The Kodaira-Spencer map for minimal toric hypersurfaces

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Abstract
In this article we study infinitesimal deformations of toric hypersurfaces. We introduce a Kodaira-Spencer map and compute its kernel. By introducing some new Laurent polynomials we make our computation as explicit as possible. This widely generalizes results of Griffiths for projective hypersurfaces.

1 Introduction
We start with an \( n \)-dimensional lattice polytope \( \Delta \) and a nondegenerate Laurent polynomial \( f \) with Newton polytope \( \Delta \) and set
\[
Z_f := \{ f = 0 \} \subset T.
\]
By results from \((\text{Bat22})\) there is (under a mild condition \( F(\Delta) \neq \emptyset \)) a projective toric variety \( \mathbb{P} \) to a simplicial fan \( \Sigma \), such that the closure \( Y_f \) of \( Z_f \) in \( \mathbb{P} \) has at most terminal singularities and the canonical divisor \( K_{Y_f} \) is nef. We say that \( Y_f \) is a minimal model of \( Z_f \).

The toric variety \( \mathbb{P} \) does not depend on \( f \). Therefore we vary \( f \) over all such polynomials, write \( f \in B := B(\Delta) \) to obtain a family of minimal toric hypersurfaces
\[
\mathcal{X} := \{ (x, f) \in \mathbb{P} \times B \mid x \in Y_f \}^{pr2} B.
\]
Next to a tangent vector
\[
\text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2) \to B
\]
at \( f \in B \) we associate a first-order infinitesimal deformation of \( Y_f \) (in \( X \)) by base change along \( v \). These infinitesimal deformations and all those in \( \mathbb{P} \) are parametrized by Kodaira Spencer maps

\[
\kappa_{\mathbb{P},f} : H^0(Y, N_{Y/\mathbb{P}}) \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y), \\
\kappa_f : H^0(Y, N_{Y/X}) \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y).
\]

In fact \( H^0(Y, N_{Y/X}) \subset H^0(Y, N_{Y/P}) \) and \( \kappa_f \) equals the restriction of \( \kappa_{\mathbb{P},f} \) (see section 9). In Theorem 6.1 we identify the kernel of \( \kappa_{\mathbb{P},f} \) with the Lie algebra of the automorphism group of \( \mathbb{P} \):

**Theorem 1.1.** Let \( \Delta \) be an \( n \)-dimensional lattice polytope, where \( n \geq 2 \), with \( l^* (\Delta) > 0 \). If \( n = 2 \) assume that \( l^* (\Delta) \geq 2 \). Then given \( f \in B \)

\[
\ker(\kappa_{\mathbb{P},f}) \cong \text{Lie Aut}(\mathbb{P})
\]

where \( l^* (\Delta) \) denotes the number of interior lattice points of \( \Delta \). The proof is cohomological. Concerning the cokernel of \( \kappa_{\mathbb{P},f} \) we prove:

**Corollary 1.2.** Given the conditions of the theorem if \( n \geq 4 \) then the following sequence is exact

\[
0 \to \text{Im}(\kappa_{\mathbb{P},f}) \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \to \text{Ext}^1_{\mathcal{O}_P}(\Omega^1_Y, \mathcal{O}_P) \to 0
\]

that is all infinitesimal deformations of \( Y_f \) arise from \( \text{Im}(\kappa_{\mathbb{P},f}) \) or from infinitesimal deformations of \( \mathbb{P} \).

A more detailed description of the second term is a topic of current research (see [ITu18], [Pet20]) and ([Mav03]) for the case that \( Y_f \) is Calabi-Yau.

To be more explicit on \( \ker(\kappa_{\mathbb{P},f}) \) we recall the definition of the **canonical closure** \( C(\Delta) \) of \( \Delta \), a polytope that might be slightly larger than \( \Delta \). Given \( P \subset M_{\mathbb{R}} \) let \( L(P) \) be the \( \mathbb{C} \)-vector space with basis \( x^m \), \( m \in P \cap M \). Then

\[
H^0(Y, N_{Y/X}) \cong L(\Delta)/\mathbb{C} \cdot f \\
H^0(Y, N_{Y/P}) \cong L(C(\Delta))/\mathbb{C} \cdot f.
\]

We apply results from ([BG99]) to identify a basis of \( \ker(\kappa_{\mathbb{P},f}) \) explicitly with certain Laurent polynomials having support on \( C(\Delta) \) (Corollary 8.4):

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Corollary 1.3. Given the conditions of Theorem 1.1, let

\[ f = \sum_{m \in M \cap \Delta} a_m x^m \in B \]

Then

\[ \text{ker}(\kappa_{P,f}) \cong \left\langle x_i \cdot \frac{\partial f}{\partial x_i} \mid i = 1, \ldots, n \right\rangle \oplus \bigoplus_{\alpha \in R(N, \Sigma)} w_{-\alpha}(f) \cdot \mathbb{C} \]

where \( R(N, \Sigma) \subset M \) denote the (Demazure) roots of \( \Sigma \) and \( w_{-\alpha}(f) \) denote certain Laurent polynomials. Concerning the kernel of \( \kappa_f \) we prove

Theorem 1.4. Given the conditions of Theorem 1.1 then

\[ \text{ker}(\kappa_f) \cong \left\langle x_i \cdot \frac{\partial f}{\partial x_i} \mid i = 1, \ldots, n \right\rangle \oplus \bigoplus_{\alpha \in R(N, \Sigma \Delta)} w_{-\alpha}(f) \cdot \mathbb{C} \]

Application: Let \( n = 3 \). Then \( Y \) is smooth and thus

\[ \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \cong H^1(Y, T_Y). \] (2)

There is a period map

\[ \mathcal{P}_{B,f} : f \mapsto [H^{2,0}(Y_f)]. \]

This map is holomorphic and by a result of Griffiths the differential \( d\mathcal{P}_{B,f} \) factors as

\[ \frac{L(C(\Delta))}{\mathbb{C}} \cdot f \xrightarrow{\kappa_{P,f}} H^1(Y, T_Y) \xrightarrow{\Phi_f} \text{Hom}(H^{2,0}(Y_f), H^{1,1}(Y_f)) \]

The infinitesimal Torelli theorem for \( Y_f \) asks if \( \Phi_f \) is injective. If we know \( \text{ker}(d\mathcal{P}_{B,f}) \) we may check if

\[ \text{ker}(d\mathcal{P}_{B,f}) \supseteq \text{ker}(\kappa_f) \] (3)

is an equality, in which case we get

\[ \text{ker}(\Phi_f|_{\text{Im} \kappa_{Y,f}}) = 0 \]
giving a partial result towards the ITT. A calculation of \( \ker(dP_{B,f}) \) should work with jacobian rings. In the dissertation (\cite{Gie23}) we worked this out for \( n = 3 \). In fact these methods should also apply to higher dimensions.

One might be also interested in subfamilies of \( X \to B \): Given \( B^0 \subseteq B \) with restriction \( \kappa_f^0 \) of \( \kappa_f \) it is obvious that

\[
x_1 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_n \cdot \frac{\partial f}{\partial x_n} \in \ker(\kappa_f^0)
\]

but hard to say when \( w_\alpha(f) \in \ker(\kappa_f^0) \). We show that if \( B^0 \) equals the set of vertices of \( \Delta \) then under some additional condition on the polytope \( \Delta \) none of the \( w_\alpha(f) \)'s contribute to \( \ker(\kappa_f) \) (see Lemma \ref{lemma:10.3}).

What is known in the direction of our results: For \( Y_f \subset \mathbb{P}^n \) a smooth degree \( d \) hypersurface by (\cite[Ch.6]{Voi03})

\[
\ker(\kappa_f) \cong \ker(\kappa_{\mathbb{P},f}) \cong J_{d,f,\text{griff}}^d,
\]

where \( J_{d,f,\text{griff}}^d \) denotes the \( d \)-th homogeneous component of

\[
J_{f,\text{griff}} := \left( \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right) \subseteq \mathbb{C}[x_0, \ldots, x_n].
\]

In Example \ref{example:8.3} we show that if \( d \geq n + 1 \) then our results specialize to this assertion. There are generalizations of this results to quasismooth hypersurfaces in weighted projective spaces (\cite[Thm.4.3.2]{Dol82}). In the dissertation (\cite{Koe91}) it is dealt with families of nondegenerate curves in toric surfaces. In (\cite{Mav03}) both the kernel and the cokernel of the Kodaira-Spencer map is dealt with for anticanonical hypersurfaces and in (\cite{Ok87}) the Kodaira-Spencer map has been studied for \( \Delta \) an \( n \)-simplex.

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## 2 Notation and Background

Let \( M \) and \( N \) be dual lattices and \( T = N \otimes \mathbb{Z} \mathbb{C}^* \cong (\mathbb{C}^*)^n \) the \( n \)-dimensional torus. We fix an \( n \)-dimensional lattice polytope \( \Delta \subset M_{\mathbb{R}} \) and denote the
normal fan of a lattice (or more generally rational) polytope $F$ by $\Sigma_F$. We denote the projective toric variety defined via $\Sigma$ by $P_F$. $\Sigma[1]$ denote the rays of the fan $\Sigma$ and $D_i$ (or $D_\nu$) the toric divisor to the ray $\nu_i$ (or $\nu$).

Throughout this article we assume $f$ to be a Laurent polynomial with Newton polytope $\Delta$, written

$$f = \sum_{m \in M \cap \Delta} a_m x^m.$$

Set $Z_F := \{f = 0\} \subset T$ and given another $n$-dimensional rational polytope $F$ let $Z_{F,f}$ or just $Z_F$ be the closure of $Z_f$ in $\mathbb{P}_F$. We repeat some results and notions introduced in ([Bat22]):

**Remark 2.1.** The linear equivalence class of $Z_{F,f}$ just depends on $\Delta$. More precisely let

$$\operatorname{Min}_\Delta(\nu) := \min_{m \in \Delta} \langle m, \nu \rangle$$

for $\nu \in N \setminus \{0\}$. Then

$$Z_{F,f} \sim_{\text{lin}} - \sum_{\nu \in \Sigma_F[1]} \operatorname{Min}_\Delta(\nu) \cdot D_\nu$$

([Bat22 Prop.7.1]). Therefore we usually omit $f$ from the notation.

The nondegeneracy of $f$, written $f \in U_{\text{reg}}(\Delta)$, means that $Z_f$ is smooth and for every $k$-dimensional face $\Gamma$ of $\Delta$

$$f|_\Gamma, \frac{\partial f|_\Gamma}{\partial x_1}, \ldots, \frac{\partial f|_\Gamma}{\partial x_n},$$

(4)

where $f|_\Gamma := \sum_{m \in M \cap \Gamma} a_m x^m$, have no common zero in the torus orbit $(\mathbb{C}^*)^k \subset \mathbb{P}_\Delta$ corresponding to $\Gamma$. Write

$$\Delta = \{x \in M_\mathbb{R} \mid \langle x, \nu \rangle \geq \operatorname{Min}_\Delta(\nu) \quad \forall \nu \in N \setminus \{0\}\}. \quad (5)$$

we consider the *Fine interior*

$$F(\Delta) := \{x \in M_\mathbb{R} \mid \langle x, \nu \rangle \geq \operatorname{Min}_\Delta(\nu) + 1 \quad \forall \nu \in N \setminus \{0\}\}.$$

In general $F(\Delta)$ is a rational polytope contained in $\Delta$ and contains all interior lattice points of $\Delta$. It is constructed by moving every hyperplane touching $\Delta$ one step into the interior of $\Delta$. Let

$$S_F(\Delta) := \{\nu \in N \setminus \{0\} \mid \operatorname{Min}_{F(\Delta)}(\nu) = \operatorname{Min}_\Delta(\nu) + 1\}$$
denote the support of $F(\Delta)$ to $\Delta$ (those hyperplanes that touch $\Delta$ and moved by one touch $F(\Delta)$).

![Figure 1: Figure: On the left a polygon $\Delta$, on the right the rays of $\Sigma_\Delta$. $F(\Delta)$ equals the interior lattice point of $\Delta$. The illustrations shows that $(0,-1) \in S_F(\Delta)$. In general the support vectors $S_F(\Delta)$ are contained in the convex span of the rays $\Sigma_\Delta[1]$ (see [Bat22, Prop.3.11])](image)

**Definition 2.2.** ([Bat22, Def.3.13])

Let $\Delta$ be a lattice polytope with presentation as in (5) and with $F(\Delta) \neq \emptyset$. The polytope

$$C(\Delta) := \{x \in M_\mathbb{R} | \langle x, \nu \rangle \geq \min_{\Delta}(\nu) \ \forall \nu \in S_F(\Delta)\}$$

is called the canonical closure of $\Delta$.

**Remark 2.3.** It is clear from the definition that $C(\Delta)$ is a rational polytope and contains $\Delta$. The operators of taking the Fine interior $F$, the canonical closure $C$ and the support $S_F$ could be defined analogously for rational polytopes.

**Proof.** The first formula follows from the exact sequence (9), the vanishing $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = 0$ due to Demazure and formula (6). The second follows from Remark 3.10. □

**Theorem 2.4.** ([Bat22, Thm.8.2])

Assume $F(\Delta) \neq \emptyset$. Then there is a complete simplicial fan $\Sigma$ with $\Sigma[1] = S_F(\Delta)$ and with associated toric variety $\mathbb{P}$ such that with $Y = Y_f$ the closure of $Z_f$ in $\mathbb{P}$

- $\mathbb{P}$ has $\mathbb{Q}$-factorial terminal singularities
- the adjoint divisor $K_\mathbb{P} + Y$ is nef.
Corollary 2.5. ([Bat22, Thm.8.2])
$Y \subset \mathbb{P}$ has at most terminal singularities and $K_Y$ is nef. We say that $Y = Y_f$ is a minimal model of $Z_f$.

Proposition 2.6. ([Bat22, Prop.7.4])
$Y \subset \mathbb{P}$ defines a nef and big $\mathbb{Q}$-Cartier divisor.

Lemma 2.7. Let $F$ be a torus orbit of $\mathbb{P}$. If
$$\emptyset \neq F \cap Y \subsetneq F$$
then $Y$ intersects $F$ transversely in a subset of codimension 1.

Proof. Choose a common refinement $\tilde{\Sigma}$ of $\Sigma$ and $\Sigma_\Delta$ in $N$ such that the toric variety $\tilde{\mathbb{P}}$ to $\tilde{\Sigma}$ is smooth. Let $\tilde{Y}$ denote the closure of $Z_f$ in $\tilde{\mathbb{P}}$. Then for $\tilde{F} \subset \tilde{\mathbb{P}}$ a torus orbit either $\tilde{Y}$ is disjoint from $\tilde{F}$ (if $\tilde{F}$ contracts to a torus fixed point on $\mathbb{P}_\Delta$) or intersects $\tilde{F}$ transversely in a subset of codimension 1 ([Bat94, Prop.3.2.1]).

There is a birational toric morphism $\tilde{\mathbb{P}} \to \mathbb{P}$ inducing a birational morphism $\tilde{Y} \to Y$. Assume that $\tilde{F}$ contracts onto $F$. If $\tilde{Y} \cap \tilde{F} \neq \emptyset$ then $Y \cap F \neq \emptyset$ and either $Y$ contains $F$ or intersects $F$ transversely in a subset of codimension 1.

Remark 2.8. The singular loci both of $\mathbb{P}$ and $Y$ are of codimension $\geq 3$ since they are terminal. It will be important to choose smooth open subset $V \subset \mathbb{P}$ and $U \subset Y$ such that $U = V \cap Y$ and $V$ is toric. This works if we define $V$ as the union of all torus orbits of dimension $\geq n - 2$.

Remark 2.9. To a toric divisor
$$D = \sum_{i=1}^r a_i D_i \subset \mathbb{P}$$
we associate a polytope
$$P_D := \{ x \in M_{\mathbb{R}} \mid \langle x, \nu_i \rangle \geq -a_i \quad \forall n_i \in \Sigma[1] \}$$
counting the global sections of $D$, that is $H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(D)) \cong L(P_D)$ (compare [CLST11, Prop.4.3.2]). By the formula in Remark 2.1 and by ([Bat22, Prop.4.3]) to $Y$ is associated the polytope $C(\Delta)$ that is
$$H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(Y)) \cong L(C(\Delta)). \quad (6)$$
It follows in the same way that to $Y + K_{\mathbb{P}}$ is associated $F(\Delta)$. 

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Corollary 2.10.  $Y \subset \mathbb{P}$ is Cartier if and only if $C(\Delta)$ is a lattice polytope.

Proof. By Remark 2.1

$$Y \sim_{\text{lin}} - \sum_{\nu \in \Sigma[1]} \text{Min}_{\Delta}(\nu)D_{\nu} = - \sum_{\nu \in \Sigma[1]} \text{Min}_{C(\Delta)}(\nu)D_{\nu}.$$  

The function $\text{Min}_{C(\Delta)} : \mathbb{N}_{\mathbb{R}} \to \mathbb{R}$ has the advantage that it is linear on the cones of $\Sigma$ since $\Sigma$ refines the normal fan of $C(\Delta)$. Thus $Y \subset \mathbb{P}$ is Cartier if and only if $\text{Min}_{C(\Delta)}$ is a support function for $Y$, that is

$$\text{Min}_{C(\Delta)}(N) \subset \mathbb{Z}.$$  

Obviously this is the case if and only if $C(\Delta)$ is a lattice polytope.

Remark 2.11. In particular if $C(\Delta) = \Delta$ then $Y$ will be Cartier. In ([Bat22, Ex.3.1.5]) there is mentioned an example of a 5-dimensional lattice polytope $\Delta$ for which $C(\Delta)$ is not a lattice polytope.

3 Reflexive sheaves and MCM-sheaves

Throughout this section $X$ denotes an $n$-dimensional normal irreducible variety with $j : U \to X$ the inclusion of the smooth locus of $X$.

Definition 3.1. A coherent sheaf $\mathcal{F}$ on $X$ is called reflexive if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, where $\mathcal{F}^{**}$ denotes the double dual of the sheaf $\mathcal{F}$. $\mathcal{F}^{**}$ is called the reflexive hull of $\mathcal{F}$.

Remark 3.2. ([CLS11, Prop.8.0.1, Thm.8.0.4])

Given an open subset $j : U \subset X$ with $\text{codim}_X(X \setminus U) \geq 2$ a reflexive sheaf is uniquely determined by its restriction to $U$, that is

$$\mathcal{F} \cong j_*(\mathcal{F}|_U).$$  

Furthermore if $\mathcal{F}$ is a coherent sheaf with $\mathcal{F}|_U$ locally free and $\text{codim}_X(X \setminus U) \geq 2$ then $j_*(\mathcal{F}|_U)$ is reflexive ([Sch08, Prop.2.12]). The dual of a coherent sheaf on a normal variety is always reflexive. In particular the reflexive hull of a coherent sheaf is reflexive.
The tensor product of two reflexive is defined by
\[ \mathcal{F} \otimes_r \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{**}. \]

Given two Weil divisors \( D, D' \) on \( X \) we have \( \mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes_r \mathcal{O}(D') \).

**Definition 3.3.** Let
\[ \Omega^p_X := j_* \Omega^p_U \quad 1 \leq p \leq n, \]
\[ T_X := (\Omega^1_X)^*, \]
be the sheaf of differential \( p \)-forms and the tangent sheaf. The sheaf
\[ N_{Y/P} := \iota_* \mathcal{O}_U(Y_{|U}) \]
is called the normal sheaf of \( Y \) in \( \mathbb{P} \).

**Remark 3.4.** There is a different method for the construction of \( T_X \): Let \( \Omega^p_{X,\text{Kähl}} \) denote the sheaf of Kähler \( p \)-differentials on \( X \) and
\[ T_{X,\text{Kähl}} := (\Omega^1_{X,\text{Kähl}})^*. \]

Then \( \Omega^1_{X,\text{Kähl}} \) is coherent and thus by Remark 3.2 its dual \( T_{X,\text{Kähl}} \) is reflexive and coincides with \( T_X \) on the smooth locus \( U \) of \( X \). Since both sheaves are reflexive
\[ T_{X,\text{Kähl}} \cong T_X. \]

By ([MuOd15 Ch.VI.1])
\[ H^0(X, T_{X,\text{Kähl}}) \cong \text{Lie}(\text{Aut}(X)), \] (8)
where \( \text{Lie}(\text{Aut}(X)) \) denotes the Lie algebra of the automorphism group of \( X \).

In order to apply Serre duality we need sheaves with the following property

**Definition 3.5.** ([CLS11 Def. before Thm.9.2.12], [Kol13 Def.5.2])
Let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then \( \mathcal{F} \) is called maximal Cohen-Macaulay (for short: MCM) if for all \( x \in X \) the stalk \( \mathcal{F}_x \) is a Cohen-Macaulay module over \( \mathcal{O}_{X,x} \) of dimension \( n \).
Proposition 3.6. [CLS11, Proof of Thm.9.2.10]
Let $V$ be an $n$-dimensional complete toric variety to a simplicial fan and $D$ a $\mathbb{Q}$-Cartier divisor on $V$. Then the sheaves $\Omega^p_V$ for $p = 0, \ldots, n$ and $\mathcal{O}_V(D)$ are MCM.

Let $\mathcal{F}$ be a MCM-sheaf on $X$. Then Serre duality ([KM98, Thm.5.71, Prop.5.75]) gives

$$H^k(X, \mathcal{F}) \cong H^{n-k}(X, \text{Hom}(\mathcal{F}, \mathcal{O}_X(K_X)))^*.$$

For $Z \subset X$ a closed subset, $\mathcal{F}$ a coherent sheaf on $X$, we work with the local cohomology groups $H^k_Z(X, \mathcal{F})$ on $X$. This allows us to extend exact sequences which exist on the smooth locus $U$ of $X$ to all of $X$. First we recall a vanishing Theorem:

Theorem 3.7. ([HaKo04, Prop.3.3])
Let $X$ be an $n$-dimensional algebraic variety and $Z \subset X$ a closed subset with $\text{codim}_X(Z) \geq r$. Let $\mathcal{F}$ be a coherent sheaf on $X$ which is MCM. Then

$$H^k_Z(X, \mathcal{F}) = 0 \quad k = 0, \ldots, r - 1.$$  

Proposition 3.8. The sequence of sheaves

$$0 \to \mathcal{O}_P \to \mathcal{O}_P(Y) \to N_{Y/P} \to 0$$  \hspace{1cm} (9)

is exact.

Proof. If $Y \subset \mathbb{P}$ is Cartier (see Corollary 2.10 for a criterion) then there is nothing to show, but in general $Y$ just defines a $\mathbb{Q}$-Cartier divisor. We may assume that $n = \dim \mathbb{P} \geq 3$. Let $\iota: V \to \mathbb{P}$ be the inclusion of the union of all torus orbits of dimension $\geq n - 2$ and $U := V \cap Y$. If we take the pushforward under $\iota_*$ of an ideal sheaf sequence for $U \subset V$ we get

$$0 \to \mathcal{O}_P \to \mathcal{O}_P(Y) \to N_{Y/P} \to R^1\iota_*(\mathcal{O}_V).$$

$R^1\iota_*(\mathcal{O}_V)$ is the sheaf associated to the presheaf

$$W \mapsto H^1(W \cap V, \mathcal{O}_{W \cap V}).$$

Assume that $W \subset \mathbb{P}$ is affine, $V \subset W$ an open subset such that $\text{codim}_W(Z) \geq 3$ for $Z := W \setminus V$. We have to show that $H^1(V, \mathcal{O}_V) = 0$. There is a local cohomology exact sequence ([Gro67, Cor.1.9])

$$\cdots \to H^1(W, \mathcal{O}_W) \to H^1(V, \mathcal{O}_V) \to H^2_Z(W, \mathcal{O}_W) \to H^2(W, \mathcal{O}_W).$$

Since $W$ is affine $H^1(W, \mathcal{O}_W) = 0$ by Serre’s criterion and $H^2_Z(W, \mathcal{O}_W) = 0$ by Theorem 3.7 since $\mathcal{O}_P$ is MCM and $\text{codim}_W(Z) \geq 3$. 

We will use this exact sequence and the vanishing $H^1(\mathbb{P}, \mathcal{O}_\mathbb{P}) = 0$ in Remark 2.9 to compute $H^0(Y, N_{Y/\mathbb{P}})$ by identifying $Y \subset \mathbb{P}$ with a torus invariant divisor.

**Corollary 3.9.** The normal sheaf $N_{Y/\mathbb{P}}$ is MCM.

**Proof.** Consider the exact sequence (9). For $y \in Y$ the $\mathcal{O}_Y,y$-module $(N_{Y/P})_y$ has dimension $n - 1$ and by ([Kol13, Cor.2.62(3)]) it is Cohen-Macaulay since both $\mathcal{O}_\mathbb{P}$ and $\mathcal{O}_\mathbb{P}(Y)$ are MCM sheaves. □

**Remark 3.10.** The normal sheaf $N_{Y/X}: = (I_{Y_f} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_f})^*$ is trivial of rank $l(\Delta) - 1$, where $\mathcal{I}_{Y_f}$ denotes the ideal sheaf of $Y_f \subset \mathcal{X}$.

**Corollary 3.11.**

$$H^0(Y, N_{Y/\mathbb{P}}) \cong L(C(\Delta))/\mathbb{C} \cdot f,$$

$$H^0(Y, N_{Y/X}) \cong L(\Delta)/\mathbb{C} \cdot f.$$

### 4 Kodaira-Spencer maps

Let $B$ be the image of $U_{reg}(\Delta)$ in $\mathbb{P}L(\Delta)$. We consider the following natural family

$$\mathcal{X} := \{(x, f) \in \mathbb{P} \times B | x \in Y_f\} \stackrel{pr_2}{\rightarrow} B.$$

**Definition 4.1.** Given a full-dimensional rational polytope $P \subset M_\mathbb{R}$ let $L(P)$ be the $\mathbb{C}$-vector space with basis $x^m$, $m \in \Delta \cap M$ and $l(P) = \#(P \cap M)$ its dimension.

Let $D := \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$ denote the dual numbers. Given a tangent vector $\phi : D \rightarrow B$ of $B$ at $f$ the pullback

$$\mathcal{X} \times_B D \rightarrow D$$

defines an *infinitesimal deformation* (of first order) of $Y_f$ in $\mathcal{X}$. Passing from $\mathcal{X} \rightarrow B$ to $\mathcal{X} \times_B D \rightarrow D$ some information gets lost. Still we can extract interesting information out of the situation (compare Kodaira [Kod86, Thm.4.3]). There are isomorphisms

$$H^0(Y, N_{Y/\mathcal{X}}) \cong \{\text{Inf. def. of } Y_f \text{ in } \mathcal{X}\}/\text{iso.}$$

$$H^0(Y, N_{Y/\mathbb{P}}) \cong \{\text{Inf. def. of } Y_f \text{ in } \mathbb{P}\}/\text{iso.}$$

$$\text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \cong \{\text{Inf. def. of } Y_f\}/\text{iso.}.$$
Let us derive two Kodaira-Spencer maps in this context: There is an isomorphism

\[ H^1(U, T_U) \cong \text{Ext}^1_{\mathcal{O}_U}(\Omega^1_U, \mathcal{O}_U). \]

since \( U \) is smooth. Besides by ([KM92, Lemma (12.5.6)])

\[ \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \cong \text{Ext}^1_{\mathcal{O}_U}(\Omega^1_U, \mathcal{O}_U) \cong H^1(U, T_U). \]

since \( \text{codim}_Y(Y \setminus U) \geq 3 \). For \( V \subset \mathbb{P} \) an open subset as in Remark (2.8) with \( V \cap Y = U \) the Kodaira-Spencer map

\[ H^0(U, N_{U/V}) \to H^1(U, T_U) \]

is the coboundary map in the tangent sheaf sequence for \( U \subset V \). We derive one Kodaira-Spencer map \( \kappa_{\mathbb{P}, f} \) from the commutative diagram

\[
\begin{array}{c}
0 \to H^0(U, T_U) \to H^0(U, T_V|_U) \to H^0(U, N_{U/V}) \to H^1(U, T_U) \\
\downarrow \cong \downarrow \cong \downarrow \cong \\
0 \to H^0(Y, T_Y) \to H^0(Y, T^*_{\mathbb{P}|Y}) \to H^0(Y, N_{Y/\mathbb{P}}) \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)
\end{array}
\]

The other Kodaira-Spencer map \( \kappa_f : H^0(Y, N_{Y/X}) \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \) is gotten similarly by working with \( X \) instead of \( \mathbb{P} \). These two maps associate two a deformation in \( \mathbb{P}(X) \) its equivalence class.

5 Mavlyutov’s vanishing Theorem

**Theorem 5.1.** ([Mav08, Thm.2.4], [CLS11, Thm.9.3.3])

Let \( V \) be an \( n \)-dimensional complete toric variety to a simplicial fan. If \( D \) is a nef Cartier divisor on \( V \), then

\[ H^p(V, \Omega^q_V \otimes \mathcal{O}(D)) = 0 \]

for \( p > q \).

**Construction 5.2.** (Multiplication morphism)

([Fuj06, 2.5, Prop.3.2], [CLS11, Lemma 9.2.6, Proof of Thm. 9.3.1])

Let \( V \) be a normal (not necessarily complete) toric variety. There is a useful construction to reduce computations of cohomology groups of \( \mathbb{Q} \)-Cartier Weil divisors \( D \) to cohomology of multiples \( mD \) of \( D \), which are Cartier.
Namely for \( l \in \mathbb{N}_{\geq 1} \) the map \( \phi_l : N \to N \) given by
\[
n \mapsto l \cdot n
\]
yields a toric morphism \( \phi_l : V \to V \). \( \phi_l \) induces an injection
\[
H^p(V, \Omega^q_V \otimes_r \mathcal{O}(D)) \to H^p(V, \Omega^q_V \otimes_r \mathcal{O}(lD)) \\
\cong H^p(V, \Omega^q_V \otimes \mathcal{O}(lD)), \quad p \geq 0,
\]
where the last isomorphism follows since \( \mathcal{O}(lD) \) is Cartier.
For us this result is powerful in connection with Theorem 5.1 above.

6 Computation of \( \ker(\kappa_{\mathbb{P},f}) \)

**Theorem 6.1.** Let \( \Delta \) be an \( n \)-dimensional lattice polytope, where \( n \geq 2 \), with \( l^*(\Delta) > 0 \). If \( n = 2 \) assume that \( l^*(\Delta) \geq 2 \). Then given \( f \in B \)
\[
\ker(\kappa_{\mathbb{P},f}) \cong \text{Lie Aut}(\mathbb{P}). \quad (10)
\]

**Proof.** We write \( Y \) and \( \kappa_{\mathbb{P}} \) for \( Y_f \) and \( \kappa_{\mathbb{P},f} \). By the exact sequence from section 4
\[
\ker(\kappa_{\mathbb{P}}) \cong H^0(Y, T_{\mathbb{P}|Y}^{**})/H^0(Y, T_Y).
\]
By Lemma (6.3) below if \( n \geq 3 \) then
\[
H^0(Y, T_Y) = 0.
\]
If \( n = 2 \) this follows from \( g(Y) \geq 2 \) since the Fine interior is not a point. By Remark 3.4 we are left to show that
\[
H^0(Y, T_{\mathbb{P}|Y}^{**}) \cong H^0(\mathbb{P}, T_\mathbb{P}).
\]

**Lemma 6.2.** There is an exact sequence
\[
0 \to T_\mathbb{P} \otimes \mathcal{O}(-Y) \to T_\mathbb{P} \to T_{\mathbb{P}|Y}^{**} \to 0.
\]

**Proof.** Let \( \iota : V \subset \mathbb{P} \) be the inclusion of the union of all torus orbits of dimension \( \geq n - 2 \) and \( U = V \cap Y \). Consider the exact sequence
\[
0 \to T_V \otimes \mathcal{O}(-Y|_V) \to T_V \to T_{V|U} \to 0
\]
and take the pushforward under $\iota$. Let

$$F := \Omega^{n-1}_P \otimes r \mathcal{O}(-Y - K_P) \cong T_P \otimes r \mathcal{O}(-Y).$$

Then $R^1_{\iota_*}(F|_V) = 0$ since given $x \in V$ choose an affine toric neighborhood $W$ of $x$ and (using Construction 5.2) replace $\mathcal{O}_W(-Y - K_P)$ by an $m$-times multiple which is Cartier in the definition of $F$. Then by Proposition 3.6 $F|_W$ is MCM and the vanishing

$$H^1(V \cap W, F|_{V \cap W}) = 0$$

follows as in the proof of Proposition 3.8 from a local cohomology sequence. \hfill \Box

Take the long exact cohomology sequence

$$0 \to H^0(P, T_P \otimes r \mathcal{O}(-Y)) \to H^0(P, T_P) \to H^0(Y, T_{\mathbb{P}|Y}) \to H^1(P, T_P \otimes r \mathcal{O}(-Y)).$$

Write

$$T_P \otimes r \mathcal{O}(-Y) \cong \Omega^{n-1}_P \otimes r \mathcal{O}(-Y - K_P).$$

By Construction 5.2 we have

$$H^k(P, \Omega^{n-1}_P \otimes r \mathcal{O}(-Y - K_P)) \subset H^k(P, \Omega^{n-1}_P \otimes \mathcal{O}(-mY - mK_P))$$

where $m \geq 1$ is such that $mY, mK_P$ are Cartier. Now use Serre duality and Theorem 5.1 to deduce

$$H^k(P, \Omega^{n-1}_P \otimes \mathcal{O}(-mY - mK_P)) \cong H^{n-k}(P, \Omega^1_P \otimes \mathcal{O}(mY + mK_P)) = 0$$

for $k = 0, 1$ if $n \geq 3$. If $n = 2$ the vanishing for $k = 1$ follows from the precise formula in ([Mav08, Cor.2.7]) (to $Y + K_P$ is associated the polytope $F(\Delta)$, see Remark 2.9). \hfill \Box

**Lemma 6.3.** Let $\Delta$ be an $n$-dimensional lattice polytope with $n \geq 3$ and $l^*(\Delta) > 0$. Then

$$H^0(Y, T_Y) = 0.$$
Proof. Using an ideal sheaf sequence we get

$$0 \to T_Y \to \Omega_{Y}^{n-1} \to \ldots$$

where we have used $\Omega_{Y}^{n-1} \cong T_Y \otimes_r \mathcal{O}(K_Y)$ and $h^0(Y, \mathcal{O}(K_Y)) = l^*(\Delta) > 0$. Thus

$$h^0(Y, T_Y) \leq h^0(Y, \Omega_{Y}^{n-1}) = 0$$

For the last vanishing: Take a common toric resolution of singularities $\mathbb{P} \xrightarrow{\pi} \tilde{\mathbb{P}} \to \mathbb{P}_\Delta$ with closure $\tilde{Y}$ of $Z_f$ in $\tilde{\mathbb{P}}$. Then $h^0(\tilde{Y}, \Omega_{\tilde{Y}}^{n-1}) = 0$ by ([DK86]) and $p_* \Omega_{\tilde{Y}}^{n-1} = \Omega_Y^{n-1}$.

We guess that the weaker assumption $F(\Delta) \neq \emptyset$ is sufficient for the above Lemma.

7 The cokernel of $\kappa_{P,f}$

Corollary 7.1. Given the conditions of the theorem if $n \geq 4$ then the following sequence is exact

$$0 \to Im(\kappa_{P,f}) \to Ext^1_{\mathcal{O}_Y}(\Omega_{Y}^1, \mathcal{O}_Y) \to Ext^1_{\mathcal{O}_{\mathbb{P}}}(\Omega_{\mathbb{P}}^1, \mathcal{O}_{\mathbb{P}}) \to 0$$

that is all infinitesimal deformations of $Y_f$ arise from $Im(\kappa_{P,f})$ or from infinitesimal deformations of $\mathbb{P}$.

Proof. Let $V$ denote the smooth locus of $\mathbb{P}$, $Z = \mathbb{P} \setminus V$ and $U = V \cap Y$. Then $Z$ has codimension $\geq 3$ in $\mathbb{P}$ and $Y \setminus U$ has codimension $\geq 2$ in $Y$. We could extend the upper exact sequence in Section 4 to

$$0 \to H^0(U, T_U) \to H^0(U, T_{V|U}) \to H^0(U, N_{U/V}) \xrightarrow{k} H^1(U, T_U)$$

$$\to H^1(U, T_{V|U}) \to H^1(U, N_{U/V})$$

By the exact sequence (9) $H^1(Y, N_{Y/\mathbb{P}}) = 0$. Relate $H^1(U, N_{U/V})$ and $H^1(Y, N_{Y/\mathbb{P}})$ via a local cohomology sequence and use (Lemma 3.9, Theorem 3.7)

$$H^k_{Z \cap Y}(Y, N_{Y/\mathbb{P}}) = 0 \quad k < n - 1.$$
It follows $H^1(U, N_{U/V}) = 0$. For $T_{V|U}$ use an ideal sheaf sequence

$$H^1(V, T_V \otimes \mathcal{O}(-Y_{|V})) \to H^1(V, T_V) \to H^1(U, T_{V|U}) \to H^2(V, T_V \otimes \mathcal{O}(-Y_{|V})).$$

$V$ is a toric variety and by Construction 5.2

$$H^k(V, T_V \otimes \mathcal{O}(-Y_{|V})) = H^k(V, \Omega^1_{V|V} \otimes \mathcal{O}(-mY_{|V} - mK_{P|V}))$$

where $mY, mK_P$ are Cartier. We show that these terms vanish for $k = 1, 2$ and the result follows since

$$H^1(V, T_V) \simeq \operatorname{Ext}^1_{\mathcal{O}_V}(\Omega^1_{P}, \mathcal{O}_P)$$

as in section 4. Use a local cohomology sequence

$$H^k(P, F) \to H^k(V, F_{|V}) \to H^{k+1}_Z(P, F)$$

where

$$F := \Omega^1_{P} \otimes \mathcal{O}(-mY - mK_P).$$

$F$ is CM by Lemma 3.9. As in the Lemma above

$$H^k(P, F) \simeq H^{n-k}(P, \Omega^1_{P} \otimes \mathcal{O}(mY + mK_P)) = 0 \quad k = 1, 2.$$ 

Further by Theorem 3.7

$$H^k_Z(P, F) = 0 \quad k < n - 1.$$

Example 7.2. If $P = P^3$, $\Delta = 4 \cdot \Delta_3$ then $\dim \operatorname{Im}(\kappa) = 19$ and $H^1(P, T_P) = 0$. But $h^1(Y, T_Y) = 20$ since $Y$ is a K3 surface. The above Corollary does not apply since $n = 3$.

8 An explicit basis for $\ker(\kappa_{P,f})$

Let

$$R(N, \Sigma) := \{ \alpha \in M \mid \langle \alpha, n(\alpha) \rangle = 1 \text{ for some } n(\alpha) \in \Sigma[1] \text{ and } \langle \alpha, n_j \rangle \leq 0 \text{ for } n_j \in \Sigma[1] \setminus \{n(\alpha)\} \}$$
denote the roots of $\Sigma$. Likewise we define $R(N, \Sigma_{C(\Delta)})$ and $R(N, \Sigma_{\Delta})$ by replacing $\Sigma$ by $\Sigma_{C(\Delta)}$ and $\Sigma_{\Delta}$.

There are inclusions

$$\Sigma_{C(\Delta)[1]} \subset \Sigma[1] \subset \text{Convhull}(\Sigma_{\Delta}[1]). \quad (11)$$

**Lemma 8.1.** Let $\Delta$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then

$$R(N, \Sigma_{\Delta}) \subset R(N, \Sigma) = R(N, \Sigma_{C(\Delta)}).$$

**Proof.** To the second equality: Let $\alpha \in R(N, \Sigma_{C(\Delta)})$, that is

$$\langle \alpha, n(\alpha) \rangle = 1, \quad \langle \alpha, n_j \rangle \leq 0 \quad \text{for } n_j \in \Sigma_{C(\Delta)}[1] \setminus \{n(\alpha)\}.$$  

$\Rightarrow \langle \alpha, n_j \rangle \leq 0$ for $n_j \in \Sigma[1] \setminus \{n(\alpha)\}$, that is $\alpha \in R(N, \Sigma)$. Conversely assume $\alpha \in R(N, \Sigma)$. If $n_i \notin \Sigma_{C(\Delta)}[1]$ then $\alpha$ would have scalar product $\leq 0$ with all vectors in $\Sigma_{C(\Delta)}[1]$ and thus would be zero since $\Sigma$ refines $\Sigma_{C(\Delta)}$, a contradiction. The first inclusion follows similarly by using (11). \[\square\]

We ask for a basis of Laurent polynomials for

$$\text{Lie Aut}(\mathbb{P}) \subset L(C(\Delta))/\mathbb{C} \cdot f.$$  

Remember the results from ([BG99]): Given $f \in B$ there is a map

$$\phi_f : T \to B$$

$$(t_1, t_2, t_3) \mapsto \left((x_1, x_2, x_3) \mapsto f(t_1x_1, t_2x_2, t_3x_3)\right).$$

By differentiating $\phi_f$ we get an injective homomorphism $(d\phi_f)_e : \text{Lie}(T) \to T_{B,f}$ where $e = (1, 1, 1)$ with

$$\text{Im}(d(\phi_f)_e) = \left\langle x_1 \cdot \frac{\partial f}{\partial x_1}, ..., x_3 \cdot \frac{\partial f}{\partial x_3} \right\rangle.$$  

For $m \in M \cap C(\Delta)$ and $\alpha \in R(N, \Sigma_{C(\Delta)})$ define

$$ht_{-\alpha}(m) := \max\{k \in \mathbb{N}_{\geq 0} \mid m - k \cdot \alpha \in C(\Delta)\}. \quad (12)$$

Given $\alpha \in R(N, \Sigma_{C(\Delta)})$ we denote by $\Gamma_{-\alpha} \leq C(\Delta)$ the facet to which $n(\alpha)$ is normal.
Remark 8.2. Assuming
\[ \Gamma_{-\alpha} = \{ x \in M_R | \langle x, n_\Gamma \rangle = b_\Gamma \} \cap C(\Delta) \]
and \( m \in M \cap C(\Delta) \) then
\[ ht_{-\alpha}(m) = \langle m, n_\Gamma \rangle - b_\Gamma. \tag{13} \]

Figure 2: A lattice polytope with a column vector \( v \) and \( ht_v(m) = 2 \).

Let \( S_{C(\Delta)} \) denote the graded semigroup \( \mathbb{C} \)-algebra over \( \text{Cone}(C(\Delta) \times \{1\}) \cap (M \times \mathbb{Z}) \)

Then the function \( ht_{-\alpha} \) continues to a map \( S_{C(\Delta)} \to S_{C(\Delta)} \) which respects the grading on \( S_{C(\Delta)} \). For \( \lambda \in \mathbb{C} \) define a graded automorphism \( e_{-\alpha}^\lambda : S_{C(\Delta)} \to S_{C(\Delta)} \) by
\[ e_{-\alpha}^\lambda(x^m) := x^m \cdot (1 + \lambda x^{-\alpha})^{ht_{-\alpha}(m)} \]

Corollary 8.3. ([BG99, Lemma 3.1, Thm.3.2b), Thm.5.4])
Lie \( \text{Aut}(\mathbb{P}) \) has a basis of derivations, which act on \( L(C(\Delta)) \) as follows

\[ x_i \frac{\partial}{\partial x_i} : x^m \mapsto m_i \cdot x^m, \quad i = 1, ..., n, \]
\[ \frac{\partial e_{-\alpha}^\lambda}{\partial \lambda} |_{\lambda=0} : x^m \mapsto ht_{-\alpha}(m) \cdot x^{m-\alpha}, \quad \alpha \in R(N, \Sigma_{C(\Delta)}). \]

By definition of the tangent sheaf sequence the homomorphism
\[ j : H^0(\mathbb{P}, T_{\mathbb{P}}) \cong H^0(Y, T_{\mathbb{P}|Y}^{**}) \to H^0(Y, N_{Y/\mathbb{P}}) \]
is given by applying the derivations from Corollary 8.3 to
\[ f = \sum_{m \in \Delta \cap M} a_m x^m \in U_{\text{reg}}(\Delta) \]
and restricting to \( Y = Y_f \).

**Corollary 8.4.** Given the conditions of Theorem 6.1 \( \ker(\kappa_{P,f}) \) has the basis
\[ x_1 \cdot \frac{\partial f}{\partial x_1}, ..., x_n \cdot \frac{\partial f}{\partial x_n}, \]
\[ w_{-\alpha}(f) := \sum_{m \in \Delta \cap M} ht_{-\alpha}(m) \cdot a_m \cdot x^{m-\alpha}, \quad \alpha \in R(N, \Sigma_{C(\Delta)}). \]

Figure 3: On the left: The column vector \(-\alpha\) with facet \(\Gamma_{-\alpha}\) and all lattice points \(m \in C(\Delta)\) with \(ht_{-\alpha}(m) > 0\). On the right: \(w_{-\alpha}(f)\) has support on the thick lattice points.

**Example 8.5.** If \( \Delta = d \cdot \Delta_n, \quad f \in U_{\text{reg}}(\Delta) \), then
\[ C(\Delta) = \Delta, \quad \Sigma = \Sigma_{\Delta}, \quad P = P^n \]
and \( Y_f \) is a smooth degree \( d \) hypersurface in \( P^n \). For such an hypersurface it is shown in ([Voi03, Lemma 6.15]) that
\[ \ker(\kappa_f) \cong J_{f,\text{griff}}^d, \]  
(14)
if we work with the family \( X \rightarrow U_{\text{reg}}(\Delta) \) (if we projectivize then we have to mod out \( f \) from the kernel). Here \( J_{f,\text{griff}}^d \) denotes the \( d \)-th homogeneous component of *Griffiths Jacobian ideal* 
\[ J_{f,\text{griff}} := (\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_n}) \leq \mathbb{C}[x_0, ..., x_n]. \]
The roots of \( \Sigma \) are given by
\[
\pm e_i, \quad i = 1, \ldots, n, \quad \pm e_i \pm e_j, \quad i, j = 1, \ldots, n, \quad i \neq j
\]
and if \( d \geq n + 1 \) then Theorem 8.4 restricts to the result (14) up to homogenization.

9 An explicit basis for \( \ker(\kappa_f) \)

**Theorem 9.1.** Given the conditions of Theorem 6.1, \( \ker(\kappa_f) \) has the basis
\[
x_i \frac{\partial f}{\partial x_i}, \quad i = 1, \ldots, n, \quad w_{-\alpha}(f), \quad \alpha \in R(N, \Sigma_\Delta).
\]

We first prove the following Proposition which reduces the proof to a combinatorial argument

**Proposition 9.2.** Let \( \Delta \) be an \( n \)-dimensional lattice polytope with \( F(\Delta) \neq \emptyset \). Then \( \kappa_f \) equals the restriction of \( \kappa_{\mathbb{P}, f} \) to \( L(\Delta)/\mathbb{C} \cdot f \) and thus
\[
\ker(\kappa_f) \cong \ker(\kappa_{\mathbb{P}, f}) \cap L(\Delta). \tag{15}
\]

**Proof.** The following reduction step is similar to ([Koe91, Ch.2.1] and [Voi03, Lemma 6.15]): Let \( V \subset \mathbb{P} \) be the union of all torus orbits of dimension \( \geq n-2 \). Then \( U = U_f := V \cap Y_f \) is smooth and \( \text{codim}_{Y_f}(Y_f \setminus U_f) \geq 2 \) for every \( f \in B \). Let
\[
W := (V \times B) \cap \mathcal{X}.
\]
Consider the differential
\[
(pr_1)_*: T_W|_U \rightarrow T_V|_U
\]
of the first projection. All the sheaves we consider are reflexive, therefore there is no difference in working with \( Y, \mathcal{X} \) and \( \mathbb{P} \). \( pr_1 \) restricts to an isomorphism
\[
Y_f \times \{f\} \rightarrow Y_f,
\]
thus \( (pr_1)_* \) restricts to the identity on \( T_Y \). The map
\[
(pr_1)_*: N_{Y/X} \cong H^0(Y, N_{Y/X}) \otimes \mathcal{O}_Y \subset H^0(Y, N_{Y/\mathbb{P}}) \otimes \mathcal{O}_Y \rightarrow N_{Y/\mathbb{P}}
\]
is given by multiplication of sections. In effect we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(Y, Ty) & \longrightarrow & H^0(Y, T^*_XY) & \longrightarrow & H^0(Y, N_{Y/X}) \\
\downarrow{id} & & \downarrow{(pr_1)_*} & & \downarrow{id} & & \downarrow{id} \\
0 & \longrightarrow & H^0(Y, Ty) & \longrightarrow & H^0(Y, T^*_YP|Y) & \longrightarrow & H^0(Y, N_{Y/P}) \\
\end{array}
\]

and

\[
\ker(\kappa_f) \cong H^0(Y, T^*_XY) \cong H^0(Y, T^*_YP|Y) \cap L(\Delta) \cong \ker(\kappa_{P,f}) \cap L(\Delta).
\]

The \(\partial f/\partial x_i\) obviously belong to \(L(\Delta)\) but the \(w_{-\alpha}(f)\) need not have support on \(\Delta\) as the following example shows:

**Example 9.3.** Consider the polytope

\[
\Delta = \langle \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \rangle
\]

\(\Delta\) has 3 interior lattice points, \(F(\Delta)\) is 1-dimensional and \(C(\Delta)\) has the additional vertex \((1, -1, 1)\). We obtain a family of elliptic surfaces \(X \rightarrow B\). There are 7 roots

\[
R(N, \Sigma) = \{ \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \}
\]

and one root \((-1, 0, -1)\) not belonging to \(R(N, \Sigma)_\Delta\). The column vector \(-\alpha = (1, 0, 1)\) belongs to the facet

\[
\Gamma_{-\alpha} = \langle \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 10 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \rangle
\]

of \(C(\Delta)\). The vertex \((-1, -1, -1)\) does not lie on \(\Gamma_{-\alpha}\) and

\[-\alpha + (-1, -1, -1) = (0, -1, 0) \notin \Delta.
\]

Thus only 6 of the roots in \(R(N, \Sigma)\) reduce the number of moduli.
(Proof of Theorem 9.1)

The proof is rather technical. By Proposition 9.2

\[ \ker(\kappa_f) = \ker(\kappa_{\varphi,f}) \cap L(\Delta) \]

and \( R(N, \Sigma_\Delta) \subset R(N, \Sigma_{C(\Delta)}) \) by Lemma 8.1. Let \( R := R(N, \Sigma_{C(\Delta)}) \setminus R(N, \Sigma_\Delta) \). The Theorem is a consequence of the three points below.

- \( \alpha \in R(N, \Sigma_\Delta) \Rightarrow w_{-\alpha}(f) \in L(\Delta) \).
- \( \alpha \in R \Rightarrow w_{-\alpha}(f) \notin L(\Delta) \).
- Varying \( \alpha \in R \) the \( w_{-\alpha}(f) \) are linearly independent in \( L(C(\Delta))/L(\Delta) \).

The necessity of the first two points is obvious and the last point assures that no linear combination of the \( w_{-\alpha}(f) \), where \( \alpha \in R \), lies in \( \ker(\kappa_f) \).

First point: To \( \alpha \in R(N, \Sigma_\Delta) \) is associated both \( \Gamma_{-\alpha} \leq \Delta \) and \( \Gamma'_{-\alpha} \leq C(\Delta) \). We show

\[ \Gamma_{-\alpha} \subset \Gamma'_{-\alpha}, \quad (16) \]

since then for \( m \in M \cap \Delta, m \notin \Gamma_{-\alpha} \) we get \( m - \alpha \in \Delta \), that is \( w_{-\alpha}(f) \in L(\Delta) \).

Concerning (16): Given \( n_i \in \Sigma_\Delta[1] \) with \( \langle \alpha, n_i \rangle = 1 \) and \( n_j \in \Sigma_{C(\Delta)}[1] \) with \( \langle \alpha, n_j \rangle = 1 \) then \( n_i = n_j \) by (11). It follows \( \Gamma_{-\alpha} \subset \Gamma'_{-\alpha} \) since

\[ \min_{C(\Delta)}(n_i) = \min_{\Delta}(n_i). \]

Thus \( \Gamma_{-\alpha} \subset \Gamma'_{-\alpha} \).

Second point: There is a facet \( \Gamma_{-\alpha} \) of \( C(\Delta) \) such that

\[ m - \alpha \in C(\Delta) \quad \text{for} \quad m \in C(\Delta) \cap M, \quad m \notin \Gamma_{-\alpha}. \]

First assume that \( \Gamma_{-\alpha} \cap \Delta \) is also a facet of \( \Delta \). There is \( n_j \in \Sigma_\Delta[1] \setminus \{ n_{\Gamma_{-\alpha}} \} \) with \( \langle \alpha, n_j \rangle > 0 \) since \( \alpha \notin R(N, \Sigma_\Delta) \). Given \( m \in \text{Vert}(\Gamma_j) \), then \( m \in \text{Supp}(f) \) and \( m - \alpha \notin \Delta \) since

\[ \langle m - \alpha, n_j \rangle < \min_{\Delta}(n_j). \]
⇒ $w_{-\alpha}(f) \notin L(\Delta)$. Assume that $\Gamma_{-\alpha} \cap \Delta$ is a face of $\Delta$ of dimension $< n - 1$. The convex span 
\[ \langle m \in Vert(\Delta) \mid m - \alpha \notin \Delta \rangle \]
is of dimension $\geq n - 1$. ⇒ there is $m \in Vert(\Delta)$ with 
\[ m \notin \Gamma_{-\alpha}, \ m - \alpha \notin \Delta, \]
that is $w_{-\alpha}(f) \notin L(\Delta)$.

**Third point:** Given a fixed facet $\Gamma = \Gamma_{-\alpha}$ of $C(\Delta)$ all 
\[ \alpha \in R(N, \Sigma_{C(\Delta)}) \setminus R(N, \Sigma_\Delta) \]
with $\Gamma_{-\alpha} = \Gamma$ build the lattice points on a lattice polytope $P \subset M_\mathbb{R}$.
Given $\alpha \in Vert(P)$ there is $m \in \text{Supp}(f)$ such that $x^{m-\alpha}$ does not appear in the support of any other $w_{-\alpha'}(f)$. Thus $w_{-\alpha}(f)$ does not appear with nonzero coefficient in any relation between the $w_{-\alpha'}(f)$. We then break down $P$ vertex by vertex.

Let $\Gamma_1, \Gamma_2$ be two different facets of $C(\Delta)$ and $\alpha_1, \alpha_2 \in R(N, \Sigma_{C(\Delta)}) \setminus R(N, \Sigma_\Delta)$ roots to these facets. Given a relation in 
\[ L(C(\Delta))/L(\Delta) \]
in which both $w_{-\alpha_1}(f)$ and $w_{-\alpha_2}(f)$ appear with nonzero coefficients there is $v \in \text{Supp}(f)$ with 
\[ \langle v - \alpha_1, n_1 \rangle < \text{Min}_\Delta(n_1), \ v - \alpha_1 + \alpha_2 \in M \cap \Delta. \]
Then 
\[ \langle v - \alpha_1 + \alpha_2, n_1 \rangle \geq \text{Min}_\Delta(n_1), \]
but $\langle \alpha_2, n_1 \rangle \leq 0$ since $\alpha_2$ is a root for $n_2 \neq n_1$, a contradiction. \hfill \Box

**Remark 9.4.** Given a common toric resolution of singularities

\[
\begin{array}{ccc}
\mathbb{P}_\Sigma & \xrightarrow{\kappa_f} \mathbb{P} & \xleftarrow{\hat{\kappa}_f} \mathbb{P}_\Delta \\
\end{array}
\]
there is a deformation of smooth toric hypersurfaces $\tilde{X} \to B$. Take the Kodaira-Spencer map $\tilde{\kappa}_f$. Then $R(N, \Sigma) \subset R(N, \Sigma_\Delta)$ and ker$(\tilde{\kappa}_f)$ is gotten as ker$(\kappa_f)$ but with $\alpha \in R(N, \bar{\Sigma})$ instead of $\alpha \in R(N, \Sigma_\Delta)$. 

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10 The number of moduli for subfamilies

Remark 10.1. Let \( \Delta \) be an \( n \)-dimensional polytope with \( F(\Delta) \neq \emptyset \) and \( A \subset M \cap \Delta \) a subset containing all vertices of \( \Delta \). Let

\[
    f := \sum_{m \in A} a_m x^m.
\]

Then for \( (a_m)_{m \in A} \) generic \( f \) is nondegenerate with respect to \( \Delta \) (see [GKZ94, Ch.10] and [Bat03, Prop.2.16]). Denote the resulting open subset of \( \mathbb{C}^{|A|} \) by \( U_A \) and the restriction of \( \mathcal{X} \) to \( U_A \) by \( \mathcal{X}_A \). Taking the quotient by the Laurent polynomials

\[
    f, \quad x_i \frac{\partial f}{\partial x_i} \quad i = 1, ..., n
\]

will reduce the number of moduli of the subfamily \( \mathcal{X}_A \to U_A \) by \( n + 1 \). Concerning the \( w_{-\alpha}(f) \) it seems to be difficult to decide in general if there are \( c_\alpha \in \mathbb{C} \) with

\[
    \sum_\alpha c_\alpha w_{-\alpha}(f) \in T_{U_A,f} \setminus \{0\}.
\]

Therefore we restrict to some special cases:

Example 10.2. Assume that \( \Delta \) is an \( n \)-dimensional simplex with \( F(\Delta) \neq \emptyset \) and \( A \) equals the set of vertices of \( \Delta \). Then by varying the coefficients to \( A \) we obtain a family with \( \kappa = 0 \). This generalizes:

Lemma 10.3. Given the conditions of Theorem 6.1 let \( A \) denote the set of vertices of \( \Delta \). Assume that for every facet

\[
    \Gamma = \Delta \cap \{ x \in M_{\mathbb{R}} \mid \langle x, n_\Gamma \rangle = b_\Gamma \}
\]

with \( n_\Gamma \in \Sigma_{\Delta}[1] \) there is no vertex of \( \Delta \) lying in the plane

\[
    H_{\Gamma,+1} := \{ x \in M_{\mathbb{R}} \mid \langle x, n_\Gamma \rangle = b_\Gamma + 1 \}
\]

Then the subfamily to \( A \) has

\[
    \#\{\text{vertices}\} - n - 1
\]

type of moduli.

Proof. Given a root \( \alpha \) and a lattice point \( m \) on \( \Delta \) then \( m - \alpha \) lies exactly one step closer to the facet \( \Gamma_{-\alpha} \) (and not closer to any other facet). Assume that \( m \) and \( m - \alpha \) are vertices of \( \Delta \). Then \( m - \alpha \) lies on \( \Gamma_{-\alpha} \) since else \( m, m - 2\alpha \in \Delta \) and \( m - \alpha \) would not be a vertex. But then \( m \in H_{\Gamma_{-\alpha},+1} \) contradicting the assumption. \( \square \)
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