Lamé problem for a multilayer cylinder made of nonlinear elastic materials under finite strains

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Abstract. The paper presents an approach to the numerical-analytical solution of the Lamé problem for a multilayer cylinder made of nonlinear elastic materials under finite strains. The dependence of the stress state of a cylinder on its inner radius after deformation is investigated and nonlinear effects are analyzed.

1. Introduction
Composite materials and structural elements made of composites are widely used now. The examples are multilayered hollow cylinders.

Along with numerical methods of analysis of the stress state in such cylinders, analytical and numerical-analytical solutions are of interest. In the theory of elasticity many analytical solutions are known, however within the framework of nonlinear elasticity under large strains the number of exact analytical solutions is small. In particular, such solutions are known for incompressible materials [1, 2] and for a special case of the Blatz-Ko material [4–7], and some nonstationary problems such as ones on the propagation of waves [8] or ones on phase transitions [9] can be solved analytically.

In the work [10] the method of numerical-analytical solution of the Lamé problem for hollow cylinders under large strains is developed. The present paper generalizes this method to the case of multilayer cylinders.

2. Problem statement and methods
The formulation of problem in the coordinates of the initial (undeformed) state includes:
the equilibrium equation
\[ \nabla \cdot P = 0, \quad (1) \]
the relations between different stress tensors
\[ P = \det \Psi \left( \Psi^{-1} \right)^T \cdot \sigma, \quad (2) \]
the constitutive relations for the compressible Mooney-Rivlin material
\[ W = \frac{1}{2} \mu (1 + \Delta)^{-2/3} I + d \Delta^2, \quad (3) \]
(here \( \mu \) is the shear modulus, the constant \( d \) characterizes the bulk stiffness of the material)
the kinematic relations
\[ G = \Psi \cdot \Psi^T, \quad (4) \]
\[ 1 + \Delta = \det \Psi, \]
the boundary conditions on external lateral area of a cylinder have the form
\[ \mathbf{n} \cdot \mathbf{P} = p(\det \Psi)\mathbf{n} \cdot (\Psi^{-1})^T. \]

This condition correspond the pulling pressure defined on the external boundary. The internal lateral cylinder surface is load-free, and on the bases of the cylinder there are no displacements in the direction of its axis. The condition of ideal contact on boundaries between layers is satisfied: condition of continuity of radial deformation and radial stress on boundaries between layers.

The following notation is used: \( W \) is an elastic potential, \( \mathbf{P} = \frac{dW}{d\Psi} \) is the first Piola-Kirchhoff stress tensor, \( \Psi = \nabla \mathbf{R} \) – a strain gradient, \( \mathbf{R} \) is the radius vector of the particle under strain, \( \mathbf{G} \) is the Green stress tensor measure, \( \mathbf{\sigma} \) is the true stress tensor, \( \Delta \) – relative change of volume, \( J \) is first invariant of the Green stress tensor measure, \( \mathbf{n} \) is the unit normal vector to the boundary in the unstrained state, \( p \) is the pulling pressure defined on the external boundary.

The numerical-analytical solution has been obtained using the semi-inverse method. This method is described in [10]. The essence of a method consists in the use of dependence between particle coordinates in initial and final states, taking into account the axial symmetry of the problem. Using this dependence, one can simplify the equations.

As a result, the problem solution reduces to solving an ordinary second-order differential equation with respect to radial coordinate of the particle in the unstrained state. After that the solution of the equation is found taking into account boundary conditions.

As an alternative of the solution of the boundary value problem, the Cauchy problems for each layer are solved. At first, the radial displacement of internal border of the inner layer or cavity radius \( R_0 \) in the deformed state is set. The radial component of deformation gradient is obtained from the assumption that this border is unloaded. This component is used as the initial condition in the Cauchy problem.

Then the Cauchy problems are solved sequentially for all layers. Stresses and displacements at the internal boundary of each layer are determined by stresses and displacements at the external boundary of the previous layer taking into account the conditions of ideal contact. As a result, the stress-strain state of each layer is obtained and, in particular, pulling stress on the external border of an external layer is computed.

3. Results
The hollow multilayer cylinder made of the compressible Mooney-Rivlin materials [11, 12] is considered. The cylinder consists of five layers with different compressibility coefficients. The thickness of each layer in the undeformed state is equal to the radius \( r_0 \) of the hollow cylinder in this state (figure 1).

The results are shown in Tables 1–3. The moduli \( \mu \) for all layers are equal. The volumetric stiffness: for the first layer \( d / \mu = 1000 \), for the second layer \( d / \mu = 0.001 \), for the third layer \( d / \mu = 1000 \), for the fourth layer \( d / \mu = 0.001 \), and for the fifth layer \( d / \mu = 1000 \).

In figures 1 and 2 the dependencies of the true circumferential stress \( \sigma_{\theta} / \mu \) and the true radial stress \( \sigma_r / \mu \) on the values of internal radius of the cylinder at the external boundary after deformation are presented, respectively.

Figure 1 shows that the dependencies for more compliant layers (with volumetric stiffness \( d / \mu = 0.001 \)) are not monotonous.
Table 1. The true circumferential stress $\sigma_{\varphi\varphi} / \mu$ at the external boundary of the $i$-th layer for different values of the internal radius $R_0$ of the cylinder after deformation.

| $R_0 / r_0$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ |
|-------------|---------|---------|---------|---------|---------|
| 1.5         | 1.8040230610 | 0.1112104036 | 0.4783516855 | 0.0633964038 | 0.2567907170 |
| 2.0         | 3.7449495630 | 0.1824732422 | 1.027218769 | 0.1220732101 | 0.5705600353 |
| 2.5         | 6.0779505900 | 0.2181412321 | 1.6361848306 | 0.1679098802 | 0.9257430607 |
| 3.0         | 8.8645045470 | 0.2307313923 | 2.3106487182 | 0.1994504003 | 1.3156090971 |
| 3.5         | 12.1240693600 | 0.230785046 | 3.0583350257 | 0.2186427369 | 1.7385587935 |
| 4.0         | 15.8630715500 | 0.2310648718 | 3.8850367444 | 0.2284709786 | 2.195139984 |
| 4.5         | 20.0828997000 | 0.230732101 | 4.7939562447 | 0.2317720769 | 2.6864137819 |
| 5.0         | 24.7822441000 | 0.230785046 | 5.7860806695 | 0.2308218532 | 3.123024851 |
| 5.5         | 29.9588690800 | 0.198645794 | 6.860414969 | 0.2272926755 | 3.775043926 |
| 6.0         | 35.6094004200 | 0.191431764 | 8.015575489 | 0.2223485205 | 4.3718925033 |

Figure 1. Dependence of true circumferential stress $\sigma_{\varphi\varphi} / \mu$ at the external boundary of the $i$-th layer with volumetric stiffness $d / \mu = 0.001$ on the internal radius $R_0$ of the cylinder after deformation.

Table 2. True radial stress $\sigma_{rr} / \mu$ at the external boundary of the $i$-th layer for different values of the internal radius $R_0$ of the cylinder after deformation.

| $R_0 / r_0$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ |
|-------------|---------|---------|---------|---------|---------|
| 1.5         | 0.4283885378 | 0.0298186576 | 0.0888159583 | 0.0111388933 | 0.0355096293 |
| 2.0         | 0.5735276333 | 0.0473545412 | 0.1562795466 | 0.0208826671 | 0.0696198324 |
| 2.5         | 0.6326144072 | 0.0550952974 | 0.2015811658 | 0.0279781291 | 0.0983573241 |
| 3.0         | 0.6597993936 | 0.0570320858 | 0.2307531388 | 0.0324384538 | 0.1202152906 |
| 3.5         | 0.6734565104 | 0.0560235950 | 0.2492892401 | 0.0348053865 | 0.1361960982 |
| 4.0         | 0.6807125644 | 0.0537277489 | 0.2612215093 | 0.0356923056 | 0.1479114818 |
| 4.5         | 0.6847233932 | 0.0510205421 | 0.2690248302 | 0.0356228514 | 0.1564240885 |
| 5.0         | 0.6870246655 | 0.0483360652 | 0.2742202196 | 0.0349757748 | 0.1626172249 |
| 5.5         | 0.6883539727 | 0.0458723632 | 0.2777483903 | 0.0340111807 | 0.1671244958 |
| 6.0         | 0.6891050989 | 0.0437049659 | 0.2801860083 | 0.0328987007 | 0.1704362974 |
As it is seen from figure 2, the true radial stress $\sigma_r / \mu$ at the external boundaries of layers at first increases with the increasing of deformations, and then begins to decrease.

**Figure 2.** Dependence of true radial stress $\sigma_r / \mu$ at the external boundary of the $i$-th layer with the volumetric stiffness $d / \mu = 0.001$ on the internal radius $R_0$ of the cylinder after deformation

In figure 3 the dependence of true radial stress $\sigma_r / \mu$ at the external boundary of the $i$-th layer with volumetric stiffness $d / \mu = 1000$ on the internal radius $R_0$ of the cylinder after deformation is presented. It is seen from figure 3 that for more rigid layers (with the volumetric stiffness $d / \mu = 1000$) stress $\sigma_r / \mu$ at first increases. Then, at increase in values of radius $R_0$, it is stabilized.

**Figure 3.** Dependence of true radial stress $\sigma_r / \mu$ at the external boundary of the $i$-th layer with volumetric stiffness $d / \mu = 1000$ on the internal radius $R_0$ of the cylinder after deformation

From table 3 it can be seen that the dependence of true circumferential stress $\sigma_{\varphi} / \mu$ at the internal boundary of the $i$-th layer of the cylinder for more compliant layers ($d / \mu = 0.001$) on the internal radius $R_0$ of the cylinder after deformation is nonmonotonic.
Table 3. True circumferential stress $\sigma_{\phi\phi} / \mu$ at the internal boundary of the $i$-th layer for different values of the internal radius $R_0$ of the cylinder after deformation.

| $R_0 / r_0$ | $i = 1$          | $i = 2$          | $i = 3$          | $i = 4$          | $i = 5$          |
|-------------|------------------|------------------|------------------|------------------|------------------|
| 1.5         | 0.9785621540     | 0.0803592293     | 0.3699599392     | 0.0520237898     | 0.2170193462     |
| 2.0         | 1.7509747485     | 0.1320144715     | 0.7791015579     | 0.1002720957     | 0.4796598934     |
| 2.5         | 2.5105295532     | 0.1580932350     | 1.2104857741     | 0.1380812415     | 0.7723217255     |
| 3.0         | 3.3228445587     | 0.1675807165     | 1.6641455141     | 0.1642254045     | 1.0863217478     |
| 3.5         | 4.219829259      | 0.1678622090     | 2.1449940882     | 0.1886095130     | 1.4189981336     |
| 4.0         | 5.2116916660     | 0.1636507646     | 2.6584795264     | 0.1882891884     | 1.7704770466     |
| 4.5         | 6.3129153013     | 0.1576523069     | 3.2086957998     | 0.1915716968     | 2.1414136906     |
| 5.0         | 7.5269577140     | 0.1513264810     | 3.7982426322     | 0.1910078330     | 2.5326375359     |
| 5.5         | 8.8569804649     | 0.1454149093     | 4.4283781409     | 0.1882891884     | 2.9446773656     |
| 6.0         | 10.30491861      | 0.1402599588     | 5.0992813581     | 0.1843745226     | 3.3776832306     |

4. Conclusion

Thus, the Lamé problem is solved for multilayer hollow cylinder made of nonlinear-elastic compressible material under finite strains. The nonlinear effects related with the nonmonotonic dependence of stresses on the radius of the hollow cylinder after deformation for layers with the smaller module of volume expansion are revealed.

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