New results on the constants in some inequalities for the Navier-Stokes quadratic nonlinearity

Carlo Morosi\textsuperscript{a}, Mario Pernici\textsuperscript{b}, Livio Pizzocchero\textsuperscript{c}\textsuperscript{(1)}

\textsuperscript{a} Dipartimento di Matematica, Politecnico di Milano, P.za L. da Vinci 32, I-20133 Milano, Italy
e–mail: carlo.morosi@polimi.it
\textsuperscript{b} Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
e–mail: mario.pernici@mi.infn.it
\textsuperscript{c} Dipartimento di Matematica, Università di Milano Via C. Saldini 50, I-20133 Milano, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy
e–mail: livio.pizzocchero@unimi.it

Abstract

We give fully explicit upper and lower bounds for the constants in two known inequalities related to the quadratic nonlinearity of the incompressible (Euler or) Navier-Stokes equations on the torus $\mathbb{T}^d$. These inequalities are “tame” generalizations (in the sense of Nash-Moser) of the ones analyzed in the previous works [Morosi and Pizzocchero: CPAA 2012, Appl.Math.Lett. 2013].

Keywords: Navier-Stokes equations, inequalities, Sobolev spaces.
AMS 2000 Subject classifications: 76D05, 26D10, 46E35.
1 Introduction

Let us consider the homogeneous incompressible Navier-Stokes (NS) equations on a torus $T^d = (\mathbb{R}/2\pi \mathbb{Z})^d$ of arbitrary dimension; the nonlinear part of these equations is governed by the bilinear map $P$ sending two sufficiently regular vector fields $v, w : T^d \to \mathbb{R}^d$ into

$$P(v, w) := \mathcal{L}(v \cdot \partial w).$$  

(1.1)

In the above $v \cdot \partial w : T^d \to \mathbb{R}^d$ is the vector field of components $(v \cdot \partial w)_s := \sum_{r=1}^d v_r \partial_s w_s$ and $\mathcal{L}$ is the Leray projection onto the space of divergence free vector fields (see Section 2 for more details). Of course the NS equations read

$$\frac{\partial u}{\partial t} = \nu \Delta u - P(u, u) + f, \quad \text{for } t > 0$$

(1.2)

where: $u = u(x, t)$ is the divergence free velocity field, depending on $x \in T^d$ and on time $t$; $\nu \geq 0$ is the kinematic viscosity, $\Delta$ is the Laplacian of $T^d$; $f = f(x, t)$ is the (Leray projected) external force per unit mass. In the inviscid case $\nu = 0$, (1.2) become the Euler equations.

In this paper we focus the attention on certain inequalities fulfilled by $P$ in the framework of Sobolev spaces. For any real $n$, we denote with $\mathbb{H}^n_{\Sigma_0}$ the Sobolev space formed by the (distributional) vector fields $v$ on $T^d$ with vanishing divergence and mean, such that $\sqrt{-\Delta}^n v$ is in $L^2$; this carries the inner product $\langle v | w \rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w \rangle_{L^2}$ and the norm $\| v \|_n := \sqrt{\langle v | v \rangle_n}$ (see the forthcoming Eqs. (2.8) (2.9)). Let $p, n$ be real numbers; it is known that

$$n > d/2, \; v \in \mathbb{H}^n_{\Sigma_0}, \; w \in \mathbb{H}^{n+1}_{\Sigma_0} \quad \Rightarrow \quad P(v, w) \in \mathbb{H}^n_{\Sigma_0}$$

(1.3)

and that there are positive real constants $K_n, G_n, K_{pn}, G_{pn}$ such that:

$$\| P(v, w) \|_n \leq K_n \| v \|_n \| w \|_{n+1} \quad \text{for } n > d/2, \; v \in \mathbb{H}^n_{\Sigma_0}, \; w \in \mathbb{H}^{n+1}_{\Sigma_0},$$

(1.4)

$$| \langle P(v, w) | w \rangle_n | \leq G_n \| v \|_n \| w \|_n^2 \quad \text{for } n > d/2 + 1, \; v \in \mathbb{H}^n_{\Sigma_0}, \; w \in \mathbb{H}^{n+1}_{\Sigma_0},$$

(1.5)

$$\| P(v, w) \|_p \leq \frac{1}{2} K_{pn}(\| v \|_p \| w \|_{n+1} + \| v \|_n \| w \|_{p+1})$$

(1.6)

for $p \geq n > d/2, \; v \in \mathbb{H}^p_{\Sigma_0}, \; w \in \mathbb{H}^{p+1}_{\Sigma_0}$,

$$| \langle P(v, w) | w \rangle_p | \leq \frac{1}{2} G_{pn}(\| v \|_p \| w \|_n + \| v \|_n \| w \|_p) \| w \|_p$$

(1.7)

for $p \geq n > d/2 + 1, \; v \in \mathbb{H}^p_{\Sigma_0}, \; w \in \mathbb{H}^{p+1}_{\Sigma_0}$.

Statements (1.3) (1.4) indicate that $P$ maps continuously $\mathbb{H}^n_{\Sigma_0} \times \mathbb{H}^{n+1}_{\Sigma_0}$ to $\mathbb{H}^n_{\Sigma_0}$ if $n > d/2$. Eq. (1.6) with $p = n$ implies Eq. (1.4), with $K_n := K_{nn}$; similarly, (1.7) with $p = n$ gives (1.5) with $G_n := G_{nn}$. 

1
Eq. (1.4) is closely related to the basic norm inequalities about multiplication in Sobolev spaces, and (1.5) is due to Kato [5]; for these reasons, in [11] [12] the inequalities (1.4) and (1.5) are referred to, respectively, as the “basic” and “Kato” inequalities for $P(2)$. Eqs. (1.6) (1.7) are tame refinements of (1.4) (1.5) (in the general sense given to tameness in studies on the Nash-Moser implicit function theorem [4]). We remark that inequalities very similar to (1.7) are used by Temam in [16], Beale-Kato-Majda in [1] and Robinson-Sadowski-Silva in the recent work [15].

From here to the end of the paper we intend

$$K_n, G_n, K_{pn}, G_{pn} := \text{the sharp constants in (1.4) (1.5) (1.6) (1.7)}$$

(i.e., the minimum constants fulfilling these inequalities). In the previous papers [11] [12], explicit upper and lower bounds were provided for $K_n$ and $G_n$. In the present work we generalize the cited results deriving upper and lower bounds for $K_{pn}$ and $G_{pn}$, for all real $p, n$ as in Eqs. (1.6) (1.7). Our derivations of the upper bounds also give, as byproducts, simple and self-consistent proofs of the related inequalities; the approach proposed follows ideas from Temam [16] and Constantin-Foias [3], making them more quantitative. The lower bounds are obtained substituting suitable trial vector fields in Eqs. (1.6) (1.7).

The relevance of a quantitative information on the constants $K_{pn}, G_{pn}$ is pointed out, e.g., in [14]. In the cited work, the inequalities (1.4) - (1.7) and the constants therein are used to give bounds on the exact $C^\infty$ solution of the NS Cauchy problem with smooth initial data (including the Euler case $\nu = 0$) via the a posteriori analysis of an approximate solution; these estimates concern the interval of existence of the exact solution and its Sobolev distance of any order from the approximate solution. Paper [14] uses systematically the known fact that the space of $C^\infty$ vector fields on $T^d$ with vanishing divergence and mean coincides with $\cap_{p \in \mathbb{R}} H^p_\Sigma^0$; the tame structure of the inequality (1.7) is essential for an efficient implementation of the a posteriori analysis since, after fixing a basic order $n > d/2 + 1$, it induces simple estimates in terms of the Sobolev norms of arbitrary order $p \geq n$. The setting of [14] is in fact a $C^\infty$ variant of the framework introduced in [10] (and inspired by Chernyshenko et al. [2]), where the exact and approximate NS solutions live in a Sobolev space of a given finite order, and the a posteriori analysis is based only on the inequalities (1.4) (1.5). For some applications of the general schemes of [10] [14], in addition to these papers we wish to mention [7] [8] [13].

---

2Due to a remark of [11], we could write the inequality (1.5) and its extension (1.7) using, in place of $P(v, w) = \mathcal{L}(v \cdot \partial w)$, the vector field (with non zero divergence) $v \cdot \partial w$. The cited reference considers the Sobolev space $H^n_\Sigma$ of vector fields $v$ on $T^d$ with vanishing mean and $\nabla \Delta v$ in $L^2$, with the inner product $(v \cdot w)_n := \langle \nabla \Delta v \cdot \nabla \Delta w \rangle_{L^2}$ for any $n > d/2$ and $v \in H^n_\Sigma$, $w \in H^{n+1}_{\Sigma^0}$, one has $v \cdot \partial w \in H^n_\Sigma$, $P(v, w) \in H^n_\Sigma$, and $(P(v, w)|w)_n = (v \cdot \partial w|w)_n$. However, these considerations will play no role in the present paper.
Organization and main results of the paper. Section 2 reviews some basic notations and presents a number of elementary facts about the bilinear map $\mathcal{P}$; one of these facts is proved in Appendix A. The subsequent Sections 3 and 4 present our upper bounds $K_{pm}^+, G_{pm}^+$ for the sharp constants (1.6) and (1.7), respectively; these are described by Theorems 3.3, 4.4 and have the form

$$
K_{pm}^+ = \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbb{Z}^d \setminus \{0\}} \mathcal{K}_{pm}(k)}, \quad G_{pm}^+ = \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbb{Z}^d \setminus \{0\}} \mathcal{G}_{pm}(k)}
$$

(1.9)

where $\mathcal{K}_{pm}, G_{pm} : \mathbb{Z}^d \setminus \{0\} \to [0, +\infty)$ are explicitly given, bounded functions. For each $k$, $\mathcal{K}_{pm}(k)$ and $\mathcal{G}_{pm}(k)$ are infinite (zeta-type) sums over the lattice $\mathbb{Z}^d$ or, to be precise, on $\mathbb{Z}^d \setminus \{0, k\}$: see Eqs. (3.2) (3.9) (1.6) (4.14). Sections 3 and 4 also propose, as preliminary results, some elementary upper bounds on the sups in Eq. (1.9); these imply elementary upper bounds $K_{pm}^{(+)} , G_{pm}^{(+)}$ for $K_{pm}^+$ and $G_{pm}^+$, much rougher than $K_{pm}^+$ and $G_{pm}^+$.

The next step along these lines is the accurate computation of the functions $\mathcal{K}_{pm}, \mathcal{G}_{pm}$ and of their sups. After some preliminaries presented in Section 5 this subject is discussed in detail in Sections 6 and 7; the basic idea is to approximate the infinite sums $\mathcal{K}_{pm}(k), \mathcal{G}_{pm}(k)$ with finite sums over the integer points of suitable balls, giving accurate reminder estimates; in the same spirit, the sups of $\mathcal{K}_{pm}$ and $\mathcal{G}_{pm}$ over $\mathbb{Z}^d$ are approximated with sups over the integer points of a ball, giving again error estimates. This construction finally produces precise upper approximants $K_{pm}^{(+)} , G_{pm}^{(+)}$ for $K_{pm}^+$ and $G_{pm}^+$.

The procedures of Sections 6 and 7 are suitable for automatic computations. Indeed, we have implemented such procedures writing a C program for the computation of the previously mentioned approximants via finite sums, and using Mathematica for some related symbolic and numerical calculations (3). In Section 8 we give some details on the overall procedures, on their computer implementation and, in particular, on the calculations of the following bounds: $K_{pm}^{(+)}$ for

$$
d = 3, \quad n = 2, \quad p = 2, 3, \ldots, 10, \quad (1.10)
$$

and $G_{pm}^{(+)}$ for the cases (1.11).

In Sections 9 and 10 we derive some lower bounds $K_{pm}^{(-)} , G_{pm}^{(-)}$ for $K_{pm}^+, G_{pm}^+$, respectively; as anticipated, these are obtained substituting for $v$ and $w$ in Eqs. (1.6) (1.7) suitable trial vector fields (which are relatively simple, since they have finitely many

$^3$Throughout the paper, an expression like $r = a.bcd\ldots$ means the following: computation of real number $r$ via C or Mathematica produces as an output $a.bcd$, followed by other digits not reported for brevity. As indicated in Section 8, some of the C computations have been validated using the Arb library [17], that gives certified roundoff errors.
nonzero Fourier components). The chosen vector fields often depend on one or more parameters, so the best lower bounds arising from them are obtained by optimization with respect to the parameters. Both Section 9 and 10 exhibit the numerical values of the above mentioned lower bounds or, to be precise, of some lower approximants for them, in the cases (1.10) or (1.11) already considered in connection with the upper bounds. For the reader’s convenience, hereafter we summarize in Tables A, B the numerical values of the bounds $K_{pn}^{(\pm)}, G_{pn}^{(\pm)}$ computed in Sections 8, 9, 10, together with the ratios of the lower to the upper bounds.

Section 9 and 10 also give rougher lower bounds $K_{pn}^{(-)}, G_{pn}^{(-)}$. These can be combined with the rough upper bounds $K_{pn}^{(+)}, G_{pn}^{(+)}$ of Sections 6 and 7 to prove the following statement on the sharp constants $K_{pn}, G_{pn}$: for fixed $(d$ and) $n$,

$$(K_{pn})^{1/p}, (G_{pn})^{1/p} \to 2 \quad \text{for } p \to +\infty ;$$

(1.12)

this concluding result is the subject of Section 11.

Table A. Upper and lower bounds $K_{pn}^{(\pm)}$ on the constants $K_{pn}$, with their ratios, in the cases (1.10) (1.11)

| $(p, n)$ | $K_{pn}^{(-)}$ | $K_{pn}^{(+)}$ | $K_{pn}^{(-)}/K_{pn}^{(+)}$ |
|----------|----------------|----------------|----------------------------|
| (2, 2)   | 0.126          | 0.335          | 0.376...                   |
| (3, 2)   | 0.179          | 0.643          | 0.278...                   |
| (4, 2)   | 0.264          | 0.831          | 0.317...                   |
| (5, 2)   | 0.463          | 1.16           | 0.339...                   |
| (6, 2)   | 0.793          | 1.94           | 0.408...                   |
| (7, 2)   | 1.33           | 3.02           | 0.440...                   |
| (8, 2)   | 2.20           | 5.07           | 0.433...                   |
| (9, 2)   | 3.60           | 8.54           | 0.421...                   |
| (10, 2)  | 5.83           | 14.5           | 0.402...                   |
| (3, 3)   | 0.179          | 0.320          | 0.559...                   |
| (4, 3)   | 0.253          | 0.539          | 0.469...                   |
| (5, 3)   | 0.418          | 0.909          | 0.459...                   |
| (6, 3)   | 0.732          | 1.54           | 0.475...                   |
| (7, 3)   | 1.25           | 2.58           | 0.484...                   |
| (8, 3)   | 2.10           | 4.28           | 0.490...                   |
| (9, 3)   | 3.48           | 7.04           | 0.494...                   |
| (10, 3)  | 5.69           | 11.5           | 0.494...                   |
Table B. Upper and lower bounds $G_{pn}^{(\pm)}$ on the constants $G_{pn}$, with their ratios, in the cases (1.11)

| $(p, n)$ | $G_{pn}^{(-)}$ | $G_{pn}^{(+)}$ | $G_{pn}^{(-)}/G_{pn}^{(+)}$ |
|----------|----------------|----------------|-----------------------------|
| (3, 3)   | 0.121          | 0.438          | 0.276...                    |
| (4, 3)   | 0.235          | 1.03           | 0.228...                    |
| (5, 3)   | 0.408          | 1.26           | 0.323...                    |
| (6, 3)   | 0.674          | 2.06           | 0.327...                    |
| (7, 3)   | 1.08           | 3.58           | 0.301...                    |
| (8, 3)   | 1.74           | 5.68           | 0.306...                    |
| (9, 3)   | 2.77           | 9.64           | 0.287...                    |
| (10, 3)  | 4.40           | 16.4           | 0.268...                    |

2 Preliminaries

Throughout the paper we work in any dimension $d \in \{2, 3, \ldots\}$.

Some notations. For $a, b \in \mathbb{C}^d$ we write $a \cdot b := \sum_{r=1}^d a_r b_r$, $\mathbf{a} := (a_r)_{r=1, \ldots, d}$ and $|a| := \sqrt{\mathbf{a} \cdot \mathbf{a}}$. We often consider the torus $T^d := (\mathbb{R}/2\pi \mathbb{Z})^d$ and the lattice $\mathbb{Z}^d$, associated to it in Fourier analysis. For $\ell, k$ in $\mathbb{Z}^d$ we put

$$Z^d_\ell := Z^d \setminus \{\ell\} ; \quad Z^d_{\ell k} := Z^d \setminus \{\ell, k\} .$$

(2.1)

Function spaces. When working on $T^d$, we often use the Fourier basis $(e_k)_{k \in \mathbb{Z}^d}$, where

$$e_k(x) := \frac{e^{i k \cdot x}}{(2\pi)^{d/2}}$$

for $x \in T^d$. (2.2)

The space of $\mathbb{R}^d$-valued distributions on $T^d$ is denoted with

$$D'(T^d, \mathbb{R}^d) \equiv \mathcal{D}'(T^d) \equiv \mathcal{D}' ;$$

(2.3)

each $v \in \mathcal{D}'$ has a weakly convergent Fourier expansion $v = \sum_{k \in \mathbb{Z}^d} v_k e_k$, with Fourier coefficients $v_k = \overline{v_{-k}} \in \mathbb{C}^d$.

We have a divergence operator $\text{div} : \mathcal{D}' \rightarrow D'$, $v \mapsto \text{div} v$ where $D' \equiv D'(\mathbb{T}^d, \mathbb{R})$ is the space of real distributions; this has the Fourier representation, of obvious meaning, $(\text{div}v)_k = k \cdot v_k$. For $v \in \mathcal{D}'$ the mean value $\langle v \rangle \in \mathbb{R}^d$ is, by definition, the action of $v$ on the constant test function $(2\pi)^{-d}$, and $\langle v \rangle = (2\pi)^{-d/2} v_0$. If $X$ is the space $\mathcal{D}'$ or any vector subspace of it we put

$$X_\Sigma := \{v \in X \mid \text{div} v = 0\} = \{v \in X \mid k \cdot v_k = 0 \text{ for } k \in \mathbb{Z}^d\} ,$$

(2.4)

$$X_0 := \{v \in X \mid \langle v \rangle = 0\} = \{v \in X \mid v_0 = 0\} ,$$

(2.5)

$$X_{\Sigma 0} := X_\Sigma \cap X_0 .$$

(2.6)
The Laplacian is an operator \( \Delta : \mathbb{D}' \to \mathbb{D}'_\Sigma, \ v \mapsto \Delta v \) with the Fourier representation \( (\Delta v)_k = -|k|^2v_k \). If \( v \in \mathbb{D}'_0 \) and \( n \in \mathbb{R} \), we define \( \sqrt{-\Delta^n} v \) to be the element of \( \mathbb{D}'_0 \) with Fourier coefficients \( (\sqrt{-\Delta^n} v)_k = |k|^nv_k \) for all \( k \in \mathbb{Z}_0^d \).

In the sequel we consider the spaces

\[
L^p(T^d; \mathbb{R}^d) \equiv L^p(T^d) \equiv L^p,
\]

most frequently in the Hilbertian case \( p = 2 \). The notations \( L^p_\Sigma, L^p_0, L^p_{\Sigma_0} \) are intended according to Eqs. (2.4)-(2.6). For any \( k \in \mathbb{R} \), we introduce the Sobolev space

\[
H^m_{\Sigma_0} = H^m_{\Sigma_0} := \{ v \in \mathbb{D}'_\Sigma | \ \text{div} v = 0, \ \langle v \rangle = 0, \ \sqrt{-\Delta^n} v \in L^2 \} \equiv \{ v \in \mathbb{D}' | k \bullet v_k = 0 \ \forall k \in \mathbb{Z}^d, \ v_0 = 0, \ \sum_{k \in \mathbb{Z}_0^d} |k|^{2n} |v_k|^2 < +\infty \};
\]

this is equipped with the inner product and with the induced norm

\[
\langle v | w \rangle_n := \langle \sqrt{-\Delta^n}v | \sqrt{-\Delta^n}w \rangle_{L^2} = \sum_{k \in \mathbb{Z}_0^d} |k|^{2n}v_k \bullet w_k, \ \| v \|_n := \sqrt{\langle v \rangle_n}. \tag{2.9}
\]

Let \( n, n', m \in \mathbb{R} \). One has \( H^m_{\Sigma_0} \subset H^n_{\Sigma_0} \) and \( \| \|_n \leq \| \|_{n'} \) if \( n \leq n' \); moreover \( \sqrt{-\Delta^m}H^m_{\Sigma_0} = H^m_{\Sigma_0} \). By the standard Sobolev lemma, \( H^m_{\Sigma_0} \) is embedded continuously in \( L^\infty_{\Sigma_0} \) if \( n > d/2 \).

**Leray projection.** This is the map

\[
\mathcal{L} : \mathbb{D}' \to \mathbb{D}'_\Sigma, \ \ v \mapsto \mathcal{L} v \tag{2.10}
\]

defined via the Fourier representation

\[
(\mathcal{L}v)_k := \mathcal{L}_kv_k \text{ for all } v \in \mathbb{D}', \ k \in \mathbb{Z}^d, \tag{2.11}
\]

\[
\mathcal{L}_k : C^d \mapsto C^d \text{ the orthogonal projection of } C^d \text{ onto } k^\perp. \tag{2.12}
\]

Of course, orthogonality in \( C^d \) is defined in terms of the inner product sending \( a, b \in C^d \) into \( a \bullet b; \ k^\perp \) is the orthogonal complement of \( k \), i.e., \( k^\perp := \{ a \in C^d | k \bullet a = 0 \} \).

If \( c \in C^d \), one has

\[
\mathcal{L}_kc = c - \frac{k \bullet c}{|k|^2} k \text{ for all } k \in \mathbb{Z}_0^d, \ \ \mathcal{L}_0c = c. \tag{2.13}
\]

From the Fourier representation it is evident that

\[
\mathcal{L} \mathbb{D} = \mathbb{D}'_\Sigma, \ \ \mathcal{L} \mathbb{D}'_\Sigma = 1_{\mathbb{D}'_\Sigma}, \ \ \mathcal{L} \mathbb{D}'_0 = \mathbb{D}'_{\Sigma_0}, \ \ \mathcal{L} L^2 = L^2_{\Sigma_0}, \ \ \mathcal{L} L^2_0 = L^2_{\Sigma_0}, \tag{2.14}
\]

\[
\| \mathcal{L}v \|_{L^2} \leq \| v \|_{L^2}, \ \ \langle \mathcal{L}v | w \rangle_{L^2} = \langle v | \mathcal{L}w \rangle_{L^2} \text{ for } v, w \in L^2. \tag{2.15}
\]
The NS bilinear map $\mathcal{P}$. We are now ready to define precisely the map (1.1). Let us consider two vector fields $v \in L^2$, $w \in \mathbb{D}'$ such that $\partial_r w \in L^2$ for $r = 1, \ldots, d$ (2.16) (which implies $w \in L^2$). Then we can define the vector field $v \partial w$ of components

\[(v \partial w)_s := \sum_{r=1}^d v_r \partial_r w_s,\]

that fulfills

\[v \partial w \in L^1; \quad (2.17)\]

we can define as well

\[\mathcal{P}(v, w) := \mathcal{L}(v \partial w) \in \mathcal{L}L^1. \quad (2.18)\]

The Fourier coefficients of these vector fields are obtained by elementary manipulations, and are as follows (see, e.g., [11]):

\[(v \partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}^d} (v_h \cdot (k - h)) w_{k-h} \quad (2.19)\]

\[\mathcal{P}(v, w)_k = (\mathcal{L}(v \partial w))_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}^d} (v_h \cdot (k - h)) \mathcal{L}w_{k-h} \quad (2.20)\]

for all $k \in \mathbb{Z}^d$. (In the above sums over $h$, one can replace $\mathbb{Z}^d$ with $\mathbb{Z}^d_k = \mathbb{Z}^d \setminus \{k\}$; if $v$ has zero mean, one can replace $\mathbb{Z}^d$ with $\mathbb{Z}^d_{0_k} = \mathbb{Z}^d \setminus \{0, k\}$.) One also proves that

\[\langle v \partial w \rangle = \langle \mathcal{P}(v, w) \rangle = 0 \quad \text{if } \text{div} v = 0 \quad (2.21)\]

(see again [11], Lemma 2.1). Of course, the maps sending $v, w$ as in Eq. (2.16) into $v \partial w$ and $\mathcal{P}(v, w)$ are bilinear. Let us go on making the stronger assumption

\[v \in L^\infty, \quad w \text{ as in } (2.16); \quad (2.22)\]

then

\[v \partial w \in L^2, \quad \mathcal{P}(v, w) \in \mathcal{L}L^2 \quad (2.23)\]

and, on account of (2.21),

\[v \partial w \in L^2, \quad \mathcal{P}(v, w) \in \mathcal{L}L^2 \quad \text{if } \text{div} v = 0. \quad (2.24)\]

Let us also mention that

\[\langle v \partial w | w \rangle_{L^2} = 0 \quad \text{if } \text{div} v = 0, \quad (2.25)\]

\[\langle \mathcal{P}(v, w) | w \rangle_{L^2} = 0 \quad \text{if } \text{div} v = 0, \text{ div} w = 0. \quad (2.26)\]
Concerning Eq. (2.25) see, e.g., Lemma 2.3 of [11]; once we have (2.25), assuming \( \text{div} \ w = 0 \) we infer \( w = Lw \) and \( 0 = \langle v \bullet \partial w | Lw \rangle_{L^2} = \langle L(v \bullet \partial w) | w \rangle_{L^2} = \langle P(v, w) | w \rangle_{L^2} \), whence Eq. (2.26).

The bilinear maps \( P_{h\ell} \) and their norms. Eq. (2.20) contains the expression \( (v_h \cdot (k - h)) L_{k-h} w \) which has the form \( (a \ell) L_{h+\ell} b \) where \( \ell := k - h \in \mathbb{Z}^d \) and \( a := v_h, b := w_{\ell} \); if \( \text{div} v = 0 \) and \( \text{div} w = 0 \) we have \( a \in h^\perp, b \in \ell^\perp \). We fix the attention on the normalized expression \( \frac{a \ell}{|\ell|} L_{h+\ell} b \) as a function of \( a, b \); more precisely we consider, for \( h, \ell \in \mathbb{Z}_0^d \), the map

\[
P_{h\ell} : h^\perp \times \ell^\perp \to (h + \ell)^\perp, \quad (a, b) \mapsto P_{h\ell}(a, b) := \frac{a \ell}{|\ell|} L_{h+\ell} b . \tag{2.27}
\]

This is a bilinear map between the finite dimensional spaces indicated above (all of them subspaces of \( \mathbb{C}^d \)); of course, Eq. (2.20) (and the remarks that follow it) indicate that

\[
\mathcal{P}(v, w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_0^d} |k - h| P_{h,k-h}(v_h, w_{k-h}) \tag{2.28}
\]

if \( v, \partial_1 w, ..., \partial_d w \in L^2, v_0 = 0 \) and \( \text{div} v = \text{div} w = 0 \).

For arbitrary \( h, \ell \in \mathbb{Z}_0^d \) we can introduce the norm

\[
|P_{h\ell}| := \min\{Q \in [0, +\infty) | |P_{h\ell}(a, b)| \leq Q |a| |b| \text{ for all } a \in h^\perp, b \in \ell^\perp\} . \tag{2.29}
\]

The above norm can be computed explicitly, as shown in Appendix A. Indeed, denoting with \( \vartheta_{qr} \in [0, \pi] \) the convex angle between any two vectors \( q, r \in \mathbb{R}^d \setminus \{0\} \), we have

\[
|P_{h\ell}| = \begin{cases} 
\sin \vartheta_{h\ell} & \text{if } d \geq 3 , \\
\sin \vartheta_{h\ell} \cos \vartheta_{h+\ell,\ell} & \text{if } d = 2 
\end{cases} \tag{2.30}
\]

(where \( \vartheta_{h+\ell,\ell} \) indicates any angle in \([0, \pi]\) if \( h + \ell = 0 \); in this situation \( \vartheta_{h\ell} = \pi \), so \( \sin \vartheta_{h\ell} \cos \vartheta_{h+\ell,\ell} = 0 \)). In any case we have the bounds

\[
|P_{h\ell}| \leq \sin \vartheta_{h\ell} \leq 1 , \tag{2.31}
\]

to be used in the sequel according to convenience. Let us also remark that Eq. (2.30) implies

\[
|P_{h}\ell| = |P_{h\ell}| \quad \text{if } d \geq 3 . \tag{2.32}
\]

An obvious remark. As already declared in the Introduction, in this paper we are mainly interested in \( \mathcal{P}(v, w) \) for \( v \in \mathbb{H}_0^p, w \in \mathbb{H}_0^{p+1} \) and \( p > d/2 \). In this case all the

\[4\text{Of course, } \cos \vartheta_{qr} = \frac{q \cdot r}{|q||r|} \text{ and } \sin \vartheta_{qr} = \sqrt{1 - \frac{(q \cdot r)^2}{|q|^2|r|^2}}.\]
conditions on $v, w$ appearing in Eqs. (2.16)–(2.22) and (2.24)–(2.26) are satisfied (in particular, $v \in L^\infty$ by the Sobolev embedding lemma); thus $v_\ast \partial w, \mathcal{P}(v, w)$ are well defined and possess all the properties listed in Eqs. (2.19)–(2.21) and (2.23)–(2.26). This suffices to infer $\mathcal{P}(v, w) \in L^2_{\Sigma_0}$; the stronger statement $\mathcal{P}(v, w) \in H^p_{\Sigma_0}$ is proved explicitly in the next section.

3 The inequality (1.6); upper bounds for its sharp constant $K_{pn}$

In this section we systematically refer to the maps $P_{h,\ell}$ defined in (2.27), and to their norms described by Eqs. (2.29)–(2.30). Moreover, we consider $p, n \in \mathbb{R}$ such that $p \geq n > d/2$.

3.1 Proposition. One can define a function

$$K_{pnd} \equiv K_{pn} : \mathbb{Z}^d_0 \to (0, +\infty),$$

$$k \mapsto K_{pn}(k) := 4|k|^{2p} \sum_{h \in \mathbb{Z}^d_0} \frac{|P_{h,k-h}|^2}{(|h|^p k^h - h^n + |h|^n |k-h|^p)^2};$$

in fact, the sum written above is finite for all $k \in \mathbb{Z}^d_0$. Moreover

$$\sup_{k \in \mathbb{Z}^d_0} K_{pn}(k) \leq 2^{2p+2} \zeta_{2n}, \quad \zeta_{2n} := \sum_{h \in \mathbb{Z}^d_0} \frac{1}{|h|^{2n}}$$

(note that $\zeta_{2n} < +\infty$, since $2n > d$).

Proof. Let $k \in \mathbb{Z}^d_0$. The sum in Eq. (3.2) and $K_{pn}(k)$ certainly exist as elements of $[0, +\infty]$; the same can be said of the other sums appearing in the proof.

Hereafter we derive an upper bound on $K_{pn}(k)$ yielding Eq. (3.3) (and ensuring, a fortiori, the finiteness of $K_{pn}(k)$). To this purpose we note the following: for all $h \in \mathbb{Z}^d$ one has $k = (k-h) + h$, whence $|k| \leq |k-h| + |h|$ and

$$|k|^{2p} \leq (|k-h| + |h|)^{2p} \leq 2^{2p-1}(|k-h|^{2p} + |h|^{2p})$$

(in the last inequality we have used the fact that $(x+y)^q \leq 2^{q-1}(x^q + y^q)$ for $q \in [1, +\infty)$ and $x, y \in [0, +\infty)$). Inserting (3.4) and the bound $|P_{h,k-h}| \leq 1$ (see Eq. (2.31)) into the definition (3.2) of $K_{pn}(k)$ we get

$$K_{pn}(k) \leq 2^{2p+1} \left( \sum_{h \in \mathbb{Z}^d_0} \frac{|k-h|^{2p}}{(|h|^p |k-h|^n + |h|^n |k-h|^p)^2} + \sum_{h \in \mathbb{Z}^d_0} \frac{|h|^{2p}}{(|h|^p |k-h|^n + |h|^n |k-h|^p)^2} \right).$$
The second sum becomes the first one after a change of variable \( h \to k - h \); thus

\[
K_{pn}(k) \leq 2^{2p+2} \sum_{h \in \mathbb{Z}^d_0} \frac{|k-h|^{2p}}{(|h|^p|k-h|^n + |h|^n|k-h|^p)^2}
\]

and since \(|h|^p|k-h|^n + |h|^n|k-h|^p \geq |h|^n|k-h|^p\) we get

\[
K_{pn}(k) \leq 2^{2p+2} \sum_{h \in \mathbb{Z}^d_0} \frac{1}{|h|^{2n}} = 2^{2p+2} \zeta_{2n}.
\]

(3.5)

This concludes the proof. \( \square \)

3.2 Remarks. (i) Let us generalize a remark presented in [12] about the constant \( K_{pn} \) for \( p = n \). To this purpose, if \( r \in \{1,...,d\} \) and \( \sigma \) is any permutation of \( \{1,...,d\} \), we define the reflection operator \( R_r \) and the permutation operator \( P_\sigma \) setting

\[
R_r, P_\sigma : \mathbb{R}^d \to \mathbb{R}^d,
\]

\[
R_r(k_1,...,k_r,...,k_d) := (k_1,...,-k_r,...,k_d), \quad P_\sigma(k_1,...,k_d) := (k_{\sigma(1)},...,k_{\sigma(d)})
\]

these are orthogonal operators (with respect to the inner product \( \cdot \) of \( \mathbb{R}^d \)), sending \( \mathbb{Z}^d_0 \) into itself. One easily checks that the function \( K_{pn} \) in (3.2) fulfills

\[
K_{pn}(R_r k) = K_{pn}(k), \quad K_{pn}(P_\sigma k) = K_{pn}(k) \quad \text{for each } k \in \mathbb{Z}^d_0
\]

(3.7) (indeed, the norms \(|k|, |h|, |k-h|, |P_{h,k-h}| \) in the definition of \( K_{pn}(k) \) do not change if an orthogonal operator is applied to \( h \) and \( k \)). Due to (3.7), the computation of \( K_{pn}(k) \) can always be reduced to the case \( k_1 \geq k_2 \geq ... \geq k_d \geq 0 \).

(ii) Typically, the bound (3.3) on \( \sup K_{pn} \) is very rough; in the subsequent Sections 6 and 8 we present much more accurate estimates on this sup, based on a lengthy analysis of the function \( K_{pn} \). As an example, let \( d = n = 3, p = 10 \). Then the bound (3.3) reads \( \sup_{k \in \mathbb{Z}^d_0} K_{10,3}(k) \leq 3.53 \times 10^7 \); on the other hand, the methods of Section 6 and their numerical implementation in Section 8 give \( \sup_{k \in \mathbb{Z}^d_0} K_{10,3}(k) \leq 3.27 \times 10^4 \).

Nevertheless, the bound (3.3) is not useless; we return to it at the end of this section (see Corollary 3.4) and, especially, in Section 11.

3.3 Theorem. Let \( v \in \mathbb{H}^p_{\Sigma_0} \) and \( w \in \mathbb{H}^{p+1}_{\Sigma_0} \). Then \( \mathcal{P}(v, w) \in \mathbb{H}^p_{\Sigma_0} \) and

\[
\|\mathcal{P}(v, w)\|_p \leq \frac{1}{2} K^+_{pn}(\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1})
\]

(3.8)

where

\[
K^+_{pnd} \equiv K^+_{pn} := \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbb{Z}^d_0} K_{pn}(k)}
\]

(3.9)
and $\mathcal{K}_{pn}$ is the function defined by (3.2). So, the inequality (1.9) holds and its sharp constant $K_{pnd} \equiv K_{pn}$ is such that

$$K_{pn} \leq K_{pn}^+.$$  

**Proof.** Let us start from the relation (2.28)

$$\mathcal{P}(v, w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_{0k}^d} |k - h| P_{h,k-h}(v_h, w_{k-h});$$

we note that $\mathcal{P}(v, w)$ has zero mean due to (2.21), and is divergence free by construction. Let us fix $k \in \mathbb{Z}_{0k}^d$ from Eqs. (2.28) (2.29) we infer

$$|\mathcal{P}(v, w)_k| \leq \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_{0k}^d} |k - h| |P_{h,k-h}| |v_h||w_{k-h}|$$

we see that $\mathcal{P}(v, w)$ has zero mean due to (2.21), and is divergence free by construction. Let us fix $k \in \mathbb{Z}_{0k}^d$ from Eqs. (2.28) (2.29) we infer

$$|\mathcal{P}(v, w)_k| \leq \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_{0k}^d} |k - h| |P_{h,k-h}| |v_h||w_{k-h}|$$

and the Cauchy inequality $\sum_h a_h b_h \leq \sqrt{\sum_h a_h^2} \sqrt{\sum_h b_h^2} (a_h, b_h \in [0, +\infty))$ gives

$$|\mathcal{P}(v, w)_k| \leq \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_{0k}^d} \frac{4|P_{h,k-h}|^2}{(|h|^p|k - h|^n + |h|^n|k - h|^{p+1})^2}$$

$$\times \sqrt{\sum_{h \in \mathbb{Z}_{0k}^d} \frac{(|h|^p|k - h|^n + |h|^n|k - h|^{p+1})^2|v_h|^2|w_{k-h}|^2}.$$ 

Multiplying by $|k|^p$ and comparing with the definition (3.2) of $\mathcal{K}_{pn}$ we see that

$$|k|^p|\mathcal{P}(v, w)_k| \leq \sqrt{\mathcal{K}_{pn}(k)} \sqrt{\sum_{h \in \mathbb{Z}_{0k}^d} (|h|^p|k - h|^n + |h|^n|k - h|^{p+1})^2|v_h|^2|w_{k-h}|^2},$$

i.e.,

$$|k|^p|\mathcal{P}(v, w)_k| \leq \frac{\sqrt{\mathcal{K}_{pn}(k)}}{2(2\pi)^{d/2}} \sqrt{\sum_{h \in \mathbb{Z}_{0k}^d} (a_{kh} + b_{kh})^2}$$

$$a_{kh} := |h|^p|k - h|^n|v_h||w_{k-h}|, \quad b_{kh} := |h|^n|k - h|^{p+1}|v_h||w_{k-h}|.$$ 

But $\sqrt{\sum_{h \in \mathbb{Z}_{0k}^d} (a_{kh} + b_{kh})^2} \leq \sqrt{\sum_{h \in \mathbb{Z}_{0k}^d} a_{kh}^2} + \sqrt{\sum_{h \in \mathbb{Z}_{0k}^d} b_{kh}^2},$ so

$$\sum_{h \in \mathbb{Z}_{0k}^d} (a_{kh} + b_{kh})^2 \leq \sum_{h \in \mathbb{Z}_{0k}^d} a_{kh}^2 + \sum_{h \in \mathbb{Z}_{0k}^d} b_{kh}^2.$$
\[ |k|^p |\mathcal{P}(v, w)_k| \leq \frac{\sqrt{K_{pn}(k)}}{2(2\pi)^{d/2}} (q_k + p_k), \quad (3.14) \]

\[
q_k := \sum_{h \in \mathbb{Z}_0^d} |h|^{2p} |k - h|^{2(n+1)} |v_h|^2 |w_{k-h}|^2, \quad p_k := \sum_{h \in \mathbb{Z}_0^d} |h|^{2p} |k - h|^{2(p+1)} |v_h|^2 |w_{k-h}|^2.
\]

To go on we note that, according to (3.9), \((2\pi)^{-d/2} \sqrt{K_{pn}(k)} \leq K_{pn}^+\); thus

\[
|k|^p |\mathcal{P}(v, w)_k| \leq \frac{K_{pn}^+}{2} (q_k + p_k), \quad (3.15)
\]

which implies

\[
\sqrt{\sum_{k \in \mathbb{Z}_0^d} |k|^{2p} |\mathcal{P}(v, w)_k|^2} \leq \frac{K_{pn}^+}{2} \left( \sqrt{\sum_{k \in \mathbb{Z}_0^d} q_k^2} + \sqrt{\sum_{k \in \mathbb{Z}_0^d} p_k^2} \right). \quad (3.16)
\]

On the other hand, the definition of \(q_k\) in (3.14) gives

\[
\sum_{k \in \mathbb{Z}_0^d} q_k^2 = \sum_{(k,h) \in \mathbb{Z}_0^d \times \mathbb{Z}_0^d, k \neq h} |h|^{2p} |k-h|^{2(n+1)} |v_{h}|^2 |w_{k-h}|^2 \quad (3.17)
\]

\[
= \sum_{(h,\ell) \in \mathbb{Z}_0^d \times \mathbb{Z}_0^d, \ell \neq -h} |h|^{2p} |v_{h}|^2 |\ell|^{2(n+1)} |w_{\ell}|^2 \leq \sum_{(h,\ell) \in \mathbb{Z}_0^d \times \mathbb{Z}_0^d} |h|^{2p} |v_{h}|^2 |\ell|^{2(n+1)} |w_{\ell}|^2 = \|v\|_p^2 \|w\|_{n+1}^2.
\]

Similarly

\[
\sum_{k \in \mathbb{Z}_0^d} p_k^2 \leq \|v\|_n^2 \|w\|_{p+1}^2 ; \quad (3.18)
\]

inserting these results into (3.16) we get

\[
\sqrt{\sum_{k \in \mathbb{Z}_0^d} |k|^{2p} |\mathcal{P}(v, w)_k|^2} \leq \frac{K_{pn}^+}{2} \left( \|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1} \right). \quad (3.19)
\]

This proves that \(\mathcal{P}(v, w) \in \mathbb{H}_0^p\) (we have already noted the vanishing of the mean and divergence of this vector field); moreover \((3.8)\) is found to hold, with \(K_{pn}^+\) as in Eq. (3.9). \(\square\)

Theorem 3.3 gives an upper bound on \(K_{pn}\) in terms of \(\sup K_{pn}\); we have anticipated that an accurate evaluation of this sup requires a lengthy analysis, occupying Section 6. However, at present we have the bound in Proposition 3.1 on \(\sup K_{pn}\), whose roughness has been emphasized in Remark 3.2 (ii). Using this rough estimate, we obtain the following from the cited propositions.
3.4 Corollary. The sharp constant $K_{pn}$ of (1.6) has the bound

$$K_{pn} \leq K_{pn}^{(+)}, \quad K_{pn}^{(+)} := \frac{2^{p+1}}{(2\pi)^{d/2}} \sqrt{\zeta_{2n}}$$

(with $\zeta_{2n}$ as in (3.3); note that $(K_{pn}^{(+)})^{1/p} \to 2$ for fixed $d, n$ and $p \to +\infty$).

Proof. In fact, Eqs. (3.9) (3.10) and (3.3) give

$$K_{pn} \leq \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbb{Z}^d} K_{pn}(k)} \leq \frac{1}{(2\pi)^{d/2}} \sqrt{2^{2p+2}\zeta_{2n}},$$

whence the thesis (3.20).

In spite of its roughness, the upper bound (3.20) on $K_{pn}$ has its own theoretical interest; in fact, as shown in Section 11, the combination of (3.20) with a suitable lower bound can be used to evaluate $\lim_{p \to +\infty} (K_{pn})^{1/p}$.

4 The inequality (1.7); upper bounds for its sharp constant $G_{pn}$

The derivation of the generalized Kato inequality (1.7) proposed hereafter is rather similar, in the special case $p = n$, to the one given in [11]. Both in [11] and herein, we refine and make a bit more quantitative some basic ideas expressed by Temam [16] and Constantin-Foias (see [3], Chapter 10).

Let us start from an elementary inequality, very similar to some relations presented in [3] [16], whose proof is reported only for completeness.

4.1 Lemma. Consider a real $p \geq 1$. Then

$$|b|^p - |a|^p \leq p |b - a| \max(|b|, |a|)^{p-1} \quad \text{for } a, b \in \mathbb{R}^d. \quad (4.1)$$

Proof. It suffices to prove the inequality (4.1) with the assumptions

$$a, b \in \mathbb{R}^d, \quad 0 \not\in [a, b], \quad (4.2)$$

where $[a, b]$ is the segment of $\mathbb{R}^d$ with endpoints $a, b$; the inequality is subsequently extended to the case $0 \in [a, b]$ by elementary continuity considerations. Assuming (4.2), let us consider the function

$$F_p : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}, \quad u \mapsto F_p(u) := |u|^p, \quad (4.3)$$

which is $C^\infty$ with

$$\nabla F_p(u) = p|u|^{p-2}u. \quad (4.4)$$
We have
\[ |b|^p - |a|^p = |F_p(b) - F_p(a)| \leq |b - a| \max_{u \in [a,b]} |\text{grad} F_p(u)| \]
\[ = p|b - a| \max |u|^{p-1} = p|b - a| \max(|b|, |a|)^{p-1}. \]

From here to the end of the section we consider
\[ p, n \in \mathbb{R} \text{ such that } p \geq n > d/2 + 1. \tag{4.5} \]

### 4.2 Proposition
One can define a function
\[ G_{pn} \equiv G_{pn} : \mathbb{Z}_0^d \to (0, +\infty), \tag{4.6} \]
\[ k \mapsto G_{pn}(k) := 4 \sum_{h \in \mathbb{Z}_0^d} \frac{(|k|^p - |k - h|^p)^2 |P_{h,k-h}|^2}{(|h|^p|k - h|^n - 1 + |h|^n|k - h|^{p-1})^2}; \]

in fact, the sum written above is finite for all \( k \in \mathbb{Z}_0^d \). Moreover
\[ \sup_{k \in \mathbb{Z}_0^d} G_{pn}(k) \leq 2^{2p} p^2 \zeta_{2n-2}, \quad \zeta_{2n-2} := \sum_{h \in \mathbb{Z}_0^d} \frac{1}{|h|^{2n-2}} \tag{4.7} \]
(note that \( \zeta_{2n-2} < +\infty, \text{ since } 2n - 2 > d \)).

**Proof.** Let \( k \in \mathbb{Z}_0^d \). The sum in Eq. (4.6) and \( G_{pn}(k) \) certainly exist as elements of \([0, +\infty]\); the same can be said of the other sums that appearing in the proof.

Hereafter we derive an upper bound on \( G_{pn}(k) \), yielding Eq. (4.7) (and ensuring, a fortiori, the finiteness of \( G_{pn}(k) \)). To this purpose we note that, for all \( h \in \mathbb{Z}^d \), the inequality (4.1) with \( b = k \) and \( a = k - h \) gives
\[ (|k|^p - |k - h|^p)^2 \leq p^2 |h|^2 \max(|k|, |k - h|)^{2p-2} \tag{4.8} \]
\[ \leq p^2 |h|^2 \max(|k - h| + |h|, |k - h|)^{2p-2} = p^2 |h|^2 (|k - h| + |h|)^{2p-2} \]
\[ \leq 2^{2p-3} p^2 |h|^2 (|k - h|^{2p-2} + |h|^{2p-2}) \]
(concerning the last inequality, see the comment after Eq. (3.1)). Inserting this inequality and the relation \( |P_{h,k-h}| \leq 1 \) (see Eq. (2.31)) into the definition (4.6) of \( G_{pn}(k) \) we get
\[ G_{pn}(k) \leq 2^{2p-1}p^2 \sum_{h \in \mathbb{Z}_+^d} \frac{|h|^2(|k-h|^{2p-2} + |h|^{2p-2})}{(|h|^p|k-h|^{n-1} + |h|^{n-1}|k-h|^{p-1})^2} \]  

(4.9)

\[ = 2^{2p-1}p^2 \left( \sum_{h \in \mathbb{Z}_+^d} \frac{|k-h|^{2p-2}}{(|h|^p|k-h|^{n-1} + |h|^{n-1}|k-h|^{p-1})^2} \right) + \sum_{h \in \mathbb{Z}_+^d} \frac{|h|^{2p-2}}{(|h|^p|k-h|^{n-1} + |h|^{n-1}|k-h|^{p-1})^2} \right) ; \]

the second sum above becomes the first one after a change of variable \( h \to k-h \), and thus

\[ G_{pn}(k) \leq 2^{2p}p^2 \sum_{h \in \mathbb{Z}_+^d} \frac{|k-h|^{2p-2}}{(|h|^p|k-h|^{n-1} + |h|^{n-1}|k-h|^{p-1})^2} . \]  

(4.10)

Since \(|h|^p|k-h|^{n-1} + |h|^{n-1}|k-h|^{p-1} \geq |h|^{n-1}|k-h|^{p-1} \), we get

\[ G_{pn}(k) \leq 2^{2p}p^2 \sum_{h \in \mathbb{Z}_+^d} \frac{1}{|h|^{2n-2}} = 2^{2p}p^2 \zeta_{2n-2} ; \]  

(4.11)

this concludes the proof. \( \square \)

4.3 Remarks. The forthcoming comments (i)(ii) are quite similar to Remarks 3.2 about the function \( K_{pn} \) and its sup.

(i) Let \( r \in \{1, \ldots, d\} \), and let \( \sigma \) denote a permutation of \( \{1, \ldots, d\} \); denoting with \( R_r \) and \( P_\sigma \) the reflection and permutation operators 3.6, we have

\[ G_{pn}(R_k k) = G_{pn}(k) , \quad G_{pn}(P_\sigma k) = G_{pn}(k) \quad \text{for each } k \in \mathbb{Z}_+^d . \]  

(4.12)

So, the computation of \( G_{pn}(k) \) can be reduced to the case \( k_1 \geq k_2 \ldots \geq k_d \geq 0 \).

(ii) The bound (4.7) on sup \( G_{pn} \) is very rough; in Sections 7 and 8 we present much more accurate estimates on this sup, based on a lengthy analysis of the function \( G_{pn} \). As an example, let \( d = n = 3, p = 10 \). Then the bound (3.3) reads \( \sup_{k \in \mathbb{Z}_+^d} G_{10,3}(k) \leq 1.74 \times 10^9 \); on the other hand, the methods of Section 7 and their numerical implementation in Section 8 give \( \sup_{k \in \mathbb{Z}_+^d} G_{10,3}(k) \leq 6.64 \times 10^4 \).

4.4 Theorem. Let \( v \in \mathbb{H}_+^{p+1} \) and \( w \in \mathbb{H}_+^{p+1} \) (so that \( \mathcal{P}(v, w) \in \mathbb{H}_+^{p+1} \)). Then

\[ \| \langle \mathcal{P}(v, w) | w \rangle_p \| \leq \frac{1}{2} G_{pn}^2(\|v\|_p \|w\|_n + \|v\|_n \|w\|_p) \|w\|_p , \]  

(4.13)
where
\[ G^+_{pm} \equiv G^+_{pn} := \frac{1}{(2\pi)^{d/2}} \sup_{k \in \mathbb{Z}^d_0} \mathcal{G}_{pn}(k) \]
(4.14)
and \( \mathcal{G}_{pn} \) is the function defined by (4.6). Therefore, the inequality (1.17) holds and its sharp constant \( G^+_{pm} \equiv G^+_{pn} \) fulfills
\[ G^+_{pm} \leq G^+_{pn} \cdot \]
(4.15)

**Proof.** We fix \( v \in \mathbb{H}^p_{\Sigma_0}, w \in \mathbb{H}^{p+1}_{\Sigma_0} \) and proceed in several steps.

**Step 1.** We have \( \mathcal{P}(v, w) \in \mathbb{H}^p_{\Sigma_0}, \sqrt{-\Delta}^p \mathcal{P}(v, w) \in \mathbb{L}^2_{\Sigma_0} \), \( \sqrt{-\Delta}^p w \in \mathbb{H}^1_{\Sigma_0} \) and \( \mathcal{P}(v, \sqrt{-\Delta}^p w) \in \mathbb{L}^2_{\Sigma_0} \); furthermore, the vector field
\[ z := \sqrt{-\Delta}^p \mathcal{P}(v, w) - \mathcal{P}(v, \sqrt{-\Delta}^p w) \in \mathbb{L}^2_{\Sigma_0} \]
(4.16)
fulfills the equality
\[ \langle \mathcal{P}(v, w) | w \rangle_p = \langle z | \sqrt{-\Delta}^p w \rangle_{L^2} \],
(4.17)
which implies
\[ |\langle \mathcal{P}(v, w) | w \rangle_p| \leq \| z \|_{L^2} \| w \|_p \]
(4.18)
The statement \( \mathcal{P}(v, w) \in \mathbb{H}^p_{\Sigma_0} \) is known after Theorem 3.3 and the statement \( \sqrt{-\Delta}^p \mathcal{P}(v, w) \in \mathbb{L}^2_{\Sigma_0} \) is just a reformulation of it. Our assumption \( v \in \mathbb{H}^p_{\Sigma_0} \) implies \( v \in \mathbb{L}^\infty_{\Sigma_0} \), by the Sobolev embedding; of course \( \sqrt{-\Delta}^p \) sends \( \mathbb{H}^{p+1}_{\Sigma_0} \) into \( \mathbb{H}^1_{\Sigma_0} \), thus \( \sqrt{-\Delta}^p w \in \mathbb{H}^1_{\Sigma_0} \). Now, applying the second result (2.24) with \( w \) replaced by \( \sqrt{-\Delta}^p w \) we obtain \( \mathcal{P}(v, \sqrt{-\Delta}^p w) \in \mathbb{L}^2_{\Sigma_0} \). To go on, we note that the definition of \( \langle v | w \rangle_p \) gives
\[ \langle \mathcal{P}(v, w) | w \rangle_p = \langle \sqrt{-\Delta}^p \mathcal{P}(v, w) | \sqrt{-\Delta}^p w \rangle_{L^2} \]
(4.19)
and that Eq. (2.26) with \( w \) replaced by \( \sqrt{-\Delta}^p w \) gives
\[ 0 = \langle \mathcal{P}(v, \sqrt{-\Delta}^p w) | \sqrt{-\Delta}^p w \rangle_{L^2} \]
(4.20)
Subtracting Eq. (4.20) from Eq. (4.19) we obtain the thesis (4.17), with \( z \) given by (4.17) and the Schwartz inequality yield \( |\langle \mathcal{P}(v, w) | w \rangle_p| \leq \| z \|_{L^2} \| \sqrt{-\Delta}^p w \|_{L^2} \).\]

**Step 2.** The vector field \( z \) in (4.10) has Fourier coefficients
\[ z_k = -\frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}^d_0} (|k|^p - |k-h|^p) |k-h| P_{h,k-h}(v_h, w_{k-h}) \quad \text{for all } k \in \mathbb{Z}^d_0 \].
(4.21)
In fact
\[ [\sqrt{-\Delta}^p \mathcal{P}(v, w)]_k = |k|^p \mathcal{P}(v, w)_k = -\frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}^d_0} |k|^p |k-h| P_{h,k-h}(v_h, w_{k-h}) \]
(4.22)
the last equality follows from Eq. \((2.28)\). Using the same equation we get

\[
\mathcal{P}(v, \sqrt{-\Delta}^p w)_k = -\frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_0^d} |k - h| P_{h,k-h}(v_h, [\sqrt{-\Delta}^p w]_{k-h})
\]

\[
= -\frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_0^d} |k - h| P_{h,k-h}(v_h, |k - h|^p w_{k-h}) .
\]

The last two equations and the definition of \(z\) yield the thesis \((4.21)\).

**Step 3. Estimating the Fourier coefficients of \(z\).** Let \(k \in \mathbb{Z}_0^d\); Eq. \((4.21)\) implies

\[
|z_k| \leq \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_0^d} |k|^p - |k - h|^p \left| P_{h,k-h} \right| |v_h| |w_{k-h}|
\]

\[
= \frac{1}{2(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}_0^d} \frac{2|P_{h,k-h}|}{|k|^p |k - h|^n + |h|^p |k - h|^p} |v_h||w_{k-h}| .
\]

Now, the Cauchy inequality \(\sum_h a_h b_h \leq \sqrt{\sum_h a_h^2} \sqrt{\sum_h b_h^2}\) \((a_h, b_h \in [0, +\infty))\) gives

\[
|z_k| \leq \frac{1}{2(2\pi)^{d/2}} \sqrt{\frac{\sum_{h \in \mathbb{Z}_0^d} 4(|k|^p - |k - h|^p)^2 |P_{h,k-h}|^2}{(|k|^p |k - h|^n + |h|^p |k - h|^p)^2}} \sum_{h \in \mathbb{Z}_0^d} (|h|^p |k - h|^n + |h|^p |k - h|^p)^2 |v_h|^2 |w_{k-h}|^2 ;
\]

comparing with the definition \((4.6)\) of \(G_{pn}\) we see that

\[
|z_k| \leq \frac{1}{2(2\pi)^{d/2}} \sqrt{G_{pn}(k) \sum_{h \in \mathbb{Z}_0^d} (|h|^p |k - h|^n + |h|^p |k - h|^p)^2 |v_h|^2 |w_{k-h}|^2} .
\]

The last inequality has the form

\[
|z_k| \leq \frac{1}{2(2\pi)^{d/2}} \sqrt{G_{pn}(k)} \sqrt{\sum_{h \in \mathbb{Z}_0^d} (a_{kh} + b_{kh})^2} ,
\]

\[a_{kh} := |h|^p |k - h|^n |v_h||w_{k-h}| , \quad b_{kh} := |h|^p |k - h|^n |v_h||w_{k-h}| .\]

But

\[
\sqrt{\sum_{h \in \mathbb{Z}_0^d} (a_{kh} + b_{kh})^2} \leq \sqrt{\sum_{h \in \mathbb{Z}_0^d} a_{kh}^2} + \sqrt{\sum_{h \in \mathbb{Z}_0^d} b_{kh}^2} ,
\]

so...
\[ |z_k| \leq \frac{1}{2(2\pi)^{d/2}} \sqrt{G_{pn}(k)(q_k + p_k)}, \quad (4.28) \]

\[ q_k := \sqrt{\sum_{h \in \mathbb{Z}_d} |h|^{2p}|k - h|^{2n}v_h w_{h-k}^2}, \quad p_k := \sqrt{\sum_{h \in \mathbb{Z}_d} |h|^{2n}|k - h|^{2p}v_h w_{h-k}^2}. \]

To go on we note that, according to (4.14), \((2\pi)^{-d/2} \sqrt{G_{pn}(k)} \leq G_{pn}^\pm;\) thus

\[ |z_k| \leq \frac{G_{pn}^\pm}{2}(q_k + p_k). \quad (4.29) \]

**Step 4. Estimating \(\|z\|_2.\)** With \(q_k\) and \(p_k\) as in Eq. (4.28), we have

\[ \|z\|_2 = \sum_{k \in \mathbb{Z}_d^d} |z_k|^2 \leq \frac{G_{pn}^\pm}{2} \sum_{k \in \mathbb{Z}_d^d} (q_k + p_k)^2 \leq \frac{G_{pn}^\pm}{2} \left( \sum_{k \in \mathbb{Z}_d^d} q_k^2 + \sum_{k \in \mathbb{Z}_d^d} p_k^2 \right). \quad (4.30) \]

On the other hand, manipulations very similar to the ones in Eq. (3.17) give

\[ \sum_{k \in \mathbb{Z}_d^d} q_k^2 \leq \|v\|_p^2 \|w\|_n^2, \quad \sum_{k \in \mathbb{Z}_d^d} p_k^2 \leq \|v\|_n^2 \|w\|_p^2; \quad (4.31) \]

inserting this result into (4.30) we get

\[ \|z\|_2 \leq \frac{G_{pn}^\pm}{2} \left( \|v\|_p \|w\|_n + \|v\|_n \|w\|_p \right). \quad (4.32) \]

**Step 5. Conclusion of the proof.** We return to Eq. (4.18), and insert therein the bound (4.32) for \(\|z\|_2.\) This gives the inequality (4.13), with \(G_{pn}^\pm\) as in (4.14). \(\square\)

Theorem 3.3 gives an upper bound on \(G_{pn}\) in terms of sup \(G_{pn}\) that, as anticipated, will be the subject of accurate estimates in Section 7. For the moment, using the rough bound of Proposition 4.2 on sup \(G_{pn}\) we obtain:

**4.5 Corollary.** The sharp constant \(G_{pn}\) of (1.7) has the bound

\[ G_{pn} \leq G_{pn}^{(+)}, \quad G_{pn}^{(+)} := \frac{2^{2p}}{(2\pi)^{d/2}} \sqrt{\zeta_{2n-2}} \quad (4.33) \]

(with \(\zeta_{2n-2}\) as in (4.7); note that \((G_{pn}^{(+)})^{1/p} \to 2\) for fixed \(d, n\) and \(p \to +\infty\)).

**Proof.** In fact, Eqs. (4.14) (4.15) and (4.7) give

\[ G_{pn} \leq \frac{1}{(2\pi)^{d/2}} \sup_{k \in \mathbb{Z}_d^d} G_{pn}(k) \leq \frac{1}{(2\pi)^{d/2}} \sqrt{2^{2p}p^2 \zeta_{2n-2}}, \]

whence the thesis (4.33). \(\square\)

Similarly to the rough bound (3.20) on \(K_{pn}\), the present bound (4.33) will be useful in Section 11 to evaluate the \(p \to +\infty\) limit of \((G_{pn})^{1/p}\).
5 Some tools preparing the analysis of the functions $K_{pn}$ and $G_{pn}$

As anticipated, in Sections 6-8 we will show how to compute accurately the functions $K_{pn}$, $G_{pn}$ of Eqs. (3.2) (4.6) and their sups; here we introduce some tools devised for this purpose.

First of all we fix some notations, to be used throughout the rest of the paper.

5.1 Definition. (i) $\delta_{ab}$ is the Kronecker delta ($\delta_{ab} := 1$ if $a = b$ and $\delta_{ab} := 0$ if $a \neq b$).
(ii) $H : \mathbb{R} \to \{0, 1\}$ is the Heaviside function such that $H(z) := 0$ if $z < 0$ and $H(z) := 1$ if $z \geq 0$.
(iii) $\Gamma$ is the Euler Gamma function, $(\cdot)$ are the binomial coefficients.
(iv) $S_{d-1}$ denotes the unit spherical hypersurface in $\mathbb{R}^d$, i.e., $S_{d-1} := \{u \in \mathbb{R}^d \mid |u| = 1\}$. For each $q \in \mathbb{R}^d \setminus \{0\}$, the versor of $q$ is $\hat{q} := \frac{q}{|q|} \in S_{d-1}$. 

In the sequel we also maintain the following notation, already introduced in Section 2: for all $q, r \in \mathbb{R}^d \setminus \{0\}$, $\vartheta_{qr} \in [0, \pi]$ denotes the convex angle between $q$ and $r$ (so that $\cos \vartheta_{qr} = \hat{q} \cdot \hat{r}$).

5.2 Lemma. For any function $f : Z_0^d \to \mathbb{R}$ and $k \in Z_0^d$, $\rho \in (1, +\infty)$, one has

$$\sum_{h \in Z_{dk}, |h| < \rho \lor |k-h| < \rho} f(h) = \sum_{h \in Z_{dk}, |h| < \rho} \left[ f(h) + H(|k-h| - \rho) f(k-h) \right]. \quad (5.1)$$

Proof. See [12].

5.3 Lemma. For any $p, n \in \mathbb{R}$ with $p \geq n > 1$, the following holds.
(i) Consider the function

$$b_{pn} : [0, 4] \times [0, 1] \to [0, +\infty), \quad (5.2)$$

$$b_{pn}(z, u) := \frac{2z(4-z)(1-u)^{2n}u^{2n}(1-zu+zu^2)^p}{[(1-u)^{2n} + u^{2n}][(1-u)^{p}u^{n} + (1-u)^{p}u^{n}]^2} \quad \text{if } u \in (0, 1),$$

$$b_{pn}(z, 0) := b_{pn}(z, 1) := \frac{2z(4-z)}{1 + 3\delta_{pn}} \cdot$$

This is well defined and continuous, which implies the existence of

$$B_{pn} := \max_{z \in [0, 4], u \in [0, 1]} b_{pn}(z, u) > 0. \quad (5.3)$$
(ii) Given \( h, \ell \in \mathbb{R}^d \setminus \{0\} \), consider the convex angle \( \vartheta_{h\ell} \) and define \( z \in [0,4] \), \( u \in (0,1) \) through the equation

\[
\cos \vartheta_{h\ell} = 1 - \frac{z}{2}, \quad |h| = \frac{u}{1-u} |\ell| ;
\]

then

\[
\frac{|h+\ell|^{2p} \sin^2 \vartheta_{h\ell}}{(\|h\|^p \|\ell\|^n + |h|^n |\ell|^p)^2} = \frac{b_{pn}(z,u)}{8} \left( \frac{1}{|h|^{2n}} + \frac{1}{|\ell|^{2n}} \right) .
\]

This implies

\[
\frac{|h+\ell|^{2p} \sin^2 \vartheta_{h\ell}}{(\|h\|^p \|\ell\|^n + |h|^n |\ell|^p)^2} \leq \frac{B_{pn}}{8} \left( \frac{1}{|h|^{2n}} + \frac{1}{|\ell|^{2n}} \right) .
\]

**Proof.** (i) Trivial (in particular it is not difficult to check that \( b_{pn}(z,0) = \lim_{u \to 0} b_{pn}(z,u) \) and \( b_{pn}(z,1) = \lim_{u \to 1} b_{pn}(z,u) \)).

(ii) Consider the quantity

\[
\frac{|h+\ell|^{2p} \sin^2 \vartheta_{h\ell}}{(\|h\|^p \|\ell\|^n + |h|^n |\ell|^p)^2} \left( \frac{1}{|h|^{2n}} + \frac{1}{|\ell|^{2n}} \right)^{-1}
\]

and express it via the relations \( \sin^2 \vartheta_{h\ell} = 1 - \cos^2 \vartheta_{h\ell} \) and

\[
|h+\ell| = \sqrt{|h|^2 + 2|h||\ell| \cos \vartheta_{h\ell} + |\ell|^2} ;
\]

subsequently, express \( \cos \vartheta_{h\ell} \) and \( |h| \) via Eq. (5.4). After tedious manipulations it is found that (5.7) equals \( \frac{b_{pn}(z,u)}{8} \), and Eq. (5.5) is proved. Eq. (5.6) is an obvious consequence. \( \Box \)

5.4 Remark. Let \( b_{pn} \) be defined as in the previous lemma. It is readily found that the derivatives \( \partial b_{pn}/\partial z, \partial b_{pn}/\partial u \) vanish at \( (z,u) = (\frac{4}{p+2}, \frac{1}{2}) \), and

\[
b_{pn} \left( \frac{4}{p+2}, \frac{1}{2} \right) = \frac{4^{p+1}}{p+2} \left( \frac{p + 1}{p + 2} \right)^{p+1} ;
\]

moreover, the hessian of \( b_{pn} \) at \( (z,u) = (\frac{4}{p+2}, \frac{1}{2}) \) is positive defined if \( p \geq n > 1 \). Considering \( B_{pn} := \max_{z \in [0,4], u \in [0,1]} b_{pn}(z,u) \), we conjecture that

\[
B_{pn} = b_{pn} \left( \frac{4}{p+2}, \frac{1}{2} \right) = \frac{4^{p+1}}{p+2} \left( \frac{p + 1}{p + 2} \right)^{p+1}
\]

for all \((p,n)\) with \( p \geq n > 1 \); note that \( B_{pn} \) does not depend on \( n \) if the above statement holds. For given \((p,n)\), statement 5.10 can be tested using a computer to plot \( b_{pn} \) or to maximize it numerically. In this way we have obtained that 5.10 holds for all \((p,n)\) as in Eqs. (1.10) (i.e., in all cases considered in the sequel to exemplify the evaluation of \( K_{pn} \)).

20
5.5 Lemma. For any \( p, n \in \mathbb{R} \) with \( p \geq n > 1 \), the following holds.

(i) Consider the function

\[
c_{pn} : [0, 4] \times [0, 1] \to [0, +\infty) ,
\]

\[
c_{pn}(z, u) := \frac{2zh(4-z)(1-z)2^{n+2}u^{2n}[(1-zu+z)^2/(1-u)^{p/2} - (1-u)^p]}{[(1-u)^2u^2 + (1-u)^2u^{2n}][(1-u)^pu^n + (1-u)^{n}u^p]} \quad \text{if } u \in (0, 1),
\]

\[
c_{pn}(z, 0) := \frac{p^2z(4-z)(2-z)^2}{2(1 + 3\delta_{pn})}, \quad c_{pn}(z, 1) := \frac{2zh(4-z)}{1 + 3\delta_{pn}}.
\]

This is well defined and continuous, which implies the existence of

\[
C_{pn} := \max_{z \in [0,4], u \in [0,1]} c_{pn}(z, u) > 0 .
\]

(ii) Given \( h, \ell \in \mathbb{R}^d \setminus \{0\} \) define \( z \in [0,4], u \in (0, 1) \) as in Eq. \( (5.4) \); then

\[
\left(\frac{|h + \ell|^p - |\ell|^p}{|h|^p|\ell|^{n-1} + |h|^n|\ell|^{p-1}}\right)^2 \leq \frac{C_{pn}(z, u)}{8} \left(\frac{1}{|h|^{2n-2}} + \frac{1}{|\ell|^{2n-2}}\right) .
\]

This implies

\[
\left(\frac{|h + \ell|^p - |\ell|^p}{|h|^p|\ell|^{n-1} + |h|^n|\ell|^{p-1}}\right)^2 \leq \frac{C_{pn}}{8} \left(\frac{1}{|h|^{2n-2}} + \frac{1}{|\ell|^{2n-2}}\right) .
\]

Proof. (i) Trivial (in particular it is not difficult to check that \( c_{pn}(z, 0) = \lim_{u \to 0} c_{pn}(z, u) \) and \( c_{pn}(z, 1) = \lim_{u \to 1} c_{pn}(z, u) \)).

(ii) Let us consider the ratio

\[
\left(\frac{|h + \ell|^p - |\ell|^p}{|h|^p|\ell|^{n-1} + |h|^n|\ell|^{p-1}}\right)^2 \left(\frac{1}{|h|^{2n-2}} + \frac{1}{|\ell|^{2n-2}}\right)^{-1}
\]

and express it using Eq. \( (5.8) \); subsequently, write \( \cos \vartheta_{h\ell} \) and \( |h| \) as in \( (5.4) \). After tedious manipulations it is found that the ratio \( (5.14) \) equals \( \frac{c_{pn}(z, u)}{8} \), and Eq. \( (5.13) \) is proved. Eq. \( (5.14) \) is an obvious consequence.

5.6 Examples. Let \( c_{pn}, C_{pn} \) be defined as in the previous lemma. For \( n = 3 \) and \( p = 3, 4, 5, 10 \) we have the following results, obtained by numerical optimization via Mathematica:

\[
C_{33} = c_{33}(0.696034..., 0.464530...) = 14.8144..., \quad \text{(5.16)}
\]

\[
C_{43} = c_{44}(0.610279..., 0.439178...) = 61.1705..., \quad \text{(5.16)}
\]

\[
C_{53} = c_{53}(0.545364..., 0.443863...) = 229.715..., \quad \text{(5.16)}
\]

\[
C_{10,3} = c_{10,3}(0.332954..., 0.489262...) = 1.36660... \times 10^5 .
\]
5.7 Lemma. Let \( p, n \in \mathbb{R} \), \( p \geq n > 0 \); then (i)(ii) hold.

(i) Let us introduce the domain
\[
\mathcal{E} := \{(c, \xi) \in \mathbb{R}^2 \mid c \in [-1, 1], \xi \in [0, +\infty), (c, \xi) \neq (1, 1)\}
\] (5.17)
and put
\[
E_{pn} : \mathcal{E} \to [0, +\infty),
\]
\[
E_{pn}(c, \xi) := \frac{1 - c^2}{\left[(1 - 2c\xi + \xi^2)^{p/2+1/2} + \xi^{p-n}(1 - 2c\xi + \xi^2)^{n/2+1/2}\right]^2}.
\]

Then the above function is well defined and continuous on \( \mathcal{E} \).

(ii) Let \( h, k \in \mathbb{R}^d \setminus \{0\} \), \( h \neq k \) and consider the convex angles \( \vartheta_{hk}, \vartheta_{h,k-h} \). Then
\[
\frac{|k|^{2p} \sin^2 \vartheta_{h,k-h}}{|h|^p |k-h|^n + |h|^n |k-h|^p} = \frac{1}{|h|^{2n}} E_{pn}\left(\cos \vartheta_{hk}, \frac{|h|}{|k|}\right).
\]
(5.19)

Proof. (i) Trivial.

(ii) The parallelograms of sides \( h \) and \( k \), \( h \) and \( k - h \) have the same area; thus
\[
|h||k| \sin \vartheta_{hk} = |h||k-h| \sin \vartheta_{h,k-h}, \text{ whence}
\]
\[
\sin \vartheta_{h,k-h} = \frac{|k|}{|k-h|} \sin \vartheta_{hk} = \frac{|k|}{|k-h|} \sqrt{1 - \cos^2 \vartheta_{hk}};
\]
moreover,
\[
|k-h| = \sqrt{|k|^2 - 2|k||h| \cos \vartheta_{hk} + |h|^2} = |k| \sqrt{1 - 2 \cos \vartheta_{hk} \frac{|h|}{|k|} + \frac{|h|^2}{|k|^2}}.
\]

Let us consider the function in the left hand side of (5.19), and reexpress it using the identities (5.20), (5.21); in this way, after some manipulations we obtain Eq. (5.19). \(\square\)

5.8 Lemma. Let \( p, n \in \mathbb{R} \), \( p \geq n > 0 \), and consider the function \( E_{pn} : \mathcal{E} \to \mathbb{R} \) of Lemma 5.7. Introduce the set
\[
\Gamma_{pn} := \{r + (p-n)s \mid r, s \in \mathbb{N}\}
\]
(5.22)
and represent it as an increasing sequence:
\[
\Gamma_{pn} = \{0 = \gamma_{pn0} < \gamma_{pn1} < \gamma_{pn2} < \ldots\}.
\]
(5.23)

There are two sequences of functions
\[
Q_{pnj} \in C([-1, 1], \mathbb{R}), \ c \mapsto Q_{pnj}(c) \quad (j \in \mathbb{N})
\]
(5.24)
$S_{pnj} \in C(\mathcal{E}, \mathbb{R}), \quad (c, \xi) \mapsto S_{pnj}(c, \xi) \quad (j \in \mathbb{N} \setminus \{0\}) \tag{5.25}$

uniquely determined by the following prescription: for each $m \in \mathbb{N}$ one has

$$E_{pn}(c, \xi) = \sum_{j=0}^{m} Q_{pnj}(c) \xi^{\gamma_{pnj}} + S_{pn,m+1}(c, \xi) \xi^{\gamma_{pn,m+1}} \quad \text{for all } (c, \xi) \in \mathcal{E} . \tag{5.26}$$

Moreover, each function $Q_{pnj}$ is of polynomial type.

**Proof.** It suffices to show the following:

(a) for each $m \in \mathbb{N}$, there is a unique family of functions $Q_{pn0}, ..., Q_{pnm} \in C([-1, 1], \mathbb{R})$, $S_{pn,m+1} \in C(\mathcal{E}, \mathbb{R})$ such that (5.26) holds. Moreover, the functions $Q_{pnj}$ ($j = 0, ..., m$) are polynomials;

(b) for $m < m' \in \mathbb{N}$, the family $Q_{pn0}, ..., Q_{pnm}, S_{pn,m+1}$ of item (a) and the family $Q'_{pn0}, ..., Q'_{pnm'}$, $S_{pn,m'+1}$ of item (a) with $m$ replaced by $m'$ are such that $Q_{pn0} = Q'_{pn0}$, ...

\[ Q_{pnm} = Q'_{pnm}. \]

Let us first prove the uniqueness statement in (a), for a given $m \in \mathbb{N}$. To this purpose we note that, given a family as in (a), Eq. (5.26) implies

$$Q_{pn0}(c) = E_{pn}(c, 0) , \tag{5.27}$$

$$Q_{pnj}(c) = \lim_{\xi \to 0} \frac{1}{\xi^{\gamma_{pnj}}} \left( E_{pn}(c, \xi) - \sum_{\ell=0}^{j-1} Q_{pn\ell}(c) \xi^{\gamma_{pn\ell}} \right) \quad \text{for } j = 1, ..., m ; \tag{5.28}$$

this set of recursive relations determines uniquely the functions $Q_{pnj}$ for $j = 0, ..., m$.

Once we have uniqueness for the sequence $(Q_{pnj})_{j=0, ..., m}$, uniqueness of $S_{pn,m+1}$ follows noting that (5.26) implies

$$S_{pn,m+1}(c, \xi) = \frac{1}{\xi^{\gamma_{pn,m+1}}} \left( E_{pn}(c, \xi) - \sum_{j=0}^{m} Q_{pnj}(c) \xi^{\gamma_{pnj}} \right) \quad \text{for } (c, \xi) \in \mathcal{E}, \xi \neq 0 \tag{5.28}$$

and that, by the continuity requirement for $S_{pn,m+1}$, $S_{pn,m+1}(c, 0)$ is the $\xi \to 0$ limit of the right hand side in the above equation.

Now, let us prove statement (b) for given $m < m' \in \mathbb{N}$. To this purpose we note that, besides the characterization (5.27) for $Q_{pn0}, ..., Q_{pnm}$ we have a similar characterization for $Q'_{pn0}, ..., Q'_{pnm'}$; these imply $Q_{pn0}(c) = E_{pn}(c, 0) = Q'_{pn0}(c)$, $Q_{pnm}(c) = \lim_{\xi \to 0} \xi^{-\gamma_{pnm}} (E_{pn}(c) - Q_{pn0}(c)) = \lim_{\xi \to 0} \xi^{-\gamma_{pnm}} (E_{pn}(c) - Q'_{pn0}(c)) = Q'_{pnm}(c)$ and so on, up to $Q_{pnm}(c) = Q'_{pnm}(c)$.

Let us pass to prove, for any $m \in \mathbb{N}$, the existence of the functions $Q_{pn0}, ..., Q_{pnm}$, $S_{pn,m+1}$ fulfilling the conditions in (a) and the polynomial nature of the functions $Q_{pnj}$; for the sake of brevity we discuss the case $p > n$, leaving to the reader the case $p = n$ which is even simpler. Let us note that Eq. (5.18) has the form

$$E_{pn}(c, \xi) = A_{pn}(c, \xi, \xi^{p-n}) , \tag{5.29}$$

23
where
\[ A_{pn} : \mathcal{E} \times [0, +\infty) \rightarrow \mathbb{R}, \] (5.30)
\[ A_{pn}(c, \xi, u) := \frac{1 - c^2}{[(1 - 2c\xi + \xi^2)^{p/2+1/2} + u(1 - 2c\xi + \xi^2)^{n/2+1/2}]^2}. \]

It is easily checked that \( A_{pn} \in C^\infty(\mathcal{E} \times [0, +\infty), \mathbb{R}) \). Now, consider any \( a \in \mathbb{N} \); by Taylor’s formula of order \( a \) in the variables \( \xi, u \) we can write
\[ A_{pn}(c, \xi, u) = \sum_{r, s \in \mathbb{N}, r + s = a} A_{pnrs}(c) \xi^r u^s + \sum_{r, s \in \mathbb{N}, r + s = a + 1} S_{pnrs}(c, \xi, u) \xi^r u^s \] (5.31)
for \((c, \xi, u) \in \mathcal{E} \times [0, +\infty)\) with suitable reminder functions \( S_{pnrs} \in C(\mathcal{E} \times [0, +\infty), \mathbb{R}) \). The coefficients \( A_{pnrs}(c) \) in the above expansions are related to the derivatives of \( A_{pn} \) at \( \xi = 0, u = 0 \); one finds by direct inspection of the definitions of \( A_{pn} \) that these coefficients are polynomial functions of \( c \). Inserting the expansions (5.31) into Eq. (5.29) we get
\[ E_{pn}(c, \xi) = \sum_{r, s \in \mathbb{N}, r + s \leq a} A_{pnrs}(c) \xi^{r+(p-n)s} + \sum_{r, s \in \mathbb{N}, r + s = a + 1} S_{pnrs}(c, \xi, \xi^{p-n}) \xi^{r+(p-n)s}. \] (5.32)
All the exponents of \( \xi \) in the above formula belong to the set \( \Gamma_{pn} = \{0 = \gamma_{pn0} < \gamma_{pn1} < \ldots\} \) defined by (5.22). Now, after fixing \( m \in \mathbb{N} \) we choose \( a \in \mathbb{N} \) so that
\[ r + (p-n)s \geq \gamma_{pn,m+1} \]
for all \( r, s \in \mathbb{N} \) such that \( r + s = a + 1 \); then Eq. (5.32) implies for \( E_{pn} \) a representation of the form (5.26) for this value of \( m \), where
\[ Q_{pnj}(c) = \sum_{r, s \in \mathbb{N}, r + s \leq a, r + (p-n)s = \gamma_{pnj}} A_{pnrs}(c) \quad (j = 0, \ldots, m), \] (5.34)
\[ S_{pn,m+1}(c, \xi) = \sum_{r, s \in \mathbb{N}, r + s = a + 1} S_{pnrs}(c, \xi, \xi^{p-n}) \xi^{r+(p-n)s-\gamma_{pn,m+1}} \] (5.35)
\[ + \sum_{r, s \in \mathbb{N}, r + s \leq a, r + (p-n)s \geq \gamma_{pn,m+1}} A_{pnrs}(c) \xi^{r+(p-n)s-\gamma_{pn,m+1}}. \]
We note that the functions \( Q_{pnj} \) \((j = 0, \ldots, m)\) are polynomials in \( c \) and \( S_{pn,m+1} \) is continuous due to the previously mentioned features of \( A_{pnrs} \) and \( S_{pnrs} \). \( \Box \)
5.9 Lemma. Let \( p, n \in \mathbb{R}, p \geq n > 0 \); then (i)(ii) hold.

(i) Let \( \mathcal{E} \) be the domain in Eq. (5.17), and put
\[
F_{pn} : \mathcal{E} \to [0, +\infty) ,
\]
\[
F_{pn}(c, \xi) := \frac{1 - c^2}{(1 - 2c\xi + \xi^2)^{p/2} + \xi^{p-n}(1 - 2c\xi + \xi^2)^{n/2}}^2 \times \left\{ \frac{(1 - \xi^p)^2}{1 - 2c\xi + \xi^2} + \left[ \frac{1 - (1 - 2c\xi + \xi^2)^{p/2}}{\xi} \right]^2 \right\} \quad \text{if } \xi \neq 0 ,
\]
\[
F_{pn}(c, 0) := \frac{(1 - c^2)(1 + p^2c^2)}{1 + 3 \delta_{pn}} .
\]

Then the above function is well defined and continuous on \( \mathcal{E} \).

(ii) Let \( h, k \in \mathbb{R}^d \setminus \{0\}, h \neq k \); then
\[
\sin^2 \vartheta_{h,k-h} \left[ \frac{(|k|^p - |k - h|^p)^2}{(|k|^p h - |k - h|^p h)^{n-1} + |h|^n|k - h|^p h^{n-1}} + \frac{(|k|^p - |h|^p)^2}{(|k - h|^p h - |k|^p h)^{n-1} + |h|^n|k - h|^p h^{n-1}} \right] = \frac{1}{|h|^{2n-2}} F_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) .
\]

Proof. (i) Trivial (in particular, it is easy to check that \( F_n(c, 0) = \lim_{\xi \to 0} F_n(c, \xi) \)).

(ii) We consider the function in the left hand side of (5.37), and reexpress it using the identities (5.20) (5.21); in this way, after some manipulations we obtain Eq. (5.37).

\[\Box\]

5.10 Lemma. Let \( p, n \in \mathbb{R}, p \geq n > 0 \), and consider the function \( F_{pn} : \mathcal{E} \to \mathbb{R} \) of Lemma 5.9. Introduce the set
\[
\Lambda_{pn} := \{ r + (p - n)s + pt \mid r, s \in \mathbb{N}, t \in \{0, 1, 2\} \}
\]
and represent it as an increasing sequence:
\[
\Lambda_{pn} = \{ 0 = \lambda_{pn0} < \lambda_{pn1} < \lambda_{pn2} < \ldots \} .
\]

There are two sequences of functions
\[
P_{pnj} \in C([-1, 1], \mathbb{R}), \quad c \mapsto P_{pnj}(c) \quad (j \in \mathbb{N})
\]
\[
R_{pnj} \in C(\mathcal{E}, \mathbb{R}), \quad (c, \xi) \mapsto R_{pnj}(c, \xi) \quad (j \in \mathbb{N} \setminus \{0\})
\]
uniquely determined by the following prescription: for each \( m \in \mathbb{N} \) one has
\[
F_{pn}(c, \xi) = \sum_{j=0}^{m} P_{pnj}(c) \xi^{\lambda_{pnj}} + R_{pn,m+1}(c, \xi) \xi^{\lambda_{pn,m+1}} \quad \text{for all } (c, \xi) \in \mathcal{E} .
\]

Moreover, each function \( P_{pnj} \) is of polynomial type.
Proof. It suffices to show the following:
(a) For each \( m \in \mathbb{N} \), there is a unique family of functions \( P_{pn0}, \ldots, P_{pnm} \in C([-1,1], \mathbb{R}) \), \( R_{pnm+1} \in C(\mathcal{E}, \mathbb{R}) \), such that (5.42) holds. Moreover, the functions \( P_{pnj} \) \((j = 0, \ldots, m)\) are polynomials;
(b) For \( m < m' \in \mathbb{N} \), the family \( P_{pn0}, \ldots, P_{pnm}, R_{pnm+1} \) of item (a) and the family \( P'_{pn0}, \ldots, P'_{pnm}, R'_{pnm+1} \) of item (a) with \( m \) replaced by \( m' \) are such that \( P_{pn0} = P'_{pn0}, \ldots, P_{pnm} = P'_{pnm} \).

Let us first prove the uniqueness statement in (a), for a given \( m \in \mathbb{N} \). To this purpose we note that, given a family as in (a), Eq. (5.42) implies
\[
P_{pnj}(c) = \frac{1}{\xi_{\lambda_{pnj}}}(F_{pn}(c, \xi) - \sum_{\ell=0}^{j-1} P_{pn\ell}(c, \xi^{\lambda_{pn\ell}})) \quad \text{for } j = 1, \ldots, m,
\]
and this set of recursive relations determines uniquely the functions \( P_{pnj} \) for \( j = 0, \ldots, m \). Once we have uniqueness for the sequence \( (P_{pnj})_{j=0,\ldots,m} \), uniqueness of \( R_{pnm+1} \) follows noting that (5.42) implies
\[
R_{pnm+1}(c, \xi) = \frac{1}{\xi_{\lambda_{pnm+1}}}(F_{pn}(c, \xi) - \sum_{j=0}^{m} P_{pnj}(c, \xi^{\lambda_{pnj}})) \quad \text{for } (c, \xi) \in \mathcal{E}, \xi \neq 0
\]
and that, by the continuity requirement for \( R_{pnm+1}, R_{pnm+1}(c, 0) \) is the \( \xi \to 0 \) limit of the right hand side in the above equation.

Now, let us prove statement (b) for given \( m < m' \in \mathbb{N} \). To this purpose we note that, besides the characterization (5.43) for \( P_{pn0}, \ldots, P_{pnm} \) we have a similar characterization for \( P'_{pn0}, \ldots, P'_{pnm} \); these imply \( P_{pn0}(c) = F_{pn}(c, 0) = P'_{pn0}(c) \), \( P_{pn1}(c) = \lim_{\xi \to 0} \xi^{-\lambda_{pn1}}(F_{pn}(c) - P_{pn0}(c)) = \lim_{\xi \to 0} \xi^{-\lambda_{pn1}}(F_{pn}(c) - P'_{pn0}(c)) = P'_{pn1}(c) \) and so on, up to \( P_{pnm}(c) = P'_{pnm}(c) \).

Let us pass to prove, for any \( m \in \mathbb{N} \), the existence of functions \( P_{pn0}, \ldots, P_{pnm}, R_{pnm+1} \) fulfilling the conditions in (a) and the polynomial nature of the functions \( P_{pnj} \); for the sake of brevity we only discuss the case \( p > n \), leaving to the reader the case \( p = n \) which is even simpler. Let us note that Eq. (5.36) has the form
\[
F_{pn}(c, \xi) = D_{pn}(c, \xi^{p-n})(1 - 2\xi^p + \xi^{2p}) + H_{pn}(c, \xi, \xi^{p-n})
\]
where
\[
26
\]
Taylor's formula of order $a$ into Eq. (5.45) we get that these coefficients are polynomial functions of $c$ with suitable reminder functions $S_{\xi}$.

All the exponents of $\lambda$ then Eq. (5.48) implies for $F_{\xi}$ that these coefficients are polynomial functions of $c$. Inserting the expansions into Eq. (5.43) we get

$$D_{pn}(c, \xi, u) := \sum_{r, s, t \in \mathbb{N}, r + s + t \leq a} D_{prs}(c) \xi^r u^s + \sum_{r, s, t \in \mathbb{N}, r + s = a + 1, t \leq 2} S_{prs}(c, \xi, u) \xi^r u^s,$$

$$H_{pn}(c, \xi, u) := \sum_{r, s, t \in \mathbb{N}, r + s + t \leq a} H_{prs}(c) \xi^r u^s + \sum_{r, s, t \in \mathbb{N}, r + s = a + 1, t \leq 2} T_{prs}(c, \xi, u) \xi^r u^s$$

for $(c, \xi, u) \in \mathcal{E} \times [0, +\infty)$, with suitable reminder functions $S_{prs}, T_{prs} \in C(\mathcal{E} \times [0, +\infty), \mathbb{R})$. The coefficients $D_{prs}(c)$ and $H_{prs}(c)$ in the above expansions are related to the derivatives of $D_{pn}, H_{pn}$ at $\xi = 0$, $u = 0$; one finds by direct inspection of the definitions of $D_{pn}, H_{pn}$ that these coefficients are polynomial functions of $c$. Inserting the expansions into Eq. (5.43) we get

$$F_{pn}(c, \xi) = \sum_{r, s, t \in \mathbb{N}, r + s + t \leq a, t \leq 2} C_{prs}(c) \xi^{r+(p-n)s+tp} + \sum_{r, s, t \in \mathbb{N}, r + s = a + 1, t \leq 2} V_{prs}(c, \xi) \xi^{r+(p-n)s+tp},$$

$$C_{prs0}(c) := D_{prs}(c) + H_{prs}(c),$$

$$C_{prs1}(c) := -2D_{prs}(c),$$

$$C_{prs2}(c) := D_{prs}(c),$$

$$V_{prs0}(c, \xi) := S_{prs}(c, \xi, \xi^{p-n}) + T_{prs}(c, \xi, \xi^{p-n}),$$

$$V_{prs1}(c) := -2S_{prs}(c, \xi, \xi^{p-n}),$$

$$V_{prs2}(c, \xi) := S_{prs}(c, \xi, \xi^{p-n}).$$

All the exponents of $\xi$ in the above formula belong to the set $\Lambda_{pn} = \{0 = \lambda_{pn0} < \lambda_{pn1} < ...\}$ defined by (3.38). Now, after fixing $m \in \mathbb{N}$ we choose $a \in \mathbb{N}$ so that

$$r + (p-n)s + tp \geq \lambda_{pn,m+1}$$

for all $r, s, t \in \mathbb{N}$ such that $r + s = a + 1$, $t \leq 2$; then Eq. (5.48) implies for $F_{pn}$, a representation of the form (5.42) for this value of
where
\[ P_{pnj}(c) = \sum_{r, s, t \in \mathbb{N}, r + s \leq a, t \leq 2, r + (p-n)s + tp = \lambda_{pnj}} C_{pnrst}(c), \quad (j = 0, \ldots, m) \] (5.50)

\[ R_{pn, m+1}(c, \xi) = \sum_{r, s, t \in \mathbb{N}, r + s = a + 1, t \leq 2} V_{pnrst}(c, \xi) \xi^{r+(p-n)s + tp - \lambda_{pn, m+1}} \] (5.51)
\[ + \sum_{r, s, t \in \mathbb{N}, r + s \leq a, t \leq 2, r + (p-n)s + tp \geq \lambda_{pn, m+1}} C_{pnrst}(c) \xi^{r+(p-n)s + tp - \lambda_{pn, m+1}}. \]

We note that the functions \( P_{pnj} \) \((j = 0, \ldots, m)\) are polynomials in \( c \) and \( R_{pn, m+1} \) is continuous due to the previously mentioned features of \( D_{pnrs}, H_{pnrs}, S_{pnrs}, T_{pnrs} \).

\[ \square \]

5.11 Lemma. Consider a real \( \nu > d \). For any real \( \rho > 2\sqrt{d} \), one has
\[ \sum_{h \in \mathbb{Z}^d, |h| \geq \rho} \frac{1}{|h|^{\nu}} \leq \frac{2\pi^{d/2}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \left( \frac{d - 1}{i} \right) \frac{d^{d/2-1/2-i/2}}{i} \left( \nu - i \right) \left( \rho - 2\sqrt{d} \right)^{\nu - 1 - i}. \] (5.52)

**Proof.** This is just Lemma C.2 of [9] (with the variable \( \lambda \) of the cited reference related to \( \rho \) by \( \lambda = \rho - 2\sqrt{d} \)). \( \square \)

The forthcoming statement uses the notation \( \hat{\cdot} \) of Definition 5.1 to indicate versors.

5.12 Lemma. Let \( \ell \in \mathbb{N}, \rho \in (1, +\infty), \varphi : [1, \rho) \to \mathbb{R} \) and \( k \in \mathbb{R}^d \setminus \{0\} \). Then
\[ \sum_{h \in \mathbb{Z}^d, |h| < \rho} \varphi(|h|) \cos^\ell \vartheta_{hk} = P_{\varphi\ell}(\hat{k}), \] (5.53)

where \( P_{\varphi\ell} \) is the following polynomial function on the spherical hypersurface \( S^{d-1} \):
\[ P_{\varphi\ell} : S^{d-1} \to \mathbb{R}, \quad u \mapsto P_{\varphi\ell}(u) := \sum_{i_1, \ldots, i_d \in \mathbb{N}, i_1 + \ldots + i_d = \ell} \frac{\ell!}{i_1! \ldots i_d!} M_{\varphi, i_1, \ldots, i_d} u_1^{i_1} \ldots u_d^{i_d}, \] (5.54)
\[ M_{\varphi, i_1, \ldots, i_d} := \sum_{h \in \mathbb{Z}^d, |h| < \rho} \varphi(|h|) \hat{h}^{i_1} \ldots \hat{h}^{i_d} \] (5.55)

(in the above \( u_r \) and \( \hat{h}_r \) stand for the \( r \)-th components of \( u \) and \( \hat{h} \); \( u_r^{i_r} \) and \( \hat{h}_r^{i_r} \) indicate their powers with exponent \( i_r \)). One has
\[ M_{\varphi, i_1, \ldots, i_d} = 0 \quad \text{if } i_r \text{ is odd for some } r \in \{1, \ldots, d\}, \] (5.56)
\[ M_{\varphi, i_{\sigma(1)}, \ldots, i_{\sigma(d)}} = M_{\varphi, i_1, \ldots, i_d} \quad \text{for each permutation } \sigma \text{ of } \{1, \ldots, d\} \] (5.57)
(so, the computation of the coefficients $M_{\varphi,i_1,...,i_d}$ can be reduced to cases with $i_1 \leq i_2 \leq ... \leq i_d$ and $i_r$ even for all $r$). The previous facts imply

$$P_{\varphi}(u) = 0 \quad \text{for all } u \in S^{d-1}, \text{ if } \ell \text{ is odd}$$

and, in the case $\ell = 2$,

$$P_{\varphi}(u) = \text{constant} = \frac{1}{d} \sum_{h \in Z_0^d, |h| < \rho} \varphi(|h|) \quad \text{for } u \in S^{d-1}.$$  (5.59)

**Proof.** We have $\cos \vartheta_{hk} = \frac{1}{d} \sum_{h \in Z_0^d, |h| < \rho} \varphi(|h|)$, which implies

$$\cos^\ell \vartheta_{hk} = (\hat{h}_1 \hat{k}_1 + ... + \hat{h}_d \hat{k}_d)^\ell = \sum_{i_1,...,i_d \in N, i_1 + ... + i_d = \ell} \frac{\ell!}{i_1!...i_d!} \hat{h}_{i_1} ... \hat{h}_{i_d} \hat{k}_{i_1} ... \hat{k}_{i_d}.$$  (5.60)

Multiplying this relation by $\varphi(|h|)$ and summing over $h$ we easily obtain Eqs. (5.53)-(5.55); the definition (5.55) of the coefficients $M_{\varphi,i_1,...,i_d}$ gives the relations (5.56) by elementary considerations of symmetry. Now, assume $\ell$ is odd; then each one of the coefficients $M_{\varphi,i_1,...,i_d}$ appearing in Eq. (5.54) is zero, because the list $(i_1,...,i_d)$ has some odd element and (5.56) can be applied, so we obtain Eq. (5.58).

Let us pass to the case $\ell = 2$; the only nonzero coefficients involved in (5.54) are

$$M_{\varphi,2,0,...,0} = M_{\varphi,0,2,0,...,0} = ... = M_{\varphi,0,...,0,2}$$ and can be determined noting that

$$M_{\varphi,2,0,...,0} + ... + M_{\varphi,0,...,0,2} = \sum_{h \in Z_0^d, |h| < \rho} \varphi(|h|) (\hat{h}_1^2 + ... + \hat{h}_d^2) = \sum_{h \in Z_0^d, |h| < \rho} \varphi(|h|).$$

In conclusion, the nonzero coefficients in (5.54) for $\ell = 2$ are

$$M_{\varphi,2,0,...,0} = M_{\varphi,0,2,0,...,0} = ... = M_{\varphi,0,...,0,2} = \frac{1}{d} \sum_{h \in Z_0^d, |h| < \rho} \varphi(|h|),$$

and Eq. (5.59) follows immediately. \qed

We remark that statement (5.59) in the above Lemma is equivalent to Lemma A.5 in the arXiv version of [12].

### 6 The function $K_{pn}$

As in Section 3, we consider $p, n \in \mathbb{R}$ such that $p \geq n > d/2$. For $k \in Z_0^d$, we recall the definition (3.2)

$$K_{pn}(k) := 4|k|^{2p} \sum_{h \in Z_0^d} \frac{|P_{h,k-h}|^2}{(|h|^n|k-h|^n + |h|^n|k-h|^p)^2}.$$
The general term of the above sum over $h$ is large when its denominator is small, which happens when $h$ is close to zero or to $k$. Therefore, it is reasonable to approximate the infinite sum in (3.2) with a finite sum over the union of two balls of centers $0$, $k$ and a suitable radius $\rho$. Such a finite sum is the main character of the forthcoming proposition, where it is denoted with $\mathcal{K}_{pn}(k)$; the proposition uses $\mathcal{K}_{pn}(k)$ and other ingredients to estimate $\mathcal{K}_{pn}(k)$ and its sup for $k \in \mathbb{Z}^d$.

Let us remark that, for $p = n$, the results presented hereafter become very similar to those appearing in Proposition B.1 of [12] (extended arXiv version).

**6.1 Proposition.** Let us choose a "cutoff"

$$\rho \in (2\sqrt{d}, +\infty) ,$$

(6.1)

a “factor”

$$\mu \in (1, +\infty)$$

(6.2)

and an “order”

$$m \in \mathbb{N} ;$$

(6.3)

then the following holds (with the functions and quantities $\mathcal{K}_{pn}$, $\delta \mathcal{K}_{pn}$, $\ldots$, $Y_{pn}$, $\mathcal{R}_{pn}$ mentioned in the sequel depending parametrically on $\rho$, $\mu$, $m$ and $d$: $\mathcal{K}_{pn}(k) \equiv \mathcal{K}_{pn}(k)$, $\delta \mathcal{K}_{pn} \equiv \delta \mathcal{K}_{pn}$, $\ldots$, $Y_{pn} \equiv Y_{pn}$, $\mathcal{R}_{pn} \equiv \mathcal{R}_{pn}$)

(i) The function $\mathcal{K}_{pn}$ fulfills the inequalities

$$\mathcal{K}_{pn}(k) < \mathcal{K}_{pn}(k) \leq \mathcal{K}_{pn}(k) + \delta \mathcal{K}_{pn} \text{ for all } k \in \mathbb{Z}^d .$$

(6.4)

Here

$$\mathcal{K}_{pn}(k) := 4|k|^{2p} \sum_{h \in \mathbb{Z}^d_0, |h| < \rho, d \leq \rho} \frac{|P_{h,k-h}|^2}{(|h|^p|k-h|^n + |h|^n|k-h|^p)^2} ;$$

(6.5)

this function can be reexpressed as

$$\mathcal{K}_{pn}(k) = 4|k|^{2p} \sum_{h \in \mathbb{Z}^d_0, |h| < \rho} \frac{|P_{h,k-h}|^2 + H(|k-h| - \rho)|P_{h,k-h}|^2}{(|h|^p|k-h|^n + |h|^n|k-h|^p)^2} ;$$

(6.6)

(with $H$ the Heaviside function, see Definition 5.1, recall that $|P_{h,k-h}| = |P_{h,k-h}|$ if $d \geq 3$, due to (2.32)). If $|k| \geq 2\rho$, in Eq. (6.6) one can replace $\mathbb{Z}^d_0$ with $\mathbb{Z}^d_0$ and $H(|k-h| - \rho)$ with 1. Moreover

$$\delta \mathcal{K}_{pn} := \frac{2\pi^{d/2} B_{pn}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \left( \begin{array}{c} d-1 \end{array} \right) \frac{d^{d/2-1/2-i/2}}{(2n-1-i)(\rho - 2\sqrt{d})^{2n-1-i}} ,$$

(6.7)

with $B_{pn}$ as in (5.3).
(ii) Consider the reflection and permutation operators $R_r$, $P_\sigma$ defined by (3.6). Then
\[ K_{pn}(R_rk) = K_{pn}(k), \quad K_{pn}(P_\sigma k) = K_{pn}(k) \quad \text{for each } k \in \mathbb{Z}_0^d. \] (6.8)

(iii) Denoting with $E_{pn}$ the function in Lemma 5.7, one has
\[ K_{pn}(k) \leq 8 \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{1}{|h|^{2n}} E_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) \quad \text{for all } k \in \mathbb{Z}_0^d. \] (6.9)

Now, consider the sequence of exponents $0 = \gamma_{pn0} < \gamma_{pn1} < \gamma_{pn2}...$ and the sequences of polynomials $Q_{pnj}$ ($j \in \mathbb{N}$) and functions $S_{pnj}$ ($j \in \mathbb{N} \setminus \{0\}$) involved in the expansion of $E_{pn}$ according to Lemma 5.8; then
\[ K_{pn}(k) \leq 8 \sum_{j=0}^{m} Q_{pnj}(\hat{k}) \frac{1}{|\hat{k}|^{|\gamma_{pnj}|}} + 8 V_{pn} Y_{pn} \frac{1}{|\hat{k}|^{2n-\gamma_{pn,m+1}}} \quad \text{for } k \in \mathbb{Z}_0^d, |k| \geq \mu \rho. \] (6.10)

Here, we recall, $\hat{k}$ is the versor of $k$ (Definition 5.1). $Q_{pnj}$ are the functions defined as follows on the spherical hypersurface $S^{d-1}$:
\[ Q_{pnj} : S^{d-1} \rightarrow \mathbb{R}, \quad u \mapsto Q_{pnj}(u) := \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{Q_{pnj}(\cos \vartheta_{hu})}{|h|^{2n-\gamma_{pnj}}}. \] (6.11)

(these are polynomials in the components of $u$, which can be computed using Lemma 5.12). Moreover
\[ V_{pn} := \max_{c \in [-1,1], \xi \in [0,1/\mu]} S_{pn,m+1}(c, \xi), \] (6.12)
\[ Y_{pn} := \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{1}{|h|^{2n-\gamma_{pn,m+1}}}. \] (6.13)

Finally,
\[ R_{pn} := 8 \max_{u \in S^{d-1}, \epsilon \in [0,1/(\mu \rho)]} \left( \sum_{j=0}^{m} \epsilon^{\gamma_{pnj}} Q_{pnj}(u) + V_{pn} Y_{pn} \epsilon^{\gamma_{pn,m+1}} \right). \] (6.14)

(iv) Items (i) and (iii) imply
\[ \sup_{k \in \mathbb{Z}_0^d} K_{pn}(k) \leq \max \left( \max_{k \in \mathbb{Z}_0^d, |k| < \mu \rho} K_{pn}(k), R_{pn} \right) + \delta K_{pn}. \] (6.15)

The proof of the above statements will be given after the following comment.
6.2 Remark. From Theorem 3.3 we know that the sharp constant $K_{pn}$ of the inequality (1.6) fulfills $K_{pn} \leq K_{pn}^+ := (2\pi)^{-d/2} \sqrt{\sup_{k \in \mathbb{Z}_d^n} \mathcal{K}_{pn}(k)}$. Of course, using for $\sup_{k \in \mathbb{Z}_d^n} \mathcal{K}_{pn}(k)$ the bound (6.15), we conclude

$$K_{pn} \leq K_{pn}^+ \leq K_{pn}^{(+) := \frac{1}{(2\pi)^{d/2}} \sqrt{\max_{k \in \mathbb{Z}_d^n} \max_{|k| < \mu \rho} \mathcal{K}_{pn}(k), R_{pn}} + \delta \mathcal{K}_{pn}. \quad (6.16)$$

The bound $K_{pn}^{(+}$ is suitable for computer implementation, a fact discussed in Section 8. From the statements in Proposition 6.1 it is clear that $K_{pn}^{(+}$ gives a good approximation of $K_{pn}^+$ under two conditions:

(a) $R_{pn}$ (which is a theoretical upper bound on $\sup_{|k| \geq \mu \rho} \mathcal{K}_{pn}(k)$), must be smaller or not too larger than $\max_{|k| < \mu \rho} \mathcal{K}_{pn}(k)$. If $R_{pn}$ is smaller, one has $\max_{|k| < \mu \rho} \mathcal{K}_{pn}(k) = \sup_k \mathcal{K}_{pn}(k)$; if $R_{pn}$ is a bit larger, $\max_{|k| < \mu \rho} \mathcal{K}_{pn}(k), R_{pn} = R_{pn}$ is a good upper approximant for $\sup_k \mathcal{K}_{pn}(k)$.

(b) $\delta \mathcal{K}_{pn}$, which binds the distance between $\mathcal{K}_{pn}(k)$ and $\mathcal{K}_{pn}(k)$ for all $k$, must be small in comparison with $\max_{|k| < \mu \rho} \mathcal{K}_{pn}(k), R_{pn}$; under this condition and the one in item (a), $\max_{|k| < \mu \rho} \mathcal{K}_{pn}(k), R_{pn}$ is a good approximation for $\sup_k \mathcal{K}_{pn}(k)$.

**Proof of Proposition 6.1** We fix $\rho, \mu, m$ as in (6.1)-(6.3). Our argument is divided in several steps; more precisely, Steps 1-4 prove the statements in (i) while Steps 5, 6-7 and 8 are about the statements in (ii), (iii) and (iv), respectively. The assumption (6.4) $\rho > 2\sqrt{d}$ is essential in Step 3.

**Step 1.** One has

$$\mathcal{K}_{pn}(k) = \mathcal{K}_{pn}(k) + \Delta \mathcal{K}_{pn}(k) \quad \text{for all } k \in \mathbb{Z}_d^n, \quad (6.17)$$

where, as in (6.2), $\mathcal{K}_{pn}(k) := 4|k|^{2p} \sum_{h \in \mathbb{Z}_d^n, |h| < \rho \text{ or } |k-h| < \rho} \frac{|P_{h,k-h}|^2}{(|h|^p |k-h|^n + |h|^n |k-h|^p)^2}$, while

$$\Delta \mathcal{K}_{pn}(k) := 4|k|^{2p} \sum_{h \in \mathbb{Z}_d^n, |h| \geq \rho \text{ or } |k-h| \geq \rho} \frac{|P_{h,k-h}|^2}{(|h|^p |k-h|^n + |h|^n |k-h|^p)^2} \in (0, +\infty). \quad (6.18)$$

The above decomposition follows noting that $\mathbb{Z}_d^n_{\rho k}$ is the disjoint union of the domains of the sums defining $\mathcal{K}_{pn}(k)$ and $\Delta \mathcal{K}_{pn}(k)$. $\mathcal{K}_{pn}(k)$ is finite, involving finitely many summands; $\Delta \mathcal{K}_{pn}(k)$ is finite as well, since we know that $\mathcal{K}_{pn}(k) < +\infty$.

**Step 2.** For each $k \in \mathbb{Z}_d^n$, one has the representation (6.6)

$$\mathcal{K}_{pn}(k) = 4|k|^{2p} \sum_{h \in \mathbb{Z}_d^n, |h| < \rho} \frac{|P_{h,k-h}|^2 + H(|k-h| - \rho)|P_{h,k-h}|^2}{(|h|^p |k-h|^n + |h|^n |k-h|^p)^2}. \quad (6.6)$$
If $|k| \geq 2\rho$, one can replace $Z_{0k}^d$ with $Z_0^d$ and $H(|k - h| - \rho)$ with 1.

To prove (6.6) we reexpress the sum in Eq. (6.5), using Eq. (5.1) with $f(h) \equiv f_k(h) := |P_{h,k-h}|^2 / (|h|^p|k-h|^n + |h|^n|k-h|^p)^2$. To go on, assume $|k| \geq 2\rho$; then, for all $h \in Z_0^d$ with $|h| < \rho$ one has $|k-h| \geq |k-h| > \rho$ whence $h \neq k$ (i.e., $h \in Z_0^d$) and $H(|k-h| - \rho) = 1$, two facts which justify the replacements indicated above.

Step 3. For each $k \in Z_0^d$ one has

$$0 < \Delta \mathcal{K}_{pn}(k) \leq \delta \mathcal{K}_{pn}, \quad (6.19)$$

with $\delta \mathcal{K}_{pn}$ as in Eq. (6.7). The obvious relation $0 < \Delta \mathcal{K}_{pn}(k)$ has been already noted; in the sequel we prove that $\Delta \mathcal{K}_{pn}(k) \leq \delta \mathcal{K}_{pn}$. The definition (6.18) of $\Delta \mathcal{K}_{pn}(k)$ contains the term

$$|k|^{2p} P_{h,k-h}^2 / (|h|^p|k-h|^n + |h|^n|k-h|^p)^2 \leq |k|^{2p} \sin \vartheta_{h,k-h} \leq (|h|^p|k-h|^n + |h|^n|k-h|^p)^2 $$

$$\leq B_{pn} \left( \frac{1}{|h|^{2n}} + \frac{1}{|k-h|^{2n}} \right). \quad (6.20)$$

The first and the second inequality (6.20) follow, respectively, from the bound (2.31) $|P_{h\ell}| \leq \sin \vartheta_{h\ell}$ and from (5.3), in both cases with $\ell = k-h$. Inserting (6.20) into (6.18), we obtain

$$\Delta \mathcal{K}_{pn}(k) \leq \frac{B_{pn}}{2} \left( \sum_{h \in Z_0^d, |h| \geq \rho, |k-h| \geq \rho} \frac{1}{|h|^{2n}} + \sum_{h \in Z_0^d, |h| \geq \rho, |k-h| \geq \rho} \frac{1}{|k-h|^{2n}} \right). \quad (6.21)$$

The domain of the above two sums is contained in each one of the sets $\{ h \in Z^d \mid |h| \geq \rho \}$ and $\{ h \in Z^d \mid |k-h| \geq \rho \}$; so,

$$\Delta \mathcal{K}_{pn}(k) \leq \frac{B_{pn}}{2} \left( \sum_{h \in Z_0^d, |h| \geq \rho} \frac{1}{|h|^{2n}} + \sum_{h \in Z_0^d, |k-h| \geq \rho} \frac{1}{|k-h|^{2n}} \right). $$

Now a change of variable $h \mapsto k-h$ in the second sum shows that the latter is equal to the former, so

$$\Delta \mathcal{K}_{pn}(k) \leq B_{pn} \sum_{h \in Z_0^d, |h| \geq \rho} \frac{1}{|h|^{2n}}. \quad (6.22)$$

Finally, Eq. (6.22) and Eq. (5.52) with $\nu = 2n$ give

$$\Delta \mathcal{K}_{pn}(k) \leq \frac{2\pi^{d/2} B_{pn}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \left( \begin{array}{c} d - 1 \\ i \end{array} \right) \frac{d^{d/2-1/2-i/2}}{(2n - 1 - i)(\rho - 2\sqrt{d})^{2n-1-i}} = \delta \mathcal{K}_{pn} \quad \text{as in (6.7).}$$

Step 4. One has the inequalities (6.7) $\mathcal{K}_{pn}(k) < \mathcal{K}_{pn}(k) \leq \mathcal{K}_{pn}(k) + \delta \mathcal{K}_{pn}$. These relations follow immediately from the decomposition (6.17) $\mathcal{K}_{pn}(k) = \mathcal{K}_{pn}(k) + \Delta \mathcal{K}_{pn}(k)$ and from the bounds (6.19) on $\Delta \mathcal{K}_{pn}(k)$.
Step 5. One has the equalities (6.8)
\[ K_n(R_r k) = K_n(k), \quad K_n(P_{\sigma} k) = K_n(k), \]
involving the reflection and permutation operators \( R_r, P_{\sigma} \). The verification is based on considerations very similar to the ones that follow Eq. (3.7). 

Step 6. For all \( k \in \mathbb{Z}_0^d \) we have the inequality (6.9)
\[ \mathcal{K}_{pn}(k) \leq 8 \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{1}{|h|^{2n}} E_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right). \]

To prove this we start from the expression (6.6) for \( \mathcal{K}_{pn}(k) \); we substitute therein the obvious inequality \( H(|k - h| - \rho) \leq 1 \), and the relations \(|P_{h,k} - h|, |P_{k,h} - h| \leq \sin \vartheta_{h,k} - h\) following from (2.31). This gives
\[ \mathcal{K}_{pn}(k) \leq 8 \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{|k|^2 \sin^2 \vartheta_{h,k-h}}{(|h|^p |k-h|^n + |h|^n |k-h|^p)^2} . \] (6.23)

On the other hand, the \( h \)-th term in the above sum equals \( \frac{1}{|h|^{2n}} E_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) \) due to Eq. (5.19), so we have the thesis (6.9). 

Step 7. Let \( k \in \mathbb{Z}^d, |k| \geq \mu \rho \); we have the inequalities (6.10)
\[ \mathcal{K}_{pn}(k) \leq 8 \sum_{j=0}^{m} \frac{Q_{pnj}(\hat{k})}{|k|^{\gamma_{pnj}}} + 8 \frac{V_{pn} Y_{pn}}{|k|^{\gamma_{pn,m+1}}} \leq \mathcal{K}_{pn} , \]
where all the objects in the right hand side are defined as indicated in item (iii). In order to prove this we start from the inequality (6.9); for each \( h \in \mathbb{Z}_0^d \) with \(|h| < \rho\), on account of Eq. (5.26) the general term of the sum in (6.9) fulfills the following:
\[ \frac{1}{|h|^{2n}} E_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) \]
\[ = \frac{1}{|h|^{2n}} \left( \sum_{j=0}^{m} Q_{pnj}(\cos \vartheta_{hk}) \left( \frac{|h|}{|k|} \right)^{\gamma_{pnj}} + S_{pn,m+1}(\cos \vartheta_{hk}, \frac{|h|}{|k|}) \left( \frac{|h|}{|k|} \right)^{\gamma_{pn,m+1}} \right) \]
\[ \leq \frac{1}{|h|^{2n}} \left( \sum_{j=0}^{m} Q_{pnj}(\cos \vartheta_{hk}) \left( \frac{|h|}{|k|} \right)^{\gamma_{pnj}} + V_{pn} \left( \frac{|h|}{|k|} \right)^{\gamma_{pn,m+1}} \right). \] (6.24)

The last inequality depends on the remark that \(|h|/|k| < 1/\mu\) and from the definition (6.12) of \( V_{pn} \). The relation (6.9), the inequality in (6.24) and the definitions (6.11) of \( Q_{pnj} \), (6.13) of \( Y_{pn} \) give the first inequality (6.10); the second inequality (6.10) is an obvious consequence of the first one and of the definition (6.14) of \( \mathcal{K}_{pn} \). 

Step 8. The previous results imply the inequality (6.15). This statement is obvious. \( \square \)
7 The function $\mathcal{G}_{pn}$

As in Section 6 we consider $p, n \in \mathbb{R}$ such that $p \geq n > d/2 + 1$. For $k \in \mathbb{Z}_0^d$, we recall the definition (4.6)

$$\mathcal{G}_{pn}(k) := 4 \sum_{h \in \mathbb{Z}_0^d, |h| < |k|} \frac{(|k|^p - |k - h|^p)^2|P_{h,k-h}|^2}{(|h|^p|k - h|^{n-1} + |h|^n|k - h|^{p-1})^2}.$$  

The forthcoming proposition presents estimates about $\mathcal{G}_{pn}(k)$ and its sup for $k \in \mathbb{Z}_0^d$, its structure is very similar to the one of Proposition 6.1 about $\mathcal{K}_{pn}$ and its sup. The proof is given only for completeness, since it is just a rephrasing of arguments employed in Section 6. Let us also remark that, for $p = n$, the results presented hereafter become very similar to those appearing in Proposition B.1 of [11].

7.1 Proposition. As in Eqs. (6.2)–(6.3), let us choose a cutoff, a factor and an order

$$\rho \in (2\sqrt{d}, +\infty), \quad \mu \in (1, +\infty), \quad m \in \mathbb{N},$$

then the following holds (with the functions and quantities $\mathcal{S}_{pn}$, $\delta \mathcal{S}_{pn}$, $\ldots$, $\mathcal{Z}_{pn}$, $\mathcal{G}_{pn}$ mentioned in the sequel depending parametrically on $\rho$, $\mu$, $m$ and $d$): $\mathcal{G}_{pn}(k) \equiv \mathcal{G}_{pn\rho\mu}(k)$, $\delta \mathcal{S}_{pn} \equiv \delta \mathcal{S}_{pn\rho\mu}$, $\mathcal{Z}_{pn} \equiv \mathcal{Z}_{pn\rho\mu\mu}$.

(i) The function $\mathcal{G}_{pn}$ fulfills the inequalities

$$\mathcal{S}_{pn}(k) < \mathcal{G}_{pn}(k) \leq \mathcal{S}_{pn}(k) + \delta \mathcal{S}_{pn} \quad \text{for all } k \in \mathbb{Z}_0^d. \quad (7.1)$$

Here

$$\mathcal{G}_{pn}(k) := 4 \sum_{h \in \mathbb{Z}_0^d, |h| < |k|} \frac{(|k|^p - |k - h|^p)^2|P_{h,k-h}|^2}{(|h|^p|k - h|^{n-1} + |h|^n|k - h|^{p-1})^2}; \quad (7.2)$$

this function can be reexpressed as

$$\mathcal{G}_{pn}(k) = 4 \sum_{h \in \mathbb{Z}_0^d, |h| < |k|} \frac{(|k|^p - |k - h|^p)^2|P_{h,k-h}|^2}{(|h|^p|k - h|^{n-1} + |h|^n|k - h|^{p-1})^2} \quad (7.3)$$

$$+ \frac{H(|k - h|)}{(|k - h|^{n+1} + |k-h|^{p-1})^2},$$

(recall that $|P_{k-h,h}| = |P_{h,k-h}|$ if $d \geq 3$, due to (2.33)). If $|k| \geq 2\rho$, in Eq. (7.3) one can replace $\mathbb{Z}_0^d$ with $\mathbb{Z}_0^d$ and $H(|k - h|)$ with 1. Moreover

$$\delta \mathcal{S}_{pn} := \frac{2\pi^{d/2}C_{pn}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \left( \frac{d - 1}{i} \right) \frac{d^{d/2-1/2-i/2}}{(2n - 3 - i)(\rho - 2\sqrt{d})^{2n-3-i}}, \quad (7.4)$$

with $C_{pn}$ as in (5.12).
(ii) Consider the reflection and permutation operators $R_r, P_\sigma$ defined by \((3.6)\). Then
\[
G_{pn}(R_rk) = G_{pn}(k), \quad G_{pn}(P_\sigma k) = G_{pn}(k) \quad \text{for each } k \in \mathbb{Z}_0^d. \quad (7.5)
\]

(iii) Denoting with $F_{pn}$ the function in Lemma \ref{lem:5.9}, one has
\[
G_{pn}(k) \leq 4 \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{1}{|h|^{2n-2}} F_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) \quad \text{for all } k \in \mathbb{Z}_0^d. \quad (7.6)
\]

Now, consider the sequence of exponents $0 = \lambda_{pn0} < \lambda_{pn1} < \lambda_{pn2} \ldots$ and the sequences of polynomials $P_{pnj}$ ($j \in \mathbb{N}$) and functions $R_{pnj}$ ($j \in \mathbb{N} \setminus \{0\}$) involved in the expansion of $F_{pn}$ according to Lemma \ref{lem:5.10}; then
\[
G_{pn}(k) \leq 4 \sum_{j=0}^{m} P_{pnj}(\hat{k}) |k|^\lambda_{pnj} + 4 |W_{pn}Z_{pn}| \leq \mathcal{G}_{pn} \quad \text{for } k \in \mathbb{Z}_0^d, |k| \geq \mu \rho. \quad (7.7)
\]

Here $P_{pnj}$ are the functions defined as follows on the spherical hypersurface $S^{d-1}$:
\[
P_{pnj} : S^{d-1} \to \mathbb{R}, \quad u \mapsto P_{pnj}(u) := \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} P_{pnj}(\cos \vartheta_{hu}) |h|^{2n-2-\lambda_{pnj}} \quad (7.8)
\]

(\text{these are polynomials in the components of $u$, which can be computed using Lemma \ref{lem:5.12}). Moreover}
\[
W_{pn} := \max_{c \in [-1,1], \xi \in [0,1/\mu]} R_{pn,m+1}(c, \xi), \quad (7.9)
\]
\[
Z_{pn} := \sum_{h \in \mathbb{Z}_0^d, |h| < \rho} \frac{1}{|h|^{2n-2-\lambda_{pn,m+1}}}. \quad (7.10)
\]

Finally,
\[
\mathcal{G}_{pn} := 4 \max_{u \in S^{d-1}, c \in [0,1/(\mu \rho)]} \left( \sum_{j=0}^{m} \epsilon^{\lambda_{pnj}} P_{pnj}(u) + \epsilon^{\lambda_{pn,m+1}} W_{pn}Z_{pn} \right). \quad (7.11)
\]

(iv) Items (i) and (iii) imply
\[
\sup_{k \in \mathbb{Z}_0^d} \mathcal{G}_{pn}(k) \leq \max \left( \max_{k \in \mathbb{Z}_0^d, |k| < \mu \rho} \mathcal{G}_{pn}(k), \mathcal{G}_{pn} \right) + \delta \mathcal{G}_{pn}. \quad (7.12)
\]

The proof of the above statements will be given after the following comment.
7.2 Remark (very similar to Remark 6.2). From Theorem 4.4 we know that the sharp constant $G_{pn}$ of (7.1) fulfills $G_{pn} \leq G_{pn}^{-} := (2\pi)^{-d/2} \sup_{k \in Z_{d}^{0}} G_{pn}(k)$. So, using for $\sup_{k \in Z_{d}^{0}} G_{pn}(k)$ the bound (7.12), we conclude

$$G_{pn}^{-} \leq G_{pn}^{+} \leq G_{pn}^{(+)} := \frac{1}{(2\pi)^{d/2}} \max_{k \in Z_{d}^{0}} \left( \max_{|k| < \mu \rho} G_{pn}(k), \mathfrak{S}_{pn} \right) + \delta \mathfrak{S}_{pn} . \quad (7.13)$$

The bound $G_{pn}^{(+)}$ is suitable for computer implementation, an aspect treated in Section 8. From the statements in Proposition 7.1, it is clear that $G_{pn}^{(+)}$ gives a good approximation of $G_{pn}^{-}$ under two conditions:

(a) $\mathfrak{S}_{pn}$ must be smaller or not too larger than $\max_{|k| < \mu \rho} G_{pn}(k)$.

(b) $\delta \mathfrak{S}_{pn}$ must be small in comparison with $\max \left( \max_{|k| < \mu \rho} G_{pn}(k), \mathfrak{S}_{pn} \right)$.

Proof of Proposition 7.1. We fix $\rho, \mu, m$ as in Eqs. (6.1)- (6.3). Our argument is divided in several steps; more precisely, Steps 1-4 prove the statements in (i) while Steps 5, 6-7 and 8 are about the statements in (ii), (iii) and (iv), respectively. The assumption (6.1) $\rho > 2\sqrt{d}$ is essential in Step 3.

Step 1. One has

$$\mathfrak{S}_{pn}(k) = \mathfrak{S}_{pn}(k) + \Delta \mathfrak{S}_{pn}(k) \quad \text{for all } k \in Z_{0}^{d} , \quad (7.14)$$

where, as in (7.2), $\mathfrak{S}_{pn}(k) := 4 \sum_{h \in Z_{d}^{0} : |h| < \rho \text{ or } |k-h| < \rho} \frac{(|k|^{p} - |k-h|^{p})^{2} |P_{h,k-h}|^{2}}{(|h|^{p} |k-h|^{n-1} + |h|^{n} |k-h|^{p-1})^{2}}$,

while

$$\Delta \mathfrak{S}_{pn}(k) := 4 \sum_{h \in Z_{d}^{0} : |h| \geq \rho, |k-h| \geq \rho} \frac{(|k|^{p} - |k-h|^{p})^{2} |P_{h,k-h}|^{2}}{(|h|^{p} |k-h|^{n-1} + |h|^{n} |k-h|^{p-1})^{2}} \in (0, +\infty) . \quad (7.15)$$

The above decomposition follows noting that $Z_{0}^{d}$ is the disjoint union of the domains of the sums defining $\mathfrak{S}_{pn}(k)$ and $\Delta \mathfrak{S}_{pn}(k)$. $\mathfrak{S}_{pn}(k)$ is finite, involving finitely many summands; $\Delta \mathfrak{S}_{pn}(k)$ is finite as well, since we know that $\mathfrak{S}_{pn}(k) < +\infty$.

Step 2. For each $k \in Z_{0}^{d}$, one has the representation (7.3)

$$\mathfrak{S}_{pn}(k) = 4 \sum_{h \in Z_{d}^{0} : |h| < \rho} \left( \frac{(|k|^{p} - |k-h|^{p})^{2} |P_{h,k-h}|^{2}}{(|h|^{p} |k-h|^{n-1} + |h|^{n} |k-h|^{p-1})^{2}} \right. \left. + \frac{H(|k-h| - \rho)(|k|^{p} - |h|^{p})^{2} |P_{k-h,h}|^{2}}{(|h|^{p} |k-h|^{n-1} + |k-h|^{n} |h|^{p-1})^{2}} \right) .$$

If $|k| \geq 2\rho$, one can replace $Z_{0}^{d}$ with $Z_{0}^{d}$ and $H(|k-h| - \rho)$ with 1.

To prove (7.3) we reexpress the sum in Eq. (7.2), using Eq. (5.1) with $f(h) \equiv f_{k}(h) := (|k|^{p} - |k-h|^{p})^{2} |P_{h,k-h}|^{2}/(|h|^{p} |k-h|^{n-1} + |h|^{n} |k-h|^{p-1})^{2}$. To go on, assume
\(|k| \geq 2\rho\); then, for all \(h \in \mathbb{Z}^d_0\) with \(|h| < \rho\) one has \(|k-h| \geq |k| - |h| > \rho\) whence \(h \neq k\) (i.e., \(h \in \mathbb{Z}^d_{\neq k}\)) and \(H(|k-h| - \rho) = 1\), two facts which justify the replacements indicated above.

**Step 3.** For each \(k \in \mathbb{Z}^d_0\) one has

\[
0 < \Delta \mathcal{G}_{pm}(k) \leq \delta \mathcal{G}_{pm},
\]

with \(\delta \mathcal{G}_{pm}\) as in Eq. (7.14). The obvious relation \(0 < \Delta \mathcal{G}_{pm}(k)\) was already noted; in the sequel we prove that \(\Delta \mathcal{G}_{pm}(k) \leq \delta \mathcal{G}_{pm}\). The definition (7.15) of \(\Delta \mathcal{G}_{pm}(k)\) contains the term

\[
\frac{(|k|^p - |k-h|^p)|P_{h,k-h}|^2}{(|h|^p|k-h|^n-1 + |h|^n|k-h|^p-1)^2} \leq \frac{(|k|^p - |k-h|^p)^2 \sin^2 \vartheta_{h,k-h}}{(|h|^p|k-h|^n-1 + |h|^n|k-h|^p-1)^2}
\]

(7.17)

The first and the second inequality (7.17) follow, respectively, from the bound (2.31) \(|P_{h\ell}| \leq \sin \vartheta_{h\ell}\) and from (5.14), in both cases with \(\ell = k - h\). Inserting (7.17) into (7.15) we obtain

\[
\Delta \mathcal{G}_{pm}(k) \leq \frac{C_{pm}}{2} \left( \sum_{h \in \mathbb{Z}^d_0, |h| > \rho, |k-h| > \rho} \frac{1}{|h|^{2n-2}} + \sum_{h \in \mathbb{Z}^d_0, |h| > \rho, |k-h| > \rho} \frac{1}{|k-h|^{2n-2}} \right).
\]

(7.18)

The domain of the above two sums is contained in each one of the sets \(\{ h \in \mathbb{Z}^d_0 \mid |h| > \rho \}\) and \(\{ h \in \mathbb{Z}^d_0 \mid |k-h| > \rho \}\); so,

\[
\Delta \mathcal{G}_{pm}(k) \leq \frac{C_{pm}}{2} \left( \sum_{h \in \mathbb{Z}^d_0, |h| > \rho} \frac{1}{|h|^{2n-2}} + \sum_{h \in \mathbb{Z}^d_0, |k-h| > \rho} \frac{1}{|k-h|^{2n-2}} \right).
\]

(7.19)

Now a change of variable \(h \mapsto k-h\) in the second sum shows that the latter is equal to the former, so

\[
\Delta \mathcal{G}_{pm}(k) \leq C_{pm} \sum_{h \in \mathbb{Z}^d_0, |h| > \rho} \frac{1}{|h|^{2n-2}}.
\]

(7.19)

Finally, Eq. (7.19) and Eq. (5.32) with \(\nu = 2n-2\) give

\[
\Delta \mathcal{G}_{pm}(k) \leq \frac{2n^d/2C_{pm}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \binom{d-1}{i} \frac{d^{d/2-1/2-i/2}}{(2n-3-i)(2\sqrt{d})^{2n-3-i}} = \delta \mathcal{G}_{pm} \text{ as in (7.14)}.
\]

(7.19)

**Step 4.** One has the inequalities (7.1) \(\mathcal{G}_{pm}(k) < \mathcal{G}_{pm}(k) \leq \mathcal{G}_{pm}(k) + \Delta \mathcal{G}_{pm}(k)\). These relations follow immediately from the decomposition (7.14) \(\mathcal{G}_{pm}(k) = \mathcal{G}_{pm}(k) + \Delta \mathcal{G}_{pm}(k)\) and from the bounds (7.16) on \(\Delta \mathcal{G}_{pm}(k)\).
Step 5. One has the equalities (7.3) \( S_{pn}(R, k) = S_{pn}(k) \), \( S_{pn}(P_\sigma k) = S_{pn}(k) \), involving the reflection and permutation operators \( R, P_\sigma \). The verification is based on considerations very similar to the ones that follow Eq. (5.4).

Step 6. For all \( k \in \mathbb{Z}_0^d \) we have the inequality (7.6)

\[
S_{pn}(k) \leq 4 \sum_{h \in \mathbb{Z}_0^d, |h| \leq \rho} \frac{1}{|h|^{2n-2}} F_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right).
\]

To prove this we start from the expression (7.3) for \( S_{pn}(k) \); we substitute therein the obvious inequality \( H(|k - h| - \rho) \leq 1 \), and the relations \(|P_{k-h,h}|, |P_{h,h-k}| \lesssim \sin \vartheta_{h,k-h}\) following from (2.31). This gives

\[
S_n(k) \leq 4 \sum_{h \in \mathbb{Z}_0^d, |h| \leq \rho} \sin^2 \vartheta_{h,k-h} \left( \frac{(|k|^p - |k| - h|^p|^2)}{(|h|^p - |h| - h|^p|^2)^2} k + 1 \right) \left( \frac{|h|^p - |h|^p}{(k - h)^{n-1} + |k - h|^n |h|^{p-1}} \right);
\]

on the other hand, the \( h \)-th term in the above sum equals \( \frac{1}{|h|^{2n-2}} F_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) \) due to Eq. (5.37), so we have the thesis (7.6).

Step 7. Let \( k \in \mathbb{Z}^d, |k| \geq \mu \rho \); we have the inequalities (7.7)

\[
S_{pn}(k) \leq 4 \sum_{j=0}^{m} P_{pnj}(k) \left( |k| \lambda_{pnj} \right) + 4 W_{pn} Z_{pn} \left( |k| \lambda_{pn,m+1} \right) \leq \mathcal{G}_{pn},
\]

where all the objects in the right hand side are defined as indicated in item (iii). In order to prove this we start from the inequality (7.6); for each \( h \in \mathbb{Z}_0^d \) with \( |h| < \rho \), the general term of the sum in (7.6) fulfills, due to (5.42):

\[
\frac{1}{|h|^{2n-2}} F_{pn} \left( \cos \vartheta_{hk}, \frac{|h|}{|k|} \right) \leq 1 \sum_{j=0}^{m} P_{pnj}(\cos \vartheta_{hk}) \frac{|h|}{|k|} \lambda_{pnj} + R_{pn,m+1}(\cos \vartheta_{hk}) \frac{|h|}{|k|} \lambda_{pn,m+1} + W_{pn} \frac{|h|}{|k|} \lambda_{pn,m+1}.
\]

The last inequality depends on the remark that \( |h|/|k| < 1/\mu \) and from the definition (7.9) of \( W_{pn} \). The relation (7.6), the inequality in (7.21) and the definitions (7.8) of \( P_{pnj} \), (7.10) of \( Z_{pn} \) give the first inequality (7.7); the second inequality (7.7) is an obvious consequence of the first one and of the definition (7.11) of \( \mathcal{G}_{pn} \).

Step 8. The previous results imply the inequality (7.12). This statement is obvious. 

\[ \square \]
8 Computation of $K^{(+)}_{pn}$, $G^{(+)}_{pn}$ in the cases (1.10) or (1.11)

Throughout this section the space dimension is

$$d = 3.$$  (8.1)

We make extensive reference to Propositions 6.1 and 7.1 and to the definitions of the upper bounds $K^{(+)}_{pn}$, $G^{(+)}_{pn}$ in Eqs. (6.16) (7.13); our purpose is to describe the computer implementation of these definitions and, in particular, the calculations performed for the cases (1.10) or (1.11).

**Computation of $K^{(+)}_{pn}$.** Eq. (6.16) reads

$$K^{(+)}_{pn} = \frac{1}{(2\pi)^{3/2}} \sqrt{\max_{k\in\mathbb{Z}^3_n, |k| < \mu \rho} K_{pn}(k), \mathfrak{R}_{pn}} + \delta K_{pn}.$$  (8.2)

We recall that the right hand side of the above equation depends on a cutoff $\rho > 2\sqrt{3}$, on a factor $\mu > 1$ and on the order $m \in \mathbb{N}$ involved in the expansion determining $\mathfrak{R}_{pn}$.

Hereafter we describe the choices of $\rho, \mu, m$ and give details on the calculation of $K_{pn}(k)$, $\mathfrak{R}_{pn}$ and $\delta K_{pn}$, for the cases (1.10) (1.11). The results obtained in these cases are summarized in Table C; here the last column contains the final values of $K^{(+)}_{pn}$ given by the overall procedure, which coincide with the ones anticipated in Table A. Any one of the necessary computations has been performed on a PC with an 8 Gb RAM, with the software utilities mentioned below.

**Choosing $\rho, \mu, m$.** As explained in Remark 6.2 in the implementation of Eq. (8.2) it is convenient to choose the cutoff $\rho$, the factor $\mu$ and the order $m$ of the expansion determining $\mathfrak{R}_{pn}$ so that $\mathfrak{R}_{pn}$ is smaller (or not too larger) than $\max_{|k| < \mu \rho} K_{pn}(k)$, and $\delta K_{pn}$ is small with respect to $\max \left( \max_{|k| < \mu \rho} K_{pn}(k), \mathfrak{R}_{pn} \right)$. In all our computations we have taken

$$\mu = 2, \quad m = 6$$  (8.3)

and we have used cutoffs $\rho$ between 20 and 100, chosen empirically so as to fulfill the previous requirements (starting from the lower value $\rho = 20$, and increasing this if necessary. Of course, the computational costs increase with $\rho$).

**The error bound $\delta K_{pn}$.** This is readily computed in terms of the cutoff $\rho$ via Eq. (6.7). The constants $B_{pn}$ in the cited equation are the maxima of certain functions, see Eqs. (5.2) (5.3) and Remark 5.4. Our actual computation of the $\delta K_{pn}$’s has been made using Mathematica.
Here and in the sequel, to give an example of our calculations we consider the case \((p,n) = (5,2)\). As indicated in Table C, we have chosen
\[
\rho = 50 \quad \text{for } p = 5, \ n = 2,
\]
that gives
\[
\delta K_{52} = 65.0229\ldots
\]  
\[\text{(8.5)}\]

**Computation of** \(K_{pn}(k)\) **for** \(k \in \mathbb{Z}_0^3, \ |k| < 2\rho\) **and of its maximum over this ball.**

The computation of \(K_{pn}(k)\) at all nonzero points in the ball \(\{|k| < 2\rho\}\) have been performed using an ad hoc C program, for each one of the cases (1.10) (1.11). These computations allow to determine the maximum point of \(K_{pn}\) over the ball; the value of \(K_{pn}\) at this point has been subsequently validated using Arb [17], a C-library that produces certified estimates on the roundoff errors. For example in the case \((p,n) = (5,2)\), treated with a cutoff \(\rho = 50\), we have found
\[
\max_{k \in \mathbb{Z}_0^3, |k| < 100} K_{52}(k) = K_{52}(2,1,0) = 263.364\ldots
\]  
\[\text{(8.6)}\]

for the other cases, see Table C. Computations for all values of \(k\) involved in Eq. (8.6) and the subsequent determination of the maximum have required a CPU time of about 3 hours on our PC; the validation of the value at the maximum point \(k = (2,1,0)\) has been been performed very quickly using Arb [3]. The analogous computations for \(n = 2\) and each one of the cases \(p = 7, 8, 9, 10\), based on the larger cutoff \(\rho = 100\), have required a CPU time of about 4 days.

**Computation of** \(R_{pn}\). This requires a rather long procedure, described by item (iii) of Proposition 6.1. First of all one should consider the function \(E_{pn}(c,\xi)\) of Lemma 5.7 and build the expansion described by Lemma 5.8. We have already mentioned the choice \(m = 6\) for the order of this expansion; for the cases that we have considered, where \(p, n\) are integers, the expansion of \(E_{pn}(c,\xi)\) involves only integer powers of \(\xi\) and takes the form
\[
E_{pn}(c,\xi) = \sum_{j=0}^{6} Q_{pnj}(c)\xi^j + S_{pn7}(c, \xi)\xi^7.
\]  
\[\text{(8.7)}\]

The coefficient \(S_{pn7}(c, \xi)\) in the reminder term is important, since its maximization (for \(c \in [-1, 1], \xi \in [0, 1/2]\)) gives a constant \(V_{pn}\) to be used later (see Eq. (6.12)). The coefficients \(Q_{pnj}(c)\) must be used to build certain functions \(Q_{pnj} : \mathbb{S}^2 \to \mathbb{R}\), see Eq. (6.11); next one should compute a finite zeta-type sum \(Y_{pn}\) (see again Eq. (6.12)). Finally, one determines \(R_{pn}\) via maximization for \(u \in \mathbb{S}^2, \epsilon \in [0, 1/2]\) of a certain function of these variables, built from the previous ingredients (see Eq. (6.14)).

\[\text{The Arb result is } K_{52}(2,1,0) = 263.36493191766936106 \pm 9.6212 \times 10^{-14}.\]
For all values of \( p, n \) in (1.10) (1.11) the coefficients \( Q_{pnj} \), the related functions \( Q_{pnj} : \mathbb{S}^2 \to \mathbb{R} \) \((j = 0, ..., 6)\) and the sums \( Y_{pn} \) have been computed symbolically using Mathematica. The maxima (for \( c \in [-1, 1], \xi \in [0, 1/2] \) or for \( u \in \mathbb{S}^2, \epsilon \in [0, 1/2\rho] \)) defining \( V_{pn} \) and \( \mathfrak{K}_{pn} \) have been computed numerically, using the internal routines of Mathematica. As an example, for \((p, n) = (5, 2)\) we have found

\[
Q_{520}(c) = 1 - c^2, \quad Q_{521}(c) = 12(c - c^3), \quad Q_{522}(c) = -6 + 90c^2 - 84c^4, \ldots \quad (8.8)
\]

\[
Q_{526}(c) = -53 + 255c + 3077c^2 - 1870c^3 - 23184c^4 + 1615c^5 + 49728c^6 - 29568c^8;
\]

\[
V_{52} = 2211.24\ldots . \quad (8.9)
\]

The other ingredients in these calculations depend on the cutoff, so we recall the choice \( \rho = 50 \) for the present case. We have recalled that one can compute from the coefficients \( Q_{52j} \) the functions \( Q_{52j} \); we report only one of the functions, namely

\[
Q_{522}(u) = 14861.4... - 10448.7... (u_1^4 + u_2^4 + u_3^4) - 20668.7... (u_1^2 u_2^2 + u_1^2 u_3^2 + u_2^2 u_3^2)
\]

for \( u = (u_1, u_2, u_3) \in \mathbb{S}^2 \).

Moreover,

\[
Y_{52} = 3.26693\ldots \times 10^{10} . \quad (8.11)
\]

Finally, maximizing for \( u \in \mathbb{S}^2 \) and \( \epsilon \in [0, 1/2\rho] = [0, 1/100] \) the function in Eq. (6.14) we obtain

\[
\mathfrak{K}_{52} = 92.5195\ldots , \quad (8.12)
\]

as indicated in Table C.

The final step: determination of \( K_{pn}^{(+)} \). After computing \( \delta K_{pn}, \max_{k \in \mathbb{Z}^3_+ |k| < 2\rho} K_{pn}(k) \) and \( \mathfrak{K}_{pn} \) one returns to Eq. (8.2) and obtains \( K_{pn}^{(+)} \). The values indicated with \( K_{pn}^{(+)} \) in Table C are in fact the roundups to 3 digits of the right hand side of Eq. (8.2). These roundups are our final upper bounds for the sharp constants \( K_{pn} \) in the inequality (1.6), in the cases under consideration.
Table C. Details on the computation of $K_{pn}^{(+)}$ in the cases (1.10) (1.11) (using Proposition 6.1 with $\mu = 2$, $m = 6$)

| $(p, n)$ | $\rho$ | $\max K_{pn}$ (*) | $k_{max}$ (**) | $\mathcal{R}_{pn}$ | $\delta K_{pn}$ | $K_{pn}^{(+)}$ |
|----------|--------|-------------------|----------------|-----------------|----------------|----------------|
| $(2, 2)$ | 20     | 22.0223...        | (9,9)          | 21.6447...      | 5.68568...     | 0.335          |
| $(3, 2)$ | 20     | 77.8597...        | (25, 23, 21)   | 84.8166...      | 17.6648...     | 0.643          |
| $(4, 2)$ | 20     | 113.227...        | (2,2,2)        | 89.5797...      | 57.7725...     | 0.831          |
| $(5, 2)$ | 50     | 263.364...        | (2,1,0)        | 92.5195...      | 65.0229...     | 1.16           |
| $(6, 2)$ | 50     | 702.295...        | (2,1,0)        | 97.2697...      | 225.357...     | 1.94           |
| $(7, 2)$ | 100    | 1884.65...        | (2,1,0)        | 96.4078...      | 376.103...     | 3.02           |
| $(8, 2)$ | 100    | 5018.97...        | (2,1,0)        | 92.5195...      | 1345.89...     | 5.07           |
| $(9, 2)$ | 100    | 13205.2...        | (2,1,0)        | 97.2697...      | 225.357...     | 7.04           |
| $(10, 2)$| 100    | 263.364...        | (2,1,0)        | 92.5195...      | 65.0229...     | 1.16           |
| $(3, 3)$ | 20     | 25.3013...        | (2,1,1)        | 11.2784...      | 0.0226087...   | 0.320          |
| $(4, 3)$ | 20     | 71.8198...        | (2,1,0)        | 44.8074...      | 0.0739415...   | 0.539          |
| $(5, 3)$ | 20     | 204.342...        | (2,1,0)        | 45.9450...      | 0.250165...    | 0.909          |
| $(6, 3)$ | 20     | 581.166...        | (2,1,0)        | 45.9450...      | 0.867027...    | 1.54           |
| $(7, 3)$ | 20     | 1636.38...        | (2,1,0)        | 46.3859...      | 3.05959...     | 2.58           |
| $(8, 3)$ | 20     | 4521.94...        | (2,1,0)        | 46.9192...      | 10.9488...     | 4.28           |
| $(9, 3)$ | 20     | 12237.3...        | (2,1,0)        | 47.5671...      | 39.6211...     | 7.04           |
| $(10, 3)$| 20     | 32495.3...        | (2,1,0)        | 48.3602...      | 144.694...     | 11.5           |

(*) $\max K_{pn}$ stands for $\max_{k \in \mathbb{Z}_3^3, |k| < 2\rho} K_{pn}(k)$.

(**) $k_{max}$ is a maximum point of the function $K_{pn}$ on the set $\{k \in \mathbb{Z}_3^3 | |k| < 2\rho\}$.

Computation of $G_{pn}^{(+)}$. We follow a general scheme very similar to the one employed to determine $K_{pn}^{(+)}$; however, in this case we refer to Proposition 7.1 and to Eq. (7.13); this reads

$$G_{pn}^{(+)} = \frac{1}{(2\pi)^{3/2}} \sqrt{\max_{k \in \mathbb{Z}_3^3, |k| < \mu \rho} \mathcal{G}_{pn}(k) + \delta \mathcal{G}_{pn}} + \mathcal{S}_{pn}$$  \hspace{1cm} (8.13)

with the right hand side depending on a cutoff $\rho > 2\sqrt{3}$, on a factor $\mu > 1$ and on the order $m \in \mathbb{N}$ in the expansion giving $\mathcal{G}_{pn}$.

Hereafter we describe the choices of $\rho, \mu, m$ and give details on the calculation of $\mathcal{G}_{pn}(k)$, $\mathcal{G}_{pn}$ and $\delta \mathcal{G}_{pn}$ in the cases (1.11). The results are summarized in Table D; here the last column contains the final values of $G_{pn}^{(+)}$ anticipated in Table B. For the necessary computations, we have used the hardware and software utilities already mentioned in relation to the constants $K_{pn}$.

43
Choosing $\rho, \mu, m$. According to Remark \[7.2\] in the implementation of Eq. \[8.13\] it is convenient to choose the cutoff $\rho$, the factor $\mu$ and the order $m$ of the expansion determining $S_{\rho \mu m}$ so that $G_{\rho \mu m}$ be smaller (or not too larger) than $\max_{|k|<\mu \rho} S_{\rho \mu m}(k)$, and $\delta S_{\rho \mu m}$ be small with respect to $\max \{ \max_{|k|<\mu \rho} S_{\rho \mu m}(k), \, G_{\rho \mu m} \}$. In all our computations we have taken

$$\mu = 2, \quad m = 6 \quad (8.14)$$

and we have used cutoffs $\rho$ between 20 and 100, chosen empirically so as to fulfill the previous requirements.

The error bound $\delta S_{\rho \mu m}$. This is readily computed in terms of the cutoff $\rho$ via Eq. \[7.4\], which has been implemented via Mathematica. The constants $C_{\rho \mu m}$ in the cited equation are the maxima of certain functions, see Eqs. \[5.11\] \[5.12\]; for the values of $(p, n)$ in \[1.11\], these constants have been determined by numerical maximization (see Eq. \[5.16\] for some examples).

Computation of $S_{\rho \mu m}(k)$ for $k \in \mathbb{Z}_0^3$, $|k| < 2\rho$ and of its maximum over this ball. The necessary computations have been performed using a C program; for each case $(p, n)$ in \[1.11\] involving the largest cutoff $\rho = 100$, the CPU time has been of 4 days approximately. For all cases in the table, the value of $S_{\rho \mu m}$ at the maximum point in the ball $\{|k| < 2\rho\}$ has been validated using Arb.

Computation of $G_{\rho \mu m}$. This is based on the procedure in item (iii) of Proposition \[7.1\]. First of all one should consider the function $F_{\rho \mu m}(c, \xi)$ of Lemma \[5.9\] and build the expansion described by Lemma \[5.10\]. We have already mentioned the choice $m = 6$ for the order of this expansion; for the cases that we have considered, where $p, n$ are integers, the expansion of $F_{\rho \mu m}(c, \xi)$ involves only integer powers of $\xi$ and takes the form

$$F_{\rho \mu m}(c, \xi) = \sum_{j=0}^{6} P_{\rho \mu m j}(c) \xi^j + R_{\rho \mu m 7}(c, \xi) \xi^7. \quad (8.15)$$

The coefficient $R_{\rho \mu m 7}$ in the reminder of this expansion determines, after a maximization for $c \in [-1, 1], \xi \in [0, 1/2]$, a constant $W_{\rho \mu m}$ to be used later (see Eq. \[7.9\]). The coefficients $P_{\rho \mu m j}$ are used to build certain functions $P_{\rho \mu m j} : S^2 \to \mathbb{R}$ (see Eq. \[7.8\]). After computing a finite zeta-type sum $Z_{\rho \mu m}$ (see again Eq. \[7.9\]), one finally determines $G_{\rho \mu m}$ via maximization for $u \in S^2$, $\epsilon \in [0, 1/2\rho]$ of a certain function of these variables, built from the previous ingredients (see Eq. \[7.11\]). For the values of $(p, n)$ in \[1.11\], all the above mentioned computations have been performed using Mathematica.

The final step: determination of $G_{\rho \mu m}^{(+)}$. After computing $\delta S_{\rho \mu m}$, $\max_{k \in \mathbb{Z}_0^3, |k|<2\rho} S_{\rho \mu m}(k)$ and $G_{\rho \mu m}$ one returns to Eq. \[8.13\] and obtains $G_{\rho \mu m}^{(+)}$. The values indicated in Table D with $G_{\rho \mu m}^{(+)}$ are in fact the roundups to 3 digits of the right hand side of Eq. \[8.13\]. These roundups are our final upper bounds for the sharp constants $G_{\rho \mu m}$ in the inequality \[1.7\], in the cases under consideration.
Table D. Details on the computation of $G_{pn}^{(+)}$ in the cases (1.11) (using Proposition 7.1 with $\mu = 2, m = 6$)

| $(p, n)$ | $\rho$  | $\max G_{pn}$ (*) | $k_{\max}$ (**) | $\mathcal{G}_{pn}$ | $\delta G_{pn}$ | $G_{pn}^{(+)}$ |
|--------|--------|-------------------|-----------------|-----------------|---------------|--------------|
| (3, 3) | 20     | 34.9016...        | (9,9,9)         | 34.4741...      | 12.4785...    | 0.438        |
| (4, 3) | 20     | 190.684...        | (23,23,23)      | 206.799...      | 51.5254...    | 1.03         |
| (5, 3) | 50     | 325.352...        | (4,4,4)         | 309.674...      | 64.3690...    | 1.26         |
| (6, 3) | 50     | 816.449...        | (2,1,0)         | 437.386...      | 230.273...    | 2.06         |
| (7, 3) | 50     | 2356.409...       | (2,1,0)         | 593.730...      | 817.263...    | 3.58         |
| (8, 3) | 100    | 6611.94...        | (2,1,0)         | 755.564...      | 1380.53...    | 5.68         |
| (9, 3) | 100    | 18068.8...        | (2,1,0)         | 965.898...      | 4977.42...    | 9.64         |
| (10,3) | 100    | 48275.0...        | (2,1,0)         | 1218.84...      | 18110.5...    | 16.4         |

(*) $\max G_{pn}$ stands for $\max_{k \in \mathbb{Z}_3^3 | |k| < 2\rho} G_{pn}(k)$.

(**) $k_{\max}$ is a maximum point of the function $G_{pn}$ on the set $\{ k \in \mathbb{Z}_3^3 | |k| < 2\rho\}$.

Comparison with previous works. We have already mentioned that the constants $K_{pn}$, $G_{pn}$ have been discussed in the previous works [11] [12] in the special case $p = n$ (using the notations $K_n, G_n$ for $K_{nn}$ and $G_{nn}$). The values of $K_{22}^{(+)}$ and $G_{33}^{(+)}$ in Tables C and D agree with the upper bounds on $K_2$ and $G_3$ computed in the cited works (for $d = 3$). In [12] it has been shown that (for $d = 3$) $K_3$ has an upper bound equal to 0.323; the result in Table C (namely, the upper bound $K_{33}^{(+)} = 0.320$) is a slight improvement, due to the use of balls with a larger value of the radius $\rho$.

9 Lower bounds for the sharp constant $K_{pn}$ of the inequality (1.6)

Consider a pair $p, n$ as in (3.1). To obtain the above mentioned lower bounds it is sufficient to use the tautological inequality

$$K_{pn} \geq \frac{2 \|P(v, w)\|_p}{\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1}} \quad \text{for } v \in \mathbb{H}_p^\circ \setminus \{0\}, \ w \in \mathbb{H}_{p+1}^\circ \setminus \{0\},$$

choosing for $v$ and $w$ two suitable non zero “trial vector fields”; rather simple choices yield the results presented hereafter. Throughout this section we denote with $a, b$ the first two elements in the canonical basis of $\mathbb{R}^d$ and, if $d \geq 3$, we write $c$ for the third element. Thus

$$a := (1, 0, \ldots, 0), \ b := (0, 1, 0, \ldots) ; \quad c := (0, 0, 1, 0, \ldots) \quad \text{if } d \geq 3. \quad (9.2)$$

Let us also recall the notation $e_k$ for the Fourier basis, see Eq. (2.2).
9.1 Proposition. For all real $p, n$ with $p \geq n > d/2$ one has

$$K_{pn} \geq K_p^{[-]} , \quad K_p^{[-]} := \frac{U_d}{(2\pi)^{d/2}} 2^{p/2} ,$$

(9.3)

$$U_d := \begin{cases} 1 & \text{if } d \geq 3, \\ 1/\sqrt{2} & \text{if } d = 2. \end{cases}$$

(9.4)

**Proof.** Step 1. For $d \geq 3$ one has

$$K_{pn} \geq \frac{2^{p/2}}{(2\pi)^{d/2}} ;$$

(9.5)

so, the thesis (9.3) (9.4) holds if $d \geq 3$. To prove this we set

$$v := ib(e_a - e_{-a}), \quad w := ic(e_b - e_{-b}) .$$

(9.6)

The above vector fields have vanishing mean and divergence (since $k \cdot v_k = k \cdot w_k = 0$ for each $k$) and clearly belong to $H^m_{\Sigma_0}$ for any real $m$, with

$$\|v\|_m = \|w\|_m = \sqrt{2} .$$

(9.7)

Using Eqs. (2.12) (2.20) one finds

$$\mathcal{P}(v, w) = -\frac{i c}{(2\pi)^{d/2}} (e_{a+b} - e_{-a-b} + e_{a-b} - e_{-a+b})$$

(note that $\mathfrak{L}_k c = c$ for $k = \pm(a + b), \pm(a - b)$); this implies

$$\|\mathcal{P}(v, w)\|_p = \frac{2^{p/2+1}}{(2\pi)^{d/2}} .$$

(9.9)

Now, using Eqs. (9.1) and (9.7) (9.9) one readily infers Eq. (9.5).

Step 2. For $d = 2$ one has

$$K_{pn} \geq \frac{2^{p/2}}{2\sqrt{2\pi}} ;$$

(9.10)

so, the thesis (9.3) (9.4) holds if $d = 2$. To prove this we define $v$ as in (9.6), and put

$$w := ia(e_b - e_{-b}) .$$

(9.11)

Again, $v, w \in H^m_{\Sigma_0}$ for any real $m$, with $\|v\|_m, \|w\|_m$ as in (9.7). Using Eqs. (2.13) (2.20) one finds

$$\mathcal{P}(v, w) = -\frac{i}{4\pi} ((a - b)(e_{a+b} - e_{-a-b}) + (a + b)(e_{a-b} - e_{-a+b})) ;$$

(9.12)
this implies
\[ \| P(v, w) \|_p = \frac{2^{p/2-1/2}}{\pi} . \]  
(9.13)

Now, using Eqs. (9.1) and (9.7) (9.13) one infers Eq. (9.10).

\[ \square \]

9.2 Remark. For \( p = n \) Eq. (9.3) becomes (writing \( K_n^{[-]} \) for \( K_{nn}^{[-]} \))
\[ K_n \geq K_n^{[-]} , \quad K_n^{[-]} := \frac{U_d}{(2\pi)^{d/2}} \cdot 2^{n/2} ; \]
(9.14)
this bound has been already proposed in [12] (6).

9.3 Proposition. Let \( p, n \) be real, with \( p \geq n > d/2 \). For each \( \ell \in \mathbb{N}_0 := \{1, 2, 3,...\} \) one has
\[ K_{pn} \geq K_{pn}^{(-)}(\ell) , \]
(9.15)
where
\[ K_{pn}^{(-)}(\ell) := \begin{cases} \sqrt{2} \frac{\sqrt{1 + (1 + 4\ell^2)^p}}{(2\pi)^{d/2} \ell^p (1 + \ell^2)^{n/2+1/2} + \ell^n (1 + \ell^2)^{p/2+1/2}} & \text{if } d \geq 3, \\
\sqrt{2} \frac{\sqrt{1 + (1 + 2\ell^2)(1 + 4\ell^2)^{p-1}}}{2\pi \ell^p (1 + \ell^2)^{n/2+1} + \ell^n (1 + \ell^2)^{p/2+1}} & \text{if } d = 2. \end{cases} \]
(9.16)
Thus
\[ K_{pn} \geq K_{pn}^{(-)} := \sup_{\ell \in \mathbb{N}_0} K_{pn}^{(-)}(\ell) . \]
(9.17)

Proof. Of course, Eq. (9.17) is an obvious consequence of (9.15); in the sequel we derive Eqs. (9.15) (9.16) for any \( \ell \in \mathbb{N}_0 \), proceeding in two steps.
Step 1. For \( d \geq 3 \) one has
\[ K_{pn} \geq \frac{\sqrt{2}}{(2\pi)^{d/2}} \frac{\sqrt{1 + (1 + 4\ell^2)^{p}}}{\ell^p (1 + \ell^2)^{n/2+1/2} + \ell^n (1 + \ell^2)^{p/2+1/2}} ; \]
(9.18)
so, the thesis (9.15) (9.16) holds if \( d \geq 3 \). To prove this we use Eq. (9.1) with
\[ v \equiv v(\ell) := ib(\epsilon_{\ell a} - e_{-\ell a}) , \quad w \equiv w(\ell) := ic(\epsilon_{\ell a + b} - e_{-\ell a - b}) . \]
(9.19)

\[ \text{6See Eq. (3.9) of [12], corresponding to Eq. (3.22) in the extended arXiv version of the same paper. The cited work states that } U_d := 1 \text{ for } d \geq 3 \text{ and } U_d := (2 - \sqrt{2})^{1/2} \text{ for } d = 2 \text{; the value of } U_2 \text{ is wrong, and reflects an error in calculations with the trial vector fields presented therein. The correct formulation of the estimate of [12] for } d = 2 \text{ requires } U_2 \text{ to be defined as in } (9.4).} \]
We have \( v, w \in \mathbb{H}^m_{\Sigma_0} \) for any real \( m \), with
\[
\|v\|_m = \sqrt{2} \ell^m, \quad \|w\|_m = \sqrt{2} (1 + \ell^2)^{m/2}.
\] (9.20)

Using Eqs. (2.12) (2.20) we find
\[
\mathcal{P}(v, w) = -\frac{ic}{(2\pi)^{d/2}} (e_{2\ell a + b} - e_{-2\ell a - b} - e_b + e_{-b})
\] (9.21)
(note that \( \mathfrak{L}_k c = c \) for \( k = \pm(2\ell a + b), \pm b \); this implies
\[
\|\mathcal{P}(v, w)\|_p = \frac{\sqrt{2}}{(2\pi)^{d/2}} \sqrt{1 + (1 + 4\ell^2)^p}.
\] (9.22)

Now, using Eqs. (9.1) and (9.20) (9.22) we infer Eq. (9.18).

Step 2. For \( d = 2 \) one has
\[
K_{pn} \geq \frac{\sqrt{2}}{2\pi} \frac{1 + (1 + 2\ell^2)^2(1 + 4\ell^2)^{p-1}}{p(1 + \ell^2)^{n/2+1} + \ell^n(1 + \ell^2)^{p/2+1}};
\] (9.23)
so, the thesis (9.15) (9.16) holds if \( d = 2 \). To prove this we use Eq. (9.1) with \( v \equiv v(\ell) \) as in (9.19), and
\[
w \equiv w(\ell) := i c (e_{\ell a + b} - e_{-\ell a - b}) , \quad c := \frac{a - \ell b}{\sqrt{1 + \ell^2}}.
\] (9.24)
Again \( v, w \in \mathbb{H}^m_{\Sigma_0} \) for any real \( m \), with \( \|v\|_m, \|w\|_m \) as in (9.20). Using Eqs. (2.13) (2.20) we find
\[
\mathcal{P}(v, w) = -\frac{i}{2\pi} \frac{(1 + 2\ell^2)(a - 2\ell b)}{\sqrt{1 + \ell^2}(1 + 4\ell^2)} (e_{2\ell a + b} - e_{-2\ell a - b}) + \frac{i}{2\pi} \frac{a}{\sqrt{1 + \ell^2}} (e_b - e_{-b});
\] (9.25)
this implies
\[
\|\mathcal{P}(v, w)\|_p = \frac{\sqrt{2}}{2\pi} \frac{1 + (1 + 2\ell^2)^2(1 + 4\ell^2)^{p-1}}{\sqrt{1 + \ell^2}}.
\] (9.26)
Now, using Eqs. (9.1) and (9.20) (9.26) we infer Eq. (9.23).

\[ \square \]

9.4 Remark. Obviously enough, Propositions 9.1 and 9.3 imply
\[
K_{pn} \geq K_{pn}^{(-)} := \max(K_{p}^{[-]}, K_{pn}^{[-]})
\] (9.27)
for all \( p \geq n > d/2 \). We anticipate that the above maximum equals \( K_{pn}^{(-)} \) for fixed \( d, n \) and \( p \) sufficiently large; this fact is suggested by the subsequent numerical examples, and is proved in the forthcoming Remark 9.6.
Some numerical examples. Let us consider the \((d = 3)\) cases \((1.10)\) \((1.11)\). In all these cases, the function \(K_p^{(-)}( ) : \ell \in \mathbb{N}_0 \mapsto K^{(-)}_{pn}(\ell)\) attains its sup at \(\ell = 1\).

In the forthcoming Table E we report for each one of the above cases the values of \(K_p^{(-)}\) (defined by (9.3)) and of \(K^{(-)}_{pn}\) (the sup of the previous function), which immediately determine \(K^{(-)}_{pn}\) according to (9.27) \((7)\). The values of \(K_{pn}\) have been anticipated in Table A of the Introduction.

It should be noted that for larger values of \(p\), not considered in these examples, the sup of the function \(K^{(-)}_{pn}\) is attained at a point \(\ell_{pn} > 1\): see the forthcoming subsection.

| \((p, n)\) | \(K_p^{(-)}\) | \(K^{(-)}_{pn}\) | \(K^{(-)}_{pn}\) |
|---|---|---|---|
| \((2, 2)\) | 0.126 | 0.0809 | 0.126 |
| \((3, 2)\) | 0.179 | 0.147 | 0.179 |
| \((4, 2)\) | 0.253 | 0.264 | 0.264 |
| \((5, 2)\) | 0.359 | 0.463 | 0.463 |
| \((6, 2)\) | 0.507 | 0.793 | 0.793 |
| \((7, 2)\) | 0.718 | 1.33 | 1.33 |
| \((8, 2)\) | 1.01 | 2.20 | 2.20 |
| \((9, 2)\) | 1.43 | 3.60 | 3.60 |
| \((10, 2)\) | 2.03 | 5.83 | 5.83 |
| \((3, 3)\) | 0.179 | 0.125 | 0.179 |
| \((4, 3)\) | 0.253 | 0.232 | 0.253 |
| \((5, 3)\) | 0.359 | 0.418 | 0.418 |
| \((6, 3)\) | 0.507 | 0.732 | 0.732 |
| \((7, 3)\) | 0.718 | 1.25 | 1.25 |
| \((8, 3)\) | 1.01 | 2.10 | 2.10 |
| \((9, 3)\) | 1.43 | 3.48 | 3.48 |
| \((10, 3)\) | 2.03 | 5.69 | 5.69 |

More on the lower bounds. Some numerical experiments performed on the function \(K^{(-)}_{pn}( ) : \ell \in \mathbb{N}_0 \mapsto K^{(-)}_{pn}(\ell)\) for \(n \leq 10\) and \(n \leq p \leq 1000\) indicate that, within this range, \(K^{(-)}_{pn}( )\) attains its maximum at a point \(\ell_{pn}\), which is approximated with good accuracy by \(\ell_{pn} \in \mathbb{N}_0\) defined hereafter. To define this approximant we

\footnote{To be precise, the table reports the rounddown to three digits of all the above mentioned quantities.}
first introduce the nonnegative real number
\[ \lambda_{pn} := \begin{cases} \\
\sqrt{\frac{p-n}{2(n+1)}} & \text{for } d \geq 3, \\
\sqrt{\frac{p-n+2}{2(n+1)}} & \text{for } d = 2
\end{cases} \] (9.28)

and then we put
\[ \hat{\ell}_{pn} := \begin{cases} \\
\llceil \lambda_{pn} \rrceil & \text{if } K_{pn}^{(-)}(\llceil \lambda_{pn} \rrceil) > K_{pn}^{(-)}(\llfloor \lambda_{pn} \rfloor) \\
\llfloor \lambda_{pn} \rfloor & \text{if } K_{pn}^{(-)}(\llfloor \lambda_{pn} \rfloor) \geq K_{pn}^{(-)}(\llceil \lambda_{pn} \rrceil).
\end{cases} \] (9.29)

In the above, for each real number \( x \geq 0 \) we intend
\[ \llfloor x \rfloor := \max(\lfloor x \rfloor, 1), \quad \llceil x \rrceil := \max(\lceil x \rceil, 1), \] (9.30)
where \( \lfloor x \rfloor \) and \( \lceil x \rceil \) are the usual lower and upper integer parts of \( x \) (i.e., the unique integers such that \( \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \) and \( \lceil x \rceil - 1 < x \leq \lceil x \rceil \); this implies, for example, \( x \leq \lceil x \rceil < x + 1 \) and \( x \leq \llceil x \rrceil \leq x + 1 \).

Let us exemplify the accuracy of the approximation of \( \ell_{pn} \) via \( \hat{\ell}_{pn} \) in a number of cases. As a matter of fact, \( \ell_{pn} \) coincides exactly with \( \hat{\ell}_{pn} \) for \( d = n = 3, \; p = 3, 4, ..., 69, 70 \) and for \( d = n = 2, \; p = 2, 3, ..., 89, 90 \); moreover, \( \ell_{pn} \) equals \( \hat{\ell}_{pn} \) or \( \hat{\ell}_{pn} + 1 \) for \( d = n = 3, \; p = 80, 90, ..., 990, 1000 \) and for \( d = n = 2, \; p = 100, 110, ..., 790, 800 \).

**The large \( p \) limit of the previous lower bounds.** We know that \( K_{pn} \geq K_{pn}^{(-)}(\ell) \) for all \( \ell \in \mathbb{N}_0 \); therefore (independently of the previous considerations and numerical experiments on the maximum over \( \ell \) of \( K_{pn}^{(-)}(\ell) \)) we are granted that
\[ K_{pn} \geq K_{pn}^{(-)}(\llceil \lambda_{pn} \rrceil), \] (9.31)

where \( \lambda_{pn} \) and \( \llceil \cdot \rrceil \) are defined via Eqs. (9.28) (9.30). This obvious remark yield the rigorous statement presented hereafter.

**9.5 Proposition.** (i) Let \( d \geq 3 \). For all real \( p, n \) with \( p \geq n > d/2 \) one has
\[ K_{pn} \geq K_{pn}^{(-)}, \] (9.32)

where
\[ K_{pn}^{(-)} := \frac{1}{(2\pi)^{d/2}} \left( \frac{n+1}{p} \right)^{(n+1)/2} \frac{\Theta_{pn}^{1/2} \cdot 2^{p+n/2+1}}{\Phi_{pn}^{p/2} \cdot \Phi_{pn}^{n/2+1/2} + \Phi_{pn}^{p/2+1/2} + \Phi_{pn}^{p/2 + 1/2}}, \] (9.33)
\[\Theta_{pn} := \left(1 - \frac{n - 1}{2p}\right)^p + \left(\frac{n + 1}{2p}\right)^p, \quad (9.34)\]

\[\Phi_{pn} := 1 + 2 \sqrt{1 - \frac{n}{p}} \sqrt{\frac{2(n + 1)}{p} + \frac{n + 2}{p}}, \quad (9.35)\]

\[\Psi_{pn} := 1 + 2 \sqrt{1 - \frac{n}{p}} \sqrt{\frac{2(n + 1)}{p} + \frac{3n + 4}{p}}. \quad (9.36)\]

(ii) Let \(d = 2\). For all real \(p, n\) with \(p \geq n > 1\) one has

\[K_{pn} \geq K_{pn}^{(-)}, \quad (9.37)\]

where

\[K_{pn}^{(-)} := \frac{1}{\pi} \left(\frac{n + 1}{p}\right)^{(n+1)/2} \frac{\Theta_{pn}^{1/2} \cdot 2^{p+n/2}}{\Phi_{pn}^{n/2+1} \Psi_{pn}^{n/2} \Phi_{pn}^{p/2+1} \Psi_{pn}^{p/2+1}}, \quad (9.38)\]

\[\Theta_{pn} := \left(1 + \frac{3}{p}\right)^2 \left(1 - \frac{n - 5}{2p}\right)^{p-1} + 4 \left(\frac{n + 1}{2p}\right)^{p+1}, \quad (9.39)\]

\[\Phi_{pn} := 1 + 2 \sqrt{1 - \frac{n - 2}{p}} \sqrt{\frac{2(n + 1)}{p} + \frac{n + 4}{p}}, \quad (9.40)\]

\[\Psi_{pn} := 1 + 2 \sqrt{1 - \frac{n - 2}{p}} \sqrt{\frac{2(n + 1)}{p} + \frac{3(n + 2)}{p}}. \quad (9.41)\]

(iii) Let \(d \geq 2\). For fixed \(n\) and \(p \to +\infty\), one has

\[K_{pn}^{(-)} \sim \frac{2}{(2\pi)^{d/2}} \frac{e^{5(n+1)/4} \cdot \Theta_{pn}^{n/2+1} \cdot \Psi_{pn}^{p/2+1}}{e^{n+1} + 1 \cdot \left(\frac{n + 1}{p}\right)^{(n+1)/2} \cdot \Phi_{pn}^{p/2+1} \cdot \Psi_{pn}^{p/2+1}} \quad (9.42)\]

and

\[(K_{pn}^{(-)})^{1/p} \to 2. \quad (9.43)\]

**Proof.** (i) Let \(d \geq 3\). We have

\[K_{pn} \geq (1) \frac{\sqrt{2}}{(2\pi)^{d/2}} \left(1 + 4 \left\| \lambda_{pn} \right\|^2 \right)^p \left(1 + \left(1 + \left\| \lambda_{pn} \right\|^2 \right)^{n/2+1/2} + \left\| \lambda_{pn} \right\|^n \left(1 + \left\| \lambda_{pn} \right\|^2 \right)^{p/2+1/2} \quad (9.44)\]

\[\geq (2) \frac{\sqrt{2}}{(2\pi)^{d/2}} \left(1 + \lambda_{pn} \right)^p \left(1 + \left(1 + \lambda_{pn} \right)^2 \right)^{n/2+1/2} + \left(1 + \lambda_{pn} \right)^n \left(1 + \left(1 + \lambda_{pn} \right)^2 \right)^{p/2+1/2} \]

\[\geq (3) K_{pn}^{(-)} \text{ as in } (9.33), (9.34). \quad (9.45)\]

In the above: the inequality \(\geq (1)\) is just the relation (9.31) with the explicit expres-
sion coming from (9.16) for \(K_{pn}^{(-)}(\|\lambda_{pn}\|)\); the inequality \(\geq (2)\) follows from \(\lambda_{pn} \leq \|\lambda_{pn}\| \leq 1 + \lambda_{pn}\); the equality \(= (3)\) follows using the explicit expression (9.28) for \(\lambda_{pn}\) and performing some elementary manipulations.

(ii) Let \(d = 2\). A chain of relations with the same structure as (9.40), justified by the same arguments employed in the proof of (i), yields the conclusion \(K_{pn} \geq K_{pn}^{(-)}\) as in (9.36) (9.37).

(iii) To prove Eq. (9.38) one starts from the explicit expression of \(K_{pn}^{(-)}\), given by Eq. (9.33) for \(d \geq 3\) and by Eq. (9.36) for \(d = 2\); both for \(d \geq 3\) and for \(d = 2\), the \(p \to +\infty\) limits are performed in an elementary way. The derivation of Eq. (9.38) for \(d \geq 3\) uses, e.g., the relations

\[
\left(1 - \frac{n - 1}{2p}\right)^p = e^{-(n-1)/2} \left(1 + O\left(\frac{1}{p}\right)\right),
\]

(9.41)

\[
\left(1 + 2\sqrt{1 - \frac{n}{p}} \sqrt{\frac{2(n+1)}{p} + \frac{n+2}{p}}\right)^{p/2} = e^{-3n/2-1} e^{2(n+1)p} \left(1 + O\left(\frac{1}{\sqrt{p}}\right)\right);
\]

for \(d = 2\) one uses, e.g., the relations

\[
\left(1 - \frac{n - 5}{2p}\right)^{p-1} = e^{-(n-5)/2} \left(1 + O\left(\frac{1}{p}\right)\right),
\]

(9.42)

\[
\left(1 + 2\sqrt{1 - \frac{n - 2}{p}} \sqrt{\frac{2(n+1)}{p} + \frac{3(n+2)}{p}}\right)^{p/2+1} = e^{-n/2+1} e^{2(n+1)p} \left(1 + O\left(\frac{1}{\sqrt{p}}\right)\right).
\]

Eq. (9.39) is a consequence of (9.38).

9.6 Remark. Let us compare the \(p \to +\infty\) asymptotics (9.38) of \(K_{pn}^{(-)}\) with the explicit expression (9.3) of the alternative lower bound \(K_{pn}^{\square}\), which has the form \(2^{p/2} \times \) a constant depending only on \(d\). From here it is evident that \(K_{pn}^{(-)} < K_{pn}^{\square}\) for fixed \(n > d/2\) and all sufficiently large \(p\). On the other hand, from the previous considerations it is evident that \(K_{pn}^{(-)} \leq K_{pn}^{\square}(\|\lambda_{pn}\|) \leq K_{pn}^{(-)}\) (where, we recall, \(K_{pn}^{\square}\) is the sup of the function \(\ell \mapsto K_{pn}^{\square}(\ell)\)).

In conclusion, for fixed \(n > d/2\) and all \(p\) sufficiently large we have \(K_{pn}^{(-)} < K_{pn}^{\square}\), which proves the last statement in Remark 9.4.
10 Lower bounds for the sharp constant $G_{pn}$ of the inequality (1.7)

Let $p, n$ be as in (1.5). Similarly to the case of $K_{pn}$, to obtain a lower bound on $G_{pn}$ we use the tautological inequality
\[
G_{pn} \geq \frac{2|\langle P(v, w) | w \rangle_p|}{\left(\|v\|_p \|w\|_n + \|v\|_n \|w\|_p\right)} \quad \text{for } v \in \mathbb{H}^p_{\Sigma_0} \setminus \{0\}, \ w \in \mathbb{H}^{p+1}_{\Sigma_0} \setminus \{0\}, \quad (10.1)
\]
choosing for $v$ and $w$ two suitable non zero trial vector fields. Hereafter we present a simple choice of $v, w$, depending on a discrete parameter and on three continuous parameters, giving rise to useful results. In the sequel we keep from Section 9 the notations
\[
a := (1, 0, ..., 0), \ b := (0, 1, 0, ... 0); \ c := (0, 0, 1, 0, ..., 0) \text{ if } d \geq 3; \ N_0 := \{1, 2, 3, ...\}.
\]

10.1 Proposition. (i) Let $d \geq 3$. For all $\omega = (\ell, \lambda, \mu, \nu) \in N_0 \times \mathbb{R}^3$, it is
\[
G_{pn} \geq G_{pn}^-(\omega), \quad G_{pn}^-(\omega) := \frac{2 \sqrt{2}}{(2\pi)^{d/2}} \frac{|S_p(\omega)|}{(\ell^p N_n(\omega) + \ell^n N_p(\omega)) N_p(\omega)},
\]
where
\[
S_p(\omega) := -\lambda + (1 + \ell^2)^p \lambda (1 - \mu) + (1 + 4\ell^2)^p \mu (\lambda - \nu) + (1 + 9\ell^2)^p \mu \nu,
\]
\[
N_m(\omega) := \sqrt{1 + 2(1 + \ell^2)^m \lambda^2 + 2(1 + 4\ell^2)^m \mu^2 + 2(1 + 9\ell^2)^m \nu^2} \quad \text{for all } m \in \mathbb{R}. \quad (10.3)
\]
(ii) Let $d = 2$. For all $\omega = (\ell, \lambda, \mu, \nu) \in N_0 \times \mathbb{R}^3$, it is
\[
G_{pn} \geq G_{pn}^-(\omega), \quad G_{pn}^-(\omega) := \frac{\sqrt{2}}{\pi} \frac{|S_p(\omega)|}{(\ell^p N_n(\omega) + \ell^n N_p(\omega)) N_p(\omega)},
\]
where $N_n(\omega), N_p(\omega)$ are defined following Eq. (10.4) and
\[
S_p(\omega) := -\frac{\lambda \ell}{\sqrt{1 + \ell^2}} + 2\lambda \ell \left(1 - \frac{3\mu \ell}{\sqrt{4 + \ell^2}}\right) (1 + \ell^2)^{p-3/2}\]
\[+ \frac{5\mu \ell^2}{\sqrt{4 + \ell^2}} \left(\frac{3\lambda}{\sqrt{1 + \ell^2}} - \frac{7\nu}{\sqrt{9 + \ell^2}}\right) (1 + 4\ell^2)^{p-1} + \frac{70\mu \ell^2}{(4 + \ell^2)(9 + \ell^2)} (1 + 9\ell^2)^{p-1}.
\]
(iii) For any $d \geq 2$, Eqs. (10.2), (10.3) imply
\[
G_{pn} \geq G_{pn}^- \quad G_{pn}^- := \sup_{\omega \in N_0 \times \mathbb{R}^3} G_{pn}^-(\omega). \quad (10.7)
\]

53
\textbf{Proof.} (i) Let $d \geq 3$. For $\omega := (\ell, \lambda, \mu, \nu) \in \mathbb{N}_0 \times \mathbb{R}^3$ we set

$$v \equiv v(\omega) := ib(e_{\ell a} - e_{-\ell a}), \quad (10.8)$$

$$w \equiv w(\omega) := e^c \left( e_b + e_{-b} + \lambda(e_{\ell a+b} + e_{-\ell a-b} - e_{\ell a-b} - e_{-\ell a+b}) + \mu(e_{2\ell a+b} + e_{-2\ell a-b} + e_{2\ell a-b} + e_{-2\ell a+b}) + \nu(e_{3\ell a+b} + e_{-3\ell a-b} - e_{3\ell a-b} - e_{-3\ell a+b}) \right);$$

incidentally, we note that $v$ is as in Eq. (9.19). Like the already considered vector field $v$, the vector field $w$ has vanishing mean and divergence ($k \cdot w_k = 0$ for each $k$), and belongs to $\mathbb{H}^m_{\mathbb{R}^3}$ for any real $m$. For each $m$ we have

$$\|v\|_m = \sqrt{2} f^m, \quad \|w\|_m = \sqrt{2} N_m(\omega), \quad (10.9)$$

with $N_m(\omega)$ as in (10.4). Using Eqs. (2.12) (2.20) we find

$$\mathcal{P}(v, w) = \frac{c}{(2\pi)^{d/2}} \left( 2\lambda(e_b + e_{-b}) + (\mu - 1)(e_{\ell a+b} + e_{-\ell a-b} - e_{\ell a-b} - e_{-\ell a+b}) \right) \quad (10.10)$$

$$+ (\nu - \lambda)(e_{2\ell a+b} + e_{-2\ell a-b} + e_{2\ell a-b} + e_{-2\ell a+b}) - \mu(e_{3\ell a+b} + e_{-3\ell a-b} - e_{3\ell a-b} - e_{-3\ell a+b})$$

$$- \nu(e_{4\ell a+b} + e_{-4\ell a-b} + e_{4\ell a-b} + e_{-4\ell a+b});$$

from here we infer

$$\langle \mathcal{P}(v, w) | w \rangle_p = - \frac{4}{(2\pi)^{d/2}} S_p(\omega), \quad (10.11)$$

with $S_p(\omega)$ as in (10.3). Now, using Eqs. (10.9) (10.11) and (10.11) we readily infer statement (10.2).

(ii) Let $d = 2$. We set

$$v \equiv v(\omega) := ib(e_{\ell a} - e_{-\ell a}), \quad (10.12)$$

$$w \equiv w(\omega) := a(e_b + e_{-b}) + \frac{\lambda}{\sqrt{1 + \ell^2}} \left( -\ell a + b \right) \left( e_{\ell a+b} + e_{-\ell a-b} + (\ell a + b) \left( e_{\ell a-b} + e_{-\ell a+b} \right) \right)$$

$$+ \frac{\mu}{\sqrt{4 + \ell^2}} \left( (\ell a - 2b) \left( e_{2\ell a+b} + e_{-2\ell a-b} \right) + (\ell a + 2b) \left( e_{2\ell a-b} + e_{-2\ell a+b} \right) \right)$$

$$+ \frac{\nu}{\sqrt{9 + \ell^2}} \left( -\ell a + 3b \right) \left( e_{3\ell a+b} + e_{-3\ell a-b} \right) + (\ell a + 3b) \left( e_{3\ell a-b} + e_{-3\ell a+b} \right),$$

where $\omega = (\ell, \lambda, \mu, \nu) \in \mathbb{N}_0 \times \mathbb{R}^3$; again, $v$ has the structure (9.19). $v, w$ have vanishing mean and divergence and belong to $\mathbb{H}^m_{\mathbb{R}^3}$ for any real $m$; their norms of any order $m$ are given again by (10.9) (10.4). Using Eqs. (2.13) (2.20) we find

54
\[ P(v, w) = -\frac{\lambda \ell}{\pi \sqrt{1 + \ell^2}} a(e_b + e_{-b}) \] 

\[ -\frac{1}{2\pi(1 + \ell^2)} \left( 1 - \frac{3\mu \ell}{\sqrt{4 + \ell^2}} \right) \left( (a - \ell b)(e_{\ell a+b} + e_{-\ell a-b}) - (a + \ell b)(e_{\ell a-b} + e_{-\ell a+b}) \right) \]

\[ + \frac{\ell}{2\pi(1 + 4\ell^2)} \left( \frac{3\lambda}{\sqrt{1 + \ell^2}} - \frac{7\nu}{\sqrt{9 + \ell^2}} \right) \left( (a-2\ell b)(e_{2\ell a+b} + e_{-2\ell a-b}) + (a+2\ell b)(e_{2\ell a-b} + e_{-2\ell a+b}) \right) \]

\[ + \frac{7\mu \ell}{2\pi(1 + 9\ell^2)\sqrt{4 + \ell^2}} \left( (-a + 3\ell b)(e_{3\ell a+b} + e_{-3\ell a-b}) + (a + 3\ell b)(e_{3\ell a-b} + e_{-3\ell a+b}) \right) \]

\[ + \frac{13\nu \ell}{2\pi(1 + 16\ell^2)\sqrt{9 + \ell^2}} \left( (a - 4\ell b)(e_{4\ell a+b} + e_{-4\ell a-b}) + (a + 4\ell b)(e_{4\ell a-b} + e_{-4\ell a+b}) \right); \]

from here we infer

\[ \langle P(v, w) \mid w \rangle_p = \frac{2}{\pi} S_p(\omega), \] 

with \( S_p(\omega) \) as in (10.6). Now, using Eqs. (10.9) (10.14) and (10.11) we readily infer statement (10.5).

(iii) Obvious. \[ \square \]

**Some numerical examples.** Let us consider the lower bounds \( G_{pn}(\omega) \) of Eq. (10.2), depending on \( \omega = (\ell, \lambda, \mu, \nu) \in \mathbb{N}_0 \times \mathbb{R}^3 \). For given \( (p, n) \), one can try to maximize this bound with respect to \( \omega \) using the routines for numerical optimization of Mathematica; these predict the maximum to be located at some point \( \omega_{pn} = (\ell_{pn}, \lambda_{pn}, \mu_{pn}, \nu_{pn}) \). Even though this is not the actual point of absolute maximum, the number

\[ G_{pn}^{(-)} := G_{pn}(\omega_{pn}) \] 

is in any case a lower bound for \( G_{pn} \). In the forthcoming Table F we report, for the \( (d = 3) \) cases (1.11), the point \( \omega_{pn} \) provided by Mathematica and the value of \( G_{pn}^{(-)} \). The numerical values \( G_{pn}^{(-)} \) have been anticipated in Table B of the Introduction.

More precisely, in the table we write \( G_{pn}^{(-)} \) for the rounddown to three digits of \( G_{pn}(\omega_{pn}) \).
Table F. Maximizing points $\omega_{pn}$ and lower bounds $G_{pn}^{(-)}$ in the cases (1.11)

| $(p, n)$ | $\omega_{pn}$ | $G_{pn}^{(-)}$ |
|--------|----------------|--------------|
| (3, 3) | (1, 0.388104..., 0.084359..., 0.0135851...) | 0.121 |
| (4, 3) | (1, 0.370907..., 0.0628525..., 0.00811876...) | 0.235 |
| (5, 3) | (1, 0.361597..., 0.0415026..., 0.00365302...) | 0.408 |
| (6, 3) | (1, 0.352601..., 0.0256793..., 0.00147754...) | 0.674 |
| (7, 3) | (1, 0.348117..., 0.0157944..., 0.000588218...) | 1.08 |
| (8, 3) | (1, 0.352603..., 0.00994449..., 0.000238632...) | 1.74 |
| (9, 3) | (1, 0.367597..., 0.00645276..., 0.0000993805...) | 2.77 |
| (10, 3) | (1, 0.392975..., 0.00430116..., 0.0000423667...) | 4.40 |

In the table we always have $\ell_{pn} = 1$. This is no more the case for larger values of $p$: for example, Mathematica gives $\ell_{pn} = 2$ for $(d = 3$ and $(p, n) = (20, 3)$.

The large $p$ limit of the previous lower bounds. We know that $G_{pn} \geq G_{pn}^{(-)}(\omega)$ for all $\omega \in N_0 \times R^3$; in this section we choose for $\omega$ the quadruple

$$\hat{\omega}_{pn} := (\ell_{pn}, 1, 2^{-p}, 0), \quad \hat{\ell}_{pn} := \lceil \sqrt{p/n} \rceil$$

where, as in the previous section, $\lceil \rceil$ denotes the upper integer part. This choice gives

$$G_{pn} \geq G_{pn}^{(-)}(\hat{\omega}_{pn})$$

for all $p \geq n > d/2 + 1$. We make the choice (10.16) because some numerical experiments seem to indicate that, for large $p$, $G_{pn}^{(-)}(\omega)$ attains its maximum for $\omega$ close to $\hat{\omega}_{pn}$ (both for $d = 2$, and for $d \geq 3$). Independently of these experiments, Eq. (10.17) yields the rigorous statement presented hereafter.

10.2 Proposition. (i) Let $d \geq 3$. For all real $p, n$ with $p \geq n > d/2 + 1$ one has

$$G_{pn} \geq G_{pn}^{(-)}$$

where

$$G_{pn}^{(-)} := \frac{\sqrt{2}}{(2\pi)^{d/2}} \left(\frac{n}{p}\right)^{n/2} \left(1 + \frac{n}{4p}\right)^p \cdot 2^p \left(1 - \frac{1}{2^p}\right) \left(1 + \frac{n}{p}\right)^p - \left(\frac{n}{p}\right)^p \left(1 + \sqrt{n/p} \Sigma_{pn}\right)^{1/2} + \left(1 + \sqrt{n/p} \Sigma_{pn}\right)^{1/2} \Sigma_{pn}^{1/2} \right),$$

$$\Sigma_{pn} := \left(1 + 2\sqrt{\frac{n}{p} + \frac{5n}{4p}}\right)^n + \left(1 + 2\sqrt{\frac{n}{p} + \frac{2n}{p}}\right)^n + \frac{1}{2\sqrt{p}} \left(1 + 2\sqrt{\frac{n}{p} + \frac{5n}{4p}}\right)^n + \frac{1}{2\sqrt{p}} \left(1 + 2\sqrt{\frac{n}{p} + \frac{2n}{p}}\right)^n,$$

$$\Sigma_{pn} := \left(1 + 2\sqrt{\frac{n}{p} + \frac{2n}{p}}\right)^n + \frac{1}{2\sqrt{p}} \left(1 + 2\sqrt{\frac{n}{p} + \frac{5n}{4p}}\right)^n + \frac{1}{2\sqrt{p}} \left(1 + 2\sqrt{\frac{n}{p} + \frac{2n}{p}}\right)^n.$$
(ii) Let again \(d \geq 3\). For fixed \(n\) and \(p \to +\infty\), one has
\[
G_{pn}^{(-)} \sim \frac{\sqrt{2}}{(2\pi)^{d/2}} \frac{e^{9n/8}}{(1 + e^{n/8} \sqrt{1 + e^{3n/4}}) \sqrt{1 + e^{3n/4}}} \left( \frac{n}{p} \right)^{n/2} \frac{2^p}{e^{2\sqrt{np}}} \tag{10.21}
\]
and
\[
(G_{pn}^{(-)})^{1/p} \to 2. \tag{10.22}
\]

(iii) Let \(d = 2\). For all real \(p, n\) with \(p \geq n > 2\) one has
\[
G_{pn} \geq G_{pn}^{(–)}, \tag{10.23}
\]
where
\[
G_{pn}^{(–)} := \frac{\left( \frac{n}{p} \right)^{n/2+1}}{\sqrt{2\pi} \sqrt{\left( 1 + \frac{n}{p} \right) \left( 1 + \frac{4n}{p} \right)}} \times 15 \left( 1 + \frac{n}{4p} \right)^{p-1} 2^{p-2} + \left( 1 + \frac{n}{p} \right)^{p-1} \left( 2 \sqrt{1 + \frac{4n}{p} - \frac{3}{2} 2^{p-1}} - \sqrt{1 + \frac{4n}{p} \left( \frac{n}{p} \right)^{p-1}} \right) \left( 1 + \sqrt{\frac{n}{p}} \right)^{\frac{1}{2}} \Sigma_{pn}^{1/2} \tag{10.24}
\]
and \(\Sigma_{pn}, \Sigma_{pn}^{1/2}\) are as in Eq. (10.20).

(iv) Let again \(d = 2\). For fixed \(n\) and \(p \to +\infty\), one has
\[
G_{pn}^{(-)} \sim \frac{15}{4\sqrt{2\pi}} \frac{e^{9n/8}}{(1 + e^{n/8} \sqrt{1 + e^{3n/4}}) \sqrt{1 + e^{3n/4}}} \left( \frac{n}{p} \right)^{n/2+1} \frac{2^p}{e^{2\sqrt{np}}} \tag{10.25}
\]
and
\[
(G_{pn}^{(-)})^{1/p} \to 2. \tag{10.26}
\]

**Proof.** (i) Let \(d \geq 3\); for \(\gamma \in [1, +\infty)\), we put
\[
S_p(\gamma) := -1 + (1 - 2^{-p})(1 + \gamma^2)^p + 2^{-p}(1 + 4\gamma^2)^p, \tag{10.27}
\]
\[
N_m(\gamma) := \sqrt{1 + 2(1 + \gamma^2)^m + 2^{1-2p}(1 + 4\gamma^2)^m} \quad (m = n, p); \tag{10.28}
\]
\(S_p, N_n, N_p\) are positive, strictly increasing functions on \([1, +\infty)\). We have
\[
G_{pn} \geq (1) \frac{2\sqrt{2}}{(2\pi)^{d/2}} \frac{S_p(\ell_{pn})}{\ell_{pn} N_n(\ell_{pn}) + \ell_{pn} N_p(\ell_{pn})} \tag{10.29}
\]
\[
\geq (2) \frac{\sqrt{2}}{(2\pi)^{d/2}} \frac{S_p(\sqrt{p/n})}{\left( (1 + \sqrt{p/n})^p N_n(1 + \sqrt{p/n}) + (1 + \sqrt{p/n})^n N_p(1 + \sqrt{p/n}) \right) N_p(1 + \sqrt{p/n})} = (3) G_{pn}^{(-)} \text{ as in } (10.19). \tag{10.29}
\]

In the above: the inequality \(\geq (1)\) is just the relation \((10.17)\) with the explicit expression coming from \((10.2)\) for \(G_{pn}^{(-)}(\omega)\) and from \((10.16)\) for \(\hat{\omega}_{pn}\); the inequality \(\geq (2)\)
is obtained noting that (10.16) implies $\sqrt{\frac{p}{n}} \leq \hat{\ell}_{pn} < 1 + \sqrt{\frac{p}{n}}$; the equality $\implies$ follows performing some elementary manipulations.

(ii) Let again $d \geq 3$. To prove Eq. (10.21) one starts from the explicit expression (10.19) of $G^{(-)}_{pn}$ and performs the $p \to +\infty$ limit in an elementary way, noting for example that

\[ (1 + \sqrt{\frac{n}{p}})^p = e^{-n/2}e^{\sqrt{np}}(1 + O\left(\frac{1}{\sqrt{p}}\right)), \tag{10.30} \]

\[ (1 + 2\sqrt{\frac{n}{p}} + \frac{5n}{4p})^p = e^{-3n/4}e^{2\sqrt{np}}(1 + O\left(\frac{1}{\sqrt{p}}\right)). \]

Eq. (10.22) is a consequence of (10.21).

(iii) Let $d = 2$; for $\gamma \in [1, +\infty)$ we put

\[ S_p(\gamma) := -\frac{\gamma}{\sqrt{1 + \gamma^2}} + 2\gamma \left(1 - 3 \cdot \frac{2 \cdot p \gamma}{4 + \gamma^2}\right)(1 + \gamma^2)^{p-3/2} \tag{10.31} \]

\[ + \frac{15 \cdot 2^{-p} \gamma^2}{\sqrt{(4 + \gamma^2)(1 + \gamma^2)}}(1 + 4 \gamma^2)^{p-1}, \]

and we define $N_m(\gamma) (m = n, p)$ as in Eq. (10.28); $S_p, N_n, N_p$ are positive, strictly increasing functions on $[1, +\infty)$. We have

\[ G_{pn} \geq (1) \frac{\sqrt{2}}{\pi} \frac{S_p(\hat{\ell}_{pn})}{\left(\hat{\ell}_{pn}^p N_n(\hat{\ell}_{pn}) + \hat{\ell}_{pn}^p N_p(\hat{\ell}_{pn})\right) N_p(\hat{\ell}_{pn})} \tag{10.32} \]

\[ \geq (2) \frac{\sqrt{2}}{\pi} \frac{S_p(\sqrt{p/n})}{\left(\sqrt{p/n}^p N_n(1 + \sqrt{p/n}) + (1 + \sqrt{p/n})^n N_p(1 + \sqrt{p/n})\right) N_p(1 + \sqrt{p/n})} = (3) G^{(-)}_{pn} \text{ as in } (10.24). \]

In the above: the inequality $\geq (1)$ is just the relation (10.17) with the explicit expression coming from (10.5) for $G^{(-)}_{pn}(\omega)$ and from (10.16) for $\hat{\omega}_{pn}, \hat{\ell}_{pn}$; the inequality $\geq (2)$ is obtained recalling that (10.16) implies $\sqrt{p/n} \leq \hat{\ell}_{pn} < 1 + \sqrt{p/n}$; the equality $\implies$ follows performing some elementary manipulations.

(iv) Let again $d = 2$. To prove Eq. (10.25) one starts from the explicit expression (10.24) of $G^{(-)}_{pn}$ and performs the $p \to +\infty$ limit, using Eq. (10.30) and similar elementary relations. Eq. (10.26) is a consequence of (10.25).

\[ \square \]

11 On the large $p$ behavior of $K_{pn}$ and $G_{pn}$

For all $p \geq n > d/2$, Corollary 3.4 and Proposition 9.5 give for the sharp constants $K_{pn}$ the bounds

\[ K_{pn}^{(-)} \leq K_{pn} \leq K_{pn}^{(+)} \tag{11.1} \]
with explicit expressions provided by Eqs. (3.20) and (9.33) (9.36). For fixed $d, n$ and $p \to +\infty$ the upper and lower bounds have similar, but not coinciding asymptotics: in fact, from Eqs. (3.20) and (9.38) we know that $K_{pn}^{(+)} = C_n^{(+)} \cdot 2^p$ while $K_{pn}^{(-)} \sim C_n^{(-)} \cdot 2^p p^{-(n+1)/2} e^{-\sqrt{2(n+1)p}}$, for suitable coefficients $C_n^{(\pm)}$ (depending also on $d$).

Due to these different behaviors, Eq. (11.1) does not determine the precise $p \to +\infty$ asymptotics of $K_{pn}$; however, we can obtain from it a weaker result on the large $p$ limit. In fact, as already mentioned after Eq. (3.20) and in Eq. (9.39), we have $(K_{pn}^{(\pm)})^{1/p} \to 2$ in this limit; thus, Eq. (11.1) yields the following result.

11.1 Proposition. For any fixed $n > d/2$, one has

$$\left( K_{pn} \right)^{1/p} \to 2 \quad \text{for } p \to +\infty .$$

(11.2)

One can make similar considerations about the sharp constants $G_{pn}$. For all $p \geq n > d/2 + 1$, Corollary 4.5 and Proposition 10.2 give the bounds

$$G_{pn}^{(-)} \leq G_{pn} \leq G_{pn}^{(+)} ,$$

(11.3)

with explicit expressions provided by Eqs. (4.33) and (10.19) (10.24). For fixed $d, n$ and $p \to +\infty$ the upper and lower bounds have similar, but not coinciding asymptotics (see again Eq. (4.33), and compare it with Eqs. (10.21) (10.25)); however, as already indicated after Eq. (4.33) and in Eqs. (10.22) (10.26), one has $(G_{pn}^{(\pm)})^{1/p} \to 2$ and these facts, combined with (11.3), yield the following result.

11.2 Proposition. For any fixed $n > d/2 + 1$, one has

$$\left( G_{pn} \right)^{1/p} \to 2 \quad \text{for } p \to +\infty .$$

(11.4)

Acknowledgments. This work has been partly supported by INdAM, INFN and by MIUR, PRIN 2010 Research Project “Geometric and analytic theory of Hamiltonian systems in finite and infinite dimensions”.

59
A Appendix. The norm of the bilinear map $P_{h\ell}$

Let $h, \ell \in \mathbb{R}^d \setminus \{0\}$. We consider the bilinear map defined by Eq. (2.27), i.e.,

$$P_{h\ell} : h^\perp \times \ell^\perp \rightarrow (h + \ell)^\perp, \quad (a, b) \mapsto P_{h\ell}(a, b) := \frac{a \cdot \ell}{|\ell|} \mathcal{L}_{h+\ell} b ;$$

we recall that $^\perp$ indicates the orthogonal complement in $\mathbb{C}^d$, and that $\mathcal{L}_{h+\ell}$ is the orthogonal projection of $\mathbb{C}^d$ onto $(h + \ell)^\perp$. We are interested in the norm

$$|P_{h\ell}| := \min\{Q \in [0, +\infty) \mid |P_{h\ell}(a, b)| \leq Q|a||b| \text{ for all } a \in h^\perp, b \in \ell^\perp\}.$$

A.1 Lemma. The above norm is given by Eq. (2.30), i.e.,

$$|P_{h\ell}| = \begin{cases} \sin \vartheta_{h\ell} & \text{if } d \geq 3, \\ \sin \vartheta_{h\ell} \cos \vartheta_{h+\ell,\ell} & \text{if } d = 2 \end{cases}.$$

Proof. Our argument is closely related to the slightly weaker one employed in [12]; in the sequel we use the abbreviations

$$\vartheta_{h\ell} \equiv \vartheta, \quad \vartheta_{h+\ell,\ell} \equiv \vartheta'.$$

Let $S$ denote a two-dimensional subspace of $\mathbb{R}^d$ containing $h$ and $\ell$ (of, course, this is unique if $h, \ell$ are linearly independent). We choose in $S$ an orthonormal basis $\eta_1, \eta_2$ so that $h$ be a positive multiple of $\eta_1$ and $\ell \cdot \eta_2 \geq 0$; then

$$h = |h| \eta_1, \quad \ell = |\ell|(\cos \vartheta \eta_1 + \sin \vartheta \eta_2).$$

In $S$ we also consider a second orthonormal basis $\eta'_1, \eta'_2$, chosen so that $h + \ell$ be a nonnegative multiple of $\eta'_1$ and $\ell \cdot \eta'_2 \geq 0$; then

$$h + \ell = |h + \ell| \eta'_1, \quad \ell = |\ell|(\cos \vartheta' \eta'_1 + \sin \vartheta' \eta'_2).$$

Finally let $\eta_3, \eta_4, \ldots, \eta_d$ be $d - 2$ vectors of $\mathbb{R}^d$, forming an orthonormal basis for the orthogonal complement of $S$ in $\mathbb{R}^d$ (obviously enough, this family is empty if $d = 2$). The orthogonal complements of $h$ and $\ell$ in $\mathbb{C}^d$ have the following representations, involving three orthonormal bases:

$$h^\perp = \langle \eta_2, \ldots, \eta_d \rangle, \quad \ell^\perp = \langle -\sin \vartheta \eta_1 + \cos \vartheta \eta_2, \eta_3, \ldots, \eta_d \rangle = \langle -\sin \vartheta' \eta'_1 + \cos \vartheta' \eta'_2, \eta_3, \ldots, \eta_d \rangle.$$

To go on we consider the cases $d \geq 3$ and $d = 2$, separately.

Case $d \geq 3$. Let us consider any two vectors $a \in h^\perp, b \in \ell^\perp$; we can write

$$a = a_{(2)}\eta_2 + \ldots + a_{(d)}\eta_d,$$
with $a_{(i)} \in \mathbb{C}$ for all $i$. From here and from Eq. (A.2) for $\ell$ we infer
\[
\frac{a \cdot \ell}{|\ell|} = a_{(2)} \sin \vartheta ,
\] (A.7) whence
\[
\frac{|a \cdot \ell|}{|\ell|} = |a_{(2)}| \sin \vartheta \leq |a| \sin \vartheta ;
\] (A.8) moreover, by a general property of orthogonal projections,
\[
|\mathcal{L}_{h,\ell} b| \leq |b| .
\] (A.9)
The last two inequalities imply
\[
|P_{h,\ell}(a, b)| \leq \sin \vartheta \, |a| \, |b| ;
\] (A.10) moreover (A.10) holds as an equality for suitable, nonzero choices of $a, b$. In fact, setting
\[
a_* := \eta_2 , \quad b_* := \eta_3
\] (A.11) we have
\[
\frac{a_* \cdot \ell}{|\ell|} = \sin \vartheta , \quad \mathcal{L}_{h,\ell} b_* = b_*
\] (A.12) (because $h + \ell$ is in the subspace $S$, spanned by $\eta_1, \eta_2$, and $b^*$ is orthogonal to it); this implies
\[
P_{h,\ell}(a_*, b_*) = \sin \vartheta b_* ,
\] (A.13) so that
\[
|P_{h,\ell}(a_*, b_*)| = \sin \vartheta = \sin \vartheta |a_*| |b_*| .
\] (A.14) From Eqs. (A.10), (A.14) we infer, as desired,
\[
|P_{h,\ell}| = \sin \vartheta .
\] (A.15)

Case $d = 2$. Let us consider any two vectors $a \in h^\perp, b \in \ell^\perp$; these can be written as
\[
a = |a| e^{i\phi} \eta_2 , \quad b = |b| e^{i\psi} (-\sin \vartheta' \eta_1' + \cos \vartheta' \eta_2') \quad (\phi, \psi \in \mathbb{R}) .
\] (A.16) From here and from the representations (A.2) for $\ell$, (A.3) for $h + \ell$ we infer
\[
\frac{a \cdot \ell}{|\ell|} = |a| e^{i\phi} \sin \vartheta , \quad \mathcal{L}_{h,\ell} b = |b| e^{i\psi} \cos \vartheta' \eta_2',
\] (A.17) so that
\[
P_{h,\ell}(a, b) = \sin \vartheta \cos \vartheta' e^{i(\phi + \psi)} |a| \, |b| \, \eta_2' , \quad |P_{h,\ell}(a, b)| = \sin \vartheta \cos \vartheta' |a| \, |b| ;
\] (A.18) this trivially yields the desired conclusion
\[
|P_{h,\ell}| = \sin \vartheta \cos \vartheta' .
\] (A.19)
References

[1] J. T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equations, Commun. Math. Phys. 94 (1984), 61-66.

[2] S.I. Chernyshenko, P. Constantin, J.C. Robinson, E.S. Titi, A posteriori regularity of the three-dimensional Navier-Stokes equations from numerical computations, J. Math. Phys. 48 (2007), 065204/10.

[3] P. Constantin, C. Foias, Navier Stokes equations, Chicago University Press (1988).

[4] R.S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 65-222.

[5] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, J.Funct.Anal. 9 (1972), 296-305.

[6] C. Morosi, M. Pernici, L. Pizzocchero, On power series solutions for the Euler equation, and the Behr-Nečas-Wu initial datum, ESAIM Math. Model. Numer. Anal. 47 (2013), 663-688.

[7] C. Morosi, M. Pernici, L. Pizzocchero, A posteriori estimates for Euler and Navier-Stokes equations, in: F. Ancona, A. Bressan, P. Marcati, A. Marson (Eds.), Hyperbolic Problems: Theory, Numerics and Applications, Proceedings of the XIV International Conference held in Padova (June 25-29, 2012), in: AIMS Series on Applied Mathematics 8 (2014), 847-855.

[8] C. Morosi, M. Pernici, L. Pizzocchero, Large order Reynolds expansions for the Navier-Stokes equations, Appl. Math. Lett. 49 (2015) 58-66. For an extended version, see arXiv:1402.0487.

[9] C. Morosi, L. Pizzocchero, An $H^1$ setting for the Navier-Stokes equations: Quantitative estimates, Nonlinear Anal. 74 (2011), 2398-2414.

[10] C. Morosi, L. Pizzocchero, On approximate solutions for the Euler and Navier-Stokes equations, Nonlinear Anal. 75 (2012), 2209-2235.

[11] C. Morosi, L. Pizzocchero, On the constants in a Kato inequality for the Euler and Navier-Stokes equations, Commun. Pure Appl. Analysis 11(2012), 557-586.

[12] C. Morosi, L. Pizzocchero, On the constants in a basic inequality for the Euler and Navier-Stokes equations, Appl. Math. Lett. 26 (2013), 277-284. For an extended version, see arXiv:1007.4412.

[13] C. Morosi, L. Pizzocchero, On the Reynolds number expansion for the Navier-Stokes equations, Nonlinear Analysis 95 (2014), 156-174.
[14] C. Morosi, L. Pizzocchero, Smooth solutions of the Euler and Navier-Stokes equations from the a posteriori analysis of approximate solutions, Nonlinear Analysis 113 (2015), 298-308.

[15] J. C. Robinson, W. Sadowski, R. P. Silva, Lower bounds on blow up solutions of the three-dimensional Navier-Stokes equations in homogeneous Sobolev spaces, J. Math. Phys. 53 (2012), 115618, 15pp.

[16] R. Temam, Local existence of $C^\infty$ solutions of the Euler equation of incompressible perfect fluids, in “Turbulence and Navier Stokes equation”, Proceedings of the Orsay Conference, Lecture Notes in Mathematics 565 (1976), 184-193.

[17] F. Yohannson, the Arb library, see http://fredrikj.net/arb/.