On the equivalence between the effective cosmology and excursion set treatments of environment

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ABSTRACT
In studies of the environmental dependence of structure formation, the large scale environment is often thought of as providing an effective background cosmology: e.g. the formation of structure in voids is expected to be just like that in a less dense universe with appropriately modified Hubble and cosmological constants. However, in the excursion set description of structure formation which is commonly used to model this effect, no explicit mention is made of the effective cosmology. Rather, this approach uses the spherical evolution model to compute an effective linear theory growth factor, which is then used to predict the growth and evolution of nonlinear structures. We show that these approaches are, in fact, equivalent: a consequence of Birkhoff’s theorem. We speculate that this equivalence will not survive in models where the gravitational force law is modified from an inverse square, potentially making the environmental dependence of clustering a good test of such models.

Key words: methods: analytical - dark matter - large scale structure of the universe

1 INTRODUCTION
One of the standard predictions of nonlinear hierarchical structure formation models is the abundance of virialized structures (Press & Schechter 1974; Sheth & Tormen 1999; Jenkins et al. 2001). Simulations show that this abundance depends on the large scale environment: the ratio of massive to low mass objects is larger in dense regions (e.g., Frenk et al. 1988). Recent measurements in galaxy surveys appear to bear this out: the virial radii of objects in underdense regions are smaller, consistent with their having smaller masses (e.g., Abbas & Sheth 2007).

This paper is motivated by the fact that there are currently in the literature three methods for estimating how the mass function of virialized halos depends on the environment which surrounds them. The first, and perhaps easiest to implement, is based on the excursion set approach (Mo & White 1996; Sheth & Tormen 2002). The second argues that halos which form in, say, voids should be thought of as forming in a less dense background cosmology, so the mass function is that in a universe with \( \Omega_{\text{void}} = \Omega_0 (1 + \Delta_{\text{void}}) \) (e.g., Gottlöber et al. 2003). The third is similar, but notes that to correctly estimate the background cosmology, one must account not only for the lower density in a void, but for the fact the effective Hubble constant of the void cosmology is larger than in the background (e.g., Goldberg & Vogeley 2004). One way of thinking about the effective Hubble constant is that it ensures that the effective cosmology has the same age as the background cosmology. (The cosmological constant is, of course, constant, but when expressed in units of the critical density in the effective model, it is modified because the critical density depends on the effective Hubble constant.) In Section 2 we use the spherical evolution model to show that the first and third methods are equivalent (although Goldberg & Vogeley 2004 state otherwise), and that both are incompatible with the second method (which incorrectly ignored the change to the Hubble constant).

There has been recent interest in the fact that the formation histories of halos of fixed mass depend on their environment (Sheth & Tormen 2004; Gao et al. 2005), an effect which is not predicted by the simplest excursion set methods (e.g., White 1998). So one might have wondered if this is where the difference between the excursion set approach and one based on the effective cosmology is manifest. In Section 3 we show that in this case also, the two approaches are equivalent.

A final section summarizes our results, and speculates that the equivalence we have shown will not survive in models models where the force law has been modified from an inverse square.
2 THE CALCULATION

The main point of the following calculation is to show explicitly that, at least for cosmologies with no cosmological constant, the environmental dependence of halo abundances can be described using the excursion set approach (e.g., Mo & White 1996; Sheth & Tormen 2002). Namely, one need not worry about the details of the effective cosmology associated with the region surrounding the perturbation (as do Goldberg & Vogel 2004); it is enough to compute an effective growth factor using the spherical collapse model. Although we have phrased our discussion in terms of an $\Omega_0 = 1$ background cosmology, it is obviously applicable to arbitrary values of $\Omega_0$. Our analysis suggests that this remains true when the background cosmology has $\Lambda \neq 0$.

For what follows, it is useful to recall that the age-redshift relation in an $\Omega_0 = 1$ cosmology is given by $H(z)t(z) = 2/3$, where $H$ is the Hubble constant. In an open universe, this relation is

$$H(z)t(z) = \frac{1}{1 - \Omega(z)} - \frac{\Omega(z)/2}{(1 - \Omega(z))^{3/2}} \cosh^{-1} \left( \frac{2}{\Omega(z)} - 1 \right)$$

where

$$\Omega(t) = \frac{\Omega_0/a(t)^3}{\Omega_0/a(t)^3 + (1 - \Omega_0)/a(t)^2},$$

with the convention that $a(t_0) \equiv 1$, so $a(t) \equiv (1 + z)^{-1}$, and $\Omega_0 \equiv \Omega(t_0)$ (Peebles 1980).

The linear theory growth factor is $D(t) = a(t)$ if $\Omega_0 = 1$, and if $\Omega_0 < 1$ then

$$D(t) = \frac{5\Omega_0/2}{1 - \Omega_0} \left( 1 + 3 \frac{x}{3} + 3 \frac{1 + x}{x^3} \ln \left[ \sqrt{1 + x} - x \right] \right)$$

where $x = a(t)(1 - \Omega_0)/\Omega_0$ (Peebles 1980).

2.1 The spherical evolution model

The spherical evolution model describes the evolution of the size $R$ of a spherical region in an expanding universe:

$$\frac{d^2R}{dt^2} = -\frac{GM(< R)}{R^2}.$$ (4)

It provides a parametric relation between the density contrast predicted by linear theory $\delta(t_0) = D(t_0)/D(t_{init}) \delta(t_{init})$, the nonlinear overdensity $\Delta$, and the infall speeds $v_{pec} \equiv (\text{Gunn & Gott 1972; Schechter 1980; Peebles 1980; Padmanabhan 1993; Bernardeau et al. 2002})$. Here $D(t_0)$ is the linear theory growth factor at time $t_0$, and we will often use the shorthand, $\delta_0 = \delta(t_0)$.

If $\Omega = 1$, then

$$\frac{M}{4\pi R^3\rho_0} \equiv 1 + \Delta = f(\theta), \quad \frac{v_{pec}}{HR} \equiv g(\theta), \quad \delta_0 \equiv h(\theta),$$

where

$$f(\theta) = \left\{ \begin{array}{ll} (9/2) (\theta - \sin \theta)^2/(1 - \cos \theta)^3 & \\
(9/2) (\sinh \theta - \theta)^2/(\cosh \theta - 1)^3 & \end{array} \right.,$$

$$g(\theta) = \left\{ \begin{array}{ll} (3/2) \sin \theta (\theta - \sin \theta)/(1 - \cos \theta)^2 & \\
(3/2) \sinh \theta (\sinh \theta - \theta)/(\cosh \theta - 1)^2 & \end{array} \right.,$$

$$h(\theta) = \left\{ \begin{array}{ll} (3/5) (3/4)^{1/3} (\theta - \sin \theta)^{3/2} & \\
-(3/5) (3/4)^{1/3} (\sinh \theta - \theta)^{3/2} & \end{array} \right.,$$

where the first expression in each pair is for initially overdense perturbations and the second is for underdense ones. Overdense perturbations eventually collapse, the final collapse being associated with the value $\theta = 2\pi$, at which time the linear theory density is $\delta_{\Delta1} \equiv (3/5)(6\pi/4)^{2/3} = 1.68647$. In this section, we use the subscript 1 to indicate that this value is associated with $\Omega_0 = 1$.

If $\Omega_0 < 1$, then only perturbations above some density $\delta_{\Omega0}$ will collapse, and

$$1 + \Delta = \frac{f(\theta)}{f(\omega)}, \quad \frac{v_{pec}}{HR} = \frac{g(\theta)}{g(\omega)} - 1, \quad \text{and} \quad \frac{\delta_0}{\delta_{\Omega0}} = -\frac{h(\theta)}{h(\omega)} + 1,$$

where

$$\omega = \left\{ \begin{array}{ll} \arccos(2/\Omega_0 - 1) & \text{if closed} \\
\arccosh(2/\Omega_0 - 1) & \text{if open} \end{array} \right.,$$

$H_\omega$ is the Hubble constant, and

$$\delta_{\Omega0} = \frac{9 \sinh \omega (\sinh \omega - \omega)}{2 (\cosh \omega - 1)^3} - 3.$$ (8)

Complete collapse is again associated with $\theta = 2\pi$, and we will write the critical linear density required for collapse as

$$\delta_{\omega0} = \delta_{\Omega0} \left[ 1 - \frac{\delta_{\omega1}/h(\omega)}{\delta_{\Omega1}/h(\omega)} \right].$$ (9)

It happens that $\delta_{\omega0}$ depends only weakly on $\Omega_0$. When $\Omega \to 0$, then $\delta_{\omega0} \to (3/5)(6\pi/4)^{2/3} = 1.68647$, and $\delta_{\omega0} \to 3/2$ when $\Omega_0 \to 0$.

The parametric solution is rather cumbersome. It happens that the relation between $\delta_0$ and $\Delta$ is rather well approximated by

$$1 + \Delta \approx (1 - \delta_0/\delta_{\omega0})^{-\delta_{\omega0}}.$$ (10)

Similarly, it is also useful to have an approximation to the exact solution for the linear theory growth factor. When $\Omega_0 \leq 1$, then the linear theory growth factor is well approximated by

$$D(t) \approx \frac{(5/2) a(t) \Omega(t)}{\Omega(t)^{4/3} + 1 + \Omega(t)/2},$$

(Carroll et al. 1992), where $a(t)$ denotes the expansion factor at time $t$, and $\Omega(t)$ is given by equation (2). This expression is normalized so that $D(t_0) = a(t_0) = 1$ if $\Omega_0 = 1$.

2.2 Environment and spherical evolution

Suppose we consider the evolution of a spherical underdense region in an $\Omega_0 = 1$ universe. Let $1 + \Delta_\omega < 1$ denote the density in this region. If we wish to think of this region as being an underdense universe, then the effective value of $\Omega$ in this region is smaller than unity for two reasons: first, because the density is lower by a factor of $1+\Delta_\omega$, and second because the region is expanding faster than the background, so it has an effective Hubble constant $H_\omega$ which is larger.

To see what equation (5) implies for the evolution, let $1 + \Delta_1$ denote the density of a small patch respect to the background density (the subscript unity denotes the fact that this is the overdensity with respect to a background which has critical density: $\Omega_0 = 1$). Now, suppose that this patch is surrounded by a region $U$ within which the average density is $1 + \Delta_\omega$ with respect to the true background. Then
the smaller patch has overdensity \((1 + \Delta_1)/(1 + \Delta_\omega)\) with respect to its local background. If we wish to describe the local environment as having its own effective cosmological parameters, then the local value of the Hubble constant \(H_\omega\) differs from the global one \(H_0\): \(H_\omega/H_0 = g(\omega)\). Thus, the expressions in equation (6) are really the statements that

\[
1 + \Delta = \frac{1 + \Delta_1}{1 + \Delta_\omega} \quad \text{and} \quad \frac{v_{\text{pec}}}{H_\omega \dot{R}} = \frac{v_{\text{pec}1}}{H_\omega \dot{R}_1} = \frac{u_{\text{pec}1} - u_{\text{pec}1}}{u_{\text{pec}1}},
\]

where \(u_{\text{pec}1}\) is the peculiar velocity of the shell \(U\) with respect to the background, had the mass within \(U\) been smoothly distributed (we know it is not because the central region has density \(1 + \Delta_1\)). Now, the local value of \(\Omega_\omega\) within \(U\) differs from the global value \(\Omega_0 = 1\) both because \(\Delta_\omega \neq 0\) and because the different expansion rate means that the local value of the critical density is different:

\[
\Omega_\omega(t_0) = \frac{\Omega_0(1 + \Delta_\omega)}{(H_\omega/H_0)^2} = \frac{f(\omega)}{g(\omega)^2},
\]

where we have used the fact that \(\Omega_0 = 1\). Notice that this relation between \(\Omega_\omega\) and \(\omega\) is the same as equation (12). In other words, we get the same description for the evolution of the small scale patch if we treat it as having overdensity \(1 + \Delta_1\) with respect to the \(\Omega_0 = 1\) background within which the Hubble constant is \(H_0\), as if we describe it with respect to the local cosmological model \(\Omega_\omega\) and \(H_\omega\), and rescale our definitions of density and peculiar velocity accordingly. In addition, using the exact expression for the age of the universe given above, we can see that these definitions also guarantee that \(t_0\) is the same in both the background and the local cosmological model.

If we write the linear theory overdensity associated with \(1 + \Delta_\omega\) as

\[
\delta_\omega = h(\omega),
\]

then

\[
\delta_0 = \frac{\delta_{\text{min}}}{\delta_\omega} \left[ h(\theta) - \delta_\omega \right] = \frac{\delta_{\text{min}}}{\delta_\omega} \left[ h(\theta) - \delta_\omega \right].
\]

The term in square brackets is simply the difference in linear theory values for the background cosmology. If we think of this as an effective linear theory overdensity in the effective cosmology, then the prefactor is the effective linear theory growth factor. It is straightforward to verify that, indeed,

\[
\frac{\delta_{\omega}}{\delta_{\text{min}}} = \frac{D_\omega}{D_1} \quad \text{or} \quad \frac{\delta_{\omega}}{\delta_{\omega}} = \frac{\delta_{\text{min}}}{\delta_1} - \frac{\delta_{\text{min}}}{\delta_\omega}.
\]

where \(D_1\) is the growth factor in the background cosmology, and \(D_\omega\) is the growth factor in the patch, at time \(t_0\). This last point is important, as the expansion factor \(a(t_0)\) in the patch cosmology is not equal to the expansion factor in the background cosmology. In particular, we know that

\[
a_\omega(t_0)/a_1(t_0) = (1 + \Delta_\omega)^{-1/3}.
\]

For completeness, we note that

\[
\delta_{\omega} = \delta_{\text{min}} \left[ 1 + \frac{(2\pi)^{2/3}}{3} \left( \sinh \omega - \omega \right)^{2/3} \right],
\]

(recall that we are in an underdense region).

In the following, take \(a_1(t_0) = 1\), so \(D_1 = 1\). The approximate solution (10) of the spherical evolution model shows similar behaviour:

\[
1 + \Delta = \frac{1 + \Delta_1}{1 + \Delta_\omega} = \left( \frac{1 - \delta_1/\delta_{\text{min}}}{1 - \delta_{\omega}/\delta_{\text{min}}} \right)^{-\delta_{\text{min}}} = \left( \frac{\delta_{\text{min}} - \delta_1}{\delta_{\text{min}} - \delta_\omega} \right)^{-\delta_{\text{min}}},
\]

\[
= \left( 1 - \frac{\delta_1 - \delta_\omega}{\delta_{\text{min}} - \delta_\omega} \right)^{-\delta_{\text{min}}} = \left( 1 - D_\omega \frac{\delta_1 - \delta_\omega}{\delta_{\text{min}} - \delta_\omega} \right)^{-\delta_{\text{min}}},
\]

where \(\delta_1\) denotes the linear theory value associated with the nonlinear density \(\Delta_1\) for \(\Omega_0 = 1\). The final approximation follows from recalling that \(\delta_{\text{min}}\) depends only weakly on cosmology. Comparison with equation (10) shows explicitly that the relevant linear theory quantity is the difference between the \(\Omega_0 = 1\) values for the perturbation and the environment, and this difference must be multiplied by the linear growth factor \(D_\omega\) in the effective cosmology.

Now, to estimate the mass function of virialized objects, we are interested in the case when \(\theta = 2\pi\). The analysis above shows that \(\delta_{\text{min}}/D_\omega = \delta_{\text{min}} - \delta_1\); the objects which form in a region of nonlinear density \(1 + \Delta_\omega\) with respect to the background, with corresponding linear overdensity \(\delta_\omega\), can either be thought of as forming in an effective \(\Omega_\omega\) cosmology (e.g., Goldberg & Vogeley 2004), or as forming in the true \(\Omega_0\) background cosmology but with an effective linear theory overdensity which is offset by \(\delta_1\) to account for the surrounding overdensity (e.g., Mo & White 1996; Sheth & Tormen 2002). The second description is easier to implement, and follows naturally from the excursion set description. In particular, the analysis above shows that approaches which do not correctly compute \(\Omega_\omega\) (e.g., Gottlöber et al. 2003) ignore the fact that \(H_\omega \neq H_0\) are incompatible with the excursion set approach. In any case, the analysis above suggests that such approaches are ill-motivated.

### 3 Formation Histories

The previous section showed that the excursion set approach results in the same expressions for the environmental dependence of the present day linear theory growth factor as one derives from thinking of the environment as defining an effective cosmology. So the question arises as to whether or not the two approaches predict the same evolution. For example, one might have wondered if the formation histories of objects are the same in these two approaches.

To see that they are, it will be convenient to modify our notation slightly. We showed that

\[
\frac{\delta_1(\Omega_\omega)}{D_{\text{c}0}} = \frac{\delta_1(\Omega_0) - \delta_i(\Delta_0)}{D_0}
\]

where the subscripts \(0\) mean the present time. The quantity \(\delta_i(\Delta_0)\) is what we previously called \(\delta_i\); it is the value of the initial overdensity extrapolated using linear theory (of the background cosmology) to the time at which the nonlinear density is \(\Delta_0\). Also, we previously had set the growth factor \(D_\omega\) in the effective cosmology at the present time to unity: \(D_0 = 1\). We have written it explicitly here to show that, had we chosen to perform the calculation for some earlier time, then we would have found

\[
\frac{\delta_1(\Omega_\omega)}{D_{\text{c}1}} = \frac{\delta_1(\Omega_0) - \delta_i(\Delta_1)}{D_1}.
\]
where the subscript 1 denotes the earlier time. I.e., $\Omega_{\omega 1}$ is the effective cosmology associated with the overdensity $\Delta_1$, which itself is related to $\Delta_0$ by the spherical evolution model (the region that is $\Delta_0$ today was a different volume in the past, but its mass was the same.) And, analogously to the previous expression, $\delta L_1(\Delta_1)$ is the initial overdensity extrapolated using linear theory to the (earlier) time at which the nonlinear density was $\Delta_1$. Since $\Delta_1$ is closer to 0 than is $\Delta_1$, $\delta L_1(\Delta_1)$ is also closer to 0 than is $\delta L_1(\Delta_0)$.

If one were to apply the excursion set approach to study formation histories in the effective cosmology, one would be interested in the difference between equations (19) and (15).

$$\frac{\delta L_1(\Omega_{\omega 1}) - \delta L_1(\Omega_{\omega 0})}{D_{\omega 1}} = \frac{\delta L_1(\Omega_1)}{D_1} - \frac{\delta L_1(\Omega_0)}{D_0} - \left[ \frac{\Delta_1(\Delta_1)}{D_1} - \frac{\Delta_1(\Delta_0)}{D_0} \right].$$

(20)

Now, the quantity in square brackets is

$$\frac{\delta L_1(\Delta_1)}{D_1} - \delta L_1(\Delta_0) \left[ \frac{\Delta_1(\Delta_1)}{D_1} - \frac{\Delta_1(\Delta_0)}{D_0} \right] D_0^{-1} = 0,$$

(21)

because $\delta L_1(\Delta_1)$ and $\delta L_1(\Delta_0)$ are the same quantity (the initial overdensity), evolved using linear theory to two different times. In particular, $\delta L_1(\Delta_1)$ is closer to 0 than is $\delta L_1(\Delta_0)$ by $\delta L_1(\Delta_1)/\delta L_1(\Delta_0) = D_1/D_0$. Thus,

$$\frac{\delta L_1(\Omega_{\omega 1}) - \delta L_1(\Omega_{\omega 0})}{D_{\omega 1}} = \left[ \frac{\delta L_1(\Omega_1)}{D_1} - \frac{\delta L_1(\Omega_0)}{D_0} \right] D_0^{-1}.$$  

(22)

Note that the expression on the right has no dependence on the effective cosmology. Moreover, it is exactly the same as the expression that one obtains when using the excursion set approach to study formation histories in the background cosmology. It is in this sense that the formation histories of objects are independent of the effective cosmology of the environment; the excursion set approach is a simple self-consistent way of exploiting this fact.

### 4 DISCUSSION

The excursion set description provides a simple, self-consistent way of estimating the effect of environment on structure formation and evolution. In particular, it is equivalent to using the fact that the large scale environment can be thought of as providing an effective background cosmology of the same age (Section 2). Estimating the parameters of the effective cosmology is slightly more involved, but useful for running simulations which mimic the formation of structure in different environments.

In essence, the equivalence between the excursion set and effective cosmology descriptions is a consequence of Birkhoff's theorem: the evolution of a perturbation does not depend on its surroundings. There has been recent interest in models with modified gravitational force laws (e.g., Shirata et al. 2002; Stabennau & Jain 2006; Shirata et al. 2007). Since Birkhoff's theorem does not apply in such models (Martino et al. 2008; Schäfer & Kovama 2008; Dai et al. 2008; Clifton 2006), it will be interesting to see if this equivalence survives. If not, the environmental dependence of clustering may be added as another constraint on such models.

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