We present some general results for the time-dependent mass Hamiltonian problem with $H = -\frac{1}{2}e^{-2\nu(t)}\partial_{xx} + h^{(2)}(t)e^{2\nu(t)}x^2$, where $\nu(t)$ is a continuous function of $t$. This Hamiltonian corresponds to a time-dependent mass ($TM$) Schrödinger equation with the restriction that there are only $P^2$ and $X^2$ terms. We give the specific transformations to a different quadratic Schrödinger ($TQ$) equation and to a different time-dependent oscillator ($TO$) equation. For each Schrödinger system, we give the Lie algebra of space-time symmetries and $(x,t)$ representations for number states, coherent states, and squeezed states. These general results include earlier work as special cases.

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1 Introduction

Recently \[1, 2\], we studied general time-dependent quadratic (TQ) Schrödinger equations and Hamiltonians

\[
S_1 \Phi(x,t) = \{-[1 + k(t)] P^2 + 2T + h(t)D + g(t)P
\]

\[
-2h^{(2)}(t)X^2 - 2h^{(1)}(t)X - 2h^{(0)}(t)I\} \Phi(x,t) = 0,
\]

\[
T = i\partial_t, \quad P = -i\partial_x, \quad X = x, \quad I = 1,
\]

\[
P^2 = -\partial_{xx}, \quad X^2 = x^2, \quad D = \frac{1}{2}(XP + PX) = -ix\partial_x - i/2.
\]

\[
H_1(x,t) = \frac{[1 + k(t)]}{2} P^2 - \frac{h(t)}{2} D - \frac{g(t)}{2} P + h^{(2)}(t)X^2 + h^{(1)}(t)X + h^{(0)}(t)I,
\]

where \{k(t), h(t), g(t), h^{(2)}(t), h^{(1)}(t), h^{(0)}(t)\} are arbitrary functions of time. We showed in Ref. [1] that these are related to time-dependent mass (TM) equations

\[
S_2 \Theta(x,t) = \{-f(t)P^2 + 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I\} \Theta(x,t) = 0
\]

by unitary transformations. We also demonstrated that the TM equations are in turn related to time-dependent oscillator (TO) equations of the form

\[
S_3 \Psi(x,t') = \{-P^2 + 2T' - 2g^{(2)}(t')X^2 - 2g^{(1)}(t')X - 2g^{(0)}(t')I\} \Psi(x,t') = 0,
\]

by time transformations \(t(t')\). In Ref. [2] we obtained the isomorphic Schrödinger algebras for these three types of systems and the general characteristics of the number states, coherent states, and squeezed states.

In the present paper we will show in detail how to obtain the number states, coherent states, and squeezed states for a certain subclass of these systems: specifically those which come from TM Hamiltonians with only time-dependent \(P^2\) and \(X^2\) terms. Some examples of this TM subclass of equations have been partially discussed in the literature [3]-[11]. Further, elsewhere we have applied our formalism to two specific Hamiltonians, obtaining very general new results [12, 13, 14]. Those examples are special cases of the more general results given here.
We begin with Section 2, where we describe this subclass of $TM$ Schrödinger equations and show the mappings from $TQ$ Schrödinger equations and to $TO$ Schrödinger equations. The main reason for mapping to $TO$ equations is that we have an algorithm for computing the operators that form a basis for the Lie, space-time symmetry algebra for $TO$ equations [15]-[18]. This algebra is called a Schrödinger algebra, and has an oscillator subalgebra, $os(1)$.

In Section 3 we construct the algebra for the $TO$ equation. Then, using the mappings discussed above, we construct the analogous $os(1)$ operators for the $TM$ equation and the $TQ$ equation. Using the structure of the oscillator subalgebra, $os(1)$, of the $TO$ Schrödinger algebra, in Section 4 we construct a set of number states that are solutions to the $TO$ Schrödinger equation. We then proceed and do the same for the $TM$ and $TQ$ Schrödinger equations.

Next, for the $TO$ equation, we can formulate displacement-operator coherent-state (DOCS) wave functions from the extremal number state and the ladder operators of the $os(1)$ algebra. Employing the transformations mentioned above, we can then transform the $TO$ coherent states into the corresponding coherent states for the $TM$ and $TQ$ Schrödinger equations. This is done in Section 5. The analogous program is done for the DOSS (squeezed-states) in Section 6.

2 The Subclass of Equations

The subclass of TM equations of interest to us have the form

\[ \hat{S}_2 \Theta(x,t) = \left\{ -2\hat{H}_2 + 2T \right\} \Theta(x,t) = 0, \]  
\[ \hat{H}_2 = \frac{1}{2}e^{-2\nu(t)}P^2 + h^{(2)}(t)e^{2\nu(t)}X^2, \] 

where $\nu(t)$ is a real function of $t$. This class of Schrödinger equations can be obtained from the subclass of TQ equations

\[ S_1 \Phi(x,t) = \{ -2H_1 + 2T \} \Phi(x,t) = 0, \] 
\[ H_1 = \frac{1}{\hbar}P^2 - \frac{1}{\hbar}h(t)D + h^{(2)}(t)X^2, \]

by the unitary transformation

\[ R(0,\nu(t),0) = \exp \left\{ i\nu(t)D \right\}, \]
\[ \nu(t) = -\frac{1}{2} \int_{t_o}^{t} ds \, h(s). \] (10)

The Hamiltonians of Refs. [12, 13] are examples of this type of TM Schrödinger equation, which were analyzed in detail there.

In order to solve this problem using algebraic methods, we transform the TM Schrödinger equation (9) into a TO Schrödinger equation

\[ S_3 \Psi(x, t') = \left\{ -2H_3 + 2T' \right\} \Psi(x, t') = 0, \] (11)

by a specific change in the “time” variable, \( t \to t' \), and with \( T' = i\partial t' \). The resulting TO Hamiltonian is

\[ H_3 = \frac{1}{2} P^2 + g^{(2)}(t') X^2, \] (12)

\[ g^{(2)}(t') = \tilde{h}^{(2)}(t') e^{4\nu(t')}, \] (13)

\[ \tilde{h}^{(2)}(t') = (h^2 \circ t)(t'), \quad \nu(t') = (\nu \circ t)(t'), \] (14)

all time dependence now being in the single function \( g^{(2)}(t') \). The operators \( X, X^2, P, P^2, \) and \( D \) and their commutation relations are given in Eq. (1).

According to Eq. (I-67), the time transformation is given by

\[ t' - t'_o = \int_{t_o}^{t} ds \, e^{-2\nu(s)}. \] (15)

We assume that the mapping \( t'(t) \), described in Eq. (15), has an inverse \( t(t') \). This was certainly true for the Hamiltonians of Refs. [12, 13].

3 The Algebras for our Problem

3.1 The TO Schrödinger algebra, \((SA)^c_1\), and its os(1) Subalgebra

Combining Eqs. (11) and (12), we obtain the equation

\[ S_3 \Psi(x, t') = \left\{ -P^2 + 2T' - 2g^{(2)}(t') X^2 \right\} \Psi(x, t') = 0. \] (16)

It is known [16, 18] that this equation has the Schrödinger algebra \((SA)^c_1\) as its Lie algebra of space-time symmetries. This algebra is spanned by six operators.
Three of the operators form an $su(1, 1)$ subalgebra,

\[
\begin{align*}
M_{3-} &= \phi_1 t' - \frac{1}{2} \dot{\phi}_1 D + \frac{1}{4} \ddot{\phi}_1 X^2, \\
M_{3+} &= \phi_2 t' - \frac{1}{2} \dot{\phi}_2 D + \frac{1}{4} \ddot{\phi}_2 X^2, \\
M_3 &= \phi_3 t' - \frac{1}{2} \dot{\phi}_3 D + \frac{1}{4} \ddot{\phi}_3 X^2.
\end{align*}
\] (17)

(The ‘dot’ over a symbol will be reserved for differentiation by $t'$.) Three others form an Heisenberg-Weyl subalgebra, $w_i^c$,

\[
\begin{align*}
J_{3-} &= i \left\{ \xi P - \dot{\xi} X \right\}, & J_{3+} &= i \left\{ -\bar{\xi} P + \dot{\bar{\xi}} X \right\}. & I = 1,
\end{align*}
\] (18)

The function, $\xi$, and its complex conjugate, $\bar{\xi}$ are solutions to the differential equation

\[
\ddot{\gamma} + 2g^{(2)}(t')\dot{\gamma} = 0,
\] (19)

where, as indicated, $\dot{\gamma} \equiv d\gamma/dt'$. These solutions satisfy the Wronskian condition [16, 19]

\[
W(\xi, \bar{\xi}) = \xi \dot{\bar{\xi}} - \dot{\xi} \bar{\xi} = -i.
\] (20)

The functions, $\phi_j$, $j = 1, 2, 3$, are defined as

\[
\begin{align*}
\phi_1(t') &= \xi^2(t'), & \phi_2(t') &= \bar{\xi}^2(t'), & \phi_3(t') &= 2\xi(t')\bar{\xi}(t').
\end{align*}
\] (21)

The $su(1, 1)$ operators satisfy the commutation relations

\[
[M_{3+}, M_{3-}] = -M_3, & \quad [M_3, M_{3\pm}] = \pm 2M_{3\pm}.
\] (22)

The $w_i^c$ operators satisfy the nonzero commutation relation

\[
[J_{3-}, J_{3+}] = I.
\] (23)

The remaining commutation relations are

\[
\begin{align*}
[M_{3-}, J_{3-}] &= 0, & [M_{3+}, J_{3-}] &= +J_{3+}, & [M_3, J_{3-}] &= -J_{3-}, \\
[M_{3-}, J_{3+}] &= -J_{3-}, & [M_{3+}, J_{3+}] &= 0, & [M_3, J_{3+}] &= +J_{3+}.
\end{align*}
\] (24)
In order to find solutions to Eq. (16), we need not consider the entire algebra \((SA)\), only
the \(os(1)\) subalgebra consisting of the operators \(\{M_3, J_{3\pm}, I\}\). From the above commutation
relations, we see that these operators close to give the nonzero commutators

\[
[M_3, J_{3\pm}] = \pm J_{3\pm}, \quad [J_{3-}, J_{3+}] = I. \tag{25}
\]

Eq. (25) has the structure of an \(os(1)\) oscillator algebra. Closure of the oscillator subalgebra
follows from the Wronskian (20) and the relationships

\[
i\xi = \phi_3 \dot{\xi} - \frac{1}{2} \ddot{\phi}_3 \xi, \tag{26}
\]

\[
i\dot{\xi} = \frac{d}{dt} \left( \phi_3 \dot{\xi} - \frac{1}{2} \ddot{\phi}_3 \xi \right) = \phi_3 \dddot{\xi} + \frac{1}{2} \dddot{\phi}_3 \xi - \frac{1}{2} \dddot{\phi}_3 \xi, \tag{27}
\]

Indeed, this subset of operators and commutation relations may be viewed as a generalization
of the usual oscillator algebra.

Although we can construct the full Schrödinger algebras for each subclass of TO (11), TM
(5), and TQ (7) equations, explicitly fully written in Eq. (16) and below,

\[
\hat{S}_2 \hat{\Theta}(x,t) = \left\{ -e^{-2\nu}P^2 + 2T - 2e^{2\nu}X^2 \right\} \hat{\Theta}(x,t) = 0, \tag{28}
\]

\[
S_1 \Phi(x,t) = \left\{ -P^2 + 2T + h(t)D - 2h(2)(t)X^2 \right\} \Phi(x,t) = 0, \tag{29}
\]

we shall henceforth consider only the \(os(1)\) subalgebras for all three systems.

### 3.2 The \(os(1)\) algebra for \(TM\) Schrödinger Equations

The operators that span the \(os(1)\) algebra for Eq. (5) are

\[
\hat{M}_2 = \hat{\phi}_3 e^{2\nu}T - \frac{1}{2} \hat{\phi}_3 D + \frac{1}{4} \hat{\phi}_3 X^2, \tag{30}
\]

\[
\hat{J}_{2-} = i \left\{ \hat{\xi}P - \hat{\xi}X \right\}, \quad \hat{J}_{2+} = i \left\{ -\hat{\xi}P + \hat{\xi}X \right\}, \quad I = 1, \tag{31}
\]

\[
\hat{\xi}(t) = (\xi \circ t')(t), \quad \hat{\xi}(t) = (\dot{\xi} \circ t')(t),
\]

\[
\hat{\phi}_3(t) = (\phi_3 \circ t')(t), \quad \hat{\dot{\phi}_3}(t) = (\ddot{\phi}_3 \circ t')(t), \quad \hat{\ddot{\phi}_3}(t) = (\dddot{\phi}_3 \circ t')(t). \tag{32}
\]

(See Eqs. (31) and (36) of Ref. [2].) The functions \(\hat{\xi}\) and \(\hat{\dot{\xi}}\) satisfy the analogue of the Wronskian \(20\)

\[
\hat{\xi} \hat{\dot{\xi}} - \hat{\dot{\xi}} \hat{\xi} = -i. \tag{33}
\]
It is important to note that, in general,
\[
\hat{\xi} = \frac{d}{dt} \hat{\xi} = \frac{dt}{dt'} \frac{d}{dt'} \hat{\xi} \neq \frac{d}{dt}. \tag{34}
\]

The operators (30) and (31) satisfy the nonzero commutator brackets
\[
[\hat{M}_2, \hat{J}_{2\pm}] = \pm \hat{J}_{2\pm}, \quad [\hat{J}_{2-}, \hat{J}_{2+}] = I. \tag{35}
\]

The structure of this algebra is isomorphic to \textit{os}(1). Because of this we refer to it as an \textit{os}(1) algebra also. Closure of the commutators in (35), is the result of Eq. (33) and
\[
i\hat{\xi} = \hat{\phi}_3 \hat{\xi} - \frac{1}{2} \hat{\phi}_3 \hat{\xi}, \quad \hat{i}\hat{\xi} = \hat{\phi}_3 \hat{\xi} + \frac{1}{2} \hat{\phi}_3 \hat{\xi} - \frac{1}{2} \hat{\phi}_3 \hat{\xi}. \tag{36}
\]

To obtain Eq. (37), as well as the earlier Eq. (27), it is helpful to realize that
\[
\hat{\xi} = -2g^{(2)}(t')\xi \Rightarrow \hat{\xi} = -2h^{(2)}(t)e^{4\nu}\hat{\xi}, \tag{38}
\]
\[
\hat{\phi}_3 = -4g^{(2)}(t')\phi_3 + 4\hat{\xi} \Rightarrow \hat{\phi}_3 = -4h^{(2)}(t)e^{4\nu}\hat{\phi}_3 + 4\hat{\xi}. \tag{39}
\]

### 3.3 The \textit{os}(1) algebra for \textit{TQ} Schrödinger Equations

The following operators are Lie symmetries for the \textit{TQ} Schrödinger equation (7):
\[
J_{1-} = i \left\{ \Xi_P P - \Xi_X X \right\}, \quad J_{1+} = i \left\{ -\Xi_P P + \Xi_X X \right\}, \quad I = 1,
\]
\[
M_1 = C_{3,T}T - C_{3,D}D - C_{3,X^2}X^2, \tag{40}
\]
\[
\Xi_P(t) = \hat{\xi}(t)e^{\nu}, \quad \Xi_X(t) = \hat{\xi}(t)e^{-\nu}. \tag{41}
\]

(See Eqs. (40)-(42) and (45)-(52) of Ref. \cite{2} with \(\kappa = 0\).) The coefficients, \(C_{3,T}\), \(C_{3,D}\), and \(C_{3,X^2}\) are
\[
C_{3,T} = \hat{\phi}_3(t)e^{2\nu} = 2\Xi_P\Xi_P, \tag{42}
\]
\[
C_{3,D} = -\frac{1}{2}h(t)\hat{\phi}_3(t)e^{2\nu} + \frac{1}{2}\hat{\phi}_3 = -\frac{1}{2}h(t)C_{3,T} + \Xi_P\Xi_X + \Xi_X\Xi_P, \tag{43}
\]
\[
C_{3,X^2} = -\frac{1}{4}\hat{\phi}_3 e^{-2\nu} = h^{(2)}(t)C_{3,T} - \Xi_X\Xi_X. \tag{44}
\]

The second equality of Eq. (44) follows from Eq. (39).
The $TQ$ symmetry operators satisfy the commutation relations

$$[M_1, J_{1\pm}] = \pm J_{1\pm}, \quad [J_{1-}, J_{1+}] = I. \quad (45)$$

This algebra is also isomorphic to the $os(1)$, and we shall refer to it as the $TQ$ oscillator algebra. Closure of the $TQ$ commutator brackets in Eq. (45) follows from

$$i\Xi_P = C_{3,T}\left(\Xi_X - \frac{1}{2}h(t)\Xi_P\right) - C_{3,D}\Xi_P,$$

$$i\Xi_X = C_{3,T}\left(-2h^{(2)}(t)\Xi_P + \frac{1}{2}h(t)\Xi_X\right) + C_{3,D}\Xi_X + 2C_{3,X^2}\Xi_P. \quad (46)$$

The corresponding relationships for complex conjugates can be obtained by taking the complex conjugate of each relationship.

Each of the three $os(1)$ algebras has a Casimir operator

$$C_j = J_{j+}J_{j-} - M_jI, \quad j = 1, \hat{2}, 3, \quad (47)$$

where $\hat{2}$ indicates that all operators with the subscript 2 should receive a hat. We refer the reader to Eqs. (62), (64), and (65) of Ref. [2] for their relationships to the corresponding Schrödinger operators.

Next, we shall exploit the $os(1)$ algebraic structure to obtain number states.

4 Number States

4.1 $\Psi_n(x, t')$ for $TO$ Systems

It is possible to construct a set of number states by first solving a first-order partial differential equation for $\Psi_n(x, t')$ in terms of $\psi_n(x)$, and then solving a first-order ordinary differential equation for the extremal state wave function, $\Psi_0(x, t')$. (See Section 4.2 of [2] on representation theory of oscillator algebras.) It is important to reemphasize that these number states are, in general, not eigenstates of any Hamiltonian. As a consequence, $\Psi_0(x, t')$, does not necessarily represent a ground state.

With the above we have

$$M_3\Psi_n(x, t') = \left(n + \frac{1}{2}\right)\Psi_n(x, t'), \quad C_3\Psi_n(x, t') = -\frac{1}{2}\Psi_n(x, t'), \quad (48)$$

$$J_{3+}\Psi_n(x, t') = \sqrt{n + 1}\Psi_{n+1}(x, t'), \quad J_{3-}\Psi_n(x, t') = \sqrt{n}\Psi_{n-1}(x, t'), \quad (49)$$
where \( n \in \mathbb{Z}_0^+ \) is the set of integers \( \geq 0 \). The constraint that the spectrum of the operator \( M_3 \), \( \text{Sp}(M_3) \), be bounded below is that

\[
J_3 - \Psi_0(x, t') = 0. 
\] (50)

Using Eqs. (17) and (48), with Eqs. (6) and (7) of [1], we obtain a first-order partial differential equation for \( \Psi_n \) which can be integrated by the method of characteristics to yield a solution of the type [16, 20]

\[
\Psi_n(x, t') = \exp \left\{ \frac{i \phi_3}{4} x^2 \right\} \psi_n \left( \frac{x}{\phi_3^{1/2}} \right) \phi_3^{-1/4} \left( \frac{\bar{\xi}}{\xi} \right)^{\frac{1}{4}}. 
\] (51)

For \( n = 0 \), Eq. (50) with \( \Psi_0 \) given by Eq. (51) yields a first-order ordinary differential equation for \( \psi_0 \) whose solution is

\[
\psi_0 \left( \frac{x}{\phi_3^{1/2}} \right) = N \exp \left( -\frac{x^2}{2\phi_3} \right), 
\] (52)

where \( N \) is an integration constant. Thus, the normalized extremal state is

\[
\Psi_0(x, t') = \exp \left\{ \frac{i \phi_3}{4} x^2 \right\} \exp \left( -\frac{x^2}{2\phi_3} \right) (\pi \phi_3)^{-1/4} \left( \frac{\bar{\xi}}{\xi} \right)^{\frac{1}{4}}. 
\] (53)

The higher-order states are obtained by the repeated application of the raising operator \( J_{3+} \). The normalized higher-order states are

\[
\Psi_n(x, t') = \sqrt{\frac{1}{n!}} (J_{3+})^n \Psi_0(x, t') 
= \sqrt{\frac{1}{n!2^n}} \exp \left\{ \frac{i \phi_3}{4} x^2 \right\} H_n \left( \frac{x}{\phi_3^{1/2}} \right) \exp \left( -\frac{x^2}{2\phi_3} \right) (\pi \phi_3)^{-1/4} \left( \frac{\bar{\xi}}{\xi} \right)^{\frac{1}{4}} \left( n+\frac{1}{2} \right), 
\] (54)

where \( H_n(x/\phi_3^{1/2}) \) is a Hermite polynomial of order \( n \).
4.2 $\hat{\Theta}_n(x,t)$ for TM Systems

An analogous set of relationships to Eqs. (48) and (49) can be written for $TM$-os(1) representation spaces of Eq. (6). They are

\[ \hat{M}_2 \hat{\Theta}_n(x,t) = \left( n + \frac{1}{2} \right) \hat{\Theta}_n(x,t), \quad \hat{C}_2 \hat{\Theta}_n(x,t) = -\frac{1}{2} \hat{\Theta}_n(x,t), \]  
(55)

\[ \hat{J}_{2+} \hat{\Theta}_n(x,t) = \sqrt{n + 1} \hat{\Theta}_{n+1}(x,t), \quad \hat{J}_{2-} \hat{\Theta}_n(x,t) = \sqrt{n} \hat{\Theta}_{n-1}(x,t), \]  
(56)

with $n \in \mathbb{Z}_0^+$. Also, we have the constraint on the extremal state

\[ \hat{J}_{2-} \hat{\Theta}_0(x,t) = 0. \]  
(57)

To obtain $\hat{\Theta}_n$, we could follow the same procedure as we did for $TO$ number states. But it is more convenient to take the composition of $\Psi_n(x,t')$ with $t'(t)$. Then

\[ \hat{\Theta}_n(x,t) = \sqrt{\frac{1}{2\pi n}} \exp \left\{ \frac{\hat{\phi}_3 x^2}{2} \right\} H_n \left( \frac{x}{\sqrt{2} \phi_3} \right) \exp \left( -\frac{x^2}{2\phi_3} \right) (\pi \hat{\phi}_3)^{-1/4} \left( \frac{\hat{\xi}}{2} \right)^{\frac{1}{2}} \left( n + \frac{1}{2} \right), \]  
(58)

where $\hat{\xi}$, $\phi_3$, and $\hat{\phi}_3$ are defined in Eq. (32). Setting $n = 0$ yields the extremal state, $\hat{\Theta}_0(x,t)$.

4.3 $\Phi_n(x,t)$ for TQ Systems

For TQ systems, we have

\[ M_1 \Phi_n(x,t) = \left( n + \frac{1}{2} \right) \Phi_n(x,t), \quad C_1 \Phi_n(x,t) = -\frac{1}{2} \Phi_n(x,t), \]  
(59)

\[ J_{1+} \Phi_n(x,t) = \sqrt{n + 1} \Phi_{n+1}(x,t), \quad J_{1-} \Phi_n(x,t) = \sqrt{n} \Phi_{n-1}(x,t), \]  
(60)

with $n \in \mathbb{Z}_0^+$. This is subject to the constraint

\[ J_{1-} \Phi_0(x,t) = 0. \]  
(61)

The wave functions, $\Phi_n(x,t)$, satisfying these relationships can be conveniently obtained as

\[ \Phi_n(x,t) = \exp \left[ -i\nu D \right] \hat{\Theta}_n(x,t) = e^{-\nu/2 \hat{\Theta}_n(xe^{-\nu},t)}, \]  
(62)
for \( n \in \mathbb{Z}_0^+ \). According to Eq. (58), the number state, \( \Phi_n(x,t) \), is

\[
\Phi_n(x,t) = \sqrt{\frac{1}{n!}} (J_{1+})^n \Phi_0(x,t)
\]

\[
= \sqrt{\frac{1}{n!2^n}} \exp \left( \frac{i}{4} \frac{\phi_3}{C_{3,T}} x^2 \right) H_n \left( \frac{x}{C_{3,T}^{1/2}} \right) \exp \left( -\frac{x^2}{2C_{3,T}} \right) \left( \pi C_{3,T} \right)^{-1/4} \left( \frac{\Xi_P}{\Xi_P} \right)^{1/2} n^{n/2} \]

(63)

where \( C_{3,T} \), which is real and positive, is given by Eq. (42). The functions \( \Xi_P \) and \( \Xi_X \) are given by Eq. (41). The extremal state, \( \Phi_0(x,t) \), is obtained by setting \( n = 0 \).

5 Coherent States

Now that we have computed the extremal states for the \( TO, TM, \) and \( TQ \) systems, we can compute explicit \((x,t)\) representations of coherent-state and squeezed-state wave functions for each of the three classes of systems. Although the wave functions themselves are not needed for the calculation of expectation values, they are independently of interest. In this section, we calculate coherent-state wave functions using a group theoretic method. Table 1 defines the CS parameters \( \alpha \) that will be used. It is obtained from Eq. (83) and Table 2 of [2], with \( \kappa = \mu = 0 \) and \( F_P = F_X = 0 \). [Here and in the following a superscript symbol \( o \) denotes that the function is evaluated at initial time.]

Table 1. Generic functions and their values according to class used to obtain the CS parameters \( \alpha = i (G_P p_o - G_X^o x_o) \).

| Function | \( TO \) | \( TM \) | \( TQ \) |
|----------|---------|---------|-------|
| \( G_P^o \) | \( \xi^o \) | \( \dot{\xi}^o \) | \( \Xi_P^o = \dot{\xi}^o \) |
| \( G_X^o \) | \( \dot{\xi}^o \) | \( \dot{\xi}^o \) | \( \Xi_X^o = \dot{\xi}^o \) |
5.1 TO Coherent-State Wave Functions

For TO systems we have

\[ \Psi_\alpha(x, t') = D(\alpha)\Psi_0(x, t') = \exp\left(-\frac{1}{2}|\alpha|^2 I\right) \exp(\alpha J_{3+})\Psi_0(x, t') \]  
\[ = \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left[i\hat{\xi}\left(x\alpha - \frac{1}{2}\xi^2 \alpha^2\right)\right]\Psi_0(x - \alpha\hat{\xi}, t') \]  
\[ = \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left[-\frac{1}{2\phi_3} \left|x - X^+_3(\alpha)\right|^2\right] \]  
\[ \times \exp\left\{ i \left[\frac{1}{4\phi_3} x^2 + \frac{1}{\phi_3} \left(x - \frac{1}{2}X^+_3(\alpha)\right) X^-_3(\alpha)\right]\right\}, \]  
\[ X^+_3(\alpha) = \alpha\hat{\xi} + \bar{\alpha}\xi = i(\xi^o\hat{\xi} - \bar{\xi}^o\xi)p_o - i(\dot{\xi}^o\hat{\xi} - \dot{\bar{\xi}}^o\xi)x_o, \]  
\[ X^-_3(\alpha) = -i(\alpha\hat{\xi} - \bar{\alpha}\xi) = (\xi^o\hat{\xi} + \bar{\xi}^o\xi)p_o - (\dot{\xi}^o\hat{\xi} + \dot{\bar{\xi}}^o\xi)x_o. \]

To obtain Eq. (64) we used the standard Baker-Campbell-Hausdorff (BCH) relation \[21, 22\] for the operator \( D(\alpha) = \exp[\alpha J_{3+} - \bar{\alpha}J_{3-}] \) and the fact that \( J_{3-} \) annihilates the extremal state. For Eq. (65), see Ref. [23]. To obtain Eq. (66), we employed Eqs. (53) and (20). The explicit definition of \( \alpha \) can be found from Table 1.

Note that both the \( X^+_3(\alpha) \) defined in Eq. (67) are real. Therefore, the final expression, Eq. (66), has been arranged in such a way that the argument of the first exponential is real while the argument of the second exponential is pure imaginary.

5.2 TM Coherent-State Wave Functions

In the case of TM coherent states, we take the composition of \( \Psi_\alpha(x, t') \) and \( t'(t) \) to obtain

\[ \hat{\Theta}_\alpha(x, t) = (\pi\phi_3)^{-1/4} \left(\frac{\hat{\xi}}{\xi}\right)^{1/4} \exp\left\{ -\frac{1}{2\phi_3} \left[x - \hat{X}^+_2(\alpha)\right]^2\right\} \]  
\[ \times \exp\left\{ i \left[\frac{1}{4\phi_3} x^2 + \frac{1}{\phi_3} \left(x - \frac{1}{2}\hat{X}^+_2(\alpha)\right) \hat{X}^-_2(\alpha)\right]\right\}, \]  
\[ \hat{X}^+_2(\alpha) = \alpha\hat{\xi} + \bar{\alpha}\xi = i(\xi^o\hat{\xi} - \bar{\xi}^o\xi)p_o - i(\dot{\xi}^o\hat{\xi} - \dot{\bar{\xi}}^o\xi)x_o, \]
\[
\dot{X}_2^-(\alpha) = -i(\alpha \dot{\xi} - \bar{\alpha} \dot{\bar{\xi}}) = (\dot{\xi}^o \xi + \bar{\xi}^o \bar{\xi})p_o - (\xi \ddot{\xi} + \bar{\xi} \ddot{\bar{\xi}})x_o. \tag{69}
\]

See Table 1 for the explicit definition of \( \alpha \). Once more both expressions in Eq. (69) are real.

5.3 \textit{TQ} Coherent-State Wave Functions

For \textit{TQ} systems, we calculate the coherent states from \textit{TM} coherent states. Starting from the definition of a \textit{TQ} coherent state, we have

\[
\Phi_\alpha(x,t) = \exp (\alpha J_{1+} - \bar{\alpha} J_{1-}) \exp (-i\nu D) \hat{\Theta}_0(x,t) \tag{70}
\]

\[
= \exp (-i\nu D) [\exp (i\nu D) \exp (\alpha J_{1+} - \bar{\alpha} J_{1-}) \exp (-i\nu D)] \hat{\Theta}_0(x,t) \tag{71}
\]

\[
= \exp (-i\nu D) \exp (\alpha \dot{J}_{2+} - \bar{\alpha} \dot{J}_{2-}) \hat{\Theta}_0(x,t). \tag{72}
\]

(Further details on the transformation to Eq. (72) can be found in the material adjoining Eqs. (40) and (41) of [2].) Therefore, we obtain

\[
\Phi_\alpha(x,t) = \exp (-i\nu D) \hat{\Theta}_\alpha(x,t) = \exp (-\frac{1}{2}\nu) \hat{\Theta}_\alpha(xe^{-\nu},t) \tag{73}
\]

\[
= (\pi C_{3,T})^{-1/4} \left( \frac{\Xi_P}{\Xi_P} \right)^{1/4} \exp \left\{ -\frac{1}{2C_{3,T}} \left[ x - X_1^+(\alpha) \right]^2 \right\} \times \exp \left\{ i \left[ \frac{1}{4} \frac{\phi^3}{C_{3,T}} x^2 + \frac{1}{C_{3,T}} \left( x - \frac{1}{2} X_1^+(\alpha) \right) X_1^-(\alpha) \right] \right\}, \tag{74}
\]

\[
X_1^+(\alpha) = \alpha \Xi_P + \bar{\alpha} \Xi_P = (\Xi_P \Xi_P - \Xi_P \Xi_P)p_0 - i(\Xi_\chi \Xi_P - \Xi_\chi \Xi_P)x_o.
\]

\[
X_1^-(\alpha) = -i(\alpha \Xi_P - \bar{\alpha} \Xi_P) = (\Xi_P \Xi_P + \Xi_P \Xi_P)p_0 - (\Xi_\chi \Xi_P + \Xi_\chi \Xi_P)x_o. \tag{75}
\]

The explicit definitions of the \( \Xi \) can be found in Table 1. Again the \( X_1^\pm(\alpha) \) are both real.

6 Squeezed States

In this section, we derive expressions for the squeezed-state wave functions by solving a first-order partial differential equation.
6.1 TO Squeezed-State Wave Functions

To calculate the squeezed state wave functions for a TO system we write

\[ \Psi_{\alpha,z}(x,t) = D(\alpha)S(z)\Psi_0(x,t), \]

where \( \{K_+ = \frac{1}{2}J_{3-}^2, \ K_- = \frac{1}{2}J_{3+}^2, \ K_3 = J_{3+}J_{3-} + \frac{1}{2} \} \) satisfy the \( su(1,1) \) squeeze algebra and the last equality is obtained using standard methods Refs. \[ 2, 18, 24 \]. The wave function (76) satisfies the following eigenvalue equation

\[ (J_{3-} - \gamma_+J_{3+}) \Psi_{\alpha,z} = (\alpha - \gamma_+\bar{\alpha})\Psi_{\alpha,z}. \] (77)

The operators \( J_{3-} \) and \( J_{3+} \) are first-order partial differential operators of the form \[ 18 \]. Hence, Eq. (77) is a first-order partial differential equation for \( \Psi_{\alpha,z} \):

\[ (\xi + \gamma_+\xi)\partial_x \Psi_{\alpha,z} - i(\xi + \gamma_+\xi)\Psi_{\alpha,z} = (\alpha - \gamma_+\bar{\alpha})\Psi_{\alpha,z}. \] (78)

We solve this equation by the method of characteristics \[ 21 \] and obtain the following solution:

\[ \Psi_{\alpha,z}(x,t') = b(t') \exp \left[ \frac{\alpha - \gamma_+\bar{\alpha}}{\xi + \gamma_+\xi} x + \frac{i}{2} (\xi + \gamma_+\xi)^2 \right]. \] (79)

The function \( b(t') \) is arbitrary. To fix \( b(t') \), we require that the wave function (79) satisfy the Schrödinger equation \( S_3\Psi_{\alpha,z} = \left\{ \partial_{xx} + 2i\partial_{t'} - 2g^{(2)}(t')x^2 \right\}\Psi_{\alpha,z} = 0. \) [See Eq. (16) with \( g^{(2)}(t') = \tilde{h}^{(2)}(t')e^{4\nu(t')} \).] This yields a first-order ordinary differential equation for \( b(t') \):

\[ \frac{db}{dt'} + \left\{ \frac{1}{2} (\xi + \gamma_+\xi)^2 - \frac{1}{2}(\alpha - \gamma_+\bar{\alpha})^2 \right\} b = 0. \] (80)

Integrating this equation, we find that

\[ b(t') = \mathcal{N} (\xi + \gamma_+\xi)^{-1/2} \exp \left[ \frac{i}{2}(\alpha - \gamma_+\bar{\alpha})^2 \int \frac{dt'}{(\xi + \gamma_+\xi)^2} \right]. \] (81)
The integral in (81) can be solved by noting that, if \( \xi \) and \( \bar{\xi} \) are solutions to Eq. (19) then so too are \( [\xi + \gamma + \bar{\xi}] \) and \( [\bar{\xi} + \bar{\gamma} + \xi] \). The latter two solutions satisfy the Wronskian
\[
W(\xi + \gamma + \bar{\xi}, \bar{\xi} + \bar{\gamma} + \xi) = -i(1 - \gamma + \bar{\gamma}).
\] (82)

With these facts in mind, we obtain
\[
b(t') = N (\xi + \gamma + \bar{\xi})^{-1/2} \exp \left[ -\frac{1}{2} \frac{(\alpha - \gamma + \bar{\alpha})^2}{1 - \gamma + \bar{\gamma}} \left( \frac{\bar{\xi} + \bar{\gamma} + \xi}{\xi + \gamma + \bar{\xi}} \right) \right],
\] (83)
and have the result
\[
\Psi_{\alpha,z}(x,t') = N (\xi + \gamma + \bar{\xi})^{-1/2} \exp \left\{ -\frac{1}{4} \frac{(\alpha - \gamma + \bar{\alpha})}{Q_3} \left[ x - \frac{1}{2} \frac{(\alpha - \gamma + \bar{\alpha})}{Q_3} (\xi + \gamma + \bar{\xi})^2 \right] \right\} 
\times \exp \left[ \frac{i}{Q_3} \frac{\bar{\xi} + \bar{\gamma} + \xi}{\xi + \gamma + \bar{\xi}} x^2 \right].
\] (84)

The integration constant \( N \) will be fixed by normalization.

A more transparent expression for \( \Psi_{\alpha,z}(x,t') \) can be obtained by first decomposing the exponential terms into their real and complex parts. After some algebra and normalizing,
\[
\Psi_{\alpha,z}(x,t') = \left( \frac{1 - \gamma + \bar{\gamma} + \gamma + \bar{\gamma}}{2\pi Q_3} \right)^{1/4} \left( \frac{\bar{\xi} + \bar{\gamma} + \xi}{\xi + \gamma + \bar{\xi}} \right)^{1/4} \exp \left\{ -\frac{1}{4} \frac{(1 - \gamma + \bar{\gamma})}{Q_3} \left[ x - \frac{1}{2} \frac{R_3}{Q_3} (\alpha, z) \right] X_3^+(\alpha, z) \right\} 
\times \exp \left\{ i \left[ \frac{1}{2} \frac{R_3}{Q_3} x^2 + \frac{1}{2} \frac{R_3}{Q_3} (x - \frac{1}{2} X_3^+(\alpha, z)) X_3^-(\alpha, z) \right] \right\},
\] (85)
\[
Q_3 = (\xi + \gamma + \bar{\xi})(\bar{\xi} + \bar{\gamma} + \xi), \quad R_3 = \bar{Q}_3,
\] (86)
\[
X_3^+(\alpha, z) = (\alpha - \gamma + \bar{\alpha})(\bar{\xi} + \bar{\gamma} + \xi) + (\bar{\alpha} - \gamma + \alpha)(\xi + \gamma + \bar{\xi}),
\]
\[
X_3^-(\alpha, z) = -i \left[ (\alpha - \gamma + \bar{\alpha})(\bar{\xi} + \bar{\gamma} + \xi) - (\bar{\alpha} - \gamma + \alpha)(\xi + \gamma + \bar{\xi}) \right].
\] (87)

In Eq. (85), the argument in the first exponential term is real, while the argument in the second is pure imaginary. The wave function (85) is clearly Gaussian. All the quantities defined in Eqs. (86) and (87) are real.

To further simplify, first define the following quantities:
\[
\gamma_- = -\frac{\bar{z}}{|z|} \tanh |z|, \quad \gamma_+ = \frac{z}{|z|} \tanh |z|, \quad \gamma_3 = -\ln (\cosh |z|),
\]
\[ z = re^{i\theta}, \quad r = |z|, \]

\[ Q_3 = \frac{Q^3}{(1 - \gamma_+^+) + (1 - \gamma_+^+)} = \frac{1}{2} \left[ \phi_3 \cosh 2r + \left( \phi_1 e^{-i\theta} + \phi_2 e^{i\theta} \right) \sinh 2r \right], \quad (88) \]

\[ R_3 = \frac{R^3}{(1 - \gamma_+^+) + (1 - \gamma_+^+)} = \dot{Q}_3, \quad (89) \]

\[ X_3^\pm (\alpha, z) = \frac{X_3^\pm (\alpha, z)}{(1 - \gamma_+^+) + (1 - \gamma_+^+)}, \quad (90) \]

where in Eq. (88) we have used Eq. (21).

The resulting squeezed-state wave function is

\[
\Psi_{\alpha,z}(x,t') = \left( \frac{1}{2\pi Q_3} \right)^{\frac{1}{4}} \left( \frac{\xi + \xi e^{-i\theta} \tanh r}{\xi + \xi e^{i\theta} \tanh r} \right) \frac{1}{2} \exp \left\{ -\frac{1}{4Q_3} \left[ x - X_3^+(\alpha, z) \right]^2 \right\} 
\times \exp \left\{ i \left[ \frac{1}{4Q_3} x^2 + \frac{1}{2Q_3} \left( x - \frac{1}{2} X_3^+(\alpha, z) \right) X_3^-(\alpha, z) \right] \right\}. \quad (91) \]

Combining Eqs. (87), (88), and (90), we can obtain the following relationships:

\[ X_3^+(\alpha, z) = X_3^+(\alpha) \quad (92) \]

\[ X_3^-(\alpha, z) = X_3^-(\alpha) \cosh 2r + Y_3^- (\alpha, \theta) \sinh 2r, \quad (93) \]

where the \( X_3^\pm (\alpha) \) are given by Eq. (77), and

\[
Y_3^- (\alpha, \theta) = -i \left( \alpha \xi e^{-i\theta} - \bar{\alpha} \bar{\xi} e^{i\theta} \right)
= \left( \xi^o \xi e^{-i\theta} + \bar{\xi}^o \bar{\xi} e^{i\theta} \right) p_o - \left( \xi^o \xi e^{-i\theta} + \bar{\xi}^o \bar{\xi} e^{i\theta} \right) x_o. \quad (94) \]

To obtain the last equation, we have used Table 1.

When \( z = 0 \), \( 2Q_3 = \phi_3 \), \( 2R_3 = \dot{\phi}_3 \), and \( X_3^- (\alpha, 0) = X_3^- (\alpha) \). Then the TO squeezed-state wave function, \( \Psi_{\alpha,z} \), reduces to the coherent-state wave function, \( \Psi_{\alpha} \), of Eq. (86).

In the Appendix we show, by way of example, how for the ordinary harmonic oscillator this can be reduced to the ordinary squeezed-state wave function.
6.2 TM Squeezed-State Wave Functions

To obtain the squeezed-state wave functions for TM systems, we note that in general

$$\hat{\Theta}_{\alpha,z}(x,t) = (\Psi_{\alpha,z}(x,t') \circ t')(t).$$

(95)

Hence, from Eq. (91), we have

$$\hat{\Theta}_{\alpha,z}(x,t') = \left( \frac{1}{2\pi \hat{Q}_2} \right)^{\frac{1}{4}} \left( \frac{\hat{\xi} + \xi e^{-i\theta} \tanh r}{\hat{\xi} + \xi e^{i\theta} \tanh r} \right)^{\frac{1}{2}} \exp \left\{ -\frac{1}{4\hat{Q}_2} \left[ x - \hat{X}_2^+(\alpha,z) \right]^2 \right\} \exp \left\{ i \left[ \frac{1}{4} \hat{R}_2 x^2 + \frac{1}{2\hat{Q}_2} \left( x - \frac{1}{2} \hat{X}_2^+(\alpha,z) \right) \hat{X}_2^-(\alpha,z) \right] \right\},$$

(96)

$$\hat{Q}_2 = \frac{1}{2} \left[ \hat{\phi}_3 \cosh 2r + \left( \hat{\phi}_1 e^{-i\theta} + \hat{\phi}_2 e^{i\theta} \right) \sinh 2r \right],$$

(97)

$$\hat{R}_2 = \frac{1}{2} \left[ \hat{\beta}_3 \cosh 2r + \left( \hat{\beta}_1 e^{-i\theta} + \hat{\beta}_2 e^{i\theta} \right) \sinh 2r \right],$$

(98)

$$\hat{X}_2^+ (\alpha,z) = \hat{X}_2^+ (\alpha),$$

(99)

$$\hat{X}_2^- (\alpha,z) = \hat{X}_2^- (\alpha) \cosh 2r + \hat{Y}_2^- (\alpha,\theta) \sinh 2r.$$  

(100)

The quantities $\hat{X}_2^+(\alpha)$ are defined in Eq. (69) and

$$\hat{Y}_2^- (\alpha,\theta) = -i \left( a \hat{\xi} e^{-i\theta} - a \hat{\xi} e^{i\theta} \right)$$

$$= \left( \hat{\xi}^{\alpha} \hat{\xi} e^{-i\theta} + \hat{\xi} \hat{\xi}^{\alpha} e^{i\theta} \right) p_\alpha - \left( \hat{\xi}^{\alpha} \hat{\xi} e^{-i\theta} + \hat{\xi} \hat{\xi}^{\alpha} e^{i\theta} \right) x_o,$$

(101)

is a real quantity.

When $z = 0$, $2\hat{Q}_2 = \hat{\phi}_3$, $2\hat{R}_2 = \hat{\phi}_3$, and $\hat{X}_2^- (\alpha,0) = \hat{X}_2^- (\alpha)$. Then the TM-squeezed-state wave function, $\hat{\Theta}_{\alpha,z}$, reduces to the coherent-state wave function, $\hat{\Theta}_\alpha$, of Eq. (68).

6.3 TQ Squeezed-State Wave Functions

To obtain the squeezed state wave function for TQ systems, we follow the same procedure that we employed for their coherent states. It is straightforward to show that

$$\Phi_{\alpha,z}(x,t) = e^{-iD} \hat{\Theta}_{\alpha,z}(x,t) = e^{-\nu/2} \hat{\Theta}_{\alpha,z}(xe^{-\nu}, t).$$

(102)
Therefore, from Eq. (96) we obtain

\[
\Phi_{\alpha,z}(x, t) = \left( \frac{1}{2\pi Q_1} \right)^{\frac{1}{4}} \left( \frac{\Xi_P + \Xi pe^{-i\theta} \tanh r}{\Xi + \Xi pe^{i\theta} \tanh r} \right)^{\frac{1}{4}} \exp \left\{ \frac{1}{4 Q_1} \left[ x - X_1^+ (\alpha, z) \right]^2 \right\} \times \exp \left\{ i \left[ \frac{1}{4 Q_1} x^2 + \frac{1}{2 Q_1} \left( x - \frac{1}{2} X_1^+ (\alpha, z) \right) X_1^- (\alpha, z) \right] \right\},
\]

(103)

\[
Q_1 = (\Xi_P + \gamma \Xi_P)(\Xi_P + \bar{\gamma} \Xi_P),
\]

(104)

\[
R_1 = (\Xi_P + \gamma \Xi_P)(\Xi_X + \bar{\gamma} \Xi_X) + (\Xi_X + \gamma \Xi_X)(\Xi_P + \bar{\gamma} \Xi_P),
\]

(105)

\[
X_1^+ (\alpha, z) = X_1^+ (\alpha),
\]

(106)

\[
X_1^- (\alpha, z) = X_1^- (\alpha) \cosh 2r + Y_1^- (\alpha, \theta) \sinh 2r.
\]

(107)

The quantities \(X_1^\pm(\alpha)\) are defined in Eq. (73) and

\[
Y_1^- (\alpha, z) = -i \left( \alpha \Xi_pe^{-i\theta} - \bar{\alpha} \Xi_pe^{i\theta} \right)
\]

\[
= \left( \Xi^{\circ}_P \Xi_P e^{-i\theta} + \Xi^{\circ}_P \Xi_P e^{i\theta} \right) p_o - \left( \Xi^{\circ}_X \Xi_P e^{-i\theta} + \Xi^{\circ}_X \Xi_P e^{i\theta} \right) x_o
\]

(108)

is a real quantity.

When \(z = 0\), \(2Q_1 = C_{3,T} \), \(2R_1 = \hat{\phi}_3\), and \(X_1^-(\alpha, 0) = X_1^- (\alpha)\). Then the \(TQ\) squeezed-state wave function, \(\Phi_{\alpha,z}\), reduces to the coherent-state wave function, \(\Phi_{\alpha}\), of Eq. (74).

7 Summary

In the above analysis, we have provided details about the construction of Lie symmetry operators for the oscillator algebras of the three classes of systems, represented by \(TO\) Hamiltonians, \(TM\) Hamiltonians, and \(TQ\) Hamiltonians, given the constraint that the \(TM\) Hamiltonians contain only \(P^2\) and \(X^2\) terms. Representation theory was then employed to construct the number states for the three systems. The number states are, in general, not eigenfunctions of the respective Hamiltonian; that is, they are not energy eigenstates. This is a consequence of the fact that the Hamiltonians are not, in general, an operator in the corresponding oscillator algebra.
Therefore, the extremal state in each set of number states is not necessarily a ground-state wave function.

Starting from the extremal states and the ladder operators of the oscillator algebras for each of the \(TO\), \(TM\), and \(TQ\) systems, the appropriate displacement and squeeze operators could then be constructed to calculate their respective coherent states and squeezed states. However, for the squeezed states it was simpler to solve first-order partial differential equations.

Our results generalize the discussions of the two specific \(TM\) time-dependent Schrödinger equations given in Refs. [12, 13]. We have shown how to handle, in general, all systems of the type given in Eq. (6). Thus, the method can be straight-forwardly applied to other special cases [11].

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Appendix: Ordinary Squeezed States

The squeezed-state wave functions for the harmonic oscillator [18, 24] can be calculated using Eq. (85) and Table 1 with \(\xi(t) = (2\omega)^{-1/2}e^{i\omega t}\). Here, for convenience, we have chosen \(t_o = 0\) and dropped the prime on \(t\). The wave functions are

\[
\Psi_{\alpha, z}(x, t) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{\cosh 2r + \cos [2\omega t - \theta] \sinh 2r}^{1/4}
\]

\[
\times \left(\frac{e^{-i\omega t} \cosh r + e^{i[\omega t - \theta]} \sinh r}{e^{i\omega t} \cosh r + e^{-i[\omega t - \theta]} \sinh r}\right)^{1/4}
\]

\[
\times \exp \left\{-\frac{1}{2} \frac{\omega}{\cosh 2r + \cos [2\omega t - \theta] \sinh 2r} \left[x - \left(\frac{\omega}{\cosh 2r + \cos [2\omega t - \theta] \sinh 2r}\right) \sin \omega t + x_o \cos \omega t\right]^2\right\}
\]

\[
\times \exp \left\{-i \frac{\omega x^2}{2} \sin [2\omega t - \theta] \sinh 2r \right\}
\]
\begin{align*}
\times \exp \left\{ \frac{i}{\omega} \left[ x - \frac{1}{2} \left( \frac{p_0}{\omega} \sin \omega t + x_0 \cos \omega t \right) \right] \right. \\
\cosh 2r + \cos [\omega t - \theta] \sinh 2r \\
\left. \frac{p_0}{\omega} \left( \cos \omega t \cosh 2r + \cos [\omega t - \theta] \sinh 2r \right) \right) \\
-x_0 \left( \sin \omega t \cosh 2r - \sin [\omega t - \theta] \sinh 2r \right) \right\} . \tag{109}
\end{align*}

For \( t = \theta = 0 \), and where \( s = \exp r \),

\[ \Psi_{\alpha,r}(x,0) = \left( \frac{\omega}{\pi s^2} \right)^{1/4} \exp \left\{ -\frac{1}{2} \frac{\omega(x-x_0)^2}{s^2} \right\} \exp \left\{ ip_0 x \right\} \exp \left\{ -\frac{i}{2}p_0 x_0 \right\} . \tag{110} \]

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\[
H = -\frac{1}{2} e^{\Upsilon(t-t_0)} \partial_{xx} + \frac{1}{2} \omega^2 e^{-\Upsilon(t-t_0)} x^2
\]

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\[
H = -\frac{1}{2} \left( \frac{t}{\tau} \right)^a \partial_{xx} + \frac{1}{2} \omega^2 \left( \frac{t}{\tau} \right)^b x^2
\]

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