ON METRIZABLE X WITH $C_p(X)$ NOT HOMEOMORPHIC TO $C_p(X) \times C_p(X)$

MIKOLAJ KRUPSKI AND WITOLD MARCISZEWSKI

ABSTRACT. We give two examples of infinite metrizable spaces $X$ such that the space $C_p(X)$, of continuous real-valued function on $X$ endowed with the pointwise topology, is not homeomorphic to its own square $C_p(X) \times C_p(X)$. The first of them is a one-dimensional continuum; the second one is a zero-dimensional subspace of the real line. Our result answers a long-standing open question in the theory of function spaces posed by A.V. Arhangel’skii.

1. Introduction

Let $C_p(X)$ denote the space of all continuous real-valued functions on a Tychonoff space $X$, equipped with the topology of pointwise convergence. One of the important questions, stimulating the theory of $C_p$-spaces for almost 30 years and leading to interesting examples in this theory, is the problem whether the space $C_p(X)$ is (linearly) homeomorphic to its own square $C_p(X) \times C_p(X)$, provided $X$ is an infinite compact or metrizable space, cf. A.V. Arhangel’skii’s articles [1, Problem 22], [2, Problem 4], [3, Problem 25]. In this note we give a complete negative answer to this problem.

The first nonmetrizable (compact) counterexamples, i.e. spaces $X$ with $C_p(X)$ not homeomorphic to $C_p(X) \times C_p(X)$, were constructed independently by Gul’ko [5] and Marciszewski [9]. However, the metrizable case seemed to be more delicate. In [14] R. Pol showed that if $M$ is a Cook continuum, then $C_p(M)$ is not linearly homeomorphic to $C_p(M) \times C_p(M)$. He also gave two other examples of metrizable spaces having the same property: a rigid Bernstein set $B$ and the A.H. Stone’s set $E$. This result, settled one part of [2, Problem 4] and [3, Problem 25] yet the question whether, for a metrizable (compact) space $X$, the space $C_p(X)$ is always homeomorphic to $C_p(X) \times C_p(X)$ remained open (see [10, Problem 4.12], [13, Problem 1029]). It was proved in [12] that if $M$ is a Cook continuum then $C_p(M)$ is not uniformly homeomorphic to $C_p(M) \times C_p(M)$, but the authors of [12] could not determine
whether the notion of uniform homeomorphism in their result can be replaced by a weaker notion of homeomorphism and left this question open (see [12, page 656]).

We show that indeed a Cook continuum $M$ can also serve as a counterexample solving a problem of Arhangel’skii for homeomorphisms. We shall prove the following:

**Theorem 1.1.** There exists a metrizable one-dimensional continuum $M$ (a Cook continuum), such that the function space $C_p(M)$ is not homeomorphic to $C_p(M) \times C_p(M)$.

We also show that the rigid Bernstein set, considered by R. Pol in the context of linear homeomorphisms, is another counterexample:

**Theorem 1.2.** There exists an infinite zero-dimensional subspace $B$ of the real line (a rigid Bernstein set), such that the function space $C_p(B)$ is not homeomorphic to $C_p(B) \times C_p(B)$.

Our proofs are based on Theorem 2.1 below, which is an easy consequence of the main result of [6] proved by the first author. Another important ingredient is Lemma 2.2 proved in the next section, which may also be of independent interest.

2. Preliminaries

Let us denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of reals. For a finite subset $A$ of a space $X$ and for $m \in \mathbb{N}$ the set

$$O_X(A; \frac{1}{m}) = \{f \in C_p(X) : \forall x \in A \ |f(x)| < \frac{1}{m}\}$$

is a basic neighborhood of the zero function on $X$ (i.e. the constant function equal to zero) in $C_p(X)$ and $\overline{O}_X(A; \frac{1}{m})$ is its closure, i.e.

$$\overline{O}_X(A; \frac{1}{m}) = \{f \in C_p(X) : \forall x \in A \ |f(x)| \leq \frac{1}{m}\}.$$

For a singleton $A = \{x\}$, we will write $\overline{O}_X(x; \frac{1}{m})$ rather than $\overline{O}_X(\{x\}; \frac{1}{m})$.

The following fact is a consequence of results proved by the first author in [6], cf. [6, Lemma 3.4 and the proof of Theorem 4.1]

**Theorem 2.1.** Suppose that $X$ and $Y$ are metrizable spaces. Let $n \in \mathbb{N}$ and suppose that $\Psi : C_p(X) \rightarrow C_p(Y)$ is a homeomorphism taking the zero function to the zero function. Then the space $Y$ can be written as countable union $Y = \bigcup_{r \in \mathbb{N}} G_r$ of closed subsets such that:

(A) For every $r \in \mathbb{N}$ there are continuous mappings $f_1^r, \ldots, f_{n^r}^r : G_r \rightarrow X$ and $m \in \mathbb{N}$ such that $\Psi(\overline{O}_X(A; \frac{1}{m})) \subseteq \overline{O}_Y(y; \frac{1}{m})$, where $A = \{f_1^r(y), \ldots, f_{n^r}^r(y)\}$.

We will need the following lemma.

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1In [6] the proof was given for $n = 1$ only, but without any changes it works also for arbitrary $n \in \mathbb{N}$. 

**Lemma 2.2.** Let $X$ and $Y$ be infinite Tychonoff spaces and let $\Psi : C_p(X) \to C_p(Y)$ be a homeomorphism. For any finite set $A \subseteq X$, there exists a finite set $B \subseteq Y$, such that, for any $y \in Y \setminus B$ and $r \in \mathbb{R}$, there is a function $f \in C_p(X)$ such that $f \upharpoonright A = 0$, and $\Psi(f)(y) = r$.

*Proof.* For a subset $A \subseteq X$, let $C_p(X, A)$ denote the subspace $\{f \in C_p(X) : f \upharpoonright A = 0\}$. It is well-known that, for any finite $A \subseteq X$, the space $C_p(X, A)$ is homeomorphic with the product $\mathbb{R}^A \times C_p(X, A)$. Indeed, we have $\mathbb{R}^A = C_p(\mathbb{R})$, and if $T : C_p(X, A) \to C_p(X)$ is a continuous extension operator (see [11, 6.6.5]), then the map $\Phi : C_p(X, A) \times C_p(X, A) \to C_p(X)$ defined by $\Phi(f, g) = T(f) + g$, for $f \in C_p(X, A)$ and $g \in C_p(X, A)$, is the required homeomorphism. Observe that $\Phi$ has the property, that

\[
\Phi(f, g) \upharpoonright A = f.
\]

Fix a finite $A \subseteq X$ and suppose that the assertion of the lemma does not hold true. Then there exist a sequence $(y_n)_{n \in \mathbb{N}}$ of distinct elements of $Y$ and a sequence $(r_n)_{n \in \mathbb{N}}$ of reals, such that

\[
\Psi(f)(y_n) \neq r_n \quad \text{for any } f \in C_p(X, A).
\]

Let $\| \cdot \|$ be the Euclidean norm in $\mathbb{R}^A$, $S$ be the unit sphere in $(\mathbb{R}^A, \| \cdot \|)$, and $G = \mathbb{R}^A \setminus \{(0, 0, \ldots, 0)\}$. Let $\iota : S \to G$ be the identity embedding. Clearly, the map $\iota$ is not homotopic in $G$ with a constant map. Put

\[
U = \{e : S \to \mathbb{R}^A : \|e(x) - \iota(x)\| < 1 \quad \text{for all } x \in S\}.
\]

Since any map $e \in U$ is homotopic in $G$ with $\iota$, it is also not homotopic in $G$ with a constant map.

Let $\iota : S \to \mathbb{R}^A \times C_p(X, A)$ be the map defined by $\iota(x) = (\iota(x), \mathbf{0})$, for $x \in S$, where $\mathbf{0}$ denotes the zero function in $C_p(X, A)$. We put $\tilde{\iota} = \Phi \circ \iota : S \to C_p(X)$.

For a topological space $Z$, by $C(S, Z)$ we denote the space of all continuous maps from $S$ into $Z$, equipped with the compact-open topology.

Let $\pi_1 : \mathbb{R}^A \times C_p(X, A) \to \mathbb{R}^A$ be the projection onto the first axis. We put $V = \{f \in C(S, \mathbb{R}^A \times C_p(X, A)) : \pi_1 \circ f \in U\}$. Clearly, $V$ is an open subset of $C(S, \mathbb{R}^A \times C_p(X, A))$, therefore the set $W = \{\Phi \circ f : f \in V\}$ is an open neighborhood of $\iota$ in $C(S, C_p(X))$.

Let $D(A) = C_p(X) \setminus C_p(Y, A)$. From property (11) it follows that $\Phi(G \times C_p(X, A)) = D(A)$. Therefore, one can easily verify that any map $g \in W$ is homotopic in $D(A)$ with $\tilde{\iota}$, hence it is not homotopic in $D(A)$ with a constant map.

The set $O = \{\Psi \circ g : g \in W\}$ is open in $C(S, C_p(Y))$. Since basic open sets in $C_p(Y)$ depend on finitely many coordinates, we can find a finite set $C \subseteq Y$ such that any $h \in C(S, C_p(Y))$ satisfying

\[
h(x) \upharpoonright C = \Psi \circ \tilde{\iota}(x) \upharpoonright C \quad \text{for all } x \in S
\]
belongs to $O$. Find $y_n \notin C$ and put $D = C \cup \{y_n\}$. Let $\Theta : \mathbb{R}^D \times C_p(Y, D) \to C_p(Y)$ be a homeomorphism such that
\begin{equation}
\Theta(f, g) | D = f \quad \text{for } f \in \mathbb{R}^D, g \in C_p(Y, D), \tag{5}
\end{equation}
cf. (1). Let $h : S \to \mathbb{R}^D$ be the map defined by
\begin{equation}
h(x) \upharpoonright C = C \circ i(x) \upharpoonright C \quad \text{and} \quad h(x)(y_n) = r_n \quad \text{for all } x \in S,
\end{equation}
$h : S \to \mathbb{R}^D \times C_p(Y, D)$ be defined by $h(x) = (h(x), 0)$, for $x \in S$, where $0$ denotes the zero function in $C_p(Y, D)$. Finally, we put $\tilde{h} = \Theta \circ h : S \to C_p(Y)$.

By \((\text{41})\), \((\text{45})\), and \((\text{43})\) we have $\tilde{h} \in O$. Let $r \in C_p(Y)$ be the constant function taking value $r_n$. Consider the homotopy $H : S \times [0, 1] \to C_p(Y)$ defined by
\begin{equation}
H(x, t) = (1 - t)\tilde{h}(x) + tr \quad \text{for } x \in S,
\end{equation}
and joining $\tilde{h}$ with the constant map. Let $h_t : S \to C_p(Y)$ be defined by $h_t(x) = H(x, t)$.

Observe that, by \((\text{46})\) and \((\text{43})\), for any $t \in [0, 1]$ and $x \in S$, $h_t(x)(y_n) = r_n$, hence from \((\text{42})\) it follows that $h_t(S) \subseteq C_p(Y) \setminus \Psi(C_p(X, A))$. Therefore the homotopy $\Psi^{-1} \circ H : S \times [0, 1] \to C_p(X)$ takes values in $D(A)$ and joins the map $\Psi^{-1} \circ h_1 \in W$ with the constant map $\Psi^{-1} \circ h_1$, a contradiction.

\end{proof}

3. Proofs

3.1. Proof of Theorem 1.1. Let $M$ be a Cook continuum, i.e. a nontrivial metrizable continuum such that, for every subcontinuum $C \subseteq M$, every continuous mapping $f : C \to M$ is either the identity or is constant. The first example of a space having indicated properties was constructed by H. Cook [4 Theorem 8]; Maćkowiak [8 Corollary 6.2] gave an example of a planar Cook continuum. Both examples are one-dimensional. Strengthening the results from [14] and [12] we shall prove that the spaces $C_p(M)$ and $C_p(M) \times C_p(M)$ are not homeomorphic. Of course $C_p(M) \times C_p(M)$ is linearly homeomorphic to $C_p(M \oplus M)$, where $M \oplus M$ is a discrete sum of two copies of $M$ and thus can be viewed as $M \times \{1, 2\}$.

It will be convenient to use the following notation:
\[ A_i = A \times \{i\} \subseteq M \oplus M, \quad i = 1, 2, \]
for a subset $A \subseteq M$. Similarly, $x_i = (x, i) \in M \oplus M$, for any $x \in M$. Thus $A_i$ is a copy of $A$ lying in the corresponding copy of $M$ in the space $M \oplus M$.

Striving for a contradiction, suppose that there is a homeomorphism $\Phi : C_p(M) \to C_p(M \oplus M)$. It is clear that without loss of generality we can assume that $\Phi$ takes the zero function to the zero function.

From Theorem 2.1 (applied with $n = 1$, $X = M \oplus M$, $Y = M$ and $\Psi = \Phi^{-1}$) it follows that $M$ is a countable union of closed subsets $G_r$, hence, by the Baire category theorem, there
is $r$ such that $G_r$ has a nonempty interior. Applying Janiszewski theorem (see [7, §47.III.1]) we can find a nontrivial subcontinuum $C' \subseteq G_r \subseteq M$. By Theorem 2.1 there are finitely many continuous functions $f'_1, \ldots, f'_{p'} : C' \to M \oplus M$ (being the restrictions of functions defined on $G_r$ provided by Theorem 2.1) to the subcontinuum $C'$.

Connectedness of $C'$ guarantees that $f'_i(C') \subseteq M_1$ or $f'_i(C') \subseteq M_2$, for $i \leq p'$. By the rigidity of $M$ each $f'_i$ is either the identity (up to identification of $C'_i$ with $C'$) or is constant. Hence, there is a finite set $J' \subseteq M \oplus M$ such that

$$\{f'_1(x), \ldots, f'_{p'}(x)\} \subseteq \{x, x_2\} \cup J', \text{ for any } x \in C'.\tag{8}$$

Property (A) from Theorem 2.1 implies that there is $k \in \mathbb{N}$ such that

$$\Phi^{-1}(O_{M \oplus M}(\{x_1, x_2\} \cup J'; \frac{1}{k})) \subseteq \overline{O}_M(x; 1), \text{ for any } x \in C'.\tag{9}$$

Now, applying Theorem 2.1 once more (with $n = 2k$, $X = M$, $Y = M \oplus M$ and $\Psi = \Phi$) together with the Baire category theorem and Janiszewski theorem, we can find a nontrivial continuum $C \subseteq C'$ and finitely many continuous functions $f^1_1, \ldots, f^1_{p_1} : C \to M$ and $f^2_1, \ldots, f^2_{p_2} : C \to M$.

By the rigidity of $M$ each $f^1_i, f^2_i$ is either the identity (up to identification of $C_i$ with $C$) or is constant. Hence there is a finite set $J \subseteq M$ such that

$$\{f^1_1(x_1), \ldots, f^1_{p_1}(x_1)\} \cup \{f^2_1(x_2), \ldots, f^2_{p_2}(x_2)\} \subseteq \{x\} \cup J, \text{ for any } x \in C.\tag{10}$$

Property (A) from Theorem 2.1 implies that there is $m \in \mathbb{N}$ such that

$$\Phi(O_M(\{x\} \cup J; \frac{1}{m})) \subseteq \overline{O}_{M \oplus M}(\{x_1, x_2\}; \frac{1}{2k}), \text{ for any } x \in C.\tag{11}$$

By the continuity of $\Phi^{-1}$, there is a finite set $I \subseteq M \oplus M$ and $\varepsilon > 0$ such that

$$\Phi^{-1}(O_{M \oplus M}(I; \varepsilon)) \subseteq O_M(J; \frac{1}{m}).\tag{12}$$

By Lemma 2.2 (where $X = M \oplus M$, $Y = M$, $\Psi = \Phi^{-1}$, $A = I \cup J'$, $r = 2$) there are $v_1, v_2 \in C_p(M \oplus M)$ and $c \in C$ such that

(i) $c_1, c_2 \notin I \cup J'$,

(ii) $v_1 \upharpoonright (I \cup J') = 0$, $v_2 \upharpoonright (I \cup J') = 0$,

(iii) $\Phi^{-1}(v_1)(c) > 2$, $\Phi^{-1}(v_2)(c) < -2$.

**Claim 1.** $|v_1(c_1)| \geq \frac{1}{k}$ or $|v_1(c_2)| \geq \frac{1}{k}$, for $i = 1, 2$.

**Proof.** If not, then by (ii) and (8) we would have $|\Phi(v_i)(c)| \leq 1$, contradicting (iii).

So let $i, j \in \{1, 2\}$ be such that

$$|v_1(c_i)| \geq \frac{1}{k} \text{ and } |v_2(c_j)| \geq \frac{1}{k}.\tag{13}$$

We shall consider two cases:
Case 1. \( v_1(c_1) \cdot v_2(c_2) = v_1(c_2) \cdot v_2(c_1) \). Let
\[
j' = j + 1 \mod 2.
\]
By the continuity of \( \Phi^{-1} \), there is \( \delta > 0 \) and \( h \in C_p(M \oplus M) \) such that
\[
\begin{cases}
  h(c_{j'}) = v_2(c_{j'}) + \delta, \\
  h \upharpoonright (I \cup J' \cup \{c_j\}) = v_2 \upharpoonright (I \cup J' \cup \{c_j\}), \\
  \Phi^{-1}(h)(c) < -1.
\end{cases}
\]
(12)
We put \( u_1 = v_1 \) and \( u_2 = h \). Using (11) one can easily verify that
\[
u_1(c_1) \cdot u_2(c_2) \neq u_1(c_2) \cdot u_2(c_1).
\]

Case 2. \( v_1(c_1) \cdot v_2(c_2) \neq v_1(c_2) \cdot v_2(c_1) \). Then we put \( u_1 = v_1, u_2 = v_2 \).

We define a mapping \( \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by the formula
\[
\varphi(t_1, t_2) = \Phi^{-1}(t_1u_1 + t_2u_2)(c),
\]
i.e. \( \varphi \) is the composition of the mapping \( (t_1, t_2) \mapsto t_1u_1 + t_2u_2 \) with \( \Phi^{-1} \) and the evaluation functional at \( c \). Consider
\[
Z = \{(t_1, t_2) \in \mathbb{R} \times \mathbb{R} : |t_1u_1(c_1) + t_2u_2(c_1)| \geq \frac{1}{k}\ \text{or}\ |t_1u_1(c_2) + t_2u_2(c_2)| \geq \frac{1}{k}\}.
\]
Let
\[
m_1 = \{(t_1, t_2) \in \mathbb{R} \times \mathbb{R} : t_1u_1(c_1) + t_2u_2(c_1) = \frac{1}{k}\},
\]
\[
m_2 = \{(t_1, t_2) \in \mathbb{R} \times \mathbb{R} : t_1u_1(c_2) + t_2u_2(c_2) = \frac{1}{k}\}.
\]
Note, that from the definition of \( u_1 \) and \( u_2 \) it follows that the above sets are nonempty, i.e. it can not happen that \( u_1(c_1) = u_2(c_1) = 0 \) or \( u_1(c_2) = u_2(c_2) = 0 \). Hence \( m_1 \) and \( m_2 \) are non-parallel lines. Indeed, by the definition of \( u_1 \) and \( u_2 \), cf. Case 1 and Case 2, we have
\[
u_1(c_1) \cdot u_2(c_2) \neq u_1(c_2) \cdot u_2(c_1),
\]
which means exactly that \( m_1 \) and \( m_2 \) are not parallel.

Since \( m_1 \) and \( m_2 \) are not parallel, the set \( Z \) is connected (being the plane with a parallelogram removed).

Claim 2. \( \varphi(Z) \subseteq \mathbb{R} \setminus (-\frac{1}{m}, \frac{1}{m}) \).

Proof. Otherwise, by (ii), (10) and (12)
\[
\Phi^{-1}\left(t_1u_1 + t_2u_2\right) \subseteq O_M(\{c\} \cup J; \frac{1}{m}),
\]
for some \((t_1, t_2) \in Z\). Hence (9) implies that
\[
|t_1u_1(c_1) + t_2u_2(c_1)| \leq \frac{1}{k^2} < \frac{1}{k},
\]
\[
|t_1u_1(c_2) + t_2u_2(c_2)| \leq \frac{1}{k^2} < \frac{1}{k}.
\]
However this contradicts the definition of \(Z\).

By (11) and (12), we have \((1, 0), (0, 1) \in Z\). Further, by (iii) and (12), we infer that
\[
\varphi(1, 0) = \Phi^{-1}(u_1)(c) > 2, \quad \varphi(0, 1) = \Phi^{-1}(u_2)(c) < -1.
\]
This means that \(\varphi(Z) \cap (-\infty, -\frac{1}{m}) \neq \emptyset\), \(\varphi(Z) \cap (\frac{1}{m}, \infty) \neq \emptyset\) and, by Claim 2, \(\varphi(Z) \cap (-\frac{1}{m}, \frac{1}{m}) = \emptyset\). Therefore the set \(\varphi(Z)\) is not connected, a contradiction with connectedness of \(Z\). This ends the proof of Theorem 1.1.

### 3.2. On the proof of Theorem 1.2

Let \(B\) be a rigid Bernstein set, i.e. a nonempty subset of the real line \(\mathbb{R}\) with the following properties:

- **(P1)** No nonempty open subset of \(B\) is meager in \(\mathbb{R}\).
- **(P2)** If \(U \subseteq B\) is open and nonempty then each continuous function \(f : U \to B\) is either the identity or is constant on a nonempty open subset of \(U\).

A set with the above properties can be constructed by fairly standard transfinite induction, cf. [14] or [11, 6.13.1]. It was proved in [14] that \(C_p(B)\) is not linearly homeomorphic to \(C_p(B) \times C_p(B)\), cf. [11, 6.13.3]. Modifying slightly the proof of Theorem 1.1 we can strengthen this result and show that \(C_p(B)\) is not homeomorphic to \(C_p(B) \times C_p(B)\). Note that the spaces \(B\) and \(M\), though both metrizable, have quite different features: \(M\) is a continuum and, by property (P2) \(B\) cannot contain any nontrivial interval, therefore it is zero-dimensional. It is well-known that no Bernstein set can be closed, or even Borel, subset of \(\mathbb{R}\). Let us briefly describe changes one has to make in the proof of Theorem 1.1 to show that \(C_p(B)\) and \(C_p(B) \times C_p(B)\) are not homeomorphic:

- **(a)** The role of nontrivial subcontinua of \(M\) is played by nonempty subsets of \(B\); observe that, by (P1), each nonempty open subset of \(B\) is infinite.
- **(b)** Instead of the Baire category theorem we use property (P1) to find a nonempty open subset of some \(G_r\).
- **(c)** To find an open subset of \(B\), being a counterpart of the subcontinuum \(C'\), we use successively the following fact:
  For every nonempty open \(U \subseteq B\) and a continuous \(f : U \to B \oplus B\), there is \(i \in \{1, 2\}\) and a nonempty open \(V \subseteq U\) such that \(f(V) \subseteq B_i\).
- **(d)** By a simple inductive argument, one can show that, for every nonempty open \(U \subseteq B\) and continuous functions \(f_1, \ldots, f_p : U \to B\), there is a nonempty open \(V \subseteq U\) such that, for each \(i \in \{1, 2\}\), \(f_i \upharpoonright V\) is the identity or is constant. We use this fact to select open counterparts of the subcontinua \(C'\) and \(C\).
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Institute of Mathematics, University of Warsaw,
ul. Banacha 2, 02–097 Warszawa, Poland

E-mail address: mkrupski@mimuw.edu.pl
E-mail address: wmarcisz@mimuw.edu.pl