Remarks on the spin-one Duffin-Kemmer-Petiau equation in the presence of nonminimal vector interactions in (3+1) dimensions

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Abstract

In a recent paper published in this journal, Hassanabadi and collaborators [Advances in High Energy Physics, vol. 2012, Article ID 489641, 10 pages, 2012] analyzed the Duffin-Kemmer-Petiau equation in the presence of nonminimal vectorial interactions (Coulomb and harmonic oscillator potentials) in (3+1) dimensions for spin-one particles. In that paper, the authors used improperly the nonminimal vector interaction endangering in their main conclusions. We present a few properties of the nonminimal vector interactions and also present the correct equations to this problem. We show that the solution can be easily found by solving a Schrödinger-like equation. As an application of this procedure, we consider spin-one particles in presence of a nonminimal vector linear potential. Additionally, we present the correct set of first-order coupled differential radial equation for the DKP equation in the presence of minimal vector and scalar interactions.

Key-words: Duffin-Kemmer-Petiau theory; vector potential; bound state

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1 Introduction

The Duffin-Kemmer-Petiau (DKP) formalism \cite{1, 2, 3, 4} describes spin-zero and spin-one particles and has been used to analyze relativistic interactions of spin-zero and spin-one hadrons with nuclei as an alternative to their conventional second-order Klein-Gordon (KG) and Proca counterparts. The DKP formalism proved to be better than the KG formalism in the analysis of $K^0$ decays, the decay-rate ratio $\Gamma(\eta \rightarrow \gamma \gamma)/\Gamma(\pi^0 \rightarrow \gamma \gamma)$, and level shifts and widths in pionic atoms \cite{5, 6, 7, 8, 9, 10, 11, 12}. The DKP formalism enjoys a richness of couplings not capable of being expressed in the KG and Proca theories \cite{17, 18}. Although the formalisms are equivalent in the case of minimally coupled vector interactions \cite{19, 20, 21}, the DKP formalism opens new horizons as far as it allows other kinds of couplings which are not possible in the KG and Proca theories. The nonminimal vector interaction refers to a kind of charge conjugate invariant coupling that behaves like a vector under a Lorentz transformation. The invariance of the nonminimal vector potential under charge conjugation means that it does not distinguish particles from antiparticles. Hence, whether one consider spin-zero or spin-one bosons, this sort of interaction cannot exhibit Klein’s paradox \cite{22}. Nonminimal vector potentials, added by other kinds of Lorentz structures, have already been used in a phenomenological context for describing the scattering of mesons by nuclei \cite{23, 24, 25, 26, 28, 30, 32}, but it should be mentioned that in Refs. \cite{23, 24, 26, 28, 30, 32} the nonminimal vector couplings have been used improperly. Nonminimal vector coupling with a quadratic potential \cite{33}, with a linear potential \cite{34}, and mixed space and time components with a step potential \cite{35, 36}, double-step potential \cite{37}, a smooth step potential \cite{38}, a linear potential \cite{22, 39, 40}, and a linear plus inversely linear potential \cite{41}, have been explored in the literature. In a recent paper published in this journal, Hassanabadi and collaborators \cite{42} analyze the DKP equation in the presence of nonminimal vectorial interactions (Coulomb and harmonic oscillator potentials) in (3+1) dimensions for spin-one particles. In that paper, the authors used improperly the nonminimal vector interaction endangering its main conclusions. The same mistake is found in recent works \cite{43, 44, 45, 46, 47, 48, 49, 50, 51}, for instance. Other misconception is found in Refs. \cite{23, 24, 26}, in which the space component of the nonminimal vector potential is absorbed into the spinor. As it is shown in \cite{22}, there is no chance to dissociate from this term. Furthermore, the space component of the nonminimal vector potential could be irrelevant for the formation of bound states for potentials vanishing at infinity, but its presence is an essential ingredient for confinement.

In view of the misconceptions on the nonminimal vector interaction propagated in the literature, the purpose of this Review Article is to review the DKP equation in the presence of a nonminimal vectorial interaction for spin-one particles in (3+1) dimensions. We present a few properties of the nonminimal vector interactions and also present the correct equations to this problem. We show that the solution can be easily found by solving a Schrödinger-like equation. As an application of this procedure, we consider spin-one particles in presence of a nonminimal vector linear potential. For this case in particular, the problem is mapped into the nonrelativistic three-dimensional harmonic oscillator. Additionally, we present the correct set of first-order coupled differential radial equation for the DKP equation with minimal vector and scalar interactions.
2 The DKP equation

The DKP equation for a free boson is given by [4] (with units in which $\hbar = c = 1$)

$$(i\beta^\mu \partial_\mu - m) \psi = 0$$ (1)

where the matrices $\beta^\mu$ satisfy the algebra

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu$$ (2)

and the metric tensor is $g^{\mu\nu} = \text{diag} (1, -1, -1, -1)$. The algebra expressed by (2) generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-zero particles and a ten-dimensional representation associated to spin-one particles. The DKP spinor has an excess of components and the theory has to be supplemented by an equation which allows us eliminate the redundant components. That constraint equation is obtained by multiplying the DKP equation by $1 - \beta^0 \beta^0$, namely

$$i\beta^i \beta^0 \beta^0 \partial_i \psi = m \left(1 - \beta^0 \beta^0\right) \psi$$ (3)

This constraint equation expresses three (four) components of the spinor by the other two (six) components and their space derivatives in the scalar (vector) sector so that the superfluous components disappear and there only remain the physical components of the DKP theory. The second-order Klein-Gordon and Proca equations are obtained when one selects the spin-zero and spin-one sectors of the DKP theory.

A well-known conserved four-current is given by

$$J^\mu = \frac{1}{2} \bar{\psi} \beta^\mu \psi$$ (4)

where the adjoint spinor $\bar{\psi}$ is given by $\bar{\psi} = \psi^\dagger \eta^0$ with $\eta^0 = 2\beta^0 \beta^0 - 1$ in such a way that $(\eta^0 \beta^\mu)^\dagger = \eta^0 \beta^\mu$ (the matrices $\beta^\mu$ are Hermitian with respect to $\eta^0$). Despite the similarity to the Dirac equation, the DKP equation involves singular matrices, the time component of $J^\mu$ given by (4) is not positive definite and the case of massless bosons cannot be obtained by a limiting process [53]. Nevertheless, the matrices $\beta^\mu$ plus the unit operator generate a ring consistent with integer-spin algebra and $J^0$ may be interpreted as a charge density. The normalization condition $\int d\tau J^0 = \pm 1$ can be expressed as

$$\int d\tau \bar{\psi} \beta^0 \psi = \pm 2$$ (5)

where the plus (minus) sign must be used for a positive (negative) charge.

3 Interactions in the DKP equation

With the introduction of interactions, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - m - U) \psi = 0$$ (6)
where the more general potential matrix $U$ is written in terms of 25 (100) linearly independent matrices pertinent to five (ten)-dimensional irreducible representation associated to the scalar (vector) sector. In the presence of interaction, $J^\mu$ satisfies the equation

$$\partial_\mu J^\mu + \frac{i}{2} \overline{\psi} \left( U - \eta^0 U^\dagger \eta^0 \right) \psi = 0$$  \hspace{1cm} (7)$$

Thus, if $U$ is Hermitian with respect to $\eta^0$ then four-current will be conserved. The potential matrix $U$ can be written in terms of well-defined Lorentz structures. For the spin-zero sector there are two scalar, two vector and two tensor terms \[17\], whereas for the spin-one sector there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms \[18\]. The tensor terms have been avoided in applications because they furnish noncausal effects \[17\]-\[18\]. For the case of scalar and minimal vectorial interactions see Appendix A.

### 3.1 Nonminimal vector couplings in the DKP equation

Considering only the nonminimal vector interaction, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - m - i[P, \beta^\mu]A_\mu) \psi = 0$$ \hspace{1cm} (8)$$

where $P$ is a projection operator ($P^2 = P$ and $P^\dagger = P$) in such a way that $\overline{\psi}[P, \beta^\mu]\psi$ behaves like a vector under a Lorentz transformation as does $\overline{\psi}\beta^\mu\psi$. One very important point to note is that this potential leads to a conserved four-current but the same does not happen if instead of $i[P, \beta^\mu]$ one uses either $P\beta^\mu$ or $\beta^\mu P$, as in \[23\]-\[26\], \[28\]-\[30\], \[32\], \[12\]-\[31\]. As a matter of fact, in \[23\] it is mentioned that $P\beta^\mu$ and $\beta^\mu P$ produce identical results.

The DKP equation is invariant under the parity operation, i.e. when $\vec{r} \rightarrow -\vec{r}$, if $\vec{A}$ changes sign, whereas $A_0$ remains the same. This is because the parity operator is $\mathcal{P} = \exp(i\delta_p)P_0\eta^0$, where $\delta_p$ is a constant phase and $P_0$ changes $\vec{r}$ into $-\vec{r}$. Because this unitary operator anticommutes with $\beta^\mu$ and $[P, \beta^\mu]$, they change the sign under a parity transformation, whereas $\beta^0$ and $[P, \beta^0]$, which commute with $\eta^0$, remain the same. Since $\delta_p = 0$ or $\delta_p = \pi$, the spinor components have definite parities. The charge conjugation operation can be accomplished by the transformation $\psi \rightarrow \psi_c = C\psi = CK\psi$, where $K$ denotes the complex conjugation and $C$ is a unitary matrix such that $C\beta^\mu = -\beta^\mu C$. The matrix that satisfies these relations is $C = \exp(i\delta_C)\eta^0\eta^1$. The phase factor $\exp(i\delta_C)$ is equal to $\pm 1$, thus $E \rightarrow -E$. Note also that $J^\mu \rightarrow -J^\mu$, as should be expected for a charge current. Meanwhile $C$ anticommutes with $[P, \beta^\mu]$ and the charge conjugation operation entails no change on $A_\mu$. The invariance of the nonminimal vector potential under charge conjugation means that it does not couple to the charge of the boson. In other words, $A_\mu$ does not distinguish particles from antiparticles. Hence, whether one considers spin-zero or spin-one bosons, this sort of interaction cannot exhibit Klein’s paradox \[22\].

If the potential is time-independent one can write $\psi(\vec{r}, t) = \varphi(\vec{r}) \exp(-iEt)$, where $E$ is the energy of the boson, in such a way that the time-independent DKP equation becomes

$$\left[ \beta^0 E + i\beta^i \partial_i - (m + i[P, \beta^\mu]A_\mu) \right] \varphi = 0 \hspace{1cm} (9)$$

In this case $J^\mu = \varphi \beta^\mu \varphi/2$ does not depend on time, so that the spinor $\varphi$ describes a stationary state.
3.2 Vectorial sector

For the case of spin-one (vectorial sector), the $\beta^\mu$ matrices are

$$\beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & 0 & -i s_i \\ 0 & 0 & 0 \\ -i e_i^T & 0 & 0 \\ -i s_i & 0 & 0 \end{pmatrix}$$

where $s_i$ are the $3 \times 3$ spin-1 matrices ($s_{i})_{jk} = -i \varepsilon_{ijk}$, $e_i$ are the $1 \times 3$ matrices ($e_i)_{1j} = \delta_{ij}$ and $0 = (0 \ 0 \ 0)$, while $I$ and $0$ designate the $3 \times 3$ unit and zero matrices, respectively, while the superscript T designates matrix transposition. In this representation $P = \beta^\mu \beta^\mu - 2 = \text{diag}(1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$, i.e. $P$ projects out the four upper components of the DKP spinor. The ten-component spinor can be written as $\varphi^T = (\varphi_1, ..., \varphi_{10})$ and partitioned as (following the notation of Ref. [42])

$$\varphi_1 = i \phi, \quad \vec{F} = \begin{pmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

$$\vec{G} = \begin{pmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix}$$

the DKP equation in (3+1) dimensions can be expressed in the compact form

$$i \vec{\nabla} \times \vec{F} - i \vec{A} \times \vec{F} = m \vec{H}$$

$$\vec{\nabla} \cdot \vec{G} + \vec{A} \cdot \vec{G} = m \phi$$

$$i \vec{\nabla} \times \vec{H} + i \vec{A} \times \vec{H} = m \vec{F} - (E - i A_0) \vec{G}$$

$$\vec{\nabla} \phi - \vec{A} \phi = m \vec{G} - (E + i A_0) \vec{F}$$

At this stage is worthwhile to mention that the eqs. (13)-(16) are completely different from those given in [42] and this fact is due to use improperly the nonminimal vector coupling. These facts should be enough to invalidate the results presented in the Refs. [42]-[51].

Using the standard procedure developed in [54], we put

$$\phi = \phi_n j (r) Y^{m_j}_{j} (\theta, \varphi)$$

$$\vec{F} = \sum_l F_{n j l} (r) \vec{Y}_{j l m_j} (\theta, \varphi)$$

$$\vec{G} = \sum_l G_{n j l} (r) \vec{Y}_{j l m_j} (\theta, \varphi)$$

$$\vec{H} = \sum_l H_{n j l} (r) \vec{Y}_{j l m_j} (\theta, \varphi)$$

where $\phi_{n j}, F_{n j l}, G_{n j l}$ and $H_{n j l}$ are radial wave functions while $Y^{m_j}_{j} (\theta, \varphi)$ is the the usual spherical harmonics of order $j$, and $\vec{Y}_{j l m_j} (\theta, \varphi)$ is the vector spherical harmonics. Then, using the notation

$$F_{n j j} = F_0, \quad F_{n j j \pm 1} = F_{\pm 1}$$
The other components are obtained through (22)-(27) and (29)-(31). A nonminimal vectorial interaction can be found by solving a Schrödinger-like equation for motion in a central field. The solution of the three-dimensional DKP equation with for spin-zero particles in (3+1) dimensions except for the term $-u / r$, finally from (16) we get (from (13) are are the radial equation obtained from (15) are where $A_k = A_0(r)$ and $\vec{A} = A_r(r) \hat{r}$, the radial differential equation obtained from (13) are

$$\left( \frac{dF_0}{dr} - \frac{j}{r} F_0 - A_r F_0 \right) = - \frac{1}{\zeta_j} m H_{+1} \tag{22}$$

$$\left( \frac{dF_0}{dr} + \frac{j + 1}{r} F_0 - A_r F_0 \right) = - \frac{1}{\alpha_j} m H_{-1} \tag{23}$$

$$- \zeta_j \left( \frac{dF_{+1}}{dr} + \frac{j + 2}{r} F_{+1} - A_r F_{+1} \right) - \alpha_j \left( \frac{dF_{-1}}{dr} - \frac{j - 1}{r} F_{-1} - A_r F_{-1} \right) = m H_0 \tag{24}$$

where $\alpha_j = \sqrt{(j + 1)/(2j + 1)}$ and $\zeta_j = \sqrt{j/(2j + 1)}$. Similarly, substituting (45) and (47) in (14), we obtain

$$- \alpha_j \left( \frac{dG_{+1}}{dr} + \frac{j + 2}{r} G_{+1} + A_r G_{+1} \right) + \zeta_j \left( \frac{dG_{-1}}{dr} - \frac{j - 1}{r} G_{-1} + A_r G_{-1} \right) = m \phi \tag{25}$$

The radial equation obtained from (15) are

$$\left( \frac{dH_0}{dr} - \frac{j}{r} H_0 + A_r H_0 \right) = - \frac{1}{\zeta_j} (m F_{+1} - (E - iA_0) G_{+1}) \tag{26}$$

$$\left( \frac{dH_0}{dr} + \frac{j + 1}{r} H_0 + A_r H_0 \right) = - \frac{1}{\alpha_j} (m F_{-1} - (E - iA_0) G_{-1}) \tag{27}$$

$$- \zeta_j \left( \frac{dH_{+1}}{dr} + \frac{j + 2}{r} H_{+1} + A_r H_{+1} \right) - \alpha_j \left( \frac{dH_{-1}}{dr} - \frac{j - 1}{r} H_{-1} + A_r H_{-1} \right) = (m F_0 - (E - iA_0) G_0) \tag{28}$$

Finally, from (16) we get

$$(E + iA_0) F_0 = m G_0 \tag{29}$$

$$\left( \frac{d\phi}{dr} - \frac{j}{r} \phi - A_r \phi \right) = - \frac{1}{\alpha_j} (m G_{+1} - (E + iA_0) F_{+1}) \tag{30}$$

$$\left( \frac{d\phi}{dr} + \frac{j + 1}{r} \phi - A_r \phi \right) = \frac{1}{\zeta_j} (m G_{-1} - (E + iA_0) F_{-1}) \tag{31}$$

Using the eqs. (22), (23) and (29) the components $H_{+1}$, $H_{-1}$ and $G_0$ can be eliminated in favor of $F_0$ then by inserting they in (28) and if one redefines $F_0(r) = u(r)/r$, the radial function $u(r)$ obeys the second-order differential equation

$$\frac{d^2 u(r)}{dr^2} + \left[ k^2 - \frac{dA_r}{dr} - \frac{j(j + 1)}{r^2} - A_r^2 \right] u(r) = 0 \tag{32}$$

where $k^2 = E^2 - m^2 + A_r^2$ and because $\nabla^2 (1/r) = -4\pi \delta(\vec{r})$, unless the potentials contain a delta function at the origin, one must impose the homogeneous Dirichlet condition $u(0) = 0$. At this stage is worthwhile to mention that (32) is very similar to DKP equation for spin-zero particles in (3+1) dimensions except for the term $-2A_r/r$. Therefore, for motion in a central field, the solution of the three-dimensional DKP equation with nonminimal vectorial interaction can be found by solving a Schrödinger-like equation. The other components are obtained through of (22)-(27) and (29)-(31).
3.3 Nonminimal vector linear potential

Having set up the spin-one equations for nonminimal vector interaction, we are now in a position to use the machinery developed above in order to solve the DKP equation for some specific form of the nonminimal interaction. As an application of this procedure, let us consider a nonminimal vector linear potential in the form

\[ A_0 = m^2 \lambda_0 r \quad A_r = m^2 \lambda_r r \]  

(33)

where \( \lambda_0 \) and \( \lambda_r \) are dimensionless quantities. Substituting (33) in (32), one finds that \( u(r) \) obeys the second-order differential equation

\[ \frac{d^2 u}{dr^2} + \left[ K^2 - \lambda^2 r^2 - \frac{j(j+1)}{r^2} \right] u = 0 \]  

(34)

where

\[ K = \sqrt{E^2 - m^2(1 + \lambda_r)} \quad \lambda = m^2 \sqrt{\lambda^2 - \lambda_0^2}. \]  

(35)

Considering \( u(0) = 0 \) and \( \int_0^\infty dr |u|^2 < \infty \), the solution for (34) with \( K \) and \( \lambda \) real is the well-known solution of the Schrödinger equation for the three-dimensional harmonic oscillator. Note that the condition \( \lambda \) real implies that \( |\lambda_r| > |\lambda_0| \), meaning that the radial component of the nonminimal vectorial potential must be stronger than its time component in order to the effective potential be a true confining potential. On the other hand, if \( \lambda_r = 0 \) or \( |\lambda_r| < |\lambda_0| \), we obtain \( \lambda = i|\lambda| \) and the effective potential in this case will be an inverted harmonic oscillator and the energy spectrum will consist of a continuum corresponding to unbound states. Therefore, the presence of radial component of the nonminimal vector potential is an essential ingredient for confinement.

An detailed study of this effective potential is done in [40]. Using the results of Ref. [40] the solution is expressed as

\[ |E| = m \sqrt{1 + \lambda_r + (2n + 3)\sqrt{\lambda^2 - \lambda_0^2}} \quad n = 0, 1, 2, \ldots \]  

(36)

\[ u(r) = N_{n,j} r^j e^{-\lambda r^2/2} L_{n-j-1}^{(j+1/2)}(\lambda r^2) \]  

(37)

where \( N_{n,j} \) is a normalization constant, \( n = 2N + j \) with \( N \) a nonnegative integer. Note that \( j \) can take values \( 0, 2, \ldots, n \) when \( n \) is an even number, and \( 1, 3, \ldots, n \) when \( n \) is an odd number and also that for each value of \( j \) there are \( 2j + 1 \) different values of \( m_j \). All the energy levels are degenerate with the exception of \( N = 0 \). The degeneracy of the level of energy for a given principal quantum number \( n \) is given by \( (n + 1)(n + 2)/2 \) as a consequence of the presence of essential and accidental degeneracies.

From (36), we can see that there is an infinite set of discrete energies (symmetrical about \( E = 0 \)) irrespective to sign of \( \lambda_0 \), although positive- and negative- energy levels do not touch, they can be very close to each other for moderately strong coupling constants without any danger of reaching the conditions for Klein`’s paradox. The absence of Klein`’s paradox for this kind of interaction is attributes to fact that the nonminimal vectorial interaction does not distinguish particles from antiparticles [36].
4 Final remarks

In this Review Article, we showed the correct use and also presented a few properties of the nonminimal vector interactions in the Duffin-Kemmer-Petiau (DKP) formalism. A relativistic wave equation must carry a conserved four-current to exhibit symmetries in physical problems. In this spirit, we showed that the four-current is not conserved when one use either the matrix potential $P\beta^\mu$ or $\beta^\mu P$ (widely used in the literature), even though the linear forms constructed from those matrices potentials behave as true Lorentz vectors. Also, we presented the correct equations for the problem addressed in [42]. In this case, we found an equation very similar to DKP equation for spin-zero particles in (3+1) dimensions, except for the term $-2A_r/r$. Therefore, the solution of the three-dimensional DKP equation with nonminimal vectorial interactions can be found by solving a Schrödinger-like equation. As an application of the procedure developed, we considered the problem of spin-one particles in the presence of a nonminimal linear vector potential and discussed the necessary conditions in order to the effective potential to be true confining potential. The absence of Klein’s paradox is attributes to fact that the nonminimal vectorial interaction does not distinguish particles from antiparticles [36].

Ours results are definitely useful because they shed some light on the understanding of the nonminimal vector interactions. Furthermore, the correct use of the nonminimal vectorial interaction may be useful due to wide applications in the description of elastic meson-nucleus scattering. Additionally, in the Appendix A, we showed the correct set of first-order coupled differential radial equation for the DKP equation in the presence of minimal vector and scalar interactions.

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Appendix A: Minimal vector and scalar couplings in the DKP equation

In this appendix, we present the correct set of first-order coupled differential radial equation for the DKP equation in the presence of minimal vector and scalar interactions. Considering the time-like minimal vector $(U^0_v)$ and scalar $(U_s)$ interactions, the time-independent DKP equation for spin-one particles becomes

$$\left[\beta^0 E + i\beta^i \partial_i - (m + U_s) - \beta^0 U^0_v\right] \varphi = 0$$

(38)

The ten-component spinor can be written as $\varphi^T = (\varphi_1, ..., \varphi_{10})$ and partitioned as

$$\varphi_1 = i\phi, \quad \vec{F} = \begin{pmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

(39)

$$\vec{G} = \begin{pmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix}$$

(40)
the DKP equation in (3+1) dimensions can be expressed in the compact form

\[ \vec{\nabla} \cdot \vec{G} = (m + U_s)\phi \]  \hspace{1cm} (41)  
\[ i\vec{\nabla} \times \vec{F} = (m + U_s)\vec{H} \]  \hspace{1cm} (42)  
\[ i\vec{\nabla} \times \vec{H} = (m + U_s)\vec{F} - (E - U_v^0)\vec{G} \]  \hspace{1cm} (43)  
\[ \vec{\nabla}\phi = (m + U_s)\vec{G} - (E - U_v^0)\vec{F} \]  \hspace{1cm} (44)  

Using the standard procedure developed in [54], we put

\[ \phi = \phi_{nj}(r)Y_j^{m_j}(\theta, \varphi) \]  \hspace{1cm} (45)  
\[ \vec{F} = \sum_l F_{njl}(r)\vec{Y}_{jlm}(\theta, \varphi) \]  \hspace{1cm} (46)  
\[ \vec{G} = \sum_l G_{njl}(r)\vec{Y}_{jlm}(\theta, \varphi) \]  \hspace{1cm} (47)  
\[ \vec{H} = \sum_l H_{njl}(r)\vec{Y}_{jlm}(\theta, \varphi) \]  \hspace{1cm} (48)  

where $\phi_{nj}$, $F_{njl}$, $G_{njl}$ and $H_{njl}$ are radial wave functions while $Y_j^{m_j}(\theta, \varphi)$ is the usual spherical harmonics of order $j$, and $\vec{Y}_{jlm}(\theta, \varphi)$ is the vector spherical harmonics. Then, using the notation

\[ F_{njj} = F_0, \quad F_{njj\pm 1} = F_{\pm 1} \]  \hspace{1cm} (49)  

and similar definitions for $G_0$, $G_{\pm 1}$, $H_0$ and $H_{\pm 1}$ together with the properties of vector spherical harmonics (see Appendix B), we can get a set of first-order coupled differential radial equations. The radial differential equation obtained from (41) are

\[ -\alpha_j \left( \frac{dG_{j+1}}{dr} + \frac{j+2}{2} r G_{j+1} \right) + \zeta_j \left( \frac{dG_{j-1}}{dr} - \frac{j-1}{2} r G_{j-1} \right) = (m + U_s)\phi \]  \hspace{1cm} (50)  

where $\alpha_j = \sqrt{(j+1)/(2j+1)}$ and $\zeta_j = \sqrt{j/(2j+1)}$.

From (42), we obtain

\[ \left( \frac{dF_0}{dr} - \frac{j}{2} F_0 \right) = -\frac{1}{\zeta_j} (m + U_s)H_{j+1} \]  \hspace{1cm} (51)  
\[ \left( \frac{dF_0}{dr} + \frac{j+1}{2} r F_0 \right) = -\frac{1}{\alpha_j} (m + U_s)H_{j-1} \]  \hspace{1cm} (52)  
\[ -\zeta_j \left( \frac{dF_{j+1}}{dr} + \frac{j+2}{2} r F_{j+1} \right) - \alpha_j \left( \frac{dF_{j-1}}{dr} - \frac{j-1}{2} r F_{j-1} \right) = (m + U_s)H_0 \]  \hspace{1cm} (53)  

The radial equation obtained from (43) are

\[ \left( \frac{dH_0}{dr} - \frac{j}{2} H_0 \right) = -\frac{1}{\zeta_j} ((m + U_s)F_{j+1} - (E - U_v^0)G_{j+1}) \]  \hspace{1cm} (54)  
\[ \left( \frac{dH_0}{dr} + \frac{j+1}{2} r H_0 \right) = -\frac{1}{\alpha_j} ((m + U_s)F_{j-1} - (E - U_v^0)G_{j-1}) \]  \hspace{1cm} (55)
\[- \zeta_j \left( \frac{dH_{+1}}{dr} + \frac{j + 2}{r} H_{+1} \right) - \alpha_j \left( \frac{dH_{-1}}{dr} - \frac{j - 1}{r} H_{-1} \right) = (m + U_s) F_0 - (E - U^0_v) G_0 \] (56)

Finally, from eq. (44) we get
\[ (E - U^0_v) F_0 = (m + U_s) G_0 \] (57)

\[ \left( \frac{d\phi}{dr} - \frac{j}{r} \phi \right) = -\frac{1}{\alpha_j} \left( (m + U_s) G_{+1} - (E + U^0_v) F_{+1} \right) \] (58)

\[ \left( \frac{d\phi}{dr} + \frac{j + 1}{r} \phi \right) = \frac{1}{\zeta_j} \left( (m + U_s) G_{-1} - (E - U^0_v) F_{-1} \right) \] (59)

At this stage is worthwhile to mention that the equations (50)-(59) are different from those given in [54].
Appendix B: The vector spherical harmonics

The properties of the vector spherical harmonics used in this work are obtained from [55]. The list of properties is the following

\[ \hat{r} \hat{Y}^{m_j} = -\alpha_j \vec{Y}_{j+1 m_j} + \zeta_j \vec{Y}_{j-1 m_j} \]  \tag{60}

\[ \vec{\nabla} \hat{Y}^{m_j} = \alpha_j \frac{j}{r} \vec{Y}_{j+1 m_j} + \zeta_j \frac{j+1}{r} \vec{Y}_{j-1 m_j} \]  \tag{61}

\[ \vec{\nabla} \cdot (f(r) \vec{Y}_{j+1 m_j}) = -\alpha_j \left( \frac{df}{dr} + \frac{j+2}{r} f \right) Y_j^{m_j} \]  \tag{62}

\[ \vec{\nabla} \cdot (f(r) \vec{Y}_{j-1 m_j}) = \zeta_j \left( \frac{df}{dr} - \frac{j-1}{r} f \right) Y_j^{m_j} \]  \tag{63}

\[ \vec{\nabla} \times (f(r) \vec{Y}_{j+1 m_j}) = i \zeta_j \left( \frac{df}{dr} - \frac{j}{r} f \right) \vec{Y}_{j+1 m_j} + i \alpha_j \left( \frac{df}{dr} + \frac{j+1}{r} f \right) \vec{Y}_{j-1 m_j} \]  \tag{64}

\[ \vec{\nabla} \times (f(r) \vec{Y}_{j-1 m_j}) = i \alpha_j \left( \frac{df}{dr} - \frac{j-1}{r} f \right) \vec{Y}_{j-1 m_j} \]  \tag{65}

\[ \hat{r} \times \vec{Y}_{j+1 m_j} = i \zeta_j \vec{Y}_{j+1 m_j} \]  \tag{66}

\[ \hat{r} \times \vec{Y}_{j-1 m_j} = i \zeta_j \vec{Y}_{j-1 m_j} \]  \tag{67}

\[ \hat{r} \cdot \vec{Y}_{j+1 m_j} = -\alpha_j Y_j^{m_j} \]  \tag{68}

\[ \hat{r} \cdot \vec{Y}_{j-1 m_j} = \zeta_j Y_j^{m_j} \]  \tag{69}

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