Type D vacuum solutions: a new intrinsic approach

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Abstract We present a new approach to the intrinsic properties of the type D vacuum solutions based on the invariant symmetries that these spacetimes admit. By using tensorial formalism and without explicitly integrating the field equations, we offer a new proof that the upper bound of covariant derivatives of the Riemann tensor required for a Cartan-Karlhede classification is two. Moreover we show that, except for the Ehlers-Kundt’s C-metrics, the Riemann derivatives depend on the first order ones, and for the C-metrics they depend on the first order derivatives and on a second order constant invariant. In our analysis the existence of an invariant complex Killing vector plays a central role. It also allows us to easily obtain and to geometrically interpret several known relations. We apply to the vacuum case the intrinsic classification of the type D spacetimes based on the first order differential properties of the 2+2 Weyl principal structure, and we show that only six classes are compatible. We define several natural and suitable subclasses and present an operational algorithm to detect them.

Keywords Type D solutions · Intrinsic classification · Invariant symmetries

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1 Introduction

Type D vacuum solutions describe physically relevant gravitational fields and an exhaustive geometrical comprehension of them is of evident interest. The subfamily of static ones was obtained by Ehlers and Kundt [1] and the full set of type D vacuum metrics by Kinnersley [2] using NP formalism. Recently, Edgar et al. [3] have revisited the type D vacuum metrics. They show that the GHP formalism
is more suitable to obtain several identities, say $I$, which need a hard computer support when using NP formalism \cite{4, 5}. They also improve previous results \cite{6, 7, 8} and show, without explicitly integrating the field equations, that the Karlhede upper bound of covariant derivatives of the Riemann tensor is two.

Here we analyze this subject by using a new approach that has the following qualities: (i) We use plain tensorial formalism and all our calculations are made without computer support. (ii) An invariant complex Killing vector $Z$ plays a central role in our study; in obtaining the Karlhede upper bound we show that $Z$ and $\nabla Z$ determine, respectively, the first and the second derivatives of the Riemann tensor, and for three different cases we give the explicit expression of $\nabla Z$ in terms of $Z$ (in one of the three cases $\nabla Z$ also depends on a constant second order scalar invariant). (iii) We easily obtain identities $I$ and give an interesting geometric interpretation of them.

In this paper we also offer a generic and invariant classification of the type D vacuum metrics. Previously known classifications derive from the integration process of the vacuum equations \cite{2}, or from the study the Karlhede upper bound \cite{3}. Our invariant classification is generic because it applies to the full set of type D metrics, and it imposes first order (tensorial) invariant conditions on the Weyl tensor. We show here that the $2^{10}$ possible classes radically decline to only 6 classes in the vacuum case.

In section 2 we present the notation used in the paper and we write the Bianchi identities and the post-Bianchi identities for a type D vacuum space-time. Moreover we introduce the invariant complex Killing vector $Z$.

In section 3 we point out the tensorial invariants which collect the 0th- and 1st-order covariant derivatives of the Riemann tensor: the 0th-order are given by the algebraic Weyl invariants, namely, the complex eigenvalue $w$ and the canonical bivector $U$, and the 1st-order are given by the complex Killing vector $Z$. By imposing the invariance of these Riemann invariants along the invariant Killing vector $Z$ we easily obtain several relevant identities, which are used in this paper. One of them is identity $I$ studied in \cite{3} which in our formalism admits a nice geometric interpretation: the projections on the two Weyl principal planes of the real and imaginary parts of $Z$ are collinear vectors.

The upper bound on the order of the Riemann covariant derivatives is studied in section 4. The 2nd-order Riemann derivatives are collected in the Killing 2–form $\nabla Z$, which can be written in terms of $w$, $U$, $Z$ and a complex scalar $m$ as a consequence of the identities obtained in the previous section. In the analysis of the scalar $m$ we distinguish three cases: the Ehlers and Kundt C-metrics, the regular generalized C-metrics, and the Kerr-NUT solutions (those type D vacuum metrics that admit a Killing tensor). In the last two cases we give the explicit expression of $m$ in terms of $w$, $U$, $Z$, and in the first one we show that $m$ depends on these invariants and on a second order constant invariant scalar.

In section 5 we present the invariant classification. Firstly we point out that two classifications of type D metrics based on first order (tensorial) invariant conditions on the Weyl tensor can be considered \cite{9}. One of them is the natural first order geometric classification of the 2+2 almost-product structure \cite{10, 11} associated to the Weyl tensor. The second one is defined by the first derivatives of the Weyl eigenvalue $w$. We show that the $2^6$ classes of the first classification notably decline to the 16 classes of the second one when the Cotton tensor vanishes. Moreover, only 6 of these 16 classes are compatible with the vacuum condition. We finish this
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section giving an operational algorithm to detect every class and certain relevant subclasses, and pointing out the relationship with previous classifications.

Section 6 is devoted to discussing and commenting our results. In appendix A and appendix B we prove some lemmas used in sections 4 and 5, respectively.

In this paper we work on an oriented space-time with a metric tensor \( g \) of signature \( \{−,+,+,+\} \). The Riemann, Ricci and Weyl tensors are defined as given in [12] and are denoted, respectively, by \( \text{Riem}, \text{Ric} \) and \( \text{W} \). For the metric product of two vectors we write \((x,y) = g(x,y)\), and we put \( x^2 = g(x,x) \). If \( A \) and \( B \) are 2-tensors, \( A \cdot B \) denotes the 2-tensor \( (A \cdot B)^{\alpha \beta} = A^{\alpha \mu}B^{\beta \nu} \). For the metric product \( A \cdot B \) of two vectors we write \((x,y) = g(x,y)\), and we put \( x^2 = g(x,x) \). If \( A \) and \( B \) are 2-tensors, \( A \cdot B \) denotes the 2-tensor \( (A \cdot B)^{\alpha \beta} = A^{\alpha \mu}B^{\beta \nu} \), and \((A,B) = \frac{1}{2}A_{\alpha \beta}B^{\alpha \beta} \).

2 Type D vacuum metrics. Basic relations

A self–dual 2–form is a complex 2–form \( F \) such that \( ^*F = iF \), where \( ^* \) denotes the Hodge dual operator. We can associate biunivocally with every real 2–form \( F \) the self-dual 2–form \( F = \frac{1}{\sqrt{2}}(F - i^*F) \). Here we refer to a self–dual 2–form as a self-dual bivector. The endowed metric on the 3-dimensional complex space of the self-dual bivectors is \( G = \frac{1}{2}G - i^*G \), \( G \) being the metric volume element of the space-time and \( G \) the metric on the space of 2–forms, \( G^{\alpha \beta \gamma \delta} = g^{\alpha \gamma}g^{\beta \delta} - g^{\alpha \delta}g^{\beta \gamma} \).

The basic elements of self-dual bivector formalism and its relationship with the formalism based on orthonormal or on null tetrads can be found in [13].

Every double 2–form, and in particular the Weyl tensor \( W \), can be considered as an endomorphism on the space of the 2–forms. The restriction of the Weyl tensor on the self-dual bivectors space is the self-dual Weyl tensor and it is given by \( W = \frac{1}{2}(W - i^*W) \). The Petrov-Bel classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial of this endomorphism. In [13] we have presented a complete study of this subject as well as the covariant determination of the geometric elements that appear in every class.

In the case of a type D spacetime the the self–dual Weyl tensor admits the canonical expression:

\[
W = 2wU \otimes U + wG_\perp = 3wU \otimes U + wG,
\]

where \( U \) is the canonical bivector, normalized eigenbivector associated with the simple eigenvalue \(-2w\), and \( G_\perp = U \otimes U + G \) is its orthogonal projector. If \( \ell \) and \( k \) are the Debever principal null directions and \( U = \ell \wedge k \), then \( U = \frac{1}{2w}(U - i^*U) \).

The canonical bivector \( U \) determines the two principal planes of a type D Weyl tensor. The projector on the time-like (resp., space-like) principal plane is \( v = U^2 \) (resp., \( h = g - v = -(^*U)^2 \)), and \( II = v - h = 2U \cdot U \) is the structure tensor. From now on \( \bar{t} \) denotes the complex conjugate of a complex tensor \( t \).

The structure tensor \( II \) will play an important role here on. On one hand in section 4 when studying the Cartan-Karlhede upper bound of type D vacuum solutions. On the other hand, in section 5 when classifying this family of solutions.

Indeed, the invariant decomposition of the covariant derivative of the structure tensor \( II \) gives rise to the classification of almost product structures assumed in differential geometry [14] [15]. In [9], [10] and [11] we have implemented these ideas to the general relativity framework and we have used them to classify type
D metrics \[9\]. In the case of 1+3 almost-product structures defined by a timeline unit vector this approach leads to the known concepts of acceleration, expansion, shear and vorticity. In the 2+2 considered case the geometric properties also have a kinematic interpretation \[10\].

**Bianchi identities**

Under the vacuum condition $Ric = 0$, Bianchi identities become $\delta W = 0$, where $(\delta W)_{\alpha \beta \gamma} = - \nabla_\lambda W^\lambda_{\alpha \beta \gamma}$. From now on, we will write $\delta t = - \nabla \cdot t$ for an arbitrary tensor $t$. Then, if we consider the canonical expression \(1\) of a type D Weyl tensor and we take into account that $2\Omega^2 = g$, Bianchi identities write for the algebraic Weyl variables $\{w, \Omega\}$ as:

\[\nabla \Omega = i(\delta \Omega)\mathcal{G}_{\perp}, \quad 3i(\delta \Omega)\Omega = d\ln w,\]

where for a vector $X$ and a $(p+1)$-tensor $t$, $i(X)t$ denotes the inner product, $[i(X)t]_\underline{p} = X^{\alpha}t_{\alpha p}$, the underline denoting multi-index.

**Post-Bianchi identities**

The integrability conditions of the second equation in \(2\) lead to the post-Bianchi equations:

\[d\chi = 0, \quad \chi \equiv i(\delta \Omega)\Omega.\]

On the other hand, we can study the integrability conditions of the first equation \(2\) by using the Ricci identities for the canonical bivector $\Omega$,

\[\nabla_\alpha \nabla_\beta \Omega_{\mu\nu} - \nabla_\beta \nabla_\alpha \Omega_{\mu\nu} = \Omega^\lambda \mathcal{R}_{\lambda \beta \alpha} - \Omega^\lambda \mathcal{R}_{\lambda \mu \beta \alpha} .\]

From the first condition in \(2\) and the canonical expression \(1\), these Ricci identities write:

\[w = -(\Omega, T), \quad T \cdot \Omega - \Omega \cdot T = 0,\]
\[T \equiv \nabla \xi - \chi \otimes \xi, \quad \xi \equiv \delta \Omega, \quad \chi \equiv i(\delta \Omega)\Omega .\]

The invariant complex Killing vector

Let $S$ be the symmetric part of $T$. The second equation in \(3\) implies $S \cdot \Omega - \Omega \cdot S = 0$. On the other hand, if we develop equation \(3\) taking into account \(2\), we obtain $S \cdot \Omega + \Omega \cdot S = 0$, and then $S = 0$:

\[\mathcal{L}_\xi g - \xi \hat{\otimes} \chi = 0 ,\]

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect the field $\xi$. From this condition and the second equation in \(2\), we recover the following result \[16\], \[17\]:

**Proposition 1** The type $D$ vacuum solutions admit the invariant complex Killing vector

\[Z = w^{-1} \xi , \quad \xi \equiv \delta \Omega.\]
There is another useful consequence of the Bianchi and post-Bianchi identities. Indeed, being $Z$ a Killing vector, (6) and (8) imply
\[ \nabla Z = dZ = w^{-\frac{1}{3}}T, \]
and from (5) we obtain that the Killing 2-form $\nabla Z$ satisfies:
\[ \langle U, \nabla Z \rangle = -w, \quad \nabla Z \cdot U - U \cdot \nabla Z = 0. \tag{9} \]
The invariant Killing vector $Z$ also exists in the charged counterpart of the type D vacuum metrics. For an in-depth study of this and other invariant symmetries and their close relation with the curvature tensor see [17] and references therein.

3 Some relevant identities: obtaining and interpretation

The Bianchi equations (2) and definition (8) imply that the first derivatives of the algebraic Weyl variables $\{w, U\}$ depend on the Killing vector $Z$. Then, we obtain the following set of zero, first and second order independent Riemann derivatives:
- 0th-order: $w, U$.
- 1st-order: $Z = w^{-\frac{1}{3}}\delta U$.
- 2nd-order: $\nabla Z$.

Moreover, in terms of $w$, $U$ and $Z$ the covariant derivatives of $\{w, U\}$ write:
\[ dw = 3w^{\frac{2}{3}}i(Z)U, \quad \nabla U = w^{\frac{1}{3}}i(Z)(U \otimes U + G). \tag{10} \]

On the other hand, the integrability condition for the Killing equation holds:
\[ \nabla \nabla Z = i(Z)Riem. \]
Then, we have for the 3rd-order Riemann derivatives:
\[ \nabla \nabla Z = w^{-\frac{2}{3}}i(Z)(3U \otimes U + G). \tag{11} \]

Consequently, we obtain the following result:

**Proposition 2** All the covariant derivatives of the Riemann tensor of a type D vacuum solution depend at most on the second order ones.

Now we study new restrictions on the Killing 2-form which allow us to improve the above result. The invariant vector $Z$ being a complex Killing vector, we have $L_Z Riem = 0$, $L_Z \nabla Riem = 0$ and, consequently, similar relations hold for the Riemann invariants $\{w, U, Z\}$ and their complex conjugate invariants:
- $L_Z w = 0$. That is $(Z, dw) = 0$. This condition is an identity as a consequence of the first expression in (10).
- $L_Z \bar{w} = 0$. That is $(\bar{Z}, dw) = 0$. From the first expression in (10) and definition (8), this condition can be written as one of the following equivalent conditions:
\[ U(Z, \bar{Z}) = 0, \quad U(\delta U, \delta \bar{U}) = 0. \tag{12} \]

The real and imaginary parts of the second condition in (12) give the scalar identities:
\[ U(\delta U, \delta \ast U) = 0, \quad U(\delta \bar{U}, \delta \ast \bar{U}) = 0. \tag{13} \]
If we write these identities in NP formalism we obtain:
\[ \pi \bar{\pi} - \tau \bar{\tau} = 0, \quad \rho \bar{\mu} - \bar{\rho} \mu = 0. \tag{14} \]
These restrictions on the NP coefficients were already known, and the difficulties in obtaining them have been outlined. Czapor and McLenaghan [4], [5] showed using computer support and they claimed that their calculations required pages and “would be virtually impossible by hand”. Recently [3] their obtaining has been widely improved by using the GHP formalism.

Here, we have used plain tensorial formalism to obtain (13). Our approach has several qualities. Firstly, the calculation is simple and straightforward and highlights the meaning of the identities: they state that the Weyl eigenvalue \( w \) is invariant under the invariant complex Killing vector \( \bar{Z} \). Secondly, it offers a nice geometric interpretation. Indeed, (13) equivalently writes as:

\[
v(\delta U) \wedge v(\delta^* U) = 0, \quad h(\delta U) \wedge h(\delta^* U) = 0,
\]

conditions which state that the projections of the first order invariant vectors \( \delta U \) and \( \delta^* U \) on the two Weyl principal planes are collinear. Thirdly, a similar geometric interpretation can be obtained in terms of the invariant Killing vector. In fact, the equivalent condition given by the first expression in (12) writes \( U(Z_1, Z_2) = \ast U(Z_1, Z_2) = 0 \), where \( Z = Z_1 + i Z_2 \). Then (13) is equivalent to:

\[
v(Z_1) \wedge v(Z_2) = 0, \quad h(Z_1) \wedge h(Z_2) = 0,
\]

conditions which state that the projections of the invariant Killing vectors \( Z_1 \) and \( Z_2 \) on the two Weyl principal planes are collinear.

\( \mathcal{L}_2 \mathcal{U} = 0 \). As a consequence of the second expression in (10), this condition is equivalent to the second constraint for the Killing 2-form given in (9).

\( \mathcal{L}_2 \bar{U} = 0 \). Taking into account the second expression in (10), this condition can be written as the following constraint for the Killing 2-form:

\[
\nabla Z \cdot \bar{U} - \bar{U} \cdot \nabla Z = \bar{w}^\perp \bar{G}(Z \wedge \bar{Z}),
\]

where, for a double 2-form \( V \) and a 2-form \( F \), \( V(F) \) denotes the action of \( V \) on \( F \) as an endomorphism, \( V(F)_{\alpha\beta} = \frac{1}{2} V_{\alpha\beta}^{\gamma\delta} F_{\gamma\delta} \).

\( \mathcal{L}_2 \bar{Z} = 0 \). This condition restricts the electric part of the Killing 2-form with respect the invariant Killing vector:

\[
i(Z) \nabla \bar{Z} - i(\bar{Z}) \nabla Z = 0.
\]

4 Upper bound on the order of the Riemann covariant derivatives

We have seen in the previous section that the second derivatives of the Riemann tensor depend on the Killing 2-form \( \nabla Z \), which are restricted by conditions (9), (17) and (18). The first condition in (9) gives the \( \mathcal{U} \)-component of \( \nabla Z \), and the second condition in (9) implies that its self-dual part orthogonal to \( \mathcal{U} \) vanishes, \( \bar{G}_\perp(\nabla Z) = 0 \). On the other hand, condition (17) determines the anti-self-dual part of \( \nabla Z \) which is orthogonal to \( \bar{U} \), \( \bar{G}_\perp(\nabla Z) = \bar{w}^\perp \bar{G}(Z \wedge \bar{Z}) \cdot \bar{U} \). Consequently, we obtain the following expression for the Killing 2-form \( \nabla Z \):

\[
\nabla Z = \bar{w}^\perp \bar{U} + \bar{m} \bar{U} + \bar{w}^\perp \bar{G}(Z \wedge \bar{Z}) \cdot \bar{U}.
\]
Note that expression (19) gives $\nabla Z$ in terms of $0$th- and $1$st-order Riemann derivatives ($w, U$ and $Z$) and a complex scalar $m$. Moreover, if we put expression (19) in (18) we obtain:

$$
\mu \Pi(\overline{Z}) - \overline{\mu} Z = \overline{\nu} \Pi(Z) - \nu \overline{Z}, \quad \mu \equiv m + \frac{1}{2} \overline{w} \overline{\nabla} (Z, \overline{Z}), \quad \nu \equiv \overline{w} \frac{\nabla}{2} \overline{w} (Z, \overline{Z}).
$$

Thus, we have a system of linear equations for the $2$nd-order differential scalar $m = -(\overline{U}, \nabla Z)$, with coefficients depending on $0$th- and $1$st-order Riemann derivatives. It is worth remarking that the first derivatives of $m$ depend on $1$st-order Riemann derivatives. Indeed, from (10), (11) and (19) we obtain:

$$
vspace{-2mm} \frac{2}{m} \frac{dm}{d\overline{m}} = \left[4 \overline{w} + \overline{w}^{2/3} (Z, \overline{Z}) \right] i (Z) \overline{U} - \frac{\overline{w}^{2/3}}{2} (Z, \overline{Z}) i (\overline{Z}) \overline{U}.
$$

Now, we study whether system (20) allows us to obtain $m$ in terms of $w, U$ and $Z$. We consider three cases and we will make use of the following result which is proven in appendix A:

**Lemma 1** For any type $D$ vacuum solution the invariant Killing vector $Z$ is a non null vector, $(Z, Z) \neq 0$.

4.1 Upper bound for the Kerr-NUT solutions

Hougston and Sommers [18] showed that, with the exception of the generalized $C$-metrics, the other type $D$ vacuum solutions admit a Killing tensor. We call Kerr-NUT metrics the type $D$ vacuum solutions where such a Killing tensor exists [17]. In a subsequent paper Hougston and Sommers [16] showed that the complex Killing vector $Z$ degenerates (it defines a unique Killing direction) if, and only if, the metric is a Kerr-NUT solution. This result can be stated in the following terms:

**Lemma 2** The Kerr-NUT vacuum metrics are the type $D$ vacuum solutions such that the complex Killing vector $Z$ given in (8) satisfies $Z \wedge \overline{Z} = 0$, or equivalently, $Z = e^{\kappa i} Z$, where $\kappa$ is a real number.

This lemma and expression (19) imply that the Killing $2$-form $\nabla Z$ is aligned with the Weyl principal structure, $\nabla Z = \overline{w} \frac{\nabla}{2} U + m \overline{U}$, in accordance with a known result [17]. From here and taking into account that $Z = e^{\kappa i} Z$ we have $m = \overline{w} e^{-\kappa i}$. Moreover lemma [1] implies in this case $(Z, \overline{Z}) \neq 0$, and we obtain the following expression for the $2$nd-order Riemann derivatives:

$$
\nabla Z = \overline{w} \frac{\nabla}{2} U + \frac{(Z, \overline{Z})}{(Z, \overline{Z})} \overline{w} \frac{\nabla}{2} \overline{U}.
$$

Consequently, we have the following result:

**Theorem 1** All the covariant derivatives of the Riemann tensor of a Kerr-NUT vacuum solution depend at most on the first order ones.

4.2 Upper bound for the regular generalized $C$-metrics

The type $D$ vacuum solutions which are not Kerr-NUT metrics have been named generalized $C$-metrics. As a consequence of lemma [2] they can be characterized by the condition $Z \wedge \overline{Z} \neq 0$. In these solutions the projection on the time-like principal plane $v(Z)$ of the complex Killing vector $Z$ is necessarily a non null vector: $v(Z, Z) \neq 0$. This fact is a consequence of the following lemma which is proven in appendix A:
Lemma 3 The type D vacuum solutions that satisfy $v(Z, Z) = 0$ are Kerr-NUT metrics, that is $Z \wedge Z = 0$.

Before studying linear system (20) for the case $Z \wedge \bar{Z} \neq 0$ we must consider the following result which is proven in appendix A:

Lemma 4 The strict (Ehlers and Kundt) C-metrics are the type D vacuum solutions that satisfy $Z \wedge \bar{Z} \neq 0$ and $Z \wedge II(\bar{Z}) = 0$. Moreover, in this case $II(\bar{Z}) = Z$, $\bar{m} = m$, $\bar{w} = w$.

We consider the strict C-metrics in the following section. Now we study the complementary set, the regular generalized C-metrics, which can be characterized by the conditions $Z \wedge \bar{Z} \neq 0$ and $Z \wedge \Pi(\bar{Z}) \neq 0$ as a consequence of lemmas 2 and 4. Under these constraints the system (20) for the scalar $m$ admits a solution given by:

$$m = m(w, U, Z) \equiv -\frac{1}{2}(Z, \bar{Z})\bar{w} + D \Delta, \quad \Delta \equiv (Z, Z)(\bar{Z}, \bar{Z}) - |II(Z, \bar{Z})|^2, \quad (23)$$

$$D \equiv II(Z, \bar{Z})[\nu(Z, Z) - \bar{\nu} II(Z, \bar{Z})] - (Z, Z)[\nu II(Z, \bar{Z}) - \bar{\nu}(Z, \bar{Z})]. \quad (24)$$

where $\nu$ is the scalar given in (20). Note that $\Delta \neq 0$ for a regular generalized C-metric. Indeed, the first order Riemann scalar $\Delta$ is the square of the 2-form $Z \wedge \Pi(\bar{Z}) \neq 0$. Thus, $\Delta = 0$ states that $Z \wedge II(\bar{Z})$ is a null 2-form, and the constraint (12) implies $v(Z, Z) = 0$, which is not possible as a consequence of lemma 3.

Thus, the Killing 2-form $\nabla Z$ takes the expression (19) where $m$ depends on first and second Riemann derivatives as (23-24). Consequently, we obtain the following result:

Theorem 2 All the covariant derivatives of the Riemann tensor of a regular generalized C-metric depend at most on the first order ones.

4.3 Upper bound for the strict (Ehlers and Kundt) C-metrics

At this point, we still have to study the case $Z \wedge \bar{Z} \neq 0$, $Z \wedge II(\bar{Z}) = 0$ which corresponds to the strict C-metrics as a consequence of lemma 1. This lemma also states that $II(\bar{Z}) = Z$, $\bar{m} = m$, $\bar{w} = w$. Under these constraints (20) becomes an identity and it does not allow us to obtain the scalar $m$ in terms of 0th- and 1st-order Riemann derivatives.

Let us consider the 2nd-order Riemann scalar:

$$K \equiv [(Z, \bar{Z}) - 2mw^{-\frac{1}{5}}]^2 + 12[w^{-\frac{1}{5}}(Z, Z) + w^{\frac{1}{5}}], \quad m \equiv -(\bar{U}, \nabla Z). \quad (25)$$

From (10), (19) and (24) we obtain $dK = 0$. Therefore $m$, and thus $\nabla Z$, can be obtained in terms of $w$, $Z$ and the invariant scalar $K$. Consequently, we arrive at the following result:

Theorem 3 All the covariant derivatives of the Riemann tensor of a strict (Ehlers and Kundt) C-metric depend at most on the first order ones and on a second order constant invariant.
5 Invariant classification

In studying the Karlhede upper bound of derivatives of the Riemann tensor in the above section, we have considered three classes of type D vacuum solutions. Every class has specific geometric properties and admits an invariant characterization in terms of 1st-order Riemann derivatives, that is, in terms of the invariant Killing vector $Z$:

- **Kerr-NUT solutions**: $Z \wedge \bar{Z} = 0$.
- **Strict (Ehlers and Kundt) C-metrics**: $Z \wedge \bar{Z} \neq 0$, $Z \wedge \Pi(\bar{Z}) = 0$.
- **Regular generalized C-metrics**: $Z \wedge \bar{Z} \neq 0$, $Z \wedge \Pi(\bar{Z}) \neq 0$.

Previous classifications have been introduced by Kinnersley [2] in integrating the Einstein vacuum solutions or by Edgar et al. [3] in studying the Karlhede upper bound.

Our aim here is to present a classification of generic type D spacetimes which is not induced by the Einstein field equations, and subsequently to study the classes which are compatible with the vacuum condition. This generic invariant classification can offer new geometrical and/or physical insight provided that it is defined by specific geometrical and/or physical restrictions.

5.1 Classifying D metrics

The Weyl tensor of a type D metric determines a 2+2 almost product structure defined by the principal 2-planes, and two real scalars defined by the complex eigenvalue. Then, naturally, we can consider two different classifications defined by first order differential conditions.

The first one corresponds to the classification of the 2+2 principal structure taking into account the invariant decomposition of the covariant derivative of the structure tensor $\Pi$ or, equivalently, according to the foliation, minimal or umbilical character of each principal plane. This classification is assumed in differential geometry [14] [15] and has been adapted to the general relativity framework [9], [10], [11]. In [9] we give the following.

**Definition 1** Taking into account the foliation, minimal or umbilical character of each principal 2-plane we distinguish $2^6 = 64$ different classes of type D metrics.

We denote the classes as $D_{pqr}^{lmn}$, where the superscripts $p, q, r$ take the value 0 if the time-like principal plane is, respectively, a foliation, a minimal or an umbilical plane, and they take the value 1 otherwise. In the same way, the subscripts $l, m, n$ collect the foliation, minimal or umbilical nature of the space-like principal plane.

The most degenerated class that we can consider is $D_{000}^{000}$ which corresponds to a product structure, and the most regular one is $D_{111}^{111}$ which means that neither the time-like plane nor the space-like plane are foliation, minimal or umbilical planes. We will put a dot in place of a fixed script (1 or 0) to indicate the set of metrics that cover both possibilities. So, for example, the metrics of type $D_{11}^{11}$ are the union of the classes $D_{111}^{11}$ and $D_{110}^{11}$; or a metric is of type $D_{0}^{0} ..$ if the time-like 2-plane is a foliation. Type $D_{0}^{0} ..$ corresponds to an umbilical structure (both planes are umbilical).
Here we use the following first order concomitants of the canonical bivector $\mathcal{U} = \frac{1}{\sqrt{2}} (U - i \ast U)$ to characterize some of these classes:

$$\Sigma \equiv \nabla \mathcal{U} - i(\delta \mathcal{U}) \mathcal{G}_\perp, \quad \Phi \equiv i(\delta \mathcal{U}) U - i(\delta \ast U) \ast U, \quad \Psi \equiv -i(\delta U) \ast U - i(\delta \ast U) U. \quad (26)$$

More precisely, we have the following results \cite{9}, \cite{11}, \cite{19}:

**Lemma 5** In a type $D$ space-time,

(a) The principal time-like plane is a foliation (type $D^{0\ldots0}$) if, and only if, $h(\Psi) = 0$.

(b) The principal space-like plane is a foliation (type $D^{\ldots0\ldots}$) if, and only if, $v(\Psi) = 0$.

(c) The principal time-like plane is minimal (type $D^{\ldots0\ldots}$) if, and only if, $h(\Phi) = 0$.

(d) The principal space-like plane is minimal (type $D^{\ldots\ldots0}$) if, and only if, $v(\Phi) = 0$.

(e) The principal structure is umbilical (type $D^{\ldots0\ldots0}$) if, and only if, $\Sigma = 0$.

The physical meaning of the types in lemma above follows from the kinematic interpretation of the geometric conditions that define them \cite{9} \cite{10}. The umbilical condition $\Sigma = 0$ implies that the two Debever null directions (lying on the time-like principal plane) are shear-free geodesics. The minimal or foliation character of the space like plane state, respectively, that both Debever null directions are expansion-free or vorticity-free. Moreover, the vectors $h(\Phi)$ and $h(\Psi)$ have been named, respectively, the expansion and the rotation of the timeline plane \cite{10}. Similarly, the vectors $v(\Phi)$ and $v(\Psi)$ are, respectively, the expansion and the rotation of the spacelike plane.

The second classification that we can consider is defined by first derivatives of the Weyl eigenvalue $w$ \cite{9}:

**Definition 2** Let $w = e^{\frac{2}{3}(\phi+i\psi)}$ be the Weyl eigenvalue. Taking into account the relative position between the gradients $d\phi$, $d\psi$ and each principal 2-plane we distinguish $2^4 = 16$ different classes of type $D$ metrics.

We denote the classes as $D[pq,rs]$, where $p,q,r,s$ take the value 0 if, respectively, the 1-form $v(d\psi)$, $v(d\phi)$, $h(d\psi)$, $h(d\phi)$ vanishes, and they take the value 1 otherwise.

The most degenerated class $D[00,00]$ is covered by the type $D$ metrics with constant eigenvalues, and the most general one $D[11,11]$ by those type $D$ spacetimes for which the gradients of both, the modulus and the argument of the Weyl eigenvalue, have non zero projection onto the principal planes. As above, a dot means that a condition is not fixed. A constant modulus, $d\phi = 0$, corresponds to type $D[0\ldots0\ldots]$, and a constant argument, $d\psi = 0$, corresponds to the metrics of type $D[0\ldots0\ldots]$.

The two invariant classifications of type $D$ metrics presented above have not, a priori, any relationship. Consequently, they define $2^{10}$ different classes. Nevertheless, we will see that the Einstein field equations or other restrictions on the Ricci tensor forbid many of these classes and correlate both classifications.

5.2 The sixteen classes of type $D$ metrics with vanishing Cotton tensor

The spacetime Cotton tensor $P$ depends on the Ricci tensor as $P_{\mu\nu,\beta} \equiv \nabla_{[\mu}Q_{\nu]\beta}$, $2Q \equiv \text{Ric} - \frac{1}{n}(\text{tr Ric})\delta$, The Bianchi identities equal the Cotton tensor with the divergence of the Weyl tensor. Consequently, if the Cotton tensor of a type $D$
metric vanishes the Bianchi identities take the same expression (2) as the vacuum case.

From the point (v) in lemma 5 the first condition in (2) means that the principal structure is umbilical (the principal directions are shear free null geodesics accordingly to the Goldberg-Sachs theorem), that is, the space-time is of type $D_0^0$. On the other hand, we have $\Phi + i \Psi = 2\chi$, and the second equation in (2) is equivalent to:

$$\Phi = d\phi; \quad \Psi = d\psi.$$ (27)

Note that in accordance with the Rainich theorem [20], this last equation states that the Weyl principal planes define a Maxwellian structure [9], [11], [19].

On the other hand, (27) and lemma 5 imply that the modulus and the argument of the Weyl eigenvalue govern, respectively, the minimal and the foliation character of the principal planes. This relation establishes a bijection between the classes of the two classifications that we have presented above. More precisely, we have [9]:

**Theorem 4** Every type D spacetime with zero Cotton tensor is of type $D_0^0$. Moreover, it is of class $D_{pq}^0$ if, and only if, it is of class $D[lm, pq]$. So we have just 16 classes of type D spacetimes with zero Cotton tensor.

5.3 The six classes of type D vacuum solutions

As a consequence of the above theorem there are at most sixteen classes $D[lm, pq]$ of type D vacuum solutions. Now we show that only six of these classes are compatible with the vacuum equations. This result is based on the following lemma which is proven in appendix B:

**Lemma 6** In a type D vacuum solution:

(i) If $v(\Phi) = 0$ then $v(\Psi) = 0$.
(ii) If $h(\Phi) = 0$ then $h(\Psi) = 0$.
(iii) If $v(\Psi) = 0$ then either $v(\Phi) = 0$ or $h(\Psi) = 0$.
(iv) If $h(\Psi) = 0$ then either $h(\Phi) = 0$ or $v(\Psi) = 0$.

Point (i) of this lemma implies that if a metric is of type $D_{0\cdot \cdot}$ then it is of type $D_{0\cdot \cdot \cdot}$. Consequently, type $D_{10\cdot \cdot}$ is forbidden. Similarly, point (ii) implies that type $D_{0\cdot \cdot \cdot}$ is forbidden. On the other hand, point (iii) states that type $D_{0\cdot \cdot \cdot}$ implies either type $D_{0\cdot \cdot \cdot}$ or type $D_{0\cdot \cdot \cdot}$. Consequently, class $D_{01,11}$ is forbidden. Similarly class $D_{11,01}$ is forbidden as a consequence of point (iv).

Finally, a metric of class $D_{00,00}$ is a product metric which in the vacuum case implies a flat spacetime.

The remaining six compatible classes are presented below in a flow chart which illustrate the degeneration paths from some classes to others. Note that no classes in the first degeneration level are compatible with the vacuum condition: the regular class can decline to three of the six classes in the second level. In the third level only two of the four classes are compatible.

Classes $D_{01,00}$, $D_{00,01}$ and $D_{01,01}$ have Weyl real eigenvalues and correspond, respectively, to the A-metrics, B-metrics and C-metrics by Ehlers and Kundt [1]. Classes $D_{11,00}$ and $D_{00,11}$ are the NUT-like generalization of the A-metrics and B-metrics, respectively. The regular class $D_{11,11}$ contains both the regular generalized C-metrics and the regular Kerr-NUT metrics. We can summarize these results in the following.
Theorem 5 Taking into account the first derivatives of the Weyl tensor we can consider six classes of type D vacuum solutions which can be characterized by the following conditions:

- $D[01,00]$ (A-metrics): $h(\Phi) = 0$, $v(\Psi) = 0$.
- $D[00,01]$ (B-metrics): $v(\Phi) = 0$, $h(\Psi) = 0$.
- $D[01,01]$ (C-metrics): $v(\Phi) \neq 0$, $h(\Phi) \neq 0$, $v(\Psi) = 0$, $h(\Psi) = 0$.
- $D[11,00]$ (A-NUT-metrics): $h(\Phi) = 0$, $v(\Psi) \neq 0$.
- $D[00,11]$ (B-NUT-metrics): $v(\Phi) = 0$, $h(\Psi) \neq 0$.
- $D[11,11]$ (Regular C and Kerr-NUT metrics): $v(\Psi) \neq 0$, $h(\Psi) \neq 0$.

5.4 Some relevant subclasses and a summary in algorithmic form

As showed in section 4 condition $Z \wedge \bar{Z} = 0$ characterizes the Kerr-NUT metrics. In terms of the vectors $\Phi$ and $\Psi$ this condition writes $N = 0$, where

$$N \equiv v(\Phi) \wedge h(\Psi) + v(\Psi) \wedge h(\Phi).$$

On the other hand, in lemma 3 we have proven that a solution which satisfies $v(Z,Z) = 0$ is, necessarily, a Kerr-NUT metric. Note that the Killing vector $Z$ is orthogonal to the null vector $v(Z)$, and using the results in [17] it is easy to prove that another Killing vector exists with the same property. Consequently, these solutions have null orbits. Moreover, $v(Z,Z) = 0$ is equivalent to $v(\Psi,\Psi) = 0$.

Thus we have:

Proposition 3 In the regular class $D[11,11]$ ($v(\Psi) \neq 0$, $h(\Psi) \neq 0$) we can distinguish three subclasses which can be characterized by the following conditions:

- $D[11,11]_C$ (Regular C-metrics): $N \neq 0$.
- $D[11,11]_K$ (Regular Kerr-NUT metrics with non null orbits): $N = 0$, $v(\Psi,\Psi) \neq 0$.
- $D[11,11]_n$ (Solutions with null orbits): $v(\Psi,\Psi) = 0$.

We can easily relate prior classifications with ours. Classes I, II III and IV by Edgar et al. [3] correspond to specific classes or types of our approach: class I to our type $D[1.00]$, class II to our class $D[11,11]_n$, class IIIA to our class $D[01,01]$, class IIIB to our class $D[11,11]_C$, class IIIC to our class $D[11,11]_K$, and class IV to our type $D[00 .1]$.

Finally, we present our results on the classification of the type D vacuum solutions in an algorithmic form by using a flow chart. We use the following Weyl
concomitants: the projectors on the principal planes $v = \frac{1}{2} g + \mathcal{U} \cdot \mathcal{U}$ and $h = g - v$, the linear first order vectors $\Phi$ and $\Psi$ given in (26) (or, equivalently, in (27)), and the quadratic first order 2-form $N$ given in (28). The explicit expression of $w$ and $\mathcal{U}$ in terms of the Weyl tensor are given by [13]:

$$w = - \frac{\mathcal{W}_{\alpha\beta}^{\mu\nu} \mathcal{W}_{\mu\nu}^{\alpha\beta}}{2 \mathcal{W}_{\alpha\beta}^{\mu\nu} \mathcal{W}_{\mu\nu}^{\alpha\beta}}; \quad \mathcal{U} = \frac{Q_{\alpha\beta}^{\mu\nu} F_{\mu\nu}}{\sqrt{3} Q_{\alpha\beta}^{\mu\nu} F_{\mu\nu} F_{\alpha\beta}}, \quad Q \equiv \frac{1}{w} \mathcal{W} - \mathcal{G},$$

where $F$ is an arbitrary 2-form such that $Q_{\alpha\beta}^{\mu\nu} F_{\mu\nu} \neq 0$. 

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Diagram:

- $v, h, \Phi, \Psi, N$
  - $v(\Psi) \neq 0 \neq h(\Psi)$
    - $N \neq 0$
      - $v(\Psi, \Psi) \neq 0$
        - $D[11,11]_{C}$ regular C-metrics
      - $D[11,11]_{K}$ regular Kerr-NUT
    - $v(\Psi, \Psi) = 0$
      - $D[11,11]_{n}$ null orbit metrics
  - $v(\Phi) \neq 0 \neq h(\Phi)$
    - $D[01,01]$ C-metrics
  - $v(\Phi) \neq 0$
    - $v(\Phi) \neq 0$
      - $D[11,00]$ A-NUT-metrics
    - $v(\Phi) = 0$
      - $D[01,00]$ A-metrics
  - $h(\Psi) \neq 0$
    - $D[00,11]$ B-NUT-metrics
  - $h(\Phi) \neq 0$
    - $D[00,01]$ B-metrics
6 Discussion and comments

The Cartan invariant scheme based on the Riemann tensor and its covariant derivatives was introduced by Brans [21] in general relativity and, after Karlhede’s work [22], this approach became more helpful within the relativistic framework. The Cartan-Karlhede method is based on working in an orthonormal (or a null) frame, fixed by the underlying geometry of the Riemann tensor. Nevertheless there are a lot of historic results which show that the determination of a Riemann canonical frame is not always necessary to label a family of metrics: theorems that characterize locally flat spaces, Riemann spaces with a maximal group of isometries, and locally conformally flat spaces are some examples. Also the well-known characterizations of the Stephani Universes or of the Friedmann-Lemaître-Robertson-Walker Universes. These examples show that the characterization conditions can involve tensorial concomitants whereas the Cartan-Karlhede scheme only uses scalar concomitants.

An example where the Cartan-Karlhede approach has shown its efficacy is the study of the covariant derivatives of the Riemann tensor for the type D vacuum metrics presented by Åman [6] and performed in [3]. But the results we present here show that a tensorial approach can bring new knowledge on this topic. Several identities with no clear sense and which have previously been acquired with hard computer support have been obtained here in an easy way and their plain geometric meaning has been outlined.

In studying the Karlhede upper bound, we have used the invariant Killing vector \( Z \) and its associated Killing 2-form \( \nabla Z \), and only three different cases have had to be considered. For the Kerr-NUT vacuum metrics, we obtain the expression \( \nabla Z \) which give \( \nabla Z \) in terms of first order invariants, and we arrive to theorem 1. For the regular C-metrics, expressions \( \nabla Z \) give \( \nabla Z \) in terms of first order invariants, and we arrive to theorem 2. Finally, for the strict Ehlers and Kundt C-metrics, expressions \( \nabla Z \) show that \( \nabla Z \) depend on first order invariants and on a second order constant invariant \( K \), and we arrive to theorem 3.

These three theorems show that the Karlhede upper bound is two. Nevertheless, there is a relevant difference between the third case and the other ones. Theorems 1 and 2 imply that a specific Kerr-NUT solution or a specific regular C-metric can be characterized by exclusively using first order Weyl concomitants. Our local intrinsic labeling of the Schwarzschild [23] and Kerr [24] black holes are examples of this fact. In particular, the mass and the angular momentum are first order constant invariants. However, theorem 3 states that in order to distinguish two different strict C-metrics, we need to calculate the second order constant invariant \( K \).

It is worth pointing out that the \( D \)-metrics (charged counterpart of the type D vacuum solutions) also admit the invariant complex Killing vector \( Z \). Thus, an invariant approach to these solutions similar to that presented here could provide a better understanding of them. Note that in this case the (non vanishing) Ricci tensor will also be an important piece in the invariant analysis. In [17], [25] we have studied invariant properties of this type of metrics, which will be useful for this forthcoming work.

This procedure could be also useful in studying the Cartan-Karlhede upper bound not only in type D solutions but also in solutions with another Petrov-Bel type. For example a similar invariant vector can be defined in type II metrics, and
in type N and type III spacetimes where all the Weyl invariants vanish, the first
derivatives of the Weyl tensor also define invariant vectors.

A generic classification of type D space times is a powerful tool in learning
generic properties of solutions to Einstein equations. In [9] we introduced a
classification based on first order Weyl constraints and where the principal planes
play a symmetric role: if a class is defined by a property of the time-like plane,
then there exists a space-like counterpart class. We find this fact in the pioneer
paper by Ehlers and Kundt [1] where the B-metrics are the time-like counterpart
of the (space-like) A-metrics. In [9] we used our classification to label the A, B and
C charged metrics. Here we apply this approach to the full set of type D vacuum
solutions and only six classes survive: the Ehlers and Kundt A and B-metrics, their
NUT generalization (the A-NUT and B-NUT metrics), the Ehlers and Kundt C-
metrics, and the most regular class which includes three subclasses, the regular
Kerr-NUT metrics, the regular generalized C-metrics, and the solutions with null
orbits.

Our classification has a hierarchic structure with five possible levels of degener-
atation when the Cotton tensor vanishes. In the vacuum case only the first, the third
and the fourth levels remain. This hierarchic structure enables a simple operational
algorithm to be build to distinguish every class.

It is worth remarking that the generic character of our classification allows
us to apply it to the full set of type D metrics or to any specific family of type
D solutions. For example it has been implemented elsewhere [9] to classify the
charged counterpart of the (static) A, B and C-metrics and to achieve an algorithm
to label every solution. In particular, an intrinsic characterization of the Reissner-
Nordström has been obtained. A similar approach could be accomplished for the
charged counterpart of the six classes of type D vacuum solutions considered here.

On the other hand, the generalization of this approach to other Petrov-Bel
solutions is also possible. Indeed, types III and II admit again a privileged 2+2
almost-product structure defined by the two null Debever directions in type III
and by a non-null Weyl eigenbivector in type II.

Appendix A: Proof of lemmas 1, 3 and 4

A.1 Proof of lemma 1

Suppose that \((Z, Z) = 0\). If we differentiate this condition and take into account
\([10]\) we obtain:

\[
2w^\dagger Z = [\bar{w}^\dagger (Z, \bar{Z}) - 2m] \Pi(Z).
\] (29)

From here we have \(Z \wedge \Pi(Z) = 0\), that is, \(Z\) is a null vector which lies on the time-
like principal plane: \(Z = \Pi(Z)\). Then \([12]\) implies \(Z \wedge \bar{Z} = 0\) and consequently
\((Z, \bar{Z}) = 0\), and \([29]\) is equivalent to:

\[
Z = \Pi(Z), \quad m + w^\dagger = 0.
\] (30)

Now if we differentiate the first constraint in \([30]\) (or equivalently \(i(Z)\mathcal{U} = i(Z)\bar{\mathcal{U}}\)),
and take into account \([10]\) and \(Z \wedge \bar{Z} = 0\), we arrive at \(m = w^\dagger\) which is not
compatible with the second constraint in \([30]\).
A.2 Proof of lemma 3

Condition \( v(Z, Z) = 0 \) equivalently states \( (Z, Z) + \Pi(Z, Z) = 0 \). If we differentiate this scalar condition and we take into account the expression of the covariant derivatives of \( U \) of \( \mathcal{U} \), we obtain the following expression for \( m \):

\[
m = -\frac{1}{2} \bar{w}^4(Z, \bar{Z}) - \nu, \quad \nu \equiv \frac{1}{2} \bar{w}^4 + \frac{1}{2} w^4(Z, Z).
\]

Then equation (20) becomes:

\[
\nu h(\bar{Z}) + \bar{\nu} h(Z) = 0.
\]

On the other hand, if we differentiate (31) and make use of (10) and (19) we obtain a new scalar condition:

\[
(Z, Z) + 2m\bar{w}^{-\frac{1}{2}} + 2\bar{w}w^{-\frac{1}{2}} = 0.
\]

Finally, if we differentiate this equation and we take into account (10) and (19) we arrive at:

\[
\nu v(\bar{Z}) + \bar{\nu} v(Z) = 0.
\]

Constraints (32) and (34) imply \( Z \wedge \bar{Z} = 0 \). Consequently \( \Pi(Z, Z) = 0 \) or \( \nu = 0 \). This last condition and (20) and (19) lead to \( (Z, Z) + \Pi(Z, Z) = 0 \), and identity (12) implies \( \Pi(Z) = Z \), which also implies, with \( \Pi(Z, Z) = 0 \).

Note that (31) gives the scalar \( m \) in terms of 0th- and 1st-order Riemann derivatives for the type D vacuum solutions satisfying \( v(Z, Z) = 0 \). Thus, we could state for them a specific theorem similar to theorems 1 and 2. Nevertheless we prefer use this lemma 3 and consider this case as included in theorem 1.

A.3 Proof of lemma 4

From the hypothesis \( Z \wedge \bar{Z} \neq 0 \) we have necessarily \( Z \wedge \Pi(Z) \neq 0 \). Then, condition \( Z \wedge \Pi(Z) = 0 \) implies that (20) is equivalent to:

\[
\bar{\mu} Z = \mu \Pi(\bar{Z}), \quad \bar{\nu} Z = \nu \Pi(\bar{Z}).
\]

On the other hand, if we differentiate \( Z \wedge \Pi(Z) = 0 \) and we make use of (10) and (19) we obtain a tensorial equation. Its trace leads to:

\[
(m + \bar{\mu}) Z = (m + \mu) \Pi(\bar{Z}), \quad (\bar{w} + 2\bar{\nu}) Z = (\bar{w} + 2\nu) \Pi(\bar{Z}),
\]

where in obtaining the second equation we have used the fact that \( Z \wedge \Pi(Z) = 0 \) implies \( \Pi(Z, \bar{Z}) = (Z, Z) \Pi(\bar{Z}) \) and \( \Pi(Z, \bar{Z}) \Pi(\bar{Z}) = (\bar{Z}, Z) \bar{Z} Z \). If we again make use of these relations, from equations (35) and (36) we obtain:

\[
\bar{m} Z = m \Pi(\bar{Z}), \quad \bar{w} \bar{Z} = w \bar{Z} \Pi(\bar{Z}),
\]

Under the hypothesis \( Z \wedge \bar{Z} \neq 0 \), at least one of the scalars \( (Z, \bar{Z}) \) and \( \Pi(Z, \bar{Z}) \) does not vanish. Consequently (35) implies \( v = 0 \). This constraint and (37) lead to \( \bar{m} = m, \bar{w} = w \) and \( \Pi(\bar{Z}) = \bar{Z} \). Moreover, the solution is a strict C-metric because it has real Weyl eigenvalues.
Appendix B: Proof of lemma 6

The conditions involved in lemma 6 can be stated by using the projections \( v(\chi) \) and \( h(\chi) \) of complex vector \( \chi = \frac{1}{2}[\Phi + i\Psi] \). In terms of \( \{w, U, Z\} \) these projections take the expression:

\[
2v(\chi) = w^\lambda i(Z)(U) + i(Z)(\bar{U}), \quad 2h(\chi) = w^\lambda i(Z)(U) - i(Z)(\bar{U}). \tag{39}
\]

Suppose that \( v(\Phi) = 0 \), that is, \( v(\chi) = -v(\bar{\chi}) \). If we calculate the covariant derivative of this equation and we take into account expressions (39) and derivatives (10) and (19) we obtain a 2-tensorial equation \( E_{\alpha\beta} = 0 \). The total projection of this equation on the time-like plane, \( v^\mu v^\nu E_{\alpha\beta} = 0 \), leads to \( v(\Psi) \otimes v(\psi) = 0 \), and so \( v(\psi) = 0 \). Consequently point (i) is proven.

Suppose now that \( v(\Phi) = 0 \), that is, \( v(\chi) = v(\bar{\chi}) \). If we calculate the covariant derivative of this equation and we take into account expressions (39) and derivatives (10) and (19) we obtain a 2-tensorial equation \( F_{\alpha\beta} = 0 \). The mixed projection of this equation on the time-like and space-like planes, \( v^\mu h^\nu F_{\alpha\beta} = 0 \), leads to \( v(\Phi) \otimes h(\Psi) = 0 \), and so either \( v(\Phi) = 0 \) or \( h(\Psi) = 0 \). Consequently point (iii) is proven.

The proof of points (ii) and (iv) of lemma 6 is similar to the proof of points (i) and (iii) by exchanging \( v \) for \( h \).

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