DIAGONAL COMPRESSED GRAPH $W^*$-PROBABILITY

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Abstract. In this paper, we will consider the compressed graph $W^*$-probability theory. In [15] and [16], we observed Graph $W^*$-probability and the properties of certain amalgamated random variables in the graph $W^*$-probability space $(W^*(G), E)$ over the diagonal subalgebra $D_G$. By using the projections $L_v, v \in V(G)$, we will consider the vertex compressed free probability on $(W^*(G), E)$. Also, for the fixed vertices $v_1, ..., v_N \in V(G)$, we will consider the diagonal compressed free probability on $(W^*(G), E)$. We can show that the diagonal compressed freeness on $(W^*(G), E)$ is preserved by the $D_G$-valued freeness on $(W^*(G), E)$.

In [16], we constructed the graph $W^*$-probability spaces. The graph $W^*$-probability theory is one of the good example of Speicher’s combinatorial free probability theory with amalgamation. In [16], we observed how to compute the moment and cumulant of an arbitrary random variables in the graph $W^*$-probability space and the freeness on it with respect to the given conditional expectation. Also, in [17], we consider certain special random variables of the graph $W^*$-probability space, for example, semicircular elements, even elements and R-diagonal elements. This shows that the graph $W^*$-probability spaces contain the rich free probabilistic objects. Roughly speaking, graph $W^*$-algebras are $W^*$-topology closed version of free semigroupoid algebras defined and observed by Kribs and Power in [10].

Throughout this paper, let $G$ be a countable directed graph and let $F^+(G)$ be the free semigroupoid of $G$, in the sense of Kribs and Power, i.e., it is a collection of all vertices of the graph $G$ as units and all admissible finite paths, under the admissibility. As a set, the free semigroupoid $F^+(G)$ can be decomposed by

$$F^+(G) = V(G) \cup FP(G),$$

where $V(G)$ is the vertex set of the graph $G$ and $FP(G)$ is the set of all admissible finite paths. Trivially the edge set $E(G)$ of the graph $G$ is properly contained in $FP(G)$, since all edges of the graph can be regarded as finite paths with their length 1. We define a graph $W^*$-algebra of $G$ by

$$W^*(G) \overset{def}{=} C[\{L_w, L^*_w : w \in F^+(G)\}]^w,$$

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where $L_w$ and $L_w^*$ are creation operators and annihilation operators on the generalised Fock space $H_G = l^2(\mathbb{F}^+(G))$ induced by the given graph $G$, respectively. Notice that the creation operators induced by vertices are projections and the creation operators induced by finite paths are partial isometries. We can define the $W^*$-subalgebra $D_G$ of $W^*(G)$, which is called the diagonal subalgebra by

$$D_G \overset{def}{=} \mathbb{C}[\{L_v : v \in V(G)\}]^w.$$

Then each element $a$ in the graph $W^*$-algebra $W^*(G)$ is expressed by

$$a = \sum_{w \in \mathbb{F}^+(G) : a} p_w L_w^u, \text{ for } p_w \in \mathbb{C},$$

where $\mathbb{F}^+(G : a)$ is a support of the element $a$, as a subset of the free semigroupoid $\mathbb{F}^+(G)$. The above expression of the random variable $a$ is said to be the Fourier expansion of $a$. Since $\mathbb{F}^+(G)$ is decomposed by the disjoint subsets $V(G)$ and $FP(G)$, the support $\mathbb{F}^+(G : a)$ of $a$ is also decomposed by the following disjoint subsets,

$$V(G : a) = \mathbb{F}^+(G : a) \cap V(G)$$

and

$$FP(G : a) = \mathbb{F}^+(G : a) \cap FP(G).$$

Thus the operator $a$ can be re-expressed by

$$a = \sum_{v \in V(G : a)} p_v L_v + \sum_{w \in FP(G : a), u_w \in \{1, \ast\}} p_w L_w^u.$$

Notice that if $V(G : a) \neq \emptyset$, then $\sum_{v \in V(G : a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Thus we have the canonical conditional expectation $E : W^*(G) \rightarrow D_G$, defined by

$$E(a) = \sum_{v \in V(G : a)} p_v L_v,$$

for all $a = \sum_{w \in \mathbb{F}^+(G) : a, u_w \in \{1, \ast\}} p_w L_w^u$ in $W^*(G)$. Then the algebraic pair $(W^*(G), E)$ is a $W^*$-probability space with amalgamation over $D_G$ (See [16]). It is easy to check that the conditional expectation $E$ is faithful in the sense that if $E(a^*a) = 0_{D_G}$, for $a \in W^*(G)$, then $a = 0_{D_G}$.

For the fixed operator $a \in W^*(G)$, the support $\mathbb{F}^+(G : a)$ of the operator $a$ is again decomposed by

$$\mathbb{F}^+(G : a) = V(G : a) \cup FP_\ast(G : a) \cup FP_\ast^c(G : a),$$
with the decomposition of $FP(G : a)$,

$$FP(G : a) = FP_*(G : a) \cup FP'_*(G : a),$$

where

$$FP_*(G : a) = \{ w \in FP(G : a) : both \ L_w \ and \ L'_w \ are \ summands \ of \ a \}$$

and

$$FP'_*(G : a) = FP(G : a) \ \setminus \ FP_*(G : a).$$

The above new expression plays a key role to find the $D_G$-valued moments of the random variable $a$. In fact, the summands $p_v L_v$’s and $p_w L_w + p_{w'} L'_{w'}$, for $v \in V(G : a)$ and $w \in FP_*(G : a)$ act for the computation of $D_G$-valued moments of $a$. By using the above partition of the support of a random variable, we can compute the $D_G$-valued moments and $D_G$-valued cumulants of it via the lattice path model $LP_k$ and the lattice path model $L'_P$ satisfying the *-axis-property. At a first glance, the computations of $D_G$-valued moments and cumulants look so abstract and hence it looks useless. However, these computations, in particular the computation of $D_G$-valued cumulants, provides us how to figure out the $D_G$-freeness of random variables by making us compute the mixed cumulants. As applications, in the final chapter, we can compute the moment and cumulant of the operator that is the sum of $N$-free semicircular elements with their covariance $2$.

Based on the $D_G$-cumulant computation, we can characterize the $D_G$-freeness of generators of $W^*(G)$, by the so-called diagram-distinctness on the graph $G$, i.e., the random variables $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ if and only if $w_1$ and $w_2$ are diagram-distinct the sense that $w_1$ and $w_2$ have different diagrams on the graph $G$. Also, we could find the necessary condition for the $D_G$-freeness of two arbitrary random variables $a$ and $b$. i.e., if the supports $FP^+(G : a)$ and $FP^+(G : b)$ are diagram-distinct, in the sense that $w_1$ and $w_2$ are diagram distinct for all pairs $(w_1, w_2) \in FP^+(G : a) \times FP^+(G : b)$, then the random variables $a$ and $b$ are free over $D_G$.

In [17], we considered some special $D_G$-valued random variables in a graph $W^*$-probability space $(W^*(G), E)$. The those random variables are the basic objects to study Free Probability Theory. We can conclude that

(i) if $l$ is a loop, then $L_l + L'_l$ is $D_G$-semicircular.

(ii) if $w$ is a finite path, then $L_w + L'_w$ is $D_G$-even.

(iii) if $w$ is a finite path, then $L_w$ and $L'_w$ are $D_G$-valued R-diagonal.

In this paper, we will observe the diagonal compressed random variables in the graph $W^*$-probability space $(W^*(G), E)$. Let $v_1, ..., v_N \in V(G)$ and let $a$ be a $D_G$-valued random variable in $(W^*(G), E)$. Define the diagonal compressed random variable of $a$ by $V = \{ v_1, ..., v_N \}$ by
Notice that if $v \in V(G)$, then $L_v a L_v$ is the compressed random variable by $L_v$ and the compressed random variable has its support contained in $\{v\} \cup \text{loop}_v(G)$, where $\text{loop}_v(G) = \{l \in \text{loop}(G) : l = vlv\}$. We will consider the $D_G$-moments, $D_G$-cumulants and $D_G$-freeness of such compressed random variables.

1. Graph $W^*$-Probability Theory

Let $G$ be a countable directed graph and let $F^+(G)$ be the free semigroupoid of $G$, i.e., the set $F^+(G)$ is the collection of all vertices as units and all admissible finite paths of $G$. Let $w$ be a finite path with its source $s(w) = x$ and its range $r(w) = y$, where $x, y \in V(G)$. Then sometimes we will denote $w$ by $w = xwy$ to express the source and the range of $w$. We can define the graph Hilbert space $H_G$ by the Hilbert space $l^2(F^+(G))$ generated by the elements in the free semigroupoid $F^+(G)$, i.e., this Hilbert space has its Hilbert basis $B = \{\xi_w : w \in F^+(G)\}$. Suppose that $w = e_1...e_k \in FP(G)$ is a finite path with $e_1,...,e_k \in E(G)$. Then we can regard $\xi_w$ as $\xi_{e_1} \otimes ... \otimes \xi_{e_k}$. So, in [10], Kribs and Power called this graph Hilbert space the generalized Fock space. Throughout this paper, we will call $H_G$ the graph Hilbert space to emphasize that this Hilbert space is induced by the graph.

Define the creation operator $L_w$, for $w \in F^+(G)$, by the multiplication operator by $\xi_w$ on $H_G$. Then the creation operator $L$ on $H_G$ satisfies that

(i) $L_w = L_{xwy} = L_x L_w L_y$, for $w = xwy$ with $x, y \in V(G)$.

(ii) $L_{w_1} L_{w_2} = \begin{cases} L_{w_1 w_2} & \text{if } w_1 w_2 \in F^+(G) \\ 0 & \text{if } w_1 w_2 \notin F^+(G), \end{cases}$

for all $w_1, w_2 \in F^+(G)$.

Now, define the annihilation operator $L_w^*$, for $w \in F^+(G)$ by

$L_w^* \xi_w' \overset{\text{def}}{=} \begin{cases} \xi_h & \text{if } w' = wh \in F^+(G) \\ 0 & \text{otherwise.} \end{cases}$

The above definition is gotten by the following observation:

$$< L_w \xi_h, \xi_{wh} > = < \xi_{wh}, \xi_{wh} > = 1 = < \xi_h, \xi_h > = < \xi_h, L_w^* \xi_w >.$$
where \(<,>\) is the inner product on the graph Hilbert space \(H_G\). Of course, in the above formula we need the admissibility of \(w\) and \(h\) in \(\mathbb{F}^+(G)\). However, even though \(w\) and \(h\) are not admissible (i.e., \(wh \notin \mathbb{F}^+(G)\)), by the definition of \(L_w^*\), we have that

\[
< L_w \xi_h, \xi_h > = < 0, \xi_h > = 0 = < \xi_h, 0 > = < \xi_h, L_w^* \xi_h > .
\]

Notice that the creation operator \(L\) and the annihilation operator \(L^*\) satisfy that

\[
(1.1) \quad L_w^* L_w = L_y \quad \text{and} \quad L_w L_w^* = L_x , \quad \text{for all} \quad w = xwy \in \mathbb{F}^+(G),
\]

under the weak topology, where \(x, y \in V(G)\). Remark that if we consider the von Neumann algebra \(W^*\{L_w\}\) generated by \(L_w\) and \(L_w^*\) in \(B(H_G)\), then the projections \(L_y\) and \(L_x\) are Murray-von Neumann equivalent, because there exists a partial isometry \(L_w\) satisfying the relation (1.1). Indeed, if \(w = xwy\) in \(\mathbb{F}^+(G)\), with \(x, y \in V(G)\), then under the weak topology we have that

\[
(1.2) \quad L_w L_w^* L_w = L_w \quad \text{and} \quad L_w^* L_w L_w^* = L_w^* .
\]

So, the creation operator \(L_w\) is a partial isometry in \(W^*\{L_w\}\) in \(B(H_G)\). Assume now that \(v \in V(G)\). Then we can regard \(v\) as \(v = vvv\). So,

\[
(1.3) \quad L_v^* L_v = L_v = L_v L_v^* = L_v^* .
\]

This relation shows that \(L_v\) is a projection in \(B(H_G)\) for all \(v \in V(G)\).

Define the graph \(W^*\)-algebra \(W^*(G)\) by

\[
W^*(G) \overset{\text{def}}{=} C[\{L_w, L_w^* : w \in \mathbb{F}^+(G)\}]^w .
\]

Then all generators are either partial isometries or projections, by (1.2) and (1.3). So, this graph \(W^*\)-algebra contains a rich structure, as a von Neumann algebra. (This construction can be the generalization of that of group von Neumann algebra.) Naturally, we can define a von Neumann subalgebra \(D_G \subset W^*(G)\) generated by all projections \(L_v\), \(v \in V(G)\). i.e.

\[
D_G \overset{\text{def}}{=} W^*(\{L_v : v \in V(G)\}) .
\]

We call this subalgebra the **diagonal subalgebra** of \(W^*(G)\). Notice that \(D_G = \Delta_{|G|} \subset M_{|G|}(\mathbb{C})\), where \(\Delta_{|G|}\) is the subalgebra of \(M_{|G|}(\mathbb{C})\) generated by all diagonal matrices. Also, notice that \(1_{D_G} = \sum_{v \in V(G)} L_v = 1_{W^*(G)}\).
If $a \in W^*(G)$ is an operator, then it has the following decomposition which is called the Fourier expansion of $a$:

$$a = \sum_{w \in F^+(G:a), u_w \in \{1, \ast\}} p_w L_{u_w}^w,$$

where $p_w \in C$ and $F^+(G:a)$ is the support of $a$ defined by

$$F^+(G:a) = \{ w \in F^+(G) : p_w \neq 0 \}.$$

Remark that the free semigroupoid $F^+(G)$ has its partition $\{V(G), FP(G)\}$, as a set. i.e.,

$$F^+(G) = V(G) \cup FP(G) \quad \text{and} \quad V(G) \cap FP(G) = \emptyset.$$

So, the support of $a$ is also partitioned by

$$F^+(G : a) = V(G : a) \cup FP(G : a),$$

where

$$V(G : a) \overset{df}{=} V(G) \cap F^+(G : a)$$

and

$$FP(G : a) \overset{df}{=} FP(G) \cap F^+(G : a).$$

So, the above Fourier expansion (1.4) of the random variable $a$ can be re-expressed by

$$a = \sum_{v \in V(G:a)} p_v L_v + \sum_{w \in FP(G:a), u_w \in \{1, \ast\}} p_w L_{u_w}^w.$$

We can easily see that if $V(G : a) \neq \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Also, if $V(G : a) = \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v = 0_{D_G}$. So, we can define the following canonical conditional expectation $E : W^*(G) \to D_G$ by

$$E(a) = E \left( \sum_{w \in F^+(G:a), u_w \in \{1, \ast\}} p_w L_{u_w}^w \right) \overset{df}{=} \sum_{v \in V(G:a)} p_v L_v,$$

for all $a \in W^*(G)$. Indeed, $E$ is a well-determined conditional expectation.

**Definition 1.1.** Let $G$ be a countable directed graph and let $W^*(G)$ be the graph $W^*$-algebra induced by $G$. Let $E : W^*(G) \to D_G$ be the conditional expectation defined above. Then we say that the algebraic pair $(W^*(G), E)$ is the graph $W^*$-probability space over the diagonal subalgebra $D_G$. By the very definition, it is one of the $W^*$-probability space with amalgamation over $D_G$. All elements in $(W^*(G), E)$ are called $D_G$-valued random variables.
We have a graph $W^*$-probability space $(W^*(G), E)$ over its diagonal subalgebra $D_G$. We will define the following free probability data of $D_G$-valued random variables.

**Definition 1.2.** Let $W^*(G)$ be the graph $W^*$-algebra induced by $G$ and let $a \in W^*(G)$. Define the $n$-th ($D_G$-valued) moment of $a$ by

$$E(d_1a d_2a ... d_n a), \text{ for all } n \in \mathbb{N},$$

where $d_1, ..., d_n \in D_G$. Also, define the $n$-th ($D_G$-valued) cumulant of $a$ by

$$k_n(d_1a, d_2a, ..., d_n a) = C^{(n)}(d_1a \otimes d_2a \otimes ... \otimes d_n a),$$

for all $n \in \mathbb{N}$, and for $d_1, ..., d_n \in D_G$, where $\tilde{C} = (C^{(n)})_{n=1}^{\infty} \in I^c (W^*(G), D_G)$ is the cumulant multiplicative bimodule map induced by the conditional expectation $E$, in the sense of Speicher. We define the $n$-th trivial moment of $a$ and the $n$-th trivial cumulant of $a$ by

$$E(a^n) \text{ and } k_n \left( a, a, ..., a \right)_{n \text{-times}} = C^{(n)}(a \otimes a \otimes ... \otimes a),$$

respectively, for all $n \in \mathbb{N}$.

To compute the $D_G$-valued moments and cumulants of the $D_G$-valued random variable $a$, we need to introduce the following new definition:

**Definition 1.3.** Let $(W^*(G), E)$ be a graph $W^*$-probability space over $D_G$ and let $a \in (W^*(G), E)$ be a random variable. Define the subset $FP_*(G : a)$ in $FP(G : a)$ by

$$FP_*(G : a) \overset{\text{def}}{=} \{ w \in \mathbb{F}^+(G : a) : \text{both } L_w \text{ and } L^*_w \text{ are summands of } a \}.$$ 

And let $FP_+^c(G : a) \overset{\text{def}}{=} FP(G : a) \setminus FP_*(G : a)$.

We already observed that if $a \in (W^*(G), E)$ is a $D_G$-valued random variable, then $a$ has its Fourier expansion $a_d + a_0$, where

$$a_d = \sum_{v \in V(G; a)} p_v L_v$$

and

$$a_0 = \sum_{w \in FP(G; a), u_w \in \{1, *\}} p_w L^u_w.$$
By the previous definition, the set $FP(G : a)$ is partitioned by

$$FP(G : a) = FP_s(G : a) \cup FP^c_s(G : a),$$

for the fixed random variable $a$ in $(W^*(G), E)$. So, the summand $a_0$, in the Fourier expansion of $a = a_d + a_0$, has the following decomposition;

$$a_0 = a(\ast) + a(non-\ast),$$

where

$$a(\ast) = \sum_{l \in FP_s(G : a)} (p_l L_l + p^*_l L^*_l)$$

and

$$a(non-\ast) = \sum_{w \in FP^c_s(G : a), u_w \in \{1, \ast\}} p_w L^u_w,$$

where $p_l$ is the coefficient of $L^*_l$ depending on $l \in FP_s(G : a)$.

1.1. $D_G$-Moments and $D_G$-Cumulants of Random Variables.

Throughout this chapter, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. In this chapter, we will compute the $D_G$-valued moments and the $D_G$-valued cumulants of arbitrary random variable

$$a = \sum_{w \in \mathbb{F}^+(G : a), u_w \in \{1, \ast\}} p_w L^u_w$$

in the graph $W^*$-probability space $(W^*(G), E)$.

1.1.1. Lattice Path Model.

Throughout this section, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. Let $w_1, \ldots, w_n \in \mathbb{F}^+(G)$ and let $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable. In this section, we will define a lattice path model for the random variable $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$. Recall that if $w = e_1, \ldots, e_k \in FP(G)$ with $e_1, \ldots, e_k \in E(G)$, then we can define the length $|w|$ of $w$ by $k$. i.e., the length $|w|$ of $w$ is the cardinality $k$ of the admissible edges $e_1, \ldots, e_k$. 
Definition 1.4. Let $G$ be a countable directed graph and $\mathbb{F}^+(G)$, the free semigroupoid. If $w \in \mathbb{F}^+(G)$, then $L_w$ is the corresponding $D_G$-valued random variable in $(W^*(G), E)$. We define the lattice path $l_w$ of $L_w$ and the lattice path $l_w^{-1}$ of $L_w^*$ by the lattice paths satisfying that:

(i) the lattice path $l_w$ starts from $* = (0, 0)$ on the $\mathbb{R}^2$-plane.

(ii) if $w \in V(G)$, then $l_w$ has its end point $(0, 1)$.

(iii) if $w \in E(G)$, then $l_w$ has its end point $(1, 1)$.

(iv) if $w \in E(G)$, then $l_w^{-1}$ has its end point $(-1, -1)$.

(v) if $w \in FP(G)$ with $|w| = k$, then $l_w$ has its end point $(k, k)$.

(vi) if $w \in FP(G)$ with $|w| = k$, then $l_w^{-1}$ has its end point $(-k, -k)$.

Assume that finite paths $w_1, ..., w_s$ in $FP(G)$ satisfy that $w_1 ... w_s \in FP(G)$. Define the lattice path $l_{w_1} ... l_{w_s}$ by the connected lattice path of the lattice paths $l_{w_1}, ..., l_{w_s}$, i.e., $l_{w_2}$ starts from $(k_{w_1}, k_{w_1}) \in \mathbb{R}^+$ and ends at $(k_{w_1} + k_{w_2}, k_{w_1} + k_{w_2})$, where $|w_1| = k_{w_1}$ and $|w_2| = k_{w_2}$. Similarly, we can define the lattice path $l_{w_1}^{-1} ... l_{w_s}^{-1}$ as the connected path of $l_{w_s}^{-1}, l_{w_{s-1}}^{-1}, ..., l_{w_1}^{-1}$.

Definition 1.5. Let $G$ be a countable directed graph and assume that $L_{w_1}, ..., L_{w_n}$ are generators of $(W^*(G), E)$. Then we have the lattice paths $l_{w_1}, ..., l_{w_n}$ of $L_{w_1}, ..., L_{w_n}$, respectively in $\mathbb{R}^2$. Suppose that $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}} \neq 0_{D_G}$ in $(W^*(G), E)$, where $u_{w_1}, ..., u_{w_n} \in \{1, *, \}$.

Define the lattice path $l_{w_1}^{u_{w_1}} ... l_{w_n}^{u_{w_n}}$ of nonzero $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}}$ by the connected lattice path of $l_{w_1}^{u_{w_1}}, ..., l_{w_n}^{u_{w_n}}$, where $l_{w_j}^1 = 1$ if $u_{w_j} = 1$ and $l_{w_j}^* = -1$ if $u_{w_j} = *$. Assume that $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}} = 0_{D_G}$. Then the empty set $\emptyset$ in $\mathbb{R}^2$ is the lattice path of it. We call it the empty lattice path. By $LP_n$, we will denote the set of all lattice paths of the $D_G$-valued random variables having their forms of $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}}$, including empty lattice path.

Also, we will define the following important property on the set of all lattice paths:

Definition 1.6. Let $l_{w_1}^{u_{w_1}} ... l_{w_n}^{u_{w_n}} \neq \emptyset$ be a lattice path of $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}} \neq 0_{D_G}$ in $LP_n$. If the lattice path $l_{w_1}^{u_{w_1}} ... l_{w_n}^{u_{w_n}}$ starts from * and ends on the *-axis in $\mathbb{R}^+$, then we say that the lattice path $l_{w_1}^{u_{w_1}} ... l_{w_n}^{u_{w_n}}$ has the *-axis-property. By $LP^*_n$, we will denote the set of all lattice paths having their forms of $l_{w_1}^{u_{w_1}} ... l_{w_n}^{u_{w_n}}$ which have the *-axis-property. By little abuse of notation, sometimes, we will say that the $D_G$-valued random variable $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}}$ satisfies the *-axis-property if the lattice path $l_{w_1}^{u_{w_1}} ... l_{w_n}^{u_{w_n}}$ of it has the *-axis-property.

The following theorem shows that finding $E(L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}})$ is checking the *-axis-property of $L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}}$. 
Theorem 1.1. (See [15]) Let $L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}} \in (W^{*}(G), E)$ be a $D_{G}$-valued random variable, where $u_{w_{1}}, ..., u_{w_{n}} \in \{1, *\}$. Then $E (L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}}) \neq 0_{D_{G}}$ if and only if $L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}}$ has the $*$-axis-property (i.e., the corresponding lattice path $l_{w_{1}}^{u_{1}}...l_{w_{n}}^{u_{n}}$ of $L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}}$ is contained in $LP_{n}^{*}$. Notice that $\emptyset \notin LP_{n}^{*}$. □

By the previous theorem, we can conclude that $E (L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}}) = L_{v}$, for some $v \in V(G)$ if and only if the lattice path $l_{w_{1}}^{u_{1}}...l_{w_{n}}^{u_{n}}$ has the $*$-axis-property (i.e., $l_{w_{1}}^{u_{1}}...l_{w_{n}}^{u_{n}} \in LP_{n}^{*}$).

1.1.2. $D_{G}$-Valued Moments and Cumulants of Random Variables. Let $w_{1}, ..., w_{n} \in F^{+}(G)$, $u_{1}, ..., u_{n} \in \{1, *\}$ and let $L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}} \in (W^{*}(G), E)$ be a $D_{G}$-valued random variable. Recall that, in the previous section, we observed that the $D_{G}$-valued random variable $L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}} = L_{v} \in (W^{*}(G), E)$ with $v \in V(G)$ if and only if the lattice path $l_{w_{1}}^{u_{1}}...l_{w_{n}}^{u_{n}}$ of $L_{w_{1}}^{u_{1}}...L_{w_{n}}^{u_{n}}$ has the $*$-axis-property (equivalently, $l_{w_{1}}^{u_{1}}...l_{w_{n}}^{u_{n}} \in LP_{n}^{*}$). Throughout this section, fix a $D_{G}$-valued random variable $a \in (W^{*}(G), E)$. Then the $D_{G}$-valued random variable $a$ has the following Fourier expansion,

$$a = \sum_{v \in V(G:a)} p_{v}L_{v} + \sum_{l \in FP_{2}(G:a)} (p_{l}L_{l} + p_{u}L_{l}) + \sum_{w \in F^{*}(G:a), \ u_{w} \in \{1, *\}} p_{w}L_{w}^{u_{w}}.$$ 

Let’s observe the new $D_{G}$-valued random variable $d_{1}ad_{2}a...d_{n}a \in (W^{*}(G), E)$, where $d_{1}, ..., d_{n} \in D_{G}$ and $a \in W^{*}(G)$ is given. Put

$$d_{j} = \sum_{v_{j} \in V(G:d_{j})} q_{v_{j}}L_{v_{j}} \in D_{G}, \text{ for } j = 1, ..., n.$$ 

Notice that $V(G : d_{j}) = F^{+}(G : d_{j})$, since $d_{j} \in D_{G} \hookrightarrow W^{*}(G)$. Then

$$d_{1}ad_{2}a...d_{n}a = \sum_{(v_{1}, ..., v_{n}) \in \Pi_{i=1}^{n} V(G : d_{j})} (q_{v_{1}}...q_{v_{n}}) \left( L_{v_{1}} \left( \sum_{w_{1} \in F^{*}(G:a), \ u_{w_{1}} \in \{1, *\}} p_{w_{1}}L_{w_{1}}^{u_{w_{1}}(w_{1})} \right) \right.$$ 

$$\left. \cdots \left( L_{v_{n}} \left( \sum_{w_{n} \in F^{*}(G:a), \ u_{w_{n}} \in \{1, *\}} p_{w_{n}}L_{w_{n}}^{u_{w_{n}}(w_{n})} \right) \right) \right).$$
(1.7)
\[
= \sum_{(v_1, \ldots, v_n) \in \Pi^n_{j=1} V(G;d_j)} (q_{v_1} \cdots q_{v_n}) \\
\sum_{(w_1, \ldots, w_n) \in \mathbb{F}^+(G:a)^n, \ u_{w_j} \in \{1, *\}} (p_{w_1} \cdots p_{w_n}) L_{v_1} L_{u_{w_1}} \cdots L_{v_n} L_{u_{w_n}}.
\]

Now, consider the random variable \( L_{v_1} L_{u_{w_1}} \cdots L_{v_n} L_{u_{w_n}} \) in the formula (1.11). Suppose that \( w_j = x_j y_j \), with \( x_j, y_j \in V(G) \), for all \( j = 1, \ldots, n \). Then

(1.8)
\[
L_{v_1} L_{u_{w_1}} \cdots L_{v_n} L_{u_{w_n}} = \delta_{(v_1, x_1, y_1; u_{w_1})} \cdots \delta_{(v_n, x_n, y_n; u_{w_n})} (L_{u_{w_1}} \cdots L_{u_{w_n}}),
\]

where

\[
\delta_{(v_j, x_j, y_j; u_{w_j})} = \begin{cases} 
\delta_{v_j, x_j} & \text{if } u_{w_j} = 1 \\
\delta_{v_j, y_j} & \text{if } u_{w_j} = *
\end{cases}
\]

for all \( j = 1, \ldots, n \), where \( \delta \) in the right-hand side is the Kronecker delta. So, the left-hand side can be understood as a (conditional) Kronecker delta depending on \( \{1, *\} \).

By (1.7) and (1.8), the \( n \)-th moment of \( a \) is determined by;

**Proposition 1.2.** Let \( a \in (W^*(G), E) \) be given as above. Then the \( n \)-th moment of \( a \) is

\[
E (d_1 a \cdots d_n a) = \sum_{(v_1, \ldots, v_n) \in \Pi^n_{j=1} V(G;d_j)} (\Pi^n_{j=1} q_{v_j}) \\
\sum_{(w_1, \ldots, w_n) \in \mathbb{F}^+(G:a)^n, \ u_{w_j} \in \{1, *\}, \ i_{w_1} \cdots i_{w_n} \in \mathbb{L}P^*} (\Pi^n_{j=1} p_{w_j}) \\
(\Pi^n_{j=1} \delta_{(v_j, x_j, y_j; u_{w_j})}) \ E (L_{u_{w_1}} \cdots L_{u_{w_n}}).
\]

\( \square \)

From now, rest of this section, we will compute the \( D_G \)-valued cumulants of the given \( D_G \)-valued random variable \( a \). Let \( w_1, \ldots, w_n \in FP(G) \) be finite paths and \( u_1, \ldots, u_n \in \{1, *\} \). Then, by the Möbius inversion, we have

(1.13)
\[
k_n (L_{u_{w_1}} \cdots L_{u_{w_n}}) = \sum_{\pi \in NC(n)} \hat{E}(\pi) (L_{u_{w_1}} \otimes \cdots \otimes L_{u_{w_n}}) \mu(\pi, 1_n),
\]
where \( \hat{E} = (E^{(n)})_{n=1}^\infty \) is the moment multiplicative bimodule map induced by the conditional expectation \( E \) (See [16]) and where \( NC(n) \) is the collection of all noncrossing partition over \( \{1, \ldots, n\} \). Notice that if \( L_{w_1}^{u_1} \cdots L_{w_n}^{u_n} \) does not have the *-axis-property, then

\[
E \left( L_{w_1}^{u_1} \cdots L_{w_n}^{u_n} \right) = 0_{D_G},
\]

by Section 2.1. Consider the noncrossing partition \( \pi \in NC(n) \) with its blocks \( V_1, \ldots, V_k \). Choose one block \( V_j = (j_1, \ldots, j_k) \in \pi \). Then we have that

\[
(1.14) \quad \hat{E}(\pi \mid V_j) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) = E \left( L_{w_{j_1}}^{u_{j_1}} d_{j_2} L_{w_{j_2}}^{u_{j_2}} \cdots d_{j_k} L_{w_{j_k}}^{u_{j_k}} \right),
\]

where

\[
d_{j_i} = \begin{cases} 1_{D_G} & \text{if there is no inner blocks between } j_{i-1} \text{ and } j_i \text{ in } V_j \\ L_{v_{j_i}} \neq 1_{D_G} & \text{if there are inner blocks between } j_{i-1} \text{ and } j_i \text{ in } V_j, \end{cases}
\]

where \( v_{j_2}, \ldots, v_{j_k} \in V(G) \). So, again by Section 2.1, \( \hat{E}(\pi \mid V_j) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) \) is nonvanishing if and only if \( L_{w_{j_1}}^{u_{j_1}} d_{j_2} L_{w_{j_2}}^{u_{j_2}} \cdots d_{j_k} L_{w_{j_k}}^{u_{j_k}} \) has the *-axis-property, for all \( j = 1, \ldots, n \).

Assume that

\[
\hat{E}(\pi \mid V_j) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) = L_{v_j}
\]

and

\[
\hat{E}(\pi \mid V_j) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) = L_{v_i}.
\]

If \( v_j \neq v_i \), then the partition-dependent \( D_G \)-moment satisfies that

\[
\hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) = 0_{D_G}.
\]

This says that \( \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) \neq 0_{D_G} \) if and only if there exists \( v \in V(G) \) such that

\[
\hat{E}(\pi \mid V_j) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) = L_{v_j}
\]

for all \( j = 1, \ldots, k \).

**Definition 1.7.** Let \( NC(n) \) be the set of all noncrossing partition over \( \{1, \ldots, n\} \) and let \( L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \in (W^*(G), E) \) be \( D_G \)-valued random variables, where \( u_1, \ldots, u_n \in \{1, *\} \). We say that the \( D_G \)-valued random variable \( L_{w_1}^{u_1} \cdots L_{w_n}^{u_n} \) is \( \pi \)-connected if the
Theorem 1.4. (See [15]) Let \( n \in 2\mathbb{N} \) and let \( L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \in (W^*(G), E) \) be \( D_G \)-valued random variables, where \( w_1, \ldots, w_n \in FP(G) \) and \( u_j \in \{1, \ast\}, j = 1, \ldots, n \). Then
1.2. $D_G$-Freeness on $(W^*(G), E)$.

Throughout this chapter, let $G$ be a countable directed graph and $(W^*(G), E)$, the graph $W^*$-probability space over its diagonal subalgebra $D_G$. In this chapter, we will consider the $D_G$-valued freeness of given two random variables in $(W^*(G), E)$. We will characterize the $D_G$-freeness of $D_G$-valued random variables $L_{w_1}$ and $L_{w_2}$, where $w_1 \neq w_2 \in FP(G)$. And then we will observe the $D_G$-freeness of arbitrary two $D_G$-valued random variables $a_1$ and $a_2$ in terms of their supports. Let

$$a = \sum_{w \in F^+(G; a), u_w \in \{1, \ast\}} p_w L_{w^u}, \quad b = \sum_{w' \in F^+(G; b), u_{w'} \in \{1, \ast\}} p_{w'} L_{w'^u}$$

be fixed $D_G$-valued random variables in $(W^*(G), E)$.

Now, fix $n \in \mathbb{N}$ and let $(a_{i_1}^{\varepsilon_{i_1}}, \ldots, a_{i_n}^{\varepsilon_{i_n}}) \in \{a, b, a^*, b^*\}^n$, where $\varepsilon_{i_j} \in \{1, \ast\}$. For convenience, put

$$a_{i_j}^{\varepsilon_{i_j}} = \sum_{w_{i_j} \in F^+(G; a), u_{i_j} \in \{1, \ast\}} p_{w_{i_j}} L_{w_{i_j}^u}, \text{ for } j = 1, \ldots, n.$$

Then, by the little modification of Section , we have that:

$$E \left( d_{i_1} a_{i_1}^{\varepsilon_{i_1}} \ldots d_{i_n} a_{i_n}^{\varepsilon_{i_n}} \right)$$

$$= \sum_{(w_{i_1}, \ldots, w_{i_n}) \in \Pi_{k=1}^n V(G; d_{i_k})} \left( \Pi_{k=1}^n q_{w_{i_k}} \right)$$

$$\times \sum_{(w_{i_1}, \ldots, w_{i_n}) \in \Pi_{k=1}^n F^+(G; a_{i_k}), w_{i_j} = x_{i_j} w_{i_j} y_{i_j}, u_{i_j} \in \{1, \ast\}} \left( \Pi_{k=1}^n p_{w_{i_k}} \right)$$

$$\times \left( \prod_{j=1}^n \delta_{(w_{i_j}, x_{i_j}, y_{i_j}, u_{i_j})} \right) E \left( L_{w_{i_1}^{u_{i_1}}} \ldots L_{w_{i_n}^{u_{i_n}}} \right).$$

Therefore, we have that

$$E \left( d_{i_1} a_{i_1}^{\varepsilon_{i_1}} \ldots d_{i_n} a_{i_n}^{\varepsilon_{i_n}} \right).$$
Proposition 1.5. Let $a / \in \epsilon$

Corollary 1.6. The $D_i \in \{1, \ldots, n\}$

where $\mu_{\Pi_{i=1}^n G} = \sum_{\pi \in C_{n_1, \ldots, n_n}} \mu(\pi, 1_n)$ and

$C_{n_1, \ldots, n_n} = \{\pi \in NC^{(even)}(n) : L_{n_1} \ldots L_{n_n}^n \text{ is } \pi\text{-connected}\}$.

So, we have the following proposition, by the straightforward computation:

\[
k_n \left(d_{i_1}^{\varepsilon_{i_1}}, \ldots, d_{i_n}^{\varepsilon_{i_n}} \right) = \sum_{(v_{i_1}, \ldots, v_{i_n}) \in \prod_{j=1}^{n} V(G : d_j)} \left( \Pi_{k=1}^{n} q_{v_k} \right)
\]

\[
\sum_{(w_{i_1}, \ldots, w_{i_n}) \in \prod_{k=1}^{n} FP \left(G : a_i \delta(w_{i_1}, x_{i_j}, y_{i_j}) \right) \left( \Pi_{k=1}^{n} p_{w_k}^{(k)} \right) \left( \Pi_{j=1}^{n} \delta(w_{i_j}, x_{i_j}, y_{i_j}) \right) \left( \Pi_{j=1}^{n} q_{w_j} \right) \text{ Pr}_{w_{i_1}} \left( L_{n_1}^{u_{i_1}} \ldots L_{n_n}^{u_{i_n}} \right)
\]

where $\mu_n = \sum_{\pi \in C_{n_1, \ldots, n_n}} \mu(\pi, 1_n)$ and

$W_{w_{i_1}, \ldots, w_{i_n}} = \{ w \in FP_{w}(G : a) \cup FP_{w}(G : b) : both L_{w_{i_1}}^{u_{i_1}} \text{ and } L_{w_{i_n}}^{u_{i_n}} \text{ are in } L_{w_{i_1}}^{u_{i_1}} \ldots L_{w_{i_n}}^{u_{i_n}} \}$.

So, we have the following $D_G$-freeness characterization:

Corollary 1.6. Let $x$ and $y$ be the $D_G$-valued random variables in $(W^*(G), E)$. The $D_G$-valued random variables $a$ and $b$ are free over $D_G$ in $(W^*(G), E)$ if
\[ FP_1(G : P(x, x^*)) \cap FP_1(G : Q(y, y^*)) = \emptyset \]
and
\[ W_1(P(x, x^*), Q(y, y^*)) = \emptyset, \]
for all \( P, Q \in \mathbb{C}[z_1, z_2]. \]

We have the above \( D_G \)-freeness characterization, but it is so hard to use the above characterization. So, we will restrict our interests to the \( D_G \)-freeness on the generator set \( \{L_w, L_w^*: w \in \mathbb{F}^+(G)\} \) of the graph \( W^*-\text{algebra} \ W^*(G) \). In this case, the \( D_G \)-freeness on the set is pictorially determined on the given graph \( G \). Now, we will introduce the diagram-distinctness of general finite paths:

**Definition 1.9.** *Diagram-Distinctness* We will say that the finite paths \( w_1 \) and \( w_2 \) are **diagram-distinct** if \( w_1 \) and \( w_2 \) have different diagrams in the graph \( G \). Let \( X_1 \) and \( X_2 \) be subsets of \( FP(G) \). The subsets \( X_1 \) and \( X_2 \) are said to be **diagram-distinct** if \( x_1 \) and \( x_2 \) are diagram-distinct for all pairs \((x_1, x_2) \in X_1 \times X_2\).

Let \( H \) be a directed graph with \( V(H) = \{v_1, v_2\} \) and \( E(H) = \{e_1 = v_1e_1v_2, e_2 = v_2e_2v_1\} \). Then \( l = e_1e_2 \) is a loop in \( FP(H) \) (i.e., \( l \in loop(H) \)). Moreover, it is a basic loop (i.e., \( l \in Loop(H) \)). However, if we have a loop \( w = e_1e_2e_1e_2 = l^2 \), then it is not a basic loop. i.e.,

\[ l^2 \in loop(H) \setminus Loop(H). \]

If the graph \( G \) contains at least one basic loop \( l \in FP(G) \), then we have

\[ \{l^n : n \in \mathbb{N}\} \subseteq loop(G) \quad \text{and} \quad \{l\} \subseteq Loop(G). \]

Suppose that \( l_1 \) and \( l_2 \) are not diagram-distinct. Then, by definition, there exists \( w \in Loop(G) \) such that \( l_1 = w^{k_1} \) and \( l_2 = w^{k_2} \), for some \( k_1, k_2 \in \mathbb{N} \). On the graph \( G \), indeed, \( l_1 \) and \( l_2 \) make the same diagram. On the other hands, we can see that if \( w_1 \neq w_2 \in loop^i(G) \), then they are automatically diagram-distinct. In [15], we found the \( D_G \)-freeness characterization on the generator set of \( W^*(G) \), as follows:

**Theorem 1.7.** (See [15]) Let \( w_1, w_2 \in FP(G) \) be finite paths. The \( D_G \)-valued random variables \( L_{w_1} \) and \( L_{w_2} \) in \((W^*(G), E)\) are free over \( D_G \) if and only if \( w_1 \) and \( w_2 \) are diagram-distinct. \( \square \)

Let \( a \) and \( b \) be the given \( D_G \)-valued random variables. We can get the necessary condition for the \( D_G \)-freeness of \( a \) and \( b \), in terms of their supports. Recall that we say that the two subsets \( X_1 \) and \( X_2 \) of \( FP(G) \) are said to be diagram-distinct if \( x_1 \) and \( x_2 \) are diagram-distinct, for all pairs \((x_1, x_2) \in X_1 \times X_2\).
Proposition 1.8. (See [15]) Let $a, b \in (W^*(G), E)$ be $D_G$-valued random variables with their supports $\mathbb{F}^+(G : a)$ and $\mathbb{F}^+(G : b)$. The $D_G$-valued random variables $a$ and $b$ are free over $D_G$ in $(W^*(G), E)$ if $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct. □

1.3. $D_G$-valued Random Variables. In this section, we will consider certain $D_G$-valued random variables. Let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. In [17], we showed that the $D_G$-semicircularity, the $D_G$-evenness and the $D_G$-valued R-diagonality of $D_G$-valued random variables can be characterized by the graphical expression on the graph $G$.

Let $B$ be a von Neumann algebra and $A$, a von Neumann algebra over $B$ and let $F : A \to B$ be a conditional expectation. Then we have a $W^*$-probability space $(A, F)$ over $B$. The $B$-valued random variable $a \in (A, F)$ is called a $B$-semicircular element if it is self-adjoint and the only nonvanishing $B$-cumulant of $a$ is the second one. i.e., a $B$-semicircular element $a$ satisfies that

$$k_n^F(a, \ldots, a) = \begin{cases} k_2^F(a, a) \neq 0_B & \text{if } n = 2 \\ 0_B & \text{otherwise,} \end{cases}$$

where $k_n^F(\ldots)$ is the $B$-cumulant bimodule map induced by the conditional expectation $F$. Suppose $x$ is a $B$-valued random variable in $(A, F)$. We say that the random variable $x$ is $B$-even if it is self-adjoint and all odd $B$-moments vanish. Equivalently, the self-adjoint operator $x$ is $B$-even if all odd $B$-moments vanish. Now, assume that $y \in (A, F)$ is a $B$-valued random variable. If the only nonvanishing mixed $B$-cumulant of $y$ and $y^*$ are alternating $B$-cumulants, i.e., if the only nonvanishing $B$-cumulants of $y$ and $y^*$ are

$$k_{2n}(y, y^*, \ldots, y, y^*) \text{ and } k_{2n}(y^*, y, \ldots, y^*, y),$$

for all $n \in \mathbb{N}$, then the $D_G$-valued random variable $y$ (and $y^*$) is called the $B$-valued R-diagonal.

The $B$-semicircular elements, $B$-even elements and $B$-valued R-diagonal elements play important role in Free Probability. The following theorem shows that the graph $W^*$-probability spaces contain such random variables. So, the graph $W^*$-probability spaces contain rich free probabilistic objects.

Proposition 1.9. (See [17]) Let $w \in \mathbb{F}^+(G)$ and let $L_w$ be the corresponding $D_G$-valued random variable in $(W^*(G), E)$. Then
(1) if \( w \) is a loop, then \( L_w + L_w^* \) is \( D_G \)-semicircular.

(2) if \( w \) is a finite path, then \( L_w + L_w^* \) is \( D_G \)-even.

(3) if \( w \) is a finite path, then \( L_w \) and \( L_w^* \) are \( D_G \)-valued \( R \)-diagonal. □

2. Vertex Compressed Graph \( W^\ast \)-Probability

In this chapter, we will consider the vertex-compressed graph \( W^\ast \)-probability. Throughout this chapter, let \( G \) be a countable directed graph and let \((W^\ast(G), E)\) be the graph \( W^\ast \)-probability space over the diagonal subalgebra \( D_G \). Let \( v_0 \in V(G) \) be a vertex and let’s fix this vertex. Then we can take the projection \( L_{v_0} \in (W^\ast(G), E) \). We will consider the compressed \( W^\ast \)-algebra \( L_{v_0} \) and observe the compressed probability space on \((L_{v_0}W^\ast(G)L_{v_0}, E_{v_0})\), where

\[
E_{v_0} : L_{v_0}W^\ast(G)L_{v_0} \to D_G
\]

is the compressed conditional expectation defined by

\[
E_{v_0} \overset{\text{def}}{=} E|_{L_{v_0}W^\ast(G)L_{v_0}}.
\]

It is easy to see that the \( v_0 \)-compressed conditional expectation \( E_{v_0} \) can be regarded as a linear functional from the \( v_0 \)-compressed graph \( W^\ast \)-algebra \( L_{v_0}W^\ast(G)L_{v_0} \) onto \( C = \mathbb{C}_{v_0} = L_{v_0}D_GL_{v_0} \). Indeed, let

\[
a = \sum_{w \in F^+(G:a), u_w \in \{1,*\}} p_w L_w^{u_w} \in (W^\ast(G), E).
\]

Then, for any summand \( L_w^{u_w} \) of \( a, w \in F^+(G:a) \),

\[
L_{v_0}L_w^{u_w}L_{v_0} = \begin{cases} 
L_{v_0} & \text{if } w = v_0 \\
L_w^{u_w} & \text{if } l = v_0l_0 \\
0_{D_G} & \text{otherwise},
\end{cases}
\]

for all \( u_w = 1, * \). Thus \( E_{v_0} \) maps \( L_{v_0}W^\ast(G)L_{v_0} \) linearly onto \( \mathbb{C}_{v_0} \). This shows that when we want to compute the \( L_{v_0}D_GL_{v_0} \)-valued moments and cumulants of a compressed random variable \( L_{v_0}aL_{v_0} \), the trivial moments and cumulants contain the full free probabilistic information of the compressed random variable \( L_{v_0}aL_{v_0} \). Also, we can easily verify that
\[ E_{v_0}(x) = \langle \xi_{v_0}, x \xi_{v_0} \rangle \in \mathbb{C} : \xi_{v_0}, \forall x \in L_{v_0} W^*(G) L_{v_0}. \]

Again, remark that
\[ L_{v_0} a L_{v_0} = p_{v_0} [L_{v_0}] + \sum_{w=v_0 w v_0 \in \text{loop}(G : a), u_w \in \{1, \ast\}} p_w L^w_{u_w}, \]

where \( [L_{v_0}] = L_{v_0} \) if \( v_0 \in V(G : a) \) and \( [L_{v_0}] = 0_{D_G} \), otherwise. Therefore, we can get that
\[ E_{v_0} (L_{v_0} a L_{v_0}) = p_{v_0} [L_{v_0}] \in D_G. \]

Hence, we can consider the \( v_0 \)-compressed graph \( W^* \)-probability space over \( D_G \cdot (L_{v_0} W^*(G) L_{v_0}, E_{v_0}) \), as a (scalar-valued) \( W^* \)-probability space. Let’s regard \( L_{v_0} D_G L_{v_0} \) as \( C \).

### 2.1. Vertex Compressed Graph \( W^* \)-Probability Spaces.

In this section, we will consider the vertex compressed graph \( W^* \)-probability space as a (scalar-valued) \( W^* \)-probability space (over \( C \)). Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable with \( FP(G : a) \subseteq \text{loop}(G) \). Then such \( D_G \)-valued random variable \( a \) is called a loop operator in \( (W^*(G), E) \). By definition, the \( v_0 \)-compressed random variable \( x \) has its form of
\[ x = p_{v_0} [L_{v_0}] + \sum_{w \in \text{loop}_{v_0}(G : a), u_w \in \{1, \ast\}} p_w L^w_{u_w}. \]

So, every \( v_0 \)-compressed random variables is a loop operator, where
\[ \text{loop}_{v_0}(G : a) \overset{\text{def}}{=} \{ l \in \text{loop}(G : a) : l = v_0 l v_0 \}. \]

**Definition 2.1.** Let \( G \) be a countable directed graph and fix \( v_0 \in V(G) \). Define the compressed \( W^* \)-algebra, \( W^*_{v_0}(G) \) by \( L_{v_0} W^*(G) L_{v_0} \), and we will call it the \( v_0 \)-compressed graph \( W^* \)-algebra. Now, define the linear functional \( E_{v_0} : W^*_{v_0}(G) \to \mathbb{C} \) by
\[ E_{v_0} (x) = \langle \xi_{v_0}, x \xi_{v_0} \rangle, \text{ for all } x \in W^*(G). \]

We will call the algebraic pair \((W^*_{v_0}(G), E_{v_0})\), the \( v_0 \)-compressed graph \( W^* \)-probability space. Let \( D_G \) be the diagonal subalgebra. Denote \( L_{v_0} D_G L_{v_0} \) by \( D^v_G \), for convenience.
Notice that we can regard the \( v_0 \)-compressed graph \( W^* \)-probability space \((W^*_{v_0}(G), E_{v_0})\) as a scalar-valued \( W^* \)-probability space. So, we can define the \( n \)-th moments and \( n \)-th cumulants, in the sense of Nica and Speicher (See [1] and [19]). In our notation, they are just trivial \( D^{v_0}_G \)-valued moments and cumulants.

**Definition 2.2.** Let \((W^*_{v_0}(G), E_{v_0})\) be a \( v_0 \)-compressed graph \( W^* \)-probability space and let \( a_{v_0} = L_{v_0}a_{v_0} \in (W^*_{v_0}(G), E_{v_0}) \) be a random variable. Then the \( n \)-th moment of \( a_{v_0} \) is \( E_{v_0}(a^n_{v_0}) \) and the \( n \)-th cumulant of \( a_{v_0} \) is \( k_n^{E_{v_0}}(a_{v_0}, \ldots, a_{v_0}) \). If there is no confusion, we will denote the \( n \)-th compressed cumulant \( k_n^{E_{v_0}}(a_{v_0}, \ldots, a_{v_0}) \), by \( k_n(a_{v_0}, \ldots, a_{v_0}) \), for all \( n \in \mathbb{N} \). Define the \( v_0 \)-compressed moment series of \( a_{v_0} \) by

\[
M_n^{(v_0)}(z) = \sum_{n=1}^{\infty} E_{v_0}(a^n_{v_0}) z^n \in \mathbb{C}[z],
\]

and, define the \( v_0 \)-compressed R-transform of \( a_{v_0} \) by

\[
R_n^{(v_0)}(z) = \sum_{n=1}^{\infty} k_n(a_{v_0}, \ldots, a_{v_0}) z^n \in \mathbb{C}[z].
\]

Observe that if \( a \in W^*(G) \) is given as above, then, by Section 1.3,

\[
(L_{v_0}a_{v_0})^n = (L_{v_0}a_{v_0})(L_{v_0}a_{v_0}) \cdots (L_{v_0}a_{v_0})
\]

\[
= L_{v_0}a_{v_0}a_{v_0}a_{v_0}a_{v_0}a_{v_0}L_{v_0}a_{v_0}a_{v_0}a_{v_0}a_{v_0}L_{v_0}a_{v_0}
\]

\[
= L_{v_0}a_{v_0}a_{v_0} \cdots L_{v_0}a_{v_0} = L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}
\]

\[
= L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}L_{v_0}a_{v_0}
\]

\[
(2.1.1)
\]

\[
= \sum_{(w_1, \ldots, w_n) \in \mathbb{F}^+(G:a)^n, w_j = v_0w_jv_0, u_{w_j} \in \{1, *\}} (\prod_{j=1}^{n} p_{w_j}) L_{w_1}^{u_{w_1}} \cdots L_{w_n}^{u_{w_n}}.
\]

One may be tempted to use loop\(_{v_0}(G : a)\), instead of using \( \mathbb{F}^+(G : a) \), in the formula (2.1.1). However, we need to consider the case when \( v_0 \in V(G : a) \), in general. (Clearly, if \( v_0 \in V(G : a) \), then \( v_0 = v_0v_0v_0. \) That’s why we used \( \mathbb{F}^+(G : a) \), in (2.1.1).

### 2.2. Vertex Compressed Moments and Cumulants.

Throughout this section, let \( G \) be a countable directed graph and let \((W^*(G), E)\) be the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \). Let \( v_0 \in V(G) \) be the fixed vertex. Then we can construct the \( v_0 \)-compressed graph \( W^* \)-probability space \((W^*_{v_0}(G), E_{v_0})\), as a scalar-valued (or \( D^{v_0}_G \)-valued) \( W^* \)-probability space with the linear functional \( E_{v_0} : W^*_{v_0}(G) \to D^{v_0}_G \simeq \mathbb{C} \).
Suppose that 

Proposition 2.1. Let \((W_{v_0}(G), E_{v_0})\) be the \(v_0\)-compressed graph \(W^*-\)probability space. Let \(a_{v_0} = L_{v_0}aL_{v_0} \in (W_{v_0}(G), E_{v_0})\) be a random variable, where \(a \in (W^*(G), E)\) and assume that loop(G : a) consists of all mutually diagram-distinct loops. Then the \(n\)-th (scalar-valued) cumulants of \(a_{v_0}\) are

\[
\kappa_n^{E_{v_0}}(a_{v_0}, ..., a_{v_0}) = \begin{cases} 
\sum_{l \in \text{loop}^{v_0}(G:a)} 2(p_lp_{l'})^2 L_{v_0} & \text{if } n = 2 \\
0 & \text{otherwise},
\end{cases}
\]

where \(\text{loop}^{v_0}(G : a) = \{l \in FP_*(G : a) : l = v_0l v_0\}\).

Proof. Suppose that \(a_{v_0} = L_{v_0}aL_{v_0}\) is the \(v_0\)-compressed random variable in \((W_{v_0}(G), E_{v_0}) \subset (W^*(G), E)\). As we have seen before, we have that

\[
a_{v_0} = [p_{v_0}L_{v_0} + \sum_{l \in \text{loop}^{v_0}(G:a)} (p_lL_l + p_{l'}L_{l'}^*) + \sum_{l = v_0wv_0 \in \text{loop}^{v_0}(G:a), u_w \in \{1, *\}} p_wL_{uw}^*,
\]

denoted by \(a^{v_0}_{(v)} + a^{v_0}_{(non-v)}\). Then

\[
k_n^{E_{v_0}}(a_{v_0},...,a_{v_0}) = k_n(a^{v_0}_{(v)},...,a^{v_0}_{(v)})
\]

\[
= k_n \left( \sum_{l \in \text{loop}^{v_0}(G:a)} (p_lL_l + p_{l'}L_{l'}^*) , ..., \sum_{l \in \text{loop}^{v_0}(G:a)} (p_lL_l + p_{l'}L_{l'}^*) \right)
\]

\[
= \sum_{l \in \text{loop}^{v_0}(G:a)} k_n ((p_lL_l + p_{l'}L_{l'}^*) , ..., (p_lL_l + p_{l'}L_{l'}^*))
\]

since \(p_{l_1}L_{l_1} + p_{l_1'}L_{l_1}^*\) and \(p_{l_2}L_{l_2} + p_{l_2'}L_{l_2}^*\) are free in \((W_{v_0}(G), E_{v_0})\), whenever \(l_1 \neq l_2\) in \(\text{loop}^{v_0}(G : a)\), by assumption.

\[
= \sum_{l \in \text{loop}^{v_0}(G:a)} \sum_{u_1, ..., u_n \in \{1,*\}^n} k_n (p_{u_1}L_{u_1}^{u_1}, ..., p_{u_n}L_{u_n}^{u_n})
\]

\[
= \sum_{l \in \text{loop}^{v_0}(G:a)} \sum_{u_1, ..., u_n \in \{1,*\}^n} (\Pi_{j=1}^{n} p_{u_j})
\]

\[
k_n (L_{u_1}^{u_1}, ..., L_{u_n}^{u_n})
\]

by the bilinearity of \(k_n(\cdot, \cdot)\), where \(p_{u_j} = p_l\) if \(u_j = 1\) and \(p_{u_j} = p_{l'}\) if \(u_j = *\)
by Section 1.3, we have that

\[
\begin{cases}
\sum_{t \in \text{loop}^n_0(G ; a)} (p_t p_{\ell_t})^2 k_2(L_{t} + L_{t}^*) & \text{if } n = 2 \\
0 & \text{otherwise}
\end{cases}
\]

Theorem 2.2. Let \((W_{v_0}^* (G), E_{v_0})\) be the \(v_0\)-compressed graph \(W^*\)-probability space, as a scalar-valued \(W^*\)-probability space. Let \(a \in \mathcal{W}^*(G)\) and let \(a_{v_0} = L_{v_0} a L_{v_0} \in W_{v_0}^*(G)\). Then

\[
E_{v_0} \left( a_{v_0}^n \right) = \sum_{(w_1, \ldots, w_n) \in \text{loop}^n_0(G ; a), \, u_j \in \{1,*\}} (\Pi_{j=1}^n p_{w_j}) \cdot L_{v_0}
\]

for all \(n \in \mathbb{N}\), in \(\mathbb{C} \xi_{v_0}\), where

\[
\text{loop}^n_0(G : a) = \{ l \in \mathcal{F} \mathcal{P}_* (G : a) : l = v_0 lv_0 \}.
\]

In particular, in this case, \(\text{loop}^0_0(G : a) = \mathcal{F} \mathcal{P}_* (G : a_{v_0})\).

Proof. Consider the \(v_0\)-compressed random variable \(a_{v_0} = L_{v_0} a L_{v_0}\), for the fixed \(D_G\)-valued random variable, \(a = a_d + a_{(*)} + a_{(\text{non-}*)} \in \mathcal{W}^*(G, E)\). Then we have that

\[
a_{v_0} = L_{v_0} a L_{v_0}
\]

\[
= L_{v_0} a_d L_{v_0} + L_{v_0} a_{(*)} L_{v_0} + L_{v_0} a_{(\text{non-}*)} L_{v_0}
\]

\[
= [p_{v_0} L_{v_0}] + \sum_{l = v_0 v_0 \in \text{loop}^0_0(G ; a)} (p_l L_l + p_{L_l^*})
\]

\[
+ \sum_{w_v = v_0 w_v \in \text{loop}^0_0(G ; a), \, u_v \in \{1,*\}} p_w L_{w_v}^{a_{w_v}},
\]

where \([p_{v_0} L_{v_0}] = p_{v_0} L_{v_0}\) if \(v_0 \in V(G : a)\) and \([p_{v_0} L_{v_0}] = 0\) if \(v_0 \notin V(G : a)\). So, this \(v_0\)-compressed random variable \(a_{v_0}\) is an addition of \([p_{v_0} L_{v_0}]\) and the loop operator \(L_{v_0} a_{(*)} L_{v_0} + L_{v_0} a_{(\text{non-}*)} L_{v_0}\), in \(\mathcal{W}^*(G)\), centered at \(v_0 \in V(G)\). Therefore, by Section 1.3, we have that
DIAGONAL COMPRESSED GRAPH

\[ E(a^n_{v_0}) = \sum_{(w_1, \ldots, w_n) \in \text{loop}^*_{v_0}(G, \alpha), \ u_j \in \{1, \ast\}} (\Pi_{j=1}^n p_{w_j}) \cdot L_{v_0}. \]

Recall that in Section 1.3, the \( D_G \)-valued cumulants of the fixed random variable \( a \) is easily gotten by multiplying

\[ \mu_{w_1, \ldots, w_n} = \sum_{\pi \in C_{w_1, \ldots, w_n}} \mu(\pi, 1_n) \]

to each summand \( E(L_{w_1} \ldots L_{w_n}) \) of the \( D_G \)-valued moments of \( a \). This happens because of the \( \ast \)-axis-property. So, we can get the following cumulants of arbitrary \( v_0 \)-compressed random variables, by the previous theorem:

**Corollary 2.3.** Let \( (W^*_v(G), E_v) \) be the \( v_0 \)-compressed graph \( W^* \)-probability space, as a scalar-valued \( W^* \)-probability space. Let \( a \in W^*(G) \) and let \( a_{v_0} = L_{v_0} a L_{v_0} \in W^*_v(G) \). Then

\[ k_n^{(E_{v_0})} = \sum_{(w_1, \ldots, w_n) \in \text{loop}^*_{v_0}(G, \alpha), \ u_j \in \{1, \ast\}} (\Pi_{j=1}^n p_{w_j}) \cdot \mu_{w_1, \ldots, w_n} L_{v_0}, \]

for all \( n \in \mathbb{N} \). \( \square \)

### 2.3. Vertex-Compressed Freeness

In this section, we will consider the vertex-compressed freeness on the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \). Let \( (W^*_v(G), E_v) \) be a graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \) and let \( (W^*_v(G), E_v) \) be the \( v_0 \)-compressed graph \( W^* \)-probability space as a scalar-valued \( W^* \)-probability space (over \( D^v_D \)).

Let \( X \) and \( Y \) be subalgebras of the \( v_0 \)-compressed \( W^* \)-algebra, \( W^*_v(G) \). We say that the subalgebras \( X \) and \( Y \) are free if all mixed cumulants of \( X \) and \( Y \) vanish (with respect to the \( v_0 \)-compressed conditional expectation or linear functional \( E_{v_0} : W^*_v(G) \to D^v_D \simeq \mathbb{C} \)). Also, two random variables \( x \) and \( y \) are free in \( (W^*_v(G), E_{v_0}) \) if \( x \notin W^*(\{y\}) \), \( y \notin W^*(\{x\}) \) and if all mixed cumulants of \( x \) and \( y \) vanish. Suppose that \( x \) and \( y \) are random variables in \( (W^*_v(G), E_{v_0}) \). Then there exists operators \( a \) and \( b \) in \( W^*(G) \) such that

\[ x = L_{v_0} a L_{v_0} \quad \text{and} \quad y = L_{v_0} b L_{v_0}. \]

Recall that the \( D_G \)-valued random variables
Theorem 2.4. Let\( a \) be random variables in \( D \) and assume \( \text{FP}(G : a) \) is preserved by the loop variables in \( \mathbb{R} \). By the above result, we have that the \( v \)-compressed random variables \( x = L_{v_0} a L_{v_0} \) and \( y = L_{v_0} b L_{v_0} \) are free in \( (W^*_{v_0}(G), E_{v_0}) \).

Proof. We have that

\[
x = p_{v_0}^{(1)}[L_{v_0}] + \sum_{l_1 \in \text{loop}^{v_0}(G : a_1), u_1 \in \{1,*\}} p_{l_1}^{(1)} L_{u_1}^{l_1}
\]

and

\[
y = p_{v_0}^{(2)}[L_{v_0}] + \sum_{l_2 \in \text{loop}^{v_0}(G : a_2), u_2 \in \{1,*\}} p_{l_2}^{(2)} L_{u_2}^{l_2}.
\]

Since \( \text{FP}(G : a_1) \) and \( \text{FP}(G : a_2) \) are diagram-distinct, \( \text{loop}^{v_0}(G : a_1) \) and \( \text{loop}^{v_0}(G : a_2) \) are diagram-distinct. Notice that

\[
\text{FP}(G : x) = \text{loop}^{v_0}(G : a_1)
\]

and

\[
\text{FP}(G : y) = \text{loop}^{v_0}(G : a_2).
\]

Thus, the \( v_0 \)-compressed random variables \( x \) and \( y \) are free over \( D_G \) in \( (W^*(G), E) \).

Remark that the \( v_0 \)-compressed random variables \( x \) and \( y \) are scalar-valued random variables in \( (W^*_{v_0}(G), E_{v_0}) \) and the compressed moments and cumulants are same as the \( D_G \)-valued moments and cumulants of \( x \) and \( y \) over \( D_G \). Therefore, the \( v_0 \)-compressed random variables \( x \) and \( y \) are free in \( (W^*_{v_0}(G), E_{v_0}) \).

We also have the following general case. This shows that the compressed freeness is preserved by the \( D_G \)-freeness.

Theorem 2.5. Let \( a \) and \( b \) be \( D_G \)-valued random variables in the graph \( W^* \)-probability space \( (W^*(G), E) \) over the diagonal subalgebra \( D_G \). If they are free over \( D_G \) in \( (W^*(G), E) \), then the corresponding \( v_0 \)-compressed random variables \( x = L_{v_0} a L_{v_0} \) and \( y = L_{v_0} b L_{v_0} \) are free in \( (W^*_{v_0}(G), E_{v_0}) \).
Proof. Suppose that there exists \( k \in \mathbb{N} \setminus \{1\} \) such that the \( k \)-th mixed cumulant \( v_0 \)-compressed cumulants of \( x \) and \( y \) does not vanish. Then since

\[
FP_\ast(G : x) = FP_\ast(G : a) \cap \text{loop}_{v_0}(G : a)
\]
and

\[
FP_\ast(G : y) = FP_\ast(G : b) \cap \text{loop}_{v_0}(G : b),
\]

The \( k \)-th mixed cumulant of \( a \) and \( b \) does not vanish. This contradict our assumption. \( \blacksquare \)

3. Diagonal Compressed Graph \( W^\ast \)-Probability

Throughout this chapter, we will let \( G \) be a countable directed graph and \( \mathbb{F}^+(G) \), the free semigroupoid of \( G \) and let \((W^\ast(G), E)\) be the graph \( W^\ast \)-probability space over the diagonal subalgebra \( D_G \). In this chapter, we will consider the diagonal compression of a \( D_G \)-valued random variable \( a \in (W^\ast(G), E) \), for the given finite subset of the vertex set \( V(G) \) of the graph \( G \). Let \( V = \{v_1, ..., v_N\} \) be a subset of the vertex set \( V(G) \) of the graph \( G \). The diagonal compressed random variable of \( a \in (W^\ast(G), E) \) by \( V \) is defined by

\[
L_{v_1}aL_{v_1} + ... + L_{v_N}aL_{v_N}
\]

in \((W^\ast(G), E)\), as a new \( D_G \)-valued random variable in \((W^\ast(G), E)\). Notice that each \( L_{v_j}aL_{v_j} \) is the \( v_j \)-compressed random variable of \( a \), for \( j = 1, ..., N \), and it can be regarded as a random variable in \((W^\ast_{v_j}(G), E_{v_j})\), the \( W^\ast \)-probability space (over \( \mathbb{C} \)). In this chapter, we will regard them as compressed \( D_G \)-valued random variables. By regarding all \( L_{v_j}aL_{v_j} \) as \( D_G \)-valued random variables in \((W^\ast(G), E)\), the diagonal compressed random variable of \( a \) by \( V = \{v_1, ..., v_N\} \subset V(G) \) is also a \( D_G \)-valued random variable in \((W^\ast(G), E)\). Of course, in the subset \( V \) of \( V(G) \), the vertices satisfy that

\[
v_i \neq v_j, \text{ whenever } i \neq j \text{ in } \{1, ..., N\}.
\]

In this chapter, we will observe the amalgamated \((D_G\)-valued\) free probability information of such diagonal compressed random variables in \((W^\ast(G), E)\).

**Definition 3.1.** Let \( V = \{v_1, ..., v_N\} \subset V(G) \). Define the diagonal compression by \( V \),

\[
P_V : W^\ast(G) \to \sum_{j=1}^{N} L_{v_j} W^\ast(G) L_{v_j} \subset W^\ast(G)
\]

by

\[
P_V(a) = \sum_{j=1}^{N} L_{v_j}aL_{v_j}, \text{ for all } a \in W^\ast(G).
\]
We say that the $D_G$-valued random variable $P_V(a)$ is the diagonal compressed random variable of $a$ by $V$.

3.1. Diagonal Compressed Moments and Cumulants.

Throughout this section, fix $N \geq 2$ in $\mathbb{N}$ and let $V = \{v_1, \ldots, v_N\}$ be the fixed subset of the vertex set $V(G)$. Let $a \in (W^*(G),E)$ be an arbitrary $D_G$-valued random variable. Then, we can get the diagonal compressed random variable of $a$ by the given set $V$, $P_V(a) \in (W^*(G),E)$. By the very definition, we have that

$$P_V(a) = L_{v_1}aL_{v_1} + \ldots + L_{v_N}aL_{v_N} = \sum_{j=1}^N \sum_{w=v_j, a(v_j) \in F^+(G:a), u_w \in \{1,\ast\}} p_w L_u a L_w,$$

where

$$a = \sum_{w \in F^+(G:a), u_w \in \{1,\ast\}} p_w L_u \in (W^*(G), E).$$

If $V \subseteq V(G:a)$, then we have that

$$P_V(a) = \sum_{j=1}^N \left( [p_{v_j} L_{v_j}] + \sum_{w \in \text{loop}_{v_j}(G:a), u_w \in \{1,\ast\}} p_w L_u a L_w \right).$$

But it is possible that $V \nsubseteq V(G:a)$ and then $\sum_{j=1}^N [p_{v_j} L_{v_j}] = 0_{D_G}$. Choose $(i,j) \in \{1, \ldots, N\}^2$ such that $i \neq j$ and assume that there is at least one loop $l = v_i l v_j \in \text{loop}_{v_i}(G:a)$ containing $v_j$ (where, $v_i \neq v_j \in V$). i.e.e, $l = v_i l v_i = v_j l v_j$. Then

$$\text{loop}_{v_i}(G:a) \cap \text{loop}_{v_j}(G:a) \neq \emptyset.$$

So, we need to be careful the intersections of $\text{loop}_{v_i}(G:a)$ are empty or not.

**Lemma 3.1.** Let $v_i \neq v_j \in V(G)$ and assume that

$$\text{loop}_{v_i}(G:a) \cap \text{loop}_{v_j}(G:a) = \emptyset.$$

Then the vertex compressed random variables $L_{v_i}aL_{v_i}$ and $L_{v_j}aL_{v_j}$ of $a$ satisfy that
(L_{v_i} a L_{v_i})^m \cdot (L_{v_j} a L_{v_j})^n = 0_{DG}, \text{ for all } m, n \in \mathbb{N}.

Proof. Assume that \( v_i \neq v_j \) in \( V \) (i.e. \( i \neq j \)) and suppose that

\[
\text{loop}_{v_i}(G : a) \cap \text{loop}_{v_j}(G : a) = \emptyset.
\]

Then we can easily conclude that there is no loop \( l = v_i v_i \in \text{loop}(G : a) \) containing \( v_j \) (or equivalently, there is no loop \( l = v_j v_j \in \text{loop}(G : a) \) containing \( v_i \)). Let \( m = 1 = n \).

\[
(L_{v_i} a L_{v_i})(L_{v_j} a L_{v_j}) = L_{v_i} a (L_{v_i} L_{v_j}) a L_{v_j} = 0_{DG}.
\]

Similarly, if \( m = n \), then

\[
(L_{v_i} a L_{v_i})^m (L_{v_j} a L_{v_j})^n = (L_{v_i} a L_{v_i} L_{v_j} a L_{v_j})^m = (L_{v_i} a (L_{v_i} L_{v_j}) a L_{v_j})^m = 0_{DG}.
\]

Now, let \( m > n \). Then

\[
(L_{v_i} a L_{v_i})^m (L_{v_j} a L_{v_j})^n = (L_{v_i} a L_{v_i})^{m-n} (L_{v_i} a L_{v_j})^n (L_{v_j} a L_{v_j})^n = (L_{v_i} a L_{v_i})^{m-n} (L_{v_i} a (L_{v_i} L_{v_j}) a L_{v_j})^n
\]

\[
= 0_{DG}.
\]

Similarly, if \( m < n \), then \( (L_{v_i} a L_{v_i})^m (L_{v_j} a L_{v_j})^n = 0_{DG} \). \( \Box \)

Lemma 3.2. Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( v_i \neq v_j \) in \( V \) and assume that

\[
\text{loop}^{v_i}_{v_j}(G : a) \overset{\text{def}}{=} \text{loop}_{v_i}(G : a) \cap \text{loop}_{v_j}(G : a) \neq \emptyset.
\]

Then the vertex compressed random variables \( L_{v_i} a L_{v_i} \) and \( L_{v_j} a L_{v_j} \) satisfy that

\[
(L_{v_i} a L_{v_i})^m (L_{v_j} a L_{v_j})^n = \sum_{(w_1, \ldots, w_{m+n}) \in \text{loop}^{v_i}_{v_j}(G : a)^{m+n}} (\prod_{k=1}^{m+n} p_{w_k}) L_{w_1}^{w_{w_1}} \cdots L_{w_{m+n}}^{w_{w_{m+n}}}.
\]

Proof. By the assumption that

\[
\text{loop}_{v_i}(G : a) \cap \text{loop}_{v_j}(G : a) \neq \emptyset,
\]

we can define the subset \( \text{loop}^{v_i}_{v_j}(G : a) \) of \( \text{loop}(G : a) \subset FP(G : a) \) by
\[
\text{loop}^v_i(G : a) \overset{\text{def}}{=} \text{loop}_v(G : a) \cap \text{loop}_{v_j}(G : a) \\
= \{ l \in \text{loop}(G : a) : l = v_i v_j = v_j v_i \}.
\]

By the very definition, we have that
\[
\text{loop}^v_j(G : a) = \text{loop}^v_i(G : a)
\]
in \text{loop}(G : a). By the observation in Section 1.3, we have that
\[
(L_{v_i} a L_{v_j})^m = \sum_{(w_1, \ldots, w_m) \in \text{loop}_{v_j}(G : a)^m} \left( \prod_{p=1}^n p_{w_p} \right) L_{w_1}^{u_{w_1}} \cdots L_{w_m}^{u_{w_m}}
\]
and
\[
(L_{v_j} a L_{v_j})^n = \sum_{(w_1', \ldots, w_n') \in \text{loop}_{v_j}(G : a)^n} \left( \prod_{p=1}^n p_{w_p}' \right) L_{w_1}'^{u_{w_1}'} \cdots L_{w_n}'^{u_{w_n}'}.
\]

Thus
\[
(L_{v_i} a L_{v_j})^m (L_{v_j} a L_{v_j})^n
\]

\[
= \sum_{(w_1, \ldots, w_m) \in \text{loop}_{v_j}(G : a)^m} \left( \prod_{p=1}^n p_{w_p} \right) L_{w_1}^{u_{w_1}} \cdots L_{w_m}^{u_{w_m}}
\]

\[
= \sum_{(w_1', \ldots, w_n') \in \text{loop}_{v_j}(G : a)^n} \left( \prod_{p=1}^n p_{w_p}' \right) L_{w_1}'^{u_{w_1}'} \cdots L_{w_n}'^{u_{w_n}'}
\]

\[
= \sum_{(w_1, \ldots, w_m, w_1', \ldots, w_n') \in \text{loop}_{v_j}(G : a)^{m+n}} \left( \prod_{p=1}^n p_{w_p} \right) \left( \prod_{p=1}^n p_{w_p}' \right) L_{w_1}^{u_{w_1}} \cdots L_{w_m}^{u_{w_m}} L_{w_1}'^{u_{w_1}'} \cdots L_{w_n}'^{u_{w_n}'}.
\]

The above lemma says that, in general, if \( \text{loop}^v_{v_i}(G : a) \neq \emptyset \), then
\[
(L_{v_i} a L_{v_j})(L_{v_j} a L_{v_j}) = \sum_{(w, w') \in \text{loop}^v_{v_i}(G : a)^2} p_{w} p_{w'} L_{w}^{u_{w}} L_{w'}^{u_{w'}}
\]

where \( u_{w}, u_{w'} \in \{1, *\} \) and \( a = \sum_{w \in \text{FP}(G : a), u_w \in \{1, *\}} p_{w} L_{w}^{u_{w}} \in (W^*(G), E) \).
Now, let \( d_k = \sum_{v(k) \in V(G : d_k)} q_{v(k)} L_{v(k)} \in D_G \), for \( k \in \mathbb{N} \). Then

\[
d_1(P_V(a)) d_2(P_V(a)) ... d_n(P_V(a))
\]

\[
= \sum_{(v(1), ..., v(n)) \in \Pi^n_{k=1} V(G : d_k)} (\Pi^n_{k=1} q_{v(k)}) (L_{v(1)} (P_V(a)) ... L_{v(n)} (P_V(a)))
\]

\[
= \sum_{(v(1), ..., v(n)) \in \Pi^n_{k=1} V(G : d_k)} (w_1, ..., w_n) \in \text{loop}_{v_0} (G : P_V(a))^n, u_w \in \{1,*\}
\]

\[
(\Pi^n_{j=1} q_{v(j)}) (\Pi^n_{j=1} P_{w_j})
\]

\[
(\Pi^n_{j=1} \delta_{v(j), x_j}) L_{w_1}^{u_1} ... L_{w_n}^{u_n}.
\]

We have that

\[ F^+(G : P_V(a)) = (\bigcup_{k=1}^n \text{loop}_{v_0} (G : a)) \cup (V \cap V(G : a)) , \]

where

\[ P_V(a) = \sum_{w \in F^+(G : P_V(a)), u_w \in \{1,*\}} p_w L_{w}^{u_w} \in (W^*(G), E) \]

and

\[ a = \sum_{w \in F^+(G : a), u_w \in \{1,*\}} p_w L_{w}^{u_w} \in (W^*(G), E) . \]

Notice that the coefficients \( p_w = \langle \xi_w, a \xi_w \rangle \)'s are not changed, because \( F^+(G : P_V(a)) \subset F^+(G : a) \). Now, we have all information to get the \( D_G \)-valued moments of the diagonal compressed random variable of \( a \) by \( V ; \)

**Theorem 3.3.** Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( V = \{ v_1, ..., v_N \} \) be the fixed finite subset of the vertex set \( V(G) \). The diagonal compressed random variable of \( a \) by \( V \), \( P_V(a) \) has the \( n \)-th moment

\[
E (d_1 P_V(a) d_2 P_V(a) ... d_n P_V(a))
\]

\[
= \sum_{\pi \in \mathcal{N}^G(n)} (v^{(1)}, ..., v^{(n)}) \in \Pi^n_{k=1} V(G : d_k)
\]

\[
(\sum_{w_1, ..., w_n} ) \in (\bigcup_{k=1}^n \text{loop}_{v_0} (G : a)) \cup (V \cap V(G : a))^n, w_j = x_j, x_j, u_w \in \{1,*\}
\]

\[
(\Pi^n_{j=1} q_{v(j)}) (\Pi^n_{j=1} P_{w_j}) (\Pi^n_{j=1} \delta_{v(j), x_j})
\]

\[
E (L_{w_1}^{u_1} ... L_{w_n}^{u_n}) ,
\]
for all $n \in \mathbb{N}$, where $d_k = \sum_{v^{(k)} \in V(G;d_k)} q_{v^{(k)}} L_{v^{(k)}} \in D_G$ are arbitrary, for $k = 1, \ldots, n$.

**Proof.** Fix $n \in \mathbb{N}$ and let $d_k = \sum_{v^{(k)} \in V(G;d_k)} q_{v^{(k)}} L_{v^{(k)}} \in D_G$ are arbitrary, for $k = 1, \ldots, n$. By using the same notations in Section 1.3, we have that

$$E(d_1 P_V(a) \ldots d_n P_V(a))$$

$$= \sum_{(v^{(1)}, \ldots, v^{(n)}) \in \prod_{k=1}^n V(G;d_k)} \sum_{(w_1, \ldots, w_n) \in F^+(G:P_V(a))^n, w_j=x_j w_j, u_{w_j} \in \{1,\ast\}} E^{\frac{u_{w_1}}{d_1 P_V(a)} \ldots \frac{u_{w_n}}{d_n P_V(a)}}$$

$$= \sum_{\pi \in NC(n)} \sum_{(v^{(1)}, \ldots, v^{(n)}) \in \prod_{k=1}^n V(G;d_k)} \sum_{(w_1, \ldots, w_n) \in F^+(G:P_V(a))^n, w_j=x_j w_j, u_{w_j} \in \{1,\ast\}}$$

$$\left(\prod_{j=1}^n q_{v^{(j)}} \right) \left(\prod_{j=1}^n p_{w_j} \right)$$

$$\left(\prod_{j=1}^n \delta_{v^{(j)}, x_j} \right) \Pr \left( L_{u_{w_1}} \ldots L_{u_{w_n}} \right),$$

where

$$F^+(G : P_V(a)) = (\cup_{k=1}^N loop_{v_k}(G : a)) \cup (V \cap V(G : a)).$$

By the Möbius inversion, we can get the $n$-th cumulants of $P_V(a)$.

**Theorem 3.4.** Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable and let $V = \{v_1, \ldots, v_N\}$ be the fixed subset of $V(G)$. Then the diagonal compressed random variable of $a$ by $V$, $P_V(a)$ has the $n$-th cumulants are

$$k_1 \left( d_1 P_V(a) \right) = \sum_{v \in V \cap (V(G;d_1) \cap V(G:a))} (q_{v} p_{v}) L_{v}$$

and

$$k_n \left( \underbrace{d_1 P_V(a), \ldots, d_n P_V(a)}_{n \text{-times}} \right)$$

$$= \sum_{(v^{(1)}, \ldots, v^{(n)}) \in \prod_{k=1}^n V(G;d_k)} \left(\prod_{j=1}^n q_{v^{(j)}} \right)$$

$$= \left(\prod_{j=1}^n q_{v^{(j)}} \right).$$
Proof. Let $k=1$.
\[
\sum_{(w_1, \ldots, w_n) \in \left( \bigcup_{k=1}^{n} \text{loop}_{v_k} (G : a) \right) \cup (V \cap V (G : a))} \left( \Pi_{j=1}^{n} P_{w_j} \right)
\]
\[
\left( \Pi_{k=1}^{n} \delta_{v(k), x_k} \right) \sum_{u_1 \ldots u_n} \mu_{u_1 \ldots u_n} = E \left( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \right),
\]
for all $n > 1$ in $\mathbb{N}$, where $d_k = \sum_{v^{(k)} \in \mathcal{V}(G : d_k)} q_{v^{(k)}} L_{v^{(k)}} \in D_G$ are arbitrary for $k = 1, \ldots, n$.

Now, fix $n > 1$ in $\mathbb{N}$. By the Möbius inversion and by Section 1.3, we have that
\[
k_n (d_1 P_V (a), \ldots, d_n P_V (a))
\]
\[
= \sum_{(v^{(1)}, \ldots, v^{(n)}) \in \Pi_{k=1}^{n} \mathcal{V}(G : d_k)} \left( \prod_{j=1}^{n} q_{v^{(j)}} \right) \left( \prod_{j=1}^{n} p_{w_j} \right) \left( \Pi_{j=1}^{n} \delta_{v^{(j)}, x_j} \right) \mu_{u_1 \ldots u_n} E \left( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \right),
\]
where
\[
\mathcal{F}^+ (G : P_V (a)) = \left( \bigcup_{k=1}^{n} \text{loop}_{v_k} (G : a) \right) \cup (V \cap V (G : a)).
\]

Notice that in the set $\bigcup_{k=1}^{n} \text{loop}_{v_k} (G : a)$, it is possible that there are subsets $\text{loop}_{v_i}^i (G : a)$, for $i \neq j$ in $\{1, \ldots, N\}$.

Now, assume that the fixed subset $V = \{v_1, \ldots, v_N\} \subset V (G)$ satisfies that, for any choice $(v_i, v_j) \in V^2$ with $i \neq j$, there is no loop $l = v_i l v_i$ containing $v_j$ in the graph.
Lemma 3.5. Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable and let $a_{v_0} = L_{v_0} a_{L_{v_0}}$ be the vertex compressed random variable of $a$ by the fixed vertex $v_0 \in V(G)$. Then $d a_{v_0} = a_{v_0} d$, for all $d \in D_G$.

Proof. Let $a_{v_0} = L_{v_0} a_{L_{v_0}} \in (W^*(G), E)$ be the $v$-compressed random variable and let $d = \sum_{v \in V(G), a} q_v L_v \in D_G$ be arbitrary. We can express $a_{v_0}$ by

$$a_{v_0} = [p_{v_0} L_{v_0}] + \sum_{w \in loop_{v_0}(G : a), u_w \in \{1, *\}} p_w L_{u_w},$$

where the $D_G$-valued random variable $a$ has the form

$$a = \sum_{w \in F^+(G : a), u_w \in \{1, *\}} p_w L_{u_w} \in (W^*(G), E)$$

and where

$$[p_{v_0} L_{v_0}] = \begin{cases} p_{v_0} L_{v_0} & \text{if } v_0 \in V(G : a) \\ 0_{D_G} & \text{otherwise.} \end{cases}$$

Suppose that $v_0 \notin V(G : d)$. Then $d a_{v_0} = 0_{D_G} = a_{v_0} d$. Now assume that $v_0 \in V(G : d)$. Then

$$d a_{v_0} = (q_{v_0} L_{v_0} + D) a_{v_0} = q_{v_0} L_{v_0} a_{v_0} + D a_{v_0}$$

$$= q_{v_0} a_{v_0} + 0_{D_G}$$

$$= q_{v_0} a_{v_0} + a_{v_0} D = a_{v_0} (q_{v_0} L_{v_0} + D)$$

$$= a_{v_0} d,$$
for all \( d = q_{v_0}L_{v_0} + D \in D_G \) having its summand \( q_{v_0}L_{v_0} \), since \([p_{v_0}L_{v_0}]\) commutes with \( d \) and 
\[
\sum_{w \in \text{loop}_{v_0}(G:a), \ u_w \in \{1, *\}} p_wL_u^{uw} \text{ also commutes with } d.
\]

By the previous lemma, we can conclude that:

**Proposition 3.6.** Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( V = \{v_1, ..., v_N\} \) be a finite subset of the vertex set \( V(G) \). Assume that

\[
\text{loop}_{v_i}^v(G : a) = \emptyset,
\]

for any \( (v_i, v_j) \in V^2 \) such that \( i \neq j \) in \( \{1, ..., N\} \). Then the diagonal compressed random variable of \( a \) by \( V \), \( P_V(a) \) has \( n \)-th cumulants

\[
k_n (d_1P_V(a), ..., d_nP_V(a))
\]

\[
= \begin{cases} 
\sum_{j=1}^{N} (d_1d_2) \left( \sum_{l \in \text{loop}_{v_i}(G:a)} 2(p_lp_l)^2 \cdot L_{v_j} \right) & \text{if } n = 2, \\
0_{D_G} & \text{otherwise},
\end{cases}
\]

for all \( nN \), where \( d_k = \sum_{v^{(k)} \in V(G:d_k)} q_{v^{(k)}}L_{v^{(k)}} \in D_G \) are arbitrary, \( j = 1, ..., n \).

**Proof.** Fix \( n \in \mathbb{N} \) and denote \( L_{v_j}aL_{v_j} \) by \( a_j \), for \( j = 1, ..., N \). Then we have that

\[
FP^\ast_G(G : a_j) = \text{loop}^\ast_G(G : a_j)
\]

and

\[
FP^\ast_G(G : a_j) = \text{loop}^\ast_G(G : a_j),
\]

for all \( j = 1, ..., N \). By the assumption that \( \text{loop}^v_{v_i}(G) = \emptyset \), for all \( i \neq j \) in \( \{1, ..., N\} \), we can conclude that

\[
FP^\ast_G(G : a_i) \cap FP^\ast_G(G : a_j) = \emptyset
\]

and

\[
FP^\ast_G(G : a_i) \cap FP^\ast_G(G : a_j) = \emptyset,
\]

for any \( i \neq j \) in \( \{1, ..., N\} \). So, \( W^\ast_G\{a_i, a_j\} = \emptyset \), by the above second intersection. This shows that the \( D_G \)-valued random variables \( a_1, ..., a_N \) are free from each other over \( D_G \), in \( (W^*(G), E) \). Thus our diagonal compressed random variable \( P_V(a) \) by \( V \) is the \( D_G \)-free sum of \( a_1, ..., a_N \). Then, for arbitrary \( d_1, ..., d_n \in D_G \),

\[
k_n (d_1P_V(a), ..., d_nP_V(a))
\]

\[
= k_n (d_1(a_1 + ... + a_N), ..., d_n(a_1 + ... + a_n))
\]
\[= \sum_{j=1}^{N} k_n (d_1 a_j, \ldots, d_n a_j)\]

by the mutual $D_G$-freeness of $a_1, \ldots, a_N$ in $(W^*(G), E)$

\[= \sum_{j=1}^{N} (d_1 \ldots d_n) \sum_{l \in \text{loop}^*_{(G:a)}} k_n (p_l L_l + p_l L_l^*, \ldots, p_l L_l + p_l L_l^*)\]

by the previous lemma and by the bimodule map property of the $D_G$-valued cumulant

\[= \sum_{j=1}^{N} (d_1 \ldots d_n) \left( \sum_{l \in \text{loop}^*_{(G:a)}} k_n (p_l L_l + p_l L_l^*, \ldots, p_l L_l + p_l L_l^*) \right)\]

by Section 3.2

\[= \sum_{j=1}^{N} (d_1 \ldots d_n) \sum_{l \in \text{loop}^*_{(G:a)}} (p_l p_l^*)^n \cdot \left( \sum_{(u_1, \ldots, u_n) \in \{1, *\}^n} k_n (L_l^{u_1}, \ldots, L_l^{u_n}) \right)\]

\[= \sum_{j=1}^{N} (d_1 \ldots d_n) \sum_{l \in \text{loop}^*_{(G:a)}} (p_l p_l^*)^n \cdot \sum_{L \in L_P^*} \mu_{L^1, \ldots, l} E (L_l^{u_1} \ldots L_l^{u_n})\]

\[= \sum_{j=1}^{N} (d_1 \ldots d_n) \left( \sum_{l \in \text{loop}^*_{(G:a)}} (p_l p_l^*)^n \cdot \sum_{L \in L_P^*} \mu_{L^1, \ldots, l} L_{v_j} \right)\]

since $E(L_l^{u_1} \ldots L_l^{u_n}) = L_{v_j}$, for the suitable $(u_1, \ldots, u_n) \in \{1, *\}^n$

\[
\begin{cases} 
\sum_{j=1}^{2} (d_1 d_2) \left( \sum_{l \in \text{loop}^*_{(G:a)}} 2(p_l p_l^*)^2 \cdot L_{v_j} \right) & \text{if } n = 2 \\
0_{D_G} & \text{otherwise},
\end{cases}
\]

by the $D_G$-semicircularity of $a_j$'s $(j = 1, \ldots, N)$. \(\blacksquare\)
3.2. Diagonal Compressed Freeness on \((W^*(G), E)\).

In this section, we will consider the \(D_G\)-freeness of two diagonal compressed random variables. Likewise the vertex compressed case, we can get that:

**Proposition 3.7.** Let \(V = \{v_1, ..., v_N\}\) be a finite subset of the vertex set \(V(G)\). Let \(a, b \in (W^*(G), E)\) be \(D_G\)-valued random variables. If \(a\) and \(b\) are free over \(D_G\), in \((W^*(G), E)\), then the corresponding diagonal compressed random variables \(P_V(a)\) and \(P_V(b)\) of \(a\) and \(b\) by \(V\) are free over \(D_G\), in \((W^*(G), E)\). □

Now, we will consider the two diagonal compressions \(P_{V_1}\) and \(P_{V_2}\) of \(D_G\)-valued random variables in the graph \(W^*\)-probability space \((W^*(G), E)\) over the diagonal subalgebra \(D_G\). The \(D_G\)-freeness of two \(D_G\)-valued random variables \(P_{V_1}(a)\) and \(P_{V_2}(a)\) is determined as follows:

**Proposition 3.8.** Let \(V_1 = \{v_1^{(1)}, ..., v_{N_1}^{(1)}\}\) and \(V_2 = \{v_1^{(2)}, ..., v_{N_2}^{(2)}\}\) be finite subsets of the vertex set \(V(G)\), where \(N_1, N_2 \in \mathbb{N}\). Let \(a \in (W^*(G), E)\) be an arbitrary \(D_G\)-valued random variable and let \(P_{V_1}(a)\) and \(P_{V_2}(a)\) be the corresponding diagonal compressed random variables by \(V_1\) and \(V_2\), respectively. If \(V_1\) and \(V_2\) satisfy that

\[
V_1 \cap V_2 = \emptyset
\]

and

\[
\text{loop}_{v_i^{(1)}}(G : a) \cap \text{loop}_{v_j^{(2)}}(G : a) = \emptyset,
\]

for all choices \((i, j) \in \{1, ..., N_1\} \times \{1, ..., N_2\}\), then \(P_{V_1}(a)\) and \(P_{V_2}(a)\) are free over \(D_G\), in \((W^*(G), E)\).

**Proof.** Let \(a = \sum_{w \in \mathcal{F}^+(G:a), u,w \in \{1, \ast\}} p_w L_w^a\) be a \(D_G\)-valued random variable in \((W^*(G), E)\) and let \(P_{V_1}\) and \(P_{V_2}\) be the diagonal compressions by \(V_1\) and \(V_2\), respectively. Suppose that \(V_1 \cap V_2 = \emptyset\) and assume that

\[
\text{loop}_{v_i^{(1)}}(G : a) \cap \text{loop}_{v_j^{(2)}}(G : a) = \emptyset,
\]

for all pairs \((i, j) \in \{1, ..., N_1\} \times \{1, ..., N_2\}\). Then \(l_1\) and \(l_2\) are diagonal-distinct, for all \((i, j) \in \{1, ..., N_1\} \times \{1, ..., N_2\}\). Thus we have that loops \(l\) and \(l'\) are diagram-distinct, for all \((l, l') \in \text{loop}(G : P_{V_1}(a)) \times \text{loop}(G : P_{V_2}(a))\). By the definition of diagonal compression, we also have that

\[
\text{loop}^c(G : P_{V_1}(a)) \cap \text{loop}^c(G : P_{V_2}(a)) = \emptyset \cap \emptyset = \emptyset.
\]
Therefore, two $D_G$-valued random variables $P_V^1(a)$ and $P_V^2(a)$ are free over $D_G$ in $(W^*(G), E)$. 

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