A NOTE ON LIMIT OF FIRST EIGENFUNCTIONS OF $p$-LAPLACIAN ON GRAPHS

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Abstract. We study the limit of first eigenfunctions of (discrete) $p$-Laplacian on a finite subset of a graph with Dirichlet boundary condition, as $p \to 1$. We prove that up to a subsequence, they converge to a summation of characteristic functions of Cheeger cuts of the graph. We give an example to show that the limit may not be a characteristic function of a single Cheeger cut.

0. Introduction

The spectral theory of linear Laplacian, called Laplace-Beltrami operator, on a domain of an Euclidean space or a Riemannian manifold is extensively studied in the literature, see e.g. [7]. The $p$-Laplacians are nonlinear generalizations of the linear Laplacian, which corresponds to the case $p = 2$. These are nonlinear elliptic operators which possess many analogous properties as the linear Laplacian.

A graph consists of a set of vertices and a set of edges. The Laplacian on a finite graph is a finite dimensional linear operator, see e.g. [6], which emerges from the discretization of the Laplace-Beltrami operator of a manifold, the Cayley graph of a discrete group, data sciences and many others. Compared to continuous Laplacians, one advantage of discrete Laplacians is that one can calculate the eigenvalues of the Laplacian on a finite graph which are in fact eigenvalues of a finite matrix.

Cheeger [5] defined an isoperimetric constant, now called Cheeger constant, on a compact manifold and used it to estimate the first non-trivial eigenvalue of the Laplacian, see also [11]. These were generalized to graphs by Alon-Milman [1] and Dodziuk [8] respectively. These estimates, called Cheeger estimates, are very useful in the spectral theory. The spectral theory for discrete $p$-Laplacians was studied by [14, 2, 13, 9, 12]. It turns out that this theory unifies these constants involved in the Cheeger estimate. The Cheeger constant of a finite graph is in fact the first non-trivial eigenvalue of 1-Laplacian, see Chang, Hein et al. [3, 4, 9]. So that the Cheeger estimate can be regarded as the eigenvalue relation for $p$-Laplacians, $p = 1$ and $p = 2$.
In this paper, we study first eigenvalues and eigenfunctions of the $p$-Laplacian on a subgraph of a graph with Dirichlet boundary condition. Let $G = (V,E)$ be a simple, undirected, locally finite graph, where $V$ is the vertex set and $E$ is the edge set. Two vertices $x, y$ are called neighbors, denoted by $x \sim y$, if $\{x, y\} \in E$. Let

$$
\mu : V \to (0, \infty), x \mapsto \mu_x,
$$

be the vertex measure on $V$ and

$$
w : E \to (0, \infty), \{x, y\} \mapsto w_{xy} = w_{yx},
$$

be the edge measure on $E$. The quadruple $(V,E,\mu,w)$ is called a weighted graph. For a subset of $V$ (a subset of $E$, resp.) we denote by $|\cdot|_\mu$ ($|\cdot|_w$, resp.) the $\mu$-measure ($w$-measure, resp.) of the set. Let $\Omega$ be a finite subset of $V$. We denote by $C_0(\Omega)$ the set of function on $V$ which vanishes on $V \setminus \Omega$.

For $p > 1$, we define the $p$-Laplacian with Dirichlet boundary condition, called Dirichlet $p$-Laplacian, on $\Omega$ as

$$
\Delta_p u(x) = \frac{1}{\mu_x} \sum_{y \in V, y \sim x} w_{xy} |u(y) - u(x)|^{p-2}(u(y) - u(x)), \quad x \in \Omega, u \in C_0(\Omega).
$$

We call the pair $(\lambda, u) \in \mathbb{R} \times C_0(\Omega)$ satisfying

\begin{equation}
- \Delta_p u(x) = \lambda |u(x)|^{p-2} u(x), \quad \forall x \in \Omega
\end{equation}

the eigenvalue and the eigenfunction of Dirichlet $p$-Laplaican. The smallest eigenvalue of Dirichlet $p$-Laplacian is called the first eigenvalue, denoted by $\lambda_{1,p}(\Omega)$, and the associated eigenfunction is called the first eigenfunction, denoted by $u_p$.

For $p \geq 1$, the $p$-Dirichlet energy is defined as

$$
E_p(u) := \sum_{x,y \in V, y \sim x} w_{xy} |u(y) - u(x)|^p, \quad u \in C_0(\Omega),
$$

and the modified $p$-Dirichlet energy is defined as

$$
\widetilde{E}_p(u) := \frac{E_p(u)}{|u|^p}, \quad \forall u \in C_0(\Omega) \setminus \{0\}.
$$

We consider the variational problem

\begin{equation}
(0.2) \quad \widetilde{E}_p : C_0(\Omega) \setminus \{0\} \to \mathbb{R}.
\end{equation}

One is ready to see that for $p > 1$, critical points and critical values of the problem satisfy (0.1). The spectral theory for 1-Laplacian was developed by Chang and Hein. Eigenvalues and eigenfunctions of Dirichlet 1-Laplacian are defined as critical values and critical points of the variational problem (0.2) for $p = 1$. We denote by $\lambda_{1,1}(\Omega)$ the first eigenvalue for Dirichlet 1-Laplacian on $\Omega$. This yields the Rayleigh quotient characterization, $p \geq 1$,

\begin{equation}
(0.3) \quad \lambda_{1,p}(\Omega) = \inf_{u \in C_0(V), u \neq 0} \frac{\widetilde{E}_p(u)}{|u|^p}.
\end{equation}
First eigenfunctions of Dirichlet $p$-Laplacian have some nice properties, see [10].

**Theorem 1.** For a finite connected subset $\Omega$ of $V$ and $p > 1$, a first eigenfunction $u$ on $\Omega$ has fixed sign, i.e. $u > 0$ on $\Omega$ or $u < 0$ on $\Omega$. Moreover, the first eigenvalue is simple, i.e. for any two first eigenfunctions $u$ and $v$ on $\Omega$, $u = cv$ for some $c \neq 0$.

By this result, there is a unique first eigenfunction on $\Omega$ satisfies
$$u > 0, \quad \text{on } \Omega, \quad |u|_p = 1,$$
which is called the normalized first eigenfunction of $p$-Laplacian.

We introduce the definition of the Cheeger constant on $\Omega$. For any subset $D \in V$, the boundary of $D$ is defined as $\partial D := \{\{(x, y) \in E : x \in D, y \in V \setminus D\}$. 

**Definition 0.1.** The Cheeger constant on $\Omega$ is defined as
$$h(\Omega) = \inf_{D \subseteq \Omega} \frac{|\partial D|_w}{|D|_\mu}.$$
A subset $D$ of $\Omega$ is called a Cheeger cut if
$$\frac{|\partial D|_w}{|D|_\mu} = h(\Omega).$$

We prove the main result in the following.

**Theorem 0.2.** Let $\Omega$ be a finite connected subset of $V$. For any sequence $\{p_i\}_{i=1}^\infty$ satisfying $p_i > 1, p_i \to 1$, let $u_i$ be the corresponding normalized first eigenfunction of $p_i$-Laplacian. Then there is a subsequence $\{p_{i_k}\}$ such that $\{u_{i_k}\}$ converges
$$\lim_{k \to \infty} u_{i_k} = \sum_{n=1}^N c_n \mathbb{1}_{A_n},$$
where $A_n, 1 \leq n \leq N$, are Cheeger cuts of $\Omega$ satisfying $A_N \subseteq A_{N-1} \subseteq \cdots \subseteq A_1$, $\mathbb{1}_{A_n}$ are characteristic functions on $A_n$ and $c_n > 0$.

We have the following corollary.

**Corollary 0.3.** For a finite connected subset $\Omega$ of $V$, suppose that the Cheeger cut of $\Omega$ is unique. Then
$$\lim_{p \to \infty} u_p = \frac{1}{|A|_\mu} \mathbb{1}_A,$$
where $A$ is the Cheeger cut of $\Omega$.

Concerning with these results, we have the following open problems.

**Problem 1.** In general case, is it true that the normalized first eigenfunction $u_p$ converges, as $p \to 1$?

**Problem 2.** What are the limits of the normalized first eigenfunctions in Theorem 0.2.
For Problem 1, since there is no uniqueness for the Cheeger cuts, see e.g. Example 1 in Section 2, one needs new ideas to prove the result. For Problem 2, one might hope that the limit of a sequence of normalized first eigenfunctions is a characterization function of a single Cheeger cut, as in Corollary 0.3. By investigating Example 1 in Section 2, we show that this is not true in general. This indicates that the result in Theorem 0.2 cannot be improved to the characterization function of a single Cheeger cut.

The paper is organized as follows: In next section, we prove the main result, Theorem 0.2. In Section 2, we construct an example to show the sharpness of Theorem 0.2.

1. Proof of Theorem 0.2

Let \((V,E,\mu,w)\) be a weighted graph and \(\Omega\) is a connected subset of \(V\), i.e. for any \(x,y \in \Omega\) there is a path, \(x = x_0 \sim x_1 \sim \cdots \sim x_k = y\), connecting \(x\) and \(y\) with \(x_i \in \Omega, \forall 1 \leq i \leq k - 1\). We need some lemmas.

Lemma 1.1.
\[
\lim_{p \to 1} \lambda_{1,p}(\Omega) = \lambda_{1,1}(\Omega).
\]

Proof. Note that for any \(u \in C_0(\Omega), u \neq 0,\)
\[
\widetilde{E}_p(u) \to \widetilde{E}_1(u), \quad p \to 1.
\]
Hence the lemma follows from the Rayleigh quotient characterization (0.3). \(\square\)

Lemma 1.2. For any \(p \geq 1,\)
\[
\lambda_{1,p}(\Omega) \leq h(\Omega).
\]

Proof. As \(\Omega\) is a finite subset, we choose \(D \subset \Omega\) such that \(\frac{|\partial D|_w}{|D|_\mu} = h(\Omega)\). Consider the characteristic function on \(D,\)
\[
\mathbb{1}_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \in V \setminus D, \end{cases}
\]
Then by (0.3),
\[
\lambda_{1,p}(\Omega) \leq \widetilde{E}_p(\mathbb{1}_D) = \frac{|\partial D|_w}{|D|_\mu} = h(\Omega).
\]
This proves the lemma. \(\square\)

Lemma 1.3. \(\lambda_{1,1}(\Omega) = h(\Omega).\)

Proof. By Lemma 1.2, it suffices to prove that \(\lambda_{1,1}(\Omega) \geq h(\Omega).\) For any \(u \in C_0(\Omega)\) with \(|u|_1 = 1,\) let \(g = |u|\). For any \(\sigma \geq 0,\) set \(\Omega_\sigma := \{x \in \Omega : g(x) > \sigma\}\) and \(G(\sigma) := |\partial \Omega_\sigma|_w.\) Then
\[
G(\sigma) = \sum_{e = \{x,y\} \in E, g(x) \leq \sigma < g(y)} w_{xy}.
\]
Hence
\[ \int_0^{+\infty} G(\sigma) \, d\sigma = \int_0^{+\infty} \sum_{e=\{x,y\} \in E, g(y) > g(x)} w_{xy} \mathbb{1}_{[g(x), g(y)]}(\sigma) \, d\sigma = \sum_{e=\{x,y\} \in E, g(y) > g(x)} w_{xy} g(x) - g(y) = E_1(g). \]

Moreover, by the definition of \( h(\Omega) \), \( \Omega_\sigma \leq h(\Omega) |\Omega_\sigma|_\mu \), \( \forall \sigma \geq 0 \). This yields that
\[ E_1(u) \geq E_1(g) = \int_0^{+\infty} G(\sigma) \, d\sigma \geq h(\Omega) \int_0^{+\infty} |\Omega_\sigma|_\mu = h(\Omega)|u|_1 = h(\Omega). \]

By taking the infimum over \( u \in C_0(\Omega) \) with \(|u|_1 = 1\) and \( h(\Omega) \), we prove the lemma by (0.3).

By the definition, \( u \) is called a first eigenfunction of 1-Laplacian on \( \Omega \) if
\[ \tilde{E}_1(u) = \inf_{v \in C_0(V), v \neq 0} \tilde{E}_1(v) = \lambda_{1,1}(\Omega). \]

Since \( \tilde{E}_1(|u|) \leq \tilde{E}_1(u) \), \(|u|_1 \) is also a first eigenfunction of 1-Laplacian on \( \Omega \).

**Lemma 1.4.** Let \( u \) be a nonnegative first eigenfunction of 1-Laplacian on \( \Omega \). Then for any \( \sigma \geq 0 \), \( \Omega_\sigma := \{ x \in \Omega : u > \sigma \} \) is a Cheeger cut. Moreover,
\[ u = \sum_{n=1}^{N} c_n 1_{A_n}. \]
where \( A_n, 1 \leq n \leq N \), are Cheeger cuts of \( \Omega \) satisfying \( A_N \subsetneq A_{N-1} \subsetneq \cdots \subsetneq A_1 \), and \( c_n > 0 \).

**Proof.** Let \( \{a_i\}_{i=0}^{M} \) be the range of the function \( u \), such that
\[ 0 = a_0 < a_1 < \cdots < a_M = \max_{x \in V} u(x). \]

Then for any \( \sigma \in [a_i, a_{i+1}) \), \( 0 \leq i \leq M - 1 \),
\[ \Omega_\sigma = \Omega_{a_i}. \]
By the proof of Lemma 1.3, see (1.1),
\[
\begin{align*}
\int_0^{+\infty} G(\sigma) d\sigma &= \sum_{i=0}^{M-1} (a_{i+1} - a_i) |\partial_{\Omega} a_i|_w \\
&\geq h(\Omega) \sum_{i=0}^{M-1} (a_{i+1} - a_i) |\Omega_{a_i}|_\mu = h(\Omega) |u_1|.
\end{align*}
\]
Hence for any \(0 \leq i \leq M-1\),
\[
|\partial_{\Omega} a_i|_w = h(\Omega) |\Omega_{a_i}|_\mu.
\]
So that for any \(\sigma \geq 0\), \(\Omega_{\sigma}\) is a Cheeger cut.

For the other assertion, note that
\[
\begin{align*}
u_{1,1}(\Omega) |u_1| = E_1(u) = \sum_{i=0}^{M-1} (a_{i+1} - a_i) |\Omega_{a_i}|_\mu = h(\Omega) |u_1|.
\end{align*}
\]
This proves the result.

Now we are ready to prove Theorem 0.2.

Proof of Theorem 0.2. Note that
\[
|\Omega_{\sigma}|_\mu = 1, \quad E_1(u) = \lambda_{1,p_i}(\Omega).
\]
For any \(x \in \Omega\), \(|u_i(x)| \leq |u_i|_{p_i} = 1\). Since \(\Omega\) is a finite set, there is a subsequence \(\{p_i\}_k\) of \(\{p_i\}_i\) and \(u \in C_0(\Omega)\) such that
\[
u_{ik} \to u, \quad \text{pointwise on } \Omega.
\]
Hence
\[
\begin{align*}
u_{ik} \to u, \quad \text{pointwise on } \Omega.
\end{align*}
\]
This yields that \(u\) is a nonnegative normalized 1-Laplacian eigenfunction. The theorem follow from Lemma 1.4.

This yields the following corollary.

Proof of Corollary 0.3. It suffices to prove that for any sequence \(p_i \to 0, p_i > 1\), there is a subsequence \(p_{i_k}\) such that
\[
\lim_{k \to \infty} u_{p_{i_k}} = \frac{1}{|A|_\mu} \mathbb{1}_A,
\]
where \(A\) is the Cheeger cut of \(\Omega\). By Lemma 1.4, there is a subsequence \(p_{i_k}\) and a nonnegative normalized first eigenfunction of 1-Laplacean \(u\) such that
\[
\lim_{k \to \infty} u_{p_{i_k}} = u.
\]
Since the Cheeger cut of $\Omega$ is unique, by (1.2), $u = c \mathbb{1}_A$. By $|u|_1 = 1$, $c = \frac{1}{|A|}$. This proves the corollary. □

2. An example

In this section, we give an example to show that the limit of the first eigenfunction of $\Delta_p$ may not be the characteristic function of a single set.

Example 1. Consider the graph $G$ as in Fig. 1 such that edge weight $w_{xy} = 1$ for any $x \sim y$ and vertex weight $\mu_{x_1} = \mu_{x_2} = 2, \mu_{y_1} = \mu_{y_2} = 4$. Let $\Omega = \{x_1, x_2, y_1, y_2\}$. By the enumeration, the Cheeger cuts for $h(\Omega)$ are 
\[
\{x_1, x_2\}, \{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}, \{x_1, x_2, y_1, y_2\}.
\]

![Figure 1. Example $G$](image)

We calculate the normalized first eigenfunctions $u_p$ of $p$-Laplacian on $\Omega$ for $p > 1$. One is ready to see that there is a symmetry, $T : \Omega \to \Omega$, such that 
\[
T(x_1) = x_2, T(x_2) = x_1, T(y_1) = y_2, T(y_2) = y_1,
\]

perserving the Dirichlet boundary condition. Since the first eigenfunction is positive and unique up to constant multiplication, $u_p(x_1) = u_p(x_2), u_p(y_1) = u_p(y_2)$, see [10]. For convenience, we write $u_p$ as a vector 
\[
u_p = (u_p(x_1), u_p(x_2), u_p(y_1), u_p(y_2)),
\]
and by scaling we set 
\[
u_p = \frac{1}{u_p(x_1)} u_p = (1, 1, t_p, t_p).
\]

Note that $\nu_p$ is also a first eigenfunction of $p$-Laplacian on $\Omega$.

Setting $x = t_p$, by the eigen-equation,
\[
-\frac{1}{p} |1 - x|^{p-2}(x - 1) = \lambda_{1,p},
\]
\[
-\frac{1}{4} (3|x|^{p-2}(-x) + \frac{1}{2} |1 - x|^{p-2}(1 - x)) = \lambda_{1,p} x^{p-1}.
\]
By the first equation, \( x < 1 \). Plugging the first equation into the second and dividing it by \( x^{p-1} \), we get

\[
2(1 - x)^q + \left(\frac{1}{x} - 1\right)^q - 3 = 0,
\]

where \( q = p - 1 \).

Set

\[
f(x, q) = 2(1 - x)^q + \left(\frac{1}{x} - 1\right)^q - 3, \quad \forall x \in (0, 1], q > 0.
\]

For any \( q > 0 \), since \( f(\cdot, q) \) is monotonically decreasing, \( f(1, q) = -3 \) and \( f(x, q) \to +\infty, x \to 0^+ \), there is a unique solution to \( f(x, q) = 0 \) denoted by \( x_q \). Note that \( x_q = t_p \).

For any \( x \in (0, 1) \), we write \( a = 1 - x, b = \frac{1}{x} - 1 \), and set

\[
g(q) = f(x, q) = 2a^q + b^q - 3.
\]

For \( q \geq 0 \),

\[
g'(q) = 2a^q \ln a + b^q \ln b, \quad g''(q) = 2a^q (\ln a)^2 + b^q (\ln b)^2 \geq 0,
\]

where the derivatives are understood as right derivatives for \( q = 0 \). Hence \( g(p) \) is convex in \((0, +\infty)\), which yields that for \( q \geq 0 \),

\[
(2.2) \quad g(q) \geq g(0) + g'(0)q = \ln(2a^2b)q.
\]

One can show that

\[
\begin{cases}
a^2b > 1, & x < \hat{x}, \\
a^2b = 1, & x = \hat{x}, \\
a^2b < 1, & x > \hat{x},
\end{cases}
\]

where \( \hat{x} \) is the real solution of \((1 - x)^3 = x \), given by

\[
\hat{x} = 1 - \sqrt[3]{\frac{\sqrt{93 + 9} - 9}{18}} + \sqrt[3]{\frac{\sqrt{93 - 9} - 9}{18}} \approx 0.31767.
\]

Hence by (2.2), for any \( x < \hat{x} \),

\[
g(q) \geq \ln(a^2b)q > 0, \quad q > 0.
\]

Hence \( x_q \geq \hat{x} \).

For any \( x > \hat{x} \), by Taylor expansion of \( g(q) \) at \( q = 0 \),

\[
g(q) = \ln(a^2b)q + o(q), \quad q \to 0.
\]

Hence by \( a^2b < 1 \), for sufficiently small \( q \),

\[
g(q) < 0,
\]

this yields that \( x_q < x \). This implies that \( \limsup_{q \to 0} x_q \leq x \). By passing to the limit \( x \to \hat{x}^+ \), we get

\[
\limsup_{q \to 0} x_q \leq \hat{x}.
\]
Combining it with $x_q \geq \hat{x}$,

$$\lim_{q \to 0} x_q = \hat{x}.$$ 

This yields that as $p \to 1$,

$$v_p = (1, 1, t_p, t_p) \to (1, 1, \hat{x}, \hat{x}).$$

This yields that

$$u_p = \frac{v_p}{|v_p|_p} \to \frac{1}{4 + 8\hat{x}}(1, 1, \hat{x}, \hat{x}) \approx (0.15287, 0.15287, 0.04856, 0.04856).$$

Hence, the limit is not a characteristic function on a single set.

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