ON THE CUBIC WEYL SUM

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Abstract. We obtain an estimate for the cubic Weyl sum which improves the bound obtained from Weyl differencing for short ranges of summation. In particular, we show that for any \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that for any coprime integers \( a, q \) and real number \( \gamma \) we have

\[
\sum_{1 \leq n \leq N} e\left(\frac{an^3}{q} + \gamma n\right) \ll (qN)^{1/4} q^{-\delta},
\]

provided \( q^{1/3+\varepsilon} \leq N \leq q^{1/2-\varepsilon} \). Our argument builds on some ideas of Enflo.

1. Introduction

Let \( g(x) = \alpha_d x^d + \cdots + \alpha_1 x \) be a polynomial of degree \( d \geq 2 \) with real coefficients and consider the problem of bounding sums

\[
\sum_{1 \leq n \leq N} e(g(n)).
\]

This was first considered by Weyl [17] who developed a technique known as Weyl differencing which shows

\[
\sum_{1 \leq n \leq N} e(g(n)) \ll N^{1+o(1)} \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{Nd}\right)^{2-d-1},
\]

provided the leading coefficient \( \alpha_d \) of \( g \) satisfies

\[
\left| \alpha_d - \frac{a}{q} \right| \leq \frac{1}{q^2},
\]

for some coprime integers \( a, q \), see [16, Lemma 2.4]. For large values of \( d \) the estimate (1) has been improved via bounds related to the Vinogradov mean value theorem while (1) remains sharpest for \( d \leq 5 \).

In this paper we consider the problem of improving the bound (1) in the case of monomials with \( d = 3 \). In particular, our sums take the form

\[
\sum_{1 \leq n \leq N} e\left(\frac{an^3}{q}\right).
\]
This problem has been considered by a number of previous authors. Enflo [8] has some nice discussions on different approaches and parts of our ideas and analysis are based on this work. We mention an open problem from [8, Section 2] to improve on the Pólya-Vinogradov bound for sums

\[
\sum_{1 \leq n \leq N} \chi(n)e\left(\frac{an^3 + bn + cn}{q}\right) \ll q^{1/2+o(1)},
\]

where \( \chi \) is a multiplicative character mod \( q \). Any improvement on (3) would give a new bound for (2) in certain ranges of parameters.

Heath-Brown [11] has applied the \( q \)-van der Corput method to the sums (2) and showed how one may improve (1) provided \( q \) has suitable factorization.

We refer the reader to Nakai [15] for some ideas related to generalizing the classic Hardy-Littlewood argument [10] from the setting of quadratic to cubic sums.

In this paper we obtain a new bound for cubic sums with a short range of summation.

**Theorem 1.** Let \( \delta > 0 \), \( a, q \) coprime integers and \( \gamma \) a real number. If \( N \) satisfies

\[
q^{1/3+\delta} \ll N \ll q^{1/2-\delta/10+o(1)},
\]

then

\[
\sum_{1 \leq n \leq N} e\left(\frac{an^3}{q} + \gamma n\right) \ll (qN)^{1/4+o(1)}q^{-\delta/20}.
\]

Note that (1) implies

\[
\sum_{1 \leq n \leq N} e\left(\frac{an^3}{q} + \gamma n\right) \ll (qN)^{1/4+o(1)} \quad \text{provided} \quad q^{1/3} \leq N \leq q^{1/2},
\]

hence we obtain a power saving over Weyl differencing in this range of parameters.

We are concerned with bounding the sums (2) for individual values of \( q \). There are also a number of interesting results and questions concerning their average or typical behaviour. Wooley’s [19] work on the cubic case of Vinogradov’s mean value theorem implies sharp bounds for moments of sums over cubic polynomials, see also [18]. More general restriction estimates for cubic sums were initiated by Bourgain [2] and further developed in [12, 13, 14]. See [5, 6] for various bounds related to the metric behaviour of large values of Weyl sums and [3] for other metric results and an asymptotic formula for minor arc sums.
2. Outline of Argument

The details in proving Theorem 1 are sometimes tedious so we first outline our argument using some heuristics to simplify things. For the purpose of this discussion we assume a heuristic form of Poisson summation, that for any $q$-periodic function $f$

$$\sum_{n \leq N} f(n) \sim \frac{N}{q} \sum_{n \leq q/N} \hat{f}(n).$$

(4)

where $\hat{f}$ is the Fourier transform

$$\hat{f}(n) = \sum_{y=1}^{q} f(y) e_q(-yn),$$

and $e_q(x) = e^{2\pi i x/q}$. Let $G(a, b; q)$ denote the Gauss sum

$$G(a, b; q) = \sum_{y=1}^{q} e_q(ay^2 + by).$$

Assuming $q$ is odd, we may complete the square to obtain

$$G(a, b; q) = e_q(-4ab^2) G(a, 0; q).$$

The sums $G(a, 0; q)$ have an explicit evaluation in terms of the Jacobi symbol. To state such results requires considering a case by case basis depending on the value of $q \mod 4$. We will ignore issues with even $q$ and for simplicity assume for any integers $a, b, q$ that

$$G(a, b; q) \sim \left( \frac{a}{q} \right) e_q(-4ab^2) q^{1/2},$$

(5)

where

$$\left( \frac{\cdot}{q} \right),$$

denotes the Jacobi symbol mod $q$.

Let $N$ satisfy

$$q^{1/3+\varepsilon} \leq N \leq q^{1/2-\varepsilon},$$

(6)

and consider the sums

$$S = \sum_{1 \leq n \leq N} e_q(an^3).$$

(7)

In this setting one may use Weyl differencing to show

$$|S|^4 \ll N^{2+o(1)} \left\{ \left| m \right| \ll N^2, \left| n \right| \ll \frac{q}{N} : am \equiv n \mod q \right\}.\]$$

(8)
If
\[ S \gg \delta(qN)^{1/4+o(1)}, \]
for some \( \delta \sim q^{-s_0} \) then
\[ \left| \{ |m| \ll N^2, |n| \ll \frac{q}{N} : am \equiv n \mod q \} \right| \gg \delta^4 \frac{q}{N}. \]  
Using lattice basis reduction we obtain integers \( \ell, s \) satisfying
\[ a \equiv \ell s \mod q, \ell \ll \frac{1}{\delta^4}, |s| \ll \frac{N^3}{q\delta^4}. \]

Returning to the sums (7), we apply the first step of Weyl differencing to get
\[ |S|^2 = \sum_{m,n \leq N} e_q(a(m^3 - n^3)) = \sum_{|m| \leq N} e_q(am^3) \sum_{n \leq N} e_q(3amn^2 + 3am^2n) \ll S_1 + S_2, \]
where
\[ S_1 = \sum_{|m| \leq N^{1-\epsilon}} \left| \sum_{n \leq N} e_q(3amn^2 + 3am^2n) \right|, \]
and
\[ S_2 = \sum_{N^{1-\epsilon} < |m| \leq N} e_q(am^3) \sum_{n \leq N} e_q(3amn^2 + 3am^2n). \]

If \( |S|^2 \ll S_1 \) then we continue with Weyl differencing and obtain an immediate improvement on (1) due to the shorter outer summation. Hence we may suppose
\[ |S|^2 \ll S_2. \]

By (4) and (5)
\[ S_2 \sim \frac{N}{q} \sum_{N^{1-\epsilon} < |m| \leq N} e_q(am^3) \sum_{n \ll q/N} G(3am, 3am^2 - n; q) \]
\[ \sim \frac{N}{q^{1/2}} \sum_{N^{1-\epsilon} < |m| \leq N} \left( \frac{m}{q} \right) e_q(4am^3) \sum_{n \ll q/N} e_q \left( -12amn^2 + 2mn \right). \]

Up to this point our analysis of \( S_2 \) is similar to ideas used by Enflo [8] and note the sums (3) arise after interchanging summation. Our new
input is to use (11) and reciprocity for modular inverses to reduce oscillations for summation over $n$ in (14). We have

$$\frac{12mn^2}{q} \equiv \frac{s12\ell mn^2}{q} \equiv -\frac{s\overline{m}n^2}{12\ell m} + \frac{sn^2}{12\ell mq} \mod 1,$$

and

$$\frac{2mn}{q} \equiv -\frac{qmn}{2} + \frac{mn}{2q} \mod 1.$$

Substituting into (14) gives

$$S_2 \sim \frac{N}{q^{1/2}} \sum_{N^{1-\varepsilon} < |m| \leq N} \left( \frac{m}{q} \right) e_q(4am^3)$$

$$\times \sum_{n \leq q/N} e_{12\ell m} (s\overline{m}n^2) e \left( -\frac{sn^2}{12\ell mq} + \frac{mn}{2q} - \frac{qmn}{2} \right).$$

If $m, n$ satisfy the conditions of summation in (15) then from (11)

$$-\frac{sn^2}{12\ell mq} + \frac{mn}{2q} \ll \frac{N^\varepsilon}{\delta^4}.$$

Using some Fourier analysis, this implies

$$S_2 \ll \frac{N^{1+\varepsilon}}{\delta^4 q^{1/2}} \sum_{N^{1-\varepsilon} < |m| \leq N} \left( \frac{m}{q} \right) e_q(4am^3) \sum_{n \leq q/N} e_{12\ell m} (s\overline{m}n^2),$$

and we have ignored the term $qmn/2$ which may be dealt with by partitioning summation depending on parity. If $N$ satisfies (6) then by (11)

$$\frac{q}{N} \gg \ell m,$$

provided $\delta$ is not too small. We see that summation over $n$ in (16) is essentially a complete sum

$$\sum_{n \leq q/N} e_{12\ell m} (s\overline{m}n^2) \sim \frac{q}{12\ell mN} G(s\overline{m}, 0; 12\ell m).$$

Substituting the above into (16) and using (5)

$$S_2 \ll \frac{q^{1/2} N^\varepsilon}{\delta^4} \sum_{N^{1-\varepsilon} < |m| \leq N} \left( \frac{m}{q} \right) \left( \frac{q}{m} \right) e_q(4am^3) \frac{1}{m^{1/2}}.$$
By quadratic reciprocity and partial summation this simplifies to

\[ S_2 \ll \frac{q^{1/2}N^{2\varepsilon}}{\delta^4N^{1/2}} \sum_{m \leq N} e_q(4am^3). \]

(18)

We arrive at a kind of recursive inequality for cubic sums. Choosing \( a \) such that the sum (7) is maximum we obtain from (13)

\[ |S| \ll \frac{q^{1/2}N^{2\varepsilon}}{\delta^4N^{1/2}}, \]

and hence by (9)

\[ \delta \ll N^{2\varepsilon/5} \left( \frac{q}{N^3} \right)^{1/20}, \]

which gives a power saving over (1) provided \( N \geq q^{1/3+3\varepsilon} \).

Our argument presented below is much more technical. One of the major issues we have ignored is that for an arbitrary modulus we need to partition summation in (12) depending on the value of \( (m, q) \). This modifies the length and modulus of summation in a way which makes it difficult to take full advantage of the recursive inequality given by (18). It is possible Theorem 1 may be improved for certain \( q \) such as prime although we have not attempted to do so here.

The reader may be curious to what extent the argument from this paper may be extended to cover a range of parameters of the form

\[ q^{1/2-\varepsilon} \leq N \leq q^{1/2+\varepsilon}. \]

The restriction to \( N \leq q^{1/2-\varepsilon} \) is used in two places. First it allows us to ignore a term \( N^3 \) on the right hand side of (8) which contributes \( N^{3/4} \) in (1). Second at (17) which implies that only the zero Fourier coefficient contributes after applying Poisson summation to (16). It would be interesting to see if one could use \( \ell, s \) as in (11) obtained only from Dirichlet’s theorem and a more careful application of Poisson summation and stationary phase at (16) to improve the Weyl differencing bound for a longer range of parameters.

3. Background on Fourier analysis

We first collect some well known results and techniques from Fourier analysis. We will work with both the Fourier transform over \( \mathbb{R} \) and the group of residues modulo an integer \( q \). We use \( \hat{f} \) to denote the Fourier transform of a function \( f \) over the space where it is defined. In particular, for

\[ f : \mathbb{R} \rightarrow \mathbb{C}, \]
we have
\[ \widehat{f}(\eta) = \int_{-\infty}^{\infty} f(x) e(-x\eta) \, dx, \]
and for
\[ g : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}, \]
we have
\[ \widehat{g}(\eta) = \frac{1}{q} \sum_{x=1}^{q} g(x) e_q(-x\eta). \]
In this context, the Poisson summation formula states:

**Lemma 2.** For \( f \in L^1(\mathbb{R}) \) smooth and \( g \) as above, we have
\[
\sum_{n \in \mathbb{Z}} f(n) g(n) = \sum_{m \in \mathbb{Z}} \widehat{f}(\frac{m}{q}) \widehat{g}(-m).
\]
The convolution of two functions \( f_1, f_2 \) is given by
\[ (f_1 * f_2)(x) = \int_{\mathbb{R}} f_1(y) f_2(x-y) \, dy. \]

**Lemma 3.** For \( f_1, f_2 \in L^1(\mathbb{R}) \) smooth we have
\[ \widehat{(f_1 * f_2)}(x) = \widehat{f_1}(x) \widehat{f_2}(x), \]
and
\[ \widehat{f_1 f_2}(x) = (\widehat{f_1} * \widehat{f_2})(x). \]
We will also use the Fourier inversion formula.

**Lemma 4.** For \( f, \widehat{f} \in L^1(\mathbb{R}) \) smooth we have
\[ f(x) = \int_{\mathbb{R}} \widehat{f}(y) e(xy) \, dy. \]

**Lemma 5.** Let \( N \) be an integer and \( F, f \) complex valued functions on the interval \([1, N]\) with \( f \) continuously differentiable and satisfying
\[
(f) \ll 1, \quad \text{and} \quad (f') \ll \frac{\Omega}{n}, \quad x \in [1, N],
\]
for some \( \Omega > 0 \). There exists \( \gamma \in [0, 1) \) such that
\[
\sum_{1 \leq n \leq N} f(n) F(n) \ll (1 + \Omega) N^{\alpha(1)} \sum_{1 \leq n \leq N} F(n) e(\gamma n).
\]
Proof. By partial summation and (19)

\[
\sum_{1 \leq n \leq N} f(n)F(n) \ll \sum_{1 \leq n \leq N} F(n) + \int_1^N f'(x) \left( \sum_{1 \leq n \leq x} F(n) \right) \, dx
\]

(20)

\[
\ll \sum_{1 \leq n \leq N} F(n) + \Omega N^{o(1)} \sum_{1 \leq n \leq x} F(n),
\]

for some \(1 \leq x \leq N\). We have

\[
\sum_{1 \leq n \leq x} F(n) = \int_0^1 \sum_{1 \leq n \leq N} F(n) \left( \sum_{1 \leq m \leq x} e(\alpha(n - m)) \right) \, d\alpha
\]

\[
\ll \int_0^1 \left| \sum_{1 \leq m \leq x} e(-\alpha m) \right| \left| \sum_{1 \leq n \leq N} F(n) e(\alpha n) \right| \, d\alpha
\]

\[
\ll \max_{\gamma \in [0, 1]} \left| \sum_{1 \leq n \leq N} F(n) e(\gamma n) \right| \int_0^1 \left| \sum_{1 \leq m \leq x} e(-\alpha m) \right| \, d\alpha,
\]

and the result follows from (20) after using

\[
\int_0^1 \left| \sum_{1 \leq m \leq x} e(-\alpha m) \right| \, d\alpha \ll \int_0^1 \min \left\{ N, \frac{1}{\|\alpha\|} \right\} \, d\alpha \ll N^{o(1)}.
\]

\[\Box\]

Lemma 6. Let \(\ell, p, N\) be positive integers with \(\ell, p \ll N^{o(1)}\). For any complex valued function \(F\) on the interval \([1, N]\) there exists some \(\gamma \in [0, 1]\) such that

\[
\sum_{1 \leq \ell n + p \leq N} F(\ell n + p) \ll N^{o(1)} \sum_{1 \leq n \leq N} F(n) e(\gamma n).
\]

Proof. We have

\[
\sum_{1 \leq \ell n + p \leq N} F(\ell n + p) = \int_0^1 \sum_{1 \leq n \leq N} F(n) \left( \sum_{1 \leq \ell m + p \leq N} e(\alpha(\ell m + p - n)) \right) \, d\alpha
\]

\[
\ll \int_0^1 \left| \sum_{1 \leq \ell m \leq (N-p)/\ell} e(\alpha \ell m) \right| \left| \sum_{1 \leq n \leq N} F(n) e(\alpha n) \right| \, d\alpha,
\]
and hence for some $\gamma \in [0, 1)$

$$
\sum_{1 \leq \ell n + p \leq N} F(\ell n + p) \ll \left| \sum_{1 \leq n \leq N} F(n)e(\gamma n) \right| \\
\times \int_0^1 \left| \sum_{(1-p)/\ell \leq m \leq (N-p)/\ell} e(\alpha m) \right| d\alpha,
$$

and the result follows after using

$$
\int_0^1 \left| \sum_{(1-p)/\ell \leq m \leq (N-p)/\ell} e(\alpha m) \right| d\alpha \ll \frac{1}{\ell} \int_0^\ell \left| \sum_{(1-p)/\ell \leq m \leq (N-p)/\ell} e(\alpha m) \right| d\alpha \\
\ll N^{o(1)}.
$$

\[ \square \]

Using a similar completion argument as above, it is possible to show the following, see [1, Lemma 2.2].

**Lemma 7.** Let $N, N_1, M, M_1$ be integers satisfying

$$
N_1 \leq M_1 \leq M \leq N.
$$

For any complex valued function $F$ on the interval $[N_1, N]$, there exists some $\gamma \in \mathbb{R}$ such that

$$
\sum_{M_1 \leq n \leq M} F(n) \ll N^{o(1)} \sum_{N_1 \leq n \leq N} F(n)e(\gamma n).
$$

The following is known as a smooth partition of unity, see [9, Lemma 2].

**Lemma 8.** There exists a sequence $V_0, V_1, \ldots$ of smooth functions satisfying

$$
supp(V_\ell) \subseteq (2^{\ell-1}, 2^\ell],
$$

(22) $$
V_\ell^{(k)}(x) \ll \frac{1}{x^k}, \quad k \geq 1,
$$

and for any $x \geq 1$ we have

$$
\sum_{\ell \geq 0} V_\ell(x) = 1.
$$

Our next result follows by combining Lemma 7, Lemma 8 and partitioning summation into dyadic intervals.
Lemma 9. Let $F$ be a complex valued function on the integers. For any $N \gg 1$ there exists some $1 \leq K \leq 2N, \gamma \in \mathbb{R}$ and smooth function $f$ satisfying

$$\text{supp}(f) \subseteq [K, 2K], \quad f^{(j)}(x) \ll \frac{1}{x^j},$$

such that

$$\sum_{1 \leq n \leq N} F(n) \ll N^{o(1)} \sum_{n \in \mathbb{Z}} f(n) F(n) e(\gamma n).$$

4. Gauss sums

Given integers $a, \ell, q$, define the complete Gauss sum

$$(24) \quad G(a, \ell; q) = \sum_{\mu=1}^{q} e_q(a\mu^2 + \ell\mu).$$

We next collect some results dating back to Gauss regarding the evaluation of $G(a, \ell; q)$.

Given integers $p, q$ we let $(p/q)$ denote the Jacobi symbol. Recall quadratic reciprocity, see [4, Equation 1.2.7].

Lemma 10. For any odd positive integers $p, q$ we have

$$\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{(p-1)(q-1)/4}.$$ 

Evaluating the Jacobi symbol at even integers requires separate treatment, see [4, pg. 17].

Lemma 11. For any odd positive integer $q$ we have

$$\left( \frac{2}{q} \right) = (-1)^{(q^2-1)/8}.$$ 

Our next result follows from completing the square in summation over $\mu$ in (24), see for example [1, Lemma 2.10].

Lemma 12. If $q \geq 1$ and $(a, q) = 1$ then

$$G(a, \ell, q) = \begin{cases} e_q \left( \frac{-a^2}{4} \right) G(a, 0; q) & \text{if } \ell \equiv 0 \mod 2, \\ e_q \left( \frac{-a^2-1}{4} \right) G(a, a; q) & \text{if } \ell \equiv 1 \mod 2. \end{cases}$$

If $q$ is odd we may complete the square in a way which does not depend on the parity of $\ell$.

Lemma 13. Let $q \geq 1$ be odd and $(a, q) = 1$. For any integer $\ell$

$$G(a, \ell, q) = e_q(-\frac{1}{4}a\ell^2)G(a, 0; q).$$
We will require explicit evaluation of the Gauss sums occuring in Lemma 12 and will consider the value of $q \mod 4$ on a case by case basis. We first note a consequence of the Chinese Remainder Theorem.

**Lemma 14.** For any $(q_1, q_2) = 1$ and $a, \ell$ we have
\[
G(a, \ell; q_1 q_2) = G(aq_1, \ell; q_2)G(aq_2, \ell, q_1).
\]

The following is [4, Theorem 1.52].

**Lemma 15.** Let $(a, q) = 1$ with $q \geq 1$ and odd. Then
\[
G(a, 0; q) = \begin{cases} 
\left(\frac{a}{q}\right) q^{1/2} & \text{if } q \equiv 1 \mod 4, \\
\left(\frac{-a}{q}\right) i q^{1/2} & \text{if } q \equiv 3 \mod 4.
\end{cases}
\]

Using Lemma 14 we may deal with the case $q \equiv 2 \mod 4$.

**Lemma 16.** Let $(a, 2q) = 1$ with $q \geq 1$ and odd. Then
\[
G(a, 0; 2q) = 0.
\]

**Proof.** By Lemma 14
\[
G(a, 0; 2q) = G(aq, 0; 2)G(2a, 0; q).
\]
Since $aq$ is odd, we see that
\[
G(aq, 0; 2) = 0,
\]
which completes the proof. \qed

The following is [4, Theorem 1.5.4].

**Lemma 17.** Let $(a, q) = 1$ with $q \geq 1$ and $q \equiv 0 \mod 4$. We have
\[
G(a, 0; q) = \left(\frac{q}{a}\right) (1 + i^a) q^{1/2}.
\]

We next consider sums of the form $G(a, a; q)$. For odd $q$ these may be evaluated using Lemma 13. Our next result deals with the case of even $q$.

**Lemma 18.** Let $q \geq 1$ be odd. For any $(a, 2q) = 1$ we have
\[
G(a, a; 2q) = 2G(2a, a; q).
\]

**Proof.** By Lemma 14
\[
G(a, a; 2q) = G(2a, a; q)G(aq, a; 2).
\]
Using that both $q, a$ are odd, we have
\[
G(aq, a; 2) = \sum_{j=0}^{q} e^{\pi i (aqj^2 + aj)} = 2,
\]
from which the result follows. \qed
Lemma 19. Let $q \geq 1$ be odd and $k \geq 2$. For any $(a, 2q) = 1$ and odd integer $b$ we have

$$G(a, b; 2^k q) = 0.$$  

Proof. By Lemma 14

$$G(a, b; 2^k q) = G(a 2^k, b; q) G(a q, b; 2^k).$$

Hence it is sufficient to show

$$G(a q, b; 2^k) = 0.$$  

We have

$$G(a q, b; 2^k) = \sum_{y=1}^{2^k} e_{2^k}(a q y^2 + b y).$$

Since $b$ is odd, the polynomial $F(x) = a q x^2 + b x$ satisfies

$$(2, F'(n)) = 1, \quad n \in \mathbb{Z}.$$  

By Hensel’s Lemma, this implies for any integer $m$

$$|\{1 \leq n \leq 2^k : F(n) \equiv m \pmod{2^k}\}| \leq 2.$$  

Since each $a, b, q$ are odd,

$$|\{1 \leq n \leq 2^k : F(n) = m\}| = 0 \quad \text{if } m \text{ odd.}$$  

By the pigeonhole principle

$$|\{1 \leq n \leq 2^k : F(n) = m\}| = \begin{cases} 2 & \text{if } m \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$G(a q, b; 2^k) = 2 \sum_{y=1}^{2^{k-1}} e_{2^{k-1}}(y) = 0.$$  

□

5. Weyl differencing

The main result of this section is Lemma 22 which will be used in an iterative manner to construct a sequence of good rational approximations to the amplitude function in the cubic exponential sums. The technique is based on Weyl differencing and lattice reduction.

We first recall [7, Lemma 6.3] which is stated for prime modulus only, the exact same proof works for arbitrary modulus.
Lemma 20. Let $q$ be a positive integer, $K \geq 1$ and $\mathcal{I}, \mathcal{J}$ two intervals containing $h$ and $H$ integers respectively. For integer $b$ satisfying $(b, q) = 1$, let $I(b)$ count the number of solutions to the congruence
\begin{equation}
yb \equiv x \mod q, \quad x \in \mathcal{I}, \quad y \in \mathcal{J}.
\end{equation}
One of the following two cases hold
\begin{equation}
I(b) \ll \frac{Hh}{q}.
\end{equation}

If $I(s) \geq K$ then there exists $\ell, s$ satisfying
\begin{equation}
b \equiv \ell s \mod q, \quad \ell \ll \frac{h}{K}, \quad |s| \ll \frac{H}{K}.
\end{equation}

We next perform some preliminary manipulations and set up notation which will be used throughout the paper.

Lemma 21. Let $g$ be a cubic polynomial with real coefficients of the form
\begin{equation}
g(x) = \frac{ax^3}{q} + \gamma x,
\end{equation}
with $(a, q) = 1$. Let $f$ be a smooth function satisfying
\begin{equation}
\text{supp}(f) \subseteq [N, 2N], \quad f^{(j)}(x) \ll \frac{1}{x^j}.
\end{equation}
Define
\begin{equation}
q_0 = \frac{q}{(q, 3)},
\end{equation}
\begin{equation}
b = \begin{cases} 3a \quad \text{if } (q, 3) = 1 \\ a \quad \text{otherwise.} \end{cases}
\end{equation}
There exists some $d | q_0$ such that
\begin{equation}
\left| \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \right|^2 \ll N
+ q^{o(1)} \sum_{1 \leq |m| < dN/q_0 \atop (m, q_0) = 1} e \left( g \left( \frac{q_0m}{d} \right) \right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{q_0m/d}n)\epsilon_d(bmn^2).
\end{equation}
where
\begin{equation}
\rho_m = \frac{bm^2}{q_0},
\end{equation}
and

\[ F_{d,m}(n) = f(q_0 m/d + n)f(n). \]

**Proof.** Let

\[ S = \sum_{n \in \mathbb{Z}} f(n)e(g(n)). \]

Expand the square and apply the change of variable \( m \to m+n \) to get

\[ |S|^2 = \sum_{m,n \in \mathbb{Z}} f(m)f(n)e(g(m) - g(n)) \]

\[ = \sum_{m,n \in \mathbb{Z}} f(m+n)f(n)e(g(m+n) - g(n)). \]

Using

\[ g(m+n) - g(n) = g(m) + \frac{3am}{q}n^2 + \frac{3am^2}{q}n, \]

and recalling (30), (31)

\[ |S|^2 = \sum_{m \in \mathbb{Z}} e(g(m)) \sum_{n \in \mathbb{Z}} f(m + n)f(n)e_{q_0}(bmn^2 + bm^2n). \]

By (29), if \( f(m+n)f(n) \neq 0 \) then \( |m| \leq N \) and hence

\[ |S|^2 = \sum_{|m| \leq N} e(g(m)) \sum_{n \in \mathbb{Z}} f(m + n)f(n)e_{q_0}(bmn^2 + bm^2n) \]

\[ \ll N + \sum_{1 \leq |m| \leq N} e(g(m)) \sum_{n \in \mathbb{Z}} f(m + n)f(n)e_{q_0}(bmn^2 + bm^2n). \]

We next partition summation over \( m \) depending on the value of \((m, q_0)\) to get

(34)

\[ S = \sum_{d|q_0} S'_d, \]

where

\[ S'_d = \sum_{1 \leq |m| \leq N} \left( e\left( g\left( \frac{q_0 m}{d} \right) \right) \sum_{n \in \mathbb{Z}} f(m + n)f(n)e_{q_0}(bm^2n) \right) \]

\[ = \sum_{1 \leq |m| \leq dN/q_0} \left( e\left( \frac{q_0 m}{d} \right) \right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{q_0 m/d} n) e_d(bmn^2). \]

Taking a maximum over \( d|q_0 \) in (34) and using estimates for the divisor function we complete the proof. \( \square \)
Lemma 22. Let notation and conditions be as in Lemma 21 and assume that
\begin{equation}
\left| \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \right| \geq \delta(qN)^{1/4+o(1)},
\end{equation}
for some
\begin{equation}
\frac{N^{1/2}}{q^{1/4}} \leq \delta \leq 1.
\end{equation}

Let $Y \leq N$ and define
\begin{equation}
S_d = \sum_{dY/q_0 \leq |m| \leq dN/q_0 \ (m,q_0)=1} e \left( g \left( \frac{q_0 m}{d} \right) \right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{q_0 m/d}n)e_d(bmn^2).
\end{equation}

Suppose $N$ satisfies
\begin{equation}
q^{1/3} \leq N \ll \delta^2 q^{1/2}.
\end{equation}

Either there exists integers $\ell_0, s_0$ satisfying
\begin{equation}
b \equiv \ell_0 s_0 \mod d, \quad |\ell_0| \ll \frac{1}{\delta^4} \frac{d}{q} \frac{Y}{N}, \quad |s_0| \ll \frac{1}{\delta^4} \frac{d^2}{q} \frac{NY^2}{q},
\end{equation}
or
\begin{equation}
\left| \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \right|^2 \ll q^{o(1)} S_d.
\end{equation}

Proof. Let
\begin{equation}
S = \sum_{n \in \mathbb{Z}} f(n)e(g(n)).
\end{equation}

By Lemma 21 and the assumptions (35) and (36) we have
\begin{equation}
|S|^2 \ll q^{o(1)} \sum_{1 \leq |m| \leq dN/q_0 \ (m,q_0)=1} e \left( g \left( \frac{q_0 m}{d} \right) \right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{q_0 m/d}n)e_d(bmn^2).
\end{equation}

With notation as in (37)
\begin{equation}
|S|^2 \ll q^{o(1)}(S' + S_d),
\end{equation}
where
\begin{equation}
S' = \sum_{1 \leq |m| < dY/q_0 \ (m,q_0)=1} e \left( g \left( \frac{q_0 m}{d} \right) \right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{q_0 m/d}n)e_d(bmn^2).
\end{equation}
Either
\[ |S|^2 \ll q^{o(1)} S', \]
or
\[ |S|^2 \ll q^{o(1)} S_d. \]

If (42) then we obtain (40). Suppose next (41). Applying the Cauchy-Schwarz inequality
\[ |S|^4 \ll \frac{dY}{q} \sum_{|m|< dY/q_0} \left| \sum_{n \in \mathbb{Z}} F_{d,m}(n) e(\rho_{qm/d} n) e_d(bmn^2) \right|^2 \]
\[ \ll \frac{dY}{q} \sum_{|m|< dY/q_0} \sum_{n_1, n_2 \in \mathbb{Z}} F_{d,m}(n_1) F_{d,m}(n_2) \]
\[ \times e_{q_0}(bm(n_1^2 - n_2^2) + \rho_{qm/d}(n_1 - n_2)). \]

Applying the change of variable \( n_1 \to n_1 + n_2 \) gives
\[ |S|^4 \ll \frac{dY}{q} \sum_{|m|< dY/q_0} \sum_{|n_1| \leq 2N} \left| \sum_{n_2 \in \mathbb{Z}} f_0(n_2) e_d(2bmn_1 n_2) \right|, \]
where
\[ f_0(n_2) = F_{d,m}(n_1 + n_2) F_{d,m}(n_2). \]

By (29) and partial summation over \( n_2 \)
\[ |S|^4 \ll \frac{dY}{q} \sum_{|m|< dY/q_0} \sum_{|n_1| \leq 2N} \min \left\{ N, \frac{1}{\|2bmn_1/d\|} \right\}. \]

Using (35), estimates for the divisor function and isolating the contribution from either \( m, n_1 = 0 \), we arrive at
\[ \delta^4(qN)^{1+o(1)} \ll \frac{dYN^{o(1)}}{q} \sum_{|\ell| \leq dNY/q} \min \left\{ N, \frac{1}{\|b\ell/d\|} \right\} + \frac{dYN^2}{q}. \]

Since \( dY/q \leq N \), by (38) this simplifies to
\[ \frac{\delta^4q^{2+o(1)}N}{dY} \ll N^{o(1)} \sum_{|\ell| \leq dNY/q} \min \left\{ N, \frac{1}{\|b\ell/d\|} \right\}. \]

For integer \( j \) define the set
\[ \mathcal{L}(j) = \left\{ |\ell| \ll dNY/q : \frac{2^j - 1}{N} \leq \|b\ell/d\| < \frac{2^j - 1}{N} \right\}, \]
so that \((43)\) implies
\[
\frac{\delta^4 q^{2+o(1)} N}{dY} \ll N^{1+o(1)} \sum_{j \leq \log N} |L(j)| \ll \frac{N^{1+o(1)} |L(j)|}{2^j},
\]
for some \(1 \leq j \ll \log N\). With suitable \(o(1)\) terms we obtain
\[(44)\]
\[
\frac{\delta^4 q^2}{dY} \ll |L(j)|.
\]
Note that \(|L(j)|\) is bounded by the number of solutions to the congruence
\[
bn \equiv m \pmod{d}, \quad |m| \ll \frac{2^j d}{N}, \quad |n| \ll \frac{dNYq}{q},
\]
to which we may apply Lemma 20. By \((38)\) we may suppose
\[
|L(j)| \gg \frac{2^j dY}{q},
\]
which implies case \((27)\) of Lemma 20. Using \((44)\) we apply Lemma 20 with
\[
K = c_0 \frac{\delta^4 q^2}{dY}, \quad h = c_1 \frac{2^j d}{N}, \quad H = \frac{c_2 dNYq}{q},
\]
for suitable absolute constants \(c_0, c_1, c_2\). We obtain integers \(\ell_0, s_0\) satisfying
\[
b \equiv \ell_0 s_0 \pmod{d}, \quad |\ell_0| \ll \frac{1}{\delta^4} \left(\frac{d}{q}\right)^2 \frac{Y}{N}, \quad |s_0| \ll \frac{1}{\delta^4} \left(\frac{d}{q}\right)^2 \frac{NY^2}{q},
\]
which completes the proof. \(\square\)

6. Duality for summation over Gauss sums

In this section we estimate the sums occurring in \((37)\) of Lemma 22. The procedure consists of Poission summation, evaluation of Gauss sums, reciprocity for modular inverses, Poission summation, evaluation of Gauss sums then quadratic reciprocity which we split into a number of stages. The main result of this section is Lemma 27.

**Lemma 23.** Let notation and conditions be as in Lemma 22 and suppose \(\ell, s\) are any integers satisfying
\[(45)\]
\[
b \equiv \ell s \pmod{d}, \quad \ell > 0.
\]
Define \(t \in \mathbb{Z}\) by
\[(46)\]
\[
sb = \ell + td,
\]
and

\[ M_d = \max \left\{ \left( \frac{|s|q_0}{\ell d^2 Y} \right)^{1/2}, \frac{N}{d} \right\}. \]

Let \( \varepsilon > 0 \) be small and suppose

\[ \ell d M_d^2 N^{11\varepsilon} \ll 1. \]

There exists polynomials \( g^*, g^{**} \) of the form

\[ g^*(x) = \frac{am^3}{4q} + \frac{b_1 dm^3}{4\ell q_0} + \gamma_1 m, \]

\[ g^{**}(x) = \frac{am^3}{4q} + \frac{b_2 dm^3}{4\ell q_0} + \gamma_2 m, \]

with \( b_1, b_2 \in \mathbb{Z}, \gamma_1, \gamma_2 \in \mathbb{R} \) and some \( s_1 | s \) such that defining \( s_2 \) by

\[ s_1 s_2 = s, \]

and \( S_d^{(1)}, S_d^{(2)} \) by

\[ S_d^{(1)} = \sum_{dY/q_0 s_1 \leq |m| \leq dN/q_0 s_1 \atop (m,q_0)=1 \atop (m,s)=1} e\left(g^*\left(\frac{q_0 s_1 m}{d}\right)\right) \frac{G(bs_1 m, 0; d)G\left(s_2 \frac{d}{m}, -\frac{s_1 t q_0 m^2}{d}; \ell m\right)}{m\ell}, \]

\[ S_d^{(2)} = \sum_{dY/q_0 s_1 \leq |m| \leq dN/q_0 s_1 \atop (m,q_0)=1 \atop (m,s)=1} e\left(g^{**}\left(\frac{q_0 s_1 m}{d}\right)\right) \frac{G(bm, bm; q)G\left(s_2 d, s_2 d - \frac{s_1 t q_0 m^2}{d}; \ell m\right)}{m\ell}, \]

we have

\[ S_d \ll \left(\frac{dM_d N^{13\varepsilon}}{N}\right)^2 \left(1 + \frac{|s|}{\ell N^2 Y}\right) (S_d^{(1)} + S_d^{(2)}). \]

**Proof.** Recall from (37) that

\[ S_d = \sum_{dY/q_0 \leq |m| \leq dN/q_0 \atop (m,q_0)=1} e\left(g\left(\frac{qm}{d}\right)\right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{q_0 m/d} n)e_d(bmn^2). \]
With $s$ as in (45), we partition summation over $m$ depending on the value of $(m, s)$ to get

$$S_d = \sum_{s_1 \mid s} S_{d, s_1},$$

where

$$S_{d, s_1} = \sum_{dY/q_0 \leq |m| \leq dN/q_0} e\left(g\left(\frac{q_0m}{d}\right)\right) \sum_{\substack{n \in \mathbb{Z} \\mid n \equiv i \mod 2}} F_{d, m}(n) e(\rho_{q_0m/d}n) e_d(bmn^2).$$

Fix $s_1 \mid s$ and consider $S_{d, s_1}$. We apply Lemma 2 to the inner summation over $n$ to get

$$\sum_{n \in \mathbb{Z}} F_{d, m}(n) e_d(bmn^2) = \frac{1}{d} \sum_{n \in \mathbb{Z}} \widehat{F}_{d, m}\left(-\left(\frac{n}{d} + \rho_{q_0m/d}\right)\right) G(bm, n; d),$$

after recalling the notation (24). Substituting the above into (55) gives

$$S_{d, s_1} = S_{d, s_1, 0} + S_{d, s_1, 1},$$

where

$$S_{d, s_1, i} = \frac{1}{d} \sum_{dY/q_0 \leq |m| \leq dN/q_0} e\left(g\left(\frac{q_0m}{d}\right)\right) \times \sum_{n \in \mathbb{Z}} \widehat{F}_{d, m}\left(-\left(\frac{n}{d} + \rho_{q_0m/d}\right)\right) G(bm, n; d).$$

We consider $S_{d, s_1, 0}$ in detail then indicate necessary modifications for $S_{d, s_1, 1}$. By Lemma 12

$$S_{d, s_1, 0} = \frac{1}{d} \sum_{dY/q_0 \leq |m| \leq dN/q_0} e\left(g\left(\frac{q_0m}{d}\right)\right) G(bm, 0; d)$$

$$\times \sum_{n \in \mathbb{Z}} \widehat{F}_{d, m}\left(-\left(\frac{2n}{d} + \rho_{q_0m/d}\right)\right) e_d(-bmn^2).$$

Recall $\ell, s$ satisfy (45), so that

$$\frac{bmn^2}{d} \equiv \frac{\ell mn^2}{d} \text{ mod } 1.$$
Using reciprocity for modular inverses
\[ \frac{\ell m}{d} + \frac{\ell d}{\ell m} \equiv \frac{1}{\ell md} \mod 1, \]
the above implies
\[ (58) \quad \frac{bmn^2}{d} \equiv -\frac{sdn^2}{\ell m} + \frac{sn^2}{\ell md} \mod 1, \]
and hence
\[ \sum_{n \in \mathbb{Z}} \hat{F}_{d,m} \left( \left( -\frac{2n}{d} + \rho_{qm/d} \right) \right) e_d(-bmn^2) \]
\[ = \sum_{n \in \mathbb{Z}} \hat{F}_{d,m} \left( \left( -\frac{2n}{d} + \rho_{qm/d} \right) \right) e \left( -\frac{sn^2}{\ell md} \right) e_{\ell m}(sdn^2). \]
Let \( h \) be a smooth function satisfying
\[ (59) \quad \text{supp}(h) \subseteq [-2, 2] \quad \text{and} \quad h(x) = 1 \quad \text{if} \quad |x| \leq 1. \]
Recalling (33), we have
\[ \hat{F}_{d,m}(x) = \int_{\mathbb{R}} f(q_m/d + y)f(y)e(-yx). \]
Repeated integration by parts shows that for any small \( \varepsilon > 0 \) and large \( C > 0 \)
\[ (60) \quad \hat{F}_{d,m}(x) \ll \frac{1}{N^{2C}x^2} \quad \text{provided} \quad |x| \geq \frac{1}{N^{1-\varepsilon}}. \]
Since
\[ \hat{F}_{d,m}(x) \ll N, \]
this implies
\[ \sum_{n \in \mathbb{Z}} \hat{F}_{d,m} \left( \left( -\frac{2n}{d} + \rho_{qm/d} \right) \right) e_d(-bmn^2) \]
\[ = \sum_{|2n/d + \rho_{qm/d}| \leq N^{-1+\varepsilon}} \hat{F}_{d,m} \left( \left( -\frac{2n}{d} + \rho_{qm/d} \right) \right) e \left( -\frac{sn^2}{\ell md} \right) e_{\ell m}(sdn^2) + O \left( \frac{1}{N^C} \right). \]
Using (59)

\[ \sum_{n \in \mathbb{Z}} \hat{F}_{d,m} \left( - \left( \frac{2n}{d} + \frac{\rho_{q_0 m/d}}{d} \right) \right) e_d(-b m n^2) = \]

\[ \sum_{|2n + \rho_{q_0 m/d}| \leq N^{-1+\varepsilon}} \hat{F}_{d,m} \left( - \left( \frac{2n}{d} + \frac{\rho_{q_0 m/d}}{d} \right) \right) \]

\[ \times h \left( \frac{N^{1-\varepsilon}}{d} (2n + d \rho_{q_0 m/d}) \right) e \left( \frac{-sn^2}{\ell m d} \right) e_{\ell m}(s d n^2) + O \left( \frac{1}{N^C} \right), \]

and by another application of (60)

\[ \sum_{n \in \mathbb{Z}} \hat{F}_{d,m} \left( - \left( \frac{2n}{d} + \frac{\rho_{q_0 m/d}}{d} \right) \right) e_d(-b m n^2) = \]

\[ \sum_{n \in \mathbb{Z}} \hat{F}_{d,m} \left( - \left( \frac{2n}{d} + \frac{\rho_{q_0 m/d}}{d} \right) \right) h_m(n)e_{\ell m}(s d n^2) + O \left( \frac{1}{N^C} \right), \]

where

\[ h_m(x) = h \left( \frac{N^{1-\varepsilon}}{d} (2x + d \rho_{q_0 m/d}) \right) e \left( \frac{-sx^2}{\ell m d} \right). \]

Define

\[ F_{d,m}^{(1)}(x) = \hat{F}_{d,m} \left( - \left( \frac{2x}{d} + \frac{\rho_{q_0 m/d}}{d} \right) \right) h_m(x). \]

By the above and (57)

\[ S_{d,s_1,0} = \frac{1}{d} \sum_{d Y/q_0 \leq |m| \leq d N/q_0} e \left( g \left( \frac{q_0 m}{d} \right) \right) G(b m, 0; d) \sum_{n \in \mathbb{Z}} F_{d,m}^{(1)}(n)e_{\ell m}(s d n^2) \]

\[ + O \left( \frac{1}{N^{C-10}} \right). \]

Hence taking \( C \) sufficiently large

\[ S_{d,s_1,0} \ll \frac{1}{d} \sum_{d Y/q_0 s_1 \leq |m| \leq d N/q_0} e \left( g \left( \frac{q_0 s_1 m}{d} \right) \right) G(bs_1 m, 0; d) \sum_{n \in \mathbb{Z}} F_{d,s_1 m}^{(1)}(n)e_{\ell m}(s_2 d n^2), \]

where \( s_2 \) is given by

\[ s_1 s_2 = s. \]
By Lemma 2
\[
\sum_{n \in \mathbb{Z}} F_{d,s_{1}m}^{(1)}(n)e_{\ell m}(s_{2}dn^{2}) = \frac{1}{\ell m} \sum_{n \in \mathbb{Z}} \hat{F}_{d,s_{1}m}^{(1)}\left(-\frac{n}{\ell m}\right)G(s_{2}d, n; \ell m).
\]

Hence from (63)
\[
S_{d,s_{1},0} \ll \sum_{dY/q_{0}s_{1} \leq |m| \leq dN/q_{0}s_{1}} e\left(g\left(\frac{q_{0}s_{1}m}{d}\right)\right) \frac{G(bs_{1}m, 0; d)}{d\ell m}
\]
\[
\times \sum_{n \in \mathbb{Z}} \hat{F}_{d,s_{1}m}^{(1)}\left(-\frac{n}{\ell m}\right)G(s_{2}d, n; \ell m).
\]

Consider the inner summation over \(n\). Recall (62) and write
\[
F_{d,s_{1}m}^{(1)}(x) = F_{2}(x)h_{s_{1}m}(x),
\]
with
\[
F_{2}(x) = \hat{F}_{d,s_{1}m}\left(-\left(\frac{2x}{d} + \rho_{q_{0}s_{1}m/d}\right)\right).
\]

By Lemma 3
\[
\hat{F}_{d,s_{1}m}^{(1)}(x) = (\hat{F}_{2} \ast \hat{h}_{s_{1}m})(x) = \int_{\mathbb{R}} \hat{h}_{s_{1}m}(y) \hat{F}_{2}(x - y) dy,
\]
and from (61)
\[
\hat{h}_{s_{1}m}(y) = \int_{\mathbb{R}} h\left(\frac{N^{1-\varepsilon}}{d}(2z + d\rho_{q_{0}s_{1}m/d})\right) e\left(-\beta_{m}z^{2} - yz\right) dz
\]
\[
\ll \int_{\mathbb{R}} h\left(\frac{2N^{1-\varepsilon}}{d}\right) e\left(-\beta_{m}z^{2} - (y - \beta_{m}d\rho_{q_{0}s_{1}m/d})z\right) dz,
\]
where
\[
\beta_{m} = \frac{s_{2}}{\ell md}.
\]

If
\[
|m| \geq \frac{dY}{q_{0}s_{1}},
\]
then
\[
|\beta_{m}| \ll \frac{|s|q_{0}}{\ell d^{2}Y}.
\]
Repeated integration by parts shows that for any integer $k$

\[
\int_{\mathbb{R}} h\left(\frac{N^{1-\varepsilon}z}{d}\right) e\left(-\beta_m z^2 - (y - \beta_m d\rho_{qym/d})z\right) \, dz
\]

\[
\ll \frac{1}{|y - \beta_m d\rho_{qym/d}|^k} \int_{\mathbb{R}} \left| d^k \left\{ h\left(\frac{2N^{1-\varepsilon}}{d}\right) e(-\beta_m z^2) \right\} \right| \, dz,
\]

which combined with (59) implies

\[
\int_{\mathbb{R}} h\left(\frac{2N^{1-\varepsilon}z}{d}\right) e\left(-\beta_m z^2 - (y - \beta_m d\rho_{qym/d})z\right) \, dz
\]

\[
\ll \frac{1}{|y - \beta_m d\rho_{qym/d}|^k} N^{(1-\varepsilon)(k+1)} \left| \beta_m \right|^k + \frac{N^{2k(1-\varepsilon)}}{d^{2k}}.
\]

Let $M_d$ be given by (47). From (69) we obtain

\[
\int_{\mathbb{R}} h\left(\frac{N^{1-\varepsilon}z}{d}\right) e\left(-\beta_m z^2 - (y - \beta_m d\rho_{qym/d})z\right) \, dz
\]

\[
\ll \frac{N^{2k\varepsilon}}{|y - \beta_m d\rho_{qym/d}|^k} \left( \frac{dM_d^2}{N} \right)^k.
\]

If

\[
|y - \beta_m d\rho_{qym/d}| \geq \frac{dM_d^2 N^{10\varepsilon}}{N},
\]

then for any $C > 0$, by choosing $k$ large enough in terms of $\varepsilon$, the above implies

\[
\tilde{h}_{s1m}(y) \ll \frac{1}{|y - \beta_m d\rho_{qym/d}|^2} \frac{1}{N^{2C}}.
\]

Substituting into (67) gives

(70)

\[
\tilde{F}_{d,s1m}^{(1)}(x) = \int_{|y| \leq dM_d^2 N^{10\varepsilon}/N} \tilde{h}_{s1m}(y + \beta_m d\rho_{q0s1m/d}) \tilde{F}_{s2}(x - y - \beta_m d\rho_{q0s1m/d}) \, dy
\]

\[
+ O\left(\frac{1}{N^C}\right).
\]
Combining the above with (65) and choosing $C$ sufficiently large,

\[
S_{d,s,1,0} \ll \int_{|y| \leq dM_d N^{10\epsilon}/N} \left( g \left( \frac{q_0 s_1 m}{d} \right) \right) G(b s_1 m, 0; d) \frac{\hat{h}_{s_1 m}(y + \beta_m d \rho_{q_0 s_1 m/d})}{md\ell} \\
\times \sum_{n \in \mathbb{Z}} \hat{F}_2 \left( \frac{n}{\ell m} - y - \beta_m d \rho_{q_0 s_1 m/d} \right) G(s_2 \overline{d}, -n; \ell m) dy.
\]

(71)

Recalling (61), we have

\[
\hat{h}_{s_1 m}(y + \beta_m d \rho_{q_0 s_1 m/d}) = \int_{\mathbb{R}} h \left( \frac{2z N^{1-\epsilon}}{d} \right) \\
\times e \left( -\beta_m (z - d \rho_{q_0 s_1 m/d}/2)^2 -(y + \beta_m d \rho_{q_0 s_1 m/d})(z - d \rho_{q_0 s_1 m/d}/2) \right) dz.
\]

Substituting the above into (71), taking a maximum over $y, z$ and using that $h$ is supported in $[-2, 2]$ we get

\[
S_{d,s,1,0} \ll \left( \frac{dM_d N^{10\epsilon}}{N} \right)^2 \\
\times \sum_{dY/q_0 s_1 \leq |m| \leq dN/q_0 s_1} e \left( g \left( \frac{q_0 s_1 m}{d} \right) + g_0(m) \right) \frac{G(b s_1 m, 0; d)}{md\ell} \\
\times \sum_{n \in \mathbb{Z}} \hat{F}_2 \left( \frac{n}{\ell m} - y - \beta_m d \rho_{q_0 s_1 m/d} \right) G(s_2 \overline{d}, -n; \ell m),
\]

(72)

for some $y, z$ satisfying

\[
|y| \leq \frac{dM_d^2 N^{10\epsilon}}{N} \quad |z| \leq \frac{dN^{\epsilon}}{N},
\]

(73)

and

\[
g_0(m) = -\beta_m (z - d \rho_{q_0 s_1 m/d}/2)^2 -(y + \beta_m d \rho_{q_0 ms_1/d})(z - d \rho_{q_0 ms_1/d}/2).
\]

(74)
We next simplify summation over $n$. Recalling (66), for any real number $y$ we have
\[
\hat{F}_2 \left( \frac{2y}{d} \right) = \int_{-\infty}^{\infty} \hat{F}_{d,s_1m} \left( -\left( \frac{2x}{d} + \rho_{q_0s_1m/d} \right) \right) e(-2xy/d)dx
\]
\[
= \frac{d}{2} e(\rho_{q_0s_1m/d}y) \int_{-\infty}^{\infty} \hat{F}_{d,s_1m}(u)e(uy)du.
\]
Hence by Lemma 4
\[
(75) \quad \hat{F}_2 \left( \frac{2y}{d} \right) = \frac{d}{2} e(\rho_{q_0s_1m/d}y)F_{d,s_1m}(y).
\]
Substituting (75) into (72) gives
\[
\sum_{n \in \mathbb{Z}} \hat{F}_2 \left( \frac{n}{\ell m} - y - \beta_m d \rho_{q_0s_1m/d} \right) G(s_2 \overline{d}, -n; \ell m)
\]
\[
= \frac{d}{2} e \left( -\rho_{q_0s_1m/d} \left( \frac{dy}{2} + \frac{\beta_m d^2 \rho_{q_0s_1m/d}}{2} \right) \right)
\times \sum_{n \in \mathbb{Z}} F_{d,s_1m} \left( \frac{dn}{2\ell m} - \frac{dy}{2} - \frac{d^2 \beta_m \rho_{q_0s_1m/d}}{2} \right) G(s_2 \overline{d}, -n; \ell m)e \left( \rho_{q_0s_1m/d} \frac{dn}{2\ell m} \right).
\]
We next show only one value of $n$ contributes to summation in (76).

Recalling (29) and (33) if
\[
F_{d,s_1m} \left( \frac{dn}{2\ell m} - \frac{dy}{2} - \frac{d^2 \beta_m \rho_{q_0s_1m/d}}{2} \right) \neq 0,
\]
then
\[
(77) \quad N \leq \frac{dn}{2\ell m} - \frac{dy}{2} - \frac{d^2 \beta_m \rho_{q_0s_1m/d}}{2} \leq 2N.
\]
By (73)
\[
dy \ll \frac{d^2 M_d^2 N^{10\varepsilon}}{N},
\]
so that if $n$ satisfies (77) then
\[
\left| \frac{dn}{\ell m} - d^2 \beta_m \rho_{q_0s_1m/d} \right| \ll N + \frac{d^2 M_d^2 N^{10\varepsilon}}{N} \ll \frac{d^2 M_d^2 N^{10\varepsilon}}{N}.
\]
By (32), (64), (68) and using that $m \ll N$
\[
(78) \quad \left| n - \frac{s_1 s b q_0 m^2}{d^2} \right| \ll \ell d M_d^2 N^{10\varepsilon}.
\]
Recalling (46), for some integer $t$ we have
\[ sb = \ell + td, \]
which substituted into (78) gives
\[ \left| n - \frac{s_1 t q_0}{d} m^2 \right| \ll \ell \left( dM_d^2 N^{10\varepsilon} + \frac{s_1 q_0 m^2}{d^2} \right). \]

Using that $m \ll dN/q_0 s_1$, we have
\[ \frac{s_1 q_0 m^2}{d^2} \ll \frac{N^2}{q} \ll dM_d^2 N^{10\varepsilon}, \]
which simplifies (79) to
\[ \left| n - \frac{s_1 t q_0}{d} m^2 \right| \ll \ell dM_d^2 N^{10\varepsilon}. \]

By (48), the only term which contributes to summation in (76) is
\[ n = \frac{s_1 t q_0}{d} m^2. \]

This implies that
\[
\sum_{n \in \mathbb{Z}} \widehat{F}_2 \left( \frac{n}{\ell m} - y - \beta_m d \rho_{q_0 s_1 m/d} \right) G(s_2 \overrightarrow{d}, -n; \ell m) = \frac{d}{2} e \left( -\rho_{q_0 s_1 m/d} \left( \frac{dy}{2} + \frac{\beta_m d^2 \rho_{q_0 s_1 m/d}}{2} \right) \right) F_{d,s_1 m} \left( -\frac{s_1 q_0 m}{2d} - \frac{dy}{2} \right) \times G \left( s_2 \overrightarrow{d}, -\frac{s_1 t q_0}{d} m^2 ; \ell m \right) e \left( \rho_{q_0 s_1 m/d} \frac{s_1 t q_0 m}{2\ell} \right).
\]

We substitute the above into (72) then simplify. This gives
\[
S_{d,s_1,0} \ll \left( \frac{dM_d N^{10\varepsilon}}{N} \right)^2 \times \sum_{d'/q_0 s_1 \leq |m| \leq dN/q_0 s_1 \atop (m,q_0)=1 \atop (m,s)=1} F_{d,s_1 m} \left( -\frac{s_1 q_0 m}{2d} - \frac{dy}{2} \right) e \left( g \left( \frac{q_0 s_1 m}{d} \right) + g_0(m) + g_1(m) \right) \times \frac{G(bs_1 m, 0; d)G \left( s_2 \overrightarrow{d}, -\frac{s_1 t q_0}{d} m^2 ; \ell m \right)}{m \ell},
\]
where
\[ g_1(m) = -\rho_{q_0 s_1 m/d} \left( \frac{dy}{2} + \frac{\beta_m d^2 \rho_{q_0 s_1 m/d}}{2} \right) + \rho_{q_0 s_1 m/d} \frac{s_1 t q_0 m}{2\ell}. \]
Recalling (32), (68) and (74)

\[ g_0(m) + g_1(m) = -\beta_m(z - dp_{q_0s_1m/d}/2)^2 - (y + \beta_md p_{q_0ms_1/d})(z - dp_{q_0ms_1/d}/2) \]
\[ - \rho_{q_0s_1m/d} \left( \frac{dy}{2} + \frac{\beta_md^2 p_{q_0s_1m/d}}{2} \right) + \rho_{q_0s_1m/d} \frac{s_1 t q_0 m}{2\ell} \]
\[ = -\beta_mz^2 - \frac{\beta_md^2 p_{q_0s_1m/d}}{4} + \rho_{q_0s_1m/d} \frac{s_1 t q_0 m}{2\ell} - yz \]
\[ = -z^2s_2 \frac{\ell m d}{4\ell q_0} \left( \frac{q_0s_1m}{d} \right)^3 + \frac{bt d}{2\ell q_0} \left( \frac{q_0s_1m}{d} \right)^3 - yz. \]

Recalling (28), (30), (31) and (46) we have

\[ g \left( \frac{q_0s_1m}{d} \right) + g_0(m) + g_1(m) \]
\[ = \frac{a}{4q} \left( \frac{q_0s_1m}{d} \right)^3 + \frac{b_1 d}{4\ell q_0} \left( \frac{q_0s_1m}{d} \right)^3 + \frac{\gamma q_0 s_1 m}{d} - \frac{z^2 s_2}{\ell m d}, \]

for some \( b_1 \in \mathbb{Z} \). Hence with \( g^* \) given by

\[ g^*(x) = \frac{am^3}{4q} + \frac{b_1 dm^3}{4\ell q_0} + \gamma m, \]

we have

\[ S_{d,s_1,0} \ll \left( \frac{dMdN^{10e}}{N} \right)^2 \]
\[ \times \sum_{dY/q_0s_1 \leq |m| \leq dN/q_0s_1} \frac{F_{d,s_1m} \left( -\frac{s_1q_0 m}{2d} - \frac{dy}{2} \right) e \left( g^* \left( \frac{q_0s_1m}{d} \right) - \frac{z^2 s_2}{\ell m d} \right) \right.}{m \ell} \]
\[ \times G(bs_1m, 0; d)G \left( s_2d, \frac{s_1 t q_0 m^2}{d}; \ell m \right). \]

Our last step is to remove the terms \( F_{d,s_1m} \) and \( e(-z^2 s_2/(\ell m d)) \) using partial summation. Recalling (33)

\[ F_{d,s_1m} \left( -\frac{s_1q_0 m}{2d} - \frac{dy}{2} \right) = f \left( \frac{q_0s_1m}{2d} - \frac{dy}{2} \right) f \left( -\frac{q_0s_1m}{2d} - \frac{dy}{2} \right), \]

hence from (29) if \( m \) satisfies

\[ dY/q_0s_1 \leq |m| \leq dN/q_0s_1, \]

then

\[ \frac{dF_{d,s_1m} \left( -\frac{q_0s_1m}{2d} - \frac{dy}{2} \right)}{dm} \ll \frac{1}{m}. \]
By (73), for \( m \) satisfying
\[
|m| \geq dY/q_0s_1,
\]
we have
\[
\frac{de \left( -\frac{s_2s}{\ell md} \right)}{dm} \ll \frac{N^{2\epsilon} q_0 |s|}{\ell N^2 Y} \frac{1}{m}.
\]
Hence by Lemma 5, for some \( \gamma'' \in [0,1] \) we have
\[
S_{d,s,1} \ll \left( \frac{dM_d N^{13\epsilon}}{N} \right)^2 \left( 1 + \frac{q_0 |s|}{\ell N^2 Y} \right)
\]
(80)
\[
\sum_{dY/q_0s_1 \leq |m| \leq dN/q_0s_1} \sum_{(m,q_0)=1} e \left( g^* \left( \frac{q_0s_1m}{d} \right) + \gamma'' m \right) G(b_1 m, 0; d) G \left( \frac{s_2d}{\ell m}, -\frac{s_1q_0m}{\ell m^2}; \ell m \right).
\]

Returning to (56), we next indicate the necessary modifications to the above argument to estimate \( S_{d,s,1} \). We have
\[
S_{d,s,1} = \frac{1}{d} \sum_{dY/q_0 \leq |m| \leq dN/q_0} \sum_{(m,q_0)=1} \sum_{(m,s)=s_1} \sum_{n \in \mathbb{Z}} \widehat{F}_{d,m} \left( -\left( \frac{2n+1}{d} + \rho_{q_0m/d} \right) \right) G(bm, 2n+1; d),
\]
and hence by Lemma 12
\[
S_{d,s,1} = \frac{1}{d} \sum_{dY/q_0 \leq |m| \leq dN/q_0} \sum_{(m,q_0)=1} \sum_{(m,s)=s_1} \sum_{n \in \mathbb{Z}} \widehat{F}_{d,m} \left( -\left( \frac{2n+1}{d} + \rho_{q_0m/d} \right) \right) e_d(-bm(n^2 + n)).
\]
Using reciprocity as in (58) gives
\[
\sum_{n \in \mathbb{Z}} \widehat{F}_{d,m} \left( -\left( \frac{2n+1}{d} + \rho_{q_0m/d} \right) \right) e_d(-bm(n^2 + n)) =
\]
\[
\sum_{n \in \mathbb{Z}} \widehat{F}_{d,m} \left( -\left( \frac{2n+1}{d} + \rho_{q_0m/d} \right) \right) e_{\ell m}(s\bar{d}(n^2 + n)) e \left( -\frac{s(n^2 + n)}{\ell md} \right).
\]
Following the argument as in the case $S_{d,s,0}$ with changes to (61) and (62) given by
\[
h_m(x) = h \left( \frac{N^{1-\varepsilon}}{d} (2x + 1 + d\rho q_{0m/d}) \right) e \left( -\frac{s(x^2 + x)}{\ell md} \right),
\]
\[
F^{(1)}_{d,m}(x) = \hat{F}_{d,m} \left( -\left( \frac{2x + 1}{d} + \rho q_{0m/d} \right) \right) h_0(x),
\]
we arrive at an analogue of (71)
\[
S_{d,s,1} < \int_{|y| \leq dM_d^{2N^{13\varepsilon}/N}} \sum_{dY / q_0 s_1 \leq |m| < qN / q_0 s_1 \atop (m,q_0)=1 \atop (m,s)=1} \frac{e \left( \left( \frac{q_0 s_1 m}{d} \right) \right) G(b s_1 m / q_0 s_1, b s_1 m / d) \hat{h}_{s_1 m}(y + \beta_m d\rho q_{0s_1m/d})}{md\ell} \hat{h}_{s_1 m}(y + \beta_m d\rho q_{0s_1m/d})
\]
\[
\times \sum_{n \in \mathbb{Z}} \hat{F}_2 \left( \frac{n}{\ell m} - y - \beta_m d\rho q_{0s_1m/d} \right) G(s_2 \overline{d}, s_2 \overline{d} - n; \ell m) dy,
\]
where
\[
F_2(x) = \hat{F}_{d,s_1 m} \left( -\left( \frac{2x + 1}{d} + \rho q_{0s_1m/d} \right) \right).
\]
This modifies (75) to
\[
\hat{F}_2 \left( \frac{2y}{d} \right) = \frac{d}{2} e \left( \left( \rho q_{0s_1m/d} + \frac{1}{d} \right) y \right) F_{d,s_1 m}(y),
\]
and the rest of the proof is similar to before. Our variant of (80) becomes
\[
S_{d,s,1} \ll \left( \frac{dM_d N^{13\varepsilon}}{N} \right)^2 \left( 1 + \frac{q_0 s}{\ell N^2 Y} \right)
\]
\[
(81)
\]
\[
\sum_{dY / q_0 s_1 \leq |m| < qN / q_0 s_1 \atop (m,q_0)=1 \atop (m,s)=1} \frac{e \left( \left( \frac{q_0 s_1 m}{d} \right) + \gamma^m m \right) G(b m, b m; q) G(s_2 \overline{d}, s_2 \overline{d} - s_1 q_0 m^2; \ell m)}{m\ell},
\]
where $g^{**}$ is a polynomial of the form
\[
g^{**}(x) = \frac{a m^3}{4q} + \frac{b_2 d m^3}{4\ell q_0} + \gamma_2 m,
\]
for suitable $b_2 \in \mathbb{Z}$ and $\gamma_2 \in \mathbb{R}$.
Combining (80), (81) and (56) then taking a maximum over $s_1|s$ in (54) we complete the proof, after suitable renaming. \(\square\)
Our next step is to use results from Section 4 to simplify the sums $S_d^{(i)}$ occurring in Lemma 23. We first consider $S_d^{(1)}$.

**Lemma 24.** With notation and conditions as in Lemma 23, there exists a positive integer $j$ and a real number $\gamma_0$ such that

$$S_d^{(1)} \ll \frac{N^{o(1)} (d\ell)^{1/2}}{2^{j/2}} \sum_{dY/2^j q_0 s_1 \leq m \leq dN/2^j q_0 s_1} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2^j q_0 s_1 m}{d} \right) + \gamma_0 m \right).$$

**Proof.** First recall (52)

$$S_d^{(1)} = \sum_{dY/q_0 s_1 \leq |m| \leq dN/q_0 s_1} e \left( g^* \left( \frac{q_0 s_1 m}{d} \right) \right) G(bs_1 m, 0; d) G \left( \frac{s_2 \overline{d}}{d} - \frac{s_1 t q_0 m^2}{d}; \ell m \right) \frac{m^\ell}{m^\ell}.$$  

We partition summation depending on the sign and largest power of 2 dividing $m$ to get

$$S_d^{(1)} \ll \sum_{j \leq \log N} S_{d,j,+}^{(1)} + S_{d,j,-}^{(1)},$$

where

$$S_{d,j,\pm}^{(1)} = \sum_{dY/q_0 s_1 \leq m \leq dN/q_0 s_1} e \left( g^* \left( \pm \frac{q_0 s_1 m}{d} \right) \right) \frac{G(\pm bs_1 m, 0; d) G \left( \frac{s_2 \overline{d}}{d} - \frac{s_1 t q_0 m^2}{d}; \ell m \right)}{m^\ell}.$$

$$= \sum_{dY/2^j q_0 s_1 \leq m \leq dN/2^j q_0 s_1} e \left( g^* \left( \pm \frac{2^j q_0 s_1 m}{d} \right) \right)$$

$$\times \frac{G(\pm 2^j bs_1 m, 0; d) G \left( \frac{s_2 \overline{d}}{d} - \frac{s_1 t q_0 m^2}{d}; \ell 2^j m \right)}{2^j m^\ell}.$$  

Fix some $j$ and consider $S_{d,j,\pm}^{(1)}$. We provide details for $S_{d,j,+}^{(1)}$ after taking complex conjugates a similar argument applies to $S_{d,j,-}^{(1)}$. For the right hand side of (83) to be nonzero, by Lemma 16 we must have either

$$d \equiv 0 \pmod{4}, \quad d \equiv 1 \pmod{4} \quad \text{or} \quad d \equiv 3 \pmod{4}.$$
If $d \equiv 0 \pmod{4}$ then for summation conditions in (83) to be nonempty we must have $j = 0$. Hence by Lemma 17

$$G(bs_1m, 0; d) = \left( \frac{d}{bs_1m} \right) (1 + i^{bs_1m}) d^{1/2}. \quad (84)$$

If either $d \equiv 1 \pmod{4}$ or $d \equiv 3 \pmod{4}$ then by Lemma 10 and Lemma 15

$$G(bs_1m, 0; d) = \varepsilon_d \left( \frac{2i^{bs_1m}}{d} \right) \left( \frac{d}{m} \right) (-1)^{(d-1)(m-1)/4} d^{1/2}, \quad (85)$$

where

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases} \quad (86)$$

In either case (84) or (85), we see that there exists some $\gamma_0 \in \mathbb{R}$ such that

$$S_d^{(1)} \ll d^{1/2} \sum_{dY/2^{j}q_0s_1 \leq m \leq dN/2^{j}q_0s_1} e \left( g^* \left( \frac{2i^{q_0s_1m}}{d} \right) + \gamma_0 m \right)$$

$$\times \left( \frac{d}{m} \right) G \left( \frac{s_2d}{2^{j}m}, -\frac{s_1q_02^{j}m^2}{d}, \ell^2 \ell m \right). \quad (87)$$

Fix $m$ satisfying conditions of summation in (87) and consider the term

$$G \left( s_2d, -\frac{s_1q_02^{j}m^2}{d}, \ell^2 \ell m \right).$$

We proceed on a case by case basis depending on whether $j = 0$ or not. If $j \geq 1$ then by Lemma 12

$$G \left( s_2d, -\frac{s_1q_02^{j}m^2}{d}, 2^j \ell m \right) = e_{2^j \ell m} \left( \frac{s_2s_1^2t^2q_02^{2j}2^4m^4}{d} \right) G(s_2d, 0; 2^j \ell m),$$

and hence there exists an integer $b_3$ such that

$$G \left( s_2d, -\frac{s_1q_02^{j}m^2}{d}, 2^j \ell m \right) = e_{\ell} (b_3m^3) G(s_2d, 0; 2^j \ell m).$$

Suppose next $j = 0$. If

$$\ell m \equiv 2 \pmod{4} \text{ and } \frac{s_1q_0m^2}{d} \equiv 0 \pmod{2},$$
then by Lemma 12 and Lemma 16
\[
G \left( \frac{s_2 \overline{d} - s_1 t q_0 2^{2j} m^2}{d}; 2^j \ell m \right) = 0.
\]

If
\[
\ell m \equiv 2 \mod 4 \quad \text{and} \quad \frac{s_1 t q_0 2^{2j} m^2}{d} \equiv 1 \mod 2,
\]
then writing \( \ell = 2 \ell_0 \), we have by Lemma 13 and Lemma 14
\[
G \left( \frac{s_2 \overline{d} - s_1 t q_0 2^{2j} m^2}{d}; 2^j \ell m \right) = 2 \epsilon(\ell) G \left( \frac{2 s_2 \overline{d} - s_1 t q_0 2^{2j} m^2}{d}; \ell_0 m \right)
\]
for some integer \( b'_3 \). Finally, if \( j = 0 \) and
\[
\ell m \equiv 1 \mod 2,
\]
then by Lemma 13
\[
G \left( \frac{s_2 \overline{d} - s_1 t q_0 2^{2j} m^2}{d}; 2^j \ell m \right) = \epsilon(\ell) G \left( \frac{2 s_2 \overline{d} - s_1 t q_0 2^{2j} m^2}{d}; \ell_0 m \right)
\]
for some integer \( b''_3 \). Combining the above with Lemma 15, Lemma 17, (87) and using that
\[
\left( \frac{d}{m} \right)^2 = 1,
\]
we get
\[
S_{d,j,+}^{(1)} \ll d^{1/2} \sum_{\substack{dY/2^j q_0 s_1 m \leq dN/2^j q_0 s_1 \\ (m, s q_0) = 1 \\ m \equiv 1 \mod 2}} \kappa_m \epsilon \left( g^* \left( \frac{2^j q_0 s_1 m}{d} \right) + \frac{b_3 m^3}{\ell} + \gamma'_0 m \right),
\]
for some integer \( b_3 \), real number \( \gamma'_0 \in \mathbb{R} \) and sequence \( \kappa_m \) satisfying \( \kappa_m \ll 1 \) which is periodic mod 4. Combining the above with (82) and taking a maximum over summation in \( j \), there exists \( b_4, \gamma''_0 \) and \( j \ll \log N \) such that
\[
S_{d}^{(1)} \ll N^{o(1)} \left( \frac{d}{\ell^2} \right)^{1/2} \sum_{\substack{dY/2^j q_0 s_1 m \leq dN/2^j q_0 s_1 \\ (m, s q_0) = 1 \\ m \equiv 1 \mod 2}} \kappa_m \epsilon \left( g^* \left( \frac{2^j q_0 s_1 m}{d} \right) + \frac{b_4 m^3}{\ell} + \gamma''_0 m \right).
\]
Recalling (49) and partitioning summation over $m$ into residue classes mod $4\ell$ and taking a maximum, we see that there exists some integer $u$ such that

\[ S_d^{(1)} \ll N^o(1) \left( \frac{d\ell}{s_1^2 \ell^j} \right)^{1/2} \times \sum_{dY/2^j q_0 s_1 \leq m \leq dN/2^j q_0 s_1 \atop (m, s_0 q_0) = 1 \atop m \equiv u \mod 4\ell} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2^j q_0 s_1 m}{d} \right) + \gamma'_0 m \right), \]

for suitable $\gamma'_0$. The result follows from Lemma 6 after renaming $\gamma'_0$. □

We next simplify the sums $S_d^{(2)}$. Inspecting the the proof of Lemma 25, it is sufficient to restrict the parameter $j$ to either 0, 1, although the statement below keeps presentation in line with Lemma 24.

**Lemma 25.** With notation and conditions as in Lemma 23, there exists a positive integer $j$ and a real number $\gamma_1$ such that

\[ S_d^{(2)} \ll N^o(1) \left( 1 + \frac{|s| q}{\ell^2 Y} \right) \left( \frac{d\ell}{2} \right)^{1/2} \times \sum_{dY/2^j q_0 s_1 \leq m \leq dN/2^j q_0 s_1 \atop (m, s_0 q_0) = 1 \atop (m, s) = 1} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2^j q_0 s_1 m}{d} \right) + \gamma_1 m \right). \]

**Proof.** Our proof is similar to Lemma 24 with some extra technical details. Recalling (53)

\[ S_d^{(2)} = \sum_{dY/\ell q_0 s_1 \leq |m| \leq dN/\ell q_0 s_1 \atop (m, q_0) = 1 \atop (m, s) = 1} e \left( g^{**} \left( \frac{q_0 s_1 m}{d} \right) \right) \frac{G(\overline{b s_1 m}, \overline{b s_1 m}; d) G \left( \frac{s_2 d, s_2 d - \frac{s_1 t q_0 m^2}{d}}{m \ell} \right)}{m \ell}, \]

and as before

\[ S_d^{(2)} \ll \sum_{j \leq \log N} S_d^{(2)}_{d,j,+} + S_d^{(2)}_{d,j,-}, \]
where

\[ S_{d,j,\pm}^{(2)} = \sum_{dY/2^{j}q_{0}s_{1} \leq m \leq dN/2^{j}q_{0}s_{1}} \sum_{(2^{m}, s_{0}) = 1}^{m \equiv 1 \mod 2} e\left( g^{*}\left( \pm \frac{2^{j}q_{0}s_{1}m}{d} \right) \right) \]

(89)

\[ G(\pm bs_{1}2^{j}m, \pm bs_{1}2^{j}m; d)G\left( s_{2}d, s_{2}d - \frac{s_{1}q_{0}2^{j}m^{2}}{d}; \pm \ell^{2}j\right) \times \left( \frac{\ell}{2m\ell} \right) . \]

We provide details only for \( S_{d,j,+}^{(2)} \). After taking complex conjugates, a similar argument applies to \( S_{d,j,-}^{(2)} \). Consider first the term

\[ G(bs_{1}2^{j}m, bs_{1}2^{j}m; d) . \]

For the right hand side of of summation in (89) to be nonzero, by Lemma 19 we may suppose either

\[ d \equiv 1 \mod 4, \quad d \equiv 3 \mod 4 \quad \text{or} \quad d \equiv 2 \mod 4 . \]

If either \( d \equiv 1 \mod 4 \) or \( d \equiv 3 \mod 4 \) then by Lemma 10, Lemma 13 and Lemma 15

\[ G(bs_{1}2^{j}m, bs_{1}2^{j}m; d) = \varepsilon_{d}(\frac{bs_{1}2^{j}m}{d})G(0, 0; d) \]

(90)

\[ = \varepsilon_{d}\varepsilon_{d}(\frac{bs_{1}2^{j}m}{d})\left( \frac{d}{m} \right) (-1)^{(m-1)(d-1)/4}d^{1/2}, \]

with \( \varepsilon_{d} \) as in (86). If \( d \equiv 2 \mod 4 \) then by Lemma 18

\[ G(bs_{1}2^{j}m, bs_{1}2^{j}m; d) = 2G\left( bs_{1}2^{j}m, bs_{1}2^{j}m; \frac{d}{2} \right) . \]

Since \( d/2 \) is odd, by Lemma 10, Lemma 13 and Lemma 15

\[ G(bs_{1}2^{j}m, bs_{1}2^{j}m; d) = 2e_{d/2}(\frac{bs_{1}2^{j}m}{d/2})G\left( \frac{bs_{1}2^{j}m}{d/2}, 0; \frac{d}{2} \right) \]

(91)

\[ = \varepsilon_{d/2}\varepsilon_{d/2}(\frac{bs_{1}2^{j}m}{d/2})(2d)^{1/2} \times (-1)^{(m-1)(d/2-1)/4}(2d)^{1/2}. \]
Define
\[
\begin{align*}
    d_0 &= \begin{cases} 
    d & \text{if } d \equiv 1 \mod 2 \\
    \frac{d}{2} & \text{if } d \equiv 2 \mod 4,
    \end{cases} \\
    \eta &= \begin{cases} 
    1 & \text{if } d \equiv 1 \mod 2 \\
    2 & \text{if } d \equiv 2 \mod 4.
    \end{cases}
\end{align*}
\]
and
\[
\begin{align*}
    \eta &= \begin{cases} 
    1 & \text{if } d \equiv 1 \mod 2 \\
    2 & \text{if } d \equiv 2 \mod 4.
    \end{cases}
\end{align*}
\]
Combining (89), (90) and (91), we see that there exists some \(\gamma_2 \in \mathbb{R}\) such that
\[
S_{d,j,+}^{(2)} \ll d^{1/2} \sum_{dY/2^j \leq m \leq dN/2^j} e \left( 2^{-j} \eta_0 s_1 m \right) + \gamma_2 m) \quad e(d_0(-4\eta bs_1 2^j m))
\]
\[
\times \left( \frac{d_0}{m} \right) G \left( \frac{s_2 \eta d_0, s_2 \eta d_0 - s_1 t q_0 2^j m^2}{\eta d_0}; \ell 2^j m \right)
\]
We have
\[
G \left( \frac{s_2 \eta d_0, s_2 \eta d_0 - s_1 t q_0 2^j m^2}{\eta d_0}; \ell 2^j m \right) = G \left( s_2 \eta d_0, 1 - s_2 \eta d_0, \ell 2^j m \right).
\]
We next proceed on a case by case basis depending on whether \(j = 0\) or not. If \(j \geq 1\) then by (95) and Lemma 12, there exists some integer \(b_4\) and real number \(\gamma_2^*\) such that
\[
G \left( \frac{s_2 \eta d_0, s_2 \eta d_0 - s_1 t q_0 2^j m^2}{\eta d_0}; \ell 2^j m \right) = e \left( \frac{b_4 m^3}{\ell} + \gamma_2^* m \right) G(s_2 \eta d_0, s_2 \eta d_0; \ell 2^j m).
\]
By Lemma 19, if summation in (94) is nonzero we must have \(j = 1\) and \(\ell m\) odd. Hence it is sufficient to estimate \(S_{j,1,+}^{(2)}\) with the condition \(\ell\) odd. By the above, Lemma 13 and Lemma 18
\[
G \left( \frac{s_2 \eta d_0, s_2 \eta d_0 - 2s_1 t q_0 m^2}{\eta d_0}; \ell m \right) = 2e \left( \frac{b_4 m^3}{\ell} + \gamma_2^* m \right) G(2s_2 \eta d_0, s_2 \eta d_0; \ell m)
\]
\[
= 2e \left( \frac{b_4 m^3}{\ell} + \gamma_2^* m - \frac{2\eta d_0 s_2}{\ell m} \right) G(2s_2 \eta d_0, 0; \ell m).
\]
Substituting into (94) and recalling $j = 1$, (45) and (51)

\[ S_{d,1,+}^{(2)} \ll d^{1/2} \sum_{dY/2q_0s_1 \leq m \leq dN/2q_0s_1 \atop (2m,sq_0) = 1 \atop m \equiv 1 \mod 2} e \left( g^{**} \left( \frac{2q_0s_1m}{d} \right) + \frac{b_4m^3}{\ell} + \gamma_2m \right) \]

(96)

\[
\times \frac{e \left( -s_2 \left( \frac{2^3\eta\ell m}{d_0} + \frac{2^3\eta d_0}{\ell m} \right) \right) (d_0)}{m}\]

Using reciprocity for modular inverses, we have

\[
\frac{2^3\eta\ell m}{d_0} + \frac{2^3\eta d_0}{\ell m} \equiv -\frac{d_0}{2^3\eta\ell m} + \frac{2^3\eta d_0}{\ell m} + \frac{1}{d_02^3\eta\ell m} \mod 1,
\]

and since

\[
\frac{d_0}{2^3\eta\ell m} \equiv \frac{2^3\eta d_0}{\ell m} + \frac{\ell ml0}{2^3\eta} \mod 1,
\]

we get

(97)

\[
\frac{2^3\eta\ell m}{d_0} + \frac{2^3\eta d_0}{\ell m} \equiv -\frac{\ell ml0}{2^3\eta} + \frac{1}{d_02^3\eta\ell m} \mod 1.
\]

Using the above in (96), recalling (93) and partitioning summation over $m$ into residue classes mod 16, there exists some integer $\nu_1$ such that

\[
\sum_{j \geq 1} S_{d,j,+}^{(2)} \ll d^{1/2} \sum_{dY/2q_0s_1 \leq m \leq dN/2q_0s_1 \atop (2m,sq_0) = 1 \atop m \equiv \nu_1 \mod 16} e \left( g^{**} \left( \frac{2q_0s_1m}{d} \right) + \frac{b_4m^3}{\ell} + \gamma_2m + \frac{s_2}{d_02^3\eta\ell m} \right) \]

(98)

\[
\times \left( \frac{d_0}{m} \right) G(2s_2\eta d_0, 0; \ell m) \frac{G}{m\ell}.
\]

Using Lemma 15 and noting summation over $m$ in (98) is restricted to a fixed congruence class mod 16 we get

\[
\sum_{j \geq 1} S_{d,j,+}^{(2)} \ll \left( \frac{d}{\ell} \right)^{1/2}
\]

\[
\times \sum_{dY/2q_0s_1 \leq m \leq dN/2q_0s_1 \atop (2m,sq_0) = 1 \atop m \equiv \nu_1 \mod 16} \frac{1}{m^{1/2}} e \left( g^{**} \left( \frac{2q_0s_1m}{d} \right) + \frac{b_4m^3}{\ell} + \gamma_2m + \frac{s_2}{d_02^3\eta\ell m} \right).
\]
Using Lemma 5 to remove the term \( s_2/(d_0 2^3 \eta \ell m) \), the above simplifies to

\[
\sum_{j \geq 1} S_{d,1,+}^{(2)} \ll N^{o(1)} \left( 1 + \frac{|s| q}{ld^2 Y} \right) \left( \frac{d}{\ell} \right)^{1/2} \times \sum_{dY/2q_0 s_1 \leq m \leq dN/2q_0 s_1 \atop (m,q_0) = 1 \quad m \equiv 1 \mod 16} \frac{1}{m^{1/2}} e \left( \frac{2q_0 s_1 m}{d} + \frac{b_4 m^3}{\ell} + \gamma_2' m \right),
\]

for some real number \( \gamma_2' \). Finally, partitioning summation into residue classes mod \( \ell \), recalling (50) and using Lemma 6 as in the proof of Lemma 24 we obtain

\[
\sum_{j \geq 1} S_{d,1,+}^{(2)} \ll N^{o(1)} (d\ell)^{1/2} \left( 1 + \frac{|s| q}{ld^2 Y} \right) \times \sum_{dY/2q_0 s_1 \leq m \leq dN/2q_0 s_1 \atop (m,q_0) = 1 \quad m \equiv 1 \mod 2} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2q_0 s_1 m}{d} \right) + \gamma_1 m \right),
\]

for some real number \( \gamma_1 \).

Returning to (94), consider next when \( j = 0 \). Using (95), we have

\[
S_{d,0,+}^{(2)} \ll d^{1/2} \sum_{dY/q_0 s_1 \leq m \leq dN/q_0 s_1 \atop (m,q_0) = 1 \quad m \equiv 1 \mod 2} e \left( g^{**} \left( \frac{q_0 s_1 m}{d} \right) + \gamma_2 m \right) e_{d_0} \left( -4\eta bs_1 m \right) \times \left( \frac{d_0}{m} \right) G \left( s_2 \eta d_0, 1 - s_1 s_2 t q_0 m^2; \ell m \right). 
\]

Let \( 2^r \) be the largest power of 2 dividing \( \ell \) and write

\[
\ell = 2^r \ell_1,
\]

with \( \ell_1 \) odd. By Lemma 14

\[
G \left( s_2 \eta d_0, 1 - s_1 s_2 t q_0 m^2, \ell m \right) = \\
G \left( s_2 \eta d_0 \ell_1 m, 1 - s_1 s_2 t q_0 m^2, 2^r \right) G \left( 2^r s_2 \eta d_0, 1 - s_1 s_2 t q_0 m^2, \ell_1 m \right).
\]
From the above and (100), there exists integers $\nu_1, \nu_2, \nu_3$ with $\nu_1, \nu_2$ odd such that

$$S_{d,0,+}^{(2)} \ll d^{1/2} G(\nu_2, \nu_3, 2^r) \sum_{dY/q_0s_1 \leq m \leq dN/q_0} e\left( \frac{q_0s_1m}{d} \right) e_{d_0}\left( -4\eta bs_1m \right)$$

\begin{align*}
\times \frac{\left( d_0 \right)}{m} & \frac{G\left( 2^r \overline{s_2} \eta d_0, 1 - s_1 \overline{s_2} t_0 m^2, \ell_1 m \right)}{m \ell_1}. \\
\end{align*}

With notation as in (86), by Lemma 13 and Lemma 15

$$G\left( 2^r \overline{s_2} \eta d_0, 1 - s_1 \overline{s_2} t_0 m^2, \ell_1 m \right)$$

\begin{align*}
&= e_{\ell_1 m}\left( s_2 \eta d_0 2^{r+2}(1 - s_1 \overline{s_2} t_0 m^2)^2 \right) G\left( 2^r \overline{s_2} \eta d_0, 0; \ell_1 m \right) \\
&= e_{\ell_1 m} e_{\ell_1 m}\left( -s_2 \eta d_0 2^{r+2}(1 - s_1 \overline{s_2} t_0 m^2)^2 \right) \left( \frac{2^r s_2 \eta d_0}{\ell_1 m} \right) (\ell_1 m)^{1/2}.
\end{align*}

Substituting into (102), there exists an integer $b_5$ and real number $\gamma'_2$ such that

$$S_{d,0,+}^{(2)} \ll \left( \frac{2^r d}{\ell_1} \right)^{1/2} \sum_{dY/q_0s_1 \leq m \leq dN/q_0} \frac{1}{m^{1/2}} e\left( \frac{q_0s_1m}{d} \right) + \frac{s_2 m^3}{\ell_1} + \gamma'_2m)$$

\begin{align*}
\times e\left( -\left( \frac{4\eta bs_1m}{d_0} + \frac{s_2 \eta d_0 2^{r+2}}{\ell_1 m} \right) \right), \\
\end{align*}

where we have used the bound

$$G(\nu_2, \nu_3, 2^r) \ll 2^{r/2}.$$

From (45), (51) and (101)

$$\frac{4\eta bs_1 m}{d_0} + \frac{s_2 \eta d_0 2^{r+2}}{\ell_1 m} \equiv s_2 \left( \frac{2^{r+2} \eta \ell_1 m}{d_0} + \frac{2^{r+2} \eta d_0}{\ell_1 m} \right) \mod 1,$$

and by reciprocity for modular inverses

$$\frac{4\eta bs_1 m}{d_0} + \frac{s_2 \eta d_0 2^{r+2}}{\ell_1 m} \equiv s_2 \left( -\frac{d_0}{2^{r+2} \eta \ell_1 m} + \frac{2^{r+2} \eta d_0}{\ell_1 m} + \frac{s_2 d_0 2^{r+2} \eta \ell_1 m}{d_0} \right) \mod 1$$

$$\equiv -\frac{s_2 d_0 \ell_1 m}{2^{r+2} \eta} + \frac{s_2}{4d_0 \eta \ell m} \mod 1.$$
Substituting into (104) and using Lemma 5 as before, we arrive at

\[ S_{d,0,+}^{(2)} \ll N^{o(1)} \left( 1 + \left| \frac{s|q}{\ell d^2 Y} \right| \left( \frac{2^r d}{\ell_1} \right)^{1/2} \right. \]

\times \sum_{dY/q_0s_1 \leq m \leq dN/q_0s_1 \atop (m,q_0)=1 \atop m \equiv \nu \mod 2^{r+3}} \frac{1}{m^{1/2}} e \left( \frac{g^{**} \left( \frac{q_0s_1m}{d} \right)}{\ell_1} + \frac{b_\gamma m^3}{\ell_1} + \gamma'' m \right),

for some real number \( \gamma'' \). Partitioning summation into residue classes mod \( \ell_1 \) and using Lemma 6 as in the proof of Lemma 24 gives

\[ S_{d,0,+}^{(2)} \ll N^{o(1)} \left( 1 + \left| \frac{s|q}{\ell d^2 Y} \right| \left( \frac{d\ell}{\ell_1} \right)^{1/2} \sum_{dY/q_0s_1 \leq m \leq dN/q_0s_1 \atop (m,q_0)=1} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{q_0s_1m}{d} \right) + \gamma'' m \right), \]

and the result follows combining the above with (88), (99) and renaming terms.

Combining Lemma 23, Lemma 24 and Lemma 25 and taking a maximum over \( S_{d}^{(1)} \) and \( S_{d}^{(2)} \) we deduce the following.

**Lemma 26.** With notation and conditions as in Lemma 23, there exists a positive integer \( j \) and some \( \gamma_1 \in \mathbb{R} \) such that

\[
S_{d} \ll \left( \frac{dM_N 14^e}{N} \right)^2 \left( 1 + \left| \frac{q|s}{\ell N^2 Y} \right| \left( 1 + \left| \frac{q|s}{\ell d^2 Y} \right| \left( \frac{d\ell}{M_1} \right)^{1/2} \right. \right.

\times \sum_{dY/2^{j}q_0s_1 \leq m \leq dN/2^{j}q_0s_1 \atop (m,q_0)=1} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2^{j}q_0s_1m}{d} \right) + \gamma_1 m \right).

**Lemma 27.** With notation and conditions as in Lemma 23, there exists integers \( a_1, d, M, y \) satisfying

\[
\frac{Y}{2s} \leq M_1 \leq 2N, \quad d|q_0, \quad (a_1, d) = 1,
\]

and some \( \gamma_1 \in \mathbb{R} \) such that writing

\[
q_1 = (q, 3)d,
\]

we have

\[
S_{d} \ll \left( \frac{dM_N 16^e}{N} \right)^2 \left( 1 + \left| \frac{q|s}{\ell N^2 Y} \right| \left( 1 + \left| \frac{q|s}{\ell d^2 Y} \right| \left( \frac{q\ell}{M_1} \right)^{1/2} \right. \right.

\times \sum_{1 \leq m \leq q_1 M_1/yq} e \left( \frac{a_1 y^3 q^2 m^3}{4q_1^3} + \gamma_1 m \right).
\]
Proof. Let

\begin{equation}
S = \sum_{dY/2^jq_0s_1 \leq m \leq dN/2^jq_0s_1, (m, q_0s_0) = 1} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2^j q_0s_1 m}{d} \right) + \gamma_1 m \right),
\end{equation}

so by Lemma 26

\begin{equation}
S_d \ll \left( \frac{dMdN^{14\varepsilon}}{N} \right)^2 \left( 1 + \frac{q|s|}{\ell N^2 Y} \right) \left( 1 + \frac{q|s|}{\ell d^2 Y} \right) \left( \frac{d\ell}{2}\right)^{1/2}. \tag{107}
\end{equation}

In (107) we partition summation over \( m \) into dyadic intervals and take a maximum to obtain some \( M \) satisfying

\begin{equation}
Y/2 \leq M \leq 2N,
\end{equation}

such that

\begin{equation}
S \ll N^{o(1)} \sum_{dY/2^jq_0s_1 \leq m \leq dN/2^jq_0s_1, dM/2^{j+1}q_0s_1 \leq m \leq dM/2^jq_0s_1, (m, q_0s_0) = 1} \frac{1}{m^{1/2}} e \left( \frac{1}{4} g \left( \frac{2^j q_0s_1 m}{d} \right) + \gamma_1 m \right).
\end{equation}

By Lemma 5 and Lemma 7

\begin{equation}
S \ll \frac{2^{j/2}q^{1/2}s_1^{1/2}N^{o(1)}}{d^{1/2}M^{1/2}} S', \tag{110}
\end{equation}

where

\begin{equation}
S' = \sum_{1 \leq m \leq dM/2^j q_0s_1, (m, q_0s_0) = 1} e \left( \frac{1}{4} g \left( \frac{2^j q_0s_1 m}{d} \right) + \gamma'_1 m \right),
\end{equation}

for some \( \gamma'_1 \in \mathbb{R} \). Detecting the condition \((m, q_0s)\) via the Möbius function

\begin{align*}
S' &= \sum_{1 \leq m \leq dM/2^j q_0s_1} \sum_{f|(m, q_0s)} \mu(f) e \left( \frac{1}{4} g \left( \frac{2^j q_0s_1 m}{d} \right) + \gamma'_1 m \right) \\
&= \sum_{f|q_0s} \mu(f) \sum_{1 \leq m \leq dM/2^j q_0s_1, m = 0 \mod f} e \left( \frac{1}{4} g \left( \frac{2^j q_0s_1 m}{d} \right) + \gamma'_1 m \right).
\end{align*}

Taking a maximum over \( f \), there exists some \( f|q_0s \) such that

\begin{equation}
S' \ll q^{o(1)} \sum_{1 \leq m \leq dM/2^j q_0s_1 f} e \left( \frac{1}{4} g \left( \frac{2^j f q_0s_1 m}{d} \right) + \gamma''_1 m \right),
\end{equation}

for some \( \gamma''_1 \in \mathbb{R} \).
for some \( \gamma'''' \in \mathbb{R} \). Recalling (28) and (30), we have

\[
S' \ll q^{o(1)} \sum_{1 \leq m \leq dM/2^jq_0s} e\left( \frac{a2^{3j}f^3q_0^2s^3m^3}{4(q,3)d^3} + \gamma''''m \right),
\]

for some \( \gamma'''' \in \mathbb{R} \). Let

\[
a_1 = as_1^3, \quad y = 2^jf, \quad q_1 = (q,3)d, \quad M_1 = \frac{M}{s_1}.
\]

Since \( s \) satisfies (45) and \( s_1 | s \), we have

\[
(a_1s_1^3, q_1) = 1,
\]

and from (109)

\[
\frac{Y}{2s} \leq \frac{Y}{2s_1} \leq M_1 \leq 2N.
\]

By (113) and (114), the conditions (131) are satisfied. Using the notation (112) and recalling (30), this gives

\[
S' \ll q^{o(1)} \sum_{1 \leq m \leq q_1M_1/yq} e\left( \frac{a_1q^2y^3m^3}{4q_1^3} + \gamma''''m \right).
\]

Combining the above with (108) and (111) results in

\[
S_d \ll \left( \frac{dMqN^{16\epsilon}}{N} \right)^2 \left( 1 + \frac{q|s|}{\ell N^2Y} \right) \left( 1 + \frac{q|s|}{\ell d^2Y} \right) \left( \frac{q\ell}{M_1} \right)^{1/2} \times \sum_{1 \leq m \leq q_1M_1/yq} e_{q_1} \left( \frac{a_1y^3q^2m^3}{q_1^2} + \gamma''''m \right),
\]

from which the desired result follows after renaming \( \gamma'''' \).

\( \square \)

7. Reduction to an iterative inequality

We next combine Lemma 22 and Lemma 27 to reduce to an iterative type inequality.

**Lemma 28.** For integers \( a, q \) satisfying \( (a, q) = 1 \) and \( \gamma \in \mathbb{R} \) let

\[
g(x) = \frac{ax^3}{q} + \gamma x.
\]

Let \( \varepsilon, \rho, \delta > 0 \) be small and suppose \( N \) satisfies

\[
\frac{N^{1/2+\varepsilon}}{q^{1/4}} \ll \delta \ll 1,
\]

and

\[
q^{1/3} \leq N \leq \left( \delta^4q \right)^{1/(2+4\rho+22\varepsilon)}.
\]
Let $f$ be a smooth function satisfying
\[ \text{supp}(f) \subseteq [N, 2N], \quad f^{(j)}(x) \ll \frac{1}{x^j}. \]
and suppose that
\[ \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \gg \delta(qN)^{1/4+o(1)}. \]

There exists integers $a_1, q_1, M, y$ satisfying
\[ \frac{q}{q_1} \leq M \leq 2N, \quad q_1 | q, \quad 1 \leq y \leq \frac{q_1 M}{q}, \quad (a_1, q_1) = 1, \]
and some $\gamma_1 \in \mathbb{R}$ such that
\[ \delta^4(NM)^{1/2+o(1)} \ll N^{32\varepsilon} \left(1 + \frac{q_1^4 N^{4\rho}}{q^4 \delta^6}\right) \left(1 + \frac{N^{2+2\rho}}{\delta^4 q}\right) \times \sum_{1 \leq m \leq q_1 M/yq} e\left(\frac{a_1 y^3 q^2 m^3}{4q_1^3} + \gamma_1 m\right). \]

Proof. Let $d$ be as in Lemma 22 and recall the notation (31). We first show the assumptions (115) and (117) imply (119). Recalling (106) it is sufficient to show
\[ d \gg \delta^2 q. \]

Apply Lemma 22 to the sum
\[ \sum_{n \in \mathbb{Z}} f(n)e(g(n)), \]
with $Y = N$. From (117), this implies either
\[ \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \ll N^{1+o(1)}, \]
or there exists integers $\ell_0, s_0$ satisfying
\[ b \equiv \ell_0 s_0 \mod d, \quad \ell_0 \ll \frac{1}{\delta^4} \left(\frac{d}{q}\right)^2, \quad |s_0| \ll \frac{1}{\delta^4} \left(\frac{d}{q}\right)^2 N^3. \]

By (115) and (117) the inequality (120) does not hold, hence we must have (121). Since $(b, d) = 1$, we have $\ell_0 \geq 1$ which implies (119).

With notation as in Lemma 12, for integer $j$ define
\[ S_{d,j} = \sum_{dN^{1-\rho}/q \leq |m| \leq dN/q} e\left(g\left(\frac{qm}{d}\right)\right) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(\rho_{qm/dn})e_d(bmn^2), \]
and let $j_0$ denote the smallest nonnegative integer such that

$$\left| \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \right|^2 \ll q^{o(1)} S_{d,j_0}. \quad (122)$$

By Lemma 21, $j_0$ exists and satisfies $j_0 \leq \rho - 1$ and $j_0 \geq 1$ since otherwise

$$\left| \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \right|^2 \ll N^{1+o(1)},$$

contradicting (115) and (117). Apply Lemma 22 to the sum

$$\sum_{n \in \mathbb{Z}} f(n)e(g(n)),$$

with

$$Y = N^{1-(j_0-1)\rho}.$$

Either there exists $\ell, s$ satisfying

$$b \equiv \ell s^{-1} \mod d, \quad \ell \ll \frac{1}{\delta^4} \left( \frac{d}{q} \right)^2 N^{-(j_0-1)\rho}, \quad |s| \ll \frac{1}{\delta^4} \left( \frac{d}{q} \right)^2 \frac{N^{3-2(j_0-1)\rho}}{q}, \quad (124)$$

or

$$\left| \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \right|^2 \ll q^{o(1)} S_{d,j_0-1}. \quad (125)$$

By definition of $j_0$, (125) does not hold and hence we must have (124).

Writing

$$S_d = S_{d,j_0} = \sum_{dN^{1-j_0}/q \leq |m| \leq dN/q \atop (m,q_0)=1} e(g \left( \frac{q_0m}{d} \right)) \sum_{n \in \mathbb{Z}} F_{d,m}(n)e(q_0m/dn)e_d(bmn^2), \quad (126)$$

we apply Lemma 27 with

$$Y = N^{1-j_0\rho},$$

and $\ell, s$ as in (124). Recalling (47), these parameters give

$$M_d = \max \left\{ \frac{1}{\ell^{1/2}\delta^2} \frac{N^{1+\rho}}{q}, \frac{N}{d} \right\}.$$

This implies that

$$\frac{dM_d}{N} \ll \left( 1 + \frac{dN^\rho}{q\ell^{1/2}\delta^2} \right). \quad (127)$$
We also have
\[(128) \quad \left(1 + \frac{q|s|}{\ell N^2 Y}\right) \ll \left(1 + \frac{d^2 N^{2\rho}}{\delta^4 \ell q^2}\right),\]
and
\[(129) \quad \left(1 + \frac{q|s|}{\ell d^2 Y}\right) \ll \left(1 + \frac{N^{2+2\rho}}{\delta^4 q}\right).\]

From (116), (124) and (127)
\[(130) \quad \ell dM_2^2 N^{11\varepsilon} \ll \frac{N^{2+2\rho+11\varepsilon}}{d} \left(\ell + \frac{1}{\delta^4}\right) \ll \frac{N^{2+2\rho+11\varepsilon}}{q\delta^4} < 1,
\]
hence the condition (48) is satisfied. By Lemma 27, (127), (128) and (129), there exists integers \(a_1, q_1, M, y\) satisfying
\[(131) \quad \frac{N^{1-j\rho}}{2|s|} \leq M \leq 2N, \quad q_1 |q, \quad 1 \leq y \leq \frac{q_1 M}{q_0}, \quad (a_1, q_1) = 1,
\]
and some \(\gamma_1 \in \mathbb{R}\) such that
\[S_d \ll N^{32\varepsilon} \left(1 + \frac{q_1^4 N^{4\rho}}{q^4 \delta^6}\right) \left(1 + \frac{N^{2+2\rho}}{\delta^4 \ell q}\right) \left(\frac{q\ell}{M_1}\right)^{1/2} \times \sum_{1 \leq m \leq q_1 M/yq} e\left(\frac{a_1 y^3 q_1^2 m^3}{4q_1^3} + \gamma_1 m\right).\]

Combining the above with (117), (124) (122) and (126) we obtain
\[\delta^4 (NM)^{1/2+o(1)} \ll N^{32\varepsilon} \left(1 + \frac{q_1^4 N^{4\rho}}{q^4 \delta^6}\right) \left(1 + \frac{N^{2+2\rho}}{\delta^4 q}\right) \times \sum_{1 \leq m \leq q_1 M/yq} e\left(\frac{a_1 y^3 q_1^2 m^3}{4q_1^3} + \gamma_1 m\right),\]
which completes the proof. \(\square\)

**Corollary 29.** For integers \(a, q\) satisfying with \((a, q) = 1\) and \(\gamma \in \mathbb{R}\) let
\[g(x) = \frac{ax^3}{q} + \gamma x.\]

Let \(\varepsilon > 0\) be small and suppose \(N\) is an integer and \(\delta\) a real number satisfying
\[(132) \quad q^{1/3} \leq N \leq \delta^2 q^{1/2-o(1)},\]
\[(133) \quad \frac{N^{1/2+o(1)}}{q^{1/4}} \leq \delta < 1.\]
Let $f$ be a smooth function satisfying
\[ \text{supp}(f) \subseteq [N, 2N], \quad f^{(j)}(x) \ll \frac{1}{x^j}, \]
and suppose that
\[ \sum_{n \in \mathbb{Z}} f(n)e(g(n)) \gg \delta(qN)^{1/4 + o(1)}. \tag{134} \]

There exists integers $a_1, M, h, q_1$ satisfying
\[ 1 \leq M \leq N, \quad h \leq M, \quad q_1 | 4q, \quad (a_1, q_1) = 1, \]
and
\[ q_1 \gg \frac{\delta^2 q}{h^3}, \tag{135} \]
such that for some $\gamma_1 \in \mathbb{R}$ and polynomial $g_1$ of the form
\[ g_1(x) = \frac{a_1 x^3}{q_1} + \gamma_1 x, \]
we have
\[ \delta^{10} (NM)^{1/2} \ll N^{o(1)} \sum_{1 \leq m \leq M/h} e(g_1(m)). \tag{136} \]

Proof. Let $\varepsilon_1$ be small and $\varepsilon, \rho, M, q_1, y, a_1$ be as in Lemma 28. Taking $\varepsilon, \rho$ sufficiently small and using (133), we obtain from Lemma 28
\[ \delta^4 (NM)^{1/2} \ll N^{40\varepsilon} \left( 1 + \frac{q_1^4}{q^2 \delta^6} \right) \sum_{1 \leq m \leq q_1 M/yq} e \left( \frac{a_1 y^3 q^2 m^3}{4q_1^3} + \gamma_1 m \right). \]

provided $N$ satisfies (132). Define $h_1, h_2$ by
\[ h_1 = y, \quad h_2 = \frac{q}{q_1}, \]
and let
\[ d = (h_1^3 h_2^2, 4q_1), \quad q_2 = \frac{4q_1}{d}. \]

With this notation, (137) becomes
\[ \delta^4 (NM)^{1/2} \leq N^{40\varepsilon} \left( 1 + \frac{1}{h_2^4 \delta^6} \right) \sum_{1 \leq m \leq M/h_1 h_2} e \left( \frac{a_2 m^3}{q_2} + \gamma_1 m \right) \]
\[ \leq \frac{N^{40\varepsilon}}{\delta^6} \sum_{1 \leq m \leq M/h_1 h_2} e \left( \frac{a_2 m^3}{q_2} + \gamma_1 m \right), \]
for some $(a_2, q_2) = 1$. Defining
\[ h = h_1 h_2, \]
and using (119), we have

\[ q_2 \gg \frac{\delta^2 q}{h^3}, \]

which establishes (136) and the result follows after taking \( \varepsilon \) sufficiently small and renaming \( a_2, q_2 \).

\[ \Box \]

8. PROOF OF THEOREM 1

Suppose \( N \) satisfies

\[ q^{1/3+\rho} \leq N \leq q^{1/2-\rho/10+o(1)}, \tag{137} \]

and define

\[ \delta = q^{-\rho/20}. \tag{138} \]

Assume for contradiction that

\[ \sum_{1 \leq n \leq N} e \left( \frac{an^3}{q} + \gamma n \right) \gg \delta(qN)^{1/4+o(1)}. \]

By Lemma 9, there exists some \( K \leq 2N \), smooth function \( f \) satisfying

\[ \text{supp}(f) \subseteq [K, 2K], \quad f^{(j)}(x) \ll \frac{1}{x^j}, \]

and real number \( \gamma' \) such that

\[ \frac{\delta N^{1/4+o(1)}}{K^{1/4}} (qK)^{1/4} \ll \sum_{n \in \mathbb{Z}} f(n)e(g(n) + \gamma'n)). \tag{139} \]

Define

\[ \delta_0 = \frac{\delta N^{1/4}}{K^{1/4}}, \tag{140} \]

and note that

\[ \delta_0 \gg \delta. \]

Let \( \varepsilon > 0 \) be small. We may suppose

\[ K \geq q^{1/3+\varepsilon}, \]

since otherwise

\[ \delta \leq q^\varepsilon \left( \frac{q}{N^3} \right)^{1/12} \leq q^{-\rho/4+\varepsilon}. \]

By Corollary 29, there exists integers \( a_1, M, h, q_1 \) satisfying

\[ 1 \leq M \leq 2K, \quad h \leq M, \quad q_1 \mid 4q, \quad (a_1, q_1) = 1, \tag{141} \]
\[ q_1 \gg \frac{\delta^2 q}{h^3} \]

such that for some \( \gamma'_1 \in \mathbb{R} \) we have

\[
\delta^{10}_0 (KM)^{1/2} \ll N^{o(1)} \sum_{1 \leq m \leq M/h} e \left( \frac{a_1 m^3}{q_1} + \gamma'_1 m \right).
\]

If

\[ h \geq q^{\rho/2}, \]

then using the trivial bound in (148)

\[ \delta^{10} \ll \delta^{10}_0 \ll \frac{N^{o(1)}}{h} \ll q^{2\varepsilon - \rho/2} \]

Hence we may suppose \( h \leq Z \). By (142), this implies

\[
q_1 \gg \delta^2 q^{1-3\rho/2}.
\]

If

\[ \frac{M}{h} \leq q^{1/3 + \varepsilon}, \]

then using the trivial bound in (148)

\[
\delta^{10} \ll N^{o(1)} \left( \frac{M}{h} \right)^{1/2} \frac{1}{N^{1/2}} \ll q^{2\varepsilon} \left( \frac{q}{N^3} \right)^{1/6} \ll q^{2\varepsilon - \rho/2}.
\]

If

\[ q^{1/3} < \frac{M}{h} \leq q_1, \]

then by (1)

\[
\delta^{10}_0 (KM)^{1/2} \ll q^{1/4} \left( \frac{M}{h} \right)^{1/4 + o(1)} + \left( \frac{M}{h} \right)^{3/4 + o(1)}.
\]

Combining the above with (140), (144) and (146)

\[ \delta^{10} \ll q^{2\varepsilon - \rho/2} + q^{2\varepsilon - 1/12 - \rho/4} \ll q^{2\varepsilon - \rho/2}, \]

since we may suppose

\[
\rho \leq 1/6.
\]

Finally consider when

\[ \frac{M}{h} \geq q_1. \]
By [3, Corollary 1.2] and (144)

\[(148) \quad \delta^{10} \ll \frac{q^\varepsilon}{d_1^{1/3}} + q^{2\varepsilon - 1/12 - \rho/4} \ll \frac{q^{\varepsilon - 1/3 + \rho/2}}{\delta^{2/3}} + q^{2\varepsilon - \rho/2},\]

which implies that either

\[\delta^{10} \ll q^{2\varepsilon - \rho/2},\]

or

\[\delta^{32/3} \leq q^{\varepsilon - 1/3 + \rho/2}.\]

Combining the above estimates we obtain the desired result after recalling (147), taking \(\varepsilon\) sufficiently small and renaming \(\rho\).

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