Escaping Saddle Points for Nonsmooth Weakly Convex Functions via Perturbed Proximal Algorithms

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Abstract

We propose perturbed proximal algorithms that can provably escape strict saddles for nonsmooth weakly convex functions. The main results are based on a novel characterization of $\epsilon$-approximate local minimum for nonsmooth functions, and recent developments on perturbed gradient methods for escaping saddle points for smooth problems. Specifically, we show that under standard assumptions, the perturbed proximal point, perturbed proximal gradient and perturbed proximal linear algorithms find $\epsilon$-approximate local minimum for nonsmooth weakly convex functions in $O(\epsilon^{-2} \log(d)^4)$ iterations, where $d$ is the dimension of the problem.

Keywords— Nonsmooth Optimization, Saddle Point, Perturbed Proximal Algorithms

1 Introduction

Nonconvex optimization plays an important role in deep learning and machine learning. Although global optimum for nonconvex optimization is not easy to obtain in general, recent studies showed that for many smooth optimization problems arising from important applications, second-order stationary points are indeed global optimum under mild conditions. We refer to [21, 39, 40, 3] for a partial list of these works. There have been many research works on this topic when the objective function is smooth. As a result, in order to find the global optimum for these problems, one only needs to avoid the saddle points. Besides these highly structured optimization problems, avoiding saddle points is also crucial for general optimization problems. For instance, in training of neural network, recent results show that the proliferation of saddle points undermines the quality of solutions, which lead to much higher error than local minimum [13]. Neural network saddle points appear even more frequently and are potentially a major bottleneck in high dimensional problems of practical interest [13]. On the other hand, nonsmooth formulations frequently appear in machine learning and modern signal processing problems. Some well studied examples include neural networks with ReLU activation [34], robust principal component analysis [42, 44], robust matrix completion [33, 6], blind deconvolution [30, 43], robust phase retrieval [18, 16, 19, 17]. It is known that neural networks with the nonsmooth ReLU activation has great expressive power [34]. Nonsmooth regularizers also provide robustness against sparse outliers for data science applications. In this paper, we consider how to escape saddle points for nonsmooth functions, which is still less developed so far.

Most existing works for finding local minimum consider smooth objective functions and derive the complexity for reaching an $\epsilon$-second order stationary point (See Definition 3.4). Nesterov
and Polyak [37] proposed the cubic regularization method, which requires $O(\epsilon^{-3/2})$ iterations for obtaining an $\epsilon$-second order stationary point. Curtis et al. [11] proved the same complexity result for the trust region method. To avoid Hessian computation required in [37, 11], Carmon et al. [8] and Agarwal et al. [1] proposed to use Hessian-vector product and achieved convergence rate of $O(\epsilon^{-7/4})$. Recently, the complexity results of pure first-order methods for obtaining local minimum have been studied (see, e.g., [21, 12, 28, 20]). Lee et al. [31] proved that gradient descent method converges to a local minimizer almost surely with random initialization by using tools from dynamical systems theory. However, Du et al. [17] showed that gradient descent (GD) method may take exponential time to escape saddle points. Recently, Jin et al. [27, 28] proved that the perturbed GD can converge to a local minimizer in a number of iterations that depends poly-logarithmically on the dimension, reaching an iteration complexity of $\tilde{O}(\epsilon^{-2} \log(d)^4)$. Jin et al. [28] also proposed perturbed stochastic gradient descent method (SGD) that requires $\tilde{O}(\epsilon^{-4})$ stochastic gradient computations to reach an $\epsilon$-second order stationary point. By utilizing the techniques developed by Carman et al. [7], Jin et al. [29] proposed the accelerated perturbed GD and improved the complexity to $\tilde{O}(\epsilon^{-7/4} \log(d)^6)$. More recently, Fang et al. [20] proved that the complexity of SGD can be further improved to $\tilde{O}(\epsilon^{-3.5})$ under the assumption that the gradient and Hessian are both Lipschitz continuous. How to escape saddle points for constrained problems are also studied in the literature. Avdiukhin et al. [2] proposed a noisy projected gradient descent method for escaping saddle points for problems with linear inequality constraints. Mokhtari et al. [36] studied a generic algorithm for escaping saddle points of smooth nonconvex optimization problems subject to a general convex set. Criscitiello and Boumal [10] and Sun et al. [41] considered the perturbed Riemannian gradient method and showed that it can escape saddle points for smooth minimization over manifolds.

The literature on escaping saddle points for nonsmooth functions, on the other hand, is relatively limited. Among few existing works, Huang and Becker [26] proposed a perturbed proximal gradient method for nonconvex minimization with an $\ell_1$-regularizer and showed that their proposed method can escape the saddle points for this particular class of problems. It is not clear how to extend the results in [26] to more general nonsmooth functions. Davis and Drusvyatskiy [14] extended the work [31] to nonsmooth problems, showing that proximal point, proximal gradient and proximal linear algorithms converge to local minimum almost surely, under the assumption that the objective function satisfies a strict saddle property. However, no convergence rate complexity for finding a local minimum was given in [14].

In this paper, we consider minimizing a nonsmooth weakly convex function. We propose perturbed proximal point, perturbed proximal gradient and perturbed proximal linear algorithms, and prove that they can escape active strict saddle points for nonsmooth weakly convex functions, under standard conditions. Our main results are based on a novel characterization on the $\epsilon$-approximate local minimum inspired by [14], and the analysis of perturbed gradient descent method for minimizing smooth functions [27, 28]. Our main contributions are summarized below.

1. We propose a novel definition of $\epsilon$-approximate local minimum for nonsmooth weakly convex problems.

2. We propose three perturbed proximal algorithms that provably escape active strict saddle points for nonsmooth weakly convex functions. The three algorithms are: perturbed proximal point algorithm, perturbed proximal gradient algorithm, and perturbed proximal linear algorithm.

3. We analyze the iteration complexity of the three proposed algorithms for obtaining an $\epsilon$-approximate local minimum for nonsmooth weakly convex functions. We show that the
iteration complexity for the three algorithms to obtain an $\epsilon$-approximate local minimum is $O(\epsilon^{-2} \log(d)^4)$. To the best of our knowledge, this is the first quantitative iteration complexity of algorithms for finding an approximate local minimum for nonsmooth weakly convex functions.

**Notation:** Let $\|\cdot\|$ denote the Euclidean norm of a vector. For a nonsmooth function $f(\cdot)$, we denote its subdifferential at point $x$ as $\partial f(x)$, and the parabolic subdifferential (defined in the appendix) at $x$ for $u \in \text{dom } df(x)$ with respect to $w$ as $df^2(x)(u|w)$. When $df^2(x)(u|w)$ is a constant with respect to $w$, we omit $w$ and denote it as $df^2(x)(u)$. We denote $\mathcal{M}$ as a manifold and $\mathcal{U}$ as a neighborhood set of some points. The tangent space of a manifold $\mathcal{M}$ at $x$ is denoted as $T_x \mathcal{M}$. We further denote a restriction $f_M(x) := f + \delta_M$, where $\delta_M$ evaluates to 0 on $\mathcal{M}$ and $+\infty$ off it. We denote the operator of three proximal algorithms as $S(\cdot)$ and its Jacobian as $\nabla S(\cdot)$. A ball centered at $x$ with radius $r$ is denoted as $B_x(r)$. We use big-$O$ notation, where $p = O(q)$ if there exists a global constant $c$ such that $|p| \leq cq$ and $p = \tilde{O}(q)$ that hides a poly-logarithmic factor of $d$ and $\epsilon$. We use $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ to denote the smallest and largest eigenvalues of matrix $Z$, respectively.

## 2 Perturbed Proximal Algorithms

We consider three types of nonsmooth and weakly convex optimization problems and their corresponding solution algorithms: proximal point algorithm (PPA), proximal gradient method (PGM), and proximal linear method (PLM). Specifically, PPA solves

$$
\min_{x \in \mathbb{R}^d} f(x),
$$

and it iterates as

$$
x_{t+1} := S(x_t), \quad \text{where } S(x) := \text{prox}_{\eta f}(x).
$$

Here we assume that $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is a nonsmooth $\ell$-weakly convex function. Recall that a function $f(x)$ is $\ell$-weakly convex if $f(x) + \frac{\ell}{2} \|x\|^2$ is convex. When choosing $\lambda \in (0, \ell^{-1})$, we define the Moreau envelope of function $f(\cdot)$, denoted as $f_{\lambda}(\cdot)$, and its corresponding proximal mapping as

$$
f_{\lambda}(x) = \min_{y \in \mathbb{R}^d} f(y) + \frac{1}{2\lambda} \|y - x\|^2, \quad \text{and } \text{prox}_{f_{\lambda}}(x) = \arg\min_{y \in \mathbb{R}^d} f(y) + \frac{1}{2\lambda} \|y - x\|^2.
$$

Note that the step size $\eta$ in (2.2) and the parameter $\lambda$ can be different.

PGM solves

$$
\min_{x \in \mathbb{R}^d} f(x) \equiv g(x) + m(x),
$$

and it iterates as

$$
x_{t+1} := S(x_t), \quad \text{where } S(x) := \text{prox}_{\eta m}(x - \eta \nabla g(x)).
$$

Here we assume $g : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is $C^2$-smooth with $\beta$-Lipschitz gradient and $m : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is closed and $\mu$-weakly convex.

PLM solves

$$
\min_{x \in \mathbb{R}^d} f(x) \equiv h(F(x)) + m(x),
$$

and it iterates as

$$
x_{t+1} := S(x_t), \quad \text{where } S(x) := \arg\min_{y \in \mathbb{R}^d} h(F(x) + \nabla F(x)(y - x)) + m(y) + \frac{1}{2\eta} \|y - x\|^2.
$$
Here we assume $F : \mathbb{R}^d \to \mathbb{R}^m$ is a $C^2$-smooth map, $h : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is nonsmooth convex, and $m : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is $\mu$-weakly convex.

Now we present a unified algorithmic framework of perturbed proximal algorithms that find $\epsilon$-approximate local minimum points (will be defined later) of (2.1), (2.3) and (2.5). Our algorithmic framework is presented in Algorithm 1 and it largely follows the perturbed gradient method for smooth problems [27, 28]. Inspired by [15], Algorithm 1 utilizes the Moreau envelope to measure the first order stationarity. When $\|\nabla f_{\lambda}(x)\|$ is close to zero, then $x$ is close to a first order stationary point. Same as [27, 28], the perturbation added to $x_t$ helps escape the saddle points. In Algorithm 1, $\xi \sim \text{Uniform}(B_0(r))$ means that $\xi$ is a random vector uniformly sampled from the Euclidean ball with radius $r$. When $S$ is chosen as in (2.2), (2.4) and (2.6), we name Algorithm 1 as perturbed PPA, perturbed PGM and perturbed PLM, respectively.

Algorithm 1 Perturbed Proximal Algorithms

Input: $x_0$, step size $\eta$, perturbation radius $r$, time interval $\mathcal{T}$, tolerance $\epsilon$.

$t_{\text{perturb}} = 0$

for $t = 0, 1, \ldots, T$ do

if $\|\nabla f_{\lambda}(x_t)\| \leq \epsilon$ and $t - t_{\text{perturb}} > \mathcal{T}$ then

$x_t \leftarrow x_t - \eta \xi_t$, $(\xi_t \sim \text{Uniform}(B_0(r)))$; $t_{\text{perturb}} \leftarrow t$

end if

$x_{t+1} \leftarrow S(x_t)$, where $S$ is one of the operators defined in (2.2), (2.4) and (2.6).

end for

Remark 2.1 Calculating $\|\nabla f_{\lambda}(x_t)\|$ in each iteration can be time consuming in the worst case. However, it is investigated by Davis et al. [15] that $\|\nabla f_{\lambda}(x_t)\|$ is proportional to more familiar quantities such as the gradient mapping: $\lambda^{-1} \|x_t - x_{t-1}\|$. In practice, we can replace the condition $\|\nabla f_{\lambda}(x_t)\| < \epsilon$ by $\lambda^{-1} \|x_t - x_{t-1}\| \leq \epsilon$, which is much easier to check.

3 Active Strict Saddles and $\epsilon$-Approximate Local Minimum

We first introduce the concepts of active manifold and strict saddles, which play an important role in characterizing nonsmooth functions.

Definition 3.1 (Active manifold [14]) Consider a closed weakly convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and fix a set $M \subseteq \mathbb{R}^d$ containing a critical point $x$ of $f$. Then $M$ is called an active $C^p$-manifold around $x$ if there exists a neighborhood $U$ around $x$ satisfying the following conditions.

1. Smoothness. The set $M \cap U$ is a $C^p$-smooth manifold and the restriction of $f$ to $M \cap U$ is $C^p$-smooth.

2. Sharpness. The lower bound holds:

$$\inf\{\|v\| : v \in \partial f(x), \ x \in U \setminus M\} > 0.$$ 

Definition 3.2 (Strict Saddles [14]) Consider a weakly convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. A critical point $x$ is a strict saddle of $f$ if there exists a $C^2$-active manifold $M$ of $f$ at $x$ and the inequality $\partial^2 f_M(x)(u) < 0$ holds for some vector $u \in T_M(x)$. A function $f$ is said to have the strict saddle property if each of its critical points is either a local minimizer or a strict saddle.
We now provide some intuitions on the geometric structure of the nonsmooth optimization. The nonsmoothness of optimization problems typically arises in a highly structured way, usually along a smooth active manifold. Geometrically, an active manifold $\mathcal{M}$ is a set that contains all points such that the objective function is smooth along the manifold, but varies sharply off it. For example, Figure 1 illustrates the surface of function $g(x, y) = |x^2 + y^2 - 1| + x$ and its active manifold is $\mathcal{M} = \{(x, y)|x^2 + y^2 = 1\}$. There is a saddle point $(1, 0)$ and a global minimum $(-1, 0)$ of function $g$ and both of them lie on the active manifold $\mathcal{M}$. The active manifold idea is widely used in many problems, for instance, the point-wise maximum of smooth function problem and the maximum eigenvalue minimization problem [32]. In [25], the authors showed that some simple algorithms such as projected gradient and proximal point methods can identify active manifold, which means all iterates after a finite number of iterations must lie on the active manifold. In this paper, we assume all critical points of the objective function lie on the active manifold (otherwise it becomes a smooth problem in the neighborhood of the critical point) and the active manifold is $C^2$ smooth in the neighborhood of each critical point.

Throughout this paper, we make the following assumption.

**Assumption 3.3** The function $f$ in (2.1), (2.3) and (2.5) satisfies the strict saddle property.

From Assumption 3.3 we know that any algorithms for solving (2.1), (2.3) and (2.5) converge to a local minimum if they can escape the strict saddle points.

We now discuss how to define an approximate local minimum for (2.1), (2.3) and (2.5). For nonconvex smooth problems, Nesterov and Polyak [37] proposed the following definition for an $\epsilon$-second-order stationary point.

**Definition 3.4** ($\epsilon$-second-order stationary point for smooth problem [37]) Assume function $f \in C^2$, and the Hessian of $f$ is Lipschitz continuous with a Lipschitz constant $\rho$. A point $x$ is called an $\epsilon$-second-order stationary point to problem $\{\min_x f(x)\}$, if it satisfies $\|\nabla f(x)\| \leq \epsilon$ and $\lambda_{\min}(\nabla^2 f(x)) > -\sqrt{\rho \epsilon}$.

This definition does not apply to our problems because we consider nonsmooth functions. In the following, we propose a novel definition of $\epsilon$-approximate local minimum for nonsmooth functions. Our definition is motivated by a recent result of Davis and Drusvyatskiy [14], which proved the following results regarding the proximal algorithms PPA (2.6), PGM (2.4) and PLM (2.6) discussed in Section 2.
Theorem 3.5 (Theorems 3.1, 4.1 and 5.3 in [14]) Assume the objective function \( f \) in problems \((2.1), (2.3)\) and \((2.5)\) is an \( \ell \)-weakly convex and \( \bar{x} \) is a critical point of \( f \). For parameters \( \lambda \in (0, \ell^{-1}) \) and sufficiently small \( \eta > 0 \), the following statements hold.

1. Consider problem \((2.1)\). Suppose that \( f \) admits a \( C^2 \) active manifold \( M \) at \( \bar{x} \). Then the proximal map \( S(\cdot) := \text{prox}_{\eta f}(\cdot) \) defined in \((2.2)\) is \( C^1 \)-smooth on a neighborhood of \( \bar{x} \). Moreover, if \( \bar{x} \) is a strict saddle point of \( f \), then \( \bar{x} \) is both a strict saddle point of \( f_\lambda \) and an unstable fixed point of the proximal map \( S \). Moreover, \( \nabla S(\bar{x}) \) has a real eigenvalue that is strictly greater than one.

2. Consider problem \((2.3)\). Suppose that \( f \) admits a \( C^2 \) active manifold \( M \) at \( \bar{x} \). Then the proximal-gradient map \( S \) defined in \((2.4)\) is \( C^1 \)-smooth on a neighborhood of \( \bar{x} \). Moreover, if \( \bar{x} \) is a strict saddle point of \( f \), then \( \nabla S(\bar{x}) \) has a real eigenvalue that is strictly greater than one.

3. Consider problem \((2.5)\). Suppose the problem admits a composite \( C^2 \) active manifold \( M \) at \( \bar{x} \). Then the proximal linear map \( S \) defined in \((2.6)\) is \( C^1 \)-smooth on a neighborhood of \( \bar{x} \). Moreover, if \( \bar{x} \) is a composite strict saddle point, then \( \nabla S(\bar{x}) \) has a real eigenvalue strictly greater than one.

Now we recall some properties of Moreau envelope.

Lemma 3.6 (Basic Properties of Moreau Envelope) Consider an \( \ell \)-weakly convex function \( f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \) and fix a parameter \( \lambda \in (0, \ell^{-1}) \). The following statements are true.

1. The envelope \( f_\lambda(\cdot) \) is \( C^1 \)-smooth with its gradient given by

\[
\nabla f_\lambda(x) = \lambda^{-1}(x - \text{prox}_{\lambda f}(x)).
\]

2. The proximal map \( \text{prox}_{\lambda f}(\cdot) \) is \( \frac{1}{1-\lambda} \)-Lipschitz continuous and the gradient map \( \nabla f_\lambda \) is Lipschitz continuous with constant \( \max\{\lambda^{-1}, \frac{\ell}{1-\lambda}\} \).

3. The critical points of \( f(\cdot) \) and \( f_\lambda(\cdot) \) coincide. In particular, they are exactly the fixed points of the proximal mapping \( \text{prox}_{\lambda f}(\cdot) \). Moreover, if \( \bar{x} \) is a local minimum of \( f_\lambda(\cdot) \), then \( \bar{x} \) is a local minimum of \( f(\cdot) \).

Theorem 3.5 and Lemma 3.6 immediately implies the following sufficient condition for local minimum of problems \((2.1), (2.3)\) and \((2.5)\).

Corollary 3.7 Assume Assumption 3.3 holds. For problems \((2.1)\) and \((2.3)\), assume function \( f(x) \) is nonsmooth \( \ell \)-weakly convex and admits a \( C^2 \) active manifold at \( \bar{x} \). For problem \((2.5)\), assume function \( f(x) \) is nonsmooth \( \ell \)-weakly convex and admits a \( C^2 \) composite active manifold at \( \bar{x} \). Then \( \bar{x} \) is a local minimum of \((2.1), (2.3)\), and \((2.5)\) if the following holds for \( \lambda \in (0, \ell^{-1}) \):

\[
\|\nabla f_\lambda(x)\| = 0, \quad \text{and} \quad \lambda_{\max}(\nabla S(\bar{x})) \leq 1.
\]

From Corollary 3.7, we can naturally define \( \epsilon \)-approximate local minimum for problems \((2.1), (2.3)\) and \((2.5)\) as follows.
Definition 3.8 (\(\epsilon\)-approximate local minimum) Assume Assumption 3.3 holds. For problems (2.1) and (2.3), assume function \(f\) is nonsmooth \(\ell\)-weakly convex and admits a \(C^2\) active manifold at \(\bar{x}\). For problem (2.5), assume function \(f\) is nonsmooth \(\ell\)-weakly convex and admits a \(C^2\) composite active manifold at \(\bar{x}\). Assume \(\nabla S(\cdot)\) is Lipschitz continuous in the neighborhood of each critical point with Lipschitz constant \(\eta\). We call \(\bar{x}\) an \(\epsilon\)-approximate local minimum of problems (2.1), (2.3) and (2.5) if the following holds for \(\lambda \in (0, \ell^{-1})\):
\[
\|\nabla f_\lambda(\bar{x})\| \leq \epsilon, \quad \lambda \max(\nabla S(\bar{x})) < 1 + \eta \sqrt{\rho \epsilon}.
\] (3.2)

Remark 3.9 Here we only assume \(\nabla S(\cdot)\) is Lipschitz continuous in the neighborhoods of all critical points. The \(\epsilon\)-approximate local minimum generalizes the concept of \(\epsilon\)-second-order stationary point [37] to nonsmooth problems. For smooth problems, the gradient descent operator is defined as \(S(x) = x - \eta \nabla f(x)\), whose Jacobian is \(\nabla S(x) = I - \eta \nabla^2 f(x)\). By replacing \(\|\nabla f_\lambda(x)\| \leq \epsilon\) in (3.2) by \(\|\nabla f(x)\| \leq \epsilon\), the conditions in (3.2) are the same as the ones required in the definition of \(\epsilon\)-second-order stationary point (Definition 3.4) for smooth problems.

4 Main Results

In this section, we provide iteration complexity results of the three perturbed proximal algorithms in Algorithm 1 for obtaining an \(\epsilon\)-approximate local minimum.

4.1 An Easy Case: Perturbed Proximal Point Algorithm

The perturbed PPA is easy to analyze, because its complexity can be obtained by directly applying the results in [28]. To see this, note that Lemma 3.6 immediately implies the following result.

Corollary 4.1 PPA (2.2) with \(\eta = \lambda\) for solving (2.1) is equivalent to
\[
x_{t+1} := x_t - \lambda \nabla f_\lambda(x_t),
\]
i.e., the gradient descent algorithm for
\[
\min f_\lambda(x),
\]
with step size \(\lambda\).

Corollary 4.1 indicates that the perturbed PPA for solving (2.1) is exactly the same as the perturbed gradient method for solving (4.1). Therefore, we can give the iteration complexity of perturbed PPA using the iteration complexity of the perturbed gradient method from [28].

Theorem 4.2 Denote the optimal value of (4.1) as \(f_\lambda^\star\). Assume function \(f(\cdot)\) is nonsmooth \(\ell\)-weakly convex and admits a \(C^2\)-smooth manifold around all its critical points. This implies that \(f_\lambda(\cdot)\) is twice differentiable and its Hessian is Lipschitz continuous around all its critical points. We use \(\rho\) to denote the Lipschitz constant of the Hessian of \(f_\lambda(\cdot)\). For any \(\epsilon \in (0, 1/\lambda^2 \rho)\) and \(\delta \in (0, 1)\), we set the parameters as
\[
\lambda < \frac{\ell^{-1}}{2}, \quad \eta = \lambda, \quad r = \frac{\epsilon}{400 \sqrt{\epsilon}} \cdot \eta, \quad \mathcal{T} = \frac{1}{\lambda \sqrt{\rho \epsilon}} \cdot \eta, \quad \mathcal{F} = \frac{1}{50 \epsilon^3 \sqrt{\rho}} \cdot \mathcal{T} = \frac{1}{4 \eta} \sqrt{\frac{\epsilon}{\rho}}.
\] (4.2)
Algorithm | Objective | Model function $f_x(y)$
---|---|---
Prox-point | $f(x)$ | $f(y)$
Prox-gradient | $g(x) + r(x)$ | $g(x) + \langle \nabla g(x), y - x \rangle + m(y)$
Prox-linear | $h(F(x)) + r(x)$ | $h(F(x) + \nabla F(x)(y - x)) + m(y)$

Table 1: The model functions $f_x(y)$ for the three problems (2.1), (2.3) and (2.5).

where $\iota = c \cdot \log(d(f_\lambda(x_0) - f_\lambda^*)/(\lambda \rho \delta))$, and $c$ is a sufficiently large absolute constant. We set the number of iterations as

$$T = 4 \max \left\{ \frac{(f_\lambda(x_0) - f_\lambda^*)}{\mathcal{F}/(2\mathcal{F})}, \frac{(f_\lambda(x_0) - f_\lambda^*)}{\eta \epsilon^2/2} \right\} = O \left( \frac{(f_\lambda(x_0) - f_\lambda^*)}{\lambda \epsilon^2} \cdot (\log d)^4 \right).$$

Then the Perturbed PPA (Algorithm 1 with $S(x) = \text{prox}_{\eta f}(x)$) has the property that with probability at least $1 - \delta$, at least one half of its iterates in the first $T$ iterations satisfy

$$\|\nabla f_\lambda(x)\| \leq \epsilon, \quad \lambda_{\min}(\nabla^2 f_\lambda(x)) > -\sqrt{\rho \epsilon}. \quad (4.3)$$

This indicates that $x$ is an $\epsilon$-approximate local minimum of problem (2.1) according to Definition 3.8.

### 4.2 Iteration Complexity of the Three Perturbed Proximal Algorithms

In this section, we provide a unified analysis of the iteration complexity of the three perturbed proximal algorithms presented in Algorithm 1 for obtaining an $\epsilon$-approximate local minimum. For the ease of presentation, we define in Table 1 some model functions for the three proximal algorithms. By making assumptions on the function model $f_x(\cdot)$, we can analyze all proximal algorithms in a unified manner.

The following assumption is from [14] and is assumed throughout this section.

**Assumption 4.3 ([14])** Function $f(\cdot)$ is nonsmooth weakly convex and admits a $C^2$ (composite) active manifold around all critical points.

For all $x \in \mathbb{R}^d$, there exists a constant $\beta > 0$ such that the function model $f_x : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ satisfies

$$\|f(y) - f_x(y)\| \leq \frac{\beta}{2} \|y - x\|^2, \quad \text{for all } y \in \mathbb{R}^d.$$  

The function model $f_x : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ itself is $\mu$-weakly convex.

When the objective function $f(\cdot)$ admits a $C^2$ active manifold around critical points, the operators $S(\cdot)$ defined in (2.2), (2.4) and (2.6) are $C^1$ smooth. Here, we make one additional assumption that $\nabla S$ is Lipschitz continuous.

**Assumption 4.4** There exists a constant $\rho > 0$ such that the following inequality holds in the neighborhood $\mathcal{U}$ of all critical points:

$$\|\nabla S(x) - \nabla S(y)\| \leq \eta \rho \|x - y\|, \quad \forall x, y \in \mathcal{U},$$

where $\eta = \lambda$ in (2.2).
Note that when \( f \) is a smooth function, this assumption reduces to the Hessian Lipschitz assumption.

Now we are ready to present our main iteration complexity result.

**Theorem 4.5** Suppose function \( f(\cdot) \) and its model function \( f_x(\cdot) \) satisfy Assumptions 4.3 and 4.4. For any \( \epsilon \in (0, L^2/\rho) \) and \( \delta \in (0, 1) \), we set the following parameters:

\[
L > \frac{7}{2} \beta + 3\mu, \quad \lambda = \left(1 - \frac{\lambda - 3 + \beta}{2L + 1 + (\beta + 2\mu)}\right)^{-1},
\]

\[
\theta_1 = \frac{(L - \lambda^{-1} + \beta) \lambda^2 L^2}{(L - \lambda^{-1} - \beta)}, \quad \theta_2 = \frac{(\lambda^{-1} - \beta - \mu)L\lambda}{(\lambda^{-1} + L - \beta - 2\mu)}, \quad \eta = \frac{1}{L}, \quad r = \frac{\theta_2}{\theta_1} \cdot \frac{\epsilon}{400\sqrt{\rho}}, \quad \mathcal{T} = \frac{L}{\sqrt{\rho\epsilon}} \cdot \iota, \quad \mathcal{F} = \frac{\theta_2}{\theta_1} \cdot \frac{1}{50\iota^3} \sqrt{\frac{\epsilon^3}{\rho}}, \quad \mathcal{J} = \frac{1}{4\iota^4} \sqrt{\frac{\epsilon}{\rho}}.
\]

where

\[
\iota = c \cdot \log(dL(f_\lambda(x_0) - f_\lambda^*))/((\rho\delta)),
\]

and \( c \) is a sufficiently large absolute constant. Moreover, we set the number of iterations as

\[
T = 4 \max \left\{ \left(\frac{(f_\lambda(x_0) - f_\lambda^*)}{\mathcal{F}/(2\mathcal{T})}, \frac{(f_\lambda(x_0) - f_\lambda^*)}{\theta_2\eta^2/2}\right) \right\} = O \left(\frac{(f_\lambda(x_0) - f_\lambda^*)}{\theta_2\epsilon^2} \cdot (\log d)^4\right).
\]

Then the Perturbed Proximal algorithms (Algorithm 1) for all three proximal operators have the property that with probability at least \( 1 - \delta \), at least one half of its iterations will be \( \epsilon \)-approximate local minimum (Definition 3.8).

### 4.3 Proof Sketch

In [28], Jin et al. analyzed the iteration complexity of the perturbed gradient descent method for minimizing smooth functions. Our proof here largely follows the one in [28]. The new ingredient is to extend the results in the smooth case to the nonsmooth case by utilizing properties of the Moreau envelope and proximal operators.

Following the main proof ideas of [28], we give the following two lemmas which guarantee sufficient decrease of the Moreau envelope.

**Lemma 4.6 (Sufficient Decrease of Moreau envelope)** With parameters chosen as in (4.4), we have

\[
\|x_{t+1} - x_t\|^2 \leq \theta_1 \eta^2 \|\nabla f_\lambda(x_t)\|^2,
\]

where \( \theta_1 = \frac{(L - \lambda^{-1} + \beta) \lambda^2 L^2}{(L - \lambda^{-1} - \beta)} \). Moreover, we have

\[
f_\lambda(x_{t+1}) \leq f_\lambda(x_t) - \frac{\theta_2}{2L} \|\nabla f_\lambda(x_t)\|^2,
\]

where \( \theta_2 = \frac{(\lambda^{-1} - \beta - \mu)L\lambda}{(\lambda^{-1} + L - \beta - 2\mu)} > 0 \).

**Lemma 4.7 (Escaping saddle point)** Suppose Assumptions 4.3 and 4.4 hold for \( f(\cdot) \), and \( \tilde{x} \) satisfies \( \|\nabla f_\lambda(\tilde{x})\| < \epsilon \), and \( \lambda_{\max}(\nabla S(\tilde{x})) > 1 + \eta\sqrt{\rho} \). Let \( x_0 = \tilde{x} + \eta \xi \) (\( \xi \sim \text{Uniform}(B(0, r)) \)). Run the update \( x_{t+1} = S(x_t) \) as in (2.2), (2.4) and (2.6) starting from \( x_0 \). We have

\[
\mathbb{P}(f_\lambda(x, \mathcal{F}) - f_\lambda(\tilde{x}) \leq -\mathcal{F}/2) \geq 1 - \frac{L\sqrt{d}}{\sqrt{\rho\epsilon}} \cdot \iota^2 2^{8-\iota},
\]

where \( x_{\mathcal{F}} \) is the \( \mathcal{F} \)-th proximal iterate starting from \( x_0 \).
Remark 4.8 Inequality (4.8) guarantees the sufficient decrease of the Moreau envelope when $\|\nabla f_\lambda(x)\|$ is large, which is deterministic. On the other hand, Lemma 4.7 shows that, after random perturbation and running the proximal updates for $T$ iterations, the Moreau envelope has sufficient decrease with high probability when stuck in a saddle point. Since the Moreau envelope can be decreased by at most $f_\lambda(x_0) - f^*_\lambda$, it is straightforward to give a bound for those iterations that are not approximate local minimums.

5 Conclusion and Future Directions

In this paper, we have presented a novel definition of $\epsilon$-approximate local minimum for nonsmooth weakly convex functions based on the recent work on active strict saddles [14]. Following the idea of perturbed gradient descent method [28], we have proposed a unified algorithm framework for perturbed proximal algorithms, namely, perturbed proximal point, perturbed proximal gradient and perturbed proximal linear algorithms for obtaining an $\epsilon$-approximate local minimum. We have proved that the proposed perturbed proximal algorithms achieve a gradient computational cost of $\tilde{O}(\epsilon^{-2})$ for finding an $\epsilon$-approximate local minimum. With this novel characterization of local minimum for nonsmooth functions in hand, we give some possible future directions:

1. Following Fang et al. [20] for smooth problems, one can attempt to extend our algorithms to their stochastic versions for nonsmooth problems. It is interesting to analyze the complexity of these stochastic proximal algorithms for obtaining an $\epsilon$-approximate local minimum of nonsmooth weakly convex functions.

2. Recently, Fang et al. [19] combined the negative curvature search with variance reduction technique for stochastic algorithms to find a local minimum and reach a remarkable rate of $\tilde{O}(\epsilon^{-3})$. It will be interesting to adopt this technique for escaping saddle points for nonsmooth problems.

3. Another line of research for finding local minimum of smooth functions is to apply second order method [37]. It will be also interesting to investigate if second order proximal methods can find local minimum with better convergence rate.
References

[1] Naman Agarwal, Zeyuan Allen-Zhu, Brian Bullins, Elad Hazan, and Tengyu Ma. Finding approximate local minima faster than gradient descent. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 1195–1199, 2017.

[2] Dmitrii Avdiukhin, Chi Jin, and Grigory Yaroslavtsev. Escaping saddle points with inequality constraints via noisy sticky projected gradient descent. Optimization for Machine Learning Workshop, 2019.

[3] Afonso S Bandeira, Nicolas Boumal, and Vladislav Voroninski. On the low-rank approach for semidefinite programs arising in synchronization and community detection. In Conference on learning theory, pages 361–382, 2016.

[4] Srinadh Bhojanapalli, Behnam Neyshabur, and Nati Srebro. Global optimality of local search for low rank matrix recovery. In Advances in Neural Information Processing Systems, pages 3873–3881, 2016.

[5] Nicolas Boumal, Vlad Voroninski, and Afonso Bandeira. The non-convex burer-monteiro approach works on smooth semidefinite programs. In Advances in Neural Information Processing Systems, pages 2757–2765, 2016.

[6] Léopold Cambier and P-A Absil. Robust low-rank matrix completion by riemannian optimization. SIAM Journal on Scientific Computing, 38(5):S440–S460, 2016.

[7] Yair Carmon, John C Duchi, Oliver Hinder, and Aaron Sidford. Convex until proven guilty: Dimension-free acceleration of gradient descent on non-convex functions. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 654–663. JMLR. org, 2017.

[8] Yair Carmon, John C Duchi, Oliver Hinder, and Aaron Sidford. Accelerated methods for nonconvex optimization. SIAM Journal on Optimization, 28(2):1751–1772, 2018.

[9] Anna Choromanska, Mikael Henaff, Michael Mathieu, Gérard Ben Arous, and Yann LeCun. The loss surfaces of multilayer networks. In Artificial intelligence and statistics, pages 192–204, 2015.

[10] Christopher Criscitiello and Nicolas Boumal. Efficiently escaping saddle points on manifolds. In Advances in Neural Information Processing Systems, pages 5985–5995, 2019.

[11] Frank E Curtis, Daniel P Robinson, and Mohammadreza Samadi. A trust region algorithm with a worst-case iteration complexity of $o(e^{-3/2})$ for nonconvex optimization. Mathematical Programming, 162(1-2):1–32, 2017.

[12] Hadi Daneshmand, Jonas Kohler, Aurelien Lucchi, and Thomas Hofmann. Escaping saddles with stochastic gradients. arXiv preprint arXiv:1803.05999, 2018.

[13] Yann N Dauphin, Razvan Pascanu, Caglar Gulcehre, Kyunghyun Cho, Surya Ganguli, and Yoshua Bengio. Identifying and attacking the saddle point problem in high-dimensional nonconvex optimization. In Advances in neural information processing systems, pages 2933–2941, 2014.
[14] Damek Davis and Dmitriy Drusvyatskiy. Active strict saddles in nonsmooth optimization. *arXiv preprint arXiv:1912.07146*, 2019.

[15] Damek Davis and Dmitriy Drusvyatskiy. Stochastic model-based minimization of weakly convex functions. *SIAM Journal on Optimization*, 29(1):207–239, 2019.

[16] Damek Davis, Dmitriy Drusvyatskiy, and Courtney Paquette. The nonsmooth landscape of phase retrieval. *arXiv preprint arXiv:1711.03247*, 2017.

[17] Simon S Du, Chi Jin, Jason D Lee, Michael I Jordan, Aarti Singh, and Barnabas Poczos. Gradient descent can take exponential time to escape saddle points. In *Advances in neural information processing systems*, pages 1067–1077, 2017.

[18] John C Duchi and Feng Ruan. Solving (most) of a set of quadratic equalities: Composite optimization for robust phase retrieval. *Information and Inference: A Journal of the IMA*, 8(3):471–529, 2019.

[19] Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. Spider: Near-optimal non-convex optimization via stochastic path-integrated differential estimator. In *Advances in Neural Information Processing Systems*, pages 689–699, 2018.

[20] Cong Fang, Zhouchen Lin, and Tong Zhang. Sharp analysis for nonconvex sgd escaping from saddle points. *arXiv preprint arXiv:1902.00247*, 2019.

[21] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points - online stochastic gradient for tensor decomposition. In *Conference on Learning Theory*, pages 797–842, 2015.

[22] Rong Ge, Chi Jin, and Yi Zheng. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 1233–1242. JMLR. org, 2017.

[23] Rong Ge, Jason D Lee, and Tengyu Ma. Matrix completion has no spurious local minimum. In *Advances in Neural Information Processing Systems*, pages 2973–2981, 2016.

[24] Quanquan Gu, Zhaoran Wang Wang, and Han Liu. Low-rank and sparse structure pursuit via alternating minimization. In *Artificial Intelligence and Statistics*, pages 600–609, 2016.

[25] Warren L Hare and Adrian S Lewis. Identifying active manifolds. *Algorithmic Operations Research*, 2(2):75–75, 2007.

[26] Zhishen Huang and Stephen Becker. Perturbed proximal descent to escape saddle points for non-convex and non-smooth objective functions. In *INNS Big Data and Deep Learning conference*, pages 58–77. Springer, 2019.

[27] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 1724–1732. JMLR. org, 2017.

[28] Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M Kakade, and Michael I Jordan. On nonconvex optimization for machine learning: Gradients, stochasticity, and saddle points. *arXiv preprint arXiv:1902.04811*, pages 1–31, 2019.
[29] Chi Jin, Praneeth Netrapalli, and Michael I Jordan. Accelerated gradient descent escapes saddle points faster than gradient descent. arXiv preprint arXiv:1711.10456, 2017.

[30] Yenson Lau, Qing Qu, Han-Wen Kuo, Pengcheng Zhou, Yuqian Zhang, and John Wright. Short-and-sparse deconvolution—a geometric approach. arXiv preprint arXiv:1908.10959, 2019.

[31] Jason D Lee, Max Simchowitz, Michael I Jordan, and Benjamin Recht. Gradient descent only converges to minimizers. In Conference on learning theory, pages 1246–1257, 2016.

[32] Adrian S Lewis. Active sets, nonsmoothness, and sensitivity. SIAM Journal on Optimization, 13(3):702–725, 2002.

[33] Xiao Li, Zhihui Zhu, Anthony Man-Cho So, and Rene Vidal. Nonconvex robust low-rank matrix recovery. SIAM Journal on Optimization, 30(1):660–686, 2020.

[34] Yuanzhi Li and Yang Yuan. Convergence analysis of two-layer neural networks with relu activation. In Advances in neural information processing systems, pages 597–607, 2017.

[35] Song Mei, Theodor Misiakiewicz, Andrea Montanari, and Roberto I Oliveira. Solving sdps for synchronization and maxcut problems via the grothendieck inequality. In Conference on learning theory, pages 1476–1515, 2017.

[36] Aryan Mokhtari, Asuman Ozdaglar, and Ali Jadbabaie. Escaping saddle points in constrained optimization. In Advances in Neural Information Processing Systems, pages 3629–3639, 2018.

[37] Yurii Nesterov and Boris T Polyak. Cubic regularization of newton method and its global performance. Mathematical Programming, 108(1):177–205, 2006.

[38] R Tyrrell Rockafellar and Roger J-B Wets. Variational analysis, volume 317. Springer Science & Business Media, 2009.

[39] Ju Sun, Qing Qu, and John Wright. Complete dictionary recovery over the sphere i: Overview and the geometric picture. IEEE Transactions on Information Theory, 63(2):853–884, 2016.

[40] Ju Sun, Qing Qu, and John Wright. A geometric analysis of phase retrieval. Foundations of Computational Mathematics, 18(5):1131–1198, 2018.

[41] Yue Sun, Nicolas Flammarion, and Maryam Fazel. Escaping from saddle points on riemannian manifolds. In Advances in Neural Information Processing Systems, pages 7274–7284, 2019.

[42] Xinyang Yi, Dohyung Park, Yudong Chen, and Constantine Caramanis. Fast algorithms for robust pca via gradient descent. In Advances in neural information processing systems, pages 4152–4160, 2016.

[43] Yuqian Zhang, Yenson Lau, Han-wen Kuo, Sky Cheung, Abhay Pasupathy, and John Wright. On the global geometry of sphere-constrained sparse blind deconvolution. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 4894–4902, 2017.
A Preliminaries on nonsmooth optimization

We review some key definitions and tools for nonsmooth nonconvex optimization.

**Definition A.1 (Subdifferential and Subderivative [38])** Consider a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. The subdifferential of $f$ at $x$, denoted as $\partial f(x)$, consists of all vectors $v$ satisfying
\[
    f(y) \geq f(x) + \langle v, x - y \rangle + o(\|x - y\|) \quad \text{as } y \to x. \tag{A.1}
\]
The subderivative of function $f$ at $x$ in the direction $\bar{u} \in \mathbb{R}^d$ is
\[
    df(x)(\bar{u}) := \liminf_{t \to 0, u \to \bar{u}} \frac{f(x + tu) - f(x)}{t}. \tag{A.2}
\]
The parabolic subderivative of $f$ at $x$ for $\bar{u} \in \text{dom } df(x)$ with respect to $\bar{w}$ is
\[
    d^2f(x)(\bar{u}|\bar{w}) := \liminf_{t \to 0, w \to \bar{w}} \frac{f(x + tu + \frac{1}{2}t^2w) - f(x) - df(x)(\bar{u})}{\frac{1}{2}t^2}. \tag{A.3}
\]

B Proofs of the Main Results

B.1 Proof of Lemma 3.6

**Proof.** Claims 1 and 2 follow from [14, Lemma 2.5]. Now we prove Claim 3. By (3.1) and the observation that function $y \mapsto f(y) + \frac{1}{2\lambda} \|y - x\|^2$, we know that all critical points of $f_\lambda(\cdot)$ and $f(\cdot)$ coincide. Moreover, if $\bar{x}$ is a local minimum of $f_\lambda(\cdot)$, then (3.1) implies that $\bar{x} = \text{prox}_{\lambda f}(\bar{x})$. Therefore,
\[
    f_\lambda(\bar{x}) = \min_{y \in \mathbb{R}^d} f(y) + \frac{1}{2\lambda} \|y - \bar{x}\|^2 = f(\text{prox}_{\lambda f}(\bar{x})) + \frac{1}{2\lambda} \|\text{prox}_{\lambda f}(\bar{x}) - \bar{x}\|^2 = f(\bar{x}). \tag{A.1}
\]
Hence, if $\bar{x}$ is a local minimum of $f_\lambda(\cdot)$, then there exists some constant $\delta > 0$ such that for any $x$ in the neighborhood of $\bar{x}$: $\mathcal{U} = \{x | 0 < \|x - \bar{x}\| \leq \delta\}$, we have $f_\lambda(x) \geq f_\lambda(\bar{x})$. This further implies that
\[
    f(x) \geq f_\lambda(x) \geq f_\lambda(\bar{x}) = f(\bar{x}),
\]
where the first inequality is due to the fact that $f_\lambda(\cdot)$ is always a lower bound to $f(\cdot)$. \qed

B.2 Proof of Theorem 4.5

**Proof.** The proof largely follows [28], which proved similar results for the smooth case. We need to extend the results to nonsmooth case. We use the Moreau envelope to measure the first-order stationarity. We show that the three perturbed proximal algorithms make sufficient decrease on the norm of the Moreau envelope with high probability if the current iterate is not a local minimum. For $x \in \mathbb{R}^d$, we have three possible cases:

1. When $\|\nabla f_\lambda(x)\| > \epsilon$, we show that the regular proximal update can guarantee a sufficient decrease on the Moreau envelope $f_\lambda$.
2. When $\|\nabla f_\lambda(x)\| \leq \epsilon$ and $\lambda_{\max}(\nabla S(x)) > 1 + \eta\sqrt{\epsilon}$, $x$ is an unstable fixed point, which corresponds to a strict saddle point or a local maximum. In this case, we show that running a perturbed proximal update will guarantee a sufficient decrease in the next $T$ iterations with high probability.
3. When \( \|\nabla f_\lambda(x)\| \leq \epsilon \) and \( \lambda_{\text{max}}(\nabla S(x)) \leq 1 + \eta\sqrt{p\epsilon} \), we have reached an approximate local minimum.

The case (i) can be proved by Lemma 4.6. When \( \|\nabla f_\lambda(x_t)\| > \epsilon \), from (4.8) we have

\[
f_\lambda(x_{t+1}) \leq f_\lambda(x_t) - \frac{\theta_2}{2L} \|\nabla f_\lambda(x_t)\|^2 \leq f_\lambda(x_t) - \frac{\theta_2\epsilon^2}{2L}.
\]

That is, there is a sufficient reduction of \( f_\lambda(x_t) \) after one proximal update. Note that \( f_\lambda \) can be decreased at most \( f_\lambda(x_0) - f_\lambda^* \). Therefore, the number of iterations in which case (i) happens is at most \( \frac{(f_\lambda(x_0) - f_\lambda^*)^2}{\theta_2\epsilon^2(2L)} \leq T/4 \), where \( T \) is defined in (4.6).

For case (ii), from Lemma 4.7 we know that with high probability, adding a stochastic perturbation can guarantee a decrease of the Moreau envelope by at least \( (f_\lambda(x_0) - f_\lambda^*) \frac{F}{\delta(T/2)} \). Choose a large enough absolute constant \( c \) such that:

\[
(TL\sqrt{d}/\sqrt{p\epsilon})\cdot 2^{8-\ell} \leq \delta,
\]

where \( T \) is defined in (4.6), and \( \ell \) is defined in (4.5). We see that with probability \( 1 - \delta \), we add the stochastic perturbation in at most \( \frac{(f_\lambda(x_0) - f_\lambda^*)^2}{\theta_2\epsilon^2(2L)} \leq T/4 \) iterations. That is, the number of iterations in which case (ii) happens is at most \( T/4 \).

Finally, we conclude that case (iii) happens in at least \( T/2 \) iterations. That is, at least \( T/2 \) of the iterates in the first \( T \) iterations must be \( \epsilon \)-approximate local minimum. As defined in (4.6), the total number of iterations \( T \) relies on the problem dimension \( d \) polylogarithmically.

### B.3 Proof of Lemma 4.6

To prove Lemma 4.6, we need to prove the following lemma first.

**Lemma B.1** Suppose Assumption 4.6 holds and parameters \( L \) and \( \lambda \) are chosen as in (4.4), we have the following three point inequality:

\[
(L - \lambda^{-1} - \beta) \|x_{t+1} - x_t\|^2 \leq (L - \lambda^{-1} + \beta) \|\bar{x}_t - x_t\|^2 - (L + \lambda^{-1} - \beta - 2\mu) \|\bar{x}_t - x_{t+1}\|^2,
\]

where \( \bar{x}_t := \text{prox}_{\lambda f}(x_t) \).

**Proof.** Under the Assumption 4.3 we know that function \( f(\cdot) \) is \((\beta + \mu)\)-weakly convex. With parameter \( \eta \) chosen in (4.4) and the model functions \( f_x \) defined in Table 1, proximal updates in (2.2), (2.4) and (2.6) can be written as:

\[
x_{t+1} = \text{argmin}_{y \in \mathbb{R}^d} \left\{ f_x(y) + \frac{L}{2} \|y - x\|^2 \right\}.
\]

From (4.4) we have \( \lambda^{-1} = \frac{1}{2}L + \frac{1}{4}(\beta + 2\mu) > \beta + \mu \), and therefore function \( y \mapsto f(y) + \frac{\lambda^{-1}}{2} \|y - x_t\|^2 \) is \((\lambda^{-1} - \beta - \mu)\)-strongly convex. We have

\[
\frac{\lambda^{-1} - \beta - \mu}{2} \|\bar{x}_t - x_{t+1}\|^2 \leq \left( f(x_{t+1}) + \frac{\lambda^{-1}}{2} \|x_{t+1} - x_t\|^2 \right) - \left( f(\bar{x}_t) + \frac{\lambda^{-1}}{2} \|\bar{x}_t - x_t\|^2 \right).
\]

(2.2)
Using part 2 of Assumption 4.3, we have

\[ f(x_{t+1}) \leq f(x_t) + \frac{\beta}{2} \| x_{t+1} - x_t \|^2, \text{ and } -f(\bar{x}_t) \leq -f(x_t) + \frac{\beta}{2} \| \bar{x}_t - x_t \|^2. \]  

(B.3)

Combining (B.3) and (B.2) yields

\[ \frac{\lambda^{-1} - \beta - \mu}{2} \| \bar{x}_t - x_{t+1} \|^2 \leq f(x_t(x_{t+1} - f(x_t(\bar{x}_t) + \frac{\lambda^{-1} + \beta}{2} \| x_{t+1} - x_t \|^2 - \frac{\lambda^{-1} - \beta}{2} \| \bar{x}_t - x_t \|^2. \]  

(B.4)

On the other hand, we have \( L > \mu \) and function \( y \mapsto f(x_t(y) + \frac{L}{2} \| y - x_t \|^2 \) is \((L-\mu)\)-strongly convex and \( x_{t+1} \) is its minimizer. So we have

\[ f(x_t(x_{t+1}) - f(x_t(\bar{x}_t)) \leq \frac{L}{2} \| \bar{x}_t - x_t \|^2 - \frac{L}{2} \| x_{t+1} - x_t \|^2 - \frac{L - \mu}{2} \| x_{t+1} - \bar{x}_t \|^2. \]  

Combining (B.4) and (B.5), we have

\[ \frac{\lambda^{-1} - \beta - \mu}{2} \| \bar{x}_t - x_{t+1} \|^2 \leq \frac{L - \lambda^{-1} + \beta}{L - \lambda^{-1} - \beta} \| \bar{x}_t - x_t \|^2 \leq \frac{L - \lambda^{-1} + \beta}{L - \lambda^{-1} - \beta} \| \bar{x}_t - x_t \|^2 \]

\[ + \frac{\lambda^{-1} + \beta}{2} \| x_{t+1} - x_t \|^2 - \frac{\lambda^{-1} - \beta}{2} \| \bar{x}_t - x_t \|^2, \]

which leads to (B.1) and completes the proof of Lemma B.1

We now prove Lemma 4.6

Proof. We note that with \( L > \frac{5}{2} \beta + \mu \), and \( \lambda^{-1} = \frac{1}{2}L + \frac{1}{4}(\beta + 2\mu) \), we have the coefficients \( L - \lambda^{-1} - \beta > 0, L - \lambda^{-1} + \beta > 0 \), and \( L + \lambda^{-1} - \beta - 2\mu > 0 \). Thus, (B.1) implies

\[ \| x_{t+1} - x_t \|^2 \leq \frac{L - \lambda^{-1} + \beta}{L - \lambda^{-1} - \beta} \| \bar{x}_t - x_t \|^2 = \frac{L - \lambda^{-1} + \beta}{L - \lambda^{-1} - \beta} \| \bar{x}_t - x_t \|^2 = \theta_1 \eta^2 \| \nabla f(x_t) \|^2, \]

where \( \theta_1 = (L - \lambda^{-1} + \beta) \lambda^2 L^2 \). This completes the proof of (4.7).

Next, we prove (4.8). Note that

\[ f_\lambda(x_{t+1}) = f(\bar{x}_{t+1}) + \frac{\lambda^{-1}}{2} \| \bar{x}_{t+1} - x_{t+1} \|^2 \leq f(\bar{x}_t) + \frac{\lambda^{-1}}{2} \| \bar{x}_t - x_{t+1} \|^2, \]

where the inequality is due to \( \bar{x}_{t+1} := \text{prox}_f(x_{t+1}) \). Combining (B.6) and (B.1) yields

\[ \| \bar{x}_t - x_{t+1} \|^2 \leq \frac{L - \lambda^{-1} + \beta}{L + \lambda^{-1} - \beta - 2\mu} \| \bar{x}_t - x_t \|^2 - \frac{L - \lambda^{-1} - \beta}{L + \lambda^{-1} - \beta - 2\mu} \| x_{t+1} - x_t \|^2 \]

\[ = \| \bar{x}_t - x_t \|^2 - \frac{2(\lambda^{-1} - \beta - \mu)}{L + \lambda^{-1} - \beta - 2\mu} \| \bar{x}_t - x_t \|^2 - \frac{(L - \lambda^{-1} - \beta) \lambda^2}{L + \lambda^{-1} - \beta - 2\mu} \| \nabla f(x_t) \|^2 \]  

(B.7)

Substituting (B.7) into (B.6) gives

\[ f_\lambda(x_{t+1}) \leq f(\bar{x}_t) + \frac{\lambda^{-1}}{2} \| \bar{x}_t - x_t \|^2 - \frac{\lambda^{-1}}{2} \frac{(L - \lambda^{-1} - \beta) \lambda^2}{L + \lambda^{-1} - \beta - 2\mu} \| \nabla f(x_t) \|^2 \]

\[ = f_\lambda(x_t) - \frac{(L - \lambda^{-1} - \beta) \lambda}{2(L + \lambda^{-1} - \beta - 2\mu)} \| \nabla f(x_t) \|^2 \]

\[ = f_\lambda(x_t) - \frac{\theta_2}{2L} \| \nabla f(x_t) \|^2, \]

where \( \theta_2 = \frac{(L - \lambda^{-1} - \beta) \lambda L}{(L + \lambda^{-1} - \beta - 2\mu)} > 0 \), which completes the proof. \( \square \)
B.4 Proof of Lemma 4.7

The main idea of Lemma 4.7 is again from [28] for the perturbed gradient descent method, and the proof also largely follows from [28]. The “improve or localize” idea shows that if the Moreau envelope does not decrease a lot, then the sequence \( x_t \) must stay in a neighborhood of \( x_0 \). Utilizing this idea, we show that the escaping area of the perturbation ball is small. We summarize the whole idea in the following two lemmas.

The first Lemma introduces the idea of improve or localize:

**Lemma B.2 (Improve or Localize)** Under the setting of Lemma 4.6, for any \( t \geq \tau > 0 \):

\[
\| x_\tau - x_0 \| \leq \sqrt{\frac{\theta_1}{\theta_2} \cdot 2\eta t (f_\lambda(x_0) - f_\lambda(x_t))}.
\]

*Proof.* We have

\[
\| x_\tau - x_0 \| \leq \sum_{\tau=1}^{t} \| x_\tau - x_{\tau-1} \| \leq \left[ \sum_{\tau=1}^{t} \| x_\tau - x_{\tau-1} \|^2 \right]^{1/2} \leq \sqrt{\frac{\theta_1}{\theta_2} \cdot 2\eta t (f_\lambda(x_0) - f_\lambda(x_t))},
\]

where the first inequality comes from the triangle inequality, the second inequality is due to \((\sum_{\tau=1}^{t} \| a_\tau \|^2)^{1/2} \leq t \sum_{\tau=1}^{t} \| a_\tau \|^2\), and the last two inequalities are from Lemma 4.6. \( \square \)

The second lemma shows that when two coupling sequences initiated along the maximal eigendirection with a small distance, at least one of them can make sufficient decrease on the Moreau envelope.

**Lemma B.3 (Coupling sequence)** Suppose \( f \) satisfies Assumptions 4.3 and 4.4, \( \hat{x} \) satisfies \( \| \nabla f_\lambda(\hat{x}) \| < \epsilon \), and \( \lambda_{\max}(\nabla S(\hat{x})) > 1 + \eta \sqrt{\rho} \). Let \( \{x_t\}, \{x'_t\} \) be two proximal update sequences that satisfy: (i) \( \max\{\| x_0 - \hat{x} \|, \| x'_0 - \hat{x} \| \} \leq \eta r_0 \); and (ii) \( x_0 - x'_0 = \eta r_0 e_1 \), where \( e_1 \) is the eigenvector corresponding to the largest eigenvalue of \( \nabla S(\hat{x}) \) and \( r_0 > \omega := 2^{-t} L^2 \). Then:

\[
\min\{f_\lambda(x_\tau) - f_\lambda(x_0), f_\lambda(x'_\tau) - f_\lambda(x'_0)\} \leq -\mathcal{F}.
\]

*Proof.* We prove it by contradiction. First assume that \( \min\{f_\lambda(x_\tau) - f_\lambda(x_0), f_\lambda(x'_\tau) - f_\lambda(x'_0)\} > -\mathcal{F} \). By Lemma B.2 we have

\[
\max\{\| x_t - \hat{x} \|, \| x'_t - \hat{x} \| \} \leq \max\{\| x_t - x_0 \|, \| x'_t - x'_0 \| \} + \{\| x_0 - \hat{x} \|, \| x'_0 - \hat{x} \| \} \leq \sqrt{\frac{\theta_1}{\theta_2} \cdot 2\eta \mathcal{F} + \eta r_0} \leq \mathcal{F}.
\]

The last inequality holds due to the choices of the parameters in [4.4] and that we can choose \( \epsilon \) to be sufficiently large so that \( \eta r_0 \) is small enough compared to the first term. On the other hand, we write the difference \( \hat{x}_t := x_t - x'_t \) and update it as:

\[
\hat{x}_{t+1} = S(x_t) - S(x'_t) = \mathcal{H}\hat{x}_t + \Delta_t \hat{x}_t
\]

\[
= \mathcal{H}^{t+1}\hat{x}_0 + \sum_{\tau=0}^{\tau=t} \mathcal{H}^{t-\tau} \Delta_{\tau} \hat{x}_{\tau},
\]

(4.5)
where \( \mathcal{H} = \nabla S(\tilde{x}) \) and \( \Delta_t = \int_0^1 |\nabla S(x_t' + \theta(x_t - x_t')) - \mathcal{H}| d\theta \). Next we show that \( \|p(t)\| \) is the dominant term by showing
\[
\|q(t)\| \leq \|p(t)\|/2, \quad t \in [\mathcal{T}].
\]
By induction, \( \|q(0)\| = 0 \leq \|q(0)\|/2 = \|p(0)\|/2 \). Therefore, \( \text{(B.10)} \) holds when \( t = 0 \). We assume it still holds for some \( t > 0 \). Denote \( \lambda_{\max}(\nabla S(\tilde{x})) = 1 + \eta\gamma \), so that \( \gamma > \sqrt{\rho} \). Since \( \tilde{x}_0 \) lies in the direction of maximum eigenvector of \( \nabla S(\tilde{x}) \), for any \( \tau \leq t \), we have:
\[
\|\hat{x}_{t}\| \leq \|p(\tau)\| + \|q(\tau)\| \leq 2\|p(\tau)\| = 2\|\mathcal{H}^t\tilde{x}_0\| = 2(1 + \eta\gamma)^t\eta r_0.
\]

By assumption \( \text{[4.4]} \), we have
\[
\|\Delta_t\| \leq \int_0^1 \|\nabla S(x_t' + \theta(x_t - x_t')) - \nabla S(\tilde{x})\| d\theta \\
\leq \eta\rho \int_0^1 \|\theta(x_t - \tilde{x}) + (1 - \theta)(x_t' - \tilde{x})\| \\
\leq \eta\rho \max\{\|x_t - \tilde{x}\|, \|x_t' - \tilde{x}\|\} \\
\leq \eta\rho \mathcal{T},
\]

hence
\[
\|q(t + 1)\| = \left\| \sum_{\tau = 0}^t \mathcal{H}^{t-\tau}\Delta_{\tau}\hat{x}_{\tau} \right\| \\
\leq \eta\rho \mathcal{T} \sum_{\tau = 0}^t \left\| \mathcal{H}^{t-\tau}\| \|\hat{x}_{\tau}\| \\
\leq 2\eta\rho \mathcal{T} \sum_{\tau = 0}^t (1 + \eta\gamma)^{t-\tau}\eta r_0 \\
\leq 2\eta\rho \mathcal{T} \|p(t + 1)\|,
\]
and we have \( 2\eta\rho \mathcal{T} \leq 1/2 \), which completes the proof.

Finally, we have
\[
\max\{\|x_t - x_0\|, \|x_t' - x_0\|\} \geq \frac{1}{2} \|\hat{x}(\mathcal{T})\| \geq \frac{1}{2} \|p(\mathcal{T})\| - \|q(\mathcal{T})\| \geq \frac{1}{4} \|p(\mathcal{T})\|
\]
\[
= \frac{(1 + \eta\gamma)^{\mathcal{T}}\eta r_0}{4} \geq 2^{t-2}\eta r_0 \geq \mathcal{T},
\]
which contradicts with \( \text{(B.8)} \). So we complete the proof. \( \square \)

Now we prove Lemma \( \text{[4.7]} \).

**Proof.** We define the stuck region on the ball \( B_\tilde{x}(\eta r) \) as
\[
\mathcal{X}_{\text{stuck}} := \{x \in B_\tilde{x}(\eta r)\}|\{x_i\} \text{is the iterate sequence with } x_0 = x, \text{ and } f_\lambda(x_\mathcal{T}) - f_\lambda(\tilde{x}) > -\mathcal{T},
\]
and show that this region is a small part of the surface of the ball. According to Lemma \( \text{[B.3]} \), we know that the width of \( \mathcal{X}_{\text{stuck}} \) along the \( e_1 \) direction is at most \( \eta\omega \). Therefore,
\[
\Pr(x_0 \in \mathcal{X}_{\text{stuck}}) = \frac{\text{Vol}(\mathcal{X}_{\text{stuck}})}{\text{Vol}(B_\tilde{x}(\eta r))} \leq \frac{\eta\omega \times \text{Vol}(B_\tilde{x}^{d-1}(\eta r))}{\text{Vol}(B_0^d(\eta r))}
\]
\[
= \frac{\omega}{r} \frac{\Gamma\left(\frac{d}{2} + 1\right)}{\Gamma\left(\frac{d}{2} + \frac{1}{2}\right)} \leq \frac{\omega}{r} \sqrt{\frac{d}{\pi}} \leq \frac{L\sqrt{d}}{\sqrt{\rho} \epsilon} \cdot 2^{d-1}.
\]

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From (4.4) we have \( \lambda^{-1} > \frac{\beta + \mu}{1 - \lambda(\beta + \mu)} \) and \( L > \lambda^{-1} \). By Lemma 3.6, \( f_\lambda \) is \( L \)-smooth. For those \( \{x_0 \notin X_{\text{stuck}}\} \), since \( f_\lambda \) is \( L \)-smooth, we have:

\[
f_\lambda(x_T) - f_\lambda(\tilde{x}) = [f_\lambda(x_T) - f_\lambda(x_0)] + [f_\lambda(x_0) - f_\lambda(\tilde{x})] \leq -\mathcal{F} + \epsilon r + \frac{L\eta^2 r^2}{2} \leq -\mathcal{F}/2, \quad (B.17)
\]

where in the last inequality, we choose \( \iota \) to be large enough so that the third term can be omitted. This completes the proof. \( \square \)