FRACTIONAL DIFFERENTIAL EQUATIONS:
\(\alpha\)-ENTIRE SOLUTIONS,
REGULAR AND IRREGULAR SINGULARITIES

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Abstract

We consider fractional differential equations of order \(\alpha \in (0, 1)\) for functions of one independent variable \(t \in (0, \infty)\) with the Riemann-Liouville and Caputo-Dzhrbashyan fractional derivatives. A precise estimate for the order of growth of \(\alpha\)-entire solutions is given. An analog of the Frobenius method for systems with regular singularity is developed. For a model example of an equation with a kind of an irregular singularity, a series for a formal solution is shown to be convergent for \(t > 0\) (if \(\alpha\) is an irrational number poorly approximated by rational ones) but divergent in the distribution sense.

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Key Words and Phrases: fractional differential equation; Riemann-Liouville derivative; Caputo-Dzhrbashyan derivative; regular singularity; irregular singularity

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1. Introduction

Fractional differential equations are widely used for modeling anomalous relaxation and diffusion phenomena; see [3, 12] for further references. Meanwhile the mathematical theory of such equations is still in its initial stage. In particular, a systematic development of the analytic theory of fractional differential equations with variable coefficients was initiated only recently, in the paper by Kilbas, Rivero, Rodríguez-Germá, and Trujillo [11] (see also Section 7.5 in [12]). For equations of order $\alpha \in (0, 1)$, of the form

$$(D_0^\alpha u)(t) = a(t)u(t), \quad t > 0,$$

(1)

where $D_0^\alpha$ is the Riemann-Liouville fractional derivative, or

$$(D^{(\alpha)} u)(t) = a(t)u(t), \quad t > 0,$$

(2)

where $D^{(\alpha)}$ is the Caputo-Dzhrbashyan fractional derivative, their main results are as follows. If $a(t) = A(t^\alpha)$, and $A(z)$ is a real function possessing an absolutely convergent Taylor expansion on an interval $|z| < \theta$, then the equation (1) possesses a solution of the form

$$u(t) = t^{\alpha-1} \sum_{n=0}^{\infty} a_n t^{\alpha n}, \quad 0 < t < \theta,$$

while the equation (2) has a solution

$$u(t) = \sum_{n=0}^{\infty} b_n t^{\alpha n}, \quad 0 \leq t < \theta.$$

In both cases the solutions are unique, if appropriate initial conditions are prescribed.

Thus, for example, the property of $\alpha$-analyticity of the coefficient $a(t)$ (defined above) implies a similar property of a solution of the equation (2). In fact, we have $u(t) = U(t^\alpha)$, where $U$ is holomorphic in a disk $\{z \in \mathbb{C}, \ |z| < \theta\}$. The coefficient $a$ may be complex-valued as well.

The above results open the way for developing a theory of $\alpha$-analytic solutions of fractional differential equations in the spirit of classical analytic theory of ordinary differential equations. Here we give some results in this direction.
If in (2) $a(t) = A(t^\alpha)$ where $A$ is an entire function, then the above results from [11] with $\theta = \infty$ (stated there in a weaker form, only for real arguments of analytic functions, than actually proved) show that the solution of the Cauchy problem for the equation (2) is of the form $u(t) = U(t^\alpha)$, where $U$ is an entire function. Following [11, 12] we call such solutions $\alpha$-entire. In particular, that is true, if $A$ is a polynomial. A natural question is about the order of $U$ (here we investigate this subject just for the equation (2) since its properties are closer to those of ordinary differential equations).

It is known (see, for example, [2] or [8]) that every nontrivial solution of the equation $u^{(k)}(z) = A(z)u(z), \ k \in \mathbb{N}$, with a polynomial coefficient $A$, is an entire function of order $1 + \deg(A)/k$. In this paper we prove that the orders of the entire functions $U$ corresponding to solutions of (2) do not exceed $(1 + \deg(A))/\alpha$. As $\alpha \to 1$, this agrees with the above differential equation result. On the other hand, if $\deg(A) = 0$, that is $A(z) = \lambda, \ \lambda \in \mathbb{C}$, then $U(z) = E_{\alpha}(\lambda z)$, where $E_{\alpha}$ is the Mittag-Leffler function whose order is $1/\alpha [3, 12]$, which shows the exactness of our general estimate.

Next, we investigate systems of fractional equations with regular singularity, that is the equations

$$t^\alpha \left( D_{0^+}^\alpha u \right)(t) = A(t^\alpha)u(t) \quad (3)$$
and

$$t^\alpha \left( D^{\alpha} u \right)(t) = A(t^\alpha)u(t), \quad (4)$$

where $A(z)$ is a holomorphic matrix-function. Under some assumptions, we prove that formal power series solutions of (3) and (4) converge near the origin and develop an analog of the classical Frobenius method of finding a solution. For scalar equations, the latter problem was considered in [12, 15].

Finally, in order to clarify characteristic features of fractional equations with irregular singularity, we study a model example, the equation

$$t^{2\alpha} \left( \mathcal{D}^{(\alpha)} u \right)(t) = \lambda u(t), \quad \lambda \in \mathbb{C}, \quad (5)$$

where, as before, $0 < \alpha < 1$. Assuming that $\alpha$ is irrational and satisfies a Diophantine condition (that is $\alpha$ is poorly approximated by rational numbers), we construct a kind of a formal solution of (5) convergent for $t > 0$. We prove that the series for the formal solution does not converge in the distribution sense, within a theory of distributions associated with the fractional calculus (see [16, 18]). Thus, the formal solution $u(t)$ cannot be interpreted as a distribution solution. It is interesting that $u(t)$ is
closely connected with a class of analytic functions with irregular behavior introduced by Hardy [6].

2. Preliminaries

2.1. Fractional derivatives and integrals [3, 12, 18]. Let $\alpha \in (0, 1)$ be a fixed number. The Riemann-Liouville fractional integral of order $\alpha$ of a function $\varphi \in L_1(0, T)$ is defined as

$$
(I^\alpha_{0+}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi(\tau) \, d\tau, \quad 0 < t \leq T.
$$

The Riemann-Liouville fractional derivative of order $\alpha$ is given by the expression

$$
(D^\alpha_{0+}\varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau) \, d\tau, \quad 0 < t \leq T,
$$

that is $(D^\alpha_{0+}\varphi)(t) = \frac{d}{dt} (I^{1-\alpha}_{0+}\varphi)(t)$, provided the fractional integral $I^{1-\alpha}_{0+}\varphi$ is an absolutely continuous function. If $\varphi$ is defined on the whole half-axis $(0, \infty)$, then $I^\alpha_{0+}\varphi$ and $D^\alpha_{0+}\varphi$ are also defined on $(0, \infty)$. Below we will consider just this case.

Note that the Riemann-Liouville derivative is defined for some functions with a singularity at the origin. For example, if $\varphi(t) = t^d$, $d > -1$, then

$$
(D^\alpha_{0+}\varphi)(t) = \frac{\Gamma(d+1)}{\Gamma(d+1-\alpha)} t^{d-\alpha},
$$

so that $D^\alpha_{0+}\varphi = 0$, if $\varphi(t) = t^{\alpha-1}$. For $\varphi(t) = t^d$, $d > -1$, we have also

$$
(I^\alpha_{0+}\varphi)(t) = \frac{\Gamma(d+1)}{\Gamma(d+1+\alpha)} t^{d+\alpha}.
$$

The Riemann-Liouville fractional differentiation and integration are inverse to each other in the following sense. If $\varphi \in L_1(0, T)$, then $D^\alpha_{0+} I^\alpha_{0+} \varphi = \varphi$. The equality $I^\alpha_{0+} D^\alpha_{0+} \varphi = \varphi$ holds under the stronger assumption that $\varphi = I^\alpha_{0+} \psi$ with some $\psi \in L_1(0, T)$. The latter is equivalent to the conditions of absolute continuity of $I^{1-\alpha}_{0+}\varphi$ on $[0, T]$ and the equality $(I^{1-\alpha}_{0+}\varphi)(0) = 0$. 

Let a function $\varphi$ be continuous on $[0, T]$ and possess the Riemann-Liouville fractional derivative of order $\alpha$. The function
\[
\left( D^{\alpha} \varphi \right) (t) = \frac{1}{\Gamma(1-\alpha)} \left[ \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau) d\tau - t^{-\alpha} \varphi(0) \right]
\]
is called the Caputo-Dzhrbashyan, or regularized, fractional derivative. If $\varphi$ is absolutely continuous on $[0, T]$, then
\[
\left( D^{\alpha} \varphi \right) (t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \varphi'(\tau) d\tau.
\]
In contrast to $D^{\alpha}_{0+}$, $D^{\alpha}$ is defined only on continuous functions and vanishes on constant functions. In most of physical applications, equations with $D^{\alpha}$ are used, because a solution of an equation with the Riemann-Liouville derivative typically has a singularity at the origin $t = 0$, so that the initial state of a system to be described by the equation is not defined.

Let $v(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function. Consider the Caputo-Dzhrbashyan derivative of the function
\[
\varphi(t) = v(t^\alpha) = \sum_{n=0}^{\infty} c_n t^{\alpha n}.
\]
It follows from (6) and (8) that
\[
\left( D^{\alpha} \varphi \right) (t) = \sum_{n=1}^{\infty} c_n \beta(n) t^{\alpha(n-1)} = \left( \mathfrak{D}_\alpha v \right) (t^\alpha)
\]
where
\[
\beta(n) = \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)},
\]
and the operator
\[
\left( \mathfrak{D}_\alpha v \right) (z) = \sum_{n=1}^{\infty} c_n \beta(n) z^{n-1}
\]
is known as the Gelfond-Leontiev (G-L) operator of generalized differentiation (see [13, pp. 72-85], [18, pp. 426-427]; in fact, $\mathfrak{D}_\alpha$ is defined for wider classes of functions). As a matter of fact, this is a generalized G-L
differentiation operator with respect to the Mittag-Leffler function (see e.g. in [13, 18]):

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad \alpha > 0, \]

and for the G-L operators with respect to (arbitrary) entire function \( \varphi(z) \), see the original work [4].

The operator, right inverse to \( D_\alpha \), has the form

\[ (I_\alpha f)(z) = z \frac{\Gamma(\alpha)}{1} \int_0^1 (1-t)^{\alpha-1} f(zt^\alpha) \, dt \]

(see Sect. 22.3 in [18]). It will be convenient to make a change of variables setting \( z = Re^{i\theta}, \, r = R^{1/\alpha} \). Then

\[ (I_\alpha f)(Re^{i\theta}) = e^{i\theta} \frac{R^{1/\alpha}}{\Gamma(\alpha)} \int_0^R \left[ \left( \frac{R}{r} \right)^{1/\alpha} - 1 \right]^{\alpha-1} f(re^{i\theta}) \, dr. \quad (11) \]

If \( f(z) = \sum_{k=0}^{\infty} f_k z^k \), then

\[ (I_\alpha f)(z) = \sum_{k=0}^{\infty} \frac{\Gamma(ak + 1)}{\Gamma(ak + 1 + \alpha)} f_k z^{k+1}, \]

and it is easy to check that

\[ (I_\alpha D_\alpha v)(z) = v(z) - v(0). \quad (12) \]

2.2. A class of distributions. Spaces of test functions and distributions behaving reasonably under the action of fractional integration operators were introduced by Rubin [16] (for a brief exposition see also [18]; both in [18] and [16] there are references regarding other approaches and earlier publications in this field). Proceeding from [16], it is easy to come to a class of distributions, where the Caputo-Dzhrbashyan derivative \( D_\alpha^{(\alpha)} \) is defined in a natural way.

Let \( S(0, \infty) \) be the Schwartz space of smooth functions on \( [0, \infty) \) with rapid decay at infinity. Denote

\[ \mathcal{S}_+ = \{ \varphi \in S(0, \infty) : \varphi^{(l)}(0) = 0, \, l = 0, 1, 2, \ldots \}, \]

\[ \Phi^{1-\alpha}_+ = \left\{ \varphi \in \mathcal{S}_+ : \int_0^{\infty} \varphi(x)x^{1-\alpha-k} \, dx = 0, \, k = 1, 2, \ldots \right\}. \]
\( \Phi_{+}^{\alpha-1} = \left\{ \varphi \in \mathcal{S}_+ : \int_0^\infty \varphi(x)x^{\alpha-1-k} \, dx = 0, \ k = 0, 1, 2, \ldots \right\} \).

These spaces are interpreted as topological vector spaces with the topologies induced from \( \mathcal{S}(0, \infty) \); see [16] for various descriptions of these topologies including the description by seminorms.

Together with the Riemann-Liouville fractional integration operator \( I_{0+}^\alpha \), it is convenient to use the operator
\[
(I_{0+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (\tau - t)^{\alpha-1} \varphi(\tau) \, d\tau, \quad t > 0.
\]

If \( \varphi, \psi \) are sufficiently good functions, for example, if \( \varphi \in L_p(0, \infty) \), \( \psi \in L_q(0, \infty) \), \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 2 - \alpha \), then
\[
\int_0^\infty \varphi(x) \left( I_{0+}^{1-\alpha} \psi \right)(x) \, dx = \int_0^\infty \psi(x) \left( I_{1-}^{1-\alpha} \varphi \right)(x) \, dx
\]
(see Section 2.5.1 in [18]). We will write this in the notation
\[
\langle \varphi, I_{0+}^{1-\alpha} \psi \rangle = \langle \psi, I_{1-}^{1-\alpha} \varphi \rangle.
\]

In particular, if \( \psi = u' \), where \( u \in \mathcal{S}(0, \infty) \), then
\[
\langle \varphi, D^{(\alpha)} u \rangle = \langle u', I_{1-}^{1-\alpha} \varphi \rangle.
\] (13)

It is known [16] that \( I_{1-}^{1-\alpha} \) acts continuously from \( \Phi_{+}^{1-\alpha} \) onto \( \Phi_{+}^{\alpha-1} \). Therefore the identity (13) can be used to define \( D^{(\alpha)} u \) as a distribution from \( (\Phi_{+}^{1-\alpha})' \), if \( u \in C^1(0, \infty) \), and \( u' \) has no more than a power-like growth near zero and infinity. This definition agrees with the classical one: if \( u \) is continuously differentiable at the origin too, then
\[
\langle u', I_{1-}^{1-\alpha} \varphi \rangle = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty u'(x) \, dx \int_x^\infty (t-x)^{-\alpha} \varphi(t) \, dt
\]
\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \varphi(t) \, dt \int_0^t (t-x)^{-\alpha} u'(x) \, dx = \left\langle \varphi, D^{(\alpha)} u \right\rangle,
\]
where \( D^{(\alpha)} u \) is understood in the sense of (9).
A typical example of a function from $\Phi^{1-\alpha}_+$ is the function

$$\kappa_\alpha(x) = x^{\alpha-2} \exp\left(-\frac{\log^2 x}{4}\right) \sin\left(\frac{\pi}{2} \log x\right).$$

It is clear that $\kappa_\alpha \in \mathcal{S}_+$. Next, if

$$\kappa(x) = \exp\left(-\frac{\log^2 x}{4}\right) \sin\left(\frac{\pi}{2} \log x\right),$$

then we can write explicitly the Mellin transform

$$\tilde{\kappa}(z) = \int_0^{\infty} x^{z-1} \kappa(x) \, dx.$$

Namely, by the formula (4.133.1) from [5],

$$\tilde{\kappa}(z) = 2 \int_0^{\infty} e^{-t^2/4} \sinh(zt) \sin\left(\frac{\pi}{2} t\right) \, dt = 2\sqrt{\pi} e^{-\frac{z^2}{4}} \sin \pi z. \quad (14)$$

In particular,

$$\int_0^{\infty} \kappa_\alpha(x)x^{1-\alpha-k} \, dx = \tilde{\kappa}^{(-k)} = 0, \quad k = 0, 1, 2, \ldots,$$

so that indeed $\kappa_\alpha \in \Phi^{1-\alpha}_+$.

In order to have a full concept of a class of distributions, one needs a result regarding density of a space of test functions in some space of integrable functions. This gives a one-to-one correspondence between ordinary functions and distributions they generate. Here we present such a result though it will not be used directly in this paper. For similar properties in other situations see [17].

**Proposition 1.** The space $\Phi^{1-\alpha}_+$ is dense in $L_p((0, \infty), t^{-1} \, dt)$, $1 \leq p < \infty$.

**Proof.** Let $R = \int_0^{\infty} \kappa_\alpha(t)t^{-1} \, dt$. We have

$$R = \tilde{\kappa}(\alpha - 2) = 2\sqrt{\pi} \exp\left((\alpha - 2)^2 - \frac{\pi^2}{4}\right) \sin \pi \alpha > 0.$$
Denote 
\[ z_N(x) = \frac{N}{R} \zeta_\alpha(x^N), \quad N = 1, 2, \ldots. \]

Then 
\[ \int_0^\infty z_N(x)x^{-1}dx = \frac{1}{R} \int_0^\infty \zeta_\alpha(t)t^{-1}dt = 1. \quad (15) \]

Suppose that \( f \in L_p((0, \infty), t^{-1}dt) \). Consider the so-called Mellin convolution
\[ (z_N *_M f)(t) = \int_0^\infty z_N(\tau)f\left(\frac{t}{\tau}\right)\tau^{-1}d\tau, \]
that is actually the convolution on the multiplicative group \((0, \infty)\) (note that \(t^{-1}dt\) is a Haar measure on that group). Obviously, \( z_N *_M f \in \Phi_1^{1-\alpha} \).

Denote by \( \| \cdot \|_p \) the norm in \( L_p((0, \infty), t^{-1}dt) \).

Using (15) we can write
\[ (z_N *_M f)(t) - f(t) = \int_0^\infty z_N(\tau)\left[f\left(\frac{t}{\tau}\right) - f(t)\right] \tau^{-1}d\tau. \]

By the generalized Minkowski inequality,
\[ \|z_N *_M f - f\|_p \leq \int_0^\infty |z_N(\tau)| \left\{ \int_0^\infty \left|f\left(\frac{t}{\tau}\right) - f(t)\right|^p t^{-1}dt \right\}^{1/p} d\tau. \]

Next we use the \( L_p \)-continuity of shifts on the multiplicative group \((0, \infty)\) (see [10], Theorem 20.4). For any \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that
\[ \left\{ \int_0^\infty \left|f\left(\frac{t}{\tau}\right) - f(t)\right|^p t^{-1}dt \right\}^{1/p} < \varepsilon, \]
if \( |\tau - 1| < \delta \). Thus,
\[ \|z_N *_M f - f\|_p \leq 2\|f\|_p \int_{|\tau - 1| \geq \delta} |z_N(\tau)|\tau^{-1}d\tau + \varepsilon \int_{|\tau - 1| < \delta} |z_N(\tau)|\tau^{-1}d\tau. \]

Note that
\[ \int_0^\infty |z_N(\tau)|\tau^{-1}d\tau = \frac{N}{R} \int_0^\infty |\zeta_\alpha(x^N)|x^{-1}dx = \frac{1}{R} \int_0^\infty |\zeta_\alpha(t)|t^{-1}dt = C_1 \]
where the constant $C_1$ does not depend on $N$. On the other hand,

$$|z_N(x)| \leq C_2N^xN^{(a-2)}\exp\left(-\frac{N^2\log^2 x}{4}\right).$$

If $|x-1| \geq \delta$, then $\log^2 x \geq b > 0$, so that

$$\int_{1+\delta}^{\infty} |z_N(x)|x^{-1}dx \leq C_3\varepsilon \frac{N^2\varepsilon}{(1+\delta)^{N(a-2)}} \to 0,$$

as $N \to \infty$, and (for $\delta < 1$)

$$\int_{-\delta}^{1-\delta} |z_N(x)|x^{-1}dx \leq C_2N \int_{-\infty}^{\log(1-\delta)} \exp\left\{Nt(a-2) - \frac{N^2t^2}{4}\right\} dt$$

$$= C_2 \int_{-\infty}^{N\log(1-\delta)} \exp\left\{s(a-2) - \frac{s^2}{4}\right\} ds \to 0,$$

as $N \to \infty$.

As a result, we see that, if $N$ is large enough, the first summand in (16) does not exceed $2\|f\|_p\varepsilon$, while the second $\leq C_1\varepsilon$. Thus,

$$\|z_N * Mf - f\|_p \to 0, \quad \text{as } N \to \infty.$$

\section{2.3. On ratios of the Gamma functions.} We will often use the function

$$\rho(t) = \frac{\Gamma(t+1)}{\Gamma(t+1-a)}, \quad -1 < t < \infty. \quad (17)$$

Here we collect some of its properties.

If $t > a - 1$, the integral representation

$$\frac{1}{\rho(t)} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-st}e^{(a-1)s}(1-e^{-s})^{a-1}ds \quad (18)$$

holds (see Chapter 4 in [14]). It follows from (18) that the function $t \mapsto \frac{1}{\rho(t)}$ is strictly monotone decreasing and $\frac{1}{\rho(t)} \to 0$, as $t \to \infty$. Since $\Gamma(t+1-a)$ has a pole at $t = \alpha - 1$, it is seen from (17) that $\rho(t) \to 0$, as $t \to \alpha - 1 + 0$. 
If \(-1 < t < \alpha - 1\), then (by a well-known identity for the Gamma function)
\[
\rho(t) = \frac{\Gamma(-t + \alpha)}{\Gamma(-t)} \cdot \frac{\sin \pi(t + 1)}{\sin \pi(t + 1 - \alpha)}.
\]
The integral representation for the ratio of the Gamma functions [14] leads, after an elementary investigation, to the conclusion that \(\rho(t)\) is strictly monotone increasing from \(-\infty\) to 0.

Thus, we conclude that on the interval \((-1, \infty)\) the function \(\rho(t)\) is strictly monotone increasing from \(-\infty\) to \(\infty\). The inverse function \(\gamma(\lambda)\) solving the equation
\[
\rho(t) = \lambda, \quad \lambda \in \mathbb{R},
\]
is a well-defined continuous function. Note that \(\rho(0) = \frac{1}{\Gamma(1 - \alpha)}\), so that
\[
\gamma(\lambda) \geq 0, \text{ if } \lambda \geq \frac{1}{\Gamma(1 - \alpha)}.
\]
It is known ([14], Chapter 4) that
\[
\frac{\Gamma(t + a)}{\Gamma(t + b)} \sim t^{a-b}(1 + O(t^{-1})), \quad t \to \infty,
\]
if \(b > a\). In particular,
\[
\rho(t) \sim t^\alpha, \quad \text{as } t \to \infty.
\]

For the sequence \(\beta(n)\) defined in (10) and appearing in the definition of the Gelfond-Leontiev generalized differentiation operator \(D_\alpha\), we have \(\beta(n) = \rho(\alpha n)\), so that the above asymptotics implies the relation
\[
\beta(n) \sim Cn^\alpha, \quad n = 0, 1, 2, \ldots.
\]

3. \(\alpha\)-Entire solutions

Let us consider \(\alpha\)-entire solutions of the equation (2) with \(a(t) = A(t^\alpha)\), where \(A\) is a polynomial of degree \(m \geq 0\). We assume the initial condition \(u(0) = u_0\).

**Theorem 1.** Under the above assumptions, the solution \(u(t)\) of the equation (2) has the form \(u(t) = v(t^\alpha)\), where \(v\) is an entire function whose order does not exceed \((1 + m)/\alpha\).
Proof. Seeking the function $v$, we have $(\mathcal{D}_\alpha v)(z) = a(z)v(z)$. Let us apply the operator $\mathcal{I}_\alpha$ (see (11)) to both sides of this equality. We get from (11) and (12) that

$$v(Re^{i\theta}) - v(0) = \frac{e^{i\theta}}{\alpha \Gamma(\alpha)} \int_0^R \left[ \left( \frac{R}{r} \right)^{1/\alpha} - 1 \right]^{\alpha-1} a(re^{i\theta})v(re^{i\theta}) \, dr,$$

which implies the inequality

$$|v(Re^{i\theta})| \leq |v(0)| + C \int_0^R \left[ \left( \frac{R}{r} \right)^{1/\alpha} - 1 \right]^{\alpha-1} |a(re^{i\theta})||v(re^{i\theta})| \, dr$$

(here and below we denote by the same letter $C$ various positive constants).

We have the asymptotic relations

$$t^{1/\alpha} - 1 \sim \frac{1}{\alpha}(t - 1), \quad \text{as } t \to 1 + 0;$$

$$\left( t^{1/\alpha} - 1 \right)^{\alpha-1} \sim t^{\frac{\alpha-1}{\alpha}}, \quad \text{as } t \to \infty.$$

Therefore

$$\left( t^{1/\alpha} - 1 \right)^{\alpha-1} \leq C(t - 1)^{\alpha-1}t^{\frac{1}{\alpha} - \frac{1}{\alpha}}, \quad t \geq 1,$$

so that

$$|v(Re^{i\theta})| \leq |v(0)| + CR^{2\frac{1}{\alpha} - \frac{1}{\alpha}} \int_0^R (R - r)^{\alpha-1}r^{\frac{1}{\alpha} - 1}|a(re^{i\theta})||v(re^{i\theta})| \, dr.$$

Since $a(z)$ is a polynomial of degree $m$, we get

$$|v(Re^{i\theta})| \leq |v(0)| + CR^{2\frac{1}{\alpha} - \frac{1}{\alpha}} \int_0^R (R - r)^{\alpha-1}r^{\frac{1}{\alpha} - 1 + m}|v(re^{i\theta})| \, dr,$$

where $C$ does not depend on $R, \theta$. Fixing $\theta$ and denoting

$$w(r) = \frac{|v(re^{i\theta})|}{r^{2\frac{1}{\alpha} - \frac{1}{\alpha}}},$$
we come to the inequality
\[ w(R) \leq \frac{|v(0)|}{R^{2-\frac{1}{\alpha}}} + C \int_0^R (R - r)^{\alpha - 1} r^{1-\alpha + m} w(r) \, dr. \]  
(21)

Now we are in a position to apply Henry’s theorem (see Lemma 7.1.2 from [9]), which states that the inequality (21) implies the inequality
\[ w(R) \leq \frac{|v(0)|}{R^{2-\frac{1}{\alpha}}} E_{\alpha,2-\alpha+m}(CR) \]
where \( E_{\alpha,\sigma}(s) \) is a certain function admitting the estimate
\[ E_{\alpha,\sigma}(s) \leq Cs^{\frac{1}{2}} \left( \frac{\alpha + 1}{\alpha + \sigma - 1} \right) \exp \left( \frac{\alpha}{\alpha + \sigma - 1} s^{\alpha + \sigma - 1} \right). \]

Thus,
\[ |v(Re^{i\theta})| \leq C \exp \left( \mu R^{\frac{1+m}{\alpha}} \right) \]
for some \( \mu \geq 0 \), as desired. \( \blacksquare \)

4. Regular singularity

4.1. Formal and \( \alpha \)-analytic solutions. Let us consider systems of equations of the form
\[ t^\alpha \left( D_0^\alpha u \right)(t) = A(t^\alpha)u(t), \]  
(22)
where
\[ A(z) = A_0 + \sum_{m=1}^{\infty} A_m z^m, \]
\( A_m \) are \( n \times n \) complex matrices and
\[ \|A_m\| \leq M\mu^m \ (\mu > 0), \quad m = 0, 1, 2, \ldots. \]

Suppose we have a formal series
\[ u(t) = \sum_{k=0}^{\infty} u_k t^{\alpha k}, \quad u_k \in \mathbb{C}^n. \]  
(23)
Let us substitute the series (23) formally into (22). We get, in accordance with (6), that
\[ \sum_{k=0}^{\infty} \beta(k) u_k t^{\alpha k} = \sum_{m,k=0}^{\infty} A_m u_k t^{\alpha(m+k)} \]
where \( \beta(k) \) is the sequence (10). Collecting and comparing the terms we find that
\[ \beta(l) u_l = \sum_{k=0}^{l} A_k u_{l-k}, \quad l = 0, 1, 2, \ldots, \]
or, equivalently,
\[ A_0 u_0 = \frac{1}{\Gamma(1 - \alpha)} u_0; \quad (24) \]
\[ [A_0 - \beta(l)] u_l = -\sum_{k=1}^{l} A_k u_{l-k}, \quad l \geq 1. \quad (25) \]

It is natural to call the formal series (23) a formal solution of the system (22) if the relations (24), (25) hold.

**Proposition 2.** If a formal series (23) is a formal solution of the system (22), then the series (23) is absolutely convergent on some neighbourhood of the origin.

**Proof.** It follows from (19) that
\[ \| [A_0 - \beta(l)]^{-1} \| \leq C l^{-\alpha}, \quad l \geq l_0. \]
In particular, we may assume that
\[ \| [A_0 - \beta(l)]^{-1} \| \leq 1, \quad l \geq l_0. \]
Considering, if necessary, \( \lambda u(t) \) instead of \( u(t) \), with \( |\lambda| \) small enough, we may assume that \( \|u_0\| \leq 1. \)

Let us choose so big \( r > 0 \) that \( \|u_l\| \leq r^l \) for \( l \leq l_0 \) and
\[ M \sum_{k=1}^{\infty} \left( \frac{\mu}{r} \right)^k \leq 1. \]
Then
\[ \|u_l\| \leq r^l \] for all \( l. \)
Indeed, if this inequality is proved up to some value of $l \geq l_0$, then
\[ \|u_{l+1}\| \leq \left\| \sum_{k=1}^{l+1} A_k u_{l+1-k} \right\| \leq M \sum_{k=1}^{l+1} \mu^kk^{l+1-k} = Mr^{l+1} \sum_{k=1}^{l+1} \left( \frac{\mu}{r} \right)^k \leq r^{l+1}, \]
and the above inequality implying local convergence in (23) has been proved.

The above arguments remain valid for systems of the form
\[ t^\alpha \left( D_{0+}^{\alpha} u \right)(t) = A(t^\alpha)u(t). \] (26)

The only difference is that, instead of (24), we get the relation $A_0u_0 = 0$, just as in the classical case (see [7]).

4.2. Model scalar equations. Consider the equation
\[ t^\alpha \left( D_{0+}^{\alpha} \varphi \right)(t) = \lambda \varphi(t), \quad \lambda \in \mathbb{R}. \] (27)

By the relation (6), a solution of the equation (27) is const $t^\gamma(\lambda)$, where $\gamma$ (the inverse function to $\rho$) was defined in Section 2.3. For example, if $\lambda = 0$, then we have $\gamma(0) = \alpha - 1$.

If we consider an equation similar to (27), but with the Caputo-Dzhrbashyan derivative, that is
\[ t^\alpha \left( D_{1-}^{\alpha} \varphi \right)(t) = \lambda \varphi(t), \] (28)
then the constant function is a solution of (28) for $\lambda = 0$. Suppose that $\lambda \neq 0$, and $\varphi$ is a solution of (28), that is $\varphi \in C[0, T]$, the function
\[ \left( I_{0+}^{1-\alpha} \varphi \right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau) d\tau \]
is absolutely continuous, and (28) is satisfied with
\[ \left( D_{\alpha}^{\alpha} \varphi \right)(t) = \frac{d}{dt} \left( I_{0+}^{1-\alpha} \varphi \right)(t) - \frac{1}{t^\alpha \Gamma(1-\alpha)} \varphi(0). \]

We have
\[ \left( I_{0+}^{1-\alpha} \varphi \right)(t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} \varphi(st) ds, \]
and since \( \varphi \) is continuous,

\[
(t_{0+}^{1-\alpha} \varphi)(t) \to 0, \quad \text{as } t \to +0.
\]

It is known [18] that in these circumstances \( I_{0+}^\alpha D_{0+}^\alpha \varphi = \varphi \). Note also that \( I_{0+}^\alpha \) transforms the function \( t^{-\alpha} \) into the constant \( \Gamma(1-\alpha) \). Dividing the equation (28) by \( t^\alpha \) and applying \( I_{0+}^\alpha \) to both sides, we find that

\[
\varphi(t) - \varphi(0) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \varphi(\tau) d\tau
\]

\[
= \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{-1+\alpha} s^{-\alpha} \varphi(ts) ds.
\]

Passing to the limit, as \( t \to 0 \), and taking into account the continuity of \( \varphi \), we obtain the identity

\[
\frac{\lambda \varphi(0)}{\Gamma(\alpha)} \int_0^1 (1-s)^{-1+\alpha} s^{-\alpha} ds = 0,
\]

whence \( \varphi(0) = 0 \).

Thus, for \( \lambda \neq 0 \), the equation (28) is equivalent to (27), if (27) is considered for continuous functions vanishing at the origin. The power solution \( Ct^{\gamma(\lambda)} \) belongs to this class, if \( \lambda > \frac{1}{\Gamma(1-\alpha)} \). It may be instructive to see these solutions, satisfying the equation

\[
\varphi(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \varphi(\tau) d\tau,
\]

as examples of non-uniqueness of solutions of linear Volterra integral equations occurring due to the singularity of a kernel.

4.3. Systems with good spectrum. Let us consider the equation (3) with

\[
A(z) = \sum_{m=0}^{\infty} A_m z^m
\]

where \( A_m \) are complex \( n \times n \) matrices, the matrix \( A_0 \) is Hermitian, and the series converges on a neighbourhood of the origin. Without restricting generality, we may assume that

\[
A_0 = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad \lambda_1, \ldots, \lambda_n \in \mathbb{R}.
\]
Following the classical method (see, for example, [1]) we look for a matrix-valued solution \((a \text{ fundamental solution})\) of the equation (3), in the form

\[
u(t) = S(t^\alpha)\psi(t)
\]

where \(\psi(t) = \text{diag}(t^{\gamma(\lambda_1)}, \ldots, t^{\gamma(\lambda_n)})\), \(S(z) = \sum_{\nu=0}^\infty \sigma_\nu z^\nu\), \(\sigma_\nu \ (\nu \geq 1)\) are some unknown matrices, \(\sigma_0 = I\).

We have

\[
u(t) = \sum_{\nu=0}^\infty \sigma_\nu \text{diag}(t^{\gamma(\lambda_1)+\alpha \nu}, \ldots, t^{\gamma(\lambda_n)+\alpha \nu}),
\]

whence

\[
u(t) = \sum_{\nu=0}^\infty \sigma_\nu R_\nu \text{diag}(t^{\gamma(\lambda_1)+\alpha \nu}, \ldots, t^{\gamma(\lambda_n)+\alpha \nu}),
\]

(30)

where

\[
R_\nu = \text{diag}
\begin{pmatrix}
\Gamma(\gamma(\lambda_1) + \alpha \nu + 1)
\Gamma(\gamma(\lambda_1) + \alpha \nu + 1 - \alpha)
\Gamma(\gamma(\lambda_2) + \alpha \nu + 1)
\Gamma(\gamma(\lambda_2) + \alpha \nu + 1 - \alpha)
\end{pmatrix}.
\]

On the other hand,

\[
A(t^\alpha)u(t) = \sum_{\nu=0}^\infty \left( \sum_{m=0}^\nu A_m \sigma_{\nu-m} \right) \text{diag}(t^{\gamma(\lambda_1)+\alpha \nu}, \ldots, t^{\gamma(\lambda_n)+\alpha \nu}).
\]

(31)

Note that \(R_0 = \text{diag}(\lambda_1, \ldots, \lambda_n) = A_0\), and since \(\sigma_0 = I\), the coefficients corresponding to \(\nu = 0\) in (30) and (31) coincide. Comparing the rest of the coefficients, we obtain the following system of equations for the matrices \(\sigma_k\):

\[
\sigma_k R_k - A_0 \sigma_k = \sum_{l=0}^{k-1} A_{k-l} \sigma_l, \quad k \geq 1.
\]

(32)

For each \(k\), the matrix equation (32) for \(\sigma_k\) has a unique solution if the spectra of the matrices \(R_k\) and \(A_0\) are disjoint (see Appendix A.1 in [1]), that is

\[
\frac{\Gamma(\gamma(\lambda_i) + \alpha k + 1)}{\Gamma(\gamma(\lambda_i) + \alpha k + 1 - \alpha)} \neq \lambda_j
\]

for all \(i, j \in \{1, \ldots, n\}\), or, equivalently, since the left-hand side of (33) equals \(\rho(\gamma(\lambda_i) + \alpha k)\),

\[
\gamma(\lambda_j) - \gamma(\lambda_i) \neq \alpha k, \quad \text{for all} \ i, j \in \{1, \ldots, n\}.
\]
We call our system (3) a system with good spectrum, if
\[ \gamma(\lambda_j) - \gamma(\lambda_i) \notin \alpha\mathbb{N}, \quad \text{for all } i, j \in \{1, \ldots, n\}. \] (34)
This definition extends the classical one [1], since for \( \alpha = 1 \) we would have \( \rho(t) = \gamma(t) = t \), and the condition (34) would mean that the eigenvalues of \( A_0 \) must not differ by a natural number.

**Theorem 2.** If a system (3) has a good spectrum, then it possesses a fundamental solution (29) where the series for \( S(z) \) has a positive radius of convergence.

**Proof.** By the asymptotic relation (19),
\[ \frac{\Gamma(\gamma(\lambda_j) + \alpha k + 1)}{\Gamma(\gamma(\lambda_j) + \alpha k + 1 - \alpha)} \sim (\alpha k)^\alpha (1 + O(k^{-1})) \quad k \to \infty, \]
for all \( j = 1, \ldots, n \). Therefore
\[ (\alpha k)^{-\alpha} R_k = I + O(k^{-1}), \quad k \to \infty. \] (35)

Let us divide both sides of the equation (32) by \((\alpha k)^\alpha\). The resulting equation, considered as a system of scalar equations for \( n^2 \) elements of the matrix \( \sigma_k \), has the coefficients bounded in \( k \) and the determinant, which is different from zero for each \( k \) and tends to 1, as \( k \to \infty \). This implies the estimate
\[ \| \sigma_k \| \leq a k^{-\alpha} \sum_{l=0}^{k-1} A_{k-l} \| \sigma_l \|, \quad k \geq 1, \] (36)
where the constant \( a > 0 \) does not depend on \( k \). It follows from (35), (36), and the convergence near the origin of the power series for \( A(z) \) that
\[ \| \sigma_k \| \leq a_1 k^{-\alpha} \sum_{l=0}^{k-1} b^{k-l} \| \sigma_l \|, \quad k \geq 1, \]
where \( a_1 \) and \( b \) are positive constants independent of \( k \).

Define a sequence \( \{ s_k \}_{0}^{\infty} \) of positive numbers, setting \( s_0 = 1 \),
\[ s_k = a_1 k^{-\alpha} \sum_{l=0}^{k-1} b^{k-l} s_l, \quad k \geq 1. \]
The induction on \( k \) yields the inequality \( \|\sigma_k\| \leq s_k \) for all \( k \geq 0 \). On the other hand,

\[
s_{k+1} = a_1(k + 1)^{-\alpha} \sum_{l=0}^{k} b^{k+1-l}s_l
\]

\[
= \frac{(k + 1)^{-\alpha}}{k^{-\alpha}} \left[ a_1k^{-\alpha} \left( b \sum_{l=0}^{k-1} b^{k-l}s_l + bs_k \right) \right]
\]

\[
= \frac{(k + 1)^{-\alpha}}{k^{-\alpha}} (bs_k + a_1k^{-\alpha}bs_k) = \frac{(k + 1)^{-\alpha}b}{k^{-\alpha}} (1 + a_1k^{-\alpha}) s_k.
\]

Therefore

\[
\frac{s_k}{s_{k+1}} \rightarrow b^{-1}, \quad \text{as} \quad k \rightarrow \infty. \quad (37)
\]

It follows from (37) that the series \( \sum_{k=0}^{\infty} s_k z^k \) has the convergence radius \( b^{-1} \) (see Section 2.6 in [21]). Moreover, the series \( \sum_{\nu=0}^{\infty} \sigma_\nu z^\nu \) converges for \( |z| < b^{-1} \).

For the equation (26), a similar construction is valid, if we assume that \( \lambda_1, \ldots, \lambda_n \geq \frac{1}{\Gamma(1 - \alpha)} \).

5. Irregular singularity: An example

5.1. A formal solution. In this section we construct a solution, in a sense to be specified, of the equation (5). Looking at classical first order equations, corresponding formally to \( \alpha = 1 \), we have to consider the equation \( t^2y'(t) = \lambda y(t) \) whose solution is \( y(t) = \exp(-\lambda t^{-1}) \). Therefore it is natural to seek a solution of the equation (5) in the form

\[
u(t) = \sum_{n=0}^{\infty} c_n t^{-n\alpha}, \quad c_n \in \mathbb{C}. \quad (38)
\]

A fractional derivative of any term in (38) with \( n > \alpha^{-1} \) does not make sense classically. However we may apply the distribution theory from Section 2.2. Below we understand the fractional derivative \( D^{(\alpha)} \) in the sense of (13).
Proposition 3. (i) If $\mu < 0$, $\mu \neq -1, -2, \ldots$, then
\[ D^{(\alpha)} t^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \alpha)} t^{\mu - \alpha}. \tag{39} \]

(ii) If $k$ is a natural number, then
\[ D^{(\alpha)} t^{-k} = \frac{(-1)^{k-1}}{(k-1)!} \frac{\Gamma(\mu + 1 - \alpha)}{\Gamma(\mu + 1 - \alpha - k)} t^{-k - \alpha} \log t. \tag{40} \]

Proof. Let $\varphi \in \Phi_{1+}^{1-\alpha}$. By (13),
\[ \langle D^{(\alpha)} t^\mu, \varphi(t) \rangle = \mu \langle t^{\mu-1}, (I_{1+}^{1-\alpha} \varphi)(t) \rangle. \tag{41} \]

It is clear that the right-hand side of (41) is an entire function of $\mu$. For $\mu > 0$, by virtue of (7),
\[ \langle t^{\mu-1}, (I_{1+}^{1-\alpha} \varphi)(t) \rangle = \frac{\Gamma(\mu)}{\Gamma(\mu + 1 - \alpha)} \langle t^{\mu-\alpha}, \varphi(t) \rangle. \]

For $\mu < 0$, $\mu \neq -1, -2, \ldots$, the analytic continuation gives the equality (39).

Next, consider the entire function
\[ F(\mu) = \langle t^{\mu-\alpha}, \varphi(t) \rangle, \quad \mu \in \mathbb{C}. \]

Note that $F(-k) = 0$, $k \in \mathbb{N}$, by the definition of the space $\Phi_{1+}^{1-\alpha}$. We have
\[ F'(\mu) = \langle \log t \cdot e^{(\mu-\alpha)\log t}, \varphi \rangle. \]

In particular,
\[ F'(-k) = \int_0^\infty t^{-\alpha-k} \log t \cdot \varphi(t) \, dt. \]

As $\mu$ belongs to a small neighbourhood of the point $-k$, $F(\mu) = F'(-k)(\mu + k) + o(\mu + k)$. Since the residue of $\Gamma(\mu + 1)$ at $\mu = -k$ equals $\frac{(-1)^{k-1}}{(k-1)!}$ (see Section 4.4.1 of [20]), we see that the function $\Gamma(\mu + 1) F(\mu)$ is holomorphic at $\mu = -k$ (in fact, it is entire) and
\[ \Gamma(\mu + 1) F(\mu) |_{\mu = -k} = \frac{(-1)^{k-1}}{(k-1)!} \int_0^\infty t^{-\alpha-k} \log t \cdot \varphi(t) \, dt. \]
Now the equality (39) implies (40).

Returning to (38), we will formally apply $D^{(\alpha)}$ termwise and find the coefficients $c_n$ comparing resulting terms in (5). It is clear from the equality (40) that such a procedure would fail if some of the numbers $n\alpha$ are integers. Thus, we have to assume that $\alpha$ is irrational. Using (39) we find that (formally)

$$ t^{2\alpha} (D^{(\alpha)} u)(t) = \sum_{n=1}^{\infty} c_n \frac{\Gamma(-n\alpha + 1)}{\Gamma(-n\alpha + 1 - \alpha)} t^{-n\alpha + \alpha}. $$

Substituting this into (5) we come to the recurrence relation

$$ c_{n+1} = \lambda \frac{\Gamma(1-(n+2)\alpha)}{\Gamma(1-(n+1)\alpha)} c_n, \quad n \geq 0, $$

and it is easy to find by induction that

$$ c_n = \lambda^n \frac{\Gamma(1-(n+1)\alpha)}{\Gamma(1-\alpha)} c_0. $$

Thus, we have found a formal solution

$$ u(t) = \frac{c_0}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \Gamma(1-(n+1)\alpha) t^{-n\alpha} $$

of the equation (5). Using the identity

$$ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} $$

we can rewrite (42) in the form

$$ u(t) = \frac{c_0\pi}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{1}{\sin(\pi(n+1)\alpha)} \frac{t^{-n\alpha}}{\Gamma((n+1)\alpha)}. $$

5.2. The convergence problem. The convergence of the series (43) depends on the arithmetic properties of the irrational number $\alpha$. It was shown by Hardy [6] that $\alpha$ can be chosen in such a way (to be well approximated by rational numbers) that the series (43) would diverge for small values of $t$. 
An irrational number $\alpha \in (0, 1)$ is said to be poorly approximated by rational numbers, if there exist such $\varepsilon > 0$, $c > 0$ that for any rational number $\frac{p}{q}$, $p, q \in \mathbb{N}$,

$$\left| \alpha - \frac{p}{q} \right| \geq cq^{-2-\varepsilon}. \quad (44)$$

By the Thue-Siegel-Roth theorem (see [19]) such are all algebraic numbers.

The first statement of the next theorem is actually contained already in the paper [6].

**Theorem 3.** If $\alpha$ is poorly approximated by rational numbers, then the series (43) converges for any $t > 0$. However this series diverges in the space of distributions $(\Phi^{1-\alpha}_+)^t$.

**Proof.** It follows from (44) that

$$|q\alpha - p| \geq cq^{-1-\varepsilon} \quad \text{for any } p \in \mathbb{N},$$

so that

$$\text{dist}((n+1)\alpha, \mathbb{Z}_+) \geq c(n+1)^{-1-\varepsilon},$$

and taking $l \in \mathbb{Z}_+$, such that $|(n+1)\alpha - l| \leq \frac{1}{2}$, we find that

$$|\sin(\pi(n+1)\alpha)| = |\sin(\pi((n+1)\alpha - l)| \geq 2|(n+1)\alpha - l|$$

$$\geq 2 \text{dist}((n+1)\alpha, \mathbb{Z}_+) \geq 2c(n+1)^{-1-\varepsilon}.$$ 

Using the Stirling formula we obtain that the series (43) converges for each $t > 0$.

To prove the second assertion, consider $(t^{-an}, \kappa_\alpha(t))$, where the function $\kappa_\alpha$ was defined in Section 2.2. We have, by (14), that

$$\langle t^{-an}, \kappa_\alpha(t) \rangle = \int_0^\infty t^{-an} \kappa_\alpha(t) \, dt = \tilde{Z}_\alpha(1 - \alpha n) = \tilde{Z}(\alpha - 1 - \alpha n)$$

$$= 2\sqrt{\pi} \exp \left( (\alpha - 1 - \alpha n)^2 - \frac{\pi^2}{4} \right) \sin(\pi(\alpha - 1 - \alpha n)).$$

Now we can give a lower estimate of the coefficients in the series (43) understood in the distribution sense: there are such $a, a_1, b > 0$ that
\[
\frac{1}{\sin(\pi(n+1)\alpha)} \left\{ \frac{(t^{-\alpha}, x_0(t))}{\Gamma((n+1)\alpha)} \right\} \\
\geq a(n+1)^{-b(n+1)}|\sin(\pi\alpha(n-1))|\exp(\alpha^2n^2 - 2\alpha(\alpha - 1)n) \\
\geq a_1(n+1)^{-b(n+1)-1-\varepsilon}\exp(\alpha^2n^2 - 2\alpha(\alpha - 1)n) \\
= a_1 \exp \left\{ \alpha^2n^2 - 2\alpha(\alpha - 1)n - (bn + b + 1 + \varepsilon) \log(n + 1) \right\} \to \infty,
\]
as \(n \to \infty\). Therefore the series (43) does not converge in \((\Phi_1^{1-\alpha})'\).

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