MIRROR SYMMETRY AND MODULI SPACES
OF SUPERCONFORMAL FIELD THEORIES

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Mirror symmetry is the remarkable discovery in string theory that certain “mirror pairs” of Calabi–Yau manifolds apparently produce isomorphic physical theories—related by an isomorphism which reverses the sign of a certain quantum number—when used as backgrounds for string propagation [13, 19, 11, 16]. The sign reversal in the isomorphism has profound effects on the geometric interpretation of the pair of physical theories. This leads to startling predictions that certain geometric invariants of one Calabi–Yau manifold (essentially the numbers of holomorphic 2-spheres of various degrees) should be related to a completely different set of geometric invariants of the mirror partner (“period” integrals of holomorphic forms).

We will discuss the applications of this mirror symmetry principle to the study of the moduli spaces of two-dimensional conformal field theories with \( N=(2,2) \) supersymmetry. Such theories depend on finitely many parameters, and for a large class of these theories the parameters admit a clear geometric interpretation. To circumvent the difficulties of trying to treat path integrals in a mathematically rigorous manner, we shall simply define the moduli spaces in terms of these geometric parameters. Other interesting physical quantities—the “topological” correlation functions—can then also be defined as asymptotic series whose coefficients have geometric meaning. The precise forms of the definitions are motivated by path integral arguments.

Mirror symmetry predicts some unexpected identifications between these moduli spaces, and serves as a powerful tool for understanding their structure. Perhaps the most striking consequence is the prediction that the moduli spaces can be analytically continued beyond the original domain of definition, into new regions, some of which parameterize conformal field theories that are related not to the original Calabi–Yau manifold, but rather to close cousins of it which differ by simple topological transformations.

In preparing this report, I have drawn on a considerable body of earlier work [1–5, 21–24], much of which was collaborative. I would like to thank my colleagues.

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1. The physics of nonlinear $\sigma$-models

We begin by describing nonlinear $\sigma$-models from the point of view of physics (see [15] and the references therein), and giving a geometric interpretation to the parameters which appear in the theory. The starting data for constructing a nonlinear $\sigma$-model consists of a compact manifold $X$, a Riemannian metric $g_{ij}$ on $X$, and a class $B \in H^2(X, \mathbb{R}/\mathbb{Z})$ (which we represent as a closed, $\mathbb{R}/\mathbb{Z}$-valued 2-form, i.e., a collection of closed, $\mathbb{R}$-valued 2-forms on the sets of an open cover of $X$ which differ by $\mathbb{Z}$-valued forms on overlaps). The bosonic version of the nonlinear $\sigma$-model is then specified, in the Lagrangian formulation, by the $\mathbb{C}/\mathbb{Z}$-valued (Euclidean) action which assigns to each sufficiently smooth map $\phi$ from an oriented Riemannian 2-manifold $\Sigma$ to $X$ the quantity

$$S[\phi] := i \int_{\Sigma} \|d\phi\|^2 d\mu + \int_{\Sigma} \phi^*(B),$$

where the norm $\|d\phi\|$ of $d\phi \in \text{Hom}(T\Sigma, \phi^*(T_X))$ is determined from the Riemannian metrics on $X$ and on $\Sigma$.

There is a variant of this theory in which additional fermionic terms are added to (1) to produce an action which is invariant under at least one supersymmetry transformation. (We will not write the fermionic terms in the action explicitly, as they do not enter into our analysis of the parameters.) The supersymmetric form of the action is also invariant under additional supersymmetry transformations when the geometry is restricted in certain ways—if the metric is Kähler then the theory has what is called $N=(2,2)$ supersymmetry, while if the metric is hyper-Kähler then the supersymmetry algebra is extended to $N=(4,4)$.

A nonlinear $\sigma$-model describes a consistent background for string propagation only if it is conformally invariant. The possible failure of conformal invariance is measured by the so-called “$\beta$-function” of the theory, and a perturbative calculation yields the result that the one-loop contribution to this $\beta$-function is proportional to the Ricci tensor of the metric. This makes Ricci-flat metrics—those with vanishing Ricci tensor—into good candidates for producing conformally invariant $\sigma$-models. In fact, supersymmetric $\sigma$-models whose Ricci-flat metric is in addition hyper-Kähler are believed to be conformally invariant, as are bosonic $\sigma$-models whose metric is flat.

When the supersymmetry algebra of the theory cannot be extended as far as $N=(4,4)$, the Ricci-flat theories fail to be conformally invariant. However, when the Ricci-flat metrics are Kähler (i.e., when the theory has $N=(2,2)$ supersymmetry), we can deduce some of the properties of the conformally invariant theory by a careful study of the Ricci-flat theories. This works as follows: renormalization

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1 We suppress the string coupling constant, and use a normalization in which the action appears as $\exp(2\pi i S)$ in the path integrals for correlation functions.
produces a flow on the space of metrics, and along a trajectory which begins at a Ricci-flat Kähler metric, the metric is expected to remain Kähler with respect to a fixed complex structure on $X$, and the Kähler class of the metric is not expected to change. Thus, if there is a conformally invariant theory in the same universality class as this trajectory, i.e., if there is a fixed point of the flow which lies in the trajectory’s closure, then any property of the conformal theory which depends only on the complex structure, the Kähler class, and the 2-form $B$ can be calculated anywhere along the trajectory, including the initial, Ricci-flat theory. Furthermore, every Ricci-flat Kähler metric whose Kähler class is sufficiently deep within the Kähler cone is expected to determine a unique conformally invariant theory (which lies in the same universality class).

We can thus define a first approximation to the parameter space for $N=(2, 2)$ superconformal field theories as follows (cf. [22]). Fix a compact manifold $X$, and define the one-loop semiclassical nonlinear $\sigma$-model moduli space of $X$ to be

$$M_\sigma := \{(g_{ij}, B)\} / \text{Diff}(X),$$

where $g_{ij}$ runs over the set of Ricci-flat metrics which are Kähler for some complex structure on $X$, $B$ is an element of $H^2(X, \mathbb{R}/\mathbb{Z})$, and Diff($X$) denotes the diffeomorphism group of $X$. Manifolds for which $M_\sigma$ is nonempty (that is, those which admit a Ricci-flat Kähler metric) are called Calabi–Yau manifolds. The 2-form $B$ should be regarded as some sort of “extra structure” (cf. [21]) which supplements the choice of metric.

It is important to keep in mind that the space $M_\sigma$ is only an approximation to the moduli space of conformal field theories, for several reasons:

- As already mentioned, not every pair $(g_{ij}, B)$ is expected to determine a conformal field theory, only those whose Kähler class is sufficiently deep within the Kähler cone.
- There may be analytic continuations of the space of conformal field theories beyond the domain where the theories have a $\sigma$-model interpretation. (We will see this in more detail in section 6.)
- There may be points of $M_\sigma$ which define isomorphic conformal field theories, even though they do not define isomorphic $\sigma$-models. This phenomenon was first observed in the case in which $X$ is a torus of real dimension $2d$, and $g_{ij}$ is a flat metric [25, 26]: in this case, $M_\sigma = \Gamma_0 \backslash \mathcal{D}$, where $\mathcal{D}$ is a certain symmetric space and $\Gamma_0 = \Lambda^2 \mathbb{Z}^d \rtimes \text{GL}(2d, \mathbb{Z})$, while the actual moduli space of conformal field theories takes the form $\Gamma \backslash \mathcal{D}$ for some $\Gamma$ containing the integral orthogonal group $\text{O}(\mathbb{Z}^{2d,2d})$ (in which $\Gamma_0$ is a parabolic subgroup).

In spite of these limitations, $M_\sigma$ provides a good arena for formulating a mathematical version of the theory, based on definitions using asymptotic expansions.

2. The correlation functions

The correlation functions of these quantum field theories will depend on the parameters in the action functional. If we construct a vector bundle over the moduli
space whose fiber over a particular point is the Hilbert space of operators in the
theory labeled by that point, then the correlation functions can be regarded as
multilinear maps from this bundle to the complex numbers. These maps and their
dependence on parameters can be studied by means of a semiclassical analysis, at
least in a certain “topological” sector of the theory. (In this sector, the dependence
of the correlation functions on the metric will always be a dependence on the Kähler
class alone.)

The semiclassical properties of the $N=(2,2)$ theory are calculated in terms
of the set of stationary values for the action (1). To find these, we pick a complex
structure on $\Sigma$ which makes its Riemannian metric Kähler, and which is compatible
with its orientation. Then the first term in the action (1) can be rewritten using
the formula:

$$\int_{\Sigma} \|d\phi\|^2 d\mu = \int_{\Sigma} \|\bar{\partial}\phi\|^2 d\mu + \int_{\Sigma} \phi^* (\omega),$$

where $\bar{\partial}\phi \in \text{Hom}(\Sigma^{(1,0)}, \phi^*(\Sigma^{(0,1)}))$ is determined by the complex structures, and
where $\omega$ is the Kähler form of the metric $g_{ij}$ on $X$. From this formula it is clear
that the stationary values are the holomorphic maps, i.e., those with $\bar{\partial}\phi \equiv 0$.
Furthermore, the action (1) evaluated on such a stationary value is the quantity

$$i \int_{\Sigma} \phi^* (\omega) + \int_{\Sigma} \phi^* (B) \in \mathbb{C}/\mathbb{Z},$$

which depends only on the homology class $\eta$ of the map $\phi$.

The path integral describing this quantum field theory has bosonic part

$$\int \mathcal{D}\phi \ e^{2\pi i S[\phi]},$$

and the correlation functions are calculated by inserting operators into this ex-
pression (see for example Witten’s address at the Berkeley ICM [31]). Such path
integrals are of course problematic for mathematicians, but it is possible to use the
outcome of the path integral manipulations as a basis for mathematical definitions.

To analyze these correlation functions, we break the path integral into a sum
over homology classes. This produces an asymptotic expansion which is expected to
converge for metrics whose Kähler class is sufficiently deep within the Kähler
cone. The terms in the asymptotic expansion are themselves path integrals whose
bosonic parts are the integrals of $\exp(2\pi i \int_{\Sigma} \|\bar{\partial}\phi\|^2 d\mu)$ over all maps of class $\eta$
(with operators inserted), weighted by the exponential of $2\pi i$ times the classical
action (4). For certain of the correlation functions, these “coefficient” path integrals
can in turn be evaluated by the methods of topological field theory (cf. [32, 34]):
upon modifying the fermionic terms in the action and introducing a parameter $t$,
the path integral with bosonic part

$$\int_{[\phi]=\eta} \mathcal{D}\phi \ e^{2\pi i t \int_{\Sigma} \|\bar{\partial}\phi\|^2 d\mu}$$
and “topological” operator insertions becomes independent of \( t \). This integral can then be evaluated by the method of stationary phase, which reduces it to a finite-dimensional integral over the set of stationary maps in class \( \eta \). Rigorous mathematical definitions for such “topological” correlation functions can be based on these finite-dimensional integrals, following ideas of Gromov [17] and Witten [32, 33]. See [20], [27], or Kontsevich’s address at this Congress for an account of these definitions and their properties.

In short, the physical quantities which can be calculated (by physicists) or defined (by mathematicians) using topological field theory will take the general form

\[
\sum_{\eta \in H_2(X,\mathbb{Z})} c_\eta e^{2\pi i (B + i\omega, \eta)}.
\]

Notice that the only dependence on the metric is through the complex structure and the Kähler class \( \omega \). The coefficient \( c_\eta \) will depend on the set of all holomorphic maps in class \( \eta \), and may well depend on the complex structure of \( X \). (It also depends on the behavior of the fermionic terms in the action which we have suppressed.) The key property of interest here is the holomorphic dependence of these functions on parameters: the coefficients \( c_\eta \) depend holomorphically on the complex structure, and the dependence of (7) on \( B + i\omega \) is also holomorphic (provided that the series converges and that \( H^{2,0}(X_J) = \{0\} \)).

3. Mirror symmetry

The analysis of the previous sections ultimately derives from the specific form of our physical theory, which is based on the geometry of Ricci-flat metrics on \( X \). We now adopt a somewhat more abstract point of view, and consider the structure of \( N=(2,2) \) superconformal field theories per se.

The algebraic approach to conformal field theories—which treats them as unitary representations of the Virasoro algebra—has been extensively studied in the mathematics literature (cf. [14], for example). When the theories are supersymmetric, the algebra which acts on the representation can be enlarged. The enlargement relevant here is the \( N=2 \) superconformal algebra (for which a convenient reference is [19]). This is a super extension of the Virasoro algebra whose even part contains a \( u(1) \)-subalgebra in addition to the Virasoro algebra itself. From this algebraic point of view, an \( N=(2,2) \) superconformal field theory is simply a unitary representation of two commuting copies of this algebra; there is thus an induced representation of the subalgebra \( u(1) \times u(1) \).

The deformations of these representations have been analyzed in the physics literature [12, 13]. The infinitesimal deformations can be identified with the finite-dimensional kernel \( V \) of a certain operator, and it is argued in [12, 13] that there should be no obstructions to deforming in the directions corresponding to \( V \).

\[2\]The arguments in [12] and [13] involve more of the physical structure than is present in the purely algebraic formulation we are discussing here. It would be desirable to have a purely algebraic proof of this statement.
The $u(1) \times u(1)$ manifests itself on $V$ in the following way: there are two commuting complex structures $J$ and $J'$ on $V$, each of which determines a natural representation of $u(1)$ on $V \otimes \mathbb{C}$ (with respect to which half of the charges\(^3\) are $+1$ and half are $-1$). The two complex structures together determine a representation of $u(1) \times u(1)$ on $V \otimes \mathbb{C}$, and we can decompose $V \otimes \mathbb{C}$ into four complex subspaces $V^{\pm 1, \pm 1}$ according to the $u(1) \times u(1)$ charges.

If we use $J$ to put a complex structure on $V$ and call the resulting space $V_J$, then we can write $V_J = V^{1,1} \oplus V^{1,-1}$. From this point of view, since $J'$ has eigenvalues $\pm i$, respectively, on the two summands while $J$ is simply multiplication by $i$, we can identify the summands as $V^{1,1} = \ker(JJ' - \text{Id}) \subset V_J$ and $V^{1,-1} = \ker(JJ' + \text{Id}) \subset V_J$.

A mirror isomorphism between two $N=(2,2)$ superconformal field theories is an isomorphism which reverses the sign of one of the $u(1)$ charges. If it is the second $u(1)$ charge which is reversed, then the isomorphism will map $V^{\pm 1, \pm 1}$ to $V^{\pm 1, \mp 1}$, and will interchange the factors in the decomposition $V_J = V^{1,1} \oplus V^{1,-1}$.

A mirror isomorphism must preserve all correlation functions, not just the topological ones. It particular, it preserves the bilinear form on $V$ which corresponds to the so-called Zamolodchikov metric on $M_\sigma$. Thanks to the preservation of this metric, a mirror isomorphism at a single point can always be extended to a local isometry between the moduli spaces. There will also be a compatible isomorphism of the bundles of Hilbert spaces which maps the topological correlation functions from one theory to those of the other, but because of the sign change in the $u(1)$ charge, the geometric interpretations of these correlation functions may be rather different. For example, Candelas, de la Ossa, Green, and Parkes [10] used a mirror isomorphism to assert that a correlation function which they could compute exactly (using period integrals) as

$$5 + 2875 \frac{q}{1-q} + 609250 \frac{2^3 q^2}{1-q^2} + 317206375 \frac{3^3 q^3}{1-q^3} + 242467530000 \frac{4^3 q^4}{1-q^4} + \cdots$$

should coincide with a generating function of the form (7) in which the coefficients represent the numbers of holomorphic 2-spheres of various degrees on a quintic hypersurface in $\mathbb{CP}^4$. (See [21] or the Givental’s address at this Congress for some of the mathematical aspects of this generating function.)

4. LOCAL ANALYSIS OF THE $\sigma$-MODEL MODULI SPACE

The abstract description of the deformations of $N=(2,2)$ theories can be made very concrete for $\sigma$-models, where it reveals the local structure of the space $M_\sigma$. The set of first-order variations $\delta g$ of a fixed Riemannian metric $g_{ij}$ on $X$ can be identified with the space of symmetric contravariant 2-tensors $\Gamma(\text{Sym}^2 T_X^*)$. If $X$ is compact and $g_{ij}$ is Ricci-flat, then according to a theorem of Berger and

\(^3\)For a representation $\rho$ of $u(1) \cong i \mathbb{R}$, the eigenvalues of $\rho(i)$ are called the \textit{charges} of the representation.
The space of first-order variations\(^4\) \(\delta g\) (modulo \(\text{Diff}(X)\)) which preserve the Ricci-flat condition can be identified with the kernel of the Lichnerowicz Laplacian \(\Delta_L\) acting on \(\Gamma(\text{Sym}^2 T^*_X)\). On the other hand, the set of first-order variations \(\delta B\) of the 2-form \(B\) can be identified with the space of harmonic 2-forms \(\ker \Delta \subset \Gamma(\Lambda^2 T^*_X)\). Since the Lichnerowicz Laplacian on 2-forms coincides with the ordinary Laplacian, the combined contravariant 2-tensor \(\delta g + \delta B \in \Gamma(\bigotimes^2 T^*_X)\) satisfies \(\Delta_L(\delta g + \delta B) = 0\). We can thus identify the tangent space to \(\mathcal{M}_\sigma\) at \((g_{ij}, B)\) with \(\ker \Delta_L \subset \Gamma(\bigotimes^2 T^*_X)\).

Let us assume that the holonomy of \(g_{ij}\) takes its “generic” value for Ricci-flat Kähler metrics, namely \(\text{SU}(n), n \geq 3\) (where \(n := \dim \mathbb{R}X\)). In this case, the two complex structures which we are expecting from our abstract analysis can be described as follows. First, if we fix a complex structure\(^5\) \(J\) on \(X\) with respect to which \(g_{ij}\) is Kähler, there is an induced operator \(J\) on \(\Gamma(\bigotimes^2 T^*_X)\) defined by \(Jh(x, y) := h(x, Jy)\). This new operator \(J\) commutes with \(\Delta_L\), and so induces an operator on the tangent space \(\ker \Delta_L\) of \(\mathcal{M}_\sigma\) whose square is \(-\text{Id}\), that is, a complex structure on \(\ker \Delta_L\).

The second complex structure \(J'\) on \(\ker \Delta_L\) is much less obvious. It can be characterized by the property that the product \(JJ'\) acts as \(-\text{Id}\) on the space of symmetric, skew-Hermitian tensors, and as \(+\text{Id}\) on the space of tensors which are either Hermitian or skew-symmetric. Explicitly, \(J'\) can be defined by the formula

\[
J'h(x, y) := \frac{1}{2} \left( -h(x, Jy) + h(y, Jx) + h(Jx, y) + h(Jy, x) \right).
\]

Using \(J\) to put a complex structure on \(\ker \Delta_L\), we can identify \(V^{1,-1}\) with the space of symmetric, skew-Hermitian tensors in \(\ker \Delta_L\). This space corresponds to that part of the moduli space of metrics which is obtained by varying the complex structure (cf. [7, Chapter 12]). The operator \(J\) preserves that space, and induces the usual complex structure on it. In fact, under our assumptions about the holonomy, the complex structure can be varied freely and we have \(V^{1,-1} \cong H^1(T^{1,0}_X, \mathbb{C})\), the latter being the space of first-order variations of complex structure.

We can similarly identify \(V^{1,1}\) as the space consisting of tensors which are either Hermitian or skew-symmetric; on this space, the operator \(J\) mixes symmetric and skew-symmetric forms, so does not have a classical interpretation in terms of metrics alone. The parameters associated to this part of the deformation space are of the form \(B + i\omega\), and \(V^{1,1} \cong H^{1,1}(X, \mathbb{C}) \cong H^2(X, \mathbb{C})\) (under our assumption that the holonomy is \(\text{SU}(n), n \geq 3\)).

A mirror isomorphism between Calabi–Yau manifolds \(X\) and \(Y\) thus identifies the space of complex deformations of \(X\) with the space of complexified Kähler deformations of \(Y\), and vice versa (at least when the holonomy is “generic”).

\(^4\)It follows from the theorem of Bogomolov [8], Tian [28] and Todorov [29] that first-order variations can always be extended to deformations of the metric.

\(^5\)When the holonomy is \(\text{SU}(n), n \geq 3\), there are precisely two such complex structures: \(J\) and \(-J\).
5. Global analysis of the σ-model moduli space

The moduli space of Ricci-flat metrics (and hence the nonlinear σ-model moduli space) can be analyzed globally as well as locally. To carry this out, we introduce a related space which includes a choice of complex structure. Define

\[ M_{N=2} := \{ (g_{ij}, B, J) \} / \text{Diff}(X) \]

where \( J \) ranges over the complex structures on \( X \) with respect to which \( g_{ij} \) is Kähler. The holonomy group of the metric \( g_{ij} \) is necessarily contained in the \( \text{SU}(n) \) specified by \( J \). The fibers of the natural map \( M_{N=2} \to M_\sigma \) depend on this holonomy group, and can be described as the set of \( \text{U}(n) \)'s which lie between the holonomy group and \( \text{O}(2n) \). Some examples:

1. If the holonomy is \( \text{SU}(n), n \geq 3 \), then the fiber consists of two points. (This is the “generic” case.)
2. If the holonomy is \( \text{Sp}(n/2, \mathbb{C}) \), then the fiber is \( \mathbb{C}P^1 \). (This is the case of an indecomposable hyper-Kähler manifold, such as a K3 surface.\(^6\))

The real dimension of the fiber is always \( \dim \mathbb{R} H^{2,0}(X, J) \).

The structure of the space \( M_{N=2} \) can be determined from the natural map \( M_{N=2} \to M_\text{complex} := \{ J \} / \text{Diff}(X) \). By the theorems of Calabi [9] and Yau [36], the fibers of this map take the form \( K_C(X, J) / \text{Aut}(X, J) \), where \( K_C(X, J) \) is the complexified Kähler cone\(^7\)

\[ K_C(X, J) := \{ B + i\omega \in H^2(X, \mathbb{C}/\mathbb{Z}) \mid \omega \in K_J \}, \]

\( K_J \) being the set of Kähler classes on \( X, J \), and \( \text{Aut}(X, J) \) being the group of holomorphic automorphisms. It is this fact which gives us access to global information about the conformal field theory moduli space, since the moduli space of complex structures can be studied by the methods of algebraic geometry. For example, by a theorem of Viehweg [30] the subspace \( M_\text{complex} \subset M_\text{complex} \) consisting of all complex structures polarized with respect to a fixed class \( L \) is a quasi-projective variety, i.e., the complement of a finite number of compact subvarieties in a compact complex manifold. (And the spaces \( M_L^\text{complex} \) are open subsets of \( M_\text{complex} \) when \( H^{2,0}(X, J) = \{ 0 \} \).) In contrast, although \( K_C \) has a canonical complex structure when \( H^{2,0}(X, J) = \{ 0 \} \), it is typically a rather small domain.

Note that the expected condition for a given pair \( (g_{ij}, B) \) to determine a conformal field theory was stated in terms of the Kähler class only and was valid for every choice of complex structure. Thus, the global description of the complex structures should be valid for the conformal field theory moduli space itself. On the other hand, the complexified Kähler directions are subject to modification.

\(^6\)K3 surfaces are “self-mirror,” and the mirror map induces an automorphism of \( M_\sigma \). Thus, as in the case of a torus, the moduli space of conformal field theories of this type is a nontrivial quotient of \( M_\sigma \) (cf. [5], where this quotient is determined precisely).

\(^7\)This definition differs slightly from ones we have given elsewhere [22, 23].
6. Beyond the Kähler cone

We now apply the mirror symmetry principle to study the moduli space in the case in which the holonomy of the Ricci-flat metrics on $X$ is $SU(n)$, $n \geq 3$.

Suppose that a mirror partner $Y$ is known for $X$. The mirror map between the moduli spaces $\mathcal{M}_\sigma(X)$ and $\mathcal{M}_\sigma(Y)$ will certainly be well-defined at points corresponding to metrics whose Kähler class is sufficiently deep within the Kähler cone, but in general we can only expect a partially defined, local isomorphism between these spaces. However, because of the global nature of the complex structure space $\mathcal{M}_{\text{complex}}(Y)$, we can deduce the structure of the Kähler moduli space $\mathcal{K}_C(X)$ from even a local knowledge of the mirror map. In principle, the mirror map should be determined essentially uniquely from the structure of the Zamolodchikov metric, once the derivative of the map is known at a single point. In practice, it is easier to approach the construction of the mirror map in other ways (based on the topological correlation functions) which determine it up to a finite number of unknown parameters. Even those parameters can often be determined. (See [24] for a recent review of this problem.)

This comparison of structure between Kähler and complex moduli spaces has been carried out in [1, 2] for cases in which a mirror partner is known (to physicists) thanks to some explicit constructions using the discrete series representation of the $N=(2,2)$ superconformal algebra [16]. The results are quite illuminating: on the one hand, the locally defined map

$$\mathcal{K}_C(X) \longrightarrow \mathcal{M}_{\text{complex}}(Y)$$

does not in general extend throughout $\mathcal{K}_C(X)$, but instead there are points where the theories become singular, and the map encounters difficulties beyond those points. On the other hand, the image of (11) is not all of $\mathcal{M}_{\text{complex}}(Y)$—as we have already suggested, $\mathcal{K}_C(X)$ is much smaller than $\mathcal{M}_{\text{complex}}(Y)$. This means that there must be a way to analytically continue the conformal field theories on $X$ beyond the theories specified by $\mathcal{K}_C(X)$ (since such theories occur in $\mathcal{M}_{\text{complex}}(Y)$). This second conclusion was independently reached by Witten [35] on somewhat different grounds.

What, then, lies beyond the Kähler cone for such theories? In some cases, the conformal field theories are $\sigma$-models on other Calabi–Yau manifolds which are obtained by a simple topological surgery from $X$ (see [1, 2] and [35], or for a more mathematical account, [23]). In these cases, as the Kähler class is varied and allowed to approach a wall of the Kähler cone, a finite number of holomorphic 2-spheres have their areas approach 0. When the Kähler class is pushed beyond that wall, the areas of those 2-spheres would apparently become negative. However, the analytically continued $\sigma$-model should instead be formulated as a $\sigma$-model on a modified manifold $X'$, which is obtained from $X$ by a surgery along the 2-spheres in such a way that the sign of their (common) homology class has been reversed (cf. [18]).

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8This phenomenon is already visible in the example considered in [10].
The collection of complexified Kähler cones of the various topological models produces a rich combinatorial structure of regions in the moduli space corresponding to the different models. But even these do not fill up the entire conformal field theory moduli space—there are additional regions whose associated conformal field theories must be described by constructions other than σ-models [35, 2]. These theories are currently under active study.

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