Spin-Depairing Transition of Attractive Fermi Gases on a Ring Driven by Synthetic Gauge Fields

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Motivated by the recent experimental realization of synthetic gauge fields in ultracold atoms, we investigate one-dimensional attractive Fermi gases with a time-dependent gauge flux on the spin sector. By combining the methods of the Bethe ansatz with complex twists and Landau-Dykhne, it is shown that a spin-depairing transition occurs, which may represent a nonequilibrium transition from fermionic superfluids to normal states with spin currents caused by a many-body quantum tunneling. For the case of the Hubbard ring at half filling, our finding forms a dual concept with the dielectric breakdown of the Mott insulator discussed in Phys. Rev. B 81, 033103 (2010). We analyze cases of arbitrary filling and continuum model, and show how filling affects the transition probability.

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I. INTRODUCTION

Owing to high controllability and flexibility of experiments, ultracold atoms have offered a testing ground to simulate a strongly correlated system in one dimension both under equilibrium and nonequilibrium conditions [1, 2]. Recent experimental efforts make it possible to study the system on a ring, which has been realized in continuum space [3, 4] and in optical lattice [5]. As far as two-component Fermi gases on a ring in equilibrium are concerned, one may obtain exact solutions by the Bethe ansatz method regardless of continuum [3] or lattice [7], and many of the properties of the systems can be examined in combination with the bosonization approach. However, as for what happens in the Fermi gases in nonequilibrium, a unified approach has yet to be established and our current knowledge is far from a comprehensive level.

Meanwhile, in recent years studies of synthetic gauge fields have attracted attention in cold-atom community [8] and the NIST group has succeeded in producing synthetic magnetic [9] and electric fields [10], and spin-orbit coupling [11] on a continuum space. In particular, the electric fields discussed in Ref. [10] have been realized with the creation of time-dependent gauge fields. In addition, very recently, Aidelsburger et al. have succeeded in creating synthetic gauge fields on an optical lattice [12], which may pave the way to areas of study in cold atomic gases. In this paper, based on the situations described above, we consider a two-component attractive Fermi gas on a ring with a time-dependent gauge flux on the spin sector at absolute zero. In the absence or presence of a time-independent gauge flux, it is well known that the ground state of the two-component attractive Fermi gas is filled with bound states of up-spin and down-spin particles and the spin excitation has a gap, which is attributed to the appearance of fermionic superfluidity [13]. Here, we demonstrate that in the presence of a time-dependent gauge flux $\Phi(t) = Ft$, a spin-depairing transition induced by a many-body quantum tunneling occurs, which may represent a breaking phenomenon of fermionic superfluidity in nonequilibrium. Our prediction relies on a recently proposed method combining the Bethe ansatz method with complex twists [16, 17] and the Landau-Dykhne method [18–20] with which a quantum tunneling probability between two states can be evaluated. In fact, for the Hubbard model, our finding builds a dual concept with the nonequilibrium transition from Mott insulators to metals in the repulsive Hubbard ring at half filling with a time-dependent charge flux discussed in the context of electronic systems [14]. In addition, we show that by using the formalism with dressed energies [13, 21], the spin-depairing transition occurs not only at half filling but also other filling cases, and in continuum model. This is peculiar to the attractive Fermi gases and is in contrast with the breakdown of the Mott insulator, which occurs in lattice systems only at half filling.

This paper is organized as follows. Section II discusses a Hamiltonian with synthetic gauge fields and shows that effects of gauge fields can be incorporated into boundary conditions. Section III examines Bethe ansatz equations and spin gap with complex spin twists. We analytically show that the spin gap closes at a critical value. In Sec. IV, we relate the real spin twist to the complex twist with the Landau-Dykhne method and evaluate the transition probability. We also comment on the ground-state decay rate. Section V summarizes the contributions and examines the continuum model.

II. HAMILTONIAN

We start by considering a Hamiltonian of two component atoms influenced by synthetic gauge fields to yield the spin-orbit coupling. In the continuum space, the kinetic term becomes the covariant form: $-(\nabla + i\tilde{A})^{2}$, where $\tilde{A} = \sum_{a=1}^{3} \tilde{A}_{a}(x,t)\tau^{a}$ represents the synthetic gauge field and $\tau^{a}$ is the generator of SU(2). We then wish to consider the situation that the particles are loaded into an optical lattice. Let us assume that the
lattice potential is so deep that the energy gap between first and second bands is much larger than the other effects such as the thermal and mean-field interaction energies per particle. If $V$ is the lattice potential and $\omega$ is the Wannier function, which is localized at each lattice point, the hopping amplitude becomes

$$
t_{ij} = \int d\bar{x} \omega^*(\bar{x} - \bar{R}_i)[-(\bar{\nabla} + i\bar{A})^2 + V(x)]\omega(\bar{x} - \bar{R}_j)
$$

$$
= \int d\bar{x} \omega^*(\bar{x} - \bar{R}_i)S^{-1}(x, x_0)[-(\bar{\nabla}^2 + V(x)]
	imes S(x, x_0)\omega(\bar{x} - \bar{R}_j) \cong S(i, j)t_{ij},
$$

(1)

where $t_{ij} = \int d\bar{x} \omega^*(\bar{x} - \bar{R}_i)[-(\bar{\nabla}^2 + V(x)]\omega(\bar{x} - \bar{R}_j)$ is the hopping amplitude at $\bar{A} = 0$.

$S(i, j) = \lim_{N \to \infty} \prod_{n=0}^{N-1} [1 + iA(\bar{R}_i + n\Delta\bar{y}, t) \cdot \Delta\bar{y}]$

$$
= P \exp \left[ i \int_{\bar{R}_i}^{\bar{R}_j} A(\bar{y}, t) \cdot d\bar{y} \right]
$$

(2)

is the so-called Wilson line in gauge theory, and $P$ denotes the path-ordered product $[22]$. Here, to obtain the last expression of Eq. (1), we used the assumption that the Wannier function is tightly localized at each lattice point.

Although the above discussion is independent of space dimensions, geometry of systems, and statistics of particles, we hereafter concentrate on a one-dimensional ring, and assume that the hopping is site-independent. Then, the Hamiltonian of two-component Fermi gases with a contact attractive interaction in units of the hopping strength is given by

$$
H = - \sum_{i, \sigma, \sigma'} \epsilon_{i\sigma}^\dagger S(i, i + 1)_{\sigma\sigma'} \epsilon_{i+1\sigma'} + \text{H.c.}
$$

$$
-4|u| \sum_i n_{i\uparrow} n_{i\downarrow}.
$$

(3)

We note that this Hamiltonian corresponds to that analyzed in Ref. [22] except for points that the attractive case is concerned and there is no U(1) charge flux in our treatment. In the following, for simplicity we consider the case that one of $\bar{A}_i$, say, $\bar{A}_2$ has a nonzero value, which is comparable to the case of SO(2) spin symmetry. This simplified gauge configuration is also considered in the continuum space in Refs. [24, 25]. Then, by considering the following transformation $[23]$:

$c_{i\sigma} = [S(i, i - 1)S(i - 1, i - 2) \cdots S(2, 1)]_{\sigma\sigma'} \epsilon_{i\sigma'}$,  

(4)

Eq. (3) is rewritten as follows:

$$
H = - \sum_{i, \sigma} \epsilon_{i\sigma}^\dagger \epsilon_{i+1\sigma} + \text{H.c.} - 4|u| \sum_i n_{i\uparrow} n_{i\downarrow}
$$

(5)

Let us briefly comment on a role of $\Phi$. As discussed in Refs. [23, 26, 27], in the repulsive Hubbard model, $\Phi$ induces spin currents since $\Phi$ affects the spin sector and there is no spin gap. In contrast, in the attractive Hubbard model without spin imbalance, since there is a spin gap, spin currents do not flow with a static $\Phi$, and therefore $\Phi$ does not play any role. In subsequent sections, however, we show that the time-dependent gauge flux $\Phi(t) = Ft$ induces spin currents through a many-body quantum tunneling as shown in Fig. [11].

![FIG. 1: (Color online) Schematic illustration of spin currents induced by the time-dependent gauge flux $\Phi(t) = Ft$ on the spin sector. Each black dot without an arrow indicates a particle with spin-up and counterclockwise flow or spin-down and clockwise flow.](image)

with

$$
c_{L+1\sigma} = \exp[i\sigma \Phi L]c_{1\sigma},
$$

(6)

where $L$ is the number of sites, $\sigma = \pm 1$ (or $\sigma$ represents $\uparrow$ or $\downarrow$), and $\Phi$ is the gauge flux per site. Equation (6) indicates that the effect of gauge fields is incorporated into twisted boundary conditions.

Let us briefly comment on a role of $\Phi$. As discussed in Refs. [23, 26, 27], in the repulsive Hubbard model, $\Phi$ induces spin currents since $\Phi$ affects the spin sector and there is no spin gap. In contrast, in the attractive Hubbard model without spin imbalance, since there is a spin gap, spin currents do not flow with a static $\Phi$, and therefore $\Phi$ does not play any role. In subsequent sections, however, we show that the time-dependent gauge flux $\Phi(t) = Ft$ induces spin currents through a many-body quantum tunneling as shown in Fig. [11].

### III. BETHE ANSATZ EQUATIONS WITH COMPLEX SPIN TWISTS

Let us consider the Hamiltonian (5) with complex spin twists, which is equivalent to the substitution $\Phi = i\Psi$ in Eq. (5) and is related to the repulsive Hubbard ring with complex charge twists. As for complex charge twists, this model has been phenomenologically introduced in Ref. [16] to describe the dissipative tunneling into the environment and analyze the breakdown of the Mott insulator. In fact, it has been shown that due to complex charge twists the Mott gap closes, which evidences the breakdown of the Mott insulator. In this section, we discuss the spin-depairing transition due to complex spin twists. The connection between the Hubbard ring with real twists and that with complex twists is shown in Sec. IV based on the analysis in Ref. [14].

Now we examine solutions of the Hubbard ring with complex spin twists. With absence of the twists, our model reduces to the one-dimensional Hubbard model with periodic boundary conditions, which can be solved
exactly by the Bethe ansatz method \[7\]. While in the presence of \[9\], the boundary condition is altered from periodic to twisted ones, we can still obtain exact solutions by the same method \[16, 27\], and the corresponding Bethe ansatz equations are then for charge \(k\) and spin \(\lambda\) rapidities,

\[
e^{i k_j L} = e^{-\Psi(L)} \prod_{\beta=1}^{N_j} \lambda_\beta - \sin k_j + i |u| = 0, \tag{7}\]

\[
\prod_{j=1}^{N} \lambda_\alpha - \sin k_j + i |u| = e^{-2\Psi L} \prod_{\beta=1}^{N_j} \lambda_\alpha - \lambda_\beta + 2i |u|, \tag{8}\]

where \(N\) and \(N_j\) are the numbers of total particles and particles with down spin, respectively. We note that the term \(e^{2\Psi L}\) in Eq. \(8\) is peculiar to the case of the spin twist. The energy of the system is then given by

\[
E = -2 \sum_j \cos k_j. \tag{9}\]

In this paper, we only consider the case of \(N = 2N_j\), which implies that there is no spin imbalance. When \(\Psi = 0\), the ground state is then filled with the spin pairs, each of which forms the bound state of a particle with spin-up and a particle with spin-down to be called the \(k - \lambda\) string

\[
\sin k_\alpha^\pm = \lambda_\alpha + i |u| + O(e^{-\eta L}), \tag{10}\]

where \(\eta\) is assumed to be a positive number of the order of unity. We see that Eq. \(10\) is consistent with Eq. \(8\). When \(\Psi \neq 0\), however, the \(k - \lambda\) string solutions \(10\) no longer satisfy Eq. \(8\). At the same time, if \(\Psi\) is less than a critical value, which is determined below, it is expected that \(k - \lambda\) string solutions still exist in the ground state. In fact, we find that the following \(k - \lambda\) string trivially satisfies Eq. \(8\):

\[
\sin k_\alpha^\mp = \lambda_\alpha - i |u| + O(e^{-\eta L}), \tag{11}\]

where \(\eta \pm \Psi\) are assumed to be positive. Namely, the modification only occurs in the order of \(e^{-\eta L}\). Taking into account the fact that it is difficult to show the mathematical existence of Eq. \(10\), the same difficulty also occurs in Eq. \(11\). However, since we can check the consistency with the Hubbard ring with complex charge twists at half filling where there is the SO(4) symmetry \[13\], we believe that Eq. \(11\) is correct at arbitrary filling. On the other hand, Eq. \(7\) becomes

\[
e^{i(k_\alpha^+ + k_\alpha^-) L} = \prod_{\beta=1}^{N_j} \frac{\lambda_\alpha - \lambda_\beta - 2i |u|}{\lambda_\alpha - \lambda_\beta + 2i |u|}. \tag{12}\]

By taking logarithm of Eq. \(12\), we obtain

\[
2L Re \left[ \sin^{-1}(\lambda_\alpha + i |u|) \right] = 2\pi J_\alpha - \sum_{\beta=1}^{N_j} \theta \left( \frac{\lambda_\alpha - \lambda_\beta}{2} \right), \tag{13}\]

where \(\theta(x) = -2 \tan^{-1}(x/|u|)\) and \(J_\alpha\) is integer or half integer. We note that since Eq. \(13\) does not depend on \(\Psi\), the configuration of spin rapidities at \(\Psi \neq 0\) is equal to that at \(\Psi = 0\). Considering that the string solutions require the tight condition between \(k_\alpha^+\) and \(\lambda_\alpha\) and therefore \(\lambda_\alpha\) cannot be changed by \(\Psi\), this may be the natural result.

Let us next consider the spin excitation above the ground state. As stated above, the spin excitation has a gap to be attributed to the appearance of fermionic superfluidity. In order to see what happens by complex twists, let us consider a spin-triplet excitation, which is accomplished by the manipulation breaking one of the \(k - \lambda\) string pairs and creating two charge rapidities \(k_1\) and \(k_2\) \[13\]. We find that the corresponding Bethe ansatz equations are given by

\[
2L Re \left[ \sin^{-1}(\lambda_\alpha + i |u|) \right] = 2\pi J_\alpha - \sum_{\beta=1}^{N_j-1} \theta \left( \frac{\lambda_\alpha - \lambda_\beta}{2} \right) - \sum_{j=1}^{2} \theta(\lambda_\alpha - \sin k_j), \tag{14}\]

\[
k_j L = 2\pi I_j + i L \Psi - \sum_{\beta=1}^{N_j-1} \theta(\sin(k_j - \lambda_\beta)), \quad (j = 1 \text{ or } 2) \tag{15}\]

where \(J_\alpha\) and \(I_j\) are integers or half integers. Compared with the Bethe ansatz equations in the ground state, it is clear that solutions of the above equations depend on \(\Psi\) explicitly. Since \(\lambda_\alpha\) represent the \(k - \lambda\) strings, it is expected that the shift of the distribution of \(\lambda_\alpha\) by \(\Psi\) is tiny and major contributions by \(\Psi\) come from \(k\) and \(\lambda\) equations are given by

\[
twist. The energy of the system is then given by

\[
E = -2 \sum_j \cos k_j. \tag{9}\]

Analysis in the Bulk Limit

We hereafter consider the bulk limit, namely, \(N \rightarrow \infty\), \(L \rightarrow \infty\), and \(N/L\) to be a constant, and introduce the ground-state distribution of \(\lambda_\alpha\) as \(\sigma_0(\lambda_\alpha) = 1/L(\lambda_\alpha+1 - \lambda_\alpha).\)
(a) $L = 2N_\downarrow = 50$

(b) $L = 50$, $N_\downarrow = 15$

FIG. 2: Possible values of $k_j$ with $|u| = 1.0$ and $\Psi = 0.2$. By comparing (a) with (b), we see that for fixed $\Psi$, the maximum value of $\text{Im}(k_j)$ at half filling is larger than that below half filling.

$|u|$). By taking derivative of Eq. (13), we have

$$\frac{1}{\pi} \text{Re} \left[ \frac{1}{\sqrt{1 - (\lambda + i|u|)^2}} \right] = \sigma_0(\lambda)$$

$$\frac{1}{\pi} \int_{-B}^{B} d\lambda' \sigma_0(\lambda') \frac{2|u|}{(2|u|)^2 + (\lambda - \lambda')^2}.$$  \hspace{1cm} (16)

This equation determines the ground-state distribution. We note that $B$ is related to filling $n$ as follows:

$$n = \frac{N}{L} = 2 \int_{-B}^{B} d\lambda \sigma_0(\lambda).$$  \hspace{1cm} (17)

It is straightforward to check that when $B = \infty$, $n = 1$, that is, half filling. Meanwhile, when $B < \infty$, we obtain $n < 1$. In addition, by performing the particle-hole transformation, the results of $n > 1$ can be obtained from those of $n < 1$. Therefore, in what follows, we only analyze the cases of $n \leq 1$.

As for the spin excitation, by introducing the counting functions $z_c(k_j) = I_j/L$ and $z_s(\lambda_\alpha) = J_\alpha/L$, Eqs. (14)}
and \((15)\) reduce to

\[
z_s(\lambda) = \frac{1}{\pi} \text{Re} \left[ \sin^{-1}(\lambda + i|u|) \right] + \frac{1}{2\pi L} \sum_{j=1}^{2} \theta(\lambda - \sin k_j)
\]
\[
+ \frac{1}{2\pi} \int_{-B}^{B} d\lambda \theta \left( \frac{\lambda - \lambda'}{2} \right) \sigma(\lambda'),
\]
\[
z_c(k) = \frac{k}{2\pi} - \frac{i\Psi}{2\pi} + \frac{1}{2\pi} \int_{-B}^{B} d\theta (\sin k - \lambda) \sigma(\lambda),
\]

where \(\sigma(\lambda) = \sigma_0(\lambda) + [\sigma^{k_1}_1(\lambda) + \sigma^{k_2}_1(\lambda)]/L\) with the shift of the distribution \(\sigma^{k_j}_1(\lambda)\). By taking derivative of Eq. \((19)\), we obtain

\[
\sigma(\lambda) = \frac{1}{\pi} \text{Re} \left[ \frac{1}{\sqrt{1 - (\lambda + i|u|)^2}} \right]
\]
\[
+ \frac{1}{\pi L} \sum_{j=1}^{2} \frac{|u|}{|u| + (\lambda - \sin k_j)^2}
\]
\[
- \frac{1}{\pi} \int_{-B}^{B} d\lambda' \frac{2|u|\sigma(\lambda')}{(2|u|)^2 + (\lambda - \lambda')^2}.
\]

We note that this equation tells us when the analytic properties of the system change. In fact, there is a possibility that the last term of the right-hand side has poles in the complex \(k\) plane. Since the poles of the nearest to the real axis are \(\pm i \sinh^{-1}|u|\), the critical value \(b_{cr}\) \((= \text{Im}(k_{cr}))\) is

\[
b_{cr} = \sinh^{-1}|u|,
\]

which is equal to the condition that the Mott insulator is broken in the repulsive Hubbard ring \((16)\). If \(b = \text{Im}(k)\) is less than \(b_{cr}\), the system is still in the spin-gapped phase in which the distribution is obtained from Eq. \((18)\). In contrast, at the critical value \(b_{cr}\), the analytic properties of the system change. Therefore the breakdown of the spin-gapped phase, namely, the spin-depairing transition occurs.

A relation between \(\Psi\) and \(b\) is obtained from Eq. \((19)\). By using the fact that the maximum of \(\text{Im}(k)\) is located on \(\text{Re}(k) = 0\) and \(z_c(k = ib) = 0\), and neglecting \(1/L\) corrections on the distribution, we have

\[
\Psi = b - i \int_{-B}^{B} d\lambda \theta(\lambda + i \sinh b)\sigma_0(\lambda),
\]

which determines the critical value of the twist, \(\Psi_{cr}\). We note that the above equation is equal to that appearing in the breakdown of the Mott insulator in the repulsive case \((16)\) if \(B = \infty\), which corresponds to the case without imbalance. Figure \(3\) shows \(\Psi_{cr}\) in several values of \(|u|\) as a function of filling and represents that \(\Psi_{cr}\) increase with decreasing filling in any \(|u|\). This implies that the spin-gapped state becomes robust as filling is decreased, and is consistent with Fig. \(2\) which indicates that \(b\) decreases with decreasing filling. In addition, while the increase of

\[
\Psi_{cr}\text{ for small }|u|\text{ is tiny at large filling and sharply-rising at small filling, that for large }|u|\text{ is always smooth. This is because for large }|u|,\text{ the main contribution of Eq. } (22)\text{ comes from the first term but for small }|u|\text{ that is not so.}
\]
Spin Gap

We now evaluate the spin gap based on the above arguments. Although the analytical expressions can be obtained at half filling with the Fourier transformation as in Ref. [10], we wish to consider arbitrary filling cases and continuum model. To this end, we adopt an approach with the dressed energy [13, 21], where the elementary excitations over the ground state are obtained by solving the following equations:

\[ \epsilon(\lambda) = -4 \text{Re}\sqrt{1 - (\lambda - i|u|^2)^2} - 2\mu \]
\[ -\frac{1}{\pi} \int_{-B}^{B} d\lambda' \frac{2|u|\epsilon(\lambda')}{(2|u|^2 + (\lambda - \lambda')^2)}, \]
\[ e(k) = -2 \cos k - \mu - \frac{1}{\pi} \int_{-B}^{B} d\lambda \frac{|u|\epsilon(\lambda)}{|u|^2 + (\sin k - \lambda)^2}. \]

(23)

(24)

Here, \( \mu \) is the chemical potential, \( -\epsilon(\lambda) \) denotes the charge (pair) excitation, which satisfies \( \epsilon(\pm B) = 0 \), and \( e(k) \) denotes the spin (particle) excitation. We note that when \( k = ib_{\text{cr}} \), Eq. (24) is nonanalytic, which is consistent with the argument below Eq. (20). This is due to the fact that Eqs. (23) and (24) are related to Eqs. (15) and (21). At half filling, \( \mu = -2|u| \) and \( B = \infty \), and it is straightforward to check that in this case, \( -\epsilon(\lambda) \) and \( e(k) \) correspond to the expressions of the charge and spin excitations at half filling, respectively. When \( B = 0 \), the system is in the empty band, and the chemical potential is then given by \( -2\sqrt{1 + |u|^2} \). Therefore, possible values of chemical potential are

\[ -2\sqrt{1 + |u|^2} \leq \mu \leq -2|u|. \]

(25)

The spin-triplet excitation is then given by

\[ \Delta E(k_1, k_2) = e(k_1) + e(k_2) = -2\mu - \sum_{j=1}^{2} \left[ 2 \cos k_j \right. \]
\[ + \frac{1}{\pi} \int_{-B}^{B} d\lambda \frac{|u|\epsilon(\lambda)}{|u|^2 + (\sin k_j - \lambda)^2} \].

(26)

Figure 4 depicts the spin gap with \( \Psi = 0 \) in several values of \( |u| \) as a function of \( n \) and shows that the spin-gapped state is robust at small \( n \) with the fixed \( |u| \). This is because the relative effect of \( |u| \) is significant and it is expected that the tightly-bound spin pairs are formed at small \( n \). In the numerical calculation, we first determine \( \epsilon(\lambda) \) by solving Eq. (23) with the iteration method, and then calculate the spin gap by assigning values to Eq. (24). By substituting \( k_j = ib \) into Eq. (26), the spin gap at \( \Psi \leq \Psi_{\text{cr}} \) is given by

\[ \Delta_{\text{spin}}(b) = -2\mu - 4 \cosh b \]
\[ - \frac{2}{\pi} \int_{-B}^{B} d\lambda \frac{|u|\epsilon(\lambda)}{|u|^2 + (i \sinh b - \lambda)^2}. \]

(27)

Gap Closing at the Critical Twist

As discussed above, since the analytic properties of the ground state change at the critical twist, the spin gap should close at the value. We analytically show this as follows. When \( b = b_{\text{cr}} - 0 \), we have

\[ \frac{|u|}{|u|^2 + (i \sinh b - \lambda)^2} = \frac{1}{2} \left( \text{p.v.} \frac{i}{\lambda} + \pi\delta(\lambda) \right. \]
\[ \left. + \frac{1}{2|u| - \eta + i\lambda} \right), \]

(28)

where \( 0 < \eta \ll 1 \), p.v. denotes the principal value, and we used \( 1/(x + i\eta) = \text{p.v.}(1/x) - i\pi\delta(x) \). By taking into account the property that the integral including the principal value vanishes, we obtain

\[ \Delta_{\text{spin}}(b_{\text{cr}} - 0) = -2\mu - 4\sqrt{1 + (|u| - \eta)^2} - \epsilon(0) \]
\[ - \frac{1}{\pi} \int_{-B}^{B} d\lambda \frac{\epsilon(\lambda)}{2|u| - \eta + i\lambda}. \]

(29)

On the other hand, Eq. (23) indicates

\[ \epsilon(0) = -2\mu - 4\sqrt{1 + |u|^2} - \frac{1}{\pi} \int_{-B}^{B} d\lambda \frac{\epsilon(\lambda)}{2|u| + i\lambda}. \]

(30)

From Eqs. (29) and (30), we conclude that \( \Delta_{\text{spin}}(b_{\text{cr}}) = 0 \) at arbitrary filling, which ensures that the spin-depairing transition occurs at the critical twist.

IV. TRANSITION PROBABILITY

As first discussed in Ref. [14], the Landau-Dykhnne method [18, 20] makes it possible to relate the original Hermitian problems (5) and (6) to those of the Hubbard ring with complex twists analyzed in Sec. III. To this end, let us consider a time-dependent gauge flux \( \Phi(t) = Ft \) with a constant \( F \) in Eqs. (5) and (6), which is switched on at \( t = 0 \). Then, based on the analytic continuation of the energy levels as a function of complex time, the Landau-Dykhnne method enables us to evaluate the spin-depairing transition probability, which is given by

\[ P = \exp \left[ -2 \text{Im} \int_{0}^{t^*} dt' \Delta_{\text{spin}}(\Phi(t')) \right], \]

(31)

where \( t^* \) is the complex time in which the level crossing occurs. As pointed out in Ref. [14], the above integral path lies on the imaginary axis in the bulk limit. This implies the relationship \( \Phi = i\Psi \) in the same limit, which reduces to the Hubbard ring with complex twists. Therefore, the transition probability becomes \( P = e^{-2\pi F \sin \Phi / F} \).
with
\[
F_{\text{th}} = \frac{2}{\pi} \int_0^{b_{\text{cr}}} \Delta_{\text{spin}}(b) \frac{d\Psi}{db} db
\]
\[
= \frac{4}{\pi} \int_0^{\sinh^{-1}|u|} \left[ 1 - \int_B^{|u|} d\lambda \frac{2|u| \cosh \sigma_0(\lambda)}{|u|^2 + (i \sinh b - \lambda)^2} \right]
\]
\[
- \mu - 2 \cosh b - \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \frac{|u| e(\lambda)}{|u|^2 + (i \sinh b - \lambda)^2} db,
\]
(32)
where \(F_{\text{th}}\) is the threshold field. Figure 5 plots the behavior of \(F_{\text{th}}\) as a function of filling and shows that the system is vulnerable to the transition toward half filling. In addition to this, the increase of \(F_{\text{th}}\) for large \(|u|\) is smooth, while that for small \(|u|\) has a large curvature around some filling. This may be because while for large \(|u|\) the spin pairs are tightly bounded at arbitrary filling, for small \(|u|\) the spin pairs are weakly bounded around half filling but tightly bounded below some filling.

We finally comment on the ground-state decay rate \(\Gamma\), which is related to the transition probability as follows
\[\Gamma/L = \frac{aF}{2\pi} \ln[1 - P],\] (33)
with an empirical factor \(a\) to be of the order of 1 and to describe the suppression of the tunneling. In the repulsive Hubbard case at half filling, this empirical factor has been attributed to the pair-annihilation processes and determined as a function of \(u\) with the time-dependent density matrix renormalization group [28]. On the other hand, in the attractive Hubbard case at arbitrary filling, \(a\) is still needed to describe the particle-annihilation processes but has yet to be determined. The determination of \(a\) may be done in the same way as [28].

V. SUMMARY AND OUTLOOK

We have investigated the attractive Hubbard ring with a time-dependent gauge flux on the spin sector combining the methods of the Bethe ansatz with complex twists and Landau-Dykhne, and shown that the spin-depairing transition occurs. We point out that this transition may represent the nonequilibrium transition from fermionic superfluids to normal states with spin currents triggered by a many-body quantum tunneling. By using the formulation with the dressed energies, it has been shown that as the spin gap increases with decreasing filling, the threshold field strength in which the spin-depairing transition occurs becomes large as filling is decreased. Although our analysis has been performed for the specific time-dependent gauge flux as \(\Phi(t) = Ft\), taking into account the fact that the breakdown of the Mott insulator occurs for another time-dependent gauge flux [15], the same consideration may be obtained in the spin-depairing transition.

\[H = \int dx \left[ -\sum_{\sigma} \phi_{\sigma}^\dagger(x) \left( \partial_x + i\sigma A(x,t) \right)^2 \phi_{\sigma}(x) - 4|u|\phi_{\sigma}^\dagger(x)\phi_{\sigma}^\dagger(x)\phi_{\sigma}(x)\phi_{\sigma}(x) \right].\] (34)

![FIG. 5: Value of the threshold field strength. In (a), \(F_{\text{th}} \neq 0\) in every filling while it takes a small value.](image)
This is because effects of gauge fields are again replaced by boundary conditions \[29\]. The continuum model is reached by passing to limits \(|u| \to 0\) and \(n \ll 1\) in the Hubbard model, which is due to the fact that mathematically the continuum model corresponds to the continuum limit of the Hubbard model \[13\]. By considering the above limits, we have in the continuum model, \(\sinh b \to b\) in Eq. \(22\), \(\text{Re}\sqrt{1 - (\lambda - i|u|)^2} \to 1 - (\lambda^2 + |u|^2)/2\) in Eq. \(23\), and \(\cos k \to 1 - k^2/2\) and \(\sin k \to k\) in Eq. \(24\). Therefore, we do not need a special consideration to calculate the transition probability in the continuum model.

We also point out that although we have analyzed the case of the spin \(SO(2)\) gauge field for experimental simplicity, the same treatment is applicable to cases of \(SU(2)\) gauge fields since effects of the gauge fields are again incorporated into boundary conditions \[23\]. As far as the continuum models are concerned, in general, we obtain exact solutions by the Bethe ansatz method for attractive \(SU(N)\) Fermi gases on a ring in which the spin excitations have gaps \[30\]. Hence, the spin-depairing transition in the \(SU(N)\) Fermi gases is also expected, which may be realized with, for example, ytterbium fermionic isotopes \[31\].

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