M-estimation in GARCH models without higher order moments
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Abstract

We consider a class of M-estimators of the parameters of the GARCH models which are asymptotically normal under mild assumptions on the moments of the underlying error distribution. Since heavy-tailed error distributions without higher order moments are common in the GARCH modeling of many real financial data, it becomes worthwhile to use such estimators for the time series inference instead of the quasi maximum likelihood estimator. We discuss the weighted bootstrap approximations of the distributions of M-estimators. Through extensive simulations and data analysis, we demonstrate the robustness of the M-estimators under heavy-tailed error distributions and the accuracy of the bootstrap approximation. In addition to the GARCH (1, 1) model, we obtain extensive computation and simulation results which are useful in the context of higher order models such as GARCH (2, 1) and GARCH (1, 2) but have not yet received sufficient attention in the literature. Finally, we use M-estimators for the analysis of three real financial time series fitted with GARCH (1, 1) or GARCH (2, 1) models.

Keywords: GARCH models, M-estimation, Weighted bootstrap, Heavy-tailed distributions.
Short title: M-estimation in GARCH models.
1 Introduction

The Generalized autoregressive conditional heteroscedastic (GARCH) models have been used extensively to analyze the volatility or the instantaneous variability of a financial time series \( \{X_t; 1 \leq t \leq n\} \). A series \( \{X_t; t \in \mathbb{Z}\} \) is said to follow a GARCH \((p, q)\) model if

\[
X_t = \sigma_t \epsilon_t, \tag{1.1}
\]

where \( \{\epsilon_t; t \in \mathbb{Z}\} \) are unobservable i.i.d. errors with symmetric distribution around zero and

\[
\sigma_t = \left(\omega_0 + \sum_{i=1}^{p} \alpha_{0i} X_{t-i}^2 + \sum_{j=1}^{q} \beta_{0j} \sigma_{t-j}^2\right)^{1/2}, \quad t \in \mathbb{Z}, \tag{1.2}
\]

with \( \omega_0, \alpha_{0i}, \beta_{0j} > 0, \forall i, j \). Mukherjee (2008) proposed a class of M-estimators for estimating the GARCH parameter

\[
\theta_0 = (\omega_0, \alpha_{01}, \ldots, \alpha_{0p}, \beta_{01}, \ldots, \beta_{0q})', \tag{1.3}
\]

based on observations \( \{X_t; 1 \leq t \leq n\} \). The M-estimators are asymptotically normal under some moment assumptions on the error distribution and are more robust than the commonly-used quasi maximum likelihood estimator (QMLE). Mukherjee (2020) considered a class of weighted bootstrap methods to approximate the distributions of these estimators and established the asymptotic validity of such bootstrap. In this paper, we apply an iteratively re-weighted algorithm to compute the M-estimates and the corresponding bootstrap estimates with specific attention to Huber’s, \(\mu\)- and Cauchy-estimators which were not considered in the literature in details. The iteratively re-weighted algorithm turns out to be particularly useful in computing bootstrap replicates since it avoids the re-computation of some core quantities for new bootstrap samples.

The class of M-estimators includes the QMLE. The asymptotic normality and the asymptotic validity of bootstrapping the QMLE were derived under the finite fourth moment assumption on the error distribution. However, there are other M-estimators such as the \(\mu\)-estimator and Cauchy-estimator which are asymptotic normal under mild assumption on the finiteness of lower order moments. Since heavy-tailed error distributions without higher order moments are common in the GARCH modeling of many real financial time series, it becomes worthwhile to use these estimators for such series but unfortunately they have not been investigated in the literature. One of the contributions of this paper is to reveal precisely the importance of such alternative M-estimators to analyze financial data instead of using the QMLE.

In an earlier work, Muler and Yohai (2008) analyzed the Electric Fuel Corporation (EFCX) time series and fitted a GARCH \((1, 1)\) model. Using exploratory analysis, they detected presence of outliers and considered estimation of parameters based on robust methods. It turned out that estimates based on different methods vary widely and so it is difficult to assess which method should be relied on in similar situations. In this paper, we use M-estimates with mild assumptions on error moments to analyze the EFCX series.
Francq and Zakoïan (2009) underscored the importance of using higher order GARCH models such as GARCH (2, 1) for some real financial time series but the computation and simulation results for such models are not available widely in the literature. We investigate the role of M-estimators for the GARCH (2, 1) model through extensive simulations and real data analysis. We also provide simulation results and analysis for the GARCH (1, 2) model.

The paper is organized as follows. Sections 2 and 3 set the background. In particular, we discuss the class of M-estimators and give examples in Section 2. Section 3 contains bootstrap formulation and the statement on the asymptotic validity of the bootstrap. Section 4 discusses computational aspects of M-estimators and its bootstrapped replicates. Section 5 reports simulation results for various M-estimators. Section 6 compares bootstrap approximation with the asymptotic normal approximation to distributions of M-estimators through simulation. Section 7 analyzes three real financial time series.

## 2 M-estimators of the GARCH parameters

Throughout this paper, for a function $g$, we use $\dot{g}$ to denote its derivative whenever it exists. Also, $\text{sign}(x) := I(x > 0) − I(x < 0)$. For $x > 0$, $\log^+(x) := I(x > 1)\log(x)$. Moreover, $\epsilon$ will denote a generic r.v. having same distribution as errors $\{\epsilon_t\}$ of (1.1).

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function which is differentiable at all but finite number of points. Let $\mathcal{D} \subset \mathbb{R}$ denote the set of points where $\psi$ is differentiable and let $\bar{\mathcal{D}}$ denote its complement. Let $H(x) := x\psi(x)$, $x \in \mathbb{R}$ so that $H$ is symmetric. The function $H$ is called the score function of the M-estimation in the scale model. Examples are as follows.

**Example 1.** QMLE score: Let $\psi(x) = x$. Then $H(x) = x^2$.

**Example 2.** LAD score: Let $\psi(x) = \text{sign}(x)$. Then $\mathcal{D} = \{0\}$ and $H(x) = |x|$.

**Example 3.** Huber’s $k$ score: Let $\psi(x) = xI(|x| \leq k) + k\text{sign}(x)I(|x| > k)$, where $k > 0$ is a known constant. Then $\bar{\mathcal{D}} = \{-k, k\}$ and $H(x) = x^2I(|x| \leq k) + k|x|I(|x| > k)$.

**Example 4.** Score function for the maximum likelihood estimation (MLE): Let $\psi(x) = -\dot{f}(x)/f(x)$, where $f$ is the true density of $\epsilon$, assumed to be known. Then $H(x) = x\{-\dot{f}(x)/f(x)\}$.

**Example 5.** $\mu$ score: Let $\psi(x) = \mu\text{sign}(x)/(1 + |x|)$, where $\mu > 1$ is a known constant. Then $\mathcal{D} = \{0\}$ and $H(x) = \mu|x|/(1 + |x|)$ is bounded.

**Example 6.** Cauchy score: Let $\psi(x) = 2x/(1 + x^2)$. Then $H(x) = 2x^2/(1 + x^2)$ is bounded.

**Example 7.** Score function for the exponential pseudo-maximum likelihood estimation: Let $\psi(x) = \delta_1|x|^{\delta_2-1}\text{sign}(x)$, where $\delta_1 > 0$ and $1 < \delta_2 \leq 2$ are known constants. Here $\bar{\mathcal{D}} = \{0\}$ and $H(x) = \delta_1|x|^{\delta_2}$.

Assume that for some $\kappa_1 \geq 2$ and $\kappa_2 > 0$,

$$
\text{E}[|\epsilon|^{\kappa_1}] < \infty \quad \text{and} \quad \lim_{t \to 0} \frac{P(\epsilon^2 < t)}{t^{\kappa_2}} = 0.
$$

(2.1)
Then $\sigma^2_t$ of (1.2) has the following unique almost sure representation:

$$\sigma^2_t = c_0 + \sum_{j=1}^{\infty} c_j X^2_{t-j}, \ t \in \mathbb{Z},$$

(2.2)

where $\{c_j; j \geq 0\}$ are defined in (2.9)-(2.16) of Berkes et al. (2003).

Let $\Theta$ be a compact subset of $(0, \infty)^{1+p} \times (0, 1)^q$. A typical element in $\Theta$ is denoted by $\theta = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'$. Define the variance function on $\Theta$ by

$$v_t(\theta) = c_0(\theta) + \sum_{j=1}^{\infty} c_j(\theta) X^2_{t-j}, \ \theta \in \Theta, \ t \in \mathbb{Z},$$

(2.3)

where the coefficients $\{c_j(\theta); j \geq 0\}$ are given in Berkes et al. (2003) (Section 3, and display (3.1)) with the property

$$c_j(\theta_0) = c_j, \ \forall j \geq 0.$$  

(2.4)

Hence the variance functions satisfy $v_t(\theta_0) = \sigma^2_t, \ t \in \mathbb{Z}$. Using (2.4), (1.1) can be rewritten as

$$X_t = \{v_t(\theta_0)\}^{1/2} \epsilon_t, \ 1 \leq t \leq n.$$  

(2.5)

Consider observable approximation $\{\hat{v}_t(\theta)\}$ of the process $\{v_t(\theta)\}$ of (2.3) defined by

$$\hat{v}_t(\theta) = c_0(\theta) + I(2 \leq t) \sum_{j=1}^{t-1} c_j(\theta) X^2_{t-j}, \ \theta \in \Theta, \ 1 \leq t \leq n.$$  

(2.6)

Then an M-estimator $\hat{\theta}_n$ is defined as the solution of $\hat{M}_{n,H}(\theta) = 0$, where

$$\hat{M}_{n,H}(\theta) := \sum_{t=1}^{n} \left\{ \frac{1}{H(\{v_t(\theta)\}^{1/2})} \right\} \{\hat{v}_t(\theta) / v_t(\theta)\}.$$  

(2.7)

Next we describe the iterative relation of $\{c_j(\theta)\}$ that is used to write computer program for their numerical evaluation. The computation is discussed in Section 4.

**Example 1.** GARCH (1, 1) model: With $\theta = (\omega, \alpha, \beta)'$,

$$c_0(\omega, \alpha, \beta) = \omega / (1 - \beta), \ c_j(\omega, \alpha, \beta) = \alpha \beta^{j-1}, \ j \geq 1.$$  

**Example 2.** GARCH (2, 1) model: With $\theta = (\omega, \alpha_1, \alpha_2, \beta)'$,

$$c_0(\theta) = \omega / (1 - \beta), \ c_1(\theta) = \alpha_1, \ c_2(\theta) = \alpha_2 + \beta c_1(\theta) = \alpha_2 + \beta \alpha_1$$

and

$$c_j(\theta) = \beta c_{j-1}(\theta), \ j \geq 3.$$  

**Example 3.** GARCH (1, 2) model: With $\theta = (\omega, \alpha, \beta_1, \beta_2)'$,

$$c_0(\theta) = \omega / (1 - \beta_1 - \beta_2), \ c_1(\theta) = \alpha, \ c_2(\theta) = \beta_1 c_1(\theta) = \beta_1 \alpha,$$
and
\[ c_j(\theta) = \beta_1 c_{j-1}(\theta) + \beta_2 c_{j-2}(\theta), \quad j \geq 3. \]

**Example 4.** GARCH (2,2) model: With \( \theta = (\omega, \alpha_1, \alpha_2, \beta_1, \beta_2)' \),
\[ c_0(\theta) = \omega/(1 - \beta_1 - \beta_2), \quad c_1(\theta) = \alpha_1, \quad c_2(\theta) = \alpha_2 + \beta_1 \alpha_1 \]
and
\[ c_j(\theta) = \beta_1 c_{j-1}(\theta) + \beta_2 c_{j-2}(\theta), \quad j \geq 3. \]

### 2.1 Asymptotic distribution of \( \hat{\theta}_n \)

The asymptotic distribution of \( \hat{\theta}_n \) is derived under the following assumptions.

**Model assumptions:** The parameter space \( \Theta \) is a compact set and its interior \( \Theta_0 \) contains both \( \theta_0 \) and \( \theta_{0H} \) of (1.3) and (2.10), respectively. Moreover, (2.1), (2.3) and (2.5) hold and \( \{X_t\} \) is stationary and ergodic.

**Conditions on the score function:**

**Identifiability condition:** Corresponding to the score function \( H \), there exists a unique number \( c_H > 0 \) satisfying
\[ E[H(\epsilon/c_H^{1/2})] = 1. \quad (2.8) \]

**Moment conditions:**
\[ E[H(\epsilon/c_H^{1/2})]^2 < \infty \quad \text{and} \quad 0 < E\{(\epsilon/c_H^{1/2}) \hat{H}(\epsilon/c_H^{1/2})\} < \infty. \quad (2.9) \]

Also various **Smoothness conditions** on \( H \) as in Mukherjee (2008) are assumed which are satisfied in all examples of \( H \) considered above. Define the score function factor
\[ \sigma^2(H) := 4 \text{ Var}\{H(\epsilon/c_H^{1/2})\}/[E\{(\epsilon/c_H^{1/2}) \hat{H}(\epsilon/c_H^{1/2})\}]^2, \]
the matrix
\[ G := E\{\hat{v}_1(\theta_{0H})\hat{v}_1'(\theta_{0H})/v_1^2(\theta_{0H})\} \]
and the transformed parameter
\[ \theta_{0H} = (c_H\omega_0, c_H\alpha_{01}, \ldots, c_H\alpha_{0p}, \beta_{01}, \ldots, \beta_{0q})'. \quad (2.10) \]

**Theorem 2.1.** Suppose that the model assumptions, identifiability condition, moment conditions and smoothness conditions hold. Then
\[ n^{1/2}(\theta_n - \theta_{0H}) \rightarrow \mathcal{N}(0, \sigma^2(H)G^{-1}). \quad (2.11) \]

Note that \( c_H \) used in above formulas are given by (i) \( c_H = E(\epsilon^2) \) for the QMLE, (ii) \( c_H = (E|\epsilon|^2) \) for the LAD while for the Huber, \( \mu \)-estimator, Cauchy and other scores, \( c_H \) does not have closed-form expression. For such score functions, \( c_H \) is calculated using (2.8) as follows. We fix a large positive integer \( I \) and generate \( \{\epsilon_i; 1 \leq i \leq I\} \) from the error
Table 1: Values of $c_H$ for M-estimators (Huber, $\mu$-, Cauchy) under various error distributions.

|                      | Huber’s $c_H$ | $\mu$-estimator $c_H$ | Cauchy $c_H$ |
|----------------------|---------------|-------------------------|--------------|
| Normal               | 0.825         | 1.692                   | 0.377        |
| DE                   | 0.677         | 1.045                   | 0.207        |
| Logistic             | 0.781         | 1.487                   | 0.316        |
| $t(3)$               | 0.533         | 0.850                   | 0.172        |
| $t(2.2)$             | 0.204         | 0.274                   | 0.053        |

distribution considered for the simulation. Then, using the bisection method on $c > 0$, we solve the equation

$$(1/I) \sum_{i=1}^{l} \left\{ H\left(\epsilon_i/c^{1/2}\right) \right\} - 1 = 0.$$  

Values of $c_H$ computed in this way were provided in Mukherjee (2008, page 1541) for some error distributions and score functions. In Table 1 we provide $c_H$ for few more error distributions and score functions such as Huber’s $k$-score and $\mu$-estimator with $k = 1.5$ and $\mu = 3$ which are used in simulations and data analysis of later sections. In the sequel, Double exponential is abbreviated as DE.

3 Bootstrapping M-estimators

Let $\{w_{nt}; 1 \leq t \leq n, n \geq 1\}$ be a triangular array of r.v.’s such that for each $n \geq 1$, $\{w_{nt}; 1 \leq t \leq n\}$ are exchangeable and independent of the data $\{X_t; t \geq 1\}$ and errors $\{\epsilon_t; t \geq 1\}$. Also, $\forall t \geq 1$, $w_{nt} \geq 0$ and $E(w_{nt}) = 1$.

Based on these weights, bootstrap estimate $\hat{\theta}_{sn}$ is defined as the solution of $\hat{M}_{n,H}^*(\theta) = 0$, where

$$\hat{M}_{n,H}^*(\theta) := \sum_{t=1}^{n} w_{nt} \left\{ 1 - H\left\{X_t/\hat{\nu}_t^{1/2}(\theta)\right\} \right\} \hat{v}_t(\theta)/\hat{\nu}_t(\theta).$$  

(3.1)

Examples. From many different choices of bootstrap weights, we consider the following three schemes for comparison.

(i) Scheme M. The sequence of weights $\{w_{n1}, \ldots, w_{nn}\}$ has a multinomial $(n, 1/n, \ldots, 1/n)$ distribution, which is essentially the classical paired bootstrap.

(ii) Scheme E. When $w_{nt} = (nE_t)/\sum_{i=1}^{n} E_i$, where $\{E_t\}$ are i.i.d. exponential r.v. with mean 1. Under scheme E, $\hat{\theta}_{sn}$ is a weighted M-estimator with weights proportional to $E_t$, $1 \leq i \leq n$.

(iii) Scheme U. When $w_{nt} = (nU_t)/\sum_{i=1}^{n} U_i$, where $\{U_t\}$ are i.i.d. uniform r.v. on $(0.5, 1.5)$. Under scheme U, $\hat{\theta}_{sn}$ is a weighted M-estimator with weights proportional to $U_t$, $1 \leq i \leq n$.

A host of other bootstrap methods in the literature are special cases of the above bootstrap formulation. Such general formulation of weighted bootstrap offers a unified way of
studying several bootstrap schemes simultaneously. See, for example, Chatterjee and Bose (2005) for details in other contexts.

We assume that the weights satisfy the following basic conditions (Conditions BW of Chatterjee and Bose (2005)) where $\sigma_n^2 = \text{Var}(w_{n1})$ and $k_3 > 0$ is a constant.

\[
E(w_{n1}) = 1, \ 0 < k_3 < \sigma_n^2 = o(n) \text{ and } \text{Corr}(w_{n1}, w_{n2}) = O(1/n) \tag{3.2}
\]

Under (3.2) and some additional smoothness and moment conditions in Mukherjee (2020), weighted bootstrap is asymptotic valid.

**Theorem 3.1.** For almost all data, as $n \to \infty$,

\[
\sigma_n^{-1} n^{1/2}(\hat{\theta}_n - \tilde{\theta}_n) \to \mathcal{N}(0, \sigma^2(H)G^{-1}) \tag{3.3}
\]

We remark that since $0 < 1/\sigma_n < 1/\sqrt{k_3}$, the rate of convergence of the bootstrap estimator is the same as that of the original estimator. The standard deviation of the weights \{\sigma_n\} at the denominator of the scaling reflects the contribution of the corresponding weights.

The distributional result of (3.3) is useful for constructing the confidence interval of the GARCH parameters as follows. Let $B$ denote the number of bootstrap replicates, $\gamma_0$ denote a generic parameter (one of $\omega_0$, $\alpha_0i$ or $\beta_0j$) and let $\hat{\gamma}_n$ and $\hat{\gamma}_{smb}$ denote its M-estimator and $b$-th bootstrap estimator (1 $\leq b \leq B$), respectively. Let $\gamma_{0H}$ be one of $c_H\omega_0$, $c_H\alpha_{0i}$ or $\beta_{0j}$, as appropriate, which has a known value for a simulation experiment. Using the approximation of $\sqrt{n}(\hat{\gamma}_n - \gamma_{0H})$ by $\sigma_n^{-1} n^{1/2}(\hat{\gamma}_{smb} - \hat{\gamma}_n)$, the bootstrap confidence interval of $\gamma_{0H}$ is of the form

\[
[\hat{\gamma}_n - n^{-1/2}\{\sigma_n^{-1} n^{1/2}(\hat{\gamma}_{smb,\alpha/2} - \hat{\gamma}_n)\}, \hat{\gamma}_n + n^{-1/2}\{\sigma_n^{-1} n^{1/2}(\hat{\gamma}_{smb,1-\alpha/2} - \hat{\gamma}_n)\}] \tag{3.4}
\]

where $\hat{\gamma}_{smb,\alpha/2}$ is the $\alpha/2$-th quantile of the numbers \{\hat{\gamma}_{smb}, 1 $\leq b \leq B$\}. Consequently, the bootstrap coverage probability is computed by the proportion of the above set of $B$ confidence intervals containing $\gamma_{0H}$.

Similarly, using (2.11) of Theorem 3.1, we can obtain the confidence interval of $\gamma_{0H}$ based on the asymptotic normality of $\hat{\gamma}_n$, and this will be called the normal confidence interval. Specifically, in view of Proposition 3.1 of Mukherjee (2008) on the estimation of the variance-covariance matrix $\sigma^2(H)G^{-1}$, we can obtain the asymptotic confidence interval of $\gamma_{0H}$ as

\[
[\hat{\gamma}_n - n^{-1/2}\hat{d}z_{1-\alpha/2}, \hat{\gamma}_n + n^{-1/2}\hat{d}z_{1-\alpha/2}] \tag{3.5}
\]

where $(\hat{d})^2$ is the estimated variance of $\hat{\gamma}_n$ obtained from the appropriate diagonal entry of the estimator of $\sigma^2(H)G^{-1}$ and $z_{1-\alpha/2}$ is the $1 - \alpha/2$-th quantile of the standard normal distribution.

In the following Section 4, we will compare the accuracy of the confidence intervals constructed by the bootstrap and asymptotic approximations.
4 Algorithm

We discuss the implementation of an iteratively re-weighted algorithm proposed in Mukherjee (2020) for computing M-estimates. In particular, we highlight $\mu$-estimate and Cauchy-estimate of the GARCH parameters in this paper as their asymptotic distributions are derived under mild moment assumptions. We also consider the bootstrap estimators based on the corresponding score functions.

4.1 Computation of M-estimates

For the convenience of writing, let $\alpha(c) = E[H(\epsilon)]$ for $c > 0$. Using a Taylor expansion of $\hat{M}_{n,H}$, we obtain the following recursive equation for computing the updated estimate $\hat{\theta}$ of $\hat{\theta}_n$ from the current estimate $\theta$ of $\hat{M}_{n,H}(\theta) = 0$:

$$\hat{\theta} = \theta + \{\hat{\alpha}(1)/2\}^{-1} \sum_{t=1}^{n} \left(\hat{v}_t(\theta)\hat{v}_t'(\theta)/\hat{v}_t^2(\theta)\right) \sum_{t=1}^{n} \left(\{X_t/\hat{v}_t^{1/2}(\theta)\} - 1\right) \left\{\hat{v}_t(\theta)/\hat{v}_t(\theta)\right\},$$

(4.1)

where $\hat{\alpha}(1) = E[\epsilon H'(\epsilon)]$ under smoothness conditions on $H$. Since the GARCH residuals $\{X_t/\hat{v}_t^{1/2}(\theta_n)\}$ estimate only $\{\epsilon_t/c_H^{1/2}\}$, in general, we cannot estimate $\hat{\alpha}(1)$ from the data. Therefore, we use ad hoc techniques such as simulating $\{\tilde{\epsilon}_t; 1 \leq t \leq n\}$ from $N(0,1)$ or standardized DE distribution and then use $n^{-1}\sum_{t=1}^{n} \tilde{\epsilon} H'(\tilde{\epsilon})$ to carry out the iteration. Note that if the iteration in (4.1) converges then $\hat{\theta} \approx \theta$. Therefore in this case from (4.1), $\hat{M}_{n,H}(\theta) \approx 0$ and hence $\hat{\theta}$ is the desired $\hat{\theta}_n$. Based on our extensive simulation study and real data analysis, the algorithm is robust enough to converge to the same value of $\hat{\theta}_n$ irrespective of different values of the unknown factor $\hat{\alpha}(1)$ used in computation.

In the following examples, we discuss (4.1) when specialized to the M-estimators computed in this paper.

QMLE: Here $H(x) = x^2$ and $\alpha(c) = c^2 E(\epsilon^2)$. Hence $\hat{\alpha}(1)/2 = E(\epsilon^2)$ and

$$\hat{\theta} = \theta + \left\{E(\epsilon^2)\right\}^{-1} \sum_{t=1}^{n} \left(\hat{v}_t(\theta)\hat{v}_t'(\theta)/\hat{v}_t^2(\theta)\right) \sum_{t=1}^{n} \left(\{X_t/\hat{v}_t^{1/2}(\theta)\} - 1\right) \left\{\hat{v}_t(\theta)/\hat{v}_t(\theta)\right\}. $$

With

$$W_t = 1/\hat{v}_t^2(\theta), x_t = \hat{v}_t(\theta), y_t = X_t^2 - \hat{v}_t(\theta),$$

$\hat{\theta}$ can be computed iteratively as

$$\hat{\theta}_{(r+1)} = \hat{\theta}_{(r)} + \left\{E(\epsilon^2)\right\}^{-1} \left\{\sum_t W_t x_t x_t'\right\}^{-1} \left\{\sum_t W_t x_t y_t\right\}. $$

Note that when $E(\epsilon^2) = 1$, this is same as the formula obtained through the BHHH algorithm proposed by Berndt et al. (1974).
\[ \dot{\theta} = \theta + \{2/\varepsilon|\varepsilon|\} \left[ \sum_{t=1}^{n} \left\{ \hat{v}_t(\theta) \hat{v}_t(\theta)' / \hat{v}_t^2(\theta) \right\} \right]^{-1} \sum_{t=1}^{n} \left[ |X_t| / \hat{v}_t^{1/2}(\theta) - 1 \right] \{ \hat{v}_t(\theta) / \hat{v}_t(\theta) \} = \theta + \{2/\varepsilon|\varepsilon|\} \left[ \sum_{t=1}^{n} \left\{ \hat{v}_t(\theta) \hat{v}_t(\theta)' / \hat{v}_t^2(\theta) \right\} \right]^{-1} \sum_{t=1}^{n} \left[ |X_t| - \hat{v}_t^{1/2}(\theta) \right] \{ \hat{v}_t(\theta) / \hat{v}_t^3(\theta) \} = \theta + \{2/\varepsilon|\varepsilon|\} \left[ \sum_{t=1}^{n} \left\{ \hat{v}_t(\theta) \hat{v}_t(\theta)' / \hat{v}_t^2(\theta) \right\} \right]^{-1} \sum_{t=1}^{n} \left\{ \hat{v}_t^{1/2}(\theta) (|X_t| - \hat{v}_t^{1/2}(\theta)) \right\} \{ \hat{v}_t(\theta) / \hat{v}_t^2(\theta) \}.
\]

With

\[ W_t = 1 / \hat{v}_t^2(\theta), x_t = \hat{v}_t(\theta), y_t = \hat{v}_t^{1/2}(\theta) (|X_t| - \hat{v}_t^{1/2}(\theta)), \]

\( \hat{\theta} \) can be computed iteratively as

\[ \hat{\theta}_{(r+1)} = \hat{\theta}_{(r)} + \{2/\varepsilon|\varepsilon|\} \left\{ \sum_{t} W_t x_t x_t' \right\}^{-1} \left\{ \sum_{t} W_t x_t y_t \right\} \]

**Huber:** Here \( H(x) = x^2 I(|x| \leq k) + k|x| I(|x| > k) \) and

\[ \alpha(c) = E [(\varepsilon c)^2 I(|\varepsilon| \leq k) + k|\varepsilon| I(|\varepsilon| > k)] \]

Hence

\[ \dot{\alpha}(1) = E [2\varepsilon^2 I(|\varepsilon| \leq k) + k|\varepsilon| I(|\varepsilon| > k)] \]

and

\[ \dot{\theta} = \theta - \left\{ \dot{\alpha}(1)/2 \right\} \left[ \sum_{t=1}^{n} \left\{ \hat{v}_t(\theta) \hat{v}_t(\theta)' / \hat{v}_t^2(\theta) \right\} \right]^{-1} \sum_{t=1}^{n} \left[ 1 - X_t^2 \hat{v}_t(\theta) I \left( \frac{|X_t|}{\hat{v}_t^{1/2}(\theta)} \leq k \right) - k \frac{|X_t|}{\hat{v}_t^{1/2}(\theta)} I \left( \frac{|X_t|}{\hat{v}_t^{1/2}(\theta)} > k \right) \right] \{ \hat{v}_t(\theta) / \hat{v}_t(\theta) \} \]

With

\[ W_t = 1 / \hat{v}_t^2(\theta), x_t = \hat{v}_t(\theta) \]

and

\[ y_t = X_t^2 I \left( |X_t| / \hat{v}_t^{1/2}(\theta) \leq k \right) + k |X_t| \hat{v}_t^{1/2}(\theta) I \left( |X_t| / \hat{v}_t^{1/2}(\theta) > k \right) - \hat{v}_t(\theta) \]

\( \hat{\theta} \) can be computed iteratively as

\[ \hat{\theta}_{(r+1)} = \hat{\theta}_{(r)} + \left\{ \dot{\alpha}(1)/2 \right\} \left\{ \sum_{t} W_t x_t x_t' \right\}^{-1} \left\{ \sum_{t} W_t x_t y_t \right\} \]

**\( \mu \)-estimator:** Here \( H(x) = \mu x / (1 + |x|) \) and \( \alpha(c) = \mu - \mu E [1/(1 + |\varepsilon|)] \). Hence

\[ \dot{\alpha}(1) = \mu E [\varepsilon/(1 + |\varepsilon|)^2] \]
and
\[
\hat{\theta} = \theta + \left\{ \frac{\mu E}{2} \left[ \frac{|\epsilon|}{(1 + |\epsilon|)^2} \right] \right\}^{-1} \left[ \sum_{t=1}^{n} \left\{ \frac{\hat{v}_t(\theta)\hat{v}_t(\theta)'}{\hat{v}_t^2(\theta)} \right\} \right]^{-1} \sum_{t=1}^{n} \left[ \frac{\mu |X_t|}{\hat{v}_t^{3/2}(\theta) + |X_t|} - 1 \right] \left\{ \frac{\hat{v}_t(\theta)}{\hat{v}_t(\theta)} \right\}. 
\]

With
\[
W_t = 1/\hat{v}_t^2(\theta), \quad x_t = \hat{v}_t(\theta), \quad y_t = \frac{\mu |X_t|\hat{v}_t(\theta)}{\hat{v}_t^{3/2}(\theta) + |X_t|} - \hat{v}_t(\theta),
\]
\(\hat{\theta}\) can be computed iteratively as
\[
\hat{\theta}_{(r+1)} = \hat{\theta}_{(r)} + \left\{ \frac{\mu E}{2} \left[ \frac{|\epsilon|}{(1 + |\epsilon|)^2} \right] \right\}^{-1} \left[ \sum_{t} W_t x_t x_t' \right]^{-1} \left[ \sum_{t} W_t x_t y_t \right].
\]

**Cauchy-estimator:** Here \(H(x) = 2x^2/(1 + x^2)\) and \(\alpha(c) = E[2c^2/(1 + c^2)^2]\). Hence
\[
\hat{\alpha}(1) = E \left[ 4c^2/(1 + c^2)^2 \right]
\]
and
\[
\hat{\theta} = \theta - \left\{ 2E \left[ \frac{c^2}{(1 + c^2)^2} \right] \right\}^{-1} \left[ \sum_{t=1}^{n} \left\{ \frac{\hat{v}_t(\theta)\hat{v}_t(\theta)'}{\hat{v}_t^2(\theta)} \right\} \right]^{-1} \sum_{t=1}^{n} \left[ 1 - \frac{2X_t^2}{\hat{v}_t(\theta) + X_t^2} \right] \left\{ \frac{\hat{v}_t(\theta)}{\hat{v}_t(\theta)} \right\}.
\]

With
\[
W_t = 1/\hat{v}_t^2(\theta), \quad x_t = \hat{v}_t(\theta), \quad y_t = \frac{2X_t^2\hat{v}_t(\theta)}{\hat{v}_t(\theta) + X_t^2} - \hat{v}_t(\theta),
\]
\(\hat{\theta}\) can be computed iteratively as
\[
\hat{\theta}_{(r+1)} = \hat{\theta}_{(r)} + \left\{ 2E \left[ \frac{c^2}{(1 + c^2)^2} \right] \right\}^{-1} \left[ \sum_{t} W_t x_t x_t' \right]^{-1} \left[ \sum_{t} W_t x_t y_t \right].
\]

### 4.2 Computation of bootstrap M-estimates

Here the relevant function is \(\tilde{M}_{n,M}(\theta)\) defined in (3.1) and the bootstrap estimate \(\hat{\theta}_*\) can be computed using the updating equation
\[
\hat{\theta}_* = \theta - \left\{ 2/\hat{\alpha}(1) \right\} \left[ \sum_{t=1}^{n} w_{nt} \left\{ \frac{\hat{v}_t(\theta)\hat{v}_t(\theta)'/\hat{v}_t^2(\theta)}{H(X_t/\hat{v}_t^{1/2}(\theta))} \right\} \right]^{-1} \\
\times \sum_{t=1}^{n} w_{nt} \left\{ 1 - H(X_t/\hat{v}_t^{1/2}(\theta)) \right\} \left\{ \hat{v}_t(\theta)/\hat{v}_t(\theta) \right\}. \tag{4.2}
\]

Notice also that weighted bootstrap is particularly computation-friendly and is easy to program in R. In particular, one can store
\[
\left\{ 1 - H(X_t/\hat{v}_t^{1/2}(\theta)) \right\} \left\{ \hat{v}_t(\theta)/\hat{v}_t(\theta) \right\}
\]
while computing M-estimates once and for all. After that, one simply needs to generate weights and compute the weighted sum while solving the above equation through iteration. Each time, the initial bootstrap estimator is taken to be the M-estimator \(\hat{\theta}_n\).
5 Simulating the distributions of M-estimators

To compare performance of various M-estimators via bias and MSE, we simulate \( n \) observations from GARCH \((p, q)\) models with specific choice of parameters and error distributions and compute M-estimates based on various score functions. This procedure is replicated \( R \)-times to enable the estimation of bias and MSE. For illustration with \( p = 1 = q \), let \( \hat{\theta}_n = (\hat{\omega}_r, \hat{\alpha}_r, \hat{\beta}_r)' \) be the M-estimator of \( \theta_0 = (\omega_0, \alpha_0, \beta_0)' \) based on a specified score function \( H \) at the \( r \)-th replication, \( 1 \leq r \leq R \). Notice that \( (\hat{\omega}_r, \hat{\alpha}_r, \hat{\beta}_r) \) estimates \( (c_H\omega_0, c_H\alpha_0, \beta_0) \), where \( c_H \) depends on both the score function and the underlying error distribution but is known in a simulation scenario. Therefore, to compare the performance for a specified error distribution across various score functions, we consider \( R \) replicates of \( (\hat{\omega}_r/c_H - \omega_0, \hat{\alpha}_r/c_H - \alpha_0, \hat{\beta}_r - \beta_0)' \) and use the following vectors to estimate the standardized bias and the standardized MSE:

\[
\begin{align*}
(R^{-1} \sum_{r=1}^{R} \{\hat{\omega}_r/c_H - \omega_0\}, R^{-1} \sum_{r=1}^{R} \{\hat{\alpha}_r/c_H - \alpha_0\}, R^{-1} \sum_{r=1}^{R} \{\hat{\beta}_r - \beta_0\})' \\
(R^{-1} \sum_{r=1}^{R} \{\hat{\omega}_r/c_H - \omega_0\}^2, R^{-1} \sum_{r=1}^{R} \{\hat{\alpha}_r/c_H - \alpha_0\}^2, R^{-1} \sum_{r=1}^{R} \{\hat{\beta}_r - \beta_0\}^2)'
\end{align*}
\] (5.1)

In Tables 2 and 3, we report the standardized bias and MSE of Huber’s and \( \mu \)-estimator to guide our choice of the corresponding tuning parameters \( k \) and \( \mu \). The underlying data generating process (DGP) is the GARCH (1, 1) model with \( \theta_0 = (1.65 \times 10^{-5}, 0.0701, 0.901)' \) under three types of innovation distributions: the normal, DE and logistic distribution. The above value of the true parameter is motivated from the estimated parameter of the GARCH (1, 1) model for the Shanghai Stock Exchange (SSE) Index data which will be analyzed later in this paper. We use (5.1) for the computation with sample size \( n = 1000 \) and \( R = 150 \) replications.

The simulation results in Table 2 and Table 3 show that the bias and MSE of Huber’s \( k \)-estimator and \( \mu \)-estimator do not vary widely for various values of \( k \) and \( \mu \). Therefore \( k = 1.5 \) and \( \mu = 3 \) are chosen for subsequent computations. Notice also that the minimum bias and MSE correspond to \( \mu = 3 \) in a number of cases.

M-estimators corresponding to different score functions for the GARCH (1, 1) models have been compared under various error distributions via simulation study in Iqbal and Mukherjee (2010). Below we focus on comparing M-estimators with the underlying DGP being GARCH (2, 1) models. We also evaluate the performance of M-estimators when the underlying DGP is the GARCH (1, 1) model but it is misspecified as the GARCH (2, 1) model. This is essentially the case where the parameter is at the boundary.

5.1 Simulation for GARCH (2, 1) models

We consider four types of innovation distributions: the normal, DE, logistic and Student’s \( t \)-distributions with 3 and 2.2 degrees of freedom (denoted by \( t(3) \) and \( t(2.2) \)). There are
Table 2: The standardized bias and MSE of the Huber’s estimator (with different $k$ values being used) under various error distributions (sample size $n = 1000$; $R = 150$ replications).

|       | Standardized bias | Standardized MSE |
|-------|-------------------|------------------|
|       | $\omega$   | $\alpha$   | $\beta$   | $\omega$   | $\alpha$   | $\beta$   |
| Normal|           |           |           |           |           |           |
| $k=1$ | 1.03$\times 10^{-5}$ | -2.44$\times 10^{-3}$ | -1.96$\times 10^{-2}$ | 2.62$\times 10^{-10}$ | 4.20$\times 10^{-4}$ | 1.54$\times 10^{-3}$ |
| $k=1.5$ | 1.22$\times 10^{-5}$ | 2.47$\times 10^{-3}$ | -1.98$\times 10^{-2}$ | 3.33$\times 10^{-10}$ | 4.55$\times 10^{-4}$ | 1.58$\times 10^{-3}$ |
| $k=2.5$ | 1.14$\times 10^{-5}$ | -4.33$\times 10^{-4}$ | -2.02$\times 10^{-2}$ | 3.10$\times 10^{-10}$ | 3.71$\times 10^{-4}$ | 1.58$\times 10^{-3}$ |
| DE    |           |           |           |           |           |           |
| $k=1$ | 7.24$\times 10^{-6}$ | 1.29$\times 10^{-3}$ | -1.57$\times 10^{-2}$ | 1.87$\times 10^{-10}$ | 4.65$\times 10^{-4}$ | 1.58$\times 10^{-3}$ |
| $k=1.5$ | 7.32$\times 10^{-6}$ | 1.67$\times 10^{-3}$ | -1.63$\times 10^{-2}$ | 2.00$\times 10^{-10}$ | 4.82$\times 10^{-4}$ | 1.68$\times 10^{-3}$ |
| $k=2.5$ | 8.27$\times 10^{-6}$ | 2.94$\times 10^{-3}$ | -1.92$\times 10^{-2}$ | 2.79$\times 10^{-10}$ | 5.60$\times 10^{-4}$ | 2.22$\times 10^{-3}$ |
| Logistic|           |           |           |           |           |           |
| $k=1$ | 9.87$\times 10^{-6}$ | 2.15$\times 10^{-3}$ | -2.03$\times 10^{-2}$ | 3.18$\times 10^{-10}$ | 5.25$\times 10^{-4}$ | 2.28$\times 10^{-3}$ |
| $k=1.5$ | 1.00$\times 10^{-5}$ | 2.04$\times 10^{-3}$ | -2.04$\times 10^{-2}$ | 3.11$\times 10^{-10}$ | 4.89$\times 10^{-4}$ | 2.22$\times 10^{-3}$ |
| $k=2.5$ | 1.06$\times 10^{-5}$ | 2.18$\times 10^{-3}$ | -2.16$\times 10^{-2}$ | 3.18$\times 10^{-10}$ | 4.84$\times 10^{-4}$ | 2.17$\times 10^{-3}$ |

Table 3: The standardized bias and MSE of $\mu$-estimator (with different $\mu$ values being used) under various error distributions (sample size $n = 1000$; $R = 150$ replications).

|       | Standardized bias | Standardized MSE |
|-------|-------------------|------------------|
|       | $\omega$   | $\alpha$   | $\beta$   | $\omega$   | $\alpha$   | $\beta$   |
| Normal|           |           |           |           |           |           |
| $\mu=2$ | 1.17$\times 10^{-5}$ | 2.97$\times 10^{-3}$ | -2.13$\times 10^{-2}$ | 4.05$\times 10^{-10}$ | 6.73$\times 10^{-4}$ | 2.16$\times 10^{-3}$ |
| $\mu=2.5$  | 1.14$\times 10^{-5}$ | 1.80$\times 10^{-3}$ | -2.12$\times 10^{-2}$ | 3.77$\times 10^{-10}$ | 5.71$\times 10^{-4}$ | 2.04$\times 10^{-3}$ |
| $\mu=3$  | 1.14$\times 10^{-5}$ | 1.36$\times 10^{-3}$ | -2.11$\times 10^{-2}$ | 3.68$\times 10^{-10}$ | 5.21$\times 10^{-4}$ | 1.97$\times 10^{-3}$ |
| DE     |           |           |           |           |           |           |
| $\mu=2$ | 7.39$\times 10^{-6}$ | 2.23$\times 10^{-3}$ | -1.49$\times 10^{-2}$ | 2.74$\times 10^{-10}$ | 7.20$\times 10^{-4}$ | 2.21$\times 10^{-3}$ |
| $\mu=2.5$  | 7.36$\times 10^{-6}$ | 1.50$\times 10^{-3}$ | -1.52$\times 10^{-2}$ | 2.68$\times 10^{-10}$ | 6.56$\times 10^{-4}$ | 2.16$\times 10^{-3}$ |
| $\mu=3$  | 7.40$\times 10^{-6}$ | 1.25$\times 10^{-3}$ | -1.53$\times 10^{-2}$ | 2.62$\times 10^{-10}$ | 6.17$\times 10^{-4}$ | 2.09$\times 10^{-3}$ |
| Logistic|           |           |           |           |           |           |
| $\mu=2$ | 7.73$\times 10^{-6}$ | 2.22$\times 10^{-3}$ | -1.37$\times 10^{-2}$ | 2.45$\times 10^{-10}$ | 6.79$\times 10^{-4}$ | 1.99$\times 10^{-3}$ |
| $\mu=2.5$  | 7.66$\times 10^{-6}$ | 9.77$\times 10^{-4}$ | -1.41$\times 10^{-2}$ | 2.48$\times 10^{-10}$ | 5.88$\times 10^{-4}$ | 1.97$\times 10^{-3}$ |
| $\mu=3$  | 7.72$\times 10^{-6}$ | 5.99$\times 10^{-4}$ | -1.42$\times 10^{-2}$ | 2.54$\times 10^{-10}$ | 5.44$\times 10^{-4}$ | 1.94$\times 10^{-3}$ |
Cauchy-estimators are relatively less sensitive to heavy-tails among these M-estimators. LAD and Huber’s estimators perform poorly compared with the $t$ and Cauchy-estimators (2) and Cauchy-estimators (3).

$\theta_0 = (4.46 \times 10^{-6}, 0.0525, 0.108, 0.832)$

...a choice motivated by the QMLE computed using the R package fGarch for the FTSE 100 data which will be analyzed later.

The standardized bias and MSE of the various M-estimators are reported in Table 4. It is worth noting that under the normal distribution, the bias and MSE of other M-estimators are generally close to those of the QMLE. However, for more heavy-tailed distributions, the QMLE produces larger bias and MSE compared with other M-estimators. Under the $t(3)$ and $t(2.2)$ distributions, which do not admit finite fourth moment, the advantage of the M-estimators over the QMLE becomes more prominent. Also, under the $t(2.2)$ distribution, the LAD and Huber’s estimators perform poorly compared with the $\mu$- and Cauchy-estimators since the former two yield significantly larger MSE than the latter two. Consequently, these provide evidence for (i) the robustness of the M-estimators for heavy-tailed distributions is not at the cost of losing much efficiency under the normal distribution and (ii) the $\mu$- and Cauchy-estimators are relatively less sensitive to heavy-tails among these M-estimators.

Table 4: The standardized bias and MSE of the M-estimators for GARCH (2, 1) models under various error distributions (sample size $n = 1000$; $R = 1000$ replications).

|               | Normal       | Logistic     | $t(3)$        | $t(2.2)$      |
|---------------|--------------|--------------|---------------|---------------|
| $\omega$      | 3.55 x 10^{-6} | 3.83 x 10^{-6} | 1.67 x 10^{-6} | 4.35 x 10^{-7} |
| $a_1$         | 1.88 x 10^{-3} | 1.38 x 10^{-3} | 2.89 x 10^{-2} | 9.90 x 10^{-2} |
| $a_2$         | 3.05 x 10^{-3} | 1.43 x 10^{-3} | 6.13 x 10^{-3} | 4.39 x 10^{-2} |
| $\beta$       | -2.02 x 10^{-2} | -1.73 x 10^{-2} | -1.63 x 10^{-2} | -1.54 x 10^{-1} |
| $\omega$      | 2.18 x 10^{-11} | 2.64 x 10^{-11} | 2.74 x 10^{-11} | 1.90 x 10^{-11} |
| $a_1$         | 1.53 x 10^{-3}  | 3.78 x 10^{-3}  | 1.37 x 10^{-2}  | 1.34 x 10^{-1}  |
| $a_2$         | 2.08 x 10^{-3}  | 3.01 x 10^{-3}  | 1.56 x 10^{-2}  | 1.48 x 10^{-1}  |
| $\beta$       | 1.36 x 10^{-3}  | 1.57 x 10^{-3}  | 8.02 x 10^{-3}  | 8.10 x 10^{-2}  |

The standardized bias and MSE of the M-estimators for GARCH (2, 1) models under various error distributions (sample size $n = 1000$; $R = 1000$ replications).
5.2 Simulation under a misspecified GARCH model

It is important to check whether the M-estimators are consistent when a GARCH model is misspecified with a higher order as over-fitting can occur in practice. In this case, we are essentially fitting a GARCH model with some component(s) of the parameter at the boundary equal to zero. We simulate below data from the GARCH (1, 1) model under various error distributions; however, the data are fitted by the GARCH (2, 1) model. In simulation, we use $R = 1000$, $n = 1000$ and $\theta_0 = (1.65 \times 10^{-5}, 0.0701, 0.901)\beta$, which is motivated by the QMLE obtained by using the fGarch package for the SSE data analyzed later in Section 7.

The standardized bias and MSE of the M-estimators are shown in Table 5. For all distributions considered, the bias are close to zero and the MSE are small indicating good performance of the M-estimators under this type of mis-specification. Similar to the results in Table 4, the QMLE is sensitive to the heavy-tailed distributions while other M-estimators are more robust.

![Table 5: The standardized bias and MSE of the M-estimators under the misspecified model (sample size $n = 1000$; $R = 1000$ replications); the underlying DGP is the GARCH (1, 1) model whereas the model is misspecified as a GARCH (2, 1).](image)

5.3 Simulation for the GARCH (1, 2) models

Since we did not come across a real data that can be fitted by the GARCH (1, 2) model, we resort to simulation results to study the performance of M-estimators for such models. We choose $\theta_0 = (0.1, 0.1, 0.2, 0.6)\beta$, $R = 1000$ replications and $n = 1000$. The standardized...
bias and MSE of the M-estimators under various error distributions are reported in Table 6. We do not report results for the QMLE when data are generated under the t(3) and t(2.2) error distributions since the algorithm for computing the QMLE did not converge for most replications. Under the normal error distribution, the LAD and Huber’s estimators produce MSE that is close to the QMLE while the µ- and Cauchy-estimators yield larger MSE corresponding for estimating ω and α. For the DE and logistic distributions, there is no significant difference between these estimators. Their difference becomes clearer under heavy-tailed distributions: the µ- and Cauchy-estimators produce smaller MSE of ω under the t(3) distribution and smaller MSE of α under the t(2.2) distribution than the LAD and Huber’s estimators.

Table 6: The standardized bias and MSE of the M-estimators for GARCH (1, 2) models under various error distributions (sample size \( n = 1000 \); \( R = 1000 \) replications).

| Error Distribution | Standardized bias | | | | Standardized MSE | | | |
|--------------------|------------------|---|---|---|-----------------|---|---|---|
|                   | \( \omega \)     | \( \alpha \) | \( \beta_1 \) | \( \beta_2 \) | \( \omega \)     | \( \alpha \) | \( \beta_1 \) | \( \beta_2 \) |
| Normal             | 5.53×10^{-2}     | 1.10×10^{-3} | 9.65×10^{-2} | -1.52×10^{-1} | 2.66×10^{-2} | 1.17×10^{-3} | 1.45×10^{-1} | 1.38×10^{-1} |
| QMLE               |                  |              |              |              | 3.21×10^{-2} | 1.31×10^{-3} | 1.55×10^{-1} | 1.45×10^{-1} |
| LAD                |                  |              |              |              | 3.72×10^{-2} | 1.37×10^{-3} | 1.56×10^{-1} | 1.47×10^{-1} |
| Huber              |                  |              |              |              | 7.41×10^{-2} | 1.84×10^{-3} | 2.16×10^{-1} | 2.01×10^{-1} |
| \( \mu \)-estimator |                  |              |              |              | 6.30×10^{-2} | 2.17×10^{-3} | 2.43×10^{-1} | 2.31×10^{-1} |
| Cauchy             | 7.51×10^{-2}     | 1.25×10^{-3} | 1.29×10^{-1} | -2.06×10^{-1} |                  |              |              |              |
| DE                 | 5.48×10^{-2}     | 2.93×10^{-3} | 1.01×10^{-1} | -1.63×10^{-1} | 3.15×10^{-2} | 1.79×10^{-3} | 1.62×10^{-1} | 1.57×10^{-1} |
| QMLE               |                  |              |              |              | 3.73×10^{-2} | 1.37×10^{-3} | 1.46×10^{-1} | 1.35×10^{-1} |
| LAD                | 4.05×10^{-2}     | 1.15×10^{-3} | 1.13×10^{-1} | -1.52×10^{-1} | 1.72×10^{-2} | 2.05×10^{-3} | 1.73×10^{-1} | 1.60×10^{-1} |
| Huber              | 4.74×10^{-2}     | 3.26×10^{-3} | 1.18×10^{-1} | -1.66×10^{-1} | 2.55×10^{-2} | 2.48×10^{-3} | 1.85×10^{-1} | 1.72×10^{-1} |
| \( \mu \)-estimator |                  |              |              |              |                  |              |              |              |
| Logistic            |                  |              |              |              |                  |              |              |              |
| QMLE               | 5.77×10^{-2}     | 2.76×10^{-3} | 1.06×10^{-1} | -1.61×10^{-1} | 3.02×10^{-2} | 1.49×10^{-3} | 1.67×10^{-1} | 1.59×10^{-1} |
| LAD                | 4.50×10^{-2}     | -5.78×10^{-3} | 7.27×10^{-2} | -1.18×10^{-1} | 1.58×10^{-2} | 1.37×10^{-3} | 1.30×10^{-1} | 1.18×10^{-1} |
| Huber              | 4.50×10^{-2}     | -2.33×10^{-3} | 8.85×10^{-2} | -1.34×10^{-1} | 1.58×10^{-2} | 1.36×10^{-3} | 1.53×10^{-1} | 1.39×10^{-1} |
| \( \mu \)-estimator | 4.52×10^{-2}     | 1.32×10^{-3} | 9.39×10^{-2} | -1.40×10^{-1} | 1.80×10^{-2} | 1.72×10^{-3} | 1.58×10^{-1} | 1.44×10^{-1} |
| Cauchy             | 5.15×10^{-2}     | 2.91×10^{-3} | 1.05×10^{-1} | -1.57×10^{-1} | 2.98×10^{-2} | 2.08×10^{-3} | 1.85×10^{-1} | 1.70×10^{-1} |
| t(3)               |                  |              |              |              |                  |              |              |              |
| QMLE               |                  |              |              |              |                  |              |              |              |
| LAD                | 2.93×10^{-2}     | 2.43×10^{-3} | 1.08×10^{-1} | -1.40×10^{-1} | 1.13×10^{-2} | 2.49×10^{-3} | 1.82×10^{-1} | 1.59×10^{-1} |
| Huber              | 2.87×10^{-2}     | 1.50×10^{-3} | 9.13×10^{-2} | -1.26×10^{-1} | 1.18×10^{-2} | 2.30×10^{-3} | 1.60×10^{-1} | 1.40×10^{-1} |
| \( \mu \)-estimator |                  |              |              |              | 5.59×10^{-3} | 1.88×10^{-3} | 1.65×10^{-1} | 1.42×10^{-1} |
| Cauchy             | 1.50×10^{-2}     | 6.44×10^{-4} | 1.38×10^{-1} | -1.54×10^{-1} | 6.50×10^{-3} | 2.15×10^{-3} | 1.90×10^{-1} | 1.65×10^{-1} |
| t(2.2)             |                  |              |              |              |                  |              |              |              |
| QMLE               |                  |              |              |              |                  |              |              |              |
| LAD                | 3.53×10^{-2}     | 2.57×10^{-2} | 1.24×10^{-1} | -1.85×10^{-1} | 1.30×10^{-2} | 1.41×10^{-2} | 2.41×10^{-1} | 2.21×10^{-1} |
| Huber              | 4.86×10^{-2}     | 3.99×10^{-2} | 7.81×10^{-2} | -1.66×10^{-1} | 1.44×10^{-2} | 1.63×10^{-2} | 1.81×10^{-1} | 1.79×10^{-1} |
| \( \mu \)-estimator | 1.72×10^{-2}     | 5.18×10^{-3} | 1.51×10^{-1} | -1.78×10^{-1} | 1.73×10^{-2} | 4.27×10^{-3} | 2.42×10^{-1} | 2.12×10^{-1} |
| Cauchy             | 2.15×10^{-2}     | 9.68×10^{-3} | 1.50×10^{-1} | -1.85×10^{-1} | 2.05×10^{-2} | 4.90×10^{-3} | 2.34×10^{-1} | 2.14×10^{-1} |

6 Simulating the bootstrap distributions

To evaluate the finite sample performance of the bootstrap approximation, here we compare the bootstrap coverage rates with the nominal levels. In particular, we generate \( R = 500 \) data of sample size \( n = 1000 \) from the GARCH (1, 1) model with parameter \( \theta_0 = (0.1, 0.1, 0.8) \) under both the normal and t(3) error distributions. For each data, we compute \( B = 2000 \)
bootstrap estimates using the bootstrap schemes M, E and U introduced in Section 3, and construct the bootstrap and asymptotic confidence intervals (CI) using (3.4) and (3.5), respectively. The coverage rates are computed as the proportions of the CIs that cover the true parameter. We report the coverage rates (in percentage) for the 90% and 95% nominal levels in Table 7.

Under the normal distribution, the coverage rates of the bootstrap approximation are generally close to the nominal levels. Also, the bootstrap approximation works better for the QMLE, LAD and Huber’s estimators than the \( \mu \)-estimator and Cauchy-estimator. However, under the \( t(3) \) distribution, the bootstrap approximation works poorly for the QMLE while the coverage rates are reasonably well for other M-estimators. For both distributions, scheme U outperforms schemes M and E. In terms of the asymptotic approximation, it works well only for few cases and is outperformed by the bootstrap coverage rates for most cases and this indicates the usefulness of the bootstrap approximation.

7 Real data analysis

In this section, we analyse daily log-returns of three financial time series: (i) the Shanghai Stock Exchange (SSE) Index from January 2007 to December 2009 with \( n = 752 \); (ii) the Electric Fuel Corporation (EFCX) data from January 2000 to December 2001 with \( n = 498 \); (iii) the FTSE 100 Index data from January 2007 to December 2009 with \( n = 783 \). Based on exploratory data analysis, GARCH (1, 1) model fits well to the SSE and EFCX data. However, we fitted GARCH (2, 1) model to the FTSE 100 data for two reasons. First, when fitted by the GARCH (2, 1) model with \texttt{fGarch} package in R, \( \alpha_2 \) is significant with p-value equal to 0.019; second, the Akaike information criterion (AIC) of the GARCH (2, 1) model is smaller than that of the GARCH (1, 1) model.

7.1 The SSE data and bootstrap estimates of the bias and MSE

Table 8 displays the QMLE computed using the R package \texttt{fGarch} and the QMLE and LAD estimates computed using the algorithm (4.1). The QMLEs given by \texttt{fGarch} and (4.1) are close. Also, the QMLE and LAD estimates of \( \beta \) are close.

To estimate the bias and MSE of the M-estimators of the GARCH (1, 1) parameters of the underlying DGP of the SSE data, notice that we know neither the underlying true parameters nor the error distribution. Moreover, a M-estimator based on \( H \) is consistent for the true parameter if and only if \( c_H = 1 \). This holds, in particular, when the QMLE is used if the underlying error distribution has unit variance. Hence to estimate population bias and MSE using simulation, we use population parameter as the one estimated from the SSE data using the QMLE computed from \texttt{fGarch}. Then we consider the DGP from GARCH (1, 1) models with four possible error distributions, namely, the normal, DE, logistic and \( t(3) \) distributions and for each scenario generate \( R \) replications of \( n \) observations. We estimate
Table 7: The coverage rates (in percentage) of the bootstrap schemes M, E and U and asymptotic normal approximations for the M-estimators QMLE, LAD, Huber’s, $\mu$- and Cauchy-; the error distributions are normal and $t(3)$.

|                   | 90% nominal level | 95% nominal level |
|-------------------|-------------------|-------------------|
|                   | $\omega$  | $\alpha$  | $\beta$ | $\omega$  | $\alpha$  | $\beta$ |
| Normal QMLE       |           |           |         |           |           |         |
| Scheme M          | 89.0     | 86.2     | 88.2    | 91.0     | 92.2     | 91.4    |
| Scheme E          | 87.2     | 83.8     | 86.8    | 90.2     | 88.4     | 91.2    |
| Scheme U          | 90.2     | 87.4     | 87.2    | 94.4     | 92.6     | 93.2    |
| Asymptotic        | 82.6     | 91.0     | 85.8    | 87.0     | 95.2     | 89.0    |
| Normal LAD        |           |           |         |           |           |         |
| Scheme M          | 86.0     | 83.4     | 84.2    | 88.2     | 87.2     | 88.4    |
| Scheme E          | 88.0     | 87.2     | 87.2    | 91.0     | 91.2     | 90.2    |
| Scheme U          | 88.6     | 88.4     | 88.0    | 93.2     | 91.8     | 91.8    |
| Asymptotic        | 94.0     | 98.8     | 87.0    | 96.4     | 99.4     | 90.4    |
| Normal Huber’s    |           |           |         |           |           |         |
| Scheme M          | 88.8     | 85.4     | 86.6    | 91.2     | 89.8     | 91.2    |
| Scheme E          | 88.2     | 89.0     | 88.0    | 91.4     | 92.4     | 90.0    |
| Scheme U          | 89.6     | 90.4     | 88.4    | 93.6     | 93.6     | 91.8    |
| Asymptotic        | 87.6     | 95.4     | 86.2    | 90.6     | 96.6     | 90.4    |
| Normal $\mu$-estimator |           |           |         |           |           |         |
| Scheme M          | 88.0     | 84.6     | 86.8    | 89.6     | 87.8     | 88.6    |
| Scheme E          | 87.4     | 84.8     | 86.6    | 89.4     | 88.4     | 88.4    |
| Scheme U          | 88.6     | 88.4     | 87.6    | 91.8     | 91.8     | 90.6    |
| Asymptotic        | 71.4     | 69.6     | 86.8    | 77.4     | 78.2     | 90.8    |
| Normal Cauchy     |           |           |         |           |           |         |
| Scheme M          | 85.6     | 84.0     | 84.4    | 87.8     | 85.8     | 87.6    |
| Scheme E          | 81.4     | 82.2     | 80.2    | 82.8     | 86.2     | 84.2    |
| Scheme U          | 88.4     | 88.2     | 87.0    | 90.4     | 91.4     | 89.4    |
| Asymptotic        | 97.8     | 99.8     | 85.0    | 98.2     | 100.0    | 89.6    |
| $t(3)$ QMLE       |           |           |         |           |           |         |
| Scheme M          | 71.0     | 75.4     | 74.8    | 75.0     | 79.0     | 78.0    |
| Scheme E          | 67.6     | 72.4     | 66.8    | 73.4     | 76.2     | 72.4    |
| Scheme U          | 75.6     | 84.6     | 75.0    | 81.6     | 87.2     | 80.0    |
| Asymptotic        | -        | -        | -       | -        | -        | -       |
| $t(3)$ LAD        |           |           |         |           |           |         |
| Scheme M          | 84.4     | 80.6     | 83.0    | 85.4     | 83.8     | 87.8    |
| Scheme E          | 84.6     | 85.0     | 81.4    | 87.6     | 87.0     | 86.6    |
| Scheme U          | 81.6     | 86.2     | 79.2    | 87.4     | 89.2     | 84.8    |
| Asymptotic        | 98.0     | 99.8     | 88.8    | 99.6     | 100.0    | 91.2    |
| $t(3)$ Huber’s    |           |           |         |           |           |         |
| Scheme M          | 83.0     | 80.6     | 81.8    | 85.6     | 83.2     | 86.6    |
| Scheme E          | 81.8     | 79.2     | 80.8    | 85.8     | 81.6     | 85.8    |
| Scheme U          | 86.2     | 88.0     | 86.0    | 90.2     | 91.4     | 90.2    |
| Asymptotic        | 96.8     | 99.0     | 88.4    | 97.8     | 99.6     | 92.8    |
| $t(3)$ $\mu$-estimator |           |           |         |           |           |         |
| Scheme M          | 82.4     | 84.8     | 83.8    | 86.2     | 88.4     | 88.2    |
| Scheme E          | 84.6     | 84.0     | 84.6    | 87.4     | 88.0     | 88.8    |
| Scheme U          | 82.6     | 83.6     | 80.4    | 88.8     | 88.2     | 86.4    |
| Asymptotic        | 86.6     | 91.8     | 80.8    | 90.6     | 95.6     | 86.4    |
| $t(3)$ Cauchy     |           |           |         |           |           |         |
| Scheme M          | 78.2     | 83.4     | 78.4    | 81.8     | 86.2     | 82.0    |
| Scheme E          | 83.4     | 85.6     | 82.6    | 85.4     | 89.0     | 87.2    |
| Scheme U          | 85.0     | 85.0     | 84.8    | 90.0     | 88.6     | 89.2    |
| Asymptotic        | 100.0    | 100.0    | 85.6    | 100.0    | 100.0    | 90.8    |
Table 8: The M-estimates (QMLE and LAD) of the GARCH (1, 1) model for the SSE data; The QMLEs are obtained by using fGarch and (4.1).

|     | \( \omega \)         | \( \alpha \)   | \( \beta \)   |
|-----|----------------------|----------------|--------------|
| QMLE | \( 1.65 \times 10^{-5} \) | \( 7.01 \times 10^{-2} \) | 0.90          |
| LAD  | \( 2.88 \times 10^{-5} \) | \( 7.97 \times 10^{-2} \) | 0.87          |

The normalized bias and normalized MSE of \( n^{1/2}(\hat{\theta}_n - \theta_0) \) by

\[
(R^{-1} \sum_{r=1}^{R} n^{1/2}(\hat{\omega}_r - c_H \omega_0), R^{-1} \sum_{r=1}^{R} n^{1/2}(\hat{\alpha}_r - c_H \alpha_0), R^{-1} \sum_{r=1}^{R} n^{1/2}(\hat{\beta}_r - \beta_0))' \quad (7.1)
\]

\[
(R^{-1} \sum_{r=1}^{R} n^{1/2}(\hat{\omega}_r - c_H \omega_0))^2, R^{-1} \sum_{r=1}^{R} n^{1/2}(\hat{\alpha}_r - c_H \alpha_0))^2, R^{-1} \sum_{r=1}^{R} n^{1/2}(\hat{\beta}_r - \beta_0))^2' \quad (7.2)
\]

Notice that the normalized bias and MSE are different from the standardized bias and MSE by some simple multiplicative factors involving \( c_H \).

To obtain the bootstrap estimates of the normalized bias and MSE, we use three different bootstrap schemes to generate weights \( \{w_{nt}; 1 \leq t \leq n\} \) \( B \) number of times. We compute the bootstrap estimates \( \{\hat{\theta}_{*nb}, 1 \leq b \leq B\} \) using (4.2) and consequently \( B \) bootstrap replicates (realizations)

\[
\{\sigma_n^{-1} n^{1/2}(\hat{\theta}_{*nb} - \hat{\theta}_n); 1 \leq b \leq B\}
\]

of the bootstrap distribution where \( \hat{\theta}_n \) is the M-estimate of the dataset computed using (4.1) based on the score function under consideration. The effect of the bootstrap scheme is reflected in the standardization through \( \sigma_n \). The bootstrap estimates of the normalized bias and MSE are computed by

\[
\text{Bias} = (1/B) \sum_{b=1}^{B} \{\sigma_n^{-1} n^{1/2}(\hat{\theta}_{*nb} - \hat{\theta}_n)\} \quad \text{and} \quad \text{MSE} = (1/B) \sum_{b=1}^{B} \{\sigma_n^{-1} n^{1/2}(\hat{\theta}_{*nb} - \hat{\theta}_n))^2\} \quad (7.3)
\]

Here the squares of vectors in the MSE above should be interpreted as entry-wise square.

Using (7.1) and (7.2) with \( n = 752 \) and \( R = 500 \), estimates of the normalized bias and MSE for the QMLE and LAD under various error distributions are shown in Table 9 and Table 10 respectively. Also, to evaluate the bootstrap approximation, we include the bootstrap estimates of the normalized bias and MSE in these tables. Note that for the LAD, all these bootstrap schemes have good approximation to the bias and MSE as they are generally of the same magnitude regardless of the underlying error distribution. For the QMLE under the DE, logistic and normal distributions, except for the bias of \( \omega \), scheme M provides good approximation while schemes E and U tend to underestimate the bias.
Table 9: The normalized bias and MSE of the QMLE and their bootstrap estimates for the SSE data.

| Error Dist. | Normalized bias | Normalized MSE |
|-------------|-----------------|----------------|
|             | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ |
| DE          | $3.80 \times 10^{-4}$ | 0.15 | -0.86 | $5.02 \times 10^{-7}$ | 0.70 | 3.51 |
| Logistic    | $4.39 \times 10^{-4}$ | 0.22 | -0.90 | $5.90 \times 10^{-7}$ | 0.65 | 3.22 |
| Normal      | $3.75 \times 10^{-4}$ | 0.12 | -0.78 | $4.14 \times 10^{-7}$ | 0.48 | 2.59 |
| $t(3)$      | $2.58 \times 10^{-4}$ | 0.49 | -1.17 | $6.56 \times 10^{-7}$ | 6.60 | 11.70 |

| Bootstrap   | Normalized bias | Normalized MSE |
|-------------|-----------------|----------------|
|             | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ |
| Scheme M    | $4.68 \times 10^{-5}$ | 0.10 | -0.18 | $7.58 \times 10^{-7}$ | 0.83 | 3.99 |
| Scheme E    | $8.39 \times 10^{-6}$ | 5.47 $\times 10^{-2}$ | -5.96 $\times 10^{-2}$ | $3.06 \times 10^{-7}$ | 0.68 | 2.35 |
| Scheme U    | $4.90 \times 10^{-6}$ | 1.58 $\times 10^{-2}$ | -3.32 $\times 10^{-2}$ | $1.23 \times 10^{-7}$ | 0.72 | 1.41 |

Table 10: The normalized bias and MSE of the LAD and their bootstrap estimates for the SSE data.

| Error Dist. | Normalized bias | Normalized MSE |
|-------------|-----------------|----------------|
|             | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ |
| DE          | $1.45 \times 10^{-4}$ | 4.54 $\times 10^{-2}$ | -0.65 | $7.01 \times 10^{-8}$ | 0.12 | 2.24 |
| Logistic    | $2.33 \times 10^{-4}$ | 9.88 $\times 10^{-2}$ | -0.85 | $1.82 \times 10^{-7}$ | 0.20 | 3.05 |
| Normal      | $2.28 \times 10^{-4}$ | 6.75 $\times 10^{-2}$ | -0.75 | $1.69 \times 10^{-7}$ | 0.21 | 2.63 |
| $t(3)$      | $6.06 \times 10^{-5}$ | 6.12 $\times 10^{-2}$ | -0.48 | $2.82 \times 10^{-8}$ | 0.12 | 2.25 |

| Bootstrap   | Normalized bias | Normalized MSE |
|-------------|-----------------|----------------|
|             | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ |
| Scheme M    | $2.44 \times 10^{-5}$ | 6.12 $\times 10^{-2}$ | -0.17 | $7.79 \times 10^{-8}$ | 0.28 | 2.19 |
| Scheme E    | $6.77 \times 10^{-5}$ | 0.11 | -0.39 | $6.66 \times 10^{-8}$ | 0.25 | 1.92 |
| Scheme U    | $1.29 \times 10^{-5}$ | 1.88 $\times 10^{-2}$ | -0.11 | $3.30 \times 10^{-8}$ | 0.22 | 1.14 |
Table 11: The M-estimates (QMLE, LAD, Huber’s, µ- and Cauchy-) of the GARCH (2, 1) model using the FTSE 100 data; the QMLEs are obtained by using fGarch and (4.1).

|        | fGarch | QMLE  | LAD   | Huber’s µ-estimator | Cauchy |
|--------|--------|-------|-------|---------------------|--------|
| ω      | 4.46×10⁻⁶ | 4.65×10⁻⁶ | 3.13×10⁻⁶ | 3.55×10⁻⁶           | 1.02×10⁻⁵ | 2.51×10⁻⁵ |
| α₁     | 5.25×10⁻² | 4.51×10⁻² | 2.46×10⁻² | 3.45×10⁻²           | 4.95×10⁻² | 6.83×10⁻³ |
| α₂     | 0.11   | 9.00×10⁻² | 5.57×10⁻² | 6.42×10⁻²           | 0.17    | 4.18×10⁻² |
| β      | 0.83   | 0.85   | 0.84   | 0.86                | 0.81    | 0.80     |

7.2 The FTSE 100 data and the GARCH (2, 1) model

Here we fit the GARCH (2, 1) model with the FTSE 100 data. The estimates given by fGarch and by our M-estimators are shown in Table 11. The QMLE (based on algorithm (4.1)) and fGarch provide similar estimates for all components of the parameter. Also, the M-estimates of β do not vary much. For ω, α₁ and α₂, the M-estimates are quite different since c_H in (2.10) depends on the score function H used for the estimation.

For a GARCH (p,q) model, using (2.6) and the formulas for \{c_j(\theta); j ≥ 0\} in Berkes et al. (2003) (Section 3), we have \hat{v}_t(\theta_0H) = c_H\hat{v}_t(\theta_0). Since a M-estimator \hat{\theta}_n estimates \theta_0H, \hat{v}_t(\hat{\theta}_n) estimates c_H\hat{v}_t(\theta_0) which is a scale-transformed estimate of the conditional variance.

To examine the behavior of the market volatility after eliminating the effect of any particular M-estimator used, we define the following normalized volatility by

\[
\hat{u}_t(\hat{\theta}_n) := \frac{\hat{v}_t(\hat{\theta}_n)}{\sum_{i=1}^{n} \hat{v}_i(\hat{\theta}_n)}; 1 ≤ t ≤ n.
\]  (7.4)

Figure 1 shows the plot of \{\hat{u}_t(\hat{\theta}_n); 1 ≤ t ≤ n\} based on various M-estimators against the squared returns. Notice that although the M-estimates in Table 11 are different, the plot of the normalized volatilities almost overlap each other based on all M-estimators. Also, large values of the normalized volatilities and large squared returns occur at the same time. In this sense, the volatilities are well-modelled by using these M-estimators.
7.3 The EFCX data

Muler and Yohai (2008) fitted the GARCH (1, 1) model to the EFCX data and noted that the QMLE and LAD estimates of the parameter $\beta$ are significantly different. Here in Table 12 we report estimates given by the fGarch and M-estimators. Note that in our previous analysis of the SSE and FTSE 100 data, fGarch estimates and our QMLE are quite close while their difference is much more significant for this data. It is also worth noting that while the LAD, Huber’s, $\mu$- and Cauchy-estimates of $\beta$ are close to each other, they are all quite different from the corresponding estimate 0.84 of the QMLE when viewed as a M-estimate. We explain below that such interesting behavior might be related to the infinite fourth moment of the underlying innovation distribution.

Table 12: The M-estimates (QMLE, LAD, Huber’s, $\mu$- and Cauchy-) of the GARCH (1, 1) model for the EFCX data; the QMLEs are obtained by using fGarch and (4.1).

|      | fGarch | QMLE       | LAD        | Huber’s $\mu$-estimator | Cauchy |
|------|--------|------------|------------|--------------------------|--------|
| $\omega$ | 1.89×10^{-4} | 6.28×10^{-4} | 6.43×10^{-4} | 8.37×10^{-4} | 1.42×10^{-3} | 2.97×10^{-4} |
| $\alpha$ | 4.54×10^{-2} | 7.20×10^{-2} | 8.87×10^{-2} | 0.10 | 0.27 | 6.35×10^{-2} |
| $\beta$ | 0.92 | 0.84 | 0.66 | 0.67 | 0.61 | 0.60 |

To examine whether the innovation distribution has finite fourth moment, we use the QQ-plots of the residuals $\{X_t/\hat{\nu}_{i}^{1/2}(\hat{\theta}_n); 1 \leq t \leq n\}$ based on the $\mu$-estimator $\hat{\theta}_n$ against the $t(d)$ distributions for various degrees of freedom $d$. We consider $\mu$-estimator since it imposes mild moment assumption on the innovation distribution. The main idea behind the QQ-plots of the residuals against the $t(d)$ distribution is simple. Recall that if $\epsilon \sim t(d)$ distribution then $E|\epsilon|^\nu < \infty$ if and only if $\nu < d$. Therefore, residuals with heavier tail than
the $t(d)$ distribution correspond to the errors with the infinite $d$-th moment while those with lighter tail than the $t(d)$ distribution have the finite $d$-th error moment.

The top-left panel of Figure 2 shows the QQ-plot of the residuals against the $t(4.01)$ distribution for the EFCX data. The residuals have heavier right tail than the $t(4.01)$ distribution which implies that the fourth moment of the error term may not exist. On the other hand, the QQ-plot against the $t(3.01)$ distribution reveals lighter tail as shown at the bottom-left panel of Figure 2 and this implies that $E|\epsilon|^3 < \infty$.

For the FTSE 100 data, the QQ-plot against the $t(4.01)$ distribution at the top-right panel of Figure 2 shows that the residuals have lighter tails than the $t(4.01)$ distribution. For the QQ-plot against the $t(12.01)$ distribution, as shown at the bottom-right panel of Figure 2, residuals fit the distribution better. Therefore, we may conclude that $E|\epsilon|^4 < \infty$ holds for the FTSE 100 data and this explains why the other M-estimates of $\beta$ in Table 11 are close.

Figure 2: The QQ-plot of the residuals against $t$ distributions for the EFCX (left column) and FTSE 100 (right column) data.
8 Conclusion

We consider a class of M-estimators and the weighted bootstrap approximation of their distributions for the GARCH models. An iteratively re-weighted algorithm for computing the M-estimators and their bootstrap replicates are implemented. Both simulation and real data analysis demonstrate superior performance of the M-estimators for the GARCH (1, 1), GARCH (2, 1) and GARCH (1, 2) models. Under heavy-tailed error distributions, we show that the M-estimators are more robust than the routinely-applied QMLE. We also demonstrate through simulations that the M-estimators work well when the true GARCH (1, 1) model is misspecified as the GARCH (2, 1) model. Simulation results indicate that under the finite sample size, bootstrap approximation is better than the asymptotic normal approximation of the M-estimators.

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References

[1] Berkes, I., Horvath L. and Kokoszka, P. (2003). GARCH processes: structure and estimation. Bernoulli 9, 201-228.

[2] Berndt, E., Hall, B., Hall, R. and Hausman, J. (1974). Estimation and Inference in Nonlinear Structural Models. Annals of Economic and Social Measurement 3, 653–665.

[3] Chatterjee, S. and Bose, A. (2005). Generalized bootstrap for estimating equations. Annals of Statistics 33, 414–436.

[4] Francq, C. and Zakoian J. (2009). Testing the nullity of GARCH coefficients: correction of the standard tests and relative efficiency comparisons. Journal of the American Statistical Association 104, 313-324.

[5] Iqbal, F. and Mukherjee, K. (2010). M-estimators of some GARCH-type models; computation and application. Statistics and Computing 20, 435-445.

[6] Mukherjee, K. (2008). M-estimation in GARCH models. Econometric Theory 24, 1530-1553.

[7] Mukherjee, K. (2020). Bootstrapping M-estimators in GARCH models. To appear in Biometrika.

[8] Muler, N. and Yohai, V. (2008). Robust estimates for GARCH models. Journal of Statistical Planning and Inference 138, 2918-2940.