Boosting First-order Methods by Shifting Objective:
New Schemes with Faster Worst Case Rates

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Abstract

We propose a new methodology to design first-order methods for unconstrained strongly convex problems, i.e., to design for a shifted objective function. Several technical lemmas are provided as the building blocks for designing new methods. By shifting objective, the analysis is tightened, which leaves space for faster rates, and also simplified. Following this methodology, we derived several new accelerated schemes for problems that equipped with various first-order oracles, and all of the derived methods have faster worst case convergence rates than their existing counterparts. Experiments on machine learning tasks are conducted to evaluate the new methods.

1 Introduction

In this paper, we focus on the following unconstrained strongly convex problem:

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where each $f_i$ is $L$-smooth and $\mu$-strongly convex and we denote $x^* \in \mathbb{R}^d$ as the solution of this problem. The $n = 1$ case covers a large family of classic strongly convex problems, where gradient descent (GD) and Nesterov’s accelerated gradient (NAG) (Nesterov, 1983, 2005, 2018) are the typical choices of optimizer. The $n \geq 1$ case is the recently popular finite-sum case, where many elegant methods that incorporate the idea of variance reduction have been proposed. Problems with a finite-sum structure arise frequently in machine learning and statistics, such as empirical risk minimization (ERM).

In this work, we tackle problem (1) from a new angle. Instead of designing methods to solve the original objective function $f$, we propose methods that are designed to solve a shifted objective $h$, which is defined as

$$\min_{x \in \mathbb{R}^d} h(x) = \frac{1}{n} \sum_{i=1}^{n} h_i(x),$$

where

$$h_i(x) = f_i(x) - f_i(x^*) - \langle \nabla f_i(x^*), x - x^* \rangle - \frac{\mu}{2} \|x - x^*\|^2,$$

and thus $h(x) = f(x) - f(x^*) - \frac{\mu}{2} \|x - x^*\|^2$. It can be easily verified that each $h_i(x)$ is $(L - \mu)$-smooth and convex, $\nabla h_i(x) = \nabla f_i(x) - \nabla f_i(x^*) - \mu(x - x^*)$, $\nabla h(x) = \nabla f(x) - \mu(x - x^*)$, $h_i(x^*) = h(x^*) = 0$ and

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The formal definitions of smoothness, strong convexity and co-coercivity are given in Section 1.1. If each $f_i(\cdot)$ is $L$-smooth, the averaged function $f(\cdot)$ is itself $L$-smooth — but probably with a smaller $L$. We keep using $L$ as the smoothness constant for a consistent analysis.
\[ \nabla h_i(x^*) = \nabla h(x^*) = 0, \] which means that problems (1) and (2) have a shared optimum \( x^* \). Let’s write the co-coercivity of \( h \) here:

\[ \forall x, y \in \mathbb{R}^d, h(x) - h(y) - \langle \nabla h(y), x - y \rangle \geq \frac{1}{2(L - \mu)} \| \nabla h(x) - \nabla h(y) \|^2. \]

This simple inequality takes both smoothness and strong convexity (of \( f \)) into consideration, which is very tight. It turns out that to design first-order methods for the shifted objective \( h \), this inequality is the only property we need. Leveraging this inequality, our proposed methods achieve faster worst case convergence rates than their counterparts that were designed for the original objective \( f \) and use smoothness and strong convexity separately.

In summary, our methodology and proposed methods have the following distinctive features:

- We show that our design methodology works for problems that equipped with various first-order oracles: deterministic gradient oracle, incremental gradient oracle and incremental proximal point oracle.
- Co-coercivity is the only objective property we need. This leads to much cleaner and tighter analysis to the proposed methods than their existing counterparts.
- For our proposed stochastic methods, we deal with shifted variance bounds / shifted stochastic gradient norm bounds, which is unlike all the previous work.
- All the proposed methods achieve faster worst case convergence rates than their counterparts (with their default parameter choices).

It is worth noting that while our work focuses on developing accelerated methods, which is to maximize the potential of this new methodology, we can also tighten the analysis of non-accelerated methods, which could lead to new algorithmic schemes.

We drew our original inspiration from a recently proposed robust momentum method (RMM) (Cyrus et al., 2018), which can be regarded as the bridge between an accelerated method called triple momentum method (TM) (Van Scoy et al., 2017) and GD. Cyrus et al. (2018) suggested that RMM converges with a Lyapunov function that contains a term of the form \( \frac{1}{2L} \| \nabla h(x) \|^2 \). We were intrigued by the special structure of this term and decided to conduct a thorough study for it.

This paper is organized as follows: In Section 2, we present technical lemmas that are the core building blocks for the proposed methods. In Section 3, we propose a new accelerated method for the deterministic \( n = 1 \) case. In Section 4, we propose an accelerated stochastic variance reduced method for the \( n \geq 1 \) case with incremental gradient oracle. In Section 5, we propose an accelerated method for the \( n \geq 1 \) case with incremental proximal point oracle. In Section 6, we present some empirical results for the proposed methods.

1.1 Notations and definitions

In this paper, we consider problems in the standard Euclidean space denoted by \( \mathbb{R}^d \), and we use \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) to denote the inner product and the Euclidean norm, respectively. \( [n] \) denotes the set \( \{1, 2, \ldots, n\} \). \( \mathbb{E} \) denotes the total expectation and \( \mathbb{E}_k \) is the conditional expectation given the information up to iteration \( k \).

We say that a convex function \( f: \mathbb{R}^d \to \mathbb{R} \) is \( L \)-smooth if it has \( L \)-Lipschitz continuous gradients, i.e., \( \forall x, y \in \mathbb{R}^d, \| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \| \). Some important consequences of this assumption can be found in the textbook (Nesterov, 2018): \( \forall x, y \in \mathbb{R}^d, \frac{1}{2L} \| \nabla f(x) - \nabla f(y) \|^2 \leq f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq \frac{L}{2} \| x - y \|^2 \). The first inequality is known as co-coercivity, which can be regarded as an enhancement to convexity. A continuously differentiable \( f \) is called \( \mu \)-strongly convex if \( \forall x, y \in \mathbb{R}^d, f(x) - f(y) - \langle \nabla f(y), x - y \rangle \geq \frac{\mu}{2} \| x - y \|^2 \).

Given a point \( x \in \mathbb{R}^d \), an index \( i \in [n] \) and \( \alpha > 0 \), deterministic oracle returns \( (f(x), \nabla f(x)) \), incremental first-order oracle returns \( (f_i(x), \nabla f_i(x)) \) and incremental proximal point oracle returns \( (f_i(x), \nabla f_i(x), \text{prox}_\alpha^i(x)) \), where the proximal operator is defined as \( \text{prox}_\alpha^i(z) = \arg \min_x \{ f_i(x) + \frac{\alpha}{2} \| x - z \|^2 \} \).

We denote \( \epsilon > 0 \) as the required accuracy for solving problem (1) (i.e., to achieve \( \| x - x^* \|^2 \leq \epsilon \)), which is assumed to be small. We let \( \kappa \triangleq \frac{\mu}{L} \), which is often called the condition ratio.
1.2 Related work

Problem \([\ell]\) with \(n = 1\) is the classic smooth strongly convex setting. Standard analysis (see the textbook (Nesterov, 2018)) shows that for this problem, GD with \(\frac{\mu}{2n}\) stepsize converges at a \(\left(\frac{n}{\mu n}\right)^2\) rate.\(^2\) The heavy-ball method (Polyak, 1964) fails to converge globally on this problem (Lessard et al., 2016). The celebrated NAG is proven to achieve a fast \(1 - \frac{1}{\sqrt{n}}\) rate (Nesterov, 2018). This rate remains the fastest one until recently, Van Scoy et al. (2017) proposed Triple Momentum method (TM) that converges at a \((1 - \frac{1}{\sqrt{n}})^2\) rate, and numerical results in Lessard and Seiler (2019) suggest that this rate is not improvable. In terms of reducing \(\|x - x^*\|^2\) to \(\epsilon\), TM is stated to have an \(O\left(\frac{\sqrt{n}}{\epsilon} \log \frac{1}{\epsilon}\right)\) iteration complexity (cf. Table 2, Van Scoy et al., 2017) compared with the \(O(\sqrt{n} \log \frac{1}{\epsilon})\) complexity of NAG. In the general convex setting, recent works (Kim and Fessler, 2016; Attouch and Peypouquet, 2016; Taylor et al., 2017; Kim and Fessler, 2018) propose several new schemes that have lower complexity than the original NAG.

For the \(n \geq 1\) case, stochastic gradient descent (SGD) (Robbins and Monro, 1951), which uses component gradients \(\nabla f_i(x)\) to estimate the full gradient \(\nabla f(x)\), achieves a lower iteration cost than GD. However, SGD only converges at a sub-linear rate. To fix this issue, the variance reduction techniques have been proposed recently, and notable methods include SAG (Roux et al., 2012; Schmidt et al., 2017), SVRG (Johnson and Zhang, 2013; Xiao and Zhang, 2014), SAGA (Defazio et al., 2014), SDCA (Shalev-Shwartz and Zhang, 2013), to mention a few. Inspired by the Nesterov’s acceleration technique, accelerated stochastic variance reduced methods have been proposed in pursuing of the lower bound \(O(n + \sqrt{n} \log \frac{1}{\epsilon})\) (Woodworth and Srebro, 2016), such as Acc-Prox-SVRG (Nitanda, 2014), APCG (Lin et al., 2014), ASDCA (Shalev-Shwartz and Zhang, 2014), APPA (Frostig et al., 2015), Catalyst (Lin et al., 2015), SPDC (Zhang and Xiao, 2015), RPDG (Lan and Zhou, 2018), Point-SAGA (Defazio et al., 2016) and Katyusha (Allen-Zhu, 2018). Among these methods, Katyusha and Point-SAGA, representing the first two directly accelerated incremental methods, achieve the fastest rates. Point-SAGA leverages a more powerful incremental proximal operator oracle. Katyusha introduces the idea of negative momentum, which serves as a variance reducer that further reduces the variance of SVRG estimator. This construction motivates many new accelerated methods (Zhou et al., 2018; Lan et al., 2019; Kulunchakov and Mairal, 2019; Zhou et al., 2019).

2 Main technical lemmas

In this section, we present some simple lemmas that serve as the main building blocks of our proposed methods. As we shall see, Lemma 1 and 2 are the keys to the improved worst case rates achieved in this paper. Lemma 1 is the classic mirror descent lemma transformed for the shifted objective.

**Lemma 1** (Shifted mirror descent lemma). **Given a gradient estimator** \(G\) **and vectors** \(z^+, z^-, y \in \mathbb{R}^d\), **define a shifted estimator** \(H = G - \mu(y - x^*)\), if \(z^+ = \arg\min_x \left\{ \langle G, x \rangle + (\alpha/2) \|x - z^-\|^2 + (\mu/2) \|x - y\|^2 \right\}\), **it holds that**

\[
\langle H, z^- - x^* \rangle = \frac{\alpha}{2} \left( \|z^- - x^*\|^2 - \left( 1 + \frac{\mu}{\alpha} \right)^2 \|z^+ - x^*\|^2 \right) + \frac{1}{2\alpha} \|H\|^2.
\]

**Proof.** Using the optimality condition,

\[
G + \alpha(z^+ - z^-) + \mu(z^+ - y) = 0,
\]

\[
H + \alpha(z^+ - z^-) + \mu(z^+ - x^*) = 0,
\]

\[
(\alpha + \mu)(z^+ - x^*) = \alpha(z^+ - x^*) - H,
\]

\[
(\alpha + \mu)^2 \|z^+ - x^*\|^2 = \alpha^2 \|z^- - x^*\|^2 - 2\alpha \langle H, z^- - x^* \rangle + \|H\|^2.
\]

Re-arranging the last equality completes the proof. \(\square\)

\(^2\)In this paper, the worst case convergence rate is measured on reducing the squared norm distance \(\|x - x^*\|^2\).
In general convex optimization, a similar lemma (for the original estimator $G$) serves as the core lemma for mirror descent\footnote{In the Euclidean case, mirror descent coincides with gradient descent, and it represents a different analysis to the same method.} (e.g., Theorem 5.3.1 in the textbook \cite{Ben-Tal2013}). This type of lemma also appears frequently in the literature of online optimization, which is used as an upper bound to the regret at current iteration. In the strongly convex setting, in order to achieve a linear rate (or logarithmic regret in online optimization), we need to create a contraction between $\|z^+ - x^*\|^2$ and $\|z^- - x^*\|^2$. Typical solutions are: (1) Using the strong convexity of objective in some sophisticated ways (e.g., the analysis of NAG \cite{Nesterov2018} in offline optimization and Theorem 1 in \cite{Shalev-Shwartz2007} in online optimization); (2) Involving a strongly convex regularizer and performing proximal mapping (e.g. Lemma 2.5 in \cite{Allen-Zhu2018}). Either way a $(1 + \frac{\mu}{L})^{-2}$ (or $1 - \frac{\mu}{L}$) contraction ratio is created instead of the $(1 + \frac{\mu}{L})^{-2}$ above. Note that except for the relation $H = G - \mu(y - x^*)$, this lemma imposes no requirement on $G$ and $H$.

**Lemma 2** (Shifted and strengthened firm non-expansiveness). Given two relations $z^+ = \text{prox}^\alpha_z(z^-)$ and $y^+ = \text{prox}^\alpha_y(y^-)$, it holds that
\[
\frac{1}{\alpha^2} \left( 1 + \frac{2(\alpha + \mu)}{L - \mu} \right) \left\| \nabla h_i(z^+) - \nabla h_i(y^+) \right\|^2 + \left( 1 + \frac{\mu}{\alpha} \right)^2 \left\| z^+ - y^+ \right\|^2 \leq \left\| z^- - y^- \right\|^2.
\]

**Proof.** Based on the first-order optimality condition and the definition of $h_i$,
\[
\nabla f_i(z^+) + \alpha(z^+ - z^-) = 0, \quad \nabla f_i(y^+) + \alpha(y^+ - y^-) = 0,
\]
\[
\nabla h_i(z^+) + \nabla f_i(x^*) + \mu(z^+ - x^*) + \alpha(z^+ - z^-) = 0,
\]
\[
\nabla h_i(y^+) + \nabla f_i(x^*) + \mu(y^+ - x^*) + \alpha(y^+ - y^-) = 0.
\]

Subtract the last two equalities,
\[
(\alpha + \mu)(z^+ - y^+) = \alpha(z^- - y^-) - (\nabla h_i(z^+) - \nabla h_i(y^+)),
\]
which implies
\[
(\alpha + \mu)^2 \left\| z^+ - y^+ \right\|^2 = \alpha^2 \left\| z^- - y^- \right\|^2 - 2\alpha \left\langle \nabla h_i(z^+) - \nabla h_i(y^+), z^- - y^- \right\rangle + \left\| \nabla h_i(z^+) - \nabla h_i(y^+) \right\|^2. \quad (3)
\]

Based on the co-coercivity of $h_i$, we have
\[
\left\langle \nabla h_i(z^+) - \nabla h_i(y^+), z^+ - y^+ \right\rangle \geq \frac{1}{L - \mu} \left\| \nabla h_i(z^+) - \nabla h_i(y^+) \right\|^2.
\]

Together with (3), it holds that
\[
\left\langle \nabla h_i(z^+) - \nabla h_i(y^+), z^- - y^- \right\rangle \geq \frac{1}{\alpha} \left( 1 + \frac{\alpha + \mu}{L - \mu} \right) \left\| \nabla h_i(z^+) - \nabla h_i(y^+) \right\|^2.
\]
It remains to use this bound in (4).

Recall the definition of a firmly non-expansive operator $T$ (e.g., Definition 4.1 in the textbook \cite{Bauschke2017}), $\forall x, y$,
\[
\|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2.
\]

Lemma 2 can be derived by choosing $T$ as the scaled proximal operator\footnote{In the strongly convex setting, $(1 + \frac{\mu}{\alpha}) \cdot \text{prox}^\alpha_x$ is firmly non-expansive (e.g., Proposition 1 in \cite{Defazio2016}).} $(1 + \frac{\mu}{\alpha}) \cdot \text{prox}^\alpha_x$ and strengthening $\langle Tx - Ty, (\text{Id} - T)x - (\text{Id} - T)y \rangle \geq 0$ using the co-coercivity of $h_i$. A similar lemma is also used in the analysis of proximal point algorithm \cite{Rockafellar1976}. In our problem setting, \cite{Defazio2016} also strengthened...
We adopt a simplified version of NAG in Algorithm 1 (1-memory accelerated methods, [Tseng, 2008]). It is known that NAG can be analyzed based on the following Lyapunov function for some $\lambda > 0$:

$$ T_k = f(x_k) - f(x^*) + \frac{\lambda}{2} \|z_k - x^*\|^2, $$

(5)

which is somehow suggested in the construction of the estimate sequence in [Nesterov, 2018]. This choice requires neither $f(x_k) - f(x^*)$ nor $\|z_k - x^*\|^2$ to be monotone decreasing over iterations, which is called the non-relaxational property in [Nesterov, 1983]. By reorganizing the proofs in [Nesterov, 2018] under the notion of Lyapunov function, we obtain the per-iteration contraction of NAG in Theorem 1. The proof is given in Appendix F for completeness.

**Theorem 1.** In Algorithm 1, suppose we choose $\alpha, \tau_x, \tau_y$ under the constraints (6), the iterations satisfy the per-iteration contraction (7) for the Lyapunov function defined in (5).

$$ \begin{align*}
\alpha & \geq \frac{L(1-\tau_y)}{1-\tau_x} \tau_x, \\
\mu & \geq \frac{L(1-\tau_y)}{1-\tau_x}, \\
(1 + \frac{\mu}{\alpha})(1 - \tau_x) & \leq 1.
\end{align*} $$

(6)

With $\lambda = (\alpha + \mu)\tau_x$,

$$ T_{k+1} \leq \left(1 + \frac{\mu}{\alpha}\right)^{-1} T_k, \text{ for } k \geq 0. $$

(7)
Algorithm 1 Nesterov’s Accelerated Gradient (NAG)

Input: Parameters $\alpha > 0, \tau_y, \tau_x \in ]0, 1[$ and initial guesses $x_0, z_0 \in \mathbb{R}^d$, iteration number $K$.

1: for $k = 0, \ldots, K - 1$ do
2:  $y_k = \tau_y z_k + (1 - \tau_y) x_k$.
3:  $z_{k+1} = \arg \min_x \left\{ \langle \nabla f(y_k), x \rangle + (\alpha/2) \| x - z_k \|^2 + (\mu/2) \| x - y_k \|^2 \right\}$.
4:  $x_{k+1} = \tau_x z_{k+1} + (1 - \tau_x) x_k$.
5: end for

Output: $x_K$.

When the inequalities in the constraints (6) (except $\tau_y \geq \tau_y$) hold as equality, we derive the standard parameter choice of NAG: $\alpha = \sqrt{L\mu} - \mu, \tau_y = (\sqrt{\kappa} - 1)^{-1}, \tau_x = (\sqrt{\kappa})^{-1}$. By substituting this choice and eliminating sequence $\{z_k\}$, we obtain the widely-used scheme (Constant Step scheme III in Nesterov (2018)):

$$x_{k+1} = y_k - \frac{1}{\sqrt{\kappa}} \nabla f(y_k), y_{k+1} = x_{k+1} + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa}} (x_{k+1} - x_k).$$

Telescoping (7), we obtain the original guarantee of NAG (cf. Theorem 2.2.3 in Nesterov (2018)),

$$f(x_K) - f(x^*) - \frac{\mu}{2} \| z_K - x^* \|^2 \leq \left( 1 - \frac{1}{\sqrt{\kappa}} \right)^K \left( f(x_0) - f(x^*) + \frac{\mu}{2} \| z_0 - x^* \|^2 \right).$$

If we regard the constraints $\{0\}$ as an optimization problem with a target of minimizing the rate factor $(1 + \frac{\mu}{\alpha})^{-1}$, the rate factor $1 - \frac{1}{\sqrt{\kappa}}$ in (5) is optimal. Combining $\alpha \geq \frac{L(1 - \tau_y)\tau_y}{1 - \tau_y}$ and $\mu \geq \frac{L(\tau_x - \tau_y)}{\sqrt{\kappa}}$, we obtain $\alpha \geq L\tau_x - \mu$. To minimize $\alpha$, we fix $\alpha = L\tau_x - \mu$, and it can be easily verified that in this case, the smallest rate factor is achieved when $(1 - \frac{\mu}{\alpha}) (1 - \tau_y) = 1$. Note that these arguments do not consider variable parameter choices and are limited to the current analysis framework only.

Denote the initial constant in (8) as $C_0^{NAG} \triangleq f(x_0) - f(x^*) + \frac{\mu}{2} \| z_0 - x^* \|^2$. This guarantee shows that in terms of reducing the squared norm distance $\| x - x^* \|^2$ to $\epsilon$, sequences $\{x_k\}$ (due to $f(x_K) - f(x^*) \geq \frac{\mu}{2} \| x_K - x^* \|^2$) and $\{z_k\}$ have the same iteration number upper bound $\sqrt{\kappa} \log \left( \frac{2C_0^{NAG}}{\mu \epsilon} \right)$. And since $\{y_k\}$ is a convex combination of them, it also converges with the same complexity.

3.2 Generalized triple momentum method

The proposed method is presented in Algorithm 2. We dub it Generalized Triple Momentum method (G-TM) due to the facts that TM can be regarded as a special parameterization of G-TM (with $\tau_y = 0$) and we were inspired by the construction of TM during the development of G-TM, especially its Lyapunov function suggested in Cyrus et al. (2018): (with $\lambda > 0$)

$$T_k = h(y_{k-1}) - \frac{1}{2(L - \mu)} \| \nabla h(y_{k-1}) \|^2 + \frac{\lambda}{2} \| z_k - x^* \|^2.$$  \hspace{1cm} (9)

However, the original TM is still imperfect for the following reasons:

- **Redundant parameters.** TM has 4 parameters and it only allows constant choices. Our results for G-TM show that 3 parameters are enough to achieve the same $(1 - \frac{1}{\sqrt{\kappa}})^2$ rate with a much smaller initial constant, and the analysis works with variable parameter choices.

- **Flawed guarantee and an additional $\log \sqrt{\kappa}$ factor.** The guarantee in Van Scoy et al. (2017), Cyrus et al. (2018) has an initial state issue: the initial constants (Equation (6) in Van Scoy et al. (2017) and (11) in Cyrus et al. (2018)) correlate to the first iterates ($x_1$ or $z_1$ in their notations), which are not initial guesses. It can be verified that the first iteration in TM is gradient descent with an $\frac{1}{\sqrt{\kappa}}$ learning rate. This learning rate exceeds the $\frac{2}{L \tau_y}$ limit, which means that we do not have $\| x_1 - x^* \|^2 \leq \| x_0 - x^* \|^2$ in general. This issue is possibly the reason for the $\log \sqrt{\kappa}$ factor stated in Van Scoy et al. (2017).
Algorithm 2: Generalized Triple Momentum (G-TM)

Input: Parameters \(\{\alpha_k > 0\}, \{\tau_k^x \in [0,1]\}, \{\tau_k > 0\}\) and initial guesses \(y_{-1}, z_0 \in \mathbb{R}^d\), iteration number \(K\).

1. for \(k = 0, \ldots, K - 1\) do
2. \(y_k = \tau_k^x z_k + (1 - \tau_k^x) y_{k-1} + \tau_k^x (\mu(y_{k-1} - z_k) - \nabla f(y_{k-1}))\).
3. \(z_{k+1} = \arg\min_x \left\{ (\nabla f(y_k), x) + \alpha_k ||x - z_k||^2 + \mu/2 ||x - y_k||^2 \right\}\).
4. end for

Output: \(z_K\).

- **Highly algebraic proof.** TM was discovered based on inspirations from integral quadratic constraints [Lessard et al., 2016]. Its analysis starts with establishing an algebraic identity. From an optimization point of view, it is generally hard to extend TM into other settings based on its existing analysis.

The proposed G-TM resolves these issues. Indeed, the framework of Algorithm 2 is general enough to cover both NAG (constant scheme) and TM, which is discussed in Appendix B. A subtlety in Algorithm 2 is that it requires storing a past gradient vector, and thus at the first iteration, two gradient computations are needed. In the following theorem, we establish the per-iteration contraction of G-TM. The proof is a simple combination of Lemma 3 and Lemma 4 and is constructive, i.e., the basic routine is to build contractions for the terms in the Lyapunov function (9) one-by-one and then coordinate them together.

**Theorem 2.** In Algorithm 2, suppose we choose \(\{\alpha_k\}, \{\tau_k^x\}, \{\tau_k\}\) under the constraints (10), the iterations satisfy the per-iteration contraction (11) for the Lyapunov function defined in (9).

\[
\begin{align*}
\tau_k^x - \mu \tau_k^x &\leq \frac{2\alpha_k}{L - \mu}, \\
(L - \mu) \tau_k^x &\geq 1 - \tau_k^x, \\
\left(1 + \frac{\mu}{\alpha_k}\right)^2 (1 - \tau_k^x) &\leq 1. 
\end{align*}
\]

With \(\lambda = \frac{(\tau_k^x - \mu \tau_k^x)(\alpha_k + \mu)^2}{2 \alpha_k}\),

\(T_{k+1} \leq \left(1 + \frac{\mu}{\alpha_k}\right)^{-2} T_k\), for \(k \geq 0\).

**Proof.** First, it is straightforward that we can introduce a contraction between \(h(y_k)\) and \(h(y_{k-1})\) using Lemma 3. Applying Lemma 3 with \(f = h\) for the recursion \(y_k = \tau_k^x z_k + (1 - \tau_k^x) y_{k-1} + \tau_k^x (\mu(y_{k-1} - z_k) - \nabla f(y_{k-1}))\) and strengthening the convexity arguments by co-coercivity, we obtain

\[
\begin{align*}
\frac{\tau_k^x}{2(L - \mu)} ||\nabla h(y_k)||^2 &- \frac{1 - \tau_k^x}{2(L - \mu)} ||\nabla h(y_{k-1}) - \nabla h(y_k)||^2. 
\end{align*}
\]

Note that \(\mu(y_{k-1} - z_k) - \nabla f(y_{k-1}) = \mu(x^* - z_k) - \nabla h(y_{k-1})\) by definition, and thus

\[
\begin{align*}
\frac{\tau_k^x}{2(L - \mu)} ||\nabla h(y_{k-1})||^2 &- \frac{1 - \tau_k^x}{2(L - \mu)} ||\nabla h(y_{k-1}) - \nabla h(y_k)||^2. 
\end{align*}
\]

Then, to build a contraction between \(||z_{k+1} - x^*||^2\) and \(||z_k - x^*||^2\), we apply Lemma 4 with \(G = \nabla f(y_k), H = \nabla h(y_k)\) and \(z^+ = z_{k+1}\), which gives

\[
(\nabla h(y_k), z_k - x^*) = \frac{\alpha_k}{2} \left( ||z_k - x^*||^2 - \left(1 + \frac{\mu}{\alpha_k}\right) ||z_{k+1} - x^*||^2 \right) + \frac{1}{2\alpha_k} ||\nabla h(y_k)||^2.
\]
We now consider the finite-sum objective (1) with
Recall that the stochastic gradient estimator of SVRG have the form (4.1 BS-SVRG et al., 2014) variant can be similarly constructed in Section 4.2.

4 Finite-sum objectives with incremental first-order oracle

∥

If

Proposition 2.1.

It remains to impose parameter constraints according to the Lyapunov function (9).

As we can see, the analysis of G-TM is much simpler and more straightforward than NAG. When the inequalities in the constraints (10) all hold as equality, we derive a simple constant parameter choice for G-TM: \( \alpha = \sqrt{L\mu} - \mu, \tau_x = \frac{\sqrt{\mu}}{\mu} - 1 \). By telescoping (11) from iteration \( K - 1 \) to 0, we obtain

\[
\frac{\mu}{2} \| z_K - x^* \|^2 \leq \left( 1 - \frac{1}{\sqrt{\kappa}} \right)^{2K} \left( h(y_1) - \frac{1}{2(L - \mu)} \| \nabla h(y_1) \|^2 \right) + \frac{\mu}{2} \| z_0 - x^* \|^2.
\]

Denoting the initial constant as \( C_0^{G-TM} \leq \frac{\sqrt{\kappa} - 1}{2\kappa} \left( h(y_1) - \frac{1}{2(L - \mu)} \| \nabla h(y_1) \|^2 \right) + \frac{\mu}{2} \| z_0 - x^* \|^2 \), if we align the initial guesses \( y_1 = x_0 \) with NAG, we have \( C_0^{G-TM} \ll C_0^{NAG} \). This guarantee concludes a \( \frac{\sqrt{\kappa}}{2} \log \left( \frac{2C_0^{G-TM}}{\mu\epsilon} \right) \)

iteration number upper bound for G-TM to achieve \( \| z_K - x^* \|^2 \leq \epsilon \), which is at least two times lower than that of NAG, and does not suffer from an additional \( \log \sqrt{\kappa} \) factor as is the case for the original TM.

3.2.1 The tightness of bound (13)

It is natural to ask how tight is the worst case guarantee (13). We show that for the quadratic function\(^5\)

\( f(x) = \frac{1}{2} x^T \begin{bmatrix} L & 0 \\ 0 & \mu \end{bmatrix} x \), G-TM converges exactly at the rate in (13). Note that for this objective, \( x^* = 0 \) and

\( h(x) - \frac{1}{2(L - \mu)} \| \nabla h(x) \|^2 \equiv 0 \), which means that this guarantee becomes \( \| z_K - x^* \|^2 \leq \left( 1 - \frac{1}{\sqrt{\kappa}} \right)^{2K} \| z_0 - x^* \|^2 \), and thus G-TM is a monotone method in this case. Expanding the recursions in Algorithm 2 we obtain the following result, and its proof is given in Appendix A.

Proposition 2.1. If \( f(x) = \frac{1}{2} x^T \begin{bmatrix} L & 0 \\ 0 & \mu \end{bmatrix} x \), the output of G-TM (with constant parameters) satisfies

\[
\| z_K - x^* \|^2 = \left( 1 - \frac{1}{\sqrt{\kappa}} \right)^{2K} \| z_0 - x^* \|^2.
\]

4 Finite-sum objectives with incremental first-order oracle

We now consider the finite-sum objective (1) with \( n \geq 1 \). We choose SVRG (Johnson and Zhang (2013) as the base algorithm to implement the acceleration trick, and we also show that an accelerated SAGA (Defazio et al., 2014) variant can be similarly constructed in Section 4.2.

4.1 BS-SVRG

Recall that the stochastic gradient estimator of SVRG have the form (\( i_k \) is sampled uniformly in \( [n] \)):

\[
G_{x_k}^{SVRG} \triangleq \nabla f_{i_k}(x_k) - \nabla f_{i_k}^\circ(x) + \nabla f^\circ(x),
\]

\(^5\)This is also the example where GD with \( \frac{\sqrt{\kappa}}{L + \mu} \) learning rate behaves exactly like its worst case analysis.
We are going to design an accelerated variant of SVRG “using” this shifted estimator. Based on the definition of the construction in G-TM, i.e.,

\[ z^k = \arg \min_x \left\{ \langle g^S_{y_k}, x \rangle + \frac{\alpha}{2} \|x - z_k^k\|^2 + (\mu/2) \|x - y_k\|^2 \right\}. \]

It is clear that we can fit this construction into Lemma 1 to build a contraction with the improved ratio, which requires us to deal with the shifted moment \( E_{i_k} \left[ \|H^S_{x_k}\|^2 \right] \). It remains to formulate the updating rules following the spirit of G-TM. For simplicity, in what follows, we only consider constant parameter choices. We dub our SVRG variant BS-SVRG, which is presented in Algorithm 3. The algorithm is designed based on the following thought experiment:

**Thought experiment** The general goal of analyzing SVRG is to establish a per-epoch contraction of the form \( T_{s+1} \leq \rho T_s \) for some Lyapunov function \( T_s \) and rate factor \( \rho < 1 \). To achieve this goal, the first step is usually to build a relation of the form \( T^s_k \leq \rho T_{s+k} \), where \( T^s_k \) is the Lyapunov function inside epoch \( s \), and then the per-epoch contraction is built by aggregating this relation in an epoch \( T_{s+1} = \frac{1}{m} \sum_{i=0}^{m-1} T^s_i \leq \rho T_s \). In view of the construction in G-TM, i.e., \( T_{k+1} \leq \rho T_k \iff y_k = \tau_s z_k + (1 - \tau_s) x_{k-1} - \tau_k (y_{k-1} - z_k) - \nabla f(y_{k-1}) \), we thus design the update of BS-SVRG (step 5 in Algorithm 3) as replacing \( y_{k-1} \) with the anchor point \( \hat{x}_s \), and we adopt a similar Lyapunov function as follows,

\[ T_s \triangleq h(\hat{x}_s) - c_1 \|\nabla h(\hat{x}_s)\|^2 + \frac{\lambda}{2} \|z^*_0 - x^*\|^2, \]

where \( c_1 \in \left[ 0, \frac{1}{2(L-\mu)} \right] \), \( \lambda > 0 \).

The analysis of BS-SVRG follows a similar routine as Theorem 2. The main difference is that there is a twist of trick on dealing with the variance. The per-epoch contraction is given in the following theorem.

**Theorem 3.** In Algorithm 3, suppose we choose \( \alpha, \tau_x, \tau_z \) under the constraints:

\[
\begin{align*}
\tau_z &= \frac{\tau_s}{\mu} - \frac{\alpha (1 - \tau_x)}{\mu (L-\mu)}, \\
(1 + \tau_x)^2 (1 - \tau_z) &\geq 4 \left( \left( \frac{\alpha}{\mu} + 1 \right) - \left( \frac{\alpha}{\mu} + \kappa \right) \tau_x \right)^2, \\
(1 + \frac{\alpha}{\mu})^2 (1 - \tau_x) &\leq 1,
\end{align*}
\]

where \( \tau_x \equiv \frac{\tau_z}{\mu} \leq 1, \) and \( \tau_x, \tau_z \) satisfy the constraints.
the following per-epoch contraction holds for the Lyapunov function defined in (16). The expectation is taken
given the information up to epoch $s$.

\[
\text{With } \lambda = \frac{\alpha^2(1 - \tau_x)}{\omega(L - \mu)} \left(1 + \frac{\mu}{\alpha}\right)^{2m}, \quad \mathbb{E}[T_{s+1}] \leq \left(1 + \frac{\mu}{\alpha}\right)^{-2m} T_s, \text{ for } s \geq 0. \tag{18}
\]

**Proof.** In order to give a clean proof, we omit the superscript $s$ for iterates in the same epoch.

Using the trick in Lemma 3 for the recursion $y_k = \tau_x z_k + (1 - \tau_x) \tilde{x}_s + \tau_x (\mu(\bar{x}_s - z_k) - \nabla f(\bar{x}_s))$ and
strengthening the convexity arguments by co-coercivity, we obtain

\[
h(y_k) \leq \frac{1 - \tau_x}{\tau_x} \langle \nabla h(y_k), \tilde{x}_s - y_k \rangle + \frac{\tau_x}{\tau_x} \langle \nabla h(y_k), \mu(\bar{x}_s - z_k) - \nabla f(\bar{x}_s) \rangle + \langle \nabla h(y_k), z_k - x^* \rangle
- \frac{1}{2(L - \mu)} \|\nabla h(y_k)\|^2.
\]

Note that here the inner product $\langle \nabla h(y_k), \tilde{x}_s - y_k \rangle$ is not upper bounded as before. This term is preserved
deal with the variance.

By the definition of $h$, $\mu(\bar{x}_s - z_k) - \nabla f(\bar{x}_s) = \mu(x^* - z_k) - \nabla h(\bar{x}_s)$. Applying Lemma 1 with $\mathcal{H} = \mathcal{H}^{SVRG}_{y_k}, \mathcal{G} = \mathcal{G}^{SVRG}_{y_k}, z^+ = z_{k+1}$ and taking expectation, we can conclude that

\[
h(y_k) \leq \frac{1 - \tau_x}{\tau_x} \langle \nabla h(y_k), \tilde{x}_s - y_k \rangle - \frac{\tau_x}{\tau_x} \langle \nabla h(y_k), \nabla h(\bar{x}_s) \rangle - \frac{1}{2(L - \mu)} \|\nabla h(y_k)\|^2
+ \frac{1 - \mu\tau_x}{\tau_x} \frac{\alpha}{2} \left(\|z_k - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}_{i_k} \left[\|z_{k+1} - x^*\|^2\right]\right)
+ \frac{1 - \mu\tau_x}{2\alpha} \mathbb{E}_{i_k} \left[\|\mathcal{H}^{SVRG}_{y_k}\|^2\right].
\]

To bound the shifted moment, we apply the co-coercivity of $h_{i_k}$, i.e.,

\[
\mathbb{E}_{i_k} \left[\|\mathcal{H}^{SVRG}_{y_k}\|^2\right] = \mathbb{E}_{i_k} \left[\|\nabla h_{i_k}(y_k) - \nabla h_{i_k}(\tilde{x}_s)\|^2\right] + 2 \langle \nabla h(y_k), \nabla h(\tilde{x}_s) \rangle - \|\nabla h(\tilde{x}_s)\|^2
\leq 2(L - \mu) \langle h(\tilde{x}_s) - h(y_k) - \langle \nabla h(y_k), \tilde{x}_s - y_k \rangle \rangle + 2 \langle \nabla h(y_k), \nabla h(\tilde{x}_s) \rangle - \|\nabla h(\tilde{x}_s)\|^2.
\]

After re-arranging the terms, we obtain

\[
h(y_k) \leq \left(1 - \frac{\mu\tau_x}{\tau_x}\right) \frac{L - \mu}{\alpha} (h(\tilde{x}_s) - h(y_k)) + \frac{1 - \tau_x}{\tau_x} \frac{L - \mu}{\alpha} \langle \nabla h(y_k), \tilde{x}_s - y_k \rangle
+ \frac{1 - \mu\tau_x}{\tau_x} \frac{\alpha}{2} \left(\|z_k - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}_{i_k} \left[\|z_{k+1} - x^*\|^2\right]\right)
+ \frac{1}{\alpha} \langle \nabla h(y_k), \nabla h(\tilde{x}_s) \rangle - \frac{1}{2(L - \mu)} \|\nabla h(y_k)\|^2
- \frac{1 - \mu\tau_x}{2\alpha} \frac{\tau_x}{\alpha} \|\nabla h(\tilde{x}_s)\|^2.
\]

To cancel $\langle \nabla h(y_k), \tilde{x}_s - y_k \rangle$, we choose $\tau_x$ (can be negative) such that $\frac{1 - \tau_x}{\tau_x} = \left(1 - \frac{\tau_x}{\tau_x}\right) \frac{L - \mu}{\alpha}$, which gives

\[
h(y_k) \leq (1 - \tau_x) h(\tilde{x}_s) + \frac{\alpha^2(1 - \tau_x)(\alpha + L)(\tau_x/2(L - \mu))}{\left(1 + \frac{\mu}{\alpha}\right)^2} \mathbb{E}_{i_k} \left[\|z_{k+1} - x^*\|^2\right]
+ \frac{\alpha + \mu}{(L - \mu)} \langle \nabla h(y_k), \nabla h(\tilde{x}_s) \rangle - \frac{\tau_x}{2(L - \mu)} \|\nabla h(y_k)\|^2
- \frac{1 - \tau_x}{2(L - \mu)} \|\nabla h(\tilde{x}_s)\|^2. \tag{19}
\]

In view of the Lyapunov function (16), there are two ways to deal with the above inequality:

**Case I (c_1 = 0):** Choosing $\tau_x$ such that $\alpha + \mu = (\alpha + L)\tau_x = 0 \Rightarrow \tau_x = \frac{\alpha + \mu}{\alpha + L}$ and dropping the negative gradient norms in (19), we arrive at (21) with $c_1 = 0$. 

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Case II ($c_1 \in \left]0, \frac{1}{2(L-\mu)}\right]$): Denoting $\gamma = \frac{\alpha + \mu - (\alpha + L)\tau_x}{L - \mu}$ and using Young’s inequality for $\langle \nabla h(y_k), \nabla h(\tilde{x}_s) \rangle$ with parameter $\beta > 0$, we can bound (19) as

$$h(y_k) \leq (1 - \tau_x)h(\tilde{x}_s) + \frac{\alpha \gamma (1 - \tau_x)}{2(L - \mu)} \left(\|z_k - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}_{i_k} \left[\|z_{k+1} - x^*\|^2\right]\right) + \left(1 - \tau_x\right) \left(\beta \gamma \frac{\tau_x}{2(L - \mu)} \|\nabla h(y_k)\|^2 - \left(\frac{1 - \tau_x}{2(L - \mu)} - \frac{\gamma}{2\beta}\right) \|\nabla h(\tilde{x}_s)\|^2\right).$$

We require $\gamma \neq 0$ and choose $\beta > 0$ such that

$$\frac{\beta \gamma}{2} - \frac{\tau_x}{2(L - \mu)} = \frac{1}{1 - \tau_x} \left(\frac{1 - \tau_x}{2(L - \mu)} - \frac{\gamma}{2\beta}\right) = c_1 > 0.$$

It can be verified that this requirement and the existence of $\beta$ are equivalent to the following constraints:

$$\begin{cases} 
\tau_x \neq \frac{\alpha + \mu}{\alpha L}, \\
(1 + \tau_x)^2(1 - \tau_x) \geq 4 \left(\frac{\alpha}{\alpha + \kappa} + 1\right) \tau_x \end{cases}.$$

Under these constraints, denoting $\Delta = \frac{(1 + \tau_x)^2}{2(L - \mu)} - \frac{4\beta^2}{\alpha - 1} \geq 0$, we can choose $\beta = \frac{1 + \tau_x}{2\gamma(L - \mu)} + \frac{\sqrt{\Delta}}{2\gamma}$, which ensures $c_1 \in \left]0, \frac{1}{2(L-\mu)}\right[$.

Let $c_2 = \frac{\alpha \gamma (1 - \tau_x)}{L - \mu}$, these two cases result in the same inequality:

$$h(y_k) - c_1 \|\nabla h(y_k)\|^2 \leq (1 - \tau_x)(h(\tilde{x}_s) - c_1 \|\nabla h(\tilde{x}_s)\|^2) + \frac{c_2}{2} \left(\|z_k - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}_{i_k} \left[\|z_{k+1} - x^*\|^2\right]\right).$$

Finally, summing the above inequality from $k = 0, \ldots, m - 1$ with weight $(1 + \frac{\mu}{\alpha})^{2k}$, we conclude that

$$\mathbb{E}\left[h(\tilde{x}_{s+1}) - c_1 \|\nabla h(\tilde{x}_{s+1})\|^2\right] = \sum_{k=0}^{m-1} \frac{1}{\omega} \left(1 + \frac{\mu}{\alpha}\right)^{2k} \mathbb{E}\left[h(y_k) - c_1 \|\nabla h(y_k)\|^2\right] \leq (1 - \tau_x)(h(\tilde{x}_s) - c_1 \|\nabla h(\tilde{x}_s)\|^2) + \frac{c_2}{2\omega} \left(\|z_0^* - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}\left[\|z_{m}^* - x^*\|^2\right]\right).$$

Imposing the constraint $(1 + \frac{\mu}{\alpha})^{2m}(1 - \tau_x) \leq 1$ completes the proof.

In what follows, we provide a simple analytical choice that satisfies constraints (17). We consider the ill-conditioned case where $\frac{m}{\kappa} \leq \frac{3}{2}$, and we fix $m = 2n$ to make it specific. In this case, Allen-Zhu (2018) derived an $O(\sqrt{\beta n} \kappa \log \frac{1}{\delta})$ expected iteration complexity\footnote{We choose the setting that is used in the analysis and experiments of Allen-Zhu (2018) to make a fair comparison.} for Katyusha (cf, Theorem 2.1, Allen-Zhu (2018)). The proof of this proposition is given in Appendix C.

**Proposition 3.1** (Ill condition). If $\frac{m}{\kappa} \leq \frac{3}{2}$, the choice $\alpha = \sqrt{c_0 \kappa L} - \mu$, $\tau_x = (1 - \frac{1}{c}) \frac{\sqrt{c_0 n}}{\sqrt{c_0 \kappa + \kappa - 1}}$, $\tau_z = \frac{c_0}{\mu} - \frac{\alpha (1 - \tau_x)}{\mu (L - \mu)}$, where $c = 2 + \sqrt{3}$, satisfies constraints (17).

Using this parameter choice in Theorem 3, we conclude an $O(\sqrt{1.87n} \kappa \log \frac{1}{\delta})$ expected iteration complexity for BS-SVRG, which is around 1.8 times lower than that of Katyusha.\footnote{We are referring to the expected number of stochastic iterations (e.g., totally $5m$ in Algorithm 3) required to achieve $\|x - x^*\|^2 \leq \epsilon$. If $m = 2n$, in average, each stochastic iteration of SVRG (and its variants) requires 1.5 oracle calls.}
Remark 3.1. We are not aware of other parameter choices of Katyusha that have faster rates. \cite{Hu et al. (2018)} made an attempt based on dissipativity theory, but no explicit rate is justified. To derive a better choice for Katyusha, significant modification to its proof is required (for its parameter $\tau_2$), which results in complicated constraints and is thus out of the scope of this paper. We believe that there could be some computer-aided ways to find better choices for both Katyusha and BS-SVRG, which we leave for future work.

For the other case where $\frac{m}{n} > \frac{3}{4}$ (i.e., $\kappa = O(n)$), almost all the accelerated and non-accelerated incremental gradient methods perform the same, at an $O(n \log \frac{1}{\epsilon})$ oracle complexity (and is indeed fast). \cite{Hannah et al. (2018)} shows that by optimizing the parameters of SVRG and SARAH, a lower $O(n + \max \{n \log (n/\kappa), \sqrt{n} \} \log \frac{1}{\epsilon})$ oracle complexity is achievable. Due to these facts, we do not optimize the parameters for this case and provide the following proposition as a basic guarantee. The proof of this proposition is given in Appendix D.

**Proposition 3.2** (Well condition). If $\frac{m}{n} > \frac{3}{4}$, suppose we choose $\alpha = \frac{2\mu}{\tau} - \mu, \tau_x = \left(1 - \frac{1}{2m}\right) \frac{3c}{2\kappa^{1/2}}, \tau_z = \frac{\tau_x}{\kappa}$, the epochs of BS-SVRG satisfy $T_{k+1} \leq \frac{1}{2} \cdot T_k$ for the Lyapunov function \cite{16} with $\lambda = \frac{2\alpha^2(1-\tau_x)}{\omega(L-\mu)}$, which implies an $O(n \log \frac{1}{\epsilon})$ expected iteration complexity.

There exists a special choice in constraints \cite{17}; by choosing $\tau_x = \frac{\alpha + \mu}{\alpha + L}$, the first inequality always holds and this leads to $c_1 = 0$ in the Lyapunov function \cite{16}. In this case, $\alpha$ can be found using numerical tools, which is summarized as follows.

**Proposition 3.3** (Numerical choice). By fixing $\tau_x = \frac{\alpha + \mu}{\alpha + L}, \tau_z = \frac{\tau_x}{\kappa}$, the optimal choice of $\alpha$ can be found by solving the equation \cite{18} $(1 + \mu)^2 (1 - \alpha/\mu - 1/\alpha + \mu/\alpha + L) = 1$ using numerical tools, and this equation has a unique positive root.

Comparing with Katyusha, BS-SVRG has a simpler scheme which only requires storing one variable vector $\{z_k\}$ similar to MiG \cite{Zhou et al. (2018)} and Varag \cite{Lan et al. (2019)}. And BS-SVRG achieves the fastest rate among these accelerated SVRG variants.

### 4.2 Accelerated SAGA variant

Recall that the stochastic gradient estimator of SAGA has the form \cite{Defazio et al. (2014)}:

$$G_{x_k}^{\text{SAGA}} \triangleq \nabla f_i(x_k) - \nabla f_i(\varphi_{i,k}^k) + \frac{1}{n} \sum_{i=1}^n \nabla f_i(\varphi_{i,k}^i),$$

where $\varphi^k \in \mathbb{R}^{d \times n}$ is a table that stores $n$ randomly previously computed anchor points. Following our methodology, we define the shifted SAGA estimator:

$$H_{x_k}^{\text{SAGA}} \triangleq \nabla h_i(x_k) - \nabla h_i(\varphi_{i,k}^k) + \frac{1}{n} \sum_{i=1}^n \nabla h_i(\varphi_{i,k}^i).$$

It can be verified that they satisfy the following relation:

$$H_{x_k}^{\text{SAGA}} = \left[ G_{x_k}^{\text{SAGA}} - \mu \left( \frac{1}{n} \sum_{i=1}^n \varphi_{i,k}^i - \varphi_{i,k}^k \right) \right] - \mu (x_k - x^*).$$

Then, we can treat $G_{x_k}^{\text{SAGA}} - \mu \left( \frac{1}{n} \sum_{i=1}^n \varphi_{i,k}^i - \varphi_{i,k}^k \right)$ as the stochastic gradient estimator to design our SAGA variant. By conducting a similar thought experiment as in the previous section, we can design the recursion (updating rule of the table) as $\varphi_{i,k}^{k+1} = \tau_x z_k + (1 - \tau_x) \varphi_{i,k}^k + \tau_z (\mu (\frac{1}{n} \sum_{i=1}^n \varphi_{i,k}^i - z_k) - \frac{1}{n} \sum_{i=1}^n \nabla f_i(\varphi_{i,k}^i))$. We found that for the resulting scheme, it is possible to build a per-iteration contraction under some parameter constraints using the following Lyapunov function (with $c_1 \in \left[0, \frac{1}{\omega(L-\mu)}\right], \lambda > 0$):

$$T_k = \frac{1}{n} \sum_{i=1}^n h_i(\varphi_{i,k}^k) - c_1 \left\| \frac{1}{n} \sum_{i=1}^n \nabla h_i(\varphi_{i,k}^k) \right\|^2 + \frac{\lambda}{2} \|z_k - x^*\|^2.$$
We dub the proposed variant BS-Point-SAGA, which is presented in Algorithm 4. Recall that the Lyapunov function used in Point-SAGA has the form (cf. Theorem 5, (Defazio, 2016)): 

\[
\sum_{i=1}^{n} \left( h_i(\phi_i^k) - c_1 \| \nabla h_i(\phi_i^k) \|^2 \right) = \frac{1}{n} \sum_{i=1}^{n} \nabla h_i(\phi_i^k) - c_1 \| \nabla h_i(\phi_i^k) \|^2 \geq 0.
\]

A similar accelerated rate can be derived for the SAGA variant and its parameter choice shows some interesting correspondence between the variants of SVRG and SAGA. However, since its updating rules require explicit knowledge of the point table, the scheme has an \(O(nd)\) memory complexity. We provide this variant in Appendix E for interested readers.

### 5 Finite-sum objectives with incremental proximal point oracle

We consider the finite-sum objective (1) and assume that the proximal operator oracle \(\text{prox}_{\alpha}^f(\cdot)\) of each \(f_i\) is available. Point-SAGA (Defazio, 2016) is the typical method that utilizes this oracle, and it achieves the same \(O((n + \sqrt{mk}) \log \frac{1}{\epsilon})\) expected iteration complexity. Although in general, the incremental proximal operator oracle is much more expensive than the incremental gradient oracle, Point-SAGA is interesting in the following aspects: (1) It has a simple scheme with only 1 parameter; (2) Its analysis is elegant and tight, which does not require any Young’s inequality; (3) For problems where the proximal point oracle has an analytical solution (e.g., ridge regression), it has a very fast rate (i.e., its expected rate factor is smaller than \(1 - \frac{1}{n}\)), which is faster than both Katyusha and BS-SVRG.

It might seem surprising that by shifting objective, the rate factor of Point-SAGA can be further boosted. We dub the proposed variant BS-Point-SAGA, which is presented in Algorithm 4. Recall that the Lyapunov function used in Point-SAGA has the form (cf. Theorem 5, (Defazio, 2016):

\[
T_{k}^{\text{Point-SAGA}} = \frac{c}{n} \sum_{i=1}^{n} \| \nabla f_i(\phi_i^k) - \nabla f_i(x^*) \|^2 + \| x_k - x^* \|^2.
\]

We adopt a shifted version of this Lyapunov function (with \(\lambda > 0\)):

\[
T_k = \lambda \cdot \frac{1}{n} \sum_{i=1}^{n} \| \nabla h_i(\phi_i^k) \|^2 + \| x_k - x^* \|^2.
\]

The analysis of BS-Point-SAGA is a direct application of Lemma 2. We build its per-iteration contraction in the following theorem.

**Theorem 4.** In Algorithm 4, if we choose \(\alpha\) as the (unique) positive root of the cubic equation

\[
2 \left( \frac{\alpha}{\mu} \right)^3 - (4n - 6) \left( \frac{\alpha}{\mu} \right)^2 - (2n\kappa + 4n - 6) \left( \frac{\alpha}{\mu} \right) - (n\kappa + n - 2) = 0,
\]

\[
\alpha = \frac{4n - 6 \pm \sqrt{16n^2 - 16n + 48n\kappa + 36n - 72}}{4(n - 1)}
\]

**Algorithm 4** Point-SAGA Boosted by Shifting objective (BS-Point-SAGA)

**Input:** Parameters \(\alpha > 0\) and initial guess \(x_0 \in \mathbb{R}^d\), iteration number \(K\).

**Initialize:** A point table \(\phi_i^0 \in \mathbb{R}^{d \times n}\) with \(\forall i \in [n], \phi_i^0 = x_0\), running averages for the point table and its gradients.

1: for \(k = 0, \ldots, K - 1\) do
2: Sample \(i_k\) uniformly in \([n]\).
3: Update \(x\):
   \[
z_k = x_k + \frac{1}{\alpha} \left( \nabla f_{i_k}(\phi_{i_k}^k) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\phi_i^k) + \mu \left( \frac{1}{n} \sum_{i=1}^{n} \phi_i^k - \phi_{i_k}^k \right) \right),
   \]
   \[
x_{k+1} = \text{prox}_{\alpha}^f(z_k).
\]
4: Set \(\phi_{i_k}^{k+1} = x_{k+1}\) and keep other entries unchanged (i.e., for \(i \neq i_k\), \(\phi_i^{k+1} = \phi_i^k\)). Update the running averages according to the change in \(\phi^{k+1}\) (note that \(\nabla f_{i_k}(\phi_{i_k}^{k+1}) = \alpha(z_k - x_{k+1})\)).
5: end for

**Output:** \(x_K\).
the following per-iteration contraction holds for the Lyapunov function defined in (23),

\[ \lambda = \frac{n}{\alpha^2} + \frac{2(\alpha + \mu)(n-1)}{\alpha^2(L-\mu)}, \]

\[ \mathbb{E}_k[T_{k+1}] \leq \left(1 + \frac{\mu}{\alpha}\right)^{-2} T_k, \text{ for } k \geq 0. \]

And the root of (24) satisfies \( \frac{\alpha}{\mu} = O(n + \sqrt{n\kappa}) \).

**Proof.** Using Lemma 2 with the relations

\[ x_{k+1} = \text{prox}_{\alpha k} \left(x_k + \frac{1}{\alpha} \left(\nabla f_i(x_k^k) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_k) + \mu \left(\frac{1}{n} \sum_{i=1}^{n} \phi_i^k - \phi_i^{k+1}\right)\right)\right), \]

\[ x^* = \text{prox}_{\alpha k} \left(x^* + \frac{1}{\alpha} \nabla f_i(x^*)\right) \text{ and } \phi_i^{k+1} = x_{k+1}, \]

based on that \( \nabla h_i(x) = \nabla f_i(x) - \nabla f_i(x^*) - \mu(x - x^*) \), we have

\[ \left(1 + \frac{2(\alpha + \mu)}{L-\mu}\right) \left\|\nabla h_i(\phi_{i}^{k+1})\right\|^2 + (\alpha + \mu)^2 \|x_{k+1} - x^*\|^2 \leq \alpha^2 \left\|x_k - x^* + \frac{1}{\alpha} \left(\nabla h_i(\phi_{i}^{k}) - \frac{1}{n} \sum_{i=1}^{n} \nabla h_i(\phi_{i}^{k})\right)\right\|^2. \]

Expanding the right side, taking expectation and using that \( \mathbb{E}\left[\|X - EX\|^2\right] \leq \mathbb{E}\left[\|X\|^2\right] \), we obtain

\[ \left(1 + \frac{2(\alpha + \mu)}{L-\mu}\right) \mathbb{E}_k \left[\left\|\nabla h_i(\phi_{i}^{k+1})\right\|^2\right] + (\alpha + \mu)^2 \mathbb{E}_k \left[\|x_{k+1} - x^*\|^2\right] \leq \alpha^2 \|x_k - x^*\|^2 + \frac{1}{\alpha} \sum_{i=1}^{n} \left\|\nabla h_i(\phi_{i}^{k})\right\|^2. \]

Note that by construction,

\[ \mathbb{E}_k \left[\sum_{i=1}^{n} \left\|\nabla h_i(\phi_{i}^{k+1})\right\|^2\right] = \frac{n-1}{n} \sum_{i=1}^{n} \left\|\nabla h_i(\phi_{i}^{k})\right\|^2 + \mathbb{E}_k \left[\left\|\nabla h_i(\phi_{i}^{k+1})\right\|^2\right]. \]

We can thus arrange the terms as

\[ \left(\frac{n}{\alpha^2} + \frac{2(\alpha + \mu)n}{\alpha^2(L-\mu)}\right) \mathbb{E}_k \left[\sum_{i=1}^{n} \left\|\nabla h_i(\phi_{i}^{k+1})\right\|^2\right] + \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}_k \left[\|x_{k+1} - x^*\|^2\right] \leq \left(\frac{n}{\alpha^2} + \frac{2(\alpha + \mu)(n-1)}{\alpha^2(L-\mu)}\right) \cdot \frac{1}{n} \sum_{i=1}^{n} \left\|\nabla h_i(\phi_{i}^{k})\right\|^2 + \|x_k - x^*\|^2. \]

In view of the Lyapunov function at (23), we choose \( \alpha \) to be the positive root of the following equation:

\[ \left(1 + \frac{\mu}{\alpha}\right)^2 \left(1 - \frac{2(\alpha + \mu)}{n(L-\mu) + 2n(\alpha + \mu)}\right) = 1. \]

Let \( q = \frac{\alpha}{\mu} > 0 \), the above is a cubic equation:

\[ s(q) \triangleq 2q^3 - (4n - 6)q^2 - (2n\kappa + 4n - 6)q - (n\kappa + n - 2) = 0, \]

which has a unique positive root (denoted as \( q^* \)).

Note that \( s(-\infty) < 0, s(-\frac{3}{2}) = \frac{1}{4} \) and \( s(0) \leq 0 \). These facts suggest that if for some \( u > 0 \), \( s(u) > 0 \), we have \( q^* < u \). It can be verified that \( s(2n + \sqrt{n\kappa}) > 0 \), and thus \( q^* = O(n + \sqrt{n\kappa}) \).
The expected worst case rate factor of BS-Point-SAGA can be minimized by solving the cubic equation \( (24) \) exactly. The analytical solution of \((24)\) is messy, but it can be easily calculated using numerical tools. In Figure 1, we numerically compare the rate factors of Point-SAGA and BS-Point-SAGA. When \( \kappa \) is large, the rate factor of BS-Point-SAGA is close to the square of the rate factor of Point-SAGA, which implies an almost 2 times lower expected iteration complexity. In terms of memory requirement, BS-Point-SAGA has an undesired \( O(nd) \) complexity since the update of \( x_{k+1} \) involves \( \phi^d_{k} \). Nevertheless, it achieves the fastest known rate for finite-sum problems (if both \( L \) and \( \mu \) are known), and we present it as an interesting instance of our design methodology.

6 Evaluations

Since a faster worst case rate does not necessarily implies a better empirical performance (it could be the case that the slower one is loose), we provide some experimental results of the proposed methods in this section. We evaluate them in the ill-conditioned case where the problem has a huge \( \kappa \). In this case, the performance gap between accelerated and non-accelerated algorithms is clear. We ran experiments on an HP Z440 machine with a single Intel Xeon E5-1630v4 with 3.70GHz cores, 16GB RAM, Ubuntu 18.04 LTS with GCC 4.8.0, MATLAB R2017b.

We start with evaluating the deterministic methods, i.e., NAG, TM and G-TM. We first did a simulation on the interesting quadratic objective \( f(x) = \frac{1}{2}x^T \begin{bmatrix} L & 0 \\ 0 & \mu \end{bmatrix} x \) mentioned in Section 3.2.1, which also serves as a justification to Proposition \((2.1)\). In this simulation, the default parameter choices were used and all the methods were initialized at \((-100, 100)\). We plot their convergences and theoretical guarantees (marked with “UB”) in Figure 2a (the bound for TM is not shown due to the initial state issue). This simulation shows that after the first iteration, TM and G-TM have the same rate, and the initial state issue of TM can make it slower than NAG. It also suggests that the guarantee of NAG is loose.

Then, we measured their performance on real world datasets from LIBSVM (Chang and Lin, 2011). The task we chose is binary logistic regression with the objective \( \frac{1}{n} \sum_{i=1}^{n} \log (1 + \exp (-b_i \langle a_i, x \rangle)) + \frac{\mu}{2} \| x \|^2 \), where \( a_i \in \mathbb{R}^d, b_i \in \{-1, +1\}, \forall i \in [n] \). We normalized the datasets and thus for this problem, \( L = 0.25 + \mu \). For real world tasks, we track function value suboptimality, which is easier to compute than \( \| x - x^* \|^2 \) in practice. The result is given in Figure 2b. In the first 30 iterations, TM is slower than G-TM due to the initial state issue. After that, they are almost identical and are faster than NAG.

We then evaluate BS-SVRG on the same problem, which can fully utilize the finite-sum structure. We evaluated two parameter choices of BS-SVRG: (1) the analytical choice given in Proposition 3.1 (marked as “BS-SVRG"); (2) the numerical choice given in (3.3) (marked as “BS-SVRG-N".). We selected SAGA \((\gamma = \frac{1}{2(\mu \kappa + L)}, \text{Defazio et al.} 2014)\) and Katyusha \((\gamma_2 = \frac{1}{2}, \gamma_1 = \sqrt{\frac{\mu}{\kappa L}}, \alpha = \frac{1}{\sqrt{\kappa L}}, \text{Zhou et al.} 2019\) with their default parameter choices as the baselines due to the following reasons: SAGA has low iteration cost and good empirical performance with support for non-smooth regularizers, and is thus implemented in machine learning libraries such as scikit-learn (Pedregosa et al. 2011); Katyusha achieves the state-of-the-art performance for ill-conditioned problems. Since SAGA and SVRG-like algorithms have different iteration complexities, we plot the curve with respect to the number of data passes. The results are given in Figure 3. In the experiment on a9a dataset, both choices of BS-SVRG perform well after 100 passes. The issue of their early stage performance can be eased by outputting the anchor point \( \tilde{x} \) instead, as shown in Figure 3a (right).

Zhou et al. (2019) shows that SSNM can be faster than Katyusha in some cases. In theory, SSNM and Katyusha achieve the same rate if we set \( m = n \) for Katyusha (both require 2 oracle calls per-iteration). In practice, if \( m = n \), they have similar performance with SSNM being slightly faster. Considering the stability and memory requirement, Katyusha still achieves the state-of-the-art performance both theoretically and empirically.
We also conducted an empirical comparison between BS-Point-SAGA and Point-SAGA in Figure 2c. Their analytical parameter choices were used. We chose ridge regression: $\frac{1}{n} \sum_{i=1}^{n} (a_i, x) + b_i^2 + \frac{\mu}{2} \|x\|^2$ as the task since its proximal operator has a closed form solution (see Appendix A in Defazio (2016)). For this objective, after normalizing the dataset, we have $L = 1 + \mu$. The performance of SAGA is also plotted as a reference. The result shows that BS-Point-SAGA is only slightly faster than Point-SAGA, which suggests that the guarantee of Point-SAGA is loose.

7 Conclusions

In this work, we focused on unconstrained strongly convex problems and proposed to design schemes for a shifted objective function. Lemma 1 and 2 are the cornerstones for the new designs. We proposed G-TM, BS-SVRG (and BS-SAGA) and BS-Point-SAGA based on these two lemmas. The new schemes achieve faster worst case rates and have simpler proofs compared with their existing counterparts. Experiments on machine learning tasks show some improvement of the proposed methods.

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A Proof of Proposition 2.1

First, we can write the $k$th-update of G-TM with constant parameter as

$$y_k = (\tau_x - \tau_z \mu) z_k + (1 - (\tau_x - \tau_z \mu)) y_{k-1} - \tau_z \nabla f(y_{k-1}),$$

$$z_{k+1} = \frac{\alpha}{\alpha + \mu} z_k + \frac{\mu}{\alpha + \mu} y_k - \frac{1}{\alpha + \mu} \nabla f(y_k).$$

Substituting the constant parameter choice, we obtain

$$y_k = \frac{2}{\sqrt{\kappa} + 1} z_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \left( y_{k-1} - \frac{1}{L} \nabla f(y_{k-1}) \right),$$

$$z_{k+1} = \left( 1 - \frac{1}{\sqrt{\kappa}} \right) z_k + \frac{1}{\sqrt{\kappa}} y_k - \frac{1}{\sqrt{L\mu}} \nabla f(y_k).$$

For the objective function $f(x) = \frac{1}{2} x^T \begin{bmatrix} L & 0 \\ 0 & \mu \end{bmatrix} x$, the update can be further expanded as

$$y_k = \frac{2}{\sqrt{\kappa} + 1} z_k + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{\kappa}} \\ 0 \end{bmatrix} y_{k-1},$$

$$z_{k+1} = \left( 1 - \frac{1}{\sqrt{\kappa}} \right) z_k + \begin{bmatrix} -\frac{1}{\sqrt{\kappa}} \\ 0 \end{bmatrix} y_k.$$

Thus,

$$z_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} z_k \implies \|z_{k+1} - x^*\|^2 = \left( 1 - \frac{1}{\sqrt{\kappa}} \right)^2 \|z_k - x^*\|^2,$$

as desired.

B Generality of the framework of Algorithm 2

First, we show that the original TM is a parameterization of Algorithm 1 (and output $z$ instead of $x$). Note that TM has the following scheme (notations follow the one in (Cyrus et al., 2018)):

$$x_{k+1} = x_k + \beta (x_k - x_{k-1}) - \alpha \nabla f(y_k),$$

$$y_{k+1} = x_{k+1} + \gamma (x_{k+1} - x_k),$$

$$z_{k+1} = x_{k+1} + \delta (x_{k+1} - x_k).$$

By casting this scheme into the framework of Algorithm 1 we obtain

$$y_k = \frac{\gamma}{\delta} z_k + \left( 1 - \frac{\gamma}{\delta} \right) x_k,$$

$$z_{k+1} = \frac{\beta (1 + \delta) - \gamma}{\delta - \gamma} z_k + \frac{\delta - \beta (1 + \delta)}{\delta - \gamma} y_k - \alpha (1 + \delta) \nabla f(y_k),$$

$$x_{k+1} = \frac{1}{1 + \delta} z_{k+1} + \frac{\delta}{1 + \delta} x_k.$$

Substituting the parameter choice of TM, we see that TM is equivalent to choosing $\alpha = \sqrt{L\mu} - \mu, \tau_y = (\sqrt{\kappa} + 1)^{-1}, \tau_x = \frac{2\sqrt{\kappa} - 1}{\kappa}$ in Algorithm 1. Interestingly, this choice and the choice of NAG only differ in $\tau_x$. Then, in Algorithm 1 by expanding the convex combinations of sequences $\{y_k\}$ and $\{x_k\}$, we can conclude that

$$y_k = \tau_x z_k + (1 - \tau_x) y_{k-1} + \tau_y (1 - \tau_x) (z_k - z_{k-1}).$$
Based on the optimality condition at iteration $k - 1$, we have

$$\alpha(z_k - z_{k-1}) = \mu(y_{k-1} - z_k) - \nabla f(y_{k-1}).$$

Now, it is clear that Algorithm 1 is an instance of the framework of Algorithm 2 with the variable parameter choice (let $y_{-1} = x_0$): at $k = 0$, $\tau^x_0 = \tau^y_0, \tau^\alpha_0 = 0$; at $k \geq 1$, $\tau^x_k = \tau^y_k = \tau^\alpha_k = \tau^{(1-\alpha)}_0$.

## C  Proof of Proposition 3.1

The choice \[
\begin{align*}
\alpha &= \sqrt{c m \mu L} - \mu \\
\tau_x &= (1 - \frac{1}{c \kappa}) \frac{\alpha + \mu}{\alpha + L} = (1 - \frac{1}{c \kappa}) \frac{\sqrt{c m \kappa}}{\sqrt{c m \kappa} + \kappa - 1}
\end{align*}
\] and the constraints are pasted here for reference.

\[
\begin{align*}
\left\{ \begin{array}{l}
(1 + \tau_x)^2 (1 - \tau_x) \geq 4 \left( \left( \frac{\alpha}{\mu} + 1 \right) - \left( \frac{\alpha}{\mu} + \kappa \right) \tau_x \right)^2 , \\
\left( 1 + \frac{4}{\alpha} \right)^{2m} (1 - \tau_x) &\leq 1.
\end{array} \right.
\end{align*}
\] (25)

(26)

Note that for $m \in (0, \frac{2}{3}]$, $\tau_x = \frac{c \kappa - 1}{c \kappa + \sqrt{4 c \kappa - 4}}$ increases monotonically and $\frac{1 + \tau_x}{m}$ decreases monotonically as $m$ increases. Thus, for the constraint (25), let

$$\phi(m, \kappa) \triangleq \frac{(1 + \tau_x)^2 (1 - \tau_x)}{ \left( \left( \frac{2}{\mu} + 1 \right) - \left( \frac{2}{\mu} + \kappa \right) \tau_x \right)^2 } = \frac{1 + \tau_x}{m} (1 - \tau_x^2) c \kappa,$$

we have $\phi(m, \kappa)$ decreases monotonically as $m$ increases.

When $m = \frac{2}{3} \kappa$, $\tau_x = \frac{c \kappa - 1}{c \kappa + \sqrt{4 c \kappa - 4}}$. For $\kappa \geq 1$, if $c + \sqrt{\frac{4 c}{3}} - c \sqrt{\frac{4 c}{3}} \leq 0 \Leftrightarrow c \geq \frac{(\sqrt{3} + \sqrt{10})^2}{16} \approx 2.319$, we have $\tau_x$ decreases monotonically as $\kappa$ increases. In this case, let $\kappa \rightarrow \infty$, we conclude that $\tau_x > \frac{c}{c + \sqrt{4 c}} \geq \frac{1}{3}$, which implies that $(1 + \tau_x)^2 (1 - \tau_x)$ increases monotonically as $\tau_x$ decreases. Thus,

$$\phi(m, \kappa) \geq \phi \left( \frac{3}{4} \kappa, \kappa \right) \geq \phi \left( \frac{3}{4}, 1 \right) = \frac{4}{3} \left( 1 - \left( \frac{c - 1}{c} \right)^2 \right) c.$$

To meet the constraint (25), we require $c \geq 2 + \sqrt{3} \approx 3.74$.

For constraint (26), define

$$\psi(m, \kappa) \triangleq \left( \frac{\alpha + \mu}{\alpha} \right)^{2m} (1 - \tau_x) = \left( 1 + \frac{1}{\sqrt{c m \kappa} - 1} \right)^{2m} \frac{\sqrt{c m \kappa} + c \kappa (\kappa - 1)}{\sqrt{c m \kappa} - 1 + \kappa) c \kappa},$$

we have $\frac{\partial \psi}{\partial m} =$

$$\left( 1 + \frac{1}{\sqrt{c m \kappa} - 1} \right)^{2m} \left[ 2 \ln \left( 1 + \frac{1}{\sqrt{c m \kappa} - 1} \right) - \frac{1}{\sqrt{c m \kappa} - 1} \right] \frac{\sqrt{c m \kappa} + c \kappa (\kappa - 1)}{\sqrt{c m \kappa} - 1 + \kappa) c \kappa} = \frac{(\kappa - 1)(c \kappa - 1)}{2 \sqrt{c m \kappa} \left( \sqrt{c m \kappa} - 1 + \kappa \right)^2}.$$ 

Denote $q = \sqrt{c m \kappa} - 1 > 0$, the roots of $\frac{\partial \psi}{\partial m}$ are identified by the following equation:

$$s(q) \triangleq 2 \ln \left( \frac{1 + \frac{1}{q}}{q} - \frac{b_0}{(q + 1)(q + \kappa)(q + b_1)} \right) = 0,$$

where $b_0 = \frac{c}{q} (\kappa - 1)(c \kappa - 1), b_1 = 1 + c \kappa (\kappa - 1)$. Taking derivative, we see that when $q \rightarrow 0, s'(q) \geq \frac{2}{q} - \frac{2}{q(1+q)} \rightarrow \infty$. We can arrange the equation $s'(q) = 0$ as finding the real roots of a polynomial. By
Descartes’ rule of signs, this equation has exactly one positive root (with \( c \geq 2 + \sqrt{3} \), we have \( \kappa b_1 - 1 - b_0 \leq 0 \) for any \( \kappa \geq 1 \) and then there is exactly one sign change in the polynomial). Thus, as \( q \) increases, \( s(q) \) first increases monotonically to the unique root and then decreases monotonically.

To see that \( s(q) \) has exactly one root, let \( q \rightarrow 0, s(q) \leq 2 \ln \left(1 + \frac{1}{q}\right) - \frac{1}{q} \rightarrow -\infty \); when \( q \) is large enough (e.g., \( q > 2 \) and \( (q + \kappa)(q + b_1) > 2b_0 \)), \( s(q) > 0 \); let \( q \rightarrow \infty, s(q) \rightarrow 0 \). These facts suggest that \( s(q) \) has a unique root. Thus, we conclude that, as \( m \) increases, \( \psi(m, \kappa) \) first decreases monotonically to the unique root and then increases monotonically, which means that for \( m \in [2, \frac{3}{4} \kappa] \), \( \psi(m, \kappa) \leq \max \{ \psi(2, \kappa), \psi \left( \frac{3}{4} \kappa, \kappa \right) \} \).

For \( \psi(2, \kappa), \psi'(2, \kappa) = \left(1 + \frac{1}{\sqrt{3} \kappa - 1}\right)^4 \left(\sqrt{2 \kappa c} + \kappa - 1\right)^{-2} \left(\sqrt{2 \kappa c} - 1\right)^{-1} \ell(\kappa) \), where \( \ell(\kappa) \) is a polynomial:

\[
\ell(\kappa) \triangleq (c - 2) \kappa - \frac{5 \sqrt{2c}}{2} \kappa^2 + (c + 1) - \left(\frac{c}{2} + \frac{1}{\sqrt{2c}}\right) \kappa^{-\frac{3}{2}} - 3 \kappa^{-1} + \frac{3}{\sqrt{2c}} \kappa^{-\frac{3}{2}}.
\]

It can be verified that with \( c \geq 2 + \sqrt{3} \), for any \( \kappa \geq \frac{3}{4}, \ell(\kappa) > 0 \), which suggests that \( \psi(2, \kappa) \leq \max \{ \psi \left( \frac{2}{4}, \kappa \right), \psi(2, \infty) \} \leq 1 \) (with \( c \geq 2 + \sqrt{3}, \psi \left( \frac{2}{4}, \kappa \right) \leq 0.953 \) and \( \psi(2, \infty) = 1 \)).

For \( \psi \left( \frac{3}{4} \kappa, \kappa \right), \psi' \left( \frac{3}{4} \kappa, \kappa \right) = \left(1 + \frac{2}{\sqrt{3} \kappa - 2}\right)^{\frac{3}{4} \kappa} \left(\left(c + \frac{4}{3} \kappa\right) \kappa - \frac{4}{3} \kappa^{-1}\right)^{-1} \omega_1(\kappa) \), where

\[
\omega_1(\kappa) \triangleq \left(\ln \left(1 + \frac{2}{\sqrt{3} \kappa - 2}\right) - \frac{2}{\sqrt{3} \kappa - 2}\right) \left(\sqrt{3} \kappa - \sqrt{3} c + \frac{3}{2}\right) + \frac{\sqrt{\frac{4c}{3} c - c - \frac{4c}{3} \kappa}}{\left(c + \frac{4c}{3}\right) \kappa - \frac{4c}{3}}.
\]

Let \( p = \sqrt{3} \kappa - 2 > 0 \), the roots of \( \omega_1(\kappa) \) are determined by the equation

\[
\omega_2(p) \triangleq \ln \left(1 + \frac{2}{p}\right) - \frac{2}{p} + \frac{3}{p + \frac{4}{2 + \sqrt{3} \kappa}} \left(\frac{4c}{3} c - c - \frac{4c}{3} \kappa\right) = 0.
\]

To ensure that \( \omega_2(p) \) increases monotonically as \( p \) increases, it suffices to set \( c \leq 3.817 \) (which ensures that \( \omega_2'(p) > 0 \)). Thus, for any \( p > 0, \omega_2(p) \leq \lim_{p \rightarrow \infty} \omega_2(p) = 0 \Rightarrow \) for any \( \kappa \geq 1, \omega_1(\kappa) \leq 0 \). Finally, we conclude that with \( 3.817 \geq c \geq 2 + \sqrt{3}, \psi \left( \frac{3}{4} \kappa, \kappa \right) \leq \psi(2, \frac{3}{4}) \leq 0.953 \), which completes the proof.

## D Proof of Proposition 3.2

The choice \( \alpha = \frac{3L}{2} - \mu, \tau_x = \left(1 - \frac{1}{6m}\right) \frac{2 + \mu}{2 + \alpha} = \left(1 - \frac{1}{6m}\right) \frac{3 \kappa}{5 \kappa - 2} \) is pasted here for reference.

We examine the constraint \( (1 + \tau_x)^2 (1 - \tau_x) \geq 4 \left(\frac{2}{p} + \mu \right) - \left(\frac{2}{p} + \kappa \right) \tau_x \geq 2 \). Let

\[
\phi(m, \kappa) \triangleq \frac{(1 + \tau_x)^2 (1 - \tau_x)}{4 \left(\frac{2}{p} + \mu \right) - \left(\frac{2}{p} + \kappa \right) \tau_x} \geq \frac{1 + \tau_x)^2 (1 - \tau_x)4m^2}{\kappa^2}.
\]

For \( m \geq \frac{3}{4} \kappa \), we have \( \tau_x \) and \( (1 - \tau_x)m \) increases monotonically as \( m \) increases. Thus, \( \phi(m, \kappa) \) increases as \( m \) increases \( \Rightarrow \phi(m, \kappa) \geq \phi \left( \frac{3}{4} \kappa, \kappa \right) \).

\( \phi \left( \frac{3}{4} \kappa, \kappa \right) = \frac{4}{3} \left(1 + \tau_x \right)^2 \left(1 - \tau_x \right) \) and \( \tau_x = \frac{9a - 2}{15m} \) in this case. Note that for \( \kappa \geq 1, \tau_x \) decreases as \( \kappa \) increases and let \( \kappa \rightarrow \infty \), we conclude that \( \tau_x > \frac{3}{4} \geq \frac{3}{4} \Rightarrow \) \( (1 + \tau_x)^2 (1 - \tau_x) \) increases as \( \tau_x \) decreases. Thus, \( \phi \left( \frac{3}{4} \kappa, \kappa \right) \geq \phi \left( \frac{3}{4}, 1 \right) > 1 \), the constraint is satisfied.

Using this choice, we can write the per-epoch contraction [22] in Theorem 3 as

\[
\begin{align*}
\mathbb{E}[h(\tilde{x}_{s+1}) - c_1 \| \nabla h(\tilde{x}_{s+1}) \|^2] &+ \alpha^2(1 - \tau_x) \frac{2m}{2c(L - \mu)} \left(1 + \frac{\mu}{\alpha}\right)^2 \mathbb{E}[\| z_{s+1} - x^* \|^2] \\
&\leq (1 - \tau_x) (h(\tilde{x}_s) - c_1 \| \nabla h(\tilde{x}_s) \|^2) + \sqrt{\alpha^2(1 - \tau_x) \frac{2m}{2c(L - \mu)} \| z_s - x^* \|^2}.
\end{align*}
\]
the following per-iteration contraction holds for the Lyapunov function defined in (27).

In Algorithm 5, suppose we choose 

Theorem E.1. In Algorithm 5, suppose we choose \( \alpha, \tau_x, \tau_z \) as

\[
\begin{align*}
\alpha & \text{ is solved from the equation } (1 + \frac{\mu}{\alpha})^2 \left( 1 - \frac{\alpha + \mu}{\alpha + L m} \right) = 1, \\
\tau_x & = \frac{\alpha + L}{\alpha + \mu}, \\
\tau_z & = \frac{\alpha (1 - \tau_x)}{\mu (L - \mu)},
\end{align*}
\]

the following per-iteration contraction holds for the Lyapunov function defined in (27) (with \( c_1 = 0 \)),

\[
E_{i_k} \left[ T_{k+1} \right] \leq \left( 1 + \frac{\mu}{\alpha} \right)^{-2} T_k, \text{ for } k \geq 0.
\]

Regard the rate, from (28), we can figure out that \( \alpha \) is the unique positive root of the cubic equation:

\[
\left( \frac{\alpha}{\mu} \right)^3 - (2n - 3) \left( \frac{\alpha}{\mu} \right)^2 - (2n + n - 3) \left( \frac{\alpha}{\mu} \right) - (n \kappa - 1) = 0.
\]
Using a similar argument as in Theorem 4, we can show that $\mathcal{L} = O(n + \sqrt{n\kappa})$, and thus conclude an $O((n + \sqrt{n\kappa}) \log \frac{1}{\delta})$ expected complexity for BS-SAGA. Interestingly, this rate is always slightly slower than that of BS-Point-SAGA.

### E.1 Proof of Theorem E.1

In order to simplify the notations in this proof, we let $\Phi^k \triangleq \frac{1}{n} \sum_{i=1}^{n} h_i(\phi^k_i)$ and $\nabla \Phi^k \triangleq \frac{1}{n} \sum_{i=1}^{n} \nabla h_i(\phi^k_i)$.

Using the trick in Lemma 3 (with $f = h_{i,k}$) for $\phi_{ik}^{k+1}$, strengthening the convexity with co-coercivity and taking expectation, we obtain

$$
\mathbb{E}_{i,k} \left[ h_{i,k}(\phi_{ik}^{k+1}) \right] \leq \frac{1 - \tau_x}{\tau_x} \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \phi_{ik}^k - \phi_{ik}^{k+1} \rangle + \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), z_k - x^* \rangle \right] \right]
+ \frac{\tau_x}{\tau_x} \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \mu(\bar{\phi}^k - z_k) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\phi_i^k) \rangle \right]
- \frac{1}{2(L - \mu)} \mathbb{E}_{i,k} \left[ \| \nabla h_{i,k}(\phi_{ik}^{k+1}) \|^2 \right].
$$

Note that by the definition of $h_{i,k}, \mu(\bar{\phi}^k - z_k) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\phi_i^k) = \mu(x^* - z_k) - \nabla \Phi^k$, and thus

$$
\mathbb{E}_{i,k} \left[ h_{i,k}(\phi_{ik}^{k+1}) \right] \leq \frac{1 - \tau_x}{\tau_x} \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \phi_{ik}^k - \phi_{ik}^{k+1} \rangle \right] - \frac{\tau_x}{\tau_x} \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \nabla \Phi^k \rangle \right]
+ \left( 1 - \frac{\mu \tau_x}{\tau_x} \right) \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), z_k - x^* \rangle \right]
- \frac{1}{2(L - \mu)} \mathbb{E}_{i,k} \left[ \| \nabla h_{i,k}(\phi_{ik}^{k+1}) \|^2 \right],
$$

which also uses Jensen’s inequality, i.e., $\mathbb{E}_{i,k} \left[ \| \nabla h_{i,k}(\phi_{ik}^{k+1}) \|^2 \right] \geq \mathbb{E}_{i,k} \left[ \nabla h_{i,k}(\phi_{ik}^{k+1}) \right]^2$.

Using Lemma 1 with $\mathcal{H} = \mathcal{H}_{\phi_{ik}^{k+1}}^{\text{SAGA}}, \mathcal{G} = \mathcal{G}_{\phi_{ik}^{k+1}}^{\text{SAGA}} - \mu(\bar{\phi}^k - \phi_{ik}^k), z^+ = z_{k+1}$ and taking expectation, we obtain

$$
\mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), z_k - x^* \rangle \right] = \frac{\alpha}{2} \left( \| z_k - x^* \|^2 - (1 + \frac{\mu}{\alpha})^2 \mathbb{E}_{i,k} \left[ \| z_{k+1} - x^* \|^2 \right] \right)
+ \frac{1}{2\alpha} \mathbb{E}_{i,k} \left[ \| \mathcal{H}_{\phi_{ik}^{k+1}}^{\text{SAGA}} \|^2 \right].
$$

Using the co-coercivity of $h_{i,k}$ to bound the stochastic moment,

$$
\mathbb{E}_{i,k} \left[ 2 \mathcal{H}_{\phi_{ik}^{k+1}}^{\text{SAGA}} \right]^2 = \mathbb{E}_{i,k} \left[ \| \nabla h_{i,k}(\phi_{ik}^{k+1}) - \nabla h_{i,k}(\phi_{ik}^k) \|^2 \right] + 2 \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \nabla \Phi^k \rangle \right] - \| \nabla \Phi^k \|^2
\leq 2(L - \mu) \left( \Phi^k - \mathbb{E}_{i,k} \left[ h_{i,k}(\phi_{ik}^{k+1}) \right] - \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \phi_{ik}^k - \phi_{ik}^{k+1} \rangle \right] \right)
+ 2 \mathbb{E}_{i,k} \left[ \langle \nabla h_{i,k}(\phi_{ik}^{k+1}), \nabla \Phi^k \rangle \right] - \| \nabla \Phi^k \|^2.
$$

Based on the updating rules of $\phi_{ik}^{k+1}$, the following relations hold

$$
\mathbb{E}_{i,k} \left[ \phi_{ik}^{k+1} \right] = \frac{1}{n} \mathbb{E}_{i,k} \left[ h_{i,k}(\phi_{ik}^{k+1}) \right] + \frac{n - 1}{n} \Phi^k,
$$
$$
\mathbb{E}_{i,k} \left[ \nabla \phi_{ik}^{k+1} \right] = \frac{1}{n} \mathbb{E}_{i,k} \left[ \nabla h_{i,k}(\phi_{ik}^{k+1}) \right] + \frac{n - 1}{n} \nabla \Phi^k.
$$
where (33) implies that
\[ \| E_{i_k} [h_{i_k}(\phi_{i_k}^{k+1})]\|^2 = n^2 \| E_{i_k} [\nabla \Phi^{k+1}]\|^2 - 2(n^2 - n) \langle E_{i_k} [\nabla \Phi^{k+1}] , \nabla \Phi^k \rangle + (n - 1)^2 \| \nabla \Phi^k \|^2 , \] (34)
\[ E_{i_k} [\langle \nabla h_{i_k}(\phi_{i_k}^{k+1}), \nabla \Phi^k \rangle] = n \langle E_{i_k} [\nabla \Phi^{k+1}] , \nabla \Phi^k \rangle - (n - 1) \| \nabla \Phi^k \|^2 . \] (35)

Then, expanding (29) using (30), (31), (34) and (35), we obtain
\[ \frac{1}{n} \sum_{i_k} [h_{i_k}(\phi_{i_k}^{k+1})] \leq \left[ 1 - \frac{\tau x}{\tau x_n} - \left( 1 - \frac{\mu \tau x}{\tau x_n} \right) \frac{L - \mu}{\alpha n} \right] \sum_{i_k} [\langle \nabla h_{i_k}(\phi_{i_k}^{k+1}), \phi_{i_k}^k - \phi_{i_k}^{k+1} \rangle]
+ \left( 1 - \frac{\mu \tau x}{\tau x_n} \right) \frac{L - \mu}{\alpha n} \sum_{i_k} \langle \nabla \Phi^k - E_{i_k} [h_{i_k}(\phi_{i_k}^{k+1})] \rangle
+ \left( 1 - \frac{\mu \tau x}{\tau x_n} \right) \frac{\alpha L}{2n} \left( \| z - x^* \|^2 - \left( 1 + \frac{\mu}{\alpha} \right)^2 \sum_{i_k} [\nabla \Phi^k, \nabla \Phi^k] \right)
+ \left[ 1 - \frac{\alpha - \mu \tau x}{\tau x_n} - \frac{\tau x}{\tau x_n} + \frac{n - 1}{n} \right] \sum_{i_k} \langle \nabla \Phi^{k+1} , \nabla \Phi^k \rangle
- \frac{(n - 1)^2}{2(L - \mu)n} \sum_{i_k} \langle \nabla \Phi^{k+1} , \nabla \Phi^k \rangle - \frac{1}{2\alpha n} \left( \frac{1 - \mu \tau x}{\alpha \tau x_n} - \frac{\tau x}{\tau x_n} \right) \frac{n - 1}{n} \| \nabla \Phi^k \|^2
- \frac{n}{2(L - \mu)} \sum_{i_k} [\nabla \Phi^{k+1}]^2 .\]

Choosing \( \tau x \) such that \( 1 - \frac{\mu \tau x}{\tau x_n} = \left( 1 - \frac{\mu \tau x}{\tau x_n} \right) \frac{L - \mu}{\alpha n} \), multiplying both sides by \( \tau x \) and using (32), we can simplify the above inequality as
\[ \sum_{i_k} [\Phi^{k+1}] \leq \left( 1 - \frac{\tau x}{n} \right) \Phi^k + \frac{(1 - \tau x)}{2(L - \mu)n} \left( \| z - x^* \|^2 - \left( 1 + \frac{\mu}{\alpha} \right)^2 \sum_{i_k} [\nabla \Phi^k, \nabla \Phi^k] \right)
+ \frac{\alpha \mu - \tau x (\alpha + L + \mu - \mu n)}{(L - \mu)\mu} \sum_{i_k} \langle \nabla \Phi^{k+1} , \nabla \Phi^k \rangle - \frac{n \tau x}{2(L - \mu)} \left( \sum_{i_k} [\nabla \Phi^{k+1}]^2 \right)
- \frac{(n - 1) \tau x}{2(L - \mu)} \left( \sum_{i_k} [\Phi^{k+1}]^2 \right) - \frac{(n - 2) \tau x + \frac{n}{\alpha}}{2(L - \mu)} \| \nabla \Phi^k \|^2 .\]

Fixing \( \tau x = \frac{\alpha + \mu}{\alpha + L} \), we obtain
\[ \sum_{i_k} [\Phi^{k+1}] \leq \left( 1 - \frac{\tau x}{n} \right) \Phi^k + \frac{(1 - \tau x)}{2(L - \mu)n} \left( \| z - x^* \|^2 - \left( 1 + \frac{\mu}{\alpha} \right)^2 \sum_{i_k} [\nabla \Phi^k, \nabla \Phi^k] \right)
+ \frac{(n - 1) \tau x}{L - \mu} \langle \sum_{i_k} [\nabla \Phi^{k+1}] , \nabla \Phi^k \rangle - \frac{n \tau x}{2(L - \mu)} \left( \sum_{i_k} [\nabla \Phi^{k+1}]^2 \right) - \frac{(n - 2) \tau x + \frac{n}{\alpha}}{2(L - \mu)} \| \nabla \Phi^k \|^2 .\]

Using Young’s inequality with \( \beta > 0 \),
\[ \sum_{i_k} [\Phi^{k+1}] \leq \left( 1 - \frac{\tau x}{n} \right) \Phi^k + \frac{(1 - \tau x)}{2(L - \mu)n} \left( \| z - x^* \|^2 - \left( 1 + \frac{\mu}{\alpha} \right)^2 \sum_{i_k} [\nabla \Phi^k, \nabla \Phi^k] \right)
+ \frac{\beta (n - 1) \tau x - n \tau x}{2(L - \mu)} \left( \sum_{i_k} [\nabla \Phi^{k+1}]^2 \right) + \frac{(n - 1) \tau x}{2(L - \mu)} \langle \sum_{i_k} [\nabla \Phi^{k+1}] , \nabla \Phi^k \rangle \|
\]
\( + \frac{(n - 2) \tau x - \frac{n}{\alpha}}{2(L - \mu)} \| \nabla \Phi^k \|^2 \).

Let \( \beta \in \left[ \frac{n - 1}{n - 2 + \frac{n}{\alpha}}, \frac{n}{n - 1} \right] \), the last two terms become non-positive, and thus we have
\[ \sum_{i_k} [\Phi^{k+1}] \leq \left( 1 - \frac{\tau x}{n} \right) \Phi^k + \frac{(1 - \tau x)}{2(L - \mu)n} \left( \| z - x^* \|^2 - \left( 1 + \frac{\mu}{\alpha} \right)^2 \sum_{i_k} [\nabla \Phi^k, \nabla \Phi^k] \right) .\]

Letting \( (1 - \frac{\tau x}{n}) \left( 1 + \frac{\mu}{\alpha} \right)^2 = 1 \) completes the proof.
F Missing proof for NAG (Theorem 1)

For the convex combination $y_k = \tau_y z_k + (1 - \tau_y) x_k$, we can use the trick in Lemma 3 to obtain

$$f(y_k) - f(x^*) \leq \frac{1 - \tau_y}{\tau_y} (\nabla f(y_k), x_k - y_k) + \langle \nabla f(y_k), z_k - x^* \rangle - \frac{\mu}{2} \|y_k - x^*\|^2$$

$$= \frac{1 - \tau_y}{\tau_y} (\nabla f(y_k), x_k - y_k) + \frac{\nabla f(y_k), z_k - z_{k+1}}{R_1} + \frac{\nabla f(y_k), z_{k+1} - x^*}{R_2} - \frac{\mu}{2} \|y_k - x^*\|^2.$$  \hfill (36)

For $R_1$, based on the $L$-smoothness, we have

$$f(x_{k+1}) - f(y_k) + \langle \nabla f(y_k), y_k - x_{k+1} \rangle \leq \frac{L}{2} \|x_{k+1} - y_k\|^2.$$  \hfill (37)

Note that $y_k - x_{k+1} = \tau_x (z_k - z_{k+1}) + (\tau_y - \tau_x) (z_k - x_k)$, we can arrange the above inequality as

$$f(x_{k+1}) - f(y_k) + \langle \nabla f(y_k), \tau_x (z_k - z_{k+1}) + (\tau_y - \tau_x) (z_k - x_k) \rangle \leq \frac{L}{2} \|x_{k+1} - y_k\|^2,$$

$$R_1 \leq \frac{L}{2\tau_x} \|x_{k+1} - y_k\|^2 + \frac{1}{\tau_x} (f(y_k) - f(x_{k+1})) - \frac{\tau_y - \tau_x}{\tau_x} \langle \nabla f(y_k), z_k - x_k \rangle.$$  \hfill (38)

For $R_2$, based on the optimality condition of the 3rd step in Algorithm 1 which is for any $u \in \mathbb{R}^d$,

$$\langle \nabla f(y_k) + \alpha (z_{k+1} - z_k) + \mu(z_{k+1} - y_k), u - z_{k+1} \rangle = 0,$$

we have (by choosing $u = x^*$),

$$R_2 = \alpha \langle z_{k+1} - z_k, x^* - z_{k+1} \rangle + \mu \langle z_{k+1} - y_k, x^* - z_{k+1} \rangle$$

$$= \frac{\alpha}{2} (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2 - \|z_k + 1 - z_k\|^2) + \frac{\mu}{2} (\|y_k - x^*\|^2 - \|z_{k+1} - x^*\|^2 - \|z_{k+1} - y_k\|^2).$$  \hfill (39)

By upper bounding (36) using (37), (38), we can conclude that

$$f(y_k) - f(x^*) \leq \frac{1 - \tau_x}{\tau_x} (\nabla f(y_k), x_k - y_k) + \frac{1}{\tau_x} (f(y_k) - f(x_{k+1})) + \frac{\alpha}{2} \|z_k - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right) \|z_{k+1} - x^*\|^2$$

$$+ \frac{L}{2\tau_x} \|x_{k+1} - y_k\|^2 - \frac{\alpha}{2} \|z_{k+1} - z_k\|^2 - \frac{\mu}{2} \|z_{k+1} - y_k\|^2,$$

$$f(x_{k+1}) - f(x^*) \leq (1 - \tau_x) (f(x_k) - f(x^*)) + \frac{\alpha\tau_x}{2} \left(\|z_k - x^*\|^2 - \left(1 + \frac{\mu}{\alpha}\right) \|z_{k+1} - x^*\|^2\right)$$

$$+ \frac{L}{2} \|x_{k+1} - y_k\|^2 - \frac{\alpha\tau_x}{2} \|z_{k+1} - z_k\|^2 - \frac{\mu\tau_x}{2} \|z_{k+1} - y_k\|^2.$$  \hfill (40)

Note that the following relation holds:

$$x_{k+1} - y_k = \tau_x \left(\frac{1 - \tau_x}{1 - \tau_y} \tau_y (z_{k+1} - z_k) + \frac{\tau_x - \tau_y}{(1 - \tau_y)\tau_x} (z_{k+1} - y_k)\right),$$

and thus if $\tau_x \geq \tau_y$, based on the convexity of $\|\cdot\|^2$, we have

$$\frac{L}{2} \|x_{k+1} - y_k\|^2 \leq \frac{L(1 - \tau_x)\tau_x\tau_y}{2(1 - \tau_y)} \|z_{k+1} - z_k\|^2 + \frac{L(\tau_x - \tau_y)\tau_x}{2(1 - \tau_y)} \|z_{k+1} - y_k\|^2.$$
Finally, suppose that the following relations hold
\[
\begin{align*}
\tau_x &\geq \tau_y \\
\alpha &\geq \frac{L(1-\tau_x)\tau_y}{1-\tau_y} \\
\mu &\geq \frac{L(\tau_x-\tau_y)}{1-\tau_y} \\
(1+\frac{\mu}{\alpha})(1-\tau_x) &\leq 1
\end{align*}
\]
we can arrange (39) as
\[
f(x_{k+1}) - f(x^*) + \frac{\alpha\tau_x}{2} \left( 1 + \frac{\mu}{\alpha} \right) \|z_{k+1} - x^*\|^2 \leq \left( 1 + \frac{\mu}{\alpha} \right)^{-1} \left( f(x_k) - f(x^*) + \frac{\alpha\tau_x}{2} \left( 1 + \frac{\mu}{\alpha} \right) \|z_k - x^*\|^2 \right),
\]
which completes the proof.