Uniform embeddings, homeomorphisms and quotient maps between Banach spaces (A short survey)

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1. Introduction

A well known theorem of Mazur and Ulam [MU] states that if $T$ is an isometry from a Banach space $X$ onto a Banach space $Y$ such that $T0 = 0$ then $T$ is linear. Another well known (and deeper) theorem due to Kadec [K] states that any two separable infinite dimensional Banach spaces are mutually homeomorphic. Thus while the linear structure of a Banach space is completely determined by its structure as a metric space, the structure of a Banach space as a topological space contains no information on the linear structure.

In the present paper I consider the situation in between those extremes. Already if we weaken the isometry assumption by just a little and consider “almost isometries” we encounter interesting problems and results (e.g. the work of F. John in the context of the theory of elasticity [J] or the work around the Hyers-Ulam problem, see [Gev] and [LS]). I shall not discuss this topic here and move somewhat more away from isometry. The topics of our discussion here will be the structure of Banach spaces as uniform spaces and the Lipschitz structure of Banach spaces. It turns out that the study of these topics leads to a rich interplay between various areas: topology, geometric measure theory, probability, harmonic analysis, combinatorics and of course the geometry of Banach spaces.

In a book [BL] which is now being written and which will (hopefully) appear in 1998 there is a detailed study of many aspects of the structure of uniformly continuous functions and in particular Lipschitz functions on Banach spaces (e.g. extension of functions, differentiability, uniformly continuous selections, approximation theorems, fixed points etc.). It also contains a study of the almost isometric topics mentioned above. Here I shall survey the main results on three topics concerning these functions

(i) Uniform and Lipschitz embeddings of one Banach space into another.

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(ii) Uniform and Lipschitz classification of Banach spaces and their balls.
(iii) Uniform and Lipschitz quotient maps.

I will just state the main results, explain them and give references to the papers in which they were originally proved. For complete proofs, additional results and further references I refer to the forthcoming book.

It is worthwhile to mention that the embedding problems treated in (i) above have discrete analogues which lead to the study of natural problems on finite metric spaces (usually graphs with their obvious metric). These problems are of a combinatorial nature and are connected to topics in computer science. This direction is however not discussed here and again I refer to [BL] for a detailed treatment of this topic.

The theory of Lipschitz and uniformly continuous functions on Banach spaces has been developing in a slow but rather steady pace over the last 35 years and by now much is known in this direction. Nevertheless many basic and natural questions remain unanswered. In the last section of this paper I present a sample of open problems (not necessarily the central ones) which are related to the material discussed in the other sections.

Finally let me mention that the recent introductory text on Banach space theory [HHZ] contains in its last chapter (chapter 12) an introduction (with proofs) to the subject matter of the present survey.

2. Embeddings

Let us start by examining Lipschitz embeddings. The main question here is the following: Assume that there is a Lipschitz embedding $f$ from a Banach space $X$ into a Banach space $Y$ (i.e. $f$ is a Lipschitz injection and $f^{-1}$ is also Lipschitz on its domain of definition). Does this imply that $X$ is actually linearly isomorphic to a subspace of $Y$?

The main tool for handling this problem is differentiation. Let us recall the definition of the two main types of differentiation.

A map $f$ defined on open set $G$ in a Banach space $X$ into a Banach space $Y$ is called Gâteaux differentiable at $x_0 \in G$ if for every $u \in X$

\[
\lim_{t \to \infty} \frac{(f(x_0 + tu) - f(x_0))/t = D_f(x_0)u}
\]
exists and $D_f(x_0)$ (= the differential of $f$ at $x_0$) is a bounded linear operator from $X$ to $Y$.

The map $f$ is said to be Fréchet differentiable at $x_0$ if the limit (*) exists uniformly with respect to $u$ in the unit sphere of $X$, or, alternatively, if

$$f(x_0 + v) = f(x_0) + D_f(x_0)v + o(\|v\|) \quad \text{as} \quad \|v\| \to 0.$$  

Note that if $f$ is a Lipschitz map and $X$ is finite dimensional the notions of Gâteaux and Fréchet derivatives coincide. If, on the other hand, $\dim X = \infty$ then there are many natural examples of Lipschitz maps which are Gâteaux differentiable at a point without being Fréchet differentiable there (this is the source of a major difficulty in the area).

It is trivial that if $f$ is a Lipschitz embedding then at every point $x_0$ where $f$ is Gâteaux differentiable $D_f(x_0)$ is a linear isomorphism (into).

We are thus naturally led to the question of existence of a Gâteaux derivative. An important notion in this context is that of the Radon Nikodym Property (RNP in short). A Banach space $Y$ is said to have RNP if every Lipschitz functions $f$: $[0, 1] \to Y$ is differentiable almost everywhere. There are many equivalent definitions of RNP (involving e.g. vector measures or extremal structure of convex sets in $Y$) but the one given above is certainly the most natural in our context. Much is known about RNP. Obviously a subspace of a space with RNP has RNP and a Banach space has RNP if all its separable subspaces have RNP and that RNP is an isomorphism invariant. A result which goes back to Gelfand [Gel] is that a separable conjugate space has RNP and therefore all reflexive Banach spaces have RNP. The typical examples of spaces which fail to have RNP are $c_0$ and $L_1$ (in $L_1$ for example consider the function $f$ from $[0,1]$ to $L_1(0,1)$ defined by $f(t) =$ the characteristic function of the interval $[0, t]$).

The main theorem on Gâteaux differentiability is the following (proved independently at about the same time in [Ar], [Chr] and [Man]).

**Theorem 1.** Let $f$ be a Lipschitz function from a separable Banach space $X$ into a space $Y$ with RNP. Then $f$ is Gâteaux differentiable almost everywhere.

Since there is no natural measure on $X$ the term a.e. in the statement of the theorem needs explanation. Actually each of the 3 papers mentioned above uses a different notion
of a.e. (and the notions are definitely not equivalent) but all will suit us here. For example
Christensen calls a Borel set $A$ a null set if there is a Radon probability measure $\mu$ on $X$ so
that $\mu(A + x) = 0$ for all $x \in X$. It is easy to see that if $\dim X < \infty$ then $A$ is null in the
above sense iff $A$ is of Lesbegue measure 0. It is also not hard to verify that a countable
union of null sets is a null set and that a null set has empty interior.

An immediate corollary of the theorem is the following statement.

Assume $X$ is separable and that there is a Lipschitz embedding of it into a space $Y$
with RNP. Then there is a linear isomorphism from $X$ into $Y$.

What happens if $Y$ fails to have RNP? Let us check first the case $Y = c_0$. The
following result was proved by Aharoni [Ah1].

*Every separable Banach space is Lipschitz equivalent to a subset of $c_0$.*

Recall that $c_0$ is a “small” space, actually a minimal space in the following sense.
Any infinite dimensional subspace of $c_0$ has in turn a subspace isomorphic to $c_0$. Thus for
example $\ell_p$, $L_p$, $1 \leq p < \infty$ or $C(0, 1)$ all are not isomorphic to a subspace of $c_0$. Hence
any Lipschitz embedding of such a space into $c_0$ is nowhere Gâteaux differentiable. It
is also interesting to note that Aharoni’s result is equivalent to the statement that every
separable metric space Lipschitz embeds into $c_0$.

We turn to the other typical example of non RNP space, namely $L_1(0, 1)$. Here the
situation is entirely different. It is very likely that every Banach space $X$ which Lipschitz
embeds (even only uniformly embeds) into $L_1(0, 1)$ is already linearly isomorphic to a
subspace of $L_1(0, 1)$. This is definitely the case if $X$ is reflexive. This follows from the
discussion following Theorem 2 below. It is interesting to note that in this case we obtain
a result on linearization of Lipschitz embeddings which apparently cannot be proved by
differentiation.

We turn now to the question of existence of a uniform embedding (i.e. a uniform
homeomorphism into) of a Banach space $X$ into a Banach space $Y$. For this question the
only case in which a significant result is known is the case $Y = \ell_2$. The following theorem
was proved by Aharoni, Maurey and Mityagin [AMM].

**Theorem 2.** A Banach space $X$ is uniformly equivalent to a subset of $\ell_2$ if and only if $X$
is linearly isomorphic to a subspace of $L_0[0, 1]$. 

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The space $L_0(0,1)$ is the space of all measurable functions on $[0,1]$ with the topology of convergence in measure. The space $L_0(0,1)$ itself is of course not a Banach space and the theorem actually holds for general topological vector spaces $X$. It is unknown if every Banach space $X$ which is isomorphic to a subspace $L_0(0,1)$ is also isomorphic to a subspace of $L_1(0,1)$. It is known that this is the case if $X$ is reflexive. In particular the space $L_p(0,1)$ or $\ell_p$ is uniformly homeomorphic to a subset of $\ell_2$ if and only if $1 \leq p \leq 2$.

If $T$ is any map from a Banach space $X$ into $\ell_2$ then $K(x, y) = \langle Tx, Ty \rangle$ is a positive definite kernel of $X$. The proof of Theorem 2 starts with this observation and then the argument shifts to examining positive definite kernels on $X$ and much of the argument is probabilistic in nature. Let us also note that $K(x, y) = e^{-\|x-y\|^2}$ is a positive definite kernel on Hilbert space. There is a uniform embedding $T$ of $H$ into itself so that $\langle Tx, Ty \rangle = K(x, y)$. Since $K(x, x) = 1$ this map $T$ gives a uniform embedding of Hilbert space into its unit sphere. Thus whenever a Banach space (or a metric space) embeds uniformly into $\ell_2$ it also embeds uniformly into the unit sphere of $\ell_2$. In this connection it is of interest to note that from the results quoted in the next section it follows that there are many examples of Banach spaces $X$ which do not embed uniformly in $\ell_2$ but whose unit balls $B(X)$ do embed uniformly in $\ell_2$ (this is the case e.g. for $X = L_p$ or $\ell_p$ with $2 < p < \infty$).

3. Uniform and Lipschitz classification of spaces and balls

As in the previous section we start with Lipschitz mappings, this time with a bi-Lipschitz map $f$ between a Banach space $X$ and a Banach space $Y$. The natural question which arises is the following: is $X$ linearly isomorphic to $Y$? If $Y$ is separable (and hence also $X$) and has the RNP we can use Theorem 1 and get a Gâteaux derivative of $f$. Unfortunately the Gâteaux derivative may be an isomorphism into (Ives [I] has constructed a Lipschitz isomorphism from $\ell_2$ onto itself which has a Gâteaux derivative at 0, say, and this derivative is an isomorphism from $\ell_2$ onto a hyperplane). So one has to use another method to verify that $X$ is isomorphic to $Y$. One approach which comes to mind is to be careful with the choice of the point in which one takes the Gâteaux derivative (we know that it exists a.e. and in Ives’ example only one point behaves badly). It is however unknown how to do such a choice (one problem here is that it is unknown if a Lipschitz homeomorphism
carries null sets to null sets as it does in finite dimensional spaces). Another possibility is to use Fréchet derivatives. It is trivial that if $f$ is Fréchet differentiable at a point $x_0$ then $D_f(x_0)$ is a linear isomorphism from $X$ onto $Y$. Actually it is enough that $f$ is $\varepsilon$-Fréchet differentiable at some point for small enough $\varepsilon$.

We say that $f$ is $\varepsilon$-Fréchet differentiable at $x_0$ if there is a bounded linear operator $T$ from $X$ to $Y$ and a $\delta > 0$ so that

\[(**)
\|f(x_0 + u) - f(x_0) - Tu\| \leq \varepsilon \|u\| \text{ for } \|u\| \leq \delta.
\]

It is trivial to check that if $\varepsilon^{-1}$ is larger than the Lipschitz constant of $f^{-1}$ and $T$ is given by $(**)$ then $T$ is an isomorphism from $X$ onto $Y$.

The trouble is that there is no general theorem which ensures existence of Fréchet derivatives (or only $\varepsilon$-Fréchet derivatives) in this situation. The only result on existence of Fréchet derivatives of Lipschitz functions is the following deep result of Preiss [P].

**Theorem 3.** Assume $X$ is a Banach space with separable dual. Then every Lipschitz function $f$ from $X$ to $R$ is Fréchet differentiable on a dense set.

The assumption that $X^*$ is separable is natural here. If $X = \ell_1$ then $f(x) = \|x\|$ is nowhere Fréchet differentiable and a similar function can be built on any separable $X$ whose dual is not separable. The trouble with Theorem 3 is that one has in its conclusion a “dense set” and not a “null set” in any sense which would allow e.g. to find a common point of Fréchet differentiability for any given sequence of functions from $X$ to $R$ (this would have strong implications related to the problem we consider in this section). At present it is unknown e.g. if every Lipschitz function from $\ell_2$ to the plane has a point of Fréchet differentiability. For finitely many functions there is however a (also quite complicated) result on $\varepsilon$-Fréchet differentiability proved in [LP].

**Theorem 4.** Assume that $X$ is a separable superreflexive space, and let $f$ be a Lipschitz function from $X$ to $R^n$. Then for every $\varepsilon > 0$ $f$ has a point of $\varepsilon$-Fréchet differentiability.

A space $X$ is called superreflexive if it has an equivalent uniformly convex norm (in particular it is reflexive).
If $X$ is $\ell_p, 1 < p < \infty$, or more generally a space with an unconditional basis $\{e_i\}_{i=1}^{\infty}$ the map $f: X \to X$ defined by

$$f \left( \sum_{i=1}^{\infty} \lambda_i e_i \right) = \sum_{i=1}^{\infty} |\lambda_i| e_i$$

is a Lipschitz map and is nowhere even 1-Fr´echet differentiable. The same map can also be considered as a map from $\ell_r$ to $\ell_s$ if $s > r$. Thus in all these situations there cannot be any existence theorem for points of $\varepsilon$-Fr´echet differentiability for general Lipschitz functions.

In the next section we shall mention some positive results on $\varepsilon$-Fr´echet differentiability of Lipschitz functions between certain pairs of infinite-dimensional Banach spaces but these results by their very nature are not of use for the problem we consider in this section.

Heinrich and Mankiewicz [HM] found however a way to deduce linear isomorphism from Lipschitz equivalence in some rather general situations. Their argument is based on differentiation but the linear isomorphism they find is not a differential of the Lipschitz homeomorphism $f$. It is constructed from differentials of $f$ and $f^{-1}$ in a rather complicated way. They showed the following.

Let $f$ be a bi-Lipschitz map from a conjugate Banach space $X$ onto a Banach space $Y$. Assume that $f$ is Gâteaux differentiable at a point $x_0$. Then $D_f(x_0)$ is an isomorphism of $X$ onto a complemented subspace of $Y$.

The complementation assertion here is the new fact. In particular it follows from this and Theorem 1 that

If $X$ and $Y$ are Lipschitz equivalent separable conjugate spaces then each is isomorphic to a complemented subspace of the other.

In other words $X \approx Y \oplus U$ and $Y \approx X \oplus W$ for some $U$ and $W$. Pelczynski showed that under a mild additional hypothesis this implies that $X$ is actually isomorphic to $Y$ (it was shown recently by Gowers that without additional assumptions this is false). Anyhow it follows from the previous result that for many concrete pairs of spaces (actually only one needs to be “concrete”) Lipschitz equivalence implies linear isomorphism. In particular we have

**Theorem 5.** If $X$ is $L_p(0, 1)$ or $\ell_p$ with $1 < p < \infty$ and if $Y$ is Lipschitz equivalent to $X$ then $Y$ is isomorphic to $X$. 

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In other words the spaces $L_p(0,1)$ and $\ell_p$, $1 < p < \infty$, are determined by their Lipschitz structure. In [AL1] an example was constructed of two Banach spaces which are Lipschitz equivalent but nonisomorphic; one space is $c_0(\Gamma)$ for $\Gamma$ uncountable and the other is a suitable subspace of $\ell_\infty$. Other examples are known by now but they are of similar nature. In particular there is no known example of a pair of separable spaces which are Lipschitz equivalent but not linearly isomorphic.

We pass now to uniform homeomorphism between Banach spaces. A very simple but useful fact here is that a uniformly continuous map $f$ defined on a Banach space (in fact on any metrically convex metric space) is a Lipschitz map for large distances in the sense that for every $\varepsilon > 0$ there is a $C(\varepsilon)$ so that $\|f(x) - f(y)\| \leq C(\varepsilon)\|x - y\|$ whenever $\|x - y\| \geq \varepsilon$. Thus if $f: X \to Y$ is a uniformly continuous map $\lim_{n \to \infty} n^{-1}f(nx)$ is a Lipschitz map if the limit exists. In general there is no reason to assume that this limit exists but this can be remedied by passing to ultraproducts.

Recall that if $X$ is a Banach space and $\mathcal{U}$ is a free ultrafilter on the integers then $X_\mathcal{U}$ is defined to be the space of all the bounded sequences $\tilde{x} = (x_1, x_2, \ldots)$ of elements of $X$ with $\|\tilde{x}\|_\mathcal{U} = \lim_{n \in \mathcal{U}} \|x_i\|$ (modulo the sequences of norm 0) with the obvious vector operations. The space $X$ isometrically embeds into $X_\mathcal{U}$ via the mapping $x \mapsto (x, x, x, \ldots)$ but in general $X_\mathcal{U}$ is much larger than $X$ (if e.g. $X$ is separable and infinite dimensional then $X_\mathcal{U}$ is nonseparable). Nevertheless the finite dimensional structure of a space is not lost by passing to an ultraproduct. Any finite dimensional subspace of $X_\mathcal{U}$ is almost isometric to a finite dimensional subspace of $X$. It is worthwhile to introduce here the notion of Banach Mazur distance which we will use a few times below. Assume that $E$ and $F$ are finite dimensional Banach spaces of the same dimension. The Banach Mazur distance $d(E, F)$ between them is defined to be $\inf\{\|T\| \cdot \|T^{-1}\|\}$ where the infimum is taken over all linear maps $T$ from $E$ to $F$. (Actually $\log d(E, F)$ is a proper distance functions but we follow the common practice of discarding the log). In this terminology what we said on ultraproducts can be expressed as follows. For every finite dimensional subspace $E$ of $X_\mathcal{U}$ and every $\varepsilon > 0$ there is a finite dimensional subspace $F$ of $X$ with $d(E, F) \leq 1 + \varepsilon$.

If we go beyond the finite dimensional subspaces and ask about the structure of $X_\mathcal{U}$ itself we encounter often very difficult questions. There are however cases where the
situation is known. For example one gets easily from abstract characterizations of $L_p$ spaces that any ultraproduct of an $L_p(\mu)$ space (e.g. $L_p(0,1)$ or $\ell_p$) is an $L_p(\varphi)$ space for a different ("huge") measure space.

Coming back to uniform homeomorphisms one gets that if $f$ is a uniform homeomorphism from a Banach space $X$ onto a Banach space $Y$ then the map $\tilde{f}: X_\mathcal{U} \to Y_\mathcal{U}$ defined by

$$\tilde{f}(x_1, x_2, \ldots, x_n, \ldots) = (f(x_1), f(2x_2)/2, \ldots, f(nx_n)/n, \ldots)$$

is a Lipschitz homeomorphism ($\tilde{f}^{-1}$ is obtained from $f^{-1}$ in the same manner). In other words,

*Uniformly homeomorphic Banach spaces have Lipschitz equivalent ultrapowers.*

At the level of the ultrapowers one can now use differentiation (i.e. Gâteaux derivatives) to obtain linear maps. That is particularly useful if we consider finite dimensional spaces since they are essentially not changed by passing to an ultrapower. Also by using duality it can be shown that in this context it is possible to bypass the condition of RNP for existence of derivatives. In this way one can prove the following result due to Ribe [Ri1].

**Theorem 6.** Let $X$ and $Y$ be uniformly homeomorphic Banach spaces. Then there is a $C < \infty$ so that for every finite dimensional subspace $E$ of $X$ there is a subspace $F$ of $Y$ with $d(E, F) < C$ (and of course vice versa).

In other words the uniform structure of a Banach space determines (up to a constant) the linear structure of its finite dimensional subspaces. In particular if $X$ is $\ell_2$ and $Y$ is uniformly homeomorphic to $X$ then every finite-dimensional subspace $E$ of $Y$ satisfies $d(E, \ell^n_2) \leq C$ for a suitable $n$ and a $C$ independent of $n$. This trivially leads to the following result due to Enflo [E2].

*A Banach space which is uniformly homeomorphic to $\ell_2$ is already linearly isomorphic to $\ell_2$.*

This approach works only for $\ell_2$. The space $\ell_2$ is probably the only separable Banach space whose global structure is determined (up to isomorphism) by the structure of its collection of finite dimensional subspaces (though this it is not known as yet there are
strong partial results in this direction which show in particular that all other common spaces fail to have this property).

What is the situation for other Banach spaces? The first question which comes to mind in this connection is the following: Are $\ell_p$ and $L_p(0,1)$ uniformly homeomorphic for a fixed $p$, $1 \leq p < \infty$ and $p \neq 2$. On the one hand these spaces have common ultraproducts (and in particular the same finite dimensional structure) but on the other hand are not isomorphic. It turns out that they are not uniformly homeomorphic. This was first proved for $p = 1$ by Enflo then for $1 < p < 2$ by Bourgain [Bo] and finally by Gorelik [Gor] for $2 < p < \infty$. The proof of Gorelik is based on a general principle which in turn is based on Brouwer’s fixed point theorem.

**Theorem 7 (Gorelik’s principle).** Let $f$ be a uniform homeomorphism from a Banach space $X$ onto a Banach space $Y$. Assume that $f$ carries a ball centered at the origin and of radius $r$ in a subspace of finite codimension in $X$ into the $\rho$ neighborhood of a subspace of infinite codimension in $Y$, then $w(2r) \geq \rho/4$ where $w$ is the modulus of uniform continuity of $f$.

Using this principle it was proved in [JLS] that several spaces (besides $\ell_2$) are determined by their uniform structure. In particular.

**Theorem 8.** Any Banach space which is uniformly homeomorphic to $\ell_p$ $(1 < p < \infty)$ is already linearly isomorphic to $\ell_p$.

Probably most common Banach spaces are determined by their uniform structure but at present the proof of this is not in sight and any class seems to require a special method. An obvious reason for this difficulty is that in general Banach spaces are not determined by their uniform structure. It was proved by Ribe [Ri2] that if $\{p_n\}_{n=1}^{\infty}$ is a sequence strictly decreasing to $p > 1$ then the spaces $X$ and $X \oplus \ell_p$ are uniformly homeomorphic but not isomorphic where $X$ is the direct sum $\left( \bigoplus_{n=1}^{\infty} \ell_{p_n} \right)$ (i.e. the direct sum is taken in the $\ell_1$ norm). In [AL2] the argument of Ribe was modified so that it works also if 1 is replaced by $s$; $1 < s < p$ and thus one gets even a superreflexive and separable example. A perhaps more striking example is presented in [JLS]. Let $T_2$ be the “2 convexified Tsirelson space”. The precise definition of this space is rather complicated and not relevant here.
What matters is that $T_2$ is a superreflexive separable space which does not contain a copy of $\ell_2$ but which is “close to $\ell_2$” so that any ultrapower of $T_2$ is isomorphic to $T_2 \oplus \ell_2(\Gamma)$ for a suitable uncountable $\Gamma$. What is proved on $T_2$ (using the method of Ribe as well as a variant of Theorem 4 the Gorelik principle and other tools) is that any space uniformly homeomorphic to $T_2$ is linearly isomorphic to either $T_2$ or $T_2 \oplus \ell_2$ and that $T_2$ and $T_2 \oplus \ell_2$ are uniformly homeomorphic but not isomorphic. In other words the spaces uniformly homeomorphic to $T_2$ represent exactly two isomorphism classes of Banach spaces.

The construction of a uniform homeomorphism between nonisomorphic spaces is based in all the examples mentioned above on the fact that unit balls in completely different spaces can be mutually uniformly homeomorphic. The uniform structure of a ball in a Banach space contains in it some information on the linear structure of the space but as we shall see below only very little information. From the technical point of view the source of this difference between balls and the entire space is that for studying uniformly continuous maps on balls we cannot “go to infinity” and transfer the study to Lipschitz maps on ultraproducts.

We pass now to the study of balls. It was already noted by Mazur [Maz] that the natural nonlinear map $\varphi_{r,s}$ from $L_r(\mu)$ to $L_s(\mu)$ ($1 \leq r, s < \infty$) defined by $\varphi_{r,s}(f) = |f|^{r/s} \text{sign } f$ is a uniform homeomorphism between the unit balls of these spaces. Thus the unit balls of separable $L_p(\mu)$ spaces $1 \leq p < \infty$ are uniformly homeomorphic to the unit ball of $\ell_2$. This is the situation for a much larger class of Banach lattices. It turns out that the only obstruction to the uniform homeomorphism of balls in Banach lattices to the unit ball of Hilbert space is the presence of large cubes. Enflo [E1] proved that it is impossible to embed the unit balls of $\ell^n_\infty$ into Hilbert space with a distortion (i.e. $\|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}}$ where $f$ is the embedding) bounded by a constant independent on $n$. Actually, he proved a somewhat stronger statement and this is one of the first results in the theory of embedding discrete metric spaces into Banach spaces to which we hinted at the end of the introduction. The following theorem was proved by Odell and Schlumprecht [OS] for discrete Banach lattices (i.e. spaces with an unconditional basis) and by Chaatit [Cha] for general lattices.

**Theorem 9.** The unit ball of a separable infinite dimensional Banach lattice $X$ is uni-
formly homeomorphic to the unit ball of $\ell_2$ if and only if $X$ does not contain finite dimensional subspaces $\{E_n\}_{n=1}^{\infty}$ with $\sup_n d(E_n, \ell_\infty^n) < \infty$.

It is interesting to recall the context in which Odell and Schlumprecht proved Theorem 9. They were interested in the following long standing open problem concerning Lipschitz (or in this context, equivalently, uniformly continuous) functions from the unit sphere $\{x_j ; \|x\| = 1\}$ of $\ell_2$ to the real line. Given such a function $f$ and given $\varepsilon > 0$, does there exist an infinite dimensional subspace $Y$ of $\ell_2$ so that the restriction of $f$ to the unit sphere of $Y$ is constant up to $\varepsilon$, i.e. has an oscillation less than $\varepsilon$ (the so-called “distortion problem”)? It is known that there are always finite-dimensional subspaces of $\ell_2$ with arbitrarily large dimension which have such a property. In [OS] this problem was solved in the negative. There is a Lipschitz function $f$ from the unit sphere of $\ell_2$ to $R$ so that its restriction to the unit sphere of every infinite dimensional subspace of $\ell_2$ has an oscillation larger than $\varepsilon_0$ for some $\varepsilon_0 > 0$. The construction of $f$ is not explicit (and till now no explicit example is known). They first work on a Tsirelson type space (like the space $T_2$ mentioned above) and then transfer the result to $\ell_2$ via a uniform homeomorphism of the unit sphere of $T_2$ with the one in $\ell_2$ (the transfer is not automatic though, since a uniform homeomorphism does not carry linear subspaces to linear subspaces).

There are many more spaces whose unit balls are known to be uniformly homeomorphic to $B(\ell_2)$ besides lattices. N.J. Kalton noted that one can apply the Pelczynski decomposition method to this question and was able e.g. to deduce from this that if $Y$ is an infinite dimensional subspace of a discrete superreflexive lattice $X$ then $B(Y)$ is uniformly homeomorphic to $B(\ell_2)$. The class of spaces having this property can be further extended by using complex interpolation of Banach spaces. For instance denote by $C_p$ the Schatten spaces of operators on $\ell_2$; it follows by interpolation that $B(C_p)$ is uniformly homeomorphic to $B(\ell_2)$ for $1 < p < \infty$ (it is known that the spaces $C_p$, $p \neq 2$, do not embed linearly into a superreflexive lattice).

Theorem 9 cannot however be extended to all Banach spaces (i.e. the lattice assumption cannot be simply dropped). Raynoud [Ra] proved that there is a separable Banach space $X$ so that $B(X)$ is not uniformly homeomorphic to a subset of $\ell_2$ but $X$ does not contain subspaces $\{E_n\}_{n=1}^{\infty}$ with uniformly bounded Banach Mazur distances from $\ell_\infty^n$. For
this space $X$, $B(X)$ is also not uniformly homeomorphic to $B(c_0)$. The proof of Raynoud is based on the so-called “stability theory” of Krivine and Maurey.

4. Quotient maps

The notion dual to embeddings (at least in the linear theory) is that of quotient maps. For the study of this notion we have first to define properly the concept of nonlinear (Lipschitz or uniform) quotient map. The most direct concept which comes to mind – that of a Lipschitz or uniformly continuous map from one Banach space onto another turns out not to be the right one in our context. Bates [Ba] proved the following result.

For every infinite dimensional Banach space $X$ there is a continuously Fréchet differentiable Lipschitz map $f$ onto any separable Banach space.

Thus one has to require more of a Lipschitz quotient map in order to get a concept which is related to the linear structure of the spaces.

A linear quotient map $T$ from a Banach space $X$ onto a Banach space $Y$ is by the open mapping theorem an open map and there exists a constant $\lambda > 0$ so that $TB_X(x, r) \supset B_Y(Tx, \lambda r)$ for every $x \in X$ and $r > 0$ ($B_X(x, r)$ is the ball in $X$ with center $x$ and radius $r$). It turns out that this property which is automatic for linear quotient maps has to be built into the definition of nonlinear quotient maps.

A Lipschitz map $f$ from a Banach space $X$ onto a Banach space $Y$ is called a Lipschitz quotient map if there is a $\lambda > 0$ so that, for all $x \in X$ and $r > 0$, $f(B_X(x, r)) \supset B_Y(f(x), \lambda r)$.

A uniformly continuous map $f$ from $X$ to $Y$ is called a uniform quotient map if there is a function $\varphi(r)$, with $\varphi(r) > 0$ for every $r > 0$, so that $f(B_X(x, r)) \supset B_Y(f(x), \varphi(r))$ for all $r > 0$.

The obvious first question to ask with these definitions is whether the existence of a uniform (resp. Lipschitz) quotient map implies the existence of a linear one.

We start with Lipschitz quotient maps. In the case of Lipschitz embeddings Gâteaux derivatives give a good linearization tool (provided the derivatives exist; for example if we are in the RNP situation). In the study of Lipschitz equivalence Gâteaux derivatives give considerable information, though not the complete answer. In the case of quotient
maps Gâteaux derivatives may give no information (unless one can find a way to use such derivatives at “good” points and not just at an arbitrary point). In fact in [BJLPS] it is shown that there is for every $1 \leq p < \infty$ a Lipschitz quotient map from $\ell_p$ onto itself whose Gâteaux derivative, at 0 say, is identically equal to 0.

On the other hand, if a Lipschitz quotient map is $\varepsilon$-Fréchet differentiable at a point and if $\varepsilon$ is small enough then it is easy to check that the linear operator appearing in the definition of $\varepsilon$-Fréchet differentiability (**) must be a linear quotient map. In some cases one can prove that every Lipschitz function from one Banach space to another is for every $\varepsilon > 0$, $\varepsilon$-Fréchet differentiable at some point.

There are pairs of infinite-dimensional Banach spaces $X$ and $Y$ so that every linear operator from $X$ into $Y$ is compact. Of course in such a case $Y$ is not a linear quotient space of $X$. It turns out that in many situations of this type one can prove the existence of points of $\varepsilon$-Fréchet differentiability for Lipschitz maps from $X$ into $Y$ and therefore $Y$ is also not a Lipschitz quotient space of $X$.

Here are some specific results of this type proved in [JLPS].

For the following pairs of spaces $X, Y$ every Lipschitz map from $X$ to $Y$ has for every $\varepsilon > 0$ points of $\varepsilon$-Fréchet differentiability and consequently there is no Lipschitz quotient map from $X$ onto $Y$

(i) $X$ a $C(K)$ space with $K$ compact countable and $Y$ a space having RNP.

(ii) $X$ has a normalized Schauder basis $\{e_i\}_{i=1}^{\infty}$ so that $\left\| \sum a_i e_i \right\| \leq C \left( \sum |a_i|^r \right)^{\frac{1}{r}}$ for some $C$ and $r$ and all choices of $\{a_i\}_{i=1}^{\infty}$ and $Y$ has a uniformly convex norm with modulus of convexity $\delta(t)$ satisfying $\delta(t) \geq \lambda t^s$ for some $\lambda > 0$ and $s < r$. (For example $X = \ell_r$, $Y = \ell_s$ with $r > s \geq 2$).

We just mention that there are also some other cases where properties weaker than (but related to) $\varepsilon$-Fréchet differentiability ensure the nonexistence of a Lipschitz quotient map from $X$ onto $Y$.

We pass now to results on Lipschitz quotient maps which are valid for uniform quotient maps as well. Therefore we turn to the discussion of uniform quotient maps. As in the case of homeomorphisms it is not hard to prove the following.

Assume that there is a uniform quotient map from a Banach space $X$ onto $Y$. Then
there is a Lipschitz quotient map from an ultrapower $X_\mathcal{U}$ of $X$ onto an ultrapower $Y_\mathcal{U}$ of $Y$.

As in the case of Ribe’s results for homeomorphism (Theorem 6 above) one can get a quite general linearization theorem for uniform quotient maps when one goes down to the “local level” i.e. to the finite dimensional setting. The following theorem is proved in [BJLPS].

**Theorem 10.** Assume that $X$ is a superreflexive Banach space and there is a uniform quotient map from $X$ onto a Banach space $Y$. Then there is a constant $C < \infty$ so that for every finite dimensional subspace $E$ of $Y^*$ there is a finite dimensional subspace $F$ of $X^*$ with $d(E, F) < C$.

This theorem could be deduced from Theorem 4 (combined with the observation above on ultrapowers). However it suffices to use the following approximation theorem for Lipschitz functions by affine functions whose proof is considerably simpler than that of Theorem 4. The proof of this result does not use differentiation and in fact the part of this statement concerning the estimate is false if we want to use a derivative as an affine approximant (even if $X$ itself is the real line).

Let $f$ be a Lipschitz function defined on a ball $B$ in a superreflexive space $X$ into a finite dimensional space $Y$. Then for every $\varepsilon > 0$ there is a ball $B_1$ with radius $\rho$ contained in $B$ and an affine function $g$ on $B_1$ so that $|g(x) - f(x)| \leq \varepsilon \rho$ for all $x \in B_1$. Moreover $\rho$ can be estimated from below in terms of $X, \varepsilon$, the radius of $B$, $\dim Y$ and the Lipschitz constant of $f$.

An immediate consequence of Theorem 10 and known facts from the linear theory of Banach spaces is.

**Theorem 11.** Assume that $Y$ is a uniform quotient of $L_p(0, 1)$, $1 < p < \infty$. Then $Y$ is isomorphic to a linear quotient of $L_p(0, 1)$. In particular every uniform quotient of a Hilbert space is isomorphic to a Hilbert space.

For obtaining further results on Lipschitz or uniform quotient maps it would be useful to have for quotient maps results in the spirit of Gorelik’s principle. Unfortunately at least in the setting of uniform quotient maps there seem to be no such results. It is
shown in [BJLPS] that for every $1 \leq p < \infty$ there is a uniform quotient map $f$ from $\ell_p$ onto $\ell_p$ that maps the unit ball of a hyperplane of $\ell_p$ to the origin. The Gorelik principle (Theorem 7) shows that such a map $f$ cannot be represented as a composition of a uniform homeomorphism with a linear quotient map (in any of the two possible orders).

5. Some open problems

(1) Which separable Banach spaces $X$ are uniformly homeomorphic to bounded subsets of themselves?

We mentioned in section 2 that this is the case for $X = \ell_2$. Hence by Theorem 2 and Mazurs map the same is true for $X = L_p(\mu)$ if $1 \leq p \leq 2$. For the same reason this is false for $X = L_p(\mu)$ with $2 < p < \infty$. Aharoni [Ah2] showed that this is true for $X = c_0$ and therefore for any Banach space containing $c_0$ (like separable $C(K)$ spaces).

(2) Can one characterize the Banach spaces $X$ which uniformly embed into a fixed space $Y$ in situations which are not immediate consequences of Theorem 2? In particular which Banach spaces embed uniformly into $L_p(0, 1)$ for some fixed $p, 2 < p < \infty$?

(3) Assume that $X$ and $Y$ are separable Lipschitz equivalent Banach spaces. Are they linearly isomorphic?

(4) Are the spaces $L_p, 1 < p < \infty, p \neq 2$ determined by their uniform structure in the sense that any space uniformly homeomorphic to them is already isomorphic to them? What about $c_0$ or $\ell_1$?

In the case of $L_p$ it is known that any Banach space uniformly homeomorphic to $L_p$ is linearly isomorphic to a complemented subspace of $L_p$. In the case of $c_0$ also much is known (the space must be an $L_\infty$ space and cannot have $C(w^w)$ as a quotient space). In the case of $\ell_1$ nothing is known besides Theorem 6 and the fact that $\ell_1$ and $L_1$ are not uniformly homeomorphic.

(5) Are there other Gorelik-like results on uniform homeomorphisms? For example can a Lipschitz homeomorphism of $\ell_2$ onto itself map the Hilbert cube $\{x = (\lambda_1, \lambda_2, \ldots); |\lambda_n| \leq 1/n\}$ into a hyperplane?

(6) Assume that $X$ is a superreflexive separable Banach space. Is $B(X)$ uniformly homeomorphic to $B(\ell_2)$?
(7) Assume that $X$ is a Banach space such that $B(X)$ is uniformly homeomorphic to a subset of $\ell_2$. Is then $B(X)$ uniformly homeomorphic to $B(\ell_2)$?

(8) Is there a Lipschitz quotient map from $\ell_\infty$ onto $c_0$?

It is known (and easy) that there is a retraction from $\ell_\infty$ onto $c_0$ which is a Lipschitz map. However this map is far from being a Lipschitz quotient map.

(9) Is every Banach space which is a uniform quotient space of $\ell_p$, $1 < p < \infty$, $p \neq 2$ isomorphic to a linear quotient space of $\ell_p$?

By Theorem 11 such a space is a linear quotient space of $L_p$.

(10) Is the Gorelik principle true for Lipschitz quotient maps? More specifically assume that $f$ is a Lipschitz quotient map from an infinite dimensional Banach space $X$ onto an infinite dimensional Banach space $Y$. Is it possible that $f$ maps a ball in a finite codimensional subspace of $X$ to a single point in $Y$?
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