INDECOMPOSABLE REPRESENTATIONS
FOR EXTENDED DYNKIN QUIVERS

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Abstract. We describe a method for an explicit determination of indecomposable preprojective and preinjective representations for extended Dynkin quivers $\Gamma$ over an arbitrary field $K$ by vector spaces and matrices. This method uses tilting theory and the explicit knowledge of indecomposable modules over the corresponding canonical algebra of domestic type. Further, if $K$ is algebraically closed we obtain all indecomposable representations for $\Gamma$. For the case that $\Gamma$ is of type $D_n$, $n \geq 4$, with a fixed orientation, we determine all indecomposable preprojective representations. Moreover, in the case $E_6$ we present the most complicated indecomposable preprojective representations of rank 3.

1. Introduction

Let $K$ be a field, $\Gamma$ a quiver and $A = KT/I$ a finite-dimensional algebra of quiver type. One of the problems in representation theory is to give normal forms for the indecomposable finite-dimensional left $A$-modules. Such a module is given by choosing a finite-dimensional vector space for each vertex of and a linear map for each arrow of the quiver such that the relations of the ideal $I$ are satisfied.

The problem to determine all indecomposable modules for an algebra explicitly by vector spaces and matrices is in general difficult. This problem is solved only in very few cases. In particular, Gabriel computed in 1972 the indecomposable representations for Dynkin quivers $[3]$.

Concerning extended Dynkin quivers only partial results are known. Already in 1890 Kronecker $[10]$ classified pairs of $n \times m$-matrices up to simultaneous equivalence, solving a problem raised by Weierstraß. In modern terminology this means to describe the finite dimensional modules over the Kronecker algebra. For the case of $D_4$ with subspace orientation, the so called 4-subspace problem, indecomposable objects were described by Nazarova $[13]$ and Gelfand and Ponomarev $[9]$.

General information about the structure of the module category of a path algebra of an extended Dynkin quiver were obtained by Donovan and Freislich $[2]$, and Nazarova $[14]$. For the characterization of the regular modules and for historical remarks we refer to $[4$, chapter 11$]$.

We recall also that Ringel $[16]$ has shown that for every finite dimensional path algebra $A = KT$ for a quiver $\Gamma$ each exceptional module can be exhibited by matrices containing as coefficients only 0 and 1. For a path algebra over an extended Dynkin quiver each indecomposable preprojective (respectively preinjective) module is exceptional, however explicit descriptions for those modules were not given in this case.

In this paper we discuss a method for the description of indecomposable representations of extended Dynkin quivers using our explicit description of indecomposable modules over domestic canonical algebras given in $[11]$ and $[9]$. We apply tilting
theory which was developed in \[1\] and \[7\]. We exploit the fact that for each path algebra \(A = K\Gamma\) of an extended Dynkin quiver \(\Gamma\) there is a canonical algebra of domestic type \(\Lambda\) and a tilting module \(T\) over \(\Lambda\) such that \(\text{End}(T) \simeq A^{op}\) \[15\] (see also \[8\] and \[12, Proposition 6.5\]). The theorem of Brenner and Butler \[1\] ensures that applying the functor \(\text{Hom}_\Lambda(T, -)\) to an indecomposable preprojective left \(\Lambda\)-module \(M\) satisfying \(\text{Ext}_\Lambda^1(T, M) = 0\) we obtain an indecomposable right \(A^{op}\)-module, thus an indecomposable representations of \(\Gamma\). Moreover, in this way we obtain all preprojective indecomposable representations of \(\Gamma\). In this paper we concentrate on the description of the preprojectives. However we remark that the regular modules can be treated in the same way, using \[11, Chapter 4\]. Finally, the indecomposable preinjective representations for \(\Gamma\) can be obtained by duality, i.e. by choosing the opposite orientation of the quiver.

2. Tilting from domestic canonical algebras to path algebras of extended Dynkin quivers

Canonical algebras were introduced by Ringel in 1984 \[15\] and play an important role in representation theory. A domestic canonical algebra of quiver type \(\Lambda\) is isomorphic to the path algebra of the quiver

\[
\begin{array}{ccc}
1 & \alpha_2 & \cdots & \alpha_{p-1} & (p-1) \\
\alpha_1 & & & & \\
0 & \beta_1 & \beta_2 & \cdots & \beta_{q-1} & \beta_q & \infty \\
\gamma_1 & & & & & & \\
1' & \gamma_2 & \cdots & \gamma_{s-1} & (s-1)' \\
\end{array}
\]

modulo the relation \(\gamma_s \cdots \gamma_1 = \alpha_p \cdots \alpha_1 + \beta_q \cdots \beta_1\), where \(p, q, s\) is the length of the upper (middle, lower, respectively) arm, and where moreover the triple \((p, q, s)\) is given by \((p, q, 1)\) (where \(p, q \geq 1\)), \((p, 2, 2)\) (where \(p \geq 2\)), \((3, 3, 2)\), \((4, 3, 2)\) or \((5, 3, 2)\).

Therefore a finite-dimensional left \(\Lambda\)-module \(M\) consists of finite-dimensional vector spaces \(M(i)\) for each point \(i\) of the quiver, and a linear map \(M(\alpha)\) for each arrow \(\alpha = \alpha_i, \beta_j\) and \(\gamma_k\), satisfying the relation

\[
M(\gamma_s) \circ \cdots \circ M(\gamma_1) = M(\alpha_p) \circ \cdots \circ M(\alpha_1) + M(\beta_q) \circ \cdots \circ M(\beta_1).
\]

The number \(\text{rk}(M) = \dim M(\infty) - \dim M(0)\) is called the rank of \(M\). Then an indecomposable module of positive rank (negative rank, rank zero, respectively) is preprojective (preinjective, regular, respectively). The global structure of the module category looks as follows: There is precisely one preprojective component and precisely one preinjective component and the indecomposable regular modules form tubes \[15\].

Indecomposable preprojective left modules over \(\Lambda\) were described by explicit matrices in \[11\] in case that the characteristic of \(K\) is different from 2 and in \[9\] for an arbitrary field, the last is relevant only for modules of rank 6 in the domestic situation \((5,3,2)\). The indecomposable \(\Lambda\)-modules appear in series and are constructed using a general principle by applying the so called method of enlargement of matrices and adding identities. We recall this general principle and provide for this the following notations.
Let $n$ and $i$ be natural numbers. Let $I_n$ be the $n \times n$-identity matrix. Define

\[ X_{n+i}^n = \begin{bmatrix} I_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad Y_{n+i}^n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ I_n \end{bmatrix} \in M_{n+i,n}(K), \]

both having $i$ zero rows of length $n$. If $Z'$ is some matrix, then we call the matrix $Z = \begin{bmatrix} Z' \\ 1 & \ddots & 1 \\ 0 & \ddots & 1 \end{bmatrix}$ with entries 1 on two diagonals each of length $m \geq 0$ the $m$-th enlargement of $Z'$.

A typical example of a series of preprojective indecomposable modules of rank 2 over a canonical algebra of type $(p, 2, 2)$ is the following. We fix $i$ and $j$ with $1 \leq i < j \leq p$ and consider the module

\[ M_{m}^{(i,j)} = \begin{bmatrix} K^m & \cdots & K^{m+i} & \cdots & K^{m+1} & \cdots & K^{m+i+1} & \cdots & K^{2m+2} & \cdots & K^{m+2} \\ & & & & & & & & & & \\ & & & & & & & & & & \\ K^m & & & & & & & & & & \\ & Y_{m+1}^m & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ K^{m+1} & & & & & & & & & & \\ & & Y_{m+1}^{m+1} & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ K^{m+2} & & & & & & & & & & \end{bmatrix} \]

where $Z_{m+2}^{m+1}$ is the $m$-th enlargement of the $2 \times 1$ matrix $Z' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Here the matrices $X_{m+1}^m$ and $X_{m+2}^{m+1}$ are associated to the arrows $\alpha_i : (i-1) \rightarrow i$ and $\alpha_j : (j-1) \rightarrow j$ respectively. It follows from [11] that each preprojective indecomposable rank 2 module for a canonical algebra of type $(p, 2, 2)$ is isomorphic to a module of this form.

The indecomposable modules for other domestic canonical algebras are defined in a similar way by enlargement of certain “small” matrices which can be found in [11, Theorem 2] and [9], respectively.

Now, let $A = K\Gamma$ be the path algebra of an extended Dynkin quiver. As we have already mentioned in the introduction there is a canonical algebra of domestic type $\Lambda$ and a left tilting module $T$ such that $\text{End}(T) \simeq A^{\text{op}}$. For general information about tilting theory we refer to [1] and [7].

We consider the functor $F = \text{Hom}(T, -) : \Lambda \text{-mod} \rightarrow \text{mod} - A^{\text{op}}$, where $\Lambda \text{-mod}$ (respectively $\text{mod} - A^{\text{op}}$) is the category of finite-dimensional left $\Lambda$ modules. (respectively finite-dimensional right $A^{\text{op}}$-modules). Obviously the last can be identified with $A \text{-mod}$, thus with the category of representations of $\Gamma$.

We recall that a homomorphism $f : M \rightarrow M'$ of $\Lambda$-modules is given by a set of linear maps $f_i : M(i) \rightarrow M'(i)$ such that for each arrow $\phi : i \rightarrow j$ of the quiver for $\Lambda$ we have $f_j M(\phi) = M'(\phi) f_i$. From [3, Lemma 4.2] we know that the linear maps $M(\phi)$ for an indecomposable preprojective module $M$ are monomorphisms. As a consequence a homomorphism between indecomposable preprojective modules $M$ and $M'$ is uniquely determined by the map $f(\infty)$ and we will always identify such a homomorphism with the matrix for $f(\infty)$.
We write the tilting module $T$ as a direct sum of pairwise non-isomorphic indecomposables $T_1, \ldots, T_l$ and choose generators for all non-zero vector spaces $\text{Hom}(T_i, T_j)$.

If such a homomorphism space is non-zero, it is 1-dimensional and in this case a generator $f$ can be identified with a single matrix $S^i_j$ describing the linear map $f(\infty)$.

Now, if $M$ is a preprojective left $\Lambda$-module the corresponding representation $N = F(M)$ of the extended Dynkin quiver $\Gamma$ can be computed as follows. The vector spaces for the vertices $i$ of $\Gamma$ are given by $N(i) = \text{Hom}(T_i, M)$. Furthermore, for each $i$ we choose a basis of $N(i)$. In case $\text{Hom}(T_i, T_j) \neq 0$ there is an arrow $\phi : i \rightarrow j$ in $\Gamma^{\text{op}}$. We again have that $\text{Hom}(T_i, M)$ (respectively $\text{Hom}(T_j, M)$) can be identified with a vector space of matrices describing the linear map in $\infty$. Then the linear map $N(\phi) : N(j) = \text{Hom}(T_j, M) \rightarrow \text{Hom}(T_i, M) = N(i)$ is the multiplication with the matrix $S^j_i$ from the right hand side. Consequently, by our choice of bases in the $N(i)$ we obtain the matrices for $N$. The rank of a representation of $\Gamma$ is by definition the rank of the corresponding $\Lambda$-module. Note that in [11] (respectively [9]) the $\Lambda$-modules $M$ are constructed as members of a series of indecomposable $\Lambda$-modules $M_m$ ($m \geq 0$). The procedure just described will be applied simultaneously to the whole series.

The theorem of Brenner and Butler [1] implies that if $M$ satisfies the condition $\text{Ext}^1(T, M) = 0$ and $M$ is indecomposable then $F(M)$ is also indecomposable. We will apply the functor $F$ to indecomposable preprojective $\Lambda$-modules satisfying the condition above. In this way we obtain the matrices of all indecomposable preprojective representations of the extended Dynkin diagram $\Gamma$.

Two kinds of data are important for our construction: the explicit knowledge of the tilting module and the explicit knowledge of the indecomposable preprojective $\Lambda$-modules, both given by vector spaces and matrices.

We note that the same method can be applied also to determine explicitly the indecomposable modules over a tame concealed algebra.

3. THE CASE $\tilde{D}_n$

3.1. For the structure of the indecomposable representations of extended Dynkin quivers of type $\tilde{A}_n$ we refer to [4, chapter 11]. In this chapter we study the representations of the extended Dynkin quiver of type $\tilde{D}_n$ where we fix the following orientation.

$$
\begin{array}{c}
\text{1} \\
\text{3} & & n - 1 \\
\text{2} & & n + 1
\end{array}
$$


The corresponding canonical algebra $\Lambda$ is of type $(n - 2, 2, 2)$. Looking at the preprojective component of $\Lambda$ we see that the following $T = \bigoplus_{k=1}^{n+1} T_k$ is a tilting module in $\Lambda - \text{mod}$ with $\text{End}(T) \cong K\Gamma^{\text{op}}$. The indecomposable direct summands $T_k$ of $T$ are given as follows:
For $3 \leq k \leq n-1$ we have
\[
T_k : \quad \begin{array}{c}
\cdots \\
K \rightarrow \cdots \\
\downarrow v_2^k \\
K^2
\end{array}
\]
with $n-k-1$ entries 0 and $k-2$ entries $K$ in the first arm (in particular for $T_{n-1}$ there is no 0 in the first arm). Moreover,
\[
T_1 : \quad \begin{array}{c}
\cdots \\
K \rightarrow K \\
\downarrow \quad \quad \\
K \quad \downarrow
\end{array}
\quad T_2 : \quad \begin{array}{c}
\cdots \\
K \rightarrow K \\
\downarrow \quad \quad \\
K \quad \downarrow
\end{array}
\]
and
\[
T_n : \quad \begin{array}{c}
\cdots \\
K \rightarrow K \\
\downarrow \quad \quad \\
K \quad \downarrow
\end{array}
\quad T_{n+1} : \quad \begin{array}{c}
\cdots \\
K \rightarrow K \\
\downarrow \quad \quad \\
K \quad \downarrow
\end{array}
\]
Here $Z_2^1$ denotes the matrix \[
\begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]
The following picture indicates the quiver for the endomorphism ring of $T$ and gives generators $S_i^j$ for the non-zero homomorphism spaces $\text{Hom}_\Lambda(T_i, T_j)$ which are always represented by the matrices for the linear maps $f_\infty : T_i(\infty) \rightarrow T_j(\infty)$.

3.2. We start with the description of the indecomposable preprojective modules of rank 2. There are precisely \((n-2)\) series of indecomposable $\Lambda$-modules of rank 2: $M_m^{(i,j)}$, $1 \leq i < j \leq n - 2$. They have been described in the previous chapter.

We now fix $i$ and $j$ with $1 \leq i < j \leq n - 2$ and compute simultaneously the representations $N_m^{(i,j)} = F(M_m^{(i,j)})$, $m \in \mathbb{N}$. We shortly write $M_m^{(i,j)} = M$.

\begin{itemize}
  \item [(n)] We assume that $i \neq 1$ and $j \neq n - 2$.
  \item [(a1)] Computation of $\text{Hom}(T_1, M)$:
  \end{itemize}

A homomorphism $f : T_1 \rightarrow M$ is given by matrices $Q = (q_l) \in M_{m+1,1}(K)$ and $S = (s_l) \in M_{m+2,1}(K)$ such that $S = Y_{m+1}^m Q$. We have already mentioned that for homomorphisms between indecomposable preprojective modules a homomorphism is uniquely determined by the matrix for the linear map of the point $\infty$, that is $S$. The matrix equation yields that $s_1 = 0$ and $s_l = q_{l-1}$ for $l = 2, \ldots, m + 2$. 

Therefore \( \dim_K \text{Hom}(T_1, M) = m + 1 \) and a basis is given by \((m + 2) \times 1\)-matrices \( w_2^{(1)}, w_3^{(1)} \ldots, w_{m+2}^{(1)} \), where \( w_i^{(1)} \) is the matrix with entries \( s_i = 1 \) and \( s_j = 0 \) for \( j \neq i \).

(a2) Computation of \( \text{Hom}(T_2, M) \):

A homomorphism \( f : T_1 \to M \) is given by matrices \( R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_j) \in M_{m+2,1}(K) \) such that \( S = Z_{m+2}^{m+1}R \).

Now, the matrix equation and the shape of \( Z_{m+2}^{m+1} \) imply that

\[
S = \begin{bmatrix}
    r_1 + r_2 \\
    r_1 + r_3 \\
    r_2 + r_4 \\
    r_3 + r_5 \\
    \vdots 
\end{bmatrix}
\]

(We formally define \( r_i = 0 \) for \( i > m + 1 \) and \( s_i = 0 \) for \( i > m + 2 \).) Observe that then

\[
r_1 + r_2 = (r_1 + r_3) + (r_2 + r_4) - (r_3 + r_5) - (r_2 + r_6) + (r_5 + r_7) + (r_6 + r_8) - \ldots
\]

and consequently

\[
s_1 = s_2 + s_3 - s_4 - s_5 + s_6 + s_7 - \ldots
\]

Therefore \( \dim_K \text{Hom}(T_2, M) = m + 1 \) and a basis is given by \((m + 2) \times 1\)-matrices \( w_2^{(2)}, w_3^{(2)} \ldots, w_{m+2}^{(2)} \), where \( w_i^{(2)} \) is the matrix with entries \( s_1 = 1, s_j = 0 \) for \( j \neq 1, i \) and \( s_1 = 1 \) or \(-1\), which is dependent on the rest of \( i \) modulo 4.

(a3) Computation of \( \text{Hom}(T_k, M) \) for \( k = 3, \ldots, n-j \):

A homomorphism \( f : T_k \to M \) is given by matrices \( P = (p_i) \in M_{m+2,1}(K), Q = (q_i) \in M_{m+1,1}(K), R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+2,2}(K) \) such that \( SX_k^1 = P, SY_k^1 = Y_{m+2}^{m+1}Q \) and \( SZ_k^1 = Z_{m+2}^{m+1}R \).

The first equation yields no condition for the coefficients of \( S \) whereas from the second equation we conclude that \( s_{1,2} = 0 \). The third condition shows as in the case (a2) that the coefficient \( s_{1,1} \) is a linear combination of the remaining coefficients.

Therefore \( \dim_K \text{Hom}(T_k, M) = m + 2 \) and a basis is given by \((m + 2) \times 2\)-matrices \( w_{2,1}^{(k)}, w_{3,1}^{(k)}, \ldots, w_{m+1,1}^{(k)}, w_{2,2}^{(k)}, w_{3,2}^{(k)}, \ldots, w_{m+2,2}^{(k)} \), where \( w_{i,j}^{(k)} \) is the matrix with entries \( s_{i,j} = 1, s_{u,v} = 0 \) for \((u, v) \neq (i, j)\) and \((1, 1)\) whereas \( s_{1,1} = 1 \) or \(-1\), which is dependent on the the rest of \( i \) modulo 4.

(a4) Computation of \( \text{Hom}(T_k, M) \) for \( k = n-j+1, \ldots, n-i \):

In this case a homomorphism \( f : T_{k+1} \to M \) is given by matrices \( P' = (p_i') \in M_{m+1,1}(K), P = (p_i) \in M_{m+2,1}(K), Q = (q_i) \in M_{m+1,1}(K), R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+2,2}(K) \) such that \( P = X_{m+2}^{m+1}P', SX_{k+1}^1 = P, SY_k^1 = Y_{m+2}^{m+1}Q \) and \( SZ_k^1 = Z_{m+2}^{m+1}R \).

In this case the first equation yields \( p_{m+2} = 0 \) and together with the second equation we get \( s_{m+2,1} = p_{m+2} = 0 \). As in the case above we conclude that \( s_{1,2} = 0 \) and that \( s_{1,1} \) is a linear combination of the remaining coefficients.

Therefore \( \dim_K \text{Hom}(T_k, M) = m + 1 \) and a basis is given by \((m + 2) \times 2\)-matrices \( w_{2,1}^{(k)}, w_{3,1}^{(k)}, \ldots, w_{m+1,1}^{(k)}, w_{2,2}^{(k)}, w_{3,2}^{(k)}, \ldots, w_{m+2,2}^{(k)} \), where \( w_{i,j}^{(k)} \) is the matrix with entries \( s_{i,j} = 1 \) and \( s_{u,v} = 0 \) for \((u, v) \neq (i, j)\) and \((1, 1)\) whereas \( s_{1,1} = 1 \) or \(-1\).

(a5) Computation of \( \text{Hom}(T_k, M) \) for \( k = n-i+1, \ldots, n-1 \):

In this case a homomorphism \( f : T_k \to M \) is given by matrices \( P' = (p_i'') \in M_{m+1,1}(K), P'' = (p_i'') \in M_{m+1,1}(K), P = (p_i) \in M_{m+2,1}(K), Q = (q_i) \in M_{m+1,1}(K), R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+2,2}(K) \) such that \( P' = X_{m+1}^mP'' \), \( P = X_{m+2}^{m+1}P', SX_{k+1}^1 = P, SY_k^1 = Y_{m+2}^{m+1}Q \) and \( SZ_k^1 = Z_{m+2}^{m+1}R \).
The first equation yields \( p_{m+1}' = 0 \) and together with the second equation we get \( p_{m+2}' = 0 \) and additionally \( p_{m+2} = 0 \). Then from \( SX_j^1 = P \) we conclude \( s_{m+1,1} = s_{m+2,1} = 0 \). As in the (a3) the other equations imply that \( s_{1,2} = 0 \) and that \( s_{1,1} \) is a linear combination of the remaining coefficients.

Therefore \( \dim_K \text{Hom}(T_k, M) = m \) and a basis is given by \( (m + 2) \times 2 \)-matrices \( w_{2,1}^{(k)}, w_{3,1}^{(k)}, \ldots, w_{m+1,1}^{(k)}, w_{2,2}^{(k)}, w_{3,2}^{(k)}, \ldots, w_{m+2,2}^{(k)}, \) where \( w_{i,j}^{(k)} \) is the matrix with entries \( s_{i,j} = 1 \) and \( s_{u,v} = 0 \) for \( (u, v) \neq (i, j) \) whereas \( s_{1,1} = 1 \) or \(-1\).

(a6) Computation of \( \text{Hom}(T_n, M) \):
A homomorphism \( f : T_n \rightarrow M \) is given by matrices \( Q = (q_i) \in M_{m+1,1}(K) \), \( R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+2,2}(K) \) such that \( S = Y_{m+1}^{m+1}Q \) and \( S = Y_{m+2}^{m+1}R \).

The first equation implies \( s_1 = 0 \) and the second equation gives

\[
S = \begin{pmatrix}
0 & r_1 + r_2 \\
1 & r_1 + r_3 \\
0 & r_2 + r_4 \\
\vdots & \vdots \\
0 & r_3 + r_5 \\
0 & r_5 + r_7 \\
\end{pmatrix}
\]

Now

\[
0 = (r_1 + r_2) = (r_1 + r_3) + (r_2 + r_4) - (r_3 + r_5) - (r_4 + r_6) + (r_5 + r_7) + \ldots,
\]

which implies

\[
0 = s_2 + s_3 - s_4 - s_5 + s_6 + \ldots
\]

and \( s_2 \) can be written as a linear combination of the remaining coefficients. (We again define formally \( r_i = 0 \) for \( i > m + 1 \) and \( s_i = 0 \) for \( i > m + 2 \).)

Therefore \( \dim_K \text{Hom}(T_n, M) = m \) and a basis is given by \( (m + 2) \times 1 \)-matrices \( w_3^{(n)}, w_4^{(n)}, \ldots, w_{m+2}^{(n)}, \) where \( w_i^{(n)} \) is the matrix with entries \( s_1 = 0, s_i = 1, s_j = 0 \) for \( j \neq 1, 2, i \) and \( s_2 = 1 \) if \( i \equiv 0, 1 \) mod 4 and \( s_2 = -1 \) if \( i \equiv 2, 3 \) mod 4.

(a7) Computation of \( \text{Hom}(T_{n+1}, M) \):
A homomorphism \( f : T_{n+1} \rightarrow M \) is given by matrices \( U = (u_i) \in M_{m,1}(K) \), \( P' = (p_i') \in M_{m+1,1}(K) \), \( P = (p_i) \in M_{m+2,1}(K) \), \( Q' = (q_i') \in M_{m+1,1}(K) \), \( Q = (q_i) \in M_{m+2,1}(K) \), \( R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+2,2}(K) \) such that \( P' = X_{m+1}^{m+1}U, P = X_{m+2}^{m+1}P', SX_j^1 = P, Q' = Y_{m+1}^{m+1}Q, Q = Y_{m+2}^{m+1}Q', SY_j^1 = Y_{m+2}^{m+1}Q, R = Y_{m+1}^{m+1}U \) and \( SZ_j^1 = Z_{m+2}^{m+1}R \).

It is easily calculated that the equations imply that \( S \) is of the form

\[
S = \begin{pmatrix}
u_1 & 0 & 0 & \cdots & 0 \\
u_2 & 0 & u_1 & \cdots & 0 \\
u_3 & \vdots & \vdots & \ddots & \vdots \\
u_m & u_{m-2} & 0 & u_{m-1} & u_m
\end{pmatrix}
\]

Therefore \( \dim_K \text{Hom}(T_{n+1}, M) = m \) and a basis is given by \( (m + 2) \times 2 \)-matrices \( w_1^{(n+1)}, w_2^{(n+1)}, \ldots, w_m^{(n+1)}, \) where \( w_{i}^{(n+1)} \) is the \( (m + 2) \times 2 \)-matrix with entries \( s_{i,1} = 1, s_{i,2} = 1 \) and all other entries are zero.

Now, in order to determine the matrices of the representation \( N = F(M) \) we have to describe the linear maps \( N(j) = \text{Hom}(T_j, M) \rightarrow \text{Hom}(T_i, M) = N(i) \) in the
given bases. As mentioned in Section 2 this map is identified with the multiplication of the matrix \( S_j \) from the right.

In particular the map \( N(n) \to N(n - 1) \) is given by the formula

\[
\begin{bmatrix}
0 \\
s_2 \\
s_3 \\
s_m+2
\end{bmatrix} 
\mapsto
\begin{bmatrix}
0 \\
s_2 \\
s_3 \\
s_m+2
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & s_2 \\
0 & s_3 \\
0 & s_m+2
\end{bmatrix}.
\]

Thus we obtain in the bases \( w^{(n)}_3, w^{(n)}_4, \ldots, w^{(n)}_{m+2} \) and \( w^{(n-1)}_1, w^{(n-1)}_3, \ldots, w^{(n-1)}_m \), \( w^{(n-1)}_{2,2}, w^{(n-1)}_{3,2}, \ldots, w^{(n-1)}_{m,2,2} \) the following matrix

\[
C = \begin{bmatrix}
-1 & 1 & 1 & -1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & 1 & 1 & \ldots
\end{bmatrix}
\]

All the other matrices for \( N \) are computed in the same way and we obtain in the case (a) the following representation

\[
N^{i,j}_m = F(M^{i,j}_m)
\]

with

\[
(3.1) \quad A = \begin{bmatrix} 0 & \mathbf{I}_{m+1} \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{I}_{m+1} & \mathbf{I}_{m+1} \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ v \\ \mathbf{I}_{m} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \mathbf{I}_{m-1} \\ 0 & \ldots & 0 \\ \mathbf{I}_{m} \end{bmatrix}
\]

(3.2) \quad \begin{bmatrix} \mathbf{I}_{m} \\ 0 & \ldots & 0 \\ \mathbf{I}_{m+1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{I}_{m-1} \\ 0 & \ldots & 0 \\ \mathbf{I}_{m+1} \end{bmatrix}

where in \( C \) the vector \( v \) is given by the first \( m \) entries of the periodic vector

\[
(-1 1 1 -1 -1 1 -1 \ldots).
\]

In the "degenerated cases" similar calculations as above lead to the following results

case (b) \( i = 1 \) and \( j < n - 2 \)
with matrices $A$, $B$ and $E$ as given in case (a) and with

\[
C = \begin{bmatrix} 0 \\ v \\ 1_m \end{bmatrix}, \quad D = \begin{bmatrix} 0 | 1_{m-1} \\ 0 \ldots 0 \\ 0 \ldots 0 \\ 1_m \end{bmatrix}
\]

with vector $v$ as in case (a).

**Case (c):** $i \neq 1$ and $j = n - 2$

with matrices $C$, $D$ and $F$ given as in case (a), and with

\[
A = \begin{bmatrix} 0 | 1_{m+1} \\ 1_m \end{bmatrix}, \quad B = \begin{bmatrix} 1_m \quad 0 \ldots 0 | 1_{m+1} \end{bmatrix}
\]

**Case (d):** $i = 1$ and $j = n - 2$

with matrices $A$, $B$ as in case (c) and and $C$, $D$ as in case (b).

**3.3.** In order to describe the representations of $Q$ of rank one we use the symmetry of the quiver. For this reason we study first by investigating the dimension vectors to which of the preprojective $\Lambda$-modules we have to apply the functor $F$ in order to get, up to symmetry, all preprojective indecomposable representations of $\Gamma$ of rank 1. Recall that a tilting module $T$ induces an isomorphism of the corresponding Grothendieck groups such that the dimension vectors of the indecomposable direct summands $T_i$ are mapped to the dimension vectors of the left indecomposable projective modules $P_i$ over $A = \text{End}(T)$ (see [7, 3.2]). In our case this isomorphism is given by

\[
f : \mathbb{Z}^{n+1} \cong K_0(\Lambda) \to K_0(A) \cong \mathbb{Z}^{n+1}
\]

where $x = a_1 + a_1' - a_c$. 
There are 4\((n - 2)\) series of rank 1-modules over \(\Lambda\) and the following table shows how their dimension vectors are mapped under the isomorphism \(f\):

| type | notation | \(\dim(M)\) | \(f(\dim(M))\) |
|------|----------|--------------|------------------|
| 1    | \((1)M_m^{(i)}\) | \(m \quad m\) | \(m \quad 2m + 1\) |
| 2    | \((2)M_m^{(i)}\) | \(m \quad m + 1\) | \(m + 1 \quad 2m + 2\) |
| 3    | \((3)M_m^{(i)}\) | \(m \quad m + 1\) | \(m \quad 2m\) |
| 4    | \((4)M_m^{(i)}\) | \(m \quad m + 1\) | \(m \quad 2m\) |

Here the growth of the dimension from \(m\) to \(m + 1\) in the first arm for \(\dim(M)\) is realized for the arrow \(\alpha_i : (i - 1) \to i\) which implies that the growth of the dimension in the middle part of \(f(\dim(M))\) is realized for the arrow \((n - i - 1) \to (n - i)\). Moreover, for all types we have \(1 \leq i \leq n - 2\) which means that in the particular cases \(i = 1\) or \(i = n - 2\) there is no growth of the dimension in the middle part of \(f(\dim(M))\).

Because each preprojective representation is exceptional and each exceptional \(A\)-module is uniquely determined, up to isomorphism, by its dimension vector, using the symmetry of the quiver \(\tilde{D}_n\) it is sufficient to describe the modules \((1)N_m^{(i)} = F((1)M_m^{(i)})\) and \((2)N_m^{(i)} = F((2)M_m^{(i)})\), both for \(1 \leq i \leq n - 2\) and \(m \in \mathbb{N}\). The calculations are done in the same way as in the case of rank 2-modules and lead to the following results:

\((1)N_m^{(i)},\ i \neq 1, n - 2\)

\[
\begin{align*}
K^m & \xrightarrow{A} K^{2m+1} & \cdots & \xrightarrow{A} K^{2m+1} & \xrightarrow{A} K^{2m} & \cdots & \xrightarrow{A} K^m \\
K^{m+1} & \xrightarrow{B} K^{2m+1} & \cdots & \xrightarrow{B} K^{2m+1} & \xrightarrow{B} K^{2m} & \cdots & \xrightarrow{B} K^{m+1}
\end{align*}
\]

with matrices

\[(3.5) \quad A = \begin{bmatrix} 0 & I_m \end{bmatrix}, \quad B = \begin{bmatrix} I_{m+1} & \begin{bmatrix} 0 \cdots 0 \\ I_m \end{bmatrix} \end{bmatrix}
\]

\[(3.6) \quad C = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad D = \begin{bmatrix} I_m \\ I_m \end{bmatrix}, \quad E = \begin{bmatrix} I_m \\ \begin{bmatrix} 0 \cdots 0 \\ I_m \end{bmatrix} \end{bmatrix}
\]

\((1)N_m^{(1)}\)

\[
\begin{align*}
K^m & \xrightarrow{A} K^{2m} & \cdots & \xrightarrow{A} K^{2m} \\
K^{m+1} & \xrightarrow{B} K^{2m} & \cdots & \xrightarrow{B} K^{2m}
\end{align*}
\]
with

(3.7) \[ A = \begin{bmatrix} 0 & I_m \end{bmatrix} \quad B = \begin{bmatrix} I_m & 0 \ldots 0 \\ 0 \ldots 0 & I_m \end{bmatrix} \]

(1) $\mathcal{N}_m^{(n-2)}$

with

(3.8) \[ C = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \quad D = \begin{bmatrix} I_m \\ 0 \ldots 0 \\ I_m \end{bmatrix} \]

(2) $\mathcal{N}_m^{(i)}$, $i \neq 1, n-2$

with

(3.9) \[ A = \begin{bmatrix} 0 & I_{m+1} \end{bmatrix} \quad B = \begin{bmatrix} I_m & 0 \ldots 0 \\ 0 \ldots 0 & I_{m+1} \end{bmatrix} \]

(3.10) \[ C = \begin{bmatrix} 0 \\ I_{m+1} \end{bmatrix} \quad D = \begin{bmatrix} I_m \\ 0 \ldots 0 \\ I_{m+1} \end{bmatrix} \quad E = \begin{bmatrix} I_m \\ 0 \ldots 0 \\ 0 \\ I_{m+1} \end{bmatrix} \]

(2) $\mathcal{N}_m^{(1)}$

with

(3.11) \[ A = \begin{bmatrix} 0 & I_{m+1} \end{bmatrix} \quad B = \begin{bmatrix} I_m \\ 0 \ldots 0 \\ I_{m+1} \end{bmatrix} \]
(2) $N_m^{(n-2)}$

\[
K^{m+1} \xrightarrow{A} K^{2m+2} \xrightarrow{B} \ldots \xrightarrow{C} K^{2m+2} \xrightarrow{D} K^m
\]

with

\[
C = \begin{bmatrix} 0 \\ I_{m+1} \end{bmatrix}, \quad D = \begin{bmatrix} I_m \\ 0 \ldots 0 \\ 0 \ldots 0 \\ I_m \end{bmatrix}
\]

(3.12)

Theorem 1. The representations given by the matrices described in (3.1) – (3.12) above form a complete list of nonisomorphic preprojective indecomposable representations for the quiver $\tilde{D}_n$ with the chosen orientation.

4. The case $\tilde{E}_6$

We consider a quiver $\Gamma$ of type $\tilde{E}_6$ with subspace orientation

\[
\begin{array}{ccccccc}
& & 2 & & \\
& & \downarrow & & \\
& & 1 & & \\
3 & \rightarrow & 4 & \leftarrow & 0 & \rightarrow & 5 & \leftarrow & 6
\end{array}
\]

In this case the corresponding domestic canonical algebra $\Lambda$ is of type $(3,3,2)$ and a tilting module $T = \bigoplus_{i=0}^{6} T_i$ such that $\text{End}(T) \simeq (K\Gamma)^{op}$ is given by

\[
\begin{array}{ccc}
T_0 & K & K^2 \\
0 & K \xrightarrow{x_2^1} K^2 & K^3 \\
& K \xrightarrow{x_2^{1}} K^2 & K^3 \\
& & K \xrightarrow{x_2^1} K^2 \\
& & K^2 \\
& & K^3 \\
& & K \\
& & K
\end{array}
\]

\[
\begin{array}{ccc}
T_1 & K & K^2 \\
0 & K \xrightarrow{y_2^1} K^2 & K \\
& K \xrightarrow{y_2^1} K^2 & K \\
& & K \xrightarrow{y_2^1} K^2 \\
& & K^2 \\
& & K \\
& & K
\end{array}
\]

\[
\begin{array}{ccc}
T_2 & K & K \\
0 & K \xrightarrow{z_2^{1}} K \\
& K \xrightarrow{z_2^{1}} K \\
& & K \\
& & 0
\end{array}
\]

\[
\begin{array}{ccc}
T_3 & K & K^2 \\
0 & 0 \xrightarrow{x_2^{1}} K & K^2 \\
& K \xrightarrow{x_2^{1}} K & K^2 \\
& & K \xrightarrow{x_2^1} K^2 \\
& & K^2 \\
& & K \\
& & K
\end{array}
\]

\[
\begin{array}{ccc}
T_4 & K & K \\
0 & 0 \xrightarrow{y_2^{1}} K & K \\
& K \xrightarrow{y_2^{1}} K & K \\
& & K \xrightarrow{y_2^1} K \\
& & K^2 \\
& & K \\
& & K
\end{array}
\]
The matrices $X^1_1$, $X^2_3$, $Y^1_2$, $Y^2_3$ are defined in the previous chapter and $Z^1_3$ denotes the matrix
\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}.
\]

As in the previous chapter we illustrate the endomorphism ring of $T$ and give generators $S^j_i$ for the non-zero homomorphism spaces $\text{Hom}_\Lambda(T_i, T_j)$.

In this chapter we determine matrices for the indecomposable preprojective representations corresponding to the indecomposable $\Lambda$-modules of rank 3. There are 2 series of indecomposable preprojective rank 3-modules over $\Lambda$. The first one is described as follows: (see [11].)

where $Z^m_{m+3}$ is the $m$-th enlargement of the $3 \times 1$ matrix $Z^1_3$.

As in chapter 3 we apply the functor $F = \text{Hom}_\Lambda(T, -)$ to the modules $M_m$. We shortly write $M_m = M$ and calculate first the the dimensions and suitable bases of the vector spaces $N(i)$ of the representation $N = \text{Hom}_\Lambda(T, M)$.

(a0) Computation of $\text{Hom}(T_0, M)$:
A homomorphism $f : T_0 \to M$ is given by matrices $P' = (p'_i) \in M_{m+1,1}(K)$, $P = (p_i) \in M_{m+2,2}(K)$, $Q' = (q'_i) \in M_{m+1,1}(K)$, $Q = (q_{ij}) \in M_{m+2,2}(K)$, $R = (r_i) \in M_{m+1,1}(K)$ and $S = (s_{ij}) \in M_{m+3,3}(K)$ such that $PX^1_2 = X^{m+1}_{m+2}P'$, $SX^2_3 = X^{m+2}_{m+3}P$, $QY^1_2 = Y^{m+1}_{m+2}Q'$, $SY^2_3 = Y^{m+2}_{m+3}Q$, and $SZ^1_3 = Z^{m+1}_{m+3}R$.

It is easy to verify that the first four conditions imply that $s_{1,2} = 0$, $s_{1,3} = 0$, $s_{2,3} = 0$, $s_{m+2,1,1} = 0$, $s_{m+3,1} = 0$ and $s_{m+3,2} = 0$. Moreover, the last equation gives that two of the other entries of $S$ (say $s_{1,1}$ and $s_{2,1}$) are linearly dependent of the remaining $s_{i,j}$. This is a consequence of the structure of the matrix.
\[ Z^{m+1}_{m+3} \] and will appear in calculations for all \( N(i) \). However in order to determine the matrices of the representation \( N \) we need the precise description of this linear dependence only for the vertex 1 for which we give the computation in detail. From this it will follow that also in the case considered here we have the two linear dependent expressions mentioned above. As a consequence we obtain that \( \dim_K \text{Hom}(T_0, M) = 3m + 1 \) and a basis is given by \( (m + 3) \times 3 \)-matrices 

\[
\begin{aligned}
 w^{(0)}_{3,1}, w^{(0)}_{4,1}, \ldots, w^{(0)}_{m+1,1}, w^{(0)}_{2,2}, w^{(0)}_{3,2}, \ldots, w^{(0)}_{m+2,2}, w^{(0)}_{3,3}, \ldots, w^{(0)}_{m+3,3},
\end{aligned}
\]

where \( w^{(0)}_{i,j} \) is the matrix with entries \( s_{i,j} = 1 \), with some possibly non-zero entries \( s_{1,1}, s_{2,1} \) and all the other entries are 0.

(11) Computation of \( \text{Hom}(T_1, M) \):

A homomorphism \( f : T_1 \rightarrow M \) is given by matrices \( P = (p_i) \in M_{m+2,1}(K) \), \( Q' = (q'_i) \in M_{m+2,2}(K) \), \( Q = (q_{i,j}) \in M_{m+2,2}(K) \), \( R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+3,2}(K) \) such that \( SX_2^1 = X^{m+2}P \), \( QY_2^1 = Y^{m+2}Q' \), \( S = Y^{m+2}Q \), and \( SZ_2' = Z^{m+1}_{m+3}R \).

From the first conditions we conclude that \( s_{1,1} = 0, s_{1,2} = 0, s_{2,2} = 0 \) and \( s_{m+3,1} = 0 \), whereas the last equation yields

\[
\begin{bmatrix}
 0 \\
 s_{2,1} \\
 s_{3,1} + s_{3,2} \\
 s_{4,1} + s_{4,2} \\
 \vdots
\end{bmatrix}
= \begin{bmatrix}
 r_1 + r_2 \\
 r_1 + r_3 \\
 r_1 + r_4 \\
 r_2 + r_5 \\
 \vdots
\end{bmatrix}.
\]

We have the two identities

\[
(2) \quad r_1 + r_2 = \sum_{k=0}^{\infty} r_{6k+1} + r_{6k+4} + \sum_{k=0}^{\infty} r_{6k+2} + r_{6k+5} - \sum_{k=0}^{\infty} r_{6k+4} + r_{6k+7} - \sum_{k=0}^{\infty} r_{6k+5} + r_{6k+8}
\]

\[
(3) \quad r_1 + r_3 = \sum_{k=0}^{\infty} r_{6k+1} + r_{6k+4} + \sum_{k=0}^{\infty} r_{6k+3} + r_{6k+6} - \sum_{k=0}^{\infty} r_{6k+4} + r_{6k+7} - \sum_{k=0}^{\infty} r_{6k+6} + r_{6k+9}
\]

(we formally define \( r_j = 0 \) for \( j > m+1 \)). According to (1) and (3) we get

\[
(4) \quad s_{2,1} = \sum_{k=0}^{\infty} s_{6k+3,1} + s_{6k+3,2} + \sum_{k=0}^{\infty} s_{6k+5,1} + s_{6k+5,2} - \sum_{k=0}^{\infty} s_{6k+6,1} + s_{6k+6,2} - \sum_{k=0}^{\infty} s_{6k+8,1} + s_{6k+8,2}
\]

(again we formally define \( s_{i,j} = 0 \) for \( i > m+3 \)). Further it follows from (1) and (2) that one more coefficient is dependent of the remaining, for instance we can write

\[
(5) \quad s_{3,1} = -s_{3,2} - \sum_{k=1}^{\infty} s_{6k+3,1} + s_{6k+3,2} - \sum_{k=0}^{\infty} s_{6k+4,1} + s_{6k+4,2} +
\]

\[
+ \sum_{k=0}^{\infty} s_{6k+6,1} + s_{6k+6,2} + \sum_{k=0}^{\infty} s_{6k+7,1} + s_{6k+7,2}.
\]

It follows that \( \dim_K \text{Hom}(T_1, M) = 2m \) and a basis \( w^{(1)}_{4,1}, w^{(1)}_{5,1}, \ldots, w^{(1)}_{m+2,1}, w^{(1)}_{3,2}, w^{(1)}_{4,2}, \ldots, w^{(1)}_{m+3,1} \) can be given in the following way: \( w^{(1)}_{i,j} \) is the matrix with entries \( s_{i,j} = 1 \) and \( s_{k,l} = 0 \) for \( (k,l) \neq (i,j), (2,1), (3,1) \) and the entries \( s_{3,1} \) and \( s_{2,1} \) have to be computed using the formulas (5) and (4) (note that first one has to calculate \( s_{3,1} \) because this coefficient appears in (4)). The following table gives these coefficients for the vectors \( w^{(1)}_{i,j} \).
We have periodicity after 6 places. Further one has to cut after the indices 
\((m + 2,1)\) and \((m + 2,2)\).

(a2) Computation of \(\text{Hom}(T_2, M)\):
A homomorphism \(f : T_2 \to M\) is given by matrices 
\(P = (p_i) \in M_{m+2,1}(K),\)
\(Q' = (q_j') \in M_{m+1,1}(K),\)
\(Q = (q_i) \in M_{m+2,1}(K),\)
\(S = (s_j) \in M_{m+3,1}(K)\) such that
\(P = X_{m+2}^1 P',\)
\(Q = Y_{m+1}^1 Q'\)
and
\(S = Y_{m+3}^1 Q.'\)

It is easy to see that this yields the vanishing conditions \(s_1 = 0, s_2 = 0\) \text{ and } \(s_{m+3} = 0\). We obtain \(\dim_K \text{Hom}(T_2, M) = m\) and a basis is given by \((m + 3) \times 1\) matrices \(w_3^{(2)}, w_4^{(2)}, \ldots, w_{m+2}^{(2)}\) where \(w_1^{(2)}\) is the matrix with entries \(s_i = 1\) and \(s_j = 0\) for \(j \neq i\).

(a3) Computation of \(\text{Hom}(T_3, M)\):
A homomorphism \(f : T_3 \to M\) is given by matrices 
\(P' = (p'_i) \in M_{m+1,1}(K),\)
\(P = (p_{i,j}) \in M_{m+2,2}(K),\)
\(Q = (q_i) \in M_{m+2,1}(K),\)
\(R = (r_j) \in M_{m+1,1}(K)\) and
\(S = (s_i) \in M_{m+3,2}(K)\) such that 
\(P X_{m+2}^1 = X_{m+2}^1 P',\)
\(S X_{m+3}^1 = X_{m+3}^1 P,\)
\(S Y_{m+3}^1 = X_{m+3}^1 Q,\)
and
\(S Z_{m+3}^1 = Z_{m+1}^1 R.\)

The first equations yields \(s_{1,2} = 0, s_{m+2,1} = 0, s_{m+3,1} = 0\). Furthermore the last equation implies, similarly as in (a2) that two further entries, say \(s_{1,1}\) and \(s_{2,1}\) are linearly dependent of the others coefficients, however the precise formula for that will not be needed. We obtain \(\dim_K \text{Hom}(T_3, M) = 2m\) and a basis is given by \((m + 3) \times 2\) matrices 
\(w_{3,1}^{(3)}, w_{4,1}^{(3)}, \ldots, w_{m+1,1}^{(3)}, w_{2,2}^{(3)}, w_{3,2}^{(3)}, \ldots, w_{m+2,1}^{(3)}\) where 
\(w_{i,j}^{(3)}\) is the matrix with entries \(s_{i,j} = 1\), all other entries are 0 except maybe \(s_{1,1}\) and \(s_{2,1}\).

(a4) Computation of \(\text{Hom}(T_4, M)\):
A homomorphism \(f : T_4 \to M\) is given by matrices 
\(P' = (p'_i) \in M_{m+1,1}(K),\)
\(P = (p_{i}) \in M_{m+2,1}(K),\)
\(Q = (q_i) \in M_{m+2,1}(K),\)
\(R = (r_j) \in M_{m+1,1}(K)\) and
\(S = (s_i) \in M_{m+3,1}(K)\) such that 
\(P = X_{m+2}^1 P',\)
\(Q = Y_{m+1}^1 Q'\)
and
\(S = Y_{m+3}^1 Q.\)

Again it is easy to verify that we get the following vanishing conditions \(s_1 = 0, s_{m+2} = 0\) \text{ and } \(s_{m+3} = 0\). Thus we have \(\dim_K \text{Hom}(T_4, M) = m\) and a basis is given by \((m + 3) \times 1\) matrices 
\(w_{2,2}^{(4)}, w_{4,3}^{(4)}, \ldots, w_{m+1,1}^{(4)}\) where \(w_i^{(4)}\) is the matrix with entries \(s_i = 1\) and \(s_j = 0\) for \(j \neq i\).

(a5) Computation of \(\text{Hom}(T_5, M)\):
A homomorphism \(f : T_5 \to M\) is given by matrices 
\(P' = (p'_i) \in M_{m+1,1}(K),\)
\(P = (p_{i}) \in M_{m+2,1}(K),\)
\(Q' = (q'_i) \in M_{m+1,1}(K),\)
\(Q = (q_i) \in M_{m+2,1}(K),\)
\(R = (r_j) \in M_{m+1,1}(K)\) and
\(S = (s_{i,j}) \in M_{m+3,2}(K)\) such that 
\(P = X_{m+2}^1 P',\)
\(Q = Y_{m+1}^1 Q'\)
and
\(S = Y_{m+3}^1 Q.\)

\(Z\)From the first equations we infer that \(s_{1,2} = 0, s_{2,2} = 0, s_{m+2,1} = 0\) \text{ and } \(s_{m+3,1} = 0\). Moreover, the last equation implies, similarly as in (a2) that two further entries, say \(s_{1,1}\) and \(s_{2,1}\) are linearly dependent of the others. We obtain \(\dim_K \text{Hom}(T_5, M) = 2m\) and a basis is given by \((m + 3) \times 2\) matrices 
\(w_{3,1}^{(3)}, w_{4,1}^{(3)}, \ldots, w_{m+1,1}^{(3)}, w_{2,2}^{(3)}, w_{3,2}^{(3)}, \ldots, w_{m+2,1}^{(3)}\) where \(w_{i,j}^{(3)}\) is the matrix with entries \(s_{i,j} = 1\), all other entries are 0, except maybe \(s_{1,1}\) and \(s_{2,1}\).

(a6) Computation of \(\text{Hom}(T_6, M)\):
A homomorphism \(f : T_6 \to M\) is given by matrices
\( T = (t_i) \in M_{m,1}(K), \; P' = (p'_i) \in M_{m+1,1}(K) \; P = (p_i) \in M_{m+2,1}(K), \; Q' = (q'_i) \in M_{m+1,1}(K), \; Q = (q_i) \in M_{m+2,1}(K), \; R = (r_i) \in M_{m+1,1}(K) \) and \( S = (s_{i,j}) \in M_{m+3,2}(K) \) such that \( P' = X_{m+1}^T, \; P = X_{m+2}^T, \; S \chi^1 = X_{m+3}^T, \; Q' = Y_{m+1} \), 
\( Q = Y_{m+2} \) and \( SY^1 = Y_{m+3} \), \( R = Y_{m+2}Q' \) and \( SZ^1 = Z_{m+3} \).

It is easily calculated that the equations imply that \( S \) is of the form

\[
S = \begin{pmatrix}
t_1 & 0 \\
t_2 & 0 \\
t_3 & 0 \\
t_4 & t_1 \\
& \ddots \\
t_m & t_{m-3} \\
0 & t_{m-2} \\
0 & t_{m-1} \\
0 & t_m \\
\end{pmatrix}
\]

Therefore \( \text{dim}_K \text{Hom}(T_{n+1}, M) = m \) and a basis is given by \((m + 2) \times 2\)-matrices \( w_{1}^{(6)}, w_{2}^{(6)}, \ldots, w_{m}^{(6)} \), where \( w_{i}^{(6)} \) is the \((m + 2) \times 2\)-matrix with entries \( s_{i,1} = 1, \; s_{i+3,2} = 1 \) and all other entries are zero.

We want to determine the matrices of the representation \( N = F(M) \) of the extended Dynkin quiver \( \Gamma \) in the bases of the vector spaces constructed above. For this reason we have to multiply with the corresponding matrices \( S_{ij} \) from the right. In particular the map \( N(1) \rightarrow N(0) \) is given by the formula

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & s_{2,1} & 0 \\
s_{3,1} & s_{3,2} & \ddots \\
s_{m+2,1} & s_{m+2,2} & 0 \\
0 & s_{m+3,2} & \end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & s_{2,1} & 0 \\
s_{3,1} & s_{3,2} & \ddots \\
s_{m+2,1} & s_{m+2,2} & 0 \\
0 & s_{m+3,2} & \end{pmatrix}
\]

This map in the bases \( w_{1}^{(1)}, w_{b,1}^{(1)}, \ldots, w_{m+2,1}^{(1)}, w_{3,2}^{(1)}, w_{4,2}^{(1)}, \ldots, w_{m+3,1}^{(1)} \) and \( w_{3,1}^{(0)}, w_{4,1}^{(0)}, \ldots, w_{m+1,1}^{(0)}, w_{2,2}^{(0)}, w_{3,2}^{(0)}, \ldots, w_{m+2,2}^{(0)}, w_{3,3}^{(0)}, w_{4,3}^{(0)}, \ldots, w_{m+3,3}^{(0)} \) has the following form

\[
A = \begin{pmatrix}
-1 & 1 & 0 & 1 & -1 & 0 & \ldots \\
-1 & 0 & 1 & 1 & 0 & -1 & \ldots \\
1 \\
1 \\
1 \\
\ddots \\
1 \\
1 \\
1 \\
\end{pmatrix}
\]
where the four indicated “half” rows with entries 0, 1 and \(-1\) have period 6 but stop on the right hand side at the position \(m + 2\) respectively \(m + 3\).

Investigating the other linear maps in the same way we get the following series of indecomposable representations \(N_m\) for \(Q\):

\[
\begin{array}{ccccccccc}
K^m & D & K^m & C & K^{3m+1} & E & K^{2m} & G & K^m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B & A & C & E & K^{2m} & G & K^m & \\
\end{array}
\]

where \(A\) is defined above and

\[
B = \begin{bmatrix}
0 & I_{m-1} \\
-I_m & 0 \\
-1_m & 0 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
I_{m-1} & 0 \\
0 & I_{m+1} \\
0 & 0 \\
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & I_{m-1} \\
-I_m & 0 \\
-1_m & 0 \\
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
I_{m-1} & 0 \\
0 & 0 \\
0 & I_{m+1} \\
\end{bmatrix}, \quad G = \begin{bmatrix}
0 & I_{m-2} \\
-1_m & 0 \\
-1_m & 0 \\
\end{bmatrix}
\]

Applying the functor \(F\) to the second series of rank 3 modules over \(\Lambda\) the same method yields the following indecomposable representations for \(\Gamma\):

\[
\begin{array}{ccccccccc}
K^m & D & K^m & C & K^{3m+1} & E & K^{2m+1} & G & K^m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B & A & C & E & K^{2m+1} & G & K^m & \\
\end{array}
\]

where

\[
A = \begin{bmatrix}
I_m & 0 \\
0 & I_{m+1} \\
0 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
I_m & 0 \\
0 & 0 \\
0 & I_{m+1} \\
\end{bmatrix}, \quad C = \begin{bmatrix}
I_m & 0 \\
0 & 0 \\
0 & I_{m+1} \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & I_{m-1} \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad E = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & I_{m+1} \\
\end{bmatrix}, \quad G = \begin{bmatrix}
I_m & -I_m \\
I_m & 0 \\
0 & I_{m+1} \\
\end{bmatrix}
\]

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