} = \frac{dp_N}{k(q_N - q_{N-1})},$$

(IV.34)

whose independent solutions are

$$H(q_i, p_i) = \frac{p_i^2}{2m} + \sum_{i=2}^{N} \left[ \frac{p_i^2}{2m} + \frac{(q_i - q_{i-1})^2}{2} \right] = C_1,$$

$$\sum_{i=1}^{N} p_i = C_2,$$

$$\sum_{i=1}^{N} q_i = C_3.$$  

(IV.35)
5. Time-dependent Conserved Quantities.

In the time-dependent case, Eq. (III.15) must be fully satisfied. There are two kinds of conserved quantities: one associated with a time-independent Hamiltonian, and another one associated with a time-dependent one. With the time-independent Hamiltonian, Eq. (III.15) still admits the trivial solution \( H(q, p) \), which means the total energy of the physical system is still conserved, as well as time-independent solutions. With the time-dependent Hamiltonian, this is no longer the case.

The subsidiary equation of Eq. (III.15) is given by Eq. (III.19), with the maximum number of its independent solutions being \( 2^n \). We shall use Eqs. (III.18) and (III.19) to derive time-dependent conserved quantities of time-independent Hamiltonians, with one-particle systems as the examples, as well as time-dependent conserved quantities associated with a time-dependent Hamiltonian. We shall only discuss the 1-dimensional cases for the non-zero potential systems since calculation difficulties arise in 2- and 3-dimensional as well as in many-particle cases.

5.1. Time-independent Physical Systems: Free Particles

Using the Hamiltonian of 3-dimensional free particle system (IV.24), we write the subsidiary equation as

\[
\frac{dq_x}{\left(\frac{p_x}{m}\right)} = \frac{dq_y}{\left(\frac{p_y}{m}\right)} = \frac{dq_z}{\left(\frac{p_z}{m}\right)} = -\frac{dp_x}{0} = -\frac{dp_y}{0} = \frac{dp_z}{0} = dt. \tag{V.36}
\]

The independent solutions for the system are

\[
p_x = C_1, \quad p_y = C_2, \quad p_z = C_3, \quad q_x - \frac{p_x}{m}t = C_4, \quad q_y - \frac{p_y}{m}t = C_5, \quad q_z - \frac{p_z}{m}t = C_6 \tag{V.37}
\]

The last 3 solutions in Eq. (V.37) are the expressions of homogeneous linear motions in 3-dimensional space. The general solutions of Eq. (III.15) for the 3-dimensional free particle systems are then

\[
f(q, p, t) = \tilde{\Phi} \left( p_x, p_y, p_z, q_x - \frac{p_x}{m}t, q_y - \frac{p_y}{m}t, q_z - \frac{p_z}{m}t \right), \tag{V.38}
\]

The angular momenta in 3-dimensional cases may be expressed by (V.38).

5.2. Time-independent Physical Systems: Central Forces

From the Hamiltonian of 1-dimensional free particle system

\[
H(q, p) = \frac{p^2}{2m} + V(q^2), \tag{V.39}
\]
one may write the associated subsidiary equation for 1-dimensional central force system as
\[ \frac{dq}{(\frac{m}{p})} = -\frac{dp}{(\frac{dV}{dq})} = dt. \] (V.40)

Unlike the derivations discussed in either Section 4 or Section 5.1, the treatment of obtaining time-dependent solution of Eq. (V.40) depends on the functional forms of the central force potentials. For the purpose of demonstration, we shall use the 1-dimensional harmonic oscillator and the 1-dimensional inverse square potential system.

For the 1-dimensional harmonic oscillator with potential \( V(q^2) = \frac{1}{2}kq^2 \), where \( k = \) constant, Eq. (V.40) becomes
\[ \frac{dq}{(\frac{p}{m})} = -\frac{dp}{kq} = dt, \] (V.41)
which is equivalent to
\[ \frac{d(p + i\sqrt{km}q)}{p + i\sqrt{km}q} = i\sqrt{\frac{k}{m}} dt. \] (V.42)
The independent solutions of (V.41) are
\[ \frac{p}{2m} + \frac{1}{2}kq^2 = H(q, p) = C_1, \]
\[ \ln(p + i\sqrt{km}q) - i\sqrt{\frac{k}{m}} t = C_2, \] (V.43)
where the second independent solution is obtained by integrating Eq. (V.42). The second solution of Eq. (V.43) gives us information of how the momentum \( p \) and coordinate \( q \) behave in the system. The general solution of (III.15) for a 1-dimensional harmonic oscillator is
\[ f(q, p, t) = \Phi \left( H(q, p), \ln(p + i\sqrt{km}q) - i\sqrt{\frac{k}{m}} t \right). \] (V.44)

With the 1-dimensional inverse square potential system, we use \( V(q^2) = -\frac{k}{q^2} \), where \( k = \) constant. Eq. (V.40) becomes
\[ \frac{dq}{(\frac{p}{m})} = -\frac{dp}{\frac{2k}{q^2}} = dt. \] (V.45)
The independent solutions of Eq. (V.45) are
\[ \frac{p}{2m} - \frac{k}{q^2} = H(q, p) = C_1, \]
\[ pq - 2H(q, p) = C_2, \] (V.46)
where the second solution in Eq. (V.46) is obtained from the equivalent relation extracted from Eq. (V.45):
\[ p dq - \frac{p^2}{m} dt = 0, \]
\[ q dp + \frac{2k}{q^2} dt = 0. \] (V.47)
The general solution of (III.15) for a 1-dimensional inverse square potential system is
\[ f(q, p, t) = \Phi(H(q, p), pq - 2H(q, p), t). \]  
(V.48)

5.3. Time-dependent Physical Systems

As mentioned in the beginning of Section 5, in the time-dependent physical systems the Hamiltonians are no longer conserved quantities, meaning that the subsidiary equations Eq. (III.19) do not have the Hamiltonians as one of their solutions.

Given the Hamiltonian of a 1-dimensional physical system as
\[ H(q, p, t) = \frac{p^2}{2m} + V(q, t), \]  
(V.49)
the subsidiary equations (III.19) may be expressed as
\[ \frac{dq}{\left(\frac{p}{m}\right)} = -\frac{dp}{\left(\frac{\partial V}{\partial q}\right)} = dt, \]  
(V.50)
where we have used partial derivative of \( V(q, t) \) instead of total derivative as in Eq. (V.40).

As in the central force case in Section 5.2, the treatment of obtaining solutions for Eq. (V.50) depends on the functional forms of the associated potentials. For the purpose of demonstration, let us consider a simple 1-dimensional system of a point mass \( m \) attached to a spring of constant \( k \), where the other end of the spring is fixed on a massless cart moving uniformly along \( q \) axis with speed \( v_0 \). The potential of the system is given by
\[ V(q, t) = \frac{k}{2}(q - v_0 t)^2. \]  
(V.51)
Inserting potential (V.51) into Eq. (V.50), we have
\[ \frac{dq}{\left(\frac{p}{m}\right)} = -\frac{dp}{k(q - v_0 t)} = dt, \]  
(V.52)
which maybe equivalently written as
\[ \frac{p}{m} dp + k(q - v_0 t) dq = 0, \]
\[ dq - \frac{p}{m} dt = 0, \]
\[ dp + k(q - v_0 t) dt = 0. \]  
(V.53)
Applying the total derivative of \( V(q, t) \) to the first equation of (V.53) and using the third equation of (V.53), we get
\[ \frac{p}{m} dp + dV - v_0 dp = 0, \]  
(V.54)
† The same problem is also discussed in Ref [7, pp. 350] using a different approach.
which can be easily integrated to obtain the associated independent solution:

\[
\frac{p^2}{2m} + \frac{1}{2} k(q - v_0 t)^2 - v_0 p = C,
\]
\[
\iff \left( \frac{(p - mv_0)^2}{2m} + \frac{1}{2} k(q - v_0 t)^2 - \frac{1}{2} mv_0^2 \right) = C,
\]
\[
\iff \left( \frac{(p - mv_0)^2}{2m} + \frac{1}{2} k(q - v_0 t)^2 \right) = C_1. \tag{V.55}
\]

Although the independent solution has the dimension of energy, it is not the total energy of the associated system, but the total energy of the point mass relative to the cart instead. The general solution of \((\text{III.15})\) for the associated system is

\[
f(q, p, t) = \bar{\Phi} \left( \frac{(p - mv_0)^2}{2m} + \frac{1}{2} k(q - v_0 t)^2 \right). \tag{V.56}
\]

6. Summary and Conclusion.

We have discussed the way of obtaining conserved quantities of a given physical system by solving Eq. \((\text{III.15})\) using the subsidiary equations \((\text{III.19})\). The maximum number of possible conserved quantities a physical system may have is given by Theorem 1. Some particular time-independent cases for one-particle and many-particle systems, as well as time-dependent cases for one-particle system have been used to demonstrate the technique and treatment of handling Eq. \((\text{III.19})\). For the same physical systems, it is found that one may not have the same results for the time-independent and time-dependent cases. More complicated treatment may be used to deal with time-dependent cases and many-particle systems.

Although the technique presented here is in the scope of classical mechanics, it may be applicable to classical fields, provided that the Poisson brackets in Eq. \((\text{III.15})\) re-formulated in an appropriate way in terms of field variables [1, 7]. Also, since the connection between classical and quantum mechanics can be established by replacing the Poisson brackets with Lie brackets or commutators, the conserved quantities in quantum mechanics may be obtained directly by replacing \((q, p)\) in the associated classical quantities with \((q, \hbar \frac{\partial}{\partial q})\).

As the symmetries of the physical system yields the existence of non-trivial conserved quantities in classical mechanics, the same argument is true for quantum system. Moreover, the existence of symmetries in quantum mechanics generates degeneracy which depends on the number of degree of freedom. For example, since there is no non-trivial conserved quantity in a time-independent one-dimensional harmonic oscillator, no degeneracy occurs in the quantum counterpart. The degeneracy of harmonic oscillator system shows up in 2- and 3-dimensional cases.

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