The $T$-Domain and Extreme Matter Phases Inside Spherically Symmetric Black Holes

A. DeBenedictis $^*$, D. Aruliah $^†$, A. Das $^{‡}$

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$^*$Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada, V5A 1S6, e-mail: adebenea@sfu.ca, adebened@langara.bc.ca

$^†$Institute of Applied Mathematics, University of British Columbia, Vancouver, British Columbia, Canada, V6T 1Z2

$^{‡}$Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada, V5A 1S6, e-mail: das@sfu.ca
Abstract

Black hole interiors (the $T$-domain) are studied here in great detail. Both the general and particular $T$-domain solutions are presented including non-singular ones. Infinitely many local $T$-domain solutions may be modeled with this scheme. The duality between the $T$ and $R$ domains is presented. It is demonstrated how generally well behaved $R$-domain solutions will give rise to exotic phases of matter when collapsed inside the event horizon. However, as seen by an external observer, the field is simply that of the Schwarzschild vacuum with well behaved mass term and no evidence of this behaviour may be observed. A singularity theorem is also presented which is independent of energy conditions.

Key Words: Black hole interiors; T-domain; Exotic matter
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1 Introduction

Over the years there have been extensive studies carried out in the literature on the subject of static, spherically symmetric stars. These regular solutions consist of some spherical matter field smoothly patched to the Schwarzschild vacuum solution at some surface outside the gravitational radius or event horizon. Many of these solutions involve bodies composed of perfect fluids [1], some of which employ equations of state to supplement the equations of relativity.

Specific models of spherically symmetric gravitational collapse have also been of much interest since the pioneering work of Oppenheimer and Snyder [2]. A collection of work regarding specific collapse models may be found in [1] and other standard references. Interestingly, some studies of collapsing spherically symmetric stars consisting of anisotropic fluid matter reveal a transition into exotic matter after collapsing past their event horizons ([3], [4]). Black holes of this type are known in the literature as exotic black holes.

Black hole interiors which are not singular at the classical level (regular black holes) have also been studied [5]-[12]. Although it is believed that any serious candidate theory for quantum gravity must eliminate the singularity, it is instructive to study what criteria must be met at the classical level to remove this singularity. We present such solutions here as well as derive the general properties non-singular solutions must possess.
For the above reasons, it is interesting to study the $T$-domain in more detail, especially addressing the issue of extreme exotic matter. It may be argued that the black hole evaporation process \cite{13} tends to reveal more of the black hole interior as time progresses. Therefore, studies of the interior region have relevance to exterior observers in this way. The $T$-domain also provides an interesting arena in which to perform theoretical studies of matter under extreme gravitational conditions.

This paper serves several purposes. First we present an investigation of the general $T$-domain. Although much work has been done regarding spherical systems in the $R$-domain, relatively little analytic work has been performed on the $T$-domain. We believe it is useful to present the general local solution to the $T$-domain problem as it sheds light on some of the origins of the exotic matter phases mentioned above. It turns out that the presence of an apparent horizon induces drastic phase transitions in the matter field. Also, it is hoped that the general solution will provide a useful starting point for future research in this area.

In the literature, analytic studies in this domain involve specific solutions which depend on the $T$ coordinate only, so called $T$-spheres which are eternal black holes. For examples of such studies the reader is referred to the interesting work by Ruban \cite{14} \cite{15} and Poisson and Israel \cite{16} as well as references above. However, physically, a $T$-domain solution is expected to result from the gravitational collapse of an $R$-domain system (some well behaved star). In other words, before the formation of an event horizon, the collapse is described by a metric depending on the exterior radial and time coordinate ($r$ and $t$ respectively). Therefore, after the system has collapsed within its Schwarzschild radius forming a black hole, the corresponding interior solution should also depend on both of these coordinates (labeled $T$ and $R$ respectively in the $T$-domain). One particular solution constructed here does possess such dependence.

The interesting particular solutions which are presented include the $T$-domain analogue of the constant density star. It is found that the corresponding matter in the $T$-domain possesses substantially different physical properties as will be discussed later in the paper. Finally, we solve the $T$-domain equations to construct non-singular solutions, one of which depends on both the radial and time coordinate. The other regular solution may represent the late stages of collapse of a polytropic star which does not form a singularity.

This study places some emphasis on the presence of exotic matter and the elimination of the central singularity is not a requirement of all solutions. These extreme matter phases are considered and studied in some detail as they are motivated by the collapse studies mentioned earlier and it is hoped that the analysis
here will shed some light on the origin of such phases. This leads us to another motivation for this study: It is interesting to note that, when considering solutions which are well behaved in the $R$-domain, their extension into the $T$-domain is usually accompanied by exotic physical properties. We show that ordinary astrophysical systems can naturally form exotic systems after collapsing in a black hole. One simple example we demonstrate, in the general solution, is that demanding a well behaved Schwarzschild mass measured by an observer in the $R$-domain requires material which possesses a local tension.

2 Physics in the $T$-domain

We begin here with a brief analysis of the $R$-domain and present the general solution in this domain. This is useful as it will later be compared to the general $T$-domain solution which possesses substantially different physical properties even though, at first glance, the $T$-coordinate chart may seem very similar to the $R$-chart except for signs. The solution presented here allows for the study of any collapse model in the $T$-domain.

The spherically symmetric line element in curvature coordinates is furnished by:

$$ds^2 = -e^{\nu(r,t)} dt^2 + e^{\lambda(r,t)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2;$$  \hspace{1cm} (1)

$$r_1 < r < r_2, \quad t_1 < t < t_2, \quad 0 < \theta < \pi, \quad 0 \leq \phi < 2\pi.$$

The geometry is governed by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu},$$  \hspace{1cm} (2)

supplemented with the conservation law

$$T^\mu_{\nu ;\nu} = 0.$$  \hspace{1cm} (3)

\footnote{Conventions here follow those of \cite{1} with $G = c = 1$. The Riemann tensor is therefore given by $R^\mu_{\rho\sigma\nu} = \Gamma^\mu_{\rho\sigma,\nu} + \Gamma^\mu_{\alpha\nu} \Gamma^\alpha_{\rho\sigma} - \ldots$ with $R_{\rho\sigma} = R^\alpha_{\rho\alpha\sigma}$.}
Explicitly, in mixed form, (2) yields the following non-trivial equations:

\[-8\pi T_t^t = \frac{1}{r^2} + \frac{e^{-\lambda(r,t)}}{r} \left( \lambda(r,t)_r - \frac{1}{r} \right), \quad (4a)\]

\[-8\pi T_r^r = \frac{1}{r^2} - \frac{e^{-\lambda(r,t)}}{r} \left( \nu(r,t)_r + \frac{1}{r} \right), \quad (4b)\]

\[-8\pi T_\theta^\phi = \frac{e^{-\lambda(r,t)}}{r}, \quad (4c)\]

\[-8\pi T_\theta^\theta = -8\pi T_\phi^\phi = \frac{e^{-\nu(r,t)}}{2} \left[ \lambda_{,tt} + \frac{1}{2} \left( \lambda(r,t)_t^2 - \lambda(r,t)_t \nu(r,t)_t \right) \right]
- \frac{e^{-\lambda(r,t)}}{2} \left[ \nu(r,t)_{,rr} + \frac{1}{2} \left( \nu(r,t)_t^2 - \nu(r,t)_r \lambda(r,t)_r \right) + \frac{1}{r} \left( \nu(r,t) - \lambda(r,t) \right)_{,r} \right]. \quad (4d)\]

The conservation law, (3), yields only two non-trivial equations

\[T_r^r + T_t^t + \left( \frac{1}{2} \nu(r,t)_{,r} + \frac{2}{r} \right) T_r^r + \frac{1}{2} \left( \lambda(r,t) + \nu(r,t) \right)_t T_t^t - \left[ \frac{1}{2} \nu(r,t)_r T_t^t + \frac{2}{r} T_\theta^\phi \right] = 0, \quad (5a)\]

\[T_t^t + T_{r,r}^t + \frac{1}{2} \lambda(r,t)_t \left( T_t^t - T_r^r \right) + \left[ \frac{1}{2} \left( \lambda(r,t) + \nu(r,t) \right)_{,r} + \frac{2}{r} \right] T_r^r = 0. \quad (5b)\]

In the system of six partial differential equations, (4a-4d) and (5a-b), there exist six unknown functions: \( \lambda(r,t) \), \( \nu(r,t) \), and the four components of the stress-energy tensor. However, there are two differential identities among the equations and therefore, one can either prescribe two functions to make the system determinate or determine them from other means.

Synge’s strategy [18] of solving the equations (4a-5b) is the following:

- Prescribe \( T_t^t \) and solve the equation (4a) for \( \lambda(r,t) \).
- Prescribe \( T_r^r \) (which may be related to \( T_t^t \) by an equation of state) and solve a linear combination of (4a) and (4b) for \( \nu(r,t) \).
- Define \( T_\theta^\phi \) by equation (4c).
• Define $T^\theta_\theta$ by the conservation equation (5a).

• At this stage, by the differential identities, one can show that all equations are satisfied.

Following this strategy, the most general solution of the system of equations in a suitable domain $D_r$ of the $r$-$t$ plane can be furnished as:

$$e^{\lambda(r,t)} = 1 + \frac{8\pi}{r} \int_{r_0}^{r} T^r_t(r',t) r'^2 dr' - \frac{f(t)}{r} =: 1 - \frac{2m(r,t)}{r},$$

$$e^{\nu(r,t)} = \left[ 1 - \frac{2m(r,t)}{r} \right] \exp \left\{ h(t) + 8\pi \int_{r_0}^{r} \left[ \frac{T^r_t(r',t) - T^t_t(r',t)}{r' - 2m(r',t)} \right] r'^2 dr' \right\}.$$  

$$T^r_t = \frac{1}{4\pi r^2 m(r,t),t} r^2 dr' dr.$$  

$$T^\theta_\theta \equiv T^\phi_\phi := \left( \frac{r}{2} \right) \left( T^r_r + T^t_t \right) + \left( 1 + \frac{r}{4} \nu(r,t) \right) T^r_r$$

$$+ \frac{r}{4} \left( \lambda(r,t) + \nu(r,t) \right) T^t_t - \frac{r}{4} \nu(r,t) T^t_t.$$  

The functions $f(t)$ and $h(t)$ are arbitrary or free functions of integration. By a coordinate transformation $\tilde{t} = \int \exp[\frac{h(t)}{2}]dt$, and subsequently dropping the hat notation, one may rewrite (6b) as:

$$e^{\nu(r,t)} = \left[ 1 - \frac{2m(r,t)}{r} \right] \exp \left\{ 8\pi \int_{r_0}^{r} \left[ \frac{T^r_t(r',t) - T^t_t(r',t)}{r' - 2m(r',t)} \right] r'^2 dr' \right\}.$$  

(7)

Usually, to avoid a singularity at $r = 0$, $f(t) \equiv 0$. However, we shall retain it for future use.

The range of the $r$-coordinate in (1), prompted by the general solution and energy conditions in (5a-d), and the domain $D_r$ are given by:

$$0 < 2m(r,t) < r_1 \leq r_0 < r < r_2$$

$$D_r := \{ (r,t) \in \mathbb{R}^2 : r_1 < r < r_2, t_1 < t < t_2 \}.$$  

In many problems the components of $T^\mu_\nu$ are due to specific fluids or fields. In these cases the number of unknowns versus the number independent equations may be different than suggested. However, the general solution presented above still contains these (which may be variationally derived and determinate) as special cases.

Otherwise, one is free to prescribe the energy density and radial pressure. This method is useful in examinations of relativistic stellar structure where one
usually prescribes a reasonable energy and pressure from nuclear theory and studies of plasmas [19].

We now turn our attention to the $T$-domain. Since this domain is physically quite different from the previous, we adopt a new, hopefully clear notation for quantities in this domain to avoid confusion. The metric for the $T$-domain is furnished by

$$ds^2 = -e^{\gamma(T,R)}dT^2 + e^{\alpha(T,R)}dR^2 + T^2 d\theta^2 + T^2 \sin^2 \theta d\phi^2,$$

with

$$T_1 < T < T_2, \quad R_1 < R < R_2, \quad 0 < \theta < \pi, \quad 0 \leq \phi < 2\pi.$$

The vacuum $T$-domain is given by the well known metric:

$$ds^2 = \left[\frac{2M}{T} - 1\right]^{-1}dT^2 + \left[\frac{2M}{T} - 1\right]dR^2 + T^2 d\theta^2 + T^2 \sin^2 \theta d\phi^2. \quad (10)$$

Einstein’s field equations, $G_{\mu\nu} = 8\pi \Theta_{\mu\nu}$, yield:

$$-8\pi \Theta_{T,T}^T = \frac{1}{T^2} + \frac{e^{-\gamma(T,R)}}{T} \left(\alpha(T,R)_T + \frac{1}{T}\right), \quad (11a)$$

$$-8\pi \Theta_{R,R}^R = \frac{1}{T^2} - \frac{e^{-\gamma(T,R)}}{T} \left(\gamma(T,R)_T - \frac{1}{T}\right), \quad (11b)$$

$$-8\pi \Theta_{T,R}^T = \frac{e^{-\gamma(T,R)}}{T} \left(\alpha(T,R)_R\right), \quad (11c)$$

$$-8\pi \Theta_{\theta,\theta}^\theta \equiv -8\pi \Theta_{\phi,\phi}^\phi = \frac{e^{-\gamma(T,R)}}{2} \left[\alpha(T,R)_T + \frac{1}{2} (\alpha(T,R)_T)^2 - \alpha(T,R)_T \gamma(T,R)_T + \frac{1}{T} (\alpha(T,R) - \gamma(T,R))_T\right]$$

$$- \frac{e^{-\alpha(T,R)}}{2} \left[\gamma(T,R)_R + \frac{1}{2} (\gamma(T,R)_R)^2 - \gamma(T,R)_R \alpha(T,R)_R\right]. \quad (11d)$$
The conservation equations, $\Theta^\mu_{\nu, \mu} = 0$ lead to

$$
\Theta^T_{T,T} + \Theta^R_{T,R} + \frac{1}{2} \left( \alpha(T, R) + \gamma(T, R) \right)_{, R} \Theta^R_T - \frac{1}{2} \alpha(T, R)_{, T} \Theta^R_R - \frac{2}{T} \Theta^\theta_{, T} = 0,
$$

(12a)

$$
\Theta^R_{R,R} + \Theta^T_{R,T} + \left[ \frac{1}{2} \left( \alpha(T, R) + \gamma(T, R) \right)_{, T} + \frac{2}{T} \right] \Theta^T_R
\frac{1}{2} \alpha(T, R)_{, T} \Theta^R_R - \frac{2}{T} \Theta^\theta_{, T} = 0.
$$

(12b)

The general solution of the system (11a-d) and (12a-b) is given by:

$$
e^{-\gamma(T, R)} = \frac{1}{T} \left[ \phi(R) - 8\pi \int^T_{T_0} T^2 \Theta^R_R(T', R) dT' \right] - 1 =: \frac{2\mu(T, R)}{T} - 1,
$$

(13a)

$$
e^{\alpha(T, R)} = \exp \left\{ \beta(R) + 8\pi \int^T_{T_0} \left[ \frac{\Theta^T_T(T', R) - \Theta^R_R(T', R)}{2\mu(T', R) - T'} \right] T'^2 dT' \right\}
\times \left[ \frac{2\mu(T, R)}{T} - 1 \right],
$$

(13b)

$$
\Theta^T_R := \frac{1}{4\pi T^2} \mu(T, R),
$$

(13c)

$$
\Theta^\theta_{\phi} := \Theta^\phi := \frac{T}{2} \left( \Theta^T_{T,T} + \Theta^R_{T,R} \right) + \left[ 1 + \frac{T}{4} \alpha(T, R)_{, T} \right] \Theta^T_T
\frac{T}{4} \left( \alpha(T, R) + \gamma(T, R) \right)_{, R} \Theta^R_R - \frac{T}{4} \alpha(T, R)_{, T} \Theta^R_R.
$$

(13d)

Here, the functions $\phi(R)$ and $\beta(R)$ are arbitrary or free functions of integration. The arbitrary function $\beta(R)$ may be eliminated by a similar transformation as in the $R$-domain:

$$
\hat{R} = \int \exp[\beta(R)/2] dR.
$$

(14)

Subsequently, dropping the hats, (13b) yields

$$
e^{\alpha(T, R)} = \left[ \frac{2\mu(T, R)}{T} - 1 \right] \exp \left\{ 8\pi \int^T_{T_0} \left[ \frac{\Theta^T_T(T', R) - \Theta^R_R(T', R)}{2\mu(T', R) - T'} \right] T'^2 dT' \right\}.
$$

(15)
A valid $T$-domain for the above solution is provided by

$$0 < T_1 < T_0 < T_2 < 2\mu(R,T)$$

$$D_T := \{(T,R) \in \mathbb{R}^2 : T_1 < T < T_2, R_1 < R < R_2\}.$$  \hfill (16)

It is evident that there exist unmistakable similarities between the solution given in (6a-d) and the solution (13a-d). These solutions, though similar, yield completely different physics as will be discussed later. These differences hinge on the fact that these are two completely different charts leading to different physical quantities in each domain. For example, (13a) shows that, to make a positive contribution to the Schwarzschild mass, a negative pressure or tension is required. There is, however a duality between the solutions in the two domains which we will state in the form of a theorem.

**Theorem: 1** Let a set of solutions of the spherically symmetric gravitational equations and of differentiability class $C^3$ in the domain $D_r$ of equation (3) be furnished by the equations in (6a-d). Then, a distinct set of solutions in the domain $D_T$ of equation (16) as provided in the equations (13a-d) by the duality transformation symbolically denoted as:

$$r \rightarrow T, \ t \rightarrow R, \ \gamma = \lambda, \ \alpha = \nu, \ \phi = f, \ \beta = h,$$

$$\Theta_R^R := T^t_t, \ \Theta_T^T := T^r_r, \ \Theta_T^R := T^r_t.$$ \hfill (17)

Proof emerges directly from equations (6a-d) and (13a-d). However, the duality transformation from domain $D_r$ into domain $D_t$ does not exist. In case, solutions in $D_r$ and $D_T$ represent two distinct coordinate systems in the same universe, the corresponding coordinate neighbourhoods are necessarily disjoint. Moreover, the domains $D_r$ and $D_T$ may correspond to two distinct universes. It is interesting to note that physically reasonable solutions in $D_r$ with, for example, $T^t_t(r,t) < 0$ yield, by (17), necessarily exotic fluid in $D_T$ with $\Theta_R^R(T,R) < 0$. Other applications of the above will be furnished later when we analyse specific solutions.

It is useful at this point to compute the components of the orthonormal Riemann tensor components in $D_T$ as this will later aid our study of the singularity structure of the solution. We choose the natural orthonormal basis from (9) as:

$$e^\mu_T = e^{-\gamma/2}\delta^\mu_T, \ e^\mu_R = e^{-\alpha/2}\delta^\mu_R, \ e^\mu_\theta = T^{-1}\delta^\mu_\theta, \ e^\mu_\phi = (T \sin \theta)^{-1}\delta^\mu_\phi.$$ \hfill (18)
In this frame, the curvature tensor possesses the following components as well as those related by symmetry:

$$ R_{(T\theta \theta \theta)} = \frac{1}{2} \left\{ e^{-\gamma(T,R)} \left[ \frac{1}{2} \gamma(T,R)_T \alpha(T,R)_T - \frac{1}{2} (\alpha(T,R)_T)^2 ight. 
\left. - \alpha(T,R)_{T,T} \right] + e^{-\alpha(T,R)} \left[ \gamma(T,R)_{R,R} + \frac{1}{2} (\gamma(T,R,R)^2 
\left. - \frac{1}{2} \alpha(T,R)_{R,R} \right] + e^{-\gamma(T,R)} \right\},$$

(19a)

$$ R_{(T\theta \theta)} = \frac{\alpha(T,R)_T}{2T} e^{-\gamma(T,R)},$$

(19b)

$$ R_{(R\theta \theta \theta)} = \frac{\alpha(T,R)_T}{2T} e^{-\gamma(T,R)},$$

(19c)

$$ R_{(\theta \phi \theta \phi)} = \frac{1 + e^{-\gamma(T,R)}}{T^2} = \frac{2\mu(T,R)}{T^3},$$

(19d)

$$ R_{(T\theta \theta \theta)} = \frac{\gamma(T,R)_{R,R} e^{-\frac{1}{2} (\gamma(T,R) + \alpha(T,R))}}{2T}. $$

(19e)

Here, indices in parentheses are used to denote expressions calculated in the orthonormal frame.

The singularity theorems of Penrose and Hawking \[20\], \[21\] are well known. These are proved under the satisfaction of certain energy conditions. However, solutions (13a-d) also produce a singularity in $D_T$, without the introduction of energy conditions, as will be demonstrated next.

**Theorem: 2** Let the metric functions $\alpha(T,R)$ and $\gamma(T,R)$ be at least of class $C^3$ and the stress-energy tensor be of class $C^1$ in the $T$-domain. Moreover, let the tension function, $\mu(T,R)$, be of class $C^3$ and satisfy the inequality $2\mu(T,R) / T > 1$ in $D_T$. In that case, $\lim_{T \to 0^+} R_{(\theta \phi \theta \phi)}$ diverges for all $R \in (R_1, R_2)$.

**Proof:** By the inequality $2\mu(T,R) / T > 1$ and the condition of thrice differentiability, it can be concluded that in the positive neighbourhood of $T = 0$,

$$ \frac{2\mu(T,R)}{T} = 1 + [H(T,R)]^2. $$

Here, $H(T,R) \neq 0$ is some function of class $C^3$. Denoting by

$$ h(R) := \lim_{T \to 0^+} H(T,R), $$

10
it is derived that
\[
\lim_{T \to 0^+} \left[ \frac{2\mu(T, R)}{T} \right] = 1 + [h(R)]^2, \quad R \in (R_1, R_2).
\]

By the above equation and (19d) it is proved that
\[
\lim_{T \to 0^+} R_{(\theta\phi\theta\phi)} = \lim_{T \to 0^+} \frac{1 + [H(T, R)]^2}{T^2} = \left\{ 1 + [h(R)]^2 \right\} \lim_{T \to 0^+} \frac{1}{T^2} \to \infty.
\]
Thus, the conclusion of the stated theorem is proved. ■

This theorem has interesting consequences. Essentially, it states that a spherically symmetric space-time can not possess a Lorentzian $T$-domain metric which is regular at $T = 0$ regardless of energy conditions. If the space-time is to contain a black hole and be regular at $T = 0$ one must either abandon an everywhere Lorentzian metric or else an “inner” Cauchy horizon must exist at some $T = T_i$ such that $0 < T_i < T_b < 2M$, with $T_b$ being the matter-vacuum boundary. This yields an $R$-domain type metric in the region $0 < T < T_i$. This last fact has been previously noted in [5] using geodesic completeness. However, it is useful to illustrate how this comes about from our local analysis. In light of the no-hair theorem, the above theorem should apply to many non-spinning, collapsing bodies even if they initially deviate from spherical symmetry.

2.1 Patching to the Vacuum Solution

In this section we address the issue of patching the interior matter solution to the vacuum Schwarzschild line element given by (10). At the junction, $T = B(R)$, the condition of Synge [18] is chosen:
\[
\Theta^\mu_n |_{T=B(R)} = 0,
\]
where $n_\mu$ are the covariant components of a unit normal vector to the boundary $T = B(R)$. This boundary may be defined by a level curve of the function $F(T, R) := B(R) - T = 0$. We may use the gradient, $F(T, R)_\mu$, of this function to define components of the normal. Namely, $n_T \propto F(T, R)_T = -1$ and $n_R \propto F(T, R)_R = B(R)_R$ with other components zero. The explicit junction conditions then read:
\[
[\Theta^R_R B(R)_R - \Theta^T_R] |_{T=B(R)} = 0,
\]
\[
[\Theta^R_T B(R)_T - \Theta^T_T] |_{T=B(R)} = 0.
\]
Solutions which cannot meet these conditions may still be useful as local solutions.

Outside the matter region, \( 2M > T > B(R) \), the solution for \( g_{TT} \) yields, via (13a),

\[
g_{TT} = -\left( \frac{2\mu(B(R), R)}{T} - 1 \right)^{-1}. \tag{22}
\]

The function \( \mu(B(R), R) \) is the total invariant Schwarzschild mass. Assuming for the moment that the junction conditions are met, the fact that \( \mu(B(R), R) \) is indeed a constant may be shown utilizing (13c) along with the condition (21a):

\[
2\mu(B(R), R) = f(R) - 8\pi \int_{T_0}^{B(R)} \Theta^R_T(T', R) T'^2 dT'. \tag{23}
\]

Therefore,

\[
2\mu(B(R), R)_R = 2 \left\{ \mu(T, R), R |_{T=B(R)} + \mu(T, R), T |_{T=B(R)} B(R), R \right\} = -8\pi [B(R)]^2 \left[ \Theta^R_R(T, R) B(R), R - \Theta^T_R(T, R) \right] |_{T=B(R)} = 0. \tag{24}
\]

When this is patched to the vacuum Schwarzschild \( T \)-domain metric (10) it is clear that the parameter \( \mu(B(R), R) = M \) is indeed identical to the Schwarzschild mass as observed by an external observer. The very interesting point is the following: Observers outside the black hole (in the \( R \)-domain) feel the usual effects of gravity for a Schwarzschild black hole or star of mass \( M \). The gravitational effects, however, are more likely generated by a tension rather than an energy density. This tension generated mass leads us to the term “exotic matter”.

At the boundary, the non-vacuum solution becomes:

\[
ds^2 |_{B(R)} = - \left[ \frac{2M}{B(R)} - 1 \right]^{-1} dT^2 + \left[ \frac{2M}{B(R)} - 1 \right] \exp [S(R)] dR^2 + [B(R)]^2 d\theta^2 + [B(R) \sin \theta]^2 d\phi^2, \tag{25}
\]

with \( S(R) \) given by:

\[
S(R) := 8\pi \int_{T_0}^{B(R)} T'^2 \left[ \frac{\Theta^R_R(T', R) - \Theta^T_T(T', R)}{T' + 8\pi \int_{T_0}^{T'} T'' \Theta^R_R(T'', R) dT''} \right] dT'. \tag{26}
\]
The \(\exp[S(R)]\) term may be absorbed into the definition of a new radial coordinate, \(\hat{R}\), so that (25) becomes:

\[
\begin{align*}
 ds^2|_{\hat{B}_-(\hat{R})} &= - \left[ \frac{2M}{B(\hat{R})} - 1 \right]^{-1} dT^2 + \left[ \frac{2M}{B(\hat{R})} - 1 \right] d\hat{R}^2 \\
 &+ [\hat{B}(\hat{R})]^2 d\theta^2 + [\hat{B}(\hat{R}) \sin \theta]^2 d\phi^2.
\end{align*}
\]

(27)

Dropping hats subsequently, it may be seen that this is indeed the limit \(T \to B_+(R)\) of the vacuum black hole solution (10).

We shall now briefly discuss the Israel [23] junction condition in the \(T\)-domain which demands continuity of the second fundamental form at the stellar boundary. The extrinsic curvature tensor of the non-null hypersurface \(T = B(R)\) is given by

\[
K_{RR} = \frac{1}{\sqrt{|e^{-\alpha(T,R)}(B'(R))^2 - e^{-\gamma(T,R)}|}} \left\{ B''(R) \\
+ \frac{1}{2} \left[ e^{\alpha(T,R) - \gamma(T,R)} \alpha(T,R)_T + (2\gamma(T,R) - \alpha(T,R))_{,R} B'(R) \\
+ (\gamma(T,R) - 2\alpha(T,R))_{,T} (B'(R))^2 \\
- e^{\gamma(T,R) - \alpha(T,R)} \gamma(T,R)_R (B'(R))^3 \right]_{T=\hat{B}(R)} \right\},
\]

(28a)

\[
K_{\theta\theta} = \frac{B(R)e^{-\gamma(B(R),R)}}{\sqrt{|e^{-\alpha(T,R)}(B'(R))^2 - e^{-\gamma(T,R)}|}},
\]

(28b)

\[
K_{\phi\phi} = \sin^2 \theta K_{\theta\theta}.
\]

(28c)

This completes the analysis of the \(T\)-domain general solution.

### 2.2 Eternal Black Holes

Consider now the special, spatially homogeneous cases, where there is an additional Killing vector given by \(\frac{\partial}{\partial R}\). This is the \(T\)-domain analogue of static
black holes. Following the previous prescription, the solution is furnished by:

\[ e^{-\gamma(T,R)} = \frac{1}{T} \left[ A - 8\pi \int_{T_0}^{T} T'^2 \Theta^R_R(T') dT' \right] - 1 = \frac{2\mu(T)}{T} - 1, \]  

\[ e^{\alpha(T,R)} = \exp \left\{ K + 8\pi \int_{T_0}^{T} \left[ \Theta^T_T(T') - \Theta^R_R(T') \right] \frac{1}{2\mu(T')} - T' \right\} \times \left[ \frac{2\mu(T)}{T} - 1 \right], \]  

\[ \Theta^T_R \equiv 0, \]  

\[ \Theta^\theta_\theta \equiv \Theta^\phi_\phi := \frac{T}{2} \Theta^T_{T,T} + \left[ 1 + \frac{T}{4} \alpha(T,T) \right] \Theta^T_T \]  

\[ - \frac{T}{4} \alpha(T,T) \Theta^R_R. \]  

Here, \( A \) and \( K \) are constants of integration.

The boundary hypersurface is given by the simple equation:

\[ T = B(R) := T_b = \text{a constant}, \]  

\[ B'(R) = B''(R) \equiv 0. \]  

The junction conditions of Synge (21a, 21b) reduce to

\[ \Theta^T_R(T_b) = 0 \]  

\[ \Theta^T_T(T_b) = 0. \]  

The first junction condition is automatically satisfied via (29c) so that the matter field must only obey (31b).

The Israel conditions boil down to the continuity of the metric as well as the following extrinsic curvature components:

\[ K_{RR}(T_b) = \frac{1}{2} e^{-\gamma(T_b)/2} (e^{\alpha(T)}),_{T,T=0} \]  

\[ K_{\theta\theta} = T_b e^{-\gamma(T_b)/2}. \]  

We next consider both spatially homogeneous and non-homogeneous particular solutions.
3 Particular Solutions

Here we investigate particular solutions of the general case described above. We first consider a special, tension generated solution provided by:

\[ \Theta_T^T(T) \equiv 0, \quad (33a) \]
\[ \Theta_R^R(T) = -\frac{1}{8\pi T^2} \delta(T) < 0, \quad (33b) \]

with \( \delta(T) \) the Dirac delta function. In this case, the equations (29a,b) yield the \( T \)-domain Schwarzschild metric of (10). (A rescaling of the \( R \)-coordinate is tacitly assumed.)

It is interesting to note that the source of this field may be interpreted as possessing a velocity which is of a tachyonic nature. The above stress-energy tensor may be written as:

\[ \Theta^\mu_\nu = -\frac{1}{8\pi T^2} \delta(T) s^\mu s_\nu \]

with

\[ s^T = s^\theta = s^\phi \equiv 0, \quad s^\alpha s_\alpha = +1. \]

The eigenvalue equation, \( \Theta^\mu_\nu s^\nu = -1/(8\pi T^2) \delta(T)s^\mu \), indicates that \( s^\mu \) may be interpreted as a space-like dynamical mean velocity of the source. This is also evident from visual inspection of the standard Kruskal-Szekeres diagram.

Next we study the well known interior solution of Schwarzschild [24]. In the \( R \)-domain it is furnished by:

\[
\begin{align*}
    ds^2 &= -\left[ \frac{3\sqrt{|1-qr^2|} - \sqrt{|1-qr^2|}}{3\sqrt{|1-qr^2|} - 1} \right]^2 dt^2 + \frac{[1-qr^2]^{-1}}{2} dr^2 \\
    &\quad + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \\
    T^r_r &= T^\theta_\theta = T^\phi_\phi = \frac{3q}{8\pi} \left[ \frac{\sqrt{|1-qr^2|} - \sqrt{|1-qr^2_b|}}{3\sqrt{|1-qr^2_b|} - \sqrt{|1-qr^2_b|}} \right] > 0, \quad (34) \\
    T_t^r &= -\frac{3q}{8\pi} < 0, \\
    T_t^r &= 0, \\
    T_r^r(r_b) &= 0, \\
    0 &< r < r_b < \frac{1}{\sqrt{q}}.
\end{align*}
\]
Here $q > 0$ is a constant proportional to the mass density and $r = r_b > 0$ is the boundary of the spherical star. We now apply the duality to the above solution and show that its $T$-domain analogue possesses quite distinct physical properties.

In the $T$-domain, the solution is a function of the corresponding time-like coordinate only. This arises from the “static” condition which is a common simplification in studies of $R$-domain stellar structure exact solutions. In the $R$-domain, the assumption that the radial pressure, $T_r$, is equal to the transverse pressure now becomes:

$$\Theta^T T = \Theta^\theta \equiv \Theta^\phi.$$  \hspace{1cm} (35)

These conditions give rise to the solution:

$$ds^2 = -(qT^2 - 1)^{-1} dT^2 + \left[\frac{3\sqrt{|qT^2_b - 1|} - \sqrt{|qT^2 - 1|}}{3\sqrt{|qT^2 - 1|} - 1}\right]^2 dR^2$$

$$+ T^2 d\theta^2 + T^2 \sin^2 \theta d\phi^2,$$

$$\Theta^R R = -3 \frac{q}{8\pi} < 0,$$

$$\Theta^T T = \Theta^\theta \equiv \Theta^\phi = -3 \frac{q}{8\pi} \left[\frac{\sqrt{|qT^2_b - 1|} - \sqrt{|qT^2 - 1|}}{3\sqrt{|qT^2 - 1|} - \sqrt{|qT^2 - 1|}}\right] < 0, \Theta^T T_b = 0,$$

$$0 < T_i < T < T_b.$$  \hspace{1cm} (36)

We now investigate the energy conditions regarding the above solution. The eigenvalue problem, $\Theta^{\mu}_{\alpha} v^{\alpha} = \lambda v^{\mu}$, is trivial since $\Theta^{\mu}_{\nu}$ is diagonal. The eigenvectors, $v^{\mu}$, define an orthonormal tetrad: $v^{\mu}_T, v^{\mu}_R, v^{\mu}_{\theta}$ and $v^{\mu}_{\phi}$ (the corresponding eigenvalues denoted by $\lambda_T, \lambda_R, \lambda_{\theta}$ and $\lambda_{\phi}$). The mixed stress-energy momentum tensor admits a decomposition in terms of its eigenvalues and eigenvectors. This decomposition is

$$\Theta^{\mu}_{\nu} = \Theta^{(\alpha)}_{(\beta)} v^{\mu}_{(\alpha)} v^{(\beta)}_{\nu}.$$  \hspace{1cm} (37)

With the notation

$$p_\perp := \lambda_{(\theta)} = \lambda_{(\phi)} = \lambda_T := -\rho = -\frac{3q}{8\pi} \left[\frac{(qT^2_b - 1)^{1/2} - (qT^2 - 1)^{1/2}}{3(qT^2_b - 1)^{1/2} - (qT^2 - 1)^{1/2}}\right] < 0$$

and

$$\lambda_R := p_i = -\frac{3q}{8\pi} < 0,$$
the equation (37) allows us to write
\[
\Theta^\mu_\nu = (\rho - |p_\nu|) s^\mu s_\nu - \rho \delta^\mu_\nu. \tag{38}
\]

Here \( s^\mu = (\nu^\mu_{(R)}, s^\mu s_\mu = 1 \) and \( s^\theta = s^\phi = s^T = 0 \). This is somewhat analogous to the case for a perfect fluid except that the isotropic pressure is replaced by a radial tension and the velocity of the fluid is space-like. This makes the fluid tachyonic in nature rather than a perfect fluid. The fluid is anisotropic since the angular tensions, \( p_\perp \), are equal in magnitude to \( \rho \) which differs from the radial tension, \( p_\parallel \).

Although the energy density is positive, the weak, strong and dominant energy conditions are not satisfied since
\[
\rho + p_\perp = 0, \tag{39a}
\]
\[
\rho + p_\parallel = -\frac{6q(qT_b^2 - 1)^{1/2}}{8\pi [3(qT_b^2 - 1)^{1/2} - (qT^2 - 1)^{1/2}]} < 0. \tag{39b}
\]

Thus, the fluid matter is exotic matter. This solution constitutes an eternal exotic black hole because the exotic matter lies entirely within the \( T \)-domain. Observers in domains \( D_I \) of the Kruskal-Szekeres spacetime see a black hole of Schwarzschild mass \( M = qT_b^3/2 \) even though the \( T \)-domain is substantially different.

At the stellar boundary, \( T = T_b \), we wish to employ the Israel boundary condition. The components of the extrinsic curvature tensor for the interior solution at \( T_b \) are given by:
\[
K_{RR|T_b^-} = -2qT_b\sqrt{qT_b^2 - 1}\left[\frac{3\sqrt{qT_b^2 - 1} - 1}{3\sqrt{qT_b^2 - 1} - 1}\right]^2, \tag{40a}
\]
\[
K_{\theta\theta|T_b^-} = T_b\sqrt{qT_b^2 - 1}. \tag{40b}
\]

Those of the \( T \)-domain vacuum solution (10) are given by:
\[
K_{RR|T_b^-} = -\frac{M}{T_b^2}\sqrt{\left(\frac{2M}{T_b} - 1\right)} \exp[S], \tag{41a}
\]
\[
K_{\theta\theta|T_b^-} = T_b\sqrt{\left(\frac{2M}{T_b} - 1\right)}. \tag{41b}
\]

\[\text{In the literature, exotic matter usually violates energy conditions because the energy density } \rho < 0 \text{ which is not the case here. However, another common feature in studies of exotic matter is that the principal stresses are tensions rather than pressures. For this reason, the matter is still called exotic.}\]
referring to (25) and (26) for the definition of $S$. Metric continuity at $T_b$ along with the result $M = \frac{4}{3} T_b^3$, immediately gives

$$\exp[S] = \frac{4}{T_b \left(3 \sqrt{\frac{2M}{T_b}} - 1 - 1\right)^2},$$

and therefore the extrinsic curvature and metric are continuous at $T = T_b$. The solution is shown schematically in figure 1.

![Figure 1: Qualitative representation of exotic matter existing in the T-domain along with the image in the Kruskal spacetime.](image)

To complete the above analysis, consider the tension generated Schwarzschild mass, $M$, as a constant and the parameter $q$ as a variable. In terms of $M$ and $q$, set the boundary parameters as

$$T_i := q^{-1/2} \quad \text{and} \quad T_b := \left(\frac{2M}{q}\right)^{1/3}. \quad (42)$$

Hence, $D_I$, is as wide as it possibly can be since $T_i$ is as small as it can be for a prescribed value of $q$. Holding $M$ constant and letting $q$ increase without bound, both parameters, $T_i$ and $T_b$, decrease towards zero. Thus, as $q$ increases, the tachyonic fluid domain shrinks down to a singularity and the entire Schwarzschild $T$-domain is recovered.
3.1 Regular Black Holes

We next attempt to construct a reasonably general non-singular $T$-domain solution. The philosophy here is to study the (curvature) singularity properties of the space-time manifold. As such, we employ a geometric method in solving the field equations since we a priori have in mind certain geometric properties (namely, a $T$-domain without singularities). The metric is prescribed here in accordance with the non-singular properties dictated by the orthonormal Riemann tensor. This, in turn, will dictate the properties the matter field must possess in order to support non-singular structure.

Solutions here are treated as local and illustrate the general properties that solutions must possess in order to be regular. We consider both the regular Lorentzian and Instanton black hole. Some very interesting work in this field regarding topology change may be found in [25]. Also, a regular black hole using a hypothesis of the existence of a limiting curvature may be found [7]. There, a deSitter universe is patched at the point of limiting curvature. In this section we are loosely using the same coordinates for all $T < 2M$. That is, the $T$ and $R$ coordinates are used for all domains “inside” the outer event horizon, even if it is an $R$-type or Euclidean domain.

The first regular solution will correspond to an eternal $T$-sphere with “hard” polytropic equation of state [19]:

$$\rho = \kappa p,$$

with $\kappa$ a constant. For such a solution consider the following metric for the matter domain:

$$ds^2 = - (2CT^{2x} - 1)^{-1}dT^2 + (2CT^{2x} - 1) dR^2 + T^2 d\theta^2 + T^2 \sin^2 \theta d\phi^2,$$

with $x > 0$ and $C > 0$. Such a metric remains everywhere Lorentzian with the inner horizon located at $T = T_i = 1/(2C)^{1/x}$. It is easy to check that the energy density is proportional to both the parallel and transverse pressures. Furthermore, in the $T$-domain, both pressures are negative or tensions. This solution respects the weak energy condition since

$$\frac{\rho + p_n}{|\rho|} \geq 0,$$

as well as $\rho \geq 0$. 

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The non-zero orthonormal Riemann components are furnished by

\[
R_{(TTR)} = -(2 + x)(1 + x)CT^x, \tag{46a}
\]

\[
R_{(T\theta\theta)} = -C(2 + x)T^x, \tag{46b}
\]

\[
R_{(RR\theta)} = C(2 + x)^2T^x, \tag{46c}
\]

\[
R_{(\theta\phi\phi)} = 2CT^x, \tag{46d}
\]

which are finite throughout the matter domain. In the special case \(x = 0\) this solution yields a matter domain of constant curvature similar to a deSitter solution. This solution provides a simple, regular polytropic \(T\)-domain model for a collapsed spherical star.

We next consider an example which is regular but may be Euclidean in part of the manifold. It is unknown how physical this situation may be. However, this case is worth discussing since it is generally believed that at near the Planck scale, the concepts of space and time may lose their meanings [28], [29]. Also, in astrophysical contexts, results in [4] found Euclidean instanton properties in the late collapse stages of a regular, anisotropic star. As well, there is the well known Euclidean cosmological instanton of Hawking and Turok [30] used to remove the big bang singularity.

We wish to include a dependence on the interior radial coordinate, \(R\), to make the model more physical than \(T\)-spheres. Unfortunately, adding non-trivial \(R\)-dependence to the mass term usually leads to a singularity at the inner horizon. Therefore, it is assumed that the mass term is independent of \(R\) and all \(R\)-dependence is incorporated through \(g_{RR}\). The Riemann component \(R_{(RR\theta)}\) dictates that, in the vicinity of \(T = 0\), \(\alpha(T, R)\) behave as

\[
\alpha(T, R) = \lambda(R)T^y + \xi(R), \tag{47}
\]

with \(y > 2\). Here \(\lambda(R)\) and \(\xi(R)\) are bounded, sufficiently differentiable functions which are otherwise arbitrary and depend on the physical model.

Following the previous example, we set the line element as:

\[
ds^2 = -(2CT^{2+x} - 1)^{-1} dT^2 + \exp [\lambda(R)T^y + \xi(R)] dR^2 + T^2 d\theta^2 + T^2 \sin^2 \theta d\phi^2. \tag{48}
\]

This solution is treated here as local within the vicinity of \(T = 0\) and the inner horizon. It is useful so study such a solution since all regular solutions subject to the restrictions above must possess the form (48) if \(T_i\) is sufficiently “near” \(T = 0\) (i.e. minimizing the Euclidean domain).
The metric \( R(T_{RTR}) \) yields the orthonormal Riemann components:

\[
R(T_{RTR}) = \frac{1}{2} \left[ 2CT^{2+x} - 1 \right] \left\{ y\lambda(R)T^{y-1} \left[ \frac{1}{2} y\lambda(R)T^{y-1} + (y-1)T^{-1} \right] \right\},
\]

\[
R(T_{\theta T\theta}) = -C'(2 + x)T^x,
\]

\[
R(R_{\theta R\theta}) = \frac{1}{2} T^y \left( 2CT^{2+x} - T^{-2} \right),
\]

\[
R(\theta_\phi \phi_\phi) = 2CT^x.
\]

Note that this solution is regular both at \( T = 0 \) and \( T = T_i = 1/(2C)^{1/(2+x)} \).

Finally, the stress-energy tensor supporting this solution is furnished by:

\[
\Theta^T_T = \frac{1}{8\pi} \left\{ \lambda(R)yT^{y-2} - 2CT^x \left[ 1 + \lambda(R)yT^y \right] \right\},
\]

\[
\Theta^R_R = -\frac{1}{4\pi} C'T^x (3 + x),
\]

\[
\Theta^\theta_\theta = -\frac{1}{32\pi} \left\{ 4C(2 + x)T^x + 2\lambda(R)yCT^{y+x} (2 + 2y + x) + \lambda(R)T^y \left( 2C\lambda(R)y^{y+2} - \lambda(R)y^{2y-2} - 2y^2T^{-2} \right) \right\}.
\]

All physical quantities are therefore bounded at \( T = 0 \) and \( T = T_i \).

4 Conclusion

In this paper we have considered spherically symmetric matter fields in the Schwarzschild \( T \)-domain. We developed the general solution as well as some interesting particular solutions. The general solution demonstrates that a locally measured tension is the source of the invariant Schwarzschild mass which govern vacuum gravitational effects felt by observers outside the black hole. Junction conditions which must be met in order to patch matter solutions to the \( T \)-domain vacuum have also been addressed. The general approach laid out here also serves as a reasonable starting point for future work in the \( T \)-domain.

Also considered was the \( T \)-domain analogue of constant density stars. We found that the matter supporting such a structure is exotic in the sense that principal stresses are tensions. Also, although the energy density is positive, the matter field is locally tachyonic in nature.
Finally, regular interiors were considered. There are two possible scenarios for non-singular black holes addressed here. One is the presence of a second horizon inside the black hole, restoring $R$-domain like structure to the spacetime near $T = 0$. The other is an “instantonic” phase which changes the spacetime signature to $+4$ as has been found in some studies on gravitational collapse. There seems to be no way to possess non-singular structure without resorting to one of the above situations.
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