ON THE HADAMARD TYPE INEQUALITIES INVOLVING PRODUCT OF TWO CONVEX FUNCTIONS ON THE CO-ORDINATES

*M. Emin Özdemir AND ♦Ahmet OcaK Akdemir

Abstract. In this paper some Hadamard-type inequalities for product of convex functions of 2-variables on the co-ordinates are given.

1. INTRODUCTION

The inequality

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

where \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) is a convex function defined on the interval \( I \) of \( \mathbb{R} \), the set of real numbers, and \( a, b \in I \) with \( a < b \), is well known in the literature as Hadamard’s inequality.

For some recent results related to this classic inequality, see [1], [8], [11], [12], and [14], where further references are given.

In [2], Hudzik and Maligranda considered, among others, the class of functions which are \( s \)-convex in the second sense. This class is defined as following:

Definition 1. A function \( f : [0, \infty) \rightarrow \mathbb{R} \) is said to be \( s \)-convex in the second sense if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)
\]

holds for all \( x, y \in [0, \infty) \), \( \lambda \in [0, 1] \) and for some fixed \( s \in (0, 1) \).

The class of \( s \)-convex functions in the second sense is usually denoted with \( K_2^s \).

It is clear that if we choose \( s = 1 \) we have ordinary convexity of functions defined on \( [0, \infty) \).

In [15], Kirmaci et al., proved the following inequalities related to product of convex functions. These are given in the next theorems.

Theorem 1. Let \( f, g : [a, b] \rightarrow \mathbb{R}, a, b \in [0, \infty), \ a < b, \) be functions such that \( g \) and \( fg \) are in \( L^1([a, b]) \). If \( f \) is convex and nonnegative on \( [a, b] \), and if \( g \) is \( s \)-convex on \( [a, b] \) for some fixed \( s \in (0, 1) \), then

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s+2} M(a,b) + \frac{1}{(s+1)(s+2)} N(a,b)
\]
where
\[ M(a, b) = f(a)g(a) + f(b)g(b) \] and \[ N(a, b) = f(a)g(b) + f(b)g(a). \]

**Theorem 2.** Let \( f, g : [a, b] \to \mathbb{R}, a, b \in [0, \infty), a < b, \) be functions such that \( g \) and \( fg \) are in \( L^1([a, b]) \). If \( f \) is \( s_1 \)-convex and \( g \) is \( s_2 \)-convex on \([a, b]\) for some fixed \( s_1, s_2 \in (0, 1) \), then
\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s_1 + s_2 + 1} M(a, b) + B(s_1 + 1, s_2 + 1)N(a, b)
\]
(1.3)
\[
= \frac{1}{s_1 + s_2 + 1} \left[ M(a, b) + s_1s_2 \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} N(a, b) \right]
\]

**Theorem 3.** Let \( f, g : [a, b] \to \mathbb{R}, a, b \in [0, \infty), a < b, \) be functions such that \( g \) and \( fg \) are in \( L^1([a, b]) \). If \( f \) is convex and nonnegative on \([a, b]\), and if \( g \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1) \), then
\[
2s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s+1}(s+2) M(a, b) + \frac{1}{s+2} N(a, b)
\]
(1.4)

For similar results, see the papers [2], [13].

In [12], Dragomir defined convex functions on the co-ordinates as follows and proved Lemma 1 related to this definition:

**Definition 2.** Let us consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b, c < d \). A function \( f : \Delta \to \mathbb{R} \) will be called convex on the co-ordinates if the partial mappings \( f_y : [a, b] \to \mathbb{R}, f_y(u) = f(a, y) \) and \( f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v) \) are convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \). Recall that the mapping \( f : \Delta \to \mathbb{R} \) is convex on \( \Delta \) if the following inequality holds,
\[
f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)
\]
for all \((x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \).

**Lemma 1.** Every convex mapping \( f : \Delta \to \mathbb{R} \) is convex on the co-ordinates, but converse is not general true.

A formal definition for co-ordinated convex functions may be stated as follow [see [16]]:

**Definition 3.** A function \( f : \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates on \( \Delta \) if the following inequality:
\[
f(tx + (1-t)y, su + (1-s)w)
\]
\[
\leq tsf(x, u) + t(1-s)f(x, w) + s(1-t)f(y, u) + (1-t)(1-s)f(y, w)
\]
holds for all \( t, s \in [0, 1] \) and \((x, u), (x, w), (y, u), (y, w) \in \Delta \).

In [12], Dragomir established the following inequalities:
**Theorem 4.** Suppose that \( f, g : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). Then one has the inequalities:
\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \\
\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}
\]

(1.5)

Similar results, refinements and generalizations can be found in [3], [5], [6], [7], [9] and [10].

In [7], Alomari and Darus defined \( s \)-convexity on \( \Delta \) as follows:

**Definition 4.** Consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \([0, \infty)^2\) with \( a < b \) and \( c < d \). The mapping \( f : \Delta \to \mathbb{R} \) is \( s \)-convex on \( \Delta \) if
\[
f(\lambda x + (1-\lambda) z, \lambda y + (1-\lambda) w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w)
\]
holds for all \( (x,y), (z,w) \in \Delta \) with \( \lambda \in [0,1] \) and for some fixed \( s \in (0,1] \).

In [7], Alomari and Darus proved the following lemma:

**Lemma 2.** Every \( s \)-convex mappings \( f : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty) \) is \( s \)-convex on the co-ordinates, but converse is not true in general.

In [4], Latif and Alomari established Hadamard-type inequalities for product of convex functions on the co-ordinates as follow:

**Theorem 5.** Let \( f, g : \Delta := [a, b] \times [c, d] \subset \mathbb{R}^2 \to [0, \infty) \) be convex functions on the co-ordinates on \( \Delta \) with \( a < b \) and \( c < d \). Then
\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dxdy \\
\leq \frac{1}{9} L(a,b,c,d) + \frac{1}{18} M(a,b,c,d) + \frac{36}{36} N(a,b,c,d)
\]

where
\[
L(a,b,c,d) = f(a,c)g(a,c) + f(b,c)g(b,c) + f(a,d)g(a,d) + f(b,d)g(b,d) \\
M(a,b,c,d) = f(a,c)g(a,d) + f(a,d)g(a,c) + f(b,c)g(b,c) + f(b,d)g(b,c) \\
+ f(b,c)g(a,c) + f(b,d)g(a,d) + f(a,c)g(b,c) + f(a,d)g(b,d) \\
N(a,b,c,d) = f(b,c)g(a,d) + f(b,d)g(a,c) + f(a,c)g(b,c) + f(a,d)g(b,c)
\]

**Theorem 6.** Let \( f, g : \Delta := [a, b] \times [c, d] \subset \mathbb{R}^2 \to [0, \infty) \) be convex functions on the co-ordinates on \( \Delta \) with \( a < b \) and \( c < d \). Then
\[
4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dxdy \\
+ \frac{5}{36} L(a,b,c,d) + \frac{7}{36} M(a,b,c,d) + \frac{2}{9} N(a,b,c,d)
\]

(1.7)
where \( L(a, b, c, d), M(a, b, c, d), N(a, b, c, d) \) as in (1.6).

The main purpose of this paper is to establish new inequalities like (1.6) and (1.7), but now for convex functions and \( s \)-convex functions of 2-variables on the co-ordinates.

2. MAIN RESULTS

**Theorem 7.** Let \( f : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty) \) be convex function on the co-ordinates and \( g : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty) \) be \( s \)-convex function on the co-ordinates with \( a < b, c < d \) and \( f_y(y)g_x(y), \ f_y(x)g_y(x) \in L^1[\Delta] \) for some fixed \( s \in (0, 1] \). Then one has the inequality:

\[
\begin{align*}
\frac{1}{(d-c)(b-a)} \int_a^b \int_c^d f(x, y)g(x, y)dxdy & \leq \frac{1}{(s+2)^2} L(a, b, c, d) + \frac{1}{(s+1)(s+2)^2} M(a, b, c, d) \\
& + \frac{1}{(s+1)^2(s+2)^2} N(a, b, c, d)
\end{align*}
\]

where

\[
\begin{align*}
L(a, b, c, d) &= \frac{1}{(s+2)^2} ([f(a, c)g(a, c) + f(b, c)g(b, c)] + [f(a, d)g(a, d) + f(b, d)g(b, d)]) \\
M(a, b, c, d) &= \frac{1}{(s+1)(s+2)^2} ([f(a, c)g(b, c) + f(b, c)g(a, c)] + [f(a, d)g(b, d) + f(b, d)g(a, d)]) \\
& + \frac{1}{(s+1)^2(s+2)^2} ([f(a, c)g(a, d) + f(b, c)g(b, d)] + [f(a, d)g(a, c) + f(b, d)g(b, c)]) \\
N(a, b, c, d) &= \frac{1}{(s+1)^2(s+2)^2} ([f(a, c)g(b, d) + f(b, c)g(a, d)] + [f(a, d)g(b, c) + f(b, d)g(a, c)])
\end{align*}
\]

**Proof.** Since \( f \) is co-ordinated convex and \( g \) is co-ordinated \( s \)-convex, from Lemma 1 and Lemma 2, the partial mappings

\[
\begin{align*}
f_y : [a, b] \to [0, \infty), \ f_y(x) &= f(x, y), \ y \in [c, d] \\
f_x : [c, d] \to [0, \infty), \ f_x(y) &= f(x, y), \ x \in [a, b]
\end{align*}
\]

are convex on \([a, b]\) and \([c, d]\), respectively, where \( x \in [a, b], \ y \in [c, d]. \) Similarly,

\[
\begin{align*}
g_y : [a, b] \to [0, \infty), \ g_y(x) &= g(x, y), \ y \in [c, d] \\
g_x : [c, d] \to [0, \infty), \ g_x(y) &= g(x, y), \ x \in [a, b]
\end{align*}
\]

are \( s \)-convex on \([a, b]\) and \([c, d]\), respectively, where \( x \in [a, b], \ y \in [c, d]. \)

Using (1.2), we can write

\[
\begin{align*}
\frac{1}{d-c} \int_c^d f_x(y)g_x(y)dy & \leq \frac{1}{s+2} [f_x(c)g_x(c) + f_x(d)g_x(d)] \\
& + \frac{1}{(s+1)(s+2)} [f_x(c)g_x(d) + f_x(d)g_x(c)]
\end{align*}
\]
That is

\[
\frac{1}{d-c} \int_c^d f(x, y)g(x, y)dy \leq \frac{1}{s+2} [f(x, c)g(x, c) + f(x, d)g(x, d)] \\
+ \frac{1}{(s+1)(s+2)} [f(x, c)g(x, d) + f(x, d)g(x, c)]
\]

Dividing both sides by \((b-a)\) and integrating over \([a, b]\), we get

\[
(2.2) \quad \frac{1}{(d-c)(b-a)} \int_a^b \int_c^d f(x, y)g(x, y)dxdy
\leq \frac{1}{s+2} \left[ \frac{1}{b-a} \int_a^b f(x, c)g(x, c)dx + \frac{1}{b-a} \int_a^b f(x, d)g(x, d)dx \right] \\
+ \frac{1}{(s+1)(s+2)} \left[ \frac{1}{b-a} \int_a^b f(x, c)g(x, d)dx + \frac{1}{b-a} \int_a^b f(x, d)g(x, c)dx \right]
\]

By applying (1.2) to each term of right hand side of above inequality, we have

\[
\frac{1}{b-a} \int_a^b f(x, c)g(x, c)dx \leq \frac{1}{s+2} [f(a, c)g(a, c) + f(b, c)g(b, c)] \\
+ \frac{1}{(s+1)(s+2)} [f(a, c)g(b, c) + f(b, c)g(a, c)]
\]

\[
\frac{1}{b-a} \int_a^b f(x, d)g(x, d)dx \leq \frac{1}{s+2} [f(a, d)g(a, d) + f(b, d)g(b, d)] \\
+ \frac{1}{(s+1)(s+2)} [f(a, d)g(b, d) + f(b, d)g(a, d)]
\]

\[
\frac{1}{b-a} \int_a^b f(x, c)g(x, d)dx \leq \frac{1}{s+2} [f(a, c)g(a, d) + f(b, c)g(b, d)] \\
+ \frac{1}{(s+1)(s+2)} [f(a, c)g(b, d) + f(b, c)g(a, d)]
\]

\[
\frac{1}{b-a} \int_a^b f(x, d)g(x, c)dx \leq \frac{1}{s+2} [f(a, d)g(a, c) + f(b, d)g(b, c)] \\
+ \frac{1}{(s+1)(s+2)} [f(a, d)g(b, c) + f(b, d)g(a, c)]
\]
Using these inequalities in (2.2), (2.1) is proved, that is

$$\frac{1}{(d-c)(b-a)} \int_a^b \int_c^d f(x, y)g(x, y)dx dy$$

\[\leq \frac{1}{(s+2)^2} \left( \left[ f(a, c)g(a, c) + f(b, c)g(b, c) \right] + \left[ f(a, d)g(a, d) + f(b, d)g(b, d) \right] \right) + \frac{1}{(s+1)(s+2)} \left( \left[ f(a, c)g(b, c) + f(b, c)g(a, c) \right] + \left[ f(a, d)g(b, d) + f(b, d)g(a, d) \right] \right) + \frac{1}{(s+2)^2(s+2)^2} \left( \left[ f(a, c)g(b, d) + f(b, c)g(a, d) \right] + \left[ f(a, d)g(b, c) + f(b, d)g(a, c) \right] \right)\]

We can find the same result using by \( f_y(x) \) and \( g_y(x) \).

\[\square\]

**Remark 1.** In (2.1), if we choose \( s = 1 \), (1.6) is obtained.

**Remark 2.** In (2.1), if we choose \( s = 1 \) and \( f(x) = 1 \) which is convex, we get the second inequality in (1.7):

$$\frac{1}{(d-c)(b-a)} \int_a^b \int_c^d g(x, y)dx dy \leq \frac{g(a, c) + g(b, c) + g(a, d) + g(b, d)}{4}$$

In the next theorem we will also make use of the Beta function of Euler type, which is for \( x, y > 0 \) defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and the Gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \text{ for } x > 0.$$ 

**Theorem 8.** Let \( f : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty) \) be \( s_1 \)-convex function on the co-ordinates and \( g : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty) \) be \( s_2 \)-convex functions on the co-ordinates with \( a < b, c < d \) and \( f_x(y)g_x(y), f_y(x)g_y(x) \in L^1[\Delta] \)
for some fixed \(s_1, s_2 \in (0, 1]\). Then one has the inequality:

\[
(2.3) \quad \frac{1}{(d-c)(b-a)} \int_{a}^{b} \int_{c}^{d} f(x, y)g(x, y)dx
dy \\
\leq \frac{1}{(s_1 + s_2 + 1)^2} [L(a, b, c, d) + B(s_1 + 1, s_2 + 1)]^2 M(a, b, c, d) \\
+ [B(s_1 + 1, s_2 + 1)]^2 N(a, b, c, d) \\
= \frac{1}{(s_1 + s_2 + 1)^2} \left[ L(a, b, c, d) + \frac{s_1 s_2 \Gamma(s_1) \Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} M(a, b, c, d) \\
+ \left[ \frac{s_1 s_2 \Gamma(s_1) \Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} \right]^2 N(a, b, c, d) \right]
\]

where

\[L(a, b, c, d) = [f(a, c)g(a, c) + f(b, c)g(b, c) + f(a, d)g(a, d) + f(b, d)g(b, d)]\]

\[M(a, b, c, d) = [f(a, c)g(b, c) + f(b, c)g(a, c) + f(a, d)g(b, d) + f(b, d)g(a, d)] + [f(a, c)g(a, d) + f(b, c)g(b, d) + f(a, d)g(a, c) + f(b, d)g(b, c)]\]

\[N(a, b, c, d) = [f(a, c)g(b, d) + f(b, c)g(a, d) + f(a, d)g(b, c) + f(b, d)g(a, c)]\]

**Proof.** Since \(f\) is co-ordinated \(s_1\)–convex and \(g\) is co-ordinated \(s_2\)–convex, from Lemma 2, the partial mappings

\[f_y : [a, b] \rightarrow [0, \infty), \quad f_y(x) = f(x, y)\]

\[f_x : [c, d] \rightarrow [0, \infty), \quad f_x(y) = f(x, y)\]

are \(s_1\)–convex on \([a, b]\) and \([c, d]\), respectively, where \(x \in [a, b], y \in [c, d]\). Similarly,

\[g_y : [a, b] \rightarrow [0, \infty), \quad g_y(x) = g(x, y)\]

\[g_x : [c, d] \rightarrow [0, \infty), \quad g_x(y) = g(x, y)\]

are \(s_2\)–convex on \([a, b]\) and \([c, d]\), respectively, where \(x \in [a, b], y \in [c, d]\).

Using (1.3), we get

\[
\frac{1}{d-c} \int_{c}^{d} f_x(y)g_x(y)dy \leq \frac{1}{s_1 + s_2 + 1} \left[ f_x(c)g_x(c) + f_x(d)g_x(d) \right] \\
+ B(s_1 + 1, s_2 + 1) \left[ f_x(c)g_x(d) + f_x(d)g_x(c) \right]
\]

Therefore

\[
\frac{1}{d-c} \int_{c}^{d} f(x, y)g(x, y)dy \leq \frac{1}{s_1 + s_2 + 1} \left[ f(x, c)g(x, c) + f(x, d)g(x, d) \right] \\
+ B(s_1 + 1, s_2 + 1) \left[ f(x, c)g(x, d) + f(x, d)g(x, c) \right]
\]
Dividing both sides of the above inequality by \((b - a)\) and integrating over \([a, b]\), we have

\[
\frac{1}{(d - c)(b - a)} \int_a^b d \int_c^d f(x, y)g(x, y)dxdy
\]

\[
\leq \frac{1}{s_1 + s_2 + 1} \left[ \frac{1}{b - a} \int_a^b f(x, c)g(x, c)dx + \frac{1}{b - a} \int_a^b f(x, d)g(x, d)dx \right]
\]

\[
+ B(s_1 + 1, s_2 + 1) \left[ \frac{1}{b - a} \int_a^b f(x, c)g(x, d)dx + \frac{1}{b - a} \int_a^b f(x, d)g(x, c)dx \right]
\]

By applying (1.3) to right side of (2.4), and we proceed similarly as in the proof of Theorem 7, we can write

\[
\frac{1}{(d - c)(b - a)} \int_a^b d \int_c^d f(x, y)g(x, y)dxdy
\]

\[
\leq \frac{1}{(s_1 + s_2 + 1)^2} \left[ f(a, c)g(a, c) + f(b, c)g(b, c) + f(a, d)g(a, d) + f(b, d)g(b, d) \right]
\]

\[
+ B(s_1 + 1, s_2 + 1) \frac{1}{s_1 + s_2 + 1} \left[ f(a, c)g(b, c) + f(b, c)g(a, c) + f(a, d)g(b, d) + f(b, d)g(a, d) \right]
\]

\[
+ B(s_1 + 1, s_2 + 1) \frac{1}{s_1 + s_2 + 1} \left[ f(a, c)g(b, d) + f(b, c)g(a, d) + f(a, d)g(b, c) + f(b, d)g(a, c) \right]
\]

That is;

\[
\frac{1}{(d - c)(b - a)} \int_a^b d \int_c^d f(x, y)g(x, y)dxdy
\]

\[
\leq \frac{1}{(s_1 + s_2 + 1)^2} \left[ L(a, b, c, d) + \frac{B(s_1 + 1, s_2 + 1)}{s_1 + s_2 + 1} M(a, b, c, d) \right.
\]

\[
+ \left[ B(s_1 + 1, s_2 + 1) \right]^2 N(a, b, c, d)
\]

\[
= \frac{1}{(s_1 + s_2 + 1)^2} \left[ L(a, b, c, d) + \frac{s_1 s_2 \Gamma(s_1) \Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} M(a, b, c, d) \right.
\]

\[
+ \left[ s_1 s_2 \Gamma(s_1) \Gamma(s_2) \right]^2 \left[ \frac{1}{\Gamma(s_1 + s_2 + 1)} \right] N(a, b, c, d)
\]

which completes the proof.

\[\square\]

**Remark 3.** In (2.3) if we choose \(s_1 = s_2 = 1\), (2.3) reduces to (1.6).

**Theorem 9.** Let \(f : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty)\) be convex function on the co-ordinates and \(g : \Delta := [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty)\) be \(s\)-convex function on the co-ordinates with \(a < b, c < d\) and \(f_x(y)g_x(y), f_y(x)g_y(x) \in L^1(\Delta)\) for some
fixed $s \in (0, 1]$. Then one has the inequality:

$$2^{2s+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dx dy$$

$$+ \frac{5}{(s+1)(s+2)^2} L(a, b, c, d) + \frac{2s^2 + 6s + 6}{(s+1)^2(s+2)^2} M(a, b, c, d)$$

$$+ \frac{2s + 6}{(s+1)(s+2)^2} N(a, b, c, d)$$

Proof. Since $f$ is co-ordinated convex and $g$ is co-ordinated $s$–convex, from Lemma 1 and Lemma 2, the partial mappings

$$f_y : [a, b] \to [0, \infty), \quad f_y(x) = f(x, y)$$
$$f_x : [c, d] \to [0, \infty), \quad f_x(y) = f(x, y)$$

are convex on $[a, b]$ and $[c, d]$, respectively, where $x \in [a, b], y \in [c, d]$. Similarly;

$$g_y : [a, b] \to [0, \infty), \quad g_y(x) = g(x, y), \quad y \in [c, d]$$
$$g_x : [c, d] \to [0, \infty), \quad g_x(y) = g(x, y), \quad x \in [a, b]$$

are $s$–convex on $[a, b]$ and $[c, d]$, respectively, where $x \in [a, b], y \in [c, d]$.

Using (1.4) and multiplying both sides of the inequalities by $2^s$, we get

$$2^{2s} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$- \frac{2^s}{b-a} \int_a^b f(x, \frac{c+d}{2}) g(x, \frac{c+d}{2}) dx$$

$$\leq \frac{2^s}{(s+1)(s+2)} \left[ f(a, \frac{c+d}{2}) g(a, \frac{c+d}{2}) + f(b, \frac{c+d}{2}) g(b, \frac{c+d}{2}) \right]$$

$$+ \frac{2^s}{s+2} \left[ f(a, \frac{c+d}{2}) g(b, \frac{c+d}{2}) + f(b, \frac{c+d}{2}) g(a, \frac{c+d}{2}) \right]$$

and

$$2^{2s} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$- \frac{2^s}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy$$

$$\leq \frac{2^s}{(s+1)(s+2)} \left[ f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, d\right) \right]$$

$$+ \frac{2^s}{s+2} \left[ f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, d\right) + f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right) \right]$$
Now, on adding (2.7) and (2.8), we get

\[
2^{s+1} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) g \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
- \frac{2^s}{b - a} \int_a^b f \left( \frac{c + d}{2}, \frac{c + d}{2} \right) dx - \frac{2^s}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) g \left( \frac{a + b}{2}, y \right) dy
\]
\[2^s f\left(\frac{a+b}{2}, c\right)g\left(\frac{a+b}{2}, c\right)\]
\[\leq \frac{1}{b-a} \int_a^b f(x, c)g(x, c)dx + \frac{1}{(s+1)(s+2)} [f(a, c)g(a, c) + f(b, c)g(b, c)]\]
\[+ \frac{1}{s+2} [f(a, c)g(b, c) + f(b, c)g(c, a)]\]

\[2^s f\left(\frac{a+b}{2}, d\right)g\left(\frac{a+b}{2}, d\right)\]
\[\leq \frac{1}{b-a} \int_a^b f(x, d)g(x, d)dx + \frac{1}{(s+1)(s+2)} [f(a, d)g(a, d) + f(b, d)g(b, d)]\]
\[+ \frac{1}{s+2} [f(a, d)g(b, d) + f(b, d)g(a, d)]\]

\[2^s f\left(\frac{a+b}{2}, c\right)g\left(\frac{a+b}{2}, d\right)\]
\[\leq \frac{1}{b-a} \int_a^b f(x, c)g(x, d)dx + \frac{1}{(s+1)(s+2)} [f(a, c)g(a, d) + f(b, c)g(b, d)]\]
\[+ \frac{1}{s+2} [f(a, c)g(b, d) + f(b, c)g(a, d)]\]

\[2^s f\left(\frac{a+b}{2}, d\right)g\left(\frac{a+b}{2}, c\right)\]
\[\leq \frac{1}{b-a} \int_a^b f(x, d)g(x, c)dx + \frac{1}{(s+1)(s+2)} [f(a, d)g(a, c) + f(b, d)g(b, c)]\]
\[+ \frac{1}{s+2} [f(a, d)g(b, c) + f(b, d)g(a, c)]\]
Using these inequalities in (2.9), we have

\[
2^{2s+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)
\]
\[
- \frac{2^s}{b-a} \int_{a}^{b} f(x, \frac{c+d}{2}) g(x, \frac{c+d}{2}) dx - \frac{2^s}{d-c} \int_{c}^{d} f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy
\]
\[
\leq \frac{1}{(s+1)(s+2)} \left\{ \int_{c}^{d} f(a, y) g(a, y) dy + \int_{c}^{d} f(b, y) g(b, y) dy \right\}
\]
\[
+ \frac{1}{(s+2)(d-c)} \left[ \int_{c}^{d} f(a, y) g(b, y) dy + \int_{c}^{d} f(b, y) g(a, y) dy \right]
\]
\[
+ \frac{1}{(s+1)(s+2)} \left[ \int_{a}^{b} f(x, c) g(x, c) dx + \int_{a}^{b} f(x, d) g(x, d) dx \right]
\]
\[
+ \frac{1}{(s+2)} \left[ \int_{a}^{b} f(x, y) g(x, d) dx + \int_{a}^{b} f(x, y) g(x, c) dx \right]
\]
\[
+ \frac{2}{(s+1)^2(s+2)^2} L(a, b, c, d) + \frac{2}{(s+1)(s+2)} M(a, b, c, d)
\]
\[
+ \frac{2}{(s+2)^{2}} N(a, b, c, d)
\]
Similarly by applying (1.4) to $2^s f(x, \frac{c+d}{2})g(x, \frac{c+d}{2})$, integrating over $[a, b]$, dividing both sides by $(b-a)$, we get

(2.12) \[
\frac{2^s}{(b-a)} \int_a^b f(x, \frac{c+d}{2})g(x, \frac{c+d}{2})dx
\]

\[- \frac{1}{(b-a)(d-c)} \int_a^d \int_c^d f(x,y)g(x,y)dxdy\]

\[\leq \frac{1}{(s+1)(s+2)} \left[ \frac{1}{(b-a)} \int_a^b f(x,c)g(x,c)dx + \frac{1}{(b-a)} \int_a^b f(x,d)g(x,d)dx \right] \]

\[+ \frac{1}{s+2} \left[ \frac{1}{(b-a)} \int_a^b f(x,c)g(x,d)dx + \frac{1}{(b-a)} \int_a^b f(x,d)g(x,c)dx \right] \]

By addition (2.11) and (2.12), we have

(2.13) \[
\frac{2^s}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right)g\left(\frac{a+b}{2}, y\right)dy + \frac{2^s}{(b-a)} \int_a^b f\left(\frac{c+d}{2}, y\right)g\left(\frac{c+d}{2}, y\right)dy
\]

\[\leq \frac{1}{(s+1)(s+2)} \left[ \frac{1}{(d-c)} \int_c^d f(a,y)g(a,y)dy + \frac{1}{(d-c)} \int_c^d f(b,y)g(b,y)dy \right] \]

\[+ \frac{1}{(b-a)} \int_a^b f(x,c)g(x,c)dx + \frac{1}{(b-a)} \int_a^b f(x,d)g(x,d)dx \]

\[+ \frac{1}{s+2} \left[ \frac{1}{(d-c)} \int_c^d f(a,y)g(b,y)dy + \frac{1}{(d-c)} \int_c^d f(b,y)g(a,y)dy \right] \]

\[+ \frac{1}{(b-a)} \int_a^b f(x,c)g(x,d)dx + \frac{1}{(b-a)} \int_a^b f(x,d)g(x,c)dx \]

From (2.10) and (2.13) and simplifying we get

\[
2^{2s+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{2}{(b-a)(d-c)} \int_a^d \int_c^d f(x,y)g(x,y)dxdy
\]

\[+ \frac{4s+6}{(s+1)^2(s+2)^2} L(a,b,c,d) + \frac{2s^2 + 6s + 6}{(s+1)^2(s+2)^2} M(a,b,c,d)
\]

\[+ \frac{2s^2 + 8s + 6}{(s+1)^2(s+2)^2} N(a,b,c,d)
\]

\[\square\]
Remark 4. In (2.5), if we choose $s = 1$, we obtained (1.7).

Remark 5. In (2.5), if we choose $s = 1$ and $f(x) = 1$ which is convex, we have the following Hadamard-type inequality like (1.5):

$$4g \left( \frac{a + b}{2}, \frac{c + d}{2} \right) - \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d g(x, y) dx$$

$\leq \frac{3 [g(a, c) + g(b, c) + g(a, d) + g(b, d)]}{4}$

Theorem 10. Let $f, g : \Delta := [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ be convex function on the co-ordinates with $a < b$, $c < d$ and $f_x(y)g_x(y)$, $f_y(x)g_y(x) \in L^1[\Delta]$. Then one has the inequality:

$$\frac{1}{(b - a)^2(d - c)^2} \left[ \int_a^b \int_c^d (x - b)(y - d)g(x, y) dy dx ight.$$

$$+ f(b, c) \int_a^b \int_c^d (a - x)(y - d)g(x, y) dy dx + f(a, d) \int_a^b \int_c^d (x - b)(c - y)g(x, y) dy dx$$

$$+ f(b, d) \int_a^b \int_c^d (a - x)(c - y)g(x, y) dy dx + g(a, c) \int_a^b \int_c^d (x - b)(y - d)f(x, y) dy dx$$

$$+ g(b, c) \int_a^b \int_c^d (a - x)(y - d)f(x, y) dy dx + g(a, d) \int_a^b \int_c^d (x - b)(c - y)f(x, y) dy dx$$

$$+ g(b, d) \int_a^b \int_c^d (a - x)(c - y)f(x, y) dy dx \left] \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)g(x, y) dy dx \right.$$

$$+ \frac{1}{9} L(a, b, c, d) + \frac{1}{18} M(a, b, c, d) + \frac{1}{36} N(a, b, c, d)$$

where $L(a, b, c, d)$, $M(a, b, c, d)$, $N(a, b, c, d)$ defined as in Theorem 6.

Proof. Since $f$ and $g$ are co-ordinated convex functions on the co-ordinates on $\Delta$, from the definition of co-ordinated convexity, we can write

$$f(ta + (1-t)b, sc + (1-s)d) \leq tsf(a, c) + t(1-s)f(a, d) + s(1-t)f(b, c) + (1-t)(1-s)f(b, d)$$

and

$$g(ta + (1-t)b, sc + (1-s)d) \leq tsg(a, c) + t(1-s)g(a, d) + s(1-t)g(b, c) + (1-t)(1-s)g(b, d)$$
holds for all \( t, s \in [0, 1] \). By using the elementary inequality, if \( e \leq f \) and \( p \leq r \), then \( er + fp \leq ep + fr \) for all \( e, f, p, r \in \mathbb{R} \), we get

\[
\begin{align*}
\int_0^1 \int_0^1 & f(ta + (1 - t)b, sc + (1 - s)d) \\
& \times \left[ tsg(a, c) + t(1 - s)g(a, d) + s(1 - t)g(b, c) + (1 - t)(1 - s)g(b, d) \right] \\
& + g(ta + (1 - t)b, sc + (1 - s)d) \\
& \times \left[ tsf(a, c) + t(1 - s)f(a, d) + s(1 - t)f(b, c) + (1 - t)(1 - s)f(b, d) \right] \\
\leq & \left[ f(ta + (1 - t)b, sc + (1 - s)d)g(ta + (1 - t)b, sc + (1 - s)d) \right] \\
& + \left[ tsf(a, c) + t(1 - s)f(a, d) + s(1 - t)f(b, c) + (1 - t)(1 - s)f(b, d) \right] \\
& \times \left[ tsg(a, c) + t(1 - s)g(a, d) + s(1 - t)g(b, c) + (1 - t)(1 - s)g(b, d) \right].
\end{align*}
\]

By integrating the above integral on \([0, 1] \times [0, 1]\), with respect to \( t, s \) and by taking into account the change of variables \( ta + (1 - t)b = x, (a - b)dt = dx \) and \( sc + (1 - s)d = y, (c - d)ds = dy \), we obtain the desired result. 

\[\square\]

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M. Emin Özdemir and Ahmet Ocak Akdemir

 Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Kampus, Erzurum, Turkey
 E-mail address: emos@atauni.edu.tr

 Ağrı İbrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, 04100, Ağrı, Turkey
 E-mail address: ahmetakdemir@agri.edu.tr