In this paper, we discuss the existence of solutions for nonlinear fractional Langevin equations with nonseparated type integral boundary conditions. The Banach fixed point theorem and Krasnoselskii fixed point theorem are applied to establish the results. Some examples are provided for the illustration of the main work.

1. Introduction

Fractional derivatives give an excellent description of memory and hereditary properties of different processes. Properties of the fractional derivatives make the fractional-order models more useful and practical than the classical integral-order models.

Several researchers in the recent years have employed the fractional calculus as a way of describing natural phenomena in different fields such as physics, biology, finance, economics, and bioengineering (for more details see [1–9] and many other references).

With the recent outstanding development in fractional differential equations, the Langevin equation has been considered a part of fractional calculus, and thus, important results have been elaborated (see [10–14]).

The Langevin equation was first introduced by Langevin in 1908; it is a fundamental theory of the Brownian motion to describe the evolution of physical phenomena in fluctuating environments [15, 16]. The fractional model of the Langevin equation as a generalization of the classical one gives a fractional Gaussian process parametrized by two indices, and this fractional model is more flexible for modeling fractal processes [17, 18].

The fractional Langevin equation is extensively studied in the literature from both the theoretical and numerical point of views (for more details see [19–25]). In [26], the authors studied a nonlinear Langevin equation involving two fractional orders in different intervals. In [27], the authors discussed the existence theory for a nonlinear Langevin equations with nonlocal multipoint and multistrip boundary conditions. In [28], fractional Langevin equations with nonlocal integral boundary conditions have been investigated by Salem et al. In [14], an antiperiodic boundary value problem for the Langevin equation involving two fractional orders has been studied.

Recently, in [29], the authors discussed the nonlinear fractional differential equations with nonseparated type integral boundary conditions; however, the fractional Langevin equations involving nonseparated integral boundary conditions have not been investigated yet; that is why, in this work and motivated by all the works cited above, we study the existence and uniqueness of the fractional Langevin equations with nonseparated integral boundary conditions as follows:

\[
\begin{align*}
\quad & cD^\beta (cD^\alpha + \lambda) x(t) = f(t, x(t), Ip^x x(t)), \quad t \in [0, 1], \\
\quad & x(0) + \mu x(1) = g(s, x(s))ds, \\
\quad & cD^\alpha x(0) + \mu cD^\alpha x(1) = h(s, x(s))ds, \\
\quad & cD^2\alpha x(0) + \mu cD^{2\alpha} x(1) = k(s, x(s))ds,
\end{align*}
\] (1)
Theorem 3 (see [5]). Let $f$ be a continuous function, then, we have
\[
\frac{cD}{f(t)} = f(t) + a_0 t + a_1 t^2 + \cdots + a_n t^{n-1}.
\]

Lemma 4 (see [5]). Let $\alpha, \beta \geq 0$; then, the following relations hold:
\[
\begin{align*}
I^\alpha f(t) &= \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s) ds, \\
\frac{cD}{f(t)} &= \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s) ds.
\end{align*}
\]

Lemma 5 (see [5]). Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If $f$ is a continuous function, then, we have
\[
I^\alpha \frac{cD}{f(t)} = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}.
\]

Lemma 6. Let $y \in C([0, 1], \mathbb{R})$. Then, a unique solution of the boundary value problem
\[
\begin{cases}
\frac{cD}{f(t)}(\frac{cD}{f(t)} + \lambda) x(t) = y(t), & t \in [0, 1], \\
x(0) + \mu x(1) = \sigma_1 \int_0^1 g(s; x(s)) ds, \\
\frac{cD}{f(t)} x(0) + \mu \frac{cD}{f(t)} x(1) = \sigma_2 \int_0^1 h(s; x(s)) ds, \\
\frac{cD}{f(t)} x(0) + \mu \frac{cD}{f(t)} x(1) = \sigma_3 \int_0^1 k(s; x(s)) ds
\end{cases}
\]
is given by
\[
\begin{align*}
x(t) &= \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} x(s) ds \\
&\quad + A_1(t) \int_0^1 h(s; x(s)) ds + A_2(t) \int_0^1 g(s; x(s)) ds \\
&\quad + A_3(t) \int_0^1 k(s; x(s)) ds + A_4(t) \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad + \frac{\mu}{(1+\mu)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha-1} x(s) ds,
\end{align*}
\]
where
\[
\begin{align*}
A_1(t) &= \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)(1+\mu)} + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)} \\
&\quad + \frac{\mu}{\Gamma(\alpha+1)(1+\mu)} + \frac{1}{(1+\mu)^2}\Gamma(2-\alpha)\lambda \sigma_2 \\
&\quad - \frac{\mu}{(1+\mu)^2}\Gamma(\alpha+1),
\end{align*}
\]
\[
\begin{align*}
A_2(t) &= \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)(1+\mu)} - \frac{\sigma_1}{1+\mu} + \frac{\mu}{(1+\mu)^2}\Gamma(\alpha+1),
\end{align*}
\]
\[
\begin{align*}
A_3(t) &= \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)(1+\mu)} + \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)} \\
&\quad + \frac{\mu}{\Gamma(\alpha+1)(1+\mu)} - \frac{\mu}{\Gamma(\alpha+1)} + \frac{1}{(1+\mu)^2}\Gamma(\alpha+1),
\end{align*}
\]
\[
\begin{align*}
A_4(t) &= \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)(1+\mu)} - \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)} + \frac{\mu}{(1+\mu)^2}\Gamma(\alpha+1),
\end{align*}
\]
\[
\begin{align*}
A_5(t) &= \frac{\Gamma(2-\alpha)}{\Gamma(\alpha+1)(1+\mu)} - \frac{\mu}{(1+\mu)^2}\Gamma(\alpha+1).
\end{align*}
\]
Proof. By applying Lemma 5, we have

\[
(D^a + \lambda)x(t) = \int_0^t y(s) \, ds + a_0 + a_1 t,
\]

\[
(D^a)x(t) = \int_0^t y(s) \, ds + a_0 + a_1 t - \lambda x(t),
\]

\[
x(t) = \int_0^t y(s) \, ds + a_0 + a_1 t - \lambda \int_0^t x(s) \, ds + a_2,
\]

(9)

where \(a_0, a_1, a_2 \in \mathbb{R}\).

Using the condition \(D^a_x x(0) + \mu D^a x(1) = \sigma_1 \int_0^1 k(s; x(s)) \, ds\), we obtain

\[
a_1 = \Gamma(2 - \alpha) \left( \frac{\sigma_1}{1 + \mu} \int_0^t k(s; x(s)) \, ds \right)
+ \frac{\lambda \sigma_1}{1 + \mu} \int_0^t h(s; x(s)) \, ds
- \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (1 - s)^{\beta - 1} y(s) \, ds.
\]

By conditions \(D^a x(0) + \mu D^a x(1) = \sigma_2 \int_0^1 k(s; x(s)) \, ds\) and \(x(0) + \mu x(1) = \sigma_2 \int_0^1 g(s; x(s)) \, ds\), we have

\[
a_2 = \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha)} \int_0^t (1 - s)^{\alpha - 1} x(s) \, ds
+ \frac{\sigma_2}{1 + \mu} - \frac{\mu \lambda \sigma_1}{1 + \mu} \left( \Gamma(\alpha + 1)/(1 + \mu)^2 \right)
+ \frac{\mu^2}{(1 + \mu) \Gamma(\alpha)} \int_0^t (1 - s)^{\alpha - 1} y(s) \, ds
+ \frac{\mu^2}{(1 + \mu) \Gamma(\alpha)} \int_0^t (1 - s)^{\alpha - 1} y(s) \, ds
+ \frac{\mu \sigma_2}{1 + \mu} \left( \Gamma(\alpha + 1)/(1 + \mu) \right)
+ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha + 1)} \int_0^t h(s; x(s)) \, ds.
\]

(10)

By Lemma 6, we transform problem (1) into a fixed point problem as \(x = P x\), where \(P : X \rightarrow X\) is given by

\[
P x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha + \beta - 1} f(s, x(s), P x(s)) \, ds
+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) \, ds + A_1(t) \int_0^1 h(s; x(s)) \, ds
+ A_2(t) \int_0^1 g(s; x(s)) \, ds + A_3(t) \int_0^1 k(s; x(s)) \, ds
+ \frac{A_4(t)}{\Gamma(\beta - \alpha)} \int_0^t (1 - s)^{\beta - 1} f(s, x(s), P x(s)) \, ds
+ A_5(t) \int_0^t (1 - s)^{\beta - \alpha} f(s, x(s), P x(s)) \, ds
+ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha)} \int_0^t (1 - s)^{\alpha - 1} x(s) \, ds
- \frac{\mu}{(1 + \mu) \Gamma(\alpha + \beta)} \int_0^t (1 - s)^{\alpha + \beta - 1} f(s, x(s), P x(s)) \, ds.
\]

(12)

Theorem 2. Suppose that \(f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}\) and \(h, g, k : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\) are continuous functions satisfying

\[(H_1) - \text{there exist positive constants } q_1, q_2 \text{ such that}
\]

\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq q_1|x_1 - x_2| + q_2|y_1 - y_2|,
\]

for all \(x_1, x_2, y_1, y_2 \in \mathbb{R}, t \in [0, 1].\)

\[(H_2) - \text{there exist positive constants } q_3, q_4, q_5 \text{ such that}
\]

\[
|g(t, x) - g(t, y)| \leq q_3|x - y|, |h(t, x) - h(t, y)| \leq q_4|x - y|,
\]

\[
|k(t, x) - k(t, y)| \leq q_5|x - y|, \quad \forall x, y \in \mathbb{R}.
\]

(13)

Then there exists a unique solution \(x\) for the boundary value problem (1) provided that \(r_1 < 1\), where

\[
r_1 = \frac{\left( q_1 + \frac{q_2}{\Gamma(p + 1)} \right) \frac{1}{\Gamma(\alpha + \beta + 1)} + \frac{A_4}{\Gamma(\alpha + \beta + 1)}}{\frac{A_5}{\Gamma(\beta - \alpha + 1)} + \frac{\mu}{\Gamma(\alpha + 1)}} + \frac{|\lambda|}{\Gamma(\alpha + 1)}
+ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha + 1)} \left( A_1 q_4 + A_2 q_3 + A_3 q_5 \right),
\]

(14)

and \(A_i = \max_{0 \leq t \leq 1} |A_i(t)| \) for \(i = 1, 2, ..., 5.\)

Proof. We set \(\sup_{0 \leq t \leq 1} |f(t, 0, 0)| = M_0, \sup_{0 \leq t \leq 1} |g(t, 0)| = M_1, \sup_{0 \leq t \leq 1} |h(t, 0)| = M_2, \sup_{0 \leq t \leq 1} |k(t, 0)| = M_3.\)

(15)

3. Main Results

Denote by \(X\) the Banach space of all continuous functions from \([0, 1] \rightarrow \mathbb{R}\) endowed with norm \(\|x\| = \sup \{|x(t)| : t \in [0, 1]\}\).
Let $B_r = \{ x \in X : \| x \| \leq r \}$ the ball with radius $r$, where $r \geq (r_2/(1-r_1))$, with

$$ r_2 = \frac{M_0}{\Gamma(\alpha + \beta + 1)} + \frac{M_0 A_1}{\Gamma(\beta - \alpha + 1)} + A_1 M_1 + A_2 M_2 + A_3 M_3 + \frac{A_3 M_0}{\Gamma(\beta + 1)} + \frac{\mu M_0}{(1 + \mu) \Gamma(\alpha + \beta + 1)}. \quad (16) $$

Then, $B_r$ is a closed, convex, and nonempty subset of the Banach space $X$.

Our aim is to prove that the operator $P$ has a unique fixed point on $B_r$. We show that $PB_r \subseteq B_r$.

For $x \in B_r$, $t \in [0, 1]$, we have

$$ |P(x)(t)| \leq \frac{1}{\Gamma(\alpha + \beta + 1)} \int_0^t (t-s)^{\alpha-\beta-1} |f(s, x(s), Pf(x(s))) - f(s, 0, 0)| \, ds $$

$$ + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| \, ds + |A_1(t)| \int_0^1 |h(s; x(s))| \, ds $$

$$ + |A_2(t)| \int_0^1 |g(s; x(s))| \, ds + |A_3(t)| \int_0^1 |k(s; x(s))| \, ds $$

$$ + \left[ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] \int_0^1 (t-s)^{\alpha-\beta-1} |x(s)| \, ds $$

$$ + \left[ \frac{\mu}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] \int_0^1 (t-s)^{\alpha-\beta-1} |f(s, x(s), Pf(x(s))) - f(s, 0, 0)| \, ds $$

$$ \leq \frac{1}{\Gamma(\alpha + \beta + 1)} \int_0^t (t-s)^{\alpha-\beta-1} |f(s, x(s), Pf(x(s))) - f(s, 0, 0)| \, ds $$

$$ + |A_1(t)| \int_0^1 |h(s; x(s)) - h(s; 0)| \, ds + |A_2(t)| \int_0^1 |g(s; x(s)) - g(s; 0)| \, ds $$

$$ + |A_3(t)| \int_0^1 |k(s; x(s)) - k(s; 0)| \, ds + \frac{\mu \lambda}{\Gamma(\beta - \alpha)} \int_0^1 (1-s)^{\beta-1} |f(s, x(s), Pf(x(s))) - f(s, 0, 0)| \, ds $$

$$ + |A_3(t)| \int_0^1 |k(s; x(s)) - k(s; 0)| \, ds $$

$$ \times \left[ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] \int_0^1 (1-s)^{\alpha-\beta-1} |x(s)| \, ds $$

$$ \times \left[ \frac{\mu}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] \int_0^1 (1-s)^{\alpha-\beta-1} |f(s, x(s), Pf(x(s))) - f(s, 0, 0)| \, ds. \quad (17) $$

which implies that

$$ \| P(x) \| \leq \frac{1}{\Gamma(\alpha + \beta + 1)} \left[ \left( \frac{q_1 + q_2}{\Gamma(\alpha + \beta + 1)} \right) \| x \| + M_0 \right] $$

$$ + \frac{\lambda}{\Gamma(\alpha + \beta + 1)} \| x \| + A_1 |q_1(\| x \| + M_1) + A_2 |q_1(\| x \| + M_2) $$

$$ + A_3 |q_1(\| x \| + M_0) + A_4 \left[ \left( \frac{q_1 + q_2}{\Gamma(\alpha + \beta + 1)} \right) \| x \| + M_0 \right] $$

$$ + A_5 \left[ \left( \frac{q_1 + q_2}{\Gamma(\beta + 1)} \right) \| x \| + M_0 \right] $$

$$ \leq \frac{1}{\Gamma(\alpha + \beta + 1)} \left[ \left( \frac{q_1 + q_2}{\Gamma(\alpha + \beta + 1)} \right) \left( \frac{1}{\Gamma(\alpha + \beta + 1)} \right) + \frac{A_5}{\Gamma(\beta + 1)} \right] $$

$$ + \left[ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] + \frac{\lambda}{\Gamma(\alpha + \beta + 1)} \| x \| $$

$$ + A_5 \left[ \left( \frac{q_1 + q_2}{\Gamma(\alpha + \beta + 1)} \right) \| x \| + M_0 \right] $$

$$ + A_5 \left[ \left( \frac{q_1 + q_2}{\Gamma(\alpha + \beta + 1)} \right) \| x \| + M_0 \right] $$

$$ \leq r_1 r + r_2 \leq r. \quad (18) $$

Now, for $x, y \in B_r$, and for $t \in [0, 1]$,

$$ |P(x)(t) - P(y)(t)| \leq \frac{1}{\Gamma(\alpha + \beta + 1)} \int_0^t (t-s)^{\alpha-\beta-1} |f(s, x(s), Pf(x(s)) - f(s, y(s), Pf(y(s))) \| ds $$

$$ - f(s, y(s), Pf(y(s))) | ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| \, ds $$

$$ + |A_1(t)| \int_0^1 |h(s; x(s)) - h(s; y(s))| \, ds + |A_2(t)| \int_0^1 |g(s; x(s)) - g(s; y(s))| \, ds $$

$$ + |A_3(t)| \int_0^1 |k(s; x(s)) - k(s; y(s))| \, ds + \left[ \frac{\mu \lambda}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] \int_0^1 (1-s)^{\alpha-1} $$

$$ \times (f(s, x(s), Pf(x(s)) - f(s, 0, 0)) \, ds $$

$$ + \left[ \frac{\mu}{(1 + \mu) \Gamma(\alpha + \beta + 1)} \right] \int_0^1 (1-s)^{\alpha-1} $$

$$ \times (f(s, x(s), Pf(x(s)) - f(s, 0, 0)) \, ds. \quad (19) $$
Thus,

$$\|Px - Py\| \leq r_1 \|x - y\|. \tag{20}$$

Since $r_1 < 1$, then operator $P$ is a contraction mapping. Therefore, boundary value problem (1) has a unique solution.

**Theorem 7.** Assume that $(H_1)$ and $(H_2)$ hold, $f : [0; 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function. Further, we suppose

$$(H_3) - \text{if } |t| \leq m(1); \quad |h(t, x)| \leq \rho(t); \quad |k(t, x)| \leq \psi(t); \quad |g(t, x)| \leq \phi(t), \forall (t, x, y) \in [0, 1] \times \mathbb{R}^2 \text{ with } m, \rho, \phi, \psi \in C([0, 1]; \mathbb{R}^+)).$$

Then, boundary value problem (1) has at least one solution on $[0, 1]$ if $Q < 1$ and $r'_1 < 1$, where

$$Q = \left[ A_1 q_1 + A_2 q_2 + A_3 q_3 + \frac{A_4}{1 + \alpha + 1}\right] q_1 + q_2 + \frac{|\mu|}{(1 + \mu)\Gamma(\alpha + 1)}\right],$$

$$r'_1 = \frac{\lambda}{\Gamma(\alpha + 1)} + \frac{|\mu|}{(1 + \mu)\Gamma(\alpha + 1)}.$$

**Proof.** Consider the closed ball $B'_r = \{ x \in X : \|x\| \leq r' \}$ with the radius $r' \geq (r'/(1 - r'))$, where

$$r'_2 = \frac{\|m\|}{\Gamma(\alpha + 1)} + \frac{A_1 |\rho| + A_2 |\phi| + A_3 |\psi|}{\Gamma(\alpha + 1)} + \frac{A_5 |m|}{\Gamma(\beta - 1)} + \frac{\|\mu\|}{(1 + \mu)\Gamma(\alpha + 1)}.$$

We introduce the decomposition $P = P_1 + P_2$, where

$$P_1 x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha \beta - 1} f(s, x(s), P^x(s)) ds$$

$$- \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s; x(s)) ds,$$

$$P_2 x(t) = A_1(t) \int_0^t h(s; x(s)) ds + A_2(t) \int_0^t g(s; x(s)) ds$$

$$+ A_3(t) \int_0^t k(s; x(s)) ds + \frac{A_4(t)}{\Gamma(\beta - 1)} \int_0^t$$

$$\cdot (1 - s)^{\beta - 1} f(s, x(s), P^x(s)) ds + \frac{\mu}{(1 + \mu)\Gamma(\alpha)}\right] q_1 + q_2 + \frac{|\mu|}{(1 + \mu)\Gamma(\alpha + 1)}.$$

$$+ A_5(t) \int_0^t k(s; x(s)) ds + \frac{\mu}{(1 + \mu)\Gamma(\alpha)}\right] q_1 + q_2 + \frac{|\mu|}{(1 + \mu)\Gamma(\alpha + 1)}.$$

$$r'_2 = \frac{\|m\|}{\Gamma(\alpha + 1)} + \frac{A_1 |\rho| + A_2 |\phi| + A_3 |\psi|}{\Gamma(\alpha + 1)} + \frac{A_5 |m|}{\Gamma(\beta - 1)} + \frac{\|\mu\|}{(1 + \mu)\Gamma(\alpha + 1)}.$$

We have

$$P_1 x(t) + P_2 y(t) \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\alpha \beta - 1} f(s, x(s), P^x(s)) ds$$

$$+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s; y(s)) ds,$$

which implies that

$$\|P_1 x + P_2 y\| \leq \frac{|m|}{\Gamma(\alpha + 1)} + \frac{\lambda}{\Gamma(\alpha + 1)} r' + A_4 |\rho|$$

$$+ A_2 |\phi| + A_3 |\psi| + \frac{A_4 |m|}{\Gamma(\beta - 1)}$$

$$+ \frac{A_5 |m|}{\Gamma(\beta - 1)} + \frac{\|\mu\|}{(1 + \mu)\Gamma(\alpha + 1)} r'$$

$$+ \frac{\|\mu\|}{(1 + \mu)\Gamma(\alpha + 1)} r' \leq 1.$$
For \( x, y \in B_r \), we have

\[
\|P_2 x(t) - P_2 y(t)\| \leq |A_1| \int_0^1 |h(s; x(s)) - h(s; y(s))| ds \\
+ |A_2| \int_0^1 |g(s; x(s)) - g(s; y(s))| ds \\
+ |A_3| \int_0^1 |k(s; x(s)) - k(s; y(s))| ds \\
+ \frac{|A_4|}{T(\beta - a + 1)} \int_0^1 (1 - s)^{\alpha - 1} f(s, x(s), p'x(s)) \\
- f(s, y(s), p'y(s))| ds \\
+ \frac{|A_5|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\n\leq A_1 \int_0^1 q_1 |x(s) - y(s)| ds + A_2 \int_0^1 q_1 |x(s) - y(s)| ds \\
+ A_3 \int_0^1 q_1 |x(s) - y(s)| ds + \frac{|A_4|}{T(\beta - a + 1)} \int_0^1 (1 - s)^{\alpha - 1} q_1 |x(s) - y(s)| ds \\
+ \frac{|A_5|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} q_1 |x(s) - y(s)| ds \\
+ q_2 |p'x(s) - p'y(s)| ds + \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
- \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
+ \frac{|\mu|}{(1 + \mu) |T(\alpha + 1)|} \int_0^1 (1 - s)^{\alpha - 1} |x(s) - y(s)| ds \\
\leq A_1 q_1 + A_2 q_1 + A_3 q_1 + A_4 q_1 + A_5 q_1 + A_6 q_1 \\
\leq Q\|x - y\|.
\]

(26)

Since \( Q < 1 \), then \( P_2 \) is a contraction.

Next, we show that \( P_1 \) is compact and continuous. Continuity of \( f \) implies that the operator \( P_1 \) is continuous.

Since, \( \|P_1 x\| \leq (\|m\|/|T(\alpha + 1)|) + (|\lambda|/|T(\alpha + 1)|) \), therefore, \( P_1 \) is uniformly bounded on \( B_r \). Suppose that \( 0 \leq t_1 < t_2 \leq 1 \). We have

\[
\|P_1 x(t_2) - P_1 x(t_1)\| \\
\leq \frac{1}{|T(\alpha + 1)|} \int_0^{t_1} \left( (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} \right) f(s, x(s), p'x(s)) ds \\
+ \frac{|\lambda|}{|T(\alpha + 1)|} \int_0^{t_1} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} x(s) ds \\
+ \int_0^{t_1} (t_2 - s)^{\alpha - 1} x(s) ds \left( \frac{|m|}{|T(\alpha + 1)|} + \frac{2(t_2 - t_1)^{\alpha - 1}}{|\lambda|} \right) \\
+ \left[ (t_2 - t_1)^{\alpha - 1} + |\lambda| \right] \int_0^{t_1} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} x(s) ds \\
\leq \frac{1}{|T(\alpha + 1)|} \int_0^{t_1} \left( (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} \right) f(s, x(s), p'x(s)) ds \\
+ \frac{|\lambda|}{|T(\alpha + 1)|} \int_0^{t_1} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} x(s) ds \\
+ \int_0^{t_1} (t_2 - s)^{\alpha - 1} x(s) ds \left( \frac{|m|}{|T(\alpha + 1)|} + \frac{2(t_2 - t_1)^{\alpha - 1}}{|\lambda|} \right) \\
+ \left[ (t_2 - t_1)^{\alpha - 1} + |\lambda| \right] \int_0^{t_1} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} x(s) ds \\
\]

(27)

As \( t_1 \to t_2 \), the above expression tends to be zero independent from \( x \in B_r \). This implies that \( P_1 \) is relatively compact on \( B_r \). Then, by the Arzelà-Ascoli theorem, operator \( P_1 \) is compact on \( B_r \).

Therefore, according to the Krasnoselskii fixed point theorem, problem (1) has at least one solution on \( B_r \).

4. Example

Example 8. Consider the following boundary value problem:

\[
\begin{align*}
\text{\( \mathcal{D}^{1/3} \left( \mathcal{D}^{1/3} + \frac{1}{300} \right) x(t) = \frac{1}{500 + t} \left( \sin x + \frac{1}{T(\alpha + 1)} \right) \int_0^t (t - s)^{-1/3} x ds \right), \quad t \in [0, 1], \\
x(0) + x(1) = \frac{1}{200} \int_0^1 \frac{|x(s)|}{300 + |x(s)|} ds, \\
\mathcal{D}^{1/3} x(0)^+ + \mathcal{D}^{1/3} x(1)^+ = \frac{1}{200} \int_0^1 \left( \frac{1}{s + 2} \right)^3 \frac{|x(s)|}{30 + |x(s)|} ds, \\
\mathcal{D}^{1/3} x(0)^+ + \mathcal{D}^{1/3} x(1)^+ = \frac{1}{200} \int_0^1 \left( \frac{1}{s + 4} \right)^2 \frac{|x(s)|}{30 + |x(s)|} ds.
\end{align*}
\]

(29)
We choose $\beta = 4/3$, $\alpha = 1/3$, $\lambda = 1/300$, $\mu = 1$, $\sigma_1 = 1/200$, $\sigma_2 = 1/200$, $\sigma_3 = 1/200$, $p = 1/4$, and the continuous functions

$$f(t, x, t^{1/4}x) = \frac{1}{500 + t^2} \left( \sin x + \frac{1}{t} \right) \left( t^{1/4} - 1/4 \right) ds,$$

$$g(t, x) = \frac{|x(t)|}{300 + |x(t)|},$$

$$h(t, x) = \left( \frac{1}{t + 2} \right)^3 \frac{|x(t)|}{30 + |x(t)|},$$

and

$$k(t, x) = \left( \frac{1}{t + 4} \right)^2 \frac{|x(t)|}{30 + |x(t)|}.$$

(30)

Clearly, $q_1 = 1/500$, $q_3 = 1/300$, $q_4 = 1/240$, and $q_5 = 1/480$.

Then, we have $r_1 \approx 0.07 < 1$.

Thus, all the assumptions of Theorem 2 hold. Then, problem (29) has a unique solution.

Example 9. Consider the following boundary value problem:

$$\begin{aligned}
&cD^{1/4} \left( cD^{1/3} + \frac{1}{300} \right) x(t) = \frac{1}{t + 10} \left( \frac{1}{5 + x^2} + \int_0^t (t - s)^{-1/4} \frac{1}{5 + x^2} ds \right), \quad t \in [0, 1], \\
x(0) + x(1) = \frac{1}{200} \int_0^1 \frac{1}{5 + 10 s + 30} |x(s)| ds, \\
cD^{1/3} x(0) + cD^{1/3} x(1) = \frac{1}{200} \int_0^1 \frac{1}{s + 3} \frac{1}{30 + 300} |x(s)| ds, \\
cD^{2/3} x(0) + cD^{2/3} x(1) = \frac{1}{200} \int_0^1 \frac{1}{s + 4} \frac{1}{30 + 300} |x(s)| ds.
\end{aligned}$$

(31)

Here, $\beta = 3/4$, $\alpha = 1/3$, $\lambda = 1/30$, $\mu = 1$, $\sigma_1 = 1/200$, $\sigma_2 = 1/200$, $\sigma_3 = 1/200$, $p = 1/4$, and the continuous functions

$$f(t, x, t^{1/4}x) = \frac{1}{(t + 10)^2} \left( \frac{1}{5 + x^2} + \int_0^t (t - s)^{-1/4} \frac{1}{5 + x^2} ds \right),$$

$$g(t, x) = \frac{1}{t + 10} \frac{|x(t)|}{30 + |x(t)|},$$

$$h(t, x) = \left( \frac{1}{t + 2} \right)^3 \frac{|x(t)|}{30 + |x(t)|},$$

$$k(t, x) = \left( \frac{1}{t + 4} \right)^2 \frac{|x(t)|}{30 + |x(t)|}.$$

(32)

Clearly, $q_1 = 1/500$, $q_3 = 1/300$, $q_4 = 1/240$, and $q_5 = 1/480$; thus, we have $Q \approx 0.057 < 1$ and $r' \approx 0.065 < 1$.

Then, problem (31) has a least one solution.

5. Conclusion

In this paper, we studied the existence and uniqueness of fractional Langevin equations with nonseparated integral boundary conditions. First of all, we transformed the problem into an equivalent fixed point problem; second of all, we utilized the Banach contraction principle and the Krasnoselskii fixed point theorem to prove the existence and uniqueness of solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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