Trapping and Steering on Lattice Strings:
Virtual Slow Waves, Directional and Non-propagating Excitations

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Abstract

Using a lattice string model, a number of peculiar excitation situations related to non-propagating excitations and non-radiating sources are demonstrated. External fields can be used to trap excitations locally but also lead to the ability to steer such excitations dynamically as long as the steering is slower than the field’s wave propagation. I present explicit constructions of a number of examples, including temporally limited non-propagating excitations, directional excitation and virtually slowed propagation. Using these dynamical lattice constructions I demonstrate that neither persistent temporal oscillation nor static localization are necessary for non-propagating excitations to occur.

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I. INTRODUCTION

Can a local excitation (source) in classical field theories be invisible to observers outside the region of excitation? This question has recently received renewed interest.

Berry et al. described a peculiar excitation case for the one-dimensional wave-equation of a perfectly elastic string under tension. They show that the response of the string can be made to be confined to a bounded region by carefully choosing a forced excitation of oscillatory type. This means that the excitation will not propagate away along the string. Denardo gives a simple and intuitive explanation by using a wave interference argument. Gbur et al. discuss conditions of finite string length and dissipation.

Other recent work investigated non-propagating excitations include Marengo and Ziolkowski who discuss the generalization of non-propagating conditions of D’Alembertian \( \Box \equiv \nabla^2 - c^{-2} \frac{\partial^2}{\partial t^2} \) operators and its temporally reduced version the Helmholtz operator \( (\nabla^2 + k^2) \) on various related classical scalar and vector fields. Marengo, Devaney and Ziolkowski give the condition for time-dependent but not necessarily time-harmonic non-radiating sources and for selective directional radiation for the inhomogeneous wave equation in three spatial dimensions. Marengo and Ziolkowski generalize these conditions to more general scalar and vector field dynamics. Marengo, Devaney and Ziolkowski also give examples in one and three spatial dimension for the time-harmonic case. Hoenders and Ferwerda discuss the relationship of non-radiating and radiating parts of the case of the reduced Helmholtz equation, which can be derived from the string equation by assuming general oscillatory time solutions (see ). Denardo and Miller discuss the related case of leakage from an imperfect non-propagating excitation on a string. Gbur provides a comprehensive recent review of this topic and the reader is referred to this review for more detailed historical context. Of the earlier work the following contributions are particularly relevant for the discussion here: Schott gave the condition for non-radiation of a spherical shell on a circular orbit. Bohm and Weinberg extended this result to more general spherical charge distributions and Goedecke showed how an asymmetrical charge distribution with spin is non-radiating. All of these works are concerned with the case of spatially moving sources. Finally it is noteworthy, that non-radiating sources play an important role in inverse problems and have been investigated in a one-dimensional electrodynamic situation by Habashy, Chow and Dudley.
In this paper our purpose is to describe this phenomena in the case of a lattice string in one dimensions by discretizing D’Alembert’s solution. This approach is used extensively to simulate vibrating strings and air tubes of musical instruments. See [18] and references therein.

This leads to explicit dynamical constructions of previously reported non-propagating excitations. Its simplicity allows for additional insight into the mechanism that allows for the local confinements and the conditions under which they occur. I will show how the basic mechanisms that provide a time-harmonic stationary non-propagating excitation in one dimension as studied by Berry et al. and Gbur et al. [1, 3] allows for a much wider class of excitations. For instance can such an excitation be relieved from the time-harmonic assumption beyond one period allowing for non-propagating sources that are short-lived. Directional excitations can easily be achieved using very simple bidirectional excitation patterns. These are explicit constructions of such waves in one spatial dimension whose general condition of existence in the three-dimensional case has been derived by Marengo, Devaney and Ziolkowski [7]. Wave propagation can be virtually slowed down. In general I will show that non-propagating excitations can be extended to steered excitation regions with basic physical restrictions imposed by the underlying field dynamics.

First I will give a quick derivation of the simple lattice model from the wave equation as can also be found in [18]. Then I will give a new argument and construction of the Berry et al. type non-propagating excitation purely based on discrete string dynamics. This will then be compared to the original approach. Then I will extend the discussion to examples of additional types of non-propagating waves, including directional and slowed waves. Finally I will discuss very general constraints on such “steered” localized excitations.

II. LATTICE STRING MODEL

The lattice string model can easily be derived from the wave-equation by discretizing the D’Alembert solution. Hence the continuous case will be discussed first.
A. Continuous Wave Solutions

The one-dimensional homogeneous wave equation of the perfectly elastic string under tension is:

\[ \mu \frac{\partial^2 y}{\partial t^2} - K \frac{\partial^2 y}{\partial x^2} = 0 \]  

(1)

where \( c^2 = K/\mu \) is derived from mass density \( \mu \) and tension \( K \). The D’Alembert solution of the homogeneous “free field” case has the well known form \[ 19 \text{, p. 596, eq. (4)} \]:

\[ y(x, t) = w^+(x - ct) + w^-(x + ct) \]  

(2)

Hence the solution of the general of the homogeneous wave-equation are two propagating waves whose content is restricted by initial and boundary conditions. As wave-equation is linear we have a connection between initial conditions and external driving forces. Driving forces can be seen as infinitesimal time frames that act on the wave dynamics by imposing a new initial condition at each point in time. Hence we need to consider the initial value problem to gain insight into both processes at once.

At a given time frame \( t_i \) let the following initial conditions hold:

\[ y(x, t_i) = f(x, t_i) \]  

(3)

\[ y_t(x, t_i) = g(x, t_i) \]  

(4)

Equation (3) with (2) gives a particular solution \( v^+ \):

\[ v^+(x - ct_i) + v^-(x + ct_i) = f(x, t_i) \]  

(5)

Taking the first temporal derivative of (2) and satisfying equation (4) we get:

\[ -cv^+_t(x - ct_i) + cv^-_t(x + ct_i) = g(x, t_i) \]  

(6)

Integrating with respect to \( x \) we get \[ 19 \text{, eq. (10) p. 596} \]:


\[-cv^+(x - ct_i) + cv^-(x + ct_i) = k(x_0) + \int_{x_0}^x g(s)ds,\]
\[k(x_0) = -cv^+(x_0) + cv^-(x_0)\quad (7)\]

From equations (5) and (7) we can solve for the traveling wave components:

\[v^+(x - ct_i) = \frac{1}{2} f(x, t_i) - \frac{1}{2c} \int_{x_0}^{x-ct_i} g(s)ds - \frac{1}{2} k(x_0)\quad (8)\]
\[v^-(x + ct_i) = \frac{1}{2} f(x, t_i) + \frac{1}{2c} \int_{x_0}^{x+ct_i} g(s)ds + \frac{1}{2} k(x_0)\quad (9)\]

We see that forced displacement \( f(\cdot) \) splits evenly between left and right traveling waves and the integrated forced velocity \( g(\cdot) \) splits with a sign inversion.

For our current discussion I will share the assumption of no initial velocity of Berry et al. \[1\] and hence the integral over \( g(\cdot) \) will vanish.

For the infinite string this is already the complete solution for any twice differentiable function of free solutions and external forced displacements.

**B. Discrete Wave Solutions**

To arrive at lattice equations we discretize the solution of the wave-equation (2) in time via the substitution \( t \to Tn \) where \( T \) is the discrete time-step and \( n \) is the discrete time index. This automatically corresponds to a discretization in space as well, because in finite time \( T \) a wave will travel \( X = cT \) distance according to (2). The spatial index will be called \( m \). The free-field discrete D’Alembert solution:

\[g(mX, nT) = w^+(mX - cnT) + w^-(mX + cnT)\quad (10)\]

In general we can always express all discrete equations in terms of finite time steps or finite spatial lengths. We chose a temporal expression and substitute \( X = cT \) and suppress
shared terms in \( cT \) to arrive at the index version of the discrete D’Alembert solution [18]:

\[
y(m, n) = w^+(m - n) + w^-(m + n)
\]  

(11)

By equations (8) and (9) we see that at an instance \( m_i, n_i \) the discrete contribution of external forced displacements splits evenly between the traveling waves and we arrive at the discrete field equations including external forced displacements:

\[
W^+(m_i - n_i) = w^+(m_i - n_i) + \frac{1}{2}f(m_i, n_i)
\]

(12)

\[
W^-(m_i - n_i) = w^-(m_i - n_i) + \frac{1}{2}f(m_i, n_i)
\]

(13)

III. NON-PROPAGATING EXCITATION

Next we will construct the non-propagating excitation from the lattice string dynamics directly.

For simplicity and without loss of generality, we will assume a region aligning with the discretization domain throughout. We want to construct an excitation which is confined to a length \(-L \leq x \leq L\). For now we will assume that the string should otherwise stay at rest. This implies that there are no incoming waves into the region \( \Omega = [-L, L] \) from the outside. We are interested in a non-trivial excitation within the region.

First we consider the contributions to the position \(-L\). As there are no incoming external waves we get:

\[
w^+(-L + n) = 0
\]

(14)

We do expect non-trivial wave \( w^-(L - n) \) to reach the boundary but we require the total outgoing wave to vanish we have:

\[
W^-(L - n) = w^-(L - n) + \frac{1}{2}f(L, n) = 0
\]

(15)

The necessary external forced displacement contribution to for cancellation needs to be:
\[
\frac{1}{2}f(-L, n) = -w^-(L - n) \quad (16)
\]

The complete incoming wave \([12]\) will see the same forced contribution \([10]\) and with equation \([14]\) we get:

\[
W^+(L + n) = \frac{1}{2}f(-L, n) = -w^-(L - n)
\]

Hence the matched forced displacement leads to a reflection with sign inversion at the region boundary at \(-L\).

Following the same line of argument at point \(L\) we get the related condition:

\[
W^-(L - n) = \frac{1}{2}f(L, n) = -w^+(L + n)
\]

With these two conditions we can study the permissible form of excitations. First we assume an initial forced displacement impulse from a position \(p\) in the interior of the domain \(\Omega \setminus \partial\Omega = (-L, L)\). Hence \(-L < p < L\) and \(f(p, 0) = a_p\) with \(a_p \in \mathbb{R}\).

It will take half the impulse \(L + p\) steps to reach the left boundary and the other half \(L - p\) steps to reach the right one.

At each boundary the respective condition \([17,18]\) needs to be satisfied and we get:

\[
f(-L, L + p) = -f(p, 0) \quad (19)
\]
\[
f(L, L - p) = -f(p, 0) \quad (20)
\]

The impulse will then reflect back and create periodic matching conditions.

\[
f(-L, L + p + 4L\nu) = f(p, 0) \quad (21)
\]
\[
f(-L, L - p + (2\nu - 1)2L) = -f(p, 0) \quad (22)
\]
\[
f(L, L - p + 4L\nu) = f(p, 0) \quad (23)
\]
\[
f(L, L + p + (2\nu - 1)2L) = -f(p, 0) \quad (24)
\]
with $\nu = 1, 2, \cdots$.

Hence we see that a single impulse will necessitate an infinite periodic series of forced external displacements at the boundaries to trap the impulse inside as each “annihilation” of a half-pulse reaching the boundary leads to a “creation” of a reflected one.

The required impulse response of a boundary forced function $f(\pm L, \cdot)$ can easily be observed from equations (21–24) to be spatially periodic in $4L$ with an initial phase factor dictated by the starting position $p$. Additionally the functional shape of the impulse responses $f(\pm L, \cdot)$ is completely defined for all time steps as $f(\pm L, \cdot) = 0$ for all times that equations (21–24) don’t apply.

A condition for stopping a non-propagating excitation can be derived from the fact that a impulse will return to its initial position every $4L$ time steps. Additionally it is easy to see that the traveling impulses will occupy the same spatial position every odd multiple of $2L$ with a sign inversion. Hence an impulsive forced displacement $f((-1)^{\mu-1}p, 4L\mu) = (-1)^{\mu-1}a_p$ with $\mu = 1, 2, \cdots$ will cancel an initial impulse $f(p, 0) = a_p$. From this we can immediately deduce the following property:

**Theorem 1** The shortest possible single impulse finite non-propagating excitation takes $2L$ time-steps.

and more generally:

**Theorem 2** The time of any single impulse excitation finite non-propagating excitation has to be $2\mu L, \mu \in \mathbb{N}$.

More importantly we observe the property: Non-propagating excitations can be finite in duration.

This is an extension beyond Berry et al. which assumes infinitely periodic temporal progressions in their derivations.

The general solution for discrete non-propagating wave functions can be derived by observing that any initial “phase” $p_i$ is orthogonal to other phases $p_j$ for $i, j \in \Omega \setminus \partial \Omega = (-L, L)$, i.e. $(f(p_i, 0), f(p_j, 0)) = 0$ for $i \neq j$. Within a $2L$ period $f(\pm L, \cdot)$ is well-defined by $\sum_i f(p_i, \cdot)$. Interestingly though this provides the only restriction to the forced boundary functions. This can be seen by Theorem After $2L$ each $p_i$ will find constructive interference and can be annihilated or rescaled to an arbitrary other value $a_i(2L)$. Hence any
arbitrary succession of $2L - 2$ force distributions with a $2L$ termination is permissible. Hence periodicity is not necessary.

The time harmonic case can be derived if the initial force distribution within the domain is not modified over time. Then a configuration will repeat after traveling left and right, being reflected at the domain boundary twice, traversing the length of the region twice. Hence the lowest permissible wave-length is $4L$. By reflecting twice the wave will have gone through a $2\pi$ phase shift, but we note that the periodicity condition is also satisfied if any number of additional $2\pi$ shifts have been accumulated. Hence we get for permissible wave-numbers:

$$k = \frac{2\pi n}{4L}, \quad \text{where } n = 1, 2, \ldots .$$

(25)

or

$$kL = \frac{n\pi}{2}$$

(26)

By allowing only even $n$ we get the Berry at al. condition [1] for an even square distribution. The odd $n$ situation corresponds to the odd-harmonic out-of-phase construction proposed by Denardo [2].

Many of these properties can be seen visually in the numerical simulation depicted in Figure 1.

It is interesting to observe that two synchronous point-sources oscillating with the above phase condition will not be completely non-propagating. They will only be non-propagating after waves created at the wave onset have escaped. This is a refinement of the argument put forward by Denardo [2] and can intuitively be described as non-interference of the first trap period. Hence the first pairs of pulses will have half-Amplitude components escaping in either direction but every subsequent period will be trapped. This behavior, which could be called imperfect trapping or trapping with transient radiation, is depicted in Figure 2. Sources presented by Berry et al. and Denardo [1, 2] do not display this behavior because the force is assumed to be oscillatory at all times and hence has no onset moment.

Non-propagating excitations can be used as generic building blocks for other unusual excitation induced behavior on the string. In particular I will next describe how to construct
an uni-directional emitter, and a virtually slowed propagation. In fact a non-propagating excitation can be seen as virtually stopping a wave at a particular position.

IV. DIRECTIONAL EXCITATIONS

A one-sided open trap immediately suggests another unusual excitation type, namely the directional excitation. The string is to be excited in such a way that a traveling wave in only one direction results.

We start with a one-sided open trap. This is a trap that uses a reflection condition (17) and (18) only on one side of an initial excitation. Evidently the wave then can only travel in the opposite direction. For the discussion we will describe a right-sided propagator (i.e. a propagator traveling with increasing negative index). The trapping condition then reads:

\[ f(m + 1 + p, n + p) = -f(m, n - 1) \]  

Hence the trapping excitation point is a \( p \) time-step lagging negative copy of the original excitation. The emitted wave will have the form

\[ \frac{1}{2}f(m + 1, n + 2p) - \frac{1}{2}f(m + 1, n) \]  

The emitting wave will show self-interference at a phase of \( 2p \) time-steps, as can be seen in the simulation depicted in Figure 3. In general the self-interference phase can be chosen by the distance \( p \) between the wave creation point and the trapping point. It is worth noting that it is possible to eliminate interference by trapping the lagging contribution and hence create a wave non-interference directional wave left of the trapping region.

V. VIRTUAL SLOW WAVES

Virtual slow waves can be achieved by alternating directional wave propagation with trapping. The slowness of the wave propagation can be controlled by the number and duration times of the traps along a propagation. The propagation characteristics of the dynamic operator has not changed at all, hence we call the this state “virtually slow”
as opposed to the case where the field itself induces a change in wave propagation speed. This also means that within a slowed or “steered” region the wave propagation is the one prescribed by the dynamic operator $(\frac{\partial}{\partial x} + c\frac{\partial}{\partial t})(\frac{\partial}{\partial x} - c\frac{\partial}{\partial t})$ on the string $y(x,t)$.

The amount of time spend in traps determines the overall slowness. One example of slow wave consists of an immediate alteration between one stage of trapping and one step of one-sided propagation is illustrated in Figure IV. The effective propagation speed of the wave can easily be read from the diagram to give $c_{\text{eff}} = \frac{X}{3T} = c/3$. As is evident from Theorem II a unit $L = 1$ trap will last 2 time-steps and will not propagate spatially and one step of free propagation will last one time-step and and make one spatial step, hence resulting in a spatial to temporal ratio of $1 : 3$.

The trapping relations are:

$$f(m - 2 - \nu, n + 1 + 6\nu) = f(m + 1 - \nu, n + 6\nu)$$
$$= -f(m, n - 1)$$

(29)

$$f(m - 3 - \nu, n + 4 + 6\nu) = f(m - \nu, n + 3 + 6\nu)$$
$$= f(m, n - 1)$$

(30)

with $\nu = 0, 2, 4, \cdots$.

VI. STEERING

The generalized interpretation of the excitation interaction lead to the general dynamical confinement of waves by external excitation. For instance following very similar arguments as for virtual slow waves a construction is possible which gives a slowed “cone of influence” by successively widening the trap boundaries at a speed slower than the the wave speed $c$. By this argument it is sufficient for the trap boundaries’ change to be less than $c$ for it to be trapping the wave. This is not a necessary condition by the following counter-example: Let the trap width be $L$ and change rapidly by some slope $dL > c$ to some new constant width $L_2$ at which it becomes constant. Obviously the wave will then be able to reach the new boundary even though a local change of the boundary exceeded the dynamical speed $c$. The necessary condition can be seen from our previous construction. At a trap boundary a wave is reflected and will propagate in the opposite direction of the domain following the linear
characteristic c. Only if this characteristics intersects with the dynamic trapping boundary will there be another externally forced reflection as illustrated in Figure 5. These may in fact have regions where no trapping is necessary and possible.

VII. INTERACTION WITH BACKGROUND FIELDS

It is important to note that while we assumed that the incoming wave vanishes, see equation (14), the outgoing wave condition (15) does not change if there is in fact an incoming wave. The “reflection wave” (17) and (18) can be rewritten for a non-zero incoming field without affecting the trapping:

\[
W^+(-L + n) = w^+(-L + n) + \frac{1}{2} f(-L, n)
\]

\[
\frac{1}{2} f(-L, n) = -w^-(-L - n)
\]

These conditions are “absorbing” in the sense that an external field entering the trapping region will not leave it.

The “non-interacting” property of a trap defined by the periodic matching conditions (21–24) can be seen by assuming a non-zero incoming wave at one point of the trap boundary \(\delta \Omega\). Then the total wave entering the trapping region the sum of the wave created by the trapping condition and the incoming wave value \(\frac{1}{2} f(\delta \Omega^1, \cdot) + w^+ (\delta \Omega^1, \cdot)\), where \(\delta \Omega^1\) denotes the first trap boundary reached. When reaching the second trapping boundary \(\delta \Omega^2\) the now outgoing wave will see a matching force \(f(\delta \Omega^2, \cdot) = -\frac{1}{2} f(\delta \Omega^1, \cdot)\) leaving an outgoing wave contribution \(w^+ (\delta \Omega^2, \cdot) = w^+ (\delta \Omega^1, \cdot)\) to escape the trapping region \(\Omega\).

In order to achieve selective radiation, only part of the content of a trapped region are trapped at the boundary as can be achieved by using a reduced force at the trapping boundary or by selectively omitting certain phases in the trapping force pattern.
A. Relationship of Traps to Non-Radiating Sources

Marengo and Ziolkowski [4] present ideas very much related to ideas presented here and in Berry et al. [1].

However, they arrive at a definition of non-radiating (NR) sources that is not obviously similar to the traps presented here. In particular they define NR sources as being non-interacting. While [4] note that a central property of NR sources is that they store non-trivial field energy, traps described here can not only store, but accumulate and selectively radiate waves.

The difference can be understood by observing that for example Berry et al. assume a simple time-harmonic driver [1, eq. (3)] throughout their discussion:

\[ f(x, t) = \text{Re} \{ f(x) e^{-i\omega t} \} \]  

(33)

By our earlier discussion we see that the temporal progression of the boundary has to match the content of the interior domain. Hence once the boundary is defined to be oscillatory the interior of the domain needs to be spatially harmonic as derived in [1, 4] and has been rederived here. Hence a NR source as noted in literature, with exception of the general orthogonality formulation for time-varying sources given by Marengo, Devaney and Ziolkowski [7], can be thought of as a time-oscillatory trap.

The arguments made here use a formalism that is discrete in nature. However, the discreteness of the arguments are not necessarily restrictive. The continuous case can be imagined with the discrete time-step made small \((T \to 0)\) or alternatively, discrete pulses can be substituted with narrow distributions of compact support. In neither case are the results of interest derived here altered.

As has already been derived in [1, 3] the critical condition for non-propagating waves lie at the boundary of the domain-range that the wave ought not to leave. In the discrete case it is easy to see how this insight can be used and generalized. In fact, the boundaries of the confining domain need not be static, nor need the condition be used in a two-sided fashion.
VIII. CONCLUSION

In summary, this paper presented constructions of a broad class of non-propagating sources on a string lattice model using trapping conditions. In particular this includes numerical demonstrations of finite-duration non-propagating excitations, directional excitations, as well as virtually slowed waves. These examples help explain the extension of non-propagating sources beyond the time-periodic case and include treatment of onset, annihilation and spatial steering. These properties ought to be observable in experiments well-described by the wave equation. This equation often arises in problems in acoustics, elasticity, optics and electromagnetics. And hence the results presented here apply to these domains of application. While here I discussed the forward problem, these results also relate to the inverse problem of finding source contributions from the one-dimensional field state as occur for example in acoustical, optical and electromagnetic detection problems.

Acknowledgments

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FIG. 1: Simulation of a non-propagating excitation of width 3 which is annihilated after 3.5 periods. The total temporal length of the excitation is 10. The excitation leaves the string at rest after it is completed. Top: Complete wave pattern. Bottom: Excitation only.
FIG. 3: Simulation of a directional excitation of width 3. The deflected component experienced a sign inversion. The temporal length of the excitation sequence is two, including the initial impulse. Top: Complete wave. Bottom: Excitations only.

FIG. 4: Simulation of a finite-duration virtual slow wave excitation of width 3. The wave is annihilated after 10 steps. Top: Complete wave. Bottom: Excitation only.

FIG. 5: A grazing propagating wave against a changing trap boundary can create regions (gray) in which no trap affect applies.

FIG. 2: Simulation of a non-propagating excitation of width 3 showing escaping waves at the onset transient. Top: Complete wave pattern. Bottom: Excitation only.
Two graphs are shown, each with a grid of discrete positions labeled as \( m \) and discrete times labeled as \( n \). The graphs are color-coded with a legend indicating values from -1 to 1. The upper graph shows a pattern of dark and light squares, with some areas having darker shades than others, indicating variations in the values. The lower graph similarly displays patterns of dark and light squares, maintaining the same color scheme and scale. The grids range from 2 to 14 for both \( m \) and \( n \), with increments of 2.