Multiple-source single-sink maximum flow in directed planar graphs in $O(n^{1.5} \log n)$ time

Philip N. Klein and Shay Mozes
Brown University
September 15, 2010

Abstract
We give an $O(n^{1.5} \log n)$ algorithm that, given a directed planar graph with arc capacities, a set of source nodes and a single sink node, finds a maximum flow from the sources to the sink. This is the first subquadratic-time strongly polynomial algorithm for the problem.

1 Introduction
The study of maximum flow in planar graphs has a long history. In 1956, Ford and Fulkerson introduced the max st-flow problem, gave a generic augmenting-path algorithm, and also gave a particular augmenting-path algorithm for the case of a planar graph where $s$ and $t$ were on the same face (that face is traditionally designated to be the infinite face). Researchers have since published many algorithmic results proving running-time bounds on max st-flow for (a) planar graphs where $s$ and $t$ are on the same face, (b) undirected planar graphs where $s$ and $t$ are arbitrary, and (c) directed planar graphs where $s$ and $t$ are arbitrary. The best bounds known are (a) $O(n)$ [11], (b) $O(n \log n)$ [6], and (c) $O(n \log n)$ [1].

Schrijver [19] has written about the history of this problem. Ford and Fulkerson, who worked at RAND, were apparently motivated by classified work of Harris and Ross on interdiction of the Soviet railroad system. (Of course, Harris and Ross were interested in the min cut, not the max flow, as seems to be true for most applications.) This article was downgraded to unclassified in 1999. It contains a diagram of a network that models the Soviet railroad system indicating “ORIGINS” (sources) and what is apparently a sink (marked “EG”).

In max-flow applied to general graphs, multiple sources presents no problem: one can reduce the problem to the single-source case by introducing an artificial source and connecting it to all the sources. However, as Miller and Naor [17] pointed out, this reduction violates planarity unless all the sources are on the same face to begin with. Miller and Naor raise the question of computing a maximum flow in a planar graph with multiple sources and multiple sinks.
Figure 1: The soviet rail network
Even when there is one sink, until now the best known algorithm for computing multiple-source max-flow in a planar graph is to use the reduction in conjunction with a max-flow algorithm for general graphs. That is, no planarity-exploiting algorithm was known for the problem.

There are workarounds. For example, in the Soviet rail network, there are two faces that together include all the sources, so solving the instance can be reduced to solving two single-source max flows in a planar graph. However, a more realistic motivation comes from selecting multiple nonoverlapping regions in a planar structure.

Consider, for example, the following image-segmentation problem. A grid is given in which each vertex represents a pixel, and edges connect orthogonally adjacent pixels. Each edge is assigned a cost such that the edge between two similar pixels has higher cost than that between two very different pixels. In addition, each pixel is assigned a weight. High weight reflects a high likelihood that the pixel belongs to the foreground; a low-weight pixel is more likely to belong to the background.

The goal is to find a partition of the pixels into foreground and background to minimize the sum

\[ \text{weight of background pixels} + \text{cost of edges between foreground pixels and background pixels} \]

subject to the constraints that, for each component \( K \) of foreground pixels, the boundary of \( K \) forms a closed curve in the planar dual that surrounds all of \( K \) (essentially that the component is simply connected).

This problem can be reduced to multiple-source, single-sink max-flow in a planar graph (in fact, essentially the grid). For each pixel vertex \( v \), a new vertex \( v' \), designated a source, is introduced and connected only to \( v \). Then the sink is connected to the pixels at the outer boundary of the grid.

**New result** We prove the following:

**Theorem 1.1** There is an \( O(n^{1.5} \log n) \) algorithm to compute multiple-source, single-sink max flow in an \( n \)-node directed planar graph.

Before our work, the best strongly polynomial bound for the problem is \( O(n^2 \log n) \), which comes from the reduction to general graphs and then use of an algorithm such as that of Goldberg and Tarjan [8]. For integer capacities less than \( U \), one could instead use the algorithm of Goldberg and Rao [7], which leads to a running time of \( O(n^{1.5} \log n \log U) \), or the planarity-exploiting min-cost flow algorithm of [12], which gives a bound of \( O(n^{1.595} \log U) \) that depends on fast matrix multiplication and interior-point methods. However, even if one assumes integer capacities and \( U = \Theta(n) \), our planarity-exploiting algorithm is asymptotically faster.

We have learned (personal communication) that Borradaile and Wulff-Nilsen have independently proved the same theorem.
1.1 Organization

The structure of the paper is as follows. In Section 2, we give some definitions and general technical background. In Section 3, we give the main algorithm. Finally, in Section 4, we describe how to efficiently convert a feasible preflow to a feasible flow.

2 Preliminaries

2.1 Embedded Planar Graphs

A planar embedding of a graph assigns each node to a distinct point onto the sphere, and assigns each edge to a simple arc between the points corresponding to its endpoints, with the property that no arc-arc or arc-point intersections occur except for those corresponding to edge-node incidence in the graph. A graph is planar if it has a planar embedding.

Assume the graph is connected, and consider the set of points on the sphere that are not assigned to any node or edge; each connected component of this set is a face of the embedding.

It is convenient to designate one face as the infinite face (by analogy to embeddings on the plane). With respect to a choice of the infinite face, we say a Jordan curve strictly encloses an edge or node if the Jordan curve separates the edge or node from the infinite face. Similarly, for a subgraph, the choice of infinite face \( f_\infty \) for the whole graph induces a choice of infinite face for each connected component of the subgraph, namely that face of the connected component that contains \( f_\infty \).

In implementations, an embedding onto the sphere can be represented combinatorially, using a rotation system.

2.2 Flow

Let \( G \) be a directed graph with arc set \( A \), node set \( V \) and sink \( t \). For notational simplicity, we assume here and henceforth that \( G \) has no parallel edges and no self-loops.

We associate with each arc \( a \) two darts \( d \) and \( d' \), one in the direction of \( a \) and the other in the opposite direction. We say that those two darts are reverses of each other, and write \( d = \text{rev}(d') \).

A flow assignment \( f(\cdot) \) is a real-valued function on darts that satisfies antisymmetry:

\[
f(\text{rev}(d)) = -f(d)
\]

A capacity assignment \( c(\cdot) \) is a function from darts to real numbers. A flow assignment \( f(\cdot) \) respects capacities if, for every dart \( d \), \( f(d) \leq c(d) \). Note that, by antisymmetry, \( f(d) \leq c(d) \) implies \( f(\text{rev}(d)) \geq -c(\text{rev}(d)) \). Thus a negative capacity on a dart acts as a lower bound on the flow on the reverse dart. In this paper, we assume all capacities are nonnegative, and therefore the all-zeroes flow respects the capacities.
For a given flow assignment \( f(\cdot) \), the net inflow (or just inflow) node \( v \) is
\[
inflow_f(v) = \sum_{a \in A : \text{head}(a) = v} f(a) - \sum_{a \in A : \text{tail}(a) = v} f(a).
\]
The outflow of \( v \) is
\[
\text{outflow}_f(v) = -\inflow_f(v).
\]
The value of \( f(\cdot) \) is the inflow at the sink, \( \inflow_f(t) \).

A flow assignment \( f(\cdot) \) is said to obey conservation if for every node \( v \) other than \( t \), \( \text{outflow}_f(v) \geq 0 \).

A supply assignment \( \sigma(\cdot) \) is a function from the non-sink nodes to \( \mathbb{R} \cup \{\infty\} \). For any node \( v \), \( \sigma(v) \) specifies the amount of flow that can originate at \( v \). A flow assignment \( f(\cdot) \) is said to respect the supplies \( \sigma(\cdot) \) if, for every node \( v \) other than the sink \( t \), \( \text{outflow}_f(v) \leq \sigma(v) \). In this paper, we assume all supply values are nonnegative.

A flow assignment is a feasible preflow if it respects both capacities and supplies. A feasible preflow is called a feasible flow if in addition it obeys conservation. In this paper, we give an algorithm to find a maximum (feasible) preflow, and then an algorithm to convert that preflow to a maximum (feasible) flow.

The residual graph of \( G \) with respect to a flow assignment \( f(\cdot) \) is the graph \( G_f \) with the same arc-set, node-set and sink, and with capacity assignment \( c_f(\cdot) \) and supply assignment \( \sigma_f(\cdot) \) defined as follows:

- For every dart \( d \), \( c_f(d) = c(d) - f(d) \).
- For every node \( v \), \( \sigma_f(v) = \sigma(v) - \text{outflow}_f(v) \).

**Single-source limited max flow**

For a particular node \( s \), a limited max \( st \)-flow is a flow assignment \( f(\cdot) \) of maximum value that obeys capacities and for which \( \inflow_f(v) = 0 \) for every node except \( s \) and \( t \) and such that \( \text{outflow}_f(s) \leq \sigma(s) \). An algorithm for ordinary max \( st \)-flow can be used to compute limited max \( st \)-flow by introducing an artificial node \( s' \) and an arc \( s' \) of capacity \( \sigma(s) \), and running the algorithm on the transformed graph. This transformation preserves planarity. Since there is an \( O(n \log n) \) algorithm for max \( st \)-flow in a planar directed graph [1], we assume a subroutine for limited max \( st \)-flow.

### 2.3 Jordan Separators for Embedded Planar Graphs

For an \( n \)-node planar embedded simple graph \( G \), we define a Jordan separator to be a Jordan curve \( S \) such that, for any arc \( a \) of \( G \), the set of points in the sphere corresponding to \( a \) either (i) does not intersect \( S \) or (ii) coincides with a subcurve of \( S \). We require in addition that, if the two endpoints of \( a \) are consecutive nodes on \( S \), then (ii) must hold. The boundary nodes of \( S \) are the nodes \( S \) goes through.

We say a Jordan separator is balanced if at most \( 2n/3 \) nodes are strictly enclosed by the curve and at most \( 2n/3 \) nodes are not enclosed.

Miller [16] gave a linear-time algorithm that, given a triangulated two-connected \( n \)-node planar embedded graph, finds a simple cycle in the graph,
consisting of at most $2\sqrt{2}\sqrt{n}$ nodes, such that at most $2n/3$ nodes are strictly enclosed by the cycle, and at most $2n/3$ nodes are not enclosed.

To find a balanced Jordan separator in a graph that is not necessarily triangulated or two-connected, add artificial edges to triangulate the graph and make it two-connected. Now apply Miller’s algorithm to find a simple cycle separator with the desired property. Viewed as a curve in the sphere, the resulting separator $S$ satisfies the requirements of a balanced Jordan separator, and it has at most $2\sqrt{2}\sqrt{n}$ boundary nodes.

3 The Algorithm

The main algorithm finds a maximum preflow in the following, slightly more general, setting.

- **Input:**
  - A directed planar embedded graph $G$,
  - a sink node $t$,
  - a nonnegative capacity assignment $c(\cdot)$, and
  - a nonnegative supply assignment $\sigma(\cdot)$.

- **Output:** A feasible preflow $G$ of maximum value.

We present the main algorithm as a recursive procedure with calls to a single-source limited-max-flow subroutine. We omit discussion of the base case of the recursion (the case where the graph size is smaller than a certain constant.) Each of the recursive calls operates on a subgraph of the original input graph. We assume one global flow assignment $f(\cdot)$ for the original input graph, one global capacity assignment $c(\cdot)$, and one global supply assignment $\sigma(\cdot)$. Whenever the single-source limited-max-flow subroutine is called, it takes as part of its input

- the current residual capacity function $c_f(\cdot)$ and
- the current residual supply function $\sigma_f(\cdot)$.

It computes a limited max flow assignment $\hat{f}(\cdot)$, and then updates the global flow assignment $f(\cdot)$ by $f(d) := f(d) + \hat{f}(d)$ for every dart in the subgraph.

In the pseudocode, we do not explicitly mention $f(\cdot)$, $c(\cdot)$, $\sigma(\cdot)$, $c_f(\cdot)$, or $\sigma_f(\cdot)$.

The pseudocode for the algorithm is given below. We assume that the sink is on the boundary of the face designated the infinite face.

The algorithm proceeds in iterations as long as the current graph, $G_i$, consists of more than $N_0$ nodes, for some constant $N_0$ to be specified later. For graphs of constant size, output the solution in constant time. At iteration $i$ it finds a small Jordan separator $S_i$ in $G_{i-1}$ as described in Section 2.3. Let $H_i$ be the subgraph of $G_{i-1}$ enclosed by $S_i$. Intuitively, one would like to think of $S_i$ as the external face of $H_i$. However, $S_i$ might cross some earlier $S_j$, so $S_i$
Algorithm 1 MultipleSourceMaxPreFlow(graph $G_0$, sink $t$)

1: triangulate $G_0$ with zero-capacity edges.
2: $i := 0$
3: while $G_i$ consists of more than $N_0$ nodes do
4: $i := i + 1$
5: find a Jordan separator $S_i$ in $G_{i-1}$
6: let $H_i$ be the subgraph of $G_{i-1}$ enclosed by $S_i$
7: let $C_i$ be the external face of $H_i$
8: let $B_i$ be the set of cycles $\{C_j : C_j$ is contained in $H_i\}$
9: for $C$ in $B_i$ do
10: designate one of the nodes of $C$ as an artificial sink $t'$ and add artificial infinite-capacity edges parallel to $C$
11: MultipleSourceMaxPreFlow($H_i$, $t'$)
12: remove the infinite-capacity artificial edges
13: let $G_i$ be the subgraph of $G_{i-1}$ that is not strictly enclosed by $S_i$
14: for $C$ in $\{C_j\}$ do
15: for every node $v$ of $C$ do
16: limited max-flow from $v$ to $t$ in $G$

3.1 Correctness of Algorithm 1

Definition 3.1 (Admissible path) A $u$-to-$v$ path $P$ is called admissible if $\sigma(u) > 0$ and if $P$ is residual.

Lemma 3.2 Fix an iteration $i$ of the while loop. At any time in that iteration after Line 11 is executed for some cycle $C'$, there are no admissible to-$C'$ paths in $H_i$.

Proof: By induction on the number of iterations of the loop in Line 9. For the base case, immediately after Line 11 is executed for cycle $C'$, by maximality of the preflow pushed when the edges of $C'$ had infinite capacity, the lemma holds. Assume the lemma holds before Line 11 is executed for cycle $C''$ and let $f$ be the flow pushed in that execution. Assume for contradiction that after the execution there exists an admissible $u$-to-$C''$ path $P$ in $H_i$ for some node $u \in H_i$. 

7
Figure 2: A possible situation in the proof of Lemma 3.2. $P$ is shown in solid blue, $Q$ in dashed red.

If $P$ was residual before $f$ is pushed, then $\sigma(u)$ must have been zero at that time. Since $P$ is admissible after the push, $\sigma(u) > 0$ after $f$ is pushed. Therefore, before the execution, there must have been an admissible $x$-to-$u$ path $R$ in $H_i$ for some $x \in H_i$. Thus, $R \circ P$ is an admissible $x$-to-$C'$ path in $H_i$ before the execution, a contradiction.

If $P$ was not residual before $f$ was pushed, there must be some dart of $P$ whose reverse is used by $f$. Let $d$ be the latest such dart in $P$. The fact that $\text{rev}(d)$ is assigned positive flow by $f$ implies that before $f$ is pushed there exists an admissible path $Q$ from some node $x \in H_i$ to $\text{head}(d)$, see Fig. 2. By choice of $d$ this implies that $Q \circ P[\text{head}(d), v]$ is an admissible $x$-to-$C'$ path in $H_i$ before Line 11 is executed for cycle $C''$, a contradiction. QED

**Lemma 3.3** Just before the loop in Line 14 is executed, for every $i$, there are no admissible to-$B_i$ paths in $H_i$.

**Proof:** Lemma 3.2 implies that, for every $i$, at the end of iteration $i$ of the while loop, there are no admissible to $B_i$ paths in $H_i$.

Since for $j > i$ $H_i \cap H_j \subseteq C_i$ and since $C_i \in B_i$, there are no admissible to-$B_i$ paths in $H_i$ at any later iteration as well. QED

**Lemma 3.4** Just before the loop in Line 14 is executed for the first time, there are no $v$-to-$t$ admissible paths for any node $v \in G_0 - \bigcup_j C_j$.

**Proof:** Let $v$ be a node of $G_0$ that does not belong to any $C_j$. Let $i$ be the unique index such that $v \in H_i$. Observe that any $v$-to-$t$ admissible flow path in $G_0$ must visit some node of $B_i$ before getting to $t$, so it consists of a $v$-to-$C'$ admissible path in $H_i$ for some $C' \in B_i$, contradicting Lemma 3.3 QED

**Lemma 3.5** For any node $u$, if there are no $u$-to-$t$ admissible paths before an execution of Line 14, then there are none after the execution as well.
Proof: If \( \sigma(u) = 0 \) before the execution then \( \sigma(u) = 0 \) after the execution as well, so there are no admissible \( u\to t \) paths.

Otherwise, there is no \( u\to T \) residual path before the execution. Let \( f \) be the \( v\to t \) flow pushed at Line 16. Assume for contradiction that after the execution there exists a \( u\to t \) residual path \( P \). Since \( P \) was not residual before \( f \) was pushed there must be some dart of \( P \) whose reverse is used by \( f \). Let \( d \) be the earliest such dart in \( P \).

The fact that \( \text{rev}(d) \) is assigned positive flow by \( f \) implies that before \( f \) is pushed there exists a residual path \( Q \) from tail(\( d \)) to \( t \). By choice of \( d \) this implies that \( P[\text{tail}(d)] \circ Q \) is a residual (and therefore admissible) \( u\to t \) path before the execution, a contradiction. QED

We can now prove the correctness of the algorithm claimed in Theorem 1.1.

Proof: (Of correctness of algorithm in Theorem 1.1) By Lemma 3.4, immediately before Line 14, there are no \( v\to t \) admissible paths in \( G_0 \) for any node \( v \in G_0 - \bigcup_j C_j \). By Lemma 3.5 there are no such paths after the loop in Line 14 terminates. Since the executions of Line 16 eliminate all \( v\to t \) admissible paths for \( v \in \bigcup_j C_j \), there are no admissible paths to \( t \) in \( G_0 \) upon termination, so the flow computed is a maximum preflow. QED

3.2 Running Time of Algorithm 1

Lemma 3.6 Every cycle \( C_j \) appears at most twice as the cycle stored by the variable \( C \) in the loop in line 9.

Proof: Consider \( C_j \). It appears as the cycle stored by the variable \( C \) in the following two cases:

1. when \( i = j \) (i.e., when \( C_j \) is the external face of \( H_i \)).
2. when \( C_j \) is contained by some \( H_i \) but is not the external face of \( H_i \).

Note that case (2) implies that \( C_j \) has some dart that is strictly enclosed by \( S_i \), so this can happen for exactly one value of \( i \) since the subgraph strictly enclosed by \( S_i \) is not part of \( G_{i+1} \). Thus, \( C_j \) is not contained by any \( H'_i \) with \( i' > i \). QED

We first consider the cost of the recursive calls in Line 11. In both cases in Lemma 3.6, the recursive call is on the graph \( H_i \), so if \( T(n) \) denotes the running time of Algorithm 1 on an input graph with \( n \) nodes, the cycle \( C_j \) contributes at most \( T(|H_j|) + T(|H_{p(j)}|) \), where \( p(j) \) is the unique value such that \( C_j \) is contained by \( H_{p(j)} \) but is not the external face of \( H_{p(j)} \). Therefore, the total time required by all recursive calls is

\[
\sum_j T(|H_j|) + T(|H_{p(j)}|).
\]
Observe that, since for every \( i \), \( \frac{1}{3} |G_{i-1}| \leq |H_i| - 2\sqrt{2} |G_{i-1}| \leq \frac{2}{3} |G_{i-1}| \), we have \( |H_i| < |H_j| \) for \( i > j \). Also note that, if \( C_j \) is not the external face of \( H_i \) and \( H_i \) contains \( C_j \), then \( i > j \). Therefore, \( \sum_j T(|H_j|) + T(|H_{p(j)}|) \) is bounded by \( \sum_j T(|H_j|) + T(|H_{j+1}|) = T(|H_i|) + 2 \sum_{j \geq 2} T(|H_j|) \).

**Lemma 3.7** \( \sum_{j=k_1}^{k_2} |H_j|^{1.5} \leq 0.7 |G_{k_1-1}|^{1.5} \).

**Proof:** By induction on \( k_2 - k_1 \). Recall that, for every \( j \), \( |H_j| \leq \frac{2}{3} |G_{j-1}| + 2\sqrt{2} |G_{j-1}| \). By inspection, for \( k_2 = k_1 \), \( (\frac{2}{3} |G_{k_1-1}| + 2\sqrt{2} |G_{k_1-1}|)^{1.5} \leq 0.7 |G_{k_1-1}| \) provided \( |G_{k_1-1}| > N_0 = 10^5 \).

Assume the claim holds for \( k_2 - k_1 = k - 1 \).

\[
\sum_{j=k_1}^{k_1+k} |H_j|^{1.5} \leq |H_{k_1}|^{1.5} + \sum_{j=k_1+1}^{k_1+k-1} |H_j|^{1.5}
\leq |H_{k_1}|^{1.5} + 0.7 |G_{k_1}|^{1.5}
\leq \left( \theta |G_{k_1-1}| + 2\sqrt{2} |G_{k_1-1}| \right)^{1.5} + 0.7 \left( (1 - \theta) |G_{k_1-1}| + 2\sqrt{2} |G_{k_1-1}| \right)^{1.5},
\]

where \( \theta \) is the balance parameter of the separator \( S_{k_1} \). In the first inequality we have used the inductive assumption. Using the convexity of the above expression, it can be bounded by setting \( \theta = \frac{2}{3} \), which satisfies the lemma provided \( N_0 = 10^5 \) nodes. QED

**Lemma 3.8** Assume \( T(n) \leq \alpha_1 n^{1.5} \log n \) for every \( N_0 \leq n < |G_0| \). Then, \( T(|H_1|) + 2 \sum_{j \geq 2} T(|H_j|) < 0.98 \alpha_1 |G_0|^{1.5} \log |G_0| \)

**Proof:** Let \( \theta_1 \) denote the balance parameter for separator \( S_1 \).

\[
T(|H_1|) + 2 \sum_{j \geq 2} T(|H_j|) \leq \alpha_1 |H_1|^{1.5} \log |G_0| + 2\alpha_1 \log |G_0| \sum_{j \geq 2} |H_j|^{1.5}
\leq \alpha_1 |H_1|^{1.5} \log |G_0| + 2 \cdot 0.7 \alpha_1 \log |G_0| |G_1|^{1.5}
\leq \alpha_1 (\theta_1 |G_0| + 2\sqrt{2} |G_0|)^{1.5} \log |G_0| + 2 \cdot 0.7 \alpha_1 \log |G_0| \left( (1 - \theta_1) |G_0| + 2\sqrt{2} |G_0| \right)^{1.5},
\]

where in the second inequality we have used Lemma 3.7. Using the convexity of the above expression, it can be bounded by setting \( \theta = \frac{1}{3} \), which yields the desired bound \( 0.98 \alpha_1 |G_0|^{1.5} \log |G_0| \) provided \( |G_0| \geq N_0 \). QED

**Lemma 3.9** for every \( i \), every node of \( C_i \) belongs to some Jordan separator \( S_j \).
Proof: $C_i$, the external face of $H_i$ consists of nodes that either belong to $S_i$ or to a face of $G_{i-1}$ that is not triangulated. To see that, consider a clockwise traversal of $S_i$. For every two consecutive nodes of $S_i$ that are connected in $G_{i-1}$ by an edge, $S_i$ coincides with that edge (see Section 2.3), so it belongs to $C_i$. The only parts of $C_i$ and $S_i$ that do not coincide correspond to $S_i$ crossing some non-triangulated face $f$ of $G_{i-1}$, say at nodes $u$ and $v$. In those cases, $C_i$ consists of the clockwise subpath $f$ between $u$ and $v$. Since $G_0$ is triangulated in the first line of the algorithm and Line 13, every face of $G_{i-1}$ that is not triangulated corresponds to regions that were strictly enclosed by previous separators, or more formally, to a union of the portions of $G_0$ that are strictly enclosed by some Jordan separators in $\{S_j : j < i\}$. Therefore, every node on these faces belongs to some Jordan separator $S_j$, which proves the lemma. QED

We can now put together the pieces to prove the running time stated in Theorem 1.1.

Proof: (Of running time in Theorem 1.1.) We have already argued that the time required for all recursive calls is bounded by $\sum_j T(|H_j|) + T(|H_{p(j)}|)$. The work done outside the recursive calls is dominated by the single-source single-sink flow computations in Line 16. Each of these computations takes $O(|G_0| \log |G_0|)$ time. The overall time required for the non-recursive calls is $O(\sum_j |S_j| |G_0| \log |G_0|)$. By lemma 3.9 this is $O(\sum_j |S_j| |G_0| \log |G_0|)$. Since $|S_i|$ is $O(\sqrt{|G_{i-1}|})$ and since the size of the $G_j$s decreases exponentially, we have

$$O \left( \sum_j |S_j| |G_0| \log |G_0| \right) \leq \alpha_2 |G_0| \log |G_0| \sum_j \sqrt{|G_{j-1}|}$$

$$\leq \alpha_2' |G_0|^{1.5} \log |G_0|$$

for some constants $\alpha_2, \alpha_2'$. The overall running time is therefore bounded by

$$T(|G_0|) \leq \sum_j T(|H_j|) + T(|H_{p(j)}|) + \alpha_2' |G_0|^{1.5} \log |G_0|.$$ 

Assume inductively $T(n) \leq \alpha_1 |G_0|^{1.5} \log |G_0|$ for some constant $\alpha_1$. Then, by Lemma 3.8 $T(|G_0| \leq 0.98 \alpha_1 |G_0|^{1.5} \log |G_0| + \alpha_2' |G_0|^{1.5} \log |G_0|$, which at most $\alpha_1 |G_0|^{1.5} \log |G_0|$ for appropriate choice of $\alpha_1$. QED

4 Converting a maximum feasible preflow into a maximum flow

In this section we describe a linear time algorithm that, given a feasible preflow in a planar graph, converts it into a feasible flow of the same value. This
algorithm can be used to convert the maximum preflow output by Algorithm 1 into a maximum flow. This section contains no novel ideas and is included for completeness. A similar procedure was used in [14], but was not described in detail.

First, use the technique of Kaplan and Nussbaum [15] to make the preflow acyclic. The running time of this step is dominated by a shortest paths computation in the dual of the residual graph. This can be done in $O(n \log n)$ using Dijkstra, or in linear time using [11].

Let $f$ denote the acyclic feasible maximum preflow in $G$. Let $p(v)$ denote the net inflow of node $v$. Let $D$ denote the DAG induced by arcs with $f(d) > 0$. Reverse every arc of $D$ and compute a topological order on the nodes of $D$. The following algorithm pushes back flow from nodes with positive net inflow to the sources and runs in linear time. Upon termination, $f$ is a feasible maximum flow.

**Algorithm 2** An algorithm that converts acyclic preflow $p$ on a DAG $D$ into a flow.

1: for $v \in D$ in topological order do
2: if $v$ is not a sink then
3: while $p(v) > 0$ do
4: let $uv$ be a dart, where $u$ comes after $v$ in topological order and $f(d) > 0$
5: $x := \min\{f(d), p(v)\}$
6: $f(d) := f(d) - x$
7: $p(v) := p(v) - x$

Acknowledgments

We thank Glencora Borradaile for pointing out that computing a maximum preflow may be useful in solving the multiple source flow problem.

References

[1] Glencora Borradaile and Philip N. Klein. An (log) algorithm for maximum $t$-flow in a directed planar graph. *J. ACM*, 56(2), 2009.

[2] E. Dinic. Algorithm for solution of a problem of maximum flow in networks with power estimation. *Soviet Mathematics Doklady*, 11:1277–1280, 1970.

[3] J. Edmonds and R. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of the ACM*, 19(2):248–264, 1972.
[4] J. Fakcharoenphol and S. Rao. Planar graphs, negative weight edges, shortest paths, near linear time. In Proceedings of the 42th Annual Symposium on Foundations of Computer Science, pages 232–241, 2001.

[5] C. Ford and D. Fulkerson. Maximal flow through a network. Canadian Journal of Mathematics, 8:399–404, 1956.

[6] G. Frederickson. Fast algorithms for shortest paths in planar graphs with applications. SIAM Journal on Computing, 16:1004–1022, 1987.

[7] A. Goldberg and S. Rao. Beyond the flow decomposition barrier. Journal of the ACM, 45(5):783–797, 1998.

[8] A. Goldberg and R. Tarjan. A new approach to the maximum-flow problem. Journal of the ACM, 35(4):921–940, 1988.

[9] R. Hassin. Maximum flow in (s,t) planar networks. Information Processing Letters, 13:107, 1981.

[10] R. Hassin and D. B. Johnson. An O(n log^2 n) algorithm for maximum flow in undirected planar networks. SIAM Journal on Computing, 14:612–624, 1985.

[11] M. R. Henzinger, P. N. Klein, S. Rao, and S. Subramanian. Faster shortest-path algorithms for planar graphs. Journal of Computer and System Sciences, 55(1):3–23, 1997.

[12] Hiroshi Imai and Kazuo Iwano. Efficient sequential and parallel algorithms for planar minimum cost flow. In Algorithms, volume 450 of Lecture Notes in Computer Science, pages 21–30. Springer Berlin / Heidelberg, 1990.

[13] A. Itai and Y. Shiloach. Maximum flow in planar networks. SIAM Journal on Computing, 8:135–150, 1979.

[14] D. B. Johnson and S. Venkatesan. Using divide and conquer to find flows in directed planar networks in O(n^{3/2} log n) time. In Proceedings of the 20th Annual Allerton Conference on Communication, Control, and Computing, pages 898–905, 1982.

[15] Haim Kaplan and Y. Nussbaum. Maximum flow in directed planar graphs with vertex capacities. In ESA 2009, pages 397–407, 2009.

[16] G. L. Miller. Finding small simple cycle separators for 2-connected planar graphs. Journal of Computer and System Sciences, 32(3):265–279, 1986.

[17] G. L. Miller and J. Naor. Flow in planar graphs with multiple sources and sinks. SIAM Journal on Computing, 24(5):1002–1017, 1995.

[18] J. Reif. Minimum s-t cut of a planar undirected network in O(n log^2 n) time. SIAM Journal on Computing, 12:71–81, 1983.
[19] A. Schrijver. On the history of the transportation and maximum flow problems. *Mathematical Programming*, 91(3):437–445, 2002.

[20] K. Weihe. Maximum (s, t)-flows in planar networks in $O(|V|\log|V|)$ time. *Journal of Computer and System Sciences*, 55(3):454–476, 1997.