Multidimensional Sticky Brownian Motions: Tail Behaviour of the Joint Stationary Distribution

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Abstract

Sticky Brownian motions, as time-changed semimartingale reflecting Brownian motions, have various applications in many fields, including queuing theory and mathematical finance. In this paper, we are concerned about the stationary distributions of a multidimensional sticky Brownian motion, provided it is stable. We will study the large deviations principle for stationary distribution and the tail behaviour of the joint stationary distribution.

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1 Introduction

Since the works of Feller \cite{Feller1951, Feller1952, Feller1971}, sticky Brownian motions have been explored extensively, for example, see Itô and McKean \cite{Ito1951, Ito1957}. Recall that a sticky Brownian motion on the half-line is the process evolving as a standard Brownian motion away from zero and reflecting at zero after spending a random amount of time there. It is known that a sticky Brownian motion arises as a time change of a semimartingale reflecting Brownian motion (SRBM), which reflects at zero instantaneously, and it describes the scaling limit of random walks on the natural numbers whose jump rate at zero is significantly smaller than that at positive sites. From a queuing theory perspective, this kind of process is quite an interesting one with many applications. Welch \cite{Welch1985} introduced an exceptional service for the first customer in each busy period and showed that a sticky Brownian motion on the half-line can be a heavy traffic limit. Later, with different exceptional service mechanisms, the same heavy traffic limit, or the sticky Brownian motion, was confirmed for other single server queuing models by Lemoine \cite{Lemoine1990}, Harrison and Lemoine \cite{Harrison1992}, Yamada \cite{Yamada1994}, and Yeo \cite{Yeo1997}.

Recently, Rácz and Shrocnikov \cite{Racz2018} introduced multidimensional sticky Brownian motions, which are a natural multidimensional extension of sticky Brownian motions on the half-line. As shown in \cite{Racz2018}, a multidimensional sticky Brownian motion can also be written as a time-changed multidimensional semimartingale reflecting Brownian motion. Multidimensional sticky Brownian motions have many potential applications in both queuing theory and mathematical finance. For example, as pointed out by Rácz and Shrocnikov \cite{Racz2018}, it can be used as an approximation of certain particle movement systems.

Stationary distributions of the multidimensional SRBM have attracted a lot of interest. When $R$ is an $\mathcal{M}$-matrix and $R^{-1}\mu < 0$, Majewski \cite{Majewski2000}, and Avram, Dai and Hasenbein \cite{Avram2004} established the large deviations principle (LDP). The $\mathcal{M}$-matrix condition can be relaxed, for example, Dupuis and Ramanan \cite{Dupuis2000}, under more general conditions, studied a time-reversed representation for the tail probabilities of an SRBM. We are interested in the tail behaviour of
stationary distributions. Many efforts have been made to study this topic and some results have been obtained, most of which are related to special multidimensional cases, including the two-dimensional case and the skew symmetric case. Intuitively, we can discuss them by the large deviations principle. Then, the problem is reduced to finding the rate function, which is formulated as a variational problem. However, it is known that, in general, with the exception of some special cases, it is very difficult to analytically solve these variational problems. Some discussions about why higher dimensional ($\geq 3$) cases are difficult have been carried out, for example, see Avram, Dai and Hasenbein [2]. Hence, additional work is needed to study this problem for the multidimensional SRBM. For the two-dimensional SRBM, Dai and Miyazawa [9], Franceschi and Raschel [21], Dai, Dawson and Zhao [5], and Franceschi and Kurkova [20] studied the tail asymptotics of the marginal distributions of the SRBM, and obtained the decay rate of the marginal distributions. At the same time, being inspired by the two-dimensional case and some special multidimensional cases, we note that some conjectures on the tail properties of the stationary marginal distributions of the multidimensional SRBM have been discussed. Miyazwa and Kobayashi [35] conjectured on the decay rate of the marginal distribution in an arbitrary direction of the multidimensional SRBM. Motivated by the above arguments, in this work, we also study some of the tail properties of the stationary distributions of the multidimensional sticky Brownian motion.

In a recent paper, Dai and Zhao [6] obtained exact tail asymptotics and asymptotic independence of a two-dimensional sticky Brownian motion. The main tools applied in [6] were the kernel method, extreme value theory and copula. In this paper, we extend those ideas for the general multidimensional case. We first discuss the LDP for the sticky Brownian motion, and then study the tail behaviour of the joint stationary distribution of the process.

The rest of this paper is organized as follows: in Section 2, we first state some preliminaries related to multidimensional sticky Brownian motions, and then we study some basic properties of the stationary distributions of multidimensional sticky Brownian motions. In Section 3, we establish the LDP for the sticky Brownian motion. In Section 4, we discuss exact tail asymptotics for some special cases of the multidimensional sticky Brownian motion.

## 2 Sticky Brownian motion

In this section, we introduce some preliminaries related to multidimensional sticky Brownian motions. We first recall the definition of the semimartingale reflecting Brownian motion (SRBM). SRBM models arise as an approximation for queuing networks of various kinds (see, for example, Williams [42, 43]).

The existence of an SRBM has been studied extensively, for example, Taylor and Williams [44], and Reiman and Williams [37]. It was proved in [37] that, for a given set of data $(\Sigma, \mu, R)$, with $\Sigma$ being positive definite, there exists an SRBM for each initial distribution of $Z(0)$, if and only if, $R$ is completely $\mathbb{S}$ (see, for example, Taylor and Williams [44] for the definitions of matrix classes). Furthermore, when $R$ is completely $\mathbb{S}$, the SRBM is unique in distribution for each given initial distribution. It is well-known that a necessary condition (see, for example, Harrison and Williams [24], or Harrison and Hasenbein [23]) for the existence of the stationary distribution for $Z$ is

\[ R \text{ is non-singular and } R^{-1} \mu < 0. \]  

(2.2)

We note that an SRBM does not spend time on the boundary. Conversely, a sticky Brownian motion would spend a duration of time on the boundary. For the one-dimensional case, Feller [17, 18, 19] first observed the sticky boundary behaviour for diffusion processes and studied the problem that describes domains of the infinitesimal generators.
associated with a strong Markov process $\tilde{X}$ in $[0, \infty)$. Moreover, $\tilde{X}$ behaves like a standard Brownian motion in $(0, \infty)$, while at 0, a possible boundary behaviour is described by

$$ f'(0+)^2 = \frac{1}{2u} f''(0+), \quad (2.3) $$

where $u \in (0, \infty)$ is a given and fixed constant and $f$ are functions belonging to the domain of the infinitesimal generator of $\tilde{X}$. The second derivative $f''(0+)$ measures the “stickiness” of $\tilde{X}$ at 0. For this reason, the process $\tilde{X}$ is called a sticky Brownian motion, which is also referred to as a sticky reflecting Brownian motion in the literature. Itô and Mckean [27] first constructed the sample paths of $\tilde{X}$. They showed that $\tilde{X}$ can be obtained from a one-dimensional SRBM $\tilde{Z}$ by the time-change $t \to T(t) := S^{-1}(t)$, where $S(s) = s + \frac{1}{u} L_s$ for $s > 0$, or $T(t) = s$ is determined by the equation $t = s + \frac{1}{u} T_s$. For more information about sticky Brownian motions on the half-line, refer to Engelbert and Peskir [16] and the references therein.

Rácz and Shrocnikov [36] introduced multidimensional sticky Brownian motions and proved the existence and uniqueness of the multidimensional sticky Brownian motion. Similar to a sticky Brownian motion on the half-line, let

$$ S(t) = t + \sum_{i=1}^{d} u_i L_i(t), \quad (2.4) $$

where $u_i \in (0, \infty)$, $i = 1, \ldots, d$, are given and fixed constants. For convenience, let $u = (u_1, \cdots, u_d)'$. Let $T(\cdot)$ be the inverse of $S(\cdot)$, that is,

$$ T(t) = S^{-1}(t). \quad (2.5) $$

Then, it follows from Kobayashi [30] Lemma 2.7] and the equation (2.4) that $T$ has continuous paths and $\lim_{t \to \infty} T(t) = \infty$. Furthermore, $0 < T(1) \leq 1$. Then, a multidimensional sticky Brownian motion can be defined as:

$$ Z(t) = \tilde{Z}(T(t)). \quad (2.6) $$

This type of process finds applications in the fields of queuing theory and mathematical finance. In the queuing field, it is well known that the SRBM is a heavy traffic limit for many queuing networks such as open queuing networks. As discussed in the introduction, in the setting for single server queues, a sticky Brownian motion on the half-line can be served as a heavy traffic limit of a queuing system with exceptional service mechanisms. It is reasonable to expect that a multidimensional sticky Brownian motion serves as a heavy traffic limit for such multidimensional queuing networks with appropriately defined exceptional service mechanisms.

In the rest of this section, we study some of the properties of the stationary distributions of multidimensional sticky Brownian motions. We first establish the so-called basic adjoint relation (BAR), which establishes some connections between the joint stationary distribution and the boundary stationary measures defined below. In particular, in the two-dimensional case, the BAR can be used to study exact tail asymptotics for the marginal stationary distributions and the boundary stationary measures of a sticky Brownian motion (see, Dai and Zhao [6]).

In the rest of this paper, we assume that $Z(0)$ follows the stationary distribution $\pi$ of $\{Z(t)\}$. Furthermore, for the stationary measure $\pi$, we define the moment generating function (MGF) $\Phi(\theta)$ by

$$ \Phi(\theta) = \int_{\mathbb{R}^d} \exp\{<\theta, x>\} \pi(dx). $$

Similar to the SRBM, $\Phi(\theta)$ is closely related to the MGFs of various boundary measures, which are defined below. For any set $A \in \mathcal{B}(\mathbb{R}^d_+)$, define

$$ V_t(A) = \mathbb{E}_\pi \left[ \int_0^{T(1)} 1_{\{Z(s) \in A\}} dL_s(s) \right]. \quad (2.7) $$

At the same time, for any Borel set $B \in \mathcal{B}(\mathbb{R}^d)$, we define the joint measure for the time-change as:

$$ V_0(B) = \mathbb{E}_\pi \left[ \int_0^1 1_{\{Z(s) \in B\}} dT(s) \right]. $$
According to Lemma 2.1 below, all \( V_i, i = 0, \cdots, d \), are finite measures on \( \mathbb{R}^d_+ \). Then, we can define MGFs \( \Phi_i(\theta) \) for \( V_i, i = 0, 1, \cdots, d \), by

\[
\Phi_i(\theta) = \int_{\mathbb{R}^d_+} \exp\{ \langle \theta, x \rangle \} V_i(dx).
\]

For these measures, we have the following BAR:

**Lemma 2.1**

1. The boundary measures \( V_i, i = 1, \cdots, d \), and the joint measure \( V_0 \) for the time-change are all finite.

2. The MGFs of \( V_i, i = 0, 1, \cdots, d \), have the following BAR: for any \( \theta \in \mathbb{R}^d = \{ \theta = (\theta_1, \cdots, \theta_d)' : \theta_i \leq 0 \} \),

\[
-\Psi_X(\theta)\Phi_0(\theta) = \sum_{i=1}^{d} \Phi_i(\theta) - \theta_i > 0,
\]

where \( R_i \) is the \( i \)th column of the reflection matrix \( R \), and \( \Psi_X(\theta) \) is the Lévy exponent of the multidimensional Brownian vector \( X(1) \), i.e.,

\[
\Psi_X(\theta) = \langle \theta, \mu \rangle + \frac{1}{2} \langle \theta, \Sigma \theta \rangle.
\]

**Proof:** Since \( Z(0) \) follows the stationary distribution \( \pi \), for any \( t \in \mathbb{R}_+ \),

\[
P(Z(t) \leq z) = P(Z \leq z).
\]

We note that \( \{Z(t)\} \) is a semimartingale. Since \( T(t) \) is continuous and \( S(t) \) is strictly increasing, it follows from Kobayashi [30, Corollary 3.4] that if \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is \( C^2 \) function, then

\[
f(Z(t)) - f(Z(0)) = \sum_{i=1}^{d} \mu_i \int_{0}^{T(t)} \frac{\partial f}{\partial x_i}(Z(u)) du + \sum_{i,j=1}^{d} \int_{0}^{T(t)} r_{ij} \frac{\partial f}{\partial x_j}(Z(u)) dL_i(u) + \sum_{i=1}^{d} \int_{0}^{T(t)} \frac{\partial^2 f}{\partial x_i \partial x_j}(Z(u)) du.
\]

Hence, we have

\[
\sum_{i=1}^{d} \mu_i \mathbb{E}_\pi \left[ \int_{0}^{T(t)} \frac{\partial f}{\partial x_i}(Z(u)) du \right] + \sum_{i,j=1}^{d} \mathbb{E}_\pi \left[ \int_{0}^{T(t)} r_{ij} \frac{\partial f}{\partial x_j}(Z(u)) dL_i(u) \right] + \frac{1}{2} \sum_{i,j=1}^{d} \Sigma_{ij} \mathbb{E}_\pi \left[ \int_{0}^{T(t)} \frac{\partial^2 f}{\partial x_i \partial x_j}(Z(u)) du \right] = 0.
\]

Next, we prove the first part of this lemma. From (2.7), we get that for all \( i = 1, \cdots, d \),

\[
V_i(\mathbb{R}^d_+) = \mathbb{E}_\pi \left[ L_i(T(1)) \right],
\]

and

\[
V_0(\mathbb{R}^d_+) = \mathbb{E}_\pi \left[ T(1) \right].
\]

Hence, it suffices to prove that for any \( i \in \{1, \cdots, d\} \),

\[
\mathbb{E}_\pi \left[ L_i(T(1)) \right] < \infty,
\]
and

\[ \mathbb{E}_\pi \left[ T(1) \right] < \infty. \] (2.13)

Since \( T(1) \leq 1 \), in order to prove (2.12), we only need to show that

\[ \mathbb{E}_\pi \left[ L_i(1) \right] < \infty. \] (2.14)

It follows from Dai and Harrison [7, Proposition 3] that (2.14) holds. Then (2.12) follows. At the same time, from the relationship between \( T(\cdot) \) and \( S(\cdot) \), we get that

\[ T(t) = t - \sum_{i=1}^{d} u_i L_i(T(t)). \] (2.15)

Hence,

\[ \mathbb{E}_\pi[T(1)] = 1 - \sum_{i=1}^{d} \mathbb{E}_\pi u_i L_i(T(1)). \] (2.16)

Combining (2.12) and (2.16) leads to (2.13).

Taking \( f(x_1, \ldots, x_d) = \exp\{\sum_{i=1}^{d} \theta_i x_i\} \) with \( \theta_i \leq 0, i = 1, \ldots, d \), in the equation (2.10) can prove the second part of this lemma. \( \square \)

**Remark 2.1** Let \( C_b^2(\mathbb{R}_+^d) \) be the set of functions \( f \) on \( \mathbb{R}_+^d \) such that \( f \), its first order derivatives, and its second order derivatives are bounded and continuous. For any \( f \in C_b^2(\mathbb{R}_+^d) \), it follows from (2.10) that

\[
\int_{\mathbb{R}_+^d} \mathcal{L}f(x) V_0(dx) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \mu_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} \mu_i \frac{\partial f}{\partial x_i}(x) = 0,
\]

where

\[ \mathcal{L}f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \Sigma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} \mu_i \frac{\partial f}{\partial x_i}(x), \]

and \( \nabla f(x) \) is the gradient of \( f \). From Dai and Kurtz [8, Theorem 1.4] (or Braverman, Dai and Miyazawa [3, Lemma 2.1]), we can get that \( V_0(\cdot) / \mathbb{E}_\pi \left[ T(1) \right] \), and \( V_i(\cdot) / \mathbb{E}_\pi \left[ L_i(T(1)) \right] \), \( i = 1, \ldots, d \), are the stationary distribution, and the boundary distributions of the corresponding reflecting Brownian motion \( \tilde{Z} \), respectively.

The following corollary immediately follows from the proof to Lemma 2.1

**Corollary 2.1** \( \mathbb{E}_\pi \left[ L_i(T(1)) \right], i = 1, \ldots, d, \) satisfy

\[
\left[ \mu u' - R^T \right] \bar{L} = \mu,
\]

where

\[
\bar{L} = \left( \mathbb{E}\left( L_1(T(1)) \right), \ldots, \mathbb{E}\left( L_d(T(1)) \right) \right)'.
\]

**Proof** Let \( f(x_1, \ldots, x_d) = \exp\{\theta x_1\} \) with \( \theta < 0 \) and \( x_1 \geq 0 \). Then we have that

\[
f_i'(x_1, \ldots, x_d) = \begin{cases} \theta \exp\{\theta x_1\}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases} \quad (2.17)
\]
and
\[ f''_{ij}(x_1, \ldots, x_d) = \begin{cases} \theta^2 \exp(\theta x_1), & \text{if } i = j \\ 0, & \text{other.} \end{cases} \] (2.18)
Hence, combining (2.10), (2.17) and (2.18) gives
\[ -\Psi_X(\theta, 0, \ldots, 0)\Phi_0(\theta, 0, \ldots, 0) = \sum_{i=1}^d \Phi_i(\theta, 0, \ldots, 0)r_{i1}\theta. \] (2.19)
Dividing \( \theta < 0 \) at both sides of (2.19) and letting \( \theta \to 0 \), we get
\[ -\mu L = \sum_{i=1}^d u_i V(B). \] (2.24)

Below, we state the main result of this section. The sticky Brownian motion defined by (2.6) satisfies the following BAR:

**Theorem 2.1**

\[ -\Psi_X(\theta)\Phi(\theta) = \sum_{i=1}^d \Phi_i(\theta) \left( < \theta, R_i > - u_i \Psi_X(\theta) \right). \] (2.23)

**Proof:** For any Borel set \( B \in \mathcal{B}(\mathbb{R}^d) \), we have
\[
\pi(B) = \mathbb{E}_\pi \left[ \int_0^1 \mathbb{1}_{\{Z(s) \in B\}} ds \right] = \mathbb{E}_\pi \left[ \int_0^{T(1)} \mathbb{1}_{\{Z(s) \in B\}} dS(s) \right] = \mathbb{E}_\pi \left[ \int_0^{T(1)} \mathbb{1}_{\{Z(s) \in B\}} dt \right] + \sum_{i=1}^d u_i \mathbb{E}_\pi \left[ \int_0^{T(1)} \mathbb{1}_{\{Z(s) \in B\}} dL_i(s) \right] = V_0(B) + \sum_{i=1}^d u_i V_i(B). \] (2.24)

From (2.8) and (2.24), we can get (2.23). The proof is completed.

**Remark 2.2** The BAR plays an important role in analyzing the asymptotic properties of various stationary distributions. For the two-dimensional case, based on the BAR, Dai and Zhao [6] applied the kernel method to get exact tail behaviours of the marginal stationary distributions and various boundary measures. On the other hand, we note that when all \( u_i = 0 \), for \( i = 1, \ldots, d \), the sticky Brownian motion \( Z \) reduces to an SRBM. In this case, the BAR still holds, which has been obtained (see, for example, Harrison and Williams [24]).
3 Large Deviations for Sticky Brownian Motion

In this section, we study the large deviations principle for the stationary distribution $\pi$ of $Z$. We first recall the definition of LDP, see, for example, Varadhan [40, Definition 2.1].

**Definition 3.1** A sequence of probability measures $\{\mu_n\}$ defined on a complete separable metric space $(\mathcal{H}, \mathcal{B})$ is said to satisfy the LDP with speed $\{\xi_k\}$ and rate function $J$ if, for all $\Theta \in \mathcal{B}$ and $\lim_{k \to \infty} \xi_k = \infty$,

$$\limsup_{n \to \infty} \frac{1}{\xi_n} \log \mu_n(\Theta) \leq - \inf_{x \in \Theta} J(x),$$

and

$$\liminf_{n \to \infty} \frac{1}{\xi_n} \log \mu_n(\Theta) \geq - \inf_{x \in \Theta} J(x),$$

where $J : \mathcal{H} \to [0, \infty]$ is a function with compact level sets, and $\Theta$ (respectively $\Theta^0$) is the closure (respectively interior) of $\Theta$. A sequence of random variables $\{X_n\}$ defined on some measure space taking values in a complete separable metric space $(\mathcal{H}, \mathcal{B})$ is said to satisfy an LDP, with rate function $J(\cdot)$, if the corresponding induced measures satisfy a LDP with the same rate function.

To reach our objective, we need the contraction principle (see, for example, Amir and Ofer [1, Theorem 4.2.1]). Here we briefly recall it. Suppose that a sequence of random variables $\{\xi_k\}$ and good rate function $J$ in the topology $\mathcal{B}$ and $f : \mathcal{H} \to \mathcal{H}'$ is a continuous and measurable mapping to the topological space $(\mathcal{H}', \mathcal{B}')$. Then the contraction principle states that the sequence $\{f(X_n)\}$ satisfies a large deviation principle with speed $\{\xi_k\}$ and good rate function $J' : \mathcal{H}' \to [0, \infty]$ given for $x' \in \mathcal{H}'$ by

$$J'(x') = \inf_{x \in E, x = f(s)} J(x),$$

in the topology $\mathcal{B}'$.

Here, we need the LDP for the SRBM. Let $\tilde{\mu}_n(B) = \pi(nB)$, where $\pi$ is the stationary distribution of the SRBM, and $\mathcal{A}(\mathbb{R}^d)$ be the corresponding sets of absolutely continuous functions on $[0, \infty)$ taking values in $\mathbb{R}^d$. We note that the LDP for an SRBM has been studied extensively (see, for example, Avram, Dai and Hasenbein [2], Majewski [34], Dupuis and Ramanan [15] and the references therein). However, there is no LDP established in other literature when $Z$ is completely-$\mathcal{S}$. In this paper, we also study the LDP for $Z$ under some mild conditions. We will study the LDP under the conditions in Dupuis and Ramanan [15]. We first recall these conditions.

**Condition 3.1** $R$ is invertible and the associated Skorohod Map $\Gamma$ is Lipschitz continuous (with respect to the topology of uniform convergence on compact sets) and is defined for every $\Psi \in \mathcal{C}_+(\mathbb{R}^d)$.

**Remark 3.1** To satisfy Condition 3.1 $R$ must, in particular, be completely-$\mathcal{S}$. Discussions about general assumptions that ensure Condition 3.1 can be found in Dupuis and Ramanan [13] and Dupuis and Ishill [12].

**Condition 3.2** Define $\mathcal{L} = \{-\sum \alpha_i R_i : \alpha_i \geq 0\}$, where $R_i$ is the $i$th column of the matrix $R$. Assume that $\mu \in \mathcal{L}$.

**Remark 3.2** If $R$ is invertible, then Condition 3.2 is equivalent to the inequality 3.2.

The following lemma comes from Dupuis and Ramanan [15].

**Lemma 3.1** Assume that the SRBM is such that $\Sigma$ is positive definite and Conditions 3.1 and 3.2 are satisfied. Let $\Gamma$ be the associated Skorohod map. Then, $\{\bar{\mu}_n\}$ satisfies the LDP with speed function $n$ and the rate function $\bar{V}(x)$ that is given by

$$\bar{V}(x) = \inf_{\phi \in \mathcal{A}(\mathbb{R}^d) : \phi(0) = 0, \phi \in \Gamma(\Psi)} \int_0^{\tau_x} L(s) \Psi(s) ds,$$
where
\[ L(\beta) = \frac{1}{2}(\beta - b)^{\prime} \Sigma^{-1}(\beta - b), \]
and
\[ \tau_x = \inf\{t \geq 0 : \phi(t) = x\}, \]
and where \( \Gamma(\psi) \) is the set of images of \( \psi \) under the Skorohod Map (SM) that is associated with \( \Sigma \).

Next, we state the LDP for the stationary distributions \( \pi \) of the sticky Brownian motion \( Z \). Let \( \mu_n(B) = \pi(nB) \).

It follows from equation (2.4) that \( S(t) \) is strictly increasing. Then it follows from Lemma 2.7 in [30] that \( T \) has continuous sample paths. Hence, let \( T : C([0, \infty), \mathbb{R}^d) \to C([0, \infty), \mathbb{R}^d) \) be a continuous function such that, for any \( \omega \),
\[ T : C([0, \infty), \mathbb{R}^d) \to C([0, \infty), \mathbb{R}^d) \]
\[ Z(t) \rightarrow Z(t) = Z(T(t)). \]
Hence, from the contraction principle and Lemma 3.1, we have the following LDP for \( \mu_n \).

**Theorem 3.1** Assume that \( \Sigma \) is positive definite and Conditions 3.1 and 3.2 are satisfied. Then \( \{\mu_n\} \) satisfies the LDP with the rate function \( V(x) \), given by
\[ V(x) = \inf_{x' \in C([0, \infty), \mathbb{R}^d)} \tilde{V}(x'). \]

**Remark 3.3** From Theorem 3.1 we can see that for any measurable set \( B \subset \mathbb{R}^d \),
\[ \limsup_{n \to \infty} \frac{1}{n} \log P(Z \in nB) \leq \alpha_B; \]
\[ \liminf_{n \to \infty} \frac{1}{n} \log P(Z \in nB) \geq \alpha_B, \]
where \( \alpha_B = -\inf_{x \in B} V(x) \) and \( \alpha_B = -\inf_{x \in \mathbb{R}^d} V(x) \).

To study the tail behaviour of the joint stationary distribution, we need the tail properties of the marginal \( Z_i, i \in \{1, \cdots, d\} \). From Theorem 3.1 and Remark 3.3 we have the following corollary.

**Corollary 3.1** Assume that \( \Sigma \) is positive definite and Conditions 3.1 and 3.2 are satisfied. Then for any \( i \in \{1, \cdots, d\} \),
\[ -\lim_{x \to \infty} \frac{1}{x} \log P(Z_i \geq x) = \alpha_i. \]

### 4 Tail Behaviour of the Joint Distribution

It is well known that if we have a multivariate Gaussian vector, where the correlation coefficients are strictly less than 1, then it is asymptotically independent (see Definition 4.2 below). On the other hand, we note that in the interior of the first quadrant \( \mathbb{R}_+^d \), the sticky Brownian motion \( Z \) behaves like the Brownian motion. Hence, it is expected that, under some mild conditions, \( Z \) is also asymptotically independent. In this section, we discuss the asymptotic independence of \( Z \). In the rest of this paper, we first assume that all the correlation coefficients \( \rho_{ij} < 1, i, j \in \{1, \cdots, d\} \) where \( X(1) = (X_1, \cdots, X_d)' \).

To study the tail behaviour of the joint stationary distribution, we mainly use the copula. For any multidiimensional distribution \( \tilde{F} \) with marginal distributions \( \tilde{F}_i, i = 1, \cdots, d \), the copula associated with \( \tilde{F} \) is a distribution function \( C : [0, 1]^d \to [0, 1] \) satisfying
\[ \tilde{F}(x) = C(\tilde{F}_1(x_1), \cdots, \tilde{F}_d(x_d)). \]
For more information on copula, we refer the reader to Joe [26]. Therefore, if $C(\cdot)$ is a copula, then it is a multivariate distribution with all univariate marginal distributions being $U(0, 1)$, or the joint distribution of a multivariate uniform random vector. It is also well known that for continuous multivariate distributions, the univariate margins and the multivariate or dependence structure can be separated, and the multivariate structure is represented by a copula.

Next, we discuss the tail behaviour of the joint stationary distribution. We first recall some definitions.

**Definition 4.1** (Domain of Attraction) Assume that $\{X_n = (X_1^{(n)}, \ldots, X_d^{(n)})^T\}$ are independent and identical distributed (i.i.d.) multivariate random vectors with common distribution $F(\cdot)$ and the marginal distributions $F_i(\cdot)$, $i = 1, \ldots, d$. If there exist normalizing constants $a_i^{(n)} > 0$ and $b_i^{(n)} \in \mathbb{R}$, $1 \leq i \leq d$, $n \geq 1$ such that, as $n \to \infty$,

$$P\left\{ \frac{M_i^{(n)} - b_i^{(n)}}{a_i^{(n)}} \leq x_i, 1 \leq i \leq d \right\} = F^n \left( a_1^{(n)} x_1 + b_1^{(n)}, \ldots, a_d^{(n)} x_d + b_d^{(n)} \right) \to G(x_1, \ldots, x_d),$$

where $M_i^{(n)} = \max_{k=1}^n X_i^{(k)}$ is the componentwise maxima, then we call the distribution function $G(\cdot)$ a multivariate extreme value distribution function, and $F$ is in the domain of attraction of $G(\cdot)$. We denote this by $F \in D(G)$.

**Definition 4.2** [Asymptotic Independence] Assume that the extreme value distribution function $G(\cdot)$ has the marginal distributions $G_i(\cdot)$, $i = 1, \ldots, d$. If

$$F^n \left( a_1^{(n)} x_1 + b_1^{(n)}, \ldots, a_d^{(n)} x_d + b_d^{(n)} \right) \to G(x_1, \ldots, x_d) = \prod_{i=1}^d G_i(x_i),$$

then we say that $F(\cdot)$ is asymptotically independent.

In the rest of this section, under some mild assumptions, we study the tail behaviour of the joint stationary distribution $F(\cdot)$ of $Z$. As is standard for Lévy-driven queueing networks, we assume below that the reflection matrix $R = I - P^T$, where $P$ is a substochastic matrix with its spectral radius strictly less than 1, and $A^T$ is the transpose of an square matrix $A$. From Condition 2.2 in Dupuis and Rananan [15], we know that $R$ satisfies conditions 3.1 and 3.2.

To study the tail behaviour of the joint stationary distribution $F$, we first need to study the extreme value distribution of the univariate marginal stationary distribution $F_i(\cdot)$. We have the following technical lemma.

**Lemma 4.1** For any $i \in \{1, \ldots, d\}$,

$$F_i(x) \in D(G_i),$$

where

$$G_i(x) = \exp\{-e^{-x}\}.$$  \hspace{1cm} (4.1)

**Proof of Lemma 4.1** It follows from (3.1) and Corollary 3.1 that

$$\alpha_i = \lim_{x \to \infty} \frac{1}{x} \left( - \log \left( 1 - F_i(x) \right) \right).$$  \hspace{1cm} (4.2)

From (4.3), we obtain

$$1 - F_i(x) = \exp\{-\alpha_i x - o(\alpha_i x)\}.$$  \hspace{1cm} (4.3)

For convenience, let $g_i(x) = \exp\{-o(\alpha_i x)\}$ and

$$\tilde{g}_i(x) = o(\alpha_i x).$$  \hspace{1cm} (4.4)

Hence,

$$g_i(x) = \exp\{-\tilde{g}_i(x)\}.$$  \hspace{1cm} (4.5)
Noting (4.5), we can furthermore assume that \( \tilde{g}_i(x) \) is twice continuously differential. In such case, (4.3) and (4.6) suggest that

\[
1 - F_i(x) \sim g_i(x) \exp\{-\alpha_i x\}, \text{ as } x \to \infty.  
\]  

(4.7)

Noting that

\[
\lim_{x \to \infty} \frac{\tilde{g}_i(x)}{\alpha_i x} = 0,  
\]  

(4.8)

we get

\[
\lim_{x \to \infty} \tilde{g}_i(x) = \begin{cases} K, \text{ where } K \text{ is a fixed and finite constant,} \\
\infty. \end{cases}  
\]  

(4.9)

If \( \lim_{x \to \infty} \tilde{g}_i(x) = \infty \), then from (4.8) and the L’Hospital rule, we get, as \( x \to \infty \),

\[
\lim_{x \to \infty} \frac{\tilde{g}_i(x)}{\alpha_i x} = \lim_{x \to \infty} \frac{\tilde{g}_i'(x)}{\alpha_i} = 0.  
\]  

(4.10)

Hence, from (4.9) and (4.10), we get

\[
\lim_{x \to \infty} \tilde{g}_i''(x) = 0.  
\]  

(4.11)

Finally, from (4.6), (4.7), (4.10), (4.11) and (4.12), we get

\[
\lim_{x \to \infty} \frac{F_i''(x) \left(1 - F_i(x)\right)}{\left(F_i'(x)\right)^2} = -1.  
\]  

(4.13)

Hence, from (4.13) and Proposition 1.1 in Resinck [38, pp. 40], we conclude that \( F_i \in D(G_1) \).

**Lemma 4.2** For the sticky Brownian motion \( Z = (Z_1, \cdots, Z_d)' \) with the stationary distribution function \( F \),

\[
F^n(a_i^{(n)} x_i + b_i^{(n)}), i = 1, \cdots, d) \to \Pi_{i=1}^d G_1(x_i), \text{ as } n \to \infty,  
\]

where \( a_i^{(n)} \) and \( b_i^{(n)} \) are normalizing constants.

**Remark 4.1** From Lemma 4.2 we can read that \( F(\cdot) \in D(G) \), with \( G(x_1, \cdots, x_d) = \Pi_{i=1}^d G_1(x_i) \), and \( F \) is asymptotically independent.

Before we can prove Lemma 4.2, we present a modified version of Proposition 5.27 in Rensick [38, pp.296], which plays a key role in the proof of Lemma 4.2.

**Lemma 4.3** Suppose that \( \{X_n = (X_1^{(n)}, \cdots, X_d^{(n)})', n \in \mathbb{N}\} \) are i.i.d. random vectors in \( \mathbb{R}^d \) with the common joint continuous distribution \( \hat{F}(\cdot) \), and the marginal distributions \( \hat{F}_i(\cdot), i = 1, \cdots, d \). Moreover, we assume that \( \hat{F}_i(\cdot), i = 1, \cdots, d \) are both in the domain of attraction of some univariate extreme value distribution \( \hat{G}_1(\cdot) \), i.e., there exist constants \( a_i^{(n)} \) and \( b_i^{(n)} \) such that

\[
\hat{F}_i\left(a_i^{(n)} x + b_i^{(n)}\right) \to \hat{G}_1(x).  
\]

Then, the following are equivalent:

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From (4.16) and (4.17), we get

\( W \) we also note that

For convenience, let

where \( \hat{\mathbf{Z}} \) is defined in (2.6). Without loss of generality, we assume \( Z(0) = 0 \). We mainly use the lemma 4.3 to prove this lemma. Let

\[
\hat{\mathbf{L}}(t) = -[(R^{-1}X(t) \wedge R^{-1}\mu t].
\]

Then, it follows from Konstantopoulos, Last and Lin [31, Proposition 1] that for any \( \mathbf{Z} = (\mathbf{Z}_1, \ldots, \mathbf{Z}_d) \in \mathbb{R}_+^d \),

\[
\mathbb{P}\{Z(t) \geq \mathbf{Z}\} \leq \mathbb{P}\{\hat{Z}(t) \geq \mathbf{Z}\},
\]

where \( \hat{Z}(t) = \hat{Z}(T(t)) \) with

\[
\hat{Z}(t) = X(t) + R\hat{L}(t).
\]

It follows from Kobayashi [30, Lemma 2.7] that

\[
0 < T^* := \sup_{\omega} \{T(1, \omega)\} \leq 1, \text{ a.s.}
\]

By the first change of variable formula (see, for example, Jacob [45, Proposition 10.21]), and the fact that \( \hat{Z}(t) \geq 0 \) and \( \hat{Z}(t) \geq 0 \) for all \( t \in \mathbb{R}_+ \), we have

\[
\hat{Z}(1) = \int_0^1 d\hat{Z}(s) = \int_0^{T(1)} d\hat{Z}(s) \leq \int_0^{T^*} d\hat{Z}(s) = \hat{Z}(T^*) \text{ a.s.},
\]

where the operations are performed component-wise. Hence, for any \( \mathbf{Z} = (\mathbf{Z}_1, \ldots, \mathbf{Z}_d) \in \mathbb{R}_+^d \),

\[
\mathbb{P}\{Z(1) \geq \mathbf{Z}\} \leq \mathbb{P}\{\hat{Z}(T^*) \geq \mathbf{Z}\}.
\]

(1) \( \tilde{F} \) is in the domain of attraction of a product measure, that is,

\[
\tilde{F}^n(a_1^{(n)}x_1 + b_1^{(n)}i, i = 1, \ldots, d) \to \Pi_{i=1}^d \tilde{G}_i(x_i);
\]

(2) For any \( 1 \leq i < j \leq d \), with \( \lim_{t \to \infty} \tilde{F}(x) = 1 \)

\[
\lim_{t \to \infty} \mathbb{P}\{X_i > t, X_j > t\}/(1 - \tilde{F}(t)) \to 0. \quad (4.14)
\]

By a slight modification of the proof to Proposition 5.27 in Rensick [38, pp.296], we can prove the above lemma, details of which are omitted here.

Now, we are ready to prove Lemma 4.2.

**Proof of Lemma 4.2.** For the readers to follow the proof below easily, we first recall some notations we introduced in the previous sections. Recall that the reflection matrix \( R \), the regulator process \( L \) and the \( d \)-dimensional Brownian motion \( X = (X_1, \ldots, X_d) \) are the components in the definition of the SRBM given in (2.1), the time-change process \( T \) is defined through (2.4), and the sticky Brownian motion \( Z \) is defined in (2.6). Without loss of generality, we assume that \( Z(0) = 0 \). We mainly use the lemma 4.3 to prove this lemma. Let

\[
\hat{L}(t) = -[R^{-1}X(t) \wedge R^{-1}\mu t].
\]

Then, it follows from Konstantopoulos, Last and Lin [31, Proposition 1] that for any \( \mathbf{Z} = (\mathbf{Z}_1, \ldots, \mathbf{Z}_d) \in \mathbb{R}_+^d \),

\[
\mathbb{P}\{Z(t) \geq \mathbf{Z}\} \leq \mathbb{P}\{\hat{Z}(t) \geq \mathbf{Z}\},
\]

where \( \hat{Z}(t) = \hat{Z}(T(t)) \) with

\[
\hat{Z}(t) = X(t) + R\hat{L}(t).
\]

It follows from Kobayashi [30, Lemma 2.7] that

\[
0 < T^* := \sup_{\omega} \{T(1, \omega)\} \leq 1, \text{ a.s.}
\]

By the first change of variable formula (see, for example, Jacob [45, Proposition 10.21]), and the fact that \( \hat{Z}(t) \geq 0 \) and \( \hat{Z}(t) \geq 0 \) for all \( t \in \mathbb{R}_+ \), we have

\[
\hat{Z}(1) = \int_0^1 d\hat{Z}(s) = \int_0^{T(1)} d\hat{Z}(s) \leq \int_0^{T^*} d\hat{Z}(s) = \hat{Z}(T^*) \text{ a.s.},
\]

where the operations are performed component-wise. Hence, for any \( \mathbf{Z} = (\mathbf{Z}_1, \ldots, \mathbf{Z}_d) \in \mathbb{R}_+^d \),

\[
\mathbb{P}\{Z(1) \geq \mathbf{Z}\} \leq \mathbb{P}\{\hat{Z}(T^*) \geq \mathbf{Z}\}. \quad (4.16)
\]

For convenience, let

\[
\tilde{F}(\mathbf{Z}) = \mathbb{P}\{Z_1 \geq \mathbf{Z}_1, \ldots, Z_d \geq \mathbf{Z}_d\}.
\]

We also note that

\[
\tilde{F}(\mathbf{Z}) = \lim_{t \to \infty} \mathbb{P}\{Z(t) \geq \mathbf{Z}\} = \inf_{t \to \infty} \mathbb{P}\{Z(t) \geq \mathbf{Z}\} \leq \mathbb{P}\{Z(1) \geq \mathbf{Z}\}. \quad (4.17)
\]

From (4.16) and (4.17), we get

\[
\tilde{F}(\mathbf{Z}) \leq \mathbb{P}\{\hat{Z}(T^*) \geq \mathbf{Z}\} \quad (4.18)
\]
Below, we apply Lemma 4.3 to prove our result. Here, for convenience, we assume that $i = 1$ and $j = 2$ in Lemma 4.3. Other cases can be discussed in the same fashion. Furthermore, for any $z = (z_1, z_2) \in \mathbb{R}_+$, let

$$F_{12}(z) = \mathbb{P}\{Z_1 \geq z_1, Z_2 \geq z_2\},$$

and for a $d$-dimensional vector $Y = (Y_1, \cdots, Y_d)^t$,

$$Y_{12} = (Y_1, Y_2)^t.$$

Therefore, from (4.18), we get

$$F_{12}(z) \leq \mathbb{P}\{\bar{Z}_1(T^*) \geq z_1, \bar{Z}_2(T^*) \geq z_2\}.$$  

(4.19)

On the other hand, for any $z = (z_1, z_2) \in \mathbb{R}_+$,

$$\mathbb{P}\{Z_{12}(T^*) \geq z\} \leq \mathbb{P}\{X_{12}(T^*) - \mu_1 T^* \geq z\}.$$  

(4.20)

It is obvious that $X_{12}(T^*) - \mu_1 T^*$ is a Gaussian vector with the correlation coefficient being less than 1.

From (4.20), we have, for large enough $z \in \mathbb{R}_+$,

$$\limsup_{z \to \infty} \frac{F_{12}(z, z)}{F_1(z)} \leq \limsup_{z \to \infty} \frac{P\{X_{12}(T^*) - \mu_1 T^* \geq (z, z)\}}{F_1(z)}.$$  

(4.21)

At the same time, we know that if a bivariate Gaussian vector has a correlation coefficient strictly less than 1, it is asymptotically independent. Hence,

$$\limsup_{z \to \infty} \frac{P\{X_1(T^*) - \mu_1 T^* \geq z, X_2(T^*) - \mu_2 T^* \geq z\}}{P\{Z_1 \geq z\}} = \limsup_{z \to \infty} \frac{P\{X_1(T^*) - \mu_1 T^* \geq z\} \cdot P\{X_2(T^*) - \mu_2 T^* \geq z\}}{P\{Z_1 \geq z\}}$$

$$\leq \limsup_{z \to \infty} \frac{P\{X_1(T^*) - \mu_1 T^* \geq z, X_2(T^*) - \mu_2 T^* \geq z\}}{P\{X_1(T^*) - \mu_1 T^* \geq z\}}$$

$$\leq \limsup_{z \to \infty} \frac{P\{X_1(T^*) - \mu_1 T^* \geq z, X_2(T^*) - \mu_2 T^* \geq z\}}{P\{X_1(T^*) - \mu_1 T^* \geq z\}} = 0,$$  

(4.22)

where the first inequality is obtained by using the fact that

$$P\{X_1(T^*) - \mu_1 T^* \geq z\}/P\{Z_1 \geq z\} \to 0, \text{ as } z \to \infty.$$

From above arguments, we obtain

$$\limsup_{z \to \infty} \frac{F_{12}(z, z)}{F_1(z)} = 0.$$  

(4.23)

From (4.23) and Lemma 4.3 the proof to the lemma follows. □

From Lemma 4.2, we can get the following result.

**Theorem 4.1** For the multidimensional sticky Brownian motion $Z = (Z_1, \cdots, Z_d)^t$,

$$\mathbb{P}\{Z_1 \geq z_1, \cdots, Z_d \geq z_d\}/\left(\prod_{i=1}^{d} g_i(z_i) \exp\{-\alpha z_i\}\right) \to 1,$$

(4.24)

as $(z_1, \cdots, z_d)^t \to (\infty, \cdots, \infty)^t$, where $g_i(\cdot)$ is given by (4.6).
To prove this theorem, we first introduce a transformation. For the multivariate extreme value distribution \( G(\cdot) \) defined in Remark 4.1:
\[
G^*(x_1, \cdots, x_d) = G\left( \left( \frac{-1}{\log(G_1)} \right)^{-1} (x_1), \cdots, \left( \frac{-1}{\log(G_1)} \right)^{-1} (x_d) \right). \tag{4.25}
\]

Then \( G^*(\cdot) \) is the joint distribution function with the common marginal Fréchet distribution \( \Phi(x) = \exp\{-x^{-1}\} \). Furthermore, for the stationary random vector \( Z = (Z_1, \cdots, Z_d)' \), define
\[
Y_i = \frac{1}{1 - F_i(Z_i)} \quad \tag{4.26}
\]

Let \( F^*(y_1, \cdots, y_d) \) be the joint distribution function of \( Y = (Y_1, \cdots, Y_d)' \). Then, it follows from Proposition 5.10 in Resnick [38] and Lemma 4.2 that
\[
F^*(y_1, \cdots, y_d) \in D(G^*(y_1, \cdots, y_d)). \tag{4.27}
\]
By (4.27), we have that for any \( Y = (y_1, \cdots, y_d)' \in \mathbb{R}_+^d \), as \( n \to \infty \),
\[
(F^*(nY))^n \to G^*(Y). \tag{4.28}
\]
It follows from (4.28) that
\[
F^*(nY) \sim \left( G^*(Y) \right)^{\frac{1}{n}}. \tag{4.29}
\]
By a simple monotonicity argument, we can replace \( n \) in the above equation by \( t \). Then we have, as \( t \to \infty \),
\[
F^*(tY) \sim \left( G^*(Y) \right)^{\frac{1}{t}}. \tag{4.30}
\]
At the same time, by Lemma 4.2 for any \( y \in \mathbb{R}_+ \),
\[
F_i^*(ty) \sim \left( G_i^*(y) \right)^{\frac{1}{t}} \text{ for any } i = 1, \cdots, d. \tag{4.31}
\]
Combining (4.29) and (4.30), we get, as \( t \to \infty \),
\[
F^*(tY) \sim \prod_{i=1}^d F_i^*(ty). \tag{4.32}
\]
It is obvious that for any \( x \in \mathbb{R}_+ \),
\[
F_i^*(tx) := 1 - F_i^*(tx) \to 0 \text{ as } t \to \infty. \tag{4.33}
\]
Let \( C(\bar{u}_1, \cdots, \bar{u}_d) \) be the copula of the random vector \( (Y_1, \cdots, Y_d)' \), i.e.,
\[
C\left( F_1^*(z_1), \cdots, F_d^*(z_d) \right) = F^*(z_1, \cdots, z_d). \tag{4.34}
\]
Furthermore, let \( \hat{C}(\bar{u}_1, \cdots, \bar{u}_d) \) be the corresponding survival copula of \( C \). Then we have (see, for example, Schmitz [39] Equation (2.46)):
\[
\hat{C}(\bar{u}_1, \cdots, \bar{u}_d) = \sum_{i=1}^d \bar{u}_i + \sum_{1 \leq i < j \leq n} C_{i,j}(1 - \bar{u}_i, 1 - \bar{u}_j) - (n - 1) \quad \tag{4.35}
\]
\[
- \sum_{1 \leq i < j < k \leq n} C_{i,j,k}(1 - \bar{u}_i, 1 - \bar{u}_j, 1 - \bar{u}_k) + \cdots + (-1)^n C_{1, \cdots, d}(1 - \bar{u}_1, \cdots, 1 - \bar{u}_n). \tag{4.36}
\]

For convenience, for any \((x_1, \cdots, x_d) \in \mathbb{R}_+^d\), let \(\tilde{u}_i(t) = \tilde{F}_i^*(tx_i)\). Hence, for any \(t \in \mathbb{R}_+\),
\[
\begin{align*}
\hat{C}(\tilde{u}_1(t), \cdots, \tilde{u}_d(t)) &= \tilde{F}_1^*(tx_1, \cdots, tx_d), \\
C(1 - \tilde{u}_1(t), \cdots, 1 - \tilde{u}_d(t)) &= F^*(tx_1, \cdots, tx_d).
\end{align*}
\]
Moreover, from (4.31), we get, as \(t \to \infty\),
\[
C(1 - \tilde{u}_1(t), \cdots, 1 - \tilde{u}_3(t)) \sim \prod_{i=1}^3 (1 - \tilde{u}_i(t)),
\]
and, for any \(1 \leq i_1 < \cdots < i_k \leq 3\) with \(k = 2, \cdots, d\),
\[
C_{i_1, \cdots, i_d}(1 - \tilde{u}_{i_1}(t), \cdots, 1 - \tilde{u}_{i_d}(t)) \sim \prod_{i=1}^k (1 - \tilde{u}_{i_d}(t)).
\]
From (4.34), (4.36) and (4.37), we can obtain that, as \(t \to \infty\),
\[
\hat{C}(\tilde{u}_1(t), \cdots, \tilde{u}_d(t)) \sim \prod_{i=1}^d \tilde{u}_i(t),
\]
which, for any \((z_1, \cdots, z_d) \in \mathbb{R}_+^d\), is equivalent to
\[
\lim_{t \to \infty} \frac{\tilde{F}_i^*(tz_1, \cdots, tz_d)}{\prod_{i=1}^d \tilde{F}_i^*(zt)} = 1.
\]
To prove our theorem, it suffices to show that
\[
\lim_{(z_1, \cdots, z_d) \to (0, \cdots, 0)} \frac{\tilde{F}_i^*(z_1, \cdots, z_d)}{\prod_{i=1}^d \tilde{F}_i^*(zt)} = 1.
\]
Note that
\[
\tilde{F}_i^*(z_1, \cdots, z_d) = \mathbb{P} \left\{ \tilde{F}_i^*(Y_1) \geq \tilde{F}_i^*(z_1), \cdots, \tilde{F}_d^*(Y_d) \geq \tilde{F}_d^*(z_d) \right\}.
\]
From (4.33), to prove (4.40), we only need to show that
\[
\lim_{(\tilde{u}_1, \cdots, \tilde{u}_d) \to (0, \cdots, 0) \text{ and } (\tilde{u}_1, \cdots, \tilde{u}_d) \in I^d} \frac{\hat{C}(\tilde{u}_1, \cdots, \tilde{u}_d)}{\prod_{i=1}^d \tilde{u}_i} = 1,
\]
where \(I = [0, 1]\). We also recall that
\[
\lim_{x \to 0} \frac{1 - \exp\{-x\}}{x} = 1.
\]
Hence, from (4.39) and (4.43), we get, for any \((\tilde{u}_1, \cdots, \tilde{u}_d) \in I^d\), that
\[
\lim_{t \to 0^+} \frac{\hat{C}(t\tilde{u}_1, \cdots, t\tilde{u}_d)}{t^d \tilde{u}_1 \cdots \tilde{u}_d} = 1.
\]
Conversely, we note that the limit (4.42) has the indeterminate form \(0^0\). Hence, we would like to apply the multivariate L'hôpital's rule (see Theorem 2.1 in (4.32)) to prove it. Without much effort, we can construct a multivariate differential function \(\hat{C}(\tilde{u}_1, \cdots, \tilde{u}_d)\), such that
\[
\hat{C}(\tilde{u}_1, \cdots, \tilde{u}_d) = \hat{C}(\tilde{u}_1, \cdots, \tilde{u}_d) \text{ for all } (\tilde{u}_1, \cdots, \tilde{u}_d) \in I^d,
\]
and
\[
\hat{C}(t\tilde{u}_1, \cdots, t\tilde{u}_d) \sim t^d \tilde{u}_1 \cdots \tilde{u}_d, \text{ as } t \to 0.
\]
Hence, it suffices to show that
\[
\lim_{(\bar{a}_1, \cdots, \bar{a}_d) \to (0, \cdots, 0) \text{ and } (\bar{a}_1, \cdots, \bar{a}_d) \in \mathcal{H}} \frac{\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)}{\Pi_{i=1}^d \bar{u}_i} = \lim_{(\bar{a}_1, \cdots, \bar{a}_d) \to (0, \cdots, 0) \text{ and } (\bar{a}_1, \cdots, \bar{a}_d) \in \mathcal{H}} \frac{\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)}{\Pi_{i=1}^d \bar{u}_i} = 1. \tag{4.45}
\]

Near the origin \((0, \cdots, 0)'\), the zero sets of both \(\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)\) and \(\bar{u}_1 \cdots \bar{u}_d\) consist of the hypersurfaces \(\bar{u}_i = 0, \ i = 1, \cdots, d\). By the multivariate L’Hôpital’s rule (see Theorem 2.1 in [32]), to prove (4.45), it is enough to show that for each component \(E_i\) of \(\mathcal{H} \setminus \mathcal{C}\), where \(\mathcal{C} = \cup_{i=1}^d \{\bar{u}_i = 0\}\), we can find a vector \(\bar{z}\), not tangent to \((0, \cdots, 0)'\), such that, \(D_{\bar{z}}(\Pi_{i=1}^d \bar{u}_i) \neq 0\) on \(E_i\) and
\[
\lim_{(\bar{a}_1, \cdots, \bar{a}_d) \to (0, \cdots, 0) \text{ and } (\bar{a}_1, \cdots, \bar{a}_d) \in E_i} \frac{D_{\bar{z}}\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)}{D_{\bar{z}}(\bar{u}_1 \cdots \bar{u}_d)} = 1. \tag{4.46}
\]

For the component \(E_1\) bounded by the hypersurfaces of \(\mathcal{H}_1 = \{(\bar{u}_1, \cdots, \bar{u}_d)': (\bar{u}_1, \cdots, \bar{u}_d) \in \mathbb{R}_+^d \text{ and } \bar{u}_i = 0\}\), for all \(i = 1, \cdots, d\), choose, say \(\bar{z} = (1, \cdots, 1)'\). Then, \(\bar{z}\) is not tangent to any hypersurfaces \(u_i = 0, \ i = 1, \cdots, d\) at the point \((0, \cdots, 0)'\). Next, we take the limit along the direction \(\bar{z} = (1, \cdots, 1)'\). It follows from (4.43) and (4.44) that
\[
\lim_{(\bar{a}_1, \cdots, \bar{a}_d) \to (0, \cdots, 0) \text{ and } (\bar{a}_1, \cdots, \bar{a}_d) \in E_1} \frac{D_{\bar{z}}\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)}{D_{\bar{z}}(\bar{u}_1 \cdots \bar{u}_d)} = 1. \tag{4.47}
\]

Similar to (4.46), for any other components \(E_i, \ i = 2, \cdots, 2^d\), we can find a vector \(\bar{z}\) such that \(\bar{z}\) is not tangent to any hypersurfaces \(u_i = 0, \ i = 1, \cdots, d\) at the point \((0, \cdots, 0)'\). Moreover, we have
\[
\lim_{(\bar{a}_1, \cdots, \bar{a}_d) \to (0, \cdots, 0) \text{ and } (\bar{a}_1, \cdots, \bar{a}_d) \in E_i} \frac{D_{\bar{z}}\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)}{D_{\bar{z}}(\bar{u}_1 \cdots \bar{u}_d)} = 1. \tag{4.48}
\]

From (4.45) to (4.47) and Theorem 2.1 in [32],
\[
\lim_{(\bar{a}_1, \cdots, \bar{a}_d) \to (0, \cdots, 0) \text{ and } (\bar{a}_1, \cdots, \bar{a}_d) \in \mathcal{H}} \frac{\hat{C}(\bar{u}_1, \cdots, \bar{u}_d)}{\bar{u}_1 \cdots \bar{u}_d} = 1. \tag{4.49}
\]

Finally, it follows from (4.26) that for any \((z_1, \cdots, z_d)' \in \mathbb{R}_+^d\),
\[
\mathbb{P}\{Z_1 \geq z_1, \cdots, Z_d \geq z_d\} = \mathbb{P}\{F_1(Z_1) \geq F_1(z_1), \cdots, F_d(Z_d) \geq F_d(z_d)\} = \mathbb{P}\{Y_1 \geq \frac{1}{1-F_1(z_1)}, \cdots, Y_d \geq \frac{1}{1-F_d(z_d)}\} = F^*(\frac{1}{F_1(z_1)}, \cdots, \frac{1}{F_d(z_d)}). \tag{4.49}
\]

Combining (4.40) and (4.49), we get
\[
\mathbb{P}\{Z_1 \geq z_1, \cdots, Z_d \geq z_d\} / \left(\Pi_{i=1}^d \bar{F}_i(z_i)\right) \to 1, \text{ as } (z_1, \cdots, z_d)' \to (\infty, \cdots, \infty)', \tag{4.50}
\]

By (4.44) and (4.50), we obtain
\[
\mathbb{P}\{Z_1 \geq z_1, \cdots, Z_d \geq z_d\} / \left(\Pi_{i=1}^d \bar{F}_i(z_i)\right) \to 1, \text{ as } (z_1, \cdots, z_d)' \to (\infty, \cdots, \infty)' . \tag{4.51}
\]

From the above arguments, the theorem is proved. \(\square\)
5 Exact Tail Asymptotic for Some Special Cases

In the previous section, we studied the rough decay rate of multidimensional sticky Brownian motion. However, we are also interested in exact tail asymptotics for a multidimensional sticky Brownian motion. In general, it is very difficult to get such results. In this section, we study exact tail asymptotics for some special cases.

Example 4.1 (Two-dimensional case): We first consider the case for $d = 2$. Dai and Zhao [6] has proved that the marginal stationary distributions have the following exact tail asymptotics:

$$
\mathbb{P}\{Z_i \geq z_i\} \sim K_i z_i^{-\beta_i} \exp\{-\alpha_i z_i\}, \ i = 1, 2,
$$

(5.1)

where $\beta_i \in \{\frac{1}{2}, 1, 0\}$ and $K_i$ is a non-zero constant. By using the same method that was used in the proof to Theorem 4.1, we can show that, for a two-dimensional sticky Brownian motion, we have, as $(z_1, z_2) \to (\infty, \infty)$,

$$
\mathbb{P}\{Z_1 \geq z_1, Z_2 \geq z_2\} / \left(\prod_{i=1}^{2} K_i z_i^{-\beta_i} \exp\{-\alpha_i z_i\}\right) \to 1.
$$

Example 4.2 (Skew symmetry case): We next consider the skew symmetric case. In Rácz and Shkolnikov [36, Theorem 5], it was demonstrated that the stationary distribution for a multidimensional sticky Brownian motion in a wedge admits a separable form if the data satisfies the conditions (14) and (15) in Rácz and Shkolnikov [36]. Following the proof to Theorem 5 in [36], and noting the skew conditions for the SRBM on a nonnegative orthant $\mathbb{R}^d_+$, (see, for example, [2][4][24][25]), we can see that the sticky Brownian motion $Z$ has a separable form if it satisfies the following skew symmetry condition:

$$
2\Sigma = R\Delta_R^{-1} \Delta_{\Sigma} + \Delta_{\Sigma} \Delta_R^{-1} R^T,
$$

(5.2)

where $\Delta_{\Sigma}$ is the diagonal matrix with diagonal entries of a square matrix $A$. In this case, we can easily get that, as $(z_1, \ldots, z_d) \to (\infty, \ldots, \infty)$,

$$
\mathbb{P}\{Z_1 \geq z_1, \ldots, Z_d \geq z_d\} / \left(\prod_{i=1}^{d} K_i z_i^{-\beta_i} \exp\{-\alpha_i z_i\}\right) \to 1.
$$

Example 4.3 (Decomposability): We first note that under the skew symmetry condition (5.2), the stationary distribution of $Z$ could be obtained explicitly via the method used in Rácz and Shkolnikov [36, pp.1169-1170]. However, the condition (5.2) may be too strong, and is not satisfied in most cases. At the same time, from the equation (2.4), we easily see that when $u_i = 0$, for all $i = 1, \ldots, d$, the sticky Brownian motion $Z$ becomes an SRBM on a nonnegative orthant $\mathbb{R}^d_+$. For the SRBM, in applications, even if the condition (5.2) is not satisfied, we can apply the product form based approximation to study the stationary distribution, see for example, [29]. This product form based approximation may be improved by the decomposability in Dai, Miyazawa and Wu [11]. At the end of this work, we consider this special case, that is, $Z$ is an SRBM with the data $(\Sigma, \mu, R)$. Let $J := \{1, \ldots, d\}$. We say that a pair $(\mathbb{K}, \mathbb{L})$ is a partition of $J$ if it satisfies $\mathbb{K} \cup \mathbb{L} = J$ and $\mathbb{K} \cap \mathbb{L} = \emptyset$. Recall that if the stationary distribution is the product of two marginal distributions associated with a partition $(\mathbb{K}, \mathbb{L})$ of the set $J$, then the stationary distribution is said to be decomposable with respect to $\mathbb{K}$ and $\mathbb{L}$. Let $A^{(K,L)}$ be the $|\mathbb{K}| \times |\mathbb{L}|$ submatrix of a $d$-dimensional square matrix $A$ whose row and column indices are taken from $\mathbb{K}$ and $\mathbb{L}$, respectively, and $x^K$ be the $\mathbb{K}$-dimensional vector with $x^K_i = x_i$ for $i \in \mathbb{K}$, where $x^K_i$ is the $i$-th entry of $x^K$. From Theorem 2 in Dai, Miyazawa and Wu [11], we get that if the covariance matrix $\Sigma$ and $R$ satisfy

$$
2\Sigma^{(K,K)} = R^{(K,K)} \Delta_{(R^{(K,K)})}^{-1} \Delta_{\Sigma^{(K,K)}} \Delta_{(R^{(K,K)})}^{-1} (R^{(K,K)})^T,
$$

(5.3)

and if the $|\mathbb{L}|$-dimensional $\left(\Sigma^{(L,K)}, \tilde{\mu}(\mathbb{L}), R^{(L,K)}\right)$-SRBM has a stationary distribution, where $\tilde{\mu}(\mathbb{L}) = \left(Q^{(L,L)}\right)^{-1} (Q \mu)^L$ with $Q = R^{-1}$, then $Z^K(0)$ and $Z^L(0)$ are independent and $Z^K(0)$ is of product form under $\pi$. From Theorem 2 in Dai, Miyazawa and Wu [11], we get that if the covariance matrix $\Sigma$ and $R$ satisfy

$$
2\Sigma^{(L,K)} = R^{(L,K)} \Delta_{(R^{(K,K)})}^{-1} \Delta_{(R^{(K,K)})}^{-1} (R^{(K,K)})^T,
$$

(5.4)
Hence, let $L \subset J$ with $|L| = 2$. If the above conditions are satisfied, then we can get, from Examples 4.1 and 4.2, that as $(z_1, \cdots, z_d)' \to (\infty, \cdots, \infty)'$,

$$
\mathbb{P}\{Z_1 \geq z_1, \cdots, Z_d \geq z_d\}/\left(\prod_{i=1}^d K_i z_i^{-\beta_i} \exp\{-\alpha_i z_i\}\right) \to 1.
$$

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