Infinite permutations of lowest maximal pattern complexity

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Abstract

An infinite permutation $\alpha$ is a linear ordering of $\mathbb{N}$. We study properties of infinite permutations analogous to those of infinite words, and show some resemblances and some differences between permutations and words. In this paper, we define maximal pattern complexity $p^*_\alpha(n)$ for infinite permutations and show that this complexity function is ultimately constant if and only if the permutation is ultimately periodic; otherwise its maximal pattern complexity is at least $n$, and the value $p^*_\alpha(n) \equiv n$ is reached exactly on the family of permutations constructed by Sturmian words.

1 Infinite permutations

Let $S$ be a finite or countable ordered set: we shall consider $S$ equal either to $\mathbb{N}$, or to some subset of $\mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $A_S$ be the set of all sequences of pairwise distinct reals defined on $S$. Define an equivalence relation $\sim$ on $A_S$ as follows: let $a, b \in A_S$, where $a = \{a_s\}_{s \in S}$ and $b = \{b_s\}_{s \in S}$; then $a \sim b$ if and only if for all $s, r \in S$ the inequalities $a_s < a_r$ and $b_s < b_r$ hold or do not hold simultaneously. An equivalence class from $A_S/\sim$ is called an $(S)$-permutation. If an $S$-permutation $\alpha$ is realized by a sequence of reals $a$, we denote $\alpha = \vec{a}$. In particular, a $\{1, \ldots, n\}$-permutation always has a representative with all values in $\{1, \ldots, n\}$, i.e., can be identified with a usual permutation from $S_n$.

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In equivalent terms, a permutation can be considered as a linear ordering of \( S \) which may differ from the “natural” one. That is, for \( i, j \in S \), the natural order between them corresponds to \( i < j \) or \( i > j \), while the ordering we intend to define corresponds to \( \alpha_i < \alpha_j \) or \( \alpha_i > \alpha_j \). We shall also use the symbols \( \gamma_{ij} \in \{<,>\} \) meaning the relations between \( \alpha_i \) and \( \alpha_j \), so that by definition we have \( \alpha_i \gamma_{ij} \alpha_j \) for all \( i \neq j \).

We are interested in properties of infinite permutations analogous to those of infinite words, for example, periodicity and complexity. A permutation \( \alpha = \{\alpha_s\}_{s \in S} \) is called \( t \)-periodic if for all \( i, j \) and \( n \) such that \( i, j, i + nt, j + nt \in S \) we have \( \gamma_{ij} = \gamma_{i+nt,j+nt} \). In particular, if \( S = \mathbb{N} \), this definition is equivalent to a more standard one: a permutation is \( t \)-periodic if for all \( i, j \) we have \( \gamma_{ij} = \gamma_{i+t,j+t} \). A permutation is called ultimately \( t \)-periodic if these equalities hold provided that \( i, j > n_0 \) for some \( n_0 \). This definition is analogous to that for words: an infinite word \( w = w_1w_2 \cdots \) on an alphabet \( \Sigma \) is \( t \)-periodic if \( w_i = w_{i+t} \) for all \( i \) and is ultimately \( t \)-periodic if \( w_i = w_{i+t} \) for all \( i \geq n_0 \) for some \( n_0 \).

In a previous paper by Fon-Der-Flaass and Frid [4], all periodic \( \mathbb{N} \)-permutations have been characterized; in particular, it has been shown that there exists a countable number of distinct \( t \)-periodic permutations for each \( t \geq 2 \). For example, for each \( n \) the permutation with a representative sequence

\[-1, 2n-2, 1, 2n, 3, 2n+2, 5, 2n+4, \ldots\]

is \( 2 \)-periodic, and all such permutations are distinct. So, the situation with periodicity differs from that for words, since the number of distinct \( t \)-periodic words on a finite alphabet of cardinality \( q \) is clearly finite (and is equal to \( q^t \)).

A set \( T = \{0, m_1, \ldots, m_{k-1}\} \) of cardinality \( k \), where \( 0 = m_0 < m_1 < \cdots < m_{k-1} \), is called a (\( k \)-)window. It is natural to define \( T \)-factors of an \( S \)-permutation \( \alpha \) as restrictions of \( \alpha \) to \( T + n, n \in \mathbb{N} \), considered as permutations on \( T \). Such a projection is well-defined for a given \( n \) if and only if \( T + n \subseteq S \), and is denoted by \( \alpha_{T+n} = \alpha_{n} \alpha_{n+1} \cdots \alpha_{n+m_{k-1}} \). We call the number of distinct \( T \)-factors of \( \alpha \) the \( T \)-complexity of \( \alpha \) and denote it by \( p_\alpha(T) \).

In particular, if \( T = \{0, 1, 2, \ldots, n-1\} \), then \( T \)-factors of an \( \mathbb{N} \)-permutation \( \alpha \) are called just factors of \( \alpha \) and are analogous to factors (or subwords) of infinite words. They are denoted by \( \alpha_{[i,i+n]} \) or, equivalently, \( \alpha_{[i,i+n-1]} = \alpha_i \alpha_{i+1} \cdots \alpha_{i+n-1} \), and their number is called the factor complexity \( f_\alpha(n) \) of \( \alpha \). This function is analogous to the subword complexity \( f_w(n) \) of infinite words which is equal to the number of different words \( w_{[i,i+n]} \) of length \( n \) occurring in an infinite word \( w \) (see [3] for a survey). However, not all the properties of these two functions are similar [3]. Consider in particular the following classical theorem.

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Theorem 1 An infinite word $w$ is ultimately periodic if and only if $f_w(n) = C$ for some constant $C$ and all sufficiently large $n$. If $w$ is not ultimately periodic, then $f_w(n)$ is increasing and satisfies $f_w(n) \geq n + 1$.

Only the first statement of Theorem 1 has an analogue for permutations; as for the second one, the situation with permutations is completely different.

**Theorem 2** Let $\alpha$ be an $\mathbb{N}$-permutation; then $f_{\alpha}(n) \leq C$ if and only if $\alpha$ is ultimately periodic. At the same time, for each unbounded nondecreasing function $g(n)$, there exists a $\mathbb{N}$-permutation $\alpha$ with $f_{\alpha}(n) \leq g(n)$ for all $n \geq N_0$ which is not ultimately periodic.

The supporting example of a permutation with low complexity can be defined by the inequalities $\alpha_{2n} < \alpha_{2n+2} < \alpha_{2n+1} < \alpha_{2n+3}$ for all $n \geq 0$, and $\alpha_{2n_k} < \alpha_{2n_k+1} < \alpha_{2n_k+2}$ for some sequence $\{n_k\}_{k=0}^{\infty}$ which grows sufficiently fast.

In this paper we study the properties of another complexity function, namely, **maximal pattern complexity**

$$p^*_\alpha(n) = \max_{\#T=n} p_\alpha(T).$$

The analogous function $p^*_w(n)$ for infinite words was defined in 2002 by Kamae and Zamboni [6] where the following statement was proved:

**Theorem 3** [6] An infinite word $w$ is not ultimately periodic if and only if $p^*_w(n) \geq 2n$ for all $n$.

Infinite words of maximal pattern complexity $2n$ include rotation words [6] and also some words built by other techniques [7]. The classification of all words of maximal pattern complexity $2n$ is an open problem [5].

In this paper, we prove analogous results for infinite permutations and furthermore, prove that in the case of permutations, lowest maximal pattern complexity is achieved only in the precisely described “Sturmian” case.

## 2 Lowest complexity

First of all, we prove a lower bound for the maximal pattern complexity of a non-periodic infinite permutation.

**Theorem 4** An infinite permutation $\alpha$ is not ultimately periodic if and only if $p^*_\alpha(n) \geq n$ for any $n$. 


**Proof.** Clearly, if a permutation is ultimately periodic, its maximal pattern complexity is ultimately constant, and thus the “if” part of the proof is obvious. Now suppose that \( p^*_\alpha(l) < l \) for some \( l \); we shall prove that \( \alpha \) is ultimately periodic.

Since \( p^*_\alpha(1) = 1 \) (there is exactly one permutation of length one), and the function \( p^* \) is non-decreasing, we see that \( p^*_\alpha(l) < l \) implies that \( p^*_\alpha(n + 1) = p^*_\alpha(n) \) for some \( n \leq l \). Consider an \( n \)-window \( T = (0, m_1, \ldots, m_{n-1}) \) such that \( p_\alpha(T) = p^*_\alpha(n) \); the equality \( p^*_\alpha(n + 1) = p_\alpha(T) \) implies that for each \( T' = (0, m_1, \ldots, m_{n-1}, m_n) \) with \( m_n > m_{n-1} \) we have \( p_\alpha(T) = p_\alpha(T') \), that is, each \( T \)-permutation can be extended to a \( T' \)-permutation in a unique way. Clearly, there exist two equal factors of length \( 2m_{n-1} \) in \( \alpha \); say,

\[
\alpha[k..k+2m_{n-1}] = \alpha[k+t..t+2m_{n-1}]
\]

for some positive \( t \) and non-negative \( k \). We shall prove that \( \alpha \) is ultimately \( t \)-periodic, namely, that \( \gamma_{ij} = \gamma_{i+t..t+j} \) for all \( i, j \) with \( k \leq i < j \). The proof will use the induction on the pair \( i, j \) starting by the pairs \( i, j \) with \( k \leq i < j < k+2m_{n-1} \), for which our statement holds since \( \alpha[k..k+2m_{n-1}] = \alpha[k+t..t+2m_{n-1}] \).

Now for the induction step: for some \( M \geq 2m_{n-1} \), suppose that \( \gamma_{ij} = \gamma_{i+t..t+j} \) for all \( k \leq i < j < k + M \), that is, \( \alpha[k..k+M] = \alpha[k+t..t+M] \). We are going to prove that \( \gamma_{i,k+M} = \gamma_{i+t..t+k+M} \) for all \( i \in \{k, \ldots, k + M - 1\} \), and thus \( \alpha[k..k+M+1] = \alpha[k+t..t+M+1] \).

Indeed, consider the case \( i \in \{k, \ldots, k + M - m_{n-1} - 1\} \) first. Then \( \alpha_{T+i} \) is a \( T \)-factor of \( \alpha[k..k+M] \) and \( \alpha_{T+i+t} \) is a \( T \)-factor of \( \alpha[k+t..t+M] \) standing at the same position. So, these \( T \)-factors of \( \alpha \) are equal, and due to the choice of \( T \), so are their extensions \( \alpha_{T+i} \) and \( \alpha_{T+i+t} \), where \( T' = (0, m_1, \ldots, m_{n-1}, M-i) \).

In particular, the first and last elements of \( \alpha_{T+i} \) and \( \alpha_{T+i+t} \) are in the same relationship: \( \gamma_{i,k+M} = \gamma_{i+t..t+k+M} \), which is what we needed.

Now if \( i \in \{k + M - m_{n-1}, \ldots, k + M - 1\} \), we consider \( \alpha_{T+i-m_{n-1}} \) which is a \( T \)-factor of \( \alpha[k..k+M] \) with the last element \( \alpha_i \), and \( \alpha_{T+i-t-m_{n-1}} \) which is a \( T \)-factor of \( \alpha[k+t..t+M] \) with the last element \( \alpha_{i+t} \). They are equal, and so are their extensions \( \alpha_{T+i-m_{n-1}} \) and \( \alpha_{T+i-t-m_{n-1}} \), where \( T' = (0, m_1, \ldots, m_{n-1}, M-i+m_{n-1}) \). In particular, the next to last and the last elements of these \( T \)-permutations are in the same relationship: \( \gamma_{i,k+M} = \gamma_{i+t..t+k+M} \).

So, \( \gamma_{i,k+M} = \gamma_{i+t..t+k+M} \) for all \( i \in \{k, \ldots, k + M - 1\} \); together with the induction hypothesis it means that \( \alpha[k..k+M+1] = \alpha[k+t..t+M+1] \). Repeating the induction step we get that \( \gamma_{ij} = \gamma_{i+t..t+j} \) for all \( k \leq i < j \), that is, the permutation \( \alpha \) is ultimately \( t \)-periodic. \( \square \)
A one-sided infinite word \( w = w_0w_1w_2 \cdots \) on the alphabet \( \{0, 1\} \) is called \textit{Sturmian} if its subword complexity \( f_w(n) \) is equal to \( n + 1 \) for all \( n \). Sturmian words have a number of equivalent definitions \cite{1}; we shall need two more of them. First, Sturmian words are exactly aperiodic \textit{balanced} words which means that for each length \( n \), the number of 1’s in factors of \( w \) of length \( n \) takes only two successive values. Second, Sturmian words are exactly irrational \textit{mechanical} words which means that there exists some irrational \( \sigma \in (0, 1) \) and some \( \rho \in [0, 1) \) such that for all \( i \) we have

\[
\begin{align*}
  w_i &= \lfloor \sigma(i + 1) + \rho \rfloor - \lfloor \sigma i + \rho \rfloor \quad \text{or} \\
  w_i &= \lceil \sigma(i + 1) + \rho \rceil - \lceil \sigma i + \rho \rceil.
\end{align*}
\]

(1)

These definitions coincide if \( \sigma i + \rho \) is never integer; if it is integer for some (unique) \( i \), the sequences built by these two formulas differ in at most two successive positions. So, we distinguish \textit{lower} and \textit{upper} Sturmian words according to the choice of \( \lfloor \cdot \rfloor \) or \( \lceil \cdot \rceil \) in the definition. A word on any other binary alphabet is called Sturmian if it is obtained from a Sturmian word on \( \{0, 1\} \) by renaming symbols. Here \( \sigma \) is called the \textit{slope} of the word \( w \).

Now let us define a \textit{Sturmian permutation} \( \alpha(w, x, y) = \alpha = \varpi \) associated with a Sturmian word \( w \) and positive numbers \( x \) and \( y \) by its representative sequence \( a \), where \( a_0 \) is a real number and for all \( i \geq 0 \) we have

\[
a_{i+1} = \begin{cases} 
  a_i + x, & \text{if } w_i = 0, \\
  a_i - y, & \text{if } w_i = 1.
\end{cases}
\]

Clearly, such a permutation is well-defined if and only if we never have \( kx \neq ly \) if \( k \) is the number of 0’s and \( l \) is the number of 1’s in some factor of \( w \); and in particular if \( x \) and \( y \) are rationally independent.

Note that a factor of \( w \) of length \( n \) corresponds to a factor of \( \alpha \) of length \( n + 1 \), and the correspondence is one-to-one. So, we have \( f_\alpha(n) = n \) for all \( n \). In fact, we are going to prove that the maximal pattern complexity of \( \alpha \) is also equal to \( n \), and thus the lower bound in Theorem 4 is precise.

**Theorem 5** For each Sturmian permutation \( \alpha \) we have \( p_\alpha^*(n) \equiv n \).

**Proof.** Let us start with the situation when \( x = \sigma \) and \( y = 1 - \sigma \). This case has been proved by M. Makarov in \cite{9}, but we give a proof here for the sake of completeness.

If we take \( a_0 = \rho \), then by the definition of the Sturmian word, \( a_i = \{\sigma i + \rho\} \) holds in the case that \( w \) is a lower Sturmian word, and \( a_i = 1 - \{1 - \sigma i - \rho\} \)
holds in the case that \(w\) is an upper Sturmian word. Here \(\{x\}\) stands for the fractional part of \(x\). In what follows, we consider lower Sturmian words without loss of generality.

Consider a \(k\)-window \(T = \{0, m_1, \ldots, m_{k-1}\}\) and the set of \(T\)-factors \(\alpha_{T+n} = \{\sigma n + \rho\}, \{\sigma(n + m_1) + \rho\}, \ldots, \{\sigma(n + m_{k-1}) + \rho\}\) for all \(n\). Since the set of \(\{\sigma n + \rho\}\) is dense in \([0, 1]\), the set of \(T\)-factors is equal to the set of all permutations \(t, \{t + \sigma m_1\}, \ldots, \{t + \sigma m_{k-1}\}\) with \(t \in [0, 1]\).

Let us arrange the points \(\{t + \sigma m_i\} (i = 0, \ldots, k - 1)\) on the unit circle, that is the interval \([0, 1]\) with the points 0 and 1 identified (recall that \(m_0 = 0\) by definition). Then, the arrangement partitions the unit circle into \(k\) arcs. Since the arrangements for different \(t\)'s are different only by rotations, the permutation defined by the points is determined by indicating the arc which contains 0 = 1. Since there exist \(k\) arcs, there are exactly \(k\) different permutations defined by the points \(\{t + \sigma m_i\} (i = 0, \ldots, k - 1)\) with different \(t\)'s. Thus, \(p_\alpha(T) = k\). Since the window \(T\) was chosen arbitrarily, we have \(p_\alpha(k) = k\).

Now consider the general case of arbitrary \(x\) and \(y\). Let us keep the notation \(\gamma_{ij}\) for the relation between \(\alpha(w, \sigma, 1 - \sigma)\), and \(\alpha(w, \sigma, 1 - \sigma)_j\), and denote the relation between \(\alpha(w, x, y)_i\) and \(\alpha(w, x, y)_j\) by \(\delta_{ij}\).

Recall that the weight of a binary word \(u\) is the number of 1's in it, denoted by \(|u|_1\). By the definition of \(\alpha\), we have \(\delta_{i,i+n} = \delta_{j,j+n}\) if \(w_{[i,i+n]}\) and \(w_{[j,j+n]}\) have the same weight. Note also that the weight of a factor of \(w\) of length \(n\) is either equal to \([n\sigma]\) or to \([n\sigma]\). In \(\alpha(w, \sigma, 1 - \sigma)\), the converse also holds: words \(w_{[i,i+n]}\) and \(w_{[j,j+n]}\) of the same length \(n\) but of different weight always correspond to \(\gamma_{i,i+n} \neq \gamma_{j,j+n}\), since \((n - [n\sigma])\sigma - [n\sigma]|(1 - \sigma) = n\sigma - [n\sigma] > 0\) and \((n - [n\sigma])\sigma - [n\sigma]|(1 - \sigma) = n\sigma - [n\sigma] < 0\). In the general case, words of different weights may correspond to the same relation. But anyway for all \(i, j,\) and \(n\) the equality \(\gamma_{i,i+n} = \gamma_{j,j+n}\) implies that \(\delta_{i,i+n} = \delta_{j,j+n}\). Thus, for any \(k\)-window \(T\) we see that \(\alpha(w, \sigma, 1 - \sigma)_{T+i} = \alpha(w, \sigma, 1 - \sigma)_{T+j}\) implies \(\alpha(w, x, y)_{T+i} = \alpha(w, x, y)_{T+j}\). So, we have \(p_\alpha(w, x, y)(T) \leq p_\alpha(w, x, y)(T)\) and thus \(p_\alpha^*(w, x, y)(k) \leq p_\alpha^*(w, x, y)(k) = k\); at the same time, \(p_\alpha^*(w, x, y)(k) \geq k\) since this permutation is not ultimately periodic. So, \(p_\alpha^*(w, x, y)(k) = k\), and the theorem is proved.

\(\Box\).

In fact, Sturmian permutations are the only \(\mathbb{N}\)-permutations of maximal pattern complexity \(n\). In the remaining part of the paper, we are going to prove it.
4 Rotation words

In what follows, we several times use the fact that Sturmian words form a particular case of so-called rotation words. Let us describe them.

Consider the interval $C = [0, 1)$ as a unit circle, which means that we identify its ends and consider it as the quotient group $\mathbb{R}/\mathbb{Z}$. When working with this group, we consider real numbers modulo one and write $x \pmod{1}$ or just $x$ as well as the fractional part $\{x\}$.

An interval $I = (x, y)$ on $C$ is defined as usual if $0 \leq x < y < 1$ and as $C \setminus (y, x)$ if $0 \leq y < x < 1$. Intervals with other combinations of parentheses are defined analogously.

Now consider a partition of $C$ into a finite number of disjoint intervals $J_0, J_1, \ldots, J_k$, $\bigcup_{j=0}^k J_j = C$. Associate with each interval $J_j$ a symbol $a_j$ from a finite alphabet $A$ (symbols for different intervals may coincide). Let $I_a$ denote the union of intervals corresponding to the symbol $a$.

Consider a sequence $(x_i)_{i=0}^{\infty}$, $x_i \in C$, given by $x_{i+1} = x_i + \xi \pmod{1}$ for some fixed $\xi$, and define an infinite word $v = v_0 \cdots v_n \cdots$ on the alphabet $A$ by $v_i = a \iff x_i \in I_a$. This word is called a rotation word on $A$ with the slope $\xi$ and the initial point $x_0$ induced by the given partition of $C$.

Thus, a Sturmian word defined by (11) is a rotation word induced by a partition of $C$ into the intervals $[0, \sigma)$ and $[\sigma, 0)$ (for a lower Sturmian word; for the upper Sturmian word, the parentheses are $(\cdot, \cdot)$); with the initial point $x_0 = \sigma + \rho$. Equivalently, we can define it by the partition into the intervals $[-\sigma - \rho, -\rho)$ and $[-\rho, -\sigma - \rho)$ with the initial point 0.

5 Proof of uniqueness: first step

Now we shall prove that the described Sturmian permutations are the only permutations of maximal pattern complexity $p^*_n(n) = n$. In the proof, we shall widely use the table of values $\gamma_{ij} \in \{<, >\}$ of a candidate permutation; for the sake of convenience, we denote the strings of that table by $\gamma_i = \gamma_{0,i} \gamma_{1,i+1} \cdots \gamma_{n,i+n} \cdots$ and the arithmetical subsequences of those strings by $\gamma_{i}^j = \gamma_{j,i+j} \gamma_{i+j, 2i+j} \cdots \gamma_{ni+j,(n+1)i+j} \cdots$

for all $i \in \mathbb{N}$ and $j \in \{0, \ldots, i - 1\}$. Thus, a string $\gamma_i$ consists of elements of $i$ disjoint sequences $\gamma_{i}^j$, each of them representing the relations between successive elements of the permutation $\{\alpha_{ni+j}\}_{n=0}^{\infty}$. 
So, each $\gamma^j_i$ is an infinite word on the alphabet \(<, >\). We also denote the subword $\gamma_{n,n+1}\gamma_{n+1,n+2} \cdots \gamma_{n+i-1,n+i}$ by $\gamma[n,n+i]$.

**Lemma 1** If $\alpha$ is an infinite permutation with $p^*_\alpha(n) \equiv n$, then for all $i > 0$ and $j \in \{0, \ldots, i-1\}$ the sequence $\gamma^j_i$ is either ultimately periodic or Sturmian.

**Proof.** Let us fix some $i$. If $p^*_\alpha(n) \equiv n$, then in particular $p_\alpha(T_n) \leq n$, where $T_n = (0, i, 2i, \ldots, (n-1)i)$. Thus, the number of different values $\alpha_{j+ik+T_n}$ for different $k$'s are at most $n$, and since the factor

$$\gamma_{j+ki,j+(k+1)i}\gamma_{j+(k+1)i,j+(k+2)i} \cdots \gamma_{j+(n-2)i,j+(n-1)i}$$

of $\gamma^j_i$ contains just a part of information contained in $\alpha_{j+ik+T_n}$, the number of such factors of length $n-1$ is at most $n$ for all $n$. Since the only non-periodic words satisfying this are Sturmian words, the lemma is proved. \(\square\)

In particular, this lemma is valid for $\gamma_1 = \gamma^0_1$. In what follows we consider the cases when $\gamma_1$ is periodic and when it is Sturmian separately.

6 Proof of uniqueness: Sturmian case

In this section we assume that the first string $\gamma_1$ of the array \{$\gamma^j_i$\}, describing the relations between successive elements of a permutation $\alpha$ with $p^*_\alpha(n) = n$, is a Sturmian word on the alphabet \(<, >\). Let us see what all the other substrings $\gamma^j_i$ are.

We say that an infinite word on \(<, >\) is *increasing* (or *decreasing*) if it is equal to $<\omega$ (or $>\omega$, respectively). It is called *monotonic* if it is either increasing or decreasing. We put “ultimately” if it holds after some point.

**Claim 1** For each $i > 0$ and $j \in \{0, \ldots, i-1\}$ the sequence $\gamma^j_i$ is either Sturmian or ultimately monotonic.

**Proof.** Due to Lemma 1 it is sufficient to prove that $\gamma^j_i$ cannot be ultimately periodic with the minimal period $t$ greater than one, that is, we cannot have for any $t$, $m_1$, and $m_2$ that $(\gamma^j_i)_{m_1+nt} = \gamma_{j+(m_1+nt)i,j+(m_1+nt+1)i}$ $<=$ for all sufficiently large $n$ and $(\gamma^j_i)_{m_2+nt} = \gamma_{j+(m_2+nt)i,j+(m_2+nt+1)i}$ $=>$ for all sufficiently large $n$. To the contrary, let us suppose this and consider the pattern $T = (0, i, i+1)$. Consider the $T$-permutations $\alpha_{k+T}$ for all $k$. Each of them is determined by the three values: $\gamma_{k,k+i}$, $\gamma_{k,k+i+1}$, and $\gamma_{k+i,k+i+1}$. Consider first $k = j + (m_1 + nt)i$ for all sufficiently large $n$. We see that $\gamma_{k,k+i}$ in this case is equal to $<$, but $\gamma_{k+i,k+i+1}$ takes both values for different $n$'s since the sequence $\gamma_1$ is Sturmian and thus any infinite arithmetic progression in it contains both symbols. Analogously, if $k = j + (m_2 + nt)i$, then $\gamma_{k,k+i}$ is ultimately equal.
to $>$ and $\gamma_{k+i,k+i+1}$ takes both values. So, $T$-permutations $\alpha_{k+T}$ take at least four values, which means that $p_\alpha^*(3) \geq 4$, contradicting to the assumption that $p_\alpha^*(n) = n$. \hfill \Box

**Claim 2** Given $i$, if $\gamma_{i}^{j_1}$ is Sturmian for some $j_1$, then $\gamma_{i}^{j}$ is Sturmian for any $j = 0, \ldots, i - 1$.

**Proof.** Due to the previous claim, the opposite would mean that some of $\gamma_{i}^{j}$ were ultimately monotonic. Suppose without loss of generality that $\gamma_{i}^{j}$ is ultimately increasing, and let $n$ be the greatest number of successive symbols $<$ in $\gamma_{i}^{j_1}$ (clearly it is finite). Consider the pattern $T_{n+2} = (0, i, \ldots, ni, (n+1)i)$ of length $n+2$. For different $k$ equal to $j_1$ modulo $i$, the number of different $\alpha_{k+T_{n+2}}$’s is at least $n+2$ since $\gamma_{i}^{j_1}$, the sequence describing the relations between the successive elements of $T_{n+2}$, is Sturmian. Moreover, since $<^{n+1}$ is not contained in $\gamma_{i}^{j_1}$, while it is contained in $\gamma_{i}^{j}$, $\alpha_{k+T_{n+2}}$ can take at least $n+3$ different values, contradicting to the assumption that $p_\alpha^*(k) = k$. \hfill \Box

**Claim 3** Suppose that $\gamma_{i}^{j_1}$ is ultimately increasing (ultimately decreasing) for some $j_1$. Then, $\gamma_{i}^{j}$ is ultimately increasing (ultimately decreasing, respectively) for any $j = 0, 1, \ldots, i - 1$.

**Proof.** Due to the previous claims, the opposite would mean that exactly $\gamma_{i}^{j}$ is ultimately decreasing for some $j_2 \in \{0, \ldots, i - 1\}$, while $\gamma_{i}^{j}$ is ultimately increasing. Now consider once again the pattern $T = (0, i, i+1)$ and like in Claim [1] observe that the pair $(\gamma_{n,n+i}, \gamma_{n+i,n+i+1})$ which contains a part of information of $\alpha_{n+T}$, takes at least two different values $(<, <)$ and $(<, >)$ when $n = j_1 \pmod{i}$. Also, it takes two values $(>, <)$ and $(>, >)$ when $n = j_2 \pmod{i}$. So, $p_\alpha^*(3) \geq p_\alpha(T) \geq 4$, a contradiction. \hfill \Box

**Claim 4** If $\gamma_{i}$ and $\gamma_{j}$ are ultimately monotonic, then they are ultimately increasing or ultimately decreasing, simultaneously.

**Proof.** Suppose the opposite: say, $\gamma_{i}$ is ultimately increasing and $\gamma_{j}$ is ultimately decreasing. It means that for a sufficiently large $k$ we have $\alpha_k < \alpha_{k+i} < \alpha_{k+2i} < \ldots < \alpha_{k+ji}$, and at the same time, $\alpha_k > \alpha_{k+j} > \alpha_{k+2j} > \ldots > \alpha_{k+j}$, a contradiction. \hfill \Box

Therefore, the set of positive integers is divided into two classes $S$ and $M$: a number $i$ belongs to $S$ if all $\gamma_{i}^{j}$ are Sturmian, and to $M$ if $\gamma_{i}$ is ultimately monotonic. Due to the previous claim, all $\gamma_{i}$ with $i \in M$ are ultimately increasing or ultimately decreasing, simultaneously, and without loss of generality we may assume that they are ultimately decreasing. Now let us specify what kind of Sturmian words $\gamma_{i}^{j}$’s are.

Let the slope of the Sturmian word $\gamma_{1}$ be equal to $\sigma$ and the initial point be $\rho$. Without loss of generality we assume that the word is lower Sturmian: this
means precisely that

\[ \gamma_{n,n+1} = \begin{cases} <, & \text{if } \{\sigma(n + 1) + \rho\} < \sigma, \\ >, & \text{otherwise}. \end{cases} \]

In other words, \( \gamma_{n,n+1} =< \) if and only if \( \sigma n \in [-\sigma - \rho, -\rho) \mod 1 \). Moreover, \( \gamma_{n+1,n+2} =< \) if and only if \( \sigma n \in [-2\sigma - \rho, -\sigma - \rho) \mod 1 \), etc.: we see that the word \( \gamma_{[n,n+i]} \) is determined by the position of the point \( \sigma n \in \mathcal{C} \) with respect to the points \(-\rho, -\sigma - \rho, \ldots, -i\sigma - \rho\).

Let us fix some \( i \). We know that

\[ \#\{\alpha_{[n..n+i]}|n \in \mathbb{N}\} = f_\alpha(i + 1) \leq p_\alpha^i(i + 1) = i + 1. \]

On the other hand, we have

\[ \#\{\alpha_{[n..n+i]}|n \in \mathbb{N}\} \geq \#\{\gamma_{[n..n+i]}|n \in \mathbb{N}\} = i + 1. \]

Hence, we have \( \#\{\alpha_{[n..n+i]}|n \in \mathbb{N}\} = \#\{\gamma_{[n..n+i]}|n \in \mathbb{N}\} \). It follows that the whole permutation \( \alpha_{[n..n+i]} \), and in particular the relation \( \gamma_{n,n+i} \), is uniquely determined by \( \gamma_{[n..n+i]} \) and thus by the position of the point \( \sigma n \in \mathcal{C} \) with respect to the points \(-\rho, -\sigma - \rho, \ldots, -i\sigma - \rho \mod 1\).

For \( i \in M \) this implies that the sequence \( \gamma_i \) is monotinic, not only ultimately monotonic.

For \( i \in S \) this means that \( \gamma_i \) is a rotation word on \( \{<, >\} \) with the slope \( \sigma \) starting at 0, and the partition of \( \mathcal{C} \) by the set of intervals of type \( [\ , \) bounded by the points \(-\rho, -\sigma - \rho, \ldots, -i\sigma - \rho \mod 1 \) and for each \( j = 0, \ldots, i - 1 \), the word \( \gamma_i^j \) is a rotation word on \( \{<, >\} \) corresponding to the same partition of \( \mathcal{C} \) by the intervals, with the slope \( i\sigma \) starting at \( j\sigma \). Here, we have not yet specified the correspondance between intervals and \( \{<, >\} \).

**Claim 5** Assume that \( i \in S \), which means that \( \gamma_i^j \) are Sturmian words for all \( j \). Let \( I_\prec \) be the union of the above intervals corresponding to \( \prec \). Then, \( I_\prec \) is an interval in \( \mathcal{C} \) of length \( \{i\sigma\} \) or \( 1 - \{i\sigma\} \).

**Proof.** Note that \( \gamma_i^j \) is a rotation word on \( \{<, >\} \) with the slope \( i\sigma \) starting at \( j\sigma \) corresponding to the partition defined by the set of points \( S = \{-\rho\}, \{-\sigma - \rho\}, \ldots, \{-i\sigma - \rho\} \). Note that \( S \cap (S + i\sigma) = \{-\rho\} \) and \( S \cap (S + ki\sigma) = \emptyset \) for any \( k = 2, 3, \ldots \).

Suppose that the conclusion in the Claim does not hold. Then, there exist \( u, v \in S \) with \( u < v \) in the boundary of \( I_\prec \) such that \( v - u \neq \{i\sigma\} \) and \( v - u \neq 1 - \{i\sigma\} \). Let \( \mathbb{P} = \{I_\prec, I_\succ\} \) be the partition of \( \mathcal{C} \) and

\[ \mathbb{P}_{k+1} = \mathbb{P} \lor (\mathbb{P} - i\theta) \lor \ldots \lor (\mathbb{P} - ki\theta) \]
be the refinement of partitions. There exists \( k > 0 \) such that \( u \) and \( v \) are in the interiors of distinct elements of the partition

\[
P' = (P - i\theta) \vee \ldots \vee (P - k i\theta).
\]

Then, we have \( \#P_{k+1} \geq \#P' + 2 \) since each of the points \( u \) and \( v \) increases \( \#P_{k+1} \) from \( \#P' \) by 1. On the other hand, \( \#P' = \#P_k = k + 1 \) and \( \#P_{k+1} = k + 2 \) hold since \( \gamma_i \) is a Sturmian word. Thus, we have a contradiction. \( \square \)

By Claim 5, only 2 cases are possible. That is, either \( I_\gamma = [\{-\rho\}, \{-i\sigma - \rho\}) \) and \( I_\gamma = [\{-i\sigma - \rho\}, \{-\rho\}) \) or \( I_\gamma = [\{-\rho\}, \{-i\sigma - \rho\}) \) and \( I_\gamma = [\{-i\sigma - \rho\}, \{-\rho\}) \). Hence, there are only two Sturmian words on \( \{<, >\} \), satisfying our properties, and they are obtained from the other by exchanging the symbols.

In fact, we can describe the Sturmian words obtained in the above in a more direct way. To do it, for each \( j = 0, \ldots, i-1 \) consider the word \( v^j = v_0 \cdots v_n \cdots \) defined by \( v_n = |\gamma_{[j+n..j+(n+1)]}| <, \) where \( |w|_a \) denotes the number of occurrences of a symbol \( a \) in the word \( w \). As it follows from the definition of \( \gamma_1 \), the word \( v \) is binary on the alphabet \( \{q_i, q_i + 1\} \), where \( q_i = |\sigma i| \).

It is not periodic since \( \gamma_1 \) is Sturmian. Moreover, the word \( v \) is balanced since \( |\gamma_{[j+n..j+k(n+1)]}| < = q_k k + |v_{[n..n+k]}|_{q_i+1} \) also takes only two values for a fixed \( k \), and so does \( |v_{[n..n+k]}|_{q_i+1} \). But non-periodic balanced words are exactly Sturmian words. So, \( v \) is Sturmian, and its symbol \( v_n \) is determined by \( \gamma_{[j+n..j+(n+1)]} \). As we have shown in the previous paragraph, it means that \( \gamma_i \) is obtained from \( v \) by renaming symbols, that is, each its symbol \( \gamma_{[j+n..j+(n+1)]} \) is determined by the number of symbols \( < \) in \( \gamma_{[j+n..j+(n+1)]} \) independently of \( j \), which is either \( q_i \) or \( q_i + 1 \).

Thus, there is a mapping, say \( \rho_i \), from \( \{q_i, q_i + 1\} \) to \( \{<, >\} \) such that \( \rho_i(|\gamma_{[m..m+i]}| <) = \gamma_{m,m+i} \) for any \( i, m \).

Consider the case \( i \in M \). Since \( \gamma_{m,m+i} \) is independent of \( m \) for any large \( m \), we have \( \rho_i(q_i) = \rho_i(q_i + 1) \). Thus, \( \rho_i \) takes only one value and it holds that \( \gamma_i \) is not only ultimately monotonic, but also monotonic for any \( i \in M \).

Consider the case \( i \in S \). Since \( i \in S \), and thus \( \gamma_{m,m+i} \) can take both values \( < \) and \( > \), \( \rho_i \) is a bijection. Suppose first that \( \rho_i(q_i) = < \). Consider a factor \( \gamma_{[m..m+i]} \) of \( \gamma_1 \) starting with \( > \) and ending with \( < \), so that \( |\gamma_{[m..m+i]}| < = q_i \) and \( |\gamma_{[m+1..m+i+1]}| > = q_i + 1 \). We have \( \gamma_{m,m+i} = < \), and thus \( \alpha_{m+1} < \alpha_{m+i} < \alpha_{m+i+1} \). At the same time, \( \gamma_{m+1,m+i+1} = > \), that is, \( \alpha_{m+1} > \alpha_{m+i+1} \). A contradiction to our assumption. Hence, we have \( \rho_i(q_i) = > \) and \( \rho_i(q_i + 1) = < \).

At last, note that \( |\gamma_{[m..m+i]}| < = q_i \) if and only if \( |\sigma(m+i) + \rho| - |\sigma m + \rho| = |\sigma i| \), which is equivalent to the inequality \( \{\sigma m + \rho\} < \{\sigma (m+i) + \rho\} \).

We have proved
Claim 6 For $i \in M, \gamma_i$ is monotonic. For $i \in S$, we have $\gamma_{m \cdot m + i} = < if and only if $|\gamma_{m \cdot m + i}| < q_i + 1$, that is, if and only if $\{\sigma(m + i) + \rho\} < \{\sigma m + \rho\}$.

Taken together, the claims above mean that a permutation $\alpha$ of maximal pattern complexity $p^\alpha_m(n) = n$, such that the upper raw $\gamma_1$ is a Sturmian word, is uniquely determined by

- the Sturmian word $\gamma_1$, and in particular its parameters $\sigma$ and $\rho$;
- the partition of $\mathbb{N}$ into $S$ and $M$;
- the type of (all the words) $\gamma_i$ with $i \in M$: in what follows we assume without loss of generality that they all are decreasing.

However, it is not difficult to see that given a word $\gamma_1$, we cannot choose the partition $\mathbb{N} = S \cup M$ arbitrarily. Let us consider restrictions which we must put on it.

Suppose first that $i, j \in M$. It means that for all large $k$ we have $\alpha_k > \alpha_{k+i} > \alpha_{k+i+j}$. Since a linear order is always transitive, this means that $\alpha_k > \alpha_{k+i+j}$ and thus $i + j \in M$, giving us the following condition:

$$i, j \in M \implies i + j \in M. \quad (3)$$

To state other conditions, let us return to the number $q_i = \lfloor i\sigma \rfloor$. Recall that the number of symbols $<$ in the factors of $\gamma_1$ of length $i$ is either $q_i$ or $q_i + 1$. Since

$$q_{i+j} + \{(i+j)\sigma\} = (i+j)\sigma = i\sigma + j\sigma = q_i + \{i\sigma\} + q_j + \{j\sigma\},$$

we have $q_{i+j} - q_i - q_j = \{i\sigma\} + \{j\sigma\} - \{(i+j)\sigma\}$. Hence, $q_{i+j} - q_i - q_j > 0$ if and only if $\{i\sigma\} + \{j\sigma\} - \{(i+j)\sigma\} > 0$. The former is equivalent to $q_{i+j} = q_i + q_j + 1$ and the latter is equivalent to $\{i\sigma\} + \{j\sigma\} > 1$. Thus, we have

$$q_{i+j} = q_i + q_j + 1 \text{ if and only if } \{i\sigma\} + \{j\sigma\} > 1.$$ 

Assume that $i, j \in S$ and $\{i\sigma\} + \{j\sigma\} > 1$. There exist infinitely many $k$’s such that $|\gamma_{[k .. k+i+j]}| < = q_i + j + 1$ since $\gamma_1$ is a Sturmian word. On the other hand, we have $q_{i+j} = q_i + q_j + 1$ since $\{i\sigma\} + \{j\sigma\} > 1$. It follows that $|\gamma_{[k .. k+i]}| < = q_i + 1$ and $|\gamma_{[k+i .. k+i+j]}| < = q_j + 1$ since $|\gamma_{[k .. k+i]}| < = q_i + 1$, $|\gamma_{[k+i .. k+i+j]}| < = q_j + 1$ and $|\gamma_{[k .. k+i+j]}| < = q_i + j + 1 = q_i + 1 + q_j + 1$. Since $i, j \in S$, this implies that $\alpha_k < \alpha_{k+i} < \alpha_{k+i+j}$ and $i + j$ cannot be in $M$. Hence, $i + j \in S$.

$$i, j \in S \text{ and } \{i\sigma\} + \{j\sigma\} > 1 \implies i + j \in S. \quad (4)$$

Now consider the situation when $i + j \in S$ and a word of length $i + j$ in $\gamma_1$ with $q_{i+j} + 1$ occurrences of $<$ ends by a suffix of length $j$ with only $q_j$ occurrences
of $<$. This is possible if and only if $q_{i+j} = q_i + q_j$, that is, $\{i\sigma\} + \{j\sigma\} = \{(i+j)\sigma\} < 1$. There exists $k$ such that $|\gamma_{[k..k+i+j]}| < q_{i+j} + 1$. Then we have $\alpha_k < \alpha_{k+i+j}$ since $i + j \in S$ and $\alpha_{k+i} > \alpha_{k+i+j}$ since $|\gamma_{[k+i..k+i+j]}| = q_j$; here it does not matter if $j \in S$ or $j \in M$. Thus, by transitivity $\alpha_k < \alpha_{k+i}$ holds, which means in particular that $i \in S$. We have proved that

$$i + j \in S \text{ and } \{i\sigma\} + \{j\sigma\} < 1 \implies i \in S. \quad (5)$$

Note that $i$ and $j$ in this condition are treated symmetrically, so in fact, $j$ also belongs to $S$.

Now using the conditions 3–5 we can prove

**Claim 7** For each $s \in S$ and $m \in M$ we have

$$\frac{1 - \{m\sigma\}}{m} < \frac{1 - \{s\sigma\}}{s}.$$

**Proof.** Suppose to the contrary that

$$\frac{1 - \{m\sigma\}}{m} \geq \frac{1 - \{s\sigma\}}{s} \quad (6)$$

for some $s \in S$ and $m \in M$, and choose a minimal counter-example, so that the sum of $s$ and $m$ is the least possible.

Suppose first that $s > m$. Then $s - m \in S$ due to (3). Moreover, since $m \in M$, we do not get into Condition (5), and thus $\{m\sigma\} + \{(s - m)\sigma\} > 1$, that is, $\{m\sigma\} + \{(s - m)\sigma\} = \{s\sigma\} + 1$. It can be checked directly using (6) that

$$\frac{1 - \{(s-m)\sigma\}}{s-m} = \frac{\{m\sigma\} - \{s\sigma\}}{s-m} \leq \frac{1 - \{m\sigma\}}{m},$$

so that $s - m \in S$ and $m \in M$ form a counter-example less than the initial one, contradicting to its minimality.

Now suppose that $m > s$. Then (6) immediately implies that $\{m\sigma\} < \{s\sigma\}$ (equality being impossible since $\sigma$ is irrational), and thus $\{s\sigma\} + \{(m-s)\sigma\} = \{m\sigma\} + 1$ (not $\{m\sigma\}$). Due to (4), we have $m - s \in M$ since otherwise we would have $m \in S$. Now we again can see that $s$ and $m - s$ give a counter-example less than the initial one since

$$\frac{1 - \{(m-s)\sigma\}}{m-s} = \frac{\{s\sigma\} - \{m\sigma\}}{m-s} \geq \frac{1 - \{s\sigma\}}{s}$$

due to (6).

Now note that $\frac{1 - \{i\sigma\}}{i} \to 0$ with $i \to \infty$. Note also that the set $S$ is not empty since $1 \in S$. So, Claim 7 means that either $S = \mathbb{N}$, or there exists some $d \in (0,1)$ such that $i \in S$ if and only if $\frac{1 - \{i\sigma\}}{i} > d$, and $i \in M$ if and only if $\frac{1 - \{i\sigma\}}{i} < d$. This parameter $d$ together with the word $\gamma_1$ and the fact
that the monotonic strings of the table $\gamma$ are decreasing, completely defines the permutation $\alpha$. Note that the situation when $S = \mathbb{N}$ just corresponds to $d = 0$.

It remains to check that $\alpha = \alpha(\gamma_1, 1 - \sigma - d, \sigma + d)$. Here, we just treat each symbol $< \alpha_1$ as 0 and $> \alpha_1$ as 1 to use the definition of a Sturmian permutation from Section 3. Indeed, $\gamma_{k,k+i} = \alpha$ if and only if $|\gamma_{[k..k+i]}|_\alpha = q_i$, which implies $\alpha(\gamma_1, 1 - \sigma - d, \sigma + d)_k > \alpha(\gamma_1, 1 - \sigma - d, \sigma + d)_{k+i}$ since $q_i(1 - \sigma - d) - (i - q_i)(\sigma + d) < 0$. On the other hand, $\gamma_{k,k+i} = < \alpha \alpha_1$ if and only if $|\gamma_{[k..k+i]}|_\alpha = q_i + 1$ and $i \in S$ which implies $\alpha(\gamma_1, 1 - \sigma - d, \sigma + d)_k < \alpha(\gamma_1, 1 - \sigma - d, \sigma + d)_{k+i}$ since we have $(q_i + 1)(1 - \sigma - d) - (i - q_i - 1)(\sigma + d) > 0$ using $\frac{1}{i - (\sigma + d)} > d$. Thus, $\alpha = \alpha(\gamma_1, 1 - \sigma - d, \sigma + d)$.

We have proved that if $\alpha$ is a permutation with maximal pattern complexity equal to $n$, and the first string $\gamma_1$ of its table $\gamma$ is a Sturmian word, then $\alpha$ is a Sturmian permutation. It remains to consider the case when $\gamma_1$ is not Sturmian and thus is ultimately periodic.

7 Proof of uniqueness: periodic case

We are going to prove that if $\alpha$ is not ultimately periodic and $\gamma_1$ is ultimately periodic, then $p^\alpha(\gamma_1) > n$ for some $n > 1$.

For $n \in \mathbb{N}$, let $\tau^n \alpha$ be the $\mathbb{N}$-permutation such that $(\tau^n \alpha)_i < (\tau^n \alpha)_j$ if and only if $\alpha_{i+n} < \alpha_{j+n}$ for any $i, j \in \mathbb{N}$ with $i \neq j$. Thus, $\tau$ is the shift on the set of $\mathbb{N}$-permutations. We use the notation $\tau$ also for the shift on the set of words on $\mathbb{N}$. Since the above statement for $\alpha$ follows from that for $\tau^n \alpha$, we'll prove it for $\tau^n \alpha$ such that $\tau^n \gamma_1$ is periodic. Denoting this $\tau^n \alpha$ by $\alpha$, we may assume that $\gamma_1$ is periodic. In the same way, every ultimately periodic sequence defined with respect to $\alpha$ can be considered as periodic.

It is convenient to consider arithmetic subpermutations of a permutation $\alpha$. Let us fix a difference $i$ and for each $j = 0, \ldots, i - 1$ denote by $S^j_i$ the subset $\{ki + j | k \in \mathbb{N}\}$ of $\mathbb{N}$, called an arithmetic progression of difference $i$. Now denote by $\alpha^j_i$ the restriction of $\alpha$ to the set $S^j_i$: $\alpha^j_i = \alpha_{S^j_i}$, and denote by $\alpha^{j,k}_i$ the union of $\alpha^j_i$ and $\alpha^k_i$, that is, the restriction $\alpha_{S^j_i \cup S^k_i}$ of $\alpha$ on $S^j_i \cup S^k_i$. Note that $\alpha$ is not obliged to be an $\mathbb{N}$-permutation: for all the definitions above, it is sufficient for it to be defined on all values of respective arithmetic progressions.

Let us say that subpermutations $\alpha^j_i$ and $\alpha^k_i$ are adjusted if $\alpha^{j,k}_i$ is $t_{j,k}$-periodic for some $t_{j,k} > 0$. (Recall that periodicity was defined for permutations on an arbitrary set, not only for $\mathbb{N}$-permutations.) Clearly, we can always choose $t_{j,k}$ divided by $i$, that is, $t_{j,k} = it_{j,k}$ for some $t_{j,k}$. It is also clear that to be adjusted
with some other subpermutation, a subpermutation must be periodic by itself.

The following lemma has been proved in [4] in slightly different notation, so we repeat its proof here.

**Lemma 2** A permutation defined on a union of infinite arithmetic progressions of difference \( i \) is periodic if and only if for all \( j, k \in \{0, \ldots, i-1\} \) the subpermutations \( \alpha_{ij}^i \) and \( \alpha_{ik}^i \) (when well-defined) are adjusted.

**Proof.** The “only if” part of the proof is obvious since \( \alpha_{ij}^i \) is just restrictions of \( \alpha_i \): if \( \alpha_i \) is \( t \)-periodic, then so do they.

To prove the “if” part, we just directly check by the definition that \( \alpha_i \) is \( t \)-periodic, where \( t = \text{lcm} j, k \), and the lcm (i.e. least common multiple) is taken over all pairs of allowed \( j \) and \( k \). Indeed, if we take \( j', k' \in S_{ij}^i \) and \( k' \in S_{ik}^i \) for any \( j \) and \( k \), we immediately see that
\[
\gamma_{ij'} = \gamma_{j'+t'_{jk},k'+t'_{jk}} = \gamma_{j'+2t'_{jk},k'+2t'_{jk}} = \cdots = \gamma_{j'+t,k'}/t \text{ which means the } t\text{-periodicity.}
\]

In particular, this lemma holds for all \( N \)-permutations.

Note also that each \( i \)-periodic permutation consists of \( i \) monotonic subpermutations since we have \( \gamma_{ij'} = \gamma_{i+j,2i+j} = \cdots = \gamma_{ni+j,(n+1)i+j} \) for all \( n \).

**Claim 8** If the maximal pattern complexity of an infinite permutation \( \alpha \) satisfies \( p_{\alpha}(n) = n \), and the sequence \( \gamma_1 \) is periodic, then for each \( i \) and \( j \) the sequence \( \gamma_i^j \) is periodic.

**Proof.** Clearly, if \( \gamma_1 \) is 1-periodic, then \( \alpha \) is monotonic, and there is nothing to be proved. So, we may assume that the minimal period \( p \) of \( \gamma_1 \) is greater than 1, and thus both symbols \(< \) and \( > \) occur in \( \gamma_1 \): moreover, there exist some \( k \) and \( l \) such that \( \gamma_{pm+k,pm+k+1} = < \) and \( \gamma_{pm+l,pm+l+1} = > \) for all \( n \in N \).

Due to Lemma [1] the sequence \( \gamma_i^j \) is either periodic or Sturmian. Suppose it is Sturmian. Then, \( \alpha_i^j \) is not periodic, and thus its maximal pattern complexity is at least \( n \). The patterns well-defined on \( S_i^j \) are exactly those of the form \( T = (0, im_1, \ldots, im_n) \) for non-negative \( m_1, \ldots, m_n \). Since the maximal pattern complexity of \( \alpha_i^j \) cannot be greater than that of \( \alpha_i \), it is equal to \( n \). But applying patterns well-defined on \( \alpha_i^j \) to \( \alpha \) as a whole must not increase the complexity, which immediately means that the language of factors of any subpermutation \( \alpha_{ij}^{i'} \) of the same difference \( i \) is equal to that of \( \alpha_i^{i'} \). In particular, for all \( j' \), the sequences \( \gamma_i^{j'} \) are Sturmian.

Now consider the pattern \( T = (0, 1, i) \). By the definition of \( k \), for any large \( n \) we have that the relation between the first two entries of \( \alpha_{T+nip+k} \) is \( < \). At the same time, the relation between \( \alpha_{pm+k} \) and \( \alpha_{pm+k+i} \) takes both values with different \( n \) since positions \( k, pi+k, 2pi+k, \ldots \) form an arithmetic progression which is
Claim 9 If the maximal pattern complexity of an infinite permutation $\alpha$ is $p_\alpha^*(n) = n$, and the sequence $\gamma_1$ is $p$-periodic, then there exists some $i'$ such that the subpermutation $\alpha_{p}^{i'}$ is monotonic.

Proof. First of all, we have $p > 1$ since otherwise $\alpha$ is monotonic and thus periodic, and its maximal pattern complexity is ultimately constant. Thus, $\gamma_1$ contains both symbols $<$ and $>$: say, the symbols $\gamma_{np+i_1, np+i_1+1}$ for all $n \in \mathbb{N}$ and some $i_1 \in \{0, \ldots, p-1\}$ are equal to $<$, and the symbols $\gamma_{np+i_2}$ for all $n \in \mathbb{N}$ and some $i_2 \in \{0, \ldots, p-1\}$ are equal to $>$. Consider the window $T = (0, 1, p)$. We must have $p_\alpha(T) \leq 3$. Since we always have $\alpha_{T+n_1p+i_1} \neq \alpha_{T+n_2p+i_2}$ for any $n_1, n_2 \in \mathbb{N}$, one of the sets $\{\alpha_{T+np+i_1} | n > N\}$ and $\{\alpha_{T+np+i_2} | n > N\}$ (and thus in particular one of the sets $\{\gamma_{np+i_1,n(p+1)+i_1} | n > N\}$ and $\{\gamma_{np+i_2,n(p+1)+i_2} | n > N\}$) is of cardinality one. So, either $\alpha_{p}^{i_1}$ or $\alpha_{p}^{i_2}$, denoted below by $\alpha_{p}^{i'}$, is monotonic. □

Claim 10 If the maximal pattern complexity of an infinite permutation $\alpha$ is $p_\alpha^*(n) = n$, and the sequence $\gamma_1$ is periodic, then there exists some $t$ such that all the subpermutations $\alpha_{p}^i$, $i = 0, \ldots, t - 1$, are monotonic.

Proof. Let $p$ be the minimal period of $\gamma_1$. Consider all the subsequences $\alpha_{p}^j$ with $j = 0, \ldots, p - 1$.

Suppose first that some $\alpha_{p}^j$ is $q(j)$-periodic (as a $S_p^j$-permutation). Then all its arithmetic subpermutations of difference $q(j)$ are monotonic.

Now consider some of $\alpha_{p}^j$ which is not periodic. However, the word of relations $\gamma_{p}^j$ has to be periodic due to the Claim. Let us denote its minimal period by $pq$ ($p$ appears here since we consider $\gamma_{p}^j$ as a word defined on $S_p^j$ not on $\mathbb{N}$). Since $\alpha_{p}^j$ is not monotonic, we have $q \geq 2$, and thus $\gamma_{p}^j$ contains both symbols $<$ and $>$ in the period. Clearly, $p_{\alpha_{p}^j}(T_{q+1}) \geq p_{\gamma_{p}^j}(T_q) = q$, where $T_n = (0, p, \ldots, (n - 1)p)$ for all $n$ and $\alpha_{p}^j$.

Suppose first that $p_{\alpha_{p}^j}(T_{q+1}) > q$. Note that among $T_{q+1}$-factors of $\alpha_{p}^j$, there are no monotonic ones since $\gamma_{p}^j$ is $pq$-periodic and contains both symbols $<$ and $>$ in the period. But $\alpha_{T_{q+1}+i'}$ is monotonic due to the previous claim, and thus $p_\alpha^*(q + 1) \geq p_\alpha(T_{q+1}) > q + 1$. A contradiction to the minimality of $p_\alpha^*$. 

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Now suppose that $p_{\alpha_p^j}(T_{q+1}) = q$. This means that each $T_{q+1}$-factor of $\alpha_{T_{q+1}+np+j}$ of $\alpha_p^j$ for $n \in \mathbb{N}$, and in particular the relation between its first entry $\alpha_{np+j}$ and last entry $\alpha_{(n+q)p+j}$, is determined by the underlying $T_q$-factor of $\gamma_p^j$ and thus just by the residue of $n$ modulo $q$. So, each of the subsequences $\alpha_{np+j}$, where $n_0 = 0, \ldots, q - 1$, is monotonic. Denote $pq = q(j)$.

Now $q(j)$ is defined for all $j = 0, \ldots, p - 1$, and all arithmetic subpermutations of $\alpha_p^j$ of difference $q(j)$ are monotonic. Defining $t = \text{lcm}_j q(j)$, we see that all the arithmetic subpermutations of $\alpha$ of difference $t$ are also monotonic, which was to be proved. \hfill $\square$

So, let $\alpha$ be an infinite permutation such that $p^*_\alpha(n) = n$, and the sequence $\gamma_1$ be periodic. Due to the previous Claim and Lemma 2, we see that there exist two subpermutations $\alpha_t^0$ and $\alpha_t^r$ which are monotonic and not adjusted. Without loss of generality we may assume that $j = 0$ and both subpermutations are increasing; indeed, if one of them is increasing and the other is decreasing, $\alpha_{0, t}$ is $t$-periodic (starting from the point when the subpermutations intersect, if it exists). If they both are decreasing, we just may consider the situation symmetrically.

It is also convenient to denote $\alpha_{0, t}$ and $\alpha_{t}^r$ by ($\mathbb{N}$-permutations) $\chi$ and $\psi$ so that $\chi_i = \alpha_{it}$ and $\psi_i = \alpha_{it+r}$ for all $i \geq 0$. Both permutations are monotonically increasing: $\psi_i < \psi_{i+1}$ and $\chi_i < \chi_{i+1}$ for all $i$.

Note that the fact that $\alpha_{0, t}^r$ is not periodic means in particular that for each $i$ there exists some $v(i)$ such that $\psi_i < \chi_{v(i)}$, and $v(i)$ is the minimal number with this property.

In particular, if $v(i) > 0$, we have $\chi_{v(i)-1} < \psi_i$.

Symmetrically, for each $j$ there exists some $w(j)$ such that $\chi_j < \psi_{w(j)}$.

Consider first the situation when the modulo $|i - v(i)|$ is bounded: for all $i$, we have $|i - v(i)| < c$.

**Lemma 3** Permutations $\alpha$ of $p^*_\alpha(n) = n$ having periodic sequence $\gamma_1$, not adjusted monotonic subpermutations $\alpha_t^0$ and $\alpha_t^r$, and $|i - v(i)| < c$ for all $i$, do not exist.

**Proof.** Suppose such a permutation exists. It follows from the property $|i - v(i)| < c$ for all $i$ that for all $i, n \geq 0$ we have $\psi_i < \chi_{i+c+n}$ and $\chi_{i-c-n} < \psi_i$ (of course, the latter inequality is valid only when $i - c - n \geq 0$).

So, for all $n$ we see that all the entries of sequences $\gamma_s$ with $s > (c + n + 1)t$ which describe the relations between elements of $\alpha_{0, t}^r$ are equal to $<$. 

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At the same time, we know from Claim 8 that all sequences $\gamma_s$, and thus their restrictions to $S_t^0 \cup S_t^r$, are periodic. Let us denote the period of $\gamma_s$ by $q_s$; then the restriction of $\gamma_s$ to $S_t^0 \cup S_t^r$ is at most $q_s$-periodic (due to the definition of periodicity involving arbitrary distance between compared periods). Denote by $q$ the least common multiple of all numbers $q_s$ with $s \leq (c + n + 1)t$. Then we can check directly that $\alpha^{0,k}_t$ is also $q$-periodic, and thus $\alpha^{0}_t$ and $\alpha^{r}_t$ are adjusted. A contradiction.

It remains to consider the case when $|i - v(i)|$ is not bounded with $i$: due to the symmetry between $\chi$ and $\psi$, it is sufficient to consider the case when for each $c$ there is some $i$ such that $i - v(i) > c$.

**Lemma 4** If increasing subpermutations $\alpha^{0}_t = \chi$ and $\alpha^{r}_t = \psi$ are not adjusted, and the difference $i - v(i)$ is not bounded with $i$, then $p_{\alpha}^*(4) \geq 5$.

**Proof.** Let us point out a pattern $T$ of length 4 such that $p_{\alpha}(T) \geq 5$. To do it, we need to prove two auxiliary statements.

**Claim 11** For each $i, j$ there exists some $k$ such that $\psi_{j+k} < \chi_{i+k}$.

**Proof.** Consider some $n$ such that $n - v(n) > j$, so that $\psi_n < \chi_{v(n)}$ and since both subpermutations are increasing, $\psi_{j+v(n)} < \psi_n < \chi_{v(n)} \leq \chi_{i+v(n)}$. So, we may take $k = v(n)$.

**Claim 12** For each $l$ such that $\chi_1 < \psi_l$ there exist some $k_1$ and $k_2$ such that

\[
\chi_{k_1} < \psi_{l+k_1} < \chi_{k_1+1}
\]

\[
\psi_{l+k_2} < \chi_{k_2} < \chi_{k_2+1}.
\]

**Proof.** The number $k_1$ can be found as the minimal number $k$ such that $\psi_{l+k} < \chi_{1+k}$: it exists due to the previous claims, and the fact that it is minimal gives us $\chi_{k_1} = \chi_{1+(k_1-1)} < \psi_{l+k_1-1} < \psi_{l+k_1}$. The number $k_2$ can be found directly from the previous claim as a number such that $\psi_{l+k_2} < \chi_{k_2}$.

**Proof of the Lemma.** Let us take an arbitrary $l$ such that $\chi_1 < \psi_l$, and choose $k_1$ and $k_2$ as described in Claim 12. Now let us choose some $m > l$ such that $\chi_{1+k_1} < \psi_{m+k_1}$ and $\chi_{1+k_2} < \psi_{m+k_2}$ (such $m$ exists since we can take just the greater of the two numbers satisfying these inequations separately).

Let us apply Claim 12 to $m$ instead of $l$ and define $k_3$ and $k_4$ so that

\[
\chi_{k_3} < \psi_{m+k_3} < \chi_{k_3+1},
\]

\[
\psi_{m+k_4} < \chi_{k_4} < \chi_{k_4+1}.
\]
Also, to unify the notation, suppose that $k_0 = 0$. Now consider the 4-window $T = (0, t, lt + r, mt + r)$ and $T$-permutations $\alpha_{T+tk_i}$, where $i = 0, 1, \ldots, 4$. By the definition, for each $i$ the permutation $\alpha_{T+tk_i}$ involves as entries exactly the elements $\chi_{k_i}, \chi_{k_i+1}, \psi_{lt+k_i}, \psi_{mt+k_i}$. Now it remains to record that all the five permutations $\alpha_{T+tk_i}$ for $i = 0, 1, \ldots, 4$ are different. Indeed, consider the 4-tuples $R_i = (\gamma_{k_it}, (k_i+1)t+r, \gamma_{k_i+1}(t,(k_i+m)t+r), \gamma_{k_i+1}(t,(k_i+m)t+r))$ and see that by the construction, $R_0 = (<, <, <, <>$, $R_1 = (<, >, <, <>$, $R_2 = (> , >, <, >)$, $R_3 = (* , >, <, >)$, and $R_4 = (> , >, > , >)$ for some value of *. But each $R_i$ contains just a part of information determining $\alpha_{T+tk_i}$. Thus $p^*_\alpha(4) \geq p_\alpha(T) \geq 5$.

We excluded all possibilities when the maximal pattern complexity of an infinite permutation with the periodic string $\gamma_1$ could be equal to $p^*_\alpha(n) \equiv n$. So, the summarizing result of the paper is the following

**Theorem 6** If an infinite permutation $\alpha$ is not periodic, then $p^*_\alpha(n) \geq n$ for any $n$. Moreover, $p^*_\alpha(n) \equiv n$ if and only if $\alpha$ is a Sturmian permutation.

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