THE PERIOD MAP FOR QUANTUM COHOMOLOGY OF $\mathbb{P}^2$

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ABSTRACT. We invert the period map defined by the second structure connection of quantum cohomology of $\mathbb{P}^2$. For small quantum cohomology the inverse is given explicitly in terms of the Eisenstein series $E_4$ and $E_6$, while for big quantum cohomology the inverse is determined perturbatively as a Taylor series expansion whose coefficients are quasi-modular forms.

1. INTRODUCTION

The results of this paper are in the settings of quantum cohomology of $\mathbb{P}^2$. Nevertheless, the problems that we solve can be given in much more general settings. Let us start by giving the general picture and providing some background and motivation for our results.

1.1. The second structure connection. We assume that the reader is familiar with the definition of a semi-simple Frobenius manifold (see [2] for some background). Let $M$ be a complex semi-simple Frobenius manifold and let $\mathcal{T}_M$ be the sheaf of holomorphic vector fields on $M$. By definition the data of Frobenius structure is given by the following list of objects

(1) A non-degenerate symmetric bi-linear pairing $(\ , \ )$ on $\mathcal{T}_M$.
(2) A commutative associative multiplication $\cdot : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M$.
(3) A flat vector field $1 \in \Gamma(M, \mathcal{T}_M)$ that is a unity, i.e., $1 \cdot v = v$ for all $v \in \mathcal{T}_M$.
(4) An Euler vector field $E \in \Gamma(M, \mathcal{T}_M)$.

Let us fix a base point $t^0 \in M$ and a basis of $\{\phi_i\}_{i=1}^N$ of the reference tangent space $H := T_{t^0}M$. Furthermore, let $(t_1, \ldots, t_N)$ be a flat coordinate system on $M$ such that $\partial/\partial t_i = \phi_i$ in $H$. Using the flat vector fields $\partial/\partial t_i$ we trivialize the tangent bundle $TM \cong M \times H$. This allows us to identify the Frobenius multiplication $\bullet$ with a family of associative commutative multiplications $\cdot_t : H \otimes H \to H$ depending analytically on $t \in M$. The operator

$$\text{ad}_E : \mathcal{T}_M \to \mathcal{T}_M, \quad v \mapsto [E, v]$$

preserves the space of flat vector fields. We will be interested in the class of semi-simple Frobenius manifolds for which this operator is diagonalizable and the eigenvalues are rational numbers. In particular, we may assume that our choices
of \{\phi_i\}_{i=1}^N and \{t_i\}_{i=1}^N are such that
\[ E = \sum_{i=1}^N ((1 - d_i)t_i + r_i)\partial/\partial t_i, \]
where \( \partial/\partial t_1 \) coincides with the unit vector field 1 and the numbers
\[ 0 = d_1 \leq d_2 \leq \cdots \leq d_N =: D \]
are symmetric with respect to the middle of the interval \([0, D]\). The number \( D \) is known as the conformal dimension of \( M \). The operator
\[ \theta : T_M \to T_M, \ v \mapsto [E,v] - \frac{1}{2}(2 - D)v \]
preserves the subspace of flat vector fields. It induces a linear operator on \( H \) which is known to be skew symmetric with respect to the Frobenius pairing ( , ). Following Givental, we refer to \( \theta \) as the Hodge grading operator.

There are two flat connections that one can associate with the Frobenius structure. The first one is usually called Dubrovin's connection. It is a connection on the \( H \)-trivial bundle on \( M \times \mathbb{C}^* \) defined by
\[
\nabla_{\partial/\partial t_i} = \frac{\partial}{\partial t_i} - z^{-1}\phi_i \bullet \\
\nabla_{\partial/\partial z} = \frac{\partial}{\partial z} + z^{-1}\theta - z^{-2}E \bullet
\]
where \( z \) is the standard coordinate on \( \mathbb{C}^* = \mathbb{C} - \{0\} \) and for \( v \in \Gamma(M, T_M) \) we denote by \( v \bullet : H \to H \) the linear operator of Frobenius multiplication by \( v \).

Our main interest is in the 2nd structure connection
\[
\nabla^{(n)}_{\partial/\partial t_i} = \partial_i + (\lambda - E \bullet_l)^{-1}(\phi_i \bullet_l)(\theta - n - 1/2) \\
\nabla^{(n)}_{\partial/\partial \lambda} = \partial_\lambda - (\lambda - E \bullet_l)^{-1}(\theta - n - 1/2),
\]
where \( n \in \mathbb{C} \) is a complex parameter. This is a connection on the trivial bundle
\[(M \times \mathbb{C})' \times H \to (M \times \mathbb{C})',\]
where
\[(M \times \mathbb{C})' = \{(t, \lambda) \mid \det(\lambda - E \bullet_l) \neq 0\}.\]
The hypersurface \( \det(\lambda - E \bullet_l) = 0 \) in \( M \times \mathbb{C} \) is called the discriminant.

1.2. Period vectors. The definition of the period map depends on the choice of a calibration of \( M \). By definition (see \[5\]), the calibration is a fundamental solution for the Dubrovin's connection near \( z = \infty \) that has the form
\[ S(t, z)z^\rho z^{-\rho}, \]
where \( \rho \in \text{End}(H) \) is a nilpotent operator and \( S = 1 + \sum_{k=1}^\infty S_k(t)z^{-k} \), \( S_k \in \text{End}(H) \) is an operator series satisfying the symplectic condition
\[ S(t, z)S(t, -z)^T = 1, \]
where $T$ denotes transposition with respect to the Frobenius pairing.

Let us fix a reference point $(t^0, \lambda^0) \in (M \times \mathbb{C})'$ such that $\lambda^0$ is a sufficiently large real number. It is easy to check that the following functions provide a fundamental solution to the 2nd structure connection

$$I^{(n)}(t, \lambda) = \sum_{k=0}^{\infty} (-1)^k S_k(t) \tilde{I}^{(n+k)}(\lambda),$$

where

$$\tilde{I}^{(m)}(\lambda) = e^{-\rho \partial_\lambda \partial_m} \left( \frac{\lambda^{\theta-m-\frac{1}{2}}}{\Gamma(\theta - m + \frac{1}{2})} \right).$$

The 2nd structure connection has a Fuchsian singularity at infinity, therefore the series $I^{(n)}(t, \lambda)$ is convergent for all $(t, \lambda)$ sufficiently close to $(t^0, \lambda^0)$. Using the differential equations we extend $I^{(n)}$ to a multi-valued analytic function on $(M \times \mathbb{C})'$. We define the following multi-valued functions taking values in $H$:

$$I^{(n)}_a(t, \lambda) := I^{(n)}(t, \lambda) a, \quad a \in H, \quad n \in \mathbb{Z}.$$ 

These functions will be called period vectors. Using analytic continuation we get a representation

$$\pi_1((B \times \mathbb{C})', (t^0, \lambda^0)) \to \text{GL}(H)$$

called the monodromy representation of the Frobenius manifold. The image $W$ of the monodromy representation is called the monodromy group.

Using the differential equations of the 2nd structure connection it is easy to prove that the pairing

$$(a|b) := (I^{(0)}_a(t, \lambda), (\lambda - E \bullet)I^{(0)}_b(t, \lambda))$$

is independent of $t$ and $\lambda$. This pairing is known as the intersection pairing. The monodromy group $W$ is generated by the monodromy transformations representing $\pi_1(M) \subset \pi_1((M \times \mathbb{C})')$ and a set of reflections

$$w_a(x) = x - (a|x)a, \quad a \in R,$$

where the set $R$ is defined as follows. Under the semi-simplicity assumption, we may choose a generic reference point, such that the Frobenius multiplication $\bullet_{\lambda^0}$ is semi-simple and the operator $E \bullet_{\lambda^0}$ has $N$ pairwise different eigenvalues $u^0_i$. Let $R$ be the set of all $a \in H$ such that $(a|a) = 2$ and there exists a simple loop in $\mathbb{C} - \{u^0_1, \cdots, u^0_N\}$ based at $\lambda^0$ such that monodromy transformation along it transforms $a$ into $-a$. Here simple loop means a loop that starts at $\lambda^0$, approaches one of the punctures $u^0_i$ along a path $\gamma$ that ends at a point sufficiently close to $u^0_i$, goes around $u^0_i$, and finally returns back to $\lambda^0$ along $\gamma$. 
1.3. **The ring of modular functions.** Our main interest is in the period map

\[ Z : ((M \times \mathbb{C})')^\sim \to H^*, \quad (t, \lambda) \mapsto Z(t, \lambda) \]

where \(((M \times \mathbb{C})')^\sim\) is the universal cover of \((M \times \mathbb{C})'\) and \(Z(t, \lambda) \in H^*\) is defined by

\[ \langle Z(t, \lambda), \alpha \rangle := Z_\alpha(t, \lambda) = (I^{-1}_\alpha)(t, \lambda), 1). \]

The flow of the unit vector field \(1\) defines a free action of \(\mathbb{C}\) on \(M\)

\[ \mathbb{C} \times M \to M, \quad (x, t) \mapsto t + x1. \]

Let us identify the orbit space \(B := M/\mathbb{C}\) with the submanifold \(\{t_1 = 0\} \subset M\). Then we have an isomorphism

\[ \mathbb{C} \times B \cong M, \quad (x, t) \mapsto t + x1. \]

The period map has the following translation symmetry

\[ (2) \quad Z(t, \lambda) = Z(t - \lambda1, 0). \]

Therefore, we will restrict our analysis to the case \(t_1 = 0\), i.e., we will assume that \(t \in B\) and that the period map is defined on the universal cover of

\[ X := (B \times \mathbb{C})' = \{(t, \lambda) \in B \times \mathbb{C} \mid \det(\lambda - E\bullet) \neq 0\}. \]

Let us denote by \(\Omega \subset H^*\) the image of the period map \(Z\). This is a \(W\)-invariant subset which will be called the period domain. In general very little is known about such period domains. For example it would be interesting to classify semi-simple Frobenius manifolds such that the action of \(W\) on \(\Omega\) is properly discontinuous and the quotient \([\Omega/W]\) is an orbifold whose coarse moduli space is isomorphic to the Frobenius manifold \(M\). Furthermore, we would like to introduce the ring of modular functions

\[ M(\Omega, W) := \{ f \in \Gamma(\Omega, \mathcal{O}_{H^*})^W \mid f \circ Z \in \mathcal{O}(B \times \mathbb{C}) \}, \]

where \(\Gamma(\Omega, \mathcal{O}_{H^*})^W\) is the ring of \(W\)-invariant holomorphic functions in \(\Omega\). Note that in general if \(f \in \Gamma(\Omega, \mathcal{O}_{H^*})^W\) is an arbitrary function, then the composition \(f \circ Z\) defines a holomorphic function on \((B \times \mathbb{C})'\). The condition in the above definition requires that \(f \circ Z\) extends analytically across the discriminant.

Our interest in the space of modular functions comes from our construction of a twisted vertex algebra representation [1]. The modular functions govern the genus-0 reduction of our \(W\)-constraints. In particular, every modular function determines a recursive relation for the genus-0 total descendant potential.

**Definition 1.1.** The period map \(Z\) is said to be invertible if there exists a set of modular functions \(f_i \in M(\Omega, W)\) \((1 \leq i \leq N)\) such that the set of holomorphic functions \(f_i \circ Z\) \((1 \leq i \leq N)\) is a coordinate system on \(B \times \mathbb{C}\). A set of such modular functions \(\{f_i\}_{i=1}^N\) is called the inverse of the period map.
If the period map is invertible then the corresponding modular functions $f_i$ will give a complete set of recursion relations, which would allow us to determine the genus-0 total descendant potential in terms of the monodromy data of the Frobenius manifold via an explicit recursion. We believe that finding an inverse for the period map of the second structure connection is a very important problem and its applications to integrable systems and representations of infinite dimensional Lie algebras are yet to be discovered.

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2. Quantum cohomology of $\mathbb{P}^2$

From now on we will work only in the settings of quantum cohomology of $\mathbb{P}^2$. The goal of this section is to introduce the necessary notation and to state our results.

2.1. Frobenius manifold structure. Let $H = H^*(\mathbb{P}^2; \mathbb{C})$ and $t_i$ ($1 \leq i \leq 3$) be the linear coordinates on $H$ corresponding to the basis

$$\phi_i := p^{i-1}, \quad i = 1, 2, 3,$$

where $p = c_1(O(1))$ is the hyperplane class. Quantum cohomology defines a Frobenius manifold structure on the space

$$M = \{ (t_1, Q, t_3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \mid |t_3 Q^{1/3}| < \epsilon \},$$

where $\epsilon$ is a sufficiently small positive real number and the coordinate $Q := e^{t_2}$ is identified with the Novikov variable. By definition, the linear coordinates $t_i$ are flat, the Frobenius pairing is given by the Poincare pairing

$$(\partial_i, \partial_j) = \delta_{i+j,4}, \quad 1 \leq i, j \leq 3,$$

where $\partial_i = \partial/\partial t_i$, while the multiplication is given by the quantum cup product. The latter is defined by

$$(\partial_i \bullet \partial_j, \partial_k) := \frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k},$$

where $F(t)$ is the genus-0 potential

$$F(t) = \sum_{l,d=0}^{\infty} \frac{Q^d}{l!} (t, \ldots, t)_{0, l, d},$$

where $t := t_1 + t_3 p^2$. Following Kontsevich and Ruan–Tian we can derive an explicit recursive formula for $F$ as follows. Using the string equation and the
dimension formula of the virtual fundamental cycle we get that \( F \) has the form
\[
F(t) = \frac{1}{2}(t_1^2 t_3 + t_1 t_2^2) + \sum_{d=1}^{\infty} \frac{N_d}{(3d-1)!} Q^d t_3^{3d-1},
\]
where the coefficient \( N_d \) can be interpreted as the number of rational curves in \( \mathbb{P}^2 \) of degree \( d \) passing through \( 3d-1 \) points in general position. The system of WDVV equations contains a single non-trivial equation
\[
F_{333} = F_{222}^2 - F_{223} F_{233},
\]
where the index \( i, i = 2, 3 \), denotes partial derivative with respect to \( t_i \). Comparing the coefficients in front of \( Q^d \) yields
\[
N_d = \frac{1}{3} \sum_{m=1}^{d-1} \left( \frac{3d-4}{3m-2} m^2 (d-m)^2 - \frac{3d-4}{3m-3} m(d-m)^3 \right) N_m N_{d-m},
\]
which together with \( N_1 = 1 \) determines \( N_d \) for all \( d > 1 \). The first few values are
\[
N_1 = N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = 87304, \quad N_6 = 26312976, \quad \ldots
\]
Let us point out that the number \( \epsilon \) in the definition of the domain \( M \) is chosen in such a way that the radius of convergence of the series (3) is \( \epsilon |Q|^{-1/3} \).

Furthermore, the Euler vector field has the form
\[
E = t_1 \partial_1 - t_3 \partial_3 + 3 \partial_2
\]
and the Hodge grading operator is
\[
\theta : H \to H, \quad \theta = \text{diag}(1, 0, -1).
\]

2.2. The \( \Gamma \)-integral structure of Iritani. Let us recall the notation of Section 1.2. Following Givental (see [5]), we equip the quantum cohomology with calibration
\[
S(t, z) = 1 + S_1(t) z^{-1} + S_2(t) z^{-2} + \cdots, \quad S_k(t) \in \text{End}(H)
\]
defined by
\[
(S(t, z) \phi_i, \phi_j) = (\phi_i, \phi_j) + \sum_{k=0}^{\infty} \langle \psi^k, \phi_j \rangle_{0,2} t z^{-k-1}.
\]
The fundamental solution corresponding to such calibration is \( S(t, z) z^\rho z^{-\rho} \), where the nilpotent operator \( \rho \) is given by classical cup product multiplication by \( c_1(T\mathbb{P}^2) = 3p \).

There is a very elegant way to describe the reflection lattice, i.e., the \( \mathbb{Z} \)-submodule of \( H \) spanned by all reflection vectors \( a \in R \). Namely, using a \( \Gamma \)-class modification of the Chern character map, Iritani has obtained an explicit description for all reflection vectors in terms of \( K^0(\mathbb{P}^2) \) – the \( K \)-ring of topological vector bundles on \( \mathbb{P}^2 \). Let us recall Iritani’s result (see [6]). Following his notation, we introduce the map
\[
\Psi : K^0(\mathbb{P}^2) \to H,
\]
defined by
\[
\Psi(E) = \frac{1}{\sqrt{2\pi}} \Gamma(1 + p)^3 e^{-p \log Q} (2\pi \sqrt{-1})^{\text{deg} \, ch(E)},
\]
where \( \text{deg} : H \to H \) is the degree operator \( \text{deg}(p^i) = ip^i \) and the \( \Gamma \) function should be expanded as a Taylor series at \( p = 0 \), i.e.,
\[
\Gamma(1 + p) = 1 + \Gamma'(1)p + \frac{1}{2} \Gamma''(1)p^2.
\]
Recall that \( K^0(\mathbb{P}^2) = \mathbb{Z}[L]/(L - 1)^3 \), where \( L = \mathcal{O}(1) \). The above formula gives
\[
\Psi(L^m) = \frac{1}{\sqrt{2\pi}} \Gamma(1 + p)^3 e^{-p \log Q} e^{2\pi \sqrt{-1} mp}.
\]
Slightly abusing the notation we identify \( L^m \) with its image \( \Psi(L^m) \). The lattice
\[
\text{Im}(\Psi) := \mathbb{Z} + \mathbb{Z} L + \mathbb{Z} L^2 \subset H
\]
coincides with the reflection lattice.

Moreover, we can describe explicitly the set of all reflection vectors as follows. Let us choose a reference point \( (t^o, \lambda^o) \in (M \times \mathbb{C})' \) such that \( t_1^o = t_2^o = t_3^o = 0 \) and \( \lambda^o \) is a sufficiently large real number. Recall that for \( t_3 = 0 \) the quantum cup product \( \bullet_t \) turns \( H \) into the following algebra
\[
(H, \bullet_t) = \mathbb{C}[p]/(p^3 - Q).
\]
We get that \( u_i^o = 3\zeta^{-i+1} \) (1 \( \leq i \leq 3 \)), where \( \zeta = e^{2\pi \sqrt{-1}/3} \). Let us denote by \( [a, b] \) (\( a, b \in \mathbb{C} \)) the straight segment in \( \mathbb{C} \) from \( a \) to \( b \). Let \( \gamma_i \) (1 \( \leq i \leq 3 \)) be the composition of the arc
\[
\lambda(s) = \lambda^o e^{-2\pi \sqrt{-1}s/3}, \quad s \in [0, i - 1]
\]

Figure 1. Reflection vectors
and the straight segment $[\zeta^{-i+1}\lambda^i, u_1^i]$ (see Figure 1). Then $L_{i-1}$ is the reflection vector corresponding to the path $\gamma_i$ and the set of all reflection vectors is given by

$$R = W \cdot 1 \cup W \cdot L \cup W \cdot L^2.$$ 

Let us point out that the intersection pairing takes the form

$$(L^m|L^n) = (\tilde{I}^{(0)}_{L^m}(\lambda), (\lambda - \rho)\tilde{I}^{(0)}_{L^n}(\lambda)) = 2 + (m - n)^2.$$ 

According to Dubrovin (see [3]) the monodromy group $W \sim \text{PSL}_2(\mathbb{Z}) \times \{\pm 1\}$. The construction of this isomorphism amounts to choosing an appropriate $\mathbb{Q}$-basis $(E_1, E_2, E_3)$ of the reflection lattice. In our notation this basis is given by

$$E_1 = 1 + 2L - L^2, \quad E_2 = -3 + 4L - L^2, \quad E_3 = 1 - 2L + L^2.$$ 

### 2.3. The period map for quantum cohomology of $\mathbb{P}^2$. 

Now we are in position to state our results. We identify $H^* \cong \mathbb{C}^3$ via the linear functions on $H^*$ corresponding to $E_i$, $i = 1, 2, 3$. The period map takes the form

$$Z(t, \lambda) = (Z_1(t, \lambda), Z_2(t, \lambda), Z_3(t, \lambda)),$$

where $Z_i(t, \lambda) := Z_{E_i}(t, \lambda)$. As we have explained in the introduction, we may restrict our analysis to parameters $t \in B := \{t_1 = 0\} \subset M$. Put $B_{\text{small}} = \{t \in B \mid t_3 = 0\} = \mathbb{C}^*$ and

$$X_{\text{small}} := (M \times \mathbb{C})'|_{t_1 = t_3 = 0} = \{(Q, \lambda) \in \mathbb{C}^* \times \mathbb{C} \mid \lambda^3 - 27Q \neq 0\}.$$ 

Let us introduce the domain

$$\Omega_{\text{small}} := \{z \in (\mathbb{C}^*)^3 \mid z_2^2 = 4z_1z_3, \ \text{Im}(-z_2/(2z_3)) > 0\}.$$ 

Let us point out that there is a natural isomorphism

$$\Phi_{\text{small}} : \mathbb{H} \times \mathbb{C}^* \to \Omega_{\text{small}}, \ (\tau, x) \mapsto (\tau^2x, -2\tau x, x)$$

under which the action of the monodromy group takes a very simple form (see Section 3.3). We will prove later on (see Lemma 4.1) that $X_{\text{small}}$ is a deformation retract of $X$. Therefore, the universal cover $\tilde{X}_{\text{small}}$ is an analytic submanifold of $\tilde{X}$ and we can introduce the restriction of the period map $Z_{\text{small}} := Z|\tilde{X}_{\text{small}}$. Recall the Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{m=1}^{\infty} \frac{mq^m}{1 - q^m},$$

$$E_4(\tau) = 1 + 240\sum_{m=1}^{\infty} \frac{m^3q^m}{1 - q^m},$$

$$E_6(\tau) = 1 - 504\sum_{m=1}^{\infty} \frac{m^5q^m}{1 - q^m},$$

where $q = e^{2\pi\sqrt{-1}\tau}$. Our first result can be stated as follows.
Theorem 2.1. a) The image of $Z_{\text{small}}$ is $\Omega_{\text{small}}$.

b) Let $\tilde{\pi}_{\text{small}} : \tilde{X}_{\text{small}} \to X_{\text{small}}$ be the universal cover and
\[
\pi_{\text{small}} : \Omega_{\text{small}} \to \mathbb{C}^* \times \mathbb{C}, \quad (\tau, x) \mapsto (Q(\tau, x), \lambda(\tau, x)),
\]
be the map defined by
\[
Q(\tau, x) := \frac{8}{27} (2\pi/x)^6 (E_4^3(\tau) - E_6^2(\tau)) \quad \text{and} \quad \lambda(\tau, x) := 2(2\pi/x)^2 E_4(\tau),
\]
where $(\tau, x)$ is the coordinate system on $\Omega_{\text{small}}$ introduced above. Then $\pi_{\text{small}} \circ Z = \tilde{\pi}_{\text{small}}$.

c) The fibers of the map $\pi_{\text{small}}$ are the $W$-orbits in $\Omega_{\text{small}}$, i.e., the small quantum cohomology $B_{\text{small}} \times \mathbb{C}$ is the coarse moduli space for the orbifold $[\Omega_{\text{small}}/W]$.

Generalizing the results of Theorem 2.1 to big quantum cohomology is a very challenging problem. We expect that $M_{\text{small}} := \mathbb{C} \times B_{\text{small}}$ is an analytic subvariety in a larger Frobenius manifold $N$ and that $M$ is just a tubular neighborhood of $M_{\text{small}}$ in $N$. The problem of constructing a global Frobenius manifold is also expected to be equivalent to constructing the manifold of stability conditions in $\mathcal{D}^b(\text{Coh} \mathbb{P}^3)$. We were able to prove an interesting result about the holomorphic thickening of $Z_{\text{small}}$, which might be viewed as the first step towards constructing a global Frobenius manifold.

The period map $Z$ maps a small open neighborhood of $\tilde{X}_{\text{small}}$ in $\tilde{X}$ into a small open neighborhood of $\Omega_{\text{small}}$ in $\mathbb{C}^3$. Therefore we have an induced map of ringed spaces
\[
(\tilde{X}_{\text{small}}, \tilde{\tau}^{-1}\mathcal{O}_{\tilde{X}}) \to \Omega := (\Omega_{\text{small}}, \iota^{-1}\mathcal{O}_{\mathbb{C}^3}),
\]
where $\tilde{\tau} : \tilde{X}_{\text{small}} \to \tilde{X}$ and $\iota : \Omega_{\text{small}} \to \mathbb{C}^3$ are the natural inclusion maps. The ring of regular functions on $\Omega$ is by definition $\Gamma(\Omega_{\text{small}}, \mathcal{O}_{\mathbb{C}^3})$, i.e., functions defined and holomorphic in an open neighborhood of $\Omega_{\text{small}}$ in $\mathbb{C}^3$. This ring is equipped with the action of the monodromy group $W$. The pullback via the period map $Z$ defines a ring homomorphism
\[
\Gamma(\Omega_{\text{small}}, \mathcal{O}_{\mathbb{C}^3})^W \to \Gamma(X_{\text{small}}, \mathcal{O}_X), \quad f \mapsto f \circ Z,
\]
where $\Gamma(\Omega_{\text{small}}, \mathcal{O}_{\mathbb{C}^3})^W$ is the subring of $W$-invariant functions. The coordinate functions $Q = e^{it^2}$, $t = t_3$, and $\lambda$ of $B \times \mathbb{C}$ are elements of $\Gamma(X_{\text{small}}, \mathcal{O}_X)$. We will prove that $Q, t,$ and $\lambda$ are pullbacks via the period map of $W$-invariant functions. Moreover, the latter have some interesting property, which can be stated as follows. Let us construct an open neighborhood of $\Omega_{\text{small}}$ as the image of the map
\[
\Phi : \mathbb{H}^2 \times \mathbb{C}^* \to \mathbb{C}^3, \quad (\tau_1, \tau_2, y) \mapsto (\tau_1 \tau_2 y, -(\tau_1 + \tau_2) y, y),
\]
where $\mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$ is the upper half-plane. The image of $\Phi$ is the coarse moduli space for the orbifold quotient $[\mathbb{H}^2 \times \mathbb{C}^* / \mu_2]$, where $\mu_2$ is the cyclic group of order 2 whose generator acts on $\mathbb{H}^2$ by permutation $(\tau_1, \tau_2) \mapsto (\tau_2, \tau_1)$. 


Theorem 2.2. a) There are $W$-invariant functions in $\Gamma(\Omega_{\text{small}}, \mathcal{O}_C^3)^W$ of the form

$$Q(\tau_1, \tau_2, y) = \frac{8}{27}(2\pi/y)^6 \sum_{n=0}^{\infty} Q_n(\tau_1)(\tau_1 - \tau_2)^{2n},$$

$$\lambda(\tau_1, \tau_2, y) = 2(2\pi/y)^2 \sum_{n=0}^{\infty} \lambda_n(\tau_1)(\tau_1 - \tau_2)^{2n},$$

$$t(\tau_1, \tau_2, y) = -\frac{1}{32}(\tau_1 - \tau_2)^2 y^2,$$

where $\tau_{12} := (\tau_1 + \tau_2)/2$, such that their pullbacks via the period map coincide with the coordinate functions $Q$, $\lambda$, and $t$.

b) The coefficients $Q_n(\tau), \lambda_n(\tau)$ ($n \geq 0$) are quasi-modular forms, i.e., they are polynomials in the Eisenstein series $E_i(\tau)$, $i = 2, 4, 6$.

Note that if $\tau_1 = \tau_2$ then we recover the formulas from Theorem 2.1. In particular

$$Q_0(\tau) = E_4(\tau)^3 - E_6(\tau)^2, \quad \lambda_0(\tau) = E_4(\tau).$$

We have computed the quasi-modular forms $\lambda_n$ and $Q_n$ for $n = 1, 2, 3$. The answer is the following

$$\lambda_1(\tau) = \frac{1}{40} \partial^2 E_4,$$

$$\lambda_2(\tau) = \frac{1}{4480} \partial_4^4 E_4 - \frac{\pi^4}{2016} \Delta,$$

$$\lambda_3(\tau) = \frac{1}{967680} \partial_6^6 E_4 - \frac{\pi^4}{209664} \partial^2_\tau \Delta - \frac{\pi^6}{101088} E_4 \Delta,$$

and

$$Q_1(\tau) = \frac{1}{104} \partial^2 \Delta + \frac{\pi^2}{26} E_4 \Delta,$$

$$Q_2(\tau) = \frac{1}{24960} \partial^4 \Delta + \frac{\pi^2}{2704} E_4 \partial^2 \Delta + \frac{\pi^2}{1040} \Delta \partial^2_\tau E_4 + \frac{17\pi^4}{20280} E_4^2 \Delta,$$

$$Q_3(\tau) = \frac{1}{10183680} \partial^6 \Delta + \frac{1611\pi^2}{3756000} (E_4^2 \partial_\tau^2 E_4)^2 + \frac{3\pi^2}{1514440} E_4 \partial^4 \Delta + \left(\frac{3\pi^2}{116480} \Delta - \frac{537\pi^2}{26499200} E_4^3\right) \partial^4_\tau E_4 + \frac{239\pi^4}{2839200} E_4 \Delta \partial^2_\tau E_4$$

$$- \frac{319\pi^6}{26732160} \Delta^2 + \frac{3977\pi^6}{202718880} E_4^3 \Delta,$$

where $\Delta := E_4^3 - E_6^2$. 
3. The period map for small quantum cohomology

The goal of this section is to prove Theorem 2.1. Let us assume that \( t_3 = 0 \) and denote by \( Z(Q, \lambda) \) the value of the period map at the point \((Q, \lambda) \in X_{\text{small}}\). We do not use an explicit notation, but we will always keep in mind that \( Z(Q, \lambda) \) depends on the choice of a reference path.

3.1. The monodromy group of the second structure connection. Let us sketch the main steps in computing the monodromy group \( W \). The matrix of the intersection form in the basis (over \( \mathbb{Q} \)) \( E_1, E_2, E_3 \) takes the form

\[
\begin{bmatrix}
0 & 0 & 4 \\
0 & -8 & 0 \\
4 & 0 & 0
\end{bmatrix}
\]

In other words the only non-vanishing pairings are \((E_1|E_3) = 4\) and \((E_2|E_2) = -8\).

Let us denote by \( R_i \) the monodromy transformation of the basis \( E = (E_1, E_2, E_3) \) corresponding to analytic continuation along the path \( \gamma_i \) (i.e. the path that turns \( L_i^{-1} \) into a reflection vector). We represent \( R_i \) by a matrix such that the monodromy transformation of the row \( E \) is \( ER_i \). A direct computation yields

\[
R_1 = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
-1 & 2 & -1 \\
-2 & 3 & -1 \\
-4 & 4 & -1
\end{bmatrix}, \quad R_3 = \begin{bmatrix}
-4 & 4 & -1 \\
-10 & 9 & -2 \\
-25 & 20 & -4
\end{bmatrix}.
\]

The matrices \( R_i \) \((1 \leq i \leq 3)\) generate reflection group \( W_R \) that can be embedded as a finite index subgroup of the modular groups as follows. Let

\[
\phi : \mathbb{C}^3 \to \text{Sym}^2(\mathbb{C}^2)
\]

be the isomorphism identifying \( \mathbb{C}^3 \) with the space of symmetric quadratic forms on \( \mathbb{C}^2 \). More precisely

\[
\phi(z)(u_1, u_2) = z_1 u_1^2 + z_2 u_1 u_2 + z_3 u_2^2.
\]

The modular group \( \Gamma := \text{PSL}_2(\mathbb{Z}) \) acts naturally on the space of quadratic forms

\[
q(u_1, u_2) \mapsto (q \cdot g)(u_1, u_2) := q(au_1 + bu_2, cu_1 + du_2), \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]

Note that the above action is a right action: \( q \cdot (g_1 g_2) = (q \cdot g_1) \cdot g_2 \). Let us define a group homomorphism

\[
\rho : \text{PSL}_2(\mathbb{Z}) \to \text{SL}_3(\mathbb{Z})
\]

such that \( \phi(z) \cdot g = \phi(z \rho(g)) \), where \( z = (z_1, z_2, z_3) \) is a row vector and the matrix \( \rho(g) \) acts on \( z \) via matrix multiplication from the right. Explicitly

\[
\rho(g) = \begin{bmatrix}
a^2 & 2ab & b^2 \\
ac & ad + bc & bd \\
c^2 & 2cd & d^2
\end{bmatrix}, \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
Note that $R_i = -\rho(g_i)$, where

$$g_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}.$$ 

The monodromy group is generated by $R_1, R_2, R_3$, and $K$, where $K$ is the monodromy transformation corresponding to the analytic continuation in the $Q$-plane along a loop around $Q = 0$ in clockwise direction. The period vectors are by definition

$$I_a^{(n)}(Q, \lambda) = I^{(n)}(Q, \lambda)\Psi(a), \quad a \in K^0(\mathbb{P}^2).$$

If we recall the Laurent series expansions near $\lambda = \infty$ we get that the monodromy of the period vectors around $Q = 0$ comes from the map $\Psi$ only. The dependence of $\Psi$ on $Q$ is very simple so we get that the monodromy operator $K$ coincides with the operator of K-theoretic multiplication by $O(1)$. Therefore $E = (E_1, E_2, E_3)$ transforms into $EK$ where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \rho(\kappa), \quad \kappa = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$ 

The relation between $W_R, W$, and the modular group can be described as follows. The matrices $g_1\kappa$ and $g_1$ have orders respectively 3 and 2 and we have $\text{PSL}_2(\mathbb{Z}) = \langle g_1\kappa \rangle \ast \langle g_1 \rangle$. Using this presentation of the modular group we define the characters

$$\chi_2, \chi_3 : \text{PSL}_2(\mathbb{Z}) \to \mathbb{C}^*,$$

such that

$$\chi_2(g_1\kappa) = \chi_3(g_1) = 1, \quad \chi_2(g_1) = -1, \quad \chi_3(g_1\kappa) = \zeta.$$ 

Let us define a group homomorphism

$$(7) \quad \text{PSL}_2(\mathbb{Z}) \times \{\pm 1\} \to \text{GL}(\mathbb{C}^3), \quad (g, \sigma) \mapsto \sigma \chi_2(g) \rho(g).$$

Using that

$$R_2 = KR_1K^{-1}, \quad R_3 = K^2R_1K^{-2}.$$ 

we get that the image of the map (7) is the monodromy group $W$. It is not very difficult to check that the map is also injective, so it gives a group isomorphism $W \cong \text{PSL}_2(\mathbb{Z}) \times \{\pm 1\}$. Finally, using that the reflection group $W_R$ is generated by $R_1, R_2, R_3$ we get that the map

$$\text{Ker}(\chi_3) \to W_R, \quad g \mapsto \chi_2(g) \rho(g)$$

is a group isomorphism.
3.2. **Quadratic relation.** The second structure connection for small quantum cohomology takes the form

\[ Q\partial_Q I^{(-1)}(Q, \lambda) = -\partial_\lambda (P \bullet I^{(-1)}(Q, \lambda)) \]

and

\[ (\lambda\partial_\lambda + 3Q\partial_Q)I^{(-1)}(Q, \lambda) = (\theta + 1/2)I^{(-1)}(Q, \lambda). \]

Using these equations, we get that the period map satisfies the following differential equations

\[ \left( (Q\partial_Q)^3 - Q\partial_\lambda^3 \right) Z(Q, \lambda) = 0 \]

and

\[ (\lambda\partial_\lambda + 3Q\partial_Q)Z(Q, \lambda) = -\frac{1}{2}Z(Q, \lambda), \]

where we used that in small quantum cohomology \( P \bullet P \bullet P = Q \) and that \( \theta(1) = 1 \).

The second equation implies that the period map has the form

\[ Z(Q, \lambda) = Q^{-1/6}z(x), \quad x := \frac{\lambda^3}{27Q} \]

while the first equation implies that the vector valued function \( z(x) \) is a solution to the hypergeometric equation of type \((3, 2)\) defined by the differential operator

\[ D(D - \rho_1)(D - \rho_2) - x(D + \alpha_1)(D + \alpha_2)(D + \alpha_3), \]

where \( D = x\partial_x \), \( \rho_1 = \frac{1}{3} \), \( \rho_2 = \frac{2}{3} \), and \( \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{6} \).

**Lemma 3.1.** The image of the period map \( Z_{\text{small}} : \tilde{X}_{\text{small}} \to \mathbb{C}^3 \) is contained in the quadratic cone \( Z_2^2 = 4Z_1Z_3 \).

**Proof.** The equation of the quadratic cone coincides with

\[ \sum_{i,j=1}^{3} \eta^{ij} Z_i Z_j = 0 \]

where \( \eta^{ij} \) are the entries of the matrix inverse to the matrix \( \eta \) whose entries are the intersection numbers \( \eta_{ij} := (E_i | E_j) \) (see formula (4)). Let us denote by \( I^{(n)} \) the matrix whose \((i, j)\)-entry is given by \( (I^{(n)}_{E_i})(t, \lambda), P^{j-1}) \). Using the equations of the second structure connection we get

\[ I^{(0)}(\lambda - E \bullet) = I^{(-1)}(-\theta + \frac{1}{2}), \]

where

\[ E \bullet = \begin{bmatrix} 0 & 0 & 3Q \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}. \]
On the other hand, the entries of the intersection pairing are
\[ \eta_{ij} = (I_{E_i}^{(0)}, (\lambda - E\bullet)I_{E_j}^{(0)}) = \sum_{k,\ell=1}^{3} (I_{E_i}^{(0)}, \phi_k)g^{k\ell}(I_{E_j}^{(0)}, (\lambda - E\bullet)\phi_\ell), \]
where \( \phi_k = P^{k-1} \) and \( g^{k\ell} \) are the entries of the matrix inverse to the matrix of the Poincare pairing. Therefore
\[ \eta = I^{(0)}g^{-1}(\lambda - E\bullet)^T(I^{(0)})^T. \]
Using that \( Z = (Z_1, Z_2, Z_3) = (I^{-1})e_1^T \) we get
\[ Z\eta^{-1}Z^T = e_1^T(I^{-1})^T\eta^{-1}I^{-1}. \]
Recalling the formula for \( \eta \) from above we get
\[ (I^{-1})^T\eta^{-1}I^{-1} = ((I^{(0)})^{-1}I^{-1})^T((\lambda - E\bullet)^T)^{-1}g(I^{(0)})^{-1}I^{-1}. \]
On the other hand
\[ (I^{(0)})^{-1}I^{-1} = (\lambda - E\bullet)(-\theta + 1/2)^{-1}. \]
Therefore \( Z\eta^{-1}Z^T \) is the (1,1)-entry of the matrix
\[ (-\theta + 1/2)^{-1} g (\lambda - E\bullet)(-\theta + 1/2)^{-1} = \begin{bmatrix} 0 & 12 & -4\lambda/3 \\ 12 & 4\lambda & 0 \\ -4\lambda/3 & 0 & -4Q/3 \end{bmatrix}. \]

3.3. **Connection Formula.** The differential equation \( [8] \) has the following basis of solutions near \( x = \infty \).
\[ z_1^\infty(x) := \sum_{n=0}^{\infty} a_n x^{-n-1/6}, \]
where the coefficients \( a_n \) are defined by
\[ a_0 = 1, \quad a_{n+1} = \frac{(n + 1/6)(n + 3/6)(n + 5/6)}{(n + 1)^3}, \quad n \geq 0. \]
Note that \( z_1^\infty(x) \) coincides with the generalized hypergeometric function
\[ _3F_2 \left[ \begin{array}{c} 1/6, 3/6, 5/6 \\ 1, 1 \end{array}; x^{-1} \right] x^{-1/6}. \]
The second solution is
\[ z_2^\infty(x) = \sum_{n=0}^{\infty} a_n x^{-n-1/6} \left( \log x - b_n \right), \]
where the constants \( b_n \) (\( n \geq 0 \)) are defined by \( b_0 = 0 \) and
\[ b_{n+1} = b_n + \frac{1}{n + 1/6} + \frac{1}{n + 3/6} + \frac{1}{n + 5/6} - \frac{3}{n + 1}. \]
Finally, the third solution is given by
\[
z_3^\infty(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{6}} \left( (\log x - b_n)^2 + c_n \right),
\]
where the constants \(c_n\) (\(n \geq 0\)) are defined by
\[
c_{n+1} = c_n + \frac{3}{(n+1)^2} - \frac{1}{(n + \frac{3}{6})(n + \frac{5}{6})} - \frac{1}{(n + \frac{1}{6})(n + \frac{7}{6})}.
\]

Let us find the transition matrix \(C^\infty\) defined by
\[
z(x) = z^\infty(x) C^\infty,
\]
where
\[
z(x) = (z_1(x), z_2(x), z_3(x)), \quad z^\infty(x) = (z_1^\infty(x), z_2^\infty(x), z_3^\infty(x)).
\]

To begin with, note that the analytic continuation of \(Z(Q, \lambda)\) along a clockwise loop around 0 in the \(Q\)-plane corresponds to analytic continuation of \(z(x)\) along an anticlockwise loop around 0 and 1 in the \(x\)-plane. Since the analytic continuation transforms \(Z(Q, \lambda)\) into \(Z(Q, \lambda) K\) where \(K\) is the matrix \([5]\) we get that \(z(x) = Q^{1/6} Z(Q, \lambda)\) transforms into
\[
\zeta^{-1/2} z(x) K.
\]

On the other hand, the analytic continuation of \(z^\infty(x)\) is
\[
\zeta^{-1/2} z^\infty(x) K^\infty, \quad K^\infty := \begin{bmatrix} 1 & 2\pi\sqrt{-1} & (2\pi\sqrt{-1})^2 \\ 0 & 1 & 4\pi\sqrt{-1} \\ 0 & 0 & 1 \end{bmatrix}.
\]

Therefore we have the following relation
\[
C^\infty K = K^\infty C^\infty.
\]

Let us denote by \(C^\infty_i\) the \(i\)-th column of \(C^\infty\). Then comparing the columns in the above relation we get
\[
C^\infty_2 = \left( K^\infty - 1 - \frac{1}{2}(K^\infty - 1)^2 \right) C^\infty_1 = \begin{bmatrix} 0 & 2\pi\sqrt{-1} & 0 \\ 0 & 0 & 4\pi\sqrt{-1} \\ 0 & 0 & 0 \end{bmatrix} C^\infty_1
\]
and
\[
C^\infty_3 = \frac{1}{2}(K^\infty - 1)^2 C^\infty_1 = \begin{bmatrix} 0 & 0 & (2\pi\sqrt{-1})^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} C^\infty_1
\]

Therefore, we need just to find the first column of \(C^\infty_1\), i.e., the coefficients in the relation
\[
z_1(x) = C^\infty_1 z^\infty_1(x) + C^\infty_{21} z^\infty_2(x) + C^\infty_{31} z^\infty_3(x).
\]
The leading order term in the expansion of the RHS at \(x = \infty\) is precisely
\[
C^\infty_{11} x^{-1/6} + C^\infty_{21} x^{-1/6} \log x + C^\infty_{31} x^{-1/6} (\log x)^2.
\]
While the leading order term of the expansion of the LHS is given from the leading order term of
\[ Q^{1/6} (\tilde{I}(-1)(\lambda) \Psi(1 + 2L - L^2), 1). \]
Recalling the definition of the period \( \tilde{I}(-1)(\lambda) \) we get that the above expression is precisely
\[ \frac{1}{\sqrt{6\pi}} (\log x + \log 1728)^2 x^{-1/6}. \]
Therefore,
\[ C_{11}^\infty = \frac{(\log 1728)^2}{\sqrt{6\pi}}, \quad C_{21}^\infty = \frac{2\log 1728}{\sqrt{6\pi}}, \quad C_{31}^\infty = \frac{1}{\sqrt{6\pi}} \]
and we get
\[ C^\infty = \frac{1}{\sqrt{6\pi}} \begin{bmatrix} \frac{(\log 1728)^2}{2\log 1728} & \frac{2(2\pi \sqrt{-1}) \log 1728}{4\pi \sqrt{-1}} & 0 \\ \frac{2\log 1728}{1} & 0 & 0 \end{bmatrix}. \]

3.4. Symmetric square of a hypergeometric equation. We are going to prove that the solution space of the generalized hypergeometric equation \( (8) \) has a basis of the form
\[ z_1 = \frac{u^2}{2}, \quad z_2 = uv, \quad z_3 = \frac{v^2}{2}, \]
where \( u, v \) is a basis of solutions for the differential equation defined by the differential operator
\[ D^2 - \frac{2 + x}{6(1 - x)} D - \frac{x}{144(1 - x)}, \quad D := x\partial_x. \]
Note that the above operator defines a differential equation equivalent to the classical hypergeometric equation defined by the differential operator
\[ (1 - x)x\partial_x^2 + (c - (a + b + 1)x)\partial_x - ab \]
with \( a = b = \frac{1}{12} \) and \( c = \frac{2}{3} \).
In order to compute the symmetric square of \( (10) \) we have to find 3 functions \( p_i(x) (1 \leq i \leq 3) \) such that
\[ D^3 z_i = p_2(x)D^2 z_i + p_1(x)D z_i + p_0(x) z_i = 0, \quad 1 \leq i \leq 3, \]
where \( z_i \) are given by \( (9) \) with \( \{u, v\} \) a basis of solutions to \( (10) \). Using that \( u \) and \( v \) are solutions to \( (10) \) we can express the equations of the above linear system for \( p_0, p_1, \) and \( p_2 \) as differential polynomials in \( u \) and \( v \). After a direct computation we get
\[ c_1(Du)^2 + c_2(uDu) + c_3u^2 = 0, \]
\[ 2c_1DuDV + c_2(uDv + vDu) + 2c_3uv = 0, \]
\[ c_1(Dv)^2 + c_2(vDv) + c_3v^2 = 0, \]
where
\[
\begin{align*}
c_1 &= p_2 - 3\alpha \\
c_2 &= p_1 + \alpha p_2 - 4\beta - \alpha^2 - D\alpha \\
c_3 &= p_0/2 - \alpha \beta - D\beta + p_2\beta,
\end{align*}
\]
and
\[
\alpha := \frac{2 + x}{6(1 - x)}, \quad \beta := \frac{x}{144(1 - x)}.
\]
From here we get that \(c_1 = c_2 = c_3 = 0\), so
\[
\begin{align*}
p_2 &= 3\alpha = \frac{2 + x}{2(1 - x)}, \\
p_1 &= 4\beta - 2\alpha^2 + D\alpha = -\frac{8 - 3x}{36(1 - x)},
\end{align*}
\]
and
\[
p_0 = -4\alpha \beta + 2D\beta = \frac{x}{216(1 - x)}.
\]
Finally, it remains only to verify that the differential operator \((1 - x)^{-1}(D^3 - p_2 D^2 - p_1 D - p_0)\) coincides with [8].

The classical hypergeometric equation \([10]\) has a basis of solutions near \(x = \infty\) of the following form
\[
v^\infty(x) = \sum_{n=0}^{\infty} v_n x^{-n - \frac{1}{12}}
\]
and
\[
u^\infty(x) = \sum_{n=0}^{\infty} v_n (\log x - u_n) x^{-n - \frac{1}{12}},
\]
where the constants \(u_n, v_n \ (n \geq 0)\) are define by
\[
v_0 = 1, \quad v_{n+1} = \frac{(n + \frac{1}{12})(n + \frac{5}{12})}{(n + 1)^2} v_n,
\]
and
\[
u_0 = 0, \quad u_{n+1} = u_n + \frac{1}{n + \frac{1}{12}} + \frac{1}{n + \frac{5}{12}} - \frac{2}{n + 1}.
\]
Comparing the leading coefficients in the Laurent series expansion near \(x = \infty\) we get
\[
\begin{align*}
z_1^\infty(x) &= (v^\infty(x))^2, \\
z_2^\infty(x) &= u^\infty(x) v^\infty(x), \\
z_3^\infty(x) &= (u^\infty(x))^2.
\end{align*}
\]
3.5. **Proof of Theorem 2.1, a.** Using the connection formula (see Section 3.3) we get

\[
\tau(x) := -\frac{z_2(x)}{2z_3(x)} = -\frac{1}{2\pi\sqrt{-1}} \left( \log 1728 + \frac{u^\infty(x)}{v^\infty(x)} \right).
\]

In order to prove part a) of Theorem 2.1 it is sufficient to construct a family of elliptic curves, such that \(\tau(x)\) is a modulus. The family is defined via the Wierstrass equation

\[
E_x : \quad Y^2 = 4X^3 - g_2(x)X^2 - g_3(x), \quad x \in \mathbb{C} - \{0,1\},
\]

where

\[
g_2(x) = g_3(x) = \frac{27}{x - 1}.
\]

The periods of the elliptic curve are by definition the following multi-valued analytic functions

\[
f_\alpha(x) = \oint_{\alpha_x} dX/Y, \quad \alpha \in H_1(E_\lambda^\circ, \mathbb{Z}),
\]

where \(\alpha_x \in H_1(E_x, \mathbb{Z})\) is the parallel transport of \(\alpha\) along a reference path. A standard computation (see [8]) shows that the periods satisfy the following second order differential equation

\[
\partial_x^2 f + \frac{1}{x} \partial_x f + \frac{31x - 4}{144x^2(1 - x)^2} f = 0.
\]

This equation is gauge equivalent to the hypergeometric equation (10). Namely, all solutions to (10) have the form

\[
u(x) = x^\frac{1}{6} (x - 1)^{-\frac{1}{4}} f(x)
\]

for some solution \(f(x)\) to (12).

Let us assume that \(x\) is close to \(\lambda^\circ\). By construction, the \(J\)-invariant of \(E_x\) is

\[
J(x) = \frac{g_2(x)^3}{g_2(x)^3 - 27g_3(x)} = x.
\]

Therefore, since \(J(x)\) has a simple pole at \(x = \infty\), we can find a symplectic basis \(\{\alpha, \beta\} \subset H_1(E_\lambda^\circ, \mathbb{Z})\), such that the parallel transport around \(x = \infty\) (along a loop going counter-clockwise around \(x = 0\) and \(x = 1\)) transforms

\[\alpha \mapsto \alpha, \quad \beta \mapsto \beta - \alpha.\]

Comparing with the monodromy near \(x = \infty\) of \(u^\infty(x)\) and \(v^\infty(x)\) we get

\[
x^\frac{1}{6} (x - 1)^{-\frac{1}{4}} f_\alpha(x) = c_1 v^\infty(x)
\]

\[
x^\frac{1}{6} (x - 1)^{-\frac{1}{4}} f_\beta(x) = c_2 v^\infty(x) - \frac{c_1}{2\pi\sqrt{-1}} u^\infty(x)
\]
for some constants $c_1$ and $c_2$. Therefore,
\[
\tau(x) - f_\beta(x)/f_\alpha(x) = (c_1/c_2) + \frac{1}{2\pi\sqrt{-1}} \log 1728 =: c.
\]
We have to prove that $c = 0$. This follows by comparing the Fourier coefficients of the $J$-invariant. According to equation (11)
\[
e^{-2\pi\sqrt{-1}\tau(x)} = 1728(1 + O(x^{-1})).
\]
On the other hand, by construction $x = J(\tau)$, where $\tau = f_\beta/f_\alpha$ and
\[
J(\tau) = \frac{1}{1728}(q^{-1} + 744 + 196884q + \cdots), \quad q := e^{2\pi\sqrt{-1}\tau}
\]
is the $J$-invariant. Put $\tau(x) = \tau + c$, then comparing the coefficients in front of $q^{-1}$ yields $c = 0$.

The above argument proves that $Z_{\text{small}} : \widetilde{X}_{\text{small}} \to \Omega_{\text{small}}$. The surjectivity of this map is easy. Indeed, we have an analytic isomorphism
\[
(13) \quad \Phi_{\text{small}} : H \times C^* \to \Omega_{\text{small}}, \quad (\tau, x) \mapsto (\tau^2 x, -2\tau x, x).
\]
The period map takes the form
\[
Z_{\text{small}} : \widetilde{X}_{\text{small}} \to H \times C^*, \quad (Q, \lambda) \mapsto (-Z_2(Q, \lambda)/(2Z_3(Q, \lambda)), Z_3(Q, \lambda)).
\]
Given a point $(\tau, x) \in H \times C^*$ we pick first $(Q, \lambda)$ such that $\lambda^3/(27Q) = J(\tau)$ and then we fix $Q$ in such a way that $Q^{1/6}z_3(\lambda^3/27Q) = x$. The surjectivity follows. □

3.6. **Proof of Theorem 2.1 b) and c).** Let us define a left action of $\text{PSL}_2(\mathbb{Z}) \times \{\pm 1\}$ on $H \times C^*$ by
\[
(g, \sigma) \cdot (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \sigma\chi_2(g)(c\tau + d)^2 z\right), \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}_2(\mathbb{Z}).
\]

**Lemma 3.2.** Let $w \in \text{GL}(C^3)$ be the monodromy transformation corresponding to an element $(g, \sigma) \in \text{PSL}_2(\mathbb{Z}) \times \{\pm 1\}$ via the map (7). Then
\[
\Phi_{\text{small}}(\tau, x) \cdot w = \Phi_{\text{small}}((sg^{-1}s^{-1}, \sigma) \cdot (\tau, x)).
\]
where $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $(\tau, x) \in H \times C^*$.

**Proof.** Put $z := \Phi_{\text{small}}(\tau, x)$, then by definition $\tau = \tau(z) := -z_2/(2z_3)$. By definition
\[
w = \sigma\chi_2(g) \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & (ad + bc) & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.
\]
After a straightforward computation we get
\[
\tau(z \cdot w) = -\frac{2abz_1 + (ad + bc)z_2 + 2cdz_3}{2(b^2z_1 + bdz_2 + d^2z_3)} = \frac{a\tau(z) - c}{-b\tau(z) + d}.
\]
where we used that \( z_1/z_3 = z_1 z_3^2/z_3^2 = \tau^2 \). It remains only to use that
\[
\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = s \begin{bmatrix} a & b' \\ c & d \end{bmatrix}^{-1} s^{-1}
\]
and \( \chi_2(sg^{-1}s^{-1}) = \chi_2(g) \).

The proof of Theorem 2.1, part b) can be completed as follows. Note that the function
\[
z_3(x) = Q^{1/6} Z_3(Q, \lambda) = \frac{1}{\sqrt[6]{6\pi}} (2\pi \sqrt{-1})^{2} v^\infty(x)
\]
can be viewed as a holomorphic function on the universal cover of \( \mathbb{C} - \{0, 1\} \). On the other hand, the map \( x \mapsto \tau(x) \) defined by (11) gives an isomorphism between the universal cover of \( \mathbb{C} - \{0, 1\} \) and the upper half-plane \( \mathbb{H} \). More precisely, using the identity
\[
q = x^{-1} \exp\left( \sum_{m=1}^{\infty} \frac{u_m v_m x^{-m}}{\sum_{m=1}^{\infty} u_m x^{-m}} \right), \quad q := e^{2\pi \sqrt{-1} \tau}
\]
we can express \( x^{-1} \) as a holomorphic function on \( \mathbb{H} \), which allows us to view the solutions \( u^\infty(x) \) and \( v^\infty(x) \) of the hypergeometric equation as holomorphic functions on \( \mathbb{H} \). According to Lemma 3.2, the following transformation law holds
\[
z_3(g \tau)^6 = (c \tau + d)^{24} z_3(\tau)^6.
\]
Moreover, since \( v^\infty(x)^{12} \) is holomorphic at \( x = \infty \) we get that \( z_3(\tau)^6 \) is a modular form of weight 12. Comparing the first few terms in the Fourier series expansions we get
\[
z_3(\tau)^6 = \frac{8}{27} (2\pi)^6 (E_4(\tau)^3 - E_6(\tau)^2).
\]
Therefore,
\[
Q = \frac{8}{27} (2\pi)^6 Z_3(Q, \lambda)^{-6} (E_4(\tau)^3 - E_6(\tau)^2).
\]
The formula for \( \lambda \) follows from the relation
\[
\frac{\lambda^3}{27Q} = x = J(\tau) = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2},
\]
which implies that
\[
\lambda = 2 (2\pi)^2 Z_3(Q, \lambda)^{-2} E_4(\tau) \xi,
\]
where \( \xi^3 = 1 \). On the other hand, \( Z_3(Q, \lambda)^2 \lambda = 8\pi^2 + O(\lambda^{-1}) \), so \( \xi = 1 \).

The proof of part c) can be completed as follows. Let us identify \( \Omega_{\text{small}} \cong \mathbb{H} \times \mathbb{C}^* \). Let us assume that \( \pi(\tau, z) = \pi(\tau', z') \). Since \( J(\tau) = J(\tau') \), there exists \( g \in \text{PSL}_2(\mathbb{Z}) \) such that \( \tau' = g(\tau) \). There are 2 cases. First, if \( J(\tau) \neq 0 \), then \( E_4(\tau) \neq 0 \) and we get
\[
z^{-2} E_4(\tau) = (z')^{-2} E_4(\tau') = (z')^{-2} (c \tau + d)^{4} E_4(\tau),
\]
so
\[ z' = \sigma'(c\tau + d)^2 z \]
for some \( \sigma' \in \{ \pm 1 \} \). Defining \( \sigma = \sigma' \chi_2(g) \) we get \( (\tau', z') = (g, \sigma) \cdot (\tau, z) \).

The second case is the case when \( J(\tau) = 0 \). We may assume that \( \tau = \tau' = \zeta \), because the point \( (\tau, z) \) (resp. \( (\tau', z') \)) is in the \( W \)-orbit of a point of the form \( (\zeta, \tilde{z}) \) (resp. \( (\zeta, \tilde{z}') \)). Using that
\[ z^{-6}E_6(\tau)^2 = (z')^{-6}E_6(\tau')^2 = (z')^{-6}E_6(\tau)^2 \]
we get that \( z' = \xi z \), for some \( \xi \) such that \( \xi^6 = 1 \). On the other hand
\[ (g_1\kappa, \sigma) \cdot (\tau, z) = (\tau, \sigma\xi^2 z). \]
Therefore, acting on \( (\tau, z) \) with an appropriate element of the form \( ((g_1\kappa)^i, \sigma) \) we can obtain \( (\tau', z') \). \( \square \)

4. Holomorphic thickening

Let us return to the general case of the period map for the big quantum cohomology. Recall that
\[ X = (B \times \mathbb{C})' \subset \mathbb{C}^* \times \mathbb{C} \times \mathbb{C} \]
and
\[ X_{\text{small}} = \{ t_3 = 0 \} \subset X. \]
Let us introduce coordinates \((Q, t, \lambda)\) on \( X \) such that \( Q = \frac{8}{27} \left( \frac{2\pi}{x} \right)^6 (E_4(\tau)^3 - E_6(\tau)^2) \), \( \lambda = 2 \left( \frac{2\pi}{x} \right)^2 E_4(\tau) \), \( t = sE_6(\tau)^2 / x^6 \).

We choose \( U \) to be the trivial disk bundle
\[ U = \{ (\tau, x, s) \mid |s| < \delta(x, \tau) \} \]
where

\[ \delta : \mathbb{H} \times \mathbb{C}^* \to \mathbb{R}_{>0} \]

is a smooth function defined as follows. We choose \( \delta \) in such a way that the preimage under \( \pi_{aux} \) of the discriminant is the analytic hypersurface \( E_6(\tau) = 0 \).

More precisely, the equation of the discriminant has the form

\[ 0 = \det(\lambda - E \bullet) = \lambda^3 - 27Q + tg(t, Q, \lambda), \]

where \( g \in \mathcal{O}(B) \) is some holomorphic function. If \( (t, Q, \lambda) = \pi_{aux}(\tau, x, s) \), then the above equation becomes

\[ E_6(\tau)^2 \left( 8(2\pi)^6 + sg \circ \pi_{aux}(\tau, x, s) \right) x^{-6} = 0. \]

For fixed \( (\tau, x) \in \mathbb{H} \times \mathbb{C}^* \) we choose \( \delta(\tau, x) \) such that \( \pi_{aux}(\tau, x, s) \in B \) and \( |sg \circ \pi_{aux}(\tau, x, s)| < 8(2\pi)^6 \) for all \( |s| < \delta(\tau, x) \).

**Lemma 4.1.** If we choose the constant \( \epsilon \) in the definition of the domain \( B \) sufficiently small, then the subvariety \( X_{small} \) is a deformation retract of \( X \).

**Proof.** A deformation retraction

\[ \Psi : X \times [0, 1] \to X \]

can be taken in the form

\[ (14) \quad \Psi(Q, t, \lambda, s) := (Q, \lambda + \sum_{i=1}^{3} \rho_i(\lambda Q^{-1/3})(u_i(Q, (1-s)t) - u_i(Q, t)), (1-s)t) \]

where \( u_i(Q, t) (1 \leq i \leq 3) \) are the eigenvalues of the quantum multiplication by \( E_{\bullet, Q, t} \), i.e., the canonical coordinates. Note that \( u_i(Q, t) = Q^{1/3}u_i(1, tQ^{1/3}) \) and \( u_i(Q, 0) = 3^i\zeta^iQ^{i/3} \), where \( \zeta = e^{2\pi \sqrt{-1}/3} \). Given a real number \( \delta > 0 \) we can always choose \( \epsilon \) sufficiently small so that \( |u_i(1, tQ^{1/3}) - u_i(1, 0)| < \delta \) for all \( (Q, t) \), s.t., \( |tQ^{1/3}| < \epsilon \). We claim that if we choose \( \delta < 3^{1/6} \| 1 - \zeta \| \) and \( \rho_i (1 \leq i \leq 3) \) to be smooth functions such that \( \rho_i(x) = 1 \) for all \( |x - 3\zeta^i| < 4\delta \) and \( \rho_i(x) = 0 \) for all \( |x - 3\zeta^i| > 8\delta \), then formula \( (14) \) defines a deformation retract, i.e., a homotopy between the identity map and a retraction \( X \to X_{small} \).

Clearly we have \( \Psi(x, 0) = x \) for all \( x \in X \) and \( \Psi(y, s) = y \) for all \( y \in X_{small} \) and for all \( s \). We have to verify that \( \Psi(Q, t, \lambda, s) \) is not a point on the discriminant. There are two cases. First, if \( |\lambda Q^{-1/3} - 3\zeta^i| > 8\delta \) for all \( i \), then \( \Psi(Q, t, \lambda, s) = (Q, \lambda, (1-s)t) \). We have

\[
|\lambda Q^{-1/3} - u_j(1, (1-s)tQ^{1/3})| \geq \left| \lambda Q^{-1/3} - u_j(1, 0) \right| - |u_j(1, (1-s)tQ^{1/3}) - u_j(1, 0)| > 8\delta - \delta = 7\delta > 0,
\]

so \( \lambda \neq u_j(Q, (1-s)t) \) for all \( j \), i.e., \( (Q, \lambda, (1-s)t) \in X \). The second case is if \( |\lambda Q^{-1/3} - 3\zeta^i| \leq 8\delta \) for some \( i \). Note that if \( j \neq i \), then

\[
|\lambda Q^{-1/3} - 3\zeta^j| \geq |3\zeta^i - 3\zeta^j| - |\lambda Q^{-1/3} - 3\zeta^i| = 3|1 - \zeta| - |\lambda Q^{-1/3} - 3\zeta^i| > 8\delta.
\]
Therefore the second component of \( \Psi(Q, t, \lambda, s) \) is
\[
\lambda + \rho_i(\lambda Q^{-1/3})(u_i(Q, (1 - s)t) - u_i(Q, t)).
\]
We have to prove that the above number does not coincide with \( u_j(Q, (1 - s)t) \) for all \( j \). Let us assume that this is not the case, i.e., the number coincides with \( u_j(Q, (1 - s)t) \) for some \( j \). Using the estimate
\[
|\lambda Q^{-1/3} - 3\zeta^j| \leq |u_j(1, (1 - s)tQ^{1/3}) - 3\zeta^j| + |u_i(1, (1 - s)tQ^{1/3}) - u_i(1, tQ^{1/3})| < 3\delta
\]
we get that we must have \( j = i \) and \( \rho_i(\lambda Q^{-1/3}) = 1 \). Therefore our assumption implies that
\[
\lambda + u_i(Q, (1 - s)t) - u_i(Q, t) = u_i(Q, (1 - s)t) \Rightarrow \lambda = u_i(Q, t).
\]
This however contradicts the fact that \((Q, t, \lambda) \in \mathcal{X}\). \(\square\)

**Proposition 4.2.** a) Let \( \pi' : \mathcal{U}' = (\pi_{\text{aux}})^{-1}(X) \to X \) be the map induced from \( \pi_{\text{aux}} \). Then the period map admits a holomorphic lift \( Z_{\text{aux}} : \mathcal{U}' \to \mathbb{C}^3 \).

b) The map \( Z_{\text{aux}} \) extends holomorphically on the entire domain \( \mathcal{U} \).

**Proof.** a) Note that our definition of \( \mathcal{U} \) implies that \( \mathcal{U}' = \mathcal{U} - \{E_6(\tau) = 0\} \). Let us define an action of the monodromy group \( W = \text{PSL}_2(\mathbb{Z}) \times \{\pm 1\} \) on \( \mathcal{U} \)
\[
(g, \sigma) \cdot (\tau, x, s) = (g(\tau), \sigma \chi_2(g) x, s).
\]
Note that the points with non-trivial stabilizers are given by the analytic hypersurfaces \( \{E_4(\tau) = 0\} \) and \( \{E_6(\tau) = 0\} \). Let \( u^\circ = (\tau^\circ, x^\circ, 0) \in \mathcal{U}' \) be a reference point, such that \( E_4(\tau^\circ) \neq 0 \) and \( \pi_{\text{aux}}(u^\circ) = y^\circ \).

Let us construct a lift \( Z' \) of the period map on \( \mathcal{U}' \). If \( u = (\tau, x, t) \in \mathcal{U}' \), then we pick a reference path \( \gamma \subseteq \mathcal{U}' \) and define \( Z'(u) = Z(\pi_{\text{aux}}(u)) \) where the value of \( Z(\pi_{\text{aux}}(u)) \) is defined via the reference path \( \pi_{\text{aux}}(\gamma) \). We claim that choosing a different reference path \( \gamma' \subseteq \mathcal{U}' \) does not change the value of \( Z'(u) \). In other words we claim that if \( L \in \pi_1(\mathcal{U}', u^\circ) \) is a loop based at \( u^\circ \), then the image \( \pi_{\text{aux}}(L) \) is in the kernel of the monodromy representation per : \( \pi_1(X) \to W \). By making a small perturbation (without changing the homotopy class) we can arrange that \( L \) is a loop in \( \mathcal{U}' - \{E_4 = 0\} \). Note that the projections \( r' : (\tau, x, s) \mapsto (\tau, x) \) and \( r'' : (Q, t, \lambda) \mapsto (Q, \lambda) \) give rise to a commutative diagram
\[
\begin{array}{ccc}
\mathcal{U}' & \xrightarrow{r'} & (\mathbb{H} \times \mathbb{C}^*)' := \mathbb{H} \times \mathbb{C}^* - \{E_6 = 0\} \\
\downarrow{\pi'} & & \downarrow{\pi''} \\
X' & \xrightarrow{r''} & X_{\text{small}}
\end{array}
\]
where \( \pi' \) and \( \pi'' \) are the maps induced from \( \pi_{\text{aux}} \),
\[
X' := \{(Q, t, \lambda) \in X : \lambda^3 - 27Q \neq 0\},
\]
and the horizontal arrows are deformation retractions. The map \( \pi'' \) induces a covering
(15) \( \mathbb{H} \times \mathbb{C}^* - \{E_6 = 0\} \cup \{E_4 = 0\} \to X'_{\text{small}} := X_{\text{small}} - \{\lambda = 0\} \).
According to Theorem 2.1 the period map provides a lift of the covering map
\[ \tau = -Z_2(Q, \lambda)/(2Z_3(Q, \lambda)), \quad x = Z_3(Q, \lambda). \]
Therefore we have a commutative diagram
\[
\begin{array}{ccc}
\pi_1(X'_\text{small}, y^\circ) & \xrightarrow{c_\text{ck}} & \pi_1(X_\text{small}, y^\circ) \\
& \searrow \downarrow \nearrow & \\
& W &
\end{array}
\]
in which the horizontal arrow is induced from the natural inclusion \( X'_\text{small} \subset X_\text{small} \) and the two diagonal arrows are given by the monodromy representations respectively of the covering and the period maps. On the other hand the lift of the loop \( r'' \circ \pi'(L) \) is \( r'(L) \), which is a loop, so the corresponding monodromy transformation
\[ w := \text{cov}(r'' \circ \pi'(L)) \in W \]
fixes the reference point \((\tau^\circ, x^\circ)\) in \( U' \). Therefore \( w = 1 \), because the stabilizer of \((\tau^\circ, x^\circ)\) is trivial. We get that the homotopy class of \( r'' \circ \pi'(L) \) is in the kernel of the monodromy representation of the period map. Using that \( r'' \) is a deformation retract, we get that \( \pi'(L) \) is homotopic to \( r'' \circ \pi'(L) \) in \( X' \). Finally, since \( X_\text{small} \) is a deformation retract of \( X \) (see Lemma 4.1) we get that the homotopy class of \( \pi'(L) \) in \( \pi_1(X, y^\circ) \) must be in the kernel of the monodromy representation of the period map.

b) It remains only to prove that \( Z' \) extends analytically to the entire domain \( U \).

The complement of \( U' \) in \( U \) is an analytic hypersurface. Recalling the Riemann extension theorem, we get that it is sufficient to prove that the values of \( Z'(u) \) are bounded in a neighborhood of an arbitrary point \( u_0 \in U - U' \). Note that \( \pi^\text{aux}(u_0) =: (Q_0, t_0, \lambda_0) \) is a point on the discriminant. Then by definition the periods
\[ I^{(-1)}(Q, t, \lambda) \sim (\lambda - u)^{1/2} \]
where \( \lambda = u \) is the local equation of the discriminant near the point \((Q_0, t_0, \lambda_0)\). Therefore, the map \( Z' \) is bounded.

4.2. The Taylor’s coefficients \( Z^{(n)} \). Recall the notation in the proof of Lemma 3.1. Let us denote by \( I^{(n)} \) the matrix whose \((i, j)\) entry is \((I^{(n)}_{E})_{ij}, p^{i-j}) \). We claim that the matrix \( I^{(-1)} \) can be expressed in terms of the Wronskian matrix
\[
\text{Wr} = \begin{bmatrix}
Z_1 & Q\partial_Q Z_1 & (Q\partial_Q)^2 Z_1 \\
Z_2 & Q\partial_Q Z_2 & (Q\partial_Q)^2 Z_2 \\
Z_3 & Q\partial_Q Z_3 & (Q\partial_Q)^2 Z_3
\end{bmatrix}.
\]
Indeed, put
\[ A = A(Q, t, \lambda) = -(-\theta + 1/2)(\lambda - E\bullet)^{-1}. \]
The differential equation of the second structure connection can be written us
\[ \partial_\lambda I^{(-1)} = -I^{(-1)} A, \quad Q\partial_Q I^{(-1)} = I^{(-1)} A\Omega_2, \quad \partial_t I^{(-1)} = I^{(-1)} A\Omega_3, \]
where $\Omega_i = P_i^{-1}\bullet$ is the matrix of quantum multiplication by $P_i^{-1}$. We get

$$Q\partial_Q \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = I^{(-1)} A \Omega_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (Q\partial_Q)^2 = I^{(-1)}(A\Omega_2 A + Q\partial_Q A) \Omega_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

Therefore, $\text{Wr} = I^{(-1)} T$, where $T$ is the matrix with columns

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A \Omega_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (A\Omega_2 A + Q\partial_Q A) \Omega_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

The proves of the next two Lemmas involve some long computations. Although they could be done by hand within acceptable amount of time, we recommend the usage of a computer software such as Mathematica or Maple.

**Lemma 4.3.** The matrix $T$ can be expressed in terms of the genus 0 potential $F$ as follows

$$T = \begin{bmatrix} 1 & \frac{1}{\Delta} (F_{23} \lambda + 9F_{33}/2) & \frac{1}{\Delta t_{13}} \\ 0 & \frac{1}{\Delta} (-\lambda^2/2 - 3F_{33}/t) & \frac{1}{\Delta t_{23}} \\ 0 & \frac{1}{\Delta} (-9\lambda/2 + 3F_{23}t) & \frac{1}{\Delta t_{33}} \end{bmatrix},$$

where $\Delta = \text{det}(\lambda - E\bullet)$,

$$t_{13} = 3F_{223} \lambda^4/4 + (-3F_{22}F_{223} + 2F_{222}F_{23} + 9F_{233} - 3F_{33}) \lambda^3/4$$

$$+ (-12F_{223}F_{23} - 9F_{22}F_{233} + 3F_{22}F_{33} + 9F_{222}F_{33} - 6F_{23}F_{233}t + 9F_{223}F_{33}t) \lambda^2/4$$

$$+ (-54F_{23}F_{233} + 27F_{223}F_{33} - 4F_{223}F_{33}^2 t + 6F_{23}F_{23}F_{233}t - 9F_{22}F_{223}F_{33}t$$

$$- 9F_{23}^2 t + 6F_{23}F_{33}(6 + F_{22}t)) /4 + (81F_{33}^2 + 36F_{23}^2 F_{233}t - 72F_{223}F_{23}F_{33}$$

$$- 12F_{23}F_{33}t + 9F_{22}F_{33}^2 t + 27F_{222}F_{33}^2 t)/4, \quad$$

$$t_{23} = -F_{223} \lambda^4/4 - 3F_{223} \lambda^3 + (-27F_{233}/4 + 2F_{223}F_{23} - 3F_{222}F_{33}/2) \lambda^2$$

$$+ (9F_{23}F_{33} t - 9F_{223}F_{33} t) \lambda + (-3F_{23}^2 F_{233} + 6F_{222}F_{23}F_{33} - 9F_{222}F_{33}^2/4) t^2, \quad$$

and

$$t_{33} = 3\lambda^4/4 - 3\lambda^3(F_{22} + 3F_{222} - 3F_{233})/4 - 3(36F_{223} + 12F_{23} + 3F_{22}F_{233} t$$

$$- 2F_{222}F_{23} t - 9F_{233} t - 3F_{33} t) \lambda^2/4 - 3(81F_{233} + 27F_{33} - 12F_{223}F_{23} t$$

$$- 4F_{23}^2 t + 9F_{22}F_{233} t + 3F_{222}F_{33} t + 9F_{222}F_{33} t + 6F_{23}F_{233} t^2 - 9F_{223}F_{33} t^2) \lambda/4$$

$$- 3(-54F_{23}F_{233} t + 81F_{223}F_{33} t + 4F_{223}F_{233} t^2 - 6F_{222}F_{23}F_{33} t^2 - 9F_{222}F_{233} F_{33} t^2$$

$$- 6F_{222}F_{23}F_{33} t^2)/4.$$ 

The homogeneous degree with respect to $Q, t, \lambda$ of the $i$th row of $T$ is $1 - i$.

**Proof.** Using that

$$\Omega_2 = \begin{bmatrix} 0 & F_{23} & F_{33} \\ 1 & F_{222} & F_{233} \\ 0 & 1 & 0 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} 0 & F_{233} & F_{33} \\ 1 & F_{223} & F_{233} \\ 0 & 1 & 0 \end{bmatrix}$$
and that $F$ is homogeneous of degree 1 we get
\[
\lambda - E \bullet = \Omega_2 = \begin{bmatrix} \lambda & -2F_{23} & -3F_{33} \\ -3 & \lambda - F_{22} & -2F_{23} \\ t & -3 & \lambda \end{bmatrix}.
\]

We can express $A$ in terms of the partial derivatives of $F$ and after some long but straightforward computation we get the formulas stated in the Lemma. \hfill $\Box$

Another long but straightforward computation yields that
\[
\det(T) = -\frac{3}{8\Delta^2} \Delta^{(1)},
\]
where
\[
\Delta^{(1)} = \lambda^3 + 3F_{223}t\lambda^2 + 3(3F_{233} + F_{33})t\lambda - 3(2F_{23}F_{233} - 3F_{223}F_{33})t^2.
\]

**Lemma 4.4.** a) There exists an operator
\[
L(Q, t, \lambda; \partial_Q) = L_0(Q, t, \lambda) + L_1(Q, t, \lambda) (Q\partial_Q) + L_2(Q, t, \lambda) (Q\partial_Q)^2
\]
whose coefficients are rational functions in $\lambda$ depending analytically on $(Q, t) \in B$ such that
\[
\partial_t Z(Q, t, \lambda) = L(Q, t, \lambda; \partial_Q) Z(Q, t, \lambda).
\]

b) The coefficients $L_i(Q, t, \lambda)$ have the form
\[
L_i(Q, t, \lambda) = \frac{1}{\Delta^{(1)}} \ell_i(Q, t, \lambda), \quad 0 \leq i \leq 2,
\]
where $\ell_i$ is a polynomial in $\lambda$ of degree $2 + i$ whose coefficients are polynomials in the partial derivatives of $F$. Moreover, the weight of $\ell_i$ with respect to the variables $Q, t, \lambda$ is 4.

**Proof.** Let $Z$ be the column with entries $Z_1, Z_2, Z_3$ and $\{e_i\}_{i=1}^3 \subset \mathbb{C}^3$ be the standard basis. We have
\[
\partial_t Z = \partial_t I^{(-1)}_e e_1 = I^{(-1)} A \Omega_3 e_1 = \text{Wr} T^{-1} A e_3.
\]
The 3rd column $A e_3$ of the matrix $A$ can be expressed in terms of the partial derivatives of $F$ as explained above. Therefore the coefficients of the differential operator are given by
\[
\begin{bmatrix} L_0 \\ L_1 \\ L_2 \end{bmatrix} = \frac{1}{\Delta} T^{-1} \begin{bmatrix} 3F_{33}\lambda/2 + (4F_{23}^2 - 3F_{22}F_{33})/2 \\ -F_{23}\lambda - 9F_{33}/2 \\ -3\lambda^2/2 + 3F_{22}\lambda/2 + 9F_{23} \end{bmatrix}.
\]
The rest of the proof is a straightforward computation. \hfill $\Box$

Let us point out that at $t = 0$ we have
\[
A = \frac{1}{\lambda^3 - 27Q} \begin{bmatrix} \lambda^2/2 & 9Q/2 & 3\lambda Q/2 \\ -3\lambda/2 & -\lambda^2/2 & -9Q/2 \\ -27/2 & -9\lambda/2 & -3\lambda^2/2 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 0 & 0 & Q \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Omega_3 = \Omega_2^2.
and

\[ T = \frac{1}{\lambda^3 - 27Q} \begin{bmatrix} 1 & 9Q/2 & \frac{3Q(2\lambda^3 + 27Q)}{4(\lambda^3 - 27Q)} \\ 0 & -\lambda^2/2 & -\frac{27Q\lambda^2}{4(\lambda^3 - 27Q)} \\ 0 & -9\lambda/2 & \frac{3\lambda(\lambda^3 - 108Q)}{4(\lambda^3 - 27Q)} \end{bmatrix}. \]

The differential operator takes the form

\[ L(Q, 0, \lambda, \partial_Q) = \lambda^{-2} \left( \frac{9Q}{2} + 36Q(Q\partial_Q) + 2(27Q - \lambda^3)(Q\partial_Q)^2 \right). \]

**Lemma 4.5.** At \( t = 0 \) the period map satisfies the following differential equation

\[ (Q\partial_Q)^3 Z(Q, \lambda) = \frac{3Q}{8(\lambda^3 - 27Q)} \left( 5 + 46(Q\partial_Q) + 108(Q\partial_Q)^2 \right) Z(Q, \lambda). \]

**Proof.** We just need to check that the substitution \( Z_i(Q, \lambda) = \frac{Q - 1}{6} z_i(x) \), with \( x = \frac{\lambda^3}{(27Q)} \) transforms the above differential equation into the generalized hypergeometric equation [5]. This however is a straightforward computation. \( \square \)

We would like to change the coordinates \((Q, \lambda) \in \mathbb{C}^* \times \mathbb{C}\) using the covering map \( \pi \) in Theorem 2.1, i.e.,

\[ \lambda = 2(2\pi/x)^2 E_4(\tau), \quad Q = \frac{8}{27} (2\pi/x)^6 (E_4(\tau)^3 - E_6(\tau)^2). \]

Recall the Ramanujan’s differential equations for the Eisenstein series

\[ q\partial_q E_2 = -\frac{1}{12} (E_2^2 - E_4), \]
\[ q\partial_q E_4 = \frac{1}{3} (E_2 E_4 - E_6), \]
\[ q\partial_q E_6 = \frac{1}{2} (E_2 E_6 - E_4^2). \]

**Lemma 4.6.** Under the change of coordinates (17) we have

\[ Q\partial_Q = \frac{E_4}{E_6} q\partial_q + \frac{1}{6E_6} (E_2 E_4 - E_6)x\partial_x. \]

**Proof.** After a short computation we get

\[ \partial_x \lambda = -4(2\pi)^2 x^{-3} E_4(\tau), \]
\[ \partial_x Q = -\frac{16}{9} (2\pi)^6 x^{-7} (E_4(\tau)^3 - E_6(\tau)^2) \]
\[ q\partial_q \lambda = \frac{2}{3} (2\pi/x)^2 (E_2 E_4 - E_6), \]
\[ q\partial_q Q = \frac{8}{27} (2\pi/x)^6 E_2(E_4^3 - E_6^2). \]

The formula for \( Q\partial_Q \) is easy to derive from here. \( \square \)
**Proposition 4.7.** Under the change of coordinates \([17]\) we have

a) The Taylor coefficient \(Z_3^{(0)}(Q, \lambda) = x\) and

\[
Z_3^{(n)}(Q, \lambda) \in x^{1-2n} E_6^{-2n} C[E_2, E_4, E_6], \quad n > 0.
\]

b) We have \(Z_2^{(0)}(Q, \lambda) = -2\tau x\) and

\[
Z_2^{(n)} + 2\tau Z_3^{(n)}(Q, \lambda) \in x^{1-2n} E_6^{-2n} C[E_2, E_4, E_6], \quad n > 0.
\]

**Proof.** Using Lemma 4.4 and 4.5 we get that

\[
Z_3^{(n)}(Q, \lambda) = M_3^{(n)}(Q, \lambda; \partial Q) Z(Q, \lambda),
\]

where \(M_3^{(n)}(Q, \lambda; \partial Q)\) is a second order differential operator of the form

\[
M_3^{(n)}(Q, \lambda; \partial Q) = \sum_{a=0}^{2} M_3^{(n)}(Q, \lambda) (Q \partial Q)^a
\]

whose coefficients are rational functions in \(Q\) and \(\lambda\) with poles only at \(\lambda = 0\) and \(\lambda^3 = 27Q = 0\). According to Theorem 2.1, under the change of coordinates \([17]\) we have

\[
Z^{(0)}(Q, \lambda) = Z(Q, \lambda) = (\tau^2 x, -2\tau x, x).
\]

Using Lemma 4.6 we also have

\[
M^{(n)} = M_0^{(n)} + M_1^{(n)} b + M_2^{(n)} (b^2 + a(q \partial_q b)) +
\left(M_1^{(n)} a + M_2^{(n)} (a(q \partial_q a) + 2ab) \right) q \partial_q + M_2^{(n)} a^2 (q \partial_q)^2,
\]

where

\[a := E_4 / E_6, \quad b := \frac{1}{6} (E_2 E_4 / E_6 - 1).\]

Note that in the above formula for \(M^{(n)}\) we have replaced \(x \partial_x\) with 1, because \(x \partial_x\) commutes with \(L\) and it acts on \(Z(Q, \lambda)\) by multiplication by 1. Hence

\[
Z_3^{(n)} = (M_0^{(n)} + M_1^{(n)} b + M_2^{(n)} (b^2 + a(q \partial_q b))) x
\]

and

\[
Z_2^{(n)} = -2Z_3^{(n)} \tau + \frac{1}{2\pi \sqrt{-1}} \left(M_1^{(n)} a + M_2^{(n)} (a(q \partial_q a) + 2ab) \right) x
\]

The statements of part a) and b), modulo the order of the poles at \(E_4 = 0\) and \(E_6 = 0\), follows from the fact that \(Z^{(n)}\) is homogeneous of degree \(n - \frac{1}{2}\), \(\tau\) has degree 0 and \(x\) has degree \(-\frac{1}{2}\). The statement that \(Z_3^{(n)}\) and \(Z_2^{(n)} + 2\tau Z_3^{(n)}\) do not have a pole at \(E_4 = 0\) and have a pole of order at most \(2n\) at \(E_6 = 0\) follows from Proposition 4.2. Indeed, according to the Proposition the series

\[
\sum_{n=0}^{\infty} Z^{(n)}(\tau, x) \frac{(E_6(\tau)^2 x - 6)^n}{n!}
\]
is convergent for all \((\tau, x, s) \in \mathcal{U}\), so in particular the coefficient in front of \(s^n\) must be holomorphic for all \((\tau, x) \in \mathbb{H} \times \mathbb{C}^*\) and for all \(n \geq 0\). □

The first component \(Z_1(Q, t, \lambda)\) of the period map is determined from the remaining two via the following relation.

**Lemma 4.8.** We have
\[
Z_2^2 - 4Z_1Z_3 = -32t.
\]

**Proof.** The argument is the same as in the proof of Lemma 3.1. Namely, we have
\[
\sum_{i,j=1}^{3} \eta_{ij} Z_i Z_j = -\frac{1}{8} (Z_2^2 - 4Z_1Z_3)
\]
and the same argument as in Lemma 3.1 proves that the LHS is the \((1,1)\)-entry of the matrix
\[
(-\theta + 1/2)^{-1} g(\lambda - E \bullet) (-\theta + 1/2)^{-1}.
\]
The entries of \(\lambda - E \bullet\) can be expressed in terms of the partial derivatives of the genus zero potential \(F\)
\[
\lambda - E \bullet = \begin{bmatrix} \lambda & -2F_{23} & -3F_{33} \\ -3 & \lambda - F_{22} & -2F_{23} \\ t & -3 & \lambda \end{bmatrix}.
\]
Note that the \((1,1)\)-entry of \(g(\lambda - E \bullet)\) is \(t\), so the \((1,1)\)-entry of (18) is \(4t\). □

4.3. **Extension of the period domain.** Recall that we have identified \(\mathbb{C}^3\) with the space of quadratic forms in two variables (see (5)). Let us define an open neighborhood of \(\Omega\) small in \(\mathbb{C}^3\) as the image of the following map:
\[
\Phi: \mathbb{H}^2 \times \mathbb{C}^* \to \mathbb{C}^3, (\tau_1, \tau_2, y) \mapsto (z_1, z_2, z_3) := \phi^{-1}(y(v - u\tau_1)(v - u\tau_2))
\]
Recalling the definition of \(\phi\) we get that
\[
z_1 = \tau_1 \tau_2 y, \quad z_2 = -(\tau_1 + \tau_2) y, \quad z_3 = y.
\]

Let us equip \(\mathbb{H}^2 \times \mathbb{C}^*\) with a left \(W\)-action. If \(w = (g, \sigma) \in W = \text{PSL}(2, \mathbb{Z}) \times \{\pm 1\}\), then we define
\[
w \cdot (\tau_1, \tau_2, y) := \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a\tau_2 + b}{c\tau_1 + d}, \sigma c\chi_2(g)(c\tau_1 + d)(c\tau_2 + d)y\right).
\]

**Lemma 4.9.** Let \(w \in \text{GL}(\mathbb{C}^3)\) be the monodromy transformation corresponding to an element \((g, \sigma) \in \text{PSL}(2, \mathbb{Z}) \times \{\pm 1\}\) via the map (7). Then
\[
\Phi(\tau_1, \tau_2, y) \cdot w = \Phi((sg^{-1}s^{-1}, \sigma) \cdot (\tau_1, \tau_2, y))
\]
where \(s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) and \((\tau_1, \tau_2, y) \in \mathbb{H}^2 \times \mathbb{C}^*\).
Proof. We prove that the quadratic forms corresponding to the LHS and the RHS (of the identity that we have to prove) coincide. The quadratic form corresponding to the LHS is
\begin{equation}
\sigma \chi_2(g)(cu + dv - (au + bv)\tau_1)(cu + dv - (au + bv)\tau_2),
\end{equation}
where \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Note that
\[
\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = s \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} s^{-1}
\]
and \( \chi_2(sg^{-1}s^{-1}) = \chi_2(g) \). Therefore, the quadratic form corresponding to the RHS is
\begin{equation}
\sigma \chi_2(g)(-b\tau_1 + d)(-b\tau_2 + d)y\left( v - u \frac{a\tau_1 - c}{-b\tau_1 + d} \right) \left( v - u \frac{a\tau_2 - c}{-b\tau_2 + d} \right).
\end{equation}
It remains only to verify that the above formula coincides with (20). \( \square \)

4.4. Proof of Theorem 2.2. We would like to invert the period map
\[ z_i = Z_i(Q, t, \lambda), \quad 1 \leq i \leq 3, \]
i.e., express \( (Q, t, \lambda) \) in terms of \( (z_1, z_2, z_3) \). Recall that
\( (Q, t, \lambda) = \pi_{aux}(\tau, x, t), \quad (z_1, z_2, z_3) = \Phi(\tau_1, \tau_2, y) \).
Since the inverse of \( \Phi \) is straightforward to find, it is sufficient to find the relation between the coordinate systems \( (\tau, x, t) \) and \( (\tau_1, \tau_2, y) \).
To begin with note that according to Lemma 4.8 we have
\[ t = -\frac{1}{32}(\tau_1 - \tau_2)^2 y^2. \]
According to Proposition 4.7 we have
\[ y = z_3 = x\left( 1 + \sum_{n=1}^{\infty} y_n(E_2, E_4, E_6)(tx^{-2})^n \right) \]
and
\[ \tau_{12} := \frac{1}{2}(\tau_1 + \tau_2) = -\frac{z_2}{2z_3} = \tau + \sum_{n=1}^{\infty} \tau_{12,n}(E_2, E_4, E_6)(tx^{-2})^n, \]
where \( y_n, \tau_{12,n} \in \mathbb{C}[E_2, E_4, E_6]E_6^{-2n} \). Using the formula for \( y \) we can express \( x \) in terms of \( y, t \), and \( E_i \) (\( i = 2, 4, 6 \)):
\[ x = y\left( 1 + \sum_{n=1}^{\infty} x_n(E_2, E_4, E_6)(ty^{-2})^n \right), \]
where \( x_n \in \mathbb{C}[E_2, E_4, E_6]E_6^{-2n} \). Substituting this into the formula for \( \tau_{12} \) we get
\[ \tau_{12} = \tau + \sum_{n=1}^{\infty} \tau_{12,n}(E_2, E_4, E_6)(ty^{-2})^n, \]
where \( \tau_{12,n} \in \mathbb{C}[E_2, E_4, E_6]E_6^{-2n} \). Using Taylor series expansion at \( \tau_{12} = \tau \) and the Ramanujan’s differential equations. We get

\[
E_i(\tau_{12}) = E_i(\tau) + \sum_{n=1}^{\infty} \tau_{12,n}^{(i)}(E_2, E_4, E_6)(ty^{-2})^n, \quad i = 2, 4, 6.
\]

Therefore, we can express \( E_i(\tau) \) \((i = 2, 4, 6)\) in terms of \( E_i(\tau_{12}) \) \((i = 2, 4, 6)\)

\[
E_i(\tau) = E_i(\tau_{12}) + \sum_{n=1}^{\infty} \tau_{n}^{(i)}(E_2(\tau_{12}), E_4(\tau_{12}), E_6(\tau_{12}))(ty^{-2})^n.
\]

Since \( ty^{-2} = - (\tau_1 - \tau_2)^2/32 \) we get inversion formulas of the following type

\[
Q = \frac{8}{27} (2\pi/y)^6 \sum_{n=0}^{\infty} Q_n(\tau_{12})(\tau_1 - \tau_2)^{2n},
\]

\[
\lambda = 2(2\pi/y)^2 \sum_{n=0}^{\infty} \lambda_n(\tau_{12})(\tau_1 - \tau_2)^{2n},
\]

\[
t = - \frac{1}{32} (\tau_1 - \tau_2)^2 y^2,
\]

where \( Q_n \) and \( \lambda_n \) are polynomial expression in \( E_2(\tau_{12}), E_4(\tau_{12}), \) and \( E_6(\tau_{12})^{\pm 1} \). We have to prove that \( Q_n \) and \( \lambda_n \) depend polynomially on \( E_i, \) i.e., there is no negative powers of \( E_6. \) This follows from the fact that the period map for the second structure connection is locally invertible.

**Lemma 4.10.** a) The value of the Jacobian determinant

\[
\frac{D(Z_1, Z_2, Z_3)}{D(\lambda, Q; t)} = \left[ \begin{array}{ccc}
\partial_{\lambda} Z_1 & \partial_{\lambda} Z_2 & \partial_{\lambda} Z_3 \\
Q \partial_Q Z_1 & Q \partial_Q Z_2 & Q \partial_Q Z_3 \\
\partial_{\lambda} Z_1 & \partial_{\lambda} Z_2 & \partial_{\lambda} Z_3
\end{array} \right].
\]

at \( t = 0 \) up to a non-zero constant coincides with \((\lambda^3 - 27Q)^{-1/2}\).

b) The value of the Jacobian determinant

\[
\frac{D(\lambda, Q, t)}{D(\tau, x, s)} = \left[ \begin{array}{ccc}
\partial_{\tau} \lambda & \partial_{\tau} Q & \partial_{\tau} t \\
\partial_x \lambda & \partial_x Q & \partial_x t \\
\partial_s \lambda & \partial_s Q & \partial_s t
\end{array} \right]
\]

at \( s = 0 \) up to a non-zero constant coincides with \( E_6(\tau)^2 x^{-6} Q(\lambda^3 - 27Q)^{1/2} \).

**Proof.** We will use the notation from section 4.2. Using the differential equations of the second structure connection we get

\[
\partial_{\lambda} Z_i = (I_1^{(0)}, 1), \quad Q \partial_Q Z_i = -(I_1^{(0)}, P), \quad \partial_{\lambda} Z_i = -(I_1^{(0)}, P^2).
\]

Therefore

\[
\det \frac{D(Z_1, Z_2, Z_3)}{D(\lambda, Q; t)} = \det I^{(0)} = \det (I^{(-1)}(-\theta + 1/2)(\lambda - E\bullet)^{-1}).
\]
The above expression should be evaluated at \( t = 0 \). We get
\[-\frac{3}{8(\lambda^3 - 27Q)} \det I^{(-1)}.\]
By definition \( \text{Wr} = I^{(-1)} T \) and the \( T(Q, 0, \lambda) \) is given by (16). Therefore,
\[\det I^{(-1)} = -\frac{8}{3} \lambda^{-3}(\lambda^3 - 27Q)^2 \det \text{Wr}.\]
The Jacobian determinant takes the form
\[\det \frac{D(Z_1, Z_2, Z_3)}{D(\lambda, Q, t)} = \lambda^{-3}(\lambda^3 - 27Q) \det \text{Wr}.\]
Recall that \( Z_i(Q, \lambda) = Q^{-1/6} z_i(x), \) with \( x = \lambda^3/(27Q) \). Therefore, the Wronskian determinant takes the form
\[-Q^{-1/2} \det \begin{bmatrix}
z_1 & (x \partial_x + 1/6)z_1 & (x \partial_x + 1/6)^2 z_1 \\
z_2 & (x \partial_x + 1/6)z_2 & (x \partial_x + 1/6)^2 z_2 \\
z_3 & (x \partial_x + 1/6)z_3 & (x \partial_x + 1/6)^2 z_3 \\
\end{bmatrix},\]
i.e.,
\[\det \text{Wr} = -Q^{-1/2} \det \begin{bmatrix}
z_1 & x \partial_x z_1 & (x \partial_x)^2 z_1 \\
z_2 & x \partial_x z_2 & (x \partial_x)^2 z_2 \\
z_3 & x \partial_x z_3 & (x \partial_x)^2 z_3 \\
\end{bmatrix}.\]
On the other hand, \( \{z_i\}_{i=1}^3 \) form a basis of solutions of the hypergeometric equation [8]. Therefore, the above determinant can be expressed easily in terms of the Wronskian of the differential equation. After a short computation we get
\[\det \begin{bmatrix}
z_1 & x \partial_x z_1 & (x \partial_x)^2 z_1 \\
z_2 & x \partial_x z_2 & (x \partial_x)^2 z_2 \\
z_3 & x \partial_x z_3 & (x \partial_x)^2 z_3 \\
\end{bmatrix} = C x(1 - x)^{-3/2} = 3\sqrt{3} C \sqrt{-1} \frac{Q^{1/2} \lambda^3}{(\lambda^3 - 27Q)^{3/2}},\]
where \( C \) is a non-zero constant. The precise value of \( C \) is irrelevant, but for the sake of completeness, let us compute it. Using the connection formulas in Section 3.3 we can compute the leading order term of the Wronskian near \( x = \infty \)
\[\det \begin{bmatrix}
z_1 & x \partial_x z_1 & (x \partial_x)^2 z_1 \\
z_2 & x \partial_x z_2 & (x \partial_x)^2 z_2 \\
z_3 & x \partial_x z_3 & (x \partial_x)^2 z_3 \\
\end{bmatrix} = -\frac{16}{3\sqrt{6}} \sqrt{-1} x^{-1/2} + O(x^{-3/2}).\]
We get \( C = -\frac{16}{3\sqrt{6}}. \) Finally, for the Jacobian determinant we get
\[\det \frac{D(Z_1, Z_2, Z_3)}{D(\lambda, Q, t)} = 8\sqrt{-2} (\lambda^3 - 27Q)^{-1/2}.\]
b) This is an elementary consequence of Ramanujan’s differential equations. \( \square \)

**Proposition 4.11.** The coefficients
\[Q_n, \lambda_n \in \mathbb{C}[E_2(\tau_{12}), E_4(\tau_{12}), E_6(\tau_{12})],\]
i.e., they are quasi-modular forms with respect to \( \tau_{12} \in \mathbb{H}. \)
Proof. Using Proposition 4.2 and Lemma 4.10 b) we get that the map \( Z^{\text{aux}} : \mathcal{U} \rightarrow \mathbb{C}^3 \) induces an isomorphism between an open neighborhood in \( \mathbb{H} \times \mathbb{C}^* \times \mathbb{C} \) of

\[
(\mathbb{H} \times \mathbb{C}^*)' := \{ (\tau, x) \in \mathbb{H} \times \mathbb{C}^* \mid E_0(\tau) \neq 0 \}
\]

and an open neighborhood in \( \mathbb{C}^3 \) of

\[
\Omega'_{\text{small}} = \{ z^* \in \Omega_{\text{small}} \mid E_0(-z_2^*/(2z_3^*)) \neq 0 \}.
\]

In particular, the coordinates \((\tau, x, s)\) and \((z_1, z_2, z_3)\) of the two neighborhoods are biholomorphic. Therefore

\[
\lambda := 2(2\pi/x)^2 E_4(\tau), \quad Q := \frac{8}{27} (2\pi/x)^6 (E_4(\tau)^3 - E_6(\tau)^2), \quad t := -\frac{1}{32} (z_2^3 - 4z_1z_3)
\]

define functions that are holomorphic in an open neighborhood in \( \mathbb{C}^3 \) of \( \Omega'_{\text{small}} \). Note that by definition the functions \((Q, t, \lambda)\) give an inversion of the period map. More precisely, if \((Q^*, t^*, \lambda^*) \in X\) with \( t^* = 0 \) and \( z^* = Z(Q^*, t^*, \lambda^*) \in \Omega_{\text{small}} \) is a value of the period map (depending on the choice of a reference path), then in a neighborhood of \( z^* \in \mathbb{C}^3 \) the functions \((Q, t, \lambda)\) coincide with the unique solution to the equations

\[
Z_i(Q, t, \lambda) = z_i, \quad 1 \leq i \leq 3,
\]

where the branch of \( Z_i \) is fixed by \( Z_i(Q^*, t^*, \lambda^*) = z_i^* \).

Clearly \( t \) is a holomorphic function on \( \mathbb{C}^3 \). We claim that \( \lambda \) and \( Q \) extend to holomorphic functions defined in a neighborhood of \( \Omega_{\text{small}} \) in \( \mathbb{C}^3 \). The statement is local, so let \((Q^*, t^*, \lambda^*)\) with \( t^* = 0 \) be a point on the discriminant and let \( z^* = Z(Q^*, t^*, \lambda^*) \in \Omega_{\text{small}} \) be a value of the period map. Let \( \lambda = u(Q, t) \) be the local equation of the discriminant at the point \((Q^*, t^*, \lambda^*)\). Locally, the components of the period map can be written as

\[
Z_i(Q, t, \lambda) = \frac{1}{2} (E_i|\alpha) Z_\alpha(Q, t, \lambda) + Z_i^{\text{inv}}(Q, t, \lambda)
\]

where \( \alpha \in H \) is a vector whose local monodromy around the discriminant is given by \( \alpha \mapsto -\alpha \), \( Z_\alpha := \langle Z, \alpha \rangle \), and \( Z_i^{\text{inv}} \) corresponds to the invariant part of \( E_i \), i.e.,

\[
Z_i^{\text{inv}} = \langle Z, E_i - (E_i|\alpha)\alpha/2 \rangle.
\]

On the other hand

\[
Z_\alpha(Q, t, \lambda) = (\lambda - u(Q, t))^{1/2} \tilde{Z}_\alpha(Q, t, \lambda)
\]

where \( \tilde{Z}_\alpha(Q, t, \lambda) \) is holomorphic in a neighborhood of \((Q^*, t^*, \lambda^*)\) and \( \tilde{Z}_\alpha(Q^*, t^*, \lambda^*) \neq 0 \). Let us choose \( i \) such that \((E_i|\alpha) \neq 0 \). Then we get

\[
\lambda = u(Q, t) + \mu^2, \quad \mu := \frac{2}{(E_i|\alpha)} (Z_i - Z_i^{\text{inv}})/\tilde{Z}_\alpha.
\]

The above equation defines a branched double covering of a neighborhood of \((Q^*, t^*, \lambda^*)\) and \((Q, t, \mu)\) is a holomorphic coordinate system on the double cover.
We claim that the local lift of the period map is an isomorphism. Indeed, the local lift is a single valued analytic map, because
\[ Z_j = \frac{1}{2} (E_j | \alpha) \mu \tilde{Z}_\alpha + Z_j^{inv}, \quad 1 \leq j \leq 3. \]

We have to check that the corresponding Jacobian determinant does not vanish at the point \((Q^*, t^*, \mu^*) = (Q^*, 0, 0)\). Using
\[
\frac{D(Q, t, \lambda)}{D(Q, t, \mu)} = 2\mu = 2(\lambda - u(Q, t))^{1/2},
\]
the chain rule, and Lemma 4.10, a) we get
\[
\frac{D(Z_1, Z_2, Z_3)}{D(Q, t, \mu)}(Q^*, 0, 0) = \frac{2C}{\sqrt{(1 - \zeta)(1 - \zeta^2)}} \frac{1}{u(Q^*, 0)} \neq 0,
\]
where \(C\) is a non-zero constant, \(\zeta = e^{2\pi \sqrt{-1}/3}\), and we used that
\[
\lambda^3 - 27Q^* = \prod_{a=0}^2 (\lambda - \zeta^a u(Q^*, 0)).
\]
Therefore, \(Q, t, \mu\) are holomorphic in a neighborhood of \(z^* \in \mathbb{C}^3\), which implies that \(Q, t, \lambda\) are also holomorphic.

To complete the proof of the proposition we note that
\[
\tau_{12} = -\frac{z_2}{2z_3}, \quad (\tau_1 - \tau_2)^2 = (z_2/z_3)^2 - 4(z_1/z_3), \quad y = z_3.
\]
Therefore, \(Q, t, \lambda\) must be holomorphic functions in \(y, \tau_{12}, (\tau_1 - \tau_2)^2\). In particular \(Q_n(\tau_{12})\) and \(\lambda_n(\tau_{12})\) must be holomorphic in \(\tau_{12}\), so the corresponding polynomial expressions in \(E_2(\tau_{12}), E_4(\tau_{12}),\) and \(E_6(\tau_{12})^{\pm 1}\) could not have negative powers of \(E_6(\tau_{12})\). \(\square\)

4.5. The ring of modular functions. Note that the ring \(\Gamma(\Omega_{\text{small}}, \mathcal{O}_{\mathbb{C}^3})\) is equipped with the action of the monodromy group \(W\). Let \(\Gamma(\Omega_{\text{small}}, \mathcal{O}_{\mathbb{C}^3})^W\) be the ring of \(W\)-invariant functions. We introduce a subring of \(W\)-invariant functions as follows. Let \(\mathbb{C}[Q, \lambda]\{t\}\) be the ring of power series in \(t\) whose coefficients depend polynomially on \(Q\) and \(\lambda\) and such that for every \((Q, \lambda) \in \mathbb{C}^* \times \mathbb{C}\) the radius of convergence is non-zero. Then we define
\[
\mathcal{M}(\Omega, W) = \{ f \in \Gamma(\Omega_{\text{small}}, \mathcal{O}_{\mathbb{C}^3})^W | f \circ Z \in \mathbb{C}[Q, \lambda]\{t\} \}.
\]
Proposition 4.11 implies that the tautological map
\[
\mathcal{M}(\Omega, W) \rightarrow \mathbb{C}[Q, \lambda]\{t\}, \quad f \mapsto f \circ Z
\]
is an isomorphism. Although the invariant functions corresponding to \(Q\) and \(\lambda\) can be find recursively, it will be nice to have a more intrinsic characterization. Unfortunately we could not achieve this goal. On the other hand we have managed
Table 1. Transformation rules

|       | $\tau \mapsto \tau + 1$ | $\tau \mapsto -1/\tau$ |
|-------|--------------------------|--------------------------|
| $\theta_{00}(\tau)$ | $\theta_{01}(\tau)$ | $(-i\tau)^{1/2}\theta_{00}(\tau)$ |
| $\theta_{01}(\tau)$ | $\theta_{00}(\tau)$ | $(-i\tau)^{1/2}\theta_{10}(\tau)$ |
| $\theta_{10}(\tau)$ | $e^{2\pi i/8}\theta_{10}(\tau)$ | $(-i\tau)^{1/2}\theta_{01}(\tau)$ |

to find explicitly invariant functions $E_4^{(2)}, \Delta^{(2)}$ that generate $\mathcal{M}(\Omega, W)$ in the following sense. If $f \in \mathcal{M}(\Omega, W)$ then

$$f = \sum_{n=0}^{\infty} c_n (E_4^{(2)}, \Delta^{(2)}) t^n,$$

where $c_n \in \mathbb{C}[E_4^{(2)}, \Delta^{(2)}]$ are some polynomials. The above equality should be interpreted as equality between formal power series in $t$.

In order to find such functions $E_4^{(2)}$ and $\Delta^{(2)}$ it is enough to construct two $W \times \mu_2$-invariant holomorphic functions in $\mathbb{H}^2 \times \mathbb{C}^*$ whose restrictions to $\mathbb{H} \times \mathbb{C}^*$ coincide with

$$8 \frac{2\pi}{27} (2\pi/z)^6 (E_4(\tau)^3 - E_6(\tau)^2) \quad \text{and} \quad 2 (2\pi/z)^2 E_4(\tau).$$

This could be done easily using the Jacobi theta constants

$$\theta_{ab}(0, \tau) = \sum_{n \in \mathbb{Z}} \exp \left( \pi \sqrt{-1} \left( (n + a/2)^2 \tau + (n + a/2)b \right) \right), \quad ab = 00, 01, 10.$$

For the reader’s convenience we have recorded in Table 1 the transformation rules for the theta constants under the two modular transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. For more details we refer to [7]. It is easy to check that

$$E_4(\tau) = \frac{1}{2} (\theta_{00}(\tau)^8 + \theta_{10}(\tau)^8 + \theta_{01}(\tau)^8)$$

$$E_4(\tau)^3 - E_6(\tau)^2 = \frac{27}{4} (\theta_{00}(\tau)\theta_{10}(\tau)\theta_{01}(\tau))^8.$$

Let us define

$$E_4^{(2)}(\tau_1, \tau_2, x) := (2\pi/x)^2 \sum_{ab \in \{00, 01, 10\}} \theta_{ab}(\tau_1)^4 \theta_{ab}(\tau_2)^4$$

and

$$\Delta^{(2)}(\tau_1, \tau_2, x) := 2(2\pi/x)^6 \prod_{ab \in \{00, 01, 10\}} \theta_{ab}(\tau_1)^4 \theta_{ab}(\tau_2)^4.$$
It is straightforward to check that $E_4^{(2)}$ and $\Delta^{(2)}$ are $W \times \mu_2$-invariant holomorphic functions on $\mathbb{H}^2 \times \mathbb{C}^*$, so they define $W$-invariant analytic functions on the domain $\mathbb{H}^2 \times \mathbb{C}^*/\mu_2$. In particular $E_4^{(2)}, \Delta^{(2)} \in \mathcal{M}(\Omega, W)$.

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