Symplectic tomography of nonlinear coherent states of a trapped ion

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Abstract

Squeezed and rotated quadrature of an ion in a Paul trap is discussed in connection with reconstructing its quantum state using symplectic-tomography method. Marginal distributions of the quadrature for squeezed and correlated states and for nonlinear coherent states of a trapped ion are obtained and the density matrices in the Fock basis are expressed explicitly in terms of these marginal distributions.

1 Introduction

Recently, such nonclassical states of a trapped ion as even and odd coherent states [1] (or Schrödinger cat states [2]) were realized experimentally [3] and their properties were discussed in Refs. [4–6]. For light modes, these states were produced in high-Q cavities [7]. The experiments
with reproducible measurements of squeezed vacuum state of light generated by an optical parametric oscillator were performed in Ref. [8]. Resonance fluorescence was proposed to reconstruct the quantum-mechanical state of a trapped ion [9].

Endoscopy method of measuring the nonclassical states, in particular, of a trapped ion was suggested in Ref. [10] and tomography method for studying the Schrödinger cat states of an ion in a Paul trap was discussed in Ref. [11]. In Refs. [12, 6], the linear integrals of motion of the parametric oscillator [13, 14], which models the motion of a trapped ion, were used to study the quantum states of the system.

New type of nonclassical states, namely, nonlinear coherent states of an ion and the method of creation of these states, as stationary states of the center-of-mass motion of a trapped and bichromatically laser-driven ion, were suggested in Ref. [15]. The nonlinear states are the particular case of f-coherent states introduced in Ref. [16] to describe a nonlinear quantum oscillator, for which the phase of vibrations depends on the energy of the vibrations. In the linear limit, these nonclassical states, which have the properties of squeezing and correlation of quadratures, become the coherent states of harmonic oscillator [17]. Thus, the problem of experimental reconstructing the nonclassical state in terms of Wigner function or density matrix (in other representations) for the nonlinear coherent state of a trapped ion is actual problem.

Recently, the symplectic tomography method was discussed for measuring a quantum state [18]. The method (extended as well for multimode case [19]) uses Fourier transform of marginal distribution for measurable squeezed and rotated quadrature instead of Radon transform [20], which is used in optical tomography to reconstruct the Wigner function; in this context, the symplectic tomography is similar to the strength-field method of Ref. [21]. The marginal distribution
for squeezed and rotated quadrature determines completely the quantum state and measuring
this distribution implies reconstructing the quantum state. It satisfies the classical-like evolution
equation introduced in Ref. [22] (see, also review [23]).

The aim of this work is to discuss the symplectic tomography scheme to measure nonlinear
coherent states of a trapped ion, following the approach considered in Ref. [11] for measuring even
and odd coherent states. Our goal is to construct explicitly the marginal distribution of squeezed
and rotated quadrature for nonlinear coherent state of an ion in a Paul trap and to compare it
with marginal distribution of optical tomography procedure. As well, the marginal distribution
for discrete oscillator levels will be considered, corresponding to photon-number tomography of
Refs. [24–26], as a procedure for measuring the quantum state of the trapped ion.

2 Nonlinear coherent states of a trapped ion

Since an ion in a Paul trap is described by the model of a parametric oscillator [12, 27] in this
section we review its properties. For a parametric oscillator with an arbitrary time dependence of
the frequency and the Hamiltonian

\[ H = -\frac{\partial^2}{2 \partial x^2} + \frac{\omega^2(t) x^2}{2}, \]  

where we put \( \hbar = m = \omega(0) = 1 \) and used expressions for the position and momentum operators
in the coordinate representation, there is the time-dependent integral of motion found in Ref. [14]:

\[ A = \frac{i}{\sqrt{2}} [\dot{\varepsilon}(t) \hat{p} - \ddot{\varepsilon}(t) \hat{q}], \]  

where

\[ \ddot{\varepsilon}(t) + \omega^2(t) \varepsilon(t) = 0; \quad \varepsilon(0) = 1; \quad \dot{\varepsilon}(0) = i, \]
which satisfies the commutation relation

\[ [A, A^\dagger] = 1. \] (4)

For the trapped ion, the time dependence of the frequency is taken to be periodic [12]:

\[ \omega^2(t) = 1 + \kappa^2 \sin^2 \Omega t. \] (5)

It is easy to show that Gaussian packet solutions to the Schrödinger equation may be introduced and interpreted as coherent states [14], since they are eigenstates of the operator \( A \) (2), of the form

\[ \Psi_\alpha(x, t) = \Psi_0(x, t) \exp \left\{ -\frac{|\alpha|^2}{2} - \frac{\alpha^2 \varepsilon^*(t)}{2\varepsilon(t)} + \frac{\sqrt{2}\alpha x}{\varepsilon} \right\}, \] (6)

where

\[ \Psi_0(x, t) = \pi^{-1/4} [\varepsilon(t)]^{-1/2} \exp \frac{i\dot{\varepsilon}(t)x^2}{2\varepsilon(t)} \] (7)
is an analog of the ground state of the oscillator and \( \alpha \) is a complex number. The variances of the position and momentum of the parametric oscillator in the state (6) are

\[ \sigma_{qq} = \frac{\varepsilon(t)^2}{2}; \quad \sigma_{pp} = \frac{\varepsilon(t)^2}{2}, \] (8)

and the correlation coefficient \( r \) of the position and momentum has a value corresponding to minimization of the Schrödinger uncertainty relation [28]:

\[ \sigma_{qq} \sigma_{pp} = \frac{1}{4} \frac{1}{1 - r^2}; \quad r = \frac{\sigma_{pq}}{\sqrt{\sigma_{qq} \sigma_{pp}}}. \] (9)

If \( \sigma_{qq} < 1/2 \) (\( \sigma_{pp} < 1/2 \)), we have squeezing in quadrature components.

Analogs of an orthogonal and complete system of number states, which are excited states of an ion in a Paul trap, are obtained by expansion of (6) into a power series in \( \alpha \). We have

\[ \Psi_m(x, t) = \left( \frac{\varepsilon^*(t)}{2\varepsilon(t)} \right)^{m/2} \frac{1}{\sqrt{m!}} \Psi_0(x, t) H_m \left( \frac{x}{\varepsilon(t)} \right), \] (10)
and these squeezed and correlated number states are eigenstates of the invariant $A^\dagger A$.

The coherent state (6), which is squeezed and correlated state for quadratures, is the superposition of number states (10)

$$
\Psi_\alpha (x, t) = \exp \left( -\frac{|\alpha|^2}{2} \right) \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \Psi_m (x, t).
$$

(11)

There exist the integrals of motion

$$
B = A f \left( A^\dagger A \right); \quad B^\dagger = f \left( A^\dagger A \right) A^\dagger,
$$

(12)

which are determined by a function $f$ of the invariants (2). These integrals of motion satisfy the commutation relations

$$
[B, B^\dagger] = F \left( A^\dagger A \right),
$$

(13)

where

$$
F \left( A^\dagger A \right) = \left( A^\dagger A + 1 \right) f^2 \left( A^\dagger A + 1 \right) - A^\dagger A f^2 \left( A^\dagger A \right).
$$

(14)

Generalizing notion of coherent states to the case of the operator, which is nonlinearly transformed annihilation operator, we introduce the eigenfunctions of the invariant $B$

$$
B \Psi_\beta (x, t) = \beta \Psi_\beta (x, t),
$$

(15)

which are the nonlinear coherent states. Such construction of the states, called f-coherent states, was suggested in Ref. [16]. For the function

$$
f(y) = L_{y+1}^1(\eta^2) \left[ y L_{y+1}^0(\eta^2) \right]^{-1},
$$

(16)

where $L_n^m(\eta^2)$ are associated Laguerre polynomials and $\eta$ is Lamb–Dicke parameter, the f-coherent states (nonlinear coherent states) have been considered in Ref. [15].
Using general scheme of constructing the normalized nonlinear coherent states of Refs. [15, 16] one can obtain the function \( \Psi_\beta(x, t) \) in the form of series

\[
\Psi_\beta(x, t) = \left( \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{n! |[f(n)]|!^2} \right)^{-1/2} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!} |[f(m)]|!} \Psi_m(x, t), \tag{17}
\]

in which we denote, e.g., \([f(m)]! = f(0)f(1) \cdots f(m)\).

For \( f(m) = 1 \), the wave function of the nonlinear coherent state (17) becomes the wave function of the coherent state (6), in which the parameter \( \beta = \alpha \). The Wigner function of the nonlinear coherent states (17) has the form [16]

\[
W_\beta(x, p) = 2 \left( \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{n! |[f(n)]|!^2} \right)^{-1} e^{-(x^2+p^2)} \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! |[f(m)]|! |[f(n)]|!} (-\beta)^n \beta^m \times \left( \sqrt{2} [x - ip] \right)^{m-n} L_n^{m-n} \left( 2 [x^2 + p^2] \right), \tag{18}
\]

where \( L_n^m \) denotes associated Laguerre polynomial. For the particular case of the function \( f \) given by (16), one has the Wigner function studied in Ref. [15], where some plots of the Wigner function were presented.

### 3 Tomography of a trapped ion

It was shown [18] that for the generic linear combination of quadratures, which is a measurable observable \((\hbar = 1)\),

\[
\hat{X} = \mu \hat{q} + \nu \hat{p}, \tag{19}
\]

where \( \hat{q} \) and \( \hat{p} \) are the position and momentum, respectively, the marginal distribution \( w(X, \mu, \nu) \) (normalized with respect to the variable \( X \)), depending on the two extra real parameters \( \mu \) and \( \nu \),
is related to the state of the quantum system expressed in terms of its Wigner function $W(q, p)$ as follows

$$w(X, \mu, \nu) = \int \exp \left[ -ik(X - \mu q - \nu p) \right] W(q, p) \frac{dk \ dq \ dp}{(2\pi)^2}. \quad (20)$$

The physical meaning of the parameters $\mu$ and $\nu$ is that they describe an ensemble of rotated and scaled reference frames, in which the position $X$ is measured. For $\mu = \cos \varphi; \nu = \sin \varphi$, the marginal distribution (20) is the distribution for homodyne output variable used in optical tomography [20]. Formula (20) can be inverted and the Wigner function of the state can be expressed in terms of the marginal distribution [18]:

$$W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp \left[ -i(\mu q + \nu p - X) \right] d\mu d\nu dX. \quad (21)$$

It was shown [22] that for systems with the Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q})$ the marginal distribution satisfies quantum time-evolution equation. For a trapped ion, the evolution equation takes the form

$$\dot{w} - \mu \frac{\partial}{\partial \nu} w + \omega^2(t) \nu \frac{\partial}{\partial \mu} w = 0. \quad (22)$$

If one uses the constraint $\mu = \cos \varphi; \nu = \sin \varphi$, this equation becomes the equation for marginal distribution of optical tomography [20]. Thus, measuring marginal distribution for scaled and rotated quadrature one can reconstruct the Wigner function of a trapped ion using the Fourier transform (21).

4 Marginal distribution for squeezed and correlated states

First we discuss marginal distribution for squeezed and correlated state (6) of a trapped ion. For these states, the Wigner function has Gaussian form [29]. Consequently, the Fourier transform (20)
of the Gaussian Wigner function determining the marginal distribution yields the Gaussian form of this marginal distribution

$$w_\alpha (X, \mu, \nu, t) = \frac{1}{\sqrt{2\pi \sigma_X(t)}} \exp \left\{ -\frac{(X - \bar{X})^2}{2\sigma_X(t)} \right\},$$

(23)
in which, in view of (19) and (2), one has

$$\bar{X} = \mu \langle q \rangle + \nu \langle p \rangle,$$

(24)
where quadrature means are

$$\langle p \rangle = \frac{1}{\sqrt{2}} (\alpha \dot{\varepsilon}^* + \alpha^* \dot{\varepsilon});$$

(25)
$$\langle q \rangle = \frac{1}{\sqrt{2}} (\alpha \varepsilon^* + \alpha^* \varepsilon).$$

(26)
The dispersion and correlation of the quadratures are

$$\sigma_X(t) = \mu^2 \sigma_{qq} + \nu^2 \sigma_{pp} + 2\mu\nu \sigma_{pq},$$

(27)
where the parameters $\sigma_{qq}$, $\sigma_{pp}$, and $\sigma_{pq}$ are given by (8) and (9). It is easy to check that the marginal distribution for the nonclassical state (23) satisfies a classical-like evolution equation for the density matrix (22) introduced in symplectic tomography scheme.

5 Marginal distribution for nonlinear coherent states

Using the decomposition of wave function of nonlinear coherent state into series (17) one can obtain by standard method the Wigner function of the trapped ion in the form

$$W_\beta (q, p, t) = 2 \left( \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[n]!} \right)^{-1} \exp \left( -\left[ q^2(t) + p^2(t) \right] \right)$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! [f(m)]!} \frac{1}{[f(n)]!} (-\beta)^n \beta^m$$

$$\times \left( \sqrt{2} [q(t) - ip(t)] \right)^{m-n} L_{m-n}^{m-n} \left( 2 \left[ q^2(t) + p^2(t) \right] \right),$$

(28)
where

\[
p(t) = \frac{\varepsilon + \varepsilon^*}{2} p - \frac{i}{2} \dot{\varepsilon} + \varepsilon^* \frac{i}{2} q; \tag{29}
\]

\[
q(t) = -\frac{\varepsilon - \varepsilon^*}{2i} p + \frac{i}{2} \dot{\varepsilon} - \varepsilon^* \frac{i}{2} q. \tag{30}
\]

Then calculating the Fourier integral (20) one obtains the marginal distribution of the trapped ion in nonlinear coherent state in the form

\[
w_{\beta}(X, \mu, \nu, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\beta^n \beta^m}{\sqrt{n!} \sqrt{m!} [f(n)]! [f(m)]!} w_{nm}(X, \mu, \nu, t). \tag{31}
\]

Here \(w_{nm}(X, \mu, \nu, t)\) is

\[
w_{nm}(X, \mu, \nu, t) = \left( \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[n]!} \right)^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\beta^n \beta^m}{\sqrt{n!} \sqrt{m!} [f(n)]! [f(m)]!} w_{nm}(X, \mu, \nu, t). \tag{32}
\]

where

\[
\nu(t) = \frac{\dot{\varepsilon} - \dot{\varepsilon}^*}{2i} \nu + \frac{\varepsilon - \varepsilon^*}{2i} \mu; \tag{33}
\]

\[
\mu(t) = \frac{\dot{\varepsilon} + \dot{\varepsilon}^*}{2} \nu + \frac{\varepsilon + \varepsilon^*}{2} \mu. \tag{34}
\]

One can check that the marginal distribution function (31) is the solution to quantum evolution equation (22). For \(f(n) = 1\) and \(\beta = \alpha\), the marginal distribution function (31) coincides with the Gaussian distribution (23).

The method of photon number tomography was discussed in Ref. [26]. In this method, the measurable marginal distribution \(w(n, \alpha)\) depends on the discrete number of photons \(n\) and on the complex amplitude \(\alpha\) of the local field oscillator, that may be scanned. Using the general relation
of symplectic marginal distribution to the distribution of the discrete photon number [30], one can obtain the discrete marginal distribution \( w_\beta(n, \alpha, t) \) for an ion in a Paul trap, which is modeled by a parametric oscillator, in terms of the marginal distribution (20) for the nonlinear coherent states in the form

\[

c_{\beta}(n, \alpha, t) = \frac{1}{2\pi} \int w_\beta(X, \mu, \nu, t) \exp \left\{ iX - \frac{\mu^2 + \nu^2}{4} + \frac{\alpha (\nu + i\mu)}{\sqrt{2}} - \frac{\alpha^* (\nu - i\mu)}{\sqrt{2}} \right\}

\times L_n \left( \frac{\mu^2 + \nu^2}{2} \right) \, dx \, d\mu \, d\nu ,
\]

(35)

where \( w_\beta(X, \mu, \nu, t) \) is given by (31).

The endoscopy method [10] was used to reconstruct matrix elements of density operator of an ion in a Paul trap \( \langle m | \rho | n \rangle \) in the Fock basis on the example of damped odd coherent state. We show how these matrix elements may be reconstructed following the approach of the symplectic tomography method [18, 19, 30]. It was demonstrated [19] that the density operator is expressed as convolution of the marginal distribution and displacement operator creating the coherent state from the vacuum state. Since the matrix elements of the displacement operator in the Fock basis are well known, the formula for the matrix elements of the density operator in the Fock basis may be expressed in terms of associated Laguerre polynomials [30]. Thus, for nonlinear coherent state of an ion in a Paul trap one has the expression for the density matrix in the Fock basis as convolution of the marginal distribution for squeezed and rotated quadrature and the associated Laguerre polynomial

\[

\langle m | \rho_\beta(t) | n \rangle = \sqrt{\frac{n!}{m!} \frac{2^{(n-m)/2}}{2\pi}} \int \exp \left( ix - \frac{\mu^2 + \nu^2}{4} \right) w_\beta(X, \mu, \nu, t) (\nu - i\mu)^{m-n}

\times L_{m-n} \left( \frac{\mu^2 + \nu^2}{2} \right) dX \, d\mu \, d\nu ; \quad m > n .
\]

(36)

Here the function \( w_\beta(X, \mu, \nu, t) \) is given by (31). For \( m < n \), we use the property of the hermitian
density matrix \( \langle m | \rho_\beta (t) | n \rangle = \langle n | \rho_\beta (t) | m \rangle^* \).

6 Conclusion

One can conclude that for an ion in a Paul trap there exist three types of different marginal distributions, which can be measured experimentally. The explicit expressions for the distributions for squeezed states and for nonlinear coherent states are the main result of our work. These marginal distributions are

(i) the distribution for squeezed and rotated quadrature;

(ii) the distribution for homodyne observable, which is partial case of the squeezed and rotated quadrature for the parameters \( \mu = \cos \varphi, \nu = \sin \varphi \);

(iii) the number distribution controlled by the scanned complex amplitude of vibrations of a local oscillator.

Measurement of any of the three distributions gives the reconstruction of the quantum state of the trapped ion. The explicit expressions for the marginal distributions obtained in this work yield the theoretical predictions for measuring experimentally the nonlinear coherent states of an ion in a Paul trap with the help of the three different methods, namely, optical tomography, symplectic tomography, and photon number tomography.

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