Chiral Spin Liquid and Quantum Criticality in Extended $S = 1/2$ Heisenberg Models on the Triangular Lattice.

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We investigate the $J_1$-$J_2$ Heisenberg model on the triangular lattice with an additional scalar chirality term and show that a chiral spin liquid is stabilized in a sizeable region of the phase diagram. This topological phase is situated in between a coplanar $120^\circ$ Néel ordered and a non-coplanar tetrahedrally ordered phase. Furthermore we discuss the nature of the spin-disordered intermediate phase in the $J_1$-$J_2$ model. We compare the groundstates from Exact Diagonalization with a Dirac spin liquid wavefunction and propose a scenario where this wavefunction describes the quantum critical point between the $120^\circ$ magnetically ordered phase and a putative $\mathbb{Z}_2$ spin liquid.

Introduction — The emergence of quantum spin liquids in frustrated quantum magnetism is an exciting phenomenon in contemporary condensed matter physics [1]. These novel states of matter exhibit fascinating properties such as long-range groundstate entanglement [2, 3] or anyonic braiding statistics of quasiparticle excitations, relevant for a potential implementation of topological quantum computation [4]. Only very recently such phases have been found to be stabilized in realistic local spin models [5–19].

Triangular lattice Heisenberg models are a paradigm of frustrated magnetism. Although the Heisenberg model with only nearest neighbour interaction is known to stabilize a regular $120^\circ$ Néel order [20–23] adding further interaction terms may increase frustration and induce magnetic disorder to the system. Experimentally, several materials with triangular lattice geometry do not exhibit any sign of magnetic ordering down to lowest temperatures [24–27]. These include for example the organic Mott insulators like $k \sim \text{(BEDT-TTF)}_2\text{Cu}_2(\text{CN})_3$ [24, 25] or $\text{EtMe}_2\text{Sb}[\text{Pd(dmit)}]_2$ [26, 27] and are thus candidates realizing spin liquid physics.

Historically Kalmeyer and Laughlin [28] introduced the chiral spin liquid (CSL) state on the triangular lattice. This state closely related to the celebrated Laughlin wavefunction of the fractional quantum Hall effect has recently been shown to be the ground state of several extended Heisenberg models on the kagomé lattice [5–7, 9]. The question arises whether a CSL can indeed be realized on the triangular lattice as originally proposed. In a recent study [10] this was shown for SU($N$) models for $N \geq 3$. In this paper we provide conclusive evidence that indeed the CSL is stabilized in a spin-$1/2$ Heisenberg model upon adding a further scalar chirality term $J_\chi \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)$ similar as in Refs. [6–8, 10]. Such a term can be realized as a lowest order effective Heisenberg Hamiltonian of the Hubbard model upon adding $\Phi$ flux through the elementary plaquettes [6, 29], either via a magnetic field or by introducing artificial gauge fields in possible cold atoms experiments [30, 31]. The coupling constants then relate to the Hubbard model parameters $t$ and $U$ as $J_1 \sim t^2/U$ and $J_\chi \sim \Phi t^3/U^2$ where $J_1$ (resp. $J_\chi$) is the nearest neighbour Heisenberg (resp. scalar chirality) coupling.

Another open question in frustrated magnetism of the triangular lattice is the nature of the intermediate phase in the phase diagram of the $S = 1/2$ Heisenberg model with added next-nearest neighbour couplings around $J_2/J_1 \approx 1/8$. Several authors [20, 32, 33] found a spin disordered state. Recently several numerical studies [34–38] proposed that a topological spin liquid state of some kind might be realized in this regime. The exact nature of this phase yet remains unclear. In this Letter we advocate the presence of a $O(4)\ast$ quantum critical point [39–41] separating the $120^\circ$ Néel order from a putative $\mathbb{Z}_2$ spin liquid. The diverging correlation length at this quantum critical point and the neighbouring first order phase transition into the stripy collinear magnetic ordered phase render the unambiguous identification of the intermediate spin liquid phase challenging however.

Model — We investigate the Heisenberg model with nearest and next-nearest neighbour interactions with an additional uniform scalar chirality term on the triangular lattice

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i \cdot \vec{S}_j + J_\chi \sum_{i,j,k \in \triangle} \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)$$

(1)

where we set $J_1 = 1$ and consider $J_2, J_\chi \geq 0$. Amongst a $120^\circ$ Néel order, a stripy and a tetrahedral magnetic order we

![FIG. 1. Approximate $T = 0$ phase diagram of the $J_1$-$J_2$-$J_\chi$ model on the triangular lattice, c.f. Eq. (1). The extent of phases is inferred from excitation spectra from ED on a periodic 36 sites triangular simulation cluster, see main text for details.](image-url)


find a CSL being realized in an extended region of the phase diagram in Fig. 1. A first study of the classical phase diagram for \( J_2 = 0 \) [20] found a three sublattice 120° Néel ordered groundstate for \( J_2 < 1/8 \) whereas for \( 1/8 < J_2 < 1 \) a two-parameter family of magnetic ground states with a four-site unit cell was found [32]. Two high-symmetry solutions within this manifold are a two-sublattice collinear stripy magnetic order breaking lattice rotation symmetry and a tetrahedral non-coplanar state with a uniform scalar chirality on all triangles. Taking into account quantum fluctuations by applying spin-wave theory, large-S perturbation theory and ED studies [20, 32, 33] the degeneracy is lifted by an order-by-disorder mechanism. The true quantum groundstate for \( J_2 \geq 0.18 \) exhibits stripy Néel order. Yet the behaviour of the system close to the classical phase transition point \( J_2 = 1/8 \) has not been fully understood.

**Phase diagram** — We performed ED calculations on a \( N_s = 36 \) sites simulation cluster with periodic boundary conditions to investigate ground state properties and order parameters of the model (1). We have also checked selected results on smaller clusters, but the \( N_s = 36 \) cluster is particularly well suited because this single cluster can harbour all phases which we were able to detect.

We present the approximate phase diagram in Fig. 1 based on the quantum numbers of the ground state level and the first excited state. The groundstate is always in the \( \Gamma . A 1 \) representation (except in the stripy phase where \( \Gamma . A 1 \) and the two \( \Gamma . E 2 \) sectors are almost degenerate). The geometry sector of the first excited state determines the phase. **Orange:** \( S = 1 \) \( K . A 1 \) (120° Néel light blue: \( S = 0 \) \( \Gamma . E 2 b \) (CSL), **Green:** \( S = 0 \Gamma . E 2 a \Gamma . E 2 b \) degenerate (Dirac/\( S/2 \) spin liquid), **Dark Blue:** \( S = 0 \Gamma . A 1 \), \( \Gamma . E 2 a \), \( \Gamma . E 2 b \) degenerate (stripic magnetic order), **Dark red/Light red:** \( S = 1 \) \( M . A 1 / S = 0 \Gamma . E 2 a \) (tetrahedral magnetic order) For the magnetically ordered phases these quantum numbers follow from a standard tower of states symmetry analysis [42, 43]. The spectral phase diagram is further corroborated by the analysis of relevant order parameters and variational energies of model wave functions, c.f. Fig. 2, where the agreement is striking. We now proceed to a detailed discussion of the phases and the corresponding order parameters.

120° Néel order: At the Heisenberg point \( J_2 = J_\chi = 0 \) the system exhibits 120° Néel order [33] for which the static spin structure factor \( S(q) = | \sum_j e^{iq\cdot r} \langle S_j \cdot S_0 \rangle |^2/N_s \) is peaked at the \( K \)-point in the Brillouin zone [44]. The Anderson tower of states for this ordering [21] yields spin-1 excitations with symmetry sectors \( K . A 1 \) and \( \Gamma . B 2 \). In the orange region in Fig. 1 the first excited state is a triplet and belongs to the \( K . A 1 \) representation. Here also the structure factor of the groundstate evaluated at the \( K \)-point is peaked, cf. Fig. 2. Thus this region determines the approximate extent of the 120° Néel phase.

Stripy order is characterized by spins aligned ferromagnetically along one direction of the triangular lattice and antiferromagnetically along the other two (c.f. illustration in Fig. 1). It breaks SU(2) spin rotation symmetry and discrete lattice rotation symmetry. According to [20, 32, 33] the stripy order is stabilized for \( J_\chi = 0 \) and \( J_2 \geq 0.18 \). The groundstate in the singlet sector is expected to be threefold degenerate with corresponding irreducible representations in the \( \Gamma . A 1 \) and the two dimensional \( \Gamma . E 2 \) singlet sectors [33]. The area where those three states are nearly degenerate is coloured dark blue in Fig. 1. The spin structure factor \( S(q) \) is peaked at the \( M \)-point for both the stripy and the tetrahedral phase [44]. We computed the nematic order parameter

\[
\mathcal{N} = \sum_{(i,j) \neq (0,1)} \left\langle \left( \vec{S}_0 \cdot \vec{S}_1 \right) \left( \vec{S}_i \cdot \vec{S}_j \right) \right\rangle_c
\]

as a sum over nearest neighbour connected dimer-dimer correlations where only parallel and non-overlapping dimer configurations are considered, Fig. 2. The region where this order parameter is large coincides with the dark blue region in Fig. 1, hence, the approximate extent of the stripy phase.

**Tetrahedral order** is a non-coplanar regular magnetic order [8, 44, 45]. The spins in a four-site unit cell are arranged in a way that their orientations form a tetrahedron and span a finite volume on each triangle. The expectation value of the scalar chirality is thus non-zero and uniform for the classical spin configuration. This implies significant summed scalar chirality correlations

\[
\mathcal{X} = \sum_{(i,j,k) \in \Delta} \langle \chi(0,1,2) \cdot \chi(i,j,k) \rangle
\]

where \( \chi(i,j,k) = S_i \cdot (S_j \times S_k) \) and the sum runs over all non-overlapping triangles. Tetrahedral order is classically degenerate with the stripy order when \( J_\chi = 0 \) and \( 1/8 < J_2 < 1 \) [20, 32, 33]. Therefore we expect that this state will be energetically favored over the stripy phase upon adding a scalar chirality term in the Hamiltonian. The tetrahedral state does
not break lattice rotation symmetry and the nematic order parameter (2) is relatively small in Fig. 2. We find a sharp transition between the nematic order parameter and scalar chirality correlations. Moreover a level crossing at a finite angle in the excitation spectra strongly indicates that this is a first order phase transition. Tower of states analysis predicts that the $S = 1$ levels belong to the irreducible representation $M.A.$ In most of the region where both the structure factor at $M$ and the summed scalar chirality correlations $X$ are large we find that the first excited level above the groundstate belongs to this representation. This region is coloured dark red in Fig. 1. Close to the stripy phase we observe that the first excited level is a $S = 0 \Gamma.E2a$ level, shown as the light red region in Fig. 1. We believe that this level is an artifact of the finite size sample and is related to the order by disorder mechanism. In neither of the groundstate correlation functions we can see a difference between the light red region and the red region and thus conclude that also this region belongs to the same tetrahedral phase.

Chiral Spin Liquids — are spin disordered chiral topological states. Hallmark features of this phase are the topology dependent ground state degeneracy, long-range entanglement, abelian anyonic excitations and gapless chiral edge modes. Several instances of this phase have recently been found in local spin models [5–15]. It has been understood that a representative lattice model wave function for the CSL is provided by Gutzwiller projected parton wavefunctions (GPWF) with a completely filled parton band with Chern number $\pm 1$ [7, 10, 46, 47]. We observe no strong magnetic structure peak in between the $120^\circ$ Néel order and the tetrahedral, cf. Fig. 2. Therefore a spin disordered state is formed in a sizeable intermediate region. The summed scalar chirality correlations $X$ in Fig. 2 are relatively large in this regime compliant with the fact that here a CSL with a uniform chirality is formed. We will now show conclusive evidence that this is indeed the case. We do so by constructing two GPWFs describing the two topological sectors of the chiral spin liquid on the torus and by computing their overlaps with the two lowest lying exact eigenstates from ED, similarly as in refs. [7, 10].

In Fig. 3 we show energy spectra for a horizontal cut in the phase diagram at $J_{\chi} = 0.24$. The first excited level above the groundstate for $J_2 \lesssim 0.16$ belongs to the irreducible representation $\Gamma.E2b$. The region where this representation is the first excited state is colored light blue in Fig. 1. The parton tight binding model for the GPWFs we choose has a two-site unit cell on the triangular lattice with $\pi/2$ flux through the triangles. This yields a bandstructure with two bands with Chern numbers $\pm 1$. The groundstate of this tight binding model at half filling is given by filling the orbitals of the lower band. After Gutzwiller projection such a state has been shown to yield a CSL wavefunction [7, 10, 48, 49]. To construct the topological partner of the CSL wavefunction the phases in the tight-binding model before projection can be tuned such that locally the flux through each triangle remains $\pi/2$ while the flux through incontractible loops around the torus changes. The set of fluxes can be chosen arbitrarily, yet after Gutzwiller projection these states only form a two dimensional space. This can be verified by computing the overlap matrix for several GPWFs with different fluxes through the torus. Indeed we find that thereby the rank of the overlap matrix is 2 with a numerical precision of $\sim 10^{-3}$ [48]. We chose two out of these wave functions spanning the CSL subspace and compute the overlaps with the lowest two numerical eigenstates from ED. We find that these two model wave functions $|\psi_{GW}^{\alpha}\rangle$ yield very high overlaps

$$O_{GW-ED}^{\alpha} = \left| \langle \psi_{ED}^{\alpha} | \psi_{GW}^{\alpha} \rangle \right|^2 + \left| \langle \psi_{ED}^{1} | \psi_{GW}^{\alpha} \rangle \right|^2$$

with the two lowest lying eigenstates of ED of up to 0.92 [50]. In Fig. 3 we plot the square overlap $|\langle \psi_{ED}^{\alpha} | \psi_{GW}^{\alpha} \rangle|^2$ with the respective exact eigenstate $(n)$ as the diameter of the red ($\alpha = 1$) and light blue ($\alpha = 2$) circles. The overlaps are large where the first excited state is in the $\Gamma.E2b$ representation and quickly decay afterwards. This region coincides approximately with the region where the CSL model wave function has a low variational energy in the upper right panel of Fig. 2. We note that the CSL phase in this phase diagram is located near a tetrahedral magnetic phase, reminiscent of a recent study of a frustrated honeycomb spin model [8]. It would be interesting to investigate the nature of the phase transitions from the tetrahedral [8] and the $120^\circ$ Néel phases into the CSL. Finally a recent purely variational study [49] also found evidence for a CSL in our model for selected values of $J_2$ and $J_{\chi}$.

Spin disordered state in the $J_1-J_2$ Heisenberg model — We now turn to the time-reversal invariant $J_1-J_2$ line with $J_{\chi} = 0$. A number of recent numerical works [34–38] involving flavors of variational Monte Carlo (VMC) [34, 38] and Density Matrix Renormalization Group (DMRG) techniques [35–37] found a spin disordered region between the $120^\circ$ magnetic order region and the stripy magnetic order at larger $J_2/J_1$. Multiple candidate phases for this intermediate parameter range have been proposed, without a consensus so far. Whereas Ref. [35] proposes a gapped spin li-
The state expectations for the region the first few levels are in agreement with the tower of order \[33\] \[52\].

In the spin liquid region seem to be states which become the comes from the observation that some of the low-lying levels not the case for our system size. An additional complication structure features gapless Dirac-like excitations before projec-

The maximum ground state overlap is attained at \( J_2 = 0.1 \). The correlations decay algebraically over distance.

Given the rather low variational energy of the DSL and to a lesser extent of the CSL model wave functions as shown in the right part of Fig. 2 (and for the DSL in Refs. \[34, 38\]) we also compute the decomposition of these model wave functions onto the exact ED eigenstates for \( J_2 = 0.12 \) and \( J_2 = 0.15 \), as shown in the lower part of Fig. 4. At \( J_2 = 0.12 \) the ground state has a sizeable overlap with the DSL model wave function of 0.56. Furthermore when going up to energies of about 0.6, we can also find four states which have non-vanishing overlap with the two different topological sectors of the CSL model wave functions, although the integrated weight is lower than for the DSL state. This might explain the findings of Ref. \[37\] and is due to the reported CSL stabilized at finite but small \( J_\chi \).

In the future one should also explore overlaps with a \( \mathbb{Z}_2 \) spin liquid model wave function in order to address the propensity to this kind of spin liquid on an equal footing the other model wave functions.

We have then explored the overlap of the exact ED ground state with the DSL model wave function in a larger range of \( J_2 \) couplings and observe the overlap to be maximal in the vicinity of the putative 120° Néel to spin liquid quantum phase transition around \( J_2 \sim 0.08 \) in Fig. 5. Motivated by this observation we have explored correlation functions in the DSL model wave function and we find likely power-law correlation functions which peak at the \( K \) point in reciprocal space (consistent with Refs. \[34, 38\]). We also investigated the spin vector chirality (twist) correlations and find them to exhibit likely power-law correlations with a real space pattern in agreement with the (ordered) pattern in the 120° Néel ordered phase cf. Fig. 5.

These nontrivial observations motivate us to conjecture that the DSL wave function should not be considered as a model wave function for an extended ASL state, but instead as a lattice wave function correctly describing the long-distance properties the quantum critical point out of the 120° Néel state into a spin liquid. The \( O(4)^* \) theory \[39–41\] is a strong contender describing this transition. Let us put this advocated picture into a broader context: It is believed that Gutzwiller projected wave functions of partons with \( SU(N) \) symmetry and a band structure with \( n_D \) Dirac points correspond to a lattice realization of \( QED_3 \): i.e. \( N_f = N \times n_D \) two-component

FIG. 4. ED spectra for \( J_\chi = 0 \) and spectral decomposition of several model wavefunctions for \( J_2 = 0.12 \) and \( J_2 = 0.15 \). Full (empty) symbols correspond to even (odd) spin. The diameter of the poles is proportional to the squareoverlap \( \langle \psi_{\text{ED}} | \psi_{\text{Model}} \rangle^2 \). Besides the CSL and Dirac spin liquid wavefunctions the three wavefunctions denoted by \( \Gamma.A1, \Gamma.A2 \) and \( \Gamma.E2 \) are the groundstates in the respective symmetry sectors at \( J_2 = 0.3 \).

FIG. 5. Overlaps of DSL wavefunction with ED eigenstates and decay of spin-spin and twist-twist correlation functions \( \langle \hat{S}_0 \times \hat{S}_1 \cdot (\hat{S}_i \times \hat{S}_j) \rangle \) of the DSL from VMC on a 144 sites lat-tice. The maximum ground state overlap is attained at \( J_2 = 0.1 \). The correlations decay algebraically over distance.

In order to shed light on this open question we present the detailed energy spectrum of the \( N_s = 36 \) site cluster along the \( J_\chi = 0 \) line in the top panel of Fig. 4. In the small \( J_2 \) region the first few levels are in agreement with the tower of state expectations for the 120° Néel state \[21\], and similarly at the largest \( J_2 \) values shown for the stripy collinear magnetic order \[33\] \[52\].

Focusing on the intermediate region \( 0.08 \lesssim J_2 \lesssim 0.16 \) we would expect to see an approximate four-fold ground state degeneracy in either a non-chiral \( \mathbb{Z}_2 \) spin liquid or two time-reversal related copies of a CSL as in Refs. \[5, 7\]. This is not the case for our system size. An additional complication comes from the observation that some of the low-lying levels in the spin liquid region seem to be states which become the ground state or low-lying levels in the stripy collinear region across the first order transition around \( J_2 \sim 0.16 \). This illustrated by calculating overlaps of several low-lying eigenstates at \( J_2 = 0.3 \) with the eigenstates at \( J_2 = 0.12 \) \( J_2 = 0.15 \) displayed in the lower left (right) panel of Fig. 4.
Dirac fermions coupled to a compact \(U(1)\) gauge field in \(2+1\) D. It has been shown that in the limit of sufficiently large \(N_f\) there are no relevant operators in the theory [53, 54], and therefore this wave function is representative for an extended ASL region at large \(N_f\). For small \(N_f < N_f^c\) on the other hand one expects QED\(_3\) to become confining in general. The DSL wave function with its power-law decaying correlation functions could then describe a (multi)critical conformal field theory fixed point in between confining phases. The precise value for \(N_f^c\) is not known, although recent work [55] bounds \(N_f^c \lesssim 10\). In the particular case of the DSL on the triangular lattice we have \(N = 2\) and \(n_D = 2\) resulting in \(N_f = 4\), substantially lower than the presently known bound. There is also an earlier observation in Ref. [56] that a different \(N_f = 4\) DSL on the honeycomb lattice describes rather accurately the deconfined quantum critical point [57] between collinear Néel order and a VBS phase, giving further evidence that \(N_f = 4\) DSLs should perhaps be seen as fixed point wave functions for exotic quantum critical points.

The quantum critical scenario naturally comes with divergent correlation lengths, which could be an explanation for the so far missing clear ground state degeneracy both in DMRG and ED. Using couplings frustrating both the 120° and the stripy Néel orders, it might be possible to widen the spin-liquid region and to reduce the correlation lengths to numerically accessible scales, allowing to identify the spin liquid unambiguously. It would also be interesting to understand whether the CSL touches the \(J_z = 0\) line at the quantum critical point.

**Conclusion** — We established the phase diagram of an extended Heisenberg model on the triangular lattice. Amongst several magnetic orderings we found a chiral spin liquid phase in an extended region. For the spin disordered region for \(J_z = 0\) we found that the DSL has sizeable overlap with ED groundstates. We proposed a scenario where this wavefunction is the quantum critical wavefunction at a transition from magnetic 120° Néel order into a putative spin liquid phase.

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Supplementary material for Chiral Spin Liquid and Quantum Criticality in extended $S = 1/2$ Heisenberg Models on the Triangular Lattice.

Remarks on Exact Diagonalization calculations: The simulation cluster we use in our calculations is the $6 \times 6$ triangular lattice in Fig. 6 with sixfold rotational and reflection symmetry. Thus the pointgroup is the full dihedral group of order 12, D6. The momentum space in Fig. 6 of this cluster features the $K$ as well as the $M$ point and is thus capable of stabilizing 120°, stripy and tetrahedral order.

The Exact Diagonalization calculations are performed exploiting the full translational, pointgroup and spinflip symmetry. We thus work in the basis of generalized Bloch states of the form

$$|\vec{\sigma}_{\text{sym}}\rangle_{\pm\text{flip},\rho,\vec{k}} = \frac{1}{N_s} \sum_{s=\pm} \sum_{p \in LG_{\vec{k}}} \sum_{\vec{T}} \chi_{\pm\text{flip}}(s) \chi_{\rho}(p)e^{i\vec{k} \cdot \vec{T}} S \circ P \circ T \langle |\vec{\sigma}\rangle \rangle$$

where $\vec{k}$ is the momentum, $s$ denotes the spinflip operation, $\chi_{\pm\text{flip}}(s)$ the character of even or odd spinflip symmetry character, $p \in LG_{\vec{k}}$ a point group element in the little group of $\vec{k}$ and $\chi_{\rho}(p)$ the character of a representation $\rho$ of the little group. This gives a valid basis for all one dimensional representations $\rho$ of the pointgroup. The two dimensional representations are considered via working with a corresponding one dimensional representation of a reduced Little group. For further details see [58]. We use the standard Mulliken notation [59] for labeling the representations of the point group. For the dihedral group we have the representations A1, A2, B1, B2, E1, E2. A1 and A2 are trivial under rotations and A2 is odd under reflections. B1 and B2 are odd under 180° rotations. E1 and E2 are two dimensional representations which split up into the one dimensional E1a, E1b and E2a, E2b of the cyclic C6 subgroup corresponding to $\pm \pi/3$ angular momentum. The E1a, E2a and E1b, E2b representations are not degenerate without time-reversal symmetry as for $J_\chi > 0$ in the main text and are then considered separately.

Anderson Tower of states for magnetic orders Magnetic orderings break continuous SU(2) symmetry of the original Heisenberg model. The breaking of continuous SU(2) symmetry implies a so called Anderson tower of states [42] whose excitation energies collapse as $1/N_s$ to the groundstate energy in the thermodynamic limit. They then form the manifold of degenerate groundstates in the thermodynamic limit and appear as low lying excitations on finite cluster size energy spectra. The quantum numbers of these states can be predicted by group representation theory. For the 120° and stripy order this has been done in Refs. [33, 60]. The method we used to reproduce their results and calculate the tower of states for the tetrahedral order is presented in Appendix B of [43]. The irreducible representations of these states in our notation for small total spin $S$ is given in table I.

| $120^\circ$ Neél | stripy order | tetrahedral order |
|------------------|-------------|------------------|
| S                | Γ.A1        | Γ.B1            |
| 0 1 0 0          | 1 1 0      | 1 0 0 0         |
| 1 0 1 1          | 0 0 1      | 0 1 0 0         |
| 2 1 0 2          | 1 1 0      | 0 1 1 1         |
| 3 1 2 2          | 0 0 1      | 1 2 0 0         |

TABLE I. Multiplicities of irreducible representations in the Anderson tower of states for the three magnetic orders on the triangular lattice defined in the main text.

FIG. 7. Parton ansatz for the DSL Gutzwiller projected wavefunction and parton bandstructure as in Refs. [38, 51]. Solid (dashed) lines denote real hopping with amplitude $+1$ ($-1$).
FIG. 8. Spectral decomposition of several model wavefunctions on the $J_x = 0$ line. The diameter of the poles is proportional to the squareoverlap $|\langle \psi_{\text{ED}} | \psi_{\text{Model}} \rangle|^2$. Top Left: overlaps with Dirac spin liquid wavefunction. Top right: overlaps with the groundstate of the $\Gamma.A1$ sector. Bottom Left: overlaps with the groundstate of the $\Gamma.E2$ sector. Bottom Right: overlaps with the groundstate of the $\Gamma.A2$ sector.

FIG. 9. Comparison of connected dimer-dimer correlation functions and static spin structure factor between DSL and ED groundstate. We see good agreement for the dimer-dimer correlations but slight deviations in the spin structure factor due to the onset of stripy order in the ED groundstate.