A Robinson characterization of finite $P\sigma T$-groups

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and let $G$ be a finite group. Then $G$ is said to be $\sigma$-full if $G$ has a Hall $\sigma_i$-subgroup for all $i$. A subgroup $A$ of $G$ is said to be $\sigma$-permutable in $G$ provided $G$ is $\sigma$-full and $A$ permutes with all Hall $\sigma_i$-subgroups $H$ of $G$ (that is, $AH = HA$) for all $i$.

We obtain a characterization of finite groups $G$ in which $\sigma$-permutability is a transitive relation in $G$, that is, if $K$ is a $\sigma$-permutable subgroup of $H$ and $H$ is a $\sigma$-permutable subgroup of $G$, then $K$ is a $\sigma$-permutable subgroup of $G$.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, p_2, \ldots\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$.

If $1 \in \mathfrak{F}$ is a class of groups, then $G^\mathfrak{F}$ denotes the $\mathfrak{F}$-residual of $G$, that is, intersection of all normal subgroups $N$ of $G$ with $G/N \in \mathfrak{F}$; $G_\mathfrak{F}$ denotes the $\mathfrak{F}$-radical of $G$, that is, the product of all normal subgroups $N$ of $G$ with $N \in \mathfrak{F}$.

In what follows, $\sigma$ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $G$ is said to be $\sigma$-full [1, 2] if $G$ has a Hall $\sigma_i$-subgroup for all $i$.

Definition 1.1. We say that a subgroup $A$ of $G$ is $\sigma$-permutable in $G$ [3] provided $G$ is $\sigma$-full and $H$ permutes with all Hall $\sigma_i$-subgroups $H$ of $G$ (that is, $AH = HA$) for all $i$.

Remark 1.2. A set $\mathcal{H}$ of subgroups of $G$ is a complete Hall $\sigma$-set of $G$ [1, 2] if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i$. By Proposition 3.1 in [1], a subgroup $A$ of $G$ is $\sigma$-permutable in $G$ if and only if $G$ possesses at least one complete Hall $\sigma$-set $\mathcal{H}$ such that $AL^x = L^xA$ for all $L \in \mathcal{H}$ and all $x \in G$.

Keywords: finite group, a Robinson $\sigma$-complex of a group, $\sigma$-permutable subgroup, $\sigma$-soluble group, $\sigma$-supersoluble group, a $\sigma$-SC-group.

Mathematics Subject Classification (2010): 20D10, 20D15, 20D30
Recall that $G$ is said to be: $\sigma$-primary \([3]\) if $G$ is a $\sigma_i$-group for some $i$, $\sigma$-decomposable (Shemetkov [5]) or $\sigma$-nilpotent (Guo and Skiba [6]) if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$.

The usefulness of $\sigma$-permutable subgroups is connected mostly with the following their property.

**Theorem A.** (See Theorem B in [3]). If $A$ is a $\sigma$-permutable subgroup of $G$, then $A^G/A_G$ is $\sigma$-nilpotent.

**Example 1.3.** (i) In the classical case, when $\sigma = \sigma^0 = \{\{2\}, \{3\}, \ldots\}$, the subgroup $A$ of $G$ is $\sigma^0$-permutable in $G$ if and only if $A$ permutes with all Sylow subgroups of $G$. Note that a $\sigma^0$-permutable subgroup is also called $S$-permutable \([7]\). Note also that for every $S$-permutable subgroup $A$ of $G$ the quotient $A^G/A_G$ is nilpotent (Kegel, Deskins) by Theorem A.

(ii) In the other classical case, when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, a subgroup $A$ of $G$ is $\sigma^\pi$-permutable in $G$ if and only if $G$ has a Hall $\pi$-subgroup and a Hall $\pi'$-subgroup and $A$ permutes with all Hall $\pi$-subgroups and with all Hall $\pi'$-subgroups of $G$. For every $\sigma^\pi$-permutable subgroup $A$ of $G$ the quotient $A^G/A_G$ is $\pi$-decomposable, that is, $A^G/A_G = O_\pi(A^G/A_G) \times O_{\pi'}(A^G/A_G)$ by Theorem A.

(iii) In fact, in the theory of $\pi$-soluble groups ($\pi = \{p_1, \ldots, p_n\}$) we deal with the partition $\sigma = \sigma^{0\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$. The subgroup $A$ of $G$ is $\sigma^{0\pi}$-permutable in $G$ if and only if $G$ has a Hall $\pi'$-subgroup and $A$ permutes with all Hall $\pi'$-subgroups and with all Sylow $p$-subgroups of $G$ for all $p \in \pi$. For every $\sigma^{0\pi}$-permutable subgroup $A$ of $G$ the quotient $A^G/A_G$ is $\pi$-nilpotent, that is, $A^G/A_G = O_\pi(F(A^G/A_G)) \times O_{\pi'}(A^G/A_G)$ by Theorem A.

We say, following [3], that $G$ is a $P\sigma T$-group if $\sigma$-permutability is a transitive relation in $G$, that is, if $K$ is a $\sigma$-permutable subgroup of $H$ and $H$ is a $\sigma$-permutable subgroup of $G$, then $K$ is a $\sigma$-permutable subgroup of $G$. In the case when $\sigma = \sigma^0$, a $P\sigma T$-group is also called a $PST$-group \([7]\).

Note that if $G = (Q_8 \times C_3) \wr (C_7 \times C_3)$ (see [8] p. 50), where $Q_8 \times C_3 = SL(2, 3)$ and $C_7 \times C_3$ is a non-abelian group of order 21, then $G$ is not a $PST$-group but $G$ is a $P\sigma T$-group, where $\sigma = \{\{2, 3\}, \{2, 3\}'\}$

The description of $PST$-groups was first obtained by Agrawal [9], for the soluble case, and by Robinson in [10], for the general case. In the further publications, authors (see, for example, the recent papers [11]–[21] and Chapter 2 in [7]) have found out and described many other interesting characterizations of $PST$-groups.

In the case when $G$ is $\sigma$-soluble (that is, every chief factor of $G$ is $\sigma$-primary) the description of $P\sigma T$-groups was obtained in the paper [22] on the base of the results and methods in [3] [23] [24] [25].

**Theorem B** (See Theorem A in [22]). If $G$ is a $\sigma$-soluble $P\sigma T$-group and $D = G^{3\sigma}$ is the $\sigma$-nilpotent residual of $G$, then the following conditions hold:

(i) $G = D \rtimes M$, where $D$ is an abelian Hall subgroup of $G$ of odd order, $M$ is $\sigma$-nilpotent and every element of $G$ induces a power automorphism in $D$;

(ii) $O_{\sigma_i}(D)$ has a normal complement in a Hall $\sigma_i$-subgroup of $G$ for all $i$. 

Conversely, if Conditions (i) and (ii) hold for some subgroups $D$ and $M$ of $G$, then $G$ is a $P\sigma T$-group.

Before continuing, we give some further definitions.

**Definition 1.4.** We say that $G$ is:

(i) $\sigma$-supersoluble if every chief factor of $G$ below $G^{\sigma_0}$ is cyclic;

(ii) a $\sigma$-$SC$-group if every chief factor of $G$ below $G^{\sigma_0}$ is simple.

**Example 1.5.** (i) $G$ is supersoluble if and only if $G$ is $\sigma$-supersoluble where $\sigma = \sigma_0$ (see Example 1.3(i)).

(ii) The group $G$ is called an $SC$-group (Robinson [10]) or a $c$-supersoluble group (Vedernikov [26]) if every chief factor of $G$ is a simple group. Note that $G$ is an $SC$-group if and only if $G$ is $\sigma$-$SC$-group where $\sigma = \sigma_0$.

(iii) Let $G = A_5 \times B$, where $A_5$ is the alternating group of degree 5 and $B = C_{29} \rtimes C_7$ is a non-abelian group of order 203, and let $\sigma = \{\{7\}, \{29\}, \{2, 3, 5\}, \{2, 3, 5, 7, 29\}\}$. Then $G^{\sigma_0} = C_{29}$, so $G$ is a $\sigma$-supersoluble group but it is neither soluble nor $\sigma$-nilpotent.

(iv) Let $G = SL(2, 7) \times A_7 \times A_5 \times B$, where $B = C_{43} \rtimes C_7$ is a non-abelian group of order 301, and let $\sigma = \{\{2, 3, 5\}, \{7, 43\}, \{2, 3, 5, 7, 43\}\}$. Then $G^{\sigma_0} = SL(2, 7) \times A_7$, so $G$ is a $\sigma$-$SC$-group but it is not a $\sigma$-supersoluble group.

In what follows, $\mathfrak{U}_0$ is the class of all $\sigma$-supersoluble groups; $\mathfrak{U}_\sigma$ is the class of all $\sigma$-$SC$-groups.

We say that $G$ is $\sigma$-perfect if $G^{\sigma_0} = G$, that is, $O^\sigma_\sigma(G) = G$ for all $i$.

From Theorem B it follows that every $\sigma$-soluble $P\sigma T$-group is $\sigma$-supersoluble. Our first observation shows that in general case every $P\sigma T$-group is a $\sigma$-$SC$-group.

**Proposition A.** Let $G$ be a $P\sigma T$-group and let $D = G^{\sigma_0}$ be the $\sigma$-soluble residual of $G$. Suppose that $G$ possesses a complete Hall $\sigma$-set $\mathfrak{H}$ whose members are $PST$-groups. Then the following conditions hold:

(i) $G$ is a $\sigma$-$SC$-group.

(ii) $D = G^{\sigma_0}$ is $\sigma$-perfect and $G/D$ is a $\sigma$-soluble $P\sigma T$-group.

(iii) $G$ satisfies $N_{\sigma_i}$ for all $i$.

In this proposition we say that $G$ satisfies $N_{\sigma_i}$ if whenever $N$ is a $\sigma$-soluble normal subgroup of $G$, $\sigma_i$-elements of $G$ induce power automorphisms in $O_{\sigma_i}(G/N)$. We say also, following [7, 2.1.18], that $G$ satisfies $N_p$ if whenever $N$ is a soluble normal subgroup of $G$, $p'$-elements of $G$ induce power automorphisms in $O_p(G/N)$.

**Corollary 1.6** (See Proposition 2.1.1 in [7]). Let $G$ be a $PST$-group. Then:

(i) $G$ is an SC-group, and

(ii) $G$ of satisfies $N_p$ for every prime $p$. 

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Definition 1.7. We say that \((D, Z(D); U_1, \ldots, U_k)\) is a Robinson \(\sigma\)-complex (a Robinson complex in the case \(\sigma = \sigma^0\)) of \(G\) if the following fold:

(i) \(D\) is a \(\sigma\)-perfect normal subgroup of \(G\),

(ii) \(D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)\), where \(U_i/Z(D)\) is a non-abelian simple chief factor of \(G\) for all \(i\),

(iii) every chief factor of \(G\) below \(Z(D)\) is cyclic, and

(iv) \(D^0 \leq D\) for every normal subgroup \(D^0\) of \(G\) satisfying Conditions (i), (ii) and (iii).

Example 1.8. Let \(G = SL(2, 7) \times A_7 \times A_5 \times B\) be the group in Example 1.5(iv) and \(\sigma = \{\{2,3,5\}, \{7,43\}, \{2,3,5,7,43\}\}\). Then

\[
(SL(2, 7) \times A_7, Z(SL(2, 7)); SL(2, 7), A_7 Z(SL(2, 7)))
\]

is a Robinson \(\sigma\)-complex of \(G\) and

\[
(SL(2, 7) \times A_7 \times A_5, Z(SL(2, 7)); SL(2, 7), A_7 Z(SL(2, 7)), A_5 Z(SL(2, 7))))
\]

is a Robinson complex of \(G\).

Being based on Theorems A and B and using some ideas in [10, 23, 24, 25], in the given paper we prove the following

Theorem C. Suppose that \(G\) possesses a complete Hall \(\sigma\)-set \(\mathcal{H}\) whose members are PST-groups. Then \(G\) is a \(P\sigma T\)-group if and only if \(G\) has a \(\sigma\)-perfect normal subgroup \(D\) such that:

(i) \(G/D\) is a \(\sigma\)-soluble \(P\sigma T\)-group.

(ii) If \(D \neq 1\), then \(G\) has a Robinson \(\sigma\)-complex of the form \((D, Z(D); U_1, \ldots, U_k)\), and

(iii) If \(\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}\), where \(1 \leq r < k\), then \(G\) and \(G/U_{i_1} \cdots U_{i_r}\) satisfy \(N_{\sigma_i}\) for all \(i\) such that \(\sigma_i \cap \pi(Z(D)) \neq \emptyset\).

Corollary 1.9 (Robinson [10]). A group \(G\) is a PST-group if and only if \(G\) has a perfect normal subgroup \(D\) such that:

(i) \(G/D\) is a soluble PST-group.

(ii) If \(D \neq 1\), then \(G\) has a Robinson complex of the form \((D, Z(D); U_1, \ldots, U_k)\), and

(iii) If \(\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}\), where \(1 \leq r < k\), then \(G\) and \(G/U_{i_1} \cdots U_{i_r}\) satisfy \(N_p\) for all \(p \in \pi(Z(D))\).

The class \(1 \in \mathcal{F}\) is said to be a formation if every homomorphic image of \(G/G\mathcal{F}\) belongs to \(\mathcal{F}\) for every group \(G\), that is, if \(G \in \mathcal{F}\), then also every homomorphic image of \(G\) belongs to \(\mathcal{F}\) and \(G/N \cap R \in \mathcal{F}\) whenever \(G/N \in \mathcal{F}\) and \(G/R \in \mathcal{F}\). The formation \(\mathcal{F}\) is said to (normally) hereditary if \(H \in \mathcal{F}\) whenever \(G \in \mathcal{F}\) and \(H\) is a (normal) subgroup of \(G\).

We prove Proposition A and Theorem C in Section 3. But before, in Section 2, we study properties of \(\sigma\)-supersoluble groups and \(\sigma\)-SC-groups. In particular, we prove the following two results.
Proposition B. For any partition $\sigma$ of $\mathbb{P}$ the following hold:

(i) The class $\mathcal{U}_{c\sigma}$ is a normally hereditary formation.
(ii) The class $\mathcal{U}_{\sigma}$ is a hereditary formation.

Theorem D Let $N = G^{\sigma G}$ and let $D = N^G$ be the soluble residual of $N$. Then $G$ is a $\sigma$-$SC$-group if and only if the following hold:

(i) $D = G^{\sigma G}$, and
(ii) if $D \neq 1$, then $G$ has a Robinson complex of the form $(D, Z(D); U_1, \ldots, U_k)$, where $Z(D) = D_0$ is the soluble radical of $D$.

Corollary 1.10 (Robinson [10]). A group $G$ is an SC-group if and only if $G$ satisfies:

(i) $G/G^G$ is supersoluble.
(ii) If $D = G^G \neq 1$, then $G$ has a Robinson complex of the form $(D, Z(D); U_1, \ldots, U_k)$.

2 Proofs of Proposition B and Theorem B

The following lemma collects the properties of $\sigma$-nilpotent groups which we use in our proofs.

Lemma 2.1 (See Corollary 2.4 and Lemma 2.5 in [3]). The class of all $\sigma$-nilpotent groups $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups.

Lemma 2.2 (See [27, 2.2.8]). If $\mathfrak{F}$ is a formation and $N, R$ are subgroups of $G$, where $N$ is normal in $G$, then

(i) $(G/N)^\mathfrak{F} = G^\mathfrak{F}N/N$, and
(ii) $G^\mathfrak{F}N = R^\mathfrak{F}N$ provided $G = RN$.

Proof of Proposition B. (i) Let $D = G^{\sigma G}$. First note that if $R$ is a normal subgroup of $G$, then $(G/R)^{\mathfrak{N}_\sigma} = DR/R$ by Lemmas 2.1 and 2.2 and so from the $G$-isomorphism $DR/R \simeq D/(D \cap R)$ we get that every chief factor of $G/R$ below $(G/R)^{\mathfrak{N}_\sigma}$ is simple if and only if every chief factor of $G$ between $D$ and $D \cap R$ is simple. Therefore if $G \in \mathcal{U}_{c\sigma}$, then $G/R \in \mathcal{U}_{c\sigma}$. Hence the class $\mathcal{U}_{c\sigma}$ is closed under taking homomorphic images.

Now we show that if $G/R, G/N \in \mathcal{U}_{c\sigma}$, then $G/(R \cap N) \in \mathcal{U}_{c\sigma}$. We can assume without loss of generality that $R \cap N = 1$. Since $G/R \in \mathcal{U}_{c\sigma}$, hence every chief factor of $G$ between $D$ and $D \cap R$ is simple. Also, every chief factor of $G$ between $D$ and $D \cap N$ is simple. Now let $H/K$ be any chief factor of $G$ below $D \cap R$. Then $H \cap D \cap N = 1$ and hence from the $G$-isomorphism

$$H(D \cap N)/K(D \cap N) \simeq H/(H \cap K(D \cap N)) = H/K(H \cap D \cap N) = H/K$$

we get that $H/K$ is simple since $D \cap N \leq K(D \cap N) \leq D$. On the other hand, every chief factor of $G$ between $D$ and $D \cap R$ is also simple. Therefore the Jordan-Hölder theorem for groups with
operators [28, A, 3.2] implies that every chief factor of \( G \) below \( D \) is simple. Hence \( G \in \mathcal{U}_{\sigma} \), so the class \( \mathcal{U}_{\sigma} \) is closed under taking subdirect products.

Finally, if \( H \leq G \in \mathcal{U}_{\sigma} \), then from Lemmas 2.1 and 2.2 and the isomorphism

\[
H/(H \cap D) \simeq HD/D \in \mathcal{U}_{\sigma}
\]

we get that \( H^{\mathfrak{m}_{\sigma}} \leq H \cap D \) and so every chief factor of \( H \) below \( H^{\mathfrak{m}_{\sigma}} \) is simple since every chief factor of \( G \) below \( D \) is simple. Hence \( H \in \mathcal{U}_{\sigma} \), so the class \( \mathcal{U}_{\sigma} \) is closed under taking normal subgroups.

(ii) See the proof of (i).

The proposition is proved.

**Lemma 2.3.** Let \( H/K \) be a non-abelian chief factor of \( G \). If \( H/K \) is simple, then \( G/H^\sigma G(H/K) \) is soluble.

**Proof.** Since \( C_G(H/K)/K = C_{G/K}(H/K) \), we can assume without loss of generality that \( K = 1 \). Then

\[
G/C_G(H) \simeq V \leq \text{Aut}(H)
\]

and

\[
H/(H \cap C_G(H)) \simeq H^\sigma G(H)/C_G(H) \simeq \text{Inn}(H)
\]

since \( C_G(H) \cap H = 1 \). Hence

\[
G/H^\sigma G(H) \simeq (G/C_G(H))/(H^\sigma G(H)/C_G(H)) \simeq W \leq \text{Aut}(H)/\text{Inn}(H).
\]

From the validity of the Schreier conjecture, it follows that \( G/H^\sigma G(H/K) \) is soluble. The lemma is proved.

**Proof of Theorem D.** First note that \( D \) is characteristic in \( N \) and \( R = D_\mathfrak{S} \) is a characteristic subgroup of \( D \), so both these subgroups are normal in \( G \).

**Necessity.** In view of Proposition B(ii), \( G/G^{\mathfrak{m}_{\sigma}} \) is \( \sigma \)-supersoluble and \( G^{\mathfrak{m}_{\sigma}} \) is contained in every normal subgroup \( E \) of \( G \) with \( \sigma \)-supersoluble quotient \( G/E \). By Lemmas 2.1 and 2.2, \( N/D = (G/N)^{\mathfrak{m}_{\sigma}} \). On the other hand, every chief factor of \( G \) between \( N \) and \( D \) is abelian and so cyclic and hence \( G/D \) is \( \sigma \)-supersoluble. Therefore \( G^{\mathfrak{m}_{\sigma}} \leq D \). Moreover, from Lemma 2.2 and Proposition B(ii) we also get that

\[
N/G^{\mathfrak{m}_{\sigma}} = (G/G^{\mathfrak{m}_{\sigma}})^{\mathfrak{m}_{\sigma}},
\]

so every chief factor of \( G \) between \( N \) and \( G^{\mathfrak{m}_{\sigma}} \) is cyclic and hence \( D \leq G^{\mathfrak{m}_{\sigma}} \). Thus \( D = G^{\mathfrak{m}_{\sigma}} \), so if \( D = 1 \), then \( G \) is \( \sigma \)-supersoluble.

Now suppose that \( D \neq 1 \). We show that in this case \( G \) has a Robinson complex of the form \((D, Z(D); U_1, \ldots, U_k)\), where \( Z(D) = R \). It is clear that every chief factor of \( G \) below \( R \) is cyclic, so \( G/C_G(R) \) is supersoluble by [28, IV, 6.10]. Hence \( D = G^{\mathfrak{m}_{\sigma}} \leq C_G(R) \), so \( R \leq Z(D) \leq D_\mathfrak{S} = R \) and therefore we have \( Z(D) = R \).
Now let $H/K$ be any chief factor of $G$ below $D$. Then $H \leq N$ and so in the case when $H/K$ is abelian, this factor is cyclic, which implies that $D = G_{\text{ab}} \leq C_G(H/K)$. On the other hand, if $H/K$ is a non-abelian simple group, then Lemma 2.3 implies that $G/H C_G(H/K)$ is soluble. Then

$$D H C_G(H/K)/H C_G(H/K) \simeq D/(D \cap H C_G(H/K)) = D/H C_D(H/K)$$

is soluble, so $D = H C_D(H/K)$ since $D$ is evidently perfect. Therefore, in both cases, every element of $D$ induces an inner automorphism on $H/K$. Therefore every element of $D$ induces an inner automorphism on $H/K$. Therefore $D$ is quasinilpotent. Hence in view of [29, X, 13.6], $G$ has a Robinson complex of the form $(D, Z(D), U_1, \ldots, U_k)$.

Sufficiency. From Conditions (i), (ii) and (iii), it follows that all factors below $N$ of any chief series of $G$ passing through $N$ are simple. Therefore the Jordan-Hölder theorem for groups with operators [28, A, 3.2] implies that every chief factor of $G$ below $N$ is simple. Therefore $G$ is a $\sigma$-$SC$-group.

The theorem is proved.

3 Proofs of Proposition A and Theorem A

Recall that a subgroup $A$ of $G$ is called $\sigma$-subnormal in $G$ [3] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is $\sigma$-primary for all $i = 1, \ldots, n$.

**Lemma 3.1** (See Remark 1.1 and [Proposition 2.6]arivII). $G$ is $\sigma$-nilpotent if and only if every subgroup of $G$ $\sigma$-subnormal in $G$.

**Lemma 3.2.** Let $A$, $K$ and $N$ be subgroups of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$.

1. $A \cap K$ is $\sigma$-subnormal in $K$.
2. $AN/N$ is $\sigma$-subnormal in $G/N$.
3. If $N \leq K$ and $K/N$ is $\sigma$-subnormal in $G/N$, then $K$ is $\sigma$-subnormal in $G$.
4. If $H \neq 1$ is a Hall $\sigma_i$-subgroup of $G$ and $A$ is not a $\sigma'_i$-group, then $A \cap H \neq 1$ is a Hall $\sigma_i$-subgroup of $A$.
5. If $A$ is a $\sigma_i$-group, then $A \leq O_{\sigma_i}(G)$.
6. If $A$ is a Hall $\sigma_i$-subgroup of $G$, then $A$ is normal in $G$.
7. If $|G : A|$ is a $\sigma_i$-number, then $O^{\sigma_i}(A) = O^{\sigma_i}(G)$.
8. If $G$ is $\sigma$-perfect, then $A$ is subnormal in $G$.
9. $A^{\sigma_i}$ is subnormal in $G$.

**Proof.** Assume that this lemma is false and let $G$ be a counterexample of minimal order. By hypothesis, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ such that either $A_{i-1} \leq A_i$
or \( A_i/(A_{i-1})A_i \) is \( \sigma \)-primary for all \( i = 1, \ldots, r \). Let \( M = A_{r-1} \). We can assume without loss of generality that \( M \neq G \).

1–7) See Lemma 2.6 in [3].

8) \( A \) is subnormal in \( M \) by the choice of \( G \). On the other hand, since \( G \) is \( \sigma \)-perfect, \( G/M_G \) is not \( \sigma \)-primary. Hence \( M \) is normal in \( G \) and so \( A \) is subnormal in \( G \).

9) \( A \) is \( \sigma \)-subnormal in \( AM_G \) by Part (1), so the choice of \( G \) implies that \( A^{\sigma_i} \) is subnormal in \( AM_G \). Hence \( G/M_G \) is a \( \sigma_i \)-group for some \( i \), so \( M_G A/M_G \simeq A/A \cap M_G \) is a \( \sigma_i \)-group. Hence \( A^{\sigma_i} \leq M_G \), so \( A^{\sigma_i} \) is subnormal in \( M_G \) and hence \( A^{\sigma_i} \) is subnormal in \( G \).

Lemma is proved.

The following lemma, in fact, is a corollary of Theorem A and Lemmas 3.1 and 3.2(3).

**Lemma 3.3.** The following statements hold:

(i) \( G \) is a \( P \sigma T \)-group if and only if every \( \sigma \)-subnormal subgroup of \( G \) is \( \sigma \)-permutable in \( G \).

(ii) If \( G \) is a \( P \sigma T \)-group, then every quotient \( G/N \) of \( G \) is also a \( P \sigma T \)-group.

**Lemma 3.4.** Let \( A \) and \( B \) be subgroups of \( G \), where \( A \) is \( \sigma \)-permutable in \( G \).

1) If \( A \leq B \) and \( B \) is \( \sigma \)-subnormal in \( G \), then \( A \) is \( \sigma \)-permutable in \( B \).

2) Suppose that \( B \) is a \( \sigma_i \)-group. Then \( B \) is \( \sigma \)-permutable in \( G \) if and only if \( O^{\sigma_i}(G) \leq N_G(B) \).

**Proof.** (1) By hypothesis, \( G \) possesses a complete Hall \( \sigma \)-set \( \mathcal{H} = \{H_1, \ldots, H_t\} \). Then \( \mathcal{H}_0 = \{H_1 \cap B, \ldots, H_t \cap B\} \) is a complete Hall \( \sigma \)-set of \( B \) by Lemma 3.2(4). Moreover, for every \( x \in B \) and \( H \in \mathcal{H} \) we have \( AH^x = H^xA \), so

\[
AH^x \cap B = A(H^x \cap B) = A(H \cap B)^x = (H \cap B)^x A.
\]

Hence \( A \) is \( \sigma \)-permutable in \( B \) by Remark 1.2.

(2) See Lemma 3.1 in [3].

The lemma is proved.

**Proof of Proposition A.** Let \( \mathcal{H} = \{H_1, \ldots, H_t\} \) and \( N = G^{\sigma_i} \) be the \( \sigma \)-nilpotent residual of \( G \). Then \( D \leq N \).

1) Statement (i) holds for \( G \).

Suppose that this is false and let \( G \) be a counterexample of minimal order. If \( D = 1 \), then \( G \) is \( \sigma \)-soluble and so \( G \) is a \( \sigma \)-SC-group by Theorem B. Therefore \( D \neq 1 \). Let \( R \) be a minimal normal subgroup of \( G \) contained in \( D \). Then \( G/R \) is a \( P \sigma T \)-group by Lemma 3.3(ii). Therefore the choice of \( G \) implies that \( G/R \) is a \( \sigma \)-SC-group. Since \( (G/R)^{\sigma_i} = N/R \) by Lemmas 2.1 and 2.2, every chief factor of \( G/R \) below \( N/R \) is simple. Hence every chief factor of \( G \) between \( G^{\sigma_i} \) and \( R \) is simple. Hence every chief factor of \( G \) between \( G^{\sigma_i} \) and \( R \) is simple. Therefore, in view of the Jordan-Hölder theorem for groups with operators [28, A, 3.2], it is enough to show that \( R \) is simple. Suppose that this is false. Let \( L \) be a minimal normal subgroup of \( R \).
Then $1 < L < R$ and $L$ is $\sigma$-permutable in $G$ by Lemma 3.3(i) since $G$ is a $P\sigma T$-group. Moreover, $L_G = 1$ and so $L$ is $\sigma$-nilpotent by Theorem A. Therefore $R$ is a $\sigma_i$-group for some $i$, so for some $k$ we have $R \leq H_k$. Now let $V$ be a maximal subgroup of $R$. Then $V$ is $\sigma$-subnormal in $G$, so $V$ is $\sigma$-permutable in $G$ and hence

$$R \leq D \leq O^{\sigma_i}(G) \leq N_G(V)$$

by Lemma 3.4(2). Thus $R$ is nilpotent, so $R$ is a $p$-group for some $p \in \sigma_i$. Now let $V$ be a maximal subgroup of $R$ such that $V$ is normal in a Sylow $p$-subgroup of $P$ of $H_k$. By hypothesis, $H_k$ is a PST-group and so $V$ is $S$-permutable in $H_k$ since it is subnormal in $H_k$. Then, by Lemma 3.4(2) (taking in the case $\sigma = \{2\}, \{3\}, \ldots$), we have $H_k = PO^p(H_k) \leq N_G(V)$. Therefore, in view of Lemma 3.4(2), we have

$$G = H_kO^{\sigma_i}(G) \leq N_G(V).$$

Hence $V = 1$ and so $|R| = p$, a contradiction. Thus we have (1).

(2) Statement (ii) holds for $G$.

It is clear that $D$ is $\sigma$-perfect and $G/D$ is $\sigma$-soluble. In view of Lemma 3.3(ii), $G/D$ is a $P\sigma T$-group. It is also clear that $D \leq G^{\Delta p}$. On the other hand, $G/D$ is $\sigma$-supersoluble by Theorem B. Therefore $G^{\Delta p} \leq D$ and so we have $D = G^{\Delta p}$. Hence we have (2).

(3) Statement (iii) holds for $G$.

Let $L$ be a $\sigma$-soluble normal subgroup of $G$ and let $x$ be a $\sigma_i$-element of $G$. Let $V/L \leq O_{\sigma_i}(G/L)$. Then $V/L$ is $\sigma$-subnormal in $G/L$, so $V/L$ is $\sigma$-permutable in $G/L$ by Lemma 3.3(i) since $G/L$ is a $P\sigma T$-group by Lemma 3.3(ii). Therefore

$$xL \in O^{\sigma_i}(G/L) \leq N_{G/L}(V/L)$$

by Lemma 3.4(2). Hence Statement (iii) holds for $G$.

The proposition is proved.

**Lemma 3.5.** Let $G$ be a non-$\sigma$-supersoluble $\sigma$-full $\sigma$-SC-group and let $(D, Z(D); U_1, \ldots, U_k)$ be a Robinson complex $G$, where $D = G^{\Delta p}$. Let $U$ be a non-$\sigma$-permutable $\sigma$-subnormal subgroup of $G$ of minimal order. Suppose that $S/Z(S)$ is $\sigma$-perfect. Then:

(i) If $U^{1_i}/U_i'$ is $\sigma$-permutable in $G/U_i'$ for all $i$, then $U$ is $\sigma$-supersoluble.

(ii) If $U$ is $\sigma$-supersoluble and $UL/L$ is $\sigma$-permutable in $G/L$ for all non-trivial nilpotent normal subgroups $L$ of $G$, then $U$ is a cyclic $p$-group for some prime $p$.

**Proof.** Suppose that this lemma is false and let $G$ be a counterexample of minimal order. By hypothesis, for some $i$ and for some Hall $\sigma_i$-subgroup $H$ of $G$ we have $UH \neq HU$.

(i) Assume that this is false. Then $U \cap D \neq 1$ since $UD/D \simeq U/(U \cap D)$ is $\sigma$-supersoluble by Proposition B(ii). Moreover, Lemma 3.2(1)(2), implies that $(U \cap D)Z(D)/Z(D)$ is $\sigma$-subnormal in $D/Z(D)$ and so $(U \cap D)Z(D)/Z(D)$ is a non-trivial subnormal subgroup of $D/Z(D)$ by Lemma
3.2(8) since $D/Z(D)$ is $\sigma$-perfect by hypothesis. Hence for some $i$ we have

$$U_i/Z(D) \leq (U \cap D)Z(D)/Z(D),$$

so $U_i \leq (U \cap D)Z(D)$. But then

$$U'_i \leq ((U \cap D)Z(D))' \leq U \cap D.$$ 

By hypothesis, $UU'_i/U'_i = U/U'_i$ is $\sigma$-permutable in $G/U'_i$ and so

$$U'U_i = (U/U'_i)(HU'_i/U'_i) = (HU'_i/U'_i)(U/U'_i) = HU/U'_i.$$ 

Hence $U'H = HU$, a contradiction. Therefore Statement (i) holds.

(ii) Let $N = U^{\sigma_i}$. Then $D$ is subnormal in $G$ by Lemma 3.2(9). Since $U$ is $\sigma$-supersoluble by hypothesis, $N < U$. By Lemmas 2.1, 2.2 and 3.2(3), every proper subgroup $V$ of $U$ with $N \leq V$ is $\sigma$-subnormal in $G$, so the minimality of $U$ implies that $VH = HV$. Therefore, if $U/V$ has at least two distinct maximal subgroups $V$ and $W$ such that $N \leq V \cap W$, then $U = \langle V, W \rangle$ is permutes with $H$ by [28, A, 1.6], contrary to our assumption on $U$ and $H$. Hence $U/N$ is a cyclic $p$-group for some prime $p$.

First assume that $p \in \sigma_i$. Lemma 3.2(4) implies that $H \cap U$ is a Hall $\sigma_i$-subgroup of $U$, so $U = N(H \cap U) = (H \cap U)N$. Hence

$$UH = (H \cap U)NH = H(H \cap U)N = HU,$$

a contradiction. Thus $p \in \sigma_j$ for some $j \neq i$.

Now we show that $U$ is a $P\sigma T$-group. Let $V$ be a proper $\sigma$-subnormal subgroup of $U$. Then $V$ is $\sigma$-subnormal in $G$ since $U$ is $\sigma$-subnormal in $G$. The minimality of $U$ implies that $V$ is $\sigma$-permutable in $G$, so $V$ is $\sigma$-permutable in $U$ by Lemma 3.4(1). Hence $U$ is a $\sigma$-soluble $P\sigma T$-group by Lemma 3.3(i), so $N$ is abelian by Theorem B.

Therefore $N$ is a $\sigma'_j$-group, so $N \leq O = O_{\sigma'_j}(F(G))$ by Lemma 3.2(5) (taking in the case $\sigma = \{2\}$, \{3\}, . . .). By hypothesis, $OU/O$ permutes with $OH/O$. By Lemma 3.2(1)(2), $OU/O$ is $\sigma$-subnormal in

$$(OU/O)(OH/O) = (OH/O)(OU/O) = OHU/O,$$

where $OU/O \simeq U/U \cap O$ is a $\sigma_j$-group and $OH/O \simeq H/H \cap O$ is a $\sigma_i$-group. Hence $UO/O$ is normal in $OHU/O$ by Lemma 3.2(6). Hence $H \leq N_G(U)$

$$H \leq N_G(O_{\sigma'_j}(OU)) = N_G(O_{\sigma'_j}(U))$$

by Lemma 3.2(7) since $p \in \sigma_j$ implies that $O_{\sigma'_j}(U) = U$. But then $HU = UH$, a contradiction. Therefore Statement (ii) holds.

The lemma is proved.
Lemma 3.6. Suppose that $G$ has a Robinson $\sigma$-complex $(D, Z(D); U_1, \ldots, U_k)$, and let $N$ be a normal subgroup of $G$.

(i) If $N = U'_i$, then

$$(D/N, Z(D/N) = U_i/N; U_1 N/N, \ldots, U_{i-1} N/N, U_{i+1} N/N, \ldots U_k N/N, U_i/N)$$

is a Robinson $\sigma$-complex of $G/N$, where $U_i/U'_i \simeq Z(D)/(Z(D) \cap U'_i)$.

(ii) If $N$ is nilpotent, then

$$(DN/N, Z(DN/N) = Z(D)N/N; U_1 N/N, \ldots, U_k N/N)$$

is a Robinson $\sigma$-complex of $G/N$.

Proof. See Remark 1.6.8 in [7].

Lemma 3.7 (See Knyagina and Monakhov [31]). Let $H$, $K$ and $N$ be pairwise permutable subgroups of $G$ and $H$ is a Hall subgroup of $G$. Then

$$N \cap HK = (N \cap H)(N \cap K).$$

Lemma 3.8. If $G$ satisfies $N_{\sigma_i}$, then $G/R$ satisfies $N_{\sigma_i}$ for every normal $\sigma$-soluble subgroup $R$ of $G$.

Proof. Let $N/R$ be a normal $\sigma$-soluble subgroup of $G/R$ and let

$$(V/R)/(N/R) \leq O_{\sigma_i}((G/R)/(N/R)).$$

Then $N$ is a normal $\sigma$-soluble subgroup of $G$ and $V/N \leq O_{\sigma_i}(G/N)$. Moreover, for every $\sigma'_i$-element $xR \in G/R$ there is a $\sigma'_i$-element $y \in G$ such that $xR = yR$ and so $yN \leq N_{G/N}(V/N)$, which implies that

$$xR(N/R) \in N_{(G/R)/(N/R)}((V/R)/(N/R)).$$

Hence $G/R$ satisfies $N_{\sigma_i}$, as required.

By the analogy with the notation $\pi(n)$, we will write $\sigma(n)$ to denote the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$.

Proof of Theorem C. First assume that $G$ is a $P\sigma T$-group and let $D = G^{G_{\sigma}}$ be the $\sigma$-soluble residual of $G$. Then $D$ is clearly $\sigma$-perfect and, by Proposition A, $G$ is a $\sigma$-SC-group and Statements (i) and (iii) hold for $G$. Moreover, Theorem B implies that $D$ coincides with the $\sigma$-supersoluble residual $G^{G_{\sigma}}$ of $G$ and if $D \neq 1$, then $G$ possesses a Robinson $\sigma$-complex of the form $(D, Z(D); U_1, \ldots, U_k)$. Therefore the necessity of the condition of the theorem holds for $G$.

Now assume that $G$ has a normal $\sigma$-perfect subgroup $D$ and $D$ satisfies Conditions (i), (ii) and (iii). We show that $G$ is a $P\sigma T$-group. Suppose that this is false and let $G$ be a counterexample of
minimal order. Then $D \neq 1$ and $G$ has a $\sigma$-subnormal subgroup $U$ such that $UH \neq HU$ for some $i$ and some Hall $\sigma_i$-subgroup $H$ of $G$ and also every $\sigma$-subnormal subgroup $U_0$ of $G$ with $U_0 < U$ is $\sigma$-permutable in $G$. Finally, note that $D = G^{d_U}$ by Condition (i) and Theorem B.

(1) $U$ is $\sigma$-supersoluble.

In view of Lemma 3.5(i), it is enough to show that the hypothesis holds on $G/U'_i$ for all $i = 1, \ldots, k$. Let $N = U'_i$. We can assume without loss of generality that $i = 1$. Then

$$(D/N)^{\sigma_i} = D^{\sigma_i} N/N = D/N$$

by Lemmas 2.1 and 2.2, so $D/N$ is a normal $\sigma$-perfect subgroup of $G/N$. Moreover, $(G/N)/(D/N) \simeq D/D$ is a $\sigma$-soluble $P\sigma T$-group. Now assume that $D/N \neq 1$. Then, by Lemma 3.6(i),

$$(D/N, Z(D/N); U_2 N/N, \ldots, U_k N/N)$$

is a Robinson $\sigma$-complex of $G/N$, where $Z(D/N) = U_1/N$. Moreover, if $\{i_1, \ldots, i_r\} \subseteq \{2, \ldots, k\}$, where $2 \leq r < k$, then the quotients $G/N = G/U'_i$ and

$$(G/N)/(U_{i_1} N/N') \cdots (U_{i_r} N/N') = (G/N)/(U'_{i_1} \cdots U'_{i_r} U'_1/N) \simeq G/U'_{i_1} \cdots U'_{i_r} U'_1$$

satisfy $N_{\sigma_i}$ for all

$$\sigma_i \in \sigma(U_1/N) = \sigma(Z(D/N)) \subseteq \sigma(Z(D)/(Z(D) \cap U'_1)).$$

Therefore the hypothesis holds for $G/R$, so we have (1).

(2) $U$ is a cyclic $p$-group for some prime $p \in \sigma_j$, where $j \neq i$.

First we show that $U$ is a cyclic $p$-group for some prime. In view of Claim (1) and Lemma 3.5(ii), it is enough to show that the hypothesis holds on $G/N$ for every normal nilpotent subgroup $N$ of $G$. First note that

$$(D/N)^{\sigma_i} = D^{\sigma_i} N/N = D/N$$

by Lemma 2.2(ii), so $D/N$ is a normal $\sigma$-perfect subgroup of $G/N$. Moreover,

$$(D/N, Z(D/N); U_2 N/N, \ldots, U_k N/N)$$

is a Robinson $\sigma$-complex of $G/N$ by Lemma 3.6(ii). Finally, if $V/N$ is a normal $\sigma$-soluble subgroup of $G/N$, then $V$ is a normal $\sigma$-soluble subgroup of $G$ and so for $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, t\}$, where $1 \leq r < k$, the quotient $G/N$ and, by Lemma 3.8, the quotient

$$(G/N)/(U_{i_1} N/N') \cdots (U_{i_r} N/N') = (G/N)/(U'_{i_1} \cdots U'_{i_r} N/N)$$

$$(G/N)/(U_{i_1} N/N') \cdots (U_{i_r} N/N') \simeq G/U'_{i_1} \cdots U'_{i_r} N \simeq (G/U'_{i_1} \cdots U'_{i_r})/(U'_{i_1} \cdots U'_{i_r} N/U'_{i_1} \cdots U'_{i_r})$$

satisfy $N_{\sigma_i}$ for all

$$\sigma_i \in \sigma(Z(D/N)) = \sigma(Z(D)N/N) \subseteq \sigma(Z(D)).$$
since $U'_i \cdots U'_{i_r} N/U'_1 \cdots U'_{i_r} \simeq N/(N \cap U'_i \cdots U'_{i_r})$ is $\sigma$-soluble.

Therefore the hypothesis holds on $G/N$, so $U$ is a cyclic $p$-group for some prime $p \in \sigma_j$. Finally, Lemma 3.2(4) implies that in the case $i = j$ we have $U \leq H$, so $UH = H = HU$. Therefore $j \neq i$. Finally, again by Lemma 3.2(4), $U \leq O_{\sigma_j}(G)$.

(3) $O_{\sigma_j}(G) \cap D = 1$.

Suppose that $L = O_{\sigma_j}(G) \cap D \neq 1$. Then, since $D/Z(D)$ is $\sigma$-perfect, $L \leq Z(D)$ and so $G$ satisfies $N_{\sigma_j}$ by Condition (iii). Therefore $H \leq N_G(U)$ since $i \neq j$, $U \leq O_{\sigma_j}(G)$ and $H$ is a $\sigma_i$-group. But then $HU = UH$, a contradiction. Hence we have (3).

Final contradiction for the sufficiency. By Lemma 3.2(2), $UD/D$ is $\sigma$-subnormal in $G/D$. On the other hand, $HD/D$ is a Hall $\sigma_i$-subgroup of $G/D$. Hence

$$(UD/D)(HD/D) = (HD/D)(UD/D) = HUD/D$$

by Condition (i) and Lemma 3.3(i), so $HU$ is a subgroup of $G$. Therefore, by Claims (2), (3) and Lemma 3.7,

$$UHD \cap HO_{\sigma_j}(G) = UH(D \cap HO_{\sigma_j}(G)) = UH(D \cap H)(D \cap O_{\sigma_j}(G))$$

$$= UH(D \cap H) = UH$$

is a subgroup of $G$ and so $HU = UH$, a contradiction.

The theorem is proved.

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