A-D-E Singularity and Prepotentials
in N=2 Supersymmetric Yang-Mills Theory

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Abstract

We calculate the instanton corrections in the effective prepotential for $N = 2$ supersymmetric Yang-Mills theory with all A-D-E gauge groups from the Seiberg-Witten geometry constructed out of the spectral curves of the periodic Toda lattice. The one-instanton contribution is determined explicitly by solving the Gauss-Manin system associated with the A-D-E singularity. Our results are in complete agreement with the ones obtained from the microscopic instanton calculations.
1 Introduction

According to Seiberg-Witten (SW) the exact solution describing the Coulomb branch of $N = 2$ supersymmetric gauge theory in four dimensions is formulated in terms of a Riemann surface together with a specific differential $\lambda_{SW}$ [1]. A non-trivial test of the SW type solution is to check if the instanton corrections to the prepotential obtained from the non-perturbative geometric description agree with the microscopic instanton calculations. So far the tests have been carried out successfully for $N = 2$ Yang-Mills theory as well as $N = 2$ QCD with various classical gauge groups [2]-[5]. For the classical groups Riemann surfaces are described in terms of hyper-elliptic curves. Toward the generalization to the case of the exceptional gauge groups, several groups attempted the hyper-elliptic type description by embedding the exceptional groups to certain classical gauge groups [6].

Martinec and Warner [10] proposed another type of Riemann surfaces based on the spectral curve of the periodic Toda lattice for any gauge groups. In the case of the exceptional gauge groups, however, the spectral curves are different from the hyper-elliptic ones and have different strong-coupling physics. Recent analysis of instanton expansions in the case of exceptional gauge groups $G_2$ [7] and $E_6$ [8] suggests that only the spectral curves yield the result consistent with the microscopic instanton calculus.

Our purpose in this paper is to perform the weak-coupling analysis of SW geometry, and in particular to examine the one-instanton correction to the prepotential for $N = 2$ Yang-Mills theory with $E_6$, $E_7$ and $E_8$ gauge groups based on the spectral curves. In fact our procedure applies systematically to all A-D-E gauge groups. As will be seen, this is by virtue of the fact that the Picard-Fuchs equations for the $N = 2$ SW period integrals are equivalent to the Gauss-Manin system for two-dimensional A-D-E topological Landau-Ginzburg (LG) models and the scaling relation for $\lambda_{SW}$, which is reported in our previous communication [9].

In section 2 the Picard-Fuchs equations formulated in terms of the flat coordinates are presented. We derive the solutions of these Picard-Fuchs equations in the weak-coupling region in section 3. We then analyze in detail the one-instanton contribution and compare with the microscopic calculations in section 4. Our results are in complete agreement with the ones obtained from the microscopic instanton calculations. As a by-product it is found
that the Gauss-Manin system for the A-D-E singularity has logarithmic solutions. This is described in section 5. Finally section 6 is devoted to discussions and conclusions.

2 Picard-Fuchs equations

We start with briefly summarizing a general scheme for SW curves. Martinec and Warner proposed that the SW Riemann surface for $N=2$ Yang-Mills theory with a simple group $G$ is given by a spectral curve of the periodic Toda lattice associated with the dual group $G^\vee$ \cite{11}. For $G = ADE$, in particular, $G = G^\vee$ and, given any irreducible representation $\mathcal{R}$ of $G$, the spectral curve is written in terms of the characteristic polynomial for $\mathcal{R}$

$$P_G^\mathcal{R}(x, u_1, \cdots, u_{r-1}, u_r + z + \frac{\mu^2}{z}) = 0,$$

(1)

which is of degree $\dim \mathcal{R}$ in $x$, and $z$ is a spectral parameter \cite{11}. Here each Casimir $u_i$ ($i = 1, \cdots, r$; $r = \text{rank} G$) has degree $q_i = e_i + 1$ where $e_i$ is the $i$-th exponent of $G$. Throughout this paper, we denote the quadratic Casimir by $u_1$ and the top Casimir of degree $h$ by $u_r$ with $h$ being the dual Coxeter number of $G$; $h = r + 1, 2r - 2, 12, 18, 30$ for $G = A_r, D_r, E_6, E_7, E_8$, respectively. When we identify (1) with the $N=2$ SW curve the Casimirs are regarded as gauge invariant moduli parameters in the Coulomb branch and $\mu^2 = \Lambda^2 h / 4$ with the dynamical scale $\Lambda$. The meromorphic SW differential is given by

$$\lambda_{SW} = \frac{x}{2\pi i} \frac{dz}{z}.$$  

(2)

To describe the moduli space of the Coulomb branch we adopt the flat coordinate system $(t_1, t_2, \cdots, t_r)$ developed in the A-D-E singularity theory instead of the conventional Casimir coordinates $(u_1, u_2, \cdots, u_r)$. The coordinate transformation is read off from the residue integral

$$t_i = c_i \oint dx W_G^\mathcal{R}(x, u)^{\hat{\Phi}}, \quad i = 1, \cdots, r$$

(3)

with a suitable constant $c_i$ \cite{13, 14}. Here $W_G^\mathcal{R}$ is obtained by solving (1) with respect to $u_r$

$$z + \frac{\mu^2}{z} + u_r = \tilde{W}_G^\mathcal{R}(x, u_1, \cdots, u_{r-1}),$$

(4)

and setting

$$W_G^\mathcal{R}(x, u_1, \cdots, u_r) = \tilde{W}_G^\mathcal{R}(x, u_1, \cdots, u_{r-1}) - u_r.$$  

(5)
Notice that the overall degree of $W^R_G$ is equal to $h$. The flat coordinates $t_i$ are expressed as polynomials in $u_i$. By degree counting we see

$$\frac{\partial}{\partial t_r} = \sum_{i=1}^r \frac{\partial u_i}{\partial t_r} \frac{\partial}{\partial u_i} = -\frac{\partial}{\partial u_r}, \quad (6)$$

and hence

$$\frac{\partial W^R_G}{\partial t_r} = 1. \quad (7)$$

Introducing the flat coordinates for the $N = 2$ moduli space is motivated by the fact that $W^{r+1}_{A_r}$ and $W^{2r}_{D_r}$ are the well-known LG superpotentials for the $A_r$- and $D_r$-type topological minimal models respectively [12]. Moreover it is demonstrated explicitly that $W^{27}_{E_6}$ describes precisely the $E_6$ topological LG model [13]. In the following, therefore, we are led to assume that $W^{56}_{E_7}$ and $W^{248}_{E_8}$ give the single-variable version of the LG superpotentials for the $E_7$ and $E_8$ models respectively.

In terms of the flat coordinates, the Picard-Fuchs equations for the SW period integrals $\Pi = \oint \lambda_{SW}$ take a succinct form [9]

$$\mathcal{L}_0 \Pi \equiv \left( \sum_{i=1}^r q_i t_i \frac{\partial}{\partial t_i} - 1 \right)^2 \Pi - 4\mu^2 h^2 \frac{\partial^2 \Pi}{\partial t_r^2} = 0,$$

$$\mathcal{L}_{ij} \Pi \equiv \frac{\partial^2 \Pi}{\partial t_i \partial t_j} - \sum_{k=1}^r C_{ij}^k(t) \frac{\partial^2 \Pi}{\partial t_k \partial t_r} = 0, \quad (8)$$

where $C_{ij}^k(t)$ are the three-point functions in the two-dimensional A-D-E topological LG models. We refer to the first equation of (8) as the scaling equation and the second ones as the Gauss-Manin system. Exactly the same form of the Gauss-Manin system has originally appeared in the study of the A-D-E singularity [17], [18], and become relevant in the context of two-dimensional topological gravity [14].

In [10] it is argued that the spectral curve (1) describes the same physics of the Coulomb branch of $N = 2$ A-D-E Yang-Mills theory irrespective of the representation $R$. Thus we expect that the explicit form of the Picard-Fuchs equations (8) does not depend on $R$. This also implies that, independently of $R$, the superpotential $W^R_G$ gives rise to the same topological field theory results with the standard A-D-E topological LG models. As an instructive example let us quote the result for the representations $5$ and $10$ of $A_4$.
The characteristic polynomials for 5 and 10 are
\[ P_{A_4}^5 = x^5 - u_1 x^3 - u_2 x^2 - u_3 x - u_4, \]
\[ P_{A_4}^{10} = \frac{1}{4}(p_2^2 - q_1^2) + u_4 q_1 - u_4^2 \]
\[ = x^{10} - 3u_1 x^8 - u_2 x^7 + 3(u_1^2 + u_3)x^6 + (2u_1 u_2 + 11u_4)x^5 + (-2u_1 u_3 - u_1^3 - u_2^2)x^4 \]
\[ + (-4u_1 u_4 - 4u_2 u_3 - u_1^2 u_2)x^3 + (-u_1^2 u_3 - 4u_2^2 + u_1 u_2^2 + 7u_2 u_4)x^2 \]
\[ + (4u_3 u_4 + u_2^3 + u_1^2 u_4)x - u_4^2 + u_2^2 u_3 - u_1 u_2 u_4, \] (9)
where
\[ q_1 = 11x^5 - 4u_1 x^3 + 7u_2 x^2 + (u_1^2 + 4u_3)x - u_1 u_2, \]
\[ p_1 = 5x^3 - u_1 x + u_2, \]
\[ p_2 = 5x^4 - 2u_1 x^2 + 4u_2 x + u_2^2 + 4u_3. \] (10)

The superpotentials are obtained as
\[ W_{A_4}^5 = P_{A_4}^5, \]
\[ W_{A_4}^{10} = \frac{1}{2}(q_1 \pm p_1 \sqrt{p_2}) - u_4. \] (11)

Applying (11) we observe that both superpotentials yield the identical formula for the flat coordinates
\[ t_1 = -\omega^3 u_1, \quad t_2 = -\omega^2 u_2, \quad t_3 = -\omega \left( u_3 + \frac{u_1^2}{5} \right), \quad t_4 = -\left( u_4 + \frac{u_1 u_2}{5} \right) \] (12)
with \( \omega = 1/5^{1/5} \). Furthermore it is shown that the chiral ring obtained from \( W_{A_4}^{10} \) has the same structure with that from \( W_{A_4}^5 \).

These \( R \)-independent results may be considered as the materialization of the universality of the special Prym variety, which is a complex rank \( G \)-dimensional sub-variety of the Jacobian of the curve, known in the theory of spectral curves [17].

Finally we note that when dealing with (8) one may need to know the explicit form of \( C_{i j}^k(t) \). It is well-known that there exists the free energy \( F(t) \) such that \( C_{i j k}(t) = \partial^3 F(t)/\partial t_i \partial t_j \partial t_k \) where \( C_{i j k}(t) = C_{i j}^k(t)\eta_{t k} \) with \( \eta_{t k} = \delta_{e_\ell + e_k, t} \), and \( C_{i j r}(t) = \eta_{i j} \) [12]. For the \( A_4 \) model the free energy is given by
\[ F_{A_4}(t) = t_2 t_3 t_4 + \frac{t_1 t_2^2}{2} + \frac{t_3^2}{6} - \frac{t_2^4}{12} - \frac{t_1 t_2^2 t_3}{2} - \frac{t_1^2 t_3^2}{4} + \frac{t_1^2 t_2^2}{6} - \frac{t_1^4}{120}. \] (13)
When we analyze the instanton expansion, however, we can proceed without using $C^k_{ij}(t)$ explicitly as will be seen in the next section.

3 Solutions of the Picard-Fuchs equations

In the semi-classical weak-coupling region where $\mu^2 \ll 1$, we can find solutions of (8) explicitly. Let $e^{(b)}$ ($b = 1, \cdots, \dim \mathcal{R}$) be a root of the characteristic polynomial, then

$$P^\mathcal{R}_G(x, u_1, \cdots, u_r) = \prod_{b=1}^{\dim \mathcal{R}} (x - e^{(b)})$$

and

$$e^{(b)} = (\lambda^{(b)}, \bar{a}),$$

where $\lambda^{(b)}$ are the weights of $\mathcal{R}$, $(\, , \,)$ stands for the inner product and

$$\bar{a} = \sum_{i=1}^{r} \bar{a}_i \alpha_i$$

with $\alpha_i$ being the simple roots of $G$.

For our discussions it suffices to take generic values of $\bar{a}_i$ at which $e^{(b)} \neq e^{(b')}$ for $b \neq b'$. It follows that $e^{(b)}$ also satisfy

$$W^\mathcal{R}_G(e^{(b)}, u_1, \cdots, u_r) = 0, \quad \partial_x W^\mathcal{R}_G(e^{(b)}, u_1, \cdots, u_r) \neq 0,$$

where the second equation is easily derived from

$$\partial_x P^\mathcal{R}_G(e^{(b)}, u_1, \cdots, u_r) \neq 0$$

with the aid of the theorem on implicit function.

For simplicity let us denote $W^\mathcal{R}_G$ as $W$ henceforth. We obtain from (2) and (3) that

$$\lambda_{SW} = \frac{1}{2\pi i} \frac{xW'}{\sqrt{W^2 - 4\mu^2}} dx,$$

where $' = \partial/\partial x$. In the classical limit $\mu^2 \to 0$, $\lambda_{SW}$ has a simple pole at $x = e^{(b)}$. When we turn on $\mu^2 \neq 0$ the pole splits into two branch points at $x = e^{(b)}_\pm$ where $e^{(b)}_+ - e^{(b)}_- = O(\mu)$. Fix a branch cut connecting $e^{(b)}_+$ and $e^{(b)}_-$, and take a closed contour $C^{(b)}$ enclosing this
branch cut anti-clockwise, then, expanding $\lambda_{SW}$ around $\mu^2 = 0$, we may evaluate the period as

$$
\pi^{(b)}(t) = \oint_{C^{(b)}} \lambda_{SW} = \sum_{n \geq 0} \left( \frac{\mu^2}{n} \right)^n \left( \frac{\partial}{\partial t} \right)_{r}^{2n} \oint_{\gamma^{(b)}} \frac{dx \ W'}{2\pi i W} = \sum_{n \geq 0} \left( \frac{\mu^2}{n} \right)^n \left( \frac{\partial}{\partial t} \right)_{r}^{2n} e^{(b)}(t),
$$

where we have used (7).

Our task now is to prove that $\pi^{(b)}(t)$ obtained above satisfies the Picard-Fuchs equations (8). We first point out that $C_{i j}^k(t)$ are independent of $t_r$ because

$$
\frac{\partial}{\partial t_r} C_{i j}^{k} = \frac{\partial}{\partial t_i} \left( \frac{\partial F(t)}{\partial t_j} \frac{\partial F(t)}{\partial t_k} \right) = \frac{\partial \eta_{j k}}{\partial t_i} = 0,
$$

which ensures

$$
[\mathcal{L}_{i j}, \partial/\partial t_r] = 0.
$$

Therefore, if the roots $e^{(b)}$ satisfy the Gauss-Manin system of (8) so do $\pi^{(b)}$'s. In fact it is not difficult to show that $e^{(b)}(t)$ is a solution of the Gauss-Manin system. The proof goes as follows. The derivative of $W(e^{(b)}(t), t) = 0$ with respect to $t_i$ yields

$$
\frac{\partial e^{(b)}}{\partial t_i} = -\frac{\phi_i}{W'},
$$

where the RHS is evaluated at $x = e^{(b)}(t)$ and

$$
\phi_i(x, t) = \frac{\partial W(x, t)}{\partial t_i}, \quad i = 1, \cdots, r
$$

are chiral primary fields in the LG model. Notice that $\phi_r = 1$ as is seen from (8). The fields $\phi_i$ obey the algebra

$$
\phi_i \phi_j = \sum_k C_{i j}^{k}(t) \phi_k + Q_{i j} W',
$$

where we have

$$
Q'_{i j} = \frac{\partial^2 W}{\partial t_i \partial t_j}
$$
by virtue of the flat coordinates. Differentiate again (23) with respect to \( t_j \) to obtain

\[
\frac{\partial^2 e^{(b)}}{\partial t_i \partial t_j} = \frac{(\phi_i \phi_j)'}{W'^2} - \frac{Q_{ij}'}{W'} - \frac{W''}{W'^2} \phi_i \phi_j
\]

\[= \sum_k C_{ij}^k(t) \left( \frac{\phi_k'}{W'^2} - \frac{W''}{W'^2} \phi_k \right), \tag{27}\]

where (23), (25) and (26) have been utilized. Notice that \( C_{r\ell}^k = C_{j\ell}^j \eta^{jk} = \eta^{j\ell} \eta^{jk} = \delta^k_\ell \), and hence putting \( i = \ell, j = r \) in (27) gives

\[
\frac{\partial^2 e^{(b)}}{\partial t_\ell \partial t_r} = \frac{\phi_\ell'}{W'^2} - \frac{W''}{W'^2} \phi_\ell. \tag{28}\]

Thus we have shown that the zeroes of the characteristic polynomial for any irreducible representation of the A-D-E groups satisfy the Gauss-Manin system (8) for A-D-E singularities.

For the fundamental representations of \( A_r \) and \( D_r \) this fact has been known to mathematicians since the forms of \( W_{A_r}^{r+1} \) and \( W_{D_r}^{2r} \) are standard in the theory of isolated singularities of types \( A_r \) and \( D_r \) \[18\]. Our present result not only extends the foregoing observation to the case of \( E \)-type singularities, but finds out the intimate relationship of the polynomial invariants with the Gauss-Manin system.

We next show that \( \pi^{(b)} \) satisfy the scaling equation of (8). For this purpose we recall the quasihomogeneous property of the superpotential

\[
x W' + \sum_{i=1}^{r} q_i t_i \frac{\partial W}{\partial t_i} = hW. \tag{29}\]

Putting \( x = e^{(b)} \) here and using (23) it is immediate to see that

\[
De^{(b)} = \left( \sum_{i=1}^{r} q_i t_i \frac{\partial}{\partial t_i} - 1 \right) e^{(b)} = 0. \tag{30}\]

From this and

\[
\left[ D^2, \frac{\partial^m}{\partial t_r^m} \right] = \frac{\partial^m}{\partial t_r^m} (m^2 h^2 - 2mhD), \tag{31}\]

one can explicitly check that (21) indeed satisfy the scaling equation.

\[\text{†We thank Kyoji Saito and Y. Yamada for informing this to us. Our proof here is based on Yamada’s unpublished note (March, 1993) on the \( A_r \)-type singularity.}\]
Finally we note that $\bar{a}_i(t)$ introduced in (16) determine the classical values of the adjoint Higgs field $\Phi$ in the $N = 2$ vector multiplet

$$\Phi = \sum_{i=1}^{r} \bar{a}_i H_i,$$  \hspace{1cm} (32)

where $H_i$ are the Cartan generators normalized as $\text{Tr} \ H_i H_j = K_{ij}$ with $K_{ij}$ being the Cartan matrix. This means that the $A$-cycles $A_i$ on the Riemann surface (II) may be fixed by the relation

$$C^{(b)} = \sum_{i=1}^{r} (\lambda^{(b)}, \alpha_i) A_i.$$  \hspace{1cm} (33)

The SW periods along the $A$-cycles are thus given by

$$a_i(t) = \int_{A_i} \lambda_{SW}, \hspace{1cm} i = 1, \ldots, r$$  \hspace{1cm} (34)

which have the weak-coupling expansion

$$a_i(t) = \bar{a}_i(t) + \sum_{k \geq 1} \tilde{a}_i^{(k)}(t) \Lambda^{bk},$$  \hspace{1cm} (35)

where $b = 2h$ is the coefficient of the one-loop beta function and

$$\tilde{a}_i^{(k)}(t) = \frac{1}{4^k (k!)^2} \left( \frac{\partial}{\partial t} \right)^{2k} \bar{a}_i(t).$$  \hspace{1cm} (36)

### 4 One-instanton corrections

Let us now turn to the analysis of instanton expansions in A-D-E Yang-Mills theory. It is known that the $N = 2$ prepotential in the weak-coupling region of the Coulomb branch takes the form

$$\mathcal{F}(a) = \frac{\tau_0}{2} \sum_{i,j=1}^{r} a_i K_{ij} a_j + \frac{i}{4\pi} \sum_{\alpha \in \Delta^+} (\alpha, a)^2 \ln \frac{(\alpha, a)^2}{\Lambda^2} + \sum_{k \geq 1} \mathcal{F}_k(a) \Lambda^{bk},$$  \hspace{1cm} (37)

where $a(t) = \sum_{i=1}^{r} a_i(t) \alpha_i$, $\tau_0$ is the bare coupling constant, $\mathcal{F}_k(a)$ stands for the $k$-instanton contribution and $\Delta^+$ is a set of positive roots of $G$.

In order to evaluate $\mathcal{F}_1(a)$ we invoke the scaling relation \[19\]

$$\sum_{i=1}^{r} a_i \frac{\partial \mathcal{F}(a)}{\partial a_i} - 2 \mathcal{F}(a) = \frac{ib}{2\pi} u_1,$$  \hspace{1cm} (38)
where the quadratic Casimir $u_1$ and $\bar{a}_i$ are related through

$$ u_1 = \frac{1}{2h} \sum_{\alpha \in \Delta^+} (\alpha, \bar{a})^2 = \frac{1}{2} \text{Tr} \Phi^2. \quad (39) $$

We substitute the weak-coupling expansion (37) together with (35) into (38). Making use of (39) and

$$ \sum_{i=1}^{r} a_i \frac{\partial F_k(a)}{\partial a_i} = (2 - bk) F_k(a), \quad (40) $$

we obtain the one-instanton contribution

$$ F_1(\bar{a}) = \frac{i}{\pi b} \sum_{\alpha \in \Delta^+} (\alpha, \bar{a})(\alpha, \bar{a}^{(1)}). \quad (41) $$

Since (36) gives

$$ \bar{a}^{(1)} = \frac{1}{4} \frac{\partial^2 \bar{a}}{\partial t_r^2} = \frac{1}{4} \frac{\partial^2 \bar{a}}{\partial u_r^2}, \quad (42) $$

further manipulations lead to

$$ F_1(\bar{a}) = \frac{i}{4\pi b} \sum_{\alpha \in \Delta^+} (\alpha, \bar{a}) \frac{\partial^2 (\alpha, \bar{a})}{\partial u_r^2} $$

$$ = \frac{i}{4\pi b} \sum_{\alpha \in \Delta^+} \left[ \frac{1}{2} \left( \frac{\partial (\alpha, \bar{a})}{\partial u_r} \right)^2 - \left( \frac{\partial (\alpha, \bar{a})}{\partial u_r} \right)^2 \right] $$

$$ = \frac{1}{4\pi i b} \sum_{\alpha \in \Delta^+} \left( \frac{\partial (\alpha, \bar{a})}{\partial u_r} \right)^2, \quad (43) $$

where in the second line we have used (39) and $\partial u_1/\partial u_r = 0$. Notice here that

$$ \frac{1}{h} \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \alpha_j) = K_{ij}, \quad (44) $$

then we find

$$ F_1(\bar{a}) = \frac{1}{8\pi i} \sum_{i,j=1}^{r} \frac{\partial \bar{a}_i}{\partial u_r} K_{ij} \frac{\partial \bar{a}_j}{\partial u_r}. \quad (45) $$

Let us introduce the Jacobian determinant $D$, as well as its $(i,j)$-th cofactor $D_{ij}$, of the map from $(\bar{a}_1, \cdots, \bar{a}_r)$ to $(u_1(\bar{a}), \cdots, u_r(\bar{a}))$:

$$ D = \det \left( \frac{\partial u_m}{\partial \bar{a}_n} \right), $$

$$ D_{ij} = (-1)^{i+j} \det_{m \neq i, n \neq j} \left( \frac{\partial u_m}{\partial \bar{a}_n} \right), \quad (46) $$
in terms of which we have
\[ \frac{\partial \bar{a}_i}{\partial u_r} = \frac{D_{ri}}{D}. \quad (47) \]

Thus we finally arrive at
\[ F_1(\bar{a}) = \frac{1}{8\pi i} \sum_{i,j=1}^r \frac{D_{ri}K_{ij}D_{rj}}{D^2}. \quad (48) \]

We now compare our results with those derived by microscopic instanton calculations. The general expression for the microscopic one-instanton term \( F_{1\text{micro}} \) in A-D-E Yang-Mills theory is found in [3]. The result reads
\[ F_{1\text{micro}} = \frac{1}{2^{b/2}\pi i} F_1(a), \quad (49) \]

where
\[ F_1(a) = \frac{\sum_{\alpha \in \Delta^+} \Pi_{\alpha^0 \in \Delta^+; (a, a^0) = 0} (a, a^0)^2 \Pi_{\alpha^1 \in \Delta^1(a)} (a, \alpha^1)(a, \alpha^1 - \alpha)}{\Pi_{\alpha \in \Delta^+ (a, \alpha)^2}}. \quad (50) \]

Here \( \Delta^1(a) \) denotes a set of roots which become the highest weights of \( SU(2) \) doublets when we embed an \( SU(2) \) into \( G \) by taking a positive root \( \alpha \) as an \( SU(2) \) direction. \( F_1(a) \) and \( F_{1\text{micro}}(a) \) should be related through
\[ F_1(a) \Lambda^b = F_{1\text{micro}}(a) \Lambda^b_{\text{micro}}, \quad (51) \]

where \( \Lambda_{\text{micro}} \) is the dynamical scale associated with the Pauli-Villas regularization scheme.

Hence what we have to show is that
\[ F_1(\bar{a}) = 2^b \pi i F_1(\bar{a}) \frac{\Lambda^b}{\Lambda^b_{\text{micro}}}. \quad (52) \]

Before calculating the explicit form of \( F_1(\bar{a}) \) we examine its singularity structure. To do this we first wish to discuss some properties of the Jacobian. The Casimirs \( (u_1(\bar{a}), \ldots, u_r(\bar{a})) \) are invariant under the Weyl reflection \( \sigma_\beta \) with respect to any root \( \beta \)
\[ u_k(\bar{a}) = u_k(\bar{b}), \quad (53) \]

where
\[ \bar{b} = \sigma_\beta(\bar{a}) = \bar{a} - \frac{2(\beta, \bar{a})}{(\beta, \beta)} \beta. \quad (54) \]
From (53) we get
\[
\frac{\partial u_k(\bar{a})}{\partial \bar{a}_i} = \sum_{j=1}^{r} \frac{\partial b_j}{\partial \bar{a}_i} \frac{\partial u_k(\bar{b})}{\partial \bar{b}_j} = \frac{\partial u_k(\bar{b})}{\partial \bar{b}_i} - 2(\alpha_i, \beta) \sum_{j=1}^{r} \beta_j \frac{\partial u_k(\bar{b})}{\partial \bar{b}_j},
\]
where \(\bar{b}_i\) is defined by \(\bar{b} = \sum_{i=1}^{r} \bar{b}_i \alpha_i\). If \(\bar{a}\) is on the hyperplane defined by \((\beta, \bar{a}) = 0\), we get \(\bar{b} = \bar{\bar{a}}\) and
\[
\sum_{j=1}^{r} \beta_j \frac{\partial u_k(\bar{\bar{a}})}{\partial \bar{a}_j} = 0.
\]
(56)

This means that the expression \(\sum_{j=1}^{r} \beta_j \frac{\partial u_k(\bar{\bar{a}})}{\partial \bar{a}_j}\), which is a polynomial in \(\bar{a}\)'s, is divided by \((\beta, \bar{\bar{a}})\), and hence takes the form
\[
\sum_{j=1}^{r} \beta_j \frac{\partial u_k(\bar{\bar{a}})}{\partial \bar{a}_j} = (\beta, \bar{\bar{a}}) f_{k,\beta}(\bar{\bar{a}})
\]
(57)
with \(f_{k,\beta}(\bar{\bar{a}})\) being a certain polynomial in \(\bar{a}\)'s \([20]\). From (57) we find that the Jacobian determinant \(D\) is also divided by \((\beta, \bar{\bar{a}})\) for any positive root \(\beta\). Since the sum of exponents \(\sum_{i=1}^{r} e_i\) is equal to the number of positive roots, \(D\) is shown to be expressed as \([21]\)
\[
D = c \prod_{\alpha \in \Delta^+} (\alpha, \bar{\bar{a}}),
\]
(58)
where \(c\) is a certain constant which depends on the choice of the normalization of the Casimirs.

It is now observed that the denominator of the RHS of (48) vanishes when \((\alpha, \bar{\bar{a}}) = \epsilon \to 0\) for a positive root \(\alpha\). Due to invariance of \(F_1(\bar{a})\) under the Weyl transformation, one may choose \(\alpha\) as a simple root \(\alpha_1\) without loss of generality. Choosing \(\beta = \alpha_1\) in (57) we find that
\[
\frac{\partial u_k(\bar{\bar{a}})}{\partial \bar{a}_1} = \epsilon f_{k,\alpha_1}(\bar{\bar{a}}) \sim O(\epsilon) \quad k = 1, \ldots, r
\]
(59)
and the other \(\frac{\partial u_k(\bar{\bar{a}})}{\partial \bar{a}_i}\) \((i \neq 1)\) are \(O(1)\). We have \(D_{r1} \sim O(1)\) and \(D_{ri} \sim O(\epsilon)\) \((i \neq 1)\). Thus, in the numerator of (18) only the term \(D_{r1} K_{11} D_{r1}\) is of order one and contributes to the most singular term in \(F_1(\bar{a})\). In the limit \(\epsilon \to 0\), \(D_{r1}\) behaves as
\[
D_{r1} = d_1 \prod_{\alpha \in \Delta^+} (\alpha, \bar{\bar{a}}) + O(\epsilon),
\]
(60)
where $d_1$ is a constant and $\Delta'$ is a subset of $\Delta^+$, which consists of the linear combinations of simple roots $\alpha_2, \cdots, \alpha_r$. We note that the positive root system can be decomposed into the sets $\Delta_i$ whose element $\alpha$ satisfies $(\alpha_1, \alpha) = i$:

$$\Delta^+ = \Delta_2 \cup \Delta_1 \cup \Delta_{-1} \cup \Delta_0. \quad (61)$$

It is easy to see that $\Delta_2 = \{\alpha_1\}$, $\Delta_1 = \Delta^1(\alpha_1)$, $\Delta_{-1} = \{\alpha - \alpha_1; \alpha \in \Delta_1\}$ and $\Delta' = \Delta_{-1} \cup \Delta_0$.

To summarize, in the limit $\epsilon \to 0$, we obtain

$$F_1(\bar{a}) = \frac{1}{8\pi i} \frac{2d_1^2}{c^2 \prod_{\alpha \in \Delta^1(\alpha_1)} (\alpha, \bar{a})^2} + \cdots. \quad (62)$$

This is the same singularity structure which is expected from the microscopic instanton result (49). In view of the holomorphic property of the prepotential, we thus conclude that the relation (52) holds up to a proportional constant. In order to determine this constant, we need to calculate the Jacobian determinant explicitly. In the following we first express $D$ and $D_{ir}$ explicitly in terms of $\bar{a}_i$ for the $A_r$ and $D_r$ gauge groups to work out (52). We next proceed to the $E_6$ gauge group.

4.1 $A_r$ gauge group

For the group $A_r$, we consider the $(r+1)$-dimensional representation, which is realized by the traceless $(r+1) \times (r+1)$ matrices. It is therefore convenient to consider the $gl(r+1)$ case, where the Cartan part $(\bar{a}_1, \cdots, \bar{a}_r, 1)$ is generic, and then impose the traceless condition $\sum_{i=1}^{r+1} \bar{a}_i = 0$. As is shown in [3], $F_1(\bar{a})$ in (54) is then expressed as

$$F_1(\bar{a}) = \frac{\sum_{i=1}^{r+1} \Delta(\bar{a}_1, \cdots, \hat{\bar{a}}_i, \cdots, \bar{a}_{r+1})}{2\Delta(\bar{a}_1, \cdots, \bar{a}_{r+1})}, \quad (63)$$

where $\hat{\bar{a}}_i$ means to exclude $\bar{a}_i$, and

$$\Delta(\bar{a}_1, \cdots, \bar{a}_n) = \prod_{1 \leq i < j \leq n} (\bar{a}_i - \bar{a}_j)^2. \quad (64)$$

On the other hand, the Casimirs $u_1, \cdots, u_{r+1}$ of $gl(r+1)$ are defined by the characteristic polynomial

$$\prod_{i=1}^{r+1} (x - \bar{a}_i) = x^{r+1} - \sum_{i=1}^{r+1} u_ix^{r+1-i}. \quad (65)$$

\[\text{In subsections 4.1 and 4.2 our use of notations } \bar{a}_i \text{ is slightly different from that in the other sections.}\]
These Casimirs are related to \( s_k \equiv \sum_{i=1}^{r+1} \bar{a}_i^k \) via Newton’s formula

\[
s_k - u_1 s_{k-1} - u_2 s_{k-2} - \cdots - u_{k-1} s_1 - k u_k = 0, \tag{66}\]

from which we get

\[
\frac{\partial u_k}{\partial \bar{a}_i} = \frac{1}{k} \frac{\partial s_k}{\partial \bar{a}_i} + \sum_{m \geq 2, j_1 + \cdots + j_m = k} d_{j_1, \cdots, j_m} s_{j_1} \cdots s_{j_{m-1}} \frac{\partial s_{j_m}}{\partial \bar{a}_i}, \tag{67}\]

where \( d_{j_1, \cdots, j_m} \) are constants. Then we obtain

\[
D = \frac{1}{(r + 1)!} \det \left( \frac{\partial s_i}{\partial \bar{a}_j} \right). \tag{68}\]

Notice that this is nothing but the Vandermonde determinant and is shown to be

\[
D = (-1)^{(r+1)r/2} \prod_{i<j} (\bar{a}_i - \bar{a}_j). \tag{69}\]

In a similar way one obtains

\[
D_{r+1} = (-1)^{r(r-1)/2} \prod_{i,j \neq k} (\bar{a}_i - \bar{a}_j). \tag{70}\]

Hence we have

\[
\mathcal{F}_1(\bar{a}) = \frac{1}{8\pi i} \frac{\sum_{i=1}^{r+1} \Delta(\bar{a}_1, \cdots, \hat{\bar{a}_i}, \cdots, \bar{a}_{r+1})}{\Delta(\bar{a}_1, \cdots, \bar{a}_{r+1})}. \tag{71}\]

Putting (63) and (71) into (62) yields

\[
\Lambda^b_{\text{micro}} = 2^{r-1} \Lambda^b. \tag{72}\]

with \( b = 2(r + 1) \). This agrees with the value evaluated in [3].

### 4.2 \( D_r \) gauge group

For the group \( D_r \), let \( \bar{a}_i \) be the skew eigenvalues of the matrix in the \( 2r \)-dimensional representation, then it is shown that \( F_1(\bar{a}) \) in (50) becomes

\[
F_1(\bar{a}) = 2 \sum_{i=1}^r \bar{a}_i^2 \frac{\Delta(\bar{a}_1^2, \cdots, \hat{\bar{a}_i^2}, \cdots, \bar{a}_r^2)}{\Delta(\bar{a}_1^2, \cdots, \bar{a}_r^2)}. \tag{73}\]
To rewrite (48) we examine the characteristic polynomial for the 2r-dimensional representation
\[ \prod_{i=1}^{r} (x^2 - \bar{a}_i^2) = x^{2r} - \sum_{i=1}^{r} \bar{u}_i x^{2r-2i}, \] (74)
where \( \bar{u}_i \)'s are related to the Casimirs \( u_1, \cdots, u_r \) of \( D_r \) by the formulas: \( \bar{u}_i = u_i \) (\( i = 1, \cdots, r-2 \)), \( \bar{u}_{r-1} = u_r \) and \( \bar{u}_r = u_{r-1}^2 \). Using \( u_{r-1} = \bar{a}_1 \cdots \bar{a}_r \), we find that
\[ \det \left( \frac{\partial \bar{u}_i}{\partial \bar{a}_j} \right) = -2u_{r-1} D. \] (75)

Since the LHS of (75) becomes \((-1)^{r(r-1)/2} 2^{r-1} u_{r-1} \prod_{1 \leq i < j \leq r} (\bar{a}_i^2 - \bar{a}_j^2) \) we obtain \( D = -(-1)^{r(r-1)/2} 2^{r-1} \prod_{1 \leq i < j \leq r} (\bar{a}_i^2 - \bar{a}_j^2). \) (76)

Similarly
\[ D_{rk} = -(-1)^{(r-1)(r-2)/2} 2^{r-2} \bar{a}_k \prod_{1 \leq i < j \leq r, i \neq k} (\bar{a}_i^2 - \bar{a}_j^2). \] (77)

Thus \( F_1(\bar{a}) \) is rewritten as
\[ F_1(\bar{a}) = \frac{1}{8\pi i} \sum_{i=1}^{r} \bar{a}_i^2 \Delta(\bar{a}_1^2, \cdots, \bar{a}_r^2) \] (78)

The ratio of the dynamical scales is determined from (73), (78) and (52). We get
\[ \Lambda_{micro}^b = 2^{2r-8} \Lambda^b \] (79)
with \( b = 2(r-1) \), in agreement with [3].

### 4.3 \( E_6 \) gauge group

For \( E_6 \), we may construct the spectral curve from the fundamental representation 27 [1].

The superpotential reads
\[ W_{E_6}^{27} = \frac{1}{x^3} (q_1 \pm p_1 \sqrt{p_2}) - u_6, \] (80)
where
\[ q_1 = 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 - 162u_1u_2x^8 \]
\[ + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 + (27u_2^2 - 9u_1u_4)x^5 \]
\[-(3u_2u_3 - 6u_1u_5)x^4 - 3u_1u_2^2x^3 - 3u_2u_5x - u_3^2,\]

\[p_1 = 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 - u_5x - u_2^2,\]

\[p_2 = 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + u_3x^4 - 4u_1u_2x^3 - 2u_4x^2 + u_5x + u_2^2.\]

(81)

The Casimirs are expressed in terms of $\bar{a}_i$ as follows:

\begin{align*}
  u_1 &= -\frac{1}{12} \chi_2, \quad u_2 = -\frac{1}{60} \chi_5, \quad u_3 = -\frac{1}{6} \chi_6 + \frac{1}{12} \chi_2^3, \\
  u_4 &= -\frac{1}{40} \chi_8 + \frac{1}{180} \chi_2 \chi_6 - \frac{1}{12} \chi_4^2, \quad u_5 = -\frac{1}{7} \cdot 6^2 \chi_9 + \frac{1}{20} \chi_2 \chi_5^2, \\
  u_6 &= -\frac{1}{60} \chi_{12} + \frac{1}{5} \cdot 6^3 \chi_6 + \frac{13}{5} \cdot 12 \chi_2 \chi_5^2 \\
  &\quad + \frac{5}{2} \cdot 12 \chi_2 \chi_8 - \frac{1}{3} \cdot 6^4 \chi_2^3 \chi_6 + \frac{29}{10} \cdot 12 \chi_2^6.
\end{align*}

(82)

where $\chi_n = \text{Tr} \Phi^n$ and $\Phi = \text{diag} \left((\lambda(1), a), \ldots, (\lambda(27), a)\right)$. The weight vectors $\lambda^{(i)}$ of 27 are listed in [22].

The ratio $\Lambda^b/\Lambda_{micro}^b$ with $b = 24$ for $E_6$ theory has not been known. Thus the present analysis predicts the value of $\Lambda^b/\Lambda_{micro}^b$ from (52). Using (82) we evaluate $F_1(\bar{a})$ and $F_{1micro}(\bar{a})$ numerically in the limit $(\alpha_1, \bar{a}) \to 0$, and find

\[\frac{\Lambda_{24}^b}{\Lambda_{24_{micro}}^b} = 2^{-8} 3^6.\]

(83)

In a similar vein, our analysis now makes it possible to determine the ratio $\Lambda^b/\Lambda_{micro}^b$ for $E_7$ and $E_8$ theory in principle. To do this we need to write down the Casimirs $u_i$ in terms of $\bar{a}_i$. However, this computation, though straightforward, requires more computer powers than those available for us at present, and hence goes beyond the scope of this paper.

5 Logarithmic solutions of the Picard-Fuchs equations

In this section we study the dual SW periods

\[a_{Di}(t) = \int_{B_i} \lambda_{SW}\]

(84)

*We thank M. Noguchi and S. Terashima for providing us with some $E_6$ data.
defined by the $B$-cycles which have intersections $\#(A_i \cap B_j) = \delta_{ij}$. In the weak-coupling region $a_{Di}$ may be evaluated by using $a_{Di} = \partial F / \partial a_i$ and (37). One obtains

$$a_{Di}(t) = \tau_0 \sum_{j=1}^{r} K_{ij} a_j + \frac{i}{2\pi} \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \bar{a}) \left( \ln \frac{(\alpha, \bar{a})^2}{\Lambda^2} + 1 \right) + \sum_{k \geq 1} \frac{\partial F_k(a)}{\partial a_i} \Lambda^{bk}$$

$$= \bar{a}_{Di}(t) + \sum_{k \geq 1} a^{(k)}_{Di}(t) \Lambda^{bk}, \quad (85)$$

where we have

$$\bar{a}_{Di}(t) = \tau_0 \sum_{j=1}^{r} K_{ij} \bar{a}_j + \frac{i}{2\pi} \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \bar{a}) \left( \ln \frac{(\alpha, \bar{a})^2}{\Lambda^2} + 1 \right),$$

$$a^{(1)}_{Di}(t) = \tau_0 \sum_{j=1}^{r} K_{ij} \bar{a}_j^{(1)} + \frac{3i}{2\pi} \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \bar{a}^{(1)})$$

$$+ \frac{i}{2\pi} \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \bar{a}^{(1)}) \ln \frac{(\alpha, \bar{a})^2}{\Lambda^2} + \frac{\partial F_1(\bar{a})}{\partial \bar{a}_i}, \quad (86)$$

upon substituting the weak-coupling expansion (35) of $a_i$. It is easy to see that

$$D^2 \bar{a}_{Di} = 0, \quad (87)$$

and hence the scaling equation of (8) yields the recursion relations among $\bar{a}_{Di}^{(k)}$. The relation at $O(\Lambda^b)$ is

$$D^2 \bar{a}_{Di}^{(1)} - h^2 \frac{\partial^2 \bar{a}_{Di}}{\partial t_r^2} = 0 \quad (88)$$

which reduces to the equation

$$\frac{\partial F_1(\bar{a})}{\partial a_i} = \frac{i}{4\pi h} \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i) \left\{ \frac{\partial^2 (\alpha, \bar{a})}{\partial t_r^2} + \frac{h}{(\alpha, \bar{a})} \left( \frac{\partial (\alpha, \bar{a})}{\partial t_r} \right)^2 \right\}. \quad (89)$$

This result is compatible with (40), (41) and (43). In view of (48), (89) provides us with non-trivial identities for the Jacobian determinant and its first derivatives.

Next we consider the Gauss-Manin system of (8) and, in particular, concentrate on checking if the logarithmic terms in $\bar{a}_{Di}$

$$\sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \bar{a}) \ln \frac{(\alpha, \bar{a})^2}{\Lambda^2}, \quad i = 1, \ldots, r \quad (90)$$

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satisfy the Gauss-Manin system. Substitute (90) in the Gauss-Manin system to obtain

\[
\sum_{\alpha \in \Delta^+} (\alpha, \alpha_i) \frac{\partial (\alpha, \bar{a})}{\partial t_m} \frac{\partial (\alpha, \bar{a})}{\partial t_n} = \sum_{\ell=1}^r C_{mn} \ell (t) \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i) \frac{\partial (\alpha, \bar{a})}{\partial t_{\ell}} \frac{\partial (\alpha, \bar{a})}{\partial t_r},
\]

(91)

where the fact that \(\bar{a}_i\) obey (8) has been taken into account. To proceed a little further we feel it more appealing to express (91) in terms of the LG variables. For this, recall that, given a representation \(R\) of \(G\), \(\bar{a}_i\) and the roots \(e^{(b)}\) of the characteristic polynomial (14) are related by (15). Choosing appropriate weights \(\lambda^{(b)}\) we express \(\bar{a}_i\) in terms of \(e^{(b)}\)

\[
\bar{a}_i = \sum_b S_{ib} e^{(b)},
\]

(92)

where \(S\) is an invertible \(r \times r\) matrix, so that

\[
\frac{\partial \bar{a}_i}{\partial t_m} = -\sum_b S_{ib} \frac{\phi_m(e^{(b)})}{W'(e^{(b)})}.
\]

(93)

with the use of (23). It is also convenient to define

\[
M_{bc}^{(i)} = \sum_{j,k=1}^r \sum_{\alpha \in \Delta^+} (\alpha, \alpha_i)(\alpha, \alpha_j)(\alpha, \alpha_k) (\alpha, \bar{a}) S_{jb} S_{kc}.
\]

(94)

Eq. (91) is then rewritten as

\[
\sum_{b,c} M_{bc}^{(i)} \frac{\phi_m(e^{(b)})}{W'(e^{(b)})} \frac{\phi_n(e^{(c)})}{W'(e^{(c)})} = \sum_{\ell=1}^r C_{mn} \ell (t) \sum_{b,c} M_{bc}^{(i)} \frac{\phi_{\ell}(e^{(b)})}{W'(e^{(b)})} \frac{\phi_{\ell}(e^{(c)})}{W'(e^{(c))}}, \quad i = 1, \ldots, r
\]

(95)

which look like operator product expansions for chiral primaries. Taking 3 of \(SU(3)\) and 10 of \(SU(5)\) as test examples, we have confirmed that this somewhat complicated relation holds correctly. These examples are non-trivial enough to convince us that the A-D-E Gauss-Manin system has the logarithmic solutions (90). The general proof of the statement is highly desirable.

6 Discussions and conclusions

We have studied the low-energy effective prepotential of \(N = 2\) supersymmetric Yang-Mills theory with A-D-E gauge groups. In the weak-coupling region we have solved the Picard-Fuchs equations for the SW periods, and constructed the solutions in the form
of a power series. Commutativity between the differential operators \( \frac{\partial}{\partial t_r} \) and \( L_{ij} \) of the Gauss-Manin system (22) is important for this construction. Then we have computed one-instanton corrections to the prepotential and shown that the present results agree with the ones obtained from the microscopic approach.

The SW curves (4) are identical with the spectral curves for the periodic Toda lattice. We stress that, for the fundamental representations, while the spectral curves for \( A_r \) and \( D_r \) are of hyper-elliptic type, they are not for \( E_6, E_7 \) and \( E_8 \). Let us show that the hyper-elliptic ansatz for the SW curves fails in producing the relation (52) for the \( E \)-type gauge groups. Take the fundamental representation \( R_f \) with dimension \( d \) of \( G = ADE \) where \( d = (r + 1), 2r, 27, 56, 248 \) for \( A_r, D_r, E_6, E_7, E_8 \), respectively. The hyper-elliptic curve proposed in [6] takes the form

\[
y^2 = P_R^G(x)^2 - \Lambda^{2h}x^{2d-2h},
\]

where the characteristic polynomial \( P_R^G(x) \) is given in (14). These curves are inferred from embedding the representation \( R_f \) into the curves for \( N = 2 \) supersymmetric QCD with the \( SU(d) \) gauge group and massless \( N_f = 2d - 2h \) fundamental flavors.

Corresponding to the curves (96), therefore, one may write down the one-instanton contribution to the prepotential

\[
F_1(a) \Lambda^{2h} = \frac{\Lambda^{2h} \sum_{i=1}^d b_i^{2d-2h} \Delta(b_1, \ldots, \hat{b}_i, \ldots, b_d)}{\Delta(b_1, \ldots, b_d)},
\]

with \( b_i = (\lambda(i), a) \), which is evaluated from the prepotential of \( N = 2 \) \( SU(N_c) \) QCD with \( N_f \) flavors [3]. It is shown that (97) agree with those obtained from microscopic calculations in the case of \( AD \)-type gauge groups. For the \( E \)-type gauge groups, however, the expression (97) behaves quite differently from the microscopic results. This can be easily seen by investigating the singularity structure of \( F_1 \). For the hyper-elliptic curves (96), the pole structure of \( F_1 \) is characterized by the discriminant of the polynomial \( P_R^G(x) \)

\[
\Delta(b_1, \ldots, b_d) = \prod_{i<j}(\lambda^{(i)} - \lambda^{(j)}, a)^2
\]

which depends explicitly on the weight vectors \( \lambda^{(i)} \) of \( R_f \). On the other hand, the spectral-
curve analysis predicts that the singularity structure of $\mathcal{F}_1$ is characterized by

$$\prod_{\alpha \in \Delta^+} (\alpha, a)^2$$

(99)
as is discussed in section 4. It is clear that (98) and (99) do not agree with each other for the $E$-type gauge groups. Therefore only the spectral curves associated with the periodic Toda lattice describe the non-perturbative corrections to the low-energy effective actions of $N = 2$ Yang-Mills theory with the $E$-type gauge groups. This is in accordance with the result obtained by the method of $N = 1$ confining phase superpotentials [20]. An analogous situation has been observed for $N = 2$ $G_2$ theory [23], [7].

In this paper we have found an interesting interplay between the A-D-E singularity theory and the non-perturbative weak-coupling properties of the SW-type solutions of $N = 2$ Yang-Mills theory. It will be worth studying the massless soliton points and $N = 2$ superconformal points [24] in the strong-coupling region of vacuum moduli space in a systematic manner based on the A-D-E singularity/LG point of view.

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References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; Nucl. Phys. B431 (1994) 484

[2] D. Finnell and P. Pouliot, Nucl. Phys. B453 (1995) 225;
    H. Aoyama, T. Harano, M. Sato and S. Wada, Phys. Lett. B388 (1996) 331;
    T. Harano and M. Sato, Nucl. Phys. B484 (1997) 167;
    N. Dorey, V.V. Khoze and M.P. Mattis, Phys. Rev. D54 (1996) 2921; 7832; Phys.
    Lett. B388 (1996) 324; B390 (1997) 205; B396 (1997) 141; Nucl. Phys. B492 (1997)
    607;
    F. Fucito and G. Travaglini, Phys. Rev. D55 (1997) 1099;
    A. Yung, Nucl. Phys. B485 (1997) 38;
    K. Ito and N. Sasakura, Mod. Phys. Lett. A12 (1997) 205;
    M.J. Slater, Phys. Lett. B403 (1997) 57

[3] K. Ito and N. Sasakura, Phys. Lett. B382 (1996) 95; Nucl. Phys. B484 (1997) 141

[4] A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929;
    K. Ito and S.-K. Yang, Phys. Lett. B366 (1996) 165; in “Frontiers In Quantum Field
    Theory” p. 331 (World Scientific, Singapore, 1996);
    Y. Ohta, J. Math. Phys. 37 (1996) 6074; 38 (1997) 682;
    T. Masuda and H. Suzuki, Int. J. Mod. Phys. A12 (1997) 3413; hep-th/9609063.
    H. Ewen, K. Foerger and S. Theisen, Nucl. Phys. B485 (1997) 63;
    H. Ewen and K. Foerger, Int. J. Mod. Phys. A12 (1997) 4725

[5] E. D’Hoker, I.M. Krichever and D.H. Phong, Nucl. Phys. B489 (1997) 179; B489
    (1997) 211

[6] U.H. Danielsson and B. Sundborg, Phys. Lett. B370 (1996) 446
    M. Alishahiha, F. Ardalan and F. Mansouri, Phys. Lett. B381 (1996) 446
    M.R. Abolhasani, M. Alishahiha and A.M. Ghezelbash, Nucl. Phys. B480 (1996)
    279
[7] K. Ito, Phys. Lett. B406 (1997) 54

[8] A.M. Ghezelbash, *Periods and Prepotential in N = 2 Supersymmetric E_6 Yang-Mills Theory*, hep-th/9710068

[9] K. Ito and S.-K. Yang, *Flat Coordinates, Topological Landau-Ginzburg Models and the Seiberg-Witten Period Integrals*, hep-th/9708017, to appear in Phys. Lett. B

[10] E.J. Martinec and N.P. Warner, Nucl. Phys. B459 (1996) 97

[11] W. Lerche and N.P. Warner, *Exceptional SW Geometry from ALE Fibrations*, hep-th/9608183

[12] R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B352 (1991) 59

[13] T. Eguchi and S.-K. Yang, Phys. Lett. B394 (1997) 315

[14] T. Eguchi, Y. Yamada and S.-K. Yang, Mod. Phys. Lett. A8 (1993) 1627

[15] K. Saito, Publ. RIMS, Kyoto University 19 (1983) 1231

[16] M. Noumi, Tokyo J. Math. 7 (1984) 1

[17] V. Kanev, Proc. Symp. Math. 49 (1989) 627;
    A. McDaniel, Duke. Math. J. 56 (1988) 47

[18] T. Ambai, *Explicit solutions of K. Saito’s system of differential equations (the case of simple singularities of type A and D)*, Master’s thesis, Tohoku Univ., March, 1987;
    T. Oda, *Introduction to Algebraic Singularities (with an Appendix by Taichi Ambai)*, preprint, 1989 (unpublished)

[19] M. Matone, Phys. Lett. B357 (1995) 342;
    J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. B367 (1996) 145;
    T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A11 (1996) 131

[20] S. Terashima and S.-K. Yang, *ADE Confining Phase Superpotentials*, hep-th/9706076, to appear in Nucl. Phys. B.
[21] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press (1990)

[22] R. Slansky, Phys. Rep. 79 (1981) 1

[23] K. Landsteiner, J.M. Pierre and S.B. Giddings, Phys. Rev. D55 (1997) 2367

[24] P.C. Argyres and M. Douglas, Nucl. Phys. B448 (1995) 93;
    P.C. Argyres, R.N. Plesser, N. Seiberg and E. Witten, Nucl. Phys. B461 (1996) 71;
    T. Eguchi, K. Hori, K. Ito and S.-K. Yang, Nucl. Phys. B471 (1996) 430