In this work, we report the control parameter dependence of the fluctuations near the jamming transition point. We show that the fluctuations do not diverge in pressure control, while it diverges in packing fraction control.

We consider purely repulsive harmonic discs in a two-dimensional $L \times L$ box with periodic boundary conditions at zero temperature\(^2\). We term this a 50 : 50 binary mixture of large radius discs $R_L = 0.7$ and small radius discs $R_S = 0.5$. The value of $V_N$ separates the jammed and unjammed phases: when the packing fraction $\varphi = N\pi(R_L^2 + R_S^2)/(2L^2)$ is smaller than the jamming transition point $\varphi_J$, one observes $V_N = 0$ after the energy minimization, while, when $\varphi > \varphi_J$, $V_N$ has a finite value. For the energy minimization, we use the fast inertial relaxation engine (FIRE)\(^2\). We terminate the energy minimization when $\sum_{i=1}^{N}(\partial_r V_N)^2/N < 10^{-25}$. In our numerical simulation, we define $\varphi$ at which the energy barely has a finite value $V_N/N \in (10^{-16}, 2 \times 10^{-16})$ after the energy minimization. We generate the configurations above $\varphi_J$ in two ways, as described below.

\begin{itemize}
  \item \textbf{a. Packing fraction control} We use $\varepsilon = \varphi - \varphi_J$ as a control parameter. Following O’Hern \textit{et al.}, we first generate the configuration at $\varphi_J$ by combining compression and decompression: we compress the system when $V_N < 10^{-16}$ and decompress when $V_N > 10^{-16}$, see Ref\(^2\) for details. After every compression/decompression, we minimize the energy by using the FIRE algorithm\(^2\). We terminate the process when $V_N/N \in (10^{-16}, 2 \times 10^{-16})$. After obtaining a configuration at $\varphi_J$, we re-compress as the amount of $\varepsilon = \varphi - \varphi_J$ to obtain a configuration above jamming. As reported in Ref\(^2\), some samples unjam after the compression (compression unjamming). We throw out such samples.
  \item \textbf{b. Pressure control} The pressure $p$ is used as the control parameter. For this purpose, we repeat the compression and decompression until the system’s pressure reaches the target pressure. In this case, the jamming transition point corresponds to $p = 0$.
\end{itemize}

\footnote{Electronic mail: harukuni.ikeda@gakushuin.ac.jp}
for $\chi_c$). In the intermediate $\varphi$ region, $\chi_{z,e}$ is well fitted with the power-law function:

$$\chi_{z,e} = A_{z,e} p^{-\beta_{z,e}},$$

where $\beta_z = 0.62$ and $\beta_e = 1.85$, see black dashed lines in Fig. 2. The power-law region increases with $N$, and in the thermodynamic limit, the fluctuations are expected to diverge at the transition point. In Fig. 3 we plot the fluctuation of $\varphi$, $\chi_{\varphi} = N \text{Var}(\varphi)/\text{Ave}(\varphi)^2$, in $p$ control and the fluctuation of $p$, $\chi_p = N \text{Var}(p)/\text{Ave}(p)^2$, in $\varphi$ control. We found that $\chi_{\varphi}$ remains finite, while $\chi_p$ exhibits a power-law divergence $\chi_p \sim (\varphi - \varphi_J)^{-\beta_p}$ with $\beta_p = 1.97$, see the dashed line in Fig. 3.

Finally, we propose a phenomenological model to explain the divergence of the physical quantities in $\varphi$ control. Fig. 3 (a) and a previous research[11] show that the variance of $\varphi$ remains finite at $\varphi_J$. Also, $p \propto \varphi - \varphi_J$ near $\varphi_J$. Therefore, $p$ and $\varphi$ have the following linear relation near $\varphi_J$: $\varphi = \varphi_J + Ap + \xi$, where $\xi$ is a random variable of zero mean and variance $\xi^2 = \Delta/N$[12], and $A$ and $\Delta$ are constants. Then, $p$ can be expressed as a function of $\delta \varphi$:

$$p = A^{-1}(\delta \varphi - \xi),$$

with $\delta \varphi = \varphi - \varphi_J$. It is straightforward to show $\chi_p \sim \delta \varphi^{-\beta_p} \sim p^{-\beta_p}$ with $\beta_p = 2$, which is close to the numerical result $\beta_p = 1.97$. Since the energy is a quadratic function of $p^2 \sim \delta \varphi^2$, we have the square-root singularity $z - z_J \sim p^{1/2} = A^{-1/2}(\delta \varphi^{1/2} - \xi \delta \varphi^{-1/2}/2 + \cdots)$, leading to $\chi_z \sim p^{-\beta_z}$ with $\beta_z = 2$. Again, this is close to the numerical result $\beta_z = 0.62$. The above mean-field argument may no longer hold when the fluctuation of the pressure $A^{-1} \delta \varphi$ becomes larger than the mean-value $A^{-1} \delta \varphi$ in Eq. [4], which defines the characteristic pressure $p \sim \delta \varphi \sim \xi \sim O(N^{-1/2})$. This consideration suggests the following scaling form:

$$\chi_{z,e} = N^\beta z f_{z,e}(N^{\frac{1}{2}} p),$$

where $f_{z,e}(x)$ denotes the scaling function such that $f_{z,e}(x) \sim x^{-\beta_e}$ for $x \ll 1$. In Fig. 4 we confirmed the above scaling ansatz using the data shown in Fig. 2 and data for larger $N$ in $\varphi$ control.

**ACKNOWLEDGMENTS**

We thank A. Ikeda, P. Urbani, and F. Zamponi for useful comments. This work was supported by KAKENHI 21K20355.

**DATA AVAILABILITY**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

1. C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Rev. E 68, 011306 (2003)
2. E. Bitzek, P. Koskinen, F. Gähler, M. Moseler, and P. Gumbsch, Phys. Rev. Lett. 97, 170201 (2006)
3. K. VanderWerf, A. Boromand, M. D. Shattuck, and C. S. O’Hern, Phys. Rev. Lett. 124, 038004 (2020)
4. C. P. Goodrich, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. 109, 095704 (2012)
5. D. Hentzer, P. Urbani, and F. Zamponi, Phys. Rev. Lett. 123, 060803 (2019)
6. F. Zamponi (private communication).