Effect of predominantly unidirectional flow of a viscous liquid between solid walls

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Abstract. The problem is considered on a flow of a viscous liquid in the presence of solid bodies – two walls and a free plate the boundary of which is permeable for a liquid – under oscillatory influences. The formulation of the problem includes the equation of the plate motion, the equations of Navier–Stokes and a continuity and conditions at solid boundaries of the liquid. The new hydro-mechanical effect is revealed which consists in that in the absence of a predominant direction in space free parts of the hydro-mechanical system at a background of oscillations perform a unidirectional, steady motion.

1. Introduction
One of actual directions of fluid mechanics is the study of the dynamics of hydro-mechanical systems under vibrational (oscillatory) influences. A significant number of works is fulfilled in this direction (see [1–19] and the references which are presented there). The realized investigations allowed to reveal a series of new hydro-mechanical effects; it is proved by obtained results that hydro-mechanical systems can behave unordinary or even paradoxically under oscillatory influences. In particular the effect of a paradoxical motion of a solid body in a vibrating liquid was discovered in [2]. The presence of the phenomenon of a predominantly unidirectional motion of compressible inclusions in a vibrating liquid was established by the theoretical and experimental works [6–8, 11]. For the first time it was introduced the division (the classification) of liquid vibrations (liquid oscillations) in uniform and non-uniform and it was shown its importance of principle in [9] (see also [12]). A hydro-mechanical analog of the “Kapitsa oscillator” effect [20] was obtained in [13]. The effect of a prescribed orientation of a solid inclusion in a viscous liquid was found in [16]. The effect of a predominantly unidirectional rotation of a solid body and a viscous liquid was found in [17]. The effect of a predominantly unidirectional rotation of a viscous liquid with a free boundary was revealed in [19].

The present work contains the formulation and the solution of the problem on the motion of a hydro-mechanical system consisting of a viscous liquid and solid bodies – two walls and a plate which boundary is permeable for a liquid. The plate is free (its motion has to be determined). The hydro-mechanical system undergoes by oscillatory influences which are characterized by the absence of a predominant direction in space. It is revealed that in response to the (non-unidirectional) influences free parts of the hydro-mechanical system – liquid layers and also the plate – perform a unidirectional, steady motion.
2. Formulation of the problem

There are a viscous incompressible liquid and contacting it absolutely solid walls \( W_1, W_2 \) and a plate \( \Xi \). The walls perform prescribed oscillations along the axis \( Y \) of the inertial rectangular coordinate system \( X, Y, Z \). The wall \( W_1 \) is confined by the plane \( Y = A_1 \), the wall \( W_2 \) – by the plane \( Y = A_2 \), the plate – by the planes \( Y = B_1 \) and \( Y = B_2 \) \((A_1 < B_1 < B_2 < A_2; B_1, B_2 \) – constants). The fields between the walls and the plate – the domains \( \Omega_1: A_1 < Y < B_1 \) and \( \Omega_2: B_2 < Y < A_2 \) \((-\infty < X < \infty, -\infty < Z < \infty)\) are filled by the liquid. The distance between the walls \( L = A_2 - A_1 \) is constant. The boundary of the plate is permeable for the liquid; the plate moves along the axis \( X \) under the action of external oscillatory forces and the forces from the liquid. It is required to determine the independent on initial data, periodical by time \( t \) motion of the hydro-mechanical system.

Let it be that \( T \) is period of the walls \( W_1, W_2 \) oscillations; \( \tau = t / T; \ x = X / L; \ y = Y / L; \ z = Z / L; \ A_1 = \bar{A} \sin 2\pi \tau \) \((\bar{A} > 0 \) – constant); \( \varepsilon = \bar{A} / L; \ a_1 = A_1 / L; \ a_2 = A_2 / L; \ b_1 = B_1 / L; \ b_2 = B_2 / L; \ z = \) part of the plate \( \Xi \) which has the length \( D_x \), the thickness \( D_y \) and is confined between the planes \( Z = Z' \) and \( Z = Z' + D_z \) \((Z', D_x > 0, D_y > 0 \) are constants); \( m \) is the body \( \xi \) mass; \( \mathbf{e}_x = \{1, 0, 0\}; \mathbf{e}_y = \{0, 1, 0\}; \mathbf{U} = \{U, 0, 0\} \) is the velocity of the body \( \xi \) (the plate \( \Xi \)); \( u = TU / L; \rho, v, V = \{V_x, V_y, V_z\} \) are the liquid density, the liquid viscosity kinematic coefficient and the liquid velocity, correspondingly; \( \mathbf{v} = TV / L \) \((\mathbf{v} = v_x \mathbf{e}_x + v_y (y, \tau) \mathbf{e}_y)\); \( P \) is the pressure in liquid; \( p = T^2 p / (\rho L^2) \) \((p = p(y, \tau))\); \( \text{Re} = L^2 / (\nu T) \) is Reynolds number; \( F_{\text{ext}} = \tilde{F} \sin (2\pi \tau + \varphi) \) is external force acting on the body \( \xi \) in the direction of the axis \( X \) \((\tilde{F} > 0, \ \varphi \) are constants); \( f_{\text{ext}} = T^2 F_{\text{ext}} / (mL)\); \( F_{\text{liq}} = \rho v \left[ (\partial V_x / \partial Y)_{Y = B_2} - (\partial V_x / \partial Y)_{Y = B_1} \right] D_x D_z \) is the force from the liquid acting on the body \( \xi \) in the direction of the axis \( X \); \( f_{\text{liq}} = T^2 F_{\text{liq}} / (mL)\).

The equation of the body \( \xi \) (the plate \( \Xi \)) motion, the equations of Navier-Stokes and a continuity and the conditions at the boundaries of the bodies \( W_1, W_2, \Xi \) have the following form:

\[
\frac{du}{d\tau} = f_{\text{ext}} + f_{\text{liq}}; \tag{1}
\]

\[
\frac{\partial \mathbf{v}}{\partial \tau} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{v} \quad \text{in} \quad \Omega_1, \Omega_2; \tag{2}
\]

\[
\nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega_1, \Omega_2; \tag{3}
\]

\[
\mathbf{v} = \frac{da_1}{d\tau} \mathbf{e}_y \quad \text{under} \quad y = a_1, \ y = 1 + a_1; \tag{4}
\]

\[
\mathbf{v} = u \mathbf{e}_x + \frac{da_1}{d\tau} \mathbf{e}_y \quad \text{under} \quad y = b_1, y = b_2. \tag{5}
\]

3. Solution of problem and analysis

It results from (3) – (5)

\[
v_y = 2\pi \varepsilon \cos 2\pi \tau. \tag{6}
\]
In accordance with (2), (4), (5), (6)

\[ p = 4\pi^2 \varepsilon (\sin 2\pi \tau) y + p' \] (7)

\( (p' \text{ depends on } \tau); \)

\[ \frac{\partial v_x}{\partial \tau} + 2\pi \varepsilon (\cos 2\pi \tau) \frac{\partial v_x}{\partial y} = \frac{1}{\text{Re}} \frac{\partial^2 v_x}{\partial y^2} \text{ in } \Omega_1, \Omega_2; \] (8)

\[ v_x = 0 \text{ under } y = a_1, \ y = 1 + a_1; \] (9)

\[ v_x = u \text{ under } y = b_1, \ y = b_2. \] (10)

The problem (1), (8) – (10) is considered further for the values of \( \varepsilon \) which are much smaller than unit. It will be assumed that

\[ u \sim u_0 + \varepsilon u_1, \quad v_x \sim v_{x0} + \varepsilon v_{x1} \text{ under } \varepsilon \to 0. \] (11)

Let it be that \( N = 0, 1 \) are the numbers of approximation (the powers of \( \varepsilon \)).

It results from (1), (8) – (11)

\[ \frac{du_N}{d\tau} = (1 - N) f_{\text{ext}} + f_{\text{liq}} N; \] (12)

\[ \frac{\partial v_{xN}}{\partial \tau} + 2N\pi (\cos 2\pi \tau) \frac{\partial v_{x0}}{\partial y} = \frac{1}{\text{Re}} \frac{\partial^2 v_{xN}}{\partial y^2} \text{ in } \Omega_1, \Omega_2; \] (13)

\[ v_{xN} = -N (\sin 2\pi \tau) \frac{\partial v_{x0}}{\partial y} \text{ under } y = 0, \ y = 1; \] (14)

\[ v_{xN} = u_N \text{ under } y = b_1, \ y = b_2. \] (15)

Here \( \Omega_1 \) and \( \Omega_2 \) are the domains respectively \( 0 < y < b_1 \) and \( b_2 < y < 1 \) \((-\infty < X < \infty, -\infty < Z < \infty); \)

\( f_{\text{liq}} N = \frac{\varepsilon}{\text{Re}} \left( \frac{\partial v_{xN}}{\partial y} \big|_{y=b_2} - \frac{\partial v_{xN}}{\partial y} \big|_{y=b_1} \right) \left( \varepsilon = \rho LD_X D_z / m \right). \)

Let it be that \( N = 0, \)

The problem (12) – (15) has the solution

\[ u_0 = \text{Real} \left( \hat{u} e^{2\pi i \tau} \right) = -\frac{\hat{f}}{2\pi} \text{Real} \left\{ \frac{e^{i(2\pi \tau + \varphi)}}{1 + \frac{Z}{X} \left[ \cosh \lambda b_1 + \cosh \lambda (1 - b_2) \right]} \right\}; \] (16)

\[ v_{x0} = \text{Real} \left( \hat{v} \frac{\sinh \lambda y}{\sinh \lambda b_1} e^{2\pi i \tau} \right) \text{ in } \Omega_1; \] (17)
\( v_{x0} = \text{Real} \left[ \bar{u} \frac{\text{sh}(1-y)}{\text{sh}(1-b_2)} e^{2\pi i r} \right] \) in \( \overline{\Omega}_2 \),

where \( \hat{f} = T^2 \bar{F}/(mL) \); \( \lambda = (1 + i)\sqrt{\pi \text{Re}} \).

Let it be that \( N = 1 \).

In accordance with (12) – (15)

\[
\frac{d\bar{v}}{dy} \bigg|_{y=b_2} - \frac{d\bar{v}}{dy} \bigg|_{y=b_1} = 0;
\]

(19)

\[
2\pi \left( \cos 2\pi r \frac{\partial v_{x0}}{\partial y} \right) = \frac{1}{\text{Re}} \frac{\partial^2 \bar{v}}{\partial y^2} \text{ in } \overline{\Omega}_1, \overline{\Omega}_2;
\]

(20)

\[
\bar{v} = - \left( \sin 2\pi r \frac{\partial v_{x0}}{\partial y} \right) \text{ under } y = 0, \quad y = 1;
\]

(21)

\[
\bar{v} = \bar{u} \quad \text{under } y = b_1, \quad y = b_2.
\]

(22)

Here \( \langle \ldots \rangle = \int_{t}^{t+1} \ldots \, dt \); \( \bar{u} = \langle u_1 \rangle \); \( \bar{v} = \langle v_{x1} \rangle \).

The problem (12) – (15) has the solution

\[
u_1 = \bar{u} + \text{Real} \left( \bar{u} e^{4\pi i r} \right),
\]

\[
v_{x1} = \bar{v} + \text{Real} \left( \bar{v}_1 e^{4\pi i r} \right) \text{ in } \overline{\Omega}_1, \quad v_{x1} = \bar{v} + \text{Real} \left( \bar{v}_2 e^{4\pi i r} \right) \text{ in } \overline{\Omega}_2,
\]

(23)

where \( \bar{u} \) – constant; \( \bar{v}_1, \bar{v}_2 \) – functions of \( y \).

It results from (16) – (22)

\[
\bar{u} = \frac{1}{2(1+b_1-b_2)} \text{Real} \{ \bar{u} \bar{u} \left[ b_1 \text{ch} \lambda(1-b_2) - (1-b_2) \text{ch} \lambda b_1 \right] \};
\]

(24)

\[
\bar{v} = \pi \text{Re} \text{Real} \left[ \bar{u} \frac{\text{ch} \lambda y - (\text{ch} \lambda b_1) b_1^{-1} y}{\lambda \text{sh} \lambda b_1} \right] + \bar{u} \frac{y}{b_1} \text{ in } \overline{\Omega}_1;
\]

(25)

\[
\bar{v} = - \pi \text{Re} \text{Real} \left[ \bar{u} \frac{\text{ch} \lambda (1-y) - (\text{ch} \lambda (1-b_2))(1-b_2)^{-1} (1-y)}{\lambda \text{sh} \lambda (1-b_2)} \right] + \bar{u} \frac{1-y}{1-b_2} \text{ in } \overline{\Omega}_2;
\]

(26)

Formulas

\[
u = u_0 + \varepsilon u_1, \quad v_x = v_{x0} + \varepsilon v_{x1}
\]

(27)
and (6), (7), (16) – (18), (23) – (26) determine the approximate solution of the problem (1) – (5). In particular it is obvious from this solution that the liquid (at a background of oscillations) performs an unidirectional, steady motion. It should be noted that the plate $\Xi$ under $b_\nu \equiv 1 - b_\mathcal{E}$ (at a background of oscillations) is motionless.

It begs to consider the question on the time-averaged liquid motion for the values of $b_\nu$, $1 - b_\mathcal{E}$ which are much smaller than unit. The use of (6), (16) – (18), (23) – (27) allows to obtain

$$
\langle u \rangle = \varepsilon \tilde{u} \sim \varepsilon (1 - \eta) Q,
$$

$$
\langle \nu \rangle = \varepsilon \tilde{\nu} e_x \sim \varepsilon Q \left( \frac{y}{b_1} - \eta \right) e_x \quad \text{in} \quad \Omega_1,
$$

$$
\langle \nu \rangle = \varepsilon \tilde{\nu} e_x \sim \varepsilon Q \left( 1 - \eta \frac{1 - y}{1 - b_2} \right) e_x \quad \text{in} \quad \Omega_2
$$

under $b_1 \to 0$, $1 - b_2 \to 0$ and constant $\eta \equiv \frac{1 - b_2}{b_1}$, $\kappa$, $\text{Re}$.

Here

$$
Q = \frac{\text{Re} \hat{\nu}}{2\kappa (1 + \eta)} \left[ \cos \varphi + \frac{2\pi \text{Re} \eta}{\kappa (1 + \eta)} (\sin \varphi) b_1 \right] = \frac{\text{Re} \hat{\nu}}{2\kappa (1 + \eta)} \left[ \cos \varphi + \frac{2\pi \text{Re} \eta}{\kappa (1 + \eta)} (\sin \varphi) (1 - b_2) \right].
$$

In accordance with (28) (at a background of oscillations), the following liquid dynamics takes place. Under $\eta < 1$ ($\eta > 0$) in the domain $\Omega_1$ there are two layers defined by the formulas $0 < y < 1 - b_2$ (the layer $l_1$) and $1 - b_2 < y < b_1$ in which the liquid moves in mutually contrary directions, moreover the liquid motion direction in the layer $l_1$ is opposite to the plate $\Xi$ motion direction; in the domain $\Omega_2$ (in the layer $b_2 < y < 1$) the liquid moves in its sole direction which coincides with the plate $\Xi$ motion direction. It is to be noted that the thickness of the layer $l_1$ equals to the “thickness” of the domain $\Omega_2$. Under $\eta > 1$ in the domain $\Omega_2$ there are two layers defined by the formulas $b_2 < y < 1 - b_1$ and $1 - b_1 < y < 1$ (the layer $l_2$) in which the liquid moves in mutually contrary directions, moreover the liquid motion direction in the layer $l_2$ is opposite to the plate $\Xi$ motion direction; in the domain $\Omega_4$ (in the layer $0 < y < b_1$) the liquid moves in its sole direction which coincides with the plate $\Xi$ motion direction. It is to be noted that the thickness of the layer $l_2$ equals to the “thickness” of the domain $\Omega_4$. Under $\eta = 1$ the plate $\Xi$ is motionless, in each of domains $\Omega_4$, $\Omega_2$ the liquid moves in its sole direction and these directions are mutually contrary.

It has to be noted that the time-averaged motion of the liquid layers takes place under any (positive) value of $\eta$.

4. Conclusion

The realized investigation allows to do the inference on the existence of the effect which consists in that in the absence of a predominant direction in space a hydro-mechanical system which undergoes by oscillatory influences produces the unidirectional responses (the reactions for the influences) which expresses in that the free parts of the system – in particular the liquid layers – at a background of oscillations perform the unidirectional, steady motion.

The results of this work can be useful for example in connection with further investigations of non-trivial dynamics of hydro-mechanical systems.
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