Coordinate-free exponentials of general multivector in $Cl_{p,q}$ algebras for $p+q=3$

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Closed form expressions in real Clifford geometric algebras $Cl_{0,3}$, $Cl_{3,0}$, $Cl_{1,2}$, and $Cl_{2,1}$ are presented in a coordinate-free form for exponential function when the exponent is a general multivector. The main difficulty in solving the problem is connected with an entanglement (or mixing) of vector and bivector components $a_i$ and $a_{jk}$ in a form $(a_i - a_{jk})^2$, $i \neq j \neq k$. After disentanglement, the obtained formulas simplify to the well-known de Moivre-type trigonometric/hyperbolic function for vector or bivector exponentials. The presented formulas may find wide application in solving GA differential equations, in signal processing, automatic control and robotics.

**KEYWORDS**
Clifford (geometric) algebra, computer-aided theory, exponentials of Clifford numbers

**MSC CLASSIFICATION**
11E88, 68U01

1 INTRODUCTION

In the complex number algebra, which is isomorphic to $Cl_{0,1}$ Clifford geometric algebra, the complex exponential may be expanded into trigonometric function sum (de Moivre’s theorem). In 2D and 3D GA algebras, similar formulas are known too under the name “polar decomposition.” In particular, if the square of the blade is equal to $\pm 1$, then GA exponential can also be expanded in de Moivre-type sum, that is, either as trigonometric or hyperbolic functions, respectively.$^{1-3}$ However, expansion of GA exponential in case of 3D or higher algebras, when the exponent is a general multivector, as we shall see is much more complicated and as far as the authors know has not been analyzed fully as yet. The authors of articles$^{4,5}$ have considered general properties of functions of MV variable for Clifford algebras $n = p + q \leq 3$, including the exponential function. For this purpose, they have made use of the property that in these algebras the pseudoscalar $I$ commutes with all MV elements and $I^2 = \pm 1$. This has allowed to introduce more general functions related to a polar decomposition of MVs. However, the analysis is not full enough. A different approach to resolve the problem is to factor, if possible, the exponential into product of simpler exponentials, for example, in the polar form.$^{6-8}$ A general bivector exponential in $Cl_{4,1}$ algebra was analyzed in previous studies$^{9,10}$ in connection with 3D conformal GA. In paper,$^{11}$ exact and closed form expressions for coefficients at basis elements to calculate GA exponentials in coordinate form are presented for all 3D GAs. However, in this form, the final MV formulas constructed in some orthogonal basis are rather complicated and inconvenient to carry a detailed analysis of the properties of GA exponential functions, although they may be useful in some practical cases, for example, for all-purpose computer programs to calculate GA exponentials with numerical coefficients.

In this paper, the exact exponential formulas$^{11}$ are transformed to coordinate-free form what allows to carry a detailed analysis and gives a clear geometric interpretation to the problem. Also, special cases where various additional
conditions and relations are imposed upon GA elements are considered what may be useful in applications of exponentials in practice. In Section 2, the notation is introduced. In Section 3, the exponential of the simplest, namely, \( Cl_{0,3} \) algebra, is considered. Since algebras \( Cl_{1,0} \) and \( Cl_{1,2} \) are isomorphic, in Section 4, the exponentials for both algebras are investigated simultaneously. In Section 5, the exponential of the most difficult \( Cl_{2,1} \) algebra is presented. In Section 6, possible applications of exponentials to solve GA linear differential equations are presented. Finally, in Section 7, we discuss further development of the problem, including the inverse function, namely, the GA logarithm.

## 2 | NOTATION AND GENERAL PROPERTIES OF GA EXPONENTIAL

In calculations below, we have intensively used our symbolic GA program written for Mathematica package.\(^1\) In the program, in GA space endowed with orthonormal basis, we expanded a general 3D MV in inverse degree lexicographic ordering: \( \{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123} \equiv I \} \), where \( e_i \) are basis vectors, \( e_{ij} \) are the bivectors and \( I \) is the pseudoscalar.\(^\dagger\) The number of subscripts indicates the grade. The scalar is a grade-0 element, the vectors \( e_i \) are the grade-1 elements, and so forth. In the orthonormalized basis used here, the geometric product of basis vectors satisfies the anti-commutation relation,

\[
e_i e_j + e_j e_i = \pm 2 \delta_{ij}.
\]

For \( Cl_{1,0} \) and \( Cl_{0,3} \) algebras the squares of basis vectors, correspondingly, are \( e_i^2 = +1 \) and \( e_i^2 = -1 \), where \( i = 1, 2, 3 \).

For mixed signature algebras such as \( Cl_{2,1} \) and \( Cl_{1,2} \) the squares are \( e_1^2 = e_2^2 = 1, e_3^2 = -1 \) and \( e_1^2 = 1, e_2^2 = e_3^2 = -1 \), respectively.

The general MV that belongs to real Clifford algebras \( Cl_{p,q} \), when \( n = p + q = 3 \) can be expressed as

\[
A = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_{12} e_{12} + a_{13} e_{13} + a_{23} e_{23} + a_{123} I
\]

\[\equiv a_0 + a + A + a_{123} I,\]  

(2)

where \( a_i, a_{ij}, \) and \( a_{123} \) are the real coefficients and \( a = a_1 e_1 + a_2 e_2 + a_3 e_3 \) and \( A = a_{12} e_{12} + a_{13} e_{13} + a_{23} e_{23} \) are, respectively, the vector and bivector. \( I \) is the pseudoscalar, \( I = e_{123} \). Similarly, the exponential of \( A \) is denoted as

\[
e^A = B = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 + b_{12} e_{12} + b_{13} e_{13} + b_{23} e_{23} + b_{123} I
\]

\[\equiv b_0 + b + B + b_{123} I.\]

(3)

The main involutions, namely, the reversion, grade inversion, and Clifford conjugation, are denoted, respectively, by tilde, circumflex, and their combination,

\[
\tilde{A} = a_0 + a - A - a_{123} I, \quad \hat{A} = a_0 - a + A - a_{123} I, \quad \check{A} = a_0 - a - A + a_{123} I.
\]

(4)

### 2.1 | General properties of GA exponential

The exponential of MV is another MV that belongs to the same geometric algebra. Therefore, we shall assume that the defining equation for exponential is \( e^A = B \), where \( A, B \in Cl_{p,q} \) and \( p + q = 3 \). The following general properties hold for MV exponential:

\[
\exp(A + B) = \exp(A) \exp(B) \text{ if and only if } AB = BA,
\]

\[
\tilde{e}^A = e^{\bar{A}}, \quad \check{e}^A = e^{\hat{A}}, \quad \check{e}^A = e^{\check{A}}.
\]

(5)

From the first formula, the exponential identity follows \( \exp A = (\exp A/m)^m, m \in \mathbb{N} \). In the numerical matrix function theory, it is frequently used where it is called the inverse scaling and squaring method.\(^13\) The middle line of (5) indicates

\(^{1}\)An increasing order of digits in basis elements is used; that is, we write \( e_{1i} \) instead of \( e_{ii} = -e_{1i} \). This convention is reflected in opposite signs when expressions are expanded in a coordinate basis.
that involution and exponentiation operations commute. In the last expression, the transformation V, for example, the rotor, has been lifted to exponent, that is, similarity transformation commutes with exponentiation.

The GA exponential $e^A$ can be expanded in a series that has exactly the same structure as a scalar exponential, from which GA trigonometric and hyperbolic GA functions as well as various other relations that are analogues of respective scalar functions follow. For example,

$$
\cos^2 A + \sin^2 A = 1, \quad \cosh^2 A - \sinh^2 A = 1, \\
\cos(2A) = \cos^2 A - \sin^2 A, \quad \sin(2A) = 2\sin A \cos A = 2\cos A \sin A.
$$

(6)

Also, it should be noted that GA functions of the same argument commute. Thus, the sine and cosine functions as well as hyperbolic GA sine and cosine functions satisfy: $\sin A \cos A = \cos A \sin A$ and $\sinh A \cosh A = \cosh A \sinh A$.

In Sections 3–5, the exact (symbolic) formulas for GA exponentials in an expanded form but coordinate-free form are presented. If the MV is in a numerical form or one is interested in a checking of GA formula, for instance, in a preliminary stage of calculation, a finite series expansion may be useful as well. It is known that GA exponential is convergent for all MVs; however, convergence is not monotone (see Figure 1). To minimize the number of multiplications, it is convenient to represent the exponential in a nested form (aka Horner’s rule)

$$
e^A = 1 + \frac{A}{1} \left( 1 + \frac{A}{2} \left( 1 + \frac{A}{3} \left( 1 + \frac{A}{4} (1 + \ldots) \right) \right) \right),
$$

(7)

which requires a minimal number of MV products to calculate the truncated series than working out each power of A. If numerical coefficients in A are not too large, the exponential $e^A$ can be approximated to high precision by (7). The series may be programmed as a simple iterative procedure repeated $k$-times that begins from the end (dots) with the initial value at $A/k = 1$ and then iteratively moving to left.‡

We start from the $C_{l,0,3}$ GA where the expanded exponential in the coordinate form has the simplest MV coefficients.

3 MV EXPONENTIALS IN $C_{l,0,3}$ ALGEBRA

3.1 Exponential in coordinate-free form

In GA, the symbolic formulas may be written in coordinate and coordinate-free forms. The latter presentation is compact and carries clear geometrical interpretation and therefore is preferred. However, the formulas written in the coordinates sometimes may be useful too, in particular, in GA numerical calculations by non-symbolic programmes. In Dargys and Acus, we have found a general MV exponentials in coordinate form for all 3D GAs. Although the expressions are rather involved, however, they acquire a simple form if coordinate-free notation (the second lines in Equations (2) and (3)) are used. Moreover, geometrical analysis of GA formulas becomes simpler and more evident when formulas are rewritten in a coordinate-free form.

‡In Mathematica the algorithm reads $\text{expHorner}[[A, n, s]] := \text{Module}[[B = 1, s = n + 1], \text{While}[(s = s - 1) > 0, B = 1. + \text{GP}[B, A/s]]; B]]$, where $n$ is the number of iterations and GP is the geometric product. If MV coefficients are large ($a_j > 3$), in addition, the formula $(\text{exp}(A/m))^m$, where $m$ is the integer, may be applied at first to accelerate the convergence and then to raise the result to the $m$th power.
In a case of $Cl_{0,3}$ algebra, after multiplication of coordinates by respective basis elements and then collection to vector, bivector, trivector, and their products, one can transform the exponential components$^{11}$ to a generic coordinate-free form.

**Theorem 1** (Exponent of $Cl_{0,3}$). Exponential of general MV $A$ can be expressed as

$$\exp(A) = \frac{1}{2} e^{a_+} \left( e^{a_{123}} (1 + I) \left( \cos a_+ + \frac{\sin a_+}{a_+} (a + A) \right) + e^{-a_{123}} (1 - I) \left( \cos a_- + \frac{\sin a_-}{a_-} (a + A) \right) \right).$$  \hspace{1cm} (8)

where $a_-$ and $a_+$ are the scalars,

$$a_- = \sqrt{-(a \cdot a + A \cdot A)} + 2I a \wedge A = \sqrt{(a_1 + a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2},$$  \hspace{1cm} (9)

$$a_+ = \sqrt{-(a \cdot a + A \cdot A)} - 2I a \wedge A = \sqrt{(a_3 - a_{12})^2 + (a_2 + a_{13})^2 + (a_1 - a_{23})^2}.$$  \hspace{1cm} (10)

The following precedence of multiplications is assumed throughout the article: inner (the highest), outer, and finally geometric.

**Proof.** An elegant proof in a coordinate-free form is unknown to us as yet. In the coordinate-full form, after rewriting Equation (8), the theorem may be proved$^{11}$ by checking the defining equation for the exponential,

$$\frac{d \exp(A t)}{d t} \big|_{t=1} = A \exp(A) = \exp(A) A,$$  \hspace{1cm} (11)

where $A$ is assumed to be independent of scalar parameter $t$. Since the MV always commutes with itself, the multiplications from left and right by $A$ coincide. After differentiation with respect to $t$ and then setting $t = 1$, one can verify that the result indeed is $A \ exp(A)$. To be sure we also checked Equation (11) by series expansions of $\exp(A t)$ up to order 6 with symbolic coefficients and up to order 20 with random integers using the Mathematica package.$^{12}$ Formulas for other Clifford algebras were verified in the same way. \hfill \Box

The scalars show how the vector and bivector components are entangled (mixed up). As we shall see the appearance of trigonometric functions in Equation (8) indicates that the exponential in $Cl_{0,3}$ (and in all remaining 3D algebras) has an oscillatory character as a function of the coefficients, similarly as it is in the de Moivre formula case. When the denominator in the formula (8), either $a_+$ or $a_-$, reduces to zero we will have a special case. Since determinant$^8$ $\det(a + A) = a_+^2 a_-^2$ the special case occurs when $\det(a + A) = 0$. The generic formula (8) then should be modified by replacing the corresponding ratios by their limits, $\lim_{a_+ \to 0} \frac{\sin a_+}{a_+} = 1$.

If either vector $a$ or bivector $A$ in (8)–(10) is absent, then $a_+ = a_- \equiv a$, where $a$ is a magnitude of the vector $a = |a| = (a \tilde{a})^{\frac{1}{2}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$, or of the bivector $a = |A| = (A \tilde{A})^{\frac{1}{2}} = \sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2}$. If, in addition, the scalar and pseudoscalar are absent, $a_0 = a_{123} = 0$, the formula (8) reduces to the well-known trigonometric expressions for exponential of vector and bivector in a polar form,$^2$ namely,

$$e^a = \cos \ |a| + \frac{a}{|a|} \sin \ |a|, \quad e^A = \cos \ |A| + \frac{A}{|A|} \sin \ |A|.$$  \hspace{1cm} (12)

Note that the appearance of trigonometric functions in (12) is due to vector and bivector properties, $a^2 < 0$ and $A^2 < 0$ in $Cl_{0,3}$. If $a_+ = a_- = 0$, then Equation (8) simplifies to

$$e^A = e^{a_0} (\cosh a_{123} + I \sinh a_{123}) (1 + a + A), \quad I^2 = 1,$$  \hspace{1cm} (13)

that yields de Moivre-like formula when $a + A = 0$.

$^8$In 3D algebras, the determinant of MV $A$ is defined by $\det(A) = \tilde{A} \tilde{A}$. This determinant should not be confused with a MV transformation determinant.$^{17}$
In the following we shall distinguish two kinds of coordinate-free formulas for exponential functions, namely, generic and special. The formula (8) is an example of generic formula since it is valid for almost all real coefficient \(a\) values, where \(J\) is a compound index: \(J = i, j, \text{or}jk\). The expression (13) represents the special formula, since in the case \(a_+ = 0\) and/or \(a_- = 0\) we have division by zero in (8) and therefore should use a modified formula (which, in this case can be obtained by computing limit of (8) when \(a_+ \to 0\), and/or \(a_- \to 0\)). On the other hand, Equation (12) represents an important in practice case of generic solution (obtained by simply equating the coefficients at scalar and pseudoscalar and, respectively, at bivector and vector, to zero). For completeness, it would be interesting to remark that in a case of logarithmic functions one may add an additional free MV to the generic or special symbolic solution.\(^{18}\) Such free terms do not appear for GA exponential functions.

**Example 1.** Exponential of MV in \(Cl_{0,3}\). Let’s compute the exponential of \(A = -8 - 6e_2 - 9e_3 + 5e_{12} - 5e_{13} + 6e_{23} - 4e_{123}\) using the coordinate-free expression (8). We find \(a_+ = \sqrt{53}\) and \(a_- = \sqrt{353}\). The exact numerical answer then is

\[
\exp(A) = \frac{e^{-A}}{2} \left( e^{-A} (1 + e_{123}) \left( \frac{\cos \sqrt{53} + \sin \sqrt{53}}{\sqrt{53}} (-6e_2 - 9e_3 + 5e_{12} - 5e_{13} + 6e_{23}) \right) + e^{A} (1 - e_{123}) \left( \frac{\cos \sqrt{53} + \sin \sqrt{53}}{\sqrt{53}} (-6e_2 - 9e_3 + 5e_{12} - 5e_{13} + 6e_{23}) \right) \right).
\]

For comparison, the calculation of the exponential series (7) by Mathematica v.12 in a floating point regime gives six significant figures of the exact solution after summation of 70 series terms (iterations). However, it should be stressed that the convergence is alternating and not monotonic (see Figure 1), and to get the first significant figure for all basis MV elements no less than 64 series terms (iterations) are needed in this particular case. The iteration number may be decreased if the inverse scaling and squaring method\(^{13}\) is applied (see Section 2.1) at a price of computing MV powers.

### 3.2 Vector-bivector entanglement in \(Cl_{0,3}\)

In Equations (9) and (10), the outer product \(a \wedge A\) in general case is a trivector. It entangles or mixes up the components of vector and bivector in the exponential (8). This is easy to see if we equate to zero either \(a\) or \(A\). Then, \(a_+ = a_- = |a|\) if \(A = 0\) and \(a_+ = a_- = |A|\) if \(a = 0\), where \(|a| = (\widetilde{a} \widetilde{a})^{\frac{1}{2}}\) and \(|A| = (\widetilde{A} \widetilde{A})^{\frac{1}{2}}\). The trivector also vanishes if \(a\) and \(A\) are unequal to zero but the vector \(a\) lies in the plane \(A\). In this case,\(^{8}\) the components satisfy the condition \(Ia \wedge A = a_1a_{23} - a_2a_{13} + a_3a_{12} = 0\). Then the entanglement (mixing) coefficients become \(a_+ = a_- \to a_m = \sqrt{|a|^2 + |A|^2}\) and the exponential (8) reduces to

\[
e^A = e^{a_m} e^{\frac{a_m}{2} (a \wedge A) \sin a_m}, \quad a_m |A|.
\]

where \(a_m|A|\) means that the vector \(a_m\) lies in the bivector \(A\) plane. Thus, the exponential \(e^A\) in this case can be factorized. It is interesting that the multiplier in round brackets now represents a disentangled de Moivre-type formula for a sum of vector and bivector, where the magnitude of \((a + A)\) is

\[
a_m = |a + A| = \sqrt{(a + A)(\overline{a} + \overline{A})} = \sqrt{|a|^2 + |A|^2}.
\]

A similar formula can be obtained in opposite case if we assumes that, for example, the vector \(a||e_1\) is perpendicular to bivector \(B||e_{12}\) and \(a_{12}a_{12} \neq 0\); that is, the vector and bivector are characterized by a single scalar term in the entanglement formula.\(^{9}\) Then the expression (8) gives

\[
e^A = -\left( \cos(a_{12} - a_3) + \frac{a_{12}e_{12} + a_3e_3}{a_{12} - a_3} \sin(a_{12} - a_3) \right).
\]

\(^{8}\)Such a situation is encountered in classical electrodynamics where magnetic field bivector and electric field vector of a free wave lie in a the same plane.

\(^{9}\)This approach reminds a popular method in physics where a judicious choice of mutual orientation of the fields and coordinate vectors allows to simplify the problem substantially.
In conclusion, apart from de Moivre-type expressions Equations (12) and (13), the generic GA exponential (8) also contains entangled MVs, which under additional conditions may be disentangled as seen from Equations (14) and (16).

4 | MV EXPONENTIALS IN $Cl_{3,0}$ AND $Cl_{1,2}$ ALGEBRAS

After multiplication of the scalar coefficients given in Dargys and Acus\cite{11} by respective basis elements and collection into a sum, and finally combining the resulting expression into a coordinate-free form, we find the generic exponential of MV $A$,

$$
\exp(A) = e^{\theta_1}(\cos a_{123} + I \sin a_{123}) (\cos a_- \cosh a_+ + I \sin a_- \sinh a_+ \\
+ \frac{1}{a_+^2 + a_-^2} (\cosh a_+ \sin a_- - I \cos a_- \sinh a_+) (a_-(a + A) + a_+ I(a + A))
$$

(17)

where scalar coefficients $a_\pm$ are

$$
a_- = \frac{-2Ia \wedge A}{\sqrt{2}\sqrt{a \cdot a + A \cdot A + \sqrt{(a \cdot a + A \cdot A)^2 - 4(a \wedge A)^2}}},
$$

$$
a_+ = \frac{\sqrt{2} \sqrt{(a \cdot a + A \cdot A)^2 - 4(a \wedge A)^2}}{\sqrt{2},
$$

\begin{align*}
&\begin{cases}
a_+ = \sqrt{2} \sqrt{(a \cdot a + A \cdot A), \quad a \cdot a + A \cdot A > 0} \\
&\begin{cases}
a_+ = 0, \quad a_- = \sqrt{-a \wedge A + A}, \quad a \cdot a + A \cdot A < 0, \quad \text{when } a \wedge A = 0.
\end{cases}
\end{cases}
\end{align*}

Since Equation (17) is in a coordinate-free form, the above formulas are valid for both mutually isomorphic $Cl_{3,0}$ and $Cl_{1,2}$ algebras. If formulas are expanded into coordinates, of course, the resulting expressions will differ by signs at some terms. Note that determinant of a vector and bivector part $a + A$ is $\det(a + A) = (a_+^2 + a_-^2)^2$. When $\det(a + A) = 0$, we have special case which again can be straightforwardly obtained by computing limit of (17), when both $a_+ \rightarrow 0$ and $a_- \rightarrow 0$. Simultaneous vanishing of $a_+$ and $a_-$ means vanishing of both the inner $a \cdot a + A \cdot A$ and outer $a \wedge A$ products. This does not require that vector and bivector should vanish (for example, consider the MV $e_1 + e_{12}$). After computing the limiting value we have $\exp A = e^{\theta_1}(\cos a_{123} + \sin a_{123} I)(1 + a + A)$. The well-known de Moivre- and Euler-type formulas from Equation (17) follow

$$
\exp A = e^{\theta_1}(\cos a_{123} + \sin a_{123} I) \quad \text{if } a = A = 0,
$$

(19)

$$
\exp A = \cos |A| + \frac{A}{|A|} \sin |A| \quad \text{if } a_0 = a_{123} = a = 0.
$$

(20)

$$
\exp a = \cosh |a| + \frac{a}{|a|} \sinh |a| \quad \text{if } a_0 = a_{123} = A = 0.
$$

(21)

Equation (19) represents a special case when $a_+ = a_- = 0$ and $a + A = 0$.

Similarly to $Cl_{0,3}$ algebra (see Section 3.2), in the exponential (17), the vector and bivector may be disentangled if we assume that $a|A|; that is, the vector $a$ lies in the plane $A$. Then, the vector-bivector sum is expressed by trigonometric and hyperbolic functions in both $Cl_{3,0}$ and $Cl_{1,2}$ algebras,

$$
\exp A = \begin{cases}
\cos \sqrt{|A|^2 - |a|^2} + \frac{a + A}{\sqrt{|A|^2 - |a|^2}} \sin \sqrt{|A|^2 - |a|^2} & \text{if } a^2 < A^2, \\
\cosh \sqrt{|a|^2 - |A|^2} + \frac{a + A}{\sqrt{|a|^2 - |A|^2}} \sinh \sqrt{|a|^2 - |A|^2} & \text{if } a^2 > A^2.
\end{cases}
$$

(22)

Example 2. Exponential of MV in $Cl_{3,0}$. Let us take the same MV $A = -8 - 6e_2 - 9e_3 + 5e_{12} - 5e_{13} + 6e_{23} - 4e_{123}$ as in Example 1 and calculate the exponential using the coordinate-free expression (17). We find $a \cdot a + A \cdot A =$
31, \(-2Ia \land A = -150\). Then \(a_- = -75\sqrt{\frac{2}{31+\sqrt{23461}}}\) and \(a_+ = \sqrt{\frac{31+\sqrt{23461}}{2}}\). Finally, the exact numerical answer is

\[
\exp(A) = \frac{1}{\sqrt{e^2}} \left( \cos\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) \cosh\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) - \sin\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) \sinh\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) \right)
\]

\[
\times \left( -75\sqrt{\frac{2}{31+\sqrt{23461}}} e_2 - 9e_3 + 5e_{12} + 5e_{23} + \sqrt{\frac{31+\sqrt{23461}}{2}} (6e_2 - 9e_3 + 5e_{12} - 5e_{23}) i \right)
\]

\[
\exp(A) = \frac{1}{\sqrt{e^2}} \left( \cos\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) \cosh\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) - \sin\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) \sinh\left(75\sqrt{\frac{2}{31+\sqrt{23461}}}\right) \right)
\]

\[
\times \left( -75\sqrt{\frac{2}{31+\sqrt{23461}}} e_2 - 9e_3 + 5e_{12} + 5e_{23} + \sqrt{\frac{31+\sqrt{23461}}{2}} (6e_2 - 9e_3 + 5e_{12} - 5e_{23}) i \right)
\]

Example 3. Exponential in Cl_{1,2} with disentanglement included. Let us take a simple MV, \(A = 3 - e_1 + 2e_{12}\), which represents disentangled case because \(a \land A = 0\). Then we have \(a_+ = \sqrt{5}\) and \(a_- = 0\). The answer is expressed in hyperbolic functions: \(\exp(A) = e^I (\cosh\sqrt{5} + (-e_1 + 2e_{12}) \frac{\sinh\sqrt{5}}{\sqrt{5}})\).

5 MV EXPONENTIAL IN Cl_{2,1}

After assembling coefficients given in Dargys and Acus\(^{11}\) into MV and then regrouping them to coordinate-free form, we find the following exponential formula in Cl_{2,1}

\[
\exp(A) = \frac{1}{2} e^{a_0} \left( e^{a_{12}} (1 + I) \left( \cos(a_+^{(2)} + si(a_+^{(2)})(a + A)) + e^{-a_{12}} (1 - I) \left( \cos(a_-^{(2)} + si(a_-^{(2)})(a + A)) \right) \right) \right),
\]

where the scalar coefficients \(a_+^{(2)}\) are

\[
a_+^{(2)} = - (a \cdot a + A \cdot A) - 2Ia \land A = (a_1 - a_{23})^2 + (a_2 + a_{13})^2 - (a_3 + a_{12})^2; \quad a_+^{(2)} \leq 0,
\]

\[
a_-^{(2)} = - (a \cdot a + A \cdot A) + 2Ia \land A = (a_1 + a_{23})^2 + (a_2 - a_{13})^2 - (a_3 - a_{12})^2; \quad a_-^{(2)} \leq 0.
\]

Contrary to \(Cl_{0,3}\) algebra case, where \(a_+^{(2)} \geq 0\) denotes a square, in the \(Cl_{2,1}\) we have to interpret superscript 2 in brackets in Equations (24)-(25) as index, which helps to discriminate two closely related quantities \(a_\pm\) and \(a_+^{(2)}\). As indicated by symbol \(\propto\), the condition in Equation (25) for \(a_+^{(2)}\) can acquire negative values and, therefore, cannot be understood as square of real number. The superscript notation in brackets, however, helps to grasp visual similarity between \(Cl_{0,3}\) (8) and \(Cl_{2,1}\) (24) formulas.

To simplify notation in Equation (24), we have introduced \(si\) and \(co\) functions that depending on sign under square root go over to either trigonometric or hyperbolic functions. All in all, this gives four cases for both \(si\) and \(co\) functions,

\[
\begin{align*}
si(a_+^{(2)}) = & \begin{cases} 
\frac{\sin \sqrt{a_+^{(2)}}}{\sqrt{a_+^{(2)}}}, & a_+^{(2)} > 0 \\
1, & a_+^{(2)} = 0; \\
\frac{\sin \sqrt{-a_+^{(2)}}}{\sqrt{-a_+^{(2)}}}, & a_+^{(2)} < 0
\end{cases} \\
co(a_+^{(2)}) = & \begin{cases} 
\cos \sqrt{a_+^{(2)}}, & a_+^{(2)} > 0 \\
1, & a_+^{(2)} = 0; \\
\cosh \sqrt{-a_+^{(2)}}, & a_+^{(2)} < 0
\end{cases}
\end{align*}
\]

(26)
\[
\sin(a_{(2)}^{(2)}) = \begin{cases} 
\frac{\sin \sqrt{-a^{(2)}/2}}{\sqrt{-a^{(2)}}}, & a_{(2)}^{(2)} > 0 \\
1, & a_{(2)}^{(2)} = 0 \\
\frac{\sinh \sqrt{-a^{(2)}}}{\sqrt{-a^{(2)}}}, & a_{(2)}^{(2)} < 0
\end{cases}
\] 
\[\cos(a_{(2)}^{(2)}) = \begin{cases} 
\cos \sqrt{a_{(2)}^{(2)}}, & a_{(2)}^{(2)} > 0 \\
1, & a_{(2)}^{(2)} = 0 \\
\cosh \sqrt{-a_{(2)}^{(2)}}, & a_{(2)}^{(2)} < 0
\end{cases} \tag{27}\]

5.1 | Special cases

If both the vector \(\mathbf{a}\) and the bivector \(\mathbf{A}\) are equal to zero the exponential (24) simplifies to \(\exp \mathbf{A} = \exp(a_0 + Ia_{123}) = e^{a_0}(\cosh a_{123} + I \sinh a_{123})\).

The exponential of vector, when \(a_0 = a_{123} = \mathbf{A} = 0\), is

\[
\exp(\mathbf{a}) = \begin{cases} 
\cos \sqrt{-a^{2}} + \frac{a}{\sqrt{-a^{2}}} \sin \sqrt{-a^{2}} & \text{if } a^{2} < 0, \\
\cosh \sqrt{a^{2}} + \frac{a}{\sqrt{a^{2}}} \sinh \sqrt{a^{2}} & \text{if } a^{2} > 0.
\end{cases} \tag{28}\]

The exponential of bivector, when \(a_0 = a_{123} = \mathbf{a} = 0\), is

\[
\exp(\mathbf{A}) = \begin{cases} 
\cos \sqrt{-A^{2}} + \frac{\mathbf{A}}{\sqrt{-A^{2}}} \sin \sqrt{-A^{2}} & \text{if } A^{2} < 0, \\
\cosh \sqrt{A^{2}} + \frac{\mathbf{A}}{\sqrt{A^{2}}} \sinh \sqrt{A^{2}} & \text{if } A^{2} > 0.
\end{cases} \tag{29}\]

If \(a_0\) and \(a_{123}\) are not equal to zero, then \(\exp(\mathbf{a})\) and \(\exp(\mathbf{A})\) should be multiplied by \(e^{a_0}(\cosh a_{123} + I \sinh a_{123})\).

**Example 4.** Exponential of MV in \(Cl_{2,1}\). Case \(a_{(2)}^{(2)} < 0\), \(a_{+}^{(2)} > 0\).

Using the same MV \(\mathbf{A} = -8 - 6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23} - 4\mathbf{e}_{123}\) for \(Cl_{2,1}\) now we have \(a_{(2)}^{(2)} = -141\), \(a_{+}^{(2)} = 159\). The answer then is

\[
\exp(\mathbf{A}) = \frac{1}{2e^8} \left( \frac{1}{e^4} (1 + I) \left( \frac{\sin \sqrt{159}}{\sqrt{159}} (-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23}) + \cos \sqrt{159} \right) \\
+ e^4 (1 - I) \left( \frac{\sinh \sqrt{141}}{\sqrt{141}} (-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23}) + \cosh \sqrt{141} \right) \right).
\]

**Example 5.** Exponential in \(Cl_{2,1}\). Case \(a_{(2)}^{(2)} < 0\), \(a_{+}^{(2)} < 0\). Multivector \(\mathbf{A} = -6\mathbf{e}_2 + 5\mathbf{e}_{12} + \mathbf{e}_{123}\) of \(Cl_{2,1}\). \(a_{(2)}^{(2)} = -11\), \(a_{+}^{(2)} = -11\). The exponential is

\[
\exp(\mathbf{A}) = \frac{1}{2} \left( (e(1 + I) + e^{-1}(1 - I)) \left( \frac{\sin \sqrt{11}}{\sqrt{11}} (-6\mathbf{e}_2 + 5\mathbf{e}_{12}) + \cosh \sqrt{11} \right) \right).
\]

**Example 6.** Exponential in \(Cl_{2,1}\). Case \(a_{(2)}^{(2)} > 0\), \(a_{+}^{(2)} > 0\). Exponential of \(\mathbf{A} = 2 + \mathbf{e}_3 + 6\mathbf{e}_{12} + 3\mathbf{e}_{123}\) of \(Cl_{2,1}\). We have \(a_{(2)}^{(2)} = 49\), \(a_{+}^{(2)} = 25\). The answer then is

\[
\exp(\mathbf{A}) = \frac{e^2}{2} \left( e^3 (1 + I) \left( \frac{\sin 5}{5} (\mathbf{e}_3 + 6\mathbf{e}_{12}) + \cos 5 \right) + e^{-3} (1 - I) \left( \frac{\sin 7}{7} (\mathbf{e}_3 + 6\mathbf{e}_{12}) + \cos 7 \right) \right).
\]
Example 7. Exponential in \( Cl_{2,1} \). Case \( a_{(+)}^{(2)} > 0, a_{(+)}^{(2)} < 0 \). \( A = 2 - 10e_2 - 10e_3 + 2e_{13} + e_{23}, a_{(+)}^{(2)} = 35, a_{(+)}^{(2)} = -45. \) The answer is

\[
\exp(A) = \frac{e^2}{2} \left( e(1 + I) \left( \frac{\sinh \sqrt{5} I}{\sqrt{5}} (-10e_2 - 10e_3 + 2e_{13} + e_{23}) + \cosh \left( \frac{3\sqrt{5}}{5} \right) \right) \right. \\
+ e^{-1}(1 - I) \left( \frac{\sin \sqrt{35}}{\sqrt{35}} (-10e_2 - 10e_3 + 2e_{13} + e_{23}) + \cos \left( \frac{3\sqrt{5}}{5} \right) \right) \right).
\]

6 | EXAMPLES OF APPLICATION: SOLUTION OF GA DIFFERENTIAL EQUATIONS

The exponential function plays an important role in solution of linear differential equations.\(^{19}\) For example, the solution of a homogeneous equation

\[
\frac{dX}{dt} = AX, \ X = X_0 \ at \ t = 0, \quad (30)
\]

with respect to MV \( X \) gives GA exponential function \( X(t) = e^{tA}X_0 \), where \( t \) is the parameter, for instance, the time. Treating \( tA \) as a new MV, after expansion of \( e^{tA} \) we will get the evolution of \( X \) in time. More generally, with suitable assumptions upon smoothness of \( e^{tA} \), the solution of the inhomogeneous system

\[
\frac{dX}{dt} = AX + f(t), \ X = X_0 \ at \ t = 0, \quad (31)
\]

may be expressed by

\[
X(t) = e^{tA}X_0 + \int_0^t e^{(t-s)A} f(s)ds. \quad (32)
\]

Some of MV differential equations, for example,

\[
\frac{dX}{dt} = AX \pm XB, \ X = X_0 \ at \ t = 0, \quad (33)
\]

have solution that consists of a product of GA exponentials

\[
X(t) = e^{tA}X_0 e^{\pm tB}. \quad (34)
\]

The answer can be easily checked by direct substitution of \( X(t) \) into Equation (33) and application of Leibniz's differentiation theorem.\(^{17}\) If \( B = A \) and the sign is negative we will get the rotor equation.

Trigonometric GA functions, as well as GA roots, arise in the solution of second-order differential equations. For example, the GA equation\(^{13}\)

\[
\frac{d^2X}{dt^2} + AX = 0, \ at \ t = 0, X = X_0 \ and \ (dX/dt)_{t=0} = X'_0, \quad (35)
\]

has the solution

\[
X(t) = \cos \left( \sqrt{A} t \right) X_0 + \left( \sqrt{A} \right)^{-1} \sin \left( \sqrt{A} t \right) X'_0, \quad (36)
\]

where \( \sqrt{A} \) is the square root of \( A \). The trigonometric functions of MV argument can be expressed by exponentials.\(^{4,11}\) Closed form expression for square root of MV when \( p + q = 3 \) are presented in Acus and Dargys.\(^{20}\) A concrete example of application of GA exponentials in physics can be found in paper.\(^{11}\)
Example 8. Solution of differential Equation (35) in Cl_{3,0}.algebra. Let us explicitly compute and check the solution (36) in case of a concrete algebra. The presence of trigonometric functions means that we are restricted to algebras Cl_{3,0} and Cl_{1,2} only. Let \( A = -1 + e_3 - e_{12} + \frac{e_{12}}{2} \) be a MV in Cl_{3,0}. Then using square root formulas from literature\(^{20,21}\) we obtain four roots \( \sqrt{A} = \pm \left\{ \frac{1}{2} - e_{12} + \frac{e_{12}}{2}, \frac{1}{2} + e_{12} - e_{123} \right\}. \) To save space, we will use only two solutions with plus sign, since solutions with minus sign can be easily obtained by proper sign changes. We have \( (\sqrt{A})^{-1} = \left\{ \frac{1}{5}(3 + 2e_3 + 4e_{12} + e_{123}), \frac{1}{5}(2 + 3e_3 + e_{12} + 4e_{123}) \right\}. \) Then, for \( t \geq 0 \)

\[
\exp \left( \sqrt{A} t \right) = \left\{ e^{t/2} \left( \cos \left( \frac{t}{2} \right) (\cos(t) - e_{12} \sin(t)) + e_3 \sin \left( \frac{t}{2} \right) \sin(t) + e_{123} \sin \left( \frac{t}{2} \right) \cos(t) \right) \right\},
\]

\[
\sin \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) + \cos \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) + e_3 \sinh \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) + e_{12} \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) + e_{123} \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) \right\}.
\]

Inverse of exponent is obtained by changing sign of \( t. \) Next, using \( \sin A = \frac{I}{2}(e^{-iA} - e^{iA}) \) and \( \cos A = \frac{1}{2}(e^{-iA} + e^{iA}) \) we find trigonometric functions

\[
\cos \left( \sqrt{A} t \right) = \left\{ \sin \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} \right) (e_{12}(\cosh(t) + 1) - e_{123} \cosh(t)) - \cos \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (e_3(\cosh(t) - 1) - \cosh(t)), \right\}
\]

\[
\sin \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) (e_{12}(\cosh(t) + 1) - e_{123} \cosh(t)) - \cos \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (e_3(\cosh(t) - 1) - \cosh(t)) \right\}
\]

and

\[
\sin \left( \sqrt{A} t \right) = \left\{ \sin \left( \frac{t}{2} \right) ( - \cosh \left( \frac{t}{2} \right) \right) (e_3(\cosh(t) - 1) - \cosh(t)) - \cos \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) (e_{12}(\cosh(t) + 1) - e_{123} \cosh(t)), \right\}
\]

\[
\cos \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) (e_{12} \cosh(t) - e_{123}(\cosh(t) + 1)) + \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) ((e_3 - 1) \cosh(t) + 1) \right\}.
\]

Note, that both entries for \( \cos \left( \sqrt{A} t \right) \) are the same, whereas for \( \sin \left( \sqrt{A} t \right) \) they differ. However the product \( (\sqrt{A})^{-1} \sin \left( \sqrt{A} t \right) \), similarly as for cosine, now gives the same answer for both entries,

\[
(\sqrt{A})^{-1} \sin \left( \sqrt{A} t \right) = \left\{ \frac{1}{5} \left( \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (\cosh(t) + 2) + \cos \left( \frac{t}{2} \right) (3 \cosh(t) + 4) \right) \right\}
\]

\[
+ \frac{1}{5} e_3 \left( \cos \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} \right) (1 - 3 \cosh(t)) - \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (\cosh(t) - 3) \right) \\
+ \frac{1}{5} e_{12} \left( \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (3 \cosh(t) + 1) - \cos \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} \right) (\cosh(t) + 3) \right) \\
+ \frac{1}{5} e_{123} \left( \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (4 - 3 \cosh(t)) + \cos \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} \right) (\cosh(t) - 2) \right),
\]

same entry as above).
To save the space we will take simple boundary conditions $X_0 = 1$ and $X'_0 = 2e_1$. After substitution of all intermediate results into (36) finally we arrive at coordinate form solution

$$X(t) = \frac{1}{5} \left( 2 \sin \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (e_1(cosh(t) + 2) + e_{13}(cosh(t) - 3) - e_2(3 \cosh(t) + 1) + e_{23}(4 - 3 \cosh(t))) + 2 \cos \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) (e_1(3 \cosh(t) + 4) + e_{13}(3 \cosh(t) - 1) + e_2(\cosh(t) + 3) + e_{23}(\cosh(t) - 2)) \right) \quad (45)$$

$$+ \sin \left( \frac{t}{2} \right) \sinh \left( \frac{t}{2} \right) (e_{12}(\cosh(t) + 1) - e_{133} \cosh(t)) - \cos \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) (e_3(3 \cosh(t) - 1) - \cosh(t)) \right).$$

Substitution of the solution (45) back to differential Equation (35) confirms that the latter is satisfied indeed. It is noteworthy that all four roots of $\sqrt{A}$ do satisfy Equation (35) which is a linear differential equation with respect to MV $X(t)$.

7 | DISCUSSION AND CONCLUSIONS

Explicit formulas of exponents in all algebras initially were derived in a coordinate form by means of pure symbolic manipulations and without any assumptions about decomposition of MV into mutually commuting parts. Coordinate expressions then were rewritten into coordinate free form.

A different approach, which allows to derive the presented formulas may be applied at least to $Cl_{0,3}$ and $Cl_{2,1}$ algebras. Indeed, both mentioned algebras have idempotent $T = \frac{1}{2}(1 \pm I)$ which belong to the center of both algebras, i.e. it commutes with all elements and has property $T^2 = T$. We can use it to construct an ideal basis \( \left\{ \frac{1}{2}(1 + I)(a + A), \frac{1}{2}(1 - I)(a + A) \right\} \), elements of which are commutative and their powers yield simple expressions. Summation of exponential series of each of basis elements then becomes almost trivial. Let us demonstrate how this approach (derivation) works for $Cl_{0,3}$ algebra.

For $Cl_{0,3}$, let us write $A_- + A_+ \equiv a + A = \frac{1}{2}(1 + I)(a + A) + \frac{1}{2}(1 - I)(a + A)$, where $A_- = \frac{1}{2}(1 + I)(a + A)$ and $A_+ = \frac{1}{2}(1 - I)(a + A)$. A Clifford product $A_\pm \wedge A_\pm$ is $\frac{1}{2} \langle 2A_\pm A_\pm \rangle_0 (1 \mp I) = \frac{1}{2} \left| \sqrt{2}A_\pm \right|^2 (1 \mp I)$, where $\left| \sqrt{2}A_\pm \right|^2 = \sqrt{(2A_\pm A_\pm)_0}$. Computing $A_\pm$ powers we have $A_\pm^2 = \frac{1}{2} \left| \sqrt{2}A_\pm \right|^2 (1 \mp I)$, $A_\pm^3 = \frac{1}{2} \left| \sqrt{2}A_\pm \right|^2 (a + A)$, $A_\pm^4 = -\frac{1}{2} \left| \sqrt{2}A_\pm \right|^4 (1 \mp I)$, $A_\pm^5 = -\frac{1}{2} \left| 2A_\pm \right|^4 (a + A)$, $A_\pm^6 = \frac{1}{2} \left| \sqrt{2}A_\pm \right|^6 (1 \mp I)$, etc. Summing separately odd and even terms of exponent series we obtain the expression

$$\exp(A_\pm) = \frac{1}{2}(1 \pm I) + \frac{1}{2} \cos \left( \left| \sqrt{2}A_\pm \right| \right) \sinh \left( \left| \sqrt{2}A_\pm \right| \right) A_\pm. \quad (46)$$

The exponential $\exp(a + A)$ then may be written as a geometric product $\exp(A_+) \exp(A_-)$, from which vector+bivector part of (8) follows. Exponent of center is obtained trivially. Similar procedure may be applied to $Cl_{2,1}$ to obtain (24), however, it fails for $Cl_{3,0} \cong Cl_{1,3}$. We think that failure in these algebras is caused by absence of primitive idempotent, which needs to commute with other elements of the algebra. For example, idempotent $\frac{1}{2}(1 + e_1)$ of $Cl_{3,0}$ is not suitable for decomposition of $a + A$ into commuting parts since $e_1$ does not commute generally with $a + A$. It is also worth noting that the product $A_\pm \wedge A_\pm$ (square of norm) is quadratic in MV coefficients, whereas $Cl_{3,0}$ algebra (see Equation (18)) requires quartic polynomial. Originally the $a_\pm$ expressions for $Cl_{3,0}$ were obtained by noting that the determinant of $(a + A)$ can be decomposed as $\det(a + A) = (a_+^2 + a_-^2)^2 = 0$, whereas for algebras $Cl_{0,3}$ and $Cl_{2,1}$ the determinant factorizes as $\det(a + A) = a_+^2 a_-^2$.

More generally the above described decomposition may considerably simplify computation of exponentials if one can find an idempotent that commutes with the exponent. For example, it may be applied to even algebra $Cl_{2,1}^+$ bivector $B_R = \frac{1}{2}(e_{12} + e_{14} - e_{23} - e_{34})$. Since $\frac{1}{2}(1 \pm I)$ is an idempotent of $Cl_{2,1}$, which commutes with even elements, we obtain decomposition into commuting bivectors $B_R = B_{R_+} + B_{R_-}$, where $B_{R_+} = \frac{1}{2}(e_{14} - e_{23})$ and $B_{R_-} = \frac{1}{2}(e_{12} - e_{34})$. Powers of

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1This explanation was suggested by anonymous referee to whom we are thankful. Note the opposite index sign convention. The quantity $A_\pm$ still represents sum of vector and bivector.
the decomposed bivectors $B_R$ again yield very simple terms in a series that can be summed up. Then the exponential of $B_R$ can be written in terms of hyperbolic functions, whereas the exponential of $B_R$ yields usual trigonometric functions.

The main results of this paper are the formulas (8), (17), and (24), where real GA exponentials are presented in an expanded coordinate-free form. Since in 3D algebras the scalar and pseudoscalar belong to GA center, the related coefficients $a_0$ and $a_{123}$ appear in scalar exponentials only. In all algebras the entanglement (mixing) of vector and bivector components takes place. The mixing is characterized by scalar coefficients $a_+$ and $a_-$, where the terms of the form $(a_i - a_j)2$, $i \neq j \neq k$ appear. The entanglement can be eliminated by equating to zero either vector or bivector, as a result the well-known trigonometric and hyperbolic de Moivre-type formulas for vector and bivector exponentials are recovered. However, more interesting case is the disentanglement where the vector is parallel to bivector plane. At this orientation we obtain the exponential which consists of a sum of scalar, vector and bivector, Equations (14) and (22). Finally, we shall note that for $n = p + q = 3$ algebras the characteristic polynomial of matrix representation is of degree 4. Since the algebraic equations of degree 4 are solvable in radicals, our results, Equations (8), (17), and (24) are consistent with this theorem. Moreover, since characteristic polynomial for $n = 4$ algebras is also of degree four one may speculate that there may exist a single signature independent exponential formula for both $n = 3$ and $n = 4$ algebras in terms of characteristic polynomials. This is indeed the case and we were able to find such a formula in an explicit form.

The exponentials are closely related to logarithms. Our attempts to find real GA logarithm from general GA exponentials revealed that the GA logarithm problem is more difficult, albeit symbolically tractable. Similarly to the complex logarithm ($C_{0,1}$ algebra), the 3D logarithm is not single-valued and therefore one must proceed with caution. In addition, it appears that to GA logarithm one may add a free MV that vanishes after exponentiation. Knowledge of explicit forms of exponentials, logarithms and square roots opens a way to compute exact symbolic expressions for trigonometric/hyperbolic functions and their inverses and to solve GA linear differential equations of single MV at least for 3D GAs as shown in Example 8.

**CONFLICT OF INTEREST**

The authors declare no potential conflict of interests.

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