PARADIGM OF NONASSOCIATIVE
HOM-ALGEBRAS AND HOM-SUPERALGEBRAS

ABDENACER MAKLHOUF

Abstract

The aim of this paper is to give a survey of nonassociative Hom-algebra and Hom-superalgebra structures. The main feature of these algebras is that the identities defining the structures are twisted by homomorphisms. We discuss Hom-associative algebras, Hom-Flexible algebras, Hom-Lie algebras, G-hom-associative algebras, Hom-Poisson algebras, Hom-alternative algebras and Hom-Jordan algebras and $\mathbb{Z}_2$-graded versions. We give an overview of the development of Hom-algebras structures which has been intensively investigated recently.

Keywords: Hom-associative algebra, Hom-Lie algebra, Hom-Lie superalgebra, Hom-Poisson algebra, Hom-Lie admissible algebra, Hom-Lie admissible superalgebra, Hom-alternative algebra, Hom-Jordan algebra.

1 Introduction

We mean by Hom-algebra or Hom-superalgebra a triple consisting of a vector space, respectively $\mathbb{Z}_2$-graded vector space, a multiplication and a homomorphism. The main feature of these algebras is that the identities defining the structures are twisted by homomorphisms. Such algebras appeared in the ninetieth in examples of $q$-deformations of the Witt and the Virasoro algebras.

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics. The first motivation to study nonassociative Hom-algebras comes from quasi-deformations of Lie algebras of vector fields, in particular $q$-deformations of Witt and Virasoro algebras [2, 7, 8, 11, 13, 25, 30, 24]. The deformed algebras arising in connection with $\sigma$-derivation are no longer Lie algebras. It was observed in the pioneering works that in these examples a twisted Jacobi identity holds. Motivated by these examples and their generalization, Hartwig, Larsson and Silvestrov introduced and studied in [22, 27, 28, 29] the classes of quasi-Lie, quasi-Hom-Lie and Hom-Lie algebras. In the class of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a homomorphism and contains three terms as in
Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map. They showed that Hom-Lie algebras are closely related to discrete and deformed vector fields and differential calculus and that some q-deformations of the Witt and the Virasoro algebras have the structure of Hom-Lie algebra.

The Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by Makhlouf and Silvestrov in [32], where it is shown that the commutator bracket of a Hom-associative algebra gives rise to a Hom-Lie algebra and where a classification of Hom-Lie admissible algebras is established. Given a Hom-Lie algebra, a universal enveloping Hom-associative algebra was constructed by Yau in [44]. See also [15, 17, 18, 19, 34, 45] for other works on Hom-associative algebras. In a similar way Yau proposed in [44] a notion of Hom-dialgebra which gives rise to Hom-Leibniz algebra. While Hom-associative superalgebras and Hom-Lie superalgebras were studied in [4].

The Hom-alternative algebras and Hom-Jordan algebras which are twisted version of the ordinary alternative algebras and Jordan algebras were introduced by the author in [31]. Their properties are discussed and construction procedures using ordinary alternative algebras or Jordan algebras are provided. Also, it is shown that a plus Hom-algebra of a Hom-associative algebra leads to Hom-Jordan algebra.

Beyond the binary algebras, a generalization of n-ary algebras to Hom situation was studied in [5], n-ary Hom-algebras of Lie type and associative type are discussed. Dualization of Hom-associative algebras leads to Hom-coassociative coalgebras. The Hom-coassociative coalgebras, Hom-bialgebras and Hom-Hopf algebras were introduced in [33, 35]. A twisted version of Yang-Baxter equation, quasi-triangular bialgebras and quantum groups were discussed in [47, 48, 49, 50, 51]. A study from the monoidal category point of view is given in [15].

The main purpose of this paper is to summarize nonassociative Hom-algebra structures extending the ordinary nonassociative algebras to Hom-algebra setting. We set the definitions and properties, and provide some examples. In Section 2, we fix the notations and some general setting of Hom-algebras. The third Section concerns Hom-associative algebras, Hom-associative superalgebras, Hom-flexible algebras and Hom-dialgebras. Section 4 deals with Hom-Lie algebras, Hom-Leibniz, Hom-Poisson algebras and Hom-Lie superalgebras. It is shown in particular that there is an adjoint pair of functors between the category of Hom-associative algebras (resp. Hom-dialgebras) and the category of Hom-Lie algebras (resp. Hom-Leibniz algebras). On the other hand the supercommutator of a Hom-associative superalgebra determines a Hom-Lie superalgebra. The Hom-Poisson algebra structure which emerges naturally from deformation theory of Hom-associative algebras (see [34]) is also given. We also present the definition of quasi-Lie algebras which offer to treat within the same framework the well known generalizations of Lie algebras, that is Lie superalgebras and color Lie algebras as well as their corresponding versions in Hom-superalgebra setting. Section 5 is dedicated to Hom-Lie admissible algebras and Hom-Lie admissible superalgebras. Their classifications lead to G-Hom-associative algebras and G-Hom-associative superalgebras, which include in particular Hom-algebra versions of left-symmetric algebras (Vinberg algebras).
or their opposite algebras, that is right symmetric algebras (pre-Lie algebras). In
Section 6, we discuss Hom-alternative algebras and study their properties. In par-
ticular, we show that the twisted version of the associator is an alternating function
of its arguments. We show that an ordinary alternative algebra and one of its al-
gebra endomorphisms lead to a Hom-alternative algebra where the twisting map
is actually the algebra endomorphism. This process was introduced in \[45\] for Lie
and associative algebras and more generally \(G\)-associative algebras (see \[32\] for this
class of algebras) and generalized to coalgebras in \[35\], \[33\] and to \(n\)-ary algebras
of Lie and associative types in \[5\]. We derive examples of Hom-alternative alge-
bras from 4-dimensional alternative algebras which are not associative and from
algebra of octonions. In the last Section we introduce a notion of Hom-Jordan
algebras and show that it fits with the Hom-associative structure, that is the plus
Hom-associative algebra leads to Hom-Jordan algebra.

2 Preliminaries

Throughout this paper \(\mathbb{K}\) is a field of characteristic 0, \(V\) is a \(\mathbb{K}\)-linear space
or, when talking about superalgebras, \(V\) is a superspace. Let \(V\) be a superspace
over \(\mathbb{K}\) that is a \(\mathbb{Z}_2\)-graded \(\mathbb{K}\)-linear space with a direct sum \(V = V_0 \oplus V_1\). The
element of \(V_j\), \(j = \{0,1\}\), are said to be homogenous and of parity \(j\). The parity
of a homogeneous element \(x\) is denoted by \(|x|\). One may consider that \(\mathbb{K}\) is any
commutative ring and \(V\) a \(\mathbb{K}\)-module.

We mean by a Hom-algebra, respectively Hom-superalgebra, a triple \((V, \mu, \alpha)\)
where \(\mu : V \times V \to V\) is a \(\mathbb{K}\)-bilinear map and \(\alpha : V \to V\) is a \(\mathbb{K}\)-linear map,
respectively an even \(\mathbb{K}\)-linear map. The type of the Hom-algebra or the Hom-
superalgebra is fixed by the identity satisfied by the elements. Let \((V, \mu, \alpha)\) and
\((V', \mu', \alpha')\) be two Hom-algebras (resp. Hom-superalgebras) of the same type, a
morphism \(f : (V, \mu, \alpha) \to (V', \mu', \alpha')\) is a linear map (resp. even linear map)
\(f : V \to V'\) such that \(f \circ \mu = \mu' \circ (f \times f)\) and \(f \circ \alpha = \alpha' \circ f\). In some statements
the multiplicativity of \(\alpha\) is required, that is \(\alpha \circ \mu = \mu \circ (\alpha \times \alpha)\). We call such
Hom-algebras, multiplicative Hom-algebras.

Let \(V\) be a \(n\)-dimensional \(\mathbb{K}\)-linear space and \(\{e_1, \cdots, e_n\}\) be a basis of \(V\).
A Hom-algebra structure on \(V\) with product \(\mu\) is determined by \(n^3\) structure
constants \(C_{ij}^k\), where \(\mu(e_i, e_j) = \sum_{k=0}^{n} C_{ij}^k e_k\) and homomorphism \(\alpha\) which is given
by \(n^2\) structure constants \(a_{ij}\), where \(\alpha(e_i) = \sum_{j=0}^{n} a_{ij} e_j\). If we require that the
Hom-algebra is of a given type then this limits the set of structure constants
\((C_{ij}^k, a_{ij})\) to an algebraic variety imbedded in \(\mathbb{K}^{n^3+n^2}\). The polynomials defining
the algebraic variety being derived from the identities. A point in such an algebraic
variety represents an \(n\)-dimensional Hom-algebra, along with a particular choice
of basis. A change of basis may give rise to a different point of the algebraic
variety. The group \(GL(n, \mathbb{K})\) acts on the algebraic varieties of Hom-structures by
the so-called "transport of structure" action defined as follows:
Let $A = (V, \mu, \alpha)$ be a Hom-algebra. Given $f \in GL(n, K)$, the action $f \cdot A$ transports the structures for $x, y \in V$ by

\begin{align*}
 f \cdot \mu(x, y) &= f^{-1}(f(x), f(y)) \\
 f \cdot \alpha(x) &= f^{-1}(f(x))
\end{align*}

The orbit of the Hom-algebra $A$ is given by $\vartheta(A) = \{ A' = f \cdot A, \ f \in GL_n(K) \}$. The orbits are in 1-1-correspondence with the isomorphism classes of $n$-dimensional algebras.

The stabilizer subgroup of $A$ ($\text{stab}(A) = \{ f \in GL_n(K) : A = f \cdot A \}$) is exactly $\text{Aut}(A)$, the automorphism group of $A$.

A subalgebra of a Hom-algebra $(V, \mu, \alpha)$ is a triple $(W, \mu, \alpha)$ where $W$ is a subspace of $V$ closed under $\mu$ and $\alpha$. A subspace $I$ of $V$ is a two-sided ideal if $\mu(I, V) \subset I$ and $\mu(V, I) \subset I$.

To any Hom-algebra $A = (V, \mu, \alpha)$, we associate a minus Hom-algebra $A^- = (V, \mu^-, \alpha)$ where $\mu^-(x, y) = \frac{1}{2} (\mu(x, y) - \mu(y, x))$ and a plus Hom-algebra $A^+ = (V, \mu^+, \alpha)$ where $\mu^+(x, y) = \frac{1}{2} (\mu(x, y) + \mu(y, x))$.

We call Hom-associator associated to a Hom-algebra $(V, \mu, \alpha)$ the trilinear map $\text{as}_\alpha$ defined for any $x, y, z \in V$ by

\begin{equation}
\text{as}_\alpha(x, y, z) = \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)).
\end{equation}

3 Hom-associative algebras, Hom-Flexible algebras, Hom-dialgebras

3.1 Hom-associative algebras and superalgebras

The notion of Hom-associative algebra was introduced in [32], while the Hom-associative algebras were discussed slightly in [4].

**Definition 3.1** ([32]). A Hom-associative algebra is a triple $(V, \mu, \alpha)$ consisting of a vector space $V$, a bilinear map $\mu : V \times V \to V$ and a linear map $\alpha : V \to V$ satisfying for all $x, y, z \in V$

\begin{equation}
\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).
\end{equation}

In terms of Hom-associator [3], the identity (4) writes $\text{as}_\alpha(x, y, z) = 0$.

**Example 3.2.** Let $\{e_1, e_2, e_3\}$ be a basis of a 3-dimensional linear space $V$ over $K$. The following multiplication $\mu$ and linear map $\alpha$ on $V$ define Hom-associative algebras over $K^3$:

\begin{align*}
\mu(e_1, e_1) &= a e_1, & \mu(e_2, e_2) &= a e_2, \\
\mu(e_1, e_2) &= \mu(e_2, e_1) = a e_2, & \mu(e_2, e_3) &= b e_3, \\
\mu(e_1, e_3) &= \mu(e_3, x_1) = b e_3, & \mu(e_3, e_2) &= \mu(e_3, e_3) = 0, \\
\alpha(e_1) &= a e_1, & \alpha(e_2) &= a e_2, & \alpha(e_3) &= b e_3,
\end{align*}
where \(a, b\) are parameters in \(K\). The algebras are not associative when \(a \neq b\) and \(b \neq 0\), since

\[
\mu(\mu(e_1, e_1), e_3) - \mu(e_1, \mu(e_1, e_3)) = (a - b)b e_3.
\]

**Example 3.3** (Polynomial Hom-associative algebra \([45]\)). Consider the polynomial algebra \(A = K[x_1, \ldots, x_n]\) in \(n\) variables. Let \(\alpha\) be an algebra endomorphism of \(A\) which is uniquely determined by the \(n\) polynomials \(\alpha(x_i) = \sum \lambda_i r_1 \cdots r_n x_1^{r_1} \cdots x_n^{r_n}\) for \(1 \leq i \leq n\). Define \(\mu\) by

\[
(5) \quad \mu(f, g) = f(\alpha(x_1), \ldots, \alpha(x_n))g(\alpha(x_1), \ldots, \alpha(x_n))
\]

for \(f, g\) in \(A\). Then, \((A, \mu, \alpha)\) is a Hom-associative algebra.

**Example 3.4** ([46]). Let \(A = (V, \mu, \alpha)\) be a Hom-associative algebra. Then \((M_n(A), \mu', \alpha')\), where \(M_n(A)\) is the vector space of \(n \times n\) matrix with entries in \(V\), is also a Hom-associative algebra in which the multiplication \(\mu'\) is given by matrix multiplication and \(\mu\) and \(\alpha'\) is given by \(\alpha\) in each entry.

We define now Hom-associative superalgebras.

**Definition 3.5** ([3]). A Hom-associative superalgebra is a triple \((V, \mu, \alpha)\) consisting of a superspace \(V\), an even bilinear map \(\mu: V \times V \to V\) and an even homomorphism \(\alpha: V \to V\) satisfying for all \(x, y, z \in V\)

\[
(6) \quad \mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))
\]

**Remark 3.6.** Some properties of Hom-associative algebras and unital Hom-associative algebras were discussed in \([4, 5, 17, 18, 19, 34, 45]\). A study from the point of view of monoidal category was provided in \([15]\) and where a new kind of unital Hom-associative algebras is introduced.

### 3.2 Hom-Flexible algebras

The flexible algebras was initiated by Albert \([1]\) and investigated by number of authors Myung, Okubo, Laufer, Tomber and Santilli, see for example \([38]\). The notion of Hom-flexible algebra was introduced in \([32]\). We summarize in the following the definition and some characterizations.

**Definition 3.7** ([32]). A Hom-algebra \(A = (V, \mu, \alpha)\) is called flexible if and only if for any \(x, y \in V\)

\[
(7) \quad \mu(\alpha(x), \mu(y, x)) = \mu(\mu(x, y), \alpha(x))
\]

The condition (7) may be written using the Hom-associator by \(\alpha_s(x, y, x) = 0\).

We recover the classical dialgebra when \(\alpha\) is the identity map.

**Lemma 3.8.** Let \(A = (V, \mu, \alpha)\) be a Hom-algebra. The following assertions are equivalent

1. \(A\) is flexible.
2. For any \(x, y \in V\), \(\alpha_s(x, y, x) = 0\).
3. For any \(x, y, z \in V\), \(\alpha_s(x, y, z) = -\alpha_s(z, y, x)\)
Proof. The first equivalence follows from the definition. To prove the last equivalence, one writes \( \alpha(x - z, y, x - z) = 0 \) which is equivalent, using the linearity, to 
\( \alpha(x, y, z) + \alpha(z, y, x) = 0 \).

\[ \square \]

Corollary 3.9. Any Hom-associative algebra is flexible.

Let \( A = (V, \mu, \alpha) \) be a Hom-algebra. Let \( A^- = (V, \mu^-, \alpha) \) (resp. \( A^+ = (V, \mu^+, \alpha) \)) be the plus Hom-algebra (resp. minus Hom-algebra) with multiplication defined for \( x, y \in V \) by \( \mu^-(x, y) = \frac{1}{2}(\mu(x, y) - \mu(y, x)) \) (resp. \( \mu^+(x, y) = \frac{1}{2}(\mu(x, y) + \mu(y, x)) \)). We have the following characterization of Hom-flexible algebras.

Proposition 3.10 (3). A Hom-algebra \( A = (V, \mu, \alpha) \) is Hom-flexible if and only if

\[ (8) \quad \mu^-(\alpha(x), \mu^+(y, z)) = \mu^+(\mu^-(x, y), \alpha(z)) + \mu^+(\alpha(y), \mu^-(x, z)) \]

Proof. Let \( A \) be a Hom-flexible algebra. Then by lemma 3.8 it is equivalent to 
\( \alpha(x, y, z) + \alpha(z, y, x) = 0 \), for any \( x, y, z \) in \( V \), where \( \alpha \) is the Hom-associator associated to \( A \). This implies

\[ (9) \quad \alpha(x, y, z) + \alpha(z, y, x) + \alpha(x, z, y) + \alpha(y, z, x) - \alpha(x, x, z) - \alpha(y, x, z) = 0 \]

By expansion, the previous relation is equivalent to \( 8 \).

Conversely, assume that we have the condition \( 8 \), by setting \( x = z \) in the equation \( 9 \), one gets \( \alpha(x, y, x) = 0 \), Therefore \( A \) is a Hom-flexible algebra. \( \square \)

3.3 Hom-dialgebras

The Hom-dialgebra structure introduced by Yau extends to Hom-algebra setting the classical dialgebra structure introduced by Loday.

Definition 3.11 (44). A Hom-dialgebra is a tuple \( (V, \lhd, \rhd, \alpha) \), where \( \lhd, \rhd : V \times V \to V \) are bilinear maps and \( \alpha : V \to V \) is a linear map such that the following five identities are satisfied for \( x, y, z \in V \)

\[ (10) \quad \alpha(x) \lhd (y \rhd z) = (x \lhd y) \lhd \alpha(z) = \alpha(x) \lhd (y \rhd z) \]
\[ (11) \quad \alpha(x) \rhd (y \lhd z) = (x \rhd y) \rhd \alpha(z) = \alpha(x) \rhd (y \lhd z) \]
\[ (12) \quad \alpha(x) \rhd (y \lhd z) = (x \rhd y) \lhd \alpha(z) \]

We recover the classical dialgebra when \( \alpha \) is the identity map. A morphism of Hom-dialgebras is a linear map that is compatible with \( \alpha \) and the two multiplications \( \rhd \) and \( \lhd \).

Note that \((V, \lhd, \alpha)\) and \((V, \rhd, \alpha)\) are Hom-associative algebras. In the classical case, Loday showed that the commutator defined for \( x, y \in V \) by \([x, y] = x \lhd y - y \rhd x\) defines a Leibniz algebra on \( V \). This result is extended to Hom-algebra setting in the next Section.\( \clubsuit \)
4 Hom-Leibniz algebras and Hom-Lie algebras

4.1 Hom-Leibniz algebras

A class of quasi Leibniz algebras was introduced in [28] in connection to general quasi-Lie algebras following the standard Loday’s conventions for Leibniz algebras (i.e. right Loday algebras).

Definition 4.1 ([28]). A Hom-Leibniz algebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a linear space \(V\), bilinear map \([\cdot, \cdot] : V \times V \to V\) and a linear map \(\alpha : V \to V\) satisfying

\[
[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]]
\]

In terms of the (right) adjoint homomorphisms \(\text{Ad}_y : V \to V\) defined by \(\text{Ad}_y(x) = [x, y]\), the identity (13) can be written as

\[
\text{Ad}_{\alpha(z)}([x, y]) = [\text{Ad}_z(x), \alpha(y)] + [\alpha(x), \text{Ad}_z(y)]
\]

or in pure operator form

\[
\text{Ad}_{\alpha(z)} \circ \text{Ad}_y = \text{Ad}_{\alpha(y)} \circ \text{Ad}_z + \text{Ad}_{\text{Ad}_z(y)} \circ \alpha
\]

4.2 Hom-Lie algebras

The Hom-Lie algebras were initially introduced by Hartwig, Larson and Silvestrov in [22] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields.

Definition 4.2 ([22]). A Hom-Lie algebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a linear space \(V\), bilinear map \([\cdot, \cdot] : V \times V \to V\) and a linear map \(\alpha : V \to V\) satisfying

\[
[x, y] = -[y, x] \quad \text{(skewsymmetry)}
\]

\[
\sum_{x,y,z} [\alpha(x), [y, z]] = 0 \quad \text{(Hom-Jacobi identity)}
\]

for all \(x, y, z\) from \(V\), where \(\sum_{x,y,z}\) denotes summation over the cyclic permutation on \(x, y, z\).

Using the skew-symmetry, one may write the Hom-Jacobi identity in the form (14). Hence, if a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

Example 4.3. Let \(\{e_1, e_2, e_3\}\) be a basis of a 3-dimensional linear space \(V\) over \(K\). The following bracket and linear map \(\alpha\) on \(V\) define a Hom-Lie algebra over \(K^3\):

\[
\begin{align*}
[e_1, e_2] &= ae_1 + be_3 & \alpha(e_1) &= e_1 \\
[e_1, e_3] &= ce_2 & \alpha(e_2) &= 2e_2 \\
[e_2, e_3] &= de_1 + 2ae_3, & \alpha(e_3) &= 2e_3
\end{align*}
\]
with \([e_2, e_1], [e_3, e_1]\) and \([e_3, e_2]\) defined via skewsymmetry. It is not a Lie algebra if and only if \(a \neq 0\) and \(c \neq 0\), since

\[[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = ace_2.\]

**Example 4.4 (Jackson \(\mathfrak{sl}_2(\mathbb{K})\)).** In this example, we will consider the Hom-Lie algebra Jackson \(\mathfrak{sl}_2(\mathbb{K})\) which is a Hom-Lie deformation of the classical Lie algebra \(\mathfrak{sl}_2(\mathbb{K})\) defined by \([e_1, e_2] = 2e_2\), \([e_1, e_3] = -2e_3\), \([e_2, e_3] = e_1\). This family of Hom-Lie algebras was constructed in [29] using a quasi-deformation scheme based on discretizing by means of Jackson \(q\)-derivations a representation of \(\mathfrak{sl}_2(\mathbb{K})\) by one-dimensional vector fields (first order ordinary differential operators) and using the twisted commutator bracket defined in [22]. The Hom-Lie algebra Jackson \(\mathfrak{sl}_2(\mathbb{K})\) is a 3-dimensional vector space over \(\mathbb{K}\) with the vector space bases \([e_1, e_2, e_3]\) and with the bilinear bracket multiplication defined on the basis by

\[[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3 - 2te_3, \quad [e_2, e_3] = e_1 + \frac{t}{2} e_1,\]

and by skew-symmetry for \([e_2, e_1], [e_3, e_1], [e_3, e_2]\). The linear map \(\alpha_t\) is defined by

\[\alpha_t(e_1) = e_1, \quad \alpha_t(e_2) = \frac{2 + t}{2(1 + t)} e_2 = e_2 + \frac{t}{2} e_2, \quad \alpha_t(e_3) = e_3 + \frac{t}{2} e_3.\]

Thus Jackson \(\mathfrak{sl}_2(\mathbb{K})\) algebra is a Hom-Lie algebra deformation of \(\mathfrak{sl}_2(\mathbb{K})\). See [34] for other examples of Hom-Lie algebra deformation of \(\mathfrak{sl}_2(\mathbb{K})\).

### 4.3 Hom-Lie and Hom-Leibniz Functors

We provide in the following a different way for constructing Hom-Lie algebras by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket.

**Theorem 4.5 ([32]).** Let \((V, \mu, \alpha)\) be a Hom-associative algebra. The Hom-algebra \((V, [\cdot, \cdot], \alpha)\), where the bracket is defined for all \(x, y \in V\) by

\[[x, y] = \mu(x, y) - \mu(y, x)\]

is a Hom-Lie algebra.

**Proof.** The bracket is obviously skewsymmetric and with a direct computation we have

\[
\begin{align*}
\alpha([x, y], z) &= \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(z), \mu(y, x)) + \mu(\alpha(x), \mu(y, z) - \mu(\alpha(z), \mu(y, x)), \\
\alpha(x, [y, z]) &= \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(z, x)) + \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(z, x)) \\
\alpha(y, [x, z]) &= \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(z, x)) + \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(z, x)) \\
\end{align*}
\]

\([\alpha(x), [y, z]] - [[x, y], \alpha(z)] - [\alpha(y), [x, z]] = 0\)

\(\square\)

A similar construction is obtained for Hom-dialgebra.
Theorem 4.6 (44). Let \((V, \cdot, \vdash, \alpha)\) be a Hom-dialgebra \((V, \mu, \alpha)\). The Hom-algebra \((V, [\cdot, \cdot], \alpha)\), where the bracket is defined for all \(x, y \in V\) by
\[ [x, y] = x \cdot y - y \cdot x \]
is a Hom-Leibniz algebra.

Therefore, we have a functor \(HLie\) (resp. \(HLeib\)) from the category of Hom-associative algebras \(\text{HomAs}\) (resp. category of Hom-dialgebras \(\text{HomDi}\)) to the category of Hom-Lie algebras \(\text{HomLie}\) (resp. category of Hom-Leibniz algebras \(\text{HomLeib}\)). Conversely, an enveloping Hom-associative algebra \(U_{HLie}(g)\) (resp. enveloping Hom-dialgebra \(U_{HLeib}(L)\)) of a Hom-Lie algebra \(g\) (resp. Hom-Leibniz algebra \(L\)) are constructed in [44]. Hence, \(U_{HLie}\) is the left adjoint functor of \(HLie\) and \(U_{HLeib}\) is the left adjoint functor of \(HLeib\).

4.4 Hom-Poisson algebras

We introduce in the following the notion of Hom-Poisson structure which emerges naturally in deformation theory of Hom-associative algebras, see [34].

Definition 4.7 (34). A Hom-Poisson algebra is a quadruple \((V, \mu, \{\cdot, \cdot\}, \alpha)\) consisting of a vector space \(V\), bilinear maps \(\mu : V \times V \rightarrow V\) and \(\{\cdot, \cdot\} : V \times V \rightarrow V\), and a linear map \(\alpha : V \rightarrow V\) satisfying

1. \((V, \mu, \alpha)\) is a commutative Hom-associative algebra,
2. \((V, \{\cdot, \cdot\}, \alpha)\) is a Hom-Lie algebra,
3. for all \(x, y, z \in V\),
\[ \{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\alpha(z), \{x, y\}) \]  

Condition (18) expresses the compatibility between the multiplication and the Poisson bracket. It can be reformulated equivalently, for all \(x, y, z \in V\), as
\[ \{\mu(x, y), \alpha(z)\} = \mu(\{x, z\}, \alpha(y)) + \mu(\alpha(x), \{y, z\}) \]

Note that in this form it means that \(\text{ad}_z(\cdot) = \{\cdot, z\}\) is a sort of generalization of a derivation of an associative algebra, and also it resembles the identity (13) in the definition of Leibniz algebra. We recover the classical Leibniz identity when \(\alpha\) is the identity map.

Example 4.8. Let \(\{e_1, e_2, e_3\}\) be a basis of a 3-dimensional vector space \(V\) over \(K\). The following multiplication \(\mu\), skew-symmetric bracket and linear map \(\alpha\) on \(V\) define a Hom-Poisson algebra over \(K^3\):
\[
\begin{align*}
\mu(e_1, e_1) &= e_1, & \{e_1, e_2\} &= ae_2 + be_3, \\
\mu(e_1, e_2) &= \mu(e_2, e_1) = e_3, & \{e_1, e_3\} &= ce_2 + de_3,
\end{align*}
\]
\[
\alpha(e_1) = \lambda_1 e_2 + \lambda_2 e_3, & \quad \alpha(e_2) = \lambda_3 e_2 + \lambda_4 e_3, & \alpha(e_3) = \lambda_5 e_2 + \lambda_6 e_3
\]
where \(a, b, c, d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\) are parameters in \(K\).
4.5 Hom-Lie Superalgebras

Hom-Lie Superalgebras is a subclass of quasi-Lie algebras introduced in [28]. They were studied in [4], where construction procedures are provided.

Definition 4.9. A Hom-Lie superalgebra is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a superspace $V$, an even bilinear map $[\cdot, \cdot] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying

\begin{align*}
(20) & \quad [x, y] = -(-1)^{|x||y|}[y, x] \\
(21) & \quad \alpha([x, y]) = [\alpha(x), [\alpha(y), z]] + [\alpha(y), [\alpha(x), z]] + [\alpha(z), [\alpha(x), y]] + [\alpha(z), [\alpha(y), x]] = 0
\end{align*}

for all homogeneous element $x, y, z$ in $V$.

The identity (21) is called Hom-superJacobi identity, while the identity (20) expresses the usual supersymmetry of the bracket.

Remark 4.10. We recover the classical Lie superalgebra when $\alpha = \text{id}$. The Hom-Lie algebras are obtained when the part of parity one is trivial.

Example 4.11 (2-dimensional abelian Hom-Lie superalgebra). Every bilinear map $\mu$ on a 2-dimensional linear superspace $V = V_0 \oplus V_1$, where $V_0$ is generated by $x$ and $V_1$ is generated by $y$ and such that $[x, y] = 0$ defines a Hom-Lie superalgebra for any homomorphism $\alpha$ of superalgebra. Indeed, the graded Hom-Jacobi identity is satisfied for any triple $(x, y, z)$.

Example 4.12 (Affine Hom-Lie superalgebra). Let $V = V_0 \oplus V_1$ be a 3-dimensional superspace where $V_0$ is generated by $\{e_1, e_2\}$ and $V_1$ is generated by $e_3$. The triple $(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie superalgebra defined by $[e_1, e_2] = e_1, [e_1, e_3] = [e_2, e_3] = [e_3, e_3] = 0$ and $\alpha$ is any homomorphism.

Example 4.13 (A q-deformed Witt superalgebra, [4]). We provide an example of infinite dimensional Hom-Lie superalgebra which is given by a realization of the q-deformed Witt superalgebra constructed in [4]. It corresponds to a superspace $\mathcal{V}$ generated by the elements $\{X_n\}_{n \in \mathbb{N}}$ of parity 0 and the elements $\{G_n\}_{n \in \mathbb{N}}$ of parity 1.

Let $q \in \mathbb{C}\backslash\{0, 1\}$ and $n \in \mathbb{N}$, we set $\{n\} = \frac{1-q^n}{1-q}$, a q-number. The q-numbers have the following properties $\{n+1\} = 1 + q\{n\} = \{n\} + q^n$ and $\{n+m\} = \{n\} + q^n\{m\}$.

Let $[\cdot, \cdot]_\sigma$ be a bracket on the superspace $\mathcal{V}$ defined by

\begin{align*}
[X_n, X_m]_\sigma &= (\{m\} - \{n\})X_{n+m} \\
[X_n, G_m]_\sigma &= (q^n\{m+1\} - q^{n+1}\{n\})G_{n+m}
\end{align*}

The others brackets are obtained by supersymmetry or are 0.

Let $\alpha$ be an even linear map on $\mathcal{V}$ defined on the generators by

\begin{align*}
\alpha(X_n) &= (1 + q^n)X_n \\
\alpha(G_n) &= (1 + q^{n+1})G_n
\end{align*}

Then the triple $(\mathcal{V}, [\cdot, \cdot]_\sigma, \alpha)$ is a Hom-Lie superalgebra.
In the following, we show that the supercommutator bracket defined using the multiplication in a Hom-associative superalgebra leads naturally to Hom-Lie superalgebra.

**Theorem 4.14 (H).** Let \((V, \mu, \alpha)\) be a Hom-associative superalgebra. The Hom-superalgebra \((V, [\cdot, \cdot], \alpha)\), where the bracket (super bracket) is defined for all homogeneous elements \(x, y \in V\) by
\[
[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(y, x)
\]
and extended by linearity to all elements, is a Hom-Lie superalgebra.

**Proof.** The bracket is obviously supersymmetric and with a direct computation we have
\[
(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = \\
(-1)^{|x||z|}\mu(x, \mu(y, z)) - (-1)^{|x||z|}\mu(\alpha(x), \mu(z, y)) \\
-(-1)^{|z||y|}\mu(y, \alpha(z)) + (-1)^{|z||y|}\mu(\mu(z, y), \alpha(x)) \\
+(-1)^{|y||x|}\mu(\alpha(y), \mu(z, x)) - (-1)^{|y||x|}\mu(\mu(y, x), \alpha(z)) \\
-(-1)^{|z||x|}\mu(\mu(z, x), \alpha(y)) + (-1)^{|z||x|}\mu(\mu(x, z), \alpha(y)) \\
+(-1)^{|x||y|}\mu(\alpha(x), \mu(y, x)) - (-1)^{|x||y|}\mu(\mu(x, y), \alpha(z)) \\
-(-1)^{|z||x|}\mu(\mu(x, y), \alpha(z)) + (-1)^{|z||x|}|x||y|\mu(\mu(y, x), \alpha(z)) = 0
\]
\(\square\)

The following theorem gives a way to construct Hom-Lie superalgebras, starting from a regular Lie superalgebra and an even superalgebra endomorphism.

**Theorem 4.15 (H).** Let \((V, [\cdot, \cdot])\) be a Lie superalgebra and \(\alpha : V \to V\) be an even Lie superalgebra endomorphism. Then \((V, [\cdot, \cdot], \alpha)\), where \([x, y]_\alpha = \alpha([x, y])\), is a Hom-Lie superalgebra.

Moreover, suppose that \((V', [\cdot, \cdot]')\) is another Lie superalgebra and \(\alpha' : V' \to V'\) is a Lie superalgebra endomorphism. If \(f : V \to V'\) is a Lie superalgebra morphism that satisfies \(f \circ \alpha = \alpha' \circ f\) then
\[
f : (V, [\cdot, \cdot], \alpha) \longrightarrow (V', [\cdot, \cdot]', \alpha')
\]
is a morphism of Hom-Lie superalgebras.

**Example 4.16 (H).** We construct an example of Hom-Lie superalgebra, which is not a Lie superalgebra starting from the orthosymplectic Lie superalgebra. We consider in the sequel the matrix realization of this Lie superalgebra.

Let \(osp(1, 2) = V_0 \oplus V_1\) be the Lie superalgebra where \(V_0\) is generated by:
\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
and \(V_1\) is generated by:
\[
F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
The defining relations (we give only the ones with non-zero values in the right hand side) are

\[
\begin{align*}
[H, X] &= 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H, \\
[Y, G] &= F, \quad [X, F] = G, \quad [H, F] = -F, \quad [H, G] = G, \\
[G, F] &= H, \quad [G, G] = -2X, \quad [F, F] = 2Y.
\end{align*}
\]

Let \( \lambda \in \mathbb{R}^* \), we consider the linear map \( \alpha_\lambda : \mathfrak{osp}(1, 2) \to \mathfrak{osp}(1, 2) \) defined by:

\[
\begin{align*}
\alpha_\lambda(X) &= \lambda^2 X, \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y, \quad \alpha_\lambda(H) = H, \quad \alpha_\lambda(F) = \frac{1}{\lambda} F, \quad \alpha_\lambda(G) = \lambda G.
\end{align*}
\]

We provide a family of Hom-Lie superalgebras \( \mathfrak{osp}(1, 2)_\lambda = (\mathfrak{osp}(1, 2), [\cdot, \cdot]_{\alpha_\lambda}, \alpha_\lambda) \) where the Hom-Lie superalgebra bracket \([\cdot, \cdot]_{\alpha_\lambda}\) on the basis elements is given, for \( \lambda \neq 0 \), by:

\[
\begin{align*}
[H, X]_{\alpha_\lambda} &= 2\lambda^2 X, \quad [H, Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2} Y, \quad [X, Y]_{\alpha_\lambda} = H, \\
[Y, G]_{\alpha_\lambda} &= \frac{1}{\lambda} F, \quad [X, F]_{\alpha_\lambda} = \lambda G, \quad [H, F]_{\alpha_\lambda} = -\frac{1}{\lambda} F, \quad [H, G]_{\alpha_\lambda} = \lambda G, \\
[G, F]_{\alpha_\lambda} &= H, \quad [G, G]_{\alpha_\lambda} = -2\lambda^2 X, \quad [F, F]_{\alpha_\lambda} = \frac{2}{\lambda^2} Y.
\end{align*}
\]

These Hom-Lie superalgebras are not Lie superalgebras for \( \lambda \neq 1 \).

Indeed, the left hand side of the super-Jacobi identity (21), for \( \alpha = \text{id} \), leads to

\[
\begin{align*}
[X, [Y, H]] - [H, [X, Y]] + [Y, [H, X]] &= \frac{2(1 - \lambda^4)}{\lambda^2} Y, \\
\end{align*}
\]

and also

\[
\begin{align*}
[H, [F, F]] - [F, [H, F]] + [F, [F, H]] &= \frac{4(\lambda - 1)}{\lambda^4} Y.
\end{align*}
\]

Then, they do not vanish for \( \lambda \neq 1 \).

### 4.6 Quasi-Lie algebras

The class of quasi-Lie algebras where introduced by Larsson and Silvestrov in order to treat within the same framework such a well-known generalizations of Lie algebras as color and Lie superalgebras, as well as Hom-Lie algebras (see [27, 28]).

Let \( \mathcal{L}_\mathbb{K}(V) \) be the set of linear maps of the linear space \( L \) over the field \( \mathbb{K} \).

**Definition 4.17** ([28]). A quasi-Lie algebra is a tuple

\( (V, [\cdot, \cdot], \alpha, \beta, \omega, \theta) \) consisting of

- \( V \) is a linear space over \( \mathbb{K} \),
- \( [\cdot, \cdot] : V \times V \to V \) is a bilinear map called a product or bracket in \( V \);
- \( \alpha, \beta : V \to V \), are linear maps,
- \( \omega : D_\omega \to \mathcal{L}_\mathbb{K}(V) \) and \( \theta : D_\theta \to \mathcal{L}_\mathbb{K}(V) \) are maps with domains of definition \( D_\omega, D_\theta \subseteq V \times V \),

such that the following conditions hold:
• (ω-symmetry) The product satisfies a generalized skew-symmetry condition
\[ [x, y] = \omega(x, y)[y, x], \quad \text{for all } (x, y) \in D_\omega; \]

• (quasi-Jacobi identity) The bracket satisfies a generalized Jacobi identity
\[ \theta(x, y, z) \{ \theta(z, x)([\alpha(x), [y, z]] + \beta[x, [y, z]]) \} = 0, \]
for all \((z, x), (x, y), (y, z) \in D_\theta.\)

Note that \((\omega(x, y)\omega(y, x) - \text{id})[x, y] = 0\), if \((x, y), (y, x) \in D_\omega\), which follows from the computation \([x, y] = \omega(x, y)[y, x] = \omega(x, y)\omega(y, x)[x, y] \).

The class of Quasi-Lie algebras incorporates as special cases hom-Lie algebras
and more general quasi-hom-Lie algebras (qhl-algebras) which appear naturally in
the algebraic study of σ-derivations (see [22]) and related deformations of infinite-
dimensional and finite-dimensional Lie algebras. To get the class of qhl-algebras
one specifies \(\theta = \omega\) and restricts attention to maps \(\alpha\) and \(\beta\) satisfying the twisting
condition \([\alpha(x), \alpha(y)] = \beta \circ \alpha[x, y] \). Specifying this further by taking \(D_\omega = V \times V, \beta = \text{id}\) and \(\omega = -\text{id}\), one gets the class of Hom-Lie algebras including Lie algebras
when \(\alpha = \text{id}\). The class of quasi-Lie algebras contains also color Lie algebras and
in particular Lie superalgebras.

5 Hom-Lie admissible algebras and G-Hom-associative algebras

The Lie-admissible algebras was introduced by A. A. Albert in 1948. Physicists attempted to introduce this structure instead of Lie algebras. For instance, the validity of Lie-Admissible algebras for free particles is well known. These algebras arise also in classical quantum mechanics as a generalization of conventional mechanics (see [1, 38]).

5.1 Hom-Lie admissible algebras

In this section, we discuss the concept of Hom-Lie-Admissible algebra, extending to Hom-algebra setting, the classical notion of Lie-admissible algebra.

Definition 5.1 ([32]). Let \(A = (V, \mu, \alpha)\) be a Hom-algebra structure. Then \(A\) is said to be Hom-Lie-admissible algebra if the bracket defined for all \(x, y \in V\) by
\[ [x, y] = \mu(x, y) - \mu(y, x) \]
satisfies the Hom-Jacobi identity ([17]).

Remark 5.2. Since the bracket is also skewsymmetric then it defines a Hom-Lie algebra.

Remark 5.3. Note that any Hom-associative and Hom-Lie algebras are Hom-Lie-admissible.
We aim now to give another characterization of Hom-Lie admissible algebras. Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-algebra. We denote by brackets the multiplication of plus Hom-algebra $\mathcal{A}^+$, that is $[x, y] = \mu(x, y) - \mu(y, x)$ and by $S$ the trilinear map defined for $x, y, z \in V$ by

$$S(x, y, z) := \alpha_{\sigma}(x, y, z) + \alpha_{\sigma}(y, z, x) + \alpha_{\sigma}(z, x, y)$$

We have the following properties

**Lemma 5.4.**

$$S(x, y, z) = [\mu(x, y), \alpha(z)] + [\mu(y, z), \alpha(x)] + [\mu(z, x), \alpha(y)]$$

**Proof.**

$$[\mu(x, y), \alpha(z)] + [\mu(y, z), \alpha(x)] + [\mu(z, x), \alpha(y)] =$$

$$\mu(\mu(x, y), \alpha(z)) - \mu(\alpha(z), \mu(x, y)) + \mu(\mu(y, z), \alpha(x)) - \mu(\alpha(x), \mu(y, z)) +$$

$$\mu(\mu(z, x), \alpha(y)) - \mu(\alpha(y), \mu(z, x)) = S(x, y, z)$$

□

**Proposition 5.5.** A Hom-algebra $\mathcal{A}$ is Hom-Lie-admissible if and only if it satisfies for any $x, y, z$ in $V$

$$S(x, y, z) = S(x, z, y)$$

**Proof.**

$$S(x, y, z) - S(x, z, y) =$$

$$[\mu(x, y), \alpha(z)] + [\mu(y, z), \alpha(x)] + [\mu(z, x), \alpha(y)] -$$

$$[\mu(x, z), \alpha(y)] - [\mu(z, y), \alpha(x)] - [\mu(y, x), \alpha(z)] =$$

$$\bigcirc_{x, y, z} [\alpha(x), [y, z]]$$

□

5.2 $G$-Hom-associative algebras

In the following, we explore some other Hom-Lie-Admissible algebras, extending to Hom-algebra setting the results obtained in [21].

**Definition 5.6.** Let $G$ be a subgroup of the permutations group $S_3$, a Hom-algebra $(V, \mu, \alpha)$ is called $G$-Hom-associative if for any $x, y, z \in V$, we have

$$(22) \quad \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \mu(\mu(x_{\sigma(1)}, x_{\sigma(2)}), \alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}), \mu(x_{\sigma(2)}, x_{\sigma(3)})) = 0$$

where $x_i$ are in $V$ and $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation $\sigma$.

The condition (22) may be written in terms of Hom-associator by

$$(23) \quad \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_{\sigma} \circ \sigma = 0$$

where $\sigma$ is the extension of the permutation, still denoted by the same notation, to a trilinear map defined by $\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. 

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Remark 5.7. If $\mu$ is the multiplication of a Hom-Lie-admissible Lie algebra then the condition \eqref{cond_hom_ass} is equivalent to the property that the bracket defined by

$$[x, y] = \mu(x, y) - \mu(y, x)$$

satisfies the Hom-Jacobi condition or equivalently to

\begin{equation}
\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} \mu(x_{\sigma(1)}, x_{\sigma(2)}), \alpha(x_{\sigma(3)}) - \mu(\alpha(x_{\sigma(1)}), \mu(x_{\sigma(2)}, x_{\sigma(3)})) = 0
\end{equation}

which may be written as

\begin{equation}
\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} a_{\sigma} \circ \sigma = 0.
\end{equation}

Theorem 5.8 \cite{[32]}. Let $G$ be a subgroup of the permutations group $S_3$. Then any $G$-Hom-associative algebra is a Hom-Lie-admissible algebra.

Proof. The skewsymmetry follows straightaway from the definition.

We have a subgroup $G$ in $S_3$. Take the set of conjugacy class $\{gG\}_{g \in I}$ where $I \subseteq G$, and for any $\sigma_1, \sigma_2 \in I, \sigma_1 \neq \sigma_2 \Rightarrow \sigma_1 G \cap \sigma_2 G = \emptyset$. Then

$$\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} a_{\sigma} \circ \sigma = \sum_{\sigma_1 \in I} \sum_{\sigma_2 \in G} (-1)^{\varepsilon(\sigma_2)} a_{\sigma_1} \circ \sigma_2 = 0.$$

The result says that for any subgroup of $S_3$ corresponds a class of $G$-Hom-associative algebra. Since the subgroups are $G_1 = \{Id\}, G_2 = \{Id, \sigma_1\}, G_3 = \{Id, \sigma_{23}\}, G_4 = \{Id, \sigma_3\}, G_5 = A_3, G_6 = S_3$, where $A_3$ is the alternating group and where $\sigma_{ij}$ is the transposition between $i$ and $j$, then we obtain the following type of Hom-Lie-admissible algebras.

- The $G_1$-Hom-associative algebras are the Hom-associative algebras defined above.
- The $G_2$-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(x, z)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(y, x), \alpha(z))$$

When $\alpha$ is the identity the algebra is called Vinberg algebra or left symmetric algebra.
- The $G_3$-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(x), \mu(z, y)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y))$$

When $\alpha$ is the identity the algebra is called pre-Lie algebra or right symmetric algebra.
- The $G_4$-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(z), \mu(y, x)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(z, y), \alpha(x))$$

- The $G_5$-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x), \mu(y, z)) + \mu(\alpha(y), \mu(z, x) + \mu(\alpha(z), \mu(x, y)) = \mu(\mu(x, y), \alpha(z)) + \mu(\mu(y, z), \alpha(x)) + \mu(\mu(z, x), \alpha(y))$$
If the product $\mu$ is skew-symmetric then the previous condition is exactly
the Hom-Jacobi identity.

- The $G_6$-Hom-associative algebras are the Hom-Lie-admissible algebras.

Special cases of $G$-Hom-associative algebras include generalization of Vinberg and pre-Lie algebras.

**Definition 5.9** ([32]). A Hom-Vinberg algebra (Hom-left-symmetric algebra) is a
triple $(V, \mu, \alpha)$ consisting of a linear space $V$, a bilinear map $\mu : V \times V \to V$ and
a homomorphism $\alpha$ satisfying

\[(26) \quad \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(x, z)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(y, x), \alpha(z))\]

**Definition 5.10** ([32]). A Hom-pre-Lie algebra (Hom-right-symmetric algebra) is
a triple $(V, \mu, \alpha)$ consisting of a linear space $V$, a bilinear map $\mu : V \times V \to V$ and
a homomorphism $\alpha$ satisfying

\[(27) \quad \mu(\alpha(x), \mu(y, z)) - \mu(\alpha(x), \mu(z, y)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y))\]

**Remark 5.11.** A Hom-pre-Lie algebra is the opposite algebra of a Hom-Vinberg
algebra.

**Remark 5.12.** The multiplicative Hom-Novikov algebras which are multiplicativ\[e Hom-Vinberg algebra with the additional identity $\mu(\mu(x, y), \alpha(z)) = \mu(\mu(x, z), \alpha(y))$, were studied in [52].

The following theorem states that $G$-associative algebras deform into $G$-Hom-associative algebras along any algebra endomorphism. Therefore, it provides a construction procedure.

**Theorem 5.13** ([45]). Let $(V, \mu)$ be a $G$-associative algebra and $\alpha : V \to V$ be an
algebra endomorphism. Then $(V, \mu', \alpha)$, where $\mu' = \alpha \circ \mu$, is a $G$-Hom-associative
algebra.

Moreover, suppose that $(V', \mu')$ is another $G$-associative algebra and $\alpha' : V' \to V'$ is an algebra endomorphism. If $f : V \to V'$ is an algebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then

\[f : (V, \mu, \alpha) \longrightarrow (V', \mu', \alpha')\]

is a morphism of $G$-Hom-associative algebras.

### 5.3 Hom-Lie-Admissible Superalgebras

We discuss in this section the concept of Hom-Lie-Admissible superalgebras studied in [4]. This study borders also an extension to graded case of the Lie-admissible algebras discussed in [21].

Let $A = (V, \mu, \alpha)$ be a Hom-superalgebra, that is a superspace $V$ with an
even bilinear map $\mu$ and an even linear map $\alpha$ satisfying eventually identities. Let

\[\langle [x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(y, x), \text{ for all homogeneous element } x, y \in V, \text{ be the associated supercommutator. The bracket is extended to all elements by linearity.} \]
**Definition 5.14** ([4]). Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-superalgebra on $V$ defined by an even multiplication $\mu$ and an even homomorphism $\alpha$. Then $\mathcal{A}$ is said to be Hom-Lie admissible superalgebra if the bracket defined for all homogeneous element $x, y \in V$ by

$$[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(y, x)$$

satisfies the Hom-superJacobi identity ([27]).

**Remark 5.15.** Since the supercommutator bracket ([28]) is always supersymmetric, this makes any Hom-Lie admissible superalgebra into a Hom-Lie superalgebra.

**Remark 5.16.** As mentioned in the proposition ([4,14]), any associative superalgebra is a Hom-Lie admissible superalgebra.

**Lemma 5.17.** Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-superalgebra and $\{\cdot, \cdot\}$ be the associated supercommutator then

$$\circ_{x,y,z} \left((-1)^{|x||z|}\alpha(x), [y, z]\right) =$$

$$( -1)^{|x||z|} \alpha_{\alpha}(x, y, z) + (-1)^{|y||z|} \alpha_{\alpha}(y, z, x)$$

$$+ ( -1)^{|z||x|} \alpha_{\alpha}(z, x, y) - ( -1)^{|x||z|+|y||z|} \alpha_{\alpha}(x, z, y)$$

$$- ( -1)^{|z||y|+|x||z|} \alpha_{\alpha}(z, x, y) - ( -1)^{|x||y|+|z||z|} \alpha_{\alpha}(y, z, x)$$

**Proof.** By straightforward calculation. □

**Remark 5.18.** If $\alpha = id$, then we obtain a formula expressing the left hand side of the classical superJacobi identity in terms of classical associator.

In the following we aim to characterize the Hom-Lie admissible superalgebras in terms of Hom-associator. We introduce a trilinear map $\tilde{S}$ defined for homogeneous elements $x, y, z \in V$ by

$$\tilde{S}(x, y, z) := (-1)^{|x||z|}\alpha_{\alpha}(x, y, z) + (-1)^{|y||x|}\alpha_{\alpha}(y, z, x) + ( -1)^{|z||y|}\alpha_{\alpha}(z, x, y).$$

Then, we have the following properties.

**Proposition 5.19** ([4]). Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-superalgebra, then $\mathcal{A}$ is a Hom-Lie admissible superalgebra if and only if it satisfies

$$\tilde{S}(x, y, z) = ( -1)^{|x||y|+|x||z|+|y||z|}\tilde{S}(x, z, y)$$

for any homogenous elements $x, y, z \in V$. 

In particular, \( \sigma \) extend a permutation \( \tau \).

**Proof.** We have

\[
S(x, y, z) - (-1)^{|x||y|+|x||z|+|y||z|} S(x, z, y)
\]

\[
= (-1)^{|x||z|} a_{x\alpha}(x, y, z) + (-1)^{|y||x|} a_{y\alpha}(y, z, x) + (-1)^{|z||y|} a_{z\alpha}(z, x, y)
\]

\[
- (-1)^{|x||z|+|y||x|+|z||y|} a_{x\alpha}(x, y, z) + (-1)^{|y||z|} a_{y\alpha}(y, z, x) + (-1)^{|z||x|} a_{z\alpha}(z, x, y)
\]

\[
+ (-1)^{|x||y|} a_{x\alpha}(y, x, z)
\]

\[
= (-1)^{|x||z|} a_{x\alpha}(x, y, z) + (-1)^{|y||x|} a_{y\alpha}(y, z, x) + (-1)^{|z||y|} a_{z\alpha}(z, x, y)
\]

\[
- (-1)^{|x||z|+|y||x|+|z||y|} a_{x\alpha}(x, y, z) + (-1)^{|y||z|} a_{y\alpha}(y, z, x) + (-1)^{|z||x|} a_{z\alpha}(z, x, y)
\]

\[
= \circ_{x,y,z} (-1)^{|x||z|} [x, [y, z]] \quad \text{(lemma 5.17)}
\]

Then the Hom-superJacobi identity \( (21) \) is satisfied if and only if the condition \( (30) \) holds.

\[\square\]

5.4 *G*-Hom-associative superalgebras

A classification of Hom-Lie admissible superalgebras using the symmetric group \( S_3 \) is provided in [4]. We extended to \( \mathbb{Z}_2 \)-graded case the notion of *G*-Hom-associative algebras and in particular *G*-associative algebras which was introduced in the classical ungraded Lie case in (21) and developed for the Hom-Lie case in (18).

Let \( S_3 \) be the permutation group generated by the transpositions \( \sigma_1, \sigma_2 \). We extend a permutation \( \tau \in S_3 \) to a map \( \tau : V^{\times 3} \to V^{\times 3} \) defined for \( x_1, x_2, x_3 \in V \) by \( \tau(x_1, x_2, x_3) = (x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) \). We keep for simplicity the same notation. In particular, \( \sigma_1(x_1, x_2, x_3) = (x_2, x_1, x_3) \) and \( \sigma_2(x_1, x_2, x_3) = (x_1, x_3, x_2) \).

We introduce a notion of a parity of transposition \( \sigma_i \) where \( i \in \{1, 2\} \), by setting

\[
|\sigma_i(x_1, x_2, x_3)| = |x_i||x_{i+1}|.
\]

We assume that the parity of the identity is 0 and for the composition \( \sigma_i \sigma_j \), it is defined by

\[
|\sigma_i \sigma_j(x_1, x_2, x_3)| = |\sigma_j(x_1, x_2, x_3)| + |\sigma_i(\sigma_j(x_1, x_2, x_3))|
\]

\[
= |\sigma_j(x_1, x_2, x_3)| + |\sigma_i(\sigma_j(x_1, x_2, x_3)| + |\sigma_i(\sigma_j(x_1, x_2, x_3))|
\]

We define by induction the parity for any composition.

For the elements \( \text{id}, \sigma_1, \sigma_2, \sigma_1 \sigma_2, \sigma_2 \sigma_1, \sigma_2 \sigma_1 \sigma_2 \) of \( S_3 \), we obtain

\[
|\text{id}(x_1, x_2, x_3)| = 0,
\]

\[
|\sigma_1(x_1, x_2, x_3)| = |x_1||x_2|,
\]

\[
|\sigma_2(x_1, x_2, x_3)| = |x_2||x_3|,
\]

\[
|\sigma_1 \sigma_2(x_1, x_2, x_3)| = |x_2||x_3| + |x_1||x_3|,
\]

\[
|\sigma_2 \sigma_1(x_1, x_2, x_3)| = |x_1||x_2| + |x_1||x_3|,
\]

\[
|\sigma_2 \sigma_1 \sigma_2(x_1, x_2, x_3)| = |x_2||x_3| + |x_1||x_3| + |x_1||x_2|.
\]
Now, we express the condition of Hom-Lie admissibility of a Hom-superalgebra using permutations.

**Lemma 5.20** (4). A Hom-superalgebra $A = (V, \mu, \alpha)$ is a Hom-Lie admissible superalgebra if the following condition holds

$$
\sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)} (-1)^{|\tau(x_1, x_2, x_3)|} \mathfrak{d}_{\alpha} \circ \tau(x_1, x_2, x_3) = 0
$$

where $x_i$ are in $V$, $(-1)^{\varepsilon(\tau)}$ is the signature of the permutation $\tau$ and $|\tau(x_1, x_2, x_3)|$ is the parity of $\tau$.

**Proof.** By straightforward calculation, the associated supercommutator bracket satisfies

$$\llangle x_1, x_2, x_3 \mid (-1)^{|x_1||x_3|}[\alpha(x_1), [x_2, x_3]] =
(-1)^{|x_1||x_3|} \sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)} (-1)^{|\tau(x_1, x_2, x_3)|} \mathfrak{d}_{\alpha} \circ \tau(x_1, x_2, x_3).$$

$\square$

It turns out that, for the associated supercommutator of a Hom-superalgebra, the Hom-superJacobi identity (21) is equivalent to

$$
\sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)} (-1)^{|\tau(x_1, x_2, x_3)|} \mathfrak{d}_{\alpha} \circ \tau(x_1, x_2, x_3) = 0.
$$

We introduce now the notion of $G$-Hom-associative superalgebra, where $G$ is a subgroup of the permutations group $S_3$.

**Definition 5.21** (4). Let $G$ be a subgroup of the permutations group $S_3$, a Hom-superalgebra on $V$ defined by the multiplication $\mu$ and a homomorphism $\alpha$ is said to be a $G$-Hom-associative superalgebra if

$$
\sum_{\tau \in G} (-1)^{\varepsilon(\tau)} (-1)^{|\tau(x_1, x_2, x_3)|} \mathfrak{d}_{\alpha} \circ \tau(x_1, x_2, x_3) = 0.
$$

where $x_i$ are in $V$, $(-1)^{\varepsilon(\tau)}$ is the signature of the permutation and $|\tau(x_1, x_2, x_3)|$ is the parity of $\tau$ defined above.

In particular, we call $G$-associative superalgebra a $G$-Hom-associative superalgebra where $\alpha$ is the identity map.

The following result is a graded version of the results obtained in (21 32).

**Theorem 5.22** (4). Let $G$ be a subgroup of the permutations group $S_3$. Then any $G$-Hom-associative superalgebra is a Hom-Lie admissible superalgebra.

**Proof.** The supersymmetry follows straightaway from the definition.

We have a subgroup $G$ in $S_3$. Take the set of conjugacy class $\{gG\}_{g \in I}$ where $I \subseteq G$, and for any $\tau_1, \tau_2 \in I, \sigma_1 \neq \tau_2 \Rightarrow \tau_1 G \cap \tau_2 G = \emptyset$. Then
\[
\sum_{\tau \in S_3} (-1)^{\epsilon(\tau)(-1)^{I(x_1,x_2,x_3)}} a_{\alpha} \circ \tau(x_1,x_2,x_3) = \\
\sum_{\tau_1 \in I} \sum_{\tau_2 \in \tau_1 G} (-1)^{\epsilon(\tau_2)(-1)^{I(x_1,x_2,x_3)}} a_{\alpha} \circ \tau_2(x_1,x_2,x_3) = 0
\]

where \((x_1, x_2, x_3) \in V\), with \(V\) the underlying superspace of the \(G\)-Hom-associative superalgebra.

It follows that in particular, we have:

**Corollary 5.23.** Let \(G\) be a subgroup of the permutations group \(S_3\). Then any \(G\)-associative superalgebra is a Lie admissible superalgebra.

Now, we provide a classification of the Hom-Lie admissible superalgebras through \(G\)-Hom-associative superalgebras.

The subgroups of \(S_3\), which are \(G_1 = \{Id\}\), \(G_2 = \{Id, \sigma_1\}\), \(G_3 = \{Id, \sigma_2\}\), \(G_4 = \{Id, \sigma_2\sigma_1\sigma_2\}\), \(G_5 = A_3\), \(G_6 = S_3\), where \(A_3\) is the alternating group, lead to the following six classes of Hom-Lie admissible superalgebras.

- The \(G_1\)-Hom-associative superalgebras are the Hom-associative superalgebras defined above.
- The \(G_2\)-Hom-associative superalgebras satisfy the identity
  \[
  \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = \\
  (-1)^{|x||y|}(\mu(\alpha(y), \mu(x, z)) - \mu(y, x), \alpha(z))
  \]
- The \(G_3\)-Hom-associative superalgebras satisfy the identity
  \[
  \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = \\
  (-1)^{|y||z|}(\mu(\alpha(x), \mu(z, y)) - \mu(x, z), \alpha(y))
  \]
- The \(G_4\)-Hom-associative superalgebras satisfy the identity
  \[
  \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = \\
  (-1)^{|x||y|+|z||y|+|z||z|}(\mu(\alpha(z), \mu(y, x)) - \mu(y, \alpha(x)))
  \]
- The \(G_5\)-Hom-associative superalgebras satisfy the identity
  \[
  -(1)^{|x||z|+|y||z|}(\mu(\alpha(z), \mu(y, x)) + \mu(\mu(z, x), \alpha(y))) = \\
  \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) \\
  +(-1)^{|y||z|+|z||z|}(\mu(\alpha(y), \mu(z, x) + \mu(y, \alpha(z)))
  \]
- The \(G_6\)-Hom-associative superalgebras are the Hom-Lie admissible superalgebras.

**Remark 5.24.** Moreover, if in the previous identities we consider \(\alpha = id\), then we obtain a classification of Lie-admissible superalgebras.
Remark 5.25. A $G_2$-Hom-associative (resp. $G_2$-associative) superalgebras might be called Hom-Vinberg superalgebras (resp. Vinberg superalgebras) or Hom-left-symmetric superalgebras (resp. left-symmetric superalgebras).

Similarly, a $G_3$-Hom-associative (resp. $G_3$-associative) superalgebras might be called Hom-pre-Lie superalgebras (resp. pre-Lie superalgebras) or Hom-right-symmetric superalgebras (resp. right-symmetric superalgebras).

Notice that a Hom-pre-Lie superalgebra is the opposite algebra of a Hom-Vinberg superalgebra. Therefore, they actually form a same class.

The following result generalizes the theorem 4.15 to any $G$-Hom-associative superalgebra.

Theorem 5.26. Let $(V, [\cdot, \cdot])$ be a $G$-associative superalgebra and $\alpha : V \to V$ be an even $G$-Hom-associative superalgebra endomorphism. Then $(V, [\cdot, \cdot], \alpha)$, where $[x, y]_\alpha = \alpha([x, y])$, is a $G$-Hom-associative superalgebra.

Moreover, suppose that $(V', [\cdot, \cdot])$ is another $G$-associative superalgebra and $\alpha' : V' \to V'$ is a $G$-associative superalgebra endomorphism. If $f : V \to V'$ is a $G$-associative superalgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then

$$f : (V, [\cdot, \cdot], \alpha) \to (V', [\cdot, \cdot]', \alpha')$$

is a morphism of $G$-Hom-associative superalgebras.

Proof. Similar to proof of theorem 4.15. □

6 Hom-alternative algebras

Now, we introduce and discuss the notion of Hom-alternative algebra, following [31].

Definition 6.1 ([31]). A left Hom-alternative algebra (resp. right Hom-alternative algebra) is a triple $(V, \mu, \alpha)$ consisting of a $\mathbb{K}$-linear space $V$, a linear map $\alpha : V \to V$ and a multiplication $\mu : V \otimes V \to V$ satisfying, for any $x, y$ in $V$, the left Hom-alternative identity, that is

$$\mu(\alpha(x), \mu(x, y)) = \mu(\mu(x, x), \alpha(y)),$$

respectively, right Hom-alternative identity, that is

$$\mu(\alpha(x), \mu(y, y)) = \mu(\mu(x, y), \alpha(y)).$$

A Hom-alternative algebra is one which is both left and right Hom-alternative algebra.

Remark 6.2. Any Hom associative algebra is a Hom-alternative algebra.

Using the Hom-associator [31], the condition (33) (resp. (34)) may be written using Hom-associator respectively

$$\alpha_\alpha(x, x, y) = 0, \quad \alpha_\alpha(y, x, x) = 0.$$
By linearization, we have the following equivalent definition of left and right Hom-alternative algebras.

**Proposition 6.3.** A triple \((V, \mu, \alpha)\) is a left Hom-alternative algebra (resp. right alternative algebra) if and only if the identity

\[ \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(y), \mu(x, z)) - \mu(\mu(y, x), \alpha(z)) = 0. \]

(respectively,
\[ \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(x), \mu(z, y)) - \mu(\mu(y, x), \alpha(z)) = 0. \]

holds.

**Proof.** We assume that, for any \(x, y, z \in V\), \(\text{as}_\alpha(x, x, z) = 0\) (left alternativity), then we expand \(\text{as}_\alpha(x + y, x + y, z) = 0\).

The proof for right Hom-alternativity is obtained by expanding \(\text{as}_\alpha(x, y + z, y + z) = 0\).

Conversely, we set \(x = y\) in (35), respectively \(y = z\) in (36).

\[ \text{as}_\alpha(\alpha(x, y, z) = \frac{1}{2} \left( \text{as}_\alpha(x, y, z) - \text{as}_\alpha(y, z, x) \right). \]

Hence, for any \(x, y, z \in V\), we have

\[ \text{as}_\alpha(x, y, z) = -\text{as}_\alpha(y, x, z) \quad \text{and} \quad \text{as}_\alpha(x, y, z) = -\text{as}_\alpha(x, z, y). \]

We have also the following property.

**Lemma 6.5.** Let \((V, \mu, \alpha)\) be an Hom-alternative algebra. Then

\[ \text{as}_\alpha(x, y, z) = -\text{as}_\alpha(z, y, x). \]

**Proof.** Using (39), we have

\[ \text{as}_\alpha(x, y, z) + \text{as}_\alpha(z, y, x) = -\text{as}_\alpha(y, x, z) - \text{as}_\alpha(y, z, x) = 0. \]

\[ \text{as}_\alpha(x, y, z) = -\text{as}_\alpha(y, x, z) = -\text{as}_\alpha(x, z, y) = -\text{as}_\alpha(z, y, x). \]

**Remark 6.4.** The multiplication could be considered as a linear map \(\mu : V \otimes V \rightarrow V\), then the condition (35) (resp. (36)) writes

\[ \mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) \circ (\text{id} \otimes_3 + \sigma_1) = 0, \]

respectively
\[ \mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) \circ (\text{id} \otimes_3 + \sigma_2) = 0. \]

where \(\text{id}\) stands for the identity map and \(\sigma_1\) and \(\sigma_2\) stands for trilinear maps defined for any \(x_1, x_2, x_3 \in V\) by

\[ \sigma_1(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_1 \otimes x_3, \quad \sigma_2(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_3 \otimes x_2. \]

In terms of associators, the identities (39) (resp. (40)) are equivalent respectively to

\[ \text{as}_\alpha + \text{as}_\alpha \circ \sigma_1 = 0 \quad \text{and} \quad \text{as}_\alpha + \text{as}_\alpha \circ \sigma_2 = 0. \]

**Remark 6.6.** The identities (39), (40) lead to the fact that an algebra is Hom-alternative if and only if the Hom-associator \(\text{as}_\alpha(x, y, z)\) is an alternating function of its arguments, that is

\[ \text{as}_\alpha(x, y, z) = -\text{as}_\alpha(y, x, z) = -\text{as}_\alpha(x, z, y) = -\text{as}_\alpha(z, y, x). \]
Proposition 6.7. A Hom-alternative algebra is Hom-flexible.

Proof. Using lemma 6.5 we have \( \alpha_\text{as}_\alpha(x, y, x) = -\alpha_\text{as}_\alpha(x, y, x) \).

Therefore, \( \alpha_\text{as}_\alpha(x, y, x) = 0 \). \( \square \)

Proposition 6.8. Let \((V, \mu, \alpha)\) be a Hom-alternative algebra and \(x, y, z \in V\).

If \(x\) and \(y\) anticommute, that is \(\mu(x, y) = -\mu(y, x)\), then we have

\[
\mu(\alpha(x), \mu(y, z)) = -\mu(\alpha(y), \mu(x, z)),
\]

and

\[
\mu(\mu(z, x), \alpha(y)) = -\mu(\mu(z, y), \alpha(x)).
\]

Proof. The left alternativity leads to

\[
\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(y), \mu(x, z)) - \mu(\mu(y, x), \alpha(z)) = 0.
\]

Since \(\mu(x, y) = -\mu(y, x)\), then the previous identity becomes

\[
\mu(\alpha(x), \mu(y, z)) + \mu(\alpha(y), \mu(x, z)) = 0.
\]

Similarly, using the right alternativity and the assumption of anticommutativity, we get the second identity. \( \square \)

The following theorem provides a way to construct Hom-alternative algebras starting from an alternative algebra and an algebra endomorphism. This procedure was applied to associative algebras, \(G\)-associative algebras and Lie algebra in [45]. It was extended to coalgebras in [35] and to \(n\)-ary algebras of Lie type respectively associative type in [31].

Theorem 6.9 ([31]). Let \((V, \mu)\) be a left alternative algebra (resp. a right alternative algebra) and \(\alpha : V \rightarrow V\) be an algebra endomorphism. Then \((V, \mu_\alpha, \alpha)\), where \(\mu_\alpha = \alpha \circ \mu\), is a left Hom-alternative algebra (resp. right Hom-alternative algebra).

Moreover, suppose that \((V', \mu')\) is another left alternative algebra (resp. a right alternative algebra) and \(\alpha' : V' \rightarrow V'\) is an algebra endomorphism. If \(f : V \rightarrow V'\) is an algebras morphism that satisfies \(f \circ \alpha = \alpha' \circ f\) then

\[
f : (V, \mu_\alpha, \alpha) \rightarrow (V', \mu'_\alpha, \alpha')
\]

is a morphism of left Hom-alternative algebras (resp. right Hom-alternative algebras).

Remark 6.10. Let \((V, \mu, \alpha)\) be a Hom-alternative algebra, one may ask whether this Hom-alternative algebra is induced by an ordinary alternative algebra \((V, \tilde{\mu})\), that is \(\alpha\) is an algebra endomorphism with respect to \(\tilde{\mu}\) and \(\mu = \alpha \circ \tilde{\mu}\). This question was addressed and discussed for Hom-associative algebras in [18, 19].

First observation, if \(\alpha\) is an algebra endomorphism with respect to \(\tilde{\mu}\) then \(\alpha\) is also an algebra endomorphism with respect to \(\mu\). Indeed,

\[
\mu(\alpha(x), \alpha(y)) = \alpha \circ \tilde{\mu}(\alpha(x), \alpha(y)) = \alpha \circ \alpha \circ \tilde{\mu}(x, y) = \alpha \circ \mu(x, y).
\]

Second observation, if \(\alpha\) is bijective then \(\alpha^{-1}\) is also an algebra automorphism. Therefore one may use an untwist operation on the Hom-alternative algebra in order to recover the alternative algebra \((\tilde{\mu} = \alpha^{-1} \circ \mu)\).
6.1 Examples of Hom-Alternative algebras

We construct examples of Hom-alternative using theorem [6.9]. We use to this end the classification of 4-dimensional alternative algebras which are not associative (see [20]) and the algebra of octonions (see [14]). For each algebra, algebra endomorphisms are provided. Therefore, Hom-alternative algebras are attached according to theorem [6.9].

Example 6.11 (Hom-alternative algebras of dimension 4). According to theorem (6.9), the linear map \mu: \mathbb{K}^4 \rightarrow \mathbb{K}^4, \mu(e_0, e_0) = e_0, \mu(e_0, e_1) = e_1, \mu(e_0, e_2) = e_2, \mu(e_0, e_3) = e_3, \mu(e_1, e_0) = e_1, \mu(e_1, e_1) = e_1, \mu(e_1, e_2) = e_2, \mu(e_1, e_3) = e_3, \mu(e_2, e_0) = e_2, \mu(e_2, e_1) = e_1, \mu(e_2, e_2) = e_2, \mu(e_2, e_3) = e_3, \mu(e_3, e_0) = e_3, \mu(e_3, e_1) = e_2, \mu(e_3, e_2) = e_1, \mu(e_3, e_3) = e_3, \mu_1(e_0, e_0) = e_0, \mu_1(e_0, e_1) = e_1, \mu_1(e_0, e_2) = e_2, \mu_1(e_0, e_3) = e_3, \mu_1(e_1, e_0) = e_1, \mu_1(e_1, e_1) = e_1, \mu_1(e_1, e_2) = e_2, \mu_1(e_1, e_3) = e_3, \mu_1(e_2, e_0) = e_2, \mu_1(e_2, e_1) = e_1, \mu_1(e_2, e_2) = e_2, \mu_1(e_2, e_3) = e_3, \mu_1(e_3, e_0) = e_3, \mu_1(e_3, e_1) = e_2, \mu_1(e_3, e_2) = e_1, \mu_1(e_3, e_3) = e_3, \mu_2(e_0, e_0) = e_0, \mu_2(e_0, e_1) = e_1, \mu_2(e_0, e_2) = e_2, \mu_2(e_0, e_3) = e_3, \mu_2(e_1, e_0) = e_1, \mu_2(e_1, e_1) = e_1, \mu_2(e_1, e_2) = e_2, \mu_2(e_1, e_3) = e_3, \mu_2(e_2, e_0) = e_2, \mu_2(e_2, e_1) = e_1, \mu_2(e_2, e_2) = e_2, \mu_2(e_2, e_3) = e_3, \mu_2(e_3, e_0) = e_3, \mu_2(e_3, e_1) = e_2, \mu_2(e_3, e_2) = e_1, \mu_2(e_3, e_3) = e_3.

These two alternative algebras are anti-isomorphic, that is the first one is isomorphic to the opposite of the second one. The algebra endomorphisms of \mu_1 and \mu_2 are exactly the same. We provide two examples of algebra endomorphisms for these algebras.

(1) The algebra endomorphism \alpha_1 with respect to the same basis is defined by
\begin{align*}
\alpha_1(e_0) &= e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \quad \alpha_1(e_1) = 0, \\
\alpha_1(e_2) &= a_4 e_2 + \frac{a_4 a_3}{a_2} e_3, \quad \alpha_1(e_3) = a_5 e_2 + \frac{a_5 a_3}{a_2} e_3,
\end{align*}
with \(a_1, \ldots, a_5 \in \mathbb{K}\) and \(a_2 \neq 0\).

(2) The algebra endomorphism \alpha_2 with respect to the same basis is defined by
\begin{align*}
\alpha_2(e_0) &= e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \quad \alpha_2(e_1) = a_4 e_1, \\
\alpha_2(e_2) &= -\frac{a_4 a_2}{a_5} e_2 - \frac{a_4 a_3}{a_5} e_3, \quad \alpha_2(e_3) = a_5 e_1 + a_6 e_2 + \frac{a_6 a_3 - a_5}{a_2} e_3,
\end{align*}
with \(a_1, \ldots, a_6 \in \mathbb{K}\) and \(a_2, a_5 \neq 0\).

According to theorem [6.9], the linear map \alpha_1 an the following multiplications
\begin{align*}
\mu_1^1(e_0, e_0) &= e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \quad \mu_1^1(e_0, e_1) = 0, \\
\mu_1^1(e_2, e_0) &= a_4 e_2 + \frac{a_4 a_3}{a_2} e_3, \quad \mu_1^1(e_2, e_3) = 0, \\
\mu_1^1(e_3, e_0) &= a_5 e_2 + \frac{a_5 a_3}{a_2} e_3, \quad \mu_1^1(e_3, e_2) = 0.
\end{align*}
\[ \mu_1(e_0, e_0) = e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \quad \mu_2(e_0, e_2) = a_4 e_2 + \frac{a_4 a_3}{a_2} e_3, \]
\[ \mu_1(e_0, e_3) = a_5 e_2 + \frac{a_5 a_3}{a_2} e_3, \quad \mu_2(e_1, e_0) = 0, \quad \mu_1(e_2, e_3) = 0, \quad \mu_2(e_3, e_2) = 0. \]

determine 4-dimensional Hom-alternative algebras.

The linear map \( \alpha_2 \) leads to the following multiplications

\[ \mu_1(e_0, e_0) = e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \quad \mu_1(e_0, e_1) = a_4 e_1, \]
\[ \mu_1(e_2, e_0) = -\frac{a_4 a_2}{a_5} e_2 - \frac{a_4 a_3}{a_5} e_3, \]
\[ \mu_1(e_2, e_3) = a_4 e_1, \quad \mu_2(e_3, e_0) = e_3, \quad \mu_2(e_3, e_2) = -a_4 e_1. \]

\[ \mu_2(e_0, e_0) = e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, \quad \mu_2(e_0, e_2) = -\frac{a_4 a_2}{a_5} e_2 - \frac{a_4 a_3}{a_5} e_3, \]
\[ \mu_2(e_0, e_3) = a_5 e_1 + a_6 e_2 + \frac{a_6 a_3 - a_5}{a_2} e_3, \]
\[ \mu_2(e_1, e_0) = a_4 e_1, \quad \mu_2(e_2, e_3) = a_4 e_1, \quad \mu_2(e_3, e_2) = -a_4 e_1. \]

**Example 6.12 (Octonions).** Octonions are typical example of alternative algebra. They were discovered in 1843 by John T. Graves who called them Octaves and independently by Arthur Cayley in 1845. See [14] for the role of the octonions in algebra, geometry and topology and see also [3] where octonions are viewed as a quasialgebra. The octonions algebra which is also called Cayley Octaves or Cayley algebra is an 8-dimensional defined with respect to a basis \( \{u, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \), where \( u \) is the identity for the multiplication, by the following multiplication table. The table describes multiplying the \( i \)th row elements by the \( j \)th column elements.

|      |  u   |  e_1 |  e_2 |  e_3 |  e_4 |  e_5 |  e_6 |  e_7 |
|------|------|------|------|------|------|------|------|------|
|  u   |  u   |  e_1 |  e_2 |  e_3 |  e_4 |  e_5 |  e_6 |  e_7 |
|  e_1 |  e_1 |  -u  |  e_4 |  e_7 | -e_2 |  e_6 | -e_5 | -e_3 |
|  e_2 |  e_2 | -e_4 |  -u  |  e_5 |  e_1 | -e_3 |  e_7 | -e_6 |
|  e_3 |  e_3 | -e_7 | -e_5 |  -u  |  e_6 |  e_2 | -e_4 |  e_1 |
|  e_4 |  e_4 | -e_1 | -e_6 | -e_7 | -u  |  e_1 |  e_3 | -e_5 |
|  e_5 |  e_5 | -e_6 |  e_3 | -e_2 | -e_7 | -u  |  e_1 |  e_4 |
|  e_6 |  e_6 | e_5  |  e_7 |  e_4 | -e_3 |  -u  |  e_2 |  e_6 |
|  e_7 |  e_7 | e_6  | e_3  |  e_6 | -e_1 |  e_5 |  e_4 | -e_2 | -u  |

The diagonal algebra endomorphisms of octonions are give by maps \( \alpha \) defined with respect to the basis \( \{u, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \) by

\[ \alpha(u) = u, \quad \alpha(e_1) = a e_1, \quad \alpha(e_2) = b e_2, \quad \alpha(e_3) = c e_3, \]
\[ \alpha(e_4) = a b e_4, \quad \alpha(e_5) = b c e_5, \quad \alpha(e_6) = a b c e_6, \quad \alpha(e_7) = a c e_7, \]

\[ a, b, c, b, a, c \]
where $a, b, c$ are any parameter in $\mathbb{K}$. The associated Hom-alternative algebra to the octonions algebra according to theorem (6.9) is described by the map $\alpha$ and the multiplication defined by the following table. The table describes multiplying the $i$th row elements by the $j$th column elements.

|   | $u$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|-----|------|------|------|------|------|------|------|
| $u$ | $u$ | $ae_1$ | $be_2$ | $ce_3$ | $abe_4$ | $bce_5$ | $abce_6$ | $ace_7$ |
| $e_1$ | $ae_1$ | $-u$ | $abe_4$ | $ace_7$ | $-be_2$ | $abce_6$ | $-bce_5$ | $-ce_3$ |
| $e_2$ | $be_2$ | $-abe_4$ | $-u$ | $bce_5$ | $ae_1$ | $-ce_3$ | $ace_7$ | $-abce_6$ |
| $e_3$ | $ce_3$ | $-ace_7$ | $-bce_5$ | $-u$ | $abce_6$ | $be_2$ | $-abe_4$ | $ae_1$ |
| $e_4$ | $abe_4$ | $be_2$ | $-ae_1$ | $-abce_6$ | $-u$ | $ace_7$ | $ce_3$ | $-bce_5$ |
| $e_5$ | $bce_5$ | $-abce_6$ | $ce_3$ | $-be_2$ | $-ace_7$ | $-u$ | $ae_1$ | $abce_6$ |
| $e_6$ | $abce_6$ | $bce_5$ | $-ace_7$ | $abe_4$ | $-ce_3$ | $-ae_1$ | $-u$ | $be_2$ |
| $e_7$ | $ace_7$ | $ce_3$ | $abce_6$ | $-ae_1$ | $bce_5$ | $-abe_4$ | $-be_2$ | $-u$ |

Notice that the new algebra is no longer unital, neither an alternative algebra since

$$
\mu(u, \mu(u, e_1)) - \mu(\mu(u, u), e_1) = (a^2 - a)e_1,
$$

which is different from 0 when $a \neq 0, 1$.


\section{Hom-Jordan algebras}

In this section, we consider, following [31], a generalization of Jordan algebra by twisting the usual Jordan identity $(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$. We show that this generalization fits with Hom-associative algebras. Also, we provide a procedure to construct examples starting from an ordinary Jordan algebras.

\textbf{Definition 7.1} ([31]). A Hom-Jordan algebra is a triple $(V, \mu, \alpha)$ consisting of a linear space $V$, a bilinear map $\mu : V \times V \to V$ which is commutative and a homomorphism $\alpha : V \to V$ satisfying

\begin{equation}
\mu(\alpha^2(x), \mu(y, \mu(x, x))) = \mu(\mu(\alpha(x), y), \alpha(\mu(x, x)))
\end{equation}

where $\alpha^2 = \alpha \circ \alpha$.

\textbf{Remark 7.2.} Since the multiplication is commutative, one may write the identity (46) as

\begin{equation}
\mu(y, \mu(x, x), \alpha^2(x)) = \mu(\mu(y, \alpha(x)), \alpha(\mu(x, x))).
\end{equation}

When the twisting map $\alpha$ is the identity map, we recover the classical notion of Jordan algebra.

The identity (46) is motivated by the following functor which associates to a Hom-associative algebra a Hom-Jordan algebra by considering the plus Hom-algebra.
Theorem 7.3 (31). Let \((V, m, \alpha)\) be a Hom-associative algebra. Then the Hom-algebra \((V, \mu, \alpha)\), where the multiplication \(\mu\) is defined for all \(x, y \in V\) by
\[
\mu(x, y) = \frac{1}{2}(m(x, y) + m(y, x)).
\]
is a Hom-Jordan algebra.

Proof. The commutativity of \(\mu\) is obvious. We compute the difference
\[
D = \mu(\alpha^2(x), \mu(y, \mu(x, x))) - \mu(\alpha(x), \mu(\alpha(x), x))
\]
A straightforward computation gives
\[
D = m(\alpha^2(x), m(y, m(x, x))) + m(m(y, m(x, x)), \alpha^2(x))
+ m(\alpha^2(x), m(m(x, x), y)) + m(m(m(x, x), y), \alpha^2(x))
- m(m(\alpha(x), y), \alpha(m(x, x))) - m(\alpha(m(x, x)), m(\alpha(x), y))
- m(m(y, \alpha(x)), \alpha(m(x, x))) - m(\alpha(m(x, x)), m(y, \alpha(x))).
\]
We have by Hom-associativity
\[
m(\alpha^2(x), m(y, m(x, x))) - m(m(\alpha(x), y), \alpha(m(x, x))) = 0
\]
\[
m(m(m(x, x), y), \alpha^2(x)) - m(\alpha(m(x, x)), m(y, \alpha(x))) = 0.
\]
Therefore
\[
D = m(m(y, m(x, x)), \alpha^2(x)) + m(\alpha^2(x), m(m(x, x), y))
- m(\alpha(m(x, x)), m(\alpha(x), y)) - m(m(y, \alpha(x)), \alpha(m(x, x))).
\]
One may show that for any Hom-associative algebra we have
\[
m(\alpha(m(x, x)), m(\alpha(x), y)) = m(m(m(x, x), \alpha(x)), \alpha(y))
= m(m(\alpha(x), m(x, x)), \alpha(y))
= m(\alpha^2(x), m(m(x, x), y)),
\]
and similarly
\[
m(m(y, \alpha(x)), \alpha(m(x, x))) = m(m(y, m(x, x)), \alpha^2(x)).
\]
Thus
\[
D = m(m(y, m(x, x)), \alpha^2(x)) + m(\alpha^2(x), m(m(x, x), y))
- m(\alpha^2(x), m(m(x, x), y)) - m(m(y, m(x, x)), \alpha^2(x))
= 0.\]

\[
\square
\]

Remark 7.4. The definition of Hom-Jordan algebra seems to be non natural one expects that the identity should be of the form
\[
\mu(\alpha(x), \mu(y, \mu(x, x))) = \mu(\mu(x, y), \alpha(\mu(x, x)))
\]
or
\[
\mu(\alpha(x), \mu(y, \mu(x, x))) = \mu(\mu(x, y), \mu(x, \alpha(x))).
\]
It turns out that these identities do not fit with the previous proposition.
Notice also that in general a Hom-alternative algebra doesn’t lead to a Hom-Jordan algebra.

The following theorem gives a procedure to construct Hom-Jordan algebras using ordinary Jordan algebras and their algebra endomorphisms.

**Theorem 7.5**. Let $(V, \mu)$ be a Jordan algebra and $\alpha : V \to V$ be an algebra endomorphism. Then $(V, \mu_{\alpha}, \alpha)$, where $\mu_{\alpha} = \alpha \circ \mu$, is a Hom-Jordan algebra.

Moreover, suppose that $(V', \mu')$ is another Jordan algebra and $\alpha' : V' \to V'$ is an algebra endomorphism. If $f : V \to V'$ is an algebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then $f : (V, \mu_{\alpha}, \alpha) \to (V', \mu'_{\alpha'}, \alpha')$ is a morphism of Hom-Jordan algebras.

**Remark 7.6**. We may give here similar observations as in the remark (6.10) concerning Hom-Jordan algebra induced by an ordinary Jordan algebra.

We provide in the sequel an example of Hom-Jordan algebras.

**Example 7.7**. We consider Hom-Jordan algebras associated to Hom-associative algebras described in example (3.2). Let $\{e_1, e_2, e_3\}$ be a basis of a 3-dimensional linear space $V$ over $\mathbb{K}$. The following multiplication $\mu$ and linear map $\alpha$ on $V$ define Hom-Jordan algebras over $\mathbb{K}$:

\[
\begin{align*}
\tilde{\mu}(e_1, e_1) &= a e_1, \\
\tilde{\mu}(e_1, e_2) &= \tilde{\mu}(e_2, e_1) = a e_2, \\
\tilde{\mu}(e_1, e_3) &= \tilde{\mu}(e_3, x_1) = b e_3, \\
\tilde{\mu}(e_2, e_2) &= a e_2, \\
\tilde{\mu}(e_2, e_3) &= \tilde{\mu}(e_3, e_2) = b e_3, \\
\tilde{\mu}(e_3, e_3) &= 0,
\end{align*}
\]

where $a, b$ are parameters in $\mathbb{K}$.

It turns out that the multiplication of this Hom-Jordan algebra defines a Jordan algebra.

**Remark 7.8**. We may define the noncommutative Jordan algebras as triples $(V, \mu, \alpha)$ satisfying the identity (46) and the flexibility condition, which is a generalization of the commutativity. Eventually, we may consider the Hom-flexibility defined by the identity $\mu(\alpha(x), \mu(y, x)) = \mu(\mu(x, y), \alpha(x))$.

**Remark 7.9**. A $\mathbb{Z}_2$-graded version of Hom-alternative algebras and Hom-Jordan algebras might be defined in a natural way.

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Laboratoire de Mathématiques, Informatique et Applications, Université de Haute-Alsace, France
E-mail address: Abdenacer.Makhlouf@uha.fr