Exact Controllability of Linear Stochastic Differential Equations and Related Problems*

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Abstract

A notion of $L^p$-exact controllability is introduced for linear controlled (forward) stochastic differential equations, for which several sufficient conditions are established. Further, it is proved that the $L^p$-exact controllability, the validity of an observability inequality for the adjoint equation, the solvability of an optimization problem, and the solvability of an $L^p$-type norm optimal control problem are all equivalent.

Keywords: Controlled stochastic differential equation, $L^p$-exact controllability, observability inequality, norm optimal control problem.

AMS subject classification: 93B05, 93E20, 60H10

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space on which a $d$-dimensional standard Brownian motion $W(\cdot) \equiv (W_1(\cdot), \cdots, W_d(\cdot))$ is defined such that $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$ is its natural filtration augmented by all the $\mathbb{P}$-null sets. Consider the following linear controlled (forward) stochastic differential equation (FSDE, for short):

\begin{equation}
\label{equation1}
dX(t) = \left[A(t)X(t) + B(t)u(t)\right]dt + \sum_{k=1}^{d} \left[C_k(t)X(t) + D_k(t)u(t)\right]dW_k(t), \quad t \geq 0,
\end{equation}

where $A, C_1, \cdots, C_d : [0, T] \times \Omega \to \mathbb{R}^{n \times n}$ and $B : D_1, \cdots, D_d : [0, T] \times \Omega \to \mathbb{R}^{n \times m}$ are suitable matrix-valued processes. Here, $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{n \times m}$ are the sets of all $(n \times n)$ and $(n \times m)$ real matrices, respectively. In the above, $X(\cdot)$ is the state process valued in the $n$-dimensional (real) Euclidean space $\mathbb{R}^n$ and $u(\cdot)$ is the control process valued in the $m$-dimensional (real) Euclidean space $\mathbb{R}^m$. We will denote system (1.1) by $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$, with $C(\cdot) = (C_1(\cdot), \cdots, C_d(\cdot))$ and $D(\cdot) = (D_1(\cdot), \cdots, D_d(\cdot))$, and denote

$$[A(\cdot), 0; B(\cdot), 0] = [A(\cdot); B(\cdot)].$$

In addition, if $A(\cdot)$ and $B(\cdot)$ are deterministic, $[A(\cdot); B(\cdot)]$ is reduced to a linear controlled ordinary differential equation (ODE, for short), for which a very mature theory is available; See, for example, Wonham [25], and references cited therein.

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For system (1.1), a control process $u : [0, T] \times \Omega \to \mathbb{R}^m$ is said to be feasible on $[0, T]$ if $t \mapsto u(t)$ is $\mathbb{F}$-progressively measurable, and for any initial state $x \in \mathbb{R}^n$, the state equation (1.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot; x, u(\cdot))$ on $[0, T]$ with the property

$$X(0) = x, \quad \mathbb{E} \left[ \sup_{t \in [0, T]} |X(t)| \right] < \infty.$$  

Let $U[0, T]$ be the set of all feasible controls on $[0, T]$, whose precise definition will be given a little later. Let $U[0, T] \subseteq U[0, T]$. System $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is said to be $L^p$-exactly controllable on $[0, T]$ by $U[0, T]$ if for any initial state $x \in \mathbb{R}^n$ and any terminal state $\xi \in L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, the set of all $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}^n$ with $\mathbb{E}|\xi|^p < \infty$, there exists a $u(\cdot) \in U[0, T]$ such that

$$X(T; x, u(\cdot)) = \xi.$$

Clearly, the above notion is a natural extension of controllability for ODE system $[A(\cdot); B(\cdot)]$ ([25]).

Controllability is one of the most important concepts in control theory. For time-invariant linear ODE systems, it is well-known that the controllability is equivalent to many conditions, among which, the Kalman’s rank condition is the most interesting one. It is also known that the controllability of a controlled linear ODE system is equivalent to the observability of its adjoint system. For controlled linear partial differential equations (PDEs, for short), the notion of controllability can further be split into the so-called exact controllability, null controllability, approximate controllability, which are not equivalent in general, and all of these three are closely related to the so-called unique continuation property, and/or the observability inequality for the adjoint equation. For extensive surveys of controllability results on deterministic systems, see Lee–Markus [13] for ODE systems, and Russell [20], Lions [14, 15] and Zuazua [29] for PDE systems.

The study of the controllability for stochastic systems can be traced back to the work of Connor [3] in 1967, followed by Sunahara–Aihara–Kishino [21], Zabczyk [28], Ehrhardt–Kliemann [6], and Chen–Li–Peng–Yong [2]. With the help of backward stochastic differential equations (BSDEs, for short), Peng [18] introduced the so-called exact terminal controllability and exact controllability\(^1\) for linear FSDE system with constant coefficients; the former was characterized by the non-degeneracy of the matrix $D$ and the latter was characterized by a generalized version of Kalman’s rank condition. Later, Liu–Peng [16] extended the results to the linear FSDE system with bounded time-varying coefficients, using a random version of Gramian matrix. On the other hand, Buckdahn–Quincampoix–Tessitore [1] and Goegec [11] studied the so-called approximate controllability (in $L^2$ sense) for linear FSDE systems, with constant coefficients and with degenerate matrix $D$ in the state equation; Some generalized Kalman type conditions are obtained to characterize the approximate controllability. In [17], Liu–Yong–Zhang established a representation of Itô’s integral as a Lebesgue/Bochner integral, which has some interesting consequences on controllability of a linear FSDE system with vanished diffusion (see below).

In this paper, for any $p \in [1, \infty)$, we propose a notion of $L^p$-exact controllability (see Definition 3.1) for FSDE systems. When $p = 2$ and all the coefficients of the system are bounded, our notion is reduced to the one studied in [18, 16]. We point out that since the coefficients $B(\cdot)$ and $D_k(\cdot)$ are allowed to be unbounded, the corresponding set of admissible controls is delicate and it makes the controllability problem under consideration more interesting (see below for detailed explanation). Inspired by the results of deterministic systems, for any $p \in (1, \infty)$, we introduce a stochastic version of observability inequality (see Theorem 4.2) for the adjoint equation, the validity of which is proved to be equivalent to the $L^p$-exact controllability of the linear FSDE (with random coefficients). This provides an approach to study the controllability of linear FSDE systems by establishing an inequality for BSDEs. Moreover, we introduce a family of optimization problems of the adjoint linear BSDE (see Problem (O) and Problem (O)’ in Section 4), and the solvability of these optimization problems is proved to be equivalent to the $L^p$-exact controllability.
of the linear FSDE. In other words, we additionally provide an approach to study the exact controllability through infinite-dimensional optimization theory.

As an application, we consider some $L^p$-type norm optimal control problems (see Problem (N) and Problem (N)' in Section 5). The norm optimal control problem for deterministic finite or infinite dimensional systems has been investigated by many researchers (see e.g. [7, 8, 12, 22, 23]). Recently, Gashi [10] studied a norm optimal control problem (in $L^2$ sense) for linear FSDE systems with deterministic time-varying coefficients by virtue of the corresponding Hamiltonian system and Riccati equation. Moreover, Wang–Zhang [24] studied a kind of approximately norm optimal control problems for linear FSDEs. In the present paper, with the help of optimization problems for BSDE, we solve the norm optimal control problem for linear FSDE systems with random coefficients (see Theorems 5.1 and 5.3, and Corollary 5.5).

The rest of this paper is organized as follows. In Section 2, we present some preliminaries. Section 3 is devoted to the introduction of the $L^p$-exact controllability for linear FSDE systems. Some sufficient conditions of the $L^p$-exact controllability are established for two types of systems: The diffusion is control-free and the diffusion is “fully” controlled. In Section 4, we establish the equivalence among the $L^p$-exact controllability, the validity of observability inequality for the adjoint equation and the solvability of optimal control problems. Finally, as an application, a norm optimization problem is considered in Section 5.

2 Preliminaries

Recall that $\mathbb{R}^n$ is the $n$-dimensional (real) Euclidean (vector) space with the standard Euclidean norm $| \cdot |$ induced by the standard Euclidean inner product $\langle \cdot , \cdot \rangle$, and $\mathbb{R}^{n \times m}$ is the space of all $(n \times m)$ (real) matrices, with the inner product

$$\langle A, B \rangle = \text{tr} [A^T B], \quad \forall A, B \in \mathbb{R}^{n \times m},$$

so that $\mathbb{R}^{n \times m}$ is also a Euclidean space. Hereafter, the superscript $^\top$ denotes the transpose of a vector or a matrix. We now introduce some spaces, besides $L^p_{\mathcal{F}_T} (\Omega; \mathbb{R}^m)$ introduced in the previous section. Let $H = \mathbb{R}^n, \mathbb{R}^{n \times m}, \text{etc.}$, and, $p,q \in [1, \infty)$.

- $L^q_p(\Omega; L^q(0,T;H))$ is the set of all $\mathcal{F}$-progressively measurable processes $\varphi(\cdot)$ valued in $H$ such that

$$\|\varphi(\cdot)\|_{L^q_p(\Omega; L^q(0,T;H))} \equiv \left[ \mathbb{E} \left( \int_0^T |\varphi(t)|^q dt \right)^{\frac{1}{q}} \right]^{\frac{1}{q}} < \infty.$$

- $L^q_p(0,T;L^p(\Omega;H))$ is the set of all $\mathcal{F}$-progressively measurable processes $\varphi(\cdot)$ valued in $H$ such that

$$\|\varphi(\cdot)\|_{L^q_p(0,T;L^p(\Omega;H))} \equiv \left[ \int_0^T \left( \mathbb{E}[|\varphi(t)|^p] \right)^{\frac{1}{p}} dt \right]^{\frac{1}{p}} < \infty.$$

- $L^p_p(\Omega; C([0,T]; H))$ is the set of all $\mathcal{F}$-progressively measurable processes $\varphi(\cdot)$ valued in $H$ such that for almost all $\omega \in \Omega$, $t \mapsto \varphi(t, \omega)$ is continuous and

$$\|\varphi(\cdot)\|_{L^p_p(\Omega; C([0,T]; H))} \equiv \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X(t)|^p \right) \right]^{\frac{1}{p}} < \infty.$$

In the similar manner, one may define

\[
\begin{align*}
L^p_p(\Omega; L^\infty(0,T;H)), & \quad L^\infty_p(0,T;L^p(\Omega;H)), & \quad L^p_p(0,T;L^\infty(\Omega;H)), & \quad L^\infty_p(\Omega;L^p(0,T;H)), \\
L^\infty_p(\Omega; L^\infty(0,T;H)), & \quad L^\infty_p(0,T;L^\infty(\Omega;H)), & \quad L^\infty_p(\Omega;C([0,T];H)).
\end{align*}
\]

We have

\[
L^p_p(\Omega; L^p(0,T;H)) = L^p_p(0,T;L^p(\Omega;H)) \equiv L^p_p(0,T;H), \quad p \in [1, \infty],
\]
Hereafter, \( \| \cdot \|_{L^p(\Omega; L^p(\cdot; H))} \) satisfies the following:
\[
L^p_2(0, T; L^p(\cdot; H)) \subseteq L^p_2(\Omega; L^p(0, T; H)), \quad 1 \leq p \leq q < \infty.
\]

In particular,
\[
L^p_2(0, T; L^p(\cdot; H)) \subseteq L^p_2(\Omega; L^1(0, T; H)), \quad 1 \leq p \leq \infty.
\]

Also, we have that
\[
L^p_2(\Omega; C([0, T]; H)) \subseteq L^p_2(\Omega; L^\infty(0, T; H)) \subseteq L^\infty_2(0, T; L^p(\cdot; H)).
\]

In fact, for \( 1 \leq p \leq q < \infty \), by Minkowski’s integral inequality, we have
\[
\| \varphi(\cdot) \|_{L^p(0, T; L^p(\cdot; H))}^p = \left[ \int_0^T \left( \mathbb{E}[|\varphi(t)|^p] \right)^{\frac{1}{p}} dt \right]^{\frac{p}{q}} \leq \mathbb{E} \left( \int_0^T |\varphi(t)|^q dt \right)^{\frac{1}{q}} = \| \varphi(\cdot) \|_{L^p(\Omega; L^q(0, T; H))}^p.
\]

This gives the first inclusion in (2.2). Other cases can be proved similarly. Now, we introduce the following definition.

**Definition 2.1.** An \( \mathbb{F} \)-progressive measurable process \( u : [0, T] \times \Omega \to \mathbb{R}^m \) is called a feasible control of system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) if under \( u(\cdot) \), for any \( x \in \mathbb{R}^n \), system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) admits a unique strong solution \( X(\cdot) \in L^2_2(\Omega; C([0, T]; \mathbb{R}^n)) \) satisfying \( X(0) = x \). The set of feasible controls is denoted by \( \mathcal{U}(0, T] \).

Now, for the state equation (1.1), we introduce the following basic hypothesis.

**H1** The \( \mathbb{R}^{n \times n} \)-valued processes \( A(\cdot), C_1(\cdot), \ldots, C_d(\cdot) \) satisfy
\[
A(\cdot), C_1(\cdot), \ldots, C_d(\cdot) \in L^\infty_2(0, T; \mathbb{R}^{n \times n}).
\]

The \( \mathbb{R}^{n \times m} \)-valued processes \( B(\cdot), D_1(\cdot), \ldots, D_d(\cdot) \) are \( \mathbb{F} \)-progressively measurable.

The following result gives a big class of feasible controls for system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\), whose proof is standard.

**Proposition 2.2.** Let (H1) hold. Let \( u : [0, T] \times \Omega \to \mathbb{R}^n \) be \( \mathbb{F} \)-progressively measurable. Suppose the following holds:

\[
p \geq 1, \quad B(\cdot)u(\cdot) \in L^1_2(\Omega; L^1(0, T; \mathbb{R}^n)), \quad D_k(\cdot)u(\cdot) \in L^2_2(\Omega; L^2(0, T; \mathbb{R}^n)), \quad 1 \leq k \leq d.
\]

Then \( u(\cdot) \in \mathcal{U}(0, T] \), and the solution \( X(\cdot) \equiv X(\cdot; x, u(\cdot)) \) of (1.1) with initial state \( x \) under control \( u(\cdot) \) satisfies the following:

\[
\| X(\cdot) \|_{L^p(\Omega; C([0, T]; \mathbb{R}^n))} \leq K \left\{ |x| + \| B(\cdot)u(\cdot) \|_{L^p_1(\Omega; L^1(0, T; \mathbb{R}^n))} + \sum_{k=1}^d \| D_k(\cdot)u(\cdot) \|_{L^p_2(\Omega; L^2(0, T; \mathbb{R}^n))} \right\}.
\]

Hereafter, \( K > 0 \) will denote a generic constant which could be different from line to line. Further, if

\[
p \geq 2, \quad B(\cdot)u(\cdot) \in L^1_2(0, T; L^p(\cdot; \mathbb{R}^n)), \quad D_k(\cdot)u(\cdot) \in L^1_2(0, T; L^p(\cdot; \mathbb{R}^n)), \quad 1 \leq k \leq d,
\]

then \( u(\cdot) \in \mathcal{U}(0, T] \), and the following holds:

\[
\| X(\cdot) \|_{L^p_2(\Omega; C([0, T]; \mathbb{R}^n))} \leq K \left\{ |x| + \| B(\cdot)u(\cdot) \|_{L^p_1(0, T; L^p(\cdot; \mathbb{R}^n))} + \sum_{k=1}^d \| D_k(\cdot)u(\cdot) \|_{L^p_2(0, T; L^p(\cdot; \mathbb{R}^n))} \right\}.
\]
The above result leads us to the following definitions.

**Definition 2.3.** A control \( u(\cdot) \in \mathcal{U}[0,T] \) is said to be \( L^p \)-feasible (respectively, \( L^p \)-restricted feasible) for system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) if (2.6) (respectively, (2.8)) holds true. The set of \( L^p \)-feasible controls (respectively, \( L^p \)-restricted feasible controls) is denoted by \( \mathcal{U}^p[0,T] \) (respectively, \( \mathcal{U}_r^p[0,T] \)).

Now, let us introduce the following two sets of hypotheses.

**(H2)** For some \( p \in [1, \infty] \) and \( \sigma \in (2, \infty] \), the following hold:

\[
B(\cdot) \subseteq \begin{cases} 
L^p_\sigma(\Omega; L^\infty(0,T; \mathbb{R}^{n \times m})), & \rho \in (1, \infty), \sigma \in (2, \infty), \\
L^p_\rho(\Omega; L^2(0,T; \mathbb{R}^{n \times m})), & \rho \in (1, \infty), \sigma = \infty.
\end{cases}
\]

\[D_1(\cdot), \ldots, D_d(\cdot) \in L^p_\sigma(\Omega; L^p(0,T; \mathbb{R}^{n \times m})).\]

**(H2)’** For some \( p, \sigma \in (2, \infty] \), the following hold:

\[
B(\cdot) \subseteq \begin{cases} 
L^\infty(0,T; L^p(\Omega; \mathbb{R}^{n \times m})), & \rho \in (2, \infty), \sigma \in (2, \infty), \\
L^2(0,T; L^p(\Omega; \mathbb{R}^{n \times m})), & \rho \in (2, \infty), \sigma = \infty.
\end{cases}
\]

\[D_1(\cdot), \ldots, D_d(\cdot) \in L^2(0,T; L^p(\Omega; \mathbb{R}^{n \times m})).\]

In what follows, (H2) will be used for problems involving \( L^p \)-feasible controls and (H2)’ will be used for problems involving \( L^p \)-restricted feasible controls. We now denote the following set of controls:

\[
\mathcal{U}^{p,\rho,\sigma}[0,T] = \begin{cases} 
L^\infty(0,T; L^p(\Omega; \mathbb{R}^m)), & p \in [1, \rho), \rho \in (1, \infty), \sigma \in (2, \infty), \\
L^p(\Omega; L^\infty(0,T; \mathbb{R}^m)), & p \in [1, \rho), \rho = \infty, \sigma \in (2, \infty), \\
L^\infty(0,T; L^p(\Omega; \mathbb{R}^m)), & p \in [1, \rho), \rho \in (1, \infty), \sigma = \infty, \\
L^p(\Omega; L^2(0,T; \mathbb{R}^m)), & p \in [1, \rho), \rho = \sigma = \infty.
\end{cases}
\]

Clearly, as \( \rho \uparrow +\infty \) and/or \( \sigma \uparrow +\infty \), the set \( \mathcal{U}^{p,\rho,\sigma}[0,T] \) is getting larger and larger. Whereas, as \( p \uparrow \rho \), the set \( \mathcal{U}^{p,\rho,\sigma}[0,T] \) is getting smaller and smaller. Similarly, we introduce

\[
\mathcal{U}_r^{p,\rho,\sigma}[0,T] = \begin{cases} 
L^\infty(0,T; L^p(\Omega; \mathbb{R}^m)), & p \in [2, \rho), \rho, \sigma \in (2, \infty), \\
L^p_\sigma(0,T; L^p(\Omega; \mathbb{R}^m)), & p \in [2, \rho), \rho = \infty, \sigma \in (2, \infty), \\
L^2(0,T; L^p(\Omega; \mathbb{R}^m)), & p \in [2, \rho), \rho \in (2, \infty), \sigma = \infty, \\
L^p(0,T; L^\infty(\Omega; \mathbb{R}^m)), & p \in [2, \rho), \rho = \sigma = \infty.
\end{cases}
\]

We have the following proposition.

**(Proposition 2.4.** (i) Let (H1) and (H2) hold. Then

\[
(2.12) \quad \mathcal{U}^{p,\rho,\sigma}[0,T] \subseteq \mathcal{U}^p[0,T], \quad \forall p \in [1, \rho).
\]

(ii) Let (H1) and (H2)’ hold. Then

\[
(2.13) \quad \mathcal{U}_r^{p,\rho,\sigma}[0,T] \subseteq \mathcal{U}_r^p[0,T], \quad \forall p \in [2, \rho).
\]
Proof. (i) Let (H1) and (H2) hold. Let \( u(\cdot) \in U^{p,\rho,\sigma}[0,T] \). We only consider the case that \( p < \infty \) and \( \sigma < \infty \). Others can be proved similarly. We make the following calculations (noting \( \sigma > 2 \) and \( 1 \leq p < \rho \)),

\[
\|B(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^1([0,T,\mathbb{R}^n]))}^p = E\left( \int_0^T |B(t)u(t)|^p dt \right)^{\frac{p}{p}} \leq E\left[ \left( \int_0^T |B(t)|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \left( \int_0^T |u(t)|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \right]^{\frac{p}{p}} \leq E\left[ \left( \int_0^T |B(t)|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \right]^{\frac{p}{p}} E\left( \int_0^T |u(t)|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \right]^{\frac{p}{p}}
\]

(2.14)

\[
\equiv \|B(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^2([0,T,\mathbb{R}^n]))}^p \equiv \|B(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^\infty([0,T,\mathbb{R}^n]))}^p \equiv \|B(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^\infty((0,T,\mathbb{R}^n)))}^p ,
\]

and for each \( k = 1, \ldots, d \),

\[
\|D_k(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^2([0,T,\mathbb{R}^n]))}^p = E\left( \int_0^T |D_k(t)u(t)|^2 dt \right)^{\frac{p}{2}} \leq E\left[ \left( \int_0^T |D_k(t)|^p dt \right)^{\frac{p-2}{p}} \left( \int_0^T |u(t)|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \right]^{\frac{p}{p}} \leq E\left[ \left( \int_0^T |D_k(t)|^p dt \right)^{\frac{p-2}{p}} \right]^{\frac{p}{p}} E\left( \int_0^T |u(t)|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \right]^{\frac{p}{p}}
\]

(2.15)

\[
\equiv \|D_k(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^\infty([0,T,\mathbb{R}^n]))}^p \equiv \|D_k(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^\infty((0,T,\mathbb{R}^n)))}^p = \|D_k(\cdot)u(\cdot)\|_{L^p_x(\Omega,L^\infty((0,T,\mathbb{R}^n)))}^p .
\]

By Definition 2.3, \( u(\cdot) \in U^p[0,T] \), proving (i).

In the similar manner, we are able to prove (ii). \( \square \)

3 Exact Controllability

We now give a precise definition of \( L^p \)-exact controllability.

**Definition 3.1.** Let \([0,T] \subseteq U[0,T] \). System \([A(\cdot),C(\cdot);B(\cdot),D(\cdot)]\) is said to be \( L^p \)-**exactly controllable** by \([U[0,T]] \) on the time interval \([0,T] \), if for any \((x,\xi) \in \mathbb{R}^n \times L^p_{xT}(\Omega;\mathbb{R}^n)\), there exists a \( u(\cdot) \in U[0,T] \) such that the solution \( X(\cdot) \in L^p_x(\Omega;C([0,T];\mathbb{R}^n)) \) of (1.1) with \( X(0) = x \) satisfies \( X(T) = \xi \).

In the above, \([U[0,T]] \) could be \( U^q[0,T] \), or \( U^p[0,T] \) for some suitable \( q \geq 1 \), and also it could be \( U^{p,\rho,\sigma}[0,T] \), or \( U^{p,\rho,\sigma}[0,T] \). We emphasize that in defining the system to be \( L^p \)-exactly controllable (for \( p \geq 1 \)) by \([U[0,T]] \), we only require \( X(\cdot) \in L^2_x(\Omega;C([0,T];\mathbb{R}^n)) \) (since \([U[0,T]] \subseteq U[0,T] \)). Depending on the choice of \([U[0,T]] \), \( X(\cdot) \) might have better integrability/regularity but we do not require any better property than \( L^2_x(\Omega;C([0,T];\mathbb{R}^n)) \). We will see shortly that this gives us a great reflexibility.

3.1 The case \( D(\cdot) = 0 \)

In this subsection, we consider system \([A(\cdot),C(\cdot);B(\cdot),0] \), i.e., the state equation reads

\[
(3.1) \quad dX(t) = \left[ A(t)X(t) + B(t)u(t) \right] dt + \sum_{k=1}^d C_k(t)X(t) dW_k(t), \quad t \geq 0.
\]

Thus, the control \( u(\cdot) \) does not appear in the diffusion. When all the coefficients in the above are constants, it was shown in [1] that the system is approximately controllable (under some additional conditions) in the following sense: For any \((x,\xi) \in \mathbb{R}^n \times L^p_{xT}(\Omega;\mathbb{R}^n)\), and any \( \varepsilon > 0 \), there exists a \( u_\varepsilon(\cdot) \in L^p_x(0,T;\mathbb{R}^m) \equiv U^{p,\rho,\sigma}[0,T] \) such that the solution \( X(\cdot) \equiv X(\cdot;x,u_\varepsilon(\cdot)) \) with \( X(0) = x \) satisfies

\[
\|X(T) - \xi\|_{L^p_{xT}(\Omega,\mathbb{R}^n)} < \varepsilon.
\]

The following is our first result which improves the results of [1] significantly.
Theorem 3.2. Let \( D(\cdot) = 0 \) and (H1) hold. Let

\[
B(t)B(t)^T \geq \delta I > 0, \quad t \in [0, T], \text{ a.s.}
\]

for some \( \delta > 0 \). Then for any \( p > 1 \), system \([A(\cdot), C(\cdot); B(\cdot), 0]\) is \(L^p\)-exactly controllable on \([0, T]\) by \(U^p[0, T]\) with any \( q \in (1, p)\).

Proof. Consider the following system:

\[
\begin{aligned}
&dX(t) = v(t)dt + \sum_{k=1}^{d} C_k(t)X(t)dW_k(t), \quad t \geq 0, \\
&X(0) = x,
\end{aligned}
\]

with \( v(\cdot) \in L^q_2(\Omega; L^1(0, T; \mathbb{R}^n)) \), \( q > 1 \). Then the unique solution \( X(\cdot) \) satisfies

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |X(t)|^q \right] \leq K \left[ |x|^q + \mathbb{E}\left( \int_0^T |v(t)|dt \right)^q \right].
\]

Let \( \Phi(\cdot) \) be the solution to the following:

\[
\begin{aligned}
&d\Phi(t) = \sum_{k=1}^{d} C_k(t)\Phi(t)dW_k(t), \quad t \geq 0, \\
&\Phi(0) = I.
\end{aligned}
\]

Then \( \Phi(\cdot)^{-1} \) exists and satisfies the following:

\[
\begin{aligned}
&d[\Phi(t)^{-1}] = \sum_{k=1}^{d} \Phi(t)^{-1}C_k(t)^2dt - \sum_{k=1}^{d} \Phi(t)^{-1}C_k(t)dW_k(t), \quad t \geq 0, \\
&\Phi(0)^{-1} = I.
\end{aligned}
\]

Therefore, for any \( q \geq 1 \),

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |\Phi(t)|^q + \sup_{t \in [0, T]} |\Phi(t)^{-1}|^q \right] \leq K(T, q),
\]

with the constant \( K(T, q) \) depending on \( T \) and \( q \) (as well as the bound of \( C_k(\cdot) \)), and we have the following variation of constants formula for \( X(\cdot) \):

\[
X(t) = \Phi(t)x + \Phi(t)\int_0^t \Phi(s)^{-1}v(s)ds, \quad t \geq 0.
\]

Now, for any \( q \in (1, p) \), we want to choose some \( v(\cdot) \in L^q_2(\Omega; L^1(0, T; \mathbb{R}^n)) \) so that \( X(T) = \xi \) which is equivalent to the following:

\[
\Phi(T)^{-1}\xi - x = \int_0^T \hat{v}(s)ds,
\]

with

\[
\hat{v}(t) = \Phi(t)^{-1}v(t), \quad t \in [0, T].
\]

Since \( \xi \in L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \), for any \( \bar{q} \in (q, p) \), we have

\[
\mathbb{E}|\Phi(T)^{-1}\xi|^q \leq \left( \frac{\mathbb{E}|\Phi(T)^{-1}|^q}{\bar{q}} \right)^{\frac{\bar{q}}{q} - \frac{q}{p}} \left( \frac{\mathbb{E}|\xi|^p}{\bar{q}} \right)^{\frac{\bar{q}}{p} - \frac{q}{p}} \leq K \left( \mathbb{E}|\xi|^p \right)^{\frac{q}{p} - \frac{q}{p}}.
\]

Thus, \( \Phi(T)^{-1}\xi - x \in L^q_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \). Then, by [17, Theorem 3.1], we can find a \( \hat{v}(\cdot) \in L^q_2(\Omega; L^1(0, T; \mathbb{R}^n)) \) such that

\[
\Phi(T)^{-1}\xi - x = \int_0^T \hat{v}(s)ds.
\]
Define \( v(t) = \Phi(t)\tilde{v}(t), \quad t \in [0,T]. \)

Since \( q < \bar{q} \), one has
\[
\mathbb{E}\left( \int_0^T |v(t)|dt \right)^q \leq \mathbb{E}\left( \sup_{t \in [0,T]} |\Phi(t)| \int_0^T |\tilde{v}(t)|dt \right)^q \
\leq K\left[ \mathbb{E}\left( \int_0^T |\tilde{v}(t)|dt \right)^q \right]^{\frac{q}{\bar{q}}} = K\|	ilde{v}(\cdot)\|_{L_{\bar{q}}^q(\Omega;L^q(0,T;\mathbb{R}^n))}^{\frac{q}{\bar{q}}}.
\]

Now, we define \( u(t) = B(t)^\top [B(t)B(t)^\top]^{-1} [v(t) - A(t)X(t)] \), \( t \geq 0 \), with \( X(\cdot) \) defined by (3.3). Then
\[
A(t)X(t) + B(t)u(t) = v(t), \quad t \geq 0,
\]
which implies that
\[
X(t) = \Phi(t)x + \Phi(t)\int_0^t \Phi(s)^{-1} \left[ A(s)X(s) + B(s)u(s) \right] ds, \quad t \geq 0.
\]
This means that \( X(\cdot) \) is the solution to (3.1), corresponding to \( (x,u(\cdot)) \) with
\[
\mathbb{E}\left( \int_0^T |B(t)u(t)|dt \right)^q = K\left[ \mathbb{E}\left( \int_0^T |v(t) - A(t)X(t)|dt \right)^q \right]^{\frac{q}{\bar{q}}} \
\leq K\left[ \mathbb{E}\left( \int_0^T |v(t)|dt \right)^q + \mathbb{E}\left( \int_0^T |X(t)|dt \right)^q \right].
\]
Therefore, \( u(\cdot) \in \mathcal{U}[0,T] \) which makes \( X(T) = \xi \). This proves our conclusion.

Let us make some comments on the above result. To this end, let us define
\[
\mathbb{L}_T(u(\cdot)) = \int_0^T u(t)dt, \quad u(\cdot) \in L_{\bar{p}}^1(0,T;\mathbb{R}^n).
\]
Then a result from [17, Theorems 3.1, 3.2] tells us that
\[
(3.4) \quad \mathbb{L}_T\left( L_{\bar{p}}^p(\Omega; L^1(0,T;\mathbb{R}^n)) \right) \supseteq \mathbb{L}_T\left( L_{\bar{p}}^p(0,T; L^p(\Omega;\mathbb{R}^n)) \right) = L_{\bar{p}}^p(\Omega;\mathbb{R}^n), \quad \forall p \in [1,\infty),
\]
and
\[
(3.5) \quad \mathbb{L}_T\left( L_{\bar{p}}^p(0,T; L^p(\Omega;\mathbb{R}^n)) \right) \subset L_{\bar{p}}^p(\Omega;\mathbb{R}^n), \quad \forall p \in (1,\infty), \ q \in (1,\infty].
\]
Thus,
\[
(3.6) \quad \mathbb{L}_T\left( L_{\bar{p}}^p(\Omega; L^q(0,T;\mathbb{R}^n)) \right) \subset L_{\bar{p}}^p(\Omega;\mathbb{R}^n), \quad \forall 1 < p \leq q \leq \infty.
\]
In particular,
\[
(3.7) \quad \mathbb{L}_T\left( L_{\bar{p}}^p(\Omega; L^2(0,T;\mathbb{R}^n)) \right) \subset L_{\bar{p}}^p(\Omega;\mathbb{R}^n), \quad \forall 1 < p \leq 2.
\]
Let us look at an implication of the above. Consider a system of the following form:
\[
dX(t) = u(t)dt, \quad t \geq 0.
\]
For terminal state \( \xi \in L_{\bar{p}}^p(\Omega;\mathbb{R}^n) \), with \( p > 1 \), if one is only allowed to use the control from \( L_{\bar{p}}^p(\Omega; L^2(0,T;\mathbb{R}^n)) \), the above system is not controllable. Whereas, by Theorem 3.2, this system is \( L^p \)-exactly controllable.
on $[0, T]$ by $\mathcal{U}^q[0, T]$ for any $q \in (1, p)$. This is a main reason that we define the $L^p$-exact controllability allowing the control taken from a larger space than $L^p_p(\Omega; L^2(0, T; \mathbb{R}^n))$, and not restricting the state process $X(\cdot)$ to belong to $L^p_p(\Omega; C([0, T]; \mathbb{R}^n))$.

Next, we notice that in Theorem 3.2, condition (3.2) implies that

$$\text{rank } B(t) = n \leq m, \quad t \in [0, T], \text{ a.s.}$$

This means that the dimension of the control process is no less than that of the state process. Now, if

$$\text{rank } B(t) < n, \quad t \in [0, T], \text{ a.s.,}$$

which will always be the case if $m < n$, then for each $t \in [0, T]$, there exists a $\theta(t) \in \mathbb{R}^n \setminus \{0\}$ such that

$$\theta(t)^\top B(t) = 0.$$

The following gives a negative result for the exact controllability, under condition (3.9) with a little more regularity conditions on $\theta(\cdot)$ and $C(\cdot)$, which is essentially an extension of (3.6).

**Theorem 3.3.** Let $D(\cdot) = 0$ and (H1) hold. Suppose there exists a continuous differentiable function $\theta : [0, T] \rightarrow \mathbb{R}^n$, $|\theta(t)| = 1$, for all $t \in [0, T]$ such that (3.9) holds. Also, let

$$C_k(\cdot) \in L^p_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^{n \times n})), \quad 1 \leq k \leq d.$$

Then for any $p > 1$, $[A(\cdot), C(\cdot); B(\cdot), 0]$ is not $L^p$-exactly controllable on $[0, T]$ by $\mathcal{U}^p[0, T]$.

**Proof.** Let

$$0 = t_0 < t_1 < t_2 < \cdots, \quad t_\ell \rightarrow T.$$

Define

$$G_0 = \bigcup_{\ell=0}^{\infty} \left( t_\ell, \frac{t_\ell + t_{\ell+1}}{2} \right), \quad G_1 = \bigcup_{\ell=0}^{\infty} \left( \frac{t_\ell + t_{\ell+1}}{2}, t_{\ell+1} \right).$$

Take

$$\zeta_0, \zeta_1 \in \mathbb{R}^n, \quad |\theta(T)^\top \zeta_1 - \theta(T)^\top \zeta_0| = 1.$$

Set

$$\zeta(s) = \zeta_0 I_{G_0}(s) - \zeta_1 I_{G_1}(s), \quad s \in [0, T),$$

and

$$\xi = x + \int_0^T \sum_{k=1}^d \zeta(s) dW_k(s).$$

We claim that the above constructed $\xi$ cannot be hit by the state $X(T)$ from $X(0) = x$ under any $u(\cdot) \in \mathcal{U}^p[0, T]$. We show this by contradiction. Suppose otherwise, then for some $u(\cdot) \in \mathcal{U}^p[0, T]$, $X(\cdot)$ satisfies

$$\begin{cases}
  dX(s) = \left[ A(s)X(s) + B(s)u(s) \right] ds + C(s)X(s) dW(s), & s \in [0, T], \\
  X(0) = x, \quad X(T) = \xi.
\end{cases}$$

Hence,

$$d[\theta(t)^\top X(t)] = \left[ \theta(t)^\top A(t) + \dot{\theta}(t)^\top \right] X(t) dt + \sum_{k=1}^d \theta(t)^\top C_k(t) X(t) dW_k(t), \quad t \geq 0.$$

Now, let

$$\eta(t) = x + \int_0^t \sum_{k=1}^d \zeta(s) dW_k(s), \quad t \in [0, T].$$

Then

$$\begin{cases}
  d\eta(s) = \left[ A(s)\eta(s) + B(s)u(s) \right] ds + C(s)\eta(s) dW(s), & s \in [0, T], \\
  \eta(0) = x, \quad \eta(T) = \xi.
\end{cases}$$

By Proposition 1.2, we have $\eta(t) = \xi$ holds for some $t \in [0, T]$. This is a contradiction.\qed
Then

\[
\begin{aligned}
\int d\left[ \theta(t)^\top (X(t) - \eta(t)) \right] & = \left[ \theta(t)^\top A(t)X(t) + \dot{\theta}(t)^\top \left( X(t) - \eta(t) \right) \right] dt \\
& + \sum_{k=1}^d \theta(t)^\top \left( C_k(t)X(t) - \zeta(t) \right) dW_k(t), \quad t \in [0, T], \\
\theta(T)^\top (X(T) - \eta(T)) & = 0.
\end{aligned}
\]

(3.13)

By a standard estimate for BSDEs and Burkholder-Davis-Gundy inequality, one obtains

\[
\begin{aligned}
\mathbb{E} \left\{ \sup_{s \in [t, T]} \left| \theta(s)^\top \left( X(s) - \eta(s) \right) \right|^p + \left[ \int_t^T \left( \sum_{k=1}^d \theta(s)^\top \left( C_k(s)X(s) - \zeta(s) \right) \right)^2 ds \right]^{\frac{p}{2}} \right\} \\
& \leq K \mathbb{E} \left( \int_t^T |\theta(s)^\top A(s)X(s) + \dot{\theta}(s)^\top X(s) - \dot{\theta}(s)^\top \eta(s)| ds \right)^p \\
& \leq K(T-t)^p \mathbb{E} \left[ \sup_{s \in [t, T]} |X(s)|^p + \sup_{s \in [t, T]} |\eta(s)|^p \right] \\
& \leq K(T-t)^p \left[ x^p + \mathbb{E} \left( \int_t^T |B(s)u(s)| ds \right)^p + \left( \int_0^T |\zeta(s)|^2 ds \right)^{\frac{p}{2}} \right].
\end{aligned}
\]

(3.14)

On the other hand,

\[
\begin{aligned}
\mathbb{E} \left[ \sup_{s \in [t, T]} |X(s) - \xi|^p \right] & \leq K \max \{|T-t|^p, |T-t|\hat{\tau}\} \left[ x^p + \mathbb{E} \left( \int_t^T |B(s)u(s)| ds \right)^p \right] \\
& + K \mathbb{E} \left( \int_t^T |B(s)u(s)| ds \right)^p,
\end{aligned}
\]

(3.15)

and for any \( f, g, h \in L^2(t, T; \mathbb{R}) \),

\[
\| f - g \|_{L^2(t, T; \mathbb{R})}^p \geq 2^{-p} \| f - h \|_{L^2(t, T; \mathbb{R})}^p - \| h - g \|_{L^2(t, T; \mathbb{R})}^p.
\]

Thus, for each \( k = 1, 2, \cdots, d \),

\[
\begin{aligned}
\mathbb{E} \left[ \int_{t_k}^T |\theta(s)^\top (C_k(s)X(s) - \zeta(s))|^2 ds \right]^{\frac{p}{2}} & \geq 2^{-4(p-1)} \mathbb{E} \left( \int_{t_k}^T |\theta(T)^\top [C_k(T)\xi - \zeta(s)]|^2 ds \right)^{\frac{p}{2}} \\
& - 2^{-4(p-1)} \mathbb{E} \left( \int_{t_k}^T \left| \theta(T)^\top \left( C_k(T)\xi + |\theta(T)^\top \zeta_1 - \theta(T)^\top C_k(T)\xi|^2 \right) \right|^2 ds \right)^{\frac{p}{2}} \\
& - 2^{-4(p-1)} \mathbb{E} \left( \int_{t_k}^T \left( \dot{\theta}(s)^\top \left( C_k(T)\xi \right) \right)^2 ds \right)^{\frac{p}{2}} \\
& - 2^{-4(p-1)} \mathbb{E} \left( \int_{t_k}^T \left( \theta(s)^\top - \theta(T)^\top \right)^2 C_k(T)\xi \right)^2 ds \right)^{\frac{p}{2}} \\
& \geq 2^{-4(p-1)} \left( \frac{T-t_k}{2} \right)^{\frac{p}{2}} \mathbb{E} \left[ \theta(T)^\top \zeta_0 - \theta(T)^\top C_k(T)\xi + |\theta(T)^\top \zeta_1 - \theta(T)^\top C_k(T)\xi|^2 \right]^{\frac{p}{2}} \\
& - 2^{-3(p-1)} \left( T-t_k \right)^{\frac{p}{2}} \mathbb{E} \left( \int_{t_k}^T |\zeta(s)|^2 ds \right)^{\frac{p}{2}} \\
& - 2^{-2(p-1)} \left( T-t_k \right)^{\frac{p}{2}} \mathbb{E} \left( \sup_{s \in [t_k, T]} |\theta(s) - \theta(T)|^p C_k(T)\xi \right) \\
& - 2^{-p} \left( T-t_k \right)^{\frac{p}{2}} \mathbb{E} \left( \sup_{s \in [t_k, T]} |C_k(T) - C_k(s)|^p \right) \\
& - \left( T-t_k \right)^{\frac{p}{2}} \mathbb{E} \left( \sup_{s \in [t_k, T]} |X(s) - \xi|^p \right).
\end{aligned}
\]
Since $C_k(\cdot) \in L_{\mathbb{F}}^\infty(\Omega; C([0,T]; \mathbb{R}^{n \times n}))$, by Lebesgue’s dominated convergence theorem, we have
\[
\mathbb{E}\left[ \sup_{s \in [t,T]} |C_k(T) - C_k(s)|^p \right] = o(1), \quad \text{a.s., } i \to \infty.
\]
From (3.15),
\[
\mathbb{E}\left[ \sup_{s \in [t,T]} |X(s) - \xi|^p \right] = o(1), \quad \text{a.s., } i \to \infty.
\]
The other two negative terms can be estimated similarly. Consequently, making use of (3.14),
\[
2^{-4(p-1)} \left( \frac{T - t_i}{2} \right)^\frac{p}{2} \mathbb{E}\left[ \theta(T)^T \zeta_0 - \theta(T)^T C_k(T) \xi_1 + |\theta(T)^T \zeta_1 - \theta(T)^T C_k(T) \xi|^2 \right] \leq \mathbb{E}\left( \int_{t_i}^T |\theta(s)^T C_k(s) X(s) - \theta(s)^T \zeta(s)|^2 ds \right)^\frac{p}{2} + o\left( |T - t_i|^{\frac{p}{2}} \right) \quad \text{a.s., } i \to \infty.
\]
This leads to
\[
\mathbb{E}\left( \theta(T)^T \zeta_0 - \theta(T)^T C_k(T) \xi_1 + |\theta(T)^T \zeta_1 - \theta(T)^T C_k(T) \xi|^2 \right)^\frac{p}{2} = 0.
\]
Thus,
\[
|\theta(T)^T \zeta_0 - \theta(T)^T C_k(T) \xi_1 + |\theta(T)^T \zeta_1 - \theta(T)^T C_k(T) \xi|^2 = 0, \quad \text{a.s.,}
\]
which is a contradiction since $|\theta(T)^T (\zeta_0 - \zeta_1)| = 1$. The above implies that the terminal state $\xi$ constructed above cannot be hit by the state under any $u(\cdot) \in \mathcal{U}^p[0,T]$. This completes the proof.

3.2 The case $D(\cdot)$ is surjective

In this subsection, we let $d = 1$. The case $d > 1$ can be discussed similarly. For system $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$, we assume the following:
\[
D(t)D(t)^\top \geq \delta I, \quad \text{a.s., a.e. } t \in [0,T].
\]
In this case, $[D(t)D(t)^\top]^{-1}$ exists and uniformly bounded. We define
\[
\begin{align*}
\hat{A}(t) &= A(t) - B(t)D(t)^\top [D(t)D(t)^\top]^{-1} C(t), \\
\hat{B}(t) &= B(t) \{ I - D(t)^\top [D(t)D(t)^\top]^{-1} D(t) \}, \\
\hat{D}(t) &= B(t)D(t)^\top [D(t)D(t)^\top]^{-1},
\end{align*}
\]
and introduce the following controlled system:
\[
\begin{align*}
dX(t) &= \left[ \hat{A}(t)X(t) + \hat{B}(t)v(t) + \hat{D}(t)Z(t) \right] dt + Z(t) dW(t), \quad t \in [0,T], \\
X(0) &= x,
\end{align*}
\]
with $X(\cdot)$ being the state and $(v(\cdot), Z(\cdot))$ being the control. Using our notation, the above system can be denoted by $[\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]$. Comparing $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ with $[\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]$, the latter has a simpler structure. For system $[\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]$, we need the following set:
\[
\hat{\mathcal{U}}^p[0,T] = \left\{ v(\cdot) \mid \hat{B}(\cdot)v(\cdot) \in L_{\mathbb{F}}^p(\Omega; L^1(0,T; \mathbb{R}^n)) \right\}.
\]
The following result is a kind of reduction.

**Theorem 3.4.** Let (H1) and (3.16) hold. Let $\hat{A}(\cdot), \hat{B}(\cdot), \hat{D}(\cdot)$ be defined by (3.17). Suppose
\[
\hat{A}(\cdot) \in L_{\mathbb{F}}^\infty(\Omega; L^{1+\varepsilon}(0,T; \mathbb{R}^{n \times n})), \quad \hat{D}(\cdot) \in L_{\mathbb{F}}^\infty(\Omega; L^2(0,T; \mathbb{R}^{n \times n})),
\]
where $\varepsilon > 0$ is a given constant. Then system $[\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]$ is $L^p$-exactly controllable on $[0,T]$ by $\hat{\mathcal{U}}^p[0,T] \times L_{\mathbb{F}}^2(\Omega; L^2(0,T; \mathbb{R}^n))$ if and only if system $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is $L^p$-exactly controllable on $[0,T]$ by $\mathcal{U}^p[0,T]$. 

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Proof. \((\Rightarrow)\). First of all, we note that for any \(Z(\cdot) \in L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))\),

\begin{equation}
\mathbb{E}\left(\int_0^T |\tilde{D}(t)Z(t)| dt\right)^p \leq \mathbb{E}\left[\left(\int_0^T |\tilde{D}(t)|^2 dt\right)^\frac{p}{2}\left(\int_0^T |Z(t)|^2 dt\right)^\frac{p}{2}\right] \\
\leq \|\tilde{D}(\cdot)\|_{L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))}^p \mathbb{E}\left(\int_0^T |Z(t)|^2 dt\right)^\frac{p}{2}.
\end{equation}

Thus, under condition (3.19), for any \(x \in \mathbb{R}^n\) and \((v(\cdot), Z(\cdot)) \in \hat{U}^p[0,T] \times L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))\), system (3.18) admits a unique solution.

Now, if system \([\tilde{A}(\cdot), 0; (\tilde{B}(\cdot), \tilde{D}(\cdot)), (0, I)]\) is \(L^p\)-exactly controllable on \([0,T]\) by \(\hat{U}^p[0,T] \times L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))\), then for any \(x \in \mathbb{R}^n\) and \(\xi \in L^p_p(\Omega; \mathbb{R}^n)\), there exists a triple \((X(\cdot), v(\cdot), Z(\cdot)) \in L^p_p(\Omega; C([0,T]; \mathbb{R}^n)) \times \hat{U}^p[0,T] \times L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))\) such that

\begin{equation}
\begin{cases}
    dX(t) = \left[\tilde{A}(t)X(t) + \tilde{B}(t)v(t) + \tilde{D}(t)Z(t)\right] dt + Z(t) dW(t), & t \in [0,T], \\
    X(0) = x, & X(T) = \xi.
\end{cases}
\end{equation}

Define

\[ u(t) = D(t)^\top [D(t)D(t)^\top]^{-1} |Z(t) - C(t)X(t)| + |I - D(t)^\top [D(t)D(t)^\top]^{-1} D(t)| v(t), \quad t \in [0,T]. \]

We have

\begin{equation}
B(t)u(t) = \tilde{D}(t)[Z(t) - C(t)X(t)] + \tilde{B}(t)v(t), \quad t \in [0,T].
\end{equation}

Since \(v(\cdot) \in \hat{U}^p[0,T]\) and

\begin{equation}
\mathbb{E}\left(\int_0^T |\tilde{D}(t)(Z(t) - C(t)X(t))| dt\right)^p \leq \mathbb{E}\left[\left(\int_0^T |\tilde{D}(t)|^2 dt\right)^\frac{p}{2}\left(\int_0^T |Z(t) - C(t)X(t)|^2 dt\right)^\frac{p}{2}\right] \\
\leq \|\tilde{D}(\cdot)\|_{L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))}^p \|Z(\cdot) - C(\cdot)X(\cdot)\|_{L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))}^p \leq \infty,
\end{equation}

one has \(B(\cdot)u(\cdot) \in L^p_p(\Omega; L^1(0,T;\mathbb{R}^n))\). Further, by

\begin{equation}
D(t)u(t) = Z(t) - C(t)X(t), \quad t \in [0,T],
\end{equation}

we obtain \(D(\cdot)u(\cdot) \in L^p_p(\Omega; L^2(0,T;\mathbb{R}^n))\). Therefore \(u(\cdot) \in U^p[0,T]\). From (3.22) and (3.23), it is easy to see

\begin{equation}
\begin{cases}
    A(t)X(t) + B(t)u(t) = \tilde{A}(t)X(t) + \tilde{B}(t)v(t) + \tilde{D}(t)Z(t), & t \in [0,T], \\
    C(t)X(t) + D(t)u(t) = Z(t),
\end{cases}
\end{equation}

and thus, (3.21) reads

\begin{equation}
\begin{cases}
    dX(t) = \left[A(t)X(t) + B(t)u(t)\right] dt + \left[C(t)X(t) + D(t)u(t)\right] dW(t), & t \in [0,T], \\
    X(0) = x, & X(T) = \xi.
\end{cases}
\end{equation}

This proves the \(L^p\)-exact controllability of system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) on \([0,T]\) by \(U^p[0,T]\).

\((\Leftarrow)\). If system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) is \(L^p\)-exactly controllable on \([0,T]\) by \(U^p[0,T]\), then for any \(x \in \mathbb{R}^n\) and \(\xi \in L^p_p(\Omega; \mathbb{R}^n)\), there exists a pair \((X(\cdot), u(\cdot)) \in L^p_p(\Omega; C([0,T]; \mathbb{R}^n)) \times U^p[0,T]\) such that

\begin{equation}
\begin{cases}
    dX(t) = \left[A(t)X(t) + B(t)u(t)\right] dt + \left[C(t)X(t) + D(t)u(t)\right] dW(t), & t \in [0,T], \\
    X(0) = x, & X(T) = \xi.
\end{cases}
\end{equation}
Let
\[
\begin{cases}
Z(t) = C(t)X(t) + D(t)u(t), \\
v(t) = u(t),
\end{cases} \quad t \in [0, T].
\]

Then \( Z(\cdot) \in L^p(\Omega; L^2(0, T; \mathbb{R}^n)) \) and
\[
u(t) = D(t)^T[D(t)D(t)^T]^{-1}[Z(t) - C(t)X(t)] + [I - D(t)^T[D(t)D(t)^T]^{-1}D(t)]v(t), \quad t \in [0, T].
\]

Further,
\[
B(t)u(t) = B(t)D(t)^T[D(t)D(t)^T]^{-1}[Z(t) - C(t)X(t)] + B(t)[I - D(t)^T[D(t)D(t)^T]^{-1}D(t)]v(t)
= \hat{D}(t)Z(t) + \left[\hat{A}(t) - A(t)\right]X(t) + \hat{B}(t)v(t), \quad t \in [0, T].
\]

Consequently,
\[
\begin{align*}
\mathbb{E}\left(\int_0^T |\hat{B}(t)v(t)|^p dt\right)^\frac{1}{p} &\leq 3^{p-1}\mathbb{E}\left[\left(\int_0^T |\hat{D}(t)Z(t)|^p dt\right)^\frac{1}{p} + \left(\int_0^T |\hat{A}(t) - A(t)|X(t)|^p dt\right)^\frac{1}{p}
+ \left(\int_0^T |B(t)u(t)|^p dt\right)^\frac{1}{p}\right] \\
&\leq K\mathbb{E}\left[\left(\int_0^T |\hat{D}(t)Z(t)|^2 dt\right)^\frac{p}{2} + \sup_{t \in [0, T]} |X(t)|^p + \left(\int_0^T |B(t)u(t)|^p dt\right)^\frac{1}{p}\right] \\
&\leq K\mathbb{E}\left[|x|^p + \mathbb{E}\left(\int_0^T |Z(t)|^2 dt\right)^{p\frac{1}{2}} + \mathbb{E}\left(\int_0^T |B(t)u(t)|^p dt\right)^{\frac{1}{p}} + \mathbb{E}\left(\int_0^T |D(t)u(t)|^2 dt\right)^{\frac{1}{2}}\right].
\end{align*}
\]

Thus, \( v(\cdot) \in \hat{U}^p[0, T] \). Also,\[
\begin{align*}
\hat{A}(t)X(t) + \hat{B}(t)v(t) + \hat{D}(t)Z(t)
&= \left[A(t) - B(t)D(t)^T[D(t)D(t)^T]^{-1}C(t)\right]X(t) + B(t)\left[I - D(t)^T[D(t)D(t)^T]^{-1}D(t)\right]u(t)
+ B(t)D(t)^T[D(t)D(t)^T]^{-1}[C(t)X(t) + D(t)u(t)] \\
&= A(t)X(t) + B(t)u(t), \quad t \in [0, T].
\end{align*}
\]

Hence, (3.24) can be rewritten as
\[
\begin{align*}
dX(t) &= \left[\hat{A}(t)X(t) + \hat{B}(t)v(t) + \hat{D}(t)Z(t)\right]dt + Z(t)dW(t), \quad t \in [0, T], \\
X(0) &= x, \quad X(T) = \xi.
\end{align*}
\]

This proves the \( L^p \)-exact controllability of system \( [\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)] \) on \([0, T]\) by \( \hat{U}^p[0, T] \times L^p(\Omega; L^2(0, T; \mathbb{R}^n)) \).

Now, for system \( [\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)] \), similar to the definition of exact controllability, we introduce the following definition.

**Definition 3.5.** System \( [\hat{A}(\cdot), 0, (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)] \) is said to be exactly null-controllable by \( \hat{U}^p[0, T] \times L^p(\Omega; L^2(0, T; \mathbb{R}^n)) \) on the time interval \([0, T]\), if for any \( x \in \mathbb{R}^n \), there exists a pair \((v(\cdot), Z(\cdot)) \in \hat{U}^p[0, T] \times L^p(\Omega; L^2(0, T; \mathbb{R}^n))\) such that the solution \( X(\cdot) \) to (3.21) under \((v(\cdot), Z(\cdot))\) satisfies \( X(T) = 0 \).

We have the following result for system \( [\hat{A}(\cdot), 0, (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)] \).

**Theorem 3.6.** Let (H1) and (3.16) hold. Let \( \hat{A}(\cdot), \hat{B}(\cdot), \hat{D}(\cdot) \) be defined by (3.17). Suppose
\[
\begin{align*}
\hat{A}(\cdot) &\in L^{2p}(\Omega; L^{1+\epsilon}(0, T; \mathbb{R}^{n \times n})), \quad \hat{B}(\cdot) \in L^{(2\epsilon p) + \epsilon}(\Omega; L^2(0, T; \mathbb{R}^{n \times n})), \\
\hat{D}(\cdot) &\in L^{2p}(\Omega; L^{2+\epsilon}(0, T; \mathbb{R}^{n \times n})).
\end{align*}
\]
where \(2 \vee p \equiv \max\{2, p\}\) and \(\varepsilon > 0\) is a given constant. Then the following are equivalent:

(i) \([\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]\) is \(L^p\)-exactly controllable on \([0, T]\) by \(\hat{U}^p[0, T] \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))\);

(ii) \([\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]\) is exactly null-controllable on \([0, T]\) by \(\hat{U}^p[0, T] \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))\);

(iii) Matrix \(G\) defined below is invertible:

\[
G = \mathbb{E} \int_0^T \gamma(t) \hat{B}(t) \hat{B}(t)\mathbb{T} \gamma(t)\mathbb{T} dt,
\]

where \(\gamma(\cdot)\) is the adapted solution to the following FSDE:

\[
\begin{aligned}
\begin{cases}
d\gamma(t) = -\gamma(t) \hat{A}(t) dt - \gamma(t) \hat{D}(t) dW(t), & t \geq 0, \\
\gamma(0) = I.
\end{cases}
\end{aligned}
\]

**Proof.** (i) \(\Rightarrow\) (ii) is trivial.

(ii) \(\Rightarrow\) (iii). First of all, under (3.25), FSDE (3.27) admits a unique strong solution \(\gamma(\cdot) \in \bigcap_{q \geq 1} L^q_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^{n \times n}))\) and the Matrix \(G\) is well defined. Now we prove the conclusion by contradiction. Suppose matrix \(G\) is not invertible, then there exists a vector \(0 \neq \beta \in \mathbb{R}^n\) such that

\[
0 = \beta^T G \beta = \mathbb{E} \int_0^T \beta^T \gamma(t) \hat{B}(t) \hat{B}(t)^\mathbb{T} \gamma(t)^\mathbb{T} \beta dt = \mathbb{E} \int_0^T \hat{B}(t)^\mathbb{T} \gamma(t)^\mathbb{T} \beta^2 dt.
\]

Therefore

\[
\beta^T \gamma(t) \hat{B}(t) = 0, \quad \text{a.e. } t \in [0, T], \quad \text{a.s.}
\]

Now, we claim that by choosing \(x = \beta \in \mathbb{R}^n\), there will be no \((v(\cdot), Z(\cdot)) \in \hat{U}^p[0, T] \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))\) such that the corresponding state process \(X(\cdot) \equiv X(\cdot; x, v(\cdot), Z(\cdot))\) satisfies

\[
X(0) = x, \quad X(T) = 0
\]

In fact, suppose there exists a pair \((v(\cdot), Z(\cdot)) \in \hat{U}^p[0, T] \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))\) such that the above is true. Then applying the Itô’s formula to \(\gamma(t) X(t)\) on the interval \([0, T]\), one obtains the following relationship:

\[
-\beta = \gamma(T) X(T) - X(0) = \int_0^T \gamma(t) \hat{B}(t) v(t) dt + \int_0^T \left[ \gamma(t) Z(t) - \gamma(t) \hat{D}(t) X(t) \right] dW(t).
\]

It is easy to check that

\[
\mathbb{E} \left( \int_0^T |\gamma(t) Z(t) - \gamma(t) \hat{D}(t) X(t)|^2 dt \right)^{1/2} < \infty.
\]

Thus,

\[
-\beta = \mathbb{E} \int_0^T \gamma(t) \hat{B}(t) v(t) dt.
\]

Making use of (3.28), we get

\[ -|\beta|^2 = \mathbb{E} \int_0^T \beta^T \gamma(t) \hat{B}(t) v(t) dt = 0, \]

a contradiction.
(iii) $\Rightarrow$ (i). Under (3.25), for any given $\xi \in L^p_{\mathbb{F}}(\Omega; \mathbb{R}^n)$, the following BSDE

$$\begin{cases}
    dX_1(t) = \left[\hat{A}(t)X_1(t) + \hat{D}(t)Z_1(t)\right]dt + Z_1(t)dW(t), & t \in [0, T], \\
    X_1(T) = \xi
\end{cases}$$

admits a unique adapted solution $(X_1(\cdot), Z_1(\cdot)) \in L^p_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$. Since $G$ is invertible, for any $x \in \mathbb{R}^n$, we may define

$$v(t) = -\hat{B}(t)^T \gamma(t)^T G^{-1} [x - X_1(0)], \quad t \in [0, T].$$

Note that $\hat{B}(\cdot) \in L^p_{\mathbb{F}}([2p+\varepsilon]; \mathbb{R}^n \times \mathbb{R}^m)$ leads to the following:

$$\begin{align*}
    \mathbb{E} \left( \int_0^T |\hat{B}(t)v(t)| dt \right)^p &\leq K \mathbb{E} \left( \int_0^T |\hat{B}(t)|^2 |\gamma(t)| dt \right)^p \\
    &\leq K \left[ \mathbb{E} \left( \int_0^T |\hat{B}(t)|^2 dt \right)^{p+\varepsilon} \right]^{\varepsilon/\varepsilon} < \infty,
\end{align*}$$

which implies $v(\cdot) \in \mathcal{H}^p[0, T]$. For this $v(\cdot)$, we define $(X_2(\cdot), Z_2(\cdot)) \in L^p_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$ to be the unique adapted solution of the following BSDE:

$$\begin{cases}
    dX_2(t) = \left[\hat{A}(t)X_2(t) + \hat{B}(t)v(t) + \hat{D}(t)Z_2(t)\right]dt + Z_2(t)dW(t), & t \in [0, T], \\
    X_2(T) = 0.
\end{cases}$$

Applying Itô’s formula to $\gamma(\cdot)X_2(\cdot)$, we have (comparing with (3.29))

$$\begin{align*}
    -X_2(0) &= \gamma(T)X_2(T) - X_2(0) \\
    &= \int_0^T \gamma(t)\hat{B}(t)v(t)dt + \int_0^T \left[\gamma(t)Z_2(t) - \gamma(t)\hat{D}(t)X_2(t)\right]dW(t) \\
    &= -\left\{ \int_0^T \mathbb{E} \left[ \gamma(t)\hat{B}(t)v(t)\gamma(t)^T \right] dt \right\} G^{-1} [x - X_1(0)] = -[x - X_1(0)],
\end{align*}$$

which implies $X_1(0) + X_2(0) = x$. Now, we define

$$X(t) = X_1(t) + X_2(t), \quad Z(t) = Z_1(t) + Z_2(t), \quad t \in [0, T].$$

Then, by linearity, we see that $(X(\cdot), v(\cdot), Z(\cdot))$ satisfies the following:

$$\begin{cases}
    dX(t) = \left[\hat{A}(t)X(t) + \hat{B}(t)v(t) + \hat{D}(t)Z(t)\right]dt + Z(t)dW(t), & t \in [0, T], \\
    X(0) = x, \quad X(T) = \xi.
\end{cases}$$

This means that system $[\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]$ is $L^p$-exactly controllable on $[0, T]$ by $\mathcal{H}^p[0, T] \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$.

The above result is essentially due to Liu–Peng [16]. We have re-organized the way presenting the result. It is worthy of pointing out that, unlike in [16], we allow the coefficients to be unbounded and allow $p$ to be different from 2. Combining Theorems 3.4 and 3.6, we have the following result.

**Theorem 3.7.** Let (H1), (3.16) and (3.25) hold with $\hat{A}(\cdot), \hat{B}(\cdot)$ and $\hat{D}(\cdot)$ defined by (3.17). Then system $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is $L^p$-exactly controllable on $[0, T]$ by $\mathcal{U}^p[0, T]$ if and only if $G$ defined by (3.26) is invertible.

As a simple corollary of the above, we have the following result for the case of deterministic coefficients.
Corollary 3.8. Let (H1), (3.16) and (3.25) hold. Let \( \hat{A}(\cdot) \) and \( \hat{B}(\cdot) \) be deterministic. Let \( \hat{\Phi}(\cdot) \) be the solution to the following ODE:

\[
\begin{align*}
(3.31) & \quad d\hat{\Phi}(t) = \hat{A}(t)\hat{\Phi}(t)dt, \quad t \geq 0, \\
& \quad \hat{\Phi}(0) = I.
\end{align*}
\]

Denote

\[
\Psi = \int_0^T \hat{\Phi}(s)^{-1} \hat{B}(s) \left( \hat{\Phi}(s)^{-1} \hat{B}(s) \right)^\top ds.
\]

Suppose that \( \Psi \) is invertible. Then system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) is \(L^p\)-exactly controllable on \([0, T]\) by \(U^p[0, T]\).

Proof. In the current case, we have

\[
\hat{\Phi}(t)^{-1} = E\mathcal{Y}(t), \quad t \in [0, T].
\]

On the other hand,

\[
0 \leq E \left[ \left( \mathcal{Y}(t) - E\mathcal{Y}(t) \right) \hat{B}(t)\hat{B}(t)^\top \left( \mathcal{Y}(t) - E\mathcal{Y}(t) \right)^\top \right] = E \left[ \mathcal{Y}(t)\hat{B}(t)\hat{B}(t)^\top \mathcal{Y}(t) - E\mathcal{Y}(t)\hat{B}(t)\hat{B}(t)^\top E\mathcal{Y}(t) \right].
\]

Hence,

\[
G \geq \Psi.
\]

Then our conclusion follows from the above theorem. \(\square\)

The invertibility of matrix \( G \) defined by (3.26) gives a nice criterion for the \(L^p\)-exact controllability of system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) through the \(L^p\)-exact controllability of system \([\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]\). However, unless \( n = 1 \), in the case of random coefficients, the solution \( \mathcal{Y}(\cdot) \) of FSDE (3.27) does not have a relatively simple (explicit) form. Thus, the applicability of condition (iii) in Theorem 3.6 is somehow limited. In the rest of this subsection, we will present another sufficient condition for the \(L^p\)-exact controllability of \([\hat{A}(\cdot), 0; (\hat{B}(\cdot), \hat{D}(\cdot)), (0, I)]\) which might have a better applicability.

Now, with random coefficients, we still let \( \hat{\Phi}(\cdot) \) be the solution to (3.31) which is a random ODE. Presumably, \( \hat{\Phi}(\cdot) \) is easier to get than \( \mathcal{Y}(\cdot) \) (the solution of (3.27)). Define

\[
\tilde{D}(t) = \hat{\Phi}(t)^{-1} \hat{D}(t)\hat{\Phi}(t), \quad t \in [0, T],
\]

and introduce the following mean-field stochastic Fredholm integral equation of first kind:

\[
(3.32) \quad Y(t) = \zeta + \Lambda \int_\tau^t E_\tau \left[ \tilde{D}(s)\tilde{Z}(s) \right] ds - \int_t^\tau \tilde{D}(s)\tilde{Z}(s) ds - \int_t^\tau \tilde{Z}(s) dW(s), \quad t \in [\tau, \bar{\tau}],
\]

where \( E_\tau [\cdot] = E[\cdot | \mathcal{F}_\tau] \). We have the following result.

Lemma 3.9. Let the following hold:

\[
(3.33) \quad \hat{A}(\cdot) \in L^\infty_\mathcal{F}(\Omega; L^{1+\varepsilon}(0, T; \mathbb{R}^{n \times n})), \quad \hat{D}(\cdot) \in L^\infty_\mathcal{F}(\Omega; L^{2+\varepsilon}(0, T; \mathbb{R}^{n \times n})),
\]

for some \( \varepsilon > 0 \). Then there exists a positive constant \( \varepsilon' \) depending only on \( \hat{D}(\cdot) \) and \( \Lambda \), such that for any \( 0 \leq \tau < \bar{\tau} \leq T \), any \( \zeta \in L^\infty_\mathcal{F}(\Omega; \mathbb{R}^n) \) and \( \Lambda \in L^\infty_\mathcal{F}(\Omega; \mathbb{R}^{n \times n}) \), satisfying \( \bar{\tau} - \tau \leq \varepsilon' \), (3.32) admits a unique solution \((Y(\cdot), \tilde{Z}(\cdot)) \in L^2_\mathcal{F}(\Omega; C([\tau, \bar{\tau}]; \mathbb{R}^n)) \times L^2_\mathcal{F}(\Omega; L^2(\tau, \bar{\tau}; \mathbb{R}^n)) \).
Proof. Let \( 0 \leq \tau < \bar{\tau} \leq T \), and let \( \widehat{z}(\cdot) \in L^p_\mathcal{F}(\Omega; L^2(\tau, \bar{\tau}; \mathbb{R}^n)) \) be given and consider the following BSDE:

\[
Y(t) = \zeta + \Lambda \int_{\tau}^T \mathbb{E}_\tau \left[ \tilde{D}(s) \widehat{z}(s) \right] ds - \int_{\tau}^T \tilde{D}(s) \tilde{Z}(s) ds - \int_{\tau}^T \tilde{Z}(s) dW(s), \quad t \in [\tau, \bar{\tau}].
\]

On the right hand side of the above, the sum of the first two terms are treated as the terminal state. By the standard theory of BSDEs ([5]), we know that the above BSDE admits a unique adapted solution \( (Y(\cdot), \tilde{Z}(\cdot)) \) and the following estimate holds:

\[
\mathbb{E} \left[ \sup_{t \in [\tau, \bar{\tau}]} |Y(t)|^p + \left( \int_{\tau}^T |\tilde{Z}(s)|^2 ds \right)^{\frac{p}{2}} \right] \leq K \mathbb{E} \left[ |\zeta| + \Lambda \int_{\tau}^T \mathbb{E}_\tau [\tilde{D}(s) \tilde{Z}(s)] ds \right]^p.
\]

Thus, for \( \widehat{z}_1(\cdot), \widehat{z}_2(\cdot) \in L^p_\mathcal{F}(\Omega; L^2(\tau, \bar{\tau}; \mathbb{R}^n)) \), if we let \( (Y_1(\cdot), \tilde{Z}_1(\cdot)) \) and \( (Y_2(\cdot), \tilde{Z}_2(\cdot)) \) be the corresponding adapted solutions, then, noting that both \( \widehat{\Phi}(\cdot) \) and \( \widehat{\Phi}(\cdot)^{-1} \) are bounded, and \( \tilde{D}(\cdot) \in L^p_\mathcal{F}(\Omega; L^{2+\varepsilon}(0, T; \mathbb{R}^{n \times n})) \), one has

\[
\mathbb{E} \left[ \sup_{t \in [\tau, \bar{\tau}]} |Y_1(t) - Y_2(t)|^p + \left( \int_{\tau}^T |\tilde{Z}_1(s) - \tilde{Z}_2(s)|^2 ds \right)^{\frac{p}{2}} \right] \leq K \mathbb{E} \left[ \left( \int_{\tau}^T |\tilde{D}(s)|^p \right) \right] \leq K \mathbb{E} \left[ \left( \int_{\tau}^T |\tilde{D}(s)|^p \right) \right] \leq K \sup_{\omega \in \Omega} \left( \int_{\tau}^T |\tilde{D}(s, \omega)|^2 ds \right)^{\frac{p}{2}} \mathbb{E} \left[ \left( \int_{\tau}^T |\tilde{Z}_1(s) - \tilde{Z}_2(s)|^2 ds \right)^{\frac{p}{2}} \right].
\]

Consequently, we can find an absolute constant \( \varepsilon' > 0 \) such that as long as \( 0 < \bar{\tau} - \tau \leq \varepsilon' \), the map \( \widehat{z}(\cdot) \mapsto \tilde{Z}(\cdot) \) is a contraction which admits a unique fixed point \( \tilde{Z}(\cdot) \). Letting \( Y(\cdot) \) be given by (3.32), one sees that \( (Y(\cdot), \tilde{Z}(\cdot)) \in L^p_\mathcal{F}(\Omega; C([\tau, \bar{\tau}]; \mathbb{R}^n)) \times L^p_\mathcal{F}(\tau, \bar{\tau}; \mathbb{R}^n) \) is the unique solution to (3.32). That completes the proof. \( \square \)

Now, we are ready to prove the following result.

**Theorem 3.10.** Let (H1), (3.16) and (3.25) hold. Let

\[
\Psi(t, \tau) = \int_{\tau}^t \mathbb{E}_\tau \left[ \left( \widehat{\Phi}(s)^{-1} \tilde{B}(s) \right) \right] ds, \quad 0 \leq \tau < t \leq T.
\]

\[
\Theta(t, \tau) = \int_{\tau}^t \mathbb{E}_\tau \left[ \left( \widehat{\Phi}(s)^{-1} \tilde{B}(s) \right) \right] ds, \quad 0 \leq \tau < t \leq T.
\]

Suppose there exists a \( \delta > 0 \) such that, for any \( T - \delta \leq \tau < t \leq T \), \( \Psi(t, \tau) \) is invertible and \( \Psi(t, \tau)^{-1} \in L^p_\mathcal{F}(\Omega; \mathbb{R}^{n \times n}) \). Moreover, suppose there exists a constant \( M > 0 \) such that

\[
|\Theta(t, \tau) \Psi(t, \tau)^{-1}| \leq M, \quad T - \delta \leq \tau < t \leq T, \text{ a.s.}
\]

Then, for any \( p > 1 \), system \( [A(\cdot), C(\cdot); B(\cdot), D(\cdot)] \) is \( L^p \)-exactly controllable on \([0, T]\) by \( U^p[0, T] \).

Note that for \( \Psi(t, \tau) \) and \( \Theta(t, \tau) \) defined in (3.34), one has

\[
\mathbb{E}_\tau \Theta(t, \tau) = \Psi(t, \tau).
\]

Thus, in the case that \( \tilde{A}(\cdot) \) and \( \tilde{B}(\cdot) \) are deterministic, condition (3.35) is automatically true with \( M = 1 \), as long as \( \Psi(t, \tau) \) is invertible. We will present an example that \( \tilde{A}(\cdot) \) is random and (3.35) holds. We also point out that condition (3.16) implies that \( m \geq n \). Further, condition that \( \Psi(t, \tau)^{-1} \) exists implies that
Therefore, getting \(X(\tau) = \xi\) is equivalent to having the following:

\[
X(t) = \hat{\Phi}(t)\hat{\Phi}(\tau)^{-1}\xi + \hat{\Phi}(t) \int_{\tau}^{t} \hat{\Phi}(s)^{-1} \left[ \hat{B}(s)v(s) + \hat{D}(s)Z(s) \right] ds + \hat{\Phi}(t) \int_{\tau}^{T} \hat{\Phi}(s)^{-1} Z(s) dW(s).
\]

This implies

\[
E_{\tau}[\hat{\Phi}(\tau)^{-1}\xi - \hat{\Phi}(\tau)^{-1}\xi] = \int_{\tau}^{T} E_{\tau}\left[ \hat{\Phi}(s)^{-1} \hat{D}(s)Z(s) \right] ds = \int_{\tau}^{T} E_{\tau}\left[ \hat{\Phi}(s)^{-1} \hat{B}(s)v(s) \right] ds.
\]

We now take

\[
v(s) = E_{\tau}\left( \hat{\Phi}(s)^{-1} \hat{B}(s) \right)^{\top} \hat{\Phi}(\tau)^{-1} \left[ E_{\tau}[\hat{\Phi}(\tau)^{-1}\xi] - \hat{\Phi}(\tau)^{-1}\xi \right] - \int_{\tau}^{T} E_{\tau}\left( \hat{\Phi}(t)^{-1} \hat{D}(t)Z(t) \right) dt,
\]

which is \(\mathcal{F}_{\tau}\)-measurable. Further, noting that \(\hat{\Phi}(\tau)^{-1}\) is bounded and \(\Psi(\tau, \tau)^{-1} \in L_{\mathcal{F}_{\tau}}^{\infty}(\Omega; \mathbb{R}^{n \times n})\), one has

\[
E\left( \int_{\tau}^{T} |\hat{B}(s)v(s)| ds \right)^{p} \leq KE\left\{ \left( \int_{\tau}^{T} |\hat{B}(s)||\hat{B}(s)||Z(t)|| \right) dt \right\}^{p},
\]

where

\[
\Gamma \equiv E_{\tau}[|\xi|^{p} + |\xi|^{p} + \left( \int_{\tau}^{T} E_{\tau}[|\hat{D}(t)||Z(t)||] dt \right)^{p}],
\]

which is \(\mathcal{F}_{\tau}\)-measurable. Since

\[
\left( \int_{\tau}^{T} |\hat{B}(s)||\hat{B}(s)||Z(t)|| \right) dt \leq \left( \frac{1}{2} \int_{\tau}^{T} |\hat{B}(s)|^{2} ds + \frac{1}{2} \int_{\tau}^{T} E_{\tau}[|\hat{B}(s)|^{2}] ds \right)^{p}
\]

\[
\leq K \left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p} + KE_{\tau}\left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p},
\]

we have

\[
E\left( \int_{\tau}^{T} |\hat{B}(s)v(s)| ds \right)^{p} \leq KE\left\{ \left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p} \Gamma + E_{\tau}\left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p} \Gamma \right\}
\]

\[
= KE\left\{ \left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p} \Gamma \right\} + KE\left\{ \left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p} \Gamma \right\} \leq KE\left\{ \left( \int_{\tau}^{T} |\hat{B}(s)|^{2} ds \right)^{p} \Gamma \right\}.
\]

Since \(\hat{B}(\cdot) \in L_{\mathcal{F}_{\tau}}^{\infty}(\Omega; L^{2}(0, T; \mathbb{R}^{n \times m}))\) and \(\hat{D}(\cdot) \in L_{\mathcal{F}_{\tau}}^{\infty}(\Omega; L^{2}(0, T; \mathbb{R}^{n \times m}))\), one has

\[
E\left( \int_{\tau}^{T} |\hat{B}(s)v(s)| ds \right)^{p} \leq KE\left\{ |\xi|^{p} + |\xi|^{p} + \left[ \int_{\tau}^{T} E_{\tau}\left( |\hat{D}(t)||Z(t)|| \right) dt \right]^{p} \right\}
\]

\[
\leq KE\left\{ |\xi|^{p} + |\xi|^{p} + \left[ \int_{\tau}^{T} E_{\tau}\left( |\hat{D}(t)||Z(t)|| \right) dt \right]^{p} \right\},
\]

\[
\leq KE\left\{ |\xi|^{p} + |\xi|^{p} + \left[ \int_{\tau}^{T} E_{\tau}\left( |\hat{D}(t)||Z(t)|| \right) dt \right]^{p} \right\} \leq KE\left\{ |\xi|^{p} + |\xi|^{p} + \left( \int_{\tau}^{T} |Z(t)|^{2} dt \right)^{\frac{p}{2}} \right\}.
\]
Thus, $Z(\cdot) \in L^p_F(\Omega; L^2(\tau, \bar{\tau}; \mathbb{R}^n))$ implies \( \hat{B}(\cdot)v(\cdot) \in L^p_F(\Omega; L^1(\tau, \bar{\tau}; \mathbb{R}^n)) \). For such a \( v(\cdot) \), (3.38) holds. Moreover, (3.37) becomes
\[
(3.39) \quad 0 = \eta(\bar{\tau}, \tau) + \int_\tau^{\bar{\tau}} \Theta(\bar{\tau}, \tau)\Psi(\bar{\tau}, \tau)^{-1}\mathbb{E}_\tau\left( \hat{D}(s)\hat{Z}(s) \right)ds - \int_\tau^{\bar{\tau}} \hat{D}(s)\hat{Z}(s)ds - \int_\tau^{\bar{\tau}} \hat{Z}(s)dW(s),
\]
where
\[
\hat{D}(s) = \Phi(s)^{-1}\hat{D}(s)\Phi(s), \quad \hat{Z}(s) = \Phi(s)^{-1}Z(s),
\]
and
\[
\eta(\bar{\tau}, \tau) \equiv \Phi(\bar{\tau})^{-1}\xi - \Phi(\tau)^{-1}\xi - \Theta(\bar{\tau}, \tau)\Psi(\bar{\tau}, \tau)^{-1}\left( \mathbb{E}_\tau[\Phi(\bar{\tau})^{-1}\xi] - \Phi(\tau)^{-1}\xi \right),
\]
which is \( \mathcal{F}_\tau \)-measurable with
\[
\mathbb{E}[\eta(\bar{\tau}, \tau)]^p \leq KE\left( |\xi|^p + |\hat{\xi}|^p \right), \quad \mathbb{E}_\tau\eta(\bar{\tau}, \tau) = 0.
\]
To summarize the above, we see that for given \( 0 \leq \tau < \bar{\tau} \leq T \) and \( \xi \in L^p_F(\Omega, \mathbb{R}^n), \hat{\xi} \in L^p_F(\Omega, \mathbb{R}^n) \), to get a pair of \( (v(\cdot), Z(\cdot)) \) so that \( X(\tau) = \xi \) and \( X(\bar{\tau}) = \hat{\xi} \) if and only if there exists an \( F \)-adapted process \( \hat{Z}(\cdot) \) such that (3.39) holds true.

By Lemma 3.9, there exists a uniform \( \varepsilon' > 0 \) such that as long as \( (T - \delta) \vee (T - \varepsilon') \leq \bar{\tau} \leq T \), the following equation
\[
(3.40) \quad Y(t) = \eta(\bar{\tau}, \tau) + \int_\tau^{\bar{\tau}} \Theta(\bar{\tau}, \tau)\Psi(\bar{\tau}, \tau)^{-1}\mathbb{E}_\tau\left( \hat{D}(s)\hat{Z}(s) \right)ds - \int_\tau^{\bar{\tau}} \hat{D}(s)\hat{Z}(s)ds - \int_\tau^{\bar{\tau}} \hat{Z}(s)dW(s), \quad t \in [\tau, \bar{\tau}]
\]
admits a unique adapted solution \( (Y(\cdot), \hat{Z}(\cdot)) \in L^p_F(\Omega; C([\tau, \bar{\tau}]; \mathbb{R}^n)) \times L^p_F(\Omega; L^2(\tau, \bar{\tau}; \mathbb{R}^n)) \). We notice that
\[
Y(\tau) = \mathbb{E}_\tau[Y(T)] = 0,
\]
which proves the existence of a \( \hat{Z}(\cdot) \in L^p_F(\tau, \bar{\tau}; \mathbb{R}^c) \) to (3.39).

Now, we can complete our proof as follows: Arbitrarily select a \( T_0 \) such that \( (T - \delta) \vee (T - \varepsilon') \leq T_0 < T \). For any \( x \in \mathbb{R}^n \) and \( \xi \in L^p_{\mathcal{F}_T}(\Omega, \mathbb{R}^n) \), let
\[
\xi = X(T_0; x, 0, 0) \in L^p_{\mathcal{F}_{T_0}}(\Omega, \mathbb{R}^n).
\]
Then by what we have proved, there exists a pair \( (v(\cdot), Z(\cdot)) \in U^p[0, T_0] \times L^p_F(\Omega; L^2(T_0, T; \mathbb{R}^n)) \) such that
\[
X(T_0) = \xi, \quad X(T) = \hat{\xi}.
\]
We obtain the \( L^p \)-exact controllability on \([0, T]\) by \( U^p[0, T] \).

The following simple example is to show that condition (3.35) is possible for random coefficient case.

**Example 3.11.** Let
\[
\hat{A}(t) = \begin{pmatrix} 0 & a(t) \\ 0 & 0 \end{pmatrix}, \quad \hat{B}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
where \( a : [0, T] \times \Omega \rightarrow \mathbb{R} \) satisfies the following conditions: \( t \mapsto a(t) \) is \( C^2 \) with
\[
a(\cdot), a'(\cdot), a''(\cdot) \in L^\infty_F(0, T; \mathbb{R}), \quad a(t) \geq 1, \quad t \in [0, T], \text{ a.s.}
\]
For example, we may choose
\[
a(t) = 1 + \int_0^t \int_0^s \frac{W(\tau)^2}{1 + W(\tau)^2}d\tau ds, \quad t \in [0, T].
\]
Then
\[ \tilde{\Phi}(t) = \begin{pmatrix} 1 & \int_0^t a(s)ds \\ 0 & 1 \end{pmatrix}, \quad t \in [0, T]. \]

Thus,
\[ \tilde{\Phi}(t)^{-1} \hat{B}(t) = \begin{pmatrix} 1 & -\int_0^t a(s)ds \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\alpha(t)) \\ 1 \end{pmatrix} \equiv \begin{pmatrix} (\alpha(t)) \\ 1 \end{pmatrix}. \]

Denoting \( \bar{\alpha}(s) = E_{\tau} \alpha(s) \), we have
\[
\Psi(t, \tau) = \int_{\tau}^t E_{\tau} \left( \tilde{\Phi}(s)^{-1} \hat{B}(s) \right) E_{\tau} \left( \tilde{\Phi}(s)^{-1} \hat{B}(s) \right) ^\top ds = \int_{\tau}^t \left( \begin{array}{c} \bar{\alpha}(s) \\ 1 \end{array} \right) \left( \begin{array}{c} \bar{\alpha}(s) \\ 1 \end{array} \right) ds
\]
\[
= \left( \int_{\tau}^t \bar{\alpha}(s)^2 ds \quad \int_{\tau}^t \bar{\alpha}(s)ds \right) \left( t - \tau \right).
\]

A direct computation shows that (denoting \( \bar{\alpha}(\tau) = E_{\tau} \alpha(\tau) \))
\[
F(t) \equiv \det \Psi(t, \tau) = (t - \tau) \int_{\tau}^t \bar{\alpha}(s)^2 ds - \left( \int_{\tau}^t \bar{\alpha}(s)ds \right)^2 = \left[ 2\bar{\alpha}(\tau)^2 + R(t) \right] (t - \tau)^4,
\]
with
\[ \lim_{t \to \tau} R(t) = 0. \]

Hence, for \( t - \tau > 0 \) small, \( \Psi(t, \tau) \) is invertible, and
\[
\Psi(t, \tau)^{-1} = \frac{1}{F(t)} \begin{pmatrix} t - \tau & -\int_{\tau}^t \bar{\alpha}(s)ds \\ -\int_{\tau}^t \bar{\alpha}(s)ds & \int_{\tau}^t \bar{\alpha}(s)^2 ds \end{pmatrix}.
\]

Also,
\[
\Theta(t, \tau) = \int_{\tau}^t \tilde{\Phi}(s)^{-1} \hat{B}(s) E_{\tau} \left( \tilde{\Phi}(s)^{-1} \hat{B}(s) \right) ^\top ds = \int_{\tau}^t \left( \begin{array}{c} \alpha(s) \\ 1 \end{array} \right) \left( \begin{array}{c} \bar{\alpha}(s) \\ 1 \end{array} \right) ds
\]
\[
= \left( \int_{\tau}^t \alpha(s)\bar{\alpha}(s)ds \quad \int_{\tau}^t \alpha(s)ds \right) \left( \frac{t - \tau}{\int_{\tau}^t \bar{\alpha}(s)ds} \right).
\]

Then
\[
\Theta(t, \tau) \Psi(t, \tau)^{-1} = \frac{1}{F(t)} \begin{pmatrix} \int_{\tau}^t \alpha(s)\bar{\alpha}(s)ds & \int_{\tau}^t \alpha(s)ds \\ \int_{\tau}^t \alpha(s)ds & \int_{\tau}^t \bar{\alpha}(s)ds \end{pmatrix} \begin{pmatrix} t - \tau & -\int_{\tau}^t \bar{\alpha}(s)ds \\ -\int_{\tau}^t \bar{\alpha}(s)ds & \int_{\tau}^t \bar{\alpha}(s)^2 ds \end{pmatrix}
\]
\[
\equiv \frac{1}{F(t)} \begin{pmatrix} \Lambda_1(t) & \Lambda_2(t) \\ 0 & F(t) \end{pmatrix},
\]

where
\[
\Lambda_1(t) = (t - \tau) \int_{\tau}^t \alpha(s)\bar{\alpha}(s)ds - \left( \int_{\tau}^t \alpha(s)ds \right) \left( \int_{\tau}^t \bar{\alpha}(s)ds \right),
\]
\[
\Lambda_2(t) = \left( \int_{\tau}^t \alpha(s)ds \right) \left( \int_{\tau}^t \bar{\alpha}(s)^2 ds \right) - \left( \int_{\tau}^t \bar{\alpha}(s)ds \right) \left( \int_{\tau}^t \alpha(s)\bar{\alpha}(s)ds \right).
\]

Some direct (lengthy) calculations show that
\[
\Lambda_1(t) = \left[ 2\alpha(\tau) + \gamma_1(t) \right] (t - \tau)^4, \quad \lim_{t \to \tau} \gamma_1(t) = 0,
\]
\[
\Lambda_2(t) = \gamma_2(t)(t - \tau)^4, \quad \lim_{t \to \tau} \gamma_2(t) = 0.
\]
Hence,
\[
\Theta(t, \tau)\Psi(t, \tau)^{-1} = \frac{1}{F(t)} \begin{pmatrix} \Lambda_1(t) & \Lambda_2(t) \\ 0 & F(t) \end{pmatrix}
\]
\[
= \frac{1}{2a(\tau) + R(t)} \begin{pmatrix} 2a(\tau) + \gamma_1(t) & \gamma_2(t) \\ 0 & 2a(\tau) + R(t) \end{pmatrix}.
\]
As a result, we obtain
\[
\left| \Theta(t, \tau)\Psi(t, \tau)^{-1} \right| \leq K, \quad \forall t > \tau, \text{ with } t - \tau \text{ small.}
\]

The above shows that (3.35) holds.

As we mentioned earlier, condition (3.16) implies that \( m \geq n \), and if \( m = n \) and (3.16) holds, then \( \hat{B}(\cdot) = 0 \). Hence, in order \( \Psi \) to be invertible, one must have \( m > n \). The following result is concerned with a case that \( D(\cdot) \) is surjective and \( m = n \).

**Proposition 3.12.** Let (H1), (3.16) and (3.25) hold and \( m = n \). Then, for any \( p > 1 \), the system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) is not \( L^p \)-exactly controllable on any \([0, T] \) by \( U^p[0, T] \) with \( T > 0 \).

**Proof.** In the current case, \( D(\cdot)^{-1} \) is bounded. If for any \( x \in \mathbb{R}^n \) and \( \xi \in L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \), one can find a \( u(\cdot) \in U^p[0, T] \) such that \( X(0) = x \) and \( X(T) = \xi \), then we let
\[
Z(t) = C(t)X(t) + D(t)u(t), \quad t \in [0, T],
\]
which will lead to
\[
u(t) = D(t)^{-1}Z(t) - C(t)X(t), \quad t \in [0, T].
\]
Hence, \((X(\cdot), Z(\cdot))\) is an adapted solution to the following BSDE:
\[
\begin{aligned}
dX(t) &= \left( A(t) - B(t)D(t)^{-1}C(t) \right)X(t) + B(t)D(t)^{-1}Z(t)dt + Z(t)dW(t), \quad t \in [0, T], \\
X(T) &= \xi.
\end{aligned}
\]
Then \( X(0) \) cannot be arbitrarily specified. Hence, \( L^p \)-exact controllability is not possible for system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\). \( \square \)

In the above two subsections, we have discussed the two extreme cases: either \( D(\cdot) = 0 \), or \( D(\cdot) \) is full rank (for the case \( d = 1 \)). The case in between remains open. Some partial results have been obtained, but they are not at a mature level to be reported. We hope to present them in a forthcoming paper.

## 4 Duality and Observability Inequality

As we know that for deterministic linear ODE systems, the controllability of the original systems is equivalent to the observability of the dual equations. We would like to see how such a result will look like for our FSDE system \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\). To this end, let us first look at an abstract result, whose proof should be standard. But for reader’s convenience, we present a proof.

**Proposition 4.1.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces, \( \mathbb{K} : \mathcal{X} \to \mathcal{Y} \) be a bounded linear operator, and \( \mathbb{K}^* : \mathcal{Y}^* \to \mathcal{X}^* \) be the adjoint operator of \( \mathbb{K} \). Then \( \mathbb{K} \) is surjective if and only if there exists a \( \delta > 0 \) such that
\[
|\mathbb{K}^* y^*|_{\mathcal{X}^*} \geq \delta |y^*|_{\mathcal{Y}^*}, \quad \forall y^* \in \mathcal{Y}^*.
\]

(4.1)
Further, if $X$ and $Y$ are reflexive and the map $x^* \mapsto |x^*|^2_X$ from $X^*$ to $\mathbb{R}$ is Fréchet differentiable, then (4.1) is also equivalent to the following: For any $y \in Y$, the functional
\begin{equation}
J(y^*; y) = \frac{1}{2} |K^* y^*|^2_X + \langle y, y^* \rangle, \quad y^* \in Y^*,
\end{equation}
adopts a minimum over $Y^*$. In addition, if the norm of $X^*$ is strictly convex, then for any $y \in Y$, the optimal solution of (4.2) is necessarily unique.

**Proof.** Suppose $\mathcal{R}(K) = Y$, i.e., $K$ is a surjection. Then $\mathcal{R}(K)$ is closed. By Banach Closed Range Theorem [(27)], $\mathcal{R}(K^*)$ is closed. Moreover,
\[ \mathcal{N}(K^*)^\perp = \left\{ y \in Y \mid \langle y^*, y \rangle = 0, \quad \forall y^* \in \mathcal{N}(K^*) \right\} = \mathcal{R}(K) = Y. \]
This implies that $\mathcal{N}(K^*) = \{0\}$. Thus, $K^*$ is injective with $\mathcal{R}(K^*)$ closed. Therefore, $K^* : Y^* \to \mathcal{R}(Y^*)$ is one-to-one and onto. Hence, $(K^*)^{-1} : \mathcal{R}(K^*) \to Y^*$ is bounded. Consequently, for any $x^* = K^* y^* \in \mathcal{R}(K^*)$,
\[ |y^*|_{Y^*} = |(K^*)^{-1} K^* y^*|_{Y^*} \leq \|(K^*)^{-1}\| |K^* y^*|_{X^*} = \frac{1}{\delta} |K^* y^*|_{X^*}, \]
which leads to (4.1).

Conversely, suppose (4.1) holds. Then $\mathcal{R}(K^*)$ is closed and $K^*$ is injective. Thus, by Banach Closed Range Theorem, $\mathcal{R}(K)$ is also closed and
\[ \mathcal{R}(K) = \mathcal{N}(K^*)^\perp = \{0\}^\perp = Y, \]
proving that $K$ is surjective.

Now, for any $y \in Y$, consider functional $y^* \mapsto J(y^*; y)$. Clearly, under condition (4.1), we have that $y^* \mapsto J(y^*; y)$ is coercive and weakly lower semi-continuous. Hence, if $\{y_k^*\}_{k \geq 1}$ is a minimizing sequence, then it is bounded. By the reflexivity of $Y^*$, we may assume that $y_k^*$ converges weakly to some $\bar{y}^* \in Y^*$. Then by the weakly lower semi-continuity of the functional $J(\cdot; y)$, $\bar{y}^*$ must be a minimum.

Conversely, for any $y \in Y$, let $\bar{y}^* \in Y^*$ be a minimum of $y^* \mapsto J(y^*; y)$. Denote the Fréchet derivative of $x^* \mapsto \frac{1}{2}|x^*|^2_X$, by $\Gamma(x^*)$, i.e.,
\[ \lim_{\delta \to 0} \frac{|x^* + \delta x^*|^2_X - |x^*|^2_X}{2\delta} = \frac{d}{d\delta} \bigg|_{\delta = 0} |x^* + \delta x^*|^2_X = \langle \Gamma(x^*), x^* \rangle, \quad \forall x^* \in X^*, \]
where we denote by $\frac{d}{d\delta} \bigg|_{\delta = 0} f(\delta)$ the derivative of function $f$ at $\delta = 0$. Then by the optimality of $\bar{y}^*$, we have
\begin{equation}
0 \leq \lim_{\delta \to 0} \frac{J(\bar{y}^* + \delta y^*; y) - J(\bar{y}^*; y)}{\delta} = \lim_{\delta \to 0} \frac{|K^* (\bar{y}^* + \delta y^*)|^2_X - |K^* \bar{y}^*|^2_X + \langle y, y^* \rangle}{2\delta}.
\end{equation}

Hence,
\[ \langle K^* \bar{y}^*, y^* \rangle + y^* = 0. \]
Since $y \in Y$ is arbitrary, we obtain that $\mathcal{R}(K) = Y$. Then by what we have proved, (4.1) holds. Finally, if we further assume that the norm of $X^*$ is strictly convex, then we must have the uniqueness of the optimal solution to $y^* \mapsto J(y^*; y)$.

For any given $1 < p < \rho \leq \infty$ and $2 < \sigma \leq \infty$, we denote $q \equiv \frac{p^p}{p-1}$, and
\begin{align*}
\hat{p} &\equiv \begin{cases} 
\frac{pp}{\rho - p}, & \rho < \infty, \\
p, & \rho = \infty,
\end{cases} \quad \hat{q} \equiv \begin{cases} 
\frac{pp}{pp - \rho + p}, & \rho < \infty, \\
\frac{p}{p - 1}, & \rho = \infty,
\end{cases} \\
\hat{\mu} &\equiv \begin{cases} 
\frac{2\sigma}{\sigma - 2}, & \sigma < \infty, \\
2, & \sigma = \infty,
\end{cases} \quad \hat{\nu} \equiv \begin{cases} 
\frac{2\sigma}{\sigma + 2}, & \sigma < \infty, \\
2, & \sigma = \infty.
\end{cases}
\end{align*}
Clearly, with the above notations, we have
\[ \frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{\bar{p}} + \frac{1}{q} = 1, \quad \frac{1}{\mu} + \frac{1}{\nu} = 1, \]
\[ 1 < \nu \leq 2 < \mu < \infty, \quad 1 < \bar{p} \leq \bar{q} < \infty, \quad 1 < \bar{q} \leq \frac{p}{p - 1}, \]
(4.5)
\[ \|p^{\mu},\sigma[0,T]\| \equiv L_{\mu}^{p}(\Omega;L^{\mu}(0,T;\mathbb{R}^{m})), \quad \|p^{\mu},\sigma[0,T]*\| \equiv L_{\mu}^{q}(\Omega;L^{\nu}(0,T;\mathbb{R}^{m})), \]
\[ \|p^{\mu},\sigma[0,T]\| \equiv L_{\mu}^{p}(0,T;L^{p}(\Omega;\mathbb{R}^{m})), \quad \|p^{\mu},\sigma[0,T]*\| \equiv L_{\mu}^{q}(0,T;L^{q}(\Omega;\mathbb{R}^{m})). \]

In the rest of this paper, we will keep the above notations. We now present the first main result of this section.

**Theorem 4.2.** Let \((H1)-(H2)\) (respectively, \((H1)\) and \((H2)'\)) hold. Then system \([A(\cdot),C(\cdot);B(\cdot),D(\cdot)]\) is \(L^{p}\)-exactly controllable on \([0,T]\) by \(U_{p,\sigma}[0,T]\) (respectively, \(U_{p,\sigma}[0,T]\)) if and only if there exists a \(\delta > 0\) such that the following, called an observability inequality, holds:

\[ \|B(\cdot)^{T}Y(\cdot) + \sum_{k=1}^{d} D_{k}(\cdot)^{T}Z_{k}(\cdot)\|_{U_{p,\sigma}[0,T]}, \quad \forall \eta \in L_{\mathcal{F}_{T}}^{q}(\Omega;\mathbb{R}^{n}), \]
\[ (4.6) \]
(4.7)
\[ \|B(\cdot)^{T}Y(\cdot) + \sum_{k=1}^{d} D_{k}(\cdot)^{T}Z_{k}(\cdot)\|_{U_{p,\sigma}[0,T]}, \quad \forall \eta \in L_{\mathcal{F}_{T}}^{q}(\Omega;\mathbb{R}^{n}), \]
where \((Y(\cdot),Z(\cdot))\) (with \(Z(\cdot) \equiv (Z_{1}(\cdot), \ldots, Z_{d}(\cdot))\)) is the unique adapted solution to the following BSDE:

\[ dY(t) = -[A(t)^{T}Y(t) + \sum_{k=1}^{d} C_{k}(t)^{T}Z_{k}(t)]dt + \sum_{k=1}^{d} Z_{k}(t)dW_{k}(t), \quad t \in [0,T], \]
\[ Y(T) = \eta. \]
\[ (4.8) \]

**Proof.** We only prove the equivalence between system’s \(L^{p}\)-exactly controllability on \([0,T]\) by \(U_{p,\sigma}[0,T]\) and the validity of the observability inequality (4.6). The other part can be proved with the similar procedure.

For any \((x,u(\cdot)) \in \mathbb{R}^{n} \times U_{p,\sigma}[0,T]\), with \(p > 1\), let \(X(\cdot) \equiv X(\cdot;x,u(\cdot))\) be the unique solution to (1.1). Then
\[ X(\cdot;x,u(\cdot)) = X(\cdot;x,0) + X(\cdot;0,u(\cdot)). \]

Define
\[ \mathbb{K}u(\cdot) = X(T;0,u(\cdot)), \quad \forall u(\cdot) \in U_{p,\sigma}[0,T]. \]
Then
\[ \|\mathbb{K}u(\cdot)||_{L_{\mathcal{F}_{T}}^{p}(\Omega;\mathbb{R}^{n})} \leq K\|u(\cdot)||_{U_{p,\sigma}[0,T]}, \quad u(\cdot) \in U_{p,\sigma}[0,T]. \]
Thus, \(\mathbb{K} : U_{p,\sigma}[0,T] \rightarrow L_{\mathcal{F}_{T}}^{p}(\Omega;\mathbb{R}^{n})\) is a bounded linear operator. Now, system \([A(\cdot),C(\cdot);B(\cdot),D(\cdot)]\) is \(L^{p}\)-exactly controllable on \([0,T]\) by \(U_{p,\sigma}[0,T]\) if and only if for any \((x,\xi) \in \mathbb{R}^{n} \times L_{\mathcal{F}_{T}}^{p}(\Omega;\mathbb{R}^{n}), \xi - X(T;x,0) = \mathbb{K}u(\cdot), \forall u(\cdot) \in U_{p,\sigma}[0,T]\), for some \(u(\cdot) \in U_{p,\sigma}[0,T]\), which is equivalent to
\[ \mathcal{R}(\mathbb{K}) = L_{\mathcal{F}_{T}}^{p}(\Omega;\mathbb{R}^{n}). \]
This means that $K : \mathbb{U}^{p,\rho,\sigma}\{0,T\} \rightarrow L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ is surjective. Hence, by Proposition 4.1, this is equivalent to the following: For some $\delta > 0$,

$$\|K^*\eta\|_{\mathbb{U}^{p,\rho,\sigma}[0,T]} \geq \delta \|\eta\|_{L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}, \quad \forall \eta \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n).$$

We now find $K^* : L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \rightarrow \mathbb{U}^{p,\rho,\sigma}[0,T]$. To this end, we consider BSDE (4.8), with $\eta \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$.

By a standard result of BSDEs ([5]), under (H1), (4.8) admits a unique adapted solution

$$(Y(\cdot), Z(\cdot)) \equiv (Y(\cdot; \eta), Z(\cdot; \eta)) \in L^p_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^p_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^{n \times d})),
$$

and the following estimate holds:

$$\mathbb{E}\left[\sup_{t \in [0,T]} |Y(t)|^q + \left( \int_0^T |Z(t)|^2 dt \right)^{\frac{q}{2}} \right] \leq KE|\eta|^q,
$$

where we denote $Z(\cdot) \equiv (Z_1(\cdot), Z_2(\cdot), \ldots, Z_d(\cdot))$. By applying Itô’s formula to $\langle X(\cdot; 0, u(\cdot)), Y(\cdot) \rangle$ on the interval $[0,T]$, we have the following duality relation:

$$\langle K u(\cdot), \eta \rangle = \mathbb{E}\langle X(T), \eta \rangle = \mathbb{E}\int_0^T \langle u(t), B(t)^T Y(t) + \sum_{k=1}^d D_k(t)^T Z_k(t) \rangle dt = \langle u(\cdot), K^* \eta \rangle.
$$

Hence,

$$\langle K^* \eta(\cdot)t \equiv B(t)^T Y(t) + \sum_{k=1}^d D_k(t)^T Z_k(t), \quad t \in [0,T].$$

Combining (4.9) with (4.12), we obtain (4.6).

Theorem 4.2 provides an approach to study the controllability of stochastic linear systems by establishing an inequality for BSDEs. The following example illustrates this approach.

**Example 4.3.** Let the dimensions of both state process and Brownian motion be 1, the dimension of control process be 2. Let

$$A(\cdot), B_1(\cdot), B_2(\cdot) \in L^p_{\mathcal{F}}(0, T; \mathbb{R}), \quad |B_1(t)|, |B_2(t)| \geq \delta, \quad t \in [0,T],$$

for some $\delta > 0$. Consider the following system:

$$dX(t) = \begin{bmatrix} A(t)X(t) + (B_1(t) & 0) \\ 0 & D_2(t) \end{bmatrix} dt + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} dW(t), \quad t \geq 0,$$

$$X(0) = x,$$

and the adjoint system is given by

$$\begin{cases}
  dY(t) = -A(t)Y(t) dt + Z(t) dW(t), & t \in [0,T], \\
  y(T) = \eta,
\end{cases}$$

where $\eta \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$. A direct calculation leads to

$$\mathbb{E}\int_0^T \left(\begin{array}{c} B_1(t) \\ 0 \end{array}\right) Y(t) + \begin{bmatrix} 0 \\ D_2(t) \end{bmatrix} Z(t) \right|^2 dt = \mathbb{E}\int_0^T \left[ |B_1(t)|^2 |Y(t)|^2 + |D_2(t)|^2 |Z(t)|^2 \right] dt \geq \delta^2 \mathbb{E}\int_0^T \left[ |Y(t)|^2 + |Z(t)|^2 \right] dt.$$
Let $\beta \in L^\infty_T(0, T; \mathbb{R})$ to be chosen later. Applying Itô’s formula to $|Y(\cdot)|^2 e^{\int_0^T \beta(s)ds}$ on the interval $[0, T]$, we deduce that

$$\mathbb{E}\left[|\eta|^2 e^{\int_0^T \beta(s)ds} - |Y(0)|^2\right] = \mathbb{E}\int_0^T \left[ (\beta(t) - 2A(t)) |Y(t)|^2 + |Z(t)|^2 \right] e^{\int_0^t \beta(s)ds} dt.$$  

(4.16)

In (4.16), selecting $\beta(\cdot) = 2A(\cdot) + 1$ leads to

$$\mathbb{E}\int_0^T \left[ |Y(t)|^2 + |Z(t)|^2 \right] e^{\int_0^t (2A(s)+1)ds} dt = \mathbb{E}\left[|\eta|^2 e^{\int_0^T (2A(s)+1)ds} - |Y(0)|^2\right].$$  

(4.17)

On the other hand, selecting $\beta(\cdot) = 2A$, we get

$$\mathbb{E}\left[|\eta|^2 e^{\int_0^T 2A(s)ds} - |Y(0)|^2\right] = \mathbb{E}\int_0^T |Z(t)|^2 e^{\int_0^t 2A(s)ds} dt \geq 0.$$  

(4.18)

Then,

$$-|Y(0)|^2 \geq -\mathbb{E}\left[|\eta|^2 e^{\int_0^T 2A(s)ds}\right].$$

Combining (4.17) with (4.18), one has

$$\mathbb{E}\int_0^T \left[ |Y(t)|^2 + |Z(t)|^2 \right] e^{\int_0^t (2A(s)+1)ds} dt \geq \mathbb{E}\left[|\eta|^2 e^{\int_0^T (2A(s)+1)ds} - |Y(0)|^2\right]$$

$$= (e^T - 1) \mathbb{E}\left[|\eta|^2 e^{\int_0^T 2A(s)ds}\right].$$

Due to the boundedness of $A$ (without loss of generality, we assume $|A(t)| \leq K$ a.s. a.e.), we obtain

$$e^{(2K+1)T} \mathbb{E}\int_0^T \left[ |Y(t)|^2 + |Z(t)|^2 \right] dt \geq (e^T - 1) e^{-2KT} \mathbb{E}|\eta|^2,$$

i.e.

$$\mathbb{E}\int_0^T \left[ |Y(t)|^2 + |Z(t)|^2 \right] dt \geq (e^T - 1) e^{-(4K+1)T} \mathbb{E}|\eta|^2.$$  

Combining with (4.15), we have proved that the observability inequality holds true for BSDE (4.14). By Theorem 4.2, system (4.13) is $L^2$-exactly controllable on $[0, T]$ by $\mathbb{U}^{2, \infty}[0, T] \equiv L^2_T(0, T; \mathbb{R}^2)$.

Now, we introduce the following definition which makes the name “observability inequality” aforementioned meaningful.

**Definition 4.4.** Let (H1) hold and $(Y(\cdot), Z(\cdot))$ be the adapted solution to BSDE (4.8) with $\eta \in L^2_T(\Omega; \mathbb{R}^n)$.

(i) For the pair $(B(\cdot), D(\cdot))$ with $B(\cdot), D_k(\cdot) \in L^1_T(0, T; \mathbb{R}^{n \times m})$ $(k = 1, 2, \cdots, d)$ and $D(\cdot) = (D_1(\cdot), \cdots, D_d(\cdot))$, the map

$$\eta \mapsto K^* \eta \equiv B(\cdot)^\top Y(\cdot) + \sum_{k=1}^d D_k(\cdot)^\top Z_k(\cdot)$$  

(4.19)

is called an $\mathcal{Y}[0, T]$-observer of BSDE (4.8) if

$$K^* \eta \in \mathcal{Y}[0, T], \quad \forall \eta \in L^2_T(\Omega; \mathbb{R}^m),$$

where $\mathcal{Y}[0, T]$ is a subspace of $L^1_T(0, T; \mathbb{R}^m)$. BSDE (4.8), together with the observer (4.19) is denoted by $[A(\cdot)^\top, C(\cdot)^\top; B(\cdot)^\top, D(\cdot)^\top]$.

(ii) System $[A(\cdot)^\top, C(\cdot)^\top; B(\cdot)^\top, D(\cdot)^\top]$ is said to be $L^q$-exactly observable by $\mathcal{Y}[0, T]$ observations if from the observation $K^* \eta \in \mathcal{Y}[0, T]$, the terminal value $\eta \in L^q_T(\Omega; \mathbb{R}^n)$ of $Y(\cdot)$ at $T$ can be uniquely determined, i.e., the map $K^*: L^q_T(\Omega; \mathbb{R}^n) \rightarrow \mathcal{Y}[0, T]$ admits a bounded inverse.

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With the above definition, we clearly have the following result:

**Theorem 4.5.** Let (H1)–(H2) (respectively, (H1)–(H2)’) hold. Then \([A(\cdot), C(\cdot); B(\cdot), D(\cdot)]\) is \(L^p\)-exactly controllable on \([0, T]\) by \(U^p,\sigma[0, T]\) (respectively, \(U^p,\sigma[0, T]\)) if and only if \([A(\cdot)^T, C(\cdot)^T; B(\cdot)^T, D(\cdot)^T]\) is \(L^q\)-exactly observable by \(U^p,\sigma[0, T]^*\) (respectively, \(U^p,\sigma[0, T]^*\) observations).

Next, for any \(x \in \mathbb{R}^n\), let \(X(\cdot; x, 0)\) be the solution to the state equation (1.1) corresponding to the initial state \(x\) and \(u(\cdot) = 0\). Denote

\[
\mathbb{K}_0 x = X(T; x, 0), \quad \forall x \in \mathbb{R}^n.
\]

Then applying Itô’s formula to \(X(\cdot; x, 0), Y(\cdot)\) with \((Y(\cdot), Z(\cdot)) \equiv (Y(\cdot; \eta), Z(\cdot; \eta))\) being the adapted solution to (4.8), we have

\[
\mathbb{E}(X(T), \eta) - \langle x, Y(0) \rangle = \mathbb{E} \int_0^T \left( (A(t)X(t), Y(t)) + \langle X(t), -A(t)^T Y(t) - \sum_{k=1}^d C_k(t)^T Z_k(t) \rangle + \sum_{k=1}^d \langle C_k(t)X(t), Z_k(t) \rangle \right) dt = 0.
\]

Hence,

\[
\langle x, Y(0) \rangle = \mathbb{E}\langle \mathbb{K}_0 x, \eta \rangle = \langle x, \mathbb{K}_0^\eta \rangle, \quad \forall x \in \mathbb{R}^n.
\]

This leads to

(4.20)

\[
\mathbb{K}_0^\eta = Y(0; \eta), \quad \forall \eta \in L^p_{\mathcal{F}_T}(\mathbb{R}^n).
\]

Now, for any \((x, \xi) \in \mathbb{R}^n \times L^p_{\mathcal{F}_T}(\mathbb{R}^n)\), we introduce a functional \(J(\cdot; x, \xi) : L^p_{\mathcal{F}_T}(\mathbb{R}^n) \rightarrow \mathbb{R}\) defined by

(4.21)

\[
J(\eta; x, \xi) = \frac{1}{2} \left\| \mathbb{K}^\eta \right\|^2_{U^p,\sigma[0, T]^*} + \langle x, \mathbb{K}_0^\eta \rangle - \mathbb{E} \langle \xi, \eta \rangle, \quad \forall \eta \in L^p_{\mathcal{F}_T}(\mathbb{R}^n),
\]

where \(\mathbb{K}^*\) and \(\mathbb{K}_0^*\) are given by (4.12) and (4.20), respectively. Equivalently,

(4.22)

\[
J(\eta; x, \xi) = \frac{1}{2} \left\| B(\cdot)^T Y(\cdot) + \sum_{k=1}^d D_k(\cdot)^T Z_k(\cdot) \right\|^2_{U^p,\sigma[0, T]^*} + \langle x, Y(0) \rangle - \mathbb{E} \langle \xi, \eta \rangle, \quad \forall \eta \in L^p_{\mathcal{F}_T}(\mathbb{R}^n),
\]

with \((Y(\cdot), Z(\cdot))\) being the adapted solution to BSDE (4.8). One can pose the following optimization problem.

**Problem (O).** Minimize (4.22) subject to BSDE (4.8) over \(L^p_{\mathcal{F}_T}(\mathbb{R}^n)\).

Note that the spaces \(U^p,\sigma[0, T]\) and \(L^p_{\mathcal{F}_T}(\mathbb{R}^n)\) are reflexive since their norms are uniformly convex. In order to apply Proposition 4.1, we need to show that

\[
\varphi(\cdot) \mapsto \frac{1}{2} \left\| \varphi(\cdot) \right\|^2_{U^p,\sigma[0, T]^*} \equiv \frac{1}{2} \left\| \varphi(\cdot) \right\|^2_{L^p_{\mathcal{F}_T}(\mathbb{R}^n)}
\]

is Fréchet differentiable. For simplicity of notation, we shall use the following notations for a while:

\[
\| \cdot \|_{L^p_{\mathcal{F}_T}(\mathbb{R}^n)} \equiv \| \cdot \|_{L^p}, \quad \text{and} \quad \| \cdot \|_{L^p(0, T; \mathbb{R}^m)} \equiv \| \cdot \|_{L^p}. \]

Now let \(\varphi(\cdot), \psi(\cdot) \in L^p_{\mathcal{F}_T}(\mathbb{R}^n)\). If \(\| \varphi(\cdot) \|_{L^p} = 0\), then

(4.23)

\[
\frac{1}{2} \frac{d}{d\delta} \left\{ \left\| \varphi(\cdot) + \delta \psi(\cdot) \right\|^2_{L^p} \right\}\bigg|_{\delta = 0} = 0.
\]

If \(\| \varphi(\cdot) \|_{L^p} \neq 0\),

(4.24)

\[
\frac{1}{2} \frac{d}{d\delta} \left\{ \left\| \varphi(\cdot) + \delta \psi(\cdot) \right\|^2_{L^p} \right\}\bigg|_{\delta = 0} = \frac{1}{2} \frac{d}{d\delta} \left\{ \left[ \mathbb{E} \left( \int_0^T |\varphi(t) + \delta \psi(t)|^q dt \right) \right]^{\frac{2}{q}} \right\}\bigg|_{\delta = 0}.
\]
provided the derivative on the right hand side exists, where
\[
f(\omega, \delta) = \left( \int_0^T |\varphi(t, \omega) + \delta\psi(t, \omega)|^\nu dt \right)^{\frac{2}{\nu}}.
\]
For simplicity of notation, we denote \( \infty \times 0 = 0 \). Then (4.23) and (4.24) can be combined into
\[
\frac{1}{2} \frac{d}{d\delta} \left\{ \|\varphi(\cdot) + \delta\psi(\cdot)\|_{L^{q, \nu}} \right\}
= \frac{1}{q} \|\varphi(\cdot)\|_{L^{q, \nu}}^2 \mathbf{1}_{\{\|\varphi(\cdot)\|_{L^{q, \nu}} \neq 0\}} \frac{d}{d\delta} \left\{ \mathbb{E}[f(\omega, \delta)] \right\}
\]
To exchange the order of derivation and expectation in (4.25), we calculate \( \frac{\partial f}{\partial \delta} \). For any \( \delta \in (-1, 1) \) and \( \omega \in \Omega \), if \( \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}} = 0 \), i.e., \( \varphi(t, \omega) + \delta\psi(t, \omega) = 0 \) a.e. \( t \in [0, T] \), then
\[
\frac{\partial f}{\partial \delta}(\omega, \delta) = \frac{1}{\Delta \delta} \left( f(\omega, \delta + \Delta\delta) - f(\omega, \delta) \right) = \frac{1}{\Delta \delta} \|\Delta\delta\psi(\cdot, \omega)\|_{L^{\nu}}^\nu = 0.
\]
On the other hand, when \( \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}} \neq 0 \),
\[
\frac{\partial f}{\partial \delta}(\omega, \delta) = \frac{\bar{q}}{\nu} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \frac{\partial}{\partial \delta} \left( \int_0^T g(t, \omega, \delta) dt \right),
\]
where
\[
g(t, \omega, \delta) = |\varphi(t, \omega) + \delta\psi(t, \omega)|^\nu.
\]
Combining (4.26) with (4.27), one has
\[
\frac{\partial f}{\partial \delta}(\omega, \delta) = \frac{\bar{q}}{\nu} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \frac{\partial}{\partial \delta} \left( \int_0^T g(t, \omega, \delta) dt \right).
\]
To exchange the order of derivation and integral in (4.28), we calculate
\[
\frac{\partial g}{\partial \delta} (t, \omega, \delta) = \nu |\varphi(t, \omega) + \delta\psi(t, \omega)|^{\nu-2} \langle \varphi(t, \omega) + \delta\psi(t, \omega), \psi(t, \omega) \rangle,
\]
and then
\[
\left| \frac{\partial g}{\partial \delta} (t, \omega, \delta) \right| \leq \nu |\varphi(t, \omega) + \delta\psi(t, \omega)|^{\nu-1} |\psi(t, \omega)|
\leq C \left( |\varphi(t, \omega)|^{\nu} + |\psi(t, \omega)|^{\nu} \right) \in L^1(0, T; \mathbb{R}),
\]
where \( C > 0 \) is a constant only depending on \( \nu \). By Theorem 2.27 in [9, Page 56], the order of derivation and integral in (4.28) can be exchanged. Then, we have
\[
\frac{\partial f}{\partial \delta}(\omega, \delta) = \bar{q} \frac{\partial g}{\partial \delta} (t, \omega, \delta) = \frac{\bar{q}}{\nu} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \int_0^T \frac{\partial g}{\partial \delta} (t, \omega, \delta) dt
\leq \bar{q} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-2} \langle \varphi(t, \omega) + \delta\psi(t, \omega), \psi(t, \omega) \rangle dt
\leq \bar{q} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \|\psi(t, \omega)\|_{L^{\nu}}^{\nu-1} \|\varphi(t, \omega)\|_{L^{\nu}}^{\nu-1} \|\psi(t, \omega)\|_{L^{\nu}}^{\nu-1}
\leq C \left( \|\varphi(\cdot, \omega)\|_{L^{\nu}}^{\nu} + \|\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu} \right) \in L^1(\Omega; \mathbb{R}),
\]
By virtue of Hölder’s inequality, we have
\[
\left| \frac{\partial f}{\partial \delta}(\omega, \delta) \right| \leq \bar{q} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \int_0^T \|\varphi(t, \omega) + \delta\psi(t, \omega)\|^{\nu-1} |\psi(t, \omega)| dt
\leq \bar{q} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \|\varphi(\cdot, \omega) + \delta\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu-1} \|\psi(t, \omega)\|_{L^{\nu}}^{\nu-1}
\leq C \left( \|\varphi(\cdot, \omega)\|_{L^{\nu}}^{\nu} + \|\psi(\cdot, \omega)\|_{L^{\nu}}^{\nu} \right) \in L^1(\Omega; \mathbb{R}),
\]
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where $C > 0$ is a constant only depending on $\hat{q}$. Theorem 2.27 in [9, Page 56] works again to exchange the order of derivation and expectation in (4.25). Then, combining with (4.29), we have

$$
\frac{1}{2} \frac{d}{dt} \left\{ \|\varphi(\cdot) + \delta\psi(\cdot)\|_{L^2}^2 \right\} |_{\delta=0} = \frac{1}{2q} \|\varphi(\cdot)\|_{L^2}^{2-q} \mathbf{1}_{\{(\varphi(\cdot)) \neq 0\}} \mathbb{E} \left\{ \frac{\partial}{\partial \delta} f(\omega, \delta) \right\} |_{\delta=0} \\
= \|\varphi(\cdot)\|_{L^2}^{2-q} \mathbf{1}_{\{(\varphi(\cdot)) \neq 0\}} \mathbb{E} \left\{ \|\varphi(\cdot, \omega)\|_{L^p}^{\nu-\nu} \int_0^T |\varphi(t)|^{\nu-2} \langle \varphi(t), \psi(t) \rangle dt \right\} \\
= \|\varphi(\cdot)\|_{L^2}^{2-q} \mathbb{E} \left\{ \|\varphi(\cdot, \omega)\|_{L^p}^{\nu-\nu} \int_0^T |\varphi(t)|^{\nu-2} \langle \varphi(t), \psi(t) \rangle dt \right\} \\
= \mathbb{E} \int_0^T \langle \|\varphi(\cdot)\|_{L^2}^{2-q} \mathbb{E} \left\{ \|\varphi(\cdot, \omega)\|_{L^p}^{\nu-\nu} \mathbf{1}_{\{(\varphi(\cdot), \omega)) \neq 0\}} \right\} |\varphi(t)|^{\nu-2} \varphi(t), \psi(t) \rangle dt \\
\equiv \mathbb{E} \int_0^T \Gamma(\varphi(\cdot))(t), \psi(t) \rangle dt,
$$

where

$$
\Gamma(\varphi(\cdot))(t) = \|\varphi(\cdot)\|_{L^2(\Omega; L^{\nu}(0,T;\mathbb{R}^m))}^{2-q} M(t)|\varphi(t)|^{\nu-2} \varphi(t), \ t \in [0, T],
$$

with

$$
M(t) = \mathbb{E}_t \left\{ \|\varphi(\cdot, \omega)\|_{L^p(0,T;\mathbb{R}^m)}^{\nu-\nu} \mathbf{1}_{\{(\varphi(\cdot), \omega)) \neq 0\}} \right\}, \ t \in [0, T].
$$

We have the following lemma.

**Lemma 4.6.** Let $p \leq \frac{2q\mu}{\nu\rho + 2\rho + 2\sigma}$. Then for any $\varphi(\cdot) \in L^q(\Omega; L^\nu(0,T;\mathbb{R}^m)) \equiv \mathbb{U}^{p,\rho,\sigma}[0,T]^*$,

$$
\Gamma(\varphi(\cdot)) \in L^p(\Omega; L^\mu(0,T;\mathbb{R}^m)) = \mathbb{U}^{p,\rho,\sigma}[0,T].
$$

We notice that, in $p \leq \frac{2q\mu}{\nu\rho + 2\rho + 2\sigma}$, $\rho$ takes values in $(1, \infty]$ and $\sigma$ takes values in $(2, \infty]$; When $\rho = \infty$ or/and $\sigma = \infty$, the right hand side of the inequality takes the limit. Moreover the inequality $p \leq \frac{2q\mu}{\nu\rho + 2\rho + 2\sigma}$ is equivalent to $\hat{q} \geq \nu$ or $\bar{p} \leq \mu$.

**Proof of Lemma 4.6.** Note that $p \leq \frac{2q\mu}{\nu\rho + 2\rho + 2\sigma}$ is equivalent to $\hat{q} \geq \nu$. It suffices to show that

$$
\hat{\Gamma}(\varphi(\cdot)) \equiv M(\cdot)|\varphi(\cdot)|^{\nu-2} \varphi(\cdot) \in L^q(\Omega; L^\nu(0,T;\mathbb{R}^m)).
$$

First of all, if $\hat{q} = \nu$, then $\bar{p} = \mu$. In this case, $M(\cdot) = 1$ and

$$
\hat{\Gamma}(\varphi(\cdot))(t) = |\varphi(t)|^{\nu-2} \varphi(t), \ t \in [0, T].
$$

Hence, (note that $L^q(\Omega; L^\nu(0,T;\mathbb{R}^m)) = L^q(0,T;\mathbb{R}^m)$ in the current case)

$$
\mathbb{E} \int_0^T \hat{\Gamma}(\varphi(\cdot))(t)^{\mu} dt = \mathbb{E} \int_0^T \left( |\varphi(t)|^{\nu-1} \right)^{\mu} dt = \mathbb{E} \int_0^T |\varphi(t)|^{\nu} dt < \infty.
$$

Now, let $\hat{q} > \nu$. Then

$$
\mu = \frac{\nu}{\nu - 1} > \frac{\hat{q} - 1}{\hat{q} - 1} = \bar{p},
$$

which leads to $\frac{\hat{q}\mu}{\nu\rho} > 1$. Note that $M(\cdot)$ is a (nonnegative valued) martingale. By Jensen's inequality,

$$
\mathbb{E} M(t)^{\frac{\hat{q}\mu}{\nu\rho}} = \mathbb{E} \left\{ \left( \int_0^T |\varphi(t)|^{\nu} dt \right)^{\frac{\hat{q}\mu}{\nu\rho}} \right\} \leq \mathbb{E} \left( \int_0^T |\varphi(t)|^{\nu} dt \right)^{\frac{\mu}{\nu}}.
$$

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Hence, using Doob’s inequality,

\[ E \left[ \sup_{t \in [0, T]} M(t)^{\frac{\sigma}{p-q}} \right] \leq \left( \frac{q}{p} \right)^{\frac{q}{p-q}} E \left[ M(T)^{\frac{\sigma}{p-q}} \right] = \left( \frac{q}{p} \right)^{\frac{q}{p-q}} \left( \int_0^T |\varphi(t)|^\mu dt \right)^{\frac{\sigma}{\mu}}. \]

Consequently,

\[ E \left( \int_0^T \left| \hat{\Gamma}(\varphi(\cdot))(t) \right|^\mu dt \right)^{\frac{\sigma}{p-q}} \leq \left( \frac{q}{p} \right)^{\frac{q}{p-q}} \left( \int_0^T |\varphi(t)|^\mu dt \right)^{\frac{\sigma}{\mu}} \]

\[ \leq \left[ E \left( \sup_{t \in [0, T]} M(t)^{\frac{\sigma}{p-q}} \right)^{\frac{\mu}{\sigma}} \left( \int_0^T |\varphi(t)|^\mu dt \right)^{\frac{\sigma}{\mu}} \right]^{\frac{\mu}{\sigma}} \]

\[ = \left( \frac{q}{p} \right)^{\frac{q}{q-q}} \left( \int_0^T |\varphi(t)|^\mu dt \right)^{\frac{\mu}{q}} \frac{\sigma pq}{\mu p - \sigma q} \leq \left( \frac{q}{p} \right)^{\frac{q}{q-q}} \left( \int_0^T |\varphi(t)|^\mu dt \right)^{\frac{\mu}{q}} \frac{\sigma pq}{\mu p - \sigma q}. \]

This proves out conclusion. □

Now, let us look at the optimal solution \( \bar{\eta} \) of Problem (O). According to the above, we see that the optimal solution \( \bar{\eta} \) of Problem (O) satisfies the following:

\[ 0 = E(\mathcal{K}(\mathcal{K}^* \bar{\eta}) + \mathcal{K}_0 x - \xi, \eta), \quad \forall \eta \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n), \]

where \( \mathcal{K}(\mathcal{K}^* \bar{\eta}) \) is given by (4.31) with \( \varphi(\cdot) = \mathcal{K}^* \bar{\eta} \). Thus,

\[ (4.35) \quad \mathcal{K} \mathcal{K}^* \bar{\eta} + \mathcal{K}_0 x - \xi = 0. \]

Now, when \( p \leq \frac{2\sigma p}{\sigma p - 2\mu + 2\sigma} \), we define

\[ (4.36) \quad \bar{u}(\cdot) = \mathcal{K}(\mathcal{K}^* \bar{\eta}) \in L^\mu_0(\Omega; L^\mu(0, T; \mathbb{R}^m)) \equiv \mathcal{U}^{p, \sigma}[0, T]. \]

Then (4.35) reads

\[ \xi = \mathcal{K} \bar{u}(\cdot) + \mathcal{K}_0 x = X(T; x, \bar{u}(\cdot)), \]

which means that \( \bar{u}(\cdot) \in \mathcal{U}^{p, \sigma}[0, T] \) is a control steering \( x \in \mathbb{R}^n \) to \( \xi \in L^\mu_0(\Omega; \mathbb{R}^n) \). Therefore, we obtain the following result, making use of Proposition 4.1 and Theorem 4.2.

**Theorem 4.7.** Let (H1)–(H2) hold and \( p \leq \frac{2\sigma p}{\sigma p - 2\mu + 2\sigma} \). Then the observability inequality (4.6) holds if and only if for any \((x, \xi) \in \mathbb{R}^n \times L^\mu_0(\Omega; \mathbb{R}^n)\), Problem (O) admits a unique optimal solution \( \bar{\eta} \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \). In this case, the control \( \bar{u}(\cdot) \in \mathcal{U}^{p, \sigma}[0, T] \) defined by (4.36) steers \( x \) to \( \xi \). Moreover, with \( \bar{u}(\cdot) \) defined by (4.36) for

\[ \mathcal{K}^* \bar{\eta} = B(\cdot)^\top \bar{Y}(\cdot) + \sum_{k=1}^d D_k(\cdot)^\top \bar{Z}_k(\cdot), \]

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the following coupled FBSDE

\[
\begin{align*}
\begin{cases}
    d\tilde{X}(t) &= \left[ A(t)\tilde{X}(t) + B(t)\tilde{u}(t) \right] dt + \sum_{k=1}^{d} \left[ C_k(t)\tilde{X}(t) + D_k(t)\tilde{u}(t) \right] dW_k(t), \\
    d\tilde{Y}(t) &= -\left[ A(t)^\top\tilde{Y}(t) + \sum_{k=1}^{d} C_k(t)^\top\tilde{Z}_k(t) \right] dt + \sum_{k=1}^{d} \tilde{Z}_k(t) dW_k(t), \\
    \tilde{X}(0) &= x, \quad \tilde{X}(T) = \xi
\end{cases}
\end{align*}
\] (4.37)

admits a unique adapted solution

\[
(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)) \in \mathbb{L}_p^p(\Omega; C([0,T]; \mathbb{R}_d^n)) \times \mathbb{L}_p^p(\Omega; C([0,T]; \mathbb{R}_d^n)) \times \mathbb{L}_p^p(\Omega; L^2(0,T; \mathbb{R}_d^{n \times d})).
\]

Proof. By the above analysis, the only remaining thing is to prove the uniqueness of FBSDE (4.37). Now, let \((\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)) \in \mathbb{L}_p^p(\Omega; C([0,T]; \mathbb{R}_d^n)) \times \mathbb{L}_p^p(\Omega; C([0,T]; \mathbb{R}_d^n)) \times \mathbb{L}_p^p(\Omega; L^2(0,T; \mathbb{R}_d^{n \times d}))\) be a solution to (4.37) with

\[
\tilde{u}(\cdot) = \Gamma^{\top}(\cdot) = \Gamma \left( B(\cdot)^\top \tilde{Y}(\cdot) + \sum_{k=1}^{d} D_k(\cdot)^\top \tilde{Z}_k(\cdot) \right),
\]

and \(\Gamma(\cdot)\) is given by (4.31). When \(p \leq \frac{2\sigma}{\sigma^{\rho}-2\rho+2\sigma}\). Lemma 4.6 implies \(\tilde{u}(\cdot) \in \mathbb{U}^{p,\rho,\sigma}[0,T]\). Next we prove that \(\tilde{\eta} = \tilde{Y}(T)\) is an optimal solution to Problem (O). For each \(\eta \in \mathbb{L}_p^p(\Omega; \mathbb{R}_d^n)\), we denote the solution to BSDE (4.8) by \((Y(\cdot), Z(\cdot))\), then

\[
J(\eta; x, \xi) - J(\tilde{\eta}; x, \xi) = \frac{1}{2} \left( \|K^\ast\eta\|^2_{\mathbb{L}_p^{p,\rho,\sigma}[0,T]} - \|K^\ast\tilde{\eta}\|^2_{\mathbb{L}_p^{p,\rho,\sigma}[0,T]} \right) + \langle x, K^\ast_0\eta - K^\ast_0\tilde{\eta} \rangle - \mathbb{E}\langle \xi, \eta - \tilde{\eta} \rangle.
\]

Due to the convexity of \(\|\cdot\|_{\mathbb{L}_p^{p,\rho,\sigma}[0,T]}\), (4.38), (12) and (4.20), we have

\[
\begin{align*}
J(\eta; x, \xi) - J(\tilde{\eta}; x, \xi) &\geq \mathbb{E} \int_0^T \langle \tilde{u}(t), B(t)^\top (Y(t) - \tilde{Y}(t)) \rangle + \sum_{k=1}^{d} \langle D_k(t)^\top (Z_k(t) - \tilde{Z}_k(t)) \rangle dt \\
&\quad + \langle x, Y(0) - \tilde{Y}(0) \rangle - \mathbb{E}\langle \xi, \eta - \tilde{\eta} \rangle.
\end{align*}
\]

We apply Itô’s formula to \(\langle \tilde{X}(\cdot), \tilde{Y}(\cdot) - \tilde{Y}(\cdot) \rangle\), and by (4.37), we obtain the right hand side of the above inequality equals zero. Hence we have \(J(\eta; x, \xi) - J(\tilde{\eta}; x, \xi) \geq 0\), which implies that \(\tilde{\eta} = \tilde{Y}(T)\) is an optimal solution to Problem (O).

Now, let \((X^i(\cdot), Y^i(\cdot), Z^i(\cdot)) \in \mathbb{L}_p^p(\Omega; C([0,T]; \mathbb{R}_d^n)) \times \mathbb{L}_p^p(\Omega; C([0,T]; \mathbb{R}_d^n)) \times \mathbb{L}_p^p(\Omega; L^2(0,T; \mathbb{R}_d^{n \times d})) (i = 1, 2)\) be two solutions to (4.37). From the above analysis, both \(Y^1(T)\) and \(Y^2(T)\) are optimal controls to Problem (O). By the uniqueness of optimal control (see Proposition 4.1), \(Y^1(T) = Y^2(T)\). Moreover, by the uniqueness of BSDE, we have \((Y^1(\cdot), Z^1(\cdot)) = (Y^2(\cdot), Z^2(\cdot))\). Furthermore, by the uniqueness of FSDE, we have \(X^1(\cdot) = X^2(\cdot)\). We obtain the uniqueness of (4.37), and complete the proof.

\[\blacksquare\]

Remark 4.8. The notion of adaptability represents a fundamental difference between deterministic and stochastic systems. From the derivation of Fréchet derivative (see the third line of (4.30)), we can obtain naturally a process:

\[
\begin{align*}
\tilde{\Gamma}(\varphi(\cdot))(t) &\equiv \|\varphi(\cdot)\|^2_{\mathbb{L}_p^p(\Omega; L^\nu(0,T; \mathbb{R}_d^n))} \|\varphi(\cdot, \omega)\|^\nu_{L^\nu(0,T; \mathbb{R}_d^n)} |\varphi(t)|^{\nu-\varphi(t)}, \quad t \in [0,T]
\end{align*}
\]

which is closely linked to our problem. But unfortunately, \(\tilde{\Gamma}(\varphi(\cdot))(\cdot)\) is not \(\mathbb{F}\)-adapted when \(p \neq \frac{2\sigma}{\sigma^{\rho}-2\rho+2\sigma}\) (equivalently \(\tilde{q} \neq \nu\)). Hence, in order to meet the requirement of adaptability, we use

\[
\Gamma(\varphi(\cdot))(t) = \mathbb{E}_t[\tilde{\Gamma}(\varphi(\cdot))(t)], \quad t \in [0,T]
\]

30
to replace $\tilde{\Gamma}(\varphi(\cdot))(\cdot)$. However, this treatment leads to some difficulty. As a matter of fact, through a direct calculation, we can obtain that the following equation

$$
\left\{ E\left( \int_0^T |\tilde{\Gamma}(\varphi(\cdot))(t)|^\beta dt \right) \right\}^{\frac{1}{\beta}} = \|\varphi(\cdot)\|_{L^p(0,T)}^\beta.
$$

holds for any $p \in (1, \infty)$. But due to the introduction of conditional expectation, we only get an inequality

$$
\left\{ E\left( \int_0^T |\Gamma(\varphi(\cdot))(t)|^\beta dt \right) \right\}^{\frac{1}{\beta}} \leq \frac{2}{\beta} \|\varphi(\cdot)\|_{L^p(0,T)}^\beta
$$

for $p \leq \frac{2\beta}{\sigma^2 + 2\beta}$. The technique involving Doob’s martingale inequality used in Lemma 4.6 is invalid for $p > \frac{2\beta}{\sigma^2 + 2\beta}$.

It is naturally to ask what happens if $p > \frac{2\beta}{\sigma^2 + 2\beta}$? To obtain a similar result as Theorem 4.7, instead of functional $J(x, \xi; \eta)$ defined by (4.21), we introduce the following functional:

$$
J'(\eta; x, \xi) = \frac{1}{2} \|K^\ast \eta\|^2_{L^p(0,T)} + \langle x, K^\ast \eta \rangle - E(\xi, \eta), \quad \forall \eta \in L^p_{2}\bar{F}_T(\Omega; \mathbb{R}^n),
$$

where $K^\ast$ and $K^\ast_0$ are given by (4.12) and (4.20), respectively. Equivalently,

$$
J'(\eta; x, \xi) = \frac{1}{2} B(\gamma)\bar{Y}(\gamma) + \sum_{k=1}^d D_k(\gamma)\bar{Z}_k(\gamma)|\|\xi, \eta|\|_{L^p(0,T)}^2 + \langle x, Y(0) \rangle - E(\xi, \eta),
$$

$$
\forall \eta \in L^p_{2}\bar{F}_T(\Omega; \mathbb{R}^n),
$$

with $(Y(\cdot), Z(\cdot))$ being the adapted solution to BSDE (4.8). Note that we have changed from $\mathbb{U}^{p,\rho,\sigma}[0,T]$ to $\mathbb{U}^{p,\rho,\sigma}[0,T]$ in the above. We now pose the following optimization problem.

**Problem (O)’.** Minimize (4.39) subject to BSDE (4.8) over $L^p_{2}\bar{F}_T(\Omega; \mathbb{R}^n)$.

Suppose $\varphi(\cdot), \psi(\cdot) \in L^p_{2}(0,T; L^q(\Omega; \mathbb{R}^m))$. A similar procedure as Problem (O) leads to

$$
\frac{1}{2} \frac{d}{dt} \left\{ \|\varphi(\cdot) + \delta \psi(\cdot)\|^2_{L^p(0,T)} \right\} \bigg|_{\delta = 0} = \|\varphi(\cdot)\|_{L^p(0,T)}^2 \int_0^T \left( E|\varphi(t)|^q \right)^{\frac{q-2}{q}} E\left[ |\varphi(t)|^q \varphi(t), \psi(t) \right] dt
$$

$$
\equiv E \int_0^T (\Gamma'(\varphi(\cdot))(t), \psi(t)) dt,
$$

where

$$
\Gamma'(\varphi(\cdot))(t) = \|\varphi(\cdot)\|_{L^p(0,T; L^q(\Omega; \mathbb{R}^m))}^2 \left( E|\varphi(t)|^q \right)^{\frac{q-2}{q}} |\varphi(t)|^q \varphi(t), \quad t \in [0, T].
$$

Unlike the case of Problem (O), in the above we do not need conditional expectation. Due to this, the following lemma holds without the constraint $p \leq \frac{2\beta}{\sigma^2 + 2\beta}$. However, due to the use of $\mathbb{U}^{p,\rho,\sigma}[0,T]^\ast$, condition $p \geq 2$ is needed.

**Lemma 4.9.** Let $p \geq 2$. Then for any $\varphi(\cdot) \in L^p_{2}(0,T; L^q(\Omega; \mathbb{R}^m)) \equiv \mathbb{U}^{p,\rho,\sigma}[0,T]^\ast$,

$$
\Gamma'(\varphi(\cdot))(\cdot) \in L^p_{2}(0,T; L^q(\Omega; \mathbb{R}^m)) = \mathbb{U}^{p,\rho,\sigma}[0,T],
$$

(4.42)
Proof. It suffices to calculate the following:
\[
\int_0^T \left\{ \mathbb{E} \left[ \left| \varphi(t) \right|^q \right] \right\}^{\frac{q}{p}} dt = \int_0^T \left[ \mathbb{E} \left[ \left| \varphi(t) \right|^q \right] \right]^{\frac{q}{p}} dt = \int_0^T \mathbb{E} \left[ \left| \varphi(t) \right|^q \right]^{\frac{q}{p}} dt < \infty.
\]
Hence, our conclusion follows. \qed

Then similar to Theorem 4.7, we have the following result.

**Theorem 4.10.** Let (H1) and (H2)' hold and \( p \geq 2 \). Then the observability inequality (4.7) holds if and only if for any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), Problem (O)' admits a unique optimal solution \( \eta' \in L^q_{F_T}(\Omega; \mathbb{R}^n) \).

In this case, the control \( \tilde{u}'(\cdot) \in U^{p,\rho,\sigma}[0, T] \) defined by the following

\begin{equation}
(4.43) \quad \tilde{u}'(\cdot) = \Gamma'(\mathbb{K}^* \eta'),
\end{equation}

with \( \Gamma'(\cdot) \) given by (4.41) steers \( x \) to \( \xi \). Moreover, with such defined \( \tilde{u}'(\cdot) \), the coupled FBSDE (4.37) admits a unique adapted solution

\[
(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)) \in L^p_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^{n \times d})).
\]

## 5 Norm optimal control problems

When the system (1.1) is \( L^p \)-exact controllability on \([0, T]\) by \( U^{p,\rho,\sigma}[0, T] \) (respectively, \( U^{p,\rho,\sigma}[0, T] \)) for any given \( p \) satisfying \( p \leq \frac{2\sigma p}{\sigma p - 2p + 2\sigma} \) (respectively, \( p \geq 2 \)), then for any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), from Theorem 4.7 (respectively, Theorem 4.10), we know that \( \tilde{u} \) (respectively, \( \tilde{u}' \)) defined by (4.36) (respectively, by (4.43)) is one of \( U^{p,\rho,\sigma}[0, T] \)- (respectively, \( U^{p,\rho,\sigma}[0, T] \)-) admissible controls which steers the state process from the initial value \( x \) to the terminal value \( \xi \). In this section, we shall restrict to the case \( p = \frac{2\sigma p}{\sigma p - 2p + 2\sigma} \) (respectively, \( p \geq 2 \)) and further show that \( \tilde{u} \) (respectively, \( \tilde{u}' \)) has a characteristic of minimum norm.

First, for any given \( 1 < \rho \leq \infty, 2 < \sigma \leq \infty \) and \( p = \frac{2\sigma p}{\sigma p - 2p + 2\sigma} \), we introduce a \( U^{p,\rho,\sigma}[0, T] \)-norm optimal control problem: for any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), minimize \( \| u \|_{U^{p,\rho,\sigma}[0, T]} \) over the \( U^{p,\rho,\sigma}[0, T] \)-admissible control set:

\[
U(x, \xi) := \left\{ u(\cdot) \in U^{p,\rho,\sigma}[0, T] \mid X(T; x, u(\cdot)) = \xi \right\}.
\]

For simplicity, we denote the above \( U^{p,\rho,\sigma}[0, T] \)-norm optimal control problem by **Problem (N)**. Note that the system (1.1) is \( L^p \)-exactly controllable on \([0, T]\) by \( U^{p,\rho,\sigma}[0, T] \) if and only if, for any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), the \( U^{p,\rho,\sigma}[0, T] \)-admissible control set \( U(x, \xi) \) is not empty. We call \( \tilde{u} \in U(x, \xi) \) a \( U^{p,\rho,\sigma}[0, T] \)-norm optimal control to Problem (N) if

\[
\| \tilde{u} \|_{U^{p,\rho,\sigma}[0, T]} = \inf_{u(\cdot) \in U(x, \xi)} \| u(\cdot) \|_{U^{p,\rho,\sigma}[0, T]}.
\]

For any given \( 2 \leq p < \rho \leq \infty \) and \( 2 < \sigma \leq \infty \), we similarly introduce the \( U^{p,\rho,\sigma}[0, T] \)-norm optimal control problem reading:

**Problem (N)'**. For any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), minimize \( \| u \|_{U^{p,\rho,\sigma}[0, T]} \) over the \( U^{p,\rho,\sigma}[0, T] \)-admissible control set

\[
U'(x, \xi) := \left\{ u \in U^{p,\rho,\sigma}[0, T] \mid X(T; x, u(\cdot)) = \xi \right\}.
\]

In the previous section, we have given some equivalent conditions for the \( L^p \)-exact controllability of system (1.1) on \([0, T]\) by \( U^{p,\rho,\sigma}[0, T] \) (respectively, \( U^{p,\rho,\sigma}[0, T] \)) (see Theorem 4.2, Theorem 4.5, Theorem 4.7 and Theorem 4.10). Now, by virtue of the related \( U^{p,\rho,\sigma}[0, T] \)- (respectively, \( U^{p,\rho,\sigma}[0, T] \)-) norm optimal control problem, we present another one.
Theorem 5.1. Let the assumptions (H1), (H2) and \( p = \frac{2\sigma}{\sigma - 2\rho + 2\epsilon} \) hold. Then the following two statements are equivalent:

- For any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), Problem (O) admits a unique optimal solution \( \bar{\eta} \in L^q_{F_T}(\Omega; \mathbb{R}^n)\);

- For any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), Problem (N) admits a unique optimal control \( \bar{u}(\cdot) \in U^{p,\rho,\sigma}[0,T] \).

Moreover, the unique optimal control \( \bar{u} \) to Problem (N) is given by (4.36), and the minimal norm is given by

\[
||\bar{u}(\cdot)||_{U^{p,\rho,\sigma}[0,T]} = \sqrt{\mathbb{E}(\xi, \bar{\eta}) - \langle x, \mathbb{K}_{\sigma}^\epsilon \eta \rangle}.
\]

The minimal value of functional \( J(\cdot; x, \xi) \) is given by

\[
J(\bar{\eta}; x, \xi) = \frac{1}{2}||\bar{u}(\cdot)||^2_{U^{p,\rho,\sigma}[0,T]} = -\frac{1}{2}(\mathbb{E}(\xi, \bar{\eta}) - \langle x, \mathbb{K}_{\sigma}^\epsilon \eta \rangle).
\]

Proof. (Sufficiency). Since for any \((x, \xi) \in \mathbb{R}^n \times L^p_{F_T}(\Omega; \mathbb{R}^n)\), Problem (N) admits an optimal control, then the corresponding \( U^{p,\rho,\sigma}[0,T] \)-admissible control set \( U(x, \xi) \) is not empty. Therefore, the system (1.1) is \( L^p \)-exactly controllable. By Theorem 4.2 and Theorem 4.7, the first statement holds true.

(Necessity). First of all, when \( p = \frac{2\sigma}{\sigma - 2\rho + 2\epsilon} \), by Lemma 4.6, \( \bar{u} \) defined by (4.36) is a \( U^{p,\rho,\sigma}[0,T] \)-admissible control to Problem (N). Let \( \bar{\eta} \in L^q_{F_T}(\Omega; \mathbb{R}^n) \) be the optimal solution to Problem (O), and \((\bar{Y}(\cdot), \bar{Z}(\cdot))\) be the solution to BSDE (4.8) with terminal condition \( \bar{Y}(T) = \bar{\eta} \). For any \( \eta \in L^q_{F_T}(\Omega; \mathbb{R}^n) \), similarly, we denote by \((Y(\cdot), Z(\cdot))\) the solution to BSDE (4.8) with terminal condition \( Y(T) = \eta \). Since \( \bar{\eta} \) is optimal, we obtain the following relation named the Euler-Lagrange equation:

\[
0 = \frac{d}{d\delta} \left[ J(\bar{\eta}; x, \xi) \right]_{\delta=0} = \mathbb{E} \int_0^T (\Gamma(K^\epsilon\eta)(t), (K^\epsilon\eta)(t))dt + \langle x, \mathbb{K}_{\sigma}^\epsilon \eta \rangle - \mathbb{E}\xi, \eta \rangle.
\]

Meanwhile, for any control \( u(\cdot) \in U(x, \xi) \), by applying Itô’s formula to \((X(\cdot; x, u(\cdot)), Y(\cdot))\) on the interval \([0,T]\), we have

\[
\mathbb{E} \int_0^T (u(t), (K^\epsilon\eta)(t))dt + \langle x, Y(0) \rangle = \mathbb{E}\xi, \eta \rangle = 0.
\]

By letting \( \eta = \bar{\eta} \) in both (5.3) and (5.4), for any \( u \in U(x, \xi) \), we obtain

\[
\mathbb{E} \int_0^T (\bar{u}(t), (K^\epsilon\eta)(t))dt = \mathbb{E}\xi, \eta \rangle - \langle x, \bar{Y}(0) \rangle = \mathbb{E} \int_0^T (\bar{u}(t), (K^\epsilon\eta)(t))dt.
\]

It is easy to calculate that

\[
\mathbb{E} \int_0^T (\bar{u}(t), (K^\epsilon\eta)(t))dt = ||K^\epsilon\eta||^2_{U^{p,\rho,\sigma}[0,T]},
\]

and

\[
||\bar{u}(\cdot)||_{U^{p,\rho,\sigma}[0,T]} = ||K^\epsilon\eta||_{U^{p,\rho,\sigma}[0,T]}.
\]

By above three equations and Hölder’s inequality, we get the optimality of \( \bar{u} \). The uniqueness of the optimal control to Problem (N) comes from the strictly convex property of \( U^{p,\rho,\sigma}[0,T] \).

Let us come back to the Euler-Lagrange equation. Letting \( \eta = \bar{\eta} \) and from the definition of \( \bar{u} \), (5.3) is reduced to

\[
||\bar{u}(\cdot)||^2_{U^{p,\rho,\sigma}[0,T]} = \mathbb{E}(\xi, \bar{\eta}) - \langle x, \bar{Y}(0) \rangle.
\]

which, together with (4.20), implies (5.1). Then, we calculate

\[
J(\bar{\eta}; x, \xi) = \frac{1}{2}||\bar{u}(\cdot)||^2_{U^{p,\rho,\sigma}[0,T]} - \left( \mathbb{E}(\xi, \bar{\eta}) - \langle x, \bar{Y}(0) \rangle \right) = \frac{1}{2}||\bar{u}(\cdot)||^2_{U^{p,\rho,\sigma}[0,T]},
\]

which is (5.2), and the proof is completed. \(\square\)
Remark 5.2. When we consider the corresponding $U_{p,\rho,\sigma}^p[0,T]$-norm optimal control problems with $p < \frac{2\rho}{\sigma + 2\rho + 2\sigma}$ in the same way, we can obtain
\[
\| \hat{u}^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]} \leq \| u^{*} \|_{U_{p,\rho,\sigma}^p[0,T]},
\]
where $u^{*} \in \mathbb{G}_{U_{p,\rho,\sigma}^p[0,T]}$ is any given $U_{p,\rho,\sigma}^p[0,T]$-admissible control. If we can also obtain
\[
\| u^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]} \leq \| \hat{u}^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]}^{*},
\]
then we will solve $U_{p,\rho,\sigma}^p[0,T]$-norm optimal control problems. However, due to the additional conditional expectation in the Fréchet derivative $\Gamma(\cdot)$, we cannot prove the above inequality. In fact, we only have (comparing (4.34))
\[
\| \hat{u}^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]} \leq \frac{q}{p} \| \hat{u}^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]},
\]

Since conditional expectation is not introduced in $\Gamma^{*}(\cdot)$, we can solve the $U_{p,\rho,\sigma}^p[0,T]$-norm optimal control problems for any $p \geq 2$, whose proof is similar to that of Theorem 5.1.

Theorem 5.3. Let the assumptions (H1), (H2)' and $p \geq 2$ hold. Then the following two statements are equivalent:

- For any $(x, \xi) \in \mathbb{R}^n \times L^p_{F,J}(\Omega; \mathbb{R}^n)$, Problem (O)' admits a unique optimal solution $\hat{u}^{*} \in L^p_{F,J}(\Omega; \mathbb{R}^n)$;
- For any $(x, \xi) \in \mathbb{R}^n \times L^p_{F,J}(\Omega; \mathbb{R}^n)$, Problem (N)' admits a unique optimal control $\hat{u}^{*}(\cdot) \in U_{p,\rho,\sigma}^p[0,T]$. Moreover, the unique norm optimal control $\hat{u}^{*}(\cdot)$ to Problem (N)' is given by (4.43), and the minimal norm is given by
\[
\| \hat{u}^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]} = \sqrt{\mathbb{E}(\xi, \hat{u}^{*}) - \langle x, \mathbb{K}^*_\rho \hat{u}^{*} \rangle}. 
\]

The minimal value of functional $J^{*}(\cdot; x, \xi)$ is given by
\[
J^{*}(\hat{u}; x, \xi) = \frac{1}{2} \| \hat{u}^{*}(\cdot) \|_{U_{p,\rho,\sigma}^p[0,T]}^{2} = \frac{1}{2} \left( \mathbb{E}(\xi, \hat{u}^{*}) - \langle x, \mathbb{K}^*_\rho \hat{u}^{*} \rangle \right). 
\]

Remark 5.4. When $p = \frac{2\rho}{\sigma + 2\rho + 2\sigma}$ (equivalently, $\bar{p} = \mu$), the spaces $U_{p,\rho,\sigma}^p[0,T]$ and $U_{p,\rho,\sigma}^p[0,T]$ coincide with $L^p_{F}(0,T; \mathbb{R}^m)$, and then both Problem (N) and Problem (N)' imply the $L^p_{F}(0,T; \mathbb{R}^m)$-norm optimal control problem. Precisely, Theorems 5.1 and 5.3 provide the same result for the $L^p_{F}(0,T; \mathbb{R}^m)$-norm optimal control problem when the system is $L^p$-exactly controllable with $p \geq 2$. However, when $1 < p < 2$, only Theorem 5.1 gives a result, and Theorem 5.3 is invalid.

More specifically, when $\sigma = \infty$ and $p = \frac{2\rho}{\sigma + 2\rho + 2\sigma}$ (equivalently, $\bar{p} = \mu = 2$), $U_{p,\rho,\sigma}^p[0,T] = U_{p,\rho,\sigma}^p[0,T] = L^p_{F}(0,T; \mathbb{R}^m)$. Problem (N) becomes the classical norm optimal control problem (see [24]), and Theorem 5.1 provides a result for the classical $L^p_{F}(0,T; \mathbb{R}^m)$-norm optimal control problem. We notice that in the present paper the matrices $B(\cdot)$ and $D_k(\cdot) (k = 1, 2, \ldots, d)$ are not necessary to be bounded (see Assumption (H2)), while in the literature, only bounded matrix cases were studied. Furthermore, instead of the standard norm $\| \cdot \|_{L^p_{F}(0,T; \mathbb{R}^m)}$, we can extend our method to minimize the following generalized weighted norm
\[
\left( \mathbb{E} \int_0^T \langle R(t)u(t), u(t) \rangle dt \right)^{\frac{1}{2}}
\]
such that \( R(\cdot) \in L^q_{F}(0,T; \mathbb{R}^{m \times m}) \) is symmetric, and there exists a constant $\delta > 0$ such that $R(t) - \delta I$ is positive semi-definite for a.e. $t \in [0,T]$.

In fact, due to the definition of $R(\cdot)$ and Denman-Beavers iteration [13] for the square roots of matrices, there exists a matrix-valued process $N(\cdot) \in L^\infty_{F}(0,T; \mathbb{R}^{m \times m})$ which is invertible and $N^{-1}$ is bounded also, such that $R(\cdot) = N^{-1}(\cdot)N(\cdot)$. Then, letting
\[
\hat{u}(\cdot) = N(\cdot)u(\cdot), \quad \hat{B}(\cdot) = B(\cdot)N^{-1}(\cdot), \quad \hat{D}_j(\cdot) = D_j(\cdot)N^{-1}(\cdot), \quad j = 1, 2, \ldots, d,
\]
we transform the original weighted norm optimal control problem (1.1) and (5.8) to the following equivalent standard one: to minimize $\|\hat{u}(\cdot)\|_{L^2_p(0,T;\mathbb{R}^m)}$ subject to

$$
\begin{cases}
    dX(t) = \left[A(t)X(t) + B(t)\hat{u}(t)\right]dt + \sum_{k=1}^{d} \left[C_k(t)X(t) + D_k(t)\hat{u}(t)\right]dW_k(t), & t \in [0,T], \\
    X(0) = x,
\end{cases}
$$

over the corresponding $L^2_p(0,T;\mathbb{R}^m)$-admissible control set

$$
\hat{U}(x,\xi) \equiv \left\{ \hat{u}(\cdot) \in L^2_p(0,T;\mathbb{R}^m) \mid X(T; x, u(\cdot)) = \xi \right\} = N^{-1}U(x,\xi)
$$

Applying the results of Theorems 4.2, 4.7 and 5.1, we can solve the generalized weighted norm optimal control problem.

**Corollary 5.5.** Let $\sigma = \infty$, $p = \frac{2p}{p+2}$ and (H1)–(H2) hold. Then the system (1.1) is $L^p$-exactly controllable on $[0,T]$ by $L^2_p(0,T;\mathbb{R}^m)$, if and only if, for any $(x,\xi) \in \mathbb{R}^n \times L^2_p(\Omega;\mathbb{R}^n)$, the weighted norm optimal control problem (1.1) and (5.8) admits a unique optimal control. In this case, with

$$
(5.9) \quad \hat{u}(t) = R^{-1}(t)\left(B^\top(t)\hat{Y}(t) + \sum_{k=1}^{d} D_k(t)\hat{Z}_k(t)\right), \quad t \in [0,T],
$$

and the following coupled FBSDE

$$
(5.10) \quad \begin{cases}
    d\bar{X}(t) = A(t)\bar{X}(t)dt + \sum_{k=1}^{d} \left[C_k(t)\bar{X}(t) + D_k(t)\hat{u}(t)\right]dW_k(t), \\
    -d\hat{Y}(t) = A(t)\hat{Y}(t)dt + \sum_{k=1}^{d} C_k(t)\hat{Z}_k(t)dt - \sum_{k=1}^{d} \hat{Z}_k(t)dW_k(t), \\
    \bar{X}(0) = x, \quad \hat{X}(T) = \xi
\end{cases}
$$

admits a unique adapted solution

$$(\bar{X}(\cdot), \hat{Y}(\cdot), \hat{Z}(\cdot)) \in L^2_p(\Omega;C(0,T;\mathbb{R}^n)) \times L^2_p(\Omega;C(0,T;\mathbb{R}^n)) \times L^2_p(\Omega;L^2(0,T;\mathbb{R}^{n \times d})).$$

Moreover, $\hat{u}(\cdot)$ defined by (5.9) is the unique weighted norm optimal control, and the minimal weighted norm is given by

$$
\left(\mathbb{E} \int_0^T \langle R(t)\hat{u}(t), \hat{u}(t)\rangle dt\right)^{\frac{1}{2}} = \sqrt{\mathbb{E}\langle \xi, \hat{Y}(T)\rangle - \langle x, \hat{Y}(0)\rangle}.
$$

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