Spin Berry phase in helical edge state: Transport signatures

Vivekananda Adak,1 Krishna Roychowdhury,2,3 and Sourin Das1
1Department of Physical Sciences, IISER Kolkata, Mohanpur, West Bengal 741246, India.
2Department of Physics, Stockholm University, SE-106 91 Stockholm, Sweden.
3Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853, USA.

Topological protection of edge state in quantum spin Hall systems relies only on time reversal symmetry, hence, $S_z$ conservation on the edge can be relaxed. This has consequence like spin Berry phase arising from a closed loop dynamics of the spin state of the electron. In distinction to most of the previous studies in this context, which have primarily been restricted to conserved $S_z$, we investigate the effects of spin Berry phase induced by $S_z$ nonconservation on transport in a pristine edge. Our work provides a minimal framework to generate and detect these effects by employing spin-polarized leads in an interferometric set-up. Naively, one would expect that the measurements involving the edge state would be jeopardized due to the presence of spin polarized leads which can induce strong backscattering by destroying the time reversal symmetry at the tunnel-junctions. Quite contrary to the expectation, these leads turn out to be advantageous as they induce sharp Fano-type antiresonances with large visibility in the two-terminal conductance. As a function of energy (of the incident electron), the position of these antiresonances gets shifted owing to the presence of spin Berry phase, hence, serving as a smoking gun signal for $S_z$ nonconserving edge state.

I. INTRODUCTION

Birth of topological insulators (TI) has marked a new realm in the field of condensed matter research and nucleated a number of experimental activities in quest for materials relevant for exploring the topological aspects of such systems in the past decade. Endowed with an exotic surface physics, these materials can be described in terms of simple band Hamiltonians with spin-orbit (SO) couplings which respect time-reversal symmetry. The surface states owe their existence to the nontrivial topology of the bulk as an implication of bulk-boundary correspondence. In two-dimension, a simplistic description of topological insulators can be captured in the so-called Bernevig-Hughes-Zhang (BHZ) model of HgTe quantum well. Exceeding a critical value of the well width, an inversion between the bands near the Fermi surface drives the system into a topological insulator state with localized edge modes on the boundary. These edge modes have conserved spin quantum number $S_z$ locked with their momentum viz. if $\uparrow$-spins ($S_z = +1$) flow along $+k$, called right movers, $\downarrow$-spins ($S_z = -1$) would flow along $-k$, called left movers, ensued from time-reversal symmetry — a phenomenon known as quantum spin Hall (QSH) effect.

The edge state in the BHZ model has linear dispersion around the point with conserved helicity ($\propto S \cdot k$), hence, known as helical edge state (HES). The spin quantization axis of the HES is aligned along the SO field operative perpendicular to the plane (along the spatial $z$ axis) that hosts the HES (i.e. the spatial $x - y$ plane), and therefore, $S_z$ serves as a good quantum number to label the HES. The dynamics of the helical edge can effectively be described in terms of a Dirac Hamiltonian of the form

$$\mathcal{H}_{\text{QSH}} = \int dx \, \Psi^\dagger \mathcal{H} \Psi; \quad \mathcal{H} = -i\hbar (a_{SO} \cdot \sigma) \partial_x,$$

where $x$ denotes the spatial coordinate along an edge, $a_{SO}$ is the SO field pointing along the spatial $z$-axis: $a_{SO} = v_F(0,0,1)$; $v_F$ being the Fermi velocity of the electrons on the edge and $\Psi \equiv (\psi_R \psi_L)^T$ denotes the annihilation operator for the right ($R$) and the left ($L$) moving electrons (they can equivalently be labeled by $\uparrow$ or $\downarrow$).

In general, the SO field along the edge can orient along any arbitrary direction destroying the conservation of $S_z$. It is only the time-reversal symmetry that suffices to preserve the HES implying that the spin rotation symmetry about the $z$ axis can be broken without influencing the topology of the bulk. Such freedom of tuning the SO field direction allows for the possibility of the generation of spin Berry (SB) phase, which can arise because of spin dynamics of the electron in addition to the dynamical phase produced due to its propagation along the edge. This phase can be understood as Aharonov-Bohm (AB) effect on the Bloch sphere, and hence, is referred to as spin AB effect. Many authors, in the last few decades, have explored the presence of such phase appearing in the context of mesoscopic transport set-ups.

There have been recent theoretical proposals which have explored the possibility of probing the helical nature of the edge state in transport set-up. In this article, we are particularly interested in interferometric signatures and manifestation of helical nature of the edge state. In this context, Maciejko et al. studied the possibility of building a spin transistor in a AB ring built into a QSH state which is sandwiched between two ferromagnetic leads. They showed that it is possible to control spin of the electron on the edge via the AB flux resulting in the spin AB effect. However, they assumed a uniform SO coupling along the edge maintaining the conservation of $S_z$.

In contrast to their work, we study a complementary situation where the electron spin on the edge is itself undergoing a nontrivial variation along the edge due to the presence of a nonuniform SO field on the edge, hence, destroying the $S_z$ conservation. We discuss the minimal scenario where such a variation could lead to a fictitious flux induced by the spin Berry phase.

Earlier theoretical study also predicted evidence of quantized geometric phase of $\pi$ where transport across a Fabry-Perot interferometer is studied using a double quantum point contact geometry in a QSH state. Our study generalizes all such results to the case of nonquantized geometric phase. Ef-
fected due to gate induced doping of the edge state resulting from the application of an electrical field along a finite patch of the edge state have also been studied. This study exploited the gate controlled dynamical phase for tuning the interference signal in QSH interferometer and was insensitive to the SO interaction induced by the electric field of the applied gate. Therefore, it could not distinguish the $S_z$ nonconserving case from the conserving one. Addressing the former is the focus of our study.

Noninterferometric signatures of scattering of electrons from a SO barrier induced by application of a local gate voltage have also been studied. But in that work also, the primary focus was on a uniform SO barrier and the possibility of realizing a nonzero geometric phase arising from the variation of the SO field along the edge was not considered.

To gain insight into the generation and detection of spin Berry phase in an interferometer set-up, let us consider a standard two-path interferometer as a prototype. Let us further assume that the interferometer arms are endowed with the possibility of rotating the electron spin due to the presence of SO coupling in the arms of the interferometer as it traverses through the respective arms of the interferometer. In this article, we will discuss specific models of SO-coupled Hamiltonians that serve as the necessary and sufficient requirement for inducing the rotation of the spin that allows it to acquire a finite SB phase in its closed loop journey around the interferometer. For further illumination, the following scenario would be useful to consider. Let us assume an electron with spin $|\uparrow\rangle$ entering the interferometer from the left lead [Fig. 1 (a)] and its wavefunction simultaneously leaking into the upper and lower arm with respective quantum mechanical amplitudes. As the amplitudes propagating along the upper and the lower arm could generically suffer different history of the SO field, the incident spinor would evolve into $|\chi_1\rangle$ in the upper arm and $|\chi_2\rangle$ in the lower arm that trace out two independent trajectories (labelled $T_1$ and $T_2$) starting from the same point corresponding to the incident state $|\uparrow\rangle$ on the Bloch sphere.

The rest of the article is organized as follows: In section II, we discuss the minimal scenario leading to a finite SB phase on the helical edge state resulting either from an intrinsic SO interaction of the spin Hall state or due to the application of an external electric field on the edge. In section III, we calculate the transfer matrix for the situation which corresponds to minimal scenario for hosting a finite SB phase. Then in section IV, we show that a two-terminal transport set-up involving spin-polarized leads provides a clear signature of the SB phase for different possible orientations of the spin polarization of the leads before we conclude in section V.

II. SCATTERING THROUGH SPIN-ORBIT BARRIERS AND SPIN BERRY PHASE

In this section, we will discuss the possibility for an electron to accumulate a finite SB phase as it traverses through a nonuniform SO region which is embedded in otherwise uniform helical edge state. The Hamiltonian for the edge state which is assumed to be extended from $x = -\infty$ to $x = +\infty$ (where $x$ represents an intrinsic one-dimensional coordinate along the edge) can be written as

$$H_{SO} = -\frac{i}{2} \hbar \{a(x), \partial_x\} \cdot \sigma,$$

where

$$a(x) = [1 - \Theta(x) + \Theta(x - L)]a_1 + [\Theta(x) - \Theta(x - L)]a_2.$$ 

Here $\Theta(x)$ denotes the Heaviside step function. To be specific, we consider a situation where the vector $a_1 = |a_1| \hat{z}$ corresponds to a uniform SO field which is pointing along $z$-axis while $a_2$, which is extended from $x = 0$ to $x = L$ [Fig. 2 (a)], can point in a direction different from $a_1$ and can also have spatial variation. This finite patch of $a_2$ can be thought of as a barrier.

We consider a simplest possible situation where $a_2$ represents a vector which is constant in space but is pointing in a direction different from $a_1$ and further assume WLOG $a_2 = |a_2| \hat{x}$. Note that, though the SO field is constant along
the barrier, the electron spin undergoes a drastic change as it enters and exits the barrier when incident either from the left or from the right side of the barrier. Hence, a priori it is not clear if such situation would lead to a finite SB phase or not.

When an electron is incident on the SO barrier from the left \((x < 0)\) [Fig. 2(a)], its spin will initially point along the \(z\)-axis (north pole on the Bloch sphere) but once it enters the barrier it will reorient itself along the \(x\)-axis and again when it exits, the spin will rotate back to the \(z\)-axis (north pole). This implies that the trajectory of electron spin on the Bloch sphere traces a single curve (geodesic path) running from the north pole to the equator when it enters the barrier and then runs back exactly along the same path during its return journey when it exits. Hence, the trajectory of the spin state on the Bloch sphere encloses zero area during its close loop journey starting from and ending at the north pole and so, a zero SB phase accumulation is expected for a constant SO barrier.

For accumulation of a finite SB phase we surely need a SO barrier which has a variation of the orientation of the SO field along the length of the barrier. The cases of nonzero SB phase can be categorized as follows:

(a) quantized SB phase of \(\pi\),

(b) nonquantized SB phase varying between 0 and \(2\pi\).

A quantized value of SB phase can be generated by means of engineering the following SO barrier. The SO field \(\hat{a}(x)\) on the edge is chosen to be such that, inside the barrier [Fig. 2(a)], rotates along the edge where the rotation is parameterized by a space dependent monotonically increasing angle \(\theta_x\) such that \(\hat{a}(x) = (\sin \theta_x, 0, \cos \theta_x)\) while outside the barrier it is \(\hat{a}(x) = (0, 0, 1)\). Then it can be shown that

\[
\phi_{SB} = \begin{cases} 
0, & \text{if } \theta_{x=0} + \Delta \theta < \pi \\
\pi, & \text{if } \theta_{x=0} + \Delta \theta > \pi,
\end{cases}
\]

where \(\theta_{x=0}, \theta_{x=L}\) specify the orientation of the SO field at \(x = 0, L\) respectively and \(\Delta \theta = \theta_{x=L} - \theta_{x=0}\). The angle \(\theta_{x=0}\) can be interpreted as the measure of the change in the orientation of the incident spinor on the Bloch sphere as it enters the SO barrier. In the case when \(\phi_{SB} = 0\), the trajectory of the incident electron spin on the Bloch sphere traces a closed loop path along the great circle defined by the intersection of the \(x-z\) plane and the Bloch sphere which goes back and forth on the Bloch sphere without encircling the center. This trajectory on the Bloch sphere is similar to the one for the case of constant SO barrier discussed previously. For the case of \(\phi_{SB} = \pi\), the electron spin on the Bloch sphere winds the great circle once as the electron traverses through the barrier and exits. This demonstrates the topological nature of this phase. An important point to note here is the fact that, obtaining a SB phase of \(\pi\) does not require rotation of the SO vector (given by \(\Delta \theta\)) to be \(\pi\); what suffices is that \(\Delta \theta\) reaches a threshold value such that \(\theta_{x=0} + \Delta \theta > \pi\).

Now we will discuss the minimal variation of the SO field within the barrier required to give rise to a finite nonquantized SB phase. We need to find a configuration of the SO field which will lead to closed loop trajectory of the electron spinor on the Bloch sphere enclosing a finite area as the electron enters and exits the SO barrier. This can be achieved if the barrier can be subdivided into two regions with their respective SO vectors pointing along \(\hat{a}_2\) first and then \(\hat{a}_3\) (starting from the left) which should be distinct from each other and also mutually distinct from \(\hat{a}_1\) [Fig. 2(b)]. The journey of the electron across such barrier, when incident from the left, can be mapped to the journey of the electron spinor on the Bloch sphere which is as follows. The incident spinor which is pointing to the north pole (\(\hat{a}_1\) being along \(z\)-axis) first moves to a point (call it \(N_1\)) on the surface of the Bloch sphere corresponding to the direction of \(\hat{a}_2\) along a geodesic path connecting the north pole and \(N_1\). Then, as the electron further moves from region 1 to region 2 inside the barrier, its spinor moves from point \(N_1\) to point \(N_2\) along the geodesic path connecting \(N_1\) and \(N_2\) on the Bloch sphere, where \(N_2\) is the point on the surface of the Bloch sphere corresponding to the direction of \(\hat{a}_3\). Finally, when the electron leaves the barrier, the electron spinor moves back to the point corresponding to \(\hat{a}_1\) along a geodesic starting from \(N_2\), hence, forming a spherical triangle on the Bloch sphere. The SB phase accumulated by the electron in this journey will be given by half the solid angle

\[
\Phi_{SB} = \frac{1}{2} \times \text{solid angle between } \hat{a}_1 \text{ and } \hat{a}_3.
\]
matrix from the region of $a_1$ to the region of $a_2$ [in Fig. 2(c)] and is of the form

$$T_{21} = \sqrt{\frac{|a_1|}{|a_2|}} \text{Exp}[i \theta_{21} \hat{D}_{21} \cdot \sigma], \quad (8)$$

where

$$\hat{D}_{21} = a_2 \times a_1, \quad \text{and} \quad \tan(2\theta_{21}) = \frac{|\hat{D}_{21}|}{(a_2 \cdot a_1)}. \quad (9)$$

Note the operator $T_{21}$ is, in general, a nonunitary operator unless $|a_1| = |a_2|$. A minimal set-up required for obtaining a nonzero SB phase corresponds to an array of such interfaces between distinct SO fields and needs to be constructed such that the electron spin, in successive steps, encounters the SO fields as $\hat{a}_1 \rightarrow \hat{a}_2 \rightarrow \hat{a}_3 \rightarrow \hat{a}_1$ as noted previously and shown in Fig. 2(b). The net transfer matrix in this process is remarkably an unitary operator of the form

$$\mathcal{T} = \prod_{ij} T_{ij} = T_{13}T_{32}T_{21}$$

$$= \text{Exp}[i \theta_{13} \hat{D}_{13} \cdot \sigma] \text{Exp}[i \theta_{32} \hat{D}_{32} \cdot \sigma] \text{Exp}[i \theta_{21} \hat{D}_{21} \cdot \sigma], \quad (10)$$

irrespective of the magnitudes of $a_1$, $a_2$, and $a_3$. The SB phase acquired by the electron as it goes once around the circle defined by $\hat{a}_1 \rightarrow \hat{a}_2 \rightarrow \hat{a}_3 \rightarrow \hat{a}_1$ [Fig. 2(b)] is given by $\phi_{\text{SB}} = \text{arg}[(n_{11}^\dagger T n_{11})]$, where the subscript 1 denotes that the spinor whose evolution is concerned is an eigenstate of $H_{\text{SO}}$ in Eq. 2 with $a = a_1$, and $\dagger$ represents the spin state of the electron aligned with the local SO field while it is moving along $\hat{a}_1 \rightarrow \hat{a}_2 \rightarrow \hat{a}_3 \rightarrow \hat{a}_1$. $\downarrow$ would correspondingly represent the spin of the electron antialigned with the local SO field while moving in the opposite direction.

To obtain an explicit expression of the SB phase, Eq. 10 can be re-expressed in a compact form as

$$\mathcal{T} \equiv \text{Exp}[i \alpha \hat{K} \cdot \sigma], \quad (11)$$

where

$$\cos \alpha = \cos \theta_{13} \cos \theta_{32} \cos \theta_{21} - (\hat{D}_{13} \cdot \hat{D}_{32}) \sin \theta_{13} \sin \theta_{32} \cos \theta_{21} - (\hat{D}_{32} \cdot \hat{D}_{21}) \cos \theta_{13} \sin \theta_{32} \sin \theta_{21} - (\hat{D}_{21} \cdot \hat{D}_{13}) \sin \theta_{13} \cos \theta_{32} \sin \theta_{21} + [\hat{D}_{13} \cdot \hat{D}_{32} \cdot \hat{D}_{21}] \sin \theta_{13} \sin \theta_{32} \sin \theta_{21}, \quad (12)$$
where these three regions are connected such that the vectors our calculations are done for a model with sudden jump be-
to take note of this point as the primary aim of this article is to
such that the entire ring is covered by three successive patches
along the edge [as in Fig. 2 (b)]. Hence we consider a model
where the SO field configuration along the edge is taken to be
the SO field along the edge \([as in Fig. 2 (b)]\). The phases \(\phi'\) and \(\phi''\) are the dynamical phases from
P1 to P2 along the upper and lower arm respectively and the total
length of the arms is \(L\). The spinors of the leads \((L_{1,2})\) are denoted by \(\uparrow\) while that of the HES are denote by \(\downarrow\).

**A. Interferometry with polarized leads**

The schematic of the proposed set-up is given in Fig. 5 where
the closed edge state is subdivided into three parts such that
the SO fields in the blue, yellow, and red region are speci-
fied by three distinct vectors \(a_1, a_2, a_3\). The Hamiltonian
for the closed edge is provided in Eq. 2 with a given spatial profile of \(a(x)\) subjected to periodic boundary
condition. We have considered two tunnel-coupled spin-
polarized leads \((L_1, L_2)\) attached to the closed edge where
\(L_1\) is coupled to a point P1 in the blue region and \(L_2\) is
coupled to a point P2 in the yellow region (the lead positions
are arbitrary and can be in any one/two of the three regions;
we, for instance, consider the case when the two leads are
placed in two different regions). We model the leads by spin-
polarized chiral edge states with linear dispersion. This way,
owing to the linear dispersion, the transport is influenced only
by the direction of spin polarization of the lead electrons and
not by the density of states of the leads. The form of the lead
Hamiltonian (for lead \(L_x, x = 1, 2\)) can be taken to be a chiral
mode (right moving with Fermi velocity \(v_F\)) and is given by

\[
\mathcal{H}_{L_i} = -i\hbar v_F \int dx \psi_{\alpha i}^\dagger \partial_x \psi_{\alpha i},
\]

where \(\psi_{\alpha i}^\dagger\) creates an electron in lead \(L_i\) with spinor \(|\alpha_i\rangle\).

\[
\quad \mathcal{H}_{T}^{(f)} = \Gamma_f \int dx \delta(x-x_f) \sum_{\alpha=\uparrow, \downarrow} \{ \mathcal{Y}_{\alpha x} \psi_{\alpha i}^\dagger \psi_{\alpha i} + \text{h.c.} \},
\]

**IV. SB PHASE AND ITS INTERFEROMETRIC MANIFESTATION**

Here we study a minimal two-terminal transport set-up (see
Fig. 4) which could lead to the detection of a finite SB phase.
Our proposed set-up is a ring (of circumference \(L\)) re-
presenting an isolated closed edge (see Fig. 4) which is tunnel-
coupled to two polarized leads \(L_1, L_2\) at the point P1
and P2 as shown in Fig. 5. As discussed earlier in the pre-
vious section, a finite SB phase requires the presence of a
spatially varying SO field and a minimal scenario demands for
the presence of at least three distinct directions of the SO field
along the edge [as in Fig. 2(b)]. Hence we consider a model
where the SO field configuration along the edge is taken to be
such that the entire ring is covered by three successive patches
of SO field pointing along \(a_1, a_2, a_3\). We have chosen
\(a_1 = (0, 0, 1)\) for calculational convenience. Note that though
our calculations are done for a model with sudden jump be-
tween different SO field directions, our qualitative results re-
main even if we replace our situation with another situation
where these three regions are connected such that the vectors
\(a_1, a_2, a_3\) go smoothly on to one another. It is important
to take note of this point as this the primary aim of this article is to

\[
\begin{align*}
\mathcal{K} \sin \alpha &= \mathbf{D}_{13} \sin \theta_{13} (\cos \theta_{12} \cos \theta_{21}) \\
&- \mathbf{D}_{32} \cdot \mathbf{D}_{21} \sin \theta_{32} \sin \theta_{21} \\
&+ \mathbf{D}_{32} \sin \theta_{32} (\cos \theta_{13} \cos \theta_{21}) \\
&+ \mathbf{D}_{13} \cdot \mathbf{D}_{13} \sin \theta_{13} \sin \theta_{21} \\
&+ \mathbf{D}_{21} \sin \theta_{21} (\cos \theta_{32} \cos \theta_{13}) \\
&- \mathbf{D}_{32} \cdot \mathbf{D}_{13} \sin \theta_{32} \sin \theta_{13} \\
&+ (\mathbf{D}_{21} \times \mathbf{D}_{13}) \sin \theta_{13} \sin \theta_{32} \sin \theta_{21} \\
&- (\mathbf{D}_{13} \times \mathbf{D}_{32}) \sin \theta_{13} \sin \theta_{32} \sin \theta_{21} \\
&- (\mathbf{D}_{32} \times \mathbf{D}_{21}) \cos \theta_{13} \sin \theta_{32} \sin \theta_{21}. \quad (13)
\end{align*}
\]

A straightforward but lengthy algebra leads to the following
expression of \(\alpha\) given by

\[
\tan \alpha = \frac{|\hat{a}_1 \cdot (\hat{a}_2 \times \hat{a}_3)|}{1 + \hat{a}_1 \cdot \hat{a}_2 + \hat{a}_2 \cdot \hat{a}_3 + \hat{a}_3 \cdot \hat{a}_1}. \quad (14)
\]

Here, \(\alpha\) has a natural interpretation as the SB phase owing to
the fact that \(\mathbf{K}\) is collinear with \(\hat{a}_1\) and \(\alpha\) appears as an over-
all phase in Eq. \(11\). The unit vector \(\mathbf{K}\) will be parallel to \(\hat{a}_1\)
\((\mathbf{K} = -\hat{a}_1)\). The explicit derivation of Eq. \(14\) is given in Appen-
dix A. Note the expression in Eq. \(5\) is exactly the same as Eq. \(14\)
with \(\hat{z} \to \hat{a}_1, \hat{a}_1 \to \hat{a}_2\) and \(\hat{a}_2 \to \hat{a}_3\) and \(\alpha\) being identified as half the solid angle sub-
tended by the area of a spherical triangle (shown in Fig. 5) on
the Bloch sphere.

**FIG. 5. Sketch of the \(a_1 - a_2 - a_3\) model with two leads connected at
P1 and P2 is shown. The SO fields in the blue, yellow, and red region
are represented by three distinct vectors \(a_1, a_2, a_3\). The respective
amplitudes of propagation are denoted at the tunnel-junctions P1 (injecting)
and P2 (receiving). The phases \(\phi'\) and \(\phi''\) are the dynamical phases from
P1 to P2 along the upper and lower arm respectively and the total
length of the arms is \(L\). The spinors of the leads \((L_{1,2})\) are denoted by \(\uparrow\) while that of the HES are denote by \(\downarrow\).**
where $\hat{a}^{\dagger}_{\alpha j}$ represents the creation operator for electrons in the HES with spinor $|n_{\alpha j}\rangle$ specified by the Hamiltonian in Eq. 2 with $a = a_j$ and $T_{\alpha j,\beta} = \langle n_{\alpha j} | n_{\beta j} \rangle$; $T_1$ represents the tunneling strength at the tunnel-junction between lead $L_1$ and the (local) edge.

Now we set up the calculation of the transmission amplitude through the ring (Fig. 5). We consider a scattering problem where an electron is incident from lead $L_1$ and transmitted into lead $L_2$. This problem can be split into three different scattering problems which are finally connected to one another via boundary conditions as follows:

a) **Scattering at $P_1$**: The scattering at point $P_1$ can be reduced to a scattering between three incoming and three outgoing chiral edges at point $P_1$. The incoming and the outgoing amplitudes at $P_1$ are connected via scattering matrix $S_1$ as

$$\begin{pmatrix} A \\ C \\ r \end{pmatrix}^T = S_1 \begin{pmatrix} D \\ B \\ i_1 \end{pmatrix}^T,$$

where $i_1 (= 1)$ and $r$ are the plane wave amplitudes of the incident and the reflected wave in lead $L_1$ at $P_1$. The incoming and outgoing amplitudes in the helical edge at $P_1$ are given by $A, B, C,$ and $D$.

b) **Scattering at $P_2$**: Similarly, at point $P_2$, the incoming and outgoing amplitudes are connected via scattering matrix $S_2$ as

$$\begin{pmatrix} N \\ F \\ t \end{pmatrix}^T = S_2 \begin{pmatrix} E \\ M \\ i_2 \end{pmatrix}^T.$$

where $E, F, M,$ and $N$ are the incoming and outgoing amplitudes in the ring and $t$ (transmission amplitude) is the outgoing amplitude in lead $L_2$. The incoming amplitude $i_2$ is zero as no incidence is considered in lead $L_2$.

c) **Connecting the amplitudes inside the ring via transfer matrices given in Eq. 10**

Now to implement the matching conditions for the various amplitudes inside the closed ring, let us divide the ring into two parts as in Fig. 5: (i) the upper arm, where the journey of the electron ($\uparrow$) starting from point $P_1 \rightarrow P_2$ in clockwise sense accumulates a dynamical phase of $\phi'$ while the geometric phase (if any) is naturally embedded inside the transfer matrix, (ii) the lower arm, where the journey of the electron ($\uparrow$) starting from point $P_2 \rightarrow P_1$ in the clockwise sense accumulates a dynamical phase of $\phi''$ while again the geometric phase (if any) is naturally embedded inside the transfer matrix. These phases are incorporated into the problem via the following boundary conditions for the upper arm:

$$E = \langle n_{\uparrow 2} | T_{21} | n_{\uparrow 1} \rangle e^{i\phi'} A,$$

$$B = \langle n_{\downarrow 1} | T_{12} | n_{\uparrow 2} \rangle e^{i\phi'} F,$$

while for the lower arm, they are given by,

$$M = \langle n_{\uparrow 2} | T_{23} T_{31} | n_{\downarrow 1} \rangle e^{i\phi''} C,$$

$$D = \langle n_{\downarrow 1} | T_{13} T_{32} | n_{\uparrow 2} \rangle e^{i\phi''} N,$$

where $|n_{\alpha j}\rangle$ ($\alpha = \uparrow / \downarrow$) represents the eigenstate with $\pm 1$ eigenvalue of $\hat{a}_i \cdot \sigma$ ($\uparrow \leftrightarrow +1$, $\downarrow \leftrightarrow -1$). Finally, these three steps (a), (b), and (c) together provide the transmission amplitudes ($t$) of the system whose explicit forms are given below. Now, three distinct physical scenarios can be realized depending upon the relative orientations of the spin polarization of the leads with respect to the orientations of the local SO fields of the edge to which the leads are being tunnel-coupled:

1. **Both leads local parallel**: The spin polarization axes of both lead $L_1$ and $L_2$ are parallel to the vectors $\hat{a}_1$ and $\hat{a}_2$ respectively.

2. **One of the leads local parallel**: The spin polarization axis of lead $L_1$ is no more parallel to the vector $\hat{a}_1$, while that of lead $L_2$ is still taken to be parallel to the vector $\hat{a}_2$.

3. **Complete deviation from local parallel condition**: Both the spin polarization axes of leads $L_1$ and $L_2$ are no more parallel to the vectors $\hat{a}_1$ and $\hat{a}_2$ respectively.

From now on, we will assume the Fermi velocity in the leads ($v_F$) and that in the ring to be the same implying $|\hat{a}_1| = |\hat{a}_2| = |\alpha_1| = v_F$. It is to be noted that such assumption does not influence the geometric phase aspect of the problem whatsoever as long as $\hat{a}_1, \hat{a}_2, \alpha_1,$ and $\alpha_2$ are distinct. An explicit calculation of scattering matrices for mutual tunneling between different chiral edges is presented in Ref. [27] by exploiting the equation of motion technique following which, here, we have calculated $S_{12}$ at tunnel-junctions $P_1$ and $P_2$ (Fig. 5) and also the transmission amplitudes for the three cases depicted above.

1. **First scenario: Both leads local parallel**

   This case corresponds to the simplest possible situation the (spin-polarized) leads $L_1$ injects only clockwise moving ($\uparrow$) electrons into the ring as the spin polarization axis of the lead
is taken to be parallel to the direction of the SO field at P1. Hence, the injected current flows only in the clockwise direction. In this case, the transmission amplitude from lead L1 to lead L2 can be straightforwardly obtained as

\[ t = \frac{16e^{i\phi_D} \Gamma_1 \Gamma_2}{-a + be^{i\phi_D}}, \]  

(21)

where \( a = (4 + \Gamma_1^2)(4 + \Gamma_2^2) \), \( b = (4 - \Gamma_1^2)(4 - \Gamma_2^2) \), \( \Gamma_{1,2} = \Gamma_{1,2} \) are dimensionless parameters, \( \Gamma_1, \Gamma_2 \) being the tunneling strength at the tunnel-junction P1 (P2) and \( \phi_D = \phi' + \phi'' \) is the total dynamical phase acquired by an electron in a full cycle of its journey along the edge (i.e. P1→P2→P1 traversing a length of \( L \)).

In presence of a finite SB phase, the transmission probability changes to \( T(\phi_D) \rightarrow T(\phi_D + \phi_{SB}) \) \( (T = |t|^2) \) where the contribution due to the SB phase enters via matching conditions which depend on the transfer matrix as given in Eq. [19] and Eq. [20]. The interference pattern observed as a function of the incident energy \( E = n\hbar v_F / L \) features resonance peaks at values of \( E = \phi_D = 2n\pi \) for integer \( n \) (we have taken \( h = v_F = L = 1 \)) when \( \phi_{SB} = 0 \) and the peaks are shifted such that at the resonance peaks, \( \phi_D + \phi_{SB} = 2n\pi \) when \( \phi_{SB} \neq 0 \). In particular, the case of quantized SB phase of \( \pi \) results in a complete swapping of the maxima and minima of the transmission probability \( T(E) \). Such special case of quantized SB phase will be very similar to the situation discussed in Ref. [30]. Hence, the shift of the maxima in transmission probability in the interference pattern would indicate the presence of a finite SB phase in our set-up.

To summarize, for the simplest possible scenario of local parallel leads, the interference pattern obtained as a function of incident energy of the electron is shown to be independent of the positions of P1 and P2. The SB phase, in this case, can be read off by measuring the shift of the resonance peaks in \( T(E) \) (see Fig. 9). In particular, the resonance, which is expected at the Dirac point \( (E = 0) \), would shift due to the presence of a finite SB phase. Hence, as long as the identification of the Dirac point could be made by some independent experimental technique, the manifestation and quantification of the SB phase can be directly related to the shift of the resonance peak from the Dirac point.

2. Second scenario: partial (one lead) deviation from local parallel condition: antiresonance

In this case, lead L1 is no more “local parallel” to the direction of the SO coupling at the tunnel-junction P1 and hence, it injects into both the clockwise and anticlockwise moving edge channels while the other lead L2 can only absorb one particular chirality (right movers). As the electron traverses from L1 to L2, the two leading contributions to the transport can be attributed to the two distinct types of path and their interference [shown in Fig. 7(a)].

(i) The first type of path [shown in Fig. 7(a) left] is related to the injection of a clockwise moving ↑-electron (↑ with respect to the local SO field) at L1 which has a finite probability amplitude to exit at L2 after a direct traversal along the upper arm without going around the ring. Rest of the subsequent paths corresponds to the electron undergoing multiple rounds of circulation along the ring before exiting. The important point to note here is the fact that, to exit, the electron must be in an “up” state (with respect to the local SO field region that holds L2) as L2 is a local parallel lead receiving only the “up” states. To ensure this, the electron circulating along the ring has two possibilities:- either it goes around the ring integer number of times as a clockwise mover (↑-electron) without suffering spin-flip backscattering at L1 via a second-order tunneling process (between the lead and the edge), or, if the electron suffers spin-flip backscattering at L1, it must undergo such scattering even number of times so that it could return back to the up state before it could exit via lead L2. (ii) The second type of path [shown in Fig. 7(a) right] is related to the injection of an anticlockwise mover (↓-electron) at lead L1. Such an electron can not exit via lead L2 unless it undergoes a spin-flip scattering. The leading process which has a finite probability amplitude to exit at L2 corresponds to a situation where the injected ↓-electron first traverses a full circle starting it journey at L1 via the lower arm of the ring and then crossing passed L2 and reaching L1 again. Then it undergoes a spin-flip scattering to bounce back as a clockwise mover and travel through the upper arm back to L2 and exits the ring. Rest of the subsequent paths corresponds to the electron undergoing multiple rounds of circulation along the ring before exiting such that the total number of spin flip scattering at L1 is odd and hence it is to be in “up” state while exiting the ring via lead L2. The total transmission amplitude, which can be thought of as the sum of amplitudes of type -(i) and

![FIG. 7. Second scenario: (a) The two distinct types of paths (described in the text) that lead to a destructive interference yielding antiresonances at \( E = 2n\pi \) when one of the polarized leads is deviated away from its local parallel configuration. (b) The resultant interference pattern in absence (solid) and presence (dashed) of the SB phase \( |\phi_{SB} = \pi/4 \) in the plot with \( \alpha_1 = (0, 0, 1) \) and \( \alpha_2 = (1, 0, 0) \). Lead L2 is kept local parallel while lead L1 is tilted by an angle of \( \pi/3 \) from \( \alpha_1 \) keeping the azimuthal angle same.](image)
discussed above is given by
\[ t = \zeta [16e^{i\phi_1^*} \hat{\Gamma}_1 \hat{\Gamma}_2 \hat{\Upsilon}_{\gamma_1,\gamma_1}(4 + \hat{\Gamma}_1^2)(e^{i\phi_D} - 1)], \tag{22} \]
where,
\[ \zeta^{-1} = (4 + \hat{\Gamma}_2^2)[16(e^{i\phi_D} - 1)^2 + 4(1 - e^{2i\phi_D})(\hat{\Gamma}_1^2 + \hat{\Gamma}_2^2) + \hat{\Gamma}_1^2\hat{\Gamma}_2(e^{2i\phi_D} - 2\hat{\Upsilon}_1 e^{i\phi_D} + 1)], \tag{23} \]
and \( \hat{\Upsilon}_1 = 2|\hat{\Upsilon}_{\gamma_1,\gamma_1}|^2 - 1 \) (the overlap \( \hat{\Upsilon}_{\gamma_1,\gamma_1} \) corresponds to lead \( L_1 \) being attached to the region with SO field \( \hat{a}_1 \).

Zero-pole analysis - appearance of antiresonances:
As can be seen from the expression for the transmission probability amplitude in Eq. (22) the case for one local parallel lead is distinct from the case of both leads being local parallel in its analytic form. Eq. (22) is carrying a term \((e^{i\phi_D} - 1)\) which represents first-order zeroes at \( E = \phi_D = 2n\pi \) resulting in Fano-type antiresonances at those points [see Fig. 7 (b)]. These antiresonances are attributed to the interference between the two types of paths shown in Fig. 7(a) and hence, are directly connected to deviation of \( L_1 \) from its local parallel condition. Also it is interesting to note that the transmission zeros are always placed symmetrically between two maxima (around \( E = 2n\pi \)). The pattern is related to the relative positions of the zeros and the poles of \( t \) in Eq. (22).

It is straightforward to verify that the poles of \( t \) obtained from Eq. (23) are given by
\[ \phi_D = 2n\pi - i\ln R \text{ where } R = \frac{16 + \hat{\Upsilon}_1 \hat{\Gamma}_1^2 \hat{\Gamma}_2^2 \pm \sqrt{\Delta}}{(4 - \hat{\Gamma}_1^2)(4 - \hat{\Gamma}_2^2)}, \tag{24} \]
where \( n \) is an integer and \( \Delta = 32 \hat{\Gamma}_1^2 \hat{\Gamma}_2^2 + 16(\hat{\Gamma}_1^4 + \hat{\Gamma}_2^4) + (\hat{\Gamma}_1^2 - 1)^2 \hat{\Gamma}_2^4 \). The quantity \( R \) in Eq. (24) turns out to be real and positive in the weak tunneling limit; \( \hat{\Gamma}_1, \hat{\Gamma}_2 < 2 \). We note that the real part of the positions of zeros and poles in the complex \( \phi_D \) plane are same. Hence, in the absence of the zeros, \(|t| \) would have maxima at \( E = \phi_D = 2n\pi \) but due to the presence of the zeros exactly at the same positions, the original maxima split into two new symmetrically placed maxima about the transmission zeros as shown in Fig. 7(b).

From Eq. (24) it is evident that the locations of the poles in the complex \( \phi_D \) plane have nontrivial dependence on the parameter \( \hat{\Upsilon}_1 \) which quantifies the deviation of the polarized lead \( L_1 \) from its local parallel condition (i.e. when \( \hat{\Upsilon}_1 = 1 \)). This parameter can thought of as a control parameter which decides the width of the antiresonance. As we bring back \( L_1 \) to local parallel, one of the values of \( R \) in Eq. (24) approaches 1, and the imaginary component of the corresponding pole vanishes, thus, exactly cancelling the zero of \( t \) in Eq. (22). Furthermore, the complex pole, left after cancellation, coincides with the pole in \( t \) for the local parallel case as expected.

In presence of a finite SB phase picked up by the electrons, the transmission probability \( T \) between the leads \( L_1 \) and \( L_2 \) gets modified by \( \phi_D \rightarrow \phi_D + \phi_{SB} \) and so, a shift of the interference pattern [see Fig. 7(b)] would render a direct evidence of the presence of SB phase as noted previously.

3. Third scenario: complete deviation (two leads) from local parallel condition: distorted interference pattern

In a realistic situation, one would expect the polarization direction of both the leads to deviate from the local parallel condition when the spatial profile of the direction of the SO field on the edge is completely unknown. Following the same procedure as for the previous cases, the transmission amplitude \( t \) is evaluated to be
\[ t = \zeta [16\hat{\Gamma}_1 \hat{\Gamma}_2 (e^{i\phi_D} - 1)(e^{i\phi_1'} \hat{\Upsilon}_{\gamma_1,\gamma_1}(4 + \hat{\Gamma}_1^2 \hat{\Gamma}_2^2)(e^{i\phi_D} - 1)](e^{i\phi_2'} \hat{\Upsilon}_{\gamma_2,\gamma_2}(4 + \hat{\Gamma}_1^2 \hat{\Gamma}_2^2)(e^{i\phi_D} - 1)], \tag{25} \]
where
\[ \zeta^{-1} = (4 + \hat{\Gamma}_1^2)(4 + \hat{\Gamma}_2^2) - 4\hat{\Gamma}_1^2\hat{\Gamma}_2^2(\gamma' e^{2i\phi_D} + \gamma'' e^{2i\phi''}) - 2(16 + \hat{\Upsilon}_1 \hat{\Gamma}_1^2 \hat{\Gamma}_2^2)(e^{i\phi_D} - 1)(4 - \hat{\Gamma}_1^2)(4 - \hat{\Gamma}_2^2)e^{2i\phi_D}, \tag{26} \]
and
\[ \hat{\Upsilon}_1 = 2|\hat{\Upsilon}_{\gamma_1,\gamma_1}|^2 - 1 \text{ and } \hat{\Upsilon}_2 = 2|\hat{\Upsilon}_{\gamma_2,\gamma_2}|^2 - 1, \gamma' = \hat{\Upsilon}_{\gamma_1,\gamma_1} \hat{\Upsilon}_{\gamma_1,\gamma_1}(4 + \hat{\Gamma}_1^2 \hat{\Gamma}_2^2)(e^{i\phi_D} - 1) \hat{\Upsilon}_{\gamma_2,\gamma_2} \hat{\Upsilon}_{\gamma_2,\gamma_2}(4 + \hat{\Gamma}_1^2 \hat{\Gamma}_2^2)(e^{i\phi_D} - 1) \hat{\Upsilon}_{\gamma_1,\gamma_1}, \]
and
\[ \hat{\Upsilon}_{\alpha_i,\beta_j} = \langle \eta_{\alpha_i} | \eta_{\beta_j} \rangle \text{ being the spinor overlap between different spinors in the HES where } \alpha, \beta = \uparrow, \downarrow \text{ and } i, j = 1, 2 \text{ and } 3; \text{ the overlap } \hat{\Upsilon}_{\alpha_i,\beta_j} \text{ (and its conjugate } \hat{\Upsilon}_{\beta_j,\alpha_i} \text{)
where \( I = 1, 2 \) is defined below Eq. \[16\] (definitions of \( \phi' \) and \( \phi'' \) are given before). The quantity \( \hat{T}' \) and \( \hat{T}'' \) geometrically represent cyclic projections which, on the Block sphere, can be identified as hexagonal and octagonal Pancharatnam loops respectively.

**Position dependency of the leads - distorted transmission pattern**: Evidently, the phases \( \phi' \) and \( \phi'' \) are dependent on the lead positions on the rings unlike their sum \( \phi_D \). For an arbitrary value of \( \phi'/\phi_D \), the poles of \( t \) can have real components other than \( 2n\pi \), but the zeroes being pinned at \( 2n\pi \) (since it depends on \( \phi_D \) only) result in asymmetric maxima around the antiresonance points as shown in Fig. \[3\](a). This is in distinction to the second scenario where the two maxima around each antiresonance point were symmetric [Fig. \[7\](b)] because of the coincidence of the zeroes and the real components of the poles at \( 2n\pi \) (see Eq. \[24\]).

**Different periodicities found in the process - calculation of a net periodicity of the envelope of the distorted transmission pattern**: The phases appearing in the individual terms in Eq. \[25\] are representatives of different closed loops formed during the spin transport from \( L_1 \) to \( L_2 \) on the interferometer [depicted in Fig. \[3\](a)] whose multiple occurrences contribute to the total transmission probability \( T (\equiv |t|^2) \). These phases have their own periodicity that could be different from each other depending on the lead positions, which determines the overall periodicity of the envelope of \( T \) when plotted as a function of \( E \). For instance, if we place our leads \( L_1 \) and \( L_2 \) such that \( \phi' \) is a rational fraction of \( \phi_D \) i.e. \( \phi'/\phi_D = p/q \) where \( p, q \) are coprime with \( p < q \), the antiresonances appear in a period of \( 2\pi q \) on the \( E \) axis [see Fig. \[8\](b)] because of the zeroes of \( t \) in Eq. \[25\] which bears a factor \((e^{i\phi_D} - 1)\), however, the overall interference pattern is periodic with a period of \( 2q\pi \) if \( q \) is odd and \( q\pi \) if \( q \) is even. In Fig. \[8\](b), we have shown the case of \( p/q = 1/3 \) rendering a periodicity of \( 6\pi \) to the interference pattern when plotted against \( E \).

In presence of the SB phase, all the phase factors in the expression of \( t \) (Eq. \[25\] and \[26\]) are modified in a nontrivial way, but the total dynamical phase goes like \( \phi_D \rightarrow \phi_D + \phi_{SB} \) as before. This is crucial for identifying the antiresonance points which are shifted from \( 2n\pi \) to \( 2n\pi - \phi_{SB} \). But note that the entire interference pattern does not experience the same overall shift unlike previously. In fact, the pattern is further distorted due to the phase factors coming from \( \hat{T}' \) and \( \hat{T}'' \) in Eq. \[26\] that depend on the spin polarizations of the leads \( L_1 \) and \( L_2 \). However, the shift of the antiresonance points would still be a concrete evidence of the presence of SB phase in the system.

**V. CONCLUSION**

Studies on edge transport in quantum spin Hall systems have primarily considered situations where the \( S_z \) component of the spin is conserved. However, it is only the time reversal symmetry which is required to protect the edge state. The present article explores the consequences of relaxing the \( S_z \) conservation by considering a generic profile of the spin-orbit (SO) field along a pristine edge. As a result, we observe spin Berry (SB) phase accumulated by the electrons flowing along the edge, a finite value of which warrants \( S_z \) nonconservation and has notable effects on edge transport.

To measure such a phase, it is essential to employ an interferometric set-up for which we consider a ring geometry of the edge state tunnel-coupled to two spin-polarized leads that serves as a two-path interferometer. Motivated by Pancharatnam’s construct of geometric phase, we present the minimal criteria for the SO profile to lead to a finite accumulation of SB phase. Furthermore, we provide an explicit derivation of the expression for the SB phase in terms of the SO field configurations using a transfer matrix approach that constitutes one of the main results of the article. The compact form of the transfer matrix presented in this article is instructive in developing an understanding of the geometric aspect of the spin dynamics of itinerant electrons.

In our set-up, introduction of the spin-polarized leads results in sharp Fano-type antiresonance in the transmission probability which, in presence of a finite SB phase, get shifted by an amount equal to the SB phase. This provides a straightforward recipe of measuring the phase. We analyze three distinct situations depending on various possible orientations of the polarization directions of the leads that leave pronounced effects on the overall pattern of the transmission probability including its periodicity, however, the features of antiresonance remain in all such cases. So, in conclusion, measurement of a finite SB phase can be regarded as a smoking gun signal for \( S_z \) nonconservation in helical edge state.

**VI. ACKNOWLEDGMENTS**

The authors thank Poonam Mehta for illuminating discussions and scrutinizing the manuscript. V.A. acknowledges financial support from University Grants Commission, India and K.R. acknowledges sponsorship, in part, by the Swedish Research Council.

**Appendix A:**

**Derivation of SB phase from the product of transfer matrices**

Here we will present the derivation of Eq. \[14\] given in the main text. In particular, we analyze the terms in Eq. \[12\] that lead to the simplified expression of Eq. \[14\].

We start with the scalar triple product in the last term of Eq. \[12\] and write it as

\[
[D_{13}, D_{32}, D_{21}] = \frac{B}{\sin 2\theta_{21} \sin 2\theta_{13} \sin 2\theta_{32}},
\]

where

\[
B = |\hat{a}_1 \cdot (\hat{a}_2 \times \hat{a}_3)|^2 = \left[ 1 - (\hat{a}_1 \cdot \hat{a}_2)^2 - (\hat{a}_2 \cdot \hat{a}_3)^2 - (\hat{a}_3 \cdot \hat{a}_1)^2 \right] + 2(\hat{a}_1 \cdot \hat{a}_2)(\hat{a}_2 \cdot \hat{a}_3)(\hat{a}_3 \cdot \hat{a}_1).
\]
This expression is obtained using $\hat{D}_{ij} = (\hat{a}_i \times \hat{a}_j)/|\hat{a}_i \times \hat{a}_j|$. The last term of Eq. 12 then becomes

$$[\hat{D}_{13}, \hat{D}_{32}, \hat{D}_{21}] \sin \theta_{13} \sin \theta_{32} \sin \theta_{21}$$

$$= \frac{B}{8 \cos \theta_{13} \cos \theta_{32} \cos \theta_{21}}$$

$$= \frac{1 - x^2 - y^2 - z^2 + 2xyz}{8 \cos \theta_{13} \cos \theta_{32} \cos \theta_{21}}, \quad (A3)$$

where

$$x \equiv \hat{a}_2 \cdot \hat{a}_1 = \cos \theta_{21}, \quad y \equiv \hat{a}_3 \cdot \hat{a}_2 = \cos \theta_{32},$$

$$z \equiv \hat{a}_1 \cdot \hat{a}_3 = \cos \theta_{13}. \quad (A4)$$

Similarly, the first term of Eq. 12 can be written as

$$\cos \theta_{13} \cos \theta_{32} \cos \theta_{21}$$

$$= \frac{1 + x + y + z + xy + yz + zx + xyz}{8 \cos \theta_{13} \cos \theta_{32} \cos \theta_{21}}, \quad (A5)$$

and similarly, the second, third, and fourth term of Eq. 12 become

$$(\hat{D}_{13} \cdot \hat{D}_{32}) \sin \theta_{13} \sin \theta_{32} \cos \theta_{21}$$

$$= \frac{xyz - x^2 + yz - x}{8 \cos \theta_{13} \cos \theta_{32} \cos \theta_{21}}, \quad (A6)$$

$$(\hat{D}_{32} \cdot \hat{D}_{21}) \cos \theta_{13} \sin \theta_{32} \sin \theta_{21}$$

$$= \frac{xyz - z^2 + xy - z}{8 \cos \theta_{13} \cos \theta_{32} \cos \theta_{21}}, \quad (A7)$$

respectively. Finally, combining all these terms, Eq. 12 simplifies to

$$\sin \theta_{13} \cos \theta_{32} \sin \theta_{21}$$

$$= \frac{xyz - y^2 + xz - y}{8 \cos \theta_{13} \cos \theta_{32} \cos \theta_{21}} \quad (A8)$$

we arrive at

$$\tan \alpha = \sqrt{1 - x^2 - y^2 - z^2 + 2xyz} \over 1 + x + y + z}, \quad (A11)$$

which, in terms of $\hat{a}_{1,2,3}$, is given in Eq. [14] of the main text.
26. A. G. Aronov and Y. B. Lyanda-Geller, Phys. Rev. Lett. 70, 343 (1993).
27. D. Wadhawan, K. Roychowdhury, P. Mehta, and S. Das, Phys. Rev. B 98, 155113 (2018).
28. C.-Y. Hou, E.-A. Kim, and C. Chamon, Phys. Rev. Lett. 102, 076602 (2009).
29. J. Maciejko, E.-A. Kim, and X.-L. Qi, Physical Review B 82, 195409 (2010).
30. W. Chen, W.-Y. Deng, J.-M. Hou, D. N. Shi, L. Sheng, and D. Y. Xing, Phys. Rev. Lett. 117, 076802 (2016).
31. X. Xiao, Y. Liu, Z. Liu, G. Ai, S. A. Yang, and G. Zhou, Applied Physics Letters 108, 032403 (2016).
32. R. Ilan, J. Cayssol, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. 109, 216602 (2012).
33. A. Aharony, O. Entin-Wohlman, B. Halperin, and Y. Imry, Physical Review B 66, 115311 (2002).
34. Y. Ji, Y. Chung, D. Sprinzak, M. Heiblum, D. Mahalu, and H. Shtrikman, Nature 422, 415 (2003).
35. S. Datta and B. Das, Applied Physics Letters 56, 665 (1990).
36. J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).
37. M. V. Berry, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 392, 45 (1984).
38. F. Eriksson, Mathematics Magazine 63, 184 (1990).
39. G. B. Lesovik and I. A. Sadovskyy, Physics-Uspekhi 54, 1007 (2011).
40. U. Fano, Physical Review 124, 1866 (1961).