MODULI OF FILTERED QUIVER REPRESENTATIONS

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Abstract. In this note, we give a construction of the moduli space of filtered representations of a given quiver of fixed dimension vector with the appropriate notion of stability. The construction of the moduli of filtered representation uses the moduli of representation of ladder quiver. The ladder quiver is introduced using the given quiver and an $A_n$-type quiver.

1. Introduction

In [Ki94], A. King has used the geometric invariant theory to construct the moduli of quiver representations. Once we fix the stability parameter, we get an abelian category of semistable representations. Since the category of filtered representations may not form an abelian category, so we can not apply the methods of King directly for the construction. We generalise the methods of King [Ki94] in the filtered representation category setup via the notion of slope stability studied by Andre [An09] for quasi-abelian categories.

In section 2, we recall the notion of filtered representations of a quiver $Q$ and also we introduce the ladder quiver $A_1 \times Q$ (see 2.3). We also define the admissible ideal which is used to describe the category of filtered representations of $Q$.

In section 3, we recall the notion of slop used by Rieneke [Re08] and Andre [An09]. As an application we get the notion of $S$-equivalence for filtered representations which is used to describe the closed points of the moduli space of filtered representations.

In section 4, we generalise the methods of King [Ki94] and moduli space is constructed following the approach of [AK07].

Notations:

Let $k$ be an algebraically closed field.

- $\mathcal{M}(A_1 \times Q)$ = the category of ($k$-linear) representation of $A_1 \times Q$.
- $\mathcal{M}_{\text{rel}}(A_1 \times Q)$ = the category of ($k$-linear) representation of $A_1 \times Q/I$; i.e., category of representation with an ideal of relations $I$.
- $\mathcal{M}_{\text{fil}}(A_1 \times Q)$ = the category of ($k$-linear) filtered representation of $Q$ of length at most $l$.

2. Filtered quiver representations

Let $Q$ be a (finite) quiver without oriented cycles, and let $A = kQ$ be its path algebra.

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Definition 2.1. A filtered representation of $Q$ of length at most $l$ is an increasing filtration of length $l$ in an abelian category of representation of the quiver $Q$. We will denote by $M_l$ the filtered representation where $M_k$ is a representation of $Q$ for $k = 2, \ldots, l$ s.t. $M_{k-1} \subseteq M_k$ is a subrepresentation.

Remark 2.2. We can realise the category of representation of quiver $Q$ inside the category of filtered representation by taking $M_k = 0$ for $k \leq (l - 1)$. The filtered representation can be related to $\mathbb{Z}$-filtered representation of Schneiders [S99], Definition 3.1.1, by putting $M_k = M_l$ for $k \geq (l + 1)$ and $M_k = 0$ for $k \leq 1$.

We shall denote by $A_l \times Q$ the ladder quiver obtained from the quivers $Q$ and $A_l$. Let $A_l$ be the path-algebra of $A_l \times Q$.

Definition 2.3. If $Q = (Q_0, Q_1)$ is a quiver and $A_l = (A_{l0}, A_{l1})$ is a linear quiver with $l$ vertices. Then the ladder quiver is defined as $A_l \times Q := (A_{l0} \times Q_0, A_{l0} \times Q_1 \sqcup A_{l1} \times Q_0)$.

To simplify the notation, let $\alpha_i := (i, \alpha) \in A_{l0} \times Q_1$ for $i = 1, \ldots, l$ (or $i \in A_{l0}$) and $\beta_j := (j, \beta) \in A_{l1} \times Q_0$ for $j = 1, \ldots, (l - 1)$ (or $j \in A_{l1}$) represents the arrows in the ladder quiver $A_l \times Q$. Now using this notation for arrows, we can define the admissible ideal $I$ as the ideal generated by $\beta_k \alpha_k - \alpha_{k+1} \beta_k$ for $k = 1, \ldots, (l - 1)$ (or $k \in A_{l1}$).

We will also identify the vertex set $Q_0$ embedded in the vertex set of ladder quiver as $\{l\} \times Q_0$ where $l$ is the sink of a linear quiver.

Example 2.4. The quiver $A_l \times A_3$ can be described as follows:

Example 2.5. We can get the ladder quiver for quiver with loop. If $L_1$ is a single loop quiver then the quiver $A_l \times L_1$ can be described as follows:

Let $\Lambda = \{1, 2, \ldots, l\}$ be a pre-ordered set. We denote by $\text{Fct}(\Lambda, A\text{-mod})$ the category of functors from $\Lambda$ to $A\text{-mod}$ [SS16]. We can immediately see the following equivalence of categories.

Lemma 2.6. The categories $\text{Fct}(\Lambda, A\text{-mod})$ and $A_l/I\text{-mod}$ are equivalent, where the ideal $I$ is an admissible ideal (see definition 2.3) in $A_l$.

Let $F_\Lambda(A\text{-mod})$ be the full subcategory of $\text{Fct}(\Lambda, A\text{-mod})$ consisting of filtered objects. Then, the natural inclusion functor $\iota: F_\Lambda(A\text{-mod}) \rightarrow \text{Fct}(\Lambda, A\text{-mod})$ is an embedding [SS16].
Under the equivalence of Lemma 2.6, to give an object $M \in \mathcal{F}_\Lambda(A\text{-mod})$ is equivalent to give a representation $M$ of $A_1 \times Q$ such that $M(r) = 0$ for all $r \in I$ and $M(a)$ is injective for any arrow in the copy of $A_i$ in $Q_{A_i}$. We shall denote by $\mathcal{M}_{\text{fil}}(A_1 \times Q)$ the full subcategory of $A_1/I\text{-mod}$ consisting of filtered quiver representation of $Q$ of length at most $l$.

**Proposition 2.7.** The category $\mathcal{M}_{\text{fil}}(A_1 \times Q)$ is a quasi-abelian category. In fact the category $\mathcal{M}_{\text{rel}}(A_1 \times Q)$ can be identified with the abelian envelope of $\mathcal{M}_{\text{fil}}(A_1 \times Q)$.

**Proof.** The proof follows from Prop. 3.1.17 of [S99] for the first part and Cor 3.1.29 of [S99] for the second part. □

3. Rank function and slope filtration

Let $Q$ be a finite quiver, and let $\theta \in \mathbb{Z}^{Q_0}$. For any dimension vector $d$, we define the slope

$$
\mu_{\theta,\dim}(d) := \frac{\theta(d)}{\dim d}
$$

where $\dim d := \sum d_i$.

We say that a representation $M$ of a quiver $Q$ is $\mu_{\theta,\dim}$-semistable, if for all non-zero subrepresentations $M'$ of $M$, we have

$$
\mu_{\theta,\dim}(\dim M') \leq \mu_{\theta,\dim}(\dim M)
$$

If the inequality is strict for all non-zero subrepresentations $M'$ of $M$, then we say that $M$ is $\mu_{\theta,\dim}$-stable.

**Remark 3.1.** Given any degree function $\deg: \mathbb{Z}^{Q_0} \to \mathbb{Z}$, we get the notion of slope. If we choose $\theta_i = \deg(d) - \deg(\epsilon_i) \dim d$, where $\epsilon_i$ is the dimension vector such that $(\epsilon_i)_j = 0$, if $i \neq j$ and $(\epsilon_i)_j = 1$, if $i = j$. Then $\theta$-semistability (defined in [Ki94]) is equivalent to $\mu_{\theta,\dim}$-semistability [Re08].

**Definition 3.2.** Define the rank function $\text{rk}: \mathcal{M}_{\text{fil}}(A_1 \times Q) \to \mathbb{Z}$ by

$$
\text{rk}(M) := \sum_{i \in Q_0} \dim M_i + \sum_{j \in A_{11}} \text{rk} \beta_j.
$$

Given a $\theta \in \Gamma := \mathbb{Z}^{A_{10} \times Q_0}$, we get a degree function $\deg_{\theta}: K_0(\mathcal{M}_{\text{rel}}(A_1 \times Q)) \to \mathbb{Z}$ and we define the another slope function as follows:

$$
\mu_{\theta,\text{rk}}(M) := \frac{\deg_{\theta}(M)}{\text{rk}(M)}.
$$

By [An09, Proposition 1.2.14], for any object $B \in \mathcal{M}_{\text{rel}}(A_1 \times Q)$, there exists a torsion object $B_{\text{tor}}$ (unique up to unique isomorphism) such that

$$
0 \to B_{\text{tor}} \to B \to M \to 0
$$
is an exact sequence. Moreover, \( \text{rk}(B_{\text{tor}}) = 0 \). Observe that the Grothendieck groups of \( \mathcal{M}_{\text{fil}}(A_1 \times Q) \) and \( \mathcal{M}_{\text{rel}}(A_1 \times Q) \) are isomorphic, we get an extension of the rank function.

**Theorem 3.3.** Let \( \theta \in \Gamma \) be such that \( \deg_\theta \) is positive on non-zero torsion classes. Then the full subcategory \( \mathcal{M}_{\text{fil}}(A_1 \times Q)(\mu) \) of \( \mathcal{M}_{\text{fil}}(A_1 \times Q) \) consisting of zero object and the \( \mu_{\theta, \text{rk}} \)-semistable objects of fixed slope \( \mu \) is an abelian category.

**Proof.** Note that \( \text{rk} \) is zero on torsion classes, and hence the result follows from [An09, Corollary 1.4.10]. \( \square \)

Note that the category \( \mathcal{M}_{\text{fil}}(A_1 \times Q)(\mu) \) is Artinian and Noetherian. Hence, the Jordan-Hölder theorem holds. More precisely, given an object \( M \in \mathcal{M}_{\text{fil}}(A_1 \times Q)(\mu) \), there exists a filtration

\[
0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = M
\]

such that all the quotients \( M_i/M_{i-1} \) are \( \mu_{\theta, \text{rk}} \)-stable objects in \( \mathcal{M}_{\text{fil}}(A_1 \times Q) \) having the same slope. By the Jordan-Hölder theorem, the associated graded object

\[
gr(M) := \bigoplus_{i=1}^k M_i/M_{i-1}
\]
depends only on \( M \).

**Definition 3.4.** We say that two objects \( M \) and \( N \) in \( \mathcal{M}_{\text{fil}}(A_1 \times Q)(\mu) \) are \( S \)-equivalent if the associated graded objects \( gr(M) \) and \( gr(N) \) are isomorphic.

Let \( M \) and \( N \) be two \( S \)-equivalent objects in \( \mathcal{M}_{\text{fil}}(A_1 \times Q)(\mu) \). It should be mentioned here that if we view the objects \( M \) and \( N \) in \( \mathcal{M}_{\text{rel}}(A_1 \times Q) \), then they remain \( S \)-equivalent in \( \mathcal{M}_{\text{rel}}(A_1 \times Q) \) with respect to the the notion of \( \mu_{\theta, \text{dim}} \)-stability. It is also possible that \( M \) and \( N \) are not \( S \)-equivalent in \( \mathcal{M}_{\text{fil}}(A_1 \times Q)(\mu) \), but they are \( S \)-equivalent in \( \mathcal{M}_{\text{rel}}(A_1 \times Q) \). This is also reflected in the GIT picture (see Section 4).

### 4. GIT and Moduli Construction

#### 4.1. GIT for quiver representation

Let \( \chi_\theta : G \to k^* \) be the character defined by

\[
\chi_\theta((g_v)) := \prod_{v \in A_1 \times Q_0} \det(g_v)^{\theta_v}
\]

As a special case of [Ki94], we have the following:

1. A point \( x \in \mathcal{R} \) is \( \chi_\theta \)-semistable iff the corresponding object \( M_x \in \mathcal{M}(A_1 \times Q) \) is \( \mu_{\theta, \text{dim}} \)-semistable.
2. Two points \( x, y \in \mathcal{R} \) are GIT-equivalent (w.r.t the \( \chi_\theta \)) iff the corresponding objects \( M_x \) and \( M_y \) are \( S \)-equivalent in \( \mathcal{M}(A_1 \times Q) \).
3. A point \( x \in \mathcal{R}_{\text{rel}} \) is \( \chi_\theta \)-semistable iff the corresponding object \( M_x \in \mathcal{M}_{\text{rel}}(A_1 \times Q) \) is \( \mu_{\theta, \text{dim}} \)-semistable.
4. Two points \( x, y \in \mathcal{R}_{\text{rel}} \) are GIT-equivalent (w.r.t the \( \chi_\theta \)) iff the corresponding objects \( M_x \) and \( M_y \) are \( S \)-equivalent in \( \mathcal{M}_{\text{rel}}(A_1 \times Q) \).
For \( x \in R_{\text{fil}} \), there is surjection from the set
\[
\{1-PS \lambda: k^* \to G \text{ such that } \lim_{t \to 0} \lambda(t) \cdot x \text{ exist in } R_{\text{fil}}\}
\]
to
\[
\{\text{admissible filtration of } M_x \text{ in } M_{\text{fil}}(A_1 \times Q)\}.
\]

For, let \( \lambda: k^* \to G \) be a one-parameter subgroup of \( G \) such that the \( \lim_{t \to 0} \lambda(t) \cdot x = y \) exist in \( R_{\text{fil}} \). Following [Ki94], we have a weight space decomposition
\[
M_x(v) = \bigoplus_{n \in \mathbb{Z}} M_x(v)^n
\]
for each vertex \( v \in A_1 \times Q_0 \), where \( \lambda(t) \) acts on the weight space \( M_x(v)^n \) as multiplication by \( t^n \). For each arrow \( a \), we have
\[
M_x(a)^{mn}: M_x(s(a))^n \to M_X(t(a))^m.
\]
Moreover, we have
\[
\lambda(t) \cdot x = \bigoplus_{m \geq n} t^{m-n} M_x(a)^{mn}.
\]
Since the limit \( \lim_{t \to 0} \lambda(t) \cdot x \) exist, we have \( M_x(a)^{mn} = 0 \) for all \( m < n \). Let
\[
M_x(v)^{\geq n} := \bigoplus_{m \geq n} M_x(v)^m.
\]
Then, \( M(a) \) gives a map \( M_x(s(a))^n \to M_x(t(a))^n \) for all \( n \). These subspaces determine subrepresentations \( M_x^n \) of \( M_x \) for all \( n \). Since \( y \in R_{\text{fil}} \), it follows that the corresponding filtration
\[
\cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots
\]
where \( M_n = M_x \) for \( n \ll 0 \) and \( M_n = 0 \) for \( n \gg 0 \), is admissible filtration of \( M_x \) in \( M_{\text{fil}}(A_1 \times Q) \). We also have
\[
M_y \cong \bigoplus_{n \in \mathbb{Z}} M_n/M_{n+1}.
\]

Conversely, suppose we have an admissible filtration
\[
\cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots
\]
of \( M_x \). This will determine a one-parameter subgroup \( \lambda \) such that the \( \lim_{t \to 0} \lambda(t) \cdot x = y \) exist and \( M_y \cong \bigoplus_{n \in \mathbb{Z}} M_n/M_{n+1} \).

4.2. Moduli functors. Let us fix \( \theta \in \mathbb{Z}^{A_1 \times Q_0} \) and the dimension vector \( d \) for the quiver \( A_1 \times Q \). Consider the functor
\[
\mathfrak{M}_{\text{ss}}: \text{Sch}^{\text{op}} \to \text{Set}
\]
defined by assigning to each \( k \)-scheme \( S \) the set of isomorphism classes of flat families over \( S \) of semistable representations of \( A_1 \times Q \) having dimension vector \( d \).

**Theorem 4.1.** [Ki94] There exist a projective variety \( M_{\text{ss}} \) over \( k \) which co-represent the moduli functor \( \mathfrak{M}_{\text{ss}} \). In particular, the closed points of \( M \) correspond to the \( S \)-equivalence classes of representations of \( A_1 \times Q \).
Consider the functor
\[ \mathcal{M}_{\text{rel}}^{\text{ss}} : \text{Sch}^{\text{op}} \to \text{Set} \]
defined by assigning to each \( k \)-scheme \( S \) the set of isomorphism classes of flat families over \( S \) of semistable representations of \( \mathbb{A}_1 \times Q_{\text{rel}} \) having dimension vector \( d \).

**Theorem 4.2.** [Ki94] There exist a projective variety \( M_{\text{rel}}^{\text{ss}} \) over \( k \) which co-represent the moduli functor \( \mathcal{M}_{\text{rel}}^{\text{ss}} \). In particular, the closed points of \( M \) correspond to the \( S \)-equivalence classes of representations of \( \mathbb{A}_1 \times Q_{\text{rel}} \).

Consider the functor
\[ \mathcal{M}_{\text{fil}}^{\text{ss}} : \text{Sch}^{\text{op}} \to \text{Set} \]
defined by assigning to each \( k \)-scheme \( S \) the set of isomorphism classes of flat families over \( S \) of semistable filtered representations of \( \mathbb{A}_1 \times Q_{\text{rel}} \) having dimension vector \( d \).

### 4.3. Representation spaces.

Let
\[ \mathcal{R} := \bigoplus_{a \in Q_0^{\mathbb{A}_1} \cup Q_0^{A_1}} \text{Hom}_k(k^{d_0(a)}, k^{d_1(a)}) \]
be the space of representations of \( \mathbb{A}_1 \times Q \) having dimension vector \( d \). Let \( \mathcal{R}_{\text{rel}} \) be the closed subset of \( \mathcal{R} \) consisting of \( (\alpha_a) \) satisfying the relation in \( I \). Let \( \mathcal{R}_{\text{fil}} \) be an open subset of \( \mathcal{R}_{\text{rel}} \) consisting of \( (\alpha_a) \in \mathcal{R}_{\text{rel}} \) such that for any \( a \in Q_0^{A_1} \), we have \( \alpha_a \) injective.

There is a natural action of the group \( G(\mathbb{A}_1 \times Q) := \prod_{v \in \mathbb{A}_1 \times Q_0} \text{GL}(k^{d_v}) \) on \( \mathcal{R} \) such that the isomorphism classes in \( M(\mathbb{A}_1 \times Q) \) corresponds to the orbits in \( \mathcal{R} \) with respect to this action. Moreover, \( \mathcal{R}_{\text{rel}} \) and \( \mathcal{R}_{\text{fil}} \) are invariant under this action. Note that \( \mathcal{R}_{\text{rel}} \) is a closed subscheme of \( \mathcal{R} \), while \( \mathcal{R}_{\text{fil}} \) is an open subscheme of \( \mathcal{R} \). Let \( \mathcal{R}^{\text{ss}} \) (resp. \( \mathcal{R}_{\text{rel}}^{\text{ss}}, \mathcal{R}_{\text{fil}}^{\text{ss}} \)) be the open subset of \( \mathcal{R} \) (resp. \( \mathcal{R}_{\text{rel}}, \mathcal{R}_{\text{fil}} \)) which is \( \mu_{\text{dim}} \)-semistable locus in \( \mathcal{R} \) (resp. \( \mathcal{R}_{\text{rel}}, \mathcal{R}_{\text{fil}} \)).

Let \( G := G(\mathbb{A}_1 \times Q)/\Delta \), where \( \Delta := \{ (t1_k)_{v \in \mathbb{A}_1 \times Q_0} \mid t \in k^* \} \).

### 4.4. Local isomorphisms.

By [Ki94], there is a natural functor \( h : \mathcal{R}^{\text{ss}}_{\text{rel}} \to \mathcal{M}^{\text{ss}} \) (defined by \( f : S \to \mathcal{R}^{\text{ss}} \to [f^*\mathbb{M}] \), where \( \mathbb{M} \) is the tautological family on \( \mathcal{R}^{\text{ss}} \), which induces a local isomorphism \( h : \mathcal{R}^{\text{ss}}_{\text{rel}}/G \to \mathcal{M}^{\text{ss}} \). This reduces the problem into the existence of good quotient of \( \mathcal{R}^{\text{ss}} \) by \( G \). It is proved in [Ki94] that the GIT quotient \( \pi : \mathcal{R}^{\text{ss}} \to M^{\text{ss}} \) exist and it is a good quotient (cf. 4.1).

Similarly, there is a local isomorphism \( h_{\text{rel}} : \mathcal{R}^{\text{ss}}_{\text{rel}}/G \to \mathcal{M}^{\text{ss}}_{\text{rel}} \) and a good quotient \( \pi_{\text{rel}} : \mathcal{R}^{\text{ss}}_{\text{rel}} \to M^{\text{ss}}_{\text{rel}} \).

The proof of following proposition is inspired from theorem 4.5 of [AK07].

**Proposition 4.3.** There is a local isomorphism \( h_{\text{fil}} : \mathcal{R}^{\text{ss}}_{\text{fil}}/G \to \mathcal{M}^{\text{ss}}_{\text{fil}} \).

**Proof.** Let \( f : S \to \mathcal{R}_{\text{fil}} \) be a morphism of \( k \)-schemes, and let \( \mathbb{M}_{\text{fil}} \) be the restriction of the tautological family to \( \mathcal{R}_{\text{fil}} \). We define
\[ h_{\text{fil}} : \mathcal{R}^{\text{ss}}_{\text{fil}} \to \mathcal{M}^{\text{ss}}_{\text{fil}} \quad \text{by} \ (f : S \to \mathcal{R}^{\text{ss}}_{\text{fil}}) \mapsto [f^*\mathbb{M}_{\text{fil}}] \]
We have the following commutative diagram

\[
\begin{array}{ccc}
R_{\text{fil}}^{ss} & \xrightarrow{i} & R_{\text{rel}}^{ss} \\
\downarrow h_{\text{fil}} & & \downarrow h_{\text{rel}} \\
M_{\text{fil}}^{ss} & \xrightarrow{j} & M_{\text{rel}}^{ss} \\
\end{array}
\]

For any \( k \)-scheme \( S \), the map \( R_{\text{fil}}^{ss}(S) \rightarrow M_{\text{fil}}^{ss}(S) \times_{M_{\text{rel}}^{ss}(S)} R_{\text{rel}}^{ss}(S) \) defined by

\[
f: S \rightarrow R_{\text{fil}}^{ss} \mapsto (h_{\text{fil}}^S(f), \iota \circ f)
\]

is a bijection. Hence, the square in the above diagram is cartesian. This will induces the following cartesian diagram

\[
\begin{array}{ccc}
R_{\text{fil}}^{ss}/G & \xrightarrow{\bar{h}_{\text{fil}}} & R_{\text{rel}}^{ss}/G \\
\downarrow h_{\text{fil}} & & \downarrow h_{\text{rel}} \\
M_{\text{fil}}^{ss} & \xrightarrow{j} & M_{\text{rel}}^{ss} \\
\end{array}
\]

This proves that \( \bar{h}_{\text{fil}}: R_{\text{fil}}^{ss}/G \rightarrow M_{\text{fil}}^{ss} \) is a local isomorphism. \( \square \)

Let \( \pi_{\text{fil}}: R_{\text{fil}}^{ss} \rightarrow M_{\text{fil}}^{ss} := R_{\text{fil}}^{ss} /_{\chi_\theta} G \) be the GIT quotient. Then we have the following commutative diagram

\[
\begin{array}{ccc}
R_{\text{fil}}^{ss} & \xrightarrow{i} & R_{\text{rel}}^{ss} \\
\downarrow \pi_{\text{fil}} & & \downarrow \pi_{\text{rel}} \\
M_{\text{fil}}^{ss} & \xrightarrow{j} & M_{\text{rel}}^{ss} \\
\end{array}
\]

where \( i \) is open embedding, \( j \) and \( \psi \) are closed embedding, but \( \phi \) may not be even injective map.

Using Hilbert-Mumford criterion, we have the following:

**Proposition 4.4.** A point \( x \in R_{\text{fil}}^{ss} \) is \( \chi_\theta \)-semistable (resp. \( \chi_\theta \)-stable) if and only if the corresponding object \( M_x \in M_{\text{fil}}(A_1 \times Q) \) is \( \mu_{\theta, \text{rk}} \)-semistable (resp. \( \mu_{\theta, \text{rk}} \)-stable).

Using GIT, we have the following:

**Proposition 4.5.** Two points \( x, y \in R_{\text{fil}}^{ss} \) are GIT equivalent, i.e.,

\[
\mathcal{G}x \cap \mathcal{G}y \cap R_{\text{fil}}^{ss} \neq \emptyset
\]

if and only the corresponding objects \( M_x \) and \( M_y \) are \( S \)-equivalent in \( M_{\text{fil}}(A_1 \times Q) \).

**Proof.** By [Ki94, Proposition 2.6], if \( x \) and \( y \) are GIT equivalent, then there exist one-parameter subgroups \( \lambda_1 \) and \( \lambda_2 \) such that the integral pairing \( < \chi_\theta, \lambda_1 > <= < \chi_\theta, \lambda_2 > = 0 \) and \( \lim_{t \rightarrow 0} \lambda_1(t) \cdot x \) and \( \lim_{t \rightarrow 0} \lambda_1(t) \cdot y \) are in the same closed \( G \)-orbit in \( R_{\text{fil}}^{ss} \). Since \( \lim_{t \rightarrow 0} \lambda_1(t) \cdot x \) corresponds to the associated graded \( \text{gr}(M_x) \) in \( M_{\text{fil}}(A_1 \times Q) \), it follows that \( M_x \) and \( M_y \) are \( S \)-equivalent. \( \square \)

Using above propositions 4.3, 4.4 and 4.5 we get following result.
Theorem 4.6. The moduli space of filtered representations of fixed dimension vector $M_{\text{fil}}^{\text{ss}}(d)$ exists. The closed points of the moduli space $M_{\text{fil}}^{\text{ss}}(d)$ corresponds to the $S$-equivalence classes of objects of the category $\mathcal{M}_{\text{fil}}(A_1 \times Q)$. Moreover there is a canonical morphism to the moduli space $M_{\text{rel}}^{\text{ss}}(d)$.

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