WIENER’S LEMMA FOR INFINITE MATRICES OF
GOHBERG-BASKAKOV-SJÖSTRAND CLASS

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Abstract. In this paper, we introduce a quasi-Banach algebra of infinite matrices, which is inverse-closed in the Banach algebra $B(ℓ^2)$ of all bounded operators on $ℓ^2$.

1. Introduction

N. Wiener showed that if $f$ is a periodic function with an absolutely convergent Fourier series and it vanishes nowhere on the real line, then $1/f$ has an absolutely convergent Fourier series too [23]. This is now called the classical Wiener’s lemma.

Define the Gohberg-Baskakov-Sjöstrand class $C(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$C(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ \left( a(i,j) \right)_{i,j \in \mathbb{Z}^d} : \sum_{k \in \mathbb{Z}^d} \left( \sup_{i-j=k} |a(i,j)| \right) < \infty \right\},$$

and the Wiener class $W(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$W(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ (a(i,j))_{i,j \in \mathbb{Z}^d} \in C(\mathbb{Z}^d, \mathbb{Z}^d) : a(i+k, j+k) = a(i,j) \text{ for all } i, j, k \in \mathbb{Z}^d \right\}.$$

Then the classical Wiener’s lemma can be reformulated as follows: $W(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $B(ℓ^2)$, the space of all bounded operators on the space $ℓ^2$ of square-summable sequences. Here a (quasi-)Banach algebra $\mathbb{B}$, which is a subalgebra of $\mathbb{A}$, is called inverse-closed if any $A \in \mathbb{B}$ with the inverse $A^{-1} \in \mathbb{A}$ implies $A^{-1} \in \mathbb{B}$.

Wiener’s lemma has various extensions and applications. Define Gröchenig-Schur class $S(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$S(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ (a(i,j))_{i,j \in \mathbb{Z}^d} : \sum_{k \in \mathbb{Z}^d} \left( \sup_{i-j=k} |a(i,j)| \right) < \infty \right\}.$$
\begin{equation}
\max \left( \sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |a(i,j)|, \sup_{j \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} |a(i,j)| \right) < \infty,
\end{equation}

and the Beurling class \( \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \) by
\begin{equation}
\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ \left( a(i,j) \right)_{i,j \in \mathbb{Z}^d} : \sum_{k \in \mathbb{Z}^d} \left( \sup_{|i-j| \geq |k|} |a(i,j)| \right) < \infty \right\},
\end{equation}
where we set \(|x| = \max(|x_1|, \ldots, |x_d|)\) for \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\). The Gröchenig-Schur class is not inverse-closed in \( \mathcal{B}(\ell^2) \) but the weighted Gröchenig-Schur class is when the weight satisfies the GRS-condition \([2, 6, 7, 9, 17, 19, 22]\). The Gohberg-Baskakov-Sjöstrand class \( \mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) \) and the Beurling class \( \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \) are inverse-closed in \( \mathcal{B}(\ell^2) \) \([3, 19, 20]\). The inverse-closed property has important applications in dual wavelet frames, dual Gabor frames, and algebra of pseudo-differential operators \([1, 5, 8, 10, 12, 13, 18, 21]\). The reader may refer to \([6, 11, 16]\) and references therein for historical remarks, recent advances and applications.

For \(0 < q \leq 1\) and a weight \(w\), Motee and Sun considered the Gröchenig-Schur class \( S_{q,w}(G) \) on a graph \(G\),
\begin{equation}
S_{q,w}(G) = \{ A = (a(i,j))_{i,j \in G} : \|A\|_{S_{q,w}} < \infty \},
\end{equation}
where
\begin{equation}
\|A\|_{S_{q,w}} := \max \left\{ \left( \sup_{i \in G} \sum_{j \in G} |(a(i,j))^q w(i,j)^q \right)^{1/q}, \right.
\end{equation}
\begin{equation}
\left. \left( \sup_{j \in G} \sum_{i \in G} |(a(i,j))^q w(i,j)^q \right)^{1/q} \right\}.
\end{equation}
The above class \( S_{q,w}(G) \) of matrices catches sparsity and localization of infinite matrices simultaneously. It does not form a Banach algebra, but it is a quasi-Banach algebra. More importantly, it is an inverse-closed subalgebra of \( \mathcal{B}(\ell^2) \) under proper assumption on the weight \(w\).

In this paper, we consider a general index set \( \Lambda \subset \mathbb{R}^d \) satisfying
\begin{equation}
\alpha = \sup_{k \in \mathbb{Z}^d} \sum_{\lambda \in \Lambda} \chi_{k+[-2,2]^d}(\lambda) < \infty.
\end{equation}
Unlike the index set \( \mathbb{Z}^d \) in \((1.1)\), our index set \( \Lambda \) may not form a group. The prime models are paraboloids
\[(x, y, z) : z = ax^2 + by^2, x, y \in \mathbb{Z}\]
and elliptical hyperboloids
\[(x, y, z) : z^2 = ax^2 + by^2, x, y \in \mathbb{Z},\]
where \(a, b > 0\). For \(0 < q \leq 1\) and a weight \(w\), we define the Gohberg-Baskakov-Sjöstrand class \( \mathcal{C}_{q,w} \), GBS class for short, on \( \Lambda \) by
\begin{equation}
\mathcal{C}_{q,w} = \{ A = (a(i,j))_{i,j \in \Lambda} : \|A\|_{\mathcal{C}_{q,w}} < \infty \},
\end{equation}
In this paper, we prove that $C_{q,w}, 0 < q \leq 1$, are inverse-closed quasi-Banach algebras of $B(\ell^2)$ under proper hypotheses on the weight $w$.

2. Quasi-Banach algebras

Let $\Lambda \subset \mathbb{R}^d$ satisfy (1.5). Then for any integer $k$

\[(2.1) \quad \max \left\{ \sup_{\lambda' \in \Lambda} \chi_{k+(-1,1)}(\lambda - \lambda'), \sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} \chi_{k+(-1,1)}(\lambda - \lambda') \right\} \leq \alpha.\]

We say that $w$ is a weight if

\[(2.2) \quad w(\lambda, \lambda') \geq 1 \quad \text{for any } \lambda, \lambda' \in \Lambda,\]

\[(2.3) \quad w(\lambda, \lambda') = w(\lambda', \lambda) \quad \text{for any } \lambda, \lambda' \in \Lambda,\]

and

\[(2.4) \quad \sup_{\lambda \in \Lambda} w(\lambda, \lambda) < \infty.\]

The Gohberg-Baskakov-Sjöstrand class $C_{q,w}$ of infinite matrices has the following basic properties.

**Proposition 2.1.** Let $0 < q \leq 1$ and $w$ be a weight.

(i) If $A \in C_{q,w}$, then $cA \in C_{q,w}$ for any $c \in \mathbb{R}$ and $\|cA\|_{C_{q,w}} = |c|\|A\|_{C_{q,w}}$.

(ii) If $A \in C_{q,w}$, then $\|A\|_{B(\ell^2)} \leq \|A\|_{C_{q,w}}$, so $A \in B(\ell^2)$.

(iii) For $A, B \in C_{q,w}$, $\|A + B\|_{C_{q,w}}^q \leq \|A\|_{C_{q,w}}^q + \|B\|_{C_{q,w}}^q$, so $A + B \in C_{q,w}$.

(iv) If there exists a positive constant $C_0$ such that

\[(2.5) \quad w(\lambda, \lambda') \leq C_0 w(\tilde{\lambda}, \tilde{\lambda})w(\lambda', \lambda') \quad \text{for all } \lambda, \lambda, \tilde{\lambda} \in \Lambda,\]

then for any $A, B \in C_{q,w}$

\[(2.6) \quad \|AB\|_{C_{q,w}}^q \leq 2^d C_0 \|A\|_{C_{q,w}}^{q/2} \|B\|_{C_{q,w}}^{q/2}.\]

**Proof.** (i) Trivial.

(ii) It is well known that

$$\|A\|_{B(\ell^2)} \leq \max \left( \sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |a(\lambda, \lambda')|, \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |a(\lambda, \lambda')| \right).$$

Since $(a + b)^q \leq a^q + b^q$ for any $a, b \geq 0$, and $w(\lambda, \lambda') \geq 1$ for any $\lambda, \lambda' \in \Lambda$, we have that

$$\sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |a(\lambda, \lambda')| \leq \left( \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')|^q \chi_{k+[0,1]^d}(\lambda - \lambda') \right)^{1/q}$$

$$\leq \left( \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k+[0,1]^d}(\lambda - \lambda') \right)^{1/q}$$
Similarly
\[
\sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |a(\lambda, \lambda')| \leq \|A\|_{C_{q,w}}.
\]

(iii) For \(A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}\) and \(B = (b(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}\),
\[
\|A + B\|_{C_{q,w}}^q = \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda') + b(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k + \{0,1\}^d}(\lambda - \lambda')
\leq \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k + \{0,1\}^d}(\lambda - \lambda')
+ \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |b(\lambda, \lambda')|^q w(\lambda, \lambda') \chi_{k + \{0,1\}^d}(\lambda - \lambda')
= \|A\|_{C_{q,w}}^q + \|B\|_{C_{q,w}}^q.
\]

(iv) Let \(A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} \in C_{q,w}, B = (b(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} \in C_{q,w}\) and \(C = AB\). Then
\[
C = \left( \sum_{\lambda \in \Lambda} a(\lambda, \tilde{\lambda})b(\tilde{\lambda}, \lambda') \right)_{\lambda, \lambda' \in \Lambda}.
\]

Observe that if \(\lambda - \lambda' \in k + \{0,1\}^d\) and \(\lambda - \tilde{\lambda} \in \ell + \{0,1\}^d\) for some \(k, \ell \in \mathbb{Z}^d\),
then \(\lambda - \lambda' \in k - \ell + (-1,1)^d\). Then we have from (2.1) that
\[
\|C\|_{C_{q,w}}^q = \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} \left| \sum_{\lambda \in \Lambda} a(\lambda, \tilde{\lambda})b(\tilde{\lambda}, \lambda') \right|^q w(\lambda, \lambda') \chi_{k + \{0,1\}^d}(\lambda - \lambda')
\leq C_0^q \alpha^2 \sum_{k \in \mathbb{Z}^d} \sum_{\lambda, \lambda' \in \Lambda} (\sup_{\lambda, \lambda' \in \Lambda} \left| a(\lambda, \tilde{\lambda}) \right|^q w(\lambda, \tilde{\lambda}) \chi_{k + \{0,1\}^d}(\lambda - \tilde{\lambda})
\times ( \sup_{\lambda, \lambda' \in \Lambda} \left| b(\tilde{\lambda}, \lambda') \right|^q w(\tilde{\lambda}, \lambda') \chi_{k - \ell + (-1,1)^d}(\tilde{\lambda} - \lambda'))
\leq 2^d C_0^q \|A\|_{C_{q,w}}^q \|B\|_{C_{q,w}}^q.
\]
This proves the conclusion (iv). \(\square\)

By Proposition 2.1, there exists a positive constant \(K\) such that
\[
\|A + B\|_{C_{q,w}} \leq K(\|A\|_{C_{q,w}} + \|B\|_{C_{q,w}}) \quad \text{for all } A, B \in C_{q,w}.
\]
So \(\|\cdot\|_{C_{q,w}}\) is a quasi-norm [4, 14]. Therefore \((C_{q,w}, \|\cdot\|_{C_{q,w}})\) forms a quasi-Banach algebra by Proposition 2.1.

**Corollary 2.2.** Let \(0 \leq q \leq 1\). Assume that \(w\) is a weight satisfying the submultiplicative condition (2.5). Then \(C_{q,w}\) is a quasi-Banach algebra.
3. Wiener’s lemma

In this section, we will show that $C_{q,w}$ is an inverse-closed subalgebra of $B(ℓ^2)$. To do it, we first establish paracompact estimate for matrices in $C_{q,w}$.

Let $w$ be a weight. A weight $u$ is called a companion matrix of $w$ if

$$w(\lambda, \lambda') = w(\lambda, \tilde{\lambda})u(\tilde{\lambda}, \lambda') + u(\lambda, \tilde{\lambda})w(\tilde{\lambda}, \lambda') \quad \text{for all } \lambda, \lambda', \tilde{\lambda} \in \Lambda.$$  \hfill (3.1)

**Proposition 3.1.** Let $0 < q < 1$, $w$ be a weight, and $u$ be a companion weight of $w$. We assume that there exist a positive constant $C_1$ and $0 < \theta < 1$ such that

$$\inf_{t \geq 0} \left\{ \alpha \sum_{|k| \leq t+1, \lambda, \lambda' \in \Lambda} \sup_{\tilde{\lambda}} u(\tilde{\lambda} - \lambda')\chi_{k+[0,1)^q}(\tilde{\lambda} - \lambda') + t \sup_{|\tilde{\lambda} - \lambda'| > t} u(\tilde{\lambda}, \lambda') \right\} \leq C_1 t^\theta$$

for all $t \geq 1$. Then there exists a positive constant $C_2$ such that for any $A, B \in C_{q,w}$

$$\|AB\|_{C_{q,w}}^q \leq C_2\|A\|_{C_{q,w}}^q\|B\|_{C_{q,w}}^q \left( \left( \frac{\|A\|_{B(ℓ^2)}}{\|A\|_{C_{q,w}}} \right)^{q(1-\theta)} + \left( \frac{\|B\|_{B(ℓ^2)}}{\|B\|_{C_{q,w}}} \right)^{q(1-\theta)} \right).$$

**Proof.** Take $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}$ and $B = (b(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}$. Then

$$AB = \left( \sum_{\lambda \in \Lambda} a(\lambda, \tilde{\lambda})b(\tilde{\lambda}, \lambda') \right)_{\lambda, \lambda' \in \Lambda}.$$

We obtain from (3.1) that

$$\|AB\|_{C_{q,w}}^q \leq \alpha \sum_{k \in \mathbb{Z}^d, \lambda, \lambda' \in \Lambda} \sup_{\tilde{\lambda} \in \Lambda} \sum_{\lambda \in \Lambda} |a(\lambda, \tilde{\lambda})|^q |b(\tilde{\lambda}, \lambda')|^q w(\lambda, \lambda')\chi_{k+[0,1)^q}(\lambda - \lambda')$$

$$\leq \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\tilde{\lambda}} \left( \sum_{\lambda \in \Lambda} |a(\lambda, \tilde{\lambda})|^q w(\lambda, \tilde{\lambda}) |b(\tilde{\lambda}, \lambda')|^q u(\tilde{\lambda}, \lambda')\chi_{k+[0,1)^q}(\lambda - \lambda') \right) + \sum_{\lambda \in \Lambda} |a(\lambda, \tilde{\lambda})|^q u(\tilde{\lambda}, \lambda') |b(\tilde{\lambda}, \lambda')|^q w(\lambda, \lambda')\chi_{k+[0,1)^q}(\lambda - \lambda')$$

$$=: I_1 + I_2.$$

Let $r \geq 0$. Since for any $\lambda, \lambda' \in \Lambda$, $|b(\tilde{\lambda}, \lambda')| \leq \|B\|_{B(ℓ^2)}$, we have that

$$I_1 \leq \alpha \|B\|_{B(ℓ^2)}^q \sum_{k \in \mathbb{Z}^d} \sup_{\tilde{\lambda}} \left( \sum_{\lambda \in \Lambda} |a(\lambda, \tilde{\lambda})|^q w(\lambda, \tilde{\lambda}) u(\tilde{\lambda}, \lambda')\chi_{k+[0,1)^q}(\lambda - \lambda') \right)$$

$$+ \alpha \sum_{k \in \mathbb{Z}^d} \sup_{\tilde{\lambda}} \left( \sum_{|\tilde{\lambda} - \lambda'| > r} |a(\lambda, \tilde{\lambda})|^q w(\lambda, \tilde{\lambda}) |b(\tilde{\lambda}, \lambda')|^q u(\tilde{\lambda}, \lambda')\chi_{k+[0,1)^q}(\lambda - \lambda') \right)$$

$$\leq \alpha^2 \|B\|_{B(ℓ^2)}^q \sum_{k \in \mathbb{Z}^d} \sum_{\tilde{\lambda}, \lambda \in \Lambda} |a(\lambda, \tilde{\lambda})|^q w(\lambda, \tilde{\lambda})\chi_{k-r+(-1,1)^q}(\lambda - \tilde{\lambda}).$$
This together with (3.2) implies that

\begin{align*}
I_1 & \leq 2^d \|A\|_{C_{q,w}}^q \|B\|_{B(L^2)}^q \sup_{|\tilde{\lambda} - \lambda' - \tau| \leq \tau} u(\tilde{\lambda}, \lambda') + (\tilde{\lambda} - \lambda')^{q(1 - \theta)} \\
I_2 & \leq 2^d \|A\|_{C_{q,w}}^q \|B\|_{B(L^2)}^q \sup_{|\tilde{\lambda} - \lambda' - \tau| > \tau} u(\tilde{\lambda}, \lambda') + (\tilde{\lambda} - \lambda')^{q(1 - \theta)} .
\end{align*}

Combining (3.4), (3.5) and (3.6) proves (3.3).

We remark that the assumption (3.2) on a weight $u$ is stronger than the submultiplicative condition (2.5). In fact, putting $\tau = 0$ and $t = 1$ in (3.2), we get

\begin{equation}
(3.7) \quad u(\lambda, \lambda') \leq C_1 w(\lambda, \lambda') \quad \text{for any } \lambda, \lambda' \in \Lambda.
\end{equation}

This together with (3.1) implies that

\begin{equation}
(3.8) \quad u(\lambda, \lambda') \leq 2C_1 w(\lambda, \tilde{\lambda}) w(\tilde{\lambda}, \lambda') \quad \text{for any } \lambda, \lambda', \tilde{\lambda} \in \Lambda,
\end{equation}

and hence (2.5) holds with $C_0' = 2C_1$. Therefore

\begin{equation}
(3.9) \quad \|AB\|_{C_{q,w}}^q \leq 2^{d+1} C_1 \|A\|_{C_{q,w}}^q \|B\|_{C_{q,w}}^q \quad \text{for all } A, B \in C_{q,w}
\end{equation}

by Proposition 2.1.

To prove the inverse-closedness of $C_{q,w}$ in $B(L^2)$, we need estimate powers of a matrix $A$ in $C_{q,w}$.
Proposition 3.2. Under the assumptions of Proposition 3.1,
\[ \|A^n\|_{C_{q,w}} \leq (2^{d+1}C_1C_2)\log_2 n \left( \frac{\|A\|_{C_{q,w}}}{\|A\|_{B(\ell^2)}} \right)^q \left( 1 + \theta \right)^{n(1+\theta)} \|A\|_{B(\ell^2)}^{nq} \]
for all \( A \in C_{q,w} \) and integers \( n \geq 1 \).

Proof. Let \( A \in C_{q,w} \) and \( n \) be a positive integer. We write \( n = \sum_{j=0}^{N} \varepsilon_j 2^j \), where \( \varepsilon_N = 1 \) and \( \varepsilon_j \in \{0,1\} \). We put
\[ n_\ell = \varepsilon_\ell + 2n_{\ell+1} \text{ and } n_N = \varepsilon_N \text{ for } \ell = 0, \ldots, N-1. \]
Without loss of generality, we assume that \( \|A\|_{B(\ell^2)} = 1 \), otherwise replace \( A \) by \( A/\|A\|_{B(\ell^2)} \). Then setting \( A = B \) in (3.3) gives
\[ \|A^2\|_{C_{q,w}}^q \leq 2C_2 \|A\|_{C_{q,w}}^{q(1+\theta)} . \]
By (3.8), (3.10) and the observation that \( N \leq \log_2 n \), we have
\[ \sum_{k=0}^{N} \varepsilon_k (1+\theta)^k \leq 1 + (1+\theta) + \cdots + (1+\theta)^N \leq \frac{1+\theta}{\theta} (1+\theta)^N, \]
and
\[ \|A^n\|_{C_{q,w}}^q \leq 2^{d+1}C_1 \|A\|_{C_{q,w}}^{q_0} \|A\|_{C_{q,w}}^{2n_1} \|A\|_{C_{q,w}}^{(1+\theta)} \]
\[ \leq 2^{d+1}C_1 (2C_2) \|A\|_{C_{q,w}}^{q_0+q_1(1+\theta)} \|A\|_{C_{q,w}}^{2n_2} \|A\|_{C_{q,w}}^{(1+\theta)^2} \]
\[ \leq (2^{d+1}C_1)^2 (2C_2)^2 \|A\|_{C_{q,w}}^{q_0+q_1(1+\theta)} \|A\|_{C_{q,w}}^{2n_2} \|A\|_{C_{q,w}}^{(1+\theta)^2} \]
\[ \cdots \]
\[ \leq (2^{d+1}C_1)^N (2C_2)^N \|A\|_{C_{q,w}}^{q_0+q_1(1+\theta) + \cdots + q_N(1+\theta)^N} . \]
(3.11)
This proves (3.9).

Finally, we prove inverse-closedness of the subalgebra \( C_{q,w} \) in \( B(\ell^2) \).

Theorem 3.3. Let \( 0 < q < 1 \). Under the assumptions of Proposition 3.1, the quasi-Banach algebra \( C_{q,w} \) is inverse-closed in \( B(\ell^2) \), that is, if \( A \in C_{q,w} \) and \( A^{-1} \in B(\ell^2) \), then \( A^{-1} \in C_{q,w} \).

Proof. Let \( A \in C_{q,w} \) and \( A^{-1} \in B(\ell^2) \). We put \( B = I - \|A\|_{C_{q,w}}^{-2} A^T A \). Then \( \|B\|_{B(\ell^2)} \leq 1 - \|A\|_{C_{q,w}}^{-2} \|A\|_{B(\ell^2)}^{-2} \leq r_0 \), where \( r_0 = 1 - \|A\|_{C_{q,w}}^{-2} \|A^{-1}\|_{B(\ell^2)}^{-2} \in [0,1) \). Since from (3.9) \( \lim_{n \to \infty} \|B^n\|_{C_{q,w}}^{q/n} \leq r_0^q < 1 \), \( \sum_{n=1}^{\infty} \|B^n\|_{C_{q,w}}^{q} < \infty \). Observing that \( A^{-1} = (A^T A)^{-1} A^T = \|A\|_{C_{q,w}}^{-2} (I - B)^{-1} A^T \), we have that
\[ \|A^{-1}\|_{C_{q,w}}^{q} \leq \|A\|_{C_{q,w}}^{-q} (\|I\|_{C_{q,w}}^{q} + \sum_{n=1}^{\infty} \|B^n\|_{C_{q,w}}^{q}) < \infty, \]
and
where \( \| I \|_{\mathcal{C}_{q,w}} = \sup_{\lambda \in \Lambda} w(\lambda, \lambda) \). Hence \( A^{-1} \in \mathcal{C}_{q,w} \). \( \square \)

We conclude this section with remarks on polynomial weights and subexponential weights that satisfy \((3.1)\) and \((3.2)\).

**Remark 3.4.** For \( \alpha > 0 \), consider polynomial weights \( w_\alpha := (1 + |i - j|)^\alpha \), \( i, j \in \mathbb{Z}^d \).

The constant weight \( u_\alpha \) with \( u_\alpha(i, j) = 2^\alpha \) for any \( i, j \in \mathbb{Z}^d \) satisfies the companion weight condition \((3.1)\). Also,

\[
\begin{align*}
\inf_{\tau \geq 0} \left\{ \sum_{|k| \leq \tau + 1} \sup_{\lambda, \lambda' \in \Lambda} u(\lambda - \lambda') \chi_{k+[0,1]^d}(\lambda - \lambda') + t \sup_{|\lambda - \lambda'| > \tau} \frac{u(\lambda, \lambda')}{w(\lambda, \lambda')} \right\} \\
\leq \inf_{\tau \geq 0} \left\{ \sum_{|k| \leq \tau + 1} 2^\alpha + t \cdot 2^\alpha (1 + \tau)^{-\alpha} \right\} \\
\leq \inf_{\tau \geq 0} \left\{ 2^\alpha \left( (2\tau + 3)^d + t(1 + \tau)^{-\alpha} \right) \right\} \\
\leq \inf_{\tau \geq 0} \left\{ 2^{\alpha + 2d} \left( (\tau + 1)^d + t(1 + \tau)^{-\alpha} \right) \right\} \\
\leq 2^{\alpha + 2d + 1} \tau^d \text{ for all } \tau \geq 1,
\end{align*}
\]

where in the last inequality \( \tau \) satisfies the equation \((\tau + 1)^d = t(\tau + 1)^{-\alpha}\). Hence the polynomial weights \( w_\alpha, \alpha > 0 \), satisfy \((3.2)\).

Next, for \( D > 0 \) and \( 0 < \delta < 1 \), we consider the subexponential weight \( e_{D,\delta} = (e^{D|\cdot|}\delta)_{i,j \in \mathbb{Z}^d} \). The weight \( e_{D(2^\delta - 1),\delta}(i, j) := u(i, j) = e^{D(2^\delta - 1)|i - j|}\delta \) satisfies the companion weight condition \((3.1)\). Choose \( C' > 1 \) and \( \tau' > 0 \) such that \( C'(2^\delta - 1) < 1 \) and \( (\tau' + 1)^\delta < C' \) for \( \tau \geq \tau' \). If \( \tau' > (\log t/D)^{1/\delta} \), that is, \( t \) is bounded, then for \( 0 < \theta < 1 \), there exists \( C_1 > 0 \) such that for \( 1 \leq t < e^{D(\tau')^\delta} \),

\[
\begin{align*}
\inf_{\tau \geq 0} \left\{ \sum_{|k| \leq \tau + 1} \sup_{\lambda, \lambda' \in \Lambda} u(\lambda - \lambda') \chi_{k+[0,1]^d}(\lambda - \lambda') + t \sup_{|\lambda - \lambda'| > \tau} \frac{u(\lambda, \lambda')}{w(\lambda, \lambda')} \right\} \\
\leq \inf_{\tau \geq 0} \left\{ \sum_{|k| \leq \tau + 1} e^{D(2^\delta - 1)|k|\delta} + t \cdot e^{D(2^\delta - 2)|\tau|\delta} \right\} \leq C_1 t^\theta,
\end{align*}
\]

where we let \( \tau = 0 \) in the third equality.

For \( t \geq 1, \tau' \leq (\log t/D)^{1/\delta} \) and

\[
\begin{align*}
\inf_{\tau \geq 0} \left\{ \sum_{|k| \leq \tau + 1} \sup_{\lambda, \lambda' \in \Lambda} u(\lambda - \lambda') \chi_{k+[0,1]^d}(\lambda - \lambda') + t \sup_{|\lambda - \lambda'| > \tau} \frac{u(\lambda, \lambda')}{w(\lambda, \lambda')} \right\} \\
\leq \inf_{\tau \geq 0} \left\{ \sum_{|k| \leq \tau + 1} e^{D(2^\delta - 1)|k|\delta} + t \cdot e^{D(2^\delta - 2)|\tau|\delta} \right\} \\
\leq e^{C't(2^\delta - 1)|\tau|\delta}(2\tau + 3)^d + t^{2^\delta - 1} \\
\leq C'(2^\delta - 1)(2\tau + 3)^d + t^{2^\delta - 1} \\
\leq C'(2^\delta - 1)((2\tau + 3)^d + 1)
\end{align*}
\]
(3.14) \[ = t^{C'(2^\delta - 1)} ((3 + 2 \left(\frac{\ln f}{\delta}\right)^{1/\delta})^\delta + 1), \]

where in the third inequality \( \tau \geq \tau' \) satisfies the equation \( \tau^\delta = \frac{\ln f}{\delta} \). Hence for any \( \theta \) with \( C'(2^\delta - 1) < \theta < 1 \) there exists \( C_1 > 0 \) such that (3.2) holds. Combining (3.12) and (3.14) proves (3.2).

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References

[1] R. Balan, P. G. Casazza, C. Heil and Z. Landau, Density, overcompleteness and localizations of frames I. theory, II. Gabor System, J. Fourier Anal. Appl. 12 (2006), no. 2, 105–143; no. 3, 309–344.
[2] B. A. Barnes, The spectrum of integral operators on Lebesgue spaces, J. Operator Theory 18 (1987), no. 1, 115–132.
[3] A. G. Baskakov, Wiener’s theorem and asymptotic estimates for elements of inverse matrices, Funktsional. Anal. i Prilozhen 24 (1990), no. 3, 64–65; translation in Funct. Anal. Appl. 24 (1990), no. 3, 222–224.
[4] M. E. Gordji and M. B. Savadkouhi, Approximation of generalized homomorphisms in quasi-Banach algebras, An. St. Univ. Ovidius Constanta 17 (2009), 203–214.
[5] K. Gröchenig, Time-frequency analysis of Sjöstrand’s class, Rev. Mat. Iberoam. 22 (2006), no. 2, 703–724.
[6] , Wiener’s lemma: theme and variations, an introduction to spectral invariance and its applications, In Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis, pp. 175–233, Edited by P. Massopust and B. Forster, Birkhauser, Boston 2010.
[7] K. Gröchenig and A. Klotz, Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices, Constr. Approx. 32 (2010), no. 3, 429–466.
[8] K. Gröchenig and M. Leinert, Wiener’s lemma for twisted convolution and Gabor frames, J. Amer. Math. Soc. 17 (2003), 1–18.
[9] , Symmetry of matrix algebras and symbolic calculus for infinite matrices, Trans. Amer. Math. Soc. 358 (2006), 2695–2711.
[10] K. Gröchenig and T. Strohmer, Pseudo-differential operators on locally compact Abelian groups and Sjöstrand’s symbol class, J. Reine Angew. Math. 613 (2007), 121–146.
[11] I. Krishtal, Wiener’s lemma: pictures at exhibition, Rev. Un. Mat. Argentina 52 (2011), no. 2, 61–79.
[12] K. Krishtal and K. A. Okoujou, Invertibility of the Gabor frame operator on the Wiener amalgam space, J. Approx. Theory 153 (2008), no. 2, 212–224.
[13] V. G. Kurbatov, Algebras of difference and integral operators, Funkt. Anal. Prilozh. 24 (1990), no. 2, 87–88.
[14] N. Motee and Q. Sun, Sparsity measures for spatially decaying systems, arXiv: 1402.4148v3.
[15] C. E. Shin and Q. Sun, Stability of localized operators, J. Funct. Anal. 256 (2009), no. 8, 2417–2439.
[16] , Wiener’s lemma: localization and various approaches, Appl. Math. J. Chinese Univ. Ser. A 28 (2013), no. 4, 465–484.
[17] Q. Sun, Wiener’s lemma for infinite matrices with polynomial off-diagonal decay, C. Acad. Sci. Paris Ser I 340 (2005), no. 8, 567–570.
[18] , Nonuniform average sampling and reconstruction of signals with finite rate of innovation, SIAM J. Math. Anal. 38 (2006/07), no. 5, 1389–1422.
[19] Wiener’s lemma for infinite matrices, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3099–3123.

[20] Wiener’s lemma for infinite matrices II, Constr. Approx. 34 (2011), no. 2, 209–235.

[21] Frames in spaces with finite rate of innovations, Adv. Comput. Math. 28 (2008), no. 4, 301–329.

[22] R. Tessera, The Schur algebra is not spectral in $B(\ell^2)$, Monatsh. Math. 164 (2010), 115–118.

[23] N. Wiener, Tauberian theorem, Ann. of Math. 33 (1932), no. 1, 1–100.

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