ANALYSIS OF THE MASS AND WIDTH OF THE $X^*(3860)$ WITH QCD SUM RULES

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Abstract

In this article, we tentatively assign the $X^*(3860)$ to be the $C\gamma_5 \otimes \gamma_5 C$ type scalar tetraquark state, study its mass and width with the QCD sum rules, special attention is paid to calculating the hadronic coupling constants $G_{X_{q\bar{q}}}$ and $G_{XDD}$ concerning the tetraquark state. We obtain the values $M_X = 3.86 \pm 0.09$ GeV and $\Gamma_X = 202 \pm 146$ MeV, which are consistent with the experimental data. The numerical result supports assigning the $X^*(3860)$ to be the $C\gamma_5 \otimes \gamma_5 C$ type scalar tetraquark state.

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Key words: Tetraquark state, QCD sum rules

1 Introduction

Recently, the Belle collaboration performed a full amplitude analysis of the process $e^+e^- \rightarrow J/\psi D\bar{D}$ based on the 980 fb$^{-1}$ data sample collected by the Belle detector at the asymmetric-energy $e^+e^-$ collider KEKB, and observed a new charmoniumlike state $X^*(3860)$ that decays to $D\bar{D}$ with a significance of 6.5$\sigma$, the measured mass is $3862^{+26}_{-32} - 13$ MeV and width is $204^{+154+88}_{-97-83}$ MeV [1]. The $J^{PC} = 0^{++}$ hypothesis is favored over the $2^{++}$ hypothesis at the level of 2.5$\sigma$. The Belle collaboration assigned the $X^*(3860)$ in stead of the $X(3915)$ to be the $\chi_{c0}(2P)$ state [1]. The mass of the state $\chi_{c0}(2P)$ from the non-relativistic potential model, the Godfrey-Isgur relativized potential model and the screened potential model is $3852$ MeV, $3916$ MeV and $3842$ MeV, respectively [2].

In 2004, the Belle collaboration observed the $X(3915)$ in the $\omega J/\psi$ mass spectrum in the exclusive $B \rightarrow K\omega J/\psi$ decays [4]. In 2007, the BaBar collaboration confirmed the $X(3915)$ in the $\omega J/\psi$ mass spectrum in the exclusive $B \rightarrow K\omega J/\psi$ decays [5]. In 2010, the Belle collaboration confirmed the $X(3915)$ in the two-photon process $\gamma\gamma \rightarrow \omega J/\psi$ [6].

In Ref. [7], Lebed and Polosa propose that the $X(3915)$ is the lightest $cs\bar{s}\bar{s}$ scalar tetraquark state based on lacking of the observed $D\bar{D}$ and $D^*\bar{D}^*$ decay modes, and attribute the single known decay mode $J/\psi\omega$ to the $\omega - \phi$ mixing effect. In Refs. [8,9], we study the $C\gamma_5 \otimes \gamma_5 C$-type, $C\gamma_5 \otimes \gamma_5 \gamma_5 \mu C$-type, $C\gamma_5 \otimes \gamma_5 C$-type, $C \otimes C$-type $cs\bar{s}\bar{s}$ scalar tetraquark states with the QCD sum rules in a systematic way, and obtain the predictions $M_{C\gamma_5 \otimes \gamma_5 C} = 3.92^{+0.12}_{-0.10}$ GeV and $M_{C\gamma_5 \otimes \gamma_5 C} = 3.89 \pm 0.05$ GeV, which support assigning the $X(3915)$ to be the $C\gamma_5 \otimes \gamma_5 C$-type or $C\gamma_5 \otimes \gamma_5 C$-type $cs\bar{s}\bar{s}$ scalar tetraquark state.

Naively, we expect the $SU(3)$ breaking effect is about $m_s - m_q = 135$ MeV, while the QCD sum rules indicate that the mass gaps $M_{cs\bar{s}} - M_{cq\bar{q}}$ are less than or much less than 90 MeV for the scalar, vector, axialvector diquark-antidiquark type hidden-charm tetraquark states [10]. If the $SU(3)$ breaking effects are small indeed for the diquark-antidiquark type hidden-charm tetraquark states, the $X^*(3860)$ and $X(3915)$ can be assigned to be the scalar tetraquark states with the symbolic quark structures $\bar{c}c\bar{u}\bar{s}d$ and $\bar{c}c\bar{s}\bar{s}$, respectively. In Ref. [11], we study the lowest $C\gamma_5 \otimes \gamma_5 C$ type scalar hidden-charm tetraquark state with the QCD sum rules and obtain the mass $M = (3.82^{+0.08}_{-0.08})$ GeV, which is consistent with the value from the Belle collaboration [1].

In Ref. [12], we update the value of the effective $c$-quark mass $M_c$ in determining the optimal energy scales of the QCD spectral densities in the QCD sum rules for the hidden-charm tetraquark states by the empirical formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2}$, where the $X$, $Y$, $Z$ denote the tetraquark states. So the predicted mass of the $C\gamma_5 \otimes \gamma_5 C$ type hidden-charm tetraquark state in Ref. [11] should be updated. In Ref. [13], we take the old value $M_c = 1.80$ GeV, now we take the updated value $M_c = 1.82$ GeV [14], and expect to extract a slightly different mass $M_X$ at a slightly different

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energy scale $\mu$ in a consistent way according to the energy scale formula $\mu = \sqrt{M_Z^2 - (2M_c)^2}$. Variations of the energy scales lead to changes of integral range $4m_c^2(\mu) - s_0$ of the variable $ds$ besides the QCD spectral density $\rho(s)$ (See Eq.(4) in Sec.2), therefore change of the Borel window and predicted mass and pole residue. Moreover, it is interesting to study the decay widths of the tetraquark states with the QCD sum rules by taking into account all the Feynman diagrams [13,16] instead of only the connected Feynman diagrams [13,17]. Furthermore, the over simplified hadron representation chosen in Ref.[13] should be modified. In this article, we assign the $C\gamma_5\otimes\gamma_5 C$ type scalar hidden-charm tetraquark state, and restudy its mass and width with the QCD sum rules in details.

The article is arranged as follows: we derive the QCD sum rules for the mass and width of the $X^*(3860)$ in section 2 and section 3 respectively; section 4 is reserved for our conclusion.

2 The mass of the $C\gamma_5\otimes\gamma_5 C$ type scalar hidden-charm tetraquark state

In the following, we write down the two-point correlation function $\Pi(p)$ in the QCD sum rules,

$$\Pi(p) = i \int d^4xe^{ip\cdot x}\langle 0|T\{J(x)J^\dagger(0)\}|0\rangle,$$

(1)

where

$$J(x) = \varepsilon^{ijk}\varepsilon^{lmn}u^i(x)C\gamma_5^e(x)d^m(x)\gamma_5^\dagger\gamma_5Ce^n(x),$$

(2)

de the $i, j, k, m, n$ are color indexes, the $C$ is the charge conjunction matrix. We choose the current $J(x)$ to interpolate the tetraquark state $X^*(3860)$ (to be more precise, the charged partner of the $X^*(3860)$, they have degenerate masses in the isospin limit).

At the phenomenological side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operator $J(x)$ into the correlation function $\Pi(p)$ to obtain the hadronic representation [13,19], and isolate the ground state contribution,

$$\Pi(p) = \frac{\lambda_X^2}{M_X^2 - p^2} + \cdots,$$

(3)

where the pole residue $\lambda_X$ is defined by $\langle 0|J(0)|X^*(3860)\rangle = \lambda_X$.

We carry out the operator product expansion to the vacuum condensates up to dimension-10, and obtain the QCD spectral density through dispersion relation, then we take the quark-hadron duality and perform Borel transform with respect to the variable $P^2 = -p^2$ to obtain the following QCD sum rule,

$$\lambda_X^2\exp\left(-\frac{M_X^2}{T^2}\right) = \int_{4m_c^2}^{s_0} ds\rho(s)\exp\left(-\frac{s}{T^2}\right),$$

(4)

where the $T^2$ is the Borel parameter and the $s_0$ is the continuum threshold parameter. The explicit expression of the QCD spectral density $\rho(s)$ is presented in Refs.[9,13].

We derive Eq.(4) with respect to $\tau = \frac{1}{T^2}$, then eliminate the pole residue $\lambda_X$ to obtain the QCD sum rule for the mass,

$$M_X^2 = -\frac{\frac{d}{d\tau}\int_{4m_c^2}^{s_0} ds\rho(s)e^{-\tau s}}{\int_{4m_c^2}^{s_0} ds\rho(s)e^{-\tau s}}.$$  (5)

We take the standard values of the vacuum condensates $\langle\bar{q}q\rangle = -(0.24\pm0.01 \text{ GeV})^3$, $\langle\bar{q}g_s\sigma Gq\rangle = m_0^2\langle\bar{q}q\rangle$, $m_0 = (0.8 \pm 0.1) \text{ GeV}^2$, $\langle\frac{\alpha_{GG}}{\pi}\rangle = 0.012 \pm 0.003 \text{ GeV}^4$ at the energy scale $\mu = 1 \text{ GeV}$.
and choose the $\overline{MS}$ mass $m_c(m_c) = (1.275 \pm 0.025)$ GeV from the Particle Data Group [21]. Moreover, we take into account the energy-scale dependence of the input parameters,

\[
\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{1}{2}},
\]

\[
\langle \bar{q}g_sGq \rangle(\mu) = \langle \bar{q}g_sGq \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{3}{2}},
\]

\[
m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{3}{2}},
\]

\[
\alpha_s(\mu) = \frac{1}{b_0} \left[ 1 - \frac{b_1}{b_0} \log \frac{t}{b_0^2} + \frac{b_2}{b_0^4} (\log^2 t - \log t - 1) + b_0 b_2 \right],
\]

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33 - 2n_f}{12\pi}$, $b_1 = \frac{153 - 19n_f}{24\pi}$, $b_2 = \frac{2857 - 45n_f + 25n_f^2}{128\pi^2}$, $\Lambda = 213$ MeV, 296 MeV and 339 MeV for the flavors $n_f = 5, 4$ and 3, respectively [21]. We tentatively take the continuum threshold parameter to be $\sqrt{s_0} = (4.4 \pm 0.1)$ GeV, i.e. $\sqrt{s_0} = M_X + (0.4 - 0.6)$ GeV. In the scenario of tetraquark states, the QCD sum rules indicate that the $Z_c(3900)$ and $Z_c(4430)$ can be tentatively assigned to be the ground state and the first radial excited state of the axialvector tetraquark states, respectively [22], the $X(3915)$ and $X(4500)$ can be tentatively assigned to be the ground state and the first radial excited state of the scalar tetraquark states, respectively [3, 9]. The energy gap between the ground state and the first radial excited state of the hidden-charm tetraquark states is about 0.6 GeV.

In Refs. [12, 23, 24], we study the acceptable energy scales of the QCD spectral densities for the hidden-charm (hidden-bottom) tetraquark states in the QCD sum rules in details for the first time, and suggest an empirical formula $\mu = \sqrt{M_X^{2}/X} - (2M_Q)^2$ to determine the optimal energy scales, where the $X, Y, Z$ denote the tetraquark states, and the $M_Q$ denotes the effective heavy quark masses. The energy scale formula works well for the $X(3872)$, $Z_c(3900)$, $X(3915)$, $Z_c(4020)/4025)$, $Y(4140)$, $Z_c(4430)$, $X(4500)$, $Y(4660)$, $X(4700)$, $Z_c(10610)$ and $Z_b(10650)$. In Ref. [13], we choose the old value $M_c = 1.80$ GeV to study the mass of the lowest scalar hidden-charm tetraquark state. In this article, we choose the updated value $M_c = 1.82$ GeV [14], and obtain the optimal energy scale $\mu = 1.3$ GeV for the QCD spectral density, the prediction is changed slightly. In fact, the empirical energy scale formula $\mu = \sqrt{M_X^{2}/X} - (2M_c)^2$ serves as a constraint to obey.

We search for the optimal Borel parameter to satisfy the two criteria (pole dominance and convergence of the operator product expansion) of the QCD sum rules, and obtain the value $T^2 = (2.5 - 2.9)$ GeV$^2$. In Fig.1, we plot the pole contribution with variations of the Borel parameter $T^2$, the pole contribution is about $(46 - 70)\%$ in the Borel window between the two vertical lines. In Fig.2, we plot the contributions of different terms in the operator product expansion with variations of the Borel parameter $T^2$ for the central value of the continuum threshold parameter $s_0$. In the Borel window, the main contributions come from the vacuum condensates of dimensions 0, 3, 5 and 6, the contributions of the vacuum condensates of dimensions 8 and 10 are about $-(3 - 6)\%$ and $< 1\%$, respectively. The two criteria of the QCD sum rules are fully satisfied, we expect to make reliable prediction.

We take into account all uncertainties of the input parameters, and obtain the values of the mass and pole residue of the $X^*(3860)$, which are shown explicitly in Fig.3,

\[
M_X = 3.86 \pm 0.09 \text{ GeV}, \\
\lambda_X = (2.02 \pm 0.34) \times 10^{-2} \text{ GeV}^5.
\]

The predicted mass $M_X = 3.86 \pm 0.09$ GeV is in excellent agreement with the experimental value $3862^{+36}_{-32}^{+40}_{-13}$ MeV within uncertainties [1]. The QCD sum rules favors assigning the $X^*(3860)$ to be the $C_{75} \otimes \gamma_5 C$ type hidden-charm tetraquark state. However, the assignment of the $X(3915)$
Figure 1: The pole contribution with variations of the Borel parameter $T^2$.

Figure 2: The contributions of different terms in the operator product expansion with variations of the Borel parameter $T^2$, where the 0, 3, 4, 5, 6, 7, 8 and 10 denote the dimensions of the vacuum condensates.
interpolate the mesons $\eta$ as the width to obtain more reliable assignment. On the other hand, the Belle collaboration observed a value 3918 GeV indicating the favored quantum numbers of the quantum numbers $J^P$.

Figure 3: The mass and pole residue of the $X^*(3860)$ with variations of the Borel parameter $T^2$.

as the $C\gamma_5 \otimes \gamma_5 C$ type hidden-charm tetraquark state with the symbolic structure $\bar{c}c \bar{u}u + \bar{d}d$ is not excluded, as the predicted mass $M_X = 3.86 \pm 0.09$ GeV is also compatible with the experimental value 3918.4 ± 1.9 MeV of the mass of the $X(3915)$ within uncertainty [21]. We can study the width to obtain more reliable assignment. On the other hand, the Belle collaboration observed the $X(3940)$ in the decays to the meson pair $D^* \bar{D}$ [25], absence of the decays $X(3940) \to D \bar{D}$ indicates the favored quantum numbers of the $X(3940)$ are $J^{PC} = 0^{-+}$, which differ from the quantum numbers $J^{PC} = 0^{++}$ of the interpolating current $J(x)$.

3 The width of the $X^*(3860)$ as scalar tetraquark state

We study the two-body strong decays $X^*(3860) \to \eta_c \pi^-$ and $D^- D^0$ with the following three-point correlation functions $\Pi_1(p, q)$ and $\Pi_2(p, q)$, respectively,

\[
\Pi_1(p, q) = i^2 \int d^4x d^4y e^{ip \cdot x} e^{iq \cdot y} \langle 0 | T \{ J_{\eta_c}(x) J_\pi(y) J(0) \} | 0 \rangle, \tag{8}
\]

\[
\Pi_2(p, q) = i^2 \int d^4x d^4y e^{ip \cdot x} e^{iq \cdot y} \langle 0 | T \{ J_D(x) J_D(y) J(0) \} | 0 \rangle, \tag{9}
\]

where the currents

\[
J_{\eta_c}(x) = \bar{c}(x)i\gamma_5 c(x), \tag{10}
\]

\[
J_\pi^0(y) = \bar{u}(y)i\gamma_5 d(y), \tag{11}
\]

\[
J_D(x) = \bar{c}(x)i\gamma_5 d(x), \tag{12}
\]

\[
J_D(y) = \bar{u}(y)i\gamma_5 c(y), \tag{13}
\]

interpolate the mesons $\eta_c, \pi^-, D^-$ and $D^0$, respectively.

At the QCD side, the correlation functions $\Pi_1(p, q)$ and $\Pi_2(p, q)$ can be written as

\[
\Pi_1(p, q) = \int_{4m_c^2}^{s_0} ds \int_{0}^{u_0} du \frac{\rho_1(s, u)}{(s - p^2)(u - q^2)} + \int_{4m_c^2}^{s_0} ds \int_{0}^{u_0} du \frac{\rho_1(s, u)}{(s - p^2)(u - q^2)} + \int_{s_0}^{\infty} ds \int_{u_0}^{\infty} du \frac{\rho_1(s, u)}{(s - p^2)(u - q^2)}, \tag{14}
\]

\[
\Pi_2(p, q) = \int_{m_c^2}^{s_0} ds \int_{m_c^2}^{u_0} du \frac{\rho_2(s, u)}{(s - p^2)(u - q^2)} + \int_{m_c^2}^{s_0} ds \int_{m_c^2}^{u_0} du \frac{\rho_2(s, u)}{(s - p^2)(u - q^2)} + \int_{s_0}^{\infty} ds \int_{u_0}^{\infty} du \frac{\rho_2(s, u)}{(s - p^2)(u - q^2)}, \tag{15}
\]
where the ρ_{1/2}(s, u) are the QCD spectral densities, the s_0 and u_0 are the continuum threshold parameters. The QCD spectral densities ρ_1(s, u) and ρ_2(s, u) are independent on the (p ± q)^2 except for some non-singular terms p · q, (p · q)^2, etc., the variables ds and du are independent, which differ from the QCD spectral densities in the QCD sum rules for the hadronic coupling constants G_{Λ,ND}, G_{Λ,NB}, G_{Ξ,ND}, G_{Ξ, NB}, G_{Λ,ND*}, G_{Λ, NB*}, G_{Ξ, ND*}, G_{Ξ, NB*} [20,21,22], G_{B^*, B, T}, G_{B^*, B, T}, G_{B^*, B, T}, G_{B^*, B, T}, G_{D_2 D}, G_{D_2 D}, G_{D_2 D}, G_{D_2 D}, G_{D_2 D}, G_{B^*, B, T} [23], in those cases the QCD spectral densities depend on the (p ± q)^2, explicitly, the variables ds and du should obey special constraints among the s, u and (p ± q)^2 according to dispersion relations or Cutkosky’s rules [28].

The strong decays X^+(3860) → η_c π^- and D^- D^0 take place through fall-apart mechanism, no quark-antiquark pair is created from the vacuum, which differs from the two-body strong decays of the conventional mesons and baryons significantly.

At the hadronic side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators into the three-point correlation functions Π_1(p, q), Π_2(p, q) and isolate the ground state contributions to obtain the following results,

\[ Π_1(p, q) = \frac{f_{ν_c} M^2_π^2 M^2_π^2 λ_X G_{Xν_cπ}}{2 m_c (m_u + m_d)} \frac{1}{(M^2_X - p^2)(M^2_{ν_c} - p^2)(M^2_π - q^2)} \]
\[ + \frac{1}{(M^2_X - p^2)(M^2_{ν_c} - p^2)} \int_{s_0}^{s_1} dt \frac{ρ_{ν_cπ}(p^2, t, p^2)}{t - q^2} \]
\[ + \frac{1}{(M^2_X - p^2)(M^2_{ν_c} - q^2)} \int_{s_0}^{s_1} dt \frac{ρ_{ν_cπ}(t, q^2, p^2)}{t - p^2} \]
\[ + \frac{1}{(M^2_{ν_c} - p^2)(M^2_π - q^2)} \int_{s_0}^{s_1} dt \frac{ρ_{ν_cπ}(p^2, q^2, t) + ρ_{ν_cπ}(p^2, q^2, t)}{t - p^2} + \cdots, \quad (14) \]

\[ Π_2(p, q) = \frac{f_{D} M^4_D λ_X G_{XDD}}{4 m_c^2} \frac{1}{(M^2_X - p^2)(M^2_{D^0} - p^2)(M^2_D - q^2)} \]
\[ + \frac{1}{(M^2_X - p^2)(M^2_{D^0} - q^2)} \int_{s_0}^{s_1} dt \frac{ρ_{XDD}(t, q^2, p^2)}{t - p^2} \]
\[ + \frac{1}{(M^2_X - p^2)(M^2_{D^0} - p^2)} \int_{s_0}^{s_1} dt \frac{ρ_{XDD}(p^2, t, p^2)}{t - q^2} \]
\[ + \frac{1}{(M^2_{D^0} - p^2)(M^2_D - q^2)} \int_{s_0}^{s_1} dt \frac{ρ_{XDD} - (p^2, q^2, t) + ρ_{XDD}(p^2, q^2, t)}{t - p^2} + \cdots, \quad (15) \]

where p' = p + q, the decay constants f_{ν_c}, f_π, f_D and the hadronic coupling constants G_{Xν_cπ}, G_{XDD} are defined by,

\[ \langle 0 | J_{ν_c}(0) | ν_c(p) \rangle = \frac{f_{ν_c} M^2_π^2}{2 m_c}, \]
\[ \langle 0 | J_π(0) | π(q) \rangle = \frac{f_π M^2_π^2}{m_u + m_d}, \]
\[ \langle 0 | J_D(0) | D(p/q) \rangle = \frac{f_D M^2_D}{m_c}, \]
\[ \langle ν_c(p) | π(q) | X(p') \rangle = i G_{Xν_cπ}, \]
\[ \langle D(p) | D(q) | X(p') \rangle = i G_{XDD}. \quad (16) \]

The eight functions ρ_{Xπ}(p^2, t, p^2), ρ_{Xπ}(t, q^2, p^2), ρ_{Xπ}(p^2, q^2, t), ρ_{Xπ}(p^2, q^2, t), ρ_{Xπ}(p^2, q^2, t), ρ_{Xπ}(p^2, q^2, t), ρ_{Xπ}(p^2, q^2, t), and ρ_{Xπ}(p^2, q^2, t) have complex dependence on the transitions between the ground states and the high resonances or continuum states. The definitions of the
hadronic coupling constants $G_{X_{\eta_3 \pi}}$, $G_{X_{DD}}$ differ from that in Ref. [13], moreover, in Ref. [13], an over simplified hadron representation is chosen.

We introduce the notations $C_{X\pi}$, $C_{Xn_c}$, $C'_{X\pi}$, $C'_{Xn_c}$, $C_{XD-}$, $C_{XD0}$, $C'_{XD-}$ and $C'_{XD0}$ to parameterize the net effects,

$$C_{X\pi} = \int_{m_c}^{\infty} dt \frac{\rho_{X\pi}(t, q^2)}{t - q^2},$$
$$C_{Xn_c} = \int_{m_c}^{\infty} dt \frac{\rho_{Xn_c}(t, q^2)}{t - q^2},$$
$$C'_{Xn_c} = \int_{m_c}^{\infty} dt \frac{\rho_{Xn_c}(p^2, q^2, t)}{t - p^2},$$
$$C_{X\pi}' = \int_{m_c}^{\infty} dt \frac{\rho_{X\pi}(p^2, q^2, t)}{t - p^2},$$
$$C_{XD-} = \int_{m_c}^{\infty} dt \frac{\rho_{XD-}(t, q^2, p^2)}{t - p^2},$$
$$C_{XD0} = \int_{m_c}^{\infty} dt \frac{\rho_{XD0}(p^2, t, q^2)}{t - q^2},$$
$$C'_{XD-} = \int_{m_c}^{\infty} dt \frac{\rho_{XD-}(p^2, q^2, t)}{t - p^2},$$
$$C'_{XD0} = \int_{m_c}^{\infty} dt \frac{\rho_{XD0}(p^2, q^2, t)}{t - p^2},$$

and rewrite the correlation functions $\Pi_1(p, q)$ and $\Pi_2(p, q)$ into the following form,

$$\Pi_1(p, q) = \frac{f_\eta M_{\eta_c}^2 f_{f_\pi} M^2 \lambda_X G_{X_{\eta_3 \pi}}}{2m_c(m_u + m_d)} \frac{1}{(M^2_X - p^2)(M_{\eta_c}^2 - p^2)(M^2_{\eta_c} - q^2)} + \frac{C_{X\pi}}{(M^2_X - p^2)(M_{\eta_c}^2 - p^2)} + \frac{C_{Xn_c}}{(M^2_X - p^2)(M_{\eta_3}^2 - q^2)} + \frac{C'_{X\pi}}{(M^2_X - p^2)(M_{\eta_3}^2 - q^2)} + \cdots,$$ (17)

$$\Pi_2(p, q) = \frac{f_D^2 M^4_{DD} \lambda_X G_{X_D D}}{4m_c^2} \frac{1}{(M^2_X - p^2)(M^2_{D-} - p^2)(M^2_{D0} - q^2)} + \frac{C_{XD-}}{(M^2_X - p^2)(M_{D0}^2 - q^2)} + \frac{C_{XD0}}{(M^2_X - p^2)(M_{D0}^2 - p^2)} + \frac{C'_{XD-}}{(M^2_X - p^2)(M_{D0}^2 - p^2)} + \frac{C'_{XD0}}{(M^2_X - p^2)(M_{D0}^2 - q^2)} + \cdots.$$ (19)

In numerical calculations, we smear the complex dependencies of the $C_{X\pi}$, $C_{Xn_c}$, $C'_{X\pi}$, $C'_{Xn_c}$, $C_{XD-}$, $C_{XD0}$, $C'_{XD-}$ and $C'_{XD0}$ on the variables $p^2$, $p^2$, $q^2$, take them as free parameters, and choose the suitable values to eliminate the contaminations from the high resonances and continuum states to obtain the stable sum rules with the variations of the Borel parameters. In the limit $M_{\pi}^2 \rightarrow 0$ and $M_{D0}^2 \rightarrow 0$, we can choose $Q^2 = -q^2$ off-shell, and match the terms proportional to $\frac{1}{Q^2}$ at the hadron side with the ones at the QCD side to obtain QCD sum rules for the momentum dependent hadronic coupling constants $G_{X_{\eta_3 \pi}}(Q^2)$ and $G_{X_{DD}}(Q^2)$, then extract the values to the mass-shell $Q^2 = -M_{\pi}^2$ or $M_{D0}^2$ to obtain the physical values. In fact, the approximations $\frac{1}{M_{\eta_c}^2 - q^2} \approx \frac{1}{Q^2}$ at the hadronic side and $\frac{1}{m_c^2 - q^2} \approx \frac{1}{Q^2}$ at the QCD side are not good. We prefer taking the imaginary parts of the correlation functions $\Pi_1(p, q)$ and $\Pi_2(p, q)$ with respect to $q^2 + i\epsilon$ through dispersion relation and obtain the physical spectral densities, then take Borel transform with respect to the $Q^2$ to obtain the QCD sum rules for the physical hadronic coupling constants.
We have to be cautious in matching the QCD side with the hadronic side of the correlation functions \( \Pi_1(p, q) \) and \( \Pi_2(p, q) \), as there appears the variable \( p^2 = (p + q)^2 \). We rewrite the correlation functions \( \Pi_1(p, q) \) and \( \Pi_2(p, q) \) at the hadronic side into the following form through dispersion relation,

\[
P_1(p, q) = \Pi_1^H(p^2, p^2, q^2) = \int_{(M_{\eta_c} + M_\pi)^2}^{s_X^0} ds' \int_{m_\pi^2}^{s_{\eta_c}^0} ds \int_0^{u_0} du \frac{\rho_1^H(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)} + \cdots, \tag{21}
\]

\[
P_2(p, q) = \Pi_2^H(p^2, p^2, q^2) = \int_{s_{\eta_c}^0}^{s_X^0} ds' \int_{m_\pi^2}^{s_{\eta_c}^0} ds \int_0^{u_0} du \frac{\rho_2^H(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)} + \cdots, \tag{22}
\]

where the \( \rho_1^H(s', s, u) \) and \( \rho_2^H(s', s, u) \) are the hadronic spectral densities,

\[
\rho_1^H(s', s, u) = \lim_{\epsilon_3 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \frac{\text{Im} \, \text{Im} \, \text{Im} \, \Pi_1^H(s' + i\epsilon_3, s + i\epsilon_2, u + i\epsilon_1)}{\pi^3}, \tag{23}
\]

\[
\rho_2^H(s', s, u) = \lim_{\epsilon_3 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \frac{\text{Im} \, \text{Im} \, \text{Im} \, \Pi_2^H(s' + i\epsilon_3, s + i\epsilon_2, u + i\epsilon_1)}{\pi^3}. \tag{24}
\]

The ground state masses have the relations \( M_X > M_{\eta_c}(2S) > M_{\eta_c} \gg M_\pi \) and \( M_X \approx 2M_D \), while the continuum threshold parameters have the relations \( \sqrt{s_X^0} \approx \sqrt{s_{\eta_c}^0} > \sqrt{u_0} \), \( \sqrt{s_X^0} \approx \sqrt{s_D^0} + 0.6 \text{ GeV} \) and \( s_D^0 = u_0^D \) \cite{15, 31}.

Now we set \( s_{\eta_c}^0 = s_X^0, p^2 = p^2 \) and carry out the integral over \( ds' \), the contribution of the \( \eta_c(2S) \) is included in, we have to take into account the contribution of the \( \rho_{X\eta_c}(t, q^2, p^2) \) explicitly. On the other hand, we set \( \sqrt{s_X^0} = \sqrt{s_D^0} + 0.6 \text{ GeV}, p^2 = 4p^2 \) and carry out the integral over \( ds' \), the contribution of the \( X(2S) \) is included in, we have to take into account the contribution of the \( \rho_{XD^*}(p^2, q^2, t) \) explicitly. The pole terms below the continuum thresholds \( s_X^0, s_{\eta_c}^0, u_0^D, s_D^0 \) and \( u_0^D \) can be written as

\[
P_1(p, q) = \frac{f_{uu} M_{\eta_c}^2 f_\pi M_X^2 G_{X\eta_c}}{2m_c( m_u + m_d)} \frac{1}{(M_X^2 - p^2)(M_{\eta_c}^2 - p^2)(M_H^2 - q^2)} + C_{X\eta_c}, \tag{25}
\]

\[
P_2(p, q) = \frac{f_D^2 M_X^2 G_{XDD}}{16m_c^2} \frac{1}{(M_X^2 - p^2)(M_{D^*}^2 - p^2)(M_{DD^*}^2 - q^2)} + C'_{XD^*}, \tag{26}
\]

where \( \tilde{M}_X^2 = \frac{M_X^2}{4} \).

We carry out the operator product expansion up to the vacuum condensates of dimension 5 and neglect the tiny contribution of the gluon condensate. In this article, we take into account both the connected and disconnected Feynman diagrams, just like in the QCD sum rules for the two-body strong decays of the \( Z_c(4200) \) and \( X(5568) \) \cite{15, 10}, which is contrary to Ref.\cite{17}, where only the connected Feynman diagrams are taken into account to study the width of the \( Z_c(3900) \). In Ref.\cite{13}, we only take into account the connected Feynman diagrams in calculating the width of the lowest scalar hidden-charm tetraquark state and obtain the value \( \Gamma \approx 21 \text{ MeV.} \)

In calculations, we observe that there appears \( \frac{Q^2}{q^2} \) in the terms associated with the \( \langle \bar{q}q \rangle \) and \( \langle \bar{g}g_\sigma Gq \rangle \) in the correlation function \( \Pi_1(p, q) \), which disappears after performing the Borel transform with respect to the variable \( Q^2 = -q^2 \), as \( \frac{Q^2}{q^2} = \frac{p^2 - p^2 - q^2}{2p^2} = -\frac{1}{2} \) by setting \( p^2 = p^2 = -p^2 \).
Once the analytical expressions of the correlation functions $\Pi_1(p, q)$ and $\Pi_2(p, q)$ at the QCD level are gotten, we can obtain the QCD spectral densities through dispersion relation, take the quark-hadron duality below the continuum thresholds, then we set $p^2 = p_0^2$ and $p'^2 = 4p_0^2$ for the correlation functions $\Pi_1(p, q)$ and $\Pi_2(p, q)$ respectively, and take the double Borel transforms with respect to the variables $F^2 = -p^2$ and $Q^2 = -q^2$ respectively to obtain the following QCD sum rules,

\[
\begin{align*}
& \frac{f_{\pi}M_\pi^2f_\pi M^2_X G_{X_{\pi}}}{2m_c(m_u + m_d)} \frac{1}{M^2_X - M^2_{q_0}} \left[ \exp\left(-\frac{M^2_{q_0}}{T^2}\right) - \exp\left(-\frac{M^2_X}{T^2}\right) \right] \exp\left(-\frac{M^2_{q_0}}{T^2}\right) \\
& + C_{X_{q_0}} \exp\left(-\frac{M^2_X}{T^2} - \frac{M^2_{q_0}}{T^2}\right) = \frac{3}{128\pi^4} \int_{4m_c^2}^{\infty} ds \int_0^{u^2} ds' du \sqrt{1 - \frac{4m_c^2}{s}} \exp\left(-\frac{s}{T^2} - \frac{u}{T^2}\right), \\
\end{align*}
\]

\[(27)\]

\[
\begin{align*}
& \frac{f_D M^4_D \lambda_X G_{XDD}}{16m_c^2} \frac{1}{M^2_X - M^2_D} \left[ \exp\left(-\frac{M^2_D}{T^2}\right) - \exp\left(-\frac{M^2_X}{T^2}\right) \right] \exp\left(-\frac{M^2_D}{T^2}\right) \\
& + C'_{XDD} \exp\left(-\frac{M^2_D}{T^2} - \frac{M^2_X}{T^2}\right) = \frac{3}{256\pi^4} \int_{m_c^2}^{\infty} ds \int_{m_c^2}^{u^2} du \frac{(s - m_c^2)^2(u - m_c^2)^2 [(3s - u)m_c^2 + 2su]}{s^2 u^2} \\
& \exp\left(-\frac{s}{T^2} - \frac{u}{T^2}\right) + m_c\langle \overline{q}q \rangle \frac{32\pi^2}{32\pi^2} \int_{m_c^2}^{u^2} du \frac{(s - m_c^2)^2(m_c^2 - 5s)}{s^2} \exp\left(-\frac{s}{T^2} - \frac{m_c^2}{T^2}\right) \\
& \exp\left(-\frac{s}{T^2} - \frac{m_c^2}{T^2}\right) + m_c\langle \overline{q}q, \sigma Gq \rangle \frac{128\pi^2}{128\pi^2} \int_{m_c^2}^{u^2} du \frac{10s^2 - 7sm_c^2 + m_c^4}{s^2} \exp\left(-\frac{s}{T^2} - \frac{m_c^2}{T^2}\right) \\
& \exp\left(-\frac{s}{T^2} - \frac{m_c^2}{T^2}\right) + m_c\langle \overline{q}q, \sigma Gq \rangle \frac{128\pi^2}{128\pi^2} \int_{m_c^2}^{u^2} du \frac{2u^2 + 5um_c^2 - 3m_c^4}{u^2} \exp\left(-\frac{m_c^2}{T^2} - \frac{u}{T^2}\right), \\
\end{align*}
\]

\[(28)\]

where the $T^2$ and $T^2_2$ are the Borel parameters. In the two QCD sum rules, the terms depend on $T^2_2$ can be factorized out explicitly,

\[
\begin{align*}
& \frac{f_{\pi}M_\pi^2f_\pi M^2_X G_{X_{\pi}}}{2m_c(m_u + m_d)} \frac{1}{M^2_X - M^2_{q_0}} \left[ \exp\left(-\frac{M^2_{q_0}}{T^2}\right) - \exp\left(-\frac{M^2_X}{T^2}\right) \right] \\
& + C_{X_{q_0}} \exp\left(-\frac{M^2_X}{T^2}\right) = \frac{3}{128\pi^4} \int_{4m_c^2}^{\infty} ds \int_0^{u^2} ds' du \sqrt{1 - \frac{4m_c^2}{s}} \exp\left(-\frac{s}{T^2} - \frac{u}{T^2}\right), \\
\end{align*}
\]

\[(29)\]
The dependence on the $T_2^2$ is rather trivial, \( \exp \left( - \frac{u-M_D^2}{T_2^2} \right) \), \( \exp \left( - \frac{u-M_D^2}{T_2^2} \right) \), \( \exp \left( - \frac{m_c^2-M_D^2}{T_2^2} \right) \), which differ from the QCD sum rules for the three-meson hadronic coupling constants greatly [29]. It is difficult to obtain $T_2^2$ independent regions in the present QCD sum rules, as no other terms to stabilize the QCD sum rules. We can take the local limit $T_2^2 \to \infty$, which is so called local-duality limit (the local QCD sum rules are reproduced from the original QCD sum rules in infinite Borel parameter limit) [30], then \( \exp \left( - \frac{u-M_D^2}{T_2^2} \right) = \exp \left( - \frac{m_c^2-M_D^2}{T_2^2} \right) = \exp \left( - \frac{m_c^2-M_D^2}{T_2^2} \right) = 1 \), the two QCD sum rules are greatly simplified.

The hadronic input parameters are chosen as $M_{cL} = 2.9836$ GeV, $f_\pi = 0.130$ GeV [21], $\sqrt{s} = 0.85$ GeV [15], $M_D = 1.87$ GeV, $f_D = 208$ MeV, $s_D = u_0^D = 6.2$ GeV$^2$ [31], $f_c = 0.387$ GeV [32], $\sqrt{s} = 4.4$ GeV, $M_X = 3.86$ GeV, $\lambda_X = 2.02 \times 10^{-2}$ GeV$^5$ (this work), and $f_\pi M_Z^2/(m_u + m_d) = -2(\bar{q}q)/f_\pi$ from the Gell-Mann-Oakes-Renner relation. The unknown parameters are chosen as $C_{Xn_c} = 0.0063$ GeV$^8$ and $C_{Xn_c} = -0.0071$ GeV$^8$ to obtain platforms in the Borel windows $T^2 = (2.5 - 2.9)$ GeV$^2$ (this work) and $T^2 = (1.3 - 1.7)$ GeV$^2$ [31], respectively. The input parameters at the QCD side are chosen as the same ones in the two-point QCD sum rules for the $X^+(3860)$. Then it is easy to obtain the values of the hadronic coupling constants,

\[
\begin{align*}
G_{Xn_c} &= 1.28 \pm 0.18 \text{ GeV}, \\
|G_{XDD}| &= 12.3 \pm 4.5 \text{ GeV}.
\end{align*}
\] (31)

In Fig.4, we plot the hadronic coupling constants $G_{Xn_c}$ and $G_{XDD}$ at much larger intervals than the Borel windows. From the figure, we can see that the values of the hadronic coupling constants $G_{Xn_c}$ and $G_{XDD}$ are rather stable with variations of the Borel parameters, so we expect to make reliable predictions. The uncertainties of the $G_{Xn_c}$ and $G_{XDD}$ lead to the uncertainties $\delta\Gamma(X^+(3860) \to \eta_c \pi^-)/\Gamma(X^+(3860) \to \eta_c \pi^-) = 26G_{Xn_c}/G_{Xn_c} = 28\%$ and $\delta\Gamma(X^+(3860) \to D^- D^0)/\Gamma(X^+(3860) \to D^- D^0) = 26G_{XDD}/G_{XDD} = 73\%$.

We choose the masses $M_X = 3.862$ GeV [1], $M_{n_c} = 2.9836$ GeV, $M_\pi = 0.13957$ GeV, $M_D = 1.8695$ GeV, $M_{D^0} = 1.8649$ GeV [21], and obtain the numerical values of the decay widths,

\[
\begin{align*}
\Gamma(X^+(3860) \to \eta_c \pi^-) &= \frac{G_{Xn_c}^2 p_{n_c}}{8\pi M_X^2} \\
&= 3.4 \pm 1.0 \text{ MeV}, \\
\Gamma(X^+(3860) \to D^- D^0) &= \frac{G_{XDD}^2 p_{DD}}{8\pi M_X^2} \\
&= 198.7 \pm 145.1 \text{ MeV},
\end{align*}
\] (32)
Figure 4: The hadronic coupling constants with variations of the Borel parameters $T^2$, where the $A$ and $B$ correspond to the $G_{X\eta_c\pi}$ and $G_{XDD}$, respectively.

where

\[
p_{\eta_c\pi} = \frac{\sqrt{[M_X^2 - (M_{\eta_c} + M_\pi)^2][M_X^2 - (M_{\eta_c} - M_\pi)^2]}}{2M_X},
\]

\[
p_{DD} = \frac{\sqrt{[M_X^2 - (M_{D^0} + M_{D^-})^2][M_X^2 - (M_{D^0} - M_{D^-})^2]}}{2M_X}.
\] (33)

If we saturate the width of the $X^*(3860)$ with the strong decays to the meson pairs $\eta_c\pi^-$ and $D^-D^0$, then $\Gamma_X = 202 \pm 146$ MeV, which is in excellent agreement with the experimental value $\Gamma_X = 201^{+154+88}_{-67-82}$ MeV from the Belle collaboration [1]. The present calculations support assigning the $X^*(3860)$ to be the $C_{\gamma_5} \otimes \gamma_5 C$ type hidden-charm tetraquark state.

4 Conclusion

In this article, we tentatively assign the $X^*(3860)$ to be the $C_{\gamma_5} \otimes \gamma_5 C$ type scalar tetraquark state, study its mass and width with the QCD sum rules, special attention is paid to calculating the hadronic coupling constants $G_{X\eta_c\pi}$ and $G_{XDD}$. We obtain the values $M_X = 3.86 \pm 0.09$ GeV and $\Gamma_X = 202 \pm 146$ MeV, which are consistent with the experimental data $M_X = 3862^{+28+40}_{-32-13}$ MeV and $\Gamma_X = 201^{+154+88}_{-67-82}$ MeV, respectively. The dominant decay mode of the neutral partner $X^0(3860)$ is $X^0(3860) \rightarrow DD$, which is also consistent with the fact that the $X^*(3860)$ is observed in the process $e^+e^- \rightarrow J/\psi DD$. The present work supports assigning the $X^*(3860)$ to be the $C_{\gamma_5} \otimes \gamma_5 C$ type hidden-charm tetraquark state.

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