A GENERALIZATION OF FORELLI’S THEOREM

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Abstract. The purpose of this paper is to present a generalization of Forelli’s theorem. In particular, we prove an all dimensional version of the two-dimensional theorem of Chirka [1] of 2005.

1. Introduction

Classical Forelli’s theorem ([2]; see also [5], [6]) states

Theorem 1.1 (Forelli). Let \( f : \mathbb{B}^n \to \mathbb{C} \) be a function. If it satisfies the following two conditions:

(i) \( f \in C^\infty(0) \),

and

(ii) for every unit vector \( v = (v_1, \ldots, v_n) \in \mathbb{C} \), \( f(\zeta v_1, \ldots, \zeta v_n) \) is holomorphic in the single complex variable \( \zeta \) with \( |\zeta| < 1 \),

then \( f \) is holomorphic.

The definition of the notation \( f \in C^\infty(0) \) in the statement is as follows: for any positive integer \( k \), there exists a polynomial \( p_k \) such that \( f(z) - p_k(z) = o(|z|^k) \).

After a long period of almost no results, the following two generalizations have been presented:

Theorem 1.2 (Chirka [1]). Let \( \{S_\tau\} \) be a foliation of a domain \( \Omega \) in a punctured ball \( \mathbb{B} \setminus 0 \) in \( \mathbb{C}^2 \) by holomorphic curves that are closed and smooth in \( \mathbb{B} \), pass through the origin, and are pairwise transversal at 0. Let \( f \) be a function at \( \mathbb{B} \) such that \( f \in C^\infty(0) \) and all restrictions \( f|_{S_\tau} \) are holomorphic. Then \( f \) is holomorphic in \( \Omega \) and, if \( \Omega = \mathbb{B} \setminus 0 \), then \( f \) is holomorphic in \( \mathbb{B} \).

Theorem 1.3 (Kim-Poletsky-Schmalz [4]). If \( n \geq 1 \) is an integer, and if \( f : \mathbb{B}^n \to \mathbb{C} \) is a function with \( f \in C^\infty(0) \), which is annihilated by

\[
\bar{X} = \sum_{j=1}^{n} \alpha_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j},
\]

where \( \alpha_1, \ldots, \alpha_n \) are real numbers, then \( f \) is holomorphic on \( \mathbb{B}^n \).

In both versions holomorphicity along straight lines through 0 has been replaced by holomorphicity along some more general family of complex curves: in case of Theorem 1.2 they are assumed to intersect transversely. On the other hand, in the case of Theorem 1.3 the family of complex curves is generated by a holomorphic vector field and its integral curves. Thus, the two theorems complement each
other in a sense; the general foliation considered by Chirka may not in general be
generated by a contracting holomorphic vector field, whereas the leaves of foliation
considered by Kim-Poletsky-Schmalz do not have to intersect mutually transversely
at the origin (even after the re-parametrization of the leaves so that they intersect
at the origin).

The purpose of this paper is to present Theorem 2.1, an all-dimensional version
of Theorem [12]. This in particular answers the question posed by Chirka in [1], p.

2. A generalization of Forelli’s theorem in \( \mathbb{C}^n \)

First we define the concept of the local \( C^k \) singular foliation by holomorphic
curves. Let \( \Delta \subset \mathbb{C} \) the unit disc and \( S^{2n-1} \) the unit sphere in \( \mathbb{C}^n \) defined by the
equation \( |z_1|^2 + \cdots + |z_n|^2 = 1 \).

**Definition 2.1.** Let \( \ell \) be a positive integer. For a point \( p \in \mathbb{C}^n \), a local \( C^\ell \) singular foliation at \( p \) by holomorphic discs is a \( C^\ell \) map \( h: \Delta \times S^{2n-1} \to \mathbb{C}^n \) satisfying the following properties:

1. For each \( v \in S^{2n-1} \), the correspondence \( h(\cdot, v): z \in \Delta \to h(z, v) \in \mathbb{C}^n \) is a
holomorphic embedding.
2. \( h(0, v) = p \) for every \( v \in S^{2n-1} \).
3. The image \( h(\Delta \times S^{2n-1}) \) contains an open neighborhood of \( p \) in \( \mathbb{C}^n \).
4. For each \( v \in S^{2n-1} \), there exists \( r_v > 0 \) such that \( \frac{\partial h}{\partial z}(0,v) = r_v v \).
5. \( h(z, e^{i\theta} v) = h(e^{i\theta} z, v) \) for any \( \theta \in \mathbb{R} \) and \( z \in \Delta \).

Throughout this paper, we shall consider the case \( \ell = 1 \) only.

We shall consider, from here on, only the case when \( p \) is the origin. This singular
foliation provides a parametrization of an open neighborhood of the origin in \( \mathbb{C}^n \)
by \( \Delta \times \mathbb{C}^{n-1} \). One can always choose coordinates around the origin in \( \mathbb{C}^n \) such
that any given direction \( v^0 \in S^{2n-1} \) becomes \( [1 : 0 \cdots : 0] \in \mathbb{C}^{n-1} \) and hence
a neighborhood of \( v^0 \in \mathbb{C}^{n-1} \) can be identified as a neighborhood of \( 0 \) in \( \mathbb{C}^{n-1} \)
with coordinates \( (\lambda_1, \ldots, \lambda_{n-1}) := \left( \frac{v_1^0}{v^0}, \ldots, \frac{v_{n-1}^0}{v^0} \right) \). Then \( h(z, v^0) \) can be understood
in this coordinate system as \( h(z, 0) \), and in a neighborhood of the corresponding
holomorphic curve, we may assume without loss of generality that it has the equation
\( h(z, 0) = (z, 0) \). Then for \( z \) and \( \lambda \), with both \( |z| \) and \( \|\lambda\| \) sufficiently small, the
curve \( z \to h(z, \lambda) \) is represented by the expression

\[
\begin{align*}
z & = z \\
w_1 & = k_1(z, \lambda_1, \ldots, \lambda_{n-1}) \\
& \vdots \\
w_{n-1} & = k_{n-1}(z, \lambda_1, \ldots, \lambda_{n-1}).
\end{align*}
\]

We use the short-hand notation \( w = k(z, \lambda) \). Then \( k \) satisfies

1. \( k(z, \lambda) \) is \( C^1 \) in \( z, \lambda \) and holomorphic in \( z \).
2. \( k(0, \lambda) = 0 \) for any \( \lambda \)
3. \( \frac{\partial k(z, \lambda)}{\partial z} \bigg|_{z=0} = \lambda \) and hence \( k(z, \lambda) = z\lambda + o(|z|) \).
Theorem 2.2. If $h$ is a local singular foliation of a domain $\Omega$ in $\mathbb{C}^n$ and $f: \Omega \to \mathbb{C}$ is a function satisfying:

(A) $f \in C^\infty(0) \cap C^1(\Omega)$; and

(B) for every leaf $h(\cdot, \lambda)$ the composition $f \circ h(\cdot, \lambda)$ is a holomorphic function, then $f$ is a holomorphic function on the intersection of $\Omega$ and some neighborhood of the origin.

Notice that the statement of this theorem in the case of $n = 2$ is weaker than what was presented in [1]. On the other hand, our proof here is not only valid for all dimensions, but also, even in dimension 2, somewhat more straightforward.

Proof. Let us use the notation $\partial f/\partial \bar{w} := (\partial f/\partial \bar{w}_1, \ldots, \partial f/\partial \bar{w}_{n-1})$. Of course the goal here is to establish that $\partial f/\partial \bar{w} = 0$ at $w = 0$.

Let $F(z, \lambda_1, \ldots, \lambda_{n-1}) = f(h(z, \lambda_1, \ldots, \lambda_{n-1})) = f(z, k(z, \lambda))$. First we prove that $\partial F/\partial \lambda_j, \partial F/\partial \bar{\lambda}_j$ and $\partial k_m/\partial \lambda_j$ and $\partial k_m/\partial \bar{\lambda}_j$ at $\lambda = 0$ are also holomorphic in $z$. In fact, let $G: \Delta \times U \to \mathbb{C}$ be a $C^1$ function that is holomorphic with respect to $z \in \Delta$, like $F, k$. Then

$$\int G(z, \lambda) \, dz \wedge d\phi = 0$$

for any function in $\phi(z) \in C_0^\infty(\Delta)$. By differentiating with respect to the parameter $\lambda_j$ or $\bar{\lambda}_j$ we get

$$\int_\Delta \frac{\partial}{\partial \lambda_j} G(z, \lambda) \, dz \wedge d\phi = 0, \quad \int_\Delta \frac{\partial}{\partial \bar{\lambda}_j} G(z, \lambda) \, dz \wedge d\phi = 0,$$

which shows that $\partial G/\partial \lambda_j$ and $\partial G/\partial \bar{\lambda}_j$ are holomorphic in $z$.

Since $F$ is holomorphic in $z$ for every $\lambda$ and since $F$ is $C^1$-smooth, $\partial F/\partial \lambda$ and $\partial F/\partial \bar{\lambda}$ at $\lambda = 0$ are also holomorphic in $z$. By the chain rule,

$$\begin{aligned}
\frac{\partial F}{\partial \lambda}_{\lambda=0} &= A \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0} + B \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0}, \\
\frac{\partial F}{\partial \bar{\lambda}}_{\lambda=0} &= B \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0} + A \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0},
\end{aligned}$$

where $\partial F/\partial \lambda$ denotes the column of partial derivatives $(\partial F/\partial \lambda_1, \ldots, \partial F/\partial \lambda_{n-1})$ etc. and $A = (A_{mj})$ and $B = (B_{mj})$ are the matrices whose entries are defined by the partial derivatives as follows: $A_{mj} = \frac{\partial k_m}{\partial \lambda_j} |_{\lambda=0}$ and $B_{mj} = \frac{\partial k_m}{\partial \bar{\lambda}_j} |_{\lambda=0}$.

From $k_m(z, \lambda) = z\lambda_m + o(|z|^2)$ we get

$$A_{mj} = \frac{\partial k_m}{\partial \lambda_j} = \delta_{mj} z + o(|z|), \quad B_{mj} = \frac{\partial k_m}{\partial \bar{\lambda}_j} = o(|z|).$$

Hence the matrix $\hat{A} := \frac{1}{z} A = \text{id} + o(1)$ is invertible in some neighbourhood of 0. Denote $(\hat{A})^{-1} = C$. Then the entries of $C$ are holomorphic functions. Let

$$H(z) := BC \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=0} - z \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=0} = (BCB - z\hat{A}) \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0}.$$

Then $H(z)$ is holomorphic on $\Delta$.  

Notice that we are using the standard complex coordinate system $(z, w_1, \ldots, w_{n-1})$ for $\mathbb{C}^n$ here.
Now we shall prove: $H(z) = 0, \forall z \in \Delta$. In order to show this, we need the following lemma. Below, the notation $\mathbb{C}[[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_k]]$ represents the local ring of formal power series in the variables $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_k$ at the origin. Of course, the unique maximal ideal is the set of all formal power series without the constant term.

**Lemma 2.3.** Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}[[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_k]]$, $\psi \in \mathbb{C}[[z_1, \ldots, z_n]]$ and $\varphi_1, \ldots, \varphi_m \in \mathfrak{M}$, where $\mathfrak{M}$ is the maximal ideal of the local ring $\mathbb{C}[[z_1, \ldots, z_k]]$. Then

$$\psi = \alpha_1 \varphi_1 + \ldots + \alpha_m \varphi_m$$

implies $\psi = 0$.

**Proof.** Assume $\psi \neq 0$ and let $\tilde{\psi}$ be the lowest degree non-vanishing polynomial in $\psi$. Then

$$\tilde{\psi} = \tilde{\alpha}_1 \tilde{\varphi}_1 + \ldots + \tilde{\alpha}_m \tilde{\varphi}_m,$$

where $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m$ are certain polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_k$ and $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m$ are polynomials in $\bar{z}_1, \ldots, \bar{z}_k$ of positive degree. Now, $\psi$ does not contain variables $\bar{z}_1, \ldots, \bar{z}_k$, whereas each monomial in $\tilde{\alpha}_1 \tilde{\varphi}_1 + \ldots + \tilde{\alpha}_m \tilde{\varphi}_m$ does contain such variables. This contradiction shows that $\psi = 0$. \qed

The following statement is then immediate:

**Corollary 2.4.** Assume that $\varphi_1, \ldots, \varphi_m$ and $\alpha_1, \ldots, \alpha_m$ are complex-valued functions defined on a domain $\Omega$ in the complex plane $\mathbb{C}$, which enjoy the properties:

(a) $\varphi_k$ has a formal Taylor expansion at $p$, and
(b) $\alpha_k$ is conjugate-holomorphic in an open neighborhood of $p$ with $\alpha_k(p) = 0$

for $k = 1, \ldots, m$. If $\psi := \alpha_1 \varphi_1 + \ldots + \alpha_m \varphi_m$ is holomorphic in $\Omega$, then $\psi$ is identically zero.

Now we return to $H(z)$. From (2.1) it follows that the anti-holomorphic terms $\frac{\partial k_m}{\partial \lambda_j}$ and $\frac{\partial k_m}{\partial \bar{\lambda}_j}$ vanish at $\zeta = 0$. Therefore, the components of $H(z)$ have the form of a function $\psi$ from the Corollary 2.3 with the $\varphi$'s being products of $\frac{\partial f}{\partial \lambda_j}$ and some holomorphic factors from the matrices $A, B, C$ and the $\alpha$'s being products of some antiholomorphic factors from the matrices $A, B$. It follows $H(z) \equiv 0$ and hence

$$(BCB - z\bar{A}) \frac{\partial f}{\partial \bar{w}} \bigg|_{w=0} \equiv 0.$$

Finally, even though $BCB - z\bar{A}$ vanishes at the origin, its determinant equals

$$\det(BCB - z\bar{A}) = (-1)^{n-1}|z|^{2n-2} + o(|z|^{2n-2})$$

and therefore has no zeroes in some punctured neighborhood of $0$. It follows that

$$\frac{\partial f}{\partial \bar{w}} \bigg|_{w=0} = 0.$$

Since $\{w = 0\}$ was an arbitrary line leaf the Cauchy-Riemann equations are satisfied transversally to each leaf. This completes the proof. \qed

Finally, for the global version of generalized Forelli’s theorem, we recall that the following definition of global singular foliation.
Definition 2.5. Let $\Omega$ be a domain in $\mathbb{C}^n$ containing the origin. By a $C^1$ singular foliation at 0 by holomorphic discs we mean a $C^1$ map $h: \Delta \times S^{2n-1} \to \Omega$ satisfying the following properties:

1. For each $v \in S^{2n-1}$, the correspondence $h(\cdot, v): z \in \Delta \to h(z, v) \in \mathbb{C}^n$ is a holomorphic embedding.
2. $h(0, v) = 0$ for every $v \in S^{2n-1}$.
3. $h(\Delta \times S^{2n-1}) = \Omega$.
4. For each $v \in S^{2n-1}$, there exists $r_v > 0$ such that $\left. \frac{\partial h}{\partial z} \right|_{(0,v)} = r_v v$.
5. $h(z, e^{i\theta} v) = h(e^{i\theta} z, v)$ for any $\theta \in \mathbb{R}$ and $z \in \Delta$.

Then we present

Corollary 2.6. Let $S_\lambda$ be the typical leaf (i.e., a holomorphic disc) of the singular foliation as above of a domain $\Omega \subset \mathbb{C}^n$ and $f$ a function that is $C^\infty(0)$ and $C^1(\Omega)$ such that the restrictions $f|_{S_\lambda}$ are holomorphic. Then $f$ is holomorphic on $\Omega$.

This follows from Theorem 2.2 and Chirka’s curvilinear Hartogs’ lemma from [1] (cf. [3]).

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