UNCONDITIONAL BASES OF SUBSPACES RELATED TO NON-SELF-ADJOINT PERTURBATIONS OF SELF-ADJOINT OPERATORS

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ABSTRACT. Assume that $T$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and that the spectrum of $T$ is confined in the union $\bigcup_{j \in J} \Delta_j$, $J \subseteq \mathbb{Z}$, of segments $\Delta_j = [\alpha_j, \beta_j] \subseteq \mathbb{R}$ such that $\alpha_{j+1} > \beta_j$ and
\[
\inf_j (\alpha_{j+1} - \beta_j) = d > 0.
\]

If $B$ is a bounded (in general non-self-adjoint) perturbation of $T$ with $\|B\| = b < d/2$ then the spectrum of the perturbed operator $A = T + B$ lies in the union $\bigcup_{j \in J} U_b(\Delta_j)$ of the mutually disjoint closed $b$-neighborhoods $U_b(\Delta_j)$ of the segments $\Delta_j$ in $\mathbb{C}$. Let $Q_j$ be the Riesz projection onto the invariant subspace of $A$ corresponding to the part of the spectrum of $A$ lying in $U_b(\Delta_j)$, $j \in J$. Our main result is as follows: The subspaces $\mathcal{L}_j = Q_j(\mathcal{H})$, $j \in J$, form an unconditional basis in the whole space $\mathcal{H}$.

1. INTRODUCTION AND MAIN RESULT

We begin with recalling some definitions (see [1] Ch.6.5). A sequence of nonzero subspaces $\{\mathcal{L}_k\}$ of a Hilbert spaces $\mathcal{H}$ is said to be a basis if any element $x \in \mathcal{H}$ is uniquely represented by the series
\[
x = \sum_k x_k, \quad \text{where } x_k \in \mathcal{L}_k,
\]
that converges in the norm of $\mathcal{H}$. A basis of subspaces $\{\mathcal{L}_k\}$ is said to be unconditional if the series
\[
\{\mathcal{L}_k\}
\]
converges to $x$ after any rearrangement of its elements. An unconditional basis of subspaces is also called a Riesz basis of subspaces. In the case where all the subspaces $\mathcal{L}_j$ are one-dimensional (finite-dimensional) we can choose elements $y_j \in \mathcal{L}_j = Q_j(\mathcal{H})$ (a basis $\{y_k\}$ in $\mathcal{L}_j$). Then the sequence of the subspaces $\{\mathcal{L}_k\}$ is a basis if and only if the corresponding system is a basis (a basis with parentheses) in $\mathcal{H}$.

When one deals with bases of subspaces, it is convenient to work in terms of projections. We will use definitions and results presented in [2] §6. Let $J$ be a finite or infinite ordered set of indices, $J \subseteq \mathbb{Z}$. A system of projections $\{Q_j\}_{j \in J}$ is said to be complete if the equalities
\[
\langle Q_j x, y \rangle = 0, \quad \text{for any } x \in \mathcal{H} \quad \text{and any } \quad j \in J,
\]
imply $y = 0$. It is easily seen that the system $\{Q_j\}_{j \in J}$ is complete if and only if any element $x \in \mathcal{H}$ can be approximated with an arbitrary accuracy by linear combinations of elements $x_k \in \mathcal{L}_k = Q_k(\mathcal{H})$, $k \in J$.

A system of projections $\{Q_j\}_{j \in J}$ is called minimal if
\[
Q_j Q_k = \delta_{k,j} Q_j \quad \text{for any } \quad j,k \in J.
\]
It follows directly from the definitions that if $\{Q_j\}_{j \in J}$ is a minimal system of projections, then the sequence of the subspaces $\mathcal{L}_j = Q_j(\mathcal{H})$ forms a basis (an unconditional basis) if and only if the series $\sum_{j \in J} Q_j$ converges (converges after any rearrangement of the indices) in the strong operator topology to the identity operator.

Further we will make use of the following results (for the corresponding proofs see [1] Ch. 6) and [2] §6).

Theorem A. Let $\{Q_j\}_{j \in J}$ be a system of projections in a Hilbert space $\mathcal{H}$. The following statements are equivalent:

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(1) A sequence of subspaces \( L_j = Q_j(\mathcal{H}) \), \( j \in J \), is an unconditional basis of the Hilbert space \( \mathcal{H} \).

(2) There exists an equivalent inner product in \( \mathcal{H} \) such that a sequence of subspaces \( L_j = Q_j(\mathcal{H}) \), \( j \in J \), is complete and mutually orthogonal (\( L_k \perp L_j \) for \( k \neq j \)).

(3) There exists a bounded and boundedly invertible operator \( K \) in \( \mathcal{H} \), and a complete and minimal system of orthogonal projections \( \{ P_j \} \) such that \( Q_j = K^{-1}P_jK \), \( j \in J \).

(4) The system of projections \( \{ Q_j \} \) is complete, minimal, and the series \( \sum_{j \in J} Q_j \) converges unconditionally.

(5) The system of projections \( \{ Q_j \} \) is complete, minimal, and
\[
\sum_{j \in J} |(Q_jx,x)| < \infty \quad \text{for any } x \in \mathcal{H}.
\] (2)

Now we are ready to formulate the main result of the paper. In what follows it is assumed that either the set of indices coincides with \( \mathbb{N} \) or with \( \mathbb{Z} \).

**Theorem 1.** Let \( T \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \). Assume that the spectrum of \( T \) is confined in the union \( \Delta := \bigcup_{j \in J} \Delta_j \) of the segments \( \Delta_j = [\alpha_j, \beta_j] \subset \mathbb{R} \) such that \( \beta_{j+1} > \beta_j \) for all \( j \in J \). Assume in addition that
\[
\inf_{j \in J} (\alpha_{j+1} - \beta_j) = d > 0.
\] (3)

Let \( B \) be a bounded (generally non-self-adjoint) operator on \( \mathcal{H} \) with \( \|B\| =: b < d/2 \). Then the spectrum the operator \( A = T + B \) lies in the union \( \bigcup_{j \in J} U_b(\Delta_j) \) of the mutually disjoint closed \( b \)-neighborhoods \( U_b(\Delta_j) \) of the segments \( \Delta_j \) in \( \mathbb{C} \). If \( Q_j, j \in J \), are the Riesz projections onto the invariant subspaces \( L_j \) of \( A \) corresponding to the isolated components of its spectrum lying in \( U_b(\Delta_j) \), then the invariant subspaces \( L_j, L_j = Q_j(\mathcal{H}), j \in J \), form an unconditional basis in \( \mathcal{H} \).

There are many papers devoted to the Riesz basis property of the root vectors of non-self-adjoint operators, which are perturbations of self-adjoint ones. The corresponding results and references can be found in the book of Markus [3] and in the paper of Shkalikov [2]. First results related to Theorem 1 were obtained by Markus [4] and Kato (see [5] Ch.5, Theorem 4.15a): Let \( T \) be a self-adjoint operator with discrete spectrum on a Hilbert space \( \mathcal{H} \) such that its eigenvalues \( \{ \lambda_j \} \) are simple and subject the condition \( \lambda_{j+1} - \lambda_j \to \infty \), as \( j \to \infty \). Then for any bounded (generally non-self-adjoint) operator \( B \) the operator \( T + B \) has discrete spectrum and its root vectors form a Riesz basis in the space \( \mathcal{H} \).

A more general result follows from the Markus-Matsaev theorem [3 Ch.1, Theorem 6.12]: Let \( T \) be a self-adjoint operator in a Hilbert space \( \mathcal{H} \), having a finite order (i.e. its eigenvalues, counting multiplicities, are subject to the condition \( |\lambda_j| \geq Cj^p \) with some constants \( C, p > 0 \)), and there exist gaps in the spectrum of \( T \) with the lengths \( \geq d \). If \( \|B\| \leq d/2 \) then the the root subspaces of the perturbed operator \( T + B \) form a Riesz basis with parentheses \( \mathcal{H} \) (or a Riesz basis consisting of finite-dimensional root subspaces).

The condition for \( T \) to be of finite order was dropped in [2]. However, the compactness of the resolvent \( (T - \lambda)^{-1} \) was essentially used in the proof. To the best of our knowledge, Theorem 1 is, apparently, the first result in this topic which deals with an unperturbed operator \( T \) possibly having a non-discrete spectrum.

We have an additional motivation to prove Theorem 1. We expect that this result might help to resolve some open problems concerning bounded perturbations of self-adjoint operators in spaces with indefinite metric (see [6,7]).

2. **Proof of Theorem 1**

We divide the proof into two parts that are called below Step 1 and Step 2, respectively. At Step 1 we prove that the sequence of subspaces \( \{ L_j \} \) forms a basis of \( \mathcal{H} \). At Step 2 we prove that it is, actually, an unconditional basis.
Step 1. For the sake of definiteness, we assume that the index set \( J \) coincides with the set of all entire numbers \( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \). This corresponds to the most general case when infinitely many segments \( \Delta_j \) lie both on \( \mathbb{R}^- \) and \( \mathbb{R}^+ \).

Under the hypothesis \( |B| = b \) with \( 0 \leq b < d/2 \) that we assume, the closed neighborhoods \( U_b(\Delta_j) \) of the segments \( \Delta_j, j \in J \), are disjoint and, surely, \( U_b(\Delta) = \bigcup_{j \in J} U_b(\Delta_j) \). The first assertion of the theorem on the inclusion of the spectrum \( \sigma(A) \) of \( A \) in the union \( \bigcup_{j \in J} U_b(\Delta_j) \) is a consequence of the well-known estimate

\[
\left\| (T - \lambda)^{-1} \right\| \leq \frac{1}{\text{dist}(\lambda, \sigma(T))} \leq \frac{1}{\text{dist}(\lambda, \Delta)}, \quad \Delta := \bigcup_{j \in J} \Delta_j,
\]

where \( \sigma(T), \sigma(T) \subset \Delta \), is the spectrum of the self-adjoint operator \( T \). For any \( \lambda \) lying outside \( U_b(\Delta) \) we have

\[
\delta := \text{dist}(\lambda, \Delta) > b.
\]

Then combining (4) and (5) with the bound

\[
\left\| (1 + B(T - \lambda)^{-1})^{-1} \right\| \leq \frac{1}{1 - b/\delta}
\]

for the resolvent \( (A - \lambda)^{-1} = (T + B - \lambda)^{-1} \) one finds

\[
\left\| (A - \lambda)^{-1} \right\| = \left\| (T - \lambda)^{-1} (1 + B(T - \lambda)^{-1})^{-1} \right\| \leq \frac{1}{\delta - b} < \infty,
\]

where the quantity \( \delta = \delta(\lambda) \) is defined in (5). Hence, any \( \lambda \in \mathbb{C} \setminus U_b(\Delta) \) belongs to the resolvent set of the perturbed operator \( A = T + B \) and then \( \sigma(A) \subset U_b(\Delta) = \bigcup_{j \in J} U_b(\Delta_j) \). Since the neighborhoods \( U_b(\Delta_j) \) for different \( j \in J \) are disjoint, namely,

\[
\text{dist}(U_b(\Delta_j), U_b(\Delta_k)) \geq d - 2b, \quad j \neq k,
\]

the spectral sets of \( A \) confined in \( U_b(\Delta) \) are also disjoint. The Riesz projections \( Q_j \) for these sets are well defined. In particular, given an arbitrary \( b' \in (b, d/2) \), one may write \( Q_j \) in the form

\[
Q_j = -\frac{1}{2\pi i} \int_{\Gamma_j} (A - \lambda)^{-1} d\lambda, \quad \Gamma_j = \partial U_{b'}(\Delta_j),
\]

where the integration along contour \( \Gamma_j \) is performed in the anti-clockwise direction.

Below we will also use the representation

\[
(A - \lambda)^{-1} = (T - \lambda)^{-1} - G(\lambda), \quad \lambda \notin U_b(\Delta),
\]

where

\[
G(\lambda) = (A - \lambda)^{-1} B (T - \lambda)^{-1} = (T - \lambda)^{-1} M(\lambda) B (T - \lambda)^{-1}
\]

and

\[
M(\lambda) = (1 + B(T - \lambda)^{-1})^{-1}.
\]

Now denote by \( R_n \) the rectangle in \( \mathbb{C} \) whose vertical sides pass through the points

\[
c_{-n} = (\beta_{n-1} + \alpha_{n-1})/2 \quad \text{and} \quad c_{n} = (\beta_{n} + \alpha_{n+1})/2
\]

while the horizontal sides coincide with the segments \([c_{-n} + i\gamma_0, c_{n} + i\gamma_0]\) where

\[
\gamma_0 = \max\{|c_{-n}|, |c_{n}|\}.
\]

Clearly, the sides of \( R_n \) do not intersect the set \( U_b(\Delta) \). By virtue of (9) one obtains

\[
\sum_{j=-n}^{n} Q_j x = -\frac{1}{2\pi i} \int_{\partial R_n} (A - \lambda)^{-1} x d\lambda = \sum_{-n}^{n} P_j x + I_n x,
\]
where $I_n$ are the respective contour integrals of the operator-valued function $G(\lambda)$ along $\partial R_n$, 

$$I_n := \frac{1}{2\pi i} \int_{\partial R_n} G(\lambda) d\lambda,$$  

(14)

and $P_j$ are the spectral projections onto the spectral subspaces of the self-adjoint operator $T$ associated with the parts of its spectrum inside the corresponding segments $\Delta_j$.

Given an arbitrary $x \in \mathcal{H}$ we have

$$\sum_{j=-n}^{n} P_j x \to x \quad \text{as} \quad n \to \infty. \quad (15)$$

Thus, in order to prove that $\sum_{j=-n}^{n} Q_j x \to x$ one only needs to show that the sequence of $I_n x$ in (13) converges to zero as $n \to \infty$.

First, let us show that the operators $I_n$ are uniformly bounded. We have $I_n = I_n^1 + I_n^2$, where $I_n^1$ and $I_n^2$ are the integrals along the horizontal and vertical sides of the rectangles $R_n$, respectively. For $\lambda$ varying on the horizontal sides of the rectangle $\partial R_n$ we have the estimate

$$\|(T - \lambda)^{-1}\| \leq \frac{1}{\gamma_n}, \quad \lambda = \xi \pm i\gamma_n, \quad \xi \in [c_n, c_n],$$

where $\gamma_n$ is defined by (12). Hence, by virtue of (7) and (10), it follows

$$\|G(\lambda)\| \leq \frac{b}{(\gamma_n - b)\gamma_n}, \quad \lambda = \xi \pm i\gamma_n, \quad \xi \in [c_n \pm i\gamma_n]. \quad (16)$$

Taking into account that the lengths of the horizontal sides do not exceed $2\gamma_n$ and $\gamma_n \to \infty$ as $n \to \infty$, we get $\|I_n^1\| \to 0$ as $n \to \infty$.

Let us estimate the norms of the operators $I_n^2$. For the sake of definiteness consider the right vertical side of the rectangle $R_n$ and divide it in three parts

$$\omega_n \cup \omega_n^+ \cup \omega_n^-, \quad \text{where} \quad \omega_n = (c_n - id, c_n + id), \quad \omega_n^+ = [c_n \pm id, c_n \pm i\gamma_n].$$

The lengths of the intervals $\omega_n$ equal $2d$ and due to (7) and (10)

$$\left\| \int_{\omega_n} G(\lambda) d\lambda \right\| \leq \frac{2bd}{(d/2-b)(d/2)}. \quad (17)$$

For $\lambda \in \omega_n^\pm$ we have

$$\delta(\lambda) - b \geq |\text{Im} \lambda| - b. \quad (18)$$

Thus, again applying (7) and (10), one obtains

$$\left\| \int_{\omega_n^\pm} G(\lambda) d\lambda \right\| \leq \int_d^{\gamma_n} \frac{b}{\tau(\tau - b)} d\tau \leq \int_d^{\gamma_n} \frac{b}{(\tau - b)^2} d\tau \leq \frac{b}{d-b} < 1. \quad (19)$$

Therefore,

$$\|I_n\| \leq C, \quad (20)$$

where the constant $C$ depends only on $d$ and $b$.

Let us show that $\|I_n x\| \to 0$ as $n \to \infty$ for any fixed $x \in \mathcal{H}$. To this end, choose some $\varepsilon > 0$ and, first, find $N \in \mathbb{N}$ such that

$$\|x - x_N\| < \frac{\varepsilon}{2C}, \quad (21)$$

where

$$x_N = \sum_{j=-N}^{N} P_j x.$$  

(22)

Obviously, from (20) and (21) it follows

$$\|I_n(x - x_N)\| < \frac{\varepsilon}{2} \quad \text{for any} \quad n \in \mathbb{N}.$$
Let us estimate \(|I_n x_N|\) as \(n \to \infty\). Denote \(e(t) := (E(t)x,x)\) where \(E(t)\) is the spectral function of the self-adjoint operator \(T\), and observe that, by the spectral theorem,
\[
\left\| (T - \lambda)^{-1} x_N \right\|^2 = \int_{c_N}^{c_N} \frac{de(t)}{(t - \zeta)^2 + \tau^2}, \quad \lambda = \zeta + i\tau \notin \sigma(T).
\] (23)

For \(N\) fixed the equality (24) implies
\[
\left\| (T - \lambda)^{-1} x_N \right\| = O(\|\lambda\|^{-1}) \quad \text{as} \ |\lambda| \to \infty.
\] (24)

Now we can modify estimate (17) and get from (7) and (24)
\[
\left\| \int_{\Gamma_n} G(\lambda) x_N d\lambda \right\| = O(\|c_n\|^{-1}) \to 0 \quad \text{as} \ n \to \infty.
\]

Analogously, by taking into account (18) we can modify the estimate (19) and obtain
\[
\left\| \int_{\Gamma_n} G(\lambda) x_N d\lambda \right\| \leq \int_d^\infty \frac{d\tau}{(\tau - b)\sqrt{(c_n - c_N)^2 + \tau^2}} = O\left(\frac{\ln c_n}{c_n}\right) = o(1) \quad \text{as} \ n \to \infty.
\]

The last two estimates together give
\[
I_n x_N \to 0 \quad \text{as} \ n \to \infty.
\]

Hence, there is \(N_1 \in \mathbb{N} \) such that \(|I_n x_N| < \varepsilon/2\) whenever \(n > N_1\). Taken together with (22) this yields \(|I_n x_N| < \varepsilon\) for \(n > N_1\). Therefore, we have proven that the sequence of \(I_n\) strongly converges to zero as \(n \to \infty\) and then from (13) and (15) it follows that for any \(x \in \mathcal{H}\) the two-sided series \(\sum_{j=-n}^{n} Q_j x\) converges to \(x\). Thus, the system of subspaces \(\mathcal{L}_j = Q_j(\mathcal{H})\), \(j \in \mathbb{Z}\) is complete. That these subspaces are linearly independent follows from the mutual orthogonality of the Riez projections (8) in the sense that \(Q_j Q_k = \delta_{jk} Q_j\) for any \(j, k \in \mathbb{Z}\) (see, e.g. [1, Ch. I §1.3]). Thus, the system \(\{\mathcal{L}_j\}_{j=-\infty}^{\infty}\) represents a basis of subspaces in \(\mathcal{H}\).

**Step 2.** By Theorem A, in order to prove that the above basis of subspaces \(\{\mathcal{L}_j\}_{j=-\infty}^{\infty}\) is unconditional, it suffices to show that the series \(\sum_{j=-\infty}^{\infty} \|Q_j x\|\) converges for any \(x \in \mathcal{H}\). First, let us transform the contours \(\Gamma_j\) in the integrals (5) into the contours \(\Gamma_j\), which surround the rectangles with the vertical sides \(\lambda = c_j + i\tau\) and \(\lambda = c_j + i\tau\) where \(\tau\) is varying in \([-d,d]\), and the horizontal sides \(\xi \pm id, \xi \in [c_j-1,c_j]\). As previously, the numbers \(c_j\) are defined by (11) and coincide with the centers of the gaps between the neighboring intervals \(\Delta_j\) and \(\Delta_{j+1}\).

Taking into account the representation (10) and estimate (6), we immediately conclude that
\[
2\pi \sum_{j=-\infty}^{\infty} \|Q_j x\| \leq \sum_{j=-\infty}^{\infty} \left| \int_{\Gamma_j} (T - \lambda)^{-1} x, x \right| d\lambda \leq \sum_{j=-\infty}^{\infty} \left| \int_{\Gamma_j} G(\lambda) x, x \right| d\lambda.
\]

The first series converges since it simply coincides with the series \(\sum_{j \in \mathbb{Z}} \|P_j x\|^2\); we recall that \(P_j\) are the spectral projections of the self-adjoint operator \(T\) associated with the segments \(\Delta_j\). By virtue of (6) and (10), the second series can be estimated as follows:
\[
\sum_{j=-\infty}^{\infty} \int_{\Gamma_j} G(\lambda) x, x \right| d\lambda \leq C_1 \int_{-\infty}^{\infty} \left| \int_{\Gamma_j} (T - \lambda)^{-1} x \right|^2 |d\lambda| \leq C_1 \left( \int_{\Gamma_{j-1}} (T - \lambda) x, x \right| d\lambda \right)^2 |d\lambda| + 2 \sum_{j \in \mathbb{Z}} \int_{\omega_j} (T - \lambda)^{-1} x, x \right| d\lambda |d\lambda|) \right),
\] (25)

where \(\Gamma_{j \pm}\) are the lines \(\lambda = \xi \pm id, \xi \in \mathbb{R}\), \(\omega_j\) are the vertical segments \(\lambda = c_j + i\tau, -d \leq \tau \leq d\), and \(C_1 = \text{const}\). Convergence of the integrals and the series in (25) can be proven by using the spectral theorem. As before, denote \(e(t) := (E(t)x,x)\) where \(E(t)\) stands for the spectral function of \(T\). Then
\[
\int_{\Gamma_{j \pm}} (T - \lambda)^{-1} x, x \right| d\lambda \right) = \int_{\Gamma_{j \pm}} |d\lambda| \int_{\mathbb{R}} \frac{de(t)}{|t - \lambda|^2},
\]
Further, notice that for \( c_j \) given by (11), the lower bound \( (3) \) yields

\[
|t - c_j| \geq \begin{cases} 
\frac{d}{2} + d(j - k) & \text{for all } t \in \Delta_k, \ k \leq j, \\
\frac{d}{2} + d(k - j - 1) & \text{for all } t \in \Delta_k, \ k \geq j + 1.
\end{cases}
\]

Hence, for \( \lambda = c_j + i\tau, \ \tau \in [-d,d] \), we have

\[
\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left\|(T - \lambda)^{-1}x\right\|^2 |d\lambda| = \sum_{j \in \mathbb{Z} \setminus [-d,d]} \int_{\mathbb{R}} \frac{d\tau}{|t - c_j - i\tau|^2} \\
= \sum_{j \in \mathbb{Z} \setminus [-d,d]} \int_{\mathbb{R}} \frac{d\tau}{|t - c_j|^2} \\
\leq 2d \sum_{j \in \mathbb{Z} \setminus [-d,d]} \int_{\mathbb{R}} \frac{d\tau}{|t - c_j|^2} \\
\leq \sum_{k \in \mathbb{Z}} \|P_kx\|^2 \left( 2 \left( \frac{2}{d} \right)^2 + \sum_{j \in \mathbb{Z}, j \neq k} \frac{1}{d^2|k - j|^2} \right) \\
\leq \frac{2C_2}{d^2} \sum_{k \in \mathbb{Z}} \|P_kx\|^2 = \frac{2C_2}{d^2} \|x\|^2, \quad (27)
\]

where \( C_2 = 4 + \sum_{j=1}^{\infty} \frac{1}{j^2} = 4 + \frac{\pi^2}{6} \). The estimate (25) together with (26) and (27) entails the bound (2). This completes the proof of Theorem 1.

The following statement is a simple corollary of Theorem 1 (combined with Theorem A).

**Corollary 2.** Assume the hypothesis of Theorem 1. Then there exists an inner product \( \langle \cdot, \cdot \rangle \) in \( \mathcal{H} \) with the following properties.

1. The product \( \langle \cdot, \cdot \rangle \) is norm-equivalent to the original inner product \( \langle \cdot, \cdot \rangle \) in \( \mathcal{H} \).
2. The subspaces \( \mathcal{L}_j = Q_j\mathcal{H} \) are mutually orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle \) and, with respect to \( \langle \cdot, \cdot \rangle \), the Hilbert space \( \mathcal{H} \) admits the orthogonal decomposition

\[
\mathcal{H} = \bigoplus_{j \in J} \mathcal{L}_j. \quad (28)
\]

3. The subspaces \( \mathcal{L}_j, \ j \in J \), are reducing for the perturbed operator \( A = T + B \) and, with respect to the decomposition (28), this operator admits a block diagonal matrix representation

\[
A = \text{diag}(\ldots, A_{-2}A_{-1}, A_0, A_1, A_2, \ldots),
\]

where \( A_j = A|_{\mathcal{L}_j}, \ j \in J \), denotes the part of \( A \) in the reducing subspace \( \mathcal{L}_j \).

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