Corners of normal matrices

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To Kalyan Sinha on his sixtieth birthday

Abstract. We study various conditions on matrices $B$ and $C$ under which they can be
the off-diagonal blocks of a partitioned normal matrix.

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The structure of general normal matrices is far more complicated than that of two special
kinds — hermitian and unitary. There are many interesting theorems for hermitian
and unitary matrices whose extensions to arbitrary normal matrices have proved to be
extremely recalcitrant (see e.g., [1]). The problem whose study we initiate in this note is
another one of this sort.

We consider normal matrices $N$ of size $2n$, partitioned into blocks of size $n$ as

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (1)$$

Normality imposes some restrictions on the blocks. One such restriction is the equality

$$\|B\|_2 = \|C\|_2 \quad (2)$$

between the Hilbert–Schmidt (Frobenius) norms of the off-diagonal blocks $B$ and $C$. If $T$
is any $m \times m$ matrix with entries $t_{ij}$, then

$$\|T\|_2 = \left( \sum_{j=1}^{m} |t_{ij}|^2 \right)^{1/2}.$$

The equality (2) is a consequence of the fact that the Euclidean norm of the $j$th column
of a normal matrix is equal to the Euclidean norm of its $j$th row.

Replacing the Hilbert–Schmidt norm by another unitarily invariant norm, we may ask
whether the equality (2) is replaced by interesting inequalities. Let $s_1(T) \geq \cdots \geq s_m(T)$
be the singular values of $T$. Every unitarily invariant norm $|||T|||$ is a symmetric gauge
function of $\{s_j(T)\}$ (see chapter IV of [1] for properties of such norms). Much of our
concern in this note is with the special norms

$$\|T\|_2 = (\text{tr} \, T^* T)^{1/2} = \left( \sum_{j=1}^{m} s_j^2(T) \right)^{1/2}.$$
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\[ \|T\| = s_1(T) = \sup_{x \in \mathbb{C}^m, \|x\| = 1} \|Tx\|. \]  

(3)

The latter is the norm of \(T\) as a linear operator on the Euclidean space \(\mathbb{C}^m\). Clearly

\[ \|T\| \leq \|T\|_2 \leq \sqrt{m} \|T\|, \]  

(4)

for every \(m \times m\) matrix \(T\).

If the matrix \(N\) in (1) is hermitian, then \(C = B^*\), and hence, \(|||C||| = |||B|||\) for all unitarily invariant norms. If \(N\) is unitary, then \(AA^* + BB^* = A^*A + C^*C = I\). Hence, the eigenvalues \(\lambda_j\) satisfy the relations

\[ \lambda_j(BB^*) = \lambda_j(I - AA^*) = 1 - \lambda_j(AA^*) \]

\[ = 1 - \lambda_j(A^*A) = \lambda_j(I - A^*A) = \lambda_j(C^*C). \]

Thus \(B\) and \(C\) have the same singular values, and again \(|||B||| = |||C|||\) for all unitarily invariant norms.

This equality of norms does not persist when we go to arbitrary normal matrices, as we will soon see. From (2) and (4) we get a simple inequality

\[ |||B||| \leq \sqrt{n} |||C|||. \]  

(5)

One may ask whether the two sides of (5) can be equal, and that is the first issue addressed in this note.

When \(n = 2\), it is not too difficult to construct a normal matrix \(N\) of the form (1) in which \(|||B||| = \sqrt{2} |||C|||\). One example of such a matrix is

\[
N = \begin{bmatrix}
0 & 0 & \sqrt{2} & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]  

(6)

When \(n = 3\), examples seem harder to come by. One that preserves some of the features of (6) is given by the matrix

\[
N = \begin{bmatrix}
0 & \sqrt{\frac{1}{3} - 1} & 0 & \sqrt{3} & 0 & 0 \\
0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\
\sqrt{\frac{1}{3} + 1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \sqrt{\frac{1}{3} + 1} \\
0 & 1 & 0 & \sqrt{\frac{1}{3} - 1} & 0 & 0 \\
1 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0
\end{bmatrix}
\]
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It can be seen that $N$ is normal and plainly $\|B\| = \sqrt{3}$ while $\|C\| = 1$. When $n = 4$, it is impossible to find such a matrix, and that is our first theorem.

The following elementary lemma (which can be verified by induction on the integer $k$) is used repeatedly in our proof.

**Lemma.** Let $V$ be an $n$-dimensional vector space and let $V_1, \ldots, V_k$ be subspaces of $V$ the sum of whose dimensions is larger than $(k - 1)n$; i.e.,

$$\sum_{j=1}^{k} \dim V_j > (k - 1)n.$$  

Then the intersection of these $k$ subspaces is nonzero.

**Theorem 1.** There exists a normal matrix $N$ of the form (1) with

$$\|B\| = \sqrt{n} \|C\|$$  

if and only if $n \leq 3$.

**Proof.** Note first that if equalities (2) and (8) hold simultaneously, then rank $B$ must be one and $C$ must be unitary. So, after applying a unitary similarity by $\begin{bmatrix} C & O \\ O & I \end{bmatrix}$, we may assume that

$$N = \begin{bmatrix} A & B \\ I & D \end{bmatrix}.  

The normality condition $N^*N = NN^*$ leads to two equations

$$A - D = A^*B - BD^*,  

2I = AA^* - A^*A + BB^* + B^*B + D^*D - DD^*.  

Since $B$ is of rank one, where $\dim X$ stands for the dimension of a space $X$. So, if $n \geq 3$, then the dimensions of $\ker B$ and $\ker B^*$ add up to more than $n$. Hence their intersection is nonzero, and we may choose a unit vector $x$ in this intersection. For this vector, we obtain from (10)

$$(A - D)x = -BD^*x,  

and

$$(A - D)^*x = B^*Ax.  

Equation (11) leads to the condition

$$2 = \|A^*x\|^2 - \|Ax\|^2 + \|Dx\|^2 - \|D^*x\|^2.  

The rest of the proof shows that if $n > 3$, then we can choose a vector $x \in (\ker B) \cap (\ker B^*)$ for which these conditions cannot be satisfied.
The two matrices $BD^*$ and $B^*A$ have rank at most 1, so their kernels have dimension at least $n - 1$. Hence

$$\dim(\ker B) + \dim(\ker B^*) + \dim(\ker BD^*) + \dim(\ker B^*A) \geq 4n - 4. \quad (15)$$

This is larger than $3n$ whenever $n > 4$. So, in this case the four kernel spaces involved in (15) have a nonzero intersection. Let $x$ be a unit vector in this intersection. Then from (12) and (13) we find that

$$ (A - D)x = 0 \quad \text{and} \quad (A - D)^*x = 0. $$

Hence, $\|Ax\| = \|Dx\|$ and $\|A^*x\| = \|D^*x\|$. This contradicts the condition (14).

Now consider the case $n = 4$. The spaces $\ker B$ and $\ker B^*$ have dimension 3 each, while the space $\ker B(A + D)^*$ has dimension at least 3. The three dimensions add up to more than 8. Hence, we can find a unit vector $x$ in the intersection of these three spaces. For this vector we have

$$ \|A^*x\|^2 - \|D^*x\|^2 = \text{Re} \langle (A + D)^*x, (A - D)^*x \rangle \\
= \text{Re} \langle (A + D)^*x, B^*Ax \rangle \\
= \text{Re} \langle B(A + D)^*x, Ax \rangle \\
= 0. \quad (16) $$

Here the second equality is a consequence of (13), and at the last step we have used the fact that $B(A + D)^*x = 0$.

Using (12) instead of (13) we get

$$ \|Dx\|^2 - \|Ax\|^2 = \text{Re} \langle (A + D)x, (D - A)x \rangle \\
= \text{Re} \langle (A + D)x, BD^*x \rangle \\
= \text{Re} \langle B^*(A + D)x, D^*x \rangle. \quad (17) $$

Since $B$ is a matrix with rank equal to 1 and norm equal to 2, we have $B^*BB^* = 4B^*$. (Use the polar decomposition $B = UP$. In some orthonormal basis $P$ is diagonal with only one nonzero entry 2 on the diagonal. So $B^*BB^* = P^*U^* = 4PU^* = 4B^*$. Hence we have

$$ 4B^*Ax = B^*BB^*Ax \\
= B^*B(A - D)^*x \quad \text{(using (13))} \\
= B^*B(A + D)^*x - 2B^*BD^*x \\
= -2B^*BD^*x \\
= 2B^*(A - D)x \quad \text{(using (12))} \\
= 4B^*Ax - 2B^*(A + D)x. $$

This shows that $B^*(A + D)x = 0$, and we get from (17)

$$ \|Dx\|^2 - \|Ax\|^2 = 0. \quad (18) $$

Clearly the relations (14), (16) and (18) cannot be simultaneously true.

We have shown that when $n \geq 4$, there cannot exist a $2n \times 2n$ normal matrix of the form (9) in which $B$ is an $n \times n$ matrix of rank one. This proves the theorem.  \[ \Box \]
Our discussion leads to some natural questions.

**Problem 1.** For \( n \geq 4 \), evaluate the quantity

\[
\alpha_n = \sup \left\{ \frac{\|B\|}{\|C\|}: \exists A, D \text{ for which } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ is normal} \right\}.
\]

We have seen \( \alpha_n < \sqrt{n} \) for \( n \geq 4 \). It would be of interest to know whether \( \alpha_n \) is a bounded sequence.

**Problem 2.** What matrix pairs \( B, C \) can be the off-diagonal entries of a normal matrix \( N \) as in (1)? In other words, when does \( \begin{bmatrix} B \\ C \end{bmatrix} \) have a normal completion?

**Example 1.** Consider the \( 2 \times 2 \) matrices

\[
B = \begin{bmatrix} 1 & \epsilon \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}.
\]

Then, \( \|B\|_2 = \|C\|_2 \). However, there do not exist any \( 2 \times 2 \) matrices \( A \) and \( D \) for which \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is normal. We leave the verification of this statement to the reader. Thus the equality (2) is only a necessary condition for normality of the matrix (1).

We consider some special cases of the question raised in Problem 2. We assume either \( B = C \), or \( B = C^* \).

For every \( B \), the matrix \( \begin{bmatrix} B \\ B \end{bmatrix} \) has a normal completion, and this completion may be chosen to be of the special type \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \). Indeed, if \( U \) is the unitary matrix \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \), then

\[
U \begin{bmatrix} A & B \\ B & A \end{bmatrix} U^* = \begin{bmatrix} A + B & 0 \\ 0 & A - B \end{bmatrix}.
\]

So \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \) is normal if and only if \( \begin{bmatrix} A + B & 0 \\ 0 & A - B \end{bmatrix} \) is normal, and this is the case if and only if \( A + B \) and \( A - B \) both are normal. The most obvious choice of \( A \) that assures this is \( A = B^* \). Thus

\[
\begin{bmatrix} B & B \\ B & B^* \end{bmatrix}
\]

is a normal completion of \( \begin{bmatrix} B \\ B \end{bmatrix} \). We have the norm inequality

\[
\|B\| \leq \|\tilde{B}\| \leq 2\|B\|.
\]

(20)

When \( B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) we have \( \|\tilde{B}\| = \|B\| \). On the other hand, if \( B \) is any hermitian matrix, then \( \|\tilde{B}\| = 2\|B\| \). In this case, and more generally when \( B \) is normal, \( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \) is normal and has norm equal to \( \|B\| \). This raises the question of finding completions of \( \begin{bmatrix} B \\ B \end{bmatrix} \) that are ‘optimal’ in various senses.
Problem 3. Given a matrix $B$ find a matrix $A$ such that

$$N = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is normal and has the least possible norm. This is equivalent to asking for a matrix $A$ such that $A + B$ and $A - B$ are normal and the quantity $\max(\|A + B\|, \|A - B\|)$ is minimised. It might be difficult to find all solutions to this problem. The following considerations lead to one solution.

We assume that $B$ is a contraction, i.e. $\|B\| \leq 1$ and ask for an $A$ so that $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is unitary. This is a unitary completion of the matrix $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. Let $B = USV$ be the singular value decomposition of $B$. Then

$$\begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} U^*AV^* & S \\ S & U^*AV^* \end{bmatrix}.$$ 

So, the problem reduces to finding an $A'$ such that $\begin{bmatrix} A' & S \\ S & A' \end{bmatrix}$ is unitary. A familiar idea from the theory of unitary dilations (p. 232 of [2]) suggests the choice $A' = i(I - S^2)^{1/2}$.

This tells us how to find for any matrix $B$ one of the least-norm normal completions of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. Assume $\|B\| = 1$ and find a unitary completion as proposed above.

Next we consider the case $B = C^*$, and ask for matrices $A$ and $D$ such that

$$N = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \quad (21)$$

is normal. A calculation shows that the matrices $A$ and $D$ must be normal and satisfy the equation

$$(A - A^*)B = B(D - D^*). \quad (22)$$

Let $A = H_1 + iK_1$ and $D = H_2 + iK_2$ be the Cartesian decompositions of $A$ and $D$. Here $(H_1, K_1)$ and $(H_2, K_2)$ are two pairs of commuting hermitian matrices. Equation (22) is equivalent to $K_1B = BK_2$. This shows that

$$B^*BK_2 = B^*K_1B = (K_1B)^*B = (BK_2)^*B = K_2B^*B.$$ 

So $K_2$ commutes with $B^*B$, and hence with the factor $P$ in the polar decomposition $B = UP$.

Thus the general solution to (22) is obtained as follows: Choose $K_0$ and $K_2$, both hermitian, satisfying the conditions

$$K_0P = PK_0, \quad K_2P = PK_2, \quad (K_0 - K_2)P = 0.$$ 

Let $K_1 = UK_0U^*$. This condition ensures

$$K_1B = UK_0U^*B = UK_0P = UK_2P = UPK_2 = BK_2.$$ 

Choose hermitian matrices $H_1$ and $H_2$ that commute with $K_1$ and $K_2$, respectively. Let $A = H_1 + iK_1$ and $D = H_2 + iK_2$. This leads to $N$ in (21) being normal.
As before, we also consider the special case \( \|B\| \leq 1 \) and ask for \( A \) and \( D \) such that the matrix (21) is unitary. This can be solved as follows: Let \( B = UP \) be any polar decomposition. Choose hermitian matrices \( K_0 \) and \( K_2 \) that commute with \( P \) and satisfy the inequalities

\[
K_0^2 \leq I - P^2, \quad K_2^2 \leq I - P^2.
\]

Then choose hermitian matrices \( H_0 \) and \( H_2 \) that commute with \( K_0 \) and \( K_2 \), respectively, and satisfy the conditions

\[
H_0^2 + K_0^2 = H_2^2 + K_2^2 = I - P^2.
\]

Let \( A = U(H_0 + iK_0)U^* \) and \( D = H_2 + iK_2 \). Then the matrix (21) is unitary.

Example 1 shows that the equality \( \|B\|_2 = \|C\|_2 \) is not a sufficient condition for the existence of a normal completion of \( [B \; C] \).

Our next proposition shows that equality between all unitarily invariant norms is a sufficient condition.

**Proposition.**

Let \( B, C \) be \( n \times n \) matrices with \( \|B\| = \|C\| \) for every unitarily invariant norm. Then the matrix \( [B \; C] \) has a completion that is a scalar multiple of a unitary matrix.

**Proof.** If \( \|B\| = \|C\| \) for every unitarily invariant norm, then \( s_j(B) = s_j(C) \) for all \( j = 1, 2, \ldots, n \). Hence, there exist unitary matrices \( U_1, U_2, V_1, V_2 \) such that \( B = U_1SU_2 \), and \( C = V_1SV_2 \). Divide \( B \) and \( C \) by \( \|S\| \), and thus assume \( \|S\| = 1 \). Then \( I - S^2 \) is positive, and has a positive square root. It is easy to see that the matrix

\[
\begin{bmatrix}
(I - S^2)^{1/2} & S \\
S & -(I - S^2)^{1/2}
\end{bmatrix}
\]

is unitary. Multiply this matrix on the left by the unitary matrix \( U_1 \oplus V_1 \), and on the right by the unitary matrix \( V_2 \oplus U_2 \). This gives a unitary matrix whose off-diagonal blocks are \( B \) and \( C \).

While the condition in the Proposition is not necessary, it is sensitive to small perturbations. The matrices \( B \) and \( C \) in Example 1 satisfy the conditions \( \|B\|_2 = \|C\|_2 \), \( \|B\| = \|C\| + O(\varepsilon) \), but for \( \varepsilon \neq 0 \), there is no possible normal completion of \( [B \; C] \).

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