Fermions and topology

Lee Smolin

Center for Gravitational Physics and Geometry
Department of Physics
The Pennsylvania State University
University Park, Pennsylvania, 16802-6360 U.S.A.

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Dedicated to John Archibald Wheeler

ABSTRACT

The canonical theory of quantum gravity in the loop representation can be extended to incorporate topology change, in the simple case that this refers to the creation or annihilation of "minimalist wormholes" in which two points of the spatial manifold are identified. Furthermore, if the states of the wormholes threaded by loop states are taken to be antisymmetrized under the permutation of wormhole mouths, as required by the relation between spin and statistics, then the quantum theory of pure general relativity, without matter but with minimalist wormholes, is shown to be equivalent to the quantum theory of general relativity coupled to a single Weyl fermion field, at both the kinematical and diffeomorphism invariant levels. The correspondence is also shown to extend to the action of the dynamics generated by the Hamiltonian constraint, on a large subspace of the physical state space, and is thus conjectured to be completely general.

* smolin@phys.psu.edu
1 Introduction

The idea that matter is made up out of the geometry of space is older than relativity theory, as it was discussed in the 19th Century by Clifford [1]. In the context of relativity theory a new possible arises, which is that matter can be constructed out of the topology of space. This idea has been championed by John Wheeler, who with Charles Misner many years ago proposed that wormholes at the Planck scale might behave as charged particles [2] and by Friedman and Sorkin, who showed that when the three manifold has non-trivial topology the quantum states of the gravitational field can have half integral angular momentum [3].

In this note I would like to show that the idea of Wheeler can be realized in a particularly straightforward way in the loop representation formulation of non-perturbative quantum gravity [4, 5, 6, 7, 8, 9]. What I will show is that a kind of minimal wormholes can be precisely identified as fermions in the sense that one can construct an isomorphism between the states and operators of the theory of general relativity coupled to fermions and a corresponding set of states of pure general relativity, with no matter, but with wormholes permitted. This correspondence can be established both at the kinematical and the dynamical levels.

The basic idea of the construction I will describe here is to make the description of the wormholes as simple as possible. The simplest possible wormhole can be constructed just by picking two points of the three manifold, Σ, which we may call x and y, and identifying them. Such a wormhole may be called a minimalist wormhole. Of course, we loose the Hausdorff property, but in such a mild way that it is still possible to define the structures that are needed to construct general relativity at both the classical and the quantum level. Indeed, spaces with such identifications are called conical spaces by the mathematicians, and have been well studied. Whether this construction is seen as fundamental, or as a simplification that can allow the details of the construction of the wormhole to be neglected and that might be eventually supplemented by details about the short distance properties of the wormhole, is left to the taste of the reader.

In the loop representation, states with fermions are represented as open loops, which stand for a holonomy element tied up with a fermion at each end [10, 11]. A closed loop that, however, goes through two identified point looks like an open loop; this is the basic idea of the identification of states with fermions with states of pure gravity on a manifold with identified points.

The first task facing such a construction is the incorporation of topology
change in a canonical quantum theory. In the case of minimalist wormholes this is particularly straightforward. For clarity, I will develop first the quantum theory of minimalist wormholes, without mentioning fermions. Then I will show that the operators that describe creation, annihilation and motion of wormholes have precisely the same algebra as those operators that are used to describe fermions in quantum gravity. Having thus established the correspondence at the kinematical level, I will then show that it can be extended to the level of diffeomorphism invariant invariant states. Following this, I show that for certain classes of states the correspondence extends to the level of the action of the Hamiltonian constraint that generates the dynamics. There may in fact be a complete equivalence between fermions and these minimalist wormholes, but to show this for general states remains a problem for future work.

2 Quantum kinematics of topology change

In order to develop the quantum theory of minimalist wormholes we need operators that take us between states defined on spaces with different topologies. This is the main task of this section\footnote{I will work here in 3 dimensions, but the construction works in other dimensions.}.

To begin with we need some notation and mathematical preliminaries. Beginning with a proper differentiable manifold $\Sigma$, we may consider the conical manifold defined by identifying two points $x$ and $y$ of $\Sigma$. I will call this $\Sigma_{(x,y)}$. It is defined by restricting the commutative algebra of $C^\infty$ of functions $f$ on $\Sigma$ to the subalgebra in which $f(x) = f(y)$. The vector fields on $\Sigma_{(x,y)}$ may be defined following the usual route from the derivations on the function algebra, from which the various other structures are defined.

Similarly, we may define the manifold $\Sigma_{(x_1,y_1)\ldots(x_n,y_n)}$ with $n$ pairs of points, $x_i, y_i$ identified by requiring that for all $i$, $f(x_i) = f(y_i)$.

In any quantum theory of interest we will have a space of states, $\mathcal{H}_{(x_1,y_1)\ldots(x_n,y_n)}$ defined on each $\Sigma_{(x_1,y_1)\ldots(x_n,y_n)}$. In some quantum field theories, such as those defined by Fock constructions, there may be subtleties concerning defining the states on the minimalist wormhole manifolds. But we will here be concerned with state spaces appropriate to the non-pertubative quantization of diffeomorphism invariant theories, principally the constructions based on using the discrete norm on the space of loops\cite{12, 13}, as formalized by Ashtekar and Isham\cite{13}. These state spaces have bases that, in the connection representation, may be represented as countable sums of products of
holonomies, or in the loop representation have support on countable sets of loops. We may note that the recent work of Ashtekar and Lewandowski\cite{14} and Baez\cite{15} establishes the existence of diffeomorphism invariant cylindrical measures on these spaces, elevating the heuristic constructions of previous work to the level of rigor of perturbative quantum field theory in Fock representations. For these kinds of representations there are, as we shall see, no difficulties associated with the non-Hausdorff behavior at minimalist wormholes.

Thus, for each $\Sigma_{(x_1,y_1),(x_n,y_n)}$ the state space $\mathcal{H}_{(x_1,y_1),(x_n,y_n)}$ will be spanned by the states $<\alpha;(x_1,y_1),(x_n,y_n)|$, which satisfy all the identities of states in the appropriate loop representation [4, 5, 6, 7, 8, 9]. In particular, the $\alpha$ must contain only closed loops, although the cases in which a loop is closed by virtue of its passage through a minimalist wormhole defined by two identified points are included.

The total Hilbert space of interest may be then written as the direct sum of all these spaces, thus

$$\mathcal{H}_{total} \equiv \bigoplus_{n=0}^{\infty} \bigoplus_{x_1}^{x_2n} \mathcal{H}_{(x_1,x_2),(x_2n-1,x_2n)}. \tag{1}$$

The continuous products here are defined using the discrete norm. We may note that in this formalism the order that the wormholes are created doesn’t matter, a point may be at the mouth of more than one wormhole and the case of a minimalist wormhole connecting a point $x$ to itself is allowed.

We now need operators that connect states in the different state spaces $\mathcal{H}_{(x_1,x_2),(x_2n-1,x_2n)}$. If $<\alpha;(x_1,x_2),(x_2n-1,x_2n)|$ denotes the bra associated with the (possibly multi-) loop $\alpha$ in $\mathcal{H}_{(x_1,x_2),(x_2n-1,x_2n)}$ we may define the operator $\hat{A}(y,z)$ by

$$<\alpha;(x_1,x_2),(x_2n-1,x_2n)|\hat{A}(y,z)=<\alpha;(x_1,x_2),(x_2n-1,x_2n)(y,z)| \tag{2}$$

We may note that there is no difficulty if the loop $\alpha$ include the points $y$ or $z$, the state on the right hand side then corresponds to loops that pass the identified point without making use of the option given by the identification of continuing "on the other side" of the wormhole. We may also note that as the classical canonical theory cannot describe topology

\footnote{From now on I will use notation appropriate to the loop representation, however all the definitions and identities may be taken as referring to the corresponding connection representation, with the connection states $\Psi_\alpha[A]$ put on the right, instead of their adjoint loop states $<\alpha;...|$ on the left.}
change these operators do not correspond to any classical observables. We
will, however, see below that they can be interpreted to describe the creation
of a pair of fermions.

The minimalist wormholes are unoriented, so that \( \hat{A}(y, z) = \hat{A}(z, y) \). It
makes no difference in which order they are created, so distinct wormhole
creation operators commute. However, it makes no sense to identify two
points twice, therefor we impose,

\[
\hat{A}(y, z)^2 = 0
\]

We may also define two other sets of useful operators. \( \hat{B}(z \rightarrow w) \) is the
operator that moves a mouth of a wormhole from \( z \) to \( w \). That is, it is a
map from \( H(x, z) \) to \( H(x, w) \) defined by

\[
< \alpha; (x_1, x_2) ... | \hat{B}(z \rightarrow w) \equiv \delta^3(z, x_1) < \alpha; (w, x_2) ... | + \delta^3(z, x_2) < \alpha; (w, x_1) ...
\]

This implies that,

\[
[\hat{B}(z \rightarrow w), \hat{A}(x, y)] = \delta^3(z, x) \hat{A}(w, y) + \delta^3(z, y) \hat{A}(w, x).
\]

We may also define a projection operator onto the space \( \mathcal{H}(x_1, x_2) ...(x_{2n-1} x_{2n}) \),
which will be denoted \( \hat{P}[(x_1, x_2) ... (x_{2n-1} x_{2n})] \). We may note that it satisfies

\[
\hat{A}(x_1, x_2) \hat{P}[(x_1, x_2) ... (x_{2n-1} x_{2n})] = 0
\]

and

\[
\hat{P}[(x_1, x_2) ... (x_{2n-1} x_{2n})] \hat{A}(x_1, x_2) = \hat{A}(x_1, x_2)
\]

Because all observables must be constructed from the field operators of
the quantum theory, a wormhole through which no loop passes will be unob-
servable. Thus, we will require that the states be restricted to superpositions
of those in which there are no such "naked wormholes". To accomplish this
we couple the action of the wormhole creation operators \( \hat{A}(x, y) \) with oper-
ators that create loops that go through the minimalist wormhole gotten by
identifying \( (x, y) \). Thus, for every open segment \( \beta \) in \( \Sigma \) we may define,

\[
\hat{T}[\beta] \equiv \hat{T}[\beta] \circ \hat{A}[\beta(0), \beta(1)]
\]

We will then restrict ourselves to states reached from states in the state
space \( \mathcal{H} \) defined on \( \Sigma \) without identifications by the action of these operators.
These spaces will be denoted by a prime as in \( \mathcal{H}'(x_1, x_2) ...(x_{2n-1} x_{2n}) \).
It is natural to have a dressed version of the operator $\hat{B}[z \to w]$ that moves a wormhole mouth from one point to another and adds an appropriate loop segment connecting the old point to the new one. This may be written as, $\hat{B}[\beta]$, where $\beta$ is an oriented loop segment in $\Sigma$, and defined by

$$\hat{B}[\beta] = \delta(\beta(1), \gamma(0))\hat{T}[\beta \circ \gamma] + \delta(\beta(1), \gamma(1))\hat{T}[\beta \circ \gamma^{-1}].$$  \hspace{1cm} (9)$$

We may note that it is difficult to write these directly in the loop representation, as there are no operators that remove loops (at least that correspond to known classical observables). But a classical observable corresponding to $\hat{B}[\beta]$ can be found in the theory in which wormhole mouths are identified as fermions, as we will see below.

To complete the definition of the loop or connection representation we need to include operators that are linear in momentum variables. This allows us to construct a closed algebra of elementary observables whose representations define the quantum theory. There are, first of all, the standard operators $\hat{T}[\alpha]$ (which may be regularized in terms of the strip operators). In addition, there are "$T^1$" operators analogous to $\hat{B}[\beta]$ associated with open curves $\beta$ in $\Sigma$ that may be defined by their actions on ordinary $T[\alpha]$’s by

$$\left[\hat{T}^a[\beta](s), \hat{T}[\alpha]\right] \equiv \int dt \delta^3(\beta(s), \alpha(t))\dot{\alpha}^a(t) \left(\hat{T}[\beta \circ \alpha] - \hat{T}[\beta \circ \alpha^{-1}]\right)$$ \hspace{1cm} (10)$$

and

$$\left[\hat{B}^a[\beta](s), \hat{T}[\alpha]\right] \equiv \int dt \delta^3(\beta(s), \alpha(t))\dot{\alpha}^a(t) \left(\hat{B}[\beta \circ \alpha] - \hat{B}[\beta \circ \alpha^{-1}]\right)$$ \hspace{1cm} (11)$$

The commutation relations not so far defined may be worked out from these definitions and naturally extend the usual $T$ algebra.

Finally, we may note that if we measure only observables constructed from local fields there will be no way to tell which wormhole mouth a particular mouth is connected to. This means that we should symmetrize or antisymmetrize over the possible identifications among the points. As dressed wormholes with one loop passing through them behave, in the neighborhood of each mouth, as spin $1/2$ excitations, it is likely that the spin statistics theorem (which has been generalized to the case of diffeomorphism invariant theories[16]) applies to this case and requires that antisymmetrization be chosen. Thus, I will choose here this option and restrict attention to the antisymmetrized states,

$$\langle \alpha; [x_1, ..., x_{2n}] \rangle \equiv \frac{1}{2n!} \sum_{\mathcal{P}[i_1, ..., i_{2n}]} (-1)^\mathcal{P} < \alpha; (x_{i_1}, x_{i_2})... |.$$ \hspace{1cm} (12)
where $\mathcal{P}$ are all the permutations of the indices among the first $2n$ integers. We may note that whenever a loop in $\alpha$ is an open segment in $\Sigma$ (that is then closed by passage through a mouth) the routing of the new loop $\alpha_P$ in each term follows from the identifications in that permutation of the points. We may note that when such states are dressed (so there are no naked wormholes), they are defined already from the open curves in $\alpha$, so they may be denoted simply by $\langle \alpha \rangle$ with the understanding that there are wormhole mouths associated with each open end in $\alpha$ and the identifications among them are summed over antisymmetrically. Finally, we may denote the spaces of these antisymmetrized dressed states as $\mathcal{H}'_{[\pi_1, ..., \pi_{2n}]}$.

We may note that once we impose these antisymmetrizations there may be at most two loops entering any wormhole mouth. This is most easily seen in the connection representation, where a wormhole with $n$ lines coming from it may be seen an object with $n$ independent spinor indices of the form, $U_{\alpha_1 A_1} ... U_{\alpha_n A_n}$, where the $\alpha_i$ are loops exiting from the other ends of the (possibly several wormholes that share a common mouth. However, by the antisymmetrization of wormholes, this must be summed antisymmetrically over the $n$ loop segments, $\alpha_i$, so that it vanishes when $n$ is greater than two.

Finally, it is particularly simple to write operators that remove a pair of wormhole acting on these antisymmetrized states. If we break the generic multiloop $\alpha$ into components $\beta_i$ that pass through wormholes, with the remaining, closed, loops denoted by $\gamma$, then we may define for each open segment, $\alpha$ an operator $\hat{X}[\alpha]$, by

$$<\beta_1, ..., \beta_n, \gamma | \hat{X}[\alpha] | \equiv \sum_i \sum_j <\beta_1, ..., \beta_i \circ \alpha \circ \beta_j, ..., \gamma | \delta^3(\alpha(0), \beta_i(1)) \delta^3(\alpha(1), \beta_j(0))$$

$$+ 3 \text{ other terms}$$

$$+ \sum_i < ... \beta_{i-1}, \beta_i \circ \alpha | \delta^3(\alpha(0), \beta_i(1)) \delta^3(\alpha(1), \beta_i(0))$$

$$+ 1 \text{ other term}$$

where $\beta_i \circ \alpha$ denotes the closed loop with $\alpha$ and $\beta$ joined on both ends, and in the other terms correspond to switches of orientation of the loops.

### 3 Connection with fermions at a kinematical level

In the last section we introduced the kinematics of pure quantum general relativity, without matter, but with topology change, in the form of addition, motion of, or removal of minimalist wormholes. In this section I will show
that this theory is identical to the kinematics of quantum general relativity coupled to one flavor of Weyl fermions. This will be done by establishing an isomorphism between the operator algebras of the two theories. This implies, and is equivalent to the demonstration of an isomorphism between the state spaces of the two theories.

The addition of fermions to general relativity or Yang-Mills fields in the loop representation has been described several places\cite{10, 11}. In this paper Weyl fermions will be denoted $\psi^A(x)$ and their canonical momenta (which are densities) by $\psi_A^\dagger(x)$. We assume the standard anticommutation relations,

$$\left[\psi^A(x) , \psi_B^\dagger(y)\right] = \delta_B^A \delta^3(y-x) \quad (14)$$

The gauge invariant states may be constructed in the connection representation by including fermions always in the combinations

$$W[\alpha] = \psi^A(\alpha(0)) U_{\alpha A}^B \psi_B(\alpha(1)) \quad (15)$$

where $\alpha$ is any curve and $U_\alpha$ is the standard path ordered holonomy operator in the spinor representation. The connection representation may then be extended to include states of the form

$$\Phi[A, \psi] = \Phi[T[\gamma], W[\alpha]] \quad (16)$$

where the dependence on the right hand side is assumed to be analytic. The corresponding states in the loop representation are functions of closed and open loops, and are spanned by the basis $< \alpha |$ where $< \alpha |$ is a multiloop containing both closed and open loops.

The correspondence between these states and those described in the previous section is obvious. Both are spanned by the basis $< \alpha |$, where closed and open lines are included and no open line is repeated. To show the isomorphism explicitly is simplest at the level of the operator algebra, the basic idea of the correspondence is that

$$\psi^A(x)\psi_B^B(y) \iff -\epsilon^{AB} \hat{A}(x,y) \quad (17)$$

The antiymmetrization among wormholes guarantees that the antiymmetrization among fermions is respected. For example, we have,

$$\psi^A(x)\psi^B(y)\psi^C(z)\psi^D(w) \iff \epsilon^{AB} \hat{A}(x,y)\epsilon^{CD} \hat{A}(z,w) - \epsilon^{AC} \hat{A}(x,z)\epsilon^{BD} \hat{A}(y,w) - \epsilon^{AD} \hat{A}(x,w)\epsilon^{BP} \hat{A}(z,y) \quad (18)$$
In terms of gauge invariant operators, we may write this correspondence as,

\[ \hat{W}[\alpha] \Longleftrightarrow \hat{T}[\alpha] \quad (19) \]

The operator \( \hat{B}[\alpha] \) that moves wormholes may then be put into correspondence with an operator containing one \( \psi^A \) and one \( \psi^A_\dagger \),

\[ Y(\alpha) \equiv \psi^A(\alpha(0))U_{\alpha A}B\psi^A_\dagger(\alpha(1)) \Longleftrightarrow \hat{B}[\alpha] \quad (20) \]

It is straightforward to show that these operators (on the spinor side), together with the standard closed loop \( T^0 \) and \( T^1 \) operators have a closed algebra\([10, 11]\). It is similarly a straightforward exercise to compute the algebra the \( \hat{T}[\alpha]'s \), \( \hat{B}[\alpha]'s \) generate together with the standard closed loop \( T's \) on the state space defined in the previous section and show that these two algebras are isomorphic. For brevity I will omit the details of these standard calculations here.

Once the isomorphism is established one may further show that the wormhole removal operator \( X[\alpha] \) corresponds to the action of the operator \( \psi^A(\alpha(0))U_{\alpha A}B\psi^A_\dagger(\alpha(1)). \)

### 4 Equivalence of fermions and wormholes at the diffeomorphism invariant level

Now that we have established the equivalence of the algebras of wormhole and fermion creation operators we must show that both sets of operators behave the same way under diffeomorphisms. As, for fermions, if \( \hat{U}[\phi] \) is the operator that generates the diffeomorphism, we have of course,

\[ \hat{U}[\phi]\psi^A(x)\psi^B(y) = \psi^A(\phi \circ x)\psi^B(\phi \circ y)\hat{U}[\phi] \quad (21) \]

But the same relation is satisfied by the operator that creates a wormhole, because by the definition of \( A(x, y) \)

\[ \hat{U}[\phi]A(x, y) = A(\phi \circ x, \phi \circ y)\hat{U}[\phi] \quad (22) \]

Because the algebra, \( C^\infty(\Sigma(x_1, y_1)\ldots(x_n, y_n)) \), of functions that acts on each of the ”almost manifolds” \( \Sigma_{(x_1, y_1)\ldots(x_n, y_n)} \) is a subalgebra of \( C^\infty(\Sigma) \), each element, \( \phi \), of the diffeomorphism group \( Diff(\Sigma) \) has a well defined action in which \( C^\infty(\Sigma(x_1, y_1)\ldots(x_n, y_n)) \) is mapped to \( C^\infty(\Sigma(\phi x_1, \phi y_1)\ldots(\phi x_n, \phi y_n)) \). Similarly, the operator \( \hat{U}[\phi] \) defines a one to one map from \( \mathcal{H}_{(x_1, y_1)\ldots(x_n, y_n)} \) to
\( \mathcal{H}_{(\phi \circ x_1, \phi \circ y_1)} \ldots (\phi \circ x_n, \phi \circ y_n) \) such that \( \hat{U}[\phi] \hat{P}[(x_1, x_2) \ldots (x_{2n-1}, x_{2n})] = \hat{P}[(\phi \circ x_1, \phi \circ y_1) \ldots (\phi \circ x_n, \phi \circ y_n)] \hat{U}[\phi] \). Thus, the diffeomorphism group of the original manifold \( \Sigma \) acts on the whole state space, \( \mathcal{H}_{total} \), of the theory. This allows us to define the diffeomorphism constraint on \( \mathcal{H}_{total} \) as \( \hat{D}(v) \equiv -\frac{d}{dt} \hat{U}[\phi_t] \), where \( \phi_t \) is the one parameter group of diffeomorphisms generated by the vector field \( v^a \).

We may then impose the diffeomorphism constraints for the original manifold on the quantum theory, and restrict ourselves to states \( \Psi \) that satisfy,

\[
\hat{U}[\phi] \circ \Psi = \Psi
\]

for all \( \phi \in \text{Diff}(\Sigma) \). Such states live in a linear subspace of \( \mathcal{H}_{total} \) we will call \( \mathcal{H}_{diffeo} \). It follows, from the action of \( \hat{U}[\phi] \) and the antisymmetrization condition, that each of these states is defined by its values on one of the subspaces \( \mathcal{H}'_{(x_1, y_1) \ldots (x_n, y_n)} \) for each \( n \). We may then write,

\[
\mathcal{H}'_{diffeo} = \sum_{n=0}^{\infty} \mathcal{H}'_{diffeo,n}
\]

where \( \mathcal{H}'_{diffeo,n} \) contains those diffeomorphism invariant states with \( n \) wormholes. This space then has a basis which is given by all diffeomorphism equivalence classes of closed curves on \( \Sigma \) that thread \( n \) minimalistic wormholes. These classes are equivalent to all diffeomorphism equivalence classes of closed and open loops on \( \Sigma \), where there are \( 2n \) distinct end points. But this is then exactly the same as the diffeomorphism equivalence classes of the loop states of the theory which includes fermions, so that the two theories are isomorphic also at the level of the diffeomorphism invariant states.

### 5 Dynamical equivalence of fermions and wormholes

When we add fermions to general relativity, we must add terms to the hamiltonian and diffeomorphism constraints of the form \[18\]

\[
C^\psi(x) = (D_a \psi)_A e^a_{\ AB} \psi_B^\dagger
\]

\[
D^\psi_a = (D_a \psi)^A \psi_A^\dagger
\]

We could use the correspondence between fermionic and wormhole operators to write the corresponding operators in the wormhole language. However, if
we are really going to implement the idea that the matter is not fundamental, but is only a consequence of space having non-trivial topology, what we would like is that the theory of pure gravity on the theory including wormholes be dynamically equivalent to the theory of gravity with matter added. There is indeed evidence that this may be exactly the case, as I will now show.

To calculate the action of the fermionic term of the hamiltonian constraint we may regulate it using the standard methods[5, 4, 11]. Following work of Morales and Rovelli[11], we may write the regulated version of (25) as

\[
\hat{C}_\delta^\psi(N) = \int d^3x N(x) \int d^3y \int d^3z f^\delta(x,y)f^\delta(x,z)(D_a\psi)^A(x)U_{\gamma_{x,y}A}E_B^C(y)U_{\gamma_{y,z}D}E_A^B\psi^\dagger(z)
\]

where \(f^\delta(x,y) = \frac{3\sqrt{\pi}}{2\delta^3}\Theta(\delta - |x-y|)\) is the standard smearing function[11] (\(\Theta\) is the step function) defined with respect to an arbitrary background flat metric \(h_{ab}\) and \(\gamma_{x,y}\) is an arbitrary curve, defined with reference to \(h_{ab}\) from \(x\) to \(y\).

It is not difficult to show [11] that the action of this on open loop states associated with pairs of fermions is, to leading order in \(1/\delta\) equivalent to the action of a diffeomorphism that acts only at the ends of the open loops to extend them outwards along their tangent vectors. That is, if \(\alpha\) is an open loop

\[
<\alpha|\hat{C}_\delta^\psi(N) = <\alpha|\int d^3x \hat{V}_h^a\hat{D}_a^\psi
\]

where \(\hat{V}\) is an operator, defined in the loop representation, and dependent on the background metric such that

\[
<\alpha|\hat{v}_h^a = \frac{1}{\delta^2} <\alpha|v^a
\]

where \(v^a\) is a vector field, defined with respect to the background metric \(h_{ab}\), such that

\[
v(\alpha(1))^a = \frac{3Nh(\alpha(1))}{4\pi} \frac{\dot{\alpha}(1)}{\dot{\alpha}(1)}
\]

with the same equation (with a negative sign) holding at the other end point.

We may note that as \(\hat{V}_h^a\) is an operator, this does not imply that arbitrary diffeomorphism invariant states are in the kernel of \(\hat{C}_\delta^\psi(N)\). But what is remarkable is that a similar equation holds in the case of the pure gravity theory with wormholes. This is because the action of the hamiltonian
constraint on any loop that goes through the wormhole is exactly like the action on a loop with a nondifferentiable point, and this action has been shown previously to be proportional, to leading order in the regulator, $\delta$, to the action of a diffeomorphism\cite{9, 20}. In particular, up to a regularization dependent numerical factor, we have, with $\beta$ a loop that goes through a wormhole identifying the points $x$ and $y$

$$< \beta | \hat{C}_{\delta}^{\text{pure gravity}}(N) =< \beta | \int d^3 x \hat{W}^a_h \hat{D}_a^{\text{pure gravity}}$$

where now $\hat{W}$ is a background dependent operator that acting at a nondifferentiable point produces the same vector field as in the fermion case

$$< \beta | \hat{W}^a_h = \frac{1}{\delta^2} < \beta | \nu^a_\beta. \quad (32)$$

Thus, we see that the pure gravity hamiltonian constraint on the states of closed loops that transverse minimalist wormholes has, at least to leading order in the regulators, precisely the same action as the added fermionic term in the hamiltonian constraints has on open lines that correspond, in the loop representation, to a pair of fermions connected by an holonomy element. However, in each case, it can be shown that the subleading terms can be eliminated so that the space of solutions is determined only by the leading terms in the action of the Hamiltonian constraint. This then establishes the dynamical equivalence of the two theories for simple states of the form we have been considering.

It remains to show the equivalence for all states, which means extending what has been done here to states in which fermions or wormholes sit at intersection points of other loops. Work in this direction is in progress.

6 Conclusions

What we have found in this paper is that if the theory of the world is generally relativity coupled to one Weyl fermion field, that world may be indistinguishable from a world whose degrees of freedom are only the geometry and topology of spacetime, at least if the possibilities for topology change are restricted to minimalist wormholes, as done here. The main thing that needs to be done is to extend this work by showing that the dynamical equivalence holds for arbitrary states involving fermions. The kinds of states that need to be considered are those in which the fermions sit at intersection points of other loops.
Beyond this, what we would like then to ask is whether this result can be extended to a more realistic theory, in such a way that we might really be able to believe that matter is a manifestation of the topology of space.

I will then close with several questions and comments about extending this result to the context of a meaningful theory.

1) Can the result be extended to incorporate Yang-Mills fields? We may consider the theory of general relativity coupled to Yang-Mills theory in the Ashtekar formulation\cite{18} with some compact gauge group $G$, taken again in the case that the theory admits creation and annihilation of minimalist wormholes, but contains no other matter. Let us quantize the theory in either the double connection representation, in which the states are functions both of the gravitational connection $A_{aAB}$ and the Yang-Mills connection $a_{aIJ}$, or in the corresponding loop representation. Taking again the same space of states, in which the connection representation states are again products of traces of holonomies, with the Yang-Mills holonomies taken in some representation $R$ of $G$. It follows directly from what has been shown here that the wormholes with one gravitational line and one gauge line will appear like a multiplet of Weyl fields, in the representation $R$ of the Yang-Mills gauge group. What has not been shown, however is that the couplings of these fermions to the Yang-Mills fields will be minimal, as there is a possibility of non-minimal couplings arising from additional terms coming from the regulated Yang-Mills terms in the Hamiltonian constraint.

2) One can also attempt to extend the present result to the non-trivial unifications of gravity and Yang-Mills theory that Peldan and Chakraborty have shown\cite{22} arise when the group in the Ashtekar form of the constraints is taken to be larger than $SU(2)$. In the cases in which the larger symmetry is broken to $SU(2) \times G$, in which Einstein-Yang-Mills theory emerges in the low energy limit, wormholes will again behave like Weyl fermions in some representation of $G$. Again, there may be non-minimal couplings at the Planck scale.

3) The same remarks hold in the simpler case in which Gambini and Pullin have shown that Einstein-Maxwell theory emerges from the Ashtekar formalism by extending the internal gauge group to $U(2)$\cite{21}.

4) It is interesting to note that, in any of these approaches, it will be impossible to generate any global symmetries of the fermions that are not at the shortest distances local symmetries.

5) The basic correspondence shown here will work also for quantum gravity in $2+1$ dimensions, in the case that the pure gravity Hamiltonian constraint is taken to be of the form $FEE$, as in $3+1$. On the other hand,
Bruegmann and Varadarajan have shown that there is a choice of representation space for 2 + 1 quantum gravity in this form such that the space of physical states is isomorphic to the state space of the Witten form (where the constraints are $F = 0$). This result implies that 2 + 1 gravity coupled to one Weyl fermion field may be an exactly solvable system whose state space is exactly the subspace of the state space of 2 + 1 gravity taken over all topologies, in which the condition of antisymmetrization over wormhole mouths has been imposed.

6) The correspondence shown here holds in the case of states of the form of

$$\Psi[A, \psi] = e^{\frac{1}{i\pi} \int_X Y \text{Chern \ Simon}^{(A)} \Phi[T[\gamma, A], W[\alpha, A, \psi]]}$$

(33)

which may be interpreted, in a semiclassical expansion, as an expansion around the DeSitter vacuum with cosmological constant $\lambda$.

6) We may also ask whether the correspondence between fermions and topology can be extended to the more studied case in which the wormholes are smooth manifolds. It seems that this is likely to be the case, at least in the semiclassical limit, given that any kind of wormhole with one Wilson loop emerging must look at low energies like a Weyl fermion. In this case there are likely to be non-minimal couplings at the scale of the wormhole size, so that the exact correspondence found here is unlikely to be found for more smooth wormholes.

Finally, one may ask whether the result found here is, in some sense, trivial, given that minimalist wormhole mouths must behave kinematically like Weyl spinors. It would certainly not be surprising to find that when one takes into account the relation between spin and statistics wormhole mouths behave in the low energy limit in which one expands around a state corresponding to a classical geometry dynamically as Weyl spinors. However, what is not required by this consideration is that the correspondence be exact at the operator level of the full non-perturbative theory, as there is no reason coming from the semiclassical analysis or the spin statistics theorem that non-minimal Planck scale couplings might not appear when wormholes were interpreted in terms of spinor fields. The fact that this is the case for general relativity, at least for a large class of states gives us perhaps reason to take seriously both the interpretation of fermions as wormholes and the conjecture that the fundamental nonperturbative dynamics at the Planck scale is the connection dynamics of general relativity.
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