SUPER-EXPONENTIAL CONDENSATION WITHOUT EXACT OVERLAPS

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Abstract. We exhibit self-similar sets on the line which are not exponentially separated and do not generate any exact overlaps. Our result shows that Hochman’s theorem for the dimension of self-similar sets on the line, which currently is the best result in this space, is still too weak to describe the full theory.

1. Introduction

A self-similar set consists of similar copies of itself. A classical result of Hutchinson [8] shows that if these copies are separated enough, then the Hausdorff dimension of the self-similar set equals the similitude dimension, a natural upper bound for the dimension. In order to handle overlaps, Simon and Pollicott [9] introduced the transversality condition. Simon and Solomyak [13] used this condition to show that in the line, for almost every choice of translations, the dimension of the self-similar set equals the similitude dimension.

In his seminal paper, Hochman [5] strengthened the estimates on the exceptional parameters for which the dimension drops below the similitude dimension. He showed that exponential separation suffices for the equality of the Hausdorff and similitude dimensions. While the transversality argument can only estimate the measure of the parameters, Hochman proved that the packing dimension of the exceptional set is zero. Furthermore, Shmerkin and Solomyak [11] used similar techniques and conditions to study the absolute continuity of self-similar measures, and Shmerkin [10] applied this approach to study the $L^q$-spectrum of self-similar measures.

A folklore conjecture proposes that the only possibility for the Hausdorff dimension to be strictly less than the similitude dimension is the existence of exact overlaps. Varjú [14] studied the dimension of Bernoulli convolutions, which is a certain class of self-similar measures. He proved that there is no dimension drop if the contraction parameter is transcendental. In particular, this means that the conjecture holds for Bernoulli convolutions. Hochman showed that the dimension drop implies the super-exponential condensation. In [6], he asked if the super-exponential condensation further implies the exact overlapping. We answer this question in negative by constructing uncountably many parametrized homogeneous self-similar sets having super-exponential condensation but no exact overlaps. Very recently, independently of us, Baker [1] showed the existence of such a self-similar set. In fact, after his result appeared online, we decided to make our considerations public as well. While Baker applied the theory of continued fractions, our proof relies on non-linear projections and the transversality condition.

The observation that the super-exponential condensation does not imply the exact overlapping means that, in order to verify or disprove the conjecture, one has to study the overlaps in a more sophisticated way. By applying Hochman [5], we characterize the dimension drop of the natural measure on a homogeneous self-similar set by means of the average exponential separation. Our results, therefore, introduce a possible roadmap to disprove the conjecture.

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2. Preliminaries and main results

We consider a tuple $\Phi = (\varphi_i)_{i \in I}$, where $I$ is a finite index set, of contracting similitudes acting on $\mathbb{R}^d$. Each of the map $\varphi_i$ has the form $\varphi_i(x) = \lambda_i O_i x + t_i$, where $0 < \lambda_i < 1$ is the contraction, $O_i$ the orthogonal part, and $t_i \in \mathbb{R}^d$ the translation of $\varphi_i$. We say that $\Phi$ is homogeneous if there exists $0 < \lambda < 1$ such that $\lambda_i = \lambda$ for all $i \in I$. A self-similar set associated to $\Phi$ is the unique non-empty compact set $X \subset \mathbb{R}^d$ for which

$$X = \bigcup_{i \in I} \varphi_i(X). \quad (2.1)$$

The existence and uniqueness of such sets was proved by Hutchinson [8]. The self-similar set $X$ is homogeneous if it is given by a homogeneous tuple. Writing $\varphi_i = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$ and $\lambda_i = \lambda_{i_1} \cdots \lambda_{i_n}$, we have $\text{diam}(\varphi_1(B)) = \lambda_1 \text{diam}(B) \leq (\max_{i \in I} \lambda_i)^n \text{diam}(B)$ for all sequences $i = i_1 \cdots i_n \in I^n$ and sets $B \subset \mathbb{R}^d$. Therefore, defining $1_{1n} = i_1 \cdots i_n$ for all $i = i_1i_2 \cdots \in I^n$, we see that $\text{diam}(\varphi_{1n}(B)) \to 0$ as $n \to \infty$ for all $i \in I^n$ and bounded sets $B \subset \mathbb{R}^d$. Each $i \in I^n$ corresponds to one point in $X$ via the canonical projection $\pi$ defined by the relation

$$\{\pi(i)\} = \{\lim_{n \to \infty} \varphi_{1n}(0)\} = \bigcap_{n=1}^{\infty} \varphi_{1n}(B(0, R)),$$

where $X \cup \varphi_i(B(0, R)) \subset B(0, R)$ for $R = \max_{i \in I} |\varphi_i(0)|/(1 - \max_{i \in I} \lambda_i)$. In fact, it is easy to see that $\pi(I^n) = X$ and hence, the canonical projection introduces an alternative way to define the self-similar set. By iterating (2.1), we see that $X = \bigcup_{i \in I^n} \varphi_i(X)$ for all $n \in \mathbb{N}$. Therefore, the family $\{\varphi_1(B(0, R))\}_{i \in I^n}$ consisting of balls as small as we wish is a natural cover for $X$. It is easy to see that $\dim_H(X) \leq \dim_{\text{sim}}(\Phi)$, where $\dim_H$ is the Hausdorff dimension and the similitude dimension $\dim_{\text{sim}}(\Phi)$ is the unique number $s \geq 0$ for which $\lim_{n \to \infty} (\sum_{i \in I^n} \lambda_i^s R^n)^{1/n} = \sum_{i \in I} \lambda_i^s = 1$.

It is well known that if the strong separation condition is satisfied, which means that $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ whenever $i \neq j$, then $\dim_H(X) = \dim_{\text{sim}}(\Phi)$. The strong separation condition can be relaxed to a slightly weaker assumption, called the open set condition, which, roughly speaking, means that the overlapping of the sets $\varphi_i(X)$ of essentially the same diameter has bounded multiplicity. It has to be emphasized that the open set condition only allows “slight overlaps”. For example, if $X$ has exact overlaps, meaning that there are finite sequences $i \neq j$ such that $\varphi_i = \varphi_j$, then $\dim_H(X) < \dim_{\text{sim}}(\Phi)$. Indeed, by denoting the length of $i$ by $|i|$ and the concatenation of $i$ and $j$ by $ij$, we may, by replacing $i$ and $j$ by $ij$ and $ji$, assume that the finite sequences $i$ and $j$ have the same length $|i| = |j| = n$. Therefore, $\dim_H(X) \leq \dim_{\text{sim}}(\Phi^n) < \dim_{\text{sim}}(\Phi)$, where $\Phi^n$ is the tuple consisting of $(\#I)^n - 1$ many $n$-length compositions of the maps $\varphi_i$ – all of them but $\varphi_j$. Currently, for self-similar sets in the real line, no other mechanism is known which drops the dimension of $X$ below the similitude dimension. The following folklore conjecture has probably first time been stated by Simon [12].

**Dimension drop conjecture.** If $\Phi$ is a tuple of contractive similitudes acting on the real line and $X \subset \mathbb{R}$ is the associated self-similar set such that $\dim_H(X) < \min\{1, \dim_{\text{sim}}(\Phi)\}$, then $X$ has exact overlaps.

There exist a version of the conjecture also in higher dimensions, see Hochman [7, Conjecture 1.3], but from now on, unless otherwise stated, we work only on the real line. In this case, the orthogonal part of the maps is just a multiplication by 1 or $-1$ and therefore, we include it in the contraction. In our first result, we characterize the dimension drop of the natural measure on homogeneous self-similar sets $X \subset \mathbb{R}$ by means of the average exponential condensation defined by

$$\Lambda(\gamma) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i \in I^n} \frac{1}{\#I^n} \log \#\{j \in I^n : |\varphi_i(0) - \varphi_j(0)| \leq \gamma^n\}.$$
for all $\gamma > 0$. The natural measure is the Borel probability measure $\mu$ on $X$ satisfying

$$\mu = \frac{1}{\# I} \sum_{i \in I} \mu \circ \varphi_i^{-1}.$$ 

Recall that the (lower) Hausdorff dimension of $\mu$ is

$$\dim_H(\mu) = \inf \{ \dim_H(A) : A \text{ is a Borel set such that } \mu(A) > 0 \}.$$ 

If $\lambda$ is the common contraction ratio of the maps $\varphi_i$, then $\dim_H(\mu) \leq \dim_H(X) \leq \dim_{\text{sim}}(\Phi) = - \log \# I / \log |\lambda|$ regardless of the translations.

**Proposition 2.1.** If $\Phi = (\varphi_i)_{i \in I}$ is a homogeneous tuple of contractive similitudes acting on the real line such that $0 < |\lambda| < 1 / \# I$ is the common contraction ratio of the maps $\varphi_i$, $X \subset \mathbb{R}$ is the associated self-similar set, and $\mu$ is the natural measure on $X$, then

$$\dim_H(\mu) = \dim_{\text{sim}}(\Phi) - \frac{\Lambda(\gamma)}{\log |\lambda| - 1}$$

for all $0 < \gamma \leq |\lambda|$. Furthermore, the limit inferior in the definition of $\Lambda(\gamma)$ is a limit and the value of $\Lambda(\gamma)$ does not depend on the choice of $0 < \gamma \leq |\lambda|$.

The quantity

$$\Delta_n = \min \{ |\varphi_1(0) - \varphi_j(0)| : i, j \in I^n \text{ such that } i \neq j \text{ and } \lambda_1 = \lambda_j \}$$

is zero for arbitrary large $n$ if and only if there is an exact overlap. It is also easy to see that $\Delta_n \to 0$ at least exponentially for every $\Phi$. We say that $\Phi$ is exponentially separated if there is $c > 0$ such that $\Delta_n \geq c^n$ for arbitrary large $n$. It is straightforward to see that if a homogeneous $\Phi$ is exponentially separated, then it has no average exponential condensation: for any $0 < \gamma < c$ it holds that $\Lambda(\gamma) = 0$. Therefore, by Proposition 2.1, the associated homogeneous self-similar set has no dimension drop.

More generally, Hochman [5 Corollary 1.2] has shown that if $\dim_H(X) < \min\{1, \dim_{\text{sim}}(\Phi)\}$, then there is super-exponential condensation, which means that $\Delta_n \to 0$ super-exponentially, $\lim_{n \to \infty} \frac{1}{n} \log \Delta_n = -\infty$. In other words, if $\Phi$ is exponentially separated, then there is no dimension drop. In particular, if $\Phi$ is defined by using algebraic parameters and there are no exact overlaps, then $\Phi$ is exponentially separated; see Hochman [5] proof of Theorem 1.5]. Therefore, the dimension drop conjecture holds for all $\Phi$ defined by using algebraic parameters.

In our main result, we show that Hochman’s theorem, as stated, is still too weak to address the full conjecture. Let us define parametrized maps $\varphi^{\lambda,t}_i : \mathbb{R} \to \mathbb{R}$ for $i \in \{1, 2, 3\}$ by

$$\varphi^{\lambda,t}_1(x) = \lambda x, \quad \varphi^{\lambda,t}_2(x) = \lambda x + t, \quad \varphi^{\lambda,t}_3(x) = \lambda x + 1$$

for all $0 < \lambda < \frac{1}{3}$ and $0 < t < \lambda/(1 - \lambda)$. Write $\Phi_{\lambda,t} = (\varphi^{\lambda,t}_1, \varphi^{\lambda,t}_2, \varphi^{\lambda,t}_3)$ and let $X_{\lambda,t}$ be the associated self-similar set. Note that the restriction $t < \lambda/(1 - \lambda) < (1 - 2\lambda)/(1 - \lambda)$ guarantees that $\varphi^{\lambda,t}_1(\text{conv}(X_{\lambda,t})) \cap \varphi^{\lambda,t}_2(\text{conv}(X_{\lambda,t})) \neq \emptyset$ and $\varphi^{\lambda,t}_2(\text{conv}(X_{\lambda,t})) \cap \varphi^{\lambda,t}_3(\text{conv}(X_{\lambda,t})) = \emptyset$, where $\text{conv}(A)$ is the convex hull of a given set $A$. Let us define three planar sets specific to this setting. The exact overlapping set is

$$\mathcal{E} = \{ (\lambda, t) : \varphi^{\lambda,t}_i = \varphi^{\lambda,t}_j \text{ for some finite sequences } i \neq j \},$$

the dimension drop set is

$$\mathcal{D} = \{ (\lambda, t) : \dim_H(X_{\lambda,t}) < \dim_{\text{sim}}(\Phi_{\lambda,t}) = -\log 3 / \log \lambda \}$$

and the super-exponential condensation set is

$$\mathcal{C} = \{ (\lambda, t) : \lim_{n \to \infty} \frac{1}{n} \log \Delta_n^{\lambda,t} = -\infty \},$$

where $\Delta_n^{\lambda,t} = \min \{ |\varphi^{\lambda,t}_1(0) - \varphi^{\lambda,t}_j(0)| : i, j \in \{1, 2, 3\}^n \text{ such that } i \neq j \}$. As discussed above, we trivially have $\mathcal{E} \subset \mathcal{D}$ and, by Proposition 2.1, $\mathcal{D} \subset \mathcal{C}$. Furthermore, by Hochman [7 Theorem 1.10], we have $\dim_H(\mathcal{E}) = \dim_p(\mathcal{C}) = 1$, where $\dim_p$ is the packing dimension. For the parametrized
tuple $\Phi_{\lambda,t}$, the dimension drop conjecture is equivalent to $D \setminus E = \emptyset$. The following result shows that there exist self-similar sets having super-exponential condensation without exact overlaps. It also answers a question of Hochman [6] in negative.

**Theorem 2.2.** For the parametrized tuple $\Phi_{\lambda,t}$ defined above, the set $C \setminus E$ is uncountable.

In Remark 4.8 we explain how the proof can be modified to show that there exist uncountably many self-similar sets having no exact overlaps but $\Delta_n$ converging to zero arbitrary fast. Finally, Proposition 2.1 and Theorem 2.2 introduce a possible way to disprove the dimension drop conjecture: If there exist $(\lambda, t) \in C \setminus E$ and $0 < \gamma \leq \lambda$ such that

$$\Lambda^{\lambda,t}(\gamma) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i \in I^n} \frac{1}{\# I^n} \log \# \{ j \in I^n : |\varphi_i^{\lambda,t}(0) - \varphi_j^{\lambda,t}(0)| \leq \gamma^n \} > 0,$$

then the dimension of the natural measure drops even though there are no exact overlaps.

3. AVERAGE EXPONENTIAL CONDENSATION

In this section, we prove Proposition 2.1. We remark that the proof strongly relies on exact dimensionality proven by Feng and Hu [3] and the behavior of the Shannon entropy described by Hochman [5]. Recall that a Borel probability measure $\nu$ on $\mathbb{R}$ is exact-dimensional if the lower/upper Hausdorff/packing dimensions of $\nu$ coincide. We refer to the book of Falconer [2] for more details on dimensions of measures. Furthermore, the Shannon entropy of $\nu$ with respect to the partition $D_n = \{ [i2^{-n}, (i+1)2^{-n}) \}_{i \in \mathbb{Z}}$ is

$$H(\nu, D_n) = - \sum_{D \in D_n} \nu(D) \log \nu(D)$$

for all $n \in \mathbb{N}$.

**Proof of Proposition 2.1.** By Feng and Hu [3, Theorem 2.8], the natural measure $\mu$ is exact-dimensional. Therefore, by Heurteaux [4, Theorem 4.1], it has dimension

$$\dim_H(\mu) = \lim_{n \to \infty} \frac{1}{n \log 2} H(\mu, D_n).$$

Define

$$\mu^n = \frac{1}{\# I^n} \sum_{i \in I^n} \delta_{\varphi_i(0)},$$

where $\delta_x$ is the Dirac measure at $x$, and let $r(n)$ be the unique integer such that $|\lambda|^{r(n)} \text{diam}(X) \leq 2^{-n} < |\lambda|^{r(n)-1} \text{diam}(X)$. By Hochman [5, Theorem 1.3], we have

$$\lim_{n \to \infty} \frac{1}{n \log 2} \left( H(\mu^{r(n)}, D_q n) - H(\mu^{r(n)}, D_n) \right) = 0$$

for all $q \in \mathbb{N}$.

Our goal is to show that a closer examination of the Shannon entropy with respect to the partition $D_q n$ leads us to the claimed formula. Indeed, we shall show that

$$\dim_H(\mu) = \lim_{n \to \infty} \frac{1}{n \log 2} H(\mu, D_n) = \lim_{n \to \infty} \frac{1}{n \log 2} H(\mu^{r(n)}, D_n) = \lim_{n \to \infty} \frac{1}{n \log 2} H(\mu^{r(n)}, D_q n)$$

$$= - \lim_{n \to \infty} \frac{1}{n \log 2} \int \log \mu^{r(n)}(B(x, 2^{-qn})) \, d\mu^{r(n)}(x) = \frac{\log \# I - \Lambda(|\lambda|^q)}{\log |\lambda|^{-1}}$$

for all $q \in \mathbb{N}$. Note that the first and third equality follow from the above mentioned results.
Observe first that we have
\[
H(\mu^{r(n)}, D_{qn}) = - \sum_{D \in D_{qn}} \mu^{r(n)}(D) \log \mu^{r(n)}(D) \geq - \int \log \mu^{r(n)}(B(x, 2^{-qn})) \, d\mu^{r(n)}(x)
\]
\[
\geq - \sum_{D \in D_{qn}} \mu^{r(n)}(D) \log(\mu^{r(n)}(D - 2^{-qn}) + \mu^{r(n)}(D) + \mu^{r(n)}(D + 2^{-qn}))
\]
\[
= - \sum_{D \in D_{qn}} \mu^{r(n)}(D) \left( \log \mu^{r(n)}(D) + \log \left( 1 + \frac{\mu^{r(n)}(D - 2^{-qn}) + \mu^{r(n)}(D + 2^{-qn})}{\mu^{r(n)}(D)} \right) \right)
\]
\[
\geq H(\mu^{r(n)}, D_{qn}) - \sum_{D \in D_{qn}} \mu^{r(n)}(D - 2^{-qn}) + \mu^{r(n)}(D + 2^{-qn})
\]
\[
\geq H(\mu^{r(n)}, D_{qn}) - 2.
\]
A similar reasoning shows that
\[
|H(\mu, D_n) - H(\mu^{r(n)}, D_{qn})| \leq 9.
\]
Indeed, this follows since
\[
\mu(D) \leq \frac{1}{\#I^{r(n)}} \#\{i \in I^{r(n)} : \varphi_1(X) \cap D \neq \emptyset\}
\]
\[
\leq \frac{1}{\#I^{r(n)}} \#\{i \in I^{r(n)} : \varphi_1(0) \in (D - 2^{-n}) \cup D \cup (D + 2^{-n})\}
\]
\[
= \mu^{r(n)}(D - 2^{-n}) + \mu^{r(n)}(D) + \mu^{r(n)}(D + 2^{-n})
\]
and \(\mu^{r(n)}(D) \leq \mu(D - 2^{-n}) + \mu(D) + \mu(D + 2^{-n})\) for all \(D \in D_n\). Finally, we also have
\[
- \int \log \mu^{r(n)}(B(x, 2^{-qn})) \, d\mu^{r(n)}(x) = - \sum_{i \in I^{r(n)}} \frac{1}{\#I^{r(n)}} \log \mu^{r(n)}(B(\varphi_1(0), 2^{-qn}))
\]
\[
= - \sum_{i \in I^{r(n)}} \frac{1}{\#I^{r(n)}} \log \#\{j \in I^{r(n)} : |\varphi_1(0) - \varphi_j(0)| \leq 2^{-qn}\}
\]
\[
= \log \#I^{r(n)} - \sum_{i \in I^{r(n)}} \frac{1}{\#I^{r(n)}} \log \#\{j \in I^{r(n)} : |\varphi_1(0) - \varphi_j(0)| \leq 2^{-qn}\}
\]
which finishes the proof. \(\square\)

4. **Super-exponential condensation**

In this section, we prove Theorem 4.2. Let us first observe that, for the parametrized tuple \(\Phi_{\lambda,t}\),
the canonical projection \(\pi_{\lambda,t} : \{1, 2, 3\}^N \to X_{\lambda,t}\) satisfies
\[
\pi_{\lambda,t}(i) = \sum_{k=1}^{\infty} (\delta^3_{ik} + t\delta^2_{ik}) \lambda^{k-1}
\]
for all \(i = i_1i_2 \cdots \in \{1, 2, 3\}^N\), where
\[
\delta^i_j = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\]
Note that \(\varphi^{\lambda,t}_1(0) = \pi_{\lambda,t}(i^{1\infty}) = \sum_{k=1}^{n} (\delta^3_{ik} + t\delta^2_{ik}) \lambda^{k-1}\) for all \(i \in \{1, 2, 3\}^n\) and \(n \in \mathbb{N}\), where \(1^{\infty}\) is the infinite sequence containing only 1's.
Lemma 4.1. Let \( n \in \mathbb{N} \), \( i = i_1 \cdots i_n, j = j_1 \cdots j_n \in \{1, 2, 3\}^n \) be such that \( i_1 \neq j_1 \), and \( 0 < \varepsilon < \frac{1}{2} \). Then

\[
|\varphi^\lambda_{i,j}(0) - \varphi^\lambda_{j,j}(0)| < \varepsilon \quad \implies \quad |t - \frac{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}| < 2\varepsilon
\]

and

\[
|t - \frac{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}| < \varepsilon \quad \implies \quad |\varphi^\lambda_{i,j}(0) - \varphi^\lambda_{j,j}(0)| < 2\varepsilon.
\]

Proof. Since

\[
\varphi^\lambda_{i,j}(0) - \varphi^\lambda_{j,j}(0) = \sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1} + t \sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1},
\]

we see that both claims follow if we can show that

\[
\frac{1}{2} \leq \frac{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}} \leq 2.
\]

The lower bound is needed in the first claim and the upper bound in the second claim. To show the lower bound, we may thus assume that \( |\varphi^\lambda_{i,j}(0) - \varphi^\lambda_{j,j}(0)| < \frac{1}{2} \). Since \( 0 < \lambda < \frac{1}{2} \), this is possible only if \( i_1 = 2 \) and \( j_1 = 1 \) or vice versa. Therefore, it follows that \( |\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}| \geq 1 - \sum_{k=1}^{\infty}\lambda^k = (1 - 2\lambda)/(1 - \lambda) > \frac{1}{2} \) as claimed. The upper bound is trivial since \( |\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}| \leq \sum_{k=0}^{\infty}\lambda^k = (1 - \lambda)^{-1} < \frac{3}{2} \).

Lemma 4.1 tells us that in order to achieve super-exponential condensation, the parameter \( t \) must be contained in a super-exponential neighbourhood of a ratio of the form

\[
\frac{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}{\sum_{k=1}^{n}(\delta^2_{i_k} - \delta^2_{j_k})\lambda^{k-1}}.
\]

We shall show that such ratios are certain non-linear projections of an induced self-similar set in the plane.

Let \( J = \{(0,0), (-1,0), (-1,-1), (1,0), (1,1), (0,-1), (0,1)\} \) and define \( S^\lambda_{(i,j)} : \mathbb{R}^2 \to \mathbb{R}^2 \) by setting

\[
S^\lambda_{(i,j)}(x, y) = (\lambda x + i, \lambda y + j)
\]

for all \((x, y) \in \mathbb{R}^2 \) and \((i, j) \in J \). Write \( \Psi^\lambda = (S^\lambda_{(i,j)})_{(i,j) \in J} \) and let \( K^\lambda \subset \mathbb{R}^2 \) be the associated self-similar set; see Figure 1 for an illustration. The map \( \beta : \{1, 2, 3\} \times \{1, 2, 3\} \to J \) defined by setting \( \beta(i, j) = (\delta^3 - \delta^3, \delta^3 - \delta^3) \) is clearly one-to-one outside the diagonal. We extend the map \( \beta \) to \( \{1, 2, 3\}^n \times \{1, 2, 3\}^n \to J^n \) for all \( n \in \mathbb{N} \) and to \( \{1, 2, 3\}^N \times \{1, 2, 3\}^N \to J^N \) in a natural way: for example, if \( i = i_1 \cdots i_n \) and \( j = j_1 \cdots j_n \) for some \( n \in \mathbb{N} \), then \( \beta(i, j) \) is defined to be \( \beta(i_1, j_1) \cdots \beta(i_n, j_n) \). Finally, we define a non-linear projection \( \text{proj} : \{(x, y) \in \mathbb{R}^2 : y \neq 0\} \to \mathbb{R} \) by setting

\[
\text{proj}(x, y) = \frac{x}{y}
\]

for all \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \setminus \{0\} \). The following lemma basically restates Lemma 4.1 in terms of the projection. If \( i = i_1 \cdots i_n \) and \( j = j_1 \cdots j_n \) are finite sequences, then we write \( i \wedge j \) for their common beginning and \( \sigma(i) = i_2 \cdots i_n \).

Lemma 4.2. Let \( n \in \mathbb{N} \), \( i, j \in \{1, 2, 3\}^n \) be such that \( i \neq j \), and \( 0 < \varepsilon < \frac{1}{2}\lambda^{(|i| + |j|)} \). Then

\[
|\varphi^\lambda_{i,j}(0) - \varphi^\lambda_{j,j}(0)| < \varepsilon \quad \implies \quad |t - \text{proj}(S^\lambda_{\beta(i,j)}(0, 0))| < 2\lambda^{-|i| + |j|}\varepsilon
\]

and

\[
|t - \text{proj}(S^\lambda_{\beta(i,j)}(0, 0))| < \lambda^{-|i| + |j|}\varepsilon \quad \implies \quad |\varphi^\lambda_{i,j}(0) - \varphi^\lambda_{j,j}(0)| < 2\varepsilon.
\]
Figure 1. Illustration for the self-similar set $K_\lambda$ and the convex hull $P_\lambda$. 

Proof. Note that $\text{proj}(S_\lambda^{\beta (i, j)}(0, 0)) = \text{proj}(S_\lambda^{\beta (i, j)}(1, 1, 0, 0))$ whenever $i \neq j$. Therefore, as $|\varphi_1^{\lambda t}(0) - \varphi_2^{\lambda t}(0)| = \lambda |\varphi_1^{\lambda t}(i, j)(0) - \varphi_2^{\lambda t}(i, j)(0)|$, the proof follows from Lemma 4.1. □

We have thus transformed the problem to a study of non-linear projections. Our first concrete goal now is to find finite words $k$ for which $\text{proj}(S_\lambda^k(K_\lambda))$ is an interval for a range of $\lambda$’s. After some preliminary lemmas we will achieve this in Lemma 4.5. Let

$P_\lambda = \text{conv}\left(\left\{\left(\frac{i}{1 - \lambda}, \frac{j}{1 - \lambda}\right)\right\}_{(i, j) \in J}\right)$

and note that $K_\lambda \subset P_\lambda$; see again Figure 1 for an illustration.

Lemma 4.3. If $(a, b) \in S_\lambda(0, 1)(P_\lambda) \cap [0, \infty) \times \mathbb{R}$ and $(0, 0)^n = (0, 0) \cdots (0, 0) \in J^n$ for all $n \in \mathbb{N}$, then

$\text{proj}(S_\lambda^k(0, 0)^n(P_\lambda) + (a, b)) = \left[\frac{a - \lambda^n}{b}, \frac{a + \lambda^n}{b}\right]$ for all $n \in \mathbb{N}$.

Proof. Since the line segment connecting $(a - \frac{\lambda^n}{1 - \lambda}, b)$ and $(a + \frac{\lambda^n}{1 - \lambda}, b)$ is contained in $S_\lambda^k(0, 0)^n(P_\lambda) + (a, b)$, the projection contains the claimed interval. Let us show that

$\frac{a - \lambda^n}{b} \leq \frac{x}{y}$

for all $(x, y) \in S_\lambda^k(0, 0)^n(P_\lambda) + (a, b)$. By differentiating, the function

$s \mapsto \frac{a - s \frac{\lambda^n}{1 - \lambda}}{b + (1 - s) \frac{\lambda^n}{1 - \lambda}}$
can easily be seen to be monotone on $[0, 1]$. It is therefore enough to check that the endpoints satisfy

$$\frac{a - \lambda^n}{b} \leq \frac{a}{b + \frac{\lambda^n}{1 - \lambda}}.$$ 

Since $0 < \lambda < \frac{1}{3}$, a simple calculation shows that this holds for every $n \in \mathbb{N}$.

Let us then show that

$$\frac{x}{y} \leq \frac{a + \frac{\lambda^n}{1 - \lambda}}{b}$$

for all $(x, y) \in S^{\lambda}_{(0, 0)^n}(P_{\lambda}) + (a, b)$. Similarly, since the function

$$s \mapsto \frac{a + s \frac{\lambda^n}{1 - \lambda}}{b - (1 - s) \frac{\lambda^n}{1 - \lambda}}$$

is monotone on $[0, 1]$, it suffices to check that

$$\frac{a}{b - \frac{\lambda^n}{1 - \lambda}} \leq \frac{a + \frac{\lambda^n}{1 - \lambda}}{b}.$$

A simple calculation shows that this holds if and only if $\frac{\lambda^n}{1 - \lambda} < \frac{1}{2}$ and for $(a, b) \in S_{(0, 1)}(P_{\lambda}) \cap [0, \infty) \times \mathbb{R}$, we have $\frac{1}{2} < \frac{1 - 2\lambda}{1 - \lambda} \leq b - a$. \hfill \Box

**Lemma 4.4.** If $(a, b) \in S^{\lambda}_{(0, 1)}(P_{\lambda}) \cap [0, \infty) \times \mathbb{R}$, $(0, 0)^n = (0, 0) \cdots (0, 0) \in J^n$ for all $n \in \mathbb{N}$, and $N \in \mathbb{N}$ is such that

$$\lambda^N \leq \min\left\{2a - \frac{1 - 3\lambda}{1 - \lambda} b, 2b - \frac{1 - \lambda}{\lambda} a\right\},$$

then

$$\text{proj}((a, b) + S^{\lambda}_{(0, 0)^n}(P_{\lambda})) = \bigcup_{(i, j) \in J} \text{proj}\left((a + \lambda^n i, b + \lambda^n j) + S^{\lambda}_{(0, 0)^{n+1}}(P_{\lambda})\right)$$

for all $n \geq N$.

**Proof.** By Lemma 4.3, it is enough to show that

$$\left[\frac{a - \lambda^n}{b}, \frac{a + \lambda^n}{b}\right] = \bigcup_{(i, j) \in J} \left[\frac{a + i\lambda^n - \frac{\lambda^{n+1}}{1 - \lambda}}{b + j\lambda^n}, \frac{a + i\lambda^n + \frac{\lambda^{n+1}}{1 - \lambda}}{b + j\lambda^n}\right].$$

In fact, it suffices to show that the consecutive intervals in the above union have non-empty intersection. The order of the intervals corresponds to the following order in $J$:

$$(-1, 0), (-1, -1), (0, 1), (0, 0), (0, -1), (1, 1), (1, 0).$$
Hence, we have to check that the following inequalities hold:

\begin{align}
\frac{a - \lambda^n - \frac{\lambda^{n+1}}{1 - \lambda}}{b - \lambda^n} & \leq \frac{a - \lambda^n + \frac{\lambda^{n+1}}{1 - \lambda}}{b}, \\
\frac{a - \frac{\lambda^{n+1}}{1 - \lambda}}{b + \lambda^n} & \leq \frac{a - \lambda^n + \frac{\lambda^{n+1}}{1 - \lambda}}{b - \lambda^n}, \\
\frac{a - \frac{\lambda^{n+1}}{1 - \lambda}}{b} & \leq \frac{a + \frac{\lambda^{n+1}}{1 - \lambda}}{b + \lambda^n}, \\
\frac{a + \lambda^n - \frac{\lambda^{n+1}}{1 - \lambda}}{b + \lambda^n} & \leq \frac{a + \lambda^n + \frac{\lambda^{n+1}}{1 - \lambda}}{b}, \\
\frac{a + \lambda^n - \frac{\lambda^{n+1}}{1 - \lambda}}{b} & \leq \frac{a + \lambda^n + \frac{\lambda^{n+1}}{1 - \lambda}}{b - \lambda^n}.
\end{align}

(4.1) \quad (4.2) \quad (4.3) \quad (4.4) \quad (4.5) \quad (4.6)

The inequality (4.1) holds if and only if \( a \leq \frac{2\lambda}{1 - \lambda} b + \lambda^n \frac{1 + \lambda}{1 - \lambda} \). Recalling that \( a \leq \frac{\lambda}{1 - \lambda} \) and \( b \geq \frac{1 - 2\lambda}{1 - \lambda} \), we see that this holds if \( \frac{1}{1 - \lambda} \leq \frac{2(1 - 2\lambda)}{(1 - \lambda)^2} \), which is true since \( 0 < \lambda < \frac{1}{3} \). The inequalities (4.2) and (4.3) hold if and only if \( 2a - \lambda^n \geq \frac{1 - 3\lambda}{1 - \lambda} b \), which is true by the assumption. The inequality (4.4) holds if and only if \( a \leq \frac{2\lambda}{1 - \lambda} b + \frac{\lambda^{n+1}}{1 - \lambda}, \) which can be seen to hold similarly as with the inequality (4.1). Finally, the inequalities (4.5) and (4.6) hold if and only if \( a \leq \frac{2\lambda}{1 - \lambda} b - \frac{\lambda^{n+1}}{1 - \lambda} \), which is true again by the assumption. \( \square \)

We are now ready to show that the projection of \( K_\lambda \) contains an interval for a range of \( \lambda \)'s.

**Lemma 4.5.** If \( \frac{1}{3} < \lambda < \frac{1}{2} \) and \( k \in \bigcup_{n=3}^{\infty} J^n \) satisfies \( k_3 = (0, 1)(1, 1)(0, -1) \) or \( k_3 = (0, 1)(1, 0)(0, 1) \), then \( \text{proj}(S_k^\lambda(K_\lambda)) \) is an interval.

**Proof.** Since \( \text{proj}(S_k^\lambda(K_\lambda)) \subset \text{proj}(S_k^\lambda(P_\lambda)) \), it is enough to show that

\[ \text{proj}(S_k^\lambda(P_\lambda)) = \bigcup_{(i,j) \in J} \text{proj}(S_k^\lambda(P_{i,j})). \]

(4.7)

for every finite sequence \( j \). Indeed, if \( (4.7) \) holds, then

\[
\text{proj}(S_k^\lambda(K_\lambda)) = \text{proj} \left( S_k^\lambda \left( \bigcap_{n=0}^{\infty} \bigcup_{J^n \setminus j} S_j(P_\lambda) \right) \right) = \bigcap_{n=0}^{\infty} \bigcup_{J^n} \text{proj}(S_k^\lambda(P_\lambda))
\]

\[
= \bigcap_{n=0}^{\infty} \text{proj}(S_k^\lambda(P_\lambda)) = \text{proj}(S_k^\lambda(P_\lambda)).
\]

To verify \( (4.7) \), it is enough to check whether the assumptions of Lemma 4.4 hold. Let \( (a, b) \) be the middle point of \( S_k^\lambda(P_\lambda) \). It is easy to see that then \( \lambda - \frac{\lambda^3}{1 - \lambda} \leq a \leq \lambda + \frac{\lambda^3}{1 - \lambda} \) and \( 1 + \lambda^2 - \frac{\lambda^3}{1 - \lambda} \leq b \leq 1 + \lambda - \lambda^2 + \frac{\lambda^3}{1 - \lambda} \). Hence, the inequality

\[
\lambda^3 \leq \min \left\{ 2 \left( \lambda - \frac{\lambda^3}{1 - \lambda} \right), \frac{1 - 3\lambda}{1 - \lambda} \left( 1 + \lambda - \lambda^2 + \frac{\lambda^3}{1 - \lambda} \right), \frac{2}{1 + \lambda^2 - \frac{\lambda^3}{1 - \lambda}} - \frac{1 - \lambda}{\lambda} \left( \lambda + \frac{\lambda^3}{1 - \lambda} \right) \right\}
\]

clearly implies the assumptions of Lemma 4.4. Numerical calculations show that the above inequality is valid for all \( \frac{1}{4} < \lambda < \frac{1}{3} \). \( \square \)
We will next show that the projection is transversal in this region of \( \lambda \)'s.

**Lemma 4.6.** There exists \( \delta > 0 \) such that for every \( \frac{1}{4} < \lambda_0 < \frac{1}{3} \) and \( k, l \in \bigcup_{n=5}^{\infty} J^n \) with \( \mathbf{k}|_{5} = (0,1)(1,1)(0,-1)(0,-1)(1,0) \) and \( \mathbf{l}|_{5} = (0,1)(1,0)(0,1)(0,1)(-1,0) \) we have
\[
\delta < \frac{d}{d\lambda}(\text{proj}(S^1_{k}(0,0)) - \text{proj}(S^1_{l}(0,0)))\big|_{\lambda = \lambda_0} < \delta^{-1}.
\]

**Proof.** The proof relies on numerical calculations. By our assumption on \( k \) and \( l \), we have
\[
\text{proj}(S^1_{k}(0,0)) - \text{proj}(S^1_{l}(0,0)) = \frac{\lambda + \lambda^3 + a(\lambda)}{1 + \lambda - \lambda^2 - \lambda^3 + b(\lambda)} - \frac{\lambda - \lambda^4 + c(\lambda)}{1 + \lambda^2 + \lambda^3 + d(\lambda)},
\]
where the functions \( a(\lambda) \), \( b(\lambda) \), \( c(\lambda) \), and \( d(\lambda) \) have the form \( \sum_{k=5}^{\infty} \delta_k \lambda^k \), where \( \delta_k \in \{ -1, 0, 1 \} \). Therefore, we see that
\[
\frac{d}{d\lambda}(\text{proj}(S^1_{k}(0,0)) - \text{proj}(S^1_{l}(0,0)))
= \frac{(b(\lambda) - \lambda^3 - \lambda^2 + \lambda + 1)(d(\lambda) + \lambda^3 + \lambda^2 + 1)^2(a'(\lambda) + 4\lambda^3)}{(1 + \lambda - \lambda^2 - \lambda^3 + b(\lambda))(1 + \lambda^2 + \lambda^3 + d(\lambda))^2}
- \frac{(a(\lambda) + \lambda^4 + \lambda)(d(\lambda) + \lambda^3 + \lambda^2 + 1)^2(b'(\lambda) - 3\lambda^2 - 2\lambda + 1)}{(1 + \lambda - \lambda^2 - \lambda^3 + b(\lambda))(1 + \lambda^2 + \lambda^3 + d(\lambda))^2}
- \frac{(b(\lambda) - \lambda^3 - \lambda^2 + \lambda + 1)^2(d(\lambda) + \lambda^3 + \lambda^2 + 1)(c'(\lambda) - 4\lambda^3)}{(1 + \lambda - \lambda^2 - \lambda^3 + b(\lambda))(1 + \lambda^2 + \lambda^3 + d(\lambda))^2}
+ \frac{(b(\lambda) - \lambda^3 - \lambda^2 + \lambda + 1)(c(\lambda) - \lambda^2 + \lambda)(d'(\lambda) + 3\lambda^2 + 2\lambda)}{(1 + \lambda - \lambda^2 - \lambda^3 + b(\lambda))(1 + \lambda^2 + \lambda^3 + d(\lambda))^2}.
\]

Let us estimate the numerator from below. Since
\[
\max\{|a(\lambda)|, |b(\lambda)|, |c(\lambda)|, |d(\lambda)|\} \leq \frac{\lambda^5}{1 - \lambda},
\]
\[
\max\{|a'(\lambda)|, |b'(\lambda)|, |c'(\lambda)|, |d'(\lambda)|\} \leq \frac{\lambda^4(5 - 4\lambda)}{(1 - \lambda)^2},
\]
the numerator is at least
\[
- \left( \frac{(5 - 4\lambda)\lambda^3}{(1 - \lambda)^2} - 4\lambda^3 + 1 \right) \left( \frac{\lambda^5}{1 - \lambda} + \lambda^3 + \lambda^2 + 1 \right) \left( \frac{\lambda^5}{1 - \lambda} - \lambda^3 - \lambda + 1 \right)^2
- \left( \frac{(5 - 4\lambda)\lambda^3}{(1 - \lambda)^2} - 3\lambda^2 - 2\lambda + 1 \right) \left( \frac{\lambda^5}{1 - \lambda} + \lambda^3 + \lambda^2 + 1 \right)^2 \left( \frac{\lambda^5}{1 - \lambda} + \lambda^4 + \lambda \right)
+ \left( \frac{\lambda^5}{1 - \lambda} - \lambda^3 - \lambda^2 + \lambda + 1 \right)^2 \left( \frac{\lambda^5}{1 - \lambda} + \lambda^4 + \lambda \right) \left( \frac{(5 - 4\lambda)\lambda^4}{(1 - \lambda)^2} + 3\lambda^2 + 2\lambda \right)
+ \left( \frac{\lambda^5}{1 - \lambda} - \lambda^3 - \lambda^2 + \lambda + 1 \right)^2 \left( \frac{\lambda^5}{1 - \lambda} + \lambda^3 + \lambda^2 + 1 \right)^2 \left( \frac{(5 - 4\lambda)\lambda^4}{(1 - \lambda)^2} + 4\lambda^3 + 1 \right),
\]
which, by numerical calculations, is bounded below by \( \frac{9}{100} \) for all \( \frac{1}{4} < \lambda < \frac{1}{3} \). Since
\[
\max\{1 + \lambda - \lambda^2 - \lambda^3 + b(\lambda), 1 + \lambda^2 + \lambda^3 + d(\lambda)\} \leq \frac{3}{2},
\]
we conclude that
\[
\frac{d}{d\lambda}(\text{proj}(S^1_{k}(0,0)) - \text{proj}(S^1_{l}(0,0))) \geq \frac{16}{81} \frac{9}{100} = \frac{4}{225}
\]
for all \( \frac{1}{4} < \lambda < \frac{1}{3} \). The other inequality is straightforward. \( \square \)

Relying on transversality, we will construct a Cantor set of super-exponentially condensed tuples. Observe that without loss of generality, we may assume that \( \delta > 0 \) in Lemma 4.6 is small, for instance \( \delta < \frac{1}{2} \).
Lemma 4.7. Let \( \frac{1}{2} < \lambda < \frac{1}{3} \), and let \( 0 < \delta < \frac{1}{2} \) be as in Lemma 4.6. Let \( k, 1 \in \bigcup_{n=5}^{\infty} J^n \) with \( k |_{5} = (0, 1)(1, 1)(0, -1)(0, -1)(1, 0) \) and \( 1|_{5} = (0, 1)(1, 0)(0, 1)(0, 1)(-1, 0) \) or vice versa. Then for every \( \varepsilon > 0 \) if \( |1|^{-1} \delta - 1 \varepsilon \) and
\[
|\text{proj}(S_k^{1}(0, 0)) - \text{proj}(S_1^{1}(0, 0))| < \varepsilon,
\]
then there exist disjoint closed intervals \( I, I' \subset [\lambda - 3\delta^{-1}\varepsilon, \lambda + 3\delta^{-1}\varepsilon] \) of length \( |1|^{-1} \delta - 1 \varepsilon \) such that
\[
\lambda \frac{1}{2}|1|^{-1} \delta < |\text{proj}(S_k^{\lambda}(0, 0)) - \text{proj}(S_1^{\lambda}(0, 0))| < \frac{\lambda}{2}\delta^{-1}|1|^{-1} \delta
\]
for all \( \lambda^* \in I \cup I' \).

Proof. By Lemma 4.6, the map \( \lambda \mapsto \text{proj}(S_k^{\lambda}(0, 0)) - \text{proj}(S_1^{\lambda}(0, 0)) \) is strictly monotone having the absolute value of the derivative between \( \delta \) and \( \delta^{-1} \). Hence, there exists unique \( \lambda_1 \in [\lambda - \delta^{-1} \varepsilon, \lambda + \delta^{-1} \varepsilon] \) such that
\[
\text{proj}(S_k^{\lambda_1}(0, 0)) - \text{proj}(S_1^{\lambda_1}(0, 0)) = 0.
\]
Choose \( I = [\lambda_1 - \frac{\lambda}{2}|1|^{-1} \delta, \lambda_1 - \frac{\lambda}{2}|1|^{-1} \delta] \) and \( I' = [\lambda_1 + \frac{\lambda}{2}|1|^{-1} \delta, \lambda_1 + \frac{\lambda}{2}|1|^{-1} \delta] \). By the mean value theorem, (4.3) holds for every \( \lambda^* \in I \cup I' \). Note that \( \lambda - 3\delta^{-1} \varepsilon \leq \lambda - \delta^{-1} \varepsilon - 2|1|^{-1} \delta \leq \lambda - 3\delta^{-1} \varepsilon \) and, similarly, \( \lambda_1 + \frac{\lambda}{2}|1|^{-1} \delta \leq \lambda + 3\delta^{-1} \varepsilon \).

Remark 4.8. Observe that the choice of the function \( 1 \mapsto |1|^{-1} \delta \) above is arbitrary: any super-exponential monotone function works here and also in the forthcoming lemmas.

Lemma 4.9. For every \( n \in \mathbb{N} \) there exists a set \( \Gamma_n \subset \bigcup_{n=1}^{\infty} J^n \) such that for each \( k \in \Gamma_n \) there is a closed interval \( I_k \subset (\frac{1}{3}, \frac{1}{2}) \) with non-empty interior. The collection \( \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \Gamma_n} \{I_k\} \) satisfies the following four conditions:

1. There exist \( I_1, I_2 \in \Gamma_{n+1} \) such that
   \[ I_1 \cup I_2 \subset I_k \text{ and } I_1 \cap I_2 = \emptyset. \]
2. \( \text{diam}(I_k) = |k|^{-|k|} \).
3. For every \( k \in \Gamma_n \) there exist \( I_1, I_2 \in \Gamma_{n+1} \) such that
   \[ \frac{1}{2}\delta |1_j|^{-1} |1| < |\text{proj}(S_k^{1}(0, 0)) - \text{proj}(S_1^{\lambda}(0, 0))| < \frac{2\lambda}{3}\delta^{-1}|1|^{-1} |1_j|. \]
   for all \( \lambda \in I_1 \) and \( j \in \{1, 2\} \).
4. For every \( k \in \Gamma_n \) there exists \( k \in \Gamma_{n-1} \) such that
   \[ \frac{1}{2}\delta |k|^{-|k|} < |\text{proj}(S_k^{\lambda}(0, 0)) - \text{proj}(S_1^{\lambda}(0, 0))| < \frac{2\lambda}{3}\delta^{-1}|k|^{-|k|} \]
   and
   \[ \min \{|\text{proj}(S_k^{\lambda}(0, 0)) - \text{proj}(S_1^{\lambda}(0, 0))| : |1| \leq |\Gamma_{n-1}| \} \geq |k|^{-|k|} \]
   for all \( \lambda \in I_k \), where \( |\Gamma_{n-1}| = \max \{|k| : k \in \Gamma_{n-1}\} \).

Proof. Let \( k = (0, 1)(1, 1)(0, -1)(0, -1)(1, 0) \) and \( 1 = (0, 1)(1, 0)(0, 1)(0, 1)(-1, 0) \). Numerical calculations show that
\[
\lambda < 0.286 \quad \implies \quad \max \text{proj}(S_k^\lambda(K_\lambda)) < \min \text{proj}(S_1^\lambda(K_\lambda)),
\]
\[
\lambda > 0.329 \quad \implies \quad \min \text{proj}(S_k^\lambda(K_\lambda)) > \max \text{proj}(S_1^\lambda(K_\lambda)).
\]
Therefore, by Lemma 4.6 there exists \( \lambda' \in (0.286, 0.329) \) such that \( \text{proj}(S_k^{\lambda'}(0, 0)) = \text{proj}(S_1^{\lambda'}(0, 0)) \).
Now choose \( k \in \mathbb{N} \) such that \( |\lambda' - \frac{1}{k}|^{-1} |k| \leq |k|^{-\frac{1}{k}} |k|^{-1} |k| \). Choose \( \lambda'' \) to be transcendental in the neighborhood of \( \lambda' \) such that
\[ \lambda'' \in [\lambda' - \frac{1}{2} \delta |k|^{-|k|}, \lambda' + \frac{1}{2} \delta |k|^{-|k|}] . \] Then choose \( m \in \mathbb{N} \) such that \( 3(1 - \frac{1}{2} \delta)^{-1}(|1| + m)^{-(|1| + m)} < |k|^{-|k|} \) and
\[ 5\delta^{-1}(|1| + m)^{-(|1| + m)} < \min\{|\text{proj}(S_1^{\lambda''}(0, 0)) - \text{proj}(S_2^{\lambda''}(0, 0))| : \text{proj}(S_1^{\lambda''}(0, 0)) \neq \text{proj}(S_2^{\lambda''}(0, 0)) \} . \]

Observe that, by Lemma 4.5, \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \) is an interval and, by Lemma 4.3, \( \text{proj}(S_1^{\lambda''}(0, 0)) \) is the middle point of \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \). Furthermore, the length of the interval \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \) is \( 2\lambda |k|/(1 - \lambda) \) and by the mean value theorem
\[ |\text{proj}(S_1^{\lambda''}(0, 0)) - \text{proj}(S_k^{\lambda''}(0, 0))| \leq \frac{1}{2} |k|^{-|k|} . \]

Since \( |k| = 5 \) and \( \lambda > \frac{1}{4} \), we get that \( \text{proj}(S_k^{\lambda''}(0, 0)) \) is an interior point of \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \). Since \( \{\text{proj}(S_1^{\lambda''}(0, 0)) : |l' | \geq m \} \) is dense in \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \), there exist \( l'_1, l'_2 \) with \( |l'_1|, |l'_2| \geq m \) such that \( l'_1 \neq l'_2 \),
\[ |\text{proj}(S_k^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| < |l'_1| + m)^{-(|1| + m)} \]
for both \( j \in \{1, 2\} \), and \( \text{proj}(S_1^{\lambda''}(0, 0)) < \text{proj}(S_k^{\lambda''}(0, 0)) < \text{proj}(S_1^{\lambda''}(0, 0)) \). Now, applying Lemma 4.7 with \( \epsilon = \delta(|k| + m)^{-|k|+m} \) and for both \( I_{11} \) and \( I_{12} \), we see that there exist disjoint closed intervals \( I_1, I_2 \) such that \( |I_j| = |11|^{-|11|} j \) for \( j \in \{1, 2\} \) and
\[ I_1, I_2 \subset [\lambda'' - 3(|1| + m)^{-(|1| + m)}, \lambda'' + 3(|1| + m)^{-(|1| + m)}] \]
with non-empty interior such that
\[ \frac{1}{2} \delta |11_j|^{-|11_j|} < |\text{proj}(S_k^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| < \frac{3}{2} \delta^{-1} |11_j|^{-|11_j|} \]
for all \( \lambda \in I_j \). Let \( i \) be a finite sequence such that \( |i| \leq |k| \). If \( \text{proj}(S_i^{\lambda''}(0, 0)) = \text{proj}(S_1^{\lambda''}(0, 0)) \), then, since \( \lambda'' \) is transcendental, we have \( \text{proj}(S_i^{\lambda''}(0, 0)) \equiv \text{proj}(S_k^{\lambda''}(0, 0)) \). Indeed, since \( \text{proj}(S_i^{\lambda''}(0, 0)) - \text{proj}(S_k^{\lambda''}(0, 0)) \) is a ratio of polynomials with integer coefficients, a transcendental \( \lambda'' \) cannot be a root of it, unless the ratio is the constant zero function of \( \lambda \). If \( \text{proj}(S_1^{\lambda''}(0, 0)) \neq \text{proj}(S_k^{\lambda''}(0, 0)) \), then
\[ |\text{proj}(S_1^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| > |\text{proj}(S_k^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| - \delta^{-1} |\lambda - \lambda''| \]
\[ > |\text{proj}(S_k^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| - \delta(|1| + m)^{-(|1| + m)} - \delta^{-1} |\lambda - \lambda''| \]
\[ > 5\delta^{-1}(|1| + m)^{-(|1| + m)} - \delta(|1| + m)^{-(|1| + m)} - 3(|1| + m)^{-(|1| + m)} \]
\[ > (|1| + m)^{-(|1| + m)} \geq |11_j|^{-|11_j|} \]
for all \( \lambda \in I_j \). Set \( I_2 = \Gamma_1 \cup \{11'_1, 11'_2\} \) and \( I_{1j'} = I_j \).

Let us then suppose that the sets \( \Gamma_1, \ldots, \Gamma_n \) have already been defined. Let \( k \in \Gamma_n \setminus \Gamma_{n-1} \). Then, by the condition (H), there exists \( i \in \Gamma_n \setminus \Gamma_{n-1} \) such that
\[ \frac{1}{2} \delta |k|^{-|k|} < |\text{proj}(S_k^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| < \frac{3}{2} \delta^{-1} |k|^{-|k|} \]
for all \( \lambda \in I_k \). Fix transcendental \( \lambda' \in I_k \) such that \( \text{dist}(\lambda', \mathbb{R} \setminus I_k) > \frac{1}{3} |k|^{-|k|} \). Choose \( m \in \mathbb{N} \) such that \( 3(1 - \frac{1}{2} \delta)^{-1}(|1| + m)^{-(|1| + m)} < |k|^{-|k|} \) and
\[ 5\delta^{-1}(|1| + m)^{-(|1| + m)} < \min\{|\text{proj}(S_1^{\lambda''}(0, 0)) - \text{proj}(S_2^{\lambda''}(0, 0))| : \text{proj}(S_1^{\lambda''}(0, 0)) \neq \text{proj}(S_2^{\lambda''}(0, 0)) \} \]
and \( |i|, |j| \leq |\Gamma_n| \).

Observe that, by Lemma 4.5, \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \) is an interval and, by Lemma 4.3, \( \text{proj}(S_1^{\lambda''}(0, 0)) \) is the middle point of \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \). Moreover, since \( |\text{proj}(S_1^{\lambda''}(0, 0)) - \text{proj}(S_1^{\lambda''}(0, 0))| < \frac{3}{2} \delta^{-1} |k|^{-|k|} < \lambda'|k|/(1 - \lambda) \), which is the half of the length of \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \), \( \text{proj}(S_1^{\lambda''}(0, 0)) \) is an interior point of \( \text{proj}(S_1^{\lambda''}(K_{\lambda''})) \).
Applying again Lemma 4.7 with \( \varepsilon \)
\[ \text{proj}(S_{\lambda}^{\lambda'}(0,0)) < \text{proj}(S_{k}^{\lambda'}(0,0)) < \text{proj}(S_{\lambda}^{\lambda'}(i_{1},i_{2},0)) \]
\[ |\text{proj}(S_{k}^{\lambda'}(0,0)) - \text{proj}(S_{\lambda}^{\lambda'}(0,0))| < \frac{\delta}{2}|i_{1} + m|^{-(\lfloor |i_{1}| + m \rfloor)} . \]

Applying again Lemma 4.7 with \( \varepsilon = \frac{1}{2}\delta(|i_{1}| + m)^{-(\lfloor |i_{1}| + m \rfloor)} \), there exist disjoint closed intervals \( I_{1}, I_{2} \) with length \( |i_{1}|^{\lfloor |i_{1}| \rfloor} \) respectively such that \( I_{1}, I_{2} \subset [\lambda' - \frac{1}{2}|i_{1}| + m, \lambda' + \frac{1}{2}|i_{1}| + m) \) and
\[ \frac{1}{2}\delta|\lambda_{I_{1}}|^{-\lfloor \lambda_{I_{1}} \rfloor} < |\text{proj}(S_{k}^{\lambda'}(0,0)) - \text{proj}(S_{\lambda}^{\lambda'}(0,0))| < \frac{\delta}{2}|\lambda_{I_{1}}|^{-\lfloor \lambda_{I_{1}} \rfloor} \]
for all \( \lambda \in I_{j} \).

Let \( i \) be a finite sequence such that \( |i| \leq |\Gamma_{n}| \). If \( \text{proj}(S_{\lambda}^{\lambda'}(0,0)) = \text{proj}(S_{k}^{\lambda'}(0,0)) \), then, since \( \lambda' \) is transcendental, we have \( \text{proj}(S_{\lambda}^{\lambda'}(0,0)) \equiv \text{proj}(S_{k}^{\lambda'}(0,0)) \). If \( \text{proj}(S_{\lambda}^{\lambda'}(0,0)) \neq \text{proj}(S_{k}^{\lambda'}(0,0)) \), then
\[ |\text{proj}(S_{\lambda}^{\lambda'}(0,0)) - \text{proj}(S_{\lambda}^{\lambda'}(0,0))| \geq |\text{proj}(S_{\lambda}^{\lambda'}(0,0)) - \text{proj}(S_{k}^{\lambda'}(0,0))| - \delta^{-1}|\lambda' - \lambda| \]
\[ > |\text{proj}(S_{\lambda}^{\lambda'}(0,0)) - \text{proj}(S_{k}^{\lambda'}(0,0))| - \frac{\delta}{2}|i_{1}| + m|^{-(\lfloor |i_{1}| + m \rfloor)} - \delta^{-1}|\lambda - \lambda'| \]
\[ \geq 5\delta^{-1}|i_{1}| + m|^{-(\lfloor |i_{1}| + m \rfloor)} - \delta(|i_{1}| + m)^{-(\lfloor |i_{1}| + m \rfloor)} - \frac{\delta}{2}|i_{1}| + m|^{-(\lfloor |i_{1}| + m \rfloor)} \]
\[ \geq |\lambda_{I_{1}}|^{-\lfloor \lambda_{I_{1}} \rfloor} \]
for all \( \lambda \in I_{j} \). Since \( k \in \Gamma_{n} \setminus \Gamma_{n-1} \) was arbitrary, we set \( \Gamma_{n+1} = \Gamma_{n} \cup \{i(k)1_{j}(k)\}_{k \in \Gamma_{n}} \) and \( I_{i(k)1_{j}(k)} = I_{j} \), where \( i(k), 1_{j}(k), \) and \( I_{j} \) for \( j \in \{1, 2\} \) are defined above. \( \square \)

**Proof of Theorem 2.2** Let \( F = \bigcap_{n=1}^{\infty} \bigcup_{k \in \Gamma_{n}} I_{k} \), where \( \{\Gamma_{n}\}_{n \in \mathbb{N}} \) and \( \{I_{k}\}_{k \in \Gamma_{n}} \) are as in Lemma 4.9. By Lemma 4.9.11, it is clear that \( F \) is compact, non-empty, and, in particular, uncountable. Let \( (k_{n})_{n \in \mathbb{N}} \) be a sequence of finite words such that \( k_{n} \in \Gamma_{n} \) and \( I_{k_{n+1}} \subset I_{k_{n}} \) for all \( n \in \mathbb{N} \). By Lemma 4.9.11(2), the set \( \bigcap_{n=1}^{\infty} I_{k_{n}} \) is a singleton. Let us denote it by \( \lambda^{*} \). Furthermore, by Lemma 4.9.3, the sequence \( \text{proj}(S_{\lambda}^{\lambda'}(0,0)) \) is a Cauchy sequence, and therefore has a limit \( t^{*} = \lim_{n \to \infty} \text{proj}(S_{\lambda}^{\lambda'}(0,0)) \). Let \( \Upsilon \subset \mathbb{R}^{2} \) be the collection of all possible pairs of \( (\lambda^{*}, t^{*}) \) obtained by such sequences. It is clear that \( \Upsilon \) is uncountable and hence, to finish the proof, it suffices to show that \( \Upsilon \subset C \setminus E \).

Let us first show that \( (\lambda^{*}, t^{*}) \in \mathcal{C} \). Recall that the function \( \beta: \{1, 2, 3\}^{n} \times \{1, 2, 3\}^{n} \to \mathbb{N} \) defined for Lemma 1.2 is invertible outside the diagonal. Let us define \( \beta^{-1}: J \to \{1, 2, 3\} \times \{1, 2, 3\} \) by \( \beta^{-1}(0,0) = (1,1) \) and extend it as before. Let \( m(n) \) be the unique integer such that \( |k_{m(n)}| = n < |k_{m(n)}'| \) and define a pair of sequences in \( \{1, 2, 3\}^{n} \) by \( (i_{n}, j_{n}) = \beta^{-1}(k_{m(n)}(0,0)) \) for all \( n \in \mathbb{N} \). Since for every \( m \) it holds that \( k_{m}|_{1} = (0,1) \), we have \( (i_{n}|_{1}, j_{n}|_{1}) \in \{((1,2), (2,1))\} \). Hence,
\[ |t^{*} - \text{proj}(S_{\lambda}^{\lambda'}_{\beta(i_{n}, j_{n})}(0,0))| = |t^{*} - \text{proj}(S_{\lambda}^{\lambda'}_{k_{m(n)-1}}(0,0))| \]
\[ \leq |\lim_{\ell \to \infty} \text{proj}(S_{k_{\ell}}^{\lambda'}(0,0)) - \text{proj}(S_{k_{m(n)-1}}^{\lambda'}(0,0))| \]
\[ \leq \sum_{\ell = m(n)}^{\infty} |\text{proj}(S_{k_{\ell}}^{\lambda'}(0,0)) - \text{proj}(S_{k_{\ell-1}}^{\lambda'}(0,0))| . \]
But, by Lemma 4.9(1), we have \( \lambda^* \in I_{k_m(n)} \) for all \( n \in \mathbb{N} \) and hence, by Lemma 4.9(3), we obtain
\[
\sum_{\ell=m(n)}^{\infty} |\text{proj}(S_{k_\ell}(0,0)) - \text{proj}(\hat{S}_{k_{\ell-1}}(0,0))| \leq \frac{3}{2} \delta^{-1} \sum_{\ell=m(n)}^{\infty} |\kappa_\ell|^{-1} \leq \frac{3}{2} \delta^{-1} \sum_{\ell=|k_m(n)|}^{\infty} \ell^{-\ell} \\
\leq \frac{3}{2} \delta^{-1} \int_{|k_m(n)|}^{\infty} e^{-x \log x} \, dx \leq 4C \int_{|k_m(n)|}^{\infty} e^{-x \log (1 + \log x)} \, dx \\
= \frac{3}{2} \delta^{-1} (|k_m(n)| - 1)^{-1} (|k_m(n)| - 1) \leq \frac{3}{2} \delta^{-1} n^{-n}.
\]
Recalling Lemma 4.2, we have now shown that \( (\lambda^*, t^*) \in \mathcal{C} \).

Let us then show that \( (\lambda^*, t^*) \notin \mathcal{E} \). Suppose to the contrary that \( (\lambda^*, t^*) \in \mathcal{E} \). Then there exists a pair of finite words such that \( \varphi_{i_1, k_1}^{\lambda^*, t^*}(0) = \varphi_{j_1, k_1}^{\lambda^*, t^*}(0) \) but \( i_1 \neq j_1 \) and, in particular, \( (i_1, j_1) \in \{(1, 2), (2, 1)\} \). Thus, we have \( t^* = \text{proj}(S_{\beta(i,j)(0,0)}) \). Choose \( n \in \mathbb{N} \) so that \( |\Gamma_n| > |\beta(1, j)| \), where we recall that \( |\Gamma_n| = \max\{|k| : k \in \Gamma_n\} \). By Lemma 4.9(1),
\[
0 = |t^* - \text{proj}(S_{\beta(1,j)}^{\lambda^*}(0,0))| = |\lim_{n \to \infty} \text{proj}(S_{\kappa_n}^{\lambda^*}(0,0)) - \text{proj}(S_{\kappa_{n+1}}^{\lambda^*}(0,0))| \\
\geq |\text{proj}(S_{\kappa_n}^{\lambda^*}(0,0)) - \text{proj}(S_{\kappa_{n+1}}^{\lambda^*}(0,0))| - \sum_{\ell=n+1}^{\infty} |\text{proj}(S_{k_\ell}(0,0)) - \text{proj}(\hat{S}_{k_{\ell+1}}(0,0))| \\
\geq |\kappa_n|^{-1} |\kappa_{n+1}| - \frac{3}{2} \delta^{-1} (|k_{n+1}| - 1)^{-1} (|k_{n+1}| - 1) > 0,
\]
which is a contradiction. \( \square \)

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References

[1] S. Baker. Iterated function systems with super-exponentially close cylinders. Preprint, available at arXiv:1909.04333, 2019.
[2] K. Falconer. Techniques in fractal geometry. John Wiley & Sons Ltd., Chichester, 1997.
[3] D.-J. Feng and H. Hu. Dimension theory of iterated function systems. Comm. Pure Appl. Math., 62(11):1435–1500, 2009.
[4] Y. Heurteaux. Estimations de la dimension inférieure et de la dimension supérieure des mesures. Ann. Inst. H. Poincaré Probab. Statist., 34(3):309–338, 1998.
[5] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy. Ann. of Math. (2), 180(2):773–822, 2014.
[6] M. Hochman. Dimension theory of self-similar sets and measures. In B. Sirakov, P. N. de Souza, and M. Viana, editors, Proceedings of the International Congress of Mathematicians (ICM 2018), pages 1943–1966. World Scientific, 2019.
[7] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy in \( \mathbb{R}^d \). To appear in Memoire of AMS, available at arXiv:1503.00943, 2019.
[8] J. E. Hutchinson. Fractals and self-similarity. Indiana Univ. Math. J., 30(5):713–747, 1981.
[9] M. Pollicott and K. Simon. The Hausdorff dimension of \( \lambda \)-expansions with deleted digits. Trans. Amer. Math. Soc., 347(3):967–983, 1995.
[10] P. Shmerkin. On Furstenberg’s intersection conjecture, self-similar measures, and the \( L^q \) norms of convolutions. Ann. of Math. (2), 180(2):319–391, 2019.
[11] P. Shmerkin and B. Solomyak. Absolute continuity of self-similar measures, their projections and convolutions. Trans. Amer. Math. Soc., 368(7):5125–5151, 2016.
[12] K. Simon. Overlapping cylinders: the size of a dynamically defined Cantor-set. In Ergodic theory of \( \mathbb{Z}^d \) actions (Warwick, 1993–1994), volume 228 of London Math. Soc. Lecture Note Ser., pages 259–272. Cambridge Univ. Press, Cambridge, 1996.
[13] K. Simon and B. Solomyak. On the dimension of self-similar sets. Fractals, 10(1):59–65, 2002.
[14] P. P. Varjú. On the dimension of Bernoulli convolutions for all transcendental parameters. Ann. of Math. (2), 189(3):1001–1011, 2019.
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