On linear systems and functions associated with Lamé’s equation
and Painlevé’s equation VI

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ABSTRACT Painlevé’s transcendental differential equation \( P_{VI} \) may be expressed as the consistency condition for a pair of linear differential equations with \( 2 \times 2 \) matrix coefficients with rational entries. By a construction due to Tracy and Widom, this linear system is associated with certain kernels which give trace class operators on Hilbert space. This paper expresses such operators in terms of Hankel operators of linear systems which are realised in terms of the Laurent coefficients of the solutions of the differential equations. Let \( P_{(t;1)} : L^2(0;1) \to L^2(t;1) \) be the orthogonal projection. For such, the Fredholm determinant \( (t) = \det(I - P_{(t;1)}) \) defines the function, which is here expressed in terms of the solution of a matrix Gelfand-Levitan equation. For suitable values of the parameters, solutions of the hypergeometric equation give a linear system with similar properties. For meromorphic transfer functions \( \hat{\tau} \) that have poles on an arithmetic progression, the corresponding Hankel operator has a simple form with respect to an exponential basis in \( L^2(0;1) \); so \( \det(I - P_{(t;1)}) \) can be expressed as a series of finite determinants. This applies to elliptic functions of the second kind, such as satisfy Lamé’s equation with \( \ell = 1 \).

Key words: random matrices, Tracy-Widom operators

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1. Introduction

Tracy and Widom [29] observed that many important kernels in random matrix theory arise from solutions of linear differential equations with rational coefficients. In particular, the classical systems of orthogonal polynomials can be expressed in such terms. In this paper, we extend the scope of their investigation by analysing kernels associated with Lamé’s equation and Painlevé’s equation VI. As these differential equations have solutions which may be expressed in terms of elliptic functions, we begin by reviewing and extending the definitions from [29].

Let \( P(x;y) \) be an irreducible complex polynomial, and let the degree of \( P(x;y) \) as a polynomial in \( y \). Then we introduce the curve \( E = \{ f(\cdot) \} \subset \mathbb{C} : P(\cdot) = 0 \), and observe that \( E = \{ f(1 ; 1)g \} \) gives a compact Riemann surface which is the \( n \)-sheeted branched cover of Riemann’s sphere \( P^1 \). Let \( K \) be splitting field of \( P(x;y) \) over \( C(x) \), so we can regard \( K \) as the space of functions of rational character on \( E \). Let \( g \) be the genus of \( E \), and introduce the Jacobian variety \( J \) of \( E \), which is the quotient of \( C^g \) by some lattice \( L \) in \( C^g \).

Definition. By a Tracy-Widom system [29] we mean a differential equation

\[
\frac{d}{dx} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}
\]

(1.1)
where ; ; belong to K or m one generally are locally rational functions on J. Then for solutions
with \( f;g \in L^1 (0;1;\mathbb{R}) \), we introduce an integrable operator on \( L^2 (0;1) \) by the kernel

\[
K (x;y) = \frac{f(x)g(y)}{xy} \quad (x \neq y; x;y \in 2 \mathbb{R})
\]  

(1.2)

The kernel \( K \) compresses to give an integral operator \( K_S \) on \( L^2 (S;dx) \) for any subinterval \( S \)
of \( (0;1) \) and it is important to identify those \( K_S \) such that \( K_S \) is of trace class and \( 0 \) \( \det K_S \) \( 1 \). In such cases, the Fredholm determinant \( \det (I + K_S) \) is defined and \( K_S \) is associated with a
determinantal random point field on \( S \). In particular, \( \det (I + K_{(x;1)}) \) gives the probability that
there are no random points on \( (0;1) \):

Definition (-function). Suppose that \( K : L^2 (0;1) \rightarrow L^2 (0;1) \) is a self-adjoint operator such
that \( K = I, K \) is trace class and \( I + K \) is invertible. For a measurable subset \( S \) of \( (0;1) \), let \( P_S : L^2 (0;1) \rightarrow L^2 (S) \) be the orthogonal projection given by \( f \mapsto fI_S \), where \( I_S \) is the indicator
function of \( S \). Then the function is

\[
(t) = \det (I + K P_{[x;1]}) \quad (t > 0);
\]  

(1.3)

The purpose of this paper is to take kernels that are given by certain Tracy-Widom system s, and show how to express the corresponding in terms of the solution of a Gelfand-Levitan integral equation. Our technique involves linear system s, and extends ideas developed in [6], and leads to a solution of the integral equation in terms of the linear system.

Let \( H \) be a complex separable Hilbert spaces, known as the state space, and let \( \{e^{tA}\}_{t \geq 0} \) a
bounded \( C_0 \)-semigroup of linear operators on \( H \); so that \( D (A) \) is a dense linear subspace of \( H \), and \( k e^{tM} k \leq M \) for all \( t > 0 \) and some \( M < 1 \). Then let \( B : C \rightarrow D (A) \) and \( C : D (A) \rightarrow C \) be bounded linear operators, and introduce the linear system

\[
\begin{align*}
\frac{dX}{dt} &= AX + BU \\
Y &= CX
\end{align*}
\]  

(1.4)

known as \( (A;B;C) \). Under further conditions to be discussed below, the integral

\[
R_x = \int_x^1 e^{tA}B Ce^{tA} dt
\]  

(1.5)

converges and defines a trace class operator on \( H \). The notation suggests that \( R_x \) is a resolvent
operator.

Definition (Hankel operator). For a linear system as above, we introduce the symbol \( (x) = Ce^{xA}B \), which gives a bounded function \( (0;1) \rightarrow \mathbb{C} \); this term should not be confused with the different usage in [26, p 6]. Generally, for \( E \) a separable complex Hilbert space and \( L^2 (0;1);E \), let be the Hankel operator

\[
h(x) = \int_0^1 (x + y)h(y) dy
\]  

(1.6)
defined on a suitable domain in \( L^2(0;1) \) into \( L^2((0;1);E) \).

By forming orthogonal sums of the state space and block operators, we can form sums of symbol functions. Likewise, by forming tensor products of state spaces and operators, we can form products of symbol functions. Using these two basic constructions, we can form some apparently complicated symbol functions, starting from the basic multiplication operator \( A : f(t) \mapsto tf(t) \) in \( L^2(0;1) \). Thus we extend the method of section 2 to a more intricate problem.

In section 3, we consider operators related to the solution of Painlevé's transcendental equation \( V_I \)

\[
\frac{d^2y}{dt^2} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{y} \right) + \frac{1}{y} \frac{dy}{dt} + \frac{1}{t} \frac{dy}{dt} + \frac{y(y - 1)(y - t)}{y^2} + \frac{y}{y^2} + \frac{y(y - 1)}{y^2} + \frac{y(y - t)}{y^2} : \quad (1.7)
\]

with constants

\[
\frac{1}{2} \left( 1 + 1 \right) = \frac{1}{2} \left( 1 + 1 \right) = \frac{1}{2} \left( 1 + 1 \right) = \frac{1}{2} \left( 1 + 1 \right) \quad (1.8)
\]

and

\[
\frac{1}{2} = 2 \left( z_0 + z_1 + z_2 \right) \quad (1.9)
\]

Jimbo, Miwa, and Ueno [15, 16] showed that the nonlinear differential equation \( P_{V_I} \) is the compatibility condition for the pair of linear differential equations

\[
\frac{d}{dt} = W_0 + W_1 + W_t \quad (1.10)
\]

\[
\frac{d}{dt} = W_t \quad (1.11)
\]

on the punctured Riemann sphere with \( W_0, W_1, W_t \) complex matrices depending upon \( t \); see (3.8) for the entries. Using the Laurent series of \( (\cdot) \), we introduce a linear system \( (A;B;C) \) that realises and deduces information about the Hankel operator \( \gamma \). In previous papers [5, 6], we have considered kernels that factorise as \( K = \gamma \) where \( \gamma \) is Hilbert-Schmidt, so that \( K = 0 \) and \( K \) is trace class. In the context of \( P_{V_I} \), we show that the prescription (1.2) gives a kernel \( K \) that admits a factorisation \( K = \gamma \), where \( \gamma \) is a constant signature matrix. In section 5 we introduce a suitable function and express this in terms of the solution of an integral equation of Gelfand-Levitan type, which we can solve in terms of the linear system. A similar approach works for suitable solutions of Gauss's hypergeometric equation with a restricted choice of parameters, as we show in section 5.

Definition (Transfer function). Given a Hilbert space \( E \), for \( 2 L^2((0;1);dt;E) \) let

\[
\mathcal{Z}^1 \left( s \right) = \int_0^1 e^{-st} (t) \, dt \quad (1.12)
\]
be the transfer function of \( s \), otherwise known as the Laplace transform, which gives an analytic function from \( s < 0 \) to \( E \).

We assume that \( ^\wedge \) is meromorphic, and that, by virtue of the Mittag-Leer theorem, one can express it as a series
\[
(\chi) = \sum_{j=1}^{\infty} \chi_j e^{ix}
\]
where we shall always assume that \( \chi_j > 0 \) and that the \( e^{ix} \) are linearly independent in \( L^2(0;1) \). We wish to express various functions in terms of the determinants
\[
D_{ST} = \det \left( \frac{1}{j + k} \chi_{jk}^{1/2} \right)
\]
where \( S \) and \( T \) are finite subsets of \( N \) of equal cardinality. In sections 6, we consider Hankel operators with symbols as in (1.13), and establish basic results about the expansions of \( \det(I - \chi) \) in terms of the bases. In particular, if \( \chi_j \) forms an arithmetic progression in the plane, then \( ^\wedge(s) = \prod_{j=1}^{\infty} \chi_j \) gives a cardinal series.

In section 7, we consider the Bessel kernel, which arises in random matrix theory as the hard edge of the eigenvalue distribution from the Jacobi ensemble [28]. Let \( J \) be Bessel's function of the first kind of order \( n \), and let \( u(x) = J_n(2 \pi x) \), which satisfies
\[
\frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \frac{1}{4x^2} u(x) = 0
\]
where \( D_{ST} = \det \left( \frac{1}{j + k} \chi_{jk}^{1/2} \right) \) and identify the determinants \( D_{NN} \) with combinatorial objects.

In section 8 we consider solutions of Lamé's equation
\[
\frac{d^2}{dz^2} + n(n + 1)k^2 \sin(njk)^2 (z) = (z)
\]
which we express as a differential equation on the elliptic curve \( z^2 = 4(x_1 - x_2)(x_1 - x_3) \). The solution gives rise to an elliptic function such that \( ^\wedge \) has poles on a bilateral arithmetic procession parallel to the imaginary axis in \( C \). Hence we can prove results concerning the Fredholm determinant of \( ^\wedge \).

2. The function associated with a linear system

In this section we introduce the basic example of the linear system which we will use in sections 3 and 5 to realise solutions of some differential equations. In [30], Tracy and Widom consider physical applications of the kernels \( R_x \) that we introduce here.
Definition (Integrable operators). Let \( f_1, \ldots, f_n \); \( g_1, \ldots, g_n \) \( L^1(0;1) \) satisfy
\[
f_j(\lambda)g_j(\lambda) = 0 \quad (\lambda > 0);
\]
Then the integral operator \( K \) on \( L^2(0;1) \) that has kernel
\[
K \frac{P_j f_j(\lambda)g_j(\gamma)}{x \ y} (2.1)
\]
is said to be an integrable operator; see [9]. One can show that \( K \) is bounded on \( L^2(0;1) \).

Let \( D(\alpha) = \frac{f^2}{2} L^2(0;1); t \ f(t) 2 L^2(0;1) \) \( g \) and for \( b_c 2 D(\alpha) \) introduce the operators:
\[
A : D L^2(0;1); L^2(0;1) ; f(\lambda) \ d f(\lambda) \ y f(\lambda) \\
B : C ! D(\alpha) ; f(\lambda) \ d b ; \\
C : D(\alpha) ! C ; f(\lambda) \ b \ d f(\lambda) c(c(s) \ ds (2.2)
\]
Then we introduce \( (s) = e^{\alpha A} B \) and \( (\alpha) (s) = (s + 2x) \), and the Hankel integral operator
\[
R \ = e^{\alpha A} B e^{\alpha A} \ dt \text{ which has kernel (s; t > 0)}: (2.3)
\]

Proposition 2.1. Suppose that \( c(t) = \frac{P_-}{t} \) and \( b(t) = \frac{P_-}{t} \) belong to \( L^2(0;1) \), and that \( c \) and \( b \) belong to \( L^1(0;1) \).

(i) Then \( (\alpha) \) and \( R \) are trace class operators for all \( x \) \( 0 \).

(ii) Suppose further that \( I + (\alpha) R \) is invertible for some \( 2 \ C \). Then
\[
T(\lambda; \gamma) = C e^{\lambda A} (I + R \lambda) \ lambda \ e^{\lambda A} B \ (2.4)
\]
gives the solution to the equation
\[
(x + y) + T(\lambda; \gamma) + \int \ T(\lambda; \zeta) (z + y) \ dz = 0 \quad (0 < x < y) \ (2.5)
\]
and
\[
T(\lambda; \gamma) = \frac{d}{d \lambda} \ det(I + (\alpha) R \lambda) \ (2.6)
\]

(iii) The operator \( R \) is an integrable operator with kernel
\[
R \ = e^{\lambda A} B(\lambda) \frac{f(\lambda)}{\lambda} c(t) e^{\lambda A} \ (2.7)
\]
where
\[
f(\lambda) = \int_0^1 b(t) c(t) e^{\lambda A} \ dt \ (2.8)
\]
(iv) If \( I + R_x \) and \( I - R_x \) are invertible, then there exists an integrable operator \( L_x (\cdot) \) such that
\[
I + L_x (\cdot) = (I - 2R_x^2)^{-1};
\]

Proof. (i) One checks that \( x \) has kernel \( \exp (t x \cdot c(t)) \) and that \( y \) has kernel \( \exp (t y \cdot b(s)) \); hence \( y \) and \( x \) are Hilbert-Schmidt operators. One verifies that their products are \( R_x = x \cdot y \) and \( x = y \cdot x \), and hence \( R_x \) and \( L_x \) are trace class.

(ii) Using (i), we can check that \( \det(I + R_x) = \det(I + R_x^{-1}) \). Then one verifies the remainder by using Lemma 5.1 (iii) of [6].

(iii) This result is essentially contained in Lemma 2.1 of [9], but we give a proof for completeness. The kernel of \( R_x^2 \) is
\[
(2.10)
\]
and one can decompose this expression by using partial fractions. By the Cauchy-Schwarz inequality, \( \int_0^1 b(s) \, dt \leq \int_0^1 t \, dt \), so \( f_x \) is bounded.

(iv) Further, \( (I + R_x)^{-1} \) is a bounded linear operator; so by Lemma 2.8 of [9], there exists an integrable operator \( L_x \) such that \( (I + L_x (\cdot))(I - 2R_x^2) = I \).

Given an integrable operator \( K \) on \( L^2(a;b) \) such that \( I - K \) is invertible, the authors of [9] show how to express \( (I - K)^{-1} \) as the solution of a Riemann-Hilbert problem on a bounded interval

3. A linear system associated with Painlevé's equation VI

The Painlevé equation \( \text{P}_VI \) is associated with the system
\[
\begin{align*}
\frac{d}{dt} &= \frac{W_0}{W} + \frac{W_1}{W} + \frac{W_t}{W} \quad (3.1) \\
\frac{d}{dt} &= \frac{W_t}{W} \quad (3.2)
\end{align*}
\]
where the fixed singular points are \( 0;1;1 \) and
\[
W = W(t) = \exp z + 2 \quad \frac{u z}{u} + z &= 2 
\]
with parameters \( u \) and \( z \) satisfying various conditions specified in [16]. The consistency condition for the system (3.1) and (3.2) reduces to the identity
\[
\frac{1}{\partial t} W_0 + \frac{1}{(t)} W_1 + \frac{1}{(t) t} W_t = \frac{W_0; W_t}{(t)} + \frac{W_1; W_t}{(1)(t)} \quad (3.4)
\]
which leads, after a lengthy computation given in Appendix C of [16], to the equation \( \text{P}_VI \).

Jimbo et al [15, 16, 17] introduced pairs of differential equations (3.1) and (3.2) such that (3.5) reduces to one of the Painlevé equations. In the present context (3.1) are known as the
deformation equations and (3.4) is associated with the names of Schlesinger and Garnier [10]. Note that \( \text{tr}W = 0 \) if and only if \( JW \) is symmetric; also \( W \) is nilpotent if and only if \( JW \) is symmetric and \( \det(JW) = 0 \).

First we introduce a linear system for the differential equation (3.4); later we introduce a linear system that realises the kernel most naturally associated with \( P_{VI} \). For notational simplicity, we often suppress the dependence of operators upon \( t \). The following result is a consequence of results of Turrittin [31, 27], who clarified certain facts about the Birkhoff canonical form for matrices.

**Lemma 3.1.** Let \( W_1 = W_0 + W_1 + W_1 \) and suppose that the eigenvalues of \( W_1 \) are \( \lambda = 2 \) where \( \lambda \) is not a positive integer, and let \( a \) be a constant \( 2 \times 1 \) vector. Then there exist \( 2 \times 2 \) complex matrices \( C_j \) for \( j = 1, 2, \ldots, \) depending upon \( t \), such that

\[
(x) = I + \sum_{j=1}^{\infty} C_j x^j \quad (|x| > t) \tag{3.5}
\]

satisfies the differential equation (3.1).

**Proof.** We can define \( x^{W_1} = \exp(W_1 \log x) \) as a convergent power series. By considering terms in the convergent Laurent series, one requires to show that there exist coefficients \( C_0 = I \) and \( C_j \) that satisfy the recurrence relation

\[
C_n(W_1 + nI) = W_1C_n + W_1(C_0 + \cdots + C_n - 1) + tW_1(t^nC_0 + t^{n-1}C_1 + \cdots + C_n - 1), \tag{3.6}
\]

where \( W_1 + nI \) and \( W_1 \) have no common eigenvalues. Sylvester showed that, given square matrices \( V;W \) and \( Z \) such that \( V \) and \( W \) have no eigenvalues in common, the matrix equation \( CV = WC = Z \) has a unique solution \( C \); see [31, Lemma 1]. Hence unique \( C_n \) exist, and one shows by induction that \( kC_n \) is at most of geometric growth in \( n \). In particular, if \( kW_1 \) \( k < 1 \), then the solution of \( W_1C_n, C_n(W_1 + nI) = D_n \) is

\[
C_n = \frac{Z^n}{n!} \int_0^1 e^{sW_1} D_n e^{s(W_1 + nI)} ds. \tag{3.7}
\]

We have proved that (3.1) has a solution in a neighbourhood of infinity, and one can show that it extends to an analytic solution on the universal cover of the punctured Riemann sphere \( P^1 \backslash \{0, 1, \infty\} \). (Jimbo, Miwa and Ueno [15] have shown that any \( C^2 \) solution of the pair (3.1) and (3.2) on \( R \) extends to a meromorphic solution on \( C \); see [10, Remark 4.7].)

Extending the construction of (2.2), we realise this solution via a linear system. We introduce the output space \( H_0 = C^2 \), then the Hilbert space \( H_1 = L^2(H_0) \), the state space \( H = L^2(\mathbb{R}I_1) \) \( ds; H_1 \) and then let \( D(A) = \mathcal{F}H : sf(s)2H \) \( g \); then we choose

\[
b_j(s) = (jI + W_1)_{1-s}^{j-1}W_1 \quad (j = 0, 1, \ldots); \tag{3.8}
\]
recalling that $(z)^{1}$ is entire. With this choice and some convergence factor $0 > 1$, we introduce linear maps

$$
A : D(A)! H : \quad f(s) \mapsto sf(s);
B_{W} : \quad \mathbb{T} : (\mathbb{P}(\mathbb{T}_{\mathbb{P}(s)})\mathbb{T}_{\mathbb{P}(s)})j_{z=0};
C : D(A)! C^{2} : \quad (f_{j})_{j_{z=0}} \mathbb{T} : (\mathbb{P}(\mathbb{T}_{\mathbb{P}(s)})\mathbb{T}_{\mathbb{P}(s)})j_{z=0} C_{W} f(s) ds;
$$

We prove below that $e^{x} 2 H$ for all sufficiently large $x$. As usual, we introduce $x : L^{2}(0,1)! H$ such that

$$
Z_{1} x = e^{x} B_{W} f(s) ds
$$

and the observability operator $x : L^{2}(0,1)! H_{0}! L^{2}(t;1)! H_{1}$ by

$$
Z_{1} x = e^{x} C_{W} f(s) ds;
$$

Proposition 3.2. (i) There exist $0; x_{0} > 0$ such that the operators $x : L^{2}(0,1)! H_{0}! H$ and $x : L^{2}(0,1)! H_{0}! H$ are Hilbert-Schmidt for $x > x_{0}$.

(ii) For $x > x_{0}$, the linear system $(A;B_{W};C_{W})$ realises the solution of (3.1), so that

$$
(x;t) = C_{W} e^{x} B_{W} 0;
$$

(iii) Let $w(x;t) = C_{W} e^{x} B_{W}$. Then the Hankel operator on $L^{2}(0,1)! H_{0}! H$ with symbol $w$ is trace class.

Proof. (i) We note that $x$ has kernel $(e^{u} j_{z=0} C_{y})j_{z=0}$, and hence the Hilbert-Schmidt norm satisfies

$$
k_{x} k_{H}^{2} = \frac{\chi_{x} Z_{1} Z_{1}}{\chi_{x} k_{H}^{2} e^{2xu} j_{z=0} k_{H}^{2} ds du k C_{y} k_{H}^{2}}
$$

so we choose $0$ so that this series converges. For notational convenience, suppose that $k_{W} < k < 1$.

Then by the functional equation of $x$, we have

$$
k_{x} (j I + W_{1})^{1} u^{1} + j^{1} k = \frac{u^{1} j (I + W_{1})^{1} (I + W_{1})^{1} k}{(j - 1)} (u > 1);
$$

Next we observe that $x : L^{2}(0,1)! H_{0}! L^{2}(t;1)! H_{1}$ has kernel $(e^{u} j_{z=0} b_{j}(u))j_{z=0}$, and hence has Hilbert-Schmidt norm

$$
k_{x} k_{H}^{2} = \frac{\chi_{x} Z_{1} Z_{1}}{\chi_{x} k_{H}^{2} e^{2xu} j_{z=0} k_{H}^{2} ds du k C_{y} k_{H}^{2}}
$$

and

$$
\frac{\chi_{x} Z_{1} Z_{1}}{\chi_{x} k_{H}^{2} e^{2xu} (2u) j_{z=0} k_{H}^{2} du}.
$$

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Theorem 3.3. In which has rank where both which satisfies the Gelfand–Levitan equation

\[ G_W(x; y) + W(x + y) \begin{pmatrix} 0 & I_j \\ I_j & 0 \end{pmatrix} G_W(x; w) W(w + y) dw = 0 \quad (t < x < y) \]  

where \( W(x; t) = C_W e^{xA} B_W \).

We also introduce

\[ jk = \begin{pmatrix} I_j & 0 \\ 0 & I_k \end{pmatrix} \]  

which has rank \( j + k \) and signature \( j, k \).

Theorem 3.3. Suppose that \( W_1 \) is as in Lemma 3.1. Let \( (t; t) \) be a bounded solution of (3.1) in \( L^2(t; 1); \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{R}^2 \) such that \( t_1 k (t; t) k^2 d < 1 \), and let

\[ K(t; t) = hJ(t; t); (t; t)i \]
(i) Then there exists \( 2L^2(0;1); d; \mathbb{R}^6 \) such that

\[
Z_1 \quad K(\cdot;\cdot;t) = h_{3j3}(\cdot+st); (\cdot+st)\text{ids} \quad (\cdot>\cdot; \emptyset) \tag{323}
\]

and hence \( K \) defines a trace class operator on \( L^2((t;1);d) \):

(ii) The kernel \( \frac{d}{dt}K(\cdot;\cdot)t \) is of finite rank in \( (\cdot) \).

Proof. Jimbo \[14\] has shown that the fundamental solution matrix to (3.1) satisfies

\[
Y(x;t) = 1 + O(x^1) \quad X^{1^2} 0 \quad x^{1^2} \; ; \tag{324}
\]

hence there exist solutions that satisfy the hypotheses.

(i) We suppress the parameter to simplify notation. From the differential equation (3.1), we have

\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial t} hJ(\cdot); (\cdot)i = \frac{1}{X} \quad D_{JW} + \frac{W^YJ(\cdot);(\cdot)}{E} \tag{325}
\]

Now

\[
JW = (z + u)^2 z + u \quad z = 2 \quad u \quad z \quad ( = 0;1;t) \tag{326}
\]

which have rank two and signature zero since \( \det W = 2^2 = 4 < 0 \). Hence \( JW = V_1^Y \ ) \ for some \ 2 \times 2 \ real \ matrix \ V \) and \( JW = V_1^Y \ ) \ . Thus we nd that (3.25) reduces to

\[
\frac{h_{1j1}V_0(\cdot);V_0(\cdot)}{E} = \frac{h_{1j1}V_1(\cdot);V_1(\cdot)}{E} = \frac{h_{1j1}V_2(\cdot);V_2(\cdot)}{E} \tag{327}
\]

Let

\[
\begin{align*}
V_0(\cdot) & = \frac{2}{3} V_0(\cdot) \\
V_1(\cdot) & = \frac{6}{4} V_1(\cdot) \\
V_2(\cdot) & = \frac{7}{5} V_2(\cdot)
\end{align*} \tag{328}
\]

which satisfies, after we permute the coordinates in the obvious way,

\[
X \quad \frac{h_{1j1}V(\cdot);V(\cdot)}{E} = h_{3j3}(\cdot); (\cdot)i \quad = \frac{\partial}{\partial t} + \frac{\partial}{\partial t} \quad Z^1 \quad h_{3j3}(\cdot+st); (\cdot+st)\text{ids}; \tag{329}
\]

We observe that both sides of (3.23) converge to zero as \( ! 1 \) and as \( ! 1 \) . By comparing the derivatives as in (3.25) and (3.29), we deduce (3.23).

Then \( K = V_{3j3}^Y \) : We observe that the Hilbert-Schmidt norm of satisfies

\[
k_k^2 \tag{330}
\]

Thus

\[
10
\]
for some \( \varepsilon > 0 \), so \( K \) gives a trace class operator on \( L^2(t;1) \).

(ii) By a similar calculation, one can compute the derivative of \( K \) with respect to the position of the critical point, and nd

\[
\frac{\partial}{\partial t} K \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \frac{1}{(t)} \begin{pmatrix} D & E \\ (z_t + \xi)u_t & z_t + \xi^2 & u_tz_t \end{pmatrix} \left( \begin{array}{c} \xi; \eta \\ \xi; \eta \end{array} \right); \quad (3.31)
\]
evidently this is a nite sum of products of functions of and functions of for each \( t \).

4. The function associated with Painlevé's equation V I

In [2], Ablowitz and Segur derived an integral equation involving the Airy kernel for the solutions of \( P_{II} \). Here we solve an integral equation and derive an expression for \( \det(I - K P_{\pi}) \), which is associated with \( P_{II} \). From Proposition 3.2, we recall the linear system \( (A_\pi; B_\pi; C_\pi) \) that realises \( \pi \), and likewise we introduce a linear system \( (A_\pi; B_\pi; C_\pi) \) that realises \( \pi = \text{diagonal}(V_0=\pi; V_1=(x - 1); V_2=(x - t)) \); then by considering

\[
( (A_\pi; I + I; A_\pi); B_\pi; B_\pi; C_\pi; C_\pi)
\]

we introduce a new linear system that realises from Theorem 3.3, so that \( (x) = Ce^{xA}B \).

Next we let \( \mathcal{L}_x \) be the Hankel integral operator with symbol \( x \) and let \( L_x \) be observability Gramian

\[
L_x = \int x e^{sA}B \int e^{sA}Y ds = \int \int x ^{y}: \quad (4.1)
\]

To take account of the signature, we introduce the modified controllability Gramian

\[
Q_x = \int x e^{sA}C \int e^{sA}C ds: \quad (4.2)
\]

We also introduce the \((6 + 1)\) block matrices

\[
G(x; y) = \begin{pmatrix} U(x; y) & V(x; y) \\ T(x; y) & (x; y) \end{pmatrix} \quad (4.3)
\]

and

\[
(x) = \begin{pmatrix} 0 \\ (x) \end{pmatrix} \quad (4.4)
\]

and the Gelfand (Levitin) integral equation

\[
G(x; y) + (x + y) + \int G(x; w) (w + y) dw = 0; \quad (4.5)
\]

where we have introduced a special matrix product to incorporate the signature, namely

\[
G(x; w) (w + y) dw
\]

\[
\times \quad \int \int V(x; w) (w + y)^3 dw = \int \int U(x; w) (w + y) dw \quad (4.6)
\]

\[
\times T(x; w) (w + y) dw
\]
Theorem 4.1. Suppose that $Q_x$ and $L_x$ are trace-class operators with operator norms less than one for all $x > t$. Then there exists a solution to the integral equation (4.5) such that $K(x) = \det(I P_{[x,t]})$ satisfies

$$\frac{d}{dx} \log K(x) = \text{trace} G(x;x): \quad (4.7)$$

Proof. By Theorem 3.3, we have $K = y_{3;3}$, and so

$$K(x) = \det(I P_{[x,t]} y_{3;3}) = \det(I y_{x \times 3;3} y_{x \times x}) = \det(I x_{3;3} y_{x \times x}) = \det(I Q_x L_x): \quad (4.8)$$

One can verify that

$$U(x;y) V(x;y) T(x;y) (x;y) \quad (4.9)$$

$$= Ce^{x^A (I L_x Q_x)} e^{y A^Y C Y_{3;3}} Ce^{x^A (I L_x Q_x)} e^{y A^Y B}$$

$$B y e^{x^A (I Q_x L_x)} e^{y A^Y C Y} B y e^{x^A (I Q_x L_x)} e^{y A^Y B}$$

$$\text{gives a solution to (4.6), so that}$$

$$\text{trace} U(x;x) = \text{trace} (I L_x Q_x)^{1} L_x e^{x A^Y C Y_{3;3}} C e^{x^A}$$

$$= \text{trace} (I L_x Q_x)^{1} L_x \frac{dQ_x}{dx}: \quad (4.10)$$

Likewise we have

$$(x;x) = \text{trace} (I Q_x L_x)^{1} Q_x e^{x A^A B} B y e^{x A^Y}$$

$$= \text{trace} (I Q_x L_x)^{1} Q_x \frac{dL_x}{dx}: \quad (4.11)$$

Adding and rearranging, we obtain

$$\text{trace} G(x;x) = (x;x) + \text{trace} U(x;x)$$

$$= \text{trace} (I L_x Q_x)^{1} L_x \frac{dQ_x}{dx}$$

$$= \frac{d}{dx} \text{trace} \log (I L_x Q_x)$$

$$= \frac{d}{dx} \log K(x): \quad (4.12)$$

$\square$
We introduce the new variable $u$ by the elliptic integral

$$u(y;t) = \frac{Z_y}{y} \int_1^y \frac{d}{(y - 1)(y - t)};$$

(4.13)

then we let $Z = \frac{\partial y}{\partial u}$ and $Y = y$, so $(Y;Z)$ lies on the elliptic curve $Z^2 = Y(Y - 1)(Y - t)$ which depends upon the parameter $t$. Soon after his discovery of $P_{VI}$, R. Fuchs showed that if $y(t)$ satisfies $P_{VI}$, then $u(t) = u(y(t);t)$ satisfies

$$t(1 - t) \frac{d^2 u}{dt^2} + (2t - 1) \frac{du}{dt} + \frac{u}{4} = \frac{y(y - 1)(y - t)}{t(1 - t)} \left[ 2 + \frac{2t}{y^2} + \frac{(t - 1)}{(y - 1)^2} + \frac{1}{t(t - 1)} \right];$$

(4.14)

where we recognise Legendre's differential operator on the left-hand side; see [32, p 304]. By analysing these solutions, Guzzetti [13] obtains various series representations and bounds on the growth of $y(t)$. We can analyse symbols that are elliptic functions of the second kind since their transfer functions have special properties.

5. Kernels associated with the hypergeometric equation

The $P_{VI}$ equation is closely related to Gauss's hypergeometric equation [32, p 283]

$$(1 - ) \frac{d^2 f}{d x^2} + (c - a - b + 1) \frac{d f}{d x} - ab f(x) = 0;$$

(5.1)

We introduce $c_0 = c$ and $c_1 = a + b + 1$, and introduce the matrix

$$W = \begin{pmatrix} c_0 & 1 \\ c_1 & 1 \\ c_1 & 1 \\ 0 & 1 \end{pmatrix};$$

(5.2)

so that we can express (5.1) in the form of a first order linear differential equation as in (5.4). For special choices of the parameters $a, b, c$, we can obtain a factorization of the corresponding kernel (5.5) which has the form of (1.2). For a separable Hilbert space $H$ we introduce the identity operator $I_H$ and

$$W = \begin{pmatrix} I_H & 0 \\ 0 & I_H \end{pmatrix};$$

(5.3)

Theorem 5.1. Suppose that $0 < c_1$ and $a + b = 0$, that $\frac{P}{ab}$ is not an integer, and that $ab > 5$; and let $x$ be a bounded solution for the equation

$$\frac{d}{dx} = W(x);$$

(5.4)

such that $\int \frac{1}{x^k} dx < 1$; then let

$$K(x;y) = \frac{hJ(x; y)}{x - y} \begin{pmatrix} x - y \\ x - y \end{pmatrix} (x \notin y, x, y > 1);$$

(5.5)
(i) Then there exists a separable Hilbert space $H$ and $(1, 1)$ ! $H^2$ such that $R_1$, $x k (x) k_{H^2} dx < 1$ and $K = \gamma_{H \phi}$ so that $K$ defines a trace class kernel on $L^2((1 + \gamma_{H \phi}); dx)$ for all $\gamma > 0$.

(ii) The statement of Theorem 4.1 applies to

$$K(s) = \det(I - K P_{(s, 1)} H\phi) = \det(I - \gamma_{(s, 1)} H\phi);$$

(5.6)

with obvious changes to notation; so $\frac{d}{d\gamma} \log K(t)$ is given by the diagonal of the solution of a Gelfand-Levitan equation.

(iii) If moreover $c$ is rational, then $K$ arises from a Trace-Wilson system as in (1.1).

Proof. Let

$$q(\gamma) = \frac{ab}{(1 - 1/4)^2} + \frac{c^2}{2c} + \frac{2c(1 - c + 1)}{1} + \frac{c^2}{1};$$

(5.7)

which is asymptotic to $(ab + 1)^{-2}$ as $1$. By the Liouville-Green transformation [25, p. 229], we can obtain solutions to (5.1) with asymptotics of the form

$$f(\gamma)^{1/2} d\gamma < 1.$$

Hence there exist solutions that satisfy the hypotheses.

(i) We observe that $c_0 + c_1 = 1$, so $0 < c_0, c_1; 1 < c_0; c_1 < 1$; we assume that $0 < c_0; c_1 < 1$, as the cases of equality are easier. Evidently the functions $c_0 (1)^{c_0 - 1}$ and $c_1 (1)^{c_1 - 1}$ are operator monotone decreasing on $(1, 1)$ in Loewner's sense and by [1, p. 577] we have an integral representation

$$c_0 (1)^{c_0 - 1} = \sin \frac{\theta}{1} Z_{\gamma_0} \frac{\gamma_{1 + u}}{\gamma_{1 + u + 1}} du (\gamma > 1);$$

(5.9)

clearly a similar representation holds for $c_1 (1)^{c_1 - 1}$ with $c_1$ instead of $c_0$. Hence there exist positive measures $\gamma_1$ and $\gamma_0$ on $[1; 1]$ such that

$$\frac{JW(x) + W(y; J)}{x y} = \frac{ab}{x^0 (1)^{x - 1}} \frac{y^{x - 1} (1)^{y - 1}}{x y} \frac{0}{0} \frac{0}{x^{y - 1} (1)^{y - 1}} \frac{0}{0}$$

(5.10)

in which $ab > 0$. The matrix kernel $(JW(x) + W(y; J))$ is a Schur multiplier on the rank one tensor $(x, y)$ in $L^2((1 + \gamma_{H\phi}); R^2)$; hence for each $\gamma > 0$, there exists $\gamma > 0$ such the Schur multiplier norm is bounded by. Since $(x + s)$ gives a Hilbert-Schmidt kernel, the operator $K_0 (x + s) (y + s) ds$ is trace class on $L^2((1 + \gamma_{H\phi}); dx)$, and it follows that

$$K(x; y) = \frac{Z_{1 D}}{E} \frac{JW(x + s) + W(y + s; J)}{x y} (x + s); (y + s) ds$$

(5.11)
is also trace class. As in Theorem 1.1 of [4], we can introduce the Hilbert space $H$, $L^2((1+1);x dx;H^2)$ and the Hankel operator with symbol such that $K = \gamma_H \nu_H \rho_H$; so

$$K(x;y) = \int_0^{\infty} h_{\nu_H}(x+s); (y+s)i_{\rho_H} ds$$

where $H_{\nu_H}$ takes account of the fact that the Schur multiplier is positive on the top left matrix block and negative on the bottom right matrix block.

(ii) We observe that

$$W(z) = \frac{1}{ab} 0 1 ab 0 + O(\rho) \quad (j \neq 1);$$

(5.12)

is analytic at infinity and the residue matrix has eigenvalues $\rho$ which do not differ by a positive integer. Hence we can repeat the proof of Lemma 3.1 and realise the solution of (5.4) by a linear system involving the coefficients in the Laurent series of $W$. Then we can realise $L^2((0;1);H^2)$ by means of a linear system $(A;B;C)$, where the state space is $L^2((0;1);H^2)$. We can now follow through the proof in section 4 as expressed in terms of the Gelfand-Levitan equation.

(iii) Let $c = k = n + 1/n$, then $f(X;Z) : Z^n = X^k (X - 1)^n k^n$ gives a $n$-sheeted cover of $P^n$, ramified at $0;1;1$. On this compact Riemann surface, the functions $\phi_n(1)^{\rho_1}$ and $\phi_{\rho}(1)^{\rho_1}$ are rational.

Remarks. (i) The Painlevé equations can be expressed as Hamiltonian systems in the canonical variables $(\bullet ; \bullet )$, where the Hamiltonian is a rational function of $(\bullet ; \bullet )$; see [24] for a list. Okamoto [24] showed that there exists a holomorphic function on the universal covering surface of $P^n$ such that $H_{\nu}(t; \tau; \tau) = \frac{dt}{\theta(t)}$: The methods of [11, 15, 16] involve complex analysis and differential geometry, and are not intended to address the operator properties of $K$.

(ii) Borodin and Dieift [7] have identified an integrable kernel $K$ involving solutions $2F_1$ of the hypergeometric equation and considered $(t) = \det(I - P(t;1)K)$; they showed that $(t) = \frac{dt}{\theta(t)}(t)$ satisfies the Jimbo-Miwa form of $P_{\nu,1}$.

6. The function associated with a Hankel operator on exponential bases.

We wish to find an explicit expression for $W(z)$, and for $(t) = \frac{dt}{\theta(t)}(t)$ for suitable $K$, especially when $K$ is diagonal.

We can obtain an explicit formula for $W(z)$ when $K$ has the exponential expansion

$$W(z) = \sum_{j=1}^{\infty} \frac{z^j}{j!} e^{xj} (6.1)$$

where the coefficients $j$ lie in some Hilbert space $E$. In this section we establish the existence of such expansions by using the theory of approximation of compact Hankel operators, whereas in subsequent sections we consider the transfer function $^*(s)$ of $^*(s)$ and use the Mittag-Leer
expansion to give explicit formulas. The Hankel operator with symbol can be expressed in terms of the exponential basis as a relatively simple matrix, so we can derive expressions for its Fredholm determinant. Our applications in sections 7 and 8 are to cases in which the poles lie on an arithmetic progression, which occurs when is a theta function or arises by a certain transformation of a power series.

We suppose that \( j > 2 \) with \( j > 0 \) are such that \( (e^{tj})_{j=1}^1 \) are linearly independent exponentials, so that

\[
D_N = \det \frac{h}{j + k} \gamma_{jk = 1}^N > 0 \quad (N = 1, 2; \ldots): \tag{6.2}
\]

Suppose that \( = (j)_{j=1}^1 \) and introduce the operators

\[
B : \quad C ! \quad e^{tA} : \quad (j)_{j=1}^1 \quad (e^{t, A}_{j=1})_{j=1}^1
\]

\[
C : \quad ! \quad C : \quad (j)_{j=1}^1 \quad (e^{t, A}_{j=1})_{j=1}^1
\]

\[
: \quad L^2(0; 1) ! \quad f \gamma (0 \to e^{t, A} f(s) ds)_{j=1}^1:
\]

Theorem 6.1. Suppose that is bounded and that there exist constants \( ; M > 0 \) such that \( \gamma_j^k \gamma_j^k + k^2 \gamma_j^k = 0 \) for all \( j \); let \( 2^k \).

(i) Then the symbol \( (x) = C e^{xA} B \) gives rise to a Hankel operator \( : L^2(0; 1) ! L^2(0; 1) \) which is trace class.

(ii) The operator

\[
R_x = e^{xA} B C e^{xA} ds \tag{6.4}
\]

on \( \gamma^k \) is trace class, and for \( x \) is an open neighbourhood of zero, the kernel \( T (x; y) = C e^{xA} (I + R_x)^{-1} e^{yA} B \) gives a solution to the integral equation

\[
T (x; y) + (x + y) + T (x; z) (z + y) dz = 0 \quad (0 < x \quad y): \tag{6.5}
\]

(iii) Suppose that \((I + R_x)\) is invertible for all \( t > 0 \). Then the Hankel operator with kernel \((x + y + 2t)\) satisfies

\[
\det (I_{\gamma^k}) = \exp \quad T_{\gamma^k} (u; u) du:
\]

Proof. (i) The kernel may be expressed as a sum of rank-one kernels

\[
\sum_{j=1}^1 \sum_{j=1}^1 e^{(x+y)}
\]

where \( \sum_{j=1}^1 j^k \) converges, so is trace class.
(iii) By considering the rows of the matrix

\[
R_x \mathbb{H} \sum_{j+k \neq 0}^{i} i_{j+k}
\]

we see that \( R_x \) is also trace class. When \( j \neq k \), \( R_x \) is well defined, and one verifies the identity (6.5) by substituting.

(iii) The operators

\[
C : \psi ! C; \quad e^{\lambda} : \psi ! \psi; \quad R_x : \psi ! \psi; \quad B : C ! \psi
\]

are all bounded, and \( R_x \) is continuous from \( \psi \) to the trace class; hence \( T(x; y) \) depends continuously on \( x \) in a neighbourhood of 0 in \( \psi \). Suppose that \( \{ n \} \) is a sequence of vectors in \( \psi \) that have only finitely many nonzero terms, and that \( n \rightarrow 1 \). Denoting the operators corresponding to \( n \) by \( R_x^{(n)} \), etcetera, we can manipulate the \( n \)-item matrices and deduce that

\[
T^{(n)}(x; x) = \frac{d}{dx} \log \det(I \quad R_x^{(n)})
\]

and hence

\[
Z^{(n)}(x; x) dx = \log \det(I \quad R_x^{(n)}) \log \det(I \quad R_x^{(n)})
\]

so letting \( n \rightarrow 1 \), we deduce that

\[
Z^*_t \quad T^{(n)}(x; x) dx = \log \det(I \quad R_t^{(n)}) \log \det(I \quad R_s^{(n)})
\]

The operator \( L^2(0; 1) ! \psi \) given by

\[
f = \int_0^1 e^{\lambda A} B f(t) dt
\]

has matrix representation

\[
\psi \rightarrow h_{j+k}^{i} i_{j+k}
\]

with respect to the standard basis \( (e_j) \), and hence \( Y \) is Hilbert-Schmidt since

\[
\sum_{j=1}^{\infty} k \psi_{e_j} k^2 < 1. \quad \text{The operator } Y \text{ is bounded by hypothesis, hence } Y \text{ is also bounded; so}
\]

\( R_0 = Y \) is also Hilbert-Schmidt.

The operator \( Y \) is trace class by (ii), and the non-zero eigenvalues of \( Y = Y \) and \( R_0 = Y \) are equal, hence

\[
\det(I \quad R_x) = \det(I \quad R_x)
\]

which when combined with (6.12), implies that

\[
\log \det(I \quad R_x) = T^{(n)}(u; u) du;
\]
Evidently \(|s|! 0\) as \(s! 1\), and hence (6.6) follows from (6.16).

\[\square\]

Theorem 6.2. Let \(K\) be an integral operator on \(L^2((0;1);d\tau;C)\) such that:

(i) \(K\) \(I\) and \(I\) \(K\) is invertible;

(ii) there exists a separable Hilbert space \(E\) and \(2 L^2((0;1);d\tau;E)\) such that \(K = \gamma\).

Then \(K\) has a function \(K\) and there exists a sequence \((K_n)_{n=1}^{\infty}\) of finite rank integral operators with corresponding functions \(K_n\) such that:

1. \(K_n! K\) in trace class norm;
2. \(K_n(x)\) \(P_K(x)\) uniformly on compact sets as \(n! 1\);
3. \(K_n(x) = \sum_{j=1}^{\infty} a_{jn} e^{-jn^2}\) for some \(a_{jn}\); \(jn > 0\) and \(jn < j\) are given in Proposition 6.4 below.

Proof. (1) For \(2 L^2((0;1);d\tau;E)\), the operator \(K\) is Hilbert-Schmidt and hence \(K\) is trace class. By the Adamyan-Arov-Krein theorem [26], there exists a sequence \((K_{n,1})_{n=1}^{\infty}\) of finite rank Hankel operators such that \(K_{n,1}! K\) in Hilbert-Schmidt norm.

Kronecker showed that a Hankel operator \(K_{n,1}\) has finite rank if and only if the transfer function \(\gamma^{(n)}(s)\) is rationally; see [26]. Hence the typical form for \(\gamma^{(n)}(s)\) is a finite sum

\[\gamma^{(n)}(s) = \sum_{k=1}^{N} a_{k,n} e^{-k^2 s}\]

where \(k\) \(2 \in \mathbb{R}\) and \(k > 0\); the term \(s\) with factor \(e^{-k^2 s}\) give poles of order \(k + 1\). To resolve the poles of order greater than one into sums of simple poles, we introduce the difference operator \(\gamma\) by \(\gamma(g) = \gamma^{(n)}(g + \gamma)\), which satisfies \(\lim_{s \to 1} \gamma(g(s)) = g^{(n)}()\) whenever \(g\) is \(k\)-times differentiable with respect to \(s\). By the dominated convergence theorem,

\[\lim_{s \to 1} \int_{0}^{1} t^{k-1} e^{-kt} e^{-jt^2} dt = 0\]

as \(s! 0\), so we can replace \(t^{k-1} e^{-jt^2}\) by \(k^{k-1} e^{-jt}\) at the cost of a small change in the operator \(\gamma^{(n)}(s)\) in Hilbert-Schmidt norm. Thus we eliminate poles of order greater than one, and we can ensure that \(0\) \(y^{(n)}(s)\) \(I\), with \(I\) \(y^{(n)}(s)\) invertible. Let \(K_n = y^{(n)}(s)\) so that \(K_n\) has finite rank and \(K_n! K\) as in trace norm as \(n! 1\).

(2) Let \((x) = (x + 2x)\) and \((x) = (x + 2x).\) We have \(y^{(n)}(s)\) \(y^{(n)}(s)\) \(y^{(n)}(s)\) \(y^{(n)}(s)\) in trace class norm as \(n! 1\) so

\[\det(I - K_{(x)} P_{(x)};1) = \det(I - y_{(x)}^{(n)};x) = \lim_{n! 1} \det(I - y_{(x)}^{(n)};x) = \lim_{n! 1} K_{n}(x)\]
since the Fredholm determinant is a continuous functional on the trace class operators.

(3) To calculate the function $K_n(x)$ in (3) of Theorem 6.2, we assume that $(n)$ has the form

$$
(n)(t) = \frac{Y_j}{j} e^{j^t} \quad (t > 0)
$$

where $j > 0$ and $X_j < 0$. Without loss of generality, we can replace $E$ by the subspace span $(j)^n_{j=1}$ and for notational simplicity we take $j > 2 M; (C)$ where $N$:

We introduce

$$a_j = \sum_{m=1}^{M} \frac{h_{j+m}^{(n)}}{j} e^{2jxj+\frac{k}{m}} \quad 2 M; N (C)
$$

and

$$b_m = \sum_{k=1}^{M} \frac{h_{j+k}^{(n)}}{j} e^{2jxj+\frac{k}{m}} \quad 2 M; N (C)
$$

Lemma 6.3. The matrix

$$K = [a_j b_m]_{j=m=1}^{n}
$$

represents the operator $Y_{(n)}^{(x)}$ with respect to the (non-orthogonal) basis $(e^{j^x})_{j=1}^{N}:

Proof. We observe that the transfer function of $(n)$ is the rational function

$$Y_{(n)}^{(x)}(s) = \sum_{j=1}^{n} \frac{Y_j}{j} e^{j^x} \quad s + j
$$

The operator $Y_{(n)}^{(x)}$ has kernel in the variables $(s,t)$

$$Z_1 h_{(n)}(2x+s+u); \quad (n)(2x+t+u)idu
$$

and hence one computes

$$Y_{(n)}^{(x)}(s) = \sum_{j=1}^{n} \frac{h_{j^x}}{j} m e^{2j^xj+k} (k+m) \quad s + j
$$

Recalling the definitions (6.21) and (6.22), one computes

$$a_j b_m = \sum_{j=1}^{n} \frac{h_{j^x}}{j} m e^{2j^xj+k} (k+m) \quad s + j
$$

and by comparing this with (6.23), one obtains the stated identity. 

\[ \square \]
We can proceed to compute the function when \((n)\) is as in Theorem 6.2. For \(S; T \iff 1; \cdots; N\ g, \let K_{S;K} \be the submatrix of \(K_n\) that is indexed by \((j; k) 2 S \times T, and \let |S| be the number of elements of \(S\).

Proposition 6.4. (i) Suppose that \((n) = (0; 1)\) \(C\) is as in (6.20). Then

\[
K_n(x) = \sum_{j = 0}^{n} \det_{T; S; j = |S|} 1 \quad X \quad Y \quad j \in \mathbb{Z}^2 \quad k \in \mathbb{Z}^2 \quad h_{j+k}^{1+i} \quad \det_{j = 1, m = 2S;j \not= K}^{1+i} \quad (6.28)
\]

(ii) Suppose that \((n) = (0; 1)\) \(E\) where \(E\) has orthonormal basis \(e_i\) and \let \(\{r\} = h_{j;e_i}. i\) Then

\[
K_n(x) = \sum_{S; T; |S| = |T|} 1 \quad X \quad Y \quad j \in \mathbb{Z}^2 \quad k \in \mathbb{Z}^2 \quad h_{j+k}^{1+i} \quad \det_{j = 1, m = 2S;j \not= K}^{1+i} \quad (6.29)
\]

and the sum is over all pairs of subsets \(S \iff 1; \cdots; N\ g, T \iff 1; \cdots; N\ g, f_1; \cdots; g\) that have equal cardinality.

Proof. (i) By the Lemma we have \(K_n(x) = \det(I - K_n)\), and by expansion of the determinant we have

\[
\det(I - K_n) = \sum_{S; T: |S| = |T|} 1 \quad X \quad Y \quad j \in \mathbb{Z}^2 \quad k \in \mathbb{Z}^2 \quad h_{j+k}^{1+i} \quad \det_{j = 1, m = 2S;j \not= K}^{1+i} \quad (6.30)
\]

where \(\det K_{S;S} = 1\) and otherwise

\[
\det K_{S;S} = \sum_{k = 1}^{\infty} \det_{j \in \mathbb{Z}^2 \quad k \in \mathbb{Z}^2 \quad h_{j+k}^{1+i} \quad (6.31)}
\]

which reduces by the Cauchy–Binet formula to

\[
\begin{align*}
X & \quad h_{j+k}^{1+i} \quad \det_{j \in \mathbb{Z}^2 \quad k \in \mathbb{Z}^2 \quad h_{j+k}^{1+i} \quad (6.32)}
T ; |T| = |S| \\
& = \sum_{j \in \mathbb{Z}^2 \quad k \in \mathbb{Z}^2 \quad h_{j+k}^{1+i} \quad (6.32)}
\end{align*}
\]

By taking the sums over both \(S\) and \(T\), we obtain the stated formula.

(ii) To prove (ii) one follows a similar route until line (6.32), except that we have \(h_{j; k} = P_{r=1}^{(r)} \{k\})\), so the indices in the Cauchy–Binet formula are over the product set \(T \iff 1; \cdots; N\ g, f_1; \cdots; g\).

\[
\square
\]

7. The function for the hard spectral edge

Our first application of section 6 is to the hard edge ensemble. The Jacobipolynomials arise when one applies the Gram–Schmidt process to \(\{x^k\}_{k=0}^\infty\) with respect to the weight \((1 - x) (1 + x)\).
The zeros of the polynomials of high degree tend to accumulate at the so-called hard edges $1$ and $(1)^+$. According to [28], the kernel that describes the limiting behaviour of the joint distribution of the scaled zeros near to the hard edges is given by

$$J \left( \frac{P}{x}, y \right) J \left( \frac{P}{y}, x \right) = \frac{Z}{J} \left( \frac{P}{tx}, y \right) J \left( \frac{P}{ty}, x \right) \ dt$$

on $L^2((0;1); dt)$; here $J$ is Bessel's function of the first kind of order. Hence we change variables and introduce the Hankel operators on $L^2((0;1); dt)$.

Proposition 7.1. For $x > 1$, let $(x) = e^{x^2} J (2e^{-x^2})$ and let be the Hankel integral operator on $L^2((0;1))$ with symbol. Then Theorem 6.2 applies to .

Proof. From the power series for $J$, we obtain a rapidly convergent series

$$x^x \left( \frac{1}{n!} \left( \frac{1}{+n+1} \right) \right) \ (x > 0) \ (7.2)$$

giving a meromorphic transfer function

$$\left( s \right) = 1 \ X_0 \left( \frac{1}{n!} \left( \frac{1}{+n+1} \right) \right) \ (7.3)$$

for which the poles form an arithmetic progression along the negative real axis. One can alternatively express $\left( s \right)$ in terms of Lommel's functions.

We choose $n = (2n + 1) = 2$, so $(n)$ gives an arithmetic progression along the positive real axis, starting at $(+1) = 2 > 0$, and $2 n < 1$. The operator $\left( s \right)$ is bounded by duality since

$$Z \left( \frac{1}{n!} \left( \frac{1}{n+m} \right) \right) \ (7.4)$$

by Hilbert's inequality. Hence is a self-adjoint trace class operator, and Theorem 6.2 applies.

We can now compute some of the finite determinants that appear in the expansion of $\det(I - (\chi)_{x})$ from Proposition 6.4.

Definition (Partition). By a partition we mean a list $n_1 \ n_2 \ \cdots \ n$ of positive integers, so that the sum $j = \sum_{j=1}^{P} n_j$, is split into $P$ parts. For each , the symmetric group on $j$ letters has an irreducible unitary representation on a complex inner product space $S$, known as the Specht module. For notational convenience, we introduce a null partition with $\chi_{0} = 0$ and write $\dim(S_{0}) = 1$.
Proposition 7.2. Suppose that \( K = 2 \) and \( (x) = \text{trace}(I \ K P_{k,l}) \). Then \( K \) is a trace class operator on \( L^2(0;1) \) such that \( 0 \leq K \leq I \) and

\[
(k) = \frac{X}{(j \ j)^2} e^{2j \ k},
\]

where the sum is over all partitions.

Proof. Let \( E_n = \text{spanf}^{(2j+1)x} j = 0; \ldots; ng \) and let \( Q_n : L^2(0;1) \to E_n \) be the orthogonal projection; likewise we introduce the closure \( E_1 \) of the subspace \( \{ 1 \} \subseteq E_n \) and the corresponding orthogonal projection \( Q_1 : L^2(0;1) \to E_1 \). Observe that \( Q_n \to Q_1 \) in the strong operator topology as \( n \to 1 \) and that \( (x) = Q_1 \); hence \( (I \ 2(x)) = \lim_{n \to 1} \det(I \ Q_n 2(x) Q_n) \).

The matrix of \( Q_n 2(x) Q_n \) with respect to \( (e^{(2j+1)x})_{j=0}^n \) satisfies

\[
Q_n 2(x) Q_n = \sum_{j=0}^{\infty} \frac{1}{j!} e^{2k (j+m+1)} \left( \begin{array}{c} x \\ k \end{array} \right)_{j+m+1} (j+k+1)(m+k+1),
\]

We observe that the corresponding matrix in \( (x) \) has entries that are sums of \( j \) and \( m \) over \( jm = 0;1; \ldots;g \) and \( \det(I \ 2(x)) \) is a determinant of Hille's type.

We consider the determinant in (6.28). We change notation so as to allow the running indices in sum to be \( j,k = 0;1; \ldots;g \) and we let \( S \) and \( T \) be subsets of \( 0;1;2; \ldots;g \) that are finite and of equal cardinality. Suppose that the elements of \( S \) are \( m_1 > m_2 > \cdots > m_g \), while the elements of \( T \) are \( k_1 > k_2 > \cdots > k_g \); next let \( N = \left( + \sum_{i=1}^{g} (m_i + k_i) \right) \). Then in Frobenius's coordinates [8, 21], there is a partition \( S = \{ m_1; \ldots; m_g; k_1; \ldots; k_g \} \) with \( j \) and \( m \) with a corresponding Specht module \( S \) such that

\[
\det Q_s 2(x) k \frac{1}{m! (m+k+1)} = \dim(S) \frac{1}{m! (m+k+1)} \]

as in the hook length formula of representation theory; see in [21]. Hence the pair of sets \( S \) and \( T \), each with \( \left( \right) \) elements give rise to the product of determinants

\[
\det \frac{1}{j! (j+1) (j+k+1)} \cdot \det \frac{1}{m! (m+1) (m+k+1)} = \dim(S) \frac{1}{(j \ j)^2}
\]

and the exponential

\[
e^{p j \ (2j+1)x} e^{p k \ (2k+1)x} = e^{2j \ k}.
\]
The zero of the discrete Bessel kernel, and derive a result vaguely similar to (7.6). The determinant det(\(K P(\varphi_1)\)) was computed by Forrester, and Forrester and Witte have considered various circular ensembles [11]. Basor and Ehrhardt have considered asymptotics of Bessel operators [3].

8. A function related to Lamé’s equation

To conclude this paper, we consider Hankel operators related to Lamé’s equation. First we review some ideas that originate with Hochstadt and are developed by M. Okkonen and van M. Orzech in [23].

Let \(E\) be a compact Riemann surface of genus \(g\), and \(J\) the Jacobi variety of \(E\), which we identify with \(C^g\) for some lattice \(L\) in \(C^g\). An abelian function is a locally rational function on \(J\), or equivalently a periodic meromorphic function on \(C^g\) with \(2g\) complex periods. A theta function (or elliptic function of the second kind) \(\phi\) with respect to \(L\) is a meromorphic function, not identically zero, such that there exists a linear map \(x: L(\kappa; u)\) for \(x \in C^g\) and \(u \in \mathbb{C}^g\), and an \(L! C\) such that \((x + u) = \phi(x + u)\) for all \(x \in C^g\) and \(u \in \mathbb{C}^g\). The pair \((L; C)\) is called the type of \(\phi\), as in [20].

Suppose that \(q: R \mapsto R\) is a non-decreasing and periodic with period one. Let \(U\) be the fundamental solution matrix for Hill’s equation

\[
\frac{d^2}{dt^2} f + q(t) f(t) = f(t)
\]

so that \(U(0) = I\), and let \(\omega = \text{trace} U(1)\) be the discriminant. Suppose in particular that \(\omega\) lies inside the Bloch spectrum of \(\frac{d^2}{dt^2} + q(t)\), but that \(\omega^2 \neq 0\). Then any non-trivial solution of (8.1) is bounded but not periodic.

We suppose that \(\omega^2\) has only finitely many simple zeros \(0 < \frac{1}{2} \omega < \frac{3}{2} \omega < \cdots < \frac{2g-1}{2} \omega\), and let \(k\) be double zeros for \(k = 1, 2, \ldots\), then

\[
4 \omega^2 = c_1 \frac{Y^g}{Y} + 1 \frac{Y}{Y} + 1 \frac{Y}{Y} + 1 \frac{Y}{Y} + \cdots
\]

Proposition 8.1. Suppose that \(q\) is a finite gap potential so that the discriminant has this form. Then Hill’s equation gives a Tracy-Widom system on a hyperelliptic curve of genus \(g\).

Proof. The equation (8.1) has non-trivial bounded solutions if and only if \(j(\omega) < 2\), so that \(\omega\) lies in an interval of stability. Hence the spectrum of \(\frac{d^2}{dt^2} + q\) in \(L^2(\mathbb{R})\) has the form

\[
\left[ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \right] + \left[ \frac{1}{2g}, \frac{3}{2g}, \cdots \right] - \cdots
\]

The zeros of \(q(t)\) consist of all the \(\frac{1}{k}\) together with zeros \(0\) that interlace the simple zeros of \(z(t)\), so \(0 < \frac{1}{2} \omega < \cdots < \frac{1}{2g} \omega\) for \(j = 1, \ldots, g\); hence

\[
\frac{4}{\omega^2}(\omega) = \frac{Q^g}{j=1} \frac{1}{j} - \frac{Q^{2g}}{j=0} \frac{1}{j} - \cdots
\]
We introduce the hyperelliptic curve

\[ E : \quad Z^2 = \sum_{j=0}^{g} \frac{X}{j} \quad ; \tag{8.5} \]

which has genus \( g \). We introduce a new variable by the integral

\[ t = \int_0^1 \frac{0(X)\,dX}{2(X)^2} \quad ; \tag{8.6} \]

so that \( 2 \cos t = (X) \), then we invert this relation by introducing a hyperelliptic function \( Q(t) \)

with local inverse \( R \) so that \( R(Q(t)) = t \) and \( 2 \cos t = (Q(t)) \). After a little reduction, Hill's equation becomes

\[ \frac{Z}{\prod_{j=1}^{g} \frac{X}{j}} \frac{d}{dX} Z^2 + q(R(X)f = f ; \tag{8.7} \]

Now by [23, p. 260], \( q(R(X)) \) is an abelian function on \( E \) and may be viewed as a locally rational

function on the Jacobian variety \( J \) over \( E \); hence we can express (8.7) as a matrix differential equation with coe cients in the \( E \) of locally rational functions on \( J \).

\( \Box \)

Suppose in particular that \( q \) is elliptic with periods \( 2K \) and \( 2K^0 \) where \( K;K^0 > 0 \). Gesztesy and W esterd [12] have shown that the spectrum has only \( \text{finitely many gaps if and only if } z \\ U(z) \)

is meromorphic (and possibly multivalued) for all \( 2 \in \mathbb{C} \). By a classical result of Picard, there

exists a nonsingular matrix \( A \) such that \( U(z + 2K) = U(z)A \). If \( A \) has distinct eigenvalues,

then there exists a solution \( f \) to (8.1) that is a theta function with respect to the lattice \( L = f2K m + 2K^0 n : m, n \in 2Zg \).

Next we describe in more detail the case of genus one. We recall Jacobi's sinus amplitudinis

of modulus \( k \) is \( \text{sn}(x \, jk) = \sin \) where

\[ x = \int_0^\frac{d}{1 \, k^2 \sin^2} : \tag{8.8} \]

For \( 0 < k < 1 \), let \( K(k) \) be the complete elliptic integral

\[ K(k) = \int_0^1 \frac{dt}{1 \, k^2 \sin^2 t} \tag{8.9} \]

Next let \( K^0(k) = K(1 - k^2) \); then \( \text{sn}(z \, jk)^2 \) has real period \( K \) and complex period \( 2iK^0 \). We introduce

\[ e_1;e_2;e_3 = \frac{2}{3} \, k^2 ; \frac{2k^2}{3} ; \frac{1}{3} ; \frac{k^2 + 1}{3} ; \tag{8.10} \]

and

\[ g_2 = \frac{4(k^4 \, k^2 + 1)}{3} ; \quad g_3 = \frac{4(k^2 \, 2k^2 \, 1)(k^2 + 1)}{27} ; \tag{8.11} \]
parame
ter. Weierstrass introduced the functions

\[ P(z) = e_1 + (e_1 - e_2) \sin(z) = \int_0^z \sin(z') dz' \quad (8.12) \]

Likewise, \( P(z) \) has periods \( 2K \) and \( 2iK \), and \( P(x + iK) \) is bounded, real and \( 2K \)-periodic. In
terms of the new variable \( x = z + iK \) and the constant \( B = (e_1 - e_2) \), Lamé’s
differential equation (1.16) transforms to

\[ \frac{d^2}{dx^2} + (\prime + 1)P(x) \frac{d}{dx} + B(x) = 0; \quad (8.13) \]

Writing \( X = P(x), Y = P(y) \) and \( Z = P^0(x) \); the point \( (X; Z) \) lies on the elliptic curve

\[ E : Z^2 = 4(X - e_1)(X - e_2)(X - e_3); \quad (8.14) \]

and the elliptic function \( \text{ell} K \) consists of the \( \text{ell} \) of rational functions of \( X \) with \( Z \) adjoined
and we think of \( B \) as a point on \( E \). For \((x_0; z_0)\) on \( E \), we introduce the function

\[ (X; Z; x_0; z_0) = \exp \frac{1}{2} \frac{Z}{X} \frac{z_0}{x_0} \frac{dx}{z} \quad (8.15) \]

which takes multiple values depending upon the path from \((x_0; z_0)\) to \((X; Z)\). Then for integers
\( 1 \), and typical values of \( B \), there exist \( 2 \) \( C \) and polynomials \( A_0(X) \) and \( A_1(X) \) such that

\[ (X) = A_0(X) + A_1(X) \frac{Z + z_0}{X} \frac{X}{x_0} \frac{dx}{z} \quad (8.16) \]

gives a solution of

\[ Z \frac{d}{dx} (X) + (\prime + 1)X (X) + B \frac{dx}{z} = 0; \quad (8.17) \]

known as a Hermite-Halphen solution. Mailer [22, Theorem 4.1] has shown how to compute \((x_0; y_0)\)
and the spectral curve in terms of \( X \) and \( B \), thus making (8.17) convenient for computation. As \( Z \)
is rational on the elliptic curve, Lamé’s equation gives rise to a Tracy-Widom system (1.1) that
closely resembles the Laguerre system of orthogonal polynomials with parameter \( \alpha \), as considered
in [5, 29].

Suppose henceforth that \( \prime = 1 \). For \( 2 \) \( \lfloor k^2; 1 \rfloor \) \( \lfloor k^2 + 1; 1 \rfloor \), all solutions to (1.16) are
bounded; however, except for the countable subset of values of \( \alpha \) that gives the periodic spectrum,
these solutions are not \( K \) or \( 2K \) periodic; see [22]. Write \( B = P(\alpha) \) where \( \alpha \) is the spectral
parameter. Weierstrass introduced the functions

\[ (z) = \sum_{l=1}^{2L} \frac{z^l}{l!} \exp \frac{z}{l!} + \frac{1}{2} \frac{z^2}{l!} \quad (8.18) \]

where \( L = \ln f(0; 0)g \), and \( \alpha = \alpha(z) = \alpha \) so that \( P = 0 \). Then by [19, (13)] the equation
(8.13) has a nontrivial solution

\[ (\alpha; \alpha) = (x) \frac{\alpha}{(\alpha)} \frac{e}{(\alpha)} \quad (8.19) \]
such that \((x;)(x;)=P(\cdot)P(x)\) and \(P(x)\) is doubly periodic.

The solutions give rise to a natural kernel for after we make the local change of independent variable \(x\), \(X\) and write \(f(X) = (x;)(x;)=0(x;),\) we have by \([19, (18)]\)

\[
\frac{f(X)g(Y)}{X} = (x + y;): \quad (8.20)
\]

The right-hand side has the shape of the kernel of Hankel integral operator. In the remainder of this section we introduce this operator, and compute the corresponding Fredholm determinant.

Lemma 8.2. Let \(= 2K (\cdot) + ( + 2K) (\cdot),\) suppose that \(> 0\) and let \(2 C\) such that \((x + 2t;\) is analytic for \(x \in [0;2K].\) Let \((t)(x) = (x + 2t;\) and \(h(s) = _{0}^{R} e^{su}(t)(u) du.\) Then \((t)\) is a theta function and has an exponential expansion

\[
(t)(x) = \sum_{m = 1}^{\infty} \frac{1}{2K} \int_{-\infty}^{\infty} e^{iK} e^{(2iK)u = (2K)} (x > 0) \quad (8.21)
\]

and \(^{(t)}\) is a meromorphic function with poles in an arithmetic progression.

Proof. We introduce \(\cdot ( + 2K) (\cdot)\) and \(( + 2K) (\cdot).\) Then \(\cdot\) is a theta function and satisfies a simple functional equation given in \([20, p. 109]\); from this we deduce that

\[
(x + 2K;\) = (x;\) e^{2K (\cdot)} ; \quad (x + 2K 0;\) = (x;\) e^{2K 0 (\cdot)} 0 ; \quad (8.22)
\]

Hence \(x\) \((x + 2t;\) is of exponential decay as \(x \to \infty\) through real values.

Due to \((8.22),\) the transfer function of \((t)(x)\) is

\[
^{(t)}(s) = \sum_{k = 0}^{\infty} e^{2K k} e^{su}(u + 2t;\) du
\]

\[
= (1 e^{2K s + 2K (\cdot)} 1) \int_{0}^{\infty} e^{su} du \quad (8.23)
\]

which is meromorphic with possible poles at the points \(s = (2K) (2K) + 2m i\) for \(m \geq 2\) which form a vertical arithmetic progression in the left half plane. The position of the poles is determined by the type of the theta function.

We can deduce the exponential expansion by inverting the Laplace transform. Let \(T = (2m + 1) = (2K)\) let \(x > 0\) and consider the contour \([iT; iT]\) \(S_{T}\), where \(S_{T}\) is the semicircular arc in the left half plane with center \(0\) that goes from \(-iT\) to \(iT;\) then by Cauchy’s Residue Theorem we have

\[
\int_{S_{T}} e^{sx} (t)(s) ds + \int_{[iT; iT]} e^{sx} (t)(s) ds = \frac{i}{K} \int_{n = 0}^{\infty} h 2ni \frac{e^{x(2ni)}(2K)}{2K} \quad (8.24)
\]

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We integrate \( \int_0^{2K} (u + 2t; e^u) du \) by parts and write
\[
e^{ax} / (x + 1) e^x = e^{2K x} L_{(x)} (2K) (x + 1) e^x (0) + e^{2K x} L_{(x)} (0) + e^{ax} / (x + 1) e^x (u) du \tag{8.25}
\]
and then use Jordan's Lemma to show that \( L_{(x)} (s) ds \) as \( T \to 1 \). Hence
\[
\int_{(x)} (s) = \frac{1}{2} \int_{i}^{Z} e^{a x} (x + 1) e^x (s) ds = \frac{X}{K} \frac{1}{2K} L_{(x)} (0) + \frac{1}{2K} L_{(x)} (1) e^{x (2 in + ) (2K)} :) \tag{8.26}
\]

**Theorem 8.3.** Let \( (x) = (x + 2t; ) \) and let \( (t) \) be the Hankel integral operator on \( L^2 (0; 1) \) with symbol \( (t) \). Then the conclusions of Theorem 6.1 hold for \( (t) \).

**Proof.** Let \( n = (2 in + ) = (2K) \) where \( K > 0 \). Then by a standard argument from the calculus of residues, we have
\[
\sum_{k=1}^{\infty} \frac{1}{j j + k k} = \frac{K^2 \cosh^2}{<} (j 2 Z) : \tag{8.27}
\]
The operator \( : L^2 (0; 1) \) is given by
\[
f (s) = e^{j s f (s)} ds \int_{0}^{1} j = 1 \tag{8.28}
\]
is bounded. Indeed, we observe that the sequence \( (e^{s x})_{n=1}^{\infty} \) forms a Riesz basic sequence in \( L^2 (0; 1) \), in the sense that there exists a constant \( C > 0 \) such that
\[
C \sum_{n=1}^{\infty} \int_{0}^{1} a_n e^{s x} dx \sum_{n=1}^{\infty} \int_{0}^{1} \hat{a}_n \hat{f} = \sum_{n=1}^{\infty} \int_{0}^{1} \hat{a}_n \hat{f} \tag{8.29}
\]
for all \( (a_n) \). To prove this, one uses a simple scaling argument and orthogonality of the sequence \( (e^{s x})_{n=1}^{\infty} \) in \( L^2 (0; 1) \); in particular, this shows that \( y : q^2 \) is bounded, \( L^2 (0; 1) \) is bounded, so is bounded.

We can now use the general Theorem 6.1. Given this rapid decay and the fact that \( (x + y + 2t; a) \) is analytic, one can easily check that \( (t) \) is trace class.

**Our nal result** gives the order of growth of the determinant
\[
D_N = \det \frac{h}{k = 1}^{i_n} \hat{f} = \sum_{j+k=1}^{i_n} \tag{8.30}
\]

**Proposition 8.4.** Suppose that \( j = (2 ij + ) = (2K) \) where \( j > 0 \) and \( K > 0 \). Let \( \lambda \) be the Haar probability measure on the unitary group \( U (N) \), and let \( \arg e^i = \) for \( 0 < \) \( 2 \).

(i) Then
\[
D_N = \sum_{j=1}^{2K} e^{2K} \sum_{k=1}^{N} \exp < \text{trace} \arg U > (dU) \tag{8.31}
\]
(ii) There exists a constant $c > 0$ such that
\[
\frac{K}{\sinh <} < N e^{(2c)^{1+3} N^{2+3} < j^{2+3}} D_N < N e^{(2c)^{1+3} N^{2+3} < j^{2+3}} : \quad (8.32)
\]
so
\[
D_N^{1+1} K \coth < < N ! 1) : \quad (8.33)
\]

Proof. (i) Let
\[
f(u) = \frac{2K e^{2c u}}{1 e^{2c}} \quad (0 < u < 1)
\]
and let the Fourier coefficients of $f$ be $a_k = R_0 f(u) e^{2 ik u} du$, which we compute and find
\[
\frac{1}{j+k} = a_{j+k} : \quad (8.35)
\]
Then we can use an identity due to Heine, and express the Toeplitz determinant of $[a_{j+k}]$ as an integral
\[
\det[a_{j+k}]_{j+k=1,\cdots,N} = \frac{1}{N!} \prod_{1 < j < k \leq N} \int_0^{2\pi} e^{2 i j} e^{2 i k} e^{2 i j} f(j) d_1 \cdots d_N ; \quad (8.36)
\]
which we regard as an integral over the group $U(N)$, and hence we convert the expression into an integral over the group $U(N)$, obtaining
\[
\det \frac{1}{j+k} \int_0^{2\pi} e^{2 i j} e^{2 i k} e^{2 i j} f(j) d_1 \cdots d_N ; \quad (8.37)
\]
(ii) Note that $\log f(\arg e^i = (2 \pi )) = \log (2K \sinh < N X_{j=1} g_j (dU) \exp \trace \log f \arg U = (2 \pi ) (dU) ; \quad (8.38)
which satisfies a central limit theorem, but we need to adjust the functions slightly to accommodate the discontinuity of $\arg$. Let $g_1, g_2 : R \mapsto R$ be Lipschitz functions with Lipschitz constant $L$, that are periodic with period 2, and satisfy $g_1(0) = g_2(0) = 0 < 2$, and
\[
\frac{1}{L} \int_0^{2\pi} g_1(\cdot) d_1 + \frac{1}{L} : \quad (8.39)
\]
By Szego's asymptotic formula [18], there exists a constant $c$ such that
\[
\frac{Z}{\exp < \chi_{j=1}^N U(N)} < \frac{Z}{\exp < \chi_{j=1}^N U(N)} < \exp \frac{Z}{\exp < \chi_{j=1}^N U(N)} < \exp \frac{Z}{\exp < \chi_{j=1}^N U(N)} < \exp N < g_2(\cdot) d_1 + c(2)^2 L^2 ; \quad (8.40)
\]
hence we have an upper bound on $D_N$ of

\[
\begin{align*}
\frac{2K}{e^{2e}} & \leq \frac{N^Z}{\prod_{j=1}^{N/2} (dU)} \leq \frac{2K}{e^{2e}} \leq N^{-L+c(\epsilon)^2L^2}.
\end{align*}
\]

Using $g_2$ instead of $g_1$, one can likewise obtain a lower bound on $D_N$. To conclude the proof, we choose $L = N^{1-\epsilon}(2\epsilon^{-1})^{1-\epsilon}$.

\[ \square \]

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