FUNCTIONAL EQUATION FOR THETA SERIES

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ABSTRACT. In this short paper, we find the transformation formula for the theta series under the action of the Jacobi modular group on the Siegel-Jacobi space. This formula generalizes the formula (5.1) obtained by Mumford in [3, p. 189].

1. Introduction

For a given fixed positive integer $g$, we let

$$
\mathbb{H}_g = \{ \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = \Omega^t, \ \text{Im} \Omega > 0 \}
$$

be the Siegel upper half plane of degree $g$ and let

$$
\Gamma_g = \{ \gamma \in \mathbb{Z}^{(2g,2g)} \mid \gamma J_g \gamma = J_g \}
$$

be the Siegel modular group of degree $g$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$, $\text{Im} \Omega$ denotes the imaginary part of $\Omega$ and $J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

For two positive integers $g$ and $m$, we consider the Heisenberg group

$$
H_{Z}^{(g,m)} := \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{Z}^{(m,g)}, \ \kappa \in \mathbb{Z}^{(m,m)}, \ \kappa + \mu \lambda \text{ symmetric} \}
$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') := (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda' \mu' - \mu' \lambda').$$

We let

$$
\Gamma_{g,m} := \Gamma_g \rtimes H_{Z}^{(g,m)} \quad \text{(semi-direct product)}
$$

be the Jacobi modular group endowed with the following multiplication law

$$(\gamma, (\lambda, \mu; \kappa)) \cdot (\gamma', (\lambda', \mu'; \kappa')) = (\gamma \gamma', (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda' \mu' - \mu' \lambda')).$$

with $\gamma, \gamma' \in \Gamma_g$, $\lambda, \lambda', \mu, \mu' \in \mathbb{Z}^{(m,g)}$, $\kappa, \kappa' \in \mathbb{Z}^{(m,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu) \gamma'$. Then $\Gamma_{g,m}$ acts on the Siegel-Jacobi space $\mathbb{H}_{g,m} := \mathbb{H}_g \times \mathbb{C}^{(m,g)}$ properly discontinuously by

$$(\gamma, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = ((A \Omega + B)(C \Omega + D)^{-1}, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1}),$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, $\lambda, \mu \in \mathbb{Z}^{(m,g)}$, $\kappa \in \mathbb{Z}^{(m,m)}$ and $(\Omega, Z) \in \mathbb{H}_{g,m}$ (cf. [8], [9], [11], [12]).

A fundamental domain for $\Gamma_{g,m} \backslash \mathbb{H}_{g,m}$ was found by the author in [10]. Let $\vartheta_{g} \bar{\vartheta}_g$ be the theta

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group consisting of all elements $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ such that the diagonal entries of matrices $t^AC$ and $t^BD$ are even integers. We set
$$\Gamma_{\vartheta,g,m} := \Gamma_{\vartheta,g} \ltimes H_{Z}^{(g,m)}.$$ 

We consider the theta series
\begin{align*}
\Theta(\Omega, Z) := \sum_{A \in \mathbb{Z}^{(m,g)}} e^{\pi i \sigma(A \Omega + 2 A^t Z)}, \quad (\Omega, Z) \in \mathbb{H}_{g,m}.
\end{align*}

Here $\sigma(T)$ denotes the trace of a square matrix $T$.

In [3, p. 189], Mumford considered the case $m = 1$ and proved the following functional equation
\begin{align*}
\Theta \left( (A \Omega + B)(C \Omega + D)^{-1}, Z(C \Omega + D)^{-1} \right) &= \zeta(\gamma) e^{\pi i \{Z(C\Omega + D)^{-1}C^t Z\}} \det(C \Omega + D)^{1/2} \Theta(\Omega, Z),
\end{align*}
where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{\vartheta,g}$ and $\zeta(\gamma)$ is an eighth root of 1. In this short article, we consider the case of an arbitrary positive integer $m$ and then prove the following functional equation.

**Theorem 1.1.** For any $\tilde{\gamma} = (\gamma, (\lambda, \mu, \kappa)) \in \Gamma_{\vartheta,g,m}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{\vartheta,g}$, we obtain the following functional equation
\begin{align*}
\Theta \left( (A \Omega + B)(C \Omega + D)^{-1}, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1} \right) &= \zeta(\tilde{\gamma}) e^{\pi i \sigma \left( (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1}C^t \left( Z + \lambda \Omega + \mu - \lambda \Omega^t - 2 \lambda^t Z \right) \right)} \det(C \Omega + D)^{m/2} \Theta(\Omega, Z),
\end{align*}
where $\zeta(\tilde{\gamma})$ is an eighth root of 1.

We observe that the formula (1.4) generalizes the formula (1.3) with $m = 1$ and $\lambda = \mu = 0$. For a positive integer $N$, we put
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$
and
$$\theta(\tau) = \sum_{r=\infty}^{-\infty} e^{2\pi i r^2 \tau}, \quad \tau \in \mathbb{H}_1.$$ 

In [2] (Werke pp. 939–940], Hecke showed that
\begin{align*}
\theta((a \tau + b)(c \tau + d)^{-1}) &= \epsilon_d^{-1} \left( \frac{c}{d} \right) (c \tau + d)^{1/2} \theta(\tau), \quad \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \in \Gamma_0(4),
\end{align*}
where $\epsilon_d = 1$ or $i$ according to $d \equiv 1$ or $3 \pmod{4}$ and $\left( \frac{c}{d} \right)$ denotes the quadratic residue symbol (cf. [6, p. 442]).

**Notations:** We denote by $\mathbb{Z}$ and $\mathbb{C}$ the ring of integers, and the field of complex numbers respectively. $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers. The symbol “$:=”$ means that the expression on the right is the definition of that on the left. For two positive integers $k$ and $l$, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in
a commutative ring $F$. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $\sigma(A)$ denotes the trace of $A$. For any $M \in F^{(k,l)}$, $^tM$ denotes the transpose matrix of $M$. $I_n$ denotes the identity matrix of degree $n$. We put $i = \sqrt{-1}$. For $z \in \mathbb{C}$, we define $z^{1/2} = \sqrt{z}$ so that $-\pi/2 < \text{arg}(z^{1/2}) \leq \pi/2$. Further we put $z^\kappa/2 = (z^{1/2})^\kappa$ for every $\kappa \in \mathbb{Z}$.

2. Proof of Theorem 1.1

Let $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa))$ be an element of $\Gamma_{g,m}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and $(\Omega, Z) \in \mathbb{H}_{g,m}$ with $\Omega \in \mathbb{H}_g$ and $Z \in \mathbb{C}^{(m,g)}$. If we put $(\Omega_*, Z_*) := \tilde{\gamma} \cdot (\Omega, Z)$, then we have

\[
\begin{align*}
\Omega_* &= \gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\
Z_* &= (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1}.
\end{align*}
\]

First of all we shall show that if the formula (1.4) holds for $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma_{g,m}$, then it hold for $\tilde{\gamma}_1 \tilde{\gamma}_2$. To prove this fact, we consider the function $J : \Gamma_{g,m} \times \mathbb{H}_{g,m} \rightarrow \mathbb{C}^\times$ defined by

\[
J(\tilde{\gamma}, (\Omega, Z)) := e^{\pi i \sigma \{ (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1}C'\Omega + \lambda \Omega \lambda - 2 \lambda^*Z - \kappa - \mu \lambda \}},
\]

where $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{g,m}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, $\lambda, \mu \in \mathbb{Z}^{(m,g)}$, $\kappa \in \mathbb{Z}^{(m,m)}$ and $(\Omega, Z) \in \mathbb{H}_{g,m}$. By a direct computation or a geometrical method (cf. [9, p. 1332]), we can show that $J$ is an automorphic factor for $\Gamma_{g,m}$ on $\mathbb{H}_{g,m}$, that is, it satisfies the following relation

\[
J(\tilde{\gamma}_1 \tilde{\gamma}_2, (\Omega, Z)) = J(\tilde{\gamma}_1, (\Omega, Z)) J(\tilde{\gamma}_2, (\Omega, Z))
\]

for any $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma_{g,m}$ and $(\Omega, Z) \in \mathbb{H}_{g,m}$. It is easy to see that the map $J_* : \Gamma_{g,m} \times \mathbb{H}_{g,m} \rightarrow \mathbb{C}^\times$ defined by

\[
J_*(\tilde{\gamma}, (\Omega, Z)) := J(\tilde{\gamma}, (\Omega, Z)) \cdot \det(C\Omega + D)^m
\]

is an automorphic factor for $\Gamma_{g,m}$ on $\mathbb{H}_{g,m}$, where $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{g,m}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and $(\Omega, Z) \in \mathbb{H}_{g,m}$. It is easily seen that $J_*(\tilde{\gamma}, (\Omega, Z))$ can be written as

\[
J_*(\tilde{\gamma}, (\Omega, Z)) = e^{-\pi i \sigma(\kappa + \mu \lambda)} \cdot e^{\pi i \sigma(\lambda \Omega + \mu) \cdot \Omega \lambda - \lambda \Omega \lambda - 2 \lambda^*Z} \cdot \det(C\Omega + D)^m.
\]

We observe that $e^{-\pi i \sigma(\kappa + \mu \lambda)} = \pm 1$ because $\sigma(\kappa + \mu \lambda)$ is an integer. Thus we see that if the formula (1.4) holds for $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma_{g,m}$, then it hold for $\tilde{\gamma}_1 \tilde{\gamma}_2$.

We recall (cf. [11, p. 326], [3, p. 210]) that $\Gamma_g$ is generated by the following elements

\[
t_0(B) := \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix} \text{ with any } B = \begin{pmatrix} B \end{pmatrix} \in \mathbb{Z}^{(g,g)},
\]

\[
g_0(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ with any } \alpha \in GL(g, \mathbb{Z}),
\]

\[
-J_g := \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}.
\]
Obviously the following matrices

\[ t_e(B) : = \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix} \]

with any \( B = t^t B \in \mathbb{Z}^{(g,g)} \) even diagonals,

\[ g_0(\alpha) : = \begin{pmatrix} t^t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \]

with any \( \alpha \in GL(g, \mathbb{Z}) \),

\[ -J_g : = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \]  

generate the theta group \( \Gamma_{\vartheta,g} \). Therefore the following elements \( s(\lambda, \mu; \kappa) \), \( t(B) \), \( g(\alpha) \) and \( \sigma_g \) of \( \Gamma_{\vartheta,g,m} \) defined by

\[ s(\lambda, \mu; \kappa) = (I_{2g}, (\lambda, \mu; \kappa)) \]

with \( \lambda, \mu \in \mathbb{Z}^{(m,g)} \) and \( \kappa \in \mathbb{Z}^{(m,m)} \),

\[ t(B) = \left( \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix}, (0, 0; 0) \right) \]

with any \( B = t^t B \in \mathbb{Z}^{(g,g)} \) even diagonals,

\[ g(\alpha) = \left( \begin{pmatrix} t^t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, (0, 0; 0) \right) \]

with \( \alpha \in GL(g, \mathbb{Z}) \),

\[ \sigma_g = \left( \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}, (0, 0; 0) \right) \]

generate the group \( \Gamma_{\vartheta,g,m} \).

**Case I.** \( \tilde{\gamma} = s(\lambda, \mu; \kappa) \) with \( \lambda, \mu \in \mathbb{Z}^{(m,g)} \) and \( \kappa \in \mathbb{Z}^{(m,m)} \).

In this case, we have

\[ \Omega_* = \Omega \quad \text{and} \quad Z_* = Z + \lambda \Omega + \mu. \]

Then we have

\[
\Theta(\Omega, Z + \lambda \Omega + \mu) = \sum_{A \in \mathbb{Z}^{(m,g)}} e^{\pi i \sigma \{ A \Omega^t A + 2 A^t (Z + \lambda \Omega + \mu) \}} \\
= e^{-\pi i \sigma (\lambda \Omega^t \lambda + 2 \lambda^t Z)} \sum_{A \in \mathbb{Z}^{(m,g)}} e^{\pi i \sigma \{ (A + \lambda) \Omega^t (A + \lambda) + 2 (A + \lambda)^t Z \}} \\
= e^{-\pi i \sigma (\lambda \Omega^t \lambda + 2 \lambda^t Z)} \Theta(\Omega, Z).
\]

Here we may take \( \zeta(\tilde{\gamma}) = 1 \). Therefore this proves the formula (1.4) in the case \( \tilde{\gamma} = s(\lambda, \mu; \kappa) \).

**Case II.** \( \tilde{\gamma} = t(B) \) with \( B = t^t B \in \mathbb{Z}^{(g,g)} \) even diagonal.

In this case, we have

\[ \Omega_* = \Omega + B \quad \text{and} \quad Z_* = Z. \]
Then we have
\[
\Theta(\Omega + B, Z) = \sum_{A \in \mathbb{Z}^{m,g}} e^{\pi i \sigma \{ A(\Omega + B)^t A + 2A^t Z \}}
\]
\[
= \sum_{A \in \mathbb{Z}^{m,g}} e^{\pi i \sigma (A t A^t Z)} \cdot e^{\pi i \sigma (A B^t A)}
\]
\[
= \sum_{A \in \mathbb{Z}^{m,g}} e^{\pi i \sigma (A t A^t Z)} \quad \text{(because } \sigma (A B^t A) \in 2 \mathbb{Z})
\]
\[
= \Theta(\Omega, Z)
\]

Here we note that \( \sigma (A B^t A) \in 2 \mathbb{Z} \) because the diagonal entries of \( B \) is even integers. Now we may take \( \zeta(\bar{\gamma}) = 1 \). Therefore this proves the formula (1.4) in the case \( \bar{\gamma} = t(B) \).

**Case III.** \( \bar{\gamma} = g(\alpha) = \left( \begin{array}{cc} t \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right) \) with \( \alpha \in GL(g, \mathbb{Z}) \).

In this case, we have
\[
\Omega_* = t \alpha \Omega \alpha \quad \text{and} \quad Z_* = Z \alpha.
\]
Then we obtain
\[
\Theta(t \alpha \Omega \alpha, Z \alpha) = \sum_{A \in \mathbb{Z}^{m,g}} e^{\pi i \sigma \{ A(t \alpha \Omega \alpha)^t A + 2A^t (Z \alpha) \}}
\]
\[
= \sum_{A \in \mathbb{Z}^{m,g}} e^{\pi i \sigma (A^t A \Omega t A + 2(A^t \alpha) Z)}
\]
\[
= \Theta(\Omega, Z).
\]

We observe that the formula (1.4) reduces to the formula
\[
(2.1) \quad \Theta(t \alpha \Omega \alpha, Z \alpha) = \zeta(\bar{\gamma}) \left( \det \alpha^{-1} \right)^{m/2} \Theta(\Omega, Z).
\]
If we take \( \zeta(\bar{\gamma}) = (\det \alpha)^{m/2} \), the formula (2.1) coincides with \( \Theta(\Omega, Z) \). Since \( \det \alpha = \pm 1 \), \( \zeta(\bar{\gamma}) \) is a fourth root of 1. Therefore this proves the formula (1.4) in the case \( \bar{\gamma} = g(\alpha) \) with \( \alpha \in GL(g, \mathbb{Z}) \).

**Case IV.** \( \bar{\gamma} = \sigma_g = \left( \begin{array}{cc} 0 & -I_g \\ I_g & 0 \end{array} \right) \).

In this case, we have
\[
\Omega_* = -\Omega^{-1} \quad \text{and} \quad Z_* = Z \Omega^{-1}.
\]
We can prove the formula (1.4) using the Poisson Summation Formula.

**Lemma 2.1.** For a fixed element \( (\Omega, Z) \in H_{g,m} \), we obtain the following
\[
\int_{\mathbb{R}^{m,g}} e^{\pi i \sigma (x \Omega t x + 2x^t Z)} dx_{11} \cdots dx_{mg} = \left( \det \left( \frac{\Omega}{2} \right) \right)^{-m} e^{-\pi i \sigma (Z \Omega^{-1} t Z)},
\]
where \( x = (x_{ij}) \in \mathbb{R}^{m,g} \).
Proof. By a simple computation, we see that
\[ e^{\pi i (x^t x + 2x^t Z)} = e^{-\pi i (Z\Omega^{-1} x^t)} \cdot e^{\pi i \sigma\{(x+Z\Omega^{-1})x + Z\Omega^{-1}\}}. \]

Since the real Jacobi group \( SP_g \times H^R_{g,m} \) acts on \( \mathbb{H}_{g,m} \) holomorphically, we may put
\[ \Omega = i A^t A, \quad Z = iV, \quad A \in \mathbb{R}^{(g,g)}, \quad V = (v_{ij}) \in \mathbb{R}^{(m,g)}. \]

\[
\begin{align*}
\int_{\mathbb{R}^{(m,g)}} e^{\pi i \sigma(x^t x + 2x^t Z)} dx_{11} \cdots dx_{mg} \\
= e^{-\pi i (Z\Omega^{-1} x^t)} \int_{\mathbb{R}^{(m,g)}} e^{\pi i \sigma\{(x+V(iA^t A)^{-1})(iA^t A)^{-1}\}} dx_{11} \cdots dx_{mg} \\
= e^{-\pi i (Z\Omega^{-1} x^t)} \int_{\mathbb{R}^{(m,g)}} e^{-\pi \sigma\{(uA)^t\}} du_{11} \cdots du_{mg} \quad (\text{Put } u = x + V(A^t A)^{-1} = (u_{ij})) \\
= e^{-\pi i (Z\Omega^{-1} x^t)} \int_{\mathbb{R}^{(m,g)}} e^{-\pi \sigma\{w^t w\}} (\det A)^{-m} \ dw_{11} \cdots dw_{mg} \quad (\text{Put } w = uA = (w_{ij})) \\
= e^{-\pi i (Z\Omega^{-1} x^t)} (\det A)^{-m} \cdot \left( \prod_{i=1}^{m} \prod_{j=1}^{g} \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} \right) \\
= e^{-\pi i (Z\Omega^{-1} x^t)} (\det A)^{-m} \quad (\text{because } \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} = 1 \quad \text{for all } i, j) \\
= e^{-\pi i (Z\Omega^{-1} x^t)} \left( \frac{\det (A^t A)}{\pi} \right)^{-\frac{m}{2}} \\
= e^{-\pi i (Z\Omega^{-1} x^t)} \left( \frac{\Omega}{i} \right)^{-\frac{m}{2}}.
\end{align*}
\]

This completes the proof of Lemma 2.1. 

For an element \((\Omega, Z) \in \mathbb{H}_{g,m}\), we define the function \( f_{\Omega,Z} \) on \( \mathbb{R}^{(m,g)} \) by
\[
(2.2) \quad f_{\Omega,Z}(x) := e^{\pi i \sigma(x^t x + 2x^t Z)}, \quad x = (x_{ij}) \in \mathbb{R}^{(m,g)}.
\]

By the Poisson summation formula, we obtain
\[
\sum_{A \in \mathbb{Z}^{(m,g)}} f_{\Omega,Z}(A) = \sum_{A \in \mathbb{Z}^{(m,g)}} \hat{f}_{\Omega,Z}(A),
\]

where \( \hat{f}_{\Omega,Z} \) is the Fourier transform of \( f_{\Omega,Z} \) given by
\[
\hat{f}_{\Omega,Z}(y) = \int_{\mathbb{R}^{(m,g)}} f_{\Omega,Z}(x) e^{2\pi i \sigma(xy)} dx_{11} \cdots dx_{mg}.
\]

Then we have
\[ \Theta(\Omega, Z) = \sum_{A \in \mathbb{Z}^{(m, g)}} \hat{f}_{\Omega, Z}(A) \]
\[ = \sum_{A \in \mathbb{Z}^{(m, g)}} \int_{\mathbb{R}^{(m, g)}} f_{\Omega, Z}(x) e^{2\pi i \sigma(\cdot^t x A)} \, dx_{11} \cdots dx_{mg} \]
\[ = \sum_{A \in \mathbb{Z}^{(m, g)}} \int_{\mathbb{R}^{(m, g)}} e^{\pi i \sigma(x_{1}^t x + 2x_{1}^t Z)} e^{2\pi i \sigma(\cdot^t x A)} \, dx_{11} \cdots dx_{mg} \]
\[ = \sum_{A \in \mathbb{Z}^{(m, g)}} \left( \det \left( \frac{\Omega}{i} \right) \right)^{-m} e^{-\pi i \sigma((Z+A)\Omega^{-1}(Z+A))} \] (by Lemma 2.1)
\[ = \left( \det \left( \frac{\Omega}{i} \right) \right)^{-m} \sum_{A \in \mathbb{Z}^{(m, g)}} e^{-\pi i \sigma(Z\Omega^{-1} Z + A\Omega^{-1} A + 2A\Omega^{-1} Z)} \]
\[ = \left( \det \left( \frac{\Omega}{i} \right) \right)^{-m} e^{-\pi i \sigma(Z\Omega^{-1} Z)} \sum_{A \in \mathbb{Z}^{(m, g)}} e^{\pi i \sigma(-A(-\Omega^{-1})^t(-A)+2(-A)^t Z \Omega^{-1})} \]
\[ = \left( \det \left( \frac{\Omega}{i} \right) \right)^{-m} e^{-\pi i \sigma(Z\Omega^{-1} Z)} \Theta(-\Omega^{-1}, Z\Omega^{-1}). \]

Therefore we obtain the formula
\[ (2.3) \quad \Theta(-\Omega^{-1}, Z\Omega^{-1}) = e^{\pi i \sigma(Z\Omega^{-1} Z)} \left( \det \left( \frac{\Omega}{i} \right) \right)^{m/2} \Theta(\Omega, Z). \]

The fact that \( \zeta(\tilde{\gamma})^8 = 1 \) follows from the formula (2.3). Indeed we may take \( \zeta(\tilde{\gamma}) = \det \left( \frac{\Omega}{i} \right)^{m/2} \). Therefore this proves the formula (1.4) in the case \( \tilde{\gamma} = \sigma_g \). Finally we complete the proof of Theorem 1.1.

**Remark 2.1.** Let \( m \) be an odd positive integer. According to the formula (1.4), we see that \( \Theta(\Omega, 0) \) is a modular form of half integral weight \( \frac{m}{2} \) with respect to \( \Gamma_{d, g} \) (cf. [8, p. 200], [6]). We may say that the theta series \( \Theta(\Omega, Z) \) is a Jacobi form of half integral weight \( \frac{m}{2} \) and index \( I_m \) with respect to \( \Gamma_{d, g} \) (cf. [8], [9]). This means that \( \Theta(\Omega, Z) \) may be regarded as an automorphic form on a two-fold covering of the Jacobi group (cf. [5]). Indeed the theta series \( \Theta(\Omega, Z) \) is closely related to the Weil representation of the Jacobi group (cf. [7], [13]). The function \( f_{\Omega, Z} \) is a covariant map for the Weil-Schrödinger representation (cf. [13]).

**Remark 2.2.** Olav K. Richter [4] obtained the transformation formula for theta functions that is more general than the formula (1.4). It is my pleasure to thank him for letting me know his paper [4]. But our proof is quite different from his. In fact, our formula (1.4) is a combination of the transformation laws (2) and (3) in [4].
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