We present an efficient technique for control of synchrony in a globally coupled ensemble by pulsatile action. We assume that we can observe the collective oscillation and can stimulate all elements of the ensemble simultaneously. We pay special attention to the minimization of intervention into the system. The key idea is to stimulate only at the most sensitive phase. To find this phase we implement an adaptive feedback control. Estimating the instantaneous phase of the collective mode on the fly, we achieve efficient suppression using a few pulses per oscillatory cycle. We discuss the possible relevance of the results for neuroscience, namely for the development of advanced algorithms for deep brain stimulation, a medical technique used to treat Parkinson’s disease.

Networks of highly-interconnected oscillatory elements are popular models for various systems, either manufactured or natural. It is well-known that, for sufficiently strong interaction, the units of the network synchronize, and the system as a whole exhibits a collective rhythm. Frequently this rhythm is detrimental and shall be suppressed: the examples include oscillation of pedestrian bridges and some pathological brain activity. On the contrary, if the interaction within the network is too weak to induce collective oscillation, enhancement of synchrony may be desirable, e.g., to ensure coherent oscillation of many low-power sources so that they produce a high-power output. These two related problems call for efficient control techniques, and various schemes have been designed for this purpose. Here we elaborate on a special case when the control action shall be pulsatile, which is a common requirement for neuroscience applications. We develop a feedback-based adaptive technique that achieves suppression of undesired collective synchrony with only one or two pulses per oscillation cycle. A slightly modified version of this technique enhances collective synchrony if required. We discuss a possible application to a clinical technique, deep brain stimulation, widely used to treat several neurological diseases.

I. INTRODUCTION

The nonlinear science community has paid a lot of attention to research on large populations of interacting self-oscillatory units. Hundreds (if not thousands) of research articles followed the pioneering publications on this topic. Many of them exploited the analytically tractable model of globally coupled phase oscillators. Theoretical, numerical, and experimental studies described and analyzed many interesting phenomena. An incomplete list includes the emergence of the collective mode, clustering, quasiperiodic dynamics, appearance of heteroclinic cycles, and chimera states, see reviews and references therein.

The most important and most studied effect is the emergence of the collective oscillation in the population due to the synchronization of individual units. Collective synchrony can be important for maintaining high-power output in a population of low-power generators and is known to play a significant role in the generation of both vital and pathological biological rhythms. Therefore, control of synchrony, i.e., either suppression or enhancement of the collective mode, is a challenging problem. In particular, the suppression task is motivated by a possible relevance to a widely used clinical procedure, deep brain stimulation (DBS). DBS implies high-frequency pulse stimulation of some brain areas and aims at an improvement of motor symptoms in Parkinsonian patients as well as in the case of some other pathologies. Though the mechanisms of DBS remain in the focus of research in neuroscience, many researchers from the nonlinear science community have adopted a working hypothesis that views DBS as a desynchronization task. This hypothesis has been exploited in a number of model studies suggesting open-loop and closed-loop techniques for suppression. In this paper, we follow this line of research and consider both the suppression and the enhancement task for a globally coupled network. We extend our previous studies on feedback-based control, concentrating on the case of pulsatile stimulation. With the goal to minimize the intervention into the controlled system, we employ precisely timed pulses, applied at a vulnerable phase that is determined on the fly. In this way, we efficiently desynchronize the oscillatory activity by a few pulses per oscillatory cycle.

The paper is organized as follows. In Section II we present the simplest model of globally coupled Bonho-
We introduce the approach using as an example a simple model of $N$ globally coupled Bonhoeffer–van der Pol oscillators:

$$
\begin{aligned}
\dot{x}_k &= x_k - x_k^3/3 - y_k + I_k + \varepsilon X + \cos \psi \cdot P(t), \\
\dot{y}_k &= 0.1(x_k - 0.8y_k + 0.7) + \sin \psi \cdot P(t),
\end{aligned}
$$

where $k$ is the oscillator index, $k = 1, \ldots, N$, and the term $\varepsilon X$ describes the global coupling. Here $X$ is the mean field, $X = N^{-1} \sum_k x_k$, and the coupling coefficient $\varepsilon$ explicitly describes the interaction between the elements of the ensemble. The oscillators are not identical: their frequencies are determined by the parameter $I_k$ that is Gaussian-distributed with the mean 0.6 and standard deviation 0.1. $P(t)$ is external pulsatile action applied to the ensemble; it will be specified below. Finally, the parameter $\psi$ describes how the external pulses act on the system. This parameter is considered to be unknown, to imitate the uncertainty in stimulation of a real-world system without any knowledge of its model.

Figure 1 illustrates the dynamics of the autonomous ensemble, $P(t) = 0$: here we plot $Y = N^{-1} \sum_k y_k$ vs. $X$ for $\varepsilon = 0.03$ and $N = 1000$. A symbol at $X \approx -0.27, Y \approx 0.55$ shows the unstable fixed point of the globally coupled system. In this representation, suppression of the collective oscillation $X(t)$ means that the system is put into and is kept in a vicinity of the unstable fixed point. Suppose the applied external pulses act along a certain direction, indicated by dashed lines in Figure 1. Obviously, the pulses applied to the system at phase angles close to $\theta_0$ and the pulses of an opposite polarity applied at approximately $\theta_0 + \pi$ are most efficient for reducing the collective oscillation, and, hence, for desynchronization. On the contrary, the oscillation amplitude is much less affected by the pulses applied around $\theta_0 \pm \pi/2$. This qualitative discussion presents the main idea of our approach: in order to achieve the control goal with minimal intervention we have to stimulate only in a small interval around the vulnerable phase $\theta_0$. For the rest of this Section we assume that $\theta_0$ is known, while in Section IV we drop this assumption and show how $\theta_0$ can be found.

**II. PULSES APPLIED AT A VULNERABLE PHASE**

**A. The basic model and the main idea**

For efficient stimulation, we have to monitor the instantaneous phase of the collective oscillation on the fly, assuming that we observe only a scalar time series. Below we suppose that $X(t)$ is measured. To this end, we follow and introduce a “device” consisting of a harmonic linear oscillator and an integrating unit

$$
\begin{aligned}
\dot{\mu} &+ \alpha \dot{\mu} + \omega_0^2 \mu = X(t), \\
\dot{\theta} &+ \mu \theta = \dot{\mu}.
\end{aligned}
$$

The role of the harmonic oscillator Eq. (2) is twofold. First, it acts as a band-pass filter and extracts the oscillatory mode of our interest from its mixture with noise. Second, it yields signal $\dot{\mu}$ which phase is close to that of the input $X(t)$, provided the frequency $\omega_0$ is chosen to be close to the mean frequency of $X(t)$. The integrating unit Eq. (3) provides a signal, shifted by $\pi/2$ with respect to $\dot{\mu}$. It is convenient to introduce two auxiliary variables $\hat{x} = \alpha \dot{\mu}$ and $\hat{y} = \alpha \omega_0 \mu \dot{\mu}$; their amplitudes are close to that of $X(t)$ while their phases are delayed by $\pi/2$, respectively, cf. Hence, we can estimate the instantaneous (proto)phase of $X(t)$ as

$$
\theta(t) = \arctan(\hat{y}/\hat{x}).
$$
In the following, we will also need the instantaneous amplitude

\[ a(t) = \sqrt{x^2 + y^2}. \]  

Figure 2 illustrates how the algorithm for phase estimation works with the system (1). The parameter values used here are: \( \omega_0 = 2\pi/32.5 \), \( \alpha = 0.3\omega_0 \), and \( \mu = 500 \).

C. Timing and strength of stimuli

To determine when and how to stimulate, we trace the instantaneous phase \( \theta(t) \) and check whether

\[ |\theta(t) - \theta_0| < \Theta_{\text{tol}} \quad \text{or} \quad |\theta(t) - \theta_0 - \pi| < \Theta_{\text{tol}}. \]  

If one of these conditions is fulfilled at time instant \( t_n \) then a pulse of a certain strength \( A_n \) is applied to all elements of the ensemble. Here \( \Theta_{\text{tol}} \) is the tolerance parameter. For a fixed width of stimulation pulses \( \Theta_{\text{tol}} \) determines whether one pulse (if \( \Theta_{\text{tol}} \) is small) or several pulses (if \( \Theta_{\text{tol}} \) is sufficiently large) are applied around \( \theta_0 \) or \( \theta_0 + \pi \), respectively. The strength of each pulse, \( A_n \), is limited by the maximal allowed value, \( |A_n| \leq A_0 \), and is determined by the current value of the instantaneous amplitude \( a(t_n) \):

\[ A = \pm \max(\varepsilon_{fb0}(t_n), -A_0), \]  

where positive and negative signs correspond to stimulation around \( \theta_0 \) and \( \theta_0 + \pi \), respectively. Here \( \varepsilon_{fb} < 0 \) is the strength of the negative feedback.

D. A numerical example

For the first illustration of the approach, we consider the model (1) with \( N = 1000 \) and \( \varepsilon = 0.03 \) and try to suppress the collective oscillation by rectangular pulses of the constant width \( \delta \) and minimal inter-pulse interval \( \Delta \), see Fig. 3. We set \( \psi = 0 \), and stimulate with negative pulses around \( \theta_0 = 0 \) and with positive pulses around \( \pi \). Other parameters are \( \varepsilon_{fb} = -0.05 \), \( \Theta_{\text{tol}} = 0.08\pi \), and \( A_0 = 0.2 \). Figure 4 demonstrates efficient suppression of the collective oscillation. For the chosen \( \Theta_{\text{tol}} \) there are three (sometimes four) stimuli in a bunch around \( \theta_0 \) or \( \theta_0 + \pi \).

Before proceeding with the further details of our approach, we discuss the meaning of the a priori unknown parameter \( \psi \). It describes the distribution of the stimulation between the equations and is related to phase shift, inherent to stimulation. The latter also depends on the property of individual oscillators and of the coupling between them, see a discussion in [11] and references therein. Thus, \( \psi \) is related to \( \theta_0 \), though is not exactly equal to it. To illustrate this and to analyse sensitivity of our technique to the choice of \( \theta_0 \) we compute the suppression coefficient \( S \) as a function of \( \theta_0 \), for \( \psi = \pm \pi/4 \) (Fig. 5). \( S \) is determined as the ratio of standard deviations of \( X \) before the stimulation is turned on and of \( X \) after suppression transient. The results show that choice of \( \theta_0 \) is crucial and therefore we need a technique for tuning \( \theta_0 \) as well as the feedback strength \( \varepsilon_{fb} \) automatically. This technique is presented in the next Section.
III. AUTOMATIC TUNING OF SUPPRESSION PARAMETERS

For a proper tuning of the feedback-based suppression algorithm we adapt the approach developed in our previous publication. Namely, we adjust parameters \( \theta_0 \), \( \varepsilon_{fb} \) after each complete cycle, according to the averaged value \( \bar{a} \) of the instantaneous amplitude \( a(t) \), see Eq. (5). To be exact, the latter is averaged over all points within one cycle, except for the interval where the system is stimulated (i.e., except for the points where \( |\theta - \theta_0| < \Theta_{tol} \) and \( |\theta - \theta_0 - \pi| < \Theta_{tol} \)). The update rules are

\[
\begin{align*}
\theta_0 &\to \theta_0 + k_1 \bar{a}(1 + \tanh[k_2(\bar{a} - a_{stop})]) , \\
\varepsilon_{fb} &\to \varepsilon_{fb} - k_3 \bar{a} / \cosh(k_4 \varepsilon_{fb}) ,
\end{align*}
\]

where \( k_i \) and \( a_{stop} \) are parameters. The initial conditions, if not said otherwise, are \( \theta_0(t_0) = 0 \), \( \varepsilon_{fb}(t_0) = 0 \).

An example of suppression with an automated tuning of parameters is illustrated in Fig. 6 for \( \psi = -\pi/4 \). We see that detected value of \( \theta_0 \) here is \( \theta_0 \approx 5.03 \), cf. Fig. 5b: the suppression factor is \( S = 52.6 \). Stimulation is turned on smoothly and its onset is followed by a temporal increase of synchrony, because \( \theta_0 \) is swept through the interval of angles that are beneficial for enhancement. For \( \psi = \pi/4 \) (not shown) the transient is shorter and there is no intermediate increase in the amplitude of the mean field. In the desynchronized state \( S = 37.7 \) and \( \theta_0 \approx 1.56 \), cf. Fig. 5b. Parameters are \( k_1 = 0.025 \), \( k_2 = 500 \), \( k_3 = 0.01 \), \( k_4 = 5 \). The parameter \( a_{stop} \) is taken as 20% of the average amplitude of the autonomous system, i.e., before the feedback is turned on.

A. An example: ensemble of chaotic Rössler oscillators

With this example we demonstrate that the approach can be also applied to more complicated models and that suppression can be achieved with only two pulses per oscillatory cycle. Next, we explore the dependence of the performance on most important parameters.

We consider an ensemble of globally coupled chaotic Rössler oscillators:

\[
\begin{align*}
\dot{x}_k &= -\omega_k y_k - z_k + \varepsilon X + \cos \psi \cdot P(t) , \\
\dot{y}_k &= \omega_k x_k + 0.15 y_k + \sin \psi \cdot P(t) , \\
\dot{z}_k &= 0.4 + z_k(x_k - 8.5) ,
\end{align*}
\]

where frequencies \( \omega_k \) are Gaussian distributed with the mean \( \omega_0 = 1 \) and standard deviation 0.02. Without stimulation the system exhibits the Kuramoto synchronization transition at the critical coupling \( \varepsilon_{cr} \approx 0.05915 \). For \( \varepsilon > \varepsilon_{cr} \) the mean-field dynamics is nearly periodic, while for \( \varepsilon < \varepsilon_{cr} \) one observes small finite-size fluctuations of \( X \).

First, in Fig. 7 we illustrate suppression of synchrony in a system of \( N = 5000 \) units, with \( \varepsilon = 0.1 \) and \( \psi = \pi/4 \). Parameters of the feedback system are: \( \omega_0 = 1 \), \( k_1 = 0.001 \), \( k_3 = 0.001 \), \( \Theta_{tol} = 0.04 \pi \), \( A_0 = 2 \), \( \delta = 0.2 \), and \( \Delta = 0.4 \) (other parameters are as given above). For the chosen value of \( \Theta_{tol} \) only two pulses per cycles are applied (as can be seen in Fig. 7) and the adaptive algorithm converges to \( \theta_0 \approx 0.47 \) and \( \varepsilon_{fb} \approx -0.55 \). The suppression coefficient is \( S = 33.5 \).

Next, we check the dependence of \( S \) on most important parameters, starting with the frequency of the linear oscillator, \( \omega_0 \), see Eq. (2). Figure 8a presents the re-
FIG. 7. Suppression of the collective mode in the ensemble of Rössler oscillators [10]. Black solid curve shows a piece of trajectory of the unforced system in the mean-field coordinates $X$ and $Y = N^{-1} \sum \Delta_{\alpha}$. Dashed curve shows trajectory of the controlled system. (The trajectory is omitted for small amplitudes for better visibility.) Black circles (red squares) indicate the points where negative (positive) pulses are applied.

FIG. 8. Suppression coefficient $S$ in dependence on the frequency $\omega_0$ of the linear oscillator Eq. (2) (a) and on the pulse width, $\delta$, for $k_4 = 5$ (circles) and $k_4 = 0.5$ (squares).

results. This plot demonstrates that the technique works for a rather broad range of $\omega_0$. This feature is important for treatment of real-world systems with drifting average frequency. The second test shows how the performance depends on the pulse width $\delta$ and on the parameter $k_4$ (Fig. 8b). The latter determines saturation level for $\varepsilon_{\beta b}$, so that we can expect that the smaller $k_4$ the larger $\varepsilon_{\beta b}$ and, correspondingly, $S$. We also expect that broadening the pulse increases efficiency of suppression. Figure 8b indicates that this expectation is correct unless the pulses become too wide and do not any more fit the interval of vulnerable phases.

Finally, we check whether the approach works for strongly coupled Rössler ensemble, $\varepsilon = 0.2$ (see Eq. 10). For the pulse width $\delta = 0.2$ and $k_4 = 5$ the technique fails, also with $A_0 = 4$. For $\delta = 0.2$, $k_4 = 0.5$, and $A_0 = 4$ we achieve suppression with $S \approx 11$. Helpful is also initial increase of the feedback, i.e. taking $\varepsilon_{\beta b}(t_0) = -1$, then the suppression works also with $k_4 = 5$. (We also compare the final values of $\varepsilon_{\beta b}$: for $k_4 = 5$ it remains $\approx 1$; for $k_4 = 0.5$ it tends to $-1.5$.)

IV. CHARGED-BALANCED PULSES

The electrical stimulation of living systems requires a special form of pulses. Since the accumulation of electrical charge in the cells can be harmful, the pulses must be bipolar and charge-balanced. Figure 9 provides the simplest example of such stimuli. In the rest of this Section, we explore desynchronization with charge-balanced stimulation. For the test system, we again take the ensemble of globally coupled Bonhoeffer – van der Pol oscillators, see Eqs. (1). If not said otherwise, the parameters are the same as in Section IIIa.

We begin with the stimuli shown in Fig. 9a. We fix $\delta = 0.2$ and the amplitude ratio $A_n/A_{n,-} = -10$, then the charge-balance condition yields $\Delta_2 = 105$. It turns out that the result of stimulation essentially depends on $\Delta_1$. If the negative part of the stimulus immediately follows the positive one (or vice versa) then their actions compensate each other. Indeed, for $\Delta_1 = 0$ and $\Delta_1 = 2$ there is no suppression, $S \approx 1$ (see 9b for a detailed model study on the suppression efficacy in dependence on the gap $\Delta_1$). However, for $\Delta_1 = 6$, to be compared with the average oscillation period $T \approx 32.5$, the suppression factor is $S \approx 40$. It means that a narrow pulse comes in the vulnerable phase while the compensating wide pulse appears close to the least sensitive phase. The efficiency of the suppression can improve if stimuli shown in Fig. 9b are used. We tested stimulation with $\delta = \Delta = 0.2$, $\Delta_1 = 6$, and $N_b = 2$ and $N_b = 3$. As expected, the
the frequency of the collective oscillation is not known in a proper way and this may be not an easy task if uncoupled. However, the frequency $\nu$ works of periodic or chaotic oscillators, even if they are

Enhancement can be also achieved via the feedback technique with slightly modified update rules:

$$\theta_0 \rightarrow \theta_0 + k_1 (A_{stop} - \bar{A})(1 + \tanh[k_2 (A_{sat} - \bar{A})]),$$  \hspace{0.5cm} (11)

$$\varepsilon_{fb} \rightarrow \varepsilon_{fb} + k_3 (A_{stop} - \bar{A}) \cosh(k_4 \varepsilon_{fb}),$$  \hspace{0.5cm} (12)

where $A_{sat}$ is the saturation value. Certainly, this approach also has a frequency parameter, namely the frequency of the linear oscillator, $\omega_0$. If the frequency of the collective oscillation is not known a priori, $\omega_0$ shall be guessed. However, the results are not very sensitive to the choice of $\omega_0$, as illustrated in Fig. [11] for the model [10] with the sub-threshold coupling $\varepsilon = 0.02$. Other parameters are $A_0 = 1$, $A_{stop} = 5$, $A_{sat} = 2$. The rectangular pulses ($\delta = 0.2$, $\Delta = 1$) were used.

VI. DISCUSSION AND CONCLUSIONS

In summary, we presented and tested a closed-loop approach for control of collective activity in a globally coupled ensemble. Its main advantage is that control is achieved by rare precisely timed pulses. So, we have shown that desynchronization can be achieved and maintained by only one stimulus per oscillatory cycle. The control parameters – the feedback coefficient $\varepsilon_{fb}$ and the value of the phase $\theta_0$ when the system is most sensitive to stimulation – are adjusted automatically. An essential feature of the approach is that phase of the collective
We especially mention the phase-specific stimulation development in the neuroscience community loop techniques for DBS that are nowadays under development or in combination with ad hoc model-free closed-loop approaches. We rely on a quite general assumption that the rhythms to be controlled emerge in a highly interconnected network. Certainly, the dynamics of the human brain is much more complex, and the synchronization hypothesis may turn out too simplistic. However, our model-based approach can be useful as it is or in combination with ad hoc model-free closed-loop techniques for DBS that are nowadays under development in the neuroscience community. In this context we especially mention the phase-specific stimulation suggested and implemented. We emphasize several properties of our technique that render it suitable for DBS application. (i) It works with realistic charge-balanced stimuli. Though we have not searched for the optimal shape of stimuli, we have shown that stimulation is efficient if pulses of opposite polarity appear at the most and least sensitive phases, respectively. It means that the condition of charge balance is fulfilled on a time scale of about one-fourth of the oscillatory cycle. (ii) Stimulation and measurement are separated in time. Indeed, the adaptive algorithm relies only on the values of the instantaneous amplitude between the epochs where stimulation is applied. (iii) Since the optimal phase for stimulation is determined automatically, it does not matter whether the mean field $X$ or its phase-shifted version is measured. This property is useful if the stimulation and measurement of brain activity are performed at different sites. (iv) The linear oscillator used for phase estimation also acts as a band-pass filter and, therefore, extracts the rhythm of interest from the raw signal. However, the bandwidth of the filter is quite large — the property required to deal with the signals with drifting frequency.

As a direction for further improvement, we mention the modification of the adaptation rule to allow for both increase and decrease of $\theta_0$. This modification will reduce the transient time for desynchronization and will help to avoid a temporal increase of synchrony in the process of adaptation.

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The fixed point can be found by simulating the ensemble for $\varepsilon = 0$.

We emphasize that phase angle $\theta$ is not the true phase of the self-sustained oscillatory system but only a protophase, see\textsuperscript{24} but this distinction is not important for our problem. We also stress, that $\theta$ is related to the parameter $\psi$ but is not equal to it, as will be discussed below.

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