Reduced critical Bellman-Harris branching processes for small populations

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Abstract

Let \( \{Z(t), t \geq 0\} \) be a critical Bellman-Harris branching process with finite variance for the offspring size of particles. Assuming that \( 0 < Z(t) \leq \varphi(t) \) as \( t \to \infty \) or \( \varphi(t) = at, a > 0 \), we study the structure of the process \( \{Z(s, t), 0 \leq s \leq t\} \), where \( Z(s, t) \) is the number of particles in the process at moment \( s \) in the initial process which either survive up to moment \( t \) or have a positive offspring number at this moment.

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1 Introduction and main results

Let \( \{Z(t), t \geq 0\} \) be a Bellman-Harris branching process with \( Z(0) = 1 \) specified by the probability generating function

\[
 f(s) = E s^Z = \sum_{k=0}^{\infty} f_k s^k
\]

and the distribution \( G(t) = P(\tau \leq t) \) of the life-length \( \tau \) of a particle.

Introduce the following hypothesis:

**Condition A1** (Criticality)

\( E\xi = 1, \quad \sigma^2 := Var\xi \in (0, \infty) \).

**Condition A2.** The support of the distribution \( G(t) \) is contained on the integer lattice \( t = 0, 1, 2, ... \) with maximal step 1 and is not degenerate.

Let \( \mu := E\tau \) and

\[
 F(t; s) := E \left[ s^{Z(t)} | Z(0) = 1 \right]
\]

be the probability generating function for the number of particles in the process at moment \( t \). It is known (see, for instance, [2]) that if Condition A1 is valid and \( t^2 (1 - G(t)) \to 0 \) as \( t \to \infty \) then

\[
 Q(t) := 1 - F(t; 0) = P( Z(t) > 0 ) \sim \frac{2\mu}{\sigma^2 t} \quad \text{as} \quad t \to \infty
\]

and, for any \( \lambda \geq 0 \)

\[
 \lim_{t \to \infty} E \left[ e^{-2\lambda Z(t)/\sigma^2 t} | Z(t) > 0 \right] = \frac{1}{1 + \lambda}
\]

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meaning that the limiting distribution of the scaled process $2\mu Z(t)/\sigma^2$ given $\{Z(t) > 0\}$ is exponential with parameter 1.

In this note we study the asymptotic properties of the so-called reduced critical Bellman-Harris process $\{Z(s, t), 0 \leq s \leq t\}$, where $Z(s, t)$ is the number of particles at moment $s$ in the initial process which either survive up to moment $t$ or have a positive offspring number at this moment.

Note that reduced processes for ordinary Galton–Watson branching processes (i.e., for the case $\mathbf{P}(\tau = 1) = 1$) were introduced by Fleischmann and Prehn [6]. Various properties of such processes were analyzed in [3], [4], [5], [7], [8], [9], [10], [11], [14] and some other papers.

Reduced critical Bellman-Harris processes were investigated by Vatutin [16] for the single-type case and by Sagitov [13] for multitype setting.

All these papers do not consider the situation when the size of the population at moment $n$ is bounded from above. Recently, Liu and Vatutin [12] study the structure of the Galton-Watson critical reduced process under the condition that the size of the population is bounded and positive at the moment of observation. In the present paper we consider a similar problem for the critical Bellman-Harris processes.

Introduce the event

$$\mathcal{H}(t) := \{0 < Z(t) \leq B\varphi(t)\}$$

where

$$B = \frac{\sigma^2}{2\mu}.$$

Our main results are contained in two theorems which we formulate below.

**Theorem 1.1** Let Conditions A1-A2 be valid,

$$\mathbf{E}\xi^2 \log(\xi + 1) < \infty, \quad \mathbf{E}\tau^3 < \infty,$$

and $\varphi(t), t > 0$, be a monotone increasing function, $\varphi(t) = o(t)$ as $t \to \infty$. If, in addition,

$$\lim_{t \to \infty} t\left(1 - G(\varepsilon\varphi(t))\right) = 0 \quad (1.4)$$

for any $\varepsilon > 0$, then for any fixed $j \geq 1$ and $y > 0$

$$\lim_{t \to \infty} \mathbf{P}(Z(t - y\varphi(t), t) = j|\mathcal{H}(t)) = \frac{y}{(j - 1)!} \int_0^y z^{j-1}e^{-z}dz. \quad (1.5)$$

**Remark 1.** For the case of the ordinary Galton-Watson processes this statement was proved in [12].

**Remark 2.** It will be shown in Lemma 2 below that

$$\mathbf{P}(\mathcal{H}(t)) \sim \frac{\varphi(t)}{Bt^2}.$$

Hence (1.4) may be rewritten as

$$\lim_{t \to \infty} \frac{t^3 \left(1 - G(\varepsilon\varphi(t))\right)}{\varphi(t)} = 0.$$
Let \( \beta(t) := \max \{0 \leq s < t : Z(s,t) = 1 \} \) be the birth moment of the so-called most recent common ancestor (MRCA) of all particles existing in the population at moment \( t \) and let \( d(t) := t - \beta(t) \) be the distance from the point of observation \( t \) to the birth moment of the MRCA.

Taking \( j = 1 \) in Theorem 1.1 and observing that \( \{d(t) \leq y \varphi(t)\} = \{Z((1-x)t,t) = j\} \) we obtain the following statement.

**Corollary 1.2** If the conditions of Theorem 1.1 are valid then for any \( y > 0 \)

\[
\lim_{t \to \infty} P(d(t) \leq y \varphi(t) | H(t)) = y \left(1 - \exp \left(-\frac{1}{y} \right)\right).
\]

(1.6)

Our next theorem deals with the case \( \varphi(t) = Bat \) for some \( a > 0 \). Here much stronger statement may be proved.

**Theorem 1.3** If Condition A1 is valid, the function \( G(t) \) is non-lattice and

\[
\lim_{t \to \infty} t^2 (1 - G(t)) = 0,
\]

(1.7)

then for any fixed \( a > 0 \), \( j \geq 1 \) and \( x \in (0,1) \)

\[
\lim_{t \to \infty} P(Z(xt,t) = j | 0 < Z(t) < Bat) = \frac{1}{(j-1)!} \int_{0}^{1-x} z^{j-1} e^{-z} dz \times \frac{(1-x)x^{j-1}}{1 - e^{-a}}.
\]

(1.8)

Taking \( j = 1 \) in Theorem 1.3 and observing that \( \{d(t) \leq xt\} = \{Z((1-x)t,t) = 1\} \) we obtain the following statement:

**Corollary 1.4** If the conditions of Theorem 1.3 are valid then for any \( x \in (0,1) \)

\[
\lim_{t \to \infty} P(d(t) \leq xt | 0 < Z(t) < Bat) = \frac{x \left(1 - e^{-a/x} \right)}{1 - e^{-a}}.
\]

(1.9)

The remaining part of the paper looks as follows. In Section 2 we prove some auxiliary results. Sections 3 and 4 contain proofs of Theorems 1.1 and 1.3 respectively. We note that Lemmas 3, 4 and Theorem 1.3 are proved by V.Vatutin, all other results are established by Wenming Hong and Yao Ji.

**2 Auxiliary results**

We write

\[
P(Z(t - By \varphi(t), t) = j | H(t)) = \frac{P(H(t) | Z(t - By \varphi(t), t) = j) P(Z(t - By \varphi(t), t) = j)}{P(H(t))}.
\]

Our aim is to investigate separately the asymptotic behavior of each probability at the right-hand side of this equality.

We start our arguments by the following lemma due to Topchii [1].
Lemma 1 If
\[ E\tau^3 < \infty, \ E\xi = 1, \ \sigma^2 > 0, \ E\xi^2 \log(\xi + 1) < \infty, \]
and \( G \) is a nondegenerate lattice distribution with span 1 then, as \( t \to \infty \)
\[ t^2 e^{\frac{1}{Bt}} P(Z(t) = k) - \frac{1}{B^2} \to 0 \]
uniformly in \( 0 < k \leq Ct < \infty \). Besides, there exists a constant \( C_1 < \infty \) such that
\[ \sup_{k > 0, t \geq 0} t^2 P(Z(t) = k) \leq C_1 < \infty. \] (2.3)

Note that the condition \( E\tau^3 < \infty \) in the lemma cannot be reduced to \( \mu = E\tau < \infty \). Indeed, if, for instance,
\[ 1 - G(t) \sim \frac{c}{t^\beta} \]
as \( t \to \infty \) then, for \( 1 < \beta \leq 2 \) and each fixed \( k \) there exists
\[ \lim_{t \to \infty} t^{\beta/2} P(Z(t) = k) \in (0, \infty) \]
(see [15]), while if \( 2 < \beta < 3 \) then
\[ \lim_{t \to \infty} t^{\beta-1} P(Z(t) = k) \in (0, \infty) \]
if \( k(\beta - 1) \leq 1 \) and
\[ \lim_{t \to \infty} t^2 P(Z(t) = k) \in (0, \infty) \]
if \( k(\beta - 1) > 1 \) (see [17]).

In what follows we agree to understand (if otherwise is not stated) the symbol \( \sim \) as \( \sim_t \).

Lemma 2 If conditions (2.1) are valid and \( G \) is a nondegenerate lattice distribution with span 1 and \( \varphi(t) = o(t) \) as \( t \to \infty \) then
\[ P(H(t)) \sim \frac{\varphi(t)}{B t^2}; \] (2.4)
2) if the conditions of Theorem 1.3 are valid then
\[ P(0 < Z(t) < Bat) \sim (1 - e^{-a}) P(Z(t) > 0) \sim \frac{1 - e^{-a}}{Bt} \] (2.5)
for any \( a > 0 \).

Proof. Using Lemma 1 we conclude that
\[ P(H(t)|Z(0) = 1) = \sum_{1 \leq k \leq B\varphi(t)} P(Z(t) = k|Z(0) = 1) \sim \frac{1}{B^2 t^2} \sum_{1 \leq k \leq B\varphi(t)} 1 \sim \frac{\varphi(t)}{B t^2} \]
proving (2.4).
To check (2.5) we recall that by (1.3)

\[
\lim_{t \to \infty} P(0 < Z(t) < B; t | Z(t) > 0) = 1 - e^{-a}
\]

and use (1.2).

Using Lemma 1 we prove the following statement.

Lemma 3 Assume that the conditions of Theorem 1.1 are valid and \( \psi(t) \to \infty \) as \( t \to \infty \) in such a way that \( \psi(t)t^{-1} \to 0 \). Then

\[
F\left(t; 1 - \frac{1}{\psi(t)} \right) - F(t; 0) \sim \frac{\psi(t)}{B^2t^2}.
\]

Proof. We have

\[
F\left(t; 1 - \frac{1}{\psi(t)} \right) = \sum_{k=0}^{\infty} P(Z(t) = k) \left( 1 - \frac{1}{\psi(t)} \right)^k.
\]

By the inequality \( 1 - x \leq e^{-x}, x \geq 0 \), we conclude that

\[
z^k = \left( 1 - \frac{1}{\psi(t)} \right)^k \leq \exp\left( -\frac{k}{\psi(t)} \right).
\]

This and (2.3) imply for any fixed \( N \) and sufficiently large \( t \) :

\[
\sum_{k>\psi(t)N} P(Z(t) = k)z^k \leq C_1 \sum_{k>\psi(t)N} \frac{1}{t^2} \exp\left( -\frac{k}{\psi(t)} \right)
\]

\[
= e^{-N} \frac{C_1}{t^2} \left( 1 - e^{-1/\psi(t)} \right)^{-1} \leq 2e^{-N} C_1 \frac{\psi(t)}{t^2}.
\]

and

\[
\sum_{0<k<\varepsilon \psi(t)} P(Z(t) = k)z^k \leq \sum_{0<k<\varepsilon \psi(t)} P(Z(t) = k) \leq \varepsilon C_1 \frac{\psi(t)}{t^2}.
\]

The intermediate term with \( \varepsilon \psi(t) < k < N\psi(t) \) is evaluated as

\[
\sum_{\varepsilon \psi(t) < k < N\psi(t)} P(Z(t) = k)z^k \sim \frac{1}{(Bt)^2} \sum_{\varepsilon \psi(t) < k < N\psi(t)} z^k
\]

\[
\leq \frac{\psi(t)}{(Bt)^2} \left( 1 - \frac{1}{\psi(t)} \right)^{\varepsilon \psi(t)} - \left( 1 - \frac{1}{\psi(t)} \right)^{N\psi(t)+1}
\]

\[
\sim \frac{\psi(t)}{(Bt)^2} \left( e^{-\varepsilon} - e^{-N} \right).
\]

Combining (2.6) - (2.8) and letting \( \varepsilon \downarrow 0 \) and \( N \uparrow \infty \) we obtain the statement of the lemma.

For convenience of references we recall Faà di Bruno’s formula for the derivatives of composite functions:
If \( i_r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( r = 1, 2, \ldots, k \), \( I_k := i_1 + \cdots + i_k \) and 
\[
D(k) := \{(i_1, \ldots, i_k) : 1 \cdot i_1 + 2 \cdot i_2 + \cdots + k \cdot i_k = k\},
\]
then for the derivatives of the composition \( H(T(z)) \) of the functions \( H(\cdot) \) and \( T(\cdot) \) we have 
\[
\frac{d^k}{dz^k}[H(T(z))] = \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} H(I_k)(T(z)) \prod_{r=1}^k \left( \frac{T^{(r)}(z)}{r!} \right)^{i_r}.
\]

(2.9)

The next lemma is crucial for the proof of Theorem 1.1.

**Lemma 4** If \( \psi(t) \to \infty \) as \( t \to \infty \) in such a way that \( \psi(t)t^{-1} \to 0 \) and the conditions of Theorem 1.1 are valid then, for any fixed \( k \in \mathbb{N} \)
\[
F^{(k)}(t; f(F(\psi(t)))) \sim \frac{(B\psi(t))^{k+1}}{B^2t^2k!}.
\]

(2.10)

**Proof.** It follows from Lemma 3 that, for any positive \( \lambda \)
\[
\lim_{t \to \infty} \frac{t^2}{\psi(t)} \left[ F \left( t; 1 - \frac{\lambda}{B\psi(t)} \right) - F(t; 0) \right] = \frac{1}{B\lambda}.
\]
Set for brevity \( F(t) := F(t; 0) \) and take a fixed \( \lambda > 0 \). Since \( f'(1) = 1 \) and \( f(F(\psi(t))) \to 1 \) as \( t \to \infty \), it follows that
\[
1 - f^\lambda(F(\psi(t))) = 1 - (1 - (1 - f(F(\psi(t))))^\lambda
\]
\[
\sim \lambda (1 - f(F(\psi(t))) \sim \lambda (1 - F(\psi(t))) \sim \frac{\lambda}{B\psi(t)}.
\]

Hence, setting for brevity \( w(t) := f(F(\psi(t))) \) we get for any positive \( \lambda \)
\[
\lim_{t \to \infty} \frac{t^2}{\psi(t)} \left[ F(t; w^\lambda(t)) - F(t; 0) \right] = \frac{1}{B\lambda}.
\]

(2.11)

Since the prelimiting and limiting functions in (2.11) are analytical in the complex domain \( \mathbb{C} \) \( \lambda > 0 \), the derivatives of any order of the prelimiting functions with respect to \( \lambda \) converge to the derivatives of the respective order of the limiting function. Hence it follows that
\[
\lim_{t \to \infty} \frac{\partial^k}{\partial \lambda^k} \left( \frac{t^2}{\psi(t)} \left[ F \left( t; w^\lambda(t) \right) - F(t; 0) \right] \right) = \lim_{t \to \infty} \frac{t^2}{\psi(t)} \frac{\partial^k}{\partial \lambda^k} F \left( t; w^\lambda(t) \right)
\]
\[
= \lim_{t \to \infty} \frac{t^2}{\psi(t)} \frac{\partial^k}{\partial \lambda^k} F \left( t; w^\lambda(t) \right) = (-1)^k \frac{k!}{B\lambda^{k+1}}.
\]

(2.12)

In particular, for \( k = 1 \)
\[
\frac{t^2}{\psi(t)} \frac{\partial}{\partial \lambda} F \left( t; w^\lambda(t) \right) = \frac{t^2}{\psi(t)} F' \left( t; w^\lambda(t) \right) w^\lambda(t) \log w(t)
\]
\[
\sim (-1) \frac{1}{B\lambda^2}.
\]

(2.13)

Using the equivalences \( \log(1 - x) \sim -x \) as \( x \downarrow 0 \) and
\[
1 - f \left( F(\psi(t)) \right) \sim 1 - F(\psi(t)) = P(Z(\psi(t)) > 0) \sim \frac{1}{B\psi(t)}
\]

(2.14)
as $t \to \infty$ it is not difficult to deduce from (2.13) with $\lambda = 1$ that

$$F'(t; w(t)) \sim \frac{\psi'(t)}{t^2} = \frac{(B\psi(t))^2}{B^2t^2} \times 1!$$

proving the lemma for $k = 1$.

Assume that the asymptotic representation

$$F^{(r)}(t; w(t)) \sim \frac{(B\psi(t))^{r+1}}{B^2t^2} r!$$

is valid for all $r < k$. By Faà di Bruno’s formula (2.9) we have

$$\frac{t^2}{\psi(t)} \frac{\partial^k}{\partial \lambda^k} [F(t; w^\lambda(t))]$$

$$= \frac{t^2}{\psi(t)} \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} F^{(I_k)}(t; w^\lambda(t)) \prod_{r=1}^k \left( \frac{1}{r!} \frac{\partial^r}{\partial \lambda^r} w^\lambda(t) \right)^{i_r}$$

$$= \frac{t^2}{\psi(t)} \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} F^{(I_k)}(t; w^\lambda(t)) \prod_{r=1}^k \left( \frac{w^\lambda(t)}{r!} \log^r w(t) \right)^{i_r}$$

$$= \frac{t^2}{\psi(t)} \log^k w(t) \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} F^{(I_k)}(t; w^\lambda(t)) w^{\lambda I_k}(t) \prod_{r=1}^k \frac{1}{(r!)^{i_r}}$$

$$\sim (-1)^k \frac{t^2}{\psi(t)} \left( \frac{1}{B\psi(t)} \right)^k \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} F^{(I_k)}(t; f(F(\psi(t)))) w^{\lambda I_k}(t) \prod_{r=1}^k \frac{1}{(r!)^{i_r}}.$$ 

Set $D'(k) = D(k) \setminus \{(k, ..., 0, 0)\}$. In view of induction hypothesis (recall (2.10)) and the estimate $I_k = i_1 + \cdots + i_k \leq k - 1$ valid for all $(i_1, ..., i_k) \in D'(k)$ we see that

$$\lim_{t \to \infty} \frac{t^2}{\psi(t)} \left( \frac{1}{B\psi(t)} \right)^k \sum_{D'(k)} \frac{k!}{i_1! \cdots i_k!} F^{(I_k)}(t; f(F(\psi(t)))) f_{I_k}(F(\psi(t))) \prod_{r=1}^k \frac{1}{(r!)^{i_r}}$$

$$\leq \sum_{D'(k)} \frac{k!}{i_1! \cdots i_k!} \lim_{t \to \infty} \frac{t^2}{\psi(t)} \left( \frac{1}{B\psi(t)} \right)^k F^{(I_k)}(t; f(F(\psi(t)))) = 0.$$ 

Hence, setting $\lambda = 1$ we conclude by (2.12) that

$$(-1)^k \frac{k!}{B} \sim \frac{t^2}{\psi(t)} \frac{\partial^k}{\partial \lambda^k} [F(t; w^\lambda(t))] \bigg|_{\lambda=1}$$

$$= o(1) + \frac{t^2}{\psi(t)} \left( \frac{1}{B\psi(t)} \right)^k (-1)^k \frac{k!}{k! \cdots 0!} F^{(k)}(t; w(t)) w^k(t)$$

$$= o(1) + \frac{t^2}{\psi(t)} \left( \frac{1}{B\psi(t)} \right)^k (-1)^k F^{(k)}(t; w(t)).$$

Therefore,

$$F^{(k)}(t; f(F(\psi(t)))) = F^{(k)}(t; w(t)) \sim \frac{(B\psi(t))^{k+1}}{B^2t^2} k!$$

that completes the induction step and proves Lemma 4.
We now consider a Bellman-Harris branching process which is initiated at time \( t = 0 \) by a *random* number of particles distributed the same as \( \xi \) specified by \( f(s) \) in (1.1). The initial particles as well as the other particles have life-length distribution \( G(t) \). Each particle of the process produces children at the end of its life in accordance with probability generating function \( f(s) \). We denote this new process as \( Y(t) \). Clearly,

\[
T(t; s) := \mathbb{E}[s^{Y(t)}] = f(F(t; s))
\]

and, as a result

\[
\mathbb{P}(Y(t) > 0) = 1 - f(F(t; 0)) \sim 1 - F(t; 0) \sim \frac{1}{Bt}
\]

and, in view of

\[
\mathbb{E}\left[e^{-2\mu \lambda t}/\sigma^2 | Y(t) > 0\right] = \frac{\mathbb{E}\left[e^{-2\mu \lambda Y(t)/\sigma^2} | Y(t) > 0\right]}{\mathbb{P}(Y(t) > 0)} = \frac{T(t; e^{-2\mu \lambda t/\sigma^2}) - T(t; 0)}{1 - T(t; 0)} = \frac{f\left(F\left(t; e^{-2\mu \lambda t/\sigma^2}\right)\right) - f(F(t; 0))}{1 - f(F(t; 0))}
\]

and (1.3)

\[
\lim_{t \to \infty} \mathbb{E}\left[e^{-2\mu \lambda t}/\sigma^2 | Y(t) > 0\right] = 1 - \lim_{t \to \infty} \frac{1 - f\left(F\left(t; e^{-2\mu \lambda t/\sigma^2}\right)\right)}{1 - f(F(t; 0))} = 1 - \lim_{t \to \infty} \frac{1 - F\left(t; e^{-2\mu \lambda t/\sigma^2}\right)}{1 - F(t; 0)} = \lim_{t \to \infty} \mathbb{E}\left[e^{-2\mu \lambda Z(t)/\sigma^2} | Z(t) > 0\right] = \frac{1}{1 + \lambda}.
\]

Hence, the limiting conditional distribution of the process \( Y(t) \) given \( \{Y(t) > 0\} \) is exponential with parameter 1.

Let \( Z^*(t, x) \) be the number of particles existing in the process at moment \( t \), which will exist at moment \( t + x \).

The following statement, showing that under the conditions of Theorem 1.3 the probability that there is a particle at time \( t \) which will survive up to moment \( t + \varepsilon t \) is negligible with \( \mathbb{P}(Z(t) > 0) \), is a particular case of Lemma 1 in [16].

**Lemma 5** If the conditions of Theorem 1.3 are valid then for any \( \varepsilon > 0 \),

\[
\lim_{t \to \infty} \frac{\mathbb{P}(Z^*(t, \varepsilon t) > 0)}{\mathbb{P}(Z(t) > 0)} = 0.
\]

We complement Lemma 5 by the following result:

**Lemma 6** If the conditions of Theorem 1.1 are valid then for any \( \varepsilon > 0 \),

\[
\lim_{t \to \infty} \frac{\mathbb{P}(Z^*(t, \varepsilon \varphi(t)) > 0)}{\mathbb{P}(\bar{H}(t))} = 0.
\]
Proof. Let \( \tilde{Z}(t, x) \) be the number of particles at moment \( t \) whose age does not exceed \( x \). Setting
\[
F(t, x; s) := \mathbb{E} \left[ s^{\tilde{Z}(t,x)} | Z(0) = 1 \right]
\]
and introducing the notation \( J(y) = 1 \) for \( y \geq 0 \) and \( J(y) = 0 \) for \( y < 0 \), we deduce by the total probability formula the integral equation
\[
F(t, x; s) = (1 - G(t))[sJ(x-t) + 1 - J(x-t)] + \int_0^t f(F(t-u, x; s))dG(u).
\]

Denoting \( A(t, x) := \mathbb{E}\tilde{Z}(t, x) \) we conclude by the previous relation that
\[
A(t, x) = (1 - G(t))J(x-t) + \int_0^t A(t-u, x)dG(u).
\]
Solving this renewal type equation gives
\[
A(t, x) = \int_0^t (1 - G(t-u))J(x-(t-u))dU(u),
\]
where \( U(t) = \sum_{k=0}^{\infty} G^k(t) \). In particular,
\[
\mathbb{E}Z(t) = A(t) = A(t, t) = 1 = \int_0^t (1 - G(t-u))dU(u).
\]

We know that
\[
\mathbb{E}[Z(t + \varepsilon \varphi(t))] = 1 = \int_0^{t+\varepsilon \varphi(t)} (1 - G(t + \varepsilon \varphi(t) - u))dU(u)
\]
and
\[
\mathbb{E}[\tilde{Z}(t + \varepsilon \varphi(t), \varepsilon \varphi(t))] = A(t + \varepsilon \varphi(t), \varepsilon \varphi(t)) = \int_0^{t+\varepsilon \varphi(t)} (1 - G(t + \varepsilon \varphi(t) - u))J(\varepsilon \varphi(t) - (t + \varphi(t)\varepsilon) - u)dU(u) = \int_t^{t+\varepsilon \varphi(t)} (1 - G(t + \varepsilon \varphi(t) - u))dU(u).
\]

Since
\[
Z^*(t, \varepsilon \varphi(t)) = Z(t + \varepsilon \varphi(t)) - \tilde{Z}(t + \varepsilon \varphi(t), \varepsilon \varphi(t))
\]
for any \( \varepsilon > 0 \), it follows by Markov inequality that
\[
\mathbb{P}(Z^*(t, \varepsilon \varphi(t) \geq 1) \leq \mathbb{E}Z^*(t, \varepsilon \varphi(t)) = \mathbb{E}[Z(t + \varepsilon \varphi(t)) - \tilde{Z}(t + \varepsilon \varphi(t), \varepsilon \varphi(t))]
\]
\[
= \int_0^t (1 - G(t + \varepsilon \varphi(t) - u))dU(u)
\]
\[
\leq U(t)(1 - G(\varepsilon \varphi(t))) \leq C \frac{1}{\mu} (1 - G(\varepsilon \varphi(t))) = o \left( \mathbb{P}(\mathcal{H}(t)) \right)
\]
in view of (1.4) and the asymptotic relation \( U(t) \sim t \mu^{-1} \) as \( t \to \infty \) being valid by the key renewal theorem for the renewal function \( U(t) \) with finite mean \( \mu \) for the increments.

Lemma 6 is proved.
3 Proof of Theorem 1.1

Let \( \zeta_i := \zeta_i(t - y\varphi(t)), i = 1, 2, \ldots, Z(t - y\varphi(t)) \) be the remaining life-lengths of the particles existing in the process at moment \( t - y\varphi(t) \). We fix \( \varepsilon > 0 \) and \( y > 0 \) and introduce the event

\[
C(t, y, \varepsilon) := \left\{ \max_{1 \leq i \leq Z(t - y\varphi(t))} \zeta_i \leq \varepsilon \varphi(t) \right\}
\]

and the event \( \bar{C}(t, y, \varepsilon) \) complementary to \( C(t, y, \varepsilon) \). In view of Lemma [3] and monotonicity of \( \varphi(t) \)

\[
\lim_{t \to \infty} \frac{P(\bar{C}(t, y, \varepsilon))}{P(\bar{H}(t))} = \lim_{t \to \infty} \frac{P(\bar{C}(t, y, \varepsilon))}{P(\bar{H}(t - y\varphi(t)))} \frac{P(H(t - y\varphi(t)))}{P(\bar{H}(t))} = 0. \tag{3.1}
\]

Thus, for any \( j \geq 1 \)

\[
P(Z(t - y\varphi(t), t) = j) = P(Z(t - y\varphi(t), t) = j; C(t, y, \varepsilon)) + o(P(H(t))). \tag{3.2}
\]

Set

\[
C_k(t, y, \varepsilon) := C(t, y, \varepsilon) \cap \{ Z(t - y\varphi(t)) = k \}.
\]

Then, for \( k \geq j \)

\[
P(C_k(t, y, \varepsilon); Z(t - y\varphi(t), t) = j)
= E[P(C_k(t, y, \varepsilon); Z(t - y\varphi(t), t) = j \mid \zeta_i, i = 1, 2, \ldots, k)]
= P(C_k(t, y, \varepsilon))
\times E\left[ \prod_{0 \leq i_1 < i_2 < \ldots < i_j \leq k \in \{i_1, \ldots, i_j\}} (1 - f(F(y\varphi(t) - \zeta_i))) \prod_{1 \leq k \leq k \in \{i_1, \ldots, i_j\}} f(F(y\varphi(t) - \zeta_i)) | C_k(t, y, \varepsilon) \right]
\geq P(C_k(t, y, \varepsilon)) C_k^j E\left[ (1 - f(F(y\varphi(t))))^j f^{k-j}(F((y - \varepsilon) \varphi(t))) | C_k(t, y, \varepsilon) \right]
= P(C_k(t, y, \varepsilon)) C_k^j(1 - f(F(y\varphi(t))))^j f^{k-j}(F((y - \varepsilon) \varphi(t)))
\geq P(Z(t - y\varphi(t)) = k) C_k^j(1 - f(F(y\varphi(t))))^j f^{k-j}(F((y - \varepsilon) \varphi(t)))
- P(Z(t - y\varphi(t) = k, \bar{C}(t, y, \varepsilon)).
\]

By the same arguments we get,

\[
P(C_k(t, y, \varepsilon), Z(t - y\varphi(t), t) = j) \leq P(C_k(t, y, \varepsilon)) C_k^j(1 - f(F((y - \varepsilon) \varphi(t))))^j f^{k-j}(F(y\varphi(t)))
\leq P(Z(t - y\varphi(t)) = k) C_k^j(1 - f(F((y - \varepsilon) \varphi(t))))^j f^{k-j}(F(y\varphi(t))).
\]

As a result we obtain

\[
P(Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon)) = \sum_{k=j}^{\infty} P(C_k(t, y, \varepsilon); Z(t - y\varphi(t), t) = j)
\leq \sum_{k=j}^{\infty} P(Z(t - y\varphi(t)) = k) C_k^j(1 - f(F((y - \varepsilon) \varphi(t))))^j f^{k-j}(F(y\varphi(t)))
= \frac{(1 - f(F((y - \varepsilon) \varphi(t))))^j}{j!} F^{(j)}(t - y\varphi(t); f(F(y\varphi(t)))) \tag{3.3}
\]
and
\[ P(Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon)) \geq \sum_{k=j}^{\infty} C_k^j (1 - f(F(y\varphi(t))))^j f^{k-j}(F((y - \varepsilon) \varphi(t))) P(Z(t - y\varphi(t)) = k) \]
\[ - \sum_{k=j}^{\infty} P(Z(t - y\varphi(t)) = k, \tilde{C}(t, y, \varepsilon)) \]
\[ \geq \frac{(1 - f(F(y\varphi(t))))^j}{j!} F^{(j)}(t - y\varphi(t); f(F((y - \varepsilon) \varphi(t)))) - P(\tilde{C}(t, y, \varepsilon)). \]

We know by Lemma 4 that
\[ F^{(j)}(t - y\varphi(t); f(F(y\varphi(t)))) \sim \frac{(B y\varphi(t))^{j+1}}{B^2 t^2} j!. \]

Thus, in view of (3.3)
\[ \limsup_{t \to \infty} \frac{P(Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon))}{P(\mathcal{H}(t))} \]
\[ \leq \limsup_{t \to \infty} \frac{B t^2}{\varphi(t)} \times \frac{1}{B} y\varphi(t) \left( \frac{y\varphi(t)}{\varphi(t)(y - \varepsilon)} \right)^j \frac{1}{t^2} = y \left( \frac{y}{y - \varepsilon} \right)^j \]

and by (3.4)
\[ \liminf_{t \to \infty} \frac{P(Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon))}{P(\mathcal{H}(t))} \]
\[ \geq \liminf_{t \to \infty} \left[ \frac{t^2 B}{\varphi(t)} \times \frac{1}{B} (\varphi(t)(y - \varepsilon)) \left( \frac{\varphi(t)(y - \varepsilon)}{y\varphi(t)} \right)^j \frac{1}{t^2} - \frac{P(\tilde{C}(t, y, \varepsilon))}{P(\mathcal{H}(t))} \right] \]
\[ = (y - \varepsilon) \left( \frac{y - \varepsilon}{y} \right)^j. \]

Hence, letting \( \varepsilon \to 0 \), we conclude
\[ \lim_{t \to \infty} \frac{P(Z(t - y\varphi(t), t) = j)}{P(\mathcal{H}(t))} = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{P(Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon))}{P(\mathcal{H}(t))} = y. \]

Let now \( Y_1^*(t), \ldots, Y_j^*(t) \) be a tuple of i.i.d. random variables distributed as \( \{Y(t) | Y(t) > 0\} \), and let \( \eta_1, \ldots, \eta_j \) be i.i.d. random variables having exponential distributed with parameter 1. It follows that
\[ \lim_{t \to \infty} P(\mathcal{H}(t) | Z(t - y\varphi(t), t) = j; C(t, y, \varepsilon)) \]
\[ = \lim_{t \to \infty} P \left( \sum_{i=1}^{j} Y_i^*(y\varphi(t) - \zeta_i) \leq B \varphi(t) | C(t, y, \varepsilon) \right) \]
\[ = \lim_{t \to \infty} P \left( \sum_{i=1}^{j} \frac{Y_i^*(y\varphi(t) - \zeta_i)}{B(y\varphi(t) - \zeta_i)} \frac{(y\varphi(t) - \zeta_i)}{y\varphi(t)} \leq \frac{1}{y} C(t, y, \varepsilon) \right). \]
Since
\[ P \left( \sum_{i=1}^{j} \frac{Y^*_i(y\varphi(t) - \zeta_i)}{B(y\varphi(t) - \zeta_i)} \frac{(y\varphi(t) - \zeta_i)}{y\varphi(t)} \leq \frac{1}{y} |C(t, y, \varepsilon)| \right) \leq P \left( \sum_{i=1}^{j} \frac{Y^*_i(y\varphi(t) - \zeta_i)}{B(y\varphi(t) - \zeta_i)} \frac{y - \varepsilon}{y} \leq \frac{1}{y} |C(t, y, \varepsilon)| \right) \]
and
\[ P \left( \sum_{i=1}^{j} \frac{Y^*_i(y\varphi(t) - \zeta_i)}{B(y\varphi(t) - \zeta_i)} \frac{(y\varphi(t) - \zeta_i)}{y\varphi(t)} \leq \frac{1}{y} |C(t, y, \varepsilon)| \right) \geq P \left( \sum_{i=1}^{j} \frac{Y^*_i(y\varphi(t) - \zeta_i)}{B(y\varphi(t) - \zeta_i)} \leq \frac{1}{y} |C(t, y, \varepsilon)| \right), \]
we conclude by (1.3) that
\[ \lim_{\varepsilon \to 0} \lim_{t \to \infty} P \left( \sum_{i=1}^{j} \frac{Y^*_i(y\varphi(t) - \zeta_i)}{B(y\varphi(t) - \zeta_i)} \frac{(y\varphi(t) - \zeta_i)}{y\varphi(t)} \leq \frac{1}{y} |C(t, y, \varepsilon)| \right) = P \left( \sum_{i=1}^{j} \eta_i \leq \frac{1}{y} \right) = \frac{1}{(j-1)!} \int_{0}^{1/y} z^{j-1} e^{-z} dz. \]

Combining this result with Lemma 2 we see that,

\[ \lim_{t \to \infty} P(Z(t - y\varphi(t), t) = j | H(t)) = \frac{P(Z(t - y\varphi(t), t) = j) P(H(t) | Z(t - y\varphi(t), t) = j)}{P(H(t))} \]
\[ = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{P(Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon)) P(H(t) | Z(t - y\varphi(t), t) = j, C(t, y, \varepsilon))}{P(H(t))} \]
\[ = \frac{y}{(j-1)!} \int_{0}^{1/y} z^{j-1} e^{-z} dz. \]

\[ \square \]

4 Proof of Theorem 1.3

The proof of the theorem follows the line of proving Theorem 1.1.

Let \( \zeta_i := \zeta_i(xt) \), \( i = 1, 2, ..., Z(xt) \) be the remaining life-lengths of the particles existing in the process at moment \( t(1 - x) \), \( x \in (0, 1) \). We fix \( \varepsilon > 0 \) and introduce the event

\[ D(t, x, \varepsilon) := \left\{ \max_{1 \leq i \leq Z(xt)} \zeta_i \leq \varepsilon t \right\}. \]

It follows from Lemma 2 in [13] and Lemma 5 of the present paper that under the conditions of Theorem 1.3

\[ \lim_{t \to \infty} \frac{P(Z(xt, t) = j)}{P(Z(t) > 0)} = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{P(Z(xt, t) = j; D(t, x, \varepsilon))}{P(Z(t) > 0)} = (1 - x)x^{j-1} \]

for any \( j \geq 1 \). Now using the arguments similar to those used in the proof of Theorem 1.1 we have
\[
\lim_{t \to \infty} P(0 < Z(t) < \text{Bat}|Z(x,t) = j) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} P(0 < Z(t) < \text{Bat}|Z(x,t) = j; \mathcal{D}(t,x,\varepsilon)) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \mathbb{E} \left( \sum_{i=1}^{j} Y_i(1-x)t - \zeta_i | \mathcal{D}(t,x,\varepsilon) \right) \leq \frac{a}{1-x} \mathbb{E}(\mathcal{D}(t,x,\varepsilon)).
\]

Combining this result with Lemma 2 we see that,

\[
\lim_{t \to \infty} P(Z(x,t) = j|0 < Z(t) < \text{Bat}) = \lim_{t \to \infty} \mathbb{E} \left( \sum_{i=1}^{j} \eta_i \leq \frac{a}{1-x} \right) = \frac{1}{(j-1)!} \int_0^{1-x} z^{j-1} e^{-z} dz.
\]

Theorem 1.3 is proved.

To prove Corollary 1.4 we set \( j = 1 \) in the preceding formula and obtain

\[
\lim_{t \to \infty} P(d(t) < xt|0 < Z(t) < \text{Bat}) = \lim_{t \to \infty} P(Z(t(1-x),t) = j|0 < Z(t) < \text{Bat}) = \frac{1 - e^{-a/x}}{1 - e^{-a}}.
\]

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