Deformations of Rational T-Varieties

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Abstract

We show how to construct certain homogeneous deformations of degree zero for rational normal varieties with codimension one torus action. This can then be used to construct homogeneous deformations of any toric variety in arbitrary degree. For locally trivial deformations coming from this construction, we calculate the image of the Kodaira-Spencer map. We then show that for a smooth complete toric variety, our homogeneous deformations span the space of infinitesimal deformations.

Keywords: Toric varieties, deformation theory, T-varieties

MSC: Primary 14D15; Secondary 14M25.

Introduction

There has been much progress made on understanding the deformation theory of toric varieties. The case of toric singularities has been studied extensively by K. Altmann, see for example [Alt95], [Alt97], and [Alt00]. There have been several results on deformations of non-affine toric varieties as well. In [Mav04] and [Mav05], A. Mavlyutov constructed certain deformations of complete weak Fano toric varieties via, respectively, regluing an open cover with automorphisms, and representing one toric variety as a complete intersection inside of a larger toric variety. Furthermore, in [Ilt09b], the first author constructed toric $\mathbb{Q}$-Gorenstein deformations for partial resolutions of toric surface singularities.

More recently, the first author has provided a combinatorial description for the space of infinitesimal deformations $T_X^1$ of a smooth complete toric variety $X$ in [Ilt09a]. Additionally, in the case that $X$ is a surface, he constructed homogeneous deformations via Minkowski decompositions of polyhedral subdivisions and showed that these deformations span $T_X^1$. In independent work, Mavlyutov presented a similar construction of certain homogeneous deformations in all dimensions [Mav09].

The goal of this paper is to generalize the results of [Ilt09a] in several directions. First of all, we construct multi-parameter deformations of varieties of arbitrary dimension which are also not necessarily smooth. Secondly, we will look not only at deformations of toric varieties but also deformations of rational $T$-varieties of complexity one, that is, rational normal varieties admitting an effective codimension one torus action.

Much as an $n$-dimensional toric variety can be described by an $n$-dimensional fan, an $n$-dimensional $T$-variety $X$ of complexity one can be described by a curve and some $n - 1$-dimensional combinatorial data. We then construct a deformation of $X$ by somehow
deforming the corresponding combinatorial data. In section 1, we give a short overview of the necessary theory of \(T\)-varieties. We then show how to construct homogeneous deformations of affine \(T\)-varieties in section 2. Here we also describe the fibers of such deformations explicitly as \(T\)-varieties. Note that the deformation theory of affine \(T\)-varieties is being further developed by the second author in [Vol09]. As a special case, we can of course consider toric varieties with an action by some subtorus. We describe this in detail in section 3 and show how to recover the deformations constructed by Altmann. In particular, we have a very natural description of toric deformations with non-negative degree, which are essential for constructing homogeneous deformations of complete toric varieties.

In section 4 we then show how to glue the deformations of affine \(T\)-varieties together to construct deformations of non-affine \(T\)-varieties. As in the affine case, we can also describe the fibers of such deformations explicitly as \(T\)-varieties. Restricting to the case of locally trivial deformations, we then calculate the Kodaira-Spencer map in section 5.

Of course, non-affine toric varieties provide again an example where our construction can be put to use. In section 6 we reformulate our Kodaira-Spencer calculation in nicer terms for this special case. For a smooth complete toric variety \(X\), we then construct certain special homogeneous deformations and show that they in fact span \(T_X^1\). Thus, at least for smooth complete toric varieties, our homogeneous deformations provide a kind of skeleton of the versal deformation.

Our approach has some aspects in common with the independent work of Mavlyutov—both approaches construct deformations via Minkowski decomposition of some combinatorial data. However, an important difference can be found in the distinct ways in which we translate our combinatorial data into deformations. His construction relies on the homogeneous coordinate ring of a toric variety. In contrast, our construction utilizes the language of \(T\)-varieties and polyhedral divisors.

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1 T-Varities

We recall several notions from [AHS08]. As usual, let \(N\) be a lattice with dual \(M\) and let \(N_\mathbb{Q}\) and \(M_\mathbb{Q}\) be the associated \(\mathbb{Q}\) vector spaces. For any polyhedron \(\Delta \subset N_\mathbb{Q}\), let \(\text{tail}(\Delta)\) denote its tailcone, that is, the cone of unbounded directions in \(\Delta\). Thus, \(\Delta\) can be written as the Minkowski sum of some bounded polyhedron and its tailcone. Now for \(u \in \text{tail}(\Delta)^\vee \cap M\), denote by \(\text{face}(\Delta, u)\) the face of \(\Delta\) upon which \(u\) achieves its minimum.

Let \(Y\) be a smooth projective variety over \(\mathbb{C}\) and let \(\delta \subset N_\mathbb{Q}\) be a pointed polyhedral cone.

**Definition.** A polyhedral divisor on \(Y\) with tail cone \(\delta\) is a formal finite sum

\[
\mathcal{D} = \sum_P \Delta_P \cdot P,
\]

where \(P\) runs over all prime divisors on \(Y\) and \(\Delta_P\) is a polyhedron with tailcone \(\delta\). Here,
finite means that only finitely many coefficient differ from the tail cone. Note that the empty set is also allowed as a coefficient.

We can evaluate a polyhedral divisor for every element \( u \in \delta^\vee \cap M \) via

\[
D(u) := \sum_{v \in \Delta_P} \min \langle v, u \rangle P
\]

in order to obtain an ordinary divisor on \( \text{Loc} \mathcal{D} \), where \( \text{Loc} \mathcal{D} := Y \setminus (\bigcup_{\Delta_P = 0} P) \).

**Definition.** A polyhedral divisor \( D \) is called proper if for all \( u \in \delta^\vee \cap M \), \( D(u) \) is semiample and if for all \( u \in \text{relint} \delta^\vee \cap M \), \( D(u) \) is big.

To a proper polyhedral divisor we associate an \( M \)-graded \( k \)-algebra and consequently an affine scheme admitting a \( \mathbb{C}^* \)-action:

\[
X(D) := \text{Spec} \bigoplus_{u \in \delta^\vee \cap M} H^0(Y, D(u)).
\]

This construction gives a normal variety of dimension \( \dim Y + \dim N_Q \) together with a \( \mathbb{T}^N \)-action.

We now wish to glue these affine schemes together; this requires some further definitions.

**Definition.** Let \( D = \sum_P \Delta_P \cdot P \), \( D' = \sum_P \Delta'_P \cdot P \) be two proper polyhedral divisors on \( Y \) with tail cones \( \delta \) and \( \delta' \).

- We define their intersection by
  \[
  D \cap D' := \sum_P (\Delta_P \cap \Delta'_P) \cdot P.
  \]

- We say \( D' \subseteq D \) if \( \Delta'_P \subseteq \Delta_P \) for every prime divisor \( P \in Y \).

- For \( y \in Y \) a not necessarily closed point, set \( \Delta_y := D_y := \sum_{y \in P} \Delta_P \), where summation is via Minkowski addition.

- \( D' \) is a face of \( D \) i.e. \( D' \prec D \) if \( D' \subseteq D \) and for each \( y \in \text{Loc}(D') \) there is a pair \((w_y, D_y) \in (\delta^\vee \cap M) \times |D(w_y)| \) such that \( y \notin \text{supp}(D_y) \), \( \Delta'_y = \text{face}(\Delta_y, w_y) \), and \( \text{face}(\Delta'_v, w_y) = \text{face}(\Delta_v, w_y) \) for all \( v \in Y \setminus \text{supp}(D_y) \).

If \( D' \subseteq D \) then we have an inclusion

\[
\bigoplus_{u \in \delta^\vee \cap M} H^0(Y, D'(u)) \supset \bigoplus_{u \in \delta^\vee \cap M} H^0(Y, D(u))
\]

which corresponds to a dominant morphism \( X(D') \to X(D) \). This is an open embedding exactly when \( D' \prec D \).

**Definition.** A divisorial fan is a finite set \( \Xi \) of proper polyhedral divisors such that for \( D, D' \in \Xi \) we have \( D \succ D' \cap D \prec D' \) with \( D' \cap D \) also in \( \Xi \). For a not necessarily closed point \( y \in Y \), the labeled polyhedral complex \( \Xi_y \) defined by the polyhedra \( D_y \), \( D \in \Xi \) is called a slice of \( \Xi \). \( \Xi \) is called complete if all slices \( \Xi_y \) are complete subdivisions of \( N_Q \).
We may glue the affine varieties $X(D)$ via

$$X(D) \leftarrow X(D \cap D') \rightarrow X(D').$$

This construction yields a normal scheme of dimension $\dim Y + \dim N_Q$ with a torus action by $T^N$; furthermore, if $\Xi$ is complete, then $X(\Xi)$ is complete as well. Note that all normal varieties with torus action can be constructed in this manner.

We will need the following lemma to construct deformation maps:

**Lemma 1.1.** Given a map $f : Y \rightarrow S$ where $S$ is affine, the composition of $f$ with the rational quotient map $X(\Xi) \rightarrow Y$ is regular.

**Proof.** The statement is local on $X(\Xi)$, hence we may assume $X = X(D)$ is affine. Then $X$ is the affine contraction of Spec$_Y \bigoplus \mathcal{O}(D(u))$ which maps to $S$ regularly, inducing a regular map $X \rightarrow S$. \qed

**Remark.** If $Y$ is a smooth projective curve, some of the above definitions simplify. We first define the degree of a polyhedral divisor by

$$\deg D := \sum_P \Delta_P$$

where summation is via Minkowski addition. A polyhedral divisor $D$ is then proper if and only if $\deg D \subset \delta$, and for all $u \in \delta$ with $\min_{v \in \deg D} \langle v, u \rangle = 0$ it follows that $u \notin \text{relint} \langle \delta \rangle$ and a multiple of $D(u)$ is principal. Likewise, $D' \prec D$ if and only if $\Delta'_P$ is a face of $\Delta_P$ for every point $P \in Y$ and $\deg D \cap \delta' = \deg D'$. 

**Remark.** Let $N'$ be an $n$-dimensional lattice with dual $M'$, $\Sigma$ a fan in $N'_Q$, and $X = TV(\Sigma)$ the associated toric variety. Choose some $R \in M'$ and let $N = N' \cap u^\perp \subset N'$. Furthermore choose some cosection $s : N' \rightarrow N$. We can thus consider $X$ as a $T$-variety with codimension one torus action by the subtorus $T^N \subset T^N'$. In this case, $Y = \mathbb{P}^1$ and the divisorial fan $\Xi$ consists of polyhedral divisors $D^\sigma$ for each $\sigma \in \Sigma$, where

$$D^\sigma = s(\sigma \cap [R = 1]) \cdot \{0\} + s(\sigma \cap [R = -1]) \cdot \{\infty\}.$$ 

Note that we define the set $[R = a]$ to be $\{v \in N_Q | \langle v, R \rangle = a\}$, that is, the set of points in $N_Q$ for which $R$ takes the value $a$.

**Example.** Consider the toric variety $X$ attained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ in all four fixpoints. The fan $\Sigma \subset N'_Q$ corresponding to this variety is pictured in figure 1(a). We choose $R = [0, 1]$ and consider the sublattice $N = R^\perp$; the dashed gray lines in figure 1(a) mark $[R = 1]$ and $[R = -1]$. Choosing the cosection $s : N' \rightarrow N$ by $s(a, b) = (a)$ leads to the divisorial fan $\Xi$ pictured in figures 1(b) and (c). $X(\Xi)$ is thus our original variety $X$ considered with an action of the subtorus $T^N$.

**Example.** Let $\Omega F_1$ be the cotangent bundle of the first Hirzebruch surface. Then $X = \mathbb{P}(\Omega F_1)$ is a $T$-variety over $\mathbb{P}^1$, see example 8.5 in [AHS08]. The corresponding divisorial fan is pictured in figure 2 where polyhedra with common tailcone are coefficients for the same polyhedral divisor.
2 Decompositions of Polyhedral Divisors

Let $Y$ be a smooth projective curve and let $\mathcal{D}$ be a proper polyhedral divisor on $Y$ with $\delta = \text{tail}(\mathcal{D})$. We describe how to construct deformations of $X = X(\mathcal{D})$ in degree 0. The construction is based on decomposing a coefficient $\Delta = D_p$ as a sum of polyhedra.

**Definition.** An $r$-parameter Minkowski decomposition of a polyhedron $\Delta$ with tailcone $\delta$ is a decomposition

$$\Delta = \Delta^0 + \ldots + \Delta^r$$

as Minkowski sum such that $\text{tail}(\Delta^s) = \delta$ for $0 \leq s \leq r$.

**Definition.** A Minkowski decomposition as above is said to be *admissible* if it satisfies one of the following equivalent properties:

(i) For each $u \in \delta^\vee \cap M$, at most one face($\Delta^s, u$) has no lattice vertices.

(ii) For each $u \in \delta^\vee \cap M$, at most one of the evaluations $\min \langle \Delta^s, u \rangle$ is not an integer.

(iii) For each vertex $v \in \Delta$, at most one of the corresponding vertices of the $\Delta^s$ is not a lattice point.

For example,

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\end{array}
\end{array}
\end{array}$$

is an admissible one-parameter decomposition of a non-lattice polyhedron with tailcone 0.
Let \( \mathcal{P} \subset Y \) be a finite set of points in \( Y \), including all those points \( P \) with nontrivial coefficient \( D_P \). Suppose now that for each \( P \in \mathcal{P} \) we have Minkowski decompositions \( D_P = \sum_{s=0}^{r_P} D_P^s \). Consider then some family \( Y^\text{tot} \to S \) of smooth projective curves with special fiber \( Y \) over \( 0 \in S \), with \( S \) affine, and with \( Y^\text{tot} \) smooth. Furthermore, let \( D^\text{tot}(P,s) \) be a pairwise different collection of relative prime divisors on \( Y^\text{tot} \) such that \( D^\text{tot}(P,s) \) restricts to \( P \) in \( Y \). From this information, we then define the polyhedral divisor

\[
\mathcal{D}^\text{tot} = \sum_{P,s} D_P^s \cdot D^\text{tot}(P,s).
\]

Note that since we required the \( D^\text{tot}(P,s) \) to restrict to \( P \) in \( Y \), we have \( \mathcal{D}^\text{tot}(u)|_Y = \mathcal{D}(u) \) for all \( u \). Indeed, for each \( P \), the coefficients of the \( D^\text{tot}(P,s) \) sum up to \( D_P \).

We assume for the moment that \( \mathcal{D}^\text{tot} \) is a proper polyhedral divisor, and call the associated \( T \)-variety \( X^\text{tot} = X(\mathcal{D}^\text{tot}) \). By lemma \([\text{Lem2.2}]\) we get a map \( \pi: X^\text{tot} \to S \). We want the central fiber to be \( X \).

**Proposition 2.1.** The map of \( T \)-varieties \( X \to X^\text{tot} \) induced by \( Y \hookrightarrow Y^\text{tot} \) embeds \( X \) as the special fiber \( \pi^{-1}(0) \) if, for each \( u \in \delta^\vee \cap M \), the following two conditions hold.

(i) \( \mathcal{D}^\text{tot}(u)|_Y = [\mathcal{D}^\text{tot}(u)]_Y \)

(ii) With \( D = [\mathcal{D}^\text{tot}(u)] \), the natural morphism \( H^0(Y^\text{tot}, D) \to H^0(Y, D|_Y) \) is surjective.

**Proof.** Let \( \mathcal{I}_Y \) be the ideal sheaf of \( Y \) in \( Y^\text{tot} \), and \( I = H^0(Y^\text{tot}, \mathcal{I}_Y) \subset H^0(Y^\text{tot}, \mathcal{O}_{Y^\text{tot}}) \) the ideal of global sections. The claim amounts to exactness of

\[
0 \to I \cdot H^0(Y^\text{tot}, \mathcal{D}^\text{tot}(u)) \to H^0(Y^\text{tot}, \mathcal{D}^\text{tot}(u)) \to H^0(Y, \mathcal{D}(u)) \to 0
\]

for each \( u \in \delta^\vee \cap M \). The second map in this sequence arises as follows (compare section 8 of \([\text{AHS08}]\)):

\[
\begin{array}{ccc}
H^0(Y^\text{tot}, \mathcal{D}^\text{tot}(u)) & \xrightarrow{\varphi} & H^0(Y, [\mathcal{D}^\text{tot}(u)]|_Y) \\
\downarrow & & \downarrow \\
H^0(Y^\text{tot}, [\mathcal{D}^\text{tot}(u)]) & \xrightarrow{\psi} & H^0(Y, [\mathcal{D}(u)])
\end{array}
\]

Since \( \mathcal{D}^\text{tot}(u)|_Y = \mathcal{D}(u) \), surjectivity of \( \psi \) follows from condition \([\text{Lem2.2}]\). Surjectivity of \( \varphi \) is condition \([\text{Lem2.2}]\). Finally, we see that the kernel is \( I \) by twisting the exact sequence

\[
0 \to \mathcal{I}_Y \to \mathcal{O}_{Y^\text{tot}} \to \mathcal{O}_Y \to 0
\]

by the locally free sheaf \( \mathcal{O}_{Y^\text{tot}}(D) \) and applying \( H^0(Y^\text{tot}, \cdot ) \).

Condition \([\text{Lem2.2}]\) is where admissibility comes in to play:

**Lemma 2.2.** Suppose \( D = \sum a_P^s D^\text{tot}(P, s) \) is a \( \mathbb{Q} \)-divisor on \( Y^\text{tot} \). Then \( [nD]|_Y = [(nD)|_Y] \) for all integers \( n \geq 0 \) if and only if, for each \( P \in Y \), at most one of the coefficients \( a_P^s \) is not an integer.
Proof. Due to our choice of divisors $D_{tot}(P,s)$, this follows easily from the following fact: Let $p,q \in \mathbb{Q} \setminus \mathbb{Z}$, $p,q \geq 0$. Then there exists an integer $n \geq 0$ such that $|np + nq| > |np| + |nq|$. \qed

Corollary 2.3. Condition (ii) of proposition 2.1 holds for each $u \in \delta^r \cap M$ if and only if the Minkowski decompositions underlying $D_{tot}$ are admissible.

From now on, we assume that our base curve $Y$ is $\mathbb{P}^1$. For each $P \in \mathcal{P}$, let $y_P$ be a local parameter at the point $P$. Likewise, let $t_{P,s}$ be coordinates on $\mathbb{A}^r$ for $P \in \mathcal{P}$ and $1 \leq s \leq r_P$, where $r = \sum r_P$. Furthermore, set $t_{P,0} = 0$. Now let $Z$ be the closed subset of $Y \times \mathbb{A}^r$ given by

$$Z = \bigcup_{P,Q \in \mathcal{P}, P \neq Q} V(y_P - t_{P,s_P}, y_Q - t_{Q,s_Q}),$$

where for any rational function $f$ on a variety $W$ we denote the effective part of $\text{div}(f)$ by $V(f)$. We can thus define $S$ to be $\mathbb{A}^r \setminus \pi_1(Z)$, where $\pi_1$ is the natural projection, and we get a trivial family of curves $Y_{tot} = \mathbb{P}^1 \times S$. As divisors, we then take $D_{tot}(P,s) = V(y_P - t_{P,s})$; these clearly satisfy the conditions we stated above. As an example, such a family is pictured in figure 3 for $Y_{tot} = \mathbb{P}^1 \times \mathbb{A}^1$, with $r_0 = 1$ and $r_\infty = 0$.

![Figure 3: A family of prime divisors $D_{tot}(P,s)$ on $\mathbb{P}^1 \times \mathbb{A}^1$](image)

For a point $\lambda \in S$, let $D_{\lambda}$ be the restriction of $D_{tot}$ to $Y_{\lambda}^{\text{tot}}$, the fiber over $\lambda$. Since $Y_{\lambda}^{\text{tot}} \cong Y$, we can view $D_{\lambda}$ as a polyhedral divisor on $Y$. In fact, we can describe $D_{\lambda}$ explicitly. Say $\lambda$ is given by the equations $t_{P,s} = \lambda_{P,s}$, and set $\lambda_{P,0} = 0$ for each $P \in \mathcal{P}$. For $0 \leq s \leq r_P$, let $D_{\lambda}(P,s)$ be the divisor on $Y$ given by the vanishing of $y_P - \lambda_s$. Then the polyhedral divisor $D_{\lambda}$ is given by

$$D_{\lambda} = \sum_{P \in \mathcal{P}, 0 \leq s \leq r_P} D_{\lambda}(P,s),$$

where the coefficients in front of prime divisors appearing multiple times are added via Minkowski sums.

Lemma 2.4. $D_{tot}$ is a proper polyhedral divisor on $Y_{tot} = \mathbb{P}^1 \times S$. Likewise, $D_{\lambda}$ is a proper polyhedral divisor on $\mathbb{P}^1$. 

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Proof. For any \( u \in \delta^v \cap M \), consider the \( \mathbb{Q} \)-divisor \( D^{\text{tot}}(u) \). One easily checks that \( D^{\text{tot}}(u) \sim D(u) \times S \), since for each \( P \in Y \) and \( s \leq r_P \) we have \( V(y_P - t_{P,s}) \sim V(y_P) \). Thus, \( D^{\text{tot}}(u) \) is semiample or big exactly when \( D(u) \) is semiample or big, so the properness of \( D^{\text{tot}} \) follows from the properness of \( D \). A similar argument holds for \( D^\lambda \).

Since \( D^{\text{tot}} \) is proper it defines an affine \( T \)-variety \( X^{\text{tot}} = X(D^{\text{tot}}) \) and we have a natural map \( \pi : X^{\text{tot}} \to S \). If the decomposition of \( D \) is admissible, this is in fact a deformation:

**Theorem 2.5.** If the decompositions of \( D_P \) for each \( P \in \mathcal{P} \) are admissible, the map \( \pi : X(D^{\text{tot}}) \to S \) gives a flat family with \( \pi^{-1}(\lambda) \cong X(D^\lambda) \) for \( \lambda \in S \). In particular, \( \pi^{-1}(0) = X(D) = X \).

The proof of the theorem relies on the following proposition:

**Proposition 2.6.** If the decomposition of \( D \) is admissible, the map \( X \to X^{\text{tot}} \) is a closed embedding given by the ideal generated by all \( t_{P,s} \) for \( P \in \mathcal{P}, 1 \leq s \leq r_P \).

**Proof.** Let \( D = [D^{\text{tot}}(u)] \) for some \( u \in \delta^v \cap M \). We twist the short exact sequence for the embedding \( Y \hookrightarrow Y^{\text{tot}} \)

\[
0 \to \mathcal{I}_Y \to \mathcal{O}_{Y^{\text{tot}}} \to \mathcal{O}_Y \to 0
\]

by the locally free sheaf \( \mathcal{O}_{Y^{\text{tot}}}(D) \). Consider the associated long exact sequence in cohomology

\[
H^0(Y^{\text{tot}}, D) \to H^0(Y, D|_Y) \to H^1(Y^{\text{tot}}, \mathcal{I}_Y(D)).
\]

We claim that \( H^1(Y^{\text{tot}}, \mathcal{I}(D)) \) vanishes, which proves the statement by proposition 2.1.

Indeed, since \( Y \hookrightarrow Y^{\text{tot}} \) is a complete intersection, we have a resolution of \( \mathcal{I} \) by the Koszul complex

\[
0 \longrightarrow \bigwedge^r \mathcal{O}_{Y^{\text{tot}}} \longrightarrow \cdots \longrightarrow \bigwedge^2 \mathcal{O}_{Y^{\text{tot}}} \longrightarrow \bigwedge^1 \mathcal{O}_{Y^{\text{tot}}} \longrightarrow \mathcal{I} \longrightarrow 0
\]

which we can split into short exact sequences

\[
0 \longrightarrow \mathcal{I}_i \longrightarrow \bigwedge^i \mathcal{O}_{Y^{\text{tot}}} \longrightarrow \mathcal{I}_{i-1} \longrightarrow 0
\]

with \( \mathcal{I}_0 := \mathcal{I} \) and \( \mathcal{I}_i := \ker d_i \). We show that \( H^p(Y^{\text{tot}}, \mathcal{I}_i(D)) = 0 \) for \( p > 0 \) by induction on \( i \). Indeed, \( H^p(Y^{\text{tot}}, \mathcal{I}_r(D)) = 0 \) since \( \mathcal{I}_r = 0 \). Suppose now that \( H^p(Y^{\text{tot}}, \mathcal{I}_i(D)) = 0 \) for some \( i \) and all \( p > 0 \). Then from the long exact sequence of cohomology, we have

\[
H^p(Y^{\text{tot}}, \mathcal{I}_{i-1}(D)) \cong H^p\left( Y^{\text{tot}}, \bigwedge^i \mathcal{O}_{Y^{\text{tot}}} \otimes \mathcal{O}(D) \right) = 0,
\]

since \( \bigwedge^i \mathcal{O}_{Y^{\text{tot}}} \otimes \mathcal{O}(D) \) is a direct sum of semiample line bundles on the product of \( \mathbb{P}^i \) and an affine variety. Thus, we can conclude that \( H^1(Y^{\text{tot}}, \mathcal{I}(D)) = 0 \).

**Proof of theorem 2.5.** By choice of \( S \subset \mathbb{A}^r \), the admissible decomposition of \( D \) also induces admissible decompositions of \( D^\lambda \) which result in the same divisor \( D^{\text{tot}} \) on \( Y^{\text{tot}} \). Thus, after coordinate change in \( S \) we can apply proposition 2.6 and get that for any \( \lambda \in S \), \( X(D^\lambda) \cong \pi^{-1}(\lambda) \). Furthermore, \( X(D^\lambda) \hookrightarrow X^{\text{tot}} \) is cut out by a regular sequence so \( \pi \) is flat.
Remark. Suppose a Minkowski summand $D^s_P$ is a multiple $kD^s_P$ of a lattice polyhedron $D^s_P$. Then replacing $D^\text{tot}(P,s) = V(y_P - t_P,s)$ by $D^\text{tot}(kP,s) = V(y^*_P - t_P,s)$ and $D^s_P$ by $D^s_P$ in $D^\text{tot}$ also gives a deformation of $X$.

Since $D^s_P \cdot D^\text{tot}(kP,s)$ restricts to $D^s_P \cdot kP$, $D^\text{tot}$ restricts to $D$ as before. The change doesn’t affect the integrality considerations since both $D^s_P$ and $D^s_P$ are lattice polytopes. The rest of the arguments carry through unchanged.

We end this section with a corollary of theorem 2.5.

**Corollary 2.7.** Let $D$ be a proper polyhedral divisor on $\mathbb{P}^1$ with affine locus. For each point $P \in \text{Loc}(D)$ let $D^s_P = \sum D^s_P$ be an admissible Minkowski decomposition. Then the general fiber of the corresponding deformation $\pi$ has exactly the analytic singularities $\text{TV}(\text{Cone}(D^s_P \times \{1\}))$.

**Proof.** This follows from the description of the general fiber from theorem 2.5 coupled with [Suß08], theorem 3.3. 

Remark. In section 4 of [Htt09b], explicit equations were used to calculate the singularities in the general fiber for toric deformations of cyclic quotient singularities. Combining this with the description of affine toric deformations in the following section, the above corollary provides a way of doing this without using the equations. Furthermore, the above corollary can be applied to see whether a toric deformation, or more generally, a deformation of a $T$-variety, is a smoothing. Note that if $D$ has complete locus, there cannot be any homogeneous degree 0 smoothings (see [Suß08], proposition 3.1).

### 3 Deformations of Affine Toric Varieties

We turn now to the case that $X = \text{TV}(\sigma)$ is a toric variety with embedded torus $T' = T^{N'}$ and show how homogeneous toric deformations fit into this picture. To treat deformations in degrees $-kR$ with $R \in M'$ primitive, we view $X$ as a $T$-variety for the subtorus $T = \ker(R: T' \to \mathbb{C}^\times) \subset T'$. As mentioned in section 1, the corresponding polyhedral divisor $D$ has coefficients

$$D_0 = s(\sigma \cap [R = 1]) \quad D_\infty = s(\sigma \cap [R = -1])$$

on $Y = \mathbb{P}^1$, where $s: N' \to N$ is a cosection. $D_0$ is Altmann’s $Q(R)$ [Alt00]. The tail cone is $\sigma \cap R^\perp = Q(R)^\infty$.

In case $R \in \sigma^\vee$, the coefficient at $\infty$ is empty, and for admissible $r$-parameter Minkowski decompositions of $D_0$, we recover Altmann’s homogeneous toric deformations in degree $-R$. If $y = y_0$ is the coordinate on $\text{Loc} D = \mathbb{A}^1$, a linear change of coordinates $t_{0,s} \mapsto t_{0,s} + y$ moves the supporting prime divisors into the invariant divisors of the toric variety $Y^\text{tot} = \mathbb{A}^{r+1}$, so we see that $X^\text{tot}$ is in fact toric. We can also see deformations in non-primitive degrees: Suppose $Q(kR) = Q_0 + \ldots + Q_r$ is a decomposition in a non-primitive degree $kR$, where $Q_1, \ldots, Q_r$ are necessarily lattice polyhedra ([Alt00] section 3.2). This gives a Minkowski decomposition

$$D_0 = Q(R) = kQ_0 + kQ_1 + \ldots + kQ_r,$$
which yields the corresponding homogeneous deformation in degree \(-kR\) by applying the remark following theorem 2.5 to \(kQ_1, \ldots, kQ_r\).

In the case \(R \notin \sigma^1\), the total space is no longer toric, since the support of \(D^{\text{tot}}\) is not a toric configuration of divisors in \(Y^{\text{tot}}\). As an example, consider the case \(r = 1\) illustrated in figure 3. While \(\mathbb{P}^1 \times \mathbb{A}^1\) is toric, the union of the three toric divisors \(\{0\} \times \mathbb{A}^1, \{\infty\} \times \mathbb{A}^1\) and \(\mathbb{P}^1 \times \{0\}\) is connected. Since this is not the case for the three divisors \(D(0,0), D(0,1)\) and \(D(\infty,0)\), \(X^{\text{tot}}\) cannot have the structure of a toric variety. Here, we recover the families which Altmann constructs by deforming partial resolutions of \(X\) (see section 3.5 of [Alt00]).

**Remark.** While we saw above that \(X^{\text{tot}}\) is not necessarily toric, it does retain an action by \(T'\). Restricting to the case of figure 3 for simplicity, we show how to obtain the corresponding polyhedral divisor by upgrading the torus action on \((D^{\text{tot}}, Y^{\text{tot}})\) by the residual \(\mathbb{C}^* = T'/T\).

This \(\mathbb{C}^*\) acts diagonally on \(Y^{\text{tot}} = \mathbb{P}^1 \times \mathbb{A}^1\) with quotient \(\mathbb{P}^1\). A choice of divisorial fan \(\Xi\) for \(Y^{\text{tot}}\) assigns to \(\infty\) the polyhedral decomposition \(\Xi_{\infty}\) of \(Q_{\geq 1}\) with vertices \(-1\) and \(0\), to \(0\) the trivial decomposition \(\Xi_0\) of \(Q_{\geq 1}\), and to all other points \(z\) the trivial decomposition of \(Q_{> 0}\). The vertices \(0\) and \(1\) in \(\Xi_{\infty}\) correspond to the toric divisors \(\{\infty\} \times \mathbb{A}^1 = D^{\text{tot}}(\infty,0)\) and \(\mathbb{P}^1 \times \{0\}\) on \(Y^{\text{tot}}\), respectively. The single vertex in \(\Xi_z\) corresponds to the line in \(\mathbb{A}^2 \subset Y^{\text{tot}}\) with slope \(z\). For a detailed treatment of invariant divisors on \(T'\)-varieties, see [PS08].

The support of \(D^{\text{tot}}\) lies over the divisors \(0, 1\) and \(\infty\) in \(\mathbb{P}^1\); note that the coefficient of \(\mathbb{P}^1 \times \{0\}\) in \(D^{\text{tot}}\) is \(\delta = \text{tail}(D)\). These data combine to form a proper polyhedral divisor \(E^{\text{tot}}\) on \(\mathbb{P}^1\) with

\[
\begin{align*}
E^{\text{tot}}_0 & = \alpha(D_0^0 \times \{1\}) + \sigma' \\
E^{\text{tot}}_1 & = \alpha(D_0^1 \times \{0\}) + \sigma' \\
E^{\text{tot}}_{\infty} & = \alpha(\text{Conv}(D_\infty \times \{-1\}, \delta \times \{0\})) + \sigma',
\end{align*}
\]

where \(\alpha: N \oplus \mathbb{Z} \to N'\) is the isomorphism induced by \(s\), and \(\sigma' = \sigma \cap [R \geq 0]\) is the positive part of \(\sigma\). This polyhedral divisor gives us the total space \(X^{\text{tot}} = X(E^{\text{tot}})\) as a \(T'\)-variety. Note that we recover the Minkowski decomposition at this level:

\[
E^{\text{tot}}_0 + E^{\text{tot}}_1 = \sigma \cap [R \geq 1] \\
E^{\text{tot}}_{\infty} = \sigma \cap [R \geq -1]
\]

The special fiber \(TV(\sigma)\) shows up in the coefficient at \(\infty\).

\[\text{Figure 4: Minkowski decomposition for an affine threefold singularity} \]

**Example.** We consider an example of a toric threefold with deformations in non-negative degrees. Let \(N' = \mathbb{Z}^3\) with standard basis \(e_1, e_2, e_3\) and \(\sigma\) generated by \((-1, 1, 1), (1, 1, 1), (-1, 1, -1),\) and \((1, 1, -1)\). \(X = TV(\sigma)\) is then the cone over the singular projective Fano surface \(X'\) whose minimal resolution is the toric surface presented in figure 1 in section 1.
Setting $N = \langle e_1, e_2 \rangle$ with cosection $s: N' \to N$ given by projection, we can consider $X$ as the $T$-variety $X(D)$ over $Y = \mathbb{P}^1$ with $D = \Delta \cdot \{0\} + \Delta \cdot \{\infty\}$ and $\Delta$ as in figure 3(a). The Minkowski decompositions $D_0 = \Delta^0 + \Delta^1$ and $D_\infty = \Delta^0 + \Delta^1$ induce a two-parameter deformation $\pi$ of $X$. Restricting to the coordinate axes of the base space gives homogeneous deformations in degrees $-e_3^*$ and $e_3^*$, neither of which lie in $\sigma^\vee$.

Note that the deformation $\pi$ has degree zero with respect to the $\mathbb{Z} = \langle e_2^* \rangle$ grading on $\mathcal{O}_X$ inducing the quotient $X'$. Thus, $\pi$ induces a two-parameter deformation $\pi'$ on $X'$ as well.

4 Decompositions of Divisorial Fans

Let $Y = \mathbb{P}^1$ and let $\Xi = \{D^1, \ldots, D^l\}$ be a divisorial fan on $Y$. We now show how to construct homogeneous deformations of the rational non-affine $T$-variety $X(\Xi)$. For simplicity’s sake, we will restrict to those deformations which correspond to primitive degrees in the toric case.

**Definition.** For $r \in \mathbb{N}$ and a closed point $P \in Y$, an $r$-parameter Minkowski decomposition of the slice $\Xi_P$ consists of polyhedra $\{D_{p}^{i,j}\}_{1 \leq i \leq l}$ such that:

(i) For fixed $i$, the $D_{p}^{i,j}$ form a Minkowski decomposition of the polyhedron $D_{p}^{i}$;

(ii) For fixed $j$, $\{D_{p}^{i,j}\}$ form a polyhedral complex in $N_\mathbb{Q}$;

(iii) If $D_{p}^{i} = D_{p}^{i'} \cap D_{p}^{i''}$, $D_{p}^{i,j} = D_{p}^{i',j} \cap D_{p}^{i'',j}$ for $0 \leq j \leq r$.

A decomposition is called admissible if for each $1 \leq i \leq l$, the corresponding decomposition of $D_{p}^{i}$ is admissible.

**Remark.** For fixed $j$, each vertex $v$ of the polyhedral complex $\Xi_0$ corresponds to exactly one vertex $v_j$ of the polyhedral complex $\{D_{0}^{i,j}\}$. Thus, a decomposition of $\Xi_0$ is admissible if and only if for all vertices $v$ of $\Xi_0$, at most one of the corresponding vertices $v_j$ is not a lattice point.

Similar to section 2 let $P \subset Y$ be a finite set of points in $Y$, this time including all those points $P$ with nontrivial slice $\Xi_P$. For each $P \in P \subset Y$, consider an admissible $r_P$-parameter Minkowski decomposition of the slice $\Xi_P$ for some $r_P \in \mathbb{Z}_{\geq 0}$. As before set $r = \sum r_P$ and let $S = \mathbb{A}^r \setminus \pi_1(Z)$, with $Z$ defined as in equation (11). From such data we shall construct a divisorial fan $\Xi^\text{tot}$ on $Y^\text{tot} := Y \times S$.

As before, for $0 \leq s \leq r_P$ let $D^\text{tot}(P, s)$ be the prime divisor on $Y^\text{tot} = Y \times S$ given by the vanishing of $y_P - t_{P,s}$. Then, for $1 \leq i \leq l$ we can define a polyhedral divisor on $Y^\text{tot}$ by

$$\mathcal{E}^i = \sum_{P \in P \atop 0 \leq s \leq r_P} D_{p}^{i,s} \cdot D^\text{tot}(P, s).$$

We then set $\Xi^\text{tot} = \langle \{\mathcal{E}^i\}_{i=1}^l \rangle$, that is, $\Xi^\text{tot}$ is the set of polyhedral divisors induced by the $\mathcal{E}^i$ via intersection. We call $\Xi^\text{tot}$ an admissible decomposition of the divisorial fan $\Xi$. 

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We also construct a family of divisorial fans $\Xi^\lambda$ on $Y$. As in section 2, fix a point $\lambda \in S \subset \mathbb{A}^r$ and define a corresponding set of divisors $D^\lambda(P, s)$ on $Y$. For each $1 \leq i \leq l$ we can define a polyhedral divisor $F_i^{\lambda, \lambda}$ on $Y$ by

$$F_i^{\lambda, \lambda} = \sum_{P \in \mathcal{P}, 0 \leq s \leq r} D_{P}^{i, s} \cdot D^\lambda(P, s),$$

where the coefficients in front of prime divisors appearing multiple times are added via Minkowski sums. We then set $\Xi^\lambda = \langle \{F_i^{\lambda, \lambda}\}_{i=1}^l \rangle$.

**Lemma 4.1.** $\Xi_{\text{tot}}$ is a divisorial fan on $Y_{\text{tot}}$. Likewise, each $\Xi^\lambda$ is a divisorial fan on $Y$.

We will prove this lemma shortly, but first we wish to construct the corresponding deformation. As before, there is a natural regular map $\pi: X(\Xi_{\text{tot}}) \to S$ by lemma 1.1.

**Theorem 4.2.** The map $\pi: X(\Xi_{\text{tot}}) \to S$ gives a flat family with $\pi^{-1}(\lambda) \cong X(\Xi^\lambda)$ for $\lambda \in S$. In particular, $\pi^{-1}(0) = X(\Xi)$.

**Proof.** Flatness can be checked locally on each $X(\mathcal{E}^i)$; this follows then directly from theorem 2.5. From this theorem, we also know $\pi^{-1}_{|X(\mathcal{E})}(\lambda) = X(F_i^{\lambda, \lambda})$, so we just need to check that everything glues properly. But from the same theorem it also follows that if $\mathcal{E} = \bigcap_{j=1}^l \mathcal{E}^j$, then $\pi^{-1}_{|X(\mathcal{E})}(\lambda) = X(F^{\lambda, \lambda})$, where $F^{\lambda, \lambda} = \bigcap_{j=1}^l F^j_{i, \lambda}$. Thus, the gluing on $X(\Xi_{\text{tot}})$ induces the gluing on $X(\Xi)$.

![Figure 5: A deformation of $\mathbb{P}(\Omega_{F_{1}})$](image)

**Example.** Consider the $T$-variety $X = X(\Xi) = \mathbb{P}(\Omega_{F_{1}})$ as described in the example at the end of section 1 and in figure 2. We construct a one-parameter deformation over $\mathbb{A}^1 \setminus \{1\}$ by decomposing the slice $\Xi_0 = \Xi_0' + \Xi_0''$ as pictured in figure 5. The general fiber of the deformation is $X(\Xi_0' \cdot \{0\} + \Xi_0'' \cdot \{\lambda\} + \Xi_1 \cdot \{1\} + \Xi_\infty \cdot \{\infty\})$.

On the other hand, $\mathbb{P}(\Omega_{F_{2}})$ is the general fiber in a certain homogeneous deformation of a toric variety. Indeed, if $\Xi$ is the divisorial fan on $\mathbb{P}^1$ pictured in figure 6, $X(\Xi)$ is in fact a toric variety and $X(\Xi)$ is the general fiber of the homogeneous deformation coming from the decomposition $\Xi_{\infty} = \Xi_\infty + \Xi_1$. This is similar to the degeneration of $\mathbb{P}(\Omega_{P^2})$ to the projective cone over the del Pezzo surface of degree six constructed in [Su08], example 5.1.

We now return to the unproven lemma.
have a unique closed point of $Y$. If possible let $\hat{\eta}$ be any closed point of $Y$. Then because of point (i) in the above definition, there is some $0 \leq s \leq r_P$ such that $\bigcap_{j=1}^l D^{\alpha,j} = \emptyset$. Thus, after coordinate change, $\text{Loc} \mathcal{E} \subset (Y \setminus P) \times \mathbb{A}^r$ and is thus affine. It follows that $\mathcal{E}'$ is automatically proper. Furthermore, the condition $\mathcal{E}' \prec \mathcal{E}$ is reduced to the condition $\mathcal{E}'_Q \prec \mathcal{E}_Q$ for all $Q \in \text{Loc} \mathcal{E}$ (cf. remark 3.5 in [AHS08]). This follows however from point (ii) in the above definition.

Suppose now instead that for each $P \in Y$, $\bigcap_{j=1}^l D^{\alpha,j} \neq \emptyset$. For some $1 \leq \alpha, \beta \leq l$ we then have $D^{\alpha} = \bigcap_{j=1}^l D^{\beta,j}$ and $D := D^{\beta} = \bigcap_{j=1}^l D^{\beta,j}$. Now, for each $P$ and for each $0 \leq k \leq r_P$ we have $D^{\alpha,k} = \bigcap_{j=1}^l D^{\beta,j,k}$ from point (iii) of the above definition. We thus know that $\mathcal{E}' = \mathcal{E}^\lambda$ is proper from the affine case, see lemma 2.4.

Now fix some $y \in \text{Loc}(\mathcal{E}') = Y^\text{tot}$, a not necessarily closed point. If possible let $\hat{y}$ be the unique closed point of $Y$ with $y \in D^\text{tot}(\hat{y}, s)$. If not, either $y$ is the general point of $Y^\text{tot}$ in which case we let $\hat{y}$ be the general point of $Y$, or else we just let $\hat{y}$ be any closed point of $Y$ not in $P$. Set $D = \bigcap_{j=1}^l D^{\beta,j}$ and let $(w_y, D_y) \in M \times |D(w_y)|$ be a lattice element and divisor on $Y$ giving the face relation $D^{\alpha} \prec D$ for the point $\hat{y}$ (see the definition of face in section 1). Let $D_y := D_y \times S$ be a divisor on $Y^\text{tot}$ and set $w_y := w_y$. Then $D_y \in |D(w_y)|$. Indeed, one easily checks that $E(w_y) \sim D(w_y) \times S$. Furthermore, we clearly have $w_y \in \text{tail}(D)^\vee \cap M$. Clearly we also have $y \notin \text{supp}(D_y)$.

We claim now that $E_y = \text{face}(E_y, w_y)$. Indeed, $E_y = D_y$ and $E_y = D_y$ unless $y$ is contained in some $D^{\text{tot}}(P, s)$. In this case, $E_y = D_y^{\alpha,s}$ and $E_y = D_y^{\alpha,s}$. Furthermore, we have

$$D^{\beta,j} = D_y^{\beta,s} + \sum_{i \neq s} D_y^{\beta,i};$$

$$D^{\alpha,j} = \text{face}(D_y^{\beta,j}, w_y);$$

$$D_y^{\alpha,s} + \sum_{i \neq s} D_y^{\alpha,i};$$

$$D^{\alpha,s} \subset \text{face}(D_y^{\beta,s}, w_y).$$

Note that for any polyhedra $\Delta_i$, if the equality $\Delta_0 = \Delta_1 + \Delta_2$ holds, then so does $\text{face}(\Delta_0, w) = \text{face}(\Delta_1, w) + \text{face}(\Delta_2, w)$. Combining this with the above yields the claim.

Finally, we claim that $\text{face}(E'_v, w_y) = \text{face}(E_v, w_y)$ for all $v \in \text{supp}(D_y)$. Indeed, this follows with arguments almost identical to the above paragraph. Thus, $E'$ is a face of $E$ and the proof is complete. \[\square\]
5 Locally Trivial Deformations

Let $Y = \mathbb{P}^1$ and let $\Xi$ be a divisorial fan on $Y$. Any locally trivial one-parameter deformation of $X(\Xi)$ can be assigned a class in $H^1(X(\Xi), T_{X(\Xi)})$ via the Kodaira-Spencer map. In this section, we will compute this image for homogeneous deformations constructed as in the last section satisfying a mild assumption.

Now for some $P \in Y$ let $\{D^i\}_{i=0,1}$ be an admissible one-parameter Minkowski decomposition of $\Xi_P$ such that for each $D^i_P$, either $D^i_P \in \mathcal{L}$ or $D^i_P$ is a lattice translate of tail($D^i_P$). Note that this is always the case if $X(\Xi)$ is smooth, see [Süß08] propositions 3.1 and 3.3. Furthermore, we assume that for any $D^i \in \Xi$ with Loc($D^i$) = $Y$, $D^i$ has trivial coefficients everywhere except $P$ and $\infty := V(y_P^{-1})$. The calculations could be carried out without this assumption, but the resulting formula would be considerably more complicated.

We consider the one-parameter deformation $\pi: X(\Xi^{\text{tot}}) \to S$ corresponding to the above decomposition. We shall describe its image in $H^1(X(\Xi), T_{X(\Xi)})$ as a Čech cocycle and thus need to choose some open cover $\mathcal{U}$ of $X(\Xi)$. Fix some order $D^1, \ldots, D^l$ such that $D^1, \ldots, D^{l_1}$ are exactly those polyhedral divisors $D^i$ with at least one slice $D^i_P$ full-dimensional; order these further such that the first $l_0$ are exactly those for which Loc($D^i$) $\neq Y$ holds. Now, for $1 \leq i \leq l_0$ let $I_i$ be the set of all points $Q \in Y \setminus P$ such that $D^i_P$ is nontrivial. For such $i$ set $U^i = X(D^i + \sum_{Q \in I_i} (0 \cdot Q))$. Likewise, set $U^{l_0+i} = X(D^i + (\emptyset \cdot P))$. Finally, for $l_0 < i < l_1$ set $U^i = X(D^i)$. We now define $\mathcal{U} = \{U^i\}_{i=1}^{l_0+l_1}$.

This is an affine open cover of $X(\Xi)$.

To the above Minkowski decomposition and for every $1 \leq i \leq l_1$ we can associate $a_i \in \{1, -1\}$ and $\lambda_i \in \mathbb{N}$ as follows. If we can write $D^i_P = \lambda_i + D^i_P^{0}$ for some $\lambda_i \in \mathbb{N}$, let $a_i = 1$. Otherwise set $a_i = -1$ and define $\lambda_i$ by $D^i_P = \lambda_i + D^i_P^{1}$. Note that one of these conditions must be fulfilled due to our above assumption. Now, for $l_1 < i \leq l_0 + l_1$ we let $a_i = 1$ and $\lambda_i = 0$. Finally, let $e_i$ be a basis for $M$.

**Theorem 5.1.** The deformation $\pi: X(\Xi^{\text{tot}}) \to S$ is locally trivial and its image under the Kodaira-Spencer map is the cocycle defined by

$$d_{i,j} = \frac{a_i - a_j}{2} \frac{\partial}{\partial y_P} + y_P^{-1} \sum_k (a_i \lambda_i - a_j \lambda_j, e_k^*) \chi^{e_k^*} \frac{\partial}{\partial \chi^{e_k^*}}.$$ 

For the proof of the theorem, we shall use the following two lemmata, which are essentially special cases of proposition 8.6 in [AH06]:

**Lemma 5.2.** Let $Y$ be a smooth variety and $\Xi$ a divisorial fan on $Y$. For some $v \in \mathbb{N}$ and $D_0, D_1 \in\text{Div}(Y)$ with $D_1 - D_0 = \text{div}(f)$ let $\tilde{\Xi} = \{D + v(D_1 - D_0) \mid D \in \Xi\}$. Then there is a canonical isomorphism $\phi_v: X(\Xi) \to X(\tilde{\Xi})$ where $\phi_v^\#$ is defined by mapping $\chi^u$ to $f^{(v,u)} \chi^u$ for $u \in M$.

**Proof.** We show that locally $\phi_v$ is an isomorphism. Consider $D \in \Xi$ with tailcone $\delta$ and set
\( \widetilde{D} = D + v(D_1 - D_0) \). Then
\[
\mathcal{O}_{X(\widetilde{D})} = \bigoplus_{u \in \delta^V \cap M} H^0(Y, \widetilde{D}(u)) \cdot \chi^u \cong \bigoplus_{u \in \delta^V \cap M} H^0(Y, D(u) + \langle v, u \rangle \text{div}(f)) \cdot \chi^u
\]
\[
\cong \bigoplus_{u \in \delta^V \cap M} H^0(Y, D(u)) \cdot f^{-\langle v, u \rangle} \chi^u
\]
and thus \( \phi^\# \) induces an isomorphism \( \mathcal{O}_{X(\widetilde{D})} \cong \mathcal{O}_{X(D)} \), since
\[
\mathcal{O}_{X(D)} = \bigoplus_{u \in \delta^V \cap M} H^0(Y, D(u)) \cdot \chi^u.
\]

**Lemma 5.3.** Let \( Y \) be a smooth variety and \( \bar{\gamma} \in \text{Aut}(Y) \). For a proper polyhedral divisor \( D \) on \( Y \), define \( \gamma(D) = \sum_D D \cdot \bar{\gamma}_+(D) \). Then there is a natural isomorphism \( \gamma: X(D) \to X(\gamma(D)) \) induced by \( \bar{\gamma} \).

**Proof.** Similar to the proof of the above lemma, we have
\[
\mathcal{O}_{X(\gamma(D))} = \bigoplus_{u \in \delta^V \cap M} H^0(Y, \gamma(D)(u)) \cdot \chi^u
\]
\[
= \bigoplus_{u \in \delta^V \cap M} (\bar{\gamma}^\#)^{-1}(H^0(Y, D(u))) \cdot \chi^u.
\]

**Proof of theorem 5.1.** We calculate the image of the Kodaira-Spencer map as described in [Ser06]. First, for each \( i, 1 \leq i \leq l_0 + l_1 \), we show that \( \pi \) is locally trivial with respect to the cover \( \mathfrak{U} \) by constructing isomorphisms
\[
\theta_i: U^i \times \text{Spec} \mathbb{C}[t]/t^2 \xrightarrow{\sim} X(\Xi_{\text{tot}})^{\mathfrak{U}}, \times S \text{ Spec} \mathbb{C}[t]/t^2.
\]
Let \( \tau_{-1}: Y^\prime \to Y^\prime \) be the automorphism induced by \( \bar{\gamma}_{-1}^\#: t \mapsto t \) and \( \bar{\gamma}_{-1}^\#: y_p \mapsto y_p + t \). Likewise, let \( \tau_1: Y^\prime \to Y^\prime \) be the identity. Furthermore, let \( \gamma_j, j = -1,1 \) be the corresponding morphisms from lemma 5.3. We also let \( \phi_v \) be as in lemma 5.2 where we take divisors \( D_0 = V(y_p), D_1 = V(y_p - t) \) on \( Y^\prime \). We claim that setting
\[
\theta_i = \phi_{a_i \lambda_i} \gamma_{a_i}
\]
gives the desired isomorphism. We can differentiate between the cases of complete and affine locus, that is between \( i \in \{l_0 + 1, \ldots, l_1\} \) and \( i \in \{1, \ldots, l_0, l_1 + 1, \ldots, l_1 + l_0\} \). For any \( i \in \{l_0 + 1, \ldots, l_1\} \) we have assumed that \( D \) is nontrivial only at \( V(y_p) = P \) and \( V(y_p^{-1}) = \infty \); let \( D_\infty = V(y_p^{-1}) \subset Y^\prime \). In this case,
\[
U^i \times \mathbb{A}^1 = X(D_i^p \cdot D_0 + D_i^\infty \cdot D_\infty) \xrightarrow{\gamma_i} X (D_i^p \cdot (D_0 + \frac{a_i - 1}{2}(D_0 - D_1)) + D_i^\infty \cdot D_\infty)
\]
\[
\xrightarrow{\phi_{a_i \lambda_i}} X(D_i^p,0 \cdot D_0 + D_i^{i,1} \cdot D_1 + D_i^\infty \cdot D_\infty) = X(\Xi_{\text{tot}})^{\mathfrak{U},i}
\]
which is the desired isomorphism when reduced modulo $t^2$. The other case is similar.

Now set $\theta_{i,j} = \theta_{i,j}^{-1}$. Define a derivation $d_{i,j}$ via $\theta_{i,j}^\# = \text{id} + t \cdot d_{i,j}$. For simplicity of notation, set $b_i = \frac{a_i - 1}{2}$. Now $\theta_{i,j}^\# = \gamma_{a_j}^\# a_{a_i} a_{a_j - a_i} (\gamma_{a_i}^\#)^{-1}$ and we can calculate

$$\gamma_{a_j}^\# a_{a_i} a_{a_j - a_i} (\gamma_{a_i}^\#)^{-1} (y_P) = \gamma_{a_j}^\# a_{a_i} a_{a_j - a_i} (y_P + b_i t) = \gamma_{a_j}^\# (y_P + b_i t) = y_P + (b_i - b_j) t = y_P + \frac{a_i - a_j}{2} \cdot t$$

and

$$\gamma_{a_j}^\# a_{a_i} a_{a_j - a_i} (\gamma_{a_i}^\#)^{-1} (\chi^u) = \gamma_{a_j}^\# a_{a_i} a_{a_j - a_i} (\chi^u) = \gamma_{a_j}^\# \left( \left( \frac{y_P - t}{y_P} \right)^{a_j a_i - a_i a_i} \chi^u \right) = \left( \gamma_{a_j}^\# \left( \frac{y_P - t}{y_P} \right)^{a_j a_i - a_i a_i} \chi^u \right) = \left( \frac{y_P - a_j t}{y_P} \right)^{a_j a_i - a_i a_i} \chi^u = \left( 1 - a_j y_P^{-1} t \right)^{a_j a_i - a_i a_i} \chi^u,$$

where the equality at the start of the last line can be seen by considering the cases $a_j = 1$ and $a_j = -1$ separately. Thus,

$$d_{i,j} (y_P) = \frac{a_i - a_j}{2}$$

$$d_{i,j} (\chi^u) = (a_i a_i - a_j a_j, u) \cdot y_P^{-1} \chi^u$$

and the theorem follows. $\square$

**Remark.** Although the above theorem is only stated for one-parameter deformations, it can easily be used to calculate the linear Kodaira-Spencer map $T_{S,0} \to H^1(X, T_X)$ for a $k$-parameter locally trivial deformation. Indeed, $T_{S,0} = T_{h^k,0}$ and each basis vector of the natural basis of $T_{h^k,0}$ corresponds to a one-parameter deformation as described above.

# 6 Deformations of Non-affine Toric Varieties

As mentioned in the remark at the end of section 1, a toric variety can be viewed as a $T$-variety by considering the action of some subtorus of the big torus. Thus, we can use the divisorial fan decompositions of section 4 to construct deformations of an arbitrary toric variety. If in particular the deformation is locally trivial, we can use theorem 5.1 to calculate the image of the Kodaira-Spencer map. We will use this result to show that the deformations we construct span the space of infinitesimal deformations for any complete smooth toric variety.

Let $N'$ be an $n$-dimensional lattice with dual $M'$; choose some basis $e_1, \ldots, e_n$ of $N'$ with corresponding dual basis $e_1^* \ldots e_n^*$. Let $\Sigma$ be a fan on $N_0^\vee$ with corresponding toric variety $X = \text{TV}(\Sigma)$. We can consider $X$ as a complexity-one $T$-variety with respect to
the subtorus $T^N$, where $N$ is generated by the first $n - 1$ basis elements and the cosection $s : N' \to N$ is simply the natural projection. In this setting, a homogeneous deformation of $X$ is given by Minkowski decompositions of the polyhedral complexes $\Xi_0 = s(\Sigma \cap [e_n^* = 1])$ and $\Xi_\infty = s(\Sigma \cap [e_n^* = -1])$.

Example. Consider the toric surface $X$ in the first example in section 1. A two parameter deformation of $X$ can be constructed by decomposing $\Xi_0 = \Xi_0' + \Xi_0''$ and $\Xi_\infty = \Xi_\infty' + \Xi_\infty''$. Note that the coefficients $D_0^{3,0}, D_0^{4,1}, D_0^{5,1}$ are all simply the lattice point 0. In this case, the base space $S \subset \mathbb{A}^2$ is the complement of the set $V(t_0t_\infty - 1)$.

Now take some homogeneous one-parameter deformation $\pi$ of $X$ coming from a Minkowski decomposition of $\Xi_0$. Suppose additionally that the decomposition of $\Xi_0$ is trivial up to lattice translation. As was previously noted, this must be the case if $X$ is smooth. Let $\mathfrak{U} = \{ U^i = TV(\sigma_i) | \sigma_i \in \Sigma^{(a)} \}$; one can check that this the same open affine covering $\mathfrak{U}$ considered in the previous section. We can define then vectors $\lambda_i \in N$ and integers $a_i \in \{-1, 1\}$ exactly as in the previous section. Furthermore, set $\lambda_i' = (\lambda_i, \frac{1}{2})$.

**Theorem 6.1.** The deformation $\pi$ is locally trivial and its image under the Kodaira-Spencer map is the cocycle defined by

$$d_{i,j} = \sum_{k=1}^{n} (a_i \lambda_i' - a_j \lambda_j', e_k^*) \chi^{e_k^* - e_i^*} \frac{\partial}{\partial \chi^{e_k^*}}.$$ 

In particular, $\pi$ is homogeneous of degree $-e_n^*$.

**Proof.** The assumption that a polyhedral divisor $\mathcal{D}$ with complete locus has at most non-trivial coefficients for $P = 0$ and $\infty$ automatically holds for any toric variety, so we can apply theorem 5.1. The formula follows then directly. \qed

Figure 7: Slice decompositions for a toric surface
Example. We revisit the example from the beginning of this section. Restricting in $S$ to either $t_\infty = 0$ or $t_0 = 0$ gives one-parameter deformations $\pi_0$ or $\pi_\infty$ corresponding, respectively, to the decomposition $\Xi_0 = \Xi'_0 + \Xi''_0$ or $\Xi_\infty = \Xi'_\infty + \Xi''_\infty$. Using the above theorem and the numbering on two-dimensional cones induced by the numbering on the polyhedral divisors yields that the image under the Kodaira-Spencer map is induced by

$$d_{i-1,i} = \begin{cases} \frac{\partial}{\partial \chi^i} e_n & i = 2 \\ -\frac{\partial}{\partial \chi^i} e_n & i = 3 \\ 0 & \text{otherwise} \end{cases}$$

for $\pi_0$ and by

$$d_{i-1,i} = \begin{cases} \frac{\partial}{\partial \chi^i} e_n & i = 4 \\ -\frac{\partial}{\partial \chi^i} e_n & i = 5 \\ 0 & \text{otherwise} \end{cases}$$

for $\pi_\infty$ where we number indices modulo 6. Thus, the deformation $\pi$ combines deformations of degree $-e^*_n$ and $e^*_n$.

Now, as mentioned in section 3, $X$ is the minimal resolution of a toric Fano surface $X'$ whose fan has rays through $(1, 1), (1, -1), (-1, 1),$ and $(-1, -1)$. We can then blow down the deformation $\pi$ to a deformation $\pi'$ of $X'$. One easily checks that this is in fact the same deformation $\pi'$ constructed in the example in section 3 by deforming the cone over $X'$ and descending to the quotient.

Assume now additionally that $X = TV(\Sigma)$ is smooth and complete. Our goal is to construct homogeneous deformations spanning $T^1_X$. Choose some $R \in M'$ and let $\rho \in \Sigma^{(1)}$ be some ray with $\langle \rho, R \rangle = 1$. Note that by abuse of notation we denote a ray and its primitive generator by the same symbol. By proper choice of the basis $e_1, \ldots, e_n$ above, we can assume that $R = e^*_n$ and that $\rho = e_n$.

Let $\Gamma_\rho(-R)$ be the graph embedded in $N'_Q$ with vertices consisting of primitive lattice generators of rays $\tau \in \Sigma^{(1)} \setminus \rho$ fulfilling $\langle \tau, R \rangle > 0$; two vertices $\tau_1$ and $\tau_2$ are connected by an edge if they are common faces of some cone in $\Sigma$. Note that by rescaling with $\mathbb{R}_{>0}$ we can consider $\Gamma_\rho(-R)$ to be embedded in the slice $\Xi_0$, with vertices of $\Gamma_\rho(-R)$ corresponding to non-zero vertices of $\Xi_0$ and with two vertices connected by an edge if they are in fact connected by a one-simplex in $\Xi_0$.

Now for $R \in M'$, define

$$\Omega(-R) = \{ \rho \in \Sigma^{(1)} | \langle \rho, R \rangle = 1 \text{ and } \Gamma_\rho(-R) \neq \emptyset \}.$$

Assume that $\rho \in \Omega(-R)$ and choose now some connected component $C$ of $\Gamma_\rho(-R)$. This leads to a decomposition of the slice $\Xi_0$ as follows. Consider $\mathcal{D}^i \in \Xi$. If $\mathcal{D}_0^i \cap C = \emptyset$, then set $\mathcal{D}_0^{i,0} = \mathcal{D}_0^i$ and $\mathcal{D}_0^{i,1} = \text{tail}(\mathcal{D}_0^i)$. If instead the intersection $\mathcal{D}_0^i \cap C$ is nonempty, set $\mathcal{D}_0^{i,0} = \text{tail}(\mathcal{D}_0^i)$ and $\mathcal{D}_0^{i,1} = \mathcal{D}_0^i$.

**Proposition 6.2.** The $\{\mathcal{D}_0^{i,j}\}$ form an admissible one-parameter Minkowski decomposition of the slice $\Xi_0$. 18
Theorem 6.3. Let $X$ be a smooth complete toric variety and $T^1_X(-R)$ the space of infinitesimal deformations in degree $-R$ for some $R \in M'$. Then the one-parameter deformations $\pi(C, \rho, R)$ span $T^1_X(-R)$, where $\rho$ ranges over all rays $\rho \in \Omega(-R)$ and $C$ ranges over all connected components of the graphs $\Gamma_\rho(-R)$.

Proof. To prove the theorem, we simply calculate the Kodaira-Spencer map for the above deformations and then use the description of $T^1_X(-R)$ from [Ht09a]. For $\rho \in \Omega(-R)$, let $\partial(R, \rho)$ be the derivation taking $\chi^v \mapsto \langle \rho, v \rangle \chi^{v-R}$. If we choose the basis $e_1, \ldots, e_n$ such that $e_n = \rho$ and $R = e_n^*$, then $\partial(R, \rho) = \frac{\partial}{\partial \chi^{e_n}}$. Applying theorem 6.1 we then have that the image
of \( \pi(C, \rho, R) \) is given by
\[
d_{i,j} = \frac{a'_i - a'_j}{2} \partial(R, \rho)
\]
where \( a'_i = 1 \) if \( D_i \cap C = \emptyset \) and \( a'_i = -1 \) otherwise. Indeed, it follows from the above construction that \( a_i = a'_i \). Furthermore, \( \lambda_i = 0 \) for all \( i \).

Now let \( f \in H^0(\Gamma_\rho(-R), \mathbb{C}) \) and \( \sigma \in \Sigma^{(n)} \). If \( \Gamma_\rho(-R) \cap \sigma = \emptyset \), set \( f(\sigma) = 1 \), otherwise set \( f(\sigma) = f(v) \) for any \( v \in \Gamma_\rho(-R) \cap \sigma \). From [Ilt09a] we then have the exact sequence
\[
0 \longrightarrow \bigoplus_{\rho \in \Omega(-R)} \mathbb{C} \longrightarrow \bigoplus_{\rho \in \Omega(-R)} H^0(\Gamma_\rho(-R), \mathbb{C}) \longrightarrow H^1(X, T_X(-R)) \longrightarrow 0
\]
where \( \Phi \) maps \( f \in H^0(\Gamma_\rho(-R), \mathbb{C}) \) to the Čech cocycle \( f_{i,j} = \frac{1}{2}(f(\sigma_i) - f(\sigma_j))\partial(R, \rho) \). Now, for \( \rho \in \Omega(-R) \) and any connected component \( C \) in \( \Gamma_\rho(-R) \) let \( f(C, \rho, R) \in H^0(\Gamma_\rho(-R)) \) be defined by \( f(C, \rho, R)|_C \equiv -1 \) and \( f(C, \rho, R)|_{\Gamma_\rho(-R)\setminus C} \equiv 1 \). Then we have that \( \Phi(f(C, \rho, R)) \) is equal to the image of \( \pi(C, \rho, R) \) in \( H^1(X, T_X^1) \) by the above calculation. Furthermore, one easily sees that the \( f(C, \rho, R) \) form a basis of \( H^0(\Gamma_\rho(-R), \mathbb{C})/\mathbb{C} \), where \( C \) ranges over all connected components of \( \Gamma_\rho(-R) \) except one. Thus, if we allow \( \rho \) to vary over the elements of \( \Omega(-R) \) as well, the \( \pi(C, \rho, R) \) span \( T_X^1(-R) \).

\[\begin{array}{c}
\begin{align*}
\Xi_0 & \\
\Xi_\infty & \\
\Xi'_0 & \\
\Xi''_0 & \\
\Xi'''_0 & \\
\end{align*}
\end{array}\]

Figure 8: Slices of \( \Xi \) for a toric threefold

Figure 9: The versal deformation of a toric threefold

**Example.** We shall now construct the versal deformation of a certain toric threefold \( X \). Let \( X \) be represented as a \( T \)-variety via the divisorial fan shown in figure 8 where the affinely shifted cones in \( \Xi_0 \) are “glued” to their tailcones in \( \Xi_\infty \) to result in a single polyhedral divisor.
We can also describe $X$ in toric terms via a fan $\Sigma$: Let $\rho_1 = (1,0,1)$, $\rho_2 = (1,1,0)$, $\rho_3 = (0,1,1)$, $\rho_4 = (-1,0,0)$, $\rho_5 = (-1,-1,1)$, $\rho_6 = \rho_9 = (0,-1,0)$, $\rho_7 = (0,0,1)$, and $\rho_8 = -\rho_7$. The fan $\Sigma$ has top-dimensional cones generated by $\rho_i, \rho_{i+1}, \rho_7$ or by $\rho_i, \rho_{i+1}, \rho_8$ for $0 \leq i < 6$.

In [Ilt09a], the first author stated that $\dim T^1_X = \dim T^1_X(-R) = 2$ for $R = [0,0,1]$. Now, consider the two-parameter deformation $\pi$ corresponding to the decomposition of $\Xi_0 = \Xi'_0 + \Xi''_0 + \Xi'''_0$ in figure 3. This is in fact the versal deformation of $X$. Indeed, restricting to $t_{0,i} = 0$ for $i = 1, 2$ gives deformations $\pi(C(i), \rho_7, R)$, where $C(1) = (-1,-1)$ and $C(2) = (0,1)$ are two of the three connected components of $\Gamma_{\rho_7}(u)$. Thus, the map $T_{h^2,0} \to T^1_X$ determined by $\pi$ is surjective and so $\pi$ is versal.

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