Dependence of acoustic surface gravity on disc thickness for accreting astrophysical black holes

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Abstract

For axially symmetric accretion maintained in hydrostatic equilibrium along the vertical direction, we investigate how the characteristic features of the embedded acoustic geometry depends on the background Kerr metric, and how such dependence is governed by three different expressions of the thickness of the matter flow. We first obtain the location of the sonic points and stationary shock between the sonic points. We then linearly perturb the flow to obtain the corresponding metric elements of the acoustic space-time. We thus construct the causal structure to establish that the sonic points and the shocks are actually the analogue black hole type and white hole type horizons, respectively. We finally compute the value of the acoustic surface gravity as a function of the spin angular momentum of the rotating black hole for three different flow thicknesses considered in the present work. We find that for some flow models, the intrinsic acoustic geometry, although in principle may be extended up to the outer gravitational horizon of the astrophysical black hole, cannot be constructed beyond a certain truncation radius as imposed by the expressions of the thickness function of the corresponding flow.

1 Introduction

Emergent gravity phenomena may be observed as a consequence of the linear perturbation of inhomogenous, inviscid, transonic fluid flow. Such perturbation leads to the formation...

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of black hole like acoustic metric embedded within the background stationary flow, and the acoustic metric can have analogue horizons from within which phonons can not escape ([Moncrief, 1980], [Unruh, 1981], [Visser, 1998], [Bilic, 1999], [Novello et al., 2002], [Unruh and Schützhold, 2007], [Barceló et al., 2011]).

Owing to the fact that accretion of hydrodynamic fluid onto astrophysical black holes are usually transonic in nature ([Liang and Thomson, 1980]), it has been recently proposed that accreting astrophysical systems may be considered as natural examples of classical analogue gravity models ([Das, 2004], [Dasgupta et al., 2005], [Abraham et al., 2006], [Das et al., 2007], [Pu et al., 2012], [Bilić et al., 2014], [Tarafrdar and Das, 2015], [Ananda et al., 2015], [Saha et al., 2016], [Bollimpalli et al., 2017], [Shaikh et al., 2017], [Tarafrdar and Das, 2018], [Shaikh and Das, 2018], [Shaikh, 2018], [Datta et al., 2018], [Shaikh et al., 2019], [Datta and Das, 2019]).

If the accretor happens to be a black hole, whether a stellar mass one or a supermassive one, then the corresponding analogue system becomes unique in the sense that such a system contains two types of horizons, viz., gravitational and acoustic. For low angular momentum axially symmetric accretion, it may also be demonstrated that the analogue space-time embedded within the accreting fluid may contain multiple acoustic horizons, two black hole type horizons and a white hole hole type analogue horizon, to be more specific ([Abraham et al., 2006]). The study of accreting astrophysical black holes, thus opens up a novel avenue to study various interesting features of emergent gravity phenomenon.

Low angular momentum black hole accretion discs may assume various geometrical configurations. Among them, one important type of flow configuration is known as, ‘accretion flow maintained in hydrostatic equilibrium along the vertical direction’ (perpendicular to the equatorial plane of the disc). Hereafter, we shall refer to such flow as flow in vertical equilibrium, in short. For general relativistic hydrodynamic accretion onto spinning (Kerr-type) black holes, three different expressions (arising out of three distinct types of force balance conditions for gravitational attraction of the black hole and pressure of the accreting fluid) for the flow thickness (disc heights) exist in literature. Astrophysical properties of multitransonic shocked accretion for flow with aforementioned three types of disc thicknesses have been discussed in great detail in one of our recent papers ([Tarafrdar et al., 2020], hereafter P1). Stationary integral accretion solutions have been constructed in P1, and the corresponding locations of the sonic points and the standing shocks has been found out as a function of the spin angular momentum of the black hole (the Kerr parameter ‘a’) for three different expressions of the disc height. The influence of the flow geometry (as represented by various disc thicknesses) on the black hole spin dependence of the multitransonic shocked accretion has been studied in P1.

As mentioned in the previous paragraphs, the sonic point of the accreting fluid, where the subsonic flow makes a smooth transition to the supersonic state, may be considered as acoustic black holes, and the discontinuous shock jump can be considered as analogue white holes. The astrophysical properties of shocked multitransonic flows, as described in P1, can thus be studied from the context of emergent gravity phenomenon where acoustic black holes and
white holes may be emerged in the same classical (non-quantum) analogue system. This is, precisely, our motivation in the present work.

For three different disc heights considered in P1, we shall calculate the location of the inner sonic points, the outer sonic points and the shock, as a function of the black hole spin ‘a’, the flow angular momentum λ, the specific energy of the flow $\mathcal{E}$, and the polytropic index of the flow $\gamma$, for accretion governed by the polytropic as well as the isothermal equation of state. By constructing the causal structure of the flow along the streamlines, it will then be proved that the sonic points and the shocks in astrophysics are actually the analogue black hole type horizons and analogue white hole type horizons respectively. We will then calculate the value of the acoustic surface gravity $\kappa$ at the corresponding locations of the acoustic black holes. We then study the dependence of the location of the acoustic black hole horizons and the value of the acoustic surface gravity on the spin angular momentum of the black hole. Through this procedure, we actually study how the actual space-time structure (described through the Kerr metric) influences the emergent space-time structure (manifested through the acoustic surface gravity). We perform our investigation for three different disc structures and hence we also study how the disc geometry (manifested through the expression of the flow thickness) determines the Kerr metric dependence of the emergent gravity space-time.

In order to accomplish such task, as mentioned, we will use the astrophysics related results (location of sonic points and shock, and the dependence of such locations on $[\mathcal{E}, \lambda, \gamma, a]$ or $[T, \lambda, a]$) as obtained in P1. In the present article, we shall not repeat the derivation and discussions on the calculations performed in P1, we shall rather borrow the results from there to calculate $\kappa$ using those results. The necessary discussions on the calculation of the value of $\kappa$ is presented in the subsequent sections.

2 Integral flow solution scheme

In the present work, low angular momentum, inviscid, axisymmetric flow of hydrodynamic accretion onto a Kerr black hole is studied. The governing equations are the general relativistic Euler equation and the equation of continuity which are to be solved in steady state, using certain thermodynamic equations of state (polytropic and isothermal), to obtain the stationary, integral flow. Solutions of multitransonic behaviour may encompass a standing shock located between the outer and the inner sonic points. The overall solution scheme to obtain such stationary accretion solutions are detailed in P1, hence we would not repeat it here. The space-time metric describing the stationary fluid flow can be described by the Boyer-Lindquist line element of the following form (normalized for $G = c = M_{BH} = 1$), where $G$, $c$ and $M_{BH}$ are the universal Gravitational constant, velocity of light in vacuum and mass of the black hole, respectively.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\frac{r^2 \Delta}{A} dt^2 + \frac{A}{r^2} (d\phi - \omega dt)^2 + \frac{r^2}{\Delta} dr^2 + dz^2,$$

(1)

where $\Delta = r^2 - 2r + a^2$, $A = r^4 + r^2 a^2 + 2ra^2$, $\omega = 2ar/A$, $a$ being the Kerr parameter.
The flow thickness for three different disc configurations considered in the present article can be expressed as follows –

**Novikov & Thorne type of discs (hereafter NT)** The earliest general relativistic formulation of gravity-pressure-balanced vertically averaged accretion disc height ([Novikov and Thorne, 1973]), is given by,

\[
H_{NT}(r) = \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \frac{a^2 r + 2a^2}{r^{\frac{3}{2}} + a} \sqrt{\frac{r^6 - 3r^5 + 2ar^2}{(r^2 - 2r + a^2)(r^4 + 4a^2r^2 - 4a^2r + 3a^4)}},
\]

where \( p \) and \( \rho \) are the pressure and the rest-mass energy density of the fluid respectively.

**Riffert & Herold type of discs (hereafter RH)** [Riffert and Herold, 1995] modified the gravity-pressure-balance condition formulated by NT, and proposed a new disc height given by,

\[
H_{RH}(r) = \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \frac{2r^4}{\sqrt{r^2 - 4ar^2 + 3a^2}}.
\]

**Abramowicz, Lanza & Percival type of discs (hereafter ALP)** Recently, [Abramowicz et al., 1997] formulated an expression for the flow thickness, given by

\[
H_{ALP}(r) = \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \sqrt{\frac{2r^4}{v_\phi^2 - a^2(v_t - 1)}},
\]

where, \( v^\mu \) denotes the four-velocity of the fluid in an azimuthally-boosted frame which co-rotates with the flow. \( v_\phi \) and \( v_t \) are the azimuthal and temporal components of the covariant 4-velocity respectively which are related by \( \lambda = -v_\phi/v_t \), where \( \lambda \) is the specific angular momentum of the flow and \( v_t \) is given by,

\[
v_t = \sqrt{\frac{\Delta}{B(1 - u^2)}},
\]

where \( B = g_{\phi\phi} + 2\lambda g_{t\phi} - \lambda^2 g_{tt} \) and \( u \) denotes advective velocity which is the three-component velocity in the co-rotating frame.\(^1\)

For three different types of flow thicknesses, we compute the location of sonic points and the shock as discussed in P1, in detail. One, however, has to establish formally that the sonic points obtained in the aforementioned way are the analogue black hole type horizon, and the shocks are actually the analogue white hole type horizon. In order to accomplish this task, in subsequent sections, we will demonstrate how one can linearly perturb the time-dependent Euler and the continuity equations to obtain the complete structure of the acoustic metric embedded within the background stationary accretion flow. We obtain the

\(^1\)we refer [Gammie and Popham, 1998] for the detailed description of expressions of various velocities in different frames for rotating accretion flow in the Kerr metric.
corresponding metric elements of such acoustic metric and finally construct the causal structure to mathematically establish that the smooth sonic transitions are the black hole type acoustic horizons and the discontinuous sonic transitions (from supersonic to subsonic flow) are the acoustic white holes.

3 Identification of horizons through causal structures

The concept of causal structure plays a crucial role in developing the correspondence between sonic points on an accretion disc and the acoustic horizons of an analogue system. Hence, understanding the construction of causal structures to identify the horizons of a general physical space-time metric turns out to be important in the present context. In this section, we follow [Wald, 1984] to introduce the idea of the construction of causal structures in general relativity.

The line interval of a radially falling particle in the equatorial plane of a non-charged axially symmetric black hole can be written in general as

\[ ds^2 = g_{tt}dt^2 + 2g_{tr}dt dr + g_{rr}dr^2. \]  

(6)

Considering a light ray, the interval \( ds^2 = 0 \), and we obtain,

\[(dt - A_+(r)dr)(dt - A_-(r)dr) = 0 \]

(7)

where,

\[ A_\pm = \frac{-g_{tr} \pm \sqrt{g_{rr}^2 - g_{rr}g_{tt}}}{g_{tt}}. \]

(8)

Thus if we try to draw the light cone in \((t, r)\) plane, then the slope of the light cone at a particular radial distance is given by \(A_\pm\). Thus one can define the horizon to be the radial distance, where one of the slopes is negative and the other is infinity. As we examine the light cone inside the horizon, the whole future light cone tilts inside the horizon or towards the centre of black hole. Thus no light can escape horizon, which is precisely the definition of horizon.

Now one can draw the trajectory of any incoming or outgoing light ray by integrating the equation

\[ \frac{dt}{dr} = A_\pm(r). \]

(9)

It is evident that no trajectory inside horizon will cross the horizon and the outgoing light rays will slow down as it reaches the horizon from inside. This fact can be shown by drawing causal structure, where we integrate null coordinate in the \((t, r)\) plane. In this case we define the null coordinates \((\chi, \omega)\) so that,

\[ d\omega = dt - A_+(r)dr \]

(10)

\[ d\chi = dt - A_-(r)dr. \]

(11)
By integrating these equations and drawing $\chi = \text{constant}$ and $\omega = \text{constant}$ lines in $(t, r)$ plane, and thus observing where the null trajectories asymptotically approaches, the horizon can be identified.

So far, we have only concentrated on the stationary solutions and obtained the sonic points in the present work. Although it may seem evident that sound waves cannot traverse through these sonic points from inside, and thus it may be justified to say that the sonic points are the acoustic horizons, nevertheless, there should be a mathematical justification when one is bringing analogy between light rays around black hole horizon and sound waves around acoustic black hole horizon. Thus, in the next section the stationary solution is linearly perturbed and the acoustic metric is obtained. Then the causal structure is drawn to show what was previously defined as sonic points can really be identified as the horizon of the acoustic black holes.

4 Construction of the acoustic metric

Once the significance of causal structures is established, we need to elaborate a scheme of calculations required to derive an acoustic metric embedded within the accretion flow. This requires an approach along the lines of perturbative analysis detailed in the following subsections.

4.1 Dynamical equations

First we provide the dynamical equations, important dynamical quantities and constant of motion which will be perturbed. The energy momentum tensor for a perfect fluid is given by

$$T^{\mu\nu} = (p + \epsilon) v^\mu v^\nu + pg^{\mu\nu},$$  \hspace{1cm} (12)

where $p$ is the dynamical pressure and $\rho$ is the rest-mass energy density of the fluid, so that $\epsilon = \rho + \epsilon_{\text{thermal}}$. The normalization condition of the four velocity $v^\mu$ is $v^\mu v_\mu = -1$.

The equation of state for adiabatic flow is given by $p = k\rho^\gamma$, where $k$ is a constant related to specific entropy of the fluid and $\gamma$ is known as the polytropic index. The sound speed for adiabatic flow (isoentropic) is given by

$$c_s^2 = \frac{\partial p}{\partial \epsilon}_{\text{entropy}} = \frac{\rho}{h} \frac{\partial h}{\partial \rho},$$ \hspace{1cm} (13)

where the specific enthalpy $h$ is given by

$$h = \frac{p + \epsilon}{\rho}$$ \hspace{1cm} (14)
The mass conservation equation and the energy-momentum conservation equations are respectively given by,

\[ \nabla_{\mu}(\rho v^{\mu}) = 0 \]  
(15)

and

\[ \nabla_{\mu}T^{\mu\nu} = 0. \]  
(16)

Using the expression for sound speed, the energy-momentum conservation equation can be written in the following form

\[ v^{\mu}\nabla_{\mu}v^{\nu} + \frac{c_{s}^{2}}{\rho}(v^{\mu}v^{\nu} + g^{\mu\nu})\partial_{\mu}\rho = 0 \]  
(17)

where \( c_{s} \) is given by eqn.(13).

Here we emphasize on a notation that will be used throughout the rest of the literature. We shall use the subscript ‘0’ to denote values of physical variables corresponding to the stationary solutions of the steady flow, e.g., \( p_{0}, \rho_{0}, v_{r}^{0} \) etc.

### 4.2 First integrals of motion

We need two first integrals of motion for the stationary flow. The first comes from the mass conservation equation and the second comes from the momentum conservation equation. It is convenient to do a vertical averaging of the flow equations by integrating over \( z \) and the resultant equation is described by the flow variables defined on the equatorial plane (\( z = 0 \)). In addition, one also integrates over \( \phi \) which gives a factor of \( 2\pi \) due to the axial symmetry of the flow. We do such vertical averaging as prescribed in ([Novikov and Thorne, 1973], [Matsumoto et al., 1984], [Gammie and Popham, 1998]) to the mass conservation equation given by eqn. (15). Thus, in case of stationary (\( t \)-independent) and axially symmetric (\( \phi \)-independent ) flow with averaged \( v^{z} \sim 0 \), eqn.(15) can be written as

\[ \frac{\partial}{\partial r}(4\pi H_{\theta}\sqrt{-\tilde{g}\rho_{0}v_{r}^{0}}) = 0 \]  
(18)

\( H_{\theta} \) arises due to the vertical averaging and is the local angular scale of flow. Thus one can relate the actual local flow thickness \( H(r) \) to the angular scale of the flow \( H_{\theta} \) as \( H_{\theta} = H(r)/r \), where \( r \) is the radial distance along the equatorial plane from the centre of the disc. \( \tilde{g} \) is the value of the determinant of the metric \( g_{\mu\nu} \) on the equatorial plane, \( \tilde{g} = \det(g_{\mu\nu})|_{z=0} = -r^{4} \).

The eqn.(18) gives the mass accretion rate \( \dot{M}_{0} \) by

\[ \dot{M}_{0} = 4\pi\sqrt{-\tilde{g}}H_{\theta}\rho_{0}v_{r}^{0} = 4\pi H(r)r\rho_{0}v_{r}^{0}, \]  
(19)

where \( v^{r} \) can be expressed in terms of \( u_{0} \) as,

\[ v_{0}^{r} = \frac{u_{0}}{\sqrt{g_{rr}(1-u_{0}^{2})}} = \frac{\sqrt{\Delta u_{0}}}{r\sqrt{1-u_{0}^{2}}} \]  
(20)
using $g_{rr} = r^2/\Delta$, with $\Delta = r(r-2)$. Thus $\dot{M}_0$ can be written as

$$\dot{M}_0 = 4\pi H(r)\Delta^{1/2}\rho_0 \frac{u_0}{\sqrt{1 - u_0^2}} = \text{constant.}$$

(21)

The second conserved quantity can be obtained from the time-component of the steady-state relativistic Euler (energy-momentum conservation) eqn.(16), which for stationary adiabatic case gives

$$\mathcal{E}_0 = h_0 v_{t0} = \text{constant.}$$

(22)

where, using eqn.(5),

$$\mathcal{E}_0 = \frac{1}{1 - n c^2 s_0} \sqrt{\frac{\Delta}{B(1 - u_0^2)}},$$

(23)

where $n = \frac{1}{\gamma - 1}$.

4.3 The linear perturbation scheme

Now, the perturbation on the stationary flow is done by following standard linear perturbation analysis ([Ananda et al., 2015], [Bollimpalli et al., 2017], [Shaikh et al., 2017], [Shaikh, 2018]). Time-dependent accretion variables, like the components of four velocity and pressure are written as small time-dependent linear perturbations added to their stationary values. Thus we can write,

$$v^t(r,t) = v^t_0(r) + v^t_1(r,t)$$
$$v^r(r,t) = v^r_0(r) + v^r_1(r,t)$$
$$\rho(r,t) = \rho_0(r) + \rho_1(r,t)$$

(24)

where the subscript ‘1’ denotes the small perturbations of some variable about the stationary value denoted by subscript ‘0’. Now we denote $\dot{M}$ as $\Psi = 4\pi \sqrt{-g}\rho(r,t)v^r(r,t)H_\theta$ which the stationary mass accretion rate of the accretion flow. Thus,

$$\Psi(r,t) = \Psi_0 + \Psi_1(r,t)$$

(25)

where $\Psi_0$ is the stationary mass accretion rate defined in eqn.($\square$). The constants can be absorbed in the definition without any loss of generality. Using the eqns.(24), we get

$$\Psi_1 = \sqrt{-g}[\rho_1 v^r_0 H_{\theta 0} + \rho_0 v^r_1 H_{\theta 0} + \rho_0 v^r_0 H_{\theta 1}]$$

(26)

The last term in the perturbation $\Psi_1$ consists of a term with the perturbation of angular height function $H_\theta$. We recall $H(r)$ as $H_\theta = H(r)/r$.

For adiabatic flow, (14) can be rewritten as

$$h = 1 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

(27)
where the corresponding perturbation $h_1$ can be written as

$$h_1 = \frac{h_0 c_s^2}{\rho_0} \rho_1. \quad (28)$$

For adiabatic flow, the irrotationality condition turns out to be ([Bilic, 1999]),

$$\partial_\mu(h v_\nu) - \partial_\nu(h v_\mu) = 0 \quad (29)$$

Now, using eqn.(29), the normalization condition $v^\mu v_\mu = -1$ and the axisymmetry of the flow, we obtain quantities needed for further perturbation. From irrotationality condition (eqn.(29)) with $\mu = t$ and $\nu = \phi$ and with axial symmetry we have,

$$\partial_t(h v_\phi) = 0, \quad (30)$$

and, for $\mu = r$ and $\nu = \phi$ and the axial symmetry, we have

$$\partial_r(h v_\phi) = 0. \quad (31)$$

So $h v_\phi$ is a constant of motion and eqn.(30) gives

$$\partial_t v_\phi = -\frac{v_\phi c_s^2}{\rho} \partial_t \rho. \quad (32)$$

Using $v_\phi = g_{\phi \phi} v^\phi + g_{\phi t} v^t$ in the previous equation gives

$$\partial_t v_\phi = -\frac{g_{\phi t}}{g_{\phi \phi}} \partial_t v^t - \frac{v_\phi c_s^2}{g_{\phi \phi} \rho} \partial_t \rho. \quad (33)$$

The normalization condition of four velocity $v^\mu v_\mu = -1$ in this case can be written as

$$g_{tt}(v^t)^2 = 1 + g_{rr}(v^r)^2 + g_{\phi \phi}(v^\phi)^2 + 2 g_{\phi t} v^\phi v^t. \quad (34)$$

The time derivative of this equation is,

$$\partial_t v^t = \alpha_1 \partial_t v^r + \alpha_2 \partial_t v^\phi \quad (35)$$

where $\alpha_1 = -\frac{v_r}{v_t}$, $\alpha_2 = -\frac{v_\phi}{v_t}$ and $v_t = -g_{tt} v^t + g_{\phi t} v^\phi$. Replacing $\partial_t v^\phi$ in eqn.(35) using eqn.(33) yields,

$$\partial_t v^t = \left( -\frac{\alpha_2 v_\phi c_s^2}{1 + \alpha_2 g_{\phi t} / g_{\phi \phi}} \right) \partial_t \rho + \left( \frac{\alpha_1}{1 + \alpha_2 g_{\phi t} / g_{\phi \phi}} \right) \partial_t v^r. \quad (36)$$

Using eqn.(24) in eqn.(36) and collecting the linear perturbation part, we yield

$$\partial_t v^t_1 = \eta_1 \partial_t \rho_1 + \eta_2 \partial_t v^r_1, \quad (37)$$

where

$$\eta_1 = -\frac{c_s^2}{\Lambda v_0^2} \frac{[\Lambda (v_0^t)^2 - 1 - g_{rr} (v_0^r)^2]}{\rho_0}, \quad \eta_2 = \frac{g_{rr} v_0^r}{\Lambda v_0^t} \text{ and } \Lambda = g_{tt} + \frac{g_{\phi t}^2}{g_{\phi \phi}} \quad (38)$$
Now $H_{\theta 1}$ can be written as

$$\frac{H_{\theta 1}}{H_{\theta 0}} = \left(\frac{\gamma - 1}{2}\right) \frac{\rho_1}{\rho_0} = \beta \frac{\rho_1}{\rho_0}$$  \hspace{1cm} (39)$$

where $\beta = \frac{\gamma - 1}{2}$ is given by (20). The continuity equation takes the form,

$$\partial_t (\sqrt{-g} \rho \nu^t H_\theta) + \partial_r (\sqrt{-g} \rho \nu^r H_\theta) = 0.$$  \hspace{1cm} (40)

Using eqn. (24) and eqn. (25) in the previous equation, and using eqn. (37) and eqn. (39) and replacing them in eqn. (40) yields

$$- \frac{\partial_t \Psi_1}{\Psi_0} = \frac{\eta_2}{v_0} \partial_t v_r^t + \frac{1}{v_0^2} \left[ 1 + \beta + \eta_1 \rho_0 \right] \partial_t \rho_1,$$  \hspace{1cm} (41)

and

$$\frac{\partial_t \Psi_1}{\Psi_0} = \frac{1}{v_0} \partial_t v_r^r + \frac{1 + \beta}{\rho_0} \partial_t \rho_1.$$ \hspace{1cm} (42)

With the two equations given by eqn. (41) and eqn. (42), we can write $\partial_t v_1^r$ and $\partial_t \rho_1$ only in terms of derivatives of $\Psi_1$ as

$$\partial_t v_1^r = \frac{1}{\sqrt{-g} H_0 \rho_0 \Lambda} \left[ (v_0^t (1 + \beta) + \rho_0 \eta_1) \partial_t \Psi_1 + (1 + \beta) v_0^r \partial_t \Psi_1 \right],$$ \hspace{1cm} (43)

$$\partial_t \rho_1 = - \frac{1}{\sqrt{-g} H_0 \rho_0 \Lambda} \left[ \rho_0 \eta_2 \partial_t \Psi_1 + \rho_0 \partial_r \Psi_1 \right],$$

where $\Lambda$ has the form

$$\Lambda = (1 + \beta) \left[ \frac{\nu_r (v_0^t)^2}{\Lambda v_0^t} - v_0^t \right] + \frac{c_\alpha^2}{\Lambda v_0^t} (\Lambda (v_0^t)^2 - 1 - g_{rr} (v_0^r)^2).$$  \hspace{1cm} (44)

Now, we linearly perturb eqn. (21) and take its time derivative, which in turn yields

$$\partial_t (h_0 \nu_r \partial_t v_1^r) + \partial_t \left( \frac{h_0 \nu_r c_\alpha^2 v_0^r}{\rho_0} \partial_t \rho_1 \right) - \partial_r (h_0 \partial_t v_{11}) - \partial_r \left( \frac{h_0 v_0 c_\alpha^2}{\rho_0} \partial_t \rho_1 \right) = 0.$$  \hspace{1cm} (45)

We can write

$$\partial_t v_{11} = \tilde{\eta}_1 \partial_t \rho_1 + \tilde{\eta}_2 \partial_r v_1^r$$ \hspace{1cm} (46)

with

$$\tilde{\eta}_1 = - \left( \Lambda \eta_1 + \frac{g_{\phi \phi} v_{00} c_\alpha^2}{g_{\phi \phi} \rho_0} \right), \quad \tilde{\eta}_2 = - \Lambda \eta_2.$$  \hspace{1cm} (47)

Using eqn. (46) in the eqn. (45) and dividing it by $h_0 v_{10}$ yields

$$\partial_t \left( \frac{\nu_r}{v_{10}} \partial_t v_1^r \right) + \partial_t \left( \frac{\nu_r c_\alpha^2 v_0^r}{\rho_0 v_{10}} \partial_t \rho_1 \right) - \partial_r \left( \frac{\tilde{\eta}_2}{v_{10}} \partial_t v_1^r \right) - \partial_r \left( \frac{\tilde{\eta}_1}{v_{10}} + \frac{c_\alpha^2}{\rho_0} \partial_t \rho_1 \right) = 0.$$ \hspace{1cm} (48)
where we use $h_0 v_{i_0} = \text{constant}$. Finally replacing $\partial_t v_t^i$ and $\partial_t \rho_1$ in eqn.(48) using eqn.(43) one obtains,

\[ \partial_t \left[ k(r) \left( -g^{tt} + (v_t^0)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \right) \right] + \partial_r \left[ k(r) \left( v_0^r v_t^0 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \right) \right] = 0 \quad (49) \]

where $k(r)$ is a conformal factor whose exact form is not required for the present analysis. Eqn.(49) can be written as

\[ \partial_\mu (f^{\mu \nu} \partial_\nu \Psi_1) = 0 \quad (50) \]

where $f^{\mu \nu}$ is obtained from the symmetric matrix

\[ f^{\mu \nu} = k(r) \begin{bmatrix} -g^{tt} + (v_t^0)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) & v_0^r v_t^0 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \\ v_0^r v_t^0 \left( 1 - \frac{1 + \beta}{c_s^2} \right) & g^{rr} + (v_r^0)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \end{bmatrix} \quad (51) \]

The eqn.(50) describes the propagation of the perturbation $\Psi_1$ in 1 + 1 dimension effectively. Eqn.(50) has the same form of a massless scalar field in curved spacetime (with metric $g^{\mu \nu}$) given by,

\[ \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \varphi) = 0 \quad (52) \]

where $g$ is the determinant of the metric $g_{\mu \nu}$ and $\varphi$ is the scalar field. Comparing eqn.(50) and eqn.(52), the acoustic spacetime $G_{\mu \nu}$ metric turns out to be

\[ G_{\mu \nu} = k_1(r) \begin{bmatrix} -g^{rr} - (1 - \frac{1 + \beta}{c_s^2}) (v_r^0)^2 & v_0^r v_t^0 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \\ v_0^r v_t^0 \left( 1 - \frac{1 + \beta}{c_s^2} \right) & g^{tt} - (1 - \frac{1 + \beta}{c_s^2}) (v_t^0)^2 \end{bmatrix} \quad (53) \]

where $k_1(r)$ is also a conformal factor arising due to the process of inverting $G^{\mu \nu}$ in order to yield $G_{\mu \nu}$. For our present purpose we do not need the exact expression for $k_1(r)$.

## 5 Acoustic surface gravity

Having established the formalism leading to the identification of the sonic points and the shocks with the acoustic black holes and the white holes, respectively, we would like to calculate the value of the acoustic surface gravity at the corresponding analogue black hole horizons, i.e., at the inner and outer sonic points. Following ([Abraham et al., 2006], [Tarafdar and Das, 2018]) usual procedures, the acoustic surface gravity $\kappa$ may be expressed as

\[ \kappa = \left. \left[ \kappa_0 \left( \frac{du}{dr} - \frac{dc_s}{dr} \right) \right] \right|_{r = r_s}, \quad (54) \]
where \( r_s \) denotes the location of the sonic point, \( u \) represents the dynamical radial advective velocity of the flow defined on the equatorial plane, \( c_s \) is the corresponding speed of sound, and

\[
\kappa_0 = \frac{\Delta^\frac{3}{2} \sqrt{B}}{r_s(r_s^3 + a^2r_s + 2a^2 - 2a\lambda)(1 - c_s^2)}. \tag{55}
\]

For a given set of flow parameters \( [\mathcal{E}, \lambda, \gamma, a] \), we use the critical point analysis to find out the critical points and then, the sonic points by integrating the stationary flow solutions starting from the critical point(s). Once the sonic points are obtained, the value of the advective flow velocity \( u \), the sound speed \( c_s \), and the respective space gradients of \( u \) and \( c_s \) will be calculated at sonic points. After we obtain the set of values of the location of the sonic points \( (r_s) \) and \( [u, c_s, du/dr, dc_s/dr] \), we calculate \( \kappa \) for polytropic accretion for a set of initial conditions prescribed by \( [\mathcal{E}, \lambda, \gamma, a] \). For isothermal accretion, the initial boundary conditions are specified by \( [T, \lambda, a] \), \( T \) is the bulk flow (ion) temperature. Isothermal sound speed is position-independent, hence the corresponding value of the acoustic surface gravity can be calculated by knowing the location of the isothermal sonic point \( r_s \) and \( [u, c_s, du/dr] \), for a fixed set of \( [T, \lambda, a] \).

Accreting matter remains subsonic at a large distance from the gravitational horizon of the astrophysical black hole. Such subsonic matter becomes supersonic after passing through the outer sonic point, encounters a stationary shock and becomes subsonic. The shock transition is a discontinuous transition. \( [u, c_s] \) changes discontinuously at the shock location. Thus, \( [du/dr, dc_s/dr] \) diverges at the shock and hence \( \kappa \) diverges at the white hole as well.

Our main goal in this work is to study how the \( \kappa - a \) dependence gets effected by different disc heights. To accomplish this task, one thus needs to have an idea about how the shock formation phenomenon is guided by the spin parameter \( a \). We need to compare \( \kappa - a \) plot for three different flow thicknesses. Hence, it is imperative to obtain the aforementioned shock parameter space for all three geometries, and then to study the solutions characterized by the parameters lying inside the mutually overlapping region. Following the methodology developed in P1, we construct shock-forming parameter space \( ([\mathcal{E}, \lambda] \text{ keeping } [\gamma, a] \text{ constant and } [\lambda, a] \text{ keeping } [\mathcal{E}, \gamma] \text{ constant for polytropic flow, and } [T, \lambda] \text{ keeping } a \text{ constant and } [\lambda, a] \text{ keeping } T \text{ constant for isothermal flow}) \) for the different flow geometries, identifying the region of overlap. Then from the obtained region, we select the range of \( a \) corresponding to a particular set of \( [\mathcal{E}, \lambda, \gamma] \) or \( [T, \lambda] \). We then calculate sonic points \( r_s \), shock locations \( r_{sh} \), and \( [u, c_s, du/dr, dc_s/dr] \) (polytropic flow) or \( [u, c_s, du/dr] \) (isothermal flow, since \( c_s \) is constant) for those values of \( a \). These obtained quantities are then substituted in eqn.\((54)\) to procure the respective \( \kappa - a \) plots.

Among the three disc thicknesses considered here, the NT and RH-types of discs do not get extended up to the gravitational horizon (radius of the outer surface of the ergosphere, or \( r_+ \)) of the Kerr black hole given in natural units (along with \( M_{BH} = 1 \)) by

\[
r_+ = 1 + \sqrt{1 - a^2}. \tag{56}
\]
Such discs have a ‘truncation radius’ \((r_T)\) given by solution of the equation

\[
(r_T)^{\frac{1}{2}} (r_T - 3) = 2a,
\]

(57)
beyond which, the height function cannot be defined, although the flow does cross the horizon, otherwise accreting material would not be able to fall onto the black hole. Hence, for the NT-type and the RH-type of flow, no sonic points located inside the truncation radius would be considered as a physical one. Since a sonic point is essentially an acoustic horizon (of black hole type), the extent upto which the embedded acoustic geometry can be formulated is constrained for accreting black hole systems governed by these two types of disc heights.

![Figure 1: Variation of truncation radius \((r_T)\) for NT/RH and radius of outer ergo-surface \((r_+)\) for ALP discs with black hole spin \(a\).](image)

Fig.(1) depicts \(r_T\) and \(r_+\) as a function of black hole spin \(a\). It is, however, to be understood that even for NT and RH-type of discs, in reality the acoustic geometry can be extended upto \(r_+\), as any acoustic perturbation will eventually propagate upto the gravitational horizon of the Kerr black hole. We will not be able to formulate the corresponding acoustic metric and the governing fluid equations because the disc height is not defined beyond \(r_T\). In this sense, the existence of the acoustic geometry will be constrained for these two types of disc heights.

6 Results

There exist two kinds of transonic accretion for which we would like to compute the value of the acoustic surface gravity as a function of the Kerr parameter. For monotransonic accretion, the integral flow solution passes either through the inner, or through the outer sonic points. The outer sonic points are located away from \(r_+\), and hence do not get significantly influenced by the spin angular momentum of the black hole. Gravity at the length scales at which the outer sonic points are formed can be sufficiently described using the
Newtonian framework due to asymptotic flattening of space-time. Hence, the variation of acoustic surface gravity at the outer sonic points is found to be almost insensitive to changes in black hole spin. Therefore, we show the dependence of $\kappa$ on $a$ for monotransonic flow at the inner sonic points only. For multitransonic accretion, one needs to calculate the value of the acoustic surface gravity at both the sonic points, the outer, and the inner, respectively. In this context as well, the variation of the location of the outer sonic point or any related physical quantity measured at the outer sonic point is not sensitive to the change of black hole spin. For shocked accretion flow, we thus show the $\kappa - a$ variation for the inner sonic points only.

It is to be noted that the surface gravity is essentially a measure of the strength of the acceleration on the horizon, and hence the value of $\kappa$ measured at the inner sonic point will be much higher compared to the value of $\kappa$ at the outer sonic point. At the inner sonic point (which always forms within a few gravitational radii measured from $r_+$), the strength of the effective gravity is much higher compared to that measured at the outer sonic point. This further demonstrates that the strength of the gravitational field influences the ‘strength’ of the effective gravity.

Formation of the analogue white hole is the consequence of the shock formation in the disc. At the shock, accretion flow makes a discontinuous transition from supersonic to the subsonic state. The space gradient of the dynamical velocity as well as the speed of propagation of the acoustic perturbation thus diverge, and hence the value of the acoustic surface gravity at the shock becomes infinite. This is a practical example (for a physical configuration naturally found in the universe) of the general situation proposed in [Liberati et al., 2000].

### 6.1 Polytropic accretion

As explained earlier, before computation and comparison of the acoustic surface gravity, we need to figure out an overlapping region of flow parameters allowing integral flow solutions with shock, corresponding to all the three flow configurations.

Fig. (2) depicts such regions of overlap in the $[E, \lambda]$ space for constant values of $a$ and $\gamma$, and the $[\lambda, a]$ space for given values of values of $E$ and $\gamma$.

Once we obtain the common parameter values across which shocked multitransonic solutions are permissible for all the three disc configurations, we choose a set of values from the respective region of overlap, construct the corresponding stationary integral flow solutions as prescribed in section 2, and then use the solutions to construct the causal structures for the respective acoustic metric derived according to the perturbative scheme devised in section 4. Fig. (3) represents the null trajectories of outgoing acoustic waves (curves along which $\chi = $ constant). In these trajectories, we see that just inside and outside the curve, null trajectories asymptotically approach the critical points (inner critical point $r_{c}^{in}$ in fig.(3(a)) and outer critical point $r_{c}^{out}$ in fig.(3(b)), both for a given set of $[E, \lambda, \gamma, a]$ specified in fig.(3(b))). $r_{c}^{in}$ and $r_{c}^{out}$, in this scenario can be considered as the sonic points $r_{s}^{in}$ and $r_{s}^{out}$, if a re-definition of
Figure 2: Comparision of parameter space diagrams for shocked solutions - (a) $\mathcal{E}$-$\lambda$, (b) $\lambda$-$a$. Shaded portions depict range of parametric overlap for integral flow solutions with shock.

The sound speed is done as $(c_{s0}^2)_{\text{effective}} = c_{s0}^2/(1 + \beta)$ ([Shaikh et al., 2019]), where $\beta = (\gamma - 1)/2$. This indicates that outgoing sound waves behave near sonic points just as light waves would do in a curved space-time around horizon. Fig.(3), thus, establishes mathematically, that the sonic points behave like acoustic horizons.

Fig.(4) represents the null trajectories of incoming acoustic waves ($\omega = \text{constant contours}$) across the inner and outer critical points $r^{in}$ and $r^{out}$, and the discontinuous shock transition from supersonic to subsonic flow regime occurring at $r_{sh}$. The trajectories correspond to the stationary integral multitransonic flow solutions for a given set of $[\mathcal{E}, \lambda, \gamma]$. It is observed that the curves remain smooth as they pass through the horizons in finite time.

For multitransonic shocked accretion, in fig.(5), we plot the location of the inner acoustic black-hole-type horizon ($r^{in}$) as a function of black hole spin angular momentum ($a$) for a fixed set of $[\mathcal{E}, \lambda, \gamma]$ as specified in the figure. The range of the Kerr parameters shown in the figure is used to obtain a common region of shocked flow for three such disc models characterised by a a fixed value of $[\mathcal{E}, \lambda, \gamma]$. Flow solutions in various other ranges of $a$ can be explored using different relevant combinations of $[\mathcal{E}, \lambda, \gamma]$ using results depicted in fig.(2). As shown in the figure, both the prograde (black hole co-rotating with disc) and retrograde (black hole counter-rotating with disc) flows have been studied. We find that the value of the acoustic surface gravity ($\kappa^{in}$) anti-correlates with the location of the inner acoustic black hole horizons ($r^{in}$). This indicates that the strength of the gravitational field enhances the ‘strength’ of the analogue effects. Thus, it clearly manifests that the actual space-time metric describing the background fluid flow determines the properties of the emergent acoustic metric.

In fig.(6(a)), we plot the location of the acoustic horizons ($r_s$) as a function of the Kerr
Figure 3: (a) $\chi = \text{constant in (t, r) plane around } r_{c}^{\text{in}}$, (b) $\chi = \text{constant contours around } r_{c}^{\text{out}}$. 

$\chi = \text{constant in (t, r) plane around } r_{c}^{\text{in}}$

(a)

$\chi = \text{constant in (t, r) plane around } r_{c}^{\text{out}}$

(b)
Figure 4: $\omega =$ constant contours in $(t, r)$ plane.

Figure 5: (a) Inner sonic point ($r_s^{in}$) vs. $a$, (b) analogue surface gravity at $r_s^{in}$ ($\kappa^{in}$) vs. $a$, for multitransonic solutions.
parameter \((a)\) for monotransonic flows passing through the inner sonic points. The values of \(r_T\) and \(r_+\) as functions of \(a\) as already depicted in fig.(1), have also been overlayed upon the plot to highlight the theoretical constraints of NT/RH type of discs with regard to obtaining sonic horizons upto the gravitational horizon \((r_+)\). It is clearly visible that for the given set of other system parameters \([\mathcal{E}, \lambda, \gamma]\) as specified in the figure, although sonic points for NT \((r_{sNT})\), shown with a blue dashed curve) could be obtained throughout the entire astrophysically relevant range of black hole spin, but the same for RH-type of discs (shown with a green dotted curve) went beyond the truncation radius \(r_T\) after a certain value of \(a\). Thus, even though there is no physical reason for the non-existence of \(r_s\) for all values of \(a\) in case of monotransonic flows, but such solutions may not be found due to the inherent mathematical limitation of the given disc models. We also note that the location of acoustic horizons is farthest (i.e., the \(r_s\) vs. \(a\) curves attain a maximum) at certain specific values of the black hole spin (note the inset depicting a magnified view of such maximum for \(r_s^\text{ALP}\)). The maximum is, however, missed in the case of \(r_s^\text{NT}\). This is not an anomalous behaviour, as the locations of maxima have been found to shift (sometimes to parameter values unrestricted mathematically, but restricted physically) while tuning the values of \([\mathcal{E}, \lambda, \gamma]\). We plot the values of the analogue surface gravity \((\kappa)\) at the corresponding acoustic horizons as a function of \(a\) in fig.(6(b)). Visibly, for polytropic flows, among all three disc prescriptions, \(\kappa\) is the highest for the ALP-type of discs. A possible correlation between the strengths of actual and emergent gravity, as elaborated earlier, can also be inferred from the plots.

### 6.2 Isothermal accretion

An isothermal equation of state of the flow implies propagation of acoustic perturbations through the fluid at a constant speed. Detailed mathematical derivations of the conditions and quantities (conservations, critical point conditions, flow velocity gradients at the critical
points, shock conditions etc.) required to obtain the multitransonic (shocked) stationary integral flow solutions have been already presented in P1. We use those previous results to scan the system parameter space and look for regions common to all three disc heights, which permit multitransonicity (i.e., the existence two sonic points signifying smooth, continuous transitions, and a shock location leading to discontinuous transition between the supersonic and subsonic regimes).

Figure 7: Comparision of parameter space diagrams for shocked solutions - (a) $T$-$\lambda$, (b) $\lambda$-$a$. Shaded portions depict range of parametric overlap for integral flow solutions with shock.

In fig.\textit{(7)}, we depict such regions of overlap for NT, RH and ALP-type of discs on the $[T, \lambda]$ and $[\lambda, a]$ spaces, for a given value of $a$ and $T$ specified in the figures, respectively. We then choose values of flow parameters from the regions of overlap in fig.\textit{(7(a))} and fig.\textit{(7(b))} in a way, such that an optimum range of black hole spin is available to serve our purpose of comparision.

Following a careful selection of the flow temperature ($T$) and the specific angular momentum ($\lambda$) as depicted in the plots, in fig.\textit{(8(a))}, we show the variation of the inner sonic point ($r_{\text{s}}^{\text{in}}$) as a function of black hole spin ($a$) for multitransonic shocked accretion flow of isothermal matter. The location of the inner acoustic horizon is found to anti-correlate with $a$. It is to be noted that for the same set of $[T, \lambda, a]$, the sonic points form closest to the gravitational horizon for RH-type of discs, whereas the sonic points are farthest for the ALP type of discs (magnified view in the inset). For all three types of disc, the values of $r_{\text{s}}$, however, are found to be extremely close to each other. The values of acoustic surface gravity $\kappa_{\text{s}}^{\text{in}}$, on the other hand are found to be distinctively different for NT, RH and ALP flows as shown in fig.\textit{(8(b))}.

Fig.\textit{(9(a))} shows the variation of $r_{\text{s}}$ for monotransonic isothermal flows with the Kerr parameter $a$. Limitations in obtaining acoustic horizons beyond $r_{T}$ is again observed in case of RH-type of discs at the given values of $T$ and $\lambda$ specified in the figure. Anti-correlation of sonic point locations with black hole spin is also seen over a considerable range of $a$. 
Figure 8: (a) Inner sonic point ($r_s^{in}$) vs. $a$, (b) analogue surface gravity at $r_s^{in}$ ($\kappa^{in}$) vs. $a$, for multitransonic solutions.

Figure 9: (a) Sonic point ($r_s$ for NT, RH & ALP) and truncation radius ($r_T$ for NT & RH) vs. $a$, (b) corresponding analogue surface gravity ($\kappa$) at the sonic points (acoustic horizons) vs. $a$, for monotransonic solutions.
The dependence of the acoustic surface gravity $\kappa$ at the corresponding sonic points on $a$ is depicted in fig.(9(b)). Comparison of fig.(9(b)) with all previous $\kappa$ vs. $a$ plots indicates that the highest value of acoustic surface gravity for both polytropic and isothermal flows, in most cases (except monotransonic polytropic flows) may be attributed to the RH-type of discs. We also find that $\kappa$ anti-correlates with black hole spin for monotransonic accretion, while it correlates with black hole spin for shocked multitransonic flows (except for isothermal RH discs). Moreover, with the only exception of monotransonic flow in isothermal RH-type of discs, there exists an interestingly consistent correlation between the strength of gravity (greater proximity to the gravitational horizon) and that of the emergent gravity phenomenon (higher values of analogue surface gravity at the acoustic horizons) among the three different classes of accretion discs.

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