REGULARITY AND ENERGY OF SECOND ORDER HYPERBOLIC 
BOUNDARY VALUE PROBLEMS ON NON-TIMELIKE 
HYPERSURFACES

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Abstract. We study the second order hyperbolic equations with initial conditions, a 
nonhomogeneous Dirichlet boundary condition and a source term. We prove the solu-
tion possesses $H^1$ regularity on any piecewise $C^1$-smooth non-timelike hypersurfaces. We 
generalize the notion of energy to these hypersurfaces, and establish an estimate of the 
difference between the energies on the hypersurface and on the initial plane where the time 
t = 0. The energy is shown to be conserved when the source term and the boundary datum 
are both zero. We also obtain an $L^2$ estimate for the normal derivative of the solution. 
In the proofs we first show these results for $C^2$-smooth solutions by using the multiplier 
methods, and then we go back to the original results by approximation.

Keywords: Hyperbolic equations, regularity, non-timelike hypersurface, energy esti-
mates, normal derivative.

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1. Introduction

Throughout the article we assume that $T$ is a fixed positive constant, $\Omega$ is an open domain 
in $\mathbb{R}^n$ ($n \geq 1$) with $C^\infty$-smooth boundary, and $A(x) = (a_{ij}(x)) \in C^{n+4}(\Omega; \mathbb{R}^n)$ is a real-valued symmetric $n \times n$ matrix function. We focus on the following second order hyperbolic 
initial/boundary value problem,

$$
\begin{align*}
\partial_t^2 u - \nabla \cdot (A(x) \nabla u) &= G & \text{in } Q := \Omega \times (0, T), \\
u &= f & \text{on } \Sigma := \partial \Omega \times (0, T), \\
u_0 &= u_0, \ u_t = u_1 & \text{on } \Omega \times \{t = 0\},
\end{align*}
$$

(1.1)

under the condition

$$
\begin{align*}
G \in L^2(Q), \ f \in H^1(\Sigma) := L^2(0, T; H^1(\partial \Omega)) \cap H^1(0, T; L^2(\partial \Omega)), \\
u_0 \in H^1(\Omega), \ u_1 \in L^2(\Omega), \ u_0(x) = f(x, 0) \text{ on } \partial \Omega.
\end{align*}
$$

(1.2)

The existence and regularity results of the solution of (1.1) under variant initial/boundary 
data can be found in the literature. The existence of the unique solution $u \in C([0, T]; H^1(\Omega))$ 
of (1.1) under the condition (1.2) is given in [LLT86 (3.5)]. In [LT81] the authors proved 
another existence result for $f \in L^2(0, T; L^2(\partial \Omega))$ and $G = 0$ by using the cosine 
operators technique, and they also showed the map from $f$ to $u$ is continuous. Then, in [LT83] 
the same authors improved the regularity of $u$ from $L^2(0, T; L^2(\Omega))$ to $C([0, T]; L^2(\Omega))$. In 
[Sak70a][Sak70b] Sakamoto studied the problem (1.1) with higher order regularities by using 
pseudo-differential operator. The book [LM72] contains a comprehensive treatments of the 
non-homogeneous boundary value problems, including hyperbolic equations. See also 
[Ava97][LT88] for related applications. These studies mentioned above treat the solution $u$ 
as maps $[0, T] \to X$ where $X$ are function spaces defined in $\Omega$, namely, the cylinder $Q$ are 
foliated horizontally and $u$ is defined in every horizontal slice $\Omega \times \{t = \tau\}$ for $\forall \tau \in [0, T]$.

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In this article, we investigate the properties of the solution of \((1.1)-(1.2)\) on non-horizontal hypersurfaces \(\Gamma_S\) described by
\[
\Gamma_S := \{(x, S(x)) ; \ x \in \Omega\}.
\]
Check Fig. 1 as an example. The restrictions of the solution on slanted hypersurfaces have already appeared in the literature. In [RU14], the authors studied the equation \((\partial_t^2 - \Delta + q)U = 0\) incited by an incident wave \(\delta(t-x_1)\). When encountered with the potential \(q\), the incident wave generates a scattered wave \(u\) such that
\[
U(x, t) = \delta(t-x_1) + u(x, t)H(t-x_1)
\]
where \(H\) is the Heaviside function. [RU14, Theorem 1] proves that the scattered wave \(u\) is also a solution of the equation \((\partial_t^2 - \Delta + q)u = 0\) and
\[
u(t, x', t) = -\frac{1}{2} \int_{-\infty}^{t} q(s, x') \, ds, \quad \forall (x', t) \in \mathbb{R}^n.
\]
The expression above involves the restriction of \(u\) on the lightlike hypersurface \(\{(t, x', t)\}\) of \(\mathbb{R}^{n+1}\). Similar situations also appeared in [RS20a, RS20b, MPMS21], and in [MS21] in the presence of a Riemannian metric.

When regarding \(\Gamma_S\) as a submanifold of \(Q\), the upward normal vector \(\nu\) of \(\Gamma_S\) is given by
\[
\nu := (\nu_x, \nu_t) = \left(\frac{-\nabla S}{\sqrt{1 + |\nabla S|^2}}, \frac{1}{\sqrt{1 + |\nabla S|^2}}\right).
\]
(1.3)

We say a hypersurface is timelike (resp. lightlike, spacelike) with respect to \(A\) at a pint \(p\) if and only if the normal vector \(\nu\) at \(p\) satisfies
\[
|\nu_x|_A < |\nu_t|, \text{ (resp. } |\nu_x|_A = |\nu_t|, \ |\nu_x|_A > |\nu_t|), \text{ where } |\xi|_A := \xi^T A(x) \xi.
\]
And the hypersurface is said to be timelike (resp. lightlike, spacelike) if it is timelike (resp. lightlike, spacelike) at every point. In this article we restrict our attentions only to non-timelike (i.e. lightlike or spacelike) hypersurfaces. Therefore, we put the following assumption.

**Assumption 1.1.** \(S \in C^1(\overline{\Omega}, [0, T])\) piecewisely, and \(\Gamma_S\) is a non-timelike hypersurface, i.e. and \(|\nabla S(x)|_A \leq 1\) for \(\forall x \in \overline{\Omega}\).

We also assume that \(-\nabla \cdot (A(x)\nabla)\) is an elliptic operator, namely,

**Assumption 1.2.** There exist two positive constants \(c_1, c_2\) such that for \(\forall \xi \in \mathbb{C}^n\) and \(\forall x \in \overline{\Omega}\),
\[
c_1|\xi|^2 \leq |\xi|^2_A := \xi^T A(x) \xi \leq c_2|\xi|^2.
\]
\(^1\text{For convenience, we shall say } "S \in C^1(\overline{\Omega}, [0, T])\text{ piecewisely}" \text{ if } S \text{ is piecewisely } C^1\text{-smooth in } \overline{\Omega} \text{ and its range is contained in } [0, T].\)
1.1. Main results. In this work we establish $H^1$ regularity and energy estimates for the solution of \eqref{1.1}-\eqref{1.2} on any non-timelike hypersurfaces. The energy is estimated not directly on the value of the energy itself, but on the difference between the energy on the hypersurface and the energy at time $t = 0$. The estimate of the difference is sharper than the estimate of the value of the energy itself, see Theorem 1.1 and the succeeding discussion below for details.

To state the main results, we introduce several notations. We define the energy $\mathcal{E}(u; \Gamma_S)$ of $u$ on $\Gamma_S$ whenever the following expression can be well-defined:

$$\mathcal{E}(u; \Gamma_S) := \int_{\Omega} \left[ |\nabla(u(x, S(x)))|^2_A + (1 - |\nabla S(x)|^2_f)|u_t(x, S(x))|^2 \right] \, dx.$$  

(1.4)

When $S(x) = \text{constant}$, \eqref{1.4} coincides with the classical energy definition. The first result involves \eqref{1.1} underneath $\Gamma_S$, so let us introduce the following notations (see Fig. 2):

$$\begin{align*}
Q &:= \{(x,t) : x \in \Omega, \tau \leq t \leq S(x) \} \subset Q, \\
\Sigma &:= \{(x, \tau) : x \in \partial \Omega, \tau \leq t \leq S(x) \} \subset \Sigma, \\
H &:= Q_0 \cap \{(x, \tau) : x \in \Omega\}, \\
\Gamma_{S, \tau} &:= \{(x, S(x)) : \tau \leq S(x) \leq T_2\}.
\end{align*}$$

Note that when $\tau \leq T_1$, $\Gamma_{S, \tau} = \Gamma_S$. And when $\Gamma_S$ is not horizontal, $Q_0$ and $\Sigma_0$ will be strict subsets of $Q$ and $\Sigma$, respectively.

In what follows we use the notation $\langle T_2 \rangle := (1 + |T_2|^2)^{1/2}$ for simplicity.

**Theorem 1.1.** Given Assumptions \ref{1.1} and \ref{1.2}, then in the system \eqref{1.1}-\eqref{1.2}, the restriction of the solution $u$ on $\Gamma_S$ is in $H^1(\Gamma_S)$, and $\mathcal{E}(u; \Gamma_S)$ is well-defined. Moreover, there holds

$$|\mathcal{E}(u; \Gamma_S) - \int_{\Omega} \left[ |\nabla u_0|^2_A + |u_1|^2 \right] \, dx|$$

$$\leq C \langle T_2 \rangle^{1/2} \left[ \|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} + \langle T_2 \rangle^{1/2}(\|f\|_{H^1(\Sigma_0)} + \|G\|_{L^2(Q_0)}) \right]$$

$$\times \left( \|f\|_{H^1(\Sigma_0)} + \|G\|_{L^2(Q_0)} \right),$$

for some constant $C$ depending only on $A$ and the dimension $n$.

**Remark 1.1.** When $\Gamma_S$ is not horizontal, we have $Q_0 \subsetneq Q$ and $\Sigma_0 \subsetneq \Sigma$, so the estimate given in Theorem 1.1 only requires parts of the data $f$ and $G$. When $\Gamma_S$ is strictly spacelike, i.e. $|\nabla S(x)|_A \leq C < 1$ for all $x \in \Omega$ for some constant $C$, Theorem 1.1 implies $u_t|_{\Gamma_S} \in L^2(\Gamma_S)$ as well.

In [Tay11] Chapter 2 Fig. 6.1, 6.2, 8.1 Taylor studied the spacelike cases where the lateral boundary is an empty set. The lightlike case is not covered by [Tay11]. Also, the results given in [Tay11] is given in a form of \eqref{1.5} below, and the inequality in Theorem 1.1 seems
to be new. In some applications it is more convenient to use the following estimate, which is a direct consequence of Theorem 1.1.

\[ \mathcal{E}(u; \Gamma_S) \leq C(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}) + C(T_2)(\|f\|^2_{H^1(\Sigma_0)} + \|G\|^2_{L^2(Q_0)}). \]  

(1.5)

But readers should note that the estimate given in Theorem 1.1 is sharper than (1.3).

The energy can also be defined on different non-timelike hypersurfaces. Denote \( S_0(x) \equiv 0 \) and \( \Gamma_0 := \{(x, S_0(x)) : x \in \overline{\Omega}\} \) so that \( \Gamma_0 = \overline{\Omega} \times \{t = 0\} \). And we define a family of hypersurfaces \( \{\Gamma_{\tau}\}_{\tau > 0} \) (see Fig. 3) by

\[ \Gamma_{\tau} := \{(x, S_{\tau}(x)) : x \in \overline{\Omega}\}, \]

with \( S_{\tau} \) satisfying the following requirement,

\[ S_{\tau} \in C^1(\overline{\Omega}; [0, T]) \]

piecewisely, such that for \( \tau_1 < \tau_2 \), \( S_{\tau_1}(x) \leq S_{\tau_2}(x) \) \( \forall x \in \overline{\Omega} \).  

(1.6)

**FIGURE 3.** Non-timelike hypersurfaces

If for every fixed \( x \in \overline{\Omega} \), the function \( S_{\tau}(x) \) is continuous with respect to \( \tau \), then the family \( \{\Gamma_{\tau}\}_{\tau > 0} \) will form another foliation of \( \overline{\Omega} \) comparing to the standard foliation \( \Omega \times \{t = \tau\} \) with \( \tau \in [0, T] \). We define the corresponding energy on \( \Gamma_{\tau} \) whenever it can be well-defined:

\[ \mathcal{E}(u; \Gamma_{\tau}) := \int_{\Omega} \left[ \|\nabla (u(x, S_{\tau}(x)))\|^2_{A} + (1 - |\nabla S_{\tau}(x)|^2) \|u(x, S_{\tau}(x))\|^2 \right] \, dx. \]

(1.7)

By Theorem 1.1, the following corollary about \( \mathcal{E}(u; \Gamma_{\tau}) \) is an immediate result.

**Corollary 1.2.** Given Assumptions 1.1 and 1.2 and assuming the family \( \{\Gamma_{\tau}\}_{\tau > 0} \) satisfies (1.6), then in the system (1.1), (1.2), when \( G = 0 \) and \( f = 0 \), the energy \( \mathcal{E}(u; \Gamma_{\tau}) \) is well-defined and is conserved, i.e.

\[ \mathcal{E}(u; \Gamma_{\tau}) = \int_{\Omega} (\|\nabla u_0\|^2_{A} + |u_1|^2) \, dx, \quad \text{for} \quad \tau \geq 0. \]

Corollary 1.2 generalizes the classical energy conservation law, which says the energy is conserved on every horizontal surface \( \Omega \times \{t = \tau\} \). It also generalizes the energy conservation in [Tay11] §2.5 & §2.6] to the complex-valued case.

We also obtain an estimate of the conormal derivative (with respect to \( A \)) \( u_{\nu, A} := \nu_{\Sigma} \cdot A(x) \nabla u \), where \( \nu_{\Sigma} \) signifies the outer unit normal vector to \( \Sigma \).

**Theorem 1.3.** Under the same assumptions as in Theorem 1.1, we have

\[ \|u_{\nu, A}\|^2_{L^2(\Sigma_0)} \leq C(T_2)(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}) + C(T_2)^2(\|G\|^2_{L^2(Q_0)} + \|f\|^2_{H^1(\Sigma_0)}), \]

for some constant \( C \) depending only on \( A \) and the dimension \( n \).
Remark 1.2. When $S(x) = \text{constant}$, the hypersurface $\Gamma_S$ will be horizontal and so $Q_0 = Q$ and $\Sigma_0 = \Sigma$. In this case, the estimate about $u_{\nu,A}$ is proved in [LLT86] (4.7) (note that $|u_{\nu,A}| \leq C|u_\nu|$). However, when $\Gamma_S$ is not horizontal, Theorem 1.3 seems to be new.

The proofs of Theorems 1.1, 1.3 and Corollary 1.2 are presented in Section 3.

1.2. The motivation. The classical result [LLT86, Remark 2.10] says that the solution $u$ of (1.1)-(1.2) satisfies

$$u \in C([0,T];H^1(\Omega)), \quad \partial_t u \in C([0,T];L^2(\Omega)),$$

which implies

$$u \in H^1(Q).$$

Therefore by the trace theorem, the restriction of $u$ to $\Gamma_S$ has regularity $H^{1/2}$. However, when $\Gamma_S$ is horizontal, e.g. when $\Gamma_S = \Omega \times \{t = \tau\}$ for some $\tau$, the restriction of $u$ on $\Gamma_S$ has $H^1$ regularity due to [LLT86] Remark 2.10. From the point of view of the relativity of simultaneity in the theory of relativity [Wal84], $\Omega \times \{t = \tau\}$ should not be more special than other non-timelike hypersurfaces $\Gamma_S$. Therefore, one would expect that the trace theorem for the solution $u$ is not sharp and $u$ shall also enjoy the same $H^1$ regularity on slanted $\Gamma_S$ as on $\Omega \times \{t = \tau\}$. This is the motivations of the work.

Let us explain the idea by a simplified example. Assume there is a sound wave $u(x,t)$ propagating inside a domain $\Omega$ which satisfies the wave equation $(\partial_t^2 - \Delta)u = 0$, and the sound speed is normalized to 1 in the medium, i.e. $A(x) = 1$ in $\Omega$. An observer $X$ is located inside $\Omega$ and he/she stands still relative to $\Omega$. Another observer $Y$ is also located inside $\Omega$ but is moving in the direction of the first axis, say, $x_1$, at a constant speed $v$ which is slower than that sound speed, $|v| < 1$. We assume the $\Omega$ is large enough such that all the events mentioned here take place inside $\Omega \times [0,T]$ for $T$ large enough. Then, from the perspective of $X$, the simultaneity at time $\tau$ is $\Omega \times \{t = \tau\}$, while from the perspective of $Y$ the simultaneity is a slanted plane $\Gamma_S$ with $S(x) = vx$, see Fig. 4. Let $(t,x)$ be the spacetime coordinate of $X$ and we denote $\gamma = (1 - v^2)^{-1/2}$ as the Lorentz factor. Then according to the theory of relativity, the spacetime coordinate $(\tilde{t},\tilde{x})$ of $Y$ should satisfy

$$\tilde{t} = \gamma(t - vx_1), \quad \tilde{x}_1 = \gamma(x_1 - vt), \quad \tilde{x}_j = x_j \ (j = 2, \ldots, n).$$

The sound wave $\tilde{u}$ that $Y$ experienced should be $\tilde{u}(\tilde{x},\tilde{t}) := u(x,t)$. The wave equation is preserved under the Lorentz transformation, namely,

$$(\partial_{\tilde{t}}^2 - \Delta_{\tilde{x}})\tilde{u}(\tilde{x},\tilde{t}) = (\partial_t^2 - \Delta_x)u(x,t).$$
This means that what Y heard is also a wave which satisfies the same physical law with the sound that X heard. Theorem 2.1 tells us the profile of the sound (the wave shape across the space at a fixed time) that Y heard has the same $H^1$ spacial regularity with the sound that X heard.

This article is organized as follows. In Section 2 we make some preparations which are necessary for the subsequent analysis. Section 3 is devoted to the proof of an intermediate result in which the solution is assumed to have $C^2$-smoothness. Then the $C^2$-smoothness constraint is lifted in Section 4 by dealing with a compatibility issue and a regularity issue consecutively. We conclude this work in Section 5 with some remarks about possible future directions.

2. Some preparations

Throughout the article we denote by $C$ a generic constant whose value may varies from line to line. We use $\text{div}_x, t(\vec{a}, b)$ to signify $\nabla \cdot \vec{a} + \partial_t b$. The following lemma shall be used in Section 3 where the constraint $u \in C$ then at every $C$ result in which the solution is assumed to have $C^2$-smoothness.

Lemma 2.1. Assume $u \in C^2$, and $\vec{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_n(x)) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ piecewisely, then at every $C^1$ continuous point of $\vec{\varphi}$, we have the following identities:

$$2 \text{Re}\{\tau_t[\partial_t^2 u - \nabla \cdot (A(x)\nabla u)]\}$$

$$= \text{Re}\text{div}_x, t[\vec{\varphi}(\nabla u)_A^2 - |u_t|^2 + |\nabla u_A|^2],$$

$$2 \text{Re}\{[\vec{\varphi} \cdot \nabla u]\partial_t^2 u - \nabla \cdot (A(x)\nabla u)]\}$$

$$= \text{Re}\text{div}_x, t[\vec{\varphi}(\nabla u)_A^2 - |u_t|^2] - 2(\vec{\varphi} \cdot \nabla u)A\nabla u, 2(\vec{\varphi} \cdot \nabla u)u_t$$

$$+ (\nabla \cdot \vec{\varphi})(|u_t|^2 - |\nabla u_A|^2) - (\vec{\varphi} \cdot \nabla)(\nabla u, \nabla u) + 2(\partial_j \varphi_k)\text{Re}(\tau_{kJ}a_{jl}u_l),$$

(2.1)

where $\text{Re}$ stands for the real part, and the summation convention is called for the last term.

Proof. A straightforward computation shows

$$2 \text{Re}\{\tau_t \partial_t^2 u\} = (|u_t|^2)'$$

$$2 \text{Re}\{\tau_t \nabla \cdot (A\nabla u)\} = 2 \text{Re}\nabla \cdot (\tau_t A\nabla u) - (|\nabla u_A|^2)'$$

which give (2.1).

Similarly, we can show

$$2 \text{Re}\{[\vec{\varphi} \cdot \nabla u]\partial_t^2 u\} = \text{Re}\partial_t[2(\vec{\varphi} \cdot \nabla u)|u_t] - \vec{\varphi} \cdot 2 \text{Re}(\tau_t \nabla u)$$

$$= \text{Re}\partial_t[2(\vec{\varphi} \cdot \nabla u)|u_t] - \vec{\varphi} \cdot (|u_t|^2)$$

$$= \text{Re}\partial_t[2(\vec{\varphi} \cdot \nabla u)|u_t] - \vec{\nabla} \cdot (\vec{\varphi} |u_t|^2) + (\nabla \cdot \vec{\varphi})|u_t|^2.$$  

(2.2)

$\vec{\varphi}$ is a real-valued vector function, so we also have

$$2 \text{Re}\{\vec{\varphi} \cdot \nabla u\nabla \cdot (A\nabla u)\} = \text{Re}\nabla \cdot [2(\vec{\varphi} \cdot \nabla u)A\nabla u] - 2 \text{Re}[2(\vec{\varphi} \cdot \nabla u) \cdot A\nabla u]$$

$$= \text{Re}\nabla \cdot (2[\vec{\varphi} \cdot \nabla u]A\nabla u) - \text{Re}(\partial_j \varphi_k)\tau_{kJ}a_{jl}u_l| - 2 \text{Re}[\varphi_k \tau_{kJ}a_{jl}u_l].$$

(2.3)

We compute $\varphi_k \tau_{kJ}a_{jl}u_l$ as follows,

$$\varphi_k \tau_{kJ}a_{jl}u_l = \varphi_k \partial_k(\tau_{kJ}a_{jl}u_l) - \varphi_k \tau_{kJ}(\partial_k a_{jl})u_l - \varphi_k \tau_{kJ}a_{lj}u_k$$

$$= \vec{\varphi} \cdot \nabla(|\nabla u_A|^2) - (\vec{\varphi} \cdot \nabla A)(\nabla u, \nabla u) - \varphi_k u_{kj}a_{lj}u_l$$

$$= \nabla \cdot (\vec{\varphi} \nabla u_A^2) - (\vec{\varphi} \cdot \nabla u)|\nabla u_A|^2 - (\vec{\varphi} \cdot \nabla A)(\nabla u, \nabla u) - \varphi_k u_{kj}a_{lj}u_l.$$

(2.4)

Because $A$ is symmetric and $\varphi$ are real-valued, we can conclude from (2.3) that

$$2 \text{Re}[\varphi_k \tau_{kJ}a_{jl}u_l] = \nabla \cdot (\vec{\varphi} \nabla u_A^2) - (\vec{\varphi} \nabla u)\nabla u_A^2 - (\vec{\varphi} \cdot \nabla A)(\nabla u, \nabla u).$$

(2.5)
Combining (2.4) and (2.6), we arrive at
\[
2 \text{Re} \left\{ (\varphi \cdot \nabla u) \nabla \cdot (A \nabla u) \right\} = \text{Re} \nabla \cdot \left[ 2(\varphi \cdot \nabla u) A \nabla u \right] - 2(\partial_j \varphi_k) \text{Re}(\overline{u}_k a_{ij} u_l) \\
- \nabla \cdot (\varphi \nabla |u|^2_A) + (\nabla \cdot \varphi)|u|^2_A + (\varphi \cdot \nabla)(\nabla u, \nabla u). \tag{2.7}
\]
Subtracting (2.7) from (2.4), we arrive at (2.6). The proof is done. \qed

In the following lemma we abbreviate \(E(u; \Gamma_S)\) as \(E(u)\) for simplicity, and we show the energy possesses a similar triangle inequality property.

**Lemma 2.2.** For any \(u_1, u_2\) such that \(E(u_1), E(u_2)\) are well-defined, we have \(E(u_1 - u_2)\) is also well-defined and

\[
|E(u_1) - E(u_2)| \leq E(u_1 - u_2) + 2\sqrt{E(u_1 - u_2)} \sqrt{E(u_j)}, \quad j = 1, 2.
\]

**Proof.** First, similar to \(|a + b| \leq |a| + |b|\), we also have \(|a + b|_A \leq |a|_A + |b|_A\). This is because

\[
|a + b|_A^2 = |a|^2_A + |b|^2_A + 2 \text{Re}\{\overline{a} \cdot b\} = |a|^2_A + |b|^2_A + 2 \text{Re}\{A^{1/2} a \cdot A^{1/2} b\}
\]

\[
\leq |a|^2_A + |b|^2_A + 2|a|_A |b|_A = (|a|_A + |b|_A)^2.
\]

By this triangle inequality, it is straightforward to check \(E(u_1 - u_2) \leq 2[E(u_1) + E(u_2)]\), so \(E(u_1 - u_2)\) is well-defined.

The triangle inequality obtained above also gives \(|a|_A - |b|_A \leq |a - b|_A\). Hence

\[
|\nabla(u_1)|^2_A - |\nabla(u_2)|^2_A = |\nabla(u_1)|_A - |\nabla(u_2)|_A|^2 + 2 |\nabla(u_1)|_A - |\nabla(u_2)|_A| |\nabla(u_2)|_A
\]

\[
\leq |\nabla(u_1 - u_2)|^2_A + 2 |\nabla(u_1 - u_2)|_A |\nabla(u_2)|_A.
\]

Similarly,

\[
|\partial_t u_1|^2 - |\partial_t u_2|^2 \leq |\partial_t(u_1 - u_2)|^2 + 2 |\partial_t(u_1 - u_2)||\partial_t u_2|.
\]

Hence,

\[
E(u_1) - E(u_2)
\]

\[
\leq E(u_1 - u_2) + 2 \int [ |\nabla(u_1 - u_2)|_A |\nabla(u_2)|_A + (1 - |\nabla S(x)|^2_A)|\partial_t(u_1 - u_2)||\partial_t u_2| ] \ dx
\]

\[
\leq E(u_1 - u_2) + 2 \int \frac{|\nabla(u_1 - u_2)|^2_A + (1 - |\nabla S(x)|^2_A)|\partial_t(u_1 - u_2)|^2}{\nabla(u_2)|_A^2 + (1 - |\nabla S(x)|^2_A)|\partial_t u_2|^2} \ dx
\]

\[
\leq E(u_1 - u_2) + 2 \sqrt{E(u_1 - u_2)} \sqrt{E(u_2)}.
\]

Similar arguments also imply

\[
E(u_1) - E(u_2) \leq E(u_1 - u_2) + 2 \sqrt{E(u_1 - u_2)} \sqrt{E(u_j)}.
\]

Therefore,

\[
|E(u_1) - E(u_2)| \leq E(u_1 - u_2) + 2 \sqrt{E(u_1 - u_2)} \sqrt{E(u_j)}, \quad j = 1, 2.
\]

The proof is done. \qed
2.1. The decomposition of the gradient. This part is devoted to the analysis of the relation between \( |\nabla u|^2_A \) and \( |w_{\nu,A}|^2 \) which will appear in (3.9). Here \( \nu \Sigma \) stands for the outer unit normal vector to \( \Sigma \), and \( w_{\nu,A} \) signifies the conormal derivative with respect to \( A \), i.e. \( w_{\nu,A} := \nu \cdot A \nabla u \). Readers can skip this part for the first time.

Recall that \( \Sigma = \partial \Omega \times (0,T) \). When \( \Omega \) is the unit ball and \( A(x) \) is the identity matrix, by straightforward computations it can be checked that on \( \Sigma \) we have the following identity:

\[
|\nu \Sigma|^2_A |\nabla u|^2_A = |w_{\nu,A}|^2 + \frac{1}{2} \sum_{i \neq j} |X_{ij} u|^2, \quad \text{where } X_{ij} = x_i \partial_j - x_j \partial_i, \ |x| = 1, \tag{2.8}
\]

see e.g. [RS20b, (1.19)]. Note that here \( X_{ij} \) are vector fields tangential to the sphere of the unit ball. Therefore, the square norm of the gradient \( |\nabla u|^2 \) is decomposed into the desired term \( |w_{\nu,A}|^2 \) along with other terms which are tangential gradients on \( \Sigma \). For general domain \( \Omega \) and general matrix \( A(x) \), we can show a similar decomposition result.

Lemma 2.3. Assume \( \Omega \) is a \( C^1 \)-smooth domain, and the matrix \( A(x) \) satisfies Assumption 1.2 with constants \( c_1 \) and \( c_2 \). Then for a \( C^1 \)-smooth function \( u \), there exist two constants \( C \) depending only on \( c_1, c_2 \) and the dimension \( n \) such that on \( \Sigma \) we have

\[
|\nu \Sigma|^2_A |\nabla u|^2_A \leq 1 \times |w_{\nu,A}|^2 + C |w_{\nu,A}| |\nabla u| + C |\nabla_{\Sigma} u|^2, \tag{2.9}
\]

where \( \nabla_{\Sigma} u \) represents the tangential gradient of \( u \) on \( \Sigma \).

Remark 2.1. In Lemma 2.3 we emphasize the coefficient of the term \( |w_{\nu,A}|^2 \) is exactly 1.

Proof of Lemma 2.3. Denote \( e_1 = \nu \Sigma \) and fix an orthonormal basis \( \{e_2, \ldots, e_{n+1}\} \) of a local chart of \( \Sigma \). Then \( \{e_1, e_2, \ldots, e_{n+1}\} \) is also an locally orthonormal basis of \( \mathbb{R}^{n+1} \). The conclusion (2.9) is a local estimate so local arguments are enough for the proof.

For simplicity we denote

\[
E(x) = (A e_1, e_2, \ldots, e_{n+1}).
\]

Assumption 1.2 guarantees that \( E(x) \) is always invertible; this is because \( e_1 \cdot A e_1 \geq c_1 |e_1|^2 = c_1 > 0 \), so \( A e_1 \) always has nonzero component in \( e_1 \) direction, and so \( \{A e_1, e_2, \ldots, e_{n+1}\} \) is always linearly independent. Hence, thanks to the existence of \( E^{-1}(x) \), we can compute

\[
|\nabla u|^2_A = (\nabla u)^T A \nabla u = (\nabla u)^T E(x) [A^{-1/2} E(x)]^{-1} [E^T(x) A^{-1/2}]^{-1} E^T(x) \nabla u
\]

\[
= (\nabla u)^T E(x) E^{-1}(x) A E^{-1, T}(x) E^T(x) \nabla u
\]

\[
= (u_{\nu,A}, e_2 \cdot \nabla u, \ldots, e_{n+1} \cdot \nabla u) E^{-1,T}(x)(u_{\nu,A}, e_2 \cdot \nabla u, \ldots, e_{n+1} \cdot \nabla u)^T.
\]

Here \( E^{-1,T}(x) \) signifies the transpose of the inverse of the matrix \( E(x) \). Let us denote

\[
M(x) = E^{-1}(x) A E^{-1, T}(x)
\]

and use \( M_{ij} \) to signify the elements of \( M \). Note that \( M \) is symmetric, then we have

\[
|\nabla u|^2_A = M_{11} |w_{\nu,A}|^2 + 2 \text{Re} \{\nabla u_A \sum_{j=2}^n M_{1j} (e_j \cdot \nabla u)\} + \sum_{k,l=2}^n M_{kl} (e_k \cdot \nabla u)(e_l \cdot \nabla u). \tag{2.10}
\]

We claim that \( e_l \cdot \nabla u \) is a component of the tangential gradient of \( u \) on \( \Sigma \), which is similar to the \( X_{ij} u \) term in (2.8). To see that, we choose a \( C^1 \)-smooth curve \( \gamma: (-1,1) \to \Sigma \) on \( \Sigma \) satisfying \( \gamma(0) = x \) and \( \dot{\gamma}(0) = e_l \), then \( e_l \cdot \nabla u(x) \) can be represented as \( \frac{d}{dt} |_{t=0} (u(\gamma(t))) \). This justifies our claim. Hence, we have

\[
|e_l \cdot \nabla u| \leq |\nabla_{\Sigma} u|, \quad l = 2, \ldots, n + 1,
\]

and so we can continue (2.10) as

\[
|\nu \Sigma|^2_A |\nabla u|^2_A = |\nu \Sigma|^2_A (M_{11} |w_{\nu,A}|^2 + C |w_{\nu,A}| |\nabla u| + C |\nabla_{\Sigma} u|^2)
\]
\( E \leq \tau \)

\( \tau \)

\( \pi \)

the following functional

\( E \)

abbreviate

\( E \)

number

for some constant \( C \) and \( n \).

It is left to show

\[
|\nu_\Sigma|^2 M_{11} = 1. \tag{2.12}
\]

To see this, let us represent the inverse matrix \( E^{-1}(x) \) as

\[
E^{-1}(x) = (f_1(x), f_2(x), \cdots, f_{n+1}(x))^T,
\]

then the matrix identity \( E^{-1}(x)E(x) = I \) gives

\[
f_1(x) \cdot e_j(x) = 0, \quad j = 2, \cdots, n + 1. \tag{2.13}
\]

\[
f_1(x) \cdot A(x)e_1(x) = 1. \tag{2.14}
\]

Because \( \{e_1, e_2, \cdots, e_{n+1}\} \) is an orthonormal basis in a chart, from (2.13) we see \( f_1(x) \) is parallel to \( e_1(x) \), i.e. \( f_1(x) = \lambda(x)e_1(x) \) for some function \( \lambda(x) \). Substitute this into (2.14), we see \( \lambda(x) = |e_1(x)|^2_{A} = |\nu_\Sigma|^2_{A}. \) Therefore, we have

\[
M_{11}|\nu_\Sigma|^2_{A} = f_1^T(x)A(x)f_1(x)|\nu_\Sigma|^2_{A} = \lambda(x)e_1^T(x)A(x)\lambda(x)e_1(x)|\nu_\Sigma|^2_{A} = \lambda^2(x)|\nu_\Sigma|^4_{A} = 1,
\]

which is (2.12). Combining (2.11) and (2.12), we can complete the proof. \( \square \)

### 3. The \( C^2 \) smooth case

Recall \( \langle T_2 \rangle := (1 + |T_2|^2)^{1/2} \). In this section we aim to prove the following result.

**Proposition 3.1.** Given Assumptions [L1] and [L2], Assume \( u \in C^2(\overline{Q}) \) solves (1.1)-(1.2). Then we have

\[
|\mathcal{E}(u; \Gamma_S) - \int_\Omega (|\nabla u_0|^2_A + |u_1|^2) \, dx|
\]

\[
\leq C \langle T_2 \rangle^{1/2} \left[ ||\nabla u_0||_{L^2(\Omega)} + ||u_1||_{L^2(\Omega)} + \langle T_2 \rangle^{1/2} \left( ||f||_{H^1(\Sigma_0)} + ||G||_{L^2(Q_0)} \right) \right]
\]

\[
\times (||f||_{H^1(\Sigma_0)} + ||G||_{L^2(Q_0)}),
\]

for some constant \( C \) depending only on \( A \) and the dimension \( n \). Especially, when \( f = 0 \) and \( G = 0 \) in (1.2), we have

\[
\mathcal{E}(u; \Gamma_S) = ||\nabla u_0||^2_{L^2(\Omega)} + ||u_1||^2_{L^2(\Omega)}. \tag{3.1}
\]

This section and a major portion of Section 4 involve the same hypersurface \( \Gamma_S \), so we abbreviate \( \mathcal{E}(u; \Gamma_S) \) as \( \mathcal{E}(u) \) for short in these parts. Also, for technical reasons we introduce the following functional \( \mathcal{E}_\tau(u) \) which we shall call it **partial energy** and which takes a real number \( \tau \) as its parameter,

\[
\mathcal{E}_\tau(u) := \int_{\pi(\Gamma_{S,\tau})} \left[ |\nabla (u(x, S(x)))|^2_A + (1 - |\nabla S(x)|^2_A)u_t(x, S(x))^2 \right] \, dx, \tag{3.2}
\]

where \( \pi: (x,t) \mapsto x \) is a projection map. Readers can compare (3.2) with (1.4), and shall distinguish the notation \( \mathcal{E}_\tau(u) \) with \( \mathcal{E}(u; \Gamma_\tau) \) defined on different hypersurfaces \( \Gamma_\tau \).

When \( \tau \leq T_1 \), we have \( \pi(\Gamma_{S,\tau}) = \Omega \), so \( \mathcal{E}_\tau(u) \) will be a constant with respect to \( \tau \) when \( \tau \leq T_1 \). Under Assumption [L1] we always have \( \mathcal{E}_\tau(u) \leq \mathcal{E}(u) \); and when \( \tau \leq T_1 \) we have \( \mathcal{E}_\tau(u) = \mathcal{E}(u) \).

The relationship between \( ||u||_{H^1(\Gamma_S)} \) and \( \mathcal{E}(u) \) are given below,

\[
\left( ||u||^2_{H^1(\Gamma_S)} \right) = \int_\Omega |\nabla (u(x, S(x)))|^2 \, dx \leq c_1^{-1} \int_\Omega |\nabla (u(x, S(x)))|^2_A \, dx
\]
where we used Assumptions 1.1 and 1.2. For readers convenience we also record the following identity,

\[ |\nabla(u)|^2_A + (1 - |\nabla S|_A^2)|u_t|^2 = |\nabla u|^2_A + |u_t|^2 + 2 \Re \{ \nabla \cdot S \cdot \nabla u \}. \]

The arguments of proving Proposition 3.1 are divided into several steps.

**Lemma 3.2.** Under the same condition as in Proposition 3.1 we have

\[ \mathcal{E}_r(u) \leq 2 \int_{Q_\tau} (|\nabla u(x, \tau)|_A^2 + |u_t(x, \tau)|^2) \, dx + 6\|f_s\|_{L^2(\Sigma)}\|u_{\nu, A}\|_{L^2(\Sigma)} + 8(T_2 - \tau)\|G\|_{L^2(Q_r)}^2. \]

**Proof.** Equation (1.1) gives \( \partial_t^2 u - \nabla \cdot (A(x) \nabla u) = G \) in \( Q_\tau \). Hence, integrating the identity (2.1) in \( Q_\tau \), we can have

\[
\begin{align*}
&\Re \int_{Q_\tau} 2\mu_t G = \Re \int_{Q_\tau} \nu Q_\tau \cdot (-2 \mu_t A \nabla u, |u_t|^2 + |\nabla u|^2_A) \\
&= \Re \int_{H^+} (\sigma, -1) \cdot (-2 \mu_t A \nabla u, |u_t|^2 + |\nabla u|^2_A) + \Re \int_{\Sigma^+} \nu^+ \cdot (-2 \mu_t A \nabla u, |u_t|^2 + |\nabla u|^2_A) \\
&\quad + \Re \int_{\Gamma_{S^+}} \nu \cdot (-2 \mu_t A \nabla u, |u_t|^2 + |\nabla u|^2_A) \quad (\nu \text{ is defined in (1.3)}) \\
&= -\int_{H^+} |u_t|^2 + |\nabla u|^2_A \, d\sigma - 2 \Re \int_{\Sigma^+} \mu_t u_{\nu, A} \, d\sigma \quad (u_{\nu, A} := \nu^- \cdot A \nabla u) \\
&\quad + \Re \int_{\Gamma_{S^+}} (2 \nabla S \cdot \mu_t A \nabla u + |u_t|^2 + |\nabla u|^2_A) \frac{d\sigma}{\sqrt{1 + |\nabla S|^2}} \\
&= -\int_{H^+} |u_t|^2 + |\nabla u|^2_A \, d\sigma - 2 \Re \int_{\Sigma^+} \mu_t u_{\nu, A} \, d\sigma \\
&\quad + \int_{\pi(\Gamma_{S^+})} (2 \Re \{ \nabla S \cdot \mu_t A \nabla u \} + |u_t|^2 + |\nabla u|^2_A) \, dx \\
&= -\int_{H^+} |u_t|^2 + |\nabla u|^2_A \, d\sigma - 2 \Re \int_{\Sigma^+} \mu_t u_{\nu, A} \, d\sigma + \mathcal{E}_r(u), \quad (3.4)
\end{align*}
\]

where \( \nu_{Q_\tau} \) signifies the outer unit normal of \( Q_\tau \), \( \nu^+ \) is the outer unit normal vector to \( \Sigma \), and \( u_{\nu, A} \) signifies the conormal derivative with respect to \( A \), i.e. \( u_{\nu, A} := \nu^- \cdot A \nabla u \).

We denote

\[ e(\tau) := \int_{H^+} (|\nabla u|^2_A + |u_t|^2) \, dx. \]  

(3.5)

Substituting (3.5) into (3.4), we have

\[
\begin{align*}
e(\tau) &= -\Re \int_{Q_\tau} 2\mu_t G + \mathcal{E}_r(u) - 2 \Re \int_{\Sigma^+} \mu_t u_{\nu, A} \, d\sigma \\
&\leq \int_{Q_\tau} \frac{1}{K} |u_t|^2 + |\nabla u|^2_A \, dx + \mathcal{E}_r(u) - 2 \Re \int_{\Sigma^+} \mu_t u_{\nu, A} \, d\sigma \\
&\leq \int_{\tau} \frac{1}{K} e(s) \, ds + K|G|_{L^2(Q_\tau)}^2 + \mathcal{E}_r(u) + 2 \int_{\Sigma^+} |u_t u_{\nu, A}| \, d\sigma.
\end{align*}
\]
where $K$ can be any positive number. Recall that under Assumption 1.1, we have $|\nabla S| \leq 1$, so $\mathcal{E}_\tau(u)$ is always non-increasing with respect to $\tau$. Hence, by Grönwall’s inequality we can obtain

$$e(\tau) \leq e(T_2 - \tau) / K \left[ K \| G \|^2_{L^2(\Sigma_\tau)} + \mathcal{E}_\tau(u) + 2 \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma \right], \quad \forall \tau \in [0, T_2]. \quad (3.6)$$

On the other hand, when $\tau \leq T_1$, from (3.4) we also obtain

$$\mathcal{E}_\tau(u) = e(\tau) + 2 \text{Re} \int_{\Sigma_\tau} \Pi_t u_{\nu,A} \, d\sigma + \text{Re} \int_{Q_\tau} 2\pi t G$$

$$\leq e(\tau) + 2 \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma + \frac{1}{\epsilon} \int_{Q_\tau} |G|^2 + e \int_{\tau}^{T_2} e(s) \, ds$$

$$\leq e(\tau) + 2 \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma + \frac{1}{\epsilon} \| G \|^2_{L^2(\Sigma_\tau)}$$

$$+ e \int_{\tau}^{T_2} e(T_2 - s) / K \left[ K \| G \|^2_{L^2(\Sigma_\tau)} + \mathcal{E}_\tau(u) + 2 \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma \right]$$

$$\leq e(\tau) + 2 \left[ 1 + eK(e(T_2 - \tau) / K - 1) \right] \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma$$

$$+ \left[ \frac{1}{\epsilon} + eK^2(e(T_2 - \tau) / K - 1) \right] \| G \|^2_{L^2(\Sigma_\tau)} + eK(e(T_2 - \tau) / K - 1) \mathcal{E}_\tau(u).$$

By setting $\epsilon = [2K(e(T_2 - \tau) / K - 1)]^{-1}$ and absorbing $\mathcal{E}_\tau(u)$ on the right-hand side (RHS) by the left-hand side (LHS), we obtain

$$\frac{1}{2} \mathcal{E}_\tau(u) \leq e(\tau) + 3 \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma + [2K(e(T_2 - \tau) / K - 1) + K/2] \| G \|^2_{L^2(\Sigma_\tau)}$$

$$\leq e(\tau) + 3 \int_{\Sigma_\tau} |u_t u_{\nu,A}| \, d\sigma + (T_2 - \tau) \left[ 2K e(T_2 - \tau) / K \frac{T_2 - \tau}{T_2 - \tau} - 3 \frac{K}{2T_2 - \tau} \right] \| G \|^2_{L^2(\Sigma_\tau)}. \quad (3.7)$$

When we treat the coefficient in front of $\| G \|^2_{L^2(\Sigma_\tau)}$ as a function of $K$, then elementary calculus shows the minimum value of the coefficient is in between 3.5 and 4, i.e.,

$$3.5 < \min_{K > 0} \left\{ 2K e(T_2 - \tau) / K \frac{T_2 - \tau}{T_2 - \tau} - 3 \frac{K}{2T_2 - \tau} \right\} < 4, \quad \text{provided } \tau < T_2.$$ 

Hence, by choosing the value of $K$ according to $T_2$ and $\tau$ properly, (3.7) can be improved to

$$\mathcal{E}_\tau(u) \leq 2e(\tau) + 6\| f_t \|_{L^2(\Sigma_\tau)} \| u_{\nu,A} \|_{L^2(\Sigma_\tau)} + 8(T_2 - \tau) \| G \|^2_{L^2(\Sigma_\tau)},$$

which is the conclusion. \(\square\)

**Remark 3.1.** The piecewise smoothness of $S$ does not cause any trouble to the proof of Lemma 3.2. This is because the integration operation does not require the integrand to be smooth everywhere.

The RHS of the inequality in Lemma 3.2 involves a norm of $u_{\nu,A}$, and to achieve an a-priori estimate of $u$ on $\Gamma_S$, we also need to estimate the term $\| u_{\nu,A} \|_{L^2(\Sigma_0)}$. This can be done by playing with (2.2). The multiplier $\varphi \cdot \nabla u$ in (2.2) is a generalization of $x \cdot \nabla u$ which has been seen in the literature, e.g., [Mor61 (12)], [Ike05 Lemma 2.1], [RS20b Lemma 3.3].
Lemma 3.3. Under the same condition as in Proposition 3.1, for vector-valued function \( \vec{\varphi} \in C^1(\Omega, \mathbb{R}^n) \) satisfying \( \vec{\varphi}(x) = A(x) \nu_{\Sigma}(x) \) on \( \partial \Omega \), we have

\[
\|u_{\nu, A}\|_{L^2(\Sigma, \tau)}^2 \leq C\|f\|_{H^1(\Sigma, \tau)}^2 + C\left(\sup_{\Omega}\|\vec{\varphi}\|\right)[\|\nabla u(\cdot, \tau)\|^2_{L^2(\Omega)} + \|u_t(\cdot, \tau)\|^2_{H^1(\Sigma, \tau)}] \\
+ C[T_2 - \tau + (\sup_{\Omega}\|\vec{\varphi}\|)](T_2 - \tau)\|G\|_{L^2(Q, \tau)}^2 + \mathcal{E}(u) \\
+ C[T_2 - \tau + (\sup_{\Omega}\|\vec{\varphi}\|)^2]\|f_t\|^2_{L^2(\Sigma, \tau)},
\]

for some constant \( C \) depending only on \( \vec{\varphi}, A \) and \( n \).

Proof. Integrating (2.2) in \( Q_{\tau} \), we have

\[
2\Re \int_{Q_{\tau}} (\vec{\varphi} \cdot \nabla \nabla A) = 2\Re \int_{Q_{\tau}} (\vec{\varphi} \cdot \nabla \nabla A) \partial^2 u - \nabla \cdot (A(x) \nabla u)
\]

\[
= \Re \int_{Q_{\tau}} \text{div}_{x,t} [\vec{\varphi}(|\nabla u_A|^2 - |u_t|^2) - 2(\vec{\varphi} \cdot \nabla \nabla A) \nabla u, 2(\vec{\varphi} \cdot \nabla \nabla A) u_t]
\]

\[
+ \int_{Q_{\tau}} [(\nabla \cdot \vec{\varphi})(|u_t|^2 - |\nabla u_A|^2) - (\vec{\varphi} \cdot \nabla A)(\nabla u, \nabla u) + 2(\partial_j \varphi_k) \text{Re}(\eta_{k, j} u_t)]
\]

\[
= \Re \int_{Q_{\tau}} [(\nu_{\Sigma} \cdot \varphi)(|\nabla u_A|^2 - |u_t|^2) - 2(\vec{\varphi} \cdot \nabla \nabla A) u_t]
\]

\[
+ \Re \int_{Q_{\tau}} [(\nu_x \cdot \varphi)(|\nabla u_A|^2 - |u_t|^2) + 2(\vec{\varphi} \cdot \nabla \nabla A) (\nu_x u_t - \nu_x \cdot A \nabla u)]
\]

\[
- \Re \int_{Q_{\tau}} [(\nu \cdot \varphi)(|u_t|^2 - |\nabla u_A|^2) - (\vec{\varphi} \cdot \nabla A)(\nabla u, \nabla u) + 2(\partial_j \varphi_k) \text{Re}(\eta_{k, j} u_t)]
\]

\[
=: I_1 + I_2 + I_3 - \Re \int_{Q_{\tau}} 2(\vec{\varphi} \cdot \nabla \nabla A) u_t,
\]

(3.8)

where \( I_1, I_2 \) and \( I_3 \) represent the integrals on \( \Sigma_{\tau}, \Gamma_{S, \tau} \) and \( Q_{\tau} \), respectively. We estimate \( I_1, I_2 \) and \( I_3 \) separately.

Recall that \( \vec{\varphi}(x) = A(x) \nu_{\Sigma}(x) \) on \( \partial \Omega \), so \( I_1 \) can be simplified as

\[
I_1 = \int_{\Sigma_{\tau}} |\nu_{\Sigma A}^2|(|\nabla u_A|^2 - |u_t|^2) - 2|u_{\nu, A}|^2 | \, d\sigma.
\]

(3.9)

By Lemma 2.3 we can obtain

\[
|\nu_{\Sigma A}^2|(|\nabla u_A|^2 - |u_t|^2) - 2|u_{\nu, A}|^2 \leq \frac{3}{2}|u_{\nu, A}|^2 + C|\nabla u|^2 \quad \text{on } \Sigma,
\]

(3.10)

for some constant \( C \) depending only on \( n \) and \( c_1, c_2 \) of Assumption 1.2, where \( \nabla \Sigma u \) represents the tangential gradient of \( u \) on \( \Sigma \). Substituting (3.10) into (3.9), we can continue

\[
I_1 \leq \int_{\Sigma_{\tau}} \left[ \left( \frac{3}{2}|u_{\nu, A}|^2 + C|\nabla u|^2 - |u_t|^2 \right) - 2|u_{\nu, A}|^2 \right] | \, d\sigma
\]

\[
\leq -\frac{1}{2}|u_{\nu, A}|^2_{L^2(\Sigma, \tau)} + C|u|^2_{H^1(\Sigma, \tau)},
\]

(3.11)

for some constant \( C \) depending on \( c_1, c_2, n \).

The integral \( I_2 \) is on \( \Gamma_S \), and on \( \Gamma_S \) we have

\[
\nabla u(x, S(x)) = \nabla \{u(x, S(x))\} - \nabla S(x) u_t(x, S(x)).
\]
For simplicity we abbreviate $\nabla (u(x, S(x)))$ as $\nabla (u)$. Then we compute

$$
(\nu_x \cdot \varphi)(|\nabla u|_A^2 - |u_t|^2)
= (\nu_x \cdot \varphi)\big[|\nabla (u) - \nabla Su_t^2|_A - |u_t|^2\big]
= (\nu_x \cdot \varphi)\big[|\nabla (u)|_A^2 - 2\Re\{\nabla Su_t \cdot A\nabla (u)\} - (1 - |\nabla S|_A^2)|u_t|^2\big],
$$
(3.12)

and

$$
2(\varphi \cdot \nabla \nabla (u_t - \nu_x \cdot A\nabla u) \times (1 + |\nabla S|_A^2)^{1/2}
= 2|\varphi \cdot \nabla (\varphi) - \varphi \cdot \nabla S|_A^2|u_t - \nu_x \cdot A\nabla u| \times (1 + |\nabla S|_A^2)^{1/2}
= 2|\varphi \cdot \nabla (\varphi) - \varphi \cdot \nabla S|_A^2|u_t + \nabla S \cdot A\nabla (u) - |\nabla S|_A^2|u_t|^2\big],
$$
(3.13)

Combining (3.12) with (3.13), we obtain

$$
\text{Re} \big[ (\nu_x \cdot \varphi)(|\nabla u|_A^2 - |u_t|^2) + 2(\varphi \cdot \nabla S)(\nu_x u_t - \nu_x \cdot A\nabla u) \big] \times (1 + |\nabla S|_A^2)^{1/2}
= - (\varphi \cdot \nabla S)\big[ |\nabla (u)|_A^2 + (1 - |\nabla S|_A^2)|u_t|^2\big]
+ 2\Re\{\varphi \cdot \nabla (\varphi)|\nabla S \cdot A\nabla (u) + (1 - |\nabla S|_A^2)|u_t|^2\big}\}.
$$
(3.14)

With the help of (3.14), we can estimate $I_2$ in the following way,

$$
|I_2| = \left| \int_{\Gamma_S} \left[ (\nu_x \cdot \varphi)(|\nabla u|_A^2 - |u_t|^2) + 2(\varphi \cdot \nabla S)(\nu_x u_t - \nu_x \cdot A\nabla u) \right] d\sigma \right|
\leq \int_{\Gamma_S} \left| \varphi \cdot \nabla S \big[ |\nabla (u)|_A^2 + (1 - |\nabla S|_A^2)|u_t|^2\big] \right| \frac{d\sigma}{\sqrt{1 + |\nabla S|_A^2}}
+ 2\int_{\Gamma_S} \left| \varphi \cdot \nabla (\varphi) \big[ |\nabla S \cdot A\nabla (u) + (1 - |\nabla S|_A^2)|u_t|^2\big] \right| \frac{d\sigma}{\sqrt{1 + |\nabla S|_A^2}}
\leq (c^{-1/2} + 2)(\sup_{\Omega} |\varphi|) \int_{\Gamma_S} \left[ |\nabla (u)|_A^2 + (1 - |\nabla S|_A^2)|u_t|^2\big] \right] \frac{d\sigma}{\sqrt{1 + |\nabla S|_A^2}}
= (c^{-1/2} + 2)(\sup_{\Omega} |\varphi|) \mathcal{E}_s (u).
$$
(3.15)

For $I_3$, we have

$$
|I_3| \leq \int_{Q_s} \big( |\nabla (\varphi)| (|u_t|^2 - |\nabla u|_A^2) - (\varphi \cdot \nabla A)(\nabla u, \nabla u) + 2(\partial_j \varphi_k) \text{Re}(\nabla a_j \nabla u) \big) \text{d}t \text{d} x
\leq C \int_{Q_s} \left[ |\nabla u|_A^2 + |u_t|^2 \right] \text{d} x \text{d} t = C \int_{\tau} e(s) \text{d}s,
$$
for some constant $C$ depending only on $\varphi$, $A$ and $n$, and the $e(s)$ is defined in (3.5). Hence by (3.9) which requires Assumption (3.1) we can have

$$
|I_3| \leq C \int_{\tau} e^{(T_2 - s)/K} \left[ K \|G\|_L^2(Q_s) + \mathcal{E}_s (u) + 2 \int_{\Sigma_a} |u_t u_{\nu A}| \text{d} \sigma \right] \text{d}s
\leq C \int_{\tau} e^{(T_2 - s)/K} \left[ K \|G\|_L^2(Q_s) + \mathcal{E}_s (u) + 2 \int_{\Sigma_a} |u_t u_{\nu A}| \text{d} \sigma \right]
\leq C K (e^{(T_2 - s)/K} - 1) \left[ K \|G\|_L^2(Q_s) + \mathcal{E}_s (u) + \frac{1}{e} \|u_t\|_L^2(\Sigma_s) + \epsilon \|u_{\nu A}\|_L^2(\Sigma_s) \right].
$$
(3.16)
In (3.8), there is a term \( \Re \int_{Q_r} (\vec{\varphi} \cdot \nabla \pi) G \), and similar to the estimation of \( I_3 \), we can also estimate this integral as follows,

\[
\left| \Re \int_{Q_r} (\vec{\varphi} \cdot \nabla \pi) G \right| \leq \frac{1}{2} \left( \sup_{\Omega} |\vec{\varphi}| \right) \int_{Q_r} \left( c_1^{-1} K^{-1} |\nabla u|^2_A + K |G|^2 \right)
\]

\[
\leq \frac{1}{2c_1} \left( \sup_{\Omega} |\vec{\varphi}| \right) \frac{1}{K} \int_{\tau}^{T_2} e(s) \, ds + \frac{1}{2} \left( \sup_{\Omega} |\vec{\varphi}| \right) K \|G\|_{L^2(Q_r)}^2
\]

\[
\leq \frac{1}{2c_1} \left( \sup_{\Omega} |\vec{\varphi}| \right) (e(T_2 - \tau)/K - 1) \left[ K \|G\|_{L^2(Q_r)}^2 + \mathcal{E}_\tau(u) + \frac{1}{\epsilon} \|u_t\|^2_{L^2(\Sigma_r)} + \epsilon \|u_{\nu A}\|^2_{L^2(\Sigma_r)} \right]
\]

\[
+ \frac{1}{2} \left( \sup_{\Omega} |\vec{\varphi}| \right) K \|G\|_{L^2(Q_r)}^2
\]

\[
\leq \frac{1}{2} \left( \sup_{\Omega} |\vec{\varphi}| \right) \cdot K (e(T_2 - \tau)/K - 1 + c_1) \cdot \|G\|_{L^2(Q_r)}^2
\]

\[
+ \frac{1}{2c_1} \left( \sup_{\Omega} |\vec{\varphi}| \right) (e(T_2 - \tau)/K - 1) \left[ \mathcal{E}_\tau(u) + \frac{1}{\epsilon} \|u_t\|^2_{L^2(\Sigma_r)} + \epsilon \|u_{\nu A}\|^2_{L^2(\Sigma_r)} \right].
\]

Combining (3.16) and (3.17) and setting \( K = T_2 - \tau \), we have

\[
|I_3| + |\Re \int_{Q_r} (\vec{\varphi} \cdot \nabla \pi) G|
\]

\[
\leq C[T_2 - \tau + (\sup_{\Omega} |\vec{\varphi}|)] \left[ (T_2 - \tau) \|G\|_{L^2(Q_r)}^2 + \mathcal{E}_\tau(u) + \frac{1}{\epsilon} \|u_t\|^2_{L^2(\Sigma_r)} + \epsilon \|u_{\nu A}\|^2_{L^2(\Sigma_r)} \right],
\]

for some constant \( C \) depending only on \( \vec{\varphi}, A \) and \( n \).

Now, by combining (3.14), (3.15) and (3.18) with (3.8), we obtain

\[
\frac{1}{2} \|u_{\nu A}\|^2_{L^2(\Sigma_r)} + \frac{1}{4} \|u_{\nu A}\|^2_{L^2(\Sigma_r)} \leq C \|u\|_{H^1(\Sigma_r)}^2 + |I_2| + 2 \left| \int_{Q_r} (\vec{\varphi} \cdot \nabla \pi) G \right| + |I_3| + \left| \int_{\Omega \times \{\tau\}} 2(\vec{\varphi} \cdot \nabla \pi) u_t \right|
\]

\[
\leq C \|u\|_{H^1(\Sigma_r)}^2 + (c_1^{-1/2} + 2)(\sup_{\Omega} |\vec{\varphi}|) \mathcal{E}_\tau(u) + (\sup_{\Omega} |\vec{\varphi}|) \left[ \|\nabla u(\cdot, \tau)\|^2_{L^2(\Omega)} + \|u_t(\cdot, \tau)\|^2_{L^2(\Omega)} \right]
\]

\[
+ C[T_2 - \tau + (\sup_{\Omega} |\vec{\varphi}|)] \left[ (T_2 - \tau) \|G\|_{L^2(Q_r)}^2 + \mathcal{E}_\tau(u) + \frac{1}{\epsilon} \|u_t\|^2_{L^2(\Sigma_r)} + \epsilon \|u_{\nu A}\|^2_{L^2(\Sigma_r)} \right].
\]

By setting \( \epsilon = \{ 4C[T_2 - \tau + (\sup_{\Omega} |\vec{\varphi}|)] \}^{-1} \), we arrive at

\[
\frac{1}{4} \|u_{\nu A}\|^2_{L^2(\Sigma_r)} \leq C \|u\|_{H^1(\Sigma_r)}^2 + C(\sup_{\Omega} |\vec{\varphi}|) \left[ \|\nabla u(\cdot, \tau)\|^2_{L^2(\Omega)} + \|u_t(\cdot, \tau)\|^2_{L^2(\Omega)} \right]
\]

\[
+ C[T_2 - \tau + (\sup_{\Omega} |\vec{\varphi}|)] \left[ (T_2 - \tau) \|G\|_{L^2(Q_r)}^2 + \mathcal{E}_\tau(u) \right]
\]

\[
+ C[T_2 - \tau + (\sup_{\Omega} |\vec{\varphi}|)]^2 \|u_t\|^2_{L^2(\Sigma_r)}.
\]

The proof is done. \( \square \)

Recall \((T_2) := (1 + |T_2|^2)^{1/2}\). With the help of Lemmas 3.2 and 3.3, we are able to bound \( \mathcal{E}(u) \) and \( \|u_{\nu A}\|_{L^2(\Sigma_0)} \) by the initial/boundary data and the source term.

**Lemma 3.4.** Under the same condition as in Proposition 3.1, we have

\[
\mathcal{E}(u) \leq C(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}) + C(T_2)(\|G\|^2_{L^2(Q_0)} + \|f\|^2_{H^1(\Sigma_0)})
\]

\[
\|u_{\nu A}\|^2_{L^2(\Sigma_0)} \leq C(T_2)(\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}) + C(T_2)^2(\|f\|^2_{H^1(\Sigma_0)} + \|G\|^2_{L^2(Q_0)}),
\]

for some constant \( C \) depending only on \( A \) and the dimension \( n \).
Proof. Using the inequality in Lemma 3.2 with \( \tau = 0 \), and noting that \( u(\cdot, 0) = u_0 \) and \( u_t(\cdot, 0) = u_1 \), we have
\[
E(u) \leq 2 \int_\Omega (|\nabla u_0|^2 + |u_1|^2) \, dx + \frac{3}{\epsilon} \| f_t \|_{L^2(\Sigma_0)}^2 + \epsilon \| u_{\nu,A} \|_{L^2(\Sigma_0)}^2 + 8T_2 \| G \|_{L^2(Q_0)}^2 .
\] (3.19)

Substituting the inequality in Lemma 3.3 with \( \tau = 0 \) into (3.19), and setting \( \epsilon \) in (3.19) to be \( \{ 2C[T_2 + (\sup_{\partial \Omega} |\varphi|)] \}^{-1} \), we obtain
\[
E(u) \leq C(\| \nabla u_0 \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2) + C[T_2 + (\sup_{\partial \Omega} |\varphi|)]^2 \| f \|_{H^1(\Sigma_0)}^2 + \| G \|_{L^2(Q_0)}^2 ,
\] (3.20)
for some constant \( C \) depending only on \( \varphi \), \( A \) and \( n \). Recall that in Lemma 3.3 we have fixed the value of \( \varphi \) on \( \partial \Omega \). But we still have the freedom to choose the value of \( \varphi \) in the interior of \( \Omega \). We can choose \( \varphi \) in such a way that \( \sup_{\partial \Omega} |\varphi| \leq 2 \sup_{\partial \Omega} |\varphi| \), and this choice guarantees
\[
\sup_{\Omega} |\varphi| \leq 2 \sup_{\partial \Omega} |A \cdot \nu_\Sigma| \leq 2c_2 ,
\]
for the constant \( c_2 \) given in Assumption 1.2. Combining this with (3.20), we arrive at the first inequality of the lemma.

For the second inequality, we substitute the inequality in Lemma 3.2 into the inequality in Lemma 3.3 with \( \tau = 0 \), and this gives
\[
\| u_{\nu,A} \|_{L^2(\Sigma_0)}^2 \leq C[T_2 + (\sup_{\partial \Omega} |\varphi|)](\| \nabla u_0 \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2) + C[T_2 + (\sup_{\partial \Omega} |\varphi|)]^2 \| f \|_{H^1(\Sigma_0)}^2
\]
\[
+ C[T_2 + (\sup_{\partial \Omega} |\varphi|)] \frac{1}{\epsilon} \| f_t \|_{L^2(\Sigma_0)}^2 + \epsilon \| u_{\nu,A} \|_{L^2(\Sigma_0)}^2 + T_2 \| G \|_{L^2(Q_0)}^2 .
\]

By letting \( \epsilon = \{ 2C[T_2 + (\sup_{\partial \Omega} |\varphi|)] \}^{-1} \) and absorbing the \( \| u_{\nu,A} \|_{L^2(\Sigma_0)}^2 \)-term on the RHS by the LHS, we obtain
\[
\| u_{\nu,A} \|_{L^2(\Sigma_0)}^2 \leq C[T_2 + (\sup_{\partial \Omega} |\varphi|)](\| \nabla u_0 \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2)
\]
\[
+ C[T_2 + (\sup_{\partial \Omega} |\varphi|)]^2 \| f \|_{H^1(\Sigma_0)}^2 + \| G \|_{L^2(Q_0)}^2 .
\]

Again, by using \( \sup_{\partial \Omega} |\varphi| \leq 2c_2 \), we obtain the second inequality. The proof is complete. \( \Box \)

Remark 3.2. When the hypersurface \( \Gamma_S \) is horizontal, the corresponding estimate of the Neumann data \( u_{\nu,A} \) given in Lemma 3.4 will be a quantitative version of [LTS86, (2.7)].

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. In (3.4), when \( \tau = 0 \), we have
\[
|E(u) - e(0)| = |2 \Re \int_{\Sigma_0} \overline{f_t} u_{\nu,A} \, d\sigma + \Re \int_{Q_0} 2\pi_i G |
\]
\[
\leq 2 \int_{\Sigma_0} |f_t| |u_{\nu,A}| \, d\sigma + 2 \int_{Q_0} |u_t| |G|
\]
\[
\leq 2 \| f_t \|_{L^2(\Sigma_0)} \| u_{\nu,A} \|_{L^2(\Sigma_0)} + 2 \| u_t \|_{L^2(Q_0)} \| G \|_{L^2(Q_0)} .
\] (3.21)

For \( \| u_t \|_{L^2(Q_0)} \), noticing \( \| u_t \|_{L^2(H_r)}^2 \leq e(\tau) \), so by (3.6) with \( K \) set to be \( T_2 \), we have
\[
\| u_t \|_{L^2(Q_0)}^2 \leq \int_{T_2}^{T_2} \| u_t \|_{L^2(H_r)}^2 \, d\tau \leq \int_0^{T_2} e(\tau) \, d\tau
\]
\[
\leq T_2 (e^{T_2/T_2} - 1)(E(u) + T_2 \| G \|_{L^2(Q_0)}^2 + 2 \int_{\Sigma_0} |u_t u_{\nu,A}| \, d\sigma)
\]
these approximate data, we obtain approximate solutions $u$. However, these two issues can be overcome by approximation. That is to say, we can find a solution that has a limit in $H^1(\Omega)$ with smooth enough initial/boundary data and to show the corresponding approximate solution has a limit in $H^1(\Omega)$, we are able to approximate the system (1.1).

The proof is complete.

\[ \square \]

4. Approximation of the solution

The results in Section 3 are based on the prerequisite $u \in C^2$. This is not true for the system (1.1) when:

1. only the compatibility condition up to order zero is satisfied;
2. and the initial/boundary data are merely $H^1$.

However, these two issues can be overcome by approximation. That is to say, we can find smooth sequences $u_0, u_1, f$, $G$, which converge to $u_0$, $u_1$, $f$, $G$, respectively, and under these approximate data, we obtain approximate solutions $u_\varepsilon$, which will converge in $H^1(\Gamma_S)$. For the compatibility conditions issue, we shall modify $f_\varepsilon$. And for the regularity issue, with the help of the estimate given in Lemma 3.3, we are able to approximate the system (1.1) with smooth enough initial/boundary data and to show the corresponding approximate solution has a limit in $H^1(\Gamma_S)$ and the corresponding energy is well-defined.

For readers convenience we reproduce [LLT86, Remark 2.10] below.

**Lemma 4.1.** Assume $A$ is a $C^4$-smooth real-valued symmetric matrix function, and $\partial \Omega$ is $C^2$. Let $u$ be a solution of the system (1.1) with $(G, f, u_0, u_1)$ satisfying the regularity assumptions (m is a non-negative integer)

\[
\begin{align*}
G & \in L^1(0, T; H^m(\Omega)), \quad \frac{d^m G}{dt^m} \in L^1(0, T; L^2(\Omega)), \\
f & \in H^{m+1}(\Sigma) := L^2(0, T; H^{m+1}(\partial \Omega)) \cap H^{m+1}(0, T; L^2(\partial \Omega)), \\
u_0 & \in H^m(\Omega), \quad u_1 \in H^m(\Omega),
\end{align*}
\]

and satisfying all necessary compatibility conditions up to order m. Then

\[ u \in C([0, T]; H^{m+1}(\Omega)), \quad \frac{d^{(m+1)} u}{dt^{(m+1)}} \in C([0, T]; L^2(\Omega)), \quad \text{and} \quad \frac{\partial u}{\partial \nu} \in H^m(\Sigma). \]

**Remark 4.1.** The case of Lemma 4.1 when $\nabla \cdot (A(x) \nabla)$ is replaced by $\Delta$ is covered by [LLT86, Remark 2.10]. And in [LLT86, Section 4] the authors discussed how to generalize from $\Delta$ to $\nabla \cdot (A(x) \nabla)$ when $m = 0, 1$. Actually, by following the same steps in the proof

\[
\leq CT_2 \left( \mathcal{E}(u) + T_2 \|G\|_{L^2(\Sigma_0)}^2 + 2\|f\|_{L^2(\Sigma_0)} \|u_{\nu,A}\|_{L^2(\Sigma_0)} \right).
\]
of [LLT86, Theorem 2.2], we can generalize the scenario to any integer \( m \) not only 0 and 1, and the proof is straightforward so we omit it.

Readers should note that, in this work, the function space for the source \( Q \) is set to be \( L^2(Q) \), which is a subset of the space \( L^1(0,T; H^m(\Omega)) \) with \( m = 0 \) used in [LLT86, Remark 2.10].

4.1. The compatibility issue. We first deal with the compatibility condition issue. Recall \( \mathcal{E}(u) \) stands for \( \mathcal{E}(u; \Gamma_S) \).

**Lemma 4.2.** Given Assumptions [1.1 and 1.2. In system (1.1), assume \( u_0, u_1 \in C^\infty(\overline{\Omega}) \), \( f \in C^\infty(\Sigma) \), \( G \in C^\infty(\overline{Q}) \), with \( u_0(x) = f(x,0) \) on \( \partial \Omega \). Then we have \( u \in H^1(\Gamma_S) \) and

\[
\left| \mathcal{E}(u) - \int_\Omega (|\nabla u_0|^2 + |u_1|^2) \, dx \right| \\
\leq C \langle T_2 \rangle^{1/2} \left[ \| \nabla u_0 \|_{L^2(\Omega)} + \| u_1 \|_{L^2(\Omega)} + \langle T_2 \rangle^{1/2} \left( \| f \|_{H^1(\Omega_0)} + \| G \|_{L^2(Q_0)} \right) \right] \\
\times \left( \| f \|_{H^1(\Omega_0)} + \| G \|_{L^2(Q_0)} \right).
\]

and

\[
\| u_{\nu, A} \|_{L^2(Q_0)} \leq C \langle T_2 \rangle \left( \| \nabla u_0 \|_{L^2(\Omega)} + \| u_1 \|_{L^2(\Omega)} \right) + C \langle T_2 \rangle^{2} \left( \| G \|_{L^2(Q_0)} + \| f \|_{H^1(\Omega_0)} \right).
\]

**Proof.** We shall find a sequence of \( C^2 \)-smooth solutions \( \{ u_\epsilon \}_{\epsilon > 0} \) of

\[
\begin{aligned}
\partial_t^2 u_\epsilon - \nabla \cdot (A(x) \nabla u_\epsilon) &= G \quad \text{in } Q, \\
u_\epsilon &= f_\epsilon \quad \text{on } \Sigma, \\
u_\epsilon &= u_0, \quad \partial_t u_\epsilon = u_1 \quad \text{on } \Omega \times \{ t = 0 \},
\end{aligned}
\]

by using the regularity result in Lemma [3.1]. For this, we need smoothness of \( u_0, u_1, f \) and \( G \), which are already assumed in this lemma, and we also need certain higher order compatibility conditions to be satisfied on \( \Omega \times \{ t = 0 \} \). To guarantee the compatibility conditions, we modify the Dirichlet boundary datum \( f \).

Let us construct a series of Dirichlet boundary data \( \{ f_\epsilon \}_{\epsilon > 0} \) in the following way. First, we define \( u_k \in \overline{\Omega} \) iteratively for \( k \geq 2 \) from \( u_0 \) and \( u_1 \) which is already given as the initial data,

\[
u_k(x) := \partial_t^{k-2} G(x,0) + \nabla \cdot (A(x) \nabla u_{k-2}(x)), \quad x \in \overline{\Omega},
\]

recall that \( u_0, u_1 \in C^\infty(\overline{\Omega}) \). Recall the smoothness of \( A \) stipulated at the beginning of the article, i.e. \( A \in C^{n+1}(\overline{\Omega}) \). This guarantees

\[
u_{2k}, \nu_{2k+1} \in C^{n+5-2k}(\overline{\Omega}), \quad \forall 1 \leq k \leq \frac{K-1}{2}, (4.2)
\]

for certain integer \( K \) which shall be determined later. Then, we fix a cutoff function \( \chi \in C_c^\infty(\mathbb{R}) \) satisfying \( \chi(t) = 1 \) when \( |t| \leq 1 \) and \( \chi(t) = 0 \) when \( |t| \geq 2 \), and we set

\[
\begin{aligned}
f_\epsilon(x,t) := \chi(t/\epsilon) \sum_{k=0}^{K} \frac{k!}{k!} u_k(x) + (1 - \chi(t/\epsilon)) f(x,t), \quad \forall (x,t) \in \Sigma. (4.3)
\end{aligned}
\]

By (4.2) and (4.3), \( f_\epsilon \in C^{n+5-K}(\Sigma) \). Now the compatibility conditions up to the order \( K \) required by Lemma [4.1] i.e.,

\[
u_k = \partial_t^k f_\epsilon \text{ on } \partial \Omega \times \{ t = 0 \}, \quad \forall k : 0 \leq k \leq K, (4.4)
\]

is satisfied (see also [Eva10, §7.2.3 (62)]). Therefore, we can use Lemma [3.1] up to order \( N := \min\{ n + 5 - K, K \} \) to conclude that the corresponding solution \( u_\epsilon \) satisfies

\[
u_\epsilon \in C([0,T]; H^N(\Omega)), \quad \partial_t u_\epsilon \in C([0,T]; H^{N-1}(\Omega)).
\]
By the Sobolev embedding theorem we know \( H^N(\Omega) \subset C^2(\mathbb{O}) \) when \( N - n/2 \geq 2 \), so we set \( K = \lfloor n/2 \rfloor + 2 \), and thus
\[
 u_\epsilon \in \mathcal{C}([0, T]; C^2(\mathbb{O})), \quad \partial_t u_\epsilon \in \mathcal{C}([0, T]; C^1(\mathbb{O})),
\] (4.5)
so \( \nabla \cdot (A \nabla u_\epsilon(\cdot, t)) \in \mathcal{C}(\mathbb{O}) \) for each \( t \). Therefore, the equation \((\partial_t^2 - \Delta)u_\epsilon = 0\) implies
\[
\partial_t^2 u_\epsilon(x, t) \text{ is continuous with respect to } t.
\] (4.6)
By (4.5) and (4.6) we obtain
\[
u_\epsilon \in C^2(\mathbb{O}).
\] (4.7)
The \( C^2 \)-smoothness of \( u_\epsilon \) guarantees us to use Proposition 3.1 to conclude
\[
\|\mathcal{E}(u_\epsilon) - \int_\Omega (\|\nabla u_0\|_A^2 + |u_1|^2) \, dx \|
\leq C(T_2)^{1/2} (\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} + \langle T_2 \rangle^{1/2} (\|f_\epsilon\|_{H^1(\Sigma_0)} + \|G\|_{L^2(Q_0)}))
\times (\|f_\epsilon\|_{H^1(\Sigma_0)} + \|G\|_{L^2(Q_0)}),
\] (4.8)
for some constant \( C \) depending only on \( A \) and \( n \). Further, by the linearity of (4.1) we obtain
\[
\mathcal{E}(u_{\epsilon_1} - u_{\epsilon_2}) \leq C(T_2)\|f_{\epsilon_1} - f_{\epsilon_2}\|^2_{H^1(\Sigma_0)}, \quad \forall \epsilon_1, \epsilon_2 > 0.
\] (4.9)
Besides the compatibility conditions, the construction (4.3) also guarantees \( f_\epsilon \rightarrow f \) in \( H^1(\Sigma) \). Indeed, it is straightforward to check that
\[
\|f - f_\epsilon\|_{L^2(\Sigma)} \rightarrow 0, \quad \|\nabla|_{\Sigma,x}(f - f_\epsilon)\|_{L^2(\Sigma)} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+,
\] (4.10)
where \( \nabla|_{\Sigma,x} \) stands for the spatial tangential gradient on \( \Sigma \). Therefore, to show \( f_\epsilon \rightarrow f \) in \( H^1(\Sigma) \), it is left to show \( \|\partial_t(f - f_\epsilon)\|_{L^2(\Sigma)} \rightarrow 0 \) as \( \epsilon \rightarrow 0^+ \). By (4.3) one can compute
\[
\partial_t(f_\epsilon - f)(x, t) = \frac{1}{\epsilon} \chi'(\frac{t}{\epsilon}) \sum_{k=0}^K \frac{t^k}{k!} u_k(x) - f(x, t) + \chi(\frac{t}{\epsilon}) \sum_{k=0}^{K-1} \frac{t^k}{k!} u_k(x) - f'(x, t)]
\]
\[
= \frac{1}{\epsilon} \chi'(\frac{t}{\epsilon}) \sum_{k=0}^K \frac{t^k}{k!} u_k(x) - u_0(x) - f'(x, \xi_\epsilon t) + \chi(\frac{t}{\epsilon}) \sum_{k=0}^{K-1} \frac{t^k}{k!} u_k(x) - f'(x, t)]
\]
\[
= \frac{1}{\epsilon} \chi'(\frac{t}{\epsilon}) \sum_{k=1}^K \frac{t^{k-1}}{k!} u_k(x) - f'(x, \xi_\epsilon t) + \chi(\frac{t}{\epsilon}) \sum_{k=0}^{K-1} \frac{t^k}{k!} u_k(x) - f'(x, t)]
\]
where we used the compatibility condition “\( u_{\epsilon_0}(x) = f(x, 0) \) on \( \partial \Omega \)” and the Taylor’s expansion with Lagrange remainder, and the \( \xi_\epsilon \) comes from the Lagrange remainder. Note that for every \( \epsilon > 0 \), \( \frac{1}{\epsilon} \chi'(\frac{t}{\epsilon}) \) and \( \chi(\frac{t}{\epsilon}) \) are uniformly bounded by \( \max\{2\sup |\chi'|, 1\} \) in \( [0, T] \), so \( |\partial_t(f_\epsilon - f)|^2 \) is dominated by an integrable function \( F \) given as follows,
\[
F(x, t) := \max\{2\sup |\chi'|, 1\}^2 (\sum_{k=1}^K \frac{t^{k-1}}{k!} u_k(x) - f'(x, \xi_\epsilon t) + \sum_{k=0}^{K-1} \frac{t^k}{k!} u_k(x) - f'(x, t)]^2.
\]
Also, from (4.3) we see \( f_\epsilon = f \) when \( t \geq 2\epsilon \), so \( \partial_t(f_\epsilon - f) = 0 \) when \( t \geq 2\epsilon \) and hence \( \partial_t(f_\epsilon - f) \rightarrow 0 \) almost everywhere in \( \Sigma \) as \( \epsilon \rightarrow 0^+ \). Therefore, by Lebesgue’s dominated convergence theorem, we obtain
\[
\|\partial_t(f - f_\epsilon)\|_{L^2(\Sigma)} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+.
\] (4.11)
Combining (4.10) with (4.11), we arrive at
\[
\|f - f_\epsilon\|_{H^1(\Sigma)} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+.
\] (4.12)
Now, combining (4.12) with (4.9), we see \( \mathcal{E}(u_{\epsilon_1} - u_{\epsilon_2}) \) goes to zero as \( \epsilon_1, \epsilon_2 \rightarrow 0^+ \), which implies \( \nabla|_{\Gamma_S} u_\epsilon \) is a Cauchy sequence in \( L^2(\Gamma_S) \) and \( \{\partial_t u_\epsilon\} \) is a Cauchy sequence in \( L^2(\Gamma_S) \).
We claim that \( \{\nabla_{\Gamma_{\epsilon}} u_{\epsilon}\} \) being a Cauchy sequence in \( L^2(\Gamma_{\epsilon}) \) is enough to conclude \( \{u_{\epsilon}\} \) is a Cauchy sequence in \( H^1(\Gamma_{\epsilon}) \). This is due to the reason that every \( u_{\epsilon} \) has the same trace on \( \partial\Gamma_{\epsilon} \). Hence, we can define \( u|_{\Gamma_{\epsilon}} \) as the limit of \( u_{\epsilon}|_{\Gamma_{\epsilon}} \), i.e.,

\[
u|_{\Gamma_{\epsilon}} := \lim_{\epsilon \to 0^+} u_{\epsilon}|_{\Gamma_{\epsilon}},
\]

and the estimate (4.9) implies the limit \( u|_{\Gamma_{\epsilon}} \) is in \( H^1(\Gamma_{\epsilon}) \). Moreover, the energy \( \mathcal{E}(u) \) is well-defined and by Lemma 2.2 and we see \( \mathcal{E}(u_{\epsilon}) \to \mathcal{E}(u) \) as \( \epsilon \to 0^+ \). Now, taking the limit of (4.8), we obtain

\[
|\mathcal{E}(u) - \int_{\Omega} (|\nabla u_{\epsilon}|^2 + |u_1|^2) \, dx| \\
\leq C(T_2)^{1/2} [\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} + \langle T_2 \rangle^{1/2} (\|f\|_{H^1(\Sigma)} + \|G\|_{L^2(Q_0)})] \\
\times (\|f\|_{H^1(\Sigma)} + \|G\|_{L^2(Q_0)}).
\]

For \( \|u_{\epsilon, A}\|_{L^2(\Sigma)} \), by combining the arguments above with Lemma 3.4 we can see \( u_{\epsilon, A} \) is also well-defined on \( \Sigma \) and

\[
\|u_{\epsilon, A}\|_{L^2(\Sigma)} \leq C(T_2) (\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}) + C(T_2)^2 (\|G\|_{L^2(Q_0)} + \|f\|_{H^1(\Sigma)}).
\]

The proof is complete. \( \square \)

4.2. The regularity issue. The \( C^\infty \)-smooth regularities requirements for \( u_0, u_1, f \) and \( G \) in Lemma 4.2 can be further improved. Recall the statement of Theorem 1.1. 

Proof of Theorem 4.2. We shall utilize Lemma 4.1. One can find sequences \( \{u_{0, \epsilon}\}_{\epsilon > 0}, \{u_{1, \epsilon}\}_{\epsilon > 0}, \{f_{\epsilon}\}_{\epsilon > 0}, \{G_{\epsilon}\}_{\epsilon > 0} \) satisfying the following requirements:

\[
\begin{cases}
\{u_{0, \epsilon}\}_{\epsilon > 0} \subset C^\infty(\Omega) & \text{such that } \|u_{0, \epsilon} - u_0\|_{H^1(\Omega)} \leq \epsilon, \\
\{u_{1, \epsilon}\}_{\epsilon > 0} \subset C^\infty(\Omega) & \text{such that } u_{1, \epsilon} \to u_1 \text{ in } L^2(\Omega), \\
\{f_{\epsilon}\}_{\epsilon > 0} \subset C^\infty(\Sigma) & \text{such that } \|f_{\epsilon} - f\|_{H^1(\Sigma)} \leq \epsilon, \\
\{G_{\epsilon}\}_{\epsilon > 0} \subset C^\infty(\Omega) & \text{such that } G_{\epsilon} \to G \text{ in } L^2(Q).
\end{cases}
\]

The condition \( u_0(x) = f(x, 0) \) on \( \partial\Omega \) mentioned in (1.2) does not guarantee \( u_{0, \epsilon}(x) = f_{\epsilon}(x, 0) \) on \( \partial\Omega \). Hence we need to modify \( f_{\epsilon} \) to \( f_{\epsilon} \) so that \( u_{0, \epsilon}, u_{1, \epsilon}, f_{\epsilon}, G_{\epsilon} \) meet the requirements of Lemma 4.2.

Similar to (4.4), we fix a cutoff function \( \chi \in C^\infty_c(\mathbb{R}) \) satisfying \( \chi(t) = 1 \) when \( |t| \leq 1 \) and \( \chi(t) = 0 \) when \( |t| \geq 2 \), and set

\[
f_{\epsilon}(x, t) := \chi(t/\epsilon)u_{0, \epsilon}(x) + (1 - \chi(t/\epsilon))\tilde{f}_{\epsilon}(x, t), \quad \forall (x, t) \in \Sigma.
\]

Then,

\[
f_{\epsilon} \in C^\infty(\Omega), \quad \text{and} \quad u_{0, \epsilon}(x) = f_{\epsilon}(x, 0) \text{ on } \partial\Omega.
\]

It can be seen that (see (4.10))

\[
\|f - f_{\epsilon}\|_{L^2(\Omega)} \to 0, \quad \|\nabla_{\Sigma, \epsilon}(f - f_{\epsilon})\|_{L^2(\Sigma)} \to 0, \quad \text{as } \epsilon \to 0^+,
\]

so, to guarantee \( f_{\epsilon} \to f \) in \( H^1(\Sigma) \), it is left to show \( \|\partial_t(f - f_{\epsilon})\|_{L^2(\Sigma)} \to 0 \) as \( \epsilon \to 0^+ \). By (4.13) we have

\[
\partial_t(f_{\epsilon} - f)(x, t) = \partial_t(\tilde{f}_{\epsilon} - f) - \chi(t/\epsilon)\tilde{f}_{\epsilon} + \frac{1}{\epsilon}\chi'(t/\epsilon)(u_{0, \epsilon}(x) - \tilde{f}_{\epsilon}(x, t)) \\
= \partial_t(\tilde{f}_{\epsilon} - f) - \chi(t/\epsilon)\tilde{f}_{\epsilon} + \frac{t}{\epsilon}\chi'(t/\epsilon)(u_{0, \epsilon}(x) - u_0(x)) + [f(x, 0) - \tilde{f}_{\epsilon}(x, 0)] + [f_{\epsilon}(x, 0) - \tilde{f}_{\epsilon}(x, t)],
\]

(4.17)
where we used the compatibility condition $u_0(x) = f(x, 0)$ on $\partial \Omega$.

The function $\frac{1}{\epsilon} \chi'(\frac{t}{\epsilon})$ is supported in the interval $[\epsilon, 2\epsilon]$ and is bounded by $2 \max |\chi'|$, and $\|u_{0, \epsilon} - u_0\|_{H^1(\Omega)} \leq \epsilon$, so

$$\int_\Omega \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \frac{u_{0, \epsilon}(x) - u_0(x)}{t} \right|^2 \leq C \int_\Omega \left( \int_\Omega \left| \frac{u_{0, \epsilon}(x) - u_0(x)}{\epsilon} \right|^2 \right) dt$$

$$= C \epsilon^{-1} \|u_{0, \epsilon} - u_0\|^2_{L^2(\partial \Omega)} \leq C \epsilon^{-1} \|u_{0, \epsilon} - u_0\|^2_{H^1(\Omega)} \leq C \epsilon,$$  \hspace{1cm} (4.18)

where we used the trace theorem. Similarly, we have

$$\int_\Omega \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \frac{f(x, 0) - \tilde{f}_\epsilon(x, 0)}{t} \right|^2 \leq C \epsilon^{-1} \|f - \tilde{f}_\epsilon\|^2_{L^2(\partial \Omega)} \leq C \epsilon^{-1} \|f - \tilde{f}_\epsilon\|^2_{H^1(\Omega)} \leq C \epsilon. \hspace{1cm} (4.19)$$

We can also estimate the last term in (4.17). Note that $\tilde{f}_\epsilon$ is a smooth function, so we have

$$\int_\Omega \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \frac{\tilde{f}_\epsilon(x, 0) - \tilde{f}_\epsilon(x, t)}{t} \right|^2 d\sigma = \int_\Sigma \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \right|^2 |\partial_t \tilde{f}_\epsilon(x, t) + O(t)|^2 d\sigma$$

$$\leq 2 \int_\Sigma \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \right|^2 |\partial_t \tilde{f}_\epsilon(x, t)|^2 d\sigma + O(\epsilon^2).$$

The function $|\frac{t}{\epsilon} \chi'(\frac{t}{\epsilon})|^2 |\partial_t \tilde{f}_\epsilon(x, t)|^2$ is dominated by $C |\partial_t \tilde{f}_\epsilon(x, t)|^2$ for certain constant $C$, whose Lebesgue integral in $\Sigma$ is bounded by $C \|\tilde{f}_\epsilon\|_{H^1(\Sigma)}$, and hence bounded by $C \|f\|_{H^1(\Sigma)} + 1$ when $\epsilon$ is small enough. Also, the function $|\frac{t}{\epsilon} \chi'(\frac{t}{\epsilon})|^2 |\partial_t \tilde{f}_\epsilon(x, t)|^2$ converges to 0 for $\forall (x, t) \in \Sigma$ as $\epsilon \to 0^+$, which is because $|\frac{t}{\epsilon} \chi'(\frac{t}{\epsilon})|^2$ converges to 0 for $\forall t \in (0, T]$ as $\epsilon \to 0^+$. Therefore, by Lebesgue’s dominated convergence theorem we have

$$\lim_{\epsilon \to 0^+} \int_\Sigma \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \frac{\tilde{f}_\epsilon(x, 0) - \tilde{f}_\epsilon(x, t)}{t} \right|^2 d\sigma \leq 2 \lim_{\epsilon \to 0^+} \int_\Sigma \left| \frac{t}{\epsilon} \chi'(\frac{t}{\epsilon}) \right|^2 |\partial_t \tilde{f}_\epsilon(x, t)|^2 d\sigma = 0. \hspace{1cm} (4.20)$$

Combining (4.18), (4.19), (4.20) with (4.17), we conclude $\|\partial_t (f_\epsilon - f)\|_{L^2(\Sigma)} \to 0$, so

$$\lim_{\epsilon \to 0^+} \|f_\epsilon - f\|_{H^1(\Sigma)} = 0, \hspace{1cm} \epsilon \to 0^+. \hspace{1cm} (4.21)$$

By (4.13) and (4.15), we see $(u_{0, \epsilon}, u_{1, \epsilon}, f_\epsilon, G_\epsilon)$ meet the requirement of Lemma 4.2 so the corresponding solution $u_\epsilon$ satisfies

$$|\mathcal{E}(u_\epsilon) - \int_\Omega \left( |\nabla u_{0, \epsilon}|^2 + |u_{1, \epsilon}|^2 \right) dx|$$

$$\leq C (T_2)^{1/2} [\|\nabla u_{0, \epsilon}\|_{L^2(\Omega)} + \|u_{1, \epsilon}\|_{L^2(\Omega)} + (T_2)^{1/2} \|f_\epsilon\|_{H^1(\Sigma_0)} + \|G_\epsilon\|_{L^2(\Sigma_0)}]$$

$$\times (\|f_\epsilon\|_{H^1(\Sigma_0)} + \|G_\epsilon\|_{L^2(\Sigma_0)}), \hspace{1cm} (4.22)$$

and

$$|\mathcal{E}(u_{1, \epsilon} - u_{2, \epsilon}) - \int_\Omega \left( |\nabla (u_{1, \epsilon} - u_{2, \epsilon})|^2 + |u_{1, \epsilon} - u_{1, \epsilon}|^2 \right) dx|$$

$$\leq C (T_2)^{1/2} [\|\nabla (u_{0, \epsilon} - u_{0, \epsilon})\|_{L^2(\Omega)} + \|u_{1, \epsilon} - u_{1, \epsilon} - u_{1, \epsilon}\|_{L^2(\Omega)} + (T_2)^{1/2} \|f_\epsilon - f_\epsilon\|_{H^1(\Sigma_0)}$$

$$+ \|G_\epsilon - G_\epsilon\|_{L^2(\Sigma_0)}] \times (\|f_\epsilon - f_\epsilon\|_{H^1(\Sigma_0)} + \|G_\epsilon - G_\epsilon\|_{L^2(\Sigma_0)}). \hspace{1cm} (4.23)$$

Combining the limits given in (4.13) and (4.21) with (4.23), we see the limit of $\{u_\epsilon\}$ and $\{\partial_t u_\epsilon\}$ exist in $H^1(T_\Sigma)$ and in $L^2(\Sigma_\Sigma)$, respectively, and the limit coincides with the solution $u$. Therefore, by taking the limit of (4.22), we arrive at the conclusion. \qed
**Proof of Corollary 1.2** Recall the notation $\mathcal{E}(u; \Gamma)\) defined in (1.7). When $f = 0$ and $G = 0$, by Theorem 1.1 we have

$$\mathcal{E}(u; \Gamma) = \int_{\Omega} (|\nabla u_0|^2_A + |u_1|^2) \, dx,$$

which is true for any $\tau$. The proof is done. \qed

We can borrow the arguments in the proof of Theorem 1.1 to show Theorem 1.3.

**Proof of Theorem 1.3** The sequences $(u_{0, \epsilon}, u_{1, \epsilon}, f_{\epsilon}, G_{\epsilon})$ constructed in the proof of Theorem 1.1 meet the requirement of Lemma 1.2, so the corresponding solution $u_{\epsilon}$ satisfies

$$\|u_{\epsilon, \omega, A}\|_{L^2(\Sigma_\tau)}^2 \leq C(T_2)(\|\nabla u_{0, \epsilon}\|_{L^2(\Omega)} + \|u_{1, \epsilon}\|_{L^2(\Omega)}^2) + C(T_2)^2(\|G_{\epsilon}\|_{L^2(Q_0)}^2 + \|f_{\epsilon}\|_{H^1(\Sigma_\tau)}^2).$$

By taking the limit, we obtain the result. \qed

5. **Concluding remark**

One of the possible future directions succeeding this work is to study the case where the initial/boundary data possess higher order regularities. The corresponding results are classical for horizontal planes, i.e. for $\Gamma_S = \Omega \times \{t = \tau\}$, see e.g. [Eva10, §7.2.3] for Dirichlet boundary condition cases. However, the method used for horizontal planes seems not to directly apply to the non-timelike case. More careful investigations are needed.

Another direction is to look at the timelike case. Physically speaking, the timelike case corresponds to supersonic waves. In this case the shock wave phenomenon could happen, which could cause additional singularities to the solution. Hence, a reasonable expectation is that the regularity of the solution on timelike hypersurfaces will not be as good as on non-timelike hypersurfaces.

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