Unconditionally Secure Qubit Commitment Scheme Using Quantum Maskers

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A commitment scheme allows one to commit to hidden information while keeping its value recoverable when needed. Despite considerable efforts, an unconditionally and perfectly secure bit commitment has been proven impossible both classically and quantum-mechanically. The situation is similar when committing to qubits instead of classical bits as implied in the no-masking theorem [K. Modi et al., Phys. Rev. Lett. 120, 230501 (2018)]. In this Letter, we find that circumvention of the no-masking theorem is possible with the aid of classical randomness. Based on this, we construct an unconditionally secure quits-commitment scheme that utilises any kind of universal quantum maskers with optimal randomness consumption, which is distributed by a trusted initializer. This shows that randomness, which is normally considered only to obscure the information, can benefit a quantum secure communication scheme. This result can be generalised to an arbitrary dimensional system.

In a commitment protocol, a sender (say Alice) commits to a secret value in the way that a receiver (Bob) cannot access the value until it is revealed. On the other hand, Bob should be able to reject Alice’s cheating of revealing a value different from the originally committed value. A commitment scheme, which has many cryptographic applications, is said to be unconditionally secure when the scheme does not depend on the computational power of both participants, while it is perfectly secure perfectly secure when the scheme works with zero probability of failure. However, an unconditionally and perfectly secure commitment protocol is impossible. If a commitment protocol is unconditionally and perfectly binding, which means that any attempt of Alice to change the already committed secret value can be detected by Bob, then there should be a unique secret value for each commitment provided by Alice to Bob. If so, the protocol, however, cannot be unconditionally and perfectly concealing, i.e., Bob could have some information about the committed value before Alice discloses it, because Bob with unlimited computational power can reveal the secret value by searching through every possible secret values and corresponding commitments.

Quantum bit-commitment is an attempt to circumvent this difficulty by using quantum mechanics. However, it has been proven that an unconditionally secure commitment of a classical value is impossible even with the aid of quantum mechanics. Meanwhile, a fully classical breakthrough was developed by Rivest introducing a partially credible mediator, “trusted initializer,” who is only involved in the beginning (setup) of the protocol and does not receive any information during the remaining protocols. In Ref. [5], it was shown that the trusted initializer enables the construction of unconditionally secure commitment schemes for classical bits.

Recently it transpired that the situation is similar for qubit commitment, in which the committed secret value is a qubit. A result recently proven by Modi et al. known as the no-masking theorem states that it is impossible to encode quantum information in a bipartite pure quantum state so that it is inaccessible to local subsystems. As a corollary, an unconditionally and perfectly secure qubit commitment is also forbidden [7]. It shows that qubit commitment schemes with straightforward protocols are vulnerable to entanglement-based attacks [7].

One might expect that the no-masking theorem can be extended to mixed states similarly as the case of the no-broadcasting theorem extended from the no-cloning theorem. In this Letter, however, we show that this is not the case. We first prove a stronger version of the no-masking theorem by showing that one additionally needs at least log_2 d bits and 2 log_2 d bits of randomness respectively when entanglement is available and when only zero one-way quantum discord is allowed in the masked bipartite quantum state. We then introduce a simple way to circumvent the strengthened no-masking theorem by constructing explicit examples of quantum masker that consumes the minimal amount of randomness.

We further show that the class of universal quantum maskers that consume uniform randomness can be used for unconditionally secure qubit commitment schemes by introducing a trusted initializer, whose role is very similar to that of Rivest’s commitment scheme. Our scheme has a security advantage over Rivest’s scheme even when applied to the bit commitment as it avoids a certain type of security failure.

Quantum Masker.— According to Modi et al.’s definition, masking quantum information is encoding quantum information in a composite quantum state with the information hidden from both the subsystems and remaining only in the quantum correlation between them. The technical definition in Ref. [7] only considers pure composite states so that the encoding of correlations are only allowed in the forms of entanglement. We note that, however, there are quantum correlations beyond entanglement in the case of mixed states. A typical example would be the states with no entanglement but having non-zero quantum discord. Therefore, we re-
define the quantum masker to encompass the processes that encode quantum information in any kind of quantum correlations.

**Definition 1.** An operator $\mathcal{M}$ from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to mask quantum information contained in states $\{\phi_{kA} \in \mathcal{B}(\mathcal{H}_A)\}$ by mapping them to $\{\Psi_{kA'B'} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)\}$ such that all marginal states of $\Psi_{kA'B'} = \mathcal{M}(\phi_{kA})$ are identical, i.e.,

$$\rho_{A'} = \text{Tr}_{B'} \Psi_{kA'B'},$$

and

$$\rho_{B'} = \text{Tr}_{A'} \Psi_{kA'B'},$$

with an unmasking operator $\mathcal{U}$ from $\mathcal{B}(\mathcal{H}_{A'} \otimes \mathcal{H}_{B'})$ to $\mathcal{B}(\mathcal{H}_A)$ such that

$$\mathcal{U} (\Psi_{kA'B'}) = \phi_{kA}$$

for all $k$. We call such $\mathcal{M}$ a (quantum) masker and we say that $\mathcal{M}$ is universal if it can mask an arbitrary quantum state.

The condition on the existence of recovery, or unmasking map is crucial, since without it, a simple erasure map would qualify as a universal masker. Besides, the extension of universal masker to arbitrary mixed state input is natural, since, from the linearity of quantum channel, a masker that can mask every pure state can also mask arbitrary mixture of pure states. Therefore, every universal quantum masker $\Phi_M$ is demanded to be an invertible quantum process. Such process can be always expressed in terms of a quantum state $\omega_S$ and a unitary transformation $M$ which maps the input system $C$ and the ancillary system $S$ to the systems $A$ and $B$ such that

$$\Phi_M(\rho) = M (\rho_C \otimes \omega_S) M^\dagger$$

for every quantum state $\rho$ with some ancillary state $\omega_S$. We will call the ancillary state $\omega_S$ of a masking process as the safe state in the sense we put quantum information $\rho$ into a secure safe. The no-masking theorem states that there is no universal masking process $\Phi_M$ with a pure safe state. This raises a natural question : is it possible to mask quantum information with non-pure safe state?

It turns out that it is indeed possible and we introduce two examples. (Fig. 1). The output systems with index $A_i$ and $B_i$ belong to Alice and Bob, respectively. In each case, discarding one party’s quantum state yields the maximally mixed state for the other party. Both maskers can be easily generalized to $d$-dimensional universal maskers by replacing $X$ and $Z$ gates with their $d$-dimensional generalizations and the maximally mixed state $\frac{1}{d} \mathbb{I}$ with its $d$-dimensional counterpart $\frac{1}{d} \mathbb{I}$.

An interesting example of universal masker is the 3-qubit masker introduced in (b) of Fig. 1. The 3-qubit masker does not form any entanglement and quantum discord with measurement on Bob’s side for any input state, but still masks a qubit of quantum information. The systems $B_1$ and $B_2$ are in classical states, therefore they can be transmitted through a classical channel. The output state has, nonetheless, non-zero quantum discord with measurement on Alice’s side for a generic input state.

We observe that a mixed safe state generated from an independent randomness source is required to mask quantum information. How much randomness is, however, required for a universal masker? Asking this question is appropriate given the recent trend in computer science and quantum information theory of treating randomness as a resource [11, 15, 16, 18]. We derive a lower bound on the randomness of the ancillary state $\omega_S$ below.

**Theorem 1.** Let $\omega_S$ be the safe state of a universal quantum masker $\Phi_M$ for a $d$-dimensional quantum system. Then, the von Neumann entropy of $\omega_S$ is lower bounded by log$_2 d$.

We note that this theorem has the no-masking theorem as its corollary, since the no-masking theorem states that there are no universal quantum maskers that consume zero amount of randomness.

This lower bound of the randomness cost is, however, achievable only if entanglement between $A$ and $B$ for the masked state is allowed. This is because of the following trade-off relation between the strength of quantum correlation allowed between $A$ and $B$ and the randomness cost of quantum masking.

**Theorem 2.** Let $\omega_S$ be the safe state of a universal quantum masker $\Phi_M$ for $d$-dimensional quantum system. If
the masked state $\Phi_M(\rho)$ for any $\rho$ is separable, then the von Neumann entropy of $\omega_S$ should be strictly larger than $\log_2 d$. If the quantum discord $D^-(A : B)$ of the masked state $\Phi_M(\rho)$ for any state $\rho$ vanishes, then the von Neumann entropy of $\omega_S$ is at least $2\log_2 d$.

This theorem shows that minimal randomness consumption of $\log_2 d$ bits is only possible with the aid of entanglement and any randomness cost saving from the $2\log_2 d$ bits bound comes from utilizing quantum discord in both directions. Theorems 1 and 2 and the examples of universal quantum maskers given in Fig. 1 show that these lower bounds are indeed minimal randomness costs that are achievable.

One may ask after Theorem 2 that if it is possible to mask quantum information without entanglement or any quantum correlations. The answer is that quantum correlations are a critical resource for masking quantum states, without which masking is impossible even with unbounded amount of randomness consumption.

**Theorem 3. Universal quantum masking is impossible without quantum correlation.**

**Qubit-Commitment.** —— A typical commitment scheme consists of two protocols COMMIT and REVEAL involving two participants, Alice (sender) and Bob (receiver). In the COMMIT protocol, Alice commits her secret value to Bob, forbidding him from learning anything about it until she allows him to. Later, Alice discloses the secret value using the REVEAL protocol. Bob should be able to reject the revealed value if it is not consistent with the information provided in the COMMIT phase. This prevents Alice from pretending to have committed a value different from the one in the COMMIT phase. For the scheme to be secure, the following requirements are demanded:

**Correctness:** If every participant honestly follows the protocol, Bob will possess the value committed by Alice after the REVEAL protocol.

**Concealing:** Bob has no information about the committed value until the REVEAL protocol begins.

**Binding:** Once the COMMIT protocol is finished, any attempt by Alice to change the committed value should be rejected by Bob.

We say that a condition holds **perfectly** when the probability of the condition failure is zero. Even if the probability of failure is non-zero, if it is possible to make it arbitrarily small by increasing the security parameter, we still say that the security condition holds.

All the requirements except the binding condition can be applied to qubit-commitment, or generally, qudit-commitment schemes in which the value being committed is a quantum state. The binding condition can be appropriately modified to meet the inherent probabilistic characteristic of quantum mechanics. Assume that Alice commits her quantum state $|\psi\rangle$ by directly giving it to Bob over a quantum channel, sacrificing the concealing requirement. Even in this case, Alice can insist that she has committed a quantum state $|\phi\rangle$, which is non-orthogonal to $|\psi\rangle$, and still could be accepted by Bob with non-zero probability, $|\langle \phi | \psi \rangle|^2$.

Indeed the best fidelity achievable by Alice cannot be lower than this, since if a quantum qubit-commitment scheme allows lower best fidelity than this, then such scheme can be used to distinguish non-orthogonal states and thus breaks the no-cloning theorem or the information causality [23]. If Alice’s best success probability of revealing the state $|\phi\rangle$ after committing the state $\rho$ is exactly $\langle \phi | \rho | \phi \rangle$, then we say that the scheme is **perfectly quantum binding**. If the best fidelity a scheme allows is not exactly $\langle \phi | \rho | \phi \rangle$, but converges to it as the security parameter increases, i.e., the parameter that characterizes the security of the scheme such as the rank of the safe state increases, then the scheme is said quantum binding.

Now we propose an unconditionally secure quantum qubit commitment scheme that utilizes a universal quantum masker of a special kind. Consider a class of quantum maskers for which the safe state $\omega_S$ is the maximally mixed state on its support. In other words, we consider the case of $\omega_S = \frac{1}{d}I_{supp(\omega_S)}$ where $r$ is the rank of $\omega_S$ and $I_T$ is the orthogonal projection operator onto a given subspace $T$. The 4-qubit masker and the 3-qubit masker introduced in this Letter are maskers of this class with $\omega_S = |0\rangle|0\rangle \otimes |0\rangle|0\rangle \otimes \frac{1}{d}I$ and $\omega_S = \frac{1}{d}I \otimes \frac{1}{d}I$, respectively. We then let $|\Omega\rangle_{SK}$ be a purification of the state $\omega_S$, where the index $S$ stands for the **safe** and $K$ stands for the **key**. From a fixed purification $|\Omega\rangle_{SK}$, we can define $\omega_K := Tr_S |\Omega\rangle_{SK} \langle \Omega |$ and find an orthonormal basis $\{ |\Omega_i\rangle \}_{i=1}^{d^2}$ on the space $supp(\omega_S) \otimes supp(\omega_K)$ that consists of purifications of $\omega_S$, just like how the Bell basis is defined.

Let us introduce a “trusted initializer,” Ted, who prepares a quantum state and classical information and distributes them to Alice and Bob at the beginning of the scheme. We trust Ted only to the extent that he cannot take advantage of the result of the commitment scheme because he never learns any information about the input state. Our scheme consists of three phases SETUP, COMMIT and REVEAL, and Ted participates only in the SETUP protocol and exits the scheme after that. (See Fig. 2)

**SETUP:** Ted prepares a bipartite state $|\Omega\rangle_{SK}$ randomly chosen from the orthonormal basis $\{ |\Omega_i\rangle_{SK}\}_{i=1}^{d^2}$. Then Ted applies a randomly chosen unitary $U_{SK}$ on the space spanned by $\{ |\Omega_i\rangle_{SK}\}_{i=1}^{d^2}$ to the state and distributes it to Alice and Bob. Ted tells
Alice the index $i$ of $|\Omega_i\rangle$ and tells Bob the unitary $U_{SK}$.

**COMMIT**: Alice prepares a secret state $\rho_C$ in the system $C$. Alice runs the masking unitary $M_{CS\rightarrow AB}$ on the systems $C$ and $S$ and gets the output state in the systems $A$ and $B$. Alice sends the subsystem $B$ to Bob and keeps the system $A$.

**REVEAL**: Alice sends the system $A$ to Bob and reveals her commitment as $|\phi_i\rangle_C$ and the index $i$ of the state $|\Omega_i\rangle_{SK}$. Bob runs the unmasking unitary $M_{AB\rightarrow CS}^\dagger$ on the systems $A$ and $B$ and successively applies the unitary $U_{CS}^\dagger$ on the systems $C$ and $S$. Bob measures the output state to check if the system $S$ and $K$ are in the state $|\Omega_i\rangle_{SK}$. If they are, Bob proceeds to check if the system $C$ is in the state $|\phi_i\rangle_C$. If either of the validations fails, Bob rejects the revelation.

We provide the security proof of the qubit commitment protocol by the following theorem:

**Theorem 4.** The initializer qudit-commitment scheme is unconditionally and perfectly concealing and unconditionally quantum binding.

Details of the proof can be found in the Supplemental Material [22]. We highlight that an advantage of our scheme over Rivest’s scheme is that it is perfectly concealing without ‘collision error’. Rivest’s scheme depends on the choice of a large prime number $p$ and with the probability of $1/p$, Alice’s secret value and the evidence provided by Ted could coincide (‘collide’), disclosing the secret value to Bob in the COMMIT protocol. In our scheme, however, regardless of Alice’s choice of the secret qubit, Alice’s secret qubit remains concealed until the REVEAL protocol begins. This fact enables the scheme proposed here to be applicable for low-dimensional cases in contrast with Rivest’s scheme. Another interesting observation is that Alice’s suboptimal strategy is to do nothing after the COMMIT protocol; the identity operation gives fidelity between the target state $|\phi\rangle$ and the originally committed state $|\psi\rangle$ arbitrarily close to the optimal value as $r \rightarrow \infty$. Moreover, any possible fidelity enhancement in bounded from above with $(1 - |\langle \phi | \psi \rangle|^2)/(1 + r^2).$ In other words, our scheme fundamentally suppresses Alice’s any kind of attempts to harm the binding requirement and successfully blocks the notorious entanglement attack [4, 5].

The initializer scheme, however, inherits the problem of the Rivest scheme that it is not perfectly binding [6]. With the security parameter $p$ for Rivest’s scheme, Alice can guess the evidence given to Bob with the probability of $1/p^2$. In the scheme proposed here, the success probability of deceiving Bob about the index $i$ of the safe and key state $U|\Omega_i\rangle_{SK}$ is generically non-zero and bounded from above with $r^2/(r^4 - 1)$. On the other hand, it is important to prevent Bob from having any access to the index $i$, otherwise Bob can infer the committed qubit by measuring the system $SK$ with the knowledge of the state $U|\Omega_i\rangle$ before the REVEAL protocol.

**Theorem 5.** Ted learns nothing about the secret qubit.

This theorem justifies the introduction of the trusted third party in to the scheme, since even if Ted is malicious, it is impossible for Ted to attain any useful information about the secret value. A certain amount of trust is required, nevertheless, for collusion of Ted with either of participants leads to failure of security conditions.

In a practical situation, Alice may not know the specification of the state $|\psi\rangle$ being committed, just as in quantum teleportation schemes. In this case, in the REVEAL
protocol, Alice cannot disclose the state \( |\psi\rangle \) over a classical channel. For such cases, a scheme in which Bob’s acceptance of Alice’s revelation with high probability implies Bob’s possession of a quantum state with high fidelity with the secret qubit of Alice would be favorable. We will say that such schemes have state-revealing property and we show here the existence of a scheme with such property. In this case, we only consider pure committed states for this property does not depend on the uncertainty of the input state, therefore one can assume the input state is in some fixed pure state just as in quantum teleportation schemes.

**Theorem 6.** The 3-qudit masker qudit commitment protocol with a trusted initializer has state-revealing property.

**Discussion.**— We have generalized the no-masking theorem suggested by Modi et al. [7] by introducing a new framework that interprets quantum masking as a task that consumes randomness in the process. In our formulation, the usage of randomness is catalytic in the sense introduced by Boes et al. [16], since after applying unmasking unitary to the masked quantum state, the safe state is completely recovered and can be used again.

Our results show that there are subtle constraints for entanglement in tripartite quantum secret sharing schemes. Theorem 3 proves that it is impossible to share a quantum secret exploiting only genuine tripartite quantum entanglement. Meanwhile, Theorem 2 implies that there is a trade-off relation between the maximum quantum correlation between \( A \) and \( B \) and the randomness cost, i.e. the entanglement for the bipartition \( AB \) and \( C \).

Quantum secret sharing scheme can be considered an isometric embedding of the space of a single party quantum state into the set of multipartite quantum states with fixed marginal states. Therefore the study of constraints and costs of quantum secret sharing scheme is important not only for practical applications but also for its potential as a tool to investigate quantum phenomena solely coming from quantum correlation by putting them in comparison with the well-understood properties of single-partite quantum states.

Recently, a qubit commitment scheme that can be implemented in fully classical communication with a fully classical receiver setting was suggested by Mahadev [26], however, with a certain degree of assumption on the computational power. Since the scheme suggested in this Letter is unconditionally secure but requires quantum control capabilities on the receiver’s side, a combination of both directions might yield a stronger result.

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APPENDIX : PROOFS OF THEOREMS

Theorem 1. Let $\omega_S$ be the safe state of a universal quantum masker $\Phi_M$ for $d$-dimensional quantum system. Then, the von Neumann entropy of $\omega_S$ is lower bounded by $\log_2 d$.

Before proving this theorem we reprove a more general result for $(k,n)$-threshold quantum secret sharing schemes given in [27] in a way that doesn’t rely on the strong subadditivity of the von Neumann entropy.

Lemma 1. Assume that $\Phi_M : \mathcal{S}(\mathbb{C}^d) \to \mathcal{S}(\mathbb{C}^d)^\otimes n$ is the quantum map implementing a $(k,n)$-threshold quantum secret sharing scheme. Then, for any $\rho \in \mathcal{S}(\mathbb{C}^d)$, the marginal state $\Phi_M(\rho)_{A_i}$ of any system obtained by tracing out the other $n-1$ parties of the $n$-partite state $\Phi_M(\rho)_{A_1,...,A_n}$ has the von Neumann entropy of $\log_2 d$ or higher.

Proof. As every $(k,n)$-threshold quantum secret sharing scheme can be purified to a pure $(k,2k-1)$-threshold quantum secret sharing scheme [17], we only prove the lemma for that case. In that case, the scheme can be implemented with an isometry $M : \mathbb{C}^d \to (\mathbb{C}^d)^\otimes 2k-1$. Consider the input state $\rho \in \mathcal{S}(\mathbb{C}^d)$ in the system $C$ and its purification $|\Psi_\rho\rangle_{EC}$ with the purification system $E$. Then the isometry $M$ outputs the state $(\mathbb{I} \otimes M) |\Psi_\rho\rangle_{EC}$ distributed among 2$k$ parties $E$ and $A_1,...,A_{2k-1}$. Let $D$ be an authorized subset of the parties $\{A_1,\ldots,A_{2k-1}\}$ of the size $k$ containing $A_i$ and $U$ be the unauthorized subset $\{A_1,\ldots,A_{2k-1}\} \setminus D$ of the size $k-1$. As a secret sharing scheme decouples any unauthorized set from environment, we have

$$H(E,U) = H(E) + H(U),$$

where all the von Neumann entropies are defined for the output state $(\mathbb{I} \otimes M) |\Psi_\rho\rangle_{EC}$. Since $(\mathbb{I} \otimes M) |\Psi_\rho\rangle_{EC}$ is a pure state we have

$$H(E,U) = H(D),$$

and from the subadditivity of the von Neumann entropy we have

$$H(D) \leq H(A_i) + H(U')$$

where $U' := D \setminus \{A_i\}$ is an unauthorized subset of $\{A_1,\ldots,A_{2k-1}\}$ of the size $k-1$. Therefore,

$$H(E) + H(U) - H(U') \leq H(A_i).$$

Since the choice of the authorized set $D$ was arbitrary barring the condition that it contains $A_i$, one can choose the new $D$ as $\{A_i\} \cup U$, which makes the new $U'$, $U$. From the same argument we have

$$H(E) + H(U') - H(U) \leq H(A_i).$$

By averaging two inequalities we have

$$H(E) \leq H(A_i).$$

As the system $E$ was defined as the purifying system of the input state $\rho$ and remained intact through the secret sharing process, $H(E) = H(\rho)$. As $\rho$ was arbitrarily chosen, however, we can take $H(E)$ as its maximum value $\log_2 d$.

Note that this result can be considered a stronger version of Theorem 4 in [17]. By noting that a quantum masking process is merely a $(2,2)$-threshold quantum secret sharing scheme and the fact that any mixed quantum secret sharing scheme can be obtained by tracing out irrelevant parties of a pure quantum secret sharing state, we can see that the following circuit represents an implementation of a pure $(2,3)$-threshold quantum secret sharing scheme among three parties, $A, B$ and $K$.

$$\begin{array}{c}
|\Psi_\rho\rangle_{EC} \\
|\Omega\rangle_{SK}
\end{array}
\begin{array}{c}
MCS\rightarrow AB \\
E
\end{array}
\begin{array}{c}
A \\
B
\end{array}
\begin{array}{c}
K
\end{array}$$

Note that $|\Psi_\rho\rangle_{EC}$ is a purification of the input state $\rho$ and $|\Omega\rangle_{SK}$ is a purification of the safe state $\omega_S$ consumed in the hiding process. The lemma says that $H(K) \geq \log_2 d$, but as $H(K) = H(\omega_S)$ we have the proof of the theorem 1.
Theorem 2. Let $\omega_S$ be the safe state of a universal quantum masker $\Phi_M$ for $d$-dimensional quantum system. If the masked state $\Phi_M(\rho)$ for any $\rho$ is separable, then the von Neumann entropy of $\omega_S$ should be strictly larger than $\log_2 d$. If the quantum discord $D^\rightarrow (A : B)$ of the masked state $\Phi_M(\rho)$ for any state $\rho$ vanishes, then the von Neumann entropy of $\omega_S$ is at least $2 \log_2 d$.

Proof. Let’s say $\Phi_M(\rho)$ is separable for any pure state input $\rho$, then the fact following equality holds is evident from the definition of quantum discord.

$$S(K) = I(A : B) + D^\rightarrow (A|K) + D^\rightarrow (B|K).$$

Here $I(A : B) = S(A) + S(B) - S(AB)$ is the quantum mutual information between $A$ and $B$. However, as at least one of the pairs $AK$ and $BK$ should be entangled for some pure state $\rho$, because if all three pairs $AB$, $AK$ and $BK$ are separable, then the bipartite state $\Phi_M(\rho)_{AB}$ should be classically correlated [28], but because of Theorem 3 below, it is impossible to mask quantum information into classically correlated systems. Note that quantum discord $D^\rightarrow (A|K)$ is zero if and only if the system $AK$ is in a quantum-classical state. Therefore for the given conditions, $S(K)$ must be strictly larger than $I(A : B)$, which is in turn no smaller than $\log_2 d$. As $S(S) = S(K)$, we get the first part of the theorem.

Since $D^\rightarrow (A : B) = 0$, the state $\Phi_M(|\psi\rangle\langle\psi|)$ is a quantum-classical (QC) state [29] that has the form, where $\text{Tr}_A(\Phi_M(|\psi\rangle\langle\psi|)) = \sum_i t_i |\sigma_i\rangle\langle\sigma_i|_B$ is the spectral decomposition of the state of the system $B$ independent of the state $|\psi\rangle\langle\psi|,$

$$\Phi_M(|\psi\rangle\langle\psi|) = \sum_i q_i \rho_i A \otimes |\sigma_i\rangle\langle\sigma_i|_B,$$

with some quantum state $\rho_i$ for each $i$. However, as $\Phi_M(|\psi\rangle\langle\psi|) = M_{CS \rightarrow AB}(|\psi\rangle\langle\psi|_C \otimes \omega_S)M_{AB \rightarrow CS}^\dag$, $\Phi_M(|\psi\rangle\langle\psi|)$ must have the same rank with the safe state $\Phi_S$. Therefore every $\rho_i$ must be a pure state, and we have

$$\Phi_M(|\psi\rangle\langle\psi|) = \sum_i q_i |\psi_i\rangle_\psi A \otimes |\sigma_i\rangle\langle\sigma_i|_B,$$

where for all $i$, $|\psi_i\rangle := M_i |\psi\rangle$ for some operator $M_i$ with $\langle\psi| M_i^\dagger M_i |\psi\rangle = 1$. However, as $\Phi_M(|\psi\rangle\langle\psi|) = M_{CS \rightarrow AB}(|\psi\rangle\langle\psi|_C \otimes \omega_S)M_{AB \rightarrow CS}^\dag$, the probability distribution $\{q_i\}$ has the same distribution with the eigenvalues of the safe state $\omega_S$. Now, whenever $|\psi\rangle$ and $|\phi\rangle$ are mutually orthogonal states, we have $\text{Tr} \sqrt{\Phi_M(|\psi\rangle\langle\psi|)\Phi_M(|\phi\rangle\langle\phi|)} = \sum_i q_i |\langle \phi_i | \psi_i \rangle|^2 = \text{Tr} \left[ M_{CS \rightarrow AB}(|\psi\rangle\langle\psi|_C \otimes \omega_S)M_{CS \rightarrow AB}^\dagger M_{AB \rightarrow CS}^\dagger (|\phi\rangle\langle\phi|_C \otimes \omega_S)M_{AB \rightarrow CS}^\dagger \right] = |\langle \phi | \psi \rangle|^2 = 0$. Therefore $|\langle \phi_i | \psi_i \rangle|^2 = |\langle \phi | M_i^\dagger M_i |\psi\rangle|^2 = 0$ for all $i$. This proves that $M_i$ is isometry for all $i$. Now, from the masking condition, the quantum channel $\text{Tr}_B[\Phi_M(\cdot)] = \sum_i q_i M_i \cdot M_i^\dagger$ is a randomization scheme [18] and therefore, from the result of the Ref. [18], the Shannon entropy of the probability distribution $\{q_i\}$, which is the von Neumann entropy of the safe state $\omega_S$, is at least $2 \log_2 d$.

Theorem 3. For arbitrary $d \geq 2$, universal quantum masker exists for $d$-dimensional quantum system that consumes $\log_2 d$ bits of randomness.

Proof. The 4-qubit masker can be easily generalized to $d$-dimensional systems by replacing the 2-dimensional controlled-X gates with its $d$-dimensional generalization given as

$$U_{C-X} |x\rangle |y\rangle = |x\rangle |x \oplus y (\text{mod } d)\rangle,$$

for $x, y = 0, ..., d - 1$ and replacing the Hadamard gate with the discrete Fourier transform gate,

$$U_{DFT} = \sum_{i=0}^{d-1} |i\rangle \langle i|,$$

where $|\hat{n}\rangle := \sum_{j=0}^{d-1} \exp(i2\pi nj/d) |j\rangle$, for $n = 0, ..., d - 1$. In this case, the output state for a pure input state $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$, is given as

$$\rho_{\text{output}} = \frac{1}{d} \sum_{i=0}^{d-1} |\psi_i\rangle\langle\psi_i|_{A_1B_1} \otimes |\Phi_i\rangle\langle\Phi_i|_{A_2B_2},$$
where
\[ |\Psi_n\rangle := \sum_{j=0}^{d-1} \alpha_{n \oplus j \mod d} |j\rangle \otimes |j\rangle, \]
and
\[ |\Phi_n\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i \frac{2\pi j n}{d}} |j\rangle \otimes |j\rangle, \]
for \( n = 0, ..., d - 1 \).

For every \( d \geq 2 \), tracing out \( B_1 B_2 \) system yields the maximally mixed state for the system \( A_1 A_2 \) and \textit{vice versa}. For general mixed states, from the linearity of quantum processes, the marginal state of the output state will be a mixture of maximally mixed marginal states, which is again the maximally mixed state. This shows that the given quantum process can mask any quantum state. As to the recoverability condition, since this operation consists of unitary operation after the attachment of ancillary system, simple inverse unitary operation followed by tracing out of the ancillary systems recovers the input system.

**Theorem 4.** Universal quantum masking is impossible without quantum correlation.

**Proof.** For a universal masking process \( \Phi_M \), having no quantum correlation in the masked quantum state implies the following expression for any input state \( \rho_C \).

\[ \Phi_M(\rho) = \sum_{ij} p_{ij}(\rho) |i\rangle_{iA} \otimes |j\rangle_{jB}, \]

where \( p_{ij}(\rho) \) is a joint probability distribution for indices \( i \) and \( j \) linear in the state \( \rho \) and \( \{|i\rangle_A\} \) and \( \{|j\rangle_B\} \) are respectively eigenbasis of \( \text{Tr}_B[\Phi_M(\rho)] \) and \( \text{Tr}_A[\Phi_M(\rho)] \) which are independent of the input state \( \rho \). From the proof of Theorem 3, however, \( \Phi_M(|\psi\rangle|\psi\rangle_C) \) also permits the following expression for any input state \( |\psi\rangle \).

\[ \Phi_M(|\psi\rangle|\psi\rangle) = \sum_i q_i M_i |\psi\rangle M_i \otimes |i\rangle_{iB}. \]

This implies that \( p_{ij}(|\psi\rangle|\psi\rangle) = q_j \) for only one \( i \) and 0 otherwise and that \( \sqrt{p_{ij}(|\psi\rangle|\psi\rangle)} |i\rangle_A = \sqrt{q_j} M_j |\psi\rangle_C \). However, the image of the right hand side of the equation with varying vector \( |\psi\rangle_C \) is a \( d \)-dimensional subspace of \( H_A \), but the image of the left hand side is included in the union of the rays spanned by basis elements \( |i\rangle_A \). As the latter is strictly smaller than the former, we have contradiction.

**Theorem 5.** The initializer qudit-commitment scheme is unconditionally and perfectly concealing.

**Proof.** The quantum state \( U|\Omega_i\rangle|\Omega_i\rangle_{SK} U^\dagger \) distributed by Ted to Alice and Bob in the SETUP protocol gets averaged to the maximally mixed state on the subspace \( \text{supp}(\omega_S) \otimes \text{supp}(\omega_K) \), \( \omega_S \otimes \omega_K = \frac{1}{r} \Pi_{\text{supp}(\omega_S)} \otimes \frac{1}{r} \Pi_{\text{supp}(\omega_K)} \), if one lacks the knowledge of the index \( i \) or the random unitary \( U \). As the system \( S \) is in the state \( \omega_S \) for both parties in the beginning of the COMMIT protocol, the quantum masker successfully masks the input state, so the perfect concealing condition holds.

**Theorem 6.** The initializer qudit-commitment scheme is unconditionally quantum binding.

**Proof.** In this case, the fidelity is given as (provided that Alice has reported the value \( j \) instead of the true value \( i \)):

\[ F_{ij} = \int dU |\langle \phi, \Omega_j |_{CSK} U_{SK}^\dagger M_{AB \rightarrow CS}^i V_A M_{CS \rightarrow AB} U_{SK} |\psi, \Omega_i \rangle_{CSK}|^2, \]

where \( M_{CS \rightarrow AB} \) is the unitary operator used in the quantum masker and \( V_A \) is a unitary operator applied by Alice in trying to deceive Bob. One can use the following expression

\[ |\langle \phi | U^\dagger MU |\psi\rangle|^2 = |\phi \otimes \langle \psi | (U^\dagger \otimes U^\dagger)(M^\dagger \otimes M)(U \otimes U) |\psi\rangle \otimes |\phi\rangle. \]

We utilize a fact about the symmetric twirling that \( [25] \)

\[ \int dU (U^\dagger \otimes U^\dagger)(M^\dagger \otimes M)(U \otimes U) = (\gamma_{GG'} + \delta F_{GG'}) \otimes (\Pi_{G' \otimes G' \perp})(M^\dagger \otimes M)(\Pi_{G' \otimes G' \perp}), \]
where $dU$ is the Haar measure over the set of unitary operators $V$ such that $V|g⟩ = |g⟩$ for all $|g⟩ \in {G}^\perp$, $Π_T$ is the orthogonal projection operator onto a given subspace $T$, and $F_{GG'}$ is the swapping operator between two subspaces $G$ and $G'$ with

$$\gamma = \frac{d|\text{Tr} M|^2 - \text{Tr}((M^\dagger \otimes M)F)}{d(d^2 - 1)},$$

$$\delta = \frac{d\text{Tr}((M^\dagger \otimes M)F) - |\text{Tr} M|^2}{d(d^2 - 1)}.$$

Note that if the unitary is only applied to a subsystem $S$ then by modifying $F \rightarrow F_S$ and $\text{Tr} \rightarrow \text{Tr}_S$ we can substitute $\gamma$ and $\delta$ with appropriate operator values. Then, by letting

$$|I^i_ψ⟩ := |ψ⟩_C \otimes |Ω_i⟩_S$$

and

$$M_V := M^i_{AB \rightarrow CS}(V_A \otimes I_B)M_{CS \rightarrow AB}$$

we have

$$F_{ij} = \langle I^i_ψ | \otimes \langle I^j_φ | (\text{Tr}_S Π_{\text{supp}(ω)}M^i_V) \otimes (\text{Tr}_{S'} Π_{\text{supp}(ω)}M_V) \otimes \left(\frac{r^2 I - F \oplus Π_{\text{supp}(ω_S)′ \oplus \text{supp}(ω_K)′}}{r^4 - 1}\right)_{SKS'K'} |I^j_φ⟩ \otimes |I^i_ψ⟩$$

$$+ \langle I^i_ψ | \otimes \langle I^j_φ | (\text{Tr}_{SKS'K'}(M^i_V \otimes M_V)(F_{CC'} \otimes F)) \otimes \left(\frac{r^2 F \oplus Π_{\text{supp}(ω)′ \oplus \text{supp}(ω_K)′}}{r^4 - 1}\right)_{SKS'K'} |I^j_φ⟩ \otimes |I^i_ψ⟩,$$

where $r$ is the rank of $ω$ and the primed subsystems are copies of their unprimed counterparts and $F$ is the swapping operator between the spaces $\text{supp}(ω_S) \otimes \text{supp}(ω_K)$ and $\text{supp}(ω_S') \otimes \text{supp}(ω_K')$. Note that

$$(I \otimes F) |I^j_φ⟩ \otimes |I^i_ψ⟩ = |I^j_φ⟩ \otimes |I^i_ψ⟩$$

and that the factor $\text{Tr}_{SKS'K'}((M^i_V \otimes M_V)(F_{CC'} \otimes F))$ can be expressed as $r \text{Tr}_S(M_VM^i_VΠ_{\text{supp}(ω_S)})$. Now we define

$$A_V := (⟨φ|_C \otimes Π_{\text{supp}(ω)}M_V(|ψ⟩_C \otimes Π_{\text{supp}(ω)}),$$

note that this is not a scalar value. For the case of $i = j$, $F_{ii} = \frac{|\text{Tr} A_V|^2}{r^2 + 1} + \frac{\text{Tr} A^i_V A_V}{r(r^2 + 1)} \leq \frac{r^2 \langle φ|ψ⟩^2 + 1}{r^2 + 1},$

where we used the facts

$$\text{Tr} A^i_V A_V = \sum_{i=0}^{r-1} ⟨ψ, a_i⟩_S M^i_V(⟨φ|_C \otimes Π_{\text{supp}(ω_S)})M_V |ψ, a_i⟩_S \leq r$$

with $\{a_i⟩_S$ being the eigenbasis of $ω_S$, which holds since $M_V$ is unitary and $\|⟨φ|_C \otimes Π_{\text{supp}(ω)}⟩_\text{op} = 1$, and

$$\max_V |\text{Tr} A_V| = \max_V |\text{Tr} B_V|$$

with

$$B_V := (V_A \otimes I_B)M_{CS \rightarrow AB}(|ψ⟩_C φ) \otimes Π_{\text{supp}(ω_S)})M^i_{AB \rightarrow CS}.$$
\[ +i(1 + \text{Im} \langle \phi | \psi \rangle) \Phi_M(|I_I\rangle) - i \Phi_M \left( \frac{\langle \psi \rangle \langle \psi \rangle + |\phi \rangle \langle \phi \rangle}{2} \right) \]

where \(|R\rangle := (2 + 2 \text{Re} \langle \phi | \psi \rangle)^{-1/2} (|\psi \rangle + |\phi \rangle)\) and \(|I\rangle := (2 + 2 \text{Im} \langle \phi | \psi \rangle)^{-1/2} (|\psi \rangle + i |\phi \rangle)\). Now, from the masking property, for any quantum state \(\rho\), \(\text{Tr}_B \Phi_M(\rho) = \sigma_A\) for a fixed quantum state \(\sigma_A\). Therefore we have

\[\text{Tr}_B(\Phi_M(|\psi\rangle \langle \phi|)) = \langle \phi | \psi \rangle \sigma_A.\]

However, from a property of the Schatten 1-norm [20],

\[\|M\|_1 = \max_V |\text{Tr}(MV)|,\]

we have

\[|\text{Tr} A_V|^2 \leq r^2 |\langle \phi | \psi \rangle|^2,\]

By letting \(V_A = I_A\) Alice can achieve the fidelity \(|\langle \phi | \psi \rangle|^2\), so the best fidelity is in the interval \([1 |\langle \phi | \psi \rangle|^2, \frac{r^2 |\langle \phi | \psi \rangle|^2 + 1}{r^4 + 1}]\), but the gap closes as \(r \to \infty\).

For the case \(i \neq j\), only the terms with the swapping operator \(F\) survive.

\[F_{ij} = \frac{r \text{Tr} A_V^j A_V - |\text{Tr} A_V|^2}{r^4 - 1} \leq \frac{r^2}{r^4 - 1} \]

As this value is bigger than that of the case of \(i = j\) when \(\langle \phi | \psi \rangle = 0\), Alice might try to deceive Bob by lying about the value of \(i\), but note that this fidelity vanishes as \(r \to \infty\). We can also see that if \(|\langle \phi | \psi \rangle|^2 \geq r^2/(r^4 - 1)\), being honest about the value of \(i\) is the best option for Alice. As the 4-qubit masker and the 3-qubit masker introduced in this Letter can be generalized to the case of arbitrarily large \(r\), we have the result.

**Theorem 7.** Ted learns nothing about the secret qubit.

**Proof.** Ted only participates in the SETUP protocol. Obviously Ted learns nothing about the secret qubit. \(\square\)

**Theorem 8.** The 3-qubit masker qudit commitment protocol with a trusted initializer has state-revealing property.

**Proof.** For the case of the 3-qubit masker, we have \(\Pi_{\text{supp}(\omega)} = I\) and \(r = d^2\). So the probability of Bob to measure the system \(S\) and \(K\) in the state \(|\Omega_i\rangle_{SK}\) in the REVEAL protocol is

\[P_i = \frac{d^2 |\text{Tr} V_A|^2 + 1}{d^4 + 1}\]

where as the probability for measurement outcome to be \(|\Omega_j\rangle_{SK}\) for any \(j \neq i\) is

\[P_j = \frac{d^2(d^2 - |\text{Tr} V_A|^2)}{d^8 - 1}.\]

Note that for \(|\text{Tr} V_A|^2 \geq d^{-6}\), \(P_i \geq P_j\), therefore reporting \(i\) to Bob in the REVEAL protocol is still the best option for Alice to be accepted for large enough \(d\). For \(|\text{Tr} V_A|^2 < d^{-6}\), Alice’s acceptance probability is less than \(d^4/(d^8 - 1)\), therefore negligible for large enough \(d\). Then the fidelity of the postmeasurement state of the system \(A\) and the initial committed state \(|\psi\rangle_A\) is given as

\[F(|\psi\rangle, \rho_{A_{\text{post}}}^\text{post}) = \frac{d^2 |\text{Tr} V_A| + \frac{1}{d^2} \sum_{ij} |\langle \psi | X^{-j} Z^{-i} V Z^{-j} X^j | \psi \rangle|^2}{d^2 |\text{Tr} V_A|^2 + 1}.\]

However, from the Cauchy-Schwarz inequality,

\[\left| \sum_{ij} \langle \psi | X^{-j} Z^{-i} V A Z A^j | \psi \rangle \right|^2 \leq \frac{1}{d^2} \sum_{ij} |\langle \psi | X^{-j} Z^{-i} V A Z A^j | \psi \rangle|^2,\]
where the left hand side reduces to
\[
\left| \text{Tr} \left( V_A \frac{1}{d^2} \sum_{ij} Z^i X^j |\psi\rangle \langle \psi| X^{-j} Z^{-i} \right) \right|^2 = \frac{1}{d^2} |\text{Tr} \ V_A|^2.
\]
So, we have
\[
F(|\psi\rangle, \rho^\text{post}_A) \geq \frac{(d^2 + d^{-2}) |\text{Tr} \ V_A|^2}{d^2 |\text{Tr} \ V_A|^2 + 1} \geq \frac{1}{d^2} |\text{Tr} \ V_A|^2.
\]
Therefore, if Alice wants \( P_i \geq 1 - \epsilon \), then \( |\text{Tr} \ V_A|^2 \geq \frac{1}{d^2} ((1 - \epsilon)(1 + d^4) - 1) \) and
\[
F(|\psi\rangle, \rho^\text{post}_A) \geq 1 - \left( 1 + \frac{1}{d^4} \right) \epsilon.
\]
However, notice that this fidelity is an average fidelity over the Haar measure \( dU \). For a given unitary \( U \) on the system \( S \) and \( K \), from the reverse Markov inequality, the probability of the fidelity between \( |\psi\rangle \) and the postmeasurement state larger than \( 1 - n\epsilon \) is given as
\[
P(F > 1 - n\epsilon) \geq 1 - \frac{1}{n} \left( 1 + \frac{1}{d^4} \right).
\]
Hence, by increasing the security parameter \( 1/\epsilon \) enough so that even for a large \( n \), \( n\epsilon \) becomes small, we have the high probability of high fidelity. \( \square \)