Mean-field dynamics of a non-hermitian Bose-Hubbard dimer

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We investigate an $N$-particle Bose-Hubbard dimer with an additional effective decay term in one of the sites. A mean-field approximation for this non-hermitian many-particle system is derived, based on a coherent state approximation. The resulting nonlinear, non-hermitian two-level dynamics, in particular the fixed point structures showing characteristic modifications of the self trapping transition, are analyzed. The mean-field dynamics is found to be in reasonable agreement with the full many-particle evolution.

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In the theoretical investigation of Bose-Einstein condensates (BEC) the mean-field approximation leading to the description via a Gross-Pitaevskii nonlinear Schrödinger equation (GPE) is almost indispensable. It is usually achieved by replacing the bosonic field operators in the multi-particle system with c-numbers (the effective single-particle condensate wave functions), and describes the system quite well for large particle numbers and low temperatures. This approach is closely related to a classicalization\cite{1} and allows for the application of semiclassical methods\cite{2}.

Recently considerable attention has been paid to effective non-hermitian mean-field theories describing the scattering and transport behavior of BECs\cite{3}, as well as the implications of decay (boundary dissipation)\cite{4, 5, 6}. The latter is closely related to atom laser, for which it is possible to go beyond the limit of decay (boundary dissipation)\cite{4, 5, 6}. The latter is closely related to a classicalization\cite{1} and allows for the ad hoc formulation of mean-field mechanics on a fundamental level\cite{8}.

However, the non-hermitian GPE has been formulated in an \textit{ad hoc} manner as a generalization of the mean-field Hamiltonian and a derivation starting from a non-hermitian many-particle system is required. This is as well interesting in a wider context of the classical limits of effective non-hermitian quantum theories. In the present letter we therefore introduce a generalized mean-field approximation and investigate the characteristic features of the dynamics resulting from the interplay of nonlinearity and non-hermiticity for a simple many-particle Hamiltonian of Bose-Hubbard type, describing a BEC in a leaking double well trap:

$$\hat{H} = (\varepsilon - 2\gamma)\hat{a}_1^\dagger \hat{a}_1 - \varepsilon \hat{a}_2^\dagger \hat{a}_2 + \nu (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger)$$

$$+ \frac{c}{2} (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)^2.$$  \hfill (1)

Here $\hat{a}_j, \hat{a}_j^\dagger$ are bosonic particle annihilation and creation operators for the $j$th mode. The onsite energies are $\pm \varepsilon$, $\nu$ is the coupling constant and $c$ is the strength of the onsite interaction. The additional imaginary part of the mode energy $\gamma$ describes the first mode as a resonance state with a finite lifetime, like, e.g., the Wannier-Stark states for a tilted optical lattice\cite{3}. A direct experimental realization could be achieved by tunneling escape of atoms from one of the wells. Even in the non-hermitian case, the Hamiltonian commutes with the total number operator $\hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$ and the number $N$ of particles is conserved. The “decay” describes not a loss of particles but models the decay of the probability to find the particles in the two sites considered here.

First theoretical results for the spectrum of the non-hermitian two-site Bose-Hubbard system\cite{1} and a closely related PT-symmetric system were presented in\cite{10, 11}. In this paper we will present first results for the dynamics of this decaying many-particle system with emphasis on the mean-field limit of large particle numbers. In order to specify the mean-field approximation in a controllable manner, we derive coupled equations for expectation values under the assumption that the system, initially in a coherent state, remains coherent for all times of interest. This is a direct extension of the frozen Gaussian approximation in flat phase space\cite{12, 13} to $SU(2)$ coherent states, relevant to the present case as discussed below. This yields classical evolution equations for the coherent states parameters.

It facilitates the analysis to rewrite the Hamiltonian\cite{1} in terms of angular momentum operators $\hat{L}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$, $\hat{L}_y = \frac{i}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$ and $\hat{L}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)$, satisfying the commutation rules $[\hat{L}_x, \hat{L}_y] = i\hat{L}_z$ and cyclic permutations, as

$$\hat{H} = 2(\varepsilon - i\gamma)\hat{L}_x + 2\nu \hat{L}_z + c\hat{L}_x^2 - \nu \hat{L}_z^2 - i\gamma \hat{N}.$$  \hfill (2)

The conservation of $\hat{N}$ appears as the conservation of $\hat{L}_z^2 = \frac{N}{2}(\frac{N}{2} + 1)$, i.e. the rotational quantum number $\ell = N/2$. The system dynamics is therefore restricted to an $(N+1)$-dimensional subspace and can be described in terms of the Fock states $|k, N-k\rangle$, $k = 0, \ldots, N$ or the $SU(2)$ coherent states\cite{14}, describing a pure BEC:

$$|x_1, x_2\rangle = \frac{1}{\sqrt{N!}} (x_1 \hat{a}_1 + x_2 \hat{a}_2^\dagger)^N |0\rangle,$$  \hfill (3)

with $x_j \in \mathbb{C}$. The norm, which may differ from unity, is $\langle x_1, x_2|x_1, x_2\rangle = n^N$, where $n = |x_1|^2 + |x_2|^2$. 

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A general discussion of the time evolution of a quantum system under a non-hermitian Hamiltonian \( \hat{H} = \hat{H} + i\hat{\Gamma} \) with hermitian \( \hat{H} \) and \( \hat{\Gamma} \) can be found in [15]. Matrix elements of an operator \( \hat{\Lambda} \) without explicit time-dependence satisfy the generalized Heisenberg equation, which in our case becomes

\[
i\hbar \frac{d}{dt} \langle \psi | \hat{\Lambda} | \psi \rangle = \langle \psi | [\hat{\Lambda}, \hat{H}] | \psi \rangle - i \langle \psi | \hat{\Lambda} \hat{\Gamma} | \psi \rangle,
\]

where \( [\cdots] \rangle \) is the anti-commutator. As an immediate consequence of the non-hermiticity, the norm of the quantum state is not conserved, \( \frac{d}{dt} \langle \psi | \psi \rangle = -2 \langle \psi | \hat{\Gamma} | \psi \rangle \), thus the survival probability decays exponentially for the simple case of a constant \( \Gamma > 0 \). The time evolution of the expectation value of an observable \( \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \) is described by the equation of motion

\[
i\hbar \frac{d}{dt} \langle \hat{A} \rangle = \langle [\hat{A}, \hat{H}] \rangle - 2i \Delta^2_{\hat{A} \Gamma},
\]

with the covariance \( \Delta^2_{\hat{A} \Gamma} = \frac{1}{i} [\langle \hat{A}, \hat{\Gamma} \rangle, \langle \hat{\Gamma} \rangle] - \langle \hat{A} \rangle \langle \hat{\Gamma} \rangle \).

For the Bose-Hubbard system [2] these evolution equations, formulated in terms of the angular momentum operators with read (units with \( \hbar = 1 \) are used in the following)

\[
\frac{d}{dt} \langle L_x \rangle = -2\epsilon \langle L_y \rangle - 2c (\langle L_y, L_z \rangle_+ + \Delta \langle L_x \rangle^2 - \Delta \langle L_z \rangle^2) + 2\gamma (\Delta \langle L_x \rangle^2 + \Delta \langle L_z \rangle^2)\]
\[
\frac{d}{dt} \langle L_y \rangle = 2\epsilon \langle L_x \rangle + 2c (\langle L_x, L_z \rangle_+ - \Delta \langle L_y \rangle^2 - \Delta \langle L_z \rangle^2) - 2\gamma (\Delta \langle L_x \rangle^2 + \Delta \langle L_z \rangle^2)\]
\[
\frac{d}{dt} \langle L_z \rangle = 2\epsilon \langle L_y \rangle - 2\gamma (\Delta \langle L_x \rangle^2 + \Delta \langle L_z \rangle^2)
\]

and the norm decays according to

\[
i\hbar \frac{d}{dt} \langle \psi | \psi \rangle = -2\gamma \{\langle L_z \rangle - \langle \hat{N} \rangle\} \langle \psi | \psi \rangle.
\]

In order to establish a mean-field description, we choose a coherent initial state \( |x_1, x_2\rangle \), i.e. a most classical state, and assume that it remains coherent for all times of interest. This assumption is, in fact, exact, if the Hamiltonian is a linear superposition of the generators of the dynamical symmetry group, i.e. for vanishing interaction \( \epsilon = 0 \) (the proof in [14] can be directly extended to the non-hermitian case). For the interacting case \( \epsilon \neq 0 \) this is an approximation and the mean-field equations of motion are obtained by replacing the expectation values in the generalized Heisenberg equations of motion [6] with their values in \( SU(2) \) coherent states [3].

The \( SU(2) \) expectation values of the \( L_i \), \( i = x, y, z \), read

\[
s_x = \frac{x_1^* x_2 + x_1 x_2^*}{2n}, \quad s_y = \frac{x_1^* x_2 - x_1 x_2^*}{2n}, \quad s_z = \frac{x_1^* x_1 - x_2^* x_2}{2n},
\]

with the abbreviations \( s_j = \langle L_j \rangle / N \) for the mean values per particle; the expectation values of the anti-commutators factorize as

\[
\langle [\hat{L}_i, \hat{L}_j]_+ \rangle = 2 \left( 1 - \frac{1}{N} \right) \langle \hat{L}_i \rangle \langle \hat{L}_j \rangle + \delta_{ij} \frac{N}{2},
\]

and \( \langle [\hat{L}_i, \hat{N}]_+ \rangle = 2N \langle \hat{L}_i \rangle \). Inserting these expressions into [6] and taking the macroscopic limit \( N \to \infty \) with \( N \epsilon = c \) fixed, we obtain the desired non-hermitian mean-field evolution equations:

\[
s_x = -2\epsilon s_y - 4cs_x s_y + 4s_2 s_x, \quad s_y = 2\epsilon s_x + 4cs_x s_y - 2s_2 + 4s_2 s_x, \quad s_z = 2s_2 s_x - 2\gamma (1 - 4s_x^2).
\]

These nonlinear Bloch equations are real valued and conserve \( s^2 = s_x^2 + s_y^2 + s_z^2 = 1/4 \), i.e. the dynamics is regular and the total probability \( n \) decays as

\[
n = -2\gamma (s_z + 1) n.
\]

Equivalently, the nonlinear Bloch equations [10] can be written in terms of a non-hermitian generalization of the discrete nonlinear Schrödinger equation, i.e. for the time-evolution of the coherent state parameters \( x_1, x_2 \). Most interestingly, these equations are canonical, \( i\dot{x}_i = \partial H / \partial x'_j \), \( i\dot{x}'_j = -\partial H / \partial x_j \), \( j = 1, 2 \), where the Hamiltonian function is related to the expectation value of the Hamiltonian \( \hat{H} \):

\[
H(x_1, x'_1, x_2, x'_2) = \langle \hat{H} \rangle n / N \quad \text{and can be conveniently rewritten in terms of the quantities} \quad \psi_j = e^{\theta x_j} \quad \text{where the (irrelevant) total phase is adjusted according to} \quad \dot{\theta} = -g \kappa^2 / n.
\]

The resulting discrete non-hermitian GPE reads

\[
i\frac{d}{dt} \langle \psi_1 \psi_2 \rangle = \begin{pmatrix} \epsilon + g \kappa & 2\gamma \langle \psi_1 \rangle \\ 2\gamma \langle \psi_1 \rangle & -\epsilon - g \kappa \end{pmatrix} \langle \psi_1 \psi_2 \rangle.
\]

Similar non-hermitian mean-field equations, with the choice \( \kappa = |\psi_1|^2 - |\psi_2|^2 \), leading to different dynamics, have been suggested and studied before [4, 5, 10, 14]. These \( \textit{ad hoc} \) nonlinear non-hermitian equations also appear for absorbing nonlinear waveguides [17].

The dynamics of the nonlinear Bloch equations [10] is organized by the fixed points which are given by the real roots

![FIG. 1: (Color online) Mean-field dynamics on the Bloch sphere for the hermitian \( \gamma = 0 \) (top) and the non-hermitian case \( \gamma = 0.75 \) (bottom) for \( c = 0 \) (left) and \( c = 2 \) (right) and \( \epsilon = 0 \) and \( \nu = 1 \).]
of the fourth order polynomial:
\[ 4(c^2 + \gamma^2)s^4 + 4c\varepsilon s^3 + (\varepsilon^2 + \nu^2 - c^2 - \gamma^2)s^2 - c\varepsilon s - \varepsilon^2/4 = 0. \]

(13)

In the following we will restrict ourselves to the symmetric case \( \varepsilon = 0 \). Then the polynomial \((13)\) becomes biquadratic and the fixed points are easily found analytically.

In parameter space we have to distinguish three different regions: (a) For \( c^2 + \gamma^2 < \nu^2 \), we have two fixed points which are both simple centers. (b) For \( |\nu| > |\gamma| \), we have again two fixed points, a sink and a source. (c) Four coexisting fixed points are found in the remaining region, namely a sink and a source (respectively two centers for \( \gamma = 0 \)), a center, and a saddle point. Note that the index sum of these singular points on the Bloch sphere must be conserved under bifurcations and equal to two \([18]\). Bifurcations occur at critical parameter values: For \( c^2 + \gamma^2 = \nu^2 \) (and \( \gamma \neq 0 \)), one of the two centers (index +1) bifurcates into a saddle (index -1) and two foci (index +1), one stable (a sink) and one unstable (a source). This is a non-hermitian generalization of the selftrapping transition for \( \gamma = 0 \). The corresponding critical interaction strength is decreased by the non-hermiticity, i.e. the decay supports selftrapping. For \( \gamma = \pm \nu \), the saddle (index -1) and the center (index +1) meet and disappear. For \( \gamma = 0 \), we observe a non-generic bifurcation at \( \gamma = \pm \nu \) (an exceptional point \([11]\)) where the two centers meet and simultaneously change into a sink and a source.

As an example, Fig. \([1]\) shows the flow \([8]\) on the Bloch sphere for \( \nu = 1 \) both for the hermitian \( \gamma = 0 \) (top) and the non-hermitian case \( \gamma = 0.75 \) (bottom). For \( \gamma = 0 \) we observe the well-known selftrapping effect: In the interaction free case \( c = 0 \) (upper left) we have two centers at \( s_x = -s_z = 0 \) and \( s_x = \pm \frac{1}{\sqrt{2}} \) and Rabi oscillations. Increasing the interaction \( c \) one of the centers bifurcates into a saddle (still at \( s_z = 0 \)) and two centers, which approach the poles (upper right for \( c = 2 \)). The corresponding nonlinear stationary states therefore favor one of the wells. In the decaying system with \( \gamma = 0.75 \) (bottom), these patterns are changed. For \( c = 0 \) (lower left) we are still in region (a) with two centers located on the equator; however, they move towards \( s_x = 0, s_z = \frac{1}{\sqrt{2}} \), approaching each other. For \( c = 2 \) (lower right), in region (b) above the bifurcation, we have a center, a sink (lower hemisphere), a source (upper hemisphere) and a saddle. The system relaxes to a state with excess population in the non-decaying well, i.e. the selftrapping oscillations are damped, which is in agreement with the effect of decoherence in a related nonlinear two mode system reported in \([19]\). Finally, in region (c) only a source and a sink survive and the flow pattern simplifies again (not shown). The manifestation of the different mean-field regimes in the many particle system is the occurrence and unfolding of higher order exceptional points in the spectrum \([11]\).

Let us finally compare the mean-field evolution with the full many-particle dynamics. The full quantum solution is obtained by numerically integrating the Schrödinger equation for the Bose-Hubbard Hamiltonian \([1]\) for an initial coherent state with unit norm. Figure \([2]\) shows the decay of the total survival probability \( \langle \psi | \psi \rangle \) as a function of time for weak interaction \( (c = 0.1) \) and weak decay \( (\gamma = 0.01) \) with \( \nu = 1 \), when initially the non-decaying site 2 is populated. The multi-particle results agree with the mean-field counterpart \( n^N \) on the scale of drawing. The deviation increases with time as can be seen on the right side. The probability shows a characteristic staircase behavior (see also \([5, 20]\)) due to the fact that the population oscillates between the two sites and the decay is fast when site 1 is strongly populated and slow if it is empty. This picture is confirmed by the populations \( \langle \psi | a_1^+ a_1 | \psi \rangle /N \) and \( \langle \psi | a_2^+ a_2 | \psi \rangle /N \) of the two sites also shown in the figure. These quantities agree with their mean-field counterparts \( (1/2 + s_z)n^N/2 \) and \( (1/2 - s_z)n^N/2 \) on the scale of drawing. The overall decay of the norm is approximately exponential, \( \frac{d}{dt} \langle \psi | \psi \rangle \approx -2\gamma N \langle \psi | \psi \rangle \) within region (a), as seen from \([11]\) with \( s_\gamma = 0 \).

The dynamics on the Bloch sphere in region (a) typically show Rabi-type oscillations. An example with parameters \( c = 0.5 \) and \( \gamma = 0.1 \) is shown in Fig. \([3]\). The mean-field oscillation follows a big loop extending over the whole Bloch sphere. The many-particle motion oscillates with the same period, however, with a decreasing amplitude. This effect, known as breakdown of the mean-field approximation in the hermitian case, is due to the spreading of the quantum phase space density over the Bloch sphere, and can be partially cured by averaging over a density distribution of mean-field trajectories \([20]\).

For strong interaction, i.e. in the selftrapping region (c), we find an attractive fixed point, a sink, in the mean-field dynamics. An example is shown in Fig. \([3]\) for \( c = 2 \) and \( \gamma = 0.5 \). The mean-field trajectory, which started at the north pole, approaches the fixed point at \( s_z = 0 = -0.433 \). The full many-particle system shows a very similar behavior.

Further numerical investigations show that the short time behavior of the many-particle dynamics, as well as characteristic quantities such as, e.g., the half-life time, are extremely well captured by the mean-field description in most parameter ranges.

In this letter, we have constructed a mean-field approximation for a non-hermitian many-particle Hamiltonian, which can directly be generalized to other effective non-Hermitian
Hamiltonians. The resulting dynamics differ from the ad hoc non-hermitian evolution equations used in previous studies. It should be noted that the nonlinear Bloch equations (10) can be derived in an alternative way, based on a recently formulated number-conserving evolution equation in quantum phase space for $M$-site Bose-Hubbard systems [20], which allows for an immediate extension to the non-hermitian case.

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