On the Tail Asymptotics of Supremum of Stationary $\chi$-Processes With Random Trend

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Abstract. Let $\chi_n(t)$, $t \geqslant 0$, be a chi-process with $n$ degrees of freedom. We derive the asymptotic exact result for

$$
\mathbb{P}\left(\sup_{t \in [0,T]} (\chi_n(t) + \eta(t)) > u\right), \text{ as } u \to \infty,
$$

where $\eta(t)$ is a certain random process independent of $\chi_n(t)$ and $T > 0$ is a constant.

1. Introduction and main results

The tail asymptotic behaviour of the supremum of chi-processes (generated by stationary, non-stationary or self-similar Gaussian process) has been a subject of numerous papers: [1, 2, 5, 11, 12]. Recently, the papers [6, 9, 10] are dealing with the asymptotic behaviour of chi-processes with a trend. We will consider a chi-process with a random trend.

Let $\xi(t)$, $t \in [0, T]$ ($T > 0$ is constant), be a centered stationary Gaussian process and let the covariance function $r(t)$ of process $\xi$ satisfies

$$
r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0,
$$

for some $\alpha \in (0, 2]$, and

$$
r(t) < 1, \text{ for all } t > 0.
$$

Let $\xi_i(t)$, $i = 1, \ldots, n$, be independent copies of process $\xi$. The process

$$
\chi_n(t) := \left(\xi_1^2(t) + \ldots + \xi_n^2(t)\right)^{1/2}, \ t \in [0, T],
$$

is called a (stationary) chi-process with $n$ degrees of freedom. Let $\eta(t)$ be another random process, independent of $\xi(t)$. The sum process $X(t) := \chi_n(t) + \eta(t)$ will be called a chi-process with random trend.

Let us first formulate the result of [6].
**Theorem 1.1 (Theorem 2.3 of [6]).** Suppose that the covariance function \( r(t) \) of the centered stationary Gaussian process \( \{ \xi(t), t \geq 0 \} \) satisfies assumptions (1) and (2). Assume further that \( g(\cdot) \) is a non-negative bounded measurable function that attains its minimum 0 over \([0, T]\) at the unique point 0, and further there exist some positive constants \( c, \beta \) such that
\[
  g(t) = ct^\beta(1 + o(1)), \quad t \to 0.
\]
Then
\[
  P\left( \max_{t \in [0, T]} \left( \chi_n(t) - g(t) \right) > u \right) = (1 + o(1)) M_{\alpha, \beta}^c t^{(\beta - 1/2)} \Gamma_{\alpha}, \quad u \to \infty,
\]
where,
\[
  M_{\alpha, \beta}^c = \begin{cases} 
  c^{-1/\beta} \Gamma(1/\beta + 1) H_{\alpha, \beta} & \text{if } \alpha < 2\beta, \\
  P_{\alpha, \alpha/2} & \text{if } \alpha = 2\beta, \\
  1 & \text{if } \alpha > 2\beta,
\end{cases}
\]
and \( \Gamma_{\alpha}(x) := \frac{2^{(n-2)/2}}{\Gamma(n/2)} x^{n-2} \exp \left\{ -\frac{x^2}{2} \right\} \).

Here, with \( \Gamma(\cdot) \) we denoted the Gamma function, \( H_{\alpha} \) denotes the Pickands constant
\[
  H_{\alpha} := \lim_{S \to \infty} \frac{1}{S} \mathbb{E} \left( \exp \left( \max_{t \in [0,S]} \left( \sqrt{2} B_{\alpha/2}(t) - t^\alpha \right) \right) \right) \in (0, \infty),
\]
and \( P_{\alpha, \alpha/2} \) is defined by
\[
  P_{\alpha, \alpha/2} := \lim_{S \to \infty} \mathbb{E} \left( \exp \left( \max_{t \in [0,S]} \left( \sqrt{2} B_{\alpha/2}(t) - t^\alpha - c \cdot t^{\alpha/2} \right) \right) \right) \in (0, \infty),
\]
where \( \{ B_{\alpha/2}(t), \ t \in \mathbb{R} \} \) is a standard fractional Brownian motion with Hurst index \( \alpha/2 \in (0, 1] \).

If there exist some positive constants \( c, \beta \) such that
\[
  g(t) = g(t_0) + ct - t_0^\beta(1 + o(1)), \quad t \to t_0,
\]
where \( t_0 = \arg\min_{t \in [0,T]} g(t) \in (0, T) \) is unique, then in the previous asymptotic relation \( u \) will be replaced by \( u + g(t_0) \), \( \Gamma(\cdot) \) will be replaced by \( \Gamma(\cdot) \) and \( P_{\alpha, \alpha/2} \) will be replaced by
\[
  \tilde{P}_{\alpha, \alpha/2} := \lim_{S \to \infty} \mathbb{E} \left( \exp \left( \max_{t \in [-S, S]} \left( \sqrt{2} B_{\alpha/2}(t) - t_0^\alpha - c \cdot t_0^{\alpha/2} \right) \right) \right).
\]

Our main results are the next two theorems.

Firstly, let us consider
\[
  \eta(t) := \lambda - \zeta t^\beta,
\]
where \( \lambda \) and \( \zeta \) are random variables independent of \( \xi(\cdot), \zeta > 0 \) almost surely, and \( \beta > 0 \) is a constant. With the notation \( o(G) := \sup \{ x : P(G \leq x) < 1 \} \) for any real valued random variable \( G \), we further assume that \( o(\lambda), o(\zeta) \) are finite.

**Theorem 1.2.** Let \( \xi(t) \) and \( \eta(t) \), \( t \in [0, T] \), be above introduced random processes and let the tail \( \tilde{F}_\lambda(x) = 1 - F_\lambda(x) \) satisfy
\[
  \tilde{F}_\lambda(\sigma - 1/u) = u^{-\tau} \mathcal{L}(u)
\]
for some \( \tau > 0 \) and \( \mathcal{L} \) is a slowly varying function.
Suppose that the functions $m_1(x) := \mathbb{E} \left( \zeta^{-\frac{1}{2}} \mid \lambda = x \right)$ and $m_2(x) := \mathbb{E} \left( \gamma_{n,\sigma \lambda / 2} \mid \lambda = x \right)$ exist and are continuous at $x = \sigma(\lambda)$.

Then
\[
\Pr \left( \max_{t \in [0,7]} (\chi_n(t) + \eta(t)) > u \right) = (1 + o(1)) W_{\alpha,\beta} \Gamma(\tau + 1) u^{-\tau(\frac{1}{2} - \frac{1}{2})} \mathcal{L}(-\bar{Y}_n(u - \sigma(\lambda))),
\]
as $u \to \infty$, where
\[
W_{\alpha,\beta} = \begin{cases} 
  m_1(\sigma(\lambda)) \Gamma \left( \frac{1}{2} + 1 \right) H_{\alpha}, & \text{if } \alpha < 2\beta, \\
  m_2(\sigma(\lambda)), & \text{if } \alpha = 2\beta, \\
  1, & \text{if } \alpha > 2\beta,
\end{cases}
\]
and $(x)_+ = \max(0, x)$.

**Example.** There are numerous examples of Gaussian processes which satisfy the assumptions of Theorem 1.2. We will give one simple example.

Let $\xi(t), t \in [0, T]$ be the Ornstein-Uhlenbeck process with a covariance function $r(t) = e^{-|t|}$, and $\eta(t) = \lambda - \zeta t$, where $\lambda$ is uniformly distributed on $(0,1)$, and $\zeta |\lambda = x$ is uniformly distributed on $(\frac{1}{2}, \infty)$. Then,
\[
\alpha = 1 < 2\beta = 2, \quad H_1 = 1
\]
\[
\sigma(\lambda) = 1, \quad F_{\lambda} \left( 1 - \frac{1}{\eta} \right) = u^{-1},
\]
and
\[
m_1(1) := \mathbb{E} \left( \zeta^{-1} \mid \lambda = 1 \right) = 2 \ln(2).
\]
It follows
\[
\Pr \left( \max_{t \in [0,7]} (\chi_n(t) + \eta(t)) > u \right) = (1 + o(1)) 2 \ln(2) \mathcal{L}_n(u - 1), \quad u \to \infty.
\]
\[\Box\]

Now, let us consider a smooth process $\eta(t)$ which satisfies the next four conditions.

**$\eta_1.$** $0 < \sigma := \sigma(\eta(t)) < \infty$.

**$\eta_2.$** For some $\varepsilon, \delta > 0$ there exists $\eta''(t)$ for all $t$ with $(t, \eta(t)) \in K(\delta, \varepsilon) := [-\delta, T + \delta] \times [\sigma - \varepsilon, \sigma]$, and that
\[
\sup_{(t, \eta(t)) \in K(\delta,\varepsilon)} |\eta''(t)| \leq c
\]
for some constant $c$. Moreover, assume that for all $t$ with $(t, \eta(t)) \in K(\delta, \varepsilon)$ $\eta''(t)$ is equicontinuous in the following sense
\[
\omega(h) := \sup_{(t, \eta(t)) \in K(\delta,\varepsilon)} \sup_{x \in [0,h]} \sigma(|\eta''(t + s) - \eta''(t)|) \to 0, \quad \text{as } h \to 0.
\]

**$\eta_3.$** For some $\varepsilon, \delta > 0$ the vector $X_t = (\eta(t), \eta'(t), \eta''(t))$ has a density $f_{X_t}(x, y, z), x \in [\sigma - \varepsilon, \sigma], $ which is bounded for any $t \in [-\delta, T + \delta]$.

**$\eta_4.$** For some $\varepsilon, \delta, \kappa > 0$ almost surely $\eta''(t) \leq -\kappa$ for any $(t, x) \in K(\delta, \varepsilon)$ such that $\eta'(t) = 0$ and $\eta''(t) < 0$. Moreover, assume that the function
\[
m(t, x) := \int_{-\kappa}^{-x} |z|^{1/2} f_{\eta''(t), \eta'(t)}(0, z) dz
\]
is continuous in $x = \sigma$ uniformly on $t$, with $\int_0^T m(t, \sigma) dt > 0$. 

Theorem 1.3. Let $\xi(t)$, $t \in [0,T]$, $T > 0$, be a stationary Gaussian process with the expectation of zero and with a covariance function $\gamma(t)$ that satisfies (1) and (2) and let $\eta(t)$ be a process being independent of the process $\xi(t)$ that satisfies conditions $\eta_1 - \eta_4$.

Let for any fixed $t \in [0,T]$ the tail $F_{\eta(t)}(x) = 1 - F_{\eta(t)}(x)$ of the distribution function of the random variable $\eta(t)$ is regularly varying at $\sigma$, i.e., $F_{\eta(t)}(\sigma - 1/u) = u^{-\gamma} \mathcal{L}(u)$ for some $\tau > 0$ and $\mathcal{L}$ is a slowly varying function.

Then

$$P\left(\sup_{t \in [0,T]} (\chi_n(t) + \eta(t)) > u\right) = (1 + o(1)) \sqrt{\pi} \Gamma(\tau + 1) H_n u^{\gamma - \frac{1}{2} - \tau} \mathcal{Y}_n(u - \sigma) \int_0^\infty \mathcal{L}(u) m(t, \sigma) dt,$$

as $u \to \infty$.

2. Proofs

2.1. Main lemma

In the proofs of Theorem 1.2 and Theorem 1.3 we will use the next lemma.

Lemma 2.1. Let $X$ be a positive random variable with the distribution function $F$ which has an upper endpoint $\sigma < \infty$. Suppose that tail $F(x) = 1 - F(x)$ satisfy $F(\sigma - 1/u) = u^{-\gamma} \mathcal{L}(u)$ for some positive $\tau$ and a slowly varying function $\mathcal{L}$. Let $h$ be a non-negative measurable function such that $E(h(X)) < \infty$ and suppose that $h$ is continuous and strictly positive at $\sigma$. Then, for any $s \in [0, \sigma)$ we have

$$\int_s^\infty h(t) \mathcal{Y}_n(u - t) dF(t) \sim \Gamma(\tau + 1) h(\sigma) \mathcal{L}(u) u^{-\gamma} \mathcal{Y}_n(u - \sigma), \quad u \to \infty.$$

Proof.

The following asymptotic relation is proved in [16] (Lemma 1)

$$\int_s^\infty h(t) \Psi(u - t) dF(t) \sim \Gamma(\tau + 1) h(\sigma) \mathcal{L}(u) u^{-\gamma} \Psi(u - \sigma), \quad u \to \infty,$$

where $\Psi(u) := \frac{1}{\sqrt{2\pi}u} \exp[-u^2/2]$ and in the proof we are using the asymptotic result of Theorem 3.1 of [7].

By using the equality

$$\mathcal{Y}_n(x) = \frac{2^{(n-2)/2} \sqrt{2\pi}}{\Gamma(n/2)} x^{n-1} \Psi(x),$$

and the first part it follows

$$\int_s^\infty h(t) \mathcal{Y}_n(u - t) dF(t) = \frac{2^{(n-2)/2} \sqrt{2\pi}}{\Gamma(n/2)} \int_s^\infty h(t) (u - t)^{n-1} \Psi(u - t) dF(t)$$

$$\sim \frac{2^{(n-2)/2} \sqrt{2\pi}}{\Gamma(n/2)} \Gamma(\tau + 1) h(\sigma) (u - \sigma)^{n-1} u^{-\gamma} \mathcal{L}(u) \Psi(u - \sigma)$$

$$\sim \Gamma(\tau + 1) h(\sigma) u^{-\gamma} \mathcal{L}(u) \mathcal{Y}_n(u - \sigma), \quad u \to \infty.$$
2.2. Proof of Theorem 1.2

By using the total probability rule we have

\[
P \left( \max_{t \in [0,T]} \left( X_n(t) + \lambda - \zeta \cdot t^\theta \right) > u \right) = E \left( P \left( \max_{t \in [0,T]} \left( X_n(t) - \zeta \cdot t^\theta \right) > u - \lambda \mid \lambda, \zeta \right) \right)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left( \max_{t \in [0,T]} \left( X_n(t) - \zeta \cdot t^\theta \right) > u - \lambda \mid \lambda = b, \zeta = a \right) f_{\lambda,\zeta}(b,a) \, db \, da
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left( \max_{t \in [0,T]} \left( X_n(t) - a t^\theta \right) > u - b \right) f_a(b) \cdot f_{\zeta | \lambda = b}(a) \, db \, da +
\]

\[
+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left( \max_{t \in [0,T]} \left( X_n(t) - a t^\theta \right) > u - b \right) f_a(b) \cdot f_{\zeta | \lambda = b}(a) \, db \, da,
\]

for some small \( \varepsilon > 0 \). Here, \( f_{\lambda,\zeta}(b,a) \) denotes density function of random vector \( (\lambda, \zeta) \), \( f_{\zeta | \lambda = b}(a) \) is a density function of random variable \( \zeta | \lambda = b \), and \( f_a(b) \) is a density function of random variable \( \lambda \).

The first integral in the previous equality we can estimate in the following way:

\[
0 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left( \max_{t \in [0,T]} \left( X_n(t) - a t^\theta \right) > u - b \right) f_a(b) \cdot f_{\zeta | \lambda = b}(a) \, db \, da \leq
\]

\[
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left( \max_{t \in [0,T]} X_n(t) > u - (\sigma - \varepsilon) \right) f_a(b) \cdot f_{\zeta | \lambda = b}(a) \, db \, da
\]

\[
= O \left( u^{\frac{2}{\beta}} \gamma(u) \right),
\]

where the last equality follows by Corollary 7.3 in [13].

By applying left inequality (3) and Theorem 2.3 of [6] we obtain

\[
P \left( \max_{t \in [0,T]} \left( X_n(t) - \zeta \cdot t^\theta \right) > u - \lambda \right) \geq
\]

\[
\left( 1 - \gamma(u) \right) \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{P}_{\alpha,\beta,d/2} \cdot \mathcal{Y}_n(u-b) \cdot f_a(b) \cdot f_{\zeta | \lambda = b}(a) \, db \, da,
\]

\[
\text{if } \alpha = 2\beta,
\]

\[
\left( 1 - \gamma(u) \right) \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{Y}_n(u-b) \cdot f_a(b) \cdot f_{\zeta | \lambda = b}(a) \, db \, da,
\]

\[
\text{if } \alpha > 2\beta,
\]

\[
\geq (1 - \gamma(u)) \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m_2(b) \cdot \mathcal{Y}_n(u-b) f_a(b) \, db,
\]

\[
\text{if } \alpha = 2\beta,
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{Y}_n(u-b) f_a(b) \, db,
\]

\[
\text{if } \alpha > 2\beta,
\]

\[
\geq (1 - \gamma(u) - \nu(u)) \mathcal{W}_{a,\beta,\gamma} \Gamma(\tau + 1) u^{-\frac{2}{\beta + 1}} \mathcal{L}(u) \mathcal{Y}_n(u - \sigma(\lambda)), \quad u \to \infty.
\]

where the last inequality follows by using Theorem 2.1. Here, \( \gamma(u), \nu(u) \to 0 \), as \( u \to \infty \).
Similarly, by using Theorem 2.3 of [6], the right inequality in (3) and Theorem 2.1 we have
\[
P \left( \max_{t \in [0,T]} \left( \chi_n(t) - \zeta \cdot t^\beta \right) > u - \lambda \right) \\
\leq O \left( u^{\frac{3}{2}} Y_n(u - \sigma(\lambda) + \varepsilon) \right) \\
+ (1 + \gamma(u) + \nu(u)) W_{n,\beta} \Gamma(\tau + 1) u^{-\tau \left( \frac{1}{4} \right)} L(u) Y_n(u - \sigma(\lambda)), \quad u \to \infty.
\]

The assertion of theorem follows. \(\Box\)

2.3. Proof of Theorem 1.3.

Upper bound.

Following the idea of the proof of Theorem 2 from [8] which was used also in papers [14, 15], we will consider the points \(t\) of local maxima of \(\eta\) such that \(\eta(t) \geq \sigma - \varepsilon(u)\) where \(0 < \varepsilon(u) < \varepsilon/2\) and \(\varepsilon(u) \to 0\) as \(u \to \infty\).

Every two points of local maxima in \(K(\delta, \varepsilon(u))\) are separated by at least \(2h\), for some small \(h > 0\). For such \(h\) and for \(s\) be a point such that \(|s - t| < h\) one can obtain
\[
\eta(t) + \frac{(s - t)^2}{2} (\eta''(t) - \omega(h')) \leq \eta(s) \leq \eta(t) + \frac{(s - t)^2}{2} (\eta''(t) + \omega(h')).
\]

Let \(s_1\) be the first local maximum of \(\eta\) in \([0,T]\) with \(\eta(s_1) \geq \sigma - \varepsilon(u)\) and \(s_M\) the last one. We introduce the random set
\[
L_+ := \left[ [0,T] \cap \bigcup_{s \in \mathcal{M}(\varepsilon(u))} [s - \delta(u), s + \delta(u)] \right] \cup [0,s_1] \cup [s_M T, T],
\]
where \(\mathcal{M}(\varepsilon(u))\) is a set of local maximum points of the process \(\eta(t)\) which are above \(\sigma - \varepsilon(u)\) and \(\delta(u) := 2 \sqrt{\frac{\varepsilon(u)}{K}}\).

If \(t \in [0,T] \setminus L_+\), then \(\eta(t) < \sigma - \varepsilon(u)\), so we have
\[
P \left( \max_{t \in [0,T] \setminus L_+} \left( \chi_n(t) + \eta(t) \right) > u \mid \eta \right) \leq P \left( \max_{t \in [0,T] \setminus L_+} \chi_n(t) > u - (\sigma - \varepsilon(u)) \right).
\]
where the last equality follows from Corollary 7.3 in [13] (or Proposition 2.1 of [6]).

By using the total probability rule and the previous inequality it follows

\[
P\left(\max_{t\in[0,T]}(\chi_n(t) + \eta(t)) > u\right) = E\left(\max_{t\in[0,T]}(\chi_n(t) + \eta(t)) > u \mid \eta\right)
\]

\[
= E\left(\max_{t\in[0,T]}(\chi_n(t) + \eta(t)) > u \mid \eta\right) + O\left(u^{\frac{2}{n}}\chi_n(u - (\sigma - \epsilon(u)))\right),
\]

so we obtain the bound

\[
P\left(\max_{t\in[0,T]}(\chi_n(t) + \eta(t)) > u\right) \leq E\left(\sum_{\eta\in\mathcal{A}(\eta)}\mathbb{P}\left(\max_{s\in[0,n-h]}(\chi_n(s) + \eta(s)) > u \mid \eta\right)\right) + E\left(\mathbb{P}\left(\max_{s\in[0,n-h]}(\chi_n(s) + \eta(s)) > u \cap A_1 \mid \eta\right)\right) + E\left(\mathbb{P}\left(\max_{s\in[0,n+h,T]}(\chi_n(s) + \eta(s)) > u \cap A_M \mid \eta\right)\right) + O\left(u^{\frac{2}{n}}\chi_n(u - (\sigma - \epsilon(u)))\right).
\]

Now, by setting

\[\mathcal{M} = \mathcal{M}(\epsilon(u)) \cap [-\delta(u), T + \delta(u)],\]

and choosing \(\epsilon(u) = \frac{\ell + \frac{2}{n}}{u - \sigma}\) \(\ln u\), with a large positive \(\ell \geq \frac{1}{2} + \frac{2}{\delta} - \frac{2}{\delta}\), such that

\[u^{\frac{2}{n}}\chi_n(u - (\sigma - \epsilon(u))) = u^{\frac{2}{n}}(u - (\sigma - \epsilon(u)))^{-1} \sim \frac{\alpha}{\kappa} h - 1\]

we get

\[
P\left(\max_{t\in[0,T]}(\chi_n(t) + \eta(t)) > u\right) \leq E\left(\sum_{\eta\in\mathcal{M}}\mathbb{P}\left(\max_{s\in[0,n+h]}(\chi_n(s) + \eta(s)) > u \mid \eta\right)\right) + O\left(u^{\frac{2}{n}}\chi_n(u - \sigma)\right).
\]

Using Theorem 2.3 in [6] we obtain

\[
P\left(\max_{t\in[0,T]}(\chi_n(s) + \eta(s)) > u \mid \eta\right)
\]

\[
\leq \mathbb{P}\left(\max_{s\in[0,n+h]}(\chi_n(s) + \eta(t) + \frac{(s-t)^2}{2}(\eta''(t) + \omega(h))) > u \mid \eta\right)
\]

\[
\leq \mathbb{P}\left(\max_{s\in[0,n-h]}(\chi_n(s) - \frac{(s-t)^2}{2}(-\eta''(t))\left(1 - \frac{\omega(h)}{\kappa}\right)) > u - \eta(t) \mid \eta\right)
\]

\[
\leq \sqrt{n} H_{\alpha} \left(-\eta''(t)\left(1 - \frac{\omega(h)}{\kappa}\right)\right)^{-\frac{1}{2}} u^{\frac{2}{n} - \frac{1}{2}} \chi_n(u - \eta(t))(1 + y(u)),
\]

where \(y(u) (\downarrow 0 \text{ as } u \to \infty)\) can be chosen to be deterministic (see [8, 14]).
Let us consider the point process of local maxima \( ((t, \eta(t), \eta''(t)), t \in \mathcal{M}(\varepsilon(u))) \) as a point process in \([-\delta(u), T + \delta(u)] \times [\sigma - \varepsilon(u), \sigma] \times [-c, -\kappa] \). Its intensity is

\[
v(t, x, z) = |z| \mathbf{1}_{[\varepsilon(u)]} f_{X}(x, 0, z)
\]

(see Chapter 3 in [3] for more details) and for any bounded function \( F(t, x, z) \) we have (Campbell’s Formula, see for instance Theorem 2.2 in [4])

\[
\mathbb{E} \left( \sum_{M(\varepsilon(u)) \cap [0, T]} F(t, \eta(t), \eta''(t)) \right) = \int_{-\delta(u)}^{T + \delta(u)} \int_{-c}^{\sigma} \int_{-c}^{\infty} F(t, x, z) v(t, x, z) dt dx dz.
\]

It follows that

\[
\Pr \left( \max_{t \in [0, T]} (\chi_n(t) + \eta(t)) > u \right) \\
\leq (1 + \gamma(u)) \sqrt{\pi} H_{\alpha} u^{\frac{\alpha - 1}{2}} \left( 1 - \frac{\alpha(h)}{\kappa} \right)^{-\frac{1}{2}} \int_{-\delta(u)}^{T + \delta(u)} \int_{-c}^{\sigma} \int_{-c}^{\infty} |z|^2 \gamma_{n}(u - x) f_{X}(x, 0, z) dt dx dz \\
+ O \left( u^{-\gamma} \gamma_{n}(u - \sigma) \right) \\
\leq (1 + \gamma(u)) \sqrt{\pi} H_{\alpha} u^{\frac{\alpha - 1}{2}} \left( 1 - \frac{\alpha(h)}{\kappa} \right)^{-\frac{1}{2}} \int_{-\delta(u)}^{T + \delta(u)} \int_{-c}^{\sigma} \int_{-c}^{\infty} |z|^2 \gamma_{n}(u - x) f_{X}(x, 0, z) dt dx dz \\
+ O \left( u^{-\gamma} \gamma_{n}(u - \sigma) \right),
\]

By the equality

\[
f_{X}(x, 0, z) = f_{\eta(t)}(x) f_{\eta''(t), \eta''(0) = 0}(x) = (0, z)
\]

and Lemma 2.1 we derive the bound

\[
\Pr \left( \max_{t \in [0, T]} (\chi_n(t) + \eta(t)) > u \right) \\
\leq (1 + \gamma(u) + \gamma_{1}(u)) \sqrt{\pi} \Gamma(\tau + 1) H_{\alpha} u^{\frac{\alpha - 1}{2}} \left( 1 - \frac{\alpha(h)}{\kappa} \right)^{-\frac{1}{2}} \gamma_{n}(u - \sigma) \int_{-\delta(u)}^{T + \delta(u)} \mathcal{L}_{n}(u \cdot m(t, \sigma)) dt \\
+ O \left( u^{-\gamma} \gamma_{n}(u - \sigma) \right),
\]

where \( \gamma_{1}(u) \to 0 \) as \( u \to \infty \).

Finally, we have

\[
\limsup_{u \to \infty} \frac{\Pr \left( \max_{t \in [0, T]} (\chi_n(t) + \eta(t)) > u \right)}{u^{\frac{\alpha - 1}{2}} \gamma_{n}(u - \sigma) \int_{0}^{T} \mathcal{L}(u \cdot m(t, \sigma)) dt} \to \sqrt{\pi} \Gamma(\tau + 1) H_{\alpha}.
\]

as \( h \to 0 \).

**Lower Bound.**

If \((s, \eta(s)), (t, \eta(t)) \in K(\delta, \varepsilon(u))\) and \( t \) and \( s \) are points of local maximum of \( \eta \), then \( |t - s| \geq 2h \). It implies that there are at most \( \lfloor \frac{T}{2h} \rfloor \) points of such local maximum in the \([0, T]\). By setting \( M_{t} := \mathcal{M}(\varepsilon(u)) \cap [\delta(u), T - \delta(u)] \) we have

\[
\mathbb{E} \left( \Pr \left( \max_{t \in [0, T]} (\chi_n(t) + \eta(t)) > u | \eta \right) \right) \geq \mathbb{E} \left( \Pr \left( \max_{t \in M_{t}} \left\{ \max_{s \in [t - 2h, t + 2h]} (\chi_n(s) + \eta(s)) > u \right\} | \eta \right) \right)
\]
\[
\begin{align*}
&\geq \mathbb{E}\left( \sum_{s \in M_0} \mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(s) + \eta(s) \right) > u \mid \eta \right) \right) \\
&- \mathbb{E}\left( \sum_{s \in M_0, s \neq \ell} \mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(v) + \eta(v) \right) > u, \max_{t \in [-h, s+h/2]} \left( \chi_n(v) + \eta(v) \right) > u \mid \eta \right) \right).
\end{align*}
\]

Using the left inequality \((4)\), and Theorem 2.3 in \([6]\), we get
\[
\mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(s) + \eta(s) \right) > u \mid \eta \right)
\geq \mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(s) + \eta(t) + \frac{(s-t)^2}{2} (\eta''(t) - \omega(h)) \right) > u \mid \eta \right)
\geq \mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(s) - \frac{(s-t)^2}{2} (-\eta''(t)) \left( 1 + \frac{\omega(h)}{\kappa} \right) \right) > u - \eta(t) \mid \eta \right)
\geq \sqrt{\pi} H_a \left( -\eta''(t) \left( 1 + \frac{\omega(h)}{\kappa} \right)^{\frac{1}{2}} u^\frac{3}{2} \Psi_n(u - \eta(t)) (1 - \nu(u)) \right),
\]
where \(\nu(u) (\to 0 \ as \ u \to \infty)\) can be chosen non-randomly. Now, by the arguments for the upper bound we get
\[
\liminf_{u \to \infty} \frac{\mathbb{E}\left( \sum_{s \in M_0} \mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(s) + \eta(s) \right) > u \mid \eta \right) \right)}{u^{\frac{3}{2} - \frac{\tau}{2}} \Psi_n(u - \alpha) \int_0^T L_n(t) \cdot m(t, \alpha) dt} \to \sqrt{\pi} \Gamma(\tau + 1) H_a,
\]
as \(h \to 0\).

Using the “Appendix” of paper \([6]\) we obtain upper bound for the double sum, i.e.
\[
\mathbb{P}\left( \max_{t \in [-h/2, h/2]} \left( \chi_n(v) + \eta(v) \right) > u, \max_{t \in [-h, s+h/2]} \left( \chi_n(v) + \eta(v) \right) > u \mid \eta \right)
\leq o\left( u^{\frac{3}{2} - \frac{\tau}{2}} \Psi_n(u - \alpha) \right) \ as \ u \to \infty.
\]

Thus, we get
\[
\liminf_{u \to \infty} \frac{\mathbb{P}\left( \max_{t \in [0, T]} \left( \chi_n(t) + \eta(t) \right) > u \right)}{u^{\frac{3}{2} - \frac{\tau}{2}} \Psi_n(u - \alpha) \int_0^T L_n(t) \cdot m(t, \alpha) dt} \to \sqrt{\pi} \Gamma(\tau + 1) H_a.
\]
as \(h \to 0\).

\[\square\]

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