FINITE TRAVELING WAVE SOLUTIONS IN A DEGENERATE CROSS-DIFFUSION MODEL FOR BACTERIAL COLONY WITH VOLUME FILLING

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Abstract. This work deals with the properties of the traveling wave solutions of a double degenerate cross-diffusion model
\[
\frac{\partial b}{\partial t} = D_b \nabla \cdot \{ n^p b(1 - b) \nabla b \} + n^q b',
\]
\[
\frac{\partial n}{\partial t} = D_n \nabla^2 n - n^q b',
\]
where \( p \geq 0, q > 1, l > 1 \). This system accounts for degenerate diffusion at the population density \( n = b = 0 \) and \( b = 1 \) modeling the growth of certain bacteria colony with volume filling. The existence of the finite traveling wave solutions is proven which provides partial answers to the spatial patterns of the colony.

In order to overcome the difficulty of traditional phase plane analysis on higher dimension, we use Schauder fixed point theorem and shooting arguments in our paper.

1. Introduction. Bacteria grown on the surface of thin agar plates develops colonies of various spatial patterns, such as fractal morphogenesis, dense-branching pattern, depending on both species and environmental conditions [7, 8]. Kawasaki et al. [12] proposed a degenerate parabolic system with cross diffusion that captures the qualitative features of the growth patterns. The model is as following:
\[
\frac{\partial b}{\partial t} = D_b \nabla \cdot \{ n^p b \nabla b \} + n b,
\]
\[
\frac{\partial n}{\partial t} = D_n \nabla^2 n - n b,
\]
with initial data
\[
b(x, y, 0) = b_0(x), \quad n(x, y, 0) = n_0,
\]
here \( b \) represents the bacterial density and \( n \) represents the nutrient density. \( b_0(x) \) is usually a compactly supported function and \( n_0 \) is a constant. \( D_b \) and \( D_n \) are positive constants that represent the diffusivity of bacterial and nutrient respectively.

When considering Model (1), (2) in one dimensional case with special case \( D_n = 0 \), Maini et al. [14] used phase plane analysis method to study the existence and uniqueness of the traveling wave solutions \((b(\xi), n(\xi))\), where \( \xi = x - ct \) is the usual

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wave coordinate. In their paper, they found that existence of the traveling wave solution depends on the traveling speed $c$ and the traveling wave with the minimum speed has sharp profile. However when the speeds are greater than the minimum speed, the traveling waves are smooth.

In 2006, J. W. Barrett and K. Deckelnick extended Model (1), (2) to the following case:

$$\frac{\partial b}{\partial t} = \nabla \cdot \{D(n)\nabla \psi(b)\} + \theta f(n)b,$$

$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - f(n)b,$$

(4)

(5)

here $f \in C_{loc}^{0,1}([0, \infty))$ is increasing with $f(0) = 0$, $D(n) \in C_{loc}^{1,1}([0, \infty))$ is only non-negative and $\psi \in C^1([0, \infty)) \cap C^2((0, \infty))$ is convex, strictly increasing with $\psi(0) = 0$ and possibly $\psi'(0) = 0$. In their case, they chose special functions $D(r) := r^q$ and $\psi(r) := r^p$ with $q \geq 2$ and $p \geq 1$. The possible degeneracy of the diffusion coefficients $D(n), \nabla \psi(b)$ make the analysis of Model (4), (5) difficult. In their paper, they established existence of a general solution to Model (4), (5) via a regularized method.

Recently, P. Feng and Z. Zhou considered the existence of the traveling wave solutions for the case:

$$\frac{\partial b}{\partial t} = D_b \nabla \cdot \{n^p b \nabla b\} + n^q b^l,$$

$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - n^q b^l,$$

(6)

(7)

where $p \geq 0, q > 1, l > 1$. In their paper, by using Schauder fixed point theorem and shooting arguments, they established the existence of the finite traveling wave solution to Problem (6), (7).

However, the movement of the bacterial may be affected by its space or volume in the neighborhood. In a recent paper by L. Bao and Z. Zhou, by using the technique of scaling of the population density and biased random walk in probability, they introduced the following single population model in one dimension:

$$b_t = [D(b)b_x]_x + g(b),$$

(8)

where $g(b)$ is a logistic like birth term and

$$D(b) \leq 0 \quad \text{for} \quad b \in [0, \alpha], \quad D(b) \geq 0 \quad \text{for} \quad b \in (\alpha, 1].$$

(9)

The same idea also appears in [1], [2] and [3].

As far as we know, high population density may cause the effect of slow diffusion which is not considered in the population systems so far. In our paper, we introduce the following population model which accounts for the slow diffusion both at low and high population density:

$$\frac{\partial b}{\partial t} = D_b \nabla \cdot \{n^p b(1 - b) \nabla b\} + n^q b^l,$$

$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - n^q b^l,$$

(10)

(11)

where $p \geq 0, q > 1, l > 1$. Here the cross diffusion coefficient $D(n,b) = n^p b(1 - b)$ means slow diffusion when the nutrient is poor, the bacterial density is small and large.
The paper is organized as follows. In section 2, we derive some properties of the traveling wave solutions such as the positivity of the speed \( c \) and the monotonicity of the traveling wave solutions. In section 3, we prove the existence of the traveling wave solution \( b \) for a given \( n \) when \( c \geq c^* \). In section 4, we study the existence of the traveling wave \( n \) for the determined traveling wave solution \( b \). In the last section, by using Schauder fixed point theorem, we derive the following existence of finite traveling wave solution result:

**Theorem 1.1.** There exists a constant velocity \( c^* \) such that the system \((10), (11)\) admits a unique traveling wave solution \((b(\xi), n(\xi))\) where \( b(\xi) \) is a finite monotone traveling wave solution and \( n(\xi) \) is a monotone classical traveling wave solution. Here \( \xi = x - ct \) is the usual wave coordinate and by finite traveling wave we mean

\[
\xi^* = \sup\{\xi : b(\xi) > 0\} < \infty.
\]

2. Preliminary results. We recall that the traveling wave solutions for the system \((10), (11)\) are solutions of the form \((b(x,t), n(x,t)) = (b(\xi), n(\xi))\), where \( \xi = x - ct \) for some constant traveling speed \( c \). The system we have to deal with is changed to

\[
D(n^p b(1 - b)b')' + cb' + n^q b = 0, \tag{12}
\]

\[
n'' + cn' - n^q b = 0, \tag{13}
\]

where \( ' \) stands for derivation with respect to the wave variable \( \xi = x - ct \). Let \( D_n = 1 \) and \( D_b = D \), which is the non-dimensional diffusion coefficient of bacteria. We denote the spatially uniform steady states by \((b, n) = (b_s, 0), (0, n_s)\), where \( b_s \) and \( n_s \) are some arbitrary constants. The laboratory set-up consists of a nutrient-enriched agar plate on which an initial inoculum of bacteria is placed. Therefore the appropriate steady state to be considered initially is the one given by

\[
n = 1, b = 0 \quad \text{for all} \quad -\infty < x < \infty.
\]

The biological interpretation of this condition is that the nutrient is at a uniform concentration level of 1 (non-dimensionalised) and there are no bacteria present \([15]\). We shall also assume that

\[
\frac{\partial n}{\partial x} \to 0 \quad \text{as} \quad |x| \to \infty, \quad \text{for all} \quad t > 0, \tag{14}
\]

\[
\lim_{\xi \to \xi_+} b(1 - b)b'(\xi) = \lim_{\xi \to \xi_-} b(1 - b)b'(\xi) = 0. \tag{15}
\]

Condition \((15)\) is motivated by the possible occurrence of sharp type or finite type profiles, that is, solutions reaching the equilibria at a finite value \( \xi_1 \) or \( \xi_2 \). However, when the existence interval is the whole real line, then Condition \((15)\) is automatically satisfied and the sharp type is reduced to the classical one which is the front type traveling wave solution.

As in \([15]\), the system \((10), (11)\) are to be solved subject to the following conditions:

\[
b(\xi), n(\xi) \geq 0, \quad -\infty < \xi < \infty,
\]

ahead of the wave

\[
b \to 0, n \to 1 \quad \text{as} \quad \xi \to +\infty, \tag{16}
\]

so that the wave is propagating into the fresh nutrient region of the plate while behind the wave

\[
b \to b_s, n \to n_s \quad \text{as} \quad \xi \to -\infty, \tag{17}
\]
Some properties of the traveling wave solutions. In this section, we derive some important properties of the traveling wave solutions which may be used in the proof of our main theorem.

**P1.** There are no traveling wave solutions with \( b(\xi) \equiv 0 \) or \( n(\xi) \equiv 1 \).

**Proof.** If not, there exists a traveling wave solution such that \( n(\xi) \equiv 1 \), then \( n' = n'' = 0 \). From Equation 13, we can get \( n^q b' = 0 \) which leads to the trivial solution \( b \equiv 0 \) and it is a contradiction. \( \square \)

**P2.** We require that \( b_s = 1 \) and \( n_s = 0 \).

**Proof.** In Equation 13, letting \( \xi \to -\infty \), we find \( n^q b' = 0 \) which leads to \( n_s = 0 \) or \( b_s = 0 \). If we integrate Equation 12 from \( -\infty \) to \( \xi \), we have

\[
Dn^p b(1 - b)b' + c(b - b_s) + \int_{-\infty}^{\xi} n^q b' = 0. \tag{18}
\]

Let \( \xi \to +\infty \), we have \( cb_s = \int_{-\infty}^{\infty} n^q b' dt > 0 \) and \( b_s \neq 0 \). So we get \( n_s = 0 \). Now we integrate Equation 13 from \( -\infty \) to \( \xi \)

\[
\int_{-\infty}^{\xi} n^q b' = n'(\xi) + cn(\xi) \tag{19}
\]

Substituting Equation 19 to 18 and passing \( \xi \to +\infty \), we have

\[
-cb_s + c = 0, \tag{20}
\]

which leads to \( b_s = 1 \). From \( cb_s = \int_{-\infty}^{\infty} n^q b' > 0 \), we obtain \( c = \int_{-\infty}^{\infty} n^q b' > 0 \). \( \square \)

**P3.** If \( n, b \) are the traveling wave solutions, then \( n \) is monotone increasing and \( b \) is monotone decreasing if \( 0 < b(\xi) < 1 \).

**Proof.** For the traveling wave solution \( b(\xi) \), we prove that

\[
b'(\xi) < 0 \quad \text{for} \quad 0 < b(\xi) < 1.
\]

Suppose there exist \( b'(\xi_0) = 0 \) for some \( \xi_0 \), then from Equation 12, we have

\[
\frac{d}{d\xi} [D(n^p b(1 - b)b')]|_{\xi = \xi_0} = -cb'(\xi_0) - n^q b'(\xi_0) < -cb'(\xi_0) \leq 0, \tag{21}
\]

hence \( D(n^p b(1 - b)b') \) is decreasing in a neighborhood of \( \xi_0 \). Since \( D(n^p b(1 - b)) > 0 \), we get \( b'(\xi) > 0 \) in \( (\xi_0 - \delta, \xi_0) \) for some \( \delta > 0 \). Let

\[
\overline{\xi} := \inf\{\xi : b'(\xi) \geq 0 \quad \text{for every} \quad t \in (\xi, \xi_0)\},
\]

\( \overline{\xi} < +\infty \). Since \( b \) is increasing on \( (\overline{\xi}, \xi_0) \) with \( \lim_{t \to -\infty} b(t) = 1 \), furthermore \( b'(\overline{\xi}) = 0 \), we can have the local minimum at \( \overline{\xi} \), which leads to

\[
b''(\overline{\xi}) = -\frac{n^q b'}{D(n^p b(1 - b))} < 0, \tag{22}
\]

which is a contradiction to the definition of minimum. \( \square \)

Now we prove the monotonicity of \( n \). From Equation 13, we obtain

\[
(n'e^{\xi})' \geq 0
\]

and \( n'e^{\xi} > 0 \) by integrating the inequality from \( -\infty \) to \( \xi \). Hence \( n' > 0 \) for all \( \xi \).
3. The existence of \( b \) for a given \( n \). In this section, we will prove the existence of the traveling wave solution \( b \) for any given \( n \). The strategies is using the monotonicity property of \( b \) for any given \( n \), then transforming the solvability of the problem to an equivalent singular boundary value problem. For the possibility of the finite or sharp type traveling wave solution for \( b \), we denote

\[
\xi_1 = \inf \{ \xi : b(\xi) < 1 \}, \quad \xi_2 = \sup \{ \xi : b(\xi) > 0 \},
\]

here \((\xi_1, \xi_2)\) is the minimum interval for \( 0 < b < 1 \). Because for the given \( a, b(\xi) \) is a decreasing function on \((\xi_1, \xi_2)\), so the inverse function \( \xi = \xi(b) \) is well defined on \((0, 1)\) and takes value on \((\xi_1, \xi_2)\). Therefore we may define

\[
n(b) := n(\xi(b)) \tag{23}
\]

and

\[
z(b) := Dn^p(b) b(1 - b)b' < 0 \quad \text{for all} \quad b \in (0, 1). \tag{24}
\]

Differentiating both sides of Equation (24) with respect to \( \xi \), we have

\[
z'(b) = \frac{dz}{db} = \left( Dn^p(b) b(1 - b)b' \right)' \frac{1}{b'(\xi)} \quad
\]

\[= \left( Dn^p(b) b(1 - b)b' \right)' \left( Dn^p(b) b(1 - b) \right) \frac{1}{z} \quad
\]

\[= -c - \frac{Dn^{p+q}(b) b^{l+1}(1 - b)}{z}. \tag{25}
\]

from (15) we can \( z(0^+) = z(1^-) \). Next we will consider the corresponding singular boundary value problem (25). Let \( n(b) \in V \) where \( V \) is the closed convex set of the Banach space \( C^\theta((0, 1)) \) defined by

\[
V := \{ n(b) \in C^\theta[0, 1], 0 \leq n \leq 1, \limsup_{b \to 1^-} \frac{n(b)}{1 - b} \leq L, n(b) \geq \frac{1}{L}(1 - b) \},
\]

where \( L \) is a sufficiently large constant and will be chosen later. We shall now consider the solvability of the singular problem \( (P^*) \)

\[
z'(b) = -c - \frac{Dn^{p+q}(b) b^{l+1}(1 - b)}{z} \quad \text{for} \quad b \in (0, 1), \tag{26}
\]

\[
z(0^+) = z(1^-) = 0, \tag{27}
\]

\[
z < 0 \quad \text{in} \quad (0, 1), \tag{28}
\]

here \( n(b) \in V \) and \( V \) is defined as above. We first introduce the following lemma.

**Lemma 3.1.** If there exists \( \nu \in C^\theta(0, 1) \) such that

\[
\nu'(b) > -c - \frac{Dn^{p+q}(b) b^{l+1}(1 - b)}{\nu} \quad \text{for} \quad b \in (0, 1), \tag{29}
\]

such that \( \nu(0^+) = 0 \) and \( \nu(b) < 0 \) for \( b \in (0, 1) \). Then Problem \( P^* \) is solvable and the solution satisfies \( \nu(b) < z(b) < 0 \)

**Proof.** First, for a fixed constant \( b_0 \in (0, 1) \), we have \( \nu(b_0) < 0 \) and let \( \alpha \in [\nu(b_0), 0) \). We consider the following initial value problem:

\[
z' = -c - \frac{Dn^{p+q}(b) b^{l+1}(1 - b)}{z} \quad \text{for} \quad b \in (0, 1),
\]

\[
z(b_0) = \alpha \geq \nu(b_0),
\]
and let $z$ represent the unique solution of the problem. We claim that $z(b) < 0$ on $(0, b_0)$. Suppose otherwise there is a $b^* \in (0, b_0)$ satisfying $z(b) < 0$ for all $b \in (b^*, b_0]$ and
\[
\lim_{b \to b^+} z(b) = 0.
\]
Since
\[
\lim_{(b, z) \to (b^*, 0)} (-cz - Dn^{p+q}(b)b^{1+l}(1 - b)) = -Dn^{p+q}(b^*)b^{1+l}(1 - b^*) < 0,
\]
hence there exists a constant $\lambda > 0$ such that
\[
z' = -c - \frac{Dn^{p+q}(b)b^{1+l}(1 - b)}{z} > \frac{-\lambda}{z(b)} > 0
\]
for all $b^* < b < b^* + \lambda$, which implies that $z > 0$ for $b^* < b < b^* + \lambda$ and it is a contraction.

Next we assert that if $\alpha > \nu(b_0)$, then
\[
\nu(b) < z(b) < 0 \quad \text{for } 0 < b \leq b_0.
\]
Since $\nu(b_0) < \alpha = z(b_0)$, we define
\[
\bar{b} = \inf\{b : \nu(s) < z(s), \quad \text{for } s \in (b, b_0)\}
\]
and suppose $\bar{b} > 0$, then $\nu(\bar{b}) = z(\bar{b})$ and
\[
z'(\bar{b}) = -c - \frac{Dn^{p+q}(\bar{b})\bar{b}^{1+l}(1 - \bar{b})}{z(\bar{b})}
= -c - \frac{Dn^{p+q}(\bar{b})\bar{b}^{1+l}(1 - \bar{b})}{\nu(\bar{b})}
< \nu'(\bar{b}),
\]
which is a contradiction.

In view of the arguments above, we have proved that $z$ is well defined in $(0, b_0)$. Now we may define the maximal existence interval $(0, b_\alpha)$ for solution $z$. The aim is to show that $b_\alpha = 1$ for some $\alpha \in [\nu(b_0), 0)$. Let $z_1$ and $z_2$ be two distinct solutions corresponding to initial value $\alpha_1$ and $\alpha_2$ respectively. Suppose for definiteness that
\[
\alpha_1 < \alpha_2,
\]
then
\[
z_1 < z_2 \quad \text{in } (0, \min\{b_{\alpha_1}, b_{\alpha_2}\}),
\]
hence
\[
b_{\alpha_1} \geq b_{\alpha_2}.
\]
We now claim that if $|\alpha|$ is sufficiently small, then
\[
b_\alpha < 1 \quad \text{and} \quad z(b_\alpha^-) = 0.
\]
Since that
\[
\lim_{(b, z) \to (b_\alpha, 0)} (-cz - Dn^{p+q}(b)b^{1+l}(1 - b)) < 0,
\]
there exists a sufficiently small constant $M > 0$ and $\lambda < 2M$ such that
\[
-c - \frac{Dn^{p+q}(b)b^{1+l}(1 - b)}{z} > \frac{M}{z},
\]
for all $-\lambda < z < 0$ and $b_0 \leq b < b_0 + \lambda$. Let $\alpha > -\lambda$ and define
\[
\psi := -\sqrt{\alpha^2 - 2M(b - b_0)}
\]
for $b_0 \leq b \leq b_0 + \frac{\alpha^2}{2M}$ which solves the following initial problem:

$$
\psi' = -\frac{M}{\psi},
$$

$$
\psi(b_0) = \alpha.
$$

By the choice of $\alpha$ and $\lambda$, we have

$$
b_0 + \frac{\alpha^2}{2M} < b_0 + \lambda < 1.
$$

Moreover, $z'(b_0) > 0$, hence $z(b) > \psi(b_0) = \alpha > -\lambda$ in a right neighborhood of $b_0$. Denote

$$
I = [b_0, \min\{b_0 + \frac{\alpha^2}{2M}, b_0\}],
$$

we deduce that

$$
z(b) > -\lambda \quad \forall \quad b \in I.
$$

Applying a similar comparison argument as before, we conclude that $z(b) > \psi(b)$ for all $b \in (b_0, b_\alpha) \cap (b_0, b_0 + \frac{\alpha^2}{2M})$. Since $\psi(b_0 + \frac{\alpha^2}{2M}) = 0$, we have

$$
b_\alpha \leq b_0 + \frac{\alpha^2}{2M} < 1.
$$

Now we let $\alpha^* = \inf\{\alpha \in (\psi(b_0), 0) : b_\alpha < 1\}$, then $b_\alpha^* = 1$, therefore the corresponding solution $z$ is defined and negative on $(0, 1)$ and $z > \nu$ in $(0, 1)$ and $z(0^+) = 0$. This complete the proof.

In the following, we prove the solvability result for Problem $P^*$.

**Theorem 3.2.** There exists $c^* > 0$ such that for all $c \geq c^*$, Problem $P^*$ has a unique negative solution.

**Proof.** We first show Problem $P^*$ is solvable for $c$ sufficiently large. To this aim, we let

$$
v = \sup_{s \in (0, 1)} Dn^{p+q}(s)s^{1+\lambda}(1-s)
$$

which is well defined. Let

$$
\nu(b) = -\sqrt{vb}.
$$

Then, for $c > 2\sqrt{v}$, we have

$$
-c - \frac{Dn^{p+q}(b)b^{1+\lambda}(1-b)}{\nu(b)} < -2\sqrt{v} + \frac{Dn^{p+q}(b)b^{1+\lambda}(1-b)}{\sqrt{vb}}
$$

$$
\leq -2\sqrt{v} + \sqrt{v} = \nu'(b)
$$

for all $b \in (0, 1)$.

Hence $\nu(b)$ satisfies condition of Lemma 3.1 and therefore Problem $P^*$ is solvable for every $c > 2\sqrt{v}$. We now show that Problem $P^*$ is not solvable for $c = 0$. Otherwise, let $z$ solves

$$
z' = -\frac{Dn^{p+q}(b)b^{1+\lambda}(1-b)}{z},
$$

that is defined on some interval $(\alpha, 1)$ with $0 < \alpha < 1$ and $z(b) < 0$ for all $b \in (\alpha, 1)$.

Integrating the equation above in $[b, \tilde{b}]$ with $a < b < \tilde{b} < 1$, we obtain

$$
z^2(\tilde{b}) = z^2(b) - 2 \int_b^{\tilde{b}} Dn^{p+q}(s)s^{1+\lambda}(1-s)ds.
$$

(32)
Therefore, if \(z(1^-) = 0\), we have
\[
z(b) = -\sqrt{2 \int_0^b Dn^{p+q}(s)s^{1+l}(1-s)ds}
\]
(33)
which implies \(z(0^+) < 0\), a contradiction.

We now let
\[c^* = \inf \{c : P^* \text{ is solvable}\}\]
which is well defined and \(c^* > 0\) based on the observation above.

First, we prove that for every \(c > c^*\), \(P^*\) is solvable. Given \(c > c^*\), take \(\tau\) such that \(P^*\) is solvable with \(\tau < c\) and the unique solution \(\tau\) for \(\tau\). Since
\[
\tau' = -\tau - \frac{Dn^{p+q}(b)b^{1+l}(1-b)}{\tau} > -c - \frac{Dn^{p+q}(b)b^{1+l}(1-b)}{\tau},
\]
hence \(\tau\) satisfies condition of Lemma 3.1 Therefore, we conclude the solvability of \(P^*\) for \(c\).

Secondly, we prove Problem \(P^*\) is solvable for \(c = c^*\). Let \((c_n)\) be a sequence of speeds decreasing to \(c^*\) and let \(z_n\) be the unique solution of Problem \(P^*\) for \(c = c_n\).

Since
\[
z_{n+1}(b) = -c_{n+1} - \frac{Dn^{p+q}(b)b^{1+l}(1-b)}{z_{n+1}} > -c_n - \frac{Dn^{p+q}(b)b^{1+l}(1-b)}{z_{n+1}}
\]
for all \(b \in (0, 1)\). Then \(z_{n+1}\) is a strict upper solution for Problem \(P^*\) with \(c = c_n\).

Hence, by using Lemma 3.1 again, we deduce that \(z_n(b) \geq z_{n+1}(b)\) for all \(b \in [0, 1]\) and \(n \in \mathbb{N}\). Therefore \((z_n)\) is a decreasing sequence. Moreover, since \(z_n(0^+) = 0\) and \(z_n'(b) \geq c_n > c_1\), it holds
\[
z_n(b) \geq -c_1 b \quad \text{for all} \quad b \in [0, 1]. \quad (34)
\]
Hence we can define \(z^*(b) = \inf_{n \in \mathbb{N}} z_n(b)\) for all \(b \in [0, 1]\). The monotone convergence theorem ensure that \(z^*(b)\) is a solution of Problem \(P^*\) for \(c = c^*\). Also from equation (34), we can have \(z^*(0^+) = 0\). Finally, we apply the shooting argument and comparison techniques to the sequence \((w_n^*)\) of solution of the Cauchy problems
\[
z' = -c^* - \frac{Dn^{p+q}(b)b^{1+l}(1-b)}{z},
\]
\[
z(1) = -\frac{1}{m},
\]
for \(m \in \mathbb{N}\), we can get that \(z^*(1^-) = 0\) and then \(z^*\) is the required solution of Problem \(P^*\) for \(c = c^*\).

Finally, we prove that \(P^*\) admits at most one solution. Suppose for contradiction that \(z_1\) and \(z_2\) are two distinct solution of \(P_1\). For definiteness, we assume
\[
z_1(b_0) > z_2(b_0) \quad \text{for} \quad b_0 \in (0, 1),
\]
it follows that
\[
z_1'(b_0) - z_2'(b_0) = - \frac{Dn^{p+q}(b_0)b_0^{1+l}(1-b_0)}{z_1(b_0)} + \frac{Dn^{p+q}(b_0)b_0^{1+l}(1-b_0)}{z_2(b_0)} > 0.
\]
Hence if \(z_1(b_0) > z_2(b_0)\), then \(z_1'(b_0) > z_2'(b_0)\). Therefore, it is impossible that \(z_1(1^-) = z_2(1^-) = 0\).

This complete the proof. \(\square\)

In the following, we will show some properties of the traveling wave solution.
Lemma 3.3. Let \( z \in C^1(0, 1) \) be the solution of Problem \( P^* \) for every \( c > c^* \). The following limit
\[
\lim_{b \to 0^+} \frac{z(b)}{b} = 0,
\]
even if \( c = c^* \),
\[
\lim_{b \to 0^+} \frac{z(b)}{b} = -c^* \quad \text{or} \quad \lim_{b \to 0^+} \frac{z(b)}{b} = 0,
\]
exists. If \( c = c^* \),
\[
\lim_{b \to 0^+} \frac{z(b)}{b} = -c^* \quad \text{or} \quad \lim_{b \to 0^+} \frac{z(b)}{b} = 0,
\]
Proof. Let \( z(b) \) be the solution of Problem \( P^* \) satisfying \( z(0^+) = z(1^-) = 0 \). Assume by contradiction that
\[
0 \geq L := \limsup_{b \to 0^+} \frac{z(b)}{b} > \liminf_{b \to 0^+} \frac{z(b)}{b} := l.
\]
Let \( \chi \in (l, L) \) and let \( \{b_n\} \) be an decreasing sequence converging to 0 such that
\[
\frac{z(b_n)}{b_n} = \chi, \quad \text{and} \quad \frac{d}{db} \left( \frac{z(b)}{b} \right) |_{b=b_n} \geq 0.
\]
Since \( \frac{d}{db} \left( \frac{z(b)}{b} \right) = \frac{1}{\chi} (\dot{z}(b) - \frac{z(b)}{b}) \), we have \( \dot{z}(b_n) - \frac{z(b_n)}{b_n} = \dot{z}(b_n) - \chi \geq 0 \), hence
\[
\dot{z}(b_n) = -c - \frac{Dn^{p+q}(b_n)b^{1+l}(1 - b_n)}{\chi b_n} \geq \chi.
\]
Passing to the limit as \( n \to +\infty \), since \( \chi < 0 \), we have \( \chi^2 + c\chi + [Dn^{p+q}(b)b^{1+l}(1 - b)]'(0^+) \geq 0 \), that is \( \chi \leq -c \). Similarly, we can choose an decreasing sequence \( \{\nu_n\} \) converging to 0, such that
\[
\frac{z(\nu_n)}{\nu_n} = \chi, \quad \text{and} \quad \frac{d}{db} \left( \frac{z(b)}{b} \right) |_{b=\nu_n} \leq 0.
\]
we can deduce \( \chi^2 + c\chi + [Dn^{p+q}(b)b^{1+l}(1 - b)]'(0^+) \leq 0 \), hence \(-c \leq \chi \leq 0 \). By the arbitrariness of \( \chi \in (l, L) \), we conclude that
\[
\lambda_1 = l = L = \frac{1}{2} (\pm c - c).
\]
Given \( c > c^* \), let \( z(b) \) and \( z^*(b) \) be the solutions of Problem \( P^* \) on \([0, 1] \) with \( z(0) = z(1) = 0 \), respectively, for \( c \) and \( c^* \). Assuming the existence of \( \overline{b} \in (0, 1) \) satisfying \( z^*(\overline{b}) \geq z(\overline{b}) \), from \( P^* \) we then have
\[
\dot{z}^*(\overline{b}) = -c^* - \frac{Dn^{p+q}(\overline{b})b^{1+l}(1 - \overline{b})}{z^*(\overline{b})} > -c - \frac{Dn^{p+q}(\overline{b})b^{1+l}(1 - \overline{b})}{z(\overline{b})} = \dot{z}(\overline{b}),
\]
This implies the contradictory conclusion \( 0 = z^*(1^-) > z(1^-) = 0 \). Hence \( z^*(0^+) < z(b) \) for all \( b \in (0, 1) \), and \( z^*(0^+) \leq \dot{z}(0^+) \). From
\[
\dot{z}^*(0^+) = \frac{1}{2} (\pm c^* - c^*),
\]
we have \( \dot{z}(0^+) = 0 \).
Because \( D[n^p b(1 - b)]'(0^+) = Dn^p(0) \neq 0 \), we also obtain
\[
\lim_{b \to 0^+} \frac{z(b)}{D[n^p b(1 - b)]} = \lim_{b \to 0^+} \frac{D[n^p b]}{b} = \frac{b}{D[n^p b(1 - b)]} = 0
\]
\( \square \)
Remark 1. Under the same conditions as Lemma 3.3, we can obtain
\[
\lim_{b \to 1^-} \frac{z(b)}{b} = 0,
\]
for all \( c \geq c^* \) by the same arguments.

Lemma 3.4. \( b \) is a finite traveling wave if and only if \( z'(0^+) = -c^* \)

Proof. If we consider the following Cauchy problem
\[
\begin{cases}
    b' = \frac{z(b)}{D_n^p b(1-b)}, & 0 < b < 1/2, \\
    b(0) = 1/2.
\end{cases}
\]
Let \( b(t) \) be the finite traveling solution of (36) defined in its maximal existence interval \((t_1, t_2)\), with \(-\infty \leq t_1 < t_2 < +\infty\). So we can see \( b(t) \) is a solution of (12) in \((t_1, t_2)\).

Observe that \( b'(t) < 0 \) for every \( t \in (t_1, t_2) \), so there exists the limit \( b(t_2^-) \in [0, 1/2) \). Since \( z(b) \neq 0 \) in \((0, 1/2)\), we deduce that \( b(t_2^-) = 0 \). Moreover, since
\[
\lim_{\xi \to t_2^-} b'(\xi) = \lim_{b \to 0^+} \frac{z(b)}{D_n^p b(1-b)},
\]
we have \( z'(0^+) = 0 \). Moreover, since \( -\infty \leq t_1 < t_2 < +\infty \) and the solution could be continued in the whole half-line \((t_2, +\infty)\), in contradiction with the maximality of the interval \((t_1, t_2)\), so \( z'(0^+) \) should be \(-c^*\).

One the other hand, if \( z'(0^+) = -c^* \), we have
\[
\lim_{\xi \to t_2^-} b'(\xi) = \lim_{b \to 0^+} \frac{z(b)}{b} \cdot \frac{1}{D_n^p(1-b)} = -\frac{c^*}{D_n^p(0)},
\]
which implies that \( b \) is a finite traveling wave solution. \(\square\)

Remark 2. From the above discussions, we can see the finite traveling wave solution occurs only at the minimum traveling speed \( c^* \).

Furthermore, we have the following lemma.

Lemma 3.5. There exist a sequence \((h_n)\) of positive real numbers decreasing to 0, and a sequence \((\psi_n)\) of continuous functions respectively defined on \([0, u_n]\) with \( u_n \leq u_{n+1} \leq 1 \) for all \( n \in \mathbb{N} \), satisfying the following properties:
(i) \( \psi_n(b) = -c_n b \) for all \( b \in [0, h_n] \);
(ii) \( \psi_n(b) \geq c^* b \) for all \( b \in [0, u_n] \);
(iii) \( \psi_n(b) = -c_n - \frac{D_n^{p+q}(b) b^{1+l}(1-b)}{\psi_n(b)} \) for every \( b \in [h_n, u_n] \);
(iv) \( \psi_n(u_n^-) = 0 \);
(v) \( \psi_n(b) \geq \psi_{n+1}(b) \) for all \( b \in [0, u_n] \).

The strategies of the proof is similar to [lemma 13, 14] with the difference in \( D_n^{p+q}(b) b^{1+l}(1-b) \) and we omit the proof here.

Applying the monotone convergence theorem to \( \psi_n \), we can prove that \( \psi \) is a solution of the equation \( z'(b) = -c^* - \frac{D_n^{p+q}(b) b^{1+l}(1-b)}{z} \) in \([0, \pi]\). Moreover, by the monotonicity of \( \psi_n \), we get
\[
-c^* \leq \frac{\psi}{b} \leq \frac{\psi_n}{b} = -c_n \quad \text{for all} \quad b \in (0, h_n).
\]
From lemma 3.4 this implies \((z^*)'(0) = -c^*\).

Using Lemma 3.4 again, we can further have the following results:
Lemma 3.6. For the solution of Problem $P^*$, there exists $C_1 < 0$, $C_2 < 0$ such that

$$C_2(1 - b)^{p+q+1} \leq z(b) \leq C_1(1 - b)^{p+q+1}$$

for $b$ sufficiently close to 1.

Proof. We only need to show that it is impossible to find negative constants $C_1, C_2$ such that $z(b) \geq C_1(1 - b)^{p+q+1} + \varepsilon$ or $z(b) \leq C_2(1 - b)^{p+q+1} - \varepsilon$ for any $\varepsilon > 0$ as $b \to 1^-$.

Otherwise, we either have

$$z'(b) = -c - \frac{Dn^{p+q}b^{1+l}(1 - b)}{z(b)} \geq -c - \frac{Dn^{p+q}b^{1+l}(1 - b)}{C_1(1 - b)^{p+q+1+\varepsilon}}$$

for $b$ sufficiently close to 1. This is a contradiction to Remark 1.

Or we have

$$z'(b) \leq -c - \frac{D(1/L)^{p+q}b^{1+l}(1 - b)^{p+q+1}}{C_2(1 - b)^{p+q+1+\varepsilon}} < 0$$

for $b \to 1^-$. This contradicts to Remark 1 again. \qed

We also have the following result.

Lemma 3.7. There exists $C_2 < 0$ such that $z(b) \geq C_2(1 - b)^{p+q+1}$ for all $b \in [0, 1]$.

Proof. Suppose that there exists $b_0 \in [0, 1]$ such that $z(b_0) < C_2(1 - b_0)^{p+q+1}$, we have

$$z'(b_0) = -c - \frac{Dn^{p+q}b_0^{1+l}(1 - b_0)}{z(b_0)} \leq -c - \frac{DL^{p+q}b_0^{1+l}(1 - b_0)^{p+q+1}}{C_2(1 - b_0)^{p+q+1}} < -C_2(p + q + 1)(1 - b_0)^{p+q},$$

which is obtained from

$$-c - \frac{DL^{p+q}b_0^{1+l}}{C_2} < -C_2(p + q + 1)(1 - b_0)^{p+q}$$

for sufficiently large $|C_2|$. We then have

$$z(b) - C_2(1 - b)^{p+q+1} < z(b_0) - C_2(1 - b_0)^{p+q+1} < 0$$

for every $b > b_0$ which is a contradiction to $z(1^-) = 0$. \qed
4. The existence of the finite traveling wave solution. We have shown that for any \( n \in V \), there exists \( c^* \) that depends on the choice of \( n \) such that \( b \) is a finite traveling wave, we may assume that

\[
b(\xi) = 0 \quad \text{for} \quad \xi \geq 0,\]

which lead to \( n \) satisfies

\[
n'' + cn' = 0 \quad \text{for} \quad \xi \geq 0.\]  

We have the following lemma:

**Lemma 4.1.** The problem

\[
\begin{align*}
n'' + cn' &= 0 \quad \text{in} \quad (0, +\infty), \\
n(0) &= n^* \quad \text{with} \quad 0 < n^* < 1, \\
n'(0) &= c(1 - n^*),
\end{align*}
\]

has a unique bounded solution which satisfies \( \lim_{\xi \to +\infty} n(\xi) = 1 \).

**Proof.** The phase plane analysis shows that every trajectory of the following ODE system:

\[
\begin{align*}
n' &= p, \\
p' &= -cp.
\end{align*}
\]

can intersect the \( n - \text{axis} \) at most once. Hence \( p' \) changes sign at most once and consequently \( n(+\infty) \) exists. Let \( n(+\infty) = v \) and we can get

\[
\int_0^{+\infty} n'' = -\int_0^{+\infty} cn'
\]

hence

\[
-c(1 - n^*) = -cv + cn^*.
\]

Therefore

\[
\lim_{\xi \to +\infty} n(\xi) = v = 1.
\]

\( \square \)

We showed before that \( b(\xi) \) decrease monotonically from 1 to 0 as \( \xi \) varies from \(-\infty \) to 0. Therefore we can define

\[
\xi = \xi(b) \quad \text{as the inverse function of} \quad b(\xi),
\]

where \( b \) varies from 0 to 1 and \( \xi \) takes value in \(( -\infty, 0)\). We define

\[
z(b) = Dn^pb(1 - b)b'(\xi(b)) < 0 \quad \forall \quad b \in (0, 1)
\]

and

\[
n(b) = n(\xi(b)).
\]

Since

\[
\frac{d}{d\xi} = b'(\xi) \cdot \frac{d}{db} = \frac{z(b)}{Dn^pb(1 - b)} \cdot \frac{d}{db},
\]

we can have the following two equivalent problems:
\textbf{P1.} Find \((c, b, n)\) with \(c > 0\) such that
\[
D(n^p b (1 - b)b') + cb' + n^q b^l = 0, \quad \xi < 0, \tag{44}
\]
\[
n'' + cn' - n^q b^l = 0, \quad \xi < 0, \tag{45}
\]
\[
n'(0) = c(1 - n(0)), n(-\infty) = 0, \tag{46}
\]
\[
b(-\infty) = 1, \quad b(0) = 0. \tag{47}
\]

\textbf{P2.} Find \((c, b, n)\) with \(c > 0\) such that
\[
z' = -c - \frac{D(n^p b (1 - b))}{Dn^p b (1 - b)} b^{1+l} (1 - b), \tag{48}
\]
\[
\frac{z}{Dn^p b (1 - b)} (\frac{z}{Dn^p b (1 - b)})' n' + \frac{z}{Dn^p b (1 - b)} n' - n^q b^l = 0, \tag{49}
\]
\[
z(0^+) = z(1^-) = 0, \tag{50}
\]
\[
n'(0^+) = -(1 - n(0))Dn(0)^p, \quad n(1^-) = 0, \tag{51}
\]
\[
z < 0 \text{ in } (0, 1). \tag{52}
\]

Also we need
\[
\int_0^1 \frac{Dn^p b (1 - b)}{z(b)} db = -\infty. \tag{53}
\]

Given \(n(b) \in V\), let \((c, z)\) be the unique solution of problem \(P^*\) such that \(z'(0^+) = -c^*\) and the corresponding \(b\) is a finite traveling wave solution. Consider the following:

\textbf{P3.} Find \(\tilde{n}(b)\) such that
\[
\frac{z}{Dn^p b (1 - b)} (\frac{z}{Dn^p b (1 - b)})' \tilde{n}' + \frac{z}{Dn^p b (1 - b)} \tilde{n}' - Dn^p \tilde{n}^q b^{1+l} (1 - b) = 0 \quad \text{in } (0, 1), \tag{54}
\]
\[
\tilde{n}' + D(1 - \tilde{n}) n^p = 0 \quad \text{at } b = 0, \tag{55}
\]
\[
\tilde{n}(1) = 0. \tag{56}
\]

We have the following theorem shows the local existence of Problem \(P_3\).

\textbf{Theorem 4.2.}

\[
\frac{z}{Dn^p b (1 - b)} (\frac{z}{Dn^p b (1 - b)})' \tilde{n}' + \frac{z}{Dn^p b (1 - b)} \tilde{n}' - Dn^p \tilde{n}^q b^{1+l} (1 - b) = 0 \quad \text{in } (0, 1), \tag{57}
\]
\[
\tilde{n}'(0) + D(1 - \tilde{n}(0)) n^p(0) = 0 \quad \text{at } b = 0, \tag{58}
\]
\[
\tilde{n}(0) = n_0, \quad 0 < n_0 < 1. \tag{59}
\]

has a local solution.

\textbf{Proof.} Take
\[
S = \{ \tilde{n}(b) | \tilde{n}(b) \in C([0, b_0]), 0 \leq \tilde{n} \leq n_0 \text{ for } 0 \leq b \leq b_0, \quad \text{and some small } b_0 > 0 \text{ to be determined later}\}.
\]

Then \(S\) is a closed convex set in \(C([0, b_0])\). Assuming that \(\tilde{n}\) is a smooth solution on the interval \([0, b_0]\) with \(b_0 \leq 1\), we integrate Equation (57) to get
\[
\frac{z}{Dn^p b (1 - b)} \tilde{n}' - \lim_{s \to 0^+} \frac{z(s)}{Dn^p b (1 - s)} \tilde{n}' + c\tilde{n}(b) - cn_0 = \int_0^b \frac{Dn^p \tilde{n}^q b^{1+l} (1 - s)}{z(s)} ds. \tag{60}
\]

Since
\[
\lim_{s \to 0^+} \frac{z(s)}{Dn^p b (1 - s)} \tilde{n}' = \frac{-c}{Dn_0^p} D(n_0 - 1)n_0^p, \tag{61}
\]

\[\text{Proof.} \]
we have
\[
\frac{z}{Dn^{p}(1-b)}\tilde{n}' + c\tilde{n}(b) - c = \int_{0}^{b} \frac{Dn^{\theta} s^{1+1}(1-s)}{z(s)} ds. \tag{62}
\]
A second integration yields
\[
\tilde{n}(b) = n_{0} + \int_{0}^{b} \frac{Dn^{p}(1-s)}{z(s)} c(1-\tilde{n}(s))ds \tag{63}
\]
\[
+ \int_{0}^{b} \frac{Dn^{p}(1-s)}{z(s)} ds \int_{0}^{s} \frac{Dn^{\theta} \tau^{1+1}(1-\tau)}{z(\tau)} d\tau. \tag{64}
\]
For small \(b_{0}\) and \(0 \leq b \leq b_{0}\), we have
\[
0 \leq \tilde{n} \leq n_{0} < +\infty,
\]
and
\[
\tilde{n}'(b) = \frac{Dn^{p}(1-b)}{z} c(1-\tilde{n}(b))
\]
\[
+ \frac{Dn^{p}(1-b)}{z} \int_{0}^{b} \frac{Dn^{\theta} s^{1+1}(1-s)}{z(s)} ds \leq C < +\infty.
\]
This shows the operator defined above is a compact operator. Therefore it has a fixed point in \(S\) and has a local solution.

To establish the existence and uniqueness of solution to \(P_{3}\), we also need the following lemmas

**Lemma 4.3.** If \(\tilde{n}(0) = n_{0}\) is sufficiently close to 1, then \(\tilde{n}(1) \geq 0\). If \(\tilde{n}(0) = n_{0}\) is sufficiently close to 0, then \(\tilde{n}(b) = 0\) for some \(0 < b < 1\).

**Proof.** We prove this lemma by contradictions. We start with the following equation:
\[
\tilde{n}' + c \frac{Dn^{p}(1-b)}{z(b)} \tilde{n}(b) = [c + \int_{0}^{b} \frac{Dn^{\theta} s^{1+1}(1-s)}{z(s)} \frac{Dn^{p}(1-b)}{z(b)}] \tag{65}
\]
Let \(h(b) = \frac{Dn^{p}(1-b)}{z(b)}\) which is non-positive and the above equation can be rewritten as
\[
(e^{\int_{0}^{b} c h(s) ds} \tilde{n})' = [c + \int_{0}^{b} h(s) \tilde{n}s ds] e^{\int_{0}^{b} c h(s) ds} h(b). \tag{66}
\]
Let \(H(b) = e^{\int_{0}^{b} c h(s) ds}\) and \(J(b) = c + \int_{0}^{b} h(s) \tilde{n}s ds\). Integrating both sides, we obtain
\[
H(b)\tilde{n} - \tilde{n}(0) = \int_{0}^{b} \frac{1}{c} H'(s) J(s) ds
\]
\[
= \frac{1}{c} H(b) J(b) - \frac{1}{c} H(0) J(0) - \int_{0}^{b} \frac{1}{c} H(s) h(s) \tilde{n}s ds
\]
\[
= \frac{1}{c} H(b) J(b) - 1 - \int_{0}^{b} \frac{1}{c} H(s) h(s) \tilde{n}s ds.
\]
Suppose that \(n_{0}\) is sufficiently close to 1 but \(\tilde{n}(b) \equiv 0\) for some \(0 < b < 1\). Then the left side of the above equation is sufficiently close to \(-1\). But since \(H(b) > 0, J(b) = \frac{z}{Dn^{p}(1-b)}\tilde{n}' + c\tilde{n}(b) > 0\) and \(\int_{0}^{b} \frac{1}{c} H(s) h(s) \tilde{n}s ds < 0\). Thus the equality above is impossible. Hence we conclude that \(\tilde{n}(1) \geq 0\) for \(n_{0}\) sufficiently close to 1.
To prove the second statement, we note that
\[ z''(0) = \lim_{b \to 0} \frac{z' + c}{b} = \lim_{b \to 0} \frac{Dn^{p+q}b^1(1) - b}{z} = 0. \]
Since
\[
\frac{z}{Dn^p b(1-b)} \frac{z''}{\tilde{\tau}''} = -\left[\left(\frac{z}{Dn^p b(1-b)}\right)'' + c] \tilde{\tau}' + \frac{Dn^p \tilde{\tau} b^1 + l (1-b)}{z} \right.
\]
\[= -\left[\frac{z'}{Dn^p b(1-b)} - \frac{z(1-2b)}{Dn^p b^2(1-b)^2} \frac{z}{Dn^p b(1-b)} \frac{z'}{z} + \frac{2z}{b} \right]
\]
\[+ \frac{Dn \tilde{\tau} b^1 + l (1-b)}{z}. \]
where
\[
\lim_{b \to 0} \frac{z'}{n^p b(1-b)} - \frac{z(1-2b)}{n^p b^2(1-b)^2} \frac{z}{n^p b(1-b)} = \lim_{b \to 0} \frac{1}{n^p(0)} \left[\frac{z'(0)}{2} - z'(0) + 2z(0) \right]
\]
\[= -\frac{c z}{n^p(0)}, \]
\[
\lim_{b \to 0} \frac{zpn'}{n^p + b(1-b)} = \frac{pc D(n(0) - 1)}{n(0)} = \frac{Dpc}{n(0)} + Dpc. \]
Thus we can have that \( \tilde{\tau}'' \) is finite on \( \{b \in [0, \delta] : \tilde{\tau} > 0 \} \), and by Taylor series, we can prove the second statement. \( \square \)

**Lemma 4.4.** Let \( \tilde{\tau}_1, \tilde{\tau}_2 \) be the solutions corresponding two different initial value \( n_{10} \) and \( n_{20} \) with \( n_{10} > n_{20} \), then \( n_1 > n_2 \) on \( [0, \min\{T(n_{10}), T(n_{20})\}] \) where \( T(n_{10}) \) represent the corresponding maximal existence interval in the sense that \( \tilde{\tau}(T) = 0 \).

**Proof.** Suppose otherwise there is a \( b_0 \in [0, \min\{T(n_{10}), T(n_{20})\}] \) such that \( \tilde{\tau}_1(b_0) = \tilde{\tau}_2(b_0) \) and \( \tilde{\tau}_1 > \tilde{\tau}_2 \) on \( (0, b_0) \), then \( \tilde{\tau}_1'(b_0) < \tilde{\tau}_2'(b_0) \). We have
\[
\frac{z(b_0)}{Dn^p(b_0)b_0(1-b_0)} \frac{\tilde{\tau}_1'(b_0) + c \tilde{\tau}_1(b_0) - c}{\tilde{\tau}_2'(b_0) + c \tilde{\tau}_2(b_0) - c} > \frac{z(b_0)}{Dn^p(b_0)b_0(1-b_0)} \frac{\tilde{\tau}_1'(b_0) + c \tilde{\tau}_1(b_0) - c}{\tilde{\tau}_2'(b_0) + c \tilde{\tau}_2(b_0) - c} \tag{67}
\]
while
\[
\int_0^{b_0} Dn \tilde{\tau}_1 s^{1+l}(1-s) ds < \int_0^{b_0} Dn \tilde{\tau}_2 s^{1+l}(1-s) ds \tag{68}
\]
which is a contradiction to \( \text{[52]} \). \( \square \)

From the above lemmas, we establish the following existence theorem.

**Theorem 4.5.** Problem \( P_4 \) admits a unique solution. The solution \( \tilde{\tau} \) satisfies the following properties:
\[
0 \leq \tilde{\tau} \leq 1,
\]
\[
\tilde{\tau} \geq \frac{1}{L} (1-b),
\]
\[
\limsup_{b \to 1^-} \frac{\tilde{\tau}}{1-b} \leq L,
\]
\[
-M \leq \tilde{\tau}' \leq 0.
\]
Proof. Since Equation (54) is only degenerate at $b = 1$. The boundedness of $\hat{n}'$ on $[0,1)$ follows from standard ODE theory. We only study the boundedness at $b = 1$. To this purpose, we let $n(b) \sim (1 - b)^\alpha$ with $\alpha \geq 1$ as $b \to 1$. Let $\hat{n}(b) \sim (1 - b)^\beta$ as $b \to 1$. In view of Equation (26), we have

$$z \sim -\frac{Dn^p q (1 - b)}{c} \quad \text{as} \quad b \to 1.$$  

Thus as $b \to 1$, $\hat{n}$ satisfies

$$\left(\frac{\eta^n q}{c} \hat{n}' + c\hat{n}' + c\hat{n}q\right) = 0.$$  

(69)

Suppose that $\beta < 1$, matching the leading singular term in the equation, we have

$$(\beta - 1) = q(\beta - \alpha)$$

or

$$\beta = \frac{\alpha q - 1}{q - 1}.$$  

Which is clearly impossible since $\alpha \geq 1$. Thus we conclude $\beta \geq 1$ or equivalently $\hat{n}'(1)$ exist. By matching the coefficients, we can have

$$\limsup_{b \to 1} \frac{\hat{n}}{1 - b} \leq L.$$  

Let $W = \frac{z}{Dn^p b (1 - b)}\hat{n}'$ and differentiate Equation (49), we have

$$\left(\frac{z}{Dn^p b (1 - b)}W'\right)' + cW' - q\hat{n}^q - l\hat{n}q = 0.$$  

(70)

Also

$$W(0) > 0, W(1) = 0.$$  

The maximum principle yields $W > 0$ in $(0,1)$.

By Lemma 4.3, we have the bound on $\hat{n}(0)$. The uniqueness follows from Lemma 4.4. This ends the proof of this theorem. ☐

Combining Lemma 4.4, Theorem 4.5 and for every $n \in V$, we define $(c,z)$ to be the solution of $P_1$ and $\hat{n}$ by Theorem 4.5 and introduce the mapping $T$ by

$$Tn = \hat{n}.$$  

Clearly, $T$ maps $V$ into itself and its image lies in a compact subset of $V$. By the uniqueness part in Lemma 4.4, it also follows that $T$ is continuous. Invoking the Schauder fixed point theorem, we conclude that there exists at least one fixed point for $T$. We shall denote it by $(c^*, z, n)$. To show that the corresponding $(c^*, z, n)$ is a solution to problem $P_2$, we shall need to show equality (53).

In fact, we also have the following conclusion.

Lemma 4.6.

$$\int_0^1 \frac{Dn^p b (1 - b)}{z(b)} \, db = -\infty.$$  

Proof. Using the fact that there exists a constant $L > 0$ such that $n(b) \geq 1 / L(1 - b)$ and a constant $C_2$ such that $z(b) \geq C_2(1 - b)^{p+q+1}$ where $(z,n)$ is the pair in the fixed point stated above. The second fact follows from Lemma 3.7. Thus

$$\int_0^1 \frac{Dn^p b (1 - b)}{z(b)} \, db \leq \int_0^1 \frac{Db(1/L)^p b^{p+q+1}}{C_2(1 - b)^{p+q+1}} \, db = -\infty.$$  

for $q > 1$. ☐
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