Phase diagram and exotic spin-spin correlations of J₁-J₂ Ising model on the Sierpiński gasket

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The J₁-J₂ antiferromagnetic Ising model on the fractal Sierpiński gasket is intensively studied, and a number of exotic properties are disclosed. The ground state phase diagram in the plane of magnetic field-interaction of the system is obtained, where three nontrivial phases including the 1/3 magnetization plateau ordered and disordered phases, and the 5/9 plateau partially ordered phase, as well as five boundaries and two intersection points are accurately identified in terms of the uniform and subset magnetization and the residual entropy as well. The thermodynamic properties of the three plateau phases are probed by exploring the temperature-dependence of magnetization, specific heat, susceptibility and spin-spin correlations, where some interesting features due to the special geometry of the Sierpiński gasket are discovered. No phase transitions are observed in this model. In the absence of a magnetic field, the unusual temperature dependence of the spin correlation length is obtained with 0 ≤ J₂/J₁ < 1, and an interesting crossover behavior between different phases at J₂/J₁ = 1 is unveiled, whose dynamics can be described by the J₁/J₂-dependence of the specific heat, susceptibility and spin correlation functions. The exotic spin-spin correlation patterns that share the same special rotational symmetry as that of the Sierpiński gasket are obtained in both the 1/3 plateau disordered phase and the 5/9 plateau partially ordered phase.

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I. INTRODUCTION

The classical models like Ising or Potts models often exhibit intriguing properties, especially when there exists geometrical frustration that may cause extensive ground-state degeneracy and nonzero residual entropy of the system at zero temperature. There are many interesting phenomena in these models that have been discovered but not yet been totally understood, for instance, the essence of the partial order or the crossover between exotic phases, etc. Extensive works have been done on the Ising models in two- and three-dimensional systems, such as the Ising models on the checkerboard lattice and kagomé lattice, as well as the Potts model on some irregular lattices where the systems possess translational invariance. There were some early works on the Ising model on fractal lattices, which show the properties quite different from the models on one-, two- and three-dimensional lattices with translational invariance even with the simplest couplings. But for the Ising models on fractals, comprehensive studies on, e.g., spin-spin correlations where there is no translational invariance, are still needed and, the questions, such as how to describe the crossover behaviors in fractals, are yet to be explored.

The Sierpiński Ising model is so defined that the Ising spin is settled on each of vertices of the Sierpiński graph. The Hausdorff dimension 𝐷_𝐻 of an Sierpiński graph satisfies 𝐷_Ｈ = log(D_E + 1)/log(2), which is not an integer, where 𝐷_𝐸 is its Euclidean dimension. Unlike other lattices such as square or kagomé lattices, a Sierpiński graph has no translational symmetries, but bears some special geometrical properties that can be seen from its recursive construction procedure. The 𝐷_𝐸 = 2 Sierpiński graph, which is also named Sierpiński gasket (SG), is shown schematically in Fig. 1(a).

Some properties of the Ising model on Sierpiński graphs were noted. For example, the order of ramification on a SG is “marginal”, and by locating the fixed point of renormalization group (RG) equations, it was confirmed that there is no phase transition at any finite temperature. With the nearest-neighbor ferromagnetic couplings, the correlation length ξ increases extremely fast with the inverse temperature β = 1/T as

$$ \xi \propto \exp\left[\frac{\ln 2}{4}\exp(4J/\beta)\right], $$

where J is a positive constant (ferromagnetic coupling) and Boltzmann constant is taken as kB = 1. This unusual relation indicates that the correlation length ξ is finite at any finite temperature. Since ξ increases so fast with β, upon lowering temperature, the correlation length ξ can be larger than the size of any finite system, resulting in a long range order.

In this paper, the J₁-J₂ antiferromagnetic Ising model on the SG is studied systematically in the thermodynamic limit. The ground state phase diagram in the plane of magnetic field-interaction (J₂) is obtained, where three nontrivial phases, including the 1/3 magnetization plateau ordered and disordered phases, and the 5/9 plateau partially ordered phase, as well as five boundaries and two intersection points are identified. The 1/3 plateau disordered phase and the 5/9 plateau partially ordered phase are found to have extensive degeneracy. The thermodynamic properties of the system, including the temperature dependence of the uniform and subset magnetization, susceptibility, specific heat and spin correlation functions, are calculated. Some interesting results due to special geometrical properties like the fractional dimensionality and the absence of symmetries on usual periodic lattices are discovered. The nonexistence of the phase transition is testified. For h = 0, the unusual relation between the correlation length and temperature is obtained for 0 ≤ J₂/J₁ < 1, and a nontrivial crossover behavior between different phases at J₂/J₁ = 1 is disclosed.
whose dynamics can be described by the $J_2/J_1$-dependence of the specific heat, susceptibility and the spin correlation functions. Exotic spin-spin correlation patterns that have the same special rotational symmetry as that of the SG are observed in the 1/3 plateau disordered phase.

This paper is organized as follows. In Sec. II, we introduce the $J_1$-$J_2$ SG Ising model and give the exact formulation of the free energy. In Sec. III, the zero temperature phase diagram is identified by calculating the ground state uniform magnetization and residual entropy. In Sec. IV, the temperature dependence of thermodynamic quantities, including the uniform and subset magnetization, susceptibility and specific heat, are studied. In Sec. V, the exotic spin-spin correlation functions of the three nontrivial plateau phases are disclosed. Finally, a summary is given.

II. FREE ENERGY OF J$_1$-J$_2$ ISING MODEL ON SIERPIŃSKI GASKET

Let us begin with discussing the exact formulation of free energy of the $J_1$-$J_2$ antiferromagnetic Ising model on the SG with $D_E = 2$. The Hamiltonian reads

$$ H(|S\rangle) = \sum_{\langle ij \rangle} -J_{ij} S_i S_j - h(S_i + S_j)/4, \quad (2) $$

where $\langle ij \rangle$ means the spins at vertices $i$ and $j$ that are nearest neighbors on the SG, each spin $S_i$ takes values of $\pm 1/2$, $h$ is the magnetic field and $J_{ij}$ is the negative coupling constant. The couplings are indicated in Fig. 1(a), where we presume $J_{ij} = J_1$ for the blue bonds and $J_{ij} = J_2$ for the red bonds. We denote the SG that contains $N^{(t)} = (3 + 3^t)/2$ spins as $\Delta^{(t)}$ and its density matrix as $\rho^{(t)}(|S\rangle) = \exp[-\beta H(|S\rangle)]$, where $t$ is the renormalization step. Then, the partition function $Z$ can be expressed as $Z^{(t)} = \sum_{|S\rangle} \rho^{(t)}(|S\rangle)$.

We begin with $\rho^{(2)}$, which is the density matrix of the model with only six Ising spins in $\Delta^{(2)}$ [Fig. 1(b)] as

$$ \rho^{(2)} = \exp[-\beta \sum_{\langle ij \rangle \in \Delta^{(2)}} (-J_{ij} S_i S_j - h(S_i + S_j)/4)]. \quad (3) $$

For the purpose of convenience in our following formulation where the tensor index should be positive integers, we ought to introduce new variable $s_i$ as the tensor index, which is defined by $s_i = S_i + 3/2$ ($i = 1, \ldots, 6$). Thus, $s_i$ only takes positive integers as 1 or 2, corresponding to the up and down states of an Ising spin, respectively. Consequently, the density matrix $\rho^{(2)}$ for $\Delta^{(2)}$ can be represented by

$$ \rho_{s_1 s_2 s_3 s_4 s_5 s_6}^{(2)} = \exp[-\beta (-J_1(s_1 - 3/2)(s_4 - 3/2) + (s_1 - 3/2)(s_2 - 3/2) + (s_1 - 3/2)(s_6 - 3/2) + (s_2 - 3/2)(s_3 - 3/2) + (s_2 - 3/2)(s_5 - 3/2) - J_2(s_3 - 3/2)(s_4 - 3/2) - J_3(S_5 - 3/2) + (s_3 - 3/2)(s_5 - 3/2) + (s_3 - 3/2)(s_6 - 3/2) - h(s_1 + s_2 + s_3 - 9/2)/2 + h(s_4 + s_5 + s_6 - 9/2)])]. \quad (4) $$

Now, we introduce the “corner density matrix” $f^{(t)}$ by summing over the degrees of freedom of $\rho^{(t)}$ except for the spins at the three vertices of the largest triangle in $\Delta^{(t)}$. For example, we have $f_{s_1 s_2 s_3}^{(2)} = \sum_{s_4 s_5 s_6} \rho_{s_1 s_2 s_3 s_4 s_5 s_6}^{(2)}$, where $s_1$, $s_2$, and $s_3$ are the spins at the three vertices of the largest triangle in $\Delta^{(2)}$. The recursive relation of the corner density matrix can be given by decimating the degrees of freedom inside the triangle (in the dash circle), the new triangle is obtained and used to construct the SG of trebled size in the next renormalization step.

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Finally, a summary is given.

Fig. 1. (Color online) (a) Schematic depiction of Sierpiński gasket with the Euclidean dimension $D_E = 2$ and the renormalization time $t = 3$, where two kinds of couplings, $J_1$ (blue) and $J_2$ (red) are shown. (b) The real space renormalization of the SG Ising model. Each time after summing over the degrees of freedom inside the triangle (in the dash circle), the new triangle is obtained and used to construct the SG of trebled size in the next renormalization step.
spin–spin correlation function $\langle S_a S_b \rangle$, defined by

$$\langle S_a S_b \rangle = \frac{\sum_{s} \{ \exp[-\beta E(s)]\} (s_a - 3/2) (s_b - 3/2) / Z}{\sum_{s} \{ \exp[-\beta E(s)]\} (s_a - 3/2) (s_b - 3/2) / Z},$$  \hfill (8)

$$\langle S_a S_b \rangle = \frac{\sum_{s} \{ \exp[-\beta E(s)]\} (s_a - 3/2) (s_b - 3/2) / Z}{\sum_{s} \{ \exp[-\beta E(s)]\} (s_a - 3/2) (s_b - 3/2) / Z},$$  \hfill (9)

can be obtained by performing the calculations similar to Eq. 5. We take $\langle S_1 \rangle$ as an example. Considering that the only difference between the equations for $Z$ and $\langle S_1 \rangle$ is the contraction that involves $s_1$, we can introduce the impurity as $f_{s_1s_2s_3}^{(2)} = f_{s_1s_2s_3}^{(0)} (s_1 - 3/2)$, and then, its renormalization can be written as

$$f_{s_1s_2s_3}^{(0+1)} = \sum_{s_1s_2s_3} f_{s_1s_2s_3}^{(0)} f_{s_1s_2s_3}^{(0)} f_{s_1s_2s_3}^{(0)} (f_{s_1s_2s_3}^{(0)})^{-3},$$  \hfill (10)

where each $f^{(i)}$ satisfies Eq. 5. Compared with Eq. 5, one of the three $f^{(i)}$'s in Eq. (10) is replaced by the impurity, and which tensor should be replaced depends on the position of $s_1$ in the SG. In the end we have $\langle S_1 \rangle = \sum_{s_1s_2s_3} f_{s_1s_2s_3}^{(0)} / \sum_{s_1s_2s_3} f_{s_1s_2s_3}^{(0)}$. The calculation of $\langle S_a S_b \rangle$ is similar. It should be mentioned that the whole formulation above can be readily extended to the N–state Potts model 1, 2.

The above formulation can be further simplified in the absence of a magnetic field ($h = 0$), and the fixed point of the renormalization can be discussed. Under this circumstance, each tensor $f^{(i)}$ only contains two inequivalent components denoted as $x^{(i)} = f_{1,1,1}^{(i)} = f_{2,2,2}^{(i)}$ and $y^{(i)} = f_{1,1,1}^{(i)} = f_{1,2,1}^{(i)} = f_{2,1,2}^{(i)} = f_{2,2,2}^{(i)}$. According to Eq. 5, we can readily get the recursive relation for $z^{(i)} = y^{(i)}/x^{(i)}$ as

$$z^{(i+1)} = 3[z^{(i)}]^3 + 4[z^{(i)}]^2 + z^{(i)},$$  \hfill (11)

with $z^{(1)} = f_{1,1,1}^{(1)}/f_{1,1,1}^{(1)}$. Consequently, the free energy becomes

$$\mathcal{F}^{(i)} = -2 \sum_{j=2}^{l-1} \left[ 3^{-j} \ln(4[z^{(j)}]^3) + 3[z^{(j)}]^2 + 1 \right] + 3^{-1} \ln f_{1,1,1}^{(i)} + 3^{-j} \ln 8 \beta. \hfill (12)
$$

Eq. 11 has two fixed points: $z = 0$ (unstable) and $z = 1$ (stable). By considering that $f_{s_1s_2s_3}^{(i)}$ represents the probability distribution of an effective Ising model of the three spins, one can see that $z = 0$ implies the effective temperature $T^{(i)} = 0$ (infinite inverse temperature), and $z = 1$ implies $T = \infty$ when all configurations of spins share the same probability, indicating that no phase transition happens at any finite temperature. This is consistent with the previous result in e.g. 17.

III. PHASE DIAGRAM AT ZERO TEMPERATURE

By calculating the residual entropy $S_0$ and the uniform magnetization $M$ in a magnetic field $h$ at sufficiently large inverse temperature [18], we have found four phases that are marked by $A_1, A_2, A_3$, and $A_4$ (Fig. 2). For $0 \leq h < 2$ and $0 \leq |J_2| < 1$ (we take $J_1 = -1$ as energy scale and $J_2 < 0$ for the antiferromagnetic coupling), the system is in the $A_1$ phase, where the residual entropy $S_0 = 0$ and the uniform magnetization $M = 1/6$. After grouping the spins into four subsets denoted as $P_1, P_2, P_3,$ and $P_4$ [Fig. 3(a)], we calculated the zero temperature subset magnetization and obtained $M_{1,3} = -1/2$ and $M_{2,4} = 1/2$ with $M_i (i = 1, 2, 3, 4)$ the subset magnetization of $P_i$. It can be seen that the system is in the Néel order, where $2/3$ of the spins that point up and belong to $P_2$ and $P_4$ subsets, and $1/3$ of them that point down and belong to $P_1$ and $P_3$ subsets, resulting in a 1/3 magnetization plateau phase.

Keeping $0 < h < 2$ and increasing the coupling from $|J_2| < 1$ to $|J_2| > 1$, we observed that the system enters into the $A_2$ phase (e.g. $|J_2| = 1.2, h = 1$) with the same magnetization $M = 1/6$, but the residual entropy is nonzero, $S_0 = 0.0257$, showing that this phase is in a disordered state with extensive ground state degeneracy. The subset magnetization in $A_2$ phase is $M_{1,4} = 0$ and $M_{2,3} = 1/4$, which is different from those in $A_1$ phase. It can be seen that the residual entropy in $A_3$ phase is associated with all spins, so $A_2$ phase is of a disordered 1/3 plateau phase. This fact indicates that by increasing $|J_2|$, the Néel order in $A_1$ phase is “melted” in $A_2$ phase by the classical frustration whereas the uniform magnetization is unchanged.

For $h > 2$ and $|J_2| > h - 1$, the system is in the $A_3$ phase with $M = 5/18$ and $S_0 = (2 \ln 3)/9$, and the subset magnetization satisfies $M_{1,3} = 1/2$ and $M_{2,4} = 1/6$. The spins belonging to $P_1$ and $P_3$ are totally polarized, and only the spins belonging to $P_2$ and $P_4$ make contributions to the residual entropy. Thus, this phase is a partially ordered 5/9 plateau phase.

For $h > 2$ and $0 \leq |J_2| < h - 1$, the system is in the $A_4$ phase with $M = 1/2$ and $S_0 = 0$, which is nothing but a polarized state.

For $h = 0$, when $0 \leq |J_2| < 1$ the system bears the Néel order with spontaneous magnetization $M = 1/6$, but on the contrary, when $|J_2| \geq 1$ [the boundary $B_1$ in Fig. 2(c)], we found the subset magnetization becomes zero, which indicates that $|J_2| = 1$ may be a crossover line. Such a crossover behavior will be re-examined by studying the specific heat, susceptibility and spin–spin correlation functions in Secs. IV and V. In addition, it appears that the magnetization in $A_2$ phase cannot appear spontaneously but can only be induced by a magnetic field.

It should be pointed out that at all phase boundaries, the magnetization as functions of $|J_2|$ and $h$ are all continuous, and no phase transitions between different phases at zero temperature are observed.

We also calculated the residual entropy $S_0$ at each of phase boundaries [Fig. 2(c)], and disclosed that $S_0$ are higher than those of the surrounding phases. At boundary $B_1$ with $h = 0$ and $|J_2| > 1$ which is the left boundary of $A_2$ phase, we got $S_0(B_1) = 0.4752$; at boundary $B_2$ between $A_1$ and $A_2$ phases with $0 < h < 2$ and $|J_2| = 1$, we found $S_0(B_2) = 0.0770$; at boundary $B_3$ between $A_2$ and $A_3$ phases with $h = 2$ and $|J_2| > 1$, we had $S_0(B_3) = 0.2738$; at boundary $B_4$ between $A_3$ and $A_4$ phases with $h = |J_2| + 1$ and $|J_2| > 1$, we obtained $S_0(B_4) = 0.3081$; at boundary $B_5$ between $A_1$ and $A_4$ phases with $h = 2$ and $0 < |J_2| < 1$, we arrived at $S_0(B_5) = 0.2311$. In
particular, we observed that $S_0$ at the two intersection points of the boundaries are larger than those at the related boundaries, namely $S_0(V_1) = 0.4930$ at the intersection point $V_1$ between $B_1$ and $B_2$, and $S_0(V_2) = 0.3843$ at the intersection point $V_2$ between $B_2$, $B_3$, $B_4$ and $B_5$.

**FIG. 2.** (Color online) (a) The residual entropy $S_0$ and (b) the uniform magnetization per site $M$ against magnetic field $h$ and coupling $|J_2|$. (c) The zero temperature phase diagram of the antiferromagnetic SG Ising model. There are four phases, including $A_1$ phase: the 1/3 magnetization plateau phase with Néel order and $S_0 = 0$; $A_2$ phase: the 1/3 plateau disordered phase with $S_0 = 0.0257$; $A_3$: the 5/9 plateau partially ordered phase with $S_0 = 2/9\ln 3 = 0.2441$; and $A_4$ phase: the polarized phase. The residual entropy at the phase boundaries: $S_0(B_1) = 0.4752$, $S_0(B_2) = 0.0770$, $S_0(B_3) = 0.2738$, $S_0(B_4) = 0.3081$, and $S_0(B_5) = 0.2311$; and the 1/3 plateau partially ordered phase with $S_0 = 2/9\ln 3 = 0.2441$. Here we take $\beta = 200$.

**FIG. 3.** (Color online) (a) The spins are divided into four subsets denoted as $P_1$, $P_2$, $P_3$, and $P_4$. (b) The $\beta$-dependence of the uniform magnetization per site at $h = 0$ and $|J_2| = 0.9$, where a 1/3 magnetization plateau appears at low temperatures. The inset shows the subset magnetization against $\beta$.

**FIG. 4.** (Color online) The $\beta$-dependence of (a) the uniform magnetization and (b) the subset magnetization in $A_2$ ($h = 1$) and $A_3$ ($h = 2.1$) phases. Here we take $|J_2| = 1.2$.

**IV. THERMODYNAMIC PROPERTIES**

In this section, we shall study the temperature dependence of the thermodynamic quantities including the uniform and subset magnetization, specific heat, susceptibility, where the nonexistence of phase transitions is testified, and the thermodynamic properties of the system will be explored.

Let us first look at the uniform magnetization [Fig. 3(b)] and subset magnetization [the inset of Fig. 3(b)] versus $\beta$ in $A_1$ phase by taking $|J_2| = 0.9$ and $h = 0$. One may see that the magnetization becomes nonzero smoothly from zero at around $\beta = 50$, which indicates that the system is crossing from the high temperature disordered paramagnetic phase to the low-temperature Néel phase, and a 1/3 magnetization plateau appears at low temperature.

The uniform and subset magnetization in $A_2$ and $A_3$ phases [Fig. 4] also have the similar behaviors, showing that the system is crossing into the low-temperature phases. In $A_2$ phase, the spins belonging to $P_1$ and $P_4$ exhibit similar behaviors, where the subset magnetization first rises from zero to its maximum at about $\beta = 4$ and then decreases to zero with increasing $\beta$, while the spins at $P_2$ and $P_3$ sites behave similarly, where the magnetization saturates to 1/4 with increasing $\beta$. In $A_3$ phase, the situation is somehow different, where the spins at $P_1$ and $P_3$ sites are gradually polarized, and the sub-
FIG. 5. (Color online) The $\beta$-dependence of (a) the specific heat $C$ and (b) the susceptibility $\chi$ at $h = 0$ for different $|J_2|$. In (a), the specific heat has two peaks, where the position of high-temperature peak is independent of $|J_2|$, while the position of the low-temperature peak, which lies in the finitely correlated region shown in Fig. 8, moves to infinite $\beta$ as $|J_2| \rightarrow 1$. In (b), there is only one peak for the susceptibility, whose position also approaches infinite as $|J_2| \rightarrow 1$. It can be seen in the inset of (b) that $\chi$ gradually becomes proportional to $\beta$ as $|J_2| \rightarrow 1$ ($|J_2| = 0.98, 0.99$). When $|J_2| \geq 1$, $\chi$ bears a linear relation with $\beta$.

FIG. 6. (Color online) The $\beta$-dependence of (a) the specific heat $C$ and (b) susceptibility $\chi$ at $h = 0$ for different $|J_2|$. In (a), the specific heat has two peaks, where the position of high-temperature peak is independent of $|J_2|$, while the position of the low-temperature peak, which lies in the finitely correlated region shown in Fig. 8, moves to infinite $\beta$ as $|J_2| \rightarrow 1$. In (b), there is only one peak for the susceptibility, whose position also approaches infinite as $|J_2| \rightarrow 1$. It can be seen in the inset of (b) that $\chi$ gradually becomes proportional to $\beta$ as $|J_2| \rightarrow 1$ ($|J_2| = 0.98, 0.99$). When $|J_2| \geq 1$, $\chi$ bears a linear relation with $\beta$.

set magnetization of $P_2$ and $P_4$ first reaches the maximum and then converges to 1/6 as $\beta$ increases.

Fig. 5 shows the specific heat $C$ and the susceptibility $\chi$ versus $\beta$ in three non-trivial $A_1, A_2$ and $A_3$ phases. It indicates that the system enters the low-temperature phases through a crossover instead of phase transitions (even though the peak in $A_1$ phase is relatively sharper, it is still round and continuous). One may notice that the temperature at which the peak of susceptibility appears is slightly higher than that of the peak of specific heat, where the fluctuation of energy reaches the maximum.

In Sec. III, we have identified $|J_2| = 1$ as a crossover line for $h = 0$. Here, we studied this crossover behavior by calculating the specific heat $C$ and susceptibility $\chi$ for different $|J_2|$, as shown in Fig. 6. For $|J_2| \gg 1$ (including the point $|J_2| = 1$), there is only one peak of $C$ at high temperature $\beta \approx 2.2$ which is independent of $|J_2|$. With gradually decreasing $|J_2|$ to $|J_2| < 1$, another peak moves from the low-temperature side to the high-temperature side, while the position of the high-temperature peak remains unchanged. This double-peak structure indicates a mixture of two kinds of excitations corresponding to the two phases. For $|J_2|$ smaller than 0.5, the two peaks merge into one, and the low-temperature peak stands dominant. For the susceptibility $\chi$, there is only one peak, whose position moves to infinite as $|J_2| \rightarrow 1$. This peak, although it looks apparently divergent, is still finite and broad. Meanwhile, as shown in Fig. 6(b), the relation between $\chi$ and $\beta$ becomes more and more linear as $|J_2| \rightarrow 1$. When $|J_2| \geq 1$, $\chi$ bears a linear relation with $\beta$. In the following section, we will show for different $|J_2|$, the positions of both the low-temperature peaks of the specific heat and the susceptibility locate within the finite correlated region.

V. EXOTIC SPIN-SPIN CORRELATIONS

Now we study the spin-spin correlation function $\xi = \langle S_i S_j \rangle$ [Eq. (9)] in the three non-trivial phases. The $A_1$ phase is manifested to be with a Ne`el order, and at $h = 0$ it has an unusual long range correlation (ULRC) which is similar to the behavior characterized by Eq. (9). The long range correlation is in the usual sense in the presence of $h$. We also discovered that the ULRC corresponds to an intermediate region which separates the $|J_2| - \beta$ diagram into two regions, the fully correlated region with the correlation length $\xi = \infty$ and the non-correlated region with $\xi < 2$. In the ULRC region where the system is finitely correlated, $\xi$ increases “super-exponentially” with $\beta$ until the system enters into the fully correlated region. When $|J_2| \rightarrow 1$, $\beta_0$ and $\beta_1$, the upper and lower boundaries of the ULRC region, both approach infinite, and so consequently, the system can only stay in the non-correlated region at finite temperature for $|J_2| \geq 1$.

The correlations in the $A_2$ and $A_3$ phases are more complicated, where exotic correlation patterns that have same special rotation symmetry as that of the SG are disclosed below. The $A_3$ phase is a fully polarized phase and will not be discussed here.

A. Unusual long range correlation for $0 \leq |J_2| < 1$ and $h = 0$

The spin-spin correlation functions $\xi$ versus $\beta$ for $0 \leq |J_2| < 1$ and $h = 0$ for different distances $L$ between two spins are studied. Fig. 7(a) shows the results at $|J_2| = 0.9$. By defining the distance where the system bears the correlation $\xi = 1/(4\epsilon)$ as the correlation length $\xi$, we determine the $\beta$-dependence of $\xi$. We discovered that in this Néel phase for $\beta$ smaller than a certain value denoted as $\beta_c$, the correlation length shares the same unusual form as Eq. (9), where $\xi$ increases extremely fast as $\beta$ increases. We show the high linearity of the relation between $\beta$ and $K = \ln(4\ln\xi/\ln 2)$, i.e. $K = k_1\beta + k_2$ [Fig. 7(b)], which gives rise to the temperature dependence of the correlation length having the simple form of

$$\xi = \exp[\frac{\ln 2}{4} c_2 \exp(c_1\beta)],$$

where the coefficients $c_1$ and $c_2$ are determined by $k_1$ and $k_2 = e^{k_2}$. The values of $k_1$ and $k_2$ for several $|J_2|$’s are shown in Table I for useful information. Also, it should be noted that
the regression coefficient $R^2$ of each linear fitting is larger than 0.999. The correlation function is independent of the distance $L$ in the presence of $h$. 

One can observe from Fig. 7(b) that for each $|J_2|$, the relation of Eq. (13) holds until the inverse temperature reaches a certain value $|\beta_1|$ that is dependent of $|J_2|$. With $|\beta > |\beta_1|$, the correlation length becomes infinite, i.e. for any distance $L$, we have the correlation $\zeta(|\beta > |\beta_1|) \geq 1/(4\epsilon)$. In this way, the system enters into a phase with long range correlation. Another interesting observation is that for different $|J_2|$, $|\beta_1|$ corresponds to one same $K$, thus consequently to one same $\xi$ which is about $2^{38(\pm 1)}$. It implies that when the correlation length reaches about $2^{38(\pm 1)}$ at $|\beta_1|$, a further decrease of temperature will instantly enable the whole system to be correlated, i.e. $\xi = \infty$. We have excluded the finite size effect by taking the system size approximately from $3^{1000}$ to $3^{5000}$.

Also, by locating the inverse temperature $|\beta_0|$ where the correlation length is $\xi = 2$, we can find where the system begins to be correlated. Fig. 8 shows the $|J_2|$-dependence of $|\beta_0|$ and $|\beta_1|$, where the whole $|\beta|/|J_2|$ region is separated into three subregions. In the subregion beyond the $|\beta_1|$ line, the system is fully correlated with the correlation length $\xi = \infty$; in the subregion below the $|\beta_0|$ line, the system is non-correlated with $\xi < 2$; in the subregion between two lines where the system is finitely correlated, the correlation length $\xi$ obeys Eq. (13) in which $\xi$ increases “super-exponentially” with $|\beta|$.

The behavior of $|\beta_0|$ and $|\beta_1|$ when $|J_2| \to 0$ as well as $|J_2| \to 1$ can be obtained. With $|J_2| \to 0$, the two curves manifest a linear relation with $|J_2|$. With $|J_2| \to 1$ from the $|J_2| < 1$ side, both $|\beta_0|$ and $|\beta_1|$ go divergent with the speed faster than the exponential divergence. Such a behavior of $|\beta_0|$ can provide a clear picture of what happens during the crossover at $|J_2| = 1$ in the absence of a magnetic field. With $|J_2| \to 1$, the system needs a much lower temperature crossover to the finitely correlated region. At $|J_2| = 1$, as $|\beta_0|$ becomes infinite, the system is forbidden to crossover to the finitely correlated region at any finite temperature. Thus, $|J_2| = 1$ is a crossover line. Meanwhile, this is coincident with the results of the low-temperature peaks of $C$ and $\chi$ shown in Fig. 6 which stand within the finitely correlated region.

For $|J_2| > 1$ and $h = 0$ [the $B_1$ boundary, Fig. 7(c)], the correlation functions $\zeta$ with $L \geq 2$ is zero even at sufficiently large $\beta$, which is coincident with previous results.

### B. Exotic correlation patterns in $A_2$ and $A_3$ phases

In this subsection, we discuss the spin-spin correlation functions in other two nontrivial $A_2$ and $A_3$ phases with sufficiently large $\beta$.

By introducing $\zeta_{ij} = \langle S_i S_b \rangle$ with $S_a \in P_1$ and $S_b \in P_j$ ($i, j = 1, \cdots, 4$), we categorize the spin-spin correlations in $A_2$ phase into ten cases ($\xi_{11}, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{21}, \xi_{22}, \xi_{23}, \xi_{24}, \xi_{31}, \xi_{32}, \xi_{33}, \xi_{34}$) and discuss them separately in the following.

(1) $\xi_{11} = 0$ for any choice of two spins that belong to the subset $P_1$.

(2) $|\xi_{12}| = 1/8$ when $S_a \in P_1$ and $S_b \in P_2$ in the same $\Delta^{(3)}$, or $|\xi_{12}| = 0$ otherwise. See Fig. 9(a).

(3) $|\xi_{13}| = 1/8$ when $S_a \in P_1$ and $S_b \in P_2$ in the same $\Delta^{(2)}$, or $|\xi_{13}| = 0$ otherwise.

(4) $|\xi_{14}| = 1/4$ when $S_a \in P_1$ and $S_b \in P_4$ in the same $\Delta^{(2)}$, or $|\xi_{14}| = 0$ otherwise.

(5) It is shown in Fig. 9(b) that $\xi_{22} = 1/16$ for any choice of $S_a$ and $S_b$ that are in two different $\Delta^{(3)}$. $\xi_{22} = 1/16$ for any choice of $S_a$ and $S_b$ that are in two different $\Delta^{(3)}$ which are not adjacent. Thus for the
long distance, we have the correlation $\xi_{22} = 1/16$ which is induced by the magnetic field.

(6) The pattern of $\xi_{23}$ within one $\Delta^{(3)}$ or two adjacent $\Delta^{(3)}$s is similar to that of $\xi_{22}$ as shown in Fig. 9(c). For the long distance, we have $\xi_{23} = 1/16$ when $S_a$ and $S_b$ are in two non-adjacent $\Delta^{(3)}$s, which is the same as $\xi_{22}$.

(7) As shown in Fig. 9(d), $|\xi_{34}|$ is equal to either 1/8 or 0 when $S_a, S_b \in P_2$, $S_a, S_b \in P_4$ are in the same $\Delta^{(3)}$ or in two different but adjacent $\Delta^{(3)}$. $\xi_{34} = 0$ when $S_a$ and $S_b$ are in two non-adjacent $\Delta^{(3)}$s.

(8) $\xi_{33} = 0$ when $S_a, S_b \in P_3$ are in the same $\Delta^{(3)}$, and $\xi_{33} = 0$ or 1/8 when $S_a$ and $S_b$ are in two adjacent $\Delta^{(3)}$s, as shown in Fig. 9(e), and $\xi_{33} = 1/16$ when $S_a$ and $S_b$ are in two non-adjacent $\Delta^{(3)}$s.

(9) The pattern of $\xi_{34}$ within one $\Delta^{(3)}$ or two adjacent $\Delta^{(3)}$s is similar to that of $\xi_{23}$ as shown in Fig. 9(f). For a longer distance, $\xi_{34} = 0$, which is the same as $\xi_{33}$.

(10) $\xi_{44} = 1/4$ only when $S_a, S_b \in P_4$ are in two different but adjacent $\Delta^{(2)}$s and also in two different but adjacent $\Delta^{(3)}$s, or otherwise $\xi_{44} = 0$ (if $S_a$ and $S_b$ are in two different $\Delta^{(2)}$s but these two $\Delta^{(2)}$s belong to the same $\Delta^{(3)}$, $\xi_{44} = 0$).

In $A_3$ phase, the spins belonging to the subset $P_1$ and $P_3$ are polarized, so any two spins $S_a$ and $S_b$ with $S_a, S_b \in P_1 \cup P_3$ are maximally correlated with $\xi = 1/4$. We also calculated the correlation associated with the spins in $P_2$ and $P_4$. The results show that with $S_a \in P_1 \cup P_3$ and $S_b \in P_2 \cup P_4$, we have $\xi = 1/12$, and with $S_a, S_b \in P_1 \cup P_4$, we have $\xi = -1/12$ when $S_a, S_b$ are the nearest neighbors, and $\xi = 1/36$ when these two spins are not the nearest neighbors.

Despite the complexity, the correlation patterns share a common feature: for any $t$, the pattern of the spin-spin correlations of $S_a$ and $S_b$ that belong to the $\Delta^{(t)}$ bears the symmetry as same as $\Delta^{(t)}$ in the SG. Specifically speaking, both the SG $\Delta^{(t)}$ and the corresponding correlation pattern are invariant under the rotations with the angles $\theta = 0, 2\pi/3, 4\pi/3, 2\pi$ around the center of $\Delta^{(t)}$ as the rotating axis.

VI. SUMMARY

In summary, we have systematically studied the $J_1$-$J_2$ antiferromagnetic Ising model on the fractal Sierpiński gasket. The zero temperature phase diagram with four phases is presented, and the thermodynamic properties of the three non-trivial phases are explored by calculating the magnetization, residual entropy, specific heat and magnetic susceptibility. The spin-spin correlations are also probed. For $h = 0$, the thermodynamic crossover behavior with $0 \leq |J_2| < 1$ and the zero-temperature crossover from $0 \leq |J_1| < 1$ to $|J_1| > 1$ are both analyzed by studying the temperature dependence of the correlation length. The exotic correlation patterns are disclosed in the 1/3 magnetization plateau disordered phase and the 5/9 plateau partially ordered phase. We would like to stress that the exotic properties of this present model are due to both the classical frustration and the special geometry of the Sierpiński gasket, and it should also be interesting to discuss the unusual properties of this model in the presence of quantum fluctuations, which is left for a future work.

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