Entropy and topology for manifolds with boundaries

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ABSTRACT

In this work a deep relation between topology and thermodynamical features of manifolds with boundaries is shown. The expression for the Euler characteristic, through the Gauss-Bonnet integral, and the one for the entropy of gravitational instantons are proposed in a form which makes the relation between them self-evident. A generalization of Bekenstein-Hawking formula, in which entropy and Euler characteristic are related in the form \( S = \chi A/8 \), is obtained. This formula reproduces the correct result for extreme black hole, where the Bekenstein-Hawking one fails \( (S = 0 \text{ but } A \neq 0) \). In such a way it recovers a unified picture for the black hole entropy law. Moreover, it is proved that such a relation can be generalized to a wide class of manifolds with boundaries which are described by spherically symmetric metrics (e.g. Schwarzschild, Reissner-Nordström, static de Sitter).

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1. Introduction

Recent works by Hawking, Horowitz and Ross\textsuperscript{1,2} have demonstrated that usual Bekenstein-Hawking law for black hole entropy fails in the case of extreme black hole. For these kinds of object we have a null entropy in spite of a non-null area of the event horizon. These authors observed that this change in extreme case in respect of non-extreme one is mainly due to the different nature of event horizon in the former. One infact finds that in these cases, the presence of the event horizon is not associated with a non-trivial topology of spacetime. Euler characteristic is infact zero (not two) for this kind of black hole. This radical difference in extreme black hole physics seems a strong hints towards a point of view that particular case of black hole solutions (for example extreme Reissner-Nordstrom black hole is “just” the case $Q^2 = M^2$ of the general solution) is to be considered a rather different object from the non-extreme one. The extreme black holes are interesting in their own right because of their very special topological structure and because they appear to be the limit in which one loses intrinsic thermodynamic of black hole. They are supposed to have a physical interpretation as elementary particles of fundamental theories of gravity\textsuperscript{3} so their understanding appears to be an important point in development of a theory of quantum gravity. Nevertheless we think that it is possible to understand extreme case as a particular case of black hole without requiring a limitation of black hole thermodynamics laws. The guiding idea (originally proposed by Gibbons and Hawking\textsuperscript{4}) of this work is that thermodynamical features of spacetimes like the Schwarzschild one are explainable as an effect due to their non-trivial topological structure and in particular to the nature of their boundaries. We will see in particular that Euler characteristic and entropy have the same dependence on the boundaries of the manifold and we will
relate them in a general formula. This relation (although demonstrated only for a certain class of metrics) would be valid for every compact manifold on which Gauss-Bonnet theorem can be extended.

2. Euler characteristic and manifold structure

The Gauss-Bonnet theorem proves that it is possible to obtain the Euler characteristic of a 4-dimensional compact riemannian manifold $M$ without boundaries by the volume integral of the 4-dimensional metric curvature:

$$S_{GB} = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd}$$

with $R$ bound to the spin-connections $\omega$ of the manifold by the relations:

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

Chern showed that the differential $n$-form $\Omega$ of Gauss-Bonnet integral:

$$\Omega = \frac{(-1)^p}{2^{2p}p!} \epsilon_{a_1 \ldots a_p} R^{a_1 a_2} \wedge \ldots \wedge R^{a_{2p+1} a_{2p}}$$

defined on $M^n$ can be defined on a manifold $M^{2n-1}$ which is the image of $M^n$ through the flux of its unitary vectors field. Then he was able to express $\Omega$ has the exterior derivative of a differential $n-1$-form in $M^{2n-1}$:

$$\Omega = d\Pi$$

He also demonstrated that the original $\Omega$ integral on $M^n$ can be performed on a submanifold $V^n$ of $M^{2n-1}$ which has as boundaries the set of singular points of the
unitary vector field previously cited. By Stoke’s theorem we then obtain:

\[ S_{GB}^{vol} = \int_{M^n} \Omega = \int_{V^n} \Omega = \int_{\partial V^n} \Pi \]

For manifold with boundaries this formula can be generalized:\[7\]:

\[ S_{GB} = S_{GB}^{vol} + S_{GB}^{bou} = \int_{M^n} \Omega - \int_{\partial M^n} \Pi = \int_{V^n} \Pi - \int_{\partial V^n} \Pi \]

(2.1)

This expression implies that the Euler characteristic of a manifold \( M^n \) with boundaries becomes null in the case that its contours would be the same as that of the submanifold \( V^n \) of \( M^{2n-1} \).

We shall now use (2.1) for black hole manifold. We always work in Euclidean manifolds after imaginary time compactification necessary in order to remove conical singularities on the horizons.

\( \chi_{euler} \) of no-extreme black hole

For no-extreme black hole the boundaries of the manifold \( V \) are set by the extreme values of the range of radius coordinate that are \( r = r_h \) and \( r = r_0 = \infty \). The physical manifold \( M \) instead has just one boundary at infinity because, after removal of conical singularity, the black hole horizon \( r = r_h \) is not a border of spacetime. So:

\[ (2.1) = \int_{r_0}^{r_h} \Pi - \int_{r_h}^{r_0} \Pi - \int_{r_0}^{r_h} \Pi = -\int_{r_h}^{r_0} \Pi \]

It is possible to use this formula for calculate Euler number and for example in Schwarzschild case one correctly obtains \( \chi_{euler} = 2 \) which is the expected value for \( S^2 \times R^2 \) topology.
χ_{euler} for extreme black hole

For extreme black hole the boundaries of the manifold $V$ are the same as that one for the ordinary case $r = r_h$ and $r = r_0 = \infty$. On the other hand the physical manifold $M$ has now two boundaries at infinity represented by the usual spatial infinity $r = r_0 = \infty$ and by the horizon $r = r_h$. In fact, in this case the time-affine Killing vector has a set of fixed points only at infinity (it becomes null only asymptotically at infinity, in this sense one says that the black hole horizon for extreme black hole is at infinity). So:

$$\int_{r_0}^{r_h} \Pi - \int_{r_h}^{r_0} \Pi + \int_{r_0}^{r_h} \Pi = 0$$

This shows that for extreme black hole the Euler characteristic is always null.

3. Entropy for manifolds with boundaries

We will follow the definition of black hole entropy adopted by Kallosh, Ortin, Peet\textsuperscript{8}.

We consider a thermodynamical system with conserved charges $C_i$ and relative potentials $\mu_i$ so we work in grandcanonical ensemble.

$$Z = \text{Tr} e^{-(\beta H - \mu_i C_i)}$$

$$Z = e^{-W}$$

$$W = E - TS - \mu_i C_i$$

We obtain:

$$S = \beta(E - \mu_i C_i) + \ln Z$$
Gibbons-Hawking demonstrated that at the tree level:

\[
Z \sim e^{-I_E}
\]

\[
I_E = \frac{1}{16\pi} \int_M (-R + L_{\text{matter}}) + \frac{1}{8\pi} \int_{\partial M} [K]
\]

Here \(I_E\) is the “on-shell” Euclidean action.

In calculating \(Z\) and so \(I_E\) it is important to correctly evaluate the boundaries of our manifold \(M\).

For no-extreme black hole we have just one boundary at infinity \(r_0 \to \infty\) (after the removal of conical singularity, the metric is regular on the horizon \(r = r_h\)).

For extreme black hole we have a drastic change in boundaries structure. Metrics do not present conical singularity so we cannot fix imaginary time value. The horizon is at an infinite distance from the external observer and so it is as an “internal” boundary of our spacetime (we can say that the coordinate of this internal boundary is \(r_b\)).

In order to determine \(S\) we have also to compute \(\beta (E - \mu_i C_i)\).

From Gibbons-Hawking [4] we know that for two fixed hypersurfaces at \(\tau = \text{cost}\) \((\tau = \text{imaginary time}), \tau_1 \text{ e } \tau_2\), one has:

\[
\langle \tau_1 | \tau_2 \rangle = e^{-(\tau_2 - \tau_1)(E - \mu_i C_i)} \approx e^{-I_E}
\]

In this case it is necessary to understand that the time-affine Killing vector \(\partial / \partial \tau\) has two sets of fixed points, one at infinity the other on the horizon. So an hypersurface at \(\tau = \text{cost}\) has two boundaries in corresponding to this sets, independently of the position of horizon (which can be at infinity for extreme black hole).

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So one obtains 
\[ S = \beta(E - \mu_i C_i) + \ln Z = \]
\[ = I_{E^\infty_{rh}} - I_{E^\infty_{boun}} \]
\[ = \frac{1}{8\pi} \left( \int_{\partial V} [K] - \int_{\partial M} [K] \right) = \]
\[ = \left( \int_{r_0} [K] - \int_{r_h} [K] - \int_{r_0} [K] + \int_{r_h} [K] \right) \]

(3.1)

The deep similarity between (3.1) and (2.1) is self-evident.

**Entropy for no-extreme black hole**

In this case we don’t have an internal boundary for \( M \) and so we don’t have \( r_{boun} \) in (3.1) and we obtain:

\[ S = \frac{1}{8\pi} \left[ \int_{r_0} [K] - \int_{r_h} [K] - \int_{\infty} [K] \right] = - \int_{r_h} [K] = A/4 \]

**Entropy for extreme black hole**

In this case the horizon is at infinity and \( M \) has two boundaries in \( r = \infty \) and \( r_b = r_h \):

\[ S = \frac{1}{8\pi} \left[ \int_{\infty} [K] - \int_{r_h} [K] - \int_{\infty} [K] + \int_{r_h} [K] \right] = 0 \]

Some comments on derivation of (3.1) are in order. It was in fact derived in a grandcanonical ensemble but for extreme black hole there is no conical singularity.

#3 Note that, for metrics under our consideration, \( V_{bulk} = M_{bulk} \) so the bulk part of the entropy always cancels also for metrics which are not Ricci-flat (as de Sitter case). All the entropy depends on boundary values of extrinsic curvature.
so there is no \( \beta \) fixing and consequently there is no intrinsic thermodynamic of
the manifold. We conjecture that the correct procedure we have to follows is
exactly the inverse. The last line of (3.1) is the general expression of entropy for
manifold with boundaries. The lack of intrinsic thermodynamics is deducible from
(3.1) by consideration of boundaries structure. It is not possible to fix \( \beta \) because
boundary changes in extreme case, not the contrary. Thus (3.1) is generalizable to a
large class of riemannian manifolds with boundaries and the similarity in boundary
dependence with Gauss-Bonnet integral is a strong hint toward the evidence of a
link between entropy and topology for gravitational instantons\(^4\).

4. Euler characteristic for Schwarzschild-like metrics

We consider metrics of the form:

\[
ds^2 = -e^{2U(r)} dt^2 + e^{2U(r)} dr^2 + R^2(r) d^2 \Omega
\]  

(4.1)

Spin connections are:

\[
\begin{align*}
\omega^{01} &= \frac{1}{2}(e^{2U})' dt \\
\omega^{21} &= e^U R' d\theta \\
\omega^{31} &= e^U R' \sin \theta d\phi \\
\omega^{32} &= \cos \theta d\phi
\end{align*}
\]  

(4.2)

One has that\(^10\):

\[
S^{vol}_{GB} = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd} = \frac{1}{4\pi^2} \int_V d(\omega^{01} \wedge R^{23}) = \frac{1}{4\pi^2} \int_{\partial V} \omega^{01} \wedge R^{23}
\]  

(4.3)

\(^4\) As a seminal work in this direction we can quote the classical paper by Gibbons and Hawking about gravitational instantons\(^9\).
For the boundary term one finds [7]:

\[ S_{Gb}^{boun} = -\frac{1}{32\pi^2} \int_{\partial M} \epsilon_{abcd} (2\Theta^{ab} \wedge R^{cd} - \frac{4}{3} \Theta^{a} \wedge \Theta^{e} \wedge \Theta^{eb}) \]  \hspace{1cm} (4.4)

From (4.4) one obtains after some standard algebra [10]:

\[ S_{Gb}^{boun} = -\frac{1}{4\pi^2} \int_{\partial M} \omega^{01} \wedge R^{23} \]  \hspace{1cm} (4.5)

Hence one has the complete result:

\[ S_{Gb} = S_{Gb}^{vol} + S_{Gb}^{boun} = \frac{1}{4\pi^2} \left( \int_{\partial M} - \int_{\partial V} \right) \omega^{01} \wedge R^{23} \]

For metrics of no-extreme black hole one has:

\[ S_{Gb} = S_{Gb}^{vol} + S_{Gb}^{boun} = -\frac{1}{4\pi^2} \left( \int_{\partial M} \omega^{01} \wedge R^{23} \right)_{r=r_h} \]

For the metrics (4.1):

\[ R^{23} = d\omega^{23} + \omega^{21} \wedge \omega^{13} = (1 - e^{2U}(R')^2) \sin \theta d\theta d\phi \]

\[ \omega^{01} \wedge R^{23} = \frac{1}{2} (e^{2U})'(1 - e^{2U}(R')^2) \sin \theta d\theta d\phi dt \]

As previously said, we shall perform our calculations for riemannian manifolds with compactification of imaginary time, \( 0 \leq \tau \leq \beta \) (generalization of conical singularity remotion condition for our class of metrics). It is easy to see that this corresponds to the choice:

\[ \beta = 4\pi \left( (e^{2U})'_{r=r_h} \right)^{-1} \]  \hspace{1cm} (4.6)

Note that condition (4.6) gives an infinite range of time (no period) for the case of extreme black hole metrics that gives \( ((e^{2U})'_{r=r_h}) = 0 \). For a general manifold \( K \),
with boundary $\partial K = (r_{\text{ext}}, r_{\text{int}})$ one has:

$$S_{\text{GB}} = \frac{1}{4\pi^2} \int_{\partial K} \omega^{01} \wedge R^{23} =$$

$$= -\frac{1}{4\pi^2} \int \frac{1}{2} (e^{2U})' \sin \theta (1 - (e^{U} R')^2) d\theta d\phi dt =$$

$$= -\frac{1}{2\pi} \beta (e^{2U})' (1 - (e^{U} R')^2) \bigg|_{r_{\text{ext}}}^{r_{\text{int}}} =$$

$$= 2 \left( (e^{2U})'_{r=r_h} \right)^{-1} \left[ \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_{\text{int}}} - \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_{\text{ext}}} \right]$$

(4.7)

if we consider the manifolds $V$ and $M$ with boundaries $\partial V = (r_0, r_h)$ and $\partial M = (r_0, r_b)$ we then obtain:

$$S_{\text{GB}} = 2 \left( (e^{2U})'_{r=r_h} \right)^{-1} \left[ \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_h} - \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_0} \right] +$$

$$- 2 \left( (e^{2U})'_{r=r_h} \right)^{-1} \left[ \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_b} - \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_0} \right] =$$

$$= 2 \left( (e^{2U})'_{r=r_h} \right)^{-1} \left[ \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_h} - \left( (e^{2U})' (1 - (e^{U} R')^2) \right)_{r_b} \right] =$$

$$= 2 \left[ 1 - (e^{U} R')^2 \right]_{r_h} - \left( (e^{2U})'_{r=r_h} \right)^{-1} \left[ (e^{2U})' (1 - (e^{U} R')^2) \right]_{r_b}$$

(4.8)

From (4.8) we can see that for no-extreme black hole (no $r_b$) one obtains:

$$S^{BH}_{\text{GB}} = 2 \left[ 1 - (e^{U} R')^2 \right]_{r_h}$$

(4.9)

For extreme black hole ($r_b = r_h$) one straightforwardly finds:

$$S^{BH_{\text{extr}}} = 0$$

(4.10)
5. Entropy for Schwarzschild-like metrics

We consider again metrics of the form:

\[ ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} dr^2 + R^2(r) d^2 \Omega \]

One has:

\[ S = \beta (E - \mu_i C_i) + \ln Z = I_E^\infty_{r_h} - I_E^\infty_{\text{boun}} = \frac{1}{8\pi} \left( \int_{\partial V} [K] - \int_{\partial M} [K] \right) \]  \hspace{1cm} (5.1)

\( \partial V \) here stays as the border of the manifold \( V \) on which we consider the surfaces at \( \tau = \text{cost} \) in order to calculate the factor \( \beta (E - \mu_i C_i) \) (these are delimited, for every kind of black hole, by \( r_h \) e \( r_0 = \infty \)).

\( \partial M \) is the boundary of space-time (in this case we have that the boundary surfaces are at \( r_h \) and \( \infty \) for extreme black hole and only at \( \infty \) for normal ones.

So we obtain:

\[ S_{BH_{nonestr.}} = \frac{1}{8\pi} \left( \int_{\infty}^{r_h} [K] - \int_{r_h}^{\infty} [K] - \int_{r_h}^{\infty} [K] \right) = - \int_{r_h}^{\infty} [K] = \frac{A}{4} \]  \hspace{1cm} (5.2)

\[ S_{BH_{estr.}} = \frac{1}{8\pi} \left( \int_{\infty}^{r_h} [K] - \int_{r_h}^{\infty} [K] - \int_{r_h}^{\infty} [K] + \int_{r_h}^{r_h} [K] \right) = 0 \]

It is well-known that one can write [8]:

\[ 2 \int_{\partial M} \sqrt{-h} [K] = \int_{\partial M} \sqrt{-h} [\omega^\mu n_\mu] \]  \hspace{1cm} (5.3)
for the metrics (4.1) under our investigation one has:

\[
\omega^\mu = \left( 0, -2e^{2U} \left( \partial_r U + 2 \partial_r \ln R \right), -\frac{2\cot \theta}{r^2}, 0 \right)
\]

\[
n_\mu = \left( 0, \frac{1}{\sqrt{g^{11}}}, 0, 0 \right)
\]

so:

\[
\omega^\mu n_\mu = \omega^1 n_1 = -2e^U \left( \partial_r U + 2 \partial_r \ln R \right)
\]

(5.5)

From the (5.5) we have to subtract the flat metric correspondent term:

\[
d s^2 = -d t^2 + d r^2 + r^2 d \Omega^2
\]

So one obtains:

\[
\omega^\mu_0 = \left( 0, -\frac{4}{r}, -\frac{2\cot \theta}{r^2}, 0 \right)
\]

\[
n^0_\mu = (0, 1, 0, 0)
\]

(5.6)

and:

\[
[\omega^\mu n_\mu] = \omega^\mu n_\mu - \omega^\mu_0 n^0_\mu =
\]

\[
= -2e^U \left( \partial_r U + 2 \partial_r \ln R \right) + \frac{4}{r}
\]

(5.7)
6. Test for the expressions for $\chi$ ed $S$ for the Schwarzschild space-time.

Here we want to check our preceding results by application to the well-known case of Schwarzschild metric:

\begin{align*}
e^{2U} &= (1 - 2M/r) \\
U &= \frac{1}{2} \ln(1 - 2M/r) \\
R &= r
\end{align*}

we find:

\begin{align*}
S &= -\frac{1}{8\pi} \int_{\mathcal{R}_h} d^3x \sqrt{-h} [K] = \\
&= -\frac{1}{16} \int_0^{\beta} d\tau e^U \int r^2 \sin \theta d\theta d\phi \left[ -2e^U \left( \frac{1}{2} \frac{2M}{(1 - 2M/r)r^2} + \frac{2}{r} \right) + \frac{4}{r} \right]_{r_h} \\
&= -\frac{1}{16} \beta 4\pi r^2 \left[ -2e^{2U} \left( \frac{M}{r^2 e^{2U}} + \frac{2}{r} \right) + e^{U} \frac{4}{r} \right]_{r_h} \\
&= \frac{8\pi M}{4} \left[ 2M + 4re^U \left( e^{U} - 1 \right) \right]_{r_h} \\
&= 4\pi M^2 = \frac{A}{4}
\end{align*}

Here we used the fact that $e^{U}|_{r_h} = 0$.

So formula (5.7) is exact. We now test the formula (4.9) for Euler characteristic.

For the Schwarzschild metric we find $\chi = 2$ which is the expected result for this space-time of topology $R^2 \times S^2$.

We can finally redo the integration of (6.1) for the general case of a general Schwarzschild-like metric obtaining:

\begin{equation}
S = \frac{\beta R}{2} \left[ (U' R + 2R') e^U - \frac{2R}{r} \right] e^U \bigg|_{r=r_h} \tag{6.2}
\end{equation}
We can also rewrite (4.9) in a more suitable form for our next purposes:

\[
\chi = \int_{r_h}^{\Pi} = \frac{1}{4\pi^2} \int_{r_h}^{\omega^0} R^{23} = \\
= \frac{\beta}{2\pi} (2U'e^{2U})(1 - e^{2U}R'^2)_{r_h}
\]

(6.3)

It can be seen in this form we leave explicit the dependence on inverse temperature.

7. Relation between gravitational entropy and Euler characteristic for different no-extreme black hole

**Schwarzschild black hole**

We have:

\[
e^{2U} = (1 - 2M/r) \\
U = \frac{1}{2} \ln(1 - 2M/r) \\
R = r
\]

In this case \( A = \beta r_h = 4\pi r_h^2 \) and the relation between \( \beta \) and \( A \) is:

\[
\beta = (4.6) = \frac{A}{r_h}
\]

From (6.2) we obtain:

\[
S = \frac{A}{4}
\]

(7.1)

We also have from (6.3):

\[
\chi = \beta r_h \frac{1}{2\pi r_h^2} = \frac{A}{2\pi r_h^2}
\]

(7.2)

If in (7.2) we pose \( A \) as a function of \( \chi \) and substituting it in (7.1), we obtain:

\[
S = \frac{\pi}{2} \chi r_h^2 = \frac{\chi A}{32\pi} = \frac{\chi A}{8}
\]

(7.3)

This is the simplest relation one can imagine for the searched relation between
entropy and Euler characteristic. Although it has been found for Schwarzschild black hole this different formulation of Bekenstein-Hawking entropy \( (7.3) \) would give the correct results \( S = 0 \) for extreme black hole without requiring that \( A = 0 \). We want to point out that this formula is valid for a large class of black hole (extreme or not) and more generally for space-times described by metric of the type \((4.1)\) and well defined in respect of Gauss-Bonnet hypothesis.

**Dilaton** \( U(1) \) **black hole with** \( 0 \leq a \leq 1 \) (case \( a = 0 \) correspond to Reissner-Nordström black hole)

We have:

\[
e^{2U} = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}}
\]

\[
U = \frac{1}{2} \ln \left[ \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} \right]
\]

\[
R = r \left(1 - \frac{r_-}{r}\right)^{\frac{a^2}{1+a^2}}
\]

\[
M = \frac{r_+}{2} + \frac{1-a^2}{1+a^2} r_-
\]

\[
Q^2 = \frac{r_+ r_-}{1+a^2}
\]

\[
r_h = r_+
\]

In this large class of black hole one finds \( A = \beta R r_h = 4\pi R^2 r_h \), \( R \) determines the caracteristic scale of distance. The relation between \( \beta \) and \( A \) is:

\[
\beta = (4.6) = \frac{A}{r_h \left(1 - \frac{r_-}{r_h}\right)}
\]

From \((6.2)\) one obtains again:

\[
S = 4\pi R^2 r_h = \frac{A}{4} \tag{7.4}
\]
From (6.3) we also find:

\[
\chi = \beta r_h \frac{1}{2 \pi R_h^2} = \frac{A}{2 \pi R_h^2}
\]  

(7.5)

Putting in (7.5) \(A\) as a function of \(\chi\) and inserting it in (7.4) we find as expected:

\[
S = \frac{\pi}{2} \chi R_h^2 = \frac{\chi A}{8}
\]

\(S^2 \times S^2\) (Schwarzschild-de Sitter)

This is another 4-dimensional gravitational instantons with a metric of the form (4.1). This is an interesting case because there is no boundary \(\partial M = 0\) for this manifold (so (6.3) does not hold). The Euler number is 4, topology being that of \(S^2 \times S^2\). We have:

\[
e^{2U} = (1 - \Lambda r^2)
\]

\[
U = \frac{1}{2} \ln(1 - \Lambda r^2)
\]

\[
R^2 = \Lambda^{-1} = \text{cost} > 0
\]

\[
r_h = \Lambda^{-1/2}
\]

By applying (7.3) to this case (\(\chi = 4\)) one finds:

\[
S = \frac{A\chi}{8} = \frac{4\pi \Lambda^{-1/4}}{8} = \frac{2\pi}{\Lambda}
\]

This is known to be [7] the exact result.\(^5\) The same result can be obtained by usual path-integral calculation. This result implies that (7.3) holds in a more general class of space-times than that described by the metrics of the form (4.1).

\(^5\) In the work of EGH [7] is given the action value which concides with our value for entropy (modulo a minus). This is correct because in the cases under our consideration the actions have the form \(I = -\alpha \beta^2\) (with \(\alpha\) an opportune constant of proportionality obtained by explicitating the physical dependence of system parameters (\(M\) or \(\Lambda\) on temperature). Then we have \(S = - (\beta \partial \beta - 1) I = 2\alpha \beta^2 - \alpha \beta^2 = \alpha \beta^2 = -I\).
General case

We have:

\[ A = 4\pi R^2(r_h) \]

\[ \beta = 4\pi ((e^{2U})')^{-1} \]

\[ S = \frac{\beta R}{2} ((U' R + 2R') e^U - \frac{2R}{r} e^U) \bigg|_{r=r_h} \]

\[ \chi = \frac{\beta}{2\pi} (2U' e^{2U}) (1 - e^{2U} R'^2) \bigg|_{r=r_h} \]

so one finds:

\[ S = 2\pi \chi ((2U' e^{2U})')^{-1} (1 - e^{2U} R'^2)^{-1} \frac{R}{r} \bigg|_{r=r_h} \]

\[ = \pi \chi ( (e^{2U})' - R^2 e^{2U} (e^{2U})'^{-1} \bigg|_{r=r_h} \frac{\beta R}{2} (e^{2U})' + 2R' e^{2U} - \frac{\beta 2R}{r} e^U \bigg|_{r=r_h} \]

by definition \( e^{2U} \big|_{r=r_h} = 0 \), so:

\[ S = \pi \chi R(r_h) \left( (e^{2U})'^{-1} \right|_{r=r_h} = \]

\[ = \frac{\pi \chi R^2(r_h)}{2} = \frac{\chi A}{8} = \]

8. Conclusions

The relation (7.3) appears to hold in a wide class of space-time. It seems that this formulation of black hole entropy area possibly clarifies the behaviour of extreme black hole entropy by interpreting gravitational entropy has a topological effect (in this sense it confirms Hawking et al. position toward its interpretation. Unfortunately at the moment it appears rather difficult to find a dynamical explanation of this “topological” entropy. We conjecture that the shown deep relation
of this entropy with boundary structure of the space-time is in a certain sense a hint towards an interpretation based on dynamical degrees of freedom associated to vacuum in topological non trivial space-times\textsuperscript{11}. Maybe that intrinsic thermodynamics of some gravitational instantons is statistically due to Casimir-like effects of vacuum. Zero-modes are known to be sensible to the topological structure of space-time but it is also not well understood how to associate them a thermodynamical interpretation.

**Note added**

When this work in its main results has been already completed, the authors became aware of recent work by\textsuperscript{12} and\textsuperscript{13} in which the relation between entropy and topology is put in evidence. In our opinion this work does not have a substantial overlapping with these already quoted. These are based on hamiltonian formulation of the problem and the thermodynamical quantities are expressed as function of the Euler number of a manifold which is not the physical one (but of course the respective results are in complete accord).

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REFERENCES

1. S.W. Hawking, G.T. Horowitz, S.F. Ross - *Entropy, Area and black hole pairs* - Preprint gr-qc 9409013.

2. S.W. Hawking, G.T. Horowitz - *The gravitational hamiltonian, action, entropy and surface terms* - Preprint gr-qc 9501014.

3. A. Sen - Preprint [hep-th/9504147](http://arxiv.org/abs/hep-th/9504147)

4. G.W. Gibbons, S.W. Hawking, *Phys.Rev.* **D15**, **10** (1977), 2752.

5. S. Chern, *Annals of Math.* **45**(1944), 4, 747.

6. S. Chern, *Annals of Math.* **46**(1945), 4, 674.

7. T. Eguchi, P.B. Gilkey, A.J. Hanson, *Phys. Rep.* **66**, **6** (1980),

8. R. Kallosh, T. Ortin, A. Peet, *Phys. Rev.* **D47** (1993), 12, 5400.

9. G.W. Gibbons, S.W. Hawking, *Commun. Math. Phys.* **66** (1979), 291

10. G.W. Gibbons, R.E. Kallosh - *Topology, entropy and Witten index of dilaton black holes* - Preprint [hep-th 9407118](http://arxiv.org/abs/hep-th/9407118)

11. S. Hacyan, A. Sarmiento, G. Cocho, F. Soto, *Phys. Rev. D* **32** (1985), 914.

12. Banados, Teitelboim, Zanelli, *Phys.Rev.Lett* **72**, **7** (1994), 957.

13. C. Teitelboim - *Action and entropy of extreme and non-extreme black holes* - Preprint [hep-th 9410103](http://arxiv.org/abs/hep-th/9410103)