CLASSICAL DEFORMATIONS OF LOCAL SURFACES
AND THEIR MODULI OF INSTANTONS

SEVERIN BARMEIER AND ELIZABETH GASPARIM

ABSTRACT. We describe the semiuniversal deformation spaces for the noncompact surfaces $Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ and prove that any nontrivial deformation $Z_k$ of $Z_k$ is affine.

It is known that the moduli spaces of instantons of charge $j$ on $Z_k$ are quasi-projective varieties of dimension $2j - k - 2$. In contrast, our results imply that the moduli spaces of instantons on any nontrivial deformation $Z_k$ are empty.

Contents

1. Motivation
2. Statement of results
3. Local surfaces
4. Geometry and topology of $Z_k$
   4.1. Line bundles on $Z_k$
   4.2. Vector bundles on $Z_k$
   4.3. Moduli
5. Classical deformations
   5.1. Deformations via the tangent bundle
   5.2. Deformations of the transition function
   5.3. Relation to deformations of Hirzebruch surfaces
6. Geometry and topology of $Z_k(\tau)$
   6.1. Line bundles on $Z_k(\tau)$
   6.2. Cohomology of $Z_k(\tau)$
   6.3. Vector bundles on $Z_k(\tau)$
   6.4. The affine structure of $Z_k(\tau)$
7. Applications to the theory of instantons
References

1. Motivation

Our interest in the deformation theory of local surfaces and their moduli of vector bundles arose in the attempt to understand how instanton moduli vary in families.

Theories of instantons and their moduli are often defined over non-compact spaces, as is the case with the instanton partition function,
defined by Nekrasov [24] and explored by various authors, for instance [25, 22, 15, 10].

In this paper we study how moduli of instantons on the local surfaces $Z_k = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ behave under deformation of the complex structure of the underlying surface. Our approach is to develop the deformation theory for the surfaces and then study holomorphic vector bundles on the deformed surfaces; the correspondence of Kobayashi–Hitchin [20] associates instantons on complex surfaces to holomorphic vector bundles, which give us the corresponding information about instantons. (Also see [27] for a version of the correspondence that fits our noncompact situation.)

In contrast to the deformation theory of compact complex manifolds developed by Kodaira and Spencer [19], a general deformation theory for noncompact complex manifolds has yet to be developed. However, under certain additional assumptions the deformation theory of noncompact complex manifolds seems to be well-behaved: for example, in case the manifold admits a global holomorphic symplectic form [17], or when the manifold compactifies holomorphically, in which case the machinery for compact manifolds may be applied [18]. In §5 we present an approach based on explicit computations of Čech cohomology, which for the study of local surfaces $Z_k$ seems to be the most economical. Relations to the deformations of Hirzebruch surfaces are given in §5.3.

The local surfaces $Z_k$ admit a rich structure of instanton moduli. Some properties of these moduli are described in [14], where it is shown that such moduli spaces are quasi-projective varieties whose dimensions increase with the topological charge. Here we show that after an infinitesimal deformation of the surface the moduli of instantons are empty, i.e. instantons disappear after an infinitesimal “classical” deformation of $Z_k$. From the point of view of mathematical physics, our results suggest that to study the instanton moduli in families, deformations should be considered in a broader framework, including noncommutative deformations. In a future paper we will pursue this more general approach.

Even though the whole story for our original motivation has yet to be told, we decided to publish the results pertaining to classical deformations separately, because these are – to the best of our knowledge – the first results in this direction and, moreover, they turned out to be also of independent interest: deformations of local surfaces (quite mysteriously) appear in an unrelated body of work concerning the homological mirror symmetry conjecture from a Lie theoretical viewpoint (see [5] and Remark 6.16).

We give a short overview of the organization of the paper. In §§3–4 we introduce local surfaces and their moduli of vector bundles. In §5 we present deformations of local surfaces and study their moduli of vector
2. Statement of results

We consider local surfaces that are total spaces of negative line bundles on the projective line, denoted by $Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$, for $k \geq 1$. Our first result (Theorem 5.3) shows that $Z_k$ admits a $(k-1)$-dimensional semiuniversal family of classical deformations. We construct this family explicitly. Denoting by $Z_k$ any nontrivial deformation of $Z_k$, our second result (Theorem 6.1) shows that $Z_k$ contains no compact complex analytic curve. Our third result (Theorem 6.10) shows that any holomorphic vector bundle on $Z_k$ splits as a direct sum of algebraic line bundles. This is somewhat surprising, given the existence of nontrivial moduli of vector bundles on the original $Z_k$ surfaces proved in [8]. Our fourth result (Theorem 6.15) shows that any nontrivial deformation $Z_k$ is affine.

These results imply that moduli of instantons on local surfaces are sensitive to the complex structure: the instanton moduli of charge $j$ over the local surfaces $Z_k$ are of dimension $2j - k - 2$, whereas $Z_k$ admits no instantons.

Let us put our results also into the context of deformations of curves and surfaces and their moduli. Grothendieck’s splitting theorem says that any holomorphic vector bundle on $\mathbb{P}^1$ splits as a direct sum of (algebraic) line bundles. Neither the curve $\mathbb{P}^1$ itself nor its discrete moduli spaces of vector bundles admit any deformations.

Curves of higher genus do admit deformations and a celebrated theorem of Narasimhan and Ramanan [23, 3] shows that classical deformations of the moduli of stable bundles on a curve come from deformations of the curve itself (case $g > 1$, $(r, d) = 1$).

The Narasimhan–Ramanan result does not generalize to higher dimensions. For instance, the projective plane $\mathbb{P}^2$ admits no classical deformations, whereas moduli of stable bundles on $\mathbb{P}^2$ have a rich structure of quasi-projective singular varieties, thus allowing for nontrivial deformations.

Our case is somewhat intermediate, in that we deal with noncompact surfaces which are neighbourhoods of a curve inside a surface. [8, Thm. 4.11] showed that the moduli of rank 2 bundles on $Z_k$ with splitting type $j$ (see Def. 4.5) and $c_1 = 0$ are quasi-projective varieties of dimension $2j - k - 2$. (See [6] for arbitrary $c_1$.) In contrast, the moduli of vector bundles on a nontrivial deformation of $Z_k$ are zero-dimensional. Thus classical deformations of $Z_k$ do not give rise to deformations of their moduli of bundles. This motivates the study of moduli of bundles over noncommutative deformations of $Z_k$, which we shall pursue in a subsequent paper.
3. Local surfaces

Let $Z_k$ be the total space of the line bundle $\mathcal{O}_\mathbb{P}^1(-k)$ for $k \geq 1$. Informally, we refer to $Z_k$ as a local surface. We also note that $Z_1$ is $\tilde{\mathbb{C}}^2$, the blowup of $\mathbb{C}^2$ at the origin, and $Z_2$ is a Calabi–Yau surface. Our main objects of study will be classical deformations of these local surfaces and their moduli spaces of vector bundles.

Remark 3.1. In this work we restrict our study to $Z_k$ for $k > 0$, in which case holomorphic bundles are algebraic (Thm. 4.1). A particularly nice consequence is that the resulting moduli spaces of vector bundles over $Z_k$ are finite-dimensional.

Notation 3.2. We fix once and for all coordinate charts on $Z_k$, which we refer to as canonical coordinates, given by $U = \mathbb{C}^2_{z,u} = \{(z, u)\}$ and $V = \mathbb{C}^2_{\xi,v} = \{(\xi, v)\}$, such that on $U \cap V = \mathbb{C}^* \times \mathbb{C}$ we identify $(\xi, v) = (z^{-1}, z^k u)$. (3.1)

4. Geometry and topology of $Z_k$

4.1. Line bundles on $Z_k$. We have that $H^1(Z_k, \mathcal{O}_{Z_k}) = H^2(Z_k, \mathcal{O}_{Z_k}) = 0$. The exponential sheaf sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$$

thus implies that $\text{Pic} Z_k \simeq H^1(Z_k, \mathcal{O}^*) \simeq H^2(Z_k, \mathbb{Z}) \simeq \mathbb{Z}$, whence line bundles on $Z_k$ are determined by their first Chern class. We write $\mathcal{O}_{Z_k}(n)$, or simply $\mathcal{O}(n)$, for the line bundle with first Chern class $n$. In canonical coordinate charts the bundle $\mathcal{O}(n)$ has the transition matrix $(z^{-n})$.

4.2. Vector bundles on $Z_k$. Recall that a rank $r$ bundle $E$ over a variety $X$ is called filtrable (by line bundles) if it can be written as successive extensions by line bundles, i.e. if there exists an increasing filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ of subbundles such that $E_i/E_{i-1} \in \text{Pic} X$, where $1 \leq i \leq r$. We now recall some properties of vector bundles on $Z_k$.

Theorem 4.1 ([11, Lem. 3.1, Thm. 3.2]). Holomorphic vector bundles on $Z_k$ are algebraic and filtrable.

In particular, any rank 2 bundle on $Z_k$ is isomorphic to an algebraic extension of line bundles. Theorem 4.1 generalizes to the case of ample conormal bundle, see [7, Thm. 3.2].

Notation 4.2. Given two vector bundles $E, E'$ over $Z_k$, an isomorphism of vector bundles is given by a pair of invertible matrices $(A_U, A_V)$, where $A_U$ (resp. $A_V$) has entries holomorphic in $U$ (resp. $V$) and such
that $A_V E = E'A_V$, or equivalently $A_V E A_U^{-1} = E'$. In particular, $\det A_U = \det A_V \in \mathbb{C}^*$.

We shall make repeated use of the following standard result.

**Lemma 4.3** (e.g. [16, III.6.3.(c), III.6.7]). There exists an isomorphism

$$\text{Ext}^1(\mathcal{O}_X(j), \mathcal{O}_X(-j)) \cong H^1(X, \mathcal{O}_X(-2j)).$$

In terms of canonical coordinates on $Z_k$, this isomorphism sends an extension class $\alpha = (\alpha_U, \alpha_V)$ to the cohomology class $\sigma = (\sigma_U, \sigma_V) = (z^{-j}p_U, \xi^{-j}p_V)$.

A rank 2 bundle with $c_1 = 0$ may thus be given by a cohomology class $\sigma \in H^1(Z_k, \mathcal{O}(-2j))$ whose general form we recall in the following lemma.

**Lemma 4.4** (cf. [8, Lem. 2.6]). Set $m = \left\lfloor \frac{n-2}{k} \right\rfloor$. A general cohomology class $\sigma \in H^1(Z_k, \mathcal{O}(-n))$ can be written in the form

$$(4.1) \quad \sigma \sim \sum_{i=0}^{m} \sum_{l=ik-n+1}^{-1} \sigma_{il} z^l u^i.$$

In particular, we have that

$$\dim H^1(Z_k, \mathcal{O}(-n)) = \frac{(m+1)(2n-mn-2)}{2} \quad \text{if } n \geq 2$$

and zero otherwise.

**Definition 4.5** ([4]). Let $E$ be a rank $r$ bundle on $Z_k$. Then the restriction of $E$ to the zero section $\ell \simeq \mathbb{P}^1$ is a rank $r$ bundle on $\mathbb{P}^1$, which by Grothendieck’s splitting lemma splits as a direct sum of line bundles. That is, $E|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(j_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(j_r)$. We call $(j_1, \ldots, j_r)$ the splitting type of $E$. When $E$ is a rank 2 bundle with first Chern class $c_1(E) = j_1 + j_2 = 0$, we set $j = |j_1| = |j_2|$ so that $j \geq 0$ and say that $E$ is of splitting type $j$.

Expressing Theorem 4.1 in canonical coordinates gives that a rank 2 bundle $E$ with first Chern class $c_1(E) = 0$ and splitting type $j$ can be defined via a transition matrix [11, Thm.3.3]

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} z^j & z^j \sigma \\ 0 & z^{-j} \end{pmatrix}$$

where $\sigma \in H^1(Z_k, \mathcal{O}(-2j))$ has the general form

$$\sigma = \sum_{i=1}^{\left\lfloor \frac{n-2}{k} \right\rfloor} \sum_{l=ik-2j+1}^{-1} \sigma_{il} z^l u^i, \quad \sigma_{il} \in \mathbb{C}$$

where in contrast to (4.1) the sum now runs from $i = 1$, so that $\sigma|_\ell = 0$ and $E|_\ell = \mathcal{O}(j) \oplus \mathcal{O}(-j)$.
4.3. **Moduli.** Moduli spaces of rank 2 bundles on $\mathbb{Z}_k$ were studied in [12, Thm. 3.5] for the case of $\mathbb{Z}_1$ and in [8, Thm. 4.11] for the cases of $k \geq 1$. One could either give an ad-hoc definition of stability and obtain quasi-projective varieties corresponding to moduli spaces of stable bundles, or else take the point of view of stacks and study the full moduli stack of bundles on $\mathbb{Z}_k$. The latter approach was taken in [9, Thm. 4.11] where two equivalent presentations of the stack of bundles on $\mathbb{Z}_k$ were given [9, Thm. 3.1]. The former approach of choosing a definition of stability allows us to describe moduli spaces of stable bundles with local second Chern class $\chi$ on $\mathbb{Z}_k$, which turn out to be smooth quasi-projective varieties of dimension $2\chi - k - 2$.

5. **Classical deformations**

Classical deformations of complex structures on a complex manifold $X$ are parametrized by $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the (holomorphic) tangent bundle, with obstructions to deformation lying in $H^2(X, \mathcal{T}_X)$. Since our surfaces are covered by only two charts, they have no second cohomologies with coefficients in coherent sheaves, whence deformations are unobstructed.

Although the general existence results in the deformation theory of Kodaira and Spencer [19] only hold for compact manifolds, the remainder of the theory still applies to families of noncompact manifolds. We construct a semiuniversal family for the local surfaces $\mathbb{Z}_k$ explicitly by a different method and show that the resulting family coincides with the informal explanation given by Kodaira as “deformations of a complex manifold $M$ is the glueing of the same polydisks $U_j$ via different identification” [19, §4.1].

5.1. **Deformations via the tangent bundle.** To construct deformations of the complex structure of $\mathbb{Z}_k$, we shall deform the tangent bundle $\mathcal{T}_{\mathbb{Z}_k}$ and reconstruct a complex manifold from the deformed bundle. The nontrivial deformations obtained in this way turn out to correspond precisely to those obtained by deforming the transition matrix by adding a cocycle of $H^1(\mathbb{Z}_k, \mathcal{T}_{\mathbb{Z}_k})$ (see §5.2).

In canonical coordinates the transition matrix for the tangent bundle of $\mathbb{Z}_k$ is given by the Jacobian matrix

\[ J = \begin{pmatrix}
\partial_z z^{-1} & \partial_u z^{-1} \\
\partial_z z^k u & \partial_u z^k u
\end{pmatrix} = \begin{pmatrix}
-z^{-2} & 0 \\
kz^{-1}u & z
\end{pmatrix} \sim \begin{pmatrix}
z^k & kz^{-1}u \\
0 & -z^{-2}
\end{pmatrix}
\]  

which shows that $\mathcal{T}_{\mathbb{Z}_k}$ fits into a short exact sequence

\[ 0 \rightarrow \mathcal{O}(-k) \rightarrow \mathcal{T}_{\mathbb{Z}_k} \rightarrow \mathcal{O}(2) \rightarrow 0. \]

By Lemma 4.3, $\text{Ext}^1(\mathcal{O}(2), \mathcal{O}(-k)) \simeq H^1(\mathbb{Z}_k, \mathcal{O}(-k-2))$ and a cohomology class $\sigma \in H^1(\mathbb{Z}_k, \mathcal{O}(-k-2))$ defines an extension of line bundles
via a transition matrix

\[
\begin{pmatrix}
  z^k & z^k \sigma \\
  0 & -z^{-2}
\end{pmatrix}
\]

Lemma 4.4 states that \( H^1(Z_k, \mathcal{O}(-k-2)) \) is generated as a \( \mathbb{C} \)-vector space by

\[
z^{-1}u, z^{-k-1}, z^{-k}, \ldots, z^{-1}.
\]

The corresponding basis for \( \text{Ext}^1(\mathcal{O}(2), \mathcal{O}(-k)) \) is

\[
z^{k-1}u, z^{-1}, 1, z, \ldots, z^{k-1}.
\]

We shall see that \( T_{Z_k} \) deforms trivially as a bundle over \( Z_k \) in the direction of \( z^{k-1}u \); the direction \( z^{-1} \) is not integrable; the direction \( z^{k-1} \) is integrable, but the corresponding family of complex manifolds is trivial. We will show that the other \( k-1 \) directions give deformations of \( T_{Z_k} \) which after integrating give nontrivial deformations of \( Z_k \).

Let

\[
\sigma = s_{1,k-1}z^{k-1}u + \sum_{l=-1}^{k-1} s_{0,l}z^l, \quad s_{1,k-1}, s_{0,-1}, \ldots, s_{0,k-1} \in \mathbb{C}
\]

be any extension class in \( \text{Ext}^1(\mathcal{O}(2), \mathcal{O}(-k)) \) and consider the bundle defined by the following transition matrix

\[
S = \begin{pmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{pmatrix} = \begin{pmatrix}
  z^k & k z^{k-1} u + \sigma \\
  0 & -z^{-2}
\end{pmatrix}
\]

so that for \( \sigma = 0 \), we have that \( S = J \), the Jacobian (5.1) of \( Z_k \).

We wish to determine under which conditions the matrix \( S \) may be the transition matrix of the tangent bundle of a different complex manifold, a deformation of \( Z_k \). For this we must have

\[
\int S_{11} \, du = \int S_{12} \, dz
\]

(5.2)

\[
\int S_{21} \, du = \int S_{22} \, dz.
\]

(5.3)

First we turn to (5.3), which after integration reads

\[
f_{21}(z) = z^{-1} + f_{22}(u),
\]

where \( f_{21}(z) \) is any function of \( z \) holomorphic on \( \mathbb{C}^* \) and \( f_{22}(u) \) is any function of \( u \) holomorphic on \( \mathbb{C} \). That is, we must have \( f_{21}(z) = z^{-1} + C' \) and \( f_{22}(u) = C' \) for some constant \( C' \in \mathbb{C} \).

Next, we turn to (5.2). Integrating the \((1, 2)\)-entry term-by-term and comparing, we get

\[
(k + s_{1,k-1}) z^{k-1} u + s_{0,-1} z^{-1} + s_{0,0} + \cdots + s_{0,k-1} z^{k-1}
\]

(5.4)

\[
\frac{k+s_{1,k-1}}{k} z^k u + s_{0,-1} \ln z + s_{0,0} z + \cdots + \frac{1}{k} s_{0,k-1} z^k + f_{12}(u)
\]

(5.5)

\[
z^k u + f_{11}(z)
\]

(5.6)
(5.4) $\mapsto$ (5.5) is integration with respect to $z$, where $f_{12}(u)$ is the integration “constant”, a holomorphic function depending only on $u$. Now (5.2) is equivalent to (5.5) = (5.6). We see immediately that we must have $s_{1,k-1} = 0$ and $f_{12}(u) = C$ for some constant $C \in \mathbb{C}$. Moreover, since $\ln z$ is not holomorphic on $\mathbb{C}^*$, we must have $s_{0,-1} = 0$ as well.

We shall rename the remaining coefficients as $t_i = \frac{1}{i}s_{0,i-1} \in \mathbb{C}$ for $1 \leq i \leq k$ and set $\tau = \sum_{i=1}^{k-1} t_i z^i$. Now (5.6) reads

$$z^k(u + t_k) + \tau + C.$$ 

and we try to construct a new complex manifold, for now denoted $Z_k(\tau, t_k, C, C')$, from the charts $U = \{(z, u)\}$ and $V = \{ (\xi, v) \}$ such that on $U \cap V \simeq \mathbb{C}^* \times \mathbb{C}$ we identify

$$(\xi, v) = (z^{-1} + C', z^k(u + t_k) + \tau + C).$$

The corresponding transition matrix from $(z, u)$- to $(\xi, v)$-coordinates would be

(5.7) $$T_{Z_k(\tau, t_k, C, C')} = \begin{pmatrix} z^{-1}(z^{-1} + C') & 0 \\ z^{-1}(\tau + t_k z^k + C) & z^k \end{pmatrix},$$

but the matrix (5.7) has to be invertible for all $z \neq 0$. Since its determinant is equal to $z^{k-1}(z^{-1} + C')$, we see that we must also require $C' = 0$. Then (5.7) indeed defines a complex manifold $Z_k(\tau, t_k, C, 0)$.

Finally, we show that the complex structure of $Z_k(\tau, t_k, C, 0)$ does not depend on $t_k$ and $C$. For this, define an isomorphism

$$\phi: Z_k(\tau, t_k, C, 0) \longrightarrow Z_k(\tau) = Z_k(\tau, 0, 0, 0).$$

given on each chart by

$$\phi|_U(z, u) = (z, u + t_k)$$

$$\phi|_V(\xi, v) = (\xi, v - C)$$

which on $U$ (resp. $V$) is clearly holomorphic in $z, u$ (resp. $\xi, v$) with holomorphic inverse

$$\phi^{-1}|_U(z, u) = (z, u - t_k)$$

$$\phi^{-1}|_V(\xi, v) = (\xi, v + C).$$

Moreover, $\phi$ and $\phi^{-1}$ are well-defined since on $U \cap V$

$$T_{Z_k(\tau)} \circ \phi|_U(z, u) = T_{Z_k(\tau)}(z, u + t_k) = (z^{-1}, z^k(u + \tau + t_k)) = (z^{-1}, z^k(u + \tau + t_k) + C - C) = \phi|_V(z^{-1}, z^k(u + \tau + t_k) + C) = \phi|_V \circ T_{Z_k(\tau,t_k,C,0)}(z, u).$$
and similarly
\[ T_{Z_k(\tau, t_0, C, 0)} \circ \phi^{-1}\vert_U = \phi^{-1}\vert_V \circ T_{Z_k(\tau)}. \]

A well-defined translation defines an isomorphism of complex manifolds since translation by a constant is clearly invertible and holomorphic in any variable and the complex structure is invariant under conformal maps.

We have thus shown that for any \( t_1, \ldots, t_{k-1} \in \mathbb{C} \), we may give a complex manifold \( Z_k(\tau) \), where \( \tau = \sum_{i=1}^{k-1} t_i z^i \).

**Notation 5.1.** We fix canonical coordinates for the surfaces \( Z_k(\tau) \), where \( \tau = \sum_{i=1}^{k-1} t_i z^i \).

Let \( Z_k(\tau) = U \cup V \) with coordinates \( U = \{(z, u)\} \), \( V = \{(\xi, v)\} \), such that on \( U \cap V \simeq \mathbb{C}^* \times \mathbb{C} \) we identify
\[
(\xi, v) = (z^{-1}, z^k u + \tau).
\]

Note that for \( \tau \) to be nonzero, we must have that \( k \geq 2 \). Moreover, we have \( Z_k(0) = Z_k \) (cf. (3.1)). However, as long as \( \tau \neq 0 \), the complex structure of \( Z_k(\tau) \) is different from the structure on \( Z_k \) as we will see when we show that \( Z_k(\tau) \) contains no compact complex analytic curves (Prop. 6.1) and that every holomorphic vector bundle on \( Z_k(\tau) \) splits as a direct sum of line bundles (Thm. 6.10).

### 5.2. Deformations of the transition function

Next, we show that the nontrivial deformations of \( Z_k \) obtained via deformations of the tangent bundle \( T_{Z_k} \) correspond to deformations obtained by deforming the transition function of \( Z_k \), by adding a cocycle in \( H^1(Z_k, T_{Z_k}) \).

**Lemma 5.2.** \( H^1(Z_1, T_{Z_1}) = 0 \). Let \( k > 1 \), then \( H^1(Z_k, T_{Z_k}) \simeq \mathbb{C}^{k-1} \) and its generators as a \( \mathbb{C} \)-vector space may be expressed in the basis \( \{ \frac{\partial}{\partial z}, \frac{\partial}{\partial u} \} \), where \( z, u \) are canonical coordinates, as
\[
\sigma_i = \begin{pmatrix} 0 \\ -k+i \end{pmatrix}
\]
with \( 1 \leq i \leq k-1 \).

**Proof.** Let \( \sigma \in T_{Z_k}(U \cap V) \) be a general 1-cocycle, which may be written as a convergent power series
\[
\sigma = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} \left( \begin{array}{c} a_{ij} \\ b_{ij} \end{array} \right) z^i u^j.
\]

Since terms with positive powers of \( z \) are holomorphic on \( U \) we obtain the cohomological equivalence
\[
\sigma \sim \sum_{i=0}^{\infty} \sum_{j=-\infty}^{-1} \left( \begin{array}{c} a_{ij} \\ b_{ij} \end{array} \right) z^i u^j.
\]
Changing coordinates we have:

\[
J\sigma = \begin{pmatrix}
-z^{-2} k z^{k-1} & 0 & \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij} b_{ij} & z^i u^j
\end{pmatrix}
\]

and since monomials of the form \(z^m u^n\) with \(m \leq nk\) are holomorphic on \(V\), we obtain the cohomological equivalence

\[
J\sigma \sim \sum_{i=0}^{\infty} \sum_{j=-k+1}^{\infty} \begin{pmatrix} 0 \\ b_{ij} \end{pmatrix} z^{k+j} u^i.
\]

Thus, the nontrivial terms in \(\sigma\) are \((0, z^{-k+\tau})^t\) with \(1 \leq i \leq k - 1\). □

Lemma 5.2 shows that the cohomology \(H^1(Z_k, T_{Z_k}) \simeq \mathbb{C}^{k-1}\) is spanned by the column vectors \((0, z^{-k+1})^t, \ldots, (0, z^{-1})^t\), where the cocycles are given in \(U\)-coordinates. A general cocycle may thus be given as

\[
\begin{pmatrix} 0 \\ z^{-k} \sum_{i=1}^{k-1} t_i z^i \end{pmatrix} = \begin{pmatrix} 0 \\ z^{-k+\tau} \end{pmatrix}.
\]

Adding \((0, z^{-k+\tau})^t\) to the coordinate function \((z, u)^t\) on \(U \cap V\) to obtain \((z, u + z^{-k+\tau})^t\) and then multiplying by the transition matrix for \(Z_k\) is another way of obtaining (5.8).

We thus have a map

\[
H^1(Z_k, T_{Z_k}) \hookrightarrow H^1(Z_k, \mathcal{O}(-k-2)) \cong \text{Ext}^1(\mathcal{O}(2), \mathcal{O}(-k))
\]

\[
\begin{pmatrix} 0 \\ z^{-k+\tau} \end{pmatrix} \mapsto z^{-k+\tau} \mapsto \tau.
\]

The image of this composition is precisely the subspace of those extension classes giving rise to the deformations \(Z_k(\tau)\), which are nontrivial as long as \(\tau \neq 0\).

We summarize the results of this section in the following theorem.

**Theorem 5.3.** Let \(k \geq 2\). Then \(Z_k\) admits a \((k-1)\)-dimensional semi-universal family \(Z_k \longrightarrow M \longrightarrow \mathbb{C}^{k-1} \simeq H^1(Z_k, T_{Z_k})\) of deformations.

**Proof.** Let \(B = \mathbb{C}^{k-1}\) with coordinates \(t_1, \ldots, t_{k-1}\) and consider the complex manifold \(M\) given by the charts

\[
U \times B = \{(z, u, t_1, \ldots, t_{k-1})\} \quad V \times B = \{ (\xi, v, t_1, \ldots, t_{k-1}) \},
\]

with transition matrix

\[
\begin{pmatrix}
-z^{-2} & 0 & 0 \\
-z^{-1} \sum_{i=1}^{k-1} t_i z^i & z^k & 0 \\
0 & 0 & I_{k-1}
\end{pmatrix}.
\]
Then $\pi : M \to B$ given by projection to the third factor is a family of noncompact manifolds with $M_0 = \pi^{-1}(0) \simeq Z_k$ and $\pi^{-1}(t_1, \ldots, t_{k-1}) \simeq Z_k(\tau)$, where $\tau = \sum_{i=1}^{k-1} t_i z_i$.

Recall from [21, Def. 1.35] that a family is semiuniversal, if it is versal (or complete) and the Kodaira–Spencer map $K : T_0B \to H^1(Z_k, T_{Z_k})$ is an isomorphism. The Kodaira–Spencer map of this family is the vector space isomorphism

\begin{equation}
(5.10) \quad \frac{\partial}{\partial t_i} \longmapsto \left( 0 \right. \left. z^{-k+i} \right)\].
\end{equation}

Moreover, by Theorem 6.15 $Z_k(\tau)$ is affine for each $\tau \neq 0$, whence $H^1(Z_k(\tau), T_{Z_k}(\tau)) = 0$ and thus $Z_k(\tau)$ admits no infinitesimal deformations. Thus for each fibre $M_t = \pi^{-1}(t)$ the family $Z_k \to M \to \mathbb{C}^{k-1}$ trivially contains all infinitesimal deformations of $M_t$ and is therefore versal.

\textbf{Remark 5.4.} When $k = 2$ the family given in Theorem 5.3 is already well-known: it is the simultaneous resolution of the $A_1$ surface singularity (the rational double point) of Atiyah [2]. The deformations of $Z_2$ may also be obtained via the methods of Kaledin–Verbitsky [17] who present a Torelli-type theorem for noncompact manifolds with a holomorphic symplectic form by giving a period map $H^1(X, T_X) \to H^2(X, \mathbb{C})$. However, of the surfaces $Z_k$, only $Z_2 \simeq T^*\mathbb{P}^1$ admits a holomorphic symplectic form.

5.3. \textbf{Relation to deformations of Hirzebruch surfaces.} Recall from [21] that the Hirzebruch surface $F_k$ is isomorphic to the subvariety of $\mathbb{P}^1 \times \mathbb{P}^{k+1}$ given as the set of points $([z_0 : z_1], [x_0 : \ldots : x_{q+1}])$ satisfying the equation

\[ z_0(x_1, \ldots, x_k) = z_1(x_2, \ldots, x_{k+1}). \]

The embedding $Z_k \hookrightarrow F_k$ may be given by

\begin{align}
(z, u) &\longmapsto ([1 : z], [1 : z^k u : z^{k-1} u : \ldots : u]) \\
(\xi, v) &\longmapsto ([\xi : 1], [1 : v : \xi v : \ldots : \xi^k v]).
\end{align}

Since $H^2(F_k, T_{F_k}) = 0$, the Theorem of Existence [19, Thm. 5.6] gives a semiuniversal family over a $(k-1)$-dimensional base, where $\dim H^1(F_k, T_{F_k}) = k - 1$. This family is given explicitly in [21, §2.3] as follows.

The family is given as $F_k \to \widetilde{M} \to \mathbb{C}^{k-1}$, where $\widetilde{M} \subset \mathbb{P}^1 \times \mathbb{P}^{k+1} \times \mathbb{C}^{k-1}$ is the set of coordinates $([z_0 : z_1], [x_0 : \ldots : x_{k+1}], (t_1, \ldots, t_{k-1}))$ satisfying the equation

\begin{equation}
(z_0(x_1, \ldots, x_k), x_k) = z_1(x_2 + t_1 x_0, \ldots, x_k + t_{k-1} x_0, x_{k+1})
\end{equation}

and $\tilde{\pi} = \pi_3|_{\tilde{M}}$ is the restriction of the projection to the third factor.
Proposition 5.5. Let $Z_k \to M \to \mathbb{C}^{k-1}$ be the family of Theorem 5.3 and let $F_k \to \tilde{M} \to \mathbb{C}^{k-1}$ be the family (5.12).

There is a commutative diagram

\[
\begin{array}{ccc}
Z_k & \to & M \\
\downarrow & & \downarrow \\
F_k & \to & \tilde{M}
\end{array}
\]

\[\mathbb{C}^{k-1} \quad \mathbb{C}^{k-1} \]

Proof. Recall from the proof of Theorem 5.3 that the family $M$ is covered by the sets $U \times B$ and $V \times B$, where $U, V$ are the canonical charts and $B \simeq \mathbb{C}^{k-1}$, such that on $(U \times B) \cap (V \times B)$ we identify

\[(\xi, v) = (z^{-1}, z^k u + \sum_{i=1}^{k-1} t_i z^i)\] (5.13)

Define the map $\tilde{M} \to M$ on charts $U \times B$ and $V \times B$ by

\[
\begin{align*}
(z, u, t_1, \ldots, t_{k-1}) &\mapsto ([1 : z], [1 : x_1 : \ldots : x_{k+1}], t_1, \ldots, t_{k-1}) \\
(\xi, v, t_1, \ldots, t_{k-1}) &\mapsto ([\xi : 1], [1 : y_1 : \ldots : y_{k+1}], t_1, \ldots, t_{k-1}),
\end{align*}
\]

(5.14)

where

\[
x_{n+1} = z^{k-n} u + \sum_{i=n+1}^{k-1} t_i z^i \\
y_{n+1} = \xi^n v - \sum_{i=1}^{n} t_i \xi^{n-i}
\]

for $0 \leq n \leq k$. (The sum is empty for $x_{k+1}$ and $y_1$.)

One checks that the map (5.14) is injective and satisfies (5.12) on each chart; moreover, the map is well defined on the intersection, which follows from using the identity (5.13).

The commutativity of the first square follows from (5.11) for $t_1 = \cdots = t_{k-1} = 0$. The commutativity of the second square is evident from (5.14). \qed

Remark 5.6. We may thus think of deformations of $Z_k$ as being induced by deformations of $F_k$. However, this is only clear in hindsight. For example, recall that the complement of the “zero section” $\ell \subset Z_k \subset F_k$ in $F_k$ is isomorphic to $Z_{-k} = \text{Tot}(\mathcal{O}_{P^1}(k))$. However, $\dim H^1(Z_{-k}, \mathcal{T}_{Z_{-k}}) = \infty$, whence $Z_{-k}$ may admit many more deformations than those induced by deformations of $F_k$, possibly even deformations that are not algebraic. Moreover, the family for $F_k$ is semiuniversal, but the same cannot be true for the family of $Z_{-k}$ because the Kodaira–Spencer map $\mathbb{C}^{k-1} \simeq T_0 \mathbb{C}^{k-1} \to H^1(Z_{-k}, \mathcal{T}_{Z_{-k}})$ can never be an isomorphism of vector spaces since the target is infinite-dimensional.
6. Geometry and topology of $Z_k(\tau)$

Deforming the complex structure does not change the topology of the manifold, thus for any $\tau$ we still have

$$H^i(Z_k(\tau), \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } i = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

The de Rham cohomology of $Z_k$ comes from the complex submanifold $\ell \simeq \mathbb{P}^1 \simeq S^2$. However, for any nontrivial deformation $Z_k(\tau)$, this is no longer the case, as the following result shows.

**Theorem 6.1.** A nontrivial deformation of $Z_k$ contains no complex analytic compact curves.

**Proof.** Let $f$ be a global holomorphic function on $Z_k(\tau)$. On $U$ it can be written as a convergent power series

$$f|_U = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} f_{il} z^l u^i.$$

Expression (5.8) gives that $u = \xi^kv - \sum_{i=1}^{k-1} t_i \xi^{k-i}$. Let $1 \leq M \leq k - 1$ be the largest integer such that $t_M \neq 0$, and use it to rewrite $f$ in $V$-coordinates as

$$f|_V = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} f_{il} \xi^{-l}(\xi^k v - \sum_{n=1}^{k-1} t_n \xi^{k-n})^i$$

$$= f_{00} + \sum_{i=1}^{\infty} \sum_{l=0}^{\infty} f_{il} \xi^{i(M-1)}(\xi^M v - \sum_{n=1}^{M-1} t_n \xi^{k-n} - t_M)^i.$$

When only $f_{00}$ is nonzero, $f$ is constant. Otherwise $f_{00} = t_M - \xi^M v - \sum_{n=1}^{M-1} t_n \xi^{k-n}$ is a factor of the global function on the $V$ chart which contains a noncompact curve, whence the curve defined by $f$ is non-compact.

By definition, any 1-dimensional submanifold of $Z_k(\tau)$ is given as the zero set of a global holomorphic function. We conclude that $Z_k(\tau)$ contains no compact 1-dimensional analytic subvariety. \qed

In particular, $Z_k(\tau)$ contains no complex submanifold with the topology of a 2-sphere for $\tau \neq 0$.

### 6.1. Line bundles on $Z_k(\tau)$.

**Lemma 6.2.** $H^1(Z_k(\tau), \mathcal{O}) = H^2(Z_k(\tau), \mathcal{O}) = 0$.

**Proof.** A general 1-cocycle can be written in the form

$$\alpha = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \alpha_{nm} z^m u^n \sim \sum_{n=0}^{\infty} \sum_{m=-\infty}^{M-1} \alpha_{nm} z^m u^n.$$
In canonical coordinates, we have \( u = \xi_k v - \sum_{i=1}^{k-1} t_i \xi^{-i} \). Thus, on the \( V \)-chart

\[
\alpha \sim \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \alpha_{n,m} \xi^m (\xi^k v - \sum_{i=1}^{k-1} t_i \xi^{-i})\sim 0
\]

since positive powers of \( \xi, v \) are holomorphic on \( V \).

\[ H^2(\mathcal{Z}_k(\tau), \mathcal{F}) = 0 \] for any coherent sheaf \( \mathcal{F} \) of coefficients, since \( \mathcal{Z}_k(\tau) \) is Leray-covered by only two open sets.

\[ \square \]

We thus get the following isomorphisms,

\[ \text{Pic} \mathcal{Z}_k(\tau) \simeq H^2(\mathcal{Z}_k(\tau), \mathbb{Z}) \simeq H^2(\mathcal{Z}_k, \mathbb{Z}) \simeq H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}, \]

where the first isomorphism follows from the exponential sheaf sequence for \( \mathcal{Z}_k(\tau) \) and the second isomorphism follows from the fact that \( \mathcal{Z}_k(\tau) \) is homeomorphic to \( Z_k \) as a real manifold. So, any line bundle on \( \mathcal{Z}_k(\tau) \) is determined by its first Chern class. We write \( \mathcal{O}_{\mathcal{Z}_k(\tau)}(n) \) or \( \mathcal{O}(n) \) for the line bundle on \( \mathcal{Z}_k(\tau) \) with first Chern class \( n \).

**Lemma 6.3.** The line bundle on \( \mathcal{Z}_k(\tau) \) with first Chern class \( n \), denoted \( \mathcal{O}(n) \), can be given the transition matrix \( (z^{-n}) \).

**First proof.** Let \( M \) be the total space of the family given in the proof of Theorem 5.3. The matrix \( (z^{-n}) \) defines a line bundle \( \mathcal{E} \) over \( M \). The restriction of this line bundle to the central fibre of the family gives \( \mathcal{O}_{\mathcal{Z}_k(\tau)}(n) \) (with transition matrix \( (z^{-n}) \) and Chern class \( n \)). Since the family is continuous and the first Chern class is a discrete topological invariant it remains constant in the family, hence \( c_1(\mathcal{E}|_{\mathcal{Z}_k(\tau)}) = n \) as well.

\[ \square \]

**Second proof.** Define a map

\[
\phi: \mathcal{Z}_k(\tau) \longrightarrow \mathbb{P}^1
\]

\[
(z, u) \longmapsto [1 : z]
\]

\[
(\xi, v) \longmapsto [\xi : 1].
\]

If we denote by \( T \) the transition function of \( \mathcal{O}_{\mathbb{P}^1}(n) \), then the transition function of the pullback bundle \( \phi^*\mathcal{O}_{\mathbb{P}^1}(n) \simeq \mathcal{O}_{\mathcal{Z}_k(\tau)}(n) \) is \( \phi^*T = T \circ \phi \), which in canonical coordinates is precisely multiplication by \( (z^{-n}) \).

\[ \square \]

6.2. **Cohomology of \( \mathcal{Z}_k(\tau) \).** We calculate sheaf cohomology with coefficients in line bundles.

**Remark 6.4.** As \( \mathcal{Z}_k(\tau) \) is covered by two open sets with acyclic intersection, we have that \( H^i(\mathcal{Z}_k(\tau), \mathcal{F}) = 0 \) for \( i \geq 2 \) and any coherent sheaf \( \mathcal{F} \).

**Lemma 6.5.** \( H^1(\mathcal{Z}_k(\tau), \mathcal{O}(-n)) = 0 \) for any integer \( n \).
Proof. As for $Z_k$, this is straightforward if $n \leq 1$. We thus assume that $n \geq 2$. As in \S5, let $\tau = \sum_{i=1}^{k-1} t_i z^i$. The idea of the proof is to use the fact that a function holomorphic on $V \subset Z_1$ is also holomorphic on $V \subset Z_k(\tau)$ (cf. Lem. 6.6). The difficult part is thus to show that all cocycles which are nontrivial in $H^1(Z_1, \mathcal{O}(-n))$ are in fact trivial in $H^1(Z_k(\tau), \mathcal{O}(-n))$, because all other terms can be removed by functions on $Z_1$. These cocycles are spanned by the terms $z^l u^i$, where $-n + i < l < 0$ and $i \leq n - 2$.

Let $\sigma = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \sigma_{il} z^l u^i$ be a general 1-cocycle, i.e. a holomorphic function in the intersection $U \cap V \simeq \mathbb{C}^* \times \mathbb{C}$. We may add any function $z^{-n} f$, where $f$ is holomorphic on $V$ without changing the cohomology class of $\sigma$.

First we remove the terms $z^l u^i$ for $-n + i < l < 0$ and $i \leq n - 2$ as follows. Let $m$ be the smallest integer such that $t_m \neq 0$. We start with $i = 0$ and add a suitable multiple of $\xi^{n-1+nm} u^n = z^{-n+1}(t_m + O(z, u))$ to remove $\sigma_{n-1} z^{-n+1}$. While removing the coefficient of $z^{-n+1}$, we only add to the coefficients of higher powers of $z$ that will be removed in subsequent steps. We continue in the same fashion for the terms $z^l$, where $-n + 1 < l < 0$ by adding suitable multiples of $\xi^{n-l+nm} u^n$ for $1 < s < n$.

For $i \geq 1$, note that $u = \xi v - \sum_{i=1}^{k-1} t_i \xi^{k-i}$ is holomorphic on $V$, so that we may use the expressions

\[
(z^k v - \sum_{i=1}^{k-1} t_i \xi^{k-i})^i \xi^{n-s+nm} u^n = z^{-n+s} u^i(1 + O(z, u)),
\]

where $1 \leq s \leq n$, to remove the remaining terms $z^l u^i$, where $-n + i < l < 0$ and $i \leq n - 2$. We have added finitely many functions to $\sigma$ and now $\sigma \sim \sigma' = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \sigma'_{il} z^l u^i$, where the coefficients $\sigma'_{il} = 0$ for $-n + i < l < 0$ and $i \leq n - 2$.

Next, we may add any function $z^{-n} f_V = z^{-n} \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} f_V(z u)^i$ to remove all remaining nonzero coefficients of $z^l u^i$ with $l \leq -n + i$ since $z u = \xi z^{k-1} - \sum_{n=1}^{k-1} t_n \xi^{k-1-n}$ is holomorphic on $V$. Finally, nonzero coefficients of the terms $z^l u^i$ for $i, l \geq 0$ may be removed by adding a suitable function holomorphic on $U$. \hfill \Box

6.3. Vector bundles on $Z_k(\tau)$. We generalize the algebraicity and filtrability result for $Z_k$ (Thm. 4.1) to its classical deformations. We start with a technical lemma.

**Lemma 6.6.** Let $f_V$ be a function holomorphic on $V \subset Z_1$. Then there is a holomorphic function $f_{\tilde{V}}$ holomorphic on $V \subset Z_k(\tau)$ such that, written in $U$-coordinates, we have that $f_{\tilde{V}|U\cap V} = f_V|U\cap \tilde{V}$.

**Proof.** To differentiate the coordinates of $V \subset Z_1$ and $V \subset Z_k(\tau)$, write $\tilde{V} = \{ (\xi, \tilde{v}) \}$ for the chart of $Z_k(\tau)$, i.e.

\[
(\xi, v) = (z^{-1}, z u) \quad \text{on } U \cap V \subset Z_1,
\]

\[
(\tilde{\xi}, \tilde{v}) = (z^{-1}, z^{k} u + \tau) \quad \text{on } U \cap \tilde{V} \subset Z_k(\tau).
\]
Now a holomorphic function $f_V$ on $V$ may be written as a convergent power series in the variables $z^{-1}$ and $zu$,

$$\sum_{i=0}^{\infty} \sum_{l=0}^{\infty} f_{il} z^{-l}(zu)^i.$$  

Now on $\tilde{V} \subset Z_k(\tau)$, we have that $z^{-1} = \tilde{\xi}$ and $zu = \tilde{\xi}^{-1} \tilde{v} - \sum_{i=1}^{k-1} t_i \tilde{\xi}^{k-i-1}$.

We may rewrite $f_V$ in $\tilde{V}$-coordinates

(6.2)  

$$f_{\tilde{V}} = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} f_{il} \tilde{\xi}^l (\tilde{\xi}^{-1} \tilde{v} - \sum_{i=1}^{k-1} t_i \tilde{\xi}^{k-i-1})^i.$$  

This is a convergent power series in the variables $\tilde{\xi}$ and $\tilde{v}$ which we may write as

$$f_{\tilde{V}} = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \tilde{f}_{il} \tilde{\xi}^l \tilde{v}^i$$

by expanding the factor $(\tilde{\xi}^{-1} \tilde{v} - \sum_{i=1}^{k-1} t_i \tilde{\xi}^{k-i-1})^i$ in (6.2) and rewriting the coefficients. For example, $\tilde{f}_{00} = \sum_{i=0}^{\infty} f_{00} t_i^{k-1}$. Thus $f_{\tilde{V}}$ is holomorphic on $\tilde{V} \subset Z_k(\tau)$. Since both $f_V$ and $f_{\tilde{V}}$ come from the same power series in $(z,u)$-coordinates, we have $f_V = f_{\tilde{V}}$ in $(z,u)$-coordinates. □

**Theorem 6.7.** Holomorphic bundles over $Z_k(\tau)$ are algebraic and filtrable.

**Proof.** As in Lemma 6.6, denote by $\tilde{V}$ the chart of $Z_k(\tau)$ with coordinates $\tilde{\xi}$ and $\tilde{v}$.

Let $\tilde{E}$ be a vector bundle of rank $r$ over $Z_k(\tau)$ with transition function $T$. The entries of $T$ are functions holomorphic in the intersection $U \cap \tilde{V}$ and may thus be written as power series

$$\sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} a_{il} z^l u^i.$$  

The same $T$ also defines a bundle $E$ on $Z_1$. As a bundle over $Z_1$, $E$ is algebraic and filtrable by Theorem 4.1. In particular, there exist matrices $A_U, A_V$ with entries holomorphic in $U, V \subset Z_1$, respectively, which are invertible for all points in $U, V$, respectively, and such that

$$A_V T A_U^{-1} = \begin{pmatrix} z^{j_1} & * & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & & z^{j_r} \end{pmatrix}$$

where each $*$ denotes an algebraic function on $U \cap V$. By Lemma 6.6, we conclude that the entries of $A_U, A_V$ are also holomorphic in $U, \tilde{V} \subset Z_k(\tau)$ and thus $A_U, A_V$ also define an isomorphism of $\tilde{E}$ with a filtered algebraic bundle. □
Remark 6.8. For the proof of Theorem 6.7 we do not need to require \( \tau \neq 0 \). In other words, to prove algebraicity and filtrability for \( Z_k \) (or any of its deformations), it is enough to prove it for \( Z_1 \).

Remark 6.9. Note that on \( \mathbb{P}^2 \) the only bundles which are filtrable by vector bundles are split, but there is a multitude of bundles which are not filtrable by vector bundles, given that the moduli spaces of stable vector bundles on \( \mathbb{P}^2 \) are quasi-projective varieties whose dimensions increase with the second Chern classes, see [26].

Theorem 6.7 implies that, as for \( Z_k \), rank 2 bundles over \( Z_k(\tau) \) with vanishing first Chern class may be written as extensions of line bundles

\[
\text{Ext}^1(\mathcal{O}_{Z_k(\tau)}(j), \mathcal{O}_{Z_k(\tau)}(-j)) \cong H^1(Z_k(\tau), \mathcal{O}(-2j)),
\]

but by Lemma 6.5, \( H^1(Z_k(\tau), \mathcal{O}(-2j)) = 0 \). In other words, \( Z_k(\tau) \), for \( \tau \neq 0 \), only admits split bundles, which gives an analogue of the splitting principle of Grothendieck of bundles over \( \mathbb{P}^1 \) for these surfaces, but not for \( Z_k \). Hence, we obtain:

**Theorem 6.10.** Let \( Z_k(\tau) \) be any nontrivial deformation of \( Z_k \). Then, every holomorphic vector bundle on \( Z_k(\tau) \) splits as a direct sum of line bundles.

**Corollary 6.11.** Any moduli space of vector bundles on \( Z_k(\tau) \) is discrete and zero-dimensional.

Remark 6.12. In contrast with \( Z_k(\tau) \), the surfaces \( Z_k \) have nontrivial moduli of vector bundles. For example, [8, Thm. 4.11] shows that the moduli of rank 2 bundles on \( Z_k \) with splitting type \((j, -j)\) has dimension \( 2j - k - 2 \).

Remark 6.13. Consider the threefold \( W_1 := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \). In [1, Thm. 4.3] it is proven that, for appropriate choices of numerical invariants, there are isomorphisms of moduli spaces of vector bundles on \( W_1 \) and moduli spaces of vector bundles on the surfaces \( Z_k \). Thus, from the point of view of moduli of bundles this threefold presents similar behaviour to our surfaces. However, from the point of view of deformation theory they are quite different. Indeed, we observe that \( H^1(W_1, \mathcal{T}_{W_1}) = 0 \), hence there are no classical deformations of \( W_1 \).

6.4. The affine structure of \( Z_k(\tau) \). As a corollary to Theorem 6.10, we are able to show that any nontrivial deformation \( Z_k(\tau) \) of \( Z_k \) admits the structure of an affine variety. We state part of the proof as a separate lemma.

**Lemma 6.14.** \( Z_k(\tau) \) has the resolution property, i.e. every coherent sheaf has a (global) resolution by locally free sheaves.

**Proof.** This follows from [29, Prop. 8.1] by noting that \( Z_k(\tau) \) has affine diagonal since intersections of affine sets are affine. \( \square \)
Theorem 6.15. Let $Z_k(\tau)$ be a nontrivial deformation of $Z_k$, for $\tau \in H^1(Z_k, T_{Z_k})$. Then $Z_k(\tau)$ is an affine algebraic variety.

Proof. Let $\mathcal{F}$ be an arbitrary coherent sheaf on $Z_k(\tau)$. By Lemma 6.14 $\mathcal{F}$ admits a (global) resolution by locally free sheaves
$$\cdots \rightarrow \mathcal{F}_n \rightarrow \cdots \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F} \rightarrow 0.$$ Since $Z_k(\tau)$ is of dimension two, we claim that the kernel of $\mathcal{F}_2 \xrightarrow{d_2} \mathcal{F}_1$ is locally free. Since locally free is a local property, it suffices to check this property on the stalks $(\ker d_1)_x$ for $x \in Z_k(\tau)$. Since the local rings $\mathcal{O}_{Z_k(\tau), x}$ are regular and of dimension $\leq 2$, they are of global dimension $\leq 2$ and free. We may thus truncate the resolution to
$$0 \rightarrow \ker d_1 \xrightarrow{i} \mathcal{F}_1 \xrightarrow{d_0} \mathcal{F}_0 \xrightarrow{\epsilon} \mathcal{F} \rightarrow 0.$$ We have short exact sequences
$$0 \rightarrow \ker d_1 \rightarrow \mathcal{F}_1 \rightarrow \ker d_0 \rightarrow 0$$
$$0 \rightarrow \ker d_0 \rightarrow \mathcal{F}_1 \rightarrow \ker \epsilon \rightarrow 0$$
$$0 \rightarrow \ker \epsilon \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F} \rightarrow 0$$
where we have used the isomorphisms coim $i \simeq \text{im } i$ and coim $d_0 \simeq \text{im } d_0$ which hold in any Abelian category, here $\text{Coh}(Z_k)$, as well as im $i \simeq \ker d_0$ and im $d_0 \simeq \ker \epsilon$ since the resolution is an exact sequence.

We have that ker $d_1$, $\mathcal{F}_1$ and $\mathcal{F}_0$ are acyclic, since they are direct sums of line bundles by Theorem 6.10, which have no higher cohomology (by Lemma 6.5). Thus in each of the above sequences, the first two terms are acyclic, so is the third by applying the long exact sequence in cohomology iteratively. This proves that $\mathcal{F}$ is acyclic. Since $\mathcal{F}$ was arbitrary, applying Serre’s criterion [16, Thm. III.3.7] we conclude that $Z_k(\tau)$ is affine. \(\square\)

Remark 6.16 (An application to mirror symmetry). A recent result of [13] shows that the adjoint orbits of semisimple Lie groups have the structure of symplectic Lefschetz fibrations. Such Lefschetz fibrations are considered in the homological mirror symmetry conjecture, and there one needs to identify details of its complex and symplectic structures. These semisimple adjoint orbits are affine varieties contained inside the Lie algebra, and a theorem of [13] shows that they have the diffeomorphism type of the cotangent bundle of a flag variety. However, the canonical complex structure of such cotangent bundle is not that of an affine variety, hence the diffeomorphism can not be made into a holomorphic isomorphism. Thus, there appears the natural question of whether this diffeomorphisms can be made into a deformation of complex structures. A result of [5] shows that this is indeed the case for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. In such case, the adjoint orbit is in fact the nontrivial deformation $Z_2$ of the local surface $Z_2$ that we study here. Deciding whether the diffeomorphism can be interpreted
as a deformation of complex structures for more general Lie algebras will require developing deformation theory of noncompact manifolds of higher dimension.

7. Applications to the theory of instantons

SU(2) instantons on $Z_k$ correspond to framed rank 2 holomorphic bundles on $Z_k$ with vanishing first Chern class [14, Prop. 5.4]. (Here the framing is taken to be a trivialization on $Z_k \setminus \ell$; this is equivalent to giving a trivialization at $\ell_\infty \subset F_k$.)

The topological charge of such an instanton is associated to the local holomorphic Euler characteristic of the bundle. We recall the definition.

**Definition 7.1** ([14]). Let $X_k$ denote the variety obtained from $Z_k$ by contracting the zero section to a point $x \in X_k$. Hence $X_k$ is singular at $x$ if $k > 1$. Let $\pi: Z_k \to X_k$ denote the contraction map. Let $E$ be a rank 2 bundle (or any reflexive sheaf) on $Z_k$, and define a skyscraper sheaf $Q$ by the exact sequence

$$0 \to (\pi_* E) \to (\pi_* E) \to Q \to 0.$$  

Then the local holomorphic Euler characteristic of $E$ is

$$\chi_{\text{loc}}(E) := \chi(\pi_* E, x) = h^0(X; Q) + h^0(X; R^1 \pi_* E).$$

In light of the Kobayashi–Hitchin correspondence for $Z_k$, we also refer to this number as the charge of a corresponding instanton on $Z_k$.

We observe that not all bundles on $Z_k$ correspond to instantons. In fact, [14, Cor. 5.5] shows that an $\mathfrak{sl}(2, \mathbb{C})$-bundle $E$ over $Z_k$ represents an instanton if and only if its splitting type (see Def. 4.5) is a multiple of $k$. In such case, [14, Prop. 4.1] implies that the restriction of $E$ to $Z_k \setminus \ell$ is trivial, and hence $E$ can be extended to a bundle $\overline{E}$ on the Hirzebruch surface $F_k$ with a trivialization on the line at infinity $\ell_\infty$.

The sum of the local charges of $\overline{E}$ is the global charge. As the restriction of $\overline{E}$ to an analytic neighbourhood of $\ell_\infty$ is trivial, the charge of $\overline{E}$ only depends on the neighbourhood of $\ell$, which is $Z_k$. That is, we have

$$\chi(\overline{E}) = \chi_{\text{loc}}(E).$$

The moduli space of instantons of charge $j \equiv 0 \mod k$ on $Z_k$ is a smooth quasi-projective variety of dimension $2j - k - 2$ [8, Thm. 4.11]. In terms of vector bundles the statement is that the moduli space of rank 2 holomorphic bundles on $Z_k$ with vanishing first Chern class and $\chi_{\text{loc}} = j$ contains an open dense subset isomorphic to $\mathbb{P}^{2j-k-2}$ minus a closed subvariety of codimension at least $k$.

Here, we do not develop a full theory of instantons on the deformed surfaces $Z_k(\tau)$. However, in light of §5.3 we may apply our results to
instantons by using the Kobayashi–Hitchin correspondence for Hirzebruch surfaces, and then restricting to $Z_k$. Also see [27] for a version of the Kobayashi–Hitchin correspondence developed for quasi-projective varieties.

**Definition 7.2.** Let $D$ be a divisor on a smooth complex surface $X$. We say that a holomorphic vector bundle $E$ on $X \setminus D$ extends trivially to $X$, if there exists a holomorphic vector bundle $\overline{E}$ on $X$ such that $\overline{E}|_{X\setminus D} = E$ and $\overline{E}|_D$ is trivial.

Now let $D$ be a divisor on the Hirzebruch surface $F_k$. The usual Kobayashi–Hitchin correspondence [20]

\[
\begin{align*}
\{ \text{irreducible SU}(2)\text{-instantons} & \} \\
of \text{charge } n & \longleftrightarrow \{ \text{stable } \mathfrak{sll}(2, \mathbb{C})\text{-bundles} & \\
& \text{with } c_1 = 0 \text{ and } c_2 = n \}
\end{align*}
\]

takes an instanton whose charge is held on $F_k \setminus D$ to a bundle which is trivial along $D$. We thus obtain a Kobayashi–Hitchin correspondence for $F_k \setminus D$ between instantons and those stable $\mathfrak{sll}(2, \mathbb{C})$-bundles on $F_k \setminus D$ which extend trivially to $F_k$.

**Theorem 7.3.** Let $Z_k(\tau)$ be a nontrivial deformation of $Z_k$. Then the moduli space of irreducible SU(2)-instantons on $Z_k(\tau)$ is empty.

**Proof.** Deformations of $Z_k$ extend to deformations of its compactification $F_k \supset Z_k$, see §5.3. The Hirzebruch surface $F_k$ deforms to a lower Hirzebruch surface $F_{k-2l}$, where $0 < l \leq \frac{k}{2}$ depends on $\tau \in H^1(F_k, T_{F_k}) \simeq H^1(Z_k, T_{Z_k})$. We choose $D$ to be the complement of $Z_k(\tau) \subset F_{k-2l}$. Under the Kobayashi–Hitchin correspondence, the moduli space of irreducible SU(2)-instantons on $Z_k(\tau)$ is in one-to-one correspondence with stable holomorphic bundles on $Z_k(\tau)$ extending trivially to $F_{k-2l}$. By Corollary 6.11, all holomorphic vector bundles on $Z_k(\tau)$ are split. But the only split bundle that extends trivially to $F_{k-2l}$ is the trivial bundle, which is semistable but not stable. □

Informally, the fact that instantons disappear after a classical deformation may also be understood as a consequence of Theorem 6.1, since there is no compact curve to hold the local charge.

**Remark 7.4.** Even though the above line of argument seems to suggest that the behaviour of instanton moduli in families might be much more well-behaved in the compact case, we stress that at least in the context of the instanton partition function [24], the noncompactness of the underlying surface is essential for the nontriviality of the theory.

**Acknowledgements.** We thank Barbara Fantechi and Pushan Majumdar for helpful discussions.
References

[1] C. C. Amilburu, S. Barmeier, B. Callander and E. Gasparim, Isomorphisms of moduli spaces, Mat. Contemp. 41 (2012), 1–16.
[2] A. F. Atiyah, On analytic surfaces with double points, P. Roy. Soc. Lond. A Math. 247 (1958), 237–244.
[3] V. Balaji and P. A. Vishwanath, On the deformations of certain moduli spaces of vector bundles, Am. J. Math. 115 (1993), 279–303.
[4] E. Ballico, Rank 2 vector bundles in a neighborhood of an exceptional curve of a smooth surface, Rocky Mt. J. Math. 29 (1999), 1185–1193.
[5] E. Ballico, S. Barmeier, E. Gasparim, L. Grama and L. A. B. San Martin, A Lie theoretical construction of a Landau–Ginzburg model without projective mirrors, preprint 2016.
[6] E. Ballico and E. Gasparim, Vector bundles on a neighborhood of an exceptional curve and elementary transformations, Forum Math. 15 (2003), 115–122.
[7] E. Ballico, E. Gasparim and T. Köppe, Local moduli of holomorphic bundles, J. Pure Appl. Algebra 213 (2009), 397–408.
[8] E. Ballico, E. Gasparim and T. Köppe, Vector bundles near negative curves: moduli and local Euler characteristic, Commun. Algebra 37 (2009), 2688–2713.
[9] O. Ben-Bassat and E. Gasparim Moduli stacks of bundles on local surfaces, in: R. Castano-Bernard, F. Catanese, M. Kontsevich, T. Pantev, Y. Soibelman and I. Zharkov (eds.), Homological Mirror Symmetry and Tropical Geometry, Springer, Berlin (2014), 1–32.
[10] A. Braverman and P. Etingof, Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg–Witten prepotential, in: Studies in Lie theory, Birkhäuser, Boston, MA (2006), 61–78.
[11] E. Gasparim, Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic, Commun. Algebra 25 (1997), 3001–3009.
[12] E. Gasparim, Rank two bundles on the blow-up of $\mathbb{C}^2$, J. Algebra 199 (1998), 581–590.
[13] E. Gasparim, L. Grama and L. A. B. San Martin, Symplectic Lefschetz fibrations on adjoint orbits, Forum Math. 28 (2016), 967–980.
[14] E. Gasparim, T. Köppe and P. Majumdar, Local holomorphic Euler characteristic and instanton decay, Pure Appl. Math. Q. 4 (2008), 161–179.
[15] E. Gasparim and C.-C. M. Liu, The Nekrasov conjecture for toric surfaces, Commun. Math. Phys. 293 (2010), 661–700.
[16] R. Hartshorne, Algebraic geometry, Springer, Berlin 1977.
[17] D. Kaledin and M. Verbitsky, Period map for non-compact holomorphically symplectic manifolds, Geom. Funct. Anal. 12 (2002), 1265–1295.
[18] Y. Kawamata, On deformations of compactifiable complex manifolds, Math. Ann. 235 (1978), 247–265.
[19] K. Kodaira, Complex manifolds and deformations of complex structures, Springer, Berlin 1986.
[20] M. Lübke and A. Teleman, The Kobayashi–Hitchin correspondence, World Scientific Publishing, River Edge, NJ 1995.
[21] M. Manetti, Lectures on deformations of complex manifolds, Rend. Mat. 24 (2004), 1–183.
[22] H. Nakajima and K. Yoshioka, Lectures on instanton counting, in: Algebraic structures and moduli spaces (Montreal 2003), CRM Proc. & Lect. Notes 38, American Mathematical Society, Providence, RI (2004), 31–101.
[23] M. S. Narasimhan and S. Ramanan, Deformations of the moduli space of vector bundles over an algebraic curve, Ann. Math. 101 (1975), 391–417.
[24] N. Nekrasov, Seiberg–Witten prepotential from instanton counting, Advances in Theoretical and Mathematical Physics 7 (2003), 831–864.
[25] N. A. Nekrasov and A. Okounkov, Seiberg-Witten theory and random partitions, in: The unity of mathematics, Birkhäuser, Boston (2006), 525–596.
[26] C. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces, Birkhäuser, Boston 1980.
[27] C. Sabbah, Théorie de Hodge et correspondance de Hitchin–Kobayashi sauvage, Séminaire Bourbaki, Volume 2011/2012, Exposé 1050, Astérisque 352 (2013), 1–36.
[28] C. S. Seshadri, Theory of moduli, in: Algebraic geometry (Arcata 1974), P. Symp. Pure Math. 29 (1974), 263–304.
[29] B. Totaro, The resolution property for schemes and stacks, J. reine angew. Math. 577 (2004), 1–22.