FUNCTIONS HOLOMORPHIC ALONG
HOLOMORPHIC VECTOR FIELDS

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Abstract. The main result of the paper is the following general-
ization of Forelli’s theorem [F]: Suppose \( F \) is a holomorphic vector
field with singular point at \( p \), such that \( F \) is linearizable at \( p \) and
the matrix is diagonalizable with the eigenvalues whose ratios are
positive reals. Then any function \( \phi \) that has an asymptotic Tay-
lor expansion at \( p \) and is holomorphic along the complex integral
curves of \( F \) is holomorphic in a neighborhood of \( p \).

We also present an example to show that the requirement for
ratios of the eigenvalues to be positive reals is necessary.

1. Introduction

Let \( F \) be a holomorphic vector field on a complex manifold \( M \). Con-
sider a function \( \phi \) on \( M \) that is holomorphic on the complex integral
curves of \( F \). In general, there is not much to say about such functions.
However, Forelli noted in [F] that if \( M = \mathbb{D}^N \) is the unit polydisk in
\( \mathbb{C}^N \) and \( F(z) = z \) is the holomorphic Euler vector field, then \( \phi \) is holo-
morphic on \( \mathbb{D}^N \) provided \( \phi \) is infinitely smooth in a neighbourhood of
the origin. In fact, (see [KW]) \( \phi \) only needs to have a Taylor series
at the origin and this is essential. Counterexamples are easy to con-
struct. Recently, Chirka in [Ch] showed that in \( \mathbb{C}^2 \) the straight lines,
which are the integral curves of \( F(z) = z \), can be replaced by families
of transversal holomorphic curves passing through the origin.

In this paper we study the question what properties of a holom orphic
vector field \( F \) with a critical point \( z_0 \) imply a Forelli type result. We
prove that if \( F \) is linearizable at \( z_0 \) and if the linearization matrix is
diagonalizable and the pairwise ratios of its eigenvalues are positive,
then any function that is holomorphic on the complex integral curves
of \( F \) and has a Taylor series at \( z_0 \) is holomorphic in the open subset

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of $M$ saturated by $z_0$ (see Section 6 for definitions). Note that the integral curves of generic vector field do not pass through the origin and Chirka’s result does not apply.

An example, presented in Section 7, of a holomorphic vector field and a non-holomorphic $C^\infty$-function holomorphic along this field shows that the imposed restrictions on the eigenvalues are necessary.

The proof follows the original steps of Forelli. The main difficulty to overcome is the absence of a multitude of separatrices in the general case. So we had to introduce the appropriate notion of an asymptotic expansion. This notion replaces the Taylor series of holomorphic functions on separatrices. An elementary theory of such expansions is contained in Sections 2 and 3. The main outcome of this theory is the estimate of the coefficients in Theorem 3.5 absolutely similar to the classical Cauchy inequalities for holomorphic functions on the disk.

## 2. Asymptotic expansions

**Definition 2.1.** Let $f$ be a function on the right half-plane $\mathbb{H}$. Suppose that \( \{\lambda_j\}, j \geq 1 \), is a strictly increasing sequence of positive numbers converging to infinity, \( \lambda_0 = 0 \) and let \( \{n_j\}, j \geq 1 \), be a sequence of positive integers. A formal series

\[
\sum_{j=0}^{\infty} n_j \sum_{k=0}^{p_{jk}} e^{-\mu_{jk} z - \nu_{jk} z}
\]

is called an asymptotic expansion of $f$ if all \( \mu_{jk}, \nu_{jk} \geq 0 \), \( \mu_{jk} + \nu_{jk} = \lambda_j \) for every $j$ and $k$ and for every $n$ we have

\[
|f(z) - \sum_{j=0}^{n} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk} z - \nu_{jk} z}| e^{\lambda_n \Re z} \to 0
\]

as $\Re z \to \infty$.

The same definition can be trivially rephrased as follows:

**Proposition 2.2.** A function $f$ on $\mathbb{H}$ has an asymptotic expansion (1) if and only if for every $\varepsilon > 0$

\[
|f(z) - \sum_{j=0}^{n+1} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk} z - \nu_{jk} z}| e^{(\lambda_{n+1} - \varepsilon) \Re z} \to 0
\]

as $\Re z \to \infty$ in $\mathbb{H}$.

**Proof.** If

\[
|f(z) - \sum_{j=0}^{n+1} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk} z - \nu_{jk} z}| e^{\lambda_{n+1} \Re z} \to 0,
\]
then
\[ f(z) - \sum_{j=0}^{n} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk}z - \nu_{jk}z} = e^{(\lambda_{n+1} - \varepsilon) Re z} \to 0 \]
as \( Re z \to \infty \).

If
\[ f(z) - \sum_{j=0}^{n} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk}z - \nu_{jk}z} = e^{(\lambda_{n+1} - \varepsilon) Re z} \to 0 \]
as \( Re z \to \infty \), then
\[ f(z) - \sum_{j=0}^{n} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk}z - \nu_{jk}z} = e^{\lambda_n Re z} \to 0. \]

\[ \square \]

We will say that two asymptotic expansions are equal if all their non-zero terms coincide. Let us state some simple properties of asymptotic expansions.

**Proposition 2.3.** 1) Every function \( f \) on \( \mathbb{H} \) has at most one asymptotic expansion (1).

2) If a holomorphic function on \( \mathbb{H} \) has an asymptotic expansion (1), then all \( \nu_{jk} = 0 \).

**Proof.** 1) Suppose that \( f \) has two asymptotic expansions:
\[ \sum_{j=0}^{\infty} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk}z - \nu_{jk}z} \quad \text{and} \quad \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} q_{jk} e^{-\alpha_{jk}z - \beta_{jk}z}, \]
where \( \mu_{jk} + \nu_{jk} = \lambda_j \) and \( \alpha_{jk} + \beta_{jk} = \gamma_j \). Let \( l \) be the first number when two non-zero terms
\[ \sum_{k=0}^{n_l} p_{lk} e^{-\mu_{lk}z - \nu_{lk}z} \quad \text{and} \quad \sum_{k=0}^{m_l} q_{lk} e^{-\alpha_{lk}z - \beta_{lk}z} \]
do not coincide.

Suppose that \( \gamma_l > \lambda_l \) and \( n \) is the least integer such that \( \lambda_n \geq \gamma_n \). Since
\[ f - \sum_{j=0}^{l} \sum_{k=0}^{m_j} q_{jk} e^{-\alpha_{jk}z - \beta_{jk}z} = e^{\gamma_l Re z} \to 0 \]
and
\[ f - \sum_{j=0}^{n} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk}z - \nu_{jk}z} = e^{\gamma_l Re z} \to 0, \]
we see that
\[
\sum_{j=l}^{n} \sum_{k=0}^{n_j} p_{jk} e^{-\mu_{jk} z - \nu_{jk} z} - \sum_{k=0}^{m_l} q_{lk} e^{-\alpha_{lk} z - \beta_{lk} z} e^{\gamma \text{Re} z} \to 0
\]
as \text{Re} z \to \infty. Hence \( p_{jk} = 0, l \leq j \leq n - 1 \). In particular, all \( p_{lk} = 0 \) and this contradicts our assumption that some of \( p_{lk} \neq 0 \).

Thus \( \lambda_l = \gamma_l \) and, letting \( z = x + iy \), we get
\[
\sum_{k=0}^{m_l} p_{lk} e^{-i(\mu_{lk} - \nu_{lk}) y} - \sum_{k=0}^{m_l} q_{lk} e^{-i(\alpha_{lk} - \beta_{lk}) y} \to 0.
\]
But this means that \( m_l = m_l \) and \( p_{lk} = q_{lk} \).

2) Let \( m \) be the least integer such that \( \nu_{mk} \neq 0 \) for some \( k \). After subtracting from \( f \) the first \( n \) holomorphic terms of its asymptotic expansion we may assume that
\[
f(z) = \sum_{k=0}^{n_l} p_{mk} e^{-\mu_{mk} z - \nu_{mk} z} + g(z),
\]
where all \( \nu_{mk} > 0 \) and \( g(z) e^{(\lambda_m + \varepsilon) \text{Re} z} \to 0 \) as \( \text{Re} z \to \infty \) for some \( \varepsilon > 0 \). The integral of the function
\[
h(z) = f(z) e^{\lambda_m z} = \sum_{k=0}^{n_m} p_{mk} e^{2i\nu_{mk} y} + g(z) e^{\lambda_m z}
\]
by \( dz \) over the boundary of the rectangle \( \{t-1 \leq x \leq t+1, -s \leq y \leq s\} \), where \( t > 1 \) and \( s > 0 \), is equal to
\[
-4i \sum_{k=0}^{n_m} p_{mk} \sin 2\nu_{mk} s + o(e^{-\varepsilon t}) = 0.
\]
But this is possible if and only if all \( \nu_{mk} = 0 \). \( \square \)

3. Asymptotic expansions of holomorphic functions

As it follows from the previous section an asymptotic expansion of a holomorphic function \( f \) on \( \mathbb{H} \) is a formal series
\[
\sum_{j=0}^{\infty} c_j e^{-\lambda_j z}.
\]

We will need a simple version of the maximum principle.

**Lemma 3.1.** Suppose that \( f \) is a bounded holomorphic function on \( \mathbb{H} \), \( \lambda > 0 \), and
\[
\limsup_{\text{Re} z \to \infty} f(z) e^{\lambda \text{Re} z} < \infty.
\]
Let $f^*(iy)$ be the non-tangential boundary values of $f$ on $i\mathbb{R}$. Then $|f(z)| \leq Me^{-\lambda \Re z}$ on $\mathbb{H}$, where $M = \sup_{y \in \mathbb{R}}|f^*(iy)|$.

Proof. Note that the function $g(z) = f(z)e^{\lambda z}$ is bounded on $\mathbb{H}$ and $|g^*(iy)| \leq M, y \in \mathbb{R}$. Hence $|g(z)| \leq M$ on $\mathbb{H}$. \hfill \Box

**Corollary 3.2.** Let $f$ be a bounded holomorphic function on $\mathbb{H}$ with an asymptotic expansion $\sum_{j=0}^{\infty} c_j e^{-\lambda_j z}$. Let $f_n(z) = f(z) - \sum_{j=0}^{n} c_j e^{-\lambda_j z}$. Then $|f_n(z)| \leq M_n e^{-\lambda_{n+1} \Re z}$ on $\mathbb{H}$, where $M_n = \sup_{y \in \mathbb{R}}|f_n^*(iy)|$.

Proof. By Lemma 3.1 for every $\varepsilon > 0$ we have $|f_n(z)| \leq M_n e^{-(\lambda_{n+1} - \varepsilon) \Re z}$ on $\mathbb{H}$. Letting $\varepsilon$ go to 0 we get our result \hfill \Box

**Corollary 3.3.** If an asymptotic expansion of $f$ has zero coefficients, then $f \equiv 0$ on $\mathbb{H}$.

The theorem below gives us estimates for the coefficients. We start with a simple lemma.

**Lemma 3.4.** Let $f$, $|f| \leq M$, be a holomorphic function on $\mathbb{H}$ with an asymptotic expansion $\sum_{j=0}^{\infty} c_j e^{-\lambda_j z}$. If $\alpha$ is real then the function
\[ g(z) = f(z + i\alpha) - f(z) \]
is holomorphic on $\mathbb{H}$, $|g| \leq 2M$ there, and $g$ has the asymptotic expansion
\[ \sum_{j=0}^{\infty} (e^{\alpha i \lambda_j} - 1)c_j e^{-\lambda_j z}. \]

Proof. Clearly, the function $g$ is holomorphic and $|g| \leq 2M$ on $\mathbb{H}$. Note that
\[ \sum_{j=0}^{k} c_j e^{-\lambda_j (z+i\alpha)} - \sum_{j=0}^{k} c_j e^{-\lambda_j z} = \sum_{j=0}^{k} (e^{\alpha i \lambda_j} - 1)c_j e^{-\lambda_j z}. \]
Hence by Corollary 3.2 we get
\[ \left| g(z) - \sum_{j=0}^{k} (e^{\alpha i \lambda_j} - 1)c_j e^{-\lambda_j z} \right| \leq \left| f(z) - \sum_{j=0}^{k} c_j e^{-\lambda_j z} \right| + \left| f(z + ia) - \sum_{j=0}^{k} c_j e^{-\lambda_j (z+ia)} \right| \leq Ce^{-\lambda_{k+1} \Re z}. \]
\hfill \Box

**Theorem 3.5.** Let $f$, $|f| < M$, be a holomorphic function on $\mathbb{H}$ with an asymptotic expansion $\sum_{j=0}^{\infty} c_j e^{-\lambda_j z}$. Then $|c_j| \leq M$. 

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Proof. The proof is by induction on the indices of the coefficients. Clearly $|c_0| \leq M$. Suppose the theorem is proved for all $c_j$, $j \leq m$, for all functions $f$ and constants $M$ as set out in the theorem. Let $a = \pi/\lambda_{m+1}$. Then $e^{-ia/\lambda_{m+1}} - 1 = -2$. If

$$g(z) = f(z + ia) - f(z)$$

and $\sum_{j=0}^{\infty} g_j e^{-\lambda_j z}$ is the asymptotic expansion of $g$, then by Lemma 3.4 $g_0 = 0$. Hence, by Lemma 3.1, the function $h(z) = e^{\lambda_1 z} g(z)$ is bounded on $\mathbb{H}$ and $|h| \leq 2M$ on $\mathbb{H}$. The asymptotic expansion $\sum_{j=0}^{\infty} h_j e^{-\nu_j z}$ of $h$ is equal to $\sum_{j=1}^{\infty} g_j e^{-(\lambda_j - \lambda_1)z}$. Thus

$$h_m = g_{m+1} = -2c_{m+1}.$$

By the induction assumption

$$|c_{m+1}| \leq \frac{|h_m|}{2} \leq M.$$

□

Corollary 3.6. Suppose that a bounded holomorphic function $f$ on $\mathbb{H}$ has an asymptotic expansion $\sum_{j=0}^{\infty} c_j e^{-\lambda_j z}$ such that the series $\sum_{j=0}^{\infty} e^{-\lambda_j d}$ converges for some $d > 0$. Then the series $\sum_{j=0}^{\infty} c_j e^{-\lambda_j z}$ converges uniformly to $f(z)$ on $\mathbb{H}_d = \{ \text{Re } z \geq d \}$.

4. Taylor series and asymptotic expansions

Let $\phi$ be a function defined in a neighborhood of the origin in $\mathbb{C}^N$. We say that $\phi$ has a Taylor series at the origin if there is a formal series

$$\sum_{j=0}^{\infty} \sum_{|k|+|m|=j} a_{km} z^k \bar{z}^m$$

such that for every $j$

$$\left| \phi(z) - \sum_{j=0}^{n} \sum_{|k|+|m|=j} a_{km} z^k \bar{z}^m \right| = o(|z|^n).$$

(Here for $k = (k_1, \ldots, k_N)$ and $z = (z_1, \ldots, z_N)$, $z^k = z_1^{k_1} \cdots z_N^{k_N}$.)

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a vector in $\mathbb{R}^N$ with positive components and $c = (c_1, \ldots, c_N)$ a vector in $\mathbb{D}^N$. We define the function $s_c : \mathbb{C} \to \mathbb{C}^N$ by

$$s_c : \zeta \mapsto (c_1 e^{-\alpha_1 \zeta}, \ldots, c_N e^{-\alpha_N \zeta}).$$

Notice that $s_c(\zeta) \in \mathbb{D}^N$ if $\zeta \in \mathbb{H}$.

Denote by $\{\lambda_j\}$ the sequence of all possible values

$$(\alpha, k) + (\alpha, m) = \alpha_1 k_1 + \cdots + \alpha_N k_N + \alpha_1 m_1 + \cdots + \alpha_N m_N.$$
where \( k, m \) are multi-indices, in ascending order.

**Proposition 4.1.** Let \( \phi \) be a function defined on \( \mathbb{D}^N \). If \( \phi \) has the Taylor series \((3)\) at the origin, then the function \( f_c = \phi \circ s_c : \mathbb{H} \to \mathbb{C} \) has the asymptotic expansion

\[
\sum_{j=0}^{\infty} \left( \sum_{(\alpha,k)+(\alpha,m)=\lambda_j} a_{km} c^{k} c^{m} e^{-(\alpha,k)\zeta-(\alpha,m)\zeta} \right)
\]

on \( \mathbb{H} \).

**Proof.** Without any loss of generality we may assume that \( \alpha_1 \) is the minimal number among those \( \alpha_j \) for which \( c_j \neq 0 \). Then \( |s_c(\zeta)| \sim e^{-\alpha_1 \text{Re} \zeta} \).

For any \( p \) choose \( n \) such that \( n\alpha_1 > \lambda_p \). By the definition of the Taylor series

\[
\left( f_c(\zeta) - \sum_{j=0}^{n} \left( \sum_{|k|+|m|=j} a_{km} c^{k} c^{m} e^{-(\alpha,k)\zeta-(\alpha,m)\zeta} \right) \right) e^{n\alpha_1 \text{Re} \zeta} \to 0
\]

as \( \text{Re} \zeta \to \infty \).

Note that if \( |k| + |m| > n \) then \( (\alpha, k) + (\alpha, m) > n\alpha_1 > \lambda_p \). Hence all multi-indices \( k \) and \( m \) for which \( (\alpha, k) + (\alpha, m) \leq \lambda_p \) are included into the sum

\[
\sum_{j=0}^{n} \left( \sum_{|k|+|m|=j} a_{km} c^{k} c^{m} e^{-(\alpha,k)\zeta-(\alpha,m)\zeta} \right).
\]

Splitting this sum into the sum over all \( k \) and \( m \) such that \( (\alpha, k) + (\alpha, m) \leq \lambda_p \) and the sum over all \( k \) such that \( (\alpha, k) + (\alpha, m) > \lambda_p \) and noticing that the terms of the second sum multiplied by \( e^{\lambda_p \text{Re} \zeta} \) go to 0 as \( \text{Re} \zeta \to \infty \), we see that

\[
\left( f_c(\zeta) - \sum_{(\alpha,k)+(\alpha,m)\leq\lambda_p} a_{km} c^{k} c^{m} e^{-(\alpha,k)\zeta-(\alpha,m)\zeta} \right) e^{\lambda_p \text{Re} \zeta} \to 0
\]

as \( \text{Re} \zeta \to \infty \).

Rearranging terms in the latter sum so that

\[
\sum_{(\alpha,k)+(\alpha,m)\leq\lambda_p} a_{km} c^{k} c^{m} e^{-(\alpha,k)\zeta-(\alpha,m)\zeta}
\]

\[
= \sum_{j=0}^{p} \left( \sum_{(\alpha,k)+(\alpha,m)=\lambda_j} a_{km} c^{k} c^{m} e^{-(\alpha,k)\zeta-(\alpha,m)\zeta} \right)
\]
we obtain the desired result.

From the second statement in Proposition 2.3 we immediately get

**Corollary 4.2.** If in Proposition 4.1 the function \( f_c \) is holomorphic, then

\[
\sum_{(\alpha,m)=\lambda} a_{km} c^m \equiv 0
\]

for all \( \lambda > 0 \).

5. **Linear systems of differential equation**

The goal of this section is to prove the following theorem.

**Theorem 5.1.** Let \( A \) be a diagonal \( N \times N \)-matrix with eigenvalues \( \alpha_1, \ldots, \alpha_N \) such \( \alpha_j/\alpha_k > 0 \) for every pair \((j,k)\) and let \( \phi \) be a function defined on a neighborhood of the origin. If \( \phi \) has the Taylor series (3) at the origin and is holomorphic on all solutions of the system \( z' = Az \), then this Taylor series does not contain anti-holomorphic variables and converges on a neighborhood of the origin to the function \( \phi \).

**Proof.** Changing the time variable \( \zeta \) to \(-\alpha_1 \zeta\) we may assume that all eigenvalues of \( A \) are real and negative. Also without any loss of generality we may assume that the coordinates on \( \mathbb{C}^N \) are chosen so that \( \phi \) is defined on the unit polydisk \( \mathbb{D}^N \).

It is well-known that for every point \( c \in \mathbb{C}^N \) there is a solution \( s_c(\zeta) : \mathbb{C} \rightarrow \mathbb{C}^N \) of our differential equation which can be written in the form \( s_c(\zeta) = (c_1 e^{\alpha_1 \zeta}, \ldots, c_N e^{\alpha_N \zeta}) \).

By Corollary 4.2

\[
\sum_{(\alpha,m)=\lambda} a_{km} c^m \equiv 0
\]

for all \( \lambda > 0 \) and all \( c \in \mathbb{C}^N \). But \( c^k \) are linearly independent monomials over \( \mathbb{C}^N \) and, therefore, the identity above holds only if all \( a_{km} = 0 \) when \( m \neq 0 \).

To prove the second part we notice that \( \phi \) is bounded in some neighborhood of the origin since it has a Taylor expansion. Therefore we may assume that \( |\phi| \leq M \) on \( \mathbb{D}^N \). By Proposition 4.1 and Theorem 3.5 the absolute values of all polynomials

\[
\sum_{(\alpha,k)=\lambda_j} a_{k0} c^k
\]

do not exceed \( M \). By the Cauchy estimates \( |a_{k0}| \leq 1 \). Thus the series \( \sum_k a_{k0} z^k \) absolutely converges on compacta in \( \mathbb{D}^N \) to a function \( \psi \).
Hence, the formal series
\[ \sum_{j=0}^{\infty} \left( \sum_{(\alpha,k)=\lambda_j} a_{k\alpha} c_k \right) e^{-\lambda_j \zeta} \]
of the asymptotic expansion for \( \phi_c(\zeta) = \phi(s_c(\zeta)) \) converges on \( \mathbb{H}_d \) for every \( d > 0 \) to a function \( \psi_c(\zeta) \). But this means that the difference \( \phi_c(\zeta) - \psi_c(\zeta) \) has an asymptotic expansion with zero coefficients. By Corollary 3.3 \( \phi_c \equiv \psi_c \) and \( \phi \equiv \psi \). Thus \( \phi \) is holomorphic on \( \mathbb{D}^N \).  

**Remark.** The first part of this theorem can be proved when there is \( \alpha \in \mathbb{C} \) such that all numbers \( \alpha_j \) have negative real parts. As the following example shows this requirement is rather essential. Consider the differential equation \( z_1' = z_1, z_2' = -z_2 \) in \( \mathbb{C}^2 \). The solutions are: \( z_1(\zeta) = c_1 e^\zeta \) and \( z_2(\zeta) = c_2 e^{-\zeta} \). The function \( \phi(z_1, z_2) = \overline{z}_1 \overline{z}_2 \) is real analytic, constant on solutions, and its Taylor series is anti-holomorphic.

### 6. F-holomorphic functions

Let \( F \) be a holomorphic vector field on a complex manifold \( M \) of dimension \( N \). We say that a function \( \phi \) on \( M \) is \( F \)-holomorphic if the restrictions of \( \phi \) to the integral curves of the \( F \) are holomorphic. If \( \phi \) is differentiable then it means that its derivative along \( F \) is equal to 0 or in local coordinates \((z_1, \ldots, z_N)\) on \( M \) if \( F = (F_1, \ldots, F_N) \), then
\[ \nabla_F \phi = \sum_{j=1}^{N} \frac{\partial \phi}{\partial z_j} F_j \equiv 0. \]

The maximal integral curves of \( F \) form a holomorphic foliation of \( M \) (see [IY, Ch. 1]). If \( U \) is an open set in \( M \) then following [IY, Def. 2.16] we denote by \( \text{Sat}(U, F) \) the union of all maximal integral curves of \( F \) intersecting \( U \). By [IY, Lemma 2.17] \( \text{Sat}(U, F) \) is open.

**Lemma 6.1.** Let \( M \) be a complex manifold and let \( F \) be a holomorphic vector field on \( M \). If an \( F \)-holomorphic function \( \phi \) on \( M \) is holomorphic on an open set \( U \subset M \), then it is holomorphic on \( \text{Sat}(U, F) \).

**Proof.** Let \( V \subset \text{Sat}(U, F) \) be the open set, where \( \phi \) is holomorphic and let \( S \) be a maximal integral curve of \( F \) in \( M \) intersecting \( U \). If \( S \cap V \neq S \) then there is a point \( z_0 \in S \) lying in the relative boundary of \( S \cap V \) in \( S \). If \( z_0 \) is a critical point of \( F \), then \( S = \{z_0\} \subset U \) and \( \phi \) is holomorphic near \( z_0 \). If \( z_0 \) is not a critical point of \( F \), then by the Rectification Theorem (see [IY, Theorem 1.18]) there are local coordinates \((z_1, \ldots, z_N)\) around \( z_0 \) in \( M \), where \( F(z) = (1,0,\ldots,0) \). We may assume that these local coordinates are defined on \( \mathbb{D}^N \) and
$S \cap \mathbb{D}^N = \{(\zeta, 0, \ldots, 0)\}$. Since $z_0$ is a relative boundary point of $S \cap V$ in $S$, there is $\zeta_0 \in \mathbb{D}$ and an open neighborhood $W$ of $\{(\zeta_0, 0, \ldots, 0)\}$ in $\mathbb{D}^N$, where $\phi$ is holomorphic. By Hartogs Lemma (see [Sh, Lemma 3.6.2.1]) there is $\epsilon > 0$ such that $\phi$ is holomorphic on an open neighborhood of $S \cap \mathbb{D}^N$. Hence $z_0 \in V$ and, by contradiction, $S \subset V$. Thus $\text{Sat}(U, F) = V$. \hfill \Box

In general, $F$-holomorphic functions need not to be holomorphic. However, an additional information on their behavior near special critical points of $F$ implies that they are holomorphic on rather large sets.

Let us recall that a vector field $F$ is linearizable at a critical point $z_0 \in M$ of $F$ if there is a biholomorphic mapping $H$ of a neighborhood $U$ of $z_0$ onto a neighborhood $V$ of $H(z_0)$ such that in new coordinates $F(w) = Aw$, where $A$ is an $N \times N$-matrix.

If $z_0$ is a critical point of $F$ then we define the set $\text{Sat}(z_0, F)$ as the union of all maximal integral curves $L$ of $F$ satisfying: $L \cap U \neq \emptyset$ for any open neighborhood $U$ of $z_0$. If $F$ is linearizable at its isolated critical point $z_0$ and there is $\alpha \in \mathbb{C}$ such that all numbers $\alpha_j \alpha$ have negative real parts, where $\alpha_j$ are the eigenvalues of the linearization matrix $A$, then there is a neighborhood $U$ of $z_0$ such that the set $\text{Sat}(z_0, F) = \text{Sat}(U, F)$ and, consequently, is open.

**Theorem 6.2.** Let $F$ be a holomorphic vector field on a complex manifold $M$. Suppose that $F$ is linearizable at its isolated critical point $z_0$, the linearization matrix $A$ diagonal and the ratios $\alpha_j/\alpha_k$ of all eigenvalues of $A$ are positive. If an $F$-holomorphic function $\phi$ on $M$ has a Taylor series at $z_0$, then $\phi$ is holomorphic on $\text{Sat}(z_0, F)$.

**Proof.** By Theorem 5.1 $\phi$ is holomorphic on a neighborhood $U$ of $z_0$ and by Lemma 6.1 on $\text{Sat}(z_0, F) \subset \text{Sat}(U, F)$. \hfill \Box

**Corollary 6.3.** Suppose that a holomorphic vector field $F$ on a complex manifold $M$ has only critical points $z_n$ described in the statement of Theorem 6.2 and $M$ is the union of the sets $\text{Sat}(z_n, F)$. Then any $F$-holomorphic $C^\infty$-function on $M$ is holomorphic. In particular, if $M$ is compact then any such function is constant.

## 7. Two counterexamples

Consider the holomorphic vector field $F(z_1, z_2) = (\alpha z_1, \beta z_2)$ in $\mathbb{C}^2$, where $\alpha$ and $\beta$ are both different from 0 and $\alpha/\beta$ is not a positive number. First we consider the case $\beta = -t\alpha$ with $t > 0$. The integral curves of $F$ do not change if we replace the complex time $\zeta$ by $\alpha \zeta$. So we may assume that $F(z_1, z_2) = (z_1, -tz_2)$. The function $|z_1|^t|z_2|$ is
constant on integral curves of $F$ and smooth when $z_1z_2 \neq 0$. To make it smooth everywhere we use

$$\phi(z_1, z_2) = \exp \left( -\frac{1}{|z_1||z_2|} \right).$$

Since $\phi$ is constant on the integral curves it is holomorphic along $F$.

If $\alpha/\beta$ is not real then replacing the complex time $\zeta$ by $\tau \zeta$ with an appropriate $\tau$ we may assume that $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 < 0$, $\alpha_2 > 0$ and $\beta = t\pi$, $t > 0$.

Let

$$\gamma = \frac{1}{2\alpha_1} - \frac{i}{2\alpha_2}.$$

Direct calculations show that

$$\gamma \text{Re} \alpha \zeta + \frac{\tau}{t} \text{Re} \beta \zeta \equiv \zeta.$$  \hspace{1cm} (4)

The sector $S = \{ \zeta \in \mathbb{C} : \zeta = r\gamma + \tau s, r, s < 0 \}$ lies in the left half-plane and has an angle strictly less than $\pi$. Hence there is $b > 1$ such that function $\xi^b$ is defined with $\text{Re} \xi^b < 0$ on $S$. Hence the function

$$\phi(z_1, z_2) = \exp \left( \gamma \log |z_1| + \frac{\tau}{t} \log |z_2| \right)^b$$

is defined when $(z_1, z_2) \in \mathbb{D}^2$ and $z_1z_2 \neq 0$. If $z_1z_2 = 0$ we let $\phi = 0$. The function $\phi$ is not holomorphic, but we shall demonstrate that it is smooth and holomorphic along every integral curve $F$.

Clearly, $\phi \in C^\infty(\mathbb{D}^2 \setminus \{z_1z_2 = 0\})$. But it also is $C^\infty$ on $\{z_1z_2 = 0\}$. To see this we note that if $\xi = \gamma \log |z_1| + \frac{\tau}{t} \log |z_2| \in S$, then

$$\text{Re} \xi^b \leq -c \max\{(-\log |z_1|)^b, (-\log |z_2|)^b\},$$

where $c > 0$. Thus if, say, $z_1 \to 0$, then

$$|\phi(z_1, z_2)| \leq e^{-c(-\log |z_1|)^b} = |z_1|^c(-\log |z_1|)^{b-1},$$

i.e., $\phi$ decreases faster than any power of $z_1$. This shows that $\phi \in C^\infty(\mathbb{D}^2)$ and its Taylor series on $\{z_1z_2 = 0\}$ is zero.

The integral curves of $F$ admit a parametrization: $z_1(\zeta) = C_1 e^{\alpha \zeta}$ and $z_2(\zeta) = C_2 e^{\beta \zeta}$. By (4)

$$\gamma \log |z_1(\zeta)| + \frac{\tau}{t} \log |z_2(\zeta)| = \zeta + \gamma \log |C_1| + \frac{\tau}{t} \log |C_2|.$$  

Hence

$$\phi(z_1(\zeta), z_2(\zeta)) = \exp \left( \zeta + \gamma \log |C_1| + \frac{\tau}{t} \log |C_2| \right)^b$$

is a holomorphic function of $\zeta$. 

11
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