We study the effect of quantum memory in a four-player quantum Minority game. The Nash equilibrium payoff of the first player is investigated for different quantum memory channels. It is seen that different memory channels affect the game differently. It is also seen that the Nash equilibrium payoff is substantially enhanced in the presence of quantum memory. Therefore, memory controls the payoff reduction due to decoherence. It is shown that the curves for depolarizing and bit-phase flip channels overlap for maximum correlations. However, amplitude damping channel influences the game more heavily as compared to the other channels. That is the behaviour of amplitude damping channel is surprising one. Furthermore, the behaviour of phase flip channel is symmetrical around 50% decoherence.

Keywords: Quantum memory; Minority game, entanglement; quantum channels.

I. INTRODUCTION

In the recent past, rapid interest has been developed in the discipline of quantum information [1] that has led to the creation of quantum game theory [2]. During last few years, number of authors have contributed to the development of quantum game theory [3-7]. Many interesting classical games have been converted into quantum domain such as quantum prisoners’ dilemma [8-10], the battle of the sexes [7, 12], the Monty Hall problem [13, 14], the rock-scissors-paper [15] and many others [16-19]. James et al. [20] have analyzed the quantum penny flip game using geometric algebra. Recently, Almeida et al. [21] have suggested that quantum correlations provide no advantage over classical correlations in a multipartite nonlocal game.

Noise effects in different quantum games have been investigated by many authors [5, 7, 10, 18, 22] with interesting results. In Ref.[18, 22] the authors have studied noise effects in quantum
magic squares game. It is shown that the probability of success can be used to determine the characteristics of quantum channels. Implementation of decoherence and quantum memory has also been extended to the novel field of quantum information theory [23, 24]. The Minority game has received much attention as a model of a population of agents repeatedly buying and selling in a market [25, 26]. First quantum version of a four player quantum Minority game (QMG) was examined by Benjamin and Hayden [27], and later generalized to N-players [28]. Flitney et al. have extended its consideration towards the implementation of decoherence [29]. Quantum channels with memory [30-32] provides a natural theoretical framework for the study of any noisy quantum communication system where correlation time is longer than the time between consecutive uses of the channel. A more general model of a quantum channel with memory was introduced by Bowen and Mancini [33] and also studied by Kretschmann and Werner [34].

Here, in this work, we study the four-player quantum Minority game influenced by different memory channels, such as amplitude damping, depolarizing, bit-flip, phase-flip and bit-phase-flip channels, parameterized by the decoherence parameter $p$ and memory parameter $\mu$. Here $p \in [0, 1]$ and $\mu \in [0, 1]$ represent the lower and upper limits of decoherence parameter and memory parameter respectively. It is seen that the players payoffs heavily depends on the memory of the channel. A similar behaviour of depolarizing and bit-phase flip channels is seen for maximum correlations. It is also seen that amplitude damping channel influences the game more heavily as compared to the other channels. The Nash equilibrium payoff is substantially enhanced due to the presence of quantum memory. Hence, memory controls the payoff reduction due to decoherence.

II. CALCULATIONS FOR FOUR-PLAYER QUANTUM MINORITY GAME

Since noise is a major hurdle in the path of efficient information transmission from one party to the other. This noise causes a distortion of the information sent through the channel. Information transmission is said to be reliable if the probability of error, in decoding the output of the channel, vanishes asymptotically in the number of uses of the channel. A basic question of information theory is whether there is any advantage in using entangled states as input states. That is, whether or not encoding the classical data into entangled rather than separable states increases the mutual information. For the case when multiple uses of the channel are not correlated, there is no advantage in using entangled states. Correlated noise, also referred as memory in the literature, acts on consecutive uses of the channels. However, in general, one may want to encode classical data into entangled strings or consecutive uses of the channel may be correlated to each other. Hence, we
are dealing with a strongly correlated quantum system, the correlation of which results from the memory of the channel itself. The initial game state consists of one qubit for each player, prepared in an entangled GHZ state by an entangling operator $\hat{J}$ acting on $|0000\rangle$. Pure quantum strategies are local unitary operators acting on a player’s qubit. After all players have executed their moves the game state undergoes a positive operator valued measurement and the payoffs are determined from the classical payoff matrix. In the Eisert protocol this is achieved by applying $\hat{J}^\dagger$ to the game state and then making a measurement in the computational basis state. Here we describe the four-player Minority game in the presence of quantum memory using the Eisert scheme in the following manner

$$\rho_0 = |\Psi_0\rangle \langle \Psi_0| \quad \text{(initial state)}$$

$$\rho_1 = \hat{J}\rho_0\hat{J}^\dagger \quad \text{(entanglement)}$$

$$\rho_2 = D(\rho_1, p, \mu) \quad \text{(partial decoherence and correlations)}$$

$$\rho_3 = \otimes_{k=1}^4 \hat{M}_k\rho_2(\otimes_{k=1}^4 \hat{M}_k)^\dagger \quad \text{(players’ moves)}$$

$$\rho_4 = D(\rho_3, p', \mu') \quad \text{(partial decoherence and correlations)}$$

$$\rho_5 = \hat{J}^\dagger\rho_4\hat{J} \quad \text{(preparation for measurement)}$$

to produce the final state $\rho_f \equiv \rho_5$ upon which a measurement is taken. Here $\hat{M}_k$ represents the kth player move. It is important to mention here that for the sake of simplicity, we have considered $p = p' = p$ and $\mu = \mu' = \mu$ in our calculations. The function $D(\rho, p, \mu)$ represents a completely positive map which can be completely described in Kraus operator formalism as studied by Macchiavello and Palma [30] a Pauli channel with partial memory. The two qubit Kraus operators for such a channel are given by

$$A_{ij} = \sqrt{\alpha_i[(1 - \mu)\alpha_j + \mu\delta_{ij}]}\sigma_i \otimes \sigma_j$$

where $\sigma_i$ ($\sigma_j$) are usual Pauli matrices, $\alpha_i$ ($\alpha_j$) represent the quantum noise and indices $i$ and $j$ runs from 0 to 3. The above expression means that with probability $\mu$ the channel acts on the second qubit with the same error operator as on the first qubit, and with probability $(1 - \mu)$ it acts on the second qubit independently. Physically the parameter $\mu$ is determined by the relaxation time of the channel when a qubit passes through it. In order to remove correlations, one can wait until the channel has relaxed to its original state before sending the next qubit, however this lowers the rate of information transfer. Thus it is necessary to consider the performance of the channel for arbitrary values of $\mu$ to reach a compromise between various factors which determine the final rate
of information transfer. Thus in passing through the channel any two consecutive qubits undergo random independent (uncorrelated) errors with probability \(1 - \mu\) and identical (correlated) errors with probability \(\mu\). This should be the case if the channel has a memory depending on its relaxation time and if we stream the qubits through it. The action of the Pauli channels on \(n\)-qubits can be generalized in Kraus operator form as given below

\[
A_{i_1 \ldots i_n} = \sqrt{\alpha_{i_n} \prod_{m=1}^{n-1} [(1 - \mu)\alpha_{i_m} + \mu\delta_{i_m, i_{m+1}}]} \sigma_{i_1} \otimes \ldots \otimes \sigma_{i_n} \tag{8}
\]

As stated above, with probability \((1 - \mu)\) the noise is uncorrelated and can be completely specified by the Kraus operators

\[
D_{ij}^u = \sqrt{\alpha_i \alpha_j} \sigma_i \otimes \sigma_j, \tag{9}
\]

and with probability \(\mu\) the noise is correlated (i.e. the channel has memory) which can be specified by the Kraus operators

\[
D_{kk}^c = \sqrt{\alpha_k} \sigma_k \otimes \sigma_k, \tag{10}
\]

A detailed list of single qubit Kraus operators for different quantum channels with uncorrelated noise is given in table 1. The action of such a channel if \(n\) qubits are streamed through it, can be described in operator sum representation as [5]

\[
\rho_f = \sum_{k_1, \ldots, k_n=0}^{n-1} (A_{k_n} \otimes \ldots \otimes A_{k_1}) \rho_{in} (A_{k_1}^\dagger \otimes \ldots \otimes A_{k_n}^\dagger) \tag{11}
\]

where \(\rho_{in}\) represents the initial density matrix for quantum state and \(A_{k_n}\) are the Kraus operators expressed in equation (8). The Kraus operators satisfy the completeness relation

\[
\sum_{k_n=0}^{n-1} A_{k_n}^\dagger A_{k_n} = 1 \tag{12}
\]

However, the Kraus operators for a quantum amplitude damping channel with correlated noise are given by Yeo and Skeen [31] as given as

\[
\begin{align*}
A_{00}^c &= \begin{bmatrix}
\cos \chi & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, & A_{11}^c &= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sin \chi & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\tag{13}
\]
where, \(0 \leq \chi \leq \pi/2\) and is related to the quantum noise parameter as
\[
\sin \chi = \sqrt{p}
\] (14)

It is clear that \(A_{00}^c\) cannot be written as a tensor product of two two-by-two matrices. This gives rise to the typical spooky action of the channel: \(|01\rangle\) and \(|10\rangle\), and any linear combination of them, and \(|11\rangle\) will go through the channel undisturbed, but not \(|00\rangle\). The action of this non-unital channel is given by
\[
\pi \rightarrow \rho = \Phi(\pi) = (1 - \mu) \sum_{i,j=0}^{1} A_{ij}^u \pi A_{ij}^{u\dagger} + \mu \sum_{k=0}^{1} A_{kk}^c \pi A_{kk}^{c\dagger}
\] (15)

The action of the super-operators provides a way of describing the evolution of quantum states in a noisy environment. In our scheme, the Kraus operators are of the dimension \(2^4\). They are constructed from single qubit Kraus operators by taking their tensor product over all \(n^4\) combinations
\[
A_k = \otimes A_{k_n}
\] (16)

where \(n\) is the number of Kraus operator for a single qubit channel. The final state of the game after the action of the channel can be obtained as
\[
\rho_f = \Phi_{\alpha,\mu}(|\Psi\rangle \langle \Psi|)
\] (17)

where \(\Phi_{\alpha,\mu}\) is the super-operator realizing the quantum channel parametrized by real numbers \(\alpha\) and \(\mu\). The players unitary operator is an \(SU(2)\) operator which represents the pure quantum strategy and is given by
\[
\hat{M}(\theta, \alpha, \beta) = \begin{pmatrix}
e^{i\alpha} \cos(\theta/2) & ie^{i\beta} \sin(\theta/2) \\
ie^{-i\beta} \sin(\theta/2) & e^{-i\alpha} \cos(\theta/2)
\end{pmatrix}
\] (18)

where \(0 \leq \theta \leq \pi\) and \(\pi \leq \{\alpha, \beta\} \leq -\pi\). Here \(\hat{M}(0,0,0) = \hat{I}\) and \(\hat{M}(\pi,0,0) = i\hat{\sigma}_x\) correspond to the two classical pure strategies. Entanglement is controlled through an entangling gate as given by
\[
\hat{J}(\gamma) = \exp(\frac{i\gamma}{2} \hat{\sigma}_x^4)
\]

where the parameter \(\gamma\) represents the degree of entanglement of the game and \(\gamma = \pi/2\) corresponds to maximal entanglement. It is shown by Benjamin and Hayden [27] that in a four player quantum Minority game an optimal strategy arises
\[
\hat{S}_{NE} = \hat{M}(\frac{\pi}{2}, \frac{-\pi}{16}, \frac{\pi}{16})
\] (19)
This strategic profile \( \{ \hat{s}_{NE}, \hat{s}_{NE}, \hat{s}_{NE}, \hat{s}_{NE} \} \) results in an NE with an expected payoff of \( 1/4 \). The expectation value of the payoff to the \( k \)th player is

\[
\langle \hat{s}^k \rangle = \sum_{\xi} \hat{P}_\xi \rho_f \hat{P}_\xi^\dagger s^k_\xi \tag{20}
\]

where \( \hat{P}_\xi = |\xi\rangle \langle \xi| \) is the projector onto the computational state \( |\xi\rangle \), \( s^k_\xi \) is the payoff to the \( k \)th player when the final state is \( |\xi\rangle \) and the summation is taken over \( \xi \) ranging from 1 to 4. We calculate here the payoff of the first player when all the players resort to their optimal strategies as given in equation (19). The expected payoff of the first player for amplitude damping channel becomes

\[
\rho_f^{AD} = 0.125\mu p^4 + [0.125(p - 1)^6 + \mu^2 (0.125p^6 - 0.625p^5 + 1.25p^4 - 1.25p^3 + 0.5p^2 + 0.125pa)] \sin(\gamma) \tag{21}
\]

The expected payoff of the first player for depolarizing channel is given by

\[
\rho_f^{Dep} = [0.25(\mu - 1)^6 p^8 - 1.625(\mu - 1.23077)(\mu - 0.998694) (\mu^2 - 2.00212\mu + 1.00212) \times
(\mu^2 - 1.99919\mu + 0.999187) p^7 + 4.25(\mu - 1.00034)(\mu - 0.999661) \times
(\mu^2 - 2.41176\mu + 1.64706) (\mu^2 - 2.\mu + 1.) p^6 - 5.75(\mu - 1.24903)(\mu - 1)^3 \times
(\mu^2 - 2.25097\mu + 1.94934) p^5 + 4.25(\mu - 1)(\mu - 1) (\mu^2 - 2.56538\mu + 1.72402) \times
(\mu^2 - 1.90521\mu + 2.3884) p^4 - 1.625(\mu - 1.19251)(\mu - 1) \times
(\mu^2 - 2.75771\mu + 2.34443) (\mu^2 - 1.35747\mu + 3.08159) p^3 + 0.25(\mu - 2.12845) \times
(\mu - 1) (\mu^2 - 2.33278\mu + 2.63081) (\mu^2 - 0.538772\mu + 5.0004) p^2 + 0.625(\mu - 1.1587) \times
(\mu^2 - 1.2413\mu + 2.76171) p + 0.25 \cos(\gamma/2) \sin(\gamma/2)] + 0.125 \tag{22}
\]

The expected payoff of the first player for bit-phase flip channel is calculated as
The expected payoff of the first player for phase flip channel is given by
\[
\rho_f^{BPF} = [-3.31371(\mu - 1.00056)(\mu + 2)((\mu - 2.00035)\mu + 1.00035)((\mu - 1.99909)\mu + 0.99909)p^7 + 11.598(\mu - 0.999197)(\mu + 2)((\mu - 2.0013)\mu + 1.0013) \times
(\mu - 1.9995)\mu + 0.999503)p^6 - 15.7401(\mu - 1.20598)(\mu - 1.00002) \times
(\mu - 0.682281)(\mu + 1.9409)((\mu - 1.99998)\mu + 0.999979)p^5 + 10.3553(\mu - 1.3349) \times
(\mu - 0.999989)(\mu - 0.202106)(\mu + 1.73701)((\mu - 2.00001)\mu + 1.00001)p^4 - 3.31371(\mu - 1.83902)(\mu - 1) \times ((\mu - 1.69669)\mu + 1.00918)(\mu(\mu + 2.03571) + 1.56825)p^3 + 0.414214(\mu - 2.62495) \times (\mu - 1)((\mu - 1.30099)\mu + 1.21191) \times
(\mu(\mu + 2.92594) + 5.86011)p^2 + 1.10355(\mu - 1.21443) \times
(((\mu - 0.597899)\mu + 1.71088)p + 0.25) \cos(\gamma/2) \sin(\gamma/2)] + 0.125 \quad (23)
\]

The expected payoff of the first player for bit flip channel can be written as
\[
\rho_f^{BF} = [19.3137(\mu - 1.00066)(\mu + 2)((\mu - 2.00041)\mu + 1.00041) \times
(\mu - 1.99894)\mu + 0.998937)p^7 - 67.598(\mu - 0.999575)(\mu + 2) \times
(\mu - 2.00069)\mu + 1.00069)((\mu - 1.99974)\mu + 0.999737)p^6 + 91.7401(\mu - 0.99995) \times
(\mu + 2.00968)((\mu - 2)\mu + 1)((\mu - 1.95705)\mu + 1.09095)p^5 - 60.3553(\mu - 1.00001) \times
(\mu + 2.03676)((\mu - 1.99999)\mu + 0.999994)((\mu - 1.83676)\mu + 1.34104)p^4 + 19.3137(\mu - 1) \times (\mu + 2.10125)((\mu - 2.05931)\mu + 1.11332)((\mu - 1.54194)\mu + 1.77849)p^3 - 2.41421(\mu - 1) \times (\mu + 2.28298)((\mu - 2.30862)\mu + 1.59107)((\mu - 0.974366)\mu + 2.65448)p^2 + 0.396447(\mu - 1.27686)((\mu - 3.76795)\mu + 7.32329)p + 0.25 \cos(\gamma/2) \sin(\gamma/2)] + 0.125 \quad (24)
\]

The expected payoff of the first player for phase flip channel is given by
\[
\rho_f^{PF} = \frac{1}{8} \left( \left( -16(\mu - 1)^3p^4 + 32(\mu - 1)^3p^3 - 4(5\mu^3 - 14\mu^2 + 15\mu - 6)p^2 + 4(\mu^3 - 2\mu^2 + 3\mu - 2)p + 1 \right) \sin(\gamma) + 1 \right) \quad (25)
\]

where the super-scripts AD, Dep, BPF, BF and PF correspond to the amplitude damping, depolarizing, bit-phase flip, bit flip and phase flip channels respectively. If we set \( \mu = 0 \), it is seen that our results are consistent with Ref. [29], as shown in figure 1.
III. DISCUSSIONS

We have computed the analytical relations for payoffs in case of amplitude damping, depolarizing, bit-phase flip, bit flip and phase flip channels in the presence of quantum memory when all the players resort to their Nash equilibrium strategy as mentioned in Ref. [27]. It is seen that the game is heavily influenced by different memory channels under consideration.

In figures 1, we plot the Nash equilibrium payoff of the first player as a function of the decoherence probability $p$ for $\gamma = \pi/2$ and $\mu = 0$. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left). Different curves indicate decoherence by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively. It is clear from the figure that our results are consistent with results presented by Flitney et al. [29]. In figures 2, 3 and 4, we plot the Nash equilibrium payoff of the first player as a function of the decoherence probability $p$ for $\gamma = \pi/2$ and $\mu = 0.3, 0.7$ and 1 respectively. The behaviour of different curves is indicated by different colour combinations in the figure illustrations. It is seen that even in the presence of decoherence, the payoffs are sufficiently enhanced from its classical counterpart due to the presence of quantum memory. The behaviours of phase flip and amplitude damping channels are surprisingly different from the other three channels. The phase flip channels shows a symmetrical behaviour around $p = 1/2$. However, the payoffs are increased as we increase the degree of memory. It is also seen that the curves for depolarizing and bit-phase flip channels overlap at $\mu = 1$.

In figures 5, 6, we plot the Nash equilibrium payoff of the first player as a function of memory parameter $\mu$ for $\gamma = \pi/2$ and $p = 0.3$ and 0.7 respectively. The memory goes from the uncorrelated quantum game at $\mu = 0$ (right) to maximum correlations at $\mu = 1$ (left). Different curves indicate memory by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively. It is seen that the payoff increases for entire range of memory with lower level of decoherence. However, the payoff for depolarizing and bit-phase flip channels starts decreasing for $\mu \geq 0.5$ for higher values of decoherence (see figure 6).

In figure 7, the Nash equilibrium payoff of the first player is plotted as a function of entanglement parameter $\gamma$ for $\mu = 0.3$ and $p = 0.3$. It is seen that entanglement can play an important role when the game is being implemented. As we increase the degree of entanglement of the game, the payoffs are sufficiently increased. However, amplitude damping channel reduces the payoffs surprisingly as compared to the other channels.
IV. CONCLUSIONS

We study the influence of quantum memory on a four-player quantum Minority game. We investigate the Nash equilibrium payoff of the first player by using different quantum channels with memory. It is seen that the players payoffs heavily depends on the memory of the channel. A similar behaviour of depolarizing and bit-phase flip channels is seen for maximum correlations. As we increase the degree of memory, the payoff reduction due to decoherence is compensated and the payoff is substantially enhanced as compared to its classical counterpart. It is also seen that amplitude damping channel influences the game more heavily as compared to the other channels. Furthermore, the behaviour of phase flip channel is symmetrical around $p = 1/2$. In conclusion, we can say that memory controls the payoff reduction due to decoherence.

[1] Nielson M A and Chuang I L 2000 Quantum computation and quantum information (Cambridge University Press)
[2] Meyer D A 1999 Phys. Rev. Lett. 82 1052
[3] Eisert J, Wilkens M and Lewenstein M 1999 Phy. Rev. Lett. 83 3077
[4] Marinatto L and Weber T 2000 Phys. Lett. A 272 291
[5] Flitney A P and Abbott D 2005 J. Phys. A: Math. Theor. 38 449
[6] Cheon T and Iqbal A 2008 J. Phys. Soc. Japan 77 024801
[7] Ramzan M et al. 2008 J. Phys. A: Math. Theor. 41 055307
[8] Iqbal A, Cheon T and Abbott D 2008 Phys. Lett. A 372 6564
[9] Eisert J and Wilkens M 2000 J. Mod. Opt., 47 2543
[10] Ramzan M and Khan M K 2008 J. Phys. A: Math. Theor. 41 435302
[11] Ramzan M and Khan M K 2009 J. Phys. A: Math. Theor. 42 025301
[12] Iqbal A and Toor A H 2001 Phys. Lett. A 280 249
[13] Flitney A P and Abbott D 2002 Phys. Rev. A 65 062318
[14] D’Ariano G M, Gill R D, Keyl M, Kuemmerer B, Maassen H and Werner RF 2002 Quant. Inf. Comp. 2 355
[15] Iqbal A and Toor A 2002 H Phys. Rev. A 65 022036
[16] Iqbal A and Toor A H 2002 Phys. Lett. A 293 103
[17] Johnson N F 2001 Phys. Rev. A 63 020302(R)
[18] Gawron P and Sladkowski J 2008 Int. J. Quant. Info. 6 667
[19] Lee C F and Johnson N 2002 Phys. Lett. A 301 343
[20] James M C, Iqbal A, Lohe M A and von Smekal L 2009 Physical Society of Japan 78 054801
[21] Almeida et al. 2010 Phys. Rev. Lett. 104 230404
[22] Ramzan M and Khan M K 2010 Quant. Info. Process. 9 667
[23] Shadman Z et al 2010 New J. Phys. 12 073042
[24] Siomau M and Fritzsche S 2010 Eur. Phys. J. D 60 397
[25] D. Challet, M. Marsili and Y-C Zhang 2000 Physica A 276 284
[26] Moro E (2004), The Minority game: an introductory guide, in Advances in Condensed Matter and Statistical Physics, eds. Korutcheva E and Cuerono R, Nova Science Publishers Inc. (Haupauge).
[27] Benjamin S C and Hayden P M 2001 Phys. Rev. A 64 030301(R)
[28] Chen Q, Wang Y, Liu J.-T and Wang K-L 2004 Phys. Lett. A 327 98
[29] Flitney A P and Hollenberg L C L 2007 Quant. Inf. Comp. 7 111
[30] Macchiavello C and Palma G M 2002 Phys. Rev. A 65 050301
[31] Yeo Y and Skeen A 2003 Phys. Rev. A 67 064301
[32] Karimipour V et. al. 2006 Phys. Rev. A 74 062311
[33] Bowen G and Mancini S 2004 Phys. Rev. A 69:01236
[34] Kretschmann D and Werner R F 2005 Phys. Rev. A 72 062323
FIG. 1: The Nash equilibrium payoff in a four-player quantum Minority game as a function of the decoherence probability $p$ for $\gamma = \pi/2$ and $\mu = 0$. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left). Different curves indicate decoherence by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively.
FIG. 2: The Nash equilibrium payoff in a four-player quantum Minority game as a function of the decoherence probability $p$ for $\gamma = \pi/2$ and $\mu = 0.3$. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left). Different curves indicate decoherence by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively.

TABLE I: Single qubit Kraus operators for amplitude damping, depolarizing, bit-phase flip, bit flip and phase flip channels where $p$ represents the decoherence parameter.

| Channel                        | $A_0$                                               | $A_1$ |
|--------------------------------|-----------------------------------------------------|-------|
| Amplitude damping channel      | $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$ | $\begin{bmatrix} 0 & \sqrt{\overline{\rho}} \\ 0 & 0 \end{bmatrix}$ |
| Depolarizing channel           | $A_0 = \sqrt{1-\frac{3p}{4}}I$, $A_1 = \sqrt{\frac{p}{4}}\sigma_x$ | $A_2 = \sqrt{\frac{p}{4}}\sigma_y$, $A_3 = \sqrt{\frac{p}{4}}\sigma_z$ |
| Bit-phase flip channel         | $A_0 = \sqrt{1-p}I$, $A_1 = \sqrt{p}\sigma_y$       |       |
| Bit flip channel               | $A_0 = \sqrt{1-p}I$, $A_1 = \sqrt{p}\sigma_x$       |       |
| Phase flip channel             | $A_0 = \sqrt{1-p}I$, $A_1 = \sqrt{p}\sigma_z$       |       |
FIG. 3: The Nash equilibrium payoff in a four-player quantum Minority game as a function of the decoherence probability $p$ for $\gamma = \pi/2$ and $\mu = 0.7$. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left). Different curves indicate decoherence by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively.
FIG. 4: The Nash equilibrium payoff in a four-player quantum Minority game as a function of the decoherence probability $p$ for $\gamma = \pi/2$ and $\mu = 1$. The decoherence goes from the unperturbed quantum game at $p = 0$ (right) to maximum decoherence at $p = 1$ (left). Different curves indicate decoherence by amplitude damping (black), depolarizing and bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively.
FIG. 5: The Nash equilibrium payoff in a four-player quantum Minority game as a function of memory parameter $\mu$ for $\gamma = \pi/2$ and $p = 0.3$. The memory goes from the uncorrelated quantum game at $\mu = 0$ (right) to maximum correlations at $\mu = 1$ (left). Different curves indicate memory by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively.
FIG. 6: The Nash equilibrium payoff in a four-player quantum Minority game as a function of memory parameter $\mu$ for $\gamma = \pi/2$ and $p = 0.7$. The memory goes from the uncorrelated quantum game at $\mu = 0$ (right) to maximum correlations at $\mu = 1$ (left). Different curves indicate memory by amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) channels respectively.
FIG. 7: The Nash equilibrium payoff in a four-player quantum Minority game as a function of entanglement parameter $\gamma$ for $\mu = 0.3$ and $p = 0.3$. Different curves indicate different channels such as amplitude damping (black), depolarizing (red), bit-phase flip (green), bit flip (blue) and phase flip (blue-green) respectively.