Information scrambling at an impurity quantum phase transition

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The two-channel Kondo impurity model realizes a local non-Fermi liquid state with finite residual entropy, and is separated from its single channel counterpart by an impurity quantum phase transition. We show that the out-of-time-ordered (OTO) commutator for the impurity spin reveals markedly distinct behaviour depending on the low energy impurity state, though it is temperature independent in both cases. For the one channel Kondo model with Fermi liquid ground state, the OTO commutator vanishes for late times, indicating the absence of the butterfly effect. For the two channel case, the impurity OTO commutator saturates quickly to its upper bound 1/4, and the butterfly effect is maximally enhanced. These compare favourably to numerics on spin chain representation of the Kondo model.

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Introduction. Non-Fermi liquids with finite residual entropy at vanishing temperature are peculiar states of matter, which hold the promise to be relevant for holography and can constitute the holographic duals of black holes [1]. Particularly interesting in this context is the Sachdev-Ye-Kitaev model [2-6], describing randomly interacting Majorana fermions, which could possibly be related to quantum gravity.

In order to diagnose information scrambling and the related quantum butterfly effect in the Sachdev-Ye-Kitaev model, the out-of-time-ordered (OTO) commutator[5] was proposed[2-4] as

$$C(t) = -\langle [V,W(t)]^2 \rangle \geq 0.\quad (1)$$

Here, $V$ and $W$ are some local hermitian operators, and $W(t) = \exp(iHt) W \exp(-iHt)$. Assuming, that the involved $V$ and $W$ operators commute at $t = 0$, the $C(t)$ measures how commutativity is destroyed during the time evolution. For a sufficiently chaotic system, the commutator is argued to exhibit exponential temporal growth [7, 8], bounded by a thermal Lyapunov exponent [9] and to become large in the long time limit, hence the butterfly effect appears. This occurs through the vanishing of the $\langle V W(t) V W(t) \rangle$ OTO correlator, investigated in a variety of systems [10-18]. In systems displaying Fermi liquid behavior [8, 17, 19], the large time limit of $C(t)$ seems to approach zero, a behaviour which is associated to the presence of the fermionic excitations at the Fermi surface. On the other hand, the quasiparticle picture is lost in a non-Fermi liquid, and understanding the behaviour of $C(t)$ is important in such exotic states.

Non-Fermi liquid phases arise typically in strongly interacting models, where enhanced quantum fluctuations destroy fermionic excitations and give way to collective modes [20-22]. Among the quantum impurity models, the two channel Kondo (2CK) model, which has already been realized experimentally [23-25], is the most promising candidate as it displays an impurity quantum phase transition (iQPT) with a non-Fermi liquid behavior and a finite residual entropy. In spite of being a traditional condensed matter model, its holographic realization has already attracted attention [26].

In constrast to the one channel Kondo (1CK) problem, where the ground state becomes a Fermi liquid and the impurity spin is completely screened at low temperatures [27], in the two channel version the physics is drastically different. When the two channels are symmetrically coupled to the impurity, the conduction electrons within each channel compete to screen the impurity: when one channel succeeds in doing so, the other channel binds to the impurity as well. This results in a spin-1/2 object, composed of two electrons and the impurity. Then the whole process starts over: the residual spin of the composite object couples again antiferromagnetically to conduction electrons and produces a new Kondo effect. The local non-Fermi liquid state borns out from this never-ending series of Kondo effects. This is reflected in its ln2/2 residual entropy as half of the impurity degrees of freedom are completely decoupled [28]. The 1CK and 2CK behaviours are separated by an iQPT, whose possible order parameter has attracted revived interest quite recently [29, 30].

Therefore, it looks relevant to address the behaviour of the OTO commutator in the anisotropic 2CK model, realizing both 1CK and 2CK physics with the hope of disentangling the effect of the Fermi vs. non-Fermi liquid character of the ground states on $C(t)$. We find that although the simple commutator of the impurity spin à la Kubo formula is temperature dependent, the OTO commutator turns out to be completely temperature indepen-
dent for the anisotropic 2CK model. While commutativity in the OTO commutator is restored at late times for the 1CK case, the 2CK model features a maximally enhanced late time value $C(t \rightarrow \infty) = 1/4$ and a maximal quantum butterfly effect. As we show, this occurs due to the decoupled Majorana mode in the 2CK case. We emphasize that unlike in chaotic models, where the OTO correlator is expected to vanish [7], for the 2CK model this correlator changes sign with respect to its $t = 0$ value during the time evolution, and the late time value of the OTO commutator is twice as big as expected in suitably chaotic systems.

The Kondo impurity model. The two-channel Kondo impurity model Hamiltonian [27] is given by

$$H_K = \sum_{j=1}^{2} \left\{ \sum_{p,s} \epsilon(p) c_{p,s,j}^+ c_{p,s,j} + \sum_{\gamma=x,y,z} J_{\gamma,j} s_{\gamma,j}(0) \right\},$$

where $S_{\gamma,j}(0) = \sum_{s,s'} \Psi_{\gamma,j}^+(0) \sigma_{s,s'} \Psi_{\gamma,j}(0)$, $s$ denotes the spin quantum number of the conduction electrons, $\epsilon(p) = v|p|$ is their kinetic energy, the $\sigma$'s are Pauli matrices, $s_{x,y,z}$ stands for the impurity spin components, $c_{k,s,j}$ and $\Psi_{s,j}(0)$ are the conduction electron annihilation operators with spin $s$ and channel $j$ in momentum and real space, respectively. In addition, we require XXZ couplings as $J_{x,j} = J_{y,j}$ and $J_{z,j} = J_{z,1}$. In case of channel isotropy, i.e. $J_{x,1} = J_{z,1}$, Eq. (2) realizes the 2CK model, otherwise the low energy physics is governed by the one channel case, and a crossover to the two-channel behaviour can occur with increasing energy, unless the model is completely anisotropic.

Upon Abelian bosonization[20, 21], this problem can be mapped onto the Majorana resonant level (MRL) model.[28] This reads as

$$H_{mrl} = H_0(\xi) + H_0(\eta) - i \frac{\alpha}{\sqrt{2\pi \alpha}} a \xi(0) - i \frac{\beta}{\sqrt{2\pi \alpha}} b \eta(0),$$

where $a = a^+, b = b^+$ are the impurity Majorana operators with $a^2 = b^2 = 1/2$, $\xi(x)$ and $\eta(x)$ stem from the Majorana representation of conduction electrons[21].

$$H_0(\xi) = \int dx \xi(x) \partial_x \xi(x), I_\pm = (J_{x,1} \pm J_{z,2})/2$$ measures the channel anisotropy and $J_{z,1} = 2\pi v$ at the Emery-Kivelson[28] or Toulouse[21] point, where the above MRL model description holds, $\alpha$ plays the role of the remnant of the lattice constant in the low energy theory, and its inverse serves as a high energy cutoff. The $(\xi, \alpha)$ part of the Hamiltonian commutes with the $(\eta, b)$ sector, therefore the dynamics of the two Majoranas decouples and can be considered separately. The $I_\pm = 0$ marks the iQPT, and the low energy physics is equivalent to 2CK, while for $I_\pm = I_-$, the low energy dynamics of the 1CK case is realized. In between, a crossover from one to two-channel behaviour takes place with increasing energy, and $I_- \ll I_\pm$. The $z$ component of the impurity spin, what we are going to investigate here, is $s_z = iab$, while the other spin components are more complicated due to the involved unitary transformations in the mapping from Eq. (2) to Eq. (3) via Refs. [20, 21, 28].

FIG. 1. Schematic "phase diagram" of the anisotropic two channel Kondo model in the time or energy domain. Only for $\Gamma_b = 0$ is the low energy fixed point governed by the two channel Kondo physics.

Majorana propagators. Since the MRL model is quadratic, the Matsubara Green’s function of the local Majorana operators are evaluated as $G_{a,b}^{-1}(i\omega_n) = \omega_n + i\Gamma_a b \text{sgn}(\omega_n)$, where $\omega_n$ is the fermionic Matsubara frequency, and $\Gamma_{a,b} = \Gamma_a/(4\pi\omega_n)$, respectively. The schematic phase diagram of the anisotropic 2CK model with the crossover regions is depicted in Fig. 1. At short times or high energies, the spin is practically not influenced by the electrons, and behaves as a free spin. With increasing time/decreasing energy, we enter into the 2CK regime, unless the model is completely anisotropic and corresponds to 1CK. At late times, the low energy fixed point is approached, which is 2CK type only in case of perfect isotropic couplings, otherwise it is dominated by 1CK physics.

We emphasize that the MRL model in Eq. (3) corresponds to the strong coupling limit of the Kondo problem, when the exchange coupling, $J_{z,1}$ is of the order of the electronic bandwidth. Consequently, $\Gamma_{a,b}$ is not the Kondo temperature but rather describes how fast we move towards the infinitely strong coupling fixed point. As noted above, at the Emery-Kivelson point, $I_- = 0$ and one of the Majorana modes decouples completely at the iQPT, and its Green’s function becomes $1/i\omega_n$.

In the following, we will need the propagator of the Majorana fermions in real time. We follow Ref. [31] to obtain

$$F_a(t) = \langle a(t)a \rangle = \int_{-\infty}^{\infty} d\omega \frac{\rho_a(\omega)}{\exp(\omega/T) + 1} \exp(-i\omega t),$$

and $\rho_a(\omega) = -\text{Im} G_a(i\omega_n \rightarrow \omega + i0^+)/\pi = \Gamma_a/(\omega^2 + \Gamma_a^2)\pi$ is the density of states, $T$ is temperature and similar expression holds for the $b$ Majorana fermion with $\Gamma_a \rightarrow \Gamma_b$ change. Since $\Gamma_b < \Gamma_a$, it is the $b$ Majorana field which decouples completely from conduction electron when $I_- = 0$, in which case its density of states becomes a Dirac-delta function as $\rho_b(\omega) = \delta(\omega)$, and $F_b(t) \equiv 1/2$.

At $t = 0$, $F_{a,b}(0) = 1/2$, in accord with the definition of Majorana operators. At $T = 0$, the above integral is
performed analytically to yield
\[ F_a(t) = \frac{1}{2} \exp(-\Gamma_a t) + \frac{i}{2\pi} \sum_{s=\pm} s \exp(-s\Gamma_a t) \text{Ei}(s\Gamma_a t), \]  
(5)
where \( \text{Ei}(x) \) is the exponential integral function\(^{[32]} \). For late times, the propagator decays as \( F_a(t) \gg 1/\Gamma_a = i/(\pi \Gamma_a t) \). It is important to note that temperature influences only the imaginary part of the Majorana propagator, the real part remains temperature independent, i.e. \( \text{Re} F_{a,b}(t) = \exp(-\Gamma_a,b t)/2 \) for all temperatures. Its imaginary part acquires an additional \( \exp(-\pi T t) \) factor to the power law decaying part.

**Simple and OTO commutator for \( s_z \).** Let us start with the simple commutator of the impurity spin, whose Fourier transform is the dynamic spin susceptibility\(^{[28]} \), \( \chi(\omega) \), accessible by e.g. electron spin resonance or neutron spectroscopy. It is given by
\[ K(t) = i\langle [s_z(t), s_z] \rangle = 2\text{Im}(F_a(t)F_b(t)). \]  
(6)
For the 2CK case, \( F_b(t) = 1/2 \), therefore is decays in a power law fashion for long times as \( K(t) \gg 1/\Gamma_a \sim 2/\pi \Gamma_a t \), which translates to a dynamic spin susceptibility as \( \text{Im} \chi_{2\text{ck}}(\omega) \sim \text{sign}(\omega) \) at low frequencies\(^{[28]} \). Away from the two channel point, the low energy fixed point is determined by the 1CK effect, therefore \( K(t) \gg 1/\Gamma_b \sim \exp(-\Gamma_b t)/\pi \Gamma_b t \), which gives rise to a spin susceptibility \( \text{Im} \chi_{1\text{ck}}(\omega) \sim \omega \) at low energies\(^{[33]} \). For \( 1/\Gamma_a \ll t \ll 1/\Gamma_b \), on the other hand, the 2CK behaviour is recovered. In the perfect one channel case with \( \Gamma_a = \Gamma_b \), no two channel crossover occurs, and the decay for \( t \gg 1/\Gamma_a \) is \( K(t) \sim 2\exp(-\Gamma_a t)/\pi \Gamma_a t \). The hierarchy of energy scales is clearly observable, the short time dynamics does not depend much on the 1CK or 2CK scenario, and depending on the anisotropies, there can be a wide enough temporal window to catch 2CK in the act.

The OTO commutator for the impurity spin reads as
\[ C(t) = -\langle [s_z(t), s_z] \rangle, \]  
(7)
which, based on the Cauchy-Schwarz inequality, is bounded from above for a spin-1/2 impurity as \( C(t) \leq 4\|s_z\|^2 = 1/4 \). Using the fact that \( s_z^2 = 1/4 \), the OTO commutator simplifies to
\[ C(t) = 2 \left( \frac{1}{4} - f(t) \right), \]  
(8)
where the OTO correlator is defined as
\[ f(t) = \langle s_z(t)s_z s_z(t)s_z \rangle. \]  
(9)
Since the effective Hamiltonian is quadratic in the Majorana operators, it is calculated exactly using Wick’s theorem as
\[ f(t) = \left( 2\langle \text{Re} F_a(t) \rangle^2 - \frac{1}{4} \right) \left( 2\langle \text{Re} F_b(t) \rangle^2 - \frac{1}{4} \right). \]  
(10)
The OTO correlator starts from \( 1/2^4 \) at \( t = 0 \), approaches and even crosses zero fast, but at late times, it recovers its initial value for 1CK while for the 2CK, it approaches \(-1/2^4 \). Since it depends only on the real part of the Majorana propagators, the OTO commutator and commutator are therefore completely temperature independent, as noted after Eq. (5).

Using the real part of Eq. (5), independent of temperature, the OTO commutator is calculated analytically for general anisotropies as
\[ C(t) = \frac{\exp(-2\Gamma_a t)}{4} + \frac{\exp(-2\Gamma_b t)}{4} - \frac{\exp(-2(\Gamma_a + \Gamma_b)t)}{2}, \]  
(11)
giving
\[ C_{2\text{ck}}(t) = \frac{1 - \exp(-2\Gamma_a t)}{4} \]  
(12)
for the 2CK case. For 1CK model, it yields \( C_{1\text{ck}}(t) = \exp(-2\Gamma_a t)(1 - \exp(-2\Gamma_a t))/2 \). Eq. (11) predicts an initial linear increase in time, though Eq. (1) only allows for a quadratic time dependence, since a high energy cutoff, accounting for the bandwidth of conduction electrons in Eq. (2), was sent to infinity. Had we retained this cutoff, the correct \( t^2 \) increase would be recovered.

For the two channel case at late times, i.e. \( t \gg 1/\Gamma_a \), the OTO commutator reaches its maximal possible value: it consists of 4 terms, each of which is bounded from above by \( 1/2^4 \), since they all contain 4 spin-1/2 operators. Each term takes on its maximal value, therefore \( C(t) \rightarrow 1/4 \) rapidly with increasing time. Note that in a suitably chaotic system, the OTO commutator is expected to saturate to \( 2/2^4 = 1/8 \), exactly half of the value for the 2CK case.

**Numerics.** In order to test the validity of the results at the Emery-Kivelson point, one can either consider additional terms in the Hamiltonian analytically using

![Figure 2](image_url)

**Figure 2.** The temperature independent \( s_z \) OTO commutator is shown in the perfect one- and two channel cases, \( \Gamma_a = \Gamma_b \) (black) and \( \Gamma_b = 0 \) (blue), respectively, exhibiting also the crossover for \( \Gamma_b = \Gamma_a/2000 \) (red).
perturbation theory, which are e.g. responsible for the $T \ln(T)$ specific heat for the 2CK case\cite{28}, or one can resort to numerics where all these additional processes are taken into account exactly. We have decided to follow the latter option. To start with, we have checked the prediction for the long time dynamics of $K(t)$ using NRG to study Eq. (2), which confirmed that even away from the Emery-Kivelson point, our temporal scaling remains intact.

The NRG calculation of the OTO correlator would require the matrix elements of $s_z$ between excited states, which are not represented accurately. Instead, when focusing on the OTO commutator, we consider the spin chain representation of the Kondo problem, consisting of two open Heisenberg chains coupled to a spin-1/2 impurity. The Hamiltonian is\cite{29,30},

$$H = \sum_{m=L,R} \left[ J_m^s \cdot J_1^s \cdot S_m^1 + J_2^s \cdot S_m^2 \right] + J_1^s \sum_{l=1}^{N_m-1} S_m^l \cdot S_m^{l+1} + J_2^s \sum_{l=1}^{N_m-2} S_m^l \cdot S_m^{l+2}, \quad (13)$$

where $s$ and $S_m^l$ represent the impurity spin and the spin at site $l$ in channel $m$, respectively, and $N_m$ is the number of spins in chain $m$ with total number of spins $N = N_L + N_R + 1$. We choose the next-nearest-neighbor coupling $J_2^s = 0.2412 J_1^s$ in order to remove marginal coupling effects\cite{29,30}. The nature of the Kondo effect is controlled by the dimensionless couplings as $J_L^s = J_R^s$ for 2CK, while 1CK emerges for $J_L^s \neq J_R^s$.

This model is treated using exact diagonalization (ED). For $J^s \leq 1$, the Kondo temperature, $T_K$, which is the typical energy scale for the formation of Kondo effect, is large and comparable to $J_1$ (see inset of Fig. 3). We use this to our favour as it allows to follow the time evolution of the system for $t \gg 1/T_K$ in Fig. 3. After an initial sharp peak, the data saturates to a constant, which is larger than what is expected for spin-1/2 operators in a chaotic system\cite{7-9} (i.e. 1/8) for 2CK, while takes on a rather small value for 1CK. In both cases, the OTO correlator starts from 1/16 at $t = 0$, but changes sign to take a negative late time value for the 2CK, while for the 1CK, it re-approaches its initial value, 1/16 at late times. These highlight the important difference between the 1CK and 2CK cases and the influence of the iQPT on the OTO commutator.

The numerical results agree with our analytical findings, even though due to the comparable $J_1$ and $T_K$ in the numerics, we cannot reach the universal $T_K \ll J_1$ limit, where probably the values at the Emery-Kivelson point would be recovered. For smaller $J^s$, a more enhanced OTO commutator (very close to its maximal value, 1/4) seems to emerge for 2CK from the numerics together with more suppressed late time value for 1CK, though the time evolution cannot be tracked due to finite size effects as we cross over to a Kondo box\cite{34}, when the level spacing becomes larger than the Kondo temperature. Nevertheless, the distinct behaviour of the local Fermi vs. non-Fermi liquid ground states is clearly observable.

By associating the Kondo temperature to the sharp peak in $C(t)$, its expected scaling, $T_K \sim \exp(-|c|/J^s)$ is confirmed in the inset of Fig. 3. We suspect that in order to reach the predicted universal values 0 and 1/4 in $C(t \gg 1/T_K)$ for 1CK and 2CK, respectively, much bigger system sizes with much smaller Kondo temperature would be required.

![FIG. 3. The ED data for the OTO commutator for the 1CK (red dashed line) and 2CK (blue solid line) is plotted for $J_L^s = 0.7$, $J_R^s = 0$, $N_L = 21$ and $J_{R,L}^s = 0.8$, $N_L = N_R = 11$, respectively. Finite size effects are already present for later times. The thick green dashed lines denotes the expected late time value when the OTO correlator vanishes in e.g. chaotic systems. The inset shows the scaling of the Kondo temperature for 2CK\cite{35} as $T_K = 1.82 J_1 \exp(-0.82/|J^s|)$ (black solid line), deduced from the position of the initial sharp peak (blue circles).](image)
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