4-Dimensional BF Theory with Cosmological Term
as a Topological Quantum Field Theory

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Abstract

Starting from a Lie group $G$ whose Lie algebra is equipped with an invariant nondegenerate symmetric bilinear form, we show that 4-dimensional BF theory with cosmological term gives rise to a TQFT satisfying a generalization of Atiyah’s axioms to manifolds equipped with principal $G$-bundle. The case $G = \text{GL}(4,\mathbb{R})$ is especially interesting because every 4-manifold is then naturally equipped with a principal $G$-bundle, namely its frame bundle. In this case, the partition function of a compact oriented 4-manifold is the exponential of its signature, and the resulting TQFT is isomorphic to that constructed by Crane and Yetter using a state sum model, or by Broda using a surgery presentation of 4-manifolds.

1 Introduction

In comparison to the situation in 3 dimensions, topological quantum field theories (TQFTs) in 4 dimensions are poorly understood. This is ironic, because the subject was initiated by an attempt to understand Donaldson theory in terms of a quantum field theory in 4 dimensions. However, this theory has never been shown to fit Atiyah’s subsequent axiomatic description of a TQFT, and it is unclear whether one should even expect it to.

Here we consider a much simpler theory, BF theory with cosmological term [16, 7, 10, 11]. This theory depends on a choice of Lie group $G$ whose Lie algebra is equipped with an invariant nondegenerate symmetric bilinear form. Given an oriented 4-manifold $M$ equipped with principal $G$-bundle $P$, the fields in this theory are a connection $A$ on $P$ together with an $\text{ad}P$-valued 2-form $B$ on $M$. The Lagrangian is given by

$$\text{tr}(B \wedge F + \frac{\Lambda}{12} B \wedge B),$$

where it is crucial for our purposes that $\Lambda$ be nonzero. In certain cases this theory can be regarded as a simplified version of general relativity [3]. This the origin of the term ‘cosmological constant’ for $\Lambda$ and the curious factor of $\frac{1}{12}$. Indeed, this one of our main reasons for studying the theory, but we defer further discussion of this aspect to another paper.
Starting with this Lagrangian and performing some nonrigorous computations, we obtain results which we then take as the definition of a TQFT-like structure satisfying the obvious generalization of Atiyah’s axioms to the case of manifolds equipped with principal $G$-bundle. Choosing $G = \text{GL}(4, \mathbb{R})$, we then obtain a TQFT satisfying Atiyah’s axioms by letting $P$ be the frame bundle of $M$. As it turns out, the partition function of any compact oriented 4-manifold $M$ is then $\exp(-36\pi^2 i\sigma(M)/\Lambda)$, where $\sigma(M)$ is the signature of $M$. This fact says the theory is uninteresting as far as new 4-manifold invariants are concerned. However, it is the key to proving a conjecture that has been discussed in the mathematical physics community for some time: namely, that the Crane-Yetter-Broda theory is isomorphic to $BF$ theory.

To understand the origins of the Crane-Yetter-Broda theory and this conjecture about it, note that one may also define $BF$ theory with cosmological term in dimension 3. This has the Lagrangian

$$\text{tr}(B \wedge F + \frac{\Lambda}{3} B \wedge B \wedge B),$$

where now $B$ is an $\text{ad}P$-valued 1-form. Although it is difficult to state a concise theorem to this effect, it is by now commonly accepted that the quantum version of this theory is isomorphic as a TQFT to a state sum model of Turaev-Viro type \cite{10, 13, 23, 24, 25, 26, 27}. The original Turaev-Viro model was defined using the quantum group $U_q\text{sl}(2)$, but it was subsequently generalized to other quantum groups \cite{1, 28}, and one expects 3-dimensional $BF$ theory for any simply-connected compact semisimple group $G$ to be isomorphic to the state sum model based on the corresponding quantum group $U_q\text{Cg}$, with the value of $q$ depending on $\Lambda$.

Using a 4-dimensional state sum model similar to that of Turaev and Viro, Crane and Yetter \cite{14} succeeded in obtaining a 4-dimensional TQFT from $U_q\text{sl}(2)$. Shortly thereafter Broda \cite{8} constructed a similar theory using a surgery presentation of 4-manifolds. Then Roberts \cite{19} showed that the Crane-Yetter and the Broda theories were essentially the same, and that when properly normalized this TQFT gives as the partition function of any compact 4-manifold just the exponential of its signature. All these results have been extended to other quantum groups by Crane, Kauffman, and Yetter \cite{13}.

While not yielding new 4-manifold invariants, the Crane-Yetter-Broda theory is still rather interesting. First, there is a close relationship between this theory and Chern-Simons theory. For example, Roberts \cite{24} has shown that at least for $U_q\text{sl}(2)$, if one takes a triangulated compact oriented 4-manifold $M$ with boundary $\Sigma$ and computes the state sum over all labellings, not holding the labels fixed on $\Sigma$, one obtains the Chern-Simons partition function $Z_{\text{CS}}(\Sigma)$. This result probably holds quite generally for other quantum groups as well. This is especially nice because no 3-dimensional state sum model for Chern-Simons theory is known. In particular, the phase of $Z_{\text{CS}}(\Sigma)$ depends on a choice of framing for $\Sigma$ in a manner that seems difficult to incorporate into a 3-dimensional state sum, but in the 4-dimensional state
sum the framing is determined by the choice of bounding 4-manifold $M$. Second, it is interesting to have a simple state-sum formula for the signature. As noted by Crane, Kauffman, and Yetter [13], this “allows us to factor 4-manifold signatures along any 3-manifold.” These two points are closely related, because two 4-manifolds with boundary equal to $\Sigma$ determine the same framing of $\Sigma$ if and only if their signatures are equal.

For various reasons, it became natural to suspect that the Crane-Yetter theory corresponds to BF theory in dimension 4 in much the same way that the Turaev-Viro theory corresponds to BF theory in dimension 3. By giving a purely differential-geometric construction of 4-dimensional BF theory with $G = \text{GL}(4, \mathbb{R})$ as a TQFT, and proving that this TQFT is isomorphic to the Crane-Yetter theory, we obtain a precise result to this effect.

We hope this result sheds some light on the correspondence between two important but quite different approaches to topological quantum field theory: the differential-geometric approach and the combinatorial approach. However, many details of this correspondence remain to be understood. We discuss some particular directions for further exploration in the Conclusions.

2 BF Theory

In this section we first review some basic properties of BF theory with cosmological term in 4 dimensions, already established by heuristic arguments due to Blau and Thompson [7], Horowitz [10], Cattaneo, Cotta-Ramusino, Fröhlich and Martellini [10, 11] and others. Then we rigorously construct a 4-dimensional TQFT having these properties, and show it satisfies a generalization of the Atiyah axioms.

Let $G$ be a Lie group, either real or complex, with an invariant nondegenerate symmetric bilinear form on its Lie algebra $\mathfrak{g}$. Let $M$ be an oriented 4-manifold equipped with a principal $G$-bundle $P_M$ over it. Let $A$ be a connection on $P$ and $B$ an $\text{ad}P$-valued 2-form. Then the 4-dimensional BF action with cosmological constant $\Lambda \neq 0$ is given by

$$S_{BF}(A, B) = \int_M \text{tr}(B \wedge F + \frac{\Lambda}{12} B \wedge B).$$

Here the ‘trace’ denotes the use of the bilinear form to turn the quantities in parentheses into an ordinary 4-form.

Heuristically we expect that this action gives rise to a TQFT $Z_{BF}$ having a one-dimensional space of states for any compact oriented 3-manifold $\Sigma$. Arguments for this have been given using both canonical and path-integral approaches. In the canonical approach, the kinematical phase space associated to theory on $\mathbb{R} \times \Sigma$ is the cotangent bundle of the space $\mathcal{A}_\Sigma$ of connections on $P|\Sigma$, with the quantity canonically conjugate to $A|\Sigma$ being $B|\Sigma$:

$$\{B^a_{ij}(x), A^b_{kl}(y)\} = \delta^a_b \epsilon_{ijk} \delta(x, y),$$
where we use spacelike indices $i, j, k, \ldots$ and internal indices $a, b, c, \ldots$, and we raise and lower internal indices using the bilinear form on $g$. However, there are constraints: the Gauss law and the constraint

$$F^a_{ij} + \frac{\Lambda}{6} B^a_{ij} = 0.$$  \hfill (2)

Thus in the Dirac approach to quantization, physical states $\psi$ are functions on $A_\Sigma$ invariant under small gauge transformations and satisfying

$$(F^a_{ij} - i \frac{\Lambda}{6} \epsilon_{ijk} \delta A^a_{ka}) \psi = 0$$  \hfill (3)

Now suppose that $P|_\Sigma$ is trivializable. Then this first-order linear PDE has one solution up to a constant factor, namely

$$\psi(A) = e^{-\frac{2i}{3} S_{CS}(A)},$$  \hfill (4)

since the Chern-Simons action

$$S_{CS}(A) = \int \Sigma \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$  \hfill (5)

satisfies

$$\frac{\delta S_{CS}}{\delta A^a_{ka}} = \epsilon_{ijk} F^a_{ik}.$$ 

Note that the definition of the Chern-Simons action, and hence the state $\psi$, depends on a choice of trivialization of $P|_\Sigma$. As well-known, the Chern-Simons action is invariant under small gauge transformations, which we may interpret as saying that $\psi$ satisfies the Gauss law. On the other hand, the Chern-Simons action changes by a constant under large gauge transformations, which changes $\psi$ by a phase. This is not a problem in the present context, since we only expect (3) to have a unique solution up to a numerical factor.

The path-integral approach is less rigorous, but it clarifies the role of the 2nd Chern class, and suggests the ideas necessary for rigorously constructing a 4d TQFT. If $M$ has (possibly empty) boundary $\partial M = \Sigma$, we expect to obtain a vector $\psi$ in the space of states on $\Sigma$ as follows:

$$\psi(A_\Sigma) = \int_{A|_\Sigma = A_\Sigma} DA \int DB \ e^{i \int_M \text{tr}(B \wedge F + \frac{\Lambda}{2} B \wedge B)}.$$  

To compute this, we can first complete the square and do the integral over $B$:

$$\psi(A_\Sigma) \propto \int_{A|_\Sigma = A_\Sigma} DA e^{-\frac{4i}{3} \int_M \text{tr}(F \wedge F)}.$$  

Now suppose again that $P|_\Sigma$ is trivializable. A choice of trivialization defines a flat connection on $P|_\Sigma$, which we extend arbitrarily to a connection $A_0$ on all of $P$, allowing
us to express any other connection on $P$ in terms of an ad$P$-valued 1-form. Then the basic relation between the 2nd Chern form and the Chern-Simons form yields

$$\int_M \text{tr}(F \wedge F) = S_{CS}(A_\Sigma) + \int_M \text{tr}(F_0 \wedge F_0)$$

(6)

where $F_0$ is the curvature of $A_0$. If we allow ourselves to neglect the volume factor due to the (heuristic) integrals over $A$ and $B$, it follows that

$$\psi(A_\Sigma) = e^{-\frac{3i}{\Lambda}(S_{CS}(A_\Sigma) + \int_M \text{tr}(F_0 \wedge F_0))}.$$

(7)

As expected, $\psi$ is proportional to the solution given in (4). But also, we find that if the boundary of $M$ is empty, its partition function equals

$$Z_{BF}(M) = e^{-\frac{3i}{\Lambda} \int_M \text{tr}(F \wedge F)}$$

where $F$ is the curvature of an arbitrary connection on $P$. Note that this partition function really depends not only on $M$ but on the bundle $P$.

While the arguments above are heuristic, we can take the results and use them to define a TQFT-like structure satisfying a generalization of Atiyah's axioms. Roughly speaking, this will be a functor $Z_{BF}$ from the category $C$ of 'cobordisms between compact oriented 3-manifolds equipped with trivializable principal $G$-bundle' to the category $\textbf{Vect}$ of vector spaces. More precisely, an object $\Sigma$ in $C$ is a compact oriented 3-manifold, also denoted $\Sigma$, together with a trivializable principal $G$-bundle $P_{\Sigma} \to \Sigma$. A morphism $M: \Sigma \to \Sigma'$ in $C$ is an equivalence class of compact oriented 4-manifolds $M$ with boundary, equipped with principal $G$-bundle $P_M \to M$ and bundle isomorphism $f_M: P_{\Sigma} \cup P_{\Sigma}' \to P_M|_{\partial M}$ lifting an orientation-preserving diffeomorphism $\bar{f}_M: \Sigma \cup \Sigma' \to \partial M$. The equivalence relation is that $M \sim M'$ if there is a bundle isomorphism $\alpha: P_M \to P_{M'}$ such that $\bar{f}_{M'} = \alpha \circ \bar{f}_M$. We will not always be so pedantic, however: usually we will work with representatives rather than equivalence classes.

We define the functor $Z_{BF}$ as follows. If $\Sigma$ is an object in $C$, let $A_{\Sigma}$ denote the space of connections on $P_{\Sigma}$, and let $Z_{BF}(\Sigma)$ be the space of functions on $A_{\Sigma}$ that are multiples of $\exp(-\frac{3i}{\Lambda}S_{CS}(A))$. If $\Sigma$ is empty we set $Z_{BF}(\Sigma) = \mathbb{C}$. While we need a trivialization of $P_{\Sigma}$ to define the Chern-Simons action, and the action may change by a constant as we change our choice the trivialization, the space $Z_{BF}(\Sigma)$ is independent of this choice.

The spaces $Z_{BF}(\Sigma)$ have some properties one expects in a TQFT. First, $Z_{BF}(\Sigma \cup \Sigma') = Z_{BF}(\Sigma) \otimes Z_{BF}(\Sigma')$. More precisely, if we choose any trivialization of $P_{\Sigma} \cup P_{\Sigma'}$, and restrict this to trivializations of $P_{\Sigma}$ and $P_{\Sigma'}$, we have

$$\exp(-\frac{3i}{\Lambda}S_{CS}(A)) = \exp(-\frac{3i}{\Lambda}S_{CS}(A|_{\Sigma})) \exp(-\frac{3i}{\Lambda}S_{CS}(A|_{\Sigma'}))$$

for any connection $A$ on $P_{\Sigma} \cup P_{\Sigma'}$. This gives an isomorphism $Z_{BF}(\Sigma \cup \Sigma') \simeq Z_{BF}(\Sigma) \otimes Z_{BF}(\Sigma')$, which one can check does not depend on the choice of trivialization. Second,
if $\Sigma$ denotes $\Sigma$ with its orientation reversed, but with the same bundle $P_\Sigma$ over it, then $Z_{BF}(\Sigma) = Z_{BF}(\Sigma)^*$. More precisely, reversing the orientation switches the sign of the Chern-Simons action, so if $\psi \in Z_{BF}(\Sigma)$ and $\phi \in Z_{BF}(\Sigma)$, the product $\psi \phi$ is a constant function on $A_\Sigma$, which can be identified with a number times the constant function 1. This gives an isomorphism $Z_{BF}(\Sigma) \simeq Z_{BF}(\Sigma)^*$, which again does not depend on the choice of trivialization.

If $M: \emptyset \to \Sigma$ is a morphism in $C$, $Z_{BF}(M)$ should be a linear map from the complex numbers to $Z_{BF}(\Sigma)$, or equivalently, a vector $\psi \in Z_{BF}(\Sigma)$. We define this vector by

$$\psi(A_\Sigma) = e^{-\frac{3i}{8} \int_M \text{tr}(F \wedge F)},$$

(8)

where $F$ is the curvature of any connection $A$ extending $A_\Sigma$ to all of $P_M$. By equation (8), $\psi$ lies in $Z_{BF}(\Sigma)$ and is independent of the choice of $A$.

More generally, given any morphism $M: \Sigma \to \Sigma'$ in $C$, since the boundary of $M$ is $\Sigma \cup \Sigma'$, we obtain a vector in $Z_{BF}(\Sigma)^* \otimes Z_{BF}(\Sigma')$ by the above procedure. We then define $Z_{BF}(M): Z_{BF}(\Sigma) \to Z_{BF}(\Sigma')$ to be this vector, reinterpreted as a linear map from $Z_{BF}(\Sigma)$ to $Z_{BF}(\Sigma')$.

It is easy to see that $C$ becomes a rigid symmetric monoidal category in a manner analogous to the usual categories of cobordisms, with the tensor product of objects and morphisms being given by disjoint union, and duality for objects being given by orientation reversal. We then have:

**Theorem 1.** $Z_{BF}: C \to \text{Vect}$ is a symmetric monoidal functor.

Proof - A straightforward computation. $\square$

Using a simple argument noticed by Crane and Yetter [15], it follows that $Z_{BF}$ preserves duals up to canonical isomorphism.

### 3 The Case $G = \text{GL}(4, \mathbb{R})$

The construction of the previous section could in fact be generalized to higher even dimensions using other characteristic classes and their secondary characteristic classes. Just as in Theorem 1, these would give functors from a category $C$ of ‘cobordisms between compact oriented $(n - 1)$-manifolds equipped with trivializable principal $G$-bundle’ to the category $\text{Vect}$. What makes the construction particularly interesting when $n = 4$ is that the tangent bundle of any compact oriented 3-manifold is trivializable. This yields various procedures for obtaining objects and morphisms in $C$ from those in $4\text{Cob}$, the category of cobordisms between compact oriented 3-manifolds. A procedure of this sort that involved no arbitrary choices might yield a functor from $4\text{Cob}$ to $C$, which we could compose with $Z_{BF}$ to obtain a functor from $4\text{Cob}$ to $\text{Vect}$, and thus, with some luck, a TQFT. In fact, the procedure we consider gives a map from $4\text{Cob}$ to $C$ which is not quite a functor, but for which the composite with
$Z_{BF}$ is still a TQFT. This is the $BF$ theory naturally associated to the frame bundle of a 4-manifold.

Let $G = \text{GL}(4, \mathbb{R})$, and equip its Lie algebra with the bilinear form

$$\langle S, T \rangle = \text{tr}(ST),$$

the trace being that of $4 \times 4$ matrices. Given any compact oriented 4-manifold with boundary $M$, the oriented frame bundle $P_M$ is a principal $\text{GL}(4, \mathbb{R})$-bundle over $M$. We can also construct an $\text{GL}(4, \mathbb{R})$-bundle over a compact oriented 3-manifold $\Sigma$ as follows. Let $T \Sigma$ be its tangent bundle and let $L \Sigma$ be the trivial line bundle $\Sigma \times \mathbb{R}$ over $\Sigma$. Then the bundle $P_\Sigma$ of oriented frames of $T \Sigma \oplus L \Sigma$ is a principal $\text{GL}(4, \mathbb{R})$-bundle. Since $T \Sigma$ is trivializable, so is $P_\Sigma$, so we have a way to get objects in $C$ from objects in $4\text{Cob}$. We do not, however, have a systematic way to get morphisms in $C$ from morphisms in $4\text{Cob}$. We may think of a morphism $M: \Sigma \to \Sigma'$ in $4\text{Cob}$ as a compact oriented 4-manifold $M$ with boundary, equipped with an orientation-preserving diffeomorphism $f_M: \Sigma \cup \Sigma' \to \partial M$. (Actually, just as in $C$, morphisms in $4\text{Cob}$ are really certain equivalence classes \[22\], but we shall work with representatives and leave the reader to check that our constructions make sense at the level of equivalence classes.) To obtain a morphism in $C$ from this morphism in $4\text{Cob}$, we need to pick a bundle isomorphism $\tilde{f}_M: P_\Sigma \cup P_{\Sigma'} \to P_M|_{\partial M}$ lifting $f_M$. There appears to be no way to do this without an arbitrary choice. Without loss of generality we can assume $\Sigma \cup \Sigma' = \partial M$ and that $f_M$ is the inclusion map. Then we can obtain $\tilde{f}_M$ from an orientation-preserving isomorphism

$$\eta: T \Sigma \oplus L \Sigma \to TM|_\Sigma$$

together with a similar isomorphism for $\Sigma'$. On $T \Sigma$ we define $\eta$ to be the inclusion

$$T \Sigma \hookrightarrow TM|_\Sigma,$$

and defining $\eta$ on $L \Sigma$ amounts to choosing a section $v$ of $TM|_\Sigma$. For $\eta$ to be an isomorphism, it is necessary and sufficient that $v$ be nowhere tangent to $\Sigma$. For it to be orientation-preserving, $v$ must be inwards-pointing. The same holds for $\Sigma'$ except that $v$ must be outwards-pointing.

So in short, this procedure does not quite give a functor from $4\text{Cob}$ to $C$, but only a functor from a category $4\text{Cob}'$ in which the morphisms $M: \Sigma \to \Sigma'$ in $4\text{Cob}$ are equipped with a bit of extra structure: a section $v$ of $TM$ over $\partial M$ that is nowhere tangent to $\partial M$, inwards-pointing on $f_M \Sigma$, and outwards-pointing on $f_M \Sigma'$. This should not be surprising: a similar structure, called the ‘lapse and shift’, plays an important role in general relativity. Let us denote this functor from $4\text{Cob}'$ to $C$ as $F$.

We then obtain an actual TQFT as follows. First we define a map $G: 4\text{Cob} \to 4\text{Cob}'$, taking objects to objects and morphisms to morphisms, but not a functor, as follows. For each object $\Sigma$ of $4\text{Cob}$, let $G(\Sigma) = \Sigma$. For each morphism $M: \Sigma \to \Sigma'$
of $4\text{Cob}$, let $G(M)$ be $M$ equipped with an arbitrary section $v$ of $TM$ over $\partial M$ with the necessary properties. Then let $Z:4\text{Cob} \to \text{Vect}$ be the composite $Z_{BF} \circ F \circ G$.

**Theorem 2.** $Z$ is a TQFT, that is, a symmetric monoidal functor from $4\text{Cob}$ to $\text{Vect}$.

First we show that $Z$ is a functor. Note that the only way $G$ fails to be a functor is that it fails to send identity morphisms to identity morphisms, so we only need to check that $Z$ has this property. Let $\Sigma$ be an object of $4\text{Cob}$ and $M: \Sigma \to \Sigma$ the identity on $\Sigma$. $Z(M): Z(\Sigma) \to Z(\Sigma)$ may be identified with a vector $\psi \in Z(\Sigma \cup \Sigma)$. Let us fix, once and for all, a trivialization of $T\Sigma$. From this we obtain a trivialization of $P\Sigma$, and, using the standard diffeomorphism $\Sigma \simeq \Sigma$, also a trivialization of $P_{\Sigma}$. Then to show that $Z(M)$ is the identity it suffices to check that

$$\psi(A_{\Sigma \cup \Sigma}) = e^{-\frac{2\pi i}{3} S_{\text{CS}}(A_{\Sigma \cup \Sigma})}$$

for any connection $A_{\Sigma \cup \Sigma}$ on $P_{\Sigma} \cup P_{\Sigma}$.

For the purposes of checking this, let us identify the manifold $M$ with $[0,1] \times \Sigma$. To construct $F(M)$ we have arbitrarily equipped $M$ with a section $v$ of $TM$ over $\partial M$ that is nowhere tangent to $\partial M = \{0,1\} \times \Sigma$, inwards-pointing on $\{0\} \times \Sigma$, and outwards-pointing on $\{1\} \times \Sigma$. This gives an isomorphism $\tilde{f}_M: P_{\Sigma} \cup P_{\Sigma} \to P_M|_{\partial M}$. By equation (8), $\psi$ is given by

$$\psi(f_M^*A) = e^{-\frac{2\pi i}{3} \int_M tr(F \wedge F)}$$

where $A$ is any connection on $P_M|_{\partial M}$, and $F$ is its curvature.

We can evaluate the right-hand side of the above equation using equation (8). Using our trivialization of $T\Sigma$ and the standard vector field $\partial_t$ on $M = [0,1] \times S$ associated with the coordinate $t$ on $[0,1]$, we obtain a trivialization of $TM$, hence of $P_M$. In equation (8) take $A_0$ to be the flat connection on $P_M$ associated with this trivialization, and restrict this trivialization to $\partial M$ to define the Chern-Simons action. Then we obtain:

$$\psi(f_M^*A) = e^{-\frac{2\pi i}{3} S_{\text{CS}}(A)}.$$

Equation (9), which we wish to check, will follow upon setting $A_{\Sigma \cup \Sigma} = f_M^*A$ if we can show

$$S_{\text{CS}}(A) = S_{\text{CS}}(f_M^*A).$$

The key point is to show that the trivialization of $P_M|_{\partial M}$ used to define the left-hand side, and the trivialization of $P_{\Sigma \cup \Sigma}$ used to define the right-hand side, are compatible. It is not true that the isomorphism $\tilde{f}_M$ carries the trivialization of $P_{\Sigma \cup \Sigma}$ to the trivialization of $P_M|_{\partial M}$. However, note $v$ is inwards-pointing on $\{0\} \times S$ and outwards-pointing on $\{1\} \times s$, and so is $\partial_t$. Thus $\tilde{f}_M$ does carry the trivialization of $P_{\Sigma \cup \Sigma}$ to that of $P_M|_{\partial M}$ up to a small gauge transformation. Since the Chern-Simons action is invariant under small gauge transformations, equation (10) holds.
It is easy to check that $Z$ is monoidal, and an argument like the above one shows that $Z$ is symmetric.

Again, while the fact that $Z$ preserves duals up to canonical isomorphism is often taken as part of the definition of a TQFT, this actually follows from $Z$ being a symmetric monoidal functor \cite{13}.

If $M$ is a compact oriented 4-manifold and $A$ is any connection on $TM$, the Hirzebruch signature theorem implies that $\int_M \text{tr}(F \wedge F) = 12\pi^2 \sigma(M)$, where $\sigma(M)$ is the signature of $M$. Thus we have

$$Z(M) = e^{-36\pi^2 i \sigma(M)/\Lambda}.$$  

\section{Relation to the Crane-Yetter-Broda Theory}

To establish an isomorphism between the Crane-Yetter-Broda theory and the TQFT of the previous section, we do not need to know much about the Crane-Yetter-Broda theory. All we need, in fact, is that it is a 4-dimensional TQFT such that the partition function of any 4-manifold equals $\exp(\alpha \sigma(M))$ for some constant $\alpha$; as we shall see, there is a unique such TQFT for any $\alpha$. Crane and Yetter’s description of the theory in terms of a state sum model is interesting nonetheless, because the existence of such a model means that we have an extended TQFT in the sense of Lawrence \cite{17}. Thus we begin with a brief review of this state sum model.

The original Crane-Yetter theory was constructed using the representation theory of the quantum group $U_q\mathfrak{sl}(2)$ when $q$ is a suitable root of unity. This was subsequently generalized by Crane, Kauffman and Yetter \cite{13} to other quantum groups, and is this generalization that we describe here. The input to the theory is a semisimple tortile tensor category $\mathbf{K}$ with trivial center. The reader may turn to the above reference for the definitions involved, but the main examples to keep in mind are certain subquotients of the categories of finite-dimensional representations of the quantum groups $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a complex semisimple Lie algebra and $q$ is a suitable root of unity \cite{22}. These subquotients are formed by first considering the full subcategory of completely reducible representations, and then quotienting by the tensor ideal generated by representations of quantum dimension 0.

Being semisimple, the category $\mathbf{K}$ has a basis $S$ of simple objects. We choose this basis so that it contains the unit 1 for the tensor product in $\mathbf{K}$, and so that $a \in S$ implies $a^* \in S$. For any objects $a, b, c \in S$, we choose a basis $B_{ab}^c$ of the vector space $\text{Hom}_{\mathbf{K}}(a \otimes b, c)$, and we write $B$ for the disjoint union of all these bases. In the quantum group case the objects in $S$ are in 1-1 correspondence with irreducible representations of nonzero quantum dimension, and we may think of $B_{ab}^c$ as a basis of intertwining operators from $a \otimes b$ to $c$, modulo intertwining operators that factor through a representation of quantum dimension zero. In general, every object $x$ of $\mathbf{K}$ has a ‘quantum dimension’ $\text{qdim}(x)$, which is typically not an integer.
Let $M$ be a triangulated compact oriented 4-manifold with an ordering of its vertices. Let $T_i$ denote the set of (nondegenerate) $i$-simplices in $M$, and let $n_i = |T_i|$. By a 'coloring' of $M$, we mean maps

$$\lambda: T_2 \cup T_3 \to S, \quad \lambda^+, \lambda^-: T_3 \to B$$

such that

$$\lambda^+(a, b, c, d) \in B^{\lambda(a,b,c,d)}_{\lambda(a,b,c,d)},$$

and

$$\lambda^-(a, b, c, d) \in B^{\lambda(a,b,c,d)}_{\lambda(a,b,c,d)},$$

where $a, b, c, d$ are vertices with $a < b < c < d$, and triples and quadruples of vertices denote triangles and tetrahedra, respectively. The reader can visualize such a coloring as follows. We form a graph in the 3-skeleton of $M$ whose intersection with any tetrahedron $\tau = (a, b, c, d) \in T_3$ appears as in Figure 1. The 4 edges of this graph intersecting the faces of $\tau$ are labelled with objects in $S$ using $\lambda|_{T_2}$, while the edge in the center of $\tau$ is labelled with an object using $\lambda|_{T_3}$. The 2 vertices are labelled with elements of $B$ using $\lambda^+$ and $\lambda^-$.

The Crane-Yetter invariant $Z_{CY}(M)$ is then given by

$$\sum_{\text{colorings}} N^{n_0-n_1} \prod_{\sigma \in T_2} \text{qdim}(\lambda(\sigma)) \prod_{\tau \in T_3} \text{qdim}(\lambda(\tau))^{-1} \prod_{\rho \in T_4} |\lambda, \rho|, \quad (11)$$

where

$$N = \sum_{a \in S} \text{qdim}(a)^2,$$

and $|\lambda, \rho|$ is a quantity obtained using the standard Reshetikhin-Turaev procedure from the portion of the graph contained in the boundary of the 4-simplex $\rho$.

The quantity $Z_{CY}(M)$ turns out to be independent of the triangulation of $M$. As shown by Roberts in the $U_q\mathfrak{sl}(2)$ case and by Crane, Kauffman and Yetter in general,

$$Z_{CY}(M) = N^{\chi(M)/2} y^{\sigma(M)}$$

where $y = (a_+/a_-)^{1/2}$, with $a_\pm$ obtained using the Reshetikhin-Turaev procedure from the $\pm 1$-framed unknot labelled with the linear combination of objects

$$\Omega = \sum_{a \in S} \text{qdim}(a)a.$$

Of course, by multiplying the state sum (11) by an appropriate exponential of Euler characteristic $\chi(M) = n_0 - n_1 + n_2 - n_3 + n_4$, we can easily obtain a modified Crane-Yetter state sum for which the partition function of $M$ contains any desired exponential of $\chi(M)$. In fact, Broda’s approach using a surgery presentation of $M$ very naturally corresponds to the modified state sum in which the partition function is just

$$Z'(M) = y^{\sigma(M)}.$$
In what follows we use this normalization.

So far we have discussed the theory only for compact 4-manifolds. One may extend the theory to a TQFT using the following procedure. Let $4\text{Cob}_{\text{PL}}$ be the piecewise-linear analog of the category $4\text{Cob}$. For each object $\Sigma$ of $4\text{Cob}_{\text{PL}}$, define $Z'(\Sigma)$ to be the vector space having as a basis all the morphisms $M: \emptyset \to \Sigma$, modulo those linear combinations $\sum c_M M$ such that

$$\sum c_M Z'(NM) = 0$$

for all morphisms $N: \Sigma \to \emptyset$. Then, for each morphism $N: \Sigma \to \Sigma'$, define $Z'(N)$ by

$$Z'(N)[\sum c_M M] = [\sum c_M NM].$$

Adapting the ideas of Blanchet et al [6] to the case of TQFTs that are not necessarily unitary, one can check that $Z'$ is well-defined on morphisms and actually gives a TQFT.

Now, while the Crane-Yetter state sum is defined in the piecewise-linear category, the categories $4\text{Cob}$ and $4\text{Cob}_{\text{PL}}$ are equivalent, so we can transfer the resulting TQFT to one in the smooth category. It will cause no confusion in what follows to also call this smooth version of the Crane-Yetter-Broda theory $Z'$. Our main result is then:

**Theorem 3.** If $y = \exp(-36\pi^2 i/\Lambda)$, then $Z: 4\text{Cob} \to \text{Vect}$ of Theorem 2 and the Crane-Yetter-Broda theory $Z': 4\text{Cob} \to \text{Vect}$ are equivalent as TQFTs. More precisely, there is a monoidal natural isomorphism $F: Z \to Z'$.

Proof - In fact, we shall show that any two TQFTs $Z, Z': 4\text{Cob} \to \text{Vect}$ having

$$Z(M) = Z'(M) = y^{\sigma(M)}$$

for all compact oriented 4-manifolds $M$ are equivalent in this sense, where $y \neq 0$. Concretely, this means, first of all, that for any object $\Sigma$ of $4\text{Cob}$ there is an isomorphism $F_{\Sigma}: Z(\Sigma) \to Z'(\Sigma)$. This isomorphism should be natural, in the sense that for any morphism $M: \Sigma_1 \to \Sigma_2$, one has $Z'(M)F_{\Sigma_1} = F_{\Sigma_2}Z(M)$. It should also be compatible with the monoidal structure. This compatibility condition is easiest to state if we use Mac Lane’s theorem to replace $4\text{Cob}$ and $\text{Vect}$ by equivalent strict monoidal categories; it then states that $F_{\Sigma_1 \cup \Sigma_2} = F_{\Sigma_1} \otimes F_{\Sigma_2}$ and $F_{\emptyset} = \text{id}$.

To construct an isomorphism with these properties, note first that for any object $\Sigma$ of $4\text{Cob}$, $Z(\Sigma)$ is 1-dimensional, and similarly for $Z'(\Sigma)$. To see this, recall that

$$\dim Z(\Sigma) = Z(S^1 \times \Sigma),$$

and that $\sigma(S^1 \times \Sigma) = 0$. Also note that for any object $\Sigma$ of $4\text{Cob}$ there is a morphism $M: \emptyset \to \Sigma$, because the oriented cobordism group vanishes in dimension 3. Moreover,
for any morphism $M: \emptyset \to \Sigma$ the vector $Z(M)1$ is nonzero, because $\overline{M}$ is a morphism from $\Sigma$ to $\emptyset$, and

$$
Z(\overline{M})Z(M)1 = Z(\overline{MM})1 = y^{\sigma(\overline{M})} \neq 0.
$$

We claim that for any $\Sigma$, there is a unique isomorphism $F_\Sigma: Z(\Sigma) \to Z'(\Sigma)$ such that $F_\Sigma(Z(M)1) = Z'(M)1$ for all $M: \emptyset \to \Sigma$. Uniqueness is clear, because the remarks above imply $Z(\Sigma)$ is spanned by $Z(M)1$. For existence, we need to check that if $M_1, M_2: \emptyset \to \Sigma$, so that $Z(M_1) = \alpha Z(M_2)$ and $Z'(M_1) = \alpha' Z'(M_2)$, then $\alpha = \alpha'$. To see this it suffices to note that

$$
Z(\overline{M_1}M_1) = \alpha Z(\overline{M_1}M_2), \quad Z'(\overline{M_1}M_1) = \alpha' Z'(\overline{M_1}M_2),
$$

and that $Z$ and $Z'$ agree and are nonzero on compact 4-manifolds.

To show that $F_\Sigma$ is natural it suffices to consider a morphism $M: \Sigma_1 \to \Sigma_2$ and check that $Z'(M)F_{\Sigma_1} = F_{\Sigma_2}Z(M)$ on a vector of the form $Z(N)1$, where $Z(N): \emptyset \to \Sigma_1$. This is clear:

$$
Z'(M)F_{\Sigma_1}Z(N)1 = Z'(M)Z'(N)1 = Z'(MN)1 = F_{\Sigma_2}Z(M)Z(N)1.
$$

To show that $F_\Sigma$ is compatible with the monoidal structure choose $M_1: \emptyset \to \Sigma_1$ and $M_2: \emptyset \to \Sigma_2$. Then we have

$$
F_{\Sigma_1 \cup \Sigma_2}Z(M_1 \cup M_2)1 = Z'(\Sigma_1 \cup \Sigma_2)1 = Z'(\Sigma_1)1 \otimes Z'(\Sigma_2)1 = F_{\Sigma_1}Z(\Sigma_1)1 \otimes F_{\Sigma_2}Z(\Sigma_2)1 = (F_{\Sigma_1} \otimes F_{\Sigma_2})Z(M_1 \cup M_2)1,
$$

so $F_{\Sigma_1 \cup \Sigma_2} = F_{\Sigma_1} \otimes F_{\Sigma_2}$. Similar manipulations show that $F_\emptyset$ is the identity. \qed

5 Conclusions

We have exhibited 4-dimensional $BF$ theory with cosmological term and quite general gauge group $G$ as a TQFT-like functor, and obtained an actual TQFT by setting $G = GL(4, \mathbb{R})$ and using the frame bundle as a natural choice of $G$-bundle. This TQFT is equivalent to the Crane-Yetter-Broda theory when the the cosmological constant $\Lambda$ and the constant $y$ appearing in the Crane-Yetter-Broda theory are related as in Theorem 3. However, the real reason for this equivalence is still somewhat mysterious, and deserves more study.

Indeed, it might seem surprising at first that the Crane-Yetter-Broda theory depends so little on the choice of category $\mathcal{C}$ — or, more concretely, on the choice of
quantum group $U_q g$. The labels involved in the Crane-Yetter state sum depend in
detail upon this choice, so different choices should give different extended TQFTs —
that is, they should assign different algebraic data to manifolds with corners, such
as the 4-simplex itself. However, regarded just as a TQFT the Crane-Yetter-Broda
theory depends on $U_q g$ only through the single constant $y$!

To understand this, note that in quantum group examples $y$ is the quantity by
which the Chern-Simons partition function of a compact oriented framed 3-manifold is
multiplied when one changes the framing by one ‘twist’ [8]. Thus one may say that as
a TQFT the Crane-Yetter-Broda theory simply keeps track of the framing-dependence
of the corresponding Chern-Simons theory. As an extended TQFT, however, it ap-
pears to contain enough information to compute the Chern-Simons partition function
— though so far this has been proved only for $U_q sl(2)$. Different extended TQFTs
may correspond to the same TQFT, as is already clear from 2-dimensional examples,
so the fact that we have found a $BF$ theory isomorphic to the Crane-Yetter-Broda
to as a TQFT does not imply that this $BF$ theory is isomorphic to it as an
extended TQFT. This is relevant to the problem of obtaining invariants of embedded
surfaces from $BF$ theory [11, 12], since obtaining such invariants uses its structure as
an extended TQFT. It is known how to obtain such invariants from the Crane-Yetter-
Broda theory [20, 29], but until we obtain an isomorphism between this theory and
a $BF$ theory as extended TQFTs, we will not know what these results imply about
$BF$ theory.

Now, just as the Crane-Yetter-Broda theory as an extended TQFT appears to
be able to compute the corresponding Chern-Simons partition function, equation (7)
suggests a similar result for $BF$ theories. So there is reason to conjecture that a
suitable $BF$ theory is isomorphic to the Crane-Yetter-Broda theory as an extended
TQFT. In fact, it seems that for $G$ simply-connected, compact and semisimple, the
Crane-Yetter-Broda theory with quantum group $U_q C g$ corresponds to a $BF$ theory
with gauge group $G \times GL(4, R)$, with each 4-manifold being equipped with a principal
bundle given by the trivial $G$-bundle times its frame bundle. This is already implicit
in Witten's original paper [27] on Chern-Simons theory, since in his approach the
framing-dependence arises by adding to the Chern-Simons Lagrangian with gauge
group $G$ a ‘gravitational’ Chern-Simons term. We hope to return to this conjecture
in future work.

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