New Light on Infrared Problems: Sectors, Statistics, Symmetries and Spectrum

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Dedicated to Roberto Longo on the occasion of his sixtieth birthday

Abstract

A new approach to the analysis of the physical state space of a theory is presented within the general setting of local quantum physics. It also covers theories with long range forces, such as Quantum Electrodynamics. Making use of the notion of charge class, an extension of the concept of superselection sector, infrared problems are avoided by restricting the states to observables localized in a light cone. The charge structure of a theory can be explored in a systematic manner. The present analysis focuses on simple charges, thus including the electric charge. It is shown that any such charge has a conjugate charge. There is a meaningful concept of statistics: the corresponding charge classes are either of Bose or of Fermi type. The family of simple charge classes is in one-to-one correspondence with the irreducible unitary representations of a compact Abelian group. Moreover, there is a meaningful definition of covariant charge classes. Any such class determines a continuous unitary representation of the Poincaré group or its covering group satisfying the relativistic spectrum condition. The resulting particle aspects are also briefly discussed.

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1 Introduction

Algebraic quantum field theory \[20\] has proved to be a powerful framework for understanding superselection structure in local quantum physics, i.e. the possible patterns of coherent subspaces (sectors) of the physical state space. For theories in Minkowskian spacetime, describing exclusively states of compactly localizable charges \[12\] or massive particles \[8\], this structure encoded in the local observables is fully understood: the sectors in any such theory correspond to the dual of some compact group, interpreted as a global gauge group; each sector carries a specific representation of the permutation group, determining its statistics; and there are charged field operators transforming as tensors under the global gauge group, satisfying Bose or Fermi commutation relations at spacelike distances and interpolating between the various sectors \[14\]. These results have been extended successfully to theories in lower dimensions, where quantum groups can appear in the description of the superselection structure and sectors can have braid group statistics \[15,17,23\].\[13\]

Yet the physically important class of theories describing long range forces mediated by massless particles, such as Quantum Electrodynamics, is not covered by these results. In fact, attempts to clarify their superselection structure \[6,16\] have failed, cf. \[7\].

The difficulties in these theories can be traced back to two related features, namely the long range effects of the forces between charged particles and the multifarious ways in which their interaction can form clouds of low energy massless particles. In Quantum Electrodynamics, for example, any configuration of electrically charged particles gives rise to a specific long range structure of the electromagnetic field, created, both by the Coulomb fields of the moving charges and by the infrared clouds caused by their acceleration. The resulting long range tails of the electromagnetic field can be discriminated by central sequences of local observables, hence giving rise to an abundance of different superselection sectors \[4,5\].

Whereas the electric charge carried by particles is a superselected quantity of fundamental physical significance, the long range features of the infrared clouds are a theoretical concept defying experimental verification. When applying the theory, this is usually taken into account by considering inclusive processes where expectation values are summed over the undetected low energy photons. This procedure effectively amounts to wiping out marginal features of the infrared sectors without destroying physically relevant information, such as their total charge. In other words, the notion of superselection sector provides too fine a resolution of the physical state space leading to unnecessary theoretical complications. It seems desirable to use concepts closer to actual experimental practice and providing a coarser resolution of the physical state space.

The aim of the present investigation is to establish such a framework and discuss its theoretical implications. Our approach is based on the fundamental fact that
both measurements and the preparation of specific states can extend well into the future, whereas it is impossible to perform these operations in the distant past. Thus the spacetime regions where actual experiments can take place are at best future directed light cones $V$ in Minkowski space whose apices are fixed by some largely arbitrary initial event. Of course, these limitations on the spacetime localization of concrete physical operations do not imply that it is impossible to obtain information about past properties of global states. But this information can only be obtained indirectly by making measurements in these states in some accessible light cone $V$ and trying to reconstruct their past values from the data on the basis of theory.

In this context, it has to be noted that there is a fundamental difference between theories describing only massive particles and theories including massless particles as well. In the massive case, the unital C*-algebras $\mathcal{A}(V)$ generated by the local observables in any given light cone $V$ are generically irreducible \[24\]. In principle, therefore, all the information on the global properties of a state can be derived from measurements made in any such region $V$. In the presence of massless particles, however, the algebra $\mathcal{A}(V)$ of observables localized in $V$ is highly reducible and time translations act on it as proper endomorphisms \[4\]. Thus, as time proceeds, one loses information about past properties of the states. This constant loss is inevitable since, by Huygens principle, outgoing massless particles created in the past of a given light cone $V$ will never enter that cone. Hence their properties cannot be determined by later measurements in $V$.

It is a remarkable fact that, by the same mechanism, the various sectors formed by infrared clouds cannot be distinguished by measurements in any fixed light cone $V$; discriminating them requires observations in larger regions of Minkowski space. On the other hand, the total charge carried by massive particles, such as the electric charge, can be determined in any given $V$. For these particles will either eventually enter this cone, or some of them will, in the course of time, be annihilated or created in pairs of conjugate charge. But the total charge is not affected and can be sharply determined by measurements in the given $V$. These features are relevant in the present context. They imply that states of equal total charge differing only by infrared clouds created in collisions are mutually normal if restricted to the algebra of observables $\mathcal{A}(V)$ of any given light cone $V$. In this way, an abundance of superselection sectors coalesce to form a charge class \[4\]. Thus restricting measurements to light cones has a similar effect to summing expectation values over undetected massless particles.

These observations suggest basing the analysis and physical interpretation of Minkowski space theories entirely on the algebras $\mathcal{A}(V)$ of observables localized in a given light cone $V$. This should avoid infrared problems right from the outset. Yet implementing this idea requires solving several conceptual problems.

(a) As already mentioned, in the presence of massless particles the algebras $\mathcal{A}(V)$ are reducible in any sector. In fact, their weak closures are generically factors
of type $\text{III}_1$, as can be inferred from results established in [3, 22]. There is then no intrinsic way of superimposing states of $\mathcal{A}(V)$ and the usual characterization of sectors fails. Here we need the notion of charge class, expressed in terms of $\mathcal{A}(V)$. The states on $\mathcal{A}(V)$, belonging to a given charge class, are distinguished by being primary (or even pure in the absence of massless particles) and can be transformed into each other by the operational effects of observables localized in the light cone $V$. More precisely, two relevant primary states $\omega_1, \omega_2$ on the algebra $\mathcal{A}(V)$ belong to the same charge class if $\omega_1$ can be mapped into any given (norm) neighbourhood of $\omega_2$ by the adjoint action of some inner automorphism of $\mathcal{A}(V)$, and \textit{vice versa}. This criterion generalizes the notion of superselection sector in the irreducible case and remains meaningful in the presence of massless particles, for the inner automorphisms of type $\text{III}_1$ factors act almost transitively on normal states [10] and can be approximated by inner automorphisms of $\mathcal{A}(V)$ by the Kaplansky density theorem.

(b) In Quantum Field Theory, local operations on states can create charges in conjugate pairs. This allows one to pass from states in the vacuum sector in Minkowski space to states in a charged sector by using a suitable charge transfer chain to create a charge in some region by shifting the conjugate charge along some path to spacelike infinity (“behind the moon”) where it evades observation. The energetic effects of this operation can be controlled by not putting a sharp restriction on the location of the chosen path, \textit{i.e.} allowing it to fluctuate within some spacelike cone. The resulting limit states are in a charged sector, but, by locality, coincide with the initial states for observations in the spacelike complement of the chosen cone [4, 8].

One can proceed similarly in the light cone $V$. To see this, regard $V$ as a globally hyperbolic spacetime with a metric induced from the ambient Minkowski space. It is foliated by hyperboloids (time–shells) parametrized by the proper time of inertial observers passing through the apex of $V$. In the spacetime $V$, Minkowskian spacelike infinity is replaced by the asymptotic lightlike boundary of $V$. The analogues of spacelike cones in Minkowski space are hypercones $C \subset V$, \textit{i.e.} the causal completions of hyperbolic cones formed by geodesics on a given time–shell emanating from a common apex. (The precise definition of these hypercones is given in the appendix.) As in Minkowski space, a limiting procedure using local unitary operators from the algebra $\mathcal{A}(C)$ of some hypercone $C$ creates charges in pairs, allowing one to pass from states in the charge class of the vacuum to states in another charge class. The resulting limit operation yields states in the new charge class, agreeing with the initial states for observations in the spacelike complement $C^c \subset V$ of the chosen hypercone by locality.

Now, whereas the electrically charged sectors in Minkowski space depend on the direction of the spacelike cone used to prepare them, this is not so for charge classes created by the analogous operations in the spacetime $V$: the disjoint infrared clouds produced using different cones cannot be sharply discriminated in $V$. Hence the
charge class of the resulting states does not depend on the choice of hypercone \( C \). So these infrared problems disappear for observers in \( V \).

As will be argued later, these heuristic considerations are reflected in the following mathematical setting. Denoting by \( \mathfrak{A}(V) \) the weak closure of the algebra \( \mathfrak{A}(V) \) relative to the vacuum class, our charge classes can be reached from the vacuum class using a morphism \( \sigma : \mathfrak{A}(V) \to \mathfrak{A}(V) \) localized in some hypercone \( C \), that is \( \sigma \) acts trivially on \( \mathfrak{A}(C^c) \), whereby \( C \subset V \) can be chosen arbitrarily. Our analysis shows that these morphisms have a rich algebraic structure, familiar from the theory of superselection sectors.

(c) Spacetime translations in \( V \) do not induce automorphisms of \( \mathfrak{A}(V) \), posing the question of how to define the energy content of states on these algebras. In a massive theory this is not a problem, since the light cone algebras are irreducible in all sectors and the spectral resolutions of the global energy–momentum operators are contained in their weak closures. It then makes physical sense to characterize states of \( \mathfrak{A}(V) \) by their spectral properties. These can in principle be checked with arbitrary precision in any light cone \( V \). But in the presence of massless particles the total energy of a state can no longer be sharply determined by such measurements since the energy carried by outgoing massless particles created in the past of \( V \) fluctuates in the corresponding statistical ensemble and cannot be deduced from measurements in \( V \). When analyzing the energy content of states one can, however, exploit the fact that the semigroup of future-directed time translations acts as endomorphisms on each algebra \( \mathfrak{A}(V) \). This allows one to introduce a notion of covariance for charge classes and to establish the existence of selfadjoint generators for the semigroup action. Even though these generators cannot be interpreted as energy observables since they include gross fluctuations of the energy in \( V \), like the Liouvillians in quantum statistical mechanics, they contain relevant information on the energy content of states. In fact, as massless particles in the past of \( V \) are not taken into account, one expects on physical grounds that they provide (fuzzy) lower bounds on the total energy of states and are bounded from below, like the global energy.

The relevant notions above are all that is required for analyzing charge classes. No global information is needed, only the algebras of local observables for fixed \( V \) and the endomorphic action of the semigroup \( \mathcal{S}_+^\uparrow = \overrightarrow{\mathcal{V}}_+ \rtimes \mathcal{L}_+^\uparrow \) on \( \mathfrak{A}(V) \), where \( \overrightarrow{\mathcal{V}}_+ \) denotes the closed semigroup of future directed translations and \( \mathcal{L}_+^\uparrow \) the group of proper orthochronous Lorentz transformations on \( V \). This input suffices to characterize the vacuum state on \( \mathfrak{A}(V) \) and to determine its properties. In particular, one can establish the existence of a continuous unitary representation of the full Poincaré group \( \mathcal{P}_+^\uparrow \) in its GNS–representation satisfying the relativistic spectrum condition whose restriction to the semigroup \( \mathcal{S}_+^\uparrow \) induces the given endomorphic action on \( \mathfrak{A}(V) \). States in the charge class of the vacuum are induced by vectors in this GNS–representation.

The analysis of all other charge classes can be based on the above hypercone
localized morphisms $\sigma : \mathfrak{A}(V) \to \mathfrak{R}(V)$. In this paper we restrict our attention to simple charge classes and simple morphisms, to be defined later. This important special case (which includes the electric charge) simplifies the exposition whilst retaining the novel features. Equivalent simple morphisms carry the same (simple) charge. Using simple morphisms and an appropriate version of Haag duality [20] in the spacetime $V$, one concludes that there is a composition law for the simple charges making the set of simple charges into an Abelian group where the inverse is charge conjugation. A simple charge and its conjugate either obey Bose or Fermi statistics. More precisely, exchanging such a pair of charges is described by charge transport operators which are 1 or $-1$ respectively, whenever the charges are localized in spacelike separated hypercones. Finally, the group of simple charges is the dual of some compact Abelian group, the global gauge group, when all charges are simple. Thus the general structure of these families resembles that of the simple sectors of localizable charges in Minkowski space [13].

As already mentioned, there is a meaningful notion of covariant charges. One requires corresponding morphisms $\sigma$ to be extendible to coherent families of morphisms $\lambda \sigma : \mathfrak{A}(V) \to \mathfrak{R}(V)$, $\lambda \in S_+^1$, as explained below. It turns out that the set of simple covariant charges is a subgroup of the set of simple charges. Moreover, each covariant simple morphism has an associated continuous unitary representation of the covering group of the Poincaré group $\mathcal{P}_+^1$, unique up to equivalence, inducing the endomorphic action of $S_+^1$ on $\mathfrak{A}(V)$. In accordance with physical expectations, these representations are shown to satisfy the relativistic spectrum condition.

These results do not depend on the choice of light cone $V$. In the present approach, analyzing the state space of local theories provides a consistent physical picture even in the presence of massless particles. The interpretation of a theory in terms of light cone data reflects experimental limitations and provides a “geometric regularization” thus avoiding the spurious infrared problems appearing in treatments based on Minkowski space. This observation may also lead to a deeper understanding of the phenomenon of quantum decoherence.

The article is organized as follows. The assumptions underlying the present analysis are stated in the next section. In Sect. 3 the concept of vacuum in a light cone is introduced and its charge class is analyzed. Section 4 establishes the concept of simple charge classes and derives their group structure and statistics. In Sect. 5, covariant charge classes are defined; they are shown to have similar properties and to admit unitary representations of the Poincaré group. Section 6 is devoted to a proof of the spectrum condition for covariant charge classes and in Sect. 7 the relation between the present approach and the usual Minkowski space interpretation of quantum field theories is discussed. The conclusions comment on the particle aspects of the present approach and on possible future developments. The appendix is devoted to defining hypercones and proving the necessary results about them. They are referred to in the main text as [A.1] to [A.13].
2 Nets, localization and covariance

In this brief section we introduce the basic objects treated in this paper and the corresponding notation.

As outlined in the introduction, fixing an open (forward) light cone $V$ in four–dimensional Minkowski space $M$ with its standard metric $x^2 \equiv (x_0^2 - x^2)$, $x \in \mathbb{R}^4$, we consider a foliation of $V$ by three–dimensional hyperboloids $H$ (time–shells). We will consider two types of causally complete regions in $V$, namely standard (relatively compact) double cones $\mathcal{O}$ and hypercones $\mathcal{C}$. A hypercone $\mathcal{C}$ is the causal completion of a hyperbolic cone on some fixed hyperboloid $H$, where a hyperbolic cone is a cone in the sense of hyperbolic geometry. The appendix contains precise definitions and proofs of the principal properties of hyperbolic cones. If $X$ is a subset of $V$ then $X^c$, the spacelike complement of $X$ in $V$, denotes the interior of the set of points $y \in V$ such that $(x - y)^2 < 0$ for all $x \in X$. In particular, $\mathcal{O}^c$ denotes the spacelike complement of the double cone $\mathcal{O}$ and $\mathcal{C}^c$ the spacelike complement of the hypercone $\mathcal{C}$ respectively.

We let $\mathcal{O} \mapsto \mathfrak{A} (\mathcal{O})$ be a net $\mathfrak{A}$ of unital $C^*$–algebras on the set $\mathcal{K}$ of double cones $\mathcal{O} \subset V$, ordered under inclusion. This net describes the observables of the underlying theory. The $C^*$–inductive limit of the $\mathfrak{A} (\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, is denoted by $\mathfrak{A} (V)$. Similarly, if $\mathcal{C}$ is a hypercone, $\mathfrak{A} (\mathcal{C})$ denotes the closed subalgebra of $\mathfrak{A} (V)$ generated by the $\mathfrak{A} (\mathcal{O})$ with $\mathcal{O} \subset \mathcal{C}$ and $\mathfrak{A} (\mathcal{C}^c)$ denotes the closed subalgebra generated by the $\mathfrak{A} (\mathcal{C}_1)$ with $\mathcal{C}_1 \subset \mathcal{C}^c$.

We assume that the net $\mathfrak{A}$ satisfies locality (Einstein causality) i.e. observables localized in spacelike separated regions commute, in short

$$\left[ \mathfrak{A} (\mathcal{O}_1), \mathfrak{A} (\mathcal{O}_2) \right] = 0 \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2^c. \quad (2.1)$$

Below, we will strengthen this to an appropriate version of Haag duality appropriate to the present geometric setting.

The semigroup $S_+^\uparrow = \mathcal{V}_+ \rtimes \mathcal{L}_+^\uparrow \subset \mathcal{P}_+^\uparrow$ of Poincaré transformations acts on $V$; its general elements are denoted by $\lambda = (x, \Lambda)$, where $x \in \mathcal{V}_+$ are the translations and $\Lambda \in \mathcal{L}_+^\uparrow$ the Lorentz transformations. This semigroup induces endomorphisms $\alpha_\lambda$ of $\mathfrak{A} (V)$ with the obvious geometric action,

$$\alpha_\lambda (\mathfrak{A} (\mathcal{O})) = \mathfrak{A} (\lambda \mathcal{O}) \quad , \quad \lambda \in S_+^\uparrow , \quad \mathcal{O} \in \mathcal{K}. \quad (2.2)$$

Clearly these features of the net extend canonically to the algebras $\mathfrak{A} (\mathcal{C})$ associated with hypercones $\mathcal{C}$.

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1. We follow the practice of using $C^*$–algebras rather than von Neumann algebras. The added generality is spurious in that only their weak closures in the vacuum representation will play a role.

2. In fact $\mathfrak{A} (\mathcal{C}^c)$ is also generated by the $\mathfrak{A} (\mathcal{O})$ with $\mathcal{O} \subset \mathcal{C}^c$, see A.4.
3 Vacuum

The input specified in the preceding section suffices to identify vacuum states on $V$ and to establish their characteristic properties.

**Definition 3.1.** A state $\omega_0$ on $\mathfrak{A}(V)$ is called a vacuum state if

(a) $\omega_0 \circ \alpha_\lambda = \omega_0$ for $\lambda \in S_+^+$,

(b) $\lambda \mapsto \omega_0(A^*\alpha_\lambda(B))$ is continuous, $A, B \in \mathfrak{A}(V)$,

(c) $x \mapsto \omega_0(A^*\alpha_x(B))$ extends continuously to a function on the complex domain $\overline{V_+} + i\overline{V_+}$ which is analytic in its interior and whose modulus is bounded by $\sqrt{\omega_0(A^*A)\omega_0(B^*B)}$, $A, B \in \mathfrak{A}(V)$.

The following result fully justifies this characterization of vacuum states in $V$.

**Proposition 3.2.** Let $\omega_0$ be a vacuum state on $\mathfrak{A}(V)$ and let $(\pi_0, \mathcal{H}, \Omega)$ be its GNS representation. Then

(i) $\Omega$ is cyclic for $\pi_0(\alpha_\lambda(\mathfrak{A}(V)))$ for any $\lambda \in S_+^+$,

(ii) there is a continuous unitary representation $U_0$ of the full Poincaré group $\mathcal{P}^+_+ = \mathbb{R}^4 \rtimes \mathcal{L}^+_+$ on $\mathcal{H}$ leaving $\Omega$ invariant and inducing the endomorphic action of the semigroup, $\text{Ad} U_0(\lambda) \circ \pi_0 = \pi_0 \circ \alpha_\lambda$, $\lambda \in S_+^+$,

(iii) the spectrum of $U_0 \restriction \mathbb{R}^4$ is contained in the closed forward light cone $\overline{V_+}$.

**Remark.** This result shows that it is meaningful for an observer in $V$ to talk about the energy–momentum content of the states in $\mathcal{H}$. It should be noticed, however, that the unitaries $U_0 \restriction \mathbb{R}^4$ are not contained in the weak closure of $\pi_0(\mathfrak{A}(V))$ in the presence of massless particles, so there is no global observable determining this energy–momentum content. Instead, $U_0 \restriction \mathbb{R}^4$ determines the energy–momentum content relative to a state (the vacuum) whose energy–momentum is not precisely known as information about processes in the past of $V$ is lacking. This is similar to the situation with KMS–states.

**Proof.** (i) As the light cone is invariant under Lorentz transformations, it suffices to prove the first statement for the semigroup of translations $x \in \overline{V}_+$. Property (c) of vacuum states implies that the functions $x \mapsto \pi_0(\alpha_\lambda(A))\Omega$, $A \in \mathfrak{A}(V)$, extend continuously to vector–valued functions which are analytic in the interior of the domain $\overline{V}_+ + i\overline{V}_+$. Now if $\Psi \in \mathcal{H}$ is a vector in the orthogonal complement of $\pi_0(\alpha_y(\mathfrak{A}(V)))\Omega$ for some $y \in \overline{V}_+$, it follows from isotony and covariance of the net that $(\Psi, \pi_0(\alpha_x(A))\Omega) = 0$ for all $x \in \overline{V}_+ + y$. The edge-of-the-wedge theorem implies $(\Psi, \pi_0(\alpha_x(A))\Omega) = 0$ for all $x \in \overline{V}_+$ and hence $(\Psi, \pi_0(A)\Omega) = 0$, $A \in \mathfrak{A}(V)$. Thus $\Psi = 0$ since the GNS vector $\Omega$ is cyclic for $\pi_0(\mathfrak{A}(V))$, proving the first part.

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The explicit form of the bound is not needed, but it simplifies matters.
(ii) Making use of property (a) of vacuum states, one can consistently define isometries $U_0(\lambda), \lambda \in \mathcal{S}^\uparrow_+$ on $\mathcal{H}$ by putting $U_0(\lambda)\pi_0(A)\Omega = \pi_0(\alpha_\lambda(A))\Omega, A \in \mathfrak{A}(V)$. These isometries are unitary since their range is dense by the preceding step. Moreover, by construction, they induce the endomorphric action of the semigroup $\mathcal{S}^\uparrow_+$ and are weakly continuous by property (b) of vacuum states.

To prove that $U_0$ extends to the full Poincaré group $\mathcal{P}^\uparrow_+$ we note that $U_0 \upharpoonright \mathcal{L}^\uparrow_+$ already defines a representation of the subgroup of Lorentz transformations. So we need only consider the subgroup of translations $\mathbb{R}^4$. Given $x, y, v, w \in \mathcal{V}_+$ with $x - y = v - w$ one has $U_0(x)U_0(w) = U_0(x + w) = U_0(v)U_0(y)$. So these unitary operators commute and $U_0(x)U_0(y)^{-1} = U_0(v)U_0(w)^{-1}$. Hence, as any $z \in \mathbb{R}^4$ can be decomposed into $z = x - y$ with $x, y \in \mathcal{V}$, one can consistently extend $U_0$ to a continuous unitary representation of $\mathbb{R}^4$, putting $U_0(z) \doteq U_0(x)U_0(y)^{-1}$. Moreover, since $U_0(\Lambda)U_0(y)^{-1}U_0(\Lambda)^{-1} = (U_0(\Lambda)U_0(y)U_0(\Lambda)^{-1})^{-1} = U_0(\Lambda y)^{-1}$ for $y \in \mathcal{V}_+$, it is also clear that the Lorentz transformations act correctly on the extended translations. Thus, putting $U_0(\lambda) \doteq U_0(z)U_0(\Lambda), \lambda = (z, \Lambda) \in \mathcal{P}^\uparrow_+$, one obtains the desired extension of $U_0$ to $\mathcal{P}^\uparrow_+$.

(iii) Picking $A, B \in \mathfrak{A}(V)$, property (c) of vacuum states implies that the function $z \mapsto (\pi_0(B)\Omega, U_0(z)\pi_0(A)\Omega) = \omega_0(\alpha_y(B^*\alpha_x(A)))$, where $z = x - y, x, y \in \mathcal{V}_+$ can be continuously extended into the tube $\mathbb{R}^4 + i\mathcal{V}_+$ and is analytic in its interior. Moreover, the modulus of this extension is bounded by $\|\pi_0(B)\Omega\|\|\pi_0(A)\Omega\|$. It then follows from standard arguments in the theory of Laplace transforms that the spectrum of $U_0 \upharpoonright \mathbb{R}^4$ is contained in the closed forward light cone $\mathcal{V}_+$. \hfill $\square$

Remark. If the eigenvalue 0 in the spectrum of $U_0 \upharpoonright \mathbb{R}^4$ is simple, the vacuum is said to be unique. This is the case if and only if $\omega_0$ is weakly clustering, i.e.

$$\lim_{\mathcal{O} \searrow \mathcal{V}} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} dx \omega_0(A\alpha_x(B)) = \omega_0(A)\omega_0(B), \quad A, B \in \mathfrak{A}(V).$$

Thus this property of the vacuum state can also be determined in $V$.

In the subsequent analysis we assume uniqueness of the vacuum state. Moreover, without loss of generality, we regard the corresponding vacuum representation as the defining representation of $\mathfrak{A}(V)$ and therefore replace in the following the morphism $\pi_0$ by the identity $\iota$.

It is worthy of note that the preceding proposition allows one to extend the given net $\mathfrak{A}$ on $V$ to a local, Poincaré covariant net $\mathfrak{A}_M$ on Minkowski space $M$. It is given by putting for any double cone $\mathcal{O}_M \subset M$

$$\mathfrak{A}_M(\mathcal{O}_M) \doteq U_0(x_M, 1)^{-1}\mathfrak{A}(\mathcal{O}_M + x_M)U_0(x_M, 1), \quad x_M \in \mathcal{V}_+, \quad \mathcal{O}_M + x_M \subset V.$$  

This definition is consistent (i.e. independent of the choice of $x_M \in \mathcal{V}_+$) because the original net is covariant and the unitaries $U_0 \upharpoonright \mathbb{R}^4$ are mutually commutative. Moreover, the vacuum state $\omega_0$ on $\mathfrak{A}(V)$ can be extended to a vacuum state $\omega_M$ on
$\mathfrak{A}_M(M)$ by $\omega_M(A_M) \doteq (\Omega, A_M\Omega)$, $A_M \in \mathfrak{A}_M(M)$. This illustrates the remark made in the introduction that theory together with data in $V$ yields information about the past. But there is a caveat: the extension of $\omega_0$ to the net $\mathfrak{A}_M$ is not unique in the presence of massless particles. For there are different extensions describing in addition outgoing massless particles created in the past of $V$ [4, Sec. 5]. Nevertheless, the above extension is of interest here. On one hand it allows us to make use of results on local nets in Minkowski space established in the literature. On the other hand it makes contact with the standard Minkowskian interpretation of quantum field theory. We shall return to this topic in Sec. [7].

In the subsequent analysis we need to consider the weak operator topology on the observable algebras in the vacuum representation. The closures of the various C*-algebras in this topology are marked by replacing the letter $\mathfrak{A}$ by $\mathfrak{R}$. Thus $\mathfrak{R}(V)$ denotes the weak closure of $\mathfrak{A}(V)$, $\mathfrak{R}(C)$ that of $\mathfrak{A}(C)$, etc. As a consequence of the preceding assumptions, cf. [22, Rem. 4], either $\mathfrak{R}(V) = \mathcal{B}(\mathcal{H})$ or $\mathfrak{R}(V)$ is a factor of type $\text{III}_1$. The former is the case if the spectrum of $U_0 \upharpoonright \mathbb{R}^4$ has a mass gap [24, Thm 2], the latter when the spectrum has non-zero weight on the boundary of $\overline{V_+ \setminus \{0\}}$, i.e. in the presence of massless particles. For then the algebra $\mathfrak{R}(V)$ has a non-trivial commutant as a consequence of Huygens principle [3, p. 161]. (This can be established in the present framework using the extended net $\mathfrak{A}_M$.) Our arguments apply to both cases, but as we are primarily interested in theories with long range forces we suppose here that $\mathfrak{R}(V)$ is a factor of type $\text{III}_1$.

We supplement the preceding results assuming the vacuum vector $\Omega$ to be cyclic and separating for the algebras $\mathfrak{A}(O)$ associated with any given double cone $O \subset V$. This Reeh–Schlieder property of the vacuum [20] is a generic feature of nets of observable algebras generated by quantum fields [2,19]. The hypercone algebras are assumed to satisfy the appropriate form of Haag duality as adapted to the present geometrical setting: in analogy to sector analysis in Minkowski space theories [8,12] we require that there is a sufficiently large family $\mathcal{F}$ of hypercones (cf. Definition 1 in the appendix) such that for each $C \in \mathcal{F}$

$$\mathfrak{A}(C)' \cap \mathfrak{R}(V) = \mathfrak{R}(C) \quad \text{and} \quad \mathfrak{A}(C)' \cap \mathfrak{R}(V) = \mathfrak{R}(C'),$$

(3.1)

where a prime $'$ on an algebra denotes its commutant in $\mathcal{B}(\mathcal{H})$. We will refer to this condition as hypercone duality. It expresses the idea that the hypercone algebras are maximal in the sense that any extension would conflict with Einstein causality. This version of duality was introduced and tested for the free Maxwell field in [9].

### 4 Charge classes and morphisms

We now start to analyze charged states. As explained in the introduction, the concept of superselection sector does not make sense in the presence of massless particles and has to be replaced by the notion of charge class.
By definition [1], the charge class $C_0$ of the vacuum $\omega_0$ consists of the set of normal states on $A(V)$ relative to the defining vacuum representation $\iota$. Thus all of these states extend to normal states on the type III$_1$ factor $\mathfrak{A}(V)$. As already mentioned, the group of inner automorphisms of a type III$_1$ factor acts almost transitively on its normal states [10]. Hence, as the group of unitaries in $\mathfrak{A}(V)$ is strongly $\ast$–dense in the group of unitaries in $\mathfrak{A}(V)$ [26, Thm. 4.11], given a state in the charge class of the vacuum, there is a sequence of inner automorphisms $\{\gamma_n \in \text{In } A(V)\}_{n \in \mathbb{N}}$ with $\omega_0 \circ \gamma_n$ converging in norm to that state. Conversely, any state that can be approximated in this way belongs to the charge class of the vacuum.

Ensembles carrying a definite global charge that can be precisely determined in $V$ are described by primary states inducing factorial representations of $A(V)$. We consider here only states carrying simple charges (defined below) where the weak closures of the algebras in the corresponding representations are factors of type III$_1$. The charge classes of these states have a characterization analogous to those in the vacuum class.

**Definition 4.1.** A state $\omega$ on $A(V)$ is said to be elemental if the weak closure of its GNS-representation is a factor of type III$_1$. The charge class $\mathcal{C}$ of an elemental state $\omega$ is the norm closure of the set of states \( \{\omega \circ \gamma : \gamma \in \text{In } A(V)\} \) and coincides with the set of normal states in its GNS–representation. (Note that any other state in $\mathcal{C}$ is elemental and belongs to the same charge class.)

**Remark.** The notion of a charge class of elemental states is a physically meaningful generalization of the concept of superselection sector of pure states in massive theories. In fact, by the Kadison transitivity theorem [26, Thm. 4.18(iii)] any state in the sector of a given pure state $\omega$ on $A(V)$ is an element of the set $\{\omega \circ \gamma : \gamma \in \text{In } A(V)\}$, a closure in norm is not needed. Hence both sectors and charge classes consist of just those states that can be reached by exploiting the quantum effects of physical operations starting from a given pure or elemental state, respectively.

As explained in the introduction in heuristic terms, the charge classes of interest here are obtained from the states in the charge class of the vacuum by composing with suitable sequences of inner automorphisms of $A(V)$. We assume that for given target charge class $\mathcal{C}$ and any hypercone $\mathcal{C} \subset V$ there is a sequence of inner automorphisms \( \{\gamma_n \in \text{In } A(V)\}_{n \in \mathbb{N}} \subset \text{In } A(V) \) such that \( \{\omega_0 \circ \gamma_n\}_{n \in \mathbb{N}} \) converges pointwise on $A(V)$ to some state $\omega \in \mathcal{C}$,

$$\lim_n \omega_0 \circ \gamma_n(A) = \omega(A), \ A \in A(V). \quad (4.1)$$

Thus the condition of norm convergence within charge classes is relaxed to weak–$\ast$–convergence, thereby allowing the limit states $\omega$ to have a different charge. We supplement this assumption by a condition expressing the heuristic idea that the process of charge creation in a given hypercone $\mathcal{C}$ and operations performed in its distant spacelike complement are only weakly correlated in the vacuum state. The
precise form of this “independence relation” is

$$\lim_{n} \sup_{B} |\omega_{0} \circ \gamma_{n}(B^{*}A)| = \sup_{B} \lim_{n} |\omega_{0} \circ \gamma_{n}(B^{*}A)|,$$

where $A \in \mathfrak{A}(V)$ is any fixed operator localized in some double cone $O$, the supremum being taken over all operators $B \in \mathfrak{A}(O_{d})$, satisfying the normalization condition $\omega_{0} \circ \gamma_{n}(B^{*}B) = \omega_{0}(B^{*}B) = 1$, where $O_{d} \subset O^{c} \cap C^{c}$ is any other distant double cone. Note that localization implies $\gamma_{n} \upharpoonright \mathfrak{A}(O_{d}) = \iota$. The following lemma shows that the limits of such sequences of inner automorphisms exist.

**Lemma 4.2.** Let $\{\gamma_{n} \in \text{In } \mathfrak{A}(C)\}_{n \in \mathbb{N}}$ be a sequence of inner automorphisms satisfying conditions (4.1) and (4.2). The limit $\sigma_{C} = \lim_{n} \gamma_{n}$ exists pointwise on $\mathfrak{A}(V)$ in the strong operator topology. Moreover,

(i) $\sigma_{C}: \mathfrak{A}(V) \to \mathfrak{R}(V)$ is a homomorphism (briefly: morphism) of algebras.

(ii) $\sigma_{C} \upharpoonright \mathfrak{A}(C^{c}) = \iota$.

(iii) $\sigma_{C}(\mathfrak{A}(C_{1})) \subset \mathfrak{R}(C_{1})$ for any hypercone $C_{1} \supseteq C$.

In virtue of the last two properties we say that $\sigma_{C}$ is localized in the hypercone $C$.

**Proof.** In the proof we make use of the Reeh–Schlieder property of the vacuum, i.e. the fact that $\Omega$ is cyclic for $\mathfrak{A}(O_{d})$ and hence separating for $\mathfrak{R}(O_{d}^{c})$, where $O_{d}$ is any double cone. Let $A \in \mathfrak{A}(O)$ for any given $O$ and let $B_{1}, B_{2} \in \mathfrak{A}(O_{d})$, where $O_{d} \subset O^{c} \cap C^{c}$. Then $\gamma_{n}(B_{1}^{*}AB_{2}) = B_{1}^{*}\gamma_{n}(A)B_{2}$. Acting with $\omega_{0}$, the limit over $n$ exists by the first condition. Since $\Omega$ is cyclic for $\mathfrak{A}(O_{d})$ and $\gamma_{n}(A)$ is uniformly bounded $\sigma_{C} = \lim_{n} \gamma_{n}$ exists in the pointwise weak operator topology and defines a linear and symmetric map $\sigma_{C}: \mathfrak{A}(V) \to \mathfrak{R}(V)$. For it to exist in the pointwise strong operator topology, we only need to show that $\gamma_{n}(A)\Omega$ converges strongly since $\gamma_{n}(A) \in \mathfrak{A}(O_{d}^{c} \cup C) \subset \mathfrak{R}(O_{d}^{c})$ and $\Omega$ is separating for $\mathfrak{R}(O_{d}^{c})$. Now according to the second condition one has for $B \in \mathfrak{A}(O_{d})$ with $(\Omega, B^{*}B\Omega) = 1$

$$\sup_{B} \lim_{n} |(\Omega, B^{*}\gamma_{n}(A)\Omega)| = \lim_{n} \sup_{B} |(\Omega, B^{*}\gamma_{n}(A)\Omega)|.$$  

This implies $\|\lim_{n} \gamma_{n}(A)\Omega\| = \lim_{n} \|\gamma_{n}(A)\Omega\|$ by the Reeh–Schlieder property of $\Omega$, proving the strong convergence of $\gamma_{n}(A)\Omega$. Hence $\lim_{n} \gamma_{n} \to \sigma_{C}$ in the pointwise strong operator topology, proving that $\sigma_{C}$ is also multiplicative, i.e. a morphism. As $\gamma_{n} \upharpoonright \mathfrak{A}(C^{c}) = \iota$, $n \in \mathbb{N}$, property (ii) of $\sigma_{C}$ is evident and (iii) follows from the inclusion $\gamma_{n}(\mathfrak{A}(C_{1})) \subset \mathfrak{R}(C_{1})$ for $C_{1} \supseteq C$ and $n \in \mathbb{N}$. 

An immediate consequence of this lemma and the Reeh–Schlieder property of the vacuum is that the GNS–representation of the (charged) limit state $\omega_{0} \circ \sigma_{C}$ is $\sigma_{C}: \mathfrak{A}(V) \to \mathfrak{R}(V) \subset \mathcal{B}(\mathcal{H})$. Thus the charged states of interest here are vector states in the defining Hilbert space of the theory $\mathcal{H}$ for representations of the observable
algebra derived from hypercone localized morphisms. Furthermore, if the state $\omega_0 \circ \sigma_C$ is elemental, composing it with the elements of $\text{In} \mathfrak{A}(V)$ generates a norm dense subset of states in its charge class. Since $\sigma_C \circ \text{In} \mathfrak{A}(V) \subseteq \text{In} \mathfrak{R}(V) \circ \sigma_C$ by the first part of the preceding lemma, the corresponding GNS-representations of $\mathfrak{A}(V)$ act on $\mathcal{H}$ too and are given by (not necessarily localized) morphisms with range in $\mathfrak{R}(V)$. Moreover, all such representations are equivalent to the initial representation $\sigma_C$ via unitary intertwiners in $\mathfrak{R}(V)$. Thus almost all states in the charge class of $\omega_0 \circ \sigma_C$ induce representations of $\mathfrak{A}(V)$, mutually equivalent to each other in this strong sense.

The starting point of sector analysis in massive theories is to consider morphisms of the (irreducible) algebra of observables whose ranges and whose intertwiners are contained in its weak closure [8]. These observations motivate our assumption that given a relevant charge class there are corresponding morphisms localized in given hypercones and mutually equivalent via unitary intertwiners in $\mathfrak{R}(V)$. In the subsequent analysis we restrict attention to the physically significant case of “simple charges”, characterized as follows.

**Criterion:** Let $\mathcal{C}$ be a charge class of elemental states on $\mathfrak{A}(V)$. The states and their charge class are said to be simple if given a hypercone $\mathcal{C} \in \mathcal{F}$ there is a morphism $\sigma_C : \mathfrak{A}(V) \to \mathfrak{R}(V)$ with $\omega_0 \circ \sigma_C \in \mathcal{C}$ and

(a) $\sigma_C \upharpoonright \mathfrak{A}(\mathcal{C}^c) = \iota$,

(b) $\sigma_C(\mathfrak{A}(\mathcal{C}_1))^- = \mathfrak{R}(\mathcal{C}_1)$ for any $\mathcal{C}_1 \supseteq \mathcal{C}$,

(c) $\text{In} \mathfrak{R}(V) \circ \sigma_{\mathcal{C}_1} \bigcap \text{In} \mathfrak{R}(V) \circ \sigma_{\mathcal{C}_2} \neq \emptyset$ for any pair of hypercones $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{F}$.

In the sequel, it will be convenient to distinguish the morphisms $\sigma_C$ not only by their action on $\mathfrak{A}(V)$ but also by a hypercone $\mathcal{C}$ of localization in the sense of conditions (a) and (b). Thus two morphisms $\sigma_{\mathcal{C}_1}, \sigma_{\mathcal{C}_2}$ with different hypercones $\mathcal{C}_1, \mathcal{C}_2$ of localization may act on $\mathfrak{A}(V)$ in exactly the same way. Condition (b), where the bar $^-$ denotes the weak closure of the respective algebras, encodes the decisive information that the charge class is simple. In particular, $\sigma_C(\mathfrak{A}(V))^- = \mathfrak{R}(V)$ follows. This generalizes how simple sectors are characterized in Minkowski space theories, where an analogous equality of algebras is implied, cf. [12, Lem. 2.2]. Condition (c) says that for any pair of morphisms $\sigma_{\mathcal{C}_1}, \sigma_{\mathcal{C}_2}$ associated with $\mathcal{C}$ there are unitary intertwiners in $\mathfrak{R}(V)$, unique up to a phase by Condition (b).

The collection of hypercone localized morphisms corresponding to simple charge classes in the theory is denoted by $\Sigma(V)$ and $\Sigma(\mathcal{C}) \subset \Sigma(V)$ is the subset of morphisms localized in a given hypercone; throughout the subsequent discussion we shall implicitly assume that $\mathcal{C} \in \mathcal{F}$. We shall see that there is a composition law for the elements of $\Sigma(V)$, reflecting the composition of charges, every element of $\Sigma(V)$ has an inverse in the conjugate charge class, and the morphisms in every charge class have definite (Bose or Fermi) statistics.

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$^4$\footnote{$\mathcal{F}$ is the family of hypercones appearing in condition (3.1) of hypercone duality.}
After fixing the framework we begin to analyze the simple charge classes described by the morphisms in \( \Sigma(V) \). The proofs are similar to those adopted in [8] for massive theories. Yet geometric complications mean that some arguments have to be modified. Since we shall have to deal with various extensions of the morphisms we introduce the following convenient notation.

**Notation:** Let \( \tilde{\mathfrak{A}}(V) \) be any given C*-algebra, \( \mathfrak{A}(V) \subset \tilde{\mathfrak{A}}(V) \subset \mathfrak{R}(V) \), and let \( \tilde{\sigma}_i : \tilde{\mathfrak{A}}(V) \to \mathfrak{R}(V) \) be morphisms, \( i = 1, 2 \). We write \( \tilde{\sigma}_1 \simeq \tilde{\sigma}_2 \) if there is a unitary intertwiner \( W \in \mathfrak{R}(V) \) between \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \), i.e.

\[
\text{Ad} W \circ \tilde{\sigma}_1(A) = W \tilde{\sigma}_1(A) W^{-1} = \tilde{\sigma}_2(A), \quad A \in \tilde{\mathfrak{A}}(V).
\]

The set of all such unitary intertwiners in \( \mathfrak{R}(V) \) is denoted by \( (\tilde{\sigma}_1, \tilde{\sigma}_2) \).

The first step is to show that any morphism \( \sigma_C \in \Sigma(V) \) can be extended (as a morphism) from \( \mathfrak{A}(V) \) to certain larger domains. These domains are fixed by specifying some funnel of hypercones \( \{ \tilde{C}_n \}_{n \in \mathbb{N}} \) where \( \tilde{C}_1 \supset \ldots \supset \tilde{C}_n \supset \ldots \) and \( \tilde{C}_n^c \not\!\subset V \), cf. [11] and are defined as C*-inductive limits \( \tilde{\mathfrak{A}}(V) = \lim \mathfrak{R}(\tilde{C}_n^c) \subset \mathfrak{R}(V) \) of the net \( \mathfrak{R}(\tilde{C}_n^c) \subset \ldots \subset \mathfrak{R}(\tilde{C}_m^c) \subset \ldots \) of von Neumann algebras associated with the spacelike complements of the hypercones in the given funnel. Since \( \mathcal{O} \subset \tilde{C}_n^c \), cf. [11] for any given double cone \( \mathcal{O} \) and sufficiently large \( n \), \( \tilde{\mathfrak{A}}(V) \supset \mathfrak{A}(V) \). The following lemma establishes the existence of such extensions and proves some of their properties.

**Lemma 4.3.** Let \( \sigma_C \in \Sigma(V) \) and let \( \{ \tilde{C}_n \}_{n \in \mathbb{N}} \) be any funnel of hypercones.

(i) \( \sigma_C \) extends to a morphism \( \tilde{\sigma}_C : \tilde{\mathfrak{A}}(V) \to \mathfrak{R}(V) \) normal on each von Neumann algebra \( \mathfrak{R}(\tilde{C}_n^c) \), \( n \in \mathbb{N} \). If \( \mathcal{C} \subset \tilde{C}_m^c \), for some \( m \in \mathbb{N} \), the domain \( \tilde{\mathfrak{A}}(V) \) is stable under \( \tilde{\sigma}_C \).

(ii) Let \( \{ \tilde{\tilde{C}}_n \}_{n \in \mathbb{N}} \) be another funnel of hypercones and \( \tilde{\tilde{\sigma}}_C : \tilde{\tilde{\mathfrak{A}}}(V) \to \mathfrak{R}(V) \) be the corresponding extension of \( \sigma_C \). If \( \mathcal{C}_0 \) is a hypercone with \( \mathcal{C}_0 \subset \tilde{\tilde{C}}_m^c \cap \tilde{\tilde{C}}_m^c \) for some \( m \in \mathbb{N} \), then \( \tilde{\tilde{\sigma}}_C \mid \mathfrak{R}(\mathcal{C}_0) = \tilde{\sigma}_C \mid \mathfrak{R}(\mathcal{C}_0) \).

(iii) Let \( \sigma_{C_1} \simeq \sigma_{C_2} \) and let \( \tilde{\sigma}_{C_1}, \tilde{\sigma}_{C_2} \) be their respective extensions to a common domain \( \tilde{\mathfrak{A}}(V) \). Then \( \tilde{\sigma}_{C_1} \simeq \tilde{\sigma}_{C_2} \) and \( (\tilde{\sigma}_{C_1}, \tilde{\sigma}_{C_2}) = (\sigma_{C_1}, \sigma_{C_2}) \), i.e. the associated sets of unitary intertwiners in \( \mathfrak{R}(V) \) coincide.

**Proof.** As morphisms can be localized arbitrarily within any given charge class there are morphisms \( \sigma_{\tilde{C}_n} \simeq \sigma_C \) and intertwiners \( W_n \in (\sigma_{\tilde{C}_n}, \sigma_C) \) with

\[
\sigma_C \mid \mathfrak{A}(\tilde{C}_n^c) = \text{Ad} W_n \circ \sigma_{\tilde{C}_n} \mid \mathfrak{A}(\tilde{C}_n^c) = \text{Ad} W_n \mid \mathfrak{A}(\tilde{C}_n^c).
\]

Hence \( \sigma_C \) extends by weak continuity to \( \mathfrak{R}(\tilde{C}_n^c) \), \( n \in \mathbb{N} \), and thus to the C*-inductive limit \( \tilde{\mathfrak{A}}(V) \) of these algebras. The algebraic and normality properties of the resulting extension \( \tilde{\sigma}_C \) are apparent from this construction. If \( \mathcal{C} \subset \tilde{C}_m^c \) for some \( m \in \mathbb{N} \),
\[ \sigma_c \mid \mathfrak{A}(\tilde{C}_n) = \iota, \text{ for } n \geq m, \text{ then } \sigma_c(\mathfrak{A}(\tilde{C}_n)) \subset \mathfrak{A}(\tilde{C}_n)' = \mathfrak{R}(\tilde{C}_n^c) \text{ by hypercone duality.} \]

But \( \sigma_c(\mathfrak{A}(\tilde{C}_n^c)) = \tilde{\sigma}_c(\mathfrak{A}(\tilde{C}_n^c)) = \tilde{\sigma}_c(\mathfrak{R}(\tilde{C}_n^c)) \), completing the proof of the first part of the statement. The second part follows from

\[ \tilde{\sigma}_c \mid \mathfrak{A}(C_0) = \sigma_c \mid \mathfrak{A}(C_0) = \tilde{\sigma}_c \mid \mathfrak{A}(C_0) \]

and the normality of \( \tilde{\sigma}_c \) and \( \tilde{\sigma}_c \) on \( \mathfrak{R}(\tilde{C}_0) \subset \mathfrak{R}(\tilde{C}_m^c) \cap \mathfrak{R}(\tilde{C}_m^c) \). The third part follows similarly from the normality properties of the extensions. \( \square \)

Lemma [4.3(i)] implies that morphisms \( \sigma_{C_1}, \ldots, \sigma_{C_n} \in \Sigma(V) \) can be extended to a common stable domain whenever there is some auxiliary hypercone \( C_0 \subset \bigcap_{k=1}^{n} C_k^c \) containing a funnel, cf. [A.1]. The (associative) product of the extended morphisms is then well defined. However, for certain pairs of hypercones there is no such auxiliary hypercone \( C_0 \), cf. the appendix. The following result ensures that this geometrical obstacle does not cause major problems.

**Proposition 4.4.** Let \( \sigma_{C_A}, \tau_{C_B} \in \Sigma(V) \).

(i) If \( \tilde{\sigma}_{C_A} \) is an extension of \( \sigma_{C_A} \) containing \( \mathfrak{R}(C_B) \) in its domain, the composed map \( \tilde{\sigma}_{C_A} \circ \tau_{C_B} : \mathfrak{A}(V) \to \mathfrak{R}(V) \) is a well defined morphism independent of the chosen extension of \( \sigma_{C_A} \).

(ii) Let \( \sigma_{C_A} \simeq \sigma_{C_a} \) and \( \tau_{C_B} \simeq \tau_{C_b} \). Then \( \tilde{\sigma}_{C_A} \circ \tau_{C_B} \simeq \tilde{\sigma}_{C_a} \circ \tau_{C_b} \), where \( \tilde{\sigma}_{C_A} \) is any extension of \( \sigma_{C_A} \) containing \( \mathfrak{R}(C_B) \) in its domain.

(iii) Given any hypercone \( C \) there is a unitary intertwiner in \( \mathfrak{R}(V) \) taking a morphism \( \rho_C \in \Sigma(C) \) to \( \tilde{\sigma}_{C_a} \circ \tau_{C_b} \). Moreover, if \( C_A, C_B \subset C, \tilde{\sigma}_{C_A} \circ \tau_{C_B} \mid \mathfrak{A}(O) = \iota \) so that if \( W \in (\tau_{C_B}, \tau_{C_b}) \),

\[ \tau_{C_b}(A) = \mathrm{Ad} W \circ \tau_{C_B}(A) = \mathrm{Ad} W(A), \quad A \in \mathfrak{A}(O). \]

Similarly, \( \tau_{C_B} \mid \mathfrak{A}(C_B^c) = \iota = \tau_{C_B} \mid \mathfrak{A}(C_B^c) \) implying \( (\tau_{C_B}, \tau_{C_b}) \subset \mathfrak{R}(C) \) by hypercone duality. Hence \( W \) is in the domain of \( \tilde{\sigma}_{C_a} \) and

\[ \tilde{\sigma}_{C_a}(W)\sigma_{C_a}(A)\tilde{\sigma}_{C_a}(W^{-1}) = \tilde{\sigma}_{C_a}(\mathrm{Ad} W(A)) = \tilde{\sigma}_{C_a}(\tau_{C_b}(A)), \quad A \in \mathfrak{A}(O). \]

As \( O \) was arbitrary, \( \tilde{\sigma}_{C_a} \circ \tau_{C_b} : \mathfrak{A}(V) \to \mathfrak{R}(V) \) is a well defined morphism. Moreover, since \( \tilde{\sigma}_{C_a}(W) \) does not depend on the chosen extension of \( \sigma_{C_a} \) by Lemma [4.3(ii)] neither does \( \tilde{\sigma}_{C_a} \circ \tau_{C_b} \).

To prove the second part of the proposition we first keep \( \tau_{C_B} \) fixed. Let \( \tilde{\sigma}_{C_a}, \tilde{\sigma}_{C_A} \) be extensions of \( \sigma_{C_a}, \sigma_{C_A} \), respectively, both having \( \mathfrak{R}(C_B) \) in their domain and let \( W \in (\sigma_{C_A}, \sigma_{C_a}) \). The normality properties of extensions, established in Lemma [4.3(i)], imply \( \mathrm{Ad} W \circ \tilde{\sigma}_{C_A} \mid \mathfrak{R}(C) = \tilde{\sigma}_{C_a} \mid \mathfrak{R}(C) \), whenever \( \mathfrak{R}(C) \) is in the domain of both extensions. Hence \( \tilde{\sigma}_{C_A} \circ \tau_{C_b} \simeq \tilde{\sigma}_{C_a} \circ \tau_{C_b} \). Next, keeping \( \sigma_{C_A} \) fixed we vary
\(\tau_{C_b}\) and pick any \(\tau_{C_B} \simeq \tau_{C_b}\) such that there is a larger hypercone \(C_0 \supset C_b \cup C_B\). Then \(\tau_{C_B} \uparrow \mathcal{A}(C_0) = \iota = \tau_{C_b} \uparrow \mathcal{A}(C_0)\), hence \((\tau_{C_B}, \tau_{C_b}) \subset \mathcal{R}(C_0)\) by hypercone duality. Choosing an extension \(\tilde{\sigma}_{C_A}\) of \(\sigma_{C_A}\) with \(\mathcal{R}(C_0)\) in its domain, we obtain 
\[\tilde{\sigma}_{C_A} \circ \tau_{C_b} = \tilde{\sigma}_{C_A} \circ \tau_{C_b} = \tilde{\sigma}_{C_A} \circ \text{Ad } W \circ \tau_{C_B} = \text{Ad } \tilde{\sigma}_{C_A}(W) \circ \tilde{\sigma}_{C_A} \circ \tau_{C_B}\] 
for \(W \in (\tau_{C_B}, \tau_{C_b})\). Hence \(\tilde{\sigma}_{C_A} \circ \tau_{C_B} \simeq \tilde{\sigma}_{C_a} \circ \tau_{C_b}\) for the restricted set of regions \(C_b, C_B\). The result for pairs of hypercones \(C_b, C_B \in \mathcal{F}\) in general position then follows by a standard interpolation argument as the family \(\mathcal{F}\) is pathwise connected, cf. [A.3]

To prove the third part of the proposition, we pick morphisms \(\sigma_{C_A} \simeq \sigma_{C_a}\), \(\tau_{C_B} \simeq \tau_{C_b}\) with \(C_A, C_B \subset \mathcal{C}\) and define \(\varrho_C \doteq \tilde{\sigma}_{C_A} \circ \tau_{C_B}\). The localization properties of \(\sigma_{C_A}, \tau_{C_B}\) imply \(\varrho_C \uparrow \mathcal{A}(\mathcal{C}_c^\circ) = \iota\), so \(\varrho_C\) satisfies point (a) of the criterion. To establish (b) we use Lemma 4.3(i) choosing, for given \(\mathcal{C}_1 \supset \mathcal{C}\), an extension \(\tilde{\sigma}_{C_A}\) of \(\sigma_{C_A}\) normal on \(\mathcal{R}(\mathcal{C}_1)\). But, as shown in the first step, \(\tilde{\sigma}_{C_A} \circ \tau_{C_B} = \tilde{\sigma}_{C_A} \circ \tau_{C_b}\), and \(\tilde{\sigma}_{C_A}(\mathcal{A}(\mathcal{C}_1)^-\rangle = \sigma_{C_A}(\mathcal{A}(\mathcal{C}_1))^- = \mathcal{R}(\mathcal{C}_1) = \tau_{C_B}(\mathcal{A}(\mathcal{C}_1)\rangle\). Hence 
\[\varrho_C(\mathcal{A}(\mathcal{C}_1)^-\rangle = \tilde{\sigma}_{C_A} \circ \tau_{C_B}(\mathcal{A}(\mathcal{C}_1))^- = \tilde{\sigma}_{C_A}(\tau_{C_B}(\mathcal{A}(\mathcal{C}_1)\rangle\) = \(\mathcal{R}(\mathcal{C}_1)\).

Thus \(\varrho_C\) also satisfies (b). But the hypercone \(\mathcal{C}\) was arbitrary so the results established in (ii) imply that the morphism \(\varrho_C\) satisfies (c), too. Hence \(\varrho_C \in \Sigma(\mathcal{C})\) and, in particular, \(\tilde{\sigma}_{C_a} \circ \tau_{C_b} \in \Sigma(\mathcal{C})\) whenever \(C_a, C_b \subset \mathcal{C}\). \(\square\)

The preceding proposition shows how pairs of morphisms \(\sigma_{C_A}, \tau_{C_B} \in \Sigma(\mathcal{V})\) can be composed inducing a composition of the corresponding simple charge classes \(\mathcal{C}_A, \mathcal{C}_B\) by applying the composed morphisms to the vacuum state. Since the composition of morphisms \(\tilde{\sigma}_{C_A} \circ \tau_{C_B}\) does not depend on the chosen extension \(\tilde{\sigma}_{C_a}\) of \(\sigma_{C_a}\) (up to limitations depending on the given localization hypercone \(C_0\) of \(\tau_{C_b}\)) we will omit the tilde in the following and simply write \(\sigma_{C_a} \circ \tau_{C_b}\) for the composed morphisms. Yet it must be remembered that care must be taken with domains in this product and that \(\sigma_{C_a} \circ \tau_{C_b} \notin \Sigma(\mathcal{V})\) unless \(C_a, C_b \subset \mathcal{C}\) for some hypercone \(\mathcal{C}\).

We show next that each simple charge class \(\mathcal{C}\) has a simple “conjugate” charge class \(\overline{\mathcal{C}}\). The following proposition says that the corresponding charges compensate (neutralize) one another by composition.

**Proposition 4.5.** Given a simple charge class there is a simple conjugate charge class in the following sense: given a morphism \(\sigma_C \in \Sigma(\mathcal{V})\) there is a morphism \(\overline{\sigma}_C \in \Sigma(\mathcal{V})\) with \(\overline{\sigma}_C \circ \sigma_C = \sigma_C \circ \overline{\sigma}_C = \iota\).

**Proof.** Given \(\sigma_C\) we pick an increasing sequence of hypercones \(C_n \supset \mathcal{C}, n \in \mathbb{N}\), with \(C_n \nearrow \mathcal{V}; \) the corresponding opposite cones \(\mathcal{C}_n \subset C_n^\circ, n \in \mathbb{N}\), then form a funnel of hypercones, cf. [A.2]. Next, we choose morphisms \(\sigma_{C_n} \simeq \sigma_C\) and unitary intertwiners \(W_n \in (\sigma_{C_n}, \sigma_C), n \in \mathbb{N}\). Thus for any given \(k \in \mathbb{N}\) 
\[\text{Ad } W_n^{-1} \circ \sigma_C \uparrow \mathcal{A}(\mathcal{C}_k) = \sigma_{C_n} \uparrow \mathcal{A}(\mathcal{C}_k) = \iota, \quad n \geq k,\]

According to (b) of the criterion, \(\sigma_C(\mathcal{A}(\mathcal{C}_k)^-) = \mathcal{R}(\mathcal{C}_k), k \in \mathbb{N}\), hence 
\[\text{Ad } W_m^{-1} \uparrow \mathcal{R}(\mathcal{C}_k) = \text{Ad } W_n^{-1} \uparrow \mathcal{R}(\mathcal{C}_k), \quad m, n \geq k.\]
So the pointwise norm limit
\[ \sigma_C = \lim_n \Ad W_n^{-1} \]
even exists on the C*-inductive limit $\mathfrak{N}(V) = \lim_n \mathfrak{R}(C_k) \supset \mathfrak{A}(V)$ of $\mathfrak{R}(C_k), k \in \mathbb{N}$ and
defines a morphism $\sigma_C : \mathfrak{N}(V) \rightarrow \mathfrak{R}(V)$ which is normal on each algebra $\mathfrak{R}(C_k), k \in \mathbb{N}$. Moreover, it is a left and right inverse of the (suitably extended) morphism $\sigma_C$ as we will show next. Let $\tilde{\sigma}_C$ be an extension of $\sigma_C$ based on the funnel $\{C_n\}_{n \in \mathbb{N}}$. Then, for any $n \geq k$, $\tilde{\sigma}_C \upharpoonright \mathfrak{R}(C_k) = \Ad W_n \uparrow \mathfrak{R}(C_k)$ and, by construction,
$\tilde{\sigma}_C \upharpoonright \mathfrak{R}(C_k) = \Ad W_n^{-1} \uparrow \mathfrak{R}(C_k)$. As $C \subset C_k$, $\mathfrak{R}(C_k) = \sigma_C(\mathfrak{R}(C_k)) = \Ad W_n(\mathfrak{R}(C_k))$, implying $\tilde{\sigma}_C(\mathfrak{R}(C_k))^{-1} = \mathfrak{R}(C_k)$, whence
\[
\tilde{\sigma}_C(\sigma(C)) = \Ad W_k^{-1}(\Ad W_k(A)) = A
\]
for $A \in \mathfrak{R}(C_k), k \in \mathbb{N}$, and these equalities extend by continuity to $\mathfrak{A}(V)$.

To proceed we need to show that the restriction $\tilde{\sigma}_C \upharpoonright \mathfrak{A}(V)$ does not depend on the initial choice of a sequence of hypercones. Another admissible sequence yields another morphism $\overline{\sigma}_C : \overline{\mathfrak{N}(V)} \rightarrow \mathfrak{R}(V)$ with the properties established above. In particular, it is a left inverse of $\sigma_C$, hence $\overline{\sigma}_C(\sigma(C)) = A = \tilde{\sigma}_C(\sigma(C))$ for $A \in \mathfrak{A}(V)$. Both $\overline{\sigma}_C$ and $\tilde{\sigma}_C$ have $\mathfrak{R}(C)$ in their domains and are normal on this algebra. It therefore follows from this equality and $\sigma_C(\mathfrak{A}(C))^{-1} = \mathfrak{R}(C)$ that $\overline{\sigma}_C \upharpoonright \mathfrak{R}(C) = \tilde{\sigma}_C \upharpoonright \mathfrak{R}(C)$. Moreover, given any double cone $O \subset V$ there is a unitary $W \in \mathfrak{R}(C)$ with $\sigma_C \upharpoonright \mathfrak{A}(O) = \Ad W \uparrow \mathfrak{A}(O)$. Thus using the above equality once more
\[
\tilde{\sigma}_C(W)\overline{\sigma}_C(A)\overline{\sigma}_C(W^{-1}) = A = \tilde{\sigma}_C(W)\overline{\sigma}_C(A)\overline{\sigma}_C(W^{-1}), \quad A \in \mathfrak{A}(O).
\]
But $\overline{\sigma}_C(W) = \mathfrak{N}(C) \upharpoonright \mathfrak{A}(V)$, hence $\overline{\sigma}_C(A) = \mathfrak{N}(C) \upharpoonright \mathfrak{A}(O)$, $A \in \mathfrak{A}(O)$. Since $O$ was arbitrary this shows $\overline{\sigma}_C \upharpoonright \mathfrak{A}(V) = \tilde{\sigma}_C \upharpoonright \mathfrak{A}(V)$.

We are now in a position to prove that the morphisms $\sigma_C : \mathfrak{A}(V) \rightarrow \mathfrak{R}(V)$ satisfy the criterion. Choosing morphisms and intertwiners as in the first step of the proof we have
\[
\Ad W_n^{-1} \upharpoonright \mathfrak{A}(C^c) = \Ad W_n^{-1} \circ \sigma_C \upharpoonright \mathfrak{A}(C^c) = \sigma_{C_n} \upharpoonright \mathfrak{A}(C^c).
\]
Since $\sigma_{C_n} \rightarrow \iota$ pointwise in norm on $\mathfrak{A}(V)$ as $n \rightarrow \infty$ it follows that $\tilde{\sigma}_C \upharpoonright \mathfrak{A}(C^c) = \iota$ proving (a). Next, given a hypercone $C_0 \supset C$, we choose an increasing sequence of hypercones $C_n, n \in \mathbb{N}$, with $C_1 \supset C_0$, yielding an extension of $\tilde{\sigma}_C$ of $\sigma_C \upharpoonright \mathfrak{A}(V)$ normal on $\mathfrak{R}(C_0)$. Thus, bearing in mind that $\sigma_C(\mathfrak{A}(C_0))^* = \mathfrak{R}(C_0) = \mathfrak{A}(C_0)^*$, we get
\[
\tilde{\sigma}_C(\mathfrak{A}(C_0))^* = \tilde{\sigma}_C(\mathfrak{A}(C_0))^* = \mathfrak{R}(C_0) = \mathfrak{A}(C_0)^* \]
where in the last equality we used (4.3). This proves (b). Finally, if $C_a, C_b$ and $C_0 \supset C_a, C_b$ are hypercones and $\sigma_{C_a} \simeq \sigma_{C_b}$ with conjugates $\overline{\sigma}_{C_a}$ and $\overline{\sigma}_{C_b}$, respectively,
we choose an increasing sequence of hypercones $C_n$, $n \in \mathbb{N}$, with $C_1 \doteq C_0$, as above. As has been shown, the corresponding extensions $\overline{\sigma}_{C_n}, \overline{\sigma}_{C_b}$ of the conjugate morphisms to the domain $\mathfrak{R}(V) \doteq \lim_{k \to \infty} \mathfrak{R}(C_k) \supset \mathfrak{A}$ are normal on each member of the net $\mathfrak{R}(C_k)$, $k \in \mathbb{N}$. Now the unitary intertwiners $W_0 \in (\sigma_{C_b}, \sigma_{C_a})$ are elements of $\mathfrak{R}(C_0)$ by hypercone duality. So like the elements of $\sigma_{C_b}(\mathfrak{A}(V))$, $\sigma_{C_a}(\mathfrak{A}(V))$, they are in the domain of $\overline{\sigma}_{C_a}$ and we can compute
\[
\overline{\sigma}_{C_a}(W_0) \overline{\sigma}_{C_a}(\sigma_{C_a}(A)) \overline{\sigma}_{C_a}(W_0^{-1}) = \overline{\sigma}_{C_a}(W_0 \sigma_{C_b}(A) W_0^{-1}) = \overline{\sigma}_{C_a}(\sigma_{C_a}(A)) = A = \overline{\sigma}_{C_b}(\sigma_{C_b}(A)), \quad A \in \mathfrak{A}(V).
\]
But $\sigma_{C_b}(\mathfrak{A}(C_k)) = \mathfrak{R}(C_k)$, $k \in \mathbb{N}$, so normality implies $\text{Ad} \overline{\sigma}_{C_a}(W_0) \overline{\sigma}_{C_a} = \overline{\sigma}_{C_b}$ on $\mathfrak{R}(V)$. Restricting this equality to $\mathfrak{A}(V)$ we conclude that the unitary operator $\overline{\sigma}_{C_a}(W_0) \in \mathfrak{R}(V)$ intertwines $\overline{\sigma}_{C_a}$ and $\overline{\sigma}_{C_b}$ for the restricted pairs of hypercones. By a standard interpolation argument, cf. [A.3], this equivalence extends to arbitrary pairs of morphisms $\overline{\sigma}_{C_a}, \overline{\sigma}_{C_b}$, so they satisfy (c), too. Thus $\overline{\sigma}_{C} \in \Sigma(V)$ for any choice of hypercone $C$. Relation (4.3) implies $\overline{\sigma}_{C} \overline{\sigma}_{C} = \overline{\sigma}_{C} \overline{\sigma}_{C} = \iota$, completing the proof of the proposition.

We now analyze the statistics of charge classes as encoded in the structure of the intertwiners of composed morphisms. Since we are just dealing with simple charges we do not need the full arsenal of categorical methods developed in [8, 12] and can rely on strategies established in [11]. But, again, geometric problems mean that some arguments have to be modified.

**Lemma 4.6.** Let $\sigma_{C_a}, \tau_{C_b}$ be morphisms.

(i) $\sigma_{C_a} \cdot \tau_{C_b} = \tau_{C_b} \cdot \sigma_{C_a}$ if $C_a$ and $C_b$ are spacelike separated.

(ii) $\sigma_{C_a} \cdot \tau_{C_b} \simeq \tau_{C_b} \cdot \sigma_{C_a}$ if $C_a$ and $C_b$ are in arbitrary position.

**Proof.** Given a double cone $O \subset V$, we choose (cf. [A.3]) hypercones $C_A \subset O^c \cap C_a$, $C_B \subset O^c \cap C_B$, morphisms $\sigma_{C_A} \simeq \sigma_{C_a}, \tau_{C_B} \simeq \tau_{C_b}$ and intertwiners $W_a \in (\sigma_{C_A}, \sigma_{C_a})$, $W_b \in (\tau_{C_B}, \tau_{C_b})$. If $C_a$ and $C_b$ are spacelike separated, the localization of the morphisms $\sigma_{C_A}, \tau_{C_B}$ imply that the intertwiners $W_a \in \mathfrak{R}(C_a), W_b \in \mathfrak{R}(C_b)$ commute, so
\[
\sigma_{C_a} \cdot \tau_{C_b}(A) = \tilde{\sigma}_{C_a} (\tau_{C_b}(A)) = \tilde{\sigma}_{C_a} (\text{Ad} W_b(A)) = \text{Ad} W_a (\text{Ad} W_b(A))
\]
\[
since O was arbitrary, (i) follows. We complete the proof by choosing spacelike separated hypercones $C_A, C_B$ and morphisms $\sigma_{C_A} \simeq \sigma_{C_a}, \tau_{C_B} \simeq \tau_{C_b}$, (ii) then follows from Proposition [4.4(iii)] and the preceding result. 

\[
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\]
We now consider equivalent morphisms \( \sigma_{c_a} \simeq \sigma_{c_b} \) associated with a given charge class. When discussing statistics it suffices to look at pairs \( C_a, C_b \) having some hypercone \( C \subset C_a \cap C_b \), as is the case if \( C_a \) and \( C_b \) are spacelike separated, cf. [A.3]. Choosing a funnel of hypercones \( \{ \tilde{C}_n \subset C \}_{n \in \mathbb{N}} \) yields extensions of \( \sigma_{c_a}, \sigma_{c_b} \) to morphisms \( \tilde{\sigma}_{c_a}, \tilde{\sigma}_{c_b} \) both acting on the common domain \( \tilde{\mathcal{A}}(V) \equiv \lim_{n \to \infty} \mathcal{A}((\tilde{C}_n)^c) \), cf. Lemma [4.3]. According to part (iii) of this lemma the spaces of intertwiners \( (\sigma_{c_a}, \sigma_{c_b}) \) and \( (\tilde{\sigma}_{c_b}, \tilde{\sigma}_{c_a}) \) coincide. Moreover, \( (\sigma_{c_b}, \sigma_{c_a}) \subset \mathcal{R}(C^c) \subset \tilde{\mathcal{A}}(V) \) using the localization properties of the morphisms and hypercone duality. Thus the unitary intertwiner \( W \in (\sigma_{c_a}, \sigma_{c_b}) \) is unique up to a phase, and

\[
\varepsilon(\sigma_{c_a}, \sigma_{c_b}) \equiv W^{-1} \tilde{\sigma}_{c_a}(W) = \tilde{\sigma}_{c_b}(W)W^{-1}
\]

is well defined. By construction, \( \varepsilon(\sigma_{c_a}, \sigma_{c_b}) \) is an intertwiner in \( (\sigma_{c_a} \circ \sigma_{c_b}, \sigma_{c_b} \circ \sigma_{c_a}) \). The following lemma shows that it is an intrinsic quantity depending only on the given morphisms.

**Lemma 4.7.** The intertwiner \( \varepsilon(\sigma_{c_a}, \sigma_{c_b}) \) is independent of the extensions of the given morphisms when chosen as above.

**Proof.** The operator \( \tilde{\sigma}_{c_a}(W) \) is independent of the choice of funnel contained in a given hypercone \( C \subset C_a \cap C_b \) as the corresponding extensions of \( \sigma_{c_a} \) coincide on \( \mathcal{R}(C^c) \) by Lemma [4.3(i)]. Next, if \( C_0 \subset C_a \cap C_b \) is another hypercone, there is a hypercone \( \tilde{C}_1 \subset C_0 \) making \( \tilde{C}_1^c \cap C^c \) hypercone connected, i.e. this region contains with any pair of hypercones a path of hypercones interpolating between them, cf. [A.7]. We take \( \tilde{C}_1 \) as initial member of a funnel \( \{ \tilde{C}_n \subset C_0 \}_{n \in \mathbb{N}} \) and consider the corresponding extension \( \tilde{\sigma}_{c_a} \) of \( \sigma_{c_a} \). By hypercone connectivity, there is an interpolating path of hypercones \( C_k \subset \tilde{C}_1^c \cap C^c \), \( k = 1, \ldots, m \), with \( C_1 = C_a \) and \( C_m = C_b \). Consequently we can write the intertwiner \( W \) as product \( W = W_1 \cdots W_m \) with \( W_k \in \mathcal{R}(C_0) \subset \mathcal{R}(\tilde{C}_1^c) \cap \mathcal{R}(C^c) \), \( k = 1, \ldots, m \). Since both \( \tilde{\sigma}_{c_a} \) and \( \tilde{\sigma}_{c_b} \) have the algebra \( \mathcal{R}(\tilde{C}_1^c) \cap \mathcal{R}(C^c) \) in their respective domains, it follows from Lemma [4.3(ii)] that \( \tilde{\sigma}_{c_a}(W_k) = \tilde{\sigma}_{c_a}(W_k), k = 1, \ldots, m \). Hence \( W^{-1} \tilde{\sigma}_{c_a}(W) = W^{-1} \tilde{\sigma}_{c_a}(W) \), as claimed.

With this information we can establish that each simple charge class has a definite “statistics parameter”.

**Proposition 4.8.** Given a simple charge class \( C \) and the corresponding family of morphisms \( \sigma_C : \tilde{\mathcal{A}}(V) \to \mathcal{R}(V) \).

(a) There is a statistics parameter \( \varepsilon_C \in \{ \pm 1 \} \), depending only on the charge class, such that \( \varepsilon(\sigma_{c_a}, \sigma_{c_b}) = \varepsilon_C \) for any pair of morphisms \( \sigma_{c_a}, \sigma_{c_b} \) localized in spacelike separated hypercones \( C_a, C_b \).

(b) The statistics parameter \( \varepsilon_{\overline{C}} \) of the corresponding conjugate charge class \( \overline{C} \) has the same value, \( \varepsilon_{\overline{C}} = \varepsilon_C \).
Proof. Let $C_a$, $C_b$ be spacelike separated hypercones. Then $\sigma_{C_a} \cdot \sigma_{C_b} = \sigma_{C_b} \cdot \sigma_{C_a}$ by Lemma 4.6(i). Since these composed morphisms are members of a simple charge class by Proposition 4.4(iii), the corresponding unitary self-intertwiners are multiples of the identity. Hence $\varepsilon(\sigma_{C_a}, \sigma_{C_b}) = \varepsilon 1$ for some phase factor $\varepsilon \in \mathbb{T}$. Choosing a hypercone $C \subset C_\varepsilon \cap C_{\tilde{\varepsilon}}$, cf. Lemma 4.7 and extensions $\tilde{\sigma}_{C_a}$, $\tilde{\sigma}_{C_b}$ based on a funnel contained in $C$, Lemma 4.6(i) gives $\varepsilon(\sigma_{C_a}, \sigma_{C_b}) = W^{-1}\tilde{\sigma}_{C_a}(W) = \tilde{\sigma}_{C_b}(W)\tilde{\sigma}_{C_a}(W)^{-1}$, where $W \in (\sigma_{C_b}, \sigma_{C_a})$. Now given a hypercone $C_A \subset C_a$ and a morphism $\sigma_{C_A} \simeq \sigma_{C_a}$ there is a unitary intertwiner $W_A \in (\sigma_{C_a}, \sigma_{C_A}) \subset \mathcal{R}(C_a)$. Hence $W_A W \in (\sigma_{C_b}, \sigma_{C_A})$ and computing gives

$$\varepsilon(\sigma_{C_A}, \sigma_{C_b}) = \tilde{\sigma}_{C_b}(W_A W)W^{-1}W_A^{-1} = W_A \tilde{\sigma}_{C_b}(W)W^{-1}W_A^{-1} = \varepsilon(\sigma_{C_a}, \sigma_{C_b}),$$

where we used the localization properties of $\sigma_{C_b}$ and the fact that $\varepsilon(\sigma_{C_a}, \sigma_{C_b})$ is a multiple of the identity. Hence $\varepsilon(\sigma_{C_a}, \sigma_{C_b})$ is independent of the choice of both $\sigma_{C_a}$ and $\sigma_{C_b}$ within their respective localization cones $C_a$ and $C_b$. The spacelike complement of a hypercone is hypercone path connected, cf. Lemma 4.7, so it follows from Lemma 4.6(i) and a standard interpolation argument that $\varepsilon = \varepsilon(\sigma_{C_a}, \sigma_{C_b})$ is independent of the choice, both of the spacelike separated hypercones $C_a$, $C_b$ and of the morphisms within the given simple charge class $\mathcal{C}$. Thus, $\varepsilon(\sigma_{C_a}, \sigma_{C_b}) = \varepsilon(\sigma_{C_b}, \sigma_{C_a})$ and, consequently,

$$\varepsilon^2(\sigma_{C_a}, \sigma_{C_b}) = \varepsilon(\sigma_{C_a}, \sigma_{C_b})^2 = \varepsilon(\sigma_{C_a}, \sigma_{C_b}) \varepsilon(\sigma_{C_b}, \sigma_{C_a}) = W^{-1}\tilde{\sigma}_{C_a}(W)\tilde{\sigma}_{C_b}(W^{-1})W = 1,$$

proving the first part of the proposition.

To prove the second, we pick a hypercone $C$ and spacelike separated hypercones $C_a, C_b \subset C$. There is then a hypercone $\tilde{C} \subset C^\varepsilon$ and a corresponding morphism $\sigma_{\tilde{C}}$ in the given charge class. Let $W_a \in (\sigma_{\tilde{C}}, \sigma_{C_a})$ and $W_b \in (\sigma_{\tilde{C}}, \sigma_{C_b})$ be unitary intertwiners. Then $\sigma_{C_a} \upharpoonright \mathcal{R}(C) = \text{Ad} W_a \upharpoonright \mathcal{R}(C)$, $\sigma_{C_b} \upharpoonright \mathcal{R}(C) = \text{Ad} W_b \upharpoonright \mathcal{R}(C)$ and $W = W_a W_b^{-1} \in (\sigma_{C_a}, \sigma_{C_b}) \subset \mathcal{R}(C)$. So

$$\varepsilon = \varepsilon(\sigma_{C_a}, \sigma_{C_b}) = W^{-1}\tilde{\sigma}_{C_a}(W) = W_b W_a^{-1}W_a(W_a W_b^{-1})W_a^{-1} = W_b W_a W_b^{-1}W_a^{-1}. $$

As far as the conjugate goes, the argument used in the first part of the proof of Proposition 4.5 gives $\sigma_{\tilde{C}} \upharpoonright \mathcal{R}(C) = \text{Ad} W_a^{-1} \upharpoonright \mathcal{R}(C)$ and $\sigma_{\tilde{C}} \upharpoonright \mathcal{R}(C) = \text{Ad} W_b^{-1} \upharpoonright \mathcal{R}(C)$. Moreover, the last part of that same proof implies $\tilde{\sigma}_{C_a}(W^{-1}) \in (\tilde{\sigma}_{C_b}, \tilde{\sigma}_{C_a})$ for $W \in (\sigma_{C_a}, \sigma_{C_b}) \subset \mathcal{R}(C)$. Hence

$$\tilde{\sigma}_{C_a}(W_b W_a^{-1}) = W_a^{-1}(W_b W_a^{-1})W_a = W_a^{-1}W_a \in (\tilde{\sigma}_{C_b}, \tilde{\sigma}_{C_a}).$$

A similar computation for the corresponding conjugate charge class $\overline{\mathcal{C}}$ yields,

$$\varepsilon = \varepsilon(\tilde{\sigma}_{C_a}, \tilde{\sigma}_{C_b}) = W_b^{-1}W_a \tilde{\sigma}_{C_a}(W_b W_a^{-1}) = W_b^{-1}W_a^{-1}W_a W_b.$$

But, by the above equality, $W_b W_a = \varepsilon W_b W_b$, hence $\varepsilon = \varepsilon$, completing the proof. \hfill $\Box$
As expressed in the criterion, the simple charge classes \( C \) of a theory are in one-to-one correspondence with equivalence classes of morphisms in \( \Sigma(V) \), modulo the equivalence relation \( \simeq \) introduced above. The preceding analysis shows that the structure of simple charge classes is analogous to that of simple sectors in superselection theory [11]. We summarize these results.

**Theorem 4.9.** Let \( \Sigma(V) \) be the family of all hypercone localized morphisms satisfying the criterion.

(i) For any given pair of morphisms \( \sigma_{C_a}, \tau_{C_b} \in \Sigma(V) \) there exists a composed morphism \( \sigma_{C_a} \cdot \tau_{C_b} : \mathfrak{A}(V) \to \mathfrak{R}(V) \). By composing on the left with \( \omega_0 \) it determines a charge class of states, the simple composite class, depending only on the charge classes of the given morphisms.

(ii) The composition of charge classes is commutative. Given any two classes, morphisms \( \sigma_{C_a} \) and \( \tau_{C_b} \) can be picked, one from each of the classes, such that \( \sigma_{C_a} \cdot \tau_{C_b} = \tau_{C_b} \cdot \sigma_{C_a} \) when \( C_a \) and \( C_b \) are spacelike separated.

(iii) A simple charge class has a simple conjugate charge class: for any morphism \( \sigma_C \in \Sigma(V) \) in the given class there is a \( \overline{\sigma}_C \in \Sigma(V) \) in the conjugate class with \( \overline{\sigma}_C \cdot \sigma_C = \sigma_C \cdot \overline{\sigma}_C = \iota \).

(iv) To any simple charge class there corresponds a statistics parameter \( \varepsilon \in \{\pm 1\} \), characteristic of Bose and Fermi statistics, respectively. The conjugate charge class has the same statistics.

**Remark.** The first three parts of this theorem imply that \( \Sigma(V)/\simeq \) is an Abelian group whose product is implemented by the composition of morphisms. This group is to be interpreted as the dual of the global gauge group deduced from the intrinsic structure of the charge classes, cf. the analogous result for superselection sectors in [11].

We conclude by pointing out that the preceding results on the structure of simple charge classes are independent of our ad hoc choice of hypercones. We have selected a family \( \mathcal{F} \) of hypercones, based on a given hyperboloid \( \mathcal{H} \). Selecting another hyperboloid \( \mathcal{H}' \), there is another family \( \mathcal{F}' \) of hypercones based on it. Now, as shown in the appendix, cf. [A.9] and [A.10] given a hypercone \( C \in \mathcal{F} \), there are hypercones \( \hat{C}, \check{C} \in \mathcal{F}' \) with \( \hat{C} \subset C \subset \check{C} \) and vice versa. Since the preceding arguments involve only the partial ordering and the causal relations between hypercones our structural results on simple charge classes do not change if the family \( \mathcal{F} \) is replaced by any other family \( \mathcal{F}' \).

The results of this section, show that it suffices in the following to denote the morphisms by \( \sigma \) rather than \( \sigma_C \), i.e. without singling out a choice \( \mathcal{C} \) of localization hypercone. If localization matters we write \( \sigma \in \Sigma(\mathcal{C}) \) to indicate that \( \sigma \) is localized in \( \mathcal{C} \).
5 Covariant morphisms

Since the semigroup $S^+_{\uparrow}$ of spacetime transformations only acts as endomorphisms on the observables in $\mathfrak{A}(V)$, the usual way of describing the transport of states and morphisms makes no sense here. Hence it is not obvious how covariant morphisms and their charge classes are to be defined.

**Definition 5.1.** A morphism $\sigma \in \Sigma(V)$ is covariant if first, for some neighbourhood of the identity $N_S \subset S^+_{\uparrow}$, there are morphisms $\lambda \sigma : \mathfrak{A}(V) \rightarrow \mathfrak{R}(V)$, $\lambda \in N_S$, looking like the original morphism on the transformed algebra, i.e.

$$\lambda \sigma \circ \alpha_\lambda = \alpha_\lambda \circ \sigma, \quad \lambda \in N_S.$$  \hfill (5.1)

Secondly, there are unitary intertwiners $\Gamma_\lambda \in (\lambda \sigma, \sigma)$, $\lambda \in N_S$, such that

$$\alpha_\lambda (\Gamma_\mu) \in (\lambda \mu \sigma, \sigma), \quad \lambda, \mu, \lambda \mu \in N_S.$$ \hfill (5.2)

This condition expresses the idea that the morphisms $\lambda \sigma$ all carry the same charge and are transported covariantly by physical operations. Finally, there is a strong operator continuous section

$$\lambda \mapsto \Gamma_\lambda \in (\lambda \sigma, \sigma)$$ \hfill (5.3)

of unitary intertwiners over $N_S$. (At the expense of additional technical complications, continuity can be relaxed to measurability.)

**Remark.** Relations (5.1) and (5.2) imply $\lambda \mu \sigma \circ \alpha_\lambda = \alpha_\lambda \circ \mu \sigma$ for $\lambda, \mu, \lambda \mu \in N_S$.

A covariant morphism $\sigma$ determines a continuous unitary local projective (ray) representation of $S^+_{\uparrow}$ on $N_S$ by putting

$$U_\sigma(\lambda) = \Gamma_\lambda U_0(\lambda), \quad \lambda \in N_S,$$ \hfill (5.4)

where $U_0$ is the continuous unitary representation of $P^+_{\uparrow}$ in the vacuum representation. Relation (5.2) implies that $\Gamma_\lambda \alpha_\lambda (\Gamma_\mu) \in (\lambda \mu \sigma, \sigma)$ and $\Gamma_\lambda \mu \in (\lambda \mu \sigma, \sigma)$ differ at most by a phase. Hence for $\lambda, \mu, \lambda \mu \in N_S$ there is a $\zeta(\lambda, \mu) \in \mathbb{T}$ such that

$$U_\sigma(\lambda) U_\sigma(\mu) = \Gamma_\lambda \alpha_\lambda (\Gamma_\mu) U_0(\lambda \mu) = \zeta(\lambda, \mu) \Gamma_\lambda \mu U_0(\lambda \mu) = \zeta(\lambda, \mu) U_\sigma(\lambda \mu).$$

Moreover, relation (5.1) gives

$$\text{Ad} U_\sigma(\lambda) \circ \sigma = \text{Ad} \Gamma_\lambda \circ \alpha_\lambda \circ \sigma = \text{Ad} \Gamma_\lambda \circ \lambda \sigma \circ \alpha_\lambda = \sigma \circ \alpha_\lambda, \quad \lambda \in N_S.$$  

Thus $U_\sigma$ is a local projective representation of $S^+_{\uparrow}$ inducing the corresponding local action on the observables in the representation $\sigma$. Its continuity follows from that of $\lambda \mapsto \Gamma_\lambda$. Applying a well known result of Bargmann [1] we can establish the following.
Proposition 5.2. Let \( \sigma \in \Sigma(V) \) be a covariant morphism. There is a continuous unitary representation \( \tilde{U}_\sigma \) of the covering group \( \tilde{P}_+ \cong \mathbb{R}^4 \rtimes SL(2, \mathbb{C}) \) of \( P_+ \) such that \( \text{Ad} \tilde{U}_\sigma(\tilde{\lambda}) \circ \sigma = \sigma \circ \alpha_\lambda \) and \( \tilde{U}_\sigma(\tilde{\lambda})U_0(\lambda)^{-1} \in (\lambda, \sigma) \) for \( \tilde{\lambda} \in \tilde{S}_+ \cong \tilde{V}_+ \rtimes SL(2, \mathbb{C}) \). Here \( \tilde{\lambda} \mapsto \lambda \) is the canonical covering map from the covering group to the Poincaré group.

Proof. The crucial step is to show that the local projective representation \( U_\sigma \) on \( N_\mathcal{S} \subset S_+^4 \), defined above, can be extended to a local projective representation on some neighbourhood of the identity \( N_\mathcal{P} \subset P_+^4 \). Without loss of generality we assume that \( N_\mathcal{S} = N_\mathcal{P}^T \times N_\mathcal{L} \) where \( N_\mathcal{P}^T = \{ x \in \mathbb{R}^4 : 2|x| \leq x_0 + |x| < 2\varepsilon \} \subset \mathbb{V}_+ \) is a double cone for any given \( \varepsilon > 0 \) and \( N_\mathcal{L} \subset L_+^1 \) is some neighbourhood of 0. Then \( N_\mathcal{P}^T = \{ x \in \mathbb{R}^4 : |x| < \varepsilon, |x_0| + |x| < 2\varepsilon \} \supset N_\mathcal{P}^T \) is a neighbourhood of 0 \( \in \mathbb{R}^4 \) and we can put \( N_\mathcal{P} = N_\mathcal{P}^T \times N_\mathcal{L} \).

The desired extension of \( U_\sigma \) requires several steps. First, we note that definition (5.4) implies \( U_\sigma(0, 1) \in (\sigma, \sigma) \), hence by adjusting phases we may assume \( U_\sigma(0, 1) = 1 \). In a second step we extend \( U_\sigma \) to the translations \( x \in N_\mathcal{P}^T \). Given any such \( x \) we write \( x = (x - \kappa(x)e) + \kappa(x)e \), where \( \kappa(x) = (x_0 - |x|) \) and \( e = (1, 0) \) denotes the time direction in the chosen coordinate system. Note that both \( (x - \kappa(x)e), \kappa(x)e \in N_\mathcal{P}^T \) so definition

\[
\tilde{U}_\sigma(x, 1) \doteq U_\sigma(x - \kappa(x)e, 1) \cdot \begin{cases} U_\sigma(\kappa(x)e, 1) & \text{if } \kappa(x) \geq 0 \\ U_\sigma(|\kappa(x)|e, 1)^{-1} & \text{if } \kappa(x) \leq 0 \end{cases}
\]

is consistent. As \( U_\sigma \upharpoonright N_\mathcal{P}^T \) is a local projective representation, the group theoretic commutators of the corresponding unitaries are multiples of the identity, and it is easy to verify that \( \tilde{U}_\sigma \) yields a local projective representation of \( N_\mathcal{P} \). Moreover \( \tilde{U}_\sigma \upharpoonright N_\mathcal{P}^T \) coincides with \( U_\sigma \upharpoonright N_\mathcal{P}^T \) up to a phase. Lastly, for \( \lambda = (x, \Lambda) \in N_\mathcal{P} \), we put \( \tilde{U}_\sigma(\lambda) \doteq \tilde{U}_\sigma, (x, 1)U_\sigma(0, \Lambda) \). Since \( U_\sigma \) is a local projective representation of \( N_\mathcal{S} \) one has \( U_\sigma(0, \Lambda)U_\sigma(y, 1) = \zeta U_\sigma(\Lambda y, 1)U_\sigma(0, \Lambda) \) for \( y, \Lambda y \in N_\mathcal{P}^T, \Lambda \in N_\mathcal{L} \) and some phase factor \( \zeta \), hence \( U_\sigma(0, \Lambda)U_\sigma(y, 1)^{-1} = \tilde{\zeta} U_\sigma(\Lambda y, 1)^{-1}U_\sigma(0, \Lambda) \). Using these equalities, another easy computation shows that \( \tilde{U}_\sigma \) is a local projective representation of \( N_\mathcal{P} \), i.e. \( \tilde{U}_\sigma(\lambda)\tilde{U}_\sigma(\mu) = \xi(\lambda, \mu) \tilde{U}_\sigma(\lambda \mu) \) for \( \lambda, \mu, \lambda \mu \in N_\mathcal{P} \) and phase factors \( \xi(\lambda, \mu) \in \mathbb{T} \). It is continuous on \( N_\mathcal{P} \) because of the continuity inherited from \( U_\sigma \) and, by construction, \( \tilde{U}_\sigma \upharpoonright N_\mathcal{S} \) coincides with \( U_\sigma \) modulo some phase factors.

Now by the results of Bargmann \[1\], exploiting the phase freedom in the definition of \( \tilde{U}_\sigma \) in some neighbourhood of the identity \( N_\mathcal{P} \subset P_+^4 \) leads to a true continuous unitary representation still denoted by \( \tilde{U}_\sigma \). As the covering group is locally isomorphic to \( P_+^4 \), its local representation induces a local continuous unitary representation \( \tilde{U}_\sigma \) of \( \tilde{P}_+^4 \), given by \( \tilde{U}_\sigma(\tilde{\lambda}) \doteq \tilde{U}_\sigma(\lambda), \tilde{\lambda} \in \tilde{N}_\mathcal{P} \). The covering group being simply connected, there is a unique extension of \( \tilde{U}_\sigma \) to a strongly continuous unitary representation of \( \tilde{P}_+^4 \) got by representing its elements as finite products of elements close to the identity (monodromy theorem). This establishes the existence
of $\tilde{U}_\sigma$. Furthermore, for any $A \in \mathfrak{A}(V)$ one has

$$\text{Ad} \tilde{U}_\sigma(\tilde{\lambda}) \circ \sigma(A) = \text{Ad} \tilde{U}_\sigma(\lambda) \circ \sigma(A) = \sigma \circ \alpha_\lambda(A), \quad \tilde{\lambda} \in \tilde{N}_S.$$ 

Thus iterating, the first and last members of this equality are equal for all $\tilde{\lambda} \in \tilde{S}_+^1$. Finally, $\tilde{U}_\sigma(\tilde{\lambda}) U_0(\lambda)^{-1} \in (\lambda \sigma, \sigma) \subset \mathfrak{R}(V)$ for $\tilde{\lambda} \in \tilde{N}_S$. Hence as for $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \tilde{N}_S$

$$\tilde{U}_\sigma(\tilde{\lambda}_n \cdots \tilde{\lambda}_1) U_0(\lambda_n \cdots \lambda_1)^{-1}$$

$$= (\tilde{U}_\sigma(\tilde{\lambda}_n) U_0(\lambda_n)^{-1}) \ U_0(\lambda_n)(\tilde{U}_\sigma(\tilde{\lambda}_{n-1} \cdots \tilde{\lambda}_1) U_0(\lambda_{n-1} \cdots \lambda_1)^{-1}) U_0(\lambda_n)^{-1}$$

and $U_0(\lambda_n) \mathfrak{R}(V) U_0(\lambda_n)^{-1} \subset \mathfrak{R}(V)$ it follows by induction that $\tilde{U}_\sigma(\tilde{\lambda}) U_0(\lambda)^{-1} \in \mathfrak{R}(V)$ for any $\tilde{\lambda} \in \tilde{S}_+^1$, completing the proof. $\Box$

Our previous, operationally inspired characterization of covariant morphisms $\sigma \in \Sigma(V)$ involved an associated covariant family of morphisms $(\lambda \sigma, \sigma)$, $\lambda \in \tilde{N}_S$. This raises the question of whether the resulting unitary representation of the Poincaré group depends on the choice of such a family. We will answer this question in the subsequent lemma, where we show that this representation is uniquely fixed by $\sigma \in \Sigma(V)$.

**Lemma 5.3.** Let $\sigma \in \Sigma(V)$ be a covariant morphism, then the associated unitary representation $\tilde{U}_\sigma$ of $\tilde{P}_+^1$ given in the preceding proposition is unique.

**Proof.** Let $\tilde{U}_j$, $j = 1, 2$, be unitary representations of $\tilde{P}_+^1$ as in the preceding proposition. Then $\text{Ad} \tilde{U}_1(\tilde{\lambda}) \circ \sigma = \sigma \circ \alpha_\lambda = \text{Ad} \tilde{U}_2(\tilde{\lambda}) \circ \sigma$ and hence $\text{Ad} \tilde{U}_2(\tilde{\lambda})^{-1} \tilde{U}_1(\tilde{\lambda}) \circ \sigma = \sigma$ for $\tilde{\lambda} \in \tilde{S}_+^1$. Recalling that $\mathfrak{R}(V)^- = \mathfrak{R}(V)$ this implies

$$\tilde{U}_2(\tilde{\lambda})^{-1} \tilde{U}_1(\tilde{\lambda}) \in \mathfrak{R}(V)', \quad \tilde{\lambda} \in \tilde{S}_+^1.$$ 

Moreover, for such $\tilde{\lambda}$, $\tilde{U}_j(\tilde{\lambda}) U_0(\lambda)^{-1} \in \mathfrak{R}(V)$, $j = 1, 2$, and consequently

$$\tilde{U}_2(\tilde{\lambda})^{-1} \tilde{U}_1(\tilde{\lambda}) \in U_0(\lambda)^{-1} \mathfrak{R}(V) U_0(\lambda), \quad \tilde{\lambda} \in \tilde{S}_+^1.$$ 

Restricting $\tilde{\lambda}$ in the preceding two relations to the subgroup $\text{SL}(2, \mathbb{C})$ and bearing in mind that in this case $U_0(\lambda)^{-1} \mathfrak{R}(V) U_0(\lambda) = \mathfrak{R}(V)$ and that $\mathfrak{R}(V)$ is a factor it follows that $\tilde{U}_2(\tilde{\lambda})^{-1} \tilde{U}_1(\tilde{\lambda}) \in \mathbb{T}1$ for $\tilde{\lambda} \in \text{SL}(2, \mathbb{C})$. Since there are no non–trivial one–dimensional representations of the Lorentz group, the restrictions of $\tilde{U}_1$, $\tilde{U}_2$ to $\text{SL}(2, \mathbb{C})$ coincide.

Turning to the translations, let $x \in \mathbf{V}_+$ and let $\Delta(x) = \tilde{U}_2(x, 1)^{-1} \tilde{U}_1(x, 1)$. Then, by the preceding step, $\Delta(x) \subset U_0(x, 1)^{-1} \mathfrak{R}(V) U_0(x, 1) \cap \mathfrak{R}(V)'$. Hence, again using $\tilde{U}_1(\tilde{\lambda}) U_0(\lambda)^{-1} \in \mathfrak{R}(V)$ for $\tilde{\lambda} \in \tilde{S}_+^1$, gives

$$\tilde{U}_1(\tilde{\lambda})^{-1} \Delta(x) \tilde{U}_1(\tilde{\lambda}) = U_0(\lambda)^{-1} \Delta(x) U_0(\lambda), \quad \tilde{\lambda} \in \tilde{S}_+^1.$$ 

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On the other hand, for $\lambda \in \text{SL}(2, \mathbb{C})$

$$\tilde{U}_1(\lambda)^{-1} \Delta(x) \tilde{U}_1(\lambda) = \tilde{U}_2(\lambda)^{-1} \tilde{U}_2(x)^{-1} \tilde{U}_2(\lambda) \tilde{U}_1(\lambda)^{-1} \tilde{U}_1(x) U_1(\lambda)$$

$$= \tilde{U}_2(\lambda^{-1} x)^{-1} \tilde{U}_1(\lambda^{-1} x) = \Delta(\lambda^{-1} x),$$

where the first equality follows since $\tilde{U}_1, \tilde{U}_2$ coincide on $\text{SL}(2, \mathbb{C})$ and the second since $\tilde{U}_1, \tilde{U}_2$ are unitary representations of $\tilde{P}_+^l$. Combining the preceding two relations yields

$$U_0(\Lambda) \Delta(x) U_0(\Lambda)^{-1} = \Delta(\Lambda x), \quad x \in \overline{\mathcal{V}}_+, \; \Lambda \in \mathcal{L}_+^l.$$  

Now, given any lightlike translation $l \in \overline{\mathcal{V}}_+$, there is a corresponding family of boosts $\{\Lambda_s \in \mathcal{L}_+^l\}_{s \in \mathbb{R}}$ scaling $l$, i.e. $\Lambda_s l = e^{-s} l$, $s \in \mathbb{R}$. The preceding equality and the continuity of $\tilde{U}_1, \tilde{U}_2$ involved in the definition of $\Delta$ show that

$$\lim_{s \to \infty} U_0(\Lambda_s) \Delta(l) U_0(\Lambda_s)^{-1} = \lim_{s \to \infty} \Delta(e^{-s} l) = 1,$$

in the strong operator topology. Taking matrix elements of this equation in the vector state given by $\Omega$, bearing in mind that $\Omega$ is invariant under the action of $U_0(\lambda)$, leads to $(\Omega, \Delta(l) \Omega) = 1$. Hence $\Delta(l) \Omega = \Omega$ since $\Delta(l)$ is unitary. But $\Omega$ is separating for $\mathcal{R}(V)'$, so $\Delta(l) = 1$ and consequently $\tilde{U}_1(l) = \tilde{U}_2(l)$ for lightlike translations $l \in \overline{\mathcal{V}}_+$. As the linear span of lightlike translations generates the subgroup of all translations and $\tilde{U}_1, \tilde{U}_2$ are representations of $\tilde{P}_+^l$, they coincide on $\mathbb{R}^4$, and hence on the whole group. \hfill $\Box$

Let $\sigma \in \Sigma(V)$ be a covariant morphism and let $\tilde{U}_\sigma$ be the associated representation of $\tilde{P}_+^l$. Then any other equivalent morphism $\sigma' \simeq \sigma$ is also covariant and the corresponding representation is given by $\tilde{U}_{\sigma'}(\lambda) = W \tilde{U}_{\sigma}(\lambda) W^{-1}$, $\lambda \in \tilde{P}_+^l$, where $W \in (\sigma, \sigma')$. Up till now the specific localization of the covariant morphisms did not matter, but to proceed further we need to have a closer look at them.

**Lemma 5.4.** Let $\mathcal{C}$ be any given hypercone and let $\sigma \in \Sigma(\mathcal{C})$ be a covariant morphism with associated representation $\tilde{U}_\sigma$ of $\tilde{P}_+^l$. There is a hypercone $\mathcal{C}_0 \supset \mathcal{C}$ (depending only on $\mathcal{C}$) and a neighbourhood of the identity $\tilde{\mathcal{N}}_S \subset \tilde{S}_+^l$ with $\tilde{U}_\sigma(\lambda) U_0(\lambda)^{-1} \in \mathcal{R}(\mathcal{C}_0)$ for $\lambda \in \tilde{\mathcal{N}}_S$.

**Proof.** We put $\tilde{\Gamma}_\lambda = \tilde{U}_\sigma(\lambda) U_0(\lambda)^{-1}$, $\tilde{\lambda} \in \tilde{P}_+^l$. These unitaries satisfy the cocycle equation $\tilde{\Gamma}_\lambda \tilde{\alpha}_\lambda(\tilde{\Gamma}_{\lambda'}) = \tilde{\Gamma}_{\lambda \lambda'}$. Moreover,

$$\text{Ad} \tilde{\Gamma}_\lambda \circ \tilde{\alpha}_\lambda \circ \sigma = \sigma \circ \tilde{\alpha}_\lambda, \quad \lambda \in \tilde{S}_+^l,$$  

(5.5)

where we have set $\Gamma_\lambda \equiv \pm \tilde{\Gamma}_\lambda$ since phases drop out in the adjoint action. In the subsequent argument we anticipate the existence of hyperbolic cones with certain specific geometric properties. This will be justified at the end of the proof. Evaluating the preceding equality, the localization of $\sigma$, for Lorentz transformations, $\Lambda \in \mathcal{L}_+^l$,  

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gives $\text{Ad} \Gamma_{\Lambda} \uparrow \mathfrak{A}(C_1^+) = \iota$ for any hypercone $C_1 \supset C \cup \Lambda C$. Thus $\pm \Gamma_{\Lambda} \in \mathfrak{R}(C_1)$ by
hypercone duality and making $C_1$ sufficiently big, this inclusion holds for all $\Lambda$ in some neighbourhood of the identity $N^L \subset L^+_1$.

As the family of hypercones based on a given hyperboloid is not stable under translations, analyzing the localization of the corresponding cocycles requires more work. Since $C_1 \supset C$, $\sigma(\mathfrak{A}(C_1))^{-1} = \mathfrak{R}(C_1)$ equation (5.5) allows us to conclude that $\text{Ad} \Gamma_{\Lambda} \circ \alpha_\Lambda(\mathfrak{R}(C_1)) \subset \mathfrak{R}(C_2)$ for $\Lambda \in S^+_1$ and any hypercone $C_2 \supset C \cup \Lambda C_1$. We now use
the cocycle equation. Since $(0, \Lambda)(x, 1) = (\Lambda x, 1)(0, \Lambda)$, $\Lambda \alpha_\Lambda(\Gamma_x) = \pm \Gamma_{\Lambda x} \alpha_{\Lambda x}(\Gamma_{\Lambda})$ for $x \in \nabla_+$, $\Lambda \in L^+_1$ and consequently

$$
\alpha_\Lambda(\Gamma_x) \Gamma_{\Lambda x}^{-1} = \pm \Gamma_{\Lambda x}^{-1} \Gamma_{\Lambda x} \alpha_{\Lambda x}(\Gamma_{\Lambda}) \Gamma_{\Lambda x}^{-1} \in \mathfrak{R}(C_2),
$$

provided $C_1 \supset C \cup \Lambda C$ and $C_2 \supset C \cup (C_1 + \Lambda x)$. We exploit this information choosing
sequences of boosts and translations $\Lambda_n \in L^+_1$, $l_n \in \nabla_+$ where $\Lambda_n l_n = l$ is a fixed (lightlike) vector, $C_1 \supset \Lambda_n C$ and $l_n$ tends to 0. Thus $\alpha_{\Lambda_n}(\Gamma_{l_n}) \Gamma_{l_n}^{-1} \in \mathfrak{R}(C_2)$, $n \in \mathbb{N}$,
where $C_2 \supset C \cup (C_1 + l)$. Now $\alpha_{\Lambda_n}(\Gamma_{l_n}) \Omega \rightarrow \Omega$ since $\Gamma_{l_n} \rightarrow 1$ in the strong operator
topology and $U_0(\Lambda_n) \Omega = \Omega$. Moreover, if $l \in \nabla_+$ is sufficiently close to 0 there is a double cone $O \subset C_1^+ \cap (C_1^c + l)$, and, after a moment’s reflection, relation (5.5) shows
that all operators $\alpha_{\Lambda_n}(\Gamma_{l_n})$, $n \in \mathbb{N}$, commute with the elements of $\mathfrak{A}(O)$. Hence, by
the Reeh–Schlieder property of the vacuum, $\alpha_{\Lambda_n}(\Gamma_{l_n}) \rightarrow 1$ in the strong operator
topology and consequently $\Gamma_1 \in \mathfrak{R}(C_2)$. Varying the direction of the chosen sequence
of boosts slightly, the convex hull $K^T \subset \nabla_+$ of the resulting lightlike vectors $l$ is the

Let $C$ be a simple charge class. The class is said to be covariant
if there is a covariant morphism $\sigma \in \Sigma(V)$ with $\omega_0 \circ \sigma \in \mathcal{C}$. The family of covariant
morphisms and their charge classes under composition and conjugation.

**Definition 5.5.** Let $\mathcal{C}$ be a simple charge class. The class is said to be covariant
if there is a covariant morphism $\sigma \in \Sigma(V)$ with $\omega_0 \circ \sigma \in \mathcal{C}$. The family of covariant
morphisms is denoted by $\Sigma_c(V) \subset \Sigma(V)$ and the subset of covariant morphisms localized in a given hypercone $C$ is denoted by $\Sigma_c(C)$.

**Theorem 5.6.** The family $\Sigma_c(V)$ of covariant morphisms is stable under composition and conjugation. More explicitly, for any pair $\sigma_1, \sigma_2 \in \Sigma_c(C)$ one has $\sigma_1 \circ \sigma_2 \in \Sigma_c(C)$ and for any $\sigma \in \Sigma_c(C)$ there is a $\tilde{\sigma} \in \Sigma_c(C)$ such that $\tilde{\sigma} \circ \sigma = \sigma \circ \tilde{\sigma} = \iota$.

**Proof.** Since all morphisms in the equivalence class of a covariant morphism are covariant there is no loss of generality in picking any hypercone $C$ and morphisms $\sigma_1, \sigma_2 \in \Sigma_c(C)$. Let $\tilde{U}_j$ be the associated unitary representations of $\tilde{P}_+^1$ and let $\tilde{\Gamma}_j(\tilde{\lambda}) = \tilde{U}_j(\tilde{\lambda})U_0(\lambda)^{-1}$, $\tilde{\lambda} \in \tilde{P}_+^1$ be the corresponding cocycles, $j = 1, 2$. By the preceding lemma there is a hypercone $C_0 \supset C$ and a neighbourhood $\mathcal{N}_S \subset S_+^1$ of the identity (the image of $\tilde{\mathcal{N}}_S$ under the covering map) such that $\Gamma_j(\lambda) \in \mathcal{R}(C_0)$ for $\lambda \in \mathcal{N}_S$, $j = 1, 2$. We choose an extension $\tilde{\sigma}_1$ of $\sigma_1$ normal on $\mathcal{R}(C_0)$ and having the range of $\sigma_2$ in its domain, c.f. Lemma 4.3(i). Then

$$\sigma_1 \circ \sigma_2 \circ \alpha_\lambda = \tilde{\sigma}_1 \circ \tilde{\alpha}_\lambda \circ \tilde{\sigma}_2 = \tilde{\alpha}_\lambda(\tilde{\sigma}_2(\tilde{\sigma}_1(\tilde{\sigma}(\tilde{\alpha}_\lambda)))) \Gamma_{1\lambda} \circ \alpha_\lambda \circ \sigma_1 \circ \sigma_2, \quad \lambda \in \mathcal{N}_S.$$ Putting $\Gamma_{12\lambda} = \Gamma(\Gamma_{1\lambda} \circ \sigma_1 \circ \Gamma_{2\lambda} \circ \sigma_2)$, $\Gamma_{1\lambda} = \Gamma(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}(\tilde{\alpha}_\lambda)))))) \Gamma_{1\lambda} \circ \alpha_\lambda \circ \sigma_1 \circ \sigma_2$, and $\lambda \in \mathcal{N}_S$. Moreover, if $\lambda, \mu, \lambda \mu \in \mathcal{N}_S$, $\Gamma_{1\lambda} \Gamma_{12\lambda} = \Gamma(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}(\tilde{\alpha}_\lambda)))))) \Gamma_{1\lambda} \circ \alpha_\lambda \circ \sigma_1 \circ \sigma_2$, and, on the algebra $\mathcal{A}(V)$,

$$\Gamma_{12\lambda} \Gamma_{12\lambda} = \Gamma(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}(\tilde{\alpha}_\lambda)))))) \Gamma_{1\lambda} \circ \alpha_\lambda \circ \sigma_1 \circ \sigma_2,$$

where we used the cocycle equations for $\Gamma_1, \Gamma_2$ (second equality) and the covariance of $\sigma_1$ (third equality). Thus $\alpha_\lambda(\Gamma_{12\lambda}) \in \Gamma(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}_1(\tilde{\sigma}_2(\tilde{\sigma}(\tilde{\alpha}_\lambda)))))) \Gamma_{1\lambda} \circ \alpha_\lambda \circ \sigma_1 \circ \sigma_2$ is covariant, i.e. $\sigma_1 \circ \sigma_2 \in \Sigma_c(C)$.

Turning to conjugation, let $\sigma \in \Sigma_c(C)$, $\tilde{U}(\lambda)$ its associated unitary representation and cocycle $\Gamma_\lambda$, $\lambda \in \mathcal{N}_S$, and let $\tilde{\sigma} \in \Sigma(C)$ be the conjugate morphism which exists by Proposition 5.3. We choose extensions $\tilde{\sigma}, \tilde{\alpha}$ of $\sigma, \alpha$ with a common stable domain and normal on $\mathcal{R}(C_0)$, c.f. Lemma 4.3(i). Note that by continuity $\tilde{\sigma} \circ \tilde{\sigma} = \tilde{\sigma} \circ \tilde{\sigma} = \iota$ on this domain and $\mathcal{R}(\mathcal{R}(C_0)) \subset \mathcal{R}(C_0)$ since $C_0 \supset C$. Putting $\Gamma_{1\lambda} = \tilde{\sigma}(\tilde{\alpha}_\lambda)$ and $\lambda \sigma = \tilde{\alpha}_\lambda \circ \sigma_\lambda \circ \tilde{\sigma}_\lambda \circ \tilde{\sigma} = \tilde{\sigma} \circ \alpha_\lambda \circ \tilde{\sigma}$. Composing this equality on the left with $\tilde{\sigma}$ gives $\tilde{\sigma} \circ \alpha_\lambda = \tilde{\sigma} \circ \tilde{\alpha}_\lambda \circ \tilde{\sigma}$ and, by construction, $\Gamma_\lambda \in \Gamma(\tilde{\alpha}_\lambda, \tilde{\sigma})$, $\lambda \in \mathcal{N}_S$. Moreover, if $\lambda, \mu, \lambda \mu \in \mathcal{N}_S$,

$$\tilde{\sigma}(\tilde{\Gamma}_{\lambda}^{-1} \Gamma_{\lambda}) = \Gamma_{\lambda} \alpha_\lambda(\Gamma_{\lambda}^{-1}) \Gamma_{\lambda}^{-1} = \tilde{\alpha}_\lambda \tilde{\sigma} \circ \tilde{\sigma} \circ \tilde{\sigma} \circ \tilde{\sigma} = \tilde{\sigma} \circ \alpha_\lambda \circ \tilde{\sigma},$$

on $\mathcal{A}(V)$, where the first equality uses the cocycle equation. Composing with $\tilde{\sigma}$ gives $\alpha_\lambda(\Gamma_{\mu}) = \alpha_\lambda(\tilde{\sigma}(\tilde{\sigma}(\Gamma_{\mu}^{-1}))) = \Gamma_{\lambda}^{-1} \Gamma_{\mu} \in \Gamma(\tilde{\alpha}_\lambda \tilde{\sigma}, \tilde{\sigma})$. The normality of $\tilde{\sigma}$ implies that $\lambda \mapsto \Gamma_\lambda$ is continuous on $\mathcal{N}_S$. Thus $\tilde{\sigma}$ is covariant, i.e. $\tilde{\sigma} \in \Sigma_c(C)$, completing the proof. \qed
The preceding results show that, even in the presence of charges, the energy and the angular momentum of the partial states in the light cone $V$ can be defined in a mathematically precise and physically meaningful way. Given a simple charge class, this information is encoded in the spectral properties of the generators of the unitary representations $\tilde{U}_\sigma$ of $\tilde{P}_\uparrow$, where $\sigma$ is any one of the equivalent morphisms associated with the class. Yet, even though these generators are uniquely fixed and, as we have seen, can be reconstructed from data in $V$, they should not be interpreted as genuine quantum observables since they are not affiliated with the algebra $\sigma(\mathfrak{A}(V))^-$. They contain not only pertinent information about the states in $V$ but also, in a consistent, though hypothetical way, some information on outgoing massless particles (radiation) created in the past that evades direct observations in $V$. This hypothetical input enters with the condition of covariance expressing, within the mathematical setting, the postulate that measurements and operations can be repeated at any time using exactly the same procedures. The generators incorporate the implicit assumption that this postulate applies to the distant past, too, involving, as it does the chosen extension of time translations from the semigroup action to the full group. Although this hypothesis seems plausible and is fully consistent with physical predictions, it cannot be verified experimentally. This explains why the generators are not described by observables in the light cone $V$.

6 Spectral properties

We now analyze the spectral properties of covariant charge classes and their associated covariant morphisms. As we shall see, the energy momentum of the corresponding states is bounded below, in accordance with the physical idea that these states describe stable elementary systems. However, the standard “additivity of the energy” argument used in sector analysis can not be applied in the present setting, because the necessary asymptotic commutativity of the transported morphisms is lacking. Yet, using the asymptotic commutation of hypercone localized operators under the action of suitable Lorentz boosts, we can establish a somewhat weaker spectral result, still enabling us to prove that all covariant charge classes satisfy the spectrum condition. We begin by stating the main technical result of this section.

**Lemma 6.1.** Let $\sigma_1, \sigma_2, \sigma_1 \cdot \sigma_2 \in \Sigma(C)$ be covariant morphisms and let $\tilde{U}_{\sigma_1}, \tilde{U}_{\sigma_2}$ and $\tilde{U}_{\sigma_1 \cdot \sigma_2}$ be the corresponding unitary representations of the (covering group of the) Poincaré group. If $\tilde{U}_{\sigma_1} \uparrow \mathbb{R}^4$ or $\tilde{U}_{\sigma_2} \uparrow \mathbb{R}^4$ (or both) violate the relativistic spectrum condition, then $\tilde{U}_{\sigma_1 \cdot \sigma_2} \uparrow \mathbb{R}^4$ also violates the spectrum condition.

Since the spectrum of translations in the vacuum class is contained in $V_+$, applying this lemma to covariant morphisms and their conjugates immediately gives the following basic result.
Theorem 6.2. Let $\sigma \in \Sigma_c(V)$ be a covariant morphism then $\widetilde{U}_\sigma$, the corresponding unitary representation of the (covering group of the) Poincaré group $\mathcal{P}_+^1$, satisfies the relativistic spectrum condition, i.e. $\text{sp}\, \widetilde{U}_\sigma \upharpoonright \mathbb{R}^4 \subset \nabla_+$. 

The proof of the preceding lemma is rather technical. We therefore first outline the idea of the argument and subsequently explain the details.

If a representation $\tilde{U}$ of the (covering group of the) Poincaré group violates the spectrum condition, then, for every positive lightlike vector $l$, the unitary one–parameter group $\theta \mapsto \tilde{U}(\theta l, 1)$ has spectrum on the negative real axis. For, otherwise, as the spectrum is Lorentz invariant, none of these groups would have spectrum on the negative real axis. Hence the spectrum of $\tilde{U} \upharpoonright \mathbb{R}^4$ would be contained in the intersection $\bigcap \{ p : pl \geq 0 \} = \nabla_+$, a contradiction. To explore the spectrum of the one parameter groups, we fix some positive lightlike vector $l$ and a corresponding one parameter group $\Lambda$ of Lorentz boosts such that $\Lambda(\beta)l = \beta^{-1}l$, $\beta \geq 1$. Note that there is always an opposite positive lightlike vector $l'$ scaling under the action of these boosts as $\Lambda(\beta)l' = \beta l'$, $\beta \geq 1$. To simplify notation in what follows, we set $T(\theta) = \tilde{U}(\theta l, 1)$, $B(\beta) = \tilde{U}(0, \Lambda(\beta))$, noting that $B(\beta)T(\theta) = T(\beta^{-1}\theta)B(\beta)$. Whereas these groups are globally defined, care is needed if one wants to determine their action on morphisms, as they just act as endomorphisms. Since the spectral properties of the unitary representations $\tilde{U}$ affiliated with a charge class do not depend on the particular choice of morphism we can choose the localization properties of the morphisms at will and adjust them to the geometric action of the given boosts and translations.

Let $T_0$, $T_1$, $T_2$, $T_{12}$ and $B_0$, $B_1$, $B_2$, $B_{12}$ be the unitary groups corresponding to the given lightlike translations and boosts in the vacuum representation and in the representations induced by the three given covariant families of morphisms. If the spectrum condition is violated in the charge class of $\sigma_2$ we choose its localization cone $C_2$ to point asymptotically in the direction of the lightlike vector $l'$ opposite to the given $l$. To analyze the spectral properties of $T_{12}$ relative to those of $T_2$ we consider the sequence of operators

$$A_n \doteq B_2(\beta_n) \int d\theta f(\theta) T_2(\theta + \theta_n) A T_0(\theta + \theta_n)^{-1} B_0(\beta_n)^{-1}, \quad n \in \mathbb{N}, \quad (6.1)$$

where $A \in \mathfrak{A}(\mathcal{C}_2)$ is any local operator, $f$ any test function and the integral is defined in the strong operator topology. These operators are designed to exploit the spectral properties of $T_2$ and $T_0$ by choosing the support of the Fourier transform of $f$ appropriately. The usual strategy in sector analysis is to look at these operators in the representation induced by $\sigma_1$ where they ought to give information on the spectral properties of $T_{12}$ relative to $T_1$. But there is a problem. The test functions $f$ are analytic, the integral extends over all of $\mathbb{R}$ and the resulting operators are not localized in the light cone $V$, i.e. they do not lie in the domain of $\sigma_1$. The sequence of shifts $\theta_n$ serves to move them asymptotically into $V$, without obliterating the
information on the spectral properties of the operators. But another problem arises: they transport the operators into the future of the localization region of any given morphism \( \sigma_1 \) so that they can interfere with the associated charges. The resulting effects are difficult to control. The solution is to let boosts in the direction of \( l' \) with an increasing sequence of rapidities act on the operators. As a result, the operators are contracted towards the boundary of \( V \) and pushed simultaneously towards spacelike infinity in the direction of \( l' \). Even though these boosts blur the information on the spectral properties of the operators, they do not mix the positive and negative spectrum allowing us sufficient control on the spectrum of \( T_{12} \). Choosing a morphism \( \sigma_1 \) whose localization cone avoids the asymptotic localization region of the resulting sequence of operators, we can show that the spectrum condition must be violated in the charge class of \( \sigma_1 \bullet \sigma_2 \) if it is violated in the charge class of \( \sigma_2 \), or interchanging the roles of \( \sigma_1 \) and \( \sigma_2 \), in that of \( \sigma_1 \). We come now to the actual proof of the lemma.

**Proof of Lemma 6.4.** If the unitary representation \( \tilde{U}_2 \) of \( \mathcal{P}_1^+ \) violates the spectrum condition, as outlined above, we fix a positive lightlike vector \( l \) and choose a morphism \( \sigma_2 \) localized in a hypercone \( C_2 \) pointing in the opposite positive lightlike direction \( l' \). We will take advantage of the localization of the cocycles corresponding to the representation \( \tilde{U}_2 \). With the above notation, Lemma 5.4 implies that there is a hypercone \( C_0 \supset C_2 \) such that \( T_2(\theta) T_0(\theta)^{-1}, B_2(\beta) B_0(\beta)^{-1} \in \mathcal{R}(C_0) \) for sufficiently small \( \theta \geq 0 \) and \( \beta \geq 1 \), respectively. As a matter of fact, since \( C_2 \) points towards \( l' \) and \( \Lambda(\beta) l' = \beta l' \), \( \beta \geq 1 \), we may in addition suppose that \( \Lambda(\beta) C_0 \subset C_0, \beta \geq 1 \), and \( \Lambda(\beta) C_0 \subset \mathcal{O}^\circ \) for any given compact region \( \mathcal{O} \subset V \) and sufficiently large \( \beta \), cf. A.11. The cocycle equation then implies \( T_2(\theta) T_0(\theta)^{-1} \in \mathcal{R}(\mathcal{T}_\theta) \) for any \( \theta \geq 0 \), where \( \mathcal{T}_\theta = \bigcup_{0 \leq \theta' < \theta} (C_0 + \partial l) \). Similarly, \( B_2(\beta) B_0(\beta)^{-1} \in \mathcal{R}(C_0) \) for any \( \beta \geq 1 \), taking account of the endomorphism action of the boosts \( \Lambda(\beta), \beta \geq 1 \), on \( C_0 \).

As explained above, \( T_2 \) has spectrum on the negative real axis \( \mathbb{R}_- \), so there is a compact set \( K_2 \subset \mathbb{R}_- \) such that \( E_2(K_2) \neq 0 \), where \( E_2 \) is the spectral resolution of \( T_2 \). We pick a test function \( f : \mathbb{R} \to \mathbb{C} \) whose Fourier transform is equal to 1 on \( K_2 \) and vanishes on the ray \( (\mathbb{R}_+ - \kappa_2) \) for some \( \kappa_2 > 0 \). We also choose a local operator \( A \in \mathcal{A}(C_2) \) with \( E_2(K_2) A \Omega \neq 0 \) by invoking the Reeh–Schlieder property of \( \Omega \). Inserting into the expression for the operators \( A_n \) of relation (6.1), where \( \theta_n > 0, \beta_n \geq 1, n \in \mathbb{N} \), gives suitable sequences to be adjusted in what follows. Since the support of \( f \) is all of \( \mathbb{R} \), the resulting operators are not localized in \( V \). We therefore pick a test function \( \chi : \mathbb{R} \to \mathbb{R} \) with support in the interval \([-1, 1]\) and equal to 1 in some neighbourhood of 0 and introduce approximating functions \( f_n \), putting \( f_n(\theta) = \chi(\theta/\theta_n) f(\theta), n \in \mathbb{N} \). Fixing a sequence \( \theta_n, n \in \mathbb{N} \), tending to infinity, \( f_n \to f \) in the Schwartz space topology. The operators

\[
A_{\nu,n} \doteq B_2(\beta_n) \int d\theta \, f_n(\theta) T_2(\theta + \theta_n) A T_0(\theta + \theta_n)^{-1} B_0(\beta_n)^{-1}, \quad n \in \mathbb{N} \tag{6.2}
\]

are elements of \( \mathcal{R}(\mathcal{T}_{2\theta_n/\beta_n}) \), bearing \( C_0 \cup \Lambda(\beta_n) \mathcal{T}_{2\theta_n} \subset \mathcal{T}_{2\theta_n/\beta_n} \) and the localization
of the cocycles $B_2(\beta)B_0(\beta)^{-1}$, $T_2(\theta)T_0(\theta)^{-1}$ in mind as well as the way boosts act on the hypercone $C_0 \supset C_2$ and the lightlike vector $l$. Moreover, $\|A_n - A_{V,n}\| \to 0$ as $n$ tends to infinity.

We first analyze the operators $A_{V,n}^*A_{V,n}$ in the limit of large $n$. Since the boosts $B_2$ cancel in these operators we have better control on their localization properties: they are localized in the region $\Lambda(\beta_n) = \bigcup_{0 \leq \theta < 2\theta_n/\beta_n} (\Lambda(\beta_n)C_0 + \partial l)$. Fixing a sequence $\beta_n$ with $\theta_n/\beta_n \to 0$ in the limit of large $n$ and bearing in mind that $\Lambda(\beta_n)C_0 \subset \mathcal{O}^c$ for any given compact region $\mathcal{O} \subset V$ and sufficiently large $\beta_n$, locality implies that $A_{V,n}^*A_{V,n}$, $n \in \mathbb{N}$, is a central sequence in $\mathcal{R}(V)$. Since the sequence is uniformly bounded and $\mathcal{R}(V)$ is a factor, its weak limit points are multiples of the identity. In fact they all coincide and can be evaluated in the vacuum state, where $(\Omega, A_{V,n}^*A_{V,n}\Omega) = \| \int d\theta f_n(\theta)T_2(\theta)A\Omega \|^2$. As a result we have $\lim_n A_{V,n}^*A_{V,n} = \| \int d\theta f(\theta)T_2(\theta)A\Omega \|^2 \cdot 1$ in the weak operator topology.

Next, we pick a morphism $\sigma_1$ localized in a hypercone $C_1$ in the spacelike complement of $\mathcal{T}_\theta$ for some $\theta > 0$. (Note that according to (A.13) of the appendix there is a hypercone $C_0 \supset \mathcal{T}_\theta$, whose opposite cone can be taken as $C_1$.) Choosing an extension $\tilde{\sigma}_1$ normal on $\mathcal{R}(\mathcal{T}_\theta)$ a routine computation gives

$$
\tilde{\sigma}_1(A_{V,n}) = \int d\theta_n f_n(\beta_n,\theta) \tilde{\sigma}_1(T_2(\theta + \theta_n/\beta_n)B_2(\beta_n)AB_0(\beta_n)^{-1}T_0(\theta + \theta_n/\beta_n)^{-1})
$$

$$
= \int d\theta_n f_n(\beta_n,\theta) T_{12}(\theta) \tilde{\sigma}_1(T_2(\theta_n/\beta_n)B_2(\beta_n)AB_0(\beta_n)^{-1}T_0(\theta_n/\beta_n)^{-1})T_1(\theta)^{-1},
$$

for sufficiently large $n$. Here the first equality uses the commutation properties of boosts and lightlike translations, given above, and the second equality the expression for the translations in the composed representation $\sigma_1 \cdot \sigma_2$,

$$
T_{12}(\theta') = \tilde{\sigma}_1(T_2(\theta')T_0(\theta')^{-1})T_1(\theta'), \quad \theta' \geq 0,
$$

established in the proof of Theorem 5.6. Note that the above integral extends over the region $(\theta + \theta_n/\beta_n) \geq 0$ by the support properties of $f_n$, so this expression may be used here. Now let $E_1$ be the spectral resolution of $T_1$, let $K_1 \subset \mathbb{R}$ be any compact set in its spectrum and let $\Phi_1 \in E(K_1)\mathcal{H}$ be any non–zero vector. Furthermore, let $E_{12}$ be the spectral resolution of $T_{12}$. Then, for any bounded operator $B \in \mathcal{B}(\mathcal{H})$, the Fourier transform (in the sense of distributions) of $\theta \mapsto E_{12}(\mathbb{R}^+)T_{12}(\theta)BT_1(\theta)^{-1}\Phi_1$ has support in the region $(\mathbb{R}^+ - \beta_n\kappa_2)$ for some $\kappa_2 \in \mathbb{R}$ (depending on the choice of $K_1$). On the other hand, with the above choice of the test function $f$, the Fourier transform of $\theta \mapsto \beta_n f(\beta_n,\theta)$ vanishes in the region $(\mathbb{R}^+ - \beta_n\kappa_2)$, where $\kappa_2 > 0$. Hence $\int d\theta_n f_n(\beta_n,\theta)E_{12}(\mathbb{R}^+)T_{12}(\theta)BT_1(\theta)^{-1}\Phi_1 = 0$ for sufficiently large $\beta_n$. Now $(\beta_n f(\beta_n,\theta) - \beta_n f_n(\beta_n,\theta)) \to 0$ in $L^1(\mathbb{R})$ in the limit of large $n$ and the sequence $B_n \equiv \tilde{\sigma}_1(T_2(\theta_n/\beta_n)B_2(\beta_n)AB_0(\beta_n)^{-1}T_0(\theta_n/\beta_n)^{-1})$ is uniformly bounded in this limit. Hence in the above expression for $\tilde{\sigma}_1(A_{V,n})$ in the second integral we can replace the function $\theta \mapsto \beta_n f_n(\beta_n,\theta)$ by $\theta \mapsto \beta_n f(\beta_n,\theta)$ since the difference tends to 0 in norm. Taking account of $\tilde{\sigma}_1(A_{V,n}) = A_{V,n}$ and the localization of $A_{V,n}$, it is then clear that $\lim_n \|E_{12}(\mathbb{R}^+)A_{V,n}\Phi_1\| = 0$.
Let us summarize the facts established so far. In the first step we have shown
\[
\lim_n \| A_{V,n} \Phi_1 \|^2 = \lim_n \langle \Phi_1, A_{V,n}^* A_{V,n} \Phi_1 \rangle = \left\| \int d\theta f(\theta) T_2(\theta) A \Phi_1 \right\|^2 \left\| \Phi_1 \right\|^2.
\]
The next step gives
\[
\lim_n \left\| (1 - E_{12}(\mathbb{R}_+)) A_{V,n} \Phi_1 \right\|^2 = \left\| \int d\theta f(\theta) T_2(\theta) A \Omega \right\|^2 \left\| \Phi_1 \right\|^2.
\]
The support properties of the Fourier transform of \( f \) and the choice of \( A \) yield
\[
\left\| \int d\theta f(\theta) T_2(\theta) A \Omega \right\|^2 \geq \left\| E_2(K_2) A \Phi_1 \right\|^2 \neq 0.
\]
Hence \((1 - E_{12}(\mathbb{R}_+)) \neq 0\), proving that \( T_{12} \) has spectrum on the negative real axis. So \( \tilde{U}_{12} \) violates the spectrum condition if \( \tilde{U}_2 \) does. Since \( \sigma_1 \cdot \sigma_2 \simeq \sigma_2 \cdot \sigma_1 \), the proof of the lemma is completed interchanging \( \sigma_1 \) and \( \sigma_2 \).

The preceding results accurately describe the energetic properties of the states of interest here. They also throw new light on the appearance of superselection rules in quantum field theory. From the present theoretical point of view, based on observations and operations performed in a light cone \( V \), the creation of a charged state in \( V \) is achieved by creating a pair and pushing the opposite charge to lightlike infinity. In practice, however, this would require an unlimited amount of energy since the opposite charge would have to be accelerated to the speed of light. The experimental creation of a charged state can therefore only be accomplished locally by moving the opposite charge sufficiently far away (“behind the moon”). In other words, superselection rules appear when the total charge in \( V \) cannot be changed by realistic physical operations due to an infinite energy barrier. Nevertheless, the theoretical limit states on \( V \) are meaningful idealizations allowing us to analyze the properties of charges. As we have seen, relative to the vacuum, these states have finite energy bounded from below. The infinite energy needed for their creation from the vacuum is carried away by the opposite charge and not visible anymore in \( V \) in the limit.

7 The Minkowskian picture

Throughout the preceding discussion, we have restricted attention to the algebra of observables in a given light cone \( V \). Whether this algebra is part of a larger algebra in Minkowski space or not did not matter. Yet interestingly enough the framework established here can be used to construct an extension of the theory to a theory on Minkowski space \( M \). Whereas this canonical extension may seem somewhat arbitrary from an observational point of view, it enables us to establish in a physically meaningful manner that our results do not depend on the choice of light cone \( V \).

The basic ingredient is the relation between our semigroup \( S^\uparrow_+ \) and the group \( P^\uparrow_+ \) it generates. \( S^\uparrow_+ \) is a semidirect product of its subsemigroup of future directed
(time) translations and the Lorentz group. The time translations are a normal sub-
semigroup generating the group of spacetime translations and \( \mathcal{P}_+^+ \) is the semidirect
product of the group of spacetime translations and the Lorentz group. This tight
relationship between \( S^+_+ \) and \( P^+_+ \) allows us to induce up structures relating to \( S^+_+ \) to
a corresponding structure relating to \( P^+_+ \). Topological aspects of this principle can
be incorporated efficiently by observing that \( S^+_+ \) contains an interior point of \( P^+_+ \).

The first observation is that \( V \) is just the quotient of \( S^+_+ \) by the Lorentz group.
The induced structure for \( P^+_+ \) is Minkowski space \( \mathcal{M} \), the quotient of \( P^+_+ \) by the
Lorentz group. We leave it to the reader to formalize the construction of the affine
space \( \mathcal{M} \) together with its Lorentzian metric as an extension of \( V \) equipped with
the analogous structure. Thus an observer in \( V \), aware of the way \( S^+_+ \) acts, can
talk about Minkowski space. The second observation is that the continuous unitary
representation \( U_0 \) of the semigroup \( S^+_+ \) determined by the vacuum state extends
canonically to a continuous unitary representation of \( P^+_+ \), as outlined in Sect. 3.
Moreover, the net \( \mathcal{A} \) of observable algebras in the given light cone \( V \) extends to a
net on Minkowski space.

We recall that the extended net is fixed by assigning to each double cone \( \mathcal{O}_M \subset M \)
the algebra \( \mathcal{A}_M(\mathcal{O}_M) \doteq U_0(x_M)^{-1} \mathcal{A}(\mathcal{O}_M + x_M) U_0(x_M) \), where \( x_M \in \mathcal{V}_+ \) is such that \( \mathcal{O}_M + x_M \subset V \). This assignment defines an (isotonous) net \( \mathcal{A}_M \) on \( M \) which is
local and covariant with regard to the adjoint action of \( U_0(\lambda), \lambda \in P^+_+ \). Moreover,
given the vacuum state \( \omega_0 \) on \( \mathcal{A}(V) \) can be extended to a (pure) vacuum state \( \omega_M \)
on \( \mathcal{A}_M(M) \doteq \bigcup_{\mathcal{O}_M \subset M} \mathcal{A}_M(\mathcal{O}_M) \), putting
\[
\omega_M(A_M) \doteq (\Omega, A_M \Omega), \quad A_M \in \mathcal{A}_M(M),
\]
where \( \Omega \) is the cyclic vector in the GNS–representation of \( \mathcal{A}(V) \) induced by \( \omega_0 \). These extensions of the net \( \mathcal{A} \) and state \( \omega_0 \) are uniquely fixed by the hypothesis of a
global (Minkowskian) vacuum state entering into the chosen extension of \( U_0 \) to \( P^+_+ \).
Yet, as already mentioned, in the presence of massless particles, other extensions
consistently describe different prehistories of \( \omega_0 \).

Nevertheless this canonical extension is useful since observations made in different
light cones \( V_1, V_2 \) may be compared, provided the respective observers use their
(partial) vacua as reference states and reconstruct the representation \( U_0 \) of \( P^+_+ \) from
their respective observations. Every pair of light cones contains some common light
cone \( V \subset V_1 \cap V_2 \), so both observers have access to \( V \) and would, in principle, be
able to reconstruct the same global net \( \mathcal{A}_M \) and vacuum state \( \omega_M \) from their partial
information. If they conventionally regard their partial vacua as restrictions of the
global vacuum state, \( \omega_{j0} = \omega_M \restriction \mathcal{A}_M(V_j), \ j = 1, 2 \), it would make sense for the
observers to compare their past data in their common future.

The properties of simple covariant charges as seen by observers in different
light cones can likewise be compared on this same basis since the corresponding

5The explicit construction of \( \mathcal{N}_P \) from \( \mathcal{N}_S \) in Proposition 5.2 can be seen as a simple application
of this general principle.
morphism can be extended to the Minkowskian net. Let \( \sigma : \mathfrak{A}(V) \to \mathfrak{R}(V) \) be any covariant morphism and let \( \tilde{U}_\sigma \) be the associated representation of \( \tilde{\mathcal{P}}^\pm_+ \). The extension of \( \sigma \) to the operators \( A_M \in \mathfrak{A}_M(M) \) is given by

\[
\sigma_M(A_M) = \text{Ad} \tilde{U}_\sigma(x_M)^{-1} \circ \sigma \circ \text{Ad} U_0(x_M)(A_M),
\]

where \( x_M \in \mathcal{V}_+ \) is so large that \( \text{Ad} U_0(x_M)(A_M) \in \mathfrak{A}(V) \). This assignment yields a well-defined morphism \( \sigma_M : \mathfrak{A}_M(M) \to \mathcal{B}(\mathcal{H}) \) transforming covariantly under the adjoint action of \( \tilde{U}_\sigma \), as is easily verified. Note that this extension is completely fixed by data available in \( V \).

Now given \( V_0 \supset V \) put \( \sigma_{V_0} = \sigma_M \mid \mathfrak{A}_M(V_0) \). If \( \sigma \) is a simple hypercone localized morphism in \( V \), then \( \sigma_{V_0} \) is a simple hypercone localized morphism in \( V_0 \), i.e. its composition with the vacuum \( \omega_{M_0} \) leads to a charge class with all properties specified in the defining criterion. We refrain from giving a detailed proof here but just indicate the main points.

Let \( t_0 \in \mathcal{V}_+ \) with \( V_0 + t_0 = V \). If the morphism \( \sigma \) is localized in the hypercone \( \mathcal{C} \subset V \), Lemma 5.4 and the cocycle equation imply \( \tilde{U}_\sigma(t_0) U_0(t_0)^{-1} \in \mathfrak{R}(\mathcal{T}_{t_0}) \), where \( \mathcal{T}_{t_0} = \bigcup_{0 \leq \varsigma \leq 1} \{ \mathcal{C}_0 + \varsigma t_0 \} \) and \( \mathcal{C}_0 \supset \mathcal{C} \) is some hypercone in \( V \). As shown in A.13, there is a hypercone \( \mathcal{T}_{t_0} \subset \mathcal{C}_0 \subset V \). Hence, denoting the weak closures of algebras in the Minkowskian net by the symbol \( \mathfrak{R}_M \), one has

\[
\tilde{U}_\sigma(t_0)^{-1} U_0(t_0) = U_0(t_0)^{-1} (\tilde{U}_\sigma(t_0) U_0(t_0)^{-1})^{-1} U_0(t_0) \in U_0(t_0)^{-1} \mathfrak{R}(\mathcal{C}_0) U_0(t_0) = \mathfrak{R}_M(\mathcal{C}_0 - t_0),
\]

and \( (\mathcal{C}_0 - t_0) \) is a hypercone in \( V_0 \), based on the shifted hyperboloid \( (H - t_0) \). But

\[
\sigma_{V_0} = \text{Ad} \tilde{U}_\sigma(t_0)^{-1} \circ \sigma \circ \text{Ad} U_0(t_0) = \text{Ad} \tilde{U}_\sigma(t_0)^{-1} U_0(t_0)^{-1} \circ \text{Ad} U_0(t_0)^{-1} \circ \sigma \circ \text{Ad} U_0(t_0),
\]

so \( \sigma_{V_0} : \mathfrak{A}_M(V_0) \to \mathfrak{R}_M(V_0) \) is localized in \( (\mathcal{C}_0 - t_0) \supset \mathcal{C} \). Moreover, if \( \sigma, \tau \) are equivalent morphisms on \( \mathfrak{A}(V) \) with intertwiners \( W \in (\sigma, \tau) \subset \mathfrak{R}(V) \), then \( \sigma_{V_0}, \tau_{V_0} \) are equivalent on \( \mathfrak{A}_M(V_0) \) with intertwiners \( W_{V_0} \in (\sigma_{V_0}, \tau_{V_0}) \subset \mathfrak{R}_M(V_0) \) given by

\[
W_{V_0} = \tilde{U}_\tau(t_0)^{-1} W \tilde{U}_\sigma(t_0) = (\tilde{U}_\tau(t_0)^{-1} U_0(t_0)) (U_0(t_0)^{-1} W U_0(t_0)) (U_0(t_0)^{-1} \tilde{U}_\sigma(t_0)).
\]

Letting the morphisms and their localization vary using Lorentz covariance one finds that the families of equivalent hypercone localized morphisms in \( V \) are restrictions of families of equivalent hypercone localized morphisms in \( V_0 \). Thus the analysis of the preceding sections and the corresponding results apply to the latter families, too.

Even though the morphisms and their intertwiners depend, in general, on the choice of light cone, important physical data, such as their statistics parameters \( \varepsilon \in \{ \pm 1 \} \) do not since the intertwiners \( W_{V_0} \) depend continuously on \( t_0 \) as can be seen from the expression given above. Hence the statistics parameters stay constant.
under changes of the light cone. Moreover, the spectral properties of the charge classes do not depend on the choice of light cone, either. Since for any pair of light cones \( V_1, V_2 \) there is a light cone \( V_0 \supset V_1 \cup V_2 \), the respective observers will agree on the intrinsic properties of the charges.

Despite the fact that the interpretation of charged states does not depend on the choice of light cone, it would not make sense, in general, to take a global view. For example, the equivalence of morphisms \( \sigma_M \upharpoonright \mathcal{A}_M(V_0) \simeq \tau_M \upharpoonright \mathcal{A}_M(V_0) \) for any light cone \( V_0 \) does not imply that \( \sigma_M \simeq \tau_M \) on \( \mathcal{A}_M(M) \). The two representations may differ by infrared clouds of massless particles superselected in \( M \). Moreover, from the Minkowskian point of view, the morphisms \( \sigma_M \) in general do not have localization properties allowing composition and an analysis of statistics. Thus, for the interpretation of theories with long range forces, the restriction to light cones is not only physical meaningful but also solves these infrared problems.

Another aspect of the preceding analysis is worth mentioning. The species of localizable charges, exhaustively treated from the Minkowskian point of view in [11–13], also fit into the present light cone setting. It is a distinctive feature of this type of charge that the intertwiners between morphisms localized in a light cone \( V \) do not change if one proceeds to larger cones \( V_0 \). In contrast, for charges related to a local gauge group, such as the electric charge, the intertwiners depend on \( V_0 \). Thus, from the present point of view, there is a clear cut distinction between these two types of charges. This prompts us to ask the intriguing question of whether this feature can be used to recover both the global gauge group and the structure of the underlying local gauge group from the structure of the charged states. This would solve a longstanding problem in the algebraic approach to local quantum physics [21].

8 Concluding remarks

The present investigation establishes a new framework for interpreting physical states in relativistic quantum field theory. Instead of implicitly supposing that the properties of physical states can be controlled in all of Minkowski space, we have from the outset taken the arrow of time into account: missing measurements and operations in the past in general mean missing information in the future. From an experimental point of view, the best physicists can hope for is to explore the properties of (partial) states in future light cones. Theory only needs to interpret and explain such data.

In a completely massive world, our present approach would not make a difference as the observables in any forward light cone would be irreducible. So one would not lose information about states by belated experiments. The situation is different, however, in the presence of massless particles. Since, as a consequence of Huygens principle and Einstein causality, there is no way of acquiring information on outgoing massless particles (radiation) created in the past of a light cone, the family
of observables in a light cone is highly reducible. Although there are numerous \textit{a priori} possibilities for the type of the algebra generated by these observables, they are generically of type III$_1$ according to the classification of Connes.

From a physical point of view, this type comes closest to the familiar irreducible case as it still allows to describe in a comprehensive way the transitive quantum effects of physical operations on states carrying the same charge. So although the superposition principle no longer holds for states on these algebras, concepts familiar from the analysis of superselection sectors such as the creation and transport of charges, their statistics and their conjugation still make good sense. Moreover, their energetic aspects can be consistently described.

The advantage of the present approach is that the infrared problems caused by infinite clouds of low energy massless particles disappear. In the traditional treatment one tries to solve these problems by splitting the massless particle content into an energetically soft and therefore unobservable part and a hard part. This method, however, breaks Lorentz invariance and is incompatible with the strict locality of the observables. In the present approach the massless particle content is split into a marginal part defying observation since it is already in the spacelike complement of the observer and an elemental part accessible to observations and operations within his lightcone. This splitting is both Lorentz invariant and compatible with Einstein causality. It thereby permits a consistent description and analysis of elementary systems even in the presence of long range forces.

Since the present approach does not aim to treat the marginal part of the outgoing massless particles it even admits unitary implementations of Lorentz transformations for charged light cone morphisms. Thus it is worthwhile posing the question of whether electrically charged states can be described as Wigner particles (irreducible representations of the Poincaré group) in the present setting. Such a result seems not to conflict with previous insights into Minkowski theories [4, 5, 18] where spacelike asymptotic properties do matter but have no counterpart in the present approach. Since in a light cone the observed massless contributions can be described by Fock states with a finite particle number there may be partial states, describing a single electrically charged particle where the (globally inevitable) accompanying radiation field has no observable effects in the cone. Such states could well contribute to an atomic part in the mass spectrum. Thus the infraparticle problem, too, may disappear if one just allows observations in a light cone.

One of the most intriguing aspects of the present approach is the endomorphic nature of time evolution, entering as it does in the interpretation of the microscopic theory. This feature, combined with the perpetual loss of control of outgoing massless particles might be relevant to a better understanding of the classical aspects of our quantum world. Observations on outgoing radiation cannot be affected anymore by later quantum experiments (unless the observer has taken timely precautions to reflect it back into his light cone). Hence those results can be taken as facts in the sense of classical physics without conflicting with the principles of quantum theory.
This aspect seems to warrant a more detailed study.

In the present analysis, we have restricted attention to simple charges satisfying Bose or Fermi statistics. Just as in the case of localizable charges in Minkowski space, substantial parts of our analysis can be extended to the more general case of charges of arbitrary finite statistics with one notable exception: we have not been able to show that the property of covariance of composite morphisms is stable under forming subrepresentations and to establish the spectrum condition. Since the physically successful theories coupling matter to the electromagnetic field do not lead to parastatistics there might be a deeper reason for our failure. But caution is needed in drawing such a conclusion and we will pursue this question elsewhere.

A Appendix

In this appendix we establish the geometric facts about hypercones used in the preceding analysis. Let $M$ be Minkowski space equipped with the metric $(+,-,-,-)$ and coordinates $x = (x_0, \mathbf{x})$. We fix the forward lightcone $V = \{ x \in \mathbb{R}^4 : x_0 > |\mathbf{x}| \}$ and regard it as a globally hyperbolic spacetime with metric inherited from Minkowski space. Recall that $X^c \subset V$ denotes the spacelike complement of any subset $X \subset V$. The lightcone $V$ is foliated by the hyperboloids $H_\tau = \{ x \in V : x_0 = \sqrt{x^2 + \tau^2} \}$ (time shells) for $\tau > 0$. Since the hyperboloids $H_\tau$ form Cauchy surfaces of $V$, the causal completions of disjoint sets on a given $H_\tau$ are spacelike separated regions in $V$.

Fixing $\tau$ and abbreviating $H = H_\tau$ we consider specific subsets of the corresponding hyperboloid as bases of causally complete regions in $V$. To have a Lorentz invariant description of these sets, we equip $H$ with the metric induced from the ambient space. Given two points $a, b \in H$, the geodesic connecting them is the segment of the “great hyperbola” got by intersecting $H$ with the 2–plane fixed by $a, b, 0$ in the ambient space. Its length is $d(a, b) = \tau \cosh^{-1}(ab/\tau^2)$, where $ab$ denotes the Lorentz scalar product of $a$ and $b$. Great hyperbolae on $H$ thus correspond to lines and will be called hyperbolic lines.

To further geometric intuition we project the hyperboloid $H$ through the origin onto the plane $x_0 = 1$ and thereby identify it with the open unit ball $B \subset \mathbb{R}^3$ about the origin. The corresponding invertible map $\mathbf{v} : H \to B$ is given by

$$\mathbf{v}(a) = a/a_0 = a/\sqrt{a^2 + \tau^2},$$

inducing on $B$ the metric $d(\mathbf{u}, \mathbf{v}) = \tau \cosh^{-1}((1-\mathbf{u} \cdot \mathbf{v})/\sqrt{(1-\mathbf{u}^2)(1-\mathbf{v}^2)})$, where $\mathbf{u} \cdot \mathbf{v}$ denotes the Euclidean scalar product. The result of this mapping is the Beltrami–Klein model of hyperbolic geometry. Its simplifying feature is that the hyperbolic lines on $H$ are mapped to chords of $B$, i.e. straight lines connecting boundary points of $B$. Note that the spherical boundary $S^2$ of $B$ corresponds to spacelike infinity on the hyperboloid.
Hyperbolic rays on $\mathbb{H}$ are fixed by specifying their apex $a$ and their asymptotic lightlike direction $l = (1, l)$, where $l \in S^2$. A union of hyperbolic rays emanating from a common apex $a$ is the analogue of a cone and is called a hyperbolic cone. The opposite hyperbolic cone results by taking the union of the corresponding opposite rays emanating from $a$. A hyperbolic cone is said to be pointed if its closure and the opposite closed cone have only the apex in common. A hyperbolic cone $C \subset \mathbb{H}$ is said to be convex if, for any two points $a, b \in C$, the geodesic connecting them is contained in $C$.

Proceeding to the Beltrami–Klein model, a hyperbolic ray on $\mathbb{H}$, fixed by its apex $a$ and asymptotic lightlike direction $l = (1, l)$, corresponds to the straight line between the apex $v(a) \in \mathbb{B}$ and the boundary point $l \in S^2$. Thus a hyperbolic cone on $\mathbb{H}$ corresponds to an ordinary (truncated) Euclidean cone $K \subset \mathbb{B}$ and the concepts of opposite, pointed and convex hyperbolic cone coincide in the Beltrami–Klein model with those familiar from Euclidean geometry. We therefore characterize the hyperbolic cones $C \subset \mathbb{H}$ by their images $K \subset \mathbb{B}$ in the Beltrami–Klein model, $C = C(K)$, where we restrict attention to pointed open convex cones $K \subset \mathbb{B}$ with elliptical base. These form a Lorentz invariant family. We will also consider hyperbolic balls $O \subset \mathbb{H}$, i.e. open balls with arbitrary hyperbolic diameter and apex. Their (ellipsoidal) images in $\mathbb{B}$ will be denoted by the same symbol.

![Figure 1: Hyperbolic cone $C$ on the hyperboloid $H$ and its image $K$ in the ball $B$](image)

**Definition 1.** Let $H \subset V$ be a fixed hyperboloid. A hyperball $O \subset V$ is the causal completion of a hyperbolic ball $O \subset H$. A hypercone $C(K) \subset V$ is the causal completion of a hyperbolic cone $C(K) \subset H$, where $K \subset B$ is any pointed open convex cone with elliptical base. The family of these hypercones is denoted by $\mathcal{F}$. 

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Note that the hypercones $C(K) \in F$, $K \subset B$, inherit the structure of a partially ordered set from the underlying ordinary cones, i.e. $C(K_1) \subset C(K_2)$ iff $K_1 \subset K_2$. Moreover, the spacelike separation of hypercones, $C(K_1) \subset C(K_2)^c$, is equivalent to disjointness of the underlying ordinary cones, $K_1 \cap K_2 = \emptyset$. It should also be noted that any double cone in $V$ is contained in some hyperball. These facts greatly simplify the subsequent discussion.

**Topology of hypercones**

In this subsection we list some topological properties of hypercones. We begin by discussing the funnels of hypercones entering into the extension of morphisms.

**A.1.** Given any hypercone $C$ there is a decreasing sequence of hypercones (a funnel) $C_n \subset C$, $n \in \mathbb{N}$ such that $C_n \cap V$.

*Proof.* Let $C = C(K)$ and pick any decreasing sequence of cones $K_n \subset K$, $n \in \mathbb{N}$ such that $\bigcap_n K_n = \emptyset$. The resulting sequence of hypercones $C(K_n)$, $n \in \mathbb{N}$ has the required properties.

**A.2.** Let $C_n$, $n \in \mathbb{N}$ be an increasing sequence of hypercones such that $C_n \cap V$. The corresponding (decreasing) sequence of opposite hypercones is a funnel in the preceding sense.

*Proof.* Let $C_n = C(K_n)$, where $K_n \subset B$ is an increasing sequence of cones such that $K_n \cap B$, $n \in \mathbb{N}$. The respective opposite cones $K_n^o$, $n \in \mathbb{N}$ form a decreasing sequence and $\bigcap_n K_n^o = \emptyset$. Since the hypercone opposite to $C_n$ is given by $C_n^o = C(K_n^o)$, $n \in \mathbb{N}$ the latter sequence forms a funnel, as claimed.

What is important when analyzing morphisms is to be flexible in choosing their localization. The following geometrical fact is then significant.

**A.3.** If $\mathcal{O}$ is a double cone and $C$ a hypercone, then there is a hypercone $C_0 \subset \mathcal{O}^c \cap C$.

*Proof.* As a double cone is contained in some hyperball we may assume that $\mathcal{O}$ is a hyperball. Let $\mathcal{O} \subset B$ be the base of $\mathcal{O}$, let $C = C(K)$ and let $L \subset S^2$ be the boundary of $K$. Since $\mathcal{O}$ is relatively compact in $B$ there is a cone $K_0 \subset K$ whose apex is sufficiently close to $L$ such that $K_0 \cap \mathcal{O} = \emptyset$. The hypercone $C_0 = C(K_0)$ has the desired properties.

The following result is of a similar nature.

**A.4.** If $C$ is a hypercone and $\mathcal{O} \subset C^c$ a hyperball then there is a hypercone $C_0$ with $\mathcal{O} \subset C_0 \subset C^c$.
A.5. The family $F$ of all hypercones is pathwise connected.

Proof. The family of cones $K \subset B$ is obviously pathwise connected. Since the corresponding family of hypercones $C(K)$ is order isomorphic to $K$, the result holds.

A.6. The family $F(C^c) \subset F$ of hypercones localized in the spacelike complement of a given hypercone $C$ is pathwise connected.

Proof. Let $C = C(K)$ for given $K \subset B$. The family $F(C^c)$ consists of all hypercones $C(K^c)$, where $K^c \subset B\backslash K$ is any pointed open convex cone with elliptical base. Let $L \subset S^2$ be the boundary of $K$. Given any pair of cones $K^c_1, K^c_2 \subset B\backslash K$, their respective boundaries $L_1, L_2$ are contained in $S^2 \backslash L$. Since $S^2 \backslash L$ is connected it is then clear that $K^c_1, K^c_2$ can be connected by an alternating path of cones whose apices stay sufficiently close to $S^2 \backslash L$ therefore not meeting $K$. Hence the family $K^c \subset B\backslash K$ of cones is pathwise connected and with it the corresponding family of hypercones $C(K^c) \in F(C^c)$.

Given any two hypercones $C_a, C_b \in F$ there may be no hypercone in their common spacelike complement $C_a^c \cap C_b^c$. (This is most easily seen in the Beltrami–Klein model.) Yet, by proceeding to smaller cones this difficulty can be overcome.

A.7. Let $C_a, C_b \in F$ be arbitrary hypercones. Then there is a hypercone $C_0 \subset C_a$ making the family of hypercones $F(C_0^c) \cap F(C_b^c)$ pathwise connected.

Proof. Let $C_a = C(K_a), C_b = C(K_b)$ for given $K_a, K_b \subset B$ and let $L_a, L_b \subset S^2$ be the (convex) boundaries of $K_a, K_b$. If $L_a \cap L_b = \emptyset$ there is a cone $K_0 \subset K_a$ whose apex lies sufficiently close to the boundary $L_a$ so that $K_0 \cap K_b = \emptyset$. Let $L_0 \subset S^2$ be the boundary of $K_0$. Then $S^2 \backslash (L_0 \cup L_b)$ is connected and, as in the preceding argument, it follows that the family of cones $K^c \subset B\backslash (K_0 \cup K_b)$ is pathwise connected. If $L_a \cap L_b \neq \emptyset$ there is a cone $K_0$ with boundary $L_0 \subset L_a \cap L_b$ and apex sufficiently close to the boundary with $K_0 \subset K_a \cap K_b$. Hence $F(C_0^c) \cap F(C_b^c) = F(C_0^c)$ and the result follows from the preceding result.
It is crucial in the statistics analysis that any given pair of spacelike separated hypercones has hypercones in its spacelike complement.

\[ \text{A.8. Let } C_a, C_b \text{ be spacelike separated hypercones, } C_a \subset C_b^\circ. \text{ Then there is a hypercone } C_0 \subset C_a^\circ \cap C_b^\circ. \]

**Proof.** Let \( C_a = \mathcal{C}(K_a), \) \( C_b = \mathcal{C}(K_b) \) where \( K_a \cap K_b = \emptyset \) and let \( L_a, L_b \subset S^2 \) be the boundaries of \( K_a \) and \( K_b, \) respectively. Since \( L_a, L_b \) are disjoint convex sets there is an open convex set \( L_0 \subset S^2 \setminus (L_a \cup L_b) \) and choosing an apex \( a_0 \) sufficiently close to \( L_0 \) one obtains a cone \( K_0 \subset B \setminus (K_a \cup K_b). \) Hence \( C_0 = \mathcal{C}(K_0) \) meets the requirements. \[ \square \]

**Hypercones based on different hyperboloids**

Having clarified the structure of hypercones based on a fixed hyperboloid, we analyze the relations between hypercones based on different hyperboloids. We put \( H_\tau = \{ x \in V : x_0 = \sqrt{x^2 + \tau^2} \}, \) \( \tau > 0 \) and denote the corresponding distance function by \( d_\tau. \) Points, hyperbolic balls and hyperbolic cones on these manifolds are denoted by \( a_\tau, O_\tau \) and \( C_\tau \) and their causal completions (the hyperballs and hypercones) by \( O_\tau \) and \( C_\tau, \) respectively. Points on different hyperboloids are identified by scaling, i.e. given \( a_\tau \in H_\tau \) we let \( a_\sigma = (\sigma/\tau) a_\tau \in H_\sigma. \) The hyperbolic balls and cones on different hyperboloids are identified in this way. The identifying map commutes with Lorentz transformations and preserves the causal relations between corresponding hypercones based on different hyperboloids. It will again be convenient to identify the hyperbolic cones \( C_\tau \subset H_\tau \) with their canonical images \( K \subset B \) in the Beltrami–Klein model, \( C_\tau = C_\tau(K). \) Their respective causal completions are denoted by \( C_\tau(K). \) The distinguished families of hypercones \( \mathcal{F}_\tau \) based on different hyperboloids \( H_\tau \) are identified in this way.

Given any \( a_\sigma \in H_\sigma, \) its future and past causal shadows on \( H_\tau \) are given by \( O_\tau(a_\sigma) = (a_\sigma \pm V_+) \cap H_\tau \) if \( \pm (\tau - \sigma) \geq 0, \) respectively. Since the distance function \( d_\tau \) is invariant under Lorentz transformations, a convenient description of this shadow follows from a straightforward computation: \( O_\tau(a_\sigma) = \{ a \in H_\tau : d_\tau(a, a_\tau) \leq \tau c_{\sigma, \tau} \}, \) where \( c_{\sigma, \tau} = \cosh^{-1}((\sigma^2 + \tau^2)/2\sigma\tau). \) Thus the causal shadow on \( H_\tau \) of any given point \( a_\sigma \in H_\sigma \) is a (closed) hyperbolic ball about \( a_\tau = (\tau/\sigma) a_\sigma \in H_\tau \) whose radius depends only on \( \sigma \) and \( \tau. \) Incidentally, this result shows that the causal completion of a hyperbolic ball about some point \( a_\tau \) is an ordinary double cone in \( V \) whose vertices lie on the timelike ray \( \mathbb{R}_+ a_\tau. \)

It follows from these remarks that the causal shadow of a hyperbolic ray on \( H_\sigma \) is the union of hyperbolic balls of fixed diameter centred on the points of the hyperbolic ray on \( H_\tau \) which is the image of the given ray by the above identification map. Similarly, the causal shadow of a hyperbolic cone on \( H_\sigma \) is the union of hyperbolic balls of fixed diameter centred on the points of the corresponding hyperbolic cone on \( H_\tau. \) Proceeding to the Beltrami–Klein model, the resulting region in \( B \) is the union of hyperbolic balls with fixed radius centred about the points of an ordinary cone \( K \subset B. \) The following result is an easy consequence.
A.9. Let $\sigma, \tau > 0$ and $C_\sigma \in \mathcal{F}_\sigma$. Then there is a $C_\tau \in \mathcal{F}_\tau$ with $C_\sigma \subset C_\tau$.

Proof. Let $C_\sigma = C_\sigma(K)$ for $K \subset B$ and let $\hat{K} = \{ u \in B : d_\tau(u, v) \leq \tau c_{\sigma, \tau}, \ v \in K \}$ be the region corresponding to its shadow on $H_\tau$. Furthermore, let $K^0$ be the cone opposite to $K$. It follows from the triangle inequality for the metric $d_\tau$ that all points in $K^0$ sufficiently close to the boundary $S^2$ of $B$ (and therefore being arbitrarily far from $K$) lie in the complement of $\hat{K}$. Hence $\hat{K}$ is contained in the interior of a spherical cap of $B$ with non-trivial complement. Picking a point $a_0$ in this complement as apex and connecting it to all points of the cap by straight lines yields a pointed convex cone $K_0 \subset B$ with spherical base containing $\hat{K}$. The corresponding hypercone $C_\tau = C_\tau(K_0)$ based on $H_\tau$ has the stated property. 

There is also a converse to this result.

A.10. Let $\sigma, \tau > 0$ and $C_\sigma \in \mathcal{F}_\sigma$. Then there is a $C_\tau \in \mathcal{F}_\tau$ with $C_\tau \subset C_\sigma$.

Proof. Let $C_\sigma = C_\sigma(K)$ for $K \subset B$. Picking any cone $K_0 \subset K$ whose hyperbolic distance from the boundary of $K$ is larger than $\tau c_{\sigma, \tau}$ ($K_0$ must have a sufficiently small opening compared to $K$ and an apex sufficiently close to the boundary $S^2$ of $B$) it follows that the causal shadow $\hat{K}_0 = \{ u \in B : d_\tau(u, v) \leq \tau c_{\sigma, \tau}, \ v \in K_0 \}$ of $K_0$ is contained in $K$ since $K$ is convex. The corresponding hypercone $C_\tau = C_\tau(K_0)$ based on $H_\tau$ fulfils the requirements.

The preceding results imply that, as is needed in the main text, the structure of hypercone localized morphisms does not depend on the choice of a hyperboloid $H_\tau \subset V$ and the corresponding family $\mathcal{F}_\tau$ of hypercones.

**Spacetime transformations of hypercones**

We now study spacetime transformations of hypercones. To this end we again fix a hyperboloid $H \subset V$ and consider the corresponding distinguished family $\mathcal{F}$ of hypercones based on it. The following results are relevant to the discussion of covariance properties of morphisms.

A.11. Let $\mathcal{C}$ be a hypercone. Then there are an open set of directions $l \in S^2$ and sequences of boosts $\Lambda_n(l)$ in these directions, i.e. $\Lambda_n(l)(1, l) = e^n(1, l)$, such that $\Lambda_n(l) \mathcal{C} \subset \mathcal{C}$, $n \in \mathbb{N}$. Moreover, given any double cone $\mathcal{O}$, $\Lambda_n(l) \mathcal{C} \subset \mathcal{O}^c$ for sufficiently large $n$.

Proof. Let $\mathcal{C} = \mathcal{C}(K)$ for given $K \subset B$. As the set of hypercones is stable under Lorentz transformations we may suppose the apex of $K$ is the centre of $B$. Let $L \subset S^2$ be the boundary of $K$ and let $l \in L$. There are boosts $\Lambda_n(l)$ with $\Lambda_n(l)(1, l) = e^n(1, l)$, $n \in \mathbb{N}$, inducing the action

$$v \mapsto \frac{1}{\text{ch}(n) + \text{sh}(n) \cdot vl} \left( \left( \text{sh}(n) + \text{ch}(n) \cdot vl \right) l + v^\perp \right),$$

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on $\mathcal{B}$, where $\mathbf{v}^\perp$ is the component of $\mathbf{v}$ orthogonal to $\mathbf{l}$. Thus the apex $\mathbf{v} = 0$ of $K$ is shifted in the direction of $\mathbf{l}$ into the cone $\mathbf{K}$ and the points of $L$ are moved along great circles towards $\mathbf{l} \in L$. Since $L$ is convex, the boosts induce contractions of $L$ and consequently map $K$ into itself. Hence $\Lambda_n(l)C \subset C$, $n \in \mathbb{N}$. It now suffices to consider hyperballs $O$. Let $O \subset B$ be the base of $O$. Since the boosts move $K$ towards the boundary $L$ and $O$ is relatively compact, the two regions become disjoint for sufficiently large $n \in \mathbb{N}$. But this implies $\Lambda_n(l)C \subset O^c$, completing the proof.

The next result is of a similar nature.

A.12. Given a hypercone $C$, there is a hypercone $C_0$ with $\Lambda C \subset C_0$ for all Lorentz transformations $\Lambda$ in some open neighbourhood $\mathcal{N}^L \subset \mathcal{L}_+^\uparrow$ of the identity.

Proof. Let $C = C(K)$ for given $K \subset B$. The action of arbitrary Lorentz transformations $\Lambda \in \mathcal{L}_+^\uparrow$ induced on $B$ is given by (choosing coordinates properly)

$$
\mathbf{v}_i \mapsto \frac{\Lambda_{00} + \sum_k \Lambda_{ik} v_k}{\Lambda_{00} + \sum_k \Lambda_{0k} v_k}, \quad i = 1, 2, 3.
$$

Since $|\Lambda_{00} + \sum_k \Lambda_{0k} v_k| \geq \left(\sqrt{1 + \sum_k \Lambda_{0k}^2} - \sqrt{\sum_k \Lambda_{0k}^2}\right) > 0$ this induced action is norm continuous on $B$ in the Euclidean topology. Hence, choosing $\Lambda$ in some sufficiently small neighbourhood $\mathcal{N}^L \subset \mathcal{L}_+^\uparrow$ of the identity, the corresponding transformed cones $K_\Lambda$ all lie in some sufficiently large cone $K_0 \subset B$. The hypercone $C_0 = C(K_0)$ therefore has the required property.

We now study the action of time translations on hypercones. The translated hypercones are no longer based on any of the hyperboloids foliating $V$, but they are still contained in sufficiently large hypercones based on the given $H$.

A.13. Let $C$ be a hypercone and let $B^T \subset V_+$ be a bounded set of translations. Then there is a hypercone $C_0$ with $C + B^T \subset C_0$.

Proof. We first suppose $C$ to be the causal completion of a special type of hyperbolic cone $C \subset H$ made up of hyperbolic rays given, choosing coordinates suitably, by $u \mapsto a(u) = (u\tau + v(u), v(u)l)$, $0 < u \leq 1$, with $v(u) = \tau(1 - u^2)/2u$ and $l \in L$, where $L \subset S^2$ is any convex set whose closure is contained in a hemisphere. The opposite cone is made up of the rays $u' \mapsto a'(u') = (u'\tau + v(u'), v(u')l')$, $0 < u' \leq 1$, with $l' \in -L$. Let $t = (t, 0)$, $t \geq 0$, be a time translation. A straightforward computation gives

$$(a(u) + t - a'(u'))^2 = t^2 + 2\tau^2 + 2t(u\tau + v(u)) - 2(t + u\tau + v(u))(u'\tau + v(u')) + 2v(u)v(u')l'.
$$

Thus, if $(u'\tau + v(u')) > (\tau + t)$, the points $(a(u) + t)$ and $a'(u')$ are spacelike separated, $(a(u) + t - a'(u'))^2 < 0$, for any $0 < u \leq 1$, $l \in L$ and $l' \in -L$; note that $ll' < 0$.  

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Phrased differently, the causal shadow of $C + t$ on $H$ is disjoint from all spacetime points in the cone opposite to $C$ for times larger than $(\tau + t)$. Proceeding to the Beltrami–Klein model this implies that the image of this shadow in $B$ is contained in the interior of a spherical cap of $B$ with non-trivial complement. As in the proof of \(A.9\), the shadow fits into a cone $K_0 \subset B$, hence $C + t \subset C(K_0) = C_0$.

Now let $C$ be any hypercone. A suitable Lorentz transformation $\Lambda$ shifts its apex to the point $(\tau, 0)$ and $\Lambda C$ is then of the special type considered in the preceding step. The given set of translations is mapped to $\Lambda B^T$. Since $B^T$ is bounded, there is a time translation $t$ as above with $\Lambda B^T \subset (t - \overline{V}_+) \cap \overline{V}_+$. Since the points $C + (t - \overline{V}_+) \cap \overline{V}_+$ on the hyperboloid $H$ have the same causal shadow as $C + t$, the hypercone $C_0$ constructed in the preceding step satisfies $\Lambda(C + B^T) \subset C_0$. Hence $\Lambda^{-1}C_0$ is as required. \(\square\)

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