Decomposition of $q$-deformed Fock spaces

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Abstract

A decomposition of the level-one $q$-deformed Fock representations of $U_q(\hat{\mathfrak{sl}}_n)$ is given. It is found that the action of $U'_q(\hat{\mathfrak{sl}}_n)$ on these Fock spaces is centralized by a Heisenberg algebra, which arises from the center of the affine Hecke algebra $\hat{H}_N$ in the limit $N \to \infty$. The $q$-deformed Fock space is shown to be isomorphic as a $U'_q(\hat{\mathfrak{sl}}_n)$-Heisenberg-bimodule to the tensor product of a level-one irreducible highest weight representation of $U'_q(\hat{\mathfrak{sl}}_n)$ and the Fock representation of the Heisenberg algebra. The isomorphism is used to decompose the $q$-wedging operators, which are intertwiners between the $q$-deformed Fock spaces, into constituents coming from $U'_q(\hat{\mathfrak{sl}}_n)$ and from the Heisenberg algebra.

1 $q$-wedges and $q$-deformed Fock space

This first introductory section describes the realization of $q$-deformed Fock space in terms of $q$-wedges. This Fock space was first constructed by a different method by Hayashi in [6], and described in terms of colored Young diagrams in [10]. A less formal version of the $q$-wedge construction was given in [11], out of which the exposition in this section first evolved.

1.1 Preliminaries on $U_q(\hat{\mathfrak{sl}}_n)$

The algebras $U_q(\hat{\mathfrak{sl}}_n)$ and $U'_q(\hat{\mathfrak{sl}}_n)$ ([4], [1]) will act on our Fock space. In this section, we will mainly work with $U'_q(\hat{\mathfrak{sl}}_n)$, which is an algebra generated by elements $E_i$, $F_i$, and $K_i^{\pm 1}$, $i = 0, 1, \ldots, n - 1$, with the following relations if $n > 2$:

$$K_i K_j = K_j K_i,$$  \hspace{1cm} (1)
The indices in all these relations are to be read modulo $n$. In (2)–(3), $a_{ij}$ is 2 if $i = j$, $-1$ if $i = j \pm 1$, and 0 otherwise.

In the case $n = 2$, we take $a_{ij} = -2$ if $i \neq j$. Moreover, the last two relations (sometimes called the $q$-Serre relations) are replaced by the following ones:

\begin{align*}
E_i E_j &= E_j E_i \quad \text{if } i \neq j \pm 1, \\
F_i F_j &= F_j F_i \quad \text{if } i \neq j \pm 1, \\
E_i^2 E_{i \pm 1} - (q + q^{-1}) E_i E_{i \pm 1} E_i + E_{i \pm 1} E_i^2 &= 0, \\
F_i^2 F_{i \pm 1} - (q + q^{-1}) F_i F_{i \pm 1} F_i + F_{i \pm 1} F_i^2 &= 0.
\end{align*}

Here we have used the standard notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$  

Throughout this paper, $q$ should be taken to be either a formal parameter or a generic complex number (specifically, not a root of unity).

$U'_q(\mathfrak{sl}_n)$ is a Hopf algebra, with coproduct given by

\begin{align*}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, \\
\Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i.
\end{align*}

In the next section we will sometimes work with the extended quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$, which is a 1-dimensional extension of $U'_q(\widehat{\mathfrak{sl}_n})$ by the “degree operator” $d$, satisfying the relations

\begin{align*}
[d, K_i] &= 0, \\
[d, E_i] &= \delta_{i,0} E_i, \\
[d, F_i] &= -\delta_{i,0} F_i.
\end{align*}

The Hopf algebra structure extends naturally to $U_q(\widehat{\mathfrak{sl}_n})$ by defining

$$\Delta(d) = d \otimes 1 + 1 \otimes d.$$
The philosophy guiding this subject is that almost any representation-theoretic construction involving the affine Lie algebra \( \widehat{\mathfrak{sl}}_n \) should have an appropriate \( q \)-analog in the representation theory of \( U'_q(\widehat{\mathfrak{sl}}_n) \). The aim of this section is to deform the infinite wedge (or “Fermionic Fock space”) level 1 representations of \( \widehat{\mathfrak{sl}}_n \) to representations of \( U'_q(\widehat{\mathfrak{sl}}_n) \). (A very friendly introduction to \( \widehat{\mathfrak{sl}}_n \) and its infinite wedge highest weight representations can be found in [3].)

We will construct infinite \( q \)-wedges in terms of an infinite tensor product of evaluation modules. Let \( V = C^n \), with basis \( v_1, \ldots, v_n \), and let \( V(z) = V \otimes C[z, z^{-1}] \), with basis \( \{z^a v_j\} \). Here \( a \in Z \) and \( j = 1, 2, \ldots, n \), so \( V(z) \) is regarded as an infinite dimensional space. \( U_q(\widehat{\mathfrak{sl}}_n) \) acts on \( V(z) \) in the following way:

\[
K_i(z^a v_j) = q^\delta_{i,j-\delta_{i+1,j}} z^a v_j, \quad (16)
E_i(z^a v_j) = \delta_{i,j-1} z^{a+\delta_{i,0}} v_{j-1}, \quad (17)
F_i(z^a v_j) = \delta_{i+1,j} z^{a-\delta_{i,0}} v_{j+1}, \quad (18)
d(z^a v_j) = az^a v_j. \quad (19)
\]

The indices in all of these relations should be read modulo \( n \).

The action of the \( E_i \)'s gives rise to a a natural ordering on the basis \( \{z^a v_j\} \). This ordering is given by

\[
\cdots > z^{a-1} v_2 > z^{a-1} v_1 > z^a v_n > z^a v_{n-1} > \cdots > z^a v_1 > z^{a+1} v_n > \cdots , \quad (20)
\]

and at times it will be convenient to relabel the basis in accordance with it. Namely, let \( u_{j-an} = z^a v_j \); then \( u_l > u_m \) just in case \( l > m \). In the course of working with \( V(z) \) and its tensor products, we will sometimes use the \( z^a v_j \) notation and sometimes the \( u_m \) notation (whichever happens to be more convenient) without further comment. The action on the basis \( \{u_m\} \) is

\[
K_i(u_m) = q^\delta(m\equiv i \mod n) - \delta(m\equiv i+1 \mod n) u_m, \quad (21)
E_i(u_m) = \delta(m-1 \equiv i \mod n) u_{m-1}, \quad (22)
F_i(u_m) = \delta(m \equiv i \mod n) u_{m+1}. \quad (23)
\]

Here, for a statement \( P \), \( \delta(P) \) is equal to 1 if \( P \) is true and 0 otherwise.

Iterating the coproduct (11)-(14) \( N-1 \) times defines a natural action of \( U_q(\widehat{\mathfrak{sl}}_n) \) on the tensor product \( V(z)^{\otimes N} \). According to a “quantum affine analog” of the usual Weyl duality between \( GL_n \) and the symmetric group, the centralizer of the action of \( U'_q(\widehat{\mathfrak{sl}}_n) \subset U_q(\widehat{\mathfrak{sl}}_n) \) on \( V(z)^{\otimes N} \) is the affine Hecke algebra \( H_N(q^2) \). (See [3] for more details.) We turn next to this algebra.
1.2 Preliminaries on the affine Hecke algebra

\( \widehat{H}_N(q^2) \) is an associative algebra generated by elements \( T_i, i = 1, \ldots, N - 1 \), and \( y_j^{\pm 1}, j = 1, \ldots, N \). These elements satisfy the following relations:

\[
T_i^2 = (q^2 - 1) T_i + q^2, \tag{24}
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{25}
\]

\[
T_i T_j = T_j T_i \quad \text{if} \ |i - j| > 1, \tag{26}
\]

\[
y_i y_j = y_j y_i, \tag{27}
\]

\[
y_j T_i = T_i y_j \quad \text{if} \ i \neq j, j + 1, \tag{28}
\]

\[
y_j T_j = T_j y_{j+1} - (q^2 - 1) y_{j+1}, \tag{29}
\]

\[
y_{j+1} T_j = T_j y_j + (q^2 - 1) y_{j+1}. \tag{30}
\]

Note that the \( T_i \) are invertible because of (24), with \( T_i^{-1} = q^{-2} T_i + (q^2 - 1) \). In light of this, relations (29) and (30) are both equivalent to the relation

\[
T_i y_j T_i = q^2 y_{j+1}.
\]

Also, the relation (24) can be written as

\[
(T_i + 1)(T_i - q^2) = 0. \tag{31}
\]

Decomposing spaces on which the \( T_i \) act into a \(-1\)-eigenspace and a \( q^2\)-eigenspace will be an important tool in what follows.

The subalgebra \( H_N(q^2) \subset \widehat{H}_N(q^2) \) generated by just the \( T_i \) is the usual (finite) Hecke algebra of type \( A \), which is a \( q \)-deformation of the symmetric group \( S_N \). The elements \( T_i \) are the \( q \)-analogs of the adjacent transpositions \( \sigma_i = (i, i + 1) \) in \( S_N \). In the same way, \( \widehat{H}_N(q^2) \) should be thought of as a \( q \)-deformation of the affine symmetric group, which is the Weyl group of the affine Lie algebra \( \widehat{\mathfrak{sl}}_N \).

\( \widehat{H}_N(q^2) \) acts on \( V(z)^{\otimes N} \) on the right in the following way. First, \( y_j \) acts as multiplication by \( z_{j-1}^{-1} \). (Having it act by \( z_j^{-1} \) rather than \( z_j \) is necessary for compatibility (29)–(30) with the action of \( T_i \) defined below.) The action of \( T_i \) is given as follows. Write elements \( z^{a_1} v_{m_1} \otimes \cdots \otimes z^{a_N} v_{m_N} \) as \( (v_{m_1} \otimes \cdots \otimes v_{m_N}) \cdot z_{1}^{a_1} \cdots z_{N}^{a_N} \). (This is the notation used in (31).) The symmetric group \( S_N \) acts (by permuting factors and variables, respectively) on both the tensor part and the polynomial part of such an expression. Write \( (v_{m_1} \otimes \cdots \otimes v_{m_N})^{\sigma_i} \) and \( (z_{1}^{a_1} \cdots z_{N}^{a_N})^{\sigma_i} \) for what results when \( \sigma_i = (i, i + 1) \) acts on \( v_{m_1} \otimes \cdots \otimes v_{m_N} \) and on \( z_{1}^{a_1} \cdots z_{N}^{a_N} \). In terms of this notation, the action of \( T_i \) on \( v_{m_1} \otimes \cdots \otimes v_{m_N} \cdot z \) (here \( z \) is
shorthand for \( z_1^{a_1} \cdots z_N^{a_N} \) is given by the following set of formulas, which are variants of the ones given in [5]:

\[
(v_{m_1} \otimes \cdots \otimes v_{m_N}) \cdot z) \cdot T_i = \begin{cases} 
-q(v_{m_1} \otimes \cdots \otimes v_{m_N})^\sigma_i \cdot z^\sigma_i \cdot \frac{z_{i+1}^i z_{i+1} - z}{z_{i+1} - z} & \text{if } m_i < m_{i+1}, \\
-(q^2 - 1)(v_{m_1} \otimes \cdots \otimes v_{m_N}) \cdot z^\sigma_i \cdot \frac{z_i (z^\sigma_i - z)}{z_i - z_{i+1}} & \text{if } m_i = m_{i+1}, \\
-q(v_{m_1} \otimes \cdots \otimes v_{m_N})^\sigma_i \cdot z^\sigma_i \cdot \frac{z_i (z^\sigma_i - z)}{z_i - z_{i+1}} & \text{if } m_i > m_{i+1}. 
\end{cases}
\]

(32)

The important fact for us is that this right action of \( \tilde{H}_N(q^2) \) on \( V(z)^{\otimes N} \) commutes with the left action of \( U'_q(\mathfrak{sl}_n) \) on \( V(z)^{\otimes N} \) given in the previous subsection in terms of the coproduct.

1.3 \( q \)-antisymmetrization and \( q \)-wedges

As a vector space, \( H_N(q^2) \subset \tilde{H}_N(q^2) \) has a natural basis \( \{T_\sigma\}_{\sigma \in S_N} \) which can be defined as follows. Given \( \sigma \in S_N \), take a minimal-length expression \( \sigma = \sigma_i, \sigma_{i_2} \cdots \sigma_i \) of \( \sigma \) in terms of adjacent transpositions. Then define \( T_\sigma = T_{i_1} T_{i_2} \cdots T_{i_l} \). It is a basic result about the Hecke algebra that \( T_\sigma \) depends only on \( \sigma \) and not on the particular expression that was used.

We define the \( q \)-antisymmetrizing operator \( A^{(N)} \) acting on \( V(z)^{\otimes N} \) to be the sum

\[
A^{(N)} = \sum_{\sigma \in S_N} T_\sigma
\]

(33)

There are no \((-1)^\sigma\) factors appearing in this sum because the sign of the permutation is already incorporated into the definition (32) of the action of \( \tilde{H}_N(q^2) \); for example, \((v_{i+1} \otimes v_i) \cdot T_i = -qv_i \otimes v_{i+1}\). (In other words, the operator \( T_i \) acting in \( V(z)^{\otimes N} \) is really a deformation of \(-\sigma_i\), rather than of \( \sigma_i \in S_N \).)

The \( q \)-antisymmetrization of a pure tensor \( z^{a_1} v_{m_1} \otimes \cdots \otimes z^{a_N} v_{m_N} \) is defined to be

\[
(z^{a_1} v_{m_1} \otimes \cdots \otimes z^{a_N} v_{m_N}) \cdot A^{(N)}.
\]

(34)

We begin our study of \( A^{(N)} \) with the following proposition, which essentially asserts that \( A^{(N)} \) is (up to scalar) an idempotent, and \( V(z)^{\otimes N} \) decomposes as a direct sum of its two eigenspaces.
Proposition 1.1

\[ V(z)^{\otimes N} = \text{Im}A^{(N)} \oplus \text{Ker}A^{(N)}. \]  

(35)

Proof. Note that for each \( i = 1, 2, \cdots, N - 1 \) we have a factorization

\[ A^{(N)} = \left( \sum_{\sigma'} T_{\sigma'} \right) (T_i + 1), \]  

(36)

where \( \sigma' \) ranges over \( S_N/\{\text{id}, \sigma_i\} \). From this and (31), it follows that

\[ A^{(N)}(T_i - q^2) = 0. \]

This means that the action of \( T_i \) on the right on \( \text{Im}A^{(N)} \) is simply multiplication by \( q^2 \). Hence, right multiplication by \( A^{(N)} \) on \( \text{Im}A^{(N)} \) is equal to multiplication by the scalar

\[ \sum_l n(l)q^{2l}, \]  

(37)

where \( n(l) \) is the number of elements of \( S_N \) having length \( l \). This sum is equal to the product

\[ \prod_{m=1}^N \frac{1 - q^{2m}}{1 - q^2}, \]

which is a non-zero scalar since \( q \) is not a root of unity. Therefore we have

\[ A^{(N)} \left( A^{(N)} - \prod_{m=1}^N \frac{1 - q^{2m}}{1 - q^2} \right) = 0, \]  

(38)

and this implies the assertion. \( \square \)

In the classical \((q = 1)\) case, there are two equivalent ways of defining the wedge product. One is as the subspace of the tensor product consisting of completely antisymmetric tensors (i.e., the image of the antisymmetrizer), and the other is as a quotient of the tensor product by relations of the form \( v \wedge w = -w \wedge v \), which generate the kernel of the antisymmetrizer.

In the quantum case \((q \neq 1)\), both approaches are again available because of Proposition 1.1. The \( q \)-wedge space can be defined either as \( \text{Im}A^{(N)} \), the subspace of \( V(z)^{\otimes N} \) consisting of \( q \)-antisymmetrized tensors, or as a quotient of this tensor product by certain relations which generate the kernel of \( A^{(N)} \). Let us now describe these relations. The first step is
Proposition 1.2 The kernel of $A^{(N)}$ is the sum of the kernels of the operators $T_i + 1$, $i = 1, 2, \ldots, N - 1$.

Proof. Equation (36) implies that $\sum_i \text{Ker}(T_i + 1) \subseteq \text{Ker}A^{(N)}$. Let us show that $\text{Ker}A^{(N)} \subseteq \sum_i \text{Ker}(T_i + 1)$. Proposition [1.7] applied to $V(z) \otimes 2$ asserts that

$$V(z) \otimes V(z) = \text{Ker}(T - q^2) \oplus \text{Ker}(T + 1).$$ \hspace{1cm} (39)

In general, by arguing as we did for that Proposition, we can conclude

$$\omega A^{(N)} \equiv \left( \prod_{m=1}^{N} \frac{1 - q^{2m}}{1 - q^2} \right) \omega \mod \sum_i \text{Ker}(T_i + 1) \hspace{1cm} (40)$$

for any $\omega \in V(z) \otimes^N$. This equation implies that if $\omega \in \text{Ker}A^{(N)}$, then $\omega \equiv 0 \mod \sum_i \text{Ker}(T_i + 1)$. \hfill \Box

Proposition 1.2 shows that to find relations generating the kernel of $A^{(N)}$, it suffices to find relations generating the kernel of each $T_i + 1$. For ease of notation, let us restrict to considering $V(z) \otimes V(z)$, on which $T = T_1$ acts. Since $T$ commutes with the action of $U_q'(\hat{sl}_n)$, $\text{Ker}(T + 1)$ is preserved by the action of $U_q'(\hat{sl}_n)$. One way of deriving relations is to start with a simple element of the kernel, say $v_1 \otimes v_1$, and then act on it by $U_q'(\hat{sl}_n)$. This gives us the following elements in $\text{Ker}(T + 1)$:

$$u_l \otimes u_m + u_m \otimes u_l \text{ if } l \equiv m \mod n, \hspace{1cm} (41)$$

$$u_l \otimes u_m + qu_m \otimes u_l + u_{m-i} \otimes u_{i+i} + qu_{i+i} \otimes u_{m-i} \text{ if } m - l \equiv i \mod n \text{ and } 0 < i < n. \hspace{1cm} (42)$$

Let $V(z) \wedge_q V(z)$ denote the quotient $(V(z) \otimes V(z))/\text{Ker}(T + 1)$, and let $u_l \wedge_q u_m$ denote the image of $u_m \otimes u_l$ under the quotient map. The space $V(z) \wedge_q V(z)$ will be called $q$-wedge space and its elements $q$-wedges. From now on, we will write $\wedge$ instead of $\wedge_q$, but it should be understood that all wedges appearing in this paper are really $q$-wedges.

The relations (41) and (42) can be understood as normal ordering rules, i.e., as prescriptions for writing a $q$-wedge whose left entry is smaller than its right in the ordering (20) (i.e., $u_l \wedge u_m$ such that $l < m$) as a linear combination of normally ordered $q$-wedges whose left entries are larger than their right. For $l \equiv m \mod n$, the rule is simply

$$u_l \wedge u_m = -u_m \wedge u_l. \hspace{1cm} (43)$$
In order to give the rule for the case \( m - l \equiv i \mod n \) and \( 0 < i < n \), let us extract from (20) the subsequence
\[
\cdots > u_m > u_{m-i} > u_{m-n} > u_{m-n-i} > \cdots > u_{l+n+i} > u_{l+n} > u_{l+i} > u_l > \cdots
\] (44)
The rule is
\[
u_l \wedge u_m = -qu_m \wedge u_l + (q^2 - 1)(u_{m-i} \wedge u_{l+i} - qu_{m-n} \wedge u_{l+n} + q^2 u_{m-n-i} \wedge u_{l+n+i} + \cdots).
\] (45)
Each wedge in the sum is obtained from the one before it by moving its left component one to the right in the sequence (44), while simultaneously moving the right component one to the left. The sum continues as long as we get normally ordered wedges.

As in the case \( N = 2 \), define \( N \)-fold \( q \)-wedge space \( \Lambda^N V(z) \) to be the quotient \( V(z)^{\otimes N}/\text{Ker}A(N) \). The next chain of arguments will enable us to conclude that \( \Lambda^N V(z) \) is equal to the quotient of \( V(z)^{\otimes N} \) by the relations (43) and (45) in each pair of adjacent factors.

The notion of normal ordering was introduced with the motivation that using the relations (43) and (45), any element of \( \Lambda^N V(z) \) can be written as a sum of normally ordered \( q \)-wedges (i.e., the terms decrease strictly from left to right with respect to the ordering (20)). This means, at the very least, that the normally ordered \( q \)-wedges span \( \Lambda^N V(z) \), but in fact more is true:

**Proposition 1.3** The elements
\[
u_{m_1} \wedge u_{m_2} \wedge \cdots \wedge u_{m_k},
\] (46)
where \( m_1 > m_2 > \cdots > m_k \), form a basis for \( \Lambda^N V(z) \).

**Proof.** In light of the previous discussion, it remains to show the normally ordered \( q \)-wedges are linearly independent. Because of Proposition 1.1 we have an isomorphism
\[
\Lambda^N V(z) \simeq \text{Im}A(N) \subset V(z)^{\otimes N}.
\]
Hence the proposition reduces to the linear independence of the vectors
\[
(u_{m_1} \otimes u_{m_2} \otimes \cdots \otimes u_{m_k})A(N)
\]
with the \( m_i \) strictly decreasing. This is easily seen by specializing at \( q = 1 \).

We can conclude from the independence of the normally ordered \( q \)-wedges that the relations (43) and (44) (applied in adjacent factors) generate the entire kernel of \( A(N) \). In other words, they are precisely the complete set of relations for the \( q \)-wedge product that we have been seeking.
1.4 The thermodynamic limit

Now consider an infinite tensor product \( V(z) \otimes V(z) \otimes V(z) \otimes \cdots \). (Physicists call this the “thermodynamic limit.”) An infinite iteration of (11)-(13) gives rise to a formal action of \( U'_q(\hat{\mathfrak{sl}}_n) \) in this tensor product. The action is only formal because when \( E_i \) or \( F_i \) from \( U'_q(\hat{\mathfrak{sl}}_n) \) acts on an element of the infinite tensor product, the result is typically an infinite sum. Consequently, it is often not possible to compose two elements of \( U'_q(\hat{\mathfrak{sl}}_n) \). For example, consider

\[
\begin{align*}
  u(m) &= u_m \otimes u_{m-1} \otimes u_{m-2} \otimes \cdots. 
\end{align*}
\]  

(47)

The action of \( F_i \) on \( u(m) \) produces infinitely many terms:

\[
F_i u(m) = q^c \sum_{k \equiv i \mod n} u_m \otimes u_{m-1} \otimes \cdots \otimes u_{k+1} \otimes u_{k+1} \otimes u_{k-1} \otimes \cdots,
\]

(48)

where \( c = 0 \) if \( m \equiv i \mod n \), and 1 otherwise. If we apply \( E_i \) to the right hand side of (48), all the terms contribute to \( u(m) \) because \( E_i u_{k+1} = u_k \) for \( k \equiv i \mod n \). Therefore \( E_i F_i \) diverges.

The affine Hecke algebra action behaves better. The formulas given by (32) define an action of the infinite affine Hecke algebra \( \hat{H}_\infty(q^2) \) (generated by \( T_i \) and \( y_{i \pm 1} \), \( i = 1, 2, 3, \ldots \) with the above relations) on \( V(z) \otimes V(z) \otimes V(z) \otimes \cdots \). This action is well-defined because each \( T_i \) acts only in a pair of adjacent factors. The action of \( \hat{H}_\infty(q^2) \) in the thermodynamic limit commutes with the formal action of \( U'_q(\hat{\mathfrak{sl}}_n) \).

Let \( U_{(m)} \) denote the linear span of all pure tensors that coincide with \( u_{(m)} \) given by (47) after finitely many factors. In other words, \( U_{(m)} \) is spanned by tensors of the form

\[
\begin{align*}
  u_{m_1} \otimes u_{m_2} \otimes u_{m_3} \otimes \cdots
\end{align*}
\]

where \( m_k = m - k + 1 \) for \( k \gg 1 \).

Define \( F_{(m)} \) to be the quotient of \( U_{(m)} \) by the space \( \sum_i \text{Ker}(T_i+1) \), or, equivalently, by the relations (13) and (15) in each pair of adjacent factors. The spaces \( F_{(m)} \) will be called \( q \)-deformed Fock spaces, or semi-infinite \( q \)-wedge spaces. Corresponding to (47), we set

\[
|m\rangle = u_m \wedge u_{m-1} \wedge u_{m-2} \wedge \cdots.
\]

(49)

Remark In [11], semi-infinite wedges were defined as completely \( q \)-antisymmetrized ‘ideal’ elements of \( U_{(m)} \), using the antisymmetrization operator \( \sum_{\sigma \in S_\infty} T_\sigma \). (\( S_\infty \) is the
group of bijections $\mathbb{Z}^+ \to \mathbb{Z}^+$ fixing all but finitely many elements; equivalently, it is the group generated by adjacent transpositions $\sigma_1, \sigma_2, \sigma_3, \ldots$. The elements $\{T_\sigma\}_{\sigma \in S_\infty}$ form a basis for the infinite Hecke algebra $H_\infty(q^2)$ inside $\hat{H}_\infty(q^2)$. We prefer to define semi-infinite wedge space as a quotient by certain relations so as not to have to work with infinite sums.

Because each $T_i$ is an intertwiner, the action of $U'_q(\hat{\mathfrak{sl}}_n)$ on $V(z) \otimes N$ given by the coproduct (iterated $N-1$ times) factors through to the quotient space $\wedge^N V(z) = V(z)^{\otimes N}/\text{Ker} A^{(N)}$. We will show that the formal action of $U'_q(\hat{\mathfrak{sl}}_n)$ on the infinite tensor space $U_{(m)}$ induces a genuine action on $F_{(m)}$.

For each vector $v \in F_{(m)}$ we have a decomposition of the form

$$v = v^{(N)} \wedge |m-N\rangle, \quad v^{(N)} \in \bigwedge^N V(z)$$

for a sufficiently large $N$. Therefore, if we determine the action of the Chevalley generators on $|m\rangle$ for all $m \in \mathbb{Z}$, the coproduct (11-13) gives the action on general vectors.

We define the action of $E_i, F_i, K_i$ ($i = 0, 1, \ldots, n-1$) on $|m\rangle$ to mirror their formal action in $U_{(m)}$, as follows:

$$E_i|m\rangle = 0,$$
$$F_i|m\rangle = \begin{cases} u_{m+1} \wedge u_{m-1} \wedge u_{m-2} \wedge \cdots & \text{if } i \equiv m \mod n; \\ 0 & \text{otherwise}, \end{cases}$$
$$K_i|m\rangle = \begin{cases} q|m\rangle & \text{if } i \equiv m \mod n; \\ |m\rangle & \text{otherwise}. \end{cases}$$

Noting that $u_l \wedge u_l = 0$, we can show the well-definedness of this action. The action of $d$ is also consistently defined by fixing the degree of $|0\rangle$ to be zero.

**Proposition 1.4** The $U'_q(\hat{\mathfrak{sl}}_n)$-module $F_{(m)}$ is isomorphic to the $q$-deformed Fock space constructed by Hayashi in [6] and Misra-Miwa in [10].

**Proof.** From Proposition 1.3 we see that the vectors

$$u_{m_1} \wedge u_{m_2} \wedge u_{m_3} \wedge \cdots$$

where $m_1 > m_2 > m_3 > \cdots$ and $m_k = m - k + 1$ for $k \gg 1$, constitute a basis of $F_{(m)}$. There is an evident one-to-one correspondence between these vectors and the colored Young diagrams in [10], and this correspondence is equivariant with respect to
the $U'_q(\widehat{\mathfrak{sl}}_n)$-actions.

Set

$$\mathfrak{h} = CH_0 \oplus CH_1 \oplus \cdots \oplus CH_{n-1} \oplus Cd,$$

where $K_i = q^{H_i}$. Let $\Lambda_i \in \mathfrak{h}^*$ ($0 \leq i \leq n-1$) be the fundamental weights, i.e., $\Lambda_i(H_j) = \delta_{i,j}$ and $\Lambda_i(d) = 0$. We define $\Lambda_m$ for $m \in \mathbb{Z}$ by requiring $\Lambda_m = \Lambda_{m-1} + \text{wt} u_m$. (If $m_1 \equiv m_2 \mod n$, then $\Lambda_{m_1}$ and $\Lambda_{m_2}$ are the same apart from the action of $d$.) Then $|m\rangle$ has weight $\Lambda_m$ and the weights of $F_m$ belong to $\Lambda_m + \sum_{i=0}^{n-1} \mathbb{Z} \leq_0 a_i$. The highest weight vector $|m\rangle \in F_m$ generates the irreducible highest weight module $V_{\Lambda_m}$ with highest weight $\Lambda_m$.

It is important to remark that $|m\rangle$ is not the only highest weight vector in $F_m$. For example, if $n = 2$, then $(F_0 F_1 - q F_1 F_0)|0\rangle$ is a highest weight vector that lies in the same Fock space as $|0\rangle$. The goal of this paper is to describe the decomposition of $F_m$ (which is completely reducible) as a $U'_q(\widehat{\mathfrak{sl}}_n)$-module.

2 Heisenberg algebra and decomposition of $F_m$

When $q = 1$, $F_m$ reduces to the ordinary infinite wedge space, and its decomposition as an $\widehat{\mathfrak{sl}}_n$-module is known. (See, for example, [1].) The decomposition comes from a Heisenberg algebra $H$ acting on wedge space and commuting with the action of $\widehat{\mathfrak{sl}}_n$. In this section, we introduce an analogous action of the Heisenberg algebra on the $q$-deformed Fock spaces, and use this action to decompose these spaces as $U'_q(\widehat{\mathfrak{sl}}_n)$-modules. We also decompose the mapping between Fock spaces that is induced by the $q$-wedging operator into the product of the vertex operator for the level-1 $U_q(\widehat{\mathfrak{sl}}_n)$-modules and that for the Heisenberg algebra.

2.1 Center of the affine Hecke algebra

The aim of this section is to define an action of a Heisenberg algebra $H$ on $F_m$, which commutes with the action of $U'_q(\widehat{\mathfrak{sl}}_n)$. This Heisenberg algebra is the limit $N \to \infty$ of the center of the finite affine Hecke algebra $\widehat{H}_N(q^2)$. Then the $q$-deformed Fock space $F_m$, regarded as a representation of $U'_q(\widehat{\mathfrak{sl}}_n) \otimes U(H)$, decomposes into the tensor product

$$F_m \simeq V_{\Lambda_m} \otimes \mathbb{C}[H_-],$$

where $\mathbb{C}[H_-]$ is the Fock space of the Heisenberg algebra $H$. 
In the previous section we constructed the $q$-deformed Fock space by starting from the $U_q(\hat{\mathfrak{sl}}_n)$-module $V(z)$. The $U'_q(\hat{\mathfrak{sl}}_n)$-action on $V(z)$ commutes with $y \in \text{End}_C V(z)$ (acting as multiplication by $z^{-1}$). Using this fact as a building block, we will construct elements in the centralizer of the $U'_q(\hat{\mathfrak{sl}}_n)$-action on $F_{(m)}$.

For $a \in \mathbb{Z}\setminus\{0\}$ define an operator $B_a$ acting formally in $V(z) \otimes V(z) \otimes V(z) \otimes \cdots$ by

$$B_a = \sum_{k=1}^{\infty} y_k^{-a}.$$  \hspace{1cm} (55)

It is clear that this formal action commutes with the formal action of $U'_q(\hat{\mathfrak{sl}}_n)$, and also that it preserves each subspace $U_{(m)}$. The following can be checked using relations (29) and (30):

**Lemma 2.1** The element

$$B^{(N)}_a = \sum_{k=1}^{N} y_k^{-a}$$

belongs to the center of $\widehat{H}_N(q^2)$.

In particular, $B_a$ commutes with each $T_i$, and therefore preserves the spaces $\text{Ker}(T_i+1)$. It therefore preserves their sum $\sum_i \text{Ker}(T_i + 1)$, which means that it acts on the quotient spaces $F_{(m)}$. In fact, this is a genuine action rather than just a formal one: the action of $B_a$ on a wedge results in a finite sum of wedges. This is because for sufficiently large $k$ we have $y_k^{-a} v = 0$, which follows from

**Lemma 2.2** Let $l \leq m$. Then, the $q$-wedges $u_m \wedge u_{m-1} \wedge \cdots \wedge u_{l+1} \wedge u_l \wedge u_m$ and $u_l \wedge u_m \wedge u_{m-1} \wedge \cdots \wedge u_{l+1} \wedge u_l$ are both equal to zero.

**Proof.** A straightforward induction using the relations (13) and (15). \hfill \Box

As already mentioned, the action of $B^{(N)}_a$ on $V(z)^{\otimes N}$ commutes with the action of $U'_q(\hat{\mathfrak{sl}}_n)$. Thus, we conclude

**Proposition 2.1** The operator $B_a$ acts on $F_{(m)}$ and commutes with $U'_q(\hat{\mathfrak{sl}}_n)$.

Next we compute the commutator $[B_{a_1}, B_{a_2}]$. For this purpose we need

**Lemma 2.3**

$$[B_a, z^b v_i] = z^{a+b} v_i.$$  \hspace{1cm} (56)
Here we consider $z^b v_i$ as an operator $F_m \to F_{m+1}$ acting as follows: if $v \in F_m$, then
$z^b v_i : v \mapsto z^b v_i \wedge v$.

**Lemma 2.4** Suppose that $\beta \in \text{End}_C \left( \oplus_{m \in \mathbb{Z}} F_m \right)$ satisfies the following:

1. $\beta F_m \subset F_m$, \hspace{1cm} (57)
2. $[d, \beta] = a\beta$ for some $a \in \mathbb{Z}$, \hspace{1cm} (58)
3. $[\beta, z^b v_i] = 0$ for any $b \in \mathbb{Z}$ and $i \in \{1, 2, \ldots, n\}$. \hspace{1cm} (59)

Then $\beta = \gamma \text{id}$ for some constant $\gamma$.

**Proof.** We use the decomposition (50). From (59) we have
$\beta v = v^{(N)} \wedge \beta |m-N\rangle$.

From (57) we have
$\beta |m-N\rangle = \sum_k \gamma_k u_{m_1,k} \wedge \cdots \wedge u_{m_M,k} \wedge |m-N-M\rangle$
for some $M > 0$, where $m_1,k > \cdots > m_M,k > m-N-M$ for all $k$. Because of (59), there exists an integer $L$ independent of $N$ such that $m_{1,k} \leq m-N+L$. By Lemma 2.4, for a sufficiently large $N$, $u^{(N)}\wedge u_{m_1,k} = 0$ for all $k$ such that $m_{1,k} \geq m-N+1$. Therefore, we can ignore these terms in $\beta v$. Suppose that $m_{1,k} \leq m-N$ and $u_{m_1,k} \wedge \cdots \wedge u_{m_M,k} \wedge |m-N-M\rangle \neq 0$. In this case, Lemma 2.4 implies $m_{l,k} = m-N-l+1$ $(1 \leq l \leq M)$. Therefore, $\beta v = \gamma v$, and it is clear that $\gamma$ is independent of $v$. \hfill $\Box$

We are now ready to show

**Proposition 2.2**

1. If $a_1 + a_2 \neq 0$, then $[B_{a_1}, B_{a_2}] = 0$.
2. $\gamma_a = [B_a, B_{-a}]$ is a non-zero constant.

**Proof.** The basic fact, which we will use to prove (i) and (ii), is that $[B_{a_1}, B_{a_2}]$ is a constant for all $a_1$ and $a_2$. This will follow from Lemma 2.4 as soon as we have verified that the operator $[B_{a_1}, B_{a_2}]$ satisfies the three conditions of the lemma. It is clear that
\([B_{a_1}, B_{a_2}]\) preserves \(F_m\). Since \([d, B_a] = aB_a\), \([d, [B_{a_1}, B_{a_2}]] = (a_1 + a_2)[B_{a_1}, B_{a_2}]\), so the second condition is satisfied. Finally, Lemma 2.3 implies
\[
[[B_{a_1}, B_{a_2}], z^b v_i] = 0,
\]
which is precisely the third condition.

Now to show (i), observe that if \(a_1 + a_2 \neq 0\), then the degree of the operator \([B_{a_1}, B_{a_2}]\) is non-zero. Since it is a constant, we must have \([B_{a_1}, B_{a_2}] = 0\).

To show (ii), note that one can compute \([B_a, B_{-a}]|m\rangle\) as a finite sum by using the formulas (43) and (45). This implies that \(\gamma_a\) is a polynomial in \(q\), and by specializing it to \(q = 1\), we can conclude that \(\gamma_a \neq 0\).

We will derive an explicit formula for \(\gamma_a\) in the next subsection. For now, let us calculate \(\gamma_1\) as an example.

Using (2), we have
\[
\begin{align*}
B_{-1}|0\rangle &= u_n \wedge |-1\rangle + u_0 \wedge u_{n-1} \wedge |-2\rangle + u_0 \wedge u_{-1} \wedge u_{n-2} \wedge |-3\rangle + \cdots + u_0 \wedge \cdots \wedge u_{-n+2} \wedge u_1 \wedge |n\rangle \\
&= u_n \wedge |-1\rangle - qu_{n-1} \wedge u_0 \wedge |-2\rangle + (-q)^2 u_{n-2} \wedge u_0 \wedge u_{-1} \wedge |-3\rangle + \cdots + (-q)^{n-1} u_1 \wedge u_0 \wedge \cdots \wedge u_{-n+2} \wedge |n\rangle.
\end{align*}
\]

Then, by applying \(B_1\) we get
\[
\begin{align*}
B_1 B_{-1}|0\rangle &= u_0 \wedge |-1\rangle - qu_{-1} \wedge u_0 \wedge |-2\rangle + (-q)^2 u_{-2} \wedge u_0 \wedge u_{-1} \wedge |-3\rangle + \cdots + (-q)^{n-1} u_{n+1} \wedge u_0 \wedge \cdots \wedge u_{-n+2} \wedge |n\rangle \\
&= (1 + q^2 + q^4 + \cdots + q^{2n-2})|0\rangle.
\end{align*}
\]

Therefore, noting that \([B_1, B_{-1}]|0\rangle = B_1 B_{-1}|0\rangle\), we conclude
\[
\gamma_1 = \frac{1 - q^{2n}}{1 - q^2}.
\]

Summing up, in this subsection we have constructed the action of the Heisenberg algebra \(H\) on \(F_m\). \(H\) is generated by the operators \(B_a\) \((a \in \mathbb{Z} \setminus \{0\})\) with the commutation relations
\[
[B_a, B_b] = \delta_{a+b,0} \gamma_a,
\]
where the constant \(\gamma_a\) is yet to be determined.
Let $C[H_-]$ be the Fock space of $H$, i.e., $C[H_-] = C[B_{-1}, B_{-2}, \ldots]$. The element $B_{-a}$ $(a = 1, 2, \ldots)$ acts on $C[H_-]$ by multiplication. The action of $B_{-a}$ $(a = 1, 2, \ldots)$ is given by (62) together with the relation
\[ B_{-a} \cdot 1 = 0 \quad \text{for} \quad a \geq 1. \quad (63) \]

By computing characters we easily obtain

**Proposition 2.3** There is an isomorphism
\[ \iota_m : F(m) \simeq V_{\Lambda_m} \otimes C[H_-] \quad (64) \]
of $U_q(\hat{\mathfrak{sl}_n}) \otimes H$-modules.

We normalize the isomorphism by requiring
\[ \iota_m(|m\rangle) = v_{\Lambda_m} \otimes 1. \]

### 2.2 Decomposition of the vertex operator

We define the vertex operator
\[ \Omega : V(z) \otimes F(m-1) \to F(m) \quad (65) \]
by $\Omega(u_a \otimes \omega) = u_a \wedge \omega$. This is an intertwiner of $U_q(\hat{\mathfrak{sl}_n})$-modules.

In this section, we decompose the vertex operator $\Omega$ into two parts corresponding to the decomposition (64): one which acts from $V_{\Lambda_{m-1}}$ to $V_{\Lambda_m}$ and the other which acts on $C[H_-]$.

The first step in carrying out this decomposition is to transfer $\Omega$ from the $F(m)$-setting to the $V_{\Lambda_m} \otimes C[H_-]$-setting. To be precise, define
\[ \Omega' : V(z) \otimes V_{\Lambda_{m-1}} \otimes C[H_-] \to V_{\Lambda_m} \otimes C[H_-] \quad (66) \]
by requiring that the following diagram commutes:
\[
\begin{array}{ccc}
V(z) \otimes F(m-1) & \xrightarrow{id_{V(z)} \otimes \iota_{m-1}} & V(z) \otimes V_{\Lambda_{m-1}} \otimes C[H_-] \\
\downarrow\Omega & & \downarrow\Omega' \\
F(m) & \xrightarrow{\iota_m} & V_{\Lambda_m} \otimes C[H_-]
\end{array}
\quad (67)
\]
We will decompose $\Omega'$ (on the level of generating series) into one part corresponding to $V_{\Lambda m}$ and another part corresponding to $C[1/H_-]$. Given $j \in \mathbb{Z}$, we associate to $\Omega$ and $\Omega'$ the generating series

$$
\Omega_j(w) = \sum_{b \in \mathbb{Z}} \Omega_{j,b} w^{-b},
$$

$$
\Omega'_j(w) = \sum_{b \in \mathbb{Z}} \Omega'_{j,b} w^{-b}.
$$

Here $\Omega_{j,b}$ is an operator $F_{(m-1)} \rightarrow F_{(m)}$ whose action is defined by

$$
\Omega_{j,b} \cdot \omega = \Omega(u_j^{m-1} \otimes \omega).
$$

Similarly, $\Omega'_{j,b}$ is an operator $V_{\Lambda m-1} \otimes C[1/H_-] \rightarrow V_{\Lambda m} \otimes C[1/H_-]$ whose action is given by

$$
\Omega'_{j,b} \cdot \omega \otimes f = \Omega'(u_j^{m-1} \otimes \omega \otimes f).
$$

The first element in the decomposition of $\Omega'$ is a $U'_q(\hat{\mathfrak{sl}}_n)$-vertex operator corresponding to $V_{\Lambda m}$. It is given by the following proposition:

**Proposition 2.4** ([2]) There exists a unique intertwiner of $U'_q(\hat{\mathfrak{sl}}_n)$-modules

$$
\tilde{\Phi}^* : V(z) \otimes V_{\Lambda m-1} \rightarrow V_{\Lambda m},
$$

such that $\tilde{\Phi}^*(u_m \otimes v_{\Lambda m-1}) = v_{\Lambda m}$.

The intertwiner $\tilde{\Phi}^*$ also has a generating series associated with it. It is given by

$$
\tilde{\Phi}^*_j(w) = \sum_{b \in \mathbb{Z}} \tilde{\Phi}^*_{j,b} w^{-b},
$$

where $\tilde{\Phi}^*_{j,b} : V_{\Lambda m-1} \rightarrow V_{\Lambda m}$ is given by

$$
\tilde{\Phi}^*_{j,b} \cdot \omega = \tilde{\Phi}^*(u_j^{m-1} \otimes \omega).
$$

**Remark** Note that the uniqueness in Proposition 2.4 is assured by the requirement that the degree with respect to $d$ is invariant. A general intertwiner of $U'_q(\hat{\mathfrak{sl}}_n)$-modules $\Phi : V(z) \otimes V_{\Lambda m-1} \rightarrow V_{\Lambda m}$ (not preserving degree) can be written as

$$
\Phi(z^a v_l \otimes \omega) = \sum_k c_k \tilde{\Phi}^*(z^{a+k} v_l \otimes \omega).
$$
The second component in the decomposition of \( \Omega' \) is a vertex operator corresponding to \( H \). This is given by the generating series

\[
\Xi(w) = \exp \left( \sum_{b \geq 1} \frac{B_{-b}w^b}{\gamma_b} \right) \exp \left( - \sum_{b \geq 1} \frac{B_{b}w^{-b}}{\gamma_b} \right).
\] (70)

Here \( \gamma_b \) is given by

\[
\gamma_b = [B_b, B_{-b}].
\] (71)

The vertex operator \( \Xi(w) \) is characterized by the commutation relation

\[
[B_a, \Xi(w)] = w^a \Xi(w).
\] (72)

Let \( \Xi_b \) denote the coefficient of \( w^{-b} \) in the expansion of (70).

We can now state the main result of this section.

**Proposition 2.5**

\[
\Omega'_j(w) = \tilde{\Phi}^*_j(w) \otimes \Xi(w).
\] (73)

**Proof.** First we observe that by Proposition 2.4 the map \( \Omega' \) is uniquely determined by the following conditions:

(i) \( \Omega'(u_m \otimes v_{\Lambda_{m-1}} \otimes 1) = v_{\Lambda_m} \otimes 1 \),

(ii) \( \Omega' \) is \( U_q(\hat{sl}_n) \)-linear,

(iii) \( \Omega' \) is \( H \)-linear.

In (iii) the action of \( B_a \) on \( V(z) \otimes V_{\Lambda_{m-1}} \otimes C[H_-] \) is given by

\[
B_a(z^b v_i \otimes v \otimes f) = z^{a+b} v_i \otimes v \otimes f + z^a v_i \otimes v \otimes B_a f.
\]

We will show that the right hand side of (73) satisfies these conditions, and hence is equal to \( \Omega'(w) \). Condition (i) follows from

\[
\tilde{\Phi}^*_{m,b} v_{\Lambda_{m-1}} = \begin{cases} 0 & \text{if } b > 0; \\ v_{\Lambda_m} & \text{if } b = 0, \end{cases}
\] (74)

\[
\Xi_b 1 = \begin{cases} 0 & \text{if } b > 0; \\ 1 & \text{if } b = 0. \end{cases}
\] (75)
Condition (ii) is satisfied automatically. Condition (iii) follows from (72).

The final step is to explicitly calculate \( \gamma_a = [B_a, B_{-a}] \) by comparing two point functions of the vertex operators appearing in Proposition 2.5. The result we are aiming for is

**Proposition 2.6**

\[
[B_a, B_{-a}] = a \frac{1 - q^{2na}}{1 - q^{2a}}. \tag{76}
\]

**Proof.** The idea of the proof is to calculate and compare the two point functions of each side of (73), then read off a formula for \( \gamma_a = [B_a, B_{a}] \). The right hand side can be done in each factor separately; the answers are given by the following two lemmas:

**Lemma 2.5**

\[
\langle 1, \Xi(w_1)\Xi(w_2)1 \rangle = \exp \left( \sum_{a>0} \frac{(w_2/w_1)^a}{\gamma_a} \right) \tag{77}
\]

**Lemma 2.6**

\[
\langle v_{\Lambda m+1}, \tilde{\Phi}^*_{m+1}(w_1)\Phi^*_{m}(w_2)v_{\Lambda m-1} \rangle = \frac{(q^{2n+2}w_2/w_1; q^{2n})_\infty}{(q^{2n}w_2/w_1; q^{2n})_\infty}. \tag{78}
\]

Here \((z, p)_\infty = \prod_{k \geq 0} (1 - zp^k)\).

The former is easy, and the latter is obtained in [3, 8]. The two point function of the right hand side of (73) is then just the product of (77) and (78).

Now on to the left hand side of (73). By (67), the two point function for \( \Omega'(w) \) is the same as the one for \( \Omega(w) \). The latter is given by the following:

**Lemma 2.7**

\[
\langle m + 1|\Omega_{m+1}(w_1)\Omega_m(w_2)|m - 1 \rangle = \frac{1 - w_2/w_1}{1 - q^2 w_2/w_1}. \tag{79}
\]
Proof. We have
\[ \Omega_m(w_2)|m - 1\rangle = \sum_{j=0}^{\infty} u_{m+nj} \wedge |m - 1\rangle w_2^j. \]
Applying \( \Omega_{m+1}(w_1) \) to this sum, and collecting the terms whose weight is equal to that of \(|m + 1\rangle\), we get
\[ \sum_{b=0}^{\infty} u_{m+1-nb} \wedge u_{m+nb} \wedge |m - 1\rangle w_1^{-b} w_2^b. \]
Using the normal ordering rule gives us
\[ \langle m + 1\rangle (w_2/w_1)^b u_{m+1-nb} \wedge u_{m+nb} \wedge |m - 1\rangle = \begin{cases} 1 & \text{if } b = 0, \\ q^{2(b-1)}(q^2-1)(w_2/w_1)^b & \text{if } b > 0. \end{cases} \]
Summing up for \( b \), we obtain (79).

The proof of Proposition 2.7 is just a matter of putting together these three lemmas. Setting the left and right two point functions equal to each other and cancelling, we obtain
\[ \exp \left( \sum_{a>0} \frac{(w_2/w_1)^a}{\gamma_a} \right) = \prod_{a\geq 0} \frac{1 - q^{2n_a}w_2/w_1}{1 - q^2 + 2n_a w_2/w_1} \] (80)
A comparison of coefficients results in the asserted formula for \( \gamma_a \).

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