Differential and complex geometry of
two-dimensional noncommutative tori

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Abstract

We analyze in detail projective modules over two-dimensional noncommutative tori and complex structures on these modules. We concentrate our attention on properties of holomorphic vectors in these modules; the theory of these vectors generalizes the theory of theta-functions. The paper is self-contained; it can be used also as an introduction to the theory of noncommutative spaces with simplest space of this kind thoroughly analyzed as a basic example.

1 Introduction

Differential geometry of noncommutative spaces, in particular, of noncommutative tori was developed by A. Connes [1], [2]. More detailed analysis of differential geometry of noncommutative tori was performed in [3], [4], [14], [15], [16], [17], [6].

Partly, the interest to this subject was motivated by the applications to string/M-theory found in [4] (see [7] for review). Complex geometry of noncommutative tori and of projective modules over them was studied in [18] in connection with noncommutative generalization of theta-functions.

The goal of present paper is to illustrate general results of [18] using the example of two-dimensional noncommutative tori, where all calculations can be performed explicitly. We repeat all basic definitions, starting with the definition of noncommutative torus. Therefore, the paper can be read independently of [18] and of other papers about noncommutative tori. However, in the rest of the introduction, we assume some knowledge of preceding work.

In Sec. 2 we describe projective modules over two-dimensional noncommutative torus and their tensor products. The results of this section are closely

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related to some results proved in multidimensional case in [18]. We will formulate these general results restricting ourselves to basic $T_\theta$-modules (modules with constant curvature connection where the endomorphism algebra is another noncommutative torus $T_\theta'$). Instead of basic modules we can work with corresponding $(T_\theta', T_\theta)$-bimodules. Tensor product of $(T_\theta, T_\theta')$-bimodule $E_1$ and $(T_\theta, T_\theta')$-bimodule $E_2$ is a $(T_\theta', T_\theta')$-bimodule $E$. We can find the bimodule $E$ using the fact that basic bimodules $E_1$ and $E_2$ can be described by means of elements $g_1, g_2 \in SO(d, d, \mathbb{Z})$ obeying $\theta' = g_1 \theta, \theta'' = g_2 \theta$. Here $g_1, g_2$ act on the parameter of noncommutativity - antisymmetric matrix $\theta$ - by means of fractional linear transformation. The description of $E$ follows from the relation $\theta'' = g_2 g_1^{-1} \theta'$. In two-dimensional case all basic right $T_\theta$-modules can be represented as modules $E_{n,m}$ (see Sec. 2) where $m$ and $n$ are relatively prime. The space $E_{n,m}$ can be considered also as a left $T_\theta$-module $E'_{a,m}$ where $an - bm = 1, a, b \in \mathbb{Z}, \theta' = (a \theta + b)(m \theta + n)^{-1}$, or as $(T_\theta', T_\theta)$-bimodule that will be denoted by $E_1$. One can use general results to calculate the tensor product $E_1 \otimes T_\theta E_2$ where $E_2$ is a $(T_\theta, T_\theta')$-bimodule, corresponding to the basic $T_\theta$-module $E'_{a,k}$. It follows from these results that this tensor product considered as a left $T_\theta$-module is isomorphic to $E'_{ak+bl, nl+mk}$. We give a direct proof of this relation constructing an explicit isomorphism. This construction is used essentially in Sec. 3, devoted to complex geometry of projective modules over two-dimensional noncommutative tori. We calculate holomorphic vectors (theta-vectors) in basic modules and tensor products of these vectors. The relations we obtain generalize well known relations for theta-functions. The appendix contains a very detailed calculation of the tensor product (we did not include the appendix into the journal version). We did not try to analyze the connection of our results to Manin’s version of the theory of noncommutative theta–functions [3 – [10]. This is an interesting problem.

2 Projective modules and their tensor products.

One can define the algebra $T^d_\theta$ of smooth functions on $d$-dimensional noncommutative torus as an algebra of formal linear combinations $f = \sum C_{\vec{n}} U_{\vec{n}}$ where $\vec{n} \in \mathbb{Z}^d$ and $C_{\vec{n}}$ are complex numbers tending to zero at infinity faster then any power and the multiplication is governed by the rule

$$U_{\vec{n}} U_{\vec{m}} = e^{\pi i \vec{n} \cdot \vec{m}} U_{\vec{n} + \vec{m}}$$

where $\theta = \theta^{\alpha \beta}$ is a real antisymmetric matrix. An antilinear involution on $T^d_\theta$ is defined by the requirement $U_{\vec{n}}^* = U_{-\vec{n}}$. These operations together with the standard structure of vector space permit us to consider $T^d_\theta$ as involutive associative algebra with unit element $1 = U_{\vec{0}}$ and with trace $\text{Tr} f = C_{\vec{0}}$. We will fix our attention on the case $d = 2$; in this case the matrix $\theta^{\alpha \beta}$ can be specified by means of one number $\theta^{12} = \theta$ and the multiplication is specified by the relation
\[ U_1 U_2 = e^{2\pi i \theta} U_2 U_1 \]

where \( U_\alpha = U_{\vec{e}_\alpha} \) are elements \( U_{\vec{e}_\alpha} \) corresponding to vectors of standard basis \( \vec{e}_1 = (1, 0), \quad \vec{e}_2 = (0, 1) \).

In the case when \( \theta \) is a natural number \( T^2_\theta \) is isomorphic to the algebra of smooth functions on the commutative two-dimensional torus. We will consider projective modules over \( T^2_\theta \) (we always assume that our modules are finitely generated; a projective module is by definition a direct summand in a free module \( (T^2_\theta)^k \)). We assume that the number \( \theta \) is irrational; in this case every projective right module over \( T^2_\theta \) is isomorphic to one of the modules \( E_{n,m} \) defined in the following way (see [2], [3], [7]). The elements of \( E_{n,m} \) are functions on the Schwartz space \( S(\mathbb{R} \times \mathbb{Z}_m) \) and the action of the generators \( U_1, U_2 \) of \( T^2_\theta \) is given by:

\[
U_1 f(x, \mu) = f(x - \frac{n + m \theta}{m}, \mu - 1)
\]

\[
U_2 f(x, \mu) = e^{2\pi i (x - \mu n/m)} f(x, \mu)
\]

We can define also left modules over \( T^2_\theta \) replacing \( \theta \) by \(-\theta\) in the above formulas; we will use the notation \( E'_{n,m} \) for left modules. If it is necessary to emphasize that we consider \( E_{n,m} \) or \( E'_{n,m} \) as a \( T_\theta \)-module we will use the notation \( E_{n,m}(\theta) \) or \( E'_{n,m}(\theta) \). We will assume that the numbers \( m \) and \( n \) are relatively prime. Corresponding modules \( E_{n,m} \) are called basic modules. Every module over \( T^2_\theta \) can be represented as a direct sum of isomorphic basic modules. In this case the algebra of endomorphisms of the module \( E = E_{n,m} \) is again a noncommutative torus \( T^2_{\theta'} \) generated by operators

\[
Z_1 f(x, \mu) = f(x - \frac{1}{m}, \mu - a)
\]

\[
Z_2 f(x, \mu) = \exp \left[ 2\pi i \left( \frac{x}{n + m \theta} - \frac{\mu}{m} \right) \right] f(x, \mu)
\]

where \( a, b \) are integers obeying \( an - bm = 1 \). This fact follows from relation

\[
Z_1 Z_2 = e^{-2\pi i \theta'} Z_2 Z_1
\]

where \( \theta' = \frac{k + a \theta}{n + m \theta} \). One can consider \( E \) as a left \( T^2_{\theta'} \)-module; this module is isomorphic to \( E_{n,m} \).

We can regard \( E \) also as \( (T^2_{\theta'}, T^2_\theta) \)-bimodule, since the action of \( T^2_{\theta'} \) commutes with the action of \( T_\theta \). Notice that the bimodule \( E \) depends on the choice of \( a \) and \( b \). The tori \( T^2_\theta \) and \( T^2_{\theta'} \) are Morita equivalent and the bimodule \( E \) is a Morita equivalence bimodule ([2], [3], [7]).

Recall that two associative algebras \( A \) and \( B \) are Morita equivalent if corresponding categories of modules are equivalent. Having an \( (A, B) \)-bimodule \( P \) we can assign to every right \( A \)-module \( E \) a right \( B \)-module \( E' = E \otimes_A P \) where tensor product over \( A \) is obtained from the standard tensor product over \( \mathbb{C} \) by
means of identification $ea \otimes p \sim e \otimes ap$. If this correspondence is invertible we say that $P$ is a Morita equivalence bimodule.

Let us calculate now the tensor product $E_{n,m} \otimes_{T_\theta} E'_{k,l}$ where $E_{n,m}$ is a right $T_\theta^2$-module and $E'_{k,l}$ is a left $T_\theta^2$-module. We will see that this product can be considered a vector space $E = \mathcal{S}(\mathbb{R} \times \mathbb{Z}_{nl+mk})$. Namely, the formula:

$$h(z, \Delta) = \sum_{q \in \mathbb{Z}} \left[ f \left( (n + m \theta)z - \frac{n + m \theta}{m}q + \frac{l(n + m \theta)}{mnl + mk} \Delta, -q + a \Delta \right) \cdot g \left( (n + m \theta)z + \frac{k - l \theta}{l}q - \frac{k - l \theta}{n + m \theta} \Delta, q \right) \right]$$

(1)

where $0 \leq \Delta \leq nl + mk - 1$, specifies a map of tensor product over $E_{n,m} \otimes_{T_\theta} E'_{k,l}$ onto $E$, so that, with the usual notation, $f \in E_{n,m}(\theta)$, $g \in E'_{k,l}(\theta')$ and $h \in E_{n,m+k,l,n+l,m+k}(\theta'')$. This map is an isomorphism between the tensor product and $E$. The formula (1) determines a bilinear map that is compatible with identification $ea \otimes p \sim e \otimes ap$ (see Appendix for details). We can consider $E_{n,m}$ as a $(T_\theta^2, T_\theta^2)$-bimodule where $\theta'' = \frac{b + a \theta}{n + m \theta}$, $an - bm = 1$ and $E_{k,l}$ as a $(T_\theta, T_\theta^2)$-bimodule where $\theta' = -\frac{c + d \theta}{k - l \theta}$, $ck - dl = 1$. Then $E$ can be regarded as $(T_\theta', T_\theta'^2)$-bimodule. It follows from (1) that $E$ considered as a left $T_\theta'$-module is isomorphic to $E'_{N', M}$ where $M = nl + mk$, $N' = ak + bl$ and $E$ considered as a right $T_\theta'^2$-module is isomorphic to $E_{N', M}$ where $N'' = -(cn + md)$ for $ck - dl = 1$. This result can be obtained also from general considerations (see Introduction).

Let us apply (1) to the special case when:

$$f(x, \mu) = e^{-\frac{1}{2} \sigma_1 x^2 - c_1 x \delta_\alpha^\mu}$$
$$g(y, \nu) = e^{-\frac{1}{2} \sigma_2 y^2 - c_2 y \delta_\beta^\nu}$$

where

$$\sigma_1 = i \tau_1 \frac{m}{n + m \theta}, \quad \mu \in \mathbb{Z}_m$$
$$\sigma_2 = i \tau_2 \frac{l}{k - l \theta}, \quad \nu \in \mathbb{Z}_l$$

and $\delta_{ij}$ is the usual Kronecker–delta. It is straightforward to check that in this
case:

$$h_{\alpha \beta}(z, \Delta) = \sum_{q \in \mathbb{Z}} \left[ \exp \left\{ -\frac{\sigma_1}{2} \left( (n + m \theta)z - \frac{n + m \theta}{m}q + \frac{l(n + m \theta)}{m(n l + m k)} \Delta \right)^2 \right. \right.

-c_1 \left( (n + m \theta)z - \frac{n + m \theta}{m}q + \frac{l(n + m \theta)}{m(n l + m k)} \Delta \right) \left. \delta_{q-a \Delta} \right. \left. \right] \cdot \exp \left\{ -\frac{\sigma_2}{2} \left( (n + m \theta)z + \frac{k-l \theta}{l}q - \frac{k-l \theta}{nl + mk} \Delta \right)^2 \right. \right.

-c_2 \left( (n + m \theta)z + \frac{k-l \theta}{l}q - \frac{k-l \theta}{nl + mk} \Delta \right) \left. \delta_{q \mod l} \right] \right]$$

Notice that the right hand side of (2) can be expressed in terms of theta–functions; first note that the set of solutions to the system of congruences:

\[
\begin{align*}
q &= a \Delta - \alpha \pmod{m} \\
q &= \beta \pmod{l}
\end{align*}
\]

can be written as \(q_o + u \frac{ml}{r}\) for some integer \(q_o, r = \gcd(m, l)\), and \(u \in \mathbb{Z}\). Hence we can substitute \(q_o + u \frac{ml}{r}\) in for \(q\) in (2), do away with Kronecker deltas, and sum over \(u\) instead of \(q\). We obtain:

$$h_{\alpha \beta}(z, \Delta) = \Theta(s, t) \cdot \xi(z, \Delta)$$

where

$$\xi_{\alpha \beta}(z, \Delta) = \exp \left[ -\frac{\sigma_1}{2} \left( (n + m \theta) z + \frac{l(n + m \theta)}{m(n l + m k)} \Delta \right)^2 - \frac{\sigma_2}{2} \left( (n + m \theta) z - \frac{(k-l \theta)}{n l + m k} \Delta \right)^2 \right]

-(c_1 + c_2) (n + m \theta) z - \left( c_1 \frac{l(n + m \theta)}{m(n l + m k)} - c_2 \frac{k-l \theta}{n l + m k} \right) \Delta

- \left( \sigma_1 \left( \frac{n + m \theta}{m} \right)^2 + \sigma_2 \left( \frac{k-l \theta}{l} \right)^2 \right) \frac{q_o^2}{2} \]$$

\[s = -\frac{\sigma_1 l^2 (n + m \theta)^2 + \sigma_2 m^2 (k-l \theta)^2}{2 \pi i r^2}\]

\[t = \frac{\sigma_1 (n + m \theta)^2}{2 \pi i m} \left( z + \frac{l \Delta}{m(n l + m k)} - \frac{l q_o}{r} \right) + c_1 \frac{n + m \theta}{2 \pi i m} - c_2 \frac{k-l \theta}{2 \pi i l} \]

\[+ \frac{\sigma_2 (k-l \theta)^2}{2 \pi i l} \left( -\frac{n + m \theta}{k-l \theta} z + \frac{\Delta}{n l + m k} - \frac{m q_o}{r} \right)\]

and \(\Theta(s, t)\) a classical theta–function. We recall that classical–theta functions are defined as:
\[ \Theta(s, t) = \sum_{u \in \mathbb{Z}} e^{\pi i s u^2 + 2\pi i t u}, \quad \text{Im } s > 0 \]

3 Connections and complex structures.

One can define a connection on a right \( T^d_\theta \)-module \( E \) as a set of \( \mathbb{C} \)-linear operators \( \nabla_1, \ldots, \nabla_d \) obeying Leibniz rule:

\[ \nabla_\alpha(ea) = \nabla_\alpha e \cdot a + e \cdot \delta_\alpha a \quad (3) \]

where \( e \in E, a \in T^d_\theta \) and the derivatives \( \delta_\alpha a \) are specified by the formula \( \delta_\alpha U_\theta = 2\pi i n U_\theta \) (see [1]). It will be convenient for us to generalize the notion of connection replacing \( \delta_\alpha \) in (3) by \( \delta_\alpha' = a_\beta' \delta_\beta \) where \( a_\beta' \) is a non-degenerate matrix. A connection \( \nabla_1, \ldots, \nabla_d \) is a constant curvature connection if

\[ [\nabla_\alpha, \nabla_\beta] = i f_{\alpha\beta} \cdot 1 \]

where \( f_{\alpha\beta} \) are numbers and 1 stands for identity operator. Similar definitions can be given for left modules. We always consider unitary connections (i.e. operators \( \nabla_\alpha \) should be anti Hermitian). It is easy to check that operators

\[ \nabla_1 = 2\pi i \frac{m}{n + m\theta} x, \quad \nabla_2 = 2\pi \frac{d}{dx} \quad (4) \]

specify a constant curvature connection of right \( T_\theta \)-module \( E_{m,n} \). The same operators determine a connection (in generalized sense) on \( E_{n,m} \) considered as a left \( T_\theta \)-module. This follows from relations \( [\nabla_i, \nabla_j] = \frac{2\pi im}{n + m\theta} \). We see that \( \nabla_1, \nabla_2 \) can be considered as a constant curvature connection on \( (T_\theta, T_\theta) \)-bimodule.

If \( \nabla_1, \nabla_2 \) is a constant curvature connection on a module \( E \) we can introduce a complex structure on \( E \) fixing \( \overline{\partial} \)-connection \( \overline{\nabla} = \lambda_1 \nabla_1 + \lambda_2 \nabla_2 \), where \( \lambda_1 \) and \( \lambda_2 \) are complex numbers and the quotient \( \tau = \lambda_1 / \lambda_2 \) is not real. This complex structure on a \( T_\theta \)-module corresponds to complex structure on \( T_\theta \)-bimodule.

Notice that the notion of holomorphicity depends only on \( \tau = \lambda_1 / \lambda_2 \), therefore we say that \( \overline{\nabla} \) and \( \rho \overline{\nabla} \) where \( \rho \neq 0 \) determine the same complex structure on \( E \). Holomorphic vectors are closely related to theta-functions, hence we call holomorphic vectors in basic modules theta-vectors.

Let us consider holomorphic vectors in \( T_\theta \)-modules \( E_{n,m} \), assuming that \( n \) and \( m \) are relatively prime. In this case all constant curvature connections have the form \( \nabla_\alpha = \nabla_\alpha^0 + c_\alpha \), where \( \nabla_\alpha^0 \) stand for the connection (3) and \( c_1, c_2 \) are constants. The equation \( \overline{\nabla} \Theta = 0 \) takes the form

\[ (i\tau \frac{m}{n + m\theta} x + \frac{\partial}{\partial x} + c)\varphi(x, \mu) = 0 \]

6
If \( \text{Im } \tau < 0 \) it has \( m \) linearly independent solutions

\[
\varphi_\alpha(x, \mu) = e^{-\frac{1}{2}\sigma x^2 - cx \delta_\alpha^\mu}
\]

(5)

where

\[
\sigma = i \tau \frac{m}{n + m \theta}, \quad \mu \in \mathbb{Z}_m
\]

The functions (5) belong to \( \mathcal{S} \) only if \( \text{Re } \sigma > 0 \). We assumed that \( n + m \theta > 0 \); in the case \( n + m \theta < 0 \), the condition of existence of holomorphic vectors is that \( \text{Im } \tau > 0 \). We see that in the case when holomorphic vectors exist the space \( \mathcal{H}_{n,m} \) of holomorphic vectors in \( E_{n,m} \) is \( m \)-dimensional; the functions (5) constitute a basis of \( \mathcal{H}_{n,m} \). Considering \( E_{n,m} \) as a \((T_{\theta}', T_{\theta})\)-bimodule and taking into account that a constant curvature connection on \( E_{n,m} \) is a constant curvature connection on the bimodule, one can define a notion of complex structure and of holomorphic vector for a bimodule. More precisely, complex structure on \( T_{\theta}\)-module \( E_{n,m} \) induces a complex structure on same space considered as left \( T_{\theta}'\)-module in such a way that the notion of holomorphic vector remains the same. These two complex structures specify a complex structure on a bimodule; the notion of holomorphic vector in a bimodule coincides with corresponding notion for both modules. If \( \mathcal{E}' \) is a complex \((T_{\theta}', T_{\theta})\)-bimodule, \( \mathcal{E}'' \) is a complex \((T_{\theta}, T_{\theta''})\)-bimodule, then the \((T_{\theta}, T_{\theta''})\)-bimodule \( \mathcal{E} = \mathcal{E}' \otimes \mathcal{E}'' \) can be equipped with complex structure. We assume that complex structure on the right \( T_{\theta} \)-module \( \mathcal{E}' \) and on the left \( T_{\theta'} \)-module \( \mathcal{E}'' \) correspond to the same complex structure on \( T_{\theta} \). One can prove \([8]\) that tensor product of two holomorphic vectors is again a holomorphic vector (i.e. we have a natural map \( \mathcal{H}' \otimes \mathcal{H}'' \) into \( \mathcal{H} \) where \( \mathcal{H}', \mathcal{H}'' \) and \( \mathcal{H} \) stand for spaces of holomorphic vectors in \( \mathcal{E}', \mathcal{E}'' \) and \( \mathcal{E} \) correspondingly). Let the basis of \( \mathcal{H}' \) consist of:

\[
\varphi'_\alpha = e^{-\frac{1}{2}\sigma_1 x^2 - x \delta_\alpha^\mu}
\]

where

\[
\sigma_1 = i \tau_1 \frac{m}{n + m \theta}, \quad \mu \in \mathbb{Z}_m, \quad \alpha \in \{1, \ldots, m\}
\]

and the basis of \( \mathcal{H}'' \) consist of:

\[
\varphi''_\beta = e^{-\frac{1}{2}\sigma_2 y^2 - y \delta_\beta^\nu}
\]

where

\[
\sigma_2 = i \tau_2 \frac{l}{k - l \theta}, \quad \nu \in \mathbb{Z}_l, \quad \alpha \in \{1, \ldots, l\}
\]

We assume that \( \mathcal{E}' \), considered as a right \( T_{\theta} \)-module, is isomorphic to \( E_{n,m} \), and that \( \mathcal{E}'' \), considered as a left \( T_{\theta'} \)-module, is isomorphic to \( E_{k,l}' \). We can use \([2]\) to calculate \( \varphi'_\alpha \otimes \varphi''_\beta \). The condition that complex structures on \( \mathcal{E}' \) and \( \mathcal{E}'' \) correspond to the same complex structure on \( \mathcal{E} \) implies \( \tau_1 = \tau_2 \). Using this, we
can check that the theta–functions that appear in (2) do not depend on $z$ in our case. Applying (2), we obtain that, $\varphi'_\alpha \otimes \varphi'_\beta$ maps to $\Xi_{\alpha \beta}(z, \Delta) \in \mathcal{H}$ which is of the form:

$$\Xi_{\alpha \beta}(z, \Delta) = \sum_{\gamma} c_{\alpha \beta}^\gamma \varphi_{\gamma}(z, \Delta)$$

where

$$\varphi_{\gamma}(z, \Delta) = \exp \left[ \frac{\pi i \tau (n l + m k)(n + m \theta)}{k - l \theta} z^2 - (c_1 + c_2)(n + m \theta) z \right] \delta^\Delta$$

consistitute a basis of $\mathcal{H}$ and the constants $c_{\alpha \beta}^\gamma$ are given by:

$$c_{\alpha \beta}^\gamma = \Theta(s, t) \cdot e^K$$

where $\Theta(s, t)$ a classical theta–function,

$$K = -\frac{\pi i \tau l}{m(n l + m k)} \gamma^2 - \frac{l(n + m \theta)c_1 - m(k - l \theta)c_2}{m(n l + m k)} \gamma + \pi i s q_o^2 + 2 \pi i t q_o$$

and:

$$s = -m l (n l + m k)$$

$$t = \frac{l \tau \Delta - q_o(n l + m k)}{r} + \frac{c_1 l (n + m \theta) - c_2 m (k - l \theta)}{2 \pi i r}$$

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4 Appendix

Our candidate function \( h(z, \Delta) \) for the map (1) is given by:

\[
h(z, \Delta) = \sum_{q \in \mathbb{Z}} \left[ f((n + m \theta)z - \frac{n + m \theta}{m}q + \frac{l(n + m \theta)}{m(nl + mk)} \Delta, -q + a \Delta)
\cdot g((n + m \theta)z + \frac{k - l \theta}{l}q - \frac{k - l \theta}{nl + mk} \Delta, q) \right]
\]

For reference, let us recall all the actions we will need. If \( f(x, \mu) \in E_{n,m}(\theta) \),
and \( g(y, \nu) \in E'_{k,l}(\theta) \), then:

\[
U_1 f(x, \mu) = f(x - \frac{n + m \theta}{m}, \mu - 1)
U_2 f(x, \mu) = e^{2\pi i (x - \mu n/m)} f(x, \mu)
U_1 g(y, \nu) = g(y - \frac{k - l \theta}{l}, \nu - 1)
U_2 g(y, \nu) = e^{2\pi i (y - \nu k/l)} g(y, \nu)
Z_1 f(x, \mu) = f(x - \frac{1}{m}, \mu - a)
Z_2 f(x, \mu) = e^{2\pi i (x/(n + m \theta) - \mu/m)} f(x, \mu)
\]

Let us check the identifications. Replacing \( f \) with \( U_1 f \) in the definition of \( h \) we obtain:

\[
\sum_{q \in \mathbb{Z}} \left[ f((n + m \theta)z - \frac{n + m \theta}{m}q + \frac{l(n + m \theta)}{m(nl + mk)} \Delta, -q + a \Delta - 1)
\cdot g((n + m \theta)z + \frac{k - l \theta}{l}q - \frac{k - l \theta}{nl + mk} \Delta, q) \right]
\]

Letting \( q \to q - 1 \) gives:

\[
\sum_{q \in \mathbb{Z}} \left[ f((n + m \theta)z - \frac{n + m \theta}{m}q + \frac{l(n + m \theta)}{m(nl + mk)} \Delta, -q + a \Delta)
\cdot g((n + m \theta)z + \frac{k - l \theta}{l}q - \frac{k - l \theta}{nl + mk} \Delta, q - 1) \right]
\]

This means that the map (1) is compatible with the identification

\[(U_1 f) \otimes g \sim f \otimes (U_1 g)\]

Similarly, replacing \( f \) with \( U_2 f \) in (1) we get:
\[
\sum_{q \in \mathbb{Z}} \exp \left[ 2\pi i \left( (n + m \theta)z - \frac{n + m \theta}{m} q + \frac{l(n + m \theta)}{m(nl + mk)} \Delta - \frac{n(-q + a \Delta)}{m} \right) \right] f \cdot g
\]
\[
= \sum_{q \in \mathbb{Z}} \exp \left[ 2\pi i \left( (n + m \theta)z - \theta q + \frac{l(n + m \theta)}{m(nl + mk)} \Delta - \frac{an\Delta}{m} \right) \right] f \cdot g
\]

using the relation \( an = 1 + bm \) gives:
\[
= \sum_{q \in \mathbb{Z}} \exp \left[ 2\pi i \left( (n + m \theta)z + \frac{k - l \theta}{l} q + \frac{k - l \theta}{nl + mk} \Delta - \frac{k}{l} q \right) \right] f \cdot g
\]

This means that the map (1) is compatible with the identification
\[
(U_2 f) \otimes g \sim f \otimes (U_2 g)
\]

So the map (1) that was defined originally as a map of the tensor product \( E_{n,m} \otimes C E'_{k,l} \) descends to a map of \( E_{n,m} \otimes T_{\theta} E'_{k,l} \).

Next, we need to check that the new product module is in fact \( E'_{a k + b l, n l + m k} \).

That is, we first need to show that the shift \( \Delta \to \Delta + nl + mk \) leaves \( h(z, \Delta) \) invariant; therefore we can consider \( \Delta \) as an element of \( \mathbb{Z}_{nl + mk} \): 

\[
h(z, \Delta + nl + mk) = \sum_{q \in \mathbb{Z}} \left[ f((n + m \theta)z - \frac{n + m \theta}{m} q + \frac{l(n + m \theta)}{m(nl + mk)} (\Delta + nl + mk), -q + a(\Delta + nl + mk)) \right. \\
\left. \cdot g((n + m \theta)z + \frac{k - l \theta}{l} q + \frac{k - l \theta}{nl + mk} (\Delta + nl + mk), q) \right]
\]

Let \( q \to q + l \). Then the extra terms in the first argument of \( f \) are:
\[-\frac{n + m \theta}{m} l + \frac{l(n + m \theta)}{m(nl + mk)} (nl + mk) = 0\]

The extra terms in the second argument of \( f \) are:
\[-l + anl + amk \equiv -l - anl \pmod{m} \]
\[\equiv l(an - 1) \pmod{m} \]
\[\equiv lbm \pmod{m} \]
\[\equiv 0 \pmod{m} \]

Similarly, the extra terms in the arguments of \( g \) are respectively:
\[(k - l \theta) - (k - l \theta) = 0\]

and

\[l \equiv 0 \pmod{l}\]

It follows that:

\[h(z, \Delta + nl + mk) = h(z, \Delta)\]

as required. Finally we need to check that the action of the endomorphism algebra generators \(Z_1, Z_2\) on \(h(z, \Delta)\) induced by the action of \(Z_1\) and \(Z_2\) on \(f\) describe the standard module \(E_{a k + b l, n l + mk}(\theta')\):

\[
Z_1 h(z, \Delta) = \sum_{q \in \mathbb{Z}} \left[ f((n + m \theta)z + \frac{n + m \theta}{m}, q + \frac{(n + m \theta)}{m(nl + mk)}(\Delta - 1, l - q - a \Delta - a) \right]
\]

where

\[
\alpha = \frac{1}{n + m \theta} \left( \frac{(n + m \theta)}{m(nl + mk)} - \frac{1}{m} \right)
\]

\[
= -\frac{(n + m \theta)}{(nl + mk)}
\]

\[
\beta
\]

and can be rewritten in a simpler way using the relation \(a n - b m = 1\):

\[
= -\frac{a k + b l}{nl + mk} + \frac{b + a \theta}{n + m \theta}
\]

\[
= -\frac{a k + b l}{nl + mk} + \hat{\theta}
\]

Therefore:

\[
Z_1 h(z, \Delta) = Z_1 h(z - \frac{a k + b l}{nl + mk} + \hat{\theta}, \Delta - 1)
\]
as required for $E'_{a,k+b,l,n,l+m,k}(\theta')$. Similarly:

\[
Z_2 h(z, \Delta) = \sum_{q \in \mathbb{Z}} \exp \left[ 2\pi i \left( z - \frac{q}{m} + \frac{l}{m(nl + mk)} - \frac{1}{m}(-q + a \Delta) \right) \right] f \cdot g
\]

substituting $-bm$ in for $1-an$ gives:

\[
\sum_{q \in \mathbb{Z}} \exp \left[ 2\pi i \left( z - \frac{a k + bl}{nl + mk} \right) \right] f \cdot g
\]

also as required.

Our considerations prove that the formula (1) specifies a map from $E_{n,m}(\theta) \otimes T_{\theta'} E'_{k,l}(\theta')$ into $E_{a,k+b,l,n,l+m,k}(\theta')$ and this map is compatible with the structure of $T_{\theta'}$ right modules on these objects. It is easy to check that this map is an isomorphism (either directly or using the general result about tensor products of modules over noncommutative tori).

One can use also the technique suggested in [17]; the latter approach has the advantage that it gives a regular way to obtain (1). Namely we can start with the description of the linear space that is dual to the tensor product $E_{n,m}(\theta) \otimes T_{\theta'} E'_{k,l}(\theta')$. We notice that a continuous linear functional on the tensor product of $E_{n,m}(\theta)$ and $E'_{k,l}(\theta')$ over $\mathbb{C}$ is specified by a generalized function (a distribution) $\varphi(x, \mu, y, \nu)$ where $x, y \in \mathbb{R}, \mu \in \mathbb{Z}_m, \nu \in \mathbb{Z}_l$; it transforms $f \otimes g$ into:

\[
\sum_{\mu \in \mathbb{Z}_m} \sum_{\nu \in \mathbb{Z}_l} \int f(x, \mu) g(y, \nu) \varphi(x, \mu, y, \nu) \, dx \, dy
\]

This functional descends to the tensor product over $T_{\theta'}$ if:

\[
\varphi(x, \mu, y, \nu) \cdot e^{2\pi i(x-\mu \, n/m)} = \varphi(x, \mu, y, \nu) \cdot e^{2\pi i(y-\nu \, k/l)}
\]

and

\[
\varphi(x + \frac{n}{m} + \theta, \mu + 1, y, \nu) = \varphi(x, \mu + \frac{k}{l} - \theta, \nu + 1)
\]

Solving this system of equations for $\varphi$, we arrive at a description of the space that is dual to the tensor product we are interested in. Using this description, it is easy to obtain (1).