Blackhole in Nonlocal Gravity: Comparing Metric from Newmann-Janis Algorithm with Slowly Rotating Solution

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The strong gravitational field near massive blackhole is an interesting regime to test General Relativity (GR) and modified gravity theories. The knowledge of spacetime metric around a blackhole is a primary step for such tests. Solving field equations for rotating blackhole is extremely challenging task for the most modified gravity theories. Though the derivation of Kerr metric of GR is also demanding job, the magical Newmann-Janis algorithm does it without actually solving Einstein equation for rotating blackhole. Due to this notable success of Newmann-Janis algorithm in the case of Kerr metric, it has been being used to obtain rotating blackhole solution in modified gravity theories. In this work, we derive the spacetime metric for the external region of a rotating blackhole in a nonlocal gravity theory using Newmann-Janis algorithm. We also derive metric for a slowly rotating blackhole by perturbatively solving field equations of the theory. We discuss the applicability of Newmann-Janis algorithm to nonlocal gravity by comparing slow rotation limit of the metric obtained through Newmann-Janis algorithm with slowly rotating solution of the field equation.

I. INTRODUCTION

Recent gravitational wave observations done by Adv. LIGO/VIRGO\(^1\) and an image captured by Event Horizon Telescope\(^2\) has rejuvenated interest about a mysterious object called 'blackhole' in scientific community all over the world. Naively speaking, blackhole is a star collapsed by its own gravity to a point where the curvature of the spacetime diverges to infinite. The solution of the Einstein equation for spacetime surrounding spherically symmetric blackhole is given by Schwarzschild metric. It is a static vacuum solution of the Einstein equation for spherical symmetry. Due to its strong gravitational field region near a blackhole can be a good laboratory for the tests of any gravity theory. Since most astrophysical objects rotate, the blackholes created from their gravitational collapses cannot be a static object instead they also rotate. The rotating blackhole solution of the Einstein equation was given by R.P.Kerr in 1963\(^3\). It is a stationary, axially symmetric vacuum solution.

In 1965, Newmann and Janis shown that by means of complex coordinate transformations operated on Schwarzschild metric one obtains a new metric. They investigated the new metric and found that the new metric corresponds to a massive ring rotating about its axis of symmetry\(^4\). This method was also shown to be working between Reissner-Nordström and Kerr-Nemann metric. The generalization of this method in the presence of cosmological constant was done by Demiański\(^5\). Gradually it became very popular to derive the rotating blackhole metric because it avoids all mathematical difficulties involved in solving Einstein equation and is generally known as Newmann-Janis (NJ) or Demiański-Janis-Newman (DJN) algorithm.

Initial developments in Nemann-Janis algorithm were majorly restricted to General Relativity (GR). But, discovery of late-time cosmological acceleration\(^6\) and quest for quantum gravity provided thrust to look for modified gravity theories beyond GR. In the absence of fundamental direction, these studies are being done mostly on trial and error basis. Various modified gravity theories are constructed and tested against astrophysical and cosmological observations. The detection of gravitational waves made strong field tests of modified gravity possible and therefore it is important to know the structure of the rotating blackhole metric for modified gravity theories.

Lately, Newmann-Janis method has been being used comfortably as a way to derive the Kerr metric in different modified gravity theories\(^7\)\(^8\). But, it should be done with a caution that the metric generated from the NJ method may not be the solution of the field equations of a particular theory because some cases of failure of NJ algorithm in non-GR theories are reported\(^9\)\(^10\). In this work, we show that the metric generated from the NJ algorithm does not match with the metric derived by solving field equations in a nonlocal gravity model.

In recent studies Nonlocal gravity has been emerged as a good candidate for cosmological constant. The first nonlocal model was studied by Deser and Woodard in 2008, they considered nonlocal correction given by $RF(\Box)\(^11\).
In this work we consider a specific model, called RR model, that has correction term proportional to \( R^{1/2} R \) proposed in \([2]\). The complete action for RR model is given by

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{m^2}{3} R^{1/2} R \right] + \mathcal{L}_m, \tag{1}
\]

where \( m \) is the mass scale associated with nonlocal correction to the Einstein-Hilbert (EH) action. In the limit \( m \rightarrow 0 \), the above action \([1] \) reduces to EH action. This model is studied substantially in \([21, 22]\). The field equations \([20]\) corresponding to action \([1] \) are

\[
\kappa^2 T_{\alpha\beta} = G_{\alpha\beta} - \frac{m^2}{3} \left\{ 2\left( G_{\alpha\beta} - \nabla_\alpha \nabla_\beta + g_{\alpha\beta} \nabla^\gamma \nabla_\gamma \right) S + g_{\alpha\beta} \nabla^\gamma U \nabla_\gamma S - \nabla_\alpha U \nabla_\beta S - \frac{1}{2} g_{\alpha\beta} U^2 \right\},
\tag{2}
\]

\[
U = -\frac{1}{\Box} R, \quad S = -\frac{1}{\Box} U
\]

\( G_{\alpha\beta} \) is the Einstein tensor and \( T_{\alpha\beta} \) is the energy-momentum tensor of the matter.

This paper is organized as follows: In Sec. II, we derive the rotating blackhole metric for the RR model of the nonlocal gravity using DJN algorithm. In Sec. III, we solve the field equations of RR model for slowly rotating blackhole by perturbing spherically symmetric static solution of the model. Comparison of metrics obtained through above mentioned two approaches are done in Sec. IIIA. Finally, we conclude our work in Sec. IV.

## II. ROTATING BLACKHOLE IN RR MODEL OF NONLOCAL GRAVITY

It is by now well known that NJ algorithm does some “trick” which transforms Schwarzschild metric into Kerr metric. The “trick” is complexifying coordinates and then performing complex coordinate transformations. But, we still do not know how complexifying coordinates gives rotation to a static blackhole and exactly generate stationary axisymmetric vacuum solution of the Einstein equation without actually solving Einstein equation. The only requirement is the spherically symmetric static vacuum solution of the Einstein equation which works as a “seed” metric. In the original NJ algorithm, we choose how to complexify the coordinates. In the original NJ algorithm the choice was such that it gives Kerr metric in the end. This prevents the generalization of the NJ method especially when the rotating metric is known apriori. Some efforts have been made to reduce the intrinsic arbitrariness of the method but still it remains elusive \([12, 17]\). Here, we apply the NJ algorithm on the spherically symmetric static solution of the field equations of the RR model of nonlocal gravity as written in \([2]\) and obtain “some” metric. We will compare this metric generated from NJ algorithm with solution of the field equations \([2]\) in slow rotation limit in the next section.

Let us first write the seed metric which is Schwartzschild solution of the RR model, derived in \([20]\).

\[
ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\tag{3}
\]

where \( A(r) \) and \( B(r) \) are given by

\[
A(r) \simeq 1 - \frac{2GM}{r} \left( 1 + \frac{m^2 r^2}{6} \right),
\tag{4}
\]

\[
B(r) \simeq A(r)^{-1}
\tag{5}
\]

The above metric is written in usual Boyer-Lindquist coordinates \((t, r, \theta, \phi)\). From cosmological point, the motivation of the RR model is to explain the late-time acceleration of the universe. Therefore the mass \( m \) of the field corresponding to the correction term (the second term in the parenthesis in \((4)\)) is of the order of Hubble length \( H_0^{-1} \). Since the distance \( r \) from the source is much smaller than the \( m^{-1} \), the authors of \([20]\) derived Schwartzschild solution in region \( r << m^{-1} \) taking low-\( m \) expansion in the field equations.

The first step of the NJ algorithm is to transform the metric in \((3)\) into Eddington-Finkelstein coordinates \((u, r, \theta, \phi)\) via

\[
dt = du + \left( 1 - \frac{2GM}{r} \left( 1 + \frac{m^2 r^2}{6} \right) \right)^{-1} dr
\tag{6}
\]
The metric (3) now can be written as
\[ds^2 = -\left[1 - \frac{2GM}{r} \left(1 + \frac{m^2r^2}{6}\right)\right] \, du^2 - 2 \, du \, dr + r^2 \, (d\theta^2 + \sin^2 \theta \, d\phi^2).\] (7)

Expressing the metric in (7) in terms of null tetrad vectors as
\[g^{\mu\nu} = \mathbf{l}^{\mu} n^{\nu} + \mathbf{l}^{\nu} n^{\mu} - m^{\mu} \overline{m}^{\nu} - m^{\nu} \overline{m}^{\mu},\] (8)
where the null tetrad vectors \(l^{\mu}, n^{\mu}, m^{\mu}\) and \(\overline{m}^{\mu}\) take the following form
\[l^{\mu} = \delta_1^{\mu}\] (9)
\[n^{\mu} = \delta_0^{\mu} - \frac{1}{2} \mathcal{A}(r) \delta_1^{\mu}\] (10)
\[m^{\mu} = \frac{1}{\sqrt{2r}} \left(\delta_2^{\mu} + \frac{i}{\sin \theta} \delta_3^{\mu}\right)\] (11)
\[\overline{m}^{\mu} = \frac{1}{\sqrt{2r}} \left(\delta_2^{\mu} - \frac{i}{\sin \theta} \delta_3^{\mu}\right)\] (12)

Now we complexify null tetrad vectors as
\[l^{\mu} = \delta_1^{\mu}\] (13)
\[n^{\mu} = \delta_0^{\mu} - \frac{1}{2} \tilde{\mathcal{A}}(r, \theta) \delta_1^{\mu}\] (14)
\[m^{\mu} = \frac{1}{\sqrt{2r}} \left(\delta_2^{\mu} + \frac{i}{\sin \theta} \delta_3^{\mu}\right)\] (15)
\[\overline{m}^{\mu} = \frac{1}{\sqrt{2r}} \left(\delta_2^{\mu} - \frac{i}{\sin \theta} \delta_3^{\mu}\right),\] (16)
where
\[\tilde{\mathcal{A}}(r) = 1 - 2GM \left(\frac{1}{r} + \frac{1}{r^2}\right) \left(1 + \frac{1}{6}m^2r^2\right).\] (17)

Apply complex transformation
\[x^\rho = x^\rho + ia \cos \theta(\delta_0^{\rho} - \delta_1^{\rho}).\] (18)

After this step tetrad vector becomes
\[l^{\mu} = \delta_1^{\mu}\] (19)
\[n^{\mu} = \delta_0^{\mu} - \frac{1}{2} \mathcal{A}(r, \theta) \delta_1^{\mu}\] (20)
\[m^{\mu} = \frac{1}{\sqrt{2(r^2 + ia \cos \theta)}} \left(i a \sin \theta (\delta_0^{\mu} - \delta_1^{\mu}) + \delta_2^{\mu} + \frac{i}{\sin \theta} \delta_3^{\mu}\right),\] (21)
\[\overline{m}^{\mu} = \frac{1}{\sqrt{2(r^2 - ia \cos \theta)}} \left(-ia \sin \theta (\delta_0^{\mu} - \delta_1^{\mu}) + \delta_2^{\mu} - \frac{i}{\sin \theta} \delta_3^{\mu}\right),\] (22)

where,
\[\mathcal{A}(r, \theta) = 1 - \frac{2GMr}{\Sigma} \left(1 + \frac{1}{6}m^2\Sigma\right)\] (23)
\[\Sigma = r^2 + a^2 \cos^2 \theta.\] (24)

One can read off the components of the contravariant metric from tetrad vectors found in the above step as
\[g^{\mu\nu} = \begin{pmatrix}
-\frac{a^2 \sin^2 \theta}{\Sigma} & 1 + \frac{a^2 \sin^2 \theta}{\Sigma} & 0 & -\frac{a}{\Sigma} \\
1 + \frac{a^2 \sin^2 \theta}{\Sigma} & -\frac{a^2 \sin^2 \theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\
0 & 0 & -\frac{1}{\Sigma} & 0 \\
-\frac{a}{\Sigma} & \frac{a}{\Sigma} & 0 & -\frac{1}{\Sigma \sin^2 \theta}
\end{pmatrix}\] (25)
Inverting metric (25) to get the covariant metric

\[
g_{\mu\nu} = \begin{bmatrix}
-\tilde{A}(r, \theta) & -1 & 0 & -a\sin^2\theta(1 - \tilde{A}(r, \theta)) \\
-1 & 0 & 0 & a\sin^2\theta \\
0 & 0 & \Sigma & 0 \\
-a\sin^2\theta(1 - \tilde{A}(r, \theta)) & a\sin^2\theta & 0 & \sin^2\theta(\Sigma + a^2\sin^2\theta(2 - A(r, \theta)))
\end{bmatrix}
\] (26)

To convert the metric (26) into Boyer-Lindquist coordinates, we perform following transformations

\[
\begin{align*}
du &= dt' - \frac{\Delta}{\tilde{A}} dr, \\
d\phi &= d\phi' - \frac{\Delta}{\tilde{A}} dr,
\end{align*}
\] (27) (28)

where we define

\[
\Delta = \Sigma \tilde{A}(r, \theta) + a^2\sin^2\theta.
\] (29)

Thus line element of the output spacetime can be written as

\[
ds^2 = -\tilde{A}(r, \theta) dt'^2 - 2a\sin^2\theta \left[1 - \tilde{A}(r, \theta)\right] dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left[\Sigma + (2 - \tilde{A}(r, \theta))a^2\sin^2\theta\right] d\phi^2.
\] (30)

The above metric is written in the form like \(g^K_{\mu\nu} + b_{\mu\nu}\), where \(g^K_{\mu\nu}\) is the Kerr metric of GR and \(b_{\mu\nu}\) is the correction terms due to modified gravity except \(g_{rr}\) component. One can also express the \(g_{rr}\) component in the same way. Finally rewriting the metric derived in (30) as

\[
ds^2 = -\tilde{A}(r, \theta) dt'^2 - 2a\sin^2\theta \left[1 - \tilde{A}(r, \theta)\right] dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left[\Sigma + (2 - \tilde{A}(r, \theta))a^2\sin^2\theta\right] d\phi^2.
\] (31)

where \(\Delta_{GR} \equiv r^2 + a^2 - 2Gr\).

### III. SLOWLY ROTATING BLACKHOLE SOLUTION OF RR MODEL

In this section, we derive the metric for the slowly rotating blackhole for the nonlocal model given in (1) and compare it with the slow rotation limit of the metric derived in (30). We consider the spherically symmetric and static solution of the field equations for our action (1) as background metric.

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\] (32)

\(A(r)\) and \(B(r)\) are given in (4) and (5). We can verify that setting \(m^2 = 0\) in Eqns. (4) and (5) the standard general relativistic forms of \(A(r)\) and \(B(r)\) are recovered,

\[
A(r) = B(r)^{-1} = \left(1 - \frac{2GM}{r}\right).
\] (33)

Now let us switch on the first order perturbation in spin \(a\). Then the metric (32) becomes

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - r^2\sin^2\theta a w(r) d\phi dt.
\] (34)

We introduce a function \(w(r)\) as a co-factor of spin \(a\). Since our desired metric is stationary it does not depend on \(t\). The reason that the first order perturbation, i.e. of the order of \(a\), only enters in \(t\phi\) component of the metric is as follows. The time reversal symmetry and symmetry in the direction of spinning dictates that the only \(t\phi\) component can have odd powers of \(a\) [20]. From the Eq. (2), one can calculate the \(t\phi\) component of the field equation as

\[
R_{t\phi} \left(1 - \frac{2m^2}{3}\right) = \frac{m^2}{3} [g_{t\phi} U - 2\nabla_i \nabla_{t\phi} S].
\] (35)
Writing the above equation up to first order in \( m^2 \) we get

\[
R_{t\phi} = \frac{m^2}{3} \left[ g_{t\phi} U - 2 \nabla_t \nabla_\phi S \right].
\] (36)

The solution of the field equations [2] give the expression \( U = -\ln \left( 1 - \frac{2GM}{r} \right) \). Since our system is independent of \( t \) and \( \phi \) the second term inside the square bracket in Eq. (36) will vanish. Calculating \( R_{t\phi} \) for the metric written in \( (34) \) and substituting it and other quantities in \( (36) \) one can obtain differential equation for \( \phi \) if

\[
w'' + \frac{4}{r} w' - \frac{w}{r} \left[ \frac{2A'}{A} - \frac{2}{rA} + \frac{2}{r} - \frac{4}{3} m^2 \ln \left( 1 - \frac{2GM}{r} \right) \right] = 0.
\] (37)

Here, prime denotes the derivative with respect to \( r \). After substituting the form of \( A(r) \) and \( A'(r) \) we obtain

\[
w'' + \frac{4}{r} w' - \frac{4}{3} m^2 w \left[ \frac{GM}{r} + \ln \left( 1 - \frac{2GM}{r} \right) \right] \left( 1 - \frac{2GM}{r} \right)^{-1} = 0.
\] (38)

For general relativity above equation reduces to

\[
w'' + \frac{4}{r} w' = 0,
\] (39)

whose solution is given by \( w_{GR} = 4GM/r^3 \). We can simplify (37) if we absorb \( r^2 \) inside \( w(r) \) in the metric \( (34) \) by defining \( W(r) = r^2 w(r) \). The differential equation \( (37) \) can be written in terms of \( W(r) \) as

\[
W'' + W \left[ \frac{2A'}{rA} - \frac{2}{r^2 A} - \frac{4}{3} m^2 \ln \left( 1 - \frac{2GM}{r} \right) \right] = 0.
\] (40)

By substituting the form of \( A(r) \), the above equation becomes

\[
W'' + W \left[ \frac{2}{r^2} - \frac{4}{3} m^2 \left( \frac{GM}{r} + \ln \left( 1 - \frac{2GM}{r} \right) \right) \left( 1 - \frac{2GM}{r} \right)^{-1} \right] = 0.
\] (41)

We can make above differential equation dimensionless by defining \( t \equiv 2GM/r \). Writing \( A(r) \) in terms of \( t \) as

\[
A(t) = 1 - t - \frac{C^2}{6t},
\] (42)

where \( C = 2GMm \) which can be considered as a new coupling constant. The fact that the Schwarzschild radius \( r_s(= 2GM) \ll m^{-1} \) implies \( C^2 \ll 1 \). Writing \( (41) \) in terms of \( t \) as

\[
t^4 \frac{d^2 W}{dt^2} + 2t^3 \frac{dW}{dt} + W \left[ -2t^2 - \frac{4}{3} C^2 \left( \frac{t}{2} + \ln(1-t) \right) (1-t)^{-1} \right] = 0.
\] (43)

The modified gravity part in \( (43) \) is coupled with usual GR part via coupling constant \( C^2 \). The solution of \( (43) \) can be split into two parts as \( W(t) = W_{GR}(t) + C^2 \tilde{W}(t) \), where \( W_{GR} = r^2 w_{GR} \) is the solution of \( (43) \) when \( C = 0 \). We can remove the GR part from the \( (43) \) and end up with equation

\[
t^4 \frac{d^2 \tilde{W}}{dt^2} + 2t^3 \frac{d\tilde{W}}{dt} - 2t^2 \tilde{W} - \frac{8}{3} t \left[ \frac{t}{2} + \ln(1-t) \right] (1-t)^{-1} = 0,
\] (44)

up to order \( C^2 \). It is extremely difficult to solve the above equation analytically. To get an approximate analytic form of \( \tilde{W}(t) \) we first solve it numerically then do curve fitting with the numerical solution. We use mathematica for this. For numerical calculation, we consider initial conditions at radial distance far away from the blackhole where \( r >> 2GM \). In this limit, \( \text{Eq.(44)} \) can be approximated as

\[
t^4 \frac{d^2 \tilde{W}}{dt^2} + 2t^3 \frac{d\tilde{W}}{dt} - 2t^2 \tilde{W} + \frac{4}{3} t^2 = 0.
\] (45)

whose solution is \( \tilde{W} = 2/3 \). Therefore, our initial conditions are \( \tilde{W}(0.0001) = 2/3 \) and \( \tilde{W}'(0.0001) = 0 \) at \( t = 0.0001 \). The fitted solution for \( \tilde{W} \) is given in the appendix. Since higher power terms in \( 2GM/r \) will be negligible for
r >> 2GM, except regions near horizon, we also fit the numerical solution with a truncated polynomial which is given by

\[ \tilde{W}_{\text{fit appx}}(t) = 0.692796 - 5.99399 t - 3.8432 t^2. \]  

(46)

For clear visibility, we have shown a plot comprising numerical solution, fitted solution and approximately fitted solution in Fig. 1 and it can be seen that all three curves match considerably well. If we consider the above solution is a good approximation then the metric for the exterior spacetime of a slowly rotating blackhole in case of RR model is given by

\[
\begin{align*}
    ds^2 &= \left[ 1 - \frac{2GM}{r} - \frac{2GMrm^2}{6} \right] dt^2 + \left[ 1 - \frac{2GM}{r} + \frac{2GMrm^2}{6 (1 - \frac{2GM}{r})^2} \right] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \\
    &\quad - a \sin^2 \theta \left[ \frac{4GM}{r} + C^2 \left( 0.692796 - 5.99399 \left( \frac{2GM}{r} \right) - 3.8432 \left( \frac{2GM}{r} \right)^2 \right) \right] d\phi dt.
\end{align*}
\]  

(47)

A. Comparing Two Metrics

Now our intention is to compare the two metrics (written in (31) and (47)) to check the validity of NJ algorithm for RR model. In order to do that we have to write down the slow rotation limit of the metric in (31). Therefore, ignoring terms of the order \( a^2 \) or higher, the metric (31) can be rewritten as

\[
\begin{align*}
    ds^2 &= \left[ 1 - \frac{2GM}{r} - \frac{2GMrm^2}{6} \right] dt^2 - \left[ \frac{4GM}{r} + \frac{2GMrm^2}{6} \right] \sin^2 \theta dt d\phi \\
    &\quad + \left[ \frac{1}{1 - \frac{2GM}{r}} + \frac{2GMrm^2}{6 (1 - \frac{2GM}{r})^2} \right] dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\end{align*}
\]  

(48)

From Eqns. (47) and (48), it is clearly visible that the two metrics do not coincide with each other. It is noteworthy that the same analysis done in linearized gravity limit shows that the metric generated from NJ algorithm and one derived by solving field equations match with each other [16].

IV. CONCLUSIONS

To conclude we have derived the spacetime metric for the exterior region of a rotating blackhole in a nonlocal gravity model called RR model. Furthermore, we have shown that the slow rotation limit of the metric generated by NJ algorithm applied to a spherically symmetric static solution of the model does not agree with the slowly rotating blackhole solution obtained by solving field equations themselves.
The two metrics differ by power of coordinate $r$ only in $g_{\phi\phi}$ component. The rotated metric via NJ method involves $g_{\phi\phi}$ term having positive power of $r$ while slowly rotating solution has negative power series in terms of $r$. This can result into a drastically divergent physical scenario. Thus, our investigation gives rise to suspicion about the applicability of NJ algorithm to modified gravity theories or at least in case of present model of nonlocal gravity.

The question that what the expected properties of the rotating blackhole metric for it to be a physically/astrophysically viable object should be, is outside the scope of this work. A thorough study in this direction and on possible modification of NJ algorithm can resolve the issue and present a trustworthy method to derive the Kerr-like solutions in modified gravity theories.

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\appendix

\section{Fitted Form of $\tilde{W}$}

\begin{equation}
\tilde{W}^{fit}(t) = 0.680682 - 5.00881 t - 12.7087 t^2 + 21.7346 t^3 + 8.60944 t^4 - 75.0625 t^5 + 49.7613 t^6 + 245.318 t^7 + 182.042 t^8 - 582.535 t^9 - 1.47589 t^{10} - 63.2133 t^{11} + 574.505 t^{12} + 82.0909 t^{13} + 104.079 t^{14} - 393.308 t^{15} - 487.026 t^{16} - 7.91134 t^{17} + 191.687 t^{18} + 841.02 t^{19} - 583.984 t^{20}, \hspace{0.4cm} (A1)
\end{equation}

where $t = 2GM/c^2$. 

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