ON BOHR SETS OF INTEGER-VALUED TRACELESS MATRICES

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Abstract. In this paper we show that any Bohr-zero non-periodic set $B$ of traceless integer-valued matrices, denoted by $\Lambda$, intersects non-trivially the conjugacy class of any matrix from $\Lambda$. As a corollary, we obtain that the family of characteristic polynomials of $B$ contains all characteristic polynomials of matrices from $\Lambda$. The main ingredient used in this paper is an equidistribution result for an $SL_d(\mathbb{Z})$ random walk on a finite-dimensional torus deduced from Bourgain-Furman-Lindenstrauss-Mozes work [7].

1. Introduction

Let us denote by $\text{Mat}_d^0(\mathbb{Z})$, $d \geq 2$, the set of integer-valued $d \times d$ matrices with zero trace, and by $\mathbb{T}^n$, $n \geq 1$, the $n$-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. Let $G$ be a countable abelian group. A set $B \subset G$ is called a non-periodic Bohr set if there exist a homomorphism $\tau : G \to \mathbb{T}^n$, for some $n \geq 1$, with $\tau(G) = \mathbb{T}^n$, and an open set $U \subset \mathbb{T}^n$ satisfying $B = \tau^{-1}(U)$. If the open set $U$ contains the zero element of $\mathbb{T}^n$, then the set $B$ is called a Bohr-zero set. We will also denote by $SL_d(\mathbb{Z})$ the group of $d \times d$ integer-valued matrices of determinant one.

The main result of this paper is the following.

Main Theorem. Let $d \geq 2$, and $B \subset \text{Mat}_d^0(\mathbb{Z})$ be a Bohr-zero non-periodic set. Then for any matrix $C \in \text{Mat}_d^0(\mathbb{Z})$ there exists a matrix $A \in B$ and a matrix $g \in SL_d(\mathbb{Z})$ such that $C = g^{-1}Ag$.

The same result has been also proved independently by Björklund and Bulinski [4]. They use the recent works of Benoist-Quint [2] and [3], instead of the work of Bourgain-Furman-Lindenstrauss-Mozes as the main ingredient in the proof. Use of Bourgain-Furman-Lindenstrauss-Mozes work enables us to prove a strong equidistribution result for the random walk of $SL_d(\mathbb{Z})$ acting on $\text{Mat}_d^0(\mathbb{R})/\text{Mat}_d^0(\mathbb{Z})$ by the conjugation (see Theorem 2.2). This result is interesting by its own, and may have other number-theoretic applications. This should be compared with Theorem 1.15 in [4], where the equidistribution is established for Cesàro average of the random walk (a weaker statement).

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Corollary 1.1. Let \( d \geq 2 \), and \( B \subset \text{Mat}_d^0(\mathbb{Z}) \) be a Bohr-zero non-periodic set. The set of characteristic polynomials of the matrices in \( B \) coincides with the set of all characteristic polynomials of the matrices in \( \text{Mat}_d^0(\mathbb{Z}) \).

The following number-theoretic statement is an immediate implication of Corollary 1.1.

Corollary 1.2. Let \( B \subset \mathbb{Z} \) be a Bohr-zero non-periodic set. Then the set of the discriminants over \( B \) defined by
\[
D := \{ xy - z^2 \mid x, y, z \in B \}
\]
satisfies that \( D = \mathbb{Z} \).

At this point we will define Furstenberg’s system corresponding to a set \( B \) of positive density in a countable abelian group \( G \). It is well known that in any (countable) abelian group we can find sequences of almost invariant finite sets, so called Følner sequences. A sequence of finite sets \( (F_n) \) in \( G \) is called Følner if it is asymptotically \( G \)-invariant, i.e. for every \( g \in G \) we have \( \frac{|F_n \cap (F_n + g)|}{|F_n|} \to 1 \), as \( n \to \infty \). We will say that \( B \) has positive density if upper Banach density of \( B \) is positive, i.e., if
\[
d^*(B) = \sup_{(F_n) \subset G} \limsup_{n \to \infty} \frac{|B \cap F_n|}{|F_n|}
\]
is positive. Furstenberg in his seminal paper [7] constructed a \( G \)-measure-preserving system \( (X, \eta, \sigma) \) and a clopen set \( \tilde{B} \subset X \) such that
\[
\bullet \ d^*(B \cap (B + h)) \geq \eta \left( \tilde{B} \cap \sigma(h) \tilde{B} \right), \text{ for any } h \in G.
\]
\[
\bullet \ \eta(\tilde{B}) = d^*(B).
\]

Moreover, we can further refine the statement of the correspondence principle and require from the system to be ergodic. We will denote Furstenberg’s (ergodic) system corresponding to \( B \) by \( X_B = (X, \eta, \sigma, \tilde{B}) \). Next, we will define the notion of the spectral measure corresponding to a set \( B \) of a countable abelian group \( G \) of positive density and its Furstenberg’s system \( X_B = (X, \eta, \sigma, \tilde{B}) \). Denote by \( 1_{\tilde{B}} \) the indicator function of the set \( \tilde{B} \). Then by Bochner’s spectral theorem [8] there exists a non-negative finite Borel measure \( \nu \) on \( \hat{G} \) (the dual of \( G \)) which satisfies:
\[
\langle 1_{\tilde{B}}, \sigma(h)1_{\tilde{B}} \rangle = \int_{\hat{G}} \chi(h) d\nu(\chi), \text{ for } h \in G.
\]

1A triple \( (X, \eta, \sigma) \) is a \( G \)-measure-preserving system, if \( X \) is a compact metric space on which acts \( G \) by a measurable action denoted by \( \sigma \), \( \eta \) is a Borel probability measure on \( X \), and the action of \( G \) preserves \( \eta \).

2The proof of Correspondence Principle II of Appendix I in [5] will work also for the finite intersections, and therefore, will imply the claim.

3A \( G \)-measure-preserving system is ergodic if any \( G \)-invariant measurable set has measure either zero or one.
The measure $\nu$ will be called the **spectral measure of the set $B$ and its Furstenberg’s system $X_B$**, and we will denote by $\tilde{\nu}(h)$ the right hand side of the last equation. We are at the position to state the main technical claim of the paper (see Section 3 for the proof).

**Theorem 1.1.** Let $d \geq 2$, and let $B \subset \text{Mat}_d^0(\mathbb{Z})$ be a set of positive density such that the spectral measure of $B$ has no atoms at non-trivial characters having finite torsion. Then for every $C \in \text{Mat}_d^0(\mathbb{Z})$ there exist $A \in B - B$ and $g \in SL_d(\mathbb{Z})$ with $C = g^{-1}Ag$.

Theorem 1.1 is an extension of the following result that has been proved in [6] by use of the equidistribution result of Benoist-Quint [1].

**Theorem 1.2.** Let $d \geq 2$, and let $B \subset \text{Mat}_d^0(\mathbb{Z})$ be a set of positive density. Then there exists $k \geq 1$ such that for any matrix $C \in k\text{Mat}_d^0(\mathbb{Z})$ there exists $A \in B - B$ and $g \in SL_d(\mathbb{Z})$ with $C = g^{-1}Ag$.

We would like to finish the introduction by stating the piecewise version of Main Theorem. We recall that a set $B \subset G$ is called **piecewise (non-periodic) Bohr set** if there is a (non-periodic) Bohr set $B_0 \subset G$ and a (thick) set $T \subset G$ of upper Banach density one, i.e., $d^*(T) = 1$ such that $B = B_0 \cap T$.

**Theorem 1.3.** Let $d \geq 2$, and let $B \subset \text{Mat}_d^0(\mathbb{Z})$ be a piecewise Bohr non-periodic set. Then for every $C \in \text{Mat}_d^0(\mathbb{Z})$ there exist $A \in B - B$ and $g \in SL_d(\mathbb{Z})$ with $C = g^{-1}Ag$.

Let us show that Theorem 1.3 implies Main Theorem.

**Proof of Main Theorem.** Let $B \subset \text{Mat}_d^0(\mathbb{Z})$ be a Bohr-zero non-periodic set. Notice that there exists $B_0 \subset \text{Mat}_d^0(\mathbb{Z})$ a Bohr-zero non-periodic set with the property that

$$B_0 - B_0 \subset B.$$ 

Now, we apply Theorem 1.3 for the set $B_0$, and as a conclusion obtain the statement of the theorem. \hfill \qed

**Organisation of the paper.** In Section 2 we establish the consequences of the equidistribution result of Bourgain-Furman-Lindenstrauss-Mozes [7] related to the adjoint action of $SL_d(\mathbb{Z})$ on $\text{Mat}_d^0(\mathbb{R})/\text{Mat}_d^0(\mathbb{Z})$. In Section 3 we prove Theorems 1.1 and 1.3.

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4We assume the existence of some Furstenberg’s system $X_B$ corresponding to the set $B$, such that the associated spectral measure satisfies the requirement of the theorem.
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2. CONSEQUENCES OF THE WORK OF BOURGAIN-FURMAN–LINDENSTRAUSS-MOZES

We start by recalling the property of strong irreducibility of an action of a discrete group. Let $\Gamma$ be a countable group, and $V$ be a finite dimensional real space. We say that an action $\rho: \Gamma \to \text{End}(V)$ is strongly irreducible if for every finite index subgroup $H$ of $\Gamma$, the restriction of the action of $\rho$ to $H$ is irreducible. We also will be using the notion of a proximal element. An operator $T \in \text{End}(V)$ will be called proximal, if there is only one eigenvalue of the largest absolute value, and corresponding to it eigenspace is one-dimensional.

Assume that a countable group $\Gamma$ acts on a compact Borel measure space $(X, \nu)$. Let $\mu$ be a probability measure on $\Gamma$. Then the convolution measure $\mu * \nu$ on $X$ is defined by:

$$
\int_X f d(\mu * \nu) = \int_X \left( \sum_{g \in \Gamma} f(gx)\mu(g) \right) d\nu(x), \text{ for any } f \in C(X).
$$

We will denote the Dirac probability measure at a point $x \in X$ by $\delta_x$. For every $k \geq 2$, we define the probability measure $\mu^*k$ on $\Gamma$ by

$$
\mu^*k(g) = \sum_{g_1 \cdots g_k = g} \mu(g_1)\mu(g_2)\cdots\mu(g_k).
$$

The main ingredient in the proofs of all our main results is the following seminal equidistribution statement due to Bourgain-Furman-Lindenstrauss-Mozes [7].

**Theorem 2.1** (Corollary B in [7]). Let $\Gamma < \text{SL}_d(\mathbb{Z})$ be a subgroup which acts totally irreducibly on $\mathbb{R}^n$, and having a proximal element. Let $\mu$ be a finite generating probability measure on $\Gamma$. Let $x \in \mathbb{T}^n$ be a non-rational point. Then the measures $\mu^*k * \delta_x$ converge in weak*-topology as $k \to \infty$ to Haar measure on $\mathbb{T}^n$.

In this note, the acting group will be $\Gamma = \text{SL}_d(\mathbb{Z})$. The group $\Gamma$ acts by the conjugation on the real vector space $V = \text{Mat}_d^0(\mathbb{R})$ of real valued $d \times d$ matrices with zero trace. So, an element $g \in \text{SL}_d(\mathbb{Z})$ acts on $v \in V$ by $\text{Ad}(g)v = g^{-1}vg$, and such action called the adjoint action of $\text{SL}_d(\mathbb{Z})$. Notice that $V$ is isomorphic to $\mathbb{R}^{d^2-1}$ and for any element $g$ of $\text{SL}_d(\mathbb{Z})$, the endomorphism of $V$ obtained by the conjugate action of $g$ has determinant one. The next claim will allow us to apply Theorem 2.1 in our setting.

**Proposition 2.1.** The adjoint action of $\text{SL}_d(\mathbb{Z})$ on $\text{Mat}_d^0(\mathbb{R})$ is strongly irreducible, and $\text{SL}_d(\mathbb{Z})$ contains an element which acts proximally.
Proof. It is proved in [6] [Corollary 5.4] that the adjoint action of $SL_d(\mathbb{Z})$ on $Mat^0_d(\mathbb{R})$ is strongly irreducible. Therefore, it is remained to prove that there is at least one element of $SL_d(\mathbb{Z})$ which acts on $Mat^0_d(\mathbb{R})$ proximally. Proposition 2.2 below finishes the proof of the statement.

**Proposition 2.2.** The matrix

$B_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

acts (by conjugation) on $Mat^0_2(\mathbb{R})$ proximally. For $d \geq 3$, the matrix

$B_d = \begin{bmatrix} 1 & -1 & 0_{2 \times (d-2)} \\ -1 & 2 & 0_{(d-2) \times (d-2)} \\ 0_{(d-2) \times (d-2)} & Id_{(d-2) \times (d-2)} \end{bmatrix}$

acts proximally on $Mat^0_d(\mathbb{R})$.

**Proof.** It is straightforward to check that the operator $B_2 : Mat^0_2(\mathbb{R}) \to Mat^0_2(\mathbb{R})$ can be written in the matrix form as

$\begin{bmatrix} 3 & -2 & 1 \\ -4 & 4 & -1 \\ 2 & -1 & 1 \end{bmatrix}$.

The characteristic polynomial of this operator is $\chi_{B_2}(\lambda) = (1 - \lambda)(\lambda^2 - 7\lambda + 1)$. Since all eigenvalues are distinct by their absolute value, it follows that the operator acts proximally.

In the case $d \geq 3$, notice that the action of $B_d$ on $Mat^0_d(\mathbb{R})$ is decomposed into 4 orthogonal spaces. The actions on the $2 \times 2$ upper left corner, $2 \times (d-2)$ upper right corner, $(d-2) \times 2$ bottom left corner, and the identity action on the bottom right $(d-2) \times (d-2)$ corner. Correspondingly, the dimensions of the spaces are $4, 2 \cdot (d-2), (d-2) : 2,$ and $(d-2)^2 - 1$.

The 4-dimensional left upper corner part can be written in the matrix form as

\[ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \rightarrow [x, y, z]^t \]

5We use the identification between $Mat^0_2(\mathbb{R})$ and $\mathbb{R}^3$, by

6We identify the vector space $Mat^0_2(\mathbb{R})$ with $\mathbb{R}^{d^2-1}$ by omitting the $(d,d)$-entry of matrices in $Mat^0_2(\mathbb{R})$

7We choose the standard basis for $Mat_2(\mathbb{R})$:

\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]
\[
\begin{bmatrix}
2 & -2 & 1 & -1 \\
-2 & 4 & -1 & 2 \\
1 & -1 & 1 & -1 \\
-1 & 2 & -1 & 2
\end{bmatrix}
\]

Its characteristic polynomial is \((\lambda - 1)^2(\lambda^2 - 7\lambda + 1)\). Therefore there is a unique highest eigenvalue by the absolute value equal to \(\frac{7+3\sqrt{5}}{2}\), and it has multiplicity one.

The operator \(B_d\) acts on the upper right corner in the following way:

\[
\begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
\vdots & \vdots \\
x_{d-2} & y_{d-2}
\end{bmatrix}^t \rightarrow \begin{bmatrix}
2x_1 + y_1 & x_1 + y_1 \\
2x_2 + y_2 & x_2 + y_2 \\
\vdots & \vdots \\
2x_{d-2} + y_{d-2} & x_{d-2} + y_{d-2}
\end{bmatrix}^t
\]

It is clear that it has two eigenvalues with multiplicity \(d - 2\). These eigenvalues correspond to the eigenvalues of the matrix

\[
C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
\]

These eigenvalues are the roots of the characteristic polynomial of the matrix \(C\) which are \(\frac{3 \pm \sqrt{5}}{2}\).

The operator \(B_d\) acts on the bottom left corner in the following way:

\[
\begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
\vdots & \vdots \\
x_{d-2} & y_{d-2}
\end{bmatrix} \rightarrow \begin{bmatrix}
x_1 - y_1 & -x_1 + 2y_1 \\
x_2 - y_2 & -x_2 + 2y_2 \\
\vdots & \vdots \\
x_{d-2} - y_{d-2} & -x_{d-2} + 2y_{d-2}
\end{bmatrix}
\]

Therefore it has two eigenvalues of the matrix \(C^{-1}\) each one having multiplicity \(d - 2\). It is immediate to check that \(C^{-1}\) has the same characteristic polynomial as \(C\), therefore the eigenvalues of the operator \(B_d\) acting on the bottom left corner are \(\frac{3 \pm \sqrt{5}}{2}\), each one having multiplicity \(d - 2\).

As the conclusion of the previous considerations we find the the eigenvalues of the operator \(B_d\) are \(\frac{7+3\sqrt{5}}{2}, \frac{7-3\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\) with corresponding algebraic multiplicities equal to \(1, 2(d - 2), [(d - 2)^2 - 1] + 2, 1, 2(d - 2)\). This implies that \(B_d\) acts proximally on \(\text{Mat}_d^0(\mathbb{R})\).

Let us denote by \(A_d = V/\Lambda\), where \(\Lambda = \text{Mat}_d^0(\mathbb{Z})\). Notice that \(A_d\) is isomorphic to \(\mathbb{T}^{d^2-1}\), and it is the dual group of \(\Lambda\). The adjoint action of \(SL_d(\mathbb{Z})\) leaves \(\Lambda\) invariant. Therefore, \(SL_d(\mathbb{Z})\) also acts on \(A_d\). Proposition \(2.1\) implies by Theorem \(2.1\) the following statement.

**Proposition 2.3.** Let \(\mu\) be a probability measure on \(SL_d(\mathbb{Z})\) with finite generating support. Let \(x \in A_d\) be a non-rational point. Then the measures \(\mu^k * \delta_x\) converge as \(k \rightarrow \infty\) in the weak* topology to the normalised Haar measure on \(A_d\).
We will be using Proposition 2.3 to prove the following:

**Theorem 2.2.** Let $\mu$ be a probability measure on $\text{SL}_d(\mathbb{Z})$ with finite generating support. Let $\nu$ be a probability measure on $A_d$ with no atoms at rational points. Then the measures $\mu^k * \nu$ converge as $k \to \infty$ in the weak*-topology to the normalised Haar measure on $A_d$.

**Proof.** Let $\nu$ be a probability measure on $A_d$ with no atoms at rational points, and let $\mu$ be a probability measure on $\Gamma = \text{SL}_d(\mathbb{Z})$ with a finite generating support. By Proposition 2.3 for every non-rational $x \in A_d$ the measures $\mu^k * \delta_x$ converge in weak*-topology as $k \to \infty$ to the Haar measure on $A_d$. Let $f$ be a continuous function on $A_d$. Denote by $A'_d$ the set of all non-rational points of $A_d$. Notice that $\nu(A'_d) = 1$. Then for every $x \in A'_d$ we have that $f_k(x) := \int_{A_d} f d(\mu^k * \delta_x) \to \int f$. We have to show that

$$\int_{A_d} f d(\mu^k * \nu) \to \int f.$$

By Egorov’s theorem, for every $\varepsilon > 0$, there exists $X' \subset A'_d$ with $\nu(X') \geq 1 - \varepsilon$ and $K(\varepsilon)$ with the property that for every $x \in X'$ and every $k \geq K(\varepsilon)$ we have

$$\left| f_k(x) - \int f \right| < \varepsilon.$$

Notice that

$$\int_{A_d} f d(\mu^k * \nu) = \sum_{g \in \Gamma} \int f(gx) \mu^k(g) d\nu(x) = \int \left( \sum_{g \in \Gamma} f(gx) \mu^k(g) \right) d\nu(x)$$

$$= \int \left( \int f d(\mu^k * \delta_x) \right) d\nu(x) = \int f_k(x) d\nu(x).$$

Let $\delta > 0$. Denote by $M = \|f\|_{\infty}$, and take $\varepsilon > 0$ so small that $\varepsilon M < 2\delta$, and $\varepsilon < \delta$. Then we have

$$\left| \int_{A_d} f d(\mu^k * \nu) - \int f \right| = \left| \int \left( f_k(x) - \int f \right) d\nu(x) \right| < (1 - \varepsilon)\varepsilon + \varepsilon M < 3\delta,$$

for $k \geq K(\varepsilon)$. Since $\delta$ can be chosen arbitrary small, we have shown that

$$\int_{A_d} f d(\mu^k * \nu) \to \int f.$$

This finishes the proof because the function $f$ was an arbitrary continuous function on $A_d$. 

\[\Box\]

### 3. Proofs of Theorems 1.1 and 1.3

First, we will prove a very useful statement.
Lemma 3.1. Let $G$ be a countable abelian group, and let $(F_n) \subset G$ be a Følner sequence. Let $\chi \in \hat{G}$ be a non-trivial character. Then we have

$$\frac{1}{|F_n|} \sum_{g \in F_n} \chi(g) \to 0, \text{ as } n \to \infty.$$ 

Proof. Let $(n_k)$ be a sequence along which the limit of

$$\frac{1}{|F_{n_k}|} \sum_{g \in F_{n_k}} \chi(g)$$

exists. Let us denote the limit by $F(\chi)$. For every $h \in G$ we have:

$$\frac{1}{|F_{n_k}|} \sum_{g \in F_{n_k}} \chi(g + h) = \chi(h) \frac{1}{|F_{n_k}|} \sum_{g \in F_{n_k}} \chi(g) \to \chi(h) F(\chi).$$

On the other hand, by use of Følner property of $(F_{n_k})$ we obtain:

$$\frac{1}{|F_{n_k}|} \sum_{g \in F_{n_k}} \chi(g + h) = \frac{1}{|F_{n_k}|} \sum_{g \in F_{n_k} + h} \chi(g) \to F(\chi).$$

Therefore, we have for every $h \in G$:

$$F(\chi) = \chi(h) F(\chi).$$

Since $\chi \not\equiv 1$, we get that $F(\chi) = 0$. The statement of the lemma follows, since our conclusion is independent of the subsequence $(n_k)$.

Recall that $\Lambda = \text{Mat}_d^0(\mathbb{Z})$, and we denote by $\Gamma = \text{SL}_d(\mathbb{Z})$. We make the identification of the dual space of $\Lambda$ with the torus $A_d = \text{Mat}_d^0(\mathbb{R})/\text{Mat}_d^0(\mathbb{Z}) \simeq \mathbb{T}^{d^2 - 1}$ by corresponding for every $x \in A_d$ the character $\chi_x$ on $\Lambda$ given by:

$$\chi_x(h) = \exp (2\pi i \langle x, h \rangle), \text{ for } h \in \Lambda.$$ 

Notice that the trivial character on $\Lambda$ corresponds to the zero element $0_{A_d}$ of $A_d$, and characters having finite torsion correspond to the rational points of $A_d$.

Proof of Theorem 1.1. Let $B \subset \Lambda$ be a set of positive density with Furstenberg’s system $X_B = (X, \eta, \sigma, B)$ and such that the spectral measure of $B$ has no atoms at non-trivial characters. Denote by $\nu$ the spectral measure of $B$, i.e., for every $h \in \Lambda$ we have

$$\langle 1_B, \sigma(h) 1_B \rangle = \int_{A_d} \exp(2\pi i \langle x, h \rangle) d\nu(x).$$

By the assumptions of the theorem, $\nu$ has no atoms at the rational points. We will show that for every $h \in \Lambda$ there exists $g \in \Gamma$ such that

$$\tilde{\nu}(g^{-1}hg) = \langle 1_B, \sigma(g^{-1}hg)1_B \rangle > 0.$$
This will imply the claim of the theorem by the first property of Furstenberg’s system $X_B$. Assume, that on the contrary, there exists $h \in \Lambda$ such that for all $g \in \Gamma$ we have

\begin{equation}
\hat{\nu}(g^{-1}hg) = 0.
\end{equation}

Since for $h = 0 \Lambda$ we have $g^{-1}0_{\Lambda}g = 0_{\Lambda}$, and $\nu(0_{\Lambda}) = \eta(\tilde{B}) > 0$, we conclude that there exists a non-zero $h \in \Lambda$ such that (2) holds for all $g \in \Gamma$.

For any Følner sequence $(F_n)$ in $\Lambda$:

\begin{equation}
\frac{1}{|F_n|} \sum_{h \in F_n} \langle 1_B, \sigma(h)1_B \rangle = \int_{A_d} \frac{1}{|F_n|} \sum_{h \in F_n} \exp \left(2\pi i \langle x, h \rangle \right) d\nu(x) \to \nu(\{o_{A_d}\}), \text{ as } N \to \infty.
\end{equation}

In the last transition, we have used Lebesgue’s dominated convergence theorem and Lemma 3.1. By ergodicity of Furstenberg’s system and von-Neumann’s ergodic theorem it follows that the left hand side of (3) satisfies

\begin{equation}
\frac{1}{|F_n|} \sum_{h \in F_n} \langle 1_B, \sigma(h)1_B \rangle \to \eta(\tilde{B})^2, \text{ as } n \to \infty.
\end{equation}

Altogether it implies that

\begin{equation}
\nu(\{o_{A_d}\}) = \eta(\tilde{B})^2 > 0.
\end{equation}

Let $\mu$ be a probability measure on $\Gamma$ having a finite generating support. By Proposition 2.2 the measures $\mu^*k \ast \nu$ converge as $k \to \infty$ in weak$^*$-topology to

\begin{equation}
\eta(\tilde{B}) \left(1 - \eta(\tilde{B}) \right) m_{A_d} + \eta(\tilde{B})^2 \delta_{o_{A_d}},
\end{equation}

where $m_{A_d}$ stands for the normalised Haar measure on $A_d$. Notice that $\Gamma$ also acts on $A_d$ by $g \cdot x = (g^*)^{-1}xg$, for $g \in \Gamma$. The action of $\Gamma$ on $A_d$ and the adjoint action of $\Gamma$ on $\Lambda$ are related by the following:

\begin{equation}
\langle (g \cdot x), h \rangle = \langle x, Ad(g)h \rangle, \text{ for every } g \in \Gamma, h \in \Lambda, x \in A_d.
\end{equation}

Notice

\begin{equation}
\mu^*k \ast \nu(h) = \int_{A_d} \exp \left(2\pi i \langle x, h \rangle \right) d \left(\mu^*k \ast \nu \right)(x) = \int_{A_d} \left(\sum_{g \in \Gamma} \exp \left(2\pi i \langle (g \cdot x), h \rangle \right) \mu^*k(g) \right) d\nu(x) = \sum_{g \in \Gamma} \left(\int_{A_d} \exp \left(2\pi i \langle x, (g^{-1}hg) \rangle \right) d\nu(x) \right) \mu^*k(g) = \sum_{g \in \Gamma} \hat{\nu}(g^{-1}hg) \mu^*k(g).
\end{equation}
Recall, we assumed that there exists a non-zero $h \in \Lambda$ such that $\hat{\nu}(g^{-1}hg) = 0$, for all $g \in \Gamma$. Therefore, we have $\mu^k * \nu(h) = 0$, for all $k \geq 1$. On other hand, since $\hat{m}_{\Lambda}(h) = 0$, and $\hat{\delta}_{\Lambda}(h) = 1$, we have:

$$\mu^k * \nu(h) \to \eta(\tilde{B})^2 > 0,$$

as $k \to \infty$.

Thus, we have a contradiction. This finishes the proof of the theorem. □

Proof of Theorem [1.3] We will use the following statement which will be proved below.

**Proposition 3.1.** Let $B \subset \Lambda$ be a non-periodic piecewise Bohr set corresponding to a Jordan measurable⁸ open set in a finite-dimensional torus. Then there exists a spectral measure associated with $B$ that does not have atoms at non-zero rational points of $A_d$.

Let $B \subset \Lambda$ be a piecewise non-periodic Bohr set given by $B = \tau^{-1}(U) \cap T$, where $\tau : \Lambda \to \mathbb{T}^n$ is a homomorphism with a dense image, $U \subset \mathbb{T}^n$ is an open set, and $T \subset \Lambda$ is a set with $d^*(T) = 1$. Then $U$ contains an open ball $U_o$, and $m_{\mathbb{T}^n}(\partial U_o) = 0$, where $m_{\mathbb{T}^n}$ denotes the Haar normalised measure on $\mathbb{T}^n$. Denote by $B' = \tau^{-1}(U_o) \cap T \subset B$. The statement of Theorem [1.3] for the non-periodic piecewise Bohr set $B'$ follows from Proposition 3.1 and Theorem [1.1]. The latter implies the statement of the theorem for the set $B$. □

Proof of Proposition 3.1 We are given a piecewise Bohr non-periodic set $B \subset \Lambda$ corresponding to a Jordan measurable open set in a finite dimensional torus. This means that $B = B_o \cap T$, where $T \subset \Lambda$ with $d^*(T) = 1$, and $B_o = \tau^{-1}(U_o) \subset \Lambda$, where $\tau : \Lambda \to \mathbb{T}^n$, for some $n \geq 1$, is a homomorphism with a dense image, and $U_o \subset \mathbb{T}^m$ is an open Jordan measurable set. We will construct an ergodic Furstenberg’s $\Lambda$-system $X_B = (X, \eta, \sigma, \tilde{B})$ corresponding to the set $B$, and will show that the spectral measure of the function $1_{\tilde{B}}$ has no atoms at the rational non-zero points of $A_d$.

Let $X = \mathbb{T}^n$, $\eta$ be the Haar normalised measure on $X$, $\sigma_h(x) := x + \tau(h)$ for $x \in X, h \in \Lambda$, and $\tilde{B} = U_o$. We will denote by $X_B := (X, \eta, \sigma, \tilde{B})$. It remains to show that

- For every $h \in \Lambda$ we have $d^*(B \cap (B + h)) \geq \eta(\tilde{B} \cap \sigma_h(\tilde{B}))$.

- $\eta(\tilde{B}) = d^*(B)$.

- The spectral measure of $1_{\tilde{B}}$ has no atoms at non-zero rational points of $A_d$.

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⁸A set $A$ in a topological space $X$ equipped with a measure $m_X$ is *Jordan measurable* if $m_X(\partial A) = 0$, where $\partial A = \overline{A} \setminus \overset{\circ}{A}$. 

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The first two properties will follow from the statement that for every \( h \in \Lambda \):
\[
d^\nu(B \cap (B + h)) = \eta(B \cap \sigma_h(B)).
\]
First, notice that for every \( h \in \Lambda \) the set \( U_o \cap \sigma_h(U_o) \) is Jordan measurable. The uniqueness of \( \nu \)-invariant probability measure on \( X \) implies the unique ergodicity of \( X_B \). Therefore, for every Følner sequence \( (F_k) \) in \( \Lambda \) and any \( h \in \Lambda \) we have
\[
\lim_{k \to \infty} \frac{1}{|F_k|} \sum_{g \in F_k} 1_{U_o \cap \sigma_h(U_o)}(\sigma_g(0x)) \int_X 1_{U_o \cap \sigma_h(U_o)}(x) d\eta(x) = \eta(B \cap \sigma_h(B)),
\]
as \( k \to \infty \). Since the left hand side of the last equation is equal to
\[
\frac{|B_o \cap (B_o + h) \cap F_k|}{|F_k|} \to \eta(B \cap \sigma_h(B)), \text{ as } k \to \infty.
\]
Since \( B \subset B_0 \), the latter implies that for every \( h \in \Lambda \) we have
\[
\eta(B \cap \sigma_h(B)) \geq d^\nu(B \cap (B + h)).
\]
On the other hand, for any Følner sequence \( (F_k) \) which lies inside the thick set \( T \) by identity (1) we have for every \( h \in \Lambda \):
\[
\frac{|B \cap (B_0 + h) \cap F_k|}{|F_k|} \to \eta(B \cap \sigma_h(B)), \text{ as } k \to \infty.
\]
By use of Følner property of the sequence \( (F_k) \), we have that
\[
\left| \frac{|B \cap (B_0 + h) \cap F_k|}{|F_k|} - \frac{|(B - h) \cap B_0 \cap F_k|}{|F_k|} \right| \to 0, \text{ as } k \to \infty.
\]
But, since \( F_k \subset T \) it follows that for every \( k \geq 1 \) we have:
\[
\frac{|(B - h) \cap B_0 \cap F_k|}{|F_k|} = \frac{|(B - h) \cap B \cap F_k|}{|F_k|}.
\]
Finally, since
\[
\frac{|(B - h) \cap B \cap F_k|}{|F_k|} = \frac{|B \cap (B + h) \cap (F_k + h)|}{|F_k|},
\]
and Følner property implies that
\[
\left| \frac{|B \cap (B + h) \cap (F_k + h)|}{|F_k|} - \frac{|B \cap (B + h) \cap F_k|}{|F_k|} \right|, \text{ as } k \to \infty,
\]
we obtain that
\[
\frac{|B \cap (B + h) \cap F_k|}{|F_k|} \to \eta(B \cap \sigma_h(B)), \text{ as } k \to \infty.
\]
This establishes that for every \( h \in \Lambda \):
\[
d^\nu(B \cap (B + h)) \geq \eta(B \cap \sigma_h(B)).
Together with the identity \((5)\) this implies that for every \(h \in \Lambda\) we have
\[
d^* (B \cap (B + h)) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})).
\]

It remains to prove that the spectral measure corresponding to \(1_{\tilde{B}}\) and the system \(X_B\) has no atoms at non-zero rational points of \(A_d\). We will be abusing the notation and will also use \(T\) to denote the Koopman operator on \(L^2(X)\) corresponding to \(\sigma\). Let us list two important properties of the system \(X_B\):  

1. \(X_B\) is \textit{totally ergodic}, i.e., every subgroup \(H < \Lambda\) of a finite index acts ergodically on \(X_B\).  

2. For every \(f \in L^2(X)\) there exists the spectral measure \(\mu_f\) of \(f\) on \(A_d\) satisfying:
\[
\hat{\mu_f}(h) := \int_{A_d} \exp(2\pi h \cdot x) d\mu_f(x) = \langle f, T_h f \rangle.
\]

Moreover, if \(f \geq 0\), then \(\mu_f\) is non-negative.

The first property follows from Lemma 3.3, while the second property is Bochner’s spectral theorem, see [8]. To prove Lemma 3.3 we will need the following result.

\textbf{Lemma 3.2.} Let \(H < \Lambda\) be a subgroup of a finite index. Then for every point \(x \in X\), the \(H\)-orbit of \(x\), i.e., \(\{\sigma_h(x) \mid h \in H\}\), is dense in \(X\).

\textit{Proof.} If \(\tau(H) \neq X\), then since \(H < \Lambda\) has a finite index, it follows that finitely many translates of \(\tau(H)\) cover \(X\). But \(X\) is connected, and we get a contradiction. \hfill \Box

\textbf{Lemma 3.3.} Let \(H < \Lambda\) be a subgroup of a finite index. The restriction of the \(\Lambda\)-action of \(X\) to \(H\) is uniquely ergodic.

\textit{Proof.} It follows from Lemma 3.2 that any \(H\)-invariant Borel probability measure on \(X\) is also \(X\)-invariant. The uniqueness of the Haar normalised measure on \(X\) implies the statement of the lemma. \hfill \Box

Let \(f \in L^2(X)\), then by the ergodicity of \(X_B\) (property (1)) it follows that for any \(\text{Følner} \) sequence \((F_k)_{k \geq 1}\) of finite sets in \(\Lambda\) we have
\[
\frac{1}{|F_k|} \sum_{h \in F_k} \langle f, T_h f \rangle \to |\langle f, 1 \rangle|^2, \text{ as } k \to \infty.
\]

On the other hand, it follows from Bochner’s spectral theorem, Lebesgue’s dominated convergence theorem and and Lemma 3.1 that
\[
\frac{1}{|F_k|} \sum_{h \in F_k} \langle f, T_h f \rangle \to \mu_f(\{o_{A_d}\}),
\]
which implies that
\begin{equation}
|\langle f, 1 \rangle|^2 = \mu_f(A_d).
\end{equation}

Let \( x_0 \in A_d \) be a non-zero rational point with the least common denominator equal to \( q \). Then the stabiliser of \( x_0 \) in \( \Lambda \) is \( H_{x_0} = q\Lambda \). Using the ergodicity of \( H_{x_0} \) action on \( X_B \) (property (1)), we obtain
\[
\frac{1}{|F_k|} \sum_{h \in F_k} \langle f, T_{qh} f \rangle \to |\langle f, 1 \rangle|^2, \quad \text{as } k \to \infty.
\]

On the other hand, we have
\[
\frac{1}{|F_k|} \sum_{h \in F_k} \exp(2\pi i \langle h, qx \rangle) \to \begin{cases} 1, & qx = o_{A_d} \\ 0, & qx \neq o_{A_d}. \end{cases}
\]

Therefore, by Lebesgue’s dominated convergence theorem we obtain
\begin{equation}
|\langle f, 1 \rangle|^2 = \sum_{qx = o_{A_d}} \mu_f(\{x\}).
\end{equation}

If we know in addition that \( f \geq 0 \), then by property (2), the spectral measure \( \mu_f \) is non-negative. Therefore, by use of equations (6) and (7) we get that for all non-zero points \( x \in A_d \) with \( qx = o_{A_d} \) we have
\[
\mu_f(\{x\}) = 0.
\]

In particular, we have that \( \mu_f(\{x_0\}) = 0 \). This finishes the proof of Proposition 2.2 if we choose \( f = 1_{\tilde{B}} \).

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