A Note on the Operator Window of Modulation Spaces

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Received: 20 August 2022 / Revised: 2 May 2023 / Accepted: 26 October 2023 / Published online: 20 November 2023
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Abstract
Inspired by the recent article Skrettingland (J. Fourier Anal. Appl. 28(2), 1–34 (2022)), this paper is devoted to the study of a suitable class of windows in the framework of bounded linear operators on $L^2(\mathbb{R}^d)$. We establish a natural and complete characterization for the window class such that the corresponding STFT leads to equivalent norms on modulation spaces. The positive bounded linear operators are also characterized by its Cohen’s class distributions such that the corresponding quantities form equivalent norms on modulation spaces. As a generalization, we introduce a family of operator classes corresponding to the operator-valued modulation spaces. Some applications of our main theorems to the localization operators are also concerned.

Keywords Modulation spaces · Equivalent norm · Operator classes

Mathematics Subject Classification 47B10 · 42B35

1 Introduction
Modulation spaces were first introduced by H. Feichtinger [6] in 1983. Now, it has been proven to be an important class of function spaces in the field of time-frequency analysis [9]. Moreover, modulation spaces have been associated with many topics of mathematics such as partial differential equations [1, 17, 19] and classical harmonic...
analysis [10, 11]. We refer to [7] for historical perspectives and background on the motivations which led to the invention of the modulation spaces.

The purpose of modulation spaces is to describe the content of the functions or distributions on the time-frequency plane. To achieve this goal, the short time Fourier transform (STFT) is used to extract the local information of functions or distributions. More precisely, the STFT can initially be defined on $L^2(R^d)$ by

$$V_{\varphi}f(z) = \langle f, \pi(z)\varphi \rangle_{L^2}, \quad (z \in \mathbb{R}^{2d}),$$

where the window $\varphi$ is a function with some good localized properties on the time-frequency plane, and $\pi(z)$ denotes the time-frequency shift for $z = (x, \xi)$ defined by

$$\pi(z)\varphi(t) = M_\xi T_x \varphi(t) = e^{2\pi i t \cdot \xi} \varphi(t - x).$$

With a suitable window $\varphi$, the STFT can be well defined for $f$ belonging to the space of tempered distributions $S'(\mathbb{R}^d)$ or the dual space of the modulation space $M^1_v(\mathbb{R}^d)$ denoted by $(M^1_v(\mathbb{R}^d))^*$. Let $g_0$ be the normalized Gaussian, i.e.,

$$g_0(t) = 2^{d/4} e^{-\pi |t|^2}.$$

We point out that $g_0$ will always work as a suitable window whether $f$ belongs to $L^2(\mathbb{R}^d)$, $S'(\mathbb{R}^d)$ or $(M^1_v(\mathbb{R}^d))^*$. The modulation space can be defined by

$$M_m^{p,q}(\mathbb{R}^d) = \{ f \in (M^1_v(\mathbb{R}^d))^* : V_{g_0} f \in L^{p,q}_m(\mathbb{R}^{2d}) \},$$

endowed with the obvious (quasi-)norm, where $L^{p,q}_m(\mathbb{R}^{2d})$ are weighted mixed-norm Lebesgue spaces with the weight $m \in \mathcal{M}_v$. Here $\mathcal{M}_v$ denotes the class of all $v$-moderate weight functions, where $v$ is a submultiplicative weight. See the precise definitions of weight functions in Sect. 2.2. Sometimes, we write $M^{p,q}_m = M^{p,q}_{m^p}$ for short.

In the above definition of modulation spaces, the Gaussian $g_0$ serves as the window. A natural problem is: can the window $g_0$ be replaced by another suitable function in the definition of modulation spaces? More precisely, can we give a characterization for all $\varphi$ satisfying the following equivalent relation:

$$\| V_{\varphi} f \|_{L^{p,q}_m} \sim_{\varphi, m, v} \| V_{g_0} f \|_{L^{p,q}_m} \quad \text{for all } f \in M^{p,q}_m(\mathbb{R}^d),$$

$$1 \leq p, q \leq \infty, m \in \mathcal{M}_v(\mathbb{R}^{2d})? \quad (1.1)$$

In fact, this problem has been perfectly answered in the previous literature. We summarize this result here. is not difficult to answer in some sense. By the fact that

$$\bigcup_{1 \leq p, q \leq \infty, m \in \mathcal{M}_v} M^{p,q}_m = M^\infty_{v^{-1}} = (M^1_v)^*,$$

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we see that in the definition of STFT, the possible largest class of windows fitting for all \( M_{p,q}^m(\mathbb{R}^d) \) mentioned above, is the modulation space \( M^1_v(\mathbb{R}^d) \), which serves as the test function of \( M_{v,1}^\infty(\mathbb{R}^d) \). On the other hand, for all \( \varphi \in M_v^1(\mathbb{R}^d) \setminus \{0\} \), \( \| V_\varphi f \|_{L_m^{p,q}(\mathbb{R}^{2d})} \) defines an equivalent norm on \( M_{p,q}^m(\mathbb{R}^d) \) (see [9, Proposition 11.4.2]). Thus, we conclude that (1.1) holds if and only if \( \varphi \in M_v^1(\mathbb{R}^d) \setminus \{0\} \).

Let \( \pi(z)^* \) be the Hilbert adjoint of \( \pi(z) \) defined by

\[
\pi(z)^* = e^{-2\pi i x \cdot \xi} \pi(-z), \quad z = (x, \xi).
\]

Note that both \( \pi(z) \) and \( \pi(z)^* \) are bounded on \( M_v^1 \) and can be extended by duality to bounded operators on \((M_v^1)^*\). Write STFT by

\[
V_\varphi f(z) = \langle f, \pi(z)\varphi \rangle_{(M_v^1)^*} = \langle \pi(z)^* f, \varphi \rangle_{(M_v^1)^*} =: L_\varphi(\pi(z)^* f). \tag{1.2}
\]

Here, \( L_\varphi \) means the obvious bounded linear functional on \((M_v^1)^*\). Denote by \( \mathcal{HS} \) the collection of all Hilbert-Schmidt operators on \( L^2(\mathbb{R}^d) \), and let \( \mathcal{N}(L^2, M_v^1) \) be the set of all nuclear operators bounded from \( L^2(\mathbb{R}^d) \) into \( M_v^1(\mathbb{R}^d) \). See Sect. 2.3 for the precise definition of nuclear operators. In [18], the author considers a class of linear operators defined by

\[
\mathcal{N}^* = \{ S \in \mathcal{HS} : S^* \in \mathcal{N}(L^2, M_v^1) \},
\]

and proves that for \( S \in \mathcal{N}^* \setminus \{0\} \) the following result is valid

\[
\| \mathfrak{N}_S f \|_{L_m^{p,q}(\mathbb{R}^{2d}; L^2)} \sim_{S,m,v} \| V_{g_0} f \|_{L_m^{p,q}} \quad \text{for all} \quad f \in M_m^{p,q},
\]

\[
1 \leq p, q \leq \infty, m \in M_v(\mathbb{R}^d), \tag{1.3}
\]

where \( \mathfrak{N}_S(f)(z) := S\pi(z)^* f \). More precisely, we recall the conclusion in [18, Theorem 5.1] as follows.

**Theorem A** Let \( S \in \mathcal{N}^* \setminus \{0\} \). For any \( 1 \leq p, q \leq \infty \) and \( m \in M_v(\mathbb{R}^d) \), we have

\[
\frac{\| S \|_{\mathcal{HS}}^2}{C^m_v \| S^* \|_{\mathcal{N}^*} \| g_0 \|_{M_v^1(\mathbb{R}^d)}} \| V_{g_0} f \|_{L_m^{p,q}} \leq \| \mathfrak{N}_S f \|_{L_m^{p,q}(\mathbb{R}^{2d}; L^2)} \leq C^m_v \| S^* \|_{\mathcal{N}^*} \| V_{g_0} f \|_{L_m^{p,q}}. \tag{1.4}
\]

To see the connection between (1.1) and (1.3), we consider a rank-one operator \( S_1 = \xi \otimes \varphi \) in (1.3), with \( \xi \in L^2 \) and \( \varphi \in M_v^1 \). Note that \( S_1 \in \mathcal{N}^* \) and

\[
\| \mathfrak{N}_{S_1} f \|_{L^2} = \| S_1 \pi(z)^* f \|_{L^2} = \| \xi \|_{L^2} \| V_\varphi f(z) \|_{L^2},
\]

\[
\| \mathfrak{N}_{S_1} f \|_{L_m^{p,q}(\mathbb{R}^{2d}; L^2)} = \| \xi \|_{L^2} \| V_\varphi f(z) \|_{L_m^{p,q}}.
\]

From this and (1.2), the equivalent relation (1.3) can be regarded as an extension for the window class of modulation space, from “bounded linear functional on \((M_v^1)^*\)” to “bounded linear operator from \((M_v^1)^*\) into \(L^2\).”
Theorem 1.2
Let \( S \in \mathcal{L}(M_1^v, L^2) \) such that (1.3) holds. More precisely, can we find the precise subset \( B \) of \( \mathcal{L}(M_1^v, L^2) \), such that \( S \in B \) if and only if (1.3) holds? Note that \( \mathcal{L}(M_1^v, L^2) \subset \mathcal{L}(L^2) \). In this paper, we will give a complete characterization for (1.1) in the framework of \( \mathcal{L}(L^2) \), that is, give the precise subset \( B \) of \( \mathcal{L}(L^2) \) such that (1.3) holds. The assumption of \( S \in \mathcal{L}(L^2) \) is convenient for our proofs of main theorems, and the reader will find that this assumption in Theorems 1.1 and 1.2 can be reduced to a weaker one, that is, \( S \in \mathcal{L}(M_1^v, L^2) \). See Proposition 4.4 and Remark 4.5 for more details.

First, we deal with the \( L^2(\mathbb{R}^d) \) case, which yields a new characterization of \( \mathcal{H}S \). In this case, we only consider the condition (1.3) with \( p = q = 2 \) and \( m = 1 \).

**Theorem 1.1** Let \( S \in \mathcal{L}(L^2(\mathbb{R}^d)) \setminus \{0\} \). The following four statements are equivalent:

1. \( \| \mathcal{W} S f \|_{L^2(\mathbb{R}^d; L^2)} \sim \| S \|_{\mathcal{H}S} \| f \|_{L^2(\mathbb{R}^d)} \) for all \( f \in L^2(\mathbb{R}^d) \);
2. \( \| \mathcal{W} S f \|_{L^2(\mathbb{R}^d; L^2)} \lesssim \| S \|_{\mathcal{H}S} \| f \|_{L^2(\mathbb{R}^d)} \) for all \( f \in L^2(\mathbb{R}^d) \);
3. \( \| \mathcal{W} S g_0 \|_{L^2(\mathbb{R}^d; L^2)} < \infty \);
4. \( S \in \mathcal{H}S \).

Furthermore, if one of the above statements holds, we have

\[
\| \mathcal{W} S f \|_{L^2(\mathbb{R}^d; L^2)} = \| S \|_{\mathcal{H}S} \| f \|_{L^2(\mathbb{R}^d)}, \quad \| S \|_{\mathcal{H}S} = \| \mathcal{W} S g_0 \|_{L^2(\mathbb{R}^d; L^2)}.
\]

If \( \| S \|_{\mathcal{H}S} = 1 \), the map \( f \mapsto \mathcal{W} S f \) is an isometry from \( L^2(\mathbb{R}^d) \) into \( L^2(\mathbb{R}^d; L^2) \).

As we will see shortly, due to the advantage of Hilbert space, the \( L^2 \) case is not difficult to deal with. However, this case is still enlightening. In fact, in the study of the general modulation case as below, one can verify \( B^v_1 \subset \mathcal{H}S \) by the logical relationship that the full version of (1.3) is stronger than the special case with \( p = q = 2 \) and \( m = 1 \). See also Proposition 4.4 for a sharper conclusion. The definition of \( B^v_1 \) can be founded in Theorem 1.2.

Next, we explore the general case. This main theorem can be stated as follows. We use \( C^m_v \) to denote the constant depending on \( v \) and \( m \), see Sect. 2.2 for more details.

**Theorem 1.2** Let \( S \in \mathcal{L}(L^2(\mathbb{R}^d)) \setminus \{0\} \) and let \( v \) be a submultiplicative weight. We put

\[
B^v_1 := \{ S \in \mathcal{L}(L^2(\mathbb{R}^d)) : \| \mathcal{W} S g_0 \|_{L^1(\mathbb{R}^d; L^2)} < \infty \}.
\]

Let \( v \) be a submultiplicative weight function on \( \mathbb{R}^d \). Denote by \( \{ e_n \}_{n=1}^{\infty} \) an orthonormal basis of \( L^2(\mathbb{R}^d) \). The following four statements are equivalent:

1. \( \| \mathcal{W} S f \|_{L^{p,q}(\mathbb{R}^d; L^2)} \sim_{S,m,v} \| f \|_{M^{p,q}_m(\mathbb{R}^d)} \) for all \( f \in M^{p,q}_m(\mathbb{R}^d) \), \( 1 \leq p, q \leq \infty \), \( m \in \mathcal{M}_v \);
2. \( \| \mathcal{W} S f \|_{L^{p,q}(\mathbb{R}^d; L^2)} \lesssim_{S,m,v} \| f \|_{M^{p,q}_m(\mathbb{R}^d)} \) for all \( f \in M^{p,q}_m(\mathbb{R}^d) \), \( 1 \leq p, q \leq \infty \), \( m \in \mathcal{M}_v \);
3. \( S \in B^v_1 \).

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Furthermore, if one of the above statements holds, for \( f \in M_{m}^{p,q}(\mathbb{R}^{d}) \) we have

\[
\frac{\|S\|_{HS}^{2}}{C_{v}^{m} \|\mathcal{U}_{S}g_{0}\|_{L_{m}^{1}(\mathbb{R}^{2d};L_{2})}} \|Vg_{0}f\|_{L_{m}^{p,q}} \leq \|\mathcal{U}_{S}f\|_{L_{m}^{p,q}(\mathbb{R}^{2d};L_{2})}
\]

\[
\leq C_{v}^{m} \|\mathcal{U}_{S}g_{0}\|_{L_{m}^{1}(\mathbb{R}^{2d};L_{2})} \|Vg_{0}f\|_{L_{m}^{p,q}}. \tag{1.5}
\]

**Remark 1.3** The reader may be confused about the definition of \( \mathcal{U}_{S}f \) for \( S \in B_{1}^{v} \) in Theorem 1.2 above. In fact, by a direct calculation

\[
\|Vg_{0}S^{*}f\|_{L_{m}^{1}(\mathbb{R}^{2d})} = \|\langle f, S\pi(z)^{*}g_{0}\rangle\|_{L_{m}^{1}(\mathbb{R}^{2d})} \leq \|f\|_{L_{2}} \|\pi(z)^{*}g_{0}\|_{L_{m}^{1}(\mathbb{R}^{2d})} \leq \|f\|_{L_{2}},
\]

we see that \( S \in B_{1}^{v} \) implies that \( S^{*} \in \mathcal{L}(L^{2}, M_{v}^{1}) \). Then the operator \( S \) can be naturally extended to a bounded operator from \( (M_{v}^{1})^{*} \) into \( L_{2}^{*} \), also denoted by \( S \). Therefore, the operator \( \mathcal{U}_{S}f \) is well-defined for all \( f \in (M_{v}^{1})^{*} \). For simplicity, we will use \( S^{*} \in \mathcal{L}(L^{2}, M_{v}^{1}) \) to denote that \( S \in \mathcal{L}(L^{2}) \) with its Hilbert adjoint \( S^{*} \) belonging to \( \mathcal{L}(L^{2}, M_{v}^{1}) \). Hence, the window class \( B_{1}^{v} \) can be re-represented by

\[
B_{1}^{v} := \{ S \in \mathcal{L}(L^{2}) : S^{*} \in \mathcal{L}(L^{2}, M_{v}^{1}), \|\mathcal{U}_{S}g_{0}\|_{L_{m}^{1}(\mathbb{R}^{2d};L_{2})} < \infty \}.
\]

**Remark 1.4** Comparing with the corresponding result in [18] (see Theorem A), the characterization in Theorem 1.2 is more natural and complete. In our approach, both the case of bounded linear functional in (1.1) and the case of bounded linear operator in (1.3) can be treated in a uniform way, that is, testing the upper bound inequality in (1.1) or (1.3) in the special case of \( p = q = 1, m = v \) and \( f = g_{0} \). Based on this method, our characterizations are derived directly from the equivalent norm conditions in (1.3), without any additional assumptions. This approach also naturally leads to the corresponding characterization associated with positive Cohen’s class distribution, giving an answer for the question posed in [18, Subsection 7.1]. On the other hand, our style of defining window classes is more convenient for further generalization, which will be demonstrated in Sect. 4.

This paper is organized as follows. In Sect. 2, we collect some basic concepts and properties used in this paper. Section 3 is devoted to the proofs of our main theorems. The corresponding problems associated with positive Cohen’s class distributions are also discussed in Sect. 3. We give a generalization of operator classes in Sect. 4, including some basic properties of general operator classes and some re-exploration of the window class \( B_{1}^{v} \). Some applications to localization operators are showed at the end of this section.

Throughout this paper, we will adopt the following notations. We use \( X \preceq Y \) to denote the statement \( X \leq CY \), with a positive constant \( C \) that may depends on...
p, q, d, but it might be different from line to line. The notation \( X \lesssim Y \lesssim X \) means the statement \( X \lesssim Y \lesssim X \). We also use \( X \lesssim_{S,m,v} Y \) and \( X \sim_{S,m,v} Y \) to denote the similar statements as above with the constant \( C \) depending on \( S, m \) and \( v \). The inverse of a function is defined by \( \tilde{g}(t) = g(-t) \).

2 Preliminaries

2.1 Time-Frequency Tools

We consider the point \( z = (x, \xi) \) in the time-frequency plane \( \mathbb{R}^{2d} \), where \( x, \xi \in \mathbb{R}^d \) denote the time and frequency variables, respectively. For any fixed \( x, \xi \), the translation operator \( T_x \), modulation operator \( M_{\xi} \) and time-frequency shift \( \pi(z) \) are defined, respectively, by

\[
T_x f(t) = f(t - x), \quad M_{\xi} f(t) = e^{2\pi i t \cdot \xi} f(t),
\]

\[
\pi(z) f(t) = M_{\xi} T_x f(t) = e^{2\pi i t \cdot \xi} f(t - x).
\]

The short-time Fourier transform (STFT) of a function \( f \) with respect to a window \( g \) is defined by

\[
V_g f(x, \xi) := \langle f, \pi(z) g \rangle_{L^2}, \quad f, g \in L^2(\mathbb{R}^d).
\]

Its extension to \( (M^1_v)^* \times M^1_v \) can be denoted by

\[
V_g f(x, \xi) = \langle f, \pi(z) g \rangle_{(M^1_v)^*, M^1_v},
\]

in which the STFT \( V_g f \) is a sesquilinear map from \( (M^1_v)^* \times M^1_v \) into \( L^\infty_{1/v} \).

A fundamental property we shall use is the following Moyal’s identity.

Lemma 2.1 [9, Proposition 4.3.2] Let \( f_1, f_2, \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \), then \( V_{\varphi_j} f_j \in L^2(\mathbb{R}^d) \) for \( j = 1, 2 \). Furthermore, we have

\[
\int_{\mathbb{R}^{2d}} V_{\varphi_1} f_1(z) V_{\varphi_2} f_2(\overline{z}) dz = \langle f_1, f_2 \rangle_{L^2} \langle \varphi_1, \varphi_2 \rangle_{L^2}.
\]

We also need the Fourier transform on a product of STFTs.

Lemma 2.2 [2, Lemma 2.1] If \( f_1, f_2, g_1, g_2 \in L^2 \), we have

\[
\mathcal{F}(V_{g_1} f_1 V_{g_2} f_2)(x, y) = (V_{f_2} f_1 V_{g_2} g_1)(-y, x).
\]

For a non-zero function \( \gamma \in M^1_v \), we write \( V^{\gamma*}_\gamma \) for the adjoint operator of \( V_\gamma \), given by

\[
\langle V^{\gamma*}_\gamma F, f \rangle = \langle F, V_\gamma f \rangle.
\]
We recall that $V_γ^*$ is bounded from $L_{p,q}^m$ into $M_{p,q}^m$ for $m \in M_v$. We also recall the inverse formula as follows.

**Lemma 2.3** [9, Theorem 11.3.7] Assume that $m \in M_v$ and let $g, γ \in M_v^1\setminus\{0\}$. Then the following inversion formula is valid

$$\langle γ, g \rangle^{-1} V_γ^* V g = I_{(M_v^1)^*}.$$

### 2.2 Function Spaces

In order to introduce the function spaces, we first recall some definitions of weights. The weights we consider here are the moderate weights, which are suitable for the time-frequency estimates [8]. More precisely, a weight function $m$ defined on $\mathbb{R}^{2d}$ is called $v$-moderate if there exists another weight function $v$ and a constant $C_v^m$ depending on $v$ and $m$, such that

$$m(z_1 + z_2) \leq C_v^m v(z_1)m(z_2), \quad z_1, z_2 \in \mathbb{R}^{2d},$$

where $v$ belongs to the class of submultiplicative weight, that is, $v$ satisfies

$$v(z_1 + z_2) \leq v(z_1)v(z_2), \quad z_1, z_2 \in \mathbb{R}^{2d}.$$

We use the notation $M_v(\mathbb{R}^{2d})$ to denote the cone of all weight functions defined on $\mathbb{R}^{2d}$ which are $v$-moderate. Without loss of generality, we assume that $v(x, \xi) = v(-x, \xi) = v(x, -\xi) = v(-x, -\xi)$. We also assume that a $v$-moderate weight is continuous since for any $m \in M_v(\mathbb{R}^{2d})$ there exists a continuous weight $m_1$ such that

$$m \sim m_1.$$

We refer to [13, Lemma 11.2.3] for more details. See also [5] for the origin of the $v$-moderate weights.

**Definition 2.4** (Weighted mixed-norm spaces) Let $1 \leq p, q \leq \infty$, $m \in M_v(\mathbb{R}^{2d})$. Then the weighted mixed-norm space $L_{p,q}^m(\mathbb{R}^{2d})$ consists of all Lebesgue measurable functions on $\mathbb{R}^{2d}$ such that the norm

$$\|F\|_{L_{p,q}^m(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \xi)m(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q}$$

is finite, with the usual modification when $p = \infty$ or $q = \infty$. We write $L_p^m(\mathbb{R}^{2d}) = L_{p,p}^m(\mathbb{R}^{2d})$ for short. If $m \equiv 1$, we write $L_p^q(\mathbb{R}^{2d}) = L_{p,q}^1(\mathbb{R}^{2d})$.

Now, we introduce the definition of (weighted) modulation space.

**Definition 2.5** Let $1 \leq p, q \leq \infty$, $m \in M_v(\mathbb{R}^{2d})$. The (weighted) modulation space $M_{p,q}^m(\mathbb{R}^{2d})$ consists of all $f \in (M_v^1)^*$ such that the norm

$$\|f\|_{M_{p,q}^m(\mathbb{R}^{2d})} := \|V_{g_0} f\|_{L_{p,q}^m(\mathbb{R}^{2d})}$$

$$= \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{g_0} f(x, \xi)m(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q}.$$
is finite, with the usual modification when \( p = \infty \) or \( q = \infty \). For simplicity, we use the notations \( M^p_0 = M^{p,p}_0 \) and \( M^{p,q} = M^{p,q}_1 \).

We recall a well-known convolution relation of modulation spaces.

**Lemma 2.6** [16, Proposition 3.3] For \( p \in [1, \infty] \), we have

\[
M^{p,\infty} \ast M^1 \subset M^p.
\]

Among the large classes of modulation spaces, a remarkable one is the Feichtinger algebra \( M^1_v \) that serves as the admissible window class in the sense of (1.1). The dual space \( (M^1_v)^\ast \) can be used as a substitute for the tempered distributions in the general case in which the weight function \( v \) grows beyond the polynomial.

**Definition 2.7** \((L^2\text{-valued weighted mixed-norm spaces})\) For \( p, q \in [1, \infty] \) and a \( v \)-moderate weight \( m \), the Banach space \( L^{p,q}_m(\mathbb{R}^{2d}, L^2) \) consists of all measurable functions \( \Psi : \mathbb{R}^{2d} \to L^2(\mathbb{R}^d) \) such that

\[
\| \Psi \|_{L^{p,q}_m(\mathbb{R}^{2d}, L^2)} := \| \| \Psi(z) \|_{L^2(\mathbb{R}^d)} \|_{L^{p,q}_m(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} \| \Psi(x, \xi) \|_{L^2(\mathbb{R}^d)}^p m^p(x, \xi) dx \right)^{q/p} d\xi \right)^{1/q}
\]
is finite, with the usual modification when \( p = \infty \) or \( q = \infty \).

### 2.3 Schatten Class Operator and Nuclear Operator

Given a separable Hilbert space \( H \) over \( \mathbb{C} \), for \( p \in [1, \infty) \), we use \( S_p \) to denote the subspace of \( \mathcal{L}(H) \) consisting of compact linear operators \( T \) with the sequence of singular values belonging to \( l^p \), that is,

\[
\| T \|_{S_p} = \left( \sum_j s_j(T)^p \right)^{1/p} < \infty,
\]

where \( s_j(T) \) denotes the singular values of \( T \). For consistency with the literature, we define \( S_\infty = \mathcal{L}(H) \) to be the space of bounded linear operators on \( H \).

If \( p = 2 \), \( S_2 \) is the space of Hilbert-Schmidt operators, also denoted by \( \mathcal{HS} \). The quantity

\[
\| T \|_{\mathcal{HS}} = \| T \|_{S_2} = \sup \left\{ \left( \sum_n \| T e_n \|^2_H \right)^{1/2} \right\} \{e_n\ \text{orthonormal basis}\}
\]
is called the Hilbert-Schmidt norm of \( T \). If \( p = 1 \), \( S_1 \) is the space of trace class operator. For a trace class operator \( T \), we define its trace by

\[
\text{tr}(T) = \sum_n (T e_n, e_n)_H.
\]
where \( \{e_n\} \) is an orthonormal basis of \( H \). In addition, the quantity \( \| T \|_{\mathcal{S}_1} \) is called the trace norm of \( T \). A basic connection between trace class operators and Hilbert-Schmidt operators is that if \( S, T \in \mathcal{S}_2 \), then \( ST \in \mathcal{S}_1 \). Specifically, we have \( tr(T^*T) = \| T \|^2_{\mathcal{S}_2} \) for \( T \in \mathcal{S}_2 \).

Next, we recall the definition of nuclear operators, see e.g., [18, Subsection 3.2]. An operator \( T \in \mathcal{L}(L^2, M^1_v) \) is said to be nuclear if it has an expansion of the form

\[
T = \sum_{n=1}^{\infty} \phi_n \otimes \xi_n,
\]

with \( \sum_{n=1}^{\infty} \| \phi_n \|_{M^1_v} \| \xi_n \|_{L^2} < \infty \). By \( \mathcal{N}(L^2, M^1_v) \) we denote the collection of all nuclear operators. Then \( \mathcal{N}(L^2, M^1_v) \) becomes a Banach space with the norm given by

\[
\| T \|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \| \phi_n \|_{M^1_v} \| \xi_n \|_{L^2} : T = \sum_{n=1}^{\infty} \phi_n \otimes \xi_n \right\}.
\]

### 2.4 Khinchin’s Inequality

**Lemma 2.8** (Khinchin’s inequality, see [12]) Let \( 0 < p < \infty \), \( \{\omega_k\}_{k=1}^{N} \) be a sequence of independent random variables taking values \( \pm 1 \) with equal probability. Denote expectation (integral over the probability space) by \( \mathbb{E} \). For any sequence of complex numbers \( \{a_k\}_{k=1}^{N} \), we have

\[
\mathbb{E} \left( \left| \sum_{k=1}^{N} a_k \omega_k \right|^p \right) \sim \left( \sum_{k=1}^{N} |a_k|^2 \right)^{p/2},
\]

where the implicit constants depend on \( p \) only.

### 3 Characterizations of Operator Windows

#### 3.1 \( L^2 \) Case

In this subsection, we deal with the \( L^2 \) case. This case reveals to us that the suitable window class in (1.3) needs to be included in the class of Hilbert-Schmidt operators.

**Proof of Theorem 1.1** It is obvious that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3). Now, we deal with (3) \( \Rightarrow \) (4). Take \( \{e_n\}_{n=1}^{\infty} \) to be an orthonormal basis of \( L^2(\mathbb{R}^d) \). By Parseval’s identity we have

\[
\| \mathcal{V} g_0(z) \|_{L^2}^2 = \sum_{n=1}^{\infty} |\langle \mathcal{V} g_0(z), e_n \rangle_{L^2}|^2.
\]
Note that for $z = (x, \xi)$, we have

$$\langle \mathcal{W}_g_0(z), e_n \rangle_{L^2} = (S\pi_0(z)^* g_0, e_n)_{L^2} = (\pi^*_0 z g_0, S^* e_n)_{L^2} = e^{-2\pi i x \cdot \xi} \langle \pi(-z) g_0, S^* e_n \rangle_{L^2}.$$  

From the above two estimates we have

$$\|\mathcal{W}_s g_0(z)\|^2_{L^2} = \sum_{n=1}^{\infty} |V_{g_0} S^* e_n(-z)|^2,$$

and

$$\|\mathcal{W}_s g_0\|^2_{L^2(\mathbb{R}^d; L^2)} = \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} |V_{g_0} S^* e_n(-z)|^2 dz = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} |V_{g_0} S^* e_n(z)|^2 dz = \sum_{n=1}^{\infty} \|S^* e_n\|^2_{L^2}.$$

where in the last equality we use Moyal’s identity (Lemma 2.1). From this and the assumption (3), we conclude that

$$\|S^*\|_{\mathcal{HS}} = \left(\sum_{n=1}^{\infty} \|S^* e_n\|^2_{L^2}\right)^{1/2} = \|\mathcal{W}_s g_0\|_{L^2(\mathbb{R}^d; L^2)} < \infty,$$

which yields that $S^* \in \mathcal{HS}$. Then, we obtain $S \in \mathcal{HS}$ with $\|S\|_{\mathcal{HS}} = \|S^*\|_{\mathcal{HS}}$.

Finally, we consider (4) $\Rightarrow$ (1). Using Parseval’s identity and the fact that

$$\langle \mathcal{W}_S f(z), e_n \rangle_{L^2} = \langle f, \pi(z) S^* e_n \rangle_{L^2} = V_{S^*} e_n f(z),$$

we have

$$\|\mathcal{W}_S f(z)\|^2_{L^2} = \sum_{n=1}^{\infty} \|\mathcal{W}_S f(z), e_n \|^2_{L^2} = \sum_{n=1}^{\infty} |V_{S^*} e_n f(z)|^2.$$  

(3.2)

Then, we conclude (1) by

$$\|\mathcal{W}_S f\|^2_{L^2(\mathbb{R}^d; L^2)} = \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} |V_{S^*} e_n f(z)|^2 dz = \sum_{n=1}^{\infty} \|S^* e_n\|^2_{L^2} \|f\|^2_{L^2} = \|S\|^2_{\mathcal{HS}} \|f\|^2_{L^2},$$

where in the second to last equality we use Moyal’s identity. \qed

It should not be difficult to see that Theorem 1.1 and its proof are still valid when $L^2(\mathbb{R}^d)$ is replaced by any other separable Hilbert space $H$. 

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3.2 $M_{m}^{p,q}$ Case

In order to deal with the general modulation space $M_{m}^{p,q}$, we first recall the following pointwise inequality of STFT. One can find the following result from Lemma 11.3.3 in [9].

**Lemma 3.1** Let $\varphi, \psi \in M_{1}^{1}$, $f \in (M_{1}^{1})^{\ast}$. We have the following inequality

$$|V_{\varphi} f| \leq \|\psi\|_{L_{2}^{2}}^{-2} |V_{\varphi} \psi| * |V_{\psi} f|.$$  

Using a randomization technique, we establish the following vector-valued inequality.

**Proposition 3.2** Let $1 \leq p, q \leq \infty$, $m \in M_{v}$ and $(\varphi_{n})_{n=1}^{\infty} \subset M_{1}^{1}$. For $f \in (M_{1}^{1})^{\ast}$, we have the following pointwise inequality

$$\left(\sum_{n=1}^{N} |V_{\varphi_{n}} f|^{2}\right)^{1/2} \lesssim \left(\sum_{n=1}^{N} |V_{\varphi_{n}} g_{0}|^{2}\right)^{1/2} * |V_{g_{0}} f|.$$

Moreover, if $\|(V_{\varphi_{n}} g_{0})_{n}\|_{L_{1}^{1}({\mathbb{R}}^{2d};l^{2})} < \infty$, then the map $f \mapsto (V_{\varphi_{n}} f)_{n=1}^{\infty}$ is bounded from $M_{m}^{p,q}({\mathbb{R}}^{d})$ to $L_{m}^{p,q}({\mathbb{R}}^{2d};l^{2})$ with

$$\|(V_{\varphi_{n}} f)_{n}\|_{L_{m}^{p,q}({\mathbb{R}}^{2d};l^{2})} \lesssim C_{m}^{v} \|(V_{\varphi_{n}} g_{0})_{n}\|_{L_{1}^{1}({\mathbb{R}}^{2d};l^{2})} \|V_{g_{0}} f\|_{L_{m}^{p,q}}.$$  

**Proof** Let $\omega_{n}$ be a sequence of independent random variables taking values $\pm 1$ with equal probability. Using Lemma 3.1, we have

$$|N \sum_{n=1}^{N} \omega_{n} V_{\varphi_{n}} f| = |V_{N} \sum_{n=1}^{N} \omega_{n} \varphi_{n} f| \leq |V_{N} \sum_{n=1}^{N} \omega_{n} \varphi_{n} g_{0}| * |V_{g_{0}} f|.$$  

Taking expectation on both sides and using the Khinchin’s inequality, we obtain

$$\left(\sum_{n=1}^{N} |V_{\varphi_{n}} f|^{2}\right)^{1/2} \sim \mathbb{E} \left(\left|\sum_{n=1}^{N} \omega_{n} V_{\varphi_{n}} f\right|\right)$$

$$\leq \mathbb{E} \left|V_{N} \sum_{n=1}^{N} \omega_{n} \varphi_{n} g_{0}| * |V_{g_{0}} f|\right)$$

$$= \mathbb{E} \left(\left|\sum_{n=1}^{N} \omega_{n} V_{\varphi_{n}} g_{0}\right| * |V_{g_{0}} f| \right) \sim \left(\sum_{n=1}^{N} |V_{\varphi_{n}} g_{0}|^{2}\right)^{1/2} * |V_{g_{0}} f|.$$  

Applying the convolution inequality $L_{m}^{p,q} * L_{v}^{1} \subset L_{m}^{p,q}$ and letting $N \to \infty$, we obtain (3.4).
Proposition 3.3 Let $1 \leq p, q \leq \infty$, $m \in \mathcal{M}_v$ and $S \in \mathcal{B}_v^1$. For $f \in (M^1_v)^*$, we have the following pointwise inequality for all $z \in \mathbb{R}^d$

$$\|\mathfrak{M}_S f(z)\|_{L^2} \lesssim \left( \|\mathfrak{M}_S g_0(\cdot)\|_{L^2} \ast |V_{g_0} f(\cdot)| \right)(z). \tag{3.5}$$

Moreover, the map $f \mapsto \mathfrak{M}_S f$ is bounded from $M^{p,q}_m(\mathbb{R}^d)$ to $L^{p,q}_m(\mathbb{R}^d; L^2)$ with

$$\|\mathfrak{M}_S f\|_{L^{p,q}_m(\mathbb{R}^d; L^2)} \lesssim C_v \|\mathfrak{M}_S g_0\|_{L^1(\mathbb{R}^d; L^2)} \|V_{g_0} f\|_{L^{p,q}_m(\mathbb{R}^d)}. \tag{3.6}$$

Proof Recall that $S \in \mathcal{B}_v^1$ implies $S^* \in \mathcal{L}(L^2, M^1_v)$, then $S$ can be extended to a bounded operator from $(M^1_v)^*$ into $L^2$, also denoted by $S$. Take $\{e_n\}_{n=1}^\infty$ to be an orthonormal basis of $L^2(\mathbb{R}^d)$. Note that for $f \in (M^1_v)^*$, $z = (x, \xi)$,

$$\langle \mathfrak{M}_S f(z), e_n \rangle_{L^2} = \langle \pi(z)^* f, e_n \rangle_{L^2} = \langle \pi(z)^* f, S^* e_n \rangle_{(M^1_v)^*} = \langle f, \pi(z) S^* e_n \rangle_{(M^1_v)^*},$$

where $S^* e_n \in M^1_v$. We have

$$\|\mathfrak{M}_S f(z)\|_{L^2}^2 = \sum_{n=1}^\infty |\langle f, \pi(z) S^* e_n \rangle_{(M^1_v)^*}|^2 = \sum_{n=1}^\infty |V_{S^* e_n} f(z)|^2.$$  

Using this and Proposition 3.2, we conclude that

$$\|\mathfrak{M}_S f(z)\|_{L^2} = \|(V_{S^* e_n} f(z))_n\|_{L^2} \lesssim \|(V_{S^* e_n} g_0(\cdot))_n\|_{L^2} \ast |V_{g_0} f(\cdot)|(z) = \left( \|\mathfrak{M}_S g_0(\cdot)\|_{L^2} \ast |V_{g_0} f(\cdot)| \right)(z).$$

Finally, (3.6) follows by the convolution inequality $L^{p,q}_m \ast L^1_v \subset L^{p,q}_m$. \hfill $\Box$

In order to obtain the lower bound for $\|\mathfrak{M}_S f\|_{L^{p,q}_m(\mathbb{R}^d; L^2)}$, we need a $(M^1_v)^*$ reconstruction formula associated with $\mathfrak{M}_S$. We establish this reconstruction by the following classical method. A similar process has also been carried out in [18]. We recall the operator $\mathfrak{M}_S^*$ for $S \in \mathcal{B}_v^1$ defined as follows:

$$\langle \mathfrak{M}_S^* F, \varphi \rangle_{(M^1_v)^*} := \int_{\mathbb{R}^d} \langle F(z), (\mathfrak{M}_S \varphi)(z) \rangle_{L^2} dz, \quad F \in L^{p,q}_m(\mathbb{R}^d; L^2), \quad \varphi \in M^1_v(\mathbb{R}^d),$$

where the right term leads to a bounded linear functional on $M^1_v(\mathbb{R}^d)$. For the boundedness of $\mathfrak{M}_S^*$ we recall the following lemma (see [18, Lemma 5.3]) with slight modification.

Lemma 3.4 Let $S \in \mathcal{B}_v^1$, $m \in \mathcal{M}_v$, and $F \in L^{p,q}_m(\mathbb{R}^d; L^2)$. For $1 \leq p, q \leq \infty$, the map $\mathfrak{M}_S^*$ is bounded from $L^{p,q}_m(\mathbb{R}^d; L^2)$ into $M^{p,q}_m(\mathbb{R}^d)$ satisfying the following inequality

$$\|V_{g_0} \mathfrak{M}_S^* F\|_{L^{p,q}_m(\mathbb{R}^d)} \leq C_v \|F\|_{L^{p,q}_m(\mathbb{R}^d; L^2)} \|\mathfrak{M}_S g_0\|_{L^1(\mathbb{R}^d; L^2)},$$

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Next, we turn to the reconstruction on \((M^1_v)^*\). First, we recall a useful result in [14, Lemma 4.1].

**Lemma 3.5** Let \(R, T \in S_1(L^2)\) be trace class operators. Then the function \(z \mapsto tr(\pi(z) R \pi(z)^* T)\) is integrable with

\[
\|tr(\pi(z) R \pi(z)^* T)\|_{L^1} \leq \|R\|_{S_1(L^2)} \|T\|_{S_1(L^2)}.
\]

Furthermore,

\[
\int_{\mathbb{R}^{2d}} tr(\pi(z) R \pi(z)^* T) dz = tr(R) tr(T).
\]

Now, we give the reconstruction on \((M^1_v)^*\) by the method of [18, Lemma 5.4] with slight modification.

**Proposition 3.6** Let \(S, T \in B^v_1\). We have

\[
\mathfrak{W}^*_T \mathfrak{W}_S = tr(T^* S) I_{(M^1_v)^*}.
\]

Specifically, we have

\[
\mathfrak{W}_S \mathfrak{W}_S = tr(S^* S) I_{(M^1_v)^*} = \|S\|^2_{S_2} I_{(M^1_v)^*}.
\]

**Proof** We need to verify that

\[
\langle \mathfrak{W}^*_T \mathfrak{W}_S f , \varphi \rangle_{(M^1_v)^*} , (M^1_v) = tr(T^* S) \langle f , \varphi \rangle_{(M^1_v)^*} , (M^1_v),
\]

for all \(f \in (M^1_v)^*\) and \(\varphi \in M^1_v\). This identity is valid for \(f \in L^2\), since

\[
\langle \mathfrak{W}^*_T \mathfrak{W}_S f , \varphi \rangle_{(M^1_v)^*} , (M^1_v) = \int_{\mathbb{R}^{2d}} \langle \mathfrak{W}_S f , \mathfrak{W}_T \varphi \rangle_L^2 dz
\]

\[
= \int_{\mathbb{R}^{2d}} \langle \pi(z) T^* S \pi(z)^* f , \varphi \rangle_L^2 dz
\]

\[
= \int_{\mathbb{R}^{2d}} tr((\pi(z) T^* S \pi(z)^* f) \otimes \varphi) dz
\]

\[
= \int_{\mathbb{R}^{2d}} tr(\pi(z) T^* S \pi(z)^* f) \otimes \varphi) dz,
\]

where by Lemma 3.5 the last term equals to

\[
tr(T^* S) tr(f \otimes \varphi) = tr(T^* S) \langle f , \varphi \rangle_{L^2} = tr(T^* S) \langle f , \varphi \rangle_{(M^1_v)^*} , (M^1_v).
\]

For \(f \in (M^1_v)^*\), recall that \(S \in B^v_1\) implies \(S^* \in L(L^2, M^1_v)\), and write

\[
\langle \mathfrak{W}^*_T \mathfrak{W}_S f , \varphi \rangle_{(M^1_v)^*} , (M^1_v) = \int_{\mathbb{R}^{2d}} \langle \mathfrak{W}_S f , \mathfrak{W}_T \varphi \rangle_L^2 dz
\]

\[
= \int_{\mathbb{R}^{2d}} \langle f , \pi(z) S^* T \pi(z)^* \varphi \rangle_{(M^1_v)^*} , (M^1_v) dz.
\]
Then, (3.7) is equivalent to

\[
\int_{\mathbb{R}^{2d}} \langle f, \pi(z)S^* T \pi(z)^* \varphi \rangle_{(M^1_\nu)^*} d\nu = tr(T^* S) \langle f, \varphi \rangle_{(M^1_\nu)^*},
\]

(3.8)

which has been verified for \( f \in L^2(\mathbb{R}^d) \). For \( f \in (M^1_\nu)^* \), there exists a sequence \( \{f_n\}_{n=1}^\infty \subset L^2(\mathbb{R}^d) \) that tends to \( f \) in the weak* topology of \((M^1_\nu)^*\), and satisfies \( \|f_n\|_{(M^1_\nu)^*} \lesssim \|f\|_{(M^1_\nu)^*} \). Then, by (3.8) we obtain

\[
\int_{\mathbb{R}^{2d}} \langle f_n, \pi(z)S^* T \pi(z)^* \varphi \rangle_{(M^1_\nu)^*} d\nu = tr(T^* S) \langle f_n, \varphi \rangle_{(M^1_\nu)^*},
\]

(3.9)

where the right term tends to \( tr(T^* S) \langle f, \varphi \rangle_{(M^1_\nu)^*} \) as \( n \to \infty \). The remaining issue is to deal with the left term by letting \( n \to \infty \). For the sequence of functions \( z \mapsto \langle f_n, \pi(z)S^* T \pi(z)^* \varphi \rangle_{(M^1_\nu)^*} \) that tends to \( \langle f, \pi(z)S^* T \pi(z)^* \varphi \rangle_{(M^1_\nu)^*} \) as \( n \to \infty \), we find the dominated function by

\[
|\langle f_n, \pi(z)S^* T \pi(z)^* \varphi \rangle_{(M^1_\nu)^*}| = \left| \langle (\mathcal{M}_S f_n)(z), (\mathcal{T} \varphi)(z) \rangle \right|_{L^2} \leq \| (\mathcal{M}_S f_n)(z) \|_{L^2} \| (\mathcal{T} \varphi)(z) \|_{L^2} = v(z)^{-1} \| (\mathcal{M}_S f_n)(z) \|_{L^2} v(z) \| (\mathcal{T} \varphi)(z) \|_{L^2} \leq \left\| \| (\mathcal{M}_S f_n)(z) \|_{L^2 \nu} \right\|_{L^1_{v}} \| (\mathcal{T} \varphi)(z) \|_{L^2} v(z) \lesssim \| f \|_{(M^1_\nu)^*} \| (\mathcal{T} \varphi)(z) \|_{L^2} v(z) \in L^1(\mathbb{R}^{2d}),
\]

where in the last inequality we use Proposition 3.3 and \( \| f_n \|_{(M^1_\nu)^*} \lesssim \| f \|_{(M^1_\nu)^*} \). Then the Lebesgue dominated convergence theorem yields that the left term in (3.9) tends to \( \int_{\mathbb{R}^{2d}} \langle f, \pi(z)S^* T \pi(z)^* \varphi \rangle_{(M^1_\nu)^*} d\nu \). We have now completed this proof. \( \Box \)

**Proof of Theorem 1.2** The proof of (1) \( \Rightarrow \) (2) is obvious, and the relation (2) \( \Rightarrow \) (3) follows by taking \( p = q = 1 \), \( f = g_0 \) and \( m = v \). Now, we consider the relation (3) \( \Rightarrow \) (1). The upper bound follows by Proposition 3.3. For the lower bound, we use Proposition 3.6 and Lemma 3.4 to deduce that

\[
\| f \|_{M^{p,q}_m} = \frac{1}{\| S \|_{\mathcal{H}^S}^2} \| \mathcal{M}^p_s \mathcal{M}^q_s f \|_{M^{p,q}_m} \leq \frac{C^m_v \| \mathcal{M}_S g_0 \|_{L^1(\mathbb{R}^{2d}; L^2)}}{\| S \|_{\mathcal{H}^S}^2} \| \mathcal{M}_S f \|_{L^{p,q}_m(\mathbb{R}^{2d}; L^2)}.
\]

Finally, (1) \( \iff \) (4) follows by (3.2). \( \Box \)

### 3.3 Positive Cohen’s Class Distributions

In this subsection, we focus on the reinterpretation of Theorems 1.1 and 1.2 by using Cohen’s class distributions. We refer to [18, Section 7] for the corresponding research on this topic. In some sense, we give an answer to the question posed in [18, Subsection 7.1]. See also [18, Example 7.3] for a discussion on the positivity assumption of \( T \).
Given a Hilbert-Schmidt operator $T$ on $L^2$, the Cohen’s class distribution $Q_T$ can be defined on $L^2$ by

$$Q_T f(z) = \langle T \pi(z)^* f, \pi(z)^* f \rangle_{L^2}, \quad f \in L^2.$$  

This definition was given in [15]. It can be regarded as a generalization of the classical Cohen’s class distribution defined by $Q_a(f) = a * W(f)$ for $a, f \in L^2$. Here, $W(f)$ denotes the Wigner distribution of $f$. Using this generalized definition of Cohen’s class distributions, we firstly give the following conclusion corresponding to Theorem 1.1.

**Theorem 3.7** Let $T \in \mathcal{L}(L^2(\mathbb{R}^d)) \setminus \{0\}$ be a positive operator. Denote by $\sqrt{T}$ the positive square root of $T$. The following five statements are equivalent:

1. $\|\sqrt{Q_T f}\|_{L^2(\mathbb{R}^d)} \sim \|f\|_{L^2(\mathbb{R}^d)}$ for all $f \in L^2(\mathbb{R}^d)$;
2. $\|\sqrt{Q_T f}\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$ for all $f \in L^2(\mathbb{R}^d)$;
3. $\|\sqrt{Q_T f}\|_{L^2(\mathbb{R}^d)} < \infty$;
4. $\sqrt{T} \in S_2$;
5. $T$ is a trace class operator.

Furthermore, if one of the above statements holds, we have

$$\|\sqrt{Q_T f}\|_{L^2(\mathbb{R}^d)} = \sqrt{\text{tr}(T)} \|f\|_{L^2}, \quad \text{tr}(T) = \|\sqrt{Q_T f}\|_{L^2(\mathbb{R}^d)}^2.$$

**Proof** For $f \in L^2$, write

$$Q_T f = \langle T \pi(z)^* f, \pi(z)^* f \rangle_{L^2} = \langle \sqrt{T} \pi(z)^* f, \sqrt{T} \pi(z)^* f \rangle_{L^2}.$$

Denote $S = \sqrt{T}$, we have

$$\sqrt{Q_T f} = \|\mathfrak{M} f\|_{L^2} \quad \text{and} \quad \|\mathfrak{M} f\|_{L^2(\mathbb{R}^d)} = \|\sqrt{Q_T f}\|_{L^2(\mathbb{R}^d)}.$$

Then, the desired equivalent relations follows by Theorem 1.1 and the fact that $\|S\|_{S_2} = \sqrt{\text{tr}(T)}$. \qed

Next, we explore the corresponding result of Theorem 1.2. We use $T^* \in \mathcal{L}((M^1_{M_1}), M^1_{M_1})$ to denote that $T \in \mathcal{L}(L^2)$ with its Hilbert adjoint $T^*$ belonging to $\mathcal{L}((M^1_{M_1}), M^1_{M_1})$. For $T^* \in \mathcal{L}((M^1_{M_1}), M^1_{M_1})$, the Cohen’s class distribution associated with $T$ can be defined on $(M^1_{M_1})^*$ by

$$Q_T f(z) = \langle \pi(z)^* f, T^* \pi(z)^* f \rangle_{(M^1_{M_1})^*}, \quad f \in (M^1_{M_1})^*.$$  \hfill (3.10)

See [18, Remark 6] for more details of the definition of Cohen’s class distributions. Now, we give the following conclusion corresponding to Theorem 1.2. Denote by $M^p,q_m(\mathbb{R}^d)$ the closure of $L^2(\mathbb{R}^d) \cap M^p,q_m(\mathbb{R}^d) \subset M^p,q_m(\mathbb{R}^d)$. Note that the space $M^p,q_m$ can be replaced by $M^p,q_m$ when $p, q < \infty$, since $M^p,q_m = M^p,q_m$ in this case.
Theorem 3.8 Let $T \in \mathcal{L}(L^2(\mathbb{R}^d)) \setminus \{0\}$ be a positive operator. Let $v$ be a submultiplicative weight function on $\mathbb{R}^{2d}$. The following statements are equivalent:

1. $\|\sqrt{T} f\|_{L^m_p(\mathbb{R}^{2d})} \lesssim_{S,m,v} \|f\|_{M^p_{m,q}(\mathbb{R}^d)}$ for all $f \in \widetilde{M^p_{m,q}(\mathbb{R}^d)}$, $1 \leq p, q \leq \infty$, $m \in \mathcal{M}_v$;
2. $\|\sqrt{T} f\|_{L^m_p(\mathbb{R}^{2d})} \lesssim_{S,m,v} \|f\|_{M^p_{m,q}(\mathbb{R}^d)}$ for all $f \in \widetilde{M^p_{m,q}(\mathbb{R}^d)}$, $1 \leq p, q \leq \infty$, $m \in \mathcal{M}_v$;
3. $\sqrt{T} \in B^v_1$.

Furthermore, if one of the above statements holds, then for $f \in \widetilde{M^p_{m,q}(\mathbb{R}^d)}$ we have

$$
\frac{\text{tr}(T)}{C_v^{m}} \|\sqrt{T} g_0\|_{L^m_1(\mathbb{R}^{2d})} \leq \|\sqrt{T} f\|_{L^m_p(\mathbb{R}^{2d})} \leq \left(\frac{C_v^m}{\sqrt{T} g_0}\right) \|\sqrt{T} f\|_{L^m_p(\mathbb{R}^{2d})}.
$$

(3.11)

**Proof** The relation (1) $\Rightarrow$ (2) is obvious, and the relation (2) $\Rightarrow$ (3) follows by taking $p = q = 1$, $f = g_0$ and $m = v$. Then we have the following estimate:

$$
\|\sqrt{T} g_0\|_{L^m_1(\mathbb{R}^{2d}; L^2)} = \left\|\sqrt{T} \pi(z) g_0, \sqrt{T} \pi(z) g_0\right\|_{L^m_1(\mathbb{R}^{2d})} \\
= \left\|\sqrt{T} \pi(z) g_0, \pi(z) g_0\right\|_{L^m_1(\mathbb{R}^{2d})} \\
= \left\|\sqrt{T} \pi(z) g_0\right\|_{L^m_1(\mathbb{R}^{2d})} \lesssim \|g_0\|_{M^1_v(\mathbb{R}^d)} < \infty.
$$

Next, we only need to verify the inverse direction (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). As in the proof of Theorem 3.7, for $f \in L^2$ and $S = \sqrt{T}$, we have

$$
\sqrt{T} f = \|\mathfrak{S} f\|_{L^2}, \quad \|\mathfrak{S} f\|_{L^m_p(\mathbb{R}^{2d}; L^2)} = \|\sqrt{T} f\|_{L^m_p(\mathbb{R}^{2d})}.
$$

(3.12)

Using this with Theorem 1.2 and the fact $S = \sqrt{T} \in B^v_1$, we conclude that the statement (2) is valid for $f \in L^2$. For $f \in \widetilde{M^p_{m,q}}$, there exists a sequence of $L^2 \cap M^p_{m,q}$ functions denoted by $(f_j)_{j=1}^\infty$ such that $f_j \to f$ in the topology of $M^p_{m,q}$. Since $f_j \in L^2$, we have

$$
\|\sqrt{T} f_j\|_{L^m_p(\mathbb{R}^{2d})} \lesssim_{S,m,v} \|f_j\|_{M^p_{m,q}(\mathbb{R}^d)}.
$$

Recalling $\sqrt{T} \in B^v_1$, we conclude $\sqrt{T} \in \mathcal{L}(L^2, M^1_v)$ by Remark 1.3. By this and that $\sqrt{T}$ is self-adjoint, the operator $S = \sqrt{T} \in \mathcal{L}(L^2)$ can be extended by duality to be a bounded operator from $(M^1_v)^*$ into $L^2$, also denoted by $\sqrt{T}$. Using this extension of $\sqrt{T}$, the operator $T = \sqrt{T} \sqrt{T}$ can be naturally extended to be a bounded operator from $(M^1_v)^*$ into $M^1_v$. Since $T$ is self-adjoint, $T^*$ is also extended automatically in this way.

Using the fact that $T^* \in \mathcal{L}((M^1_v)^*, M^1_v)$, we conclude that $T^* \pi(z) f_j$ tends to $T^* \pi(z) f$ in $M^1_v$. From this and the continuity of the sesquilinear map $\langle \cdot, \cdot \rangle_{(M^1_v)^*, M^1_v}$, we obtain
\[ Q_T f_j(z) = \langle \pi(z)^* f_j, T^* \pi(z)^* f_j \rangle_{(M^1_\alpha)^*, M^1_\beta} \rightarrow \langle \pi(z)^* f, T^* \pi(z)^* f \rangle_{(M^1_\alpha)^*, M^1_\beta} \]
\[ = Q_T f(z), \]  
(3.13)

where the convergence process is valid for each point \( z \in \mathbb{R}^{2d} \) as \( j \to \infty \). By applying Fatou’s lemma, we conclude that

\[
\| \sqrt{Q_T f} \|_{L^p_m, q_m(\mathbb{R}^{2d})} = \liminf_{j \to \infty} \sqrt{Q_T f_j} \|_{L^p_m, q_m(\mathbb{R}^{2d})} \leq \liminf_{j \to \infty} \| Q_T f_j \|_{L^p_m, q_m(\mathbb{R}^{2d})} \lesssim S, m, v \liminf_{j \to \infty} \| f_j \|_{M^p_m, q_m(\mathbb{R}^d)} = \| f \|_{M^p_m, q_m(\mathbb{R}^d)}. 
\]

This completes the proof of statement (3) \( \Rightarrow \) (2).

Next, we turn to the proof (3) \( \Rightarrow \) (1). Using Theorem 1.2 with the fact (3.12), we obtain

\[
\| \sqrt{Q_T f_j} \|_{L^p_m, q_m(\mathbb{R}^{2d})} \sim S, m, v \| f_j \|_{M^p_m, q_m(\mathbb{R}^d)},
\]

where \( f_j \subset L^2 \cap M^p_m, q_m \) is the approximating sequence mentioned above. We claim that

\[
\| \sqrt{Q_T f_j} \|_{L^p_m, q_m(\mathbb{R}^{2d})} \to \| \sqrt{Q_T f} \|_{L^p_m, q_m(\mathbb{R}^{2d})} \quad (j \to \infty),
\]

then the desired conclusion follows by this claim and the fact that \( \| f_j \|_{M^p_m, q_m(\mathbb{R}^d)} \to \| f \|_{M^p_m, q_m(\mathbb{R}^d)} \) as \( j \to \infty \).

Now, we verify the claim. Using the fact in (3.12), we conclude that

\[
|\sqrt{Q_T f_j} - \sqrt{Q_T f_i}| = |\| \mathfrak{M}_S f_j \|_{L^2} - \| \mathfrak{M}_S f_i \|_{L^2}| \leq \| \mathfrak{M}_S (f_j - f_i) \|_{L^2} = \sqrt{Q_T (f_j - f_i)}. 
\]

Letting \( l \to \infty \) and using (3.13), we conclude that

\[
|\sqrt{Q_T f_j} - \sqrt{Q_T f}| \leq \sqrt{Q_T (f_j - f)}. 
\]

Taking the \( L^p_m, q_m \) norm on both sides, we have

\[
\| \sqrt{Q_T f_j} - \sqrt{Q_T f} \|_{L^p_m, q_m(\mathbb{R}^{2d})} \leq \| \sqrt{Q_T (f_j - f)} \|_{L^p_m, q_m(\mathbb{R}^{2d})}. 
\]
By the conclusion in statement (2), we obtain that \( \| \sqrt{Q_T (f_j - f)} \|_{L^p_m(\mathbb{R}^{2d})} \lesssim S, m, v \) \( \| f_j - f \|_{M^p_m(\mathbb{R}^d)} \). The claim follows by

\[
\left| \| \sqrt{Q_T f_j} \|_{L^p_m(\mathbb{R}^{2d})} - \| \sqrt{Q_T f} \|_{L^p_m(\mathbb{R}^{2d})} \right| \leq \| Q_T (f_j - f) \|_{L^p_m(\mathbb{R}^{2d})} \lesssim \| f_j - f \|_{M^p_m(\mathbb{R}^d)},
\]

where the last term tends to zero as \( j \to \infty \).

Finally, if one of the statements (1 – 3) is valid, by using Theorem 1.2 and the fact \( \| \sqrt{T} \|_{S^2} = \sqrt{\text{tr}(T)} \), we conclude that (3.11) is valid for \( f \in L^2 \). Then the desired conclusion follows by a similar limiting argument used above.

3.4 The Relation Between \( \mathcal{B}_1^v \) and \( \mathcal{N}^* \)

**Proposition 3.9** *The following embedding relation is valid*

\[ \mathcal{N}^* \subset \mathcal{B}_1^v. \]

**Proof** Although this conclusion is implied in the logical relationship by

\[ S \in \mathcal{N}^* \Rightarrow (1.3) \iff S \in \mathcal{B}_1^v, \]

we would like to give a direct proof here. Let \( S \in \mathcal{N}^* \), then

\[ S = \sum_{n=1}^{\infty} \xi_n \otimes \phi_n \quad \text{with} \quad \sum_{n=1}^{\infty} \| \xi_n \|_{L^2} \| \phi_n \|_{M^1_v} < \infty. \]

By a direct calculation, we have

\[
\| \mathcal{G} S g_0 \|_{L^2} = \left\| \sum_{n=1}^{\infty} \xi_n \langle \pi(z)^* g_0, \phi_n \rangle_{L^2} \right\|_{L^2} \leq \sum_{n=1}^{\infty} \| \xi_n \|_{L^2} \| \langle \pi(z)^* g_0, \phi_n \rangle_{L^2} \|_{L^2}.
\]

Then,

\[
\| \mathcal{G} S g_0 \|_{L^1_v(\mathbb{R}^{2d}; L^2)} \leq \sum_{n=1}^{\infty} \| \xi_n \|_{L^2} \| \langle \pi(z)^* g_0, \phi_n \rangle_{L^2} \|_{L^1_v} = \sum_{n=1}^{\infty} \| \xi_n \|_{L^2} \| V g_0 \phi_n \|_{L^1_v} = \sum_{n=1}^{\infty} \| \xi_n \|_{L^2} \| \phi_n \|_{M^1_v}.
\]

We have now completed this proof. Moreover, we obtain that \( \| \mathcal{G} S g_0 \|_{L^1_v(\mathbb{R}^{2d}; L^2)} \lesssim \| S \|_{\mathcal{N}(L^2, M^1_v)} \) by the definition of \( \mathcal{N}(L^2, M^1_v) \). \( \square \)
Remark 3.10 As we see, the window class $B^v_1$ give a complete characterization of (1.3), while Proposition 3.9 tells us that $N^*$ is a subspace of $B^v_1$. However, we are still confused about whether $N^*$ is a proper subset of $B^v_1$. We refer to a recent article [4] for a positive answer.

4 The $B^m_{p,q}$ Operator Classes

As mentioned above, the $B^v_1$ class is the optimal window class in the framework of bounded operators on $L^2$. Here, we will introduce some more general classes of operators that may be of independent interest.

4.1 Start With the $B^{v^{-1}}_\infty$ Class

In order to introduce our general classes of operators, we would like to start with the weakest assumption. Let

$$\mathcal{H}_0 = \text{span}\{\pi(z)g_0 : z \in \mathbb{R}^{2d}\}$$

be the linear space of all finite linear combination of time-frequency shifts of the Guassian function $g_0$. For a submultiplicative weight $v$, we use $B^{v^{-1}}_\infty$ to denote the collection of linear operators $T$ defined on $\mathcal{H}_0$ satisfying

$$\|T\|_{B^{v^{-1}}_\infty} := \sup_{z \in \mathbb{R}^{2d}} \|T\pi(z)^*g_0\|_{L^2} v(z)^{-1} < \infty.$$ 

We claim that

$$B^{v^{-1}}_\infty = \mathcal{L}(M^1_v, L^2).$$

We first verify $\mathcal{L}(M^1_v, L^2) \subset B^{v^{-1}}_\infty$ by

$$\|T\pi(z)^*g_0\|_{L^2} = \|T\pi(-z)g_0\|_{L^2} \leq \|T\|_{\mathcal{L}(M^1_v, L^2)} \|\pi(-z)g_0\|_{M^1_v}$$

$$\leq v(z)\|T\|_{\mathcal{L}(M^1_v, L^2)} \|g_0\|_{M^1_v}.$$ 

This also implies $T \in B^{v^{-1}}_\infty$ with

$$\|T\|_{B^{v^{-1}}_\infty} \leq \|T\|_{\mathcal{L}(M^1_v, L^2)} \|g_0\|_{M^1_v}.$$ 

On the other hand, if $T \in B^{v^{-1}}_\infty$, for any Gabor expansion

$$f = \sum_{n=1}^{\infty} c_n \pi(z_n)g_0$$
with \( \sum_{n=1}^{\infty} |c_n|v(z_n) < \infty \), we have

\[
\| T(\sum_{n=1}^{N} c_n \pi(z_n)g_0) \|_{L^2} \leq \sum_{n=1}^{N} |c_n| \| T(\pi(z_n)g_0) \|_{L^2} \leq \| T \|_{B_{\infty}^{-1}} \sum_{n=1}^{N} |c_n|v(z_n).
\]

Then, the operator \( T \in B_{\infty}^{-1} \), first defined on \( \mathcal{H}_0 \) can be uniquely extended to a bounded linear operator from \( M_{v}^{1} \) into \( L^2 \) with

\[
\| T \|_{\mathcal{L}(M_{v}^{1}, L^2)} \lesssim \| T \|_{B_{\infty}^{-1}}.
\]

The claim is proved. Now, we take \( B_{\infty}^{-1} \) as the largest operator class in our discussion, just like the status of \( M_{v}^{\infty} \) in the class of modulation spaces (with \( p, q \in [1, \infty) \)).

### 4.2 The \( B_{p,q}^m \) Classes and Their Relations

With \( p, q \in [1, \infty] \) and \( m \in \mathcal{M}_v \), the \( B_{p,q}^m \) is defined as

\[
B_{p,q}^m = \{ T \in B_{\infty}^{-1} : \| T \pi(z)^*g_0 \|_{L^\frac{p}{q}(\mathbb{R}^{2d})} < \infty \}
\]

with the obvious norm. Since \( B_{\infty}^{-1} = \mathcal{L}(M_{v}^{1}, L^2) \), the \( B_{p,q}^m \) class can be also defined by

\[
B_{p,q}^m = \{ T \in \mathcal{L}(M_{v}^{1}, L^2) : \| T \pi(z)^*g_0 \|_{L^\frac{p}{q}(\mathbb{R}^{2d})} < \infty \}.
\]

We write \( B_{p}^m = B_{p,p}^m \). If \( m \equiv 1 \), denote \( B_{p,q} = B_{p,q}^m \).

At first view, this definition with \( p = q = 1 \) and \( m = v \) coincides with the corresponding definition in Theorem 1.2, except the description of \( T \in \mathcal{L}(M_{v}^{1}, L^2) \). We point out that these two descriptions lead to the same operator class. This fact will be clarified in Proposition 4.4 and Remark 4.5.

Observe that the modulation space \( M_{m}^{p,q} \) can be naturally isometric embedded into \( B_{p,q}^m \) by

\[
f \mapsto g_0 \otimes f,
\]

with

\[
\| g_0 \otimes f \|_{B_{p,q}^m} = \left\| \langle g_0, \pi(z)^* g_0 \rangle_{M_{v}^{1}, M_{v}^{1}} \|_{L^\frac{p}{q}(\mathbb{R}^{2d})} \right\|
= \left\| V g_0 f(-z) \right\|_{L^\frac{p}{q}(\mathbb{R}^{2d})} = \| f \|_{M_{m}^{p,q}}.
\]

In this sense, modulation space \( M_{m}^{p,q} \) can be regarded as a closed subspace of \( B_{p,q}^m \). Here, we use the notation \( \tilde{m}(z) = m(-z) \).
Like the case of modulation spaces, the $B_{p,q}^m$ classes also have some similar embedding relations.

**Proposition 4.1** Let $p_i, q_i \in [1, \infty]$, $m_i \in \mathcal{M}_v$, $i = 1, 2$. If $p_2 \geq p_1$, $q_2 \geq q_1$ and $m_2 \leq m_1$, we have

$$B_{p_2,q_2}^m \subset B_{p_1,q_1}^m \quad \text{with} \quad \| T \|_{B_{p_2,q_2}^m} \lesssim \| T \|_{B_{p_1,q_1}^m}$$

for all $T \in B_{p_1,q_1}^m$.

**Proof** Without loss of generality, we assume $m_1 = m_2 = m$. Let $\{e_n\}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. For $T \in \mathcal{L}(M^1_v, L^2_v)$, we have

$$\| T \pi(z)^* g_0 \|_{L^2_v} = \left( \sum_{n=1}^{\infty} |\langle T \pi(z)^* g_0, e_n \rangle_{L^2_v}|^2 \right)^{1/2} = \left( \sum_{n=1}^{\infty} |\langle T^* e_n, \pi(z)^* g_0 \rangle_{(M^1_v)^*, M^1_v}|^2 \right)^{1/2} \quad (4.1)$$

By Lemma 3.1, we have

$$|V_{g_0}(T^* e_n)| \leq |V_{g_0} g_0| \ast |V_{g_0}(T^* e_n)|.$$

Using this and a randomization method as in the proof of Propositions 3.3 and 3.2, we find that

$$\| T \pi(z)^* g_0 \|_{L^2_v} \lesssim \| T \pi(z)^* g_0 \|_{L^2_v} \ast |V_{g_0} g_0|. \quad (4.2)$$

By the mixed-norm Young’s inequality $L^{p_1,q_1}_m \ast L^{p_3,q_3}_v \subset L^{p_2,q_2}_m$ with

$$1 + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_3}, \quad 1 + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q_3},$$

we obtain

$$\left\| \| T \pi(z)^* g_0 \|_{L^2_v} \right\|_{L^{p_2,q_2}_m} \lesssim \left\| \| T \pi(z)^* g_0 \|_{L^2_v} \right\|_{L^{p_1,q_1}_m} \| V_{g_0} g_0 \|_{L^{p_3,q_3}_v}.$$

This proof has been finished. \qed

Recall that $v^{-1}(z) \lesssim m(z) \lesssim v(z)$, we have an immediate conclusion as follows.
Corollary 4.2 Let $p, q \in [1, \infty]$ and $m \in \mathcal{M}_v$. We have the continuous embedding relation

$$\mathcal{B}_1^v \subset \mathcal{B}_{p,q}^m \subset \mathcal{B}_\infty^{-1} = \mathcal{L}(M_1^v, L^2).$$

Proposition 4.3 Let $p, q \in [1, \infty]$ and $m \in \mathcal{M}_v$, the $\mathcal{B}_{p,q}^m$ class is a Banach space.

Proof For a Cauchy sequence $\{T_n\}_n$ in $\mathcal{B}_{p,q}^m$, by the continuous embedding $\mathcal{B}_{p,q}^m \subset \mathcal{L}(M_1^v, L^2)$, $\{T_n\}_n$ is also Cauchy in $\mathcal{L}(M_1^v, L^2)$. Since $\mathcal{L}(M_1^v, L^2)$ is a Banach space, there exists an operator $T \in \mathcal{L}(M_1^v, L^2)$ such that $T_n$ tends to $T$ in the topology of operator norm in $\mathcal{L}(M_1^v, L^2)$. For every $z \in \mathbb{Z}^d$, we have

$$\lim_{n \to \infty} \|T_n \pi(z)^* g_0\|_{L^2} = \|T \pi(z)^* g_0\|_{L^2}.$$

By using Fatou’s lemma, we verify $T \in \mathcal{B}_{p,q}^m$ by

$$\left\|T \pi(z)^* g_0\right\|_{L^2} \leq \liminf_{n \to \infty} \left\|T_n \pi(z)^* g_0\right\|_{L^2} \leq \lim_{n \to \infty} \left\|T_n\right\|_{\mathcal{B}_{p,q}^m} \leq C.$$

For sufficiently large $n, m$, we have

$$\|T_m - T_n\|_{\mathcal{B}_{p,q}^m} = \left\|(T_m - T_n) \pi(z)^* g_0\right\|_{L^2} \leq \epsilon.$$

Letting $m \to \infty$, we obtain for sufficiently large $n$

$$\left\|(T - T_n) \pi(z)^* g_0\right\|_{L^2} \leq \liminf_{m \to \infty} \left\|(T_m - T_n) \pi(z)^* g_0\right\|_{L^2} \leq \lim_{m \to \infty} \left\|(T_m - T_n) \pi(z)^* g_0\right\|_{L^2} < \epsilon.$$

From this, we have that $T_n$ tends to $T$ in the topology of $\mathcal{B}_{p,q}^m$. This proof is completed.

4.3 Connection With the Schatten Class

Proposition 4.4 Let $1 \leq p \leq 2$. The following embedding relation is valid,

$$\mathcal{B}_p \subset \mathcal{S}_p$$

in the sense that every $T \in \mathcal{B}_p$ can be extended to a bounded operator on $\mathcal{L}(L^2)$ with

$$\|T\|_{\mathcal{S}_p} \leq \|T\|_{\mathcal{B}_p}.$$
In particular, for $p = 2$ we have

$$B_2 = S_2.$$  

Proof First, we verify $B_2 = S_2$. This fact has been proved in Theorem 1.1 with $T \in \mathcal{L}(L^2)$. The only remaining thing we have to do is checking the proof for $T \in \mathcal{L}(M^1_v, L^2)$. As in the proof of Theorem 1.1, we find that for $T \in \mathcal{L}(M^1_v, L^2)$,

$$\|T\|_{B_2} = \left( \sum_{n=1}^{\infty} \|T^* e_n\|_{L^2}^2 \right)^{1/2}. \quad (4.3)$$

From this, we conclude that $T^* \in S_2$. Using this and the fact that $M^1_v$ is dense in $L^2$, we conclude that the operator $T \in B_2$ can be uniquely extended to a bounded operator on $L^2(\mathbb{R}^d)$ satisfying

$$\|T\|_{S_2} = \|T^*\|_{S_2} = \|T\|_{B_2}.$$  

This completes the proof of $B_2 \subset S_2$. The inverse direction follows directly by (4.3).

Now, we turn to the proof of $B_p \subset S_p$. Using Proposition 4.1, we conclude that $B_p \subset B_2 = S_2$, $1 \leq p \leq 2$.

Then, the operator $T \in B_p$ is compact, so it can be decomposed by

$$T = \sum_{j=1}^{\infty} \lambda_j \xi_j \otimes \eta_j,$$

where $(\lambda_j)$ denotes the singular values of $T$, $(\xi_j)_j$ and $(\eta_j)_j$ are two orthonormal systems on $L^2(\mathbb{R}^d)$. With this decomposition, we have

$$\|T \pi(z)^* g_0\|_{L^2} = \left( \sum_{j=1}^{\infty} \lambda_j^2 \|\langle \pi(z)^* g_0, \eta_j \rangle\|^2 \right)^{1/2} \geq \left( \sum_{j=1}^{\infty} \lambda_j^p \|\langle \pi(z)^* g_0, \eta_j \rangle\|^2 \right)^{1/p},$$

where we use the Hölder inequality with the fact

$$\sum_{j=1}^{\infty} \|\langle \pi(z)^* g_0, \eta_j \rangle\|^2 \leq \|\pi(z)^* g_0\|_{L^2}^2 = \|g_0\|_{L^2}^2 = 1.$$  

Then

$$\|T\|_{B_p} = \left( \int_{\mathbb{R}^{2d}} \|T \pi(z)^* g_0\|_{L^2}^p dz \right)^{1/p}.$$
\[
\left( \int_{\mathbb{R}^{2d}} \sum_{j=1}^{\infty} \lambda_j^p |\langle \pi(z)^* g_0, \eta_j \rangle|^2 \, dz \right)^{1/p} 
\]
\[
= \left( \sum_{j=1}^{\infty} \lambda_j^p \int_{\mathbb{R}^{2d}} |\langle \pi(z)^* g_0, \eta_j \rangle|^2 \, dz \right)^{1/p} 
\]
\[
= \left( \sum_{j=1}^{\infty} \lambda_j^p \| V_{g_0 \eta_j} \|_{L^2(\mathbb{R}^{2d})}^2 \right)^{1/p} = \left( \sum_{j=1}^{\infty} \lambda_j^p \right)^{1/p}. 
\]

We have \( B_p \subset S_p \) with \( \| T \|_{S_p} \leq \| T \|_{B_p} \).

\[\square\]

**Remark 4.5** Let \( v \) be a submultiplicative weight. Recall that \( v(z) \geq 1 \). Using this proposition, we have \( B^v_1 \subset B_1 \subset S_1 \). Then the operator classes \( B^v_1 \) can be re-represented as

\[
B^v_1 := \{ S \in S_1 : \| S \pi(z)^* g_0 \|_{L^2} \| L^1(\mathbb{R}^{2d}) < \infty \}
\]

or as the definition in Theorem 1.2, that is, \( B^v_1 := \{ S \in L(L^2) : \| S \pi(z)^* g_0 \|_{L^2} \| L^1(\mathbb{R}^{2d}) < \infty \} \).

**Proposition 4.6** Let \( 2 < p \leq \infty \). We have the following embedding relation

\[
S_p \subset B_p
\]

with

\[
\| T \|_{B_p} \leq \| T \|_{S_p}.
\]

**Proof** The case of \( p = \infty \) follows by

\[
S_\infty = L(L^2) \subset L(M^1, L^2) = B_\infty.
\]

For \( T \in S_p \) with \( p \in (2, \infty) \) we write

\[
T = \sum_{j=1}^{\infty} \lambda_j \xi_j \otimes \eta_j
\]

with singular values \( (\lambda_j) \), where \( (\xi_j)_j \) and \( (\eta_j)_j \) are two orthonormal systems on \( L^2(\mathbb{R}^d) \).

Using a similar way as in the proof of Proposition 4.4, we conclude that

\[
\| T \pi(z)^* g_0 \|_{L^2} = \left( \sum_{j=1}^{\infty} \lambda_j^2 |\langle \pi(z)^* g_0, \eta_j \rangle|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{\infty} \lambda_j^p |\langle \pi(z)^* g_0, \eta_j \rangle|^2 \right)^{1/p}.
\]

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Then
\[
\|T\|_{B_p} = \left( \int_{\mathbb{R}^{2d}} \| T \pi(z)^* g_0 \|_{L^2}^p dz \right)^{1/p} \leq \left( \sum_{j=1}^{\infty} \lambda_j^p \| V_{\eta_j} \|_{L^2}^2 \right)^{1/p} = \left( \sum_{j=1}^{\infty} \lambda_j^p \right)^{1/p}.
\]

We have \( S_p \subset B_p \) with \( \|T\|_{B_p} \leq \|T\|_{S_p} \).

\[\text{Remark 4.7}\] For \( p \in [1, 2) \), take \( T_1 = f \otimes g \) with \( f \in L^2, g \in L^2 \setminus M^p \). One can verify that \( T_1 \in S_1 \setminus B_p \subset S_p \setminus B_p \). From this and the relation \( B_p \subset S_p \), we conclude that \( B_p \) is a proper subset of \( S_p \) for \( p \in [1, 2) \). If \( p \in (2, \infty] \), take \( T_2 = f \otimes g \) with \( f \in L^2, g \in M^p \setminus L^2 \). We find that \( T_2 \in B_p \setminus S_{\infty} \subset B_p \setminus S_p \). This and the relation \( S_p \subset B_p \) implies that \( S_p \) is a proper subset of \( B_p \) for \( p \in (2, \infty] \). We also point out that for \( p \in (2, \infty] \), \( B_p \) is no longer a subset of \( \mathcal{L}(L^2) \), which is quite different from the Schatten class.

\subsection{4.4 \( B_{p,q}^m \) as the Operator-Valued Modulation Spaces}

According to the previous description in this paper, the window class \( B_1^1 \) is an extension of the classical window class \( M_1^1 \), and the operator class \( B_{p,q}^m \) is a generalization of \( B_1^1 \). In Sect. 4.2, we point out that the classical modulation space \( M_{m,q}^p \) can be isometric embedded into \( B_{p,q}^m \). Here, from another perspective, we will make clear that the operator classes \( B_{p,q}^m \) can be exactly regarded as the modulation spaces in the level of operators on \( \mathcal{L}(M_1^1, L^2) \).

For any \( f \in (M_1^1)^* = \mathcal{L}(M_1^1, \mathbb{C}) \), recall the classical STFT by
\[
V_{g_0} f(z) = \langle f, \pi(z)g_0 \rangle_{(M_1^1)^*, M_1^1}.
\]

We extend the STFT by
\[
\mathcal{V}_{g_0} T(z) = T(\pi(z)g_0), \quad T \in \mathcal{L}(M_1^1, L^2).
\]

Then the \( B_{p,q}^m \) norm can be re-represented by
\[
\|T\|_{B_{p,q}^m} = \left\| T(\pi(z)^*g_0) \right\|_{L_m^{p,q}(\mathbb{R}^{2d})} = \left\| \mathcal{V}_{g_0} T(z) \right\|_{L_m^{p,q}(\mathbb{R}^{2d})}.
\]

Define the operator-valued modulation spaces \( \mathcal{M}_{m,q}^p \) by
\[
\mathcal{M}_{m,q}^p = \{ T \in \mathcal{L}(M_1^1, L^2) : \left\| \mathcal{V}_{g_0} T(z) \right\|_{L_m^{p,q}(\mathbb{R}^{2d})} < \infty \}.
\]

We have the equivalent relation
\[
B_{p,q}^m = \mathcal{M}_{m,q}^p.
\]
From this, the basic properties of $\mathcal{B}_{p,q}^m$ established above can be naturally transferred to $\mathcal{M}_m^{p,q}$. Specifically, we have

$$\mathcal{M}_v^1 \subset \mathcal{M}_m^{p,q} \subset \mathcal{M}_v^\infty = \mathcal{L}(M_v^1, L^2), \quad 1 \leq p, q \leq \infty.$$ 

Note that the distribution space $(M_v^1)^* = \mathcal{L}(M_v^1, C)$ in classical modulation spaces is replaced by the “operator-valued distribution space” $\mathcal{L}(M_v^1, L^2)$.

As mentioned above, the $\mathcal{B}_{p,q}^m$ class can be regarded as the modulation spaces in the level of operators on $\mathcal{L}(M_v^1, L^2)$. From this viewpoint, all the properties of classical modulation spaces are expected to be represented in the corresponding operator-valued modulation spaces or operator classes. Here, we only point out the independence of window functions.

**Proposition 4.8** (The window functions of $\mathcal{B}_{p,q}^m$) The definition of $\mathcal{B}_{p,q}^m$ is independent of the window $\varphi \in M_v^1 \setminus \{0\}$. Different windows yield equivalent norms. For all $\varphi \in M_v^1 \setminus \{0\}$, we have

$$\mathcal{B}_{p,q}^m = \{ T \in \mathcal{L}(M_v^1, L^2) : \| T \pi(z)^* \varphi \|_{L^p,q} < \infty \}.$$ 

**Proof** For $T \in \mathcal{L}(M_v^1, L^2)$, $\varphi \in M_v^1$, by a similar calculation as in the proof of Proposition 4.1, we have

$$\| T \pi(z)^* g_0 \|_{L^2} = \left( \sum_{n=1}^{\infty} | V_{g_0}(T^* e_n)(-z) |^2 \right)^{1/2},$$ 

$$\| T \pi(z)^* \varphi \|_{L^2} = \left( \sum_{n=1}^{\infty} | V_{\varphi}(T^* e_n)(-z) |^2 \right)^{1/2}.$$ 

Using Lemma 3.1 and a randomization method, we conclude that

$$\| T \pi(z)^* \varphi \|_{L^2} \lesssim \| T \pi(z)^* g_0 \|_{L^2} \cdot | V_{g_0} \varphi |$$ 

and

$$\| T \pi(z)^* g_0 \|_{L^2} \lesssim \| \varphi \|_{L^2}^{-2} \| T \pi(z)^* \varphi \|_{L^2} \cdot | V_{\varphi} g_0 |.$$ 

By the Young inequality $L_m^{p,q} \ast L_v^1 \subset L_m^{p,q}$, we conclude that

$$\| T \pi(z)^* \varphi \|_{L^2} \lesssim \| T \pi(z)^* g_0 \|_{L^2} \cdot V_{g_0} \varphi \|_{L_v^1}$$ 

and

$$\| T \pi(z)^* g_0 \|_{L^2} \lesssim \| \varphi \|_{L^2}^{-2} \| T \pi(z)^* \varphi \|_{L^2} \cdot V_{\varphi} g_0 \|_{L_v^1}.$$ 

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We have now completed this proof.

4.5 Application to the Localization Operators

The localization operator $A_{\varphi_1, \varphi_2}^a$ with symbol $a \in S(\mathbb{R}^{2d})$, analysis window $\varphi_1$ and synthesis window $\varphi_2$ is defined formally by means of the STFT as

$$A_{\varphi_1, \varphi_2}^a f = V_{\varphi_2}^* (a V_{\varphi_1} f) = \int_{\mathbb{R}^{2d}} a(x, \xi) V_{\varphi_1} f(x, \xi) M_\xi T_x \varphi_2 \, dx \, d\xi$$

whenever the vector-valued integral makes sense. Usually, it is more convenient to interpret the definition of localization operator in a weak sense as follows

$$\langle A_{\varphi_1, \varphi_2}^a f, g \rangle_{S'({\mathbb{R}^d}), S({\mathbb{R}^d})} = \langle a, \overline{V_{\varphi_1} f} V_{\varphi_2} g \rangle_{S'({\mathbb{R}^{2d}}), S({\mathbb{R}^{2d}})}, \quad f, g \in S({\mathbb{R}^d}),$$

where the right term makes sense for $a \in S'({\mathbb{R}^{2d}})$ and $\varphi_1, \varphi_2, f, g \in S({\mathbb{R}^d})$. From this, one can find that $A_{\varphi_1, \varphi_2}^a$ is a well-defined continuous operator from $S({\mathbb{R}^d})$ into $S'({\mathbb{R}^d})$.

Here, we focus on the property of symbol $a$ when the corresponding localization operator $A_{\varphi_1, \varphi_2}^a$ belongs to the operator classes defined in this section. First, we recall a classical theorem connecting the symbols and the corresponding localization operators of $S_p$ classes. See also [2].

**Lemma 4.9** [3, Theorem 1] Let $1 \leq p \leq \infty$.

1. The mapping $(a, \varphi_1, \varphi_2) \mapsto A_{\varphi_1, \varphi_2}^a$ is bounded from $M^{p, \infty}({\mathbb{R}^{2d}}) \times M^1({\mathbb{R}^d}) \times M^1({\mathbb{R}^d})$ into $S_p$ with a norm estimate

$$\| A_{\varphi_1, \varphi_2}^a \|_{S_p} \leq B \| a \|_{M^{p, \infty}({\mathbb{R}^{2d}})} \| \varphi_1 \|_{M^1({\mathbb{R}^d})} \| \varphi_2 \|_{M^1({\mathbb{R}^d})}.$$  

2. Conversely, if $A_{\varphi_1, \varphi_2}^a \in S_p$ for all windows $\varphi_1, \varphi_2 \in M^1_v({\mathbb{R}^d})$ with

$$\| A_{\varphi_1, \varphi_2}^a \|_{S_p} \leq B \| \varphi_1 \|_{M^1({\mathbb{R}^d})} \| \varphi_2 \|_{M^1({\mathbb{R}^d})},$$

then $a \in M^{p, \infty}({\mathbb{R}^{2d}})$.

For the connection of the symbols and the corresponding localization operators of $B_p$ classes, we give the following proposition.

**Theorem 4.10** Let $1 \leq p \leq \infty$, $a \in S'({\mathbb{R}^{2d}})$.

1. The mapping $(a, \varphi_1, \varphi_2) \mapsto A_{\varphi_1, \varphi_2}^a$ is bounded from $M^{p, \infty}({\mathbb{R}^{2d}}) \times M^1({\mathbb{R}^d}) \times M^1({\mathbb{R}^d})$ into $B_p$ with a norm estimate

$$\| A_{\varphi_1, \varphi_2}^a \|_{B_p} \leq B \| a \|_{M^{p, \infty}({\mathbb{R}^{2d}})} \| \varphi_1 \|_{M^1({\mathbb{R}^d})} \| \varphi_2 \|_{M^1({\mathbb{R}^d})}.$$
2. Conversely, if $A_a^\varphi_1, \varphi_2 \in B_p$ for all windows $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$ with
\[
\|A_a^\varphi_1, \varphi_2\|_{B_p} \leq B_a \|\varphi_1\|_{M^1(\mathbb{R}^d)} \|\varphi_2\|_{M^1(\mathbb{R}^d)},
\]
then $a \in M^{p,\infty}(\mathbb{R}^{2d})$ with $\|a\|_{M^{p,\infty}} \lesssim B_a$.

**Proof** We first verify the statement (1). Using Lemma 4.9 and Proposition 4.6, we conclude that for $p \in [2, \infty]$,
\[
\|A_a^\varphi_1, \varphi_2\|_{B_p} \lesssim \|A_a^\varphi_1, \varphi_2\|_{S_p} \leq C \|a\|_{M^{p,\infty}(\mathbb{R}^{2d})} \|\varphi_1\|_{M^1(\mathbb{R}^d)} \|\varphi_2\|_{M^1(\mathbb{R}^d)}.
\]

For the case $p \in [1, 2)$, since $S_p \subset B_p$ is not valid, we need to deal with this case directly. For $a \in M^{p,\infty}(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, we find $A_a^\varphi_1, \varphi_2 \in S_\infty = \mathcal{L}(L^2)$ by using Lemma 4.9. For $z = (z_1, z_2) \in \mathbb{R}^{2d}$ and $z' = (z'_1, z'_2) \in \mathbb{R}^{2d}$, we write
\[
\langle A_a^\varphi_1, \varphi_2 \pi(z)g_0, \pi(z')g_0 \rangle_{L^2} = \langle a, V_{\varphi_1} \pi(z)g_0 V_{\varphi_2} \pi(z')g_0 \rangle_{M^\infty, M^1}.
\]

By Lemma 2.2, we have
\[
\mathcal{F}(V_{\varphi_1} \pi(z)g_0 V_{\varphi_2} \pi(z')g_0)(x, y) = \left(V_{\varphi_1} \varphi_2 \pi(z)g_0 \pi(z')g_0\right)(-y, x).
\]

Let $b(x, y) = a(y, -x)$. We have $\|b\|_{M^{p,\infty}} = \|a\|_{M^{p,\infty}}$. Write
\[
\langle a, V_{\varphi_1} \pi(z)g_0 V_{\varphi_2} \pi(z')g_0 \rangle_{M^\infty, M^1} = \langle \mathcal{F}a, \mathcal{F}(V_{\varphi_1} \pi(z)g_0 V_{\varphi_2} \pi(z')g_0) \rangle_{M^\infty, M^1}
\]
\[
= \langle \mathcal{F}b, V_{\varphi_1} \varphi_2 V_{\pi(z)g_0} \pi(z')g_0 \rangle_{M^\infty, M^1} = \langle \mathcal{F}b \cdot V_{\varphi_1} \varphi_2, V_{\pi(z)g_0} \pi(z')g_0 \rangle_{M^\infty, M^1}.
\]

By a direct calculation, we conclude that
\[
V_{\pi(z)g_0} \pi(z')g_0 = e^{2\pi i z'_1 (z'_2 - z_2)} M_{(z_2, -z'_1)} T(z'_2 - z_1) V_{g_0} g_0.
\]

The above two estimates imply that
\[
\langle a, V_{\varphi_1} \pi(z)g_0 V_{\varphi_2} \pi(z')g_0 \rangle_{M^\infty, M^1} = \langle \mathcal{F}b \cdot V_{\varphi_1} \varphi_2, M_{(z_2, -z'_1)} T(z'_2 - z_1) V_{g_0} g_0 \rangle_{M^\infty, M^1}
\]
\[
= \langle \mathcal{F}b \cdot V_{\varphi_1} \varphi_2, M_{(z_2, -z'_1)} T(z'_2 - z_1) \Phi_0 \rangle_{M^\infty, M^1}
\]
\[
= \langle \mathcal{F}b \cdot V_{\varphi_1} \varphi_2, (z'_2 - z_2, -z_1) \rangle_{M^\infty, M^1}.
\]

Here, we denote $\Phi_0 = V_{g_0} g_0$. Combining the convolution relation $M^{p,\infty} \ast M^1 \subset M^p$ (see Lemma 2.6) with
\[
\|V_{\varphi_1} \varphi_2\|_{M^1(\mathbb{R}^{2d})} \lesssim \|\varphi_1\|_{M^1(\mathbb{R}^d)} \|\varphi_2\|_{M^1(\mathbb{R}^d)},
\]

\(\square\)
we conclude that
\[ \| \mathcal{F} b \cdot V_{\varphi_1} \varphi_2 \|_{M^p(\mathbb{R}^{2d})} = \| b \ast (\mathcal{F}^{-1} V_{\varphi_1} \varphi_2) \|_{M^p(\mathbb{R}^{2d})} \]
\[ \leq \| b \|_{M^{p,\infty}(\mathbb{R}^{2d})} \| \mathcal{F}^{-1} V_{\varphi_1} \varphi_2 \|_{M^1(\mathbb{R}^{2d})} \]
\[ = \| b \|_{M^{p,\infty}(\mathbb{R}^{2d})} \| V_{\varphi_1} \varphi_2 \|_{M^1(\mathbb{R}^{2d})} \]
\[ \leq \| b \|_{M^{p,\infty}(\mathbb{R}^{2d})} \| \varphi_1 \|_{M^1(\mathbb{R}^4)} \| \varphi_2 \|_{M^1(\mathbb{R}^d)} . \]

Denote by \( F = \mathcal{F} b \cdot V_{\varphi_1} \varphi_2 \in M^p(\mathbb{R}^{2d}) \). Let
\[ \mathcal{A} : (z_1', z_2', z_1, z_2) \mapsto (z' - z, z_2, -z_1') \]
be a inverse linear transform from \( \mathbb{R}^{4d} \) into \( \mathbb{R}^{4d} \). We write
\[ |V_{\Phi_0} (\mathcal{F} b \cdot V_{\varphi_1} \varphi_2) (z' - z, (z_2, -z_1'))| = |(V_{\Phi_0} F \circ \mathcal{A})(z_1', z_2', z_1, z_2)|. \]

Form the following inequality,
\[ |V_{\Phi_0} F| \lesssim |V_{\Phi_0} \Phi_0| \ast |V_{\Phi_0} F| , \]
we obtain
\[ |V_{\Phi_0} F \circ \mathcal{A}| \lesssim |V_{\Phi_0} \Phi_0 \circ \mathcal{A}| \ast |V_{\Phi_0} F \circ \mathcal{A}| . \]

Using Young’s inequality \( L^{r,1}(\mathbb{R}^{2d} \times \mathbb{R}^{2d}) \ast L^{p,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d}) \subset L^{2,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d}) \) with \( 1 + 1/2 = 1/r + 1/p \), we conclude that
\[ \| V_{\Phi_0} F \circ \mathcal{A} \|_{L^{2,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})} \lesssim \| V_{\Phi_0} \Phi_0 \circ \mathcal{A} \|_{L^{r,1}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})} \| V_{\Phi_0} F \circ \mathcal{A} \|_{L^{p,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})} \]
\[ \lesssim \| V_{\Phi_0} F \circ \mathcal{A} \|_{L^{p,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})} \sim \| V_{\Phi_0} F \|_{L^{p,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})} \]
\[ = \| F \|_{M^p(\mathbb{R}^{2d})} . \]

The desired conclusion follows by
\[ \| A_{\alpha}^{\varphi_1,\varphi_2} \|_{B^p} = \| A_{\alpha}^{\varphi_1,\varphi_2} \pi(z) g_0 \|_{L^2(\mathbb{R}^{d})} \|_{L^p(\mathbb{R}^{2d})} \]
\[ = \| (A_{\alpha}^{\varphi_1,\varphi_2} \pi(z) g_0 , \pi(z') g_0) \|_{L^2(\mathbb{R}^{2d})} \|_{L^p(\mathbb{R}^{2d})} \]
\[ = \| V_{\Phi_0} (\mathcal{F} b \cdot V_{\varphi_1} \varphi_2) (z' - z, (z_2, -z_1')) \|_{L^2(\mathbb{R}^{2d})} \|_{L^p(\mathbb{R}^{2d})} \]
\[ = \| V_{\Phi_0} F \circ \mathcal{A} \|_{L^{2,p}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})} \]
\[ \lesssim \| F \|_{M^p(\mathbb{R}^{2d})} = \| \mathcal{F} b \cdot V_{\varphi_1} \varphi_2 \|_{M^p(\mathbb{R}^{2d})} \]
\[ \lesssim \| b \|_{M^{p,\infty}(\mathbb{R}^{2d})} \| \varphi_1 \|_{M^1(\mathbb{R}^d)} \| \varphi_2 \|_{M^1(\mathbb{R}^d)} \]
\[ = \| a \|_{M^{p,\infty}(\mathbb{R}^{2d})} \| \varphi_1 \|_{M^1(\mathbb{R}^d)} \| \varphi_2 \|_{M^1(\mathbb{R}^d)} . \]
Next, we turn to the proof of statement (2). Take $Vg_0 = \|Vg_0\|^2$. A direct calculation (see also [3, Lemma 1]) yields that

$$|V\varphi_0(\alpha, \zeta)| = |\langle a, M_{\alpha} T_{\zeta}(\overline{Vg_0Vg_0})\rangle| = |\langle a, Vg_0(M_{z_T^{-0}} 0) V M_{\alpha} T_{-\zeta_T 0} (M_{\alpha} T_{-\zeta_T 0} M_{z_T^{-0}} 0)\rangle| = |\langle A_{a} \pi(0) M_{\alpha} T_{-\zeta} 0 M_{z_T^{-0}} 0, M_{\alpha} T_{-\zeta} 0 M_{z_T^{-0}} 0\rangle| \leq \|A_{a} \pi(0) M_{\alpha} T_{-\zeta} 0 M_{z_T^{-0}} 0\|_{L^2}.$$

For any $\zeta \in \mathbb{R}^{2d}$, we conclude that

$$\|V\varphi_0(\alpha, \zeta)\|_{L^p} \leq \|A_{a} \pi(0) \|_{L^p} \leq \|A_{a} \pi(0) \|_{B^p} \leq B \|g_0\|_{M^1} \|M_{\alpha} T_{-\zeta} 0 M_{z_T^{-0}} 0\|_{M^1} = B \|g_0\|^{2}_{M^1}.$$

We have now completed the proof of statement (2). \[\square\]

**Remark 4.11** In contrast to Lemma 4.9, the conclusion in Theorem 4.10 is somewhat surprising, since in general we have $B_p \not\subset S_p$ for $p < 2$, and $S_p \not\subset B_p$ for $p > 2$. A reason for this surprising conclusion may be from the overly nice property of the window functions.

Next, we turn our attention to the Cohen class $Q_T$ for $T = A_{a}^{\Phi, \Phi}$. In this case, we assume that $\alpha \geq 0$. Then $T$ is a positive operator.

Denote by $Q_0 = [-1/2, 1/2]^{2d}$ the unit cube of $\mathbb{R}^{2d}$ centered at the origin. Let $p, q \in (0, \infty]$. We recall that the Wiener amalgam space $W(L^p(q), L^q)(\mathbb{R}^{2d})$ consists of all measurable functions for which the following norm are finite:

$$\|f\|_{W(L^p(q), L^q)(\mathbb{R}^{2d})} := \left( \sum_{k,n \in \mathbb{Z}^d} \|f T(k,n)\|_{L^q}^{q} \chi_{Q_0(v(k,n))}^{q} \right)^{1/q},$$

with the usual modification when $q = \infty$.

**Lemma 4.12** Let $\Phi(z) = |Vg_0g_0(z)|^2 = e^{-\pi|z|^2}$ for $z \in \mathbb{Z}^{2d}$. We have the following equivalent norm of $W(L^1(q), L^1(q)^{1/2})$

$$\|a\|_{W(L^1(q), L^1(q)^{1/2})} \sim \|a T\zeta \Phi\|_{L^1(q)^{1/2}}.$$

If $v$ grows at most polynomial, the window function $\Phi$ can be replaced by any nonzero Schwartz function.
Proof. For $z \in Q_0 + (k, n)$, we have $T_{(k,n)}\chi_{Q_0} \lesssim T_z\Phi$. Then 

$$\|aT_{(k,n)}\chi_{Q_0}\|_{L^p(\mathbb{R}^{2d})} \lesssim \|aT_z\Phi\|_{L^p(\mathbb{R}^{2d})}, \quad z \in Q_0 + (k, n).$$

From this and the fact $v(z) \sim v(k, n)$ for $z \in Q_0 + (k, n)$. We conclude that 

$$\left( \sum_{k,n \in \mathbb{Z}^d} \|aT_{k,n}\chi_{Q_0}\|_{L^1(\mathbb{R}^{2d})}^{1/2} v(k, n)^{1/2} \right)^2 \lesssim \|aT_z\Phi\|_{L^1} \|aT_z\Phi\|_{L^{1/2}(\mathbb{R}^{2d})}^2.$$

This estimate is also valid when $\Phi$ is replaced by any non-zero Schwartz function, since that any non-zero continuous function has a positive lower bound on a sufficiently small cube.

Let $\Phi_1(z) = e^{-\frac{|z|^2}{2}}$ and $\Phi_2(z) = e^{-\frac{|z|^2}{4}}$. For the inverse direction, notice that for $z \in Q_0 + (k, n)$ we have 

$$T_z\Phi \lesssim T_{(k,n)}\Phi_1 \quad \text{and} \quad \|aT_z\Phi\|_{L^1(\mathbb{R}^{2d})} \lesssim \|aT_{(k,n)}\Phi_1\|_{L^1(\mathbb{R}^{2d})}.$$ 

Using a similar method as above, we find that 

$$\|aT_z\Phi\|_{L^1} \|aT_z\Phi\|_{L^{1/2}(\mathbb{R}^{2d})} \lesssim \left( \sum_{k,n \in \mathbb{Z}^d} \|aT_{(k,n)}\Phi_1\|_{L^1(\mathbb{R}^{2d})}^{1/2} v(k, n)^{1/2} \right)^2.$$

Next, we write 

$$\|aT_{(k,n)}\Phi_1\|_{L^1} \leq \sum_{j,l \in \mathbb{Z}^d} \|aT_{(k,n)}\Phi_1 \cdot T_{(j,l)}\chi_{Q_0}\|_{L^1} \lesssim \sum_{j,l \in \mathbb{Z}^d} \|aT_{(j,l)}\chi_{Q_0}\|_{L^1} \|T_{(k,n)}\Phi_1 \cdot T_{(j,l)}\chi_{Q_0}\|_{L^\infty} \lesssim \sum_{j,l \in \mathbb{Z}^d} \|aT_{(j,l)}\chi_{Q_0}\|_{L^1} \Phi_2((k, n) - (j, l)).$$

From this and a convolution inequality $l^{1/2}_v \ast l^{1/2}_v \subset l^{1/2}_v$, we conclude that 

$$\left( \sum_{k,n \in \mathbb{Z}^d} \|aT_{(k,n)}\Phi_1\|_{L^1(\mathbb{R}^{2d})}^{1/2} v(k, n)^{1/2} \right)^2 \lesssim \left( \sum_{k,n \in \mathbb{Z}^d} \|aT_{(k,n)}\chi_{Q_0}\|_{L^1(\mathbb{R}^{2d})}^{1/2} v(k, n)^{1/2} \right)^2 \Phi_2 l^{1/2}_v \lesssim \left( \sum_{k,n \in \mathbb{Z}^d} \|aT_{(k,n)}\chi_{Q_0}\|_{L^1(\mathbb{R}^{2d})}^{1/2} v(k, n)^{1/2} \right)^2.$$

For the case of submultiplicative weight $v$ with at most polynomial growth, we notice that 

$$\Phi(z) \leq C_{\Phi,N} (1 + |z|)^{-N}, \quad z \in \mathbb{Z}^{2d}, \quad N \geq 1,$$
for any Schwartz function $\Phi$. Then the above argument still works in this case. We have now completed the whole proof.

**Proposition 4.13** Let $1 \leq p, q \leq \infty$. Suppose that $a \geq 0$ be a measurable function on $\mathbb{R}^{2d}$. Let $v$ be a submultiplicative weight. For any $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and $\Psi = |V_{\varphi}g_0|^2$, we have

$$\sqrt{A_{a}^{\varphi, \varphi}} \in \mathcal{B}_1^v \text{ and } a \in M^\infty \iff \|aT_2^z\Psi L^1_{L^{1/2}_v}(\mathbb{R}^{2d}) < \infty$$

with

$$\|\sqrt{A_{a}^{\varphi, \varphi}}\|_{\mathcal{B}_1^v} \sim \sqrt{\|aT_2\Psi L^1_{L^{1/2}_v}(\mathbb{R}^{2d})\|}.$$

Furthermore, if $\varphi = g_0$ or $v$ has at most polynomial growth, we have

$$\sqrt{A_{a}^{\varphi, \varphi}} \in \mathcal{B}_1^v \text{ and } a \in M^\infty \iff a \in W(L^1, L^{1/2}_v).$$

**Proof** Checking the proof of Lemma 4.12, we find that $\|\sqrt{aT_2\Psi L^1_{L^{1/2}_v}(\mathbb{R}^{2d}) < \infty}$ implies

$$a \in W(L^1, L^{1/2}_v) \subset W(L^1, L^{1/2}) \subset W(L^1, L^1) = L^1 \subset M^\infty.$$

If $a \in M^\infty$ and $a \geq 0$, by Lemma 4.9 we have $A_{a}^{\varphi, \varphi}$ be a positive operator in $\mathcal{L}(L^2)$. Write

$$\|\sqrt{A_{a}^{\varphi, \varphi}} \pi(z)^*g_0\|_{L^2}^2 = \langle A_{a}^{\varphi, \varphi} \pi(z)^*g_0, \pi(z)^*g_0 \rangle_{L^2} = \langle \pi(z)A_{a}^{\varphi, \varphi} \pi(z)^*g_0, g_0 \rangle_{L^2} = \langle A_{T_2^z a g_0, g_0}^{\varphi, \varphi}, g_0 \rangle_{L^2} = \langle T_2a, \Psi \rangle_{S', S} = \|aT_2\Psi L^1_{L^1}. $$

The desired conclusion follows by

$$\|\sqrt{A_{a}^{\varphi, \varphi}}\|_{\mathcal{B}_1^v} = \|\sqrt{A_{a}^{\varphi, \varphi}} \pi(z)^*g_0\|_{L^1_{L^{1/2}_v}(\mathbb{R}^{2d})} = \|aT_2\Psi L^1_{L^{1/2}_v(\mathbb{R}^{2d})} = \sqrt{\|aT_2\Psi L^1_{L^{1/2}_v(\mathbb{R}^{2d})}.}$$

If $\varphi = g_0$, notice that $\Psi(z) = |V_{g_0}g_0(z)|^2 = e^{-\pi |z|^2}$, we obtain $\|a\|_{W(L^1, L^{1/2}_v)} \sim \|aT_2\Psi L^1_{L^{1/2}_v(\mathbb{R}^{2d})}$ from Lemma 4.12. If $v$ has at most polynomial growth, the same conclusion also follows by Lemma 4.12. Thus, the equivalent relation $\sqrt{A_{a}^{\varphi, \varphi}} \in \mathcal{B}_1^v \iff a \in W(L^1, L^{1/2}_v)$ is valid.
As a corollary, we have the following improvement of [18, Proposition 8.2].

**Corollary 4.14** Suppose that $1 \leq p, q < \infty$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Let $a \in W(L^1, L^{1/2} \nu^2)$ be a non-negative function. Suppose that $\nu$ is a submultiplicative weight with at most polynomial growth and $m \in \mathcal{M}_\nu$. We have

$$
\|f\|_{M_p^q, \nu \mathcal{M}(\mathbb{R}^d)} \sim \|\sqrt{\mathcal{A}_a^{\varphi, \varphi} f(z)}\|_{L_p^q, \nu \mathcal{M}(\mathbb{R}^{2d})}.
$$

**Proof** By Proposition 4.13, we find that $\mathcal{A}_a^{\varphi, \varphi}$ is a positive operator in $\mathcal{L}(L^2)$, and $\sqrt{\mathcal{A}_a^{\varphi, \varphi}} \in \mathcal{B}_1$. Then the desired conclusion follows by Theorem 3.8. 

**Acknowledgements** This work was supported by the National Natural Science Foundation of China [12371100] and Natural Science Foundation of Fujian Province [2021J011192, 2022J011241].

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