NOTES ON THE GEOGRAPHY OF THE PLANE AT INFINITY

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Abstract. This note is a kind of sequel to [19]: the questions behind it have their start in early work of Kapranov on Drinfel’d’s asymptotic zones for the KZ equations, and they grew under the influence of Consani and Marcolli’s insight that completely decomposed curves are natural basepoints in Arakelov geometry. It has benefitted from conversations with J. Baez, S. Devadoss, M. Davis, P. Etingof, A. Henriques, T. Januszkiewicz, Y. Manin, J. Stasheff, D. Stevenson, and others, but it remains incomplete and inconclusive. I have compiled it as evidence of a rich and mysterious structure, some kind of fundamental groupoid at infinity on the projective line, whose existence is coming slowly into focus.

1. Variations on the fundamental group

Given a nice space $A$ with a finite subset $B$ of base points, the classical fundamental groupoid $\pi (A \text{ rel } B)$ is the category with elements of $B$ as objects, and equivalence classes of paths, e.g. from $v \in B$ to $v' \in B$, as morphisms between them.

1.1 If $A \subset X$ is contained in some reasonable larger space, this construction can be elaborated to define a two-groupoid (ie, a two-category in which all morphisms and two-morphisms are invertible). If for simplicity $B$ consists of a single basepoint, and $A$ is connected, then $\Pi (X \text{ rel } A)$ will have one object, with $\pi_1(A, b)$ as its self-maps. The boundary homomorphism

$$\partial : \pi_2(X, A, b) \to \pi_1(A, b)$$

in the homotopy exact sequence of a pair defines a two-category with

$$\{\phi \in \pi_2(X, A, b) \mid \partial(\phi) = \sigma_1^{-1} \cdot \sigma_0 \in \pi_1(A, b)\}$$

as two-morphisms between $\sigma_0, \sigma_1 \in \pi_1(A, b)$. In this two-category, paths in $A$ are the morphisms, and the two-morphisms are ‘bubbles’ in

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X bounding these paths; to check the composition axioms it is probably easiest to think of \( \pi_n(X, A, b) \) as defined by maps of the \( n \)-cube, relative to its boundary, to \((X, A)\). This definition continues to make sense when \( B \) has more than one element.

1.2 The case in which \( A = V(\mathbb{R}) \) and \( X = V(\mathbb{C}) \) are the real and complex points of an algebraic variety seems particularly interesting: real bubbles (i.e. morphisms \( \beta : \mathbb{P}_1 \to V \) defined over \( \mathbb{R} \)) have invariants in the second relative homotopy group. The Galois group of \( \mathbb{C} \) over \( \mathbb{R} \) defines an involution on \( V(\mathbb{C}) \) with \( V(\mathbb{R}) \) as fixed points; restricting a map

\[
(\mathbb{P}_1(\mathbb{C}), \mathbb{P}_1(\mathbb{R})) \to (V(\mathbb{C}), V(\mathbb{R}))
\]

to the lower hemisphere of \( \mathbb{P}_1(\mathbb{C}) \) defines an element

\[
\deg(\beta) \in \pi_0 \text{Maps}(D^2, \partial D^2; V(\mathbb{C}), V(\mathbb{R})) = \pi_2(V(\mathbb{C}), V(\mathbb{R}))
\]

based at 0 \in \mathbb{P}_1. This group fits in an exact sequence

\[
\cdots \to \pi_2 V(\mathbb{C}) \to \pi_2(V(\mathbb{C}), V(\mathbb{R})) \to \pi_1 V(\mathbb{R}) \to \cdots ;
\]

complex conjugation \( \beta \mapsto \beta^c \) in the domain acts trivially in the group on the right, and by reversing signs on the left. The class \( \deg(\beta - \beta^c) \) lifts to \( \pi_2 V(\mathbb{C}) \), where it represents the class of \( \beta_\ast [\mathbb{P}_1(\mathbb{C})] \).

One of the purposes of this note is to propose

\[
\Pi (V(\mathbb{C}) \text{ rel } V(\mathbb{R}) \text{ rel } F) := Q\pi_1(V \text{ rel } F)
\]

as an ad hoc kind of quantum fundamental (two-)group(oid) of \( V \) (with respect to some suitable collection \( F \subset V(\mathbb{R}) \) of basepoints).

Projective space is an interesting example: when \( n > 1 \), \( \Pi (\mathbb{P}_n, \infty) \) has one object, one morphism, and \( \mathbb{Z} \) as its two-morphisms; but \( \Pi (\mathbb{P}_1, \infty) \) has one object, \( \mathbb{Z} \) as its morphisms, and \( 2\mathbb{Z} \) as two-morphisms. In both cases the Galois group of \( \mathbb{C} \) over \( \mathbb{R} \) acts as \( \pm 1 \) on the two-morphisms, trivially on the morphisms.

When \( V \) is defined over \( \mathbb{Q} \) or \( \mathbb{Z} \), the set \( V(\mathbb{Q}) \) presents itself as a potential supply of basepoints for this construction; in the case of \( \mathbb{P}_1 \), the points 0, 1 and \( \infty \) stand out. There is an algebraic version of the fundamental groupoid, which involves the specification of suitable tangent vectors at the base points, cf. eg. [14], but I will neglect that refinement here. I suspect that related higher invariants of this sort can be defined using the quaternionic, and perhaps octonionic, points of a variety defined over \( \mathbb{Q} \), but I have not pursued that either (yet).
2. Genus zero moduli spaces

2.1 One way to explore the complex line at infinity is to send out a finite collection of points as probes [18]:

The moduli spaces of stable genus zero curves, marked with suitably many distinct ordered smooth points, have been extensively studied; they form a (slightly incomplete, in that \( n \geq 2 \)) operad \( \{\mathcal{M}_{0,n+1}\} \) of smooth algebraic varieties over \( \mathbb{Z} \). Algebras over the operad by the homology of its points (enlarged slightly to accommodate the Novikov ring) are fundamental to the study of quantum cohomology: these are the polycommutative algebras of Manin [8, 13]. On the other hand, the spaces \( \{\mathcal{M}_{0,n+1}(\mathbb{R})\} \) are acyclic, and their fundamental groups define an operad in discrete groups [5, 6]. Recently, Etingof et al [7] have identified algebras over \( \{H_\ast(\mathcal{M}_{0,n+1}(\mathbb{R}), \mathbb{Q})\} \) as 2-Gerstenhaber algebras: these are commutative algebras endowed with a suitably compatible structure as a 2-Lie algebra (in the sense of Hanlon and Wachs. Such a structure can be interpreted as a restricted kind of \( L_\infty \)-algebra (carrying a triple bracket satisfying a generalized Jacobi identity), or [2 §2.3 Cor. 17] as a trivial Lie algebra together with an additional gerbe-looking 3-cohomology class.) One thus expects the existence of such a triple bracket on \( H^\ast(V(\mathbb{R}), \mathbb{Q}[\pi_2 V(\mathbb{C})]) \).

Note that the Hodge cohomology of an algebraic variety defined over the real is more than just a Galois representation; it is naturally a representation of the Weil group of \( \mathbb{C}/\mathbb{R} \) [21 §4.4].

All these operads are in fact cyclic: the action of the symmetric group \( \Sigma_n \) on \( \mathcal{M}_{0,n+1} \) extends naturally to an action of \( \Sigma_{n+1} \). My impression is that this cyclic structure, on the real points, is still not completely understood.

2.2 The real genus zero moduli spaces have a natural tessellation by Stasheff polyhedra, and thus a canonical CW-decomposition. There is a set \( B_n \) of \( (2n-3)!! \) distinguished zero-cells in \( \mathcal{M}_{n+1}(\mathbb{R}) \), corresponding to configurations whose dual graphs are rooted binary planar trees with \( n \) labeled leaves, modulo a certain equivalence relation which allows twigs with precisely two leaves to rotate [6, 19]. I am indebted to John Baez [1] for pointing out that this collection of zero-cells is itself an operad, whose algebras are commutative magmas (sets with a
commutative but not necessarily associative binary product) and that its generating function

\[ |B|(T) = \sum |B_n| \frac{T^n}{n!} = 1 - (1 - 2T)^{1/2} \]

satisfies an equation

\[ |B|(T) = T + \frac{1}{2}|B|(T)^2 \]

which asserts that if an algebra over this operad is not a one-element set, then it is the union of an unordered pair of sets, each with such a structure.

From another point of view such binary trees define ‘totally decomposed’ configurations of genus zero marked curves, with three points on each irreducible component; thus when \( n = 3 \) we recover configurations in \( \overline{M}_{0,4} \) corresponding to the three canonical points \( 0, 1, \infty \) on \( \mathbb{P}_1 \). In fact in general such maximally degenerate configurations are all defined over \( \mathbb{Q} \), and indeed over \( \mathbb{Z} \); they define a natural ‘operad of basepoints’ for \( \{ \overline{M}_{*+1}(\mathbb{Z}) \} \).

2.3 Applying the constructions of the section above defines a (cyclic) operad

\[ \{ \Pi (\overline{M}_*(\mathbb{C}) \text{ rel } \overline{M}_*(\mathbb{R}) \text{ rel } B_*) \} \]

in two-categories. Because \( \overline{M}_{0,n}(\mathbb{R}) \) has trivial higher homotopy groups, and because \( \overline{M}_{0,n}(\mathbb{C}) \) has vanishing odd cohomology, there is an exact sequence

\[ 0 \to \pi_2(\overline{M}_{0,n}, \mathbb{Z}) \to \pi_2(\overline{M}_{0,n}(\mathbb{C}), \overline{M}_{0,n}(\mathbb{R})) \to \pi_1(\overline{M}_{0,n}(\mathbb{R})) \to 1 \]

(with trivial action of the quotient group, as far as I can see). This two-category then has Baez’s operad \( B_* \) of commutative magmas (or, more precisely, its cyclic enrichment) as objects; automorphisms of an object correspond to elements of the pure cactus group [identified abstractly in [5] and in terms of generators and relations in [10 §3.4 ]], while the two-endomorphisms of a morphism correspond to elements of \( \pi_2 \cong H_2(\overline{M}_{0,n}(\mathbb{C}), \mathbb{Z}) \). According to Keel, this group is free abelian, with generators corresponding to unordered partitions of \( n \) into two sets, both of cardinality at least two, ie to trees with exactly two internal nodes.
3. Coda: strings of pearls

3.1 A genus zero curve with only two marked points has too many automorphisms to be stable, so the operad of DKM moduli spaces lacks a good ‘unary’ operation. More generally, Manin and Losev study stacks

\[ \tilde{L}_{g,m} = \coprod_{n \geq 1} \mathcal{T}_{g,m+1,n} \]

of genus \( g \) curves marked with \( m+1 \) marked smooth points, required to be distinct, and \( n \) further auxiliary marked smooth points, which can collide with one another; they show that the resulting moduli objects admit the clutching morphisms needed for the structure of a complete operad. In particular, the monoid \( \tilde{L}_{0,1} \) (which now has countably many components) provides a replacement for the nonexistent \( \overline{M}_{0,2} \).

The configurations parametrized by this new space might be visualized as chains of bubbles, exactly as seen in physicists’ experiments, or as strings of pearls: sequences of \( \mathbb{P}_1 \)’s laid end to end, with ‘heavy’ marked points (corresponding to 0 and \( \infty \)) anchoring the ends of the chain; intermediate nodes are marked by the ephemeral points of the second class. Just as the usual genus zero operad captures the WDVV equations for associativity, the resulting extended modular operad accounts for certain commutativity equations arising in physics [3].

The operads \( \tilde{L}_{0,*} \) and their cohomology have been extensively studied [15 §3.3.1, 16 §4.6.1]; they, and their algebras, are relatively well-understood. In particular, the components of \( \tilde{L}_{0,2,*} \) are toric varieties, with permutohedra as associated quotients, and the cohomology of \( \tilde{L}_{0,2,n} \) has a combinatorial interpretation related to the Stanley-Reisner ring of (the barycentric subdivision of) the \( n \)-cube [20 §4.3]. The real points \( \tilde{L}_{0,1} (\mathbb{R}) \) are again aspherical [11] but as far as I know their fundamental groups have yet to be calculated explicitly.

3.2 The construction of moduli stacks \( \tilde{L}_{*,*}(V) \) of stable maps from such configurations to some reasonable class of (algebraic, or more generally, symplectic) varieties, seems to be within reach [17]; another objective of this note is to draw attention to the interest of these objects.

\( \tilde{L}_{0,1}(V) \), in particular, is a space over \( V \times V \): it is an algebraic analog of the classical space of (unrestricted) paths in \( V \), and one expects the existence of glueing morphisms making it into a monoid in such a category, with respect to the concatenation of bubble chains; but (just
as in the classical case) the relevant monoidal structure on spaces over $V \times V$ is not symmetric.

If we fix a basepoint $\infty \in V$, then the fiber $\tilde{L}_{0,1}(V \text{ rel } \infty)$ of $\tilde{L}_{0,1}(V)$ over $(\infty, \infty) \in V \times V$ is the analog of the classical space of based loops in $V$, which is an algebra over the $A_\infty$ operad. [The space of unbased loops is similarly an algebra over this operad, in the category of spaces over $V$, with its natural symmetric monoidal structure]. This construction is a good candidate for an algebra over $\{\tilde{L}_{0,*}\}$. If $V$ is defined over $\mathbb{Q}$ or $\mathbb{R}$, this Manin - Losev loopspace will be as well, and its (rational) homology can be expected to be a $W(\mathbb{C}/\mathbb{R})$-equivariant $\{H_*(\tilde{L}_{0,*})\}$ algebra. This entails more than the existence of a nice action of the Galois group on the homology: in particular it means that the various fixed point sets possess compatible operad actions.

Manin and Losev [15 §3.3.1] describe such algebras in an essentially Tannakian language, so we may imagine thinking of them as representations of some motivic pro-algebraic group. The remarks above suggest that it may be natural to enlarge this group slightly, to include $\mathbb{Q}\pi_1(\tilde{L}_{0,1})$ as something like its group of components.
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