q-MOMENT ESTIMATES FOR THE SINGULAR p-LAPLACE EQUATION AND APPLICATIONS

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Abstract. We provide q-moment estimates on annuli for weak solutions of the singular p-Laplace equation where p and q are conjugates. We derive q-uniform integrability for some critical parameter range. As a application, we derive a mass conservation as well as a weak convergence result for a larger critical parameter range. Concerning the latter point, we further provide a rate of convergence of order \( t^{\frac{q}{q-1}} \) of the solution in the q-Wasserstein distance.

Keywords: Singular p-Laplace Equation; Moment Estimates; Mass Conservation; Convergence Rate in Wasserstein Distance.

1. Introduction

We consider the Cauchy problem for the parabolic p-Laplace equation posed in the whole Euclidean space \( \mathbb{R}^N \)

\[
\begin{cases}
\partial_t u = \text{div}(|\nabla u|^{p-2}\nabla u), \\
u(0) = \mu,
\end{cases}
\]

with a positive Radon measure \( \mu \) as initial data and \( p_c < p < 2 \) where the critical value \( p_c \) is defined as

\[ p_c := \frac{2N}{N + 1}. \]

Solutions are understood in the weak sense on the whole space and finite time horizon \( \mathcal{S}_T := \mathbb{R}^N \times (0, T] \) where \( N \geq 2 \).

For \( 1 < p < 2 \), such Cauchy problem is known as the fast diffusion or singular equation. Existence and uniqueness and regularity of weak solutions with various conditions for initial data are studied by Chen and Di Benedetto [5], Di Benedetto and Herrero [6], Junning [10]. For further insights about parabolic p-Laplace equation, we refer to [3, 15]. While for a Dirac measure \( \delta_0 \) as initial data, this Cauchy problem does not admit a positive solution for any \( 1 < p \leq p_c \), see [10]. It is known that for \( p_c < p < 2 \), there exists a fundamental solution called Barenblatt source-type solution, see [15].

In the context of optimal matching problem, Ambrosio et al. [1] draw a link between 2-Wasserstein distance and Poisson equation through linearization of Monge-Ampère equation and regularize empirical measures with heat semigroup. They suggest that such a linearization for the q-Wasserstein distance leads to p-Laplacian where p and q are conjugates. Such an approach has been used in a different context by Evans and Gangbo [2]. Based on these ideas, it seems natural to regularize empirical measures with p-Laplacian semigroup. And further, to connect the Cauchy problem (1.1) to flows in the q-Wasserstein space, as done in Kell [12]. A key point in this connection concerns the behavior of q-moment of weak solutions \( u(x, t) \) of Cauchy problem (1.1), namely,

\[ \int_{\mathbb{R}^N} |x|^q u(x, t) dx. \]
For the degenerate case where \( p > 2 \), that is, \( 1 < q < 2 \), such \( q \)-moments are easily estimated, at least for initial data with finite mass and compact support, due to the finite propagation property, see \([11]\). However, for the singular case \( p_c < p < 2 \), solutions diffuse everywhere even with a Dirac measure as initial data. Hence, the focus on the parameter range \( p_c < p < 2 \).

Following similar methods as in \([6, 10]\), our main result are the following \( q \)-moment estimate for weak solutions of \((1.1)\) on annuli

\[
\sup_{0 < \tau \leq t} \int_{r \leq |x| \leq R} |x|^q u(x, \tau) dx \leq C \int_{\frac{1}{2} r \leq |x| \leq 2R} |x|^q d\mu + C t^{\frac{1}{q-p} - \frac{1}{q} + \frac{3}{p-1} - \frac{1}{2} + \frac{1}{2N} + 2 + \frac{2}{p-1} - \frac{1}{2}}.
\]

(1.2)

for any \( p_c < p < 2 \), where \( C \) is a constant depending only on \( N \) and \( p \) and \( p(N) \) is the polynomial

\[ p(N) = (N + 2)p^2 - (3N + 3)p + 2N. \]

This estimate shows in particular that with a finite \( q \)-moment Radon measure as initial data, there exists a weak solution which has uniformly bounded \( q \)-moments on all compact time interval for any \( p_N < p < 2 \), where the critical value \( p_c < p_N < 2 \) is the largest root of the polynomial \( p(N) \). This critical range \([p_N, 2)\) is sharp in the sense that the Barenblatt source-type solution has finite \( q \)-moment if and only if \( p_N < p < 2 \).

As applications, we show that for such a weak solution with a finite \( q \)-moment Radon measure \( \mu \) as initial data for any \( p_N < p < 2 \), namely,

\[ W_q^2(\mu, u(x, t)dx) \leq C t^{q-1}, \]

(1.3)

where \( C \) is a constant depending only on \( N, p, \mu \) and \( T \). In particular, when \( p \) converges to 2, the convergence rate coincides with the well-known heat flow case in \([1]\).

To the best of our knowledge, there does not exists \( q \)-moment estimates for singular \( p \)-Laplacian equation. However, similar results in terms of mass conservation and connection to \( q \)-Wasserstein space has been done. The mass conservation of singular parabolic \( p \)-Laplace equation for \( p_c < p < 2 \) is proved by Fino et al. \([8]\) for any weak solution where the initial data are Radon measures with compact support. Our result only requires finite \( q \)-moment instead of compact support for initial data, but for any weak solution constructed through mollification. As for the connection to the flow in the \( q \)-Wasserstein space, the work of Kell \([12]\) is the closest to the present one. There, the author shows that a smooth solution of Cauchy problem \((1.1)\) solves a generalized gradient flow problem of Rényi entropy functional in the \( q \)-Wasserstein space based on gradient flow and functional analysis methods. The author further provides a condition for the mass conservation of the flow with an integrable function as initial data for \( 3/2 < p < \infty \) in a general metric space. In contrast, our \( q \)-moment estimate \((1.2)\) is a local estimate based on pure PDE and analysis method, and, while holding for \( \mathbb{R}^N \), our result shows that mass conservation holds for a different range \( p_c < p < 2 \).

The paper is organized as follows: in Section 2, we introduce notations and definitions and auxiliary Lemmas as well as the critical parameters \( p_c, p(N) \) and \( p_N \). Section 3, is dedicated to the main theorem for the local \( q \)-moment estimate \((1.2)\). Section 4 addresses the mass conservation and the weak convergence. Finally, in Section 5 we prove the Wasserstein convergence rate.

2. Notation and preliminary results

On \( \mathbb{R}^N \) equipped with the Euclidean norm \(|\cdot|\), we denote by \( B_\rho(x) = \{ y : |y - x| \leq \rho \} \) the closed ball centered at \( x \) with radius \( \rho \), and set \( B_\rho := B_\rho(0) \). We denote by \( A^R_\tau = \{x : r \leq \} \)}
Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set. For $1 \leq p \leq \infty$, we denote by $L^p(\Omega)$ the space of $p$-integrable functions with respect to the Lebesgue measure and set $L^p(\mathbb{R}^N) = L^p$. We denote by $W^{1,p}(\Omega)$ the space of first-order Sobolev space and set $W^{1,p}(\mathbb{R}^N) = W^{1,p}$. For a function $u$ in $W^{1,p}(\Omega)$, we denote by $\nabla u$ the weak derivative of $u$. For $T > 0$ and $S_T = \mathbb{R}^N \times (0, T]$, we denote by $L^\infty(0, T; W^{1,p}(\Omega))$ the space of functions which belong to $L^\infty(K \times [s, t])$ for all intervals $[s, t] \subseteq (0, T]$ and compact subsets $K$ of $\mathbb{R}^N$. For $X = L^p(\Omega)$ or $X = W^{1,p}(\Omega)$, we denote by $C([0, T]; X)$, $C((0, T); X)$ and $L^p(0, T; X)$ in a Bochner sense. We further denote by $\mathcal{M}^+$ the set of positive Radon measures on $\mathbb{R}^N$.

**Definition 2.1.** We say that a measurable function $u : S_T \to \mathbb{R}$ is a weak solution of the Cauchy problem (1.1) with $L^1$-initial data, that is

$$u_0 = \mu \in L^1,$$

(2.1)

if for every bounded open set $\Omega \subseteq \mathbb{R}^N$, it holds

$$u \in C^1((0, T); L^1(\Omega)) \cap L^{p-1}(0, T; W^{1,p-1}(\Omega)) \cap L^\infty(0, T; W^{1,p}(\Omega)),$$

(2.2)

and for all $0 < s < t < T$ and $\phi \in C^1(\mathbb{R}^N \times [0, T])$ such that $\text{supp}(\phi(\cdot, \tau)) \subseteq B_\rho$ for all $\tau \in [0, T]$ for some $\rho > 0$,

$$\int_{B_\rho} (\phi u)(x, t)dx + \int_0^t \int_{B_\rho} (-u\partial_\tau \phi + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi) dx d\tau = \int_{B_\rho} (\phi u)(x, s)dx,$$

(2.3)

and for all $R > 0$,

$$\lim_{\tau \searrow 0} \int_{B_R} |u(x, t) - u_0(x)| dx = 0.$$

(2.4)

The measurable function $u : S_T \to \mathbb{R}^N$ is a weak solution of the Cauchy problem (1.1) with positive Radon measure as initial data, that is,

$$u_0 = \mu \in \mathcal{M}^+,$$

(2.5)

if condition (2.4) is replaced by

$$\lim_{\tau \searrow 0} \int_{\mathbb{R}^N} \psi(x)u(x, \tau)dx = \int_{\mathbb{R}^N} \psi(x)\mu(dx)$$

(2.6)

for all $\psi \in C_c(\mathbb{R}^N)$.

For the existence and uniqueness of weak solution of (1.1) with initial data $u_0$ in $L^1$, we refer readers to [3]. In particular, if the initial data $u_0$ is in $C^\infty_c(\mathbb{R}^N)$, then the weak solution $u$ is unique and satisfies that $u \in C([0, T]; L^1(\mathbb{R}^N))$, $\partial_\tau u \in L^2(0, T; L^2(\mathbb{R}^N))$ and $|\nabla u| \in L^p(0, T; L^p(\mathbb{R}^N))$ and $u \in L^\infty(S_T)$. In this case, the condition (2.3) is equivalent to the following condition:

$$\int_{\mathbb{R}^N} (\phi u)(x, t)dx + \int_0^t \int_{\mathbb{R}^N} (-u\partial_\tau \phi + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi) dx d\tau = \int_{\mathbb{R}^N} \phi(x, 0)u_0 dx.$$

(2.7)

Furthermore, $u$ is locally $\alpha$-Hölder continuous on $\Omega^\varepsilon_T := \Omega \times [\varepsilon, T]$ for all bounded subsets $\Omega \subseteq \mathbb{R}^N$ and $\varepsilon > 0$, with $\alpha \in (0, 1)$ depending on $N, p$ and supremum norm $\|u\|_{L^\infty(\Omega^\varepsilon_T)}$ on $\Omega^\varepsilon_T$, see [3]. Moreover, if initial data $u_0 \geq 0$, then the weak solution $u(x, t) \geq 0$ for all $(x, t) \in S_T$.

For the existence of weak solution of (1.1) with a positive Radon measure as initial data, we refer readers to [10, Theorem 1] and [6, Theorem III.8.1]. However, to our knowledge,
in this case the uniqueness of weak solution is unknown and depends on the choice of the definition of the solution (viscosity, entropy, distributional, etc.).

Recall that a sequence $\langle \mu_n \rangle$ of positive Radon measures converges vaguely, or weakly, to $\mu$ in $\mathcal{M}^+$ if $\int_{\mathbb{R}^N} \phi d\mu_n \to \int_{\mathbb{R}^N} \phi d\mu$ for every $\phi$ in $C_c(\mathbb{R}^N)$, or for every $\phi$ in $C_0(\mathbb{R}^N)$, respectively.

We briefly recall two important Lemmas for the proofs that are formulated in our notations.

**Lemma 2.2.** (A priori estimate, Junning [10]) Let $\mu$ in $\mathcal{M}^+$ if $\int_{\mathbb{R}^N} \phi d\mu_n \to \int_{\mathbb{R}^N} \phi d\mu$ for every $\phi$ in $C_c(\mathbb{R}^N)$, or for every $\phi$ in $C_0(\mathbb{R}^N)$, respectively.

Let $u$ be the weak solution of (1.1) with $u_0 \in C_c^\infty(\mathbb{R}^N)$ as initial data. Then there exists $C_1 := C_1(N, p)$ such that for all $R > 0$, it holds

$$
\sup_{0 < \tau \leq t} \int_{B_R} |u(x, \tau)| \, dx \leq C_1 \int_{B_{2R}} |u_0| \, dx + C_1 R^{-\frac{1}{p-1}} t^{\frac{1}{p-1}},
$$

(2.8)

and for any $R_0$, there exists $C_2 := C_2(N, p, R_0)$ such that for all $R > R_0$, it holds

$$
\sup_{x \in B_R} |u(x, t)| \leq C_2 \left( t^{-\frac{N}{p}} \left( \int_{B_{2R}} |u_0| \, dx \right)^\frac{2}{N} + R^{-\frac{1}{p-1}} t^{\frac{1}{p-1}} \right).
$$

(2.9)

**Lemma 2.3.** ([9, Lemma 3.1]) Let $f : [r_0, r_1] \to [0, \infty)$ be a bounded function with $0 \leq r_0 < r_1$. Suppose that for any $\sigma, \sigma'$ with $r_0 \leq \sigma < \sigma' \leq r_1$, we have

$$
f(\sigma) \leq A (\sigma' - \sigma)^{-a} + B + \theta f(\sigma'),
$$

where $A, B, a, \theta$ are nonnegative constants with $0 \leq \theta < 1$. Then for any $\lambda, \lambda'$ with $r_0 \leq \lambda < \lambda' \leq r_1$, it holds

$$
f(\lambda) \leq C(a, \theta) \left( A (\lambda' - \lambda)^{-a} + B \right),
$$

for some positive constant $C(a, \theta)$ depending only on $a$ and $\theta$.

Finally, recall the following critical values

$$
p_c := \frac{2N}{N + 1},
$$

$$
p(N) := (N + 2)p^2 - (3N + 3)p + 2N, \quad p_N := \text{largest root of } p(N).
$$

(2.10)

Note that since $N \geq 2$, it holds

$$
\frac{4}{3} \leq p_c < p_N < 2.
$$

### 3. Moments estimates

In this section, we derive $q$-moment estimates for a weak solution of (1.1) with finite positive Radon measure as initial data. We denote by $q$ the Hölder conjugate number of $p$, that is, $\frac{1}{p} + \frac{1}{q} = 1$, and denote by $k$ the constant $N(p-2) + p$. Given a $\mu \in \mathcal{M}^+$, we say that $\mu$ has finite $q$-moment if $\int_{\mathbb{R}^N} |x|^q d\mu < \infty$. We begin to state our main result in this section.

**Theorem 3.1.** ($q$-moment estimates.) Let $p_c < p < 2$ and $\mu$ be in $\mathcal{M}^+$. Then for any $0 < r < R$ and $0 < t \leq T$, there exists a weak solution $u$ of (1.1) with $\mu$ as initial data satisfying

$$
\sup_{0 < \tau \leq t} \int_{r \leq |x| \leq R} |x|^q u(x, \tau) \, dx \leq C \int_{\frac{1}{2}r \leq |x| \leq 2R} |x|^q d\mu
$$

$$
+ Ct^{\frac{1}{p-1}} \left( r^{-\frac{p(N)}{p-1}} + R^{-\frac{p(N)}{p-1}} \right),
$$

(3.1)
for some constant $C := C(N, p)$ depending only on $N$ and $p$. Furthermore, if $\mu$ in $\mathcal{M}^+$ has finite total mass and finite $q$-moment, and $p(N) > 0$, then for any $0 < t \leq T$, the family $\{u(x, \tau)dx : 0 \leq \tau \leq t\}$ has uniformly bounded $q$-moment on $[0, t]$ and that

$$\lim_{r \to \infty} \sup_{0 < \tau \leq t} \int_{|x| \geq r} |x|^q u(x, \tau)dx = 0.$$  \hspace{1cm} (3.2)

**Remark 3.2.** Let $\{X_\tau: 0 \leq \tau \leq t\}$ be a family of $\mathbb{R}^N$-valued random variables such that the law of $X_\tau$ is $u(x, \tau)dx$. Then equality (3.2) is equivalent to say that $\{|X_\tau|^q : 0 \leq \tau \leq t\}$ is uniformly integrable for any $0 < t \leq T$.

**Lemma 3.4.** For the first term on the right hand side of (3.1), applying Young’s inequality

$$p \leq \frac{p}{2} + \frac{q}{2},$$

for some constant $\xi$ and $\tau = \tau(x_{\xi})$.

**Proof.** Let $\xi > 0$ be fixed and $u_{\xi} = u + \epsilon$ and $\psi(x, \tau) = \tau^{\beta q} u_{\xi}^{1-\alpha q} \xi$. Note that $u_{\xi}^{1-\alpha q}$ is bounded since $-1 < 1 - \alpha q < 0$.

On the one hand, by the regularity of $u$ on $(0, T)$ it follows that

$$\partial_\tau (\tau^{\beta q} u_{\xi}^{2-\alpha q} \xi^p) = (2 - \alpha q) \psi p \partial_\tau u + \beta q \tau^{\beta q-1} u_{\xi}^{2-\alpha q} \xi^p \geq (2 - \alpha q) \psi p \partial_\tau u.$$  \hspace{1cm} (3.5)

By the regularity of $u$ it follows that

$$|\nabla u|^2 \nabla u \nabla \psi = p \epsilon^{p-1} |\nabla u|^p u_{\xi}^{2-\alpha q} \xi^p \nabla u \div \nabla \xi - (\alpha q - 1) |\nabla u|^p \xi^p \tau^{\beta q} u_{\xi}^{2-\alpha q}. \hspace{1cm} (3.5)$$

For the first term on the right hand side of (3.5), applying Young’s inequality $a \cdot b \leq \epsilon |a|^q + \epsilon^{-\frac{1}{q'}} |b|^p$ for $\epsilon = (\alpha q - 1)/(2p)$ and

$$a = \xi^{p-1} |\nabla u|^p \tau^{\beta q} u_{\xi}^{2-\alpha q} \xi^p \nabla u \text{ and } b = \tau^{\beta(q-1)} u_{\xi}^{1-\alpha(q-1)} \nabla \xi.$$
Equation (3.5) yields
\[
|\nabla u^{p-2} \nabla u \nabla \psi| \leq \frac{1}{2} (\alpha q - 1) \xi^{p} |\nabla u|^{p} u_{e}^{-\alpha q} + p \left( \frac{2p}{\alpha q - 1} \right)^{\frac{1}{p-1}} \tau^{\beta q} u_{e}^{-\alpha q} |\nabla \xi|^{p} . \tag{3.6}
\]
Since $|\nabla u|$ is in $L^{p}(0,T;L^{p}(\mathbb{R}^{N}))$ and $u_{e}^{-\alpha q}$ is bounded and $0 < p - \alpha q < 1$, it follows that
\[
\int_{0}^{t} \int_{\Omega} \xi^{p} |\nabla u|^{p} u_{e}^{-\alpha q} dxd\tau < \infty \quad \text{and} \quad \int_{0}^{t} \int_{\Omega} \tau^{\beta q} u_{e}^{-\alpha q} |\nabla \xi|^{p} dxd\tau < \infty .
\]
Hence, integrating (3.6) over $\Omega \times (0,t)$ yields
\[
\int_{0}^{t} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi dxd\tau \leq \frac{1}{2} (\alpha q - 1) \int_{0}^{t} \int_{\Omega} \xi^{p} |\nabla u|^{p} u_{e}^{-\alpha q} dxd\tau + p \left( \frac{2p}{\alpha q - 1} \right)^{\frac{1}{p-1}} \int_{0}^{t} \int_{\Omega} \tau^{\beta q} u_{e}^{-\alpha q} |\nabla \xi|^{p} dxd\tau . \tag{3.7}
\]
On the other hand, multiplying (1.1) by $\psi$ and integrating by part, it follows that
\[
\int_{0}^{t} \int_{\Omega} \psi \partial_{\tau} u dxd\tau + \int_{0}^{t} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi dxd\tau = 0 .
\]
Hence, adding (3.4) and (3.7) together yields
\[
\int_{0}^{t} \int_{\Omega} \xi^{p} |\nabla u|^{p} u_{e}^{-\alpha q} dxd\tau \leq \left( \frac{2p}{\alpha q - 1} \right)^{p} \int_{0}^{t} \int_{\Omega} \tau^{\beta q} u_{e}^{-\alpha q} |\nabla \xi|^{p} dxd\tau + \int_{\Omega} \tau^{\beta q} u_{e}^{-\alpha q} |\nabla \xi|^{p} dxd\tau .
\]
Taking
\[
C(N,p) = \max \left\{ \left( \frac{2p}{\alpha q - 1} \right)^{-p}, \frac{2}{(\alpha q - 1)(2 - \alpha q)} \right\}
\]
yields the result. \[\square\]

We address now the proof of Theorem 3.1. In the following proof, we denote by $C$ a generic positive constant depending only on $p$ and $N$.

**Proof of Theorem 3.1** In a first step, we show inequality (3.1) in the case where $u$ is the positive weak solution of (1.1) with initial data $u_{0}$ in $C^{\infty}_{c}(\mathbb{R}^{N})$ and $u_{0} \geq 0$. Let $R > r > 0$ and $1 \leq \sigma < \sigma' \leq 2$. Let $A_{\sigma}, A_{\sigma'}$ denote closed annulus $A_{R}^{\sigma}$ and $A_{r}^{\sigma'}$, respectively. Let $\xi$ be a smooth cut-off function such that $\xi = 1$ in $A_{\sigma}$ and $\xi = 0$ in $A_{\sigma}^{c}$, and $|\nabla \xi| \leq \frac{\sigma}{(\sigma - \sigma')^{2}}$ in $A^{\sigma} := A_{r}^{\sigma}$ and $|\nabla \xi| \leq \frac{1}{(\sigma - \sigma')^{2}}$ in $A^{R} := A_{R}^{\sigma}$. Let $\phi(x,\tau) = |x|^{q} \chi_{[0,T]}(\tau)$ be the test function where $\chi_{[0,T]}$ is indicator function. Then for any $0 < t \leq T$, by (2.7), it...
follows that

\[
\int_{A_\sigma} |x|^q u(x,t) dx \leq \int_{A_\sigma} |x|^q u_0 dx + p \int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi x |\nabla \xi| dx d\tau
\]

\[
+ q \int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi x |\nabla \xi|^{\frac{1}{p-1}} dx d\tau. \quad (3.8)
\]

For the second term on the right hand side of (3.8), since \(|\nabla \xi| = 0\) in \(A_\sigma\) and \(|\nabla \xi| \leq \frac{1}{(\sigma' - \sigma) R} \) in \(A'\) and \(A^n\) respectively, and note that \(\sigma, \sigma' \leq 2\), it follows that

\[
\int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi x |\nabla \xi| dx d\tau = \int_0^t \int_{A' \cup A^n} |\nabla u|^{p-1} \xi x |\nabla \xi| dx d\tau \leq \frac{4}{\sigma' - \sigma} \int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi x |\nabla \xi|^{\frac{1}{p-1}} dx d\tau. \quad (3.9)
\]

Plugging (3.9) into (3.8) and use the fact that \(\xi^p \leq \xi^{p-1} \) and \(1 \leq 1/(\sigma' - \sigma)\) yields

\[
\int_{A_\sigma} |x|^q u(x,t) dx \leq \int_{A_\sigma} |x|^q u_0 dx + \frac{C}{(\sigma' - \sigma)^p} \int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi x |\nabla \xi|^{\frac{1}{p-1}} dx d\tau. \quad (3.10)
\]

Let \(u_\varepsilon := u + \varepsilon\) for \(\varepsilon > 0\). We consider the second term on the right hand side of (3.10) for different cases of \(p\):

**Case 1:** If \(p > 1/(p - 1)\), then by Hölder’s inequality, it follows that

\[
\int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi x |\nabla \xi|^{\frac{1}{p-1}} dx d\tau
\]

\[
\leq \left( \int_0^t \int_{A_{\sigma'}} \tau^{\beta} |\nabla u|^{p} \xi^{p} u_\varepsilon^{-\alpha p} dx d\tau \right)^{\frac{1}{p}} \left( \int_0^t \int_{A_{\sigma'}} \tau^{-\beta p} |x|^q u_\varepsilon^{\alpha p} dx d\tau \right)^{\frac{1}{q}}, \quad (3.11)
\]

where \(\alpha = 1/p\) and \(\beta = 1/(2p)\).

For the first term on the right hand of (3.11), by Lemma 3.4 and upper bounds of \(|\nabla \xi|\) in \(A'\) and \(A^n\), it follows that

\[
\int_0^t \int_{A_{\sigma'}} \tau^{\beta} |\nabla u|^{p} \xi^{p} u_\varepsilon^{-\alpha p} dx d\tau \leq \frac{C}{[\sigma' - \sigma) R]^p} \int_0^t \int_{A'} \tau^{\beta} u_\varepsilon^{-\alpha p} dx d\tau
\]

\[
+ \frac{C}{(\sigma' - \sigma) R^p} \int_0^t \int_{A^n} \tau^{\beta} u_\varepsilon^{-\alpha p} dx d\tau + C \int_{A_{\sigma'}} \tau^{\beta} u_\varepsilon^{-\alpha p} dx. \quad (3.12)
\]
For the first term on the right hand side of (3.12), since $0 < p - 1/(p - 1) < 1$, Hölder’s inequality yields
\[
\int_0^t \int_{A_r} \frac{u_{x}}{A_r^{p-1}} dx \leq \int_0^t \left( \int_{A_r} \left( \frac{|x|^{-q} |u_{x}|^{p}}{A_r^{q-1/2}} \right) dx \right)^{1-p+q} \int_0^t \left( \int_{A_r} |u_{x}|^{q} dx \right)^{p-1} \leq Ct^{q-\frac{2}{p}N(q-p)^{-q}} \left( \sup_{0 < r \leq t} \int_{A_r} |u_{x}|^q dx \right)^{p-1}. \tag{3.13}
\]

The second term on the right hand side of (3.12) is the same as the first term, replacing $r$ by $R$. As for the third term on the right hand side of (3.12), Hölder’s inequality yields
\[
\int_{A_r} \frac{u_{x}}{A_r^{p-1}} dx \leq \int_{A_r} \left( \int_{A_r} \left( \frac{|x|^{-q} |u_{x}|^{p}}{A_r^{q-1/2}} \right) dx \right)^{1-p+q} \int_{A_r} |u_{x}|^{q} dx \leq Ct^{q-\frac{2}{p}N(q-p)^{-q}} \left( \sup_{0 < r \leq t} \int_{A_r} |u_{x}|^q dx \right)^{p-1}. \tag{3.14}
\]

We now turn to the second term on right hand side of (3.11) for which holds
\[
\left( \int_0^t \int_{A_r} |x|^q u_{x} dx \right)^{\frac{1}{p}} \leq Ct^{\frac{q}{p}} \left( \sup_{0 < r \leq t} \int_{A_r} |x|^q u_{x} dx \right)^{\frac{1}{p}}. \tag{3.15}
\]

Plugging (3.13), (3.14) into (3.12), and then applying inequality $(a + b)^{1/q} \leq a^{1/q} + b^{1/q}$ for $a, b > 0$ to (3.12), and plugging it into (3.11) together with (3.15) and taking $\varepsilon \searrow 0$, it follows that
\[
\frac{C}{\sigma' - \sigma} \int_0^t \int_{A_{r,\sigma}} \nabla u|^{p-1} \xi^{p-1} |x|^\frac{p}{p-1} dx \leq C(\sigma' - \sigma)^{-p} \left( r^{1-k} \left( -p - \frac{1}{p} \right) + R^{1-k} \left( -p - \frac{1}{p} \right) \right) M_q(\sigma')^{p-1} + C(\sigma' - \sigma)^{-\frac{p}{p-1}} \max \left\{ r^{1-k} \left( -p - \frac{1}{p} \right) , R^{1-k} \left( -p - \frac{1}{p} \right) \right\}^{2-p} M_q(\sigma')^{\frac{2(p-1)}{p}}. \tag{3.16}
\]

where $M_q(\sigma') = \sup_{0 < r \leq t} \int_{A_{r, \sigma}} |x|^q dx$. Applying Young’s inequality $ab \leq \varepsilon b^{1/(p-1)} + C_\varepsilon a^{1/(2-p)}$ and $ab \leq \varepsilon b^{1/(2-p)} + C_\varepsilon a^{p/(2-p)}$ to the first and second terms of the right hand side of (3.16), respectively, whereby $\varepsilon = 1/4$, and using identity $1 - k - p + 1/(p - 1) = -p(N)/(p - 1)$, it follows that
\[
\frac{C}{\sigma' - \sigma} \int_0^t \int_{A_{r,\sigma}} \nabla u|^{p-1} \xi^{p-1} |x|^\frac{p}{p-1} dx \leq \frac{1}{2} \sup_{0 < r \leq t} \int_{A_{r,\sigma}} |x|^q u dx + C(\sigma' - \sigma)^{-\frac{p}{p-1}} \int_{A_{r,\sigma}} \left( r^{1-k} \left( -p - \frac{1}{p} \right) + R^{1-k} \left( -p - \frac{1}{p} \right) \right). \tag{3.17}
\]
Case 2: If $p \leq 1/(p-1)$, by Hölder’s inequality, it follows that
\[
\int_0^t \int_{A_{\sigma'}} |\nabla u|^{|p-1|} |x|^{\frac{1}{p-1}} \, dx \, d\tau \leq \left( \int_0^t \int_{A_{\sigma'}} \tau^{\beta q} |\nabla u|^{|p-1|} |x|^{\frac{1}{p-1}} \, dx \, d\tau \right)^{\frac{p}{p-1}} \left( \int_0^t \int_{A_{\sigma'}} \tau^{-\beta q} |x|^p u_{\xi} \, dx \, d\tau \right)^{\frac{1}{p}},
\]
(3.18)

where $\alpha = p - 1$ and $\beta = (p - 1)/2$.

For the first term on the right hand side of (3.18), taking $\tilde{\xi} = \xi |\xi|^{(1/(p-1)-p)/(p-1)}$ and applying Lemma 3.4, yields
\[
\int_0^t \int_{A_{\sigma'}} \tau^{\beta q} |\nabla u|^{|p-1|} |x|^{\frac{1}{p-1}} \, dx \, d\tau \leq C \int_0^t \int_{A_{\sigma'}} \tau^{\beta q} u_{\xi}^{-\alpha q} |\nabla \tilde{\xi}| |\tilde{\xi}| \, dx \, d\tau + C \int_0^t \tau^{\beta q} u_{\xi}^{-\alpha q} \tilde{\xi} \, dx. \quad (3.19)
\]

For the first term on the right hand side of (3.19), note that we have
\[
\nabla \tilde{\xi} = |x|^{\frac{1}{p-1}} \tilde{\xi} |x|^{\frac{1}{p-1}} \nabla |x| + \xi \left( \frac{1}{p-1} - p \right) \frac{1}{p-1} |x|^{\frac{1}{p-1}} |x|^{\frac{1}{p-1}} \tilde{\xi}^{\frac{1}{p-1}} - 2x.
\]

Hence, applying inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ and upper bounds of $|\nabla \xi|$ in $A^r$ and $A^R$ and $|\xi| \leq 1$, it follows that
\[
\int_0^t \int_{A_{\sigma'}} \tau^{\beta q} |\nabla \tilde{\xi}| \, dx \, d\tau \leq C t^{\beta q + 1} \left\{ \int_{A_{\sigma'}} \left[ |x|^{\frac{1}{p-1}} |\nabla \tilde{\xi}|^p + |x|^{\frac{1}{p-1}} \right] \, dx \right\}.
\]
\[
\leq C t^{\beta q + 1} (\sigma' - \sigma)^{-p} \left( r^{-p} \int_{A_{\sigma'}} |x|^{\frac{1}{p-1}} \, dx + R^{-p} \int_{A_{\sigma'}} |x|^{\frac{1}{p-1}} \, dx \right) + C t^{\beta q + 1} \left\{ r^{N+(\frac{1}{p-1})q-p} + R^{N+(\frac{1}{p-1})q-p} \right\}.
\]
(3.20)

By the fact that $N + (1/(p-1)-p)q-p \geq 0$ when $N \geq 2$ and $r \leq R$ and $1 \leq (\sigma' - \sigma)^{-p}$, it follows from (3.20), that
\[
\int_0^t \int_{A_{\sigma'}} \tau^{\beta q} |\nabla \tilde{\xi}| \, dx \, d\tau \leq C t^{\beta q + 1} (\sigma' - \sigma)^{-p} R^{N+(\frac{1}{p-1})q-p}. \quad (3.21)
\]
As for the second term on the right hand side of (3.19), note that $|\tilde{\xi}|^p \leq |x|^{(1/(p-1)-p)q}$, which by Hölder’s inequality yields

$$\int_{A_{t'}} t^\frac{2}{p} u_\varepsilon^{2-p} \tilde{\xi}^p dx \leq t^\frac{2}{p} \left( \int_{A_{t'}} |x|^{\left(\frac{p}{p-1} - p\right)q} u_\varepsilon^{2-p} |x|^{\left(2-p\right)q} dx \right)$$

$$\leq t^\frac{2}{p} \left( \int_{A_{t'}} |x|^{\left(\frac{p}{p-1} - 2\right)q} dx \right)^{p-1} \left( \int_{A_{t'}} |x|^q u_\varepsilon^q dx \right)^{2-p}$$

$$\leq C t^\frac{2}{p} \max \left\{ \left( \int_{A_{t'}} |x|^q u_\varepsilon^q dx \right)^{p-1} \left( \sup_{0 < \tau \leq t} \int_{A_{t'}} |x|^q u_\varepsilon^q dx \right)^{2-p} \right\}. \quad (3.22)$$

Note that if $p \leq 3/2$, it is obvious that $N + p(3 - 2p) / (p - 1)^2 \geq 0$. If $p > 3/2$, since $N \geq 2$, then it holds that

$$N(p-1)^2 + p(3 - 2p) \geq \frac{2}{2} (p - 1)^2 + p(3 - 2p) = -p^2 + p + 1 \geq 0.$$ 

Taking $\varepsilon \searrow 0$ on the right hand side of (3.22), yields

$$\int_{A_{t'}} t^\frac{2}{p} u_\varepsilon^{2-p} \tilde{\xi}^p dx \leq C t^\frac{2}{p} R^{(N + p(3 - 2p) / (p - 1)^2)} M_q(\sigma')^{2-p}. \quad (3.23)$$

Plugging (3.21) and (3.23) into (3.19) and applying inequality $(a + b)^{1/q} \leq a^{1/q} + b^{1/q}$ yields

$$\left( \int_0^t \left( \int_{A_{t'}} \tau^q |\nabla u|^p u_\varepsilon^{\alpha q} |\tilde{\xi}|^{\left(\frac{p}{p-1} - p\right)q} dx d\tau \right)^\frac{1}{q} \right)^{\frac{1}{p}}$$

$$\leq C t^\frac{2-p}{p} \left( \sigma' - \sigma \right)^{1-p} R^{\frac{N}{p} + \frac{2}{p} - p - \frac{2}{q}} + C t^\frac{2}{p} R^{(N + p(3 - 2p) / (p - 1)^2)} M_q(\sigma')^{2-p}. \quad (3.24)$$

As for the second term on the right hand side of (3.18), Hölder’s inequality and then letting $\varepsilon \searrow 0$, yields

$$\left( \int_0^t \left( \int_{A_{t'}} \tau^{-\frac{2(p-1)}{p}} |x|^p u_\varepsilon^{(p-1)} dx d\tau \right)^\frac{1}{p} \right)^{\frac{1}{q}}$$

$$\leq \left\{ \left( \int_0^t \left( \int_{A_{t'}} |x|^p u_\varepsilon^{(p-1)} dx \right)^{1-p + p} \right)^{\frac{1}{p}} \right\} \left( \int_0^t \left( \int_{A_{t'}} \tau^{-\frac{1}{p}} |x|^q u_\varepsilon^q dx \right)^{p^2 - p} \right)^{\frac{1}{p}}$$

$$\leq C t^{\frac{2-p}{p}} R^{-\frac{2(p-1)}{p} + \frac{2}{q}} M_q(\sigma')^{p-1}. \quad (3.25)$$

Plugging (3.24) and (3.25) into (3.18), it follows that

$$\frac{C}{\sigma' - \sigma} \int_0^t \int_{A_{t'}} |\nabla u|^{p-1} \tilde{\xi}^{p-1} |x|^{\left(\frac{p}{p-1} - p\right)q} dx d\tau \leq C (\sigma' - \sigma)^{-p} t R^{(N + p(3 - 2p) / (p - 1)^2)} M_q(\sigma')^{p-1}$$

$$+ C (\sigma' - \sigma)^{-1} t^\frac{2}{p} R^{-\frac{(N+2p-3)}{p} + \frac{2}{q}} M_q(\sigma')^{2(p-1) + \frac{3-2p}{p}}. \quad (3.26)$$
Applying Young’s inequality \( ab \leq \varepsilon b^{1/(p-1)} + C_{\varepsilon} a^{1/(2-p)} \) as well as \( ab \leq \varepsilon b^{p/(2p-2)} + C_{\varepsilon} a^{p/(2p-2)} \) to the first and second terms on the right hand side of (3.26), respectively, where \( \varepsilon = 1/4 \), it follows that

\[
\frac{C}{\sigma' - \sigma} \int_0^t \int_{A_{\sigma'}} |\nabla u|^{p-1} \xi \frac{1}{\sigma'} dx \, dt \leq \frac{1}{2} \sup_{0 < \tau \leq t} \int_{A_{\sigma'}} |x|^q u \, dx + C (\sigma' - \sigma) \frac{p}{\sigma' - \tau} \frac{1}{\tau^{p-1}} R^{\frac{p(N)}{p - (p-1)(2-p)}}. \tag{3.27}
\]

Together with (3.11) and (3.22), we deduce from (3.10) that

\[
\int_{A_{\sigma'}} |x|^q u(x, t) \, dx \leq \int_{A_{\sigma'}} |x|^q u_0 \, dx + \frac{1}{2} \sup_{0 < \tau \leq t} \int_{A_{\sigma'}} |x|^q u \, dx + C (\sigma' - \sigma) \frac{p}{\sigma' - \tau} \frac{1}{\tau^{p-1}} \left( R^{\frac{p(N)}{p - (p-1)(2-p)}} + R^{-\frac{p(N)}{p - (p-1)(2-p)}} \right). \tag{3.28}
\]

Inequality (3.28) holds by replacing \( t \) by any \( \tau \in (0, t] \) implying that

\[
\sup_{0 < \tau \leq t} \int_{A_{\sigma'}} |x|^q u \, dx \leq \int_{A_{\sigma'}} |x|^q u_0 \, dx + \frac{1}{2} \sup_{0 < \tau \leq t} \int_{A_{\sigma'}} |x|^q u \, dx + C (\sigma' - \sigma) \frac{p}{\sigma' - \tau} \frac{1}{\tau^{p-1}} \left( R^{\frac{p(N)}{p - (p-1)(2-p)}} + R^{-\frac{p(N)}{p - (p-1)(2-p)}} \right). \tag{3.29}
\]

By Lemma 2.2 it holds that

\[
\sup_{\sigma \in [1, 2]} M_\sigma(\sigma) \leq (2R)^q \sup_{0 < \tau \leq t} \int_{B_{2R}} u \, dx \leq C (2R)^q \left( \int_{\mathbb{R}^N} u_0 \, dx + R^{-\frac{q}{2}} \frac{1}{\tau^{p-1}} \right) < \infty.
\]

Hence \( M_\sigma(\sigma) \) has bounded value on \( \sigma \in [1, 2] \). Applying Lemma 2.3 for \( \lambda = 1 \) and \( \lambda' = 2 \), it follows that

\[
\sup_{0 < \tau \leq t} \int_{r \leq |x| \leq R} |x|^q u \, dx \leq C \int_{r \leq |x| \leq 2R} |x|^q u_0 \, dx + C t^{1-\frac{p}{p'}} \left( R^{\frac{p(N)}{p - (p-1)(2-p)}} + R^{-\frac{p(N)}{p - (p-1)(2-p)}} \right). \tag{3.30}
\]

for some \( C := C(N, p) \). Thus we prove inequality (3.1) for the case where the initial data \( u_0 \) is in \( C^\infty_c (\mathbb{R}^N) \).

Let us finally address the inequality (3.1) in the general case where \( \mu \in \mathcal{M}^+ \) is taken as initial data. Let \( m \in C^\infty_c (\mathbb{R}^N) \) be a mollifier function with \( \text{supp}(m) \subset B_1, m \geq 0 \) and \( \int_{\mathbb{R}^N} m \, dx = 1 \) and \( m_n(x) = n^N m(nx) \). Let \( \xi_n \in C^\infty_c (\mathbb{R}^N) \) be a smooth cut-off function such that \( \xi_n = 1 \) in \( B_n \) and \( \xi = 0 \) in \( B_{\frac{3}{2}n}^c \). Let \( u_{0n} \in C^\infty_c (\mathbb{R}^N) \) be defined as follows:

\[
u_{0n}(x) = \xi_n(x) (m_n \ast \mu)(x) = \xi_n(x) \int_{\mathbb{R}^N} m_n(x - y) \, d\mu(y).
\]

Obviously, \( u_{0n} \geq 0 \). Moreover, for any \( \rho > 0 \), since \( \text{supp}(m_n(x - \cdot)) \subset B_{1/n}(x) \), it follows that

\[
\int_{B_{\rho}} u_{0n} \, dx \leq \int_{y \in B_{\frac{3}{2}n}^c} \int_{B_{1/n}(y) \cap B_{\rho}} m_n(x - y) \, d\mu(y) \leq \mu(B_{\rho+1}). \tag{3.31}
\]
Furthermore, for any $\phi \in C_c^\infty(\mathbb{R}^N)$, take $n$ large enough such that $\text{supp}(\phi) \subset B_n$, where $\text{supp}(\phi)_1 := \{x : d(x, \text{supp}(\phi)) \leq 1\}$. Then it follows that

$$\int_{\mathbb{R}^N} \phi u_{0n} dx = \int_{\text{supp}(\phi)} \int_{B_{1/n}(y)} \xi_n(x) \phi(x)m_n(x-y)d\mu(y) = \int_{\text{supp}(\phi)} (\phi * \tilde{m}_n)(y) d\mu(y),$$

where $\tilde{m}_n(x) = m_n(-x)$. By the uniform convergence of $\phi * \tilde{m}_n$ to $\phi$ on compact sets, see [4, Proposition 4.21], it follows that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \phi u_{0n} dx = \int_{\mathbb{R}^N} \phi d\mu.$$  \hfill (3.32)

Let $u_n$ be the unique positive weak solution of (1.1) and (2.1) with initial data $u_{0n}$ for $n \geq 1$. Then by inequality (2.3) of Lemma 2.2 and (3.31), it follows that $(u_n)_n$ is locally equibounded in $S_T$, that is, for any $\varepsilon > 0$ and bounded domain $\Omega \subset \mathbb{R}^N$, $(u_n)_n$ is equibounded in $\Omega_\varepsilon := \Omega \times [\varepsilon, T]$. Then by [5, Theorem 1], it follows that $(u_n)_n$ is uniformly equicontinuous in $\Omega_\varepsilon$ for all $\varepsilon > 0$ and all bounded domains $\Omega \subset \mathbb{R}^N$. By diagonalization procedure, we can find a subsequence of $(u_n)_n$, which is relabelled by $n$, such that

$$u_n, \nabla u_n \to u, \nabla u \quad \text{uniformly on every compact subset of } S_T,$$  \hfill (3.33)

and $u$ is a positive weak solution of (1.1) and (2.5) with initial data $\mu$, see [6, Theorem III.8.1].

Now for any $R > r > 0$ and $0 < t \leq T$, by inequality (3.30), it follows that for any $\tau \in (0, t]$,

$$\int_{r \leq |x| \leq R} |x|^q u_n(x, \tau) dx \leq C \int_{\frac{r}{2} \leq |x| \leq 2R} |x|^q u_{0n} dx + C t^\frac{1}{q-p} \left( r^{-\frac{p(N)}{p-1}(\frac{p}{p-1})} + R^{-\frac{p(N)}{p-1}(\frac{p}{p-1})} \right).$$  \hfill (3.34)

For the left hand side of (3.34), by dominated convergence and (3.33), it follows that

$$\lim_{n \to \infty} \int_{r \leq |x| \leq R} |x|^q u_n(x, \tau) dx = \int_{r \leq |x| \leq R} |x|^q u(x, \tau) dx.$$  \hfill (3.35)

For the first term on the right hand side of (3.34), let $\varepsilon > 0$ and $\phi \in C_c(\mathbb{R}^N)$ such that $\phi = |x|^q$ on $\left\{ \frac{1}{2} r \leq |x| \leq 2R \right\}$ and $\phi = 0$ on $\left\{ \frac{1}{2} r - \varepsilon \leq |x| \leq 2R + \varepsilon \right\}$. Then taking $n \to \infty$ and by (3.32), it follows that

$$\lim_{n \to \infty} \int_{\frac{1}{2} r \leq |x| \leq 2R} |x|^q u_{0n} dx \leq \int_{\mathbb{R}^N} \phi d\mu \leq \int_{\frac{1}{2} r - \varepsilon \leq |x| \leq 2R + \varepsilon} |x|^q d\mu.$$  \hfill (3.36)

For $\varepsilon \searrow 0$ together with (3.35) and (3.36), it follows that for any $\tau \in (0, t]$,

$$\int_{r \leq |x| \leq R} |x|^q u(x, \tau) dx \leq C \int_{\frac{1}{2} r \leq |x| \leq 2R} |x|^q d\mu + C t^\frac{1}{q-p} \left( r^{-\frac{p(N)}{p-1}(\frac{p}{p-1})} + R^{-\frac{p(N)}{p-1}(\frac{p}{p-1})} \right).$$

Taking the supremum over $\tau \in (0, t]$ on the left hand side yields the desired result.

We are left to show uniform boundedness of $q$-moment and inequality (3.2). Assume that $\mu$ is in $\mathcal{M}^+$ with finite total mass and finite $q$-moment, and $p \in (p_c, 2)$ is such that...
\( p(N) > 0 \). By inequality (2.3) in Lemma 2.2 and similar arguments, see also [10, Theorem 1], it follows that for any \( t \in (0, T] \) and \( r > 0 \),
\[
\sup_{0 < \tau \leq t} \frac{1}{B_r} \int_{B_r} u(x, \tau) \, dx \leq C \int_{B_{2r}} d\mu + C r^{-\frac{2}{p-1}} t^{1 - \frac{1}{p-1}}. \tag{3.37}
\]

Taking \( r = 1 \) and together with (3.1), it follows that for any \( R > 1 \)
\[
\sup_{0 < \tau \leq t} \int_{B_R} |x|^q u(x, \tau) \, dx \leq C \int_{|x| < 1} |x|^q u(x, \tau) \, dx + \sup_{0 < \tau \leq t} \int_{1 \leq |x| \leq R} |x|^q u(x, \tau) \, dx
\]
\[
\leq C \int_{|x| \leq 2} |x|^q d\mu + C \int_{|x| \leq 1} |x|^q d\mu + C t^{\frac{1}{2-p}} \left( 1 + R^{-\frac{p(N)}{2-p}} \right). \tag{3.38}
\]

Since \( p(N) > 0 \), taking \( R \to \infty \) and it follows that
\[
\sup_{0 < \tau \leq t} \int_{\mathbb{R}^N} |x|^q u(x, \tau) \, dx \leq C \int_{|x| \geq \frac{1}{2}} |x|^q d\mu + C \int_{|x| \leq 2} |x|^q d\mu + C t^{\frac{1}{2-p}}.
\]

From this we obtain that \( \{\mu_{\tau} = u(x, \tau) \, dx : \tau \in [0, t]\} \) has uniformly bounded \( q \)-moment for any \( t > 0 \). For inequality (3.2), taking \( R \to 0 \) for both side of (3.1) and it follows that
\[
\sup_{0 < \tau \leq t} \int_{|x| \geq r} |x|^q u(x, \tau) \, dx \leq C \int_{|x| \geq \frac{1}{2}} |x|^q d\mu + C t^{\frac{1}{2-p}} r^{-\frac{p(N)}{2-p}} \tag{3.39}
\]

Since \( \mu \) has finite \( q \)-moment, taking \( r \to \infty \) and it follows that
\[
\lim_{r \to \infty} \sup_{0 < \tau \leq t} \int_{|x| \geq r} |x|^q u(x, \tau) \, dx = 0.
\]
\[
\square
\]

\section*{4. Mass Conservation and Weak Convergence}

In [8], the authors show that mass conservation of any positive weak solution of (1.1) and (2.5) with initial data \( \mu \in M_c^+ \) with compact support. In this section, we will show that the weak solution constructed in Theorem 3.1 with initial data \( \mu \) in \( M_c^+ \) with finite total mass and finite \( q \)-moment, preserves mass. As a by-product, measure \( \mu_t := u(x, t) \, dx \) converges to \( \mu \) weakly.

\textbf{Theorem 4.1 (Mass Conservation).} Let \( p \in (p_c, 2) \) and \( \mu \) be a positive finite Radon measure in \( \mathbb{R}^N \) and \( u \) is the positive weak solution in Theorem 3.1 If \( \mu \) has \( q \)-finite moment, then \( u \) preserves mass, that is,
\[
\int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} d\mu, \quad \text{for all } 0 < t \leq T. \tag{4.1}
\]

\textbf{Corollary 4.2 (Weak Convergence).} Let \( \mu \) be a positive finite Radon measure in \( \mathbb{R}^N \) with finite \( q \)-moment and \( u \) be the positive weak solution in Theorem 3.1 Then the measure \( \mu_t := u(x, t) \, dx \) converges weakly to the initial data \( \mu \) as \( t \) goes to 0.

We start with a lemma providing gradient estimate of the weak solution of (1.1) with respect to \( q \)-moment of smooth and compact-supported initial data on the annulus.
Lemma 4.3. Let $u$ be the positive weak solution of (1.1) with initial data $u_0$ in $C^\infty_0(\mathbb{R}^N)$ and $u_0 \geq 0$. Then there exists a positive constant $C := C(N, p)$ such that for any $0 < t \leq T$ and $R > r > 0$, it holds that

$$
\int_0^t \int_{r \leq |x| \leq R} |\nabla u|^{p-1} dx \, d\tau 
\leq Ct^\frac{2}{p} r^{-\frac{2p(N)}{p(p-1)}} \left( r - \frac{p(N)}{p-1} + R - \frac{p(N)}{p-1} \right) \left( M_q \left( u_0, A R/4 \right) \right)^\frac{2(p-1)}{p} 
+ Ct^\frac{1}{p} R^{-\frac{2p(N)}{p(p-1)}} \left( r - \frac{p(N)}{p-1} + R - \frac{p(N)}{p-1} \right),
$$

(4.2)

where $M_q(u_0, A) := \int_A |x|^q u_0 dx$.

Proof. Let $R > r > 0$ and denote by $A_{r/\sigma}$ the annulus $A_{r/\sigma}$ for $\sigma \geq 1$. Let $\xi$ be a smooth cut-off functions such that $\xi = 1$ on $A_1$ and $\xi = 0$ on $A_2$, and $|\nabla \xi| \leq 2r^{-1}$ on $A_{r/2}$ and $|\nabla \xi| \leq R^{-1}$ on $A_{r}^2$. Since $|x| \geq r$ on $A_1$, it follows that

$$
\int_0^t \int_{A_2} |\nabla u|^{p-1} dx \, d\tau \leq r^{-\frac{2p(N)}{p-1}} \int_0^t \int_{A_1} |\nabla u|^{p-1} |x|^{\frac{2p}{p-1}} dx \, d\tau.
$$

(4.3)

We consider the right hand side of (4.3) for different cases of $p$:

Case 1: If $p > 1/(p-1)$, then by inequality (3.16) in which we choose $\sigma = 1$ and $\sigma' = 2$, it follows that

$$
\int_0^t \int_{A_2} |\nabla u|^{p-1} |x|^{\frac{2p}{p-1}} dx \, d\tau 
\leq Ct \left( r^{1-k-p+\frac{2}{p-1}} + R^{1-k-p+\frac{2}{p-1}} \right) M_q(2)^{p-1} 
+ Ct^\frac{1}{p} \left( \max \left\{ r - \frac{p(N)}{p-1}, R - \frac{p(N)}{p-1} \right\} \right)^\frac{2(p-1)}{p} M_q(2)^{\frac{2(p-1)}{p}},
$$

(4.4)

where $M_q(2) = \sup_{0 < r \leq 1} \int_{A_r} |x|^q u dx$. Since that $p \leq 2(p-1)$, applying Young’s inequality $ab \leq \alpha^\frac{p}{p-1} + b^\frac{p}{p-1}$ to the first term on the right hand of (4.4), yields

$$
t r^{1-k-p+\frac{2}{p-1}} M_q(2)^{p-1} = \left( t^\frac{k}{p} r^{1-k-p+\frac{2}{p-1} \frac{p(N)}{p-1}} \right) \left( t^\frac{1}{p} r^{-\frac{p(N)}{p-1}} M_q(2)^{p-1} \right) 
\leq t^\frac{k}{p} r^{1-k-p+\frac{2}{p-1} \frac{p(N)}{p-1}} + t^\frac{1}{p} r^{-\frac{p(N)}{p-1}} M_q(2)^{\frac{2(p-1)}{p}},
$$

which plugged into (4.3) yields

$$
\int_0^t \int_{A_2} |\nabla u|^{p-1} |x|^{\frac{2p}{p-1}} dx \, d\tau 
\leq Ct \left( r^{-\frac{p(N)}{p-1} + R^{-\frac{p(N)}{p-1}}} \right) M_q(2)^{p-1} 
+ Ct^\frac{1}{p} \left( r^{-\frac{p(N)}{p-1} + R^{-\frac{p(N)}{p-1}}} \right) M_q(2)^{\frac{2(p-1)}{p}}.
$$

(4.5)

Case 2: If $p \leq 1/(p-1)$, then by inequality (3.26) in which we choose $\sigma = 1$ and $\sigma' = 2$, it follows that

$$
\int_0^t \int_{A_2} |\nabla u|^{p-1} |x|^{\frac{2p}{p-1}} dx \, d\tau 
\leq Ct R^{-\frac{p(N)}{p-1}} M_q(2)^{p-1} + Ct^\frac{1}{p} R^{-\frac{p(N)}{p-1}} M_q(2)^{\frac{2(p-1)}{p}}.
$$

(4.6)
Applying the same Young’s inequality as in the previous step to the first term on the right hand side, yields

\[ tR^\frac{\mu(N)}{\mu(N) - 1} M_q(2)^{p - 1} = \left( tR^\frac{-\mu(N)}{\mu(N) - 1} \right) \left( tR^\frac{-\mu(N)}{\mu(N) - 1} M_q(2)^{p - 1} \right) \]

\[ \leq tR^\frac{-\mu(N)}{\mu(N) - 1} + tR^\frac{-\mu(N)}{\mu(N) - 1} M_q(2)^{2(p - 1)}, \]

which plugged into (4.6), implies

\[
\int_0^t \int_{A_2} |\nabla u|^{p - 1} \xi |x|^{\frac{2}{p - 1}} \, dx \, d\tau \leq Ct^{\frac{1}{2\gamma}} R^\frac{1}{(p - 1)(4 - p)} + Ct^{\frac{1}{2\gamma}} R^\frac{1}{(p - 1)(4 - p)} M_q(2)^{2(p - 1)}.
\]

(4.7)

Hence, from (4.5) and (4.7), we obtain that for any \( p \in (p_c, 2) \), it holds that

\[
\int_0^t \int_{A_2} |\nabla u|^{p - 1} \xi |x|^{\frac{2}{p - 1}} \, dx \, d\tau \leq Ct^{\frac{1}{2\gamma}} \left( R^\frac{-\mu(N)}{\mu(N) - 1} + R^\frac{-\mu(N)}{\mu(N) - 1} M_q(2)^{2(p - 1)} \right)
\]

\[ + Ct^{\frac{1}{2\gamma}} \left( R^\frac{-\mu(N)}{\mu(N) - 1} + R^\frac{-\mu(N)}{\mu(N) - 1} \right) M_q(2)^{2(p - 1)}. \]

(4.8)

Let \( \alpha_1 = -p(N)/[(p - 1)(2 - p)] \) and \( \alpha_2 = -p(N)/[p(p - 1)] \) and \( \alpha_3 = -2p(N)/[p(2 - p)] \) and denote by \( F_i \) the terms \( r^\alpha + R^\alpha \), for \( i = 1, 2, 3 \). By applying inequality (3.1) in Theorem 3.1 to \( M_q(2) \) and inequality \((a + b)\gamma \leq a^\gamma + b^\gamma \) for \( \gamma = 2(p - 1)/p \), it follows that

\[
\int_0^t \int_{A_2} |\nabla u|^{p - 1} \xi |x|^{\frac{2}{p - 1}} \, dx \, d\tau \leq Ct^{\frac{1}{2\gamma}} F_1 + Ct^{\frac{1}{2\gamma}} F_2 \left( \int_{A_t} |x|^2 u_0 \, dx \right)^{\frac{2(p - 1)}{p}} \]

\[ \leq Ct^{\frac{1}{2\gamma}} \left( F_1 + F_2 (F_1)^{\frac{2(p - 1)}{p}} \right) + Ct^{\frac{1}{2\gamma}} F_2 \left( \int_{A_t} |x|^2 u_0 \, dx \right)^{\frac{2(p - 1)}{p}}. \]

(4.9)

By using inequality \((a + b)\gamma \leq a^\gamma + b^\gamma \) for \( \gamma = (2 - p)/p \) and \( \gamma = 2(p - 1)/p \) respectively, we have \((F_1)^{2-p}/p \leq F_2 \) and \((F_1)^{2p-1}/p \leq F_3 \). Then for the first two terms on the right hand side in (4.9), it follows that

\[ F_1 + F_2 (F_1)^{\frac{2(p - 1)}{p}} = (F_1)^{\frac{2(p - 1)}{p}} + (F_1)^{\frac{2(p - 1)}{p}} \leq 2F_2 F_3. \]

Plugging it into (4.9), we obtain

\[
\int_0^t \int_{A_2} |\nabla u|^{p - 1} \xi |x|^{\frac{2}{p - 1}} \, dx \, d\tau \leq Ct^{\frac{1}{2\gamma}} F_2 F_3 + Ct^{\frac{1}{2\gamma}} F_2 \left( \int_{A_t} |x|^2 u_0 \, dx \right)^{\frac{2(p - 1)}{p}}.
\]

which together with (4.3) yields the result. □

**Proof of Theorem 3.7** Let \( \mu \in M^+ \) with finite total mass be fixed and \((u_0)\) be the sequence of \( C_c^\infty \) functions constructed in the proof of Theorem 3.1 and \( u_0 \) be the positive weak solution of (1.1) with \( u_0 \) as initial data, and \( u \) be the positive weak solution constructed in the proof of Theorem 3.1. Let \( \rho > 0 \) and \( \xi \) be a smooth cut-off function such that \( \xi = 1 \) on \( B_\rho \) and \( \xi = 0 \) on \( B_\rho^c \) and \( |\nabla \xi| \leq \rho^{-1} \) on \( A_\rho^c \). By using \( \phi(x, \tau) = \xi(x) \chi_{[0, \tau]}(\tau) \)
as the test function in (2.7), it follows that for any $0 < t \leq T$ and $n \geq 1$,
\[
\left| \int_{B_{2\rho}} \xi(x)u_n(x,t)dx - \int_{B_{2\rho}} \xi(x)u_0(x)dx \right| \leq \int_0^t \int_{B_{2\rho}} |\nabla u_n|^{p-1} |\nabla \xi|dxdt
\]
\[
\leq \rho^{-1} \int_0^t \int_{0 \leq |x| \leq 2\rho} |\nabla u_n|^{p-1} dxdt. \tag{4.10}
\]
By Lemma 4.3 with $r = \rho$ and $R = 2\rho$, it follows that
\[
\left| \int_{B_{2\rho}} \xi(x)u_n(x,t)dx - \int_{B_{2\rho}} \xi(x)u_0(x)dx \right| \leq \int_{B_{2\rho}} |x|^q u_0(x)dx
\]
\[
= \int_{\rho \leq |x| \leq 8\rho} \left( C t^\frac{1}{p} \rho^{-1} \frac{1}{\rho - 1} \int_{\rho \leq |x| \leq 8\rho} |x|^q dx \right)^{\frac{2(p-1)}{p}}
\]
\[
+ C t^\frac{1}{p - 1} \rho^{-1} \frac{1}{\rho - 1} \int_{\rho \leq |x| \leq 8\rho} |x|^q d\mu. \tag{4.11}
\]
By (3.33) in the proof of Theorem 3.1 it holds that $u_n$ converges to $u$, up to a subsequence, uniformly on each compact subset of $S_T$. Together with equality (3.32), taking $n \to \infty$ on both side of (4.11) and it follows that
\[
\left| \int_{B_{2\rho}} \xi(x)u(x,t)dx - \int_{B_{2\rho}} \xi(x)dx \right| \leq \int_{B_{2\rho}} |x|^q u_0(x)dx
\]
\[
= \int_{\rho \leq |x| \leq 8\rho} \left( C t^\frac{1}{p} \rho^{-1} \frac{1}{\rho - 1} \int_{\rho \leq |x| \leq 8\rho} |x|^q dx \right)^{\frac{2(p-1)}{p}}
\]
\[
+ C t^\frac{1}{p - 1} \rho^{-1} \frac{1}{\rho - 1} \int_{\rho \leq |x| \leq 8\rho} |x|^q d\mu. \tag{4.12}
\]
For the left hand side of (4.12), by letting $r \to \infty$ on both side of (3.37) in the proof of Theorem 3.1 it follows that $\int_{\mathbb{R}^N} u(x,t)dx < \infty$. Together with $|\xi(x)u(x,t)| \leq u(x,t)$, by dominated convergence theorem, it follows that
\[
\lim_{\rho \to \infty} \int_{B_{2\rho}} \xi(x)u(x,t)dx - \int_{B_{2\rho}} \xi(x)dx = \int_{\mathbb{R}^N} u(x,t)dx - \mu(\mathbb{R}^N). \tag{4.13}
\]
For the right hand side of (4.12), it is easy to check that for $p \in (p_c, 2)$,
\[
-1 - \frac{1}{p - 1} = \frac{p(N)}{p(p - 1)} - \frac{(N + 3)p - 2N}{p} < 0,
\]
\[
-1 - \frac{1}{p - 1} = \frac{2p(N)}{p(2 - p)} - \frac{p(N)}{p(p - 1)} = \frac{(N + 1)p - 2N}{2 - p} < 0.
\]
Since $\mu$ has finite $q$-moment, the term on the right hand side of (4.12) converges to 0 as $\rho \to \infty$. So we obtain that for any $0 < t \leq T$,
\[
\int_{\mathbb{R}^N} u(x,t)dx = \mu(\mathbb{R}^N).
\]

\[\square\]

**Proof of Corollary 4.2** By Theorem 3.1 and definition of weak solution of (1.1) and (2.5), it follows that measure $\mu_t = u(x,t)dx$ converges vaguely to initial data $\mu$. Since $\mu_t(\mathbb{R}^N) = \mu(\mathbb{R}^N)$, it follows that \[\square\]
We address now the convergence rate of the constructed weak solution \( u(t, \cdot) \) to \( \mu \) in the \( q \)-Wasserstein distance. We recall that the \( q \)-Wasserstein distance \( W_q(\mu, \nu) \) between finite Borel measures \( \mu, \nu \) in \( \mathbb{R}^N \) with equal mass is defined as

\[
W_q(\mu, \nu) = \min \left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^q d\pi(x, y) : \pi \in \Pi(\mu, \nu) \right\},
\]

(5.1)

where \( \Pi(\mu, \nu) \) is the family of all Borel measures on \( \mathbb{R}^N \times \mathbb{R}^N \) having \( \mu \) and \( \nu \) as their first and second marginal measures respectively.

**Theorem 5.1.** Let \( \mu \) be a finite Radon measure on \( \mathbb{R}^N \) and \( u \) be the weak solution of (1.1) constructed in Theorem 3.7 with \( \mu \) as initial data. If \( p \in (p_N, 2) \) and \( \mu \) has finite \( q \)-moment, then there exist constant \( C := C(N, p, \mu, T) \) such that for all \( t \in [0, T] \),

\[
W_q(\mu_t, \mu) \leq Ct^{q-1}.
\]

(5.2)

To address the proof of Theorem 5.1, we first establish an auxiliary lemma, prove the theorem in the case where the initial data is in \( C_c^\infty(\mathbb{R}^N) \), and then show the general case by an approximation procedure. The key ingredients of the proof are to use Brenier-Benamou formulation for \( q \)-Wasserstein distance and write the parabolic \( p \)-Laplace equation as a law of mass conservation \( \partial_t u = \text{div}(\nabla u) \) where \( \nabla = |\nabla u|^{p-2}(-\nabla u)/u \).

**Lemma 5.2.** Let \( p \in (p_N, 2) \) and \( u \) be the positive weak solution of Cauchy problem (1.1) with \( u_0 \in C_c^\infty(\mathbb{R}^N) \) and \( u_0 \geq 0 \) as initial data. Then for any \( t > 0 \), it holds that

\[
\int_0^t \int_{\mathbb{R}^N} |\nabla u|^p \frac{-\rho}{\nabla u} \, dx \, d\tau \leq C \left( \int_{|x| \leq 2} u_0 \, dx \right)^{2 - \frac{1}{\rho}} + C \left( \int_{|x| \geq 1/2} |x|^q u_0 \, dx \right)^{2 - \frac{1}{q}} + C t \left( \frac{2p - 2}{p - q} \right)
\]

(5.3)

for some \( C := C(N, p) \).

**Proof.** Let \( \epsilon > 0 \) and \( R > 0 \). Let \( \xi \) be a smooth cut-off function such that \( \xi = 1 \) on \( B_R \) and \( \xi = 0 \) on \( B_{2R}^c \). Let \( u_\epsilon := u + \epsilon \) and \( u_{0\epsilon} := u_0 + \epsilon \) and \( \psi := u_\epsilon^{1/(p-1)} \xi \). Multiplying \( \psi \) on (1.1) and integrating it over \( B_{2R} \times [0, t] \), by integral by part and similar argument in Lemma 3.4, it follows that

\[
\int_0^t \int_{B_{2R}} |\nabla u|^p u_\epsilon^{-\frac{p-1}{p}} \xi^p \, dx \, d\tau \leq C \int_0^t \int_{B_{2R}} u_\epsilon^{-\frac{p-1}{p}} |\nabla \xi|^p \, dx \, d\tau
\]

\[
+ C \left( \int_{B_{2R}} u_\epsilon^{2 - \frac{1}{p-1}} \xi \, dx - \int_{B_{2R}} u_{0\epsilon}^{2 - \frac{1}{p-1}} \xi \, dx \right)
\]

\[
\leq C \int_0^t \int_{B_{2R}} u_\epsilon^{-\frac{p-1}{p}} |\nabla \xi|^p \, dx \, d\tau + C \int_{B_{2R}} u_\epsilon^{2 - \frac{1}{p-1}} \, dx
\]

(5.4)

Note that \( p \in (p_N, 2) \) implies that \( p - 1/(p-1) > 1 \). Hence, for the first term on the right hand side of (5.4), by using the similar argument as in the proof of Theorem 3.1, it follows
that
\[
\int_0^t \int_{B_{2R}} u_\varepsilon^{p - \frac{1}{p - 1}} |\nabla \xi|^p dx d\tau \leq D^p \int_0^t \int_{A_{2R}^2} |x|^q (p - \frac{1}{p - 1}) u_\varepsilon^{p - \frac{1}{p - 1}} dx d\tau
\]
and \( \phi \) is the identity. We first show that inequality (5.2) holds in the case where
\[
\int_0^t \int_{A_{2R}^2} |x|^q u_\varepsilon^{p - \frac{1}{p - 1}} dx d\tau \leq C \left( \sup_{0 < \tau \leq t} \int_{A_{2R}^2} |x|^q u_\varepsilon dx \right) \left( \sup_{0 < \tau \leq t} \int_{A_{2R}^2} |x|^q u_\varepsilon dx \right)^{1 - \frac{k}{p - 1}}.
\]
(5.5)

As for the second term on the right hand side of (5.4), we take \( \phi_1(x) = 1_{|x| < 1} + |x|^{q(2 - 1/(p - 1))} 1_{A_{2R}^2} \) where \( \phi_2(x) = 1_{|x| < 1} + |x|^{-q(2 - 1/(p - 1))} 1_{A_{2R}^2} \). By Hölder’s inequality and \( (a + b)^\gamma \leq a^\gamma + b^\gamma \) for \( \gamma = 2 - 1/(p - 1) \), it follows that
\[
\int_{B_{2R}} u_\varepsilon^{2 - \frac{1}{p - 1}} dx \leq \left( \int_{B_{2R}} \phi_2^{\frac{p - 1}{p}} dx \right)^{\frac{p - 1}{p}} \left( \int_{B_{2R}} \phi_1^{\frac{p - 1}{p}} u_\varepsilon^{p - \frac{1}{p - 1}} dx \right)^{\frac{p - 1}{p}} \leq C \left( \int_{B_1} u_\varepsilon dx \right)^{\frac{p - 1}{p}} + C \left( \int_{A_{2R}^2} |x|^q u_\varepsilon dx \right)^{\frac{p - 1}{p}}.
\]
(5.6)

Plugging (5.5) and (5.6) into (5.4), taking \( \varepsilon \to 0 \) and using monotone convergence theorem as well as identity \( 1 - k - p + 1/(p - 1) = -p(N)/(p - 1) \) yields
\[
\int_0^t \int_{B_{2R}} |\nabla u|^p u^{\frac{1}{p - 1}} \xi dx d\tau \leq C D^p \left( \sup_{0 < \tau \leq t} \int_{A_{2R}^2} |x|^q u(x, \tau) dx \right)^{\frac{p - 1}{p - 1}} + C \left( \int_{B_1} u dx \right)^{\frac{p - 1}{p}} + C \left( \int_{A_{2R}^2} |x|^q u_0 dx \right)^{\frac{p - 1}{p}}.
\]
(5.7)

where \( M_q(t, A) := \sup_{0 < \tau \leq t} \int_A |x|^q u(x, \tau) dx \). Using Inequality (3.1) from Theorem 3.1 and (2.3) from (2.2) in the right hand side of (5.7), it follows that
\[
\int_0^t \int_{B_{2R}} |\nabla u|^p u^{\frac{1}{p - 1}} \xi dx d\tau \leq C \left( \int_{B_1} u dx \right)^{\frac{p - 1}{p}} + C D^p \left( \sup_{0 < \tau \leq t} \int_{A_{2R}^2} |x|^q u(x, \tau) dx \right)^{\frac{p - 1}{p - 1}} + C \left( \int_{A_{2R}^2} |x|^q u_0 dx \right)^{\frac{p - 1}{p}}.
\]
(5.8)

where \( M_q(0, A) := \int_A |x|^q u_0 dx \). Since \( p(N) > 0 \), taking \( R \to \infty \) on both side of (5.8), monotone convergence theorem and \( q \)-finite moment of \( u_0 \) yields the result. \( \square \)

**Proof of Theorem 5.7** We first show that inequality (5.2) holds in the case where \( u(x, \tau) \) is the positive weak solution of (L1) with \( u_0 \) in \( C^\infty_0(\mathbb{R}^N) \) and \( u_0 \geq 0 \) as initial data. Let \( \mu = u_0 dx \) and \( \mu_t = u(x, t) dx \). By Benamou-Brenier formulation for \( q \)-Wasserstein, see [14, Theorem 5.28] and [2], it follows that
\[
W_q^*(\mu, \mu_t) = \inf \left\{ \frac{1}{N} \int_{\mathbb{R}^N} |v|^{q} \rho dx d\tau : \partial_\tau \rho + \text{div}(\rho \nabla v) = 0, \rho_0 = u_0, \rho_1 = u(t) \right\}.
\]
Rescaling as \( \tilde{\rho}(x, \tau) = \rho(x, t^{-1}\tau) \) and \( \tilde{v}(x, \tau) = t^{-1}v(x, t^{-1}\tau) \) and changing variable yields

\[
W^q_w(\mu, \mu_t) = \inf \left\{ t^{q-1} \int_0^t \int_{\mathbb{R}^N} |\nabla \rho_\tau - \nabla \tilde{v}_\tau| d\tau \right\}.
\]

Note that by the property of singular \( p \)-Laplace equation, \( u(x, \tau) > 0 \) for all \( \tau \in (0, t) \). So choosing \( \tilde{v} = -|\nabla u| p^{-2}v \) and \( \tilde{\rho} = u \) for \( \tau \in (0, t] \), it follows that

\[
W^q_w(\mu, \mu_t) \leq t^{q-1} \int_0^t |\nabla u| p^{-2}v dx d\tau.
\]

By Lemma 5.2, the inequality (5.2) follows.

We are left to show that inequality (5.2) holds in the case where \( u \) is the positive weak solution of the Cauchy problem (1.1) with finite Radon measure with finite \( q \)-moment as initial data, constructed in the proof of Theorem 3.1. Let \( (u_n) \) be the sequence of smooth initial data in \( C^\infty_c(\mathbb{R}^N) \) defined in the proof of Theorem 3.1 and \( u_n \) be the corresponding positive weak solution, and \( (u_{nk}, (u_{nk})_n) \) be the subsequence of \( (u_n), (u_{nk}) \) such that \( u_{nk} \) and \( \nabla u_{nk} \) converges to \( u \) and \( \nabla u \) uniformly on all compact subset of \( S_T \). Let \( \varepsilon > 0 \) be fixed. For \( W^q_w(\mu, \mu_t) \), by triangle inequality, it follows that

\[
2^{1-q}W^q_w(\mu, \mu_t) \leq W^q_w(\mu, \mu_{0n}^n) + W^q_w(\mu_{0n}^n, \mu_{t}^n) + W^q_w(\mu_{t}^n, \mu_t), \quad (5.9)
\]

where \( d\mu_{0n}^n = u_{0n} dx \) and \( d\mu_t^n = u_{n}(x, t) dx \).

As for the first term on the right hand side of (5.10), we claim that \( \mu_{0n}^n \) converges to \( \mu \) vaguely and that \( \sup_n \int_{|x| \geq R} |x|^p u_{0n} dx \to 0 \) as \( R \to \infty \). Indeed, for all \( n \in \mathbb{N} \) and \( R > 0 \), by construction of \( u_n \) and Fubini’s theorem, it follows that

\[
\int_{|x| \geq R} |x|^p u_{0n} dx = \int_{|x| \geq R} |x|^p \xi_n(x) \int_{B_1/n(x)} m_n(x - y) d\mu(y) dx
\]

\[
= \int_{|y| \geq (R-1)^+} \int_{B_1/n(y)} m_n(x - y) \xi_n(x) |x - y + y|^q dx d\mu(y)
\]

\[
\leq 2^{q-1} \int_{|y| \geq (R-1)^+} |y|^q d\mu(y) + 2^{q-1}n^{-q} \int_{|y| \geq (R-1)^+} d\mu(y). \quad (5.11)
\]

Hence, \( \sup_n \int_{|x| \geq R} |x|^p u_{0n} dx \to 0 \) as \( R \to \infty \). Choose \( R > 0 \) large enough such that \( \sup_n \int_{|x| \geq R} |x|^p u_{0n} dx < \varepsilon \) and \( \mu(B_R^\varepsilon) < \varepsilon \) and let \( \xi \in C^\infty_c(\mathbb{R}^N) \) such that \( \xi = 1 \) on \( B_R \) and \( \xi = 0 \) on \( B_R^\varepsilon \). Since \( \mu_{0n} \) converges to \( \mu \) vaguely by (3.32), choose \( n \) large enough such that

\[
\left| \int_{\mathbb{R}^N} u_{0n} dx - \mu(\mathbb{R}^N) \right| \leq \int_{\mathbb{R}^N} \xi u_{0n} dx - \int_{\mathbb{R}^N} \xi d\mu + \int_{|x| \geq R} u_{0n} dx + \mu(B_R^\varepsilon) \leq 3\varepsilon. \quad (5.12)
\]

Hence, \( \int_{\mathbb{R}^N} u_{0n} dx \to \mu(\mathbb{R}^N) \) as \( n \to \infty \). By classical result [13, Theorem 13.16], \( u_{0n} dx \) converges to \( \mu \) weakly. Together with (5.11), by [14, Theorem 7.12], it follows that \( W^q_w(\mu_{0n}^n, \mu) \to 0 \) as \( n \to \infty \).

For the third term on the right hand side of (5.10), we claim that \( \mu_{nk}^n \) converges to \( \mu_t \) weakly and that \( \sup_n \int_{|x| \geq R} |x|^p u_{nk} dx \to 0 \) as \( R \to \infty \). Indeed, by inequality (5.1) of
Theorem 5.1 it follows that
\[
\sup_k \int_{|x| \geq R} |x|^q u_{n_k} (x,t) dx \leq C \sup_k \int_{|x| \geq R} |x|^q u_{0n_k} dx + Ct^\frac{1}{p} R^{- \frac{q}{p(\frac{p}{2} - 1)}}. \tag{5.13}
\]

Applied to (5.11), it follows that \(\sup_k \int_{|x| \geq R} |x|^q u_{n_k} dx \to 0\) as \(R \to \infty\). Furthermore, using inequality (5.13) as well as (3.2), choose \(R > 0\) large enough such that \(\sup_k \int_{|x| \geq R} |x|^q u_{n_k}(x,t) dx < \varepsilon\) and \(\int_{|x| \geq R} |x|^q u(x,t) dx < \varepsilon\). Then since \(u_{n_k}\) converges to \(u\) uniformly on all compact subsets on \(S_T\), choose \(k \in \mathbb{N}\) large enough, such that
\[
\left| \int_{\mathbb{R}^N} u_{n_k}(x,t) dx - \int_{\mathbb{R}^N} u(x,t) dx \right| \leq \left| \int_{\mathbb{R}^N} u_{n_k}(x,t) dx - \int_{B_R} u(x,t) dx \right| + \int_{|x| \geq R} |x|^q u_{n_k}(x,t) dx + \int_{|x| \geq R} |x|^q u(x,t) dx \leq 3\varepsilon.
\]

By the same argument as previously, it follows that \(W_q(u_{n_k}, \mu_t) \to 0\) as \(k \to \infty\).

As for the second term on the right hand side of (5.10), by result from the first step for smooth initial data and inequality 5.11, it follows that for any \(t \in [0,T]\) it holds
\[
W_q^q(u_{0n_k}, \mu_{1n_k})
\leq C(N,q) \left\{ \left( \int_{\mathbb{R}^N} u_{0n_k} dx \right)^{2 - \frac{1}{p}} + \left( \int_{|x| \geq 1/2} |x|^q u_{0n_k} dx \right)^{2 - \frac{1}{p}} + t^{\frac{q(p-2)}{p(p-1)}} \right\} t^{q-1}
\leq C(N,q) \left\{ \left( \mu(\mathbb{R}^N) \right)^{2 - \frac{1}{p-1}} + \left( \int_{\mathbb{R}^N} |x|^q d\mu \right)^{2 - \frac{1}{p-1}} + t^{\frac{q(p-2)}{p(p-1)}} \right\} t^{q-1}. \tag{5.14}
\]

Thus, taking \(k \to \infty\) on both side of (5.10) and together with inequality (5.14), the result follows. \(\square\)

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