DEGENERATE INTEGRABILITY OF QUANTUM SPIN CALOGERO–MOSER SYSTEMS.

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ABSTRACT. The main result of this note is the proof of degenerate quantum integrability of quantum spin Calogero–Moser systems and the description of the spectrum of quantum Hamiltonians in terms of the decomposition of tensor products of irreducible representations of corresponding Lie algebra.

INTRODUCTION

Recall the definition of quantum spin Calogero–Moser systems. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. We fix a choice of the Borel subalgebra in $\mathfrak{g}$, which gives the choice of the Cartan subalgebra $\mathfrak{h}$. Let $V_\mu$ be an irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight $\mu$ and $V_\mu[0]$ be zero weight subspace.

The Hamiltonian for the spin Calogero-Moser system corresponding to these data is the following differential operator acting on $V_\mu[0]$-valued functions on the Cartan subgroup $H \subset G$:

\begin{equation}
H_{CM} = -\Delta + \sum_{\alpha \in \Delta^+} \frac{X_\alpha X_{-\alpha}}{(h_\alpha/2 - h_{-\alpha}/2)^2}
\end{equation}

Here $\Delta$ is the Laplacian on $\mathfrak{h}^*$ corresponding to the scalar product given by the Killing form, $X_\alpha$ are root elements of the Chevalley basis in $\mathfrak{g}$, and $h_\alpha$ is the function on $H$ corresponding to the root $\alpha$, $Ad_h(X_\alpha) = h_\alpha X_\alpha$.

There is an extensive literature on both classical and quantum spin Calogero–Moser systems. The original Calogero–Moser system corresponds to $sl_n$ and $\mu = (m, 0, \ldots, 0)$. For Calogero-Moser system see for example [1][10][15][12][13][14][24] and references therein.

1Calogero-Moser model was initially discovered for $G = SL_n$, [3][1] [23] for rational and trigonometric potentials. The generalization to other root systems is due to Olshanetsky and Perelomov [13].

2See Appendix A for examples.

3 Note that operators $X_{\pm\alpha}$ act on $V_\mu$ but their product acts on $V_\mu[0]$.
Classical degenerate integrability\textsuperscript{4} generalizes the notion of Liouville integrability to the case when the dimension of Liouville tori is less than half of the dimension of the phase space of the system. The notion goes back to analysis of the Kepler system, also known as the model of hydrogen atom in the quantum case. It can be attributed to Pauli \textsuperscript{19}. After a number of new examples discovered in the 60’s, see for example \cite{9}, it was formulated in modern mathematical terms by Nekhoroshev in \cite{17}. Then it was used in a series of papers by Mischenko, Fomenko and co-authors, for details see \cite{8}. Some of the work in this direction is summarized in \cite{22}.

Instead of Lagrangian fibration by level sets of Poisson commuting Hamiltonians as in Liouville integrable systems, a degenerately integrable system on a symplectic manifold $\mathcal{M}_{2n}$ is defined by two projections

\begin{equation}
\mathcal{M}_{2n} \to P_{2n-k} \to B_k
\end{equation}

where $P_{2n-k}$ is a Poisson manifold, and the connected components of generic fibers of the last projection are symplectic leaves in $P_{2n-k}$. Functions on $B_k$ form a Poisson commutative subalgebra in the algebra of functions on $\mathcal{M}_{2n}$. This subalgebra is Poisson central in the Poisson subalgebra of functions on $P_{2n-k}$. The subalgebra of functions on $B_k$ (the subalgebra of Hamiltonians) is the Poisson center of the subalgebra of function on $P_{2n-k}$ (the subalgebra of integrals).

Similar to quantization of Liouville integrable systems, a quantization of a degenerate integrable system \textsuperscript{2} consists of a sequence of three embedded associative algebras:

$$C_h(\mathcal{M}_{2n}) \supset C_h(P_{2n-k}) \supset C_h(B_k)$$

Here $C_h(\mathcal{M}_{2n})$ is an associative deformation of the algebra of functions\textsuperscript{5} on $\mathcal{M}_{2n}$. The subalgebra $C_h(P_{2n-k})$ is an associative deformation of functions on $P_{2n-k}$, and $C_h(B_k)$ is a commutative algebra which is a deformation of functions on $B_k$. The algebra $C_h(B_k)$ must be the center of $C_h(P_{2n-k})$. Also, the algebra $C_h(B_k)$ must be the centralizer of $C_h(P_{2n-k})$ in $C_h(\mathcal{M}_{2n})$. The algebra $C_h(P_{2n-k})$ is called the algebra of quantum integrals, the algebra $C_h(B_k)$ is the algebra of quantum Hamiltonians.

\textsuperscript{4} It is also known as non-commutative integrability or as superintegrability. Though terms superintegrability and non-commutative integrability are more accepted in the literature, we will use the term degenerate integrability to avoid confusion with supergeometry and quantum integrability.

\textsuperscript{5} In the real smooth, or real analytic case this is the space smooth, or real analytic functions respectively. In the complex algebraic setting this is the space of algebraic functions, etc.
Classical degenerate integrability of spin Calogero–Moser models was proved in [20] where the duality between the spin Calogero–Moser model and the spin Ruijsenaars model was also noted. For latest developments on this duality see [2]. The main result of this paper is the proof of quantum degenerate integrability of the quantum spin Calogero–Moser system. The description of eigenfunctions and of the spectrum of quantum Hamiltonians can be found in [5] [7] respectively. We also present it here for completeness.

In Section 1 we describe the sequence of algebras responsible for degenerate integrability of the quantum spin Calogero–Moser systems. Eigenfunctions of quantum spin Calogero–Moser Hamiltonians and described in Section 2 where we also describe the spectrum of quantum Hamiltonians in the compact case. Conclusion contains a few remarks on semiclassical asymptotic of degenerate integrable systems, a comment on the relation to earlier works [24] and [14].

1. Quantum degenerate integrability of spin Calogero–Moser systems.

In this paper we will consider only quantum systems that are quantizations of classical systems. The core structure of a degenerate quantum integrable system consists of an algebra $C$ with trivial center and two subalgebras:

$$C \supset A \supset B$$

where $B = Z(A)$ (the center of $A$) and $B = Z(A, C)$ (the centralizer of $A$ in $C$).

1.1. Algebra of quantum integrals. Let $\text{Diff}(G)$ be the algebra of differential operators on $G$. We have a natural embedding of the left- and the right-invariant vector fields by Lie derivatives. Polynomials in these Lie derivatives can be identified with two copies of the universal enveloping algebras $U(g_L)$ and $U(g_R)$, where $g_L$ and $g_L$ are two copies of $g$ realized by right and left Lie derivatives respectively. It is clear that $Ad_G$-invariant polynomials in left and right Lie derivatives coincide. Therefore, we have the natural embedding:

$$\text{Diff}(G) \supset U(g) \otimes_{Z(U(g))} U(g)$$

Here the first factor in the tensor product corresponds to left-invariant vector fields and the second factor, to right-invariant vector fields. The tensor product is taken over the ring of $Ad_G$-invariant polynomials in Lie derivatives. The space of such polynomials can be naturally identified with the center of $U(g)$. 
Let $\text{Diff}_G(G)$ be the algebra of differential operators invariant with respect to the adjoint $G$-action. Denote by $(U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g}))_G$ the subalgebra of $U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g})$ of $G$-invariant elements, where $G$ acts diagonally on the tensor product. We have the following commutative diagram:

$$
\begin{array}{c}
\text{Diff}(G) & \xrightarrow{L} & U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g}) & \xleftarrow{R} & U(\mathfrak{g}) \\
\text{Diff}_G(G) & \xrightarrow{i} & (U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g}))_G \\
\end{array}
$$

Here $L(a) = a \otimes 1$, $R(a) = 1 \otimes a$ and the map $i$ comes from the natural isomorphism $Z(U(\mathfrak{g})) \simeq (Z(U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g})))_G$. It acts as $z \mapsto z \otimes Z(U(\mathfrak{g})) 1 = 1 \otimes z$.

This diagram is the quantum version of the corresponding diagram for Poisson projections which prove the degenerate integrability of spin Calogero–Moser systems in the classical case.

The lower horizontal sequence of Poisson maps lies at the heart of degenerate integrability of the spin Calogero–Moser systems [20].

Recall that classical spin Calogero–Moser systems are parameterized by co-adjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$. If $\mathcal{O}$ is passing through $t \in \mathfrak{h}^*$, then it is also passing through each $w(t)$, where $w \in W$ is an element of the Weyl group. We will denote such orbit passing through $t$ by $\mathcal{O}[t]$ where $[t] \in \mathfrak{h}^*/W$ is the orbit of $t$ with respect to the Weyl group action. When $\mathfrak{g}$ is a real compact form of a simple Lie algebra, we can identify $\mathfrak{h}^*/W$ with $\mathfrak{h}^*_{\geq 0} = \{ \sum_{i=1}^r x_i \omega_i | x_i \in \mathbb{R}_{\geq 0} \}$, where $\omega_i$ are the fundamental weights of $\mathfrak{g}$, and $r$ is the rank of $\mathfrak{g}$.

For a generic co-adjoint orbit $\mathcal{O}[t]$ the phase space of the corresponding spin Calogero–Moser system is the symplectic leaf $S_{[t]} = \mu^{-1}(\mathcal{O}[t])/G$, where $\mu : T^*G \to \mathfrak{h}^*$ is the moment map for the adjoint
action of $G$:

$$\mu(x, g) = x - \text{Ad}_g^*(x) \in \mathfrak{g}^*$$

Here $x \in \mathfrak{g}^*$, $g \in G$.

The sequence of projections from the diagram above produces the sequence of projections

$$S[t] \rightarrow \sqcup_{[s] \in \mathfrak{h}^*/W} \mathcal{M}_{[s],-[s][t]} \rightarrow \mathcal{B}[t] \subset \mathfrak{h}^*/W$$

Here the moduli space $\mathcal{M}_{[s_1],[s_2][t]}$ is defined as

$$\mathcal{M}_{[s_1],[s_2][t]} = \{(x_1, x_2) \in \mathcal{O}_{[s_1]} \times \mathcal{O}_{[s_2]} | x_1 + x_2 \in \mathcal{O}_{[t]}\}$$

and $\mathcal{B}[t] = \{[s] \in \mathfrak{h}^*/W | \mathcal{M}_{[s],-[s][t]} \neq \emptyset\}$. Note that $\mathcal{B}[t]$ is unbounded but if $t \neq 0$ it does not contain the vicinity of zero. The series of projections ([1]) describes the degenerate integrability of classical spin Calogero-Moser model. The Hamiltonian of the classical spin Calogero-Moser system is the pull-back of the quadratic Casimir function on $\mathfrak{h}^*/W$ to $S[t]$. Taking into account the isomorphism $S_{[t]}^{reg} \simeq (T^*\mathfrak{h} \times \mathcal{O}_{[t]}/\mathcal{H})/\mathcal{W}$, where $\mathcal{O}_{[t]}/\mathcal{H}$ is the Hamiltonian reduction of $\mathcal{O}_{[t]}$ with respect to the coadjoint action of $\mathcal{H}$, the Hamiltonian of the classical spin Calogero-Moser system can be written as

$$H_{CM} = <p, p> + \sum_{\alpha \in \Delta_+} \frac{x_\alpha x_{-\alpha}}{(h_\alpha/2 - h_{-\alpha}/2)^2}$$

where $p, h_\alpha$ are coordinate functions on $T^*\mathfrak{h}$ and $x_\alpha x_{-\alpha}$ is a function on $\mathcal{O}_{[t]}/\mathcal{H}$, see [20] for details.

The sequence of embeddings of algebras

$$\text{Diff}_G(G) \leftarrow \longrightarrow (U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g}))_G \leftarrow i \longrightarrow Z(U(\mathfrak{g}))$$

quantizes the sequence of Poisson projections from the low row of the commutative diagram (3).

We will use this sequence to demonstrate degenerate quantum integrability of quantum spin Calogero-Moser systems. The subalgebra $(U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g}))_G \subset \text{Diff}_G(G)$ gives the algebra of quantum integrals, and the subalgebra $Z(U(\mathfrak{g})) \subset \text{Diff}_G(G)$ gives the subalgebra of quantum integrals. The resulting system of embedded algebras will quantize Poisson projections (1).

1.2. Quantum Spin Calogero–Moser Hamiltonians.
1.2.1. Recall that in order to define quantum spin Calogero–Moser system we need to fix an irreducible finite dimensional representation of $\mathfrak{g}$. Let $\mu$ be the highest weight of such representation and $V_{\mu}[0]$ be its zero weight subspace. Denote by $W_{\mu}(H)$ the space of $V_{\mu}[0]$-valued smooth functions on the Cartan subgroup $H \subset G$ which are equivariant with respect to the action of the Weyl group of $G$:

$$f(wh) = wf(h)$$

Here the action of $W$ on $V_{\mu}[0]$ is induced by the action of $N(H)$, the normalizer of $H \subset G$.

We have a natural embedding of $W_{\mu}(H)$ into the space $C_{\mu}(G)$ of $V_{\mu}$-valued smooth functions on $G$. The image of this imbedding we denote by $W_{\mu}(G)$:

$$W_{\mu}(G) = \{ f \in C_{\mu}(G) | f(g^{-1}h g) = \pi_{\mu}(g) f(h), g \in G, h \in H, f(h) \in V_{\mu}[0] \subset V_{\mu} \}$$

For functions from $W_{\mu}(G)$ we have:

$$f(g^{-1}g') = \pi_{\mu}(g) f(g')$$

1.2.2. Let $x \in \mathfrak{g}$ and $x^L$ and $x^R$ be corresponding left and right invariant vector fields on $G$ which we identify with the corresponding Lie derivatives. If $f \in W_{\mu}(G)$

$$((x^L + x^R)f)(g) = \pi_{\mu}(x)f(g).$$

Let $\{X_\alpha, H_i\}$, where $\alpha$ are roots, and $i = 1, \ldots, r = \text{rank}(G)$, be a Chevalley basis in $\mathfrak{g}$. For the right and left action of these basis elements on functions from $W_{\mu}(G)$, followed by the restriction to $H \subset G$, we have:

$$(X_\alpha^R f)(h) = \frac{d}{dt} f(he^{-tX_\alpha})|_{t=0} = \frac{d}{dt} f(e^{-th_\alpha X_\alpha} h)|_{t=0} = -h_\alpha (X_\alpha^L f)(h)$$

$$(H^R_i f)(h) = -(H^L_i f)(h) = h_i \frac{\partial}{\partial h_i} f(h).$$

Here $h_\alpha$ the value of $h$ on the root $\alpha$:

$$\text{Ad}_h(X_\alpha) = h_\alpha X_\alpha$$

In combination with (5) we have:

$$((X_\alpha^L - h_\alpha X_\alpha^L)f)(h) = \pi_{\mu}(X_\alpha)f(h).$$

Thus, we have the following explicit action of left and right invariant vector fields on $G$ on the space $W_{\mu}(H)$:

$$(X_\alpha^L f)(h) = (1 - h_\alpha)^{-1} \pi_{\mu}(X_\alpha)f(h)$$

$$(X_\alpha^R f)(h) = -h_\alpha (1 - h_\alpha)^{-1} \pi_{\mu}(X_\alpha)f(h)$$
The algebra \( \text{Diff}_G(G) \) of \( G \)-invariant differential operators on \( G \) acts naturally on the space \( W_\mu(G) \). Because the algebra \((U(\mathfrak{g}) \otimes Z(U(\mathfrak{g})) U(\mathfrak{g}))_G\) can be identified with a subalgebra of \( \text{Diff}(G) \), it acts naturally on the space \( W_\mu(G) \). Let \( I_\mu \) be the ideal in \((U(\mathfrak{g}) \otimes Z(U(\mathfrak{g})) U(\mathfrak{g}))_G\) defined by the highest weight \( \mu \) considered as an irreducible \( Z(U(\mathfrak{g})) \)-character \( \mu \). We have the homomorphism of algebras

\[
\phi : (U(\mathfrak{g}) \otimes Z(U(\mathfrak{g})) U(\mathfrak{g}))_G/I_\mu \rightarrow \text{Diff}(H) \otimes \text{End}(V_\mu[0])
\]

The formulae (6) describe this homomorphism explicitly.

Denote the image of this homomorphism \( A_\mu \). Images of elements \( i(Z(U(\mathfrak{g})) \) in \( \text{Diff}(H) \otimes \text{End}(V_\mu[0]) \) form commutative subalgebra \( B_\mu \subset A_\mu \). This commutative subalgebra is generated by images of Casimir elements (whose degrees are exponents of \( \mathfrak{g} \)). These elements remain independent in \( B_\mu \) for generic \( \mu \).

The Hamiltonian of quantum spin Calogero–Moser systems is the image of the quadratic Casimir in \( Z(U(\mathfrak{g}) \otimes Z(U(\mathfrak{g})) U(\mathfrak{g}))_G \simeq Z(U(\mathfrak{g})) \):

\[
\hat{H}_{CM} = \frac{1}{2} \phi \circ i(c_2)
\]

Here \( \phi \) is defined above and \( i \) is defined in the last diagram of section 1.1.

Quantum degenerate integrability of the spin Calogero–Moser system is given by the sequence of subalgebras

\[
\text{Diff}(H) \otimes \text{End}(V_\mu[0]) \hookrightarrow A_\mu \hookrightarrow B_\mu
\]

These subalgebras quantize the system of Poisson projections (4) ensuring the degenerate integrability of the classical spin Calogero–Moser system.

The algebra \( B_\mu \) can be described as follows. Consider canonical elements (mixed Casimirs):

\[
X^L = \sum_i H^i \otimes H^i_i + \sum_\alpha e_\alpha \otimes X^L_\alpha = - \sum_i H^i \otimes h_i \frac{\partial}{\partial h_i} + \sum_\alpha X_\alpha \otimes \pi^\mu(X_\alpha) \frac{1}{1 - h_\alpha}
\]

\[
X^R = \sum_i H^i \otimes H^i_i + \sum_\alpha e_\alpha \otimes X^R_\alpha = \sum_i H^i \otimes h_i \frac{\partial}{\partial h_i} - \sum_\alpha X^R_\alpha \otimes \pi^\nu(X_\alpha) h_\alpha \frac{1}{1 - h_\alpha}
\]

The algebra \( B_\mu \) is generated by elements

\[
H^V_\mu = (tr_V \otimes id)(X^L)^n = (tr_V \otimes id)(X^R)^n
\]

In the semiclassical limit, when \( h \rightarrow 0 \), we have \( \text{Spec}(B_\mu) \rightarrow B[\hbar] \), in the appropriate sense.
Here we assume that elements are acting on $W_\mu(H)$.

The elements

$$(tr \otimes id)((X_L)^{n_1}(X_R)^{n_2}(X_L)^{m_1} \ldots)$$

acting on $W_\mu(H)$ are typical elements from $A_\mu$. The exact description of elements of $A_\mu$ is part of the geometric invariant theory and we will not discuss it here.

2. Eigenfunctions of quantum Calogero–Moser systems

In this case the Hamiltonian (1) can be written in terms of $q$-coordinates, $h = \exp(iq)$, as follows:

$$H = -\Delta + \frac{1}{4} \sum_{\alpha > 0} \frac{X_\alpha X_{-\alpha}}{\sin^2(q_\alpha^2)}$$

where $0 < q_\alpha < \pi$ are coordinates on $h^* \mod \mathbb{Z}^r$, $r = \text{rank}(\mathfrak{g})$. When $G$ is a compact real form of a simple Lie group, $q_\alpha$ are real. For split real form they are imaginary. This construction of eigenvectors and of quantum Hamiltonians follows [5].

2.1. Construction of eigenfunctions. Here we follow [5]. Let $(L_\lambda, \pi^\lambda)$ be a representation of the Lie group $G$ with an irreducible central character $\lambda$. Here $\pi^\lambda : G \to \text{Aut}(L_\lambda)$ is the corresponding group homomorphism. It induces naturally a representation of the universal enveloping algebra $U(\mathfrak{g})$. Central elements of $U(\mathfrak{g})$ acts as $\pi^\lambda(z) = z(\lambda)I_{V_\lambda}$ where $z(\lambda)$ is the value of $\lambda$ on $z$. Let $\hat{z} \in \text{Diff}_G(G)$ be the image of $z \in Z(U(\mathfrak{g}))$ in the algebra of differential operators on $G$. Then

$$\hat{z} \pi^\lambda(g) = z(\lambda) \pi^\lambda(g)$$

Thus, $\pi^\lambda$ is a joint eigenfunction of the subalgebra $Z(U(\mathfrak{g})) \subset \text{Diff}_G(G)$.

Let $K^\lambda_\mu(a) : L_\lambda \to L_\lambda \otimes V_\mu$ be $G$-linear map. Here we assume that indices $a$ enumerate all such maps, i.e. $a$ enumerates a basis in $\text{Hom}_G(L_\lambda \to L_\lambda \otimes V_\mu)$. Define the function

$$(7) \quad f_{\lambda, \mu}(g, a) = tr_{L_\lambda}(K^\lambda_\mu(a) \circ \pi_\lambda(g))$$

when the trace converges. This function is an element of $W_\mu(G)$. It is a linear combination of matrix elements of $\pi^\lambda(g)$ and therefore it satisfies the differential equation

$$\hat{z} f_{\lambda, \mu}(g, a) = z(\lambda) f_{\lambda, \mu}(g, a)$$

for each $z \in Z(U(\mathfrak{g}))$. Restricting this to $H \subset G$ we have

$$\phi \circ i(z)f_{\lambda, \mu}(h, a) = z(\lambda) f_{\lambda, \mu}(h, a)$$
Here $f_{\lambda,\mu}(h, a) \in V_{\mu}[0]$. When $z = c_2$ the operator on the left is the quantum spin Calogero-Moser Hamiltonian.

Thus we constructed joint eigenfunctions of $B_{\mu}$, and in particular, eigenfunctions for the quantum spin Calogero–Moser Hamiltonians. A generalization of this construction to the case when conjugation action of $G$ is replaced by a twisted conjugation action was found in [7].

2.2. The spectrum of quantum Hamiltonians for the compact real form of a simple Lie algebras. In order to construct the spectrum of quantum Hamiltonians we need a Hilbert space structure on $W_{\mu}(G)$. Now let us consider spin Calogero–Moser systems corresponding to compact real forms of simple Lie groups. We assume that representations $V_{\mu}$ and $L_{\lambda}$ from previous constructions are irreducible finite dimensional representations.

Define the Hilbert space structure on the space $W_{\mu}(G)$ (on its $L_1$-completion) by the Hermitian scalar product

$$(f_1, f_2) = \int_G \langle f_1(g), f_2(g) \rangle \, dg$$

where $\langle x, y \rangle$ is the Hermitian invariant scalar product on $V_{\mu}$ and $dg$ is the Haar measure on $G$.

In what follow both $L_{\lambda}$ and $V_{\mu}$ are finite dimensional irreducible representations of $G$. We will use two different letters for them to emphasize their different roles in the construction.

Choose a basis in the space $\text{Hom}(L_{\lambda}^* \otimes L_{\lambda} \to V_{\mu})$ enumerated by indices $a, b, \ldots$. Choose a basis in $L_{\lambda}$ enumerated by $\alpha, \beta, \ldots$. Let $K_{\mu}^{\lambda}(a)$ be a $G$-linear map corresponding to the index $a$. Its Hermitian conjugate $(K_{\mu}^{\lambda}(a))^*$ is a $G$-linear map $V_{\mu} \to L_{\lambda}^* \otimes L_{\lambda}$. Assume that this basis is orthonormal:

$$K_{\mu}^{\lambda}(a)(K_{\mu}^{\lambda}(a'))^* = \delta_{a,a'}id_{V_{\mu}}, \quad \sum_{a,\mu}(K_{\mu}^{\lambda}(a))^*K_{\mu}^{\lambda}(a) = id_{L_{\lambda}^* \otimes L_{\lambda}}$$

The first identity here is orthonormality and the second, is completeness. The eigenfunctions of Calogero–Moser Hamiltonians constructed in the previous section can be written as

$$f_{\lambda,\mu}(g, a) = \langle K_{\mu}^{\lambda}(a), \pi^{\lambda}(g) \rangle = \sum_{\alpha,\beta} K_{\mu}^{\lambda}(a)_{\alpha,\beta} \pi^{\lambda}(g)_{\alpha,\beta} \in V_{\mu}$$

Let us check the orthogonality and completeness of eigenfunctions of Calogero–Moser Hamiltonians constructed in the previous section. We
have
\[
\int_G \sum_{\alpha, \beta, \gamma, \delta} K_\mu^\lambda(a)_{\alpha, \beta, \gamma, \delta} K_\mu^{\lambda'}(a')_{\gamma, \delta, i, j} \bar{\pi}^\lambda(g)_{\alpha, \beta} \bar{\pi}^{\lambda'}(g)_{\gamma, \delta} dg = \delta_{\lambda, \lambda'} \delta_{a, a'} \delta_{i, j}
\]

Here we used the Schur’s lemma and (3), indices \(i, j, ..\) enumerate a basis in \(V_\mu\). This proves the orthogonality. We also have
\[
\sum_\lambda \sum_a f_{\lambda, \mu}(g, a) f_{\lambda, \mu}(g', a) = \delta_{g, g'} \delta_{i, j}
\]

which proves completeness.

Here we used the completeness of the basis of functions \(\pi^\lambda(g)_{\alpha, \beta}\) in \(L_2(G)\):
\[
\sum_\lambda \pi^\lambda(g)_{\alpha, \beta} \bar{\pi}^\lambda(g')_{\gamma, \delta} = \delta_{g, g'} \delta_{\alpha, \gamma} \delta_{\beta, \delta}
\]

and (3).

Thus, in the compact case the system of eigenvectors \(f_{\lambda, \mu}(g, a)\) is orthonormal and complete. The multiplicity of \(\lambda\)'s joint eigenvalue is equal to \(\text{dim}(\text{Hom}(L_\lambda^* \otimes L_\lambda \to V_\mu))\).

Note that in terms of \(q\)-coordinates the orthogonality of eigenfunctions can be written as
\[
\int_{t^*} f_{\lambda, \mu}(e^{i q}, a) f_{\lambda, \mu}(e^{i q}, b) \prod_{\alpha > 0} \left| e^{ \frac{i q_{\alpha}}{2} } - e^{ - \frac{i q_{\alpha}}{2} } \right|^2 dq = \delta_{a, b} \delta_{\lambda, \lambda'} \delta_{i, j}
\]

Here \(dq\) is the Euclidean measure on \(t^* = h^* \mod \mathbb{Z}^r\) corresponding to the Killing form, and \(i, j\) enumerate a basis in \(V_\mu[0]\). This is an immediate consequence of the ”radial” structure of the Haar measure on simple compact groups.

Now algebras \(A_\mu\) and \(B_\mu\) can be described in terms of the spectral decomposition:
\[
B_\mu \simeq \bigoplus_{\lambda \in C_\mu} E_\lambda
\]

where \(E_\lambda\) is a complex one-dimensional, and \(C_\mu\) is the set of all integral dominant weights \(\lambda\) such that \(\text{dim}(\text{Hom}_G(V_\lambda, V_\lambda \otimes V_\mu)) \neq 0\). The algebra \(A_\mu\) has the following decomposition:
\[
A_\mu \simeq \bigoplus_{\lambda \in C_\mu} \text{End}(\text{Hom}_G(V_\lambda, V_\lambda \otimes V_\mu))
\]

The decomposition also describes the action of \(A_\mu\) on eigenfunctions of quantum Hamiltonians.
3. Conclusion

First, let us say few words about semiclassical limit when components of all weights $\lambda$, $\mu$, ... go to infinity so that components of weights remain projectively finite, i.e. they are all proportional to $N \to \infty$ with finite coefficients. The quantity $1/N$ plays the role of the Planck constant. In this limit the quantum Calogero–Moser system becomes a classical one. Note that the multiplicity of eigenvalues grows as a power of $N$.

In the appropriate sense the endomorphism algebra of the multiplicity space becomes the algebra of functions on the corresponding symplectic leaf of $P_{2n-k}$. In particular, in the compact case

$$\dim((\text{Hom}(V_\lambda^* \otimes V_\lambda \to V_\mu)) = h^{-n/2} \text{vol}(\mathcal{M})(1 + O(h))$$

when $\mu = t/h, \lambda = s/h$, $h \to 0$, $s, t \in \mathfrak{h}_{\geq 0}^*$, and we identify $\mathfrak{h}^*/W$ with $\mathfrak{h}_{\geq 0}^*$ by choosing corresponding representatives in $W$-orbits. Here $n$ is the dimension and $\text{vol}(\mathcal{M})$ is the symplectic volume of the moduli space $\mathcal{M}_{s,-\lambda_0(s)|t}$ with its natural symplectic structure.

This is a common feature that distinguishes Liouville integrable systems from degenerately integrable systems. In the first case, generic semiclassical joint eigenstates of Hamiltonians are non-degenerate, while in the second case, they are either infinitely degenerate (in the non-compact case) or become infinitely degenerate as $h \to 0$. The algebras of endomorphisms of multiplicity spaces can be regarded as a quantization of algebras of functions on symplectic leaves of $P_{2n-k}$.

The proof of degenerate integrability of $q$-spin Calogero–Moser models and corresponding Ruijsenaars type models is completely parallel. The difference is that is the $q$-case one has to use quantized universal enveloping algebras. This will be done in a separate publication.

Degenerate classical and quantum integrability of spin Calogero-Moser systems is part of a more general construction involving affine loop groups. For the elliptic case, see [13].

We discussed in details the structure of the spectrum in the compact case. Similarly, one can study non-compact real forms of simple complex algebraic Lie groups. Eigenfunctions are still given by (7) but the orthogonality and completeness property will be different and will depend on the structure of corresponding unitary irreducible representations.

Here we made an assumption that $\mu$ is generic. If it is not, i.e. if its stabilizer is greater then $H$, the dimension of $\mathcal{B}_{[t]}$ in the classical case drops by the rank of the stabilizer and the system becomes even more
degenerately integrable. For $SL_n$ the "extreme" case is when the stabilizer is $SL_{n-1}$. In this case our results give the Liouville integrability of the usual Calogero-Moser system [12] and of its quantum counterpart.

The results of this paper can be generalized to spin q-Calogero-Moser systems and q-Ruijsenaars systems. For degenerate classical integrability of these systems see [21]. The spectrum of corresponding quantum Hamiltonians is constructed in [6].

Finally, a few words about the relation to works [24] and [14]. In [24] a maximal degenerate integrability of classical (non-spin) Calogero-Moser system was established. Essentially, the result of [24] is that any trajectory of classical Calogero-Moser system is periodic. This was extended to quantum case in [14]. We conjecture that the spin Calogero-Moser systems have the same property.

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APPENDIX A. MORE ON QUANTUM SPIN CALOGERO-MOSER SYSTEMS

A.1. For $\mathfrak{g} = sl_N$, when $\mu$ is a highest weight corresponding to a partition of $N$, the space $V_{\mu}[0]$ can be realized as a subspace of the $N$-th tensor power of the $n$-dimensional representation. Here we will remind that construction which is based on Howe and Schur-Weyl dualities (it is well known and details can be found in many textbooks).

The Howe duality states Lie groups $SL_n$ and $SL_N$, acting on the space of polynomials on matrices $n \times N$ in a natural way, centralize each other. Moreover, the subspace of homogeneous polynomials of degree $m$ decomposes as $SL_n \times SL_N$ module as follows:

$$Pol(M_{n,N})_m \simeq \bigoplus_{\mu \in P(m,n)} V^{SL_n}_\mu \times V^{SL_N}_\mu$$

Here we assume that $n \leq N$, $P(m,n)$ is a set of partitions of $m$ with at most $n$ nonempty rows, and $V^{SL_n}_\mu$ is an irreducible finite dimensional $SL_n$ module corresponding to the partition $\mu$.

When $m = N$ the subspace

$$\bigoplus_{\mu \in P(N,m)} V^{SL_n}_\mu \times V^{SL_N}_\mu [0]$$
corresponds to the subspace of $\text{Pol}(M_{n,N})_N$ spanned by monomials of the form $M_{\alpha i}$ where each $i = 1, \ldots, N$ appear exactly once. This subspace can be identified with $(\mathbb{C}^n)^{\otimes N}$ as follows. We identify monomials with the tensor product basis as

\[ M_{\alpha_1,1} \cdots M_{\alpha_N,N} \mapsto e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N} \]

The group $SL_N$ does not act anymore on $V^\mu_{\mu[0]}$, its Cartan subgroup $H$ acts trivially on it, and the normalizer of the Cartan $N(H) \subset SL_N$ acts by permutations of indices $\alpha_i$. In other words, it induces the action of the symmetric group $S_N$ on $V^\mu_{\mu[0]}$, which can also be identified with permutations of factors in the tensor product. This is how the Schur-Weyl duality follows from Howe duality. Now let us look at how the action of operators $t_{ij} = e_{ij} e_{ji}$ on $V^\mu_{\mu[0]}$ translates to the action on the tensor product when $\mu$ is a partition of $N$.

Computing in the Gelfand-Zeitlin basis it is easy to show that when $\mu$ is a partition of $N$

\[ t_{ij} = 1 + P_{ij} \]

where $P_{ij}$ is the permutation of $i$-th and $j$-th factors in the tensor product. Note that the identity

\[ [t_{ij} + t_{kj}, t_{ik}] = 0 \]

holds in $V^\mu_{\mu[0]}$ for any $\mu$.

In this case the Hamiltonians of quantum spin Calogero-Moser model can be written as

\[ H = -\Delta + \sum_{i<j} \frac{l(l + P_{ij})}{\sin(x_i - x_j)^2} \]

which acts in $L^2(\mathbb{R}^n) \otimes (\mathbb{C}^n)^{\otimes N}$ with $l = 1$. This is the family of Hamiltonians in physics literature as quantum spin Calogero-Moser systems, see for example [11].

A.2. When $\mu$ is a one row partition, the subspace $V^\mu_{\mu[0]}$ is non-zero only if $\mu = Nk$. In this case $t_{ij}$ act by multiplication on the same scalar (independent of $i$ and $j$). The value of this scalar is easy to compute from the action of the Casimir element and we have in this case

\[ t_{ij} = 2k(k + 1) \]

which corresponds to the usual, non-spin quantum Calogero-Moser system. In this case $\dim(\text{Hom}(V^\lambda \otimes V^\lambda, V^m_\omega)) = 1$. 

A.3.
REFERENCES

[1] J. Avan, O. Babelon and E. Billey. The Gervais-Neveu-Felder Equation and the Quantum Calogero-Moser System. Comm. in Math. Physics 178:281-299(1996)
[2] Superintegrability of rational Ruijsenaars-Schneider systems and their action-angle duals. V. Ayadi, L. Feher, T.F. Gorbe. J. Geom. Symmetry Phys. 27 (2012) 27-44.
[3] F. Calogero, Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1971) 419-436.
[4] Moser, J. Three integrable Hamiltonian systems connected with isospectral deformations. Advances in Math. 16 (1975), 197220.
[5] Etingof P., Kirillov A. (Jr), On a unified representation theoretical approach to special functions, Funk. Anal. i Prilozh., 28 (1994), 91-94; Etingof, P., Frenkel, I., Kirillov, A., Spherical functions on affine Lie groups, Duke Math Journal, 80 (1995), 79-90; Etingof, P.I., Quantum integrable systems and representations of Lie algebras, Journal of Mathematical Physics v. 36 (1995) pp.2636-2651.
[6] Etingof, Pavel, Varchenko, Alexander Traces of intertwiners for quantum groups and difference equations. I. Duke Math. J. 104 (2000), no. 3, 391432; Etingof, P., Schiffmann, O., Varchenko, A. Traces of intertwiners for quantum groups and difference equations. Lett. Math. Phys. 62 (2002), no. 2, 143158; Etingof, P., Varchenko, A. Orthogonality and the QKZB-heat equation for traces of Uq(g)-intertwiners. Duke Math. J. 128 (2005), no. 1, 83117.
[7] Feher, L.; Pusztai, B. G. Twisted spin Sutherland models from quantum Hamiltonian reduction. J. Phys. A 41 (2008), no. 19, 194009
[8] A.T. Fomenko. Symplectic geometry. Advanced Studies in Contemporary Mathematics, 5. New York: Gordon and Breach Science Publishers, 1988
[9] J. Frish, V. Mandrosov, Y.A. Smorodinsky, M. Uhlir and P. Winternitz. On higher symmetries in quantum mechanics Physics Letters 16:354-356 (1965)
[10] Gibbons J., Hermsen T., A generalization of the Calogero-Moser system., Physica, 11D(1984), 337
[11] K. Hikami, M. Wadati, Integrability of Calogero-Moser spin system, Journal of the Physics Society of Japan, v. 62, n. 2 (1993) 469-472.
[12] D. Kazhdan, B. Kostant and S. Sternberg. Hamiltonian group actions and dynamical systems of Calogero type. Comm. Pure Appl. Math. 31:no.4, 481-507(1978)
[13] I. Krichever, O. Babelon, E. Billey and M. Talon, Spin Generalization of Calogero-Moser system and the matrix KP equation. Translations of AMS, series 2, v. 170, Advances in Mathematical Science, Topics in Topology and Math. Phys., 1975[hep-th/9411160]
[14] Kuznetsov, Vadim B., Hidden symmetry of the quantum Calogero-Moser system. Phys. Lett. A 218 (1996), no. 3-6, 212222.
[15] L.C. Li, P. Xu, Spin Calogero-Moser systems associated with simple Lie algebras C.R. Acad. Sci. Paris, Serie I , 331 : n1, 55-61(2000)
[16] Ryota Nakai, Yusuke Kato, Particle Propagator of Spin Calogero-Sutherland Model J. Phys. A: Math. Theor. 47 (2014) 305205
[17] N.N. Nekhoroshev. Action-angle variables and their generalizations. Trans. Moscow Math. Soc. 26:180-197 (1972)
[18] M.A. Olshanetsky and A.M. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rept. 94 (1983) 313-404
[19] W. Pauli, On the hydrogen spectrum from the standpoint of the new quantum mechanics, Zeitschrift fur Physik, 36, 336-363 (1926).
[20] N. Reshetikhin, Degenerate integrability of the spin Calogero-Moser systems and the duality with the spin Ruijsenaars systems. Lett. Math. Phys. 63 (2003), no. 1, 5571.
[21] N. Reshetikhin, Degenerately Integrable Systems, preprint, arXiv, 2015.
[22] Superintegrability in Classical and Quantum Systems, Edited by: P. Tempesta, P. Winternitz, J. Harnad, W. Miller, Jr., G. Pogosyan, M. Rodriguez, CRM Proceedings and Lecture Notes, Volume: 37, 2004.
[23] B. Sutherland, Exact results for a many-body problem in one dimension. II. Phys. Rev. A, v. 5, n 3 (1972), 1372-1376.
[24] S. Wojciechowski, 1983 Superintegrability of the Calogero-Moser system Phys.Lett.A, 95, 279

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[18] M.A. Olshanetsky and A.M. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rept. 94 (1983) 313-404
[19] W. Pauli, On the hydrogen spectrum from the standpoint of the new quantum mechanics, Zeitschrift fur Physik, 36, 336-363 (1926).
[20] N. Reshetikhin, Degenerate integrability of the spin Calogero-Moser systems and the duality with the spin Ruijsenaars systems. Lett. Math. Phys. 63 (2003), no. 1, 5571.
[21] N. Reshetikhin, Degenerately Integrable Systems, preprint, arXiv, 2015.
[22] Superintegrability in Classical and Quantum Systems, Edited by: P. Tempesta, P. Winternitz, J. Harnad, W. Miller, Jr., G. Pogosyan, M. Rodriguez, CRM Proceedings and Lecture Notes, Volume: 37, 2004.
[23] B. Sutherland, Exact results for a many-body problem in one dimension. II. Phys. Rev. A, v. 5, n 3 (1972), 1372-1376.
[24] S. Wojciechowski, 1983 Superintegrability of the Calogero-Moser system Phys.Lett.A, 95, 279

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