Rational, Replacement and Local Invariants of a Group Action

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Abstract

The paper presents a new algorithmic construction of a finite generating set of rational invariants for the rational action of an algebraic group on the affine space. The construction provides an algebraic counterpart of the moving frame method in differential geometry. The generating set of rational invariants appear as the coefficients of a Gröbner basis, reduction with respect to which allows to express a rational invariant in terms of the generators. The replacement invariants, introduced in the paper, are tuples of algebraic functions of rational invariants. Any invariant, whether rational, algebraic or local, can be rewritten in terms of a replacement invariant by a simple substitution.

Key words: rational and algebraic invariants, algebraic and Lie group actions, cross-section, Gröbner basis, moving frame method, smooth and differential invariants.

1 Introduction

We present algebraic constructions for invariants of a rational group action on an affine space, and relate them to their counterparts in differential geometry. The constructions are algorithmic and can easily be implemented in general purpose computer algebra systems or software specialized in Gröbner basis computations. This is illustrated by the MAPLE worksheet available at http://www.inria.fr/cafe/Evelyne.Hubert/Publi/rrl_invariants.html where the examples of the paper are treated.

The first construction is for the computation of a generating set of rational invariants. This generating set is endowed with a simple algorithm to express any
rational invariant in terms of them. The construction comes into two variants. In the first one we consider the ideal of the graph of the action as did Rosenlicht \[20\], Vinberg & Popov \[35\]\(^1\), and Müller-Quade & Beth \[24\]\(^2\). We point out the connections with these previous works in the text. Our proofs are independent and provide an original approach. We show that the coefficients of a reduced Gröbner basis of the ideal of the graph of the action are invariant. We prove that these coefficients generate the field of rational invariants by exhibiting an algorithm for rewriting any rational invariant in terms of them. The second variant provides a purely algebraic formulation of the geometric construction of a fundamental set of local invariants on a smooth manifold proposed by Fels and Olver \[11\], as a generalization of Cartan’s moving frame method. It is also computationally more effective as we reduce to zero the dimension of the polynomial ideal for which a reduced Gröbner basis is computed. This is achieved by adding the ideal of a cross-section to the ideal of the graph.

That latter construction allows to introduce replacement invariants, the algebraic counterpart of normalized invariants appearing in the geometric construction. A replacement invariant is a tuple of algebraic functions of rational invariants. Any invariant can be trivially rewritten in their terms by substituting the coordinate functions by the corresponding invariants from this tuple. An invariantization map, a computable isomorphism from the set of algebraic functions on the cross-section to the set of algebraic invariants, is defined in terms of replacement invariants.

We use invariantization process to make explicit the connection between the present algebraic construction and the geometric construction of Fels and Olver \[11\]. We introduce an alternative definition of smooth invariantization which, on one hand, generalizes the one given in \[11\] and, on the other hand, matches the algebraic construction. We thus provide a bridge between the theory of rational and algebraic invariants \[35\] \[9\] and the theory of smooth local invariants in differential geometry.

Diverse fields of application of algebraic invariant theory are presented in \[9\] Chapter 5. Some of the applications can be addressed with rational invariants. Their present construction together with the simple rewriting algorithm can bring computational benefits. An application of the moving frame method to classical invariant theory \[11\] \[17\] \[33\] was proposed in \[27\] \[19\] \[8\] \[20\]. In these works, however, the geometric formulation of the method is used without adapting it to the algebraic nature of the problem. A purely algebraic formulation of the moving frame method opens new possibilities of its application in classical invariant theory.

The present algebraic formulation provides a new tool for the investigation of the differential invariants of Lie group actions and their applications to differential

\(^1\)We are indebted to a referee of the MEGA conference for pointing out this reference that motivated us to push further some of the results.

\(^2\)We would like to thank H. Derksen for suggesting comparison with this reference after we made public a first preprint.
systems in the line of \[11, 18, 21, 23\]. This larger project motivates our choice to consider rational actions. Even if we start with an affine or even linear action on the zeroth order jet space, the prolongation of the action to the higher order jet spaces is usually rational.

The paper is structured as follows. In Section 2 we introduce the action of an algebraic group on the affine space and the graph of the action. This leads to a first construction of a set of generating rational invariants. A second version of the construction is given after the introduction of the cross-section to the orbits in Section 3. This second construction gives rise to the replacement invariants in Section 4 which are used to define a computable invariantization map. In Section 5 we present a geometric construction of local smooth invariants that generalizes the construction of \[11\] and explicitly relates it to the algebraic construction of the previous sections. Section 6 provides additional examples.

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2 Graph of a group action and rational invariants

We give a definition of a rational action of an algebraic group over a field \(K\) on an affine space, and formulate two additional hypotheses necessary to our construction. We recall the definition for the graph of the action. It plays a central role in our constructions. The first variant of the algorithm for constructing a generating set of rational invariants, together with an algorithm for expressing any rational invariant in terms of them, is presented in this section.

For exposition convenience we assume that the field \(K\) is algebraically closed. The construction proposed in this section relies only on Gröbner basis computations and thus can be performed in the field of definition of the data (usually \(\mathbb{Q}\) or \(\mathbb{F}_p\)). Outside of Section 3 the terms open, close and closure refer to the Zariski topology.

2.1 Rational action of an algebraic group

We consider an algebraic group that is defined as an algebraic variety \(G\) in the affine space \(K^n\). The group operation and the inverse are given by polynomial maps. The neutral element is denoted by \(e\). We shall consider an action of \(G\) on an affine space \(Z = K^n\).
Throughout the paper $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $z = (z_1, \ldots, z_n)$ denote indeterminates while $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_l)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ denote points in $G \subset \mathbb{K}^l$ and $Z = \mathbb{K}^n$ respectively. The coordinate ring of $Z$ and $G$ are respectively $\mathbb{K}[z_1, \ldots, z_n]$ and $\mathbb{K}[\lambda_1, \ldots, \lambda_l]/G$ where $G$ is a radical unmixed dimensional ideal. By $\bar{\lambda} \cdot \bar{\mu}$ we denote the image of $(\bar{\lambda}, \bar{\mu})$ under the group operation while $\bar{\lambda}^{-1}$ denotes the image of $\bar{\lambda}$ under the inversion map.

**Definition 2.1** A rational action of an algebraic group $G$ on the affine space $Z$ is a rational map $g : G \times Z \to Z$ that satisfies the following two properties

1. $g(e, \bar{z}) = \bar{z}, \forall \bar{z} \in Z$

2. $g(\bar{\mu}, g(\bar{\lambda}, \bar{z})) = g(\bar{\mu} \cdot \bar{\lambda}, \bar{z})$, whenever both $(\bar{\lambda}, \bar{z})$ and $(\bar{\mu} \cdot \bar{\lambda}, \bar{z})$ are in the domain of definition of $g$.

A rational action is thus uniquely determined by a $n$-tuple of rational functions of $\mathbb{K}(\lambda, z)$ whose domain of definition is a dense open set of $G \times Z$. We can bring these rational functions to their least common denominator $h \in \mathbb{K}[\lambda, z]$ without affecting the domain of definition. In the rest of the paper the action is thus given by

$$g(\bar{\lambda}, \bar{z}) = (g_1(\bar{\lambda}, \bar{z}), \ldots, g_n(\bar{\lambda}, \bar{z})) \text{ for } g_1, \ldots, g_n \in h^{-1}\mathbb{K}[\lambda, \ldots, \lambda_l, z_1, \ldots, z_n] \quad (1)$$

**Assumption 2.2** We make the additional assumptions

1. for all $\bar{z} \in Z$, $h(\lambda, \bar{z}) \in \mathbb{K}[\lambda]$ is not a zero-divisor of $G$. This says that the domain of definition of $g_{\bar{z}} : \lambda \mapsto g(\lambda, \bar{z})$ contains a non-empty open set of each component of $G$.

2. for all $\bar{\lambda} \in Z$, $h(\bar{\lambda}, z) \in \mathbb{K}[z]$ is different from zero. In other words, for every element $\bar{\lambda} \in G$ there exists $\bar{z} \in Z$, such that $(\bar{\lambda}, \bar{z})$ is in the domain of definition $g$.

The following three examples serve as illustration throughout the text.

**Example 2.3** Scaling. Consider the multiplicative group given by $G = (1 - \lambda_1 \lambda_2) \subset \mathbb{K}[\lambda_1, \lambda_2]$. The neutral element is $(1, 1)$ and $(\bar{\mu}_1, \bar{\mu}_2) \cdot (\bar{\lambda}_1, \bar{\lambda}_2)^{-1} = (\bar{\mu}_1 \bar{\lambda}_2, \bar{\mu}_2 \bar{\lambda}_1)$. We consider the scaling action of this group on $\mathbb{K}^2$. It is given by the following polynomials of $\mathbb{K}[\lambda_1, \lambda_2, z_1, z_2]$: $g_1 = \lambda_1 z_1$, $g_2 = \lambda_1 z_2$.

**Example 2.4** Translation+Reflection. Consider the group that is the cross product of the additive group and the group of two elements $\{1, -1\}$, its defining ideal in $\mathbb{K}[\lambda_1, \lambda_2]$ being $G = (\lambda_2^2 - 1)$. The neutral element is $(0, 1)$ while $(\bar{\mu}_1, \bar{\mu}_2) \cdot (\bar{\lambda}_1, \bar{\lambda}_2)^{-1} = (\bar{\mu}_1 - \bar{\lambda}_1, \bar{\mu}_2 \bar{\lambda}_2)$. We consider its action on $\mathbb{K}^2$ as translation parallel to the first coordinate axis and reflection w.r.t. this axis. It is defined by the following polynomials of $\mathbb{K}[\lambda_1, \lambda_2, z_1, z_2]$: $g_1 = z_1 + \lambda_1$, $g_2 = \lambda_2 z_2$.  

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Example 2.5 Rotation. Consider the special orthogonal group given by 
\[ G = (\lambda_1^2 + \lambda_2^2 - 1) \subset K[\lambda_1, \lambda_2] \] with \( e = (1, 0) \) and \((\mu_1, \mu_2) \cdot (\lambda_1, \lambda_2)^{-1} = (\mu_1 \lambda_1 + 
\mu_2 \lambda_2, \mu_2 \lambda_1 - \mu_1 \lambda_2)\). Its linear action on \( K^2 \) is given by the following polynomials of \( K[\lambda_1, \lambda_2, z_1, z_2] \):

\[ g_1 = \lambda_1 z_1 - \lambda_2 z_2, \quad g_2 = \lambda_2 z_1 + \lambda_1 z_2. \]

An element of the group acts as a rotation around the origin.

2.2 Graph of the action and orbits

The graph of the action is the image \( O \subset Z \times Z \) of the map \( (\bar{\lambda}, \bar{z}) \mapsto (\bar{z}, g(\bar{\lambda}, \bar{z})) \) 
that is defined on a dense open set of \( G \times Z \). We have \( O = \{(\bar{z}, \bar{z}') \mid \exists \lambda \in G \) s.t. \( \bar{z}' = g(\bar{\lambda}, \bar{z})\}\subset Z \times Z \).

We introduce a new set of variables \( Z = (Z_1, \ldots, Z_n) \) and the ideal \( J = G + (Z - g(\lambda, z)) \subset h^{-1}K[\lambda, z, Z] \), where \( (Z - g(\lambda, z)) \) stands for \((Z_1 - g_1(\lambda, z), \ldots, Z_n - g_n(\lambda, z))\). The set \( O \) is dense in its closure \( \overline{O} \), and \( \overline{O} \) is the algebraic variety of the ideal:

\[ O = J \cap K[z, Z] = (G + (Z - g(\lambda, z))) \cap K[z, Z]. \]

Since \( G \) is radical and unmixed dimensional so is \( J \) because of the linearity in \( Z \). If \( G = \bigcap_{i=0}^n G^{(i)} \) is the prime decomposition of \( G \) then we have the following prime decomposition of \( J \):

\[ (G + (Z - g(\lambda, z))) = \bigcap_{i=0}^n \left(G^{(i)} + (Z - g(\lambda, z))\right). \]

The prime ideal \( O^{(i)} = (G^{(i)} + (Z - g(\lambda, z))) \cap K[z, Z] \) is therefore a component of \( O \). The ideals \( O^{(i)} \), however, need not be all distinct.

The set \( O \) is symmetric: if \( (\bar{z}, \bar{z}') \in O \) then \( (\bar{z}', \bar{z}) \in O \). By the Nullstellensatz the ideal \( O \) is also symmetric: \( p(Z, z) \in O \) if \( p(z, Z) \in O \). Since \( J \cap K[z] = (0), O \cap K[z] = (0) \) and therefore \( O \cap K[Z] = (0) \) also.

A set of generators, and more precisely a Gröbner basis \( \overline{O} \), for \( O \subset K[z, Z] \) can be computed.

Proposition 2.6 Let \( g' \) be the \( n \)-tuple of numerators of \( g \), that is \( g' = hg = (hg_1, \ldots, hg_n) \in (K[\lambda, z])^n \). Consider a term order s.t. \( z \cup Z \ll \lambda \cup \{y\} \) where \( y \) is a new indeterminate. If \( Q \) is a Gröbner basis for \( G + (hZ - g') + (yh - 1) \) according to this term order then \( Q \cap K[z, Z] \) is a Gröbner basis of \( O \) according the induced term order on \( z \cup Z \).

Proof: Take \( J' = (G + (Z - g')) \cap K[\lambda, z, Z] \) and note that \( J' = (G + (hZ - g')) : h^\infty \) where \( g' \) is the numerator of \( g \). Given a basis \( \Lambda \) of \( G \) and \( g \) explicitly, a
Gröbner basis of $J$ is obtained thanks to [2] Proposition 6.37, Algorithm 6.6. We recognize that $O$ is an elimination ideal of $J'$, namely $O = J' \cap \mathbb{K}[z, Z]$. A Gröbner basis for $O$ is thus obtained by [2] Proposition 6.15, Algorithm 6.1. □

We mainly use the extension $O^e$ of $O$ in $\mathbb{K}(z)[Z]$. If $Q$ is a Gröbner basis of $O$ w.r.t. a term order $z \ll Z$ then $Q$ is also a Gröbner basis for $O^e$, for the term order induced on $Z$ [2 Lemma 8.93]. It is nonetheless often preferable to compute a Gröbner basis of $O^e$ over $\mathbb{K}(z)$ directly (see Example 6.1).

The orbit of $\bar{z} \in Z$ is the image $O_{\bar{z}}$ of the rational map $g_{\bar{z}}: G \mapsto Z$ defined by $g_{\bar{z}}(\lambda) = g(\lambda, \bar{z})$. We then have the following specialization property (see for instance [6 Exercise 7]).

**Proposition 2.7** Let $Q$ be a Gröbner basis for $O^e$ for a given term order on $Z$. There is a closed proper subset $W$ of $Z$ s.t. for $\bar{z} \in Z \setminus W$ the image of $Q$ under the specialization $z \mapsto \bar{z}$ is a Gröbner basis for the ideal whose variety is the closure of the orbit of $\bar{z}$.

Therefore, for $\bar{z} \in Z \setminus W$, the dimension of the orbits of $\bar{z}$ is equal to the dimension of $O^e \subset \mathbb{K}(z)[Z]$ [6 Section 9.3, Theorem 8]. In the rest of the paper this dimension is denoted by $s$.

**Example 2.8 Scaling.** Consider the group action of Example 2.3. The set of orbits consists of 1-dimensional punctured straight lines through the origin and a single zero-dimensional orbit, the origin. By elimination on the ideal $J = (1 - \lambda_1 \lambda_2, Z_1 - \lambda_1 z_1, Z_2 - \lambda_1 z_2)$ we obtain $O = (z_1 Z_2 - z_2 Z_1)$. Take $W$ to consist solely of the origin. For $\bar{z} \in Z \setminus W$ the closure of the orbit of $\bar{z}$ is the algebraic variety of $(\bar{z}_1 Z_2 - \bar{z}_2 Z_1)$

**Example 2.9 Translation+Reflection.** Consider the group action of Example 2.4. By elimination on the ideal $J = (\lambda_2^2 - 1, Z_1 - z_1 - \lambda_1, Z_2 - \lambda_2 z_2)$ we obtain $O = (Z_2^2 - z_2^2)$. The orbit of a point $\bar{z} = (\bar{z}_1, \bar{z}_2)$ with $\bar{z}_2 \neq 0$ consists of two lines parallel to the first coordinate axis, while the latter is the orbit of all points with $\bar{z}_2 = 0$.

**Example 2.10 Rotation.** Consider the group action of Example 2.5. The orbits consist of the origin and the circles with the origin as center. By elimination on the ideal $J = (\lambda_1^2 + \lambda_2^2 - 1, Z_1 - \lambda_1 z_1 + \lambda_2 z_2, Z_2 - \lambda_2 z_1 - \lambda_1 z_2)$ we obtain $O = (Z_1^2 + Z_2^2 - z_1^2 - z_2^2)$.

### 2.3 Rational invariants

We construct a finite set of generators for the field of rational invariants. Our construction brings out a simple algorithm to rewrite any rational invariant in terms of them. The required operations are restricted to computing a Gröbner basis and normal forms. Those are implemented in most computer algebra systems. We provide a comparison with related results in [24, 29, 35].
Definition 2.11 A rational function \( r \in \mathbb{K}(z) \) is a rational invariant if
\[ r(q(\lambda, z)) = r(z) \mod G. \]

The set of rational invariants forms a field\(^3\) \( \mathbb{K}(z)^G \). We show that the coefficients of the Gröbner basis for \( O^e \) are invariant and generate \( \mathbb{K}(z)^G \). The basis is computed using Proposition 2.6.

Lemma 2.12 If \( q(z, Z) \) belongs to \( O \) then \( q(g(\lambda, z), Z) \) belongs to \( O^e \) for all \( \lambda \in G \).

Proof: A point \((\bar{z}, \bar{z}') \in O\) if there exists \( \bar{\mu} \in G \) s.t. \( \bar{z}' = g(\bar{\mu}, \bar{z}) \). Then for a generic \( \bar{\lambda} \in G \), \( \bar{z}' = g(\bar{\mu} \cdot \bar{\lambda}^{-1}, g(\bar{\lambda}, \bar{z})) \). Therefore \( g(\bar{\lambda}, \bar{z}, \bar{z}') \in O \). Thus if \( q(z, Z) \in O \) then \( q(g(\bar{\lambda}, \bar{z}), \bar{z}') = 0 \) for all \((\bar{z}, \bar{z}') \) in \( O \). By the Hilbert Nullstellensatz the numerator of \( q(g(\bar{\lambda}, z), Z) \) belongs to \( O \) and therefore \( q(g(\bar{\lambda}, z), Z) \in O^e \). □

Following [2], Definition 5.29, a set of polynomials is reduced, for a given term order, if the leading coefficients of the elements are equal to 1 and each element is in normal form with respect to the others. Given a term order on \( Z \), a polynomial ideal in \( \mathbb{K}(z)|Z| \) has a unique reduced Gröbner basis [2, Theorem 5.3].

Theorem 2.13 The reduced Gröbner basis of \( O^e \) with respect to any term order on \( Z \) consists of polynomials in \( \mathbb{K}(z)^G[Z] \).

Proof: Let \( Q = \{q_1, \ldots, q_k\} \) be the reduced Gröbner basis of \( O^e \) for a given term order on \( Z \). By Lemma 2.12 \( q_i(g(\bar{\lambda}, z), Z) \) belongs to \( O^e \). It has the same support\(^4\) as \( q_i \). As \( q_i(g(\bar{\lambda}, z), Z) \) and \( q_i(z, Z) \) have the same leading monomial, \( q_i(g(\bar{\lambda}, z), Z) - q_i(z, Z) \) is in normal form with respect to \( Q \). As this difference belongs to \( O^e \), it must be 0. The coefficients of \( q_i \) are therefore invariant. □

Let us note the construction of a generating set of rational invariants proposed by Rosenlicht [24]. In the paragraph before Theorem 2, Rosenlicht points out that the coefficients of the Chow form of \( O^e \) over \( \mathbb{K}(z) \) form a set of separating rational invariants. By [24, Theorem 2] or [35, Lemma 2.1] this set is generating for \( \mathbb{K}(z)^G \).

Vinberg and Popov showed the existence of a subset of \( \mathbb{K}(z)^G[Z] \) that generates \( O^e \) [35, Lemma 2.4]. We propose the construction of such a set. They showed furthermore that the set of the coefficients of such a family of generators separates generic orbits \( \mathbb{K}(z)^G \) [24, Theorem 2], [35, Lemma 2.1]. From those results we deduce that the set of coefficients of a reduced Gröbner basis of \( O^e \) generates \( \mathbb{K}(z)^G \). The next theorem provides an alternative proof of this result, providing additionally a

\(^3\)Though we do not use this fact but rather retrieve it otherwise, it is worth noting that, as a subfield of \( \mathbb{K}(z) \), the field of rational invariants is always finitely generated [35].

\(^4\)The support here is the set of terms in \( Z \) with non zero coefficients.
rewriting algorithm. To prove generation we indeed exhibit an algorithm that allows to rewrite any rational invariant in terms of the coefficients of a reduced Gröbner basis.

In the case of linear actions Müller-Quade and Beth [24] showed that the coefficient of the Gröbner basis of \( O^e \) generate the field of rational invariants. Their proof is based on more general results about the characterization of subfields of \( \mathbb{K}(z) \) obtained in [25]. Our approach is quite different and more direct. The rewriting algorithm we propose, although it was obtained independently, is nonetheless reminiscent of [25, Algorithm 1.10].

**Lemma 2.14** Let \( \xi \) be a rational invariant, \( p,q \in \mathbb{K}[z] \). Then \( p(Z) q(z) − q(Z) p(z) \in O \).

**Proof:** Since \( \xi \) is an invariant \( \frac{p(z)}{q(z)} = \frac{p(\lambda,z)}{q(\lambda,z)} \) for all \( (\lambda, \tilde{z}) \) where this expression is defined. Thus \( a(\tilde{z}',z) = p(\tilde{z}') q(z) − q(\tilde{z}') p(\tilde{z}) = 0 \) for all \( (\tilde{z}, \tilde{z}') \) in \( O = \{(\tilde{z}, z) \mid \exists \lambda \in G \ s.t. \, \tilde{z}' = g(\lambda, \tilde{z})\} \subset Z \times \tilde{Z} \). In other words the polynomial \( a(Z,z) = p(Z) q(z) − q(Z) p(z) \in \mathbb{K}[Z,z] \) is zero at each point of \( O \). Since the algebraic variety of \( O \) is the closure \( \bar{O} \) of \( O \) and that \( \bar{O} \) is dense in \( \bar{O} \) we can conclude that \( a(Z,z) \in \bar{O} \) by Hilbert Nullstellensatz. \( \square \)

Assume a polynomial ring over a field is endowed with a given term order. A polynomial \( p \) is in *normal form* w.r.t. a set \( Q \) of polynomials if \( p \) involves no term that is a multiple of a leading term of an element in \( Q \). A *reduction* w.r.t. \( Q \) is an algorithm that given \( p \) returns a polynomial \( p' \) in normal form w.r.t. \( Q \) s.t. \( p = p' + \sum_{q \in Q} a_q q \) and no leading term of any \( a_q q \) is larger than the leading term of \( p \). Such an algorithm is detailed in [24, Algorithm 5.1]. It consists in rewriting the terms that are multiple of the leading terms of the elements of \( Q \) by polynomials involving only terms that are lower. Note that if the leading coefficients of \( Q \) are 1 then no division occurs. When \( Q \) is a Gröbner basis w.r.t. the given term order, the reduction of a polynomial \( p \) is unique in the sense that \( p' \) is then the only polynomial in normal form w.r.t. \( Q \) in the equivalence class \( p + (Q) \).

**Theorem 2.15** Consider \( \{r_1, \ldots, r_k\} \in \mathbb{K}(z)^G \) the coefficients of a reduced Gröbner basis \( Q \) of \( O^e \). Then \( \mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_k) \) and we can rewrite any rational invariant \( \frac{\xi}{q} \), with \( p,q \in \mathbb{K}[z] \), in terms of those as follows.

Take a new set of indeterminates \( y_1, \ldots, y_k \) and consider the set \( Q_y \subset \mathbb{K}[y,Z] \) obtained from \( Q \) by substituting \( r_i \) by \( y_i \). Let \( a(y,Z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(y) Z^\alpha \) and \( b(y,Z) = \sum_{\alpha \in \mathbb{N}^n} b_\alpha(y) Z^\alpha \) in \( \mathbb{K}[y,Z] \) be the reductions\(^5\) of \( p(Z) \) and \( q(Z) \) w.r.t. \( Q_y \). There exists \( \alpha \in \mathbb{N}^n \) s.t. \( b_\alpha(r) \not= 0 \) and for any such \( \alpha \) we have \( \frac{a_\alpha(z)}{b_\alpha(z)} = \frac{q(z)}{q(r)} \).

**Proof:** It is sufficient to prove the second part of the statement. The Gröbner basis \( Q \) is reduced and therefore monic. The sets of leading monomials of \( Q \)

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\(^5\)For those reductions in \( \mathbb{K}[y,Z] \) the term order on \( Z \) is extended to a block order \( y \ll Z \) so that the set of leading term of \( Q_y \) is equal to the set of leading terms of \( Q \).
and of $Q_y$ are equal. If $a(y, Z)$ is the reduction of $p(Z)$ w.r.t. $Q_y$ then $a(r, Z)$, obtained by substituting back $y_i$ by $r_i$, is the normal form of $p(Z)$ w.r.t. $Q$. Similarly for $b(y, Z)$ and $q(Z)$.

As $O^e \cap K[Z] = \{0\}$, neither $p(Z)$ nor $q(Z)$ belong to $O^e$ and therefore both $a(r, Z)$ and $b(r, Z)$ are different from 0. By Lemma 2.14 $q(z)p(Z) \equiv p(z)q(Z)$ mod $O^e$ and thus the normal forms of the two polynomials modulo $O^e$ are equal: $q(z)a(r, Z) = p(z)b(r, Z)$. Thus $a(r, Z)$ and $b(r, Z)$ have the same support and this latter is non empty since $a, b \neq 0$. For each $\alpha$ in this common support, we have $q(z)a_{\alpha}(r) = p(z)b_{\alpha}(r)$ and therefore $\frac{p(z)}{q(z)} = \frac{a_{\alpha}(r)}{b_{\alpha}(r)}$. □

Example 2.16 Scaling. We consider the group action given in Example 2.3. A reduced Gröbner basis of $O^e$ is $Q = \{Z_2 - \frac{3}{z_1}Z_1\}$. By Theorem 2.13 $K(z_1, z_2)^G = K(\frac{z_2}{z_1})$.

Let $p = z_1^2 + 4z_1z_2 + z_2^2$ and $q = z_1^2 - 3z_2^2$. We can check that $p/q$ is a rational invariant and we set up to write $p/q$ as a rational function of $r = z_2/z_1$. To this purpose consider $P = Z_1^2 + 4Z_1Z_2 + Z_2^2$ and $Q = Z_1^2 - 3Z_2^2$ and compute their normal forms $a$ and $b$ w.r.t. $\{Z_2 - yZ_1\}$ according to a term order where $Z_1 < Z_2$. We have $a = (1 + 4y + y^2)Z_1^2$ and $b = (1 - 3y^2)Z_1^2$. Thus

$$\frac{z_1^2 + 4z_1z_2 + z_2^2}{z_1^2 - 3z_2^2} = \frac{1 + 4r + r^2}{1 - 3r^2}$$

where $r = \frac{z_2}{z_1}$.

Example 2.17 Translation+Reflection. We consider the group action given in Example 2.3. A reduced Gröbner basis of $O^e$ is $Q = \{Z_2^3 - z_3^2\}$. By Theorem 2.13 $K(z_1, z_2)^G = K(z_2^3)$.

Example 2.18 Rotation. We consider the group action given in Example 2.3. A reduced Gröbner basis of $O^e$ is $Q = \{Z_1^2 + Z_2^2 - (z_1^2 + z_3^2)\}$. By Theorem 2.13 $K(z_1, z_2)^G = K(z_1^2 + z_2^2)$.

### 3 Cross-section and rational invariants

Given a cross-section we construct a generating set of rational invariants endowed with a rewriting algorithm. The method is the same as the one presented in previous section but applies to only a section of the graph. In previous section we considered an ideal of the dimension of the generic orbits. Here we consider a zero dimensional ideal. This is computationally advantageous when Gröbner bases are needed.

We use Noether normalization to prove the existence of a cross-section. The construction thus relies on selecting elements of an open subset of a certain affine space. Therefore the construction does not entail a deterministic algorithm for the computation of rational invariants. Yet the freedom of choice is extremely fruitful in applicative examples.
Though the presentation is done with an algebraically closed field $K$ that is therefore infinite, the construction is meant to be realized in characteristic zero (i.e. over $\mathbb{Q}$) or over a sufficiently large field.

3.1 Cross-section

Geometrically speaking a cross-section of degree $d$ is a variety that intersects generic orbits in $d$ simple points. We give a definition in terms of ideals for it is closer to the actual computations. We give its geometric content in a proposition afterward.

**Definition 3.1** Let $P$ be a prime ideal of $K[Z]$ of complementary dimension to the generic orbits, i.e. if $O^e$ is of dimension $s$ then $P$ is of codimension $s$. $P$ defines a cross-section to the orbits of the rational action $g : G \times Z \to Z$ if the ideal $I^e = O^e + P$ of $K(z)[Z]$ is radical and zero dimensional. If $d$ is the dimension of $K(z)[Z]/I^e$ as a $K(z)$-vector space, we say that $P$ defines a cross-section of degree $d$.

Indeed the algebra $K(z)[Z]/I^e$ is a finite $K(z)$-vector space since $I^e$ is zero dimensional [2, Theorem 6.54]. A basis for it is provided by the terms in $Z$ that are not multiple of the leading terms of a Gröbner basis of $I^e$ [2, Proposition 6.52]. Let us note here that an ideal of $K(z)[Z]$ is zero dimensional iff any Gröbner basis of it has an element whose leading term is $Z_i^d$, for all $1 \leq i \leq n$ [2, Theorem 6.54].

The cross-section is thus the variety $\mathcal{P}$ of $P$. The geometric properties of this variety are explained by the following proposition.

**Proposition 3.2** Let $P$ define a cross-section $\mathcal{P}$ of degree $d$. There is a closed set $S \subset Z$ s.t. the closure of the orbit of any $\bar{z} \in Z \setminus S$ intersects $\mathcal{P}$ in $d$ simple points.

**Proof:** Let $Q$ be a reduced Gröbner basis for $I^e = O^e + P$. Similarly to Proposition [2,7] the image $Q_{\bar{z}}$ of $Q$ under the specialization $z \mapsto \bar{z}$ is a Gröbner basis for $O_{\bar{z}} + P$ in $K[Z]$ for all $\bar{z}$ in $Z$ outside of a closed set $\mathcal{W}$. Thus $I_{\bar{z}} = O_{\bar{z}} + P$ is zero dimensional and the dimension of $K[Z]/I_{\bar{z}}$ as a vector space over $K$ is $d$.

By the Jacobian criterion for regularity and the prime avoidance theorem [10, Corollary 16.20 and Lemma 3.3] there is a $n \times n$ minor $f$ of the Jacobian matrix of $Q$ that is not included in any prime divisor of $I^e$. Therefore $f$ is not a zero divisor in $K(z)[Z]/I^e$ which is a product of fields. There exists thus $f' \in K(z)[Z]$ s.t. $ff' \equiv 1 \mod I^e$.

Provided that $\bar{z}$ is furthermore chosen so that the denominators of $f$ and $f'$ do not vanish, $f$ specializes into a $n \times n$ minor $f_{\bar{z}}$ of the Jacobian matrix of $Q_{\bar{z}}$ and we have $f_{\bar{z}}f'_{\bar{z}} \equiv 1 \mod I_{\bar{z}}$ for the specialization of $f'$ of $f$. So $f_{\bar{z}}$ belongs to no prime divisors of $I_{\bar{z}}$ and thus $I_{\bar{z}}$ is radical [10, Corollary 16.20]. We take $S$
to be the union of $\mathcal{W}$ with the algebraic set associated to the product of the denominators of $f$ and $f'$. That the number of points of intersection is $d$ is shown by [10, Proposition 2.15]. □

That property shows that the cross-sections of degree $d = 1$ and $d > 1$ are respectively the sections and the quasi-sections defined in [35, Paragraph 2.5]. The existence of quasi-section is insured by [35, Proposition 2.7], while a criterion for the existence of a section is described in [35, Paragraph 2.5 and 2.6]. Our terminology elaborates on the one used in [29] and [11].

The discussion of [35, Section 2.5] shows that $K(P)$ is isomorphic to $K(z)^G$ when $P$ is a cross-section of degree 1. If $P$ is a cross-section of degree $d > 1$ then $K(P)$ is an algebraic extension of $K(z)^G$ of degree $d$. In Section 4 we shall come back to those points with a constructive angle that relies on the choice of a cross-section. The viewpoint adopted here is indeed the geometric intuition of the moving frame construction in [11]: almost any algebraic variety of complementary dimension provides a cross-section (of some degree).

The existence of a cross-section is proved by Noether normalization theorem and is linked to an alternative definition of the dimension of an ideal [30, Section 6.2].

**THEOREM 3.3** A linear cross-section to the orbit is associated to each point of an open set of $K^{s(n+1)}$, where $s$ is the dimension of the generic orbits and $n$ the dimension of $Z$.

**PROOF:** Assume that a Gröbner basis $Q$ of $O^e$ w.r.t. a term order $Z_1, \ldots, Z_s \ll Z_{s+1}, \ldots, Z_n$ is s.t. an element of $Q$ has leading term $Z_i^{d_i}$ for some $d_i \in \mathbb{N} \setminus \{0\}$ for all $s + 1 \leq i \leq n$ and there is no element of $Q$ independent of $\{Z_{s+1}, \ldots, Z_n\}$. Then $Q$ is a Gröbner basis for the extension of $O^e$ to $K(z)(Z_1, \ldots, Z_s)[Z_{s+1}, \ldots, Z_n]$ [2, Lemma 8.93]. For $(a_1, \ldots, a_s)$ in an open set of $K^s$ the specialization $Q_a \subset K[Z_{s+1}, \ldots, Z_n]$ of $Q$ under $Z_i \mapsto a_i$ is a Gröbner basis [6, Exercise 7]. Therefore $Q_a \cup \{Z_1 - a_1, \ldots, Z_s - a_s\}$ is a Gröbner basis by Buchberger’s criteria [2, Theorem 5.48 and 5.66]. It is a Gröbner basis of a zero dimensional ideal [2, Theorem 6.54]. We can thus take $P$ to be generated by $\{Z_1 - a_1, \ldots, Z_s - a_s\}$.

We can always retrieve the situation assumed above by a change of variables thanks to Noether normalization theorem [16, Theorem 3.4.1]. Inspecting the proof we observe that we can choose a change of variables given by a matrix $(m_{ij})_{1 \leq i,j \leq n}$ with the vector of entries $m_{ij}$ in $\mathbb{K}^n$ outside of some algebraically closed set. The set $\{a_i - \sum_{1 \leq j \leq n} m_{ij} Z_j \mid 1 \leq i \leq s\}$ thus defines a cross-section. □

The choice of a cross section introduces a non deterministic aspect to the algebraic construction proposed in next section. An analysis of the probability of success in characteristic 0 would be based on the measure of a correct test sequence [13, Theorem 3.5 and 3.7.2], [14, Section 3.2], [22, Section 4.1].
We can computationally test if $\mathcal{P}$ is a cross-section by checking the properties of $I^e = (G + P + (Z - g(\lambda, z))) \cap K(z)[Z]$, starting with the computation of its Gröbner basis. It is nonetheless worth performing the preliminary necessary test of transversality detailed in Section 5.3. It relies on computing the rank of a matrix.

**Proposition 3.4** Assume that $P \subset K[Z]$ defines a cross-section and that $O = \cap_{i=0}^T O(i)$ is the prime decomposition of $O$. Then

$$O + P = \bigcap_{i=0}^T (O(i) + P) \quad \text{and} \quad (O(i) + P) \cap K[Z] = P.$$

**Proof:** We can easily check that $\bigcap_{i=0}^T (O(i) + P) \subset O + P$ because $O + P$ is radical. The converse inclusion is trivial.

For the second equality, note first that $P \subset (O(i) + P) \cap K[z, Z]$. The projection of the variety of $O(i) \subset Z \times Z$ is thus contained in $\mathcal{P}$. We show that the projection is exactly $\mathcal{P}$. We can assume that the numbering is such that $O(i) = ((G^{(i)} + (z - g(\lambda, Z))) \cap K[z, Z]$ where $G^{(i)}$ is a minimal prime of $G$ (see Section 2.3). By Assumption 2.2, for any $\bar{z}$ in $Z$ and therefore in $\mathcal{P}$, there exists $\bar{\lambda}$ in the variety of $G^{(i)}$ s.t. $g(\bar{\lambda}, \bar{z})$ is defined. Above each point of $\mathcal{P}$ there is a point in the variety of $O(i)$. $\square$

### 3.2 Rational invariants revisited

The following theorems provide a construction of a generating set of rational invariants together with an algorithm to rewrite any rational invariant in terms of generators. The method is the same as in Section 2.3 but applied to the ideal $I^e$ rather than to $O^e$. The computational advantage comes from the fact that $I^e$ is zero dimensional.

If $G$ is a prime ideal we can actually choose a coordinate cross-section that is $P$ can be taken as the ideal generated by a set of the following form: $\{Z_j, - \alpha_1, \ldots, Z_j, - \alpha_s\}$ for $(\alpha_1, \ldots, \alpha_s)$ in $K^e$. In this case we can remove $r$ variables for the computation.

**Theorem 3.5** The reduced Gröbner basis of $I^e$ with respect to any term ordering on $Z$ consists of polynomials in $K(z)^G[Z]$.

**Proof:** The union of a reduced Gröbner basis of $O^e$ and $P$ forms a generating set for $I^e = O^e + P$. The coefficients of a basis for $P$ are in $K$, while the coefficients of a reduced basis for $O^e$ belong to $K(z)^G$ due to Theorem 2.3. Since the coefficients of a generating set for $I^e$ belong to $K(z)^G$, so do the coefficients of the reduced Gröbner basis with respect to any term ordering. $\square$
THEOREM 3.6. Consider \( \{ r_1, \ldots, r_n \} \in \mathbb{K}(z)^G \) the coefficients of a reduced Gröbner basis \( Q \) of \( I^c \). Then \( \mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_n) \) and we can rewrite any rational invariant \( p/z \), with \( p, q \in \mathbb{K}[z] \) relatively prime, in terms of those as follows.

Take a new set of indeterminates \( y_1, \ldots, y_k \) and consider the set \( Q_y \subset \mathbb{K}[y, Z] \) obtained from \( Q \) by substituting \( y_i \) by \( y_i \). Let \( a(y, Z) = \sum_{\alpha \in \mathbb{N}^N} a_{\alpha}(y)Z^\alpha \) and \( b(y, Z) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha}(y)Z^\alpha \) in \( \mathbb{K}[y, Z] \) be the reductions of \( p(Z) \) and \( q(Z) \) w.r.t. \( Q_y \). There exists \( \alpha \in \mathbb{N}^m \) s.t. \( b_\alpha(r) \neq 0 \) and for any such \( \alpha \) we have \( \frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)} \).

PROOF: We can proceed just as in the proof of Theorem 2.19. We only need to argue additionally that if \( r = \frac{z}{\bar{z}} \in \mathbb{K}(z)^G \), \( p \) and \( q \) being relatively prime, then \( p(Z), q(Z) \notin I^c \). We prove the result for \( p \), the case of \( q \) being similar.

By hypothesis \( p(z) q(g(\lambda, z)) \equiv q(z) p(g(\lambda, z)) \mod G \). Since \( p \) and \( q \) are relatively prime, \( p(z) \) divides \( p(g(\lambda, z)) \) modulo \( G \), that is there exists \( \alpha \in \mathbb{N}[z, \lambda] \) s.t. \( p(g(\lambda, z)) = p(z) \mod G \). Therefore if \( p \) vanishes at \( \bar{z} \in \mathbb{Z} \), then it vanishes on \( \mathcal{O}_z \). Thus if \( p \in P \), or equivalently if \( p \) vanishes on \( \mathcal{P} \), it vanishes on an open subset of \( \mathbb{Z} \) (Proposition 3.2). So \( p \) must be zero. This is not the case and thus \( p \notin P \). Since \( I^c \cap \mathbb{K}[Z] = P \), it is the case that \( p(Z) \notin I^c \). □

When \( P \) defines a cross-section of degree 1, the rewriting trivializes into a replacement. Indeed, if the dimension of \( \mathbb{K}(z)[Z]/I^c \) as a \( \mathbb{K}(z) \) vector space is 1 then, independently of the chosen term order, the reduced Gröbner basis \( Q \) for \( I^c \) is given by \( \{ Z_i - r_i(z) | 1 \leq i \leq n \} \) where the \( r_i \in \mathbb{K}(z)^G \). In view of Theorem 3.6 \( \mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_n) \) and any rational invariant \( r(z) \in \mathbb{K}(z)^G \) can be rewritten in terms of \( r_i \) by replacing \( z_i \) by \( r_i \):

\[
r(z_1, \ldots, z_n) = r(r_1(z), \ldots, r_n(z)), \quad \forall r \in \mathbb{K}(z)^G.
\]

In the next section we generalize this replacement to cross-section of any degree by introducing some special algebraic invariants.

EXAMPLE 3.7. SCALING. We carry on with the action considered in Example 2.6 and 2.16.

Choose \( P = (Z_1 - 1) \). A reduced Gröbner basis of \( I^c \) is given by \( \{ Z_1 - 1, Z_2 - z_1 \} \).

We can see that Theorem 3.6 is verified and that \( P \) defines a cross-section of degree 1. By Theorem 3.6 we know that \( r = z_2/z_1 \) generates the field of rational invariants \( \mathbb{K}(z)^G \). In this situation, the cross section is of degree 1 and we see that the rewriting algorithm of Theorem 3.6 is a simple replacement. For all \( p \in \mathbb{K}(z)^G \) we have \( p(z_1, z_2) = p(1, r) \).

EXAMPLE 3.8. TRANSLATION+REFLECTION. We carry on with the action considered in Example 2.6 and 2.17.

Choose \( P = (Z_1 - Z_2) \) to define the cross-section. A reduced Gröbner basis of \( I^c \) is given by \( \{ Z_1 - Z_2, Z_2^2 - z_2^2 \} \). The cross-section is thus of degree 2.
Example 3.9 Rotation. We carry on with the action considered in Examples 2.5 and 2.18.

Choose \( P = (Z_2) \). The reduced Gröbner basis of \( I^e \) w.r.t. any term order is \( \{ Z_2, Z_1^2 - (z_2^2 + z_1^2) \} \). We can see that Theorem 2.13 is verified and that \( P \) defines a cross-section of degree 2. By Theorem 3.6 we know that \( r = z_1^2 + z_2^2 \) generates the field of rational invariants \( \mathbb{K}(z)^G \). In this situation, the rewriting algorithm of Theorem 3.6 consists in substituting \( z_2 \) by 0 and \( z_1^2 \) by \( r \).

4 Replacement invariants and invariantization

Given a cross-section \( P \) of degree \( d \) we introduce \( d \) distinct \( n \)-tuples of elements that are algebraic over the field of rational invariants. Each \( n \)-tuple has an important replacement property: any rational invariant can be rewritten in terms of its components by a simple substitution of the variables by the corresponding elements from the tuple.

The replacement invariants are used to define a process of invariantization, that is a projection from the algebraic functions onto the field of algebraic invariants. This projection can be explicitly computed by algebraic elimination. It gives a constructive approach to the isomorphism \( \mathbb{K}(P) \cong \mathbb{K}(z)^G \).

4.1 Replacement invariants

Let \( P \) be a cross-section of degree \( d \) defined by a prime ideal \( P \) of \( \mathbb{K}[Z] \). The field of rational functions on \( P \) is denoted by \( \mathbb{K}(P) \). It is the fraction field of the integral domain \( \mathbb{K}[Z]/P = \mathbb{K}[P] \). We introduce \( d \) replacement invariants associated to \( P \). We use them to show that \( \mathbb{K}(P) \) is an algebraic extension of degree \( d \) of the field of rational invariants \( \mathbb{K}(z)^G \).

Definition 4.1 An algebraic invariant is an element of the algebraic closure \( \overline{\mathbb{K}(z)^G} \) of \( \mathbb{K}(z)^G \).

A reduced Gröbner basis \( Q \) of \( I^e = O^e + P \) is contained in \( \mathbb{K}(z)^G[Z] \) (Theorem 3.5) and therefore is a reduced Gröbner basis of \( I^G = I^e \cap \mathbb{K}(z)^G[Z] \). The dimension of \( \mathbb{K}(z)^G[Z]/I^G \) as a \( \mathbb{K}(z)^G \)-vector space is therefore equal to the dimension \( d \) of \( \mathbb{K}(z)[Z]/I^e \) as a \( \mathbb{K}(z) \)-vector space. Consequently the ideal \( I^G \) has \( d \) zeros \( \xi = (\xi_1, \ldots, \xi_n) \) with \( \xi_i \in \mathbb{K}(z)^G \) \([10\text{, Proposition 2.15}]\). We call such a tuple \( (\xi_1, \ldots, \xi_n) \) a \( \overline{\mathbb{K}(z)^G} \) -zero of \( I^G \). A \( \overline{\mathbb{K}(z)^G} \) -zero of \( I^G \) is a \( \mathbb{K}(z)^G \) -zero of \( I^e \) and conversely.

Definition 4.2 A replacement invariant is a \( \overline{\mathbb{K}(z)^G} \) -zero of \( I^G = I^e \cap \mathbb{K}(z)^G[Z] \), i.e. a \( n \)-tuple \( \xi = (\xi_1, \ldots, \xi_n) \) of algebraic invariants that forms a zero of \( I^e \).
Thus $d$ replacement invariants $\xi^{(1)}, \ldots, \xi^{(d)}$ are associated to a cross-section of degree $d$. The name owes to next theorem which can be compared with Thomas replacement theorem discussed in [11] page 38.

**Theorem 4.3** Let $\xi = (\xi_1, \ldots, \xi_n)$ be a replacement invariant. If $r \in \mathbb{K}(z)^G$ then $r(z_1, \ldots, z_n) = r(\xi_1, \ldots, \xi_n)$ in $\mathbb{K}(z)^G$.

**Proof:** Write $r = \frac{p}{q}$ with $p, q$ relatively prime. By Lemma 4.4 $p(z)q(Z) - q(z)p(Z) \in O^e \subset I^e$ and therefore $p(Z) - \frac{q(z)}{q(Z)}q(Z) = p(Z) - r(z)q(Z) \in I^e$. Since $\xi$ is a zero of $I^e$, we have $p(\xi) - r(\xi)q(\xi) = 0$. In the proof of Theorem 4.3 we saw that $p(Z), q(Z)$ can not belong to $P$ and therefore cannot be zero divisors modulo $I^e$. Thus $q(\xi) \neq 0$ and the conclusion follows. □

The field $\mathbb{K}(\xi)$, for any replacement invariant $\xi$, is an algebraic extension of $\mathbb{K}(z)^G$. Indeed $\mathbb{K}(z)^G \subset \mathbb{K}(\xi)$ and $\xi$ is algebraic over $\mathbb{K}(z)^G$. This leads to the following results.

**Lemma 4.4** $I^G = I^e \cap \mathbb{K}(z)^G[Z]$ is a prime ideal of $\mathbb{K}(z)^G[Z]$.

**Proof:** Let $I^{(1)}$ and $I^{(2)}$ be prime divisors of $I^G$ in $\mathbb{K}(z)^G[Z]$ and consider replacement invariants $\xi^{(1)}$ and $\xi^{(2)}$ that are $\mathbb{K}(z)^G$-zeros of $I^{(1)}$ and $I^{(2)}$ respectively. Due to Theorem 4.3 $\mathbb{K}(\xi^{(i)}) = \mathbb{K}(z)^G(\xi^{(i)})$. There is therefore a $\mathbb{K}(z)^G$-isomorphism $\mathbb{K}(z)^G[Z]/I^{(i)} \cong \mathbb{K}(\xi^{(i)})$ for $i = 1$ or 2. On the other hand we have $\mathbb{K}(\xi^{(i)}) \cong \mathbb{K}(P)$ since $P$ is the ideal of all relationships on the components of $\xi^{(i)}$ over $\mathbb{K}$ (Proposition 3.4). Thus

\[ \mathbb{K}(z)^G[Z]/I^{(1)} \cong \mathbb{K}(\xi^{(1)}) \cong \mathbb{K}(P) \cong \mathbb{K}(\xi^{(2)}) \cong \mathbb{K}(z)^G[Z]/I^{(2)}. \]

We have an isomorphism between $\mathbb{K}(z)^G[Z]/I^{(1)}$ and $\mathbb{K}(z)^G[Z]/I^{(2)}$ that leaves $\mathbb{K}(z)^G$ fixed and maps the class of $Z$ modulo $I^{(1)}$ to the class of $Z$ modulo $I^{(2)}$. Therefore $I^{(1)} = I^{(2)}$ so that $I^G$ is prime. □

**Theorem 4.5** The field $\mathbb{K}(P)$ is an algebraic extension of $\mathbb{K}(z)^G$ of degree $d$, the degree of the cross-section $P$.

**Proof:** For any replacement invariant $\xi$ we have $\mathbb{K}(z)^G[Z]/I^G \cong \mathbb{K}(\xi) \cong \mathbb{K}(P)$. Since the dimension of $\mathbb{K}(z)^G[Z]/I^G$ as $\mathbb{K}(z)^G$-vector space is $d$, the field $\mathbb{K}(P)$ is an algebraic extension of $\mathbb{K}(z)^G$ of degree $d$. □

In particular if $P$ is a cross-section of degree one we have $\mathbb{K}(P) \cong \mathbb{K}(z)^G$. In all cases we have the isomorphism $\mathbb{K}(P) \cong \mathbb{K}(z)^G$ obtained in [11] Section 2.5 by different means.

**Example 4.6** Scaling. Consider the multiplicative group from Example 2.6. We considered the cross-section of degree 1 defined by $P = (Z_1 - 1)$. There is single replacement invariant $\xi = (1, \frac{a}{z_1})$ with rational components,
which can be read off the reduced Gröbner basis of \( I^e = (Z_1 - 1, Z_2 - \frac{z_1}{z_2}). \) One can check that \( r(z_1, z_2) = r(1, \frac{z_2}{z_1}) \) for any \( r \in \mathbb{K}(z)^G = \mathbb{K}\left(\frac{z_2}{z_1}\right). \)

**Example 4.7 Translation+Reflection.** Consider the group action from Example 2.14, 2.17. We chose the cross-section defined by \( P = (Z_1 - Z_2) \) and found that \( \mathbb{K}(z_2^2) \) was the field of rational invariants. Generic orbits have two components and the cross-section is of degree 2. Since \( I^e = (Z_1 - Z_2, Z_2^2 - z_2^2) \), the two replacement invariants are \( \xi^{(1)} = (z_2, z_2) \) and \( \xi^{(2)} = (-z_2, -z_2). \) Though rational functions, their components are not rational invariants but only algebraic invariants. Also \( I^c = (Z_1 - z_2, Z_2 - z_2) \cap (Z_1 + z_2, Z_2 + z_2) \) is a reducible ideal of \( \mathbb{K}(z)[Z] \), while \( I^G \) is an irreducible ideal of \( \mathbb{K}(z)^G[Z]. \)

**Example 4.8 Rotation.** Consider the group action from Example 2.14, 2.17. We chose the cross-section defined by \( P = (Z_2) \). Here the cross-section is again of degree 2 but the generic orbits have a single component. Since \( I^c = (Z_2, Z_2^2 - z_2^2) \) the two replacement invariants associated to \( P \) are \( \xi^{(\pm)} = (0, \pm \sqrt{z_2^2 + z_2^2}). \)

### 4.2 Invariantization

In this section we introduce invariantization as a projection from the ring of univariate polynomials over \( \mathbb{K}[z] \) to the ring of univariate polynomials over \( \mathbb{K}(z)^G. \) It depends on the choice of a cross-section and is computable by algebraic elimination. As this projection extends to univariate polynomials over \( \mathbb{K}(\mathcal{P}) \) it can be understood as the computable counterpart to the isomorphism \( \mathbb{K}(\mathcal{P}) \cong \mathbb{K}(z)^G \) that follows from Proposition 4.5.

We assume throughout this section that the field \( \mathbb{K} \) is of characteristic zero. The ideal of the cross-section \( \mathcal{P} \) is taken alternatively in \( \mathbb{K}[z] \) and in \( \mathbb{K}[Z]. \) To avoid confusion we shall use in this section \( P_z \) and \( P_Z \) to distinguish the two cases. The localization of \( \mathbb{K}(z) \) at \( P_z \) is denoted by \( \mathbb{K}(z)[P]. \) In the proof of Theorem 5.6 we have shown that \( \mathbb{K}(z)^G \subset \mathbb{K}(z)[P]. \)

The first approach for invariantization that draws directly on [11] is to consider a replacement invariant \( \xi \) associated to \( \mathcal{P} \) and the following chain of homomorphisms:

\[
\mathbb{K}[z]_{\mathcal{P}} \xrightarrow{\pi} \mathbb{K}(\mathcal{P}) \xrightarrow{\phi_{\xi}} \mathbb{K}(z)^G \xrightarrow{r} r(\xi)
\]

(2)

The restriction of \( \iota_{\xi} = \phi_{\xi} \circ \pi : \mathbb{K}[z]_{\mathcal{P}} \to \mathbb{K}(z)^G \) to \( \mathbb{K}(z)^G \) is the identity map by Theorem 4.6. We call the image of a rational function \( r(z) \in \mathbb{K}[z]_{\mathcal{P}} \) under \( \iota_{\xi} \) its \( \xi \)-invariantization.

If \( \mathcal{P} \) is a cross-section of degree \( d \) there are \( d \) distinct associated replacement invariants \( \xi^{(1)}, \ldots, \xi^{(d)} \). The image \( \iota_{\xi}(r(z)) = r(\xi) \) depends on the chosen replacement invariant \( \xi \). Such is not the case of the minimal polynomial of \( r(\xi) \)
over $\mathbb{K}(z)^G$ which depends only on $\mathcal{P}$ as we shall see. We therefore define the $\mathcal{P}$-invariantization as a map taking a univariate polynomial over $\mathbb{K}[z]_P$ to a univariate polynomial over $\mathbb{K}(z)^G$. The connection to the smooth invariantization of $\mathcal{P}$ is developed in Section 5.

**Definition 4.9** The $\mathcal{P}$-invariantization $\zeta$ of a monic univariate polynomial $\alpha \in \mathbb{K}[z]_P[\zeta]$ is the squarefree part of $\prod_{i=1}^d \alpha(\xi^{(i)}, \zeta)$, where $\xi^{(1)}, \ldots, \xi^{(d)}$ are the $d$ distinct roots of the zero dimensional prime ideal $I^G$ of $\mathbb{K}(z)^G[Z]$. By a transcription of the primitive element theorem, see for instance [34, Proposition 4.2.2-3], they are thus the images by a polynomial map $\psi : \theta \mapsto (\psi_1(\theta), \ldots, \psi_n(\theta))$ over $\mathbb{K}(z)^G$ of the roots $\theta^{(1)}, \ldots, \theta^{(d)} \in \mathbb{K}(z)^G$ of an irreducible univariate polynomial of degree $d$ with coefficients in $\mathbb{K}(z)^G$. The coefficients of the polynomial

$$\prod_{i=1}^d \alpha(\xi^{(i)}, \zeta) = \prod_{i=1}^d \alpha(\psi(\theta^{(i)}), \zeta)$$

are elements of the field extension $\mathbb{K}(z)^G(\theta^{(1)}, \ldots, \theta^{(d)})$ of $\mathbb{K}(z)^G$ that are invariant under all permutations of the $\theta^{(i)}$. By [34] Section 8.1 or [12] Theorem 8.15, that polynomial belongs to $\mathbb{K}(z)^G[\zeta]$ and thus so does its squarefree part $\zeta \alpha$. [34] Section 8.1.

For a Galois theory oriented reader the details are given below. By definition $\zeta \alpha$ belongs to the extension $\mathbb{K}(\xi^{(1)}, \ldots, \xi^{(d)})$, which we denote by $\mathbb{K}_\xi$. Due to Theorem 4.3, $\mathbb{K}_\xi = \mathbb{K}(\xi^{(1)}, \ldots, \xi^{(d)})$. In order to prove that $\alpha \in \mathbb{K}(z)^G[\zeta]$ we will show that this polynomial is preserved by the Galois group of the extension $\mathbb{K}_\xi \supset \mathbb{K}(z)^G$. We need the following proposition.

**Proposition 4.10** Let $\{\xi^{(1)}, \ldots, \xi^{(d)}\}$ be the set of replacement invariants corresponding to a cross-section $\mathcal{P}$ of degree $d$. Then the field $\mathbb{K}_\xi = \mathbb{K}(\xi^{(1)}, \ldots, \xi^{(d)})$ is a splitting field of a univariate polynomial $\beta(z, \zeta) \in \mathbb{K}(z)^G[\zeta]$ of degree $d$. The Galois group of the extension $\mathbb{K}_\xi \supset \mathbb{K}(z)^G$ permutes the n-tuples $\xi^{(1)}, \ldots, \xi^{(d)}$.

**Proof:** Due to the replacement Theorem 4.3 one has the equality $\mathbb{K}(\xi^{(1)}) = \mathbb{K}(z)^G(\xi^{(1)})$. From Corollary 4.4 it follows that $\mathbb{K}(z)^G(\xi^{(1)})$ is an extension of degree $d$ of $\mathbb{K}(z)^G$ for $i = 1, \ldots, d$. Since $\mathbb{K}$ assumed to be of characteristic zero, the components $\xi^{(1)}_1, \ldots, \xi^{(1)}_n$ of n-tuple $\xi^{(1)}$ are separable over $\mathbb{K}(z)^G$. Hence there exists a primitive element $\theta_1 \in \mathbb{K}(\xi^{(1)})$, such that $\mathbb{K}(\xi^{(1)}) = \mathbb{K}(z)^G(\xi^{(1)}) = \mathbb{K}(z)^G(\theta_1)$, where $\theta_1$ is a root of an irreducible univariate polynomial $\beta(z, \zeta) \in \mathbb{K}(z)^G[\zeta]$ of degree $d$. [5] Theorem 5.4.1.

Let $\sigma_j : \mathbb{K}(\xi^{(i)}) \to \mathbb{K}(\xi^{(j)})$ be the $\mathbb{K}(z)^G$-isomorphism induced by exchanging $\xi^{(i)}$ and $\xi^{(j)}$. Then $\theta_j = \sigma_j(\theta_1)$ is a primitive element of the extension $\mathbb{K}(\xi^{(j)})$. 

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In addition, we proved that the following proposition that is used in Section 5.
Indeed, since the field \( \beta \) defines \( \theta \) show that the elements of the Galois group, since it has the same roots in \( K(z)^G \), and so \( K_\xi = K(z)^G(\theta_1, \ldots, \theta_d) \).

In addition, we proved that \( n \)-tuples \( \xi^{(1)}, \ldots, \xi^{(d)} \) are images of \( \theta_1, \ldots, \theta_d \) under the polynomial map \( \psi = (\psi_1, \ldots, \psi_n) : K(z)^G \rightarrow \left[ K(z)^G \right]^n \), where the coefficients of the univariate polynomials \( \psi_1, \ldots, \psi_n \) are in \( K(z)^G \). Since \( \xi^{(1)}, \ldots, \xi^{(d)} \) are distinct tuples, then \( \theta_1, \ldots, \theta_d \) are distinct elements of \( K(z)^G \). We will now show that \( \theta_1, \ldots, \theta_d \) are roots of the minimal polynomial \( \beta \in K(z)^G[\zeta] \) that defines \( \theta_1 \).

Indeed, since the field \( K(z)^G \) is fixed under \( \sigma_j \), for \( j = 1.d \), then so is the polynomial \( \beta \). Thus \( \theta_j = \sigma_j(\theta_1) \) are roots of the polynomial \( \beta \). It follows that \( K_\xi = K(z)^G(\theta_1, \ldots, \theta_d) \) is the splitting field of an irreducible univariate polynomial \( \beta \in K(z)^G[\zeta] \) of degree \( d \).

The elements of the \( \text{Gal}(K_\xi/K(z)^G) \) permute the roots \( \theta_1, \ldots, \theta_d \) of the polynomial \( \beta \), and therefore it permutes the tuples \( \xi^{(j)} = \psi(\theta_j) \) for all \( j = 1.d \).

**Corollary 4.11** Let \( \alpha(z, \zeta) \in K[z]_P \) be a univariate polynomial over \( K[z]_P \). Then its \( P \)-invariantization \( \iota \alpha \) is a polynomial over \( K(z)^G \).

**Proof:** The Galois group of the extension \( K_\xi \supset K(z)^G \) induces permutations of the \( n \)-tuples \( \xi^{(1)}, \ldots, \xi^{(d)} \). Thus the polynomial \( p(\zeta) = \prod_{i=1}^d \alpha(\xi^{(i)}, \zeta) \in K_\xi[\zeta] \) is fixed under \( \text{Gal}(K_\xi/K(z)^G) \). Hence its coefficients belong to \( K(z)^G \). By definition \( \alpha \) is the square-free part of \( p(\zeta) \), and hence it is also fixed under the Galois group, since it has the same roots in \( K_\xi \) as \( p(\zeta) \) itself [5, Proposition 5.3.8], and the Galois group permutes these roots. Thus its coefficients of \( \iota \alpha \) are in \( K(z)^G \). □

The following properties follow directly from the definition of the map \( \iota \):

1. A \( K(z)^G \)-zero of \( \iota \beta \) is a \( K(z)^G \)-zero of a \( \beta(\xi^{(i)}, \zeta) \) and conversely.
2. If \( \beta \in K(z)^G[\zeta] \) then \( \iota \beta = \beta \) since \( \beta(\xi^{(i)}, \zeta) = \beta(z, \zeta) \) by Theorem 4.3
3. If \( \alpha \equiv \beta \mod P_z \) then \( \iota \alpha = \iota \beta \) since the elements of \( P_z \) vanish on all \( \xi^{(i)} \).

The last property shows that \( \iota \) induces a map \( \phi \) from the set of monic polynomials of \( K(\mathcal{P})[\zeta] \) to the set monic polynomials of \( K(z)^G[\zeta] \) s.t. \( \iota = \phi \circ \pi \).

From the first property it follows that \( \beta(\xi^{(i)}, \zeta) \) divides \( \iota \beta(z, \zeta) \) in \( K(\xi^{(i)})[\zeta] \supset K(z)^G[\zeta] \) when \( \beta(\xi^{(i)}, \zeta) \) is squarefree. Since \( K(\mathcal{P}) \cong K(\xi^{(i)}) \) this amounts to the following proposition that is used in Section 5.
Proposition 4.12 Let $\beta$ be a monic polynomial of $\mathbb{K}[z]p[\zeta]$. If $\beta$ is squarefree when considered in $\mathbb{K}(P)[\zeta]$ then it divides $\iota\beta(z, \zeta)$ in $\mathbb{K}(P)[\zeta]$, that is there exists $g(z, \zeta) \in \mathbb{K}[z]p[\zeta]$ s.t. $\iota\beta(z, \zeta) \equiv q(z, \zeta)\beta(z, \zeta) \mod P_z$.

Also we recognize in the definition of the invariantization map the norm of a polynomial in an algebraic extension [12], Section 8.8. We reformulate the results extending those of that text namely:

- $\iota\beta$ can be computed by algebraic elimination.

- if $\beta(\xi^{(i)}, \zeta)$ is the minimal polynomial over $\mathbb{K}(\xi^{(i)}) \subset \mathbb{K}(z)^G$ of an element in $\mathbb{K}(z)^G$, then $\iota\beta$ is the minimal polynomial of this element over $\mathbb{K}(z)^G$.

The algebraic elimination to compute $\iota\beta$ can be performed by several techniques. For a strict generalization of [12, Section 8.8] one could introduce a resultant formula, as developed in [7]. We propose here a formulation in terms of elimination ideals.

Proposition 4.13 Let $\beta \in \mathbb{K}[z]p[\zeta]$ be a monic polynomial. Then its $P$-invariantization $\iota\beta$ is the squarefree part of the monic generator of $(I^G + \alpha(Z, \zeta)) \cap \mathbb{K}(z)^G[\zeta]$ where $\alpha(z, \zeta) \in \mathbb{K}[z][\zeta]$ is the numerator of $\beta$.

Proof: The leading coefficient of $\alpha(Z, \zeta) \in \mathbb{K}[Z][\zeta]$ does not belong to $P_z$, and therefore it does not belong to $I^G$. It follows that $(I^G + \alpha(Z, \zeta)) \cap \mathbb{K}(z)^G[\zeta] \neq (0)$ since $I^G$ is zero-dimensional.

Let $\gamma(z, \zeta)$ be the monic generator of $(I^G + \alpha(Z, \zeta)) \cap \mathbb{K}(z)^G[\zeta]$. We first prove that $\iota\beta$ divides the squarefree part of $\gamma(z, \zeta)$. The fact that $\gamma(z, \zeta)$ belongs to $I^G + \alpha(Z, \zeta)$ can be written as $\gamma(z, \zeta) \equiv q(z, Z, \zeta)\alpha(Z, \zeta) \mod I^G$ where $q(z, Z, \zeta) \in \mathbb{K}(G)[Z, \zeta]$. Substituting $\xi^{(i)}$ for $Z$ we have $\gamma(z, \zeta) = q'(z, \xi^{(i)}, \zeta)\beta(\xi^{(i)}, \zeta)$ where $q(z, \xi^{(i)}, \zeta)$ and $q'(z, \xi^{(i)}, \zeta)$ differ by the factor in $\mathbb{K}[\xi^{(i)}]$ that distinguishes $\alpha(\xi^{(i)}, \zeta)$ from $\beta(\xi^{(i)}, \zeta)$. Therefore all the factors $\beta(\xi^{(i)}, \zeta)$ of $\iota\beta$ divide $\gamma(z, \zeta)$. Since $\iota\beta$ is the squarefree product of $\beta(\xi^{(i)}, \zeta)$ it divides the squarefree part of $\gamma(z, \zeta)$.

Conversely, we prove that the squarefree part of $\gamma(z, \zeta)$ divides $\iota\beta$. The $\mathbb{K}(z)^G$-zeros of $\alpha(Z, \zeta) + I^G$ are the $(n+1)$-tuples $(\xi^{(i)}, f_{i,j})$, where $f_{i,j}, 1 \leq j \leq \deg \beta$, are the roots of $\beta(\xi^{(i)}, \zeta)$. Since $\gamma(z, \zeta)$ belongs to $\alpha(Z, \zeta) + I^G$ its set of $\mathbb{K}(z)^G$-roots includes all the $f_{i,j}$. Thus $\gamma$ and $\iota\beta$ have the same set of roots. Therefore the squarefree part of $\gamma$ divides $\iota\beta$. \square

Note that the monic generator of $(I^G + \alpha(Z, \zeta)) \cap \mathbb{K}(z)^G[\zeta]$ is the monic generator of $(I^G + \alpha(Z, \zeta)) \cap \mathbb{K}(z)[\zeta]$. This latter is an element of the reduced Gröbner basis of $(\alpha(Z, \zeta) + I^G)$ w.r.t a term order that eliminates $Z$. It follows from Proposition 4.3 that it belongs to $\mathbb{K}(z)^G[\zeta]$. Therefore computations over $\mathbb{K}(z)$ lead to the correct result over $\mathbb{K}(z)^G$.
The last proposition provides the computable counterpart of the isomorphism $\mathbb{K}(P) \cong \mathbb{K}(z)^G$, elements of $\mathbb{K}(P)$ or $\mathbb{K}(z)^G$ being represented by irreducible monic polynomials over $\mathbb{K}(P)$ or $\mathbb{K}(z)^G$ respectively.

**Proposition 4.14** Let $\alpha$ be a monic polynomial of $\mathbb{K}[z]_P[\zeta]$. The polynomial $\alpha \in \mathbb{K}(z)^G[\zeta]$ is irreducible if and only if $\alpha$ is a power of an irreducible polynomial when considered in $\mathbb{K}(P)[\zeta]$.

**Proof:** Note that $\iota(\beta \gamma)$, for $\beta, \gamma \in \mathbb{K}[z]_P[\zeta]$, is the squarefree part of the product $\iota \beta \iota \gamma$. So if $\alpha$ considered in $\mathbb{K}(P)[\zeta]$ is the product of two relatively prime factors then $\alpha \iota$ cannot be irreducible.

We can replace $\alpha$ by its squarefree part when considered in $\mathbb{K}(P)[\zeta]$ without loss of generality and thus assume for the converse implication that $\alpha(z, \zeta)$ is irreducible there. Let $\bar{\alpha} \in \mathbb{K}[z][\zeta]$ be obtained from $\alpha$ by cleaning up the denominators. Then $\bar{\alpha}(Z, \zeta)$ is irreducible modulo $I^G$ so that $(\bar{\alpha}(Z, \zeta) + I^G)$ is prime. The monic generator $\alpha \iota$ of $(\alpha(Z, \zeta) + I^G) \cap \mathbb{K}(z)[\zeta]$ is thus irreducible. \(\Box\)

The following example illustrates various properties of the $P$-invariantization map $\iota$.

**Example 4.15** Scaling. We consider the scaling action defined in Example 2.8 and the cross-section defined by the ideal $P_Z = (Z_1^2 + Z_2^2 - 1)$. It is a cross-section of degree 2. We have $I^c = (Z_1^2 - \frac{z_1^2}{z_1^2 + z_2^2}, Z_2 - \frac{z_2}{z_1}Z_1)$ and therefore the two replacement invariants are

$$\xi^{(\pm)} = \left( \frac{\pm z_1}{\sqrt{z_1^2 + z_2^2}}, \frac{\pm z_2}{\sqrt{z_1^2 + z_2^2}} \right).$$

The invariance of $\alpha = \zeta - z_1$ is $\iota \alpha = \zeta^2 - \frac{z_1^2}{z_1^2 + z_2^2}$. We have $\iota \alpha = (\zeta + z_1) \alpha + \frac{z_1^2}{z_1^2 + z_2^2}(z_1^2 + z_2^2 - 1) \equiv (\zeta + z_1) \alpha \mod P_Z$. We obtained $\iota \alpha$ by computing the reduced Gröbner basis of the ideal $(\zeta - Z_1, Z_1^2 - \frac{z_1^2}{z_1^2 + z_2^2}, Z_2 - \frac{z_2}{z_1}Z_1)$ with a term order that eliminates $Z_1$ and $Z_2$. Note that, although $\alpha$ defines a polynomial function, its invariantization defines two algebraic invariants $\pm \frac{z_1}{\sqrt{z_1^2 + z_2^2}}$.

The invariance of $\beta = \zeta^3 + \zeta^2 + 22 \zeta + 1$ is $\iota \beta = \zeta^6 + 2 \zeta^5 + \zeta^4 + 2 \zeta^3 + \frac{z_1^2 + 2z_2^2}{z_1^2 + z_2^2} \zeta^2 + 1$. We have $\iota \beta \equiv (\zeta^3 + \zeta^2 - 22 \zeta + 1) \beta \mod P_Z$.

In the next two instances the monic polynomial is equal modulo $P_Z$ to a polynomial in $\mathbb{K}(z)^G[\zeta]$. As a consequence, the invariantization equals to the original polynomial modulo $P_Z$.

The polynomial $\gamma = \zeta - \frac{z_1^2}{z_1^2 + z_2^2}$ is equal to its $P$-invariantization $\iota \gamma = \zeta - \frac{z_1^2}{z_1^2 + z_2^2} \equiv \gamma \mod P_Z$. 

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The irreducible polynomial \( \delta = \zeta^2 - \frac{2}{\zeta^2} \) becomes a reducible modulo \( P_{\zeta} \): \( \delta \equiv \zeta^2 - \frac{2}{\zeta^2} \mod P_{\zeta} \). Its invariantization is thus reducible: \( \iota \delta = (\zeta - \frac{2}{\zeta})(\zeta + \frac{2}{\zeta}) \equiv \delta \mod P_{\zeta} \).

5 Local invariants and the moving frame construction

In this section we connect the algebraic algorithms presented in this paper with their original source of inspiration, the Fels-Olver moving frame construction [11]. It is shown in [11] that in the case of a locally free smooth action of a Lie group \( G \) on a manifold \( Z \), a choice of local cross-section corresponds to a local \( G \)-equivariant map \( \rho: Z \to G \). This map provides a generalization of the classical geometrical moving frames\(^6\) [4]. A moving frame map gives rise to an invariantization process, a projection from the set of smooth functions to the set of local invariants.

We introduce an alternative definition of the smooth invariantization process which, on one hand, generalizes the definition given in [11] to non-free, semi-regular actions and, on the other hand, can be effectively reformulated in the algebraic context. We make explicit comparisons with both the moving frame and the algebraic constructions in Section 5.6 and Section 5.5 respectively.

In this section we consider real smooth manifolds. All statements and constructions from this section are applicable to complex manifolds. In the latter case all maps and functions are assumed to be meromorphic.

5.1 Local action of a Lie group on a smooth manifold

We consider a Lie group \( G \), with identity denoted \( e \), and a smooth manifold \( Z \) of dimension \( n \). We first review the necessary facts and terminology from the theory of Lie group actions on smooth manifolds. Our presentations is based on [15, 26].

**Definition 5.1** A local action of a Lie group \( G \) on a smooth manifold \( Z \) is a smooth map \( g: \Omega \to Z \), where \( \Omega \supset \{ e \} \times Z \) is an open subset of \( G \times Z \), and the map \( g \) satisfies the following two properties:

1. \( g(e, \bar{z}) = \bar{z}, \forall \bar{z} \in Z \).
2. \( g(\tilde{\mu}, g(\bar{\lambda}, z)) = g(\tilde{\mu} \cdot \bar{\lambda}, z) \), for all \( \bar{z} \in Z \) and \( \bar{\lambda}, \tilde{\mu} \in G \) s. t. \( (\bar{\lambda}, \bar{z}) \) and \( (\tilde{\mu} \cdot \bar{\lambda}, \bar{z}) \) are in \( \Omega \).

\(^6\)For this reason the map \( \rho \) is called moving frame in [11]. We adopt the term a moving frame map.
The orbit of $\bar{z} \in Z$ is the image $O_{\bar{z}}$ of the smooth map $g_{\bar{z}} : G \to Z$ defined by $g_{\bar{z}}(\bar{\lambda}) = g(\bar{\lambda}, \bar{z})$. The domain of $g_{\bar{z}}$ is an open subset of $\mathcal{G}$ containing $e$.

For every point $\bar{z} \in Z$ the differential $dg_{\bar{z}} : T_G|_e \to T_Z|_{\bar{z}}$ maps the tangent space of $\mathcal{G}$ at $e$ to the tangent space of $Z$ at the point $\bar{z}$. The tangent space $T_G|_e$ can be identified with the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Let $\hat{v} \in \mathfrak{g}$ then $v(\bar{z}) = dg_{\bar{z}}(\hat{v})$ is a smooth vector field on $Z$, called the infinitesimal generator of the $\mathcal{G}$-action corresponding to $\hat{v}$. The set of all infinitesimal generators for a $\mathcal{G}$-action form a Lie algebra, such that the map $\hat{v} \to v$ is a Lie algebra homomorphism.

By $\exp(\epsilon v, \bar{z}) : \mathbb{R} \times Z \to Z$ we denote the flow of $v$. The flow is defined as an integral curve of the vector field $v$ with the initial condition $\bar{z}$. One can prove that every point of the connected component of the orbit $O_{\bar{z}}^\mathcal{G}$ of $\bar{z}$ can be reached from $\bar{z}$ by a composition of flows of a finite number of infinitesimal generators.

Let $\hat{v}_1, \ldots, \hat{v}_\kappa$, where $\kappa \geq s$ is the dimension of the group, be a basis of the Lie algebra of $\mathcal{G}$. Then the infinitesimal generators $v_1, \ldots, v_\kappa$ span the tangent space to the orbits at each point of $Z$.

**Definition 5.2** An action of a Lie group $\mathcal{G}$ on a smooth manifold $Z$ is semi-regular if all orbits have the same dimension.

Throughout this section the action is assumed to be semi-regular. The dimension of the orbits is denoted by $s$.

### 5.2 Local invariants

We give definitions of local invariants and fundamental sets of those. We prove that the existence of a fundamental set of local invariants follows from the existence of a flat coordinate system. The proof is based on standard arguments from differential geometry.

**Definition 5.3** A smooth function $f$, defined on an open subset $U \subset Z$, is a local invariant if $v(f) = 0$ for any infinitesimal generator $v$ of the $\mathcal{G}$-action on $U$.

Equivalently $f(\exp(\epsilon v, \bar{z})) = f(\bar{z})$ for all $\bar{z} \in U$, all infinitesimal generator $v$, and all real $\epsilon$ sufficiently close to zero. If the group $\mathcal{G}$ is connected, the function $f$ is continuous on $Z$, and the condition of Definition 5.3 is satisfied at every point of $Z$ then $f$ is a global invariant on $Z$ due to [26, Proposition 2.6]. In what follows we neither assume $f$ to be continuous outside of $U$, nor $\mathcal{G}$ to be connected.

A collection of smooth functions $f_1, \ldots, f_l$ are functionally dependent on a manifold $Z$ if for each point $\bar{z} \in U$ there exists an open neighborhood an $U$ and a non-zero differentiable function $F$ in $l$ variables such that $F(f_1, \ldots, f_l) = 0$ on $U$. From the implicit function theorem it follows that $f_1, \ldots, f_l$ are functionally
dependent on $\mathcal{U}$ if and only if the rank of the corresponding Jacobian matrix is less than $l$ at each point of $\mathcal{Z}$. We say that functions $f_1, \ldots, f_l$ are independent on $\mathcal{Z}$ if they are not dependent when restricted to any open subset of $\mathcal{Z}$. As it is commented in [26, p85] functional dependence and functional independence on $\mathcal{Z}$ do not exhaust the range of possibilities, except for analytic functions. Throughout the section the term independent functions means functionally independent functions. Finally we say that $f_1, \ldots, f_n$ are independent at a point $\bar{z} \in \mathcal{Z}$ if the rank of the corresponding Jacobian matrix is maximal at $\bar{z}$. Independence at $\bar{z}$ implies independence on some open neighborhood of this point. If $\mathcal{U}$ is an open subset of $\mathcal{Z}$ and $f_1, \ldots, f_n$ are independent at each point of $\mathcal{Z}$, then these functions provide a coordinate system on $\mathcal{U}$.

**Definition 5.4** A collection of local invariants on $\mathcal{U}$ forms a fundamental set if they are functionally independent, and any local invariant on $\mathcal{U}$ can be expressed as a smooth function of the invariants from this set.

The Lie algebra of infinitesimal generators provides an integrable distribution\(^7\) of smooth vector-fields on $\mathcal{Z}$, whose integral manifolds are orbits. For a semi-regular action this distribution is of constant rank $s$, the dimension of the orbits. It follows from Frobenius theorem that in an open neighborhood $\mathcal{U}$ of each point there exists a coordinate system $x_1, \ldots, x_s, y_1, \ldots, y_{n-s}$ such that the connected components of the orbits on $\mathcal{U}$ are level sets of the last $n-s$ coordinates [31, p. 262] and [26, Theorem 1.43]. Such coordinate system is called flat, or straightening. The proof of the following theorem establishes that $y_1, \ldots, y_{n-s}$ form a fundamental set of local invariants.

**Theorem 5.5** Let $\mathcal{G}$ be a Lie group acting semi-regularly on an $n$-dimensional manifold $\mathcal{Z}$. Let $s$ be the dimension of the orbits. In the neighborhood of each point $\bar{z} \in \mathcal{Z}$ there exists a fundamental set of $n-s$ local invariants.

**Proof:** By Frobenius theorem there exists a flat coordinate system $x_1, \ldots, x_s, y_1, \ldots, y_{n-s}$ in a neighborhood $\mathcal{U} \ni \bar{z}$. The connected components of the orbits on $\mathcal{U}$ coincide with the level sets of the last $n-s$ coordinate functions. Thus $y_1, \ldots, y_{n-s}$ are constant on the connected components of the orbits, and therefore they are local invariants, being smooth and functionally independent by definition of a coordinate system. It remains to show that any other invariant is locally expressible in terms of them. Let $v$ be an infinitesimal generator of the group action. Since $v(y_i) = 0$ for $i = 1..(n-s)$ then $v = \sum_{i=1}^{n-s} v(x_i) \frac{\partial}{\partial x_i}$ is a linear combination of the first $s$ basis vector fields. Let $v_1 = \sum_{i=1}^{s} a_1 \frac{\partial}{\partial x_i}, \ldots, v_s = \sum_{i=1}^{s} a_s \frac{\partial}{\partial x_i}$ be a basis of infinitesimal generators of the group action. Without loss of generality we may assume that the first $s$ generators $v_1, \ldots, v_s$ are linearly independent at each point of $\mathcal{U}$. Let $f(x_1, \ldots, x_s, y_1, \ldots, y_{n-s})$ be a local invariant, then $v_j(f) = \sum_{i=1}^{s} a_j \frac{\partial f}{\partial x_i} = 0$ for

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\(^7\)An integrable distribution is a collection of smooth vector fields, whose span over the ring of smooth functions is closed with respect to Lie bracket.
This is a homogeneous system of \( s \) linear equation with \( s \) unknowns \( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_s} \). Since \( v_1, \ldots, v_s \) are linearly independent at each point, the rank of the system is maximal. Thus \( \left( \frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_s} = 0 \right) \) is the only solution. Hence \( f \) is a function of invariants \( y_1, \ldots, y_{n-s} \). □

The existence of a fundamental set of local invariants, therefore, follows from the existence of a flat coordinate system. The proof is not constructive however. The invariantization process, introduced in next section, leads to a different characterization of a fundamental set of invariants. Invariantization, and therefore fundamental invariants, can be effectively computed either by the algorithms of Section 4 in the case of a rational action of an algebraic group (see Section 5.5), or by the moving frame method of [11], in the case of a locally free action of a Lie group (see Section 5.6).

### 5.3 Local cross-section and smooth invariantization

We define local cross-sections to the orbits and show that a local cross-section passing through any given point can easily be constructed. A local cross-section gives rise to an equivalence relationship on the ring of smooth functions such that any class has a single representative that is a local invariant. This leads to an invariantization map, a projection from the ring of smooth functions to the ring of local invariants. It generalizes the invariantization process defined in [11] to semi-regular actions. Although a possibility of such generalization is indicated in the remarks of [11, Section 4], the precise definitions and theorems, appearing in this section, are new.

**Definition 5.6** An embedded submanifold \( \mathcal{P} \) of \( \mathcal{Z} \) is a local cross-section to the orbits if there is an open set \( \mathcal{U} \) of \( \mathcal{Z} \) such that

- \( \mathcal{P} \) intersects \( \mathcal{O}_0^\circ \cap \mathcal{U} \) at a unique point \( \forall \vec{z} \in \mathcal{U} \), where \( \mathcal{O}_0^\circ \) is the connected component of \( \mathcal{O}_0 \cap \mathcal{U} \), containing \( \vec{z} \).

- for all \( \vec{z} \in \mathcal{P} \cap \mathcal{U} \), \( \mathcal{O}_0^\circ \) and \( \mathcal{P} \) are transversal and of complementary dimensions.

The second condition in the above definition is equivalent to the following condition on tangent spaces: \( T_{\vec{z}} \mathcal{Z} = T_{\vec{z}} \mathcal{P} \oplus T_{\vec{z}} \mathcal{O}_0, \forall \vec{z} \in \mathcal{P} \cap \mathcal{U} \).

An embedded submanifold of codimension \( s \) is locally given as the zero set of \( s \) independent functions. Assume that \( h_1(z), \ldots, h_s(z) \) define \( \mathcal{P} \) on \( \mathcal{U} \). The tangent space at a point of \( \mathcal{P} \) is the kernel of the Jacobian matrix \( J_h \) at this point. A basis of infinitesimal generators \( v_1, \ldots, v_\kappa \), where \( \kappa \geq s \) is the dimension of the group, span the tangent space to the orbits at each point of \( \mathcal{P} \). Therefore the submanifold \( \mathcal{P} \) is a local cross-section if and only if the span of the infinitesimal generators \( v_1, \ldots, v_\kappa \) has a trivial intersection with the kernel of \( J_h \) on \( \mathcal{P} \).
Equivalently:

the rank of the \( s \times \kappa \) matrix \( (v_j(h_i))_{i=1..s}^{j=1..\kappa} = J_h \cdot V \) equals to \( s \) on \( P \), \( \text{(3)} \)
where \( V \) is the \( n \times \kappa \) matrix, whose \( i \)-th column consists of the coefficients of the infinitesimal generator \( v_i \) in a local coordinate system. In the next theorem we prove the existence of a local cross-section through every point. The first paragraph of the proof provides a simple practical algorithm to construct a coordinate local cross-section through a point. An algebraic counterpart of this statements is given by Theorem 3.

**THEOREM 5.7** Let \( G \) act semi-regularly on \( \mathcal{Z} \). Through every point \( \bar{z} \in \mathcal{Z} \) there is a local cross-section that is defined as the level set of \( s \) coordinate functions.

**PROOF:** Let \( V \) be the \( n \times \kappa \) matrix of the coefficients of the infinitesimal generators \( v_1, \ldots, v_\kappa \) relative to a coordinate system \( z_1, \ldots, z_\kappa \). The rank of \( V \) equals to the dimension of the orbits \( s \). Thus there exist \( s \) rows of \( V \) that form an \( s \times \kappa \) submatrix \( \hat{V} \) of rank \( s \) at the point \( \bar{z} \), and therefore it has rank \( s \) on an open neighborhood \( U_1 \ni \bar{z} \). Assume that these rows correspond to coordinate \( z_{1i}, \ldots, z_{si} \). Let \( (c_1, \ldots, c_\kappa) \) be coordinates of the point \( \bar{z} \), then functions \( h_1 = z_{1i} - c_{1i}, \ldots, h_s = z_{si} - c_{si} \) satisfy condition \( \text{(3)} \). The common zero set \( \mathcal{P} \) of these functions contains \( \bar{z} \).

It remains to prove that there exists a neighborhood \( U \ni \bar{z} \) such that \( \mathcal{P} \) intersects each connected component of the orbits on \( U \) at a unique point. Let \( x_1, \ldots, x_s, y_1, \ldots, y_{n-s} \) be a flat coordinate system in an open neighborhood \( U_2 \ni \bar{z} \). Due to Theorem 1.35 \( y_1, \ldots, y_{n-s} \) are independent local invariants. We will show that functions \( z_1, \ldots, z_s, y_1, \ldots, y_{n-s} \) provide a coordinate system an open set \( U = U_1 \cap U_2 \) containing \( \bar{z} \). Without loss of generality we may assume that \( \{z_1, \ldots, z_s\} = \{1, \ldots, s\} \) are the first \( s \) coordinates. In terms of flat coordinates \( z_i = F_i(x, y) \), \( i = 1..s \), where \( F_i \) are smooth functions on \( U_2 \). Since \( v_i(y_j) = 0 \) for \( i = 1..\kappa, j = 1..n-s \), then

\[
(v_j(z_i))_{i=1..s}^{j=1..\kappa} = (J_{x_i}^F)_{r=1..s}^{j=1..\kappa} \cdot (v_j(x_r))_{r=1..s}^{j=1..\kappa}.
\]

We note that \( (v_j(z_i))_{i=1..s}^{j=1..\kappa} = \hat{V} \) is \( s \times \kappa \) matrix of rank \( s \) at each point of \( U \). Matrix \( (v_j(x_r))_{r=1..s}^{j=1..\kappa} \) also has maximal rank \( s \) on \( U \). Therefore the matrix \( (J_{x_i}^F)_{r=1..s}^{j=1..\kappa} \) is invertible on \( U \). By looking at the rank of the corresponding Jacobian matrix in flat coordinates, we conclude that functions \( z_1, \ldots, z_s, y_1, \ldots, y_{n-s} \) are independent at each point of \( U \), and therefore define a coordinate system on \( U \).

By construction all points on \( \mathcal{P} \) have the same \( z \)-coordinates. Thus two distinct points of \( \mathcal{P} \) must differ by at least one of the \( y \)-coordinates. Since \( y \) coordinates are constant on the connected components of the orbits on \( U \), distinct points of \( \mathcal{P} \) belong to distinct connected components of the orbits. \( \square \)
Given a cross-section on $U$ one can define a projection from the set of smooth functions on $U$ to the set of local invariants.

**Definition 5.8** Let $P$ be a local cross-section to the orbits on an open set $U$. Let $f$ be a smooth function on $U$. The invarianization $\bar{f}$ of $f$ is the function on $U$ that is defined, for $\bar{z} \in U$, by $\bar{f}(\bar{z}) = f(\bar{z}_0)$, where $\bar{z}_0 = O^f \cap P$.

In other words, the invarianization of a function $f$ is obtained by spreading the values of $f$ on $P$ along the orbits. The next theorem shows that $\bar{f}$ is the unique local invariant with the same values on $P$ as $f$.

**Theorem 5.9** Let a Lie group $G$ act semi-regularly on a manifold $Z$, and let $P$ be a local cross-section. Then $\bar{f}$ is the unique local invariant defined on $U$ whose restriction to $P$ is equal to the restriction of $f$ to $P$. In other words $\bar{f}|_P = f|_P$.

**Proof:** For any $\bar{z} \in U$ and small enough $\varepsilon$ the point $\exp(\varepsilon v, \bar{z})$ belongs to the same connected component $O^f_\varepsilon$. Let $\bar{z}_0 = O^f_\varepsilon \cap P$. Then $\bar{f}(\exp(\varepsilon v, \bar{z})) = f(\bar{z}_0) = \bar{f}(\bar{z})$, and thus $\bar{f}$ is a local invariant. By definition $\bar{f}(\bar{z}_0) = f(\bar{z}_0)$ for all $\bar{z}_0 \in P$.

In order to show its smoothness we write $\bar{f}$ in terms of flat coordinates $x_1, \ldots, x_s, y_1, \ldots, y_{n-s}$. By probably shrinking $U$, we may assume that $P$ is given by the zero-set of smooth independent functions $h_1(x_1, \ldots, x_s, y_1, \ldots, y_{n-s}), \ldots, h_s(x_1, \ldots, x_s, y_1, \ldots, y_{n-s})$. From the transversality condition and local invariance of $y$'s, it follows that the first $s$ columns of the Jacobian matrix $J_h$ form a submatrix of rank $s$. Thus the cross-section $P$ can be described as a graph $x_1 = p_1(y_1, \ldots, y_{n-s}), \ldots, x_s = p_s(y_1, \ldots, y_{n-s})$, where $p_1, \ldots, p_s$ are smooth functions. Then the function

$$\bar{f}(x_1, \ldots, x_s, y_1, \ldots, y_{n-s}) = f(p_1(y_1, \ldots, y_{n-s}), \ldots, p_s(y_1, \ldots, y_{n-s}), y_1, \ldots, y_{n-s})$$

is smooth, as a composition of smooth functions.

To prove the uniqueness, assume that an invariant function $q$ has the same values on $P$ as $f$, then the invariant function $h = \bar{f} - q$ has zero value on $P$. A point $\bar{z} \in U$ can be reached from $\bar{z}_0 = P \cap O^f_\varepsilon$ by a composition of flows defined by infinitesimal generators. Without loss of generality, we may assume that it can be reached by a single flow $\bar{z} = \exp(\varepsilon v, \bar{z}_0)$, where $\exp(\varepsilon v, \bar{z}_0) \subset O^f_\varepsilon$ for all $0 \leq \varepsilon \leq \varepsilon$. From the invariance of $h$ it follows that $h(\exp(\varepsilon v, \bar{z}_0)) = h(\bar{z}_0) = 0$. Thus $q(z) = \bar{f}(z)$ on $U$. $\Box$

Theorem 5.9 allows us to view the invarianization process as a projection from the set of smooth functions on $U$ to the equivalence classes of functions with the same value on $P$. Each equivalence class contains a unique local invariant. The algebraic counterpart of this point of view is described in Section 4.2.

The invarianization of differential forms can be defined in a similar implicit manner. It has been shown in [11, 21] that the essential information about
the differential ring of invariants and the structure of differential forms can be computed from the infinitesimal generators of the action and the equations that define the cross-section, without explicit formulas for invariants.

5.4 Normalized and fundamental invariants

The normalized invariants introduced in \[11\] are the invariantizations of the coordinate functions. They have the replacement property. In the algebraic context they correspond to replacement invariants defined in Section \[11\]. This correspondence is made precise by Proposition 5.14. We show that a set of normalized invariants contains a fundamental set of local invariants.

All results of this subsection are stated under the following assumptions. A manifold \( \mathcal{P} \) is a local cross-section to the \( s \)-dimensional orbits of a semi-regular \( G \)-action on an open \( U \subset Z \), and \( \iota \) is the corresponding invariantization map. The set \( U \) is a single coordinate chart on \( Z \) with coordinate functions \( z_1, \ldots, z_n \). By possibly shrinking \( U \) we may assume that \( \mathcal{P} \) is the zero set of \( s \) independent smooth functions.

Since our definition of invariantization differs from \[11\] we restate and prove the replacement theorem.

**Theorem 5.10** If \( f(z_1, \ldots, z_n) \) is a local invariant on \( U \) then \( f(\iota z_1, \ldots, \iota z_n) = f(z_1, \ldots, z_n) \).

**Proof:** Since \( \iota z_1|_\mathcal{P} = z_1|_\mathcal{P}, \ldots, \iota z_n|_\mathcal{P} = z_n|_\mathcal{P} \), then \( f(\iota z_1, \ldots, \iota z_n)|_\mathcal{P} = f(z_1, \ldots, z_n)|_\mathcal{P} \). Thus functions \( f(\iota z_1, \ldots, \iota z_n) \) and \( f(z_1, \ldots, z_n) \) are both local invariants and have the same value on \( \mathcal{P} \). By Theorem 5.9 they coincide.

**Lemma 5.11** Let \( \mathcal{P} \) be a local cross-section on \( U \), given as the zero set of \( s \) independent functions \( h_1, \ldots, h_s \). Then \( h_1(\iota z_1, \ldots, \iota z_n) = 0, \ldots, h_s(\iota z_1, \ldots, \iota z_n) = 0 \) on \( U \). If for a differentiable \( n \)-variable function \( f \) we have \( f(\iota z_1, \ldots, \iota z_n) \equiv 0 \) on an open subset of \( U \), then there exists open \( W \subset U \) such that \( W \cap \mathcal{P} \neq \emptyset \) and at each point of \( W \cap \mathcal{P} \) functions \( f, h_1, \ldots, h_s \) are not independent.

**Proof:** Since \( h(\iota z)|_\mathcal{P} = \iota h(z)|_\mathcal{P} \) and both functions are invariants, one has \( h(\iota z) = \iota h(z) \) by Theorem 5.2. The latter is zero since \( h|_\mathcal{P} = 0 \). Assume now that there exists a differentiable function \( f \) and an open subset of \( V \subset U \) such that \( f(\iota z_1, \ldots, \iota z_n) \equiv 0 \) on \( V \). Since \( f(\iota z) = \iota f(z) \) is invariant, there exists an open \( W \supset V \) such that \( f(\iota z_1, \ldots, \iota z_n) \equiv 0 \) on \( W \) and \( W \cap \mathcal{P} \neq \emptyset \). We conclude that \( f(z_1, \ldots, z_n) \equiv 0 \) on \( \mathcal{P} \cap W \). In this case \( f \) cannot be independent of \( h_1, \ldots, h_s \) at any point of \( \mathcal{P} \cap W \) since otherwise this would imply that \( \mathcal{P} \) is of dimension less then \( n - s \).

**Theorem 5.12** Let \( \mathcal{P} \) be a local cross-section on \( U \), given as the zero set of \( s \) independent functions. The set \( \{\iota z_1, \ldots, \iota z_n\} \) of the invariantizations of the co-
ordinate functions \( z_1, \ldots, z_n \) contains a fundamental set of \( n-s \) local invariants on \( \mathcal{U} \).

**Proof:** Due to the implicit function theorem, after a possible shrinking \( \mathcal{U} \) and renumbering of the coordinate functions, we may assume that \( \mathcal{P} \) is the zero set of the functions \( h_1(z) = z_1 - p_1(z_{s+1}, \ldots, z_n), \ldots, h_s(z) = z_s - p_s(z_{s+1}, \ldots, z_n) \).

Therefore \( \bar{z}_1 = p_1(\bar{z}_{s+1}, \ldots, \bar{z}_n), \ldots, \bar{z}_s = p_s(\bar{z}_{s+1}, \ldots, \bar{z}_n) \) by Theorem 5.9. From Theorem 5.10 we can conclude that any local invariant can be written in terms of \( \bar{z}_{s+1}, \ldots, \bar{z}_n \). Since for every differentiable non-zero \( n - s \)-variable function \( f(z_{s+1}, \ldots, z_n), h_1(z), \ldots, h_s(z) \) are independent at every point of \( \mathcal{U} \), then by Lemma 5.11, \( \bar{z}_{s+1}, \ldots, \bar{z}_n \) are functionally independent on \( \mathcal{U} \). \( \square \)

### 5.5 Relation between the algebraic and the smooth constructions

We establish a connection between the smooth and the algebraic constructions. We show that the normalized invariants (Section 5.4) can be viewed as smooth representatives of the replacement invariants (Section 4.1), and that algebraic invariantization (Section 4.2) provides a constructive approach to smooth invariantization (Section 5.3).

To be at the intersection of the hypotheses of the smooth and the algebraic settings we consider a real algebraic group, that is the set of real points of an algebraic group defined over \( \mathbb{R} \). It is a real Lie group [32, the Proposition in Chapter 3, Section 2.1.2]. Lie groups appearing in applications often satisfy this property. We also assume that the local action is given by a rational map (1), in Section 2.1, that satisfies Assumption 2.2. This guarantees semi-regularity of the action on an open set \( \mathcal{Z} \) of \( \mathbb{R}^n \) as the orbits of non-maximal dimension are contained in an algebraic set defined by minors of the matrix \( V \) of (3), in Section 5.3.

In Section 2 to 4 we assumed for convenience of writing that the field of coefficients \( K \) was algebraically closed. Yet the algebraic constructions of those sections require no extension of the field of definition of the group or the action. With the initial data described above, Theorem 2.13 produces a set of rational invariants in \( \mathbb{R}(z)^G \) that generate \( \mathbb{R}(z)^G \) by Theorem 2.15.

Rational invariants are obviously local invariants. We show that so are smooth representatives of algebraic invariants. The following definition formalizes the notion of a smooth representative of an algebraic function.

**Definition 5.13** A smooth map \( F : \mathcal{U} \rightarrow \mathbb{R}^k \) is a smooth zero of \( \{p_1, \ldots, p_n\} \subset \mathbb{R}(z)[\zeta_1, \ldots, \zeta_k] \) if the coefficients of the \( p_i \) are well defined on \( \mathcal{U} \) and \( p_i(\bar{z}, F(\bar{z})) = 0 \).
0 for all $\bar{z} \in \mathcal{U}$. In this case we also say that $F$ is a smooth zero of the ideal $(p_1, \ldots, p_n)$.

Proposition 5.14 Assume $F : \mathcal{U} \to \mathbb{R}^k$ is a smooth zero of \{p_1, \ldots, p_n\} $\subset \mathbb{R}(z)^G[\zeta_1, \ldots, \zeta_k]$. If $(p_1, \ldots, p_n)$ is a zero dimensional ideal then the components of $F$ are local invariants.

Proof: Let $\bar{p} \in \mathbb{R}(z)^G[\zeta]$, that is $p(z, \zeta) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z)\zeta^\alpha$, where $a_\alpha(z) \in \mathbb{R}(z)^G$. Assume that $p(\bar{z}, F(\bar{z})) = 0$ for all $\bar{z} \in \mathcal{U}$. For any $\bar{z} \in \mathcal{U}$ and an infinitesimal generator $v$ there exits $\epsilon > 0$, such that $\exp(\epsilon v, \bar{z}) \in \mathcal{U}$ whenever $|\epsilon| < \epsilon$. Then $p(\exp(\epsilon v, \bar{z}), F(\exp(\epsilon v, \bar{z}))) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(\exp(\epsilon v, \bar{z})) F(\exp(\epsilon v, \bar{z}))^\alpha = 0$. Since the coefficients $a_\alpha$ are invariant $\sum_{\alpha \in \mathbb{N}^n} a_\alpha(\bar{z}) F(\exp(\epsilon v, \bar{z}))^\alpha = 0$ for all $\bar{z} \in \mathcal{U}$ and small enough $\epsilon$. Thus for a fixed point $\bar{z}$ all the values $F(\exp(\epsilon v, \bar{z}))$ for all sufficiently small $\epsilon$ are the common roots of the set of polynomials \{p_1, \ldots, p_n\}. Since by the assumption the number of roots is finite, we conclude that $F(\exp(\epsilon v, \bar{z})) = F(\exp(0v, \bar{z})) = F(\bar{z})$ and thus the components of $F(\bar{z})$ are local invariants. $\square$

It follows from Theorem 5.12 that, through every point of $\mathcal{Z}$, there exists a local cross-sections defined by linear equations over $\mathbb{R}$. Conversely, we can consider a cross-section $\mathcal{P}$, defined over $\mathbb{R}$, that has non singular real points, meaning that the real part has the same dimension as the complex part. For any point $\bar{z} \in \mathcal{Z} \cap \mathcal{P}$ where the rank of the matrix does not drop, there is a neighborhood $\mathcal{U}$ on which $\mathcal{P}$ defines a local cross-section, and such points are dense in $\mathcal{P}$.

The $\mathbb{R}(z)^G$-zero of the zero dimensional ideal $I^G = (G + P + (z - g(\lambda, z))) \cap \mathbb{R}(z)^G[\mathcal{Z}]$ are precisely the replacement invariants. According to the previous proposition the smooth zeros of this ideal are local invariants. We characterize the tuple of normalized invariants as one of them.

Theorem 5.15 Let $\mathcal{P}$ be an algebraic cross-section which, when restricted to an open set $\mathcal{U}$, defines a smooth cross-section. The tuple of normalized invariants $\bar{\mathcal{I}} z = (\bar{\mathcal{I}} z_1, \ldots, \bar{\mathcal{I}} z_n)$ is the smooth zero of the ideal $I^G$ whose components agree with the coordinate functions on $\mathcal{P} \cap \mathcal{U}$.

Proof: Let $\bar{z} \in \mathcal{U}$ be an arbitrary point, and let $\bar{z}_0$ be the point of intersection of $\mathcal{P}$ with the connected component of $O_\bar{z} \cap \mathcal{U}$, containing $\bar{z}$. Then there exists $\lambda$ in the connected component of the identity of $G$, such that $\bar{z}_0 = \lambda \bar{z}$ so that $(\bar{z}, \bar{z}_0)$ is a zero of the ideal $I = O + P$. By definition $\bar{\mathcal{I}} z(\bar{z}) = \bar{z}_0$ and therefore $(\bar{z}, \bar{\mathcal{I}} z(\bar{z}))$ is a zero of the ideal $I$ for all $\bar{z} \in \mathcal{U}$. Equivalently $\bar{\mathcal{I}} z$ is a smooth zero of $I^G$. By Theorem 5.14 it is the unique tuple of local invariants that agree with the coordinate functions on $\mathcal{P} \cap \mathcal{U}$. $\square$

Therefore a replacement invariant not only generates algebraic invariants but their smooth representatives also generate local invariants.
Example 5.16 Scaling. The action defined in Example 2.3 corresponds to the following action of the multiplicative group $\mathbb{R}^*$:

$$g : \mathbb{R}^* \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(\lambda, z_1, z_2) \mapsto (\lambda z_1, \lambda z_2).$$

The action is semi-regular on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

In Example 3.7 we chose the cross-section $\mathcal{P}$ defined by $z_1 = 1$. The cross-section being of degree 1 there is a single associated replacement invariant that corresponds to the tuple $(1, \frac{z_2}{z_1})$ of rational invariants.

Let $\mathcal{U} = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \neq 0\}$. The components of the smooth map $F : \mathcal{U} \rightarrow \mathbb{R}^2$ s.t. $F(z_1, z_2) = (1, \frac{z_2}{z_1})$ are the normalized invariants for the local cross-section $\mathcal{P} \cap \mathcal{U}$.

Example 5.17 Translation+Reflection. The action defined in Example 2.4 corresponds to the following action of the Lie group $\mathbb{R} \times \{1, 1\}$ given by

$$g : \mathbb{R} \times \{1, 1\} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(\lambda_1, \lambda_2, z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2).$$

The action is semi-regular on $\mathbb{R}^2$.

In Example 3.8 we chose the cross-section $\mathcal{P}$ defined by $z_2 = z_1$. There are two replacement invariants associated to $\mathcal{P}$: $\xi^{(\pm)} = (\pm z_2, \pm z_2)$. They both correspond to smooth maps $F^{(\pm)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the components of which are local invariants.

Only $(z_2, z_2)$ coincides with the coordinate functions on $\mathcal{P}$, that defines a local cross-section on $\mathcal{U} = \mathbb{R}^2$. The normalized invariants are thus $(z_2, z_2)$.

Example 5.18 Rotation. The action defined in Example 2.5 corresponds to the following action of the additive group $\mathbb{R}$ given by

$$g : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(t, z_1, z_2) \mapsto \left(1 - t^2 z_1 - \frac{2t}{1+t^2}z_2, \frac{2t}{1+t^2}z_1 + \frac{1-t^2}{1+t^2}z_2\right).$$

The action is semi-regular on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

In Example 3.9 we chose the cross-section $\mathcal{P}$ defined by $z_2 = 0$. The replacement invariants associated to the cross-section $\mathcal{P}$ are the $\mathbb{R}(z_1)$-zeros of the ideal $I^G = (Z_2, Z_2^2 - (z_1^2 + z_2^2))$.

The smooth maps $F^{(\pm)} : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ s.t. $F^{(\pm)}(z_1, z_2) = (0, \pm \sqrt{z_1^2 + z_2^2})$ are smooth zeros of $I^G$. Their components are thus local invariants.

The cross-section $\mathcal{P}$ defines a local cross-section for instance on $\mathcal{U} = \mathbb{R}^2 \setminus \{(z_1, z_2) \mid z_1 = 0, z_2 \leq 0\}$. As $F^{(+)}|_{\mathcal{P}\cap\mathcal{U}} = z_1$, the tuple of normalized invariants are $(0, \sqrt{z_1^2 + z_2^2})$ on $\mathcal{U}$.
We conclude this section by linking the smooth invariantization and the algebraic invariantization introduced in Section 4.2. Recall that the algebraic invariantization was a map that associated a univariate polynomial over $\mathbb{R}(z)^{G}$ to univariate polynomials over $\mathbb{K}[z]$. (Definition 4.9).

**Theorem 5.19** Let $\mathcal{P}$ be an algebraic cross-section which, when restricted to an open set $\mathcal{U}$, defines a local cross-section. Let $f : \mathcal{U} \to \mathbb{R}$ be a smooth zero of a univariate polynomial $\beta \in \mathbb{K}(z)[\zeta]$. The smooth invariantization $\bar{\iota}f$ of $f$ is a smooth zero of the algebraic $\mathcal{P}$-invariantization $\iota \beta \in \mathbb{R}(z)^{G}[\zeta]$ of $\beta$.

**Proof:** The polynomial $\iota \beta(z, \zeta) = \sum_{i=1}^{k} b_i(z) \zeta^i$, where $b_i \in \mathbb{K}(z)^{G}$. Any point $\bar{z} \in \mathcal{U}$ can obtained from the point $\bar{z}_0 \in \mathcal{P}$ by a composition of flows along infinitesimal generators of the group action. The argument will not change if we assume that $\bar{z} = \exp(\varepsilon v, \bar{z}_0)$ is obtained by the flow along a single vector field. Then from the invariance of $b_i(z)$ and local invariance of $\bar{\iota}f(z)$ it follows that $\forall \bar{z} \in \mathcal{U}$:

$$\iota \beta(\bar{z}, \bar{\iota}f(\bar{z})) = \sum_{i=1}^{k} b_i(\exp(\varepsilon v, \bar{z}_0)) f(\exp(\varepsilon v, \bar{z}_0))^i$$

$$= \sum_{i=1}^{k} b_i(\bar{z}_0) \bar{\iota}f(\bar{z}_0)^i = \iota \beta(\bar{z}_0, \bar{\iota}f(\bar{z}_0)),$$

where $\bar{z}_0 \in \mathcal{P} \cap \mathcal{U}$.

From Proposition 4.12 it follows that $\iota \beta$ is divisible by $\beta$ when restricted to $\mathcal{P}$. Thus $\iota \beta(\bar{z}_0, f(\bar{z}_0)) = 0$, $\forall \bar{z}_0 \in \mathcal{P} \cap \mathcal{U}$, since $\beta(\bar{z}, f(\bar{z})) \equiv 0$ on $\mathcal{U}$. It follows that $\bar{\iota}f(z)$ is a smooth zero of a polynomial $\iota \beta(z, \zeta) \in \mathbb{K}(z)^{G}[\zeta]$. $\Box$

In particular if $r(z)$ is a rational function that is well defined on $\mathcal{U}$, then its smooth invariantization $\bar{\iota}r(z)$ is a smooth zero of the $\mathcal{P}$-invariantization $\iota(\zeta - r(z))$ of the polynomial $\zeta - r(z)$. To discriminate the right one we only need to check that its value coincide with the one of $r(z)$ on $\mathcal{P} \cap \mathcal{U}$.
5.6 Moving frame map

We show that the invariantization map described in Section 4.2 generalizes the invariantization process described in [11]. The latter is restricted to locally-free actions, and is based on the existence of a local $G$-equivariant map $\rho: U \to G$. Although local freeness of the action guarantees the existence of $\rho$, due to the implicit function theorem, it might not be explicitly computable. We review the Fels-Olver construction, and prove that in the case of locally free actions it is equivalent to the one presented in Section 5.3.

**Definition 5.20** An action of a Lie group $G$ on a manifold $Z$ is locally free if for every point $\bar{z} \in Z$ its isotropy group $G_{\bar{z}} = \{ \bar{\lambda} \in G | \bar{\lambda} \cdot \bar{z} = \bar{z} \}$ is discrete.

Local freeness implies semi-regularity of the action, the dimension of each orbit being equal to the dimension of the group. Theorem 4.4 from [11], can be restated as follows in the case of locally free actions.

**Theorem 5.21** A Lie group $G$ acts locally freely on $Z$ if and only if every point of $Z$ has an open neighborhood $\mathcal{U}$ such that there exists a map $\rho: \mathcal{U} \to G$ that makes the following diagram commute. Here the map $\bar{\mu} \mapsto \bar{\mu} \cdot \bar{\lambda}^{-1}$ is chosen for the action of $G$ on itself, and $\bar{\lambda}$ is taken in a suitable neighborhood (depending on the point of $\mathcal{U}$) of the identity in $G$.

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\bar{\lambda}} & \mathcal{U} \\
\rho \downarrow & & \downarrow \rho \\
G & \xrightarrow{\bar{\lambda}} & G
\end{array}
\]

The map $\rho$ is locally $G$-equivariant, $\rho(\bar{\lambda} \cdot \bar{z}) = \rho \cdot \bar{\lambda}^{-1}$ for $\bar{\lambda}$ sufficiently close to the identity, and is called a moving frame map. If $\mathcal{P}$ is a cross-section, then the equation

\[\rho(\bar{z}) \cdot \bar{z} \in \mathcal{P},\]

uniquely defines $\rho(\bar{z})$ in a sufficiently small neighborhood of the identity. In particular, $\rho(\bar{z}_0) = e$ for all $\bar{z}_0 \in \mathcal{P}$. Reciprocally, a moving frame map defines a local cross-section to the orbits: $\mathcal{P} = \{ \rho(\bar{z}) \cdot \bar{z} | \bar{z} \in U \} \subset U$.

In local coordinates, Condition (5) gives rise to implicit equations for expressing the group parameters in terms of the coordinate functions on the manifold. When the group acts locally freely, the local existence of smooth solutions is guaranteed by the transversality condition and the implicit function theorem. Since the implicit function theorem is not constructive, we might nonetheless not be able to obtain explicit formulas for the solution.

In [11, Definition 4.6] the invariantization of a function $f$ on $\mathcal{U}$ is defined as the function whose value at a point $\bar{z} \in \mathcal{U}$ is equal to $f(\rho(\bar{z}) \cdot \bar{z})$. Next proposition
shows that this moving frame based definition of invariantization is equivalent to Definition 5.8 given in terms of cross-section. The advantage of the latter definition is that it is not restricted to locally free actions.

**Proposition 5.22** Let \( \rho \) be a moving frame map on \( U \). Then \( \bar{\iota}f(\bar{z}) = f(\rho(\bar{z}) \cdot \bar{z}) \).

**Proof:** Local invariance of \( f(\rho(z) \cdot z) \) follows from the local equivariance of \( \rho \), i.e. for \( \lambda \) sufficiently close to the identity:

\[
f(\rho(\lambda \cdot \bar{z}) \cdot (\lambda \cdot \bar{z})) = f(\rho(\bar{z}) \cdot \lambda^{-1} \cdot (\lambda \cdot \bar{z})) = f(\rho(\bar{z}) \cdot \bar{z}).
\]

Since \( \rho(z_0) = e \) then \( f(\rho(\bar{z}_0) \cdot \bar{z}_0) = f(\bar{z}_0) \) for all \( \bar{z}_0 \in \mathcal{P} \). Thus \( f(\rho(z) \cdot z) \) is locally invariant and equals to \( f \), when restricted to \( \mathcal{P} \). The conclusion follows from Theorem 5.9. \( \square \)

Thus the moving frame map offers an approach to invariantization that is constructive up to the resolution of the implicit equations given by (5). In the algebraic case the moving frame map is defined by the ideal

\[
M^c = (G + P + (Z - g(\lambda, z))) \cap \mathbb{R}(z)[\lambda].
\]

Indeed, if \( (\bar{z}, \lambda) \) is a zero of \( M = M^c \cap \mathbb{R}[z, \lambda] \), in an appropriate open set of \( Z \times G \), then \( \lambda \cdot \bar{z} \in \mathcal{P} \). The action is locally free if and only if \( M^c \) is zero dimensional.

In this case, the smooth zero \( F : U \to G \) of \( M^c \), that is the identity of the group when restricted to \( \mathcal{P} \), provides a moving frame map \( \rho \) on \( U \).

If one can obtain the map \( \rho \) explicitly, the invariantization map can be computed using Proposition 5.22. Even in this favorable case, the expression for \( \rho \) often involves algebraic functions which can prove difficult to manipulate symbolically. The purely algebraic approach proposed in Section 4 is more suitable for symbolic computation.

### 6 Additional examples

We first consider a linear action of \( SL_2 \) on \( \mathbb{K}^7 \) taken from [8]. That latter paper presents an algorithm to compute a set of generators of the algebra of polynomial invariants for the linear action of a reductive group. The ideal \( O = (G + (Z - g(\lambda, z))) \cap \mathbb{K}[z, Z] \), where now \( g \) is a polynomial map that is linear in \( z \), is also central in the construction as a set of generators of \( \mathbb{K}[z]^G \) is obtained by applying the Reynolds operator, which is a projection from \( \mathbb{K}[z] \) to \( \mathbb{K}[z]^G \), to generators of \( O + (Z_1, \ldots, Z_n) \), the ideal of the null cone.

The fraction field of \( \mathbb{K}[z]^G \) is included in \( \mathbb{K}(z)^G \) but does not need to be equal. Conversely there is no known algorithm to compute \( \mathbb{K}[z]^G = \mathbb{K}(z)^G \cap \mathbb{K}[z] \) from the knowledge of a set of generators of \( \mathbb{K}(z)^G \).
Example 6.1 We consider the linear action of $SL_2$ on $\mathbb{K}^7$ given by the following polynomials of $\mathbb{K}[\lambda_1, \ldots, \lambda_4, z_1, \ldots, z_7]$:

$$
\begin{align*}
g_1 &= \lambda_1 z_1 + \lambda_2 z_2, \\
g_2 &= \lambda_3 z_1 + \lambda_4 z_2 \\
g_3 &= \lambda_1 z_3 + \lambda_2 z_4, \\
g_4 &= \lambda_3 z_3 + \lambda_4 z_4 \\
g_5 &= \lambda_3^2 z_5 + 2\lambda_1 \lambda_2 z_6 + \lambda_2^2 z_7, \\
g_6 &= \lambda_3 \lambda_1 z_5 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 z_6 + \lambda_2 \lambda_4 z_7, \\
g_7 &= \lambda_3^2 z_5 + 2\lambda_3 \lambda_4 z_6 + \lambda_4^2
\end{align*}
$$

the group being defined by $G = (\lambda_1 \lambda_4 - \lambda_2 \lambda_3 - 1) \subset \mathbb{K}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$. The cross-section defined by $P = (Z_1 + 1, Z_2, Z_3)$ is of degree one. The reduced Gröbner basis (for any term order) of the ideal $I^c \subset \mathbb{K}(z)[Z]$ is indeed given by $\{Z_1 + 1, Z_2, Z_3, Z_4 - r_2, Z_5 - r_3, Z_6 - r_4, Z_7 - r_1\}$ where

$$
\begin{align*}
r_1 &= z_7 z_1^2 - 2 z_2 z_6 z_1 + z_2^2 z_5, \\
r_2 &= z_3 z_2 - z_1 z_4, \\
r_3 &= \frac{z_3^2 z_7 - z_2 z_5 z_3 + z_5 z_4^2}{(z_1 z_2 - z_3 z_2)^2}, \\
r_4 &= \frac{z_1 z_6 z_4 - z_1 z_3 z_7 + z_4 z_2 z_6 - z_2 z_5 z_4}{z_1 z_4 - z_3 z_2}
\end{align*}
$$

By Theorem 3.6, $\mathbb{K}(z)^G = \mathbb{K}(r_1, r_2, r_3, r_4)$. In this case the rewriting of any rational invariant in terms of $r_1, r_2, r_3, r_4$ consists simply of the substitution of $(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ by $(-1, 0, 0, r_2, r_3, r_4, r_1)$. We illustrate this by rewriting the five generating polynomial invariants computed in [S] in terms of $r_1, r_2, r_3, r_4$:

$$
\begin{align*}
z_2^2 z_5 - 2 z_2 z_6 z_1 + z_7 z_1^2 &= r_1, \\
z_3 z_2 - z_1 z_4 &= r_2, \\
z_3^2 z_7 - 2 z_6 z_4 z_3 + z_5 z_4^2 &= r_3^2, \\
z_1 z_3 z_7 - z_3 z_2 z_6 + z_2 z_5 z_4 - z_1 z_6 z_4 &= r_4 r_2, \\
z_6^2 - z_7 z_5 &= r_4^2 - r_1 r_3
\end{align*}
$$

The reduced Gröbner basis of $O^c$, relative to the total degree order with ties broken by reverse lexicographical order, has 9 elements:

$$
\begin{align*}
Z_6^2 - Z_7 Z_5 + r_1 r_3 - r_4^2, \\
Z_6 Z_4 + r_3 r_2 Z_2 - r_4 Z_4 - Z_3 Z_7, \\
Z_5 Z_4 - Z_3 Z_6 + r_3 r_2 Z_1 - r_4 Z_3, \\
Z_3 Z_2 - Z_1 Z_4 - r_2, \\
Z_2 Z_6 - Z_1 Z_7 + r_4 Z_2 - \frac{r_4}{r_3} Z_4, \\
Z_2 Z_5 + Z_1 r_4 - Z_6 Z_1 - \frac{r_4}{r_3} Z_3, \\
Z_2 Z_1 - \frac{r_4}{r_3} Z_6 + \frac{r_1}{r_3 r_2} Z_4 Z_3 - 2 \frac{r_1}{r_3 r_2} Z_4 Z_1
\end{align*}
$$

Though this Gröbner basis is obtained without much difficulty, the example illustrates the advantage obtained by considering the construction with a cross-section: $I^c$ has a much simpler reduced Gröbner basis than $O^c$.

We finally take a classical example in differential geometry: the Euclidean action on the second order jets of curves. The variables $x, y_0, y_1, y_2$ stand for the independent variable, the dependent variable, the first and the second derivatives respectively. We shall recognize the curvature as the non constant component of a replacement invariant.

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Example 6.2 We consider the group defined by $G = (\alpha^2 + \beta^2 - 1, \epsilon^2 - 1) \subset \mathbb{K}[\alpha, \beta, a, b, \epsilon]$. The neutral element is $(1, 0, 0, 0, 1)$, the group operation is $(\alpha', \beta', a', b', \epsilon') \cdot (\alpha, \beta, a, b, \epsilon) = (\alpha a' - \beta b', \beta a' + \alpha b', a + a' - \beta b', b + \alpha b', \epsilon')$ and the inverse map $(\alpha, \beta, a, b)^{-1} = (\alpha, -\beta, -\alpha a - b\beta, \beta a - \alpha b, \epsilon)$. The rational action on $\mathbb{K}^4$ we consider is given by the rational functions:

$$g_1 = \alpha x - \beta y_0 + a, \quad g_2 = \epsilon \beta x + \epsilon \alpha y_0 + b,$$
$$g_3 = \frac{\beta + \alpha y_1}{\alpha - \beta y_0}, \quad g_4 = \frac{y_2}{(\alpha - \beta y_0)^3}.$$

We have

$$O = \left( (1 + y_1^2)^3 Y_2^2 - (1 + Y_1^2)^3 y_2^2 \right)$$

and if we consider the the cross section defined by $P = (X, Y_0, Y_1)$ the reduced Gröbner basis of $I^c = O^c + P$ is

$$\left\{ X, Y_0, Y_1, Y_2^2 - \frac{y_2^2}{(1 + y_1^2)^3} \right\}.$$

According to Theorem 2.15 or Theorem 3.6

$$\mathbb{K}(z)^G = \mathbb{K} \left( \frac{y_2^2}{(1 + y_1^2)^3} \right).$$

The two replacement invariants $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ associated to the cross-sections are given by

$$\xi_1 = 0, \xi_2 = 0, \xi_3 = 0, \xi_4 = \pm \sqrt{\frac{y_2^2}{(1 + y_1^2)^3}}.$$

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