The $q$-component static model: modeling social networks

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(Dated: February 15, 2022)

We generalize the static model by assigning a $q$-component weight on each vertex. We first choose a component $(\cdot)$ among the $q$ components at random and a pair of vertices is linked with a color according to their weights of the component $(\cdot)$ as in the static model. A $(1 - \varepsilon)$ fraction of the entire edges is connected following this way. The remaining fraction $\varepsilon$ is added with $(q + 1)$-th color as in the static model but using the maximum weights among the $q$ components each individual has. This model is motivated by social networks. It exhibits similar topological features to real social networks in that: (i) the degree distribution has a highly skewed form, (ii) the diameter is as small as and (iii) the assortativity coefficient $\varepsilon$ is as positive and large as those in real social networks with $\varepsilon$ reaching a maximum around $\varepsilon = 0.2$. PACS numbers: 89.65.-s, 89.75.Hc, 89.75.Da

I. INTRODUCTION

Recently there have been considerable efforts to understand complex systems in terms of random graph, consisting of vertices and edges, where vertices (edges) represent individuals (acquaintances or their interactions) [1, 2, 3, 4]. In such complex networks, the emergence of a power-law degree distribution, $P(k) \sim k^{-\gamma}$, $k \to \infty$, is an interesting feature. Such networks are called scale-free (SF) networks. To illustrate such SF behavior, many in silico models have been introduced, whose examples include the Barabási and Albert model [5], the Huberman and Adamic model [6], etc. In those models, the number of vertices grows with time.

The static model [7,8] is another type of in silico model designed to generate SF networks, where the number of vertices is fixed. Each vertex is indexed by an integer $i = 1, 2, \ldots, N$, and assigned its own weight $w_i$. Next, two different vertices $(i, j)$ are selected with probabilities equal to normalized weights, $w_i w_j / \sum_k w_k$ and $w_j w_i / \sum_k w_k$, respectively, and are connected via an edge unless one already exists. This process is repeated until $m < N$ edges are present in the system, so that the mean degree is $2m/N$. Then it follows that the degree distribution is SF with the exponent $\gamma = 1 + 2m/N$. Thus, tuning the parameter in $0 < 1 + 2m/N < 3$, we can obtain a continuous spectrum of the exponent in the range $2 < \gamma < 3$, for which the degree distribution has finite mean and diverging variance. Since the number of vertices does not grow, one may wonder if this model can be applied to evolving real world network. However, since the model network can be easily generated and exhibits little hump in the degree distribution, it can be useful to study many aspects of SF networks.

In this paper, we generalize the static model by allowing a $q$-component weight $(w_1^{(1)}, w_2^{(2)}, \ldots, w_q^{(q)})$ to each vertex. We suppose that the $i$-th component $w_i^{(1)}$ of a vertex $i$ represents its own weight or fitness to a subgroup $(\cdot)$, $(1 \leq i \leq q)$ in a society. For example, we suppose that two persons $i$ and $j$ are alumni of a high school, a subgroup $(\cdot)$. They would have different weights $w_i^{(1)}$ and $w_j^{(1)}$ in the subgroup $(\cdot)$, determined by their school activities. The person $i$ and another person $k$ are colleagues in a company, another subgroup $(\cdot)$. They have also different weights, $w_i^{(2)}$ and $w_k^{(2)}$, by their positions in the company, the subgroup $(\cdot)$. Then the person $i$ has weights $w_i^{(1)}$ and $w_i^{(2)}$ in different subgroups, which are not the same in general. We make an edge between the pair $(i, j)$ in one color representing the subgroup $(\cdot)$ and the pair $(i, k)$ in another color for the subgroup $(\cdot)$. Vertices in the system are connected with edges in $q$ different colors representing different subgroups. Subgroups are then connected each other by weak ties as explained later. Since our society comprises many different subgroups and a person can be acquainted with other people belonging to diverse subgroups, this generalized static model is useful for modeling social networks. Meanwhile, it is noteworthy that our model is reminiscent of the generalization of the Ising model to the $q$-component cubic model [9] in equilibrium spin systems. So far, there have been many attempts to explain the structures and the properties of social networks [10]. Recently, Watts et al. [11] introduced a hierarchical model for social network. In the model, individuals belong to groups that in turn belong to groups of groups and so on, creating a tree-structure.
like hierarchical structure of social organization. Here an individual can belong to more than one group, as a result of which the distance between two persons is shorter than the ultrametric distance between them. Such hierarchical model illustrates well the small-world property of social network as implied in the Milgram’s “six degrees of separation” \cite{12}. Another simple social network model, introduced by Newman and Park \cite{13}, is based on the concept of bipartite graph \cite{14} and community structure \cite{15}. This has an advantage toward explaining the non-trivial high clustering of real-world social networks. While our \( q \)-component model is similar to the hierarchical model and the community model in the spirit of dividing people into subgroups, however, we assign weights to each person for each subgroup, and connections are made following those weights.

Social network exhibits an interesting feature in the degree-degree correlation function, different from biological or information networks. Newman \cite{16} studied the degree-degree correlation in terms of the correlation function between the remaining degrees of the two vertices on each side of an edge, where the remaining degree means the degree of that vertex minus one. He introduced the assortativity coefficient \( r \), defined as

\[
 r = \frac{1}{\langle q \rangle} \sum_{jk} w_{jk} \langle q_{jk} \rangle ;
\]

where \( w_{jk} \) is the joint probability that the two vertices on each side of a randomly chosen link have \( j \) and \( k \) remaining degrees, respectively. \( q_{jk} \) is the normalized distribution of the remaining degree \( q_{jk} = (k + 1)p(k + 1) - jP(j) \), and \( \frac{1}{\langle q \rangle} = k^2 q_{jk}P(k) \). Interestingly, complex networks can be classified into three types, having \( r < 0 \), \( r = 0 \) and \( r > 0 \), called the disassortative, the neutral, and the assortative network, respectively \cite{16}. Most social networks are assortative as shown in TABLE I, while the Internet and the protein interaction network are dissortative. While many in silico models have been introduced, most of them are neutral. Thus it would be interesting to introduce an in silico model having the assortativity coefficient as positive and large as empirical values, which would enable one to understand a basic mechanism of social network formation. We will show that such assortative networks can be generated via the \( q \)-component static model.

### II. MODEL

The \( q \)-component static model network is constructed as follows. Initially, \( N \) vertices are present in the system, representing \( N \) people in a society. Each vertex is assigned a \( q \)-component weight \( w_{i}^{(1)}; \cdots ; w_{i}^{(q)} \), where \( i \) is the vertex index. \( w_{i}^{(1)} \), the \( 1 \)-th weight of a vertex \( i \) is given as \( \nu_{ij} \), where \( \nu_{ij} \) is the rank of the vertex \( j \) in the \( 1 \)-th subgroup. We take \( \ell_{1}, \cdots ; \ell_{N} \), \( q \) be a random permutation of the integers \( \ell_{1}, \cdots ; \ell_{N} \). \( q \) is also taken to be a real random number distributed uniformly in the range \( 1 \leq q \leq \frac{1}{2} \). In general, ranks of a person for different subgroups should be correlated in real society; however, we take them as independent in this work for simplicity. As the number of people \( N \) becomes large, the number of distinct subgroups \( q \) can increase in real world. Thus, we set \( q \) to be \( q = sN \) with \( s \leq 1 \). Then \( 1 \leq s \) is the average number of people belonging to one subgroup.

Edges are connected as follows: First, among the \( q \) components, we choose a component at random. Second, we choose two vertices \( (i, j) \) with probabilities equal to normalized weights, \( P_{1}^{(1)} \) \( w_{i}^{(1)} = \sum_{j} w_{ij}^{(1)} \) and \( P_{2}^{(2)} \) \( w_{ij}^{(2)} = \sum_{j} w_{ij}^{(2)} \), and attach an edge between them with the \( 1 \)-th color unless an edge in that color exists already. Edge color is distinct for each component. Note that the pair \( (i, j) \) can be connected via more than one edge in different colors. Edges in different component are distinguished by their own colors. The process of attaching such edges is repeated until \( 1 \leq \ell \leq m \) \( \frac{N}{q} \) edges are added to the system. \( \ell \) is a parameter between \( 0 \) and \( 1 \). We will see that \( m \) is related to the average degree. Since the component was chosen randomly, the number of edges in one color is \( \ell \frac{N}{q} \) on average.

To construct a minimal model mimicking social relations, we need elements playing the role of “weak ties” \cite{20}. So, to take into account of social relationships among people having different backgrounds, we suppose that additional social relationships are formed following the maximum weights among the \( q \) components each individual has. Let \( w_{ij}^{(1)} \) be the normalized maximum weight of vertex \( i \). Then two distinct vertices \( i \) and \( j \) are chosen with probabilities, \( w_{ij} = \frac{\max \{ \nu_{ij}^{(1)}; \cdots \nu_{ij}^{(q)} \} }{\sum_{j} \nu_{ij}^{(1)} \cdots \nu_{ij}^{(q)} \} } \), and are linked by a new color different from the previous \( q \) colors unless such an edge exists already. This process is repeated until \( \ell m N \) edges are formed. Edges formed by such maximum weights can be regarded as weak ties, introduced by Granovetter \cite{20} which play an important role in social

| Name            | \( N \) | \( \ell \) | \( c \) | \( d \) | \( r \) | Ref.       |
|-----------------|-------|-------|------|------|------|-----------|
| cond-mat        | 16,264| 5.85  | 6.628| 0.185|      |           |
| arXiv.org       | 52,909| 9.27  | 6.188| 0.363|      |           |
| Mathematics     | 78,835| 4.16  | 8.455| 0.672|      |           |
| Neuroscience    | 205,202| 11.79 | 5.532| 0.604|      |           |
| Video movies    | 29,824| 33.69 | 4.789| 0.222|      |           |
| TV miniseries   | 33,980| 73.04 | 3.845| 0.379|      |           |
| TV cable movies | 117,655| 55.48 | 3.796| 0.135|      |           |
| TV series       | 79,663| 118.44| 4.595| 0.529|      |           |

TABLE I: The size \( N \), the mean degree \( \langle k \rangle \), the diameter \( d \), and the assortativity coefficient \( r \) obtained under selected conditions of \( N, m \) and \( q \) with \( \ell = 0.2 \).

TABLE II: Typical simulation results of the diameter \( d \) and the assortativity coefficient \( r \) for a number of social networks.
networks, connecting different subgroups. We find that the assortativity coefficient is enhanced by the presence of such weak ties.

Networks constructed in this way have a N edges with $(q+1)$ colors representing internal structure of subgroups. So, some pair of vertices are linked by more than one edges in different colors, albeit such incidences are not so frequent when $q$ is large. However, when we measure the network properties such as the shortest pathways, the degree of a vertex, the assortativity coefficient, and so on, we regard those multiple edges as a single one. Thus the mean degree $\langle k \rangle$ is slightly less than $2m$ by about 5% for typical networks we consider below. A small size network constructed in this way is shown in FIG. 1.

III. SIMULATION RESULTS

We perform numerical simulations for various values of $q$, $f$, $m$ and $N$, and examine the diameter $d$ and the assortativity coefficient $r$ as functions of those parameters. Here the diameter is the average distance between a connected pair of vertices along the shortest pathways.

First, the shape of the degree distribution depends on the number of subgroups. For small $q$, the degree distribution follows a power law with $2.85$, however, for large $q$, it has a highly skewed form, approximately obeying a power law for a part of its range, and having an apparently exponential cutoff for larger $k$. The exponential cutoff for large $k$ originates from the randomness of ranks of each subgroup, and the SF behavior for intermediate $k$ does from the SF behavior of each subgroup. A similar crossover behavior can be found in real social networks, for example, the collaboration networks of physicists, biologists and movie stars [21].

Second, we examine the assortative coefficient $r$ as a function of $f$ for a fixed $N$ and several values of $m$ and $q$. As shown in FIG. 3, the assortativity coefficient exhibits a peak around $f = 0.2$, meaning that the connections among subgroups are mostly optimized. Thus we limit our further consideration to the case $f = 0.2$. Meanwhile, the diameter gradually decreases with increasing $f$ as shown in the inset of FIG. 3.

Third, we study the assortativity coefficient $r$ as a function of $N$ for various $m$ and $q$. It is likely that $r$ increases with increasing $N$ as $r \sim \ln \ln N$ as shown in FIG. 4, but it seems to saturate for larger $N$. It also increases with increasing $m$ and $q$ as shown in FIG. 4. Thus the $q$-component static model exhibits $r$ values as large as empirical values listed in TABLE I. Some numerical results of $r$ listed in TABLE II show a quantitative agreement with the ones obtained in real social networks.

Fourth, the diameter $d$ is investigated as a function of the number of vertices $N$ for various $m$ and $q$ as shown in FIG. 5. It is found that the diameter is proportional to $\ln \ln N$ as in the case of random graph, in which $d \sim \ln \ln N$ [22]. Furthermore, the diameter becomes smaller as $m$ increases, which is like the case of random graphs. However, the diameter is almost insensitive to $q$. To test the so-called “six degrees of separation”, we extrapolate the straight line in the semi-logarithmic plot of FIG. 5 to large $N$ for $m = 10$ and $q = N/100$. We obtain $d \sim 6\ln N = 10^9$ and $d \sim 6.8$ for $N = 10^9$, in reasonable agreement with the Milgram’s “six degrees of separation” [12]. Here the choice of $m = 10$ and $q = N/100$ comes from the facts that a person knows about 20 other people on average (see Chapter 5 of Ref. [23]), and there are about 100 members on average in a subgroup [24].
FIG. 6: Plot of the slopes of the fit lines are drawn for the eye. Inset: Plot of the clustering coefficient $C(N)$ vs degree $k$ for various $N$ with fixed $m = 2, q = N/20$ and $f = 0.2$.

Fifth, one of the properties well studied for complex networks is the clustering coefficient $C$, which is defined as the average over all vertices of the ratio of the number of triangles connected to a given vertex to the number of triples centered on that vertex. It is known that for the neutral network, the clustering coefficient behaves as $C(N) \propto N^{\frac{1}{3}}$. Thus when $q = 3, C(N) \propto N^{\frac{1}{4}}$. For the $q$-component static model, while $r$ is not close to zero, the rule of attaching edges is such that no explicit degree-degree correlation enters, so that it is natural to expect the behavior $C(N) \propto N^{\frac{1}{4}}$. Indeed, the measured behavior is close to the expected one as shown in FIG. 6, but the slope in FIG. 6 exhibits small deviations for smaller $q$. For neutral networks, it is known that the clustering coefficient of a vertex with $k$ is almost independent of degree $k$. Even in our case, we find that indeed $C(k)$ is independent of $k$ for different $N$ as shown in the inset of FIG. 6.

IV. CONCLUSIONS

We have introduced the $q$-component static model assigning a $q$-component weight to each vertex. The weight of a given component of a vertex mimics a weight or fitness of that person in that subgroup. Through this model, we obtained the diameter of the acquaintance network as small as the Milgram’s “six degrees of separation” and the assortativity coefficient as positive and large as empirical values for a variety of social networks. Moreover, we obtain the degree distribution in a skewed form, which is also similar to those of real world social networks. The clustering coefficients $C(N)$ and $C(k)$ behave as those of a neutral network, being due to the absence of intrinsic degree-degree correlation. Such deficiency of the present model may be improved by introducing the hierarchical structure among subgroups, or correlated ranks of a person for different subgroups.

Acknowledgments

The authors would like to thank K.-I. Goh for sharing his code for the static model and valuable discussions. This work is supported by the KOSEF Grant No. R14-2002-059-0100-0 in the ABRL program and BK21 program of Ministry of Education, Korea.

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