Geometry/Algebra

The equivariant Riemann–Roch theorem and the graded Todd class

Le théorème de Riemann–Roch équivariant et la classe de Todd graduée

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A B S T R A C T

Let $G$ be a torus with Lie algebra $\mathfrak{g}$ and let $M$ be a $G$-Hamiltonian manifold with Kostant line bundle $\mathcal{L}$ and proper moment map. Let $\Lambda \subset \mathfrak{g}^*$ be the weight lattice of $G$. We consider a parameter $k \geq 1$ and the multiplicity $m(\lambda, k)$ of the quantized representation $RR_G(M, \mathcal{L}^k)$. Define $\langle \Theta(k), f \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) f(\lambda/k)$ for $f$ a test function on $\mathfrak{g}^*$. We prove that the distribution $\Theta(k)$ has an asymptotic development $\langle \Theta(k), f \rangle \sim k^{\dim M/2} \sum_{\lambda \in \Lambda \cap \mathbb{Z} \mathfrak{g}^*} k^{-\pi(DH_{\lambda}, f)}$ where the distributions $DH_{\lambda}$ are the twisted Duistermaat–Heckman distributions associated with the graded equivariant Todd class of $M$. When $M$ is compact, and $f$ polynomial, the asymptotic series is finite and exact.

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R É S U M É

Soit $G$ un tore d’algèbre de Lie $\mathfrak{g}$ agissant de manière hamiltonienne sur une variété $M$. Soit $\mathcal{L}$ un fibré de Kostant tel que l’application moment associée soit propre. Soit $\Lambda \subset \mathfrak{g}^*$ le réseau des poids de $G$. On considère un paramètre $k \geq 1$ et la multiplicité $m(\lambda, k)$ de la représentation quantifiée $RR_G(M, \mathcal{L}^k)$. On définit la distribution $\langle \Theta(k), f \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) f(\lambda/k)$ pour $f$ une fonction test sur $\mathfrak{g}^*$. La distribution $\Theta(k)$ admet un développement asymptotique $\langle \Theta(k), f \rangle \sim k^{\dim M/2} \sum_{\lambda \in \Lambda \cap \mathbb{Z} \mathfrak{g}^*} k^{-\pi(DH_{\lambda}, f)}$ où les distributions $DH_{\lambda}$ sont des distributions associées aux composantes homogènes de la classe de Todd équivariante de $M$. Lorsque $M$ est compacte et $f$ polynomiale, cette série est finie et exacte.

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1. Introduction

Let $G$ be a torus with Lie algebra $\mathfrak{g}$. Identify $\hat{G}$ to a lattice $\Lambda$ of $\mathfrak{g}^*$. If $\lambda \in \Lambda$, we denote by $g^\lambda$ the corresponding character of $G$. If $g = \exp(X)$ with $X \in \mathfrak{g}$, then $g^\lambda = e^{(\lambda, X)}$.

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1631-073X/© 2017 Published by Elsevier Masson SAS on behalf of Académie des sciences. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Let $M$ be a prequantizable $G$-Hamiltonian manifold with symplectic form $\Omega$, Kostant line bundle $\mathcal{L}$, and moment map $\Phi : M \to \mathfrak{g}^*$. Assume $M$ compact and of dimension $2d$. The Riemann–Roch quantization $RR_G(M, \mathcal{L})$ is a virtual finite dimensional representation of $G$, constructed as the index of a Dolbeaut–Dirac operator on $M$. The dimension of the space $RR_G(M, \mathcal{L})$ will be called the Riemann–Roch number of $(M, \mathcal{L})$. The character of the representation of $RR_G(M, \mathcal{L})$ is a function on $G$, denoted by $RR_G(M, \mathcal{L})(g)$. We write

$$RR_G(M, \mathcal{L})(g) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) g^\lambda.$$ 

The typical example is the case where $M$ is a projective manifold, and $\mathcal{L}$ the corresponding ample bundle. Then

$$RR_G(M, \mathcal{L})(g) = \sum_{i=0}^d (-1)^i \text{Tr } H^i(M, \mathcal{O}(\mathcal{L}))(g)$$

is the alternate sum of the traces of the action of $g$ in the cohomology spaces of $\mathcal{L}$. In particular $\dim RR_G(M, \mathcal{L}) = \sum_{i=0}^d (-1)^i \dim H^i(M, \mathcal{O}(\mathcal{L}))$ is given by the Riemann–Roch formula.

It is natural to introduce the $k$th power $\mathcal{L}^k$ of the line bundle $\mathcal{L}$. Thus

$$RR_G(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) g^\lambda.$$ 

Assume $k \geq 1$. We associate with $(M, \mathcal{L})$ the distribution on $\mathfrak{g}^*$ given by

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) f(\lambda/k)$$

where $f$ is a test function.

**Example.** When $M$ is a toric manifold associated with the Delzant polytope $P$, then $\dim RR_G(M, \mathcal{L})$ is the number of integral points in the convex polytope $P$, and $\frac{1}{n!} \langle \Theta_M(k), f \rangle$ is the Riemann sum of the values of $f$ on the sample points $\frac{\lambda}{n} \cap P$.

We prove that $\Theta_M(k)$ has an asymptotic behavior when the integer $k$ tends to $\infty$ of the form

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n$$

where $DH_n$ are distributions on $\mathfrak{g}^*$ supported on $\Phi(M)$. We determine the distributions $DH_n$ in terms of the decomposition of the equivariant Todd class $\text{Todd}(M)$ of $M$ in its homogeneous components $\text{Todd}_n(M)$ in the graded equivariant cohomology ring of $M$. The distribution $DH_0$ is the Duistermaat–Heckman measure. The asymptotics are exact when $f$ is a polynomial. This generalizes the weighted Ehrhart polynomial for an integral polytope, and the asymptotic behavior of Riemann sums over convex integral polytopes established by Guillemin–Sternberg [8].

We then consider the case where $M$ is a prequantizable $G$-Hamiltonian manifold, not necessarily compact, but with proper moment map $\Phi : M \to \mathfrak{g}^*$. The formal quantization of $(M, \mathcal{L}^k)$ [19] is defined by

$$RR_G(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) g^\lambda.$$ 

Here $m_{\text{geo}}(\lambda, k)$ is the geometric multiplicity function constructed by Guillemin–Sternberg in terms of the Riemann–Roch number of the reduced fiber $M_\lambda = \Phi^{-1}(\lambda)/G$ of the moment map. When $M$ is compact, Meinrenken–Sjamaa [10] proved that $m_{\text{rep}}(\lambda, k) = m_{\text{geo}}(\lambda, k)$, so this purely geometric definition extends the definition of $RR_G(M, \mathcal{L}^k)$ given in terms of index theory when $M$ is compact.

Similarly, we construct distributions $DH_n$ on $\mathfrak{g}^*$ using the equivariant cohomology classes $\text{Todd}_n(M)$ and push-forwards by the proper map $\Phi$. The main result of this announcement is that the distribution $\Theta_M(k)$ defined by

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) f(\lambda/k),$$

is asymptotic to $k^d \sum_{n=0}^{\infty} k^{-n} DH_n$.

A similar result holds for Dirac operators twisted by powers of a line bundle $\mathcal{L}^k$.

Recall that we introduced a truncated Todd class (of the cotangent bundle $T^* M$) for determining the multiplicities of the equivariant index of any transversally elliptic operator on $M$ [17]. Here the use of the parameter $k$ allows us to have
families of such equivariant indices, and the full series $\sum_{n=0}^{\infty} \text{Todd}_n(M)$ enters in the description of the asymptotic behavior. This is similar to the Euler–Maclaurin formula evaluating sums of the values of a function at integral points of an interval involving all Bernoulli numbers. We finally give some information on the piecewise polynomial behavior of the distributions $DH_n$.

2. Equivariant cohomology

Let $N$ be a $G$-manifold and let $\mathcal{A}(N)$ be the space of differential forms on $N$, graded by its exterior degree. Following [3] and [20], an equivariant form is a $G$-invariant smooth function $\alpha : g \to \mathcal{A}(N)$, thus $\alpha(X)$ is a differential form on $N$ depending differentiably of $X \in g$. Consider the operator

$$d_\theta \alpha(X) = d\alpha(X) - \iota(v_X)\alpha(X)$$

(2.1)

where $\iota(v_X)$ is the contraction by the vector field $v_X$ generated by the action of $-X$ on $N$. Then $d_\theta$ is an odd operator with square $0$, and the equivariant cohomology is defined to be the cohomology space of $d_\theta$. It is important to note that the dependence of $\alpha$ on $X$ may be $C^\infty$. If the dependence of $\alpha$ in $X$ is polynomial, we denote by $H^*_c(N)$ the corresponding $\mathbb{Z}$-graded algebra. By definition, the grading of $P(X) \otimes \mu$, $P$ a homogeneous polynomial and $\mu$ a differential form on $N$, is the exterior degree of $\mu$ plus twice the polynomial degree in $X$.

The Hamiltonian structure on $M$ determines the equivariant symplectic form $\Omega(X) = \langle \Phi, X \rangle + \Omega$.

Choose a $G$-invariant Riemannian metric on $M$. This provides the tangent bundle $TM$ with the structure of a Hermitian vector bundle. Let $J(A) = \det_{C^d} e^{A - 1 \over n}$, an invariant function of $A \in \text{End}(C^d)$. Then, $J(0) = 1$. Consider $\frac{1}{J(A)}$ and its Taylor expansion at 0:

$$\frac{1}{J(A)} = \det_{C^d} (e^A - 1) = \sum_{n=0}^{\infty} B_n(A).$$

Each function $B_n(A)$ is an invariant polynomial of degree $n$ on $\text{End}(C^d)$ and by the Chern–Weil construction, $B_n$ determines an equivariant characteristic class $\text{Todd}_n(M)(X)$ on $M$ of homogeneous degree $2n$. Remark that $\text{Todd}_0(M) = 1$. We define the formal series of equivariant cohomology classes:

$$\text{Todd}(M)(X) = \sum_{n=0}^{\infty} \text{Todd}_n(M)(X).$$

For $X$ small enough, the series is convergent, and $\text{Todd}(M)(X)$ is the equivariant Todd class of $M$. In particular, $\text{Todd}(M)(0)$ is the usual Todd class of $M$.

In the rest of this note, using the Lebesgue measure $d\xi$ determined by the lattice $\Lambda$, we may identify distributions and generalized functions on $g^*$, and we may write $\langle \theta, f \rangle = \int_{g^*} \theta(\xi) f(\xi) d\xi$ for the value of a distribution $\theta$ on a test function $f$ on $g^*$.

3. The compact case

Let $M$ be a compact $G$-Hamiltonian manifold. Recall (see [2]) the “delocalized Riemann–Roch formula.” For $X \in g$ sufficiently small, we have

$$RR_G(M, \mathcal{L})(\exp X) = \frac{1}{(2\pi)^d} \int_M e^{i\Omega(X)} \text{Todd}(M)(X).$$

Here $i = \sqrt{-1}$.

For each integer $n$, consider the analytic function on $g$ given by

$$\theta_n(X) = \frac{1}{(2\pi)^d} \int_M e^{i\Omega(X)} \text{Todd}_n(M)(X).$$

There is a remarkable relation between the Riemann–Roch character associated with $\mathcal{L}^k$ and the dilation $X \mapsto X/k$ on $g$.

**Lemma 3.1.** When $X \in g$ is sufficiently small, then for any $k \geq 1$, one has

$$RR_G(M, \mathcal{L}^k)(\exp(X/k)) = \sum_{n=0}^{\infty} k^{d-n} \theta_n(X).$$
Theorem

For $X \in \mathfrak{g}$ is small, then $\sum_{n=0}^{\infty} \text{Todd}_n(M)(X)$ is a convergent series with sum the equivariant Todd class. Thus we obtain

$$RR_G(M, \mathcal{L}^k)(\exp(X/k)) = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^d} \int_M e^{i\omega + i\Phi(X/k)} \text{Todd}_n(M)(X/k)$$

$$= \sum_{n,m} \frac{1}{(2\pi)^d} \int_M e^{i\Phi(X)} \frac{1}{m!} k^m (i\Omega)^m \text{Todd}_n(M)(X/k).$$

For each $m$, only the term of differential degree $2d - 2m$ of $\text{Todd}_n(M)$ contributes to the integral, and this term is homogeneous in $X$ of degree $n + m - d$. This implies the result. \Box

When $n = 0$,

$$\theta_0(X) = \frac{1}{(2\pi)^d} \int_M e^{i\Omega(X)}$$

is the equivariant volume of $M$, and the Fourier transform $DH_0$ of $\theta_0$ is the Duistermaat–Heckman measure of $M$, a piecewise polynomial measure on $\mathfrak{g}^*$. **Theorem 3.2.** Let $DH_n$ be the Fourier transform of $\theta_n$. Then $DH_n$ is a distribution supported on $\Phi(M)$. For any polynomial function $P$ of degree $N$ on $\mathfrak{g}^*$, we have

$$\sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) P(\lambda) = \sum_{n \leq N + d} \int_{\mathfrak{g}^*} DH_n(\xi) P(\xi) d\xi.$$

In particular, we have the following Euler–Maclaurin formula for the Riemann–Roch number of $(M, \mathcal{L})$:

$$\dim RR_G(M, \mathcal{L}) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) = \int_{\mathfrak{g}^*} \sum_{n \leq d} DH_n(\xi) d\xi.$$

We now give a theorem for smooth functions.

**Theorem 3.3.** When the integral parameter $k$ tends to $\infty$, the distribution $\Theta_M(k)$ admits the asymptotic expansion

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n.$$

Let us sketch the proof of Theorems 3.2 and 3.3. It is easy to see that the distributions $DH_n$ are supported on the image $\Phi(M)$ of $M$ by the moment map. Furthermore, it follows from the piecewise quasi-polynomial behavior of the function $m_{\text{rep}}(\lambda, k)$ that for $P$ a homogeneous polynomial of degree $N$, the sum $\sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) P(\lambda)$ is a quasi-polynomial function of $k \geq 1$ of degree less than or equal to $N + d$. Thus Theorem 3.2 will be a consequence of Theorem 3.3, which we now prove.

The Fourier transform of $\Theta_M(k)$ is

$$\sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) e^{i\lambda, X/k} = RR_G(M, \mathcal{L}^k)(\exp(X/k)).$$

Against a test function $\phi$ of $X$, this is

$$\frac{1}{(2\pi)^d} \int_M RR_G(M, \mathcal{L}^k)(\exp(X/k)) \phi(X) dX.$$

For $k$ large, and $X$ in the support of $\phi$, $X/k$ is small, and we use Lemma 3.1.

Let us give an example of the asymptotic expansion.

Let $P_1(\mathbb{C})$ equipped with the torus action $g([z_1, z_2]) = [gz_1, z_2]$ of $g = e^{i\theta}$, in homogeneous coordinates. We consider $M = P_1(\mathbb{C}) \times P_1(\mathbb{C})$ with diagonal action, and let $\mathcal{L}$ be its Kostant line bundle. Then we have

$$RR(M, \mathcal{L}^k)(g) = \sum_{j \in \mathbb{Z}} m_{\text{rep}}(j, k) g^j.$$
with

$$m_{\text{rep}}(j, k) = \begin{cases} 0 & \text{if } j < -2k, \\ 2k + 1 + j & \text{if } -2k \leq j \leq 0, \\ 2k + 1 - j & \text{if } 0 \leq j \leq 2k, \\ 0 & \text{if } j > 2k. \end{cases}$$

We have

$$\Theta(k) \sim k^2 (DH_0 + \frac{1}{k} DH_1 + \frac{1}{k^2} DH_2 + \frac{1}{k^3} DH_3 + \cdots).$$

Let us give the explicit formulae for $DH_0, DH_1, DH_2, DH_3$.

$$\langle DH_0, f \rangle = \int_{-2}^{2} m(\xi) f(\xi) d\xi$$

with

$$m(\xi) = \begin{cases} 2 + \xi & \text{if } -2 \leq \xi \leq 0, \\ 2 - \xi & \text{if } 0 \leq \xi \leq 2. \end{cases}$$

$$\langle DH_1, f \rangle = \int_{-2}^{2} f(\xi) d\xi,$$

$$\langle DH_2, f \rangle = \frac{5}{12} f(-2) + \frac{1}{6} f(0) + \frac{5}{12} f(2),$$

$$\langle DH_3, f \rangle = -\frac{1}{12} f'(-2) + \frac{1}{12} f'(2).$$

We now sketch another proof of Theorem 3.3, which can be extended to the non-compact case. We use Paradan’s decomposition ([11,12], see also [18]) of $RR_G(M, L)$ in a sum of simpler characters supported on cones. Let us consider a generic value $r$ of the moment map, and choose a scalar product on $\mathfrak{g}^\ast$. Then there exists a certain finite subset $B(r)$ of $\mathfrak{g}^\ast$, and for each $\beta \in B(r)$, a cone $C(\beta)$ in $\mathfrak{g}^\ast$ and an (infinite dimensional) representation $P_{\beta, k}$ such that

$$RR_G(M, \mathcal{L}^k) = \sum_{\beta \in B(r)} P_{\beta, k}.$$ 

Here $P_{\beta, k}(g) = \sum_{\lambda \in \Lambda_\beta \cap \Lambda_G(\beta)} m_{\text{rep}}(\lambda, k) g^\lambda$. Thus $\Theta(k)$ is decomposed in $\sum_{\beta \in B(r)} \Theta_\beta(k)$. Similarly, each distribution $DH_n$ is decomposed as $DH_n = \sum_{\beta \in B(r)} DH_{n, \beta}$ and the support of $DH_{n, \beta}$ is contained in the cone $C(\beta)$. It is easily verified that, for each $\beta$, the distribution $\Theta_\beta(k)$ is asymptotic to $k^d \sum_{n=0}^{\infty} k^{-n} DH_{n, \beta}$. Here we use the explicit Euler–Maclaurin expansion on half lines, and convolutions of such distributions. The proof is entirely similar to that in the case of a polytope given in [4].

Let us return to the example of the case of $M = P_1(\mathbb{C}) \times P_1(\mathbb{C})$, for $r < 0$ a small negative number. Then $B(r) = \{-2, r, 0, 2\}$. We have

$$P_{\beta=-2, k}(g) = -\sum_{j=-2k}^{j=\infty} (2k + 1 + j) g^j,$$

$$P_{\beta=r, k}(g) = \sum_{j=-\infty}^{j=\infty} (2k + 1 + j) g^j,$$

$$P_{\beta=0, k}(g) = -2 \sum_{j=0}^{j=0} j g^j,$$

$$P_{\beta=2, k}(g) = \sum_{j=2k}^{j=\infty} (j - (2k + 1)) g^j.$$ 

Consider, for example, the asymptotic development of the distribution

$$\langle \Theta_{\beta=2}(k), f \rangle = \sum_{j=2k}^{j=\infty} (j - (2k + 1)) f(j/k).$$
It is easy to see that this distribution is the convolution \( K(k) * K(k) \) where \( K(k) \) is the distribution defined by \( \langle K(k), f \rangle = \sum_{j \geq k} f(j/k) = \sum_{j \geq k} f(j/k) - f(1) \). We then use the explicit exact Euler–Maclaurin formula to evaluate the distribution \( K(k) \), thus its convolution. In particular, the Fourier transform of \( K(k) * K(k) \) coincides with the analytic function \( \frac{e^{ikx}}{(1-e^{-ik)})^2} \) for \( (1-e^{-ik)}) \neq 0 \). As is natural, the asymptotic series of distributions \( q^{-d} \sum_{n=0}^{\infty} q^n DH_{β=2,n} \) is the unique series of distributions supported on \( \xi \geq 2 \) and with Fourier transform, for \( x \neq 0 \), the Laurent series in \( q \) of \( \frac{e^{ikx}}{(1-e^{-ix})^2} \) at \( q = 0 \).

4. Proper moment maps

Consider the case where \( M \) is non-necessarily compact, but \( \Phi : M \to g^* \) is a proper map. One can then define [19,13] the formal geometric quantization of \( M \) with respect to the line bundle \( L^k \) to be

\[
RR_{G, \text{geo}}(M, L^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) g^\lambda,
\]

using a function \( m_{\text{geo}}(\xi) \) on \( g^* \). The definition of the function \( m_{\text{geo}}(\xi) \) is due to Guillemin–Sternberg [7]. Let us recall its delicate definition ([10], see also [16]). There is a closed set \( A \), union of affine hyperplanes, such that if \( r \) is in the complement of \( A \), then either \( r \) is not in \( \Phi(M) \) or \( r \) is a regular value of \( \Phi \). Consider the open subset \( g_{\text{reg}} = g^* \setminus A \). If \( \xi \in g_{\text{reg}} \) but not in \( \Phi(M) \), \( m_{\text{geo}}(\xi) \) is defined to be 0. If \( \xi \in g_{\text{reg}} \cap \Phi(M) \), the reduced fiber \( M_\xi = \Phi^{-1}(\xi)/G \) is a compact symplectic orbifold, and \( m_{\text{geo}}(\xi) \) is defined to be a sum of integrals on the various strata of the compact orbifold \( M_\xi \). When \( \lambda \in A \), \( m_{\text{geo}}(\lambda) \) is the Riemann–Roch number of any \( M_\lambda \) equipped with its Kostant–orbifold line bundle. Let \( \lambda \in \Lambda \) be any point in \( \Phi(M) \). Choose a vector \( \epsilon \) such that \( \lambda + t\epsilon \) is in \( \Phi(M) \cap g_{\text{reg}} \) for any \( t > 0 \) and sufficiently small. It can be proved, using the wall crossing formulæ of Paradan [15], that \( \lim_{t \to 0} m_{\text{geo}}(\lambda + t\epsilon) \) is independent of the choice of such an \( \epsilon \). This allows us to define \( m_{\text{geo}}(\lambda) \) by “continuity on \( \Phi(M) \)” for any \( \lambda \in \Lambda \).

The \( [Q, R] = 0 \) theorem [10,9,14] asserts that \( RR_{G, \text{geo}}(M, L) \) coincides with a representation of \( G \) defined using index theory. In particular, \( RR_{G, \text{geo}}(M, L) \) coincides with \( RR_{C, \text{geo}}(M, L) \) when \( M \) is compact. However, in the rest of this note, we only use the geometric definition of \( RR_{G, \text{geo}}(M, L) \).

Replacing \( L \) by \( L^k \), and the moment map \( \Phi \) by \( k\Phi \), define the distribution, with parameter \( k \),

\[
\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) f(\lambda/k).
\]

As in the compact case, the asymptotic behavior of \( \Theta_M(k) \) is determined by the graded Todd class, using push-forwards by the proper map \( \Phi \). Indeed if \( \alpha \) is an equivariant cohomology class with polynomial coefficients, then the Duistermaat–Heckman twisted distribution \( DH(M, \Phi, \alpha) \) is well defined by the formula

\[
\langle DH(M, \Phi, \alpha), f \rangle = \frac{1}{(2\pi)^d} \int_{M \times g} e^{i\Omega(X)}\alpha(X) \hat{f}(X)dX
\]

where \( \hat{f}(X) = \int_{g^*} e^{i\langle Xi, X \rangle} f(\xi)d\xi \) is the Fourier transform of the test function \( f(\xi) \) (see [6]). It is a distribution supported on \( \Phi(M) \).

**Definition 4.1.** We define \( DH_n \) to be the distribution on \( g^* \) associated with the equivariant cohomology class \( \text{Todd}_n(M) \):

\[
\langle DH_n, f \rangle = \frac{1}{(2\pi)^d} \int_{M \times g} e^{i\Omega(X)}\text{Todd}_n(M)(X) \hat{f}(X)dX.
\]

The distribution \( DH_0 \) is the Duistermaat–Heckman measure, a locally polynomial function.

The distribution \( DH_n \) is given by a polynomial function on each connected component of the open set \( g_{\text{reg}}^* \). Its restriction to \( g_{\text{reg}}^* \) vanishes when \( n > d - \dim G \). Furthermore, if all stabilizers of points of \( M \) are connected, it follows from Witten non-Abelian localization theorem that

\[
m_{\text{geo}}(\lambda, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_n(\lambda/k)
\]

when \( \lambda/k \) is a regular value of \( \Phi \). Otherwise, it can be defined by the limit of the function \( m_{\text{geo}}(\xi, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_n(\xi/k) \) along \( \xi = \lambda + t\epsilon \), and \( t > 0, t \to 0 \), where the direction \( \epsilon \) is chosen to be arbitrary if \( \lambda \) does not belong to \( k\Phi(M) \), or in such a way that \( \lambda + t\epsilon \) stays in \( k\Phi(M) \) if \( \lambda \in k\Phi(M) \). Similar formulæ can be given without assumption on connected stabilizers.

We can see that, for any \( n \), the distributions \( DH_n \) can be expressed (but not uniquely) as derivatives of locally polynomial functions associated with symplectic submanifolds \( M^T \) where \( T \) are subtori of \( G \).
The main result of this note is the following theorem.

**Theorem 4.2.** When the integer $k$ tends to $\infty$,

$$
\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n.
$$

Let us sketch the proof of this theorem, in the case where each stabilizer is connected. We use Paradan’s decomposition formula \cite{12,11}. We choose $\alpha$ a generic element of $\mathfrak{g}_{\text{reg}}^\alpha$. As in the compact case, there is a locally finite set $\mathcal{B}(r) \subset \Phi(M)$, cones $C_{r}$, and decompositions

$$
DH_n = \sum_{\beta \in \mathcal{B}(r)} DH_{n,\beta}
$$

where $DH_{n,\beta}$ are supported on $C_{r}$. The functions $DH_{n,\beta}$ are given by polynomial functions on each connected component of $\mathfrak{g}_{\text{reg}}^\alpha$ and vanishes on $\mathfrak{g}_{\text{reg}}^\alpha$ when $n > d - \dim G$. Thus the locally polynomial function $A_{\beta}(\xi, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_{n,\beta}(\xi/k)$ is well defined when $\xi/k \in \mathfrak{g}_{\text{reg}}^\alpha$. For each $\beta \in \mathcal{B}(r)$, choose a direction $\epsilon_{\beta}$ such that $\beta + t \epsilon_{\beta}$ is in $\Phi(M) \cap \mathfrak{g}_{\text{reg}}^\alpha$ for $t > 0$ small. Then $w_{\beta}(\lambda, k) = \lim_{t \to 0, t \to 0} A_{\beta}(\lambda + t \epsilon_{\beta}, k)$ is well defined. Define

$$
P_{\beta,k}(g) = \sum_{\lambda \in \Lambda} w_{\beta}(\lambda, k) g^\lambda
$$

and

$$
\langle \Theta_{\beta,\text{geo}}(k), f \rangle = \sum_{\lambda \in \Lambda} w_{\beta}(\lambda, k) f(\lambda/k).
$$

As before, it is easy to see that $\Theta_{\beta,\text{geo}}(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_{n,\beta}$. Here we use the following “continuity” result on partition function (see for example \cite{5}). Let $\Delta$ be a unimodular list of non-zero vectors in $\Lambda$, and $\gamma \in \mathfrak{g}$ generic. There is a unique function $K$ (the Kostant partition function) on $\Lambda$ supported on the half space $\langle \xi, \gamma \rangle \geq 0$ and such that $\sum_{\lambda \in \Lambda} K(\lambda) g^\lambda = \prod_{\alpha \in \Delta} \frac{1}{1 - g^\alpha}$ for $g$ in the open set $\prod_{\alpha \in \Delta} (1 - g^\alpha) \neq 0$. Let $d = |\Delta|$. Consider the Laurent series expansion in $q$

$$
\prod_{\alpha \in \Delta} \frac{1}{1 - q^\alpha} = q^{-d} \sum_{n=0}^{\infty} q^n U_n(X)
$$

and the distributions $D_n$ on $g^\alpha$ supported on the half space $\langle \xi, \gamma \rangle \geq 0$, such that

$$
\int_{g^\alpha} D_n(\xi) e^{i \langle \xi, X \rangle} = U_n(X)
$$

when $\prod_{\alpha \in \Delta} (\alpha, X) \neq 0$. Define $T(\xi) = \sum_{n=0}^{\infty} D_n(\xi)$, which is well defined outside of a system of hyperplanes. Then for any $\lambda \in \Lambda$, and $\epsilon_{\lambda}$ generic and belonging to the cone $\text{Cone}(\Delta)$ generated by $\Delta$, we have $K(\lambda) = \lim_{\gamma \to 0, t \to 0} T(\lambda + t \epsilon_{\lambda})$.

Define $P_{\alpha,k} = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}$. It remains to see that $P_{\alpha,k}$ is independent of $r$, using \cite{15}. This is very similar to the technique used in \cite{1} to establish decompositions à la Paradan of characteristic functions of polyhedra. It then follows that $P_{\alpha,k} = RR_{G,\text{geo}}(M, L^k)$. Indeed for each connected component $\epsilon$ of $\mathfrak{g}_{\text{reg}}^\alpha$ contained in $\Phi(M)$, we choose $r \in \epsilon$. In the decomposition $P_{\alpha,k} = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}$, the term $w_{\beta}(\lambda, k)$ for $\beta = r \in \mathcal{B}(r)$ is the polynomial function coinciding with $m_{\text{geo}}(\lambda, k)$ for $\lambda \in \epsilon k$. The other terms $w_{\beta}(\lambda, k)$ for $\beta \neq r$ vanishes when $\lambda \in \epsilon k$ \cite{12,18}.

A quicker route, but less instructive, for determining asymptotics of $\Theta_{M,\text{geo}}$ would be to take a test function with small support around a point $r \in \mathfrak{g}^*$. Then we can choose $\epsilon_{\lambda}$ coinciding with $\epsilon_{\beta}$ for all $\beta \in \mathcal{B}(r)$ and in the support of the test function $f$. The additivity is immediate on those $\beta$.

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