FOLDING OPERATORS, ROOT GROUPS AND RETRACTIONS

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Abstract. It is shown that the folding operators introduced by Gaussent and Littelmann can be expressed purely in terms of retractions of the building which in turn are tightly connected to root groups and root data with valuations. The relation shown here highlights the deep connection between folding operators and to certain kinds of double coset intersections of subgroups of a semi-simple algebraic group.

1. Introduction

Folding operators for paths in an apartment were first introduced by Littelmann [Lit94, Lit95] in the context of the path model for finite dimensional representations of connected complex semisimple algebraic groups $G$. Later Gaussent and Littelmann [GL05] defined a version for galleries while establishing a connection between the path model and the geometry of the affine Grassmanian $G$. Milicevic, Thomas and the author recently used root operators in their new approach to affine Deligne Lusztig varieties in [MST15].

It is mentioned in several places in the literature that there is a close relationship with folding operators and retractions in the affine building. In this small note we will make this connection explicit and prove that the folding operators can be expressed purely in terms of the buildings’ retractions.

Apart from assuming thickness the proof does not depend on the branching of the building. In particular, we do not need to assume that the building is associated to an algebraic group and if it comes from a group, then the proof does not depend on the underlying field. The heuristic reason is that root operators themselves are defined on galleries in a single apartment only and do not “see” the branching of the building.

Retractions are an essential tool in the theory of buildings. They are for example used to show that buildings are CAT(0) spaces and appear in all kinds of applications. In case that the building comes from an algebraic group they are strongly linked to two kinds of decompositions of the group: the Iwasawa and Cartan decomposition.

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The explicit connection between the folding operators and retractions, which is subject of Theorem 5.3 and 5.4 of this note, highlights their deep connection to certain kinds of double coset intersections of subgroups of a semisimple algebraic group. Numerous works make use of this connection, such as [Hit10], [PRS09], [GL05], [KM08], [KLM04], [KLM08] or [MST15], to name just a few.

We will start in Section 2 by recalling the definition of buildings, their root groups and Bruhat and Iwasawa decomposition. This section does not contain any original work but collects material from the literature that will be used later. After that, in Section 3 we introduce the two types of retractions in an affine building and give their interpretation in terms of subgroups of $G$. We include the definitions of the root operators in Section 4. The main results of the paper are contained in Section 5 where we discuss the connection between retractions and root operators. Lastly in Section 6 we comment on the relationship between the root operators defined on galleries and the one on paths, which were introduced in [Lit94].

2. PRELIMINARIES

We will quickly recall the definition and some basic properties of buildings. For more details compare standard textbooks such as [AB08] [Bro89] or [Ron89].

2.1. Buildings and root systems. A building $X$ is a simplicial complex which is the union of subcomplexes, called apartments, such that each apartment is isomorphic to some (geometric realization of) a Coxeter complex $\Sigma$ and the following two axioms are satisfied:

(B1) For any two simplices there exists an apartment containing both.
(B2) If $A$ and $A'$ are two apartments containing simplices $\sigma$ and $\tau$, then there exists an isomorphism $A \rightarrow A'$ fixing their intersection point-wise.

The set of all apartments is called an atlas $\mathcal{A}$ of $X$. It is easy to see that all apartments in $\mathcal{A}$ are pairwise isomorphic and hence of the same type, that is the type of the associated Weyl group is the same.

A building is thick if each panel, i.e. co-dimension one face of a chamber, is contained in at least three chambers. This is not a strong assumption as each building has a canonical thickening (compare [Sch87] and [KL97, Sec 3.7]) and affine buildings of higher rank are automatically thick.

Suppose now that $X$ is an affine building, that is $W$ is infinite and $\Sigma$ is a Euclidean Coxeter complex.

The affine Weyl group $W$ of the Coxeter complex $\Sigma$ is a semi-direct product of the spherical Weyl group $W_0$ with the co-root lattice $R^\vee$ of the
underlying root system $\Phi$. Recall further, that every apartment $A$ can be identified with the $\mathbb{R}$-span of the simple co-roots in $\Phi^\vee$. We enumerate the simple roots by $\alpha_1, \ldots, \alpha_n$ and denote the corresponding co-roots by $\alpha_1^\vee, \ldots, \alpha_n^\vee$.

The chambers of the spherical building $\partial X$ at infinity of $X$ are the parallel classes of Weyl chambers in $X$. The map that sends an apartment $A$ of $X$ to the union $\partial A$ of all parallel classes of Weyl chambers in $A$ is a bijection between the set of apartments in $X$ and the set of apartments in $\partial X$.

Let $A$ be a fixed apartment in the building $X$ and identify $A$ with the $\mathbb{R}$-span of the simple co-roots in $\Phi^\vee$. Then for every pair of a root $\alpha$ and integer $m \in \mathbb{Z}$, there is a wall $H_{\alpha,m} = \{ x \in A | \langle x, \alpha \rangle = m \}$ in $A$, which determines a positive (with respect to $\alpha$) half-space $H_{\alpha,m}^+ = \{ x \in A | \langle x, \alpha^\vee \rangle \geq m \}$ and a negative half-space $H_{\alpha,m}^- = \{ x \in A | \langle x, \alpha^\vee \rangle \leq m \}$ as illustrated in Figure 2. If we restrict to positive roots the set of hyperplanes is in bijection with the pairs of roots and elements of $\mathbb{Z}$, that is with $\Phi^+ \times \mathbb{Z}$. The closures of the connected components of $A \setminus \bigcup_{m \in \mathbb{Z}, \alpha \in \Phi^+} H_{\alpha,m}$ are called alcoves and are the maximal simplices in the simplicial structure of $\Sigma$. The 0-simplices will be called vertices and the codimension one simplices will be referred to as panels.

The origin $v_0$ in $A$ is the intersection $\bigcap_{i=1}^n H_{\alpha_i,0}$. All other vertices may be identified with the elements of the weight lattice. The apartment $A$ carries in a natural way the structure of a chamber complex, where the maximal simplices, called alcoves, are the closures of the connected components of $A \setminus \{ H_{\alpha,m} \}_{\alpha \in \Phi, m \in \mathbb{Z}}$. The alcoves are in one-to-one correspondence with the elements of the affine Weyl group $W$. The fundamental alcove is the set

$$c_r = \{ x \in A | 0 \leq \langle x, \alpha \rangle \leq 1, \forall \alpha \in \Phi \}$$

and corresponds to the identity. Here we choose as a generating set of $W$ the set of reflection $S = \{ s_0, s_1, \ldots, s_n \}$, where $s_i$ is the reflection on the wall $H_{\alpha_i,0}$ for all $i \neq 0$ and where we put $s_0$ to be the reflection on the wall of index one perpendicular to the highest root $\alpha$ in $\Phi$. With this setup the set $S = \{ s_1, \ldots, s_n \}$ generates the associated spherical Weyl group $W_0$.

As they are defined via equivalence classes of Weyl chambers in $X$ we will refer to the maximal simplices in the spherical building at infinity of $X$ as chambers.

2.2. **Root groups.** In case the affine building in question comes from an (semisimple) algebraic group $G$ the root group datum of $G$ together with a valuation of this datum fully determines $X$. A crucial role is played by the root groups. Moreover $G$ admits Cartan, Bruhat and Iwasawa decompositions. The last two mentioned are tightly related to the two types of

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1We use the definition of root systems and the notation as given in [Bou02].
retractions that can be defined in an affine building. We define these retractions in Section 3.

In the present section we recall the decompositions mentioned, introduce roots groups and valuations of root data. We will not give much details but refer the reader to Section 11 of [AB08], Section 6,7 and in particular 7.3 of [BT72] or Chapters 3 and 13 of [Wei08].

We always suppose that the buildings considered are irreducible.

Let in the following $G$ be an algebraic group over a field $F$ with a discrete valuation $\nu$. Suppose $G$ has an affine Tits system (or BN-pair) $(B,N)$ and write $X$ for the associated affine building. We fix an affine apartment $A$ in $X$ together with an origin $v_0$ and write $\partial A$ for the spherical apartment that is the boundary of $A$ at infinity. We will write $c_f$ for the fundamental alcove in $A$ which contains $v_0$ and denote by $C_f$ be the fundamental Weyl chamber, that is the unique Weyl chamber based at $v_0$ containing $c_f$.

The group $N$ of the Tits system is the stabilizer in $G$ of the apartment $A$ and the group $B$ is the subgroup of $G$ stabilizing the fundamental alcove $c_f$. The spherical Weyl group $W_0$ equals the stabilizer $\operatorname{Stab}_W(v_0)$ of the origin in the affine Weyl group $W$, while $K$ is the stabilizer of $v_0$ in $G$. In case that $F$ is locally compact the group $K$ is a maximal compact subgroup of $G$. We write $T$ for the sub-group of translations in $W$ and $T_{C_f}$ for the translations $t$ in $W$ with $tv_0 \in C_f$ and $U$ for the stabilizer in $G$ of the chamber $\partial C_f$ at infinity.

We get the following properties:

**Proposition 2.1.** With $C_f$, $c_f$ and $v_0$ as above and subgroups $K = \operatorname{Stab}_G(v_0)$, $U = \operatorname{Stab}_G(\partial C_f)$ and $B = \operatorname{Stab}_G(c_f)$ in $G$, the group $G$ decomposes as follows:

1. **Bruhat decomposition**
   $$G = \bigsqcup_{w \in W} BwB.$$

2. **Iwasawa decomposition**
   $$G = \bigsqcup_{t \in T} UtK,$$

3. **Cartan decomposition**
   $$G = \bigsqcup_{t \in T_{C_f}} KtK,$$

For proofs see (4.4.3) of [BT72] and Lemma 5.1 in [Ron89].

Let in the following $\alpha$ be a half-apartment\(^2\) of $\partial A$ in the spherical building $\partial X$ at infinity of $X$. Define $U_\alpha$, the root group of $\alpha$, to be the subgroup of

\(^2\)In [Ron89] or [Wei08] half-apartments in spherical buildings are referred to as roots.
the full automorphism group of $\partial X$ that fixes every chamber having a panel contained in $\alpha \setminus \partial \alpha$. For a Weyl chamber $C$ in $X$ let $U_C$ be the product of all root groups $U_\alpha$, where $\alpha$ contains $\partial C$. Thus the group $U$ from above satisfies

$$U = U_C = \prod_{\partial C \in \alpha} U_\alpha.$$ 

One can define two ordered sets of roots, the closed interval $[a, b] = (\alpha_0, \alpha_1, \ldots, \alpha_s)$ and the open interval $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_{s-1})$ of roots $\alpha$ and $\beta$ which are not opposite each other. For a precise definition see [Wei08, Definition 3.1]. As a set the closed interval $[\alpha, \beta]$ consists of all roots of $\partial A$ containing $\alpha \cap \beta$.

**Proposition 2.2.** [Wei08, Prop. 3.2] With notation as above let $[\alpha, \beta]$ be a closed interval of non-opposite roots. Then

1. if $s \geq 3$ one has $[U_1, U_s] \subset U_2 U_3 \cdots U_{s-1}$, where $U_i$ denotes the root group $U_{\alpha_i}$ for all $i \in \{1, 2, \ldots, s\}$.
2. If $s = 2$ one has $[U_1, U_s] = 1$.
3. Every element of $\langle U_1, U_2, \ldots, U_s \rangle$ can be written uniquely as a product of the form $u_1 u_2 \cdots u_s$ with $u_i \in U_i$ for all $i \in \{1, 2, \ldots, s\}$.

We will now define root data with valuation.

There is a natural one-to-one correspondence between the half-apartments of $\partial A$ and the elements of the underlying root system $\Phi$. In the following we will once and for all fix such an identification and will index the set of half-apartments in $\partial A$ by elements of $\Phi$ and call both of them roots. We refer the reader to Definition 3.12 and Proposition 3.14 of [Wei08] where this identification is made precise and where it is shown that, except for types $B_l, C_l, F_4$ and $G_2$, there is only one possible identification up to isometry. In the mentioned exceptions there will be two possible cases where the long and short roots are flipped. Compare also Remark 13.16 of [Wei08].

**Definition 2.3.** Let $G^\dagger$ be the subgroup of $\partial X$ generated by all the root groups $U_\alpha$ for all roots $\alpha$ in $\partial A$. Let $\xi$ denote the map from $\Phi$ to $G^\dagger$ that assigns to a root $\alpha$ the root group $U_\alpha$. We call the triple $(G^\dagger, \{U_\alpha\}_{\alpha \in \Phi}, \xi)$ root datum of $\partial X$ based at $\partial A$.

It is shown in [Wei08, 3.4 and 29.15.iii] that $G^\dagger$ acts transitively on the set of apartments of $\partial X$ and hence the root datum is unique up to conjugation by an element of $G^\dagger$. In fact Moufang spherical buildings are uniquely determined by their root datum, see [Wei08, 3.6].

Recall that we had fixed an apartment $A \subset X$ and an origin $v_0 \in A$. Thus the roots $\alpha \in \Phi$ are in one to one correspondence with all the half-apartments in $A$ determined by all the walls through the origin, that is walls $H_{\alpha, 0}$ with $\alpha$ positive. Recall that the set of all half-apartments in $A$ equals $\left\{ H_{\alpha, k}^\pm \mid k \in \mathbb{Z}, \alpha \in \Phi^+ \right\}$. 


By Propositions 13.2 and 13.18.ii and Notation 13.17 of [Wei08] we have

**Proposition 2.4.** Let \( \alpha \) be a root and \( u \in U_\alpha^* \). Then the fixed point set \( a_u := A \cap A^u \) of \( u \) in \( A \) is a half-apartment of \( A \) with \( \partial a_u = \alpha \). In particular elements of \( U_\alpha^* \) are special automorphisms of \( \partial A \) and for each of them exists \( k \in \mathbb{Z} \) such that \( a_u = H_{\alpha,k}^+ \), the positive half-apartment of index \( k \) with respect to \( \alpha \).

We are now ready to give the valuations of the root datum that are of interest to us. Compare 3.21 in [Wei08].

**Definition 2.5.** The following family of maps \( \Phi_\alpha : U_\alpha^* \rightarrow \mathbb{Z} \), for all \( \alpha \in \Phi \), is a valuation of the root datum \( (G^\dagger, \{U_\alpha\}_{\alpha \in \Phi}, \xi) \) as defined [Wei08, 3.21]: For all \( \alpha \in \Phi \) and all \( u \in U_\alpha^* \) put \( \Phi_\alpha(u) = -k \) where \( k \) is as in 2.4. We let \( \Phi_\alpha(1) = \infty \) for all \( \alpha \) to extend \( \Phi_\alpha \) to all of \( U_\alpha \).

Note that \( \Phi_\alpha \) is in fact surjective for all \( \alpha \).

**Definition 2.6 (Affine root groups).** For \( \alpha \in \Phi \) we set \( U_{\alpha,k} := \{ u \in U_\alpha | \Phi(a) \geq k \} \) for each \( k \in \mathbb{R} \) (or in \( \mathbb{Z} \)).

**Remark 2.7.** One can show [Wei08, 13.18.iii] that \( u \in U_{\alpha,k}^* \) if and only if \( H_{\alpha,-k}^+ \subset a_u \). Thus the group \( U_{\alpha,k}^* \) can be written as

\[
U_{\alpha,k}^* = \{ u \in U_\alpha | a_u = H_{\alpha,l}^+ \text{ with } l \leq -k \}
\]

Moreover, the elements in \( U_{\alpha,k}^* \setminus U_{\alpha,k-1}^* \) are the ones with \( a_u = H_{\alpha,-k}^+ \) and \( \phi_\alpha(u) = k \).

**Definition 2.8.** Let \( \alpha \) be a root. Then, see 3.8 of [Wei08], for each \( u \in U_\alpha^* \) there exists a unique element in \( U_{-\alpha,u}^* \alpha \) that maps \( \alpha \) to \( -\alpha \). We call this element \( m(u) \).

One can show that

\[
m(u)^{-1} = m(u^{-1}) \text{ for all } u \in U_\alpha^*.
\]

Therefore every \( g \in m(U_\alpha^*) \) induces the reflection \( s_\alpha \) on \( \partial A \) which interchanges \( \alpha \) and \( -\alpha \). For elements \( u \in U_{\alpha,k}^* \setminus U_{\alpha,k-1}^* \) we get that \( m(u) \) induces the reflection \( s_{\alpha,-k} \) on \( A \) which switches the half spaces \( H_{\alpha,-k}^+ \) and \( H_{\alpha,-k}^- \).

3. Retractions

In the first subsection we will recall the definitions of the two types of retractions of an affine building onto a fixed apartment as well as their connection to the Bruhat and Iwasawa decompositions of the underlying group.
3.1. **Definition of retractions.** Before we are able to define the retractions let us recall the following Lemma which ensures that the retractions will be well defined.

**Lemma 3.1.** Let $c$ be any alcove in an affine building $X$ and $\partial C$ an arbitrary chamber in the building $\partial X$ at infinity. Then $X$ is (as a set) the union of

1. all apartments containing $c$ or
2. all apartments $A$ whose apartment $\partial A$ at infinity contains $\partial C$.

**Remark 3.2.** The first item of Lemma 3.1 can, on the alcove level, be deduced from the Bruhat decomposition. Compare 2.1 item (1). To see this we argue as follows. First recall that, as $B$ stabilizes the fundamental alcove, the set of alcoves in $X$ is in bijection with the cosets in $G/B$. The Weyl group $W$ leaves the fixed apartment $A$ invariant and acts transitive on the set of alcoves therein. The group $B$ acts transitive on all apartments containing the fundamental alcove as otherwise $G$ could not be at the same time transitive on the alcoves in $X$ and admit the Bruhat decomposition. With this one can see that as a set of alcoves $X$ is the union of all apartments containing the fundamental alcove $c_f$. By conjugation with elements of $G$ one obtains the more general statement of the Lemma.

Item (2) of the same Lemma is in a similar way related to the Iwasawa decomposition of $G$. The main difference is that we now look at the set of vertices in $X$ of the same type as $v_0$ instead of the set of all alcoves. We use the fact that these vertices are in one to one correspondence with the cosets in $G/K$ and the fact that the elements of $T$ leave the apartment $A$ invariant. By transitivity of $G$ on the vertices of a fixed type in $X$ we obtain that $U$ needs to be transitive on the set of apartments containing $\partial c_f$ at infinity. Thus $X$ may as a set of vertices of type the type of $v_0$ be seen as the union of apartments containing $\partial c_f$. Again by conjugation the full statement is obtained.

Each of the decompositions of Lemma 3.1 allows us to define a retraction of $X$ onto a fixed apartment. The first kind of retraction depends on the choice of the alcove $c$ in $X$.

**Definition 3.3.** Let $c$ an alcove in a fixed apartment $A$. For an alcove $d$ in $X$ let $B \in A$ be an apartment containing both $c$ and $d$, which exists by 3.1 item (1). Define

$$r_{A,c}(d) := \varphi(d),$$

where $\varphi : B \to A$ is the unique isomorphism mapping $B$ to $A$ fixing their intersection. We refer to $r_{A,c}$ as the retraction onto $A$ based at $c$.

The second kind of retraction depends on a choice of a chamber in $\partial X$.

**Definition 3.4.** Let $\partial C$ be a chamber at infinity contained in a fixed apartment $\partial A$. For an alcove $d$ in $X$ choose $B \in A$ such that $d \in B$ and $\partial C \subset \partial B$,
which exists by Lemma 3.1[2]. Define
\[ \rho_{A,\partial C}(d) := \varphi(d), \]
where \( \varphi : B \to A \) is the unique isomorphism mapping \( B \) to \( A \) and fixing their intersection. We call \( \rho_{A,\partial C} \) the retraction from infinity onto \( A \) based at \( \partial C \).

Lemma 3.1(1) implies that the retraction based at \( c \) can be defined and item (2) allows us to define the retraction from infinity. From the definition of buildings one can conclude that for any two such apartments \( B \) as in 3.3 and 3.4 there is an isomorphism fixing the intersection of the two. Thus these definitions do not depend on the choice of the apartments \( B \) and the retractions are well defined.

There is a tight connection between the two retractions which can easily be deduced from Exercise 2 on page 171 in [Bro89].

**Proposition 3.5.** Fix an apartment \( A \) in \( X \) and a chamber \( \partial C \) in its boundary. For all \( d \in X \) there exists then \( c \in A \) such that \( \rho_{A,\partial C}(d) = r_{A,c}(d) \).

### 3.2. Group theoretic interpretation

The retractions \( \rho_{A,\partial C} \) can also be interpreted in terms of root groups. Compare also Proposition 1 of [GL05].

**Proposition 3.6.** The fibers of \( \rho_{A,\partial C} : X \to A \) are the \( U_C \) orbits on \( X \).

**Proof.** Recall that for an arbitrary Weyl chamber \( C \) the group \( U_C \) is the product of all root groups containing \( \partial C \) at infinity. The action of \( U_C \) on the set of all apartments containing a sub-Weyl chamber of \( C \) is transitive by Remark 3.2. Moreover for each \( u \in U_C \) the restriction of \( u \) to \( uA \) is the unique isomorphism mapping \( uA \) to \( A \) and fixing their intersection pointwise. Therefore for all points \( x' \) in the preimage \( \rho_{A,\partial C}^{-1}(x) \), for some \( x \in A \), there exists \( u \in U_C \) with \( x' = ux \). We thus have the assertion. \( \Box \)

- also give interpretation in terms of \( U_{\alpha,k} \).

Similarly we may interpret the retractions \( r_{A,c} \) in terms of group orbits.

**Proposition 3.7.** The fibers of \( r_{A,c} : X \to A \) are the \( B \) orbits on \( X \).

**Proof.** Recall that the action of \( B = \text{Stab}_G(c_F) \) is transitive on all apartments containing \( c_F \) by Remark 3.2. Thus, by conjugation, the stabilizer \( B_c \) of an arbitrary alcove \( c \) is transitive on all apartments containing \( c \).

Let \( c \) be an alcove in \( A \). Then for each \( b \in B_c \) the restriction of \( b \) to \( bA \) is the unique isomorphism mapping \( bA \) to \( A \) and fixing their intersection pointwise. Therefore for all points \( x' \) in the preimage \( r_{A,c}^{-1}(x) \), for some \( x \in A \), there exists \( b \in B_c \) with \( x' = bx \). We thus have the assertion. \( \Box \)
4. Folding operators

In this section we recall the definition of the operators $e_\alpha$, $f_\alpha$ and $\tilde{e}_\alpha$ for simple roots $\alpha$ in a fixed apartment $A$ as introduced in [GL05]. They act on the set of positively folded combinatorial galleries of a fixed type as introduced therein.

We consider galleries in the broader, combinatorial sense of [GL05]

**Definition 4.1.** A combinatorial gallery $\gamma$ is a sequence of simplices in $X$,

$\gamma = (p_0 \subset c_0 \supset p_1 \subset \cdots c_l \supset p_{l+1})$,

where $p_0, p_{l+1}$ are vertices in $X$, the $c_i$ are simplices all of the same dimension and the $p_i$, $i \neq 0, l + 1$, are codimension one faces of $c_{i-1}$ and $c_i$.

A combinatorial gallery where all $c_i$ are alcoves is an ordinary gallery of alcoves where in addition a start and end vertex as well as for each pair of consecutive alcoves a shared codimension one face is specified. Such a codimension one face is unique in case that $c_i \neq c_{i-1}$.

**Notation 4.2.** Let $\gamma = (p_0 \subset c_0 \supset \cdots c_l \supset p_{l+1})$ be a combinatorial gallery of type $\gamma_\lambda$ that starts in the origin and ends in a co-character $\nu \preceq \lambda$, i.e. $\gamma \in \Gamma(\gamma_\lambda, \nu)$. Let $\alpha$ be a simple root, and define $m \in \mathbb{Z}$ to be minimal such that there exists $q$ with $p_q$ contained in the hyperplane $H_{\alpha,m}$. Note that $m \leq 0$ as $p_0$ is the origin. There are the following cases:

(I) $m \leq -1$. In this case let $k$ be minimal with $p_k \subset H_{\alpha,m}$, and let $0 \leq j \leq k$ be maximal with $p_j \subset H_{\alpha,m+1}$.

(II) $m \leq \langle \nu, \alpha \rangle - 1$. In this case let $j$ be maximal with $p_j \subset H_{\alpha,m}$, and let $j \leq k \leq l + 1$ be minimal with $p_k \subset H_{\alpha,m+1}$.

(III) $\gamma$ crosses $H_{\alpha,m}$. In this case fix $j$ minimal such that $p_j \subset H_{\alpha,m}$ and $H_{\alpha,m}$ separates $c_i$ from $C_f$ for all $i < j$. Let $k > j$ be maximal such that $p_k \subset H_{\alpha,m}$.

Observe that cases (I) – (III) are not disjoint.

**Definition 4.3.** With all notation as in 4.2 we define folding operators $e_\alpha$, $f_\alpha$ and $\tilde{e}_\alpha$ as follows:

- In case (I) let $e_\alpha(\gamma)$ be the combinatorial gallery defined by
  $e_\alpha(\gamma) = (\nu = p_0 \subset c'_0 \supset p'_1 \subset c'_1 \supset \ldots \subset c'_l \supset p'_{l+1} = \lambda)$,

  where
  \[
  c'_i = \begin{cases} 
  c_i & \text{for } i < j - 1, \\
  s_{\alpha,m+1}(c_i) & \text{for } j \leq i < k, \\
  t_{\alpha,\nu}(c_i) & \text{for } i \geq k.
  \end{cases}
  \]

- In case (II) let $f_\alpha(\gamma)$ be the combinatorial gallery defined by
  $f_\alpha(\gamma) = (\nu = p_0 \subset c'_0 \supset p'_1 \subset c'_1 \supset p'_1 \subset \ldots \subset c'_l \supset p'_{l+1} = \lambda)$,
where
\[
c'_i = \begin{cases} 
  c_i & \text{for } i < j, \\
  s_{\alpha,m+1}(c_i) & \text{for } j \leq i < k, \\
  t_{-\alpha}(c_i) & \text{for } i \geq k.
\end{cases}
\]

- In case (III) let \( \tilde{e}_\alpha \) be the combinatorial gallery defined by
  \[
  \tilde{e}_\alpha(\gamma) = (\nu = p_0 \subset c'_0 \supset p'_1 \subset c'_1 \supset \ldots \subset c'_l \supset p'_{l+1} = \lambda),
  \]
  where
  \[
c'_i = \begin{cases} 
  c_i & \text{for } i \leq j - 1 \text{ and } i \geq k, \\
  s_{\alpha,m+1}(c_i) & \text{for } j \leq i < k.
\end{cases}
\]

The \( e \) and \( f \) operators are partial inverses to each other and have many nice combinatorial properties listed in Lemma 6 and 7 of [GL05]. We continue with an example in type \( \tilde{C}_2 \).

**Example 4.4.** In Figure 1 we illustrate an example of applications of the \( e \) and \( f \) folding operators. The galleries are depicted using a red path that walks through the alcoves of the gallery. The sharp bends on codimension one faces illustrate folds which happen at this face. The gallery in the middle picture is unfolded by \( f_\alpha \) to the minimal gallery shown in the picture on the right. The gallery in the first picture is the image of this minimal gallery under a 2-fold application of \( e_\alpha \) or, equivalently as \( e \) and \( f \) are (partial) inverse, the image of \( \gamma \) under \( e_\alpha \).

![Figure 1](image-url)

**Figure 1.** An illustration of the root operators \( e_\alpha \) and \( f_\alpha \) in type \( \tilde{C}_2 \).

5. **Expressing folding operators in terms of retractions**

The goal of this section is to prove the main result of this note and show that one can express root operators in terms of retractions and analogously as images under products of elements of root groups.

Let \( X \) be a thick affine building. Fix an apartment \( A \) in \( X \), a simple root \( \alpha \), and an integer \( m \in \mathbb{Z} \). Then there exists a wall \( H_{\alpha,m} = \{ x \in \)
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\[ A|\langle x, \alpha \rangle = m \} \text{ in } A, \] which determines a positive half-space \( H_{\alpha,m}^+ \) and a negative half-space \( H_{\alpha,m}^- \).

For fixed \( \alpha \) and \( k \in \mathbb{Z} \) let \( A_k \) be an apartment in \( X \) such that
\[ A \cap A_k = H_{\alpha,k}^- = H_{\alpha,k}^+ \]
and write \( \rho_k \) for the restriction of \( \rho_{A,C_{f_0}} : X \to A \) to the apartment \( A_k \).

It is easy to see that \( \rho_k \) is an isometry that fixes the intersection \( A_k \cap A \).

Similarly write \( \rho_k^{op} \) for the restriction of \( \rho_{A,C_f} \) to the apartment
\[ B_k := (A_k \setminus A) \cup (A \setminus A_k). \]

We may now define isometries between two apartments \( B_k \) and \( B_l \).

**Lemma 5.1.** With notation as above the map
\[ \iota_{k,l,B}^{op} := (\rho_k^{op})^{-1} \circ \rho_l^{op} : B_l \to B_k \]
is an isometry from \( B_k \) to \( B_l \) fixing \( B_l \cap B_k \) pointwise and
\[ \iota_{k,l,A} := (\rho_k)^{-1} \circ \rho_l : A_l \to A_k \]
is an isometry from \( A_l \) to \( A_k \) fixing \( A_l \cap A_k \) pointwise.

**Proof.** This is an easy consequence of the fact that the restriction of \( \rho_{A,C_f} \), respectively \( \rho_{A,C_{f_0}} \), to an apartment \( A' \) containing a sub-Weyl chamber of \( C_f \), respectively \( C_{f_0} \), is an isometry between \( A' \) and \( A \) that fixes their intersection pointwise. \( \square \)

**Lemma 5.2.** For any alcove \( c \in A_k \cap B_k \) it’s two retracted images \( d = \rho_k(c) \) and \( d^{op} = \rho_k^{op}(c) \) are reflected images of one another along \( H_{\alpha,k} \). That is
\[ d^{op} = s_{\alpha,k}(d) = m(u)(d) \text{ for all } u \in . \]
Proof. First observe that for any Weyl chamber $C$ in $A$ ans any apartment $B$ such that the intersection $A \cap B \neq \emptyset$ contains a sub-chamber of $C$ we have that the restriction of $\rho_{A,C}$ to $B$ is an isometry of apartments fixing $A \cap B$ pointwise.

Thus the half space $B_k \setminus A$ get isometrically mapped onto $A \setminus B_k$ by $\rho_k^{op}$. The same statement holds for the half-space $A_k \setminus A$ and $A \setminus A_k$ via the retraction $\rho_k$.

Combining the two we obtain an isometry $A \setminus A_k \rightarrow A \setminus b_k$ that fixes the common wall. This isometry has to be the reflection along the wall $H_{\alpha,k}$. □

We first consider an operator $e_\alpha$ for a fixed simple root $\alpha$. The next theorem shows that $e_\alpha$ first unfolds $\gamma$ along $H_{\alpha,m}$ and then folds the unfolded image along $H_{\alpha,m+1}$, where $m$ is as in 4.2. This folding process is possible only if $\gamma$ has a face contained in the hyperplane $H_{\alpha,m+1}$. However this is equivalent to the operator being defined.

Let $\gamma = (f_0 \subset c_1 \supset f_1 \subset \ldots \subset c_n \supset f_n)$ be a combinatorial gallery. For arbitrary $0 \leq k \leq n$ one can write $\gamma$ as a concatenation of two sub-galleries

$$\gamma_k^- = (f_0 \subset c_1 \supset \ldots \subset c_{k-1} \supset f_k)$$

and

$$\gamma_k^+ = (f_k \subset c_k \supset \ldots \subset c_n \supset f_n).$$

That is $\gamma = \gamma_k^- \star \gamma_k^+$. Using this decomposition we can show the following theorem.

**Theorem 5.3.** Let $C_{w_0}$ be the unique Weyl chamber in $A$ opposite $C_f$. Let $\gamma = (f_0 \subset c_1 \supset f_1 \subset \ldots \subset c_n \supset f_n)$ be a combinatorial gallery and let $k \geq 1$, $m \leq -1$ and $j$ be as in 4.2.

Suppose that $e_\alpha$ is defined for $\gamma$ and decompose $\gamma$ as $\gamma = \gamma_k^- \star \gamma_k^+$. Then

$$e_\alpha(\gamma) = \rho_{A,C_{w_0}} \left(t_{m+1,m,B} \left(\gamma_k^- \star \rho_m^{-1}(\gamma_k^+)\right)\right).$$

Suppose $f_\alpha$ is defined for $\gamma$ and decompose $\gamma$ as $\gamma = \gamma_j^- \star \gamma_j^+$. Then

$$f_\alpha(\gamma) = \rho_{A,C_{w_0}} \left(t_{m-1,m,B} \left(\gamma_j^- \star \rho_m^{-1}(\gamma_j^+)\right)\right).$$

Proof. Retractions in buildings preserve adjacency and dimension of simplices, therefore

$$\tilde{\gamma} := \rho_{A,C_{w_0}} \left(t_{m+1,m,B} \left(\gamma_k^- \star \rho_m^{-1}(\gamma_k^+)\right)\right)$$

is again a combinatorial gallery.

Let $c$ be one of the $c_i$ in $\gamma$ and let $\tilde{c}$ denote the image of $c$ in the gallery $\tilde{\gamma}$. In order to prove the statement we need to see that $\tilde{c}$ is the same as the image of $c$ under the operator $e_\alpha$. There are three cases: either $c$ is in $\gamma_k^+$, or $c$ is in $\gamma_k^-$ where we distinguish between the case that $c$ is in the strip between $H_{\alpha,m}$ and $H_{\alpha,m+1}$, or $c \in \gamma_k^-$ is in the half space $H_{\alpha,m+1}^-$. 

Suppose first that \( c \in \gamma^{-}_k \cap H^{+}_{\alpha,m+1} \). In this case \( c = c_j \) for some \( j < k \) and the root operator \( e_{\alpha} \) thus fixes \( c \). As \( c \in H^{+}_{\alpha,m+1} \) one can see that \( \rho^{op}_i(c) = c \), for \( i = m, m+1 \) and \( \rho_{A,Cw_0}(c) = c \). Hence
\[
\rho_{A,Cw_0}(i^{op}_{m+1,m,B}(c)) = \rho_{A,Cw_0}((\rho^{op}_{m+1})^{-1} \circ \rho^{op}_m(c)) = \rho_{A,Cw_0}(c) = c.
\]

Suppose now that \( c \in \gamma^+_k \) is in the strip between \( H_{\alpha,m} \) and \( H_{\alpha,m+1} \). In this case \( e_{\alpha} \) reflects \( c \) along \( H_{\alpha,m+1} \). To see what the map on the right hand side of the equation does argue as follows: as \( \rho^{op}_m(c) = c \) we have that
\[
\rho_{A,Cw_0}(i^{op}_{m+1,m,B}(c)) = \rho_{A,Cw_0}((\rho^{op}_{m+1})^{-1}(c)).
\]

Lemma 5.2 implies that in fact \( \rho_{A,Cw_0}(i^{op}_{m+1,m,B}(c)) = c \).

In the last case \( c \in \gamma^+_k \). The image of \( c \) by \( e_{\alpha} \) is then the translate \( t_{\alpha}(c) \). We need to verify that \( \check{c} = t_{\alpha}(c) \). As \( c \in \gamma^+_k \) we may conclude by Lemma 5.2 that
\[
\check{c} = \rho_{A,Cw_0}(((\rho^{op}_{m+1})^{-1} \circ \rho^{op}_m)(\rho^{-1}_m(c))) = \rho_{A,Cw_0}((\rho^{op}_{m+1})^{-1}(s_{\alpha,m}(c))).
\]

The reflected image \( s_{\alpha,m}(c) \) is contained in \( A \setminus B_{m+1} \) and thus the simplex \( c' := (\rho^{op}_{m+1})^{-1}(s_{\alpha,m}(c)) \) is contained in \( B_{m+1} \setminus A \) which implies that \( \rho_{A,Cw_0}(c') = \rho_{m+1}(c') \). Another application of Lemma 5.2 thus implies that \( \check{c} = s_{\alpha,m+1}(s_{\alpha,m}(c)) = t_{\alpha}(c) \), which completes the proof in the first case. The presentation of \( f_\alpha \) in terms of retractions is along the same lines. \( \square \)

Finally we study the operator \( \check{e}_{\alpha} \).

**Theorem 5.4.** Suppose that \( \gamma = (f_0 \subset c_1 \supset f_1 \subset \ldots \subset c_{n-1} \supset f_n) \) is a gallery for which \( \check{e}_{\alpha} \) is defined. Let \( m,j,k \) be as in \( \gamma \) case (III), that is \( j \) is minimal such that \( f_j \subset H_{\alpha,m} \) and \( k > j \) minimal such that \( f_k \) is also contained in \( H_{\alpha,m} \). We define three sub-galleries
\[
\gamma_j = (f_0 \subset c_1 \supset f_1 \subset \ldots \subset c_{j-1} \supset f_j),
\]
\[
\gamma_{jk} = (f_j \subset c_j \supset \ldots \subset c_{k-1} \supset f_k) \quad \text{and}
\]
\[
\gamma_k = (f_k \subset c_k \supset \ldots \subset c_{n-1} \supset f_n)
\]
of \( \gamma \) and write \( \gamma \) as a concatenation of them, that is \( \gamma = \gamma_j \star \gamma_{jk} \star \gamma_k \). Then
\[
\check{e}_{\alpha}(\gamma) = \gamma_j \star \rho^{op}_m(\rho^{-1}_m(\gamma_{jk})) \star \gamma_k.
\]

**Proof.** Convince yourself that the sub-galleries \( \gamma_j \) and \( \gamma_k \) remain untouched under an application of \( \check{e}_{\alpha} \) and that the gallery \( \gamma_{jk} \), which lies in between \( H_{\alpha,m} \) and \( H_{\alpha,m-1} \) gets reflected along \( H_{\alpha,m} \) by \( e_{\alpha} \). It is then easy to see that the application of \( \rho^{op}_m \circ \rho^{-1}_m \) does just that. \( \square \)
Example 5.5. In Figure 3 we illustrate the statement of Theorem 5.3 and show how the retractions are used to express the operator $e_\alpha$ with. We abbreviate the map $\iota_{m+1,m,B}$ by $\iota_m$.

We start with the gallery $\gamma$ which is the concatenation of the bold blue gallery $\gamma_k^-$ and the bold red gallery $\gamma_k^+$. In the first step we keep the initial blue part and take a preimage of the red piece under the retraction $\rho_m$. The second step consists of an application of a the map $\iota_m = \iota_{m+1,m,B}$ to the concatenation $\gamma_k^- \star \rho_m^{-1}(\gamma_k^+)$. This yields a gallery which coincides with $\gamma$ now up to the wall $H_{\alpha,m+1}$. A final application of the opposite retraction, namely $\rho_{A,C_{w_0}}$, gives us the image of $\gamma$ under the root operator $e_\alpha$.

Interpretation in terms of root groups. We use the fact that one can write pre-images of retractions in terms of groups to also rewrite the statements of Theorem 5.3 and 5.4.

Corollary 5.6. Let $\gamma = (f_0 \subset c_1 \supset f_1 \subset \ldots \subset c_n \supset f_n)$ be a combinatorial gallery and let $k \geq 1$, $m \leq -1$ and $j$ be as in 4.2.

Suppose we are in case (I), that is $e_\alpha$ is defined for $\gamma$ and $\gamma$ is decomposed as $\gamma = \gamma_k^- \star \gamma_k^+$. Then there exist elements

\[
  u_1 \in U^*_{-\alpha,-m} \backslash U^*_{-\alpha,-m-1},
  u_2 \in U^*_{-\alpha,-(m+1)} \backslash U^*_{-\alpha,-m-2}
\]

and

\[
  v \in U^*_{\alpha,-(m+1)} \backslash U^*_{\alpha,-m-2}
\]

such that

\[
e_\alpha(\gamma) = u_2 \left( v \left( \gamma_k^- \star u_1(\gamma_k^+) \right) \right).
\]
With \(u_1\) and \(u_2\) as above and \(m(u_i)\) as defined in 2.8, one has
\[
e_\alpha(\gamma) = m(u_2)\left(\gamma_k^- \star m(u_1)(\gamma_k^+)\right).
\]

Suppose we are in case (II), that is \(f_\alpha\) is defined for \(\gamma\) and \(\gamma\) is decomposed as \(\gamma = \gamma_j^- \star \gamma_j^+\). Then there exist elements
\[
\begin{align*}
    u_1 &\in U^*_{-\alpha,-(m+1)} \setminus U^*_{\alpha,-m-2} \\
u_2 &\in U^*_{\alpha,-m} \setminus U^*_{-\alpha,-m-1}
\end{align*}
\]
such that
\[
f_\alpha(\gamma) = m(u_2)\left(\gamma_j^- \star m(u_1)(\gamma_j^+)\right).
\]

Suppose we are in case (III), that is \(\tilde{e}_\alpha\) is defined. Decompose \(\gamma\) as a concatenation \(\gamma_j \star \gamma_{jk} \star \gamma_k\) as in 5.4. With \(u \in U^*_{\alpha,m} \setminus U_{\alpha,m-1}\) and \(m(u)\) as in 2.8, we can write
\[
\tilde{e}_\alpha(\gamma) = \gamma_j \star m(u)(\gamma_{jk}) \star \gamma_k.
\]

Proof. To see the statement in case (I) convince yourself that both \(\rho_k\) and \(u_1\) stabilize the half-apartment \(H^\pm_{-\alpha,m}\) pointwise. Similarly \(u_2\) and the retraction based at \(C_{w_0}\) do fix \(H^+_{\alpha,m+1}\) and the half-apartment \(H^+_{\alpha,m+1}\) is fixed by \(v\) and the map \(\iota_{m+1,m,B}\). With this we can read off the statement of Theorem 5.3 and the definition of the groups \(U_{\beta,k}\). The remaining statements are shown accordingly. \(\square\)

6. Galleries versus paths

One can show, see Lemma 6.1 below, that every geodesic in a building is contained in a (not necessarily unique) minimal gallery. The converse statement, that a minimal combinatorial gallery contains a geodesic connecting the same endpoints is obviously not true. In case that the geodesic \(\pi\) is contained in a wall there is a minimal combinatorial gallery in the sense of [GL05], that is also contained in the wall and contains \(\pi\).

This connection allows to deduce the same results we obtained for the operators on galleries for the operators defined on paths, see [Lit95]. More explicitly one can show Proposition 6.3.

The (simplicial) support of a point \(x \in X\) is the smallest simplex containing \(x\) in its interior. A proof of the following lemma in case of galleries of alcoves was for example given in Lemma 3.1 of [Mar14].

Lemma 6.1. Suppose \(\pi : [0,1] \rightarrow X\) is a geodesic in a building. Then \(\pi\) is contained in a (not necessarily unique) minimal combinatorial gallery connecting the support of \(\pi(0)\) with the support of \(\pi(1)\).
As an immediate consequence we have that every geodesic $\pi : [0, 1] \rightarrow X$ in a building $X$ is contained in an apartment and contained in the convex hull in an apartment $\text{conv}_A(\{\pi(0), \pi(1)\})$ for each apartment $A$ containing $\pi$.

**Notation 6.2.** In the following we refer to the root operators defined on paths, as in [Lit94, Lit95], as *path folding operators* and to the folding operators defined on galleries as *gallery folding operators*.

Lemma 6.1 allows us to consider the image of a geodesic under a sequence of gallery folding operators. Here we consider the image $\pi([0, l])$ of the path $\pi$. This is, by the lemma, a subset of $\gamma$. The image of $\pi$ under a sequence of gallery folding operators is then the image of the set $\pi([0, l])$ inside $\gamma$ under this sequence of operators applied to $\gamma$.

To prove the following proposition it thus remains to compare the definition of path folding operators with the one of gallery folding operators.

**Proposition 6.3.** Let $\pi : [0, l] \rightarrow A$ be a geodesic in $A$ for which a sequence of path folding operators is defined and let $\gamma$ be a minimal combinatorial gallery containing $\pi$ such that the same sequence of gallery folding operators is defined. Then the image of $\pi$ under a sequence of path folding operators is the same as the image of $\pi$ under the sequence of gallery folding operators.

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