Evolution of the System
with Singular Multiplicative Noise

Alexander I. Olemskoi and Dmitrii O. Kharchenko

Sumy State University
2, Rimskii-Korsakov St., 244007 Sumy UKRAINE

Abstract

The governed equations for the order parameter, one-time and two-time correlators are obtained on the basis of the Langevin equation with the white multiplicative noise which amplitude $x^a$ is determined by an exponent $0 < a < 1$ ($x$ being a stochastic variable). It turns out that equation for autocorrelator includes an anomalous average of the power-law function with the fractional exponent $2a$. Determination of this average for the stochastic system with a self-similar phase space is performed. It is shown that at $a > 1/2$, when the system is disordered, the correlator behaves non-monotonically in the course of time, whereas the autocorrelator is increased monotonically. At $a < 1/2$ the phase portrait of the system evolution divides into two domains: at small initial values of the order parameter, the system evolves to a disordered state, as above; within the ordered domain it is attracted to the point having the finite values of the autocorrelator and order parameter. The long-time asymptotes are defined to show that, within the disordered domain, the autocorrelator decays hyperbolically and the order parameter behaves as the power-law function with fractional exponent $-2(1-a)$. Correspondingly, within the ordered domain, the behavior of both dependencies is exponential with an index proportional to $-t \ln t$.

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alexander@olem.sumy.ua
dikh@ssu.sumy.ua
1 Introduction

Within the framework of the ordinary thermodynamic approach, it is postulated that the bath is passive with respect to a variation of the system state parameter $x$ which is an amplitude of the hydrodynamic mode [1]. In such a case the noise of the corresponding stochastic process $x(t)$ is additive one in the course of time $t$ to have the temperature as intensity, which is independent on the variable $x$. On the contrary, the amplitude $g(x)$ of a multiplicative noise varies with the stochastic variable $x$. Such type examples represent a population dynamics [2], directed percolation [3], Lévy flights [4] and so on.

According to Ref.[5], the multiplicative function $g(x, t)$ represents the homogeneous function for the systems with a self-similar phase space. In such a case we can write the noise amplitude as the power-law function

$$g(x) = x^a, \quad a \in [0, 1]$$

(1)

that is singular in character because $g = 0$ at $x = 0$. This kind assumption allows us to consider such models as ordinary thermodynamic system with an additive noise ($a = 0$), directed percolation process ($a = 1/2$), population dynamics ($a = 1$) and so on.

This work is organized as follows. In Sec.2 the governed equations for the first moment (order parameter) of the stochastic variable $x(t)$ as well as the one-time and two-time correlators are obtained on the basis of the Langevin equation with the white multiplicative noise. It turns out that equation for autocorrelator gains an anomalous average of the squared power function (1) with fractional exponent $2a$ (the fractional average). Sec.3 deals with the determination of this average for the stochastic system with a self-similar phase plane, whose distribution function is a homogeneous one and can be approximated by a power function [5]. Within the framework of this approach, the system behavior is governed by the value of the exponent $a$ in Eq.(1) [5,6]. At $a > 1/2$ the system is disordered to be represented by correlator and autocorrelator (Sec.4). At $a < 1/2$ the evolution of the order parameter ought to take into account (Sec.5). Sec.6 contains a short conclusions.

2 Basic equations

As usually, let us start with the Langevin equation

$$\frac{dx}{dt} = f(x) + g(x)\xi(t)$$

(2)

for a stochastic variable $x(t)$. In right-hand side of Eq.(2), $f(x)$ is the deterministic evolution force, the second term defines the multiplicative noise with the amplitude $g(x)$. The
statistical properties of the Langevin force $\xi(t)$ are standard:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = \delta(t - t')$$  \hspace{1cm} (3)

where the angle brackets denote the averaging. In order to study the simplest system properties, we consider the order parameter $\eta(t) \equiv \langle x(t) \rangle$, the two-time correlator $G(t, t') \equiv \langle \delta x(t) \delta x(t') \rangle$, $\delta x(t) \equiv x(t) - \langle x(t) \rangle$ and the autocorrelator (structure factor) $S(t) \equiv \langle (\delta x(t))^2 \rangle$. Within the framework of the white-noise approximation that is expressed by Eqs.(3) and the Ito calculus, we can treat $g(x)$ and $\xi(t)$ as statistically independent functions. Then, at averaging of Eq.(2) we can set $\langle g(x) \xi(t) \rangle = 0$ so that the evolution of the first moment is defined by the equation $\dot{\eta} = \langle f(x) \rangle$ where $\langle f(x) \rangle \neq f(\eta)$, dot stands for the derivative with respect to the time $t$. Without loss of generality, the deterministic part of the evolution force can be chosen in the polynomial Landau form:

$$f(x) = -\frac{\partial V(x)}{\partial x}, \quad V(x) = -\frac{\varepsilon}{2} x^2 + \frac{1}{4} x^4$$  \hspace{1cm} (4)

where $-\varepsilon$ is an external driven parameter type of the dimensionless temperature counted from a critical value. Performing the averaging under the force definition (4), we get term $\langle x^3 \rangle$ that is reduced to $\eta \langle x^2 \rangle \equiv \eta(\eta^2 + S)$, in accordance with the cumulant expansion. As a result, the evolution equation for order parameter takes the form

$$\dot{\eta} = \eta \left( \varepsilon - \eta^2 \right) - 3\eta S$$  \hspace{1cm} (5)

where the values $\eta, S$ are dependent on the time $t$.

By analogy, accounting the equation

$$\langle g(x(t)) x(t') \xi(t) \rangle = \langle g(x(t)) x(t') \rangle \langle \xi(t) \rangle + \langle g(x(t)) \xi(t) \rangle \eta(t') + \langle g(x(t)) \rangle \langle x(t') \xi(t) \rangle = 0$$

that follows from Eq.(3) and above-mentioned property of the white noise, we obtain the equation for two-time correlator

$$\frac{\partial}{\partial t} G(t, t') = \{\varepsilon - 3[\eta^2(t) + S(t)]\} G(t, t').$$  \hspace{1cm} (6)

The problem lies now in obtaining an evolution equation for the variance $S \equiv \langle x^2 \rangle - \eta^2$. It can be performed if we use the relation $dx^2 \equiv (x + dx)^2 - x^2$ where, in accordance with Eq.(3), the differential $dx$ is written as follows:

$$dx = f(x)dt + g(x)dw, \quad dw \equiv \xi(t)dt, \quad (dw)^2 = dt.$$  \hspace{1cm} (7)

Then, up to the first derivative, the equation for $\langle x^2 \rangle$ takes the form

$$\frac{d}{dt} \langle x^2 \rangle = 2\langle xf(x) \rangle + \langle g^2(x) \rangle.$$  \hspace{1cm} (8)
Here the last term is the average intensity of the multiplicative noise being a result of the interaction of the variable \( x \) with a bath, whose variables have been appropriately eliminated. Inserting definition given by Eq.(1) into Eq.(8), we come up against the average \( \langle x^{2a} \rangle \) of the stochastic variable with a fraction exponent \( 2a \). Further, we are coming to the obtaining an expression for such fractional average in terms of the cumulant expansion.

3 Calculation of the fractional average

Let us admit that, apart from the initial distribution \( P(x) \), there is another distribution \( P_q(x) \) specified by a positive parameter \( q < 1 \). Moreover, for \( P(x) \) and \( P_q(y) \) we assume the following relation:

\[
x^q P(x) dx \equiv y P_q(y) dy.
\]

Then, the probability density \( P_q(y) \) for the new stochastic variable \( y \equiv x^q \) can be rewritten as the normalized distribution

\[
P_q(y) = y^{-1} y^{(1-q)/q} P(y^{1/q}).
\]

Denoting the averaging over the distribution \( P_q(y) \) as \( \langle \ldots \rangle_q \) and using the designation \( \langle \ldots \rangle \) for the initial distribution \( P(x) \), we obtain

\[
\langle x^q \rangle = \langle y \rangle_q; \\
\langle x^q \rangle \equiv \int x^q P(x) dx, \quad \langle y \rangle_q \equiv q^{-1} \int y y^{(1-q)/q} P(y^{1/q}) dy.
\]

Thus, the distribution (10) allows us to use the usual cumulant expansion for any average \( \langle x^q \rangle \) with the fractional exponent \( q \).

For self-similar stochastic systems the distribution function can be written in a power-law form

\[
P(x) \simeq A x^{-2a}, \quad A \equiv \frac{1}{2} |1 - 2a| b^{1-2a}
\]

to be a homogeneous function [5], [7]. Here factor \( A \) is responsible for a cut-off procedure in normalization condition

\[
2 \int_b^{1/b} P(x) dx = 1
\]

with cut-off parameter \( b \to 0 \). The integrating with the distribution function (12) gives
\[ \langle x^{nq} \rangle \equiv A \int x^{nq} x^{-2a} dx = A(1 - 2a + nq)^{-1} x^{1 - 2a + nq}, \quad (14) \]
\[ \langle x^n \rangle \equiv A \int x^n x^{-2a} dx = A(1 - 2a + n)^{-1} x^{1 - 2a + n}. \quad (15) \]

As a result, using Eq.(11), we obtain the relation
\[ \langle x^{nq} \rangle = \alpha_n(q) \langle x^n \rangle p_n(q) \quad (16) \]
where the exponent \( p_n(q) \) and the multiplier \( \alpha_n(q) \) are introduced as follows:
\[ p_n(q) = \frac{1 - 2a + nq}{1 - 2a + n}, \quad \alpha_n(q) = A^{\frac{n(1-q)}{1-2a+n}} p_n^{-1}(q) (1 - 2a + n)^{p_n(q)-1}. \quad (17) \]

Now, we are ready to formulate the equation for the autocorrelator \( S = \langle x^2 \rangle - \eta^2 \) on the basis of Eqs.(8), (1), (16), (17). According to [6] a keypoint of the system with the multiplicative noise is that its behavior is governed by the magnitude of the exponent \( a \) in Eq.(1). At \( 1/2 < a < 1 \), when the fractal dimension of the phase space \( D = 2(1-a) \) is less than 1, the system is always disordered and its evolution is represented by the correlator \( G(t, t') \) and the structure factor \( S(t) \). The former is governed by Eq.(6), for the latter it follows
\[ \dot{S} = 2S(\varepsilon - 3S) + \alpha_2 S^{p_2}, \quad (18) \]
\[ \alpha_2 \equiv \alpha_2(a) = A^{2(1-a)p_2} p_2^{-p_2}, \quad p_2 \equiv p_2(a) = (3 - 2a)^{-1} \]
from Eqs.(3), (8), (1), (13), (17) if we put \( q = a, n = 2 \). Within another domain \( 0 < a < 1/2 \), the above fractal dimension \( D > 1 \) so that the system can be ordered and instead of Eq.(18) we obtain
\[ \dot{S} = 2S \left[ \varepsilon - 3(\eta^2 + S) \right] + \alpha_1 \eta^{p_1}, \quad (19) \]
\[ \alpha_1 \equiv \alpha_1(2a) = A^{(1-2a)p_1} p_1^{-p_1}, \quad p_1 \equiv p_1(a) = [2(1-a)]^{-1} \]
at \( q = 2a, n = 1 \) in Eqs.(16), (17).

4 Evolution of disordered system

As pointed out above, in the case when the exponent \( a > 1/2 \) (the fractal dimension \( D < 1 \)), the system is governed by Eqs.(18), (1) for the one-time and two-time correlators \( S(t), G(t, t') \) being the structure factor and Green response function. The form of the
time-dependence for the former is shown in Fig.1a. It is seen that $S(t)$ monotonically increases to the stationary magnitude $S_0$ determined by the equation

$$\varepsilon - 3S_0 + (\alpha_2/2)S_0^{p_2-1} = 0. \quad (20)$$

In the limit $S \ll 1$ when $S^{p_2} \gg S^2$, Eq.(18) gives the power-law time dependence

$$S(t) = \left( \frac{A^{2(1-a)}}{p_2(1-p_2)} \right)^{p_2/(1-p_2)}t^{1/(1-p_2)}, \quad p_2 \equiv (3-2a)^{-1} \quad (21)$$

where we put $S(t = 0) = 0$. In opposite case $S_0 - S \ll S_0$ one has the exponential dependence $S - S_0 \propto e^{-\lambda t}$, $\lambda \equiv 6(2 - p_2)S_0 - 2(1 - p_2)\varepsilon$. According to Eq.(20) the stationary value $S_0$ raises with $\varepsilon$ increase from the minimal magnitude $(\alpha_2/6)^{1/(2-p_2)}$ (see Fig.1b).

The solutions of Eq.(6) for different values $a$ and $\varepsilon$ are shown in Fig.2. We plot correspondent dependencies of the Green function $G(t,0)$ at identical initial conditions. It is interesting to observe that the correlator $G(t,0)$ reaches firstly its maximum and then monotonically decreases to zero. The function $G(t,0)$ attains its maximum more sharply if the parameter $\varepsilon$ increases (cf. curves 1, 2). The exponent $a$ increasing drives to the similar effect (cf. curves 1, 3).

5 Evolution of ordering system

Now one has $a < 1/2$, $D > 1$ and the system behavior is governed by Eqs.(3), (4), (19), from which the first and third state the enclosed system of differential equation. To analyze the latter, it is convenient to use the phase plane method. As is seen from the corresponding phase portrait in Fig.3a, at small values $\varepsilon$ there is only one attractive point $\eta_0 = 0$, $S_0 = \varepsilon/3$ (Fig.3a). With $\varepsilon$ increase at the point

$$\varepsilon_0 = \frac{4 - p_1}{2 - p_1} \frac{3}{8(2 - p_1)\alpha_1}^{2/(4-p_1)} \quad (22)$$

a bifurcation creates new saddle and attractive points (see Fig.4) with coordinates $\eta_c = [(2 - p_1)(4 - p_1)^{-1}\varepsilon_0]^{1/2}$, $S_c = (2/3)(4 - p_1)^{-1}\varepsilon_0$. It is seen from Fig.5 that bifurcation temperature $\varepsilon_0$ is increased infinitely with the exponent $a$ growth to the critical value $a = 1/2$. The coordinates of the new stationary points are determined by equations

$$\varepsilon - \eta_0^2 - (3/4)\alpha_1\eta_0^{p_1-2} = 0, \quad S_0 = (4\alpha_1)^{-1}(\varepsilon - 3S_0)(p_1/2)^{-1}, \quad (23)$$

that is obtained from Eqs.(3), (19) at $\dot{\eta} = 0$, $\dot{S} = 0$. The corresponding dependencies $\eta_0(\varepsilon)$, $S_0(\varepsilon)$ are depicted in Fig.4 where the dashed curves are respective for the saddle $S$ and solid ones – for the attractive point $C$. 
It is interesting to note that the system undergoes the phase transition of the first order, despite of the bare $x^4$-potential (4) corresponds to the continuous one. Thus, at small values of the exponent of the multiplicative noise (1) ($a < 1/2$) the fluctuations transform order of the phase transition, whereas at $a > 1/2$ ones suppress the ordering process at all.

Let us return now to analysis of the time dependencies of the main averages under consideration. Firstly, we analyze the system evolution to the disordered state (a vicinity of the point $C_0$ in Fig.3) for extremely large time $t \to \infty$. The keypoint of our consideration is that the fractional average appearance in Eq. (19) does not allow to use the ordinary Lyapunov’s method because the exponential time-dependence becomes invalid. Instead, let us introduce the generalized exponential form:

$$e^{qt} \to E_q(t) \equiv [1 + (1 - q)t]^{1/1-q}$$ (24)

where $q$ is a generalized Lyapunov index. First, such a type of generalization was used by Tsallis [8] to obtain the ordinary Gibbs-Boltzmann exponent in the limit $q \to 1$. In our case, the latter is arbitrary so that function $E_q(t)$ acquires the power-law character, in particular the follow derivation rule is fulfilled:

$$\partial E_q(t)/\partial t = (E_q(t))^q \equiv E_q^q(t).$$ (25)

In the limits of the short and long times this function has the asymptotic behavior:

$$\lim_{t \to 0} E_q(t) \to 1 + t, \quad \lim_{t \to \infty} E_q(t) \to ((1 - q)t)^{1/1-q}. $$ (26)

Below, the first of the asymptotes will be used for extraction of the Lyapunov-type multipliers, the second one allows us to set an index $q$ (see after Eq.(30)).

Let us define the solutions of Eqs.(5), (19) in the form

$$\eta(t) = m E_\mu(t), \quad S(t) = S_0 + n E_\nu(t)$$ (27)

where $S_0 = \epsilon/3$ to correspond to the point $C_0$; the exponents $\mu$, $\nu$ and coefficients $m, n$ must be determined. Inserting Eqs.(27) into Eqs.(5), we obtain up to the first power of $m, n \ll 1$

$$3nE_\mu^{1-\mu}(t)E_\nu(t) = -1.$$ (28)

From this, within the long-time approximation (29), one obtains

$$n^{-1} = 3(1 - \mu), \quad \nu = 2.$$ (29)

It is worth to note that the parameter $n \ll 1$ to be a second term in expansion (27), whereas a magnitude $m$ can be arbitrary because it stands as a first term there. Physically, it means that the order parameter behaves in non-linear manner, in contrast to a linear regime of the structure factor.
Respectively, the inserting Eq.\((27)\) into Eq.\((19)\) gives
\[
E_{\nu}^{1-\nu}(t) \left[ 2\varepsilon n - \alpha_{1} m^{p_{1}} E_{\mu}^{p_{1}}(t) E_{\nu}^{-1}(t) \right] = -n. \tag{30}
\]
As has been pointed out above, in the short-time limit the function \(E_{\nu}^{1-\nu}(t)\) can be taken as 1 to correspond to extraction of the Lyapunov multiplier. Respectively, in the long-time limit there is \(E_{\mu}^{p_{1}}(t) E_{\nu}^{-1}(t) = \text{const} \equiv p_{1}^{-1}\) and we obtain
\[
3\alpha_{1} m^{p_{1}} = -(1 + 2\varepsilon), \quad \mu = 1 + p_{1} \equiv 1 + [2(1 - a)]^{-1}. \tag{31}
\]
Thus, within the long time-approximation, the structure factor
\[
S(t) = S_{0} + \left( \frac{2}{3} \right)(1-a)t^{-1}, \quad t \to \infty \tag{32}
\]
tends to the stable magnitude \(S_{0}\) hyperbolically. The order parameter is decreased according to the power dependence
\[
\eta(t) = \eta_{0} - [2(1-a)]^{2(1-a)}|m|t^{-2(1-a)}, \quad t \to \infty \tag{33}
\]
with the exponent decreasing with the parameter \(a\); the amplitude \(m\) is given by the first equation \((31)\).

Thus, in accordance with the phase portraits in Fig.3a, at \(\varepsilon < \varepsilon_{0}\) the order parameter \(\eta(t)\) decreases monotonically in the course of time, whereas the structure factor can vary non-monotonically, in contrast to the case \(a > 1/2\) (cf. Fig.1). More complex behavior appears within the domain \(\varepsilon > \varepsilon_{0}\) when the ordered state occurs due to the bifurcation. As is seen from Fig.3b the phase plane divides into two domains corresponding to small and large values of the order parameter. Within the former, the system behaves as at the above case \(\varepsilon < \varepsilon_{0}\), but if an initial magnitude of the order parameter is more a critical value, the system passes to the attractive point \(C\). The corresponding dependencies of the time are depicted in Fig.6. It is characterically that these dependencies display a critical slowing-down near the separatrix \(C_{0}SC\) in Fig.3b (see curves 1, 2 in Figs.6a, 6b).

In order to analyze the long-time behavior in the vicinity of the point \(C\), we can not use the solution like the generalized exponent \((24)\). The latter is applicable when nonlinearity effects are sufficient to fix the above mentioned amplitudes \(m, n\) by Eqs.\((29), (31)\). In the case under consideration, the linear conditions are satisfied and instead of the generalized exponent \((24)\) we ought to use the Mellin transformation. The principle difference is that the former is defined by the single index \(q\), whereas the later contains a set of \(q\). Inserting the definitions (cf. Eqs.\((27)\))
\[
\eta(t) = \eta_{0} + \int m_{q} t^{q} dq, \tag{34}
\]
\[
S(t) = S_{0} + \int n_{q} t^{q} dq \tag{35}
\]
into Eqs. (5), (19) being written within the linear approximation, we obtain the equations for specific amplitudes $m_q, n_q \ll 1$:

$$
\begin{align*}
(q/t + 2\eta_0^2)m_q - 3\eta_0 n_q &= 0, \\
[4\eta_0(\varepsilon - \eta_0^2) - \alpha_1 \eta_0^{p_1-1} p_1]m_q + [q/t + 2(\varepsilon + \eta^2)]n_q &= 0.
\end{align*}
$$

(36) (37)

This system has solutions provided the ratio $c \equiv -q/t$ is determined by equation

$$
c = (\varepsilon + \eta_0^2) \left[ 1 \pm \sqrt{1 - \frac{8\eta_0^2(2\varepsilon - \eta_0^2) - 3\alpha_1 \eta_0^{p_1} p_1}{\varepsilon + 2\eta_0^2}} \right].
$$

(38)

Thus, in a vicinity of the ordered point $C$ the dependencies of the order parameter and structure factor (34), (35) take the forms

$$
\begin{align*}
\eta(t) &= \eta_0 + m \exp(-ct \ln t), \\
S(t) &= S_0 + n \exp(-ct \ln t),
\end{align*}
$$

(39) (40)

where the amplitudes $m, n$ correspond to the index $q = -ct$.

Finally, the time dependencies of the Green function $G(t, 0)$ are determined by Eqs. (5), (6), (19) as is shown in Fig. 7. At $\varepsilon < \varepsilon_0$, the monotonic decrease appears for large value of initial magnitude $\eta(0)$ of the order parameter (see Fig. 7a). In the case $\varepsilon > \varepsilon_0$, a maximum of dependence $G(t, 0)$ disappears at condition $\eta(0) > \eta_c$ corresponding to the domain of the ordering state (see Fig. 7b).

6 Conclusion

Summarizing, we have derived and analyzed the equations for order parameter, structural factor and correlation function to describe the evolution of the system with multiplicative noise. In representation of the noise amplitude as the power-law function $g(x) = x^a$, we have discussed a way to apply the cumulant expansion for average $\langle x^{2a} \rangle$ in the case of a self-similar phase space.

We have shown that the system behavior is given by magnitude of the exponent $a$: at $a > 1/2$ the system is disordered; at $a < 1/2$ a phase transition to the ordered state is observed. In the first case the obtained time dependencies for the structure factor show the monotonic increasing to a stationary state, whereas the correlator shows the non-monotonic behavior. In the second case ($a > 1/2$) we would have to investigate the system behavior on the phase plane given by order parameter $\eta$ and structure factor $S$. It was shown that the phase portrait falls into two domains characterized by disordered and
ordered states (the former corresponds to small initial values of $\eta$, the latter – to finite values of both $\eta$ and $S$). We have found how the system attains the steady states at long-time asymptotics. Within disordered domain, the structure factor decays hyperbolically and the order parameter dependence is described by the power-law function with the exponent $-2(1 - a)$. In the ordered phase, the order parameter and structure factor exhibit exponential behavior with index proportional to $-t \ln t$.

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Captions

Fig. 1. Structure factor behavior at $a > 1/2$:

a) time dependencies $S(t)$ (curves 1, 2, 3 correspond to $a = 0.6, \varepsilon = 0.2$; $a = 0.6, \varepsilon = 0.4$; $a = 0.9, \varepsilon = 0.2$);

b) stationary point $S_0$ vs temperature $\varepsilon$ for several values of exponent $a$.

Fig. 2. Correlator $G(t,0)$ vs time at $a > 1/2$ for several exponent $a$ and temperature $\varepsilon$ (curves 1, 2, 3 correspond to: $a = 0.6, \varepsilon = 0.4$; $a = 0.6, \varepsilon = 0.2$; $a = 0.9, \varepsilon = 0.4$).

Fig. 3. Phase portrait at $a > 1/2$:

a) $a = 0.3, \varepsilon = 0.2$;

b) $a = 0.3, \varepsilon = 0.4$.

Fig. 4. Stationary states of the system at $a > 1/2$:

a) order parameter $\eta$ vs temperature $\varepsilon$ for several exponent $a$;

b) structure factor $S$ vs temperature $\varepsilon$ for several exponent $a$.

Fig. 5. Phase diagram $\varepsilon_0(a)$.

Fig. 6. Time dependencies corresponded to different trajectories on the phase portrait in Fig. 3:

a) $\eta$ vs ln $t$ at $a = 0.3, \varepsilon = 0.2$ and $S(0) = 0$ (curves 1, 2, 3 correspond to $\eta(0) = 0.057$, $\eta(0) = 0.066$, $\eta(0) = 1.0$);

a) $S$ vs ln $t$ at $a = 0.3, \varepsilon = 0.4$ and $S(0) = 0$ (curves 1, 2, 3 correspond to $\eta(0) = 0.057$, $\eta(0) = 0.066$, $\eta(0) = 1.0$).

Fig. 7. Correlator $G(t,0)$ vs time $t$ at:

a) $a = 0.3, \varepsilon = 0.2$, $S(0) = 0$ (curves 1, 2, 3 correspond to $\eta(0) = 0.098$, $\eta(0) = 0.4$, $\eta(0) = 0.97$);

b) $a = 0.3, \varepsilon = 0.4$, $S(0) = 0$ (curves 1, 2, 3 correspond to $\eta(0) = 0.057$, $\eta(0) = 0.066$, $\eta(0) = 1.0$).
\[ a = 0.6 \]
\[ a = 0.9 \]

Fig. 1
$G(t,0)$

Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7