Entanglement production with Bose atoms in optical lattices

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Abstract. A method of entanglement production is suggested, based on the resonant generation of topological modes in systems with Bose-Einstein condensates trapped in optical or magnetic lattices. The method makes it possible to regulate the strength of entanglement production as well as to govern its time variation. This method can serve as a practical tool for quantum information processing and quantum computing.

1. Introduction
Systems with Bose-Einstein condensate (BEC) possess a number of unusual properties (see book [1] and review articles [2–7]), which can be employed in a variety of applications. One of such important applications is the possibility of creating massive entanglement in Bose-condensed systems trapped in lattice potentials [8,9]. The most known examples of the lattices are the optical lattices formed by a combination of laser beams [10–12], though magnetic fields could also be used for creating magnetic lattices. In experiments, one can form optical lattices of varying spacing, depth, and different filling factors, ranging between one and $10^4$ atoms in each lattice site [13,14]. When a lattice is sufficiently deep and the filling factor is large, then a lattice site represents a microtrap, in which at low temperature BEC can be formed.

A BEC in a trap possesses a set of discrete or quasidiscrete atomic energy levels. If different trapping sites are completely separated from each other, the energy spectrum is purely discrete. When atoms can tunnel between the sites, then a line of the discrete spectrum widens into a band. In what follows, we consider the situation, when the linewidths are not too large, so that the bands are always well separated from each other, that is, when the linewidths are much smaller than the spectrum gaps. The latter situation corresponds to a quasidiscrete spectrum. Note that we are talking here about the atomic BEC energy levels in a trap, which should not be confused with the spectrum of elementary excitations.

In an equilibrium system, BEC sets in the lowest energy level. But if the system is subject to an alternating modulating field, with a frequency in resonance with one of the transition frequencies, then a nonground-state BEC can be realized, as was first proposed in Ref. [15]. The condensate functions, describing the standard ground-state BEC and the nonground-state condensates have different spatial shapes, because of which the condensate wave functions, pertaining to different energy levels, can be termed topological modes. The properties of these
modes have been theoretically investigated in a series of papers [15–35] and a dipole topological mode was generated in experiment [36]. A simple example of such a mode is a vortex that can be excited in a rotating BEC.

Dynamics of BEC are usually considered in the frame of the Gross-Pitaevskii equation (see Refs. [1,2]), which presupposes the case of zero temperature and very weak atomic interactions. The Gross-Pitaevskii equation was also the basis for describing the resonant excitation of topological modes [15]. In the present paper, we show that topological modes can be created as well at finite temperature, which is due to the resonant mechanism of their generation.

Another our goal is to demonstrate how the regulated generation of topological modes in optical lattices can be used for the controlled entanglement production. The possibility of effectively varying the state of a complex quantum system and of controlling its entanglement are the two key points for realizing quantum information processing and quantum computing [37–42].

2. Topological modes

In order to give a general correct definition of topological coherent modes, we need, first, to write down the exact equation for the condensate wave function. The latter, for a Bose-condensed system, is introduced by means of the Bogolubov shift [43–45] for the field operator

\[ \psi(r, t) \rightarrow \hat{\psi}(r, t) = \eta(r, t) + \psi_1(r, t), \]

in which \( \eta(r, t) \) is the condensate wave function and \( \psi_1(r, t) \) is the field operator of uncondensed particles. The Bogolubov shift (1) explicitly breaks the gauge symmetry of the Bose system. It is worth emphasizing that the gauge symmetry breaking is the necessary and sufficient condition for the occurrence of BEC [46]. After introducing the Bogolubov shift (1), it is necessary to resort to a representative statistical ensemble for the Bose system with broken gauge symmetry [47,48], defining the appropriate grand Hamiltonian \( H[\hat{\psi}] = H[\eta, \psi_1] \). The theory of Bose-condensed systems is self-consistent only with the correctly defined grand Hamiltonian [49–51]. The general equation for the condensate wave function has the form

\[ i \frac{\partial \eta(r, t)}{\partial t} = \left( -\frac{\nabla^2}{2m} + U - \mu_0 \right) \eta(r, t) + \Phi_0 \left\{ \rho_0(r, t) + 2\rho_1(r, t) \right\} \eta(r, t) + \sigma_1(r, t) \eta^*(r, t) + \xi(r, t) \right\}. \]

Assuming the standard energy Hamiltonian with the local interaction potential

\[ \Phi(r) = \Phi_0 \delta(r), \quad \Phi_0 = 4\pi \frac{a_s}{m}, \]

where \( a_s \) is the scattering length, we obtain from Eq. (2) the exact equation for the condensate wave function

\[ i \frac{\partial \eta(r, t)}{\partial t} = \left( -\frac{\nabla^2}{2m} + U - \mu_0 \right) \eta(r, t) + \Phi_0 \left\{ \rho_0(r, t) + 2\rho_1(r, t) \right\} \eta(r, t) + \sigma_1(r, t) \eta^*(r, t) + \xi(r, t) \right\}. \]

Here \( U = U(r, t) \) is an external potential and the notation is used for the condensate density

\[ \rho_0(r, t) = |\eta(r, t)|^2, \]

the density of uncondensed atoms

\[ \rho_1(r, t) = <\psi_1^\dagger(r, t)\psi_1(r, t)>, \]
the anomalous average
\[ \sigma_1(r, t) \equiv \langle \psi_1(r, t)\psi_1(r, t) \rangle, \quad (7) \]
and the triple anomalous average
\[ \xi(r, t) \equiv \langle \psi_1^\dagger(r, t)\psi_1(r, t)\psi_1(r, t) \rangle. \quad (8) \]

Looking for the stationary solutions of Eq. (4) in the common form
\[ \eta_n(r, t) = \eta_n(r) e^{-i\omega_n t}, \]
we come to the stationary equation for the condensate wave function
\[ \left[ -\frac{\nabla^2}{2m} + U(r) \right] \eta_n(r) + \Phi_0 \left\{ \left| \eta_n(r) \right|^2 + 2\rho_1(r) \right\} \eta_n(r) + \sigma_1(r)\eta_n(r) + \xi(r) = E_n \eta_n(r), \quad (9) \]
in which
\[ E_n \equiv \mu_0 + \omega_n. \quad (10) \]

In equilibrium, BEC corresponds to the lowest energy level
\[ E_0 \equiv \min_n E_n = \mu_0, \quad \omega_0 \equiv \min_n \omega_n = 0. \]

But, generally, the nonlinear eigenvalue problem (9) possesses a set of energy levels \( E_n \) and of the related eigenfunctions \( \eta_n(r) \), labelled by a multi-index \( n \). The solutions \( \eta_n(r) \) to the eigenvalue problem (9) are the topological coherent modes. It is only in the limiting case of zero temperature and asymptotically weak interactions, when we can neglect \( \rho_1(r) \), \( \sigma_1(r) \), and \( \xi(r) \) in Eq. (9), we come to the stationary Gross-Pitaevskii equation
\[ \left[ -\frac{\nabla^2}{2m} + U(r) + \Phi_0 |\eta_n(r)|^2 \right] \eta_n(r) = E_n \eta_n(r), \]
where, actually, solely the ground-state level \( E_0 = \mu_0 \) is to be considered [1].

3. Resonant generation
Equation (9) defines the topological modes as stationary solutions. In order to realize transitions between modes, it is necessary to include a time-dependent external potential and to consider the temporal equation (4).

Suppose that at the initial time \( t = 0 \) the system is in equilibrium and the condensate wave function corresponds to the standard ground-state condensate,
\[ \eta(r, 0) = \eta_0(r) \equiv \eta(r). \quad (11) \]
Let us wish to generate a topological mode labelled by the index \( n = n_1 \), with the related energy \( E_1 = E_{n_1} \). Hence, the transition frequency is given by
\[ \omega_{10} \equiv E_1 - E_0 = E_{n_1} - \mu_0. \quad (12) \]
To generate this mode, it is necessary to apply an alternating field
\[ V(r, t) = V_1(r) \cos(\omega t) + V_2(r) \sin(\omega t) \quad (13) \]
with a frequency \( \omega \) tuned close to the transition frequency (12), so that the resonance condition

\[
\left| \frac{\Delta \omega}{\omega} \right| \ll 1 \quad (\Delta \omega \equiv \omega - \omega_0) \tag{14}
\]

be valid. Then the total external potential in Eq. (4) is the sum

\[
U(r, t) = U(r) + V(r, t) \tag{15}
\]

of a trapping potential and of the alternating field (13).

The condensate wave function is normalized to the total number of condensed atoms

\[
N_0 = \int |\eta(r, t)|^2 \, dr = \int |\eta(r)|^2 \, dr . \tag{16}
\]

Let us introduce a function \( \varphi_n(r) \), defined by the equality

\[
\eta_n(r) \equiv \sqrt{N_0} \varphi_n(r) , \tag{17}
\]

which is normalized to one

\[
\int |\varphi_n(r)|^2 \, dr = 1 . \tag{18}
\]

We shall look for the solution of Eq. (4) in the form

\[
\eta(r, t) = \sum_n C_n(t) \eta_n(r) e^{-i\omega_n t} , \tag{19}
\]

where \( \omega_n \equiv E_n - E_0 \) is in agreement with Eq. (10). The coefficient function \( C_n(t) \) is treated as a slow function of time, such that

\[
\frac{1}{\omega_n} \frac{dC_n}{dt} \ll 1 . \tag{20}
\]

This allows us to consider \( C_n \) as a quasi-integral of motion and to employ the averaging method [52] and the scale separation approach [53,54]. For instance, substituting form (19) into the normalization condition (16), and averaging the latter over time, with \( C_n \) kept as quasi-integrals, we get

\[
\sum_n |C_n(t)|^2 = 1 . \tag{21}
\]

The quantity

\[
p_n(t) \equiv |C_n(t)|^2 \tag{22}
\]

plays the role of the mode probability, or fractional mode population, which is normalized to one, according to Eq. (21).

Substituting expansion (19) into Eq. (4), we use the averaging techniques [52–54]. The resonant field (13) can be written as

\[
V(r, t) = \frac{1}{2} \left[ B(r) e^{i\omega t} + B^*(r) e^{-i\omega t} \right] , \tag{23}
\]

where

\[
B(r) \equiv V_1(r) - i V_2(r) . \tag{24}
\]

We need the notation for the interaction amplitude

\[
\alpha_{mn} \equiv N_0 \Phi_0 \int |\varphi_m(r)|^2 \left[ 2|\varphi_n(r)|^2 - |\varphi_m(r)|^2 \right] \, dr , \tag{25}
\]
pumping-field amplitude

\[ \beta_{mn} \equiv \int \varphi_m^*(r) B(r) \varphi_n(r) \, dr , \]  

(26)

where \( B(r) \) is given by Eq. (24), and for the quantity

\[ \gamma_{nn} \equiv \alpha_{nn} - \Phi_0 \int \varphi_n(r) \left\{ 2 \left[ \rho_1(r) - \rho_1(r, t) \right] \varphi_n(r) + \sigma_1(r) \varphi_n^*(r) + \frac{\xi(r)}{\sqrt{N_0}} \right\} \, dr . \]  

(27)

Also, we introduce the effective detuning

\[ \Delta_{mn} \equiv \Delta \omega + \alpha_{mm} - \alpha_{nn} , \]  

(28)

in which

\[ \alpha_{nn} = N_0 \Phi_0 \int |\varphi_n(r)|^4 \, dr . \]

From expression (27) it follows that there exists an effective time \( t_{eff} \), during which \( \gamma_{nn} \) can be treated as a real quantity, such that

\[ |\text{Im} \gamma_{nn}| t_{eff} \ll 1 , \]  

(29)

being weakly dependent on time, in the sense that

\[ \left| \frac{t_{eff}}{\gamma_{nn}} \frac{d\gamma_{nn}}{dt} \right| \ll 1 . \]  

(30)

This effective time is of the order

\[ t_{eff} \sim \frac{1}{\rho_1 \Phi_0} , \]  

(31)

where \( \rho_1 \) is the density of uncondensed atoms. When practically all atoms are condensed, so that \( \rho_1 \to 0 \), then \( t_{eff} \to \infty \). Defining

\[ c_n(t) \equiv C_n(t) \exp(i \gamma_{nn} t) , \]  

(32)

we see that, for the times shorter than \( t_{eff} \), the fractional mode population (22) can be written as

\[ p_n(t) \cong |c_n(t)|^2 \quad (0 \leq t < t_{eff}) . \]  

(33)

The initial condition (11) for the mode amplitude (32) takes the form

\[ c_n(0) = \delta_{n0} . \]  

(34)

With this initial condition, we obtain the equations

\[ i \frac{dc_0}{dt} = \alpha_{01} |c_1|^2 c_0 + \frac{1}{2} \beta_{01} c_1 e^{i \Delta_{01} t} , \]

\[ i \frac{dc_1}{dt} = \alpha_{10} |c_0|^2 c_1 + \frac{1}{2} \beta_{01} c_0 e^{-i \Delta_{01} t} , \]  

(35)

where \( \Delta_{01} = \Delta \omega + \alpha_{00} - \alpha_{11} \).

Equations (35) can be simplified by separating the absolute values \( |c_n| \) and the phases \( \pi_n \) of the complex quantities

\[ c_n = |c_n|e^{i \pi_n t} . \]  

(36)
Let us also introduce the following parameters
\[ \alpha \equiv \frac{1}{2}(\alpha_{01} + \alpha_{10}) , \quad \beta \equiv |\beta_{01}| = \beta_{01}e^{-i\gamma} , \quad \delta \equiv \Delta_{01} + \frac{1}{2}(\alpha_{01} - \alpha_{10}) . \] (37)

By defining the population imbalance
\[ s \equiv |c_1|^2 - |c_0|^2 \] (38)
and the phase difference
\[ x \equiv \pi_1 - \pi_0 + \gamma + \Delta_{01} , \] (39)
we can transform Eqs. (35) to the two-dimensional dynamical system
\[ \frac{ds}{dt} = -\beta \sqrt{1 - s^2} \sin x , \quad \frac{dx}{dt} = \alpha s + \frac{\beta s}{\sqrt{1 - s^2}} \cos x + \delta . \] (40)

Solving these equations defines the fractional mode populations (33) as
\[ p_0(t) = \frac{1 - s(t)}{2} , \quad p_1(t) = \frac{1 + s(t)}{2} . \]

Equations (35) and (40) were derived earlier [15–17] for a purely coherent system at zero temperature, when all atoms were in BEC, so that \( N_0 = N \). Here we have showed that the same equations can be obtained for a system at finite temperature, when \( N_0 < N \). The main difference is that, when the density of uncondensed atoms is not zero, then Eqs. (35) and (40) are valid not for all times, but in the time interval \( 0 \leq t < t_{\text{eff}} \), limited by the effective critical time (31).

In the same way, we could derive the equations for the dynamics of several topological modes, generated by the quasiperiodic modulating field
\[ V(\mathbf{r}, t) = \frac{1}{2} \sum_n \left[ B_n(\mathbf{r})e^{i\varepsilon_n t} + B_n^*(\mathbf{r})e^{-i\varepsilon_n t} \right] , \]
with several frequencies \( \varepsilon_n \) tuned to the resonance with different transition frequencies \( \omega_{mn} \equiv E_m - E_n \). For example, the equations for three topological modes would have the form as in Refs. [31,32], or similar to the equations for three coupled BEC [55]. The systems with multiple generated topological modes display a variety of interesting effects, such as interference patterns and interference currents [20,24], mode locking [15,24,26], dynamical transitions and critical phenomena [17,20,21,24], chaotic motion [31,32], harmonic generation and parametric conversion [31,32] that are analogous to these effects in optics and for elementary excitations in Bose-condensed systems [56–58], atomic squeezing [24,27,28], which can also be called spin squeezing, Ramsey fringes [59], and massive entanglement production, which, being similar to the entanglement of two atoms, differs from the latter by occurring for multiatomic condensates.

In the following sections, we explain how it is possible to create and regulate entanglement in a Bose-condensed system with topological modes in optical lattices.

4. Coherent states
First of all, we need to define a basis of states that we shall consider in what follows. It is convenient to use the basis of coherent states.

Let us consider an optical lattice with \( N_L \) sites, each site representing a deep well with a large filling factor \( \nu_j \gg 1 \) and with the number of condensed atoms in a well \( N_j \), so that
\[ N_0 = \sum_j N_j , \quad N = \sum_j \nu_j , \] (41)
where \( j = 1, 2, \ldots, N_L \). Suppose that each well is subject to the action of a modulating field \( V_j(r, t) \) generating topological modes inside that well. Let a multi-index \( n_j \) label the topological modes in the \( j \)-th site well, and let \( \eta_n(r) \) be the related coherent modes, normalized to the number of condensed atoms in the well,

\[
N_j = \int |\eta_n_j(r)|^2 dr .
\]

(42)

For generality, we consider the case when \( N_j \) and \( \nu_j \) can be different for differing lattice sites. This can happen, e.g., if the lattice is perturbed by a disordering potential.

Topological modes are the solutions to nonlinear equations of type (9), because of which they are not necessarily orthogonal to each other, so that the scalar product

\[
N_{ij} \equiv \int \eta_{n_i}^*(r) \eta_{n_j}(r) \, dr
\]

(43)

is, generally, not zero for \( i \neq j \). The diagonal elements of \( N_{jj} = N_j \) are the numbers of condensed atoms (42).

For an \( n_j \)-mode, we may construct the coherent states in the Fock space as

\[
|n_j> = \left[ \exp\left( -\frac{N_j}{2} \right) \frac{1}{\sqrt{k!}} \prod_{l=1}^{k} \eta_{n_j}(r_l) \right],
\]

(44)

which is a column with respect to \( k = 0, 1, 2, \ldots \). The coherent states (44) are not orthogonal to each other, yielding the scalar product

\[
<n_i|n_j> = \exp\left( -\frac{N_i + N_j}{2} + N_{ij} \right),
\]

(45)

but each of them is normalized to one, so that \( <n_j|n_j> = 1 \).

Let us define the correlation factor

\[
\lambda_{ij} \equiv \frac{N_{ij}}{N_i N_j},
\]

(46)

for which \( \lambda_{jj} = 1 \). From the Cauchy-Schwartz inequality for Eq. (43), we have

\[
|N_{ij}|^2 < N_i N_j \quad (i \neq j).
\]

(47)

Hence, for factor (46), we get

\[
|\lambda_{ij}| < 1 \quad (i \neq j).
\]

(48)

The expression in the exponential of the right-hand side of Eq. (45) can be rewritten by using the equality

\[
\frac{1}{2}(N_i + N_j) - N_{ij} = \frac{1}{2}(N_i - N_j)^2 + (1 - \lambda_{ij})N_i N_j.
\]

The latter diverges if either \( N_i \) or \( N_j \), or both, tend to infinity and \( i \neq j \). Thus, we come to the conclusion that the coherent states (44) are asymptotically orthogonal,

\[
<n_i|n_j> \simeq \delta_{ij} \quad (N_i + N_j \gg 1).
\]

(48)

They also are asymptotically complete (or overcomplete) in the weak sense,

\[
\sum_{n_j} |n_j><n_j| \simeq 1 \quad (N_i + N_j \gg 1).
\]

(49)
Therefore, the set \( \{ |n_j > \} \) forms a basis, which is asymptotically orthogonal and complete. The closed linear envelope of this basis forms a Hilbert space \( \mathcal{H}_j \). The tensor products

\[
|n > \equiv \otimes_j |n_j > \quad (n \equiv \{ n_j \})
\]

(50)

compose an asymptotically orthogonal and complete basis \( \{ |n > \} \), whose closed linear envelope is the Hilbert space \( \mathcal{H} \equiv \otimes_j \mathcal{H}_j \). The states of BEC, which is a coherent subsystem of the physical system, can be interpreted as the vectors of the space \( \mathcal{H} \).

5. Lattice register

By varying the resonant modulating field acting on the lattice, it is feasible to govern the creation and behavior of the topological coherent modes and to regulate entanglement produced in the system. To quantify the level of the produced entanglement, we shall use the measure of entanglement production introduced in Ref. [60].

The density operator, characterizing the coherent modes, can be represented as an expansion over the basis \( \{ |n > \} \),

\[
\hat{\rho} = \sum_n p_n |n > < n| ,
\]

(51)

with the normalization

\[
\text{Tr}_\mathcal{H} \hat{\rho} = \sum_n p_n = 1 .
\]

Let us define a single-partite operator

\[
\hat{\rho}_j = \text{Tr}_{\mathcal{H}\setminus \mathcal{H}_j} \hat{\rho} .
\]

(52)

From this definition and Eq. (51), we have

\[
\hat{\rho}_j = \sum_n p_n |n_j > < n_j| .
\]

(53)

Then we define the factor operator

\[
\hat{\rho}^\otimes \equiv \otimes_j \hat{\rho}_j ,
\]

(54)

for which

\[
\text{Tr}_\mathcal{H} \hat{\rho}^\otimes = \prod_j \text{Tr}_{\mathcal{H}_j} \hat{\rho}_j = 1 .
\]

The measure of entanglement, produced by the density operator (51) is defined [60] as

\[
\varepsilon(\hat{\rho}) \equiv \log \frac{||\hat{\rho}||_D}{||\hat{\rho}^\otimes||_D} ,
\]

(55)

where the logarithm is to the base 2 and \( || \cdot ||_D \) implies the norm over the disentangled set

\[
D \equiv \{ f = \otimes_j \varphi_j | \varphi_j \in \mathcal{H}_j \} .
\]

(56)

Here the norms are defined as follows. The set \( D \) is assumed to be unitary, that is, for any two vectors \( f \in D \) and \( f' \in D \), one can introduce the scalar product \( (f, f') \), which is the standard requirement for any physical system. Having the scalar product makes it straightforward to define the vector norm

\[
||f||_D \equiv \sqrt{(f, f)} \quad (f \in D) ,
\]
generated by this scalar product. Then, for any linear operator $\hat{A}$ on $D$, the operator norm is given as

$$||\hat{A}||_D \equiv \sup_{f,f'} \frac{|(f, \hat{A}f')|}{||f||_D||f'||_D} \quad (f \neq 0, f' \neq 0).$$

This can also be represented as

$$||\hat{A}||_D \equiv \sup_{f,f'} |(f, \hat{A}f')| \quad (||f||_D = ||f'||_D = 1).$$

Vectors $f \in D$ and $f' \in D$ have the product form similar to Eq. (50).

For the norms of operators (51), (53), and (54), we have

$$||\hat{\rho}||_D = \sup_{n} p_n, \quad ||\hat{\rho}_j||_{\mathcal{H}_j} = \sup_{n_j(n \neq n_j)} p_n, \quad ||\hat{\rho}^\otimes||_D = \prod_j ||\hat{\rho}_j||_{\mathcal{H}_j}.$$

So that for measure (55), we obtain

$$\varepsilon(\hat{\rho}) = \log \frac{\sup_{n} p_n}{\prod_j \sup_{n_j} \sum_{n(\neq n_j)} p_n} . \quad (57)$$

Entanglement is generated in the system if and only if

$$\sup_{n} p_n \neq \prod_j \sup_{n_j} \sum_{n(\neq n_j)} p_n . \quad (58)$$

For example, if all lattice sites would be completely independent, such that $p_n$ would be a product of some $p_{n_j}$, $p_n \rightarrow \prod_j p_{n_j}$. Then, since

$$\sup_{n} \prod_j p_{n_j} \rightarrow \prod_j \sup_{n_j} p_{n_j} ,$$

measure (57) would be zero, $\varepsilon(\hat{\rho}) \rightarrow 0$, that is, no entanglement would be produced.

The opposite case would be if all lattice sites were correlated, so that

$$p_n = p_n \prod_j \delta_{nn_j} . \quad (59)$$

This can be realized if all the lattice is shaken synchronically, with the same topological mode being generated in all lattice sites. In that case,

$$\sup_{n} p_n = p_n , \quad \sum_{n(\neq n_j)} p_n = p_n \delta_{nn_j} ,$$

because of which

$$||\hat{\rho}||_D = p_n , \quad ||\hat{\rho}^\otimes||_D = p_n^{N_L} .$$

As a result, the measure of entanglement production (57) becomes

$$\varepsilon(\hat{\rho}) = (1 - N_L) \log \sup_{n} p_n . \quad (60)$$

If $M$ topological modes are simultaneously generated in each lattice site, then the maximal entanglement production is achieved for $p_n = 1/M$. In such a case, keeping in mind that $N_L \gg 1$, one gets

$$\varepsilon(\hat{\rho}) = N_L \log M .$$
For the two-mode case, this reduces to $\varepsilon(\tilde{\rho}) = N_L \log 2$. By varying the resonant modulating fields, it is possible to regulate entanglement in a wide diapason between zero and $N_L \log M$.

The fractional mode populations are given by $p_n = p_n(t)$, which is defined in Eq. (33).

The resonant process of mode generation is a fast process, occurring on the time scale $1/\alpha$. The latter is much shorter than the thermal effective time $\tau_{\text{th}}$, provided that the number of condensed atoms $N_0 \gg N_1$ is essentially larger than the number of uncondensed atoms $N_1$.

Another temporal restriction is imposed by the power broadening, defining the resonance time $t_{\text{res}}$, after which nonresonant levels become excited, even though the modulating field is resonant. The resonance time can be estimated [24] as

$$t_{\text{res}} = \frac{\alpha^2 \omega}{\beta^2 (\alpha^2 + \beta^2)}.$$

For $\beta \leq \alpha$ and $\alpha \ll \omega$, the resonance time $t_{\text{res}} \sim \omega/\beta^2$, is much longer than $1/\alpha$. It looks, therefore, feasible to achieve sufficiently long decoherence times allowing for the functioning of the lattice register that can be used for quantum information processing and the creation of a boson lattice quantum computer.

In conclusion, we have shown that the generation of topological coherent modes is feasible in Bose systems not only at zero temperature and under asymptotically weak interactions, when the whole system would be almost completely condensed, but also at finite temperatures and interactions. This becomes possible because of the resonant character of the suggested mode generation. An important feature of the resonant mode generation is the feasibility of controlling the process, thus, allowing one to govern the level of entanglement production realized in an optical lattice. Such a possibility of regulating entanglement production in a lattice could be employed for creating boson lattice registers for quantum information processing.

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