Merons, conjectured as a semiclassical mechanism for color confinement in QCD, are topological charge-1/2, singular solutions to the classical Yang-Mills equations of motion. I will discuss how lattice techniques can extend the study of merons to nonsingular stationary solutions without destroying properties believed to be essential for confinement, and how zero modes can be used to identify these gauge field configurations in stochastic evaluations of the lattice QCD path integral.

1 Introduction

Merons are one of the earliest proposed semiclassical mechanisms for confinement, but have received less attention than other approaches due to analytical limitations. I will present a new direction for meron studies, in which lattice gauge theory will be able to determine the presence and role of merons in QCD.

Chiral symmetry breaking and confinement of color charge, the two essential features of low-energy QCD, cannot be understood perturbatively in the coupling constant. Two complementary nonperturbative approaches are analytical approximation using semiclassical analysis of the QCD path integral (analogous to the WKB approximation in quantum mechanics) and numerical solution using lattice gauge theory. Combining physical understanding from the semiclassical approximation with systematic rigor of the lattice has provided valuable insight into hadron structure and can be extended to study confinement.

Most semiclassical treatments of QCD have focused on instantons, leading to a qualitative, and in some cases even quantitative, understanding of chiral symmetry breaking and the structure of the QCD vacuum. However, simple configurations of instantons, such as a dilute gas or a random superposition, do not confine color charges. It is possible the complicated dynamics of confinement escapes any explanation save a full solution of QCD at large coupling. However, analogy with lower dimensional models seems to indicate

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otherwise.

In 2+1 dimensions, the Georgi-Glashow model, which couples a triplet Higgs to an SU(2) vector boson field, contains magnetic monopoles as solutions to the Euclidean classical equations of motion. Taking the Higgs to have an expectation value along the third isospin direction \( \phi_a = \delta_a^3 v \), the residual massless photon exhibits a monopole solution of form \( A_\mu^3 \sim \epsilon_{3\mu\nu} x_\nu / x^2 \) at large distances. Polyakov showed these monopoles condense, leading to confinement of electric charge. Similarly, the Schwinger model (QED in 1+1 dimensions) exhibits confinement through the dominance of vortex configurations of form \( A_\mu = \epsilon_{\mu\nu} x_\nu / x^2 \). These vortices have a topological charge of \( \frac{1}{2} \), which also shows up in the quark correlator at finite temperature.

These semiclassical examples of confinement are intriguing in that the effect is generated by the same gauge-field configuration interpreted in different dimensions. The meron is the (3+1)-dimensional nonabelian extension of the \('t\) Hooft–Polyakov monopole configuration with \( A_\mu^a = \eta_{a\mu\nu} x_\nu / x^2 \). The important property shared by each of these configurations is \( A \sim 1/r \) and is not pure gauge. Then the integral \( \oint A_\mu dx^\mu \) over a loop of area \( R \times T \) will have a contribution of order unity as long as the source of the field is within \( R \) of this loop in any direction. If the vacuum is made up of a uniform distribution of these semiclassical configurations, the energy between charged particles will behave like \( E(R) \propto R^d \) in \( d+1 \) dimensions. Interactions between monopoles in 2+1 dimensions reduce this down to a linearly rising potential and it is conjectured that merons will do the same in 3+1 dimensions.

However, a purely analytic study of merons has proven intractable for several reasons. Unlike instantons, no exact solution exists for more than two merons, gauge fields for isolated merons fall off too slowly (\( A \sim 1/x \)) to superpose them, and the field strength is singular. A patched \( \textit{Ansatz} \) configuration that removes the singularities does not satisfy the classical Yang-Mills equations, preventing even calculation of gaussian fluctuations around a meron pair.

The properties of merons are reviewed in Sec. 2. The discussion is restricted to SU(2) color, since topological objects live in embedded SU(2) subgroups of larger gauge groups. As work with instantons has shown, fermionic zero modes are a powerful tool for identifying topological objects on the lattice. Therefore, in Sec. 3, the zero mode of the continuum meron pair \( \textit{Ansatz} \) is analytically derived. In Sec. 4, a smooth, finite action configuration corresponding to a meron pair is presented, which with its associated quantum fluctuations, should be present in Monte Carlo lattice QCD calculations. These exact numerical results along with the associated zero mode are compared with analytical results of the meron pair \( \textit{Ansatz} \).
2 Analytic Meron Review

Two known solutions to the classical Yang-Mills equations in four Euclidean dimensions are instantons and merons. Both have topological charge, can be interpreted as tunneling solutions, and can be written in the general form (for the covariant derivative $D_{\mu} = \partial_{\mu} - i A_{\mu}^a \sigma^a / 2$)

$$A_{\mu}^a(x) = \frac{2 \eta_{\mu\nu} x^\nu}{x^2} f(x^2),$$

with $f(x^2) = x^2 / (x^2 + \rho^2)$ for an instanton and $f(x^2) = \frac{1}{2}$ for a meron.

Conformal symmetry of the classical Yang-Mills action, in particular under inversion $x_{\mu} \rightarrow x_{\mu} / x^2$, shows that in addition to a meron at the origin, there is a second meron at infinity. These two merons can be mapped to arbitrary positions, defined to be the origin and $d_\mu$, by the conformal transformation

$$x_{\mu} \rightarrow d_\mu + d^2 (x - d)^\mu (x - d)^2 .$$

(2)

After a gauge transformation, the gauge field for the two merons takes the simple form

$$A_{\mu}^a(x) = \eta_{\mu\nu} \left( \frac{x^\nu}{x^2} + \frac{(x - d)^\nu}{(x - d)^2} \right) .$$

(3)

Similar to instantons, a meron pair can be expressed in singular gauge by performing a large gauge transformation about the mid-point of the pair, resulting in a gauge field that falls off faster at large distances ($A \sim x^{-3}$). A careful treatment of the singularities shows that the topological charge density is

$$Q(x) \equiv \frac{\text{Tr}}{16\pi^2} \left( F_{\mu\nu} \tilde{F}^{\mu\nu} \right) = \frac{1}{2} \delta^4(x) + \frac{1}{2} \delta^4(x - d) ,$$

(4)

yielding total topological charge $Q = 1$, just like the instanton.

The gauge field Eq. (3) has infinite action density at the singularities.

\footnote{A more general conformal transformation only differs from this choice by either a rotation, a translation, or a dilatation.}
$x_\mu = \{0, d_\mu\}$. Hence, a finite action Ansatz has been suggested

$$A^\mu_\mu(x) = \eta_{\mu\nu} x^\nu \begin{cases} \frac{2}{x^2 + r^2}, & \sqrt{x^2} < r, \text{ region I,} \\ \frac{1}{x^2}, & r < \sqrt{x^2} < R, \text{ region II,} \\ \frac{2}{x^2 + R^2}, & R < \sqrt{x^2}, \text{ region III,} \end{cases}$$

(5)

with arbitrary radii $r$ and $R$. Here, the singular meron fields for $\sqrt{x^2} < r$ and $\sqrt{x^2} > R$ are replaced by instanton caps, each containing topological charge $\frac{1}{2}$ to agree with Eq. (4). The action for this configuration is

$$S = \frac{8\pi^2}{g^2} + \frac{3\pi^2}{g^2} \ln \frac{R}{r},$$

(6)

which shows the divergence in the $r \to 0$ or $R \to \infty$ limit. There is no angular dependence in this patching, and so the conformal symmetry of the meron pair is retained. For example, under a dilatation $x_\mu \to \lambda x_\mu$, both $r$ and $R$ get multiplied by $1/\lambda$ but the ratio and hence the action Eq. (6) remains invariant. Although this patching of instanton caps is continuous, the derivatives are not, and so the equations of motion are violated at the boundaries of the regions in Eq. (5).

Applying the same transformations used to attain Eq. (3) to the instanton cap solution, regions I and III become four-dimensional spheres each containing half an instanton. The geometry of the instanton caps is shown in Fig. 1 for the symmetric choice of displacement $d = \sqrt{Rr}$ along the $z$-direction. Note

Figure 1. Meron pair separated by $d = \sqrt{Rr}$ regulated with instanton caps, each containing $\frac{1}{2}$ topological charge.
that the action for the complicated geometry of Fig. 1 is still given by Eq. (6), which for \( d = \sqrt{Rr} \) can be rewritten as

\[
S = S_0 \left( 1 + \frac{3}{4} \ln \frac{d}{r} \right),
\]

with \( S_0 = 8\pi^2/g^2 \). This case will be used below, and generalization to a different \( d \) is straightforward.

The original positions of the two merons \( x_\mu = \{0, d_\mu\} \) are not the centers of the spheres, nor are they the positions of maximum action density, which occurs within the spheres with \( S_{\text{max}} = (48/g^2)(R + r)^4/d^8 \) at

\[
(M_I)_\mu = \frac{r^2}{r^2 + d^2} d_\mu, \quad (M_{III})_\mu = \frac{R^2}{R^2 + d^2} d_\mu.
\]

However, in the limit \( r \to 0 \) and \( R \to \infty \) holding \( d \) fixed, the spheres will shrink around the original points reducing to the bare meron pair in Eq. (3). In the opposite limit \( R \to r \), the radii of the spheres increase to infinity, leaving an instanton of size \( \rho = d \).

An instanton can therefore be interpreted as consisting of a meron pair. This complements the fact that instantons and antiinstantons have dipole interactions between each other. If there exists a regime in QCD where meron entropy contributes more to the free energy than the logarithmic potential between pairs, like the Kosterlitz-Thouless phase transition, instantons will break apart into meron pairs, with the intrinsic size of the instanton caps determined by the original instanton scale \( \rho \). This will occur when the entropy to create meron pairs, which is \( R/r \) in each space-time direction with the general scales \( r \) and \( R \) given in Eq. (5), overcomes the logarithmic interaction energy Eq. (6) to dominate the QCD path integral, or schematically,

\[
\left( \frac{R}{r} \right)^4 e^{-3\pi^2 \ln(R/r)/g^2(s)} = \left( \frac{R}{r} \right)^{4-3\pi^2/g^2(s)} \gtrsim 1.
\]

How the scale \( s \) in the renormalized coupling constant depends on \( r \) and \( R \) is not known. It is assumed that as longer and longer distances are probed, \( s \) also increases, causing entropy to win out over energy in Eq. (9). This occurs for relatively weak coupling, \( g^2(s)/8\pi^2 > \frac{3}{16\pi^2} \), so the semiclassical approximation should still be valid. Lattice calculations can test this physical picture by increasing the coupling constant and seeing if the predominant semiclassical degrees of freedom change from a dilute gas of instantons into a plasma of merons.
3 Analytic Zero Modes

The Atiyah-Singer index theorem states that a fermion in the presence of a gauge field with topological charge $Q$, described by the equation

$$\mathcal{D} \psi = \lambda \psi , \quad \text{with} \quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} ,$$

has $n_R$ right-handed and $n_L$ left-handed zero modes (defined by $\lambda = 0$) such that $Q = n_L - n_R$. This can sometimes be strengthened by a vanishing theorem. Applying $\mathcal{D}$ twice to $\psi$ decouples the right- and left-hand components, and focusing on the equation for $\psi_R$, gives

$$\left( D^2 + \frac{1}{2} \tilde{\eta}_{a\mu\nu} \sigma^a F^{\mu\nu} \right) \psi_R = 0 ,$$

where $\tilde{\eta}_{a\mu\nu}$ denotes the 't Hooft symbol, $\sigma^a$ acts on the spin indices of $\psi_R$, and $F^{\mu\nu}$ acts on the color indices. For a self-dual gauge field (like an instanton), the second term in Eq. (11) is zero; and since $D^2$ is a negative definite operator, there are no normalizable right-handed zero modes, implying $Q = n_L$.

Although the meron pair is not self-dual, the second term can be shown to be negative definite as well, leading to the same conclusion.

In general, a gauge field in Lorentz gauge with $Q = 1$ can be written in the form

$$A^a_\mu(x) = -\eta_{a\mu\nu} \partial_\nu \ln \Pi(x) .$$

This has a fermion zero mode given by

$$\psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix} , \quad \phi^{a} = \mathcal{N} \Pi^{3/2} \varepsilon^{a} ,$$

with normalization $\mathcal{N}$ and $\varepsilon = i\sigma_2$ coupling the spin index $\alpha$ to the color index $a$ (both of which can be either 1 or 2) in a singlet configuration. The gauge field for the meron pair with instanton caps can be expressed in the form Eq. (12) with

$$\Pi(x) = \begin{cases} \frac{2\xi_d d^2}{x^2 + \xi_i^2(x - d)^2} , & \text{for regions } i = I, III , \\ \frac{d^2}{\sqrt{x^2(x - d)^2}} , & \text{region } II , \end{cases}$$

for regions $i = I, III$,
with \( \xi_1 = r/d \) and \( \xi_{II} = R/d \). The normalized solution to the zero mode is then Eq. (13) with

\[
N^{-1} = 2\pi d^2 \left[ 2 - \sqrt{\frac{r}{R}} \right]^{1/2}.
\]

(15)

Note that the unregulated meron pair \((r \to 0, R \to \infty)\) has a normalizable zero mode itself \(\phi^a_\alpha(x) = \frac{d \epsilon^a_\alpha}{2\sqrt{2\pi} (x^2(x - d)^2)^{3/4}}\).

(16)

The gauge-invariant zero mode density \(\psi^\dagger \psi(x)\) has a bridge between two merons that falls off like \(x^{-3}\) in contrast to the \(x^{-6}\) fall-off in all other directions. This behavior can be used to identify merons when analyzing their zero modes on the lattice, similar to what was done for instantons in Ref. [16].

### 4 Merons on the Lattice

As mentioned above, the patching of instanton caps to obtain an explicit analytic solution has unavoidable and unphysical discontinuities in the action density. Therefore, in order to study this solution further, the gauge field is put on a space-time lattice of spacing \(a\) in a box of size \(L_0 \times L_1 \times L_2 \times L_3\). The gauge-field degrees of freedom are replaced by the usual parallel transport link variable,

\[
U_\mu(x) = P \exp \left[ -i \int_{x-a}^{x} A_\nu(z) dz^\nu \right].
\]

(17)

The exponentiated integral can be performed analytically within the instanton caps, producing arctangents. Outside of the caps, the integral is evaluated numerically by dividing each link into as many sub-links as necessary to reduce the \(O(a^3)\) path-ordering errors below machine precision.

Calculating the action density \(S(x)\) for a meron pair with instanton caps using the improved action of Ref. [17] and the topological charge density \(Q(x)\) using products of clovers, the lattice results (open circles) are compared with the patched Ansatz results (solid line) in Fig. 2 for \(r = 9\) and \(d = 20.8\) (in units of the lattice spacing) on a \(32^4 \times 40\) lattice. The Arnoldi method is used to solve for the zero mode of this gauge configuration and hence the density \(\psi^\dagger \psi(x)\), which is also compared with the analytic result in the same figure, showing excellent agreement.

Two important features of the patched Ansatz also evident with the lattice representation are the discontinuities in the action density at the boundary.
Figure 2. The action density $S(x)$ for the regulated meron pair, normalized by the instanton action $S_0 = 8\pi^2/g^2$, sliced through the center of the configuration in the direction of separation. Shown are the analytic (solid), initial lattice (open circles) and cooled lattice (filled circles) configurations. The same is shown for the topological charge density $Q(x)$ and fermion zero mode density $\psi^\dagger \psi(x)$.

The discontinuity in the action density is unphysical and can be smoothed out by using a relaxation algorithm to iteratively minimize the lattice action. On a sweep through the lattice, referred to as a cooling step, each link is chosen to locally minimize the action density. Since a single instanton is already a minimum of the action, this algorithm would leave the instanton unchanged (for a suitably improved lattice action). The result for the regulated meron pair after ten cooling steps is represented in Fig. 2 by the filled circles, showing the discontinuities in the action density of the instanton caps and vanishing of the topological charge density outside of the caps as given by Eq. (4).
Figure 3. The topological charge density $Q(x)$ on the left and fermion zero mode density $\psi^\dagger \psi(x)$ on the right for a cooled meron pair with $r = 9$, separated by distance $d = 20.8$, 17.7, and 14.1 respectively, in the $(z,t)$-plane.

have already been smoothed out. Figure 3 shows side-by-side the cooled $Q(x)$ and $\psi^\dagger \psi(x)$ in the $(z,t)$-plane for three different meron pair separations. As the separation vanishes, the zero mode goes over into the well-known instanton zero mode result.

Like instanton-antiinstanton pairs, however, a meron pair is not a strict minimum of the action, since it has a weak attractive interaction Eq. (6) and under repeated relaxation will coalesce to form an instanton. This is analogous to the annihilation of an instanton-antiinstanton pair through cooling. Nevertheless, just as it is important to sum all the quasi-stationary instanton-
antiinstanton configurations to obtain essential nonperturbative physics, meron pairs may be expected to play an analogous role. A precise framework for including quasi-stationary meron pairs is to introduce a constraint \( Q[A] \) on a suitable collective variable \( q \) (here chosen to be the quadrupole moment \( 3z^2 - x^2 - y^2 - t^2 \) of the topological charge) as follows

\[
Z = \int DA \exp \left\{ -S[A] \right\} = \int dq \, Z_q, \tag{18}
\]

\[
Z_q = \int DA \exp \left\{ -S[A] - \lambda (Q[A] - q)^2 \right\}. \tag{19}
\]

The meron pair is then a true minimum of the effective action with constraint, allowing for a semiclassical treatment of \( Z_q \). Afterwards, \( q \) is integrated to obtain the full partition function \( Z \).

The criterion for a good collective variable \( q \) of the system is that the gradient in the direction of \( q \) is small compared to the curvature associated with all the quantum fluctuations. In this case, an adiabatic limit results in which relaxation of the unconstrained meron pair slowly evolves through a sequence of quasi-stationary solutions, each of which is close to a corresponding stationary constrained solution. Detailed comparison of the quasi-stationary and constrained solutions shows this adiabatic limit is well satisfied. Therefore, the action of a meron pair as it freely coalesces into an instanton is presented here. To compare with the patched Ansatz, the meron separation \( d \) and radii \( r \) and \( R \) of the lattice configuration are required. These are found by first measuring the separation of the two maxima in the action density \( \Delta \equiv |M_I - M_{III}| \) (using cubic splines) and their values (which are the same in the symmetric case, denoted by \( S_{\text{max}} = 48/g^2 w^4 \)), and then using Eq. (8) to give

\[
d^2 = \Delta^2 + 4w^2, \quad r = \frac{2wd}{d + \Delta}, \quad R = \frac{2wd}{d - \Delta}. \tag{20}
\]

In Fig. 4, the total action of a meron pair is plotted as a function of \( d/r \) (solid line), which for the analytic case is given by Eq. (8). Also shown are cooling trajectories for four lattice configurations with different initial meron pair separations. Each case starts with a patched Ansatz of a given separation and the lattice action agrees with that of the analytic Ansatz.

The first few cooling steps primarily decrease the action without changing the collective variable (which is now effectively \( d/r \), as observed above in Fig. 2). Further cooling gradually decreases the collective variable, tracing out a new logarithmic curve for the total action (dashed line in Fig. 4), which is about 0.25 smaller than the analytic case. This curve represents the total action of the smooth adiabatic or constrained lattice meron pair solutions.
The essential property of merons that could allow them to dominate the path integral and confine color charge is the logarithmic interaction Eqs. (6,7) which is weak enough to be dominated by the meron entropy. Hence, the key physical result from Fig. 4 is the fact that smooth adiabatic or constrained lattice meron pair solutions clearly exhibit this logarithmic behavior.

In summary, stationary meron pair solutions have been found on the lattice exhibiting a logarithmic interaction and fermion zero mode localized about the individual merons in agreement with analytic results. The presence and role of merons in numerical evaluations of the QCD path integral should be investigated, and the study of fermionic zero modes and cooling of gauge configurations are possible tools to do so.
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