YANGIAN CONSTRUCTION OF THE VIRASORO ALGEBRA

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Abstract. We show that a Yangian construction based on the algebra of an infinite number of harmonic oscillators (i.e. a vibrating string) terminates after one step, yielding the Virasoro algebra.

In this paper we consider a somewhat more general notion of Yangian than the usual one (as defined by e.g. Chari and Pressley [1, Chapter 12]). Let $\mathfrak{g}$ be a Lie algebra, with universal enveloping algebra $U(\mathfrak{g})$, and suppose $\mathfrak{g} \otimes \mathfrak{g}$ contains an element $\Omega$ which commutes with $\Delta_0(X) = X \otimes 1 + 1 \otimes X$ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, for all $X \in \mathfrak{g}$. We define a Yangian of $\mathfrak{g}$ to be a bialgebra $Y(\mathfrak{g})$ containing $U(\mathfrak{g})$ and another copy of $\mathfrak{g}$, so that $Y(\mathfrak{g})$ is generated by $\mathfrak{g} \oplus \theta(\mathfrak{g})$ where $\theta$ is a vector space isomorphism, with coproducts

$$\Delta(X) = \Delta_0(X) = X \otimes 1 + 1 \otimes X$$
$$\Delta(\theta(Y)) = \theta(X) \otimes 1 + 1 \otimes \theta(X) + [\Omega, X \otimes 1]$$

where the square brackets denote the commutator.

Usually one imposes relations

$$[X, \theta(Y)] = \theta([X, Y])$$

as is justified by the following simple calculation:

Proposition. The coproducts (1) and (2) satisfy

$$[\Delta(X), \Delta(\theta(Y))] = \Delta \circ \theta([X, Y]) \mod R \otimes Y(\mathfrak{g}) + Y(\mathfrak{g}) \otimes R$$

where $R$ is the subspace of $Y(\mathfrak{g})$ spanned by the relations (3) (i.e. by all elements $[X, \theta(Y)] - \theta([X, Y])$).

Proof.

$$[\Delta(X), \Delta(\theta(Y))] = [X \otimes 1 + 1 \otimes X, \theta(Y) \otimes 1 + 1 \otimes \theta(Y)] + [X \otimes 1 + 1 \otimes X, [\Omega, Y \otimes 1]].$$

The first term on the right-hand side is

$$[X, \theta(Y)] \otimes 1 + 1 \otimes [X, \theta(Y)] = \theta([X, Y]) \otimes 1 + 1 \otimes \theta([X, Y]) \mod R \otimes Y(\mathfrak{g}) + Y(\mathfrak{g}) \otimes R.$$ 

The second, using the Jacobi identity and the fact that $\Omega$ commutes with $X \otimes 1 + 1 \otimes X$, is

$$[\Omega, [X \otimes 1 + 1 \otimes X, Y \otimes 1]] = [\Omega, [X, Y] \otimes 1].$$
Hence their sum is as stated.

The commutators $[\theta(X), \theta(Y)]$ are constrained but not completely determined by the requirement of compatibility with the coproducts (4), and in general they involve new linearly independent elements. Further linearly independent elements are introduced by higher commutators, so that the size of the Yangian so defined is considerably larger than the polynomial algebra on $\mathfrak{g} \oplus \theta(\mathfrak{g})$.

The purpose of this note is to point out that when $\mathfrak{g} = \mathfrak{s}$, the Lie algebra associated with the oscillations of a string, a slight change in the above construction leads to a Yangian algebra which closes at the first level $\theta(\mathfrak{s})$, where it yields the Virasoro algebra.

By the string algebra $\mathfrak{s}$ we mean the algebra of an infinite number of oscillators whose frequencies are integer multiples of a basic frequency. Thus we take $\mathfrak{s}$ to have generators $a_m$ ($m \in \mathbb{Z}$) (raising operators if $m > 0$, lowering operators if $m < 0$, central if $m = 0$) and $H$ (the Hamiltonian), with commutators

$$[a_m, a_n] = ma_0\delta_{m+n,0}$$
$$[H, a_m] = ma_m.$$  

This Lie algebra has a Casimir tensor

$$\Omega = \sum_{m \in \mathbb{Z}} a_m \otimes a_{-m} + a_0 \otimes H + H \otimes a_0.$$  

We introduce the second copy $\theta(\mathfrak{s})$ with generators $b_m = \theta(a_m)$, $K = \theta(H)$ and coproducts

$$\Delta(b_m) = b_m \otimes 1 + 1 \otimes b_m + \varepsilon[\Omega, a_m \otimes 1]$$
$$= b_m \otimes 1 + 1 \otimes b_m + \varepsilon m(a_m \otimes a_0 - a_0 \otimes a_m)$$
$$\Delta(K) = K \otimes 1 + 1 \otimes K + \delta[\Omega, H \otimes 1]$$
$$= K \otimes 1 + 1 \otimes K - \varepsilon \Omega$$

where $\varepsilon \in \mathbb{C}$ is a parameter. The Lie brackets of the generators $b_m$, $K$ with each other and with the string generators $a_m$, $H$ are required to be compatible with the coproducts (4) and with the usual Lie algebra coproducts $\Delta_0(a_m)$ and $\Delta_0(H)$ as in (4). We note that the only elements of $\mathfrak{s} \oplus \theta(\mathfrak{s})$ with these Lie algebra-like coproducts are combinations of $a_m$, $H$ and $b_0$: in Hopf algebra terminology, the primitive Lie subalgebra of $\mathfrak{s} \oplus \theta(\mathfrak{s})$ is $\mathfrak{s} \oplus \langle b_0 \rangle$.

The requirement of compatibility gives

$$\Delta([a_m, b_n]) = [\Delta(a_m), \Delta(b_n)]$$
$$= [a_m, b_n] \otimes 1 + 1 \otimes [a_m, b_n],$$
so that \([a_m, b_n] \) is Lie algebra-like, and
\[
\Delta([H, b_n]) = [\Delta(H), \Delta(b_n)] = [H, b_n] \otimes 1 + 1 \otimes [H, b_n] + \varepsilon n^2 (a_n \otimes a_0 - a_0 \otimes a_n)
\]
\[
= [H, b_n] \otimes 1 + 1 \otimes [H, b_n] + n\Delta(b_n) - n(b_n \otimes 1 + 1 \otimes b_n)
\]
so that \([H, b_n] - nb_n \) is Lie algebra-like. The simplest choice is to take it to vanish:
\[
[H, b_n] = nb_n.
\] (8)
Then \([a_m, b_n] \), which belongs to \( \mathfrak{s} \oplus \langle b_0 \rangle \), must be a multiple of \( a_m + n \); suppose \([a_m, b_n] = \gamma_{mn} a_m + n \). The Jacobi identity between \( a_l \), \( a_m \) and \( b_n \) gives
\[
\lambda \gamma_{mn} = m \gamma_{ln}
\]
so that \( \gamma_{mn} = m \gamma_n \), where the \( \gamma_n \) are either all zero or all non-zero. Taking them to be non-zero, we can redefine the \( b_n \) by dividing by \( \gamma_n \) to get
\[
[a_m, b_n] = ma_{m+n}
\]
(but note that this requires the coproduct (7) to be modified by replacing \( \varepsilon \) by \( \varepsilon_m = \varepsilon / \gamma_m \)).

Now the Jacobi identity between \( a_l \), \( b_m \) and \( b_n \) gives
\[
[a_l, [b_m, b_n]] = l(m - n)a_{l+m+n}
\]
\[
= [a_l, (m - n)b_{m+n}].
\]
Put
\[
[b_m, b_n] = (m - n)b_{m+n} + c_{mn};
\]
then \( c_{mn} \) commutes with all \( a_l \), and \([H, c_{mn}] = (m + n)c_{mn} \), so if we are to avoid introducing new generators \( c_{mn} \) must be a function of \( a_0 \) and vanish unless \( m + n = 0 \). The Jacobi identity between \( b_l \), \( b_m \) and \( b_n \) with \( l + m + n = 0 \) gives
\[
(m - n)c_{m+n,-m-n} - (m + 2n)c_{m,-m} + (2m + n)c_{n,-n} = 0
\]
which has the two independent solutions
\[
c_{m,-m} = m \quad \text{and} \quad c_{m,-m} = m^3;
\]
hence the commutator between \( b_m \) and \( b_n \) must be of the form
\[
[b_m, b_n] = (m - n)b_{m+n} + \delta_{m+n,0}(m^3 F + mG)
\] (9)
where \( F \) and \( G \) are functions of \( a_0 \).

Finally, applying the coproduct (7) to this commutator gives
\[
m^3 \Delta(F) + m\Delta(G) = m^3 (F \otimes 1 + 1 \otimes F) + m(G \otimes 1 + 1 \otimes G)
\]
\[
- \varepsilon_m^2 m^3 (a_0 \otimes a_0^2 + a_0^2 \otimes a_0).
\]
Write
\[ \Delta'(X) = \Delta(X) - X \otimes 1 - 1 \otimes X, \]
\[ A = a_0 \otimes a_0^2 + a_0^2 \otimes a_0; \]
then
\[ m^3 \Delta'(F) + m \Delta'(G) + m^3 \varepsilon_m^2 A = 0. \]
We can solve these equations with \( m = 1 \) and \( 2 \) to get \( \Delta'(F) \) and \( \Delta'(G) \) as multiples of \( A \); then, since \( A \neq 0 \), the remaining equations give \( \varepsilon_m \) in terms of \( \varepsilon_1 \) and \( \varepsilon_2 \). The result is
\[ \Delta'(F) = \frac{1}{3}(\varepsilon_1^2 - 4\varepsilon_2^2)A, \]
\[ \Delta'(G) = \frac{4}{3}(\varepsilon_2^2 - \varepsilon_1^2)A, \]
\[ \varepsilon_m^2 = \varepsilon_1^2 + \sum_{r=1}^{m} \frac{2r - 1}{3r^2}(\varepsilon_2^2 - \varepsilon_1^2). \]
Expanding \( F \) and \( G \) in powers of \( a_0 \) and using
\[ \Delta(a_0^0) = (a_0 \otimes 1 + 1 \otimes a_0)^r = \sum_k \binom{r}{k} a_0^k \otimes a_0^{r-k}, \]
we find
\[ F = \alpha a_0 + \frac{1}{9}(\varepsilon_1^2 - 4\varepsilon_2^2)a_0^3, \]
\[ G = \beta a_0 + \frac{4}{9}(\varepsilon_2^2 - \varepsilon_1^2)a_0^3 \]
where \( \alpha \) and \( \beta \) are arbitrary.

At this stage we see that we have a bialgebra generated by \( a_n, b_n \) and \( H \), so there is no need to consider the generator \( K \). In terms of our definition, it is a subbialgebra of the Yangian which contains the Virasoro algebra (9).

In order to exhibit a simple example, we take \( \alpha = \beta = 0 \) and \( \varepsilon_1 = \varepsilon_2 \), so that \( \varepsilon_m^2 \) is independent of \( m \); say \( \varepsilon_m = \varepsilon \). Then we have a bialgebra generated by \( a_m, b_m \) and \( H \) with relations
\[ [a_m, a_n] = ma_0\delta_{m,n-m} \]
\[ [a_m, b_n] = ma_{m+n} \]
\[ [b_m, b_n] = (m - n)b_{m+n} - \frac{1}{3}\varepsilon^2 m^3 a_0^3 \delta_{m+n,0} \]
\[ [H, a_m] = ma_m \]
\[ [H, b_m] = mb_m \]
and coproducts
\[ \Delta(a_m) = a_m \otimes 1 + 1 \otimes a_m \]
\[ \Delta(b_m) = b_m \otimes 1 + 1 \otimes b_m + \varepsilon m(a_m \otimes a_0 - a_0 \otimes a_m) \]
\[ \Delta(H) = h \otimes 1 + 1 \otimes H. \]

To summarise, our general result is
Theorem. Let \( s \) be the Lie algebra with generators \( a_m, (m \in \mathbb{Z}) \) and \( H \) and Lie brackets (4) and (5), and let \( U(s) \) be the universal enveloping algebra of \( s \), with the usual bialgebra structure. Then there is a four-parameter family of bialgebras containing \( U(s) \) and additional generators \( b_m (m \in \mathbb{Z}) \), with coproducts of the form

\[
\Delta(b_m) = b_m \otimes a + 1 \otimes b_m + m \varepsilon_m (a_m \otimes a_0 - a_0 \otimes a_m)
\]

and relations including \([H, b_m] = m b_m\). In any such bialgebra the \( b_m \) satisfy the Virasoro-like relations (9).

References

[1] V. Chari and A. Pressley, Quantum Groups, Cambridge University Press, 1994.

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