$C^\infty$ Functions on the Stone-Čech Compactification of the Integers

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Abstract

We construct an algebra $A = \ell^{\infty}\!\!\!\!\,_{\infty}(\mathbb{Z})$ of smooth functions which is dense in the pointwise multiplication algebra $\ell^\infty(\mathbb{Z})$ of sup-norm bounded functions on the integers $\mathbb{Z}$. The algebra $A$ properly contains the sum of the algebra $A_c = \ell^{\infty}\!\!\!\!\,_{c}(\mathbb{Z})$ and the ideal $S(\mathbb{Z})$, where $A_c$ is the algebra of finite linear combinations of projections in $\ell^\infty(\mathbb{Z})$ and $S(\mathbb{Z})$ is the pointwise multiplication algebra of Schwartz functions. The algebra $A$ is characterized as the set of functions whose “first derivatives” vanish rapidly at each point in the Stone-Čech compactification of $\mathbb{Z}$. 

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1 Introduction

In a previous paper [Sch, 1998], a notion of smooth functions on the Cantor set was developed. Recall that a totally disconnected topological space has a basis of clopen sets. The Cantor set is such a space. In this paper, we attempt to construct smooth functions on the Stone-Čech compactification of the integers. In addition to being totally disconnected, this space is extremely
disconnected, which means that the closure of every open set is clopen.

We will be working with the $C^*$-algebra of all bounded complex-valued functions (or sequences) on the integers $\mathbb{Z}$, under pointwise multiplication, with pointwise complex-conjugation for involution. We denote this algebra by $\ell^\infty(\mathbb{Z})$. None of the theorems will use the additive and multiplicative structure of $\mathbb{Z}$, so that any countable discrete set can be substituted for $\mathbb{Z}$. However, when specific objects are constructed in the examples, we may make reference to the underlying set of integers. Recall that the integers $\mathbb{Z}$ consists of all whole numbers from $-\infty$ to $\infty$, whereas the natural numbers $\mathbb{N}$ contains only the whole numbers from 0 to $\infty$. The norm on $\ell^\infty(\mathbb{Z})$ is the sup-norm $\|\cdot\|_\infty$, defined by $\|\varphi\|_\infty = \sup_{n \in \mathbb{Z}} |\varphi(n)|$.

We will use the notation $c_0(\mathbb{Z})$ for the complex-valued functions (sequences) on $\mathbb{Z}$ which vanish at infinity.

2 The Stone-Čech Compactification of $\mathbb{Z}$

We recall the standard definition of the Stone-Čech compactification from [Roy, 1968]. Let $F$ be the set of all real-valued functions from $\mathbb{Z}$ into the closed interval $I = [-1, 1]$. (So $F$ is the set of real-valued functions in the
unit ball of $\ell^\infty(Z)$. Let 

$$X = \prod_{F} I$$

be the $F$-fold cartesian product of unit intervals $I$. By the Tychonoff theorem [Roy, 1968, Chapter 9, Theorem 19], this is a compact Hausdorff space. Let

$$i : Z \hookrightarrow X \quad \text{where} \quad i(n) \mapsto \{f(n)\}_{f \in F}$$

be the natural inclusion map.

**Definition 1.1.** We define the *Stone-Čech Compactification of $Z$*, denoted by $\beta(Z)$, to be the closure of the image $i(Z)$ in $X$. We let $C(\beta(Z))$ denote the continuous complex-valued functions on $\beta(Z)$. See [Roy, 1968] for the basic properties of $\beta(Z)$.

**Definition 1.2.** Let $n_0 \in \beta(Z)$, and let $\{n_\alpha\}_{\alpha \in \Lambda}$ be a net converging to $n_0$, where $\Lambda$ is some directed set, and $n_\alpha \in Z$ for each $\alpha \in \Lambda$. Let $\varphi \in \ell^\infty(Z)$. Then $\varphi(n_0)$ is defined as the limit of the net $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$. In this way $\varphi$ defines a continuous function on $\beta(Z)$, giving an isomorphism of commutative $C^*$-algebras $\ell^\infty(Z) \cong C(\beta(Z))$. (The fact that $\varphi$ is continuous, and that the limit defining $\varphi(n_0)$ converges, can be verified directly from the definition of the Stone-Čech compactification above, using the fact that the real and imaginary parts of $\varphi/\|\varphi\|_\infty$ are functions in $F$.)
Let \( p \in \ell^\infty(\mathbb{Z}) \) be a projection. In other words for each \( n \in \mathbb{Z}, \ p(n) = 0 \) or \( p(n) = 1 \). Then \( p \) also defines a projection on \( \beta(\mathbb{Z}) \). In fact, if \( n_0 \in \beta(\mathbb{Z}) \) and \( \{n_\alpha\}_{\alpha \in \Lambda} \) are as above, then \( p(n_\alpha) \) is either eventually equal to 1 (in the case \( p(n_0) = 1 \)) or eventually equal to 0 (in the case \( p(n_0) = 0 \)). Thus each point \( n_0 \in \beta(\mathbb{Z}) \) defines a family \( \mathcal{F}_{n_0} \) of subsets of \( \mathbb{Z} \), where \( S \in \mathcal{F}_{n_0} \) if and only if the projection \( p \) whose support is equal to \( S \) satisfies \( p(n_0) = 1 \).

**Definition 1.3.** A family of subsets \( \mathcal{F} \) of \( \mathbb{Z} \) is a filter on \( \mathbb{Z} \) if it is closed under finite intersections

\[
S, T \in \mathcal{F} \implies S \cap T \in \mathcal{F}
\]

and supersets

\[
S \in \mathcal{F} \quad \text{and} \quad T \supset S \implies T \in \mathcal{F},
\]

and if the empty set \( \emptyset \) in *not* in \( \mathcal{F} \). A filter \( \mathcal{U} \) on \( \mathbb{Z} \) is an ultrafilter on \( \mathbb{Z} \) if

\[
S \subseteq \mathbb{Z} \implies S \in \mathcal{F} \quad \text{or} \quad S^c \in \mathcal{F}.
\]

The map \( n_0 \in \beta(\mathbb{Z}) \mapsto \mathcal{F}_{n_0} \) from the previous paragraph defines a map \( \mathcal{U}\mathcal{F} \) from \( \beta(\mathbb{Z}) \) to the set of ultrafilters on \( \mathbb{Z} \). (Use the definition of \( \beta(\mathbb{Z}) \) to check this.)

A principal filter is of the form \( \mathcal{U}_n = \langle S \subseteq \mathbb{Z} \mid n \in S \rangle \) for some \( n \in \mathbb{Z} \). Such a filter is also an ultrafilter. The image of \( \mathbb{Z} \subseteq \beta(\mathbb{Z}) \) under the map \( \mathcal{U}\mathcal{F} \)
is precisely the set of principal ultrafilters on $\mathbb{Z}$.

We show that the map $\mathcal{U}\mathcal{F}$ is an isomorphism by constructing an inverse map $\mathcal{F}U$. Let $U$ be any ultrafilter on $\mathbb{Z}$. Define a directed set $\Lambda$ to be $U$ with superset order. That is $\alpha \leq \beta \iff \beta \subseteq \alpha$. Thus smaller sets are “bigger” in this order. In the case of the principal ultrafilter $\Lambda = U_n$, the singleton $\{n\}$ is the biggest element. In general, $\Lambda$ has a biggest element if and only if it comes from a principal ultrafilter. Next, for $\alpha \in \Lambda$ choose any $n_\alpha \in \alpha \subseteq \mathbb{Z}$. We show that $\{n_\alpha\}_{\alpha \in \Lambda}$ converges to an element of $\beta(\mathbb{Z})$.

Let $\varphi \in F$, where $F$ is the defining family of functions for $\beta(\mathbb{Z})$ from Definition 1.1. We wish to show that $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$ converges. Define

$$\varphi_+(n) = \begin{cases} \varphi(n) & \text{if } \varphi(n) \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\varphi_-(n) = \begin{cases} -\varphi(n) & \text{if } \varphi(n) \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi = \varphi_+ - \varphi_-$, and $\varphi_+, \varphi_- \in F$. It suffices to show that $\{\varphi_+(n_\alpha)\}_{\alpha \in \Lambda}$ and $\{\varphi_-(n_\alpha)\}_{\alpha \in \Lambda}$ each converge. So without loss of generality, we take $\varphi$ with range in the unit interval $[0, 1]$.

If $\varphi$ were a projection $p$, we would be done. Simply let $S$ be the support of
If $S \in U = \Lambda$, then $\{p(n_\alpha)\}_{\alpha \in \Lambda}$ is eventually 1 (when $\alpha \geq S$). Otherwise, it is eventually 0, and in either case they converge. We proceed by writing $\varphi$ as an infinite series of projections. Define the set of integers

$$S_{1/2} = \{n \in \mathbb{Z} \mid \frac{1}{2} \leq \varphi(n) \leq 1\}.$$

Let $p_{1/2}$ be the projection corresponding to $S_{1/2}$. Then $\varphi - \frac{1}{2}p_{1/2}$ has its range in the interval $[0, \frac{1}{2}]$. We repeat the process to get a new function with range in the interval $[0, \frac{1}{4}]$, etc, until we get:

$$\varphi = \frac{1}{2}p_{1/2} + \frac{1}{4}p_{1/4} + \cdots = \sum_{q=1}^{\infty} \frac{1}{2^q}p_q,$$

an infinite series that converges absolutely (and geometrically fast) in sup norm to $\varphi$. To see that $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$ converges, now use a standard series argument. Let $\epsilon > 0$ be given, and find $N$ large enough so that the sup norm of the tail

$$\left\| \sum_{q=N}^{\infty} \frac{1}{2^q}p_q \right\|_\infty$$

is less than $\epsilon$. Then find $\alpha \in \Lambda$ sufficiently large so that $\alpha \subseteq S_{1/2^q}$ or $\alpha \subseteq S_{1/2^q}^c$ for each $q = 1, \ldots, N$. (One could first find an $\alpha$ that works for each $S_{1/2^q}$ separately, using the ultrafilter condition (1.4c). Then, find an $\alpha$ that works for all $S_{1/2^q}, q = 1, \ldots, N$ using the finite intersection property (1.4a).) For any $\beta \in \Lambda$ beyond this $\alpha$, the first $N$ projections in (1.5) have settled down.
to their final value (either 0 or 1) on the net \(\{n_\alpha\}_{\alpha \in \Lambda}\). Thus the net of real numbers \(\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}\) converges. Since \(\{n_\alpha\}_{\alpha \in \Lambda}\) was an arbitrary net from an arbitrary ultrafilter on \(\mathbb{Z}\), this shows that \(FU\) is a well-defined map from the ultrafilters on \(\mathbb{Z}\) into \(\beta(\mathbb{Z})\). One easily checks that the compositions \(UF \circ FU\) and \(FU \circ UF\) are identity maps, and so we have proved:

**Proposition 1.6.** The map \(UF\) is an isomorphism of the Stone-Čech compactification \(\beta(\mathbb{Z})\) with the set of ultrafilters on \(\mathbb{Z}\). Under this map, a point \(n_0 \in \beta(\mathbb{Z})\) is taken to the ultrafilter of sets that any net of integers converging to \(n_0\) is eventually in.

### 3 Definition of the Smooth Functions \(\ell_\infty(\mathbb{Z})\)

( also denoted by \(C^\infty(\beta(\mathbb{Z}))\) )

**Definition 2.1.** Define \(\ell^\infty_c(\mathbb{Z})\) to be the finite span of projections in \(\ell^\infty(\mathbb{Z})\). This is a dense \(*\)-subalgebra of \(\ell^\infty(\mathbb{Z})\), and plays an analogous role to \(\ell^\infty(\mathbb{Z})\) as \(c^\infty_c(\mathbb{Z})\), the compact (or finite) support functions on \(\mathbb{Z}\), does to \(c^0(\mathbb{Z})\). (The series expansion (1.5) proves the density.)

For \(n_0 \in \beta(\mathbb{Z})\), choose a net \(\{n_\alpha\}_{\alpha \in \Lambda}\) of integers converging to \(n_0\). Define the *smooth functions on \(\beta(\mathbb{Z})\), denoted by \(\ell^\infty(\mathbb{Z})\) or \(C^\infty(\beta(\mathbb{Z}))\), to be those
functions $\varphi$ in $\ell^\infty(Z)$ that satisfy

$$\lim_{\alpha \in \Lambda} n^d_\alpha \left( \varphi(n_\alpha) - \varphi(n_0) \right) = 0 \quad (2.2)$$

for each $d = 0, 1, 2, \ldots$ and for each $n_0 \in \beta(Z)$. This set of functions $\ell^{\infty\infty}(Z)$ is a dense $\ast$-subalgebra of $\ell^\infty(Z)$, which contains $\ell^\infty_c(Z)$, and plays an analogous role to $\ell^\infty(Z)$ as $S(Z)$, the Schwartz functions on $Z$, does to $c_0(Z)$. If $p$ is a projection in $\ell^\infty(Z)$, we noticed above that $p(n_0) = 0$ or $p(n_0) = 1$ for any $n_0 \in \beta(Z)$. In either case, the quantity in parentheses in (2.2) eventually becomes 0, so that $\varphi = p$ satisfies (2.2). This shows that $\ell^\infty_c(Z) \subseteq \ell^{\infty\infty}(Z)$.

**Lemma 2.3.** The limit (2.2) holds independently of the choice of net $\{n_\alpha\}_{\alpha \in \Lambda}$ converging to $n_0$.

**Proof:** Let $\epsilon > 0$ be given and find a $\beta$ such that $|n^d_\alpha(\psi(n_\alpha) - \psi(n_0))| < \epsilon$ for $\alpha \geq \beta$. The set $S = \bigcup \{n_\alpha \mid \alpha \geq \beta\}$ is in the ultrafilter associated with $n_0$ and $|m^d(\psi(m) - \psi(n_0))| < \epsilon$ for $m \in S$. If $\{m_\alpha\}_{\alpha \in \Gamma}$ is another net tending to $n_0$, then it is eventually in $S$. So we have $|m^d_\alpha(\psi(m_\alpha) - \psi(n_0))| < \epsilon$ for $\alpha \geq \gamma$, for some $\gamma \in \Gamma$. QED

To see that $\ell^{\infty\infty}(Z)$ is closed under products, let $\varphi, \psi \in \ell^{\infty\infty}(Z)$. Then evaluate the quantity in parentheses in (2.2), namely the difference

$$\varphi(n_\alpha)\psi(n_\alpha) - \varphi(n_0)\psi(n_0) = \left( \varphi(n_\alpha) - \varphi(n_0) \right) \psi(n_\alpha) + \varphi(n_0) \left( \psi(n_\alpha) - \psi(n_0) \right).$$
So the absolute value of the quantity in the limit (2.2) is

$$\left| n_\alpha^d \left( \varphi \psi(n_\alpha) - \varphi(n_0) \right) \right| \leq \left| n_\alpha^d \left( \varphi(n_\alpha) - \varphi(n_0) \right) \right| \| \psi \|_\infty + \| \varphi \|_\infty \left| n_\alpha^d \left( \psi(n_\alpha) - \psi(n_0) \right) \right|$$

Clearly, this tends to zero as $n_\alpha$ tends to $n_0$.

Next, we note that $\ell^{\infty\infty}(\mathbb{Z})$ is actually bigger than $\ell_\infty^c(\mathbb{Z})$. For example, any function in $S(\mathbb{Z})$ satisfies the limit (2.2), so $\ell^{\infty\infty}(\mathbb{Z}) \supseteq \ell_\infty^c(\mathbb{Z}) + S(\mathbb{Z})$. For $\varphi \in S(\mathbb{Z})$, note that $\varphi(n_0) = 0$ for any $n_0 \in \beta(\mathbb{Z}) - \mathbb{Z}$. (Any nonprincipal ultrafilter eventually leaves every finite set.) Therefore

$$\left| n_\alpha^d \left( \varphi(n_\alpha) - \varphi(n_0) \right) \right| = \left| n_\alpha^d \varphi(n_\alpha) \right| = 1/n_\alpha^2 \left| n_\alpha^{d+2} \varphi(n_\alpha) \right| \leq 1/n_\alpha^2 \| \varphi \|_{d+2},$$

where $\| \|_{d+2}$ denotes the $d + 2$th Schwartz seminorm on $S(\mathbb{Z})$.

**Proposition 2.4.** The inclusions $\ell_\infty^c(\mathbb{Z}) + S(\mathbb{Z}) \subseteq \ell^{\infty\infty}(\mathbb{Z}) \subseteq \ell_\infty^c(\mathbb{Z})$ are proper.

**Proof:** The function $\frac{1}{n^2 + 1}$ in $c_0(\mathbb{Z}) - S(\mathbb{Z})$ does not satisfy (2.2), so the second inclusion is proper. The first inclusion is proper since $e^{S(\mathbb{Z})} \subseteq \ell^{\infty\infty}(\mathbb{Z})$. QED

Let $\varphi \in \ell^{\infty\infty}(\mathbb{Z})$. We may write $\varphi$ (in fact any function in $\ell_\infty^c(\mathbb{Z})$) uniquely in the form

$$\varphi = \sum_{q=1}^\infty c_q p_q,$$

where the $p_q$’s are pairwise disjoint projections, and the $c_q$’s are distinct constants.
Proposition 2.6. At most finitely many projections in the series (2.5) have infinite support.

Proof: We assume for a contradiction that there are infinitely many projections in (2.5), each having infinite support. Note that the coefficients of the infinite projections must have an accumulation point. By dropping to a subsequence, and getting rid of all the finite projections, we may assume that \( c_q \to c_0 \in \mathbb{C} \) as \( q \to \infty \). (Multiply \( \varphi \) by an appropriate projection, and renumber the \( c_q \)'s.) Let \( S_q \subseteq \mathbb{Z} \) denote the support of the projection \( p_q \). Then each set \( S_q \) is infinite. Define a decreasing sequence of infinite subsets of \( \mathbb{Z} \) by the disjoint unions

\[
U_n = \bigcup_{q \geq n} \left( S_q \cap \left\{ m \in \mathbb{Z} \mid m \geq \frac{1}{|c_q - c_0|} \right\} \right)
\]

for \( n = 1, 2, \ldots \). Since finite intersections of the sets \( U_n \)'s are non-empty, there exists an ultrafilter \( \mathcal{U} \) for which \( U_n \in \mathcal{U} \) for each \( n \). Let \( \{n_\alpha\}_{\alpha \in \mathcal{U}} \) be a net of integers that is eventually in this ultrafilter. This net must therefore also eventually be in each of the sets \( U_n \). By construction, the point \( n_0 \in \mathcal{F}(\mathcal{U}) \subseteq \beta(\mathbb{Z}) \) that \( \{n_\alpha\}_{\alpha \in \mathcal{U}} \) converges to must satisfy \( \varphi(n_0) = c_0 \).
If \( n_\alpha \in S_q \cap \{ m \in \mathbb{Z} \mid m \geq \frac{1}{|c_q - c_0|} \} \), then

\[
\left| n_\alpha^d \left( \psi(n_\alpha) - \psi(n_0) \right) \right| = \left| n_\alpha^d (c_q - c_0) \right| \quad \text{since} \quad n_\alpha \in S_q \\
\geq \left| n_\alpha^d \times \frac{1}{n_\alpha} \right| = \left| n_\alpha^{d-1} \right| \geq 1,
\]

contradicting the fact that \( \varphi \) must satisfy (2.2) for \( d \geq 2 \) at the point \( n_0 \in \beta(\mathbb{Z}) \) we constructed. \textbf{QED}

It follows from Proposition 2.6 that for any \( \varphi \in \ell^{\infty}(\mathbb{Z}) \), we may substract an element of \( \ell^\infty_c(\mathbb{Z}) \) to force the expansion (2.5) to have only projections of finite support.

\textbf{Remark 2.7.} Note that \( S(\mathbb{Z}) \) is an ideal in \( \ell^{\infty}(\mathbb{Z}) \) and \( \ell^\infty(\mathbb{Z}) \). The closure \( \overline{S(\mathbb{Z})}^{\| \cdot \|_\infty} \) is equal to \( c_0(\mathbb{Z}) \), so \( S(\mathbb{Z}) \) is not dense in \( \ell^\infty(\mathbb{Z}) \). The algebra \( \ell^\infty_c(\mathbb{Z}) \), being unital, is not an ideal in either algebra \( \ell^{\infty}(\mathbb{Z}) \) or \( \ell^\infty(\mathbb{Z}) \), and for the same reason \( \ell^{\infty}(\mathbb{Z}) \) is not an ideal in \( \ell^\infty(\mathbb{Z}) \). This is in contrast to the case \( c_c(\mathbb{Z}) \subseteq S(\mathbb{Z}) \subseteq c_0(\mathbb{Z}) \), where every algebra is a dense ideal in every algebra above it.
4 References

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