Extreme coefficients of Jones polynomials and graph theory

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October 27, 2018

Abstract

We find families of prime knot diagrams with arbitrary extreme coefficients in their Jones polynomials. Some graph theory is presented in connection with this problem, generalizing ideas by Yongju Bae and Morton [3] and giving a positive answer to a question in their paper.

1 Introduction.

Let $L$ be an oriented link, and $V_L(t)$ its Jones polynomial with normalization one. We are interested in exhibit examples of links with arbitrary extreme coefficients in their Jones polynomials. Consider an unoriented diagram $D$ of $L$. We denote by $\langle D \rangle$ its Kauffman bracket with normalization $\langle \emptyset \rangle = 1$ (see [1]). Since $V_L(t) = (-A)^{-3w(D)} \langle D \rangle$ after the substitution $A = t^{-1/4}$, we have that $span(V_L(t)) = span(\langle D \rangle)/4$ and the coefficients of both polynomials are the same, maybe up to sign.

Hence we will work with the Kauffman bracket of unoriented diagrams. We recall the definition of this polynomial using the states sum:

$$\langle D \rangle = \sum_s \langle D, s \rangle$$

where the sum is taken over all states $s$ of $D$ and $\langle D, s \rangle = A^{a(s)-b(s)}(-A^{-2} - A^2)^{|s|-1}$.

A state $s$ of $D$ is a labelling of each crossing of $D$ by either an A-chord or a B-chord. We write $a(s)$ (resp. $b(s)$) for the number of A-chords (resp. B-chords) of the state $s$, and $|s|$ for the number of components of the diagram $sD$, which is $D$ after the $s$-smoothing of $D$. Precisely $sD$ is obtained smoothing every crossing in $D$ according to the type of chord associated to the crossing by the state, as shown in Figure 1. We will draw a small chord with the letters $A$ or $B$ to remember which was the state. In this way we can reconstruct the diagram $D$ from $sD$ and the chords, by just reversing the smoothings shown in Figure 1. An example is shown in Figure 2. For a state $s$ we denote by $\max(s)$ (resp. $\min(s)$)
the highest (resp. the lowest) degree of $\langle D, s \rangle$. The extreme states $s_A$ and $s_B$ are defined by the equalities $a(s_A) = c(D)$ and $b(s_B) = c(D)$ respectively, where $c(D)$ is the number of crossings of the diagram $D$. Write $m = \min(s_B)$ and $M = \max(s_A)$. Clearly $m = -c(D) - 2|s_B| + 2$ and $M = c(D) + 2|s_A| - 2$. These numbers $m$ and $M$ will be called the extreme states degrees of $\langle D \rangle$ and their corresponding coefficients $a_m$ and $a_M$ in $\langle D \rangle$ will be called the extreme states coefficients of $\langle D \rangle$. It turns out that (see Proposition 1)

$$\langle D \rangle = a_m A^m + a_{m+4} A^{m+4} + \cdots + a_{M-4} A^{M-4} + a_M A^M.$$

In this paper we deal with the question of finding arbitrary extreme coefficients for $\langle D \rangle$. Two different approachings will be given:

The first approaching follows Yonju Bae and Morton [3]. In their paper a connection between $a_M$ and graph theory is given. The coefficient $a_M$ appears to be, up to sign, the value $f(G_D)$ of a certain graph $G_D$, constructed from $s_A D$ and called the Lando’s graph of $sD$. In general, for any graph $G$, $f(G) = \sum_C (-1)^{|C|}$ where $C$ runs over all the independent subsets of vertices of $G$, where independent means that there is no edge joining two vertices of $C$. In [3] it arises the question of if any integer can be realized as $f(G)$ for some graph $G_D$. We have found a positive answer to this question. In fact, we will exhibit examples (in fact complete families) of graphs, and from these we will reconstruct prime knot diagrams with arbitrary extreme states coefficients $a_m$ and $a_M$. This will be done in the second section. The third section will be dedicated to explain further constructions of graphs, which in many cases give easier examples of knots with the wanted extreme states coefficients. Here “easier” means much fewer crossings.
The second approaching is treated in the fourth and final section. Here we exhibit examples of prime knot diagrams for which the extreme states coefficients \( a_m \) and \( a_M \) are zero, and the next coefficients \( a_{m+4} \) and \( a_{M-4} \) take arbitrary values. The idea is just to look at the more general circle graph defined by \( sD \), rather than the Lando’s graph \( G_D \), and use a very simple trick for counting \( a_{M-4} \) in special cases. We do not know if there is a complete nice interpretation of \( a_{M-4} \) in terms of graph theory, parallel to that one in which \( a_M \) is described in terms of \( G_D \) in the second section. A really more interesting question is the following: is it possible to get any extreme coefficient when the spread of the Jones polynomial is previously fixed?

2 Extreme states coefficients of Jones polynomials and graph theory.

We begin by recalling some very basic facts about the Kauffman bracket \( \langle D \rangle \) of an unoriented diagram \( D \).

**Proposition 1**  
(i) All degrees in \( \langle D \rangle \) are congruent modulo four.

(ii) \( \max(s) \leq M = \max(s_A) \) with equality if and only if \( |s| = |s_A| + b(s) \).

(iii) \( \min(s) \geq m = \min(s_B) \) with equality if and only if \( |s| = |s_B| + a(s) \).

(iv) The highest (resp. lowest) degree of \( \langle D \rangle \) is less (resp. great) or equal than \( M \) (resp. \( m \)).

(v) A state \( s \) contributes to \( a_M \) if and only if \( s \in \Gamma_A = \{s/|s| = |s_A| + b(s)\} \). The contribution of \( s \in \Gamma_A \) to \( a_M \) is \( (-1)^{|s_A|} (-1)^{b(s)} \).

(vi) A state \( s \) contributes to \( a_m \) if and only if \( s \in \Gamma_B = \{s/|s| = |s_B| + a(s)\} \). The contribution of \( s \in \Gamma_B \) to \( a_m \) is \( (-1)^{|s_B|} (-1)^{a(s)} \).

(vii) \( a_M = (-1)^{|s_A|} \sum_{s \in \Gamma_A} (-1)^{b(s)} \) and \( a_m = (-1)^{|s_B|} \sum_{s \in \Gamma_B} (-1)^{a(s)} \).

**Proof.** We calculate the difference

\[
\max(s_A) - \max(s) = c(D) + 2|s_A| - 2 - a(s) + b(s) - 2|s| + 2
\]

\[
= 2b(s) + 2|s_A| - 2|s|
\]

Now the key point is that if two states \( s \) and \( s' \) differ in the label of only one crossing, then either \( |s'| = |s| + 1 \) or \( |s'| = |s| - 1 \), depending on whenever the two strings that appear after \( s \)-smoothing the crossing belong or not to the same component of \( sD \). It follows that for an arbitrary state \( s \) we have that there are non-negative integers \( p \) and \( n \) such that \( b(s) = p + n \) and \( |s| = |s_A| + p - n \). Then \( b(s) + |s_A| - |s| = p + n + |s_A| - |s_A| - p + n = 2n \) is an even number great or equal than zero. The completion of the proof is left to the reader. \( \blacksquare \)
Remark In order to find $sD$, one can start with $s_AD$, choose an order in the set of the $b(s)$ crossings labelled with a B-chord in the state $s$ and perform the opposite smoothing in these crossings following this order. In this way, one associate $+1$ (resp. $-1$) to each one of the $b(s)$ crossings if after the $s_B$-smoothing of this crossing we get one more (resp. fewer) component. By definition $p$ (resp. $n$) is the number of associated $+1$ (resp. $-1$). This non-negative integers does not depend on the chosen order of the $b(s)$ crossings, although the association of $+1$ and $-1$ to each one of the $b(s)$ crossings does.

Now we recall the connection between $a_M$ and graph theory given in [3]. In order to obtain the Lando’s graph of $D$ start with $s_AD$ and delete the A-chords joining two different components. In this way we obtain a bipartite circle graph (BCG). Now define the Lando’s graph of $D$ as the graph that appears taking a vertex for every A-chord of the BCG, and joining two vertices with an edge if and only if the endpoints of the corresponding A-chords in the BCG alternate in the same component of the BCG. An example of $s_AD$, the corresponding BCG and the Lando’s graph is shown in Figure 3. A subset $C$ of vertices of a

\[
\text{Figure 3: } s_AD, \text{ the BCG and the Lando’s graph } G_D.
\]

graph $G$ is said to be independent if there is no edge in $G$ joining two vertices of $C$. We define $f(G) = \sum_C (-1)^{|C|}$ where $C$ runs over the independent sets of vertices of $G$.

Then

\[
a_M = (-1)^{|s_A|} \sum_{s \in \Gamma_A} (-1)^{b(s)} = (-1)^{|s_A|-1} \sum_C (-1)^{|C|} = (-1)^{|s_A|-1} f(G_D).
\]

The first equality is given by Proposition [1] and the third equality is just the definition of $f(G_D)$. In order to check the second equality, we think of a state $s$ as the set of $b(s)$ A-chords of $s_AD$ which correspond to the $b(s)$ B-chords of $s$. Then we have that $s \in \Gamma_A$ if and only if the two following conditions occur:

(1) The endpoints of every A-chord lies in the same component.
(2) The endpoints of two A-chords lying in the same component do not alternate.

The example in Figure 4 is exhibited in [3], where the corresponding Lando’s graph \( G_D \) has \( f(G_D) = 3 \). The question formulated in [3] is if any integer \( n \)

![Figure 4: A Lando’s graph \( G_D \) with \( f(G_D) = 3 \) and the corresponding BCG and link diagram.](image)

can be realized as \( f(G_D) \) for a graph \( G_D \) arising from a diagram \( D \). Since these graphs arise from BCG, we will call them “graphs convertible in BCG”.

**Some graph theory.**

Since the graphs \( G_D \) must be always convertible in BCG, the examples shown in Figure 5 are not allowed. On the other hand the graphs in consideration are not necessarily planar graphs. Figure 6 exhibits such an example.

![Figure 5: Examples of graphs non-convertible in BCG.](image)

![Figure 6: Non-planar Lando’s graph.](image)

Calculation of \( f(G) \) can be simplified using some readily established properties, as explained in [3]. Here is a very brief description of these properties:

- Law of recursion. If \( G \) is a graph and \( v \) is a vertex of \( G \), we will denote by \( G - v \) the graph obtained from \( G \) by deleting the vertex \( v \) and its incident edges.
Let \( \{v_1, \ldots, v_k\} \) be the set of the neighbour vertices of \( v \) in \( G \) (by definition these are the vertices of \( G \) joined to \( v \) by an edge). We will denote by \( G - Nv \) the graph \( \ldots ((G - v) - v_1) \ldots - v_k \). Then the law of recursion says that \( f(G) = f(G - v) - f(G - Nv) \).

- Law of multiplication. If a graph \( G \) is the disjoint union of two graphs \( G_1 \) and \( G_2 \), then \( f(G) = f(G_1)(G_2) \).

- Law of duplication. Suppose that \( v \) and \( w \) are two non-neighbour vertices of a graph \( G \), and the set of neighbour vertices of \( v \) is included in the set of neighbour vertices of \( w \). Then we say that \( G \) is a duplication of \( G - w \) and we have \( f(G) = f(G - w) \).

We consider examples that will be used later. Let \( L_n \) be the graph with \( n \) vertices shown in Figure 7. Clearly \( f(L_2) = -1 \). By using duplication in the third vertex and then multiplication we get the formula \( f(L_n) = -f(L_{n-3}) \).

Let \( C_n \) be the polygon with \( n \) vertices, \( n \geq 3 \). By recursion we obtain \( f(C_n) = f(L_{n-1}) - f(L_{n-3}) \). In particular, for the hexagon \( H = C_6 \) we have \( f(H) = 2 \).

We now explain how to find different families \( \{G_r\}_{r \in \mathbb{Z}} \) of graphs with \( f(G_r) = r \) for any integer \( r \). From now on, we will write \( G^v \) to mean the pair \((G, v)\), where \( G \) is a graph and \( v \) is a particular vertex of \( G \).

**Definition** Let \( G \) be a planar graph convertible in BCG and let \( v \) be a vertex of \( G \). We say that \( G^v \) a brick of type \((n, k)\) if \( f(G) = n \) and \( f(G - v) = k \).

Our main example will be the hexagon \( H \) with any arbitrary vertex chosen, which is a brick of type \((2, 1)\). It is a graph convertible in BCG, as we show in Figure 8. Now we describe the basic construction that will be used for giving our examples: Let \( G_1 \) and \( G_2 \) be two graphs and \( v_1 \) and \( v_2 \) two vertices of \( G_1 \) and \( G_2 \) respectively. Then \( G_1^{v_1} * G_2^{v_2} \) will denote the new graph obtained from the disjoint union of \( G_1 \) and \( G_2 \) by joining \( v_1 \) and \( v_2 \) with an extra edge (see Figure 9). Note that \( G_1^{v_1} * G_2^{v_2} \) is convertible in BCG if both \( G_1 \) and \( G_2 \) are.
Lemma 1 Let $G$ be a graph and $v$ a vertex of $G$. If $G^v$ is a brick of type $(n, k)$, then $(G', v')$ is a brick of type $(n + k, k)$, where $G' = G^w * H^w$ and $v'$ is a vertex of $H$ adjacent to $w$ (see Figure 10).

Proof. We have to show that $f(G') = n + k$ and $f(G' - v') = k$.

We have $f(G' - v') = (f(L_2))^2 f(G - v) = (-1)^2 k = k$ where we have used duplication, first in the vertex $v_1$ and then in the vertex $v$, multiplication and the fact that $f(L_2) = -1$ (see Figure 11).

Figure 11: $G' - v'$.

On the other hand

\[ f(G') = f(G' - v') - f(G' - Nv') \quad (\text{recursion}) \]
\[ = k - f(L_3) f(G') \quad (\text{multiplication}) \]
\[ = k - (-1)n \quad (f(L_3) = -1) \]
\[ = n + k. \]
Finally, $G'$ is convertible in BCG since $G$ and $H$ are. 

**Lemma 2** Let $G$ be a graph. Then $f(G^-) = -f(G)$ where $G^- = G^v + L^v_3$, $v$ being any vertex of $G$ and $w$ being an extreme vertex of $L_3$. Moreover, $G^-$ is convertible in BCG if $G$ is (see Figure 12).

![Figure 12: Graph $G^-$.](image)

**Proof.** We have

$$f(G^-) = f(G - w) \quad \text{(duplication)}$$

$$= f(L_2) f(G) \quad \text{(multiplication)}$$

$$= -f(G) \quad (f(L_2) = -1).$$

Finally, $G^-$ is convertible in BCG since $G$ and $L_3$ are. 

**Theorem 3** For any integer $n$ there is a planar graph $G_{n-1}$ convertible in bipartite circle graph such that $f(G_{n-1}) = n$.

**Proof.** Consider the brick $H^w$ of type $(2,1)$ and apply $r$ times Lemma 1 to get a planar graph $G_{r+1}$ convertible in BCG such that $f(G_{r+1}) = r + 2$ (see Figure 13).

On the other hand $L_4$, $L_1$ and $L_2$ are planar graphs convertible in BCG with $f(L_4) = 1$, $f(L_1) = 0$ and $f(L_2) = -1$ respectively.

Finally, for all integer $r \geq 1$ we have that $G^r_{r+1}$ is a planar graph convertible in BCG with $f(G^r_{r+1}) = -(r + 2)$ according to Lemma 2 (see Figure 14). 

We will now construct a prime knot diagram with arbitrary extreme states coefficients, starting with the graphs $G_r$. This process is illustrated in figures 15, 16, 17 and 18. We first reconstruct the associated bipartite circle graph (Figure 15). Changing every A-chord $\overline{1}$ by a crossing $\overline{1}$ we see that this BCG is $s_A D'_r$ where the (3–components) diagram $D'_r$ is shown in Figure 16. Figure 17 shows $s_B D'_r$, proving that $D'_r$ is quite far from being a adequate diagram (the coefficient $a_m$ of $\langle D'_r \rangle$ is not $\pm 1$ in general since there are other states in $\Gamma_B$ apart from $s_B$). Because of this we modify $D'_r$ to produce the diagram $D_r$ in Figure 18.
Theorem 4  \( \langle D_r \rangle = A^m + \cdots + (r + 1)A^M \) where \( m = -24r - 4 \) and \( M = 12r \). In particular \( \text{span}(\langle D_r \rangle) = 4(9r + 1) \). Moreover, \( D_r \) is a prime knot diagram with \( 12r \) crossings.

Proof. First we fix our attention in \( D_r \) after \( s_A \)-smoothing. Note that \( \left| s_A D_r \right| = 1 \) hence \( M = c(D) = 12r \). On the other hand the Lando’s graph of \( D_r \) appear to be a duplication of \( G_r \), hence \( a_M = (-1)^{|s_A| - 1} f(G_r) = r + 1 \).

Now apply the \( s_B \)-smoothing to \( D_r \). We have \( \left| s_B D_r \right| = 6r + 3 \) (6\( r \) components are given by the small circles \( \odot \) and the other three components are those appearing in Figure 17), hence \( m = -12r - 2(6r + 3) + 2 = -24r - 4 \). On the other hand the Lando’s graph (respect to the \( s_B \)-smoothing) is the empty set, hence \( a_m = (-1)^{|s_B| - 1} f(\emptyset) = 1 \).

We now join in a very particular way \( D_r \) and \( \bar{D}_s \) (the mirror image of \( D_s \)) in order to control simultaneously both extreme states coefficient (we avoid the obvious solution given by the appropriate connected sum in order to get a prime diagram):

Theorem 5 Let \( D_{rs} \) be the diagram shown in Figure 19. Then \( \langle D_{rs} \rangle = (s + 1)A^m + \cdots + (r + 1)A^M \) where \( m = -24r - 12s - 6 \) and \( M = 12r + 24s + 6 \). In particular, \( \text{span}(\langle D_{rs} \rangle) = 36(r + s) + 12 \). Moreover, \( D_{rs} \) is a prime knot diagram with \( 12(r + s) + 2 \) crossings.

Proof. Note first that \( c(D_{rs}) = c(D_r) + c(\bar{D}_s) + 2 = 12r + 12s + 2 \).

Now, the very special way in which we join \( D_r \) and the mirror image \( \bar{D}_s \) of \( D_s \) gives the equalities \( |s_A| = |s_A(\bar{D}_s)| = |s_B(D_s)| = 6s + 3 \) and the fact that
the Lando’s graph is still a duplication of $G_r$. It follows that $M = 12r + 24s + 6$ and $a_M = (-1)^{|s_A|} f(G_r) = r + 1$.

Analogously, $|s_B| = |s_B(D_r)| = 6r + 3$ and the Lando’s graph is still a duplication of $G_s$, hence $m = -24r - 12s - 6$ and $a_m = (-1)^{|s_B|} f(G_s) = s + 1$. ■

A small refinement of the last result allows us to modify the signs of the extreme states coefficients:

**Theorem 6** Let $\alpha$ be an odd integer great than 1. Let $D_{rs}^\alpha$ be the diagram $D_{rs}$ with a modification in the way in which $D_r$ and $\bar{D}_s$ are joined on the left, as shown in Figure 20. Then $\langle D_{rs}^\alpha \rangle = -(s + 1)A^m + \cdots + (r + 1)A^M$ where $m = -24r - 12s - \alpha - 8$ and $M = 12r + 24s + 3\alpha + 4$. In particular $\text{span}(\langle D_{rs}^\alpha \rangle) = 36(r + s) + 4\alpha + 12$. Moreover, $D_{rs}^\alpha$ is a prime knot diagram with $12(r + s) + 2 + \alpha$ crossings.

**Proof.** We have that $c(D_{rs}^\alpha) = c(D_{rs}) + \alpha$, $|s_A| = |s_A(D_{rs})| + \alpha - 1 = 6s + \alpha + 2$ and $|s_B| = |s_B(D_{rs})| + 1 = 6r + 4$. In addition the $s_A$ and $s_B$ Lando’s graphs are still the duplications of $G_r$ and $G_s$ respectively. ■

### 3 More graph theory.

Let us consider the specific example provided by Theorem 6 for the extreme state coefficient $a_M = 41$. The knot diagram $D_{40}$ has $12 \times 40 = 480$ crossings! If we
take the non-adequate link diagram with three components $D'_r$, we still have $6 \times 40 = 240$ crossings! In this section we develop other graph constructions, providing easier examples for many possible extreme states coefficients. Here “easier” means always that the diagram has fewer crossings. In many cases, easier means as well that the corresponding span is lower.

**Building with bricks**

Let $\{G^{v_1}_1, \ldots, G^{v_k}_k\}$ a set of graphs with an a chosen vertex $v_i \in G_i$ for every $1 \leq i \leq k$. We denote by $S = S(G^{v_1}_1, \ldots, G^{v_k}_k)$ the graph shown in Figure 21a, and we call it “simple building” constructed with the bricks $G^{v_1}_1, \ldots, G^{v_k}_k$. Precisely, $S$ can be defined in steps using the operation $*$ introduced in the second section. Let $W$ be the only vertex of $L_1$. Then

$$S = ((\ldots((L^w_w \ast G^{v_1}_1)^w \ast G^{v_2}_2)^w \ast \ldots)^w) \ast G^{v_k}_k.$$ 

Other related construction can be obtained by introducing $k$ extra vertices $w_1, \ldots, w_k$, one in every edge joining $v_i$ and $w$ in the graph $S$. The new graph is denoted by $C = C(G^{v_1}_1, \ldots, G^{v_k}_k)$ and it is shown in Figure 21b. We will call it “complicated building” constructed with the bricks $G^{v_1}_1, \ldots, G^{v_k}_k$. Its precise definition using the operation $*$ is left to the reader. In both constructions the vertex $w$ is called the central vertex. In the second construction, the vertices $w_i$ are called the intermediate vertices.

**Lemma 7** Let $G^{v_i}_i$ be a brick of type $(n_i, m_i)$ for every $i \in \{1, \ldots, n\}$. Let
Figure 19: The diagram $D_{rs}$.

Figure 20: A small modification of $D_{rs}$ produces $D_{\alpha rs}^\alpha$.

$S = S(G_1^{v_1}, \ldots, G_k^{v_k})$ and $C = C(G_1^{v_1}, \ldots, G_k^{v_k})$. Then:

1. $S^w$ is a brick of type $(\prod_{i=1}^k n_i - \prod_{i=1}^k m_i, \prod_{i=1}^k n_i)$, where $w$ is the central vertex of $S$.
2. $S^{v_j}$ is a brick of type $(\prod_{i=1}^k n_i - \prod_{i=1}^k m_i, m_j \prod_{j \neq i=1}^k n_i - \prod_{i=1}^k m_i)$.
3. $C^w$ is a brick of type $(\prod_{i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i, \prod_{i=1}^k (n_i - m_i))$, where $w$ is the central vertex of $C$.
4. $C^{v_j}$ is a brick of type $(\prod_{i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i, -m_j \prod_{j \neq i=1}^k n_i)$.
5. $C^{w_j}$ is a brick of type $(\prod_{i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i, n_j \prod_{j \neq i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i)$, where $w_j$ is any intermediate vertex of $C$.

Proof. We prove (1) and leave the other proofs to the reader:

$$f(S - w) = \prod_{i=1}^k f(G_i^{v_i}) \quad \text{(multiplication)}$$

$$= \prod_{i=1}^k n_i$$
Let us come back to the example with $a_M = 41$. Note that $41 = 5 	imes 5 	imes 2 - 3 	imes 3 	imes 1$. Hence $S = S(G_{v1}^1, G_{v2}^2, H^v)$ has $f(S) = 41$ if $G_{v1}^1$ and $G_{v2}^2$ are two copies of a brick of type $(5,3)$. Two different bricks of type $(5,3)$ are shown in Figure 22. It follows from (1) in above lemma that the planar graph $S$ convertible in BCG shown in Figure 23 has $f(S) = 41$. The corresponding diagram with $a_M = 41$ in its Kauffman bracket has 43 crossings.

\begin{align*}
  f(S) &= f(S - w) - f(S - Nw) \quad \text{(recursion)} \\
  &= \prod_{i=1}^{k} n_i - \prod_{i=1}^{k} f(G_i - v_i) \quad \text{(multiplication)} \\
  &= \prod_{i=1}^{k} n_i - \prod_{i=1}^{k} m_i.
\end{align*}

It can be easily checked that any prime number less or equal than 50, different
from 41, can be realized as $f(S)$ where $S$ is a simple building constructed with at most three hexagons (a brick of type $(2,1)$).

Finally in this section we present a sequence $\{F_r\}_{r \in \mathbb{N}}$ of planar graphs, all of them convertible in BCG, such that the corresponding sequence of integers $\{f(F_r)\}_{r \in \mathbb{N}}$ is the Fibonacci sequence $2, 3, 5, 8, 13, 21, \ldots$. The graph $F_r$ is shown in Figure 24. If we compare the link diagram arising from $F_{r+1}$ and $F_r$, we note that while $a_M$ increases by $f(F_{r-1})$, the number of crossings of the diagram increases only by 7.

![Figure 23: The graph $F_r$.](image)

4 Non-states extreme coefficients of Jones polynomials.

In this final section we will exhibit examples of prime knot diagrams for which the extreme states coefficients $a_m$ and $a_M$ are zero and the next coefficients $a_{m+4}$ and $a_{M-4}$ take arbitrary values. The idea is just to look at the more general bipartite circle graph defined by $sD$, rather than the Lando’s graph $G_D$, and use a very simple trick for counting $a_{M-4}$ in special cases.

We start with the natural enlargement of proposition given in the second section. Recall that $\langle D \rangle$ denotes the Kauffman bracket of an unoriented diagram $D$ with normalization one.

**Proposition 2** (i) A state $s$ contribute to $a_{M-4}$ if and only if either $s \in \Gamma_A = \{s/|s| = |s_A| + b(s)\}$ or $s \in \Gamma_A^1 = \{s/|s| = |s_A| + b(s) - 2\}$. The contribution of $s \in \Gamma_A$ to $a_{M-4}$ is $(-1)^{|s_A| - 1}(-1)^{b(s)}(|s_A| + b(s) - 1)$. The contribution of $s_1 \in \Gamma_A^1$ to $a_{M-4}$ is $(-1)^{|s_A| - 1}(-1)^{b(s_1)}$.

(ii) $a_{M-4} = (-1)^{|s_A| - 1}[(|s_A| - 1) \sum_{s \in \Gamma_A} (-1)^{b(s)} + \sum_{s \in \Gamma_A} (-1)^{b(s)} b(s) + \sum_{s_1 \in \Gamma_A^1} (-1)^{b(s_1)}]$.

(iii) If $a_M = 0$ then the whole contribution of $\Gamma$ to $a_{M-4}$ is given by

$(-1)^{|s_A| - 1} \sum_{s \in \Gamma_A} (-1)^{b(s)} b(s)$. 

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Proof. A state \( s \) contributes to \( a_{M-4} \) if and only if \( \max(s) \geq M - 4 \), hence if \( s \) contributes to \( a_{M-4} \) but is not in \( \Gamma_A \), we have that \( \max(s) = M - 4 \) by (i) in Proposition \( \Box \). Now the statements follow from the fact that \( \{s_1/|s_1| = |s_A| + b(s_1) - 2\} \).

Consider now the generalized bipartite circle graph \( s_AD \) obtained from an unoriented diagram \( D \) by \( s_A \)-smoothing. Suppose that locally this has the very special aspect shown in Figure 25, and suppose that only the A-chords \( x_1, \ldots, x_k \) have their endpoints in the same component. The Lando’s graph is then given by \( k \) parallel A-chords, hence by duplication we have that \( a_M = 0 \) in \( \langle D \rangle \).

![Figure 24: A very special \( s_A D \).](image)

Now on, we will identify a state \( s \) with the set of the \( b(s) \) A-chords in \( s_AD \) that correspond to the \( b(s) \) crossings of \( D \) in which the label associated by \( s \) is a B-chord. Let \( \mathcal{X} = \{x_1, \ldots, x_k\} \), \( \mathcal{A} = \{a_1, \ldots, a_n\} \) and \( \mathcal{A}' = \{a'_1, \ldots, a'_m\} \) (see Figure 25).

The states in \( \Gamma_A \) are the subsets of \( \mathcal{X} \), hence by (iii) in Proposition \( \Box \) we have that the contribution of \( \Gamma_A \) to \( a_{M-4} \) is given up to sign by

\[
\sum_{s \in \Gamma_A} (-1)^{b(s)} b(s) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j j = 0.
\]

We now analyze the contribution to \( a_{M-4} \) of the states in \( \Gamma_A^1 \) which contain at least one A-chord of \( \mathcal{A} \bigcup \mathcal{A}' \). Note first that these states are necessarily contained in \( \mathcal{X} \bigcup \mathcal{A} \bigcup \mathcal{A}' \).

Suppose first that in \( s_1 \) there are no A-chords of both sets \( \mathcal{A} \) and \( \mathcal{A}' \). Then the union of \( s_1 \setminus \mathcal{X} \) and any subset of \( \mathcal{X} \) is a state which lies in \( \Gamma_A^1 \), and all of their contributions to \( a_{M-4} \) cancel.

By the contrary, suppose that \( s_1 \) has at least one A-chord of \( \mathcal{A} \) and one of \( \mathcal{A}' \). Then \( s_1 \setminus \mathcal{X} \) is the empty set, and the contribution of all these states is \( (-1)^{|s_1\setminus \mathcal{X}| - 1} \) in total.

As a consequence we have found a geometrical reason for some of the results algebraically obtained in \( \Box \) about the Kauffman bracket of pretzel link diagrams. Let \( P \) be the pretzel link diagram \( P(a, b_1, \ldots, b_s) \) where \( a \geq 2 \), and \( b_i \leq -2 \) for any \( i = 1, \ldots, s \) with \( s \) at least 2 . Let \( \alpha \) be the cardinal of the set
\{i/b_i = -2\} and assume that \(\alpha \geq 1\). Then \(P\) is a \(-\)-adequate diagram, \(a_M = 0\) and \(a_{M-4} = \alpha\).

Finally, in order to get a prime knot diagram with extreme states coefficient equal to zero and arbitrary values for \(a_{m+4}\) and \(a_{M-4}\), we manipulate two pretzel link diagrams \(P(2, -2, \ldots, -2)\) and \(P(2, \ldots, 2, -2, 2)\). We join these two diagrams in the way shown in Figure 26, using two extra columns with \(\alpha\) and \(\beta\) crossings respectively, both \(\alpha\) and \(\beta\) great or equal than two. Note that

\[\langle L \rangle = (-1)^{r+s+\beta-1} r A^{-4r-4s-\alpha-3\beta+2} + \cdots + (-1)^{r+s+\alpha-1} s A^{4r+4s+3\alpha+\beta-2}\]

In particular \(\text{span}(\langle L \rangle) = 8(r + s) + 4(\alpha + \beta) - 4\). The highest degree of this polynomial is \(M - 4\), and the lowest degree is \(m + 4\). Moreover, \(L\) has \(r + s + 2 - \frac{1}{2}(-1)^n - \frac{1}{2}(-1)^n\) components.

**Proof.** Note first that \(c(L) = 2 + 2s + 2 + 2r + \alpha + \beta\).
After $s_A$-smoothing (see Figure 28) we have $|s_A| = r + s + \alpha$ (hence $M = 4r + 4s + 3\alpha + \beta + 2$), and since the Lando’s graph is given by two parallel A-chords, we have $a_M = 0$. The above discussion applied to Figure 28 gives $a_{M-4} = (-1)^{|s_A|-1}s = (-1)^{r+s+\alpha-1}s$. Analogously, after $s_B$-smoothing (see Figure 29) we have $|s_B| = r + s + \beta$ (hence $m = -4r - 4s - \alpha - 3\beta - 2$), and since the Lando’s graph is given by two parallel A-chords, we have $a_m = 0$. Finally, from the above discussion applied to Figure 29 we deduce that $a_{m+4} = (-1)^{|s_B|-1}r = (-1)^{r+s+\beta-1}r$. $\blacksquare$

We now modify the link diagram $L(r, s; \alpha, \beta)$ in order to get a prime knot diagram $K(r, s; \alpha, \beta)$. This is shown in Figure 30.
Theorem 9 Let $K = K(r, s; \alpha, \beta)$ be the link diagram shown in Figure 30. Then

$$
\langle K \rangle = (-1)^{r+s}rA^{-5r-7s-\alpha-3\beta-5} + \ldots + (-1)^s A^{7r+5s+3\alpha+\beta+3}
$$

In particular \(\text{span}(\langle K \rangle) = 12(r + s) + 4(\alpha + \beta) + 8\). The highest degree of this polynomial is \(M - 4\), and the lowest degree is \(m + 4\). Moreover, \(K\) is a prime knot diagram.

Proof. We first consider the modification \(L' = L'(r, s; \alpha, \beta)\) of \(L = L(r, s; \alpha, \beta)\) shown in Figure 31. Looking at the differences between \(L\) and \(L'\) after \(s_A\) and \(s_B\)-smoothings, we can prove that

$$
\langle L' \rangle = (-1)^{r+s}rA^{-5r-7s-\alpha-3\beta+2} + \ldots + (-1)^s A^{7r+5s+3\alpha+\beta-2}
$$

where the highest and lowest degrees are \(M - 4\) and \(m + 4\) respectively.

But \(L'\) is not a knot diagram when the parity of \(\alpha\) and \(\beta\) is the same. For this reason we introduce a small change in \(L'\) in order to get the prime knot diagram \(K\). This change is shown in Figure 32. Note then that \(c(K) = c(L') + 1\) and

$$
|s_A| \text{ does not change, hence } M(K) = M(L') + 1 = 7r + 5s + 3\alpha + \beta + 3. \text{ Respect to the coefficients we have } a_M = 0 \text{ and } a_{M-4} = (-1)^{|s_A|-1}s = (-1)^s A^{7r+5s+3\alpha+\beta+3}.
$$

But after \(s_B\)-smoothing we have \(|s_B| = |s_B(L')| + 1\), hence \(m(K) = m(L') - 1 - 2 \times 1 = -5r - 7s - \alpha - 3\beta - 5\). Respect to the coefficients we have \(a_m = 0\) and \(a_{m+4} = (-1)^{|s_B|-1}r = (-1)^{r+s}r\). \(\blacksquare\)
As we said in the introduction, we do not know if there is a nice interpretation of $a_{M-4}$ in terms of graph theory, parallel to that one in which $a_M$ is given in terms of the graph $G_D$ as described in the second section. Apart from the examples, this last section can be seen as a partial interpretation of $a_{M-4}$ in terms of graph theory. More important, it remains to be answered the following question: how arbitrary can be the extreme coefficients of the Jones polynomial when the value for the spread is previously fixed?

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