APPENDIX TO V. MATHAI AND J. ROSENBERG’S PAPER “A NONCOMMUTATIVE SIGMA-MODEL”

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This short note is an appendix to [6].

Let \( \theta \in \mathbb{R} \). Denote by \( A_\theta \) the rotation \( C^* \)-algebra generated by unitaries \( U \) and \( V \) subject to \( UV = e^{2\pi i \theta} VU \), and by \( A_\theta^\infty \) its canonical smooth subalgebra. Denote by \( \text{tr} \) the canonical faithful tracial state on \( A_\theta \) determined by \( \text{tr}(U^m V^n) = \delta_{m,0}\delta_{n,0} \) for all \( m, n \in \mathbb{Z} \). Denote by \( \delta_1 \) and \( \delta_2 \) the unbounded closed \( * \)-derivations of \( A_\theta \) defined on some dense subalgebras of \( A_\theta \) and determined by \( \delta_1(U) = 2\pi i U \), \( \delta_1(V) = 0 \), and \( \delta_2(U) = 0 \), \( \delta_2(V) = 2\pi i V \). The energy \( E(u) \), of a unitary \( u \) in \( A_\theta \) is defined as

\[
E(u) = \frac{1}{2} \text{tr}(\delta_1(u)^*\delta_1(u) + \delta_2(u)^*\delta_2(u))
\]

when \( u \) belongs to the domains of \( \delta_1 \) and \( \delta_2 \), and \( \infty \) otherwise.

Rosenberg has the following conjecture [9, Conjecture 5.4].

Conjecture 1. For any \( m, n \in \mathbb{Z} \), in the connected component of \( U^m V^n \) in the unitary group of \( A_\theta^\infty \), the functional \( E \) gets minimal value exactly at the scalar multiplies of \( U^m V^n \).

For a \( * \)-endomorphism \( \varphi \) of \( A_\theta^\infty \), its energy \( \mathcal{L}(\varphi) \), is defined as \( 2E(\varphi(U)) + 2E(\varphi(V)) \). Mathai and Rosenberg’s Conjecture 3.1 in [6] about the minimal value of \( \mathcal{L}(\varphi) \) follows directly from Conjecture 1.

Denote by \( H \) the Hilbert space associated to the GNS representation of \( A_\theta \) for \( \text{tr} \), and denote by \( \| \cdot \|_2 \) its norm. We shall identify \( A_\theta \) as a subspace of \( H \) as usual.

Then (1) can be rewritten as

\[
E(u) = \frac{1}{2}(\|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2).
\]

Now we prove Conjecture 1 and hence also prove Conjecture 3.1 of [6].

Theorem 2. Let \( \theta \in \mathbb{R} \) and \( m, n \in \mathbb{Z} \). Let \( u \in A_\theta \) be a unitary whose class in \( K_1(A_\theta) \) is the same as that of \( U^m V^n \). Then \( E(u) \geq E(U^m V^n) \), and “=” holds if and only if \( u \) is a scalar multiple of \( U^m V^n \).

Proof. We may assume that \( u \) belongs to the domains of \( \delta_1 \) and \( \delta_2 \). Set \( a_j = u^*\delta_j(u) \) for \( j = 1, 2 \). For any closed \( * \)-derivation \( \delta \) defined on a dense subset of a unital \( C^* \)-algebra \( A \) and any tracial state \( \tau \) of \( A \) vanishing on the range of \( \delta \), if unitaries \( v_1 \) and
and \(v_2\) in the domain of \(\delta\) have the same class in \(K_1(A)\), then \(\tau(v_1^*\delta(v_1)) = \tau(v_2^*\delta(v_2))\) \cite{page 281}. Thus

\[
\text{tr}(a_j) = \text{tr}((U^mV^n)^j\delta_j(U^mV^n)) = \begin{cases} 2\pi im & \text{if } j = 1; \\ 2\pi in & \text{if } j = 2. \end{cases}
\]

We have

\[
\|\delta_j(u)\|_2^2 = \|a_j\|_2^2 = \|\text{tr}(a_j)\|_2^2 + \|a_j - \text{tr}(a_j)\|_2^2 \\
\geq \|\text{tr}(a_j)\|_2^2 = |\text{tr}(a_j)|^2 \\
= \begin{cases} 4\pi^2m^2 & \text{if } j = 1; \\ 4\pi^2n^2 & \text{if } j = 2, \end{cases}
\]

and “==” holds if and only if \(a_j = \text{tr}(a_j)\). It follows that \(E(u) \geq 2\pi^2(m^2 + n^2)\), and “==” holds if and only if \(\delta_1(u) = 2\pi imu\) and \(\delta_2(u) = 2\pi inu\). Now the theorem follows from the fact that the elements \(a\) in \(A_\theta\) satisfying \(\delta_1(a) = 2\pi ima\) and \(\delta_2(a) = 2\pi ina\) are exactly the scalar multiples of \(U^mV^n\). \(\square\)

When \(\theta \in \mathbb{R}\) is irrational, the \(C^*\)-algebra \(A_\theta\) is simple \cite[Theorem 3.7]{10}, has real rank zero \cite[Theorem 1.5]{11}, and is an \(A\mathcal{T}\)-algebra \cite[Theorem 4]{5}. It is a result of Elliott that for any pair of \(A\mathcal{T}\)-algebras with real rank zero, every homomorphism between their graded \(K\)-groups preserving the graded dimension range is induced by a \(*\)-homomorphism between them \cite[Theorem 7.3]{4}. The graded dimension range of a unital simple \(A\mathcal{T}\)-algebra \(A\) is the subset \(\{(g_0, g_1) \in K_0(A) \oplus K_1(A) : 0 \leq g_0 \leq [1_A]_0 \} \cup (0, 0)\) of the graded \(K\)-group \(K_0(A) \oplus K_1(A)\) \cite[page 51]{8}. It follows that, when \(\theta\) is irrational, for any group endomorphism \(\psi\) of \(K_1(A_\theta)\), there is a unital \(*\)-endomorphism \(\varphi\) of \(A_\theta\) inducing \(\psi\) on \(K_1(A_\theta)\). It is an open question when one can choose \(\varphi\) to be smooth in the sense of preserving \(A_\theta^\infty\), though it was shown in \cite{2, 3} that if \(\theta\) is irrational and \(\varphi\) restricts to a \(*\)-automorphism of \(A_\theta^\infty\), then \(\psi\) must be an automorphism of the rank-two free abelian group \(K_1(A_\theta)\) with determinant 1. When \(\psi\) is the zero endomorphism, from Theorem\cite{2} one might guess that \(L(\varphi)\) could be arbitrarily small. It is somehow surprising, as we show now, that in fact there is a common positive lower bound for \(L(\varphi)\) for all \(0 < \theta < 1\). This answers a question Rosenberg raised at the Noncommutative Geometry workshop at Oberwolfach in September 2009.

**Theorem 3.** Suppose that \(0 < \theta < 1\). For any unital \(*\)-endomorphism \(\varphi\) of \(A_\theta\), one has \(L(\varphi) \geq 4(3 - \sqrt{5})\pi^2\).

**Theorem 3** is a direct consequence of the following lemma.

**Lemma 4.** Let \(\theta \in \mathbb{R}\) and let \(u, v\) be unitaries in \(A_\theta\) with \(uv = \lambda vu\) for some \(\lambda \in \mathbb{C} \setminus \{1\}\). Then \(E(u) + E(v) \geq 2(3 - \sqrt{5})\pi^2\).

**Proof.** We have

\[
\text{tr}(uv) = \text{tr}(\lambda vu) = \lambda \text{tr}(uv),
\]
and hence $\text{tr}(uv) = 0$. Thus

$$-\text{tr}(u)\text{tr}(v) = \text{tr}(uv - \text{tr}(u)\text{tr}(v)) = \text{tr}((u - \text{tr}(u))v + \text{tr}(u)(v - \text{tr}(v))) = \text{tr}((u - \text{tr}(u))v).$$

We may assume that both $u$ and $v$ belong to the domains of $\delta_1$ and $\delta_2$. For any $m, n \in \mathbb{Z}$, denote by $a_{m,n}$ the Fourier coefficient $\langle u, U^m V^n \rangle$ of $u$. Then $a_{0,0} = \text{tr}(u)$, and

$$(2\pi)^2\|u - \text{tr}(u)\|^2_2 = \sum_{m,n \in \mathbb{Z}, m^2+n^2>0} |2\pi a_{m,n}|^2 \leq \sum_{m,n \in \mathbb{Z}, m^2+n^2>0} |2\pi a_{m,n}|^2(m^2 + n^2) = \|\delta_1(u)\|^2_2 + \|\delta_2(u)\|^2_2 = 2E(u).$$

Thus

$$|\text{tr}(u)|^2 = \|\text{tr}(u)\|^2_2 = \|u\|^2_2 - \|u - \text{tr}(u)\|^2_2 \geq 1 - \frac{1}{2\pi^2}E(u),$$

and

$$|\text{tr}((u - \text{tr}(u))v)| \leq \|(u - \text{tr}(u))v\|_2 = \|u - \text{tr}(u)\|_2 \leq \left(\frac{1}{2\pi^2}E(u)\right)^{1/2}.$$ 

Similarly, $|\text{tr}(v)|^2 \geq 1 - \frac{1}{2\pi^2}E(v)$.

Write $\frac{1}{2\pi^2}E(u)$ and $\frac{1}{2\pi^2}E(v)$ as $t$ and $s$ respectively. We just need to show that $t + s \geq 3 - \sqrt{5}$. If $t \geq 1$ or $s \geq 1$, then this is trivial. Thus we may assume that $1 - t, 1 - s > 0$. Then

$$(1 - t)(1 - s) \leq |\text{tr}(u)\text{tr}(v)|^2 \leq t.$$ 

Equivalently, $t(1 - s) \geq 1 - (t + s)$. Without of loss generality, we may assume $s \geq t$. Write $t + s$ as $w$. Then

$$t(1 - w/2) \geq t(1 - s) \geq 1 - (t + s) = 1 - w,$$

and hence

$$w = t + s \geq \frac{1 - w}{1 - w/2} + \frac{w}{2}.$$ 

It follows that $w^2 - 6w + 4 \leq 0$. Thus $w \geq 3 - \sqrt{5}$. \hfill \Box

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