Abstract

If \( \varphi \) is an analytic self-map of the open unit disc \( \mathbb{D} \) in the complex plane, the composition operator \( C_\varphi \) on the Hardy space \( H^p \) is defined as \( C_\varphi(f) = f \circ \varphi \). In this paper we prove an equivalent condition for the composition operator \( C_\varphi \) on \( H^p(1 < p < \infty) \) to have closed range. Actually, we show that the already known results for \( C_\varphi \) to have closed range on \( H^2 \) (Cima, Thomson, and Wogen (1974), Zorboska (1994)) can be extended to \( H^p \) for \( 1 < p < \infty \).

1 Introduction

Let \( \mathbb{D} \) denote the open unit disk in the complex plane, \( \mathbb{T} \) the unit circle, \( A \) the normalized area Lebesgue measure in \( \mathbb{D} \) and \( m \) the normalized length Lebesgue measure in \( \mathbb{T} \). For \( 1 < p < \infty \) the Hardy space \( H^p \) is defined as the set of all analytic functions in \( \mathbb{D} \) for which

\[
\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p dm(\theta) < +\infty
\]

and the corresponding norm in \( H^p \) is defined by

\[
\|f\|_{H^p}^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p dm(\theta).
\]

In this work we will mainly make use of the following equivalent norm:

\[
\|f\|_{H^p}^p = |f(0)|^p + \iint_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z).
\]
If \( \varphi \) is a non-constant analytic self-map of the unit disk \( \mathbb{D} \), then the composition operator \( C_\varphi : H^p \to H^p \) is defined as \( C_\varphi(f) = f \circ \varphi \) and the Nevanlinna counting function \( N_\varphi \) is defined as
\[
N_\varphi(w) = \begin{cases} 
\sum_{\varphi(z) = w} \log \frac{1}{|z|}, & \text{if } w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\} \\
0, & \text{otherwise}
\end{cases}
\] (1)

Let \( \rho(z, w) \) denote the pseudo-hyperbolic distance between \( z, w \in \mathbb{D} \),
\[
\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|
\]
and \( D_\eta(a) \) denote the pseudo-hyperbolic disk of center \( a \in \mathbb{D} \) and radius \( \eta < 1 \):
\[
D_\eta(a) = \{ z \in \mathbb{D} : \rho(a, z) < \eta \}.
\]

In the following, \( C \) denotes a positive and finite constant which may change from one occurrence to another. Moreover, by writing \( K(z) \simeq L(z) \) for the non-negative quantities \( K(z) \) and \( L(Z) \) we mean that \( K(z) \) is comparable to \( L(z) \) if \( z \) belongs to a specific set: there are positive constants \( C_1 \) and \( C_2 \) independent of \( z \) such that
\[
C_1 K(z) \leq L(z) \leq C_2 K(z).
\]

In [1] and [7] the case of closed range composition operators in Hardy space \( H^2 \) is studied. In [1] Cima, Thomson, and Wogen gave an equivalent condition for \( C_\varphi : H^2 \to H^2 \) to have closed range that depends only on the behavior of the function \( \varphi \) on the boundary \( \mathbb{T} \) of the open unit disk \( \mathbb{D} \). First, they extend \( \varphi \) in \( \mathbb{T} \) as the boundary limit of points in the disk, that is \( \varphi(\zeta) = \lim_{r \to 1} \varphi(r\zeta) \).

It is well known that this limit exists almost everywhere in \( \mathbb{T} \) with respect to Lebesgue measure \( m \). Then they define the measure \( \nu_\varphi \) on Borel sets \( E \subset \mathbb{T} \) by
\[
\nu_\varphi(E) = m(\varphi^{-1}(E)).
\] (2)

This measure \( \nu_\varphi \) is absolutely continuous to Lebesgue measure \( m \) on \( \mathbb{T} \) and its Radon-Nikodym derivative \( \frac{d\nu_\varphi}{dm} \) is in \( L^\infty(\mathbb{T}) \).

**Theorem 1 (Cima, Thomson, and Wogen).**

\( C_\varphi : H^2 \to H^2 \) has closed range if and only if the Radon-Nikodym derivative \( \frac{d\nu_\varphi}{dm} \) is essentially bounded away from zero.

In [7] Zorboska proved a criterion for \( C_\varphi \) to have closed range on \( H^2 \) based upon properties of \( \varphi \) on pseudo-hyperbolic disks. She defines the function
\[
\tau_\varphi(z) = \frac{N_\varphi(z)}{\log \frac{1}{|z|}}
\]
for \( z \in \varphi(\mathbb{D}) \setminus \varphi(0) \) and, for \( c > 0 \), the set
\[
G_c = \{ z \in \mathbb{D} : \tau_\varphi(z) > c \}.
\] (3)
Theorem 2 (Zorboska).

\( C_\varphi : H^2 \to H^2 \) has closed range if and only if there exist constants \( c > 0 \), \( \delta > 0 \) and \( \eta \in (0, 1) \) such that the set \( G_c \) satisfies

\[
A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))
\]

for all \( a \in \mathbb{D} \).

2 Main result

We are going to prove that the results of theorems 1 and 2 hold, not only for the case of \( H^2 \) space, but for every space \( H^p, 1 < p < \infty \). Actually, we are going to prove the following result.

Theorem 3

Let \( 1 < p < \infty \). Then the following assertions are equivalent:

(i) \( C_\varphi : H^p \to H^p \) has closed range.

(ii) The Radon-Nikodym derivative \( \frac{d\nu_\varphi}{dm} \) is essentially bounded away from zero.

(iii) There exist \( c > 0 \), \( \delta > 0 \) and \( \eta \in (0, 1) \) such that the set \( G_c \) satisfies

\[
A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))
\]

for all \( a \in \mathbb{D} \).

We will make use of the following theorem and lemma proved in [2], as well as theorem proved in [4]. Let \( \Delta \) be the usual Laplacian \( \Delta = \frac{\partial^2}{\partial z \partial \overline{z}} \) and, for \( \zeta \in \mathbb{T} \) and \( 0 < h < 1 \), let \( W(\zeta, h) \) be the usual Carleson square

\[
W(\zeta, h) = \{ z \in \overline{\mathbb{D}} : 1 - h < |z| \leq 1, |\arg(z\overline{\zeta})| \leq \pi h \}.
\]

We also will make use of the measure \( m_\varphi \) defined on Borel sets \( E \subset \overline{\mathbb{D}} \) by

\[
m_\varphi(E) = m(\varphi^{-1}(E) \cap \mathbb{T}).
\]

Actually, \( \nu_\varphi \) defined in (2) is the restriction of \( m_\varphi \) on \( \mathbb{T} \).

Lemma 1

For every \( g \in C^2(\mathbb{C}) \) we have

\[
\iint_{\mathbb{D}} g(z)dm_\varphi(z) = g(\varphi(0)) + \frac{1}{2} \iint_{\mathbb{D}} \Delta g(w)N_\varphi(w)dA(w).
\]

Theorem 4

For \( 0 < c < \frac{1}{8} \), \( \zeta \in \mathbb{T} \) and \( 0 < h < (1 - |\varphi(0)|)/8 \), we have

\[
\sup_{z \in W(\zeta, h) \cap \mathbb{D}} N_\varphi(z) \leq \frac{100}{c^2} m_\varphi[W(\zeta, (1 + c)h)].
\]
Theorem 5 Let $1 < p < \infty$ and let $\mu$ be a positive measure in $\overline{D}$. Then the following assertions are equivalent.

(i) There exists $C > 0$ such that for every $f \in H^p \cap C(\overline{D})$,
\[
\iint_{\overline{D}} |f(z)|^p d\mu(z) \geq C \|f\|^p_{H^p}.
\]

(ii) There exists $C > 0$ such that for every $\lambda \in D$
\[
\iint_{\overline{D}} |K_\lambda(z)|^p d\mu(z) \geq C,
\]
where $k_\lambda(z) = \frac{1}{1-\lambda z}$ is the reproducing kernel in $H^p$ and $K_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{H^p}}$ is its normalised companion.

(iii) There exists $C > 0$ such that for $0 < h < 1$ and $\zeta \in T$ we have
\[
\mu(W(\zeta, h)) \geq C h.
\]

(iv) There exists $C > 0$ such that the Radon-Nikodym derivative of $\mu|_T$ (the restriction of measure $\mu$ on $T$) with respect to $m$ is bounded below by $C$.

In the proof of theorem 5 we will make use of a theorem of D. Luecking in [5] which we restate here as

Theorem 6 Let $\tau$ be a non-negative, measurable, bounded function in $D$ and for $c > 0$ let $G_c$ be the set $G_c = \{z \in D : \tau(z) > c\}$. Then the following assertions are equivalent.

(i) There exists $C > 0$ such that
\[
\iint_{\overline{D}} |f'(z)|^2 |\tau(z)| \log \frac{1}{|z|} dA(z) \geq C \iint_{D} |f'(z)|^2 \log \frac{1}{|z|} dA(z)
\]
for every $f \in H^2$.

(ii) There exist $C > 0$ and $c > 0$ such that
\[
\iint_{G_c} |f'(z)|^2 \log \frac{1}{|z|} dA(z) \geq C \iint_{D} |f'(z)|^2 \log \frac{1}{|z|} dA(z)
\]
for every $f \in H^2$.

(iii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that $G_c$ satisfies
\[
A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))
\]
for all $a \in D$. 

4
In the following we will also use the following non-univalent change of variable formula (see [6], section 4.3). If \( g \) is a measurable, non-negative function in \( \mathbb{D} \), we have

\[
\iint_D g(\varphi(z))|\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) = 2 \iint_D g(w)N_\varphi(w) dA(w). \tag{5}
\]

**Proof of theorem 3.** Without loss of generality we may assume that \( \varphi(0) = 0 \) and \( f(0) = 0 \) for all functions \( f \) in \( H^p \).

**Proof of (i) \( \Rightarrow \) (iii).** If \( C_\varphi \) has closed range then, there exist \( C > 0 \) (we may suppose \( C < 2 \)) such that, for every \( f \in H^p \) we have

\[
\|C_\varphi f\|_H^p \geq C\|f\|_H^p
\]
i.e.

\[
\iint_D |f(\varphi(z))|^{p-2} |f'(\varphi(z))|^2 |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) \geq C \int |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z).
\]

By (5) we have

\[
\iint_D |f(w)|^{p-2} |f'(w)|^2 N_\varphi(w) dA(w) \geq C \iint_D |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z)
\]
i.e.

\[
\iint_D |f(w)|^{p-2} |f'(w)|^2 \tau_\varphi(w) \log \frac{1}{|w|} dA(w)
\]

\[
\geq C \iint_D |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z). \tag{6}
\]

Let \( f \in H^p \) with \( f(z) \neq 0 \) for every \( z \in \mathbb{D} \). We define the analytic function

\[ g(z) = f(z)^{p/2}. \]

Obviously, \( g \in H^2 \) and \( g(z) \neq 0 \) for every \( z \in \mathbb{D} \). Then from (6) we have

\[
\iint_D |g'(w)|^2 \tau_\varphi(w) \log \frac{1}{|w|} dA(w) \geq C \iint_D |g'(z)|^2 \log \frac{1}{|z|} dA(z)
\]

and, because of Luecking’s theorem [6]

\[
\iint_{G_c} |g'(w)|^2 \log \frac{1}{|w|} dA(w) \geq C \iint_D |g'(z)|^2 \log \frac{1}{|z|} dA(z)
\]

5
or, equivalently,

\[ \iint_{G_c} |g'(w)|^2 (1 - |w|^2) dA(w) \geq C \iint_{D} |g'(z)|^2 (1 - |z|^2) dA(z). \tag{7} \]

for every \( g \in H^2 \) with \( g(z) \neq 0 \) for every \( z \in \mathbb{D} \).

Let \( a \in \mathbb{D} \). Choosing \( g \in H^2 \) such that \( |g'(z)|^2 = \frac{(1 - |a|^2)^3}{|1 - az|^6} \) and with \( g(z) \neq 0 \) for every \( z \in \mathbb{D} \), we get

\[ \iint_{G_c \cap D(a)} \frac{(1 - |a|^2)^3}{|1 - az|^6} (1 - |z|^2) dA(z) \geq \iint_{G_c} \frac{(1 - |a|^2)^3}{|1 - az|^6} (1 - |z|^2) dA(z) \]

\[ - \iint_{D \setminus D(a)} \frac{(1 - |a|^2)^3}{|1 - az|^6} (1 - |z|^2) dA(z). \tag{8} \]

From (7) we get

\[ \iint_{G_c} \frac{(1 - |a|^2)^3}{|1 - az|^6} (1 - |z|^2) dA(z) \geq C \iint_{D} \frac{(1 - |a|^2)^3}{|1 - az|^6} (1 - |z|^2) dA(z) = \frac{C}{2}. \tag{9} \]

We have

\[ \iint_{D} (1 - |z|^2) dA(z) = \frac{1}{2}, \]

and we choose \( \eta \in (0, 1) \) such that

\[ \iint_{|z| < \eta} (1 - |z|^2) dA(z) \geq \frac{1}{2} - \frac{C}{4} = \frac{2 - C}{4} > 0. \]

By the change of variable \( z = \frac{w - a}{1 - aw} = \psi_a(w) \) we get

\[ \iint_{D(a)} \frac{(1 - |\psi_a(w)|^2)^2}{|1 - aw|^4} (1 - |a|^2)^2 dA(w) \geq \frac{2 - C}{4} \]

and hence

\[ \iint_{D \setminus D(a)} \frac{(1 - |\psi_a(w)|^2)^2}{|1 - aw|^4} dA(w) \leq \frac{C}{4}. \tag{10} \]

Combining (8), (9) and (10), we find

\[ \iint_{G_c \cap D(a)} \frac{(1 - |a|^2)^3}{|1 - az|^6} (1 - |z|^2) dA(z) \geq \frac{C}{4}. \]
Using the fact that if \( z \in D_{\eta}(a) \) then \( (1 - |a|^{2}) \asymp (1 - |z|^{2}) \asymp |1 - \pi a| \), we get
\[
\frac{C' A(G_{c} \cap D_{\eta}(a))}{(1 - |a|^{2})^{2}} \geq C \frac{1}{4}
\]
and, finally,
\[
A(G_{c} \cap D_{\eta}(a)) \geq \delta A(D_{\eta}(a)),
\]
**Proof of (iii) \( \Rightarrow \) (i).** We consider the measure \( m_{\varphi} \) as defined in (5) and we will show that (iii) of theorem 3 implies (iii) of theorem 5 with \( \mu = m_{\varphi} \). Now we consider \( \zeta \in \mathbb{T} \) and \( 0 < h < 1 \) and the corresponding Carleson box \( W(\zeta, h) \). Having in mind to apply theorem 4 we take \( c = \frac{1}{10} \) and \( h' = \frac{h}{3} \). Then there exists \( a \in W(\zeta, h') \) so that
\[
D_{\eta}(a) \subset W(\zeta, h') \subset W(\zeta, (1 + c)h') \subset W(\zeta, h), \quad 1 - |a|^{2} \geq Ch,
\]
where \( C \) depends upon \( \eta \). We have \( A(G_{c} \cap D_{\eta}(a)) \geq \delta A(D_{\eta}(a)) \) and hence \( G_{c} \cap D_{\eta}(a) \neq \emptyset \). Let \( b \in G_{c} \cap D_{\eta}(a) \). Then \( 1 - |b|^{2} \geq Ch \) and \( N_{\varphi}(b) \geq c \log \frac{1}{|b|} \) (since \( b \in G_{c} \)). Applying theorem 4 (recalling that \( \varphi(0) = 0 \), we find
\[
m_{\varphi}(W(\zeta, h)) \geq m_{\varphi}(W(\zeta, (1 + c)h')) \geq C \sup_{z \in W(\zeta, h) \cap \mathbb{D}} N_{\varphi}(z) \geq CN_{\varphi}(b)
\]
\[
\geq C \log \frac{1}{|b|} \geq C(1 - |b|^{2}) \geq Ch.
\]
Therefore we get (iii) of theorem 5 which is equivalent to (i) of the same theorem, with \( \mu = m_{\varphi} \). Now we take any \( f \) which is analytic in a disk larger than \( \mathbb{D} \) and so that \( f(0) = 0 \) and use (i) of theorem 5 together with lemma 3 to find
\[
||C_{\varphi}f||_{H^{p}}^{p} = \int_{\mathbb{D}} |f(\varphi(z))|^{p} - |f'(\varphi(z))|^{2} |\varphi'(z)|^{2} \log \frac{1}{|z|} dA(z)
\]
\[
= \int_{\mathbb{D}} |f(w)|^{p} - |f'(w)|^{2} N_{\varphi}(w) dA(w)
\]
\[
= \frac{1}{p} \int_{\mathbb{D}} \Delta(|f|^{p}) N_{\varphi}(w) dA(w) = C \int_{\mathbb{D}} |f(w)|^{p} dm_{\varphi}(w)
\]
\[
\geq C ||f||_{H^{p}}^{p},
\]
Now, if \( f \) is the general function in \( H^{p} \) with \( f(0) = 0 \), we apply the result to the functions \( f_{r}, 0 < r < 1 \), defined by \( f_{r}(z) = f(rz) \), \( z \in \mathbb{D} \), and take the limit as \( r \to 1^- \). Therefore \( C_{\varphi} \) has closed range.

**Proof of (i) \( \Rightarrow \) (ii).** Let’s suppose that \( C_{\varphi} \) has closed range on \( H^{p} \) and \( E \subset \mathbb{T} \). For \( n \in \mathbb{N} \) we can choose a function \( f_{n} \in H^{p} \) (theorem 4.4, page 63, in [3]) such that
\[
|f_{n}(\zeta)|^{p} = \begin{cases} 1, & \zeta \in E \\
\frac{1}{|n|}, & \zeta \in \mathbb{T} \setminus E \end{cases}
\]
Then $\|C_\varphi f_n\|_{H^p} \geq C\|f_n\|_{H^p}$ and hence

$$m(\varphi^{-1}(E)) + \frac{1}{2^n} m(T \setminus \varphi^{-1}(E)) \geq C m(E) + C \frac{1}{2^n} m(T \setminus E).$$

Taking limit as $n \to +\infty$, we get $m(\varphi^{-1}(E)) \geq C m(E)$ and, finally,

$$\nu_\varphi(E) \geq C m(E).$$

Thus the Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ is bounded below by $C$.

**Proof of (iii) $\Rightarrow$ (i).** Let’s suppose that the Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ is bounded below by $C$. For $\lambda > 0$ we consider the set

$$E_f(\lambda) = \{ e^{i\theta} : |f(e^{i\theta})| > \lambda \}.$$

Then,

$$m(E_{f\circ\varphi}(\lambda)) = \nu_\varphi(E_f(\lambda)) \geq C m(E_f(\lambda))$$

for all $\lambda > 0$, and finally

$$\|C_\varphi f\|_{H^p}^p = \int_0^{2\pi} |f(\varphi(e^{i\theta}))|^p dm(\theta) = \int_0^{+\infty} p\lambda^{p-1} m(E_{f\circ\varphi}(\lambda)) d\lambda$$

$$\geq C \int_0^{+\infty} p\lambda^{p-1} m(E_f(\lambda)) d\lambda = C \int_0^{2\pi} |f(e^{i\theta})|^p dm(\theta) = C \|f\|_{H^p}^p.$$

Hence $C_\varphi$ has closed range.

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