CRITICAL EXPONENT FOR EVOLUTION EQUATIONS IN MODULATION SPACES

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Abstract. In this paper, we propose a method to find the critical exponent for certain evolution equations in modulation spaces. We define an index \( \sigma(s, q) \), and use it to determine the critical exponent of the fractional heat equation as an example. We prove that when \( \sigma(s, q) \) is greater than the critical exponent, this equation is locally well posed in the space \( C(0, T; M^{s}_{p,q}) \); and when \( \sigma(s, q) \) is less than the critical exponent, this equation is ill-posed in the space \( C(0, T; M^{s}_{p,q}) \). Our method may further be applied to some other evolution equations.

1. Introduction and main results

As we all know, many evolution equations have their critical exponents on either Sobolev spaces or Besov spaces, or both. For example, the critical exponent of nonlinear Schrödinger equation (NLS) in Besov spaces \( \dot{B}^{s}_{p,2} \) is \( \frac{n}{p} - \frac{2}{k-1} \), where \( k \) is the power of the nonlinear term \( u^{k} \). Cazenave and Weissler [4] showed that NLS is locally well-posed in \( C([-T, T]; \dot{H}^{s}) \) when \( s \geq 0 \) and \( s \geq \frac{n}{2} - \frac{2}{k-1} \). In [6], Christ, Colliander and Tao proved that when \( s < \max\{0, \frac{n}{2} - \frac{2}{k-1}\} \), NLS is ill-posed in \( \dot{H}^{s} \). In [11], Miao, Xu and Zhao proved similar results for the nonlinear Hartree equation. We observe that both works in [6] and [11] are heavily based on the scaling invariance of the work space. On the other hand, the modulation space \( M^{s}_{p,q} \) is lack of the scaling property, although this space emerges in recent years and plays a significant role in the study of certain nonlinear evolution equations. (We will describe more details of the modulation space in the following content.) Since we are not able to find in literature any study on critical exponent for evolution equation in the modulation space, the aim of this paper is to propose a different method from [6] and [11] to find the critical exponents. Particularly we find the critical exponent for the fractional heat equation on the modulation space, without the scaling invariance. This exponent satisfies the well and ill posedness property on the modulation space, which is quite similar to that for NLS in the Sobolev space.

Modulation spaces was introduced by Feichtinger in [7] to measure smoothness of a function or distribution in a way different from \( L^{p} \) spaces, and they are now recognized as a useful tool for studying pseudo-differential operators (see [2] [5] [12] [14] [15]). The original definition of the modulation space is based on the short-time Fourier transform and window function. In [18], Wang and Hudzik
gave an equivalent definition of the discrete version on modulation spaces by the frequency-uniform-decomposition. With this discrete version, they are able to study the global solution for nonlinear Schrödinger equation and nonlinear Klein-Gordon equation. After then, there are many studies on nonlinear PDEs in modulation spaces followed their work. Below we list some of them, among many others. In [9], Guo and Chen proved the Stricharz estimates on $\alpha$-modulation spaces. For well-posedness in modulation space, Wang, Zhao, and Guo [19] studied the local solution for nonlinear Schrödinger equation and Navier-Stokes equations. In [17], Wang and Huang studied the local and global solutions for generalized KdV equations, Benjamin-Ono and Schrödinger equations. In [13], Ruzhansky, Sugimoto and Wang stated some new progress and open questions in modulation spaces. Also, for the ill posedness in modulation spaces, Iwabuchi studied well and ill posedness for Navier-Stokes equations and heat equations (see [10]). Iwabuchi’s result can be stated in the following theorem:

**Theorem A [10]**: When $s - \frac{n}{q} > -\frac{2}{k}$, the Heat equation

$$\begin{align*}
(H) \ u(t) &= e^{t\Delta}u_0 + \int_0^t e^{t-s\Delta}u^k d\tau
\end{align*}$$

is locally well-posed in $C([0, T, M^{s}_{p,q}])$. When $s < -\frac{2}{k}$ or $s - \frac{n}{q} < -\frac{n+2}{k}$, equation (H) is ill-posed in $C([0, T, M^{s}_{2,q}])$

Since Iwabuchi’s result is not a sharp one, a natural question is if there are some critical exponents for this equation in modulation spaces based on the well and ill posedness. In this paper, we will answer this question.

First, we recall some important properties of Besov spaces [8]. The first one is a Sobolev embedding that says $B^{s_1}_{p_1,q} \subset B^{s_2}_{p_2,q}$ if and only if

$$s_2 \leq s_1 \text{ and } s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}.$$  

The second one says that the Besov space $B^{s}_{p,q}$ forms a multiplication algebra if $s - \frac{n}{p} > 0$. By comparing these properties to the algebra property of modulation spaces and (2.2) in Section 2, we observe that the index $s - \frac{n}{p}$ in the Besov space is an analog of the index $s - n(1 - \frac{1}{q})$ in the modulation space. Motivated by such an observation, heuristically, we may use the index $s - n(1 - \frac{1}{q})$ to describe the critical exponent in the modulation space. Of course, this heuristic idea will be technically supported in our following discussion. For convenience in the discussion, we denote $\sigma(s,q) = s - n(1 - \frac{1}{q})$, and use the inequality

$$A(u,v,w...) \lesssim B(u,v,w...)$$

to mean that there is a positive number $C$ independent of all main variables $u,v,w...$, for which $A(u,v,w...) \leq CB(u,v,w...)$. 

Now we state a general theorem for well posedness.

**Theorem 1.** Let $U(t)$ be the dispersive semigroup:

$$U(t) := \mathcal{F}^{-1}e^{it\xi(\zeta)}\mathcal{F}$$
where \( p(\xi) : \mathbb{R}^n \to C, \mathcal{F} \) denotes the Fourier transform. Assume that there exists a \( \theta > 0 \) such that \( U(t) \) satisfies the estimate: for \( 1 \leq p < \infty \),

\[
\|U(t)f\|_{M^s_0,1} \leq t^{-\theta(\sigma(q_1,q_2)^-)\sigma(q_2,q_2)}\|f\|_{M^s_0,2} \tag{1.1}
\]

for all \(-\infty < s_2 \leq s_1 < +\infty, 1 \leq q_1 \leq q_2 < +\infty \) and \( 0 < t < \infty \). Then the general dispersive equation

\[
u(t) = U(t)u_0 + \int_0^t U(t - \tau)u^k d\tau, \ k \in \mathbb{Z}^+
\tag{1.2}
\]
is locally and globally well-posed for any \( s > 0, q \geq 1 \) and \( \sigma(s, q) > -\frac{1}{(k-1)\theta} \). More precisely, we have the following statements.

(i) Let \( 1 \leq p, q < \infty, s > 0 \) and \( \sigma(s, q) > -\frac{1}{(k-1)\theta} \). For any \( u_0 \in M^s_0,q \), there exists a \( T > 0 \) such that the equation (1.2) has an unique solution in \( C(0, T; M^s_0,q) \).

(ii) There exists a small number \( \nu > 0 \) such that for any \( \|u_0\|_{M^s_0,2} \leq \nu \), the equation (1.2) has an unique solution in the space

\[
L^\infty(R; M^s_0,2) \cap \bigcup_{1 \leq q < 2} L^\gamma(q)(R; M^s_0,q),
\]

where \( \frac{1}{\gamma(q)} = \left(\frac{1}{q} - \frac{1}{2}\right)n\theta < \frac{1}{2} \), \( s > 0 \) and \( \sigma(s, 2) > -\frac{1}{(k-1)\theta} \).

**Remark 1.** In this theorem, we can see that the index \( q' \) in modulation spaces plays a similar role as the index \( p \) in Besov spaces (see [8]). The only difference is that \( s - \frac{n}{2} \) can not be equal to \( -\frac{1}{(k-1)\theta} \) in the global case. This is because that the equality does not hold in the condition in (2.2), since Wang and Hudzik in [18] proved that the condition in (2.2) is sharp. The reader can find this condition in Section 2.

**Remark 2.** It is well known that the Schrödinger semigroup \( S(t) \) has the following estimate in Besov spaces for \( 2 \leq p < \infty \):

\[
\|S(t)f\|_{B^{s}_{p,2}} \leq t^{-\frac{\gamma(q)}{p}n}\|f\|_{B^{s}_{p,2}}
\tag{1.3}
\]

If we rewrite above inequality as following:

\[
\|S(t)f\|_{B^{s}_{p,2}} \leq t^{-\frac{1}{p}\sigma(s,q)^-}\|f\|_{B^{s}_{p,2}}
\tag{1.4}
\]

we find that its critical exponent is \( s_c = \frac{n}{p} - \frac{2}{k-1} \). From this observation we see that \( \theta \) in Theorem 1 just likes \( \frac{1}{2} \) in (1.4).

Now, as an application of Theorem 1, we consider the Cauchy problem for the fractional heat equation

\[
u(t) + (-\Delta)^\frac{q}{2}u = u^k, \ u(0) = u_0
\tag{1.5}
\]
The following two theorems show that in this equation, \( \theta = \frac{1}{\alpha} \) is the minimum number in the inequality (1.1) for the fractional heat equation, and \( -\frac{\alpha}{k-1} \) is critical.

**Theorem 2.** Let \( 1 \leq p, q < \infty, s \geq 0 \) and \( \sigma(s, q) > -\frac{\alpha}{k-1} \). There exists a \( T > 0 \) such that the equation (1.5) is locally well-posed in \( C(0, T; M^s_0,q) \).

**Theorem 3.** Let \( 1 \leq q < \infty \). When \( \sigma(s, q) < -\frac{\alpha}{k-1} \) or \( s < -\frac{\alpha}{k-1} \) for any \( q \), then there exists a \( T > 0 \) for which the equation (1.5) is ill-posed in \( C(0, T; M^s_0,q) \).
Comparing above results (Theorem 2, Theorem 3) in the case $\alpha = 1$ and Iwabuchi’s result (Theorem A), For the area $s > 0$, our result is a sharp one which Theorem A is not. On the other hand, our result not only works for $\alpha = 1$, but it gives the critical exponent of (1.5) for all $\alpha > 0$. By the same way, We can get the similar sharp result for incompressible Navier-Stokes equations which is also better than Iwabuchi’s (see [10]).

It is known that for each evolution equation, we have a set $\Theta$ of indices $\theta$ for which the time-spaces estimate (1.1) holds for $0 < t \leq 1$, and we have a critical exponent for its Cauchy problem with nonlinear term $u^k$. It is reasonable to guess that if the set $\Theta$ has the positive minimum value and if we obtain the critical exponent

$$\sigma(s, q) > -\frac{1}{(k-1)\theta_0}$$

on the modulation space $M_{p,q}^s$, then this $\theta_0$ must be the minimum value of $\Theta$. Although these results seem to work for the case $\theta_0 > 0$, if $\rho(\xi)$ in the symbol of the fundamental semi-group is real, the method used in our proof may also work in the case $\theta_0 = 0$. For instance, we look the Schrödinger equation, for which we can not obtain the time-space estimate as (1.1). Actually we only have the estimate:

$$\|S(t)f\|_{M_{2,q}^s} \leq \|f\|_{M_{2,q}^s}.$$  

(1.6)

We can still use the same method to obtain partial conclusion as that for the fractional heat equation. The following theorem is our result for Schrödinger equation:

**Corollary 1.** Let $1 \leq q < \infty$. When $s > 0$ and $\sigma(s, q) > 0$, the Schrödinger equation is locally well-posed in $C(0, T; M_{2,q}^s)$ for some $T > 0$. When $s = 0$ and $\sigma(s, q) > 0$, the Schrödinger equation is locally well-posed in $C(0, T; M_{2,q}^0)$ for some $T > 0$. When $\sigma(s, q) < -\frac{2}{k-1}$ or $s < -\frac{2}{k-1}$, the Schrödinger equation is ill-posed in $C(0, T; M_{2,q}^s)$.

Also, for Klein-Gordon equation, if we write in this form:

$$u(t) = K(t)(u_0 + \tan t \omega^2 u_1) + \int_0^t \frac{K_1(t - \tau)}{\omega^{\frac{q}{2}}} u^k d\tau$$  

(1.7)

where

$$K(t) = \cos t \omega^\frac{q}{2}, \quad K_1(t) = \sin t \omega^\frac{q}{2}, \quad \omega = (I - \Delta),$$

then choose $u_0 = \tan t \omega^\frac{q}{2} u_1$ in the proof of ill posedness, we can obtain following corollary:

**Corollary 2.** Let $1 \leq q < \infty$. When $s \geq 0$ and $\sigma(s, q) > -\frac{1}{k-1}$, the equation (1.7) is locally well-posed in $C(0, T; M_{2,q}^s)$. When $\sigma(s, q) < -\frac{2}{k-1}$ or $s < -\frac{2}{k-1}$ for any $q$, then the equation (1.5) is ill-posed in $C(0, T; M_{2,q}^s)$.

In Corollary 1, there is a gap in the interval $[-\frac{2}{k-1}, 0]$ for the Schrödinger equation. This is an unsolved problem. Similar gaps exist for the Klein-Gordon equation in Corollary 2.

It is interesting to see that the index $q'$ plays a crucial role in the study of modulation space $M_{p,q}^s$, while it plays almost no role in the study of the Besov space $B_{p,q}^s$. The essence of this phenomenon is that they have different geometric regions in decompositions on the frequency space, so that the Bernstein inequality gives quite different estimates in the proofs of their embedding and algebra properties.
This paper is organized as follows. In Section 2, we will introduce some basic knowledge on the modulation space, as well as some useful estimates that will be used in our proofs. All proofs of main theorems will be presented in Section 3.

2. Preliminaries

In this section, we give the definition and discuss some basic properties of modulation spaces. Also, we will prove some estimates which are described by the index $\sigma(s, q)$.

**Definition 1.** (Modulation spaces) Let $\{\varphi_k\} \subset C^\infty_0(R^n)$ be a partition of the unity satisfying the following conditions:

$$\text{supp} \varphi \subset \{\xi \in R^n : |\xi| \leq \sqrt{n}\}, \quad \sum_{k \in Z^n} \varphi(\xi - k) = 1, \varphi_k(\xi) := \varphi(\xi - k)$$

for any $\xi \in R^n$, and let

$$\square_k := F^{-1} \varphi_k F.$$

By this frequency-uniform decomposition operator, we define the modulation spaces $M^s_{p,q}(R^n)$, for $0 < p, q \leq \infty$, $-\infty < s < \infty$, by

$$M^s_{p,q}(R^n) := \{f \in S^\prime : \|f\|_{M^s_{p,q}(R^n)} = (\sum_{k \in Z^n} < k >^s \|\square_k f\|_p^q)^{\frac{1}{q}} \leq \infty\},$$

where $\langle k \rangle = \sqrt{1 + |k|^2}$. See [18] for details.

**Proposition 1.** (Isomorphism) [18] Let $0 < p, q \leq \infty, s, \sigma \in R$. $J_{\sigma} = (I - \Delta)^{\frac{\sigma}{2}} : M^s_{p,q} \rightarrow M^{s-\sigma}_{p,q}$ is an isomorphic mapping, where $I$ is the identity mapping and $\Delta$ is the Laplacian.

**Proposition 2.** (Embedding) [18] We have

(i) $M^{s_1}_{p_1,q_1} \subset M^{s_2}_{p_2,q_2}$, if $s_1 \geq s_2, 0 < p_1 \leq p_2, 0 < q_1 \leq q_2$. \hspace{1cm} (2.1)

(ii) $M^{s_1}_{p_1,q_1} \subset M^{s_2}_{p_2,q_2}$, if $q_1 > q_2, s_1 > s_2, s_1 - s_2 > n/q_2 - n/q_1$. \hspace{1cm} (2.2)

**Lemma 1.** Let $s \geq 0, k \in Z^+, 1 \leq p,q, q_1, q_2 \leq \infty$, and $\frac{1}{q} + k - 1 = \frac{1}{q_1} + \frac{k-1}{q_2}$. We have

$$\|u^k\|_{M^s_{p,q}} \leq \|u\|_{M^s_{p,q_1}} \|u\|_{M^s_{p,q_2}}^{k-1}. \hspace{1cm} (2.3)$$

**Proof:** We only consider the case when $k = 2$ for simplicity, since the proof for $k \geq 2$ is similar. Note the project operators $\{\square_k\}$ satisfying

$$\sum_{k \in Z^n} \square_k = I.$$

By the Minkowski inequality, we may write

$$\langle i \rangle^s \|\square_i u^2\|_{L^p} \leq \langle i \rangle^s \sum_{i_1, i_2 \in Z^n} \|\square_i(\square_{i_1} u \square_{i_2} u)\|_{L^p}.$$

We observe that the support condition of $\square_k$ in the frequency space implies that

$$\square_i(\square_{i_1} u \square_{i_2} u) = 0 \text{ if } |i - i_1 - i_2| \geq k_0(n),$$

where $k_0(n)$ is an integer which depends only on $n$ (see [18]). So we have

$$\langle i \rangle^s \|\square_i u^2\|_{L^p} \leq \langle i \rangle^s \sum_{i_1, i_2 \in Z^n, |i - i_1 - i_2| \leq k_0(n)} \|\square_i(\square_{i_1} u \square_{i_2} u)\|_{L^p}.$$
By the Bernstein and Hölder’s inequalities, we obtain that
\[
(i)^{\sigma} \| \Box_i u \|_{L^p} \leq \sum_{i_1, i_2 \in \mathbb{Z}^n, |i_1 - i_2| \leq k_0(n)} (i_1 + i_2)^{\sigma} \| \Box_{i_1} u \|_{L^{p_1}} \| \Box_{i_2} u \|_{L^{p_2}}
\]
\[
\leq \sum_{i_1, i_2 \in \mathbb{Z}^n, |i_1 - i_2| \leq k_0(n)} (i_1)^{\sigma} \| \Box_{i_1} u \|_{L^{p_1}} \| \Box_{i_2} u \|_{L^{p_2}}
\]
\[
+ \sum_{i_1, i_2 \in \mathbb{Z}^n, |i_1 - i_2| \leq k_0(n)} (i_2)^{\sigma} \| \Box_{i_1} u \|_{L^{p_1}} \| \Box_{i_2} u \|_{L^{p_2}},
\]
where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Thus, by (2.1) and Young’s inequality of series, we have
\[
\| u^2 \|_{M_{p,q}^{r_2}} \leq \| u \|_{M_{p,q}^{r_1}} \| u \|_{M_{p,q}^{r_2}}.
\] (2.4)
By the induction and (2.4), we can easily obtain the desired result.

**Remark 3.** In [3], Cazenave proved \( \| u^k \|_{B_{p,\infty}^{r_2}} \leq \| u \|_{B_{p,\infty}^{r_1}} \| u \|_{L_\infty}^{k-1} \) when \( \frac{1}{p} = \frac{1}{p_1} + \frac{k-1}{p_2} \). Also, we can see that the condition of Lemma 1 is equivalent to \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \).
This again indicates that the index \( \frac{1}{q} \) in modulation spaces behaves like the index \( \frac{1}{p} \) in the Besov spaces.

**Lemma 2.** Let \( 1 \leq p \leq \infty \), \( 1 \leq q_1 \leq q_2 \), \( 0 \leq s_2 \leq s_1 \) and \( \sigma(s_1, q_1) - \sigma(s_2, q_2) = R \).
If \( s_1 > 0 \) and \( \sigma(s_1, q_1) > -\frac{R}{k-1} \), then we have
\[
\| u^k \|_{M_{p,q}^{r_2}} \leq \| u \|_{M_{p,q}^{r_1}}^k.
\] (2.5)

**Proof:** Fix a small \( \epsilon > 0 \) and pick \( q_3 \) such that
\[
\frac{n}{q_3} = \frac{n}{q_1} - R + \epsilon.
\]
It is easy to check that \( s_1 + \frac{n}{q_3} > s_2 + \frac{n}{q_2} \). Using (2.2), we have
\[
\| u^k \|_{M_{p,q_2}^{r_1}} \leq \| u \|_{M_{p,q_3}^{r_2}}^k.
\] (2.6)
Since \( s_1 > 0 \), by Lemma (2.3), we obtain that
\[
\| u^k \|_{M_{p,q_2}^{r_1}} \leq \| u \|_{M_{p,q_3}^{r_1}}^k \| u \|_{M_{p,q_3}^{r_1}}^{k-1},
\] (2.7)
where
\[
\frac{n}{q_5} = n - \frac{R}{k-1} + \frac{\epsilon}{k-1},
\] (2.8)
and \( \epsilon \) is small enough to ensure \( s_1 + \frac{n}{q_5} > s_2 + \frac{n}{q_5} \). Using (2.2) again, we have
\[
\| u \|_{M_{p,q_2}^{r_1}} \leq \| u \|_{M_{p,q_3}^{r_1}}.
\] (2.9)
Inserting (2.9) into (2.7), we now obtain (2.5). This completes the proof.

3. **Proof of the main theorems**

**Proof of Theorem 1** We first prove the local case. Consider the integral equation
\[
\Phi(u) = U(t)u_0 + \int_0^t U(t - \tau)u^k d\tau.
\]
It is well known that this equation is equivalent to the Cauchy problem (1.5). To prove the above equation has a unique solution, we will use the standard contraction method. To this end, we define the space

$$X_1 = \{ u : \| u \|_{L^\infty(0, T; M_{p,q}^*)} \leq C_0 \}$$

with the metric

$$d(u, v) = \| u - v \|_{L^\infty(0, T; M_{p,q}^*)},$$

where the positive numbers $C_0$ and $T$ will be chosen later when we invoke the contraction. We now choose numbers $\bar{\sigma}$ and $\bar{\sigma}$ for which

$$\sigma(s, q) - \sigma(\bar{\sigma}, \bar{\sigma}) = \frac{1}{\bar{\sigma} + \epsilon},$$

where $\epsilon$ is a small positive number such that $\sigma(s, q) > -\frac{1}{(\bar{\sigma} + \epsilon)(s-1)}$. By (1.1) and Lemma 2, we have

$$\| \Phi(u) \|_{X_1} \leq \| u_0 \|_{M_{p,q}^*} + \int_0^t U(t - \tau) u^k d\tau \|_{X_1}$$

$$\leq \| u_0 \|_{M_{p,q}^*} + \sup_{\tau \in (0, T)} \left| \int_0^t (t - \tau)^{-\frac{q(s, q) - \sigma(\bar{\sigma}, \bar{\sigma})}{\bar{\sigma} + \epsilon}} \| u \|_{M_{p,q}^*} d\tau \right|$$

$$\leq \| u_0 \|_{M_{p,q}^*} + \sup_{\tau \in (0, T)} \left| \int_0^t (t - \tau)^{-\frac{q(s, q) - \sigma(\bar{\sigma}, \bar{\sigma})}{\bar{\sigma} + \epsilon}} d\tau \right| \| u \|_{X_1}$$

$$\leq \| u_0 \|_{M_{p,q}^*} + T^{1 - \frac{q}{2}} \| u \|_{X_1}. \quad (3.2)$$

By the contraction mapping argument, we obtain (i) in Theorem 1 after choosing suitable $T$ and $C_0$.

Next, we consider the global case. Choosing $s_1 = s_2$, $\frac{1}{q_1} + \frac{1}{q_2} = 1$ in (1.1), we have

$$\| U(t) f \|_{M_{s,q}^*} \leq t^{-\theta[n\left(\frac{2s}{q} - 1\right)]} \| f \|_{M_{s,q}^*} \quad (3.3)$$

for $1 \leq q \leq 2$, $s > 0$. When $\theta[n(\frac{2s}{q} - 1)] < 1$, we can obtain the following estimates by standard dual methods (see [16]):

$$\| U(t) f \|_{L^{\frac{2n}{n(\frac{2s}{q} - 1)}}(R; M_{s,q}^*)} \leq \| f \|_{M_{s,2}^*}, \quad (3.4)$$

$$\| \int_0^t U(t - \tau) f d\tau \|_{L^{\frac{2n}{n(\frac{2s}{q} - 1)}}(R; M_{s,q}^*)} \leq \| f \|_{L^1(R; M_{s,2}^*)}, \quad (3.5)$$

$$\| \int_0^t U(t - \tau) f d\tau \|_{L^\infty(R; M_{s,2}^*)} \leq \| f \|_{L^\infty(R; M_{s,2}^*)}, \quad (3.6)$$

$$\| \int_0^t U(t - \tau) f d\tau \|_{L^{\frac{2n}{n(\frac{2s}{q} - 1)}}(R; M_{s,q}^*)} \leq \| f \|_{L^{\frac{2n}{n(\frac{2s}{q} - 1)}}(R; M_{s,q}^*)}, \quad (3.7)$$

By interpolation among (3.5), (3.6) and (3.7), we obtain that

$$\| \int_0^t U(t - \tau) f d\tau \|_{L^{\frac{2n}{n(\frac{2s}{q} - 1)}}(R; M_{s,q}^*)} \leq \| f \|_{L^{\frac{2n}{n(\frac{2s}{q} - 1)}}(R; M_{s,q}^*)}, \quad (3.8)$$

for any $1 \leq q, r \leq 2$. 
We choosing $\frac{1}{q} = \frac{1}{2} + \frac{1}{(k-1)}\theta$ and let 

$$X_2 = L^\infty(R; M^n_{2,2}) \cap L^{\frac{2}{\theta(n(\frac{2}{q} - 1))}}(R; M^n_{2,q})$$

with the metric 

$$d(u, v) = \|u - v\|_{L^\infty(R; M^n_{2,2})} + \|u - v\|_{L^{\frac{2}{\theta(n(\frac{2}{q} - 1))}}(R; M^n_{2,q})}$$

by (1.1), (3.4), (3.8) and Lemma 2, we have 

$$\|\Phi(u)\|_{X_2} \leq \|u_0\|_{M^n_{2,2}} + \|u^k\|_{L^{\frac{2}{\theta(n(\frac{2}{q} - 1))}}(R; M^n_{2,q})}$$

$$\leq \|u_0\|_{M^n_{2,2}} + \|u\|^k_{L^{\frac{2}{\theta(n(\frac{2}{q} - 1))}}(R; M^n_{2,q})}$$

$$= \|u_0\|_{M^n_{2,2}} + \|u\|_{X_2}^k.$$

(3.9)

it is easy to check that 

$$k(\frac{2}{\theta(n(\frac{2}{q} - 1))})^\prime = \frac{2}{\theta(n(\frac{2}{q} - 1))}$$

(3.10)

and 

$$s + \frac{n}{q} > n - \frac{n}{q} \frac{-n}{k-1}.$$ 

(3.11)

From (3.10), we have 

$$\frac{nk}{q} - \frac{n}{q} \frac{1}{\theta} - \frac{n}{2} + \frac{nk}{2}.$$ 

(3.12)

then insert (3.12) into (3.11), we can obtain 

$$s > \frac{n}{2} - \frac{1}{(k-1)\theta}.$$ 

(3.13)

Using the standard contraction mapping argument in (3.9), we can find unique solution in $X_2$. Then by (3.4) and (3.8), we can obtain the conclusion of (ii) in Theorem 1.

**Proof of Theorem 2.** We first prove 

$$\|e^{-t((-\Delta)^{\frac{1}{2}})} f\|_{M^n_{2,q}} \leq (1 + t^{-\frac{1}{\theta}(s_1 - s_2)}) \|f\|_{M^n_{2,q}}$$ 

(3.14)

for any $s_1 \geq s_2$. For the low frequency part $|k| \leq 100\sqrt{\bar{n}}$, we have 

$$\sum_{|k| \leq 100\sqrt{\bar{n}}} \langle k \rangle^{s_1} \|\Box_k e^{-t((-\Delta)^{\frac{1}{2}})} f\|_{L^p}^p$$ 

$$\leq \sum_{|k| \leq 100\sqrt{\bar{n}}} \langle k \rangle^{s_2} \|\Box_k f\|_{L^p}^p \leq \|f\|_{M^n_{2,q}}.$$ 

For the high frequency part, note that the operator $\Box_k e^{-t((-\Delta)^{\frac{1}{2}})}$ can be written as 

$$\Box_k e^{-t((-\Delta)^{\frac{1}{2}})} = \sum_{|\ell| \leq 1} \Box_{k+\ell} e^{-t((-\Delta)^{\frac{1}{2}})} \Box_k$$

and $\Box_{k+\ell} e^{-t((-\Delta)^{\frac{1}{2}})}$ are convolution operators with the kernels 

$$\Omega_{k+\ell}(y) = e^{i<k+\ell,y>} \int_{\mathbb{R}^n} e^{-t(|\xi+k+\ell|^2)} e^{i<y,\xi>} \varphi(\xi) d\xi.$$
Hence, when $|k| \geq 100\sqrt{n}$ it is easy to prove
\[ \| \Box k e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^p} \leq e^{-\frac{k}{4}|k|^\alpha} \| \Box k f \|_{L^p}. \]

Now, we have
\[ < k >^{\frac{1}{3}} \| \Box k e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^p} \leq < k >^{\frac{1}{3}} \| \Box k e^{-\frac{k}{4}|k|^\alpha} f \|_{L^p} < k >^{\frac{1}{3}} \| \Box k f \|_{L^p} \]
\[ \leq \frac{1}{t^{\frac{1}{3}}(s_1-s_2)} < k >^{\frac{1}{3}} \| \Box k f \|_{L^p}. \]

Taking $L^p$ norm in both sides, we obtain (3.14) from the definition of the modulation space.

Next, we estimate the case $1 \leq q_1 < q_2$ and $s_1 \geq s_2$. For any $\varepsilon > 0$, by (2.2) and (3.14), we have
\[ \| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \|_{M^{s_1,q_1}_{p,q}} \leq \| e^{-t(-\Delta)^{\frac{\alpha}{2}}} f \|_{M^{s_1-s_2+\varepsilon,q_1}_{p,q}} \]
\[ \leq (1 + t^{-\frac{1}{\alpha}(s_1-s_2-s_2+\varepsilon)}) \| f \|_{M^{s_1,q_1}_{p,q}} \]
\[ = (1 + t^{-\frac{1}{\alpha}(s_1-q_1)}) \| f \|_{M^{s_1,q_1}_{p,q}}, \]
where $\varepsilon_1 \to 0^+$ as $\varepsilon \to 0^+$. Notice that the behavior of $1+t^{-\frac{1}{\alpha}(s_1-q_1)}$ likes $t^{-\frac{1}{\alpha}(s_1-q_1)}$ when $t$ is finite. So, by Theorem 1, we can obtain that equation (1.5) is locally well-posed in $C(0,T;M^{s_1,q_1}_{p,q})$, when $\sigma(s, q) > -\frac{1}{(\alpha-1)(k-1)}$. Since $\varepsilon_1 > 0$ is arbitrary, we obtain the conclusion.

**Proof of Theorem 3** By the Bejenaru and Tao’s conclusion (see Theorem 4 of [1]), it suffices to show that the map from $M^s_{2,q}$ to $L^\infty([0,T];M^s_{2,q})$ defined by
\[ u_0 \to \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} (e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} u_0)^k d\tau \quad (3.15) \]
is discontinuous for $s < -\frac{\alpha}{k-1}$ or $\sigma(s, q) < -\frac{\alpha}{k-1}$. Actually, if the map is continuous, we will have
\[ \sup_{t \in (0,T)} \| \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} (e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} u_0)^k d\tau \|_{M^s_{2,q}} \leq \| u_0 \|_{M^s_{2,q}}^k. \quad (3.16) \]

So, we only need to find a $u_0$ such that (3.16) fails.

We first consider the case $s < -\frac{\alpha}{k-1}$. In this case, choose $u_0$ such that
\[ \mathcal{F} u_0 = \chi_N = \chi(\xi - N\varepsilon) + \chi(\xi + N\varepsilon) = \chi_+(\xi) + \chi_-(\xi) \]
where $N$ is a large natural number, $\varepsilon = (1,1,...,1)$, and $\chi$ is the characteristic function of the cube
\[ E = [-1,1]^n. \]

This $\mathcal{F} u_0$ is a non-negative even function. By the choice of $\mathcal{F} u_0$ and the definition of the modulation space, using the Plancherel formula we have
\[ \| u_0 \|_{M^s_{2,q}}^k \leq N^{ks}. \quad (3.17) \]

Now, we estimate
\[ \| \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} (e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} u_0)^k d\tau \|_{M^s_{2,q}}. \]
By taking $t = \frac{1}{N\tau}$, we get
\[
\| \int_0^{\frac{1}{N\tau}} e^{-\frac{1}{N\tau}(-\Delta)^{\frac{\alpha}{2}}}(e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}u_0)^k d\tau \|_{M_{s,q}}^q
\]
\[
= \sum_{j \in \mathbb{Z}^n} < j >^q \| \phi_j(\xi) \int_0^{\frac{1}{N\tau}} e^{-\frac{1}{N\tau}(-\Delta)^{\frac{\alpha}{2}}}(e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}u_0)^k d\tau \|_{L^2}^q
\]
We denote the convolution of $k$ functions of $\chi_N$ by $\chi_N * \cdots * \chi_N$. It is easy to find that the cube $E_N = [kN - k, kN + k]^n$ is a subset of the support of $\chi_N * \cdots * \chi_N$. Also, notice that
\[
e^{-\frac{(1+\tau)}{N\tau}} \xi^n \geq C > 0
\]
for $\tau \in [0, \frac{1}{N\tau}]$ and $\xi \in E_N$, and that
\[
e^{-\tau|\xi|^n} \geq C > 0
\]
for $\tau \in [0, \frac{1}{N\tau}]$ and $\xi \in \text{supp} \chi_N$. By the Plancherel theorem, we have that, for $j = kN\xi$,
\[
\| \phi_j(\xi) \int_0^{\frac{1}{N\tau}} e^{-\frac{1}{N\tau}(-\Delta)^{\frac{\alpha}{2}}}(e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}u_0)^k d\tau \|_{L^2}^q \leq C \int_0^{\frac{1}{N\tau}} \| (\chi_+ * \cdots * (\chi_+)) \|_{L^2(E_N \cap \text{supp} \phi_j)}^q d\tau.
\]
Moreover, because the Lebesgue measure of $E_N$ is a constant, we have that for $j = kN\xi$
\[
(j)^{\alpha q} \| \phi_j(\xi) \int_0^{\frac{1}{N\tau}} e^{-\frac{1}{N\tau}(-\Delta)^{\frac{\alpha}{2}}}(e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}u_0)^k d\tau \|_{L^2}^q \geq N^{(s-\alpha)q}.
\]
It leads to the inequality
\[
\| \int_0^{\frac{1}{N\tau}} e^{-\frac{1}{N\tau}(-\Delta)^{\frac{\alpha}{2}}}(e^{-\tau(-\Delta)^{\frac{\alpha}{2}}}u_0)^k d\tau \|_{M_{s,q}} \geq CN^{s-\alpha}
\]
which contradicts to (3.16) and (3.17). So equation (1.5) is ill-posed in $M_{s,q}$ when $s < -\frac{\alpha}{2\tau}$.

Now, we consider the case $s, q < -\frac{\alpha}{2\tau}$. For convenience, we let
\[
F u_0 = \chi_N^\alpha = \chi(\frac{1}{N}(\xi - 100kN\tau)),
\]
where $\chi$ is the characteristic function of the set $E = [-1, 1]^n$. If we want $u_0$ to be a real function, we can make an even extension just like what we did in the previous case, the result should be the same. So, by the Plancherel theorem and the definition of the modulation spaces, we have
\[
\| u_0 \|_{M_{s,q}} \leq N^k(s + \frac{\alpha}{2})
\]
(3.20).

On the other hand, choosing $t = N^{-\alpha}$ again, by the similar method as we did previously, when $j$ is the center (or very close to the center of the support of
\[ \chi_N^* \cdots \cdots \chi_N^* , \] we have
\[ \| \Box \| \int_0^\varpi e^{-(\frac{\varpi}{\alpha} - \tau)(-\Delta)^{\frac{\alpha}{2}}} (e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} u_0)^k d\tau \|_L^q_x \]
\[ \geq C \| \int_0^\varpi e^{-(\frac{\varpi}{\alpha} - \tau)|\xi|^\alpha \chi_N^*} \cdots \cdots (e^{-\tau|\xi|^\alpha \chi_N^*}) d\tau \|_{L^2(\text{supp} \varphi_j)} \]
\[ \geq C N^{-\alpha} \| \chi_N^* \cdots \cdots \chi_N^* \|_{L^2(\text{supp} \varphi_j)}. \]
Notice that the Lebesgue measure of \(\text{supp} \chi_N^*\) is \(N^n\) times that of \(\text{supp} \Box_i\), and \((\chi_N^* \cdots \cdots \chi_N^*)\) is constructed by \((k-1)\) convolutions. Therefore, we have
\[ \| \chi_N^* \cdots \cdots \chi_N^* \|_{L^2(\text{supp} \varphi_i)} \geq N^{(k-1)n}. \] (3.21)
Moreover, the support of \(\chi_N^* \cdots \cdots \chi_N^*\) is the cube
\[ E_N^* := [100k^2N - kN, 100k^2N + kN]^n \]
of Lebesgue measure \((2kN)^n\). Therefore, the number of summands, in right side of (3.18) is \(CN^m\) for some constant \(C > 0\). We now obtain
\[ \| \int_0^\varpi e^{\frac{\varpi}{\alpha} - \tau)(-\Delta)^{\frac{\alpha}{2}}} (e^{-\tau(-\Delta)^{\frac{\alpha}{2}}} u_0)^k d\tau \|_{L^q_x} \geq C N^{s+\frac{\alpha}{2}+(k-1)n-\alpha}. \] (3.22)
The last inequality gives a contradiction to (3.16) and (3.20). So equation (1.5) is ill-posed in \(M^s_{\delta,q}\) when \(\sigma(s, q) < -\frac{\alpha}{k-1}\). This completes the proof of Theorem 3.

The proofs for Corollary 1 and Corollary 2 are similar to the above proof. We leave them to the reader.

**References**

[1] I. Bejenaru, T. Tao, \textit{Sharp well-posedness and ill-posedness result for a quadratic nonlinear Schrödinger Equation. J. Funct. Anal.}, 239(2006), 228-259.

[2] A. Bényi, K. Gröchenig, K.A. Okoudjou, et al. \textit{Unimodular Fourier multipliers for modulation spaces. J. Funct. Anal.}, 246(2007), 366-384.

[3] T. Cazenave, \textit{Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics. Vol. 10. New York University Courant Institute of Mathematical Sciences 2003.}

[4] T. Cazenave, F. B. Weissler, \textit{Critical nonlinear Schrödinger Equation. N. Anal. TMA}, 14(1990), 807-836.

[5] J. Chen, D. Fan, L. Sun \textit{Asymptotic estimates for unimodular Fourier multipliers on modulation space. Discret. Contin. Dyn. Dyn. Syst.}, 32(2012), 467-485.

[6] M. Christ, J. Colliander, T. Tao, \textit{Ill-posedness for nonlinear Schrödinger and wave equations. arXiv:math.AP/0311048.}

[7] H. G. Feichtinger, \textit{Modulation space on locally compact Abelian group. Technical Report}, (1983) University of Vienna.

[8] L. Grafakos, \textit{Classical and Modern Fourier Analysis. Prentice Hall, NJ 2003.}

[9] W. C. Guo, J. C. Chen, \textit{Strichartz estimates on 0-modulation spaces. Electron. J. Differential Equations}, 118 (2013) 1-13.

[10] T. Iwabuchi, \textit{Navier-Stokes equations and nonlinear heat equations in modulation space with negative derivative indices. J. Differential Equations}, 248 (2010), 1972-2002.

[11] C. X. Miao, G. X. Xu, L. F. Zhao \textit{The Cauchy problem of the Hartree equation. J. Partial Diff. Eqns}, 21 (2008), 22-44.

[12] A. Miyachi, F. Nicola, S. Rivetti \textit{Estimates for unimodular Fourier multipliers on modulation spaces. Proc Amer Math Soc}, 137(2009), 3869-3883.
[13] M. Ruzhansky, M. Sugimoto, B. X. Wang. Modulation spaces and nonlinear evolution equations. Progress in Mathematics, Volume 301 (2012), 267-283.

[14] J. Sjöstrand An algebra of pseudo-differential operators. Math Res Lett, 1(1994), 185-192.

[15] J. Toft Continuity properties for modulation spaces. Basel: Birkhäuser, 1983.

[16] B. Wang, C. Hao, C. Huo. Harmonic Analysis Method for Nonlinear Evolution Equations I. Hackensack, NJ: World Scientific (2011).

[17] B. Wang, C. Huang, Frequency-uniform decomposition method for the generalized BO, Kdv and NLS equations. J. Differential Equations 239 (2007), 213-250.

[18] B. Wang, H. Hudzik. The global Cauchy problem for NLS and NLKG with small rough data. J. Differential Equations, 232 (2007), 36-73.

[19] B. Wang, L. Zhao, B. Guo. Isometric decomposition operators, function space $E^\lambda_{p,q}$ and applications to nonlinear evolution equations. J. Funct. Anal, 233 (2006), 1-39.

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