ALGEBRAIC SURFACES WITH LOG-TERMINAL SINGULARITIES AND nef ANTICANONICAL CLASS AND REFLECTION GROUPS IN LOBACHEVSKY SPACES.

I. (BASICS OF THE DIAGRAM METHOD)

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Abstract. In our preprint: "Algebraic surfaces with log terminal singularities and nef anticanonical class and reflection groups in Lobachevsky spaces", Preprint Max-Planck-Institut für Mathematik, Bonn, (1989) MPI/89-28 (Russian), we had extended the diagram method to algebraic surfaces with nef anticanonical class (before, it was applied to Del Pezzo surfaces). The main problem here is that Mori polyhedron may not be finite polyhedral in this case. In the §4 of this preprint we had shown that this method does work for algebraic surfaces with nef anticanonical class and log terminal singularities.

Recently, using in particular these our results, V.A. Alexeev (Boundedness and $K^2$ for log surfaces, Preprint (1994), alg-geom/942007), obtained strong and important results for surfaces of general type with semi-log canonical singularities. The reason is that Del Pezzo surfaces (the diagram method is especially effective for them) and surfaces of general type "are connected" through surfaces with numerically zero canonical class. This was the main reason that we decided to make this English translation of the first part (§1—§3) of our preprint above with some extensions (where we give omitted proofs) and minor corrections.

Thus, this part contains basics of the diagram method for generalized reflection groups in Lobachevsky spaces and for surfaces with nef anticanonical class.

Introduction

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0. Introduction

In fact, this is the English translation with some extensions (where we give proofs which were omitted) and minor corrections of the first part §1—§3 of our preprint [33]. This part contains basics of the diagram method for generalized reflection groups in Lobachevsky spaces and for algebraic surfaces with nef anticanonical class. We hope to translate the rest part of the preprint [33] later.

In the series of papers of the author [10], È.B. Vinberg [6], A.G. Khovanski [15] and M.N. Prokhorov [14], it was proved that discrete reflection groups in Lobachevsky spaces with the fundamental polyhedron of finite volume exist in dimension < 996 only. The proof was based on the investigation of geometrical properties of the fundamental polyhedra \( \mathcal{M} \) of the groups. They are closed convex polyhedra characterized by the property that any facial angle (the angle between faces of highest dimension) has the form \( \pi/n, \ n \in \mathbb{N}, \ n > 1 \). In [11], [12] we remarked that in the above series of papers a general result about polyhedra with acute angles in Lobachevsky space was proved. We formulate this result now.

Let \( \mathcal{M} \) be a convex locally-finite polyhedron in a Lobachevsky space \( \mathcal{L} \) (defined by a hyperbolic form \( \Phi \) over \( \mathbb{R} \) of signature \( (1, n), \ n = \dim \mathcal{L} \) and \( P(\mathcal{M}) \subset \Phi \) be the set of vectors orthogonal to faces of \( \mathcal{M} \) of highest dimension and directed outside (one vector for each face). Let \( \Gamma(P(\mathcal{M})) = (v_i \cdot v_j), \ v_i, v_j \in P(\mathcal{M}) \) be the Gram matrix of \( \mathcal{M} \). We can also correspond to \( P(\mathcal{M}) \) (and subsets of \( P(\mathcal{M}) \) as well) the Gram graph \( \Gamma(P(\mathcal{M})) \) with the set of vertices \( P(\mathcal{M}) \) and edgers \( v_i v_j \) if \( v_i \cdot v_j \neq 0 \) with the weight \( v_i \cdot v_j \). Evidently, the Gram graph is equivalent to the Gram matrix, and we denote them by the same letter. Using the Gram graph, we can consider paths, length of a path, and so on. A subset \( \mathcal{E} \subset P(\mathcal{M}) \) is called elliptic if its Gram matrix is negative definite. A subset \( \mathcal{L} \subset P(\mathcal{M}) \) is called Lanner if its Gram matrix is hyperbolic, but the Gram matrix of any proper subset of \( \mathcal{L} \) is not hyperbolic (or \( \mathcal{L} \) is a minimal hyperbolic subset).

A polyhedron \( \mathcal{M} \) has acute angles if \( v_i \cdot v_j \geq 0 \) for any \( v_i \neq v_j \in P(\mathcal{M}) \). A polyhedron \( \mathcal{M} \) is called closed if it is a convex envelope of a finite set of points (with some of them at infinity) in \( \mathcal{L} \). The polyhedron \( \mathcal{M} \) is called non-degenerate if it is not contained in a hyperplane. In fact, in papers [10], [6], [15], [14], the following general result about polyhedra with acute angles in a Lobachevsky space was obtained.

**Lemma 1.** Let \( \mathcal{M} \) be a closed non-degenerate polyhedron with acute angles in a Lobachevsky space \( \mathcal{L} \) and suppose there exist some constants \( d, C_1, C_2 \) such that...
(a) diam $\Gamma(L) \leq d$ for any Lanner subset $L \subset P(M)$;

(b) $$\#\{v_1, v_2\} \subset E \mid 1 \leq \rho(v_1, v_2) \leq d \leq C_1 \#E$$

and

$$\#\{v_1, v_2\} \subset E \mid d + 1 \leq \rho(v_1, v_2) \leq 2d + 1 \leq C_2 \#E$$

for any elliptic subset $E \subset P(M)$ and the distance in $\Gamma(E)$. Then

$$\dim \mathcal{L} < 96(C_1 + C_2/3) + 68.$$
We prove (Proposition 2.2) that in this case $M$ is elliptic if $L'$ is a finite point, $M$ is parabolic relative to $L'$ if $L'$ is an infinite point, and $M$ is hyperbolic relative to $L'$ if $L'$ is a finite subspace of $L$. Thus, we can apply the diagram method to the fundamental polyhedron $M$ (since it has acute angles). We get

**Theorem 2.3.** Let $G$ be a generalized crystallographic reflection group in a Lobachevsky space $L$. Then

(a) $\dim L < 996$ if $L'$ is a finite point (this is the result of A.G. Khovanskii [15] and M.N. Prokhorov [14]);

(b) $\dim L < 997$ if $L'$ is an infinite point;

(c) $\dim L < 996 + \dim L'$ if $L'$ is a finite subspace of $L$.

We mention that if $M$ is bounded, ´E.B. Vinberg [6] had proven that $\dim L < 30$. Before, the author had proven [10] that $\dim L < 10$ if $G$ is an arithmetic reflection group and the degree of the ground field is sufficiently large. Thus, Theorem 2.3 generalizes these results of the author, ´E.B. Vinberg, A.G. Khovanskii and M.N. Prokhorov to wider class of groups. One can find non-trivial examples of generalized crystallographic reflection groups in [9].

We don't know whether there exists an absolute estimate of $\dim L$ (which does not depend from $\dim L'$) for the case (c).

In §3 we show that the diagram method above may be applied to appropriate classes of algebraic surfaces.

In the first place, we consider non-singular projective algebraic surfaces $Y$ over an algebraically closed field. Let $\text{NS}(Y)$ be the Neron-Severi lattice of $Y$ (generated by algebraic curves by the numerical equivalence). By Hodge index theorem, the lattice $\text{NS}(Y)$ is hyperbolic (i.e. it is non-degenerate and has exactly one positive square) and defines the Lobachevsky space $L(Y) = V(Y)/\mathbb{R}^+$ where $V(Y) = \{x \in \text{NS}(Y) \otimes \mathbb{R} | x^2 > 0\}$ and $V(Y)$ is the half-cone containing the class of a hyperplane section of $Y$.

We recall that a curve $C$ of $Y$ is called exceptional if $C$ is irreducible and $C^2 < 0$. Let $\text{Exc}(Y)$ be the set of all exceptional curves of $Y$. Any exceptional curve $F$ defines a half-space $H^+_F = \{\mathbb{R}^+ x \in L(Y) | x \cdot F \geq 0\}$ bounded by the hyperplane $H_F = \{\mathbb{R}^+ x \in L(Y) | x \cdot F = 0\}$ in $L(Y)$. The set $\text{Exc}(Y)$ defines a convex polyhedron

$$M(Y) = \bigcap_{F \in \text{Exc}(Y)} H^+_F$$

in $L(Y)$ with $P(M(Y)) = \text{Exc}(Y)$. The $M(Y)$ has acute angles since $F \cdot F' \geq 0$ if $F \neq F' \in \text{Exc}(Y)$.

From the classical results of Mori [23] for the case of surfaces, we deduce

**Proposition 3.1.** Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field. Assume that the anticanonical class $-K_Y$ is pseudoeffective (i.e. $-K_Y$ belongs to the Mori cone $\overline{\text{NE}}(Y)$) and $-K_Y \neq 0$. Let

$$-K_Y \equiv P + \sum_{i=1}^{n} \alpha_i F_i$$

be Zariski decomposition of $-K_Y$ (here $P$ is nef; $P \cdot F_i = 0$ and $\alpha_i > 0$ for all $i$, curves $F_i$, $i = 1, 2, \ldots, n$, have a negative definite Gram matrix). Then $M(Y)$
is elliptic if $P^2 > 0$, $\mathcal{M}(Y)$ is parabolic relative to $\mathbb{R}^+P$ if $P^2 = 0$ and $P \neq 0$, and $\mathcal{M}(Y)$ is hyperbolic relative to the subspace $\mathcal{T} = \cap \mathcal{H}_{F_i} \subset \mathcal{L}$ (intersection of hyperplanes $\mathcal{H}_{F_i}$ orthogonal to $F_i$, $i = 1, ..., n$) of codim $\mathcal{T} = n$ if $P \equiv 0$.

Thus, we can apply the diagram method above to non-singular projective algebraic surfaces with pseudo-effective anti-canonical class $-K_Y$. See exact formulations in Theorems 3.4—3.6 and 3.5', 3.6'.

These applications are especially important for projective algebraic surfaces with normal (or isolated) singularities and nef anti-canonical class.

Let $Z$ be a normal projective algebraic surface, and $\sigma : Y \to Z$ the minimal resolution of singularities of $Z$. Let the anticanonical class $-K_Z$ is nef. Then Zariski decomposition of $-K_Y$ has the form

\[(0.1)\]

$$-K_Y \equiv -\sigma^*K_Z + \sum \alpha_j F_j,$$

where $F_j$ are the components of the exceptional divisor of $\sigma$. Here all $\alpha_j \geq 0$ and $\alpha_j = 0$ iff $\sigma(F_j)$ is a double rational point (Du Val singularity). Thus, to get Zariski decomposition, we should just take away these zero summands in (0.1).

By Proposition 3.1, $\mathcal{M}(Y)$ is elliptic if $(K_Z)^2 > 0$, $\mathcal{M}(Y)$ is parabolic if $(K_Z)^2 = 0$ and $K_Z \not\equiv 0$, $\mathcal{M}(Y)$ is hyperbolic if $K_Z \equiv 0$. Hence we can apply the diagram method to $\mathcal{M}(Y)$. Thus, using the diagram method we get an estimate for $\dim \text{NS}(Y)$ if $K_Z \not\equiv 0$. And we get an estimate for the number $n$ of curves $F_j$ with positive $\alpha_j$ in the formula (0.1) if $K_Z \equiv 0$. At any case, we get the non-trivial restriction on singularities of $Z$.

It is an interesting problem to find classes of singularities of $Z$ for which the results above (the diagram method) give non-trivial and interesting applications. This paper contains only basics of the diagram method, and we don’t discuss these applications. Here in Introduction we can only give references where one can find these applications.

In our papers [11], [12], [13] and of V.A. Alexeev [2] the diagram method was applied very effectively to Del Pezzo surfaces (when $-K_Z$ is numerically ample) with log terminal singularities (for surfaces over $\mathbb{C}$ ”log terminal” is the same as ”quotient”). For this case, the polyhedron $\mathcal{M}(Y)$ is elliptic, and one can apply Lemma 1 directly.

For projective algebraic surfaces with log-terminal singularities and nef anti-canonical class, the diagram method presented here was first applied in the rest part (§4) of the preprint [33]. We hope to translate and prepare for publishing this part later.

Recently, V.A. Alexeev [37] (see also [36]) found very interesting and important applications of the diagram method for algebraic surfaces with nef anticanonical class which we present here, to algebraic surfaces of general type. Probably, the reason is that surfaces of positive canonical class ”are connected” with surfaces of negative canonical class through surfaces with numerically zero canonical class. Actually, this results of V.A. Alexeev were the reason that we decided to make this English translation of our preprint [33].

Some generalization of the diagram method to 3-folds one can find in [34] and [35].

I am grateful to Professor V.A. Alexeev for his interest to this translation.
§1. Convex polyhedra with acute angles in Lobachevsky spaces and the diagram method

Here we obtain some general results about polyhedra in Lobachevsky space. They give the basics of diagram method we will later apply to reflection groups in Lobachevsky spaces and to algebraic surfaces.

1.1. Klein model of Lobachevsky space.

Let $\Phi$ be a hyperbolic linear space (i.e. $\Phi$ is a $\mathbb{R}$-linear space equipped with a non-degenerate symmetric $\mathbb{R}$-bilinear form with inertia indexes $(1,n)$, where $n + 1 = \dim(\Phi)$). For $x, y \in \Phi$, we denote as $x \cdot y$ the value at the pair $(x, y)$ of the corresponding to $\Phi$ symmetric bilinear form. One can associate to $\Phi$ an open cone

$$V = \{ x \in \Phi \mid x^2 > 0 \}.$$ 

Since the bilinear form of $\Phi$ is hyperbolic, the $V$ is the disjoint union of two open convex half-cones. We denote by $V^+$ one of them. Thus, $V = V^+ \cup -V^+$. The corresponding to $\Phi$ Lobachevsky space is the set of rays

$$\mathcal{L} = V^+ / \mathbb{R}^+ = \{ \mathbb{R}^+ x \mid x \in V^+ \},$$

where $\mathbb{R}^+$ denote the set of positive real numbers. The distance $\rho$ in $\mathcal{L}$ is defined by the formula

$$\cosh(\rho(\mathbb{R}^+ x, \mathbb{R}^+ y)) = (x \cdot y)/(x^2 y^2)^{1/2}.$$ 

Then the curvature of $\mathcal{L}$ is equal to $(-1)$.

Let $O(\Phi)$ be the group of automorphisms of the hyperbolic linear space $\Phi$. Then the group of motions of $\mathcal{L}$ is identified with the subgroup $O_+(\Phi) \subset O(\Phi)$ of index 2 of automorphisms which preserve the half-cone $V^+$.

A half-space of $\mathcal{L}$ is the set

$$\mathcal{H}_\delta^+ = \{ \mathbb{R}^+ x \in \mathcal{L} \mid x \cdot \delta \geq 0 \},$$

where $\delta \in \Phi$ and $\delta^2 < 0$. The half-space $\mathcal{H}_\delta^+$ is bounded by the hyperplane

$$\mathcal{H}_\delta = \{ \mathbb{R}^+ x \in \mathcal{L} \mid x \cdot \delta = 0 \}.$$ 

The element $\delta \in \Phi$ is defined by the half-space $\mathcal{H}_\delta^+$ (respectively, the hyperplane $\mathcal{H}_\delta$) up to multiplying by elements of $\mathbb{R}^+$ (respectively, by elements of the set $\mathbb{R}^*$ of non-zero real numbers). The $\delta$ is called orthogonal to the half-space $\mathcal{H}_\delta^+$ (respectively, the hyperplane $\mathcal{H}_\delta$).

Assume that intersection $\mathcal{H}_{\delta_1}^+ \cap \mathcal{H}_{\delta_2}^+$ of two half spaces $\mathcal{H}_{\delta_1}^+, \mathcal{H}_{\delta_2}^+$ orthogonal to $\delta_1, \delta_2 \in \Phi$ contains a non-empty open subset of $\mathcal{L}$. Then we have two cases:

a) $\mathcal{H}_{\delta_1}^+ \cap \mathcal{H}_{\delta_2}^+$ is the angle of the value $\phi$ where

$$\cos \phi = (\delta_1 \cdot \delta_2)/((\delta_1^2 \delta_2^2)^{1/2})$$

if $-1 \leq (\delta_1 \cdot \delta_2)/((\delta_1^2 \delta_2^2)^{1/2}) \leq 1$.

b) Intersection of the hyperplanes $\mathcal{H}_{\delta_1}, \mathcal{H}_{\delta_2}$ is empty (they are hyperparallel), and the distance $\rho$ between them is defined by

$$\cosh \rho = (\delta_1 \cdot \delta_2)/((\delta_1^2 \delta_2^2)^{1/2})$$
if $1 \leq (\delta_1 \cdot \delta_2)/(\delta_1^2 \delta_2^2)^{1/2}$.

As usual, we complete $\mathcal{L}$ by the set of infinite points $\mathbb{R}^+ c$ where $c \in \Phi$, $c^2 = 0$, $c \cdot V^+ > 0$. Then $\mathcal{L}$ with its infinite points is the closed ball

$$\overline{\mathcal{L}} = (\overline{V^+} - \{0\})/\mathbb{R}^+,$$

where $\overline{V^+}$ is the closure of $V^+$ in the linear space $\Phi$). The boundary $\mathcal{L}_\infty = \overline{\mathcal{L}} - \mathcal{L}$ is the sphere of the dimension $n - 1$. It is called the $\infty$-sphere. We also add infinite points to half-spaces and hyperplanes considering their closure in $\overline{\mathcal{L}}$. Thus,

$$\mathcal{H}_\delta^+ = \{\mathbb{R}^+ x \in \overline{\mathcal{L}} \mid x \cdot \delta \geq 0\},$$

and

$$\mathcal{H}_\delta = \{\mathbb{R}^+ x \in \overline{\mathcal{L}} \mid x \cdot \delta = 0\}$$

are the half-space $\mathcal{H}_\delta^+$ and the hyperplane $\mathcal{H}_\delta$ with their infinite points respectively.

1.2. Polyhedra in Lobachevsky space.

A non-degenerate convex locally finite polyhedron $\mathcal{M}$ in a Lobachevsky space $\mathcal{L}$ is an intersection of a set of half-spaces:

$$\mathcal{M} = \bigcap_{\delta \in P(\mathcal{M})} \overline{\mathcal{H}}_\delta^+.$$ 

The $\mathcal{M}$ is called non-degenerate if $\mathcal{M}$ contains a non-empty open subset of $\mathcal{L}$. The $\mathcal{M}$ is called locally finite if for any $X \in \mathcal{L}$, there exists its open neighborhood $U \subset \mathcal{L}$ such that $\mathcal{M} \cap U$ is intersection with $U$ of a finite set of half-spaces in $\mathcal{L}$. Here $P(\mathcal{M}) \subset \Phi$ is a finite or countable subset of elements with negative square. We assume that neither two of these elements are proportional and every half-space $\mathcal{H}_\delta^+$, $\delta \in P(\mathcal{M})$ defines a face of $\mathcal{M}$ of the highest dimension $\dim \mathcal{L} - 1$ (it is clear what this means, since $\mathcal{M}$ is locally finite). Then, the polyhedron $\mathcal{M}$ defines the subset $P(\mathcal{M}) \subset \Phi$ uniquely up to multiplying by $\mathbb{R}^+$ elements of $P(\mathcal{M})$.

Below, we are only considering locally-finite and convex polyhedra.

We recall that a face $\gamma$ of a polyhedron $\mathcal{M}$ is intersection

$$\gamma = \mathcal{M} \cap (\bigcap_{\delta \in S} \mathcal{H}_\delta)$$

where $S$ is a non-empty subset of $P(\mathcal{M})$. One can consider $\gamma$ as a polyhedron is the subspace $[\gamma]$ of $\overline{\mathcal{L}}$ generated by $\gamma$ if $\gamma$ is finite (i.e. contains a finite point of $\overline{\mathcal{L}}$). Otherwise, $\gamma$ is either infinite (is a point of $\mathcal{L}_\infty$) or is empty.

1.2.1. Elliptic polyhedra.

A convex non-degenerate polyhedron $\mathcal{M}$ in Lobachevsky space $\mathcal{L}$ is called elliptic (equivalently closed) if $P(\mathcal{M})$ is finite (in particular, $\mathcal{M}$ is finite) and $\mathcal{M} \cap \mathcal{L}_\infty$ is finite. Equivalently, $\mathcal{M}$ is the convex envelope of a finite set of points of $\overline{\mathcal{L}}$ which is not contained in a hyperplane. The last definition is valid for Euclidean space too.

A face $\gamma$ of $\mathcal{M}$ is called elliptic if $\gamma$ is an elliptic polyhedron in $[\gamma]$. We say that a polyhedron $\mathcal{M}$ in $\mathcal{L}$ is elliptic (equivalently, closed) in a neighborhood of its face $\gamma$ if there exists a neighborhood $\mathcal{T} \subset U$ in $\overline{\mathcal{L}}$ and an elliptic polyhedron $\mathcal{M}'$ in $\mathcal{L}$ such that $\mathcal{M} \cap U = \mathcal{M}' \cap U$. Obviously, the face $\gamma$ is an elliptic polyhedron in $[\gamma]$ and the set

$$P(\gamma) = \{\delta \in P(\mathcal{M}) \mid \overline{\mathcal{H}}_\delta \cap \mathcal{T} \neq \emptyset\}$$

is finite if $\mathcal{M}$ is elliptic in a neighborhood of its face $\gamma$. 
1.2.2. Parabolic polyhedra.

Let \( P = \mathbb{R}^+ c \in \mathcal{L}_\infty \) (\( c \in \Phi \) and \( c^2 = 0 \) but \( c \neq 0 \)) is a point at infinity of \( \mathcal{L} \).

We remind that the horosphere \( \mathcal{E}_P \) with the center \( P \) is the set of all lines in \( \mathcal{L} \) containing the \( P \). The line \( l = P\mathbb{R}^+ h \in \mathcal{E}_P \), \( \mathbb{R}^+ h \in \mathcal{L}_c \), is the set \( \mathcal{L} = \{ \mathbb{R}^+(tc + h) \mid t \in \mathbb{R} \text{ and } (tc + h)^2 > 0 \} \). We fix a constant \( R > 0 \). Then there exists a unique \( \mathbb{R}^+ h \in l \) such that \( h \cdot c = R \) and \( h^2 = 1 \). Let \( l_1, l_2 \in \mathcal{E}_P \) and \( h_1, h_2 \) the corresponding elements respectively we had defined above. Let

\[
(2.1) \quad \rho(l_1, l_2) = \sqrt{-(h_1 - h_2)^2}.
\]

The horosphere \( \mathcal{E}_P \) with this distance is an affine Euclidean space. If one changes the constant \( R \), the distance \( \rho \) will be multiplied by a constant. The set

\[
\mathcal{E}_{P,R} = \{ \mathbb{R}^+ h \in \mathcal{L} \mid h \cdot c = R \text{ and } h^2 = 1 \} \cup \{ P \}
\]

is a sphere in \( \mathcal{L} \) touching \( \mathcal{L}_\infty \) at the \( P \). Besides, the sphere \( \mathcal{E}_{P,R} \) is orthogonal at a line \( l \in \mathcal{E}_P \) at the corresponding to the \( l \) point \( \mathbb{R}^+ h, h \in \mathcal{E}_{P,R} \). The distance of \( \mathcal{L} \) induces an Euclidean distance in \( \mathcal{E}_{P,R} \) which is similar to the distance (2.1). The set \( \mathcal{E}_{P,R} \) is identified with \( \mathcal{E}_P \) and called horosphere too.

Let \( K \subset \mathcal{E}_P \). The set

\[
C_K = \bigcup_{l \in K} \mathcal{L}
\]

is called the cone with the vertex \( P \) and the base \( K \). Evidently, the cone \( C_K \) is a subspace of the Lobachevsky space \( \mathcal{L} \) if the \( K \) is a subspace of the Euclidean space \( \mathcal{E}_P \). The dimension \( \dim C_K = \dim K + 1 \).

A non-degenerate locally finite polyhedron \( \mathcal{M} \) in \( \mathcal{L} \) is called parabolic (relative to the point \( P \in \mathcal{L}_\infty \)), if the conditions 1) and 2) below are valid:

1) \( \mathcal{M} \) is finite at the point \( P \), i.e. the set \( \{ \delta \in P(\mathcal{M}) \mid c \cdot \delta = 0 \} \) is finite.

2) For any elliptic polyhedron \( \mathcal{N} \subset \mathcal{E}_P \) (i.e. \( \mathcal{N} \) is a convex envelope of a finite set of points in \( \mathcal{E}_P \)), the polyhedron \( \mathcal{M} \cap C_N \) is elliptic if it is non-degenerate.

By the definition, a parabolic polyhedron \( \mathcal{M} \) is elliptic if and only if either \( P \notin \mathcal{M} \) or

\[
(2.2) \quad \mathcal{M}_P = \bigcap_{\delta \in P(\mathcal{M}), \delta \cdot c = 0} \mathcal{E}_P \cap H_\delta^+
\]

is an elliptic polyhedron in \( \mathcal{E}_P \).

We need

**Lemma 1.2.1.** Let \( \mathcal{M} \) be a non-degenerate locally-finite convex polyhedron in a Lobachevsky space \( \mathcal{L} \) and \( \mathcal{M} \) is parabolic relative to a point \( P \in \mathcal{L}_\infty \). Then there exists a face \( \gamma \) of \( \mathcal{M} \) such that \( \dim \gamma = \dim \mathcal{L} - 1 \), \( \mathcal{L} \) does not contain \( P \) and \( \mathcal{M} \) is elliptic in a neighborhood of \( \gamma \) (in particular, \( \gamma \) is elliptic in the hyperplane \( [\gamma] \)).

**Proof.** Assume that \( P \notin \mathcal{M} \). Then, there exists \( \delta \in P(\mathcal{M}) \) such that \( P \notin H_\delta^+ \). The set \( D = \{ PX \in \mathcal{E}_P \mid X \in H_\delta^+ \} \) is a ball in \( \mathcal{E}_P \). There exists an elliptic polyhedron \( \mathcal{N} \subset \mathcal{E}_P \) which contains the \( D \). Then \( \mathcal{M} = \mathcal{M} \cap C_N \) is elliptic, and the face \( \gamma \) of \( \mathcal{M} \) we are looking for does evidently exist.

Assume that \( P \in \mathcal{M} \). Let \( Q \) be a point inside of \( \mathcal{M} \). There exists an elliptic polyhedron \( \mathcal{N} \subset \mathcal{E}_P \) such that \( \mathcal{N} \subset \mathcal{M}_P \) (see (2.2)) and \( PQ \in \mathcal{N} \) is an internal point.
of \( \mathcal{N} \). A polyhedron \( C_N \cap \mathcal{M} \) is elliptic, since \( \mathcal{M} \) is parabolic. By the construction, the \( C_N \cap \mathcal{M} \) and the cone \( C_N \) coincide at the point \( P \). It follows that there exists a codimension one face \( \beta \) of \( C_N \cap \mathcal{M} \) such that \( P \notin [\beta] \). By construction, the \([\beta]\) is the hyperplane of a codimension one face \( \gamma \) of \( \mathcal{M} \), \( \beta \subset \gamma \) and \( [\gamma] = [\beta] = \mathcal{H}_\delta \) for \( \delta \in P(\mathcal{M}), \delta \cdot c \neq 0 \) (here \( P = \mathbb{R}^+c \)). The set

\[
C_\gamma = \{ l \in \mathcal{E}_P \mid l \cap \mathcal{H}_\delta \neq \emptyset \}
\]

is a closed ball of the horosphere \( \mathcal{E}_P \). There exists an elliptic polyhedron \( T \subset \mathcal{E}_P \) such that the ball \( C_\gamma \) is contained in interior of the \( T \). The polyhedra \( \mathcal{M} \) and \( \mathcal{M} \cap C_T \) coincide in a neighborhood of \( \mathcal{H}_\delta \). Since \( \mathcal{M} \cap C_T \) is elliptic (because the \( \mathcal{M} \) is parabolic), the polyhedron \( \mathcal{M} \) is elliptic in the neighborhood of \( \mathcal{H}_\delta \). \( \square \)

1.2.3. Hyperbolic polyhedra.

Let \( \mathcal{T} \) be a Lobachevsky subspace (i.e. intersection of a finite set of hyperplanes) of a Lobachevsky space \( \mathcal{L} \), and \( 1 \leq \dim \mathcal{T} \leq \dim \mathcal{L} - 1 \). Let \( K \subset \mathcal{T} \). We denote

\[
C_K = \bigcup_{l \perp \mathcal{T}, l \cap T \in K} l
\]

where \( l \) is a line of \( \mathcal{L} \). Thus, \( C_K \) is the cylinder over the \( K \).

A non-degenerate locally-finite polyhedron \( \mathcal{M} \) in \( \mathcal{L} \) is called hyperbolic (relative to \( \mathcal{T} \)) if the conditions 1) and 2) below hold:

1) \( \mathcal{M} \) is finite in infinite points, i.e. the set \( \{ \delta \in P(\mathcal{M}) \mid \delta \cdot c = 0 \} \) is finite for any point \( \mathbb{R}^+c \in \mathcal{L}_\infty \);

2) For any compact elliptic polyhedron \( \mathcal{N} \subset \mathcal{T} \), the polyhedron \( C_N \cap \mathcal{M} \) is elliptic if it is non-degenerate.

We need

Lemma 1.2.2. Let \( \mathcal{M} \) be a non-degenerate locally-finite convex polyhedron in a Lobachevsky space \( \mathcal{L} \) and \( \mathcal{M} \) is hyperbolic relative to a subspace \( \mathcal{T} \subset \mathcal{L} \) where \( 0 < \dim \mathcal{T} < \dim \mathcal{L} \). Then there exists an elliptic face \( \gamma \) of \( \mathcal{M} \) such that \( \text{codim} \gamma \leq \dim \mathcal{T} \) and \( \gamma \) is cut out by several hyperplanes \( \mathcal{H}_\delta, \delta \in P(\mathcal{M}) \), which don't contain the \( \mathcal{T} \).

Proof. We prove this by induction on \( \dim \mathcal{T} \).

Let \( \dim \mathcal{T} = 1 \). Assume that there exists a hyperplane \( \mathcal{H}_\delta, \delta \in P(\mathcal{M}) \), such that either \( \mathcal{H}_\delta \cap \mathcal{T} = \emptyset \), or \( \mathcal{H}_\delta \) and \( \mathcal{T} \) intersect in a finite point of \( \mathcal{L} \). Let \( D \subset \mathcal{T} \) be the image of \( \mathcal{H}_\delta \) by the orthogonal projection on \( \mathcal{T} \). Under the conditions above, the \( D \) is a compact interval. Let \( \mathcal{N} \) be an elliptic compact polyhedron in \( \mathcal{T} \) such that \( D \) is contained inside of \( \mathcal{T} \). By the construction, the polyhedra \( C_N \cap \mathcal{M} \) and \( \mathcal{M} \) coincide in a neighborhood of \( \mathcal{H}_\delta \). Since \( \mathcal{M} \) is hyperbolic, it follows that \( C_N \cap \mathcal{M} \) and \( \mathcal{M} \) are both elliptic in the neighborhood of \( \mathcal{H}_\delta \). Obviously, the codimension one face \( \gamma = \mathcal{M} \cap \mathcal{H}_\delta \) is the required one. Now, assume that any hyperplane \( \mathcal{H}_\delta, \delta \in P(\mathcal{M}) \), either contains \( \mathcal{T} \) or intersects \( \mathcal{T} \) in infinite point. In both cases, \( \mathcal{H}_\delta \) contains an infinite point of \( \mathcal{T} \). Since \( \mathcal{M} \) is finite at infinite points (by the condition 1 above) and \( \mathcal{T} \) contains exactly two infinite points, it follows that \( P(\mathcal{M}) \) is finite. Since \( \mathcal{M} \) is hyperbolic relative to \( \mathcal{T} \), one can easily see that \( \mathcal{M} \) is then elliptic. Obviously, then there exists \( \mathcal{H}_\delta, \delta \in P(\mathcal{M}) \), such that \( \mathcal{T} \not\subset \mathcal{H}_\delta \). The codimension one face \( \gamma \) of \( \mathcal{M} \) containing in the \( \mathcal{H}_\delta \) is required.
Let $\dim \mathcal{T} > 1$. We consider several cases.

Assume that there exists a hyperplane $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, such that $\mathcal{T} \cap \overline{\mathcal{H}}_\delta = \emptyset$. Arguing like above, we find the required face $\gamma \subset \mathcal{H}_\delta$ of $\mathcal{M}$ with $\operatorname{codim} \gamma = 1$.

Assume that there exists a hyperplane $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, such that $\mathcal{T} \cap \overline{\mathcal{H}}_\delta$ is an infinite point $P \in \mathcal{L}_\infty$. Let $\beta \subset \mathcal{H}_\delta$ be a face of $\mathcal{M}$ of codimension one. Let $A$ be an internal point of $\beta$. There exists a half-space $\mathcal{H}_\epsilon^+$ of $\mathcal{L}$ such that $P \notin \mathcal{H}_\epsilon^+$ but $A \in \mathcal{H}_\epsilon^+$ and the boundary $\mathcal{H}_\epsilon$ is orthogonal to $\mathcal{T}$. Let $D$ be the image by the orthogonal projection onto $\mathcal{T}$ of the set $\overline{\mathcal{H}}_\epsilon^+ \cap \overline{\mathcal{H}}_\delta$. The $\mathcal{D} \subset \mathcal{T}$ is compact, and there exists an elliptic compact polyhedron $\mathcal{N} \subset \mathcal{D}$ such that $D$ is contained inside of $\mathcal{N}$. The polyhedron $\mathcal{C}_\mathcal{N} \cap \mathcal{M}$ is elliptic, since $\mathcal{M}$ is hyperbolic. Then the polyhedron $\mathcal{H}_\epsilon^+ \cap \mathcal{C}_\mathcal{N} \cap \mathcal{M}$ is elliptic either. By the construction, the $\mathcal{H}_\epsilon^+ \cap \mathcal{C}_\mathcal{N} \cap \mathcal{M}$ and $\mathcal{H}_\epsilon^+ \cap \mathcal{M}$ coincide in a neighborhood of $\overline{\mathcal{H}}_\epsilon^+ \cap \overline{\mathcal{H}}_\delta$. The $\beta \cap \overline{\mathcal{H}}_\epsilon^+$ is the face of $\mathcal{H}_\epsilon^+ \cap \mathcal{M}$ of codimension one (since the point $A$ is contained inside of $\beta$ and $\mathcal{H}_\epsilon^+$), and this face is contained in $\overline{\mathcal{H}}_\delta$. Thus, the polyhedron $\mathcal{H}_\epsilon^+ \cap \mathcal{M}$ is elliptic in a neighborhood of its face $\beta \cap \overline{\mathcal{H}}_\epsilon^+$. Moving the point $A$ inside $\beta$ a little, we can suppose that the line $PA \subset \mathcal{H}_\delta$ contains internal points (for example, the point $A$) of the elliptic polyhedron $\beta \cap \mathcal{H}_\epsilon^+$ and does not contain its infinite points. Then $PA \cap \mathcal{H}_\epsilon^+ \cap \beta = [R, S]$ where $R \neq S$ and $R \in [P, S]$. There exists a face $\gamma'$ of $\mathcal{H}_\epsilon^+ \cap \beta$ which intersects the line $PA$ at the point $S$ and $\operatorname{codim} \gamma' = 2$. In particular, $P \notin \overline{\gamma'}$. The subspace $\overline{\gamma'}$ is different from $\mathcal{H}_\epsilon \cap \mathcal{H}_\delta$ because the last subspace intersects the line $PM$ at the point of $[P, R]$. It follows that $\gamma' \subset \gamma$ where $\gamma$ is a face of the face $\beta$ of $\mathcal{M}$, and $\operatorname{codim} \gamma = 2$. Thus, $P \notin \overline{\gamma'} = \overline{\gamma}$. It follows that the image of $\overline{\gamma}$ by the orthogonal projection onto $\mathcal{T}$ is compact in $\mathcal{T}$. Since $\mathcal{M}$ is hyperbolic, it follows like above that $\mathcal{M}$ is elliptic in a neighborhood of $\overline{\gamma}$. Let $\gamma \subset \mathcal{H}_\alpha$, $\alpha \in P(\mathcal{M})$. Let us suppose that $\mathcal{T} \subset \mathcal{H}_\alpha$. Then $P \in \mathcal{H}_\alpha$. Since $\operatorname{codim} \gamma = 2$ and $P \notin \overline{\gamma}$, it follows that $\mathcal{H}_\alpha = \mathcal{H}_\delta$. We get a contradiction since $\mathcal{H}_\delta$ does not contain $\mathcal{T}$. Thus, $\gamma$ is the required face of $\mathcal{M}$.

Assume that there exists $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$ such that $\mathcal{H}_\delta$ intersects $\mathcal{T}$. Then $\mathcal{T}' = \mathcal{T} \cap \mathcal{H}_\delta \subset \mathcal{H}_\delta$ is a subspace of $\dim \mathcal{T}' = \dim \mathcal{T} - 1 \geq 1$. One can easily see that the codimension one face $\mathcal{M}'$ of $\mathcal{M}$ containing in $\mathcal{H}_\delta$ is a hyperbolic polyhedron relative to $\mathcal{T}'$. By induction hypothesis, there exists an elliptic face $\gamma$ of the $\mathcal{M}'$ such that $\dim \mathcal{H}_\delta - \dim \gamma = \dim \mathcal{L} - 1 - \dim \gamma \leq \dim \mathcal{T}' = \dim \mathcal{L} - 1$ and $\gamma$ is intersection of some codimension one faces of the polyhedron $\mathcal{M}'$ in $\mathcal{H}_\delta$ which don’t contain $\mathcal{T}'$. It follows that $\gamma$ is an elliptic face of $\mathcal{M}$, $\operatorname{codim} \gamma \leq \dim \mathcal{T}$, and $\overline{\gamma}$ is intersection of codimension one faces $\mathcal{H}_\alpha$, $\alpha \in P(\mathcal{M})$, which don’t contain $\mathcal{T}$.

If $\mathcal{M}$ does not have a hyperplane $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, which satisfies one of conditions above, then all hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, contain the subspace $\mathcal{T}$. Obviously, then $\mathcal{M}$ is not hyperbolic relative to $\mathcal{T}$ since the condition 2) above is not valid for any compact polyhedron $\mathcal{N} \subset \mathcal{T}$. ■

1.3. Polyhedra with acute angles in Lobachevsky space.

We recall basic facts about polyhedra with acute angles in a Lobachevsky space (for example, one can find a very elementary exposition in [7]).

Below we assume that all polyhedra we are considering are non-degenerate, locally-finite and convex.

We say that a polyhedron $\mathcal{M}$ in a Lobachevsky space $\mathcal{L}$ has acute angles if $\delta \cdot \delta' \geq 0$ for any $\delta \neq \delta' \in P(\mathcal{M})$. By (1.1) and (1.2), facial angles (angles between codimension one faces) of $\mathcal{M}$ are really acute ($\leq \pi/2$). The inverse statement is
finite in infinite points of $L$. Lemma 1.3.1.

important. We have hyperbolic subset by properties of elliptic and connected parabolic subsets mentioned above, a non-connected component of $\Gamma(T)$ subset. It is clear that a subset $E \subset P(M)$ is hyperbolic (i.e. it has not more than one positive square), and has rank $\leq \dim L + 1$ because $P(M)$ is a subset of the hyperbolic space $\Phi$ of $\dim \Phi = \dim L + 1$.

We also consider the corresponding to $M$ Gram graph $\Gamma(P(M))$ or $\Gamma(M)$. Vertices of $\Gamma(M)$ correspond to elements of $P(M)$. A vertex $\delta \in P(M)$ has the weight $(-\delta^2)$. Two vertices $\delta \neq \delta'$ are joined by an arrow of the weight $\delta \cdot \delta'$ if $\delta \cdot \delta' > 0$. Evidently, the Gram graph is equivalent to the Gram matrix, and we identify them. Similarly, we consider Gram matrix and graph for any subset of $P(M)$. Using Gram graph, we can consider connected components, paths, length of paths, and so on.

A subset $E \subset P(M)$ is called elliptic if its Gram matrix (or graph) is negative definite. These subsets are important because they are in one to one correspondence with finite (i.e. having a finite point of $L$) faces of $M$. Every elliptic subset $E \subset P(M)$ defines a finite face $\gamma(E) = (\cap_{\delta \in E} \mathcal{H}_\delta) \cap M$ of $M$ of the codimension $\#E$. Any finite face $\gamma$ of $M$ defines an elliptic subset $P(\gamma) = \{\delta \in P(M) \mid \gamma \subset \mathcal{H}_\delta\}$ with $\#E = \text{codim } \gamma$ elements.

A subset $P \subset P(M)$ is called connected parabolic if $P$ has negative semi-definite (i.e. with no positive and at least one zero square) Gram matrix and the Gram graph of $P$ is connected. One can show that the rank of the Gram matrix of $P$ is equal to $\#P - 1$, and it has exactly one zero square. Connected parabolic subsets of $P(M)$ correspond to some infinite points of $M$. If $P$ is a connected parabolic subset, then $\cap_{\delta \in P} \mathcal{H}_\delta$ is an infinite point of $M$. By this construction, two different connected parabolic subsets $P_1, P_2 \subset P(M)$ define the same infinite point of $M$ if and only if subgraphs $\Gamma(P_1)$ and $\Gamma(P_2)$ are disjoined in $\Gamma(P(M))$ (i.e. $P_1 \cdot P_2 = 0$). If for a subset $P' \subset P(M)$ the intersection $\cap_{\delta \in P'} \mathcal{H}_\delta$ is an infinite point, then this point belongs to $M$ and one of connected components of Gram graph of $P'$ is parabolic.

A subset $T \subset P(M)$ is called hyperbolic if its Gram matrix has at least one positive square (then it has exactly one positive square). A subset $L \subset P(M)$ is called Lanner if it is minimal hyperbolic. Thus, Gram matrix of $L$ is hyperbolic, but Gram matrix of any proper subset of $L$ is not hyperbolic (i.e. it is negative definite or negative semi-definite). It is not difficult to see that Gram graph $\Gamma(L)$ is connected (see Lemma 1.5.1.2 below). Any proper subset of $L$ is either elliptic or connected parabolic.

Obviously, a subset $T \subset P(M)$ is hyperbolic if and only if $T$ contains a Lanner subset. It is clear that a subset $T \subset P(M)$ is not hyperbolic if and only if any connected component of $\Gamma(T)$ is either elliptic or connected parabolic. In particular, by properties of elliptic and connected parabolic subsets mentioned above, a non-hyperbolic subset $T$ is orthogonal to some point of $M$. The last property is very important. We have

Lemma 1.3.1. A polyhedron $M$ with acute angles in a Lobachevsky space $L$ is finite in infinite points of $L$. 
Proof. Let $P = \mathbb{R}^+ c \in \mathcal{L}$ where $c \in \Phi$ and $c \neq 0$ but $c^2 = 0$. The set

$$Q = \{ \delta \in P(M) \mid \delta \cdot c = 0 \}$$

is a subset of $c^\perp$. Since $\Phi$ is a hyperbolic space, the linear subspace $c^\perp \subset \Phi$ is negative semi-definite. It follows that Gram matrix of $Q$ is not hyperbolic. Thus, connected components of $\Gamma(Q)$ are either negative definite or negative semi-definite. By results mentioned above, $\#Q \leq 2\text{rk} \left( \Gamma(Q) \right) \leq 2(\dim \mathcal{L} + 1)$. In particular, $Q$ is finite. ■

1.4. Diagram method.

Series of articles of the author [10], É.B. Vinberg [6], A.G. Khovanskii [15] and M.N. Prokhorov [14] was devoted to getting a bound for dimension of Lobachevsky space $L$ admitting a reflection group (i.e. a discrete group generated by reflections in hyperplanes of $\mathcal{L}$) with a fundamental polyhedron $M$ of a finite volume. It was shown that $\dim L < 996$. We remarked in [11], [12] that in fact, in these papers, there was obtained a general result valid for arbitrary elliptic polyhedra with acute angles in a Lobachevsky space, and applied these results to Del Pezzo surfaces with log terminal singularities.

Here we want to extend these results to parabolic and hyperbolic polyhedra with acute angles. To apply results of Section 1.2, we prove a general diagram method estimate for dimension of an elliptic face $\gamma$ of a polyhedron $M$ with acute angles.

**Theorem 1.4.0.** Let $M$ be a non-degenerate locally-finite convex polyhedron with acute angles in a Lobachevsky space $\mathcal{L}$, and $\gamma$ an elliptic face of $M$. Let

$$P(\gamma, M) = \{ \delta \in P(M) \mid \overline{H}_\delta \cap \overline{\gamma} \neq \emptyset \}$$

and

$$P(\gamma^\perp) = \{ \delta \in P(M) \mid \gamma \subset \mathcal{H}_\delta \}.$$

Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset P(\gamma, M)$ such that $L$ contains at least two elements which don’t belong to $P(\gamma^\perp)$ and for any proper subset $L' \subset L$ the set $P(\gamma^\perp) \cup L'$ is not hyperbolic,

$$\text{diam} \, \Gamma(L) \leq d.$$

(b) For any elliptic subset $\mathcal{E}$ such that $P(\gamma^\perp) \subset \mathcal{E} \subset P(\gamma)$ and $\mathcal{E}$ has $\dim \mathcal{L} - 1$ elements, we have for the distance in the graph $\Gamma(\mathcal{E})$:

$$\# \{ \{ \delta_1, \delta_2 \} \subset \mathcal{E} - P(\gamma^\perp) \mid 1 \leq \rho(\delta_1, \delta_2) \leq d \} \leq C_1 \#(\mathcal{E} - P(\gamma^\perp)),$$

and

$$\# \{ \{ \delta_1, \delta_2 \} \subset \mathcal{E} - P(\gamma^\perp) \mid d + 1 \leq \rho(\delta_1, \delta_2) \leq 2d + 1 \} \leq C_2 \#(\mathcal{E} - P(\gamma^\perp)).$$

Then $\dim \gamma < 96(C_1 + C_2/3) + 68$.

**Proof.** See the proof in Sect. 1.5. ■

In particular, for elliptic $M$, we get the following statement which is a variant of the similar statement we had formulated in [11] and [12] (in [11] and [12] we used $\#L$ instead of $\text{diam} \, \Gamma(L)$).
Theorem 1.4.1. Let \( M \) be an elliptic convex polyhedron with acute angles in a Lobachevsky space \( L \). Assume that there are some constants \( d, C_1, C_2 \) such that the conditions (a) and (b) below hold:

(a) For any Lanner subset \( L \subset P(M) \)

\[
diam \Gamma(L) \leq d.
\]

(b) For any elliptic subset \( \mathcal{E} \subset P(M) \) such that \( \mathcal{E} \) has \( \dim L - 1 \) elements, we have for the distance in the graph \( \Gamma(\mathcal{E}) \):

\[
\sharp\{\{\delta_1, \delta_2\} \subset \mathcal{E} \mid 1 \leq \rho(\delta_1, \delta_2) \leq d\} \leq C_1 \sharp\mathcal{E};
\]

and

\[
\sharp\{\{\delta_1, \delta_2\} \subset \mathcal{E} \mid d + 1 \leq \rho(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2 \sharp\mathcal{E}.
\]

Then \( \dim L < 96(C_1 + C_2/3) + 68 \).

For a parabolic (relative to an infinite point \( P \in L_\infty \)) or hyperbolic (relative to a subspace \( T \subset L \)) polyhedron \( M \subset L \) with acute angles, by Lemmas 1.2.1 and 1.2.2, there exists an elliptic face \( \gamma \subset M \). We have mentioned in Sect. 1.3 that \( \gamma \) has acute angles (see the proof of Theorems 1.4.2 and 1.4.3 below). There are two possibilities to apply results above to get an estimate for \( \dim L \) using this elliptic face \( \gamma \). In the first place, we can apply Theorem 1.4.1 to the elliptic polyhedron \( \gamma \) with acute angles. Secondly, we can directly apply Theorem 1.4.0 to the pair \( \gamma \subset M \). Both these variants give good results which are almost the same in practice. We formulate results of both.

At first, we apply Theorem 1.4.1. For a subset \( Q \) of the set of vertices of a graph \( \Gamma \) and vertices \( v_1, v_2 \) of \( \Gamma \) which both don’t belong to \( Q \), we consider the distance

\[
\rho_Q(v_1, v_2) = \min\{\min_s (\rho(s) - \#(s \cap Q)), +\infty\},
\]

where \( s \) is a path in \( \Gamma \) joined \( v_1, v_2 \), the distance \( \rho(s) \) is the length of \( s \) in \( \Gamma \), and the \( \#(s \cap Q) \) is the number of vertices of \( s \) which belong to \( Q \).

Theorem 1.4.2. Let \( M \) be a convex parabolic relative to a \( P \in L_\infty \) polyhedron with acute angles in a Lobachevsky space \( L \). Let \( \gamma \) be the elliptic codimension one face of \( M \) from Lemma 1.2.1, and \( \gamma \subset \mathcal{H}_e, P \notin \overline{\mathcal{H}_e} \) where \( e \in P(M) \). Assume that there are some constants \( d, C_1, C_2 \) such that the conditions (a) and (b) below hold:

(a) For any Lanner subset \( L \subset P(\gamma, M) \) such that \( L \) contains at least two elements different from \( e \) and for any proper subset \( L' \subset L \) the set \( \{e\} \cup L' \) is not hyperbolic,

\[
diam \Gamma(L) \leq d.
\]

(b) For any elliptic subset \( \mathcal{E} \subset P(M) \) such that \( \mathcal{E} \) has \( \dim L - 1 \) elements and \( e \in \mathcal{E} \), we have for the distance in the graph \( \Gamma(\mathcal{E}) \):

\[
\sharp\{\{\delta_1, \delta_2\} \subset \mathcal{E} \setminus \{e\} \mid 1 \leq \rho_e(\delta_1, \delta_2) \leq d\} \leq C_1 (\#\mathcal{E} - 1);
\]

and

\[
\sharp\{\{\delta_1, \delta_2\} \subset \mathcal{E} \setminus \{e\} \mid d + 1 \leq \rho_e(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2 (\#\mathcal{E} - 1).
\]

Then \( \dim L < 96(C_1 + C_2/3) + 69 \).
Theorem 1.4.3. Let $\mathcal{M}$ be a convex hyperbolic relative to a subspace $\mathcal{T} \subset \mathcal{L}$ polyhedron with acute angles in a Lobachevsky space $\mathcal{L}$. Let $\gamma$ be the elliptic and of codimension $\leq \dim \mathcal{T}$ face of $\mathcal{M}$ from Lemma 1.2.2, and $Q = P(\gamma^\perp)$ where $Q \subset P(M)$, $P$ has codim $\gamma \leq \dim \mathcal{T}$ elements and $\mathcal{T} \not\subset \mathcal{H}_\delta$ for any $\delta \in Q$. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset P(\gamma, \mathcal{M})$ such that $L$ contains at least two elements which don't belong to $Q$ and for any proper subset $L' \subset L$ the set $Q \cup L'$ is not hyperbolic,

$$\text{diam} \Gamma(L) \leq d.$$ 

(b) For any elliptic subset $\mathcal{E} \subset P(\mathcal{M})$ such that $\mathcal{E}$ has $\dim \mathcal{L} - 1$ elements and $Q \subset \mathcal{E}$, we have for the distance in the graph $\Gamma(\mathcal{E})$:

$$\sharp \{\delta_1, \delta_2 \subset \mathcal{E} - Q \mid 1 \leq \rho_Q(\delta_1, \delta_2) \leq d\} \leq C_1 \sharp(\mathcal{E} - Q);$$

and

$$\sharp \{\delta_1, \delta_2 \subset \mathcal{E} - Q \mid d + 1 \leq \rho_Q(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2 \sharp(\mathcal{E} - Q).$$

Then $\dim \mathcal{L} < 96(C_1 + C_2/3) + 68 + \dim \mathcal{T}$.

Proof of Theorems 1.4.2 and 1.4.3. Let $e \in \Phi$ and $e^2 < 0$. Let us consider the orthogonal projection of $\Phi$ along $e$. By this projection, the image $a'$ of $a \in \Phi$ is equal to

$$a' = a - e(e \cdot a)/e^2.$$ 

(4.1)

For images $a', b'$ of $a, b \in \Phi$, we then get

$$a' \cdot b' = a \cdot b + (e \cdot a)(e \cdot b)/(-e^2).$$ 

(4.2)

We mention, that using these formulae, one can easily deduce that a face of a convex polyhedron with acute angles is a polyhedron with acute angles too.

We put $Q = \{e\}$ for the Theorem 1.4.2. Thus, for both Theorems 1.4.2 and 1.4.3, $Q = P(\gamma^\perp)$ and $\#Q = \text{codim} \gamma$ (by results mentioned in Sect. 1.3). The set $P(\gamma)$ of the polyhedron $\gamma \subset [\gamma]$ is the orthogonal projection along $Q$ of the set

$$P'(\gamma, \mathcal{M}) = \{\delta \in P(\mathcal{M}) \mid Q \cup \{\delta\} \text{ is an elliptic subset of } P(\mathcal{M})\}.$$ 

In particular, using (4.2) $\#Q$ times, we then get that $\gamma$ is a polyhedron with acute angles. A subset $U \subset P(\gamma)$ is elliptic, parabolic or hyperbolic if and only if it is a projection along $Q$ of an elliptic, parabolic or hyperbolic subset respectively of $U' \subset P'(\gamma, \mathcal{M})$ such that $Q \subset U'$. Let $a, b \in U' - Q$ and $a', b'$ their images in $U$. Applying (4.2) several times, we get

$$\rho(a', b') = \rho_Q(a, b)$$ 

(4.3)

where $\rho(a', b')$ is the distance in $\Gamma(U)$ and $\rho_Q(a, b)$ uses the distance in $\Gamma(U')$.

Now, we apply Theorem 1.4.1 to the elliptic polyhedron $\gamma \subset [\gamma]$ and the constants $d, C_1, C_2$ of the corresponding Theorem 1.4.2 or 1.4.3. By (4.3), the condition (b)
of Theorem 1.4.1 is equivalent to the condition (b) of the corresponding Theorem 1.4.2 or 1.4.3.

Now, let $R \subset \mathcal{P}(\gamma)$ be a Lanner subset. Then $R$ is the image by the projection of the hyperbolic subset $M \subset \mathcal{P}^{\prime}(\gamma, \mathcal{M})$ such that $M$ contains $Q$, $M$ contains at least two elements which don’t belong to $Q$ (because $R$ has at least two elements), and any proper subset $M' \subset M$ which contains $Q$ is not hyperbolic. Since $M$ is hyperbolic, there exists a Lanner subset $L \subset M$. Let $T \subset R$ be the image of $L$ by the projection. Since $T$ is hyperbolic and $R$ is Lanner, we get $T = R$. By (4.3),

$$\text{diam } \Gamma(R) \leq \text{diam } \Gamma(L).$$

Thus, the condition (a) of the corresponding Theorem 1.4.2 or 1.4.3 implies the condition (a) of Theorem 1.4.1 for $\gamma$. Then, we can apply Theorem 1.4.1 to $\gamma \subset [\gamma]$ with constants $d, C_1, C_2$ of the corresponding Theorem 1.4.2 or 1.4.3 to get the estimate for $\text{dim}[\gamma]$.

By Lemma 1.2.1 for the parabolic case, $\text{dim } \mathcal{L} = \text{dim}[\gamma] + 1$. By Lemma 1.2.2, for the hyperbolic case, $\text{codim } [\gamma] = \text{dim } \mathcal{L} - \text{dim}[\gamma] \leq \text{dim } \mathcal{T}$, and $\text{dim } \mathcal{L} \leq \text{dim } [\gamma] + \text{dim } \mathcal{T}$. Thus, by Theorem 1.4.1, we get Theorems 1.4.2 and 1.4.3. ■

Applying Theorem 1.4.0 and Lemmas 1.2.1 and 1.2.2, we get

**Theorem 1.4.2**'. Let $\mathcal{M}$ be a convex parabolic relative to a $P \in \mathcal{L}_\infty$ polyhedron with acute angles in a Lobachevsky space $\mathcal{L}$. Let $\gamma$ be the codimension one face of $\mathcal{M}$ from Lemma 1.2.1, and $\gamma \subset \mathcal{H}_e$, $P \notin \mathcal{H}_e$ where $e \in P(\mathcal{M})$. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \mathcal{P}(\gamma, \mathcal{M})$ such that $L$ contains at least two elements different from $e$ and for any proper subset $L' \subset L$ the set $\{e\} \cup L'$ is not hyperbolic,

$$\text{diam } \Gamma(L) \leq d.$$

(b) For any elliptic subset $\mathcal{E} \subset P(\mathcal{M})$ such that $\mathcal{E}$ has $\text{dim } \mathcal{L} - 1$ elements and $e \in \mathcal{E}$, we have for the distance in the graph $\Gamma(\mathcal{E})$:

$$\sharp\{\delta_1, \delta_2 \in \mathcal{E} - \{e\} \mid 1 \leq \rho(\delta_1, \delta_2) \leq d\} \leq C_1(\sharp \mathcal{E} - 1);$$

and

$$\sharp\{\delta_1, \delta_2 \in \mathcal{E} - \{e\} \mid d + 1 \leq \rho(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2(\sharp \mathcal{E} - 1).$$

Then $\text{dim } \mathcal{L} < 96(C_1 + C_2/3) + 69$.

**Theorem 1.4.3**'. Let $\mathcal{M}$ be a convex hyperbolic relative to a subspace $\mathcal{T} \subset \mathcal{L}$ polyhedron with acute angles in a Lobachevsky space $\mathcal{L}$. Let $\gamma$ be the codimension \textless; $\text{dim } \mathcal{T}$ face of $\mathcal{M}$ from Lemma 1.2.2, and $Q = P(\gamma)$ where $Q \subset P(\mathcal{M})$, $Q$ has codim $\gamma \leq \text{dim } \mathcal{T}$ elements and $\mathcal{T} \notin \mathcal{H}_s$ for any $\delta \in Q$. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \mathcal{P}(\gamma, \mathcal{M})$ such that $L$ contains at least two elements which don’t belong to $Q$ and for any proper subset $L' \subset L$ the set $Q \cup L'$ is not hyperbolic,

$$\text{diam } \Gamma(L) \leq d.$$

(b) For any elliptic subset $\mathcal{E} \subset P(\mathcal{M})$ such that $\mathcal{E}$ has $\text{dim } \mathcal{L} - 1$ elements and $Q \subset \mathcal{E}$, we have for the distance in the graph $\Gamma(\mathcal{E})$:

$$\sharp\{\delta_1, \delta_2 \in \mathcal{E} - Q \mid 1 \leq \rho(\delta_1, \delta_2) \leq d\} \leq C_1(\sharp \mathcal{E} - Q);$$

and

$$\sharp\{\delta_1, \delta_2 \in \mathcal{E} - Q \mid d + 1 \leq \rho(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2(\sharp \mathcal{E} - Q).$$
and
\[ \sharp\{\{\delta_1, \delta_2\} \subset \mathcal{E} - Q \mid d + 1 \leq \rho(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2 \sharp(\mathcal{E} - Q). \]

Then \( \dim \mathcal{L} < 96(C_1 + C_2/3) + 68 + \dim \mathcal{T}. \)

**1.5. The proof of Theorem 1.4.0.**

1.5.1. The compact case.

For it would be easier for a reader, we at first prove similar statement for the case when \( \gamma \) does not have an infinite vertex. This statement is important as itself. One can also apply this to reflection groups and algebraic surfaces. We don’t do this, but a reader can find appropriate applications himself. We prove

**Theorem 1.5.1.1.** Let \( \mathcal{M} \) be a non-degenerate locally-finite convex polyhedron with acute angles in a Lobachevsky space, and \( \gamma \) be an elliptic compact (i.e. without infinite vertices) face of \( \mathcal{M} \). Let

\[ P(\gamma, \mathcal{M}) = \{ \delta \in P(\mathcal{M}) \mid \mathcal{H}_\delta \cap \gamma \neq \emptyset \} \]

and

\[ P(\gamma^\perp) = \{ \delta \in P(\mathcal{M}) \mid \gamma \subset \mathcal{H}_\delta \}. \]

Assume that there are some constants \( d, C_1, C_2 \) such that the conditions (a) and (b) below hold:

(a) For any Lanner subset \( L \subset P(\gamma, \mathcal{M}) \) such that \( L \) contains at least two elements which don’t belong to \( P(\gamma^\perp) \) and for any proper subset \( L' \subset L \) the set \( P(\gamma^\perp) \cup L' \) is not hyperbolic,

\[ \text{diam} \ \Gamma(L) \leq d. \]

(b) For any elliptic subset \( \mathcal{E} \) such that \( P(\gamma^\perp) \subset \mathcal{E} \subset P(\gamma) \) and \( \mathcal{E} \) has \( \dim \mathcal{L} \) elements, we have for the distance in the graph \( \Gamma(\mathcal{E}) \):

\[ \sharp\{\{\delta_1, \delta_2\} \subset \mathcal{E} - P(\gamma^\perp) \mid 1 \leq \rho(\delta_1, \delta_2) \leq 2d + 1\} \leq C_2 \sharp(\mathcal{E} - P(\gamma^\perp)). \]

Then \( \dim \gamma < 8C + 6. \)

**Proof (Compare with [6] and also [34]).**

**Lemma 1.5.1.2.** Let \( \mathcal{M} \) be a polyhedron with acute angles. Then any Lanner subset \( L \subset P(\mathcal{M}) \) has a connected graph \( \Gamma(L) \), and any two Lanner subsets \( L, N \subset P(\mathcal{M}) \) either have a common element or are joined by an edge in \( \Gamma(L \cup N) \).

**Proof.** Let \( L = L_1 \cup L_2 \) where \( L_1 \cdot L_2 = 0 \) (equivalently, \( \Gamma(L) \) is the disjoint union of \( \Gamma(L_1) \) and \( \Gamma(L_2) \)). Since \( L \) is hyperbolic, one of the subsets \( L_1, L_2 \) is hyperbolic. If this is \( L_1 \), one gets \( L = L_1 \) since \( L \) is minimal hyperbolic.

Let \( L \cdot N = 0 \). Since \( L \) and \( N \) are both hyperbolic, the \( L \cup N \) has Gram matrix with at least two positive squares. We get a contradiction since Gram matrix of \( P(\mathcal{M}) \) has not more than one positive square. ■

A convex elliptic (i.e. closed) polyhedron is called *simplicial* if all its proper faces are simplexes. A convex elliptic polyhedron is called *simple* (equivalently, it has simplicial angles) if it is dual to a simplicial one. In other words, any its face of codimension \( k \) is contained exactly in \( k \) faces of the highest dimension (the last
definition is valid not only for elliptic polyhedron). By results mentioned in Sect.
1.3, any elliptic compact convex polyhedron with acute angles is simple. This is the
very important combinatorial property of a compact polyhedron with acute angles.
Thus, the face $\gamma \subset M$ is a simple polyhedron.

Besides, since any face $\gamma_1 \subset \gamma$ is a finite face (since $\gamma$ has no vertices at infinity)
of the polyhedron $M$ and $M$ has acute angles, we also have:

\[(1.5.1.1) \quad \sharp P(\gamma_1^\perp) - \sharp P(\gamma^\perp) = \text{codim}_{\gamma_1} \gamma_1 \]

for any face $\gamma_1$ of $\gamma$, and $\sharp P(\gamma^\perp) = \text{codim}_{M\gamma_1} \gamma_1$.

We use the following Lemma 1.5.1.3 which was proved in [10]. The lemma was
used in [10] to get a bound ($\leq 10$) for the dimension of a Lobachevsky space
admitting an action of an arithmetic reflection group with a field of definition of
the degree $> N$. Here $N$ is some absolute constant.

**Lemma 1.5.1.3.** Let $N$ be a convex elliptic simple polyhedron of a dimension $n$, and $A_{i,k}^n$ the average number of $i$-dimensional faces of $k$-dimensional faces of $N$. Then for $n \geq 2k - 1$

\[ A_{i,k}^n < \frac{(n-i)}{(n-k)} \cdot \left( \left(\left[\frac{n}{2}\right]\right) + \left(\left[\frac{n}{2}\right]\right) \right) \]

In particular:

If $n \geq 3$

\[ A_{0,2}^n < \left\{ \begin{array}{ll}
  \frac{4(n-1)}{n-2} & \text{if } n \text{ is even,} \\
  \frac{4n}{n-1} & \text{if } n \text{ is odd.}
\end{array} \right. \]

If $n \geq 5$

\[ A_{1,3}^n < \left\{ \begin{array}{ll}
  \frac{12(n-1)}{n-4} & \text{if } n \text{ is even,} \\
  \frac{12n}{n-3} & \text{if } n \text{ is odd.}
\end{array} \right. \]

and

\[ A_{2,3}^n < \left\{ \begin{array}{ll}
  \frac{6(n-2)}{n-4} & \text{if } n \text{ is even,} \\
  \frac{6(n-1)}{n-3} & \text{if } n \text{ is odd.}
\end{array} \right. \]

If $n \geq 7$,

\[ A_{3,4}^n < \left\{ \begin{array}{ll}
  \frac{8(n-3)}{n-6} & \text{if } n \text{ is even,} \\
  \frac{8(n-2)}{n-5} & \text{if } n \text{ is odd.}
\end{array} \right. \]

**Proof.** See [10]. We mention that the right side of the inequality of the Lemma
1.5.1.3 decreases and tends to the number $2^{k-i} \binom{k}{i}$ of $i$-dimensional faces of $k$-
dimensional cube if $n$ increases. ■

From the estimate of $A_{n,2}^0$ of the Lemma, it follows the following Vinberg’s
Lemma from [6]. This Lemma was used by É.B. Vinberg to get an estimate
(dim < 30) for the dimension of a Lobachevsky space admitting an action of a
discrete reflection group with a compact fundamental polyhedron.

By definition, a 2-angle of a polyhedron $T$ is an angle of a 2-dimensional face
of $T$. Thus, the 2-angle is defined by a vertex $A$ of $T$, a plane containing $A$ and
a 2-dimensional face $\gamma_2$ of $T$, and two rays with the beginning at $A$ which contain
two corresponding sides of the $\gamma_2$.
Lemma 1.5.1.4. Let $N$ be a convex elliptic simple polyhedron of a dimension $n$. Let $C$ and $D$ are some numbers. Assume that 2-angles of $N$ are supplied with weights and the following conditions (1) and (2) hold:

1. The sum of weights of all 2-angles at any vertex of $N$ is not greater than $Cn + D$.
2. The sum of weights of all angles of any 2-dimensional face of $N$ is at least $5 - k$ where $k$ is the number of vertices of the 2-dimensional face.

Then

$$n < 8C + 5 + \begin{cases} 1 + 8D/n, & \text{if } n \text{ is even,} \\ (8C + 8D)/(n-1), & \text{if } n \text{ is odd.} \end{cases}$$

In particular, for $C \geq 0$ and $D = 0$, we have

$$n < 8C + 6.$$

Proof. Since the proof of Lemma is very short, we give the proof here.

Let $\Sigma$ be the sum of weights of all 2-angles of the polyhedron $N$. Let $\alpha_0$ be the number of vertices of $N$ and $\alpha_2$ the number of 2-dimensional faces of $N$. Since $N$ is simple,

$$\alpha_0 n(n-1)/2 = \alpha_2 A_n^{0,2}.$$

From this equality and conditions of the Lemma, we get inequalities

$$(Cn + D)\alpha_0 \geq \Sigma \geq \sum \alpha_{2,k} (5 - k) = 5\alpha_2 - \alpha_2 A_n^{0,2} =$$

$$= \alpha_2 (5 - A_n^{0,2}) = \alpha_0 (n(n-1)/2)(5/A_n^{0,2} - 1).$$

Here $\alpha_{2,k}$ is the number of 2-dimensional faces with $k$ vertices of $N$. Thus, from this inequality and the inequality for $A_n^{0,2}$ of Lemma 1.5.1.3, we get

$$Cn + D \geq (n(n-1)/2)(5/A_n^{0,2} - 1) > \begin{cases} n(n-6)/8, & \text{if } n \text{ is even,} \\ (n-1)(n-5)/8, & \text{if } n \text{ is odd.} \end{cases}$$

From this calculations, Lemma 1.5.1.4 follows. $\blacksquare$

The proof of Theorem 1.5.1.1. (Compare with [6] and also [34].) Let $\angle$ be a 2-angle of $\gamma$. Let $P(\angle) \subset P(\gamma)$ be the elliptic set of all $\delta \in P(M)$ which are orthogonal to the vertex $R + h \in L$ of the $\angle$. By (1.5.1.1), the set $P(\angle)$ is the disjoint union

$$P(\angle) = P(\angle^\perp) \cup \{ \delta_1(\angle) \} \cup \{ \delta_2(\angle) \}$$

where $P(\angle^\perp)$ contains all $\delta \in P(M)$ orthogonal to the plane of the 2-angle $\angle$, and $\delta_1(\angle)$ and $\delta_2(\angle)$ are orthogonal to two sides of the 2-angle $\angle$. And by (1.5.1.1), $\dim P(\angle) = \dim L$, and elements $\delta_1(\angle), \delta_2(\angle)$ are defined uniquely. Backward, an elliptic subset $P(\angle) \subset P(M)$ with dim $L$ elements and a pair of its different elements $\{ \delta_1(\angle), \delta_2(\angle) \} \subset P(\angle)$ uniquely define the 2-angle $\angle$ of $M$. We define the weight $\sigma(\angle)$ by the formula:

$$\sigma(\angle) = \begin{cases} 1, & \text{if } 1 \leq \rho(\delta_1(\angle), \delta_2(\angle)) \leq 2d + 1, \\ 0, & \text{if } 2d + 2 \geq \rho(\delta_1(\angle), \delta_2(\angle)). \end{cases}$$
Here we take the distance in the graph $\Gamma(P(\angle))$. Let us prove conditions of the Lemma 1.5.1.4 with the constant $C$ of Theorem 1.5.1.1 and $D = 0$.

The condition (1) follows from the condition (b) of the theorem. We remark that $\delta_1(\angle), \delta_2(\angle)$ do not belong to the set $P(\gamma_4^\perp)$ since $P(\gamma_4^\perp) \subset P(\angle^\perp)$.

Let us prove the condition (2).

Let $\gamma_3$ be a 2-dimensional triangle face (triangle) of $\gamma$. The set $P(\gamma_3)$ of all $\delta \in P(\mathcal{M})$ orthogonal to some points of $\gamma_3$ is the union of the set $P(\gamma_3^\perp)$ of $\delta \in P(\mathcal{M})$, which are orthogonal to the plane of the triangle $\gamma_3$, and $\delta_1, \delta_2, \delta_3$, which are orthogonal to the sides of the triangle $\gamma_3$. Union of the set $P(\gamma_3^\perp)$ with any two elements from $\delta_1, \delta_2, \delta_3$ is elliptic, since it is orthogonal to a vertex of $\gamma_3$. On the other hand, the set $P(\gamma_3) = P(\gamma_3^\perp) \cup \{\delta_1, \delta_2, \delta_3\}$ is hyperbolic, since it is not orthogonal to a point of $\mathcal{M}$. Indeed, the set of all points of $\mathcal{M}$, which are orthogonal to the set $P(\gamma_3^\perp) \cup \{\delta_2, \delta_3\}, P(\gamma_3^\perp) \cup \{\delta_1, \delta_3\},$ or $P(\gamma_3^\perp) \cup \{\delta_1, \delta_2\}$ is the vertex $A_1, A_2$ or $A_3$ respectively of the triangle $\gamma_3$, and the intersection of these sets of vertices is empty. Thus, there exists a Lanner subset $L \subset P(\gamma_3)$, which contains the set $\{\delta_1, \delta_2, \delta_3\}$. By the condition (a), the graph $\Gamma(L)$ contains a shortest path $s$ of the length $0 \leq d \leq 2d + 1$ which connects $\delta_1, \delta_3$. If this path does not contain $\delta_2$, then the angle of $\gamma_3$ defined by the set $P(\gamma_3^\perp) \cup \{\delta_1, \delta_3\}$ and the pair $\{\delta_1, \delta_3\}$ has the weight 1. If this path contains $\delta_2$, then the angle of $\gamma_3$ defined by the set $P(\gamma_3^\perp) \cup \{\delta_1, \delta_2\}$ and the pair $\{\delta_1, \delta_2\}$ has the weight 1. Thus, we proved that the side $A_2A_3$ of the triangle $\gamma_3$ defines an angle of the triangle with the weight 1 and one side $A_2A_3$ of the angle. The triangle has three sides. Thus, there are at least two angles of the triangle with the weight 1. It follows the condition (2) of the Lemma 1.5.1.4 for the triangle.

Let $\gamma_4$ be a 2-dimensional quadrangle face (quadrangle) of $\gamma$. In this case,

$$P(\gamma_4) = P(\gamma_4^\perp) \cup \{\delta_1, \delta_2, \delta_3, \delta_4\}$$

where $P(\gamma_4^\perp)$ is the set of all $\delta \in P(\mathcal{M})$ which are orthogonal to the plane of the quadrangle and $\delta_1, \delta_2, \delta_3, \delta_4$ are orthogonal to the consecutive sides of the quadrangle. As above, one can see that the sets $P(\gamma_4^\perp) \cup \{\delta_1, \delta_3\}, P(\gamma_4^\perp) \cup \{\delta_2, \delta_4\}$ are hyperbolic, but the sets $P(\gamma_4^\perp) \cup \{\delta_1, \delta_2\}, P(\gamma_4^\perp) \cup \{\delta_2, \delta_3\}, P(\gamma_4^\perp) \cup \{\delta_3, \delta_4\},$ and $P(\gamma_4^\perp) \cup \{\delta_4, \delta_1\}$ are elliptic. It follows that there are Lanner subsets $L, N$ such that $\{\delta_1, \delta_3\} \subset L \subset P(\gamma_4^\perp) \cup \{\delta_1, \delta_3\}$ and $\{\delta_2, \delta_4\} \subset N \subset P(\gamma_4^\perp) \cup \{\delta_2, \delta_4\}$. By Lemma 1.5.1.2, there exist $\alpha \in L$ and $\beta \in N$ such that $\alpha \beta$ is an edge in $\Gamma(M)$. By the condition (a) of the theorem, one of $\delta_1, \delta_3$ is joined by a path $s_1$ of the length $\leq d$ with $\alpha$ and this path does not contain another element from $\delta_1, \delta_3$ (here $\alpha$ is the terminal of the path $s_1$). We can assume that this element is $\delta_1$ (otherwise, one should replace the $\delta_1$ by the $\delta_3$). As above, we can assume that the element $\beta$ is joined by the path $s_2$ of the length $\leq d$ with the element $\delta_2$ and this path does not contain the element $\delta_4$. The path $s_1 \alpha \beta s_2$ is a path of the length $\leq 2d + 1$ in the graph $\Gamma(P(\gamma_4^\perp) \cup \{\delta_1, \delta_2\})$. It follows that the angle of the quadrangle $\gamma_4$, such that two sides of this angle are orthogonal to the elements $\delta_1$ and $\delta_2$, has the weight 1. It proves the condition (2) of the Lemma 1.5.1.4 and the theorem.

1.5.2. The non-compact case.

The proof of Theorem 1.4.0. This proof is very similar to the proof of Theorem 1.5.1.1 above but details are more complicated. And we only outline the proof. One can find omitted details in A.G Khovanskii [15] and M.N. Prokhorov [14]: we don’t want to rewrite these papers here.
In the first place, for the general case when $\gamma$ is only an elliptic polyhedron with acute angles with some vertices at infinity, the $\gamma$ may not be a simple polyhedron. But it is "almost" simple.

A convex elliptic (i.e. closed) polyhedron is called simplicial in codimension $k$ if all its proper faces of codimension $\geq k + 1$ are simplex. Thus, a simplicial polyhedron is simplicial in codimension 0. A convex closed polyhedron is called simple in dimension $k$ (equivalently, it has simplicial angles in dimension $k$) if it is dual to a polyhedron which is simplicial in codimension $k$. In other words, a polyhedron $T$ is simple in dimension $k$ if any its proper face of a dimension $m \geq k$ is contained exactly in $\dim T - m$ faces of highest dimension (the last definition is valid not only for an elliptic polyhedron). By results mentioned in Sect. 1.3, any convex polyhedron with acute angles is simple in dimension 1. This is the very important combinatorial property of a polyhedron with acute angles. Thus, the face $\gamma \subset M$ is an elliptic simple in dimension 1 polyhedron.

Besides, since any face $\gamma_1 \subset \gamma$ of $\dim \gamma_1 \geq 1$ is a finite face of the polyhedron $M$ with acute angles, we also have:

\[(1.5.2.1) \quad \sharp P(\gamma_1^+) - \sharp P(\gamma^+) = \text{codim}_\gamma \gamma_1\]

for any face $\gamma_1$ of $\gamma$ if $\dim \gamma_1 \geq 1$, and $\sharp P(\gamma_1^+) = \text{codim}_M \gamma_1$.

A.G. Khovanskii [15] generalized Lemma 1.5.1.3 to polyhedra which are simple in dimension 1.

**Lemma 1.5.2.3.** Lemma 1.5.1.3 is valid for convex elliptic polyhedra which are simple in dimension 1.

*Proof.* See [15]. We should say that the proof of this Lemma required new ideas comparing with the proof of Lemma 1.5.1.3 in [10]. We remark that for a rational polyhedron (actually, only this case is important for surfaces) one can prove Lemma 1.5.2.3 similarly to [10] using known properties of the combinatorial polynomial of the polyhedron. 

Since the property (1.5.2.1) is only true for $\dim \gamma_1 \geq 1$, one cannot use 2-dimensional angles. One has to consider 3-dimensional angles (3-angles).

By definition, a 3-angle of a polyhedron $T$ is a facial angle of a 3-dimensional face of $T$. Thus, the 3-angle is defined by an edge (i.e. 1-dimensional face) $\gamma_1$ of $T$, a 3-dimensional face $\gamma_3$ of $T$ containing $\gamma_1$ and two different 2-dimensional faces $\gamma_2^{(1)}, \gamma_2^{(2)}$ of $\gamma_3$ with the common edge $\gamma_1$.

To formulate the analog of Lemma 1.5.1.4, we need an additional definition. Let us take two 3-dimensional tetrahedra and glue together along their two-dimensional triangle face. This polyhedron is called triangle bipyramid. Thus, a triangle bipyramid has 5 vertices: two vertices of the order 3 and three vertices of the order 4 (here the order of a vertex is equal to the number of edges which contain this vertex). A 3-dimensional polyhedron which is combinatorially equivalent to a triangle bipyramid is called bad. A 3-dimensional polyhedron is good if it is not a bad one.

Using estimates for $A_{3,4}^3, A_{1,3}^1, A_{2,3}^2$ of Lemma 1.5.2.3, M.N.Prokhorov [14] proved the following analog of Vinberg’s Lemma 1.5.1.4.

**Lemma 1.5.2.4.** Let $N$ be an elliptic convex polyhedron with acute angles in a Lobachevsky space of dimension $n$. Let $C > 0$ be a constant. Assume that 3-angles of $N$ are supplied with weights and the following conditions (1) and (2) hold:

*Proof.* See [15].
(1) The sum of weights of all 3-angles at any edge $\gamma_1$ of $\gamma$ is not greater than $C(n - 1)$.

(2) The sum $\sigma(\gamma_3)$ of weights of all 3-angles of any good 3-dimensional face $\gamma_3$ of $\gamma$ is at least $7 - k$ where $k$ is the number of 2-dimensional faces of the 3-dimensional face $\gamma_3$.

Then

$$n < 96C + 68.$$ 

Proof. The proof uses estimates for $A_{n}^{3,4}, A_{n}^{1,3}, A_{n}^{2,3}$ of Lemma 1.5.2.3 and is similar to the proof of Lemma 1.5.1.4. But the proof requires more calculations. Mainly, one should analyze how many bad 3-dimensional faces may have a 4-dimensional convex elliptic polyhedron with acute angles in Lobachevsky space which has $\leq 9$ three-dimensional faces. See details in [14] in the proof of [14, Lemma 2].

Now we are ready to discuss the proof of Theorem 1.4.0.

The proof of Theorem 1.4.0. (Compare with [14, §3].) Let $\angle$ be a 3-angle of $\gamma$, and $\angle$ is defined by faces $\gamma_1 \subset \gamma_2 \subset \gamma_3$ of $\gamma$ (see the definition above of a 3-angle). Let $P(\angle) \subset P(\gamma)$ be the elliptic set of all $\delta \in P(\mathcal{M})$ which are orthogonal to the edge $[\gamma_1]$ of the $\angle$. By (1.5.2.1), the set $P(\angle)$ is the disjoint union

$$P(\angle) = P(\angle^\perp) \cup \{\delta_1(\angle)\} \cup \{\delta_2(\angle)\}$$ 

where $P(\angle^\perp)$ contains all $\delta \in P(\mathcal{M})$ orthogonal to the 3-dimensional subspace $[\gamma_3]$ of the 3-angle $\angle$, and $\delta_1(\angle)$ and $\delta_2(\angle)$ are orthogonal to two sides $[\gamma_2^{(1)}]$ and $[\gamma_2^{(2)}]$ of the $\angle$. And, by (1.5.2.1), $\# P(\angle) = \dim \mathcal{L} - 1$, and elements $\delta_1(\angle), \delta_2(\angle)$ are defined uniquely. Backward, an elliptic subset $P(\angle) \subset P(\mathcal{M})$ with $\dim \mathcal{L} - 1$ elements and a pair of its different elements $\{\delta_1(\angle), \delta_2(\angle)\} \subset P(\angle)$ uniquely define the 3-angle $\angle$. We define the weight $\sigma(\angle)$ by the formula:

$$\sigma(\angle) = \begin{cases} 
1, & \text{if } 1 \leq \rho(\delta_1(\angle), \delta_2(\angle)) \leq d, \\
1/3, & \text{if } d + 1 \leq \rho(\delta_1(\angle), \delta_2(\angle)) \leq 2d + 1, \\
0, & \text{if } 2d + 2 \leq \rho(\delta_1(\angle), \delta_2(\angle)). 
\end{cases}$$

Here we take the distance in the graph $\Gamma(P(\angle))$. Let us prove conditions of the Lemma 1.5.2.4 with the constant $C = C_1 + C_2/3$ where $C_1, C_2$ are constants from the condition of Theorem 1.4.0.

The condition (1) follows from the condition (b) of the theorem. We remark that $\delta_1(\angle), \delta_2(\angle)$ do not belong to the set $P(\gamma^\perp)$ since $P(\gamma^\perp) \subset P(\angle^\perp)$.

Let us prove the condition (2). Let $\gamma_3 \subset \gamma$ be a good 3-dimensional face of $\gamma$ and $\gamma_3$ contains $\leq 6$ two-dimensional faces. (We want to show that this case is similar to the case of triangle or quadrangle for the proof of Theorem 1.5.1.1.)

A set $\{\gamma^{(1)}, \ldots, \gamma^{(k)}\}$ of two-dimensional faces of a 3-dimensional face $\gamma_3$ is called good if it has one of the types 1—4 below (see [14, Figure 2]):

Type 1: $k = 4$ and $\gamma^{(1)}, \ldots, \gamma^{(4)}$ define the configuration of two-dimensional faces of a three-dimensional tetrahedron (then $\gamma_3$ is also tetrahedron).

Type 2: $k = 3$ and $\gamma^{(1)} \cap \gamma^{(2)} \cap \gamma^{(3)}$ is empty but $\gamma^{(i)} \cap \gamma^{(j)}$ is an edge of $\gamma_3$ for any $1 \leq i < j \leq 3$.

Type 3: $k = 3$ and $\gamma^{(1)} \cap \gamma^{(2)} \cap \gamma^{(3)}$ is empty, $\gamma^{(1)} \cap \gamma^{(2)}$ and $\gamma^{(2)} \cap \gamma^{(3)}$ are edges of $\gamma_3$, and $\gamma^{(3)} \cap \gamma^{(1)}$ is a vertex at infinity of $\gamma$.
Type 4: $k = 4$, intersection of any three different faces from $\gamma^{(1)}, ..., \gamma^{(4)}$ is empty, intersection of any two different faces from $\gamma^{(1)}, ..., \gamma^{(4)}$ is empty except intersections $\gamma^{(1)} \cap \gamma^{(2)}, \gamma^{(2)} \cap \gamma^{(3)}, \gamma^{(3)} \cap \gamma^{(4)}, \gamma^{(4)} \cap \gamma^{(1)}$ which define edges of $\gamma_3$.

For a good subset $\mathcal{K} = \{\gamma^{(1)}, ..., \gamma^{(k)}\}$ of two-dimensional faces of a 3-dimensional face $\gamma_3$ we consider the sum $\sigma(\mathcal{K})$ of all 3-angles with both sides which belong to $\mathcal{K}$. Using Lemma 1.5.1.2, like for the proof of Theorem 1.5.1.1, one can prove that

$$\sigma(\mathcal{K}) \geq \begin{cases} 3, & \text{if } \mathcal{K} \text{ has the type 1}, \\ 2, & \text{if } \mathcal{K} \text{ has the type 2}, \\ 1, & \text{if } \mathcal{K} \text{ has the type 3}, \\ 1/3, & \text{if } \mathcal{K} \text{ has the type 4}. \end{cases}$$

(1.5.2.2)

Here the types 1, 2 and 3 are similar to triangle and the type 4 is similar to quadrangle.

It is shown in [14] that any good face $\gamma_3$ with $k \leq 6$ two-dimensional faces contains good subsets of two-dimensional faces. Using these subsets and (1.5.2.2), one can prove that $\sigma(\gamma_3) \geq 7 - k$. In fact, one should draw pictures of all 3-dimensional polyhedra with acute angles and $k \leq 6$ (see [14, Figure 3]) to find good subsets of two-dimensional faces. For example, if $\gamma_3$ is a cube, $\gamma_3$ evidently has three different good subsets of the type 4. By (1.5.2.2), $\sigma(\gamma_3) \geq 3 \cdot (1/3) = 1 = 7 - k$.

We remark that a bad 3-dimensional face does not have a good subset of two-dimensional faces. It is why we had to exclude bad faces in Lemma 1.5.2.4.

This finishes the proof of Theorem 1.4.0. ■

§2. Discrete reflection groups in Lobachevsky spaces and the diagram method

Let $G$ be a discrete group in a Lobachevsky space $\mathcal{L}$ with a fundamental domain of finite volume (i.e. crystallographic group). Let $W \triangleleft G$ be the reflection subgroup of $G$, (i.e. $W$ is generated by all reflections in hyperplanes which belong to $G$). We choose a fundamental polyhedron $\mathcal{M}$ of $W$. Then one can consider the quotient group $A = G/W$ as an automorphism group of $\mathcal{M}$. Thus, $G = W \times A$ is a semi-direct product. As in [26, Sect. 3], we say that $G$ is a generalized (hyperbolic) crystallographic reflection group if there exists a subgroup $A' \subset A$ of a finite index and a non-trivial subspace $\mathcal{L}' \subset \mathcal{L}$ (including the case when $\mathcal{L}'$ is an infinite point of $\mathcal{L}$) such that $\mathcal{L}'$ is $A'$-invariant. Here ”non-trivial” means that $\mathcal{L}' \neq \emptyset$ and $\mathcal{L}' \neq \mathcal{L}$.

In the first place, we have the following preliminary

Proposition 2.1. Let $G$ be a generalized crystallographic reflection group in a Lobachevsky space $\mathcal{L}$. Then there exist a subgroup $A' \subset A$ of a finite index and a non-trivial $A'$-invariant subspace $\mathcal{T} \subset \mathcal{L}$ such that $\mathcal{T}$ is generated by $\mathcal{T} \cap \mathcal{M}$.

Proof. If $A$ is finite, $A'$ is a unite subgroup and $\mathcal{T}$ is any point of $\mathcal{M}$.

Assume that $A$ is infinite. Let $A' \subset A$ be a finite index subgroup and let $\mathcal{L}' \subset \mathcal{L}$ be a non-trivial $A'$-invariant subspace. Let $m \in \mathcal{M}$ be a finite point. Since $A'$ is discrete and infinite, there is a sequence $a_1, ..., a_n, ..$ in $A'$ such that the sequence $a_1(m), ..., a_n(m), ..$ tends to a point $P \in \mathcal{M} \cap \mathcal{L}_\infty$. If $\mathcal{L}' = Q$ is an infinite point, any horosphere $\mathcal{E}_{Q,R}$ with the center $Q$ is $A'$-invariant. It follows that the sequence $a_1(m), ..., a_n(m), ..$ tends to $Q$ and $Q \in \mathcal{L}$ belongs to $\mathcal{M}$. If $\mathcal{L}'$ is a finite subspace
of \( L \), the distances \( \rho(a_i(m), L') \) are equal. It follows that \( P \in L' \). Thus, at any case, \( P \in \mathcal{M} \cap L' \) and \( \mathcal{M} \cap L' \neq \emptyset \). Thus, we can set \( T = [\mathcal{M} \cap L'] \).

Using Proposition 2.1, we divide generalized crystallographic reflection groups \( G \) in three types. The group \( G \) is called \textit{elliptic} if \( G \) satisfies Proposition 2.1 with \( T \) which is a finite point \( P \in \mathcal{M} \). The group \( G \) is called \textit{parabolic relative to} \( P \) if \( G \) satisfies Proposition 2.1 with \( T \) which is an infinite point \( P \in \mathcal{M} \cap L_\infty \). The group \( G \) is called \textit{hyperbolic relative to} \( T \) if \( G \) satisfies Proposition 2.1 with \( T \) which is a finite subspace of \( L \) generated by \( T \cap \mathcal{M} \) and \( 0 < \dim T < \dim L - 1 \).

This definition is justified by the following

**Proposition 2.2.** Let \( G \) be a generalized crystallographic reflection group in a Lobachevsky space \( L \). Then:

(a) \( \mathcal{M} \) is elliptic if \( G \) is elliptic;
(b) \( \mathcal{M} \) is parabolic relative to \( P \in \mathcal{M} \cap L_\infty \) if \( G \) is parabolic relative to \( P \);
(c) \( \mathcal{M} \) is hyperbolic relative to a subspace \( T \subset L \) if \( G \) is hyperbolic relative to \( T \).

**Proof.** Considering the subgroup \( G' = W \rtimes A' \subset G \) of a finite index, we can assume that the subspace \( T \subset L \) of Proposition 2.1 is \( A \)-invariant, and for the case (a), \( A \) is trivial and \( G = W \).

We use the following well-known statement about discrete groups \( G \) in Lobachevsky space with a fundamental domain of finite volume (see [27]): the \( G \) has a fundamental domain \( D \) which is an elliptic (i.e. closed) convex polyhedron. Besides, we use the following standard statement about discrete reflection groups: the fundamental polyhedron \( \mathcal{M} \) of the group \( W \) is a convex locally-finite polyhedron with acute angles and facial angles of the form \( \pi/n, n \in \mathbb{N}, n \geq 2 \), and any codimension one face \( \gamma \) of \( \mathcal{M} \) defines a reflection in the hyperplane \([\gamma]\) which belongs to \( G \). It follows that if \( D \cap \mathcal{M} \) contains an open non-empty subset, then \( D \subset \mathcal{M} \) and \( D \) is a fundamental domain for \( A \) acting in \( \mathcal{M} \).

For the case (a), \( D = \mathcal{M} \) is an elliptic polyhedron.

For the case (b), simple standard considerations using facts above imply that \( D = \mathcal{M} \cap \mathcal{C}_N \) where \( \mathcal{N} \subset \mathcal{E}_P \) is an elliptic polyhedron on the horosphere \( \mathcal{E}_P \) and \( \mathcal{N} \) is a fundamental domain for the action of \( A \) in

\[
\mathcal{M}_P = \bigcap_{\delta \in P(\mathcal{M}), P \in \mathcal{H}_\delta} \mathcal{E}_P \cap \mathcal{H}_\delta^\perp.
\]

Here, by Lemma 1.3.1, the set \( \{ \delta \in P(\mathcal{M}) \mid P \in \mathcal{H}_\delta \} \) is finite and the polyhedron \( \mathcal{M}_P \subset \mathcal{E}_P \) is bounded by a finite set of hyperplanes \( \mathcal{H}_\delta \cap \mathcal{E}_P \). Since \( D \) is an elliptic polyhedron, it easily follows that \( \mathcal{M} \) is parabolic relative to \( P \).

For the case (c), similarly, \( D = \mathcal{M} \cap \mathcal{C}_N \) where \( \mathcal{N} \subset \mathcal{T} \) is an elliptic polyhedron in \( \mathcal{T} \) and \( \mathcal{N} \) is a fundamental domain for the action of \( A \) in \( \mathcal{M} \cap \mathcal{T} \). Since \( D \) is an elliptic polyhedron in \( L \), it easily follows that \( \mathcal{M} \) is hyperbolic relative to \( \mathcal{T} \).

Using Theorems 1.4.1–1.4.3 (or 1.4.1, 1.4.2', 1.4.3') and Proposition 2.2, we get

**Theorem 2.3.** Let \( G \) be a generalized crystallographic reflection group in a Lobachevsky space \( L \). Then

(a) \( \dim L < 1056 \) if \( G \) is elliptic;
(b) \( \dim L < 1057 \) if \( G \) is parabolic;
(c) \( \dim L < 1058 \) if \( G \) is hyperbolic.
(c) \( \dim \mathcal{L} < 1056 + \dim \mathcal{T} \) if \( G \) is hyperbolic relative to a subspace \( \mathcal{T} \subset \mathcal{L} \).

Proof. It is well-known that \( \text{diam} \Gamma(L) \leq 8 \) for any Lanner subset \( L \subset P(M) \) (for example, see [14]). For any elliptic subset \( \mathcal{E} \subset P(M) \), every connected component of the graph \( \Gamma(\mathcal{E}) \) has the form \( A_n, D_n, E_6, E_7 \) or \( E_8 \). It easily follows that we can take \( C_1 = 8, \ C_2 = 9 \) for Theorems 1.4.1–1.4.3 (and 1.4.2', 1.4.3').

Remark 2.4. With reference to the assertion (c) of Theorem 2.3, we don’t know if there exists an absolute estimate of \( \dim \mathcal{L} \) which does not depend from \( \dim \mathcal{T} \).

Remark 2.5. A.G. Khovanskii [15] and M.N. Prokhorov [14] have proven that \( \dim \mathcal{L} < 996 \) for the case (a) (see considerations in [14]). Similarly, we can improve estimates above in this special case of reflection groups: \( \dim \mathcal{L} < 997 \) for the parabolic case, and \( \dim \mathcal{L} < 996 + \dim \mathcal{T} \) for the hyperbolic case.

If \( \mathcal{M} \) is bounded, É.B. Vinberg [6] had proven that \( \dim \mathcal{L} < 30 \). Before, the author had proven [10] that \( \dim \mathcal{L} < 10 \) if \( G \) is an arithmetic reflection group (elliptic) and the degree of the ground field is sufficiently large.

§3. Diagram method and algebraic surfaces

Let \( Y \) be a non-singular projective algebraic surface over an algebraically closed field and \( \text{NS}(Y) \) be the Neron–Severi lattice (i.e. an integral symmetric bilinear form) of \( Y \). By Hodge index theorem, the lattice \( \text{NS}(Y) \) is hyperbolic (i.e. it is non-degenerate and has exactly one positive square) and defines the Lobachevsky space \( \mathcal{L}(Y) = V^+(Y)/\mathbb{R}^+ \) where \( V(Y) = \{ x \in \text{NS}(Y) \otimes \mathbb{R} \mid x^2 > 0 \} \) and \( V^+(Y) \) is the half-cone containing the class of a hyperplane section of \( Y \).

We recall that a curve \( C \) of \( Y \) is called \emph{exceptional} if \( C \) is irreducible and \( C^2 < 0 \). Let \( \text{Exc}(Y) \subset \text{NS}(Y) \) be the set of classes in \( \text{NS}(Y) \) of all exceptional curves of \( Y \) and

\[
\mathcal{M}(Y) = \bigcap_{\delta \in \text{Exc}(Y)} \mathcal{H}^\perp_\delta
\]

the corresponding convex polyhedron in \( \mathcal{L}(Y) \) with \( P(\mathcal{M}(Y)) = \text{Exc}(Y) \). The \( \mathcal{M}(Y) \) is a polyhedron with acute angles since \( \delta \cdot \delta' \geq 0 \) if \( \delta, \delta' \in \text{Exc}(Y) \) and \( \delta \neq \delta' \).

One can apply the diagram method — results of Sect. 1.4— to \( \mathcal{M}(Y) \) (and \( Y \)) if \( \mathcal{M}(Y) \subset \mathcal{L}(Y) \) is locally finite and is elliptic, parabolic or hyperbolic. Below we consider an important case when this is true.

We recall that a formal finite linear combination \( D = \sum_i a_i C_i \) of irreducible curves \( C_i \) of \( Y \) is called \emph{divisor}, \( \mathbb{Q} \)-\emph{divisor} and \( \mathbb{R} \)-\emph{divisor} if \( a_i \in \mathbb{Z}, a_i \in \mathbb{Q} \), and \( a_i \in \mathbb{R} \) respectively. An \( \mathbb{R} \)-divisor \( D = \sum_i a_i C_i \) is called \emph{effective} if all \( a_i \geq 0 \). An \( \mathbb{R} \)-divisor \( D \) is called numerically effective (equivalently, \emph{nef}) if \( D \cdot F \geq 0 \) for any effective divisor \( F \). An \( \mathbb{R} \)-divisor \( F \) is called \emph{pseudo-effective} if \( F \cdot D \geq 0 \) for any \emph{nef} divisor \( D \). Here it is sufficient considering \( D \) which are irreducible curves \( D \) with \( D^2 \geq 0 \). The same definition is valid for classes of divisors in \( N(Y) = \text{NS}(Y) \otimes \mathbb{R} \).

One can also use Mori (or Kleiman–Mori) cone to give these definitions. Let \( \text{NE}(Y) \subset N(Y) \) be a convex cone generated by all classes of effective divisors. The \textit{Mori cone} \( \text{NE}(Y) \) is the closure of \( \text{NE}(Y) \) in Euclidean topology of \( N(Y) \) and is equal to the set of all pseudo-effective divisors. The dual cone \( \overline{\text{NE}(Y)}^* \subset N(Y) \) (with respect to the intersection pairing) is the set of all \( \text{nef} \) divisors. Since \( N(Y) \) is a hyperbolic space with respect to the intersection pairing, we have

\[
\mathcal{M}(Y) = \overline{\text{NE}(Y)}^*/\mathbb{R}^+.
\]
We recall Zariski decomposition (see [31], [28]). Let $D$ be a pseudo-effective (i.e. $D \in \overline{NE}(Y)$) $\mathbb{Q}$-divisor. Then $D$ has the Zariski decomposition (in $N(Y)$):

\[(1) \quad D \equiv P + N,\]

where

(i) $P$ is a numerically effective (i.e. nef) $\mathbb{Q}$-divisor,

(ii) $N = \sum_{i=1}^{n} \alpha_i F_i$ where $\alpha_i \in \mathbb{Q}$ and $\alpha_i > 0$, and $F_1, ..., F_n$ are exceptional curves with a negative definite Gram matrix $(F_i \cdot F_j)$, $1 \leq i, j \leq n$;

(iii) $P \cdot F_i = 0$ for any $F_i$, $1 \leq i \leq n$.

It is known that the Zariski decomposition is unique: the $P \in N(Y)$, the divisor $N$ are unique if one has properties (i), (ii), (iii). If $D$ is an effective $\mathbb{Q}$-divisor, Zariski decomposition is defined for $\mathbb{Q}$-divisors: $D = P + N$.

Zariski decomposition defines the numerical Kodaira dimension $\nu(D, Y)$ of $D$:

\[\nu(D, Y) = \begin{cases} 
2, & \text{if } P^2 > 0, \\
1, & \text{if } P^2 = 0 \text{ and } P \neq 0, \\
0, & \text{if } P \equiv 0 \\
-\infty & \text{if } D \text{ is not pseudo-effective.}
\end{cases}\]

Let $K = K_Y$ be the canonical class of $Y$. Then $\nu(K, Y)$ is called the numerical Kodaira dimension of $Y$ and $\nu(-K, Y)$ is called the numerical anti-Kodaira dimension of $Y$.

We use Mori theory [23] to prove

**Proposition 3.1.** Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field. Let $K = K_Y \neq 0$, $\nu(-K, Y) \geq 0$ and

\[-K \equiv P + N\]

be the Zariski decomposition of $-K$ where $N = \sum_{i=1}^{n} \alpha_i F_i$. Then we have statements (a), (b), (c) and (d) below:

(a) All exceptional curves of $Y$ are curves $F_1, ..., F_n$, non-singular rational curves with square $(-2)$ and exceptional curves of the first kind (i.e. non-singular rational curves with square $(-1)$). The polyhedron $\mathcal{M}(Y)$ is locally finite.

(b) $\mathcal{M}(Y)$ is elliptic and $\mathbb{R}^+ P \in \mathcal{M}(Y)$ if $\nu(-K, Y) = 2$ (i.e. $P^2 > 0$).

(c) $\mathcal{M}(Y)$ is parabolic relative to the point $P = \mathbb{R}^+ P \in \mathcal{M}(Y) \cap \mathcal{L}_\infty$ if $\nu(-K, Y) = 1$ (i.e. $P^2 = 0$ but $P \neq 0$).

(d) $\mathcal{M}(Y)$ is hyperbolic relative to the subspace $\mathcal{T} = \bigcap_{i=1}^{n} \mathcal{H}_{F_i}$ of $\text{codim} \mathcal{T} = n$ in $\mathcal{L}(Y)$ if $\nu(-K, Y) = 0$ (i.e. $P \equiv 0$).

**Proof.** (a) Let $L$ be an exceptional curve of $Y$ and $L$ is different from $F_1, ..., F_n$. The arithmetic genus $p_a(L) = (L^2 + L \cdot K)/2 + 1 \geq 0$. Since $L$ is different from $F_1, ..., F_n$, then $L \cdot K = -L \cdot P - L \cdot N \leq 0$. Hence, either $L^2 = -2$ and $L \cdot K = 0$ or $L^2 = L \cdot K = -1$. For both cases $p_a(L) = 0$. Then $p_a(L) = 0$ since $p_a(L) \geq p_a(L) \geq 0$, and $L$ is non-singular rational. This proves the first statement of (a).

Elements $\delta \in \text{NS}(Y)$ with $\delta^2 = -1, -2$ define reflections

\[x \mapsto x + 2(-x \cdot \delta)/\delta^2, \quad x \in \text{NS}(Y)\].
Evidently, $s_\delta \in O_+(\text{NS}(Y))$ and $s_\delta$ in $\mathcal{L}(Y)$ is the reflection in the hyperplane $\mathcal{H}_\delta$. The group $O_+(\text{NS}(Y))$ is discrete in $\mathcal{L}(Y)$ (this is true for the automorphism group of any hyperbolic lattice $S$ acting in $\mathcal{L}(S)$, for example, see [27] for the general statement). It follows that the set of hyperplanes $\mathcal{H}_\delta$, $\delta \in \text{NS}(Y)$ with $\delta^2 = -1$, or $-2$, is locally finite in $\mathcal{L}(Y)$. The set of the hyperplanes $\mathcal{H}_{F_1}, ..., \mathcal{H}_{F_n}$ is finite. It follows that $\mathcal{M}(Y)$ is locally finite.

Let us prove (b), (c) and (d). The polyhedron $\mathcal{M}(Y)$ is obviously elliptic if $\rho(Y) = \dim \text{NS}(Y) = 1$ or 2 because $\mathcal{L}(Y)$ is 0 or 1-dimensional respectively.

Assume $\rho(Y) \geq 3$. Let

$$\overline{\mathcal{N}\mathcal{E}_-}(Y) = \{x \in \overline{\mathcal{N}\mathcal{E}(Y)} | x \cdot K < 0\}$$

be the ”negative” part of Mori cone $\overline{\mathcal{N}\mathcal{E}(Y)}$. By Mori theory [23], the projectivization $P\overline{\mathcal{N}\mathcal{E}_-}(Y) = \overline{\mathcal{N}\mathcal{E}_-}(Y)/\mathbb{R}^+$ is locally polyhedral and has as its vertices (equivalently, extremal rays of the cone $\overline{\mathcal{N}\mathcal{E}_-}(Y)$) rays $\mathbb{R}^+E$ where $E$ is an exceptional curve of the first kind.

By Riemann–Roch Theorem for surfaces, we have

$$\overline{\mathcal{V}^+(Y)} \subset \overline{\mathcal{N}\mathcal{E}(Y)}.$$ 

Let $P\overline{\mathcal{N}\mathcal{E}(Y)} = (\overline{\mathcal{N}\mathcal{E}(Y)} - \{0\})/\mathbb{R}^+$ is the projectivization of the Mori cone. Obviously, the cone $\overline{\mathcal{V}^+(Y)}$ is self-dual: $\overline{\mathcal{V}^+(Y)}^* = \overline{\mathcal{V}^+(Y)}$. Thus, we get the sequence of embeddings of projectivizations of the convex cones:

(2) \hspace{1cm} $\mathcal{M}(Y) \subset \overline{\mathcal{L}(Y)} \subset P\overline{\mathcal{N}\mathcal{E}(Y)}$

which is self-dual:

(3) \hspace{1cm} $\mathcal{M}(Y)^* = P\overline{\mathcal{N}\mathcal{E}(Y)}$, $P\overline{\mathcal{N}\mathcal{E}(Y)}^* = \mathcal{M}(Y)$ and $\overline{\mathcal{L}(Y)}^* = \overline{\mathcal{L}(Y)}$.

Using these facts, we get

**Lemma 3.2.** With conditions of Proposition 3.1, assume that $\dim \text{NS}(Y) \geq 3$ and $Q \in \mathcal{L}(Y)_\infty \cap \mathcal{M}(Y) \cap P\overline{\mathcal{N}\mathcal{E}_-}(Y)$. Let $U$ be an open neighborhood of $Q$ in the infinite sphere $\mathcal{L}(Y)_\infty$. Then $U$ is not contained in $\mathcal{M}(Y)$.

**Proof.** Let $U \subset \mathcal{M}(Y)$. Considering smaller neighborhood, we can assume that $U \subset P\overline{\mathcal{N}\mathcal{E}_-}(Y)$ since $\mathcal{L}(Y)_\infty \subset P\overline{\mathcal{N}\mathcal{E}(Y)}$. We claim that $U$ is contained in the boundary of $P\overline{\mathcal{N}\mathcal{E}_-}(Y)$. Indeed, let $\mathbb{R}^+c \in U$ where $c \in \text{NS}(Y) \otimes \mathbb{R}$, $c^2 = 0$ and $c \neq 0$. Since $P\overline{\mathcal{N}\mathcal{E}(Y)} = \mathcal{M}(Y)^*$ and $\mathbb{R}^+c \in \mathcal{M}(Y)$, we have

$$P\overline{\mathcal{N}\mathcal{E}_-}(Y) \subset P\overline{\mathcal{N}\mathcal{E}(Y)} \subset \overline{\mathcal{H}^+_c}$$

where

$$\overline{\mathcal{H}^+_c} = \{\mathbb{R}^+x | x \in \text{NS}(Y) \otimes \mathbb{R}, x \neq 0 \text{ and } x \cdot c \geq 0\}$$

is the projectivization of a half-space in $\text{NS}(Y) \otimes \mathbb{R}$. The point $\mathbb{R}^+c$ belongs to the boundary of $\overline{\mathcal{H}^+_c}$ because $c \cdot c = c^2 = 0$. It follows that $\mathbb{R}^+c$ belongs to the boundary of $P\overline{\mathcal{N}\mathcal{E}(Y)}$ and $P\overline{\mathcal{N}\mathcal{E}_-}(Y)$ because $P\overline{\mathcal{N}\mathcal{E}_-}(Y) \subset P\overline{\mathcal{N}\mathcal{E}(Y)} \subset \overline{\mathcal{H}^+_c}$. Thus, all points of $U$ belong to the boundary of $P\overline{\mathcal{N}\mathcal{E}_-}(Y)$. The sphere $\mathcal{L}(Y)$ is contained in the boundary of $P\overline{\mathcal{N}\mathcal{E}_-}(Y)$.
is strongly convex in $P(\text{NS}(Y) \otimes \mathbb{R})$. It follows that any point of $U$ is a vertex of $P\overline{\text{NE}_-}(Y)$ (equivalently, an extremal ray of $\overline{\text{NE}_-}(Y)$). We get the contradiction with Mori theory. Thus, $U$ is not contained in $\mathcal{M}(Y)$. ■

Now we prove (b), (c) and (d). By (a), the set $\text{Exc}(Y)$ is divided on three subsets: $\text{Exc}(Y)_3 = \{F_1, \ldots, F_n\}$ and $\text{Exc}(Y)_2$, $\text{Exc}(Y)_1$. The last two contain all non-singular rational exceptional curves with square $(-2)$ and $(-1)$ respectively which do not belong to $\text{Exc}(Y)_3$. If $F \in \text{Exc}(Y)_2$, then $F \cdot K = F \cdot P = F \cdot F_i = 0$ ($i = 1, \ldots, n$). If $E \in \text{Exc}(Y)_1$, we get $-E \cdot K = E \cdot (P + \sum_{i=1}^n \alpha_i F_i) = 1$ where $E \cdot P \geq 0$ and $E \cdot F_i \geq 0$. In particular, $\mathbb{R}^+ P \in \mathcal{M}(Y)$ for cases (b) and (c).

Let us consider the case (b). Let $\mathbb{R}^+ h \in \mathcal{M}(Y)$. Then $-h \cdot K = h \cdot P + h \cdot \sum_{i=1}^n \alpha_i F_i > 0$, because $h \cdot F_i \geq 0$, $\alpha_i > 0$ and $h \cdot P > 0$ since $\mathbb{R}^+ h \in \mathcal{L}(Y)$ and $\mathbb{R}^+ P \in \mathcal{L}(Y)$. Thus, $\mathcal{M}(Y) \subset P\overline{\text{NE}_-}(Y)$. Moreover, the set $\text{Exc}(Y)_3 \cup \text{Exc}(Y)_2$ is finite. Indeed, the set of elements $\delta \in \text{NS}(Y)$ with properties $0 \leq \delta \cdot P \leq 1$ and $\delta^2 = -1$ or $-2$ is finite for a hyperbolic lattice $\text{NS}(Y)$ and the fixed element $P \in \text{NS}(Y) \otimes \mathbb{Q}$ with $P^2 > 0$. Thus, the set $\text{Exc}(Y)$ is finite and $\mathcal{M}(Y)$ is bounded by a finite set of hyperplanes $\mathcal{H}_\delta$, $\delta \in \text{Exc}(Y)$. If $\mathcal{M}(Y)$ is not elliptic, there exists a non-empty open subset $U \subset \mathcal{M}(Y) \cap \mathcal{L}(Y)_\infty$. Since $\mathcal{M}(Y) \subset P\overline{\text{NE}_-}(Y)$, we get the contradiction with Lemma 3.2.

Case (c). Similarly to (b), one can prove that $\mathcal{M}(Y) - \{P\} \subset P\overline{\text{NE}_-}(Y)$ where $P = \mathbb{R}^+ P$. Let

$$R = \{\delta \in \text{Exc}(Y) \mid \delta \cdot P = 0\}.$$ 

By Lemma 1.3.1, the set $R$ is finite, since $\mathcal{M}(Y)$ has acute angles. The set $R$ contains $\{F_1, \ldots, F_n\}$, $\text{Exc}(Y)_2$ and a finite set of elements of $\text{Exc}(Y)_1$. Let $\text{Exc}(Y)'_1 = \text{Exc}(Y)_1 - R$. Let $\mathcal{M}(Y)_P = \bigcap_{\delta \in R} \mathcal{H}_\delta^P$. The polyhedron $\mathcal{M}(Y)_P$ is a cone with the vertex $P$ and the base $T = \{e \in \mathcal{E}_P \mid l \subset \mathcal{M}(Y)_P\}$ on the horosphere $\mathcal{E}_P$. Obviously, $\mathcal{M}(Y)_P \subset \mathcal{M}(Y)_P$. Let $K \subset \mathcal{E}_P$ be an elliptic polyhedron on the horosphere. If $C_K \cap \mathcal{M}(Y)_P = C_{K \cap T}$ is degenerate (equivalently, the polyhedron $K \cap T \subset \mathcal{E}_P$ is degenerate), the polyhedron $C_K \cap \mathcal{M}(Y)_P$ is degenerate either. Let $K \cap T$ be a non-degenerate polyhedron. Then the polyhedron $\mathcal{M}(Y) \cap C_K = \mathcal{M}(Y) \cap C_{K \cap T}$ is bounded by a finite set of hyperplanes of codimension one faces of $C_{K \cap T}$ which all contain the point $P$ and by hyperplanes $\mathcal{H}_E$, $E \in \text{Exc}(Y)'_1$, which have a non-empty intersection with the cone $C_{K \cap T}$ over the compact set $K \cap T$ of the horosphere $\mathcal{E}_P$. Moreover, $0 < E \cdot P \leq 1$ because $-E \cdot K = E \cdot P + E \cdot \sum_i \alpha_i F_i = 1$ and $E \cdot P > 0$, $E \cdot F_i \geq 0$, $\alpha_i > 0$. This is a simple purely arithmetic statement valid for any hyperbolic lattice (here, $\text{NS}(Y)$), any element $0 \neq P \in \text{NS}(Y) \otimes \mathbb{Q}$ with $P^2 = 0$ and any compact set $K \cap T \subset \mathcal{E}_P$, $P = \mathbb{R}^+ P$, that the set

$$\{e \in \text{NS}(Y) \mid e^2 = -1, \ 0 < e \cdot P \leq 1, \ \mathcal{H}_e \cap C_{K \cap T} \neq \emptyset\}$$

is finite. (One should remark that hyperplanes $\mathcal{H}_e$ with fixed $\lambda = e \cdot P > 0$ are touching some horosphere $\mathcal{E}_{P, R(\lambda)}$, and the set of tangent points is compact in $\mathcal{L}(Y)$ if additionally $\mathcal{H}_e \cap C_{K \cap T} \neq \emptyset$. See the proof of a similar statement below.) Thus, the polyhedron $\mathcal{M}(Y) \cap C_{K \cap T}$ is bounded by a finite set of hyperplanes in $\mathcal{L}(Y)$. If this polyhedron is not elliptic, there exists an open subset $U \subset \mathcal{L}(Y)_\infty \cap \mathcal{M}(Y) \cap C_{K \cap T}$ which does not contain $P$. Since $\mathcal{M}(Y) - \{P\} \subset P\overline{\text{NE}_-}(Y)$, we get the contradiction with Lemma 3.2. Thus, the polyhedron $\mathcal{M}(Y) \cap C_K = \mathcal{M}(Y) \cap C_{K \cap T}$ is elliptic.
Case (d). Let \( \mathcal{K} \subset \mathcal{T} \) be a compact elliptic polyhedron in \( \mathcal{T} \) and \( \mathcal{C}_\mathcal{K} \cap \mathcal{M}(Y) \) is non-degenerate.

We know that \( \mathcal{M}(Y) \) is a locally finite polyhedron with acute angles and \( P(\mathcal{M}(Y)) = \text{Exc}(Y) \). The subset \( \{F_1, ..., F_n\} = \text{Exc}(Y)_3 \subset \text{Exc}(Y) \) has a negative definite Gram matrix. It follows (see Sect. 1.3) that

\[
\mathcal{T} = \bigcap_{F \in \text{Exc}(Y)_3} \mathcal{H}_F
\]

is a subspace of \( \mathcal{L} \) of the codimension \( n \) and \( \gamma = \mathcal{T} \cap \mathcal{M}(Y) \) is a face of \( \mathcal{M}(Y) \) of the codimension \( n \). Moreover, \( \mathcal{M}(Y) - \gamma \subset P^\mathcal{M}(Y) - \gamma \), since \( h \cdot F_i > 0 \) for at least one \( i, 1 \leq i \leq n \), if \( \mathcal{R}^+ h \in \mathcal{M}(Y) - \gamma \), and \( -K = \sum \alpha_i F_i \) where \( \alpha_i > 0 \) and \( h \cdot F_i \geq 0 \).

Let \( E = \text{Exc}(Y)_2 \). Since \( E \notin \text{Exc}(Y)_3 \), \( -K = \sum \alpha_i F_i \) where \( \alpha_i > 0 \), and \( E \cdot K = 0 \), we have \( E \cdot \text{Exc}(Y)_3 = 0 \). Thus, hyperplanes \( \mathcal{H}_E \), \( E \in \text{Exc}(Y)_2 \) are orthogonal to \( \mathcal{T} \). Since \( \mathcal{M}(Y) \) is locally-finite, it follows that \( \mathcal{C}_\mathcal{K} \cap \mathcal{M}(Y) \) is bounded by a finite set of hyperplanes \( \mathcal{H}_E \), \( E \in \text{Exc}(Y)_3 \cup \text{Exc}(Y)_2 \). Let \( E \in \text{Exc}(Y)_1 \). Then

\[
1 = E \cdot (-K) = \sum_{i=1}^{n} \alpha_i (E \cdot F_i).
\]

Since \( \alpha_i \) are rational, \( \alpha_i > 0 \) and \( E \cdot F_i \geq 0 \), we have \( 0 \leq E \cdot F_i \leq N \) for some constant \( N \) depending from the rational numbers \( \alpha_i \) (\( N \) is not greater than the least common multiple of denominators of \( \alpha_1, ..., \alpha_n \)). Since the lattice \( \text{NS}(Y) \) is hyperbolic, we can prove purely arithmetically the following

**Statement.** The set

\[
\{e \in \text{NS}(Y) \mid e^2 = -1, \ 0 \leq e \cdot F_i \leq N \ \text{for all} \ 1 \leq i \leq n, \ \text{and} \ \mathcal{H}_e \cap \mathcal{C}_\mathcal{K} \neq \emptyset\}
\]

is finite.

**Proof.** Let \( S \) be a hyperbolic lattice (i.e. a hyperbolic integral symmetric bilinear form) and elements \( f_i \in S, 1 \leq i \leq n \), have a negative definite Gram matrix \( (f_i, f_j) \).

Let \( \mathcal{T} = \bigcap_{i=1}^{n} \mathcal{H}_{f_i} \subset \mathcal{L}(S) \) be the corresponding subspace in \( \mathcal{L}(S) \) and \( \mathcal{K} \subset \mathcal{T} \) a ball with a center \( Q \in \mathcal{T} \) and a radius \( R \). We should prove that the set

\[
\{e \in S \mid e^2 = -1, \ 0 \leq e \cdot f_i \leq N \ \text{for all} \ 1 \leq i \leq n, \ \text{and} \ \mathcal{H}_e \cap \mathcal{C}_\mathcal{K} \neq \emptyset\}
\]

is finite.

The Gram matrix of elements \( e, f_1, ..., f_n \) determines the configuration of hyperplanes \( \mathcal{H}_e, \mathcal{H}_{f_1}, ..., \mathcal{H}_{f_n} \) up to motions of the Lobachevsky space \( \mathcal{L}(S) \). By our conditions, the set of possible Gram matrices of \( e, f_1, ..., f_n \) is finite. So, we can assume that the configuration of the hyperplanes \( \mathcal{H}_e, \mathcal{H}_{f_1}, ..., \mathcal{H}_{f_n} \) is fixed up to motions of \( \mathcal{L}(S) \). Thus, we can assume that the configuration of the hyperplane \( \mathcal{H}_e \) and the subspace \( \mathcal{T} \) is fixed up to motions of \( \mathcal{L}(S) \). This configuration is defined by either the angle or the distance between \( \mathcal{H}_e \) and \( \mathcal{T} \).

Let us assume that the hyperplane \( \mathcal{H}_e \) intersects the subspace \( \mathcal{T} \) in a finite or an infinite point. Considering a bigger ball, we can suppose that the center \( Q \) of the ball is \( Q = \mathbb{R}^+ h \) where \( h \in S \) and \( h^2 > 0 \). Let us consider a 2-dimensional Lobachevsky plane with the same curvature \( (-1) \) and a line \( AO \) in the plane where
$A$ is a point at infinity and $O$ is a finite point. Let $B \in [A, O]$ and $\rho(B, O) = R$. Let $BC$ be the perpendicular line to the line $AO$ where $C$ is a point at infinity. We put $\rho(R) = \rho(AC, O)$ the distance between the line $AC$ and the point $O$. Obviously, the constant $\rho(R)$ only depends from $R$. Elementary geometrical considerations show that

$$\rho(\mathcal{H}_e, Q) \leq \rho(R).$$

It follows that $|(e \cdot h)/(e^2h^2)^{1/2}| \leq \cosh \rho(R)$. Since $e^2 = -1$ and $h$ is a fixed element, we get $|e \cdot h| < M$ for some constant $M$. Since $S$ is a hyperbolic (integral) lattice, the element $h \in S$ with $h^2 > 0$ is fixed, and $e^2 = -1$ is fixed, the set of these elements $e \in S$ is finite.

Now we assume that the distance $\rho(\mathcal{H}_e, \mathcal{T}) = r > 0$ is fixed. Let us consider the set

$$G(r) = \{X \in \mathcal{L}(S) \mid \rho(X, \mathcal{T}) = r\}.$$

This is the boundary of the strongly convex closed in $\mathcal{L}(S)$ set

$$\widetilde{G}(r) = \{X \in \mathcal{L}(S) \mid \rho(X, \mathcal{T}) \leq r\}.$$

The set $\widetilde{G}(r) = G(r) \cup \mathcal{T}_\infty$ is a closed hypersurface in $\overline{\mathcal{L}(S)}$ tangent the $\mathcal{L}(S)_\infty$ exactly in points of $\mathcal{T}_\infty$. Obviously, $\mathcal{H}_e$ is a hyperplane touching $G(r)$ exactly in one point of $G(r)$.

We consider an orthogonal basis $\xi_0, \xi_1, \ldots, \xi_m$ in $S \otimes \mathbb{R}$ such that $\xi_0^2 = 1$ and $\xi_i^2 = -1$ for $1 \leq i \leq m$. We can assume that $\mathcal{T}$ is orthogonal to $\xi_{m-n+1}, \ldots, \xi_m$, and the point $Q = \mathbb{R}^+\xi_0$. Then a point of $\overline{\mathcal{L}(S)}$ is uniquely defined by a ray $\mathbb{R}^+v$ where $v = \xi_0 + x_1\xi_1 + \cdots + x_m\xi_m$, and one can consider $(x_1, \ldots, x_m)$, $x_i \in \mathbb{R}$, as coordinates in $\overline{\mathcal{L}(S)}$.

In these coordinates, the $\overline{\mathcal{L}(S)}$ is the ball

$$\overline{\mathcal{L}(S)}: \quad x_1^2 + \cdots + x_m^2 \leq 1,$$

the subspace $\mathcal{T}$ is defined by the system of equations in $\mathcal{L}(S)$

$$\mathcal{T}: \quad x_{m-n+1} = \cdots = x_m = 0;$$

the hypersurface $G(r) \cup \mathcal{T}_\infty$ is the ellipsoid

$$G(r): \quad x_1^2 + \cdots + x_{m-n}^2 + x_{m-n+1}^2/\lambda(r) + \cdots + x_m^2/\lambda(r) = 1$$

where $\lambda(r) < 1$. The cylinder $C_\mathcal{K}$ is defined by the inequality in $\overline{\mathcal{L}(S)}$

$$C_\mathcal{K}: \quad x_1^2 + \cdots + x_{m-n}^2 \leq \mu(R)$$

where $\mu(R) < 1$. For this model, a hyperplane in $\mathcal{L}(S)$ is a section of $\mathcal{L}(S)$ by an affine linear hyperplane in the coordinates $x_1, \ldots, x_m$. Thus, hyperplanes $\mathcal{H}_e$ are tangent to the ellipsoid $G(r)$ hyperplanes which intersect the cylinder $C_\mathcal{K}$ in $\overline{\mathcal{L}(S)}$. Using the above coordinate description, one can easily find that the set of common points of the hyperplanes $\mathcal{H}_e$ and $G(r)$ is contained in the compact subset $A \subset G(r)$ which is defined by the condition: $x \in A$ if and only if $x \in G(r)$ and the tangent to $G(r)$ at the point $x$ a hyperplane has a common point with the cylinder $C_\mathcal{K} \cap \overline{\mathcal{L}(S)}$. 
Let $A_ε$ be the open $ε$-neighborhood of $A$ in $L(S)$. Then all hyperplanes $H_ε$ intersect the open subset $A_ε \subset L(S)$ with the compact closure $\overline{A_ε}$ in $L(S)$. Since the set of hyperplanes $H_ε$ with $ε \in S$ and $ε^2 = -1$ is locally finite in $L(S)$, it follows, that the set of hyperplanes $H_ε$ under consideration is finite. ■

We continue the proof of Proposition 3.1. By Statement above, the polyhedron $C_K \cap M(Y)$ is bounded by the finite set of hyperplanes $H_ε$, $ε \in \text{Exc}(Y)_1$. By considerations before the statement, it follows that the polyhedron $C_K \cap M(Y)$ is bounded by the finite set of hyperplanes. If $C_K \cap M(Y)$ is not elliptic, there is a non-empty open subset $D \subset C_K \cap M(Y) \cap L_∞$. By construction, the polyhedron $C_K \cap M(Y)$ is elliptic in a neighborhood of $\gamma$. Thus, we can assume that $D \cap \gamma = \emptyset$. Since $M(Y) - M(Y) \cap \gamma \subset P\overline{NE}_-(Y)$, it follows that $D \subset M(Y) \cap P\overline{NE}_-(Y) \cap L(Y)_∞$. This contradicts to Lemma 3.2. Thus, $M(Y) \cap C_K$ is elliptic. ■

We can apply Proposition 3.1, to describe Mori cone $\overline{NE}(Y)$. We recall that a ray $\mathbb{R}^+δ \subset \overline{NE}(Y), 0 \neq δ \in \overline{NE}(Y)$, is called extremal if $δ = δ_1 + δ_2$ where $0 \neq δ_1 \in \overline{NE}(Y)$ and $0 \neq δ_2 \in \overline{NE}(Y)$ implies that $δ_1, δ_2 \in \mathbb{R}^+δ$. Obviously, $\overline{NE}(Y)$ is generated by extremal rays: for any $0 \neq x \in \overline{NE}(Y)$ there are extremal rays $\mathbb{R}_1+δ_1, ..., \mathbb{R}_k+δ_k$ of $\overline{NE}(Y)$ such that $x = δ_1 + ... + δ_k$.

An extremal ray $\mathbb{R}^+δ$ of $\overline{NE}(Y)$ gives rise for the dual polyhedron $\mathbb{R}^+M(Y)$ the so called supporting half-space $H_δ^\perp$. That is the half-space

$$H_δ^\perp = \{ \mathbb{R}^+x \subset \text{NS}(Y) \otimes \mathbb{R} \mid x \neq 0 \text{ and } δ \cdot x \geq 0 \}$$

which contains $M(Y)$ and there don’t exist non-zero $δ_1, δ_2 \in \text{NS}(Y) \otimes \mathbb{R}$ such that $H_{δ_1}^\perp \neq H_{δ_2}^\perp$, $H_{δ_1}^\perp \neq H_{δ_2}^\perp$, $M(Y) \subset H_{δ_1}^\perp \cap H_{δ_2}^\perp$ and $H_{δ_1}^\perp \cap H_{δ_2}^\perp \subset H_{δ_2}^\perp$. Evidently, $\mathbb{R}^+δ$ is an extremal ray of $\overline{NE}(Y)$ if and only if $H_δ^\perp$ is a supporting half-space of $M(Y)$. Thus, the description of extremal rays of $\overline{NE}(Y)$ is equivalent to the description of supporting half-spaces of $M(Y)$.

In particular, since $M(Y) \subset PV(Y)$, we get that $δ^2 \leq 0$ for an extremal ray $\mathbb{R}^+δ$ of $\overline{NE}(Y)$. Using the description of extremal rays as supporting half-spaces of $M(Y)$ and Proposition 3.1, we get

**Corollary 3.3.** Let $Y$ be a non-singular projective algebraic surface. Then Mori polyhedron $\overline{NE}(Y)$ is generated by extremal rays $\mathbb{R}^+δ$ where $δ \in \text{NS}(Y) \otimes \mathbb{R}$ and $δ^2 \leq 0$.

Assume that the canonical class $K = K_Y \neq 0$ and the numerical anti-Kodaira dimension $ν(-K, Y) \geq 0$. Let $-K \equiv P + N$, $N = \sum_{i=1}^n α_i F_i$, be Zariski decomposition of $-K$. Then

(a) Extremal rays $\mathbb{R}^+δ$ of $\overline{NE}(Y)$ where $δ^2 < 0$ are exactly extremal rays $\mathbb{R}^+E$ where $E$ is an exceptional curve.

(b) If $ν(-K, Y) = 2$ (i.e. $P^2 > 0$), there is no an extremal ray $\mathbb{R}^+δ$ with $δ^2 = 0$; the set $\text{Exc}(Y)$ is finite and generates $\overline{NE}(Y)$.

(c) If $ν(-K, Y) = 1$ (i.e. $P^2 = 0$ but $P \neq 0$), there may exist the only extremal ray $\mathbb{R}^+c$ with $c^2 = 0$: this is $\mathbb{R}^+P$. The ray $\mathbb{R}^+P$ is extremal if and only if $P \neq \sum_{j=1}^m α_j E_j$ with $α_j > 0$ and $E_j \in \text{Exc}(Y)$. In particular, $\overline{NE}(Y)$ is generated by $\text{Exc}(Y) \cup \{P\}$.

(d) If $ν(-K, Y) = 0$ (i.e. $P \equiv 0$) and $\mathbb{R}^+c$ an extremal ray of $\overline{NE}(Y)$ with $c^2 = 0$, then $\mathbb{R}^+c \in M(Y)$ and $c \cdot F_i = 0$ for all $i = 1, ..., n$ (thus, $\mathbb{R}^+c \in M(Y) \cap T$ where $T = \mathbb{Q}^n \setminus U_0$). backwards, a ray $\mathbb{R}^+c \in M(Y) \cap T$ is extremal if and only if...
Proof. We don’t need this statement for the diagram method and leave details of the proof to reader.

It is the most important for us that by Proposition 3.1, we can apply the diagram method (results of Sect. 1.4) to $Y$ if $-K = -K_Y$ is pseudo-effective. We only remark that if $\nu(-K,Y) = 1$, then $\mathcal{M}(Y)$ is parabolic relative to $\mathbb{R}^+P$ and the element $e$ of Theorem 1.4.2 (or Theorem 1.4.2') has property $e \cdot P > 0$. It follows that $e$ is the class of an exceptional curve $E$ of the first kind. If $\nu(-K,Y) = 0$, when $\mathcal{M}(Y)$ is hyperbolic relative to the subspace $\mathcal{T} = \bigcap_{i=1}^n \mathcal{H}_{F_i}$ of $\dim \mathcal{T} = \dim \mathcal{L}(Y) - n$, the subset $Q$ is a set of classes of non-singular rational curves $E_1, \ldots, E_m$ with square $(-1)$ or $(-2)$, different from $F_1, \ldots, F_n$, and $m \leq \dim \mathcal{L} - n$. Theorems 1.4.3 and 1.4.3’give estimates for the number $n$ of exceptional curves $F_1, \ldots, F_n$.

Thus, applying Proposition 3.1 and results of Section 1.4, we get the following Diagram method Theorems for surfaces $Y$ with $\nu(-K,Y) \geq 0$.

**Theorem 3.4.** Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field, and numerical anti-Kodaira dimension $\nu(-K,Y) = 2$. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \text{Exc}(Y)$

$$\text{diam } \Gamma(L) \leq d.$$ 

(b) For any elliptic subset $\mathcal{E} \subset \text{Exc}(Y)$ such that $\mathcal{E}$ has $\dim \mathcal{NS}(Y) - 2$ elements, we have for the distance in the graph $\Gamma(\mathcal{E})$:

$$\sharp \{ \{E_1, E_2\} \subset \mathcal{E} \mid 1 \leq \rho(E_1, E_2) \leq d \} \leq C_1 \sharp \mathcal{E};$$

and

$$\sharp \{ \{E_1, E_2\} \subset \mathcal{E} \mid d + 1 \leq \rho(E_1, E_2) \leq 2d + 1 \} \leq C_2 \sharp \mathcal{E}.$$

Then $\dim \mathcal{NS}(Y) < 96(C_1 + C_2/3) + 69$.

**Theorem 3.5.** Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field, and numerical anti-Kodaira dimension $\nu(-K,Y) = 1$. Let $-K \equiv P + N$ be Zariski decomposition where $P^2 = 0$ and $P \neq 0$. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \text{Exc}(Y)$

$$\text{diam } \Gamma(L) \leq d.$$ 

(b) For any exceptional curve $E$ of the first kind such that $E \cdot P > 0$ and any elliptic subset $\mathcal{E} \subset \text{Exc}(Y)$ such that $\mathcal{E}$ has $\dim \mathcal{NS}(Y) - 2$ elements and $E \in \mathcal{E}$, we have for the distance in the graph $\Gamma(\mathcal{E})$:

$$\sharp \{ \{E_1, E_2\} \subset \mathcal{E} - \{E\} \mid 1 \leq \rho_E(E_1, E_2) \leq d \} \leq C_1(\sharp \mathcal{E} - 1);$$

and

$$\sharp \{ \{E_1, E_2\} \subset \mathcal{E} - \{E\} \mid d + 1 \leq \rho_E(E_1, E_2) \leq 2d + 1 \} \leq C_2(\sharp \mathcal{E} - 1).$$

Then $\dim \mathcal{NS}(Y) < 96(C_1 + C_2/3) + 70$. 
Theorem 3.6. Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field and numerical anti-Kodaira dimension $\nu(-K, Y) = 0$. Let $-K \equiv \sum_{i=1}^{n} \alpha_i F_i$ be Zariski decomposition of $-K$ (i.e. all $\alpha_i > 0$ and the Gram matrix of $F_1, \ldots, F_n$ is negative definite). Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \text{Exc}(Y)$

\[
\text{diam } \Gamma(L) \leq d.
\]

(b) For any elliptic subset $Q \subset \text{Exc}(Y)$ such that $Q$ contains only non-singular rational curves with square $(-1)$ or $(-2)$ which are different from $F_1, \ldots, F_n$ above and $\sharp Q \leq \dim \text{NS}(Y) - n - 1$, and for any elliptic subset $E \subset \text{NS}(Y)$ which contains $\dim \text{NS}(Y) - 2$ elements and $Q \subset E$, we have for the distance in the graph $\Gamma(E)$:

\[
\sharp\{\{E_1, E_2\} \subset E - Q \mid 1 \leq \rho(Q, E_2) \leq d\} \leq C_1 \sharp(E - Q);
\]

and

\[
\sharp\{\{E_1, E_2\} \subset E - Q \mid d + 1 \leq \rho(Q, E_2) \leq 2d + 1\} \leq C_2 \sharp(E - Q).
\]

Then $n < 96(C_1 + C_2/3) + 68$.

Theorem 3.5'. Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field and numerical anti-Kodaira dimension $\nu(-K, Y) = 1$. Let $-K \equiv P + N$ be Zariski decomposition where $P^2 = 0$ and $P \neq 0$. Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \text{Exc}(Y)$

\[
\text{diam } \Gamma(L) \leq d.
\]

(b) For any exceptional curve $E$ of the first kind such that $E \cdot P > 0$ and any elliptic subset $E \subset \text{Exc}(Y)$ such that $E$ has $\dim \text{NS}(Y) - 2$ elements and $E \in E$, we have for the distance in the graph $\Gamma(E)$:

\[
\sharp\{\{E_1, E_2\} \subset E - \{E\} \mid 1 \leq \rho(E_1, E_2) \leq d\} \leq C_1 (\sharp E - 1);
\]

and

\[
\sharp\{\{E_1, E_2\} \subset E - \{E\} \mid d + 1 \leq \rho(E_1, E_2) \leq 2d + 1\} \leq C_2 (\sharp E - 1).
\]

Then $\dim \text{NS}(Y) < 96(C_1 + C_2/3) + 70$.

Theorem 3.6'. Let $Y$ be a non-singular projective algebraic surface over an algebraically closed field and numerical anti-Kodaira dimension $\nu(-K, Y) = 0$. Let $-K = \sum_{i=1}^{n} \alpha_i F_i$ be the Zariski decomposition of $-K$ (i.e. all $\alpha_i > 0$ and the Gram matrix of $F_1, \ldots, F_n$ is negative definite). Assume that there are some constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) For any Lanner subset $L \subset \text{Exc}(Y)$

\[
\text{diam } \Gamma(L) \leq d.
\]
(b) For any elliptic subset $Q \subset \text{Exc}(Y)$ such that $Q$ contains only non-singular rational curves with square $(-1)$ or $(-2)$ which are different from $F_1, ..., F_n$ above and $\sharp Q \leq \dim \text{NS}(Y) - n - 1$, and for any elliptic subset $E \subset \text{NS}(Y)$ which contains $\dim \text{NS}(Y) - 2$ elements and $Q \subset E$, we have for the distance in the graph $\Gamma(E)$:

$$\sharp\{(E_1, E_2) \in E - Q \mid 1 \leq \rho(E_1, E_2) \leq d\} \leq C_1\sharp(E - Q);$$

and

$$\sharp\{(E_1, E_2) \in E - Q \mid d + 1 \leq \rho(E_1, E_2) \leq 2d + 1\} \leq C_2\sharp(E - Q).$$

Then $n < 96(C_1 + C_2/3) + 68$.

**Remark 3.7.** Probably, it is the most interesting applying Theorems 3.4–3.6 and 3.5’, 3.6’ to normal projective algebraic surfaces $Z$ with numerically effective anti-canonical class. We refer to Mumford [25] and Sakai [28] on intersection theory of Weil divisors on a normal surface.

Let $Z$ be a normal projective algebraic surface and $\sigma : Y \to Z$ be a resolution of singularities of $Z$. Then canonical classes are connected by the formula:

$$K_Y \equiv \sigma^*K_Z + \sum_{i=1}^n \alpha_i F_i,$$

where $F_i$ are irreducible components of the exceptional divisor of the resolution (i.e. $\sigma(F_i)$ is a singular point). One says that $\sigma$ is minimal if any $F_i$ is not an exceptional curve of the first kind. Then $\alpha_i \leq 0$ for all $i$. This property is very important for us. So, we name a resolution of singularities *almost minimal* if in the formula (4) corresponding to this resolution all $\alpha_i \leq 0$.

Let us assume that the surface $Z$ has nef anti-canonical class $-K_Z$. Then $(-\sigma^*K_Z)$ is nef, and for the almost minimal resolution of singularities $\sigma$, the formula (4) gives the Zariski decomposition of $-K_Y$ (if we take away zero summands in the $\sum$):

$$-K_Y \equiv -\sigma^*K_Z + \sum_{i=1}^n (-\alpha_i) F_i,$$

where $P = -\sigma^*K_Z$ is nef (since $-K_Z$ is nef) and $N = \sum_{i=1}^n (-\alpha_i) F_i$. Here $(-\alpha_i) \geq 0$ since $\sigma$ is almost minimal, the Gram matrix of $F_1, ..., F_n$ is negative definite by Mumford’s Theorem (see [25]) and evidently $P$ is orthogonal to all $F_i$. Since $P^2 = (-\sigma^*K_Z)^2 = (K_Z)^2$, we obtain

$$\nu(-K_Y, Y) = \begin{cases} 2, & \text{if } (K_Z)^2 > 0; \\ 1, & \text{if } (K_Z)^2 = 0 \text{ and } K_Z \not\equiv 0; \\ 0, & \text{if } K_Z \equiv 0; \end{cases}$$

Applying Theorems 3.4–3.6 and 3.5’, 3.6’ to the surface $Y$, we get a bound for $\dim \text{NS}(Y)$ for elliptic and parabolic cases (when $K_Z \not\equiv 0$). And we get a bound for $n$ in the formula (5) if we take away all summands in (5) with $\alpha_i = 0$. In particular, for the minimal resolution of singularities of $Z$, it is well-known that $\alpha_i = 0$ if and only if $\sigma(F_i)$ is a double rational point (Du Val singularity) of the type $A_n$, $D_n$, $E_6$, $E_7$ or $E_8$. Thus, for $K_Z \equiv 0$ we get a restriction on singularities of $Z$ (in particular, on the number of non-Du Val singularities).
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