Comment on “ADM reduction of IIB on $\mathcal{H}^{p,q}$ to dS braneworld”

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Abstract

We make a comment on “ADM reduction of IIB on $\mathcal{H}^{p,q}$ to dS braneworld” by E. Hatefi, A. J. Nurmagambetov and I. Y. Park, [arXiv:1210.3825]

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In [1, 2, 3], we showed that the effective action of D-brane, the Born-Infeld action with the Wess-Zumino terms, is a solutions to the Hamilton-Jacobi equation in supergravity. In [4] we also generalized this to the case of M2-brane and M5-brane. To derive the Hamilton-Jacobi equation where the radial direction is regarded as the time direction, we made a consistent truncation of supergravity over $S^{8-p}$ in the case of the $p$-brane and obtained a $p+2$-dimensional theory. Here, as usual, the consistent truncation means that all the classical solutions of the truncated theory are also classical solutions of the original theory.

Recently, the authors of [5] have studied a similar Hamilton-Jacobi equation in a five-dimensional theory. In obtaining the five-dimensional theory from ten-dimensional supergravity by a consistent truncation over $H^5$ or $S^5$ and deriving the Hamilton-Jacobi equation in the five-dimensional theory, they use our calculation for the $p = 3$ case in [1] as a reference. However, they claim that there is an error in our consistent truncation over $S^5$. They obtained a five-dimensional theory and a Hamilton-Jacobi equation that differ from ours. They show that the effective action of D3-brane is not a solution to their Hamilton-Jacobi equation.

In this brief article, by deriving our truncated theory from a different viewpoint, we verify that our results in [1] are correct. We also point out an error in [5]. More specifically, the truncated theory obtained in the Einstein frame in [5] is consistent with ours, but there is an error in transforming this truncated theory to the one in the string frame.

As in [1, 5], we consider ten-dimensional type IIB supergravity. The discrepancy between [1] and [5] exists for the part of the metric and the dilaton. Also, ignoring all the fields other than the metric and the dilaton is consistent. We therefore concentrate on the metric and the dilaton throughout this article. Then, the ten-dimensional action in the string frame takes the form

$$I_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G} e^{-2\Phi} \left( R_G + 4 \partial_M \Phi \partial^M \Phi \right).$$

(1)

In order to make a consistent truncation over $S^5$, we split the ten-dimensional coordinates $X^M$ into two parts, as $X^M = (\xi^\alpha, \theta_i)$ ($\alpha = 0, \cdots, 4$, $i = 1, \cdots, 5$), where the $\xi^\alpha$ are five-dimensional coordinates and the $\theta_i$ parametrize $S^5$. We make the following ansatz for the metric and the dilaton:

$$ds_{10}^2 = G_{MN}dX^M dX^N.$$
where \( h_{\alpha \beta} \) is a five-dimensional metric, and \( d \Omega_5 \) is the \( SO(6) \)-invariant metric of unit \( S^5 \). We set the other modes to zero.

(2) is the most general form of the metric and the dilaton that is invariant under the \( SO(6) \) transformation. Namely, we discard all the \( SO(6) \)-variant modes. (1) is \( SO(6) \)-invariant so that the terms in (1) including these \( SO(6) \)-variant modes are at least quadratic order in those modes. Hence, the equations of motion for these \( SO(6) \)-variant modes are automatically satisfied under (2). This implies that the desired five-dimensional theory can be obtained by substituting (2) into (1) (see, for instance, argument in \cite{6}). Thus, by substituting (2) into (1) and using the formula in appendix, we obtain the five-dimensional theory

\[
I_5 = \frac{1}{2\kappa_5^2} \int d^5 \xi \sqrt{-h} \left[ e^{-2\phi + \frac{\partial \phi}{4}} \left( \mathcal{R}^{(5)} + 4 \partial_\alpha \phi \partial^\alpha \phi + \frac{5}{4} \partial_\alpha \rho \partial^\alpha \rho - 5 \partial_\alpha \phi \partial^\alpha \rho \right) + e^{-2\phi + \frac{\partial \phi}{4}} \mathcal{R}^{(S^5)} \right],
\]

(3)

where

\[
\frac{1}{2\kappa_5^2} = \frac{\text{volume of } S^5}{2\kappa_{10}^2}, \quad \mathcal{R}^{(S^5)} = 20.
\]

This indeed agrees with the five-dimensional theory in \cite{1} with the Kalb-Ramond field and the Ramond-Ramond fields set to zero. Thus, our consistent truncation in \cite{1} is correct.

Let us see that the above result is consistent with (2.6) of \cite{5}. The authors of \cite{5} work in the Einstein frame where ten-dimensional action takes the form

\[
I_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10} X \sqrt{-G} \left( R_G - \frac{1}{2} \partial_M \Phi \partial^M \Phi \right).
\]

(4)

This action is obtained from (1) by making a replacement

\[
G_{MN} \to e^{\frac{1}{2}\phi} G_{MN}.
\]

(5)

\footnote{1This procedure is still valid when the Kalb-Ramond field and the Ramond-Ramond fields except the self-dual 4-form are included and the \( SO(6) \) invariant ansatz is made for those fields.}

\footnote{2We have verified that the equations of motion of (3) reproduce the equations obtained by substituting (2) into the equations of motion of (1). This is another check for the correctness of (3). The claim in footnote 4 of \cite{5} that this is not the case is wrong.}
They make the following ansatz for this new metric and the dilaton:

\[
\begin{align*}
\tilde{s}_{10}^2 &= e^{2\tilde{\rho}(\xi)} \tilde{h}_{\alpha \beta}(\xi) \, d\xi^\alpha d\xi^\beta + e^{-6\tilde{\rho}(\xi)/5} \, d\Omega_5, \\
\Phi &= \phi(\xi).
\end{align*}
\]  

(6)

From (2), (5) and (6), we find the relationship

\[
\begin{align*}
h_{\alpha \beta} &= e^{\frac{1}{2} \phi + 2\tilde{\rho}} \tilde{h}_{\alpha \beta}, \\
\rho &= -\frac{12}{5} \tilde{\rho} + \phi.
\end{align*}
\]

(7)

By substituting (7) into (3) and using the formula in appendix, we obtain

\[
I_5 = \frac{1}{2\kappa_5^2} \int d^5 \xi \sqrt{-\tilde{h}} \left[ \tilde{R}^{(5)} - \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{24}{5} \partial_\alpha \tilde{\rho} \partial^\alpha \tilde{\rho} + e^{\frac{4}{5} \tilde{\rho}} R^{(S^5)} \right].
\]

(8)

This agree with (2.6) of [5] with the Kalb-Ramond field and the Ramond-Ramond fields set to zero.

In deriving the Hamilton-Jacobi equation, the authors of [5] move to the string frame\(^3\) by making a replacement:

\[
\begin{align*}
\tilde{\rho} &= -\frac{5}{12} \hat{\rho}, \\
\tilde{h}_{\alpha \beta} &= e^{-\frac{1}{2} \phi + \frac{7}{2} \tilde{\rho}} \hat{h}_{\alpha \beta}.
\end{align*}
\]

(9)

Then, (8) leads to

\[
I_5 = \frac{1}{2\kappa_5^2} \int d^5 \xi \sqrt{-\tilde{h}} \left[ e^{-\frac{1}{2} \phi + \frac{7}{2} \tilde{\rho}} \left( \tilde{R}^{(5)} - \frac{10}{3} \hat{\nabla}^2 \hat{\rho} - \frac{35}{12} \partial_\alpha \hat{\rho} \partial^\alpha \hat{\rho} + 2\hat{\nabla}^2 \phi - \frac{5}{4} \partial_\alpha \phi \partial^\alpha \phi + \frac{5}{2} \partial_\alpha \phi \partial^\alpha \hat{\rho} \right) + e^{-\frac{1}{2} \phi + \frac{7}{2} \tilde{\rho}} R^{(S^5)} \right].
\]

(10)

However, the term \(\frac{5}{2} \partial_\alpha \phi \partial^\alpha \hat{\rho}\) is missing in (3.2) of [5]. Thus, (3.2) of [5] is unfortunately incorrect. The Hamilton-Jacobi equation derived from (3.2) of [5] is of course different from ours. Hence, their claim that the D3-brane effective action is not a solution to the Hamilton-Jacobi equation is wrong.

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\(^3\)The definition of the string frame in [5] is different from ours.
Appendix: Some useful formulae

First, we write down the Wely transformation in $D$ dimensions. Under $\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}$,

$$
\tilde{R} = e^{-2\omega}(R - 2(D - 1)\nabla^2 \omega - (D - 1)(D - 2)\partial_\mu \omega \partial^\mu \omega).
$$

Next, we consider the following reduction of the ten-dimensional space-time on $S^{8-p}$:

$$
ds_{10}^2 = G_{MN} dX^M dX^N = h_{\alpha\beta}(\xi) d\xi^\alpha d\xi^\beta + e^{\sigma(\xi)/2} d\Omega_{8-p}.
$$

Here, the $\xi^\alpha$ are $(p+2)$-dimensional coordinates, and $S^{8-p}$ is parametrized by $\theta_1, \ldots, \theta_{8-p}$.

The ten-dimensional curvature is represented by the $(p+2)$-dimensional curvature and the $(8-p)$-dimensional curvature as

$$
R_G = R^{(p+2)} - \frac{8-p}{2} \nabla^{(p+2)} \nabla^{(p+2)} \sigma - \frac{(8-p)(9-p)}{16} \partial_\alpha \sigma \partial_\beta \sigma + e^{-\sigma/2} R^{(S^{8-p})},
$$

where $R^{(S^{8-p})}$ is the constant curvature of $S^{8-p}$.

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