Lyapunov-type inequalities for differential equation with Caputo–Hadamard fractional derivative under multipoint boundary conditions

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Abstract
In this work, we establish Lyapunov-type inequalities for the fractional boundary value problems with Caputo–Hadamard fractional derivative subject to multipoint and integral boundary conditions. As far as we know, there is no literature that has studied these problems.

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1 Introduction
The well-known Lyapunov inequality [1] states that if \( u(t) \) is a nontrivial solution of the boundary value problem

\[
\begin{align*}
  u''(t) + q(t)u(t) &= 0, \quad t \in (a, b), \\
  u(a) &= 0 = u(b),
\end{align*}
\]

where \( q(t) \in C([a, b]; \mathbb{R}) \), then

\[
\int_a^b |r(t)| \, dt > \frac{4}{b-a}.
\]

The Lyapunov inequality (1.2) is a useful tool in various branches of mathematics, including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of inequality (1.2) have appeared in the literature; see [2–13] and references therein.

The study of Lyapunov-type inequalities for fractional differential equations has begun recently. The first result in this direction is due to Ferreira [14]. He obtained a Lyapunov
inequality for Riemann–Liouville fractional differential equations; his main result is as follows.

**Theorem 1.1** If the fractional boundary value problem

\[
\left(D_a^\alpha u\right)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leq 2, \tag{1.3}
\]

\[
u(a) = 0 = u(b), \tag{1.4}
\]

has a nontrivial solution, where \( q \) is a real continuous function, then

\[
\int_a^b \left| q(s) \right| ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}, \tag{1.5}
\]

where \( D_a^\alpha \) is the Riemann–Liouville fractional derivative of order \( \alpha \).

One year later, the same author Ferreira [15] obtained a Lyapunov-type inequality for the Caputo fractional boundary value problem.

**Theorem 1.2** If the fractional boundary value problem

\[
\left(^C D_a^\alpha u\right)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leq 2, \tag{1.6}
\]

\[
u(a) = 0 = u(b), \tag{1.7}
\]

where \( q \) is a real continuous function, has a nontrivial continuous solution, then

\[
\int_a^b \left| q(s) \right| ds > \frac{\Gamma(\alpha) \alpha^\alpha}{[(\alpha - 1)(b-a)]^{\alpha-1}}, \tag{1.8}
\]

where \(^C D_a^\alpha \) is the Caputo fractional derivative of order \( \alpha \).

After the publication of [14, 15], the research on Lyapunov inequalities for fractional differential equations has become a hot topic. The results in the literature can be divided into two categories. The first one is using other fractional derivatives instead of the Caputo fractional derivatives or Riemann–Liouville fractional derivatives in equation (1.3) or (1.6). Secondly, the boundary conditions (1.4) or (1.7) are replaced by multipoint boundary conditions or integral boundary conditions. For instance, in [16–18], Lyapunov inequalities for Hadamard fractional differential equations are given. Lyapunov-type inequalities regarding sequential fractional differential equations are obtained in [19–21]. The first paper considering integral boundary conditions is also due to Ferreira [22]. For the results of multipoint boundary conditions, see [23, 24].

Motivated by the above works, in this paper, we establish Lyapunov-type inequalities for the fractional boundary value problems with Caputo–Hadamard fractional derivative under multipoint boundary condition

\[
\left(^C D_a^\alpha u\right)(t) + q(t)u(t) = 0, \quad 0 < a < t < b, \ 1 < \alpha < 2, \tag{1.9}
\]

\[
u(a) = 0, \quad \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{1.10}
\]

where \(^C D_a^\alpha \) denotes the Caputo–Hadamard fractional derivative of order \( \alpha \).
In this paper, we assume that $\beta_i \geq 0$ ($i = 1, 2, \ldots, m - 2$), $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, and $0 \leq \sum_{i=1}^{m-2} \beta_i < 1$.

2 Preliminaries

In this section, we recall the concepts of the Riemann–Liouville fractional integral, the Riemann–Liouville fractional derivative, the Caputo fractional derivative of order $\alpha \geq 0$, and the definition of the Caputo–Hadamard fractional derivative.

**Definition 2.1** ([25]) Let $\alpha \geq 0$, and let $f$ be a real function on $[a,b]$. The Riemann–Liouville fractional integral of order $\alpha$ is defined by $(I_a^0 f) \equiv f$ and

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds, \quad \alpha > 0, t \in [a,b].$$

**Definition 2.2** ([25]) The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined by $(D_a^0 f) \equiv f$ and

$$(D_a^\alpha f)(t) = \left(D^m I_{a+}^{m-\alpha} f\right)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s) \, ds$$

for $\alpha > 0$, where $m$ is the smallest integer greater than or equal to $\alpha$.

**Definition 2.3** ([25]) The Caputo fractional derivative of order $\alpha \geq 0$ is defined by $(^C D_a^0 f) \equiv f$ and

$$(^C D_a^\alpha f)(t) = \left(^C D_{a+}^{m-\alpha} D^m f\right)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^m(s) \, ds,$$

for $\alpha > 0$, where $m$ is the smallest integer greater than or equal to $\alpha$.

**Definition 2.4** ([25]) The Hadamard fractional integral of order $\alpha \in \mathbb{R}$, for a continuous function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$(^H I_a f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t}{s}\right)^{\alpha-1} f(s) \, ds, \quad \alpha > 0, t \in [a,b].$$

**Definition 2.5** ([25]) The Hadamard fractional derivative of order $\alpha \in \mathbb{R}$, for a continuous function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$(^H D_a f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \, ds,$$

where $n - 1 < \alpha < n, n = [\alpha] + 1$.

**Definition 2.6** ([25]) The Caputo–Hadamard fractional derivative of order $\alpha \in \mathbb{R}$, for a function $f \in AC_a^{[\alpha]}[a,b]$ is defined as

$$(^H \delta D_a f)(t) = \left(^H I_a^{m-\alpha} \delta^n f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \delta^n f(s) \, ds,$$

where $n = [\alpha] + 1$, and $f \in AC_a^{[\alpha]}[a,b] = \{\varphi : [a,b] \to \mathbb{C} : \delta^{[n-1]} \varphi \in AC[a,b], \delta = t \frac{d}{dt}\}$. 
Lemma 2.7 ([25]) Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. If $f \in AC^n_{\alpha}[a, b]$ or $f \in C^n_{\alpha}[a, b]$, then

$$(nI_{a+}^{C}D_{a}^{\alpha}f)(t) = f(t) - \sum_{k=1}^{n} \frac{\delta^{k-1}f(a)}{(k-1)!} \left( \ln \frac{t}{a} \right)^{k-1}.$$  

3 Main results

We begin by writing problem (1.9)–(1.10) in an equivalent integral form.

Lemma 3.1 A function $u \in C[a, b]$ is a solution to the boundary value problem (1.9)–(1.10) if and only if it satisfies the integral equation

$$u(t) = \int_{a}^{b} G(t, s)q(s)u(s) \, ds + \frac{\ln \frac{t}{a}}{\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}} \int_{a}^{b} \sum_{i=1}^{m-2} \beta_i G(\xi_i, s)q(s)u(s) \, ds,$$

where $G(t, s)$ is defined as

$$G(t, s) = \frac{1}{s \ln \frac{b}{a} \Gamma(\alpha)} \begin{cases} \ln \frac{t}{a} (\ln \frac{b}{s})^\alpha - \ln \frac{b}{a} (\ln \frac{b}{s})^\alpha, & 0 < a \leq s \leq t \leq b, \\ \ln \frac{t}{a} (\ln \frac{b}{s})^\alpha, & 0 < a \leq t \leq s \leq b. \end{cases}$$

Proof By Lemma 2.7 $u \in C[a, b]$ is a solution to the boundary value problem (1.9)–(1.10) if and only if

$$u(t) = c_0 + c_1 \left( \ln \frac{t}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \ln \frac{t}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s},$$

where $c_0$ and $c_1$ are real constants. Since $u(a) = 0$, we immediately get that $c_0 = 0$, and thus

$$u(t) = c_1 \left( \ln \frac{t}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \ln \frac{t}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}.$$  

The boundary condition $u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i)$ yields

$$c_1 \left( \ln \frac{b}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s} = \sum_{i=1}^{m-2} \beta_i \left[ c_1 \left( \ln \frac{\xi_i}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{a}^{\xi_i} \left( \ln \frac{\xi_i}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s} \right],$$

so,

$$c_1 = \frac{\int_{a}^{b} \ln \frac{b}{s} \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s} - \sum_{i=1}^{m-2} \beta_i \int_{a}^{\xi_i} \ln \frac{\xi_i}{s} \left( \ln \frac{\xi_i}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}}{(\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}) \Gamma(\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \ln \frac{b}{s} \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s} - \sum_{i=1}^{m-2} \beta_i \int_{a}^{\xi_i} \ln \frac{\xi_i}{s} \left( \ln \frac{\xi_i}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}$$

$$+ \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a} \int_{a}^{b} \ln \frac{b}{s} \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}.$$
Hence

\[
  u(t) = c_1 \left( \ln \frac{t}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_a^t \left( \ln \frac{s}{t} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}
\]

\[
  = \frac{\ln \frac{t}{a}}{\Gamma(\alpha)} \int_a^b \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s} - \frac{\ln \frac{b}{a}}{\Gamma(\alpha)} \sum_{i=1}^{m-1} \beta_i \int_a^b \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}
\]

\[
  + \frac{\ln \frac{b}{a}}{\Gamma(\alpha)} \sum_{i=1}^{m-2} \beta_i \ln \frac{b}{a} \int_a^b \left( \ln \frac{b}{s} \right)^{\alpha-1} q(s)u(s) \frac{ds}{s}
\]

\[
  = \int_a^b G(t, s)q(s)u(s) ds + \frac{\ln \frac{b}{a}}{\sum_{i=1}^{m-2} \beta_i \ln \frac{b}{a}} \int_a^b \sum_{i=1}^{m-2} \beta_i G(\xi_i, s)q(s)u(s) ds,
\]

which concludes the proof. 

\[
\square
\]

**Lemma 3.2** Let \( 0 < a \leq s \leq b \) and \( 1 < \alpha < 2 \). Then

\[
  0 \leq \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha-1}{\alpha}} \leq (2 - \alpha)(\alpha - 1) \left( \frac{b}{a} \right)^{\frac{1}{\alpha}}.
\]

**Proof** Let

\[
  f(s) = \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha-1}{\alpha}}, \quad s \in [a, b].
\]

Clearly, \( f(a) = f(b) = 0 \), and \( f(s) > 0 \) on \( (a, b) \). By Rolle’s theorem there exists \( s^* \in (a, b) \) such that \( f(s^*) = \max f(s) \) on \( (a, b) \), that is, \( f'(s^*) = 0 \). Note that

\[
  f'(s) = \frac{1}{s} \left( \ln \frac{b}{s} \right)^{\frac{\alpha-1}{\alpha}} \left[ \ln \frac{b}{s} - \frac{\alpha - 1}{2 - \alpha} \ln \frac{s}{a} \right].
\]

Letting \( f'(s) = 0 \), we obtain \( s^* = a^{\alpha-1}b^{2-\alpha} \). It is easy to show that \( \frac{s}{a} = \left( \frac{b}{a} \right)^{2-\alpha} > 1 \), \( \frac{b}{s} = \left( \frac{b}{a} \right)^{\alpha-1} > 1 \), and \( s^* \in (a, b) \), and thus

\[
  \max f(s) = f(s^*) = \ln \frac{s}{a} \left( \ln \frac{b}{s^*} \right)^{\frac{\alpha-1}{\alpha}} = (2 - \alpha)(\alpha - 1) \left( \frac{b}{a} \right)^{\frac{1}{\alpha}},
\]

which concludes the proof. 

\[
\square
\]

**Lemma 3.3** Let \( 0 < a \leq s \leq b \) and \( 1 < \alpha < 2 \). Then

\[
  0 \leq \frac{1}{s} \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\alpha-1} \leq \frac{1}{a} \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\alpha-1} \leq \frac{1}{a} \frac{(\alpha - 1)^{\alpha-1}}{a^\alpha} \left( \ln \frac{b}{a} \right)^{\alpha}.
\]

**Proof** Let

\[
  g(s) = \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\alpha-1}, \quad s \in [a, b].
\]
As \( g(a) = g(b) = 0 \) and \( g(s) > 0 \) on \((a, b)\). So, there exists \( s^* \in (a, b) \) such that \( g(s^*) = \max g(s) \) on \((a, b)\), that is, \( g'(s^*) = 0 \). Note that

\[
g'(s) = \frac{1}{s} \left( \ln \frac{b}{s} \right) \left( \ln s - (\alpha - 1) \ln \frac{s}{a} \right).
\]

Letting \( g'(s) = 0 \), we obtain \( s^* = a^{\frac{\alpha - 1}{\alpha}} b^{\frac{1}{\alpha}} \), \( \frac{s^*}{a} = (\frac{b}{a})^{\frac{1}{\alpha}} > 1 \), and \( \frac{b}{s^*} = (\frac{b}{a})^{\frac{\alpha - 1}{\alpha}} > 1 \), which imply that \( s^* \in (a, b) \), and thus

\[
\max g(s) = g(s^*) = \ln \frac{s^*}{a} \left( \ln \frac{b}{s^*} \right)^{\alpha - 1} = \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha} \left( \ln \frac{b}{a} \right)^\alpha,
\]

which concludes the proof. \( \square \)

**Lemma 3.4** Let \( 0 < \alpha \leq s \leq b(a/b)^{\alpha - 1} \) and \( 1 < \alpha < 2 \). Then the function

\[
h(s) = (2 - \alpha)(\alpha - 1)^{\frac{\alpha - 1}{\alpha}} \frac{1}{s^\alpha} \left( \ln \frac{b}{s} \right)^{\frac{1}{\alpha}} - \frac{1}{s} \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\alpha - 1},
\]

satisfies

\[
\max_{s \in [a, b(a/b)^{\alpha - 1}]} h(s) = (2 - \alpha)(\alpha - 1)^{\frac{\alpha - 1}{\alpha}} \frac{1}{a^\alpha} \ln \frac{b}{a}.
\]

**Proof** For \( 0 < \alpha \leq s \leq b(a/b)^{\alpha - 1} \), we have \( (\alpha - 1) \ln \frac{s}{a} \leq \ln \frac{b}{s} \leq \ln \frac{b}{a} \), \( 0 < \ln \frac{s}{a} < (2 - \alpha) \ln \frac{b}{a} \).

Define the new function

\[
r(s) = sh(s) = (2 - \alpha)(\alpha - 1)^{\frac{\alpha - 1}{\alpha}} \frac{1}{s^\alpha} \left( \ln \frac{b}{a} \right)^{\frac{1}{\alpha}} - \frac{1}{s} \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\alpha - 1} - \frac{(\alpha - 1) \ln \frac{s}{a} \ln \frac{b}{s}}{\alpha - 1} - \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha} \left( \ln \frac{b}{a} \right)^\alpha
\]

By Lemma 3.2, \( r(s) \geq 0 \), and we easily obtain

\[
r'(s) = \frac{1}{s} \left[ (\alpha - 1)^{\frac{\alpha - 1}{\alpha}} \left( \ln \frac{b}{a} \right)^{\frac{1}{\alpha}} - \left( \ln \frac{b}{s} \right)^{\alpha - 1} + (\alpha - 1) \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\alpha - 2} \right]
\]

\[
\leq \frac{1}{s} \left[ (\alpha - 1)^{\frac{\alpha - 1}{\alpha}} \left( \ln \frac{b}{a} \right)^{\frac{1}{\alpha}} - (\alpha - 1)^{\alpha - 1} \left( \ln \frac{b}{a} \right)^{\alpha - 1} \left( \ln \frac{b}{a} \right) + (\alpha - 1)(2 - \alpha) \ln \frac{b}{a} (\alpha - 1)^{\alpha - 2} \left( \ln \frac{b}{a} \right)^{\alpha - 2} \right]
\]

\[
= \frac{1}{s} \left[ (\alpha - 1)^{\alpha - 1} \left( \ln \frac{b}{a} \right)^{\alpha - 1} + (\alpha - 1)(2 - \alpha) \ln \frac{b}{a} (\alpha - 1)^{\alpha - 2} \left( \ln \frac{b}{a} \right)^{\alpha - 2} \right]
\]

\[= 0.\]
So,

\[ h'(s) = \left( \frac{r(s)}{s} \right)' = \frac{sr'(s) - r(s)}{s^2} < 0. \]

Therefore

\[
\max_{s \in [a, b]} h(s) = h(a) = (2 - \alpha)(\alpha - 1) \frac{a^1}{a} \frac{1}{a} \left( \ln \frac{b}{a} \right)^\alpha.
\]

Lemma 3.5  If \( 1 < \alpha < 2 \), then

\[
(2 - \alpha)(\alpha - 1) \frac{a^1}{\alpha^\alpha} \leq \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha}.
\]

Proof  A proof of this lemma can be found in [15]. Here we give a new proof. Let \( 0 < a \leq s \leq b \). It is easy to check that

\[
\ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}} \left( \ln \frac{b}{a} \right)^{\frac{(\alpha - 1)^2}{\alpha^2}} - \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}} = \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}} \left( \ln \frac{b}{a} \right)^{\frac{(\alpha - 1)^2}{\alpha^2}} - 1
\]

so,

\[
\ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}} \left( \ln \frac{b}{a} \right)^{\frac{(\alpha - 1)^2}{\alpha^2}} \leq \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}}
\]

and thus

\[
\max_{0 < a \leq s \leq b} \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}} \left( \ln \frac{b}{a} \right)^{\frac{(\alpha - 1)^2}{\alpha^2}} \leq \max_{0 < a \leq s \leq b} \ln \frac{s}{a} \left( \ln \frac{b}{s} \right)^{\frac{\alpha - 1}{\alpha}}.
\]

By Lemmas 3.2 and 3.3 we obtain

\[
(2 - \alpha)(\alpha - 1) \frac{a^1}{\alpha^\alpha} \left( \ln \frac{b}{a} \right)^\alpha \leq \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha} \left( \ln \frac{b}{a} \right)^\alpha.
\]

Thus the proof is completed. \( \square \)

Lemma 3.6  The function \( G \) defined in Lemma 3.1 satisfies the following property:

\[
|G(t, s)| \leq \frac{1}{a} \cdot \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha \Gamma(\alpha)} \left( \ln \frac{b}{a} \right)^{\alpha - 1}.
\]
Proof. The Green’s function $G(t, s)$ can be rewritten as the following form:

$$\left( \ln \frac{b}{a} \right) \Gamma(a) G(t, s) = \begin{cases} \frac{1}{a} \ln \frac{t}{s} (\ln \frac{b}{a})^{a-1} - \frac{1}{a} \ln \frac{b}{a} (\ln \frac{t}{s})^{a-1}, & a \leq s \leq t \leq b, \\ \frac{1}{a} \ln \frac{t}{s} (\ln \frac{b}{a})^{a-1}, & a \leq t \leq s \leq b. \end{cases}$$

Define two functions

$$g_1(t, s) = \frac{1}{s} \ln \frac{t}{s} \left( \ln \frac{b}{s} \right)^{a-1} - \ln \frac{b}{s} \left( \ln \frac{t}{s} \right)^{a-1}, \quad a \leq s \leq t \leq b,$$

$$g_2(t, s) = \frac{1}{s} \ln \frac{t}{s} \left( \ln \frac{b}{s} \right)^{a-1}, \quad a \leq t \leq s \leq b.$$

Obviously, $g_2(t, s)$ is an increasing function in $t$, and $0 \leq g_2(t, s) \leq g_2(s, s)$. By Lemma 3.3, we obtain

$$g_2(t, s) \leq g_2(s, s) \leq \frac{1}{a} \frac{(a - 1)^{a-1}}{a^a} \left( \ln \frac{b}{a} \right)^a.$$

Now we turn our attention to the function $g_1(t, s)$. We start by fixing an arbitrary $s \in [a, b]$. Differentiating $g_1(t, s)$ with respect to $t$, we get

$$\frac{\partial g_1(t, s)}{\partial t} = \frac{1}{s t} \left( \ln \frac{b}{s} \right)^{a-1} - (a - 1) \ln \frac{b}{s} \left( \ln \frac{t}{s} \right)^{a-2}.$$

It follows that $\frac{\partial g_1(t^*, s)}{\partial t} = 0$ if and only if $t^* = s e^{\frac{(a - 1) (s - b \ln \frac{b}{a})}{s - b \ln \frac{b}{a}}}$, provided that $t^* \leq b$, that is, as long as $s \leq b(a/b)^{a-1}$. So, if $s > b(a/b)^{a-1}$, then $t^* > b$ and $t < t^* = s e^{\frac{(a - 1) (s - b \ln \frac{b}{a})}{s - b \ln \frac{b}{a}}}$, and therefore $\frac{\partial g_1(t, s)}{\partial t} < 0$, $g_1(t, s)$ is strictly decreasing with respect to $t$, and thus we have

$$0 = g_1(b, s) \leq g_1(t, s) \leq g_1(s, s) = g_2(s, s).$$

From this we conclude that

$$|g_1(t, s)| \leq g_2(s, s) \leq \frac{1}{a} \frac{(a - 1)^{a-1}}{a^a} \left( \ln \frac{b}{a} \right)^a, \quad s \in [b(a/b)^{a-1}, b].$$

It remains to verify the result for $s \leq b(a/b)^{a-1}$, that is, for $t^* \leq b$. It is easy to check that $\frac{\partial g_1(t^*, s)}{\partial t} < 0$ for $t < t^*$ and $\frac{\partial g_1(t^*, s)}{\partial t} \geq 0$ for $t \geq t^*$. This, together with the fact that $g_1(b, s) = 0$, implies that $g_1(t^*, s) \leq 0$, and we only have to show that

$$|g_1(t^*, s)| \leq \frac{1}{a} \frac{(a - 1)^{a-1}}{a^a} \left( \ln \frac{b}{a} \right)^a, \quad s \in [a, b(a/b)^{a-1}].$$

Indeed, by Lemmas 3.4 and 3.5, we obtain

$$|g_1(t^*, s)| = \left| \frac{1}{s} \ln \frac{t^*}{s} \left( \ln \frac{b}{s} \right)^{a-1} - (a - 1) \ln \frac{b}{s} \left( \ln \frac{t^*}{s} \right)^{a-1} \right|$$

$$= (2 - a)(a - 1)^{a-1} \left( \ln \frac{b}{a} \right)^{a-1} \frac{1}{s} \left( \ln \frac{b}{a} \right)^a - \frac{1}{s} \ln \frac{b}{s} \left( \ln \frac{b}{s} \right)^{a-1}.$$
\[
\leq (2-\alpha)(\alpha - 1) \frac{1}{a} \left( \frac{b}{a} \right)^{\alpha} \\
\leq \frac{1}{a} \frac{(\alpha - 1)^{\alpha - 1}}{\alpha} \left( \frac{b}{a} \right)^{\alpha}.
\]

The proof is completed. \(\Box\)

Now we are ready to prove our Lyapunov-type inequality.

**Theorem 3.7** If a nontrivial continuous solution of the Caputo–Hadamard fractional boundary value problem

\[
(\frac{1}{a} D^\alpha_a u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b, 1 < \alpha < 2,
\]

\[
u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),
\]

exists, where \(\beta_i \geq 0 \quad (i = 1, 2, \ldots, m - 2), \quad a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b, \quad 0 \leq \sum_{i=1}^{m-2} \beta_i < 1, \quad \text{and} \quad q \text{ is a real continuous function on } [a, b], \text{then}

\[
\int_a^b |q(s)| ds \geq \frac{\Gamma(\alpha) a^{\alpha}}{[(\alpha - 1)(\ln b - \ln a)]^{\alpha-1}} \cdot \frac{\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}}{\ln \frac{b}{a} + \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}}
\]

(3.1)

**Proof** Let \(B = C[a, b]\) be the Banach space endowed with norm \(\|u\| = \sup_{t \in [a, b]} |u(t)|\). It follows from Lemma 3.1 that a solution \(u\) to the boundary value problem satisfies the integral equation

\[
u(t) = \int_a^b G(t, s)q(s)u(s) \, ds + \frac{\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}}{\ln \frac{b}{a} + \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}} \int_a^b \sum_{i=1}^{m-2} \beta_i G(\xi_i, s) q(s) u(s) \, ds.
\]

Now an application of Lemma 3.6 yields

\[
\|u\| \leq \frac{1}{a} \frac{(\alpha - 1)^{\alpha - 1}}{\alpha \Gamma(\alpha)} \left( \frac{b}{a} \right)^{\alpha - 1} \left( 1 + \frac{\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}}{\ln \frac{b}{a} + \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}} \right) \int_a^b |q(s)| \, ds \|u\|
\]

\[
= \frac{1}{a} \frac{(\alpha - 1)^{\alpha - 1}}{\alpha \Gamma(\alpha)} \left( \frac{b}{a} \right)^{\alpha - 1} \frac{\ln \frac{b}{a} + \sum_{i=1}^{m-2} \beta_i \ln \frac{b}{a} \ln \frac{\xi_i}{a}}{\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}} \int_a^b |q(s)| \, ds \|u\|,
\]

which implies that (3.1) holds. \(\Box\)

Letting \(\beta_i = 0 \quad (i = 1, 2, \ldots, m - 2)\) in Theorem 3.7, we have the following result.

**Corollary 3.8** If a nontrivial continuous solution of the Caputo–Hadamard fractional boundary value problem

\[
(\frac{1}{a} D^\alpha_a u)(t) + q(t)u(t) = 0, \quad 0 < a < t < b, 1 < \alpha < 2,
\]

\[
u(a) = 0, \quad u(b) = 0,
\]

\[
\frac{\Gamma(\alpha) a^{\alpha}}{(\alpha - 1)(\ln b - \ln a)]^{\alpha-1}} \cdot \frac{\ln b - \ln a}{\ln b + \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}}
\]

holds.
exists, where q is a real continuous function in \([a, b]\), then

\[
\int_{\alpha}^{b} |q(s)| \, ds \geq \frac{\Gamma(\alpha) \alpha^{\alpha}}{[(\alpha - 1)(\ln b - \ln a)]^{\alpha - 1}}.
\]  

\[\text{(3.2)}\]

4 Remarks

Applying the Green’s approach, we can also obtain Lyapunov-type inequalities for Caputo–Hadamard fractional differential equations under integral boundary conditions,

\[
\left( {^{\alpha}D_{a}^{\alpha}u(t)} \right) + q(t)u(t) = 0, \quad 0 < a < t < b, 1 < \alpha < 2,
\]

\[\text{(4.1)}\]

\[
u(a) = 0, \quad u(b) = \lambda \int_{\alpha}^{b} h(s)u(s) \, ds, \quad \lambda \geq 0.
\]

\[\text{(4.2)}\]

where \(h : [a, b] \rightarrow [0, \infty)\) with \(h \in L^{1}(a, b)\).

**Lemma 4.1** A function \(u \in C[a, b]\) is a solution to the boundary value problem (4.1)–(4.2) if and only if it satisfies the integral equation

\[
u(t) = \int_{\alpha}^{b} G(t, s)q(s)u(s) \, ds + \frac{\lambda \ln \frac{t}{a}}{\ln \frac{b}{a} - \lambda \sigma} \int_{\alpha}^{b} \left( \int_{\alpha}^{b} G(t, s)h(t) \, ds \right) q(s)u(s) \, ds,
\]

where \(h : [a, b] \rightarrow [0, \infty)\) with \(h \in L^{1}(a, b)\), \(\sigma = \int_{\alpha}^{b} h(t) \ln \frac{t}{a} \, dt, 0 \leq \lambda \sigma < \ln \frac{b}{a}\), and \(G(t, s)\) is defined in Lemma 3.1.

**Proof** By Lemma 2.7 \(u \in C[a, b]\) is a solution to the boundary value problem (4.1)–(4.2) if and only if

\[
u(t) = c_{0} + c_{1} \left( \ln \frac{t}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} \left( \ln \frac{t}{s} \right)^{\alpha - 1} q(s)u(s) \, ds,
\]

where \(c_{0}\) and \(c_{1}\) are real constants. Since \(u(a) = 0\), we immediately get that \(c_{0} = 0\), and thus

\[
u(t) = c_{1} \left( \ln \frac{t}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} \left( \ln \frac{t}{s} \right)^{\alpha - 1} q(s)u(s) \, ds.
\]

The boundary condition \(u(b) = \lambda \int_{\alpha}^{b} h(s)u(s) \, ds\) yields

\[
c_{1} \left( \ln \frac{b}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{b} \left( \ln \frac{b}{s} \right)^{\alpha - 1} q(s)u(s) \, ds = \lambda \int_{\alpha}^{b} h(t)u(t) \, dt,
\]

so,

\[
c_{1} = \frac{1}{(\ln \frac{b}{a}) \Gamma(\alpha)} \int_{\alpha}^{b} \left( \ln \frac{b}{s} \right)^{\alpha - 1} q(s)u(s) \, ds + \frac{\lambda}{\ln \frac{b}{a}} \int_{\alpha}^{b} h(t)u(t) \, dt.
\]
and therefore the solution of the boundary value problem (4.1)–(4.2) is

\[ u(t) = c_1 \left( \frac{\ln t}{a} \right) - \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{\ln s}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \]

\[ = \frac{(\ln \frac{t}{a})}{\ln \frac{b}{a} \Gamma(\alpha)} \int_a^b \left( \frac{\ln \frac{s}{b}}{s} \right)^{\alpha-1} q(s)u(s)\frac{ds}{s} + \frac{\lambda (\ln \frac{t}{a})}{\ln \frac{b}{a}} \int_a^b h(t)u(t) dt \]

\[ - \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{\ln \frac{t}{s}}{s} \right)^{\alpha-1} q(s)u(s)\frac{ds}{s} \]

\[ = \int_a^b G(t,s)q(s)u(s)\frac{ds}{s} + \frac{\lambda \ln \frac{t}{a}}{\ln \frac{b}{a}} \int_a^b h(t)u(t) dt. \]

Multiplying both sides of this equality by \( h(t) \) and integrating from \( a \) to \( b \), we obtain

\[ \int_a^b h(t)u(t) dt = \int_a^b \left( \int_a^b G(t,s)q(s)u(s)\frac{ds}{s} \right) h(t) dt + \frac{\lambda \sigma}{\ln \frac{b}{a}} \int_a^b h(t)u(t) dt \]

and

\[ \int_a^b h(t)u(t) dt = \frac{\ln \frac{b}{a}}{\ln \frac{b}{a} - \lambda \sigma} \int_a^b \left( \int_a^b G(t,s)q(s)u(s)\frac{ds}{s} \right) h(t) dt, \]

and thus

\[ u(t) = \int_a^b G(t,s)q(s)u(s)\frac{ds}{s} + \frac{\lambda \ln \frac{t}{a}}{\ln \frac{b}{a} - \lambda \sigma} \int_a^b \left( \int_a^b G(t,s)q(s)u(s)\frac{ds}{s} \right) h(t) dt, \]

which concludes the proof.

\[ \square \]

**Theorem 4.2** If a nontrivial continuous solution of the Caputo–Hadamard fractional boundary value problem

\[ (\frac{1}{2}D^\alpha_{a+}u)(t) + g(t)u(t) = 0, \quad 0 < a < t < b, 1 < \alpha < 2, \]

\[ u(a) = 0, \quad u(b) = \lambda \int_a^b h(s)u(s) ds, \quad \lambda \geq 0, \]

exists, where \( q : [a, b] \to \mathbb{R} \) is a continuous function, \( h : [a, b] \to [0, \infty) \) with \( h \in L^1(a, b) \), \( \sigma = \int_a^b h(t) ln \frac{t}{a} dt \), and \( 0 \leq \lambda \sigma < ln \frac{b}{a} \), then we have

\[ \int_a^b |q(s)| ds \geq \frac{a [ln \frac{b}{a} - \lambda \sigma]}{ln \frac{b}{a} + \lambda [(ln \frac{b}{a}) \int_a^b h(t) dt - \sigma]} \cdot \left( \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(ln b - ln a)]^{\alpha-1}} \right). \]

\[ (4.3) \]

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