Nonextensive statistics of relativistic ideal gas

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Abstract

We obtain the specific heat in the third constraint scenario for a canonical ensemble of a nonextensive extreme relativistic ideal gas in a closed form. The canonical ensemble of $N$ particles in $D$ dimensions is well-defined for the choice of the deformation parameter in the range $0 < q < 1 + \frac{1}{DN}$. For a relativistic ideal gas with particles of arbitrary mass a perturbative scheme in the nonextensivity parameter $(1 - q)$ is developed by employing an infinite product expansion of the $q$-exponential, and a direct transformation of the internal energy from the second to the third constraint picture. All thermodynamic quantities may be uniformly evaluated to any desired perturbative order.

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I Introduction

Tsallis [1] proposed nonextensive statistical mechanics by generalizing the functional form of the Boltzmann-Gibbs entropy as

\[ S_q = k \left[ \frac{W^{1-q} - 1}{1-q} \right] \equiv k \ln_q W, \quad q \in \mathbb{R}^+, \quad (1.1) \]

where the deformation parameter \( q \) is taken to be a real positive number as this ensures the stability of the Tsallis entropy. In (1.1) the quantity \( k \) is the generalized Boltzmann constant, and \( W \) denotes the weight. The entropy (1.1) satisfy a nonlinear, inhomogeneous relation

\[ S_q(A + B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B), \quad (1.2) \]

where \( A \) and \( B \) refer to statistically independent systems. The nonextensivity of the entropy manifest in (1.2) is governed by the parameter \((1-q)\). The Boltzmann-Gibbs statistics is recovered in the \( q \to 1 \) limit.

The nonextensive statistical mechanics has found wide-ranging applications in studies of the systems exhibiting long range interactions [3], long time microscopic memory effects [4], anomalous diffusion [5], nonequilibrium phenomena [6] and so on. For instance, the formation of a new hadronic state of matter known as the quark-gluon plasma that occurs in the early stage of the relativistic hadronic collisions exemplifies long range interactions as well as long time memory effects [7], and, consequently, the nonextensive statistical mechanics is expected to be more appropriate there than the classical Boltzmann-Gibbs statistics. The rapidity spectrum obtained by using the Tsallis distribution is found [8] to be in good agreement with the experimental data. The data on the distribution of transverse momentum of hadrons as well as the differential cross sections in high energy \( e^+ e^- \) collisions bear close resemblance with the theoretical analysis [9, 10] based on the Tsallis nonextensive statistical mechanics. In the context of many body systems endowed with self-gravitating long range interactions it has been observed [11] that the power law distributions may be achieved using a \( q \)-kinetic theory based on the Tsallis statistics. As a groundwork for complete theoretical understanding of the relativistic heavy ion collisions in the high energy physics regime, and also for possible applications in astrophysics, it is imperative to study the relativistic ideal gas in the context of nonextensive statistical mechanics. This investigation has been initiated in [12]. These authors observe that the grand canonical partition function exhibits an essential singularity for \( q > 1 \) region, and, consequently, they claim that, in the said region, the nonextensive relativistic ideal gas does not exist. Making a slight departure from the arguments in [12], we, in the present work, consider a canonical ensemble of a fixed number of \( N \) molecules of mass \( m \) of a relativistic ideal gas subject to the Tsallis statistics. We produce an exact evaluation of the canonical specific heat in the extreme relativistic case \((m \to 0)\). In \( D \) dimensions the generalized partition function is nonsingular in the region \( 0 < q < 1 + \frac{1}{DN} \). For an arbitrary mass \( m \) we obtain the generalized partition function [13], and the thermodynamic quantities in the second and the third constraint pictures as perturbative series in the nonextensivity parameter \((1-q)\). Towards this purpose, we, as a calculational tool, disentangle the \( q \)-exponential (2.4) employing a technique developed in [14], and previously used [15] to obtain thermodynamic quantities of a nonrelativistic ideal gas obeying
the Tsallis statistics. In addition, we employ a direct transformation linking the internal 
energies in the second and the third constraint pictures that allows us to evaluate the 
thermodynamic quantities uniformly to any arbitrary prescribed perturbative order in 
\((1 - q)\). For the sake of simplicity we produce them till the order \((1 - q)^2\).

The plan of this article is as follows. The extreme or ultra relativistic gas is discussed in 
Sec. II. This is followed by our perturbative evaluation of the thermodynamic quantities 
for the general case of a relativisitic ideal gas in Sec. III. We conclude in Sec. IV.

II Ultra relativistic ideal gas

The Hamiltonian of a relativistic ideal gas with particles possessing \(D\)-dimensional mo-
menta \(p_i (i = 1, \cdots, N)\) reads

\[
H(p) = \sum_{i=1}^{N} mc^2 \left( \sqrt{1 + \left( \frac{p_i}{mc} \right)^2} - 1 \right), \quad p_i = |p_i|, \quad (2.1)
\]

where \(c\) is the velocity of light. In the extreme relativistic case \(m \ll p_i\) the Hamiltonian 
\((2.1)\) reduces to

\[
H(p) = c \sum_{i=1}^{N} p_i. \quad (2.2)
\]

The generalized partition function in the third constraint [13] approach is given by

\[
\overline{Z}_q^{(3)}(\beta, V, N) = \frac{1}{N!} \frac{1}{h^{D_N}} \int d^{D_N} x \exp_q \left( -\beta H(p) - U_q^{(3)} \right), \quad \beta = \frac{1}{kT}, \quad (2.3)
\]

where \(h\) refers to the elementary cell of the one-dimensional phase space, and the deformed 
\(q\)-exponential is the inverse of the \(q\)-logarithm introduced in (1.1):

\[
\exp_q(z) = (1 + (1 - q)z)^{\frac{1}{1-q}}. \quad (2.4)
\]

The series expansion for the deformed exponential reads [16]

\[
\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad [n]_q = \frac{n}{1 - (1-q)(n-1)}. \quad (2.5)
\]

We briefly remark here that the following \(q\)-derivative

\[
D_q(x) = \frac{1}{(1 - (1-q)(x \frac{d}{dx}))} \frac{d}{dx} \quad (2.6)
\]

acts on the monomials \(x^n\) to produce the \(q\)-number \([n]_q\) defined in (2.5):

\[
D_q(x) x^n = [n]_q x^{n-1}. \quad (2.7)
\]
Consequently, the $q$-exponentials represented by the infinite series (2.5) are the eigenfunctions of the $q$-derivative introduced in (2.6):

$$D_q(x) \exp_q(\alpha x) = \alpha \exp_q(\alpha x).$$

(2.8)

Incidentally, other $q$-derivatives were introduced in Ref. [17].

The ensemble probability for an arbitrary energy $E_j$ in the third constraint framework

$$p_j^{(3)}(\beta, V, N) = \frac{1}{Z_q^{(3)}} \exp_q \left( -\beta \frac{E_j - U_q^{(3)}}{c^{(3)}} \right)$$

leads [13] to the following sum of the $q$-weights, referred to in (2.3):

$$c^{(3)} = \sum_j (p_j^{(3)}(\beta, V, N))^q = \left( \frac{Z_q^{(3)}}{\beta} \right)^{1-q}. \quad (2.10)$$

For the extreme relativistic Hamiltonian (2.2) the generalized partition function (2.3) reads

$$Z_q^{(3)}(\beta, V, N) = \mathcal{G} Z_{m=0}(\beta, V, N) \left( 1 + (1 - q) \frac{\beta}{c^{(3)}} U_q^{(3)} \right)^{\frac{1}{1-q} + D\pi} (c^{(3)})^{D\pi}, \quad (2.11)$$

where $Z_{m=0}(\beta, V, N)$ is the classical partition function for the ultra relativistic gas in arbitrary dimension $D$

$$Z_{m=0}(\beta, V, N) = \mathcal{W} \beta^{-D\pi}, \quad (2.12)$$

and the parameters $\mathcal{W}$ and $\mathcal{G}$ are given by

$$\mathcal{W} = \frac{1}{N!} \left( \frac{2 V \pi^2 \Gamma(D)}{(e h)^D \Gamma(D/2)} \right)^N, \quad \mathcal{G} = \frac{\Gamma\left(\frac{1}{1-q} + 1\right)}{(1-q)^{D\pi} \Gamma\left(\frac{1}{1-q} + D\pi + 1\right)}.$$

The generalized partition function (2.11) has simple poles for the values of the deformation parameter $q = 1 + \frac{1}{n}, \ n = 1, ..., D\pi$. The number of singularities equals the number of degrees of freedom of the system. The generalized partition function is, therefore, well defined in the interval $0 < q < 1 + \frac{1}{D\pi}$. As the number of particles $N$ increases, we observe that the poles accumulate towards $q = 1$, the limiting value where statistical mechanics becomes extensive. The internal energy is defined [13] via the escort probability as

$$U_q^{(3)}(\beta, V, N) = \left( c^{(3)} \right)^{-1} \sum_j (p_j^{(3)}(\beta))^q E_j. \quad (2.13)$$

Converting the above sum to the phase space integration à la (2.3) and employing (2.10, 2.11) we now get

$$U_q^{(3)}(\beta, V, N) = D\pi \frac{c^{(3)}}{\beta}. \quad (2.14)$$

Applications of the relations (2.10, 2.13) produce the explicit solution for the sum of the $q$-weights:

$$c^{(3)} = \left( \mathcal{W} \mathcal{G} \right)^{\frac{1-q}{1-(1-q)D\pi}} \left( 1 + (1 - q)D\pi \right)^{\frac{1+(1-q)D\pi}{1-(1-q)D\pi}} \beta^{-\frac{(1-q)D\pi}{1-(1-q)D\pi}}. \quad (2.15)$$
Above equation in conjunction with (2.14) now produces the internal energy as

$$U_q^{(3)}(\beta, V, N) = DN \left(\mathcal{W} G\right)^{\frac{1-q}{(1-q)DN}} \left(1 + (1 - q)DN\right)^{\frac{1+(1-q)DN}{(1-q)DN}} \beta^{-\frac{1}{(1-q)DN}}. \quad (2.16)$$

The specific heat defined as

$$C_q^{(3)} \equiv \frac{\partial U_q^{(3)}}{\partial T} \quad (2.17)$$

reads

$$C_q^{(3)} = \frac{1}{DNk} = \frac{1}{1 - (1 - q)DN} \left(\mathcal{W} G\right)^{\frac{1-q}{(1-q)DN}} \left(1 + (1 - q)DN\right)^{\frac{1+(1-q)DN}{(1-q)DN}} \beta^{-\frac{1}{(1-q)DN}}. \quad (2.18)$$

The extensive limit of the specific heat is readily obtained: \( C_q^{(3)} \bigg|_{q=1} = DNk \). The behaviour of the specific heat with respect to the dimensionless scaled temperature

$$t = \left(\frac{V^{1/D}}{ch} kT\right)^{\frac{(1-q)DN}{1-(1-q)DN}}, \quad (2.19)$$

for various values of \( N \) and \( q \) values are shown in the Figs. (1) and (2), respectively. Following [20] we now obtain the specific heat at the physically important thermodynamic limit \( N \to \infty, V \to \infty, \rho = \frac{N}{V} \to \text{finite}: \)

$$C_q^{(3)} \to -\frac{ch}{\sqrt{\pi} \left(1 - q\right)} \exp \left(- \left(1 + \frac{1}{D}\right) \right) \left(\frac{\Gamma\left(\frac{D}{2}\right)}{2\Gamma(D)}\right)^{\frac{1}{D}} \frac{\rho^{\frac{1}{2}}}{T}, \quad (2.20)$$

Figure 1: Temperature dependence of specific heat for fixed \( q = 0.95 \) and various \( N \).
which implies that the classical $q \rightarrow 1$ limit and the thermodynamic limit do not commute. As remarked in [20] the $N$-independent negative specific heat realized in (2.20) for the extreme relativistic perfect gas may be of consequence in astrophysical problems.

We have also calculated the specific heat in the third constraint picture using a detour via the second constraint framework. The results obtained through the direct evaluation of the generalized partition function and the internal energy transformation method are identical. Further discussions will appear in Sec. III.

### III Relativistic ideal gas: molecules of arbitrary mass

The relativistic ideal gas containing particles of arbitrary mass $m$, and described by the Hamiltonian (2.1) has possible applications in heavy ion collisions [7-10] in nuclear physics. Moreover, in the context of self-gravitating systems in astrophysics they are relevant [11]. In the opposite limits $m \gg p_i$ and $m \ll p_i$ the general relativistic perfect gas reduces to the nonrelativistic case and the extreme relativistic case, respectively. For the relativistic ideal gas we obtain the thermodynamic quantities perturbatively by treating $(1 - q)$ as the series parameter. In the present section we employ the second constraint approach [13] as an intermediate step for evaluating the thermodynamic quantities. The physical variables obtained in the second constraint method are transformed [13] to the respective quantities in the third constraint framework by introducing a fictitious temperature $\beta'$ that provides for the correspondence between the alternate constraints. The ensemble
probability and the partition function in the second constraint \[18\] approach read
\[
\begin{align*}
\mathbf{p}_j^{(2)}(\beta, V, N) &= \frac{\exp_q(-\beta E_j)}{Z_q^{(2)}}, \\
Z_q^{(2)} &= \sum_j \exp_q(-\beta E_j).
\end{align*}
\] (3.1)

The above probability is linked \[13\] to that of the third constraint approach as
\[
\mathbf{p}_j^{(3)}(\beta) = \mathbf{p}_j^{(2)}(\beta'),
\] (3.2)
where the general transformation rule for the temperature reads
\[
\beta = \beta' \frac{c^{(2)}(\beta')}{1 - (1 - q)\beta' \frac{U_q^{(2)}(\beta')}{c^{(2)}(\beta')}}, \quad c^{(2)}(\beta) = \sum_j \left(\mathbf{p}_j^{(2)}(\beta)\right)^q.
\] (3.3)

In the above equation the internal energy in the second constraint approach is defined \[18\] as
\[
U_q^{(2)}(\beta) = \sum_j \left(\mathbf{p}_j^{(2)}(\beta)\right)^q E_j.
\] (3.4)

Converting the sum over states in (3.1) to an integration over the phase space for the Hamiltonian (2.1) the partition function in the second constraint approach may be recast as
\[
Z_q^{(2)}(\beta, V, N) = \frac{1}{N!} \frac{1}{h^{DN}} \int d^D N x d^D N p \ exp_q(-\beta H(p)).
\] (3.5)

The integral over the phase-space in (3.5) may be performed exactly in the extreme relativistic case in the \(m \to 0\) limit:
\[
Z_q^{(2)}(\beta, V, N) \bigg|_{m=0} = Z_{m=0}(\beta, V, N) \left(\frac{\Gamma\left(\frac{2-q}{1-q}\right)}{(1-q)\Gamma\left(\frac{2-q}{1-q} + DN\right)}\right),
\] (3.6)

where the classical partition function \(Z_{m=0}(\beta, V, N)\) for the extreme relativistic perfect gas is given in (2.12). In the general case with particles of arbitrary mass we proceed towards evaluating the integral (3.5) perturbatively. Assuming the partition function (3.5) to be well behaved in the neighborhood of \(q = 1\), we follow [13,15] to disentangle the \(q\)-exponential (2.4) as an infinite product series of ordinary exponentials:
\[
\exp_q(-\beta H(p)) = \exp\left(-\sum_{n=1}^{\infty} \frac{(1-q)^{n-1}}{n} \beta^n H^n(p)\right) \equiv \hat{D}(d(\beta)) \ exp(-\beta H(p)).
\] (3.7)

In (3.7) the operator valued series \(\hat{D}(d(\beta))\) reads
\[
\hat{D}(d(\beta)) = 1 - \frac{(1-q)}{2} d^{(2)}(\beta) + \frac{(1-q)^2}{3} \left(d^{(3)}(\beta) + \frac{3}{8} d^{(4)}(\beta)\right) + \ldots,
\] (3.8)

where \(d^{(n)}(\beta) = \beta^n \frac{\partial^n}{\partial \beta^n}\). The operator (3.8) links the partition function (3.5) in the second constraint approach with the classical Boltzmann-Gibbs partition function \(Z(\beta, V, N)\) as
\[
Z_q^{(2)}(\beta, V, N) = \hat{D}(d(\beta)) Z(\beta, V, N), \quad Z(\beta, V, N) = \frac{1}{N!}(Z(\beta, V, 1))^N.
\] (3.9)
Henceforth, for the purpose of simplicity, we consider $D = 3$. The phase-space integral of the classical partition function for a single particle reads

$$Z(\beta, V, 1) = \frac{1}{h^3} \int d^3x \, d^3p \exp \left( -\beta mc^2 \left( \sqrt{1 + \left( \frac{p}{mc} \right)^2} - 1 \right) \right), \quad (3.10)$$

and may be expressed \[19\] via the modified Bessel function of the second kind $K_n(z)$, $n = 2$:

$$Z(\beta, V, 1) = \frac{4 \pi V \exp(u)}{\lambda^3} \frac{K_2(u)}{u}, \quad \lambda = \frac{h}{mc}, \quad u = \beta mc^2. \quad (3.11)$$

For future use we express $Z(\beta, V, N)$ in a factorized form:

$$Z(\beta, V, N) = g(V) \, f(u), \quad g(V) = \frac{(4 \pi V)^N}{\lambda^3}, \quad f(u) = \frac{1}{N!} \left( \frac{\exp(u)}{u} \right)^N. \quad (3.12)$$

The recipe \((3.9, 3.11)\) now produces the partition function in the second constraint approach as an infinite perturbative series in the nonextensivity parameter $(1 - q)$:

$$Z_q^{(2)}(\beta, V, N) = Z(\beta, V, N) \left( \sum_{n=0}^{\infty} (-1)^n (1 - q)^n \sum_{\ell=0}^{2n} \alpha_{n\ell}(u) (K(u))^\ell \right), \quad (3.13)$$

where we have defined

$$K(u) \equiv u \frac{K_1(u)}{K_2(u)}. \quad (3.14)$$

The coefficients $\alpha_{n\ell}(u)$ for the first few orders are listed below:

$$\begin{align*}
\alpha_{00}(u) &= 1,
\alpha_{10}(u) &= \frac{3}{2}(1 + 3N)N - 3N^2u + \frac{1}{2}(1 + N)Nu^2,
\alpha_{11}(u) &= -\frac{3}{2}(1 - 2N)N - N^2u,
\alpha_{12}(u) &= -\frac{1}{2}(1 - N)N,
\alpha_{20}(u) &= \frac{1}{8}(2 + 27N + 90N^2 + 81N^3)N - \frac{9}{2}(1 + 3N)N^3u \\
&\quad + \frac{1}{8}(11 - 42N + 48N^2 + 54N^3)Nu^2 + \frac{1}{6}(15 - 25N - 9N^2)N^2u^3 \\
&\quad - \frac{1}{2}(2 - 3N - 6N^2 - N^3)Nu^4,
\alpha_{21}(u) &= -\frac{1}{4}(11 - 45N + 63N^2 - 54N^3)N - \frac{3}{2}(4 - 10N + 9N^2)N^2u \\
&\quad + \frac{1}{12}(44 - 93N + 15N^2 + 54N^3)Nu^2 + \frac{1}{2}(2 - 3N - N^2)N^2u^3,
\alpha_{22}(u) &= \frac{3}{8}(25 - 63N + 56N^2 - 18N^3)N - \frac{1}{2}(11 - 20N + 9N^2)N^2u \\
&\quad + \frac{1}{4}(4 - 7N + 3N^3)Nu^2,
\alpha_{23}(u) &= -\frac{1}{12}(62 - 129N + 85N^2 - 18N^3)N - \frac{1}{2}(2 - 3N + N^2)N^2u, \\
\alpha_{24}(u) &= -\frac{1}{8}(6 - 11N + 6N^2 - N^3)N.
\end{align*} \quad (3.15)$$
In the second constraint framework the internal energy (3.4) may be recast as

$$U_q^{(2)}(u) = -mc^2 \frac{\partial}{\partial u} \ln_q Z_q^{(2)}(u).$$  \hspace{1cm} (3.16)

The expressions (3.13) and (3.15) may now be employed to produce the following series expansion for the internal energy $U_q^{(2)}(u)$:

$$\frac{U_q^{(2)}(u)}{NkT} = (Z(u, V, N))^{1-q} \left( \sum_{n=0}^{\infty} (1 - q)^n \sum_{\ell=0}^{2n+1} \rho_{n\ell}(u) (K(u))^\ell \right),$$  \hspace{1cm} (3.17)

where the first few coefficients read

$$\rho_{00}(u) = 3 - u, \quad \rho_{01}(u) = 1, \quad \rho_{10}(u) = -3Nu + \frac{1}{2}(5 - 4N)u^2 + Nu^3,$$

$$\rho_{11}(u) = 6(2N - 1) - 5Nu + (1 - N)u^2, \quad \rho_{12}(u) = \frac{14N - 11}{2} - Nu, \quad \rho_{13}(u) = N - 1,$$

$$\rho_{20}(u) = -\frac{9}{2}(1 + 3N)N + \frac{3}{2}(1 + 9N)Nu - \frac{1}{2}(11 - 48N + 48N^2)u^2$$
$$-\frac{1}{2}(26 - 41N)Nu^3 + \frac{1}{6}(28 - 48N - 9N^2)u^4 + (1 - N)Nu^5,$$

$$\rho_{21}(u) = \frac{1}{2}(22 - 102N + 81N^2) + \frac{1}{2}(57 - 102N)Nu - \frac{1}{4}(163 - 352N + 138N^2)u^2$$
$$-3(6 - 7N)Nu^3 + (2 - 3N)u^4,$$

$$\rho_{22}(u) = \frac{1}{4}(311 - 786N + 516N^2) + \frac{1}{2}(110 - 147N)Nu - \frac{1}{6}(175 - 312N + 90N^2)u^2$$
$$-4(1 - N)Nu^3,$$

$$\rho_{23}(u) = \frac{1}{4}(323 - 664N + 350N^2) + 3(8 - 9N)Nu - (5 - 8N + 2N^2)u^2,$$

$$\rho_{24}(u) = \frac{5}{2}(11 - 20N + 9N^2) + 3(1 - N)Nu, \quad \rho_{25}(u) = (3 - 5N + 2N^2).$$  \hspace{1cm} (3.18)

In constructing the transformation leading to the thermodynamic quantities pertaining to the third constraint approach we first express the weight factor in (3.3) in terms of an integral over the phase space, and subsequently use a perturbative approach à la (3.9):

$$c^{(2)}(\beta) = \frac{1}{(Z_q^{(2)}(\beta, V, N))^q} \hat{R}(d_\beta) Z(\beta, V, N),$$  \hspace{1cm} (3.19)

where the operator-valued series reads

$$\hat{R}(d_\beta) = 1 - (1 - q) \left( d^{(1)}_\beta + \frac{1}{2}d^{(2)}_\beta \right) + (1 - q)^2 \left( d^{(2)}_\beta + \frac{5}{6}d^{(3)}_\beta + \frac{1}{8}d^{(4)}_\beta \right) + \ldots.$$  \hspace{1cm} (3.20)

We follow (3.19) to compute the weight factor $c^{(2)}(\beta)$, and subsequently substitute it in the transformation equation (3.3). To eliminate a trivial kinematical dependence on the volume of the nonextensive system in our evaluation of its specific heat, we introduce an
appropriately scaled variable. Using the factorized form of the classical partition function (3.12) we define

\[ u = \frac{u}{(g(V))^{1-q}} \]  \hspace{1cm} (3.21)

and compute the inverse transformation in a series as

\[ u' = u \left( 1 + \sum_{n=1}^{\infty} (1-q)^n \sum_{\ell=0}^{2n-1} g_n \ell(u) \left( K(u) \right)^\ell \right), \]  \hspace{1cm} (3.22)

where the first few perturbative coefficients read

\[
\begin{align*}
g_{10}(u) &= -(6N + \ln f(u)) + 2Nu, \\
g_{11}(u) &= 2N, \\
g_{20}(u) &= \frac{3}{2} (1 + 9N + 2 \ln f(u))N + \frac{1}{2} (\ln f(u))^2 - 3(5N + \ln f(u))Nu \\
&\quad - \frac{1}{2} (9 + 5N - 4 \ln f(u))Nu^2 + 2N^2u^3, \\
g_{21}(u) &= \frac{3}{2} (7 + 22N + 6 \ln f(u))N - 13N^2u - 2(1+N)Nu^2, \\
g_{22}(u) &= \frac{1}{2} (21 + 31N + 4 \ln f(u))N - 2N^2u, \\
g_{23}(u) &= 2(1+N)N. \hspace{1cm} (3.23)
\end{align*}
\]

The internal energies in the second and the third constraint pictures may be directly interrelated. Employing the respective definitions (3.4) and (2.13) in conjunction with the ensemble probabilities (3.1) and (2.9) pertaining to these two pictures, the internal energy in the third constraint scenario may be expressed in terms of the fictitious temperature \( \beta' \) as

\[ U_q^{(3)}(\beta') = \frac{U_q^{(2)}(\beta')}{c^{(2)}(\beta')} \hspace{1cm} (3.24)\]

The above transformation method may be readily extended to any other thermodynamic average. The compendium of structures described in (3.17-3.24) now produce the internal energy of an arbitrary relativistic gas in the third constraint picture as a perturbative series in \((1-q)\)

\[ \frac{U_q^{(3)}(u)}{NkT} = \sum_{n=0}^{\infty} (1-q)^n \left( \sum_{\ell=0}^{2\ell+1} g_{n\ell}(u) \left( K(u) \right)^\ell \right). \]  \hspace{1cm} (3.25)

To derive the above expression for the internal energy we have employed rescaling of the argument given in Eq. (A.1) in the Appendix. The first few coefficients \( g_{n\ell} \) in (3.25) are
listed below:

\[
\begin{align*}
\varrho_{00}(u) &= 3 - u, & \varrho_{01}(u) &= 1, \\
\varrho_{10}(u) &= 3N(3 - u) + \frac{1}{2}(5 + 6N)u^2 - Nu^3 + (3 + u^2)\ln f(u), \\
\varrho_{11}(u) &= -6(1 + N) + 3Nu + (1 + N)u^2 - 3\ln f(u), \\
\varrho_{12}(u) &= -\frac{11}{2} - 6N + Nu - \ln f(u), & \varrho_{13}(u) &= -(1 + N), \\
\varrho_{20}(u) &= -\frac{9}{2}(1 - 9N)N - 18N^2u - \frac{1}{2}(11 + 57N - 3N^2)u^2 \\
&\quad + \frac{1}{2}(17 + 12N)Nu^3 + \frac{1}{6}(28 + 15N + 6N^2)u^4 - (1 + N)Nu^5 \\
&\quad + \left(3N(6 - u) - \frac{1}{2}(17 + 12N)u^2 + 4Nu^3 + (1 + N)u^4\right)\ln f(u) \\
&\quad + \frac{3}{2}(\ln f(u))^2 - 2u^2(\ln f(u))^2, \\
\varrho_{21}(u) &= 11 + 63N + \frac{9}{2}N^2 - 6(3 + 2N)Nu \\
&\quad - \frac{1}{4}(163 + 214N + 114N^2)u^2 + 16(1 + N)Nu^3 + (2 + N)u^4 \\
&\quad + \left(6(3 + 2N) - (16 + 9N)u - 16Nu^2 + 2Nu^3\right)\ln f(u) \\
&\quad + \frac{1}{2}(9 - 2u^2)(\ln f(u))^2, \\
\varrho_{22}(u) &= \frac{1}{4}(311 + 600N + 258N^2) - \frac{1}{2}(89 + 84N)Nu - \frac{1}{6}(175 + 162N + 90N^2)u^2 \\
&\quad + 4(1 + N)Nu^3 + \frac{1}{2}\left(89 + 84N - 20Nu - 8(1 + N)u^2\right)\ln f(u) \\
&\quad + 5(\ln f(u))^2, \\
\varrho_{23}(u) &= \frac{1}{4}(323 + 430N + 258N^2) - 22(1 + N)Nu - (5 + 4N + 2N^2)u^2 \\
&\quad + (22(1 + N) - 2Nu)\ln f(u) + (\ln f(u))^2, \\
\varrho_{24}(u) &= \frac{1}{2}(55 + 61N + 40N^2) - 3(1 + N)Nu + 3(1 + N)\ln f(u), \\
\varrho_{25}(u) &= 3 + 3N + 2N^2. \\
\end{align*}
\]

With the internal energy (3.25, 3.26) at hand, the definition of the specific heat (2.17) in the third constraint scenario may be employed in conjunction with the scaling equation (3.21) for obtaining the following perturbative series in the nonextensivity parameter \((1 - q)\):

\[
\frac{C_q^{(3)}(u)}{Nk} = C_0(u) + (1 - q) C_1(u) + (1 - q)^2 C_2(u) + \ldots, 
\]

(3.27)

where the extensive Boltzmann-Gibbs limit is given by \(C_0(u)\). We now enlist the first few
coefficients in the rhs of (3.27):

\[ C_0(u) = 3 + u^2 - 3K(u) - (K(u))^2, \]
\[ C_1(u) = 3N(6 - u) - \frac{1}{2}(17 + 12N)u^2 + 4Nu^3 + (1 + N)u^4 + (3 - 4u^2) \ln f(u) \]
\[ + \left(3 \left(6 + 4N + 3 \ln f(u)\right) - 9Nu - 2 \left(8 + 8N + \ln f(u)\right)u^2 + 2Nu^3\right)K(u) \]
\[ + \frac{1}{2} \left(89 + 84N + 20 \ln f(u)\right) - 10Nu - 4(1 + N)u^2 \right) (K(u))^2 \]
\[ + 2 \left(11(1 + N) + \ln f(u)\right) - Nu \right) (K(u))^3 + 3(1 + N)(K(u))^4, \]
\[ C_2(u) = -\frac{9}{2} (1 - 21N)N - 27N^2u + \frac{3}{2}(11 + 44N - 8N^2)u^2 - \frac{1}{2}(53 + 12N)Nu^3 \]
\[ - \frac{1}{4} (219 + 232N + 130N^2)u^4 + 19(1 + N)Nu^5 + (2 + N)u^6 \]
\[ + \frac{1}{2} \left(54N - 6Nu + (53 + 12N)u^2 - 26Nu^3 - 38(1 + N)u^4 + 4Nu^5\right) \ln f(u) \]
\[ + \frac{1}{2} \left(3 + 13u^2 - 2u^4\right) \left(\ln f(u)\right)^2 - \frac{1}{2} \left(66 + 270N - 81N^2 + 12(9 + N) \ln f(u) \right) \]
\[ + 27(\ln f(u))^2 \right) + 3(18 + 2N + 9 \ln f(u))Nu + \frac{1}{4} \left(1437 + 2044N + 906N^2 \right) \]
\[ + 4(169 + 154N) \ln f(u) + 60(\ln f(u))^2\right)u^2 - (169 + 154N + 30 \ln f(u))Nu^3 \]
\[ - \frac{1}{3} \left(217 + 180N + 93N^2 + 24(1 + N) \ln f(u)\right)u^4 + 8(1 + N)Nu^5 \right)K(u) \]
\[ - \frac{1}{4} \left(2221 + 1318 \ln f(u) + 158(\ln f(u))^2 + 2 \left(1923 + 534 \ln f(u) + 636N\right)N \right) \]
\[ - 2(659 + 534N + 158 \ln f(u))Nu - 2 \left(1091 + 236 \ln f(u) + 8(\ln f(u))^2 \right) \]
\[ + 4(1182 + 232 \ln f(u) + 678N)N\right)u^2 + 4(118 + 8 \ln f(u) + 116N)Nu^3 \]
\[ + 4(17 + 13N + 6N^2)u^4\right) (K(u))^2 - \frac{1}{4} \left(4175 + 5488N + 2922N^2 \right) \]
\[ +(1324 + 1240N) \ln f(u) + 84(\ln f(u))^2\right) - (331 + 310N + 42 \ln f(u))Nu \]
\[ - \frac{1}{3} \left(700 + 672N + 402N^2 + 60(1 + N) \ln f(u)\right)u^2 + 20(1 + N)Nu^3 \right) (K(u))^3 \]
\[ - \left(\frac{1}{4} \left(2619 + 2996N + 1850N^2 + 4(111 + 109N) \ln f(u) + 12(\ln f(u))^2 \right) \right) \]
\[ -(111 + 109N + 6 \ln f(u))Nu - (30 + 27N + 16N^2)u^2 \right) (K(u))^4 \]
\[ - \left(167 + 176N + 115N^2 + 12(1 + N) \ln f(u) - 12(1 + N)Nu \right) (K(u))^5 \]
\[ - 5(3 + 3N + 2N^2)(K(u))^6. \]
The dependence of the specific heat \((3.28)\) on the dimensionless variable \(u\) for numerous values of \(N\) and \(q\) are displayed in the Figs. (3) and (4), respectively.

Above perturbative evaluation of the specific heat hinges on the series \((3.25)\) for the internal energy in the third constraint picture, and the thermodynamic transformation property \((3.24)\) that is based on the equivalence of the ensemble probabilities \((3.2)\). An alternate determination of the specific heat utilizing the generalized partition function \((2.3)\) is well-known \[13\]. Integration of the following thermodynamic relation \[13\]

\[
 u \frac{\partial U_q^{(3)}}{\partial u} = \frac{\partial}{\partial u} \ln_q Z_q^{(3)}
\]  

(3.29)

allows one to determine the internal energy in the third constraint picture, which, in turn, produces the specific heat via the definition \((2.17)\). In the present case, an explicit analytical integration of the differential equation \((3.29)\) even as a perturbative series turns out to be difficult. The specific heat, however, may be directly extracted \[15\] from the generalized partition function as follows:

\[
 C_q^{(3)}(u) = -ku \frac{\partial}{\partial u} \frac{(Z_q^{(3)}(u))^{1-q} - 1}{1 - q} \equiv -ku \frac{\partial}{\partial u} \frac{c_q^{(3)} - 1}{1 - q},
\]  

(3.30)

even though an explicit evaluation of the internal energy \(U_q^{(3)}\) by integrating \((3.29)\) may not be feasible. The equivalence property in \((3.30)\) is based on the equality \((2.10)\). A perturbative calculation of the specific heat employing \((3.30)\) is now accomplished by using the sum \(c_q^{(3)}\) of the \(q\)-weights in the third constraint picture. The connection formula \((3.2)\) for the probabilities along with the transformation equation \((3.22)\) allows a systematic
perturbative evaluation of the relevant quantity \((\overline{Z}_q^{(3)})^{1-q}\) appearing in the rhs of (3.30). A shortcoming of the method, however, is that it is imperative to evaluate the said quantity, say, at the order \((1 - q)^3\) for obtaining the specific heat, via (3.30), at the perturbative order \((1 - q)^2\).

Following the above description we now obtain a perturbative series for the generalized partition function raised to the exponent \((1 - q)\) by using the equations (3.2, 2.10, 3.22):

\[
[\overline{Z}_q^{(3)}(u)]^{1-q} = Z(u)^{1-q} \left( \sum_{n=0}^{\infty} (1 - q)^n \sum_{\ell=0}^{2n-1} 3_{n\ell}(u)(\mathcal{K}(u))^{\ell} \right). \quad (3.31)
\]
The first few coefficients $z_{n\ell}(u)$ in the above series read as follows:

$$
\begin{align*}
\hat{z}_{00}(u) &= 1, & \hat{z}_{10}(u) &= N(3 - u), & \hat{z}_{11}(u) &= N, \\
\hat{z}_{20}(u) &= \frac{3}{2}(9N - 1)N - 6N^2u + \frac{1}{2}(4 + 7N)Nu^2 - N^2u^3 + N(3 + u^2)\ln f(u), \\
\hat{z}_{21}(u) &= -\frac{3}{2}(3 + 2N)N + 2N^2u + (1 + N)Nu^2 - 3N\ln f(u), \\
\hat{z}_{22}(u) &= -\frac{1}{2}(10 + 11N)N + N^2u - N\ln f(u), & \hat{z}_{23}(u) &= -(1 + N)N, \\
\hat{z}_{30}(u) &= \frac{1}{4}(1 - 27N + 234N^2)N + \frac{3}{2}(1 - 21N)Nu^2 - \frac{1}{8}(33 + 168N - 72N^2)Nu^2 \\
&\quad\quad\quad\quad\quad\quad\quad+ \frac{1}{6}(39 + 17N)Nu^3 + \frac{1}{12}(53 + 33N + 18N^2)Nu^4 - (1 + N)Nu^5 \\
&\quad\quad\quad\quad\quad\quad\quad+ (3N^2(6 - u) - \frac{1}{2}(17 + 12N)Nu^2 + 4N^2u^3 + (1 + N)Nu^4)\ln f(u) \\
&\quad\quad\quad\quad\quad\quad\quad- \frac{5}{2}Nu^2(\ln f(u))^2, \\
\hat{z}_{31}(u) &= \frac{1}{4}(33 + 174N + 18N^2)N - \frac{9}{2}(3 + 2N)Nu^2 - \frac{1}{12}(445 + 600N + 318N^2)Nu^2 \\
&\quad\quad\quad\quad\quad\quad\quad+ 15(1 + N)Nu^3 + (N + 2)Nu^4 + \left(6(3 + 2N)N - 9N^2u - 16(1 + N)u^2 \\
&\quad\quad\quad\quad\quad\quad\quad+ 2N^2u^3\right)\ln f(u) + (6 - u^2)N(\ln f(u))^2, \\
\hat{z}_{32}(u) &= \frac{1}{8}(547 + 1050N + 432N^2)N - \frac{1}{2}(79 + 73N)Nu^2 \\
&\quad\quad\quad\quad\quad\quad\quad- \frac{1}{6}(169 + 159N + 90N^2)Nu^2 + 4(1 + N)Nu^3 \\
&\quad\quad\quad\quad\quad\quad\quad+ \left(\frac{1}{2}(89 + 84N)N - 10N^2u - 4(1 + N)Nu^4\right)\ln f(u) + \frac{11}{2}N(\ln f(u))^2, \\
\hat{z}_{33}(u) &= \frac{1}{12}(907 + 1212N + 722N^2)N - 21(1 + N)Nu^2 - (5 + 4N + 2N^2)Nu^2 \\
&\quad\quad\quad\quad\quad\quad\quad+ (22(1 + N)N - 2N^2u)\ln f(u) + N(\ln f(u))^2, \\
\hat{z}_{34}(u) &= \frac{1}{4}(107 + 119N + 78N^2)N - 3(1 + N)Nu^2 + 3(1 + N)Nu\ln f(u), \\
\hat{z}_{35}(u) &= (3 + 3N + 2N^2)N. \\
\end{align*}
$$

The alternate method of evaluation of the specific heat may now be completed by substituting the perturbative series (3.31, 3.32) in the defining relation (3.30). The results obtained in this parallel procedure completely agrees with our previous evaluation presented in (3.27) and (3.28). However, as we remarked earlier, an evaluation of the specific heat at the perturbative order $(1 - q)^n$ using the property (3.30) necessitates evaluation of the relevant sum $c^{(3)}$ of the $q$-weights at the succeeding order $(1 - q)^{n+1}$. Our prior evaluation of the specific heat based on direct determination (3.25) of the internal energy that follows from the thermodynamic transformation property (3.24) requires computation of all pertinent quantities only up to the level $(1 - q)^n$, and, therefore, has an advantage in this respect.

We now obtain various physically relevant limits of the specific heat (3.27). For this
purpose we express the specific heat (3.27) in terms of the variable $u$ introduced in (3.11), rather than the scaled quantity $u$ defined in (3.21). The translation of our results (3.27, 3.28) to the corresponding quantities expressed in terms of the variable $u$ may be most succinctly expressed by the functional form

$$C_q^{(3)}(u, \ln Z(u)) = C_q^{(3)}(u, \ln f(u)),$$

(3.33)

where the classical partition function $Z(u)$ is given in (3.12). The lhs of (3.33) is easily obtained by replacing the polynomials in $u$ in the coefficients (3.28) by identical polynomials in the variable $u$, and simultaneous substitution of $\ln f(u)$ factors therein with the corresponding quantities in the variable $\ln Z(u)$. For the sake of brevity we do not reproduce the full expression of the lhs in (3.33).

(i) For the nonrelativistic gas the limiting value is given by $u \gg 1$. We substitute the asymptotic expansion for the ratio of Bessel functions given in the Appendix (A.2) in the expression of the specific heat obtained via the replacement (3.33) in the equations (3.27, 3.28). We also note that in the $u \gg 1$ regime the classical relativistic partition function defined in (3.9) reduces [19] to its nonrelativistic analog $Z_{NR}(T, V, N)$:

$$Z(T, V, N)|_{u \gg 1} = Z_{NR}(T, V, N) \equiv \frac{V^N}{N!} \left( \frac{2\pi mkT}{h^2} \right)^{\frac{3N}{2}},$$

(3.34)

where we have used the dimension $D = 3$. With the above facts in mind, we now obtain the nonrelativistic limit of the perturbative series for the specific heat of the ideal gas in the third constraint picture as

$$\frac{2}{3} \frac{C_q^{(3)}}{Nk} \bigg|_{NR} = 1 + (1 - q) (3N + \ln Z_{NR})$$

$$- \frac{(1 - q)^2}{8} \left( 3N(2 - 21N) - 18N \ln Z_{NR} - 4(\ln Z_{NR})^2 \right) + \ldots. \quad (3.35)$$

The above series reproduces the specific heat of the nonrelativistic ideal gas [20] in the neighborhood of $q \rightarrow 1$.

(ii) Another interesting limit is the extreme relativistic case that has been discussed in detail in Sec. II, where we have obtained the corresponding specific heat (2.18) in a closed form. Here, following the transition to the variable $u$ that has been explained in the context of (3.33), we consider the massless $m \rightarrow 0$ limit of the perturbative series for the specific heat (3.27, 3.28). In the $m \rightarrow 0$ limit we observe that the classical partition function (3.9) reduces [19] to its nonrelativistic analog $Z_{m=0}(T, V, N)$:

$$Z(T, V, N)|_{u \rightarrow 0} = Z_{m=0}(T, V, N),$$

(3.36)

where the classical partition function $Z_{m=0}(T, V, N)$ in the massless case is given in (2.12). The $u \rightarrow 0$ limiting value of the specific heat obtained from (3.27, 3.28) reads

$$\frac{C_q^{(3)}}{3Nk} \bigg|_{u \rightarrow 0} = 1 + (1 - q)(6N + \ln Z|_{m=0})$$

$$- \frac{(1 - q)^2}{2} \left( 3N(1 - 21N) - 18N \ln Z|_{m=0} + (\ln Z|_{m=0})^2 \right) + \ldots. \quad (3.37)$$
As a further check on our results (3.27, 3.28) the limiting series (3.37) in the vicinity of \( q \to 1 \) completely agrees with a perturbative expansion of the exact value (2.18) of the specific heat for the extreme relativistic ideal gas for the dimension \( D = 3 \).

IV Remarks

We have considered a canonical ensemble of \( N \) particles of a relativistic ideal gas, and found its specific heat in the third constraint scenario. In the extreme relativistic limit the generalized partition function, the internal energy, and, therefore, the specific heat may be exactly evaluated. This makes it possible for us to observe the nature of the singularities. The generalized partition function exhibits simple poles on the \( q \)-plane at \( q = 1 + \frac{1}{n}, \ n = 1, 2, \ldots, DN \), where the factor \( DN \) equals the number of degrees of freedom. The canonical ensemble is, consequently, well-defined in the parametric range \( 0 < q < 1 + \frac{1}{DN} \). As \( N \) increases, the singularities approach the limit \( q \to 1^+ \) that represents extensive statistical mechanics. This agrees with the earlier observation [12] that a grand canonical ensemble of a nonextensive relativistic ideal gas does not exist in the \( q > 1 \) regime. We also notice that in the present case of relativistic gas the thermodynamic limit and the extensive limit do not commute. For other systems similar results also hold [20,15].

For the general case of a relativistic ideal gas with arbitrary mass of the molecules we used a perturbative mechanism developed in [15]. The specific heat was obtained as a perturbative series in the nonextensivity parameter \( (1 - q) \) up to the second order. The evaluation was performed by using two different methods: a procedure that directly links the internal energies in the second and third constraint pictures via a transformation, and the traditional generalized partition function based approach. To our knowledge the former method has not been used earlier. The second approach requires the evaluation of the sum of the \( q \)-weights \( c^{(3)} \) defined in (2.10) up to a perturbative order higher than the prescribed order of the specific heat, whereas in the first method based on the transformation of internal energies all the quantities are uniformly computed up to the required perturbative order of evaluation of the specific heat. Both procedures generate identical perturbative series for the specific heat. As a bonus, the former approach easily produces the series for the internal energy in the third constraint picture. The known series of internal energy, in turn, produce the specific heat. The nonrelativistic and the extreme relativistic limits of the said perturbative series of the specific heat agree with known respective expressions.

Using the integral representations of the gamma function, Prato [21] observed that in the context of the second constraint picture, it is possible to connect the generalized partition function for a nonextensive statistical system with the partition function of the corresponding extensive \( (q = 1) \) system for both \( q > 1 \) and \( q < 1 \) domains. We now demonstrate such relations for the extreme relativistic ideal gas that has been exactly solved in our work. Moreover, such integral representations provide alternate derivation of the perturbative expansion scheme for the generalized partition function of a relativistic
ideal gas with particles of arbitrary mass. The Hilhorst integral for \( q > 1 \) region reads

\[
Z^{(2)}(\beta, V, N) = \int_0^\infty dt \, \mathfrak{G}(t) \, Z(t\beta, V, N), \tag{4.1}
\]

where the kernel \( \mathfrak{G}(t) \) is given by \[21\]

\[
\mathfrak{G}(t) = \frac{1}{(q - 1)^{\frac{1}{q-1}} \Gamma(\frac{1}{q-1})} \cdot \frac{1}{t^{\frac{1}{q-1}-1}} \exp \left(- \frac{t}{q-1} \right). \tag{4.2}
\]

The classical partition function \( Z_{m=0}(\beta, V, N) \) for extreme relativistic ideal gas is given in (2.12). The corresponding nonextensive partition function in the second constraint picture is obtained via (4.1) for the region \( q > 1 \):

\[
Z^{(2)}(\beta, V, N) \bigg|_{m=0} = Z_{m=0}(\beta, V, N) \left( \frac{\Gamma\left(\frac{1}{q-1} - DN\right)}{(q - 1)^{DN} \Gamma\left(\frac{1}{q-1}\right)} \right) \prod_{n=1}^{DN} \frac{1}{1 + (1 - q) n}. \tag{4.3}
\]

For an arbitrary mass of the molecules we consider the kernel (4.2) perturbatively. In the limit \( q \to 1^+ \), it behaves as a distribution comprising of delta function and its derivatives having support at \( t = 1 \):

\[
\mathfrak{G}(t) = \delta(t - 1) + \frac{1}{2} (q - 1) \frac{\partial^2}{\partial t^2} \delta(t - 1) - \frac{1}{3} (q - 1)^2 \left( \frac{\partial^3}{\partial t^3} - \frac{3}{8} \frac{\partial^4}{\partial t^4} \right) \delta(t - 1) + \ldots. \tag{4.4}
\]

Substituting the above series of distributions in the Hilhorst integral (4.1), we derive a perturbative connection formula between the generalized partition function in the second constraint scenario and the classical Boltzmann-Gibbs partition function. This connection formula agrees precisely with (3.9) obtained by using an infinite product expansion of the \( q \)-exponential.

For the complementary \( q < 1 \) region, the generalized partition function in the second constraint framework is expressed \[21\] in terms of the Boltzmann-Gibbs partition function as a contour integral on a complex plane:

\[
Z^{(2)}(\beta, V, N) = \frac{i}{2\pi} \oint_C dt \, \overline{\mathfrak{G}}(t) \, Z(t\beta, V, N), \tag{4.5}
\]

where the kernel reads

\[
\overline{\mathfrak{G}}(t) = \frac{\Gamma\left(\frac{2-q}{1-q}\right)}{(q - 1)^{-\frac{2-q}{1-q}+1}} \left(-t\right)^{\frac{2-q}{1-q}} \exp\left(-\frac{t}{q-1}\right). \tag{4.6}
\]

The contour \( C \) on the complex plane in (4.5) is comprised \[21\] of the segments \( \{ (\infty, \varepsilon), (t = \varepsilon \exp(i \vartheta), 0 < \vartheta < 2\pi), (\varepsilon \exp(i 2\pi), \infty \exp(i 2\pi)) \ | \varepsilon \to 0 \} \). Employing the kernel
as before, we compute the generalized partition function for the extreme relativistic ideal gas in the \( q < 1 \) domain:

\[
Z^{(2)}(\beta, V, N) \bigg|_{m=0} = Z_{m=0}(\beta, V, N) \left( \frac{\Gamma\left(\frac{2-q}{1-q}\right)}{(1-q)^{DN}\Gamma\left(\frac{2-q}{1-q} + DN\right)} \right)
\]

\[
= Z_{m=0}(\beta, V, N) \prod_{n=1}^{DN} \frac{1}{1 + (1-q)n}. \tag{4.7}
\]

As expected, the expressions (4.3) and (4.7) produce identical results for the generalized partition function for the extreme relativistic ideal gas signalling continuity at the extensive parametric value \( q = 1 \).

**APPENDIX**

1. The translation of the argument of \( K(u) \) defined in (3.14) is given by the following Taylor series:

\[
K(u) = K(u) + (1-q) \left( 6N + \ln f(u) \right) u^2 - 2Nu^3 - \left( 4\left(6N + \ln f(u)\right) - 8Nu \right. \\
-2Nu^2 \right) K(u) - \left( 14N + \ln f(u) \right) \left( K(u) \right)^2 - 2N(K(u))^3 \\
+ (1-q)^2 \left( -3 \left( \frac{1}{2}(1 + 69N + 22\ln f(u))N + (\ln f(u))^2 \right) u^2 + (75N + 13\ln f(u))Nu^3 + \frac{1}{2}(9 - 15N + 4N\ln f(u))Nu^4 - 2N^2u^5 + \left( 6(1 + 45N + 14\ln f(u))N + 8(\ln f(u))^2 \right) u^2 - 12(17N + 3\ln f(u))Nu - \frac{1}{2}(57N + 230N^2 + 78N\ln f(u)) + 78N\ln f(u) \right) \\
\left( K(u) + 2(\ln f(u))u^2 + (65N + 4\ln f(u))Nu^3 - 2N(N-1)u^4 \right) K(u) \\
+ \left( \frac{1}{2}(87 + 975N + 258\ln f(u))N + 6(\ln f(u))^2 \right) \left( K(u) \right)^2 + \left( \frac{1}{2} \right) \left( 105 + 574N + 102\ln f(u) \right) \left( K(u) \right)^3 \\
+ 2\left( 1 + 3N \right) \left( \ln f(u) \right) N - 10N^2u \right) \left( K(u) \right)^4 \\
\right)
\]

\[+2(1 + 3N) N (K(u))^5 \] + \ldots. \tag{A.1}
2. The asymptotic expansion of the ratio of Bessel functions in the \( u \gg 1 \) region reads

\[
\frac{K_1(u)}{K_2(u)} \approx 1 - \frac{3}{2u} + \frac{15}{8u^2} - \frac{15}{8u^3} + \frac{135}{128u^4} + \frac{45}{32u^5} - \frac{7425}{1024u^6} + \frac{375}{32u^7} - \frac{1095525}{32768u^8} + \ldots \quad (A.2)
\]

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**References**

1. C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
2. S. Abe, Phys. Rev. **E66**, 046134 (2002).
3. H.N. Nazareno, P.E. de Brito, Phys. Rev. **B60**, 4629 (1999).
4. T. Rohlf, C. Tsallis, Physica **A379**, 465 (2007).
5. T.D. Frank, J. Math. Phys. **43**, 344 (2002).
6. A.K. Rajagopal, Physica **A253**, 271 (1998).
7. W.M. Alberico, A. Lavagno and P. Quarati, Nucl. Phys. **A680**, 94 (2001).
8. A. Lavagno, Physica **A305**, 238 (2002).
9. I. Bediaga, E.M.F. Curado and J.M. de Miranda, Physica **A286**, 156 (2000).
10. C. Beck, Physica **A286**, 164 (2000).
11. D. Jiulin, Astrophys. Space Sci. **312**, 47 (2007).
12. C.E. Aguiar and T. Kodama, Physica **A320**, 371 (2003).
13. C. Tsallis, R.S. Mendes, A.R. Plastino, Physica **A261**, 534 (1998).
14. C. Quesne, Int. J. Theo. Phys. **43**, 545 (2004).
15. R. Chakrabarti, R. Chandrashekar and S.S. Naina Mohammed, Physica **A387**, 4589 (2008).
16. R. Jaganathan and S. Sinha, Phys. Lett. **A388**, 277 (2005).
17. E.P. Borges, Physica **A340**, 95 (2004).
18 E.M.F. Curado, C. Tsallis, J. Phys. A24, L69 (1991).

19 W. Greiner, L. Neise and H. Stöcker, *Thermodynamics and statistical mechanics*, Springer-Verlag, New-York (1995).

20 S. Abe, Phys. Lett. A263, 424 (1999).

21 D. Prato, Phys. Lett. A203, 165 (1995).