Eigen problem Sensitivity Analysis of Continuous Parametric Periodic Systems

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Abstract. The paper concerns sensitivity analysis of the complex eigensolutions of monodromy matrix (Floquet transient matrix) for continuous parametric periodic systems. The first and the second derivatives of monodromy matrix and its multipliers which are the complex eigenvalues of monodromy matrix have been calculated. The method’s innovation is the idea to achieve the sensitivity equation by evaluating the derivative of the parametric equation of motion. Then, by solving the sensitivity equation obtained in this way, to evaluate the first and second derivative of monodromy matrix and finally the first and second derivatives of multipliers. Furthermore, the sensitivity analysis method was improved and generalized to allow to correctly determine the eigenderivatives also with respect to those system parameters, on which the parametric excitation period depends. In particular, it becomes possible to use the parametric excitation period as a design parameter.

1. Introduction

The sensitivity analysis of parametric periodic systems can be an interesting theoretical problem in itself. However, the most important feature of parametric periodic systems is the instability phenomenon, which can be observed for particular values of the system parameters. Resonance vibration in unstable parametric systems are very dangerous. Thus, stabilization of unstable systems is usually the most important practical problem. In this article the first and the second order sensitivity analyses with respect to those parametric system parameters which can influence stability/instability of the system were performed. Finally, it stays possible to determine those parameters of the system whose influence on the stabilisation procedure of such a system could be the greatest. The results of this analysis could be useful in parametric resonance vibration stabilization procedures. There are two ways of performing stability analysis of parametric periodic systems. Both of them are connected with Floquet theory [1]. The first way is to use Lyapunov characteristic exponents, the second – to use multipliers which are the complex eigenvalues of monodromy matrix. There are some papers in which the sensitivity of parametric periodic systems is analysed. For instance, in [2, 3] derivatives of eigenexponents are calculated, whereas the sensitivity analysis of multipliers is used in [4]. In paper [3] the method of determination of first order derivatives of characteristic exponents is presented. The paper contains the improvement of the method presented in [2]. This improvement allows to determine correctly the derivatives of characteristic exponents with respect to those system parameters, on which the parametric excitation period depends or is itself a design parameter.
In the current paper the original method of first- and second-order sensitivity analysis of parametric periodic systems was formulated. This method is based on sensitivity analysis of absolute values of multipliers. From the mathematical point of view, sensitivity analysis of multipliers is the calculation of eigenderivatives with the use of derivatives of the monodromy matrix. Eigenderivatives are extremely useful for determining the sensitivities of dynamic response to the system parameters variations. The innovation of method is the idea to achieve the sensitivity equation by evaluating the derivative of the parametric equation of motion. Then, by solving the sensitivity equation obtained in this way, to evaluate the first and second derivative of monodromy matrix and finally the first and second derivatives of multipliers. The example of this method implementation is also presented.

2. Linear parametric periodic system and its stability
A linear non-homogeneous periodic parametric system of an n linear second order differential equation of motion can be written as a first order non-homogeneous periodic coefficient system

$$\dot{x}(t) = A(t) x(t) + f(t)$$

$$A(t) = \begin{bmatrix} 0 & 1 \\ -B^{-1}(t)K(t) & -B^{-1}(t)C(t) \end{bmatrix}$$

$$f(t) = \begin{bmatrix} 0 \\ B^{-1}F(t) \end{bmatrix}$$

where an 2n×2n matrix A(t) is periodic with period T and 2n dimensional vectors x(t) and f(t) are a vector of state variables and an external excitation respectively, B(t), C(t), K(t) are square n×n real matrices of inertia, damping and stiffness periodic with a period T and F(t) is an n dimensional excitation column vector.

A homogenous system corresponding to system (1) is

$$\dot{x}(t) = A(t) x(t)$$

On the basis of Floquet theory [1] a solution of equation (3) has the form

$$x(t) = X(t) x(0) \quad X(0) = I \quad X(t) = I + \int_{0}^{t} A(\tau) X(\tau) d\tau$$

where X(t) is a standard fundamental solution matrix or state transition matrix normalized at zero. A steady 2n×2n monodromy matrix D is defined as the value of the fundamental matrix X(t) at time point t = T i.e. D = X(T).

Solving the right- and left-side eigenproblem of monodromy matrix i.e.

$$(D - \rho I) r = 0 \quad I^T (D - \rho I) = 0^T$$

one can find 2n the right- and the left-side modal vectors r and l and the 2n multipliers i.e. eigenvalues of monodromy matrix D

$$\{ \rho \} = \text{diag}(\rho_1, \rho_2, \ldots, \rho_{2n})$$

Since, the monodromy matrix D is the real, non-singular and asymmetrical one, the multipliers (6) are complex numbers.
The stability of the trivial solution of a homogeneous equation (3) depends on the absolute values of multipliers i.e. if:

- the absolute value of each multiplier is less than 1, the parametric periodic homogeneous system (3) is asymptotically stable;
- the absolute value of at least one multiplier is greater than 1, the parametric periodic homogeneous system (3) is unstable.

### 3. First order sensitivity analysis

#### 3.1. First derivative of multipliers with respect to design parameter

As in the papers [2-4, 9] to simplify the analysis, there are considered only the case when the eigenvalue of monodromy matrix are non-repeated. However, in this paper is assumed the multipliers can be repeated provided then the Jordana’s form of monodromy matrix has to be diagonal.

Equations (5)₁ and (5)₂ for all multipliers can be written in the form

\[
\{ \rho \} = \mathbf{D} \mathbf{R} = \mathbf{R} \{ \rho \}
\]

\[
\{ \rho \} = \mathbf{L}^T \mathbf{D} \mathbf{L} = \{ \rho \} \mathbf{L}^T
\]

where \( \mathbf{R} \) (\( \det \mathbf{R} \neq 0 \)) is right-side modal matrix, whose columns are right-sides eigenvectors \( \mathbf{r}_k \) of equation (5)₁ corresponding to the eigenvalue \( \rho_k \) (to simplify writing the index \( k \) is omitted henceforth). The left-side modal matrix \( \mathbf{L} \) (\( \det \mathbf{L} \neq 0 \)) whose columns are left-sides eigenvectors \( \mathbf{l}_k \) of equation (5)₂ corresponding to the eigenvalue \( \rho_k \) has to satisfy the condition

\[
\mathbf{L}^T \mathbf{R} = \mathbf{I} \quad \text{i.e.} \quad \mathbf{L}^T = \mathbf{R}^{-1}
\]

After operation of left-side multiplication of the equation (7)₁ by \( \mathbf{L}^T \) or of right-side multiplication of the equation (7)₂ by \( \mathbf{R} \), and taking into account the equations (8), the equations (7) can be written in the more convenient form

\[
\begin{align*}
\{ \rho \} &= \mathbf{L}^T \mathbf{D} \mathbf{R} \\
\{ \rho \} &= \mathbf{L}^T \mathbf{D} \mathbf{r} \\
\end{align*}
\]

By calculating the derivative \( \rho_p' = \partial \rho / \partial p \) of both sides, for example, of the equation (7)₁ with respect to design parameter \( p \) and taking into account the equality (5)₂, the formula for the first derivative of multipliers is received in form

\[
\{ \rho_p' \} = \mathbf{L}^T \mathbf{D}_p' \mathbf{R} + \mathbf{L}^T \mathbf{D} \mathbf{R}_p' \mathbf{r} - \mathbf{L}^T \mathbf{R}_p' \{ \rho \}
\]

where the symbol \( \{ \rho_p' \} \) denotes the diagonal matrix of the first derivatives of the monodromy matrix’s multipliers with respect to the design parameter \( p \).

Finally equation (10) can be described in the matrix notation

\[
\rho_p' = \mathbf{1}^T \mathbf{D}_p' \mathbf{r} = \text{diag}(\mathbf{L}^T \mathbf{D}_p' \mathbf{R}) \]

Where symbol \( \text{diag}(\mathbf{L}^T \mathbf{D}_p' \mathbf{R}) \) indicates the diagonal matrix whose non-zero elements are the elements of the main diagonal of the matrix \( \mathbf{L}^T \mathbf{D}_p' \mathbf{R} \). To calculate the derivative of the multipliers of the monodromy matrix \( \mathbf{D} \), there is no need now to know the derivative of the right-hand modal matrix \( \mathbf{R} \) with respect to the design parameter \( p \), i.e. \( \mathbf{R}_p' = \partial \mathbf{R} / \partial p \), but only the derivative of the monodromy matrix, i.e. \( \mathbf{D}_p' = \partial \mathbf{D} / \partial p \).
3.2. First derivative of a monodromy matrix with respect to design parameter

It could be proofed that the first derivative of monodromy matrix with respect to parameter $p$ is the solution of the non-homogeneous equation (12) with zero initial conditions

$$\dot{y}(t) = A(t) y(t) + f(t)$$

(12)

where

$$y(t) = \frac{\partial x(t)}{\partial p} \quad \dot{y}(t) = \frac{\partial \dot{x}(t)}{\partial p} \quad f(t) = A'_p(t) x(t) \quad A'_p = \frac{\partial A}{\partial p}$$

and finally has the form

$$X'_p(t) = X(t) \int_0^T X^{-1}(\tau) A'_p(\tau) X(\tau) \, d\tau \quad D'_p = D \int_0^T X^{-1}(\tau) A'_p(\tau) X(\tau) \, d\tau$$

(14)

This formula can be calculated analytically or numerically, and the result may be used to calculate derivatives of multipliers in accordance with the formula (11). The same result was obtained in [4] for the first order sensitivity analysis.

3.3. The case when period of parametric excitation depends on design parameter

The algorithm discussed in the previous paragraph does not include a case in which the period $T$ of the parametric excitation depends on design parameters. This is a special case. This problem was examined theoretically in paper [3]. The method presented there is, unfortunately, very complicated, and the algorithm completely useless from the point of view of the method proposed in this work. However, the results obtained in work [3] are valuable, due to the possibility of their comparison with results obtained in this paper.

The starting point to obtain the more general formula of a special case in which the period $T$ of the parametric excitation depends on design parameters is a formula which formally describes the monodromy matrix [1]

$$D = I + \int_0^T A(t) X(t) \, dt$$

(15)

In the calculation of the derivative of the monodromy matrix (15) with respect to parameter $p$ it is assumed that not only matrices $A = A(t, p)$ and $X = X(t, p)$, but also period $T = T(p)$ is a function of the design parameter $p$. As a consequence of this assumption, the integration limit in the definite integral (15) becomes functionally dependent on parameter $p$.

When calculating the derivative of the monodromy matrix, in this case, one must use the formula for the derivative of the integral with respect to the parameter [7]

$$\frac{d}{dp} \left( \int_{t_1}^{t_2} f(t, p) \, dt \right) = \int_{t_1}^{t_2} \frac{\partial f(t, p)}{\partial p} \, dt + f(\beta(p), p) \frac{d\beta}{dp} - f(\alpha(p), p) \frac{d\alpha}{dp}$$

(16)

In accordance with rule (16), one can calculate the derivative of the matrix $D$ describes by formula (15)

$$\dot{D}_p =\frac{dD}{dp} = \int_0^{T(p)} \frac{\partial}{\partial p} \left[ A(t, p) X(t, p) \right] dt + A(T(p), p) X(T(p), p) \frac{dT}{dp} =\int_0^{T(p)} \left[ A'_p(t, p) X(t, p) + A(t, p) X'_p(t, p) \right] dt + A(T) D T'_p$$

(17)
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\[ D'_p = \int_0^{T_p} \left[ A'_p(t, p) X(t, p) + A(t, p) X'_p(t, p) \right] dt \] (18)

and

\[ D'_p = A(T) D T'_p \] (19)

and comparing (18) and (14) one can state that the formula (18) is different apparently from the formula (27), however both of them have to be in fact equivalent. The formula (18) is the result of differentiation with respect to design parameter \( p \) of the matrix \( D \) described by formula (15), while the formula (14) is a solution of sensitivity equation achieved by differentiation of equation (3) with respect to the same design parameter \( p \).

However, since there is an unknown matrix \( X'_p(t, p) \) in the formula (18), this formula is practically useless. Such a method (with the use of formula (18)) of computing the derivative of a monodromy matrix can only be used if the analytical form of the fundamental matrix of solutions \( X(t) \) is known. Then the first derivative of this matrix can be calculated analytically. Thus, in the general case, the matrix \( D'_p \) has to be designated using the relationship (14) instead of (18).

Finally, a more general formula than (14) for the first derivative of monodromy matrix can be written as

\[ \tilde{D}'_p = D'_p + \overline{D}'_p \] (20)

It can be concluded that the pattern (20) is different from the earlier formula (27) with the component \( \overline{D}'_p \) (19) that assumes values different from zero only when the period of parametric excitation is a function of the design parameter, or is itself a design parameter. In other cases, the formula (14) remains valid.

4. Second order sensitivity analysis

4.1. Second derivative of multipliers with respect to design parameter

Calculating a derivative of both sides of equation (10) with respect to parameter \( p_j \) and left-hand multiplication by the left-hand eigenvector \( l^T \) of monodromy matrix leads to equation

\[ l^T D'_{p_j p_j} r - \rho'_{p_j} l^T r + l^T D'_{r_j p_j} r - \rho'_{p_j} l^T r = -l^T D'_{r_j p_j} r - \rho'_{p_j} l^T r - l^T (D - \rho I) r'_{r_j p_j} \] (21)

The last component of the equation (21) is equal to zero, because under the validity of (5)_2, the vector \( l^T \) is the left-hand eigenvector of the monodromy matrix corresponding to its eigenvalue \( \rho \). Moreover, on the basis of equation (8) scalar product

\[ l^T r = 1 \] (22)

If additionally, one requests that the scalar product of the vectors satisfy the orthogonality condition

\[ l^T r'_{r_j p_j} = 0 \] (23)
it is possible from the equation (21) to obtain the formula for calculating the second derivative of the multiplier in simpler form

$$\rho_{p,p_j}^{\prime\prime}=l^{\top}D_{p,p_j}^{\prime\prime}+l^{\top}D_{p,p_j}^{\prime}\cdot r_{p}^{\prime}+l^{\top}D_{p_j}^{\prime}\cdot r_{p_j}^{\prime}$$

(24)

4.2. Second order derivatives of a monodromy matrix with respect to design parameters

The second derivatives of the monodromy matrix can be determined by calculating the derivative of the derivative of the monodromy matrix defined by formula (20) taking into account (18) and (19) i.e.

$$\frac{d}{dp_j} D_{p_j} = \frac{d^2}{dp dp_j} D_{p_j} = \bar{D}_{p,p_j}^{\prime}\ + \bar{D}_{p,p_j}^{\prime\prime}\$$

(25)

The first component on the right side of the equation (25) can be calculated using the formula (18),

$$\bar{D}_{p,p_j}^{\prime\prime} = \frac{d}{dp_j} \left( \frac{d}{dp} [A_{p_j}(t, p_j) \mathbf{X}(t, p_j) + A(t, p_j) \mathbf{X}_{p_j}(t, p_j)] dt + \ [A_{p_j}(T) \mathbf{D} + A(T) \mathbf{D}_{p_j}^{\prime}] \frac{dT}{dp_j} \right) = \bar{D}_{p,p_j}^{\prime}\ + \bar{D}_{p,p_j}^{\prime\prime}\$$

(26)

where \( \mathbf{X}_{p_j}(t, p_j) \) is derived from the expression described by formula (14).

The second component on the right side of the equation (25), i.e. the matrix \( \bar{D}_{p,p_j}^{\prime}\ ), is determined by counting the derivative of the second component on the right side of the equation (20) described by the formula (19)

$$\bar{D}_{p,p_j}^{\prime} = \frac{d}{dp_j} \left[ A(T(p_j), p_j) \mathbf{D}(T(p_j), p_j) \mathbf{T}_{p_j}(p_j) \right] = \left[ A_{p_j}(T) \mathbf{D} T_{p_j}^{\prime} + A(T) \mathbf{D}_{p_j}^{\prime} T_{p_j}^{\prime} + A(T) \mathbf{D} T_{p_j}^{\prime\prime} \right]$$

(27)

Considering that the matrix \( \bar{D}_{p_j}^{\prime}\ ) in formula (27) is defined by (18), (19) and (20) the matrix \( \bar{D}_{p,p_j}^{\prime}\ ) is obtained lastly in form

$$\bar{D}_{p,p_j}^{\prime}\ = \left[ A_{p_j}(T) \mathbf{D} T_{p_j}^{\prime} + A(T) \mathbf{D}_{p_j}^{\prime} T_{p_j}^{\prime} + A(T) \mathbf{D} T_{p_j}^{\prime\prime} \right]$$

(28)

Finally, the second derivative of the monodromy matrix can be written as a sum of two matrices

$$\bar{D}_{p,p_j}^{\prime\prime} = \bar{D}_{p,p_j}^{\prime}\ + \bar{D}_{p,p_j}^{\prime\prime}\$$

(29)

The matrix \( \bar{D}_{p,p_j}^{\prime}\ ) calculated on the basis of formulas (27) and (28) is

$$\bar{D}_{p,p_j}^{\prime}\ = \left[ A_{p_j}(T) \mathbf{D} + A(T) \mathbf{D}_{p_j}^{\prime} T_{p_j}^{\prime} + A_{p_j}(T) \mathbf{D} + A(T) \mathbf{D}_{p_j}^{\prime} T_{p_j}^{\prime} + A(T) \mathbf{D} T_{p_j}^{\prime\prime} \right]$$

(30)

and contains a complement that extends the ability to use an algorithm to calculate the second derivative of a monodromy matrix for cases where the period of parametric excitation is a design parameter or depends on the design parameter with respect to which the sensitivity is analysed.

The matrix \( \bar{D}_{p,p_j}^{\prime}\ ) in formula (29) is defined by
and denotes the second mixed derivative of the monodromy matrix in cases when the period of a parametric excitation is not a design parameter or does not depend on the design parameter.

In order to determine practically useful formulas for the calculation of the matrix $D_{p,p_j}^\ast$

The second derivative of the monodromy matrix $D_{p,p_j}^\ast$ can be derived from the relation

$$D_{p,p_j}^\ast = \frac{\partial}{\partial p} \left[ A_{p_j}^\prime(t, p_j) X(t, p_j) + A(t, p_j) X_{p_j}^\prime(t, p_j) \right] dt = $$

$$= \int_0^T \left[ A_{p,p_j}^\ast(t) X(t, p_j) + A_{p_j}^\prime(t) X_{p_j}^\ast(t) + A_{p_j}^\prime(t) X_{p_j}^\prime(t) + A(t) X_{p,p_j}^\ast(t) \right] dt$$

Changing the order of integration and the designation of variables in the second integral (32) is obtained

$$D_{p,p_j}^\ast = \int_0^T \left[ A_{p_j}^\prime(t) X(t) \right] \int_0^\tau \left[ X^{-1}(\tau) A_{p_j}^\prime (\tau) X(\tau) d\tau \right] dt + $$

$$+ \int_0^T \left[ A_{p_j}^\prime(t) X(t) \right] \left[ X^{-1}(\tau) A_{p_j}^\prime (\tau) X(\tau) d\tau \right] dt + $$

If matrix factor $X^{-1}(\tau) A_{p_j}^\prime(\tau) X(\tau)$ is commutative with matrix factor $X^{-1}(\tau) A_{p_j}^\prime(\tau) X(\tau)$ it becomes possible to write a sum of two first double integral over the triangle in formula (33) by one double integral over a rectangle, and this one by product of two integrals. The formula (32) takes then the form

$$D_{p,p_j}^\ast = \int_0^T \left[ A_{p_j}^\prime(t) X(t) \right] \int_0^\tau \left[ X^{-1}(\tau) A_{p_j}^\prime (\tau) X(\tau) d\tau \right] dt + $$

Finally, the calculation of the second derivatives of the monodromy matrix can be founded with the use of formula
\[ D_{p,p_j}^* = D_{p_j}^* D_{p}^* D_{p}^* + \mathcal{D} \int_{0}^{T} X^{-1}(t) A_{p,p_j}^* (t) X(t) dt \]  

Unfortunately, in the general case, the factors that occur under the integral are not commutative and need to use the formula (33).

Finally in general formulas (11) and (24) for of first and second order derivatives of multipliers instead of matrices \( D_{p}^* \) and \( D_{p,p_j}^* \), matrices \( \tilde{D}_{p}^* \) and \( \tilde{D}_{p,p_j}^* \) ought to be substituted, which are different from the matrices \( D_{p}^* \) and \( D_{p,p_j}^* \) about matrix components \( \tilde{D}_{p} \) and \( \tilde{D}_{p,p_j} \) described by formulas (19) and (30) respectively.

5. Example

The method presented in this paper was verified on the same example that was analysed in [3]. This example, for the parametric system, is unique. There is an analytical solution for all mathematical operations associated with the computational algorithm presented in this work. This is a great advantage of this example. It is possible to objectively verify the correctness of the theory and to determine the efficiency of the method. In addition, one can directly compare the results with those obtained at work [3].

A linear parametric system described by equation (3) is considered in which the matrix

\[
A(t) = \begin{bmatrix}
     a + i(a + b) \cos 2at & -ab + ib(a + b) \sin 2at \\
     a + i \cdot a \cdot b / b & \sin 2at & a - i(a + b) \cos 2at
\end{bmatrix}
\]  

is periodic with a period \( T = \pi / a \) and constants \( a \) and \( b \) are system parameters. Consequently, the period \( T \) is a function of the design parameter \( T = T(a) \). The fundamental matrix of the solutions of this system, monodromy matrix, modal matrices and first order sensitivity analysis results have the form

\[
X(t) = \begin{bmatrix}
    \cos at \cdot e^{iat} & -b \sin at \cdot e^{-iat} \\
    \frac{1}{b} & \sin at \cdot e^{iat} & \cos at \cdot e^{-iat}
\end{bmatrix} \begin{bmatrix}
    e^{(a+b)t} & 0 \\
    0 & e^{(a-b)t}
\end{bmatrix} = \begin{bmatrix}
    \rho_1 & 0 \\
    0 & \rho_2
\end{bmatrix} = \begin{bmatrix}
    e^{(a+b)T} & 0 \\
    0 & e^{(a-b)T}
\end{bmatrix} \begin{bmatrix}
    \frac{e^{(a+b)\pi / a}}{a} & 0 \\
    0 & \frac{e^{(a-b)\pi / a}}{a}
\end{bmatrix} = \{p\}
\]  

\[
D = X(T, a, b) = \begin{bmatrix}
    \rho_1 & 0 \\
    0 & \rho_2
\end{bmatrix}
\]  

\[
R = L = I
\]  

\[
D_a' = \begin{bmatrix}
    \rho_{1a}' & 0 \\
    0 & \rho_{2a}'
\end{bmatrix} = \begin{bmatrix}
    -\frac{b}{a}T \cdot \rho_1 & 0 \\
    0 & \frac{b}{a} \cdot T \cdot \rho_2
\end{bmatrix}
\]  

\[
D_b' = \begin{bmatrix}
    \rho_{1b}' & 0 \\
    0 & \rho_{2b}'
\end{bmatrix} = \begin{bmatrix}
    -iT \cdot \rho_1 & 0 \\
    0 & iT \cdot \rho_2
\end{bmatrix}
\]  

A comparison of results with those obtained in the work [3] requires the calculation of the Lapunov’s characteristic exponents and its first derivatives.
\[
\text{Re} \lambda_k = \frac{1}{T} \ln|\rho_k| = \frac{1}{2T} \ln[(\text{Re} \rho_k)^2 + (\text{Im} \rho_k)^2]
\]  
(42)

\[
\lambda'_k = -\frac{1}{T^2} \frac{\partial T}{\partial \rho_k} \ln \rho_k + \frac{1}{T} \frac{\partial \rho_k}{\partial \rho_k}
\]  
(43)

Substituting the results obtained in (38), (39), (40) and (41) into formula (43) and taking into account that \( \partial T/\partial a = -\pi/a^2 \) and \( \partial T/\partial b = 0 \) finally

\[
\lambda'_a = -\frac{1}{T^2} \left( \frac{\pi}{a^2} \right) (a \pm ib)T + \frac{1}{T} \frac{b}{a} \left( \mp i \right) T \rho_k = 1, \quad \lambda'_b = \frac{1}{T} \frac{i}{\rho_k} (\pm iT \rho_k) = \pm i
\]  
(44)

These results are consistent with those received at work [3].

Further testing of the algorithm presented in this paper is carried out first in the case of sensitivity analysis with respect to the parameter \( b \). This parameter does not affect the period of the parametric excitation. Correct should be, therefore, the results obtained in accordance with the formula (14). The algorithm described in this formula was programmed in the Mathematica computer system. This allowed conducting analytical calculations using the symbolic procedures. The results achieved were consistent with those obtained in work [3].

The next calculations were carried out with the use of the same procedure (in accordance with the formula (14)) by studying the sensitivity with respect to the parameter \( a \), on which depends the period of the parametric excitation. The following form of solution was obtained

\[
D'_a = \begin{bmatrix}
(1 + i) \frac{\pi}{a} e^{(1+i) \frac{b}{a}} & -b \frac{\pi}{a} e^{(1-i) \frac{b}{a}} \\
\frac{1}{ab} \frac{\pi}{a} e^{(1+i) \frac{b}{a}} & (1 - i) \frac{\pi}{a} e^{(1-i) \frac{b}{a}}
\end{bmatrix}
\]  
(45)

This is a wrong result! The matrix \( D'_a \) is not even a diagonal one.

The calculations were repeated using a generalized algorithm described by the formula (20). The obtained results are identical to those achieved by analytical calculation of the derivatives.

Similar calculations were made with respect to second derivatives. Results obtained with the use of the algorithm presented in this work were again consistent with the analytical calculations. Among other results, mixed derivatives of multipliers were obtained with respect to the parameters \( a \) and \( b \) in form

\[
D''_{ab} = \begin{bmatrix}
\rho''_{ab} & 0 \\
0 & \rho''_{ab}
\end{bmatrix}
= \begin{bmatrix}
\frac{T}{a^2} (\pi b - ia) \rho_1 & 0 \\
0 & \frac{T}{a^2} (\pi b + ia) \rho_2
\end{bmatrix}
\]  
(46)

The same results were obtained by solving the eigenproblem for the matrix \( D''_{ab} \) on the basis of the conclusion in Chapter 4.1.
6. Conclusion
This paper presents a method of sensitivity analysis of the eigenproblem of discrete linear parametric systems. The method is much more general than the one presented in papers [2, 3] and [4] because it relates the second-order sensitivity analysis. The method allows to calculate derivatives of multipliers of the monodromy matrix. It is therefore an alternative approach to that proposed in [2] and [3] and is similar to that which was presented in paper [4]. The equation of sensitivity with respect to the design parameter is funded first. The solutions of this equation allow to find eigenderivatives of multipliers. The method has been tested on the example giving the correct results consistent with the ones obtained at work [2, 3] and [4] for the first order sensitivity analysis. The method gives correct results also when the parametric period depends on the design parameter or is itself the design parameter for which the sensitivity is analysed.

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