Localized exact solutions of $\mathcal{PT}$ symmetric nonlinear Schrödinger equation with space and time modulated nonlinearities

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Abstract

Using canonical transformations we obtain localized (in space) exact solutions of the nonlinear Schrödinger equation (NLSE) with space and time modulated nonlinearity and in the presence of an external potential depending on space and time. In particular we obtain exact solutions of NLSE in the presence of a number of non Hermitian $\mathcal{PT}$ symmetric external potentials.

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I. INTRODUCTION

Solitons are solitary waves which preserve their shapes during propagation as well as after collisions. They emerge as solutions of various nonlinear equations, the nonlinear Schrödinger equation (NLSE) (in different contexts also being called Gross-Pitaevskii equation) being one of them. Over the years NLSE has found applications in a wide range of fields [1]. In particular, it plays a crucial role in Bose-Einstein condensation [2] and nonlinear optics [3, 4]. It may be noted that the nonlinearities may be of various types e.g., cubic, quartic, quintic etc and the couplings or strength management associated with these nonlinearities may either be a constant or may depend on space, time or both. Here we shall be concerned with a NLSE with cubic nonlinearity (CNLSE) and strength management depending on space and time.

In recent years non Hermitian quantum mechanics has been studied in great detail. The Hamiltonians of many of these systems, in particular the $\mathcal{PT}$ symmetric [5] and the $\eta$ pseudo Hermitian [6] ones admit real eigenvalues despite being non Hermitian. The possibility of realizing $\mathcal{PT}$ symmetric structures in various fields e.g., optics has been suggested [7]. It may be mentioned that lately the study of NLSE with complex $\mathcal{PT}$ symmetric potentials has drawn a lot of attention because of intrinsic interest as well as for possible applications. In particular, soliton solutions of the NLSE with constant nonlinearity and complex potentials have been obtained by many authors [8]. In this context it may also be mentioned that schemes of experimental observation of $\mathcal{PT}$ symmetry has also been reported [9].

Here our objective is to obtain exact localized solutions of NLSE with space-time modulated nonlinearities [10] and in the presence of complex $\mathcal{PT}$ symmetric potentials. However, in general obtaining exact solutions of NLSE with space-time modulated nonlinearities are considerably more difficult in comparison to NLSE with constant nonlinearities. On the other hand, many solutions of NLSE with constant nonlinearity are known to exist. So, as a method of solution we shall employ canonical transformation to map NLSE with space-time modulated nonlinearity and an external potential onto a NLSE with constant nonlinearity and complex time independent $\mathcal{PT}$ symmetric potential for which exact solutions are known. This method was recently employed [11] to reduce a NLSE with space and time modulated cubic and quintic nonlinearities into a stationary NLSE with constants coefficients. Here, once we are interested to reach a stationary NLSE with a complex $\mathcal{PT}$ symmetric potential, we resort to a slight modification of the method due to [12].

The organization of the paper is as follows: in section II we discuss the method to transform NLSE with variable nonlinearities to one with constant nonlinearity and a time independent non
Hermitian potential; in section III we use the formalism described in section II to obtain solutions of NLSE with $\mathcal{PT}$ symmetric external potentials and finally section IV is devoted to a conclusion.

II. METHOD OF SOLUTION

In this section we present the method of solution by focusing on the non-autonomous CNLSE, namely

\[
\frac{i}{\partial Z} \Psi + m(X, Z) \frac{\partial^2 \Psi}{\partial X^2} + v(X, Z) \Psi + g_3(X, Z) |\Psi|^2 \Psi + i w(X, Z) \Psi = 0, \tag{1}
\]

where $\Psi = \Psi(X, Z)$, $m(X, Z)$ is the dispersion parameter, $v(X, Z)$ is the trapping potential, $g_3(X, Z)$ is the strength management of the cubic nonlinearity and $w(X, Z)$ is the gain/loss coefficient. The specific forms of $m(X, Z), v(X, Z), g_3(X, Z)$ and $w(X, Z)$ are taken to be

\[
m(X, Z) = \frac{\zeta(Z)}{(\gamma(Z) h)^2}, \tag{2}
\]

\[
v(X, Z) = \omega_1(X, Z) X^2 + f_1(X, Z) X + f_2(X, Z) + \zeta(Z) V[F(h)] - \frac{\zeta(Z)}{h^2} \left( \left( \frac{h \xi}{2h} \right)^2 - \frac{d}{d\xi} \left( \frac{h \xi}{2h} \right) \right), \tag{3}
\]

\[
g_3(X, Z) = g \zeta(Z) h, \tag{4}
\]

\[
w(X, Z) = f_3(X, Z) X + f_4(X, Z) + \zeta(Z) W[F(h)], \tag{5}
\]

where $h$ is an invertible, differentiable and positive function: $h = h[\gamma(Z) X + \delta(Z)]$, $h_\xi = \frac{dh(\xi)}{d\xi}|_{\xi=\gamma(Z) X + \delta(Z)}$ and $F(h)$ is a function of $h[\xi]$. The reason for choosing the inhomogeneous coefficients and potential in this way is going to be clarified below.

We now perform the following coordinate transformation and time rescaling

\[
X = \frac{\xi}{\gamma(z)} - \frac{\delta(z)}{\gamma(z)}, \quad Z - Z_0 = \int_0^z \frac{dz'}{\xi(z')}, \tag{6}
\]

with $\gamma(z) = \gamma(Z)$, $\delta(z) = \delta(Z)$, $\xi(z) = \zeta(Z)$ and, in order to remove the first derivative of $\Psi$ with respect to $\xi$ which arises from the transformation (5), we redefine the wavefunction $\Psi$ as

\[
\Psi[X(\xi, z), Z(z)] = e^{-i\alpha(\xi, z)} \Phi(\xi, z), \tag{7}
\]
where $\alpha (\xi, z) = -a(z) + \frac{1}{2} \int_0^\xi h^2(\xi') \left( \frac{\partial}{\partial \xi'} (\xi' - \bar{\delta}) + \bar{\delta} \right) d\xi'$, with $a(z)$ being an arbitrary function.

By substituting the Eqs. (2)–(7) into Eq. (1) one gets

$$
\begin{align*}
\iota \frac{\partial \Phi}{\partial z} + \frac{\gamma}{2h} \frac{\partial^2 \Phi}{\partial \xi^2} &+ \left\{ \left( \omega_1 + \frac{\gamma}{4} \frac{h^2}{\gamma} \right) \left( \frac{\xi - \bar{\delta}}{\gamma} \right)^2 + \left( \frac{f_1 + \gamma}{4} \frac{h^2}{\gamma} \right) \left( \frac{\xi - \bar{\delta}}{\gamma} \right) \\
&+ f_2 + \frac{\gamma}{4} \frac{h^2}{\gamma} \left( \frac{\xi}{\gamma} \right)^2 + \frac{\xi}{\gamma} \frac{\partial}{\partial z} \left( -a + \frac{1}{2\xi} \int_0^\xi h^2(\xi') \left( \frac{\partial}{\partial \xi'} (\xi' - \bar{\delta}) + \bar{\delta} \right) d\xi' \right) + \iota \gamma V \left[ F(h) \right] \\
&- \frac{\gamma}{h^2} \left( \frac{h_\xi}{2h} \right)^2 - \frac{\partial}{\partial \xi} \left( -a + \frac{1}{2\xi} \int_0^\xi h^2(\xi') \left( \frac{\partial}{\partial \xi'} (\xi' - \bar{\delta}) + \bar{\delta} \right) d\xi' \right) \right\} \Phi + g \gamma h |\Phi|^2 \Phi + i \left\{ \left( f_3 - h_\xi \frac{\gamma}{2h} \frac{\partial}{\partial \xi} \right) \left( \frac{\xi - \bar{\delta}}{\gamma} \right) \right. \\
&+ f_4 - h_\xi \frac{\gamma}{h^2} \frac{\partial}{\partial z} \left( \frac{\gamma}{2\gamma} \right) + \iota \gamma W \left[ F(h) \right] \right\} \Phi = 0,
\end{align*}
$$

(8)

where $f_k = f_k(\xi, z) \; (k = 1, 2, 3, 4)$. From the last equation one can see why the factors involving $\gamma(\xi), \zeta(\xi)$ and $h \gamma(\xi) X + \delta(\xi)$ are present in the expressions of $V(X, Z), m(X, Z), g_3(X, Z)$ and why we have chosen the specific dependence of $h, V$ and $W$ on $\xi = \gamma(\xi) X + \delta(\xi)$. Now, one may choose $\gamma(z), \delta(z), \zeta(z)$ and $h(\xi)$ such that

$$
\begin{align*}
\omega_1 &= \frac{\gamma}{4} \frac{h^2}{\gamma} \frac{\gamma}{\gamma^2}, & f_1 &= \frac{\gamma}{4} \frac{h^2}{\gamma^2} \frac{\gamma}{\gamma^2}, \\
f_2 &= \frac{\gamma}{4} \frac{h^2}{\gamma^2} \frac{\gamma}{\gamma^2} - \frac{\gamma}{\gamma^2} \frac{\partial}{\partial z} \left( -a + \frac{1}{2\xi} \int_0^\xi h^2(\xi') \left( \frac{\partial}{\partial \xi'} (\xi' - \bar{\delta}) + \bar{\delta} \right) d\xi' \right), \\
f_3 &= \frac{h_\xi}{h} \frac{\gamma}{\gamma^2} \frac{\gamma}{\gamma^2} \text{ and } f_4 &= \frac{h_\xi}{h} \frac{\gamma}{\gamma^2} \frac{\gamma}{\gamma^2} + \frac{\gamma}{\gamma^2} \frac{\gamma}{\gamma^2}.
\end{align*}
$$

(9)

In terms of the original variables $(X, Z)$, the functions $\omega_i, f_i, \; i = 1, 2, 3, 4$ are given by

$$
\begin{align*}
\omega_1 (X, Z) &= \frac{\gamma}{4} \frac{h^2}{\gamma^2} \frac{\gamma}{\gamma^2} \frac{\gamma}{\gamma^2}, & f_1 (X, Z) &= \frac{\gamma}{4} \frac{h^2}{\gamma^2} \frac{\gamma}{\gamma^2} \frac{\gamma}{\gamma^2}, \\
f_2 (X, Z) &= \frac{\gamma}{4} \frac{h^2}{\gamma^2} \frac{\gamma}{\gamma^2} - \frac{\gamma}{\gamma^2} \frac{\partial}{\partial Z} \left( -a + \frac{1}{2\xi} \int_0^X h^2(\xi') \left( \gamma Z X' + \delta Z \right) dX' \right) - \frac{h^2}{4\xi(\xi)} \frac{\partial^2}{\partial Z^2}, \\
f_3 (X, Z) &= \frac{h_\xi}{h} \frac{\gamma}{\gamma Z} \text{ and } f_4 &= \frac{h_\xi}{h} \frac{\gamma}{\gamma Z} + \frac{\gamma Z}{2\gamma}.
\end{align*}
$$

(10)

where $\gamma Z = \frac{d\gamma}{dZ}, \delta Z = \frac{d\delta}{dZ}$ revealing the intrinsic connection between $\omega_1 (X, Z), f_k (X, Z) \; (k = 1, 2, 3, 4)$ on the functions $\gamma(\xi), \delta(\xi), \zeta(\xi)$ and $h [\xi = \gamma(\xi) X + \delta(\xi)]$. Thus, Eq. (8) takes the form

$$
\begin{align*}
\iota \frac{\partial \Phi}{\partial z} + \frac{\gamma}{2h} \frac{\partial^2 \Phi}{\partial \xi^2} + gh |\Phi|^2 \Phi + i W \left[ F(h) \right] \Phi + \\
+ \left\{ V \left[ F(h) \right] - \frac{1}{h^2} \left( \frac{h_\xi}{2h} \right)^2 - \frac{\partial}{\partial \xi} \left( \frac{h_\xi}{2h} \right) \right\} \Phi = 0,
\end{align*}
$$

(11)

and the wavefunction (1) is written as

$$
\Psi (X, Z) = e^{-i \alpha(X, Z)} \Phi [\xi(X, Z), \xi(Z)],
$$

(12)
where \( \alpha (X, Z) = -a (Z) + \frac{\gamma (Z)}{\xi (x)} \int_0^X h^2 (\xi') (\gamma Z X' + \delta Z) dX'. \)

Since we still have a NLSE with inhomogeneous nonlinearity, we are going to make further transformations in order to arrive at a NLSE with constant nonlinearity. For that we redefine \( \xi \) as a function of another variable \( x \)

\[
\xi - \xi_0 = \int_{x_0}^x dx' h'(x') ,
\]

where \( h(x) = h [\xi (x)] \) and \( x - x_0 = F(h) = \int_{\xi_0}^x h(\xi')d\xi' \). In order to remove the first derivative of \( \Phi \) with respect to \( x \) which arises from the transformation (13), we redefine the wavefunction \( \Phi \) as

\[
\Phi (\xi (x), z) = \frac{\psi (x, z)}{\sqrt{h(x)}} .
\]

By substituting Eqs. (13) and (14) into Eq. (11) one gets

\[
i \partial \psi \partial Z + m (Z) \partial^2 \psi \partial X^2 + v (X, Z) \psi + g_3 (Z) \psi |\psi|^2 \psi + i w (X, Z) \psi = 0,
\]

where \( \psi = \psi (x, z), V (x) = V [F(h)] \) and \( W (x) = W [F(h)] \). Note that the variable \( z \) can be formally identified with time \( t \) and Eq. (15) can be interpreted as a NLSE with complex potential \( V(x) + iW(x) \). We are now going to consider cases for which the potential \( V(x) + iW(x) \) is invariant under the \( \mathcal{PT} \) transformation, that is \( x \to -x, z \to -z \) and \( i \to -i \).

By returning to the original space-time coordinates \( (X, Z) \), the wavefunction can be obtained from Eqs. (14) and (12), that is

\[
\Psi (X, Z) = \frac{e^{-i \alpha (X, Z)}}{\sqrt{h [\xi (X, Z)]}} \psi (x, Z) .
\]

Thus, we have shown, by means of point canonical transformations, how the non-autonomous NLSE with cubic nonlinearity management, Eq. (1), can be mapped onto a NLSE with cubic homogeneous nonlinearity, Eq. (15).

A. Particular case \( h [\xi] = 1 \).

In this case the non-autonomous CNLSE (1) is written as

\[
i \partial \Psi \partial Z + m (Z) \partial^2 \Psi \partial X^2 + v (X, Z) \Psi + g_3 (Z) \psi |\psi|^2 \psi + i w (X, Z) \Psi = 0 .
\]

The coefficients and the potential reduce to

\[
m (Z) = \frac{\zeta (Z)}{\gamma^2 (Z)} ,
\]
\[ g_3(Z) = g \zeta(Z), \]  

\[ w(X, Z) = \frac{\gamma Z}{2\gamma} + \zeta(Z) W(\xi), \]  

\[ v(X, Z) = -\left( \frac{\gamma^2}{\zeta} + \frac{\partial}{\partial Z} \left( \frac{\gamma \gamma Z}{\zeta} \right) \right) \frac{X^2}{4} - \left( \frac{\gamma Z \delta Z}{\zeta} + \frac{\partial}{\partial Z} \left( \frac{\gamma \delta Z}{\zeta} \right) \right) \frac{X}{2} + \frac{da}{dZ} - \frac{\delta^2}{4 \zeta} + \zeta(Z) V(\xi), \]  

where \( \xi = \gamma(Z) X + \delta(Z) \). From Eq. (13) we deduce that \( \xi = x \). Thus, by means of the Eq. (6) and the wavefunction (16), which can be rewritten as  

\[ \Psi(X, Z) = \psi(x, Z) \exp \left\{ -i \left( \frac{\gamma \gamma Z}{4\zeta} X^2 + \frac{\gamma \delta Z}{2\zeta} X - a(Z) \right) \right\}, \]  

we can map the non-autonomous CNLSE with cubic nonlinearity management (17) onto a NLSE with cubic homogenous nonlinearity (15).

### III. Examples

In this section we present some examples. We deal with some specific cases for which the exact solutions. Explicitly, we take \( h[\xi] = e^{\xi^2/b^2}, \zeta(Z) = \gamma^2, \delta(Z) = a(Z) = 0 \) and \( \gamma(Z) = \frac{\gamma_0}{\varepsilon + \cos(\nu Z)} \) with \( |\varepsilon| > 1 \).

These functions are related to the dispersion parameter \( m(X, Z) \) and cubic nonlinearity \( g_3(X, Z) \) by Eqs. (2) and (11), that is  

\[ m(X, Z) = e^{-\frac{\nu_0^2}{2(\varepsilon + \cos(\nu Z))^2} X^2}, \quad g_3(X, Z) = \frac{\nu_0^2}{(\varepsilon + \cos(\nu Z))^2} e^{\frac{\nu_0^2}{2(\varepsilon + \cos(\nu Z))^2} X^2}. \]  

It is important to remark that with this choice and from Eqs. (6) and (13), we obtain the relationship between the original variables \((X, Z)\) and the variables \((x, z)\), namely

\[ x = \frac{b \sqrt{\pi}}{2} \text{Erfi} \left[ \frac{\gamma_0}{b(\varepsilon + \cos(\nu Z))} X \right], \]

\[ z = \gamma_0^2 \left( \frac{2 \varepsilon}{\nu (\varepsilon^2 - 1)^{3/2}} \arctan \left[ \frac{\varepsilon - 1}{\sqrt{\varepsilon^2 - 1}} \tan \left( \frac{\nu}{2} Z \right) \right] - \frac{\sin(\nu Z)}{\nu (\varepsilon^2 - 1)(\varepsilon + \cos(\nu Z))} \right) , \]

where Erfi is the imaginary error function [14]. Then, we notice that \((X, Z) \rightarrow (-X, -Z)\) implies into \((x, z) \rightarrow (-x, -z)\), such that we can establish the \( \mathcal{P} \mathcal{T} \) symmetry in the examples we are going to treat below.
In the particular case $h[\xi] = 1$, and with the same functions $\gamma(Z) = \frac{n}{\epsilon + \cos(\nu Z)}$, $\zeta(Z) = \gamma^2$, $\delta(Z) = a(Z) = 0$, the relationship between the original variables $(X, Z)$ and the variables $(x, z)$ are

$$x = \frac{\gamma_0}{\epsilon + \cos(\nu Z)} X,$$

$$z = \gamma_0 \left( \frac{2\epsilon}{\nu (\epsilon^2 - 1)^{3/2}} \arctan \left[ \frac{\epsilon - 1}{\sqrt{\epsilon^2 - 1}} \tan \left( \frac{\nu Z}{2} \right) \right] - \frac{\sin(\nu Z)}{\nu (\epsilon^2 - 1) (\epsilon + \cos(\nu Z))} \right).$$

Thus, when $(X, Z) \rightarrow (-X, -Z)$ then $(x, z) \rightarrow (-x, -z)$ too. Moreover, from Eqs. (18) and (19) we get

$$m(Z) = 1, \quad g_3(Z) = \frac{g \gamma_0^2}{(\epsilon + \cos(\nu Z))^2},$$

such that, from (23) and (24), one has: $m(X, Z) = m(-X, -Z)$ and $g_3(X, Z) = g_3(-X, -Z)$.

**A. Example 1**

As a first example, we consider a $\mathcal{PT}$ symmetric potential of Scarf II type [15]

$$V(x) = V_0 \text{sech}^2(x), \quad W(x) = W_0 \text{sech}(x) \tanh(x),$$

where the coupling constants satisfy the condition

$$W_0 \leq V_0 + 1/4.$$ (26)

Then Eq. (15), with $g = 1$, admits a solution of the form

$$\psi(x, z) = \sqrt{2 - V_0 + \left( \frac{W_0}{3} \right)^2} \text{sech}(x) e^{i \left( z + \frac{W_0}{3} \arctan[\sinh(x)] \right)},$$ (27)

corresponding to zero boundary condition at $x \rightarrow \pm \infty$. Substituting the Scarf II type potential, Eq. (25), in Eqs. (3) and (5) we obtain the trapping potential $v(X, Z)$

$$v(X, Z) = - \left( \frac{4\gamma_0^4 e^{-b^2(\epsilon + \cos(\nu Z))^2} X^2 + 3b^4 \nu^2 \sin^2(\nu Z) (\epsilon + \cos(\nu Z))^2 e^{b^2(\epsilon + \cos(\nu Z))^2} X^2}}{4b^4 (\epsilon + \cos(\nu Z))^4} X^2 + \frac{\gamma_0^2 e^{-b^2(\epsilon + \cos(\nu Z))^2} X^2}{b^2 (\epsilon + \cos(\nu Z))^2} - \frac{b^2 \nu^2 (\epsilon \cos(\nu Z) + \cos(2\nu Z))}{8\gamma_0^2} e^{b^2(\epsilon + \cos(\nu Z))^2} X^2 - 1 \right) + \frac{V_0 \gamma_0^2}{(\epsilon + \cos(\nu Z))^2} \text{sech}^2 \left[ \frac{\sqrt{\pi} b}{2} \text{Erfi} \left[ \frac{\gamma_0}{b(\epsilon + \cos(\nu Z))} X \right] \right],$$

(28)
and the gain/loss coefficient \(w(X, Z)\)

\[
w(X, Z) = \frac{2\gamma_0^2 \nu \sin (\nu Z)}{b^2 (\epsilon + \cos (\nu Z))^3} X^2 + \frac{\nu \sin (\nu Z)}{2 (\epsilon + \cos (\nu Z))} + \frac{W_0 \gamma_0^2}{(\epsilon + \cos (\nu Z))^2} \times \\
\times \text{sech} \left[ \frac{\sqrt{\pi} b \text{Erfi}}{2} \frac{\gamma_0}{b (\epsilon + \cos (\nu Z))} X \right] \text{tanh} \left[ \frac{\sqrt{\pi} b \text{Erfi}}{2} \frac{\gamma_0}{b (\epsilon + \cos (\nu Z))} X \right]. \quad (29)
\]

The latter expressions satisfy the following equalities \(v(X, Z) = v(-X, -Z)\) and \(w(X, Z) = -w(-X, -Z)\), i.e. the trapping potential and gain/loss coefficient are even and odd respectively, with regard to sign reversal of \(X\) and \(Z\). Therefore, we can say that \(v(X, Z) + i w(X, Z)\) works as a complex \(\mathcal{PT}\) symmetric potential and, due to the specific choices of \(h[\xi], \gamma(Z), \zeta(Z)\) and \(\delta(Z)\), the non-autonomous NLSE in Eq. (11) is invariant under “time” \((Z \rightarrow -Z, i \rightarrow -i)\) and space \((X \rightarrow -X)\) reversals.

The wavefunction \(\Psi(X, Z)\), which is solution of Eq. (11), is obtained by substituting Eq. (27) into Eq. (16)

\[
\Psi(X, Z) = \sqrt{2 - V_0 + \left(\frac{W_0}{3}\right)^2} \text{sech} \left[ \frac{\sqrt{\pi} b \text{Erfi}}{2} \frac{\gamma_0}{(\epsilon + \cos (\nu Z))} X \right] \times \\
\times e^{-\frac{\nu^2}{3b^2 (\epsilon + \cos (\nu Z))^2} X^2} e^{i\varphi(X, Z)},
\]

where

\[
\varphi(X, Z) = \int_0^Z \frac{\gamma_0^2}{(\epsilon + \cos (\nu Z'))} dZ' + \frac{W_0}{3} \arctan \left[ \sinh \left[ \frac{\sqrt{\pi} b \text{Erfi}}{2} \frac{\gamma_0}{(\epsilon + \cos (\nu Z))} X \right] \right] + \\
+ \frac{b^2 \nu (\epsilon + \cos (\nu Z)) \sin (\nu Z)}{8 \gamma_0^2} \left( e^{\frac{\nu^2}{3b^2 (\epsilon + \cos (\nu Z))^2} X^2} - 1 \right). \quad (31)
\]

The plot of \(|\Psi(X, Z)|^2\) is shown in Fig. (11).

In the case when \(h[\xi] = 1, m(Z) = 1\), the trapping potential \(v(X, Z)\) and the gain/loss coefficient \(w(X, Z)\) become

\[
v(X, Z) = \frac{\nu^2 (\cos (2\nu Z) - 3 - 2\epsilon \cos (\nu Z))}{8 (\epsilon + \cos (\nu Z))^2} X^2 + \frac{V_0 \gamma_0^2}{(\epsilon + \cos (\nu Z))^2} \text{sech}^2 \left[ \frac{\gamma_0 X}{\epsilon + \cos (\nu Z)} \right],
\]

\[
w(X, Z) = \frac{\nu \sin (\nu Z)}{2 (\epsilon + \cos (\nu Z))} + \frac{W_0 \gamma_0^2}{(\epsilon + \cos (\nu Z))^2} \text{sech} \left[ \frac{\gamma_0 X}{\epsilon + \cos (\nu Z)} \right] \text{tanh} \left[ \frac{\gamma_0 X}{\epsilon + \cos (\nu Z)} \right]. \quad (32)
\]

These last expressions also satisfy \(v(X, Z) = v(-X, -Z)\) and \(w(X, Z) = -w(-X, -Z)\) i.e. the trapping potential and gain/loss coefficient are even and odd respectively, with regard to \(X\) and \(Z\). Thus the potential \(v(X, Z) + i w(X, Z)\) is \(\mathcal{PT}\) symmetric. The plots of the \(v(X, Z)\) and \(w(X, Z)\) are shown in Fig. (2).
The wavefunction $\Psi(X, Z)$, which is solution of Eq. (17), is obtained from Eq. (22)
\[ \Psi(X, Z) = \sqrt{2 - V_0 + \left(\frac{W_0}{3}\right)^2 \text{sech} \left[ \frac{\gamma_0}{\epsilon + \cos(\nu Z)} X \right]} e^{i\varphi(X, Z)}, \] (33)
where
\[ \varphi(X, Z) = -\frac{\nu \sin(\nu Z)}{4(\epsilon + \cos(\nu Z))} X^2 + \int_0^Z \frac{\gamma_0^2}{(\epsilon + \cos(\nu Z'))^2} dZ' + \frac{W_0}{3} \arctan \left[ \frac{\gamma_0}{\epsilon + \cos(\nu Z)} X \right]. \]

The evolution and behavior of $|\Psi(X, Z)|^2$, in this case, is similar to that shown in Fig. (1).
B. Example 2

Now we consider a $\mathcal{PT}$ symmetric periodic potential of the form

$$V(x) = V_0 \text{sn}^2(x, k) + W_0^2 k^2 \text{sn}^4(x, k), \quad W(x) = W_0 \text{sn}(x, k) \left(4 \text{dn}^2(x, k) - 1 + k^2\right).$$

(34)

The solution $\psi(x, z)$ of Eq. (15), for such Jacobi periodic potentials and $g = 1$, is of the form

$$\psi(x, z) = \sqrt{V_0 + W_0^2 k^2 + W_0^2 + 2k^2} \text{cn}(x, k) e^{i[(V_0 + W_0^2 k^2 + 2k^2 - 1)z + W_0 \text{sn}(x, k)]},$$

(35)

and it exists in the branch $V_0 > -(W_0^2 k^2 + W_0^2 + 2k^2)$. By substituting the Jacobian periodic potentials, Eq. (34), in Eqs. (3) and (5) we obtain the trapping potential $v(x, z)$ and the gain/loss coefficient $w(x, z)$

$$v(X, Z) = -\left(\frac{4\gamma_0^4 e^{-\frac{2\gamma_0^2}{b^2(\epsilon + \cos(\nu Z))^2}X^2} + 3b^2 \nu^2 \sin^2(\nu Z)(\epsilon + \cos(\nu Z))^2 e^{\frac{2\gamma_0^2}{b^2(\epsilon + \cos(\nu Z))^2}X^2}}{4b^4(\epsilon + \cos(\nu Z))^4}\right)X^2 +$$

$$+ \frac{\gamma_0^2 e^{-\frac{2\gamma_0^2}{b^2(\epsilon + \cos(\nu Z))^2}X^2}}{b^2(\epsilon + \cos(\nu Z))^2} - \frac{b^2 \nu^2 (\epsilon \cos(\nu Z) + \cos(2\nu Z))}{8\gamma_0^2} \left(\frac{\gamma_0^2 e^{-\frac{2\gamma_0^2}{b^2(\epsilon + \cos(\nu Z))^2}X^2}}{b^2(\epsilon + \cos(\nu Z))^2} - 1\right) +$$

$$+ \frac{\gamma_0^2}{(\epsilon + \cos(\nu Z))^2} \left(V_0 + W_0^2 k^2 \text{sn}^2 \left[\frac{\sqrt{\pi}b}{\sqrt{\pi}b(\epsilon + \cos(\nu Z))}X\right], k\right) \times$$

$$\times \text{sn}^2 \left[\frac{\sqrt{\pi}b}{\sqrt{\pi}b(\epsilon + \cos(\nu Z))}X\right], k\right),$$

(36)

$$w(X, Z) = \frac{2\gamma_0^2 \nu \sin(\nu Z)}{b^2(\epsilon + \cos(\nu Z))^3}X^2 + \frac{\nu \sin(\nu Z)}{2(\epsilon + \cos(\nu Z))} + \frac{W_0^2 \gamma_0^2}{(\epsilon + \cos(\nu Z))^2} \text{sn} \left[\frac{\sqrt{\pi}b}{2} \text{Erfi} \left[\frac{\gamma_0}{b(\epsilon + \cos(\nu Z))}X\right], k\right] \times$$

$$\times \left(4d_4^2 \left[\frac{\sqrt{\pi}b}{2} \text{Erfi} \left[\frac{\gamma_0}{b(\epsilon + \cos(\nu Z))}X\right], k\right] + k^2 - 1\right).$$

The wavefunction $\Psi(X, Z)$, which is solution of Eq. (1) is obtained by substituting Eq. (36) into the Eq. (16)

$$|\Psi(X, Z)| = \sqrt{V_0 + W_0^2 k^2} \text{sn}^2(X, k) e^{-\frac{\gamma_0^2}{2b^2(\epsilon + \cos(\nu Z))^2}X^2} \times$$

$$\times \left|\text{cn} \left[\frac{\sqrt{\pi}b}{2} \text{Erfi} \left[\frac{\gamma_0}{b(\epsilon + \cos(\nu Z))}X\right], k\right]\right|. \quad \text{(37)}$$

The plot of the $|\Psi(X, Z)|^2$ is shown in Fig. 3.

In the particular case $h[\xi] = 1$ the trapping potential $v(X, Z)$ and the gain/loss coefficient


The plots of the $v(X,Z)$ and $w(X,Z)$ are shown in Fig. (4).

The wavefunction $\Psi(X,Z)$, which is solution of Eq. (17) is obtained by substituting Eq. (35) into the Eq. (39).

\[ |\Psi(X,Z)| = \sqrt{V_0 + W_0^2 k^2 + W_0^2 + 2k^2} \left| \text{cn} \left( \frac{\gamma_0}{\epsilon + \cos (\nu Z)} X, k \right) \right|. \]  

The plot of the $|\Psi(X,Z)|^2$ is shown in Fig. (5).

IV. CONCLUSION

In this paper we have presented a method of obtaining exact solutions of the NLSE with a cubic inhomogeneous nonlinearity and in the presence of an external potential. The method has been used to obtain localized exact solutions CNLSE with inhomogeneous nonlinearity in the presence...
FIG. 4: Periodic evolution of (a) the even trapping potential \( v(X,Z) \) and of (b) the odd gain/loss coefficient \( w(X,Z) \), Eq. \((38)\) with \( \nu = 2, \epsilon = 2, V_0 = 0.5, W_0 = 0.5, k = 0.4 \) and \( \gamma_0 = 2.5 \).

FIG. 5: Profile (a) and Density plot (b) of \( |\Psi|^2 \), Eq. \((39)\) with \( \nu = 2, \epsilon = 2, \gamma_0 = 2.5, V_0 = 0.5, W_0 = 0.5 \) and \( k = 0.4 \).

of a couple of \( \mathcal{PT} \) symmetric external potentials. However the method is general enough and can be applied to obtain exact solutions of other potentials if the solution associated to NLSE with constant cubic nonlinearity is known. We would like to mention that whether the original CNLSE Eq. \((1)\) will be \( \mathcal{PT} \) symmetric or not depends on several factors, the crucial among them being the choice of the function \( \gamma(Z) \). Here we have chosen a particular form of \( \gamma(Z) \) although there
are many other choices which would ensure $\mathcal{PT}$ symmetry of Eq. (1). In this context it may be noted that besides $\mathcal{PT}$ symmetric potentials, there are also $\eta$ pseudo Hermitian potentials for which the linear Schrödinger Hamiltonian has real eigenvalues [6]. We feel it would be interesting to try to find exact solutions of NLSE in the presence of external $\eta$ pseudo Hermitian potentials. Finally we would like to mention that it will be of interest to extend the present formalism to higher dimensional [17] or radially symmetric NLSE with variable strength management and in the presence of a $\mathcal{PT}$ symmetric potential.

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