A REMARK ON THE PAPER “A UNIFIED PIETSC H DOMINATION THEOREM”

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Abstract. In this short communication we show that the Unified Pietsch Domination proved in [1] remains true even if we remove two of its apparently crucial hypothesis.

1. Introduction

Let $X$, $Y$ and $E$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X$ to $Y$, $G$ be a Banach space and $K$ be a compact Hausdorff topological space. Let $R: K \times E \times G \to [0, \infty)$ and $S: \mathcal{H} \times E \times G \to [0, \infty)$ be arbitrary mappings.

A mapping $f \in \mathcal{H}$ is said to be $RS$-abstract $p$-summing if there is a constant $C > 0$ so that

\[
\left( \sum_{j=1}^{m} S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{j=1}^{m} R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}},
\]

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$.

The main result of [1] proves that under certain assumptions on $R$ and $S$ there is a quite general Pietsch Domination-type Theorem. More precisely $R$ and $S$ must satisfy the three properties below:

1. For each $f \in \mathcal{H}$, there is a $x_0 \in E$ such that $R(\varphi, x_0, b) = S(f, x_0, b) = 0$ for every $\varphi \in K$ and $b \in G$.
2. The mapping $R_{x,b}: K \to [0, \infty)$ defined by $R_{x,b}(\varphi) = R(\varphi, x, b)$ is continuous for every $x \in E$ and $b \in G$.
3. For every $\varphi \in K, x \in E, 0 \leq \eta \leq 1, b \in G$ and $f \in \mathcal{H}$, the following inequalities hold: $R(\varphi, x, \eta b) \leq \eta R(\varphi, x, b)$ and $\eta S(f, x, b) \leq S(f, x, \eta b)$.

The Pietsch Domination Theorem from [1] reads as follows:

**Theorem 1.1.** If $R$ and $S$ satisfy (1), (2) and (3) and $0 < p < \infty$, then $f \in \mathcal{H}$ is $RS$-abstract $p$-summing if and only if there is a constant $C > 0$ and a Borel probability measure $\mu$ on $K$ such that

\[
S(f, x, b) \leq C \left( \int_{K} R(\varphi, x, b)^p \, d\mu \right)^{\frac{1}{p}}
\]

for all $x \in E$ and $b \in G$.

The aim of this note is to show that, surprisingly, the hypothesis (1) and (3) are not necessary. So, Theorem 1.1 is true for arbitrary $S$ (no hypothesis is needed) and the map $R$ just needs to satisfy (2).
2. A recent approach to PDT

In a recent preprint \cite{3} we have extended the Pietsch Domination Theorem from $\Pi$ to a more abstract setting, which allows to deal with more general nonlinear mappings in the cartesian product of Banach spaces. In the present note we shall recall the argument used in \cite{3} and a combination of this argument with an interesting argument due to M. Mendel and G. Schechtman (used in $\Pi$) will help us to show that Theorem 1.1 is valid without the hypothesis (1) and (3) on $R$ and $S$.

The first step is to prove Theorem 1.1 without the hypothesis (1). This result is proved in \cite{3} in a more general setting. Since the paper \cite{3} is unpublished and we just need a very particular case, we prefer to sketch the proof for this particular case. The proof of this particular case is essentially Pietsch’s original proof on a nonlinear disguise.

**Theorem 2.1.** Suppose that $R$ and $S$ satisfy (2) and (3). A map $f \in \mathcal{H}$ is RS-abstract $p$-summing if and only if there is a constant $C > 0$ and a Borel probability measure $\mu$ on $K$ such that

\[ S(f, x, b) \leq C \left( \int_K R(\varphi, x, b)^p d\mu \right)^{1/p} \]

for all $x \in E$ and $b \in G$.

**Proof.** If (2.1) holds it is easy to show that $f$ is RS-abstract $p$-summing. For the converse, consider the (compact) set $P(K)$ of the probability measures in $C(K)^*$ (endowed with the weak-star topology). For each $(x_j)_{j=1}^m$ in $E$, $(b_j)_{j=1}^m$ in $G$ and $m \in \mathbb{N}$, let $g : P(K) \to \mathbb{R}$ be defined by

\[ g(\mu) = \sum_{j=1}^m \left[ S(f, x_j, b_j)^p - C_p \int_K R(\varphi, x_j, b_j)^p d\mu \right] \]

and $\mathcal{F}$ be the set of all such $g$. Using (3), one can prove that the family $\mathcal{F}$ is concave and each $g \in \mathcal{F}$ is convex and continuous. Besides, for each $g \in \mathcal{F}$ there is a measure $\mu_g \in P(K)$ such that $g(\mu_g) \leq 0$. In fact, from (2) there is a $\varphi_0 \in K$ so that

\[ \sum_{j=1}^m R(\varphi_0, x_j, b_j)^p = \sup_{\varphi \in K} \sum_{j=1}^m R(\varphi, x_j, b_j)^p \]

and, considering the Dirac measure $\mu_g = \delta_{\varphi_0}$, we have $g(\mu_g) \leq 0$. So, Ky Fan Lemma asserts that there exists a $\overline{\mu} \in P(K)$ so that

\[ g(\overline{\mu}) \leq 0 \]

for all $g \in \mathcal{F}$ and by choosing an arbitrary $g$ with $m = 1$ the proof is done. \hfill $\square$

3. The main result

Note that if each $\lambda_j$ is a positive integer, by considering each $x_j$ repeated $\lambda_j$ times in (1.1) one can easily see that (1.1) is equivalent to

\[ \left( \sum_{j=1}^m \lambda_j S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{j=1}^m \lambda_j R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}} \]

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$, positive integers $\lambda_j$ and $m \in \mathbb{N}$. Then it is possible to show that (1.1) holds for positive rationals and finally extend to positive real numbers $\lambda_j$ (using an argument of density). The essence of this argument appears in $\Pi$ \cite{2} and is credited to M. Mendel and G. Schechtman.

Now using (3.1) and invoking Theorem 2.1 we can prove a Pietsch Domination-type theorem with no hypothesis on $S$ and just supposing that $R$ satisfies (2):
Theorem 3.1. Suppose that $S$ is arbitrary and $R$ satisfies (2). A map $f \in H$ is $RS$-abstract $p$-summing if and only if there is a constant $C > 0$ and a Borel probability measure $\mu$ on $K$ such that

$$S(f, x, b) \leq C \left( \int_K R(\varphi, x, b)^p \, d\mu \right)^{1/p}$$

for all $x \in E$ and $b \in G$.

Proof. It is clear that if $f$ satisfies (3.2) then $f \in H$ is $RS$-abstract $p$-summing. Conversely, if $f \in H$ is $RS$-abstract $p$-summing, then

$$\left( \sum_{j=1}^{m} \lambda_j S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{j=1}^{m} \lambda_j R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}}$$

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$, $\lambda_1, \ldots, \lambda_m \in [0, \infty)$ and $m \in \mathbb{N}$. Let

$$E_1 = E \times G \quad \text{and} \quad G_1 = K$$

and define

$$\overline{R}: K \times E_1 \times G_1 \rightarrow [0, \infty) \quad \text{and} \quad \overline{S}: \mathcal{H} \times E_1 \times G_1 \rightarrow [0, \infty)$$

by

$$\overline{R}(\varphi, (x, b), \lambda) = |\lambda| R(\varphi, x, b) \quad \text{and} \quad \overline{S}(f, (x, b), \lambda) = |\lambda| S(f, x, b).$$

From (3.3) we conclude that

$$\left( \sum_{j=1}^{m} \overline{S}(f, (x_j, b_j), \eta_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{j=1}^{m} \overline{R}(\varphi, (x_j, b_j), \eta_j)^p \right)^{\frac{1}{p}}$$

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$, $\eta_1, \ldots, \eta_m \in K$ and $m \in \mathbb{N}$.

Since $\overline{R}$ and $\overline{S}$ satisfy (2) and (3), from Theorem 2.1 we conclude that there is a measure $\mu$ so that

$$\overline{S}(f, (x, b), \eta) \leq C \left( \int_K \overline{R}(\varphi, (x, b), \eta)^p \, d\mu \right)^{1/p}$$

for all $x \in E$, $b \in G$ and $\eta \in K$. Hence it easily follows that, for all $x \in E$ and $b \in G$, we have

$$S(f, x, b) \leq C \left( \int_K R(\varphi, x, b)^p \, d\mu \right)^{1/p}.$$

$\blacksquare$

References

[1] G. Botelho, D. Pellegrino and P. Rueda, A unified Pietsch Domination Theorem, J. Math. Anal. Appl. 365 (2010), 269-276.

[2] J. Farmer and W. B. Johnson, Lipschitz $p$-summing operators, Proc. Amer. Math. Soc. 137 (2009), 2989-2995.

[3] D. Pellegrino and J. Santos, On summability of nonlinear maps: a new approach, preprint, arXiv:1004.2643v2.