AN INEQUALITY CHARACTERIZING CONVEX DOMAINS

STEFAN STEINERBERGER

Abstract. A property of smooth convex domains $\Omega \subseteq \mathbb{R}^n$ is that if two points on the boundary $x, y \in \partial \Omega$ are close to each other, then their normal vectors $n(x), n(y)$ point roughly in the same direction and this direction is almost orthogonal to $x - y$ (for ‘nearby’ $x$ and $y$). We prove there exists a constant $c_n > 0$ such that if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $C^1$-boundary $\partial \Omega$, then

$$\int_{\partial \Omega \times \partial \Omega} \frac{|\langle n(x), y - x \rangle \langle y - x, n(y) \rangle|}{\|x - y\|^{n+1}} d\sigma(x)d\sigma(y) \geq c_n |\partial \Omega|$$

and equality occurs if and only if the domain $\Omega$ is convex.

1. Introduction and Result

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^1$-boundary. The normal vectors $n(x), n(y)$ of two elements of the boundary, $x, y \in \partial \Omega$, will point roughly in the same direction which is roughly orthogonal to $y - x$ if $x$ and $y$ are close. In regions of large curvature the normal vector changes quickly but convex domains whose boundary has regions with large curvature are ‘flatter’ in other regions and it might all average out in the end. We prove a quantitative version of this notion.

**Theorem.** There exists $c_n > 0$ so that for any bounded $\Omega \subseteq \mathbb{R}^n$ with $C^1$-boundary

$$\int_{\partial \Omega \times \partial \Omega} \frac{|\langle n(x), y - x \rangle \langle y - x, n(y) \rangle|}{\|x - y\|^{n+1}} d\sigma(x)d\sigma(y) \geq c_n |\partial \Omega|$$

with equality if and only if the domain $\Omega$ is convex.

Integration is carried out with respect to the $(n - 1)$-dimensional Hausdorff measure and the size of the boundary $|\partial \Omega|$ is measured the same way. Somewhat to our surprise, we were unable to find this statement in the literature. It can be interpreted as a global conservation law for convex domains or as a geometric functional with an extremely large set of minimizers (all convex domains). The requirement of the boundary $\partial \Omega$ being $C^1$ can presumably be somewhat relaxed.

![Figure 1](image)

**Figure 1.** If $x$ and $y$ are close, then $n(x)$ and $n(y)$ are nearly orthogonal to $x - y$ unless $x$ and $y$ are far apart.

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If $\Omega$ is the unit ball in $\mathbb{R}^n$, then $\partial \Omega = \mathbb{S}^{n-1}$ and for any $x, y \in \mathbb{S}^{n-1}$, we have $n(x) = x$ and $\|x - y\|^2 = 2 - 2\langle x, y \rangle$. This simplifies the expression since

$$|\langle n(x), y - x \rangle \langle y - x, n(y) \rangle| = (1 - \langle x, y \rangle)^2.$$ 

Moreover, using rotational symmetry and $w = (1, 0, 0, \ldots, 0)$ for the north pole,

$$\int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{(1 - \langle x, y \rangle)^2}{\|y - x\|^{n+1}} \, d\sigma(x) \, d\sigma(y) = \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{(1 - \langle x, y \rangle)^2}{(2 - 2\langle x, y \rangle)^{\frac{n+1}{2}}} \, d\sigma(x) \, d\sigma(y)$$

$$= \frac{|\mathbb{S}^{n-1}|}{2^{\frac{n+1}{2}}} \int_{\mathbb{S}^{n-1}} (1 - \langle x, w \rangle)^{\frac{n-3}{2}} \, d\sigma(x)$$

which implies

$$c_n = \frac{1}{2^{\frac{n+1}{2}}} \int_{\mathbb{S}^{n-1}} (1 - x_1)^{-\frac{n-3}{2}} \, d\sigma(x) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |x_1| \, d\sigma(x).$$

The constant has a simple form in low dimensions where $c_2 = 2$ and $c_3 = \pi$. Our proof can be best described as an application of Integral Geometry; we formulate and use a bilinear version of the Crofton formula. The proof tells us a little bit more: if the domain $\Omega$ is convex, stronger statements can be made.

**Corollary.** For any convex, bounded $\Omega \subset \mathbb{R}^n$ with $C^1$-boundary and all $x \in \partial \Omega$

$$\int_{\partial \Omega} \frac{|\langle n(x), y - x \rangle \langle y - x, n(y) \rangle|}{\|x - y\|^{n+1}} \, d\sigma(y) = c_n.$$ 

If $x \in \Omega \setminus \partial \Omega$ and $w \in \mathbb{S}^{n-1}$ is an arbitrary unit vector, then

$$\int_{\partial \Omega} \frac{|\langle w, y - x \rangle \langle y - x, n(y) \rangle|}{\|x - y\|^{n+1}} \, d\sigma(y) = 2 \cdot c_n.$$ 

We note that $c_n$ is the exact same constant as above (which can be seen by integrating the first equation over $\partial \Omega$ with respect to $d\sigma(x)$). Some of the conditions can presumably be relaxed a little. The Crofton formula is known to hold in a very general setting (see Santaló [2]). It is an interesting question whether any of these results could be generalized to more abstract settings.

**2. PROOF OF THE THEOREM**

The Crofton formula in $\mathbb{R}^n$ (see, for example, Santaló [1]) states that for rectifiable $S$ of co-dimension 1 one has

$$|S| = \alpha_n \int L n_\ell(S) \, d\mu(\ell),$$

where the integral runs over the space of all oriented lines in $\mathbb{R}^n$ with respect to the kinematic measure $\mu$ (which is invariant under all rigid motions of $\mathbb{R}^n$) and $n_\ell(S)$ is the number of times the line $\ell$ intersects the surface $S$. The constant $\alpha_n$ can be computed by picking $S = \mathbb{S}^{n-1}$ but will not be needed for our argument.

**Lemma.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^1$-boundary. Almost all lines $\ell$ (with respect to the kinematic measure) intersect the boundary $\partial \Omega$ either never or in exactly two points if and only if $\Omega$ is convex.

**Proof.** If $\Omega$ is convex, the result is immediate. Suppose now $\Omega$ is not convex; then there exists a boundary point $x \in \partial \Omega$ such that the supporting hyperplane does not contain all of the domain on one side (note that because the boundary of $\Omega$ is $C^1$, the supporting hyperplane is unique). In particular, there exists $y \in \Omega$ that is
on the other side of the supporting hyperplane. The line \( \ell \) that goes through \( x \) and \( y \) satisfies \( n_\ell(\partial \Omega) \geq 4 \), moreover, this is stable under some perturbations of the line (and thus a set of kinematic measure larger than 0) because \( \partial \Omega \) is \( C^1 \).

**Proof of the Theorem.** We first note for almost all lines \( \ell \) (with respect to the kinematic measure) the number of intersections \( n_\ell(\partial \Omega) \) is either 0 or at least 2: if a line enters the domain, it also has to exit the domain (lines that are tangential to the boundary are a set of measure 0). This implies

\[
|\partial \Omega| = \alpha_n \int L n_\ell(\partial \Omega) \, d\mu(\ell) \leq \frac{\alpha_n}{2} \int L n_\ell(\partial \Omega)^2 \, d\mu(\ell)
\]

with equality if and only if \( \Omega \) is convex. At this point we pick a small \( \varepsilon > 0 \) and decompose the boundary \( \partial \Omega = \bigcup_i \partial \Omega_i \) into small disjoint regions that have diameter \( \leq \varepsilon \ll 1 \) (and \( \varepsilon \) will later tend to 0).

We will now evaluate the integral over such a product. The diagonal terms \( i = j \) behave a little bit differently than the non-diagonal terms and we start with those.

As \( \varepsilon \to 0 \), the fact that the boundary is \( C^1 \) implies that a ‘random’ line will hit any such infinitesimal segment at most once and thus the Crofton formula implies

\[
\frac{\alpha_n}{2} \int L \sum_i n_\ell(\partial \Omega_i)^2 \, d\mu(\ell) = (1 + o(1)) \frac{\alpha_n}{2} \int L \sum_i n_\ell(\partial \Omega_i) \, d\mu(\ell)
\]

\[
= (1 + o(1)) \frac{\alpha_n}{2} \int L n_\ell(\partial \Omega) \, d\mu(\ell) = (1 + o(1)) \frac{|\partial \Omega|}{2}
\]

which is nicely behaved as \( \varepsilon \to 0 \) (the error could be made quantitative in terms of the modulus of continuity of the normal vector). It remains to analyze the off-diagonal terms. Let us assume that \( \partial \Omega_x \subset \partial \Omega \) is a small segment centered around \( x \in \partial \Omega \) and \( \partial \Omega_y \subset \partial \Omega \) is a small segment centered around \( y \in \partial \Omega \) and that both are scaled to have surface area \( 0 < \varepsilon \ll 1 \). We can also assume, because the surface is \( C^1 \) and we are allowed to take \( \varepsilon \) arbitrarily small, that they are approximately given by hyperplanes (and, as above, the error is a lower order term coming from curvature). The quantity to be evaluated,

\[
\int_L n_\ell(\partial \Omega_x)^2 \, d\mu(\ell),
\]

as the likelihood that a ‘random’ line (random as induced by the kinematic measure \( \mu \)) intersects both \( \partial \Omega_x \) and \( \partial \Omega_y \). Appealing to the law of total probability

\[
\mathbb{P}(n_\ell(\partial \Omega_x) n_\ell(\partial \Omega_y) = 1) = \mathbb{P}(n_\ell(\partial \Omega_y) = 1 | n_\ell(\partial \Omega_x) = 1) \cdot \mathbb{P}(n_\ell(\partial \Omega_x) = 1).
\]

The last quantity is easy to evaluate: by Crofton’s formula

\[
\mathbb{P}(n_\ell(\partial \Omega_x) = 1) = \frac{1}{\alpha_n} |\partial \Omega_x| = \frac{\varepsilon}{\alpha_n}.
\]

It remains to compute the second term: the likelihood of a ‘random’ line hitting \( \partial \Omega_y \) provided that it has already hit \( \partial \Omega_x \). For this purpose, we first consider what we can say about random lines that have hit \( \partial \Omega_x \). The distribution of \( \partial \Omega_x \cap \ell \), provided it is not empty, is, to leading order, uniformly distributed over \( \partial \Omega_x \) because \( \partial \Omega_x \)
and let $x$. Theorem in two additional settings leading to the two identities. Let $\Omega$ be convex.

Proof. The proof of the Corollary is using the same computation as the proof of the Theorem in two additional settings leading to the two identities. Let $\Omega$ be convex and let $x \in \partial\Omega$. We start by considering an infinitesimal hyperplane segment $\partial\Omega_x$.

This establishes the inequality with constant $c_n = \frac{1}{2} \int_{S^{n-1}} |\langle w, n \rangle| \, d\sigma(w) = \frac{1}{2} \int_{S^{n-1}} |w_1| \, d\sigma(w)$. 

\[ |\partial\Omega| = \frac{1}{2} \int_L n_\ell(\partial\Omega) \, d\mu(\ell) \leq \frac{\alpha_n}{2} \int_L n_\ell(\partial\Omega)^2 \, d\mu(\ell) = \frac{\alpha_n}{2} \int_L \sum_i n_\ell(\partial\Omega_i) \, d\mu(\ell) + \frac{\alpha_n}{2} \int_L \sum_{i \neq j} n_\ell(\partial\Omega_i)n_\ell(\partial\Omega_j) \, d\mu(\ell) \]

The inequality is an equation if and only if $\Omega$ is convex. As already discussed above, the first term tends to $|\Omega|/2$ as $\varepsilon \to 0$. Thus, for arbitrary $a \in S^{n-1}$,

\[ |\partial\Omega| \leq \lim_{\varepsilon \to 0} \frac{\alpha_n}{2} \int_L \sum_{i \neq j} n_\ell(\partial\Omega_i)n_\ell(\partial\Omega_j) \, d\mu(\ell) = \left( \int_{S^{n-1}} |\langle w, a \rangle| \, d\sigma(w) \right)^{-1} \int_{\partial\Omega \times \partial\Omega} \frac{|\langle n(x), y - x \rangle (y - x, n(y))|}{\|y - x\|^{n+1}} \, d\sigma(x)\,d\sigma(y) \]

This establishes the inequality with constant $c_n = \frac{1}{2} \int_{S^{n-1}} |\langle w, n \rangle| \, d\sigma(w) = \frac{1}{2} \int_{S^{n-1}} |w_1| \, d\sigma(w)$.

3. Proof of the Corollary

Proof. The proof of the Corollary is using the same computation as the proof of the Theorem in two additional settings leading to the two identities. Let $\Omega$ be convex and let $x \in \partial\Omega$. We start by considering an infinitesimal hyperplane segment $\partial\Omega_x$. 

Plugging in the definition of $\Psi$ this simplifies to

\[ P = \frac{2 \langle n(x), \phi \rangle}{\int_{S^{n-1}} |\langle w, n(x) \rangle| \, d\sigma(w)} \]

where the factor 2 comes from the fact that each line creates two directions of intersections. This allows us to perform a change of measure: we may assume that the lines are oriented uniformly at random provided that we later weigh the end result by $\Psi$. If the lines are oriented in all directions uniformly, then it is easy to see the relative likelihood is then proportional to the size of the projection of $\partial\Omega_y$ onto the sphere of radius $\|x - y\|$ centered at $x$. The projection shrinks the area by a factor of $\frac{\|n(y) - n(x)\|^2}{\|x - y\|^n}$. The relative likelihood is then proportional to

\[ P = \Psi \left( \frac{x - y}{\|x - y\|} \right) \left| \langle n(y), \frac{x - y}{\|x - y\|} \rangle \right| \frac{\varepsilon}{\|x - y\|^{n-1}}. \]
centered around $x$. By convexity of $\Omega$, almost all lines intersecting $\partial \Omega_x$ will intersect $\partial \Omega$ in exactly one other point. This implies, as the size of $\partial \Omega_x$ tends to 0, that

$$\int_L n_\ell(\partial \Omega_x)n_\ell(\partial \Omega \setminus \partial \Omega_x)d\mu(\ell) = (1 + o(1)) \cdot \mu(\partial \Omega_x).$$

At the same time, by Crofton’s formula, the likelihood of a line hitting $\partial \Omega_x$ is only a function of the surface area of $\partial \Omega_x$ and independent of everything else. Finally, using linearity, we can decompose $\partial \Omega \setminus \partial \Omega_x$ into small hyperplane segments and use the computation above to deduce that

$$\int_{\partial \Omega} \frac{|\langle n(x), y-x \rangle \langle y-x, n(y) \rangle|}{\|x-y\|^{n+1}} d\sigma(y) = \text{const}.$$  

 Integrating once more and applying the Theorem immediately implies that the constant has to be $c_n$. As for the second part, we can consider an infinitesimal hyperplane segment $H_x$ centered at $x \in \Omega \setminus \partial \Omega$ with normal direction given by $w \in S^{n-1}$. Every line hitting $H_x$ intersects $\partial \Omega$ in exactly two points and thus

$$\int_L n_\ell(H_x)n_\ell(\partial \Omega)d\mu(\ell) = 2 \cdot \mu(H_x).$$

By Crofton’s formula, the right-hand side does not depend on the shape or location of $H_x$ and is only a function of the surface area of the infinitesimal segment. As for the left-hand side, using the computation done in the proof of the Theorem shows

$$\int_L n_\ell(H_x)n_\ell(\partial \Omega)d\mu(\ell) = (1 + o(1)) \int_{\partial \Omega} \frac{|\langle w, y-x \rangle \langle y-x, n(y) \rangle|}{\|x-y\|^{n+1}} d\sigma(y)$$

where the error term is with respect to the diameter of $H_x$ shrinking to 0. □  

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[2] L. Santaló, Integral Geometry and Geometric Probability, Cambridge University Press, 2004.