A variant of the Bombieri-Vinogradov theorem with explicit constants

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Abstract

In this paper we improve the result of [1] with getting \((\log x)^{7/2}\) instead of \((\log x)^{9/2}\). In particular we obtain a better version of Vaughan’s inequality by applying the explicit variant of an inequality connected to the Möbius function from [2].

1 Introduction

For integer number \(a\) and \(q \geq 1\), let

\[
\psi(x; q, a) = \sum_{n \leq x, n \equiv a \pmod{q}} \Lambda(n),
\]

where \(\Lambda(n)\) is the von Mangoldt function. The Bombieri-Vinogradov theorem is an estimate for the error terms in the prime number theorem for arithmetic progressions averaged over all \(q \leq x^{1/2}\).

**Theorem. (Bombieri-Vinogradov)** Let \(x\) be a given positive number and \(Q \leq x^{1/2}(\log x)^{-B}\) where \(B = B(A)\), then

\[
\sum_{2 \leq y \leq x} \max_{\substack{q \leq Q \atop (a, q) = 1}} \left| \psi(y, q, a) - \frac{y}{\varphi(q)} \right| \ll A \frac{x}{(\log x)^A}.
\]

The implied constant in this theorem is not effective, since we have to take care of characters, associated with those \(q\) that have small prime factors. The main result of this paper is

**Theorem 1. (Bombieri-Vinogradov theorem with explicit constants)** Let \(x \geq 4, 1 \leq Q_1 \leq Q \leq x^\frac{1}{4}\). Let also \(l(q)\) denote the least prime divisor of \(q\). Define \(F(x, Q, Q_1)\) by

\[
F(x, Q, Q_1) = \frac{14x}{Q_1} + 4x^\frac{1}{2}Q + 15x^\frac{3}{6}Q^\frac{1}{2} + 4x^\frac{5}{6} \log \frac{Q}{Q_1}.
\]
Then
\[ \sum_{q \leq Q, \nu(q) > Q_1} \max_{2 \leq y \leq x} \max_{(a, q) = 1} \left| \frac{\psi(y; q, a) - \psi(y)}{\phi(q)} \right| < c_1 F(x, Q, Q_1) (\log x)^{\frac{7}{2}}, \]

where
\[ c_1 = \frac{5}{4} E_0 c_0 + 1 = 42.140461 \ldots, \]
\[ E_0 = \prod_p \left( 1 + \frac{1}{p(p - 1)} \right) = 1.943596 \ldots, \]
\[ c_0 = (2A_0)^\frac{1}{2} \frac{2^5}{3^2 \pi (\log 2)^2} \left( 2 + \frac{\log(\log 2)}{\log \frac{4}{3}} \right) = 16.93375 \ldots, \]
\[ A_0 = \max_{x > 0} \left( \frac{\psi(x)}{x} \right) = \frac{\psi(113)}{113} = 1.03883 \ldots. \]

Previously the best result obtained by these methods in the literature is due to Akbary, Hambrook (see [1, Theorem 1.3]), where they proved that under assumptions of Theorem [1] we have
\[ \sum_{q \leq Q, \nu(q) > Q_1} \max_{2 \leq y \leq x} \max_{(a, q) = 1} \left| \frac{\psi(y; q, a) - \psi(y)}{\phi(q)} \right| < c_1 F(x, Q, Q_1) (\log x)^{\frac{7}{2}}, \]

where \( F(x, Q, Q_1) \) is defined by
\[ F(x, Q, Q_1) = \frac{4x}{Q_1} + 4x^\frac{1}{2} Q + 18x^\frac{2}{3} Q^\frac{1}{2} + 5x^\frac{5}{6} \log Q \frac{Q}{Q_1}. \]

Here we reduce this power to \( (\log x)^{\frac{7}{2}} \) by applying an explicit version for an upper bound for
\[ b_k = \sum_{d \in V} \mu(d), \]

where \( \mu(d) \) is Mobi\( \text{u} \) function, \( V \) is a given number. This version can be found in [2], namely we have

**Lemma 1. (Helfgott, [2, (6.9), (6.10)])** For \( V \) large enough we have
\[ \sum_{k \leq Y} |b_k|^2 = Y(L + O^*(C)) + O^*(V^2), \text{ where } C = 0.000023, L = 0.440729 \]

and \( O^*(x) \) means that it is less in absolute value than \( x \).

This Lemma is a variant of the sum considered in [3], where it is shown that
\[ \sum_{d_1, d_2 \leq V} \frac{\mu(d_1) \mu(d_2)}{\gcd(d_1, d_2)} \]
1 INTRODUCTION

tends to a positive constant as $Y \to \infty$. It is also suggested without proving that $L$ can be about 0.440729.

Notice, that by sharpening the inequality in Lemma 1 we will not be able to reduce the power of $\log x$, since the upper bound is optimal there, so by these methods the power $\frac{7}{2}$ is the best possible. Going further seems to be a hard problem which involves among simpler things a very careful analysis of the logarithmic mean of Möbius function twisted by a Dirichlet character.

**Remark.** Let $Q = \frac{x^{\frac{7}{2}}}{(\log x)^B}$, where $B > \frac{7}{2}$. Then Theorem 1 gives us the following bound

$$\sum_{\substack{q \leq Q \\ll q \gg Q_1}} \max_{2 \leq y \leq x \atop (a,q)=1} \left| \psi(y; q, a) - \psi(y) \frac{\phi(q)}{\phi(q)} \right| < c_1 \left( \frac{14x}{Q_1} (\log x)^{\frac{7}{2}} + 19x(\log x)^{\frac{7}{2}-B} \right).$$

**Remark.** It would be very good for applications to get $(\log x)^2$ in Theorem 1, however it seems impossible to get by present methods.

**Remark 1.** Define

$$\pi(x) = \sum_{p \leq x} 1 \quad \text{and} \quad \pi(x; q, a) = \sum_{p \leq x, p \equiv a \mod q} 1.$$

Then Theorem 1 under the same assumptions can be also formulated for $\pi(x)$, $\pi(x; q, a)$:

$$\sum_{\substack{q \leq Q \\ll q \gg Q_1}} \max_{2 \leq y \leq x \atop (a,q)=1} \left| \pi(y; q, a) - \frac{\pi(y)}{\phi(q)} \right| < c_2 F(x, Q, Q_1)(\log x)^{\frac{5}{2}},$$

where $c_2 = 1 + \frac{9}{2 \log 2}$.

Proof of the remark is exactly the same as in [1], we just have to change the power of log.

The key tool for the proof of Theorem 1 is Vaughan’s identity, which we have to get in an explicit version for our goal. Define

$$\psi(y, \chi) = \sum_{n \leq y} \Lambda(n) \chi(n),$$

the twisted summatory function for the von Mangoldt function $\Lambda$ and a Dirichlet character $\chi$ modulo $q$. One of two main results of this paper is

**Proposition 1. (Vaughan’s inequality in an explicit form)** For $x \geq 4$

$$\sum_{\substack{q \leq Q \\ll q \gg Q_1}} \frac{q}{\phi(q)} \sum_{\chi(q) \geq x} \max_{y \leq x} |\psi(y, \chi)| < c_0 (7x + 2Q^2 x^{\frac{1}{2}} + 5Q^3 x^{\frac{1}{3}} + 4Q x^{\frac{5}{3}})(\log x)^{\frac{5}{2}},$$

where $Q$ is any positive real number and $\sum_{\chi(q) \geq x}$ means a sum over all primitive characters $\chi \mod q$.

The goal is to get an explicit version of $f(x, Q)$ by applying an improved version of Pólya-Vinogradov inequality (see [4]), that will reduce the coefficients of $f(x, Q)$ and then we can apply Lemma 1.
2 Proof of Proposition 1

Fix arbitrary real numbers $Q > 0$ and $x \geq 4$. In this section, we shall establish Proposition 1, which is the main ingredient in the proof of Theorem 1. Here we follow the ideas of [1] and applying the results from [2]. The main tool in the proof is the large sieve inequality (see, for example [5, p.561])

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \left| \sum_{m=m_0+1}^{m_0+M} a_m \chi(m) \right|^2 \leq (M + Q^2) \sum_{m=m_0+1}^{m_0+M} |a_m|^2,$$

from which it follows (see [1, Lemma 6.1]) that

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_y \left| \sum_{m=m_0}^M \sum_{n=n_0}^N a_m b_n \chi(mn) \right| \leq c_3 (M' + Q^2)^{1/2} (N' + Q^2)^{1/2} \left( \sum_{m=m_0}^M |a_m|^2 \right)^{1/2} \left( \sum_{n=n_0}^N |b_n|^2 \right)^{1/2} L(M, N),$$

where $c_3 = 2.64\ldots$, $L(M, N) = \log(2MN)$ and $M' = M - m_0 + 1$, $N' = N - n_0 + 1$ are the number of terms in the sums over $m$ and $n$ respectively. Here the $a_m$, $b_n$ are arbitrary complex numbers.

2.1 Sieving and Vaughan’s identity

We reduce to the case $2 \leq Q \leq x^{1/2}$. If $Q < 1$, then the sum on the left-hand side of (1) is empty and we are done. Next, $1 \leq Q < 2$ then only the $q = 1$ term exists and we have

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{x \leq y} \left| \psi(y, \chi) \right| = \max_{y \leq x} \left| \sum_{n \leq y} \Lambda(n) \right| = \psi(x) \leq A_0 x,$$

which is better than the theorem. Finally, if $Q > x^{1/2}$, Theorem 1 follows from (2) with $M = m_0 = n_0 = 1$, $N = \lfloor x \rfloor$, $a_m = 1$, $b_n = \Lambda(n)$ by the estimate

$$\sum_{n \leq x} \Lambda(n)^2 \leq \psi(x) \log x \leq A_0 x \log x.$$

From now on we assume $2 \leq Q \leq x^{1/2}$. Notice that the fact that we can restrict ourselves to the range $2 \leq Q \leq x^{1/2}$ allows us to apply Lemma 1 otherwise it would make less sense, since the main term in Lemma 1 would be smaller than $O^*$-term). As in [1] we will use Vaughan’s identity (see also [3])

$$\Lambda(n) = \lambda_1(n) + \lambda_2(n) + \lambda_3(n) + \lambda_4(n),$$
2 PROOF OF PROPOSITION 1

where

\[
\begin{align*}
\lambda_1(n) &= \begin{cases} 
\Lambda(n), & \text{if } n \leq U, \\
0, & \text{if } n > U,
\end{cases} \\
\lambda_2(n) &= \sum_{h \leq n, d \leq V} \mu(d) \log h, \\
\lambda_3(n) &= -\sum_{mdr = n, m \leq U,d \leq V} \Lambda(m)\mu(d), \\
\lambda_4(n) &= -\sum_{mk = n, m > U,k > V} \Lambda(m)\sum_{d \mid k, d \leq V} \mu(d).
\end{align*}
\]

Assume \( y \leq x, \ q \leq Q, \) and \( \chi \) is a character mod \( q. \) We use the above decomposition to write

\[
\psi(y, \chi) = S_1 + S_2 + S_3 + S_4,
\]

where

\[
S_i = \sum_{n \leq y} \lambda_i(n)\chi(n).
\]

Let \( U, V \) be non-negative functions of \( x \) and \( Q \) to be set later and denote the contributions to our main sum by

\[
S_i = \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)} \max_{y \leq x} |S_i|.
\]

Easily we obtain

\[
\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)} \max_{y \leq x} |\psi(y, \chi)| \leq S_1 + S_2 + S_3 + S_4.
\]

The heart of the proof of Theorem 1.3 in \[1\] are the following estimates:

Lemma. (Akbary, Hambrook, \[1, \text{Section 7}\]) We have

\[
\begin{align*}
S_1 &\leq A_0 UQ^2, \quad S_2 < (x + Q^2V)(\log xV)^2, \quad S_3 < S_3' + S_3'', \\
S_3' &< (x + Q^2U)(\log xU)^2, \\
S_3'' &< \frac{c_3}{\log 2} \left( x + Qx^{1/4}U^{1/4}V^{1/2} + 2^{1/4}QxU^{-1/4} + Q^2x^{1/4} \right) (\log 2UV)^2(\log 4x), \\
S_4 &< \frac{2^{1/4}A_1^{1/3}}{\log 2} \left( x + QxV^{-1/4} + 2^{1/4}QxU^{-1/4} + Q^2x^{1/4} \right) \left( \frac{2x}{V} \right)^{3/4} (\log e^3V)(\log 4x).
\end{align*}
\]

where \( c_3 \) as in (77).

We estimate \( S_4 \) contribution with the use of Lemma \[1\] Writing \( S_4 \) as a dyadic sum we have

\[
S_4 = -\sum_{M = 2^r}^{2^{r+1}} \sum_{U \leq m \leq x/V} \sum_{V < k \leq x/M} \Lambda(m) \left( \sum_{d \mid k, d \leq V} \mu(d) \right) \chi(mk).
\]
Using the triangle inequality

\[ S_4 \leq \sum_{M=2^a}^{2^a} \sum_{1 \leq q \leq Q} \phi(q) \sum_{x \leq \alpha_1^{u<U<M} \leq x/V} \left( \sum_{V \leq k \leq x/M} \frac{\chi(q)}{M} \right) \sum_{M \leq m \leq 2M} \sum_{mk \leq y} a_m b_k \chi(mk) , \]

where \( a_m = \Lambda(m) \), and, as it was defined in the introduction \( b_k = \sum_{d|k, d \leq V} \mu(d) \).

Now apply the large sieve inequality (2) to get

\[ S_4 \leq c_3 \sum_{M=2^a}^{2^a} \left( M' + Q^2 \right) \left( K' + Q^2 \right) \sigma_1(M) \sigma_2(M) L(M) \]

where

\[ \sigma_1(M) = \sum_{V < k \leq x/M} |b_k|^2, \quad \sigma_2(M) = \sum_{V < m \leq x/M} |a_m|^2, \]

and

\[ L(M) = \log \left( \frac{2x}{M} \min \left( \frac{x}{V}, 2M \right) \right) \leq \log 4x, \]

where \( M' \) and \( K' \) denote the number of terms in the sums over \( m \) and \( k \), respectively. From the definition of \( M' \) and \( N' \) we conclude

\[ M' = \min \left( 2M, \frac{x}{V} \right) - \max (M + 1, U + 1) \leq M, \]
\[ K' = \frac{x}{M} - (V + 1) + 1 \leq \frac{x}{M}. \]

By Chebyshev estimate we have an upper bound

\[ \sigma_2(M) \leq \sum_{m \leq 2M} \Lambda(m)^2 \leq \psi(2M) \log 2M \leq 2A_0 M \log 2M. \]

Thus by Cauchy inequality

\[ S_4 \leq c_3 (\log 4x) \sum_{M=2^a}^{2^a} \left( M + Q^2 \right) \left( \frac{x}{M} + Q^2 \right) \left( 2A_0 M \log 2M \right) \sigma_1(M) \frac{1}{2}. \] (4)

Further

\[ M(M + Q^2) \left( \frac{x}{M} + Q^2 \right) = Mx + Q^2x + M^2Q^2 + MQ^4 \]

and

\[ (\log 2M)^{\frac{1}{2}} \leq \left( \log \frac{2x}{V} \right)^{\frac{1}{2}}. \]

Using Lemma [1] we get

\[ (\sigma_1(M))^{\frac{1}{2}} \leq \frac{x}{M} (L + C) - V(L + C) + 2V^2, \]
that implies
\[ S_4 \leq c_3 (2A_0)^{\frac{1}{2}} \left( x + 2^{\frac{1}{2}} Q^{\frac{1}{2}} x U^{-\frac{1}{2}} + Q x V^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}} \right) \left( \log \frac{2x}{V} \right)^{\frac{1}{4}} \sum_{\frac{1}{2} U < M \leq x/V} 1. \]

Since
\[ \sum_{\frac{1}{2} U < M \leq x/V} 1 \leq \frac{\log \frac{2x}{V}}{\log 2}, \]
then
\[ S_4 \leq \frac{c_3}{\log 2} (2A_0)^{\frac{1}{2}} \left( x + 2^{\frac{1}{2}} Q^{\frac{1}{2}} x U^{-\frac{1}{2}} + Q x V^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}} \right) \left( \log \frac{2x}{V} \right)^{\frac{1}{4}}. \]

Combining it with results of Lemma 2.1 we get
\[ S = \sum_{q \leq Q} q \sum_{y \leq x} \max |\psi(y, \chi)| \leq c_4 \text{Rat}(x, Q, U, V) \log(x, V, U), \tag{5} \]
where
\[ c_4 = \max \left\{ A_0, \frac{c_3}{\log 2}, \frac{c_3}{\log 2} (2A_0)^{\frac{1}{2}} \right\} = \frac{c_3}{\log 2} (2A_0)^{\frac{1}{2}}, \]

\[ \text{Rat}(x, Q, U, V) = 4x + 2Q^2 x^{\frac{1}{2}} + UQ^2 + Q^2 (U + V) + \]
\[ + 2^\frac{1}{2} Q^\frac{1}{2} x U^{-\frac{1}{2}} + 2^\frac{1}{2} Q x U^{-\frac{1}{2}} + Q x U^{\frac{1}{2}} V^{\frac{1}{2}} + Q x V^{-\frac{1}{2}}, \]

\[ \log(x, V, U) = \max \left\{ (\log xV)^2, (\log xU)^2, (\log 2UV)^2 \log x, \left( \log \frac{2x}{V} \right)^{\frac{1}{4}} \log 4x \right\}. \]

Now let’s specify \( U \) and \( V \). If \( x^{\frac{1}{4}} \leq Q \leq x^{\frac{1}{3}} \), then \( U = V = x^{\frac{1}{3}} Q^{-1} \). Then putting that into previous expression we get for the factor
\[ \text{Rat}_1(x, Q) = 4x + 2Q^2 x^{\frac{1}{2}} + Q x^{\frac{1}{2}} (1 + 2^\frac{1}{2}) + Q^2 x^{\frac{1}{2}} (2 + 2^\frac{1}{2} + 1) + x^{\frac{1}{2}} \leq \]
\[ \leq 4x + 2Q^2 x^{\frac{1}{2}} + 2Q x^{\frac{1}{2}} + Q^2 x^{\frac{1}{2}} (2 + 2^\frac{1}{2} + 1), \]

where we used the fact that \( x^{\frac{1}{2}} \leq Q x^{\frac{1}{3}} \) and \( Q x^{\frac{1}{3}} \leq Q x^{\frac{1}{2}} \). Working in the same manner with \( \log \) and keeping in mind the condition \( x \geq 4 \) we find that
\[ \log_1(x, V, U) \leq \left( \frac{4}{3} \log x \right)^{\frac{1}{2}} 2 \log x = \frac{2^4}{3^2} (\log x)^{\frac{5}{2}}. \]

If \( Q \leq x^{\frac{1}{4}} \), we let \( U = V = x^{\frac{1}{4}} \) and get
\[ \text{Rat}_2(x, Q) = 4x + 2Q^2 x^{\frac{1}{2}} + Q^2 x^{\frac{1}{2}} + 2Q x^{\frac{1}{2}} + 2^{\frac{1}{2}} Q^\frac{1}{2} x^{\frac{1}{2}} + Q x^{\frac{1}{2}} (2 + 2^\frac{1}{2}) \leq \]
\[ \leq x(5 + 2^\frac{1}{2}) + 2Q^2 x^{\frac{1}{2}} + 2Q^\frac{1}{2} x^{\frac{1}{2}} + Q x^{\frac{1}{2}} (2 + 2^\frac{1}{2}), \]
where we used $Q^2\sqrt{x} \leq x$, $Q\sqrt{x}\sqrt{\log x} \leq Q\sqrt{x}\sqrt{\log x}$ and $Q\sqrt{x}\sqrt{\log x} \leq x$.

Similarly we get for

$$\log_2(x, V, U) \leq 2 \left( \frac{7}{6} \right)^{\frac{3}{2}} \log x.$$ 

Finally, we have in (5)

$$S \leq c_4 \frac{2^4}{3^2} (7x + 2Q^2\sqrt{x} + 5Q\sqrt{x}\sqrt{\log x} + 4Q\sqrt{x}\sqrt{\log x})(\log x)^{\frac{3}{2}},$$

as demanded.

### 3 Proof of Theorem 1

Let $y \geq 2, (a, q) = 1$. By orthogonality of characters modulo $q$, we have

$$\psi(y; q, a) = \frac{1}{\phi(q)} \sum_{\chi} \chi(a) \psi(y, \chi).$$

Define $\psi'(y, \chi) = \psi(y, \chi)$ if $\chi \neq \chi_0$ and $\psi'(y, \chi) = \psi(y, \chi) - \psi(y)$ otherwise, $\chi_0$ is the principal character mod $q$. Then

$$\psi(y, q, a) - \frac{\psi(y)}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi} \chi(a) \psi'(y, \chi).$$

For a character $\chi \pmod{q}$, we let $\chi^*$ be the primitive character modulo $q^*$ inducing $\chi$. Follow the way of [1] we obtain

$$\psi'(y, \chi^*) - \psi'(y, \chi) = \psi(y, \chi^*) - \psi(y, \chi) = \sum_{p^k \leq y} (\log p) (\chi^*(p^k) - \chi(p^k)).$$

If $p|q$ then $(p^k, q^*) = 1$, and hence $\chi^*(p^k) = \chi(p^k)$. If $p|q$ then $\chi(p^k) = 0$. Therefore

$$|\psi'(y, \chi^*) - \psi'(y, \chi)| \leq \sum_{p^k \leq y \atop p|q} (\log p) \leq (\log y) \sum_{p|q} 1 \leq (\log qy)^2.$$

Denote the quantity we want to estimate as

$$\mathcal{M} = \sum_{\substack{q \leq x \\atop \nu(q) > \nu_1}} \max_{a \pmod{q}} \max_{2 \leq y \leq x} \frac{1}{\phi(q)} \left| \psi(y; q, a) - \frac{\psi(y)}{\phi(q)} \right|.$$

Since

$$\left| \psi(y, q, a) - \frac{\psi(y)}{\phi(q)} \right| \leq \frac{1}{\phi(q)} \sum_{\chi} |\psi'(y, \chi)| \leq (\log qy)^2 + \frac{1}{\phi(q)} \sum_{\chi} |\psi'(y, \chi^*)|,$$
then
\[ M \leq Q(\log Qx)^2 + \sum_{q \leq Q \mu(q) > Q_1} \frac{1}{\phi(q)} \sum_{\chi \max 2 \leq y \leq x} |\psi(y, \chi^*)|. \]

We have to take care just of the second term in the inequality above, since the first one is smaller than the desired bound. It remains to prove
\[ N = \sum_{q \leq Q \mu(q) > Q_1} \frac{1}{\phi(q)} \sum_{\chi(q^*) \max 2 \leq y \leq x} |\psi(y, \chi^*)| \leq (c_1 - 1)F(x, Q, Q_1)(\log x)^4, \]

where \( F(x, Q, Q_1) \) is the function from Theorem \( \Box \) A primitive character \( \chi^* \mod q^* \) induces characters of moduli \( dq^* \) and \( \psi(y, \chi^*) = 0 \) for \( \chi \) principal, we observe
\[ N = \sum_{q \leq Q \mu(q) > Q_1} \frac{1}{\phi(q)} \sum_{\chi(q^*) \max 2 \leq y \leq x} |\psi(y, \chi^*)| \sum_{k \leq Q} \frac{1}{\phi(kq^*)}. \]

As it was noted in \( \Box \) for \( x > 0 \)
\[ \sum_{k \leq x} \frac{1}{\phi(k)} \leq E_0 \log(ex) \]
and as \( q^* \leq Q \leq x^{1/2}, \phi(k)\phi(q^*) \leq \phi(kq^*) \) and \( x \geq 4 \), we have
\[ \sum_{k \leq Q} \frac{1}{\phi(kq^*)} < \frac{5E_0}{4\phi(q^*)} \log x. \]

For \( q > 1 \) and \( \chi \) primitive character \( \mod q \), we know that \( \chi \) is non-principal and \( \psi(y, \chi) = \psi(y, \chi^*) \). Since we assumed \( Q_1 \geq 1 \) then we can can replace \( \psi(y, \chi) \) by \( \psi(y, \chi) \) inside the internal sum for \( N \). Combining it with an expression for \( N \) we get
\[ N \leq \frac{5E_0}{4}(\log x) \sum_{q \leq Q \mu(q) > Q_1} \frac{1}{\phi(q)} \sum_{\chi(q) \max 2 \leq y \leq x} |\psi(y, \chi)| = R. \]

Thus it remains to show that
\[ R \leq \frac{4(c_1 - 1)}{5E_0}F(x, Q, Q_1)(\log x)^2. \]

Let
\[ R(q) = \frac{q}{\phi(q)} \sum_{\chi(q) \max 2 \leq y \leq x} |\psi(y, \chi)|. \]

Partial summation gives us
\[ \sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \sum_{\chi(q) \max 2 \leq y \leq x} |\psi(y, \chi)| = \frac{1}{Q} \sum_{q \leq Q} \sum_{q \leq Q_1} R(q) - \frac{1}{Q_1} \sum_{q \leq Q_1} R(q) + \int_{Q_1}^{Q} \left( \sum_{q \leq t} R(q) \right) \frac{dt}{t}. \]
Now we apply Theorem 1

\[ \sum_{q \leq Q} \mathcal{R}(q) < c_0 f(x, Q)(\log x)^{\frac{2}{3}}, \]

where \( f(x, Q) = 7x + 2Q^2 x^{\frac{1}{3}} + 5Q^2 x^{\frac{2}{3}} + 4Qx^{\frac{2}{5}}. \) Then

\[ \sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \sum_{x(q)} \max_{2 \leq \ell \leq x} |\psi(y, \chi)| < c_0 \left( \Delta f(Q, Q_1) + \int_{Q_1}^Q f(x, t) \frac{dt}{t} \right) (\log x)^{\frac{2}{3}}, \]

where

\[ \Delta f(Q, Q_1) = \frac{f(x, Q)}{Q} - \frac{f(x, Q_1)}{Q_1} \leq \frac{7x}{Q_1} + 2x^{\frac{1}{3}}Q + 5x^{\frac{2}{3}}Q^2. \]

Calculating the integrals gives us

\[ \int_{Q_1}^Q f(x, t) \frac{dt}{t} < \frac{7x}{Q_1} + 2x^{\frac{1}{3}}Q + 10x^{\frac{2}{3}}Q^\frac{1}{2} + 4x^{\frac{5}{6}} \log Q \frac{Q}{Q_1}. \]

Finally

\[ \mathcal{N} \leq \frac{4(c_1 - 1)}{5E_0} \left( 14x^{\frac{1}{2}}Q + 4x^{\frac{5}{2}}Q + 15x^{\frac{2}{3}}Q^\frac{1}{2} + 4x^{\frac{5}{6}} \log Q \frac{Q}{Q_1} \right) (\log x)^{\frac{2}{3}}. \]

### 3.1 Proof of Remark 1

Define two functions

\[ \pi_1(y) = \sum_{2 \leq n \leq y} \frac{\Lambda(n)}{\log n} \quad \text{and} \quad \pi_1(y; q, a) = \sum_{\substack{2 \leq n \leq y \\mod q}} \frac{\Lambda(n)}{\log n}. \]

Since

\[ \pi_1(y; q, a) - \pi(y; q, a) = \sum_{2 \leq k < \frac{x}{\log y}} \sum_{p^k \leq y \\mod q} \frac{1}{k} \leq \sum_{2 \leq k < \frac{x}{\log y}} \left( \frac{\pi(y^\frac{1}{2})}{2} < 2y^{\frac{1}{2}}, \right. \]

where we used the fact that for \( x > 1 \) (see for example [1, Lemma 3.1])

\[ \pi(x) < 1.25506 \frac{x}{\log x}. \]

Similarly, \( \pi_1(y) - \pi(y) < 2y^{\frac{1}{2}}. \) Thus by partial summation we obtain the bound

\[ \left| \frac{\pi_1(y; q, a) - \pi(y; q, a)}{\phi(q)} \right| = \left| \frac{\psi(y; q, a) - \psi(y)/\phi(q)}{\log y} \right| - \left( \int_{\log y}^y \frac{\psi(y; q, a) - \psi(t)/\phi(q)}{t \log^2 t} dt \right| \]

\[ \leq \frac{1}{\log 2} \left| \frac{\psi(y; q, a) - \psi(y)}{\phi(q)} \right| + \max_{2 \leq \ell \leq y} \left| \frac{\psi(t; q, a) - \psi(t)/\phi(q)}{\phi(q)} \right| \left( \frac{1}{\log 2} - \frac{1}{\log y} \right). \]
We have
\[
\sum_{q \leq Q} \max_{\nu(q) > Q_1} \max_{2 \leq y \leq x} \max_{a, (a,q) = 1} \left| \pi(y; q, a) - \frac{\pi(y)}{\phi(q)} \right|
\leq \frac{2}{\log 2} \sum_{q \leq Q} \max_{\nu(q) > Q_1} \max_{2 \leq y \leq x} \max_{a, (a,q) = 1} \left| \psi(y, q, a) - \frac{\psi(y)}{\phi(q)} \right| + 2x^{\frac{1}{2}} \sum_{\nu(q) > Q_1} \left( 1 + \frac{1}{\phi(q)} \right)
\leq \frac{2c_1}{\log 2} F(x, Q, Q_1)(\log x)^{\frac{3}{2}} + 2x^{\frac{1}{2}} \sum_{\nu(q) > Q_1} \left( 1 + \frac{1}{\phi(q)} \right),
\]
where we used Theorem \( \square \) to estimate the first summand. For \( x \geq 4 \)
\[
2x^{\frac{1}{2}} \sum_{\nu(q) > Q_1} \left( 1 + \frac{1}{\phi(q)} \right) < \frac{2c_1}{\log 2} F(x, Q, Q_1)(\log x)^{\frac{3}{2}}.
\]
and we are done.

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