Manifolds with boundary and of bounded geometry

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Last edited: Jan 19, 2000 — Last complied: May 29, 2018

Abstract

For non-compact manifolds with boundary we prove that bounded geometry defined by coordinate-free curvature bounds is equivalent to bounded geometry defined using bounds on the metric tensor in geodesic coordinates.

We produce a nice atlas with subordinate partition of unity on manifolds with boundary of bounded geometry, and we study the change of geodesic coordinate maps.

1 Introduction

Manifolds of bounded geometry arise naturally when one deals with non-compact Riemannian manifolds, and are studied extensively in the literature. So far, the focus was on manifolds without boundary.

One main source of examples are coverings of compact manifolds, which are particularly important in the context of $L^2$-cohomology and other $L^2$-invariants. These invariants are studied frequently also for manifolds with boundary. Therefore, it is natural to look at more general manifolds with boundary and bounded geometry.

There are mainly two ways to define manifolds of bounded geometry: either one uses bounds on the curvature (and its covariant derivatives) — this is the coordinate-free description — or one uses geodesic charts and bounds on the metric tensor and its derivatives in these coordinates — the coordinate approach. A proof of the equivalence of these two definitions for manifolds without boundary can be found in Eichhorn [4], using Jacobi fields. Related but different
results are obtained in [1] using synchronous frames. The case of manifolds with
boundary causes additional technical difficulties and seems not to be covered in
the literature. Therefore we give a proof here, using synchronous frames.

Dealing with manifolds with boundary, in addition to the usual requirements
in the interior we must impose boundary regularity conditions. These involve the
second fundamental form (in the coordinate-free description) or special charts
for the boundary (in the coordinate description).

In the last section, we show that the functions given by a change of geodesic
coordinates and their derivatives admit uniform bounds on manifolds of boun-
ded geometry. And we provide one technical tool, namely a nice atlas with
subordinate nice partition of unity. This was introduced and used by Shubin [1]
1.2 and 1.3 if the boundary is empty.

This paper grew out of part of the Dissertation [7] of the author, and the re-
sults obtained here are used in [8]. I thank my advisor, Prof. Wolfgang Lück, for
his constant support and encouragement. I also thank the referees for valuable
comments and suggestions concerning the exposition of the paper.

2 Coordinate-free versus coordinate-wise curva-
ture bounds

2.1. Definition. On a Riemannian manifold \((M^m, g)\) with boundary \(\partial M\), \(R\) denotes the curvature tensor of \(M\), \(l\) the second fundamental form of \(\partial M\), and \(\bar{R}\) the curvature tensor of \(\partial M\) (with its induced metric). The (Levi-Civita)-
covariant derivative of \(M\) is denoted with \(\nabla\), the one of \(\partial M\) with \(\bar{\nabla}\). We use \(\nu\) for the unit inward normal vector field at
\(\partial M\).

If not stated otherwise, a manifold \(M\) will always have dimension \(m\).

Given an open subset \(U \subset M\) and a chart \(x = (x_1, \ldots, x_m): U \rightarrow \mathbb{R}^m\), we
consider the corresponding derivations \(\frac{\partial}{\partial x_i}\) as derivations on \(U\), or as elements
in the tangent bundle \(T M\). We abbreviate \(\partial_i := \frac{\partial}{\partial x_i}\). We let \(g_{ij} := g(\partial_i, \partial_j)\)
be the metric tensor in the given coordinates and \(g^{ij}\) be the coefficients of the
inverse matrix.

We use the notation of multi-indices throughout: Let \(\alpha = (\alpha_1, \ldots, \alpha_m), \beta =
(\beta_1, \ldots, \beta_m)\) be multi-indices (with \(\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}\)). Then

\[
D^\alpha := D^\alpha_x := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}};
\]

and we set \(\beta \leq \alpha\) if and only if \(\beta_i \leq \alpha_i\) for \(i = 1, \ldots, n\). Define \(|\alpha| := \sum_{i=1}^m \alpha_i\).

For \(V \subset M\) and \(r > 0\) set \(U_r(V) := \{x \in M| d(x, V) < r\}\). If \(p \in \partial M\), \((B(p, r) \subset \partial M)\) means the corresponding set
for \(\partial M\) with the induced Riemannian metric.

We use the normal geodesic flow \(K : \partial M \times [0, \infty) \rightarrow M : (x', t) \mapsto
\exp_M^p(tu'x')\). For \(p \in \partial M\) set \(Z(\rho, r_1, r_2) := K((B(p, r_1) \subset \partial M) \times [0, r_2)) \subset M\).

Set \(N(s) := K(\partial M \times [0, s])\) if \(s \geq 0\).
2.2. Definition. Suppose $M$ is a manifold with boundary $\partial M$ (possibly empty). It is of (coordinate-free defined) bounded geometry if the following holds:

(N) Normal collar: there exists $r_C > 0$ so that the geodesic collar

$$\partial M \times [0, r_C) \to M : (t, x) \mapsto \exp_x(t \nu_x)$$

is a diffeomorphism onto its image ($\nu_x$ is the unit inward normal vector).

(TIC) The injectivity radius $r_{\text{inj}}(\partial M)$ of $\partial M$ is positive.

(I) Injectivity radius of $M$: There is $r_i > 0$ so that if $r \leq r_i$ then for $x \in M - N(r)$ the exponential map is a diffeomorphism on $B(0, r) \subset T_x M$. Hence, if we identify $T_x M$ with $\mathbb{R}^m$ via an orthonormal frame we have Gaussian coordinates $\mathbb{R}^m \supset B(0, r) \to M$ around every point in $M - N(r)$.

(B) Curvature bounds: For every $k \in \mathbb{N}$ there is $C_k > 0$ so that $|\nabla^i R| \leq C_k$ and $|\bar{\nabla}^i l| \leq C_k$ for $0 \leq i \leq k$.

The injectivity radius and curvature bounds are what one is used to for manifolds without boundary (compare e.g. [3, Section 3]). The embedding of the boundary is described by the second fundamental form. Because the injectivity radius does not make sense near the boundary, we replace it by the geodesic collar.

To give the coordinate-wise definition of bounded geometry, we have to explain which charts we want to use:

2.3. Definition. Let $M$ be a Riemannian manifold with boundary $\partial M$. Fix $x' \in \partial M$ and an orthonormal basis of $T_{x'} \partial M$ to identify $T_{x'}\partial M$ with $\mathbb{R}^{m-1}$. For $r_1, r_2 > 0$ sufficiently small (such that the following map is injective) define normal collar coordinates

$$\kappa_{x'} : B(0, r_1) \times [0, r_2) \to M : (v, t) \mapsto \exp_{\exp_{x'}^M(v)}(tv).$$

(We compose the exponential maps of $\partial M$ and of $M$, and $\nu$ is the inward unit normal vector field). The tuple $(r_1, r_2)$ is called the width of the normal collar chart $\kappa_{x'}$.

We adopt the convention that the boundary defining coordinate is the last (i.e. $m^{\text{th}}$) coordinate.

For $x \in M - \partial M$ and $r_3 > 0$ sufficiently small the exponential map yields Gaussian coordinates (identifying $T_x M$ with $\mathbb{R}^m$ via an orthonormal base)

$$\kappa_x : B(0, r_3) \to M : v \mapsto \exp_x^M(v).$$

We call $r_3$ the radius of the Gaussian chart $\kappa_x$.

We use the common name normal coordinates for normal collar coordinates as well as Gaussian coordinates.
2.4. Definition. A Riemannian manifold \( M \) with boundary \( \partial M \) has \((coordinate-wise
defined)\) bounded geometry if and only if (N), (IC), (I) of Definition 2.2 hold and (instead of (B))

(B1) There exist \( 0 < R_1 \leq r_{inj}(\partial M) \), \( 0 < R_2 \leq r_C \) and \( 0 < R_3 \leq r_1 \) and \( C_K > 0 \) (for each \( K \in \mathbb{N} \)) such that whenever we have normal boundary coordinates of width \((r_1, r_2)\) with \( r_1 \leq R_1 \) and \( r_2 \leq R_2 \), or Gaussian coordinates of radius \( r_3 \leq R_3 \) then in these coordinates

\[
|D^\alpha g_{ij}| \leq C_K \quad \text{and} \quad |D^\alpha g^{ij}| \leq C_K \quad \forall |\alpha| \leq K.
\]

The numbers \( R_1, R_2, R_3 \) and \( C_K \) are called the bounded geometry constants of \( M \).

The main result of the paper is the following:

2.5. Theorem. Let \((M^n, g)\) be a Riemannian manifold with boundary \( \partial M \). To given \( C > 0 \), \( k \in \mathbb{N} \), and dimension \( m \) there are \( R_1, R_2, R_3 > 0 \) and \( D > 0 \) such that the following holds:

(a1) If \( x \in M - \partial M, 0 < r_3 \leq R_3 \) and \( \kappa_x : B(0, r_3) \to (M - \partial M) \) is a Gaussian chart, and if \( |\nabla^i R| \leq C \) for \( i = 0, \ldots, k \) on the image of \( \kappa_x \) then in these coordinates

\[
|D^\alpha g_{ij}| \leq D \quad \text{and} \quad |D^\alpha g^{ij}| \leq D \quad \forall |\alpha| \leq k.
\]

(a2) If on the other hand

\[
|D^\alpha g_{ij}| \leq C \quad \text{and} \quad |D^\alpha g^{ij}| \leq C \quad \text{for } |\alpha| \leq k + 2
\]

then on the image of \( \kappa_x \) we have

\[
|\nabla^i R| \leq D \quad \text{for } i = 0, \ldots, k.
\]

(b1) If \( x' \in \partial M, 0 < r_1 \leq R_1, 0 < r_2 \leq R_2 \) and \( \kappa_{x'} : B(0, r_1) \times [0, r_2) \to M \) is a normal boundary chart, and if \( |\nabla^i R| \leq C \) and \( |\nabla^i l| \leq C \) for \( i = 0, \ldots, k \) on the image of \( \kappa_{x'} \), then in these coordinates we get

\[
|D^\alpha g_{ij}| \leq D \quad \text{and} \quad |D^\alpha g^{ij}| \leq D \quad \forall |\alpha| \leq k.
\]

(b2) If, on the other hand,

\[
|D^\alpha g_{ij}| \leq C \quad \text{and} \quad |D^\alpha g^{ij}| \leq C \quad \text{for } |\alpha| \leq k + 2
\]

then, on the image of \( \kappa_{x'} \),

\[
|\nabla^i R| \leq D \quad \text{and} \quad |\nabla^i l| \leq D \quad \text{for } i = 0, \ldots, k.
\]
(c) $M$ has (coordinate-wise defined) bounded geometry if and only if it has (coordinate-free defined) bounded geometry. In particular, we can drop the prefix in notation. The bounded geometry constants of Definition 2.4 can be chosen to depend only on $r_i$, $r_C$, $r_mj(\partial M)$ and $C_k$ of Definition 2.2.

Observe that (c) follows from (a1)-(b2). Moreover, (a2) and (b2) are immediate consequences of the formulas for $R$ and $l$ (and their covariant derivatives) in local coordinates in terms of $g_{ij}$, $g^{ij}$ and their partial derivatives (compare 2.54, 3.16 and 5.1 of [5] — note that our charts near the boundary are adapted to the embedding $\partial M \hookrightarrow M$). The statement (a1) about internal points is already included in [4, Theorem A and Proposition 2.3]. It remains to establish (b1). Since in the course of this proof we have to set up most of the notation necessary for the synchronous-frame-proof of (a1), we include a complete proof also of (a1).

The proof is done in four steps. First, we give the argument for $k = 0$, using the Rauch comparison theorem. Secondly, we prove (a1). In the third step, we establish bounds on the curvature tensor of the boundary. Lastly, we derive (b1).

**Step 1: Proof of Theorem 2.5(a1) and 2.5(b1) for $k = 0$**

2.6. Proposition. Suppose we are in the situation of Theorem 2.5(a1) or (b1) and $k = 0$. Suppose $(x_i) := \kappa^{-1} : U \subset M \to \mathbb{R}^m$ is the normal coordinate system. There are $R_1, R_2, R_3 > 0$ and $C_1, C_2 > 0$ (depending only on $C$ and $m$) such that if for width or radius we have $(r_1, r_2) \leq (R_1, R_2)$ or $r_3 \leq R_3$, respectively, then

$$C_1 \leq \left| \sum \lambda_i \frac{\partial}{\partial x_i} \right|_{TM} \leq C_2, \quad \text{if } \sum \lambda_i^2 = 1, \quad (2.7)$$

where $|v|_{TM} := \sqrt{g(v, v)}$ for $v \in TM$.

Moreover, $g_{ij}$ and $g^{ij}$ are bounded with a bound depending only on $C$ and the dimension.

The numbers $R_1$, $R_2$ and $R_3$ of Theorem 2.5 are determined by Proposition 2.6 and (IC), (I), (N).

**Proof.** The last statement is a reformulation of Inequality (2.7). To prove (2.7), we apply Warner’s generalization of the Rauch comparison theorem [10, 4.3]. We compare with two complete manifolds of constant sectional curvature $-C$ and $C$, respectively. To compare with normal collar coordinates, choose a hypersurface in this manifold so that all the eigenvalues of its second fundamental form at one (comparison) point are equal to $C$ in the first case and to $-C$ in the second case. Inequality (2.7) for vectors orthogonal to $R := \sum x_i \partial_i$ (in Gaussian coordinates), or orthogonal to $\partial_m$ (in normal boundary coordinates) is just the statement of the comparison theorem, with $C_1$ and $C_2$ depending only on the manifold we compare with (i.e. on $C$ and on $m$). Here, $r_1$, $r_2$ and $r_3$ must be sufficiently small (again depending only on the manifolds we compare with).
The comparison theorem says nothing about \( R \) or about \( \partial_m \), respectively. But for these vectors Euclidean length and length in \( TM \) as well as the orthogonal complements coincide by the following Proposition 2.8. Therefore, the inequality is true in general.

In the proof of Proposition 2.6 we used the Gauss lemma:

**2.8. Proposition.** Let \((M, g)\) be a Riemannian manifold and \( \exp : B(0, R) \to M \) a Gaussian chart. Pull the metric \( g \) back to \( B(0, R) \). Then \( g(R, R) = r^2 \), \((R = \sum_i x_i \partial_i)\), and \( g(R, v) = 0 \) if and only if \( v \) is a tangent vector to a sphere with center the origin 0.

Let \( K : \partial M \times [0, r_C) \to M \) be the geodesic collar and pull \( g \) back to \( \partial M \times [0, r_C) \). Then \( g(\partial_m, \partial_m) = 1 \) and \( g(\partial_m, v) = 0 \) if and only if \( v \) is tangent to a translate \( \partial M \times \{t\} \).

**Proof.** Compare [5, 2.93] — the proof there works also for the collar.

**Step 2: Proof of 2.5(a1).**

Suppose we are in the situation of 2.5(a1) with \( p \in M - \partial M \) and Gaussian coordinates \( x = (x_1, \ldots, x_m) = \kappa_p^{-1} : B(p, r_3) \to \mathbb{R}^m \). We will state a (differential) equation for \( g_{ij} \) in terms of the curvature tensor, so that a bound on partial derivatives of the components of the curvature tensor will give corresponding bounds for the metric. Partial and covariant derivative are related by the Christoffel symbols, so we will compute them, too.

Choose an orthonormal base \( \{s_i\} \) for \( T_p M \). Using parallel transport along geodesics emanating from \( p \), construct a synchronous orthonormal frame \( \{s_i(x)\} \) of the tangent space restricted to \( B(p, r_3) \). Let \( \{\theta^i\} \) be the frame of 1-forms dual to \( \{s_i\} \) (therefore orthonormal). The connection forms \( \theta^i_j \) for this frame are defined by

\[
\nabla s_j = \sum_i \theta^i_j s_i,
\]

with associated Christoffel symbols \( \Gamma^i_{jk} \) and curvature tensor \( R^i_{jkl} \) given by

\[
\theta^i_j = \sum_k \Gamma^i_{jk} dx_k; \quad d\theta^i_j - \sum_k \theta^i_k \wedge \theta^j_k = \sum_{k,l} R^i_{jkl} dx_k \wedge dx_l.
\]

We can express the curvature entirely in terms of \( s_i \) and \( \theta^i \), which defines \( K^i_{jk} \):

\[
d\theta^i_j - \sum_k \theta^i_k \wedge \theta^j_k = \sum_{k,l} K^i_{jkl} \theta^k \wedge \theta^l.
\]

Define functions \( a^i_j \) and \( b^i_j \) via the equations

\[
\theta^i = \sum_j a^i_j dx_j; \quad dx_i = \sum_j b^i_j \theta^j. \tag{2.9}
\]
Then \( R^i_{jkl} = \sum_{\alpha,\beta} K^i_{j\alpha\beta} a^\alpha_k a^\beta_l \) and \( g_{ij} = \sum_{\alpha} a^\alpha_i a^\alpha_j; \quad g^{ij} = \sum_{\alpha} b^\alpha_i b^\alpha_j. \) (2.10)

As matrix, \((g_{ij})\) is the product of \((a^i_j)\) and its adjoint, and accordingly for \((g^{ij})\) and \((b^i_j)\). Hence

2.11. Lemma. There are bounds on \( a^i_j \) and \( b^i_j \) corresponding to the bounds on \( g_{ij} \) and \( g^{ij} \) given by Proposition 2.4.

The Christoffel symbols \( \tilde{\Gamma}^i_{jk} \) of the covariant differentials of \( \partial_i \) are given by

\[
\nabla_{\partial_i} \partial_j = \sum_i \tilde{\Gamma}^i_{jk} s_i.
\]

Dualizing (2.3) we see that \( \partial_j = \sum_{\alpha} a^\alpha_j s_{\alpha} \), hence

\[
\tilde{\Gamma}^i_{jk} = \partial_k a^j_i + \sum_{\alpha} a^\alpha_j \Gamma^i_{\alpha k}.
\]

Atiyah, Bott, and Patodi \([1, a6 \text{ and } a10]\) derive the following equations (note that our definition of \( R^i_{jkl} \) takes care of the problems described in \([2]\)), where \( \mathcal{R} = \sum_i x_i \partial_i \):

\[
\mathcal{R} \Gamma^i_{jk} + \Gamma^i_{jk} = \sum_l 2x_l R^i_{jkl} \quad \forall i, j, k; \quad (2.13)
\]

\[
(R^2 + \mathcal{R}) a^i_j = -2 \sum_{j,k} R^i_{jkl} x_j x_k \quad \forall i, l. \quad (2.14)
\]

Set \( f_x(t) := t \Gamma^i_{jk}(tx) \). Let \( ' \) denote differentiation with respect to \( t \). Then

\[
f'_x(t) = \Gamma^i_{jk}(tx) + t \sum_l x_l \partial_l \Gamma^i_{jk}(tx) \overset{(2.14)}{=} \sum_l 2t x_l R^i_{jkl}(tx).
\]

\[
\Rightarrow \quad \Gamma^i_{jk}(x) = f_x(1) = \int_0^1 \sum_l 2\tau x_l R^i_{jkl}(\tau x) \, d\tau \quad \text{and}
\]

\[
D^\alpha_x \Gamma^i_{jk}(x) = \int_0^1 \tau^{|\alpha|} \left( D^\alpha_x (x \mapsto \sum_l x_l R^i_{jkl}(x)) \right)(\tau x) \, d\tau. \quad (2.15)
\]

Set \( f_{il}(t, x) := a^i_l(tx) \). Then \( tf_{il}(t, x) = \mathcal{R} a^i_l(tx) \) and \( t^2 f''_{il}(t, x) + tf'_{il}(t, x) = \mathcal{R}^2 a^i_l(tx) \). By (2.14)

\[
t^2 f''_{il} + 2tf'_{il} = -2t^2 \sum_{k,j} R^i_{jkl}(tx)x_j x_k.
\]

With \( w_{il}(t, x) := t^2 f''_{il}(t, x) \) we get \( w'_{il} = t^2 f''_{il} + 2tf'_{il} \). Since \( w_{il}(0) = 0 \),

\[
t^2 f''_{il}(t, x) = -2 \int_0^t \tau^2 \sum_{j,k} R^i_{jkl}(\tau x)x_j x_k \, d\tau \quad \overset{\tau = ty}{\Rightarrow}
\]
\[ f'(t, x) = -2t \int_0^1 u^2 \sum_{j, k, \alpha, \beta} K_{j\alpha\beta}^i (tux) a^2 \alpha \beta (tux) a^2 \alpha \beta (tux) x_j x_k du. \] (2.16)

Now we are in the position to explain how the bounds on \( R \) and its covariant derivatives up to order \( k \) give rise to bounds on \( g_{ij} \), \( g^{ij} \) and their partial derivatives up to order \( k \). Because of (2.10) we can consider \( a_j^i \) and \( b_j^i \) instead of the metric tensor. Moreover, the case \( k = 0 \) is done by Proposition 2.6.

**Lemma 2.17.** Let \( A, B \) be matrix valued functions which are inverse to each other. Then

\[ \frac{\partial}{\partial x_i} B = \frac{\partial}{\partial x_i} (A^{-1}) = -A^{-1}(\frac{\partial}{\partial x_i} A) A^{-1} = -B (\frac{\partial}{\partial x_i} A) B. \]

Iterated application of this and of the product rule yields

\[ D_x^\alpha B = P_{\alpha,ijkl} (B, D_x^\beta A; \beta \leq \alpha), \]

where \( P_{\alpha} \) is a fixed polynomial in non-commuting variables. Bounds for the partial derivatives of \( A \) up to order \( k \) and on \( B \) yield bounds for the partial derivatives of \( B \).

Lemma 2.17 applies to the matrices \( A = (a_j^i) \) and \( B = (b_j^i) \). Moreover, by Proposition 2.6, we have a bound for \( (b_{ij}) \). Hence it remains to find bounds for the derivatives of \( (a_j^i) \).

**Lemma 2.18.** For \( \alpha = (\alpha_1, \ldots, \alpha_n) \) there is a polynomial \( P_{\alpha,i,j,k} \) (only depending on \( \alpha, i, j, k, l \)) in partial derivatives up to order \( (|\alpha| - 1) \) of \( K_{i,j,k}^l \), \( \Gamma_{i,j,k}^l \), and \( a^* \) such that as functions on the set \( B(p, r) \)

\[ (\nabla_{\partial_i} \nabla_{\partial_i})^{|\alpha|} R(s_i, s_j, \partial_k, \partial_l) = D_x^\alpha K_{i,j,k}^l + P_{\alpha,i,j,k,l}. \]

**Proof.** This follows from the formula for covariant differentials in coordinates. Note that for \( |\alpha| = 1 \) only \( \Gamma_{i,j}^l \) shows up (since \( K_{i,j,k}^l \) is defined entirely in terms of \( s_i \)). But if we iterate the covariant differentials, we have to take into account that we contracted \( \nabla R \) with \( \partial_i \) and not with \( s_i \). This yields (via \( \nabla \partial_i \) \( \Gamma_{i,j}^l \) and, since we iterate the covariant differentials, their partial derivatives up to order \( |\alpha| - 2 \). Since \( \Gamma_{i,j}^l = \partial_k a^i j + \sum_{\alpha} a^\alpha \Gamma_{\alpha,i,j,k}^l \), the result follows.

Now we proceed by induction on the order of derivatives \( |\alpha| \). For \( |\alpha| = 0 \) observe that by assumption we have a bound on the curvature. Since \( \{s_i\} \) is orthonormal this gives bounds on \( K_{i,j,k}^l \). By Proposition 2.6 the same is true for \( a_j^i \).

Assume by induction that for \( r \geq 0 \) we have found bounds on the partial derivatives up to order \( r \) of \( K_{i,j,k}^l \) and \( a_j^i \) and on the derivatives up to order \( (r - 1) \) of \( \Gamma_{i,j}^l \). The assumptions of the Theorem give bounds on \( |R|, \ldots, |\nabla^{r+1} R| \).

From equation (2.10), relating \( K_{i,j,k}^l \) and \( R_{i,j,k}^l \), we get bounds on the partial derivatives up to order \( r \) of \( R_{i,j,k}^l \). Then Equation (2.15) yields bounds for the
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derivatives of order $r$ of $\Gamma^j_{ik}$. Lemma 2.18 and the bound on $\nabla^{r+1} R$ yield bounds on $(r+1)$-order partial derivatives of $K^j_{kl}$ (since by Proposition 2.4 the length of $\partial_i$ is controlled). In all instances the new bounds are given in terms of the old ones.

It remains to deal with the derivatives of order $(r+1)$ of $a^j_i$. Remember Equation (2.16) for $f_{il}(t, x) = a^j_i(t x)$:

\[ f'_{il}(t, x) = -2t \int_0^1 u^2 \sum_{j, k, \alpha, \beta} \left( K^j_{i\alpha\beta} a^0_i a^\beta_k \right) (t u x) x_j x_k \, du. \]

Let $\alpha$ be a multi-index with $|\alpha| = r + 1$. We differentiate the equation with respect to $x$ to get an equation for $D^\alpha f_{il}(t, x) = t^{\alpha}(D^\alpha a^j_i)(t x)$. This yields

\[
(D^\alpha f_{il})'(t, x) = -2t \int_0^1 (u^2 \sum_{j, k, \beta, \gamma} K^j_{l\beta\gamma} (t u x). \\
((D^\alpha f_{ik}) f_{jl} + (D^\alpha f_{jkl}) (t u x) x_j x_k \, du) - 2t \int_0^1 P_{il} \, du. \tag{2.19}
\]

Here $P_{il}$ is a polynomial in $t, u, x$, partial derivatives up to order $(r+1)$ of $K^*_{il}$ at $tux$, and partial derivatives up to order $r$ of $f_{il}$ at $(t u x)$. The left and right hand side of (2.19) are equal as function of $x$ and $t$. The induction hypothesis implies for $0 \leq t \leq 1$ with suitable $C_1, C_2 > 0$ the inequality

\[
|D^\alpha f_{ij}'(t, x)| \leq C_1 \sup_{0 \leq \tau \leq t} \{ |D^\alpha f_{ij}(\tau, x)| \} + C_2, \tag{2.20}
\]

Moreover, $D^\alpha f(0, x) = 0$ since $|\alpha| \geq 1$.

Let $h(t) := C_2 (\exp(C_1 t) - 1)/C_1$ be the unique solution of $h'(t) = C_1 h(t) + C_2$ with $h(0) = 0$. This is a positive monotonous increasing function, with an explicit bound $h(t) \leq C := C_2 (\exp(C_1) - 1)/C_1$ for $0 \leq t \leq 1$.

Abbreviate $u^j_i(t) := D^\alpha f_{ij}(t, x)$. We will prove $|u^j_i(t)| \leq h(t)$ and therefore

\[
|D^\alpha a^j_i(x)| = |u^j_i(1)| \leq h(1) \leq C. \tag{2.21}
\]

This then finishes the induction step. To show $|u^j_i(t)| \leq h(t)$, let $h_n$ be the unique solution of

\[
h_n'(t) = C_1 h_n(t) + C_2 + 1/n \quad \text{with} \quad h_n(0) = 0.
\]

Then $h_n(t) \xrightarrow{n \to \infty} h(t)$ uniformly for $0 \leq t \leq 1$. Therefore, it suffices to show $|u^j_i(t)| \leq h_n(t)$. For a contradiction, assume $|u^j_i(t)| > h_n(t)$ for some $n$ and $t$. Set $t_0 := \inf_{0 \leq t \leq 1} \{ |u^j_i(t)| > h_n(t) \}$. Then $|u^j_i(t_0)| = h(t_0)$, since $u^j_i(0) = 0 = h(0)$, $h_n$ is monotonous, and $|u^j_i(t)| \leq h_n(t) \forall t \leq t_0$. Consequently, $\sup_{t \leq t_0} |u^j_i(t)| = |u^j_i(t_0)|$. Then (2.20) shows

\[
|(u^j_i)'(t_0)| \leq C_1 |u^j_i(t_0)| + C_2 < h_n'(t_0).
\]
Moreover, $d/dt |u_j^i(t_0)| \leq |(u_j^i)'(t_0)|$ (compare §3.1.3.2 for the difficult case $u_j^i(t_0) = 0 - d/dt$ is understood to be the right derivative). It follows $|u_j^i(t)| < h_0(t)$ for $t \in [t_0, t_0 + \epsilon)$ and $\epsilon > 0$ sufficiently small. But this contradicts the choice of $t_0$.

**Step 3: Curvature of $\partial M$.**

We adopt the notation of Definition 2.1.

In the following we consider $(0, p)$-tensors $T$ on $\partial M$ and their restriction to $\partial M$, given by the inclusion $T\partial M \hookrightarrow TM$. We will use the same notation for $T$ and its restriction, the meaning will be clear from the context.

We compute the covariant derivatives $\nabla^k R$ using the following rules:

**2.22. Lemma.** Suppose $T$ is a $(0, q)$-tensor on $M$, $S$ a $(0, p)$-tensor on $\partial M$, and $S^{*1}$ the $(1, p-1)$-tensor on $\partial M$ given by $g(S^{*1}_{x}(v_2, \ldots, v_p), v_1) = S_x(v_1, \ldots, v_p)$ for $v_1, \ldots, v_p \in T_x \partial M$, where $x \in \partial M$ and $S_x, S^{*1}_x$ are the values of $S$ and $S^{*1}$, respectively, at $x$. Let $\sigma$ be a permutation (operating on a multiple tensor product by permutation of the factors) with $\sigma^{-1}(1) \leq p$. Let $c$ denote the contraction of a $(0, r)$-tensor with a $(1, s)$-tensor which contracts the $r$-th entry of the $(0, r)$-tensor. The covariant derivative is understood to be a map $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$. Then the following holds:

1. $\nabla T = \nabla T - \sum_i c(T \otimes l \circ \sigma_i, \nu)$, where $\sigma_i$ are appropriate permutations.

2. $\nabla ((T \otimes S) \circ \sigma) = ((\nabla T) \otimes S) \circ \sigma + (T \otimes \nabla S) \circ \sigma$.

3. $\nabla c((T \otimes S) \circ \sigma, \nu) = c(\nabla T) \otimes S \circ \sigma', \nu) + c(T \otimes \nabla S \circ \sigma'', \nu) + \sum_i c(T \otimes S \circ \sigma_i, \nu) \otimes l \circ \sigma_i, \nu) + c(T \otimes S \circ \sigma, l^{*1})$, with $\sigma', \sigma''$, and $\sigma_i$ appropriate permutations.

4. $\nabla c(T, (\nabla^k l)^{*1}) = c(\nabla T, (\nabla^k l)^{*1}) + c(T, (\nabla^{k+1} l)^{*1})$.

**Proof.** Formulas 2 and 3 are well known. Let $v_1, \ldots, v_p$ and $X$ be vector fields on $\partial M$. For 1, we compute:

\[
\nabla T(v_1, \ldots, v_p, X) = X.T(v_1, \ldots, v_p) - T(\nabla_X v_1, \ldots, v_p) - \cdots - T(v_1, \ldots, \nabla_X v_p)
\]

For 3, we have:

\[
\nabla c(T \otimes S \circ \sigma, \nu) = c(\nabla T) \otimes S \circ \sigma', \nu) + c(T \otimes \nabla S \circ \sigma'', \nu) + \sum_i c(T \otimes S \circ \sigma_i, \nu) \otimes l \circ \sigma_i, \nu) + c(T \otimes S \circ \sigma, l^{*1})
\]

For 4, we obtain:

\[
\nabla c(T, (\nabla^k l)^{*1}) = c(\nabla T, (\nabla^k l)^{*1}) + c(T, (\nabla^{k+1} l)^{*1}).
\]

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For $\mathfrak{3}$ set $v_1 := \nu$ and calculate:
\[
\bar{\nabla}_X c(T \otimes S \circ \sigma, \nu)(v_2, \ldots, v_{p+q}) v_1 = \nu = (X.T(v_{\sigma_1}, \ldots, v_{\sigma_p})) S(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}) \\
+ T(v_{\sigma_1}, \ldots, v_{\sigma_p}) (X.S(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})) \\
- \sum_{i=1}^{p+q} T \otimes S(v_{\sigma_1}, \ldots, \bar{\nabla}_X v_{\sigma i}, \ldots, v_{\sigma(p+q)}) \\
= c(T \otimes \bar{\nabla}_X S \circ \sigma, \nu)(v_2, \ldots, v_{p+q}) \\
+ \left( X.T(v_{\sigma_1}, \ldots, v_{\sigma_p}) - \sum_{i=1}^{p} T(v_{\sigma_1}, \ldots, \nabla_X v_{\sigma_1}, \ldots, v_{\sigma_p}) \right) S(\ldots) \\
- \sum_{i=1}^{p} T(v_{\sigma_1}, \ldots, l(X, v_{\sigma_i}) \nu, \ldots, v_{\sigma_p}) S(\ldots) \\
+ T(v_{\sigma_1}, \ldots, \nabla_X \nu, \ldots, v_{\sigma_p}) S(\ldots) \\
= c(T \otimes \bar{\nabla}_X S \circ \sigma, \nu)(\ldots) + c((\nabla_X T) \otimes \sigma, \nu)(\ldots) \\
- \sum_i c(T \otimes S \circ \sigma, \nu) \otimes l \circ \sigma_i, v)(\ldots, X) + c(T \otimes S \circ \sigma, \nabla_X \nu)(\ldots).
\]

If $Y \in C^\infty(T \partial M)$ then
\[
0 = X.g(\nu, Y) = g(\nabla_X \nu, Y) + g(\nu, \nabla_X Y) \\
\implies g(\nabla_X \nu, Y) = l(X, Y) = l(Y, X) \\
0 = X.g(\nu, \nu) = 2g(\nabla_X \nu, \nu). \\
\implies \nabla_X \nu = l^{*1}(X) \implies \nabla \nu = l^{*1}.
\]

This finishes the proof.

\[\square\]

2.23. Corollary. $\bar{\nabla}^k \bar{R}$ is a finite sum of tensor products and possibly iterated contractions, composed with permutations, involving (i) $\nabla^j R$ for $j \leq k$; (ii) $\bar{\nabla}^j l$ for $j < k$; (iii) $(\bar{\nabla}^l)^*$ for $j < k - 1$; and (iv) $\nu$.

Bounds for the building blocks (i) and (ii) yield a bound for $\bar{\nabla}^k \bar{R}$.

Proof. The first statement follows by iterated application of Lemma 2.22. The last statement follows since tensor products and contractions of tensors are bounded in terms of the bounds on the factors, and because permutations are isometric. Note that $|\nu| = 1$ and $|S^*| = |S|$ for an arbitrary tensor $S$. Moreover, restriction to the boundary only decreases the norm of a tensor.

\[\square\]

2.24. Corollary. If $M$ is a Riemannian manifold of (coordinate-free defined) bounded geometry, the same is true for its boundary.
Step 4: Proof of Theorem 2.5(b1).

Suppose we are in the situation of 2.5(b1) with \( p \in \partial M \) and normal collar coordinates \((x_1, \ldots, x_m) = \kappa_p^{-1} : U \to \mathbb{R}^m \) around \( p \). By our convention \( x_m \) is the boundary defining coordinate, i.e. \( \partial_{\partial M} x_m = \nu \).

First consider \( \partial M \) as a Riemannian \((m - 1)\)-dimensional manifold of its own. Corollary 2.23 shows that bounds on the covariant derivatives \( \nabla^j R \) and \( \nabla^j l \) give rise to bounds on \( \nabla^j R \) \((0 \leq j \leq k)\). As in Step 2 (applied to \( \partial M \)) construct the orthonormal frame \( \{s_i\}_{1 \leq i \leq m-1} \) of \( T\partial M \). Extend this to an orthonormal frame of \( TM|_{\partial M} \) by setting \( s_m := \nu \). By parallel transport along geodesics with initial speed \( \nu \) we get a synchronous orthonormal frame of \( TM \) on the normal collar neighborhood. Define the dual frame \( \{\theta^i\} \), the Christoffel symbols \( \Gamma^i_{jk} \) and \( \tilde{\Gamma}^i_{jk} \), the curvature coefficients \( R^i_{jkL} \) and \( K^i_{jkL} \), and \( a^i_j \) and \( b^i_j \) in exactly the same way as in Step 2. Note that (2.10) and (2.12) remain true.

Now we come to the differential equations which relate these quantities. By construction, \( \{s_i\} \) translates to
\[
\partial_m \theta^i_j = 0, \quad \text{i.e.} \quad \Gamma^i_{jm} = 0 \quad \forall i, j \quad (2.25)
\]
\( (\partial_m) \) denotes contraction with \( \partial_m \). The Lie derivative along \( \partial_m \) (denoted by \( \partial_m \)) acts on differential forms via \( \partial_m = c(\partial_m)d + dc(\partial_m) \). Hence
\[
\partial_m \theta^i_j = c(\partial_m)(d\theta^i_j - \sum_{k=1}^m \theta^i_k \wedge \theta^j_k) = c(\partial_m)(\sum_{k,l} R^i_{jkl}dx_k \wedge dx_l = 2 \sum_k R^j_{ikm}dx_k.
\]

On the other hand, \( \partial_m \theta^i_j = \sum_k \partial_m(\Gamma^i_{jk})dx_k \). Hence, applying \( D^\alpha \) yields
\[
\partial_m (D^\alpha \Gamma^i_{jk}) = 2D^\alpha R^i_{jmk} \quad \forall i, j, k . \quad (2.26)
\]

Additionally, we need an equation for \( a^i_j \). We apply \( \partial_m \) twice to the dual frame. Since \( \partial_m = s_m \) we have \( c(\partial_m) \theta^i = \delta^i_{im} \) \((\delta \text{ the Kronecker symbol})\). Then
\[
\partial_m \theta^i = c(\partial_m)d\theta^i + dc(\partial_m)\theta^i = c(\partial_m)d\theta^i.
\]

The connection is torsion free. This means
\[
d\theta^i = \sum_j \theta^i_j \wedge \theta^j
\]
\[
\Rightarrow \partial_m (\theta^i) = \sum_j c(\partial_m)(\theta^i_j \wedge \theta^j) - \theta^i_m \quad (2.27)
\]
\[
\Rightarrow \partial_m^2 (\theta^i) = -\partial_m (\theta^i_m) = -2 \sum_k R^i_{mmk}dx_k.
\]

The left hand side can be computed in terms of \( a^i_j \)
\[
\partial_m (\theta^i) = \sum_j \partial_m (a^i_j)dx_j \quad \Rightarrow \quad \partial_m^2 (\theta^i) = \sum_j \partial_m^2 (a^i_j)dx_j . \quad (2.28)
\]
Equating coefficients, applying $D^\alpha$, and expressing $R^i_{jk}|x_m=0$ in terms of $K^i_{jk}$ yields
\[
\partial_m^2 D^\alpha a^i_j = -2 \sum_{k,l} D^\alpha (K^i_{mkkl} a^k_m a^l_j) \quad \forall i,j.
\] (2.29)

For $|\alpha| > 0$ this is (for each point in the boundary) a system of inhomogeneous linear ordinary differential equations for $D^\alpha a^i_j$, with coefficients given by partial derivatives of $K^i_{mkkl}$ up to order $|\alpha|$ and of $a^i_j$ up to order $|\alpha| - 1$.

To make use of the differential equation (2.26) and (2.29) we have to determine the initial values of the equation for $D^\alpha a^i_j$, giving in particular bounds on the right hand sides of (2.26) and (2.29) (using the bounds on the initial values), namely $\partial M = 0$ (if $\alpha = (\alpha_1, \ldots, \alpha_m)$). Later, we will by induction on $|\alpha|$ get bounds on the right hand sides of (2.26) and (2.29) (using the bounds on the initial values), giving in particular bounds on $\partial_m D^\alpha a^i_j|_{x_m=0}$ and $\partial_m^2 D^\alpha a^i_j|_{x_m=0}$ which are the initial values of the equation for $D^\beta \Gamma^i_{jk}$ and $D^\beta a^i_j$ where $\beta = (\alpha_1, \ldots, \alpha_m + 1)$. We will therefore, inductively, get the required bounds
\[
|D^\beta \Gamma^i_{jk}|_{x_m=0} \leq C, \quad |D^\beta a^i_j|_{x_m=0} \leq C, \quad |\partial_m D^\alpha a^i_j|_{x_m=0} \leq C.
\] (2.30)

We proceed with a bootstrap argument similar to the one in Step 2. We have to find bounds on $g_{ij}$, $g^{ij}$ and their derivatives. Because of (2.9) it suffices to
look at $a_i^j$ and $b_i^j$ and their derivatives. Bounds on $a_i^j$ and $b_i^j$ are given by Lemma 2.11. As in the proof of Step 2, because of Lemma 2.17 we only have to control the derivatives of $a_i^j$. We do this by induction on the order of these derivatives. To carry out the induction step, we also have to control the derivatives of $K_i^{jkl}$, $R_i^{jkl}$ and $\Gamma_i^{jk}$ (and the initial values in (2.30)).

To conclude the start of the induction, Lemma 2.18 and the assumptions give bounds on $K_i^{jkl}$ and (since the length of $\partial_i$ is bounded by Proposition 2.6) on $R_i^{jkl}$. Integrating Equation (2.26), we find bounds for $\Gamma_i^{jk}$ (depending also on the given width $R_1$ of the normal boundary charts).

Assume now by induction that we have bounds on $D_\alpha a_i^j$, $D_\alpha \Gamma_i^{jk}$, $D_\alpha K_i^{jkl}$, and $D_\alpha R_i^{jkl}$ for $|\alpha| \leq r$. By Lemma 2.18, the assumed bound on $\nabla^{r+1} R$ therefore gives bounds on $\nabla^\alpha K_i^{jkl}$, $\nabla^\alpha a_i^j$ ($|\alpha| \leq r$) and on the initial values (2.30) we can apply [6, IV.4.2] to (2.26) to obtain bounds on $\nabla^\alpha a_i^j$ ($|\alpha| = r + 1$). This in turn, together with the relation (2.13) between $R_i^{jkl}$ and $K_i^{jkl}$ yields bounds on $\nabla^\alpha R_i^{jkl}$ ($|\alpha| = r + 1$).

Now we integrate (2.26) to get bounds on $\nabla^\alpha \Gamma_i^{jk}$ ($|\alpha| = r + 1$) to finish the induction step and to conclude the proof of Theorem 2.5.

The bounds we obtain, inductively, depend only on the bounds we started with.

3 Technical properties of manifolds of bounded geometry

We use the notation of Definition 2.1.

3.1. Lemma. Let $(M^n, g)$ be a Riemannian manifold with boundary and with bounded geometry as in Definition 2.4. We find $r_0 > 0$ such that for all $r, s \leq r_0$ the following holds:

1. If $x, x' \in \partial M$ and $Z(x', s, 2\frac{R_2}{3}) \cup U(r)(Z(x, \frac{R_2}{3}, 2\frac{R_2}{3})) \neq \emptyset$ then $Z(x', s, 2\frac{R_2}{3}) \subset Z(x, \frac{R_2}{3}, 2\frac{R_2}{3})$.

2. We find $0 < D_1(r) < D_2(r)$ for $r \geq 0$ such that

$$D_1(r) \leq \text{vol}(B(x, r)) \leq D_2(r) \quad \forall x \in M - N(\frac{R_2}{3}), \text{ if } r < R_3$$

$$D_1(r) \leq \text{vol}(Z(x', r, 2\frac{R_2}{3})) \leq D_2(r) \quad \forall x' \in \partial M, \text{ if } r < R_1.$$ 

$D_1(r)$, $D_2(r)$ and $r_0$ can be chosen to depend only on the bounded geometry constants.

Proof. Bounded geometry implies the existence of $C_1, C_2 > 0$ so that in normal coordinates

$$\|(g^{ij})_{i,j}\| < C_1, \quad \|(g_{ij})_{i,j}\| < C_1, \quad \text{and } C_2 \leq \sqrt{|\text{det}(g_{ij})|} \leq C_1.$$
Observe that $d(Z(x, R_i, 2R_i/3), M - Z(x, 3R_i/4, 9R_i/10))$ is bounded independent of $x \in \partial M$, using the bounds on the metric tensor. With all sets and distances in $\partial M$, $d(B(x, R_i/2), \partial M - B(x, 3R_i/4)) \leq R_i/10$. Choose $r_0$ smaller than half the minimum of these two bounds. If $r, s < r_0$ and $Z(x', s, 2R_i/3) \cap U_i, (Z(x, R_i/2, 2R_i)) \neq \emptyset$ for $x, x' \in \partial M$ then $Z(x', s, 2R_i/3) \cap Z(x, 3R_i/4, 9R_i/10) \neq \emptyset$ which in turn implies $Z(x', s, 2R_i/3) \subset Z(x, 3R_i/4, 9R_i/10)$ by the choice of $r_0$. This proves the first assertion.

The assertion about the volume bounds follows immediately from the upper and lower bounds of $|\det(g_{ij})|$. We can choose all constants to depend only on the bounded geometry constants.

The following is important to do analysis on manifolds of bounded geometry. The corresponding result for empty boundary is due to Shubin [9, A1.2 and A1.3].

3.2. Proposition. (Partition of unity)

Let $M$ be a manifold with boundary and of bounded geometry as in Definition 2.3. There are $r_m > 0$ and, for $0 < r < r_m$ constants $C_K > 0$ ($K \in \mathbb{N}$), $M_f \in \mathbb{N}$, all depending only on the bounded geometry constants (and $r$) such that a covering of $M$ exists by sets $\{U(x_i, r)\}_{i \in I} \subset \mathbb{R}$ which has the following properties:

1. $x_i \in \partial M$ for $i \geq 0$ and $U(x_i, r) = Z(x_i, r, 2R_i)$;
   
   $x_i \in M - N(R_i)$ for $i < 0$ and $U(x_i, r) = B(x_i, r)$.

2. If $s < r_m$ and $x \in M$ then $B(x, s) \cap U(x_i, r) \neq \emptyset$ for at most $M_f$ of the $x_i$.

3. $\{U(x_i, r/2)\}_{i \in I}$ is a covering of $M$, too.

Denote with $\kappa_i : B(0, r) \to U(x_i, r)$ ($i < 0$) and $\kappa_i : B(0, r) \to U(x_i, r)$ for $i \geq 0$ the corresponding normal charts.

To this covering, a subordinate partition of unity $\{\varphi_i\}$ exists such that

$|D^\alpha \varphi_i| \leq C_K \quad \forall i \in I \quad \forall |\alpha| \leq K$ (in normal coordinates).

Proof. Set $r_m := \min\{R_l/2, R_2/12, R_3, r_0/2\}$, where $r_0$ is given by Lemma 3.1. Let $0 < r < r_m$. First choose a maximal set of points $\{x_i \in \partial M; i = 0, 1, 2, \ldots\}$ such that all $(B(x_i, r/4) \in \partial M)$ are disjoint. Next, choose a maximal set of points $\{x_i \in M - N(R_i); i = -1, -2, \ldots\}$ such that all $B(x_i, r/4) (i < 0)$ are disjoint. Note that the set $I$ of $i$ obtained this way may be a proper subset of $I$. For $0 < s \leq r_0$ set $U(x_i, s) := B(x_i, s)$ if $i < 0$ and $U(x_i, s) := Z(x_i, s, 2R_i/3)$ if $i \geq 0$. Then

$$\bigcup_{i < 0} U(x_i, r/2) = \bigcup_{i < 0} B(x_i, r/2) \} \text{ covers } M - N(R_2/2).$$

This is true because else we find $z \in M - N(R_2/2)$ which has distance $\geq r/2$ to all of the $x_i$. Then $B(z, r/4) \cap B(x_i, r/4) = \emptyset \forall i < 0$, violating the maximality...
of \( \{x_i\}_{i<0} \). Similarly, \( \{B(x_i,r/2) \subset \partial M\}_{i\geq 0} \) covers \( \partial M \implies \{U(x_i,r/2)\}_{i\geq 0} \) covers \( N(\frac{R_2}{2}) \).

Now we have to show that the covering \( \{U(x_i,r)\}_{i\in I} \) has Property 3. So fix \( 0<s<r_m \) and \( x \in M \).

- If \( x \in N(\frac{R_2}{3}) \) and \( i<0 \) then \( B(x,s) \cap U(x_i,r) = \emptyset \) since \( d(N(\frac{R_2}{3}), M - N(\frac{R_2}{2})) = \frac{R_2}{6} > r + s \).
- If \( x \in M - N(\frac{R_2}{3}) \) then the number \( N_1 \) of \( x_i \) (\( i<0 \)) with \( U(x_i,r) \cap B(x,s) \neq \emptyset \) is by Lemma 3.1 bounded by
  \[
  N_1 \leq \frac{\text{vol}(B(x,s + r))}{\inf_{x_i \in M - N(\frac{R_2}{3})} \text{vol}(B(x_i,r/4))} \leq \frac{D_2(2r_m)}{D_1(r/4)}
  \]
since for such \( x_i \) we have \( B(x_i,r/4) \subset B(x, s + r) \) and all of these are disjoint.
- If \( x \in M - N(\frac{R_2}{2}) \) then \( B(x,s) \cap U(x_i,r) = \emptyset \) for \( i \geq 0 \) since \( d(N(\frac{2R_2}{3}), M - N(\frac{R_2}{2})) = \frac{R_2}{3} > s \).
- If \( x \in N(\frac{R_2}{2}) \) then the number \( N_2 \) of \( x_i \) (\( i \geq 0 \)) with \( B(x,s) \cap U(x_i,r) \neq \emptyset \) is bounded by
  \[
  N_2 \leq \frac{\sup_{x_i \in \partial M} \text{vol}(Z(x_i, \frac{R_2}{10}, \frac{2R_2}{3}))}{\inf_{x_i \in \partial M} \text{vol}(Z(x_i, \frac{R_2}{4}, \frac{2R_2}{3}))} \leq \frac{D_2(9R_1/10)}{D_1(r/4)}
  \]
since if there is one such \( i_0 \) then for all other such \( i \) by Lemma 3.1
  \( Z(x_i, \frac{R_2}{4}, \frac{2R_2}{3}) \subset Z(x_{i_0}, 9R_1/10, \frac{2R_2}{3}) \), and all the \( Z(x_i, \frac{R_2}{4}, \frac{2R_2}{3}) \) are disjoint.

It follows in all cases
\[
M_f(r) \leq \frac{D_2(9R_1/10) + D_2(2r_0)}{D_1(r/4)} < \infty.
\]

It remains to construct the subordinate partition of unity. Choose a smooth cutoff function \( \varphi : \mathbb{R}^m \to [0,1] \) with \( \varphi(x) = 1 \) if \( |x| \leq r/2 \) and \( \varphi(x) = 0 \) if \( |x| \geq r \). Denote the restriction to \( \mathbb{R}^{m-1} \) also with \( \varphi \). Choose smooth \( \psi : \mathbb{R} \to [0,1] \) with \( \psi(x) = 0 \) if \( x \geq 2R_2/3 \) and \( \psi(x) = 1 \) if \( x \leq R_2/2 \). Via the normal coordinates this yields cutoff functions \( f_i \) on \( U(x_i,r) \) with \( f_i \circ \kappa_i(y',t) = \varphi(y')\psi(t) \) if \( i \geq 0 \) and \( f_i \circ \kappa_i = \varphi \) if \( i < 0 \). Therefore, if \( \kappa \) is any normal chart, \( f_i \circ \kappa = \varphi \circ (\kappa_i^{-1} \circ \kappa) \) (\( i < 0 \)) and \( f_i \circ \kappa = (\varphi \cdot \psi) \circ (\kappa_i^{-1} \circ \kappa) \) (\( i \geq 0 \)). The chain rule shows that the bounds on derivatives up to order \( K \) of the coordinate changes (Proposition 3.3) yield bounds on the partial derivatives up to order \( K \) of \( f_i \) in normal coordinates. To construct the partition of unity, set \( F = \sum_{i \in I} f_i \) (at each point there are at most \( M_f \) non-zero summands).

Since \( M - N(\frac{R_2}{2}) \subset \bigcup_{i<0} U(x_i,r/2) \) and \( N(\frac{R_2}{2}) \subset \bigcup_{i\geq 0} U(x_i,r/2) \),
for each \(z \in M\) at least one of \(f_i(z) = 1 \implies F \geq 1\). Define
\[
\varphi_i := \frac{f_i}{F}.
\]
Obviously, \(\{\varphi_i\}_{i \in I}\) is a smooth partition of unity subordinate to our covering. Pick one \(\varphi_i\) and one normal chart \(\kappa\). For partial derivatives up to order \(K\) in normal coordinates observe
\[
|D^\alpha(\varphi_i \circ \kappa)| = \left|\frac{D^\alpha f_i \circ \kappa}{\kappa}\right| = \frac{|P_\alpha(D^{\beta}(f_i \circ \kappa), D^{\gamma}(F \circ \kappa); |\beta|, |\gamma| \leq |\alpha|)|}{|F \circ \kappa|^{2|\alpha|}} \\
\leq \frac{|P_\alpha(D^{\beta}(f_i \circ \kappa), D^{\gamma}(F \circ \kappa))|}{|F|^{\geq 1}}.
\]
\(P_\alpha\) is a polynomial entirely determined by \(\alpha\). At every point \(x \in M\), \(D^{\gamma}(F \circ \kappa)|_x\) is the sum of at most \(M_\gamma\) summands of the type \(D^{\gamma}(f_i \circ \kappa)|_x\). Therefore, we have bounds for all the entries of \(P_\alpha\). This yields a bound \(C_K\), depending only on the bounded geometry constants, for \(|D^\alpha(\varphi_i \circ \kappa)| \) if \(|\alpha| \leq K\).

Changes of normal coordinates

3.3. Proposition. Suppose \(M\) is a Riemannian manifold with boundary and of bounded geometry. More precisely, suppose \(C > 0\) is a bound for partial derivatives up to order \(k + 1\) of \(g^{ij}\) and \(g_{ij}\) in normal coordinates. Then \(D > 0\) exists, depending only on \(C\) so that, if \(\kappa_1 : U_1 \subset \mathbb{R}^m \to M\) and \(\kappa_2 : U_2 \subset \mathbb{R}^m \to M\) are normal charts as in 2.3, the following holds for \(f := \kappa_1^{-1} \circ \kappa_2 : U_0 \subset \mathbb{R}^m \to \mathbb{R}^m\) (\(U_0\) the domain of definition of the composition):
\[
|D^\alpha f| \leq D \quad \forall |\alpha| \leq k.
\]

Since the maps \(\kappa_i\) are solutions of certain ordinary differential equation, namely the equation for geodesics, we first recall a result about differential equations.

3.4. Lemma. Let \(x'(t) = F(t, x(t))\) be a system of ordinary differential equations \((t \in \mathbb{R}, x(t) \in \mathbb{R}^n), F \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)\). Suppose \(\varphi(t, x)\) is the flow of this equation. We find a universal expression \(\text{Expr}_\alpha\), only depending on \(\alpha\) such that for all \(t \geq 0\) where \(\phi(t, x_0)\) is defined
\[
|D^\alpha \phi(t, x_0)| \leq \text{Expr}_\alpha \left( \sup_{0 \leq \tau \leq t} \{|D^\beta F(\tau, \phi(\tau, x_0))| | \beta \leq \alpha, t \} \right). \tag{3.5}
\]

Proof. The theory of ordinary differential equations [3, V.3.1.] tells us that we have the linear differential equation
\[
\alpha'(t) = \frac{\partial F}{\partial x}(t, \varphi(t, x)) \cdot \alpha(t); \quad \alpha(0) = e_k = (0, \ldots, 1, \ldots, 0)
\]
for \(\partial_k \varphi(t, x)\). For linear differential equations [3, IV.4.2] gives inequalities which directly imply (3.5) if \(|\alpha| = 1\).
Inductively one shows that higher derivatives fulfill the linear differential equation
\[ (D^2_x \varphi)'(t, x) = (D_x F)(t, \varphi(t, x)) \cdot D^2_x \varphi(t, x) + P_\alpha(D_x \varphi, (D^2_x F)(t, \varphi(t, x))), \gamma < \alpha, \beta \leq \alpha \] (3.6)
with \( D^\alpha_x \varphi(0, x) = 0 \) if \(|\alpha| > 1\). Here \( P_\alpha \) is a polynomial matrix which depends only on \( \alpha \). By induction and using [3, IV.4.2] again, the proposition follows.

Reduction of order implies:

3.7. Corollary. A statement corresponding to Lemma 3.4 holds for ordinary differential equations of order \( k \).

3.8. Lemma. Let \( p \in U \subset \mathbb{R}^m \), \( g_{ij} \) a Riemannian metric on \( U \) and \( \exp_p : B(r, 0) \to U \) the exponential map at \( p \) (we identify \( T_p U \) with \( \mathbb{R}^m \) via an orthonormal frame). If \( r \) is sufficiently small then \( \exp_p \) is a diffeomorphism onto some open subset of \( U \), and the derivatives up to order \( k \) of \( \exp_p \) and its inverse are bounded in terms of \( g_{ij}, g^{ij} \), their derivatives up to order \( k + 1 \), and \( r \).

Proof. We have \( \exp_p(x) = \varphi(x, p, 1) \), where \( \varphi \) with \( \varphi(x, q, 0) = q, \varphi'(x, q, 0) = x \) is the flow of the differential equation for geodesics. Corollary 3.7 applies to this equation \( x'' = F(x) \), and \( F(x) = -\sum_{i,j} \Gamma_{ij}(x)x'_{i}x'_{j} \) is given by \( g_{ij} \) and its first order derivatives.

For the inverse, by Lemma 2.17 it suffices to study its first order derivatives. Bounds on these follow from Proposition 2.6.

3.9. Lemma. Suppose \( U', V \subset \mathbb{R}^{m-1} \), \( \kappa : U' \times [0, r_C) \to V \times [0, r_C) \) is a normal boundary chart centered at \( p \in V \) on the Riemannian manifold \( V \times [0, r_C) \) with metric \( g_{ij} \) \( (g_{im} = \delta_{im} = g_{mi}) \). Then the derivatives of \( \kappa \) and its inverse up to order \( k \) are bounded in terms of \( g_{ij}, g^{ij} \) and their derivatives up to order \( k + 1 \).

Proof. \( \kappa(q, s) = \varphi_1(s \cdot \partial_m, \varphi_2(q, p, 1), 0, 1) \), where \( \varphi_1 \) is the flow of the differential equation for the geodesics in \( V \times [0, r_C) \) \( (\varphi_1(v, p, \tau, 0) = (p, \tau), \varphi_1'(v, (p, \tau), 0) = v) \), and \( \varphi_2 \) is the flow of the differential equation for geodesics on \( V \). Hence \( \kappa \) is the composition of two flows to which Corollary 3.7, and then Lemma 2.17 and Proposition 2.6 applies exactly as in the previous lemma.

We prove Proposition 3.3 using these Lemmas as follows: By Theorem 2.5 we have bounds for \( g_{ij} \) and their derivatives up to order \( k + 1 \) in normal coordinates. Write
\[ \kappa_1^{-1} \circ \kappa_2 = (\kappa_1^{-1} \circ \kappa_0) \circ (\kappa_0^{-1} \circ \kappa_2). \]
with \( \kappa_0 \) either being an exponential map or a normal boundary map with suitable range, respectively. If we use \( \kappa_1 \) or \( \kappa_2 \) to pull back the given Riemannian metric to the domain of the charts, \( \kappa_0^{-1} \circ \kappa_2 \) and \( \kappa_1^{-1} \circ \kappa_0 \) each fulfill exactly the assumptions of one of the two lemmas. The conclusion of these lemmas is then true for there composition, as well, and the Proposition follows.
3.10. Corollary. To check condition (B1) in Definition 2.4 it suffices to do this for an atlas of such charts.

References

[1] Atiyah, M., Bott, R., and Patodi, V.: “On the heat equation and the index theorem”, Inventiones Mathematicae 19, 279–330 (1973)

[2] Atiyah, M., Bott, R., and Patodi, V.: “Errata to the paper: On the heat equation and the index theorem”, Inventiones Mathematicae 28, 277–280 (1975)

[3] Cheeger, J., Gromov, M., and Taylor, M.: “Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds”, Journal of Differential Geometry 17, 15–53 (1982)

[4] Eichhorn, J.: “The boundedness of connection coefficients and their derivatives”, Math. Nachr. 152, 145–158 (1991)

[5] Gallot, S., Hulin, D., and Lafontaine, J.: “Riemannian geometry”, Universitext, Springer (1987)

[6] Hartman, P.: “Ordinary differential equations”, Wiley (1964)

[7] Schick, T.: “Analysis on $\partial$-manifolds of bounded geometry, Hodge-de Rham isomorphism and $L^2$-index theorem”, Shaker Verlag, Aachen, Dissertation, Johannes Gutenberg-Universität Mainz (1996)

[8] Schick, T.: “Geometry and Analysis on $\partial$-manifolds of bounded geometry”, preprint, Münster (1998)

[9] Shubin, M.A.: “Spectral theory of elliptic operators on non-compact manifolds”, in: Méthodes semi-classiques, Vol.1, vol. 207 of Astérisque, 35–108, Société mathématique de France (1992)

[10] Warner, F.W.: “Extension of the Rauch comparison theorem to submanifolds”, Transactions of the AMS 122, 341–356 (1966)