Optimum unambiguous identification of $d$ unknown pure qudit states

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We address the problem of unambiguously identifying the state of a probe qudit with the state of one of $d$ reference qudits. The $d$ reference states are assumed pure and linearly independent but we have no knowledge of them. The state of the probe qudit is assumed to coincide equally likely with either one of the $d$ unknown reference states. We derive the optimum measurement strategy that maximizes the success probability of unambiguous identification and find that the optimum strategy is a generalized measurement. We give both the measurement operators and the optimum success probability explicitly. Technically, the problem we solve amounts to the optimum unambiguous discrimination of $d$ known mixed quantum states.

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I. INTRODUCTION

It has been shown that $N$ given pure quantum states can be unambiguously discriminated provided they are linearly independent \(^1\), at the expense of a certain fraction of inconclusive results if the states are nonorthogonal. Unambiguous state identification is a variant of the pure-state discrimination problem. Here we assume that a quantum system of a $d$-dimensional Hilbert space is prepared in a definite pure state out of a set of $N$ linearly independent states, but that we do not know the states in the set. Instead, we are given $N$ reference systems each being prepared in another one of the $N$ unknown pure states. Our task is to determine the optimum measurement for unambiguously identifying the state of the probe system with the state of one of the reference systems, that is to find the particular measurement which maximizes the overall probability of a successful identification. Despite the complete lack of knowledge about the pure states, their probabilistic identification is possible due to symmetry properties inherent in the quantum mechanical description.

Up until now state identification of unknown pure states has only been considered for the case $N = 2$. The problem has been introduced by Bergou and Hillery \(^2\) for the measurement strategy of optimum unambiguous identification and was first solved for two unknown qubit states, that is for $d = 2$ \(^2\). The identification of two unknown pure states has been shown to be equivalent to the discrimination of two known mixed quantum states \(^3\) \(^4\). The treatment has been extended to the case that the relevant states are each encoded into a certain number of identical copies of the respective quantum systems \(^3\) \(^4\) \(^5\) \(^6\), and the measurement strategy of state identification with minimum error has been derived, as well \(^3\) \(^4\). Moreover, it has been assumed that only one out of two possible qubit states is unknown and the other is known \(^4\). In addition, a number of schemes for implementing the unambiguous identification of two unknown qubit states have been proposed \(^8\) \(^9\) \(^10\). Encoding of the unknown states in quantum systems with $d$ Hilbert space dimension $d > 2$ has also been considered for $N = 2$, treating both minimum error \(^6\) \(^11\) and optimum unambiguous identification strategies of two states \(^3\) \(^12\).

In this paper we focus on the measurement strategy of optimum unambiguous identification. We extend the previous investigations to allow for an arbitrary number $N$ of linearly independent unknown pure states. Clearly, in order to encode these states we need quantum systems with dimensionality $d \geq N$. Here we shall consider the simplest case, assuming that $d = N$, and that the probe qudit is equally likely prepared in any of the $d$ reference states. In Sec. II we establish the connection between the unambiguous identification of $d$ unknown pure qudit states and the unambiguous discrimination of $d$ known mixed quantum states. Sec. III provides a general representation of the detection operators describing the measurement that unambiguously identifies the unknown pure states. Based on this result, in Sec. IV the optimum structure of the detection operators and thus the optimum measurement, realizing the maximum success probability, is obtained. Sec. V concludes the paper with a brief discussion of the results.

II. UNAMBIGUOUS IDENTIFICATION AS MIXED-STATE DISCRIMINATION

The total quantum system we are considering consists of one probe qudit, labeled by the index $0$, and $d$ reference qudits, labeled by the indices $1, \ldots, d$. Let the states $|i\rangle_n$ denote orthonormal basis vectors for the $n$-th qudit. Then the identity operator in the $d$-dimensional Hilbert space of the $n$-th qudit is given by

$$I_n = \sum_{i=0}^{d-1} |i\rangle_n \langle i|_n \quad (n = 0, 1, \ldots, d). \quad (1)$$

The projector $P_{0,n}$ onto the two-qudit subspace of dimension $d^2$ jointly spanned by the eigenstates of the probe
qudit and the n-th reference qudit can be written as

\[ P_{0,n} = I_0 \otimes I_n = P_{0,n}^{sym} + P_{0,n}^{as} \quad (n \neq 0), \quad (2) \]

where the projectors \( P_{0,n}^{sym} \) and \( P_{0,n}^{as} \) refer to the antisymmetric and symmetric part of the joint subspace, respectively. In the following we shall sometimes omit the symbol denoting the tensor product. From the explicit expressions for the antisymmetric and symmetric basis states in the two-qudit subspace we obtain the representations

\[ P_{0,n}^{as} = \sum_{j=1}^{d-1} \sum_{i=0}^{j-1} \frac{|i⟩⟨j|_n - |j⟩⟨i|_n}{\sqrt{2}} \quad (3) \]

and

\[ P_{0,n}^{sym} = \sum_{i=0}^{d-1} |i⟩⟨i|_n - \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \frac{|i⟩⟨j|_n + |j⟩⟨i|_n}{\sqrt{2}} \quad (4) \]

which show that the ranks of \( P_{0,n}^{as} \) and \( P_{0,n}^{sym} \) are \( d(d-1)/2 \) and \( (d+1)/2 \), respectively.

Let us now assume that the state of the probe qudit coincides with the state of the n-th reference qudit, so that the state of the total \((d+1)\)-qudit system can be written as

\[ |Ψ_n⟩ = |ψ_n⟩|ψ_1⟩ \ldots |ψ_n⟩ \ldots |ψ_d⟩. \quad (5) \]

On the right-hand-side the order of qudits from left to right is probe qudit (zeroth position) followed by the \( d \) reference qudits. If no information is available about the reference states other that they are linearly independent, the discrimination strategy must be independent of the actual reference states. In order to emphasize this state independence, we introduce the density operator \( |Ψ_n⟩⟨Ψ_n| \) and take its average over the unknown reference states. It will uniformly span the symmetric subspace of the probe and n-th reference qudit and uniformly span the Hilbert spaces of the remaining \( d-1 \) reference systems. From symmetry considerations it then follows that, for the case when the probe matches the n-th reference state, the total \((d+1)\)-qudit system is characterized by the average density operator

\[ \rho_n = \frac{2}{(d+1)d} P_{0,n}^{sym} \bigotimes_{i \neq n} I_i \quad (n = 1, \ldots, d), \quad (6) \]

in analogy to the treatment introduced for the identification of two unknown states \[3, 4\]. Here the pre-factor arises due to normalization, taking into account the ranks of \( P_{0,n}^{sym} \) and of the \( d-1 \) identity operators. Obviously, since the state of the probe can coincide with any one of the \( d \) reference states, there are \( d \) resulting average density operators.

In order to identify the state of the probe qudit, we have to discriminate among the \( d \) different density operators \( \rho_n \). A measurement suitable to accomplish this task can be formally described with the help of the \((d+1)\) positive detection operators \( \Pi_1, \ldots, \Pi_d \) and \( \Pi_{17} \) which together span the identity,

\[ \sum_{n=1}^{d} \Pi_n + \Pi_{17} = I \equiv I_0 \otimes I_1 \otimes \ldots \otimes I_d. \quad (7) \]

The detection operators have the property that \( \text{Tr}(\rho_n \Pi_n) \) is the probability of successfully identifying the density operator as \( \rho_n \), while \( \text{Tr}(\rho_m \Pi_n) \) \( (m \neq n) \) describes the probability to get an erroneous result and \( \text{Tr}(\rho_n \Pi_{17}) \) is the probability that the measurement result is inconclusive, i.e. that the attempt at discrimination fails to give a definite answer \[13, 14\]. For unambiguous discrimination we have to require that \( \text{Tr}(\rho_n \Pi_m) = 0 \) for \( m \neq n \), and from the positivity of the operators \( \rho_n \) and \( \Pi_m \) it follows that this requirement can be only met when \( \rho_n \Pi_m = 0 \) if \( m \neq n \)

\[ \rho_n \Pi_m = 0 \quad (m \neq n) \quad (8) \]

\[ [13, 14]. \] When the \( d \) mixed states occur with equal prior probability, given by 1/\( d \), the overall success probability of the discrimination measurement takes the form

\[ P_{\text{succ}} = \frac{1}{d} \sum_{n=1}^{d} \text{Tr}(\rho_n \Pi_n). \quad (9) \]

For determining the optimum measurement, we want to find the detection operators \( \Pi_n \) that maximize the success probability \( P_{\text{succ}} \) under the constraint that the eigenvalues of the operator \( \Pi_{17} \equiv I - \sum_{n=1}^{d} \Pi_n \) have to be non-negative.

### III. GENERAL STRUCTURE OF THE DETECTION OPERATORS

#### A. Three unknown qutrit states

In the following we proceed along the lines of our previous investigations on the unambiguous discrimination of two mixed quantum states \[12, 16, 17\] and start by deriving the general structure of the detection operators \( \Pi_n \). It is instructive to consider first only three density operators \( \rho_1, \rho_2 \) and \( \rho_3 \) given by Eq. \(3\) with \( d = 3 \) and referring to the unambiguous identification of three unknown pure qutrit states.

To be specific, let us focus on the detection operator \( \Pi_1 \) that unambiguously discriminates the state \( \rho_1 \) and therefore fulfills the requirement \( \Pi_1 \rho_2 = \Pi_1 \rho_3 = 0 \). Clearly, this requirement is only met when the support of \( \Pi_1 \) is orthogonal to the supports of \( \rho_2 \) and \( \rho_3 \). In other words, it belongs simultaneously to the kernel of \( \rho_2 \) and to the kernel of \( \rho_3 \). From Eqs. \(2\) and \(6\) we can write the projectors onto these two kernels as

\[ P_{K_2} = P_{0,2}^{as} \otimes I_1 \otimes I_3, \quad P_{K_3} = P_{0,3}^{as} \otimes I_1 \otimes I_2. \quad (10) \]
Since $I_1$ is common to both, the projector $P_{1}$ onto the support of $\Pi_1$ can be written as
\begin{equation}
P_{1} = I_1 \otimes P'_{1},
\end{equation}
where $P'_{1}$ projects onto the subspace spanned by all states $|\phi\rangle$ that are linear combinations of the eigenstates of the operator $P_{0,2}^{a} \otimes I_3$, on the one hand, and also linear combinations of the eigenstates of the operator $P_{0,3}^{a} \otimes I_2$, on the other hand. We denote the corresponding eigenstates by $|a\rangle$ for $P_{0,2}^{a} \otimes I_3$, and by $|b\rangle$ for $P_{0,3}^{a} \otimes I_2$.

\begin{align}
2^{-1/2}(|i_0j_2 - i_2j_0\rangle) |k\rangle_3 & \rightarrow |a\rangle, \\
2^{-1/2}(|i_0j_3 - i_3j_0\rangle) |k\rangle_2 & \rightarrow |b\rangle,
\end{align}
with $i < j$ and $i, j, k = 0, 1, 2$, where $l = 1, \ldots, 9$ labels the nine triples $\{i, j, k\}$. For determining $|\phi\rangle$ we put
\begin{equation}
|\phi\rangle = \sum_{i=1}^{9} a_i |a_i\rangle = \sum_{i=1}^{9} b_i |b_i\rangle,
\end{equation}
where $a_i$ and $b_i$ are some complex coefficients. Since $a_l = \sum_{i} b_i \langle a_i |b_i\rangle$ and $b_l = \sum_{i} a_i \langle a_i |b_i\rangle$, we obtain
\begin{equation}
b_l = \sum_{i' = 1}^{9} b_{i'} \left( \sum_{i} |a_{i'}\rangle \langle a_i |b_{i'}\rangle \right) = \sum_{i' = 1}^{9} b_{i'} B_{i'i},
\end{equation}
where $B_{11} = B_{22} = B_{33} = 1/2$, $B_{12} = B_{23} = B_{23} = B_{32} = B_{31} = 1/4$, and $B_{l' - 1} = 0$ if $l, l' \geq 4$. Here, Eqs. (12) and (13) have been used, with $l = 1, l = 2$, and $l = 3$ standing for $\{i, j, k\}$ equal to $\{1, 2, 0\}$, $\{2, 0, 1\}$, and $\{0, 1, 2\}$, respectively. The only non-trivial solutions of the system of equations given by Eq. (14) with $l = 1, \ldots, 9$ reads $b_1 = b_3 = -c$, $b_2 = c$ with $c$ being an arbitrary constant, while $b_l = 0$ if $l \geq 4$. Upon inserting these values into Eq. (14) and applying Eq. (13) we arrive at the normalized state
\begin{align}
|\phi_1(3)\rangle = (-1)^{1} \frac{1}{\sqrt{6}} (|0\rangle_0|1\rangle_1|2\rangle_2 - |0\rangle_0|2\rangle_1|1\rangle_3 \\
+ |2\rangle_0|0\rangle_2|1\rangle_3 - |2\rangle_0|1\rangle_2|0\rangle_3 \\
+ |1\rangle_0|2\rangle_1|0\rangle_3 - |1\rangle_0|0\rangle_1|2\rangle_3),
\end{align}
which is the single eigenstate of $P_{11}$. In $|\phi_n(d)\rangle$ the index $n = 1$ refers to the fact that it is the eigenstate of this projector and the argument $d = 3$ to the fact that we are dealing with the qutrit case. It should also be noted that $|\phi_n(d)\rangle$ is the completely antisymmetric state of all qudits with the one corresponding to the index omitted.

In a completely analogous way we can represent the projectors on the other two detection operators as $P_{2} = I_1 \otimes |\phi_3(3)\rangle \langle \phi_3(3)|$ with $n = 2, 3$, where
\begin{align}
|\phi_2(3)\rangle = (-1)^{2} \frac{1}{\sqrt{6}} (|0\rangle_0|1\rangle_1|2\rangle_2 - |0\rangle_0|2\rangle_1|1\rangle_3 \\
+ |2\rangle_0|0\rangle_2|1\rangle_3 - |2\rangle_0|1\rangle_2|0\rangle_3 \\
+ |1\rangle_0|2\rangle_1|0\rangle_3 - |1\rangle_0|0\rangle_1|2\rangle_3),
\end{align}
while we shall show that, generalizing Eq. (19),
\begin{align}
|\phi_3(3)\rangle = (-1)^{3} \frac{1}{\sqrt{6}} (|0\rangle_0|1\rangle_1|2\rangle_2 - |0\rangle_0|2\rangle_1|1\rangle_3 \\
+ |2\rangle_0|0\rangle_2|1\rangle_3 - |2\rangle_0|1\rangle_2|0\rangle_3 \\
+ |1\rangle_0|2\rangle_1|0\rangle_3 - |1\rangle_0|0\rangle_1|2\rangle_3).
\end{align}
Thus we obtain for $n = 1, 2, 3$.
\begin{equation}
P_{11} = I_1 \otimes |\phi_n(3)\rangle \langle \phi_n(3)| = \sum_{k=0}^{2} |\pi_n^{(k)}\rangle \langle \pi_n^{(k)}|,
\end{equation}
where we introduced the normalized states
\begin{equation}
|\pi_n^{(k)}\rangle = |k\rangle_n \otimes |\phi_n(3)\rangle \quad (k = 0, 1, 2).
\end{equation}
Since the states $|\pi_n^{(k)}\rangle$ are orthonormal basis states of the support of $\Pi_n$, it follows that $\Pi_n$ can be represented as
\begin{equation}
\Pi_n = \sum_{k, k' = 0}^{2} \alpha_n^{(k, k')} |\pi_n^{(k)}\rangle \langle \pi_n^{(k')}| \quad (n = 1, 2, 3),
\end{equation}
where the coefficients $\alpha_n^{(k, k')}$ are some complex constants that have to guarantee the positivity of the detection operators. Eq. (20) yields $\langle \pi_n^{(k)} | \pi_n^{(k')} \rangle = \delta_{k, k'} \langle \pi_n^{(k)} | \pi_n^{(k)} \rangle$ and
\begin{equation}
\langle \pi_1^{(k)} | \pi_2^{(k)} \rangle = \langle \pi_2^{(k)} | \pi_3^{(k)} \rangle = \langle \pi_1^{(k)} | \pi_3^{(k)} \rangle = -\frac{1}{3}
\end{equation}
for $k = 0, 1, 2$, where Eqs. (16) - (18) have been used.

A note is in place here about the choice of the overall signs of the states $|\phi_n(3)\rangle$. Although they do not affect the projectors $P_{1n}$, they lead to sign changes in the overlaps of the $|\pi_n^{(k)}\rangle$ states. For any choice of the overall signs either one or all three of the overlaps are negative. For the purpose of this paper we made the choice given by Eq. (20) together with Eqs. (16) - (18). With this all three overlaps are negative in Eq. (22), reflecting the intrinsic symmetry of the identification problem.

B. d unknown qudit states

Now we return to the general task of identifying $d$ unknown qudit states or, equivalently, of discriminating the $d$ density operators given by Eq. (6). By the same reasoning that, for $d = 3$, led to Eq. (11), the projector onto the support of the detection operator $\Pi_n$ is of the form
\begin{equation}
P_{1n} = I_1 \otimes P'_{1n},
\end{equation}
Here $P'_{1n}$ projects onto the subspace spanned by all states that can be simultaneously written as linear combinations of the eigenstates of any one of the operators $P_{0,m}^{a} \otimes I_1$, where $i \neq m, n$ and $m \neq n$. In the following we shall show that, generalizing Eq. (19),
\begin{equation}
P_{1n} = I_1 \otimes |\phi_n(d)\rangle \langle \phi_n(d)| = \sum_{k=0}^{d-1} |\pi_n^{(k)}\rangle \langle \pi_n^{(k)}|
\end{equation}
with
\[ |\pi_n^{(k)}⟩ = |k⟩_n ⊗ |φ_n(d)⟩ \quad (k = 0, 1, \ldots, d - 1), \]
where in analogy to Eqs. (16) - (18) in Eq. (26) the sum is taken over all \( \sigma \) distributing the excitation numbers \( \sigma_j = 0, 1, \ldots, d - 1 \) over the system of \( d \) qudits that we obtain by omitting the \( n \)-th reference qudit from the total system of \( d \) qudits. The latter are written in fixed order, and \( \text{sgn}(\sigma) \) is the sign of the permutation.

Since \( \langle \pi_m^{(k)} | \pi_n^{(k)} ⟩ = δ_{k,k'} \), Eq. (24) is equivalent to the fact that the detection operators can be written as
\[ \Pi_n = \sum_{k,k'}^{d-1} a_n^{(k,k')} |\pi_n^{(k)}⟩⟨\pi_n^{(k')}| \quad (n = 1, \ldots, d), \]
where the coefficients \( a_n^{(k,k')} \) are complex constants subject to the constraints \( \Pi_n \geq 0 \) and \( \Pi^2 = I - \sum_n \Pi_n \geq 0 \).

In order to prove Eqs. (24) and (27), respectively, we first use Eq. (10) to calculate \( ρ_m |\pi_n^{(k)}⟩ \), for \( m \neq n \), yielding
\[ ρ_m |\pi_n^{(k)}⟩ \propto |k⟩_n \sum_{σ}^{d} \text{sgn}(σ) \bigotimes_{j,m,n} |σ_j⟩ |σ⟩_{j,m,n} = 0. \]

Here the equality sign holds because the contributions of any two terms \( P^\text{sym}_0 |i⟩_0 |j⟩_m \) and \( P^\text{sym}_0 |j⟩_m |i⟩_0 \) (\( i \neq j \)) canceled due to the opposite sign of the respective permutations. Hence the requirement for unambiguous discrimination, Eq. (3), is met and Eq. (24) is sufficient for the operator \( P^\Pi_0 \) to be the projector onto the support of the detection operator \( \Pi_n \). It remains to be shown that Eq. (24) is also necessary. Because of Eq. (28) this holds true if Eq. (20) is the only eigenstate of \( P^\Pi_0 \), that is if \( P^\Pi_0 \) is an operator of rank one. From Eq. (10) it follows that for any \( m \) with \( m \neq n \) we get the representation \( ρ_m = I_n ⊗ ρ_m' \), where all operators \( ρ_m' \) together span a certain Hilbert space \( \mathcal{H}_n \). Because of Eq. (25) the eigenstates of \( P^\Pi_0 \) also lie in \( \mathcal{H}_n \), and the requirement \( P^\Pi_0 ρ_m = 0 \) is equivalent to \( P^\Pi_0 ρ_m' = 0 \). This holds for any \( m \). If \( P^\Pi_0 \) is an operator of rank \( r \), this implies that to each of the basis states of the Hilbert space \( \mathcal{H}_n \) jointly spanned by the operators \( ρ_m' \) there belong \( r \) states that are orthogonal to it and lie also in \( \mathcal{H}_n \). Clearly this is a contradiction if \( r \geq 2 \). Thus Eq. (24) with \( r = 1 \) is not only sufficient, but also necessary for \( P^\Pi_0 \) to be the projector onto the support of \( \Pi_n \).

Before proceeding, we derive two important properties of the \((d+1)\)-qudit states \(|\pi_n^{(k)}⟩\). First, Eqs. (25) and (26) show that for arbitrary \( n \) the total number of excitations in the \((d+1)\)-qudit system described by \(|\pi_n^{(k)}⟩\) is equal to \( k + \sum_{i=1}^{d-1} i \). Hence for any two states with different \( k \) the excitation numbers of at least one of the qudits composing the total system are different and therefore these states are orthogonal, which leads to
\[ \langle \pi_m^{(k)} | \pi_n^{(k)} ⟩ = δ_{k,k'} \langle \pi_m^{(k)} | \pi_n^{(k)} ⟩. \]

Second, while \( \langle \pi_m^{(k)} | \pi_n^{(k)} ⟩ = 1 \), for \( m \neq n \) we find that
\[ \langle \pi_m^{(k)} | \pi_n^{(k)} ⟩ = -\frac{1}{d} \] since the expression
\[ \frac{(-1)^{(n+m)}}{d!} \sum_{σ,σ'} \text{sgn}(σ) \text{sgn}(σ') \prod_{j,m,n} (k|σ_m⟩_m |σ_j⟩_j |σ'_n⟩_n |k⟩_n) \]
is equal to \(-1/d\). Here we took into account that the inner products vanish unless \( σ_j = σ_j' \) for \( j \neq m, n \) and \( σ'_m = k = σ_m \), reducing the double sum to a single one over \((d-1)!\) permutations which each contribute the value \( 1 \). Eq. (30) reflects the intrinsic symmetry of our identification problem.

**IV. THE OPTIMUM MEASUREMENT**

Now we apply the general representation of the detection operators \( P^\Pi_0 \), Eq. (24), in order to find the special coefficients \( a_n^{(k,k')} \) that determine the optimum operators \( P^\Pi_0 \) which maximize the overall success probability \( P^\text{succ} \), given by Eq. (9). Using Eqs. (6) and (27) a straightforward calculation shows that
\[ \text{Tr}(ρ_n P^\Pi_0) = \sum_{k,k'}^{d-1} a_n^{(k,k')} \langle \pi_n^{(k')} | ρ_n |\pi_n^{(k)}⟩ \]
\[ = \frac{2}{(d+1)d^2} \sum_{k,k'}^{d-1} a_n^{(k,k')} (d-1)! R^{(d)}_{kk'}, \]
with
\[ R^{(d)}_{kk'} = \sum_{i=0}^{d-1} \langle i|_n P^\text{sym}_0 |k⟩_n |i⟩_0 = \frac{d+1}{2} δ_{k,k'}, \]
where Eq. (31) has been applied in Eq. (32). Thus, we arrive at
\[ P^\text{succ} = \frac{1}{d} \sum_{n=1}^d \text{Tr}(ρ_n P^\Pi_0) = \frac{1}{d^2+2} \sum_{k=0}^{d-1} \left( \sum_{n=1}^d a_n^{(k)} \right), \]
where \( a_n^{(k)} \equiv a_n^{(k,k)} \). In order to maximize \( P^\text{succ} \) we need to determine the largest values \( a_n^{(k)} \) which are still in accordance with the constraint \( P^\Pi_0 = I - \sum_n P^\Pi_n = 0 \), or \( \sum_n P^\Pi_n ≤ I \), respectively, where the operators \( P^\Pi_n \) are
given by Eq. (27). Since P_{suc} does not depend on the non-diagonal elements α_n^{(k,k')} (k \neq k'), for maximizing P_{suc} on the given constraint we have to put
\[ α_n^{(k,k')} = α_n^{(k)} \delta_{k,k'} . \] (34)
The constraint then reduces to
\[ \sum_{n=1}^{d} \Pi_n = \sum_{k=0}^{d-1} \left( \sum_{n=1}^{d} α_n^{(k)} \langle π_n^{(k)} | π_n^{(k)} \rangle \right) ≤ I , \] (35)
where from Eq. (29) it follows that for different values of k the operators within the bracket in Eq. (35) act in orthogonal subspaces. Maximizing the success probability given by Eq. (33) therefore amounts to solving d independent maximization problems in the d orthogonal subspaces belonging to different values of k. Moreover, since according to Eq. (30) the mutual overlaps \( \langle π_n^{(k)} | π_n^{(k)} \rangle \) do not depend on k, all these optimization problems are mathematically identical and the index k can be omitted. Therefore our task is to maximize \( \sum_{n=1}^{d} α_n \equiv S \) with
\[ \sum_{n=1}^{d} α_n | π_n⟩⟨ π_n | \equiv \Pi S ≤ I , \] (36)
\[ ⟨ π_n | π_m ⟩ = 1 , \quad ⟨ π_m | π_n ⟩ = - \frac{1}{d} \quad (m \neq n) . \] (37)
For this purpose we make use of the method developed by Chefles and Barnett [18] for the optimum unambiguous discrimination of linearly independent symmetric pure states. We first observe that d state vectors, with mutual overlaps given by Eq. (38), can be represented as
\[ | π_n ⟩ = \frac{1}{d} | u_0 ⟩ + \sqrt{\frac{d+1}{d}} \sum_{l=1}^{d-1} \exp \left( 2πi \frac{n-l}{d} \right) | u_l ⟩ , \] (38)
where the states \( \{| u_l ⟩\} \) with \( ⟨ u_l | u_{l'} ⟩ = δ_{ll'} \) denote some orthonormal basis in the d-dimensional subspace spanned by the set of states \( \{| π_n ⟩\} \). Indeed, from Eq. (38) we get
\[ ⟨ π_m | π_n ⟩ = - \frac{1}{d} + \frac{d+1}{d^2} \sum_{l=0}^{d-1} \left[ \exp \left( 2πi \frac{n-m}{d} \right) \right]^{l} , \] (39)
and Eq. (38) is immediately recovered using the sum rule for a geometric series. With the help of Eq. (38) it is easy to check that the states \( \{| π_n ⟩\} \) belong to the class of symmetric states that are covariant with respect to the unitary operator \( U = \sum_{l=0}^{d-1} \exp \left( 2πi \frac{l}{d} \right) | u_l ⟩⟨ u_l | \) and transform according to \( U | π_d ⟩ = | π_1 ⟩ \) and \( U | π_{d+1} ⟩ = | π_1 ⟩ \) for \( n = 1, \ldots , d - 1 \). It has been shown [18] that then there exists an optimum operator \( Π_n^{opt} \) maximizing \( S \) and possessing the same symmetry with respect to \( U \), leading to \( α_1 = \ldots = α_d \equiv α \). The constraint given by Eq. (36) thus takes the form
\[ α \sum_{n=1}^{d} | π_n ⟩⟨ π_n | ≤ I = \sum_{l=0}^{d-1} | u_l ⟩⟨ u_l | . \] (40)
On the other hand, from Eq. (38) we obtain
\[ \sum_{n=1}^{d} | π_n ⟩⟨ π_n | = | u_0 ⟩⟨ u_0 | + \frac{d+1}{d} \sum_{l=1}^{d-1} | u_l ⟩⟨ u_l | , \] (41)
where the relation \( \sum_{n=1}^{d} \exp \left( 2πi(l-l')n/d \right) = d δ_{ll'} \) has been taken into account. Inserting Eq. (41) into Eq. (40) we find immediately that the largest value of \( α \), allowed by the constraint, is given by \( α^{opt} = d/(d+1) \). With \( α_n^{(k)} = α^{opt} \), Eqs. (33), (34) and (27) yield both the maximum success probability for unambiguously identifying \( d \) unknown pure qudit states,
\[ P^{opt}_{suc} = \frac{1}{(d+1)^{d-1}} , \] (42)
and the optimum detection operators, with \( | π_n^{(k)} ⟩ \) given by Eq. (35),
\[ Π_n^{opt} = \frac{d}{d+1} \sum_{k=0}^{d-1} | π_n^{(k)} ⟩⟨ π_n^{(k)} | \quad (n = 1, \ldots , d) . \] (43)

Equations (12) and (13) represent the main results of this paper, generalizing previous results from two qubits to \( d \) qudits. For \( d = 2 \) Eq. (42) reproduces the success probability 1/6 obtained previously [2, 4] for the optimum unambiguous identification of two unknown pure qudit states. The optimum detection operators \( Π_n^{opt} \) are not projectors and, therefore, the optimum measurement strategy is a generalized measurement.

V. CONCLUSIONS

In this paper we derived the optimum measurement for the unambiguous identification of \( d \) unknown pure qudit states. Our treatment is based on Eqs. (29) and (30), resulting from the special structure of the density operators to be discriminated, and reduces the optimization problem to \( d \) independent maximization problems in orthogonal subspaces of dimension \( d \). The procedure is analogous to determining the optimum measurement for the discrimination of two mixed quantum states in cases where the optimization reduces to maximization problems in orthogonal two-dimensional subspaces [15, 16, 17]. It should be noted in this context that the correct detection operators have previously been derived by Zhang et al., using a slightly different method and without explicitly working out the optimal success and failure probabilities [23].

Quantum systems with \( d > 2 \) have been proposed as carriers of quantum information in various contexts like e. g. quantum cryptography [19] and methods for realizing general linear transformations on single-photon qudits have been theoretically described [20]. Moreover, the manipulation of biphotonic qutrits [21] and ququarts [22] has been experimentally demonstrated.
possible applications in quantum information, our investigations are also of interest with respect to the theory of optimum unambiguous discrimination of more than two mixed quantum states\textsuperscript{23, 24}, where no examples of explicit solutions have been given before.

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