On realizations of polynomial algebras with three generators via deformed oscillator algebras

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Abstract
We present the most general polynomial Lie algebra generated by a second order integral of motion and one of order $M$, construct the Casimir operator, and show how the Jacobi identity provides the existence of a realization in terms of deformed oscillator algebra. We also present the classical analogue of this construction for the most general polynomial Poisson algebra. Two specific classes of such polynomial algebras are discussed that include the symmetry algebras observed for various 2D superintegrable systems.

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1. Introduction
The study of quadratically superintegrable systems using quadratic algebra was initiated by Zhedanov et al [1] and since, various examples of quadratically superintegrable systems have been studied using this approach [1–11]. These quadratic algebras that are quadratic extensions of Lie algebras appear to be very rich objects that allow one to obtain algebraically the energy spectrum of superintegrable systems and explain the total number of degeneracies for a given level through representation theory even beyond two-dimensional cases [7, 8]. These algebraic structures can also be used to classify superintegrable Hamiltonians [4] using the Casimir operators. In addition, there is a very interesting and recent connection between quadratic algebras of superintegrable systems in two-dimensional conformally flat spaces with the full Askey scheme of orthogonal polynomials [11]. For a contemporary review on the topic of classical and quantum systems and related algebraic structures with, in addition, an extensive list of references, we refer the reader to [12].
There exist realizations of quadratic associative algebra in terms of deformed oscillator algebra [3]. A deformed oscillator algebra generated by \( \{N, b, b^\dagger, 1\} \) has the following form [13, 14]:

\[
[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad b^\dagger b = \Phi(N), \quad bb^\dagger = \Phi(N + 1).
\]

It was extended for specific cases to take into account reflection and grading [15, 16] and many papers were devoted to deformed oscillator beyond q-algebras and their applications [17–22].

Initiated by Daskaloyannis [3] for quadratic associative algebra related to superintegrable systems and generated by two second order integrals of motion, the approach of constructing realizations via deformed oscillator algebras can be used to obtain finite-dimensional unitary representations. The quadratic algebra considered in [3] includes the quadratic Racah algebra [1, 9] as a special case which possesses a certain duality property in the generators. This construction of Daskaloyannis has been extended to cubic [23] and quartic [24] associative algebras in order to study their representations and apply them to various superintegrable systems with higher order integrals of motion. Polynomial algebras of arbitrary order have been obtained for specific examples of superintegrable quantum systems using various approaches [25, 26] involving recurrence relations and ladder operators. From these works and specific examples considered, it was demonstrated that to obtain the full spectrum from well-defined integrals of motion via the approach of Daskaloyannis [3], we need to allow the structure function to be a polynomial of degree greater than one. Moreover, the existence of realizations as deformed oscillator algebras for general polynomial Lie algebras beyond the quartic case is an unexplored subject. The purpose of this paper is to study the existence of a Casimir operator and realizations via deformed oscillator algebras for polynomial algebras of arbitrary order with three generators and the connection with the Jacobi identity.

Let us describe the organization of the paper. In section 2, we present the most general polynomial Poisson algebra with three generators associated with a superintegrable systems with two integrals of motion of order 2 and \( M \). We present constraints from the Jacobi identity and calculate the Casimir operator. We obtain realizations in terms of the classical analogue of the deformed oscillator algebra. In section 3, we study the corresponding polynomial Lie algebras with three generators and using the Jacobi identity obtain constraints on the structure constants and for the existence of a Casimir operator. We present an algorithm to solve these constraints at any order \( M \) from linear recurrence equations. We show that the Jacobi identity provides the existence of a realization via deformed oscillator algebras and present the Casimir operator in terms of the parafermionic number only. In section 4, we discuss more explicitly two families of polynomial algebras of arbitrary order that are particularly relevant to superintegrable systems.

2. Polynomial Poisson algebras with three generators

Here we begin with the classical case and consider only two-dimensional systems with canonical position \( q_1, q_2 \) and canonical momenta \( p_1, p_2 \). We seek to introduce the most general polynomial Poisson algebra \( \mathcal{P}_M \) equipped with Poisson bracket denoted \( \{,\}_p \), and generated by a function \( A \) which is second order polynomial in the momenta and another function \( B \) of arbitrary order \( M \). Here, both \( A \) and \( B \) are well-defined integrals of motion for some classical Hamiltonian system. That is,

\[
A = \sum_{0 \leq i+j \leq 2} f_{ij}(q_1, q_2)p_i^1p_2^j,
\]
\[ B = \sum_{0 \leq i+j \leq M} g_{ij}(q_1, q_2)p_1^i p_2^j, \]  

(2.2)

where the \( f_{ij} \) and \( g_{ij} \) are sufficiently smooth functions of the canonical position coordinates only. Note that on all suitably smooth functions of the canonical coordinates, the Poisson bracket has the well known form

\[ \{X, Y\}_p = \sum_{i=1}^{\lfloor \frac{M}{2} + 1 \rfloor} \left( \frac{\partial X}{\partial q_i} \frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial Y}{\partial q_i} \right). \]  

(2.3)

We note that the Poisson bracket is antisymmetric, i.e. \( \{X, Y\}_p = -\{Y, X\}_p \), and the Jacobi identity satisfied by \( \{,\}_p \) is in general given by

\[ \{A, \{B, C\}_p\}_p + \{B, \{C, A\}_p\}_p + \{C, \{A, B\}_p\}_p = 0, \]  

(2.4)

for all relevant functions \( X, Y, Z \). We also remark that the Poisson bracket satisfies the derivation rule,

\[ \{X, YZ\}_p = \{X, Y\}_p Z + \{X, Z\}_p Y. \]  

(2.5)

### 2.1. Explicit Poisson bracket

For convenience, we define a third function \( C \) by

\[ C = [A, B]_p, \]

and observe from (2.3) that it is of order \( M + 1 \) in the momenta. We now seek to define a polynomial Poisson algebra, denoted by \( \mathcal{P}_M \), with Poisson bracket of the form

\[ \{A, C\}_p = \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \alpha_i A^i + \delta B + \epsilon + 2\beta AB, \]  

(2.6a)

\[ \{B, C\}_p = \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \lambda_i A^i + \rho B^2 + \eta B + \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} 2\omega_i A^i B + \zeta, \]  

(2.6b)

where we use the standard notation \( \lfloor y \rfloor \) to denote the integer part of \( y \). It is clear, however, that there will be some constraints on the coefficients in order for this Poisson bracket to satisfy the Jacobi identity. In fact, for the case under consideration, the only non-trivial expression for the Jacobi identity in equation (2.4) that we need to consider is

\[ \{A, \{B, C\}_p\}_p = \{B, \{A, C\}_p\}_p. \]  

(2.7)

This leads to the following result.

**Proposition 1.** The polynomial Poisson algebra \( \mathcal{P}_M \) generated by functions \( A \) and \( B \) of order \( 2 \) and \( M \) respectively in the momenta, has Poisson bracket given by

\[ \{A, B\}_p = C, \]  

(2.8a)

\[ \{A, C\}_p = \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \alpha_i A^i + \delta B + \epsilon + 2\beta AB, \]  

(2.8b)

\[ \{B, C\}_p = \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \lambda_i A^i - \beta B^2 - \alpha_1 B + \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} (i + 1)\alpha_{i+1} A^i B + \zeta. \]  

(2.8c)
Proof. We begin with the arbitrary forms of appropriate order for the brackets $[A, C]_p$ and $[B, C]_p$ given in $(2.6a)$ and $(2.6b)$. The forms of the right-hand sides of equations $(2.6a)$ and $(2.6b)$ are determined by allowing the most general polynomial in generators $A$ and $B$ constrained by the order of the Poisson bracket on the left side. These expressions do not depend on the generator $C$. The Jacobi identity $(2.7)$ provides constraints on the structure constants of the polynomial Poisson algebra, which are shown to be given by the linear relations

$$\eta = -\alpha_1, \quad \rho = -\beta, \quad 2\omega_i = -(i + 1)\alpha_{i+1}, \quad (2.9)$$

thus proving the form of the Poisson algebra as stated in the proposition. □

2.2. Casimir element

Let $K$ denote the Casimir element, defined by

$$\{K, A\}_p = 0 = \{K, B\}_p \quad (2.10)$$
in this case. For polynomial Poisson algebras of low orders, the Casimir element is known [3, 23, 24] to have the form

$$K = C^2 + P(A, B),$$

where $P(A, B)$ is a polynomial of the same order as $C^2$ in terms of momenta. Consequently, the following result is established immediately.

Proposition 2. The Casimir element $K$ for $P_M$ is given by

$$K = C^2 - \sum_{i=1}^{\left[\frac{M}{2}\right]} 2\alpha_i A^i B - 2\beta AB^2 + 2\xi A + \sum_{i=1}^{M} \frac{2}{i+1} \lambda_i A^{i+1} = 2\epsilon B - \delta B^2.$$

Proof. The most general form of the Casimir element with terms up to the same order as $C^2$ is given by

$$K = C^2 + \sum_{i=1}^{\left[\frac{M}{2}\right]} m_i A^i B + nAB^2 + \sum_{i=1}^{M+1} k_i A^i + \sum_{i=1}^{2} l_i B^i.$$

Substituting this form into the definition given in equation $(2.10)$, the parameters $m_i, n, k_i$ and $l_i$ are found to be expressible in terms of the structure constants as

$$m_i = -2\alpha_i, \quad l_1 = -2\epsilon, \quad l_2 = -\delta, \quad n = -2\beta,$$

$$k_1 = 2\xi, \quad k_{i+1} = \frac{2}{i+1} \lambda_i. \quad (2.11)$$

Following [27], we make the remark that by setting $F = -\frac{1}{2}P(A, B)$, the Poisson bracket is expressible in terms of $F$ as

$$[A, B]_p = C, \quad [A, C]_p = \frac{\partial F}{\partial B}, \quad [B, C]_p = -\frac{\partial F}{\partial A}. $$
2.3. Oscillator realization

Let us now investigate realizations of $\mathcal{T}_M$ in terms of the classical analogue of the deformed oscillator algebra $\{N, b^+, b\}$ with relations

$$[N, b]_p = -b, \quad [N, b^+]_p = b^+, \quad b b^+ = b^+ b = G(N), \quad [b, b^+]_p = \Phi(N),$$  \hspace{1cm} (2.12)

where $G(N)$ and $\Phi(N)$ are as yet undetermined functions. Let us note that the first and second equation of (2.12) provide the third one using the Jacobi identity and the fourth one using the derivation rule. Furthermore, it can be shown that

$$\Phi(N) = G'(N).$$  \hspace{1cm} (2.13)

We start with a straightforward lemma that will be useful in our calculations.

**Lemma 1.** Let $x(N)$ be a function of $N$ expressible as a formal power series. Then

$$\{x(N), b\}_p = -x'(N)b,$$

$$\{x(N), b^+\}_p = x'(N)b^+,$$

where $x'(N)$ denotes the usual derivative of $x(N)$.

**Proof.** Using induction and the derivation rule (2.5), it is a straightforward matter to establish the relations

$$[N^k, b]_p = -kN^{k-1}b,$$

$$[N^k, b^+]_p = kN^{k-1}b^+,$$

for any $0 \leq k \in \mathbb{Z}$. The result is then obtained by expressing $x(N)$ as a formal power series. \qed

Our goal is to consider realizations of the form

$$A = A(N), \quad B = B_0(N) + \rho(N)b + \rho(N)b^+,$$

and to then determine constraints on the functions $A(N), B_0(N)$ and $\rho(N)$, along with $G(N)$ and $\Phi(N)$. To derive such constraints, we begin by imposing the relations of theorem 1, and make repeated use of the result of lemma 1.

Firstly, equation (2.8a) gives

$$[A(N), B_0(N) + \rho(N)b + \rho(N)b^+]_p = C,$$

from which we obtain

$$C = \rho(N)A(N)'(b^+ - b).$$

Equation (2.8b) then gives

$$[A(N), \rho(N)A'(N)(b^+ - b)]_p = \sum_{i=1}^{[\frac{p}{2}+1]} \alpha_i A(N)' + \delta(B_0(N) + \rho(N)b + \rho(N)b^+) + \epsilon$$

$$+ 2\beta A(N)(B_0(N) + \rho(N)b + \rho(N)b^+)$$

$$\Rightarrow \rho(N)A'(N)(b^+ - b) = \sum_{i=1}^{[\frac{p}{2}+1]} \alpha_i A(N)' + \delta B_0(N) + \epsilon + 2\beta A(N)B_0(N)$$

$$+ (\delta + 2\beta A(N))\rho(N)(b + b^+).$$
By equating coefficients of \((b + b^+)\) and the remaining functions of \(N\), assuming \(\rho(N) \neq 0\), we obtain the following two constraints on the unknown functions \(A(N)\) and \(B_0(N)\).

\[
A'(N)^2 = \delta + 2\beta A(N), \tag{2.14}
\]

\[
\sum_{i=1}^{\frac{N}{2} + 1} \alpha_i A(N)^i + \delta B_0(N) + 2\beta A(N) B_0(N) + \epsilon = 0. \tag{2.15}
\]

Assuming that \(A(N)\) is non-trivial, particularly that \(A'(N) \neq 0\), we note that differentiating these two constraints respectively gives

\[2A'(N)A''(N) = 2\beta A'(N) \Rightarrow A''(N) = \beta, \tag{2.16}\]

\[
\sum_{i=1}^{\frac{N}{2} + 1} i\alpha_i A'(N) A(N)^{i-1} + \delta B_0'(N) + 2\beta (A'(N) B_0(N) + A(N) B_0'(N)) = 0. \tag{2.17}
\]

Finally, equation (2.8c) gives rise to three identities, obtained by equating the coefficients of \((b^2 + (b^+)^2), (b + b^+)\) and the remaining functions of \(N\). Assuming \(\rho(N) \neq 0\), the coefficients of \((b^2 + (b^+)^2)\) and \((b + b^+)\) give, respectively, the identities

\[
A''(N) = \beta, \tag{2.18}
\]

\[
A'(N) B_0'(N) = -2\beta B_0(N) - \alpha_1 - \sum_{i=1}^{\frac{N}{2}} (i + 1) \alpha_{i+1} A(N)^i. \tag{2.19}
\]

Equations (2.18) and (2.16) are precisely the same. Moreover, if we multiply equation (2.19) by \(A'(N)\) and impose the constraint (2.14), we may easily arrive at equation (2.17). It is clear, then, that equations (2.18) and (2.19) give no new information. Equating the remaining functions of \(N\) in equation (2.8c), however, gives the only constraint so far involving the structure functions \(G(N)\) and \(\Phi(N)\), namely

\[
4\rho'(N) \rho(N) A'(N) G(N) + 2\rho(N)^2 A''(N) G(N) + 2\rho(N)^2 A'(N) \Phi(N)
\]

\[
= \sum_{i=1}^{M} \lambda_i A(N)^i - \beta B_0(N)^2 - 2\beta \rho(N)^2 G(N) - \alpha_1 B_0(N)
\]

\[
- B_0(N) \sum_{i=1}^{\frac{N}{2}} (i + 1) \alpha_{i+1} A(N)^i B_0(N) + \zeta. \tag{2.20}
\]

Taking equation (2.19) and multiplying through by \(B_0(N)\) (which we assume to be non-zero) gives

\[
A'(N) B_0'(N) = -\beta B_0(N)^2 - B_0(N) \alpha_1 - B_0(N) \left[ \sum_{i=1}^{\frac{N}{2}} (i + 1) \alpha_{i+1} A(N)^i \right].
\]

We can use this, along with equations (2.18) and (2.13), to further simplify equation (2.20).

The discussion of this section is summarized in the following, noting that the differential equation (2.14) may be easily solved.

**Proposition 3.** The Poisson algebra \(\mathcal{P}_\mathfrak{g}\) has the realization

\[
A = A(N),
\]

\[
B = B_0(N) + \rho(N)(b + b^+),
\]

\[
C = \rho(N) A'(N)(b^+ - b),
\]

\[
\]
in terms of the classical analogue of the deformed oscillator algebra with relations given by (2.12), where

\[ A(N) = \begin{cases} \sqrt{2N} + c_1, & \beta = 0, \\ \frac{\delta}{2\beta} + \frac{\beta}{2}(N + c_1)^2, & \beta \neq 0, \end{cases} \]

with \( c_1 \) an arbitrary constant,

\[ B_0(N) = -\frac{1}{A(N)^2} \left[ \sum_{i=1}^{[\frac{N}{2}]+1} a_i A(N)^i + e, \right. \]

and the function \( \rho(N) \) along with the structure functions \( G(N) \) and \( \Phi(N) \) must satisfy the constraint

\[ [A'(N)(2\rho(N)^2G(N))]' = \sum_{i=1}^{M} \lambda_i A(N)^i + A'(N)B_0(N) + \beta B_0(N)^2 + \zeta. \]

**Corollary 1.** The Casimir element \( K \) for \( \mathcal{P}_M \) has a realization in terms of the classical analogue of the deformed oscillator algebra (satisfying relations (2.12)) given by

\[ K = -2\rho(N)^2 A'(N)^2 G(N) - \sum_{i=1}^{[\frac{N}{2}]+1} 2\alpha_i A(N)^i B_0(N) - 2\beta A(N)B_0(N)^2 + 2\rho(N)^2 G(N)) 
\]

\[ + 2\zeta A(N) \sum_{i=1}^{M} \frac{2}{i+1}\lambda_i A(N)^{i+1} - 2\epsilon B_0(N) - \delta(B_0(N)^2 + 2\rho(N)^2 G(N)). \]

**Proof.** The abstract form of the Casimir element was determined in proposition 2. Substituting the realization from proposition 3,

\[ A = A(N), \]

\[ B = B_0(N) + \rho(N)(b + b^+), \]

\[ C = \rho(N)A'(N)(b^+ - b), \]

into the expression from proposition 2 for \( K \) gives

\[ K = \rho(N)^2(A'(N)^2 - 2\beta A(N) - \delta)(b^2 + (b^+)^2) \]

\[ - 2\rho(N) \left( \sum_{i=1}^{[\frac{N}{2}]+1} \alpha_i A(N)^i + \delta B_0(N) + 2\beta A(N)B_0(N) + \epsilon \right) (b + b^+) \]

\[ - 2\rho(N)^2 A'(N)^2 G(N) - \sum_{i=1}^{[\frac{N}{2}]+1} 2\alpha_i A(N)^i B_0(N) \]

\[ - 2\beta A(N)B_0(N)^2 + 2\rho(N)^2 G(N)) 
\]

\[ + 2\zeta A(N) \sum_{i=1}^{M} \frac{2}{i+1}\lambda_i A(N)^{i+1} - 2\epsilon B_0(N) - \delta(B_0(N)^2 + 2\rho(N)^2 G(N)). \]

The coefficients of \((b^2 + (b^+)^2)\) and \((b + b^+)\) are both zero due to the imposed constraints (2.14) and (2.15), hence the result.

We remark that the function \( \rho(N) \) can be chosen so that the structure function \( \Phi(N) \) is a polynomial.
3. Polynomial Lie algebras with three generators

Let us now consider the quantum case. We replace the Poisson bracket by the commutator, and make use of the quantum operators \( A \) and \( B \), being analogues of their classical counterparts. Quadratic terms such as \( AB \) are then symmetrized using the anticommutator \( \{A, B\} \). The explicit calculations to any given order can be done but are much more involved even for low order (i.e. small values of \( M \)). We will ultimately show that a realization exists when we impose the Jacobi identity.

3.1. Explicit Lie bracket

By generalization of the classical case covered in the previous section, we define an operator \( C \) by

\[
C = [A, B],
\]

where \([A, B] = AB - BA\) denotes the commutator. To be a Lie algebra, the structure constants must be defined so as to satisfy the Jacobi identity, which has the general form

\[
[X, [Y, Z]] - [Y, [X, Z]] = [[X, Y], Z],
\]

for operators \( X, Y, Z \). For the case at hand, the only non-trivial situation required for consideration is

\[
[A, [B, C]] = [B, [A, C]].
\]

To prove the following proposition, it is a simple matter of substituting the forms of the given Lie brackets into this Jacobi identity. We also use the notation \( \{X, Y\} = XY + YX \) for the anticommutator.

**Proposition 4.** The polynomial Lie algebra, \( \mathcal{L}_M \), which is the \( M \)th order analogue of the classical Poisson algebra \( \mathcal{P}_M \) of proposition 1, has bracket operation given by

\begin{align*}
[A, B] &= C, \quad (3.1a) \\
[A, C] &= \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor + 1} \alpha_i A^i + \delta B + \epsilon + \beta [A, B], \quad (3.1b) \\
[B, C] &= \sum_{i=1}^{M} \lambda_i A^i - \beta B^2 + \eta B + \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \omega_i [A^i, B] + \zeta \quad (3.1c)
\end{align*}

subject to the constraint

\[
\eta C + \sum_{i=1}^{\lfloor \frac{M}{2} \rfloor} \omega_i [A^i, C] + \sum_{k=1}^{\lfloor \frac{M}{2} \rfloor + 1} \alpha_k [A^k, B] = 0. \quad (3.2)
\]

The constraint in proposition 4 is difficult to evaluate in general. Here we seek to develop an algorithm to determine conditions on the coefficients \( \eta, \omega_i \) and \( \alpha_i \) so that equation (3.2) is satisfied. To this end, the following result provides valuable insight.

**Lemma 2.** For all positive integers \( n \), \( [A^n, B] = \sum_{i=1}^{n} A^{n-i} CA^{i-1} \).
**Proof.** The result follows from induction, noting that \( n = 1 \) recovers the relation \([A, B] = C\). □

Alternatively, the result of lemma 2 may be expressed as a sum of symmetric terms as

\[
[A^n, B] = \frac{1}{2} \sum_{i=1}^{n} (A^{n-i}CA^{i-1} + A^{i-1}CA^{n-i}).
\]

Without loss of generality, we may focus only on the expressions \(A^{n-i}CA^{i-1} + A^{i-1}CA^{n-i}\) for which \(n - i \geq i - 1\). Defining

\[
P(m, \ell) = A^mCA^\ell + A^\ell CA^m,
\]

\[
Q(m, \ell) = A^mBA^\ell - A^\ell BA^m,
\]

and noting that

\[
P(m, \ell) = P(\ell, m),
\]

\[
Q(m, \ell) = -Q(\ell, m),
\]

and

\[
P(m, 0) = [A^m, C],
\]

\[
Q(m, 0) = [A^m, B],
\]

we may then write the result of lemma 2 as

\[
Q(n, 0) = \frac{1}{2} \sum_{i=1}^{n} P(n - i, i - 1),
\]

or alternatively as

\[
Q(2k, 0) = \sum_{i=1}^{k} P(2k - i, i - 1),
\]

\[
Q(2k + 1, 0) = \sum_{i=1}^{k} P(2k + 1 - i, i - 1) + \frac{1}{2} P(k, k).
\]

In this notation, using the commutation relations it is straightforward to verify that

\[
P(m, \ell) = P(m + 1, \ell - 1) - \delta Q(m, \ell - 1) - \beta Q(m + 1, \ell - 1) - \beta Q(m, \ell),
\]

\[
Q(m, \ell) = Q(m + 1, \ell - 1) - P(m, \ell - 1),
\]

leading to a system of two variable recurrence relations. Further substituting equation (3.8) into equation (3.7) then gives

\[
P(m, \ell) = P(m + 1, \ell - 1) - \delta Q(m, \ell - 1) - 2\beta Q(m + 1, \ell - 1) + \beta P(m, \ell - 1),
\]

\[
Q(m, \ell) = Q(m + 1, \ell - 1) - P(m, \ell - 1),
\]

the significance being that equations (3.9) and (3.10) reduce the second variable by one on each iteration. This inspires the following lemma.
Lemma 3. Let $x_{i,j}$ and $y_{i,j}$ be numbers satisfying the system of recurrence relations

$$
x_{i,j} = x_{i-1,j-1} + \beta x_{i,j-1} + y_{i,j-1},
$$
$$
y_{i,j} = \delta x_{i-1,j-1} + 2\beta x_{i-1,j-1} + y_{i-1,j-1},
$$

with

$$
x_{0,1} = \beta, \quad x_{1,1} = 1, \quad y_{0,1} = \delta, \quad y_{1,1} = 2\beta,
$$

and where we adopt the convention

$$
x_{-1,j} = 0 = y_{-1,j}, \quad x_{j+1,j} = 0 = y_{j+1,j},
$$

for any $j \geq 1$. Then for $m \geq \ell \geq 1$, we have

$$
A^mCA^\ell + A^\ell CA^m = \sum_{i=0}^{\ell} x_{i,\ell}[A^{m+i}, C] - \sum_{n=0}^{\ell} y_{n,\ell}[A^{m+n}, B].
$$

Proof. Exploiting the notation of equations (3.3) and (3.4), we first use induction to prove the result

$$
P(m, \ell) = \sum_{i=0}^{j} x_{i,j}P(m+i, \ell-j) - \sum_{k=0}^{j} y_{k,j}Q(m+k, \ell-j)
$$

(3.11)

for any $j = 1, 2, \ldots, \ell$. Clearly this holds for $j = 1$, recovering equation (3.9). The induction hypothesis gives

$$
P(m, \ell) = \sum_{i=0}^{j} x_{i,j}P(m+i, \ell-j) - \sum_{k=0}^{j} y_{k,j}Q(m+k, \ell-j)
$$

$$
= \sum_{i=0}^{j} x_{i,j}(P(m+i+1, \ell-j-1) + \beta P(m+i, \ell-j-1) - \delta Q(m+i, \ell-j-1)
$$

$$
- 2\beta Q(m+i+1, \ell-j-1))
$$

$$
- \sum_{k=0}^{j} y_{k,j}(Q(m+k+1, \ell-j-1) - Q(m+k, \ell-j-1))
$$

$$
= \sum_{i=0}^{j} (x_{i-1,j} + \beta x_{i,j} + y_{i,j})P(m+i, \ell-j-1) + x_{j,j}P(m+j+1, \ell-j-1)
$$

$$
\quad - \sum_{k=0}^{j+1} \delta x_{k,j} + 2\beta x_{k-1,j} + y_{k-1,j})Q(m+k, \ell-j-1)
$$

$$
\quad - (2\beta x_{j,j} + y_{j,j})Q(m+j+1, \ell-j-1)
$$

$$
= \sum_{i=0}^{j+1} (x_{i-1,j} + \beta x_{i,j} + y_{i,j})P(m+i, \ell-j-1)
$$

$$
\quad - \sum_{k=0}^{j+1} \delta x_{k,j} + 2\beta x_{k-1,j} + y_{k-1,j})Q(m+k, \ell-j-1)
$$

$$
= \sum_{i=0}^{j+1} x_{i,j+1}P(m+i, \ell-j-1) - \sum_{k=0}^{j+1} y_{k,j+1}Q(m+k, \ell-j-1),
$$

where we have made use of the conventions highlighted in the statement of the lemma. This proves the identity (3.11). Setting $j = \ell$ as a special case then proves the lemma. \qed
In the context of lemma 2, particularly the form given in equations (3.5) and (3.6), the result of lemma 3 implies that

\[ Q(2k, 0) = \sum_{i=1}^{k} \sum_{j=0}^{i-1} (x_{j,i-1} P(2k - i + j, 0) - y_{j,i-1} Q(2k - i + j, 0)), \quad (3.12) \]

\[ Q(2k + 1, 0) = \sum_{i=1}^{k} \sum_{j=0}^{i-1} (x_{j,i-1} P(2k + 1 - i + j, 0) - y_{j,i-1} Q(2k + 1 - i + j, 0)) \]

\[ + \frac{1}{2} \sum_{i=0}^{k} (x_{i,k} P(k + i, 0) - y_{i,k} Q(k + i, 0)). \quad (3.13) \]

In other words, we can see that the result of corollary 2 implies that \( Q(n, 0) \) reduces to a linear combination of \( P(u, 0) \) and \( Q(v, 0) \) terms, in particular where \( v \) is less than \( n \). Using this idea recursively gives rise to the following consequence of lemma 3.

**Corollary 2.** For any \( j = 1, 2, \ldots \), we have

\[ [A^j, B] = \sum_{k=0}^{j-1} s_k^{(j)} [A^k, C]. \]

Making use of the result of corollary 2 allows us to express equation (3.12) as

\[ Q(2n, 0) = \sum_{i=1}^{2n-1} \sum_{m=0}^{2n-i-1} T_{m}^{2n-i-1} P(m, 0), \quad (3.14) \]

where

\[ T_m^{j-i-1} = \begin{cases} 
- \sum_{k=0}^{i-1} y_{k,i-1} s_m^{(j-i+k)}, & m = 0, 1, \ldots, j - i - 1, \\
\sum_{k=m-j+i+1}^{i-1} y_{k,i-1} s_m^{(j-i+k)}, & m = j - i, j - i + 1, \ldots, j - 2, \\
x_{i-1,i-1,}, & m = j - 1, 
\end{cases} \quad (3.15) \]

for \( i = 2, 3, \ldots, n \), and \( T_m^{2n-1} = \delta_m^{2n-1} \) (i.e. the Kronecker delta). Interchanging the order of summation in (3.14) trivially gives

\[ Q(2n, 0) = \sum_{m=0}^{2n-1} \sum_{i=1}^{n} T_m^{2n-i-1} P(m, 0). \]

corollary 2, however, implies

\[ Q(2n, 0) = \sum_{m=0}^{2n-1} s_m^{(2n)} P(m, 0), \]

from which we conclude that

\[ s_m^{(2n)} = \sum_{i=1}^{n} T_m^{2n-i-1}. \quad (3.16) \]
Regarding (3.13), in a similar way we may use corollary 2 to write

\[ Q(2n + 1, 0) = \sum_{m=0}^{2n} \left( \sum_{i=1}^{n} T^{2n+1-i, i-1}_m + \frac{1}{2} R_{m,n} \right) P(m, 0), \tag{3.17} \]

from which we conclude that

\[ s^{(2n+1)}_m = \sum_{i=1}^{n} T^{2n+1-i, i-1}_m + \frac{1}{2} R_{m,n}. \tag{3.18} \]

In order to actually determine the \( s^{(k)}_m \) coefficients introduced in corollary 2, we first need to find the \( x_{i,j} \) and \( y_{i,j} \) of lemma 3, and then treat equations (3.16) and (3.18) as a system of recurrence relations for the \( s^{(k)}_m \). We have as an ‘initial condition’ to this system

\[ s^{(1)}_0 = \frac{1}{2}, \]

which arises from the definition of \( C \) in equation (3.1a). For a given degree \( M \) of the polynomial algebra, these equations are solved recursively for \( s^{(k)}_m \).

We are now in a position to give more detail about the structure constants satisfying constraint (3.2). Having (in principle) determined the \( s^{(k)}_m \), we use the result of corollary 2 to express equation (3.2) as

\[ \eta C + \sum_{i=1}^{L} \omega_i [A^i, C] + \sum_{k=1}^{L+1} \sum_{i=0}^{k-1} \alpha_k s^{(k)}_i [A^i, C] = 0, \]

where we have set \( L = \left\lfloor \frac{M}{2} \right\rfloor \). Furthermore, for convenience we set \( \omega_0 = \eta/2 \) and interchange the order of the double sum, so that the constraint becomes

\[ \sum_{i=0}^{L} \omega_i [A^i, C] + \sum_{k=0}^{L+1} \sum_{i=k+1}^{L+1} \alpha_k s^{(k)}_i [A^i, C] = 0. \]

Equating the coefficients of the linearly independent \( [A^i, C] \) then gives the following result.

**Proposition 5.** The constraint (3.2) is satisfied by

\[ \omega_i = -\sum_{k=i+1}^{L+1} \alpha_k s^{(k)}_i, \]

for \( i = 0, 1, \ldots, L \), where \( L = \left\lfloor \frac{M}{2} \right\rfloor \) and \( \omega_0 = \frac{\eta}{2} \).

### 3.1.1. Summary

We now summarize the algorithm that we use to determine the structure constants of the degree \( M \) polynomial Lie algebra.

Step 1. Find \( x_{i,j} \) and \( y_{i,j} \) satisfying the recurrence relations of lemma 3.

Step 2. Using the result of step 1, determine the coefficients \( s^{(1)}_i \) of corollary 2 by solving the recurrence relations given by equations (3.16) and (3.18).

Step 3. Using the result of step 2, give the explicit expressions for the structure constants \( \eta \) and the \( \omega_i \) in terms of the \( \alpha_i, \beta \) and \( \delta \) via proposition 5.

We remark that while we have not given closed form solutions for the structure constants for arbitrary \( M \), we are able to find the structure constants using the recursive algorithm outlined in steps 1, 2 and 3 above for a specified value of \( M \). The results of our methods agree with the low order cases presented in [3, 23, 24].
3.2. Casimir operator for the Lie algebra case

By analogy with the classical case of section 2.2, we consider a Casimir operator $K$ defined by

$$[A, K] = 0 = [B, K].$$

(3.19)

As in the previous section, we set $L = \left\lfloor \frac{M}{2} \right\rfloor$.

From the low order polynomial algebras studied in [3, 23, 24] and the classical case we consider the Casimir operator to be of the form

$$K = C^2 + \sum_{i=1}^{L+1} \frac{1}{2} m_i [A^i, B] + \frac{n}{2} [A, B^2] + \sum_{i=1}^{M+1} k_i A^i + \ell_1 B + \ell_2 B^2.$$

(3.20)

We now seek constraints on the coefficients by imposing the two conditions stated above in (3.19). Firstly, after some straightforward manipulation involving the Lie bracket given in equations (3.1a)–(3.1c) of proposition 4, we have

$$[A, K] = \sum_{i=1}^{L+1} \alpha_i [A^i, C] + \delta [B, C] + 2\epsilon C + \beta [(A, B), C]$$

$$+ \sum_{i=1}^{L+1} \frac{1}{2} m_i [A^i, C] + \frac{n}{2} [A, [B, C]] + \ell_1 C + \ell_2 [B, C].$$

Furthermore, one may verify that

$$[A, [B, C]] = [(A, B), C] + [[A, C], B]$$

$$= [(A, B), C] + \sum_{i=1}^{L+1} \alpha_i [A^i, B] + \beta [B, C],$$

which then gives

$$[A, K] = \sum_{i=1}^{L+1} \alpha_i [A^i, C] + \delta [B, C] + 2\epsilon C + \beta [(A, B), C] + \sum_{i=1}^{L+1} \frac{1}{2} m_i [A^i, C] + \frac{n}{2} [(A, B), C]$$

$$+ \sum_{i=1}^{L+1} \frac{n\alpha_i}{2} [A^i, B] + \frac{n\beta}{2} [B, C] + \ell_1 C + \ell_2 [B, C].$$

Setting the coefficients of $[(A, B), C]$ and $[B, C]$ to zero then gives

$$n = -2\beta, \quad \ell_2 = \beta^2 - \delta$$

respectively, and leaves us with

$$[A, K] = \sum_{i=1}^{L+1} \alpha_i [A^i, C] + 2\epsilon C + \sum_{i=1}^{L+1} \frac{1}{2} m_i [A^i, C] - \sum_{i=1}^{L+1} \beta \alpha_i [A^i, B] + \ell_1 C.$$

(3.21)

The constraint (3.2) from proposition 4 implies that we can make the substitution

$$- \sum_{i=1}^{L+1} \alpha_i [A^i, B] = \eta C + \sum_{i=1}^{L} \omega_i [A^i, C]$$
in (3.21), leading to
\[ [A, K] = \sum_{i=1}^{L+1} \alpha_i [A^i, C] + 2\epsilon C + \sum_{i=1}^{L+1} \frac{1}{2} m_i [A^i, C] + \beta \eta C + \sum_{i=1}^{L} \beta \omega_i [A^i, C] + \epsilon_1 C. \]

Therefore the Casimir property \([A, K] = 0\) can be obtained by setting the coefficients of \(C\) and \([A^i, C]\) to zero. Namely,
\[ \epsilon_1 = -2\epsilon - 2\eta, \]
\[ m_i = -2\alpha_i - 2\beta \omega_i, \quad i = 1, \ldots, L \]
\[ m_{L+1} = -2\alpha_{L+1}. \]
The form of the Casimir operator \(K\) is then
\[ K = C^2 - \sum_{i=1}^{L+1} \alpha_i [A^i, B] - \sum_{i=1}^{L} \beta \omega_i [A^i, B] - \beta [A, B^2] \]
\[ + \sum_{i=1}^{M+1} k_i A^i - (2\epsilon + \beta \eta) B + (\beta^2 - \delta) B^2. \]  
(3.22)

Now there remain constraints to be determined by applying \([B, K] = 0\). We have
\[ [B, K] = [B, C^2] - \sum_{i=1}^{L+1} \alpha_i [B, [A^i, B]] - \sum_{i=1}^{L} \beta \omega_i [B, [A^i, B]] - \beta [B, [A, B^2]] + \sum_{i=1}^{M+1} k_i [B, A^i]. \]

We make use of the easily established relations
\[ [B, B^2] = \sum_{i=1}^{M} \lambda_i [A^i, C] - \beta [B^2, C] + \eta [B, C] + \sum_{i=1}^{L} \omega_i ([A^i, B], C) + 2\zeta C, \]
\[ [B, [A^i, B]] = -([A^i, B], \]
\[ B), [B, [A, B^2]] = -[B^2, C], \]
along with
\[ [A^i, B], C) = ([A^i, C], B) - \beta ([A^i, B], B) + \eta [A^i, B] + \sum_{j=1}^{L} \omega_j [A^i, [A^i, B]], \]
to simplify this expression. That is,
\[ [B, K] = \sum_{i=1}^{M} \lambda_i [A^i, C] + \eta [B, C] + \sum_{i=1}^{L} \omega_i ([A^i, C], B) + \sum_{i=1}^{L} \omega_i \eta [A^i, B] \]
\[ + \sum_{i=1}^{L} \sum_{j=1}^{L} \omega_i \omega_j [A^i, [A^j, B]] + 2\zeta C + \sum_{i=1}^{L} \alpha_i ([A^i, B], B) - \sum_{i=1}^{M+1} k_i [A^i, B] \]
\[ = \sum_{i=1}^{M} \lambda_i [A^i, C] + \sum_{i=1}^{L} \omega_i \eta [A^i, B] + \sum_{i=1}^{L} \sum_{j=1}^{L} \omega_i \omega_j [A^i, [A^j, B]] + 2\zeta C \]
\[ - \sum_{i=1}^{M+1} k_i [A^i, B] + \left\{ B, \eta C + \sum_{i=1}^{L} \omega_j [A^i, C] + \sum_{i=1}^{L} \alpha_i [A^i, B] \right\}. \]  
(3.23)

Clearly the last anti-commutator term in the above expression vanishes due to constraint (3.2). We also have
\[ [A^i, [A^j, B]] = [A^{i+1}, B] + A^i B A^j - A^j B A^i. \]
The following result is helpful in further simplifying these expressions.
Lemma 4. Let $\tilde{x}_{i,j}$ and $\tilde{y}_{i,j}$ be numbers satisfying the system of recurrence relations

\[
\tilde{x}_{i,j} = \tilde{x}_{i-1,j-1} + \beta \tilde{x}_{i,j-1} + \tilde{y}_{i,j-1},
\]
\[
\tilde{y}_{i,j} = \delta \tilde{x}_{i,j-1} + 2\beta \tilde{x}_{i-1,j-1} + \tilde{y}_{i-1,j-1},
\]

with

\[
\tilde{y}_{0,1} = 0, \quad \tilde{y}_{1,1} = 1, \quad \tilde{x}_{0,1} = 1,
\]

and where we adopt the convention

\[
\tilde{y}_{-1,j} = 0 = \tilde{x}_{-1,j}, \quad \tilde{x}_{j} = 0 = \tilde{x}_{j+1},
\]

for any $j \geq 1$. Then for $i > j \geq 1$,

\[
A'B^i A - A'B^j A = \sum_{k=0}^{j-1} \tilde{x}_{k} \{ A^{i+k}, B \} - \sum_{k=0}^{j-1} \tilde{x}_{k} \{ A^{i+k}, C \}.
\]

(3.24)

Proof. Using the notation of equations (3.3) and (3.4), the result we seek to prove is expressed as

\[
Q(i, j) = \sum_{k=0}^{j} \tilde{y}_{k} Q(i+k, 0) - \sum_{k=0}^{j-1} \tilde{x}_{k} P(i+k, 0).
\]

The details of the proof follow the same lines as that of lemma 3. □

We remark that the system of recurrence relations are the same as those occurring in lemma 3, but with different boundary conditions.

Substituting the result of corollary 2 into equation (3.24) gives the following.

Corollary 3. For $i, j \geq 1$,

\[
A'B^i A - A'B^j A = \sum_{k=0}^{i+j-1} W_{i,j}^{k} \{ A^{k}, C \},
\]

where

\[
W_{i,j}^{k} = \begin{cases}
\sum_{m=0}^{j} \tilde{y}_{m,k} s_{k}^{(i+m)}, & k = 1, 2, \ldots, i - 1 \\
\sum_{m=k-i+1}^{j} \tilde{y}_{m,k} s_{k}^{(i+m)} - \tilde{x}_{k-i,j}, & k = i, i + 1, \ldots i + j - 1.
\end{cases}
\]

One may verify that $W_{i,j}^{k} = 0$ for all $k$ as expected.

The outcome of corollary 3 is that we may write

\[
[A^i, [A^j, B]] = [A^{i+j}, B] + A'B^i A - A'B^j A = \sum_{k=0}^{i+j-1} W_{i,j}^{k} \{ A^{k}, C \}.
\]
Setting $\lambda_0 = \zeta$, equation (2.3) then becomes

$$[B, K] = \sum_{k=0}^{M} \lambda_k [A^k, C] + \sum_{i=1}^{L} \sum_{k=0}^{i-1} \omega_i \eta_i s_k^{(i)} [A^k, C] + \sum_{i=1}^{M+1} \sum_{k=0}^{i-1} \alpha_i s_k^{(i)} [A^k, C]$$

$$+ \sum_{i=1}^{L} \sum_{j=1}^{L} \sum_{k=0}^{i+j-1} \omega_i \omega_j \left( s_k^{(i+j)} + W_k^{(i+j)} \right) [A^k, C] - \sum_{i=1}^{M+1} \sum_{k=0}^{i-1} k_i s_k^{(i)} [A^k, C].$$

Interchanging the order of all the multiple summations so that the sums over $k$ are on the outside leads to

$$[B, K] = \sum_{k=0}^{M} \lambda_k [A^k, C] + \sum_{k=0}^{L-1} F_k [A^k, C] + \sum_{k=0}^{L} G_k [A^k, C] + \sum_{k=0}^{2L-1} Z_k [A^k, C] - \sum_{k=0}^{M} H_k [A^k, C],$$

where

$$F_k = \sum_{i=k+1}^{L} \omega_i \eta_i s_k^{(i)}, \quad G_k = \sum_{i=k+1}^{L+1} \alpha_i s_k^{(i)},$$

$$H_k = \sum_{i=k+1}^{M+1} k_i s_k^{(i)}, \quad (3.25)$$

and

$$Z_k = \begin{cases} \sum_{i=1}^{L} \sum_{j=1}^{L} \omega_i \omega_j \left( s_k^{(i+j)} + W_k^{(i+j)} \right), & k = 0, 1, \\ \sum_{i=k}^{L} \sum_{j=1}^{L} \omega_i \omega_j \left( s_k^{(i+j)} + W_k^{(i+j)} \right) + \sum_{i=k+1}^{k-1} \sum_{j=1}^{L} \omega_i \omega_j \left( s_k^{(i+j)} + W_k^{(i+j)} \right), & k = 2, 3, \ldots, L, \\ \sum_{i=k-L+1}^{L} \sum_{j=k-L+1}^{L} \omega_i \omega_j \left( s_k^{(i+j)} + W_k^{(i+j)} \right), & k = L + 1, \ldots, 2L - 1. \end{cases}$$

Setting $[B, K] = 0$ and equating coefficients of the linearly independent $[A^k, C]$ then leads to the system of linear equations (keeping in mind $\lambda_0 = \zeta$)

$$\lambda_i + F_i + G_i + Z_i = H_i, \quad i = 0, 1, \ldots, L - 1,$$

$$\lambda_L + G_L + Z_L = H_L, \quad (3.26)$$

$$\lambda_i + Z_i = H_i, \quad i = L + 1, \ldots, 2L - 1,$$

$$\lambda_i = H_i, \quad i = 2L, M. \quad (3.27)$$

Since there are clearly $L$ equations in (3.26), 1 equation in (3.27), $L - 1$ equations in (3.28) and $M - 2L + 1 (= 1$ if $M$ is even and 2 otherwise) equations in (3.29), there are $M + 1$ linear equations that can be solved for the coefficients $k_1, k_2, \ldots, k_{M+1}$. We have therefore shown the following.

**Proposition 6.** The Casimir operator $K$ is of the form given in (3.22), with coefficients $k_i$ determined by first solving the system of linear equations (3.26)-(3.29) for the $H_k$, and then solving (3.25).
3.3. Oscillator realization

Let us consider the following deformed oscillator algebra:

\[ [N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad b^\dagger b = \Phi(N), \quad bb^\dagger = \Phi(N + 1) \] (3.30)

and realizations of the form

\[ A = A(N), \quad B = B_0(N) + b^\dagger \rho(N) + \rho(N)b. \] (3.31)

We refer to \( \Phi(N) \) as a structure function, and note that \( \Phi(N) \) and \( \rho(N) \) are as yet unspecified. We will find, however, that there are constraints that will determine the form of these functions. Defining

\[ \Delta A(N) = A(N + 1) - A(N), \]

by analogy with lemma 1 in the classical case, we have the easily established relations

\[ [A(N), b^\dagger] = b^\dagger \Delta A(N), \]
\[ [A(N), b] = -\Delta A(N)b, \]
\[ [A(N), b^\dagger] = b^\dagger (A(N + 1) + A(N)), \]
\[ [A(N), b] = (A(N + 1) + A(N))b. \]

We may obtain the realization of the generator \( C \) by applying the first relation (3.1a) of the polynomial algebra, giving

\[ C = [A, B] = b^\dagger \Delta A(N) \rho(N) - \rho(N) \Delta A(N) b. \] (3.32)

The second relation (3.1b) provides two equations to constrain \( A(N) \) and \( B_0(N) \) as in the classical case. Applying the realization, one has

\[ [A, C] = b^\dagger (\Delta A(N))^2 \rho(N) + \rho(N)(\Delta A(N))^2 b \]

\[ = \sum_{i=1}^{L+1} \alpha_i (A(N))^i + \delta B_0(N) + b^\dagger \rho(N) + \rho(N)b + \epsilon + \beta(2A(N)B_0(N) + b^\dagger (A(N + 1) + A(N))\rho(N) + \rho(N)(A(N + 1) + A(N))b). \]

This gives two functional equations for the functions \( A(N) \) and \( B_0(N) \):

\[ (\Delta A(N))^2 = \delta + \beta(A(N) + A(N + 1)), \] (3.33)
\[ 0 = \sum_{i=1}^{L+1} \alpha_i A(N)^i + \delta B_0(N) + \epsilon + 2\beta A(N)B_0(N). \] (3.34)

As in the classical case, the conditions (3.33) and (3.34) can be solved for \( A(N) \) and \( B_0(N) \), for two distinct cases. We have

\[ A(N) = \begin{cases} \sqrt{\delta N} + c_1, & \beta = 0, \\ -\frac{\beta}{8} - \frac{\delta}{2\beta} + \frac{\beta}{2}(N + c_1)^2, & \beta \neq 0, \end{cases} \] (3.35)

with \( c_1 \) an arbitrary constant,

\[ B_0(N) = -\frac{4}{4A'(N)^2 - \beta^2} \sum_{i=1}^{L+1} \alpha_i A(N)^i + \epsilon, \] (3.36)

where \( A' \) denotes the usual derivative of the function \( A \).

At this stage we point out that equation (3.2), which gives a relationship between the structure constants of the Lie algebra, also has a corresponding expression in terms of the
realization. This expression, which actually arises as the coefficient of both $b$ and $b^\dagger$ upon substitution of the realization into (3.2), is given by

$$\eta \Delta A(N) + \sum_{i=1}^{L} \omega_i \Delta A(N)(A(N)^i + A(N + 1)^i) + \sum_{i=1}^{L+1} \alpha_i (A(N + 1)^i - A(N)^i) = 0. \quad (3.37)$$

Now applying the realization to the third relation (3.1c) of the Lie algebra leads to

$$[B, C] = (b^\dagger)^2 \rho(N) \rho(N + 1)(\Delta A(N) - \Delta A(N + 1))$$
$$+ \rho(N) \rho(N + 1)(\Delta A(N) - \Delta A(N + 1))b^2$$
$$+ b^\dagger \rho(N) \Delta A(N)(B_0(N + 1) - B_0(N)) + \rho(N) \Delta A(N)(B_0(N + 1) - B_0(N))b$$
$$- 2\Phi(N) \rho(N - 1)^2 \Delta A(N - 1) + 2\Phi(N + 1) \rho(N)^2 \Delta A(N)$$
$$= \zeta + \sum_{i=1}^{M} \lambda_i A(N)^i + \eta (B_0(N) + b^\dagger \rho(N + \rho(N)b)$$
$$- \beta ((b^\dagger)^2 \rho(N) \rho(N + 1) + \rho(N) \rho(N + 1)b^2)$$
$$- \beta (b^\dagger (B_0(N + 1) + B_0(N)) \rho(N) + \rho(N)(B_0(N + 1) + B_0(N))b)$$
$$- \beta (B_0(N)^2 + \rho(N - 1)^2 \Phi(N) + \rho(N)^2 \Phi(N + 1))$$
$$+ \sum_{j=1}^{L} \omega_j (b^\dagger (A(N + 1)^j + A(N)^j) \rho(N) + \rho(N)(A(N + 1)^j + A(N)^j)b$$
$$+ 2A(N)^j B_0(N))$$.

This gives rise to the three constraint relations

$$\Delta A(N) - \Delta A(N + 1) = -\beta \quad (3.38)$$
$$\Delta A(N)(B_0(N + 1) - B_0(N)) = -\beta (B_0(N + 1) + B_0(N))$$
$$+ \sum_{i=1}^{L} \omega_i (A(N + 1)^i + A(N)^i) + \eta \quad (3.39)$$

$$- 2\Phi(N) \rho(N - 1)^2 \Delta A(N - 1) + 2\Phi(N + 1) \rho(N)^2 \Delta A(N)$$
$$= \zeta + \sum_{i=1}^{M} \lambda_i A(N)^i - \beta B_0(N)^2 - \beta (\rho(N - 1)^2 \Phi(N) + \rho(N)^2 \Phi(N + 1))$$
$$+ \eta B_0(N) + \sum_{j=1}^{L} 2\omega_j A(N)^j B_0(N). \quad (3.40)$$

The first two relations, however, can be seen to be redundant given (3.33), (3.34) and (3.37). This can be seen from the following calculation.

In the first instance, let us consider (3.33) at $N$ and $N + 1$ and subtract the two equations. This leads directly to equation (3.38).

Now let us multiply the equation (3.39) by $\Delta A(N)$ and replace the expression for $(\Delta A(N))^2$ from (3.33). We obtain
The functions \( \alpha \) however, remain with equation (3.30) for the generators \( A, B, \) and \( C \) given by (3.31) and have already discussed, however, such a proposition, which serves as the quantum analogue of proposition 3.

We summarise our results on the realization of the polynomial Lie algebra in the following proposition, which serves as the quantum analogue of proposition 3.

**Proposition 7.** The polynomial Lie algebra \( \mathcal{L}_M \), has realization in terms of the deformed oscillator algebra with relations (3.30) for the generators \( A, B, \) and \( C \) given by (3.31) and (3.32). The functions \( \Lambda(N) \) and \( B_0(N) \) are given by (3.35) and (3.36) respectively, and the constraint (3.40) is satisfied by \( \Phi(N) \) and \( \rho(N) \).

Now we turn to investigate the Casimir operator in the deformed oscillator realization. Using the form of the Casimir operator \( K \) of equation (3.21), we have the following realization for \( K \):

\[
\begin{align*}
K &= (b^1)^2(\Delta A(N + 1)\Delta A(N) - \beta(A(N + 1) + A(N)) + \beta^2 - \delta)\rho(N)\rho(N + 1) \\
&+ (\Delta A(N + 1)\Delta A(N) - \beta(A(N + 1) + A(N)) + \beta^2 - \delta)\rho(N)\rho(N + 1)b^2 \\
&+ b^1\left(-\beta(A(N + 1) + A(N))(B_0(N + 1) + B_0(N)) \\
- \sum_{i=1}^{L+1} a_i(A(N + 1)^i + A(N)^i) - \sum_{i=1}^{L+1} \beta \omega_i(A(N + 1)^i + A(N)^i) - 2\epsilon - \beta \eta \\
+ (\beta^2 - \delta)(B_0(N + 1) + B_0(N)) \right)\rho(N) \\
+ \left(-\beta(A(N + 1) + A(N))(B_0(N + 1) + B_0(N)) \\
- \sum_{i=1}^{L+1} a_i(A(N + 1)^i + A(N)^i) - \sum_{i=1}^{L+1} \beta \omega_i(A(N + 1)^i + A(N)^i) - 2\epsilon - \beta \eta \\
+ (\beta^2 - \delta)(B_0(N + 1) + B_0(N)) \right)\rho(N)b - (\Delta A(N - 1))^2\rho(N - 1)^2\Phi(N) \\
- (\Delta A(N))^2\rho(N)^2\Phi(N + 1) - \sum_{i=1}^{L+1} 2a_iA(N)^iB_0(N) - \sum_{i=1}^{L} 2\beta \omega_iA(N)^iB_0(N)
\end{align*}
\]
\[-2\beta A(N)B_0(N)^2 - 2\beta A(N)\rho(N-1)^2\Phi(N) - 2\beta A(N)\rho(N)^2\Phi(N+1) + \sum_{i=1}^{M+1} k_i A(N)^i - (2\epsilon + \beta \eta)B_0(N) + (\beta^2 - \delta)(B_0(N)^2 + \rho(N-1)^2\Phi(N)) + \rho(N)^2\Phi(N+1)\].

The condition that $K$ should not depend explicitly on $b$, $b^i$ or their powers can be achieved by setting the coefficients of those terms to zero in the above expression for $K$. This implies the following two constraints, along with the actual form of $K$:

\[0 = \Delta A(N + 1)\Delta A(N) - \beta(A(N + 2) + A(N)) + \beta^2 - \delta \tag{3.43}\]

\[0 = -\beta(A(N + 1) + A(N))(B_0(N + 1) + B_0(N)) - \sum_{i=1}^{L+1} \alpha_i(A(N + 1)^i + A(N)^i) - \sum_{i=1}^{L} \beta \omega_i(A(N + 1)^i + A(N)^i) - 2\epsilon - \beta \eta + (\beta^2 - \delta)(B_0(N + 1) + B_0(N)) \tag{3.44}\]

\[K = -\Delta(A(N)N - 1)^2\rho(N - 1)^2\Phi(N) - (\Delta A(N))^2\rho(N)^2\Phi(N+1) - \sum_{i=1}^{L+1} 2\alpha_i A(N)^i B_0(N) - 2\beta A(N)B_0(N)^2 - 2\beta A(N)\rho(N-1)^2\Phi(N) - 2\beta A(N)\rho(N)^2\Phi(N+1) + (\beta^2 - \delta)(B_0(N)^2 + \rho(N-1)^2\Phi(N) + \rho(N)^2\Phi(N+1)). \tag{3.45}\]

It turns out that the constraints (3.43) and (3.44) are also redundant given the result of proposition 7 above. We clarify this in what follows.

Expressing equation (3.38) as

\[\Delta A(N) = \Delta A(N + 1) - \beta, \tag{3.46}\]

we substitute (3.46) in for $\Delta A(N)$ in equation (3.43), and then make use of equation (3.33) at $N + 1$ to substitute in for $(\Delta A(N + 1))^2$. The expression then simplifies to equation (3.38), which implies the constraint (3.43) provides no new information.

Finally, taking relation (3.34) in the form

\[-\sum_{i=1}^{L+1} \alpha_i A(N)^i - \epsilon - \delta B_0(N) - \beta A(N)B_0(N) = \beta A(N)B_0(N), \]

and also considering this form at $N + 1$, we substitute these expressions into equation (3.44) which then becomes

\[-\sum_{i=1}^{L} \beta \omega_i(A(N + 1)^i + A(N)^i) + \beta \Delta A(N)(B_0(N + 1) - B_0(N)) - \beta \eta + \beta^2(B_0(N + 1) + B_0(N)) = 0. \]

Multiplying by $\Delta A(N)$ and using (3.37), (3.33) and (3.34) at $N$ and $N + 1$ shows that the identity is trivially satisfied. In summary, we have proved the following result.
Proposition 8. The Casimir operator $K$ of the polynomial Lie algebra $\mathcal{L}_M$ can be realized in terms of the deformed oscillator algebra with relations (3.33). The explicit expression for $K$ is given by (3.45).

We remark that in some sense the form of $K$ given in (3.45) can be considered a further constraint on the functions $\Phi(N)$ and $\rho(N)$. With this form of $K$ in addition to the constraint (3.40), we may determine $\Phi(N)$, with $\rho(N)$ chosen in a such way that $\Phi(N)$ is a polynomial function of $N$ with degree greater than one as in previous work for $M = 2$ [3], $M = 3$ [23] and $M = 4$ [24]. The possibility for the structure function to be a polynomial is related with existence of more complicated patterns for the spectrum of quantum systems as one considers systems beyond quadratic superintegrability. Let us now illustrate the necessity of taking a polynomial structure function in $N$. There are ways to construct polynomial algebras other than adopting a direct approach for a given 2D superintegrable Hamiltonian via polynomial Heisenberg algebras formed by ladder operators [28, 29] of two 1D components. These polynomial Heisenberg algebras are themselves direct consequences of the fact that the ladder operators are of higher order which allow the possibility of many zero modes. This implies the existence of many infinite and finite sequences of levels. Thus the linearization of the deformed oscillator algebras using other choices for the function $\rho(N)$ would essentially be equivalent to linearization of the corresponding polynomial Heisenberg algebra without singlet states [29]. Furthermore, the structure function that takes a polynomial form (equation (36.18) of [31]) has been shown to be useful for the anisotropic harmonic oscillator with ratio $m:n$ in order to obtain an algebraic derivation of the energy spectrum using the well-defined integrals of motion expressed in terms of polynomials in the momenta.

3.4. Application to superintegrable systems

These realizations as deformed oscillator algebras of polynomial associative algebras can be used to calculate algebraically the energy spectrum and total number of degeneracies per level of quantum superintegrable systems. Before discussing how these constructions can be applied, let us present some definitions concerning superintegrability [12].

Definition 1. A classical Hamiltonian system in $n$ dimensions is (polynomially) superintegrable if it admits $n + k$ (with $k = 1, \ldots, n - 1$) functionally independent constants of the motion that are polynomial in the momenta and are globally defined except possibly for singularities on a lower dimensional manifold. It is minimally (polynomially) superintegrable if $k = 1$ and maximally (polynomially) superintegrable if $k = n - 1$.

Note that as mentioned in [12], many distinct $n$-subsets of the $2n - 1$ polynomial constants of the motion for a superintegrable system could be in involution and in a such case the system would be called multi-integrable.

Definition 2. A quantum system in $n$ dimensions is superintegrable (of finite-order) if it admits $n + k$, $k = 1, \ldots, n$ algebraically independent finite-order partial differential operators $L_1 = H, \ldots, L_n + k$ in the variables $x$ globally defined, such that $[H, L_j] = 0$. Again, it is minimally superintegrable (of finite-order) if $k = 1$ and maximally superintegrable (of finite-order) if $k = n - 1$.

A direct consequence of these definitions is that all two-dimensional superintegrable systems are maximally superintegrable. This means that for a two-dimensional superintegrable system, two other integrals of motion $A$ and $B$ can commute with the Hamiltonian (i.e. $[H, A] = [H, B] = 0$) and they will generate a non-Abelian algebra. This algebra, if it closes
under polynomial relations, will be of the form studied in section 3 and given in proposition 4. In such a case, the structure constants will be polynomial in the Hamiltonian, however as it commutes with generators, this does not modify the results obtained. The maximal order of these polynomials is bounded by the order of the left side of second and third relation of polynomial associative algebra. The Casimir can be written only in terms of the Hamiltonian of these polynomials is bounded by the left side of second and third relations in this paper would remain valid. The structure constants of the polynomial associative algebra, however, will be polynomial functions not only of the Hamiltonian but also of these other integrals of motion (\(F_i\)) are also central elements and form with the Hamiltonian an Abelian subalgebra i.e.

\[
[F_i, H] = [F_i, F_j] = [F_i, A] = [F_i, B] = [F_i, C] = 0. \tag{3.47}
\]

These supplementary integrals take for example the form of a monopole charge or angular momentum for which we know eigenvalues. In such a case, the form of the commutation relations in this paper would remain valid. The structure constants of the polynomial associative algebra, however, will be polynomial functions not only of the Hamiltonian but also of these other integrals of motion (\(F_i\)). Thus in the study of the algebra’s realization in terms of deformed oscillator algebras and its representations, we will fix the energy (\(H\psi = E\psi\)) and these other integrals (\(F_i\psi = f_i\psi\)). The Casimir operator \(K\) of this quadratic algebra is thus given in terms of the generators and will be also rewritten as a polynomial of \(H\) and \(F_i\). In this case the order of these polynomials is bounded by the left side of the second and third relations of the polynomial associative algebra. However, as these others form an Abelian subalgebra, the form of the Casimir in terms of generators \(A, B\) and \(C\) is also not affected. We can thus introduce an energy dependent Fock space of dimension \(p + 1\) defined by

\[
H|f_1, \ldots, f_n; E, n\rangle = E|f_1, \ldots, f_n; E, n\rangle,
\]

with

\[
N|f_1, \ldots, f_n; E, n\rangle = n|f_1, \ldots, f_n; E, n\rangle
\]

\[b|f_1, \ldots, f_n; E, 0\rangle = 0,
\]

and the action of operators \(b\) and \(b^\dagger\) are given by

\[
b^\dagger|n\rangle = \sqrt{\Phi(f_1, \ldots, f_n; E, n + 1, u)}|f_1, \ldots, f_n; E, n + 1\rangle,
\]

\[
b|n\rangle = \sqrt{\Phi(f_1, \ldots, f_n; E, n, u)}|f_1, \ldots, f_n; E, n - 1\rangle.
\]

We furthermore have the existence of finite dimensional unitary representations if we impose the following constraint

\[
\Phi(f_1, \ldots, f_n; E, p + 1, u) = 0, \quad \Phi(f_1, \ldots, f_n; E, 0, u) = 0,
\]

\[
\Phi(f_1, \ldots, f_n; E, n, u) > 0, \quad \forall n > 0.
\]

The energy \(E\) and the constant \(u\) can be obtained from this set of constraints that are algebraic equations. The dimension of the finite-dimensional unitary representations is given by \(p + 1\). Let us point out that the fact that the structure function is a polynomial allows for the existence of many solutions and the possibility to obtain algebraically more complicated patterns.

In the case of three-dimensional systems [7], it was shown in an extended Kepler system that other integrals do not form an Abelian subalgebra and thus do not commute with \(A\) and \(B\). We can, however, identify many structures of quadratic algebra with three generators in which other integrals can play the role of the Hamiltonian in the structure constant.
Let us point out that this method is very convenient for determining the energy spectrum of various superintegrable systems. Generally it is not guaranteed that integrals of motion close at a given order and it was pointed out one needs to sometimes to take further commutators between integrals to generate integrals that would close.

Furthermore, it was observed that in some cases integrals of motion can close at a given order but the finite dimensional unitary representations do not provide all the levels and total number of degeneracies [28]. It was shown, however, that it is still possible to generate higher order polynomial algebras that provide all the appropriate levels. Through direct but involved calculation, it can be shown that the degeneracies are correct but for a fixed level given by the union of finite dimensional unitary representations [29]. This algebraic technique can also generate non-physical solutions that one needs to remove. We point out that careful analysis should be performed and further study on the method itself needs to be done.

The polynomial Poisson algebra can be calculated in the context of superintegrable classical systems. However, this is not clear how to obtain information on the systems from this algebraic structure in the classical context. It can be used in context of the classification of these systems [4, 11]. We can also observe by looking at the classical and quantum analogue how the polynomial Poisson algebra is deformed into a polynomial associative algebra with higher order correction terms in the Planck constant.

4. Examples of polynomial lie algebra

4.1. Systems associated with Cartesian coordinates

Let us investigate further the case of the polynomial algebra of arbitrary order related to many examples of 2D superintegrable systems in Euclidean space with separation of variables in Cartesian coordinates [25]

\[ [A, B] = C, \quad [A, C] = \delta B, \quad [B, C] = \sum_{i=1}^{M} \lambda_i A^i + \zeta. \]

The Jacobi identity is trivially satisfied since in proposition 4

\[ \omega_i = 0, \quad \eta = 0, \quad \alpha_i = 0, \quad \beta = 0, \quad \epsilon = 0. \]

Constraint (3.40) for the Casimir operator reduces to

\[ -2\Phi(N)\rho(N-1)^2\sqrt{\delta} + 2\Phi(N+1)\rho(N)^2\sqrt{\delta} = \zeta + \sum_{i=1}^{M+1} k_i A(N)^i. \]

The solutions for the functions \( A(N) \), \( B_0(N) \) are

\[ A(N) = \sqrt{\delta}N + c_1, \quad B_0(N) = 0 \]

and the Casimir operator is given by

\[ K = -2\delta\rho(N-1)^2\Phi(N) - 2\delta\rho(N)^2\Phi(N+1) + \sum_{i=1}^{M+1} k_i A(N)^i, \]

which can be computed explicitly to any given order \( M \) using the result of proposition 6 (i.e proposition 6 is used to the find the \( k_i \) for a fixed order \( M \)). The function \( \rho(N) \) can be taken to be a constant in this case.
4.2. Systems associated with polar coordinates

Let us describe another class of polynomial algebra of arbitrary order that includes many 2D superintegrable systems in Euclidean space and allowing separation of variables in polar coordinates, such as the Tremblay–Turbiner–Winternitz (TTW) systems [26, 30].

\[ [A, B] = C, \]
\[ [A, C] = \alpha_1 A + \alpha_2 A^2 + \delta B + \epsilon + \beta [A, B], \]
\[ [B, C] = \sum_{i=1}^{M} \lambda_i A^i - \beta B^2 + \eta B + \omega_1 [A, B] + \zeta. \]

The constraint equation (3.2) of proposition 4 implies
\[ \eta = -\alpha_1, \quad \omega_1 = -\alpha_2. \]

The constraints on the parameters of the Casimir operator are
\[ k_1 = 2\zeta - \alpha_1 \alpha_2, \quad k_2 = \lambda_1 - \alpha_2^2, \]
\[ \sum_{i=2}^{M} \lambda_i [C, A^i] = \sum_{i=3}^{M+1} k_i [A^i, B]. \]

This is a system of linear equations in \( k_i \) that can be solved at any order \( M \). The solution takes the form
\[ A(N) = \frac{\beta}{2} \left( \left( (N + c_1)^2 - \frac{1}{4} \right) - \frac{\delta}{\beta^2} \right), \]
\[ B_0(N) = -\frac{\alpha_2}{4} \left( (N + c_1)^2 - \frac{1}{4} \right) + \left( -\frac{\beta \alpha_1 + \alpha_2 \delta}{2 \beta^2} \right) \]
\[ -\frac{-2\beta \alpha_1 \delta + \alpha_2 \delta^2 + 4\beta^2 \epsilon}{4\beta^2} \frac{1}{(N + c_1)^2 - \frac{1}{4}}. \]

We have two equations to obtain the structure functions, namely
\[ -2\Phi(N) \rho (N-1)^2 \Delta A(N-1)^2 + 2\Phi(N+1) \rho (N)^2 \Delta A(N)^2 = \sum_{i=1}^{M} \lambda_i A(N)^i \]
\[ -\alpha_1 B_0(N) + 2\alpha_2 B_0(N) A(N) - \beta \rho^2 (N-1) \Phi(N) + \rho^2 (N) \Phi(N+1), \]
along with the expression for \( K \) from equation (3.45), which for this case becomes
\[ K = -(\Delta A(N-1))^2 \rho (N-1)^2 \Phi(N) + (\Delta A(N))^2 \rho (N)^2 \Phi(N+1) - 2\alpha_1 A(N) B_0(N) \]
\[ -2\alpha_2 A(N)^2 B_0(N) + 2\beta \alpha_2 A(N) B_0(N) - 2\beta A(N) B_0(N)^2 \]
\[ -2\beta A(N) \rho (N-1)^2 \Phi(N) - 2\beta A(N) \rho (N)^2 \Phi(N+1) + \sum_{i=1}^{M+1} k_i A(N)^i \]
\[ -(2\epsilon + \beta \eta) B_0(N) + (\beta^2 - \delta)(B_0(N)^2 + \rho (N-1)^2 \Phi(N) + \rho (N)^2 \Phi(N+1)). \]

In particular, an algebraic derivation of the spectrum of the TTW Hamiltonian could be obtained using deformed oscillator algebra this is an open problem that would need to be investigated. As mentioned this paper is devoted to showing the existence of a realization beyond the quartic case and the construction of the Casimir operator, however it is clear that many examples with integrals of motion of higher order could be studied using deformed oscillator algebras and results of this paper.
5. Conclusion

In this paper, we presented the most general polynomial Lie algebra generated by integrals of order two and $M$, and presented constraints on the structure constants arising from the Jacobi identity. We also constructed the Casimir operator and obtained further constraints on the parameters. Explicit formulae could be obtained at an arbitrary, fixed order $M$ and can be applied to a given example using the algorithm outlined in section 3.1.1. We found, however, that the general solution is difficult and it is clear that further work on this problem could be undertaken. We derived many identities concerning various commutators and anti-commutators involving the generators of the polynomial Lie algebra. We showed that realizations of the Polynomial Lie algebra via a deformed oscillator algebra exist as a result of the Jacobi identity. We also pointed out that in the classical case the most general polynomial Poisson algebra can be put in the form of the classical analogue of a deformed oscillator algebra when the Jacobi identity is imposed. We also explicitly constructed the Casimir element for the classical case.

As described in section 3, the deformed oscillator algebra is a very convenient tool to study representations and algebraically obtain the energy spectrum of superintegrable systems. We discussed two particular classes of such polynomial algebras that were observed in the context of superintegrable systems.

It is worth pointing out that the polynomial algebras investigated in this article are in fact extensions of the Askey–Wilson algebra [1] that do not possess the duality property in the generators.

The case of higher dimensional systems would necessitate the study of polynomial algebra with more than three generators and have investigation of the structure of the subsets of integrals that commute together. The application of realizations via deformed oscillator algebras beyond a class of systems in which other integrals form an Abelian subalgebra with the Hamiltonian is a relatively unexplored subject for the case of deformed Kepler systems [7]. This could provide insight to a more systematic study of the algebraic derivation of energy spectrum of superintegrable systems in more than two dimensions.

Let us point out that quadratic algebras can also be used in the context of systems involving reflection operators [10]. It is likely that higher order polynomial associative algebras could have application also in this context and thus formulae that provide realization as oscillator algebras. We further speculate that continued research into the algebraic structures we have introduced and studied in this paper is likely to reveal applications beyond superintegrable systems in quantum mechanics.

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