Imaginary spin-orbital coupling in parity-time symmetric systems with momentum-dependent gain and loss

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Abstract
Spin-orbital coupling (SOC) and parity-time ($PT$) symmetry both have attracted paramount research interest in condensed matter physics, cold atom physics, optics and acoustics to develop spintronics, quantum computation, precise sensors and novel functionalities. Natural SOC is an intrinsic relativistic effect. However, there is an increasing interest in synthesized SOC nowadays. Here, we show that in a $PT$-symmetric pseudo spin-1/2 system (in other words, two-state system), the momentum-dependent balanced gain and loss can synthesize a new type of SOC, which we call imaginary SOC. The imaginary SOC can substantially change the energy spectrum of the system. Firstly, we show that it can generate a pure real energy spectrum with a double-valleys structure. Therefore, it has the ability to generate supersolid stripe states. Especially, the imaginary SOC stripe state can have a high contrast of one. Moreover, the imaginary SOC can also generate a spectrum with tunable complex energy band, in which the waves are either amplifying or decaying. Thus, the imaginary SOC would also find applications in the engineering of $PT$-symmetry-based coherent wave amplifiers/absorbers. Potential experimental realizations of imaginary SOC are proposed in cold atomic gases and systems of coupled waveguides constituted of nonlocal gain and loss.

1. Introduction
Spin–orbit coupling (SOC) arises from the relativistic-induced coupling between a particle’s spin degree of freedom and its momentum. It plays a significant role in various physical systems. For relativistic elementary particles, SOC leads to their Zitterbewegung oscillation [1]. For atoms, SOC gives rise to the fine structure spectra [2]. And in condensed matter physics, the investigation of SOC has led to fruitful achievements (such as the spin-Hall effect [3], topological insulator [4], just name a few) with potential applications in spintronics [5] and quantum computations [6]. Currently, SOC researches are drastically expanding to the fields of cold atom physics [7–22], optics [23–29] and acoustics [30–34]. Cold atom systems have no natural SOC, whereas using the Raman coupling scheme [11] (and also some others, see the review article [12]) artificial SOC has been experimentally realized, furthermore supersolid stripe states [13–20] and momentum-space Josephson oscillations [21, 22] can be generated. In photonics, the subwavelength scales and additional degrees of freedom of structured optical field are explored, and in such fields, spin and orbital properties are strongly coupled with each other [23]. In a birefringent optical fiber, traveling of the light pulses with mutually orthogonal linear polarizations is described by a set of nonlinear Schrödinger equations with a SOC Hamiltonian [24, 25]. In dual-core waveguides, SOC can be synthesized by dispersively coupling the light field in different cores [26–28], and stripe solitons can be produced [29]. In acoustical systems, SOC has also been successfully synthesized [30–34].

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Non-Hermitian parity-time ($\mathcal{PT}$) symmetry physics is another active researching field these days [35–38]. The seminal work by Bender and Boettcher in 1998 states that a non-Hermitian Hamiltonian with $\mathcal{PT}$-symmetry can support complete real value eigenenergies [39], and this has boosted the investigation of non-Hermitian quantum mechanics [40–45]. Considering the analogy of classical wave equation to the quantum mechanics Schrödinger equation, $\mathcal{PT}$-symmetric Hamiltonian can be used to model the balanced gain and loss, and has been realized in systems of optics [46, 47], acoustics [48–51], and many others [52–58]. Fruitful applications such as invisible acoustic sensors [59], microcavity sensors [60, 61], unidirectional transportation [62–64], light-light switch [65], and laser amplifier/absorber [66–71] were born out.

Cold atomic gas systems are also promising for non-Hermitian physics studying, a good many phenomena such as soliton [72–74], vortex [75, 76], Bose–Hubbard model [77], double-well problem [78, 79], many-body physics [80], skin effect [81–83] have been explored, and recently experimentally non-Hermitian Hamiltonians have also been realized [84–86]. In these systems, the gain is usually realized by injecting atoms into the condensate using an atom laser [87], while the loss can be realized by exciting atoms firstly to an excited state with a laser beam and then ejecting them out from the condensate via photon recoil [84]. Due to the momentum distribution of the atom laser [88, 89], and the Doppler effect in atom-light interaction [90], gain and loss realized in these ways have a distinct momentum-dependence. Besides, momentum-dependence of gain and loss also occurs in optical mediums, where spatially nonlocal gain and loss after a Fourier transformation are wavevector-dependent [91, 92] (since in quantum mechanics, momentum equals Planck constant multiplying wavevector, that is equivalently momentum-dependent), and this feature has been proposed to explore the topological physics in photonic system [93].

Here, we show that in a $\mathcal{PT}$-symmetric pseudo spin-1/2 system (in other words, a two-state system), momentum-dependent gain and loss can synthesize an imaginary interaction between the spin degree of freedom and the orbital motion, thus an imaginary SOC. Next, we present the imaginary SOC Hamiltonian, and study its properties at first, leaving the discussion on potential experimental realizations in cold atom physics and optical systems to the end of the paper.

2. Model

We study a pseudo spin-1/2 system (in other words, a two-state system) with Hamiltonian in momentum representation given by

$$H = \frac{\vec{p}^2}{2m} \sigma_0 + i\hbar \gamma \Theta (\vec{p}) \sigma_z + \hbar \Omega \sigma_x,$$

(1)

where $\sigma_0, \sigma_{x,z}$ are the conventional $2 \times 2$ unitary and Pauli matrices, $m$ is the particle mass, $\vec{p}$ is the momentum operator, $\hbar$ is the Planck constant, $\gamma$ represents the rate of the balanced gain and loss, $\Theta(\vec{p})$ is a positive dimensionless function with its maximum normalized to 1 [i.e., $0 \leq \Theta(\vec{p}) \leq 1$], and it is used to describe the profile of momentum-dependence of the balanced gain and loss, at last $\Omega$ is the Rabi coupling strength between the two pseudo spin states.

If $\Theta (\vec{p})$ is an imaginary function, the second term in Hamiltonian (1) represents a real interaction between the spin and a momentum-dependent magnetic field along the $z$-direction. Thus, it is a conventional Hermitian SOC term. For example, when $\Theta (\vec{p}) = ip_xp_y/(m\hbar \gamma)$, the Hamiltonian (1) has the same form as that in the SOC cold atom system realized with the Raman coupling scheme [11] (see the appendix). However, in the present paper, $\Theta (\vec{p})$ is a profile function describing the momentum-dependence of the gain and loss, that is to say, it is a real function, hence the second term of the Hamiltonian is imaginary, thus termed as ‘imaginary SOC’. This imaginary SOC Hamiltonian is $\mathcal{PT}$-symmetric, with the parity operator $P$ exchanging the two spin components, and the operator $\mathcal{T}$ performing the complex conjugation [35]. We also note that there already exist some researches combining SOC and $\mathcal{PT}$ symmetry [81–83, 86, 94–96], however in these works it is a conventional Hermitian SOC term and a $\mathcal{PT}$-symmetric term been stiffly glued. In contrast, here the SOC term itself is non-Hermitian.

To analytically understand the properties of imaginary SOC, we solve the eigenvalue problem of Hamiltonian (1). The eigenenergies and corresponding eigenvectors are

$$E_{\pm} = \frac{|\vec{p}|^2}{2m} \pm \hbar \sqrt{\Omega^2 - \gamma^2 \Theta^2 (\vec{p})},$$

(2)
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Figure 1. Energy spectra $E_\pm (p_x)$ for imaginary SOC with momentum-dependence profile $\Theta (p_x)$ given by equation (4). The real (Re) parts of $E_\pm$ and $E_\pm$ are plotted as the solid violet and green lines, while their imaginary (Im) parts are plotted as the cyan dashed and brown dotted lines. Panels (a) and (b) Pure real spectra for $\Omega = 1.1 \gamma > \gamma$. Under these parameters, the critical gain/loss momentum-space width separates the single- and double-valleys spectrum is $\sigma = 2.09 p_x$ with $p_x = \sqrt{m/\gamma}$. Panel (a) Single-valley energy spectrum for $\sigma = 2.5 p_x > \sigma$. Panel (b) Double-valleys energy spectrum for $\sigma = 0.5 p_x < \sigma$. Panel (c) Pure real gapless energy spectrum for $\Omega = \gamma$ and $\sigma = 0.5 p_x$. Panel (d) Complex energy spectrum for $\Omega = 0.9 \gamma < \gamma$ and $\sigma = 0.5 p_x$. In the gray color filled band, the spectrum has complex eigenenergy.

and

$$\psi_{\pm} = \frac{1}{A_{\pm}} \left[ \left( i \gamma \Theta (\vec{p}) \pm \sqrt{\Omega^2 - \gamma^2 \Theta^2 (\vec{p})} \right) / \Omega, 1 \right],$$

with $A_{\pm}$ being normalization constants. When $\Omega \geq \gamma$, the eigenenergies $E_{\pm}$ always take real values, and the eigenvectors can be simplified to $\psi_{\pm} = \left[ \pm \exp (\pm i \theta), 1 \right] / \sqrt{2}$ with $\theta = \arcsin \left( \gamma \Theta (\vec{p}) / \Omega \right)$, and $^T$ denoting the transpose operation. It is seen that these eigenvectors are $\mathcal{PT}$-symmetric with equal spin amplitudes $|\psi_{\pm,1}| = |\psi_{\pm,1}|$. Considering the dynamical factor $\exp [-i E_{\pm} t / \hbar]$, the states will evolve with their norms conserved due to the exact balance between loss and gain. When $\Omega < \gamma$, the states with momentum fulfilling $\gamma^2 \Theta^2 (\vec{p}) \leq \Omega^2$ are also $\mathcal{PT}$-symmetric with pure real eigenenergies. However, the states with momentum fulfilling $\gamma^2 \Theta^2 (\vec{p}) > \Omega^2$ have complex eigenenergies $E_{\pm} = E_R \pm i E_I$ with

$E_R = |\vec{p}|^2 / (2m)$ and $E_I = h \sqrt{\gamma^2 \Theta^2 (\vec{p}) - \Omega^2}$. Now, the dynamical factor becomes $\exp [-i E_{\pm} t / \hbar] = \exp [-i E_R t / \hbar] \cdot \exp [\pm E_I t / \hbar]$, for $\exp [-E_I t / \hbar]$ the states will automatically decay, while for $\exp [+ E_I t / \hbar]$ the states will automatically be amplified during the evolution. The decay and amplify can be understood by examining the corresponding eigenvectors. In this case, the eigenvectors are simplified to $\psi_{\pm} = \left[ \exp (\pm \theta), \right] / \sqrt{A_{\pm}}$ with $\theta = \arccosh \left( \gamma \Theta (\vec{p}) / \Omega \right)$ and $A_{\pm} = \sqrt{1 \pm \exp (\pm 2 \theta)}$, which have unequal spin amplitudes $|\psi_{\pm,1}| \neq |\psi_{\pm,1}|$. Thus, the $\mathcal{PT}$-symmetry is broken, the gain and loss no longer can cancel each other.

Next, we examine the effect of imaginary SOC on the shape of energy spectrum curve. For facilitation, we particularly consider the one-dimensional case ($\vec{p} = p_x \hat{x}$), and assume that the momentum-dependence profile function $\Theta (p_x)$ takes the Lorentzian shape

$$\Theta (p_x) = \frac{\sigma^2}{p_x^2 + \sigma^2},$$

where $\sigma$ is the momentum-space width of the balanced gain and loss.

Firstly, we found that the imaginary SOC can generate spectra with both single-valley and double-valleys structures when $\Omega \geq \gamma$, as shown in panels (a) and (b) of figure 1. The minima and maxima of the spectrum can be found by letting the derivatives of $E_{\pm} (p_x)$ equal to zero, $dE_{\pm} / dp_x = 0$. The upper branch $E_+$ of the spectrum always has only one minimum at $p_x = 0$. However, the lower branch $E_-$ can have either one or two minima, depending on the value of $\sigma$. When $\sigma$ is larger than the critical value...
\[
\sigma_x = \sqrt{2m\hbar\gamma/(\Omega^2 - \gamma^2)}^{1/4}, \quad \text{the lower branch also has only one minimum at } p_x = 0. \quad \text{While for } \sigma < \sigma_x, \quad \text{the point } p_x = 0 \quad \text{becomes a maximum, and the lower branch spectrum exhibits a double-valleys structure with two minima located at } p_x = \pm p_x^0, \quad \text{which are the solutions of equation } dE/\ dp_x = 0.
\]

Usually, there is an energy gap between the upper and lower spectrum branches, \(\Delta E = 2\hbar \sqrt{\Omega^2 - \gamma^2}\). However, due to the energy band attraction effect in non-Hermitian systems [97, 98], as the strength of the balanced gain and loss increases, the gap becomes narrow, and when \(\gamma = \Omega\), it disappears, see panel (c) of figure 1. Interestingly, even the spectrum becomes gapless, the two spin components are still coupled with each other [see eigenstates (3)], in sharp contrast with the conventional Hermitian SOC case where the gapless system is trivial due to the decoupling of the two spin components, see the appendix or references [11, 15, 99–102].

We also found that the imaginary SOC can generate spectrum with a complex energy band when \(\Omega < \gamma\), as shown by the gray color filled rectangle in panel (d) of figure 1. In this case, there exist a critical momentum \(p_c = \sigma \sqrt{\gamma/\Omega - 1}\) [the solution of \(\Omega^2 - \gamma^2 \Theta^2 (p_c) = 0\)], the state with momentum \(|p_c| \geq p_c\) still has pure real eigenenergy. However, the eigenenergy corresponding to state with momentum \(|p_c| < p_c\) becomes complex, thus forming the complex energy band in the figure. And according to formula \(p_c = \sigma \sqrt{\gamma/\Omega - 1}\), this complex energy band can be conveniently tuned by both the gain/loss or Rabi coupling strength.

The above discussion indicates that imaginary SOC would find applications in the fields of energy spectrum engineering and coherent wave amplifying/absorbing. Similar to conventional Hermitian SOC, imaginary SOC also can generate spectra with single-valley or double valleys structures. Additionally, imaginary SOC also generates non-trivial gapless spectrum and spectrum with complex energy band, which are absent in the case of conventional Hermitian SOC. It has been proposed and experimentally demonstrated that the complex energy mode of a \(\mathcal{PT}\)-symmetric optical system can be applied to realize laser amplifiers and absorbers [66–71]. Here we show that imaginary SOC can control the complex energy spectrum, thus provides it the ability to manage \(\mathcal{PT}\)-symmetry-based laser amplifiers/absorbers.

### 3. Imaginary SOC in harmonic trap

For the single-valley spectrum, it is easy to imagine that the ground state of a imaginary SOC system will fall into the bottom of this valley. However, for the double-valleys spectrum, the ground state may fall in either the \(p_0\) valley or the opposite \(-p_0\) valley. Thus, when an external trapping potential is included, the superposition of these two momentum states can be induced to produce a supersolid stripe state. In this vein of thought, now we consider the case that the imaginary SOC particles are trapped in a harmonic potential \(V(x) = \frac{1}{2}m\omega^2 x^2\), or equivalently \(V(p_x) = -\frac{1}{2}m\hbar^2 / \omega^2 \left|p_x\right|^2\) in the momentum representation. We first numerically diagonalize the full Hamiltonian in the momentum representation to obtain the eigenwavefunctions \(\psi_n(p_x) = \left[\psi_{\uparrow,n}(p_x), \psi_{\downarrow,n}(p_x)\right]^{\dagger}\) with \(n = 1, 2, \ldots\), and further calculate the wavefunction in position representation by Fourier transform, \(\phi_n(x) = \int \psi_n(p_x) e^{i p_x x / \hbar} d p_x / \sqrt{2\pi}\).

In figure 2, for SOC parameters chosen corresponding to the free spectra previously shown in panels (a)–(c) of figure 1, we show the probability densities of the minimal eigenenergy states (i.e., ground states) in both momentum and position space for a weak harmonic trap with frequency \(\omega = 0.05\gamma\). Panels (a1) and (a2) correspond to the single-valley spectrum. We see that in momentum space the probability density is centered at \(p_x = 0\), which is just the center of the spectrum valley; and in position space, the density is centered at the bottom of the harmonic trap (since it is similar to the ground state of a normal harmonic oscillator, we call it ‘normal’ state). When the free spectrum has a double-valleys structure, we found the probability density of the bound state exhibits two peaks centered around the bottom of the two spectrum valleys; and in position space, the expected supersolid stripe structure is observed (‘stripe’ state), see panels (b1) and (b2); (c1) and (c2). Especially, the free spectrum corresponding to panels (c1) and (c2) is gapless, but the stripe state is also observed. This further demonstrates the non-trivialness of the gapless free spectrum which has been discussed in the previous section.

For the stripe states in figure 2, we see that the density minima can drop to zero, i.e., they have a high contrast of one. By comparison, the supersolid stripe realized by the conventional Hermitian SOC usually has poor contrast, leading to difficulty in experimental observation [103–105]. This can be understood as follows: in the case of conventional Hermitian SOC, matterwaves with momentum \(p_x = +p_0\) and \(-p_0\) are oppositely spin polarized—one state is spin-\(\downarrow\) favored, while the other one is spin-\(\uparrow\) favored (see the appendix). In quantum mechanics, only the identical particles can spatially interfere with each other. Therefore, only a part of the atoms have a contribution to the interference stripe pattern formation, others...
Figure 2. $\mathcal{PT}$-symmetric normal and supersolid stripe eigenstates generated by imaginary SOC in a harmonic trap. The left panels (a1)–(c1) show the probability density in momentum space, while the right panels (a2)–(c2) show the probability density in position space. The spin-$\uparrow$ and spin-$\downarrow$ components are plotted using violet solid and green dashed line respectively (since the states are $\mathcal{PT}$-symmetric with $|\psi\rangle = |\psi\rangle$, the two lines overlap with each other). The harmonic trap frequency is $\omega = 0.05\gamma$ for all the panels. Panels (a1) and (a2) Normal state under parameters $\Omega = 1.1\gamma$, $\sigma = 0.5p_1$. Panels (b1) and (b2) Supersolid stripe state under parameters $\Omega = 0.9\gamma$, $\sigma = 0.5p_1$. Panels (c1) and (c2) Supersolid stripe state under parameters $\Omega = \gamma$, $\sigma = 0.5p_1$. The pure real eigenenergies of these states are $E = -0.44\hbar\gamma$ (a), $-0.81\hbar\gamma$ (b), $-0.70\hbar\gamma$ (c). The corresponding free energy spectra are shown in panels (a)–(c) of figure 1.

Figure 3. $\mathcal{PT}$-symmetry broken eigenstates generated by imaginary SOC in a harmonic trap. The probability in momentum space for spin-$\uparrow$ and spin-$\downarrow$ components are plotted as the violet solid and green dashed lines respectively. Panel (a) Decaying eigenstate with complex eigenenergy $E = (0.08 - 0.36i)\hbar\gamma$. Panel (b) Amplifying eigenstate with complex energy $E = (0.08 + 0.36i)\hbar\gamma$. The parameters used are $\omega = 0.05\gamma$, $\Omega = 0.9\gamma$ and $\sigma = 0.5p_1$ ($p_1 = \sqrt{m\hbar\gamma}$). The corresponding free energy spectrum has a complex energy band around $p_x = 0$, as shown in panel (d) of figure 1.

only contribute a smooth background. As a result, the contrast of the stripe pattern is poor in this case. However, for the imaginary SOC, as having been shown in section 2, when $\Omega > \gamma$, the spin state is a balanced one, $\psi_{\pm} = \left[ \pm \exp(\pm i\theta), 1 / \sqrt{2} \right]^{T}$ with $\theta = \arcsin \left( \gamma \Theta \left( p_x \right) / \Omega \right)$ and $\Theta \left( p_x \right) = \sigma^2 / (p_x^2 + \sigma^2)$. This means that matterwaves with momentum $p_x = +p_0$ and $-p_0$ have equal spin amplitudes [this is also confirmed by panels (b1) and (c1) in figure 2]. And, the equal amplitudes interference of two waves can produce a high contrast stripe pattern. This explanation also indicates that high contrast of the stripe state is a universal phenomenon for imaginary SOC, has little to do with the specific choice of parameters [in fact, although panels (b2) and (c2) correspond to different parameters, they both show a high contrast stripe].

Besides, corresponding to the complex energy band free spectrum [panel (d) of figure 1], imaginary SOC can also generate $\mathcal{PT}$-symmetry broken bounded states in the harmonic trap. The $\mathcal{PT}$-symmetry
broken states appear in pairs with opposite spin polarization and opposite imaginary parts of eigenenergies, thus one of them is a decaying state, while the other is an amplifying one. In figure 3, the maximal decaying/amplifying states are shown.

4. Experimental realizations

Firstly, we propose that imaginary SOC can be realized in cold atom systems. We consider a system of one-dimensional pseudo spin-1/2 cold atomic gas with the two spin components coupled by a laser field with Rabi frequency $\Omega$. The momentum-dependent gain in the spin-$\uparrow$ component can be realized by injecting spin-$\uparrow$ atoms into the system using an atom laser with appropriate momentum distribution [87–89]. Especially, reference [89] proposed a procedure to produce cold-atom beams with the Lorentzian profile of momentum distribution. And according to reference [90], exploiting the Doppler shift technique, atoms in the spin-$\downarrow$ state can be velocity-selectively excited to narrow Rydberg or metastable states with a laser, and the following photon recoil could produce a momentum-dependent atomic loss. Denoting the momentum-dependent gain and loss by $\pm i\hbar \gamma \Theta (\vec{p})$, the system can be readily described by the Hamiltonian (1).

Another possible system to realize the imaginary SOC is the coupled planar waveguides with nonlocal gain and loss [91, 92]. We consider a system of two coupled planar ($x$–$z$ plane) waveguides, where light propagates along the $z$-axis, and $x$ is the transverse direction. In the paraxial approximation, the light fields in the two waveguides $E_{\uparrow,\downarrow}$ follow equations [46, 47, 106, 107]

\[
\frac{i}{\lambda_0} \frac{\partial E_{\uparrow,\downarrow} (z,x)}{\partial z} = -\frac{\lambda_0}{4\pi n_0} \frac{\partial^2 E_{\uparrow,\downarrow} (z,x)}{\partial x^2} + \Theta E_{\uparrow,\downarrow} (z,x) - \frac{2\pi}{\lambda_0} \int i\gamma \Xi (x - x') E_{\uparrow,\downarrow} (z,x') \, dx',
\]

where $\lambda_0$ is the vacuum wavelength, $n_0$ is the background refractive index of the waveguides, $\Theta$ is the coupling between the two waveguides, and $\pm i\gamma \Xi (x - x')$ describes the nonlocal gain and loss in the two waveguides. Performing Fourier transform on both sides of equation (5), we get

\[
\frac{i}{\lambda_0} \frac{\partial}{\partial z} \left[ \begin{array}{c} E_{\uparrow} (z,k_x) \\
E_{\downarrow} (z,k_x) \end{array} \right] = \mathcal{H} \left[ \begin{array}{c} E_{\uparrow} (z,k_x) \\
E_{\downarrow} (z,k_x) \end{array} \right],
\]

with

\[
\mathcal{H} = -\frac{\lambda_0 k_x^2}{4\pi n_0} \sigma_0 + \frac{2\pi}{\lambda_0} i\gamma \Theta (k_x) \sigma_z + \Theta \sigma_x,
\]

where $\Theta (\cdot)$ is the Fourier transform of function $\Xi (\cdot)$. Now, it is obvious that equation (6) is analogous to an Schrödinger equation in momentum representation, where the ‘Hamiltonian’ $\mathcal{H}$ has the same form as that of equation (1), with wavevector $k_x$ playing the role of momentum. Therefore, we propose that the coupled optical waveguides system with nonlocal gain and loss can be used to emulate the imaginary SOC. And noting that equations governing the propagation of lights in fibers [107–110], are similar to that in waveguides, optical fibers would also be a possible system to realize imaginary SOC.

5. Summary

In summary, we have proposed a new type of SOC—imaginary SOC, which is induced by momentum-dependent gain and loss, and possesses the property of $\mathcal{PT}$-symmetry. Imaginary SOC has the ability of energy spectrum engineering. As examples, we firstly show that it can generate pure real energy spectra with double-valleys structures. The interference between waves corresponding to different valleys can produce supersolid stripe states. Compared to the conventional Hermitian SOC supersolid stripe state which usually has low contrast, the supersolid stripe state realized by imaginary SOC can have a high contrast of one. What is more, we have also shown that the imaginary SOC can generate spectra with tunable band of complex energy, in which the waves are either decaying or amplifying during the time evolution. Thus, the imaginary SOC can also have potential applications in engineering the $\mathcal{PT}$-symmetry-based coherent wave amplifiers/absorbers. For experimental realization, we propose that the imaginary SOC can be implemented in pseudo spin-1/2 cold atomic gases and also in systems of coupled waveguides.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

For the convenience of comparison between imaginary SOC and conventional Hermitian SOC, here we present a brief review on the properties of spectrum and eigenstates of a one-dimensional SOC cold atomic gases realized by the Raman coupling scheme [11]. The Hamiltonian for such a system is \[ H_{SO} = \frac{p_x^2}{2m} \sigma_0 + \frac{p_y p_x}{m} \sigma_z + \hbar \Omega \sigma_y, \] (A.1)

where \( m \) is the atom mass, \( \hbar \) is the Planck constant, \( p_x \) is the one-dimensional momentum operator, \( p_c \) describes the strength of SOC, and \( \Omega \) is the Rabi coupling strength. By diagonalizing the Hamiltonian, we can obtain the eigenvalues and eigenvectors, they are

\[ E_{SO, \pm} = \frac{p_x^2}{2m} \pm \sqrt{\frac{p_y^2 p_x^2}{m^2} + \hbar^2 \Omega^2}, \] (A.2)

and

\[ \psi_{SO, \pm} = \begin{bmatrix} \sin (\phi) \\ \cos (\phi) \end{bmatrix}, \] (A.3)

with

\[ \phi = \arctan \left( \frac{-p_y p_x}{m \hbar \Omega} \pm \sqrt{\frac{p_y^2 p_x^2}{m^2 \hbar^2 \Omega^2} + 1} \right). \] (A.4)

The energy spectrum splits into two branches. The upper branch \( E_{SO, +} \) always has only one minimum at \( p_x = 0 \). The lower branches \( E_{SO, -} \) also has only one minimum at \( p_x = 0 \) when the SOC strength is weak \( (p_c < \sqrt{m \hbar \Omega}) \), see panel (a) of figure A1. However, in the case of strong SOC strength \( (p_c > \sqrt{m \hbar \Omega}) \), \( p_x = 0 \) becomes a maximum point of the lower branche, and there are two minima at \( p_x = \pm p_0 = \sqrt{\frac{p_y^4}{m^2 \hbar^2 \Omega^2} - p_c^2}. \) That is, the lower branch spectrum exhibits a double-valleys structure in this case, as shown in panel (b) of figure A1.

For the eigenvector (A.3), we see that the two spin components have unequal amplitude \( |\psi_{SO, +, \uparrow}| \neq |\psi_{SO, +, \downarrow}| \). For the lower branch states, when \( p_x > 0 \), we have \( |\psi_{SO, -, \uparrow}| > |\psi_{SO, -, \downarrow}| \), the state is spin-\( \uparrow \) favored; while \( p_x < 0 \), we have \( |\psi_{SO, -, \uparrow}| < |\psi_{SO, -, \downarrow}| \), the state is spin-\( \downarrow \) favored. For the upper branch, it is the opposite. To shown this spin polarization feature, in figure A1 the value \( S_{SO, +} = \langle \psi_{SO, +} | \sigma_z | \psi_{SO, +} \rangle \) is shown as the color of the lines, with red meaning spin-\( \uparrow \) favored, and blue meaning spin-\( \downarrow \) favored.

Usually, there is an energy gap between the two branches of the spectrum, \( \Delta E = 2 \hbar \Omega \). This energy gap disappears when the Rabi coupling is absent, \( \Omega = 0 \). In this case, according to equation (A.1), the two spin components are decoupled, and the spectrum simply consists of two parabolas with their centers shifted to \( p_x = \pm p_y \), see panel (c) of figure A1. And, the system is trivial in such a gapless case.
Figure A1. Spectra of one-dimensional spin–orbit coupled cold atomic gas. The color of the lines represents the spin polarization along $z$-direction $S_{0z}$ of the corresponding eigenstates. Panel (a) Single-valley spectrum for parameter $p_z = \sqrt{3}m\Omega/2$. Panel (b) Double-valleys spectrum for parameter $p_z = \sqrt{3}m\Omega$. Panel (c) Gapless spectrum for $\Omega = 0$.

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