ON FRACTIONAL SMOOTHNESS OF MODULUS OF FUNCTIONS

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ABSTRACT. We consider the Nemytskii operators $u \to |u|$ and $u \to u^+$ in a bounded domain $\Omega$ with $C^2$ boundary. We give elementary proofs of the boundedness in $H^s(\Omega)$ with $0 \leq s < 3/2$.

1. INTRODUCTION

Let $\Omega$ be a nonempty open bounded set in $\mathbb{R}^d$. For $0 < \gamma < 1$ and $f \in C^1(\Omega)$, define the nonlocal $H^\gamma$ semi-norm as

$$
\|f\|_{H^\gamma(\Omega)}^2 = \int_{x, y \in \Omega, x \neq y} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2\gamma}} \, dx \, dy.
$$

(1.1)

For $f \in C^2(\Omega)$ and $0 < \gamma < 1$, we define

$$
\|f\|_{H^{1+\gamma}(\Omega)} = \|f\|_{H^1(\Omega)} + \|\partial f\|_{H^\gamma(\Omega)},
$$

(1.2)

where (and throughout this note) $\partial f = (\partial_{x_1} f, \ldots, \partial_{x_d} f)$ denotes the usual gradient. Throughout this note we shall only be concerned with real-valued functions, however with some additional work the results can be generalized to complex-valued functions. Define the Nemytskii operators

$$
T_1 u = |u|, \quad T_2 u = u^+ = \max\{u, 0\}, \quad T_3 u = u^- = \max\{-u, 0\}.
$$

(1.3)

The purpose of this note is to give an elementary proof of the following.

**Theorem 1.1** (Boundedness in $H^{\frac{d}{2}-}(\Omega)$). Let $d \geq 1$ and $0 \leq s < \frac{d}{2}$. Assume $\Omega$ is a nonempty open bounded set in $\mathbb{R}^d$ with $C^2$ boundary, i.e. locally it can be written as the graph of a $C^2$ function on $\mathbb{R}^{d-1}$. Then $T_i$, $i = 1, 2, 3$ are bounded on $H^s(\Omega)$. More precisely,

$$
\sum_{i=1}^{3} \|T_i u\|_{H^s(\Omega)} \leq \alpha_1 \|u\|_{H^s(\Omega)},
$$

(1.4)

where $\alpha_1 > 0$ depends on $(s, \Omega, d)$.

**Remark 1.1.** In Theorem 1.1 the case $0 < s < 1$ is trivial thanks to the simple inequality $|x| - |y| \leq |x - y|$ for any $x, y \in \mathbb{R}$. The case $s = 1$ corresponds to the well-known distributional calculation $\partial(|u|) = \text{sgn}(u) \partial u$ for $u \in H^1$. Thus only the case $1 < s < \frac{d}{2}$ requires some work. The obstruction $s < \frac{3}{2}$ is clear since there are jump discontinuities of the gradient along manifolds of dimension $d - 1$. In 1D one can take a smooth compactly supported function $\phi$ such that $\phi(x) \equiv x$ for $x$ near the origin. It is trivial to verify that $|\phi| \notin H^2$.

**Remark 1.2.** It follows from our proof that for $1 < s < \frac{d}{2}$, $T_i$ maps bounded sets in $H^s(\Omega)$ to pre-compact sets in $H^1(\Omega)$. This fact has important applications in the convergence of approximating solutions in some nonlinear PDE problems.

There is by now an enormous body of literature on extension, composition, regularity and stability of nonlocal operators and we shall not give a survey on the state of art in this short note. For $C^\infty$ boundary $\partial \Omega$, one can use interpolation to define the fractional spaces $H^s(\Omega)$ which can be regarded as restrictions of functions in $H^s(\mathbb{R}^n)$ (cf. [4]). In [6] Bourdaud and Meyer proved the boundedness of $T_1$ in Besov spaces $B^{s,p}_{p,q}(\mathbb{R}^d)$, $0 < s < 1 + \frac{1}{p}$, $1 \leq p \leq \infty$. By using linear spline approximation theory, Oswald [5] showed that $T_1$ is bounded in $B^{s,p}_{p,q}(\mathbb{R})$, $1 \leq p, q \leq \infty$ if and only if $0 < s < 1 + \frac{1}{p}$. In [9], Savaré showed the regularity of $T_2$ in the space
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Lemma 2.1. Let $0 < \alpha < 1$ and $0 < \delta < 1 - \alpha$. Consider

$$F(k) = \int_0^\infty \frac{1 - y^{-\delta}}{(k + |1 - y|^2)^{\frac{1 + \alpha}{2}}} dy, \quad k > 0.$$  \hfill (2.1)

Then $F(k)$ is uniformly bounded and

$$\lim_{k \to 0} F(k) = F(0) > 0.$$  \hfill (2.2)

Proof. By using Lebesgue Dominated Convergence we have $\lim_{k \to 0} F(k) = F(0)$. Note that

$$F(0) = \int_0^1 \frac{(1 - y^{-\delta})(1 - y^{\alpha + \delta - 1})}{|1 - y|^{1+\alpha}} dy > 0.$$  \hfill (2.3)

Lemma 2.2. Let $0 < \alpha < 1$. Assume $u$ is bounded on $[0, \infty)$ and $|u(x)| \lesssim x^{-2}$ for $x \geq 1$. Then

$$\frac{u(x) - u(y)^2}{|x - y|^{1+\alpha}} \leq \frac{u^2(x)}{x^{\alpha}}.$$  \hfill (2.4)

Proof. We use by now the standard super-harmonic approach (cf. [2, 3]). Observe for $w > 0$,

$$(u(x) - u(y))^2 \geq u^2(x) \frac{w(x) - w(y)}{w(x)} + u^2(y) \frac{w(y) - w(x)}{w(y)}.$$  \hfill (2.5)

For $x > 0$, take $w(x) = x^{-\delta}$ with $0 < \delta < 1 - \alpha$. By Lemma 2.1, it is not difficult to check that

$$\sup_{x > 0, \epsilon > 0} \frac{x^{\alpha}}{w(x)} \left[ \int_0^\infty \frac{w(x) - w(y)}{\epsilon^2 + |x - y|^2} \frac{1}{|x|^{1+\alpha}} dy \right] \lesssim 1.$$  \hfill (2.6)

Thus

$$\int \frac{(u(x) - u(y))^2}{(\epsilon^2 + |x - y|^2)^{\frac{1+\alpha}{2}}} dxdy \geq 2 \int u(x)^2 \frac{1}{w(x)} \int \frac{w(x) - w(y)}{(\epsilon^2 + |x - y|^2)^{\frac{1+\alpha}{2}}} dy dx.$$  \hfill (2.7)

Sending $\epsilon \to 0$ then yields the result. \hfill \Box

Lemma 2.3. Let $0 < \alpha < 1$. Assume $u$ is bounded on $[0, 1]$. Suppose $\int_0^1 u(x)dx = 0$. Then

1. $\int_0^1 \int_0^1 (u(x) - u(y))^2 dxdy = 2\|u\|_2^2$,
2. $\int_0^1 \int_0^1 \frac{(u(x) - u(y))^2}{|x - y|^{1+\alpha}} dxdy \geq \int_0^1 u(x)^2 \min\{x^n, (1-x)^n\} dx$.  

2. The One Dimensional Case

Lemma 2.1. Let $0 < \alpha < 1$ and $0 < \delta < 1 - \alpha$. Consider

$$\int_0^\infty \frac{1 - y^{-\delta}}{(k + |1 - y|^2)^{\frac{1 + \alpha}{2}}} dy, \quad k > 0.$$  \hfill (2.1)

Then $F(k)$ is uniformly bounded and

$$\lim_{k \to 0} F(k) = F(0) > 0.$$  \hfill (2.2)

Proof. By using Lebesgue Dominated Convergence we have $\lim_{k \to 0} F(k) = F(0)$. Note that

$$F(0) = \int_0^1 \frac{(1 - y^{-\delta})(1 - y^{\alpha + \delta - 1})}{|1 - y|^{1+\alpha}} dy > 0.$$  \hfill (2.3)

Lemma 2.2. Let $0 < \alpha < 1$. Assume $u$ is bounded on $[0, \infty)$ and $|u(x)| \lesssim x^{-2}$ for $x \geq 1$. Then

$$\int_0^\infty \int_0^\infty \frac{(u(x) - u(y))^2}{|x - y|^{1+\alpha}} dx dy \gtrsim \int_0^\infty \frac{u(x)^2}{x^{\alpha}} dx.$$  \hfill (2.4)

Proof. We use by now the standard super-harmonic approach (cf. [2, 3]). Observe for $w > 0$,

$$(u(x) - u(y))^2 \geq u^2(x) \frac{w(x) - w(y)}{w(x)} + u^2(y) \frac{w(y) - w(x)}{w(y)}.$$  \hfill (2.5)

For $x > 0$, take $w(x) = x^{-\delta}$ with $0 < \delta < 1 - \alpha$. By Lemma 2.1, it is not difficult to check that

$$\sup_{x > 0, \epsilon > 0} \frac{x^{\alpha}}{w(x)} \left[ \int_0^\infty \frac{w(x) - w(y)}{\epsilon^2 + |x - y|^2} \frac{1}{|x|^{1+\alpha}} dy \right] \lesssim 1.$$  \hfill (2.6)

Thus

$$\int \frac{(u(x) - u(y))^2}{(\epsilon^2 + |x - y|^2)^{\frac{1+\alpha}{2}}} dxdy \geq 2 \int u(x)^2 \frac{1}{w(x)} \int \frac{w(x) - w(y)}{(\epsilon^2 + |x - y|^2)^{\frac{1+\alpha}{2}}} dy dx.$$  \hfill (2.7)

Sending $\epsilon \to 0$ then yields the result. \hfill \Box

Lemma 2.3. Let $0 < \alpha < 1$. Assume $u$ is bounded on $[0, 1]$. Suppose $\int_0^1 u(x)dx = 0$. Then

1. $\int_0^1 \int_0^1 (u(x) - u(y))^2 dxdy = 2\|u\|_2^2$,
2. $\int_0^1 \int_0^1 \frac{(u(x) - u(y))^2}{|x - y|^{1+\alpha}} dxdy \gtrsim \int_0^1 u(x)^2 \min\{x^n, (1-x)^n\} dx$.  

More generally, there are constants $\beta_1 > 0$, $\beta_2 > 0$ depending only on $\alpha$, such that for any finite interval on $\mathbb{R}$, it holds that (below $|I|$ denote the length of the interval)

$$
\int_{x \neq y \in I} \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} dx + \beta_1 |I|^{-\alpha} \|f\|_{L^2(I)}^2 \geq \beta_2 \int_I \frac{f(x)^2}{\text{dist}(x,I^c)^\alpha} dx.
$$

If $\int_I f dx = 0$, then we have

$$
\int_{x \neq y \in I} \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} dx \geq \beta_3 \int_I \frac{f(x)^2}{\text{dist}(x,I^c)^\alpha} dx,
$$

where $\beta_3 > 0$ depends only on $\alpha$.

Remark 2.1. Remarkably, the Hardy inequality (2) does not hold if we assume $u \in C_c^\infty((0,1))$. As a counterexample, one can take $u_n \in C_c^\infty((0,1))$ such that $u_n(x) = 1$ for $\frac{1}{n} \leq x \leq 1 - \frac{1}{n}$, $u_n(x) = 0$ for $x \leq \frac{1}{n}$ and $1 - \frac{1}{n} \leq x \leq 1$, and $|u'| \leq n$. Then we have

$$
\int_0^1 \frac{u_n(x)^2}{x^\alpha (1-x)^\alpha} dx \sim 1;
$$

$$
\int_0^1 \int_0^1 \frac{(u_n(x) - u_n(y))^2}{|x-y|^{1+\alpha}} dx dy \lesssim n^{-(1-\alpha)}.
$$

One can see [3] for an extensive discussion on general fractional order Hardy inequalities and counterexamples.

Proof. The first identity is obvious. For the second inequality, one can apply Lemma 2.2 to $u(x)\chi(x)$ where $\chi$ is a smooth cut-off function supported in $[0,2/3]$. This then yields

$$
\int_0^1 \frac{u(x) - u(y)}{|x-y|^{1+\alpha}} dx + \text{const} \cdot \int_0^1 u(x)^2 dx \gtrsim \int_0^1 \frac{u(x)^2}{x^\alpha} dx.
$$

The desired inequality follows easily by using the first identity. To get the extra factor $(1-x)^{-\alpha}$ one can invoke the symmetry $x \to 1-x$. The inequality (2.8) follows from rescaling and reducing to the case $I = (0,1)$.

The following theorem is a special case of Bourdaud-Meyer [6]. We reproduce the proof here to highlight the needed changes for the finite domain case (see Theorem 2.2).

Theorem 2.1. Let $0 < s < 1/2$. We have

$$
\|u' \cdot 1_{u>0}\|_{H^s(\mathbb{R})} \lesssim \|u'\|_{H^s(\mathbb{R})};
$$

$$
\|T_2 u\|_{H^{1+s}(\mathbb{R})} \lesssim \|u\|_{H^{1+s}(\mathbb{R})}.
$$

Proof. Set $\alpha = 2s$. Write $I = \{x : u(x) > 0\}$ as a countable disjoint union of intervals $I_j$ such that each $I_j = (a_j, b_j)$ satisfies $u(a_j) = u(b_j) = 0$. Here if $a_j$ or $b_j$ are infinity the value of $u(a_j)$ or $u(b_j)$ are understood in the limit sense. By using Lemma 2.3 we have

$$
\|u' 1_{u>0}\|_{H^s(\mathbb{R})}^2 \lesssim \|u'\|_{H^s(\mathbb{R})}^2 + \int_{x \in \mathbb{R} : u(x) > 0} \frac{(u'(x))^2}{u(y) < 0} \frac{1}{|x-y|^{1+\alpha}} dy dx
$$

$$
\lesssim \|u'\|_{H^s(\mathbb{R})}^2 + \sum_j \int_{I_j} \frac{(u'(x))^2}{(\text{dist}(x,I_j^c))^{\alpha}} dx
$$

$$
\lesssim \|u'\|_{H^s(\mathbb{R})}^2 + \sum_j \int_{I_j \times I_j} \frac{(u'(x) - u'(y))^2}{|x-y|^{1+\alpha}} dx dy
$$

$$
\lesssim \|u'\|_{H^s(\mathbb{R})}^2.
$$

Corollary 2.1. Let $0 < s < 1/2$ and $d \geq 2$. Then

$$
\|T_2 u\|_{H^{1+s}(\mathbb{R}^d)} \lesssim \|u\|_{H^{1+s}(\mathbb{R}^d)}.
$$
Lemma 3.1. Let \( Q \) be a positive continuous function. Suppose \( f \) is a nonempty compact set which is properly contained in \( Q \). In yet other words, the set \( \bar{K} = \{(y_1, y_2) : (y_1, y_2) \in K\} \) is in the interior of \( Q = \{(y_1, y_2) : |y_1| < 1, |y_2| < f(y_1)\} \). Assume \( g \) vanishes on \( Q \setminus \bar{K} \). We have
\[
\int_{x \in Q, y \in Q} \frac{|g(y)|^2}{(|x_1 - y_1| + |x_2 + y_2|)^{2+2s}} dy dx \leq C_1 \|g\|_{H^s(Q)}^2,
\]
where \( C_1 > 0 \) depends only on \((s, \bar{K}, f)\).

Proof. Denote \( \tilde{g} \) as an extension of \( g \) to \( \mathbb{R}^2 \) such that \( \tilde{g} \) has compact support and \( \|\tilde{g}\|_{H^s(\mathbb{R}^2)} \lesssim \|g\|_{H^s(Q)} \). It is rather easy to construct such an extension by using reflection and smooth
truncation. Clearly

\[
\text{LHS of (3.1)} \lesssim \int_{y \in Q_+} \frac{|g(y)|^2}{y_2^2} dy \\
\lesssim \int_{|y_1| < 1} \left( \int_{0 < y_2 < 1} |g(y_1, y_2)|^2 dy_2 + \int_{0 < y_2, \tilde{y}_2 < 1} \frac{|g(y_1, y_2) - g(y_1, \tilde{y}_2)|^2}{|y_2 - \tilde{y}_2|^{1+2s}} dy_2 d\tilde{y}_2 \right) dy_1
\lesssim \|\tilde{g}\|_{H^{s}(\mathbb{R}^2)}^2 \lesssim \|\tilde{g}\|_{H^{s}(Q_+)}^2. \tag{3.2}
\]

Lemma 3.2. Let \(0 < s < 1/2\). Suppose \(K\) is a nonempty compact set which is properly contained in \(Q_+ = \{(y_1, y_2) : |y_1| < 1, 0 \leq y_2 < f(y_1)\}\) where \(f\) is a positive continuous function. Assume \(u\) vanishes on \(Q_+ \setminus K\). Define \(Q_- = \{(y_1, y_2) : |y_1| < 1, -f(y_1) < y_2 < 0\}\) and

\[
\tilde{u}(x_1, x_2) = \begin{cases} 
    u(x_1, x_2), & x \in Q_+; \\
    -3u(x_1, -x_2) + 4u(x_1, -\frac{x_2}{2}), & x \in Q_-; \\
    0, & \text{otherwise}.
\end{cases} \tag{3.3}
\]

Then

\[
\|\partial \tilde{u}\|_{H^{s}(\mathbb{R}^2)} \leq C_2\|\partial u\|_{H^{s}(Q_+)}, \tag{3.4}
\]

where \(C_2 > 0\) depends only on \((s, K, f)\).

Proof. Observe that

\[
\partial_1 \tilde{u}(x_1, x_2) = \begin{cases} 
    \partial_1 u(x_1, x_2), & x \in Q_+; \\
    -3\partial_1 u(x_1, -x_2) + 4\partial_1 u(x_1, -\frac{x_2}{2}), & x \in Q_-; \\
\end{cases} \tag{3.5}
\]

\[
\partial_2 \tilde{u}(x_1, x_2) = \begin{cases} 
    \partial_2 u(x_1, x_2), & x \in Q_+; \\
    3(\partial_2 u)(x_1, -x_2) - 2(\partial_2 u)(x_1, -\frac{x_2}{2}), & x \in Q_-.
\end{cases} \tag{3.6}
\]

By Lemma 3.1 we have

\[
\|\partial \tilde{u}\|_{H^{s}(\mathbb{R}^2)}^2 \lesssim \|\partial u\|_{H^{s}(Q_+)}^2 + \|\partial \tilde{u}\|_{H^{s}(Q_-)}^2 + \int_{x \in Q_+, y \in Q_-} \frac{|(\partial u)(x) - (\partial \tilde{u})(y)|^2}{|x - y|^{2+2s}} dxdy \\
\lesssim \|\partial u\|_{H^{s}(Q_+)}^2 + \int_{x \in Q_+, y \in Q_+} \frac{|(\partial u)(y_1, y_2) - (\partial \tilde{u})(y_1, \frac{y_2}{2})|^2}{(|x_1 - y_1| + |x_2 + y_2|)^{2+2s}} dydx \\
\lesssim \|\partial u\|_{H^{s}(Q_+)}^2. \tag{3.7}
\]

\[\square\]
Lemma 3.3. Let $0 < s < \frac{1}{2}$ and $r_0 > 0$. Suppose $\gamma : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function such that $\gamma(0) = 0$, and the region $\Omega = \{(x_1, x_2) : |x| < r_0, x_2 > \gamma(x_1)\}$ can be exactly prescribed by $\Omega = \{(x_1, x_2) : x_2 < \frac{\sqrt{r_0^2 - x_1^2} - \delta}{r_0^2 - x_1^2}\}$ for some $-r_0 < x < x_0 < r_0$. Assume $u$ is compactly supported in $F_0 = \{(x_1, x_2) : x_0 - \delta_0 < x_1 < x_0 + \delta_0, \gamma(x_1) \leq x_2 < \sqrt{r_0^2 - x_1^2} - \delta_0\}$ for some small $\delta_0 > 0$. Then there exists an extension $\tilde{u}$ of $u$ which is compactly supported in $B(0, r_0)$, such that

$$
\|\tilde{u}\|_{L^2(\mathbb{R}^2)} + \|\partial u\|_{H^s(\mathbb{R}^2)} \leq C_3 \cdot (\|u\|_{L^2(\Omega)} + \|\partial u\|_{H^s(\Omega)}),
$$

where $C_3 > 0$ depends on $(\delta_0, \gamma, s)$.

Proof. Define the usual boundary straightening map by $y = \psi(x)$ by $y_1 = x_1$, $y_2 = x_2 - \gamma(x_1)$. The inverse map is $x = \phi(y) : x_1 = y_1, x_2 = y_2 + \gamma(y_1)$. Define $v(y) = u(\phi(y))$, for $y \in W = \psi(\Omega)$. Note that $(\partial y)(y) = (\partial y)(\phi(y))(\partial \phi)(y)$. Since $\gamma \in C^2$, it is not difficult to check that

$$
\int_{y, \tilde{y} \in W} \frac{|(\partial y)(y) - (\partial y)(\tilde{y})|^2}{|y - \tilde{y}|^{2+2s}} dy d\tilde{y} \lesssim \int_{\Omega} \frac{|(\partial u)(x) - (\partial u)(\tilde{x})|^2}{|x - \tilde{x}|^{2+2s}} dxd\tilde{x} + \|\partial u\|_{L^2(\Omega)},
$$

where the second term on the RHS arises from the difference $(\partial \phi)(y) - (\partial \phi)(\tilde{y})$. Denote $\Omega_0 = \{(x_1, x_2) : x_2 < \frac{\sqrt{r_0^2 - x_1^2}}{r_0^2 - x_1^2}, W_0 = \psi(\Omega_0), and K_1 = \psi(F_0)$. Note that $K_1$ is properly contained in $W_0$. By a minor adjustment of constants, we can then apply Lemma 3.2 to obtain an extension $v$ to the whole $\mathbb{R}^2$. The map $\psi$ then provides the needed extension of $u$ which is compactly supported in $B(0, r_0)$.

\[\square\]

Theorem 3.1 (Extension of $H^{\frac{d}{2} -}$ functions). Let $d \geq 1$ and $0 < s < \frac{1}{2}$. Assume $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^2$ boundary. Select a bounded open set $V$ such that $\Omega \subset \subset V$. Then there exists a bounded linear operator

$$
E : H^{1+s}(\Omega) \to H^{1+s}(\mathbb{R}^d)
$$

such that for each $f \in H^{1+s}(\mathbb{R}^d)$, we have

1. $Ef = f$ a.e. in $\Omega$;
2. $Ef$ has support within $V$;
3. Denote $\tilde{f} = Ef$, then

$$
\|\tilde{f}\|_{H^{1+s}(\mathbb{R}^d)} \lesssim \|f\|_{H^{1+s}(\Omega)}; \quad \|\partial \tilde{f}\|_{H^{s}(\mathbb{R}^d)} \lesssim \|\partial f\|_{H^{s}(\Omega)} = \|\partial f\|_{H^{s}(\Omega)} + \|\partial f\|_{L^2(\Omega)}.
$$

Proof. With no loss we consider the two dimensional case. The argument then follows from a standard partition of unity. The extension for the interior piece is quite straightforward. The localized boundary piece follows from (after rotation and relabelling coordinate axes if necessary) Lemma 3.3.

\[\square\]
Corollary 3.1 (Boundedness of $T_2$ on a finite domain). Let $d \geq 1$ and $0 < s < \frac{1}{2}$. Assume $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^2$ boundary. Then
\[
\| \partial f I_{f>0} \|_{H^s(\Omega)} \lesssim \| \partial f \|_{H^s(\Omega)} = \| \partial f \|_{H^s(\Omega)} + \| f \|_{L^2(\Omega)}, \tag{3.13}
\]
\[
\| T_2 f \|_{H^{1+s}(\Omega)} \lesssim \| f \|_{H^{1+s}(\Omega)}. \tag{3.14}
\]

Proof. By Theorem 3.1, we extend $f$ to $\tilde{f}$ defined on the whole $\mathbb{R}^d$. The result then follows from the boundedness in the whole space case (see Corollary 2.1). \qed

Finally we remark that Theorem 1.1 follows from Corollary 3.1 since the proofs for $T_1$ and $T_3$ are similar.

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