Exact results in the large system size limit for the dynamics of the Chemical Master Equation, a one dimensional chain of equations

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We apply the Hamilton-Jacobi equation (HJE) formalism to solve the dynamics of the Chemical Master Equation (CME). We found exact analytical expressions (in large system-size limit) for the probability distribution, including explicit expression for the dynamics of variance of distribution. We also give the solution for some simple cases of the model with time-dependent rates. We derived the results of Van Kampen method from HJE approach using a special ansatz. Using the Van Kampen method, we give a system of ODE to define the variance in 2-d case. We performed numerics for the CME with stationary noise. We give analytical criteria for the disappearance of bi-stability in case of stationary noise in 1-d CME.

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I. INTRODUCTION

The statistical physics of a living cell requires a theory for chemical reactions with few molecules \(^1\),\(^2\). One of mathematical tools here is the Chemical Master Equation (CME) \(^3\),\(^4\), describing the dynamics of probability \(P(X,t)\) of having different (integer) number \(X\) of molecules. The same CME equation could be applied to other areas of science as well, even in the financial market theory \(^5\). Recently in \(^6\) has been considered a kinetic model \(^7\) for the phosphorylation-dephosphorylation cycle in the cell, and the corresponding CME was investigated. The authors considered the model with the bi-stability phenomenon, and have been derived some results for the existence of bi-stability. Here we solve exactly the dynamics of CME, a very important topics according to \(^6\). The accurate (exact) solution of the CME dynamics is important for financial market modelling \(^5\). In this article we will derive exact dynamics for the CME.

The master equation is formulated as a system of linear differential equations for \(P(X,t)\) of having \(X\) molecules, \(0 \leq X \leq N\), where \(N\) is a large integer:

\[
\frac{dP(X,t)}{Ndt} = R_+(X-1)\frac{1}{N}P(X-1,t) + R_-(X+1)\frac{1}{N}P(X+1,t) - (R_+(X+1)\frac{1}{N} + R_-(X-1)\frac{1}{N})P(X,t) \quad (1)
\]

Here \(R_+\) is the growth rate and \(R_-\) is the degradation rate. Actually we should modify the equation at the border:

\[
\frac{dP(0,t)}{Ndt} = R_-(1)P(1,t) - R_+(0)P(0,t) \quad (2)
\]

and

\[
\frac{dP(N,t)}{Ndt} = R_+(1-1)P(1-1,t) - R_-(1)P(N,t) \quad (3)
\]

The large parameter \(N\) describes the system volume.

A close master equations have been considered in evolution theory in \(^9\)-\(^14\).

Let us introduce the coordinate \(x\) and function \(p(x,t)\):

\[
x = X/N, \quad p(x,t) = NP(X,t) \quad (4)
\]

Assuming that the probability distribution is a smooth function of \(x\),

\[
P(X+1,t) - P(X\pm 1,t) \ll 1 \quad (5)
\]

one gets the Fokker-Plank equation for the model by Eq.(1).

The investigation of CME via Fokker-Plank equations meet some problems \(^8\). An alternative approach is to assume that \(p(x,t)\) is not a smooth function of \(x\), but the function \(u(x,t)\) is, where

\[
p(x,t) = \exp[Nu(x,t)] \quad (6)
\]

Thus the \(p(x,t)\) might be un-smooth in the limit of \(N \to \infty\), but still have smooth \(u(x,t)\). Such an ansatz has been assumed for the statics in \(^9\), and for the dynamics in \(^12\),\(^13\), while considering the evolution models. This ansatz gives the solution of the dynamics and steady state with the accuracy \(O(1/N)\), while the approximation of the master equation via Fokker-Plank equation, assuming the smoothness of \(p(x,t)\), is certainly wrong. This topic we already discussed in \(^14\), then in \(^18\).

In section II we calculate the dynamics of variance for population distribution, using HJE. In section III we solve the CME dynamics for the simple case of time dependent rates. In appendix we re-derive the variance of distribution using Van Kampen method, also used that method to calculate the variance for general 1-d multi-step CME, as well as give ODE to derive the variance in case of 2-d CME.
II. THE MASTER EQUATIONS WITH CONSTANT RATES

A. Hamilton-Jacobi equation for the Chemical Master Equation

Using ansatz by Eq. (6), the model equations (1) can be written as Hamilton-Jacobi equations for

$$\frac{\partial u}{\partial t} + H(u', x) = 0,$$

(7)

where $u' = \frac{\partial u}{\partial x}$,

$$H(u', x) = R_+(x) + R_-(x) - R_+(x)e^{-u'} - R_-(x)e^{u'}$$

(8)

Eqs. (7),(8) has been derived in [16,18], where has been investigated mainly the corresponding Hamilton equations to calculate the time of extinction from meta-stable state. We are interested in the investigation of the whole distributions, using the traditional mathematical method of characteristics. To solve the HJE, we consider the Hamilton equation for $x$ and corresponding momentum, getting the system for the characteristics [19,20]

$$\dot{x} = H_v(x, v) = R_+(x)e^{-v} - R_-(x)e^v,$$

$$\dot{v} = -H_x(x, v),$$

$$\dot{u} = v H_v(x, v) - H(x, v) = v\dot{x} + q,$$

(9)

subject to initial conditions: $x(0) = x_0$, $v(0) = v_0(x_0)$, $u(0) = u_0(x_0)$. Here $v := \frac{\partial u}{\partial x}$, $v_0(x) := u_0'(x)$, $q := \frac{\partial u}{\partial t}$.

The respective solution to Eq. (9) in ($x, t$) - space is called the characteristic of Eq. (7).

Our Hamiltonian is time independent. Then Eq. (7) and Eq. (9) result in

$$\dot{q} = 0$$

(10)

Along the characteristic $x = x(t)$ the variable $q$ is constant, so $q$ is selected to parameterize these curves.

Consider the equation

$$q = -(R_+(x) + R_-(x)) + R_+(x)e^{-v} + R_-(x)e^{v}$$

(11)

It has a solution for

$$q \geq 0,$$

(12)

if at some point

$$R_+(x) = R_-(x),$$

(13)

and we take $q \geq 0$.

Using Eq. (11), we transform the first equation in (9) into

$$\dot{x} = \pm\sqrt{(q + R_+ + R_-)^2 - 4R_+ R_-}$$

(14)

Consider the following initial distribution

$$u_0(x) = -a(x - x_0)^2$$

(15)

with large $a$.

The maximum of the distribution corresponds to the point $u' = 0$, therefore $q = 0$.

Thus for the maximum of distribution we should consider a characteristic with $q = 0$.

Integrating the Eq.(14), we derive:

$$T = \int_{x_0}^{x} \frac{dy}{R_+(y) - R_-(y)}$$

(16)

Such equation was derived in [3].

We can define the dynamics of the full distribution. Let us define the function:

$$T(x, q) = \int_{x_0}^{x} \frac{dy}{\sqrt{(q + R_+(y) + R_-(y))^2 - 4R_+(y)R_-(y)}}$$

(17)

To calculate $u(x, t)$ we first calculate $q$ for the given $x$ from the equation

$$T(x, q) = t$$

(18)

Eq.(18) defines an implicit function

$$q = Q(x, t)$$

(19)

B. The elasticity

It is important to calculate $u''(x, t)$ at the point of the maximum of distribution, the “elasticity” [6].

We calculate $q''_x \equiv \frac{\partial^2 u}{\partial x \partial t}$ from Eq.(19):

$$\frac{\partial q}{\partial x} = \frac{\partial Q(x, t)}{\partial x}$$

(20)

We can calculate the last derivative at fixed $t$ from the expression:

$$\frac{\partial T(x, q)}{\partial x} + \frac{\partial T(x, q)}{\partial q} q''_x = 0$$

(21)

It is equivalent to calculate $q''_x$ from Eq. (17) at the fixed $t$. Thus we get the following equation for $q''_x$ at the point of maximum ($u'(x, t) = 0$):

$$\frac{1}{R_+(x) - R_-(x)} - q'' \int_{x_0}^{x} \frac{dy(R_+(y) + R_-(y))}{[(R_+(y) - R_-(y))]^3} = 0$$

(22)

From Eq.(11) we can obtain:

$$q''_x = -(R_+(x) - R_-(x))u'$$

(23)

Thus eventually we get:

$$\frac{-1}{v''_x} = (R_+(x) - R_-(x))^2 \int_{x_0}^{x} \frac{dy(R_+(y) + R_-(y))}{[(R_+(y) - R_-(y))]^3}$$

(24)

Eq. (24) is the main result of our work.

In Fig. 1 is given the comparison of analytical result with the numerics.
C. Probability distribution

Eq.(17) defines the $q(x,t)$ for given $x,t$, we can define $v(x,t)$ also using Eq.(7).

To calculate $u(x,t)$ at the given $x,t$, let us consider the trajectory of points $x(\tau), \tau$ connecting that point with the starting point $(x_0,0)$.

We have chosen the trajectory to have $q(x(\tau),\tau) = q$. We take $x(\tau)$ just as a solution of the equation

$$\tau = T(x(\tau), q)$$

(25)

At any point of our trajectory $x(\tau), \tau$, we can calculate $v(x(\tau), \tau)$, while $q(x(\tau), \tau) = q$ is constant. Eq.(11) gives

$$R_+ e^{2u} - (q + R_+ + R_-) e^u + R_+ = 0,$$

$$v(y) = -\ln 2R_+ + \ln [(q + R_+(y) + R_-(y) ±\sqrt{(q + R_+(y) + R_-(y))^2 - 4R_+(y)R_-(y)}])$$

(26)

where we denoted $x(\tau) = y$.

We derive the solution of the original Eq. (2), integrating the equation $\dot{u} = v + q$ along our trajectory (the characteristics connecting the points $(x,t)$ and $(x_0,0)$):

$$u(x,t) = u(x,0) + \int_{x_0}^{x} dy v(y) + qt =$$

$$\int_{x_0}^{x} dy [-\ln 2R_-(y) + \ln [(q + R_+(y) + R_-(y) ±\sqrt{(q + R_+(y) + R_-(y))^2 - 4R_+(y)R_-(y)})] + qt$$

(27)

Having the expression $u(x,t)$ we can calculate $p(x,t)$.

D. The restricted meaning of probability distributions in master equation

In case of evolution models [9-13], we have master equation similar to Eqs.(1)-(3), only the negative term $\sim P(X,t)$ has other coefficient than $- (R_+ + R_-)$, and therefore there is no a balance condition.

Contrary to the case of master equations in evolution models [9,14] where all the initial distributions have a meaning, now there are some restrictions. We should clarify well the meaning of the probabilities $P(X,t)$. At every moments of time the system has only ONE value of $X$, and $P(X,t)$ just gives such probabilities. We should solve the system (1),(2),(3) for the given initial value $P(X_0,t) = 1$ and $P(X,t) = 0$ for other $X$, other initial distributions have not meaning.

Another difference is connected with the stable point solutions. Eq. (13) gives the steady state solution. If that equation has two stable solutions $x_1, x_2$ and the probability of these two positions is the same, i.e.

$$\int_{x_1}^{x_2} dx \log \frac{R_+(x)}{R_-(x)} = 0$$

(28)

we again should accurately interpret the HJE results [6].

FIG. 1: The graphics for the elasticity $V(t) \equiv \frac{1}{\tau} \frac{d\tau}{dt}$ for the model with $N = 100, R_+(x) = \exp(-x), R_-(x) = \exp(x), x(0) = 0.5$. The smooth line is the analytical result by Eq.(24) and the dashed line is the numerical result. The difference is less than 0.5%.

at $k_1 = 43.1274$. One can use the HJE method to calculate the mean period of time that solution, initially located at one stationary point, will move to the other stationary point [17]. Then it again should return back, as every moment the system could exists only with one value of X.

In case of evolution model, the system goes to the equilibrium state instead of oscillation between two stable solutions.

E. The dynamics for the stationary but random rates.

Consider now the case when the rates are smooth functions of $x$ plus some random noises. The noise in the rates is well confirmed experimentally [22]. We took the case of rates from [6],

$$R_+(k_1, x) = (1 - x)(0.5 + k_1 x^2),$$

$$R_-(k_2, x) = x(k_2 + 0.01x^2)$$

(29)

where now $k_1$ and $k_2$ are random variables,

$$k_1 = K_1 \exp(a \xi_1(x) - a^2/2)x^2, \quad k_2 = K_2 \exp(a \xi_2(x) - a^2/2)$$

(30)

$\xi_1(x), \xi_2(x)$ have normal distributions. We consider the model with $N = 100$, and performed numerics for different values of parameters $K_1, K_2, a$. For $K_1 = 50, K_2 = 10$, at $a \approx 0.9$ there is a phase transition: instead of two steady state solution we get one steady state $x \approx 0.088$.

We can analytically estimate the transition point (from one stabile point to bi-stability), considering the behavior of

$$U(x) = -\ln \frac{R_+(k_1, x)}{R_-(k_2, x)} > |k_1, k_2|$$

(31)

We found that the function $U(x)$ changes its behavior with the level of the noise. At $a = 0$ it has only three roots $U(x) = 0$, while for $a > 0.75$ there is one root.

III. MASTER EQUATION WHEN THE RATES VARY WITH TIME

A. The simplest solvable case

Consider the case when birth and death rate coefficients in CME change with the time as $g(t)$ and $f(t)$, while they are
For the maximum of distribution we have
\[ H = (ax + g(t))e^{-v} + (bx + f(t))e^v - (a + b)x - f(t) - g(t) \]

We have
\[ \frac{dv}{dt} = ae^{-v} + be^v - (a + b) \]

We can solve this case:
\[ \int_{v_0}^{v} \frac{dv}{ae^{-v} + be^v - (a + b)} = t \]

We can define the function \( v = V(v_0, t) \) from the latter equation
\[ V(v_0, t) = \text{Log} \left[ \frac{ae^{at} - ae^{bt} - be^{at+v_0} + ae^{bt+v_0}}{ae^{at} - be^{at-v_0} + be^{bt+v_0}} \right] \]

We have also
\[ \frac{dV}{dv_0}(t) = \frac{\text{Log} \left[ \frac{ae^{at} - ae^{bt} - be^{at+v_0} + ae^{bt+v_0}}{ae^{at} - be^{at-v_0} + be^{bt+v_0}} \right]}{ae^{at} - be^{at-v_0} + be^{bt+v_0}} - \frac{be^{at+v_0} + be^{bt+v_0}}{ae^{at} - be^{at-v_0} + be^{bt+v_0}} \]

Now we solve the equation for \( x \):
\[ \frac{dx}{dt} = (b - a)x + f(t)e^V - e^{-V}g(t) \]

Its solution gives
\[ x - x_0 = e^{(b-a)t} \int_{0}^{t} [f(\tau)e^{V(v_0, \tau)} - g(\tau)e^{-V(v_0, \tau)}]e^{-(b-a)\tau} d\tau \]

Eqs.(42),(46) together define the trajectory of characteristics.

To get the solution for the maximum, we put \( v = 0 \) on the left hand side of Eq. (42). In this way we get a simple explicit equation for \( v_0 \) as a function of time \( t \).
\[ \frac{ae^{at} - ae^{bt} - be^{at+v_0} + ae^{bt+v_0}}{ae^{at} - be^{at-v_0} + be^{bt+v_0}} = 1 \]

Putting that solution into Eq.(46), we find the trajectory of the maximum of distribution.

To calculate \( \nu' \) we should differentiate the expression in Eq.(46) via \( x \). Using the relation \( V'(v_0, t) = 1 \), we derive
\[ \frac{1}{\nu'} = e^{(b-a)t} \times \int_{0}^{t} \left[ f(\tau)e^{V(v_0, \tau)} + g(\tau)e^{-V(v_0, \tau)} \right] \frac{dV}{dv_0}(\tau)e^{-(b-a)\tau} d\tau \]

where \( v_0 \) is defined by Eq.(46) as a function of \( t \).
IV. CME IN MULTI-DIMENSIONAL SPACE

Consider now the CME in multi-dimensional space, when we have d-dimensional $\vec{X}$.

$$\frac{dP(\vec{X}, t)}{N dt} =$$

$$\sum_{n_1=-K}^{K} R_n(\vec{X} - \vec{n}/N)P(\vec{X} - \vec{n}, t) - \sum_{n} R_n(\vec{n}/N)P(\vec{X}, t)$$

(49)

We get HJE for the function $u(\vec{x})$ with following Hamiltonian:

$$-H(\vec{x}, \vec{p}) = \sum_{n_1=-K}^{K} R_n(\vec{x})[\exp[-\vec{n}\vec{p}] - 1]$$

(50)

Let us investigate the motion of the maximum of distribution. We denote $\vec{y}(t) = \vec{x}$ the maximum of distribution at moment of time $t$, and assume the following ansatz for the $u(x, t)$ near the maximum of distribution:

$$u(\vec{x}, t) = \frac{1}{2} \left| t; \vec{x} - y(t) \right| V| x - y(t) | dt$$

(51)

Putting this ansatz into

$$u_{xxt} + H'_{\vec{x}}(\vec{x}, \vec{p}) + \sum_{m} H'_{\vec{p}m}(\vec{x}, \vec{p}) \frac{dp_m}{dx_l} = 0$$

(52)

Consider the point $\vec{x} = \vec{y}(t), \vec{p} = 0$, and use the identity:

$$\frac{dp_m}{dx_l} = -V_{ml}$$

(53)

We get:

$$\sum_{m} V_{lm} \frac{dy_l}{dt} - \left( \sum_{n_1=-K}^{K} R_{n_1,\ldots,n_d} \sum_{m} n_{m} \right) V_{lm} = 0$$

(54)

We get the following ODE for the dynamics of the maximum of distribution:

$$\frac{dy_l(t)}{dt} = \sum_{m} Q_{m} y_{m}(t)),$$

$$Q_{m} = \sum_{1 \leq h \leq d - K \leq n_h \leq K} \sum_{m} R_{n_1,\ldots,n_d} n_{m}$$

(55)

V. DERIVATION OF ELASTICITY IN 1-D CASE

Let us get system of ODE for the $V$. We consider the multi-step version of CME in 1-d.

$$\frac{dP(X, t)}{N dt} =$$

$$\sum_{n=-K}^{K} R_n(X - n/N)P(X - n, t) - \sum_{n} R_n(X/N)P(X, t)$$

(56)

Now we have the following Hamilton-Jacobi equation:

$$-H(u', x) = \sum_{n=-K}^{K} R_n(x)[e^{nu'} - 1]$$

(57)

Now we have to consider the higher terms in expansion of $u(x, t)$ near the maximum.

$$u = -V(x - y(t))^2/2 - T(x - y(t))^3/6$$

(58)

Eq.(55) gives

$$\frac{dy(t)}{dt} = y(t)b$$

(59)

We also denote:

$$a(x) = \sum_{-K \leq n \leq K} R_n(x)n^2$$

(60)

therefore

$$\dot{y} = b$$

With the ansatz by Eq.(56) we have:

$$u''_{xxt} = -V + T b,$$

$$u''_{xxt} = 2V b - T b^2 + V \dot{y}$$

$$2V b - T b^2 + V b'$$

(61)

From the other hand we have differentiating the right hand side of Eq.(50):

$$- u''_{xxt} = H''_{pp} V^2 - 2H''_{xp} V - TH'_{p} = aV^2 - 2bV - bT,$$

$$- u''_{xxt} = -H''_{xp} V \dot{y} + H''_{pp} bV^2 - H'_{p} V + H''_{p} T b^2 =$$

$$-bbV + abV^2 - bV + T b$$

(62)

We derived Eqs.(58),(59) putting $x = y(t)$.

Then

$$- V + 2T b = aV^2 - 2b'V,$$

$$3V b - 2T b^2 = aV^2 b - 2b'V$$

Removing T we get

$$2b \frac{d}{dt} V = -4bb'V + 2baV^2$$

(63)

or

$$\frac{d}{dt} \frac{1}{V} = b' \frac{1}{V} - 2a$$

(64)

Which gives Eq.(24) for 1-step CMW case.

For the multi-step CME we have the following expression for the elasticity:

$$\frac{-1}{V_x} = (b(x))^2 \int_{x_0}^{x} \frac{dy a(y)}{[b(y)]^3}$$

(65)
VI. CONCLUSION

In conclusion, we calculated exact probability dynamics Eq.(27) for the Chemical Master Distribution using Hamilton Jacobi equation method. Using the methods of characteristics for solution of HJE, we gave explicit expression for the variance of distribution Eq.(24). The latter is important both for chemical [2] and financial applications. We derived the variance also directly from HJE, using a special ansatz. The latter is equivalent to Van Kampen method [4]. Using Van Kampen method, in Appendix we give an exact expression for the variance for general 1-d model, as well as a system of ODE to define the variance in 2-d case. Both HJE and Van Kampen methods gave identical results for the variance and the maximum point dynamics of CME: HJE gives directly differential equation for the variance, while the Van Kampen method originally gives a differential equation for the probability distributions, Eq.(A.5), which later proved the differential equation for the variance. HJE gives also exact steady state distribution, and is more adequate for investigation of meta-stable points [16] [18]. Using HJE, we derived exact dynamics Eqs.(44)-(46) for the case of 1-d CME when the rates are linear functions of the coordinate (number of molecules) plus some time dependent functions.

We performed some numerics in case of static noise, also give an analytical criteria for the level of the noise when the bistability disappears. Our choice of potential for the averaging Eq.(31) is rather arbitrary (contrary to all other results in this article, which are derived rigorously and are exact). A further investigation of the problem is necessary. Perhaps it is possible to investigate the more realistic case of non-stationary noise [23].

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Appendix A: The calculation of variance dynamics using the Van Kampen method

1. 1-d multi-step models.

Consider the CME defined trough the system of equations:

\[
\frac{dP(X, t)}{dt} = \frac{N}{N} \cdot \sum_{l=-K}^{K} R_l \left( \frac{X + l}{N} \right) P(X + l, t) - \sum_{l} R_l \left( \frac{X}{N} \right) P(X, t)
\]

(A.1)

Following to Van Kampen, we assume that near the maximum

\[
X = N\phi(t) + \sqrt{N}\xi(t), \\
P(X, t) = \Pi(\xi, t)
\]

(A.2)

For the maximum point \(\phi(t) = \sum_i i P(i, t)\) we will derive an equation (4) later.

We consider terms with different scaling by \(N\) in the equation:

\[
\frac{d\Pi(\xi, t)}{dt} = -\sqrt{N}\phi'(t)\frac{\partial \Pi}{\partial \xi} = \\
N \sum_{l=-K}^{K} R_l(\phi(t) + \frac{\xi}{\sqrt{N}} - \frac{1}{N})P(X - l, t) \\
- N \sum_{l} R_l(\phi(t) + \frac{\xi}{\sqrt{N}})P(X, t)
\]

(A.3)

Collecting together the \(\sim \sqrt{N}\) terms, we get

\[
\frac{d\phi(t)}{dt} = \sum_{l=-K}^{K} l R_l(\phi(t))
\]

(A.4)

The \(N^0\) terms give an equation

\[
\frac{d\Pi(t)}{dt} = -b' \frac{\partial}{\partial \xi}(\xi)\Pi + \frac{a}{2} \frac{\partial^2}{\partial \xi^2} \Pi
\]

\[
b = \sum_{l} l R_l, a = \sum_{l} l^2 R_l
\]

(A.5)

We derive the following equations for \(<\xi_i>\):

\[
\frac{d <\xi>}{dt} = b'(t) <\xi_i>
\]

(A.6)

The initial condition \(<\xi(0)> = 0\) gives \(<\xi(t)> = 0\).

We get for the variance

\[
\frac{d}{dt} <\xi(t)^2> = a(\psi(t)) + 2b'((\psi(t)) <\xi(t)>^2
\]

(A.7)

We can consider ODE via \(\psi\) instead of \(t\). Then Eq. (A.4) gives \(\frac{d\psi}{dt} = b\), then:

\[
\frac{b}{d\psi} <\xi^2> = a(\psi) + 2b'(\psi) <\xi >^2
\]

Eventually we get Eq.(24), with \(R_+ - R_- \rightarrow b \equiv \sum_i R_i l, R_+ + R_- \rightarrow a \equiv \sum_i l^2 R_i\).

2. 2-d case

We will apply the Van Kampen method, giving ODE to calculate the variance of distribution in 2-d case.

Consider the following system of equations for \(P(X,Y, t), 0 \leq X \leq N, 0 \leq Y \leq N:\)

\[
\frac{dP(X,Y, t)}{dt} = -R_{1+} \left( \frac{X}{N}, \frac{Y}{N} \right) + R_{1-} \left( \frac{X}{N}, \frac{Y}{N} \right) \\
+ R_{2+} \left( \frac{X}{N}, \frac{Y}{N} \right) + R_{2-} \left( \frac{X}{N}, \frac{Y}{N} \right) \right) P(X,Y, t) + \\
+ R_{1+} \left( \frac{X - 1}{N}, \frac{Y - 1}{N} \right) P(X - 1, Y, t) + \\
+ R_{1-} \left( \frac{X + 1}{N}, \frac{Y - 1}{N} \right) P(X + 1, Y, t) + \\
+ R_{2+} \left( \frac{X}{N}, \frac{Y - 1}{N} \right) P(X, Y - 1, t) + \\
+ R_{2-} \left( \frac{X}{N}, \frac{Y + 1}{N} \right) P(X, Y + 1, t)
\]

(A.9)
Following to [4], we introduce the fluctuating variables $\xi_1, \xi_2$ and replace $P(X, Y, t)\text{ } by \Pi(\xi_1, \xi_2, t)$, see [4]:

\[
X = N\phi_1(t) + \sqrt{N}\xi_1(t),
\]

\[
Y = N\phi_2(t) + \sqrt{N}\xi_2(t),
\]

\[
P(X, Y, t) = \Pi(\xi_1, \xi_2, t) \tag{A.10}
\]

where $\psi_1(t), \psi_2(t)$ give the solution in case of infinite $N$:

\[
d\psi_1(t)/dt = b_1(\psi_1(t), \psi_2(t))
\]

\[
d\psi_2(t)/dt = b_2(\psi_1(t), \psi_2(t))
\]

\[
b_1(\psi_1, \psi_2) = R_{1+}(\psi_1, \psi_2) - R_{1-}(\psi_1, \psi_2),
\]

\[
b_2(\psi_1, \psi_2) = R_{2+}(\psi_1, \psi_2) - R_{2-}(\psi_1, \psi_2) \tag{A.11}
\]

We can solve Eq.(A.11) and calculate $\psi_1(t), \psi_2(t)$. Following to the methods of [4], we derive an equation:

\[
\frac{d\Pi(t)}{dt} = - \sum_\alpha \left( R'_{\alpha+} - R'_{\alpha-} \right) \frac{\partial}{\partial \xi_\alpha} (\xi_\alpha \Pi(\xi_1, \xi_2, t)) + \sum_\alpha \frac{R_{\alpha+} + R_{\alpha-}}{2} \frac{\partial^2}{\partial \xi_\alpha^2} \Pi(\xi_1, \xi_2, t)
\]

\[
= - \sum_\alpha \frac{\partial b_{\alpha+}}{\partial \xi_\alpha} \frac{\partial}{\partial \xi_\alpha} \Pi + \frac{1}{2} \sum_\alpha a_\alpha \frac{\partial^2}{\partial \xi_\alpha^2} \Pi \tag{A.12}
\]

where we denoted $R'_{\alpha\pm} = \frac{\partial R_{\alpha\pm}(\psi_1(t), \psi_2(t))}{\partial \xi_\alpha}$, $R_{\alpha\pm} = R_{\alpha\pm}(\psi_1(t), \psi_2(t))$.

We derive the following equations for $<\xi_i>$:

\[
\frac{d}{dt} <\xi_i(0)> = 0 \implies <\xi_i(t)> = 0.
\]

We get for the variance

\[
\frac{d}{dt} <\xi_i(t)^2> = a_i(\psi_1(t), \psi_2(t)) + \frac{d}{dt}
\]

\[
\partial \xi_\alpha \Pi(\xi_1, \xi_2, t) <\xi_i(t)>^2 \tag{A.13}
\]

We can calculate the variance numerically as function of $t$, using the solution of Eq.(A.11).

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