Perfect porcupines: ideal networks for low frequency gravitational wave astronomy

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Perfect porcupines are specially-configured networks of gravitational wave detectors, in the limit that the individual detectors and the distances between them are short relative to the gravitational wavelengths of interest. They have beautiful properties which make them ideal gravitational wave telescopes. I present the most important cases explicitly. For a network of one-arm detectors (like “AGIS” [1]), the minimal perfect porcupine has 6 detectors, oriented along the 6 diameters of a regular icosahedron. For a network of two-arm detectors (like the equal-arm Michelson interferometers LIGO/VIRGO) the minimal perfect porcupine is a certain 5 detector configuration.

INTRODUCTION

In this decade, we hope and expect that gravitational waves will be directly detected for the first time, and a new field will be born: gravitational wave astronomy. Attention will shift, from individual gravitational wave detectors, to neworks of multiple detectors which function together as a gravitational wave telescope. (For an introduction to gravitational wave detector networks, see [2] and references therein.) Here I consider the regime in which (by choice or necessity) the individual detectors and the distances between them are short relative to the gravitational wavelengths of interest, so that the arms of the various detectors in the network may be thought of as emanating from nearly the same point in space, like the famed quills of a frightened porcupine. In this limit, the relative positions of the detectors are unimportant, and it is their relative orientations that matter. For special configurations (particular choices for the number of detectors and their orientations), such networks exhibit beautiful properties that make them, in certain respects, ideal astronomical instruments. These “perfect porcupines” are the subject of this Letter.

Motivation comes from recent developments in gravitational wave detection. The ground based LIGO/VIRGO detectors [3] have successfully reached their (“initial” or “stage I”) design sensitivity and will be upgraded over the next several years to a higher (“advanced” or “stage II”) sensitivity which will likely allow them to detect gravitational waves. But the initial LIGO/VIRGO experiments are only sensitive to high frequency (30 Hz \( \lesssim f \lesssim 10^3 \) Hz) radiation; they are analogous to gamma ray telescopes, in that their expected signal consists of occasional brief bursts which represent only the high frequency tip of the astrophysical gravitational wave spectrum. It would be tremendously valuable to have complementary detectors (analogous to optical or radio telescopes) to study the rich array of sources and physical effects at lower frequencies (\( f \lesssim 10 \) Hz). To this end, several space missions have been proposed; but, unfortunately, there is uncertainty about when and whether the first generation mission (LISA [4], optimized near \( f \sim 10^{-3} \) Hz) will launch, and even greater uncertainty surrounding the proposed second generation missions (BBO [5] and DECIGO [6], optimized near \( f \sim 0.3 \) Hz). Thus, it makes sense to give some consideration to the possibility of pursuing low frequency (\( f \lesssim 10 \) Hz) gravitational wave astronomy from the ground – as best we can, and hopefully in parallel with the space missions mentioned above. But then we quickly find ourselves in the “porcupine” regime, because frequencies \( f \lesssim 50 \) Hz correspond to wavelengths longer than the radius of the Earth!

Recent developments have invigorated the possibility of gravitational wave astronomy at low frequencies (\( f \lesssim 10 \) Hz) from the ground. In particular, there are ideas for how to design detectors that are less susceptible to the two noise sources (seismic noise and gravity gradient noise) that limit LIGO/VIRGO at low frequencies. One idea is to operate underground, where seismic noise and gravity gradient noise are intrinsically weaker and, moreover, may be more effectively monitored (with seismometers) and removed [7]. A second idea is gravitational wave detection using atom interferometry [1, 8, 9], which may be more immune to seismic noise, and has many possible avenues for future development. In the Discussion, I suggest another idea which may be relevant to gravity gradient noise in certain contexts.

PROPERTIES OF PERFECT PORCUPINES

In this section I define perfect porcupines and highlight some of their good features. In the next two sections, I present the most important perfect porcupines explicitly.

First let me fix notation. My fourier conventions are:

\[
g(t) = \int_{-\infty}^{+\infty} df \ e^{-2\pi if t} \hat{g}(f), \quad \hat{g}(f) = \int_{-\infty}^{+\infty} dt \ e^{2\pi if t} g(t). \quad (1)
\]

I work in transverse-traceless (TT) gauge [10]. The lower case latin indices \( \{i, j\} \) label the 3 spatial directions: \( i, j = 1, 2, 3 \). The upper case latin indices \( \{A, B\} \) label the 2 gravitational wave polarizations: \( A, B = 1, 2 \). The lower case greek indices \( \{\alpha, \beta\} \) label the \( N \) detectors...
in the network: $\alpha, \beta = 1, \ldots, N$. I use the Einstein summation convention: repeated indices (one upper and one lower) are to be summed. Hats denote unit vectors.

When gravitational waves reach us from a distant astronomical source, they appear as a gravitational plane wave travelling in the $\hat{n}$ direction

$$h_{ij}(\tau) = \sum_{A=1,2} h_A(\tau) P_{ij}^A(\hat{n}),$$  \hspace{1cm} (2)

where the two polarization waveforms $h_A(\tau)$ are arbitrary functions of $\tau = t - \hat{n} \cdot \vec{x}$, and the polarization tensors $P_{ij}^A(\hat{n})$ form an orthonormal basis on the 2-dimensional space of symmetric, transverse, traceless $3 \times 3$ matrices:

\begin{align}
P_{ij}^A(\hat{n}) - P_{ji}^A(\hat{n}) &= 0 \hspace{1cm} (3a) \\
\hat{n}^i P_{ij}^A(\hat{n}) &= 0 \hspace{1cm} (3b) \\
\delta_{ij} P_{ij}^A(\hat{n}) &= 0 \hspace{1cm} (3c) \\
[P_{ij}^A(\hat{n})]^t P_{ij}^B(\hat{n}) &= \delta_{AB}. \hspace{1cm} (3d)
\end{align}

The output $s_\alpha(t)$ of detector $\alpha$ has two parts, gravitational wave signal $h_\alpha(t)$ and noise $n_\alpha(t)$:

$$s_\alpha(t) = h_\alpha(t) + n_\alpha(t).$$  \hspace{1cm} (4)

Let us first consider $h_\alpha(t)$ and the information it contains. In the porcupine limit, all of the detectors in the network live at essentially the same spatial location ($\vec{x} \approx \hat{0}$), and hence only measure $h_{ij}(t, \vec{x})$ at that point: $h_{ij}(t, \hat{0})$. The gravitational wave signal is

$$h_\alpha(\tau) = \int_{-\infty}^{+\infty} dT W_\alpha(T) h_{ij}(\tau - T) A_{ij}^\alpha,$$  \hspace{1cm} (5)

where the window function $W_\alpha(T)$ describes the temporal response of detector $\alpha$, and the “antenna pattern” $A_{ij}^\alpha$ [normalized as $(A_{ij}^\alpha)^* A_{ij}^\beta = 1$] is a projector describing its spatial orientation. Since each detector $\alpha$ is sensitive to a particular linear combination $A_{ij}^\alpha h_{ij}$ of the components of $h_{ij}$, the collection of detectors in the network measure linearly independent combinations if and only if the determinant of the $N \times N$ gramian matrix of the antenna projectors is non-zero:

$$\text{Det}[A_{ij}^\alpha A_{ij}^\beta] \neq 0.$$  \hspace{1cm} (6)

We can have up to 6 linearly independent antenna projectors, and in this maximal case the network measures all 6 components of $h_{ij}(t, \hat{0})$. Then, even though these measurements are confined to $\vec{x} = \hat{0}$, they are enough to determine the properties of the plane wave as follows. First, the propagation direction $\hat{n}$ is the zero eigenvector of $h_{ij}(t, \hat{0})$. (Actually, this only determines $\hat{n}$ up to a sign, since $-\hat{n}$ is also a zero eigenvector.) Then, by projecting $h_{ij}(t, \hat{0})$ onto the polarization tensors $P_{ij}^A(\hat{n})$, we obtain the two polarization waveforms: $h_A(t) = [P_{ij}^A(\hat{n})]^* h_{ij}(t, \hat{0})$. We started with 6 functions of time $h_{ij}(t, \hat{0})$; so far we have used these to completely determine the properties of the gravitational plane wave [two functions $h_A(t)$ and two angles to specify $\hat{n}$]; and the remaining information may now be used to perform the following cross checks that we are really observing a standard gravitational plane wave:

\begin{align}
d\hat{n}/dt &= 0 \hspace{1cm} (7a) \\
\text{Det}[h_{ij}(t, \hat{0})] &= 0 \hspace{1cm} (7b) \\
\text{Tr}[h_{ij}(t, \hat{0})] &= 0. \hspace{1cm} (7c)
\end{align}

Now let us turn to the noise $n_\alpha(t)$, which we model as stationary and gaussian, with zero mean, so that it is characterized by its correlation function $C_{\alpha\beta}(T)$ or, equivalently, its spectral density $S_{\alpha\beta}(f) = C_{\alpha\beta}(f)$:

$$C_{\alpha\beta}(T) = n_\alpha(t + T) n_\beta(t) \hspace{1cm} (8a)$$

$$S_{\alpha\beta}(f) = \delta(f - f') S_{\alpha\beta}(f') \hspace{1cm} (8b)$$

$S_{\alpha\beta}(f)$ induces a natural inner product on the space of signals (or noise) in the network:

\begin{equation}
(g^{(1)}|g^{(2)}) = \int_{-\infty}^{+\infty} df \tilde{g}_{\alpha}^{(1)*}(f) [S^{-1}(f)]^{\alpha\beta} \tilde{g}_{\beta}^{(2)}(f). \hspace{1cm} (9)
\end{equation}

A particular noise fluctuation has probability proportional to $\exp[-(n/n)^2/2]$ and the expected signal-to-noise ratio (SNR) of a gravitational wave is $(h|h)^{1/2}$. If a gravitational wave signal (which depends on various parameters $\theta_k$) is detected, and the likelihood function may be approximated as a gaussian $\propto \exp[-(1/2)\theta_k \Gamma^{kl} \theta_l]$ near its peak, then the expected inverse covariance matrix is

$$\Gamma^{kl} = \left( \frac{\partial h}{\partial \theta_k} \frac{\partial h}{\partial \theta_l} \right).$$  \hspace{1cm} (10)

Consider a network of identical uncorrelated detectors: $W_\alpha(T) = W(T)$ and $S_{\alpha\beta}(f) = S(f) \delta_{\alpha\beta}$. Such a network is a “perfect porcupine” if $(h|h)$ simplifies to the form

$$\langle h|h \rangle = C \int_{-\infty}^{+\infty} df \frac{|\tilde{W}(f)|^2}{S(f)} \sum_{A=1}^{2} \left| \tilde{h}_A(f) \right|^2$$  \hspace{1cm} (11)

where $C$ is a constant. This says that the network’s gravitational wave sensitivity is independent of the direction or polarization of the wave.

Perfect porcupines also have other nice properties. First, as we shall see, they: (i) determine the propagation direction $\hat{n}$ and both polarization waveforms $h_A(t)$ of a gravitational plane wave, as described above; and (ii) permit the systematic checks (7a-7c). In addition, perfect porcupines built from one-arm detectors (like AGIS) also permit the other systematic check (6). We return to this point in the Discussion. Next recall that $h_\alpha(t)$ depends on two sorts of parameters: (i) the two angles
in \( \hat{n} \), which we denote by the labels \( \mu \) and \( \nu \); and (ii) all other parameters (such as the masses and spins and inclinations in an inspiraling binary black hole) which we denote by the label \( \sigma \). With this notation, if we choose angular coordinates that are “nice” near the point \( \hat{n} \), in the sense that they run along two perpendicular great circles through \( \hat{n} \) (like ordinary polar coordinates \( \theta \) and \( \phi \) at the equator), then we find:

\[
\Gamma^{\mu\nu} = (h|\hat{h})\delta^{\mu\nu} = \text{SNR}^2 \delta^{\mu\nu} \quad (12a)
\]

\[
\Gamma^{\mu\sigma} = 0. \quad (12b)
\]

This says that the expected uncertainties in the two angular coordinates of the source are equal to each other, mutually uncorrelated, and independent of \( \hat{n} \); and also that they are uncorrelated with the uncertainties in all of the other parameters characterizing the source. Furthermore, the perfect porcupine’s angular resolution \( \delta \theta = 1/\text{SNR} \) should be compared with the angular resolution from triangulation: \( \delta \theta \sim (\lambda/L)(1/\text{SNR}) \), where \( \lambda \) is the gravitational wavelength and \( L \) is the distance between the detectors in the network. Thus, in the (porcupine) regime where \( L \) is short relative to \( \lambda \), the angular resolution of a perfect porcupine is parametrically better than the angular resolution from triangulation.

**NETWORKS OF ONE-ARM DETECTORS**

One-arm detectors (such as AGIS [1]) have antenna projectors of the form:

\[
A_{ij}^\alpha = \hat{m}_i^\alpha \hat{m}_j^\alpha. \quad (13)
\]

The minimal perfect porcupine built from such detectors has 6 arms, oriented along the 6 directions connecting opposite vertices of a regular icosahedron (or, equivalently, the 6 directions connecting opposite faces of a regular dodecahedron):

\[
\hat{m}^\alpha = \begin{cases}
\{0,0,1\} & (\alpha = 0) \\
\sqrt{4/5} \{\cos \frac{2\pi \alpha}{5}, \sin \frac{2\pi \alpha}{5}, \frac{1}{\sqrt{5}}\} & (\alpha = 1, \ldots, 5)
\end{cases} \quad (14)
\]

This network has \( C = 4/5 \). These 6 detectors are independent in the sense of (8); so in addition to determining the direction of a gravitational plane wave \( \hat{n} \), and both polarization waveforms \( h_A(t) \), they also permit the three cross checks (13) to be performed.

A larger perfect porcupine has 10 detectors, oriented along the 10 directions connecting opposite vertices of a regular dodecahedron (or, equivalently, the 10 directions connecting opposite faces of an icosahedron):

\[
\hat{m}^\alpha_{25} = \frac{2\varphi^{1/2}}{\sqrt{3\sqrt{5}}} \left\{\cos \frac{2\pi \alpha}{5}, \sin \frac{2\pi \alpha}{5}, \frac{1 + \varphi^{1/2}}{2}\right\} \quad (15)
\]

where \( \alpha = 1, \ldots, 5 \), \( \varphi = (1 + \sqrt{5})/2 \) is the golden ratio, and \( C = 4/3 \). Of course, these 10 detectors are not independent in the sense of Eq. (8): they measure all 6 independent components of \( h_{ij} \), but do so redundantly. All else being equal, this redundancy makes the 10-arm perfect porcupine better (but also more expensive) than its 6-arm counterpart: the redundancy allows us to cross correlate two noisy data streams which contain the same gravitational wave signal, but different and uncorrelated noise, and thereby extract the true signal better than we could from either data stream individually.

An even larger and more redundant perfect porcupine consists of 15 detectors, oriented along the 15 directions connecting opposite edges of a regular icosahedron (or, equivalently, the 15 directions connecting opposite edges of a regular dodecahedron). This has \( C = 2 \).

If one relaxes the requirement that all of the detectors be identical, more options become available. For example, for \( k \geq 5 \), the configuration

\[
\hat{m}^\alpha = \begin{cases}
\{0,0,1\} & (\alpha = 0) \\
\sqrt{4/5} \{\cos \frac{2\pi \alpha}{5}, \sin \frac{2\pi \alpha}{5}, \frac{1}{\sqrt{5}}\} & (\alpha = 1, \ldots, 5)
\end{cases} \quad (16)
\]

is a perfect porcupine with \( C = 4k/25 \). The \( k = 5 \) case is just the minimal perfect porcupine (14).

**NETWORKS OF TWO-ARM DETECTORS**

Two-arm detectors (Michelson interferometers like LIGO/VIRGO, with equal and orthogonal arms) have antenna projectors of the form

\[
P_{ij}^\alpha = \frac{1}{\sqrt{2}} (\hat{p}_i^\alpha \hat{p}_j^\alpha - \hat{q}_i^\alpha \hat{q}_j^\alpha). \quad (18)
\]

If we define \( \kappa = \frac{1}{2} \arccos \sqrt{3/5} \) and

\[
\hat{a}^\alpha = \{ -\sin \frac{2\pi \alpha}{5}, \cos \frac{2\pi \alpha}{5}, 0 \} \\
\hat{b}^\alpha = \{ -\cos \frac{2\pi \alpha}{5}, -\sin \frac{2\pi \alpha}{5}, \sqrt{2} \} \sqrt{1/3}
\]

then, for \( \alpha = 1, \ldots, N \) and \( N \geq 5 \), the configuration

\[
\hat{p}^\alpha = +\cos \kappa \hat{a}^\alpha + \sin \kappa \hat{b}^\alpha \\
\hat{q}^\alpha = -\sin \kappa \hat{a}^\alpha + \cos \kappa \hat{b}^\alpha
\]

is a perfect porcupine with \( C = N/5 \). This network measures the 5 traceless components of \( h_{ij} \) (with greater redundancy when \( N \) is larger). It therefore determines the direction \( \hat{n} \) of a gravitational plane wave, and both polarization waveforms \( h_A(t) \); and it permits the first two cross checks (16) but not the third (13). When \( N = 5 \) this is the minimal perfect porcupine.

Another perfect porcupine is worth mentioning. Consider the 15 directions connecting opposite edges of a
regular icosahedron (or, equivalently, the 15 directions connecting opposite edges of a regular dodecahedron). These 15 directions separate into 5 orthonormal triads. From each orthonormal triad, we can select 3 different orthonormal pairs \{\hat{p}_\alpha, \hat{q}_\alpha\}. In this way, we obtain a perfect porcupine with 15 detectors, and \( C = 3 \).

**DISCUSSION**

We have seen that a perfect porcupine built from one-arm detectors will monitor \( \text{Tr}[h_{ij}] \). As mentioned above, the vanishing of this channel is a check that one is observing standard gravitational waves, as opposed to noise, or something more exotic. Alternatives to general relativity often give the graviton a zero-helicity component; and zero-helicity gravitational waves would show up as fluctuations in \( \text{Tr}[h_{ij}] \). Also, as one goes underground, and to lower frequencies, the coherence length of the fluctuations in the Newtonian gravitational potential becomes longer: if the various detectors in a perfect porcupine can be placed sufficiently close together, so that they all see the same (or similar) fluctuations in the Newtonian potential, then these fluctuations will also look like fluctuations in \( \text{Tr}[h_{ij}] \), and monitoring this channel may even be helpful in subtracting gravity gradient noise. On the other hand, if it is necessary, e.g., to build the detectors parallel to Earth’s local gravitational field (so that the various detectors in the porcupine must be located at widely separated points on the Earth), then this method for removing gravity gradient noise won’t work.

I have focused on a perfect porcupine’s ability to measure a single plane wave; but this analysis also applies to a sum of many plane waves, as long as they are separable in the time-frequency (or template) domain. For example, two plane waves of different frequencies are not a problem (they may be cleanly separated in frequency space, and then handled independently); but two plane waves moving in different directions at the same frequency would be a problem: the cross check \[ \text{Tr} \] and the porcupine’s direction-finding algorithm would fail. Fortunately, although source “blending” (non-separability) does occur for LISA sources when \( f \lesssim 10^{-3} \) Hz, it is unlikely that a ground-based perfect porcupine would reach sufficiently low frequencies and sufficiently good sensitivities for this to be a practical concern.

An earth-bound porcupine must contend with (in order of importance) the spin of the Earth, the Moon’s orbit around the Earth, and the Earth’s orbit around the Sun; but above we considered perfect porcupines which move in straight line, without rotation. As long as we are looking at gravitational waves with periods much shorter than a day (which is the realistic case), then this is a good starting point, for the same reason that for many purposes LIGO/VIRGO may be modeled as moving through flat space: over timescales containing many wave cycles, the detectors are moving along a nearly straight non-rotating trajectory. Nevertheless, it is important to extend the above analysis to include the rotation of the Earth, etc., especially for the purposes of studying sources that are detectable over timescales longer than a day. It will also probably be best to align one of the perfect porcupine’s symmetry axes with the Earth’s rotation axis, if possible.

Finally, the considerations in this paper were motivated by recent developments (outlined in the Introduction), but I should also mention a more futuristic possibility: we may eventually be led to build low-frequency gravitational wave detectors on the Moon, where the seismic and gravity gradient noise levels are much lower than on Earth. This would again lead us to porcupines.

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