Counting on Rectangular Areas
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Abstract

In the first section of this paper we prove a theorem for the number of columns of a rectangular area that are identical to the given one. A special case, concerning (0, 1)-matrices, is also stated.

In the next section we apply this theorem to derive several combinatorial identities by counting specified subsets of a finite set. This means that the obtained identities will involve binomial coefficients only. We start with a simple equation which is, in fact, an immediate consequence of Binomial theorem, but it is derived independently of it. The second result concerns sums of binomial coefficients. In a special case we obtain one of the best known binomial identity dealing with alternating sums. Klee's identity is also obtained as a special case as well as some formulae for partial sums of binomial coefficients, that is, for the numbers of Bernoulli's triangle.

1 A counting theorem

The set of natural numbers \{1, 2, \ldots, n\} will be denoted by \([n]\), and by \(|X|\) will be denoted the number of elements of the set \(X\).

For the proof of the main theorem we need the following simple result:

\[
\sum_I (-1)^{|I|} = 0,
\]

where \(I\) run over all subsets of \([n]\) (empty set included). This may be easily proved by induction or using Binomial theorem. But the proof by induction makes all further investigations independent even of Binomial theorem.

Let \(A\) be an \(m \times n\) rectangular matrix filled with elements which belong to a set \(\Omega\).

By the \(i\)-column of \(A\) we shall mean each column of \(A\) that is equal to \([c_1, c_2, \ldots, c_m]^T\), where \(c_1, c_2, \ldots, c_m\) of \(\Omega\) are given. We shall denote the number of \(i\)-columns of \(A\) by \(\nu_A(c)\) or simply by \(\nu(c)\).

For \(I = \{i_1, i_2, \ldots, i_k\} \subset [m]\), by \(A(I)\) will be denoted the maximal number of columns \(j\) of \(A\) such that

\[a_{ij} \neq c_j, \quad (i \in I).\]
We also define
\[ A(\emptyset) = n. \]

**Theorem 1.** The number \( \nu(c) \) of i-columns of \( A \) is equal
\[ \nu(c) = \sum_I (-1)^{|I|}A(I), \tag{2} \]
where summation is taken over all subsets \( I \) of \([m]\).

Proof. Theorem may be proved by the standard combinatorial method, by counting the contribution of each column of \( A \) in the sum on the right side of (2).

We give here a proof by induction. First, the formula will be proved in the case \( \nu(c) = 0 \) and \( \nu(c) = n \). In the case \( \nu(c) = n \) it is obvious that for \( I \neq \emptyset \) we have \( A(I) = 0 \), which implies
\[ \sum_I (-1)^{|I|}A(I) = n + \sum_{I \neq \emptyset} (-1)^{|I|}A(I) = n. \]

In the case \( \nu(c) = 0 \) we use induction on \( n \). If \( n = 1 \) then the matrix \( A \) has only one column, which is not equal \( c \). It yields that there exists \( i_0 \in \{1, 2, \ldots, m\} \) such that \( a_{i_0, 1} \neq c_{i_0} \). Denote by \( I_0 \) the set of all such numbers. Then \( A(I) = 1 \) if and only if \( I \subset I_0 \). From this and (2) we obtain
\[ \sum_I (-1)^{|I|}A(I) = \sum_{I \subset I_0} (-1)^{|I|} = 0. \]

Suppose now that the formula is true for matrices with \( n \) columns and that \( A \) has \( n + 1 \)-columns, and \( \nu_A(c) = 0 \). Omitting the first column, the matrix \( B \) with \( n \) columns remains. By the induction hypothesis theorem is true for \( B \).

\[ \sum_I (-1)^{|I|}A(I) = \sum_{I \subset I_0} (-1)^{|I|} = 0, \]
\[ \sum_{I \notin I_0} (-1)^{|I|}B(I) + \sum_{I \subset I_0} (-1)^{|I|} = 0, \]
\[ = \sum_{I \subset I_0} (-1)^{|I|}B(I) + \sum_{I \subset I_0} (-1)^{|I|} = 0, \]
since the first sum is equal zero by the induction hypothesis, and the second by (2).

For the rest of the proof we use induction on \( n \) again. For \( n = 1 \) the matrix \( A \) has only one column which is either equal \( c \) or not. In both cases theorem is true, from the preceding.

Suppose that theorem holds for \( n \), and that the matrix \( A \) has \( n + 1 \) columns. We may suppose that \( \nu(c) \geq 1 \). Omitting one of the \( i \)-columns we obtain the matrix \( B \) with \( n \) columns. By the induction hypothesis theorem is true for \( B \).
On the other hand it is clear that \( A(I) = B(I) \) for each nonempty subset \( I \). Furthermore \( A \) has one i-column more then \( B \), which implies

\[
\nu(c) = \nu_A(c) = \nu_B(c) + 1 = 1 + \sum_{I} (-1)^{|I|} B(I) = 1 + n + \sum_{I \neq \emptyset} (-1)^{|I|} B(I) = 1 + \sum_{I \neq \emptyset} (-1)^{|I|} A(I).
\]

Thus

\[
\nu(c) = \sum_{I} (-1)^{|I|} A(I),
\]

and theorem is proved.

If the number \( A(I) \) does not depend on elements of the set \( I \), but only on its number \( |I| \) then the equation (2) may be written in the form

\[
\nu(c) = \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} A(i),
\]

where \( |I| = i \).

Our object of investigation will be \((0, 1)\) matrices. Let \( c \) be the i-column of a such matrix \( A \). Take \( I_0 \subseteq [m] \), \( |I_0| = k \) such that

\[
c_i = \begin{cases} 
1 & i \in I_0 \\
0 & i \notin I_0 
\end{cases}
\]

Then the number \( A(I) \) is equal to the number of columns of \( A \) having 0’s in the rows labelled by the set \( I \cap I_0 \), and 1’s in the rows labelled by the set \( I \setminus I_0 \). Suppose that the number \( A(I) \) depends only on \( |I \cap I_0|, |I \setminus I_0| \). If we denote \( |I \cap I_0| = i_1, |I \setminus I_0| = i_2 \), \( A(I) = A(i_1, i_2) \), then (2) may be written in the form

\[
\nu(c) = \sum_{i_1=0}^{k} \sum_{i_2=0}^{m-k} (-1)^{i_1+i_2} \binom{k}{i_1} \binom{m-k}{i_2} A(i_1, i_2).
\]

2 Counting subsets of a finite set

Suppose that a finite set \( X = \{x_1, x_2, \ldots, x_n\} \) is given. Label by \( 1, 2, \ldots, 2^n \) all subsets of \( X \) arbitrary and define an \( n \times 2^n \) matrix \( A \) in the following way

\[
a_{ij} = \begin{cases} 
1 & \text{if } x_i \text{ lies in the set labelled by } j \\
0 & \text{otherwise}
\end{cases}
\]

Take \( I_0 \subseteq [n] \), \( |I_0| = k \), and form the submatrix \( B \) of \( A \) consisting of those rows of \( A \) which indices belong to \( I_0 \). Let \( c \) be arbitrary i-column of \( B \). Define

\[
I_0^c = \{ i \in I_0 : c_i = 1 \}, \\
I_0'' = \{ i \in I_0 : c_i = 0 \}
\]

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The number $\nu(c)$ is equal to the number of subsets that contain the set \{\(x_i, \ i \in I_0^c\}\}, and do not intersect the set \{\(x_i : i \in I_0^c\}\}. There are obviously $\nu(c) = 2^{n-k}$, such sets.

Furthermore, if $I \subseteq I_0$ then the number $B(I)$ is equal to the number of subsets that contain the set \{\(x_i : i \in I \cap I_0^c\}\}, and do not meet the set \{\(x_i : i \in I \cap I_0\}\}. It is clear that there are

$$B(I) = 2^{n-|I|}$$

such subsets, so that the formula (2) may be applied. It follows

$$2^{n-k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2^{n-i}.$$

Thus we have

**Proposition 2.1.** For each nonnegative integer $k$ holds

$$1 = \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2^{k-i}.$$

**Note 2.1.** The preceding equation is a trivial consequence of Binomial theorem. But here it is obtained independently of this theorem.

The preceding Proposition shows that counting i-columns over all subsets of $X$ always produce the same result.

We shall now make some restrictions on the number of subsets of $X$. Take $0 \leq m_1 \leq m_2 \leq n$ fixed, and consider the submatrix $C$ of $A$ consisting of rows whose indices belong to $I_0$, and columns corresponding to those subsets of $X$ that have $m$, \((m_1 \leq m \leq m_2)\) elements.

Let $c$ be an i-column of $C$. Define $I_0' = \{i \in I_0 : c_i = 1\}$, \(|I_0'| = l\).

The number $\nu(c)$ is equal to the number of sets that contain \{\(x_i : i \in I_0 \setminus I_0'\}\}, and do not intersect the sets \{\(x_i : i \in I_0 \setminus I_0'\}\}. We thus have

$$\nu = \sum_{i=m_1-|I_0'|}^{m_2-|I_0'|} \binom{n-|I_0|}{i}.$$

On the other hand, for $I \subseteq I_0$ the number $C(I)$ corresponds to the number of sets that contain \{\(x_i : i \in I \setminus I_0'\}\}, and do not intersect \{\(x_i : i \in I \cap I_0'\}\}. Its number is equal

$$\sum_{i_3=m_1-|I_0'|}^{m_2-|I_0'|} \binom{n-|I|}{i_3}.$$

It follows that the formula (3) may be applied. We thus have
Proposition 2.2. For $0 \leq m_1 \leq m_2 \leq n$, and $0 \leq l \leq k$ holds

$$\sum_{i=m_1-l}^{m_2-l} \binom{n-k}{i} = \sum_{i=0}^{l} \sum_{i_2=0}^{i} \sum_{i_3=m_1-i_2}^{m_2-i_2} (-1)^{i_1+i_2} \binom{l}{i_1} \binom{k-l}{i_2} \binom{n-i_1-i_2}{i_3}$$  \hspace{1cm} (8)

In the special case when one takes $k = l$, $m_1 = m_2 = m$ we obtain

Corollary 2.1. For arbitrary nonnegative integers $m, n, k$ holds

$$\binom{n-k}{m-k} = \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} \binom{n-i}{m-i}.$$  \hspace{1cm} (9)

Note 2.2. The preceding is one of the best known binomial identities. It appears in the book [1] in many different forms.

Taking $m_1 = m_2 = m$, in (8) one gets

Corollary 2.2. For arbitrary nonnegative integer $m, n, k, l$, $(l \leq k)$ holds

$$\binom{n-k}{m-l} = \sum_{i=0}^{l} \sum_{i_1=0}^{k-l} (-1)^{i_1+i_2} \binom{l}{i_1} \binom{k-l}{i_2} \binom{n-i_1-i_2}{m-i_2},$$  \hspace{1cm} (10)

For $l = 0$ we obtain

$$\binom{n-k}{m} = \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} \binom{n-i}{m-i},$$  \hspace{1cm} (11)

which is only another form of (9).

Taking $n = 2k$, $l = k$ in (10) we obtain

$$\binom{k}{m-k} = \sum_{i_1=0}^{k} (-1)^{i_1} \binom{k}{i_1} \binom{2k-i_1}{m}.$$  \hspace{1cm} (12)

Substituting $k - i_1$ by $i$ we obtain

Corollary 2.3. Klee’s identity,([2],p.13)

$$(-1)^k \binom{k}{m-k} = \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} \binom{k+i}{m}.$$  

From (12) we may obtain different formulae for partial sums of binomial coefficients, that is, for the numbers of Bernoulli’s triangle. For instance, taking $l = 0$, $m_1 = 0$, $m_2 = m$ we obtain

Corollary 2.4. For any $0 \leq m \leq n$ and arbitrary nonnegative integer $k$ holds

$$\sum_{i=0}^{m} \binom{n}{i} = \sum_{i_1=0}^{k} \sum_{i_2=0}^{m-i_1} (-1)^{i_1+i_2} \binom{k}{i_1} \binom{n+k-i_1}{i_2}.$$  \hspace{1cm} (12)
Note 2.3. The number $k$ in the preceding equation may be considered as a free variable that takes nonnegative integer values. Specially, for $k = 1$ the equation represents the standard recursion formula for the numbers of Bernoulli’s triangle.

Taking $k = l = m_1$, $m_2 = m$ one obtains

$$
\sum_{i=0}^{m} \binom{n}{i} = \sum_{i_1=0}^{k} \sum_{i_2=k}^{m+k} (-1)^{i_1} \binom{k}{i_1} \binom{n + k - i_1}{i_2} \tag{13}
$$

Note 2.4. The formulae (12) and (13) differs in the range of the index $i_2$.

References

[1] J. Riordan, Combinatorial Identities. New York: Wiley, 1979.