FULLY DEGENERATE MONGE AMPÈRE EQUATIONS

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ABSTRACT. In this paper, we consider the following nonlinear eigenvalue problem for the Monge-Ampère equation: find a non-negative weakly convex classical solution \( f \) satisfying
\[
\begin{cases}
\det D^2 f = f^p & \text{in } \Omega \\
f = \varphi & \text{on } \partial \Omega
\end{cases}
\]
for a strictly convex smooth domain \( \Omega \subset \mathbb{R}^2 \) and \( 0 < p < 2 \). When \( \{ f = 0 \} \) contains a convex domain, we find a classical solution which is smooth on \( \{ f > 0 \} \) and whose free boundary \( \partial \{ f = 0 \} \) is also smooth.

1. INTRODUCTION

We consider in this work the following nonlinear eigenvalue problem for the Monge-Ampère equation: find a non-negative weakly convex classical solution \( f \) satisfying
\[
\begin{cases}
\det D^2 f = f^p & \text{in } \Omega \\
f = \varphi & \text{on } \partial \Omega
\end{cases}
\]
for a strictly convex bounded smooth domain \( \Omega \subset \mathbb{R}^2 \), with \( \varphi > 0 \) on \( \partial \Omega \) and smooth, and \( 0 < p < 2 \).

The study of problem (MA) is motivated by the general Minkowski problem in differential geometry, asking to find the manifold whose Gauss curvature has been prescribed. More generally, the Gauss curvature itself may depend on the graph \( z = f(P) \) of the manifold, namely
\[
\det D^2 f(P) = h(P, f(P), \nabla f(P)).
\]
For a positive bounded \( h \), this problem has been discussed by many authors and the \( C^{1,1} \)-regularity of \( f \) has been established (c.f. in [GT]). When \( h \) is allowed to

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be zero, \( f \) is not always a \( C^{1,1} \) function, as it will be discussed in the sequel. The regularity of \( f \) is an open problem (c.f. Aubin [Au]).

One of the interesting cases is when \( h = 0 \) on the vanishing set of \( f \), especially when \( h = f^p \), as in problem (MA). For \( p = 1 \), this problem corresponds to an eigenvalue problem describing the asymptotic behavior, as \( t \to T \), of the parabolic Monge-Ampère equation

\[
\begin{cases}
  f_t = \det D^2 f & \text{in } \Omega \times (0, T) \\
  f = \eta(t) \varphi & \text{on } \partial \Omega
\end{cases}
\]

where \( \eta(t) = 1/(T - t) \).

For \( f < 0 \), \( f = 0 \) on \( \partial \Omega \) and \( h = (-f)^{-(n+2)} \) in \( \Omega \), problem (MA) was considered by Cheng and Yau in [CY]. When, \( h = (-\lambda f)^n \) problem (MA) corresponds to the eigenvalue problem for the concave operator \((\det D^2 f)^+\) and has been studied in [Li]. The exponential nonlinearity, \( h = e^{-2f} \) has been studied by Cheng and Yau in [CY2]. Equation \( \det D^2 f = h \) with a degenerate source term \( h \) has been studied at [G]. The limiting case \( p \to 0^+ \), \( f(x)^p \to \chi_{\{f > 0\}} \) was considered by O. Savin, [S] as the obstacle problem for Monge-Ampère equation, where the obstacle stays below the graph of \( f \). The second author also considered the case where the obstacle stays above the graph of \( f \).

Since \( f(x)^p \to \chi_{\{f > 0\}} \) as \( p \to 0^+ \), (MA) corresponds to a perturbation problem for the obstacle problem

\[
\det D^2 f = \chi_{\{f > 0\}}
\]

and \( f \geq 0 \) in \( \Omega \).

Depending on the boundary values \( \varphi \) one of the three possibilities may occur in (MA):

i. \( f > 0 \) in \( \Omega \): the equation (MA) is then strictly elliptic and by the regularity theory of fully-nonlinear equations, \( f \) is \( C^\infty \) smooth in \( \Omega \) (cf. [CC]).

ii. \( f \equiv 0 \) on a convex sub-domain \( \Lambda(f) \subset \Omega \): equation (MA) becomes degenerate on \( \Lambda(f) \) and \( \Gamma(f) = \partial \Lambda(f) \) is the free-boundary associated to this problem. The function \( f \) is \( C^\infty \) smooth on \( \Omega(f) = \Omega \setminus \Lambda(f) \) (cf. [CC]). The optimal regularity of \( f \) up to the interface will be discussed in this work.

iii. \( f(P_0) = 0 \) at a single point \( P_0 \in \Omega \) and \( f > 0 \), on \( \Omega \setminus \{P_0\} \): equation becomes degenerate at the point \( P_0 \). The function \( f \) is \( C^\infty \) smooth on \( \Omega \setminus \{P_0\} \) (cf. [CC]). However, the regularity of \( f \) at \( P_0 \) is an open question.
We will restrict our attention from now on to the case (ii) above, where the solution $f$ of (MA) vanishes on a domain $\Lambda(f)$.

By looking at radial solutions $z = f(r)$ on $\Omega = B_2(0)$ which vanish on $B_1(0)$, we find that the expected behavior of $f$ near the interface $r = 1$ is $f(r) \sim (r-1)^q$, with $q$ given in terms of $p$ by $q = \frac{3}{2-p}$. This motivates the introduction of the pressure function

$$g = q^\frac{2}{n} f^\frac{4}{n}, \quad q = \frac{3}{2-p}. \tag{1.1}$$

A direct calculation shows that $g$ satisfies the problem

$$\text{(MAP)} \begin{cases} g \det D^2 g + \theta(g g_{xx} + 2g_x g_y g_{xy} + g g_{yy}) = \chi_{\{g>0\}} & \text{in } \Omega \\ g = \bar{\phi} & \text{on } \partial \Omega \end{cases}$$

with

$$\theta = \frac{1+p}{2-p}$$

and $\bar{\phi}(x) = q^\frac{2}{n} \phi^\frac{4}{n}$. One observes that $\theta > 0$ iff $p < 2$ which explains our assumption on $p$.

A similar concept of pressure plays an important role in obtaining the optimal regularity of solutions to another degenerate equation, this time parabolic, the porous medium equation, namely the flow of a density function $f$ of a gas through a porous medium given by

$$f_t = \nabla f^m \text{ on } \mathbb{R}^n. \tag{PME}$$

The corresponding pressure $g = f^{m-1}$ of the gas satisfies

$$g_t = (m-1)g \Delta g + |\nabla g|^2, \quad \text{on } \mathbb{R}^n. \tag{1.2}$$

The pressure $g$ is more natural in terms of the regularity. For a classical solution, the expanding speed of the free boundary $\partial \Omega(g) = \partial \{g > 0\}$ is $|\nabla g|$. If we observe that the free boundary expands with finite non-degenerate speed, $g$ grows linearly away from the free boundary $\partial \Omega(g)$, while the density $f$ grows like a Hölder function whose Hölder coefficient depends on $m$. The pressure $g$ is a kind of normalization of $f$. Then, $g$ is Lipschitz on $\mathbb{R}^n$ and smooth on $\overline{\Omega(g)}$.

A pressure-like function $g = \sqrt{2f}$ for the parabolic Monge Ampré equation

$$f_t = \det D^2 f \tag{1.3}$$
has also been shown to be Lipschitz globally and smooth on $\Omega(f)$ in [DH2], [DL1] and [DL3].

Let us now turn our attention back to equation (MA). Our objective in this work is to establish the existence of a classical solution $f$ of the problem (MA), when the boundary data $\varphi$ is such that the solution $f$ vanishes on a region $\Lambda(f) \subset \Omega$ and therefore the equation becomes degenerate near the interface $\Gamma(f) = \partial \Lambda(f)$. The concept of a classical solution will be discussed in section 2.1. To guarantee that such vanishing region exists, we assume that there is a classical super-solution $\psi$ of (MA) vanishing on a non-empty domain $\Lambda(\psi) \subset \Omega$. In section 2.3 we will actually present an example which shows that this is indeed possible.

**Theorem 1.1.** Assume that $\Omega \subset \mathbb{R}^2$ is a strictly convex bounded smooth domain and let $\varphi \in C^2(\Omega)$, $\varphi > 0$ on $\partial \Omega$ and $0 < p < 2$. Assume that there is a classical super-solution $\psi$ of (MA) vanishing on a non-empty domain $\Lambda(\psi) \subset \Omega$. Then, there is a classical solution of (MA) and its pressure $g$, given in terms of $f$ by (1.1), is $C^\infty$ smooth on $\Omega(g)$ up to the interface $\Gamma$. Consequently, $f$ enjoys the optimal regularity $f \in C^{k,\alpha}$ with $k = \lfloor \frac{3}{2-p} \rfloor$, $\alpha = \frac{3}{2-p} - k$, ( $C^{k-1,1}$, if $k := \frac{3}{2-p}$ is an integer) and the interface $\Gamma(g)$ is $C^\infty$ smooth.

A brief outline of the paper is as follows: in section 2 the concept of classical solutions of (MA) is introduced and the proof of its existence via the method of continuity is outlined. Section 3 will be devoted to the derivation of sharp a’priori derivative estimates for classical solutions of equation (MAP). These estimates play crucial role in establishing the $C^{2,\alpha}$ regularity of classical solutions of (MAP) which will be shown in section 4 (see in section 2.2 for the definition of this space). Based on the estimates in section 4, we will conclude, in section 5, the proof of the existence of a $C^{2,\alpha}$ up to the interface solution $g$ of (MAP), via the method of continuity. We will also show that the pressure $g$ is $C^\infty$ smooth up to the interface.

**Notation:**

- $\Omega \subset \mathbb{R}^2$ denotes a strictly convex bounded smooth domain in $\mathbb{R}^2$.
- $\varphi$ denotes a smooth strictly positive function defined on $\partial \Omega$.
- For any $g \geq 0$ on $\Omega$, we denote $\Omega(g) = \{x \in \Omega \mid g(x) > 0\}$, $\Lambda(g) = \{x \in \Omega \mid g(x) = 0\}$ and $\Gamma(g) = \partial \Lambda(g)$.
- $ds^2$ denotes the singular metric defined in section 2.2.
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• $\|g\|_{C^2_{\alpha}} = \max_{\partial \Omega} (|D_{ij} g| + |D_i g| + g)$.
• $C^{2,\alpha}_s(\Omega)$ will be defined in section 2.2 and $C^{2+\alpha}_s(\Omega)$ will be defined in section 5.
• $\nu, \tau$ denote the outward normal and tangential directions to the level sets of a function $g$.
• $g_\nu, g_\tau, g_{\nu\nu}, g_{\nu\tau}, g_{\tau\tau}$ denote the derivatives of $g$ with respect to $\nu, \tau$.

2. Classical solutions and the method of continuity

In this section we will define the concept of a classical solution of equation (MA) [resp. of (MAP)] and sketch the proof of its existence via a method of continuity.

2.1. The concept of classical solutions and the comparison principle. We consider the following generalization of equation (MAP), namely

(MAPh) $g \det D^2 g + \theta (g_x^2 g_{yy} - 2 g_x g_y g_{xy} + g_y^2 g_{yy}) = h \chi_{\{g > 0\}}$

where $h \in C^2(\Omega)$ and satisfies the bounds

(2.1) $0 < \lambda < h < \lambda^{-1} < \infty$

for some constant $\lambda > 0$.

We recall the notation $\Omega(g) = \{x \in \Omega \mid g(x) > 0\}$ and $\Lambda(g) = \Omega \setminus \Omega(g)$. On the free-boundary $\Gamma(g) := \partial \Lambda(g)$, where $g = 0$, we then have, from (MAPh),

$\theta (g_x^2 g_{yy} - 2 g_x g_y g_{xy} + g_y^2 g_{yy}) = \theta g_\nu^2 g_{\tau\tau} = \theta g^{3}_\nu \kappa = h$

where $\nu$ and $\tau$ are inward normal and tangential unit directions to $\Gamma(g)$ respectively and where $\kappa = g_{\tau\tau}/g_\nu$ denotes the curvature of $\Gamma(g)$.

More generally, denote by $\nu, \tau$ the outward normal and tangential directions to the level sets of the function $g$.

**Definition 2.1.** We say that $g \in C^2_{\nu}(\Omega(g))$ iff

$g, g_\nu, g_\tau, g g_{\nu\nu}, \sqrt{g} g_{\nu\tau}, g_{\tau\tau}$

extend continuously up to $\Omega(g)$ and are bounded on $\Omega(g)$.

Define the non-linear operator

(2.2) $\mathcal{P}[g] := g \det D^2 g + \theta (g_x^2 g_{yy} - 2 g_x g_y g_{xy} + g_y^2 g_{yy})$. 
Definition 2.2. Assume that \( g \in C^{0,1}(\Omega) \cap C^2_s(\Omega(g)) \) and that \( f = \left(q^{-\frac{2}{3}} g\right)^q \), \( q = \frac{3}{2-p} \) is convex in \( \Omega \). The function \( g \) is called a classical super-solution (sub-solution) of equation (MAPh) if

\[
\begin{cases}
P[g] \leq h (\geq h) & \text{in } \Omega(g) \\
P[g] = \theta g^3 \kappa \leq h (\geq h) & \text{on } \Gamma(g).
\end{cases}
\]

The function \( g \) is called a classical solution if it is both a classical super-solution and sub-solution.

If \( g \) satisfies (MAPh), then the corresponding convex function \( f = \left(q^{-\frac{2}{3}} g\right)^q \), with \( q = \frac{3}{2-p} \), satisfies the equation (MAh)

\[\det D^2 f = h f^p.\]

Definition 2.3. A convex function \( f \) is called a classical super-solution, sub-solution, or solution of (MAPh) if the corresponding pressure \( g \) belongs to \( C^{0,1}(\Omega) \cap C^2_s(\Omega(g)) \) and is a classical super-solution, sub-solution, or solution of (MAPh) respectively.

Lemma 2.4. Let \( g_1 \) be a classical super-solution and \( g_2 \) be a classical sub-solution of (MAPh) such that \( g_2 < g_1 \) on \( \partial \Omega \). Assuming that \( \Omega(g_2) \subset \Omega(g_1) \), we have \( g_2 \leq g_1 \) in \( \Omega \).

Proof. Choose \( \epsilon > 0 \) sufficiently small so that \( g_2^\epsilon := (1 + \epsilon)g_2 < g_1 \) on \( \partial \Omega \) and \( g_2^\epsilon \) is a strict sub-solution of (MAPh). We claim that \( g_2^\epsilon := (1 + \epsilon)g_2 \leq g_1 \) in \( \Omega \).

Indeed, let us assume that \( g_2^\epsilon \) touches \( g_1 \) at a point \( P_0 \). If \( P_0 \in \Omega(g_2^\epsilon) \), then

\[h(P_0) \geq P[g_1](P_0) \geq P[g_2^\epsilon](P_0) > h(P_0) \]

which is a contradiction.

Hence, we may assume that \( P_0 \in \partial \Omega(g_2^\epsilon) \). Clearly \( \partial \Omega(g_2^\epsilon) \) will also touch \( \partial \Omega(g_1) \) at \( P_0 \). Then, at \( P_0 \), we have \( (g_2^\epsilon)_\nu \leq (g_1)_\nu \) and \( \kappa_2 \leq \kappa_1 \), where \( \kappa_1, \kappa_2 \) denote the curvatures of \( \partial \Omega(g_1), \partial \Omega(g_2^\epsilon) \) respectively. Thus at \( P_0 \)

\[(g_1)_\tau \tau = (g_1)_\nu \kappa_1 \geq (g_2^\epsilon)_\nu \kappa_2 = (g_2^\epsilon)_\tau \tau \]

and then

\[\frac{h}{\theta} = (g_1)_\nu (g_1)_\tau \tau \geq (g_2^\epsilon)_\nu (g_2^\epsilon)_\tau \tau = \frac{(1 + \epsilon)h}{\theta} \]

which is a contradiction. This finishes the proof of our claim.
Since \((1 + \varepsilon)g_2 = g_2^\varepsilon \leq g_1\) for any small \(\varepsilon > 0\), letting \(\varepsilon \to 0\) we conclude that \(g_2(P) \leq g_1(P)\).

**Theorem 2.5** (Comparison Principle for Classical Solutions). Let \(g_1\) be a classical super-solution and \(g_2\) be a classical sub-solution of \([\text{MAPh}]\) such that \(g_2 \leq g_1\) on \(\partial \Omega\). Assuming that \(\Omega(g_2) \subset \Omega(g_1)\), we have \(g_2 \leq g_1\) in \(\Omega\).

**Proof.** The function \(g_2^\varepsilon = (1 + \delta \varepsilon)(g_2 - \varepsilon)_+\) is also a strict sub-solution of \([2.3]\) such that \(g_2^\varepsilon < g_1\) on \(\partial \Omega\), for a small \(\delta \varepsilon\) depending \(\varepsilon\). For large \(\varepsilon > 0\), \(\Omega(g_2^\varepsilon) \subset \Omega(g_1)\). The same argument as in the lemma above shows that \(P_0\) can not be a point in \(\Omega(g_2^\varepsilon)\), since \(g_2^\varepsilon\) is a strict sub-solution. Also \(P_0 \notin \partial \Omega(g_2^\varepsilon)\); otherwise \(\partial \Omega(g_2^\varepsilon)\) would touch \(\partial \Omega(g_1)\), for \(\varepsilon > 0\), which leads to contradiction similarly as in the proof of the previous lemma.

### 2.2. The linearized operator near the free-boundary and sharp a’ priori estimates.

In sub-section 2.3 we will outline the proof of the existence of a classical solution of problem \([\text{MAP}]\) via the method of continuity. Our approach relies on the observation that one can obtain sharp a priori estimates for classical solutions \(g\) of the degenerate equation \([\text{MAPh}]\) if one scales the estimates according to the natural singular metric corresponding to problem.

To illustrate this better, assume that \(g\) is a classical solution of equation \([\text{MAPh}]\) and that \(P_0 \in \Gamma(g)\) is a free-boundary point. We will show in section 3, that \(g\) satisfies the a’priori bounds \([2.10]\) and \([2.11]\) near the free-boundary, which in particular imply the bound
\[
\alpha < |Dg| \leq c^{-1}
\]
for some \(c > 0\). We may assume, without loss of generality, that \(g_x > 0, g_y = 0\) at \(P_0\) so that it is possible to solve the equation \(z = g(x, y)\) near \(P_0\) with respect to \(x\) yielding to a map \(z = q(z, y)\) defined for all \((z, y)\) sufficiently close to \(Q_0 = (0, y_0)\).

The function \(q\) satisfies the equation
\[
(2.4) \quad \frac{-z \det D^2 q + \theta q_z q_{yy}}{q_z^2} = -H(z, y)
\]
where \(H(z, y) := h(x, y), x = q(x, y)\). Based on the a’priori estimates, we will show in section 4 that the linearized operator of equation \([2.4]\) near a function \(q\) satisfying the bounds \([4.8]\) and \([4.9]\) is of the form
\[
(2.5) \quad L(\tilde{q}) = z \alpha_{11} \tilde{q}_{zz} + 2\sqrt{z} \alpha_{12} \tilde{q}_{zy} + \alpha_{22} \tilde{q}_{yy} + \tilde{b} \tilde{q}_z + c \tilde{q}
\]
with \((\alpha_{ij})\) strictly positive and \(b \geq \nu > 0\).

To apply the method of continuity one needs to establish sharp a priori estimates for linear degenerate equations of the form \((2.5)\). These estimates become optimal when scaled according to the singular metric

\[(SM) \quad ds^2 = \frac{dz^2}{z} + dy^2\]

which is the natural metric corresponding this problem.

Denote by \(B_\eta\) the box \(B_\eta = \{ 0 \leq z \leq \eta^2, \ |y - y_0| \leq \eta \} \) and for any two points \(Q_1 = (z_1, y_1)\) and \(Q_2 = (z_2, y_2)\) in \(B_\eta\), by \(s\) the distance function

\[(DF) \quad s(Q_1, Q_2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2|\]

with respect to the singular metric \(ds^2\). Let \(C^\alpha_s(B_\eta)\) be the space of all Hölder continuous functions on \(B_\eta\) with respect to the distance function \(s\). Suppose that the function \(q\) belongs to the class \(C^\alpha_s(B_\eta)\) and has continuous derivatives \(q_t, q_z, q_y, q_{zz}, q_{zy}, q_{yy}\) in the interior of \(B_\eta\), and that

\[(2.6) \quad q, q_z, q_y, \sqrt{z} q_{zy}, q_{yy} \in C^\alpha_s(B_\eta)\]

extend continuously up to the boundary, and the extensions are Hölder continuous on \(B_\eta\) of class \(C^\alpha_s(B_\eta)\) as before. We denote by \(C^{2+\alpha}_s(B_\eta)\) the Banach space of all such functions with norm

\[
\|q\|_{C^{2+\alpha}_s(B_\eta)} = \|f\|_{C^\gamma(B_\eta)} + \|Dq\|_{C^\gamma(B_\eta)} + \|q_t\|_{C^\gamma(B_\eta)} + \|q_z\|_{C^\gamma(B_\eta)} + \|q_y\|_{C^\gamma(B_\eta)} + \|\sqrt{z} q_{zy}\|_{C^\gamma(B_\eta)} + \|q_{yy}\|_{C^\gamma(B_\eta)}.
\]

**Definition 2.6.** We say that \(g \in C^{2+\alpha}_s(\Omega(g))\) if \(g\) is of class \(C^{2,\alpha}\) in the interior of \(\Omega(g)\) and its transformation \(q \in C^{2,\alpha}_s(B_\eta)\) near any free-boundary point \(P_0\). We denote by \(\|g\|_{C^{2,\alpha}_s}\) the corresponding norm.

The following result follows as an easy modification of Theorem 5.1 in [DH2].

**Theorem [DH]** (Schauder estimate). Assume that the coefficients of the operator \(L\) given by \((2.5)\) belong to the class \(C^\alpha_s(B_\eta)\), for some \(\eta > 0\), and \((a_{ij})\) is strictly positive. Then, for any \(r < \eta\)

\[
\|\tilde{q}\|_{C^{2,\alpha}_s(B_r)} \leq C (\|\tilde{q}\|_{C^\alpha_s(B_\eta)} + \|h\|_{C^\gamma(B_\eta)})
\]

for all smooth functions \(\tilde{q}\) on \(B_\eta\) for which \(L\tilde{q} = h\).

The following result was shown in [DL2].
Theorem [DL] (Hölder regularity). Assume that the coefficients of the operator $L$ given by (2.5) are bounded measurable on $B_\eta$, $\eta > 0$, with $(a_{ij})$ strictly positive and $b \geq \nu > 0$. Set $d\mu = x^{\frac{\nu}{2} - 1} dx dy$. Then, there exist a number $0 < \alpha < 1$ so that, for any $r < \eta/2$

$$\|\tilde{q}\|_{C^\alpha(B_r)} \leq C \left( \|\tilde{q}\|_{C^\infty(B_\eta)} + \left( \int_{B_\eta} h^2 d\mu \right)^{1/2} \right)$$

for all smooth functions $\tilde{q}$ on $B_\eta$ for which $L\tilde{q} = h$.

Based on Theorem [DL] and the sharp a priori bounds Theorem 2.8, the following a priori estimate will be shown in section 4.

Theorem 2.7 ($C^{2,\alpha}_x$-estimate). Assume that $g \in C^4(\Omega(g))$ is a classical solution of problem (MAP), with $0 < p < 2$, and that $B_\rho(0) \subset \Lambda(g)$. Then, there exists a constant $C = C(\|\phi\|_{C^{3,\alpha}(\theta, \rho)}, \rho) > 0$ such that $\|g\|_{C^{2,\alpha}_x(\Omega(g))} \leq C$.

Based on Theorem [DH] and Theorem 2.7, a $C^{2,\alpha}_x$ solution of the problem (MAP) will be constructed via the method of continuity. It follows from Theorem [DH] and an inductive argument that the pressure $g$ is $C^\infty$ smooth up to the interface $\Gamma(g)$, which readily implies that the interface is smooth (c.f. section 4).

2.3. Existence of solutions via the method of continuity. We will now outline the basic steps of the proof of the existence of a classical solution of (MAP) via the method of continuity. The proofs of these steps will be given in the following sections.

According to our assumption in Theorem 1.1, there exists a super-solution $\psi$ of (MA), i.e., $\psi$ satisfies

$$\det(D^2 \psi) \leq \psi^p, \quad \text{in } \Omega$$

which vanishes on a non-empty domain $\Lambda(\psi) \subset \Omega$. We define

$$\overline{h} := \frac{\det(D^2 \psi)}{\psi^p} \leq 1.$$

Before we proceed with the outline of the method of continuity, let us give an example which shows that there exist boundary values $\phi$ for which such a super-solution can be found.

Example. Set

$$\psi_1(P) = c_1(|P|^2 - \rho^2)_+, \quad q = \frac{3}{2 - p}$$
and pick a $c_1 > 0$ so that
\[ 2\lambda < \frac{\det(D^2\psi_1(P))}{\psi_1(P)^p} < \frac{1}{2} \]
for some $\lambda \in (0, 1)$ in $\Omega(\psi_1)$. When the boundary data $\varphi$ in (MA) is such that
\[ \varphi \geq \psi_1, \quad \text{on } \partial \Omega \]
we can modify $\psi_1$ to a convex function $\psi(P)$, keeping the decay rate to zero on $\partial \Omega(\psi)$, so that $\psi(P) = \varphi(P)$ on $\partial \Omega$ and
\[ \lambda < \frac{\det(D^2\psi(P))}{\psi(P)^p} := \bar{h}(P) < 1. \]
Hence, $\psi$ is the desired super-solution.

Going back to the method of continuity, we consider the following boundary value problems depending on a parameter $t \in [0, 1]$:
\[
\begin{aligned}
(MAt) \quad & \begin{cases}
\det(D^2f(P)) = ((1-t)\bar{h} + t)f^p & \text{in } \Omega \\
f = \varphi & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]
Set $h_t := (1-t)\bar{h} + t$ and observe that
\[ \lambda < h_t \leq 1 \]
since $\bar{h}$ satisfies (2.7). Hence, $h_t$ satisfies condition (2.1). Also, since $h_t \geq \bar{h}$, a classical solution $f(P; t)$ of (MAt) is a sub-solution of
\[
\begin{aligned}
(2.8) \quad & \begin{cases}
\det(D^2f(P)) = \bar{h} f^p & \text{in } \Omega \\
f = \varphi & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]
while the given $\psi(x)$ is a super-solution of (2.8). Hence, by the comparison lemma 2.15, if $\{\psi(P) = 0\} \subset \{f(P; t) = 0\}$, then $f(P; t) \leq \psi(P)$ in $\Omega$. We are going to carry out the method of continuity starting with $f_0 = \psi(x)$ at $t = 0$, keeping
\[ \{\psi(P) = 0\} \subset \{f(P; t) = 0\}, \quad \text{for } 0 \leq t \leq 1 \]
so that $f(P; t)$ has a non-empty vanishing region $\Lambda(f(P; t))$, for every $t \in [0, 1]$. This justifies our assumption (H-2) below.

Assume that $f$ is a classical solution of (MAt) (we will drop the index $t$ on $f$ for the rest of the section). Then, the corresponding pressure function $g$, defined
in terms of \( f \) by \([1.1]\), satisfies

\[
\begin{aligned}
\text{(MAPt)} \quad \begin{cases}
g \det D^2 g + \theta (g^2_{xy}g_{yy} - 2g_x g_y g_{xy} + g^2_{y} g_{xx}) = h_t & \text{in } \Omega \\
g = \varphi & \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]

for \( \varphi = q^* \varphi^* \).

We make the following assumptions:

(H-1) \( \Omega \subset B_1(0) \).

(H-2) \( f \) and \( g \) vanish on a non-empty sub-domain \( \Lambda(f) = \Lambda(g) \subset \subset \Omega \) and \( B_\rho(0) = \{ x \in \mathbb{R}^2 : |x| < \rho \} \subset \Lambda(f) \), for some \( \rho > 0 \).

(H-3) \( f \) is strictly positive and strictly convex on \( \Omega(f) = \{ x \in \Omega | f > 0 \} \).

(H-4) The pressure \( g \) satisfies \( g \in C^4(\Omega(g)) \), i.e., in particular it is \( C^4 \)-smooth up to \( \partial \Omega(g) \).

To simplify the notation, we will set from now on

\[
\|g\|_{C^2_{\partial \Omega}} = \max_{\partial \Omega}(|D_{ij} g| + |D_t g| + g).
\]

In the next section we will establish sharp a-priori bounds on the first and second derivatives of the pressure \( g \) up to the interface \( \partial \Omega \), as stated in the sequel.

**Theorem 2.8 (C^2\text{-estimate}).** Assume that \( g \) is a classical solution of equation \([\text{MAPt}]\) in \( \Omega \) with \( 0 < p < 2 \) and \( h \in C^2(\Omega) \) satisfying \([2.1]\). Assume in addition that \( g \) satisfies the assumptions (H-1)–(H-4). Define the matrix

\[
M = (\mu_{ij}) = \begin{pmatrix}
g g_{\nu \nu} + \theta g^2_{\nu \tau} \\
\sqrt{g} g_{\nu \tau}
g_{\tau \tau}
\end{pmatrix}
\]

with \( \nu, \tau \) denoting the outer normal and tangent direction to the level sets of \( g \) respectively. Then, there exists \( c = c(\|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2}, \theta, \lambda, \rho) > 0 \), for which the bounds

\[
(2.10) \quad c \leq |Dg| \leq c^{-1}
\]

and

\[
(2.11) \quad c |\xi|^2 \leq \mu_{ij} \xi_i \xi_j \leq c^{-1} |\xi|^2, \quad \forall \xi \neq 0
\]

hold on \( \Omega(g) \).

Combining Theorem 2.8 with the Hölder regularity Theorem \([DL]\), we will show in section 4 the following a priori estimate.
**Theorem 2.9** \((C^2,\alpha)-estimate\). Under the same conditions as in Theorem 2.8, there is a uniform \(0 < \alpha < 1\) and \(C = C(\|g\|_{C^2},\|h\|_{C^2}, \theta, \lambda, \rho) < \infty\), such that
\[
\|g\|_{C^2,\alpha}(\Omega(g)) \leq C.
\]
In addition the curvature \(\kappa(g)\) of the free-boundary \(\Gamma(g)\) is of class \(C^\alpha\).

The above result shows that the coefficients of the matrix (2.9) are uniformly Hölder. This will be combined in section 4 with the Schauder estimate, Theorem [DH], to obtain the following regularity of \(g\).

**Theorem 2.10** \((Higher\ regularity)\). Under the same conditions as in Theorem 2.8 and the additional assumption that \(h \in C^\infty(\Omega)\), the solution \(g\) of (MAP\(h\)) is smooth on \(\Omega(g)\) up to the interface \(\Gamma(g)\) which means that for every positive integer, there exists \(C_k = C(\|g\|_{C^2h},\|h\|_{C^{k+2}}, \theta, \lambda, \rho, k) < \infty\) for which
\[
\|g\|_{C^{k+2,\alpha}(\Omega(g))} \leq C_k
\]
and the curvature \(\kappa(g)\) of \(\Gamma(g)\) is \(C^{k,\alpha}\). It follows that \(g\) is \(C^\infty\)-smooth up to the interface \(\Gamma(g)\) and that the interface is smooth.

To implement the method of continuity, we next set
\[
I = \{t \in [0, 1] | (MAP_t) has a classical solution satisfying (H-1)-(H-4)\}.
\]
Clearly \(I\) is nonempty since by the assumption of Theorem 1.1, \(\psi\) is a solution of (MAP\(t\)) for \(t = 0\). The existence of classical solution of (MAP\(h\)) is equivalent to that \(1 \in I\). The method of continuity relies on showing that the nonempty set \(I\) is both open and closed in \([0, 1]\) in the relative topology, which means that \(I = [0, 1]\) and hence \(1 \in I\).

The closedness of \(I\) easily follows from Theorems 2.8 and 2.9 as shown next.

**Lemma 2.11.** The set \(I\) is closed.

**Proof.** Let \(\{t_k\} \subset I\) be a sequence converging to \(t_0\). Then, there is a sequence of solutions \(\{g_k\}\) of (MAP\(t\)), \(t = t_k\), and their free-boundaries \(\Gamma(g_k)\) which have uniform estimates depending only on the boundary data and the domain \(\Omega\). First we can extract a converging subsequence of the free boundaries \(\Gamma(g_k)\) to \(\Gamma_0\) and, among them, extract converging subsequence \(g_{k_i}\) converging to a function \(g_0\). The non-degeneracy estimate in (2.10) implies that \(\Gamma_0 = \Gamma(g_0)\) and the uniform \(C^2,\alpha\)-estimate in Theorem 2.9 implies that \(g_0\) is a solution of (MAP\(t\)) with \(t = t_0\). Hence \(t_0 \in I\).
The openness of $I$ will be proved in Section 5 through the stability in the parameter $t$, Theorem 5.1, which is similar to Theorem 8.5 in [DH2].

The method of continuity then implies the following existence of classical solutions.

**Theorem 2.12 (Existence of a classical solution).** Under the assumptions of Theorem 1.1 there is a classical solution $g$ of (MAP) which satisfies the estimates in Theorems 2.8, 2.9, and 2.10.

The rest of the paper will be devoted to the proof of Theorems 2.8 - Theorem 2.10 which, in particular, imply Theorem 1.1.

### 3. Optimal Estimates

In this section we are going to prove the optimal a’priori estimates stated in Theorem 2.8. We will assume, throughout this section, that $g \in C^4(\Omega(g))$ is a classical solution of equation (MAP) in $\Omega$ with $0 < p < 2$ and $h \in C^2(\Omega)$ satisfying (2.1). In addition, we will assume that $g$ satisfies the assumptions (H-1)–(H-4) introduced in section 2.3. We recall the notation $\Omega(g) = \{x \mid g(x) > 0\}$ and $\|g\|_{C^2_{\partial \Omega}} = \max_{\partial \Omega}(\|D_{ij}g\| + |D_i g| + g)$.

We will first establish an upper bound on the first order derivative $|Dg|$.

**Lemma 3.1.** Under the assumptions of Theorem 2.8 we have

$$\max_{\Omega} |Dg| \leq C(\rho, \theta, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^1}).$$

**Proof.** We set $M := r^2 |Dg|^2 = (x^2 + y^2)(g_x^2 + g_y^2)$. We will show that $M$ attains its maximum at $\partial \Omega$. This readily implies the desired bound, since $r^2 = x^2 + y^2 \geq \rho^2$ on $\Omega(g)$. (Notice that we cannot bound $|Dg|^2$ from above by the maximum principle, if $h \neq 1$, so we need to multiply by $r^2$).

Let $P_0$ be the maximum point of $M$ on $\Omega(g)$. Assume first that $P_0 \in \Omega(g)$. We may also assume, by rotating the coordinates, that

$$g_y = 0 \quad \text{and} \quad g_x > 0 \quad \text{at } P_0.$$  

(3.1)

Also, since $M_x = M_y = 0$ at $P_0$, we have

$$g_{xx} = -\frac{x g_x}{r^2}, \quad \text{and} \quad g_{xy} = -\frac{y g_x}{r^2} \quad \text{at } P_0$$

(3.2)

which combined with (3.1) and (MAP) gives that

$$\left(\theta g_x^2 - \frac{x g y g_x}{r^4}\right) g_{yy} - \frac{y^2 g y^2}{r^4} = h$$
and hence

\[(3.3) \quad g_{yy} = \frac{r^4 h + y^2 g_{g_x}^2}{r^2 g_x(\theta r^2 g_x - x g)} \quad \text{at } P_0.\]

Let

\[(3.4) \quad A = (a_{ij}) = (g g_{ij} + \theta g_i g_j)^t\]
denote the transpose of the matrix \(G = (G_{ij}) = (g g_{ij} + \theta g_i g_j).\) This is the second order derivative coefficient matrix of the linearization of equation (MAPh). Differentiating equation (MAPh) to eliminate the third order derivatives on \(a_{ij} M_{ij}\) and using (3.1), (3.2) and (3.3), we find, after a direct calculation, that

\[(3.5) \quad a_{ij} M_{ij} = \sum_{i=0}^{6} b_i M^i \frac{D}{D} \quad \text{with} \quad D = M r^5 (M \theta r - x g) \quad \text{and} \quad b_6 = 2 \theta (\theta x^2 + y^2)\]

and

\[|b_i| \leq C(\theta, \|h\|_1), \quad i = 1, \ldots, 5.\]

Since \(r \geq x,\) assuming that \(M > \theta^{-1} \max \Omega g\) we conclude that \(D > 0\) at \(P_0.\) Since the leading order term in (3.5), when \(M\) is sufficiently large, is \((b_6 M^6)/D\) and \(b_6 > 0\) we conclude that either \(M \leq C(\rho, p, \lambda, \|g\|_{C^1}, \|h\|_{C^1})\) at \(P_0\) or \(a_{ij} M_{ij} > 0.\) In the latter case \(P_0\) cannot be a maximum point, contradicting our assumption.

Assume next that \(P_0 \in \Gamma(g)\) and that \(M > 0\) at \(P_0.\) We may assume again that (3.1) holds at \(P_0,\) i.e. \(y\) is a tangential direction to \(\Gamma(g).\) Hence, \(M_y = 0, M_x \leq 0\) and \(M_{yy} \leq 0\) also hold at \(P_0.\) In addition, since \(g = 0\) at \(P_0,\) equation (MAPh) and (3.1) imply that \(\theta g_{x}^2 g_{yy} = h\) at \(P_0.\) We conclude, after some direct calculations, that

\[(3.6) \quad g = 0, \quad g_{xx} \leq -\frac{g_{x}^2}{r^2}, \quad g_{xy} = -\frac{g_{x} g_{y}}{r^2}, \quad g_{yy} = \frac{h}{\theta g_{x}^2} \quad \text{at } P_0.\]

and

\[(3.7) \quad M_{yy} = 2 r^2 g_x g_{x y} + \frac{2(x^2 - 2y^2) g_x^2}{r^2} + \frac{2r^2 h^2}{g_y g_x^4} \leq 0, \quad \text{at } P_0.\]

On the other hand, differentiating equation (MAPh) with respect to \(x\) and using (3.1) and (3.6) we find that

\[\theta g_{x}^2 g_{x y} - \frac{(1 + 2\theta) y^2 g_x^3}{r^4} - h_x - \frac{(1 + 2\theta) x h}{r^2} = 0, \quad \text{at } P_0\]
which implies that $g_{yy} = -\theta^{-1}(1 + 2\theta)\frac{g_{xx}}{g_{x}^2}$. Substituting in (3.7) gives that

$$M_{yy} = \frac{2(\theta x^2 + y^2)M^2}{\theta r^4} + \frac{2(1 + 2\theta)rxh + r^3h_x}{\theta\sqrt{M}} + \frac{2r^6 h^2}{M^2} \leq 0$$

which is impossible, if we assume that $M$ is sufficiently large, depending on the data. This finishes the proof. □

We will next provide a bound from below on $|Dg|$.

**Lemma 3.2.** Under the assumptions of Theorem 2.8, we have

$$|Dg| \geq c(\rho, \theta, \lambda, \max_{\partial\Omega}|Dg|, \|h\|_{C^1}) > 0,$$  on $\Omega(g)$.

**Proof.** For $q > 0$ we set

$$M := (x^2 + y^2)^{-q} (xg_x + yg_y) = r^{-2q} g_r,$$  $(x, y) \in \Omega(g)$

with $g_r$ denoting the radial derivative of $g$.

**Claim:** There exists an integer $q > 1$ which depends only on data (on $\|h\|_{C^1}$ and $\theta$) and such that $M \geq c(\rho, \theta, \max_{\partial\Omega}|Dg|, \|h\|_{C^1}) > 0$ on $\Omega(g)$. Since, from condition (H-2) we have $r^2 = x^2 + y^2 > \rho^2$ for any $(x, y) \in \Omega(g)$, the claim readily implies the desired bound from below on $|Dg|$.

We will next prove the claim by the maximum principle. Let $P_0 = (x_0, y_0)$ be an interior minimum point of $M$ in $\Omega(g)$. We may assume, by rotating the coordinates, that

$$(3.8) \quad y = 0 \text{ and } x > \rho > 0 \quad \text{at } P_0.$$  

Since $M_x = 0$ and $M_y = 0$ at $P_0$ we have

$$xg_{xx} - (2q - 1)g_x = 0 \quad \text{and} \quad xg_{xy} + g_y = 0 \quad \text{at } P_0,$$

and hence

$$(3.9) \quad g_{xx} = \frac{(2q - 1)g_x}{x} \quad \text{and} \quad g_{xy} - \frac{g_y}{x} = 0 \quad \text{at } P_0.$$  

Substituting the above to equation (MAPh), using also (3.8), gives

$$(3.10) \quad g_{yy} = \frac{x^2(1 + h) + g_y^2}{xg_x [g - (2q + 1)\theta xg_x]} \quad \text{at } P_0.$$  

Let $A = (a_{ij})$ be the matrix defined in (3.4). Differentiating equation (MAPh) and (3.8) - (3.10) we find, after several direct calculations, that

$$L := a_{ij} M_{ij} = \sum_{i=0}^{4} b_i \frac{M^i}{D}$$
with $D = (M \theta x^{2q} + (2q - 1)g) > 0$ and

$$|b_i| \leq C(\rho, \theta, \lambda, \max_{\partial\Omega} |Dg|, \|h\|_{C^1})$$

and

$$b_0 = -(2q - 1)x^{-2q-2}g[2(q - 2)x^2(h + 1) + 2qgg_y - x^3h_x].$$

By choosing $q > 1$ sufficiently large (depending on $\|h\|_{C^1}$) so that

$$2(q - 2)(x - h) > 0,$$

we can make $b_0 < 0$. We conclude from the above that $L \leq 0$ unless $M(P_0) \geq c > 0$, for some constant $c = c(\rho, \theta, \lambda, \max_{\partial\Omega} |Dg|, \|h\|_{C^1})$. This shows that an interior minimum of $M$ must satisfy $\min M \geq c(\rho, \theta, \lambda, \max_{\partial\Omega} |Dg|, \|h\|_{C^1}) > 0$.

Assume next that $P_0 \in \Gamma(g)$ is a minimum point for $M$. We may assume this time that (3.11) holds at $P_0$. Hence,

$$M_y = \frac{-2qxyg_x + r^2(yg_y + xg_{xy})}{r^2(q+1)} = 0$$

and

$$M_x = \frac{[(1 - 2q)x^2 + y^2]g_x + r^2(yg_{xy} + xg_{xx})}{r^2(q+1)} \geq 0$$

and also, by equation (MAPh), $\theta g_x^2 g_{yy} = h$ at $P_0$. Substituting $g_{yy} = h/(\theta g_x^2)$ in (3.11) and solving with respect to $g_{xy}$ gives

$$g_{xy} = -\frac{yh}{\theta x g_x^2} + \frac{2pg g_x}{r^2}.$$

Substituting this in (3.12) and solving with respect to $g_{xx}$ gives

$$g_{xx} \geq \frac{y^2 h}{\theta x^2 g_x^2} + \frac{[(2q - 1)x^2 - (2q + 1)y^2]g_x}{r^2 x}.$$

Here we have used that $x > 0$ at $P_0$. This follows from assumption (3.11), (H-2) and the convexity of $\Lambda(g)$.

We next differentiate equation (MAPh) with respect to $x$ and use that $g = 0$, $g_y = 0, g_{yy} = h/(\theta g_x^2)$ and (3.13) to conclude that

$$g_{yyy} = \frac{h_y}{\theta g_x^2}, \quad \text{at } P_0.$$

Also, we differentiate equation (MAPh) with respect to $y$ and use that $g = 0$, $g_y = 0, g_{yy} = h/(\theta g_x^2)$, (3.13) - (3.15), to conclude that

$$g_{xxy} \leq \frac{h_x}{\theta g_x^2} + \frac{(1 + 2\theta)(1 - 2q)h}{\theta^2 x g_x^2} + \frac{4(1 + 2\theta)q^2 y^2 g_x}{\theta r^4}.$$
We next differentiate $M$ twice with respect to $y$ and use (3.13) - (3.16), $g_y = 0$, $g = 0$ and $g_{yy} = h/(\theta g_x^2)$. We obtain, after several direct calculations, that

$$M_{yy} = \frac{x g_{xxy}}{y^2} - \frac{2q x(x^2 + (2q - 1)y^2)g_x}{r^2(y^2+1)} + \frac{2h + y h y}{\theta r^2 g_x^2}$$

at $P_0$. Substituting (3.16) and $g_x = M r^{2q - 1}$ in the above gives, after several calculations, that

$$M_{yy} \leq b_1 M + \frac{b_0}{M}$$

with

$$b_1 = \frac{2q(2q + \theta + \theta q)g^2 - \theta r^2}{\theta r^4}$$

and

$$b_0 = \frac{x^2 ([1 + 4\theta - 2q(1 + 2\theta)] h + \theta y h y + \theta x h x)}{\theta^2 r^6 q}.$$ 

Observe that since $h \geq \lambda > 0$ we may choose $q$ sufficiently large, depending on $\|h\|_{C^1}$, to make $b_0 < -1$. Since, $M_{yy} \geq 0$ at $P_0$ and $r \geq \rho$ (by our assumption (H-2)) we conclude that $M \geq c(\rho, \theta, \lambda, \max_{\partial \Omega} |Dg|, \|h\|_{C^1}) > 0$, finishing the proof. □

We will next establish sharp upper bounds on the second order derivatives of $g$. We begin by an upper bound on the rotationally invariant quantity

$$G := g_x^2 g_{yy} - 2g_x g_y g_{xy} + g_y^2 g_{xx} = g_\nu^2 g_\tau \tau$$

where $\nu$, $\tau$ denote the outer normal and tangential directions to the level sets of $g$ respectively. Since the level sets of $g$ are convex (because the function $f$ is convex) we have $G \geq 0$.

**Lemma 3.3.** Under the assumptions of Theorem 2.8, the quantity $G = g_\nu^2 g_\tau \tau$ satisfies

$$\max_{\Omega(g)} G \leq C(\theta, \rho, \lambda, \max_{\partial \Omega} |Dg|, \|h\|_{C^2}).$$

**Proof.** Set $M := G + |Dg|^2$. We will estimate $M$ by the maximum principle. Since $g$ is assumed to be in $C^4(\overline{\Omega(g)})$, and hence $g g_{ij} = 0$ at $\Gamma(g)$, it follows from equation (MAPh) that $M = \theta^{-1} h + |Dg|^2$ at $\Gamma(g)$. Hence, we only need to control $M$ in the interior of $\Omega(g)$. Assuming that the maximum of $M$ is attained at an interior point $P_0 \in \Omega(g)$, we will show that

$$a_{ij} M_{ij} = \frac{1}{M^2 (1 + g_x^2)} \sum_{i=0}^4 A_i M^i, \quad \text{at } P_0$$

with $A = (a_{ij})$ given by (MAPh), $A_i \geq c(\theta, \rho, \lambda, \max_{\partial \Omega} G, \|h\|_{C^2}) > 0$ and $|A_i| \leq C(\theta, \rho, \lambda, \max_{\partial \Omega} G, \|h\|_{C^2})$, for $i = 0, ..., 3$. Since $a_{ij} M_{ij} \leq 0$ at a maximum point,
this will imply the inequality $M \leq C(\theta, \rho, \lambda, \max_{\partial\Omega} G, \|h\|_{C^2})$, showing that $\max_{\Omega(g)} M \leq C(\max_{\partial\Omega} M, \theta, \rho, \lambda, \max_{\partial\Omega} G, \|h\|_{C^2})$, as desired.

To prove (3.17), we begin by noticing that since $M$ is rotationally invariant we may assume that (3.1) holds at $P_0$, i.e., $g_x > 0$, $g_y = 0$ and $M = g_x^2 (g_{yy} + 1)$ at $P_0$. Also, by a standard change of variables (see in the proof of Proposition 4.1 in [S]), we may also assume that $g_{xy} = 0$ at $P_0$. Using (3.1) we compute that

$$M_x = g_x^2 g_{xy} + 2 g_x g_{xx}(1 + g_{yy}) = 0, \quad M_y = g_x^2 g_{yyy} = 0 \quad \text{at } P_0$$

implying that

$$(3.18) \quad g_{yyy} = -\frac{2 g_{xx}(1 + g_{yy})}{g_x}, \quad g_{xyy} = 0, \quad \text{at } P_0.$$ 

Differentiating equation (MAPh) in $y$, using (3.18) and solving with respect to $g_{xyy}$ we obtain

$$(3.19) \quad g_{xyy} = \frac{h_y}{g_y} g_{yy} \quad \text{at } P_0.$$ 

since $g_y = g_{xy} = 0$ at $P_0$. Also, differentiating equation (MAPh) in $x$, using (3.18)-(3.19) and solving with respect to $g_{xxx}$ we obtain

$$(3.20) \quad g_{xxx} = \frac{h_x g_x + g_{xx} [(2 \theta - g_{yy}) g_x^2 + 2 g (1 + g_{yy}) g_{xx}]}{g_y g_x g_{yy}} \quad \text{at } P_0.$$ 

We next differentiate equation (MAPh) twice in $y$, multiply it by $g_x^2$ and subtract it from $a_{ij} M_{ij}$ to eliminate fourth order derivatives, while use (3.18)-(3.20) to eliminate third order derivatives. After several direct calculations, using also that $g_y = 0 = g_{xy} = 0$ at $P_0$, we obtain that

$$a_{ij} M_{ij} = \frac{1}{M^2 (1 + g_x^2)} \sum_{i=0}^4 A_i M^i$$

with

$$A_4 = 3 \theta (1 + 4 \theta) g_x^4$$

and

$$|A_i| \leq C(\theta, \rho, \lambda, \|h\|_{C^2}, \|g\|_{C^1}), \quad i = 0, \ldots, 3.$$ 

By the previous two Propositions, $0 < c \leq g_x \leq C < \infty$. Hence, $A_4 > 0$, while $A_i, \ i = 0, \ldots, 3$ bounded. This shows that at an interior maximum point, $M \leq C(\theta, \rho, \lambda, \max_{\partial\Omega} G, \|h\|_{C^2})$, hence finishing the proof of the Lemma. □
We will now bound
\[(3.21)\]
\[Q := \max_{\gamma} (g D_{\gamma\gamma} g + \theta |D_{\gamma} g|^2), \quad \theta = \frac{1 + p}{2 - p}\]
from above, where the maximum in (3.21) is taken over all directions \(\gamma\). Note, that in terms of the function \(f\), we have
\[Q = \max_{\gamma} (q^{1/3} f^{1/3} f_{\gamma\gamma}), \quad q = \frac{3}{2 - p} \]
In particular, since \(f\) is convex, \(Q \geq 0\).

**Lemma 3.4.** Under the assumptions of Theorem 2.8, we have
\[(3.22)\]
\[\max_{\Omega(g)} Q \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2}).\]

**Proof.** We begin by observing that since \(g \in C^4(\Omega(g))\), by Lemma 3.1, the bound
\[Q = \theta |Dg|^2 \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^1})\]
holds.

Assume next that the maximum of \(Q\) is attained at an interior point \(P_0 \in \Omega(g)\) and at a direction \(\gamma\), so that
\[(3.23)\]
\[Q(P_0) = g D_{\gamma\gamma} g + \theta |D_{\gamma} g|^2.\]
Let \(\nu, \tau\) denote the outward normal and tangential directions to the level sets of \(g\) respectively.

**Claim.** Either \(Q(P_0) \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2})\) or \(\gamma = \nu\).

To prove the claim, we begin by expressing the maximum direction \(\gamma\) as \(\gamma = \lambda_1 \nu + \lambda_2 \tau\), with \(\lambda_1^2 + \lambda_2^2 = 1\) so that
\[(3.23)\]
\[Q(P_0) = g [\lambda_1^2 g_{\nu\nu} + 2 \lambda_1 \lambda_2 g_{\nu\tau} + \lambda_2^2 g_{\tau\tau}] + \lambda_1^2 g_{\nu}^2.\]
Next, we use the equation (MAPh) expressed in the form
\[(g g_{\nu\nu} + \theta g_{\tau}^2) g_{\tau\tau} = g g_{\nu\tau} + h\]
and the bounds in Lemmas 3.1 - 3.3 to first conclude the bound
\[Q(P_0) \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2})\]
unless \(gg_{\nu\nu}\) is sufficiently large at \(P_0\). If in particular \(gg_{\nu\nu} > \theta g_{\nu}^2\) at \(P_0\), we then conclude from (MAPh) and Lemmas 3.2 and 3.3 that
\[g g_{\nu\tau}^2 \leq 2 g g_{\nu\nu} g_{\tau} \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2}) g g_{\nu\nu}\]
showing the bound \(g_{\nu\tau} \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2}) \sqrt{g_{\nu\nu}}\). Using once more the bound \(g_{\tau\tau} \leq C(\theta, \rho, \lambda, \|g\|_{C^2_{\partial \Omega}}, \|h\|_{C^2}, \theta)\), we readily deduce from (3.23) that \(Q(P_0)\)
becomes maximum when $\lambda_2 = 0$, provided it is sufficiently large, depending only on $\|g\|_{C^{2}_0}, \theta, \rho$. This proves the Claim.

If $Q(P_0) \leq C(\theta, \rho, \lambda, \|g\|_{C^{2}_0}, \|h\|_{C^2})$ then the proof of the Proposition is complete. Otherwise, from the previous claim we may assume that $Q(P_0) = gg_{\nu\nu} + \theta g^2_{\nu}$ and also, since $\nu$ is the maximum direction, that $gg_{\nu\tau} + \theta g_{\nu}g_{\tau} = 0$ at $P_0$, implying that $g_{\nu\tau} = 0$ at $P_0$, since $g > 0$ and $g_{\tau} = 0$ at $P_0$. Also, by rotating the coordinates, we may assume that \(3.1\) holds at $P_0$, i.e., the direction of the vector $\nu$ is that of the $x$-axis.

We will show that

\[
0 \geq a_{ij} Q_{ij} = g \left[ (1 + h) Q + C(\theta, \rho, \lambda, \|g\|_{C^{2}_0}, \|h\|_{C^2}) \right] \quad \text{at } P_0
\]

with $A = (a_{ij})$ given by \(3.1\). Since $h > 0$, this implies the bound $Q \leq C(\theta, \|h\|_{C^2}, \|g\|_{C^1})$ at $P_0$, which combined with Lemma 3.2 implies the desired estimate.

To prove \(3.24\) let us first summarize that

\[
g_x > 0, \quad g_y = 0, \quad g_{xy} = 0 \quad \text{at } P_0.
\]

Also, since $Q = gg_{xx} + \theta g^2_x$ at $P_0$, equation \(\text{MAPh}\) together with conditions \(3.25\) imply that

\[
g_{yy} = h Q^{-1} \quad \text{at } P_0.
\]

We next differentiate $Q$ is $x$ and $y$ and use \(3.25\) to deduce the equalities

\[
Q_x = gg_{xxx} + (1 + 2 \theta) g_x g_{xx}, \quad G_y = gg_{xyy} = 0 \quad \text{at } P_0
\]

from which we conclude that

\[
g_{xxx} = \frac{(1 + 2 \theta) g_x g_{xx}}{g}, \quad g_{xyy} = 0, \quad \text{at } P_0.
\]

Also, differentiating equation \(\text{MAPh}\) in $x$, using \(3.25\) and \(3.27\) gives

\[
g_{xyy} = h x Q^{-1} \quad \text{at } P_0.
\]

We next differentiate twice the equation \(\text{MAPh}\) in $x$ to eliminate the fourth order derivatives from $a_{ij} Q_{ij}$ and use \(3.27\)-\(3.28\) to eliminate third order derivatives and also \(3.25\)-\(3.28\) to finally conclude, after several calculations, that

\[
a_{ij} Q_{ij} = g \left[ (1 + h) G + C(\theta, \rho, \lambda, \|g\|_{C^{2}_0}, \|h\|_{C^2}) \right]
\]

by the previous Lemmas and our assumptions. This finishes the proof of the Lemma.
We are now going to combine the estimates in Lemmas 3.1-3.4 to give the proof of Theorem 2.8.

Proof of Theorem 2.8. We begin by expressing (MAPh) in the form
\[ \det M = (g g_{\nu \nu} + \theta g_\nu^2) g_{\tau \tau} - g g_\nu^2 = h. \]

Hence, it is enough to establish the bounds
\[ c \leq g g_{\nu \nu} + \theta g_\nu^2 \leq c^{-1} \quad \text{and} \quad c \leq g_{\tau \tau} \leq c^{-1} \]
for a constant \( c = c(\theta, \rho, \lambda, \|g\|_{C^2}, \|h\|_{C^2}) > 0. \)

The bounds from above readily follow from Lemmas 3.3 and 3.4 combined with Lemma 3.2. The bounds from below follow from \((g g_{\nu \nu} + \theta g_\nu^2) g_{\tau \tau} = h + g g_\nu^2 \geq \lambda > 0 \) (from our assumption on \( h \)) and the corresponding bounds from above. \( \square \)

We next re-state Theorem 2.8 in terms of the solution \( f \) of (MAh).

Corollary 3.5. Assume that \( f \) is a non-negative weakly convex classical solution \( f \) of the boundary value problem (MAh) in \( \Omega \), with \( 0 < p < 2 \), which satisfies assumptions (H-1)-(H-4). Define the matrix
\[ (3.29) \quad M = (\mu_{ij}) = \begin{pmatrix} q^{\frac{1}{2}} f^{\frac{1-p}{2}} f_{\nu \nu} & f^{\frac{1}{2}} f_{\nu \tau} \\ f^{-\frac{1}{2}} f_{\nu \tau} & q^{-\frac{1}{2}} f^{-\frac{1+p}{2}} f_{\tau \tau} \end{pmatrix} \]
with \( \nu, \tau \) denoting the outer normal and tangent direction to the level sets of \( f \) respectively and \( q = 3/(2-p) \). Then, there exist a constant \( c = c(\theta, \rho, \lambda, \|f\|_{C^2_{\Omega}}, \|h\|_{C^2}) > 0 \) for which
\[ (3.30) \quad c |\xi|^2 \leq \mu_{ij} \xi_i \xi_j \leq c^{-1} |\xi|^2, \quad \forall \xi \neq 0. \]

We will finish this section with the following lower bound on \( \sqrt{g} \det D^2 g \), which will be used in the next section.

Proposition 3.6. Under the assumptions of Theorem 2.8, there exists a constant \( C = C(\|g\|_{C^2_{\Omega}}, \|h\|_{C^2}, \theta, \lambda, \rho) > 0 \), for which the bound
\[ (3.31) \quad \sqrt{g} \det D^2 g \geq -C \]
holds on \( \Omega(g) \).

Proof. Set \( Z := (x^2 + y^2) \sqrt{g} \det D^2 g \). We will use the maximum principle to establish the bound \( Z \geq -C \), which readily implies \((3.31)\), since \( x^2 + y^2 \geq \rho^2 \) on \( \Omega(g) \).
Clearly, \( Z \geq -C \) on \( \partial \Omega(g) \). Assume that the maximum of \( Z \) is attained at an interior point \( P_0 \in \Omega(g) \). Since \( Z \) is rotationally invariant, we may assume, without loss of generality, that

\[
\begin{align*}
g_x > 0, \quad g_y = 0 \quad \text{and} \quad g_{xy} = 0, \quad \text{at} \quad P_0.
\end{align*}
\]

Differentiating equation once and twice in \( x, y \) and using that \( Z_x = Z_y = 0 \) at \( P_0 \), we find, after several direct calculations, that at the minimum point \( P_0 \) where (3.33) holds, we have

\[
\begin{align*}
0 \leq a_{ij} Z_{ij} &= \frac{1}{4 \sqrt{g} g_1 g_2} \sum_{i=1}^{3} A_i Z^i, \quad \text{at} \quad P_0
\end{align*}
\]

with \( A = (a_{ij}) \) given by (3.4), and

\[
A_1 = 13 (x^2 + y^2)^2 g_1^4 g_22 + C \sqrt{g}
\]

and

\[
A_2 = -\sqrt{g} (117 (x^2 + y^2)^{3/2} g_1^2 g_22 + C g) \quad \text{and} \quad A_3 = -\frac{4g^2 x_2^2}{(x^2 + y^2)^2 g_22}.
\]

The constants \( C = C(g_1, g_22, x, y) \) depend only on \( g_1, g_22, x, y \) and hence they are bounded, by Theorem 2.8.

We will show that \( a_{ij} Z_{ij} < 0 \) at \( P_0 \) provided that \( Z < 0 \) is sufficiently large in absolute value and \( P_0 \) is sufficiently close to the free-boundary \( \Gamma(g) \), establishing a contradiction to \( a_{ij} Z_{ij} \geq 0 \) at the minimum point \( P_0 \) of \( Z \).

It is clear from the estimates in Theorem 2.8 that \( A_1 Z < 0 \) and \( A_2 Z^2 < 0 \), provided \( P_0 \) is sufficiently close to the free-boundary \( \partial \Omega \), i.e. \( g \) is sufficiently close to zero. The term \( A_3 Z^3 \) is nonnegative, however we observe that

\[
A_3 Z^3 = -\frac{4x_2^2 g (\sqrt{g} Z)^2}{(x^2 + y^2)^2 g_22} Z = CgZ
\]

with \( C \) bounded, since \( \sqrt{g} Z \) is bounded by the estimates in Theorem 2.8. Hence, \( \sum_{i=1}^{3} A_i Z^i \leq A_1 + A_3 < 0 \) at \( P_0 \), provided that \( Z < 0 \) is sufficiently large and \( P_0 \) is sufficiently close to the free-boundary \( \Gamma(g) \), which concludes the proof of the Proposition. \(\square\)

4. \( C^{2,\alpha}_s \)-Regularity

We will assume throughout this section that \( g \in C^4(\Omega(g)) \) is a classical solution of the boundary value problem \( \text{MAPh} \) in \( \Omega \), with \( 0 < p < 2 \) and \( h \in C^2(\Omega) \).
satisfying (2.1). In addition, we assume that $g$ satisfies the assumptions (H-1)–(H-4). Our goal is to establish a uniform estimate on the norm $\|g\|_{C^{2,\alpha}_{\Omega(g)}}$, as defined in section 2.2, by combining the a-priori estimates in Theorem 2.8 with the Hölder Regularity result Theorem [DL]. We will obtain estimates which depend only on the data $\|g\|_{C^{2,\alpha}_{\Omega}}, \|h\|_{C^{2}}, \theta, \lambda, \rho$.

Since the regularity theorem [DL] concerns with solutions on a fixed domain, we will first perform a change of coordinates, near the interface, which transforms the free-boundary problem (MAPh) to a nonlinear degenerate problem with fixed-boundary. The same coordinate change was used in [DH2]. We refer the reader to that paper for the detailed computations.

Let $P_0 = (x_0, y_0) \in \Gamma(g)$ be a free-boundary point. We may assume, by rotating the coordinates, that at the point $P_0$,

\[(4.1) \quad n_0 := \frac{P_0}{|P_0|} = e_1.\]

Then, by Theorem 2.8, $g_x(P) > 0$, for all points $P = (x, y)$ sufficiently close to $P_0$. Hence, we can solve around the point $P_0$, the equation $z = g(x, y)$ with respect to $x$, yielding to a map

\[x = q(z, y)\]

defined for all $(z, y)$ sufficiently close to $Q_0 = (0, y_0)$. Using the identities

\[g_x = \frac{1}{q_z}, \quad g_y = -\frac{q_y}{q_z}, \quad g_{xx} = -\frac{1}{q_z^2} q_{zz}\]

\[g_{xy} = -\frac{1}{q_z} \left( -\frac{q_y}{q_z^2} q_{zz} + \frac{1}{q_z} q_{zy} \right), \quad g_{yy} = -\frac{1}{q_z} \left( \frac{q_y^2}{q_z^2} q_{zz} - 2 \frac{q_y}{q_z} q_{zy} + q_{yy} \right)\]

which yield to

\[g_{xx} g_{yy} - g_{xy}^2 = \frac{1}{q_z^2} \left( q_{zz} q_{yy} - q_{zy}^2 \right)\]

and

\[g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_x^2 g_{yy} = -\frac{1}{q_z^2} q_{yy}\]

we find that $q$ satisfies the equation

\[(4.2) \quad -z \det D^2 q + \theta q_z q_{yy} = -H\]

with

\[(4.3) \quad H(z, y) = h(x, y), \quad x = q(z, y).\]

In addition, $q$ is a concave function, since $g$ is convex.
Consider the non-linear operator
\[ L_q := -z \det D^2 q + \theta q_z q_{yy} \]
The linearization \( \tilde{L} \) of \( L \) around a point \( q \) has the form
\[ L_q(\tilde{q}) = -z q_{yy} \tilde{q}_{zz} + 2z q_{zy} \tilde{q}_{zy} + (\theta q_z - z q_{zz}) \tilde{q}_{yy} + \frac{4z \det D^2 q - 3 \theta q_z q_{yy}}{q_z^5} \tilde{q}_z. \] (4.4)

Let us denote by \( B_\eta \) the box
\[ B_\eta = \{ 0 \leq z \leq \eta^2, |y - y_0| \leq \eta \} \] (4.5)
around \( Q_0 = (0, y_0) \) and by \( C^{2,\alpha}(B_\eta) \) the spaces defined in section 2.2.

Our goal in this section is to establish the following result:

**Theorem 4.1.** Assume that \( g \in C^4(\Omega(g)) \) is a non-negative classical solution of the boundary value problem \( (\text{MAPh}) \) on \( \Omega \), with \( 0 < p < 2 \) and \( h \in C^2(\Omega) \) satisfying condition (2.1). In addition, assume that \( g \) satisfies the assumptions (H-1)–(H-4).
Then, there exist constants \( 0 < \alpha < 1, C < \infty \) and \( \eta > 0 \), depending only on the data \( \|g\|_{C^4(\Omega(g))}, \|h\|_{C^2, \theta, \lambda, \rho} \), such that for any free-boundary point \( P_0 = (x_0, y_0) \), satisfying condition (4.1), the function \( x = q(z, y) \) satisfies the estimate
\[ \|q\|_{C^{2+\alpha}(B_\eta)} \leq C \]
on \( B_\eta = \{ 0 \leq z \leq \eta^2, |y - y_0| \leq \eta \} \).

Consider the matrix
\[ \mathcal{A} = (\alpha_{ij}) := q_z^{-4} \begin{pmatrix} -q_{yy} & \sqrt{z} q_{zy} \\ \sqrt{z} q_{zy} & \theta q_z - z q_{zz} \end{pmatrix} \] (4.6)
and the coefficient
\[ b := \frac{4z \det D^2 q - 3 \theta q_z q_{yy}}{q_z^5}. \] (4.7)

A direct consequence of Theorem 2.8 is the following a-priori bounds on \( \mathcal{A} \) and \( b \).

**Lemma 4.2.** Under the assumptions of Theorem 4.1, there exist positive constants \( c = c(\|g\|_{C^4(\Omega)}, \|h\|_{C^2, \theta, \lambda, \rho}) \) and \( \eta_0 \), for which the bounds
\[ 0 < c \|\xi\|^2 \leq \alpha_{ij} \xi_i \xi_j \leq c^{-1} \|\xi\|^2, \quad \forall \xi \neq 0 \] (4.8)
and

\begin{equation}
0 < c \leq b \leq c^{-1}
\end{equation}

hold on the box $B_{\eta}$, provided $\eta \leq \eta_0$.

Proof. By direct calculation

\begin{equation}
det A = \frac{z \det D^2 q - \theta q_z q_{yy}}{q_z^4} = h
\end{equation}

and

\begin{equation}
tr A = \frac{1}{g_z^3} \left[ \left( g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_x^2 g_{yy} \right) + \left( g g_{xx} + \theta g_x^2 \right) \right].
\end{equation}

By \((2.1)\), $\lambda < \det A < \lambda^{-1}$. The bound $c < tr A < c^{-1}$ follows from Theorem 2.8 and \((4.11)\). These two bounds yield to \((4.8)\).

Next, we observe that $b = \frac{4z \det D^2 q - 3\theta q_z q_{yy}}{q_z^4} = g_x (3h + g \det D^2 q)$. Theorem 2.8 shows that $b \leq c^{-1}$ on $B_{\eta}$. The bound from below $b \geq c > 0$ on $B_{\eta}$, with $\eta$ sufficiently small, readily follows from \((2.1)\) and \((3.31)\).

We are now in position to show the uniform Hölder bounds of the first order derivatives $h_y$ and $h_z$ of $h$ on $B_{\eta}$.

Lemma 4.3. Under the assumptions of Theorem 4.1, there exists a number $\alpha \in (0, 1)$, and positive constants $\eta$ and $C$, depending only on the data $\|g\|_{C^2(\Omega)}$, $\|h\|_{C^2}$, $\theta$, $\lambda$, $\rho$, such that

\begin{equation}
\|q_z\|_{C^\alpha(B_{\eta})} \leq C \quad \text{and} \quad \|q_y\|_{C^\alpha(B_{\eta})} \leq C.
\end{equation}

Proof. We will first establish the bound for $\tilde{q} = q_y$. Differentiating equation \((4.2)\) with respect to $y$ we find that $\tilde{q} = q_y$ satisfies the equation $L_q(\tilde{q}) = \tilde{H}$ with $\tilde{H} = -\partial_y H$, with $L_q$ given by \((4.3)\). Since $\partial_y H = h_y + h_x q_y$, using the notation

\begin{equation}
H_y(z, y) = h_y(x, y) \quad \text{and} \quad H_z(z, y) = h_z(x, y), \quad x = q(x, y)
\end{equation}

we conclude that $\tilde{q}$ satisfies the equation

\begin{equation}
z \alpha_{11} \tilde{q}_{zz} + 2\sqrt{z} \alpha_{12} \tilde{q}_{sz} + \alpha_{22} \tilde{q}_{yy} + b \tilde{q}_z + c \tilde{q} = -H_y
\end{equation}

with $\alpha_{ij}$ and $b$ given by \((4.6)\) and \((4.7)\) respectively and $c = h_z(x, y) = H_z(z, y)$. In addition, Lemma 4.2 and our conditions on the function $h$, imply that equation \((4.12)\) satisfies all the assumptions of our $C^\alpha$-regularity result, Theorem [DL].
Hence, there exists a number $\alpha$ in $0 < \alpha < 1$, such that the Hölder norm $\| \tilde{q} \|_{C^\alpha(B_{\eta}^2)}$ is bounded in terms of $\| \tilde{h} \|_{C^0(B_{\eta})}$ and $\| H_y \|_{C^0(B_{\eta})}$. Since $\| \tilde{q} \|_{C^0(B_{\eta})}$ is uniformly bounded, the bound $\| q_y \|_{C^\alpha(B_{\eta}^2)} \leq C$ readily follows from our assumptions on the function $h$.

We will now establish the $C^\alpha$ bound for $\tilde{q} = q_z$. Differentiating equation (4.2) with respect to $z$ we find that $\tilde{q} = q_z$ satisfies the equation

$$z q_{yy} \tilde{q}_{zz} - 2 z q_{zy} \tilde{q}_{zy} + (z q_{zz} - \theta q_z) \tilde{q}_{yy}$$

$$q_z^2$$

$$\frac{4 z \det D^2 q - (3 \theta + 1) q_z q_{yy}}{q_z^5} \tilde{q}_z + \frac{q_{zy}^2}{q_z^5} = H_1$$

with $H_1 = \partial_z H = h_z q_z = H_z q_z$. We wish to apply the regularity Theorem [DL] shown in [DL2] to control the $C^\alpha$ norm of $\tilde{q} = q_z$. However, our a-priori bounds in Theorem 2.8 do not imply that the term $q_{zy}^2 / q_z^5$ is bounded, since the bounds (4.8) only control $\sqrt{z} h_{zy}$.

To control the $C^\alpha$ norm of $h_z$, we will apply Theorems 3.6 and Theorem 3.7 in [DL1] on certain super-solutions and sub-solutions of equation (4.13).

We begin by noticing that since the term $q_{zy}^2 / q_z^5$ is nonnegative, (4.13) implies that $\tilde{q} = q_z$ is a super-solution of equation

$$z q_{yy} \tilde{q}_{zz} - 2 z q_{zy} \tilde{q}_{zy} + (z q_{zz} - \theta q_z) \tilde{q}_{yy}$$

$$q_z^2$$

$$\frac{4 z \det D^2 q - (3 \theta + 1) q_z q_{yy}}{q_z^5} \tilde{q}_z \leq H_1.$$  

Let us denote by $(a_{ij})$ the matrix in (4.6) and by

$$b_1 := \frac{4 z \det D^2 q - (3 \theta + 1) q_z q_{yy}}{q_z^5}$$

and set

$$L_1(\tilde{q}) := z a_{11} \tilde{q}_{zz} + 2 \sqrt{z} a_{12} \tilde{q}_{zy} + a_{22} \tilde{q}_{yy} + b_1 \tilde{q}_z.$$ 

A similar argument to that used in the proof of (4.9) shows that $b_1$ satisfies the bounds

$$c < b_1 < c^{-1}, \quad \text{on } B_{\eta}$$

with $c = c(\| g \|_{C^2(\Omega)}, \| h \|_{C^2(\Omega), \theta, \lambda, \rho}) > 0$.

Following very similar computations to those in the proof of Lemma 5.9 in [DL1], we conclude:
\[ \tilde{q} = q_z \text{ is a super-solution of equation} \]
\[ L_1(\tilde{q}) \leq \tilde{H}_1, \quad \text{on } B_\eta \]
with \( \tilde{H}_1 = H_z \tilde{q} \).

- There exists a number \( \beta > 1 \), depending only on the a priori bounds, for which if \( (h_z - m) > 0 \) on \( B_\eta \), for some positive constant \( m \), then \( \tilde{q}_2 := (h_z - m)^\beta \) is a sub-solution of the equation
  \[ L_1(\tilde{q}_2) \geq H_2. \]

- There exists a number \( \beta > 1 \), depending only on the a priori bounds, so that \( \tilde{q}_3 := h_\beta^2 \) is a sub-solution of the equation
  \[ L_1(\tilde{q}_3) \geq H_3. \]

- There exists a number \( \beta > 1 \), depending only on the a priori bounds, so that for any constant \( M \), \( \tilde{q}_4 := (M^\beta - h_\beta^2) \) is a super-solution of the equation
  \[ L_1(\tilde{q}_4) \leq H_4. \]

It can be shown, as in the proof of Lemma 5.9 in [DL1], that the functions \( H_i \), \( i = 1, \ldots, 4 \) satisfy the bounds
\[
\|H_i\|_{L^\infty(B_\eta)} \leq C(\|g\|_{C^2} + \|h\|_{C^2}, \theta, \rho, \lambda).
\]

The Hölder regularity of the function \( \tilde{h} = h_z \) on \( B_\eta \) follows by combining the above with the Harnack estimate, Theorem 3.6, and the local maximum principle, Theorem 3.7 in [DL2], along the lines of the proof of Lemma 5.9 in [DL1]. This yields to the bound
\[
\|q_z\|_{C^2_r(B_{\eta})} \leq C(\|g\|_{C^2}, \|h\|_{C^2}, \theta, \rho, \lambda). \]

We will next combine Lemmas 4.2 and 4.3 with the classical regularity results for strictly elliptic linear and fully nonlinear equations, to obtain the \( C^{2,\alpha} \) regularity of the solution \( q \) on the box \( B_\eta \) defined by (4.5) around the boundary point \( Q_0 = (0, y_0, t_0) \), where Lemma 4.2 holds.

Let \( Q^r = (r^2, y_r) \) be a point in \( B_\eta \), where the index \( r \) indicates that the \( z \) coordinate of \( Q \) is of distance \( r \) from the boundary \( z = 0 \). For \( 0 < \mu < 1 \), denote by \( D_\mu \) the disk \( D_\mu = \{ z^2 + y^2 \leq \mu^2 \} \). Define the dilation \( q^r \) of \( q \) on \( D_\mu \), namely the function
\[
q^r(z, y) := \frac{q(r^2 + r^2z, y_r + ry)}{r^2}. \]
A direct computation shows that the function $q^r$ satisfies the equation
\[(4.18) \quad -\tilde{z} \det D^2 q^r + \theta q^r q_{yy} = -H^r\]
with $\tilde{z} = 1 + \tilde{z}$ and $H^r(z, y) = H(r^2 + r^2 z, y + r y)$. When $P = (z, y) \in D_\mu$, with $0 < \mu < 1$, then $\tilde{z} \geq 1 - \mu^2 > 0$. It follows by the bounds of Lemma 4.2 and the bound $0 < \lambda \leq H \leq \lambda^{-1}$, that (4.18) is uniformly elliptic on $D_\mu$. Hence, by the known results on the regularity of solutions to strictly elliptic fully-nonlinear equations (see in [CC]), one obtains uniform $C^\infty$ bounds for $q^r$ on $D_\mu$, in terms of $\|q^r\|_{L^\infty(D_\mu)}$. The above discussion leads to the following lemma:

**Lemma 4.4.** For any $0 < \mu_0 < 1$, there exists a constant $C(\mu_0)$ depending also on $\|g\|_{C^2_{\partial \Omega}}$, $\|h\|_{C^2, p, \lambda, \rho}$, such that
\[\|q^r\|_{C^\infty(D_\mu)} \leq C(\mu_0)\]
for all $0 < \mu < \mu_0$.

One may now combine Lemma 4.4 with Lemma 4.3 along the lines of the proof of Lemma 6.8 in [DL1] to establish the $C^\alpha_s$ regularity of $z h_{zz}$ and $\sqrt{z} h_{zy}$, as stated next:

**Lemma 4.5.** Under the assumptions of Theorem 4.1, there exists a number $\alpha$ in $0 < \alpha < 1$ and constants $C$, $\eta$ depending only on the data $\|g\|_{C^2_{\partial \Omega}}$, $\|h\|_{C^2, p, \lambda, \rho}$, such that for any two points $Q_1 = (z_1, y_1)$ and $Q_2 = (z_2, y_2)$ in $B_\eta$, we have
\[|z_1 q_{zz}(Q_1) - z_2 q_{zz}(Q_2)| + |\sqrt{z_1} q_{zz}(Q_1) - \sqrt{z_2} q_{zz}(Q_2)| \leq C s(Q_1, Q_2)^\alpha.\]

Finally, the Hölder estimate for $q_{yy}$ can be derived from the Hölder estimates of $q_z, q_y$ and $z q_{zz}, \sqrt{z} q_{zy}$ and the regularity of $H$.

**Lemma 4.6.** Under the assumptions of Theorem 4.1, there exists a number $\alpha \in (0, 1)$ and constants $C$, $\eta$ depending only on the data $\|g\|_{C^2_{\partial \Omega}}$, $\|h\|_{C^2, p, \lambda, \rho}$, such that for any two points $Q_1 = (z_1, y_1)$ and $Q_2 = (z_2, y_2)$ in $B_\eta$, we have
\[|q_{yy}(Q_1) - q_{yy}(Q_2)| \leq s(Q_1, Q_2)^\alpha.\]

Following an inductive argument as in Theorem 7.3 in [DH2], we can show higher regularity, as stated next.
Theorem 4.7. Assume that \( g \in C^{2, \alpha}_x \) is a solution of (MAPh) which also satisfies the assumptions of Theorem 4.1 and the additional assumption that \( h \in C^{k+2}(\Omega) \), there exist constants \( 0 < \alpha < 1, C < \infty \) and \( \eta > 0 \), depending only on the data \( \| g \|_{C^2(\partial \Omega)}, \| h \|_{C^{k+2}(\Omega)}, p, \lambda, \rho \), such that for any free-boundary point \( P_0 = (x_0, y_0) \), satisfying condition (4.1), the function \( x = q(z, y) \) satisfies the estimate

\[
\| q \|_{C^{k+\alpha}(B_\eta)} \leq C
\]
on \( B_\eta = \{ 0 \leq z \leq \eta^2, |y - y_0| \leq \eta \} \) for any positive integer \( k \).

We are now in position to give the proof of Theorem 2.10.

Proof of Theorem 2.10. Let \( \eta \) denote the uniform constant in Theorem 4.7. Consider the sub-domains \( \Omega^*_\eta(g) = \{ x \in \Omega(g) | d(x, \Gamma(g)) > \eta \} \) and \( \Omega_\eta(g) = \{ x \in \Omega(g) | d(x, \Gamma(g)) < \eta \} \).

The estimate in Theorem 4.7 implies the bound

\[
\| g \|_{C^{k+\alpha}(\Omega^*_\eta(g))} \leq C(\| g \|_{C^2(\partial \Omega)}, \| h \|_{C^2(\Omega)}, p, \lambda, \rho).
\]

It remains to show that \( g \in C^\infty(\Omega^*_\eta(g)) \). Indeed, on \( \Omega^*_\eta(g) \) we have

\[
0 < \delta_0(\eta) \leq \det D^2 f = h f^p \leq C(\lambda, \max \varphi_{\partial \Omega})
\]

for a positive constants \( \delta_0 \) and \( C(\lambda, \max_{\partial \Omega} \varphi) \). Hence, \( f \) satisfies a Monge-Ampère equation as those considered in [CKN].

The bounds in Corollary 3.5 imply the upper bound on any second derivative \( f_{ii} \) on \( \Omega^*_\eta(g) \), and the lower bound of \( f_{ii} \) follows from the balance of the second derivatives \( \det(D^2 f) \approx 1 \) on \( \Omega^*_\eta \). Therefore \( f \) satisfies a uniformly elliptic equation and \( \det^{1/2}(D^2 f) \) is a concave operator. Hence, the \( C^\infty \) regularity of \( f \), satisfying \( \det D^2 f = h f^p \), on \( \Omega^*_\eta(g) \) follows from the regularity theory for uniformly convex or concave fully-nonlinear operators ([CC]).

The proof of Theorem 4.1 readily follows from Lemmas 4.3, 4.5 and 4.6.

5. Stability: \( I \) is open

In this section, we will utilize the estimates of previous sections to show the following stability of solutions of (MAPt) in the parameter \( t \). This will conclude the proof of the Theorem 1.1 as discussed in section 2.3.
**Theorem 5.1.** Assume that $g_0$ is a classical solution of \([\text{MAP}_t]\) for $t = t_0$, satisfying conditions (H-1)–(H-4) and such that $\|g_0\|_{C^{2,\alpha}_S} \leq C(\|\phi\|_{C^{2,\alpha}_S}, p, \lambda, \rho)$. Then, there is a $\delta > 0$ such that for any $t$ with $|t - t_0| < \delta$, the problem \([\text{MAP}_t]\) admits a $C^{2,\alpha}_S$-solution $g(\cdot, t)$.

We will use the corresponding elliptic argument to the parabolic one which was used in section 8 of [DH2]. Since the two arguments are quite similar, we will only outline the proofs, referring the reader to [DH2] (see also in [DH1]) for the details.

We pick a smooth surface $S$, sufficiently close to the $f_0 = (q^{-2/3} g_0)^a$, such that its inner boundary $\partial S$ lies on the $z = 0$ plane and its outer boundary is $\partial \Omega$. Denoting by $D$ a ring

$$D = \{(u, v) \in \mathbb{R}^2 : 1 \leq u^2 + v^2 \leq 2\}$$

we let $S : D \to \mathbb{R}^2$ be a smooth parameterization for the surface $S$ which maps $\partial^{in} D = \{(u, v) : u^2 + v^2 = 1\}$ to $\partial^{in} S = S \cap \{z = 0\}$ and $\partial^{out} D = \{(u, v) : u^2 + v^2 = 2\}$ to $\partial^{out} S = \partial \Omega$. We can find a smooth vector vector field

$$T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

which is transverse to the surface $S \cap \{z \geq \delta\}$ while it is parallel to the $z = 0$ plane when $0 \leq z \leq \delta$. Now we define the change of coordinate $\varphi : D \to \mathbb{R}^3$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \varphi \begin{pmatrix} u \\ v \\ w \end{pmatrix} = S \begin{pmatrix} u \\ v \\ w \end{pmatrix} + wT \begin{pmatrix} u \\ v \end{pmatrix}.$$  

Via this coordinate change, the solution $z = f(x, y; t)$ of \([\text{MAT}]\) will be mapped onto the graph

$$\left\{ \begin{pmatrix} u \\ v \\ w(u, v; t) \end{pmatrix} : \begin{pmatrix} u \\ v \end{pmatrix} \right\}$$

if $z = f(x, y; t)$ is close to the surface $S$. By the choice of the parameterization $S$ of $S$, we have

$$(u, v) \in \partial^{in} D \text{ iff } z = 0.$$  

In the other words, the interfaces $\Gamma(g(x, y; t)) = \partial\{(x, y) : g(x, y; t) > 0\}$ will be always mapped to the fixed boundary $\partial^{in} D$. 

Definition 5.2. We say \( g(x,y;t) \) is of class \( C^{k,2+\alpha}_x \) if the function \( w(x,y;t) \) belongs to the class \( C^{k,2+\alpha}_x(D) \). Finally, we say that \( g(x,y;t) \) are smooth up to the interface \( \Gamma(g(x,y;t)) \) if \( w(u,v;t) \) is smooth on \( D \).

In addition, the equation (MAPt) will be transformed to the boundary value problems

\[
\begin{cases}
Mw(u,v;t) = 0 & (u,v) \in D \\
w(u,v;t) = \psi(u,v) & (u,v) \in \partial out D
\end{cases}
\]

where \( \psi(u,v) \) is the function, uniquely determined by \( \varphi(x,y) \), after the change of variables and \( Mw = F(D^2w,Dw,w,u,v;t) \) is a fully nonlinear equation whose linearized equation at \( t = 0 \) has the form \( \text{(4.10)} \) satisfying \( \text{(4.8)}, \text{(4.9)} \).

Theorem 5.1 follows by combining Theorem 8.4 in [DH2] and Theorems 4.1 and 4.7.

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References

[A] A.D. Alexandrov; Existence and uniqueness of a convex surface with a given integral curvature, Dokl. Acad. Nauk Kasah SSSR 36 (1942), pp. 131–134.

[Au] Thierry Aubin; Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics. Springer-Verlag, Berlin, (1998).

[C1] L.A. Caffarelli; Some regularity properties of solutions of Monge Ampère equation. Comm. Pure Appl. Math. 44 (1991), no. 8-9, pp. 965–969.

[CC] L.A. Caffarelli, X. Cabré; Fully Nonlinear Elliptic equations AMS 43 (1991).

[CKN] L.A. Caffarelli, L. Nirenberg and J. Spruck The Dirichlet problem for nonlinear second order elliptic equations I. Monge-Ampère equation Comm. Pure Appl. Math. 37 (1984) pp. 369–402.

[CVW] L.A. Caffarelli, J.L. Vázquez, N.I. Wolanski, Lipschitz continuity of solutions and interfaces of the \( N \)-dimensional porous medium equation, Indiana Univ. Math. J. 36 (1987), no. 2-3, pp. 373–401.

[CW] L.A. Caffarelli, N. Wolanski; \( C^{1+\alpha} \) regularity of the free boundary for the \( N \)-dimensional porous medium equation, Comm. Pure Appl. Math.. 43 (1990) no. 7, pp. 885–902.

[CY] S.Y. Cheng, St.T. Yau; On the regularity of the Monge-Ampère equation \( \det(\partial^2 u/\partial x_i \partial x_j) = F(x,u) \), Comm. Pure Appl. Math. 30 (1977) no. 1, pp. 41–68.
[CY2] S.Y. Cheng and S.T. Yau; The real Monge-Ampere equation and affine flat structures. Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, 1,2,3 (Beijing, 1980) pp. 339–370, Science Press, Beijing, 1982.

[DH1] P. Daskalopoulos and R. Hamilton; The Free Boundary for the N-dimensional Porous Medium Equation, Internat. Math. Res. Notices 17, (1997) pp. 817-831.

[DH2] P. Daskalopoulos and R. Hamilton; The Free Boundary on the Gauss Curvature Flow with Flat Sides, J. Reine Angew. Math., no 510, (1999) pp. 187-227.

[DL1] P. Daskalopoulos and K. Lee; Worn stones with flat Sides: all time regularity of the interface, Invent. Math. 156 (2004), no. 3, pp. 445–493.

[DL2] P. Daskalopoulos and K. Lee; H"{o}lder Regularity of Solutions to Degenerate Elliptic and Parabolic Equations, J. Funct. Anal. (2002), no. 5, pp. 633–653.

[DL3] P. Daskalopoulos and K. Lee; Free-boundary regularity on the focusing problem for the Gauss curvature flow with flat sides, Math. Z. 237 (2001), no. 4, pp. 847–874.

[DL4] P. Daskalopoulos and K. Lee; A parabolic approach to eigenvalue problems for Fully Degenerate Monge Ampère equations, in preparation.

[K] H. Koch; Non-Euclidean Singular Integrals and the Porous Medium Equation, Habilitation thesis, University of Heidelberg, (1999).

[P] Pengfei Guan, $C^2$ A Priori Estimates for Degenerate Monge-Ampere Equations, Duke Mathematical Journal, 86, (1997), pp. 323–346.

[GT] D. Gilbarg and N.S. Trudinger; Elliptic Partial Differential Equations of Second Order, Classics in Mathematics. Springer-Verlag, Berlin, (2001).

[GTW] P. Guan, N.S. Trudinger, X-J. Wang; On the Dirichlet problem for degenerate Monge-Ampère equations Acta Math. 182 (1999), no. 1, pp. 87–104.

[G] C.E. Gutiérrez; The Monge Ampère equation, Progress in Nonlinear Differential Equations and their Applications, 44, Birkhäuser, Inc., Boston, MA, (2001).

[Li] P.-L. Lions; Two remarks on Monge-Ampère equations. Ann. Mat. Pura Appl. (4) 142 (1985), pp. 263–275.

[S] O. Savin; The obstacle problem for Monge Ampère equation. Calc. Var. Partial Differential Equations 22 (2005), no. 3, pp. 303–320.

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