THE A-POLYNOMIAL 2-TUPLE OF TWISTED WHITEHEAD LINKS

ANH T. TRAN

Abstract. We compute the A-polynomial 2-tuple of twisted Whitehead links. As applications, we determine canonical components of twisted Whitehead links and give a formula for the volume of twisted Whitehead link cone-manifolds.

1. Introduction

The A-polynomial of a knot in the 3-sphere $S^3$ was introduced by Cooper, Culler, Gillet, Long and Shalen [CCGLS] in the 90’s. It describes the $SL_2(\mathbb{C})$-character variety of the knot complement as viewed from the boundary torus. The A-polynomial carries a lot of information about the topology of the knot. For example, it distinguishes the unknot from other knots [BoZ, DG] and the sides of the Newton polygon of the A-polynomial give rise to incompressible surfaces in the knot complement [CCGLS].

The A-polynomial was generalized to links by Zhang [Zh] about ten years later. For an $m$-component link in $S^3$, Zhang defined a polynomial $m$-tuple link invariant called the A-polynomial $m$-tuple. The A-polynomial 1-tuple of a knot is nothing but the A-polynomial defined in [CCGLS]. The A-polynomial $m$-tuple also carries important information about the topology of the link. For example, it can be used to construct concrete examples of hyperbolic link manifolds with non-integral traces [Zh].

Finding an explicit formula for the A-polynomial is a challenging problem. So far, the A-polynomial has been computed for a few classes of knots including two-bridge knots $C(2n, p)$ (with $1 \leq p \leq 5$) in Conway’s notation [HS, Pa, Ma, HL], $(-2, 3, 2n + 1)$-pretzel knots [TY, GM]. It should be noted that $C(2n, p)$ is the double twist knot $J(2n, -p)$ in the notation of [HS]. Moreover, $C(2n, 1)$ is the torus knot $T(2, 2n + 1)$ and $C(2n, 2)$ is known as a twist knot. A cabling formula for the A-polynomial of a cable knot in $S^3$ has recently been given in [NZ]. Using this formula, Ni and Zhang [NZ] has computed the A-polynomial of an iterated torus knot explicitly.

In this paper we will compute the A-polynomial 2-tuple for a family of 2-component links called twisted Whitehead links. As applications, we will determine canonical components of twisted Whitehead links and give a formula for the volume of twisted Whitehead link cone-manifolds. For $k \geq 0$, the $k$-twisted Whitehead link $W_k$ is the 2-component link depicted in Figure 1. Note that $W_0$ is the torus link $T(2, 4)$ and $W_1$ is the Whitehead link. Moreover, $W_k$ is the two-bridge link $C(2, k, 2)$ in Conway’s notation and is $b(4k+4, 2k+1)$ in Schubert’s notation. These links are all hyperbolic except for $W_0$.

The A-polynomial 2-tuple of the twisted Whitehead link $W_k$ is a polynomial 2-tuple $[A_1(M, L), A_2(M, L)]$ given as follows.

\begin{itemize}
  \item \textbf{2000 Mathematics Subject Classification.} Primary 57M27, Secondary 57M25.
  \item \textbf{Key words and phrases.} canonical component, cone-manifold, hyperbolic volume, the A-polynomial, twisted Whitehead link, two-bridge link.
\end{itemize}
Theorem 1.1. If \( k = 2n - 1 \) then \( A_1(M, L) = (L - 1)F(M, L) \) where

\[
F(M, L) = \sum_{i=0}^{n} \left\{ \binom{n+1+i}{2i+1} - \binom{n-1+i}{2i+1} \right\} (M - M^{-1})^{2i} \left( \frac{L-1}{L+1} \right)^{2i} \\
+ \sum_{i=0}^{n-1} \binom{n+i}{2i+1} (M + M^{-1})(M - M^{-1})^{2i+1} \left( \frac{L-1}{L+1} \right)^{2i+1}.
\]

If \( k = 2n \) then \( A_1(M, L) = (LM^2 - 1)G(M, L) \) where

\[
G(M, L) = \sum_{i=0}^{n} \left\{ \binom{n+1+i}{2i+1} + \binom{n+i}{2i+1} \right\} (M - M^{-1})^{2i+1} \left( \frac{LM^2 - 1}{LM^2 + 1} \right)^{2i} \\
+ \sum_{i=0}^{n} \binom{n+i}{2i} (M + M^{-1})(M - M^{-1})^{2i} \left( \frac{LM^2 - 1}{LM^2 + 1} \right)^{2i}.
\]

In both cases we have \( A_1(M, L) = A_2(M, L) \).

It should be noted that in the definition of the A-polynomial \( m \)-tuple of an \( m \)-component link, we discard the irreducible component of the \( SL_2(\mathbb{C}) \)-character variety containing only characters of reducible representations and therefore consider only the nonabelian \( SL_2(\mathbb{C}) \)-character variety of the link group.

For a link \( \mathcal{L} \) in \( S^3 \), let \( E_\mathcal{L} = S^3 \setminus \mathcal{L} \) denote the link exterior. The link group of \( \mathcal{L} \) is defined to be the fundamental group \( \pi_1(E_\mathcal{L}) \) of \( E_\mathcal{L} \). It is known that the link group of the twisted Whitehead link \( W_k \) has a standard two-generator presentation of a two-bridge link group \( \pi_1(E_{W_k}) = \langle a, b \mid aw = wa \rangle \), where \( a, b \) are meridians depicted in Figure 1 and \( w \) is a word in the letters \( a, b \). More precisely, we have \( w = (bab^{-1}a^{-1})^n a(a^{-1}b^{-1}ab)^n \) if \( k = 2n - 1 \) and \( w = (bab^{-1}a^{-1})^n bab(a^{-1}b^{-1}ab)^n \) if \( k = 2n \).

For a presentation \( \rho : \pi_1(E_{W_k}) \rightarrow SL_2(\mathbb{C}) \) we let \( x, y, z \) denote the traces of the images of \( a, b, ab \) respectively. We also let \( v \) denote the trace of the image of \( bab^{-1}a^{-1} \). Explicitly, we have \( v = x^2 + y^2 + z^2 - xyz - 2 \). It was shown in [Tr1] that the nonabelian \( SL_2(\mathbb{C}) \)-character variety of \( W_k \) has exactly \( \left\lfloor \frac{k}{2} \right\rfloor \) irreducible components. For a hyperbolic link, there are distinguished components of the \( SL_2(\mathbb{C}) \)-character variety called the canonical components. They contain important information about the hyperbolic structure of the link. By using the formula of the A-polynomial in Theorem 1.1 we can determine the canonical component of the hyperbolic twisted Whitehead link \( W_k \) (with \( k \geq 1 \)) as follows.
Theorem 1.2. If $k = 2n - 1$ then the canonical component of $W_k$ is the zero set of the polynomial $(xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v)$.

If $k = 2n$ then the canonical component of $W_k$ is the zero set of the polynomial $zS_n(v) - (xy - z)S_{n-1}(v)$.

Here the $S_k(v)$ are the Chebychev polynomials of the second kind defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$ for all integers $k$. Explicitly, we have

$$S_k(v) = \sum_{0 \leq i \leq k/2} (-1)^i \binom{k - i}{k - 2i} v^{k - 2i}$$

for $k \geq 0$, $S_k(v) = -S_{-k-2}(v)$ for $k \leq -2$, and $S_{-1}(v) = 0$.

For a hyperbolic link $L \subset S^3$, let $\rho_{\text{hol}}$ be a holonomy representation of $\pi_1(E_L)$ into $\text{PSL}_2(\mathbb{C})$. Thurston [Th] showed that $\rho_{\text{hol}}$ can be deformed into an $m$-parameter family $\{\rho_1, \ldots, \rho_m\}$ of representations to give a corresponding family $\{E_L(\alpha_1, \ldots, \alpha_m)\}$ of singular complete hyperbolic manifolds, where $m$ is the number of components of $L$. In this paper we will consider only the case where all of $\alpha_i$'s are equal to a single parameter $\alpha$. In which case we also denote $E_L(\alpha_1, \ldots, \alpha_m)$ by $E_L(\alpha)$. These $\alpha$'s and $E_L(\alpha)$'s are called the cone-angles and hyperbolic cone-manifolds of $L$, respectively.

We consider the complete hyperbolic structure on a link complement as the cone-manifold structure of cone-angle zero. It is known that for a two-bridge knot or link $L$ there exists an angle $\alpha_L \in \left[\frac{\pi}{3}, \pi\right]$ such that $E_L(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_L)$, Euclidean for $\alpha = \alpha_L$, and spherical for $\alpha \in (\alpha_L, \pi)$ [HLM, Ko1, Po, PW]. Explicit volume formulas for hyperbolic cone-manifolds of knots were known for $4_1$ [HLM, Ko1, Ko2, MR], 52 [Me1], two-bridge knots [HMP] and two-bridge knots $C(2n, 3)$ [HL]. Recently, the volume of double twist knot cone-manifolds has been computed in [Tr3]. We should remark that a formula for the volume of the cone-manifold of $C(2n, 4)$ has just been given in [HLMR1]. However, with an appropriate change of variables, this formula has already obtained in [Tr3], since $C(2n, 4)$ is the double twist knot $J(2n, -4)$.

In this paper we are interested in hyperbolic cone-manifolds of links. Explicit volume formulas for hyperbolic cone-manifolds of links have been only known for $5_2^1$ [MV], $6_2^1$ [Me2], $6_3^3$ [DMM] and $7_3^2$ [HLMR2]. From the proof of Theorem 1.2 we can compute the volume of the hyperbolic cone-manifold of the twisted Whitehead link $W_k$ as follows.

Let

$$R_{W_k}(s, z) = \begin{cases} x^2 S_{n-1}(v) - z S_n(v) - (x^2 - z) S_{n-2}(v) & \text{if } k = 2n - 1 \\ z S_n(v) - (x^2 - z) S_{n-1}(v) & \text{if } k = 2n, \end{cases}$$

where $x = s + s^{-1}$ and $v = 2x^2 + z^2 - x^2 z - 2$.

Theorem 1.3. For $\alpha \in (0, \alpha_{W_k})$ we have

$$\text{Vol } E_{W_k}(\alpha) = \int_\alpha^{\pi} 2 \log \left| \frac{z - (s^{-2} + 1)}{z - (s^2 + 1)} \right| \, d\omega$$

where $s = e^{i\omega/2}$ and $z$ (with $\text{Im } z \geq 0$) satisfy $R_{W_k}(s, z) = 0$.

Note that the above volume formula for the hyperbolic cone-manifold $E_{W_k}(\alpha)$ depends on the choice of a root $z$ of $R_{W_k}(e^{i\omega/2}, z) = 0$. In practice, we choose the root $z$ which gives the maximal volume.

As a direct consequence of Theorem 1.3 we obtain the following.
Corollary 1.4. The hyperbolic volume of the $r$-fold cyclic covering over the twisted Whitehead link $W_k$, with $r \geq 3$, is given by the following formula
\[
\text{rVol } E_{W_k}(\frac{2\pi}{r}) = r \int_{2\pi}^\pi 2\log \left| \frac{z - (s^2 + 1)}{z - (s^{-2} + 1)} \right| d\omega
\]
where $s = e^{i\omega/2}$ and $z$ (with $\text{Im } z \geq 0$) satisfy $R_{W_k}(s, z) = 0$.

Remark 1.5. The link $7_3^2$ is the twisted Whitehead link $W_3$. With an appropriate change of variables, we obtain the volume formula for the hyperbolic cone-manifold of $7_3^2$ in [HLMR2, Theorem 1.1] by taking $k = 3$ in Theorem 1.4.

The paper is organized as follows. In Section 2 we review the definition of the A-polynomial $m$-tuple of an $m$-component link in $S^3$. In Section 3 we review the nonabelian $SL_2(\mathbb{C})$-representations of a two-bridge link and compute them explicitly for twisted Whitehead links. We also give proofs of Theorems 1.1–1.3 in Section 3.

2. The A-polynomial

2.0.1. Character varieties. The set of characters of representations of a finitely generated group $G$ into $SL_2(\mathbb{C})$ is known to be a complex algebraic set, called the character variety of $G$ and denoted by $\chi(G)$ (see [CS, LM]). For a manifold $Y$ we also use $\chi(Y)$ to denote $\chi(\pi_1(Y))$. Suppose $G = \mathbb{Z}^2$, the free abelian group with 2 generators. Every pair of generators $\mu, \lambda$ will define an isomorphism between $\chi(G)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs $(M, L)$ and $\tau : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$ is the involution defined by $\tau(M, L) := (M^{-1}, L^{-1})$, as follows. Every representation is conjugate to an upper diagonal one, with $M$ and $L$ being the upper left entry of $\mu$ and $\lambda$ respectively. The isomorphism does not change if we replace $(\mu, \lambda)$ with $(\mu^{-1}, \lambda^{-1})$.

2.0.2. The A-polynomial. Suppose $L = K_1 \sqcup \cdots \sqcup K_m$ be an $m$-component link in $S^3$. Let $E_L = S^3 \setminus L$ be the link exterior and $T_1, \ldots, T_m$ the boundary tori of $E_L$ corresponding to $K_1, \ldots, K_m$ respectively. Each $T_i$ is a torus whose fundamental group is free abelian of rank two. An orientation of $K_i$ will define a unique pair of an oriented meridian $\mu_i$ and an oriented longitude $\lambda_i$ such that the linking number between the longitude $\lambda_i$ and the knot $K_i$ is 0. The pair provides an identification of $\chi(\pi_1(T_i))$ and $(\mathbb{C}^*)_i^2/\tau_i$, where $(\mathbb{C}^*)_i^2$ is the set of non-zero complex pairs $(M_i, L_i)$ and $\tau_i$ is the involution $\tau(M_i, L_i) = (M_i^{-1}, L_i^{-1})$, which actually does not depend on the orientation of $K_i$.

The inclusion $T_i \hookrightarrow E_L$ induces the restriction map
\[
\rho_i : \chi(E_L) \longrightarrow \chi(T_i) \equiv (\mathbb{C}^*)_i^2/\tau_i.
\]
For each $\gamma \in \pi_1(E_L)$ let $f_\gamma$ be the regular function on the character variety $\chi(E_L)$ defined by $f_\gamma(\rho) = (\chi_\rho(\gamma))^2 - 4$, where $\chi_\rho$ denotes the character of a representation $\rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$. Let $\chi_i(X)$ be the subvariety of $\chi(X)$ defined by $f_{\mu_j} = 0$, $f_{\lambda_j} = 0$ for all $j \neq i$. Let $Z_i$ be the image of $\chi_i(X)$ under $\rho_i$ and $\tilde{Z}_i \subset (\mathbb{C}^*)_i^2$ the lift of $Z_i$ under the projection $(\mathbb{C}^*)_i^2 \rightarrow (\mathbb{C}^*)_i^2/\tau_i$. It is known that the Zariski closure of $\tilde{Z}_i \subset (\mathbb{C}^*)_i^2 \subset \mathbb{C}_i^2$ in $\mathbb{C}_i^2$ is an algebraic set consisting of components of dimension 0 or 1 ([ZL]). The union of all the 1-dimension components is defined by a single polynomial $A'_i \in \mathbb{Z}[M_i, L_i]$ whose coefficients are co-prime. Note that $A'_i$ is defined up to $\pm 1$. It is also known that $A'_i$ always contains the factor $L_i - 1$ coming from the characters of reducible representations, hence $A'_i = (L_i - 1)A_i$.
for some \( A_i \in \mathbb{Z}[M_i, L_i] \). As in [Zh], we will call \([A_1(M_1, L_1), \ldots, A_m(M_m, L_m)]\) the \( A \)-polynomial \( m \)-tuple of \( \mathcal{L} \). For brevity, we also write \( A_i(M, L) \) for \( A_i(M_i, L_i) \). We refer the reader to [Zh] for properties of the \( A \)-polynomial.

Recall that the Newton polygon of a two-variable polynomial \( \sum a_{ij}M^iL^j \) is the convex hull in \( \mathbb{R}^2 \) of the set \{\((i, j) : a_{ij} \neq 0\)\}. The slope of a side of the Newton polygon is called a boundary slope of the polygon. The following proposition is useful for determining canonical components of hyperbolic links.

**Proposition 2.1.** [Zh] Theorem 3(2) Suppose \( \mathcal{L} \subset S^3 \) is a hyperbolic \( m \)-component link. For each \( j = 1, \ldots, m \), the factor \( A_{ij}^\text{can}(M, L) \) of the \( A \)-polynomial \( A_j(M, L) \) corresponding to a canonical component has an irreducible factor whose Newton polygon has at least two distinct boundary slopes. In particular, this irreducible factor contains at least 3 monomials in \( M, L \).

### 3. Proofs of Theorems 1.1–1.3

In this section we review the nonabelian \( SL_2(\mathbb{C}) \)-representations of a two-bridge link from [Ri] and compute them explicitly for twisted Whitehead links. Finally, we give proofs of Theorems 1.1–1.3.

#### 3.1. Two-bridge links

Two-bridge links are those links admitting a projection with only two maxima and two minima. The double branched cover of \( S^3 \) along a two-bridge link is a lens space \( L(2p, q) \), which is obtained by doing a \( 2p/q \) surgery on the unknot. Such a two-bridge link is denoted by \( \mathfrak{b}(2p, q) \). Here \( q \) is an odd integer co-prime with \( 2p \) and \( 2p > |q| \geq 1 \). It is known that \( \mathfrak{b}(2p', q') \) is ambient isotopic to \( \mathfrak{b}(2p, q) \) if and only if \( p' = p \) and \( q' \equiv q \pm 1 \) (mod \( 2p \)), see e.g. [BuZ]. The link group of the two-bridge link \( \mathfrak{b}(2p, q) \) has a standard two-generator presentation \( \pi_1(\mathfrak{b}(2p, q)) = \langle a, b \mid wa = aw \rangle \) where \( a, b \) are meridians, \( w = b^i a^{s_2} \cdots a^{s_p-2} b^{s_{2p-1}} \) and \( \varepsilon_i = (-1)^{[q/(2p)]} \) for \( 1 \leq i \leq 2p - 1 \).

We now study representations of link groups into \( SL_2(\mathbb{C}) \). A representation is called nonabelian if its image is a nonabelian subgroup of \( SL_2(\mathbb{C}) \). Let \( \mathcal{L} = \mathfrak{b}(2p, q) \). Suppose \( \rho : E_\mathcal{L} \to SL_2(\mathbb{C}) \) is a nonabelian representation. Up to conjugation, we may assume that

\[
\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}
\]

where \((u, s_1, s_2) \in \mathbb{C}^3\) satisfies the matrix equation \( \rho(aw) = \rho(aw) \).

For any word \( r \), we write \( \rho(r) = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \). By induction on the word length, we can show that \( r_{21} \) is a multiple of \( u \) in \( \mathbb{C}[u, s_1^{\pm 1}, s_2^{\pm 1}] \). Hence we can write \( r_{21} = ur'_{21} \) for some \( r'_{21} \in \mathbb{C}[u, s_1^{\pm 1}, s_2^{\pm 1}] \). A word is said to be palindromic if it reads the same backward or forward. By Lemma 1 in [Ri] we have

\[
r_{22} - r_{11} + (s_1 - s_1^{-1})r_{12} = (s_2 - s_2^{-1})r'_{21}
\]

for any palindromic word \( r \) of odd length.

The matrix equation \( \rho(aw) = \rho(aw) \) is easily seen to be equivalent to the two equations \( uw'_{21} = 0 \) and \( w_{22} - w_{11} + (s_1 - s_1^{-1})w_{12} = (s_2 - s_2^{-1})w'_{21} \). Since \( w \) is a palindromic word of odd length, we conclude that the matrix equation \( \rho(aw) = \rho(aw) \) is equivalent to a single equation \( w'_{21} = 0 \). The polynomial \( w'_{21} \) is called the Riley polynomial of a two-bridge link, and it determines the nonabelian representations of the link.
Remark 3.1. Other approaches, using character varieties and skein modules, to the Riley polynomial of a two-bridge link can be found in [Qa, IT2, LT].

3.2. Chebyshev polynomials. Recall that the $S_k(v)$ are the Chebychev polynomials of the second kind defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = v S_{k-1}(v) - S_{k-2}(v)$ for all integers $k$. The following results are elementary, see e.g. [IT3].

Lemma 3.2. For any integer $k$ we have

$$S_k^2(v) + S_{k-1}^2(v) - v S_k(v) S_{k-1}(v) = 1.$$ 

Lemma 3.3. Suppose $V \in SL_2(\mathbb{C})$ and $v = \text{tr} V$. For any integer $k$ we have

$$V^k = S_{k-1}(v) V - S_{k-2}(v) \mathbf{1}$$

where $\mathbf{1}$ denotes the $2 \times 2$ identity matrix.

We will need the following lemma in the proof of Theorem 1.1.

Lemma 3.4. Suppose $v = 2 + q$. For $k \geq 0$ we have

$$S_k(v) = \sum_{i=0}^{k} \binom{k + 1 + i}{2i + 1} q^i.$$ 

Proof. We use induction on $k \geq 0$. The cases $k = 0, 1$ are clear. Suppose $k \geq 2$ and holds true for $k - 2$ and $k - 1$. Since $S_k(v) = v S_{k-1}(v) - S_{k-2}(v)$, we have

$$S_k(v) = (2 + q) \sum_{i=0}^{k-1} \binom{k + i}{2i + 1} q^i - \sum_{i=0}^{k-2} \binom{k - 1 + i}{2i + 1} q^i$$

$$= \sum_{i=0}^{k} \left\{ 2 \binom{k + i}{2i + 1} + \binom{k + i - 1}{2i - 1} - \binom{k - 1 + i}{2i + 1} \right\} q^i.$$ 

It remains to show the following identity

$$\binom{k + 1 + i}{2i + 1} = 2 \binom{k + i}{2i + 1} + \binom{k + i - 1}{2i - 1} - \binom{k - 1 + i}{2i + 1}.$$ 

This follows by applying the equality $\binom{c}{d} + \binom{c+1}{d+1} = \binom{c+1}{d+1}$ three times. \qed

3.3. Twisted Whitehead links. In this subsection we compute nonabelian representations of twisted Whitehead links explicitly.

We first consider the case of $W_{2n-1}$. Recall that the link group of $W_{2n-1}$ has a presentation $\pi_1(E_{W_{2n-1}}) = \langle a, b \mid a^w = w a \rangle$, where $a, b$ are meridians depicted in Figure II and $w = (bab^{-1}a^{-1})^n a(a^{-1}b^{-1}ab)^n$.

Suppose $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}$$

where $(u, s_1, s_2) \in \mathbb{C}^3$ satisfies the matrix equation $\rho(aw) = \rho(wa)$.

Recall from the Introduction that $x = \text{tr} \rho(a) = s_1 + s_1^{-1}$, $y = \text{tr} \rho(b) = s_2 + s_2^{-1}$, $z = \text{tr} \rho(ab) = u + s_1 s_2 + s_1^{-1} s_2^{-1}$ and

$$v = \text{tr} \rho(bab^{-1}a^{-1}) = u(u + s_1 s_2 + s_1^{-1} s_2^{-1} - s_1 s_2^{-1} - s_1^{-1} s_2) + 2.$$
Let $c = bab^{-1}a^{-1}$ and $d = a^{-1}b^{-1}ab$. We have

$$\rho(c) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \text{and} \quad \rho(d) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

where

\begin{align*}
c_{11} &= 1 - s_1^{-1}s_2u, \\
c_{12} &= -s_1 + s_1s_2^2 + s_2u, \\
c_{21} &= u(-s_1^{-2}s_2^{-1} + s_2^{-1} - s_1^{-1}u), \\
c_{22} &= 1 + (s_1^{-1}s_2^{-1} - s_1s_2^{-1} + s_1s_2)u + u^2,
\end{align*}

\begin{align*}
d_{11} &= 1 + (s_1^{-1}s_2^{-1} - s_1s_2^{-1} + s_1s_2)u + u^2, \\
d_{12} &= s_1^{-1}s_2^{-2} - s_1^{-1} + s_2^{-1}u, \\
d_{21} &= u(s_2 - s_1^2s_2 - s_1u), \\
d_{22} &= 1 - s_1s_2^{-1}u.
\end{align*}

Since $v = \text{tr} \rho(c) = \text{tr} \rho(d)$, by Lemma 3.3 we have

\begin{align*}
\rho(c^n) &= \begin{bmatrix} c_{11}S_{n-1}(v) - S_{n-2}(v) & c_{12}S_{n-1}(v) \\ c_{21}S_{n-1}(v) - S_{n-2}(v) & c_{22}S_{n-1}(v) - S_{n-2}(v) \end{bmatrix}, \\
\rho(d^n) &= \begin{bmatrix} d_{11}S_{n-1}(v) - S_{n-2}(v) & d_{12}S_{n-1}(v) \\ d_{21}S_{n-1}(v) - S_{n-2}(v) & d_{22}S_{n-1}(v) - S_{n-2}(v) \end{bmatrix}.
\end{align*}

This implies that $\rho(w) = \begin{bmatrix} w_{11} & \ast \\ w_{21} & \ast \end{bmatrix} = \rho(c^{n}d^{n})$ with

\begin{align*}
w_{11} &= (c_{12}d_{21}s_1^{-1} + c_{11}d_{21} + c_{11}d_{11}s_1)S_{n-1}(v) + s_1S_{n-2}(v) \\
&\quad - (d_{21} + c_{11}s_1 + d_{11}s_1)S_{n-1}(v)S_{n-2}(v), \\
w_{21} &= [(c_{22}d_{21}s_1^{-1} + c_{21}d_{21} + c_{21}d_{11}s_1)S_{n-1}(v) - (d_{21}s_1^{-1} + c_{21}s_1)S_{n-2}(v)]S_{n-1}(v).
\end{align*}

By direct calculations we have

\begin{align*}
c_{22}d_{21}s_1^{-1} + c_{21}d_{21} + c_{21}d_{11}s_1 &= u(xy - vz), \\
d_{21}s_1^{-1} + c_{21}s_1 &= u(xy - 2z).
\end{align*}

Hence $w_{21} = uu_{21}'$ where

$$w_{21}'(W_{2n-1}) = ((xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v))S_{n-1}(v),$$

which is the Riley polynomial of $W_{2n-1}$.

Similarly, the Riley polynomial of $W_{2n}$ is given by the following formula

$$w_{21}'(W_{2n}) = (zS_{n}(v) - (xy - z)S_{n-1}(v))(S_{n}(v) - S_{n-1}(v)).$$

**Remark 3.5.** The above formulas for the Riley polynomials of twisted Whitehead links were already obtained in [Tr1] using character varieties. It was also shown in [Tr1] that the Riley polynomial $w_{21}'(W_{2n-1}) \in \mathbb{C}[x, y, z]$ is factored into exactly $n$ irreducible factors $(xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v)$ and $v - 2\cos \frac{2\pi}{n} (1 \leq j \leq n - 1)$. Similarly, $w_{21}'(W_{2n}) \in \mathbb{C}[x, y, z]$ is factored into exactly $n + 1$ irreducible factors $zS_{n}(v) - (xy - z)S_{n-1}(v)$ and
$v - 2 \cos \frac{(2j-1)\pi}{2n+1} (1 \leq j \leq n)$. Note that

$$S_{n-1}(v) = \prod_{j=1}^{n-1} (v - 2 \cos \frac{j\pi}{n}) \quad \text{and} \quad S_n(v) - S_{n-1}(v) = \prod_{j=1}^{n} (v - 2 \cos \frac{(2j-1)\pi}{2n+1}).$$

3.4. Proof of Theorem 1.1. We first consider the case of $W_{2n-1}$. The canonical longitudes corresponding to the meridians $a$ and $b$ are respectively $\lambda_a = wa^{-1}$ and $\lambda_b = \overline{w}b^{-1}$, where $\overline{w}$ is the word obtained from $w$ by exchanging $a$ and $b$. Precisely, we have $\overline{w} = (aba^{-1}b^{-1})^n b(b^{-1}a^{-1}ba)^n$.

Suppose $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ is a nonabelian representation. With the same notations as in the previous subsection, we have

$$\rho(w) = \begin{bmatrix} w_{11} & \ast \\ 0 & (w_{11})^{-1} \end{bmatrix} \quad \text{and} \quad \rho(\overline{w}) = \begin{bmatrix} \overline{w}_{11} & 0 \\ \ast & (\overline{w}_{11})^{-1} \end{bmatrix}.$$

Proposition 3.6. On $v - 2 \cos \frac{j\pi}{n} = 0$ we have

$$w_{11} = s_1 \quad \text{and} \quad \overline{w}_{11} = s_2.$$

Proof. Recall from the previous subsection that

$$w_{11} = (c_{12}d_{21}s_{1}^{-1} + c_{11}d_{21} + c_{11}d_{11}s_{1})S_{n-1}^{2}(v) + s_1S_{n-2}^{2}(v) - (d_{21} + c_{11}s_1 + d_{11}s_1)S_n(v)S_{n-2}(v).$$

If $v = 2 \cos \frac{j\pi}{n}$ then $S_{n-1}(v) = 0$ then $S_{n-2}(v) = \pm 1$. Hence $w_{11} = s_1$.

The proof for $\overline{w}_{11} = s_2$ is similar. \qed

Proposition 3.7. On $(xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v) = 0$ we have

$$w_{11} = \frac{-1 - s_2 + s_1s_2z}{s_1 + s_1s_2^2 - s_2z} \quad \text{and} \quad \overline{w}_{11} = \frac{-1 - s_2^2 + s_1s_2z}{s_2 + s_1s_2^2 - s_2z}.$$

Proof. By Lemma 3.2 we have $S_k^2(v) + S_{k-1}^2(v) - vS_k(v)S_{k-1}(v) = 1$ for all integers $k$. Hence, on $(xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v) = 0$ we have

$$S_{n-1}^2(v) = (1 - rv + r^2)^{-1}$$

where $r = (xy - vz)/(xy - 2z)$. By a direct calculation we have

$$w_{11} = \frac{[(c_{12}d_{21}s_1^{-1} + c_{11}d_{21} + c_{11}d_{11}s_1) + s_1r^2 - (d_{21} + c_{11}s_1 + d_{11}s_1)r]}{s_1s_2(s_1^2s_2 + s_1u - s_2)} \frac{(1 - rv + r^2)^{-1}}{s_1 + s_1s_2^2 - s_2z}.$$

The proof for $\overline{w}_{11}$ is similar. \qed

We now prove Theorem 1.1 for $W_{2n-1}$. Let $\rho(\lambda_a) = \begin{bmatrix} L_a & \ast \\ 0 & L_a^{-1} \end{bmatrix}$. Since $\lambda_a = wa^{-1}$, we have $L_a = w_{11}s_1^{-1}$. On

$$C_j := \left\{(v - 2 \cos \frac{j\pi}{n} = 0) \cap \{s_2^2 = (\overline{w}_{11})^2 = 1\}\right\}$$

by Proposition 3.6 we have $w_{11} = s_1$. Hence $L_a - 1 = 0$ on $C_j$ for all $1 \leq j \leq n - 1$.

By Proposition 3.7 on

$$C := \left\{((xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v) = 0) \cap \{s_2^2 = (\overline{w}_{11})^2 = 1\}\right\}$$
we have \( w_{11} = \frac{-1-s_2^2+s_1 s_2}{s_1+s_1^2-s_2} \). Note that on \( C \), we have \( s_2^2 = 1 \) implies \( (\overline{w}_{11})^2 = 1 \).

With \( s_2 = 1 \) we then have \( w_{11} = \frac{-2+2 s_1}{2 s_1 + z} \). Hence \( z = \frac{2(1+s_1 w_{11})}{s_1 + w_{11}} \). Note that \( v = 2+(z-x)^2 \).

Let \( z-x = t \). Then

\[
t = \frac{2(1 + s_1 w_{11})}{s_1 + w_{11}} - (s_1 + s_1^{-1}) = (s_1 - s_1^{-1}) \left( \frac{w_{11} - s_1^{-1}}{w_{11} + s_1} \right) = (s_1 - s_1^{-1}) \left( \frac{L_a - 1}{L_a + 1} \right)
\]

and \( v = 2 + t^2 \). Since \( y = s_2 + s_2^{-1} = 2 \) and \( v S_{n-1}(v) = S_a(v) + S_{n-2}(v) \) we have

\[
0 = 2x S_{n-1}(v) - z S_n(v) - (2x - z) S_{n-2}(v)
= 2x S_{n-1}(v) - (x + t) S_n(v) - (x - t) S_{n-2}(v)
= -x t S_{n-1}(v) - t (S_n(v) - S_{n-2}(v)).
\]

If \( t = 0 \) then \( w_{11} = \frac{-2+2 s_1}{2 s_1 + z} = s_1 \). In this case we have \( L_a = 1 \). We now consider the case \( x t S_{n-1}(v) + S_a(v) - S_{n-2}(v) = 0 \). Then, by Lemma 3.4 we have

\[
0 = \sum_{i=0}^{n} \left\{ \left( \frac{n+1+i}{2i+1} \right) - \left( \frac{n-1+i}{2i+1} \right) \right\} t^{2i} + x \sum_{i=0}^{n-1} \left( \frac{n+i}{2i+1} \right) t^{2i+1}
= \sum_{i=0}^{n} \left\{ \left( \frac{n+1+i}{2i+1} \right) - \left( \frac{n-1+i}{2i+1} \right) \right\} (s_1 - s_1^{-1})^{2i} \left( \frac{L_a - 1}{L_a + 1} \right)^{2i}
+ \sum_{i=0}^{n-1} \left( \frac{n+i}{2i+1} \right) (s_1 + s_1^{-1})(s_1 - s_1^{-1})^{2i+1} \left( \frac{L_a - 1}{L_a + 1} \right)^{2i+1}.
\]

The last expression in the above equalities is exactly the polynomial \( F(M_a, L_a) \), with \( M_a = s_1 \), defined in Theorem 1.1. Hence, with \( s_2 = 1 \), we have \( (L_a - 1)F(M_a, L_a) = 0 \) on \( C \). Similarly, with \( s_2 = -1 \), we obtain the same equation \( (L_a - 1)F(M_a, L_a) = 0 \) on \( C \).

This proves the formula of the A-polynomial for the component of \( W_{2n-1} \) corresponding to the meridian \( a \). The one for the component corresponding \( b \) is exactly the same.

For \( W_{2n} \) and its component corresponding to \( a \), we have \( L_a M_a^2 - 1 = 0 \) on

\[
D_j := \left\{ \{ v - 2 \cos \left( \frac{2j-1}{2n+1} \pi \right) = 0 \} \cap \{ s_2^2 = (\overline{w}_{11})^2 = 1 \} \right\}
\]

for \( 1 \leq j \leq n \). Moreover, on

\[
D := \left\{ \{ z S_n(v) - (xy - z) S_{n-1}(v) = 0 \} \cap \{ s_2^2 = (\overline{w}_{11})^2 = 1 \} \right\}
\]

we have \( G(M_a, L_a) = 0 \) where \( G \) is the polynomial defined in Theorem 1.1. The same formulas can be obtained for the component of \( W_{2n} \) corresponding to \( b \).

This completes the proof of Theorem 1.2.

3.5. Proof of Theorem 1.2. We make use of Proposition 2.1 which implies that for each \( j = 1, 2 \) the factor \( A_j^{\text{can}}(M, L) \) of the A-polynomial \( A_j(L, M) \) corresponding to a canonical component of \( W_k \) (with \( k \geq 1 \)) has an irreducible factor containing at least 3 monomials in \( M, L \). This irreducible factor cannot be \( L - 1 \) or \( LM^2 - 1 \). Hence, from the proof of Theorem 1.1 we conclude that the canonical components of \( W_{2n-1} \) and \( W_{2n} \) are respectively the zero sets of the polynomials \((xy - v) S_{n-1}(v) - (xy - 2z) S_{n-2}(v) \) and \( z S_n(v) - (xy - z) S_{n-1}(v) \). This completes the proof of Theorem 1.2.
3.6. **Proof of Theorem 1.3.** Recall that \( E_{W_k}(\alpha) \) is the cone-manifold of \( W_k \) with cone angles \( \alpha_1 = \alpha_2 = \alpha \). There exists an angle \( \alpha_{W_k} \in [\frac{2\pi}{3}, \pi) \) such that \( E_{W_k}(\alpha) \) is hyperbolic for \( \alpha \in (0, \alpha_{W_k}) \), Euclidean for \( \alpha = \alpha_{W_k} \), and spherical for \( \alpha \in (\alpha_{W_k}, \pi) \).

We first consider the case of \( W_{2n-1} \). By Theorem 1.2 the canonical component of \( W_{2n-1} \) is determined by \((xy-vz)S_{n-1}(v) - (xy-2z)S_{n-2}(v) = 0\). Moreover, by Proposition 3.7 on this component we have

\[
w_{11} = \frac{-1 - s_2^2 + s_1 s_2 z}{s_1 + s_1 s_2^2 - s_2 z}.
\]

Here we use the same notations as in the previous subsections.

For \( \alpha \in (0, \alpha_{W_{2n-1}}) \), by the Schlafli formula we have

\[
\text{Vol } E_{W_{2n-1}}(\alpha) = \int_\alpha^{\pi} 2 \log |w_{11}| \, d\omega = \int_\alpha^{\pi} 2 \log \left| \frac{-1 - s_2^2 + s_1 s_2 z}{s_1 + s_1 s_2^2 - s_2 z} \right| \, d\omega
\]

where \( s_1 = s_2 = s = e^{i \omega / 2} \) and \( z \) (with \( \left| \frac{-1 - s_2^2 + s_1 s_2 z}{s_1 + s_1 s_2^2 - s_2 z} \right| > 1 \)) satisfy

\((xy - vz)S_{n-1}(v) - (xy - 2z)S_{n-2}(v) = 0\).

We refer the reader to [DMM] for the volume formula of hyperbolic cone-manifolds of links using the Schlafli formula.

Note that \( \left| \frac{-1 - s_2^2 + s_1 s_2 z}{s_1 + s_1 s_2^2 - s_2 z} \right| = \left| \frac{z-(s^{-2}+1)}{z-(s^2+1)} \right| \geq 1 \) is equivalent to \( \text{Im } z \geq 0 \). Since \( x = y = s + s^{-1} \) and \( v = x^2 + y^2 + z^2 - xyz - 2 = 2x^2 + z^2 - x^2 z - 2 \), the proof of Theorem 1.3 for \( W_{2n-1} \) is complete. The proof for \( W_{2n} \) is similar.

**Acknowledgements**

The author has been partially supported by a grant from the Simons Foundation (#354595 to Anh Tran).

**References**

[BoZ] S. Boyer and X. Zhang, *Every non-trivial knot in \( S^3 \) has non-trivial A-polynomial*, Proc. Amer. Math. Soc. 133 (2005), no. 9, 2813–2815 (electronic).

[BuZ] G. Burde and H. Zieschang, *Knots*, de Gruyter Stud. Math., vol. 5, de Gruyter, Berlin, 2003.

[CCGLS] D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. 118 (1994), pp. 47–84.

[CS] M. Culler and P. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. (2) 117 (1983), no. 1, 109–146.

[DG] N. Dunfield and S. Garoufalidis, *Non-triviality of the A-polynomial for knots in \( S^3 \)*, Algebr. Geom. Topol. 4 (2004), 1145–1153 (electronic).

[DMM] D. Derevnin, A. Mednykh and M. Mulazzani, *Volumes for twist link cone-manifolds*, Bol. Soc. Mat. Mexicana (3) 10 (2004), Special Issue, 129–145.

[GM] S. Garoufalidis and T. Mattman, *The A-polynomial of the \((-2,3,3+2n)\)-pretzel knots*, New York J. Math. 17 (2011), 269–279.

[HL] J. Ham and J. Lee, *An explicit formula for the A-polynomial of the knot with Conway’s notation C(2n, 3)*, preprint 2016, [arXiv:1601.05860](https://arxiv.org/abs/1601.05860)
[HLM] H. Hilden, M. Lozano, and J. Montesinos-Amilibia, Volumes and Chern-Simons invariants of cyclic coverings over rational knots, in Topology and Teichmüller spaces (Katinkulta, 1995), pages 31–55. World Sci. Publ., River Edge, NJ, 1996.

[HLMR1] J. Ham, J. Lee, A. Mednykh and A. Rasskazov, On the volume and the Chern-Simons invariant for the 2-bridge knot orbifolds, preprint 2016, arXiv:1607.08044.

[HLMR2] J. Ham, J. Lee, A. Mednykh and A. Rasskazov, An explicit volume formula for the link $7_2^3(\alpha, \alpha)$ cone-manifolds, preprint 2016, arXiv:1607.08047.

[HMP] J.-Y. Ham, A. Mednykh, and V. Petrov, Identities and volumes of the hyperbolic twist knot cone-manifolds, J. Knot Theory Ramifications 23 (2014) 1450064.

[HS] J. Hoste and P. Shanahan, A formula for the A-polynomial of twist knots, J. Knot Theory Ramifications 14 (2005), 91–100.

[Ko1] S. Kojima, Deformations of hyperbolic 3-cone-manifolds, J. Differential Geom. 49 (1998) 469–516.

[Ko2] S. Kojima, Hyperbolic 3-manifolds singular along knots, Chaos Solitons Fractals 9 (1998) 765–777.

[LM] A. Lubotzky and A. Magid, Varieties of representations of finitely generated groups, Memoirs of the AMS 336 (1985).

[LT] T. Le and A. Tran, The Kauffman bracket skein module of two-bridge links, Proc. Amer. Math. Soc. 142 (2014), no. 3, 1045–1056.

[Ma] Daniel Mathews, An explicit formula for the A-polynomial of twist knots, J. Knot Theory Ramifications 23 (2014), no. 9, 1450044, 5 pp.

[Me1] A. Mednykh, The volumes of cone-manifolds and polyhedra http://mathlab.snu.ac.kr/~top/workshop01.pdf, 2007. Lecture Notes, Seoul National University.

[Me2] A. Mednykh, Trigonometric identities and geometrical inequalities for links and knots, Proceedings of the Third Asian Mathematical Conference, 2000 (Diliman), 352–368, World Sci. Publ., River Edge, NJ, 2002.

[MR] A. Mednykh and A. Rasskazov, Volumes and degeneration of cone-structures on the figure-eight knot, Tokyo J. Math. 29 (2006) 445–464.

[MV] A Mednykh and A. Vesnin, On the volume of hyperbolic Whitehead link cone-manifolds, Geometry and analysis. Sci. Ser. A Math. Sci. (N.S.) 8 (2002), 1–11.

[NZ] Y. Ni and X. Zhang, Detection of knots and a cabling formula for A-polynomials, preprint 2014, arXiv:1411.0353.

[Pe] K. Petersen, A-polynomials of a family of two-bridge knots, New York J. Math. 21 (2015), 847–881.

[Po] J. Porti, Spherical cone structures on 2-bridge knots and links, Kobe J. Math. 21 (2004) 61–70.

[PW] J. Porti and H. Weiss, Deforming Euclidean cone 3-manifolds, Geom. Topol. 11 (2007) 1507–1538.

[Qa] K. Qazaqzeh, The character variety of a family of one-relator groups, Internat. J. Math. 23 (2012), no. 1, 1250015, 12 pp.

[Ri] R. Riley, Algebra for Heckoid groups, Trans. Amer. Math. Soc. 334 (1992), 389–409.

[Th] W. Thurston, The geometry and topology of 3-manifolds, http://library.msri.org/books/gt3m, 1977/78. Lecture Notes, Princeton University.

[Tr1] A. Tran, Character varieties of $(-2,2m+1,2n)$-pretzel links and twisted Whitehead links, J. Knot Theory Ramifications 25 (2016), no. 2, 1650007, 16 pp.

[Tr2] A. Tran, The universal character ring of the $(-2,2m+1,2n)$-pretzel link, Internat. J. Math. 24 (2013), no. 8, 1350063, 13 pp.
[Tr3] A. Tran, *Volumes of double twist knot cone-manifolds*, preprint 2015, arXiv:1512.08105.

[TY] N. Tamura and Y. Yokota, *A formula for the A-polynomials of \((-2, 3, 1+2n)\)-pretzel knots*, Tokyo J. Math. 27 (2004), no. 1, 263–273.

[Zh] X. Zhang, *The A-polynomial \(n\)-tuple of a link and hyperbolic 3-manifolds with non-integral traces*, J. Knot Theory Ramifications 15 (2006), 279–287.

Department of Mathematical Sciences, The University of Texas at Dallas, Richardson, TX 75080, USA

E-mail address: att140830@utdallas.edu