Geodesic growth in virtually abelian groups

Alex Bishop

Abstract. We show that no finitely generated virtually abelian group has intermediate geodesic growth, and that the language of geodesics for such a group is blind multicounter.

2010 Mathematics Subject Classification: 20K35, 68Q45.
Keywords: virtually abelian group, geodesic language, geodesic growth, blind multicounter automata.

1. Introduction

The concept of growth in groups is well-studied, with famous results including Gromov’s classification of groups with polynomial growth [12], and Grigorchuk’s example of a group of intermediate growth [10] which initiated a great interest in such groups [1, 5, 13, 14]. In this paper, we are interested in analogous results for geodesic growth.

Bridson et al. [3] asked if there exists a group with intermediate geodesic growth, and if there is a characterisation of groups with polynomial geodesic growth. Towards the second question, [3] furnished an example of a virtually \( \mathbb{Z}^2 \) group with polynomial geodesic growth with respect to a certain finite generating set. They showed that there is no nilpotent group with intermediate geodesic growth, and provided a sufficient condition for a virtually abelian group to have polynomial geodesic growth; in particular, they showed that \( \mathbb{Z}^2 \) itself can only have exponential geodesic growth with respect to any finite generating set. In this paper, we extend this work by showing that no virtually abelian group can have intermediate geodesic growth with respect to any finite generating set.

In the process of understanding the geodesics of virtually abelian groups, we prove that the language of geodesics is a blind multicounter language, as defined by Greibach [9]. This result is of particular interest as it was shown in [7] that a group has a word problem which is blind multicounter if and only if it is virtually abelian. Thus, this paper is also a step in extending the known results of formal language classification of geodesics [4, 6, 8, 11].

After establishing notation in Section 2, we begin by introducing the concept of polyhedral sets in Section 3 which will become crucial in the proof of our main theorems. In Section 4, we introduce a special form for the geodesics in virtually abelian groups, which we show to be polyhedral, and give a deterministic algorithm to slide geodesics into this standard form. The content in Sections 3 and 4 follow Benson [2]. Then, in Sections 5 and 6 we make use of these properties to prove our main theorems.

This research is supported by an Australian Government Research Training Program Scholarship, and by Australian Research Council grant DP160100486 (chief investigator Murray Elder, University of Technology Sydney).
2. Notation

In this paper, we will write $\mathbb{N} = \{0, 1, 2, \ldots\}$ to denote the nonnegative integers, including zero, and $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ for the positive integers.

Let $G$ be a group generated as a monoid by a finite subset $S$ where each $s \in S$ has a positive integer weight given by $\omega(s) \in \mathbb{Z}_+$. Then, we write $S^*$ to denote the free monoid generated by $S$, and we define the \textit{weighted length} for words $\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in S^*$, denoted as $\omega(\sigma)$, such that

$$\omega(\sigma) = \omega(\sigma_1) + \omega(\sigma_2) + \cdots + \omega(\sigma_k).$$

Moreover, we write $|\sigma|_S = k$ for the \textit{word length} of $\sigma$, and $\varpi \in G$ for the group element corresponding to $\sigma$. Notice here that the word length and weighted length of the empty word, $\varepsilon \in S^*$, is zero.

We define a weighting on elements of $G$, given by $\omega' : G \to \mathbb{N}$, to be the minimal weighted length of a word representing such an element, that is,

$$\omega'(g) = \min\{\omega(\sigma) \mid \sigma = g \text{ where } \sigma \in S^*\}$$

for each $g \in G$. Notice here that $\omega'(1) = 0$ as the identity element of $G$ can be represented by the empty word in $S^*$.

Hence, we say that a word $\sigma \in S^*$ is a \textit{geodesic} if $\omega(\sigma) = \omega'(\sigma)$. Thus, we define the \textit{geodesic growth function} $\Gamma_S : \mathbb{N} \to \mathbb{N}$ such that $\Gamma_S(n)$ counts the number of geodesic words with weighted length $n$ or less, that is,

$$\Gamma_S(n) = \#\{\sigma \in S^* \mid \omega(\sigma) \leq n \text{ and } \sigma \text{ is geodesic}\}.$$  

We say $G$ has \textit{polynomial geodesic growth} with respect to $S$ if there are constants $C, d \in \mathbb{Z}_+$ such that $\Gamma_S(n) \leq C \cdot n^d$ for each $n > 0$; \textit{exponential geodesic growth} with respect $S$ if there is a positive constant $\alpha \in \mathbb{R}_+$ such that $\Gamma_S(n) \geq \alpha^n$; or \textit{intermediate geodesic growth} with respect to $S$ if its geodesic growth is neither polynomial nor exponential.

3. Polyhedral Sets

In this section we introduce the concept of polyhedral sets. Our notation follows that of Benson [2]. Moreover, this section also provides two technical lemmas which will become crucial in the proof of Theorem 5.1.

A subset $E \subseteq \mathbb{Z}^m$ is called an \textit{elementary region} if it can be expressed as

$$\{z \in \mathbb{Z}^m \mid a \cdot z = b\}, \{z \in \mathbb{Z}^m \mid a \cdot z > b\} \text{ or } \{z \in \mathbb{Z}^m \mid a \cdot z \equiv b \pmod{c}\}$$

for some $a \in \mathbb{Z}^m$ and $b, c \in \mathbb{Z}$ with $c > 0$. A \textit{basic polyhedral set} is a finite intersection of elementary regions; and a \textit{polyhedral set} is a finite disjoint union of basic polyhedral sets. It can be seen from this definition that the sets $\emptyset$, $\mathbb{N}^m$ and $\mathbb{Z}^m$ are polyhedral. We have the following closure properties which will be used throughout this paper.

\textbf{Proposition 3.1} ([2, Proposition 13.1]). \textit{Polyhedral sets are closed under finite union, finite intersection and set difference.}

We say a map $E : \mathbb{Z}^m \to \mathbb{Z}^n$ is an integer affine transform if it can be written as $E(v) = Av + b$ where $A \in \mathbb{Z}^{n \times m}$ is a matrix and $b \in \mathbb{Z}^n$ is a vector. Then, we have the following additional closure property.
**Proposition 3.2** ([2, Propositions 13.7 and 13.8]). Suppose that \( \mathcal{P} \subseteq \mathbb{Z}^m \) and \( \mathcal{Q} \subseteq \mathbb{Z}^m \) are polyhedral sets and \( E: \mathbb{Z}^m \to \mathbb{Z}^n \) is an integer affine transformation. Then, \( E(\mathcal{P}) \) and \( E^{-1}(\mathcal{Q}) \) are both polyhedral sets.

### 3.1. Positive polyhedral sets

We will call a polyhedral set \( \mathcal{P} \) positive if \( \mathcal{P} \subseteq \mathbb{N}^m \). Suppose that \( \mathcal{B} \subseteq \mathbb{N}^m \) is a positive basic polyhedral set, then for any non-empty subset \( I \subseteq \{1, 2, \ldots, m\} \), we will say that \( \mathcal{B} \) is \( I \)-unbounded if for any given \( N \in \mathbb{N} \) there is a vector \( p = (p_1, p_2, \ldots, p_m) \in \mathcal{B} \) such that \( p_i > N \) for each \( i \in I \). Then, we have the following result.

**Lemma 3.3.** Suppose that \( \mathcal{B} \subseteq \mathbb{N}^m \) is a positive basic polyhedral set which is \( I \)-unbounded, then there exists a vector \( v = (v_1, v_2, \ldots, v_m) \in \mathbb{N}^m \) with \( v_i > 0 \) for each \( i \in I \), such that \( nv + u \in \mathcal{B} \) for each \( n \in \mathbb{N} \) and \( u \in \mathcal{B} \).

**Proof.** To prove this statement, we will construct an infinite sequence of vectors \( (w_j)_{j \in \mathbb{N}} \) such that for any choice of \( s, t \in \mathbb{N} \) with \( s < t \) we obtain the desired properties with \( v = w_t - w_s \).

Since \( \mathcal{B} \) is \( I \)-unbounded, then there exists an infinite sequence of distinct vectors \( (w_j)_{j \in \mathbb{N}} \), where each \( w_j = (w_{j,1}, w_{j,2}, \ldots, w_{j,m}) \in \mathcal{B} \), with the property that \( w_{j+1,i} > \max\{w_{j,k} \mid k \in I\} \) for each \( i \in I \) and \( j \in \mathbb{N} \).

Since \( \mathcal{B} \) is a basic polyhedral set, it can be written as an intersection

\[
\mathcal{B} = \bigcap_{\ell=1}^{L} \{ z \in \mathbb{Z}^m : \alpha_{\ell} \cdot z > \beta_{\ell} \} \\
\cap \bigcap_{x=1}^{X} \{ z \in \mathbb{Z}^m : \gamma_x \cdot z = \delta_x \} \cap \bigcap_{y=1}^{Y} \{ z \in \mathbb{Z}^m : \zeta_y \cdot z \equiv \eta_y \text{ (mod } \theta_y) \}
\]

where \( \alpha_{\ell}, \gamma_x, \zeta_y \in \mathbb{Z}^m \) are vectors, \( \beta_{\ell}, \delta_x, \eta_y, \theta_y \in \mathbb{Z} \) and each \( \theta_y > 1 \).

Now suppose that we chose \( v = w_t - w_s \) where \( s < t \). Then, for each \( x \) and \( y \) we have \( \gamma_x \cdot v = 0 \) and \( \zeta_y \cdot v \equiv 0 \text{ (mod } \theta_y) \). Thus, for each \( n \in \mathbb{N} \) and \( u \in \mathcal{B} \), the vector \( p_{n,u} = nv + u \) has the properties that \( \gamma_x \cdot p_{n,u} = \gamma_x \cdot u = \delta_x \) and \( \zeta_y \cdot p_{n,u} \equiv \zeta_y \cdot u \equiv \eta_y \text{ (mod } \theta_y) \) for each \( x \) and \( y \). Thus, if \( p_{n,u} \notin \mathcal{B} \) for some given \( n \in \mathbb{N} \), then it must be the case that \( \alpha_{\ell} \cdot p_{n,u} \leq \beta_{\ell} \) for some \( \ell \) with \( 1 \leq \ell \leq L \). Hence, we will now guarantee that situation cannot occur by performing the following procedure for each \( \ell \) from 1 to \( L \).

For each \( j \in \mathbb{N} \) we have the lower bound \( \alpha_{\ell} \cdot w_j \geq \beta_{\ell} \) and thus there exists an infinite subsequence \( (w_j')_{j \in \mathbb{N}} \) of \( (w_j)_{j \in \mathbb{N}} \) such that the sequence of integers \( (\alpha_{\ell} \cdot w_j')_{j \in \mathbb{N}} \) is monotone non-decreasing; and thus we have \( \alpha_{\ell} \cdot (w'_t - w'_s) \geq 0 \) for each \( s, t \in \mathbb{N} \) with \( s < t \). Hence, after choosing vectors \( u \in \mathcal{B} \) and \( v = w'_t - w'_s \) with \( s < t \), we have \( \alpha_{\ell} \cdot p_{n,u} \geq \alpha_{\ell} \cdot u > \beta_{\ell} \) for each \( n \in \mathbb{N} \). Thus, we will now replace the sequence \( (w_j)_{j \in \mathbb{N}} \) with \( (w'_j)_{j \in \mathbb{N}} \).

Let the vertex \( v = (v_1, v_2, \ldots, v_m) \) be defined as \( v = w_2 - w_1 \). Then, after performing the previously described procedure, we know that \( nv + u \in \mathcal{B} \) for each \( n \in \mathbb{N} \) and \( u \in \mathcal{B} \). Furthermore, \( v \in \mathbb{N}^m \) as otherwise we would have a component \( v_k < 0 \) and thus for a sufficiently large \( n \) we would have \( nv + u \notin \mathbb{N}^m \) which is a contradiction. Thus completes our proof. \( \square \)

For each \( I \subseteq \{1, 2, \ldots, m\} \), we say that a non-empty positive basic polyhedral set \( \mathcal{B} \subseteq \mathbb{N}^m \) is \( I \)-dominating if it is \( I \)-unbounded (or finite in the case
that $I = \emptyset$) and there exists a constant $C_B \in \mathbb{Z}_+$ such that $v_j \leq C_B$ for each $v = (v_1, v_2, \ldots, v_m) \in B$ and $j \notin I$. We have the following result.

**Lemma 3.4.** Given a positive basic polyhedral set $B \subseteq \mathbb{N}^m$, there is a subset $I \subseteq \{1, 2, \ldots, m\}$ such that $B$ is $I$-dominating; and there exists a vector $v = (v_1, v_2, \ldots, v_m) \in \mathbb{N}^m$ such that $v_i > 0$ for each $i \in I$, and $v_j = 0$ for each $j \notin I$ with the property that $nv + u \in B$ for each $n \in \mathbb{N}$ and $u \in B$.

**Proof.** For each $k \in \{1, 2, \ldots, m\}$, either $B$ is $\{k\}$-unbounded or there is a constant $C_{B,k} \in \mathbb{N}$ for which $v_k \leq C_{B,k}$ for every $v \in B$. Define the subset $I \subseteq \{1, 2, \ldots, m\}$ such that $i \in I$ if and only if $B$ is $\{i\}$-unbounded; and let $C_B \in \mathbb{Z}_+$ be an upper bound on all $C_{B,j}$ where $j \notin I$.

From Lemma 3.3, it follows that for each index $i \in I$ there is a vector $v^{(i)} = (v^{(i)}_1, v^{(i)}_2, \ldots, v^{(i)}_m) \in \mathbb{N}^m$ with $v^{(i)}_i > 0$ such that $nv^{(i)} + u \in B$ for each $n \in \mathbb{N}$ and $u \in B$. Thus, after defining $v = \sum_{i \in I} v^{(i)}$, we see that $v_i > 0$ for each $i \in I$, and $nv + u \in B$ for each $n \in \mathbb{N}$ and $u \in B$.

We then have $v_j = 0$ for each $j \notin I$ as otherwise we have $(C_B + 1)v + u \notin B$ for each $u \in B$ which is a contradiction. Thus completes the proof. $\square$

4. Special Form

Henceforth, we will suppose that $G$ is a finitely generated virtually abelian group. Then, by definition, $G$ has a finite index subgroup $H$ isomorphic to $\mathbb{Z}^n$ for some $n$. Moreover, we may assume without loss of generality that $H$ is normal in $G$ where generality is maintained by taking the normal core of $H$ which must also be free abelian [15, pp. 100-1].

Fix a finite index, free abelian, normal subgroup $H \cong \mathbb{Z}^n$, and write $d$ for the index of $H$ in $G$. Fix coset representatives $T = \{t_1 = 1, t_2, t_3, \ldots, t_d\}$, and a standard basis $h_1, h_2, \ldots, h_n$ for $H$ such that the elements of $H$ can be written as $(u_1, u_2, \ldots, u_n) \in H$ with each factor $u_j \in \mathbb{Z}$. Define the maps $\phi: G \to H$ and $\rho: G \to T$ such that $g = \phi(g) \cdot \rho(g)$ for each $g \in G$. That is, for each $g \in H \cdot t$ with $t \in T$, we have $\rho(g) = t$ and thus $\phi(g) = g \cdot \rho(g)^{-1}$.

From the weighted generating set $S$, we define

$$
Y = \left\{ \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in S^* \mid 1 \leq k \leq d, \sigma \in H, \text{ and } \sigma_i \sigma_2 \cdots \sigma_q \notin H \text{ when } 1 \leq q < k \right\}
$$

and

$$
P = \left\{ \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in S^* \mid 1 \leq k \leq d, \text{ and } \sigma_i \sigma_{i+1} \cdots \sigma_j \notin H \text{ when } 1 \leq i \leq j \leq k \right\}
$$

where we extend the weighted length $\omega$ to $Y$ and $P$ in the natural way. Henceforth, we write $m$ to denote the number of elements in $Y$ and we fix a labelling $\{y_1, y_2, \ldots, y_m\} = Y$ of its elements.

**Definition 4.1.** We call a word $\pi = \pi_1 \pi_2 \cdots \pi_k \in P^*$ a pattern if $k \leq d + 1$. Then, we write $N^\pi$ to denote the set of vectors $\mathbb{N}^{(k+1)m}$ and for each vector $v = (v_1, v_2, \ldots, v_{(k+1)m}) \in N^\pi$ we write $v^\pi$ to denote the word

$$
v^\pi = \left( y_1^{v_1} y_2^{v_2} \cdots y_m^{v_m} \right) \pi_1 \left( y_1^{v_{m+1}} y_2^{v_{m+2}} \cdots y_m^{v_{2m}} \right) \pi_2 \cdots \pi_k \left( y_1^{v_{mk+1}} y_2^{v_{mk+2}} \cdots y_m^{v_{(k+1)m}} \right)
$$

which we refer to as a word in special form which is patterned by $\pi$. 
Notice that if \( v \in N^\epsilon \), then \( v^\pi = y_1^{v_1} y_2^{v_2} \cdots y_m^{v_m} \) and thus \( \overline{v} \in H \).

4.1. Sliding window algorithm. Given two words \( \sigma \) and \( \zeta \), we will write \( \sigma \simeq \zeta \) to denote that both \( \overline{\sigma} = \overline{\zeta} \) and \( \omega(\sigma) = \omega(\zeta) \). In this section we describe a deterministic algorithm to convert a word \( \sigma \in S^* \) to a patterned word in special form \( v^\pi \) such that \( v^\pi \simeq \sigma \). The description of this algorithm is essential to the proof of our main theorems, i.e., Theorems 5.1 and 6.2.

The following observation enables this algorithm to be deterministic.

**Lemma 4.2.** Suppose that \( \Xi \in S^* \) with \( 1 \leq |\Xi|_S \leq d \), then either \( \Xi \in P \) and \( |\Xi|_S < d \), or there exists a unique \( y \in Y \) and \( p \in P \cup \{ \epsilon \} \) such that \( \Xi \) factors as \( \Xi \in pyS^* \) and \( |p|_S \) is minimal.

**Proof.** It is clear that if \( \Xi \in P \), then \( |\Xi|_S < d \); as otherwise, by the pigeonhole principal on the cosets in \( G \), we would have a non-empty factor \( y \) of \( \Xi \) such that \( \overline{y} \in H \) which is a contradiction.

Suppose now that \( \Xi = \xi_1 \xi_2 \cdots \xi_t \notin P \) with \( 1 \leq t \leq d \). Then, by the definition of the set \( P \), there is at least one non-empty factor \( y = \xi_i \xi_{i+1} \cdots \xi_j \) of \( \Xi \), for which \( \overline{y} \in H \).

Without loss of generality we will assume that \( y \) appears as early as possible in \( \Xi \); that is, after writing such factors as \( \xi_1 \xi_2 \cdots \xi_j \), then we choose the factor with smallest \( i \). It then follows that this choice is unique.

We now write \( p = \xi_1 \xi_2 \cdots \xi_{j-1} \), then \( p \) cannot contain a factor in \( Y \) as this would contradict our earlier choice of \( y \). Thus, \( p \in P \cup \{ \epsilon \} \).

It is clear then that if \( \Xi \) factors as \( \Xi \in p'y' S^* \) with \( y' \in Y \), \( p' \in P \cup \{ \epsilon \} \) and \( |p' y'|_S \leq |p|_S \), then \( p' = p \) and \( y' = y \) as otherwise we contradict our earlier choice of \( y \). Thus completes our proof.

We are now ready to describe our procedure. Throughout this algorithm we will keep track of a word \( \sigma' \in S^* \), a pattern \( \varphi = \varphi_1 \varphi_2 \cdots \varphi_k \in P^* \) and vectors \( u_0, u_1, u_2, \ldots, u_k \in N^m \); where we will have the invariant that \( (u_0, u_1, u_2, \ldots, u_k)^\varphi \sigma' \simeq \sigma \); and at each step of the algorithm we decrease the word length \( |\sigma'|_S \in N \). Thus, the algorithm will terminate when \( \sigma' = \epsilon \).

Throughout this algorithm, we will also keep track of a word \( \Xi \in S^* \) called the window. The window, \( \Xi \), is the longest prefix of \( \sigma' \) with length no greater than \( d \). Thus, if \( \sigma' = \sigma_1 \sigma_2 \cdots \sigma_t \), then \( \Xi = \sigma_1^\tau \sigma_2 \cdots \sigma_t \) with \( \tau = \min(t, d) \).

We begin this algorithm by setting \( \sigma' \) to the input word \( \sigma \), we set \( \varphi \in P^* \) to the empty word, and \( u_0 \in N^m \) to the zero vector. Thus, at the beginning of this algorithm we have \( (u_0)^\varphi \sigma' \simeq \sigma \) and thus we have the invariant \( (u_0)^\varphi \sigma' \simeq \sigma \) as required. To complete this algorithm, we repeat the following procedure.

Suppose that \( \varphi = \varphi_1 \varphi_2 \cdots \varphi_k \) and \( \sigma' = \sigma_1^\tau \sigma_2 \cdots \sigma_t \) with the window given by \( \Xi = \xi_1 \xi_2 \cdots \xi_{\tau} \) where \( \tau = \min(t, d) \).

1. If \( \Xi = \epsilon \), then after setting \( v = (u_0, u_1, u_2, \ldots, u_k) \in N^{(k+1)m} \) and \( \pi = \varphi \), we have \( v^\pi \simeq \sigma \), and thus we end the procedure.

2. If \( \Xi \in P \) and thus \( |\Xi|_S < d \) by Lemma 4.2, then we add \( \Xi \in P \) as a letter to the end of \( \varphi \) and set \( u_{K+1} \in N^m \) to be the zero vector. Thus, after replacing \( \sigma' \) with the empty word, we have the algorithm invariant \( (u_0, u_1, u_2, \ldots, u_k, u_{K+1})^\varphi \sigma' \simeq \sigma \), and the length \( |\sigma'|_S \) has been decreased to zero.
3. Otherwise, we factor $\Xi$ as $pyS^*$ where $p = \xi_1 \xi_2 \cdots \xi_{i-1} \in P \cup \{\varepsilon\}$ and $y = \xi_i \xi_{i+1} \cdots \xi_j \in Y$ are the unique factors given by Lemma 4.2. Thus, in this part of the procedure we move the factor $y$ out of the window $\Xi$ and into one of the vectors $u_\ell$ as follows.

Considering that we are currently representing the word $(u_0, u_1, u_2, \ldots, u_k)^p y \sigma_j \sigma_{j+1} \cdots \sigma_t,$ we see that $y \in H$ contributes to $\sigma$ from within the coset $r = \rho(\varphi p).$

Thus, we consider the following two sub-cases.

(a) If there is an $\ell \in \mathbb{N}$ such that $r = \rho(\varphi_1 \varphi_2 \cdots \varphi_\ell),$ then we can move $y$ into the vector $u_\ell = (u_{\ell,1}, u_{\ell,2}, \ldots, u_{\ell,m}).$ In particular, if $y = y_q \in Y,$ then we replace $u_\ell$ with $(u_{\ell,0}, u_{\ell,1}, \ldots, u_{\ell,q-1}, u_{\ell,q} + 1, u_{\ell,q+1}, \ldots, u_{\ell,m})$

and replace $\sigma'$ with $p\sigma_j' \sigma_{j+1} \cdots \sigma_t'.$ Thus, it can be seen that we still have the invariant $(u_0, u_1, \ldots, u_k)^p \sigma' \simeq \sigma$ as required, and the length $|\sigma'|_S$ has decreased by $|y|_S \geq 1.$

(b) Otherwise, we must add a new vector $u_{k+1} \in \mathbb{N}^m.$ In particular, if $y = y_q \in Y,$ then we add $p$ as a letter to the end of $\varphi$ and set $u_{k+1} \in \mathbb{N}^m$ to the $q$-th standard basis vector of $\mathbb{N}^n.$

Thus, after replacing $\sigma'$ with $\sigma_j' \sigma_{j+1} \cdots \sigma_t'$, we still have the invariant $(u_0, u_1, \ldots, u_k)^p \sigma' \simeq \sigma,$ and the length $|\sigma'|_S$ has been decreased by $|py|_S \geq 1.$

Thus, at the end of the previously described procedure, we have the desired special form $v^\pi \simeq \sigma.$ Consider that in previously describe procedure, we repeat case 3 some number of times, followed potentially by case 2 (at most once) and finishing with case 1; and further, repeating case 3 will not increase the length of $\varphi$ to more than $d,$ and cases 2 can add at most one letter to $\varphi.$ Thus, we see that the word $\pi = \varphi = \varphi_1 \varphi_2 \cdots \varphi_k \in P^*$ is a pattern, that is, $k \leq d + 1.$ Hence, we have the following lemma.

**Lemma 4.3.** If $\sigma \in S^*$ is a geodesic word, then there is a patterned word in special form, $v^\pi,$ such that $v^\pi \simeq \sigma.$

4.2. **Computations in special form words.** Given a word $v^\pi$ in special form we will need to determine the group element it represents and its weighted length with respect to $\omega.$ In the following we see that these problems reduce to a computation in linear algebra.

Let $3 \in \mathbb{Z}^{m \times n}$ be the matrix defined such that we have $v3 = \overline{v^\pi} \in H$ for each $v \in \mathbb{N}^m.$ For each $g \in G,$ let $G^g \in \mathbb{Z}^{n \times n}$ be the integer matrix defined such that $h G = gh g^{-1}$ for each $h \in H.$

For each pattern $\pi = \pi_1 \pi_2 \cdots \pi_k \in P^*$ and $v = (v_1, v_2, \ldots, v_{(k+1)m}) \in \mathbb{N}^\pi,$ we may write the element corresponding to $v^\pi$ as

$$\overline{\pi} = \left[ (v_1, v_2, \ldots, v_m)3 + (v_{m+1}, v_{m+2}, \ldots, v_{2m})3 G^{v_1} + \cdots + (v_{km+1}, v_{km+2}, \ldots, v_{(k+1)m})3 G^{v_{v_1}} G^{v_{v_2}} \cdots G^{v_{v_k}} \right] \cdot \pi.$$
Thus, after defining
\[ \mathcal{M}^\pi = \begin{pmatrix} 3 \\ 3 \mathfrak{g}^\pi_1 \\ \vdots \\ 3 \mathfrak{g}^\pi_1 \mathfrak{g}^\pi_2 \cdots \mathfrak{g}^\pi_k \end{pmatrix}, \]
we can write this as \( v^\pi = e^{\mathcal{M}^\pi} \cdot \pi \). Thus, if we write \( a_i^\pi \in \mathbb{Z}^{(k+1)m} \) to denote the vector corresponding the \( i \)-th column of the matrix \( \mathcal{M}^\pi \), and we define integers \( b_i^\pi \in \mathbb{Z} \) such that \((b_1^\pi, b_2^\pi, \ldots, b_n^\pi) = \phi(\pi) \in H\), then we have
\[ v^\pi = (a_1^\pi \cdot v + b_1^\pi, a_2^\pi \cdot v + b_2^\pi, \ldots, a_n^\pi \cdot v + b_n^\pi) \cdot \rho(\pi). \quad (4.1) \]

For each pattern \( \pi = \pi_1 \pi_2 \cdots \pi_k \in P^* \), define a vector \( a_{n+1}^\pi \in \mathbb{Z}^{(k+1)m} \) such that its components consist of the sequence \( \omega(y_1), \omega(y_2), \ldots, \omega(y_m) \) repeated \( k + 1 \) times, that is,
\[ a_{n+1}^\pi = (\omega(y_1), \omega(y_2), \ldots, \omega(y_m), \cdots, \omega(y_1), \omega(y_2), \ldots, \omega(y_m)); \]
and we define a constant \( b_{n+1}^\pi = \omega(\pi) \).

Thus, the weighted length of the word \( v^\pi \) is given by
\[ \omega(v^\pi) = a_{n+1}^\pi \cdot v + b_{n+1}^\pi. \quad (4.2) \]

Hence, we have the following lemma.

**Lemma 4.4.** For each pattern \( \pi = \pi_1 \pi_2 \cdots \pi_k \in P^* \), there exists vectors \( a_i^\pi, a_2^\pi, \ldots, a_{n+1}^\pi \in \mathbb{Z}^{(k+1)m} \) and integers \( b_1^\pi, b_2^\pi, \ldots, b_{n+1}^\pi \in \mathbb{Z} \) such that equations (4.1) and (4.2), given above, hold for each \( v \in \mathcal{N}^\pi \).

4.3. **Geodesics in special form.** For each pattern \( \pi \), we will now construct a set \( \mathcal{F}^\pi \subseteq \mathcal{N}^\pi \) such that \( v \in \mathcal{F}^\pi \) if and only if \( v^\pi \) is a geodesic when written as a word in \( S^* \).

Suppose that \( v^\pi \) is a word in special form, then from Lemma 4.3 we know that \( v^\pi \) is geodesic if and only if there is no other word \( w^\pi \) in special form for which \( v^\pi = w^\pi \) and \( \omega(v^\pi) > \omega(w^\pi) \). Notice that it would then follow that \( \rho(\pi) = \rho(\pi') \) since \( \rho(v^\pi) = \rho(w^\pi) \).

For each pattern \( \pi = \pi_1 \pi_2 \cdots \pi_k \in P^* \), we define \( E^\pi : \mathbb{Z}^{(k+1)m} \to \mathbb{Z}^{n+1} \) as
\[ E^\pi(v) = (a_1^\pi \cdot v + b_1^\pi, a_2^\pi \cdot v + b_2^\pi, \ldots, a_n^\pi \cdot v + b_n^\pi, a_{n+1}^\pi \cdot v + b_{n+1}^\pi), \]
where the \( a_i^\pi \) and \( b_i^\pi \) are as in Lemma 4.4. Thus, each map \( E^\pi \) is an integer affine transformation such that,
\[ E^\pi(v) = (w_1, w_2, \ldots, w_n, w_{n+1}) \]
for each vector \( v \in \mathcal{N}^\pi \) where \( \phi(v^\pi) = (w_1, w_2, \ldots, w_n) \) and \( \omega(v^\pi) = w_{n+1} \).

Thus, we define a set \( \Delta \subseteq \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \) as
\[ \Delta = \left\{ (v, u) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \middle| v_1 = u_1, v_2 = u_2, \ldots, v_n = u_n \text{ and } v_{n+1} > u_{n+1} \right\}; \]
and write \( f : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1} \) for the projection onto the first \( \mathbb{Z}^{n+1} \) factor of \( \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \), that is, \( f(z) = v \) for each \( z = (v, u) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \).

For each coset representative \( t_i \in T \), we write \( \pi^{i,1}, \pi^{i,2}, \ldots, \pi^{i,p_t} \) to denote a list of all the patterns which lie in the \( t_i \) coset, i.e., \( \rho(\pi^{i,j}) = t_i \). Notice then that a word \( v^{i,j} \) in special form fails to be geodesic if and only if there
exists a word \( u^{\pi_i,k} \) in special form for which \( (E^{\pi_i,j}(v), E^{\pi_i,k}(u)) \in \Delta \), which we then write as \( v \in \mathcal{G}^{\pi_i,j,\pi_i,k} \). Thus, each set \( \mathcal{G}^{\pi_i,j,\pi_i,k} \) can be written as
\[
\mathcal{G}^{\pi_i,j,\pi_i,k} = \mathcal{N}^{\pi_i,j} \cap \left[ \left( E^{\pi_i,j} \right)^{-1} \left( E^{\pi_i,j} \left( \mathcal{N}^{\pi_i,j} \right) \times E^{\pi_i,k} \left( \mathcal{N}^{\pi_i,k} \right) \right) \cap \Delta \right].
\]

Hence, we can now write each subset \( \mathcal{G}^{\pi_i,j} \subseteq \mathcal{N}^{\pi_i,j} \) as
\[
\mathcal{G}^{\pi_i,j} = \mathcal{N}^{\pi_i,j} \setminus \bigcup_{k=1}^{p_i} \mathcal{G}^{\pi_i,j,\pi_i,k}
\]
and thus we obtain the following result.

**Lemma 4.5.** Each set \( \mathcal{G}^{\pi} \) is positive polyhedral.

**Proof.** The set \( \Delta \) is polyhedral, as it can be written as the finite intersection
\[
\Delta = \bigcap_{j=1}^{n} \left\{ (v, u) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \mid (e_j, -e_j) \cdot (v, u) = 0 \right\}
\]
\[
\cap \left\{ (v, u) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \mid (e_{n+1}, -e_{n+1}) \cdot (v, u) > 0 \right\}
\]
where \( e_1, e_2, \ldots, e_{n+1} \) is the standard basis for \( \mathbb{Z}^{n+1} \).

Thus, the statement follows from Propositions 3.1 and 3.2. \( \square \)

5. Geodesic Growth

Using the structure of polyhedral sets along with the sliding window algorithm, we prove the following.

**Theorem 5.1.** There is no finitely generated virtually abelian group with intermediate geodesic growth.

**Proof.** Let \( G \) be a virtually abelian group with a finite weighted generating set \( S \). Let \( H \triangleleft G \) be an index \( d \), normal subgroup with \( H \cong \mathbb{Z}^n \), let \( P \) and \( Y \) be the weighted generating sets as in Section 4, with \( m = |Y| \), and let \( \mathcal{G}^{\pi} \) be the positive polyhedral sets as defined in Section 4.3.

**Notation.**

For \( t \in \mathbb{Z}_+ \), define the norm \( \| \cdot \|_1 : \mathbb{N}^t \rightarrow \mathbb{N} \) such that
\[
\|v\|_1 = v_1 + v_2 + \cdots + v_t
\]
for each \( v = (v_1, v_2, \ldots, v_t) \in \mathbb{N}^t \). Thus, there are constants
\[
C = \max \{ \omega(y) \mid y \in Y \} \quad \text{and} \quad D = \max \{ \omega(p_1 p_2 \cdots p_\ell) \mid p_j \in P \text{ and } \ell \leq d + 1 \}
\]
such that
\[
\|v\|_1 \leq \omega(v^{\pi}) \leq C\|v\|_1 + D \tag{5.1}
\]
for each pattern \( \pi \in \mathbb{P}^* \) and vector \( v \in \mathcal{G}^{\pi} \).

For each positive polyhedral set \( \mathcal{G}^{\pi} \), fix a finite list of positive basic polyhedral sets \( \mathcal{B}^{\pi}_1, \mathcal{B}^{\pi}_2, \ldots, \mathcal{B}^{\pi}_{J_\pi} \) such that \( \mathcal{G}^{\pi} = \bigcup_{j=1}^{J_\pi} \mathcal{B}^{\pi}_j \). Suppose that each set \( \mathcal{B}^{\pi}_j \) is \( I_\pi \)-dominating. Moreover, for each set \( \mathcal{B}^{\pi}_j \), fix a vector
\[
v^{\pi,j} = (v^{\pi,j}_1, v^{\pi,j}_2, \ldots, v^{\pi,j}_{(k+1)m}) \in \mathcal{N}^{\pi},
\]
with $v_{i,j}^\pi > 0$ if and only if $i \in I_j^\pi$, such that $\kappa v_{i,j}^\pi + u \in B_j^\pi$ for each $\kappa \in \mathbb{N}$ and $u \in B_j^\pi$. Notice that such a choice exists by Lemma 3.4.

Overview.

The idea of this proof is to separate the sliding window algorithm into finitely many pieces, representing each piece by some graph $G_j^\phi$. We then see that if any such graph, $G_j^\phi$, has the property as described in (5.5), then there exists an exponential family of geodesics which the sliding window algorithm converts to special forms with vectors lying on a line in the set $B_j^\phi$. Otherwise, if no graph $G_j^\phi$ has the property described in (5.5), then we will be able to construct a polynomial upper bound on the geodesic growth for the group.

Constructing graphs $G_j^\phi$.

For each set $B_j^\phi$ we define a finite, directed, edge-labelled graph $G_j^\phi$ with vertices and edges described as follows.

The vertices of each $G_j^\phi$ correspond to potential windows, $\Xi$, in the sliding window algorithm. That is, each word $\Xi \in S^\star$ with length $|\Xi|_S \leq d$ is a vertex of $G_j^\phi$. Thus, $G_j^\phi$ has finitely many vertices.

The edges of $G_j^\phi$ correspond to reversing one step of case 3a in the sliding window algorithm. In particular, the edges are defined as follows.

Let $\sigma \in S^\star$ be a geodesic word for which the sliding window algorithm, at some point, has a current pattern $\phi$, current vector $u \in N^\phi$ and current window $\Xi$. Suppose that, from this configuration, the next step is to perform case 3a of the sliding window algorithm by adding the $i$-th standard basis element, $e_i \in N^\phi$, to $u$ and replace the window with $\Xi'$. Further, suppose that, if we continue the sliding window algorithm from such a configuration, then we eventually have a current vector in $B_j^\phi$. Then, the graph $G_j^\phi$ has an edge, $(\Xi', i, \Xi) \in E(G_j^\phi)$, from vertex $\Xi'$ to $\Xi$ with label $i$; which we denote as $\Xi' \rightarrow^i \Xi$. Notice that we allow parallel edges if they have distinct labels, and further, $G_j^\phi$ has finitely many edges.

Algebra of the sliding window algorithm.

Each step of the sliding window algorithm, as presented in Section 4.1, belongs to one of four sub-cases, i.e., case 1, 2, 3a or 3b. Thus, suppose that at some point in the execution of the sliding window algorithm we have a current pattern $\phi$ and window $\Xi$, then we write

- $A_{\Xi}^\phi$ if the next step is to perform case 3a;
- $B_{\Xi}^\phi$ if the next step is to perform case 2 or 3b; and
- $C$ to denote that the next step is to perform case 1.

Thus, we can faithfully and uniquely represent an execution of the sliding window algorithm by sequences of symbols of the form $A_{\Xi}^\phi$, $B_{\Xi}^\phi$, $C$ read from left to right. In particular, we have the following observation.

Suppose that $\sigma \in S^\star$ is a geodesic for which the sliding window algorithm returns the special form $v^\pi$ with $\pi = \pi_1 \pi_2 \cdots \pi_k$ and $v \in G^\pi$. Then, when
given $\sigma$ as input, the execution of the algorithm corresponds to a sequence
\[
\left( \begin{array}{c}
\mathcal{A}^\pi_{\Xi_1} \\
\mathcal{A}^\pi_{\Xi_2} \\
\cdots \\
\mathcal{A}^\pi_{\Xi_{l-1}}
\end{array} \right) 
\mathcal{B}^\pi_{\Xi_1} 
\left( \begin{array}{c}
\mathcal{A}^\pi_{\Xi_2} \\
\mathcal{A}^\pi_{\Xi_3} \\
\cdots \\
\mathcal{A}^\pi_{\Xi_{l-1}}
\end{array} \right) 
\mathcal{B}^\pi_{\Xi_2} 
\cdots 
\mathcal{B}^\pi_{\Xi_{k-1}} 
\left( \begin{array}{c}
\mathcal{A}^\pi_{\Xi_{k-1}} \\
\mathcal{A}^\pi_{\Xi_{k-2}} \\
\cdots \\
\mathcal{A}^\pi_{\Xi_1}
\end{array} \right) 
\mathcal{C} 
\] (5.2)
where each factor of the form
\[
\mathcal{A}^\pi_{\Xi_1} \mathcal{A}^\pi_{\Xi_2} \cdots \mathcal{A}^\pi_{\Xi_{l-1}} 
\] (5.3)
corresponds to a unique path in some graph $\mathcal{G}^{\pi_1 \pi_2 \cdots \pi_l}$ of the form
\[
\Xi_{l-1}^i \Xi_{l-2}^i \cdots \Xi_1^i 
\] (5.4)
Furthermore, each edge-label $i_s$ in (5.4) corresponds to the index of the current vector $u$ that is incremented after performing step $\mathcal{A}^\pi_{\Xi_{l-1}}$.

Notice that the choice of graph $\mathcal{G}^{\pi_1 \pi_2 \cdots \pi_l}$ in (5.4) is such that the current vector $u$ lies in $\mathcal{B}^{\pi_1 \pi_2 \cdots \pi_l}$ after performing the step $\mathcal{A}^\pi_{\Xi_{l-1}}$ in (5.3).

Reversing the sliding window algorithm.
An important observation here is that we can use paths in the graph $\mathcal{G}^\sigma$ to construct geodesics by reversing the steps of the sliding window algorithm. For example, suppose that there is a path in $\mathcal{G}^\sigma$ of the form
\[
\mathcal{P} : \Xi_0 \xrightarrow{i_1} \Xi_1 \xrightarrow{i_2} \cdots \xrightarrow{i_l} \Xi_l 
\]
for which there is a geodesic word $v^\sigma \Xi_0$ where $u = v - \sum_{s=1}^l c_{i_s}$ lies in the set $\mathcal{N}^\sigma$. Then, there is a geodesic word $\sigma' \in S^\ast$ for which $u^\sigma \sigma' \simeq v^\sigma \Xi_0$, and we can apply the steps
\[
\mathcal{A}^\sigma_{\Xi_l} \mathcal{A}^\sigma_{\Xi_{l-1}} \mathcal{A}^\sigma_{\Xi_2} \cdots \mathcal{A}^\sigma_{\Xi_1} 
\]
of the sliding window algorithm to recover $v^\sigma \Xi_0$ from $u^\sigma \sigma'$. Notice that since each step of the sliding window algorithm is deterministic, we see that for each such path, there is a distinct and well-defined geodesic $\sigma'$. Thus, we say that $\sigma'$ is the reversal of $v^\sigma \Xi_0$ with respect to the path $\mathcal{P}$.

Exponential geodesic growth.
Let $\pi$ be a pattern, and let $\mathcal{B}^\pi_j$ and $I^\pi_j$ be sets as described earlier. Suppose that the graph $\mathcal{G}^\pi_j$ has a vertex $\Xi$ with two distinct non-trivial circuits
\[
\mathcal{C}_1 : \Xi \xrightarrow{i_1} \nu_1 \xrightarrow{i_2} \nu_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{l-1}} \nu_{l-1} \xrightarrow{i_l} \Xi 
\]
(5.5)
\[
\mathcal{C}_2 : \Xi \xrightarrow{i'_1} \nu'_1 \xrightarrow{i'_2} \nu'_2 \xrightarrow{i'_3} \cdots \xrightarrow{i'_{l-1}} \nu'_{l-1} \xrightarrow{i'_l} \Xi 
\]
such that each $i_s, i'_t \in I^\pi_j$, and each $\nu_s, \nu'_t$ is distinct from $\Xi$.

Then, by the definition of edges in the $\mathcal{G}^\pi_j$, there must be a geodesic $u^\sigma \sigma'$ with $u \in \mathcal{B}^\pi_j$ and a path in $\mathcal{G}^\pi_j$ of the form
\[
\mathcal{P} : \xi \xrightarrow{h_0} \mu_1 \xrightarrow{h_1} \mu_2 \xrightarrow{h_2} \cdots \xrightarrow{h_{r-1}} \mu_r \xrightarrow{h_r} \Xi 
\]
such that there is a reversal of $u^\sigma \sigma'$ with respect to the path $\mathcal{P}$. 

\[\]
For each $\kappa \in \mathbb{N}$ we define a vector $p(\kappa) \in \mathcal{B}_j^\pi$ as

$$p(\kappa) = (\kappa \cdot \max(\ell_1, \ell_2)) v^\pi_j + u.$$ 

where $u$ is as before. Thus, it can be seen that $(p(\kappa))^\pi \sigma'$ can be reversed with respect to each path

$$\mathcal{P} e_{x_1} e_{x_2} e_{x_3} \cdots e_{x_n}$$

with each $x_s \in \{1, 2\}$. Thus, for each $\kappa \in \mathbb{N}$, there are $2^\kappa$ such paths, and thus $2^\kappa$ distinct geodesics $\sigma'$. Furthermore, from (5.1) we see that the weighted length of each such word is bound from above as

$$\omega(\sigma') \leq C\|p(\kappa)\|_1 + D = C (\kappa \cdot \max(\ell_1, \ell_2)\|v^\pi_{-j}\|_1 + \|u\|_1) + D = C'\kappa + D'$$

for some $C', D' \in \mathbb{Z}_+$. Hence, for each $\kappa \in \mathbb{N}$ there are at least $2^\kappa$ geodesics with weighted length less than or equal to $C'\kappa + D'$. Therefore, $G$ has exponential geodesic growth with respect to the generating set $S$.

**Polynomial geodesic growth.**

Earlier in our proof we saw that if there is a graph $G_j^\pi$ which contains a vertex $\Sigma$ with two distinct non-trivial circuits as in (5.5), then $G$ has exponential geodesic growth. Thus, in the remainder of this proof we consider the case where, for each graph $G_j^\pi$ and $\Sigma \in V(G_j^\pi)$, there is at most one non-trivial circuit of the form

$$\Xi \xrightarrow{i_1} v_1 \xrightarrow{i_2} v_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{\ell-1}} v_{\ell} \xrightarrow{i_{\ell}} \Sigma$$

(5.6)

with each $i_s \in I_j^\pi$, and each $\nu_s$ distinct from $\Xi$. Thus, we show that under such a condition, the geodesic growth of $G$ has a polynomial upper bound.

Let $\Gamma_{S, \pi}^\omega(x)$ count the geodesics $\sigma \in S^*$, with weight $\omega(\sigma) \leq x$, for which the sliding window algorithm return a special form $v^\pi \simeq \sigma$ with $v \in \mathscr{G}^\pi$. Thus, we can write the geodesic growth as the sum

$$\Gamma_{S, \pi}^\omega(x) = \sum_{\pi} \Gamma_{S, \pi}^\omega(x).$$

(5.7)

Since each $\mathcal{B}_j^\pi$ is $I_j^\pi$-dominating, there exists a choice of constants $C_j^\pi \in \mathbb{N}$ such that $v_i \leq C_j^\pi$ for each $v = (v_1, v_2, \ldots, v_{\ell}) \in \mathcal{B}_j^\pi \subseteq \mathbb{N}^\ell$ and $i \notin I_j^\pi$. Fix a constant $K = (d + 1)m \cdot \max\{C_j^\pi : \pi, j\}$. Thus, we see that

$$\sum_{i \notin I_j^\pi} v_i \leq K$$

(5.8)

for each set $\mathcal{B}_j^\pi$ and each vector $v = (v_1, v_2, \ldots, v_{\ell}) \in \mathcal{B}_j^\pi$.

For each graph $G_j^\pi$, define a map $r_j^\pi : \mathbb{N} \to \mathbb{N}$ such that $r_j^\pi(x)$ is the number of distinct paths in $G_j^\pi$ of the form

$$\mu_0 \xrightarrow{i_1} \mu_1 \xrightarrow{i_2} \mu_2 \xrightarrow{i_3} \cdots \xrightarrow{i_\ell} \mu_\ell$$

with $\ell \leq x$, and $i_s \in I_j^\pi$ for all except at most $K$ values $s \in \{1, 2, \ldots, \ell\}$. Notice that $r_j^\pi(x)$ considers paths with one vertex and no edges.

Suppose that $\sigma \in S^*$ is a geodesic for which the sliding window algorithm returns a special form $v^\pi \simeq \sigma$. Then, as we saw earlier in our proof, the execution of the sliding window algorithm can be faithfully represented by a sequence as in (5.2) where each factor of the form (5.3) corresponds to a
path, (5.4), in some graph $G^\pi_j$; where each such path is included in the count given by $r^\pi_j(x)$ with $x = \|v\|_1$. From (5.1) we have the bound $\|v\|_1 \leq \omega(\sigma)$, and thus we have the upper bound

$$\Gamma^\pi_{S,\pi}(x) \leq \sum_{j_1,j_2,\ldots,j_k} \left[ r^{\pi_1\cdot,\pi_{j_1}}_{j_1}(x) \cdot r^{\pi_2\cdot,\pi_{j_2}}_{j_2}(x) \cdot r^{\pi_3\cdot,\pi_{j_3}}_{j_3}(x) \cdots r^{\pi_k\cdot,\pi}(x) \right] \cdot |S^{(k+1)}(d+1)|$$

for each pattern $\pi = \pi_1\pi_2\cdots\pi_k \in P^*$, where $|S^{(k+1)}(d+1)|$ is an upper bound on the choices of factors $\mathfrak{A}^{\pi_1\pi_2\cdots\pi_k}_{\pi}$ in (5.2).

Thus, if we can find a polynomial upper bound for each $r^\pi_j$, then each $\Gamma^\pi_{S,\pi}$ has a polynomial upper bound and thus $G$ has polynomial geodesic growth.

In order to show a polynomial upper bound on each map $r^\pi_j$, we define mappings $R^\pi_j : \mathbb{N}^3 \to \mathbb{N}$ such that $R^\pi_j(x; q, k)$ is the number of distinct paths in $G^\pi_j$ of the form

$$\mu_0 \xrightarrow{i_1} \mu_1 \xrightarrow{i_2} \mu_2 \xrightarrow{i_3} \cdots \xrightarrow{i_\ell} \mu_\ell$$

with $\ell \leq x$, at most $q$ distinct vertices, and $i_s \in I^\pi_j$ for all except at most $k$ values of $s \in \{1, 2, \ldots, \ell\}$. Thus, we see that

$$r^\pi_j(x) = R^\pi_j(x; |V(G^\pi_j)|, K).$$

We will show by induction with respect to a well-ordering on the parameters $q$ and $k$, that for any fixed parameters $(q, k) \in \mathbb{N}^2$ the function $R^\pi_j(x; q, k)$ is bound from above by a polynomial in $x$ of degree $2q + k - 1$.

Define a well-ordering $\preceq$ on the set $\mathbb{N}^2$ such that $(q_1, k_1) \preceq (q_2, k_2)$ if we have $(q_1 + k_1) \leq (q_2 + k_2)$ with respect to the lexicographic ordering. Notice here that $(0, 0)$ is the minimum element of $\mathbb{N}^2$ with respect to $\preceq$.

From our earlier definition, it follows that $R^\pi_j(x; 0, k) = 0$ as a path must have at least one vertex. Thus, we have the required upper bound for each $R^\pi_j(x; 0, k)$. Hence, all that remains is the case where $q \geq 1$.

Let $(q, k) \in \mathbb{N}^2$ with $q \geq 1$. Suppose for each $(q_0, k_0) \in \mathbb{N}^2 \setminus \{(q, k)\}$ with $(q_0, k_0) \preceq (q, k)$, that the function $R^\pi_j(x; q_0, k_0)$ is bound from above by a polynomial in $x$ of degree $2q_0 + k_0 - 1$. Then, we have the upper bound

$$R^\pi_j(x; q, k) \leq R^\pi_j(x - 1; q, k) + |E(G^\pi_j)| \cdot R^\pi_j(x - 1; q - 1, k) + 1$$

where $R^\pi_j(x - 1; q, k)$ is an upper bound on the number of paths which begin with the unique circuit described in (5.6); $|E(G^\pi_j)|$ is an upper bound on the number of outgoing edges from the initial vertex;

$$|E(G^\pi_j)| \cdot R^\pi_j(x - 1; q - 1, k) + 1$$

is an upper bound on the number of paths that do not revisit their initial vertex (include the trivial path); and

$$\begin{cases} |E(G^\pi_j)| \cdot R^\pi_j(x - 1; q - 1, k) \cdot R^\pi_j(x; q, k - 1) & \text{if } k > 0 \\ 0 & \text{if } k = 0 \end{cases}$$

is an upper bound on the number of paths that do not revisit their initial vertex (include the trivial path); and
is an upper bound on the number of paths which begin with a circuit different to one as described in (5.6), that is, a circuit which contains an edge-label $i \notin I_j$, and thus the rest of the path may contain no more than $k - 1$ labels $i' \notin I_j$. Thus, there is an upper bound given by

$$R_{\pi j}^\pi(x; q, k) \leq R_{\pi j}^\pi(x - 1; q, k) + f_{\pi q, k}^\pi(x)$$

where $f_{\pi q, k}^\pi(x)$ is bound from above by a polynomial of degree $2q + k - 2$. Hence, by solving the recurrence relation, we see that $R_{\pi j}^\pi(x; q, k)$ is bound from above by a polynomial in $x$ of degree $2q + k - 1$ as required.

By induction with respect to the well-ordering $\leq$, we have our required polynomial upper bound on $R_{\pi j}^\pi(x; q, k)$ for each $(q, k) \in \mathbb{N}^2$. Thus, $G$ has polynomial geodesic growth with respect to the generating set $S$. □

6. Blind Multicounter Automata

For any fixed $k \in \mathbb{N}$, a blind $k$-counter automaton, as studied by Greibach in [9], is a nondeterministic finite state acceptor with a one-way input tape and $k$ integer counters; where such a machine is allowed to increment and decrement its counters by fixed amounts only during transitions. A computation of such a machine begins with zero on all its counters and accepts when in an accepting state with all input consumed and zero on all its counters.

Formally, a blind $k$-counter automaton can be defined as follows.

**Definition 6.1.** For any given $k \in \mathbb{N}$, a blind $k$-counter machine is a 6-tuple of the form $M = (Q, \Sigma, \delta, q_0, F, \varepsilon)$ where

1. $Q$ is a finite set of states;
2. $\Sigma$ is a finite input alphabet;
3. $\delta$ is a finite subset of $$(Q \times (\Sigma \cup \{\varepsilon, \varepsilon\})) \times (Q \times \mathbb{Z}^k)$$
called the transition relation;
4. $q_0 \in Q$ is the initial state;
5. $F \subseteq Q$ is the set of final states; and
6. $\varepsilon \notin \Sigma$ is the end of tape symbol.

Let $M = (Q, \Sigma, \delta, q_0, F, \varepsilon)$ be a $k$-counter automaton. Then, $M$ will begin in state $q_0$ with zero on all its counters. Suppose that there is a transition relation $((q, a), (p, v)) \in \delta$ with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon, \varepsilon\}$ and $v \in \mathbb{Z}^k$; if $M$ is in state $q$ with $a$ on its input tape, then it can transition to state $p$ after adding $v$ to its counters and consuming $a$ from its input tape. The machine will accept when it’s in some state $q_{\text{accept}} \in F$ with all letters on it input tape consumed and zero on all its counters.

**Theorem 6.2.** The geodesic language of any finitely generated virtually abelian group is blind counter.

**Proof.** Let $G$ be a virtually abelian group with finite weighted generating set $S$. Let $H \triangleleft G$ be an index $d$ normal subgroup with $H \cong \mathbb{Z}^n$, let $P$ and $Y$ be the weighted generating sets as in Section 4, $m = |Y|$, and $\mathcal{G}^\pi$ be the polyhedral sets of vectors as described in Section 4.3.
Thus, we construct a blind $k$-counter automaton $M = (Q, \Sigma, \delta, q_0, F, \epsilon)$ that accepts precisely the geodesic language of $G$ with respect to $S$ where $k$ is chosen to be sufficiently large (as we will see later in this proof). The machine $M$ will have an input alphabet given by $\Sigma = S$ and one accepting state $\{q_{\text{accept}}\} = F$.

Before describing our construction we establish some notation as follows.

**Notation.**

For each pattern $\pi = \pi_1 \pi_2 \cdots \pi_k$ we fix a finite list of positive basic polyhedral sets $B_j^\pi, B_2^\pi, \ldots, B_j^\pi$ such that $\mathcal{G}^\pi = \bigcup_{j=1}^J B_j^\pi$. Notice that this is possible as each $\mathcal{G}^\pi$ is positive polyhedral by Lemma 4.5.

For each basic polyhedral set $B_j^\pi$ we fix an intersection

$$B_j^\pi = \bigcap_{d=1}^{D_j^\pi} \left\{ z \in \mathbb{Z}^{(k+1)m} \mid \alpha_d^\pi, \cdot \cdot \cdot, \alpha_d^\pi, \cdot \cdot \cdot > \beta_d^\pi, \cdot \cdot \cdot \right\}$$

$$\cap \bigcap_{d=1}^{D_2^\pi} \left\{ z \in \mathbb{Z}^{(k+1)m} \mid \gamma_d^\pi, \cdot \cdot \cdot, \gamma_d^\pi, \cdot \cdot \cdot = \lambda_d^\pi \right\}$$

$$\cap \bigcap_{d=1}^{D_3^\pi} \left\{ z \in \mathbb{Z}^{(k+1)m} \mid \zeta_d^\pi, \cdot \cdot \cdot, \zeta_d^\pi, \cdot \cdot \cdot \equiv \eta_d^\pi \pmod{\theta_d^\pi} \right\}$$

where $\alpha_d^\pi, \gamma_d^\pi, \zeta_d^\pi \in \mathbb{Z}^{(k+1)m}$ and $\beta_d^\pi, \lambda_d^\pi, \eta_d^\pi, \theta_d^\pi \in \mathbb{Z}$ with $\theta_d^\pi > 1$.

To simplify notation, we will write the counters of $M$ as

$$(z_1, z_2, \ldots, z_{(d+1)m} \mid a_1, a_2, \ldots, a_{D_1}, b_1, b_2, \ldots, b_{D_2} \mid c_1, c_2, \ldots, c_{D_3}),$$

where $k = (d+1)m + D_1 + D_2 + D_3$ and $D_1, D_2, D_3 \in \mathbb{N}$ are the smallest values such that $D_1 \geq D_1^\pi, D_2 \geq D_2^\pi, D_3 \geq D_3^\pi$ for each $\pi$ and $j$.

To simplify notation when presenting transition relations, given vectors

$$v^{i,j} = \left( v_1^{i,j}, v_2^{i,j}, \ldots, v_{(d+1)m}^{i,j} \right) \in \mathbb{Z}^{D_3},$$

we will write

$$\left( v^{i,1}, v^{i,2}, \ldots, v^{i,c} \mid v^{2,1}, v^{2,2}, \ldots, v^{2,d} \mid v^{3,1}, v^{3,2}, \ldots, v^{3,e} \mid v^{4,1}, v^{4,2}, \ldots, v^{4,f} \right)$$

to denote the vector

$$\left( v_{1,1}, v_{1,2}, \ldots, v_{1,c}^{i,1}, v_{1,c}^{i,2}, \ldots, v_{1,c}^{i,1} \right)$$

$$\left( v_{2,1}, v_{2,2}, \ldots, v_{2,c}^{i,1}, v_{2,c}^{i,2}, \ldots, v_{2,c}^{i,1} \right)$$

$$\left( v_{3,1}, v_{3,2}, \ldots, v_{3,c}^{i,1}, v_{3,c}^{i,2}, \ldots, v_{3,c}^{i,1} \right)$$

$$\left( v_{4,1}, v_{4,2}, \ldots, v_{4,c}^{i,1}, v_{4,c}^{i,2}, \ldots, v_{4,c}^{i,1} \right)$$

in $\mathbb{Z}^k$ such that the components match that given in (6.2).

**Overview.**

Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_t \in S^*$ be an input word given to $M$. Then, in the remainder of our proof, we write $\pi = \pi_1 \pi_2 \cdots \pi_k$ and $v \in \mathbb{N}^{(k+1)m}$ to denote the
values obtained after performing the sliding window algorithm as described in Section 4.1. Thus, $\sigma$ is geodesic if and only if $v \in \mathcal{G}$.

The execution of $M$ is separated into three stages as follows.

1. We perform the sliding window algorithm on the input word $w$, ending in state $q_\mathcal{E}^\xi \in \mathcal{Q}$ with $v$ on the counters $z_1, z_2, \ldots, z_{(k+1)m}$ and zero on all other counters.

2. After nondeterministically choosing a basic polyhedral set $B_j^\xi$, at the end of this stage we will have subtracted a vector $u \in \mathbb{N}^{(k+1)m}$ from the counters $z_1, z_2, \ldots, z_{(k+1)m}$ and have

$$a_{d_1} = \alpha_{d_1}, b_{d_2} = \beta_{d_2}, c_{d_3} = \gamma_{d_3} \cdot u$$

for each $d_i$ with $1 \leq d_i \leq D^\xi_i$. This stage of the computation will end with a transition to a state labelled $q_{\mathcal{E}^\xi}$.

3. We will confirm that the vector $u$ lies in $B_j^\xi$ by nondeterministically verifying (by subtraction) that

$$a_{d_1} = \beta_{d_1} + 1, b_{d_2} = \alpha_{d_2}, c_{d_3} = \gamma_{d_3} + \delta_{d_3}$$

for each $d_i$ with $1 \leq d_i \leq D^\xi_i$, where each $a'_{d_1}, b'_{d_2}, c'_{d_3} \in \mathbb{N}$ and $c'_{d_3} \in \mathbb{Z}$.

The details of these stages are given below.

Stage 1: performing the sliding window algorithm.

In order to perform the sliding window algorithm, as in Section 4.1, we will introduce states to keep track of the window $\Xi \in S^\ast$. In particular, we introduce states $q^\xi_\mathcal{E} \in \mathcal{Q}$ where $\varphi \in P^\ast$ is a pattern, and $\xi \in S^\ast \cup S^\ast \epsilon$ with $|\xi| \leq d$; where we define $|u| = |u|_S + 1$ for $u \in S^\ast$. Notice here that such a $\xi$ denotes a prefix of the sliding window $\Xi$ where $\epsilon$ is included for the case where the window is less than $d$ in length.

Thus, to begin the sliding window algorithm we introduce a transition

$$((q_0, \epsilon), (q^\xi_\mathcal{E}, 0)) \in \delta$$

from the initial state. We will then ensure that we consume the entire sliding window by introducing transitions

$$((q^\xi_\mathcal{E}, a), (q^\xi_{a\mathcal{E}}, 0)) \in \delta$$

for each $\xi \in S^\ast$ with $|\xi|_S < d$, and each $a \in S \cup \{\epsilon\}$. Hence, a state $q^p_\xi$ represents the current window if and only if either $|\xi|_S = d$ or $\xi \in S^\ast \epsilon$.

Notice that the sliding window algorithm is complete when we enter a state of the form $q^p_\xi$, i.e., when the sliding window is empty.

Note that in the remainder of this stage of the proof, we denote the standard basis of $\mathbb{Z}^{(d+1)m}$ as $e_1, e_2, \ldots, e_{(d+1)m}$. Thus, we continue our construction as follows.

Let $q^\xi_\mathcal{E} \in \mathcal{Q}$ be a state with $\varphi = \varphi_1 \varphi_2 \cdots \varphi_k$ such that either $|\xi|_S = d$ or $\xi \in S^\ast \epsilon$. That is, the word $\xi$ in the subscript of $q^\xi_\mathcal{E}$ corresponds to the current window $\Xi$. We now implement the sliding window algorithm by considering the following cases for the state $q^\xi_\mathcal{E}$.

1. If we have $\xi = \epsilon$, i.e., we are in a state of the form $q^\xi_\mathcal{E}$, then we have successfully performed the sliding window algorithm and thus end this stage of the computation.
2. If \( \xi = pe \) with \( p \in P \), then we introduce a transition
\[
((q^p_\xi, \varepsilon), (q^p_\xi, 0)) \in \delta.
\]

3. Otherwise, we write \( \xi = \xi_1 \xi_2 \cdots \xi_\tau \) with each \( \xi_i \in S \cup \{ \varepsilon \} \). Thus, if neither of the previous cases apply, then using Lemma 4.2 we factor \( \xi_1 \xi_2 \cdots \xi_\tau \) as \( py \xi_j+1 \xi_j+1 \cdots \xi_\tau \) where \( p = \xi_1 \xi_2 \cdots \xi_{i-1} \in P \cup \{ \varepsilon \} \) and \( y = \xi_i \xi_{i+1} \cdots \xi_j \in Y \). Let \( r \in T \) be the coset representative given by \( r = r(\overline{PP}) \), and consider the following sub-cases.

(a) If there is an \( \ell \) such that \( r = r(\overline{\tau_1 \tau_2 \cdots \tau_j}) \), then we introduce a transition
\[
((q^p_\xi, \varepsilon), (q^p_\xi, (e_{\ell m+f} | 0 | 0 | 0))) \in \delta
\]
where \( \xi' = p \xi_j+1 \xi_j+2 \cdots \xi_\tau \) and \( f \in \mathbb{N} \) is such that \( y = yf \in Y \).

(b) Otherwise, we introduce a transition
\[
((q^p_\xi, \varepsilon), (q^p_\xi, (e_{(k+1)m+f} | 0 | 0 | 0))) \in \delta
\]
where \( \xi'' = \xi_{i+1} \xi_{i+2} \cdots \xi_\tau \) and \( f \in \mathbb{N} \) is such that \( y = yf \in Y \).

Comparing the above cases to those given in Section 4.1, we see that we have implemented the sliding window algorithm as required.

**Stage 2a: subtracting a vector** \( u \in \mathbb{N}^{(k+1)m} \).

Beginning in state \( q^p_\xi \), we nondeterministically choose a positive basic polyhedral set \( B^p_\pi \subseteq \mathcal{B}^p \). In particular, we do this by introducing states \( q^{\pi,j} \in Q \), to correspond to each sets \( B^p_\pi \), and transitions
\[
((q^p_\xi, \varepsilon), (q^{\pi,j}, 0)) \in \delta
\]
for each pattern \( \pi \), and each \( j \) with \( 1 \leq j \leq J_\pi \).

Given a pattern \( \pi = \pi_1 \pi_2 \cdots \pi_k \in P^\pi \), we write \( e_1, e_2, \ldots, e_{(k+1)m} \) to denote the standard basis for \( \mathbb{Z}^{(k+1)m} \). We then write \( e_i^{\pi,j} \) to denote
\[
e_i^{\pi,j} = \left( -e_i \right) \begin{vmatrix}
\alpha_{1,j}^{\pi,j} & e_i, & \alpha_{2,j}^{\pi,j} \cdot e_i, & \ldots, & \alpha_{D_\phi,j}^{\pi,j} \cdot e_i \\
\gamma_{1,j}^{\pi,j} \cdot e_i, & \gamma_{2,j}^{\pi,j} \cdot e_i, & \ldots, & \gamma_{D_j,j}^{\pi,j} \cdot e_i \\
\zeta_{1,j}^{\pi,j} \cdot e_i, & \zeta_{2,j}^{\pi,j} \cdot e_i, & \ldots, & \zeta_{D_j,j}^{\pi,j} \cdot e_i
\end{vmatrix}
\]
and introduce transitions
\[
((q^{\pi,j}, \varepsilon), (q^{\pi,j}, e_i^{\pi,j})) \in \delta,
\]

Thus, the machine is able to nondeterministically subtract any particular vector \( u \in \mathbb{N}^{(k+1)m} \) from the counters labelled \( z_1, z_2, \ldots, z_{(k+1)m} \) by repeated subtraction of basis vectors.

To complete this stage, we introduce states \( q^{\pi,j}_{\text{fin}} \in Q \) and transitions
\[
((q^{\pi,j}, \varepsilon), (q^{\pi,j}_{\text{fin}}, 0)) \in \delta
\]
that is, this stage of the computation is complete when the machine enters a state of the form \( q^{\pi,j}_{\text{fin}} \).
Notice that at the end of this stage, we have subtracted a nondeterministically chosen vector \( u \in \mathbb{Z}^{(k+1)m} \) from \( z_1, z_2, \ldots, z_{(k+1)m} \), and we have
\[
a_{d_1} = \alpha_{d_1} \cdot u, \quad b_{d_2} = \gamma_{d_2} \cdot u, \quad c_{d_3} = \delta_{d_3} \cdot u
\]
for each \( d_i \) with \( 0 \leq d_i \leq D_i^{\pi,j} \).

Note that the counters \( z_1, z_2, \ldots, z_{(k+1)m} \) will not be touched in the final stage of our construction. Thus, the machine will accept from its current configuration only if each \( z_1, z_2, \ldots, z_{(k+1)m} \) is zero and thus \( u = v \).

Stage 2b: check that \( u \in \mathcal{B}^\pi_j \).

In order to verify that \( u \) belongs to the set \( \mathcal{B}^\pi_j \), we will nondeterministically verify (by subtraction) that we have
\[
a_{d_1} = \beta^\pi_{d_1} + 1 + a'_{d_1}, \quad b_{d_2} = \lambda^\pi_{d_2}, \quad c_{d_3} = \eta^\pi_{d_3} + c'_d \theta^\pi_{d_3}
\]
for each \( d_i \) where \( 1 \leq d_i \leq D_i^{\pi,j} \), with \( a'_{d_1} \in \mathbb{N} \) and \( c'_d \in \mathbb{Z} \).

To check this, we will first introduce states \( q_{\text{check}}^\pi, j \in Q \) and transitions
\[
((q_{\text{fin}}^\pi, \varepsilon), (q_{\text{check}}^\pi, \psi^\pi_j)) \in \delta
\]
where
\[
\psi^\pi_j = (0 | -\beta^\pi_1 - 1, -\beta^\pi_2 - 1, \ldots, -\beta^\pi_{D_j^{\pi,j}} - 1 |
\]
\[
-\lambda^\pi_1, -\lambda^\pi_2, \ldots, -\lambda^\pi_{D_j^{\pi,j}} |
\]
\[
-\eta^\pi_1, -\eta^\pi_2, \ldots, -\eta^\pi_{D_j^{\pi,j}}
\]}

Thus, after following such a transition, we have \( u \in \mathcal{B}^\pi_j \) if and only if
\[
a_{d_1} = a'_{d_1}, \quad b_{d_2} = 0, \quad c_{d_3} = c'_d \theta^\pi_{d_3}
\]
for each \( d_i \) where \( 1 \leq d_i \leq D_i^{\pi,j} \), with \( a'_{d_1} \in \mathbb{N} \) and \( c'_d \in \mathbb{Z} \). Thus, we will not touch the counters \( b_{d_2} \) at any other point in this stage, that is, we accept only if each counter \( b_{d_2} \) is zero.

Writing the standard basis of \( \mathbb{Z}^{D_j^\pi} \) as \( e_1, e_2, \ldots, e_{D_j^\pi} \), we introduce
\[
((q_{\text{check}}^\pi, \varepsilon), (q_{\text{check}}^\pi, (0 | -e_i | 0 | 0))) \in \delta
\]
such that the machine is able to nondeterministically subtract \( a'_{d_1} \in \mathbb{N} \) from counter \( a_{d_1} \) for each \( d_1 \) with \( 1 \leq d_1 \leq D_1^{\pi,j} \).

After writing \( f_i \in \mathbb{Z}^{D_j^\pi} \) to denote the vector with zero in all components except the \( i \)-th where it will instead be \( \theta_i^\pi \), we introduce transitions
\[
((q_{\text{check}}^\pi, \varepsilon), (q_{\text{check}}^\pi, (0 | 0 | 0 | \pm f_i))) \in \delta
\]
such that the machine is able to nondeterministically subtract some value \( c'_d \theta^\pi_{d_3} \), with \( c'_d \in \mathbb{Z} \), from counter \( c_{d_3} \) for each \( d_3 \) with \( 1 \leq d_3 \leq D_3^{\pi,j} \).

We will end this stage of the computation by following a transition
\[
((q_{\text{check}}^\pi, \varepsilon), (q_{\text{accept}}, 0)) \in \delta.
\]

Thus, it is possible to follow such transitions to the accepting state \( q_{\text{accept}} \) with zero on all counters of the form \( a_i, b_i, c_i \) if only if \( u \in \mathcal{B}^\pi_j \).
Soundness and Completeness.

Suppose that some word $\sigma \in S^\star$ is accepted by $M$. Then, at the end of the computation we must have zero on all counters of the form $a_i, b_i, c_i$ and thus we know from Stage 2b that $u \in B^x_j$, for some $j$. Further, since we must have zero on all counters of the form $z_i$, we then know from Stage 2a that $u = v$. Therefore, $v \in B^x_j$ where we have $v^x \simeq \sigma$ from Stage 1. Therefore, $\sigma$ is a geodesic as $v^x \simeq \sigma_j$ for some $v \in G^x$.

Let geodesic $\sigma \in S^\star$ be a geodesic word, and let $v^x \simeq \sigma$ be the special form obtained from the sliding widow algorithm. Then, at the end of Stage 1 the machine will be in state $q^e_n$ with $v$ on the counters of the form $z_i$. Since $\sigma$ is geodesic, there must exists some $B^x_j$ that contains $v$. Thus, without loss of generality we may assume that the machine chooses $B^x_j$ in Stage 2a and subtracts $u = v \in B^x_j$ from the counters of the form $z_i$, with Stage 2a ending with the machine in state $q^e_{u,j}$. Thus, we see that Stage 2b is able to accept by subtracting the appropriate values from the counters of the form $a_i, b_i, c_i$. Hence, $M$ will accept the word $\sigma$.

Therefore, the machine $M$ accepts a word $\sigma \in S^\star$ if and only if $\sigma$ is a geodesic with respect to the weighted generation set $S$. □

References

[1] Laurent Bartholdi, Rostislav Grigorchuk, and Volodymyr Nekrashevych, From fractal groups to fractal sets, Fractals in Graz 2001, 2003, pp. 25–118. MR2091700
[2] M. Benson, Growth series of finite extensions of $\mathbb{Z}^n$ are rational, Invent. Math. 73 (1983), no. 2, 251–269. MR714092
[3] Martin R. Bridson, José Burillo, Murray Elder, and Zoran Šunić, On groups whose geodesic growth is polynomial, Internat. J. Algebra Comput. 22 (2012), no. 5, 1250048, 13. MR2949213
[4] Sean Cleary, Murray Elder, and Jennifer Taback, Cone types and geodesic languages for lamplighter groups and Thompson’s group $F$, J. Algebra 303 (2006), no. 2, 476–500. MR2255118
[5] Pierre de la Harpe, Topics in geometric group theory, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000. MR1786869
[6] Murray Elder, Regular geodesic languages and the falsification by fellow traveler property, Algebr. Geom. Topol. 5 (2005), 129–134. MR2135549
[7] Murray Elder, Mark Kambites, and Gretchen Ostheimer, On groups and counter automata, Internat. J. Algebra Comput. 18 (2008), no. 8, 1345–1364. MR2483126
[8] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, Word processing in groups, Jones and Bartlett Publishers, Boston, MA, 1992. MR1161694
[9] S. A. Greibach, Remarks on blind and partially blind one-way multicontainer machines, Theoret. Comput. Sci. 7 (1978), no. 3, 311–324. MR513714
[10] R. I. Grigorchuk, On the Milnor problem of group growth, Dokl. Akad. Nauk SSSR 271 (1983), no. 1, 30–33. MR712546
[11] M. Gromov, Hyperbolic groups, Essays in group theory, 1987, pp. 75–263. MR919829
[12] Mikhail Gromov, Groups of polynomial growth and expanding maps, Inst. Hautes Études Sci. Publ. Math. 53 (1981), 53–73. MR623534
[13] Avinoam Mann, How groups grow, London Mathematical Society Lecture Note Series, vol. 395, Cambridge University Press, Cambridge, 2012. MR2894945
[14] Volodymyr Nekrashevych, Self-similar groups, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005. MR2162164
[15] Derek J. S. Robinson, A course in the theory of groups, Second, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR1357169