A NOTE ON THE SIGNATURE REPRESENTATIONS OF THE SYMMETRIC GROUPS

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Abstract. For a partition \( \lambda \) and a prime \( p \), we prove a necessary and sufficient condition for there exists a composition \( \delta \) such that \( \delta \) can be obtained from \( \lambda \) after rearrangement and all the partial sums of \( \delta \) are not divisible by \( p \).

Let \( p \) be a prime. In the representation theory of the symmetric groups over a field of characteristic \( p \), the Young permutation and Young modules play a central role. The Young modules are the indecomposable summands of the Young permutation modules up to isomorphism and labelled by partitions (see [5]). In [1], Donkin generalised these objects and obtained the signed Young modules as the indecomposable summands of the signed Young modules up to isomorphism and they are labelled by certain bipartitions. In his paper, the listing modules were also obtained as a generalisation of the tilting modules for the Schur algebras and they are, up to isomorphism, the indecomposable summands of the tensor products of certain symmetric and exterior powers. The original constructions of the Young and signed Young modules used the representation theories of the Schur algebras and superalgebras respectively (as in [5, 1]). Constructions of these objects using only the representation theory of the symmetric groups can be found in [2, 3].

Suppose further that \( p \) is an odd prime. Let \( \lambda \) be a partition and \( c_{\lambda}^{(p)} \) be the number of compositions \( \delta \) such that \( \delta \) can be obtained from \( \lambda \) after rearrangement and all the partial sums of \( \delta \) are not divisible by \( p \). In [6, Corollary 4.6], in particular, the first author showed that, in the Green ring of the symmetric groups (respectively, Schur algebras) over a field of characteristic \( p \), the signature representation of a symmetric group (respectively, exterior power of certain natural module of a Schur algebra) can be written as a linear combination of the signed Young permutation modules (respectively, mixed powers) labelled by bipartitions of the form \((\lambda|p(s))\) with coefficients \( c_{\lambda}^{(p)} \) up to signs. In this paper, in Theorem 4, we give a necessary and sufficient condition for \( c_{\lambda}^{(p)} \neq 0 \).

Let \( \mathbb{N}_0 \) be the set of non-negative integer and let \( n \in \mathbb{N}_0 \). A composition \( \delta \) of \( n \) is a sequence of positive integers \((\delta_1, \ldots, \delta_s)\) such that \( \sum_{i=1}^{s} \delta_i = n \). In this case, we write \( \ell(\delta) = s \) and \( n = |\lambda| \). By convention, the unique composition of 0 is denoted as \( \emptyset \) and \( \ell(\emptyset) = 0 \). Let \( \eta \) be another composition. We define the concatenation

\[
\delta \# \eta = (\delta_1, \ldots, \delta_s, \eta_1, \ldots, \eta_t)
\]

2010 Mathematics Subject Classification. 11P81, 20C30, 20G43.

The first author is supported by Singapore MOE Tier 2 AcRF MOE2015-T2-2-003.
if \( t = \ell(\eta) \). For each \( 1 \leq j \leq s \), we write
\[
\delta_j^+ = \sum_{i=1}^{j} \delta_i
\]
for the partial sum so that \( \delta_s^+ = n \). For each positive integer \( d \), the number of parts of \( \delta \) equal to \( d \) is denoted as \( n_d(\delta) \), i.e.,
\[
n_d(\delta) = |\{i : 1 \leq i \leq s \text{ and } \delta_i = d\}|.
\]
The composition \( \delta \) is called a partition if \( \delta_1 \geq \cdots \geq \delta_s \).

Let \( q \) be a positive integer, \( \lambda \) be a partition of \( n \) and \( \mathcal{C}(\lambda) \) be the set consisting of all compositions which can be rearranged to \( \lambda \). A composition \( \delta \) is called \( q' \)-cumulative if \( q \nmid \delta_j^+ \) for all \( 1 \leq j \leq \ell(\delta) \). We write \( c^{(q)}_\lambda \) for the number of compositions \( \delta \in \mathcal{C}(\lambda) \) such that \( \delta \) is \( q' \)-cumulative. Clearly, the number \( c^{(q)}_\lambda \) depends only on the parts of \( \lambda \) modulo \( q \). For each composition \( \delta \) and \( 0 \leq j \leq q-1 \), let
\[
r_j(\delta) = \sum_{i \equiv j (\text{mod } q)} n_i(\delta)
\]
and \( r(\delta) = (r_1(\delta), \ldots, r_{q-1}(\delta)) \). For any \( r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}_{\text{even}}^{q-1} \), we write
\[
\mathcal{W}_r^{(q)} = \{ \delta \in \mathcal{C}(\mu) : \delta \text{ is } q'-\text{cumulative} \}
\]
where \( \mu = ((q-1)r_{q-1}, \ldots, 1r_1) \). The following lemma is easy.

**Proposition 1.** Let \( q \) be a positive integer and \( \lambda \) be a partition. Then
\[
c^{(q)}_\lambda = |\mathcal{W}_r^{(q)}| \prod_{i=0}^{q-1} r_i(\lambda)! \prod_{s \geq 1} n_s(\lambda)! \left( \frac{\ell(\lambda) - 1}{r_0(\lambda)} \right).
\]
In particular, we have \( c^{(q)}_\lambda \neq 0 \) if and only if \( \mathcal{W}_r^{(q)}_{\ell(\lambda)} \neq \emptyset \).

**Proof.** We only check the converse of the final assertion. If \( \mathcal{W}_r^{(q)}_{\ell(\lambda)} \neq \emptyset \) then \( r_i(\lambda) \neq 0 \) for some \( 1 \leq i \leq \ell(\lambda) \). So \( \ell(\lambda) - 1 \geq r_0(\lambda) \) and hence \( \left( \frac{\ell(\lambda) - 1}{r_0(\lambda)} \right) \neq 0 \). \( \square \)

For example, \( c^{(1)}_\lambda = 0 \) for all \( \lambda \) (including when \( \lambda = \emptyset \)). When \( q = 2 \), \( \mathcal{W}_r^{(2)}_{\ell(\lambda)} \neq \emptyset \) if and only if \( r(\lambda) = (1) \), i.e., \( \lambda \) has exactly one part with odd size and the rest with even sizes. In this case,
\[
c^{(2)}_\lambda = \frac{(\ell(\lambda) - 1)!}{\prod_{s \text{ even}} n_s(\lambda)!}.
\]

To prove our Theorem 4, we will need the following two lemmas. We begin with some notations.

Let \( r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}_{\text{even}}^{q-1} \). We set
\[
\|r\| = \sum_{i=1}^{q-1} ir_i
\]
which is the size of any composition belonging to the set \( \mathcal{W}_r^{(q)} \) so that \( q \mid \|r\| \) if \( \mathcal{W}_r^{(q)} \neq \emptyset \). Also, set

\[
|r|_q = (q - 1) + \sum_{i=2}^{q-1} (q - i)r_i,
\]

\[
\max r = \max\{r_1, \ldots, r_{q-1}\}.
\]

**Lemma 2.** Let \( q \) be a positive integer, \( r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}_0^{q-1} \) and suppose that \( r_1 = \max r \). Then \( \mathcal{W}_r^{(q)} \neq \emptyset \) if and only if

(i) \( q \not\mid \|r\| \), and

(ii) \( \max r \leq |r|_q \).

**Proof.** We argue by induction on \( |r|_q \). If \( |r|_q = q - 1 \) then \( r_i = 0 \) for all \( 2 \leq i \leq q - 1 \). In this case, \( \mathcal{W}_r^{(q)} \neq \emptyset \) if and only if \( \mathcal{W}_r^{(q)} = \{(1^q)\} \) and \( 0 < r_1 \leq q - 1 \), and that is if and only if \( q \not\mid \|r\| \), \( r_1 = \max r \leq |r|_q = q - 1 \). Fix a positive integer \( N \geq q \). Suppose now that the equivalent statement in the lemma holds true for any \( r \in \mathbb{N}_0^{q-1} \) such that \( r_1 = \max r \) and \( |r|_q < N \). Let \( r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}_0^{q-1} \), \( |r|_q = N \) and \( r_1 = \max r \).

Assume that \( \mathcal{W}_r^{(q)} \neq \emptyset \) (so part (i) is satisfied). Suppose on the contrary that \( r_1 = \max r > |r|_q \). Let \( \delta \in \mathcal{W}_r^{(q)} \), \( s = \ell(\delta) \) and \( c \) be maximum such that \( b = \delta_c > 1 \) and \( \delta_i = 1 \) for all \( c + 1 \leq i \leq s \). Since \( \delta \) is \( q' \)-cumulative, we have \( s - c < q - 1 \) (otherwise, \( \delta^{s+c}_{c+d} = \delta^s_c + d \equiv 0(\text{mod} \ q) \) for some \( 1 \leq d \leq q - 1 \leq s - c \)). Consider the following two cases.

(A) Suppose that \( s - c < q - b \). Let \( \delta' \) be the composition such that

\[
\delta = \delta'(b, 1^{s-c}).
\]

Clearly, \( \delta' \) is \( q' \)-cumulative and hence \( \delta' \in \mathcal{W}_{r'}^{(q)} \) for some \( r' = (r'_1, \ldots, r'_{q-1}) \). Notice that

\[
r'_i = r_i - (s - c) > |r|_q - (q - b) = |r'|_q \geq r'_j
\]

for any \( 2 \leq j \leq q - 1 \).

(B) Suppose that \( q - b \leq s - c \). Let \( \eta \) be the composition such that

\[
\delta = \eta'(b, 1^{q-b})#(1^{(s-c)-(q-b)})
\]

and let \( \delta' = \eta'(1^{(s-c)-(q-b)}) \). Notice that

\[
\delta'^{s+c}_{c+d} = \begin{cases} 
\delta^{s+c}_{c+d} & \text{if } j > \ell(\eta), \\
\delta^{s+c}_{c+d} + q & \text{otherwise.}
\end{cases}
\]

So \( \delta' \) is \( q' \)-cumulative and hence \( \delta' \in \mathcal{W}_{r'}^{(q)} \) for some \( r' = (r'_1, \ldots, r'_{q-1}) \). Also,

\[
r'_i = r_i - (q - b) > |r|_q - (q - b) = |r'|_q \geq r'_j
\]

for any \( 2 \leq j \leq q - 1 \).
In both cases, we have \( q \nmid \|r'\| \) and \( r'_1 = \max r' > |r'_1|_q < |r|_q = N \) but yet \( \mathcal{H}'_r^{(q)} \neq \emptyset \). This contradicts to our induction hypothesis.

Conversely, assume that both parts (i) and (ii) in the statement hold. In particular, \( q > 1 \). Consider the following two cases.

(A) Suppose there exists \( 2 \leq b \leq q - 1 \) such that \( r_b > 0 \) and \( b \nmid \|r'\| \pmod{q} \). Let \( r' = (r'_1, \ldots, r'_{q-2}) \) where \( r'_j = r_j \) if \( j \neq b \) and \( r'_b = r_b - 1 \). Then \( r'_1 = \max r' \), \( |r'_1|_q = |r_q| - (q - b) < N \) and

\[
\|r'\| = \|r\| - b \not\equiv 0 \pmod{q}.
\]

By induction hypothesis, there exists \( \delta' \in \mathcal{H}'_r^{(q)} \). It is easy to check that \( \delta' \not\equiv (b, 1) \in \mathcal{H}'_r^{(q)} \).

(B) Suppose that, for any \( 2 \leq b \leq q - 1 \) with \( r_b > 0 \), we have \( b \equiv \|r'\| \pmod{q} \). Since \( |r|_q \geq q \), there exists a unique \( 2 \leq b \leq q - 1 \) such that \( r_b > 0 \). By assumption, \( b \equiv \|r\| \pmod{q} \). We further consider two cases.

(a) Suppose that \( r_1 \leq |r|_q - (q - b) \). Let \( r' = (r'_1, \ldots, r'_{q-1}) \) such that \( r'_i = 0 \) for all \( i \neq 1, b \) and \( r'_1 = r_1 - 1 \) if \( i = 1, b \). Then

\[
\|r'\| = r'_1 + br'_q = \|r\| - 1 - b \equiv -1 \pmod{q},
\]

\[
\max r' = r'_1 \leq |r|_q - (q - b) - 1 < |r'|_q < N.
\]

Since \( q > 2 \), we have \( -1 \not\equiv 0 \pmod{q} \) and hence \( q \nmid \|r'\| \). By induction hypothesis, there exists \( \delta' \in \mathcal{H}'_r^{(q)} \). Define \( \delta = \delta' \not\equiv (b, 1) \). Notice that \( \delta'^{-1}_{i(b')} + 1 \equiv b - 1 \pmod{q} \) and \( \delta'^{-1}_{i(b') + 2} \equiv b \pmod{q} \).

(b) Suppose that \( |r|_q - (q - b) \leq r_1 \). Let \( s = r_1 - (q - b)(r_b - 1) - (q - 1) \).

By assumption, \( 0 < s \leq q - b \). Let \( \eta = (b, 1^{q-b}) \) and

\[
\delta = (1^{q-1}) \not\equiv \eta \equiv \overbrace{\eta \cdots \eta}^{(r_b - 1) \text{ times}} \not\equiv (b, 1^s) \pmod{q}.
\]

In both cases, it is easy to check that \( \delta \) is \( q' \)-cumulative and hence \( \delta \in \mathcal{H}'_r^{(q)} \).

□

To state the next lemma, we introduce a notation. Suppose that \( a \) is multiplicatively invertible in \( \mathbb{Z}/q\mathbb{Z} \) and \( r = (r_1, \ldots, r_{q-2}) \). We write

\[
a^r = (r'_1, \ldots, r'_{q-2})
\]

where \( r'_j = r_j \) if \( j \equiv ai \pmod{q} \) for any \( 1 \leq j \leq q - 1 \). So \( a^r \) is obtained from \( r \) by a permutation determined by \( a \).

Lemma 3. Let \( q \) be a positive integer, \( a \) be multiplicatively invertible in \( \mathbb{Z}/q\mathbb{Z} \) and \( r = (r_1, \ldots, r_{q-1}) \in \mathbb{N}_0^{q-1} \). Then \( |\mathcal{H}'_r^{(q)}| = |\mathcal{H}'_{a^r}^{(q)}| \).

Proof. Define \( \phi : \mathcal{H}'_r^{(q)} \to \mathcal{H}'_{a^r}^{(q)} \) as, for any \( \delta = (\delta_1, \ldots, \delta_s) \in \mathcal{H}'_r^{(q)} \), \( \phi(\delta) = \delta' = (\delta'_1, \ldots, \delta'_s) \) where \( 1 \leq \delta'_t \leq q - 1 \) and \( \delta'_t \equiv a\delta_t \pmod{q} \) for all \( 1 \leq t \leq s \). Notice that, for each \( 1 \leq j \leq q - 1 \), the number of parts of \( \delta' \) with size \( j \) is precisely the number of parts of \( \delta \) with size \( j \) where \( j \equiv ai \pmod{q} \) and \( \delta'_t+ \equiv a\delta_t+ \pmod{q} \). So
Each element of $C$ is well-defined. Since $a$ is invertible in $\mathbb{Z}/q\mathbb{Z}$, $\phi$ is invertible and hence we have $|\mathcal{W}_{\mathbf{r}}^{(q)}| = |\mathcal{W}_{a \mathbf{r}}^{(q)}|$.

We are now ready to prove our main theorem.

**Theorem 4.** Let $p$ be a prime number, $\lambda$ be a partition and $r(\lambda) = (r_1, \ldots, r_{p-1})$. Then $c_{\lambda}^{(p)} \neq 0$ if and only if

1. $p \nmid |\lambda|$, and
2. for some $r_a = \max r(\lambda)$, $\max r(\lambda) \leq |b r(\lambda)|_p$ where $ab \equiv 1 \pmod{p}$.

**Proof.** By Proposition 1 and Lemma 3 we have $c_{\lambda}^{(p)} \neq 0$ if and only if $\mathcal{W}_{\mathbf{r}}^{(p)} \neq \emptyset$ if and only if $\mathcal{W}_{\mathbf{r}_a}(\lambda) \neq \emptyset$ for some $1 \leq b \leq p - 1$. Let $1 \leq a, b \leq p - 1$ be such that $r_a = \max r(\lambda)$, $ab \equiv 1 \pmod{p}$ and $b r(\lambda) = r' = (r'_1, \ldots, r'_{p-1})$. Notice that $r'_1 = r_a = \max r(\lambda) = \max \mathbf{r}$. By Lemma 2 $\mathcal{W}_{\mathbf{r}_a}(\lambda) = \mathcal{W}_{\mathbf{r}'}^{(p)} \neq \emptyset$ if and only if

1. $p \nmid \|\mathbf{r}'\|$, and
2. $\max \mathbf{r}' \leq |\mathbf{r}'|_p$.

Notice that

$$
\|\mathbf{r}'\| = \sum_{j=1}^{p-1} j r'_j = \sum_{i=1}^{p-1} i b r_i = b \sum_{i=0}^{p-1} i r_i = b \sum_{i=0}^{p-1} i \sum_{j \equiv i \pmod{p}} n_j(\lambda)
\leq b \sum_{i=0}^{p-1} \sum_{j \equiv i \pmod{p}} j n_j(\lambda) = b |\lambda| \pmod{p}.
$$

Therefore, $p \nmid \|\mathbf{r}'\|$ is equivalent to $p \nmid |\lambda|$. The proof is now complete.

We end the paper with the following remark.

**Remark 5.** Keep the notation as in Theorem 4 and suppose that there is another $1 \leq c \leq p - 1$ such that $c \neq a$ and $r_a = r_c = \max r(\lambda)$. Then $|b r(\lambda)|_p = (p - 1) + \sum_{i=2}^{p-1} (p - i) r'_i \geq (p - j) r'_j = (p - j) r_c \geq r_c$ where $j \equiv bc \pmod{p}$. If $p \nmid |\lambda|$, by Theorem 4 $c_{\lambda}^{(p)} \neq 0$. In other words, as soon as $p \nmid |\lambda|$ and $r(\lambda)$ attains its maximum at least 2 distinct places, we have $c_{\lambda}^{(p)} \neq 0$.

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