INTEGRAL POINTS IN TWO-PARAMETER ORBITS

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Abstract. Let $K$ be a number field, let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a nonconstant rational map of degree greater than 1, let $S$ be a finite set of places of $K$, and suppose that $u, w \in \mathbb{P}^1(K)$ are not preperiodic under $f$. We prove that the set of $(m, n) \in \mathbb{N}^2$ such that $f^m(u)$ is $S$-integral relative to $f^n(w)$ is finite and effectively computable. This may be thought of as a two-parameter analog of a result of Silverman on integral points in orbits of rational maps.

This issue can be translated in terms of integral points on an open subset of $\mathbb{P}^2_1$; then one can apply a modern version of the method of Runge, after increasing the number of components at infinity by iterating the rational map. Alternatively, an ineffective result comes from a well-known theorem of Vojta.

1. Introduction

In 1929, Siegel \cite{Sie29} proved that if $C$ is an irreducible affine curve defined over a number field $K$ and $C$ has at least three points at infinity, then there are at most finitely many $K$-rational points on $C$ that have integral coordinates. When $C$ has positive genus, something stronger is true: any affine curve has defined over a number field has at most finitely many $K$-rational points that have integral coordinates. Silverman \cite[Theorem A]{Sil93} later gave a dynamical variant of Siegel’s theorem, proving that if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function such that $f^2$ is not a polynomial and $u \in K$ is not preperiodic for $f$, there are only finitely many $n$ such that $f^n(u)$ is integral relative to the point at infinity (we will give a full definition of what it means to be integral relative to a point in Section 2). Moreover, \cite[Theorem A]{Sil93} can be made effective, as we shall see Section 5.

Recently, various authors (see \cite{Den92, Bel06, GT09}) have proposed a dynamical analog of the Mordell-Lang conjecture for semiabelian varieties. The Mordell-Lang conjecture for semiabelian varieties, which was proved by Faltings \cite{Fal94} and Vojta \cite{Voj96}, states that if a finitely generated subgroup $\Gamma$ of a semiabelian variety $A$ over $\mathbb{C}$ intersects a subvariety $V \subseteq S$ in infinitely many points, then $V$ must contain a positive-dimensional algebraic subgroup of $A$. One dynamical analog asserts that if one has a morphism of varieties $\Phi : X \to X$ defined over $\mathbb{C}$, a subvariety $V \subseteq X$, and a point $\alpha$ in $X(\mathbb{C})$, then the forward orbit of $\alpha$ under $\Phi$ (that is, the set of distinct iterates $\Phi^n(\alpha)$) may intersect $V$ infinitely often only if $V$ contains
a $\Phi$-periodic subvariety of $X$ (that is, a subvariety of $W$ of $X$ such that $\Phi^n(W) = W$ for some $n > 0$) having positive dimension. Note, however, that since the forward orbit of a point under a single map is parametrized by the positive integers, it is more analogous to a cyclic group $\Gamma$ than it is to an arbitrary finitely generated group $\Gamma$. Thus, one might ask for a “multi-parameter” dynamical conjecture concerning the forward orbit of $\alpha$ under a finitely generated semigroup of commuting maps. In [GTZ], this problem is considered and results are obtained in the case where the subvarieties $V$ are lines in $\mathbb{A}^2$ and the semigroup of morphisms is the set of all $(f^m, g^n)$ for fixed polynomials $f$ and $g$.

The dynamical variants of the Mordell-Lang conjecture described above all pose questions about the intersection of forward orbits with subvarieties. Here we consider the related problem of integral points in forward orbits. The main theorem of this paper is the following, which may be thought of as a two-parameter version of [Sil93, Theorem A].

**Theorem 1.1.** Let $K$ be a number field and $S$ a finite set of primes in $K$. Let $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function with degree $d \geq 2$ that is not conjugate to a powering map $x^{\pm d}$, and let $u, w \in \mathbb{P}_1(K)$ be points that are not preperiodic for $f$. Then the set of $(m, n) \in \mathbb{N}^2$ such that $f^m(u)$ is $S$-integral relative to $f^n(w)$ is finite and effectively computable.

Clearly the set $(m, n)$ such that $f^m(u)$ is $S$-integral relative to $f^n(w)$ depends on $u$ and $w$. It is possible, however, to prove an effective degeneracy result for integral points depending only on $f$, $S$, and $K$. This is stated in Theorem 4.1, which is phrased in terms of the $S$-integrality of points $(f^m(u), f^n(w))$ relative to inverse images of the diagonal in $\mathbb{P}_2^1$.

The outline of the paper is as follows. In Section 2, after introducing some notation, we give some equivalent notions of integrality. This will reduce our problem to the study of integral points on the complement in $\mathbb{P}_2^1$ of suitable divisors. Then, we show that a noneffective version of Theorem 1.1 can be obtained very quickly by combining [BGK12, Appendix] with [Voj87, Theorem 2.4.1]; this is Theorem 3.3.

In Section 4, we prove Theorem 1.1 an effective degeneracy result for $K$-rational in $\mathbb{P}_2^1$ that are $S$-integral relative to inverse images of the diagonal under $(f, f)$.

The technique here originates from Runge’s theorem [Run87]: Runge treated only the case of curves, but see [BG06, Section 9.6] and [Lev08] for a modern account and higher dimensional generalizations. Since inverse images of the diagonal under $(f, f)$ have several components, one can construct many rational functions $\psi$ whose pole divisors are supported on these inverse images. The main difficulty is dealing with the points at which many components of these pole divisors meet; this can be overcome by introducing rational functions that vanish to a high degree at these intersection points.
Then, in Section 3, we use Theorem 1.1 and some simple facts about periodic curves to finish the proof of Theorem 1.1. We end with a few remarks about what happens when $f$ is conjugate to a powering map or $u$ or $w$ is preperiodic.

2. Notation

Let $\mathbb{P}_n$ denote the projective $n$-space and write $\mathbb{P}_n^m$ for $m$-fold Cartesian product of $\mathbb{P}_n$. We fix projective coordinates $[z_1 : \cdots : z_n]$. When convenient, we regard $\mathbb{P}_1(K)$ as $K \cup \{\infty\}$ and work in affine coordinates.

Let $P = [x_0 : \cdots : x_n]$ and $Q = [y_0 : \cdots : y_n]$ be points in $\mathbb{P}_n$, and define

$$\|P\|_v = \begin{cases} \max(|x_0|_v, \ldots, |x_n|_v) & \text{if } v \text{ is non-archimedean} \\ \sqrt{|x_0|^2_v + \cdots + |x_n|^2_v} & \text{if } v \text{ is archimedean}. \end{cases}$$

Then the Chordal Distance on $\mathbb{P}_n$ is given by

$$\Delta_v(P, Q) = \frac{\max_{0 \leq i \leq j \leq n}\{|x_iy_j - x_jy_i|_v\}}{\|P\|_v \|Q\|_v}.$$ 

This definition is independent of the choice of projective coordinates for $P$ and $Q$, and satisfies $0 \leq \Delta_v(P, Q) \leq 1$. Additionally, $\Delta_v$ changes by a uniformly bounded factor under an automorphism of $\mathbb{P}_n$. Viewing $\mathbb{P}_1^2$ as a projective subvariety of $\mathbb{P}_3$ via the Segre embedding, we see that $P = (x, y) \in \mathbb{P}_1^2$ satisfies $\Delta_v(P, (\infty, \infty)) < \delta$ if and only if $|x|_v > 1/\delta$ and $|y|_v > 1/\delta$.

Let $f(x) = p(x)/q(x)$, with $p(x), q(x)$ coprime, be a rational function of degree $d$, and write $f^{(n)}(x) = p_n(x)/q_n(x)$ where $p_n(x)$ and $q_n(x)$ are also coprime. The homogenization of $f^{(n)}$ gives the rational function $F^{(n)}([x_0 : x_1]) = [P_n(x_0, x_1) : Q_n(x_0, x_1)]$ on $\mathbb{P}_1$. Let $D_n$ be divisor of zeros for $P_n(x_0, x_1)Q_n(y_0, y_1) - P_n(y_0, y_1)Q_n(x_0, x_1)$ and $B_i := D_i - D_{i-1}$. We note that $B_i$ is an effective divisor. In fact, put $x = x_0/x_1$ and $y = y_0/y_1$ and let

$$\beta_i = \frac{f^{(i)}(x) - f^{(i)}(y)}{f^{(i-1)}(x) - f^{(i-1)}(y)}.$$ 

Then $B_i$ is the zero divisor of $\beta_i$, in particular it is effective.

Having fixed coordinates on $\mathbb{P}_1$, we have models $(\mathbb{P}_1)_K$ and $(\mathbb{P}_1^2)_K$ for $\mathbb{P}_1$ and $\mathbb{P}_1^2$ over the ring of integers $\mathfrak{o}_K$ for a number field $K$. Then, for any finite set of places of $K$ including all the archimedean places of $K$, we may define $S$-integrality in the usual ways. We say that a point $Q \in \mathbb{P}_1(K)$ is $S$-integral with respect to a point $P \in \mathbb{P}_1(K)$ if the Zariski closures of $Q$ and $P$ in $(\mathbb{P}_1)_K$ do not meet over any primes $v \notin S$; similarly, we say that a point $Q \in \mathbb{P}_1^2(K)$ is $S$-integral with respect to a subvariety $V$ of $\mathbb{P}_1^2$, defined over $K$, if if the Zariski closures of $Q$ and $V$ in $(\mathbb{P}_1^2)_K$ do not meet over any primes $v \notin S$.

Note that the diagonal $D_0$ in $\mathbb{P}_1^2$ is defined by the equation $x_0y_1 - y_0x_1 = 0$. So, more concretely, we say that $([a_0 : a_1], [b_0 : b_1])$ is $S$-integral relative to
the diagonal $D_0$ if
\begin{equation}
|a_0b_1 - a_1b_0|_v \geq \max(|a_0|_v, |a_1|_v) \cdot \max(|b_0|_v, |b_1|_v)
\end{equation}
for all $v \notin S$. We say $[a_0 : a_1] \in \mathbb{P}^1(K)$ is $S$-integral relative to $[b_0 : b_1] \in \mathbb{P}^1(K)$ if $([a_0 : a_1], [b_0 : b_1])$ is $S$-integral relative to $D_0$. This definition of integrality is consistent with the previous one given that involved models for $\mathbb{P}_1$ and $\mathbb{P}_1^2$. Hence, given two points $P, Q \in \mathbb{P}_1^2$, $P$ is integral with respect to $Q$ if and only if the pair $(P, Q) \in \mathbb{P}^2_1(K)$ is integral with respect to the diagonal.

We may suppose, after enlarging our set of places $S$, that our rational function $f$ has good reduction at all primes outside of $S$, that is that $p$ and $q$ have no common root at any place outside of $S$. Then we see that $([a_0 : a_1], [b_0 : b_1])$ is $S$-integral relative to $D_n$ if
\begin{equation}
|P_n(a_0, a_1)Q_n(b_0, b_1) - P_n(b_0, b_1)Q_n(a_0, a_1)|_v \geq \max(|a_0|_v, |a_1|_v)^n \cdot \max(|b_0|_v, |b_1|_v)^n
\end{equation}
for all $v \notin S$. Note that from the definition above, it is clear that if a point is $S$-integral relative to $D_n$ then it is also $S$-integral relative to $D_m$ for any $m \leq n$ (as one would expect given that the support of $D_m$ is contained in the support of $D_n$ when $m \leq n$).

Furthermore, if $S$ contains all the places of bad reduction for $f$, we have
\begin{equation}
([a_0 : a_1], [b_0 : b_1]) \text{ is } S\text{-integral relative to } D_n
\iff (f^n([a_0 : a_1]), f^n([b_0 : b_1])) \text{ is } S\text{-integral relative to } D_0.
\end{equation}

We will often use coordinates $(x, y)$ on $\mathbb{P}_1^2$ where $x = x_0/x_1$ and $y = y_0/y_1$ for projective coordinates $([x_0 : x_1], [y_0 : y_1])$. We write $(\infty, \infty)$ for the point $([1 : 0], [1 : 0])$.

We say that point $z$ is exceptional for $f$ if $z$ is a totally ramified fixed point of $f^n$.

3. Ineffective finiteness

Applying a result of Vojta, we have the following.

\textbf{Theorem 3.1.} Let $K$ be a number field and $S$ a finite set of primes in $K$. Let $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function of degree $d \geq 2$. Then the set of points in $\mathbb{P}_1^2(K)$ that are $S$-integral relative to $D_4$ lies in a proper closed subvariety $Z$ of $\mathbb{P}_1^2$.

\textbf{Proof.} The divisor $D_4$ has at least five irreducible components, since it contains $B_0, \ldots, B_4$. A theorem of Vojta \cite{Voj87} Theorem 2.4.1] asserts that for any divisor $W$ on a nonsingular variety $V$, the points in $V(K)$ that are $S$-integral points relative to $W$ are not dense in $V$ if $W$ has at least $\rho + r + \dim V + 1$ components, where $\rho$ and $r$ are the ranks of $\text{Pic}^0(V)$ and the Néron-Severi group of $V$, respectively. Since $\text{Pic}^0(\mathbb{P}_1^2)$ is trivial and $\mathbb{P}_1^2$
has a Néron-Severi group of rank 2 (see [Har77, Example 6.6.1]), it follows that the set of points in \( \mathbb{P}^2_1(K) \) that are \( S \)-integral relative to \( D_4 \) lies in a proper closed subvariety of \( \mathbb{P}^2_1 \).

**Corollary 3.2.** Let \( K \) be a number field and \( S \) a finite set of primes in \( K \). Let \( f : \mathbb{P}_1 \to \mathbb{P}_1 \) be a rational function with degree \( d \geq 2 \). There is a proper closed subvariety of \( Y \) of \( \mathbb{P}^2_1 \) such that for any \( u, w \in \mathbb{P}_1(K) \), the subvariety \( Y \) contains all but at most finitely many points \((f^m(u), f^n(w))\) for which \( f^m(u) \) is \( S \)-integral relative to \( f^n(w) \).

**Proof.** Let \( Z \) be as in Theorem 3.1; let \( z_1, \ldots, z_e \) be the set of preperiodic points of \( f \) in \( K \) (note that this set must be finite), and let

\[
Y = Z \cup \left( \bigcup_{i=1}^{e} \mathbb{P}_1 \times \{z_i\} \right) \cup \left( \bigcup_{i=1}^{e} \{z_i\} \times \mathbb{P}_1 \right).
\]

If \( u \) or \( w \) is preperiodic for \( f \), then \((f^m(u), f^n(w)) \in Y \) for all \( m, n \), so we may assume that neither \( u \) nor \( w \) is preperiodic.

Now, by [Sil93, Theorem A], for any fixed \( n \), there are at most finitely many \( m \) such that \( f^m(u) \) is \( S \)-integral relative to \( f^n(w) \), because no \( f^n(w) \) is exceptional; likewise, there are at most finitely many \( m \) such that \( f^m(u) \) is \( S \)-integral relative to \( f^m(u) \). Thus, there are at most finitely many \((m, n)\) with \( \min(m, n) \leq 4 \) such that \((f^m(u), f^n(w))\) is \( S \)-integral relative to \( D_0 \).

By (2.0.2), we see that if \( m, n \geq 4 \), then \((f^{m-4}(u), f^{n-4}(w))\) is \( S \)-integral relative to \( D_4 \) if and only if \((f^m(u), f^n(w))\) is \( S \)-integral relative to \( D_0 \). Applying Theorem 3.1, one sees then that the set of points of the form \((f^m(w), f^n(u))\) that are \( S \)-integral relative to \( D_0 \) is contained in \( Y \).

**Corollary 3.3.** Let \( K \) be a number field and \( S \) a finite set of primes in \( K \). Let \( f : \mathbb{P}_1 \to \mathbb{P}_1 \) be a rational function with degree \( d \geq 2 \) that is not conjugate to a powering map \( x^{\pm d} \), and let \( u, w \in \mathbb{P}_1(K) \) be points that are not preperiodic for \( f \). Then the set of \((m, n) \in \mathbb{N}^2 \) such that \( f^m(u) \) is \( S \)-integral relative to \( f^n(w) \) is finite.

**Proof.** Let \( Y \) be as in Theorem 3.2. The set of points in \( Y \) that are \( S \)-integral relative to \( D_0 \) is finite by the main theorem of the Appendix of [BGK12], since \( f \) is not conjugate to a powering map \( x^{\pm d} \).

4. Effective degeneracy

We will now prove the following theorem.

**Theorem 4.1.** Let \( K \) be a number field and \( S \) a finite set of places of \( K \) including all the archimedean places. Let \( f : \mathbb{P}_1 \to \mathbb{P}_1 \) be a rational function of degree \( d \geq 2 \) that is not conjugate to a powering map \( x^{\pm d} \). Then there is a computable integer \( k_0 \) such that the set of points in \( \mathbb{P}^2_1(K) \) that are \( S \)-integral relative to \( D_{k_0} \) lies in an effectively computable proper closed subvariety of \( \mathbb{P}^2_1 \).
It suffices to show that there is a computable constant $A$ and a computable set $\Psi$ of nonzero rational functions $\psi$ on $\mathbb{P}^2_0(K)$ such that all the points in $\mathbb{P}^2_0(K)$ that are $S$-integral relative to $D_k$ lie on a curve of the form $\psi(x, y) = \gamma$ with $h(\gamma) < A$ and $\gamma \in K$ (note that there are finitely many such $\gamma$ and they can be effectively determined). The remainder of this section is devoted to constructing such a set $\Psi$ of rational functions.

Much of the proof is devoted to exploring the behavior of the functions $\psi$ near points that lie on the intersections of their pole divisors. We begin with a lemma about these intersection points.

**Lemma 4.2.** If the intersection of $2d$ distinct divisors $B_{m_0}, \ldots, B_{m_{2d-1}}$ contains a point $p = (\xi, \eta)$, then there are distinct $i$ and $j$ such that $f^{m_i-1}(\xi) = f^{m_j-1}(\eta)$ is a periodic critical point with period dividing some $m_j - m_i$.

**Proof.** First, we note that if $(c, c) \in B_1 \cap B_0$, then $(c, c)$ has multiplicity greater than 1 on $D_1$. Since $D_1$ is defined by the equation $f(x) = f(y)$, this means that $c$ must be a ramification point of $f$. Now suppose that $(\xi, \eta)$ are in $B_m$ and $B_n$ for $m < n$. Then $(f^{o(n-1)}(\xi), f^{o(n-1)}(\eta)) \in B_1 \cap B_0$ so $f^{o(n-1)}(\xi) = f^{o(n-1)}(\eta) = c$ for $c$ a ramification point of $f$. Thus, if $(\xi, \eta) \in B_{m_0} \cap \cdots \cap B_{m_{2d-1}}$ for $m_0 < m_1 < \cdots < m_{2d-1}$, then $f^{o(m_k-1)}(\xi) = f^{o(m_k-1)}(\eta)$ is a ramification point of $f$ for $k = 1, \ldots, 2d - 1$. Since $f$ has at most $2d - 2$ ramification points, we must have $f^{o(m_i-1)}(\xi) = f^{o(m_j-1)}(\xi)$ for some $i \neq j$ with $i, j \in \{1, \ldots, 2d - 1\}$. \hfill $\square$

Since $f$ has only finitely many ramification points, we may choose an $M$ such that the period of each periodic ramification point divides $M$. Note that a point $x$ is periodic for $f$ if and only if it is periodic for $f^M$, since if $(f^M)^{ok}(x) = x$, then $f^{Ok}(x) = x$ and if $f^{ok}(x) = x$, then $(f^{oM})^{ok}(x) = (f^{ok})^{oM}(x) = x$. Thus, every periodic ramification point of $f^M$ is a periodic ramification point for $f$ and is a fixed point of $f^M$. Proving Theorem [4.1] for an iterate $f^M$ of $f$ is equivalent to proving it for $f$ itself, so we may suppose, in view of the previous remark, that all the periodic ramification points of $f$ are fixed points for $f$. Since $f$ is not conjugate to a powering map and can therefore have at most one exceptional point, then, after possibly switching coordinates and passing to another higher iterate of $f$, we can make two further assumptions:

(i) If $f$ has an exceptional point, then that point is the point at infinity.

(ii) If $f$ does not have an exceptional point, then the point at infinity is fixed and unramified.

**Remark 4.3.** The conditions above ensure that if $(\infty, y)$ or $(x, \infty)$ appears in the intersection of $2d$ or more divisors $B_i$, then $\infty$ must be an exceptional point for $f$.

We now note that if two rational functions take on a large value at a point, then that point must lie near a point in the intersection of the two
rational functions’ pole divisors. The following lemma is quite similar to [Lev08, Lemma 2.1].

**Lemma 4.4.** Let $\phi_1, \ldots, \phi_m$ be rational function on $\mathbb{P}_1^2$. Let $Z_1, \ldots, Z_m$ denote their pole divisors. Suppose that for $i \neq j$, we have that $Z_i$ and $Z_j$ do not share a component; let $\Pi$ denote the (finite) set of points $p$ such that $p \in Z_i \cap Z_j$ for some $i \neq j$. Then for any $\delta > 0$ and any place $v$ of $K$, there is a computable constant $\gamma_{\delta,v}$ such that if $R \in \mathbb{P}_1^2(K_v) \setminus \bigcup_{i=1}^m Z_i(K_v)$ and $\Delta_v(R,p) \geq \delta$ for all $p \in \Pi$, then

$$|\phi_i(R)|_v \leq \gamma_{\delta,v}$$

for at least $(m - 1)$ of $i \in \{1, \ldots, m\}$.

**Proof.** For each $i$ and any $\delta > 0$, one can effectively bound $|\phi_i|_v$ on the set of points that are at a distance of at least $\delta$ from the pole divisor $Z_i$. To see this, note that it is clear when the poles are contained in fibers of $\mathbb{P}_1^2$ and that otherwise there is an explicit embedding (a composition of Segre and $d$-uple embeddings) $\iota: \mathbb{P}_1^2 \rightarrow \mathbb{P}_n$ that sends $Z_i$ into a hyperplane $H$. Moreover, one can effectively compute a generator $h$ for the ideal sheaf of $H$ and a regular function $\psi$ on $\mathbb{P}_n \setminus H$ that restricts to $\phi_i$ on $\iota(\mathbb{P}_1^2 \setminus Z_i)$, using the effective form of Hilbert’s Nullstellensatz (see [MWS3]). One can then explicitly bound $\phi_i$ away from $Z_i$ in terms of the coefficients of $h$ and $\psi$.

Now, let $\mathcal{U}$ denote the set of all points $R$ such that $\Delta_v(R,p) \geq \delta$ for all $p \in \Pi$. Then, for any $\phi_i, \phi_j$ with $i \neq j$, there is a computable lower bound $\epsilon_{i,j}$ on $\mathcal{U}$ for the minimum of the distances of a point in $\mathcal{U}$ to $Z_i$ and $Z_j$. Thus, there is some computable $\gamma_{i,j}$ such that $\min(|\phi_i(R)|_v, |\phi_j(R)|_v) \leq \gamma_{i,j}$ for all $R \in \mathcal{U}$. Letting $\gamma_{\delta,v}$ be the maximum of these $\gamma_{i,j}$ gives the desired bound.

In the case where $f$ has an exceptional point we need to treat this point, which we have assumed is $(\infty, \infty)$, differently than other points. For an integer $i$, we let $d_i$ equal the degree of $\beta_i(x, y)$, which is $d^i - d^{i-1}$ when $f$ has an exceptional point. We begin with a lemma about the behavior of the $\beta_i$ near $(\infty, \infty)$ in the case where $\infty$ is an exceptional point of $f$. The idea here is simple: since any distinct $B_i$ and $B_j$ meet transversally at $(\infty, \infty)$, at least one of $|\beta_i(x, y)|_v$ and $|\beta_j(x, y)|_v$ can be bounded below in terms of its degree near $(\infty, \infty)$.

**Lemma 4.5.** Let $p = (\infty, \infty)$ and suppose that $\infty$ is exceptional for $f$. Then, for any $M \neq N$, we have computable constants $C_{M,N,v}$ and a computable $\delta > 0$ such that

$$\left( \frac{|\beta_M(x, y)|_v}{\max(|x|_v^{d_M}, |y|_v^{d_M})}, \frac{|\beta_N(x, y)|_v}{\max(|x|_v^{d_N}, |y|_v^{d_N})} \right) \geq C_{M,N,v}$$

whenever $\Delta_v(P, p) < \delta$. 
**Proof.** Since $\infty$ is exceptional, $f$ is a polynomial. Assume $(x, y) \in K^2_v$ with $|x|^v \leq |y|^v$. The largest degree homogeneous term in $\beta_N$ is a constant times $\frac{x^{dN} - y^{dN}}{x^{dN_1} - y^{dN_1}}$ and thus we have

\begin{equation}
(4.5.2) \quad |\beta_N(x, y)|_v = |ey^{dN} \tilde{\beta}_n(t)|_v + O(|y|^{dN_1})
\end{equation}

for some constant $\epsilon$, where $t = x/y$ and $\tilde{\beta}_n$ is the monic polynomial whose roots are exactly the roots of unity of order dividing $dN$ but not $dN_1$. The $\tilde{\beta}_n$ are clearly pairwise coprime; thus, for fixed $M \neq N$, we have

$$\max(|\tilde{\beta}_M(t)|_v, |\beta_N(t)|_v) \geq W_{M,N,v}$$

for all $|t|_v \leq 1$ for some computable $W_{M,N,v}$. Combining this with (4.5.2) gives

$$\max \left( \frac{|\beta_M(x, y)|_v}{|y|^{dM}_v}, \frac{|\beta_N(x, y)|_v}{|y|^{dM}_v} \right) \geq W_{M,N,v} + O(1/|y|_v)$$

for large $|y|_v$. Similarly, when $|x|_v \geq |y|_v$, we obtain a bound

$$\max \left( \frac{|\beta_M(x, y)|_v}{|x|^{dM}_v}, \frac{|\beta_N(x, y)|_v}{|x|^{dM}_v} \right) \geq W_{M,N,v}' + O(1/|x|_v)$$

for large $|x|_v$. Combing these two bounds give (4.5.1) for all $|x|_v, |y|_v > 1/\delta$ for some computable $\delta > 0$. When $\Delta(x, y) < \delta$, we have $|x|_v, |y|_v > 1/\delta$, so our proof is complete.

We now treat the case where $(\xi, \eta)$ is contained in the intersection of at least $2d$ divisors $B_i$ but neither $\xi$ nor $\eta$ is exceptional for $f$. Again we take advantage of the fact that the $B_i$ meet transversally at these $(\xi, \eta)$ to show that for $i \neq j$, at least one of $|\beta_i(P)|_v$ and $|\beta_j(P)|_v$ can be bounded below in terms of the multiplicity of its pole divisor at $(\xi, \eta)$ for points $P$ near $(\xi, \eta)$.

**Lemma 4.6.** Let $p = (\xi, \eta) \neq (\infty, \infty)$ be contained in the intersection of $2d$ divisors $B_{m_0}, \ldots, B_{m_{2d-1}}$. Let $P = (\xi + t, \eta + u)$. There is an increasing sequence $r_i$ with $\lim_{i \to \infty} r_i/(d^i - d^{i-1}) = 0$ such that for any $M > N > m_{2d-1}$, we have computable constants $C_{M,N,v}$ and a computable $\delta > 0$ such that

\begin{equation}
(4.6.1) \quad \max \left( \frac{|\beta_M(P)|_v}{\max(|t|^{r_M}_v, |u|^{r_M}_v)}, \frac{|\beta_N(P)|_v}{\max(|t|^{r_N}_v, |u|^{r_N}_v)} \right) \geq C_{M,N,v}
\end{equation}

whenever $\Delta_v(P, p) < \delta$.

**Proof.** Since the point $p = (\xi, \eta)$ is contained in the intersection of $2d$ divisors $B_{m_0}, \ldots, B_{m_{2d-1}}$, Lemma 4.2 implies that $f^{m_{i-1}}(\xi) = f^{m_{i-1}}(\eta)$ is a periodic, and hence fixed, critical point for $f$. If that fixed critical point is the totally ramified point, then $\xi = \eta = \infty$. Since $p \neq (\infty, \infty)$, we may assume $f^{m_{i-1}}(\xi) = f^{m_{i-1}}(\eta) = \theta$ is not a totally ramified point for $f$ (see Remark 4.3); let $r$ be its ramification index. Then, for any $Q \geq 1$, we have

$$f^{cQ}(\theta + t_1) = \theta + \gamma t_1^Q + \ldots$$
with $r < d$ and
\[ f^{\circ Q}(\theta + t_1) - f^{\circ Q}(\theta + u_1) = \gamma t_1^{\circ Q} - u_1^{\circ Q} + \ldots \]
Write $h = m_{2d-1}$. Taking $N = Q + h$ and denoting $f^{\circ h}(\xi + t) = \theta + \lambda t^n + \ldots$ and $f^{\circ h}(\eta + u) = \theta + \lambda' u^d + \ldots$ with $\lambda, \lambda' \neq 0$, we obtain
\[ |f^{\circ N}(\xi + t) - f^{\circ N}(\eta + u)|_v = |f^{\circ Q}(f^{\circ h}(\xi + t)) - f^{\circ Q}(f^{\circ h}(\eta + u))|_v \]
\[ = |\epsilon t^{\sigma Q} - \epsilon' u^{srQ}|_v + o(1) \]
for some constants $\epsilon, \epsilon'$. Suppose that $|t^\sigma|_v \leq |u^s|_v$. Then we have
\[ |\beta_N(\xi + t, \eta + u)|_v = |\epsilon''|_v \left| \frac{t^{\sigma rQ} - u^{srQ}}{t^{\sigma rQ-1} - u^{srQ-1}} \right|_v + o(1) \]
\[ = |u^{(srQ-rQ-1)}(|\beta^*_N(\rho)|_v + o(1)) \]
where $\rho = t^\sigma/u^s$, and $\beta^*_N$ is polynomial whose roots are the roots of unity of order dividing $rQ$ but not $rQ-1$. Therefore, the distinct $\beta^*_N$ are pairwise coprime, and so for $M \neq N$, we have
\[ \max(|\beta^*_M(\rho)|_v, |\beta^*_N(\rho)|_v) \geq W_{M,N,v} > 0 \]
for $|\rho|_v \leq 1$. If $|t^\sigma|_v \geq |u^s|_v$, we divide instead by $t^\sigma$ and obtain a bound like (4.6.2) for $|\beta_N(\xi + t, \eta + u)|_v$ in $|t|_v^{\sigma(rQ-rQ-1)}$, also in terms of coprime polynomials for which a bound analogous to (4.6.3) holds. Letting $r_N = \max(\sigma, s)rQ$, and specifying that $|t|_v, |u|_v < 1$ gives (4.6.1). Finally, since $r < d$ and $Q = N - h$, we have
\[ \lim_{N \to \infty} \frac{(rQ - rQ-1)/(d^N - d^{N-1})}{0,} \]
so $\lim_{N \to \infty} r_N/(d^N - d^{N-1}) \to 0$.}

Now, we introduce new rational functions $\phi_N$, on $\mathbb{P}^2_1$ having poles where $\beta_N$ have zeros, but where the behavior of $|\psi_N(P)|_v$ can be controlled near certain points in the intersection of at least $2d$ pole divisors $B_N$. We will write
\[ \Phi_N(x, y) = \frac{\alpha_N(x, y)}{\beta_N(x, y)} \]
where $\alpha_N$ vanishes to a high degree at these points and where $\alpha_N$ and $\beta_N$ have the same pole divisors.

We begin by writing $\beta_N(x, y)$ as a quotient of degree $2(d^N - d^{N-1})$ bihomogeneous polynomials as
\[ (4.6.4) \]
\[ \frac{G_N(x_0, x_1; y_0, y_1)}{H_N(x_0, x_1; y_0, y_1)}. \]
If $f$ is not a polynomial, then there are no points in the intersection of at least $2d$ divisors $B_i$ at which any $H_N$ vanishes. To see this, note that the
product of all $\beta_N$ up to $M$ is
\[ \frac{P_M(x_0, x_1)Q_M(y_0, y_1) - Q_M(x_0, x_1)P_M(y_0, y_1)}{Q_M(x_0, x_1)Q_M(y_0, y_1)} \]
where $P_M$ and $Q_M$ are the homogeneous quotients of $f$, as in Section 2. Thus, if $H$ vanishes at $(a, b)$, then some iterate of $a$ or $b$ under $f$ is the point at infinity. On the other hand, if $(a, b)$ is in the intersection of at least $2d$ divisors, then some iterate of $a$ and some iterate of $b$ is a periodic ramification point. But we have chosen coordinates so that no iterate of the point at infinity is a periodic ramification point, proving the claim. We can rewrite (4.6.4) as
\[ \frac{g_N(x, y)}{h_N(x, y)} \]
where $x = x_0/x_1$ and $y = y_0/y_1$ as before. Then $\deg g_N \geq d^N - d^{N-1}$, since $\deg g_N = d^N - d^{N-1}$ if $f$ is a polynomial and $\deg g_N \geq 2(d^N - d^{N-1}) - 1$ otherwise, since $f$ fixes the point at infinity. We see that if $f$ is a polynomial, then $h(x, y)$ is a constant (and hence vanishes nowhere). We let $d_N$ denote $\deg g_N$ (note that is is consistent with the definition of $d_N$ before Lemma 4.5).

**Lemma 4.7.** Let $L > 0$ be a positive integer, and $\Pi_\infty$ a finite set of points contained in at least $2d$ of the divisors $B_i$. Then, there are rational functions $\phi_{M_1}, \ldots, \phi_{M_L}$ on $\mathbb{P}^2$, with $\phi_{M_i}$ having pole divisor $B_{M_i}$, where $M_L > \cdots > M_1$, and $A_v > 0$, $\delta > 0$ such that whenever $\Delta_v(P, p) < \delta$ for some $p \in \Pi_\infty$, we have $|\phi_{M_i}(P)|_v < A_v$ for all but at most one $\ell \in \{1, \ldots, L\}$.

**Proof.** Write $\Pi_\infty = \{p_1, \ldots, p_k\}$. Take $N$ large enough so that the conclusion of Lemma 4.6 holds for each point $p_j = (\xi_j, \eta_j) \neq (\infty, \infty)$ in $\Pi_\infty$, and write $r_{j, N}$ for the maximum of the ramification indices of $f^N$ at $\xi_j$ and $\eta_j$. Let $e(j)$ be the product of the degrees of $\xi_j$ and $\eta_j$ over $K$, and let $g_{\xi_j}(x)$ and $g_{\eta_j}(y)$ denote their respective minimal polynomials over $K$. Since $r_{j, N}/d_N \to 0$, increase $N$ so that $\sum_{j=1}^k 2e(j)r_{j, N} < d_N$ by Lemma 4.6. Then
\[ \prod_{j=1}^k (g_{\xi_j}(x)g_{\eta_j}(y))^{r_{j, N}} \]
is a polynomial with coefficients in $K$ vanishing at each of the points $p_j = (\xi_j, \eta_j) \neq (\infty, \infty)$ with multiplicity at least $r_{j, N}$. Furthermore, denoting $e_N$ for the degree of $\alpha_N$, we find that $e_N \leq \sum_{j=1}^k 2e(j)r_{j, N} < d_N$. We let
\[ \alpha_N(x, y) = \prod_{j=1}^k (g_{\xi_j}(x)g_{\eta_j}(y))^{r_{j, N}}(xy)^{d_N - \sum_{j=1}^k 2e(j)r_{j, N}} h_N(x, y) \]
where $h(x, y)$ is as in (4.6.5). Then the pole divisor of $\alpha_N(x, y)/\beta_N(x, y)$ is simply the zero divisor of $\beta_N(x, y)$.
Let $\phi_{M_1}, \ldots, \phi_{M_L}$ be the rational function $\frac{\alpha_{M_i}(x,y)}{\beta_{M_i}(x,y)}$, where $M_i > \cdots > M_1$ are sufficiently large as above. Let $p_j \neq (\infty, \infty)$ be a point in $\Pi_\infty$. By \cite{4.7.1}, we have a computable constant $E_{M_i,j,v}$ such that
\[
|\alpha_{M_i}(\xi + t, \eta + u)|_v \leq E_{M_i,j,v}|t|_{\nu,v}^{\nu, N}|u|_{\nu,v}^{\nu, N}
\]
for small $|t|_v, |u|_v$ by \cite{4.7.1}. Thus, by \cite{4.6.1}, we have for any $i \neq k$ that
\[
\min(|\phi_{M_i}(\xi + t, \eta + u)|_v, |\phi_{M_k}(\xi + t, \eta + u)|_v) \leq A_{M_i,M_k,j,v}
\]
for some computable $A_{M_i,M_k,j,v}$. Letting $A_{j,v}$ be the maximum of these $A_{M_i,M_k,j,v}$ gives $|\phi_{M_\ell}(P)|_v < A_{j,v}$ for all but at most one $\ell \in \{1, \ldots, L\}$ when $\Delta_v(P,p_j) < \delta$, for some computable $\delta > 0$.

Now, suppose that $p_j = (\infty, \infty)$. Then $f$ is a polynomial so $\beta_{N_i}$ and $\alpha_{N_i}$ are each polynomials of degree $d^{N_i} - d^{N_i-1}$. Thus, we have
\[
\frac{|\alpha_{N_i}(x,y)|_v}{\max(|x|_{\nu,v}, |y|_{\nu,v})} < U_{j,v}
\]
for some computable constant $U_{j,v}$. Then, \cite{4.5.1} implies that for any $i \neq k$, there is a constant $A_{M_i,M_k,j,v}$ we have
\[
\min(|\phi_{M_i}(x,y)|_v, |\phi_{M_k}(x,y)|_v) \leq A_{M_i,M_k,j,v}
\]
for all sufficiently large $|x|_v, |y|_v$. Taking $A_{j,v}$ to be the maximum of these $A_{M_i,M_k,j,v}$ gives $|\phi_{M_\ell}(P)|_v < A_{j,v}$ for all but at most one $\ell \in \{1, \ldots, L\}$ when $\Delta_v(P,p_j) < \delta$.

Finally, taking $A_v$ to be the maximum of all the $A_{j,v}$ gives the desired $A_v$ and our proof is complete. \hfill $\Box$

Now, we use the Lemma \cite{4.6} to produce a set of rational functions, each having a pole divisor supported on some $B_i$, that are bounded away from a finite set of points contained in the intersection of at least $2d B_i$ at each place $v \in S$. We use induction on the size of $S$.

**Proposition 4.8.** There exists a set $\Psi$ of rational functions $\psi_{N_i}$ with $i \in \{1, \ldots, 2ds + 1\}$, each having pole divisor with support on $B_{N_i}$, and a finite set of points $\Pi = \{p_1, \ldots, p_k\}$, each contained in at least $2d$ distinct $B_{N_i}$ such that, for any $\delta > 0$, there are effectively computable constants $T_\delta$ with the property that for any $Q$ that is $S$-integral relative to $\sum B_{N_i}$, one of the following conditions holds:

(i) there is a $\psi \in \Psi$ such that $|\Psi(Q)|_v \leq T_\delta$ for all $v \in S$;

(ii) for each $v \in S$, there is a $p_v \in \Pi$ such that $\Delta_v(Q,p_v) < \delta$.

**Proof.** Let $s = |S|$. We say that Property (*) holds for $s_0 \leq s$ if there exists a set $\Psi$ of rational functions $\psi_{N_i}$, each having pole divisor with support on $B_{N_i}$, and a finite set of points $\Pi = \{p_1, \ldots, p_k\}$, each contained in at least $2d$ distinct $B_{N_i}$ such that, for any $\delta > 0$, there are effectively computable proper constants $T_\delta$ with the property that for any $Q$ that is $S$-integral relative to all of the $B_{N_i}$, one of the following conditions holds:

(i) there is a $\psi \in \Psi$ such that $|\psi(Q)|_v \leq T_\delta$ for all $v \in S$;
(ii) there is a subset \( S_0 \subseteq S \) with \( |S_0| = s_0 \) such that for each \( v \in S_0 \), there is a \( p_v \in \Pi \) such that \( \Delta_v(Q, p_v) < \delta \).

We will prove, by induction, that for every \( s_0 \leq s \), Property (*) holds for \( s_0 \). If \( s_0 = 0 \), then (ii) holds since it is vacuously true for the empty set. Now suppose that Property (*) holds for some \( s_0 \leq s - 1 \), and let \( \Pi \) and \( T_\delta \) be the set and constants for which (i) or (ii) holds. By Lemma 4.7, there is a computable proper \( \delta_0 > 0 \) such that we may construct a set \( \Phi \) of \( 2ds + 1 \) rational functions \( \phi_{M_1}, \ldots, \phi_{2ds+1} \), with \( \phi_{M_i} \) having pole divisor \( B_{M_i} \), such that if \( \Delta_v(Q, p) < \delta_0 \) for some \( p_v \in \Pi \), then

\[
|\phi_{M_i}(R)|_v < A_v \quad \text{for all but at most one } i \in \{1, \ldots, 2ds + 1\}.
\]

Let \( \Pi^* \) denote the finite set of points at which the \( B_{M_i} \) intersect. We let \( \Pi_0 \) denote those at which at most \( 2d - 1 \) meet. Choose \( \delta_1 \) small enough such that if \( p_v \in \Pi_0 \) is in \( B_{M_i} \cap \cdots \cap B_{M_{\delta_1}} \) for distinct \( \{i_1, \ldots, i_{\delta_1} \} \) and is in no other \( B_{M_i} \) (note that \( e \leq (2d - 1) \)), then \( \Delta_v(R, p_v) > \delta_1 \) for all \( R \in B_{M_k} \) for \( k \notin \{i_1, \ldots, i_{\delta_1} \} \). Take any \( R \in \mathbb{P}_1(K) \) outside the support of the \( B_{M_i} \). If \( \Delta_v(R, p_v) < \delta_1 \) for some \( p_v \in \Pi_0 \), where \( p_v \in B_{M_{\delta_1}} \cap \cdots \cap B_{M_{\delta_1}} \) for distinct \( \{i_1, \ldots, i_{\delta_1} \} \), then \( |\phi_{M_i}(R)|_v \) is bounded by some computable \( \omega_v \) for all but at most one \( i \notin \{i_1, \ldots, i_{\delta_1} \} \) by our choice of \( \delta_1 \), by Lemma 4.3 so in particular \( |\phi_{M_i}(R)|_v \leq \omega_v \) for all but at most \( 2d \) of the \( M_i \).

Now, let \( \delta < \min(\delta_0, \delta_1) \). Suppose further that \( \delta \) is less than the minimum of the distances between distinct points of \( \Pi_0 \). If \( \Delta_v(R, p_v) \geq \delta \) for all \( p_v \in \Pi^* \), then for all but at most one of \( \phi_{M_1}, \ldots, \phi_{2ds+1} \) we have \( |\phi_{M_i}(R)|_v \leq \gamma_v \) for some computable \( \gamma_v \) by Lemma 4.3. Putting this together, we see that since there are \( s \) places in \( S \) and \( 2ds + 1 \) functions \( \phi_{M_i} \), this means that there is some \( \phi_{M_i} \) such that \( |\phi_{M_i}(R)|_v \) is bounded by \( \max(\gamma_v, \omega_v, A_v) \) for all \( v \in S \) unless \( \Delta_v(R, p_v^*) < \delta \) for some \( v \in S \) and some \( p_v^* \in \Pi^* \setminus (\Pi_0 \cup \Pi) \).

Let \( Q \) be a point that is \( S \)-integral relative all of the \( B_{M_i} \) and all of the \( B_{M_j} \). Suppose that there is no \( \psi_N \) such that \( |\psi_N(Q)|_v \) is bounded by \( T_\delta \), where \( (T_\delta \) is as in the statement of Property (*)) for all \( v \in S \). Then there is a subset \( S_0 \subseteq S \) with \( |S_0| = s_0 \) such that for each \( v \in S_0 \), there is a \( p_v \in \Pi \) such that \( \Delta_v(Q, p_v) < \delta \), by the inductive hypothesis. Suppose that there is no \( |\phi_{M_i}(R)|_v \) that is bounded by \( \max(\gamma_v, \omega_v, A_v) \) for all \( v \in S \). Then, as above, there is a \( v \in S \) such that \( \Delta_v(Q, p_v^*) < \delta \) for some \( p_v^* \in \Pi^* \setminus (\Pi \cup \Pi_0) \); no two elements of \( \Pi^* \) are within \( 2\delta \) of each other, we must have \( v \notin S_0 \). Thus there is a set \( S' \) of size \( s_0 + 1 \) such that for each \( v \in S' \), there is \( p_v \in \Pi \cup (\Pi^* \setminus \Pi_0) \) such that \( \Delta_v(Q, p_v) < \delta \). Hence, letting \( \Psi' = \Psi \cup \Phi \) and \( \Pi' = \Pi \cup (\Pi^* \setminus \Pi_0) \), we see that Property (*) holds for \( s_0 + 1 \). This completes the inductive step and our proof is done.

Finally, we use Proposition 4.8 to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let \( \Psi \) and \( \Pi \) be the sets given by Proposition 4.8. Recall that each \( \psi_N \in \Psi \) has a pole divisor with support on \( B_{N_i} \). Applying Lemma 4.7 we obtain \( s + 1 \) rational functions \( \phi_{M_i} \) each having
pole divisor with support on $B_{M_i}$ along with computable $\delta > 0$ and constants $\gamma_v$ such that whenever $\Delta_v(P, p_v) < \delta$ for some $p_v \in \Pi$, we have $|\phi_{N_i}(P)|_v < \gamma_v$ for all but at most one $i \in \{1, \ldots, s + 1\}$. Take any $Q$ that is $S$-integral for $D_{k_0}$ where $k_0$ is the maximum of all the $N_i$ and $M_i$ above. Then we have a constant $\kappa$ such that $\sum_{v \notin S} \max \{|\psi_{N_i}(Q)|_v, 0\} \leq \kappa$ and $\sum_{v \in S} \max \{|\phi_{M_i}(Q)|_v, 0\} \leq \kappa$ for all $v \in S$. Furthermore, by Proposition 4.31 we have one of the following:

(i) there is a $\psi_{N_i} \in \Psi$ such that $|\psi_{N_i}(Q)|_v \leq T_\delta$ for all $v \in S$;

(ii) for each $v \in S$, there is a $p_v \in \Pi$ such that $\Delta_v(Q, p_v) < \delta$.

If (i) holds, then there is some $\psi_{N_i}$ such that

$$\sum_{\text{places } v \text{ of } K} \max \{|\log \psi_{N_i}(Q)|_v, 0\} \leq \kappa + \sum_{v \in S} T_\delta.$$ 

If (ii) holds, then for each $|\phi_{M_i}(P)|_v < \gamma_v$ for all but at most one $i \in \{1, \ldots, s + 1\}$. Since there are only $s$ places in $S$, this means there is some $\phi_{M_i}$ such that $|\phi_{M_i}(P)|_v < \gamma_v$ for all $v \in S$. Hence, there is some $\phi_{M_i}$ such that

$$\sum_{\text{places } v \text{ of } K} \max \{|\phi_{M_i}(Q)|_v, 0\} \leq \kappa + \sum_{v \in S} \gamma_v.$$ 

Since there are only finitely many points $z \in K$ of bounded height and $Q$ lies on a curve of the form $\psi_{N_i}(x, y) = z$ or $\phi_{M_i}(x, y) = z$, with $z \in K$ having bounded height, we see then that $Q$ lies on an effectively computable proper subvariety of $\mathbb{P}^2$, as desired. $\square$

5. Effective finiteness

Silverman mentions that [Sil93, Theorem A] can be made effective. For the sake of completeness, we give a quick proof of this fact.

**Theorem 5.1.** Let $K$ be a number field, let $S$ a finite set of primes in $K$, let $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function with degree $d \geq 2$, let $a$ be a point that is not periodic for $f$, and let $b$ be a point that is not exceptional for $f$. Then the set of $n$ such that $f^n(a)$ is integral relative to $b$ is finite and effectively computable.

**Proof.** Since $b$ is not exceptional, $f^{-4}(b)$ contains at least three distinct points. To see this note that $f^{-2}(b)$ contains at least two points, since $b$ is not exceptional. If $f^{-2}(b)$ contains exactly two points, then there is a totally ramified point in $f^{-1}(b)$ or $f^{-2}(b)$. This point cannot be fixed by $f$ so it cannot be in both $f^{-3}(b)$ and $f^{-4}(b)$. If $f^{-3}(b)$ contains only two points, then they must both be totally ramified, so $f^{-4}(b)$ must contain a point that is not totally ramified (because $f$ has at most two totally ramified points, by Riemann-Hurwitz), which means that $f^{-4}(b)$ contains at least three points.
For \( n \geq 3 \), we have that \( f^m(a) \) is \( S \)-integral relative to \( b \) if and only if \( f^{(n-3)}(a) \) is \( S \)-integral relative to the points in \( f^{-2}(b) \). Changing coordinates, these \( f^{(n-3)}(a) \) are solutions to the \( S \)-unit equation, which has an effective solution (see \[BG06\] Theorem 5.4.1, for example). \( \square \)

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 4.1 delivers an effectively computable one-dimensional subvariety \( Z \) such that the \( (m,n) \) with \( m,n \geq k_0 \) for which \( f^m(u) \) is \( S \)-integral relative to \( f^n(w) \) are effectively computable for all \( (f^m(u), f^n(w)) \) outside of \( Z \).

Let \( c \) be the number of components of \( Z \). Let \( I_{u,w} \) denote the set of \( (m,n) \) such that \( f^m(u) \) is \( S \)-integral relative to \( f^n(w) \). By Theorem 5.1 we know set of \( (m,n) \in I_{u,w} \) with \( \min(m,n) \leq c + k_0 \) is effective computable. Thus, it suffices to show that the set of \( (m,n) \in I_{u,w} \) with \( \min(m,n) \geq c + k_0 \) and \( (f^m(u), f^n(w)) \in Z \) is effective computable. Note that if \( m,n \geq c \geq r \), then \( f^m(u) \) can be \( S \)-integral relative to \( f^n(w) \) only when \( f^{(m-r)}(u) \) is \( S \)-integral relative to \( f^{(n-r)}(w) \). Hence, it suffices to find all \( (m,n) \in I_{u,w} \) such that \( (f^m(u), f^n(w)) \) is in

\[
Z \cap (f,f)^{-1}(Z) \cap \cdots \cap (f,f)^{-c}(Z).
\]

If this is finite, we are done. Otherwise, there is a common component \( X \) among \( Z, (f,f)^{-1}(Z), \ldots, (f,f)^{-c}(Z) \). Then \( (f,f)^i(X) \) is a component of \( Z \) for \( i = 0, \ldots, c \). Therefore, \( (f,f)^i(X) = (f,f)^j(X) \) for some \( c \geq j > i \geq 0 \). So \( (f,f)^i(X) \) is a periodic component of \( Z \).

Thus, we are left to show that the points of the form \( (f^m(u), f^n(w)) \) which \( S \)-integral relative to \( D_0 \) on any periodic curve \( X \) for \( (f,f) \) can be computed. Now, since \( X \) admits a self-map of positive degree, it must have genus 0 or 1. Since \( X \cap D_0 \) contains at least one point, we see that if \( X \) has genus 1, then the integral points on \( X \) relative to \( D_0 \) can be effectively computed (see \[BC70, Bil97\]). If \( X \) has genus 0, and \( X \cap D_0 \) contains a nonexceptional point, then we are done by Theorem 5.1. If \( X \cap D_0 \) contains only an exceptional point \( z \), then after changing coordinates, we may write the restriction of \( (f,f)^2 \) to \( X \) as a polynomial \( P(t) \), where \( z \) is the point at infinity. If \( X \cap D_0 \) contains two exceptional points for \( P(t) \), the, after changing coordinates, we may write the restriction of \( (f,f)^2 \) to \( X \) as a polynomial \( t^n \), where \( z_1 \) is zero and \( z_2 \) is the point at infinity. In either case, after expanding \( S \) to a possibly larger set of primes \( S' \) we have that for any \( S' \)-integral point \( \gamma \) on \( X \), each iterate \( (f,f)^{\alpha}(\gamma) \) is \( S' \)-integral relative to \( D_0 \). This means that there are infinitely many \( S' \)-integral points relative to \( D_0 \) on \( X \), which contradicts the the main theorem of the Appendix of \[BGK+12\]. \( \square \)

6. Cyclic and Exceptional Cases

When \( f \) is conjugate to a powering map, we do not obtain a finiteness result. This can be seen, for example, by considering the map \( f(x) = x^3 \) and
the points $u = 2, w = -2$. Then, if $S$ is the set containing the archimedean place and the place 2, we have that $f^{om}(u)$ is $S$-integral relative to $f^{on}(w)$ for all $m$. On the other hand, it is possible to give a reasonable description of the $(m, n)$ such that $f^{om}(u)$ is $S$-integral relative to $f^{on}(w)$.

In [GTZ], it is proved that if $\deg a, \deg b > 2$ for polynomials $a$ and $b$, then the set of $(m, n)$ such that $a^{om}(u) = b^{on}(v)$ forms a finite union of cosets of subsemigroups of $\mathbb{N}^2$ (that is a finite union of additive translates of subsets of $\mathbb{N}^2$ that are closed under addition). Here, $\mathbb{N}$ is considered to include 0 so any finite set of $(m, n)$ is a finite union of cosets of $(0, 0)$.

For a set of places $S'$ containing all the archimedean places, we define

$$I_{u,w,S'} = \{(m, n) \in \mathbb{N}^2 \mid f^{om}(u) \text{ is } S'-\text{integral relative to } f^{on}(w)\}.$$ 

**Proposition 6.1.** Let $f$ be conjugate to $x^{\pm d}$, let $S$ be a finite set of places of $K$. Then, for some finite set of places $S' \subseteq S$, the set $I_{u,w,S'}$ is a finite union of effectively computable cosets of subsemigroups of $\mathbb{N}^2$. Furthermore, the set $I_{u,w,S'}$ is finite if $u$ and $w$ are multiplicatively independent or if $u$ and $w$ are in the cyclic group generated by a non-torsion element of $K^*$.

**Proof.** After changing coordinates by an automorphism $\sigma \in \text{PGL}_2(K')$, for $K'$ a finite extension of $K$, we can write $\sigma f \sigma^{-1}(x) = x^{\pm d}$. Choose a set $S'$ of primes that includes both all the primes appearing in the coefficients or determinant of $\sigma$ as well as all the primes lying over primes in $S$; then for any $P, Q \in \mathbb{P}_1(K')$, we have that $P$ is $S'$-integral relative to $Q$ if and only if $\sigma P$ is $S'$-integral relative to $\sigma Q$ (note that this choice of $S'$ depends only on $f$, not on $u$ or $w$). Thus, it suffices to prove the theorem when $f(x) = x^{\pm d}$.

If $f(x) = x^{\mp d}$, then by considering the orbit of $(u, w)$ along with those of $(f(u), w), (u, f(w)), \text{ and } (f(u), f(w))$, we reduce to the case where $f(x) = x^{d^2}$ for some $d$. If $u$ or $w$ is zero or infinity, the conclusion is obvious. If neither $u$ nor $v$ is 0 or infinity, we may assume that $u$ and $w$ are both $S'$-units after expanding $S'$. Then $f^{om}(u) - f^{on}(w)$ is an $S'$-unit if and only if $\frac{f^{om}(u)}{f^{on}(w)} - 1$ is an $S'$-unit. Thus, if $(f^{om}(u), f^{on}(w))$ is $S'$-integral relative to $D_0$, then it lies on a curve of the form $x - y = \tau y$ where $\tau$ is an $S'$-unit such that $\tau + 1$ is also an $S'$-unit. By [BG06] Theorem 5.4.1, the set of such $\tau$ is finite and effectively computable. Thus, if $u$ and $w$ are multiplicatively independent, then $\frac{f^{om}(u)}{f^{on}(w)}$ takes on any such value $\tau$ at most once, so there are at most finitely many $(m, n)$ such that $f^{om}(u)$ is $S'$-integral relative to $f^{on}(w)$. For any fixed value of $1 + \tau$, the set of $m, n$ such that $u^{d_m} w^{d_n} = 1 + \tau$ clearly forms a finite union of cosets of subsemigroups of $\mathbb{N}^2$.

If $u$ and $w$ are both in the subgroup of $K^*$ generated by a single element $z$ that is not a root of unity, then we may write $u = z^A, w = z^B$. Then, we have $z^{Ad_m - Bd_n} = (1 + \tau)$ for one of the finitely many $1 + \tau$ above whenever $f^{om}(u)$ is $S'$-integral relative to $f^{on}(w)$. Now, for any constant $C$, the set of $(m, n)$ such that $Ad_m - Bd_n = C$ is finite unless $C = 0$ (since $\gcd(Ad_m, Bd_n) \to \infty$ if $m$ and $n$ are both infinite).
to infinity as \( \min(m, n) \to \infty \), but when \( C = 0 \), we have \( \tau = 0 \), which is not an \( S \)-unit. Hence, in this case there are at most finitely many \((m, n)\) such that \( f^{on}(u) \) is \( S' \)-integral relative to \( f^{on}(w) \).

When at least one of \( u \) or \( w \) is preperiodic, but neither of \( u \) or \( w \) is exceptional, it is easy to see from Theorem 4.1 that the set of \((m, n) \in \mathbb{N}^2 \) such that \( f^{on}(u) \) is \( S \)-integral relative to \( f^{on}(w) \) forms a finite union of effectively computable cosets of subsemigroups of \( \mathbb{N}^2 \). When \( u \) or \( w \) is exceptional, however, one should not expect there to be a particularly nice pattern to the set of \((m, n) \) such that \( f^{on}(u) \) is \( S \)-integral relative to \( f^{on}(w) \). Benedetto-Briend-Perdry \cite{BBP07} show that if \( f(x) = x^2 + \frac{x}{p} \), and \( v \) is the point at infinity, then for any set \( U \) of positive integers, there is a point \( u \in \mathbb{Q}_p \) such that \( f^{on}(u) \in \mathbb{Z}_p \) if and only if \( m \in U \); although this is only stated over \( \mathbb{Q}_p \), it is very likely that one can find examples for many complicated infinite \( U \) over \( \mathbb{Q} \). This problem can be overcome by enlarging \( S \) to a finite set of primes \( S' \) including all the primes of bad reduction for \( S \).

**Proposition 6.2.** Suppose that \( w \) is exceptional and that there is no \( m \) such that \( f^m(u) = w \). Then for some finite set of places \( S' \) with \( S \subseteq S' \), the set \( I_{u, w, S'} \) is all of \( \mathbb{N}^2 \).

**Proof.** Arguing as in Proposition 6.1 we may change coordinates so that \( f^{o2} \) is a polynomial and \( w \) is the point at infinity and enlarge \( S \) to some \( S' \) where our notion of \( S' \)-integrality is not affected by the coordinate change. If we enlarge \( S' \) further to include all of the places at which \( u \), \( f(u) \), or a coefficient of \( f^{o2} \) has a pole, then \( f^{2m}(u) \) and \( f^{2m}(f(u)) \) are \( S' \)-integral relative to \( w \) for all \( m \), so \( I_{u, w, S'} \) is all of \( \mathbb{N}^2 \).

7. Further questions

If \( f \) and \( g \) are two rational functions of degree \( d > 1 \) such that there are no \( z_1, z_2 \) such that \( f^{o2}(z_1) = g^{o2}(z_2) \) with \( f^{o2} \) ramifying at \( z_1 \) and \( g^{o2} \) ramifying at \( z_2 \) (a reasonably “generic” condition), then \( f^{o2}(x) - g^{o2}(y) = 0 \) gives a nonsingular curve corresponding to a divisor \( D_2 \) of type \((2d, 2d)\) on \( \mathbb{P}_1^2 \). Since \( d \geq 2 \), we have that \( D_2 + K_X \) is ample for \( K_X \) a canonical divisor of \( \mathbb{P}_1^2 \). Thus, Vojta’s conjecture \cite{Voj87} [Conjecture 3.4.3] would imply that the set of \( S \)-integral points relative to \( D_2 \) must be degenerate. Hence, we may expect that an analog of Theorem 3.1 holds in this case. It may be that the method of this paper allows one to prove such general statements.

**References**

\cite{BBP07} R. Benedetto, J.-Y. Briend, and H. Perdry, *Dynamique des polynômes quadratiques sur les corps locaux*, J. Théor. Nombres Bordeaux 19 (2007), no. 2, 325–336.

\cite{BC70} A. Baker and J. Coates, *Integer points on curves of genus 1*, Proc. Cambridge Philos. Soc. 67 (1970), 595–602.

\cite{Bel06} J. P. Bell, *A generalised Skolem-Mahler-Lech theorem for affine varieties*, J. London Math. Soc. (2) 73 (2006), no. 2, 367–379.
[BG06] E. Bombieri and W. Gubler, Heights in Diophantine geometry, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006.

[BGK+12] R. L. Benedetto, D. Ghioca, P. Kurlberg, T. J. Tucker, and U. Zannier, A case of the dynamical Mordell-Lang conjecture, Math. Ann. 352 (2012), no. 1, 1–26.

[Bil07] Y. F. Bilu, Quantitative Siegel’s theorem for Galois coverings, Compositio Math. 106 (1997), no. 2, 125–158.

[Den92] L. Denis, Hauteurs canoniques et modules de Drinfeld, Math. Ann. 294 (1992), no. 2, 213–223.

[Fal94] G. Faltings, The general case of S. Lang’s conjecture, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., no. 15, Academic Press, San Diego, CA, 1994, pp. 175–182.

[GT09] D. Ghioca and T. J. Tucker, Periodic points, linearizing maps, and the dynamical Mordell-Lang problem, J. Number Theory 129 (2009), no. 6, 1392–1403.

[GTZ] D. Ghioca, T. J. Tucker, and M. E. Zieve, Linear relations between polynomial orbits, to appear in Duke Math J., available online at arxiv.org/abs/0807.3576, 27 pages.

[Har77] R. Hartshorne, Algebraic geometry, Springer-Verlag, New York, 1977.

[Lev08] A. Levin, Variations on a theme of Runge: effective determination of integral points on certain varieties, J. Théor. Nombres Bordeaux 20 (2008), no. 2, 385–417.

[MW83] D. W. Masser and G. Wüstholz, Fields of large transcendence degree generated by values of elliptic functions, Invent. Math. 72 (1983), no. 3, 407–464.

[Run87] C. Runge, Über ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen, J. Reine Angew. Math. 100 (1887), 425–435.

[Sie29] C. L. Siegel, Über einige anwendungen diophantischer approximationen, Abh. Preuss. Akad. Wiss. Phys. Math. Kl. (1929), 41–69.

[Sil93] J. H. Silverman, Integer points, Diophantine approximation, and iteration of rational maps, Duke Math. J. 71 (1993), no. 3, 793–829.

[Voj87] P. Vojta, Diophantine approximations and value distribution theory, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987.

[Voj96] ________, Integral points on subvarieties of semiabelian varieties. I, Invent. Math. 126 (1996), no. 1, 133–181.

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