TOWARDS HEIM AND NEUHAUSER’S UNIMODALITY CONJECTURE ON THE NEKRASOV-OKOUNKOV POLYNOMIALS

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Abstract. Let \( Q_n(z) \) be the polynomials associated with the Nekrasov-Okounkov formula

\[
\sum_{n \geq 1} Q_n(z) q^n := \prod_{m=1}^{\infty} (1 - q^m)^{-z-1}.
\]

In this paper we partially answer a conjecture of Heim and Neuhauser, which asks if \( Q_n(z) \) is unimodal, or stronger, log-concave for all \( n \geq 1 \). Through a new recursive formula, we show that if \( A_{n,k} \) is the coefficient of \( z^k \) in \( Q_n(z) \), then \( A_{n,k} \) is log-concave in \( k \) for \( k \ll n^{1/6} / \log n \) and monotonically decreasing for \( k \gg \sqrt{n \log n} \). We also propose a conjecture that can potentially close the gap.

1. Introduction

In their groundbreaking work [6], Nekrasov and Okounkov showed the hook length formula

\[
\sum_{\lambda} q^{\lvert \lambda \rvert} \prod_{h \in \mathcal{H}(\lambda)} \left(1 + \frac{z}{h^2}\right) = \prod_{m=1}^{\infty} (1 - q^m)^{-z-1},
\]

where \( \lambda \) runs over all Young tableaux, \( \lvert \lambda \rvert \) denotes the size of \( \lambda \), and \( \mathcal{H}(\lambda) \) denotes the multiset of hook lengths associated to \( \lambda \). We define

\[
Q_n(z) = \sum_{\lvert \lambda \rvert = n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 + \frac{z}{h^2}\right).
\]

For example, we can calculate that

\[
Q_0(z) = 1, \\
Q_1(z) = 1 + z, \\
Q_2(z) = 2 + \frac{5}{2}z + \frac{1}{2}z^2, \\
Q_3(z) = 3 + \frac{29}{6}z + 2z^2 + \frac{1}{6}z^3.
\]

The polynomials \( Q_n(z) \) are of degree \( n \) with positive coefficients and satisfy

\[
\sum_{n \geq 1} Q_n(z) q^n = \prod_{m=1}^{\infty} (1 - q^m)^{-z-1}.
\]

The study of \( Q_n(z) \) was initiated by D’Arcais in [1]. More recently in [4] and [5], Heim and Neuhauser have been investigating the number theoretic and distributional properties of the \( Q_n(z) \).

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1This formula was also obtained concurrently by Westbury (see Proposition 6.1 and 6.2 of [9].)

2D’Arcais defined the polynomials via the infinite product, not the hook number expression.
They proved the identity ([4], Conjecture 1)

\[ Q_n(z) = \sum_{|\lambda|=n} \prod_{h \in \mathcal{H}(\lambda)^0} \left( 1 + \frac{z}{h} \right) \]

where \( \mathcal{H}(\lambda)^0 \) denotes the multiset of hook lengths associated to \( \lambda \) with trivial legs.

In [4], Heim and Neuhauser conjectured that the polynomials \( Q_n(z) \) are unimodal. In other words, let the coefficient of \( z^k \) in \( Q_n(z) \) as \( A_{n,k} \); then there exists some integer \( k_1 \in [0, n] \) such that \( A_{n,i} < A_{n,i+1} \) when \( 0 \leq i < k_1 \) and \( A_{n,i} > A_{n,i+1} \) when \( k_1 \leq i < n \). They verified via computation that up to \( n \leq 1000 \), the polynomials \( Q_n(z) \) are in fact log-concave, which means that \( A_{n,k}^2 \geq A_{n,k-1}A_{n,k+1} \) for all \( 1 \leq k \leq n - 1 \). In this paper we make partial progress towards Heim and Neuhauser’s conjecture. We show that the polynomial \( Q_n(z) \) is log-concave at the start, and monotone decreasing at the tail. Throughout the rest of the paper, the constants in \( O, \gg \) and \( \ll \) are absolute unless otherwise stated.

**Theorem 1.1.** For \( n \) sufficiently large, we have

1. For \( k \ll n^{1/6} \log n \), we have \( A_{n,k}^2 \geq A_{n,k-1}A_{n,k+1} \).

2. For \( k \gg \sqrt{n} \log n \), we have \( A_{n,k} \geq A_{n,k+1} \).

We also reduce Heim and Neuhauser’s conjecture to a more “explicit” conjecture. For positive integers \( n \), define

\[ \sigma_{-1}(n) = \sum_{d|n} d^{-1} \]

and define \( f(q) \) to be its generating function

\[ f(q) := \sum_{n \geq 1} \sigma_{-1}(n) q^n. \]

We are interested in the behavior of \( c_{n,k} \), the coefficient of \( q^n \) in \( f^k(q) \).

**Conjecture 1.2.** There exists a constant \( C > 1 \) such that for all \( k \geq 2 \) and \( n \leq C^k \), we have

\[ c_{n,k}^2 \geq c_{n-1,k}c_{n+1,k}. \]

**Remark.** In the last section we offer some numerics with regard to this conjecture. We believe that \( C = 2 \) is a viable value in the Conjecture.

We show that Conjecture 1.2 implies Heim and Neuhauser’s conjecture for \( n \) sufficiently large.

**Theorem 1.3.** If Conjecture 1.2 is true, then for all \( n \gg \log^{-7} C + 1 \), the polynomial \( Q_n(z) \) is unimodal.

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2. Proof of Theorem 1.1(1)

2.1. A recursive formula for $A_{n,k}$. Our proof is centered around the following observation.

**Lemma 2.1.** For any positive integers $a < b$, and any $n \geq b$, we have

$$A_{n,b} = \frac{a!}{b!} \sum_{i=0}^{n} A_{n-i,a} c_{i,b-a}.$$ 

**Proof.** We first note that $f(q)$ is equal to the log derivative of

$$\prod_{m=1}^{\infty} (1 - q^m)^{-1}.$$ 

In (1.3), taking derivative with respect to $z$ for $k$ times, then setting $z = 0$, we get

$$\sum_{n=0}^{\infty} A_{n,b} q^n = \frac{1}{b!} f^b(q) \prod_{m=1}^{\infty} (1 - q^m)^{-1}.$$ 

Therefore, we obtain

$$\sum_{n=0}^{\infty} A_{n,b} q^n = \frac{a!}{b!} f^{b-a}(q) \sum_{n=0}^{\infty} A_{n,a} q^n.$$ 

The lemma then follows from the definition of $c_{n,k}$.

2.2. An asymptotic for $c_{n,k}$ and $A_{n,k}$. We now apply Lemma 2.1 on $(a, b) = (0, k)$, giving

$$A_{n,k} = \frac{1}{k!} \sum_{i=0}^{n} A_{n-i,0} c_{i,k}.$$ 

We observe that $A_{n-i,0} = p(n-i)$, where $p(n)$ is the partition function, which satisfies a well-known asymptotic obtained by Hardy and Ramanujan that we shall use below. Thus, to understand the behavior of $A_{n,k}$, it suffices to estimate $c_{i,k}$. However, we are only able to obtain a very crude estimate.

**Lemma 2.2.** For any positive integers $n, k$ with $n \geq k^2$, we have

$$\sum_{m \leq n} c_{m,k} = \left( \frac{\pi^2}{6} \right)^k \binom{n}{k} \left( 1 + O \left( \frac{k^2 \log n}{n} \right) \right).$$

---

Throughout this paper, we define $A_{n,k}$ or $c_{n,k}$ to be 0 for all undefined subscripts.
Proof. We first note that
\[ \sum_{m \leq n} c_{m,k} = \sum_{a_1 + \cdots + a_k \leq n} \sigma_1(a_1) \cdots \sigma_1(a_k) \]
(2.4)
\[ = \sum_{d_1, \ldots, d_k = 1}^{n} \frac{1}{d_1 d_2 \cdots d_k} \# \{(x_1, \ldots, x_k) \in \mathbb{N}^+ : d_1 x_1 + \cdots + d_k x_k \leq n \}. \]

We now show that
\[ \# \{(x_1, \ldots, x_k) \in \mathbb{N}^+ : d_1 x_1 + \cdots + d_k x_k \leq n \} \leq \frac{n^k}{k! d_1 \cdots d_k} \]
(2.5)
\[ - n^{k-1} (d_1 + \cdots + d_k) \leq \frac{n^k}{(k-1)! d_1 \cdots d_k}. \]

For each \((x_1, \ldots, x_k) \in \mathbb{N}^+\) that satisfies \(d_1 x_1 + \cdots + d_k x_k \leq n\), we place a unit cube with the uppermost vertex at \((x_1, \ldots, x_k)\). The union of the cubes are contained in the region
\[ \# \{(x_1, \ldots, x_k) \in \mathbb{R}^+ : d_1 x_1 + \cdots + d_k x_k \leq n \} \]
and contain the region
\[ \# \{(x_1, \ldots, x_k) \in \mathbb{R}^+ : d_1 x_1 + \cdots + d_k x_k \leq n - d_1 - \cdots - d_k \}. \]
Calculating the volume of the regions, we get (2.5). Plugging (2.5) to (2.4), we conclude that
\[ \sum_{m \leq n} c_{m,k} = R + \sum_{d_1, \ldots, d_k = 1}^{n} \frac{1}{d_1 d_2 \cdots d_k} \frac{n^k}{k! d_1 \cdots d_k} \]
\[ = R + \left( \frac{\pi^2}{6} \right)^k \binom{n}{k} \left( 1 + O \left( \frac{k^2}{n} \right) \right) \]
where the error term \(R\) is controlled by
\[ |R| \leq \sum_{d_1, \ldots, d_k = 1}^{n} \frac{1}{d_1 d_2 \cdots d_k} \frac{n^{k-1} (d_1 + \cdots + d_k)}{(k-1)! d_1 \cdots d_k} \]
\[ = k \cdot \frac{n^{k-1}}{(k-1)!} \sum_{d_1, \ldots, d_k = 1}^{n} \frac{1}{d_1^2 d_2 d_3 \cdots d_k^2} \]
\[ \ll \frac{n^{k-1}}{(k-1)!} \cdot k \left( \frac{\pi^2}{6} \right)^{k-1} \log n. \]
Combining the error terms, we conclude the lemma. \(\square\)

With this estimate we can get an asymptotic expression for \(A_{n,k}\) when \(n\) is much larger than \(k\).

Lemma 2.3. For \(n \geq k^4 \log^4 k\), we have
\[ A_{n,k} = \left( 1 + O \left( \frac{k^2 \log^2 n}{\sqrt{n}} \right) \right) \frac{k}{k!} \left( \frac{\pi^2}{6} \right)^k \sum_{i=0}^{n} p(n-i) \binom{i-1}{k-1}. \]

Proof. We recall the Hardy-Ramanujan formula for \(p(n)\) in [3],
\[ p(n) \sim \frac{e^{\pi \sqrt{2/3 \sqrt{n}}}}{4 \sqrt{3n}}. \]
(2.6)
From (2.3), we obtain
\[ A_{n,k} = \frac{1}{k!} \sum_{i=0}^{n} (p(n-i) - p(n-i-1)) \sum_{j \leq i} c_{j,k} \]
where we let \( p(-1) = 0 \). We first cut off the part of small \( i \), and we obtain
\[ \sum_{i \leq n^{1/2}/\log n} (p(n-i) - p(n-i-1)) \sum_{j \leq i} c_{j,k} \leq p(n) \sum_{j \leq n^{1/2}/\log n} c_{j,k}, \]
while we also have
\[ \sum_{i \in [2\sqrt{n},3\sqrt{n}]} (p(n-i) - p(n-i-1)) \sum_{j \leq i} c_{j,k} \geq (p(n - \lceil 2\sqrt{n} \rceil) - p(n - \lfloor 3\sqrt{n} \rfloor)) \sum_{j \leq 2\sqrt{n}} c_{j,k}. \]
By (2.6), we obtain
\[ p(n) \asymp p(n - \lceil 2\sqrt{n} \rceil) - p(n - \lfloor 3\sqrt{n} \rfloor), \]
while by Lemma 2.2, when \( n \geq k^4 \log^4 k \), we get
\[ \sum_{j \leq \sqrt{n}/\log n} c_{j,k} \ll (2 \log n)^{-k} \sum_{j \leq 2\sqrt{n}} c_{j,k}. \]
Therefore, we obtain that
\[ A_{n,k} = \left( 1 + O((2 \log n)^{-k}) \right) \frac{1}{k!} \sum_{i \in [n^{1/2}/\log n, n]} (p(n-i) - p(n-i-1)) \sum_{j \leq i} c_{j,k}. \]
Applying Lemma 2.2 we conclude that
\[ A_{n,k} = \left( 1 + O((2 \log n)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}}) \right) \frac{1}{k!} \sum_{i \in [n^{1/2}/\log n, n]} (p(n-i) - p(n-i-1)) \left( \frac{\pi^2}{6} \right)^k \binom{i}{k}. \]
Simplifying, we get the desired conclusion
\[ A_{n,k} = \left( 1 + O((2 \log n)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}}) \right) \frac{1}{k!} \left( \frac{\pi^2}{6} \right)^k \sum_{i=0}^{n} p(n-i) \binom{i-1}{k-1}. \] □

Remark. From this lemma it is easy to derive an explicit asymptotic for \( A_{n,k} \) as \( n \to \infty \) by simply plugging in the asymptotic for \( p(n) \). However, for our application it is simpler to leave \( A_{n,k} \) unsimplified in the current form.

2.3. Proof of Theorem 1.1(1). We now note that Lemma 2.3 essentially tells us that \( A_{n,k} \) is close to a log-concave sequence. This directly implies the desired result.

Lemma 2.4. For \( n \geq k^4 \log^4 k \), we have
\[ A_{n,k} = \left( 1 + O((2 \log n)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}}) \right) \tilde{A}_{n,k}, \]
where \( \tilde{A}_{n,k} \) is a log-concave sequence in \( n \).

\(^4\)Note that \( p(n) \) is monotone increasing, thus all the terms are positive.
Proof. Lemma \[2.3\] says that

\[ A_{n,k} = \left(1 + O \left( \left(2 \log n\right)^{-k} \right) \right) \frac{1}{k!} \sum_{i=0}^{n} p(n-i) \binom{i-1}{k-1}. \]

By \[(2.6)\], we have \( p(\sqrt{n}) \gg e^{n^{1/5}} p(25), \) while \( \binom{n-1}{k-1} \propto \binom{n-\frac{\sqrt{n}}{k-1}-1}{k-1} \). Thus, it follows that

\[ A_{n,k} = \left(1 + O \left( \left(2 \log n\right)^{-k} \right) \right) \frac{1}{k!} \sum_{i=0}^{n} p(n-i) \binom{i-1}{k-1}. \]

By \[2\], the sequence \( p(n)_{n \geq 25} \) is log-concave, and the binomial polynomial \( \binom{n}{k} \) is log-concave in \( n \). Therefore, the series

\[ \bar{A}_{n,i} = \frac{1}{k!} \left( \pi^2 \frac{2}{6} \right)^{k-n} \sum_{i=0}^{n-1} p(n-i) \binom{i-1}{k-1}. \]

is the convolution of two log-concave series, and is therefore log-concave; we thus obtain the lemma.

\[ \square \]

Remark. Numerical evidence suggests that for all \( k \geq 2 \), the sequence \( A_{n,k} \) is log-concave in \( n \). Unfortunately, this statement seems to be even harder than Heim and Neuhauser’s conjecture.

Proof of Theorem \[7.1(1)\]. For convenience, we replace \( k \) with \( k+1 \). We use Lemma \[2.1\] for \((a, b) = (k, k+1)\) and \((a, b) = (k, k+2)\), and get

\[ (2.7) \quad A_{n,k+1} = \frac{1}{k} \sum_{i=0}^{n} A_{n-i,k} \sigma_{-1}(i) \]

and

\[ (2.8) \quad A_{n,k+2} = \frac{1}{k(k+1)} \sum_{i,j \geq 0, i+j \leq n} A_{n-i-j,k} \sigma_{-1}(i) \sigma_{-1}(j). \]

By \((2.6)\), for any \( 0 \leq x \leq n/2 \), we have

\[ p(\lfloor n/2 \rfloor + x) \gg e^{0.5\sqrt{n}} p(x). \]

Thus, comparing \((2.3)\) term by term, for all \( i \leq n/2 \), we have

\[ A_{n-1,k} \gg e^{0.5\sqrt{n}} A_{i,k}. \]

Since \( \sigma_{-1}(i) \in [1, 2 + \log |i|] \), we conclude that the terms in \((2.7)\) with \( i \leq \frac{n}{2} \) are all negligible compared to the term \( i = 1 \). More precisely, we get

\[ A_{n,k+1} = \left(1 + O \left( e^{-0.4\sqrt{n}} \right) \right) \frac{1}{k} \sum_{i=0}^{n/2} A_{n-i,k} \sigma_{-1}(i). \]

Applying Lemma \[2.4\] we get

\[ A_{n,k+1} = \left(1 + O \left( \left(2 \log n/2\right)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}} \right) \right) \frac{1}{k} \sum_{i=0}^{n/2} \bar{A}_{n-i,k} \sigma_{-1}(i). \]

Similarly, from \((2.8)\) and Lemma \[2.4\] we get

\[ A_{n,k} = \left(1 + O \left( \left(2 \log n/2\right)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}} \right) \right) \bar{A}_{n,k}. \]
and

\[ A_{n,k+2} = \left(1 + O \left(\frac{(2 \log n/2)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}}}{k(k+1)}\right)\right) \frac{1}{k(k+1)} \sum_{i,j=0}^{\lfloor n/2 \rfloor} \tilde{A}_{n-i-j,k} \sigma_{-1}(i) \sigma_{-1}(j). \]

We note that by the log-concavity of \( \tilde{A}_{n,k} \), we have

\[ \tilde{A}_{n,k} \sum_{i,j=0}^{\lfloor n/2 \rfloor} \tilde{A}_{n-i-j,k} \sigma_{-1}(i) \sigma_{-1}(j) \leq \sum_{i,j=0}^{\lfloor n/2 \rfloor} \tilde{A}_{n-i,k} \tilde{A}_{n-j,k} \sigma_{-1}(i) \sigma_{-1}(j) = \left( \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{A}_{n-i,k} \sigma_{-1}(i) \right)^2. \]

Thus, it follows that

\[ \frac{A_{n,k}A_{n,k+2}}{A_{n,k+1}^2} \leq \frac{k}{k+1} \left(1 + O \left(\frac{(2 \log n/2)^{-k} + \frac{k^2 \log^2 n}{\sqrt{n}}}{k(k+1)}\right)\right). \]

Since we assume that \( n \gg k^6 \log^6 k \), with the implicit constant sufficiently large, the big-O term is at most \( \frac{1}{k} \). In this case, we get \( A_{n,k}A_{n,k+2} \leq A_{n,k+1}^2 \) as desired. \( \square \)

3. Proof of Theorem 1.1(2)

3.1. Unsigned Stirling numbers of the first kind. Let \( \left[ \begin{array}{c} n \\ m \end{array} \right] \) denote the absolute values of the Stirling numbers of the first kind, i.e. they satisfy

\[ \sum_m \left[ \begin{array}{c} n \\ m \end{array} \right] t^m = t(t+1) \cdots (t+n-1), \]

and let \( H_n \) denote the \( n \)-th harmonic number. Sibuya [7] proved the following inequality:

\[ \left[ \begin{array}{c} n \\ m \end{array} \right] \leq \frac{n-m+1}{(n-1)(m-1)} H_{n-1} \leq \frac{H_m}{m-1}. \]

which gives us, for \( m \geq 2H_n + 1 \), that

\[ \left[ \begin{array}{c} n+1 \\ m+1 \end{array} \right] \leq \left( \frac{H_m}{m} \right)^t \leq 2^{-t}. \]

The following lemma is useful to our proof.

Lemma 3.1. Let \( r = \lfloor \log_2 n \rfloor \), and we are given a sequence \( \{k_j\} \). Define \( s_j = 2[H_{k_j}] + r + 1 \) for \( k_j \neq 0 \) and 0 otherwise, and take their sum \( s = \sum_j s_j \). We have

\[ \sum_{l_1 + \cdots + l_n = s, \ j} \prod_{j} k_j + 1 \left\lfloor \begin{array}{c} k_j + 1 \\ l_j \end{array} \right\rfloor \leq \sum_{l_1 + \cdots + l_n = s-r, \ j} \prod_{j} k_j + 1 \left\lfloor \begin{array}{c} k_j + 1 \\ l_j \end{array} \right\rfloor. \]

Proof. Let \( p \) be the index of the first term satisfying \( l_p \geq s_p \). We write \( l_j' = l_j \) for \( j \neq p \) and \( l_p' = l_p - r \). Recall that

\[ \left\lfloor \begin{array}{c} n+1 \\ m+t+1 \end{array} \right\rfloor \leq 2^{-t} \left\lfloor \begin{array}{c} n+1 \\ m+1 \end{array} \right\rfloor, \]

when \( m \geq 2H_n + 1 \). We note that

\[ l_p' = l_p - r \geq s_p - r = 2[H_{k_p}] + 1, \]
and so we obtain
\begin{equation}
\prod_j \left[ k_j + 1 \right] \leq 2^{-\left\lfloor \log_2 n \right\rfloor} \prod_j \left[ k_j + 1 \right] \leq \frac{1}{n} \prod_j \left[ k_j + 1 \right].
\end{equation}

For each tuple \((l_1, l_2, \ldots, l_n)\) such that \(\sum_j l_j = s\) and \(l_j \leq k_j\), we let \(i_0\) be the first \(i\) such that \(l_i \geq s_i\), and send it to \((l_1, \cdots, l_{i_0-1}, l_{i_0} - r, l_{i_0+1}, \cdots, l_n)\). Combining (3.2) and this correspondence and noting that each term of the right hand side’s summation of (3.1) has at most \(n\) preimages, we obtain our result. 

**Proof of Theorem 1.1(2).** We let 
\[
k_j := \# \{ i \mid \lambda_i = j \}.
\]
By Corollary 2 of [4], we have
\[
\sum_{k=0}^{n} A_{n,k} z^k = \sum_{|\lambda|=n} \prod_{j=1}^{n} \binom{k_j + z}{k_j}.
\]

Since all the roots of the polynomial
\begin{equation}
\prod_{j=1}^{n} \binom{k_j + z}{k_j} := \sum_k q_k z^k
\end{equation}
are real, the coefficients \(\{q_k\}\) form a log-concave thus unimodal sequence [8]. We next prove that the mode is at most \(O(\sqrt{n \log n})\) for these \(k_j\) satisfying the obvious identity \(k_1 + 2k_2 + \cdots + nk_n = n\).

Using (3.3), we directly calculate
\[
q_k = C_0 \sum_{l_1 + \cdots + l_{i_0} = k, \ l_j \leq k_j} \prod_j \left[ k_j + 1 \right],
\]
where the constant \(C_0 = \prod_j \frac{1}{k_j!}\). By Lemma 3.1 we can compare that \(q_s \leq q_{s-r}\).

\[
\sum jk_j = n, \quad \text{sum } s = \sum s_j = \sum_{k_j \neq 0} O(\log n) \quad \text{is of the asymptotic } O(\sqrt{n \log n}). \quad \text{The unimodality of } \{q_k\} \quad \text{implies that } k \gg \sqrt{n \log n} \quad \text{exceeds the mode. Therefore, the coefficients } \{A_{n,k}\} \quad \text{are monotonically decreasing in } k \quad \text{as we desire.}
\]

4. **On Conjecture 1.2 and Theorem 1.3**

4.1. **Proof of Theorem 1.3** As we have seen in Section 2, the main setback in our method is that we are unable to derive a good asymptotic for \(c_{n,k}\). Conjecture 1.2 is based on numerics, and its truth represents the obstruction for establishing Heim and Neuhauser’s Conjecture for large \(n\). In particular, its truth leads to a vastly improved form of Lemma 2.4.

**Lemma 4.1.** Assume Conjecture 1.2 Then for all \(k \geq 2\) and \(k^{3/2} \leq n \leq C^k\), we have
\begin{equation}
A_{n,k} = \left( 1 + O \left( e^{-n^{0.1}} \right) \right) \tilde{A}_{n,k}
\end{equation}
where \(\tilde{A}_{n,k}\) is a log-concave sequence in \(n\).
Proof. Recall that
\[ A_{n,k} = \frac{1}{k!} \sum_{i=0}^{n} p(n-i)c_{i,k}. \]
The sequence is thus almost the convolution of two log-concave sequences. It suffices to trim away the terms \( i \leq 25 \). We first recall from the proof of Lemma 2.2 that
\[ c_{n,k} \leq \frac{n^k}{k!} \left( \frac{\pi^2}{6} \right)^{k}. \]
Since \( c_{n,k} \) is log-concave for all \( n \leq C^k \), for any \( 0 \leq l \leq n - k \) we have
\[ \frac{c_{n,k}}{c_{n-l,k}} \leq \left( \frac{c_{n,k}}{c_{k,k}} \right)^{\frac{l}{n-k}}. \]
While by (2.6),
\[ p(l) \gg e^{0.5\sqrt{l}}. \]
Taking \( l = \lfloor n^{1/3} \rfloor \), we conclude that
\[ p(l)c_{n-l,k} \gg e^{n^{1/10}}c_{n,k}. \]
Since \( l > 25 \) for \( n > 25^3 \), it follows that
\[ A_{n,k} = \left( 1 + O \left( e^{-n^{1/10}} \right) \right) \frac{1}{k!} \sum_{i=0}^{n-25} p(n-i)c_{i,k}. \]
Since both \( c_{i,k}(i \leq n) \) and \( p(i)(i \geq 25) \) are log-concave sequences, the sequence
\[ \hat{A}_{n,k} = \frac{1}{k!} \sum_{i=0}^{n-25} p(n-i)c_{i,k} \]
is log-concave, so we have shown the lemma. \( \square \)

Proof of Theorem 1.3 By Theorem 1.1 it suffices to show, for all sufficiently large \( n \) and \( \frac{n^{1/6}}{\log n} \ll k \ll \sqrt{n} \log n \), that
\[ A_{n,k+1}^2 \geq A_{n,k}A_{n,k+2}. \]
Since \( n \gg \log^{-7} C + 1 \), for all \( k \) in this range we have \( 2k^{3/2} \leq n \leq C^k \). By the proof of Theorem 1.1 we get
\[ A_{n,k+1} = \left( 1 + O \left( e^{-n^{1/10}} \right) \right) \frac{1}{k} \sum_{i=0}^{\lfloor n/2 \rfloor} A_{n-i,k}\sigma_1(i). \]
By Lemma 4.1 we conclude that
\[ A_{n,k+1} = \left( 1 + O \left( e^{-n^{0.1}} \right) \right) \frac{1}{k} \sum_{i=0}^{\lfloor n/2 \rfloor} \hat{A}_{n-i,k}\sigma_1(i). \]
Similarly, we have
\[ A_{n,k} = \left( 1 + O \left( e^{-n^{0.1}} \right) \right) \hat{A}_{n,k} \]
and
\[ A_{n,k+2} = \left( 1 + O \left( e^{-n^{0.1}} \right) \right) \frac{1}{k(k+1)} \sum_{i,j=0}^{\lfloor n/2 \rfloor} \hat{A}_{n-i-j,k}\sigma_1(i)\sigma_1(j). \]
We note that by the log-concavity of \( \hat{A}_{n,k} \) for \( k^{3/2} \leq n \leq C^k \), we have
\[
\hat{A}_{n,k} \sum_{i,j=0}^{n/2} \hat{A}_{n-i-j,k} \sigma_1(i) \sigma_1(j) \leq \left( \sum_{i=0}^{n/2} \hat{A}_{n-i,k} \sigma_1(i) \right)^2.
\]
Thus, we obtain the desired conclusion. Namely, we have that
\[
A_{n,k} A_{n,k+2} \leq k \left( 1 + O\left(e^{-n^{0.1}}\right) \right) A_{n,k+1}^2.
\]

4.2. Numerical Evidence for Conjecture 1.2. We are unable to show Conjecture 1.2. Numerical evidence does suggest that Conjecture 1.2 is likely to hold for \( C = 2 \). Let \( n_0(k) \) denote the smallest \( n \) such that \( c_{n,k}^2 < c_{n-1,k} c_{n+1,k} \). The following table shows the value of \( n_0(k) \) for \( 2 \leq k \leq 13 \).

| \( k \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-------|---|---|---|---|---|---|---|---|----|----|----|----|
| \( n_0(k) \) | 6 | 21 | 39 | 73 | 135 | 251 | 475 | 917 | 1801 | 3595 | 7259 | 14787 |

**Remark.** We also note that Conjecture 1.2 seems to generalize to other series whose terms display a similar behavior, such as
\[
f(z) = \frac{z}{1 - z} + \frac{z^2}{2(1 - z^2)}.
\]
Investigating this phenomenon might be interesting on its own.

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