A TWO–EXPONENT MASS-SIZE POWER LAW
FOR
CELESTIAL OBJECTS

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Abstract

The Universe that we know is populated by structures made up of aggregated matter that organizes into a variety of objects; these range from stars to larger objects, such as galaxies or star clusters, composed by stars, gas and dust in gravitational interaction. We show that observations support the existence of a composite (two–exponent) power law relating mass and size for these objects. We briefly discuss these power laws and, in view of the similarity in the values of the exponents, ponder the analogy with power laws in other fields of science such as the Gutenberg–Richter law for earthquakes and the Hutchinson–MacArthur or Damuth laws of ecology. We argue for a potential connection with avalanches, complex systems and punctuated equilibrium, and show that this interpretation of large scale–structure as a self–organized critical system leads to two predictions: (a) the large scale structures are fractally distributed and, (b), the fractal dimension is $1.65 \pm 0.25$. Both are borne out by observations.

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Each point in Figure 1 represents typical (average) observational data for each of the structures known to exist in the Universe whose mass is approximately the mass of the Sun or larger. The abscissa of this log–log plot represents the characteristic linear size of the structure normalized to the size of the Sun; in the ordinate we have plotted one over the mass of the object normalized to the mass of the Sun, or, what is the same, one over the number of suns that would fit in the structure if all its mass was made up of suns. The actual data is presented in Table I.

A very clear pattern of a composite power law emerges from the figure: a power law for “star–like” objects (s) and a different power law for what we can generically call “multistellar” objects (ms). The two exponents are radically different and they are given by \( \tau_S = -0.0999 \) and by \( \tau_{MS} = -2.21 \). These values are obtained by a simple chi–squared fit to the logarithm of the data of Table I to two different straight lines: one line for the s–class of objects and another for the ms–class of objects. The fit properties are described in Tables II (for s–objects) and III (for ms–objects); we see that the fit quality is passable for s–objects and excellent for ms–objects. The fit dependence on \( \Omega \) and \( h \) is very mild: for \( \Omega = 0.1 \) and \( h = 1 \), \( \tau_{MS} = -2.1213 \) and \( a_{MS} = 34.877 \), whereas for \( \Omega = 1.0 \) and \( h = 0.5 \), \( \tau_{MS} = -2.1836 \) and \( a_{MS} = 36.4538 \); with errors very close to what is shown in Table III.

Thus we have the relation

\[
\log \left( \frac{M_{\text{object}}}{M_\odot} \right) = a_{\text{class}} + \tau_{\text{class}} \log \left( \frac{l_{\text{object}}}{l_\odot} \right)
\]

with \( M_{\text{object}} \) the typical mass of the object, \( l_{\text{object}} \) its longitudinal size. The two parameters \( a_{\text{class}} \) and \( \tau_{\text{class}} \) have the values quoted in the table for each class.

The fact that there are two different power laws for the two classes of objects is a good

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1Because they are aggregates of “star–like” objects together with dust, gas, etc..
indication that objects within each class share a common physical origin which is, in turn, different for each of the two classes.

It is a remarkable coincidence that a similar, composite power law, has been found to hold for the number of species of all different kinds of multicellular terrestrial animals versus their body length. Furthermore, the exponents for multicellular animals are very close to what we find in this paper, especially the exponent for the larger–sized species, which is approximately equal to $-2$, and would then correspond to what we have called the ms–class of objects.

The existence of the power law for the s–class of objects has been known for many years, and is the celebrated mass-radius law. It can be understood using basic features of the astrophysics of stellar objects and dimensional considerations, as was done by Eddington who related it to the mass-luminosity ratio. It codifies a great deal of information on the physics of gravitational collapse and the nuclear physics of stellar material. This power law holds for about one order of magnitude in mass and eight orders of magnitude in size.

The rest of this paper is devoted to the power law for ms–objects. This power law is valid from open stellar clusters, the simplest type of objects in this class (i.e., the one with the smallest number of components), to the full Universe, obviously the most complex of the objects in its class, and spans twenty orders of magnitude in mass and almost nine orders of magnitude in linear size as can be seen from Figure 2. The relationship is clearly non-gaussian (which would show in this log–log plot as a sharp, cut–off curve), and therefore implies the existence of some scale–free phenomena, perhaps revealing the presence of a critical regime, manifesting in the variety of structures we list under the category of multistellar objects in Table I. All these objects would then have their origin in a common mechanism,

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2This also extends to bacteria and prokaryotes, as found in Reference.

3Since power laws are scale–invariant, they are construed in statistical mechanics as the quintessential indicators of criticality.
possibly related to their complex many-body nature (in the sense that they are made up of many star-like objects), which is a property shared by all of them, and establishes a clear difference between these objects and the objects in the other class, characterized by being compact objects. The presence of “gaps” between the structures (we do not know of any structures intermediate between, e.g., “globular clusters” and “galaxies”) points in the direction of “avalanche” behavior and, perhaps, some form of “punctuation” in the sense of Elredge and Gould [10].

The ordinate in these plots can also be interpreted, for example, as the cumulative dynamical–mass fraction of suns that would be contained in structures with longitudinal size equal or greater than \( l_{\text{object}} \). This interpretation of the ordinate together with the results of our fit, specially for the ms-objects, remind one of the Gutenberg–Richter relation [11] for earthquakes, another power law which describes the cumulative number of earthquakes, \( N \), of magnitude \( M \) greater than or equal to a value \( M \) through the expression

\[
\log_{10} N = a - bM
\]

with exponent \( b \) typically slightly greater than 1. This, like punctuated equilibrium, is again a phenomena which can be related to a form of avalanche behavior [11].

Shifting the ordinate by \( \log(M_\odot/M_{\text{Universe}}) \), it represents the \( \log(M_{\text{Universe}}/M_{\text{object}}) \), i.e., the logarithm of the total number of objects of a given dynamical mass fitting in the Universe.

\[4\] An understanding of this power law can be gained by using the hydrodynamics of many bodies in gravitational interaction in an expanding Universe. What emerges is an avalanche picture of the large scale structure in the Universe as a particular case, of a very general class of phenomena. See below.

\[5\] The term compact is not used here in the sense which is used for completely collapsed objects like black holes.

\[6\] The magnitude of an earthquake is proportional to the logarithm of the energy released in the quake or contained in the seismic wave amplitude.
Thus one can think of an analogy with Damuth’s law \[12\] of ecology which relates the population density, \(D\), of moving life–forms\(^7\) on the surface of the Earth to their longitudinal size \(L\), according to

\[
\log D \sim \tau_{\text{Damuth}} \log L + \text{constant} \tag{2}
\]

with power law exponent \(\tau_{\text{Damuth}}\). It is very intriguing that as Damuth found, \(\tau_{\text{Damuth}} \approx -2.25\), very close to our \(\tau_{\text{MS}} = -2.21\) for multistellar objects\(^8\).

Can one gain quantitative understanding of this power law? Unlike in geophysics or in ecology, where a mathematical framework is less well defined, in cosmology there is a well defined formalism which at least permits one to attack the description of structure formation \[1\], \[13\]. Within this framework, the equations that describe density perturbations and structure formation in an expanding Universe can be written in a form similar to the Directed Percolation Depinning (DPD) model in 3 + 1 dimensions \[7\]. It is known from computer simulations of DPD–models that they display self–organized critical behavior \[14\], \[15\] (i.e. they lead to power laws) and, in particular, that the probability that an avalanche of time duration \(t\) survives scales as

\[
P(t) \sim t^{-\tau_{\text{survival}}} \tag{3}
\]

with \[15\] \(\tau_{\text{survival}} \approx 2.54\), which is reasonably close to the exponent found here for the MS–objects. Notice that the (comoving) longitudinal size of the object is related to the time it takes light to cross it by \(l = ct\); this is the time entering in the above equation. Furthermore, since the “number of objects of a given dynamical–mass fitting in the Universe” is the same

\(^7\)For comments on the extension of a similar law to plants, see p. 45 of the book by Bonner in Ref. \[4\].

\(^8\)This, of course, was interpreted by Damuth, using Kleiber’s law of metabolism, as evidence that there exists a “pyramid of metabolism”.

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as the probability that an object of a given dynamical–mass is present in the Universe, we see that our phenomenological ms–power law represents Equation (3) with $\tau_{\text{survival}} \approx 2.21$.

If one adopts this “avalanche” interpretation of the ms–power law, there is a prediction that arises from it. As shown by the authors of Ref. [15], in DPD–models the “shell of unblocked cells in the interface forms a fractal dust”, which can be related to the distribution of survival times for the avalanches, Equation (3). This fractal dust is packed into “moving blocks” which behave as quasi–particles and are distributed like a fractal of dimension $d_{\text{dust}}$. The dimension of the fractal (or Levy) dust is related to $\tau_{\text{survival}}$ through

$$d_{\text{dust}} = z(\tau_{\text{survival}} - 1),$$

where the dynamical exponent $z$ is defined as the exponent relating typical time scales to typical length scales, i.e. $t \sim L^z$. In 3+1 dimensions, computer simulations of DPD–models [15] give $z = 1.36 \pm 0.05$, and therefore the predicted fractal dimension for the dust is $d_{\text{dust}} = 1.65 \pm 0.25$. The equations of cosmological hydrodynamics are best approximated by the DPD–model for times in the History of the Universe not very much after the decoupling era, and thus one expects that the predictions for scales in the realm of the galaxies and larger of “avalanche” physics, as described here, will be the most accurate. It is very tantalizing (a) that the “avalanche” physics interpretation leads to a fractal distribution of the largest structures in the Universe, a fact which has been known since the time of the first large scale surveys and (b) that the value predicted for $d_{\text{dust}}$ is so close to what is inferred from observations [16] for galaxies, rich clusters and clusters of galaxies, $d_{\text{obs}} = 1.65 \pm 0.15$.

We end by making several remarks. It is conceivable that the reason why power laws with exponents so similar, but for such diverse and different systems, are present, is because the essential underlying physics is the laws of hydrodynamics together with some form of noise which models the huge number of “unforeseebles” in a very complex system. The closeness in the values of the exponents, together with the intuitive connection to systems exhibiting Self–Organized Critical behavior, such as earthquakes, the ecology and DPD, together with the correctness of the predictions derived from it leads one to surmise whether the Universe
is yet another example of a Self–Organized Critical system, and that this is the general
principle behind this phenomenology.

A final question is that one must understand how the two power laws are connected
together or if they can be understood in terms of a single law encompassing both, and
why the change of exponent takes place. This requires new ways of looking at gravitational
phenomena and the large scale structure in the Universe and, like other aspects of large scale
structure physics [17], points in the direction of phase transitions and critical phenomena
where a change of value in the exponent of the power law can take place as a consequence
of a phase change.

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FIGURES

Log of Mass Fraction vs. Log of Linear Size for Stellar and Multistellar Objects. (Normalized to the Sun and for $\Omega=1$, $h=1$)

FIG. 1. Plot of the data in Table I.
Log of Mass Fraction vs. Log of Linear Size for Multi-Stellar Objects. Fit and Residues.
(Normalized to the Sun and for $\Omega=1$, $h=1$)

FIG. 2. Power–law fit and residues for ms–objects.
TABLES

TABLE I. Characteristic longitudinal sizes and dynamical masses for typical celestial objects. These observational data are taken from Reference [1]. To help the reader’s intuition we have also included in this table the data on Jupiter. The labels refer to Figure 1. Here $\Omega$ and $h$ are the density parameter for the Universe and the normalized (to $100 \text{ km sec}^{-1} \text{ Mpc}^{-1}$) Hubble parameter, respectively.

| OBJECT TYPE   | CLASS | RADIUS (cm) | MASS (g)     | LABEL |
|---------------|-------|-------------|--------------|-------|
| Jupiter       | –     | $6 \times 10^9$ | $2 \times 10^{30}$ | -     |
| Neutron Star  | S     | $10^6$      | $3 \times 10^{33}$ | E     |
| White Dwarf   | S     | $10^8$      | $2 \times 10^{33}$ | D     |
| Sun           | S     | $7 \times 10^{10}$ | $2 \times 10^{33}$ | B     |
| Red Giant     | S     | $10^{14}$   | $(2 - 6) \times 10^{34}$ | C     |
| Open Cluster  | MC    | $3 \times 10^{19}$ | $5 \times 10^{35}$ | G     |
| Globular Cluster | MC | $1.5 \times 10^{20}$ | $1.2 \times 10^{39}$ | F     |
| Elliptical Galaxy | MC | $(1.5 - 3) \times 10^{23}$ | $2 \times (10^{43} - 10^{45})$ | I     |
| Spiral Galaxy | MC    | $(6 - 15) \times 10^{22}$ | $2 \times (10^{44} - 10^{45})$ | H     |
| Group of Galaxies | MC | $3 \times 10^{24}$ | $4 \times 10^{46}$ | J     |
| Cluster of Galaxies | MC | $1.2 \times 10^{25}$ | $2 \times 10^{48}$ | K     |
| Universe      | MC    | $10^{28}/h$  | $7.5 \times 10^{55}\Omega/h$ | L     |
TABLE II. Basic Information from the Analysis of Data for Stellar (s) objects assuming a Simple Linear Regression equation.

(a) Predictive equation

$$\log \left( \frac{M_{\text{object}}}{M_\odot} \right) = a_s + \tau_s \log \left( \frac{t_{\text{object}}}{t_\odot} \right)$$

(b) Estimates of Regression Coefficients and their standard errors in power law for Stellar objects.

| Parameter estimate | $a_s$ | $\tau_s$ |
|--------------------|-------|---------|
| Parameter estimate  | $-0.9379$ | $-0.0999$ |
| Standard error of parameter estimate | $0.493505$ | $0.0665$ |

(c) Analysis of variance.

| Source of variation | D.F | S.S. | M.S. | $F$  | $p$ |
|---------------------|-----|------|------|------|-----|
| Regression          | 1   | 1.9258 | 1.9258 | 2.2563 | 0.2719 > 0.01 |
| Residual            | 2   | 1.7071 | 0.8535 |      |    |
| Total               | 3   | 3.6329 |      |      | 14 |

(c) Percentage of variation explained by explanatory variable.

$$R^2 = 53\%$$

$$R^2_{\text{adj}} = 29.5\%$$
TABLE III. Basic Information from the Analysis of Data for Multistellar (ms) Objects assuming a Simple Linear Regression equation and that $\Omega = 1$ and $h = 1$. See also Figure 2.

(a) Predictive equation

$$\log \left( \frac{M_{\text{object}}}{M_\odot} \right) = a_{\text{MS}} + \tau_{\text{MS}} \log \left( \frac{l_{\text{object}}}{l_\odot} \right)$$

(b) Estimates of Regression Coefficients and their standard errors in power law for Multistellar objects.

| Parameter | $a_{\text{MS}}$ | $\tau_{\text{MS}}$ |
|-----------|-----------------|-------------------|
| Parameter estimate | 35.1723 | -2.21274 |
| Standard error of parameter estimate | 4.10181 | 0.139667 |

(c) Analysis of variance.

| Source of variation | D.F | S.S. | M.S. | $F$ | $p$ |
|---------------------|-----|------|------|-----|-----|
| Regression          | 1   | 1331.74 | 1331.74 | 251.01 | $2 \times 10^{-5} << 0.01$ |
| Residual            | 5   | 26.5287 | 5.30574 |      |      |
| Total               | 6   | 1358.27 |      |      |      |

(c) Percentage of variation explained by explanatory variable.

$$R^2 = 98.0\%$$

$$R^2_{\text{adj}} = 97.7\%$$