Effective theory for the Goldstone field in the
BCS-BEC crossover at $T = 0$

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Abstract

We perform a detailed study of the effective Lagrangian for the Goldstone mode of a superfluid Fermi gas at zero temperature in the whole BCS-BEC crossover. By using a derivative expansion of the response functions, we derive the most general form of this Lagrangian at the next to leading order in the momentum expansion in terms of four coefficient functions. This involves the elimination of all the higher order time derivatives by careful use of the leading order field equations. In the infinite scattering length limit where conformal invariance is realized, we show that the effective Lagrangian must contain an unnoticed invariant combination of higher spatial gradients of the Goldstone mode, while explicit couplings to spatial gradients of the trapping potential are absent. Across the whole crossover, we determine all the coefficient functions at the one-loop level, taking into account the dependence of the gap parameter on the chemical potential in the mean-field approximation. These results are analytically expressed in terms of elliptic integrals of the first and second kind. We discuss the form of these coefficients in the extreme BCS and BEC regimes and around the unitary limit, and compare with recent work by other authors.
I. INTRODUCTION

The last few years have witnessed a renewed interest in the physics of the BCS-BEC crossover \([1, 2, 3]\), partly motivated by the availability of tunable interactions in the realm of interacting Fermi gases \([4]\). Recent experimental work \([5, 6, 7, 8, 9]\) has shown evidence for condensation of fermionic atom pairs, suggesting the formation of a fermionic superfluid. From the theoretical point of view, the qualitative description of the BCS-BEC crossover has been based on the mean-field theory of Leggett \([2]\) and its extension to finite temperature by Nozières and Schmitt-Rink \([10]\) and Sá de Melo, Randeria and Engelbrecht \([11]\). In this description of the superconducting system, the effective action in terms of a complex order parameter which couples to the pairing field plays a central role. Some recent developments \([12, 13, 14, 15]\) in the crossover problem beyond mean field theory have improved our understanding of the equilibrium state at a quantitative level. In particular, Diener et al. \([15]\), by computing the complete quadratic part of the effective action, have obtained the correction to the mean field result which arises from the integration of the Gaussian fluctuations, finding excellent agreement with calculations based on quantum Monte Carlo techniques in the unitary limit where the scattering length \(a \to \infty\).

The effective action can also used, in principle, to derive an effective Lagrangian which captures the low-energy behavior of the system in terms of the Goldstone mode phase of the order parameter. At zero temperature, one expects that an expansion of the effective Lagrangian in derivatives of the Goldstone field can be used in order to study the low-energy behavior of the system. The leading order (LO) in this expansion was evaluated by Greiter et al. \([16]\) and by Aitchison et al. \([17]\) some years ago, and since then, there have been various microscopic derivations \([18, 19, 20, 21, 22]\) of effective models, some of which \([19, 20]\) have included more or less explicitly some derivative corrections.

In a recent work, Son and Wingate \([23]\) have systematically studied the form of the effective Lagrangian in the unitary limit at the next-to-leading order (NLO) in the derivative expansion, when the effective theory is formulated in terms of the Goldstone mode coupled to external gauge and gravitational fields. At unitarity, it turns out that, besides general coordinate and gauge invariance, the theory exhibits conformal invariance \([24]\), which puts constraints on the form of the NLO Lagrangian and restricts to two the number of independent NLO parameters.

In this paper, we extend these studies. We evaluate the effective parameters of the NLO Lagrangian in the unitary limit at the mean-field level, and also obtain the most general form of this Lagrangian away from this limit, where the symmetry under conformal transformations is not realized. By computing all the necessary functions at the one-loop level in terms of elliptic integrals, we have obtained the simplest approximation to the low energy effective theory for the whole crossover region at zero temperature.

We find that, at unitarity, the effective Lagrangian is specified by two constants, but its actual form differs from that given in Ref. \([23]\), and includes a new contribution \((\partial_i \partial_j \theta)^2\) of higher spatial derivatives of the Goldstone mode, while the NLO contribution of the external trapping potential proportional to \(\nabla^2 V_{\text{ext}}\) is absent. We show that these features are a necessary consequence of the conformal invariance of the NLO field equations. As an application, we derive the energy density functional in the unitary limit, and compare it with the computation of Rupak and Schäffer \([25]\), which is based on an epsilon expansion around \(d = 4 - \varepsilon\) spatial dimensions and is, to our knowledge, the only one in the literature. Although the coefficients computed by these two methods show discrepancies of the order of
30% in general, we find a surprisingly good agreement for the coefficient of the quantum pressure.

For the whole crossover region, we obtain the NLO Lagrangian in terms of four functions which are given in closed form in terms of elliptic integrals. The BCS and BEC limits as well as the near unitarity limits of the NLO Lagrangian are worked out in detail. In the BEC limit, we recover the known features of the hydrodynamic description of superfluidity at zero temperature.

The plan of this paper is as follows. In Section II we formulate the problem in the framework of linear response theory and derive a linearized equation for the Goldstone field in terms of derivatives of response functions. In Section III we show how to construct a Lagrangian, including second order time derivatives of $\theta$, by considering all the available Galilean invariants consistent with required general properties. Then we present a careful procedure of reduction, and show how to use the LO field equation in order to eliminate undesired higher order time derivatives without changing its perturbative contents. We also argue the need to compute two three-point functions in order to determine all the coefficient functions in the effective Lagrangian. In Section IV we present an analytical expression for the thermodynamical potential at the one-loop level, and the analytical expressions of the NLO coefficients in the two (BCS and BEC) limits and near unitarity. In Section V we compute the energy density functional in the unitary limit, and compare our result with other approaches. Section VI gives our conclusions. Details of the calculations as well as additional material on the invariance properties under conformal transformations in the unitary limit are given in a series of four appendices.

II. DERIVATIVE EXPANSION OF THE RESPONSE FUNCTIONS AT $T = 0$

The system is conveniently described by the BCS Lagrangian density in terms of a Nambu spinor field and a complex field which decouples the short-range interaction in the Cooper channel

$$\mathcal{L} = \Psi^\dagger \left[ i \partial_t + \tau^3 \frac{1}{2m} \nabla^2 + \frac{1}{2} (\tau^1 + i\tau^2) \Delta + \frac{1}{2} (\tau^1 - i\tau^2) \Delta^* \right] \Psi - \frac{1}{g_\Lambda} \Delta^* \Delta, \quad (2.1)$$

where $\Psi^\dagger = (\psi^\dagger_\uparrow, \psi^\dagger_\downarrow)$ and $\tau^i$ is the corresponding Pauli matrix. The bare coupling parameter $g_\Lambda$ depends on the details of the short range interaction and on a regulator that truncates some loop integrals at some very large scale $\Lambda$. As usual in the framework of effective field theories, the result of adding the appropriate loop corrections to this bare coupling will be matched with the measured low energy scattering properties encoded in the $s$-wave scattering length $a$. Other physical parameters such as the $s$-wave effective range or the $p$-wave scattering length are related to operators with more derivatives in the effective Lagrangian, and hence, they are generically subleading in the expansion in powers of $R \partial$, where $R$ is some length setting the size of the interaction region. The Lagrangian is invariant under the $U(1)$ symmetry of phase independent spacetime-independent transformations $\Psi \rightarrow e^{i\gamma_3 \theta} \Psi$, $\Delta \rightarrow e^{i2\theta} \Delta$, which is spontaneously broken down to $\hat{H} = Z_2$ below a critical temperature. In order to compute the effective action for the resulting gapless collective mode it is convenient to express the fields as a $U(1)$ transformation acting on fields $\tilde{\Psi}$ and $\sigma$ which do not contain the Goldstone mode

$$\Psi(x) = e^{i\tau^3 \theta(x)} \tilde{\Psi}(x), \quad \Delta(x) = e^{i2\theta(x)} (\Delta_0 + \sigma(x)), \quad (2.2)$$
where the constant amplitude $\Delta_0$ and its fluctuation $\sigma(x)$ are real numbers. Since we shall hereafter use the fermion fields $\Psi$, the tildes will be dropped to simplify the notation. With this choice, the Lagrangian involving the couplings between the fermion field and the other fields becomes, after an integration by parts,

$$L_I = -\Psi^\dagger \tau^3 \Psi \left( \partial_t \theta + \frac{1}{2m} (\nabla \theta)^2 + V_{\text{ext}} \right) - \frac{1}{2mi} (\Psi^\dagger \nabla \Psi - \nabla \Psi^\dagger \Psi) \cdot \nabla \theta + \Psi^\dagger \tau^1 \Psi \sigma, \quad (2.3)$$

where $V_{\text{ext}}$ is an external potential. The quadratic part $L_2$ of the Lagrangian including the chemical potential coupled to conserved particle density $\Psi^\dagger \tau^3 \Psi$ is given by Eq. (2.1) with the replacements $\nabla^2 / (2m) \rightarrow \nabla^2 / (2m) + \mu$, $\Delta \rightarrow \Delta_0$ and $\Delta^* \Delta \rightarrow \sigma^2$.

Under an infinitesimal Galilean transformation

$$x \rightarrow x' = x + vt, \quad t \rightarrow t' = t, \quad (2.4)$$

the Goldstone field changes inhomogeneously as

$$\delta \theta(t, x) = -vt \cdot \nabla \theta + mv \cdot x. \quad (2.5)$$

Once the auxiliary field $\sigma$ has been eliminated, the invariance of the effective action for $\theta$ requires a dependence through the scalar quantity

$$X \equiv \mu - \partial_t \theta - \frac{1}{2m} (\nabla \theta)^2 - V_{\text{ext}}, \quad \delta X = -vt \cdot \nabla X, \quad (2.6)$$

and some appropriate derivatives of $X$ and $\nabla \theta$. According to the power counting scheme of Ref. [23], these quantities are $O(p^0)$. This can also be seen by noting that, at equilibrium, the gradient of the phase $v_s = m^{-1} \nabla \theta$ is the constant superfluid velocity, and $\partial_t \theta$ goes with the chemical potential. To next-to-leading order, we have the following $O(p)$ galilean invariant derivatives

$$\nabla^2 \theta, \quad \left[ \partial_i + \frac{\nabla \theta}{m} \cdot \nabla \right] X, \quad \nabla X, \quad \partial_i \partial_j \theta, \quad (2.7)$$

where we have only written derivatives of the quantities $X$ and $\nabla \theta$, which are coupled to the particle density and the current. The signature under time reversal ($t \rightarrow -t$, $\theta \rightarrow -\theta$) of the two scalar terms (and the tensorial term) is $-1$, and hence there is no contribution of $O(p)$ to the effective Lagrangian. In the next Section we will list all possible scalar terms of $O(p^2)$ potentially contributing to the effective Lagrangian.

A possible way to derive the effective Lagrangian for the Goldstone mode is to compute the appropriate linear response at low frequency and momentum for an external perturbation given by a Hamiltonian

$$H^{\text{ext}}(t) = \int d^3 x \left[ \Psi^\dagger \tau^3 \Psi (\partial_t \theta + V_{\text{ext}}) + \frac{1}{2mi} (\Psi^\dagger \nabla \Psi - \nabla \Psi^\dagger \Psi) \cdot \nabla \theta - \Psi^\dagger \tau^1 \Psi \sigma \right]$$

$$= \int d^3 x \left[ n(\partial_t \theta + V_{\text{ext}}) + j^p \cdot \nabla \theta - \Psi^\dagger \tau^1 \Psi \sigma \right], \quad (2.8)$$

1 A quantity containing $N[\partial_t]$ time derivatives, $N[\partial_i]$ spatial derivatives and $N[\theta]$ powers of the Goldstone field is counted as $O(p^N)$, where $N = N[\partial_t] + N[\partial_i] - N[\theta]$.
where \( j^0 \) is the current density operator in the absence of \( \nabla \theta \) and the total current operator is \( j = j^0 + \Psi^\dagger \tau^3 \Psi m^{-1} \nabla \theta \). The induced changes to be computed are \( \delta \langle n(Q) \rangle, \delta \langle j(Q) \rangle \) and the change in the expectation value of the pairing field \( \delta (\Psi^\dagger \tau^1 \Psi) \) in terms of \( \theta(Q) \) and \( \sigma(Q) \), where \( Q = (q, \omega) \). By combining the conservation of the particle number with the gap equation for \( \delta (\Psi^\dagger \tau^1 \Psi) \), one can obtain a single linear equation of motion for \( \theta \) and, consequently, a quadratic Lagrangian to be consistently matched with the form of the Galilean invariants listed above.

In what follows, we use \( \chi_{AB}(Q) \) for the Fourier transform of the retarded response function \( \chi_{AB}(X, X') = -i\langle [A(X), B(X')] \rangle \theta(t - t') \). For the Hamiltonian given in Eq. (2.8), the required set of linear response equations is

\[
\begin{align*}
\delta \langle n(Q) \rangle &= \chi_{nn}(Q) (-i\omega \theta(Q) + V_{\text{ext}}(Q)) + \chi_{nj}^k(Q) iq^k \theta(Q) - \chi_{n1}(Q) \sigma(Q), \\
\delta \langle j^k(Q) \rangle &= \chi_{jn}^k(Q) (-i\omega \theta(Q) + V_{\text{ext}}(Q)) + \chi_{jj}^{kl}(Q) iq^l \theta(Q) - \chi_{j1}^k(Q) \sigma(Q), \\
\delta \langle \Psi^\dagger \tau^1 \Psi \rangle &= \chi_{1n}(Q) (-i\omega \theta(Q) + V_{\text{ext}}(Q)) + \chi_{1j}^k(Q) iq^k \theta(Q) - \chi_{11}(Q) \sigma(Q),
\end{align*}
\]

where there is a summation over upper repeated indices. It is useful to collect some symmetry properties that follow from the behavior under time reversal and parity:

\[
\begin{align*}
\chi_{jn}^k(Q) &= \chi_{nj}^k(Q), \\
\chi_{j1}^{kl}(Q) &= \chi_{j1}^{lk}(Q), \\
\chi_{n1}(Q) &= \chi_{1n}(Q).
\end{align*}
\]

When the operators \( A \) and \( B \) have the same (opposite) signature under time reversal, \( \text{Im} \chi_{AB}(Q) \) is odd (even) in \( \omega \). This implies that \( \text{Re} \chi_{jn}(Q), \text{Re} \chi_{j1}(Q) \) are odd in \( \omega \), while \( \text{Re} \chi_{n1}(Q), \text{Re} \chi_{nn}(Q) \) and \( \text{Re} \chi_{11}(Q) \) are even functions of \( \omega \). The current-current response function can be written as

\[
\chi_{j1}^{kl}(Q) = \chi_{L}(Q)q^k q^l + \chi_{T}(Q) (\delta^{kl} - q^k q^l),
\]

where \( \text{Re} \chi_{L,T}(Q) \) are even in \( \omega \). At \( q = 0 \), they must obey the usual sum rules giving the static responses at superfluidity

\[
\begin{align*}
\chi_{L}(0) &= \lim_{q \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\text{Im} \chi_{L}(q, \omega)}{\omega} = \frac{\langle n \rangle}{m}, \\
\chi_{T}(0) &= \lim_{q \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\text{Im} \chi_{T}(q, \omega)}{\omega} = \frac{\langle n \rangle - n_s}{m},
\end{align*}
\]

where \( n_s \) is the superfluid particle density.

The results of this Section do not depend on the specific approximation used to compute the response functions, but in order to gain some physical insight on them, we write the kind of integrals to be computed at the one-loop level.

\[
\int \frac{d\nu_1 d^3k}{(2\pi)^4} \left\{ \begin{array}{c}
1 \\
(2k + q)^k \\
(2k + q)^k (2k + q)^l
\end{array} \right\} \text{tr} \left[ \tau^\mu G(k + q, i\nu_1 + z) \tau^\nu G(k, i\nu_1) \right]\bigg|_{z = \omega + i\epsilon}.
\]

Here \( \tau^0 \) is the identity matrix and \( G \) is the Nambu-Gorkov Green’s function

\[
G(k, z) = \frac{z + \tau^3 \xi_k - \tau^1 \Delta_0}{z^2 - E_k^2},
\]

\( E_k = \sqrt{\xi^2 + m^2} \).
with \( \xi_k = \epsilon_k - \mu = k^2 / 2m - \mu \) and \( E_k = \sqrt{\xi_k^2 + \Delta_0^2} \). Note that, together with the contribution coming from \([j^p, j^p]\), the current-current response function \( \chi_{jj}^{kl}(Q) \) includes the longitudinal piece \( \langle n \rangle m^{-1} \hat{q}^k \hat{q}^l \), where \( \langle n \rangle \) is the total particle density. The integration over the imaginary frequency produces the denominators \( \omega + i\varepsilon \pm (E_k + E_{k+q}) \), which for small frequency \( |\omega| < 2\Delta_0 \) do not contribute to the imaginary part of the response function. In addition, the expansions about \( \omega = 0 \) and \( q = 0 \) are well behaved, in marked contrast to the expansions of the denominators \( \omega + i\varepsilon \pm (E_k - E_{k+q}) \) which arise from Landau damping at \( T \neq 0 \). Thus, at \( T = 0 \) the needed response functions are regular real functions near \( Q = 0 \). After some calculation, the one-loop expressions for \( \chi_{AB}(q = 0, \omega = 0) \) become

\[
\chi_{nn}(0) = -\int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0^2}{E_k^3},
\]

\[
\chi_{n1}(0) = -\int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0 \xi_k}{E_k^3},
\]

\[
\chi_{11}^{(\Lambda)}(0) = -\int_\Lambda \frac{d^3k}{(2\pi)^3} \frac{\xi_k^2}{E_k^3},
\]

\[
\chi_{jj}^{kl}(0) = \frac{\langle n \rangle}{m} \hat{q}^k \hat{q}^l,
\]

and we see that static current transverse response \( \chi_T(0) \) vanishes which, according to Eq. (2.17), shows that the entire system is superfluid, \( \langle n \rangle = n_s \), at \( T = 0 \).

The linear ultraviolet divergence in \( \chi_{11} \) has been regulated by the cut-off \( \Lambda \). As it is well known, the renormalization of this divergence is performed by the substitution of the bare coupling constant \( g_\Lambda \) in terms of the \( s \)-wave scattering length \( a \). Putting together the leading order interaction of the form \( g_\Lambda^{-1} \) and the first correction in the vacuum, one obtains the measured coupling constant \( g \equiv -4\pi a m^{-1} \),

\[
\frac{1}{g} = \frac{1}{g_\Lambda} - \int_\Lambda \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k},
\]

and this relation allows the determination of \( g_\Lambda \) in terms of \( a \). As all the calculations will be \( \Lambda \) independent, it is convenient to define the `renormalized’ \( \chi_{11} \) as

\[
\chi_{11}(0) \equiv -\int \frac{d^3k}{(2\pi)^3} \left( \frac{\xi_k^2}{E_k^3} - \frac{1}{\epsilon_k} \right) = \chi_{11}^{(\Lambda)}(0) + 2 \frac{\Delta_0 + m}{2\pi a}.
\]

As mentioned above, at \( T = 0 \) all response functions are real in the region of small \( Q \) and their lowest-order derivatives at \( Q = 0 \) must determine the form of the effective Lagrangian at next-to-leading order in the derivative expansion. We next show how to compute this Lagrangian. From the gap equation

\[
\delta \langle \Psi^\dagger \tau^1 \Psi \rangle = \frac{2}{g_\Lambda} \sigma(Q),
\]

and the equation for the response \( \delta \langle \Psi^\dagger \tau^1 \Psi \rangle \), we can write the change \( \sigma(Q) \) in the local amplitude in terms of \( \theta(Q) \) and \( V_{ext}(Q) \). If we replace the result for \( \sigma(Q) \) in the continuity equation

\[
- \omega \delta \langle n(Q) \rangle + q^k \delta \langle j^k(Q) \rangle = 0,
\]

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and expand in powers of \( \omega \) and \( q \), we find the following equation for \( \theta(Q) \)
\[
0 = (b_1 \omega^2 + b_2 q^2 + b_3 \omega^2 q^2 + b_4 \omega^4 + b_5 q^4 + \ldots) \theta(Q) + (b_1 + b_6 q^2 + b_4 \omega^2 + \ldots) i \omega V_{\text{ext}}(Q),
\]
where
\[
b_1 = -\chi_{nn}(0) + \frac{g \chi_{n1}(0)^2}{2 + g \chi_{11}(0)},
\]
\[
b_2 = -\chi_{L}(0) = -\langle n \rangle / m,
\]
\[
b_3 = 2 \frac{\partial \chi_{jn}}{\partial \omega} - \frac{\partial \chi_{nn}}{\partial q^2} - \frac{\partial \chi_{n1}}{\partial \omega^2} - \frac{2 g \chi_{n1}}{2 + g \chi_{11}} \left( \frac{\partial \chi_{j1}}{\partial \omega} - \frac{\partial \chi_{n1}}{\partial q^2} \right) - \frac{g^2 \chi_{n1}^2}{(2 + g \chi_{11})^2} \frac{\partial \chi_{11}}{\partial q^2},
\]
\[
b_4 = -\frac{\partial \chi_{nn}}{\partial \omega^2} + \frac{2 g \chi_{n1}}{2 + g \chi_{11}} \frac{\partial \chi_{n1}}{\partial \omega^2} - \frac{g^2 \chi_{n1}^2}{(2 + g \chi_{11})^2} \frac{\partial \chi_{11}}{\partial \omega^2},
\]
\[
b_5 = -\frac{\partial \chi_{L}}{\partial q^2},
\]
\[
b_6 = \frac{\partial \chi_{jn}}{\partial \omega} - \frac{\partial \chi_{nn}}{\partial q^2} - \frac{2 g \chi_{n1}}{2 + g \chi_{11}} \left( \frac{1}{2} \frac{\partial \chi_{j1}}{\partial \omega} - \frac{\partial \chi_{n1}}{\partial q^2} \right) - \frac{g^2 \chi_{n1}^2}{(2 + g \chi_{11})^2} \frac{\partial \chi_{11}}{\partial q^2}.
\]
All these parameters are computable from the appropriate two-point retarded functions and their derivatives. As the gap equation in terms of the thermodynamic potential \( \Omega(\mu, \Delta_0) \) at \( T = 0 \)
\[
\frac{\partial \Omega(\mu, \Delta_0)}{\partial \Delta_0} = 0
\]
implicitly determines the function \( \Delta_0(\mu) \), the coefficients \( b_i(\mu) \) depend on the chemical potential and, parametrically, on the scattering length \( a \).

III. THE HIGHER-ORDER AND THE REDUCED EFFECTIVE LAGRANGIANS

A. The higher-order effective Lagrangian

We now write an effective Lagrangian for the Goldstone mode which leads to the above equation of motion for \( \theta \). This Lagrangian contains higher-order time derivatives, which will be dealt with in the second part of this Section. Once the auxiliary field \( \sigma \) has been eliminated through the use of the gap equation, the NLO Lagrangian is a linear combination of all the independent scalar operators of \( O(p^2) \) constructed from derivatives of \( X \) and \( \nabla \theta \), with coefficients which are functions of \( X \), the only scalar of \( O(p^0) \). Note that, as mentioned above, \( O(p) \) scalar terms such as \( r_1(X) \nabla^2 \theta \) and \( r_2(X) X_t \) are excluded by invariance under time reversal.

There are only five independent NLO terms, \( (\nabla X)^2, (\nabla^2 \theta)^2, (\partial_t \partial_\theta)^2, X_t^2 \), and \( X_t \nabla^2 \theta \), where
\[
X_t \equiv \partial_t X + m^{-1} \nabla \theta \cdot \nabla X.
\]
Other candidate terms can be shown to be dependent on these five. For instance, we have excluded a term \( h(X) \nabla^2 X \), which becomes proportional to \( (\nabla X)^2 \) after an integration by parts. Another potential contribution of \( O(p^2) \), \( (\partial_t + m^{-1} \nabla \theta \cdot \nabla) \nabla^2 \theta \), which by
\[
(\partial_t + m^{-1} \nabla \theta \cdot \nabla) \nabla^2 \theta = -\nabla^2 X - \nabla^2 V_{\text{ext}} - m^{-1} (\partial_t \partial_\theta)^2
\]
is equivalent to $\nabla^2 V_{\text{ext}}$, must also be excluded, since the identity

$$G'(X)X_t \nabla^2 \theta + G'(X)(\nabla X)^2 - G(X)\nabla^2 V_{\text{ext}} - \frac{1}{m} G(X) [(\partial_i \partial_j \theta)^2 - (\nabla^2 \theta)^2] =$$

$$\partial_t (G(X)\nabla^2 \theta) + \nabla \cdot \left( \frac{\nabla \theta}{m} G(X)\nabla^2 \theta + G(X)\nabla X \right)$$

(3.2)

shows that this term is in fact redundant. Thus, the most general Lagrangian up to next-to-leading order in the derivative expansion is given in terms of six coefficient functions

$$\mathcal{L} = P(X) + g_1(X)(\nabla X)^2 + g_2(X)(\nabla^2 \theta)^2 + g_3(X)(\partial_i \partial_j \theta)^2 + g_4(X)X_t^2 + g_5(X)X_t \nabla^2 \theta,$$

(3.3)

to be evaluated from the low-energy behavior of the response functions.

The linearized field equation following from this Lagrangian has the form

$$0 = -P''(\mu) \left( \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial V_{\text{ext}}}{\partial t} \right) + \frac{P'(\mu)}{m} \nabla^2 \theta + 2 [g_1(\mu) - g_5(\mu)] \frac{\partial^2 (\nabla^2 \theta)}{\partial t^2}$$

$$+ 2 [g_2(\mu) + g_3(\mu)] \nabla^4 \theta + [2g_1(\mu) - g_5(\mu)] \frac{\partial (\nabla^2 V_{\text{ext}})}{\partial t} + 2g_4(\mu) \left( \frac{\partial^4 \theta}{\partial t^4} + \frac{\partial^3 V_{\text{ext}}}{\partial t^3} \right).$$

(3.4)

Comparison with Eq. (2.28) yields the relations

$$g_1 = -\frac{b_2}{2} + b_6,$$

(3.5a)

$$g_2 + g_3 = \frac{b_6}{2},$$

(3.5b)

$$g_4 = \frac{b_1}{2},$$

(3.5c)

$$g_5 = -b_3 + b_6,$$

(3.5d)

and

$$P''(\mu) = b_1,$$

(3.6a)

$$P'(\mu) = -mb_2 = \langle n \rangle.$$  

(3.6b)

These expressions match the coefficients of the effective Lagrangian with the response functions. Note that from the response functions we can only determine the sum $g_2(X) + g_3(X)$, but not the individual functions. This is not surprising since, in the quadratic approximation to the Lagrangian, the terms $(\nabla^2 \theta)^2$ and $(\partial_i \partial_j \theta)^2$ are not independent, but differ by a total derivative. Hence, in order to evaluate $g_2$ and $g_3$ separately, one must resort to the computation of three-point functions. Details of this computation at the one-loop level are given in Appendix D, where we find the remarkably simple result $g_3(X)/g_2(X) = 2$. This implies:

$$g_2(X) = \frac{1}{6} b_5, \quad g_3(X) = \frac{1}{3} b_5.$$

(3.7)

Regarding the expressions for the leading order coefficients $b_1(\mu)$ and $b_2(\mu)$, it is possible to check their consistency without actually computing them. As the authors of Refs. [14, 23] have shown, the leading order effective Lagrangian at $T = 0$ is a function of the invariant
which precisely coincides with the pressure \( P(\mu) \) as a function of the chemical potential. Obviously, the expression for \( b_2 \) in (2.29b) satisfies this requirement. To see the agreement of \( b_1 \) in (2.29a) with \( P''(\mu) \), we note that the pressure is given by \( P(\mu) = -V^{-1} \Omega(\mu, \Delta_0(\mu)) \), where \( \Delta_0(\mu) \) satisfies the gap equation (2.30). The thermodynamic potential can be written as the sum of a ‘tree’ level renormalized piece and the full quantum contribution \( \Phi \) from the perturbative expansion

\[
\frac{\Omega(\mu, \Delta_0)}{V} = \frac{\Delta_0^2}{g} + \Phi(\mu, \Delta_0).
\]

By using \( \Delta_0'(\mu) = -(\partial^2 \Omega / \partial \Delta_0^2)^{-1} \partial^2 \Omega / \partial \mu \partial \Delta_0 \), a direct evaluation of \( P''(\mu) \) yields

\[
P''(\mu) = -V^{-1} \frac{\partial^2 \Omega}{\partial \mu^2} + V^{-1} \left( \frac{\partial^2 \Omega}{\partial \mu \partial \Delta_0} \right)^2 \left( \frac{\partial^2 \Omega}{\partial \Delta_0^2} \right)^{-1} \bigg|_{\Delta_0=\Delta_0(\mu)},
\]

which agrees with the expression for \( b_1 \) when one identifies properly the static response functions with the susceptibilities

\[
\chi_{nn}(0) = V^{-1} \frac{\partial^2 \Omega}{\partial \mu^2},
\]

\[
\chi_{n1}(0) = V^{-1} \frac{\partial^2 \Omega}{\partial \mu \partial \Delta_0},
\]

\[
\chi_{11}(0) = V^{-1} \frac{\partial^2 \Omega}{\partial \Delta_0^2} - \frac{2}{g}.
\]

Note that the condition of thermodynamic stability implies that \( b_1 \) must be positive. This analysis shows the important role played by the response functions of the pairing field \( \Psi^\dagger \tau \Psi \) in the construction of an effective Lagrangian satisfying known general properties. Some recent work [22] has overlooked this point.

### B. Reduction of the higher-order Lagrangian

To proceed further, we must find a “reduced effective Lagrangian” without higher-order time derivatives, but perturbatively equivalent to Eq. (3.3). The LO field equations can be used, in principle, to eliminate terms with higher order time-derivatives in favour of terms with spatial derivatives but, in doing so, one must be very careful that the perturbative content of the original Lagrangian is preserved. In this regard, it is important to note that the LO equations

\[
X_t + \frac{P'(X)}{mP''(X)} \nabla^2 \theta = O(p^3)
\]

are satisfied only up to terms of \( O(p^3) \) coming from the NLO Lagrangian.

The term proportional to \( X_t^2 \), which gives rise to fourth-order time derivatives of \( \theta \) in the field equation, can be eliminated by adding and subtracting \( g_4(X) \) times the square of the
we are left with the following reduced Lagrangian
\[ \mathcal{L} = P(X) + g_1(X)(\nabla X)^2 + g_2(X)(\partial_i \partial_j \theta)^2 \]
\[ + \left[ g_3(X)\left( \frac{P'(X)}{mP'(X)} \right)^2 \right](\nabla^2 \theta)^2 \]
\[ + \left[ g_4(X) - 2g_4(X)\frac{P'(X)}{mP'(X)} \right]X_t \nabla^2 \theta + g_4(X)\left( X_t + \frac{P'(X)}{mP'(X)} \nabla^2 \theta \right)^2. \] (3.14)

Now, the term proportional to the square of the LO field equation, when evaluated on a perturbative solution, is of \(O(p^6)\), and thus highly suppressed. Furthermore, one can easily check that its contribution to the field equation is of \(O(p^5)\). It can thus be safely dropped and we are left with the following reduced Lagrangian
\[ \tilde{\mathcal{L}} = P(X) + g_1(X)(\nabla X)^2 + \tilde{g}_2(X)\nabla^2 \theta + g_3(X)(\partial_i \partial_j \theta)^2 + \tilde{g}_5(X)X_t \nabla^2 \theta. \] (3.15)

We still have to get rid of the second-order time derivative in the last term of \(\tilde{\mathcal{L}}_{\text{NLO}}\). But in this case we can not use the leading field equation
\[ \tilde{g}_5(X)X_t \nabla^2 \theta \equiv \tilde{g}_5(X) \left( X_t + \frac{P'(X)}{mP'(X)} \nabla^2 \theta \right) \nabla^2 \theta - \tilde{g}_5(X) \frac{P'(X)}{mP'(X)}(\nabla^2 \theta)^2 \] (3.16)
to this end since, even though the numerical correction to the classical effective action introduced by dropping the first term in the RHS of Eq. (3.16) would be of \(O(p^4)\), and thus acceptable, the corresponding correction to the field equation would be \(O(p^3)\), which is of the same order as the NLO contributions to the field equations. This can be checked explicitly by computing the linearized correction to the field equation coming from the first term of the RHS of Eq. (3.16)
\[ - 2\tilde{g}_5(\mu) \nabla^2 \frac{\partial \theta}{\partial t^2} + \frac{\partial V_{\text{ext}}}{\partial t} - \frac{P'(\mu)}{mP'(\mu)} \nabla^2 \theta \] (3.17)

While, by Eq. (3.13), the first term is \(O(p^5)\) on a classical solution and can be dropped, the second term yields a contribution of \(O(p^3)\) which produces an unacceptable change in the perturbative field equations.

Fortunately, we can instead use integration by parts, which is guaranteed to exactly preserve the numerical value of \(\tilde{\mathcal{L}}_{\text{NLO}}\) and the perturbative field equations. With the help of identity (3.2), we perform the replacement
\[ \tilde{g}_5(X)X_t \nabla^2 \theta \rightarrow -\tilde{g}_5(X)(\nabla X)^2 + \tilde{G}_5(X) \nabla^2 V_{\text{ext}} + \frac{1}{m} \tilde{G}_5(X) \left[ (\partial_i \partial_j \theta)^2 - (\nabla^2 \theta)^2 \right], \] (3.18)

where \(\tilde{G}_5'(X) = \tilde{g}_5(X)\). Thus, the reduced effective Lagrangian becomes
\[ \mathcal{L} = P(X) + f_1(X)(\nabla X)^2 + f_2(X)(\nabla^2 \theta)^2 + f_3(X)(\partial_i \partial_j \theta)^2 + f_4(X) \nabla^2 V_{\text{ext}}, \] (3.19)

\[ \text{By this we mean the effective action } \int d^4xdt \tilde{\mathcal{L}} \text{ evaluated on a classical solution.} \]
where the coefficient functions are given by

\[
\begin{align*}
  f_1(X) &= g_1(X) - \tilde{G}_5'(X), \\
  f_2(X) &= g_2(X) - g_4(X) \left( \frac{P'(X)}{m P''(X)} \right)^2 - \frac{1}{m} \tilde{G}_5(X), \\
  f_3(X) &= g_3(X) + \frac{1}{m} \tilde{G}_5(X), \\
  f_4(X) &= \tilde{G}_5(X),
\end{align*}
\]

with

\[
\tilde{G}_5'(X) = g_5(X) - 2 g_4(X) \frac{P'(X)}{m P''(X)}.
\]

The constant of integration in \( \tilde{G}_5(X) \) is irrelevant, since it enters the effective Lagrangian as the coefficient of a total divergence, as follows from Eq. (3.2) when \( G(X) \) is a constant.

Note that the term proportional \((\partial_i \partial_j \theta)^2\), which has not been considered previously in the literature, is an essential ingredient of the effective Lagrangian and can not be eliminated without introducing unacceptable changes in the NLO field equations: Although we could use Eq. (3.2) to eliminate this term in favour of the other three invariants, in doing so we would reintroduce the term proportional to \( X_l \nabla^2 \theta \) which, as we have seen, should not be eliminated through the use of the LO field equations.

An alternative way to understand this issue is by noting that using the LO field equation in Eq. (3.16) would be equivalent, up to \( O(p^4) \) terms arising from the second variation of \( P(X) \), to performing the field redefinition\(^3\) \( \theta = \theta + \delta \theta \)

\[
\delta \theta = \frac{\tilde{g}_5(X)}{P''(X)} \nabla^2 \theta
\]

The variation of \( P(X) \) under this redefinition cancels the last term in Eq. (3.15) and we are left with an effective NLO Lagrangian depending on only three coefficient functions. Since a field redefinition can change neither the value of the action evaluated at its extrema (the classical action) nor the result of any functional integration (the generating functional), it is clear that, for some applications, one can use an effective NLO Lagrangian with only three coefficient functions. However, the field redefinition (3.22) involves a change of \( O(p^2) \), making \( \theta \) and \( \tilde{\theta} \) non-equivalent at NLO. Thus, with hydrodynamical applications in view where the field \( \theta \) has a concrete physical meaning — through the relation to the superfluid velocity \( \mathbf{v}_s = m^{-1} \nabla \theta \) — one is forced to keep all four NLO coefficient functions in (3.19). Only this way can we preserve, at NLO, the interpretation of \( \theta \) as \( (1/2) \) the phase of the condensate, the canonical structure of the theory and the perturbative field equations. Note that the situation is different with the use of the LO field equation to eliminate the term proportional to \( X_l^2 \) in Eq. (3.14), which is equivalent to a field redefinition with

\[
\delta \theta = \frac{g_4(X)}{P''(X)} \left( X_l + \frac{P'(X)}{m P''(X)} \nabla^2 \theta \right)
\]

In this case the difference between \( \theta \) and \( \tilde{\theta} \) is of \( O(p^4) \) and they are equivalent at NLO.

\(^3\) We are indebted to the authors of Ref. [23] for this observation.
The effective theory of the previous Section contains a set of functions depending on the chemical potential and the scattering length once \( \Delta_0(\mu) \) is inserted. Calculating these functions in the mean field approximation is relatively straightforward but tedious and, as explained in Appendix A, all the integrals (3.15) are computable in closed form in terms of complete elliptic integrals of the first and second kinds; the results are given in this Section and in Appendix C. Other physical quantities, such as the length for pair correlation, have been expressed in terms of elliptic integrals in Ref. [27], but here we will focus on the thermodynamical potential and the coefficients \( f_i(X) \). Henceforth the particle density is written as \( \langle n \rangle \equiv k_F^3/3\pi^2 \), where \( k_F \) is the Fermi wave vector.

**A. Leading order results**

Here, we briefly collect the results for the pressure and the static, zero-momentum susceptibilities in the mean field approximation. We start with the one-loop thermodynamic potential [15]

\[
\frac{\Omega}{V} = -\frac{m\Delta_0^2}{4\pi a} - \int \frac{d^3k}{(2\pi)^3} \left( E_k - \epsilon_k + \mu - \frac{1}{2} \frac{\Delta_0^2}{\epsilon_k} \right),
\]

which, as mentioned above, can be differentiated to yield Eqs. (2.20), (2.21) and (2.22) in the same approximation. The integral can be done analytically, giving

\[
\frac{\Omega}{V} = -\frac{m\Delta_0^2}{4\pi a} - \frac{4m^{3/2}|\mu|^{5/2}}{15\pi^2} \left[ (1 - 3\alpha^2)\sqrt{-\text{sign}(\mu) + \sqrt{1 + \alpha^2} E(-\gamma)} 
\right.
\]

\[
\left. + \frac{\text{sign}(\mu)(1 + \alpha^2) + (3\alpha^2 - 1)\sqrt{1 + \alpha^2}}{\sqrt{-\text{sign}(\mu) + \sqrt{1 + \alpha^2}}} K(-\gamma) \right],
\]

where \( K(n) \), \( E(n) \) are the complete elliptic integrals of the first and second kind respectively (see Appendix A for details and notation), and \( \alpha^2 = \Delta_0^2/\mu^2 \). The parameter \( \gamma \) is

\[
\gamma = \frac{\sqrt{1 + \alpha^2 + \text{sign}(\mu)}}{\alpha^2}.
\]

The susceptibilities are given by

\[
\chi_{nn}(0) = m^{3/2}|\mu|^{1/2} \sqrt{-\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ -E(-\gamma) + K(-\gamma) \right]},
\]

\[
\chi_{n1}(0) = -\frac{m^{3/2}\Delta_0}{\pi^2|\mu|^{1/2}} \frac{1}{\sqrt{-\text{sign}(\mu) + \sqrt{1 + \alpha^2}}} K(-\gamma),
\]

\[
\chi_{11}(0) = m^{3/2}|\mu|^{1/2} \frac{1}{\alpha^2\sqrt{1 + \alpha^2}} \sqrt{-\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ 3\alpha^2\sqrt{1 + \alpha^2} E(-\gamma) 
\right.}
\]

\[
\left. - \left( 2\text{sign}(\mu)(1 + \alpha^2) + (2 + 3\alpha^2)\sqrt{1 + \alpha^2} \right) K(-\gamma) \right].
\]
The expression for the thermodynamic potential (4.12), the gap Eq. (2.30) and the number equation
\[ \langle n \rangle = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu} \] (4.7)
summarize the properties of the crossover at \( T = 0 \) in the one-loop approximation. From these one can find the quantities \( \Delta_0 / \epsilon_F \) and \( \mu / \epsilon_F \) as functions of the parameter \( (k_F a)^{-1} \), where \( \epsilon_F = k_F^2 / 2m \) is the Fermi energy. The energy density of the ground state is given by \( \epsilon = \mu \langle n \rangle + V^{-1} \Omega \).

Next we collect results, some of then well known, which follow easily from Eq. (4.2) for specific ranges of the parameter \( (k_F a)^{-1} \).

1. Near unitarity

Here we present the results for the gap parameter, the chemical potential and the ground state energy per particle near unitarity, expressed as power series in \( (k_F a)^{-1} \):
\[
\begin{align*}
\frac{\Delta_0}{\epsilon_F} &= 0.6864 + \frac{0.6368}{k_F a} + \frac{0.0959}{(k_F a)^2} + \ldots, \\
\frac{\mu}{\epsilon_F} &= 0.5906 - \frac{0.7401}{k_F a} - \frac{0.5150}{(k_F a)^2} + \ldots, \\
\frac{\epsilon}{\langle n \rangle} &= 3 \epsilon_F \left( \frac{0.5906}{k_F a} - \frac{0.9251}{(k_F a)^2} - \frac{0.8582}{(k_F a)^3} + \ldots \right).
\end{align*}
\] (4.8) (4.9) (4.10)

The numerical coefficients have been obtained by simultaneous power series solution of the gap and number equations using the analytic solution (4.2) for \( \Omega \). It is also easy to write a few terms for the pressure at large scattering length
\[ P(\mu) = 0.0842 m^{3/2} \mu^{5/2} + 0.1075 \frac{m^2 \mu^2}{a} + 0.1274 \frac{m^{1/2} \mu^{3/2}}{a^2} + 0.1006 \frac{\mu}{a^3} + \ldots \] (4.11)

2. BCS limit

In the BCS limit \( (k_F a)^{-1} \to -\infty \). By using the following asymptotic expansions for the complete elliptic integrals, valid when \( z \to -\infty \)
\[
\begin{align*}
K(z) &\sim \frac{\ln(-16z)}{2(-z)^{1/2}} + \frac{2 - \ln(-16z)}{8(-z)^{3/2}} + \mathcal{O}((-z)^{-5/2} \ln(-z)), \\
E(z) &\sim (-z)^{1/2} + \frac{1 + \ln(-16z)}{4(-z)^{1/2}} + \frac{3 - 2 \ln(-16z)}{64(-z)^{3/2}} + \mathcal{O}((-z)^{-5/2} \ln(-z)),
\end{align*}
\] (4.12) (4.13)
we obtain the solution to the gap equation \( \alpha = 8e^{-2}\exp(\pi/2\sqrt{2m\mu a}) \). This produces the pressure
\[ P(\mu) = \frac{25/2 m^{3/2} \mu^{5/2}}{15\pi^2} \left( 1 + 60 \exp(-4 + \pi a^{-1}(2m\mu)^{-1/2}) + \ldots \right), \] (4.14)
and

\[ \frac{\Delta_0}{\epsilon_F} = 8e^{-2e^{\pi/2k_Fa}}, \]

\[ \frac{\mu}{\epsilon_F} = 1 + \frac{8\pi e^{-4e^{\pi/k_Fa}}}{k_Fa} + \ldots = 1 + \frac{\Delta_0^2}{4\epsilon_F^2} \left[ 2 + \ln \left( \frac{\Delta_0}{8\epsilon_F} \right) \right] + \ldots \]

We find that the ground state energy per particle and the fermion density are given by

\[ \frac{\epsilon}{\langle n \rangle} = \frac{3\epsilon_F}{5} - \frac{3\Delta_0^2}{8\epsilon_F} + \ldots, \]

\[ \langle n \rangle = \frac{k_F^3}{3\pi^2} \left[ 1 + \frac{3\Delta_0^2}{16\epsilon_F^2} \left( 1 - 2\ln \left( \frac{\Delta_0}{8\epsilon_F} \right) \right) + \ldots \right]. \]

3. BEC limit

The other extreme regime, the BEC limit \((k_Fa)^{-1} \rightarrow \infty\), is obtained when \(\mu\) is close to \(-1/2ma^2\). In this regime \(|\alpha|\) is small and we can use the Maclaurin series for the elliptic integrals. The thermodynamic potential is thus

\[ \frac{\Omega}{V} = -\frac{m\Delta_0^2}{4\pi a} + \frac{m^{3/2}(-\mu)^{1/2}\Delta_0^2}{2\pi \sqrt{2}} + \frac{m^{3/2}\Delta_0^4}{64\pi \sqrt{2}(-\mu)^{3/2}} + \ldots \]

and one finds

\[ \mu = -\frac{1}{2ma^2} + \delta \mu = \epsilon_F \left( -\frac{1}{(k_Fa)^2} + \frac{2k_Fa}{3\pi} + \ldots \right), \]

\[ \Delta_0 = \epsilon_F \frac{4}{(3\pi k_Fa)^{1/2}} + \ldots, \]

\[ \frac{\epsilon}{\langle n \rangle} = \epsilon_F \left( -\frac{1}{(k_Fa)^2} + \frac{k_Fa}{3\pi} + \ldots \right). \]

The solution of the gap equation yields a power series in \(ma^2\delta \mu\) for \(\Delta_0\)

\[ \Delta_0 = \frac{2\sqrt{\delta \mu}}{a\sqrt{m}} \left( 1 + \frac{5}{8}ma^2\delta \mu + O((ma^2\delta \mu)^2) \right), \]

which can be used to write the pressure as

\[ P = \frac{m\delta \mu^2}{2\pi a} + \frac{m^2a\delta \mu^3}{4\pi} + \ldots \]

Eliminating the chemical potential leads to a pressure \(P(n) = (2m)^{-1}\pi an^2 + \ldots\). Noting that \(m_B = 2m\) and \(n_B = n/2\), a comparison with the pressure of a weakly interacting Bose gas in the lowest-order approximation \(P(n_B) = 2\pi a_B m_B^{-1}n_B^2\), yields the mean-field result for the scattering length between bosons, \(a_B = 2a\). But, as Diener, Sensarma and Randeria have recently shown [15], these mean-field results in the BEC limit are poor approximations to the results which are obtained when the contribution from the quantum fluctuations is included in the computation.
B. Next-to-leading order results in the one-loop approximation

The derivatives of the one-loop response functions at \( Q = 0 \) are collected in Appendix C. From these expressions we can determine the coefficient functions \( f_i(X) \) in the effective NLO Lagrangian

\[
\mathcal{L}_{\text{NLO}} = f_1(X)(\nabla X)^2 + f_2(X)(\nabla^2 \theta)^2 + f_3(X)(\partial_i \partial_j \theta)^2 + f_4(X)\nabla^2 V_{\text{ext}}.
\] (4.25)

1. Unitarity

It is possible to express the complete elliptic integrals in the unitary limit through two useful formulas obtained by simultaneous solution of the pressure \( P = c_0 m^{3/2} \mu^{5/2} \) and gap equations. These are given by

\[
K(-\gamma) = \frac{15\pi^2 \sqrt{-1 + \sqrt{1 + \alpha^2}}}{4(1 + \alpha^2)} c_0, \quad (4.26)
\]
\[
E(-\gamma) = \frac{15\pi^2}{4\sqrt{1 + \alpha^2} \sqrt{-1 + \sqrt{1 + \alpha^2}}} c_0. \quad (4.27)
\]

Substituting these expressions into the derivatives of Appendix C produces the remarkable result that \( \tilde{G}''_5(\mu) = 0 \). This is due to the fact that, when the coupling \( g \to \infty \), the expression for this quantity is proportional to the gap equation, which gives rise to the exact cancellation of \( \tilde{G}''_5(\mu) \). The complete list of coefficient functions that we find is

\[
f_1(\mu) = -\frac{35c_0}{192} m^{1/2} \mu^{1/2} \simeq -0.0153 \frac{m^{1/2}}{\mu^{1/2}}, \quad (4.28a)
\]
\[
f_2(\mu) = -\frac{c_0}{18\alpha(\mu)^2} \mu^{1/2} \simeq -0.0035 \frac{\mu^{1/2}}{m^{1/2}}, \quad (4.28b)
\]
\[
f_3(\mu) = \frac{c_0}{6\alpha(\mu)^2} \mu^{1/2} \simeq 0.0104 \frac{\mu^{1/2}}{m^{1/2}}, \quad (4.28c)
\]
\[
f_4(\mu) = 0. \quad (4.28d)
\]

where the numerical values are obtained by substitution of the one-loop numerical values \( \alpha(\mu) \simeq 1.1622 \) and \( c_0 \simeq 0.0842 \). Noting that \( f_3/f_2 = -3 \), our result for the NLO effective Lagrangian at unitarity can be written as

\[
\mathcal{L}_{\text{NLO}} = c_1 m^{1/2} X^{-1/2} (\nabla X)^2 + c_2 m^{-1/2} X^{1/2} \left[(\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2\right], \quad (4.29)
\]

where \( c_1 \simeq -0.0153 \) and \( c_2 \simeq -0.0035 \). This is one of the main results in this paper.

One might wonder whether the cancellation of the \( f_4 \) coefficient and the ratio \( f_3/f_2 = -3 \) found here are mere accidents of the mean field approximation. Actually, these are exact consequences of the conformal invariance displayed by the system at the unitarity limit. In other words, any approximation scheme that respects conformal invariance must necessarily yield a result of the form given by Eq. (4.29). This is explained in Appendix B, where the reason for the discrepancy with the form of the effective action given in [23] is also analyzed.
The following expansions are valid for the NLO coefficient functions near unitarity

\[ f_1(\mu) = -\frac{0.0153m^{1/2}}{\mu^{1/2}} + \frac{0.0014}{\alpha \mu} - \frac{0.0016}{a^2m^{1/2}\mu^{3/2}} + \ldots, \quad (4.30a) \]

\[ f_2(\mu) = -\frac{0.0035\mu^{1/2}}{m^{1/2}} + \left[ 0.0023 + 0.0022 \ln \left( \frac{\mu}{\mu_0} \right) \right] \frac{1}{am} + \frac{0.0004}{a^2m^{3/2}\mu^{1/2}} + \ldots, \quad (4.30b) \]

\[ f_3(\mu) = \frac{0.0104\mu^{1/2}}{m^{1/2}} - \left[ 0.0113 + 0.0022 \ln \left( \frac{\mu}{\mu_0} \right) \right] \frac{1}{am} + \frac{0.0097}{a^2m^{3/2}\mu^{1/2}} + \ldots, \quad (4.30c) \]

\[ f_4(\mu) = -\frac{0.0022}{a} \ln \left( \frac{\mu}{\mu_0} \right) - \frac{0.0037}{a^2m^{1/2}\mu^{1/2}} + \frac{0.0004}{a^3m\mu} + \ldots, \quad (4.30d) \]

where \( \mu_0 \) is an arbitrary scale (\( \ln \mu_0 \) is multiplied by a total divergence).

3. BCS limit

The determination of the leading behavior of the coefficient functions in the BCS limit is more involved. By using the asymptotic expansions for the complete elliptic integrals and the perturbative solution of the gap equation \( \alpha(\mu) \approx 8e^{-2}\exp(\pi/2a\sqrt{2m\mu}) \), one finds the lowest-order approximation

\[ \frac{g\chi_{n1}(0)}{2 + g\chi_{11}(0)} = \frac{\alpha(\mu)}{2} \ln \left( \frac{\alpha(\mu)}{8} \right) + O(\alpha^3(\ln \alpha)^2), \quad (4.31) \]

to be used in the coefficients of some derivatives in the asymptotic expressions for the \( b_i \) coefficients. The pressure \( (4.14) \) gives

\[ \frac{P'(\mu)}{mP''(\mu)} = \frac{2\mu}{3m} - \frac{\mu}{6m} \alpha(\mu)^2 \ln \left( \frac{\alpha(\mu)}{8} \right) \left[ 2 + \ln \left( \frac{\alpha(\mu)}{8} \right) \right] + O(\alpha^4(\ln \alpha)^4), \quad (4.32) \]

and the substitution of these results into Eqs. \((3.20)\) produces

\[ \tilde{G}_5'(\mu) = \frac{(\ln(\alpha(\mu)/8))^2 m^{1/2}}{36\sqrt{2}\pi^2} \mu^{1/2} + \ldots = \frac{1}{288\sqrt{2} m^{1/2} a^2 \mu^{3/2}} + \ldots, \quad (4.33) \]

and

\[ f_1(\mu) = -\frac{(\ln(\alpha(\mu)/8))^2 m^{1/2}}{24\sqrt{2}\pi^2} \mu^{1/2} + \ldots = -\frac{1}{192\sqrt{2} m^{1/2} a^2 \mu^{3/2}} + \ldots, \quad (4.34a) \]

\[ f_2(\mu) = -\frac{23/2}{135\pi^2 \alpha(\mu)^2 m^{1/2}} + \ldots = -\frac{\exp(4 - \pi a^{-1}(2m\mu)^{-1/2})}{2160\sqrt{2}\pi^2} \mu^{1/2} + \ldots, \quad (4.34b) \]

\[ f_3(\mu) = \frac{23/2}{45\pi^2 \alpha(\mu)^2 m^{1/2}} + \ldots = \frac{\exp(4 - \pi a^{-1}(2m\mu)^{-1/2})}{720\sqrt{2}\pi^2} \mu^{1/2} + \ldots, \quad (4.34c) \]

\[ f_4(\mu) = -\frac{1}{144\sqrt{2} m^{1/2} a^2 \mu^{1/2}} + \ldots \quad (4.34d) \]
It is worth noting that \( f_4/(m f_{2,3}) = O(\alpha^2 (\ln \alpha)^2) \), which shows that \( f_4 \) can be safely neglected. Hence, an important feature of this regime is the dominance of the \( f_2 \) and \( f_3 \) terms in the next-to-leading order effective Lagrangian. The coefficients \( b_4 \) and \( b_5 \) are at the origin of the leading behavior of \( f_{2,3} \), while \( b_2 \) and \( b_4 \) govern the expression for \( f_1 \).

Note that these expressions cannot be trusted for \( k_F a \) arbitrarily small, because then \( \Delta_0 \to 0 \), and the condition for the validity of the derivative expansion \( |\omega| < 2\Delta_0 \) cannot be satisfied. Indeed, when \( a \to 0^- \) the Goldstone mode can hardly be considered a propagating mode due the arbitrarily small two-particle states threshold, and this renders this effective field description meaningless.

4. BEC limit

The BEC limit can be obtained by substituting the gap parameter (4.23) into the appropriate derivatives of the response functions and then expanding in powers of \( \delta \mu \). We obtain the following leading behavior for the coefficient functions

\[
\tilde{G}_5''(\mu) = -\frac{ma}{48\pi} + O(\delta \mu), \tag{4.35a}
\]

\[
f_1(\mu) = -\frac{1}{8\pi m a^3 \Delta_0(\mu)^2} + O(\delta \mu^0) = -\frac{ma}{16\pi(1 + 2ma^2\mu)} + O(\delta \mu^0), \tag{4.35b}
\]

\[
f_2(\mu) = -\frac{1}{96\pi ma} + O(\delta \mu^2), \tag{4.35c}
\]

\[
f_3(\mu) = \frac{1}{96\pi ma} + O(\delta \mu^2), \tag{4.35d}
\]

\[
f_4(\mu) = -\frac{ma}{48\pi} + O(\delta \mu^2). \tag{4.35e}
\]

These results reveal that, in this limit, \( \tilde{G}_5''(\mu)/f_1(\mu) = O(ma^2\delta \mu) \), and the leading derivative contribution comes from the \( f_1 \) term of the effective Lagrangian. Now, the coefficient \( b_3 \) is at the origin of the leading behavior of \( f_1 \). Note that the constant coefficients \( f_2 \) and \( f_3 \) play no role in this limit, because an integration by parts produces their mutual cancellation.

V. DENSITY FUNCTIONAL IN THE UNITARY LIMIT. RELATION TO OTHER APPROACHES

From the effective Lagrangian one can easily derive the energy density \( \mathcal{E} \) depending on the number density and spatial derivatives of the Goldstone mode. The corresponding first-order equations are the continuity equation and the London equation for \( \theta \). Here we outline the computation of \( \mathcal{E} \).

Since \( s \mathcal{L}^\text{eff} = n(-\partial_t \theta - (\nabla \theta)^2/2m) + \ldots \) one sees that the number density \( n \) is conjugate to \(-\theta\), and the energy density is given by

\[
\mathcal{E} = -n\partial_t \theta + \mu n - \mathcal{L}, \tag{5.1}
\]

where \( \partial_t \theta \) is determined in terms of \( n, \nabla \theta \) and \( V_{\text{ext}} \) by assuming a derivative expansion for \( \partial \mathcal{L}/\partial (\partial_t \theta) = -n \). From the effective Lagrangian at unitarity obtained in the previous Section

\[
\mathcal{L} = c_0 m^{3/2} X^{5/2} + c_1 m^{1/2} X^{-1/2} (\nabla X)^2 + c_2 m^{-1/2} X^{1/2} \left[ (\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2 \right], \tag{5.2}
\]
one finds the energy density

$$\mathcal{E} = n V_{\text{ext}} + \frac{3 \cdot 2^{2/3}}{5^{5/3} c_0^{2/3} m} n^{5/3} + n \left(\nabla \theta\right)^2 - \frac{8c_1}{45c_0 m} \frac{(\nabla n)^2}{n}$$

$$- \frac{2^{1/3} c_2}{5^{1/3} c_0^{1/3} m} n^{1/3} \left[ (\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2 \right].$$

(5.3)

Note that the next-to-leading order contribution to $\mathcal{E}$ is exactly $-\mathcal{L}_{\text{NLO}}$ if $\partial_t \theta$ is replaced by

$$\partial_t \theta = \mu - V_{\text{ext}} - \frac{2^{2/3} n^{2/3}}{5^{2/3} c_0^{2/3} m} - \frac{(\nabla \theta)^2}{2m},$$

(5.4)

which is the solution of $\partial \mathcal{L}_{\text{LO}}/\partial (\partial_t \theta) = -n$.

The variation of the energy functional $H[n, \theta] = \int d^3 x \mathcal{E}$ with respect to $\theta$ yields the continuity equation for the particle number

$$\partial_t n = \frac{\delta H}{\delta \theta(x)} = -\nabla \cdot \frac{\partial \mathcal{E}}{\partial (\nabla \theta(x))} + \ldots = -\nabla \cdot \left( \frac{n}{m} \nabla \theta + \ldots \right),$$

(5.5)

and the hydrodynamic equation for $\theta$ is given by

$$\partial_t \theta = -\frac{\delta H}{\delta n(x)}.$$  

(5.6)

These equations describe the irrotational hydrodynamics of the superfluid at zero temperature. The equilibrium state in the presence of an external potential corresponds to $\theta = -\mu_0 t + \text{cons}$, which gives a stationary density profile and zero superfluid velocity, $\nabla \theta = 0$. The equilibrium particle density satisfies the condition

$$\mu_0 = V_{\text{ext}} + \frac{2^{2/3}}{5^{2/3} c_0^{2/3} m} n^{2/3} + \frac{32c_1}{45c_0 m} \frac{\nabla^2 (\sqrt{n})}{\sqrt{n}},$$

(5.7)

where $\mu_0$ is the chemical potential. Thus the leading behavior of $n$, which is obtained by dropping all the $c_i$, is given by a Thomas-Fermi approximation, whereas $c_1$ determines the quantum kinetic energy correction and, as shown in Appendix B, the last term is proportional to the square of the traceless strain rate tensor.

In writing Eq. (5.2) we have used the results of the previous Section, namely $f_4 = 0$ and $f_3/f_2 = -3$. It is worth mentioning that, had we found a nonvanishing value for $f_4$, the effective Lagrangian (5.2) would have contained an additional term proportional to $m^{1/2} X^{1/2} \nabla^2 V_{\text{ext}}$, giving rise to a term proportional to $n^{-2/3} \nabla^2 V_{\text{ext}}$ in (5.7). Such a term would be incompatible with the boundary condition for $n$ at infinity, due to the negative power of $n$. If we consider for instance the isotropic harmonic trap, where $\nabla^2 V_{\text{ext}}$ is a constant, it is not possible to satisfy the asymptotics $n \to 0$ as $r \to \infty$. But, as mentioned above and shown explicitly in Appendix B, the condition $f_4 = 0$ is an exact consequence of the conformal invariance of the field equations and the problem with the boundary conditions does not arise.

Very recently, Rupak and Schäfer have derived an energy density functional using an epsilon expansion around $d = 4 - \varepsilon$ spatial dimensions. They find an expression depending only on $n$ given by

$$\mathcal{E}_{\text{Rupak, Schäfer}} = V_{\text{ext}} n + 1.364 \frac{n^{5/3}}{m} + 0.032 \frac{(\nabla n)^2}{mn} + O(\nabla^4 n),$$

(5.8)
which follows from their value of \( c_1 \approx -0.0209 \) up to \( O(\varepsilon \ln \varepsilon) \), together with \( \xi \approx 0.4754 \) (or \( c_0 \approx 0.1166 \) in the same approximation [28]). Our result for \( \mathcal{E} \) is obtained by inserting the mean field results \( c_0 = 2^{5/2}/(15\pi^2\xi^{3/2}) = 0.0842 \), \( c_1/c_0 = -35/192 \) and \( c_2 = -0.0035 \) into Eq. (5.3). We find that the energy density takes the form

\[
\mathcal{E} = V_{\text{ext}} n + 1.6956 \frac{n^{5/3}}{m} + n \frac{(\nabla \theta)^2}{2m} + 0.0324 \frac{(\nabla n)^2}{mn} + 0.0059 \frac{n^{1/3}}{m} \left( (\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2 \right). \tag{5.9}
\]

It is remarkable the agreement between the terms corresponding to the quantum kinetic energy or quantum pressure, although they have been computed using two very different approaches. Although the values for the individual coefficients \( c_0 \) and \( c_1 \) show differences of the order of 30%, surprisingly these differences cancel in the ratio \( c_1/c_0 \) giving the quantum pressure.

It is interesting to consider the expression for the energy density in the BEC limit which follows from the effective Lagrangian. It can be written in the form

\[
\mathcal{E}_{\text{BEC}} = n V_{\text{ext}} + \frac{\pi}{2m} n^2 + n \frac{(\nabla \theta)^2}{2m} + \frac{1}{32} \frac{(\nabla n)^2}{mn} + \frac{1}{96\pi a} (-1 + 2\pi a^3 n) \nabla^2 V_{\text{ext}}, \tag{5.10}
\]

where the most important derivative term is determined by the expression for \( f_1(X) \) given in Eq. (4.35b). If we make the replacements \( n \rightarrow 2n_B , m \rightarrow m_B/2 \) and \( a \rightarrow a_B/2 \), this expression becomes

\[
\mathcal{E}_{\text{BEC}} = n_B (2V_{\text{ext}}) + \frac{4\pi a_B}{2m_B} n_B^2 + n_B \frac{(\nabla (2\theta))^2}{2m_B} + \frac{1}{2m_B} (\nabla \sqrt{n_B})^2 + \ldots , \tag{5.11}
\]

and one recovers the correct quantum pressure in the last term. Thus, the derivative part of the effective Lagrangian in the BEC limit fits in with the hydrodynamic description of a superfluid at zero temperature with bosonic constituents of mass \( 2m \). It is worth pointing out the excellent numerical agreement between the coefficient \( 1/32 \approx 0.031 \) in the BEC limit and the corresponding coefficient \( 7/216 \approx 0.032 \) in the unitary limit. This is probably a reflection of the fact that, in the \( d = 4 \) limit about which the epsilon expansion is taken, the fermion-fermion scattering amplitude is saturated by the propagator of a boson with mass \( 2m \).

Finally, in order to check the quality of the above NLO results in the BCS limit, a comparison with the predictions from the approach of Furnstahl et al. [29] would be very insightful.

**VI. CONCLUSION**

In this paper we have considered the derivative expansion of the effective action for the Goldstone field of a nonrelativistic superfluid Fermi gas at zero temperature in the whole BCS-BEC crossover. Based on the pioneering analysis of symmetries in Ref. [23], we have shown that the NLO action can be given in terms of the four coefficient functions in

\[
\mathcal{L} = P(X) + f_1(X)(\nabla X)^2 + f_2(X)(\nabla^2 \theta)^2 + f_3(X)(\partial_i \partial_j \theta)^2 + f_4(X) \nabla^2 V_{\text{ext}}, \tag{6.1}
\]
and we have given the precise relationships —see Eqs. (3.20)— between these functions and the derivatives of the response functions. It turns out that the computation of the NLO action relies also on the ratio of a pair of three-point functions, whose value we have determined by a calculation at the one-loop level.

An important step towards Eqs. (3.20) has been the reduction of the initial Lagrangian (3.3) to its final form (6.1) without higher-order time derivatives. In this regard, we have shown how the proper use of the LO field equations to eliminate higher-order time derivatives insures the consistency of the reduction process. As novel consequences of this critical analysis, we find the presence of a term proportional to \((\partial_i \partial_j \theta)^2\) in (6.1) and the form of this action in the unitary limit

\[
L = c_0 m^{3/2} X^{5/2} + c_1 m^{1/2} X^{-1/2} (\nabla X)^2 + c_2 m^{-1/2} X^{1/2} \left[ (\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2 \right],
\]

(6.2)

which, as we have shown, is dictated by conformal invariance. In particular, conformal invariance of the field equation prevents the existence of an explicit coupling to the external field.\(^4\) This Lagrangian determines uniquely the form of an energy density functional depending on the particle density \(n\) and the Goldstone mode \(\theta\),

\[
E = n V_{\text{ext}} + \frac{3 \cdot 2^{2/3}}{5^{5/3} c_0^{2/3} m^{5/3}} n^{5/3} + n \frac{(\nabla \theta)^2}{2m} - \frac{8 c_1}{45 c_0 m} \frac{(\nabla n)^2}{n} - \frac{2^{1/3} c_2}{5^{1/3} c_0^{1/3} m} n^{1/3} \left[ (\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2 \right].
\]

(6.3)

It is worth mentioning that these aspects of our work do not rely on the specific approximation used to compute the 2-point functions. Rather, they arise as an application of effective field theory ideas and techniques.

By resorting to the one-loop approximation and taking into account the \(\mu\) dependence of \(\Delta_0\) in the mean-field approximation, we have also obtained analytic, closed expressions in terms of complete elliptic integrals for the coefficient functions of the NLO effective Lagrangian in the whole BCS-BEC crossover. Having closed expressions for these functions, rather than purely numerical results, makes the analytic study of different limits feasible. We have obtained series expansions near the unitary limit, and in the extreme BCS and BEC regimes. In particular, we have determined the mean-field values for the coefficients \(c_1\) and \(c_2\) at unitarity, and thus we have explicitly checked the form of the Lagrangian (6.2). In this regard, it is interesting how the one-loop result found for the ratio of three-point functions combines with the result for \(g_4\) to yield the conformally invariant combination \(X^{1/2}[ (\nabla^2 \theta)^2 - 3(\partial_i \partial_j \theta)^2 ] \) —see Eqs. (3.7) and (3.20). Furthermore, the good agreement in the extreme regimes with known results from other approaches suggests that our mean field approximation can be taken as a reliable, first qualitative estimate for the coefficients of the effective theory.

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\(^4\) There remains, of course, an implicit coupling through the dependence of \(X\) on \(V_{\text{ext}}\)
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**APPENDIX A: ON THE REDUCTION OF THE INTEGRALS TO CANONICAL FORMS**

Here we outline the method used to evaluate all the momentum integrals in this paper in terms of elliptic integrals of the first and second kind. For the sake of concreteness, we consider the integral for $\chi_{n1}(0)$

$$\chi_{n1}(0) = -\int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0 \xi_k}{E_k^3},$$

and assume $\mu > 0$. The change of variables $x = \frac{k^2}{2\mu}$ brings the integral into the following form

$$\chi_{n1}(0) = \frac{m^{3/2}\Delta_0}{\sqrt{2\pi^2\mu^{1/2}}} \int_0^\infty dx \frac{R(x)}{y},$$

where $y^2 = x[(x-1)^2 + \alpha^2]$ and $R(x)$ is the rational function

$$R(x) = \frac{x(x-1)}{(x-1)^2 + \alpha^2}.$$

Now, the integral of the quotient of a rational function by the square root of a cubic or a quartic polynomial is, by definition, an elliptic integral. But we still have to reduce this integral to a combination of canonical forms. The fact that the integrand has no other singularities than the branch points of $y$ implies that only elliptic integrals of the first ($F$) and second kind ($E$) can be involved. These are given by [30]

$$F(\varphi|n) = \int_0^{\sin \varphi} dt \frac{1}{\sqrt{(1-t^2)(1-nt^2)}},$$

$$E(\varphi|n) = \int_0^{\sin \varphi} dt \frac{\sqrt{1-nt^2}}{\sqrt{1-t^2}}.$$

The change of variables

$$t = \frac{x - \sqrt{1 + \alpha^2}}{x + \sqrt{1 + \alpha^2}},$$

brings the integration interval to $(-1, 1)$ and $y$ into the canonical form

$$\frac{dx}{y} = \frac{\sqrt{2}}{\sqrt{1+\sqrt{1+\alpha^2}}} \frac{dt}{\sqrt{(1-t^2)(1+\gamma t^2)}},$$

where

$$\gamma = \frac{[\sqrt{1+\alpha^2}+1]^2}{\alpha^2}.$$

Finally, after using standard “reduction formulae” [30] we obtain

$$\chi_{n1}(0) = -\frac{m^{3/2}\Delta_0}{\pi^2 \mu^{1/2}} \frac{1}{\sqrt{-1+\sqrt{1+\alpha^2}}} K(-\gamma),$$
where \( K(\gamma) = F(\pi/2 | \gamma) \) is the complete elliptic integral of the first kind.

The evaluation of the other integrals proceeds along the same lines. In all cases we have an odd power of \( E_k = \mu[(x-1)^2 + \alpha^2] \) that combines with \( \sqrt{x} \) from the measure \( d^3k \) to give an integrand of the form \( R(x)/y \), where only the rational function \( R(x) \) changes from case to case. As the only singularities are the branch points of \( y \) — there are no additional poles — only complete elliptic integrals of the first and second kind can result.

**APPENDIX B: CONFORMAL INVARIANCE OF THE NLO LAGRANGIAN**

In this Appendix we investigate the constraints imposed by conformal invariance on the form of the NLO Lagrangian in the unitary limit. As shown in Refs. [23, 24], in that limit the original fermionic action\(^5\) is invariant under a special set of time-dependent transformations. Actually, Son and Wingate [23] have discussed the invariance properties of the action when the system is put in a curved manifold and an external gauge field. In order to adapt their transformations to our case, where the metric is euclidean and the gauge field is set to zero, we have to use a “gauge fixed” version which includes, besides the purely conformal transformation, “compensating” coordinate and gauge transformations. Its infinitesimal form is given by

\[
\delta \psi = \beta''(t) \frac{imx^2}{4} \psi - \frac{\beta'(t)}{2} x \cdot \nabla \psi - \beta(t) \partial_t \psi - \frac{3\beta'(t)}{4} \psi, \\
\delta V_{\text{ext}} = -\beta''(t) \frac{mx^2}{4} - \frac{\beta'(t)}{2} x \cdot \nabla V_{\text{ext}} - \beta(t) \partial_t V_{\text{ext}} - \beta'(t)V_{\text{ext}},
\]

where \( \beta(t) \) is an arbitrary function of \( t \). In the notation of Ref. [23], this transformation can be seen as a combination of general, gauge and conformal transformations with parameters \( \xi^k = \beta'(t)x^k/2, \alpha = m\beta''(t)x^2/4 \) and \( \beta(t) \), respectively. With this choice, the euclidean metric and the zero external gauge field are untouched. We assume that the variation of the chemical potential is given by \( \delta \mu = -\beta'(t)\mu \), which assigns to \( V_{\text{ext}} \) all the change in the variation \( \delta(V_{\text{ext}} - \mu) \) under a gauge transformation. In the effective theory the relevant field is the Goldstone mode \( \theta \), which transforms inhomogeneously according to\(^6\)

\[
\delta \theta = \beta''(t) \frac{mx^2}{4} - \frac{\beta'(t)}{2} x \cdot \nabla \theta - \beta(t) \partial_t \theta.
\]

A “scale transformation” is a particularly simple conformal transformation where \( \beta(t) \) is a linear function. As shown in Ref. [23], scale invariance alone determines the form of the functions \( f_i(X) \) in the NLO Lagrangian (4.25)

\[
\mathcal{L}_{\text{NLO}} = c_1 m^{1/2} X^{-1/2} (\nabla X)^2 + m^{-1/2} X^{1/2} [c_2 (\nabla^2 \theta)^2 + c_3 (\partial_i \partial_j \theta)^2 + c_4 m \nabla^2 V_{\text{ext}}] \tag{B4}
\]

The change under a general conformal transformation is then given by

\[
\delta \mathcal{L}_{\text{NLO}} = (3c_2 + c_3) m^{1/2} X^{1/2} \nabla^2 \theta \beta''(t) - \frac{3}{2} c_4 m^{3/2} X^{1/2} \beta''(t) \\
- \partial_t \left( \mathcal{L}_{\text{NLO}} \beta(t) \right) - \frac{1}{2} \nabla \cdot (x \mathcal{L}_{\text{NLO}} \beta'(t)). \tag{B5}
\]

---

\(^5\) By this we mean the fermion action before the Hubbard-Stratanovich transformation leading to Eq. (2.1) is applied.

\(^6\) The Goldstone field here corresponds to \( \mu t - \theta_{\text{Son,Wingate}} \) of Ref. [23].
Note that, as expected, the action is automatically invariant under scale transformations, for which $\beta''(t) = \beta'''(t) = 0$. Invariance under general conformal transformations requires
\[
c_3 = -3c_2, \quad c_4 = 0, \tag{B6}
\]
while the value of $c_1$ is unrestricted. This constrains the NLO Lagrangian to the form given by Eq. (4.29).

Note that we could integrate the variation of the $c_4$ term in (B5) by parts to get a contribution proportional to $\beta''(t)$. Using the LO field equations to eliminate the time derivative $\partial_t X$ would then give the following result
\[
\delta L_{\text{NLO}} = (3c_2 + c_3 + c_4)m^{1/2}X^{1/2}\nabla^2\theta \beta''(t) \tag{B7}
\]
up to total derivatives. This suggests that the action is conformally invariant as long as
\[
c_3 = -3c_2 - c_4, \tag{B8}
\]
which is a weaker constraint than (B6). But here, as was the case with the elimination of the $\tilde{g}_5(X)X_t\nabla^2\theta$ term at the end of Section III, the use of the LO field equation is not fully legitimate. The reason is that the elimination of the time derivative $\partial_t X$ in favour of $\nabla^2\theta$ would involve dropping a term proportional to the LO field equation, namely
\[
\frac{3c_4}{4}m^2X^{-1/2}\beta''(t) \left(X_t + \frac{2X}{3m}\nabla^2\theta \right).
\]

It is easy to check that the Euler-Lagrange equation arising from this term gives a contribution to the linearized field equation proportional to
\[
3m\partial_t [\beta''(V_{\text{ext}} + \partial_t \theta)] + 10\mu\beta''\nabla^2\theta - 4\mu\beta'''\nabla^2 V_{\text{ext}}, \tag{B9}
\]
which can not be neglected. In other words, the field equations derived from a Lagrangian subject only to the weaker constraints (B8) will not be invariant under conformal transformations. Thus, the coefficient of $\beta''(t)$ in Eq. (B5) must be zero irrespective of the values of $c_2$ and $c_4$. This fact, which seems to have been overlooked by the authors of Ref. [23], would explain the discrepancies with their results.

We end this Appendix by noting that the the conformally invariant combination $(\nabla^2\theta)^2 - 3(\partial_i\partial_j\theta)^2$ can be written as the square of an $l = 2$ irreducible rank-two tensor of conformal dimension one
\[
(\nabla^2\theta)^2 - 3(\partial_i\partial_j\theta)^2 = -3\left(\partial_i\partial_j\theta - \frac{1}{3}\delta_{ij}\nabla^2\theta\right)^2. \tag{B10}
\]
This allows a natural interpretation in terms of superfluid hydrodynamics. Writing the RHS of (B10) in terms of the superfluid velocity $v_s = m^{-1}\nabla\theta$ shows that the new invariant is proportional to the square of the traceless part of the shear rate tensor, also known as the shear rate tensor
\[
\frac{1}{2}(\partial_i v_{s,j} + \partial_j v_{s,i}) - \frac{1}{3}\delta_{ij}\nabla \cdot v_s. \tag{B11}
\]
7 See also comments at the end of Section III.
Here we give the derivatives of the response functions used in the computation of the NLO action at the one-loop level. The results are

\[
\frac{\partial \chi_{nn}}{\partial \omega^2} = - \int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0^2}{4E_k^5} \left( \frac{1}{m^{3/2}} \frac{1}{\mu^{3/2} \alpha^2 + \alpha^4} \right) \sqrt{\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ (-4 - 3\alpha^2) E(-\gamma) + \left( 3 + 3\alpha^2 - \text{sign}(\mu) \sqrt{1 + \alpha^2} \right) K(-\gamma) \right]}.
\]

\[
\frac{\partial \chi_{nn}}{\partial q^2} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{3\Delta_0^2 \xi_k}{8mE_k^5} + \frac{\Delta_0^2(\Delta_0^2 - 4\xi_k^2)k^2}{24m^2E_k^7} \right) \left( \frac{1}{m^{1/2}} \frac{1}{\mu^{1/2} \alpha^2 + \alpha^4} \right) \sqrt{\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ \text{sign}(\mu)E(-\gamma) + \sqrt{1 + \alpha^2} K(-\gamma) \right]}.
\]

\[
\frac{\partial \chi_{nn}}{\partial q^2} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{\Delta_0(2\xi_k^2 - \Delta_0^2)}{8mE_k^5} + \frac{\Delta_0 \xi_k(3\Delta_0^2 - 2\xi_k^2)k^2}{24m^2E_k^7} \right) \left( \frac{1}{m^{1/2}} \frac{1}{\mu^{1/2} \alpha^2 + \alpha^4} \right) \sqrt{\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ \alpha^2 E(-\gamma) + \left( 1 + \alpha^2 + \text{sign}(\mu) \sqrt{1 + \alpha^2} \right) K(-\gamma) \right]}.
\]

\[
\frac{\partial \chi_{11}}{\partial \omega^2} = - \int \frac{d^3k}{(2\pi)^3} \frac{\xi_k^2}{4E_k^5} \left( \frac{1}{m^{3/2}} \frac{1}{\mu^{3/2} \alpha^2 + \alpha^4} \right) \sqrt{\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ (-2 - 3\alpha^2) E(-\gamma) + \left( 3 + 3\alpha^2 + \text{sign}(\mu) \sqrt{1 + \alpha^2} \right) K(-\gamma) \right]}.
\]

\[
\frac{\partial \chi_{11}}{\partial q^2} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{\xi_k(\xi_k^2 - 2\Delta_0^2)}{8mE_k^5} + \frac{5\Delta_0^2 \xi_k^2 k^2}{24m^2E_k^7} \right) \left( \frac{1}{m^{1/2}} \frac{1}{72\pi^2 \mu^{1/2} \alpha^2 + \alpha^4} \right) \sqrt{\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[ (4 + \alpha^2) \text{sign}(\mu) E(-\gamma) + \left( 6 \text{sign}(\mu)(1 + \alpha^2) + (10 + 7\alpha^2) \sqrt{1 + \alpha^2} \right) K(-\gamma) \right]}.
\]
\[
\frac{\partial \chi_L}{\partial q^2} = -\int \frac{d^3 k}{(2\pi)^3} \frac{\Delta_0^2 k^4}{20m^4 E_k^5} \\
= \frac{|\mu|^{1/2}}{30\pi^2m^{1/2} + \alpha^2} \left[-\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[(-4 - 3\alpha^2)E(-\gamma) + \left(3 + 3\alpha^2 - \text{sign}(\mu)\sqrt{1 + \alpha^2}\right)K(-\gamma)\right]\right],
\]
(C7)

\[
\frac{\partial \chi_L}{\partial \omega^2} = 0,
\]
(C8)

\[
\frac{\partial \chi_{jn}}{\partial \omega} = -\int \frac{d^3 k}{(2\pi)^3} \frac{\Delta_0^2 k^2}{12m^2 E_k^5} \\
= \frac{m^{1/2}}{36\pi^2|\mu|^{1/2} + \alpha^2} \left[-\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[-4 \text{sign}(\mu)E(-\gamma) + \left(3 \text{sign}(\mu) - \sqrt{1 + \alpha^2}\right)K(-\gamma)\right]\right],
\]
(C9)

\[
\frac{\partial \chi_{j1}}{\partial \omega} = -\int \frac{d^3 k}{(2\pi)^3} \frac{\Delta_0^2 k^2}{12m^2 E_k^5} \\
= \frac{m^{1/2}}{12\pi^2|\mu|^{1/2} + \alpha^2} \left[-\text{sign}(\mu) + \sqrt{1 + \alpha^2} \left[-E(-\gamma) + K(-\gamma)\right]\right].
\]
(C10)

The derivatives of the transverse current response are in the same approximation

\[
\frac{\partial \chi_T}{\partial q^2} = \frac{1}{3} \frac{\partial \chi_L}{\partial q^2},
\]
(C11)

\[
\frac{\partial \chi_T}{\partial \omega^2} = 0.
\]
(C12)

APPENDIX D: ONE-LOOP THREE-POINT FUNCTIONS AND THE RATIO \(g_3/g_2\).

In this Appendix we show how the computation of a pair of three-point functions in the one-loop approximation leads to the result \(g_3(X)/g_2(X) = 2\). In order to identify the coefficients of \((\nabla^2 \theta)^2\) and \((\partial_i \partial_j \theta)^2\), we first write down all the third-order terms proportional to \(Y \theta \theta\) in the effective Lagrangian (3.3)

\[
L^{(3)} = -\frac{1}{m} g_5(\mu) \nabla \cdot Y \nabla^2 \theta - g'_2(\mu) Y(\nabla^2 \theta)^2 - g'_3(\mu) (\partial_i \partial_j \theta)^2,
\]
(D1)

where \(Y = \mu - X\). This cubic Lagrangian gives rise to a coupling \(Y \theta \theta\) with vertex proportional to

\[
1 \frac{g_5(\mu)}{m} [q_1 \cdot q_2(q_1^2 + q_2^2) + 2q_1^2 q_2^2] - 2g'_2(\mu)q_1^2 q_2^2 - 2g'_3(\mu)(q_1 \cdot q_2)^2,
\]
(D2)

where \(q_1\) and \(q_2\) are the momenta of the pair of Goldstone fields. Note that the coefficient of the first term is known in terms of two-point functions

\[
g_5 = b_6 - b_3 = -\frac{\partial \chi_{jn}}{\partial \omega} + \frac{g_{\chi_{n1}}}{2 + g_{\chi_{11}}} \frac{\partial \chi_{j1}}{\partial \omega}.
\]
(D3)
Next, we compute the Fourier transforms of the one-loop three-point functions that induce the effective action proportional to $\Gamma^{(3)}$. These are given by $\langle (\nabla \cdot j^p(y) \nabla \cdot j^p(z)) \rangle$, where $j^p$ is the paramagnetic fermionic current that couples to $\nabla \theta$ in Eq. (2.3). The two triangle diagrams that contribute read

$$
\Gamma_{1,3}^{(3)}(Q_1, Q_2) = \int_K \text{tr} [\mathcal{G}(K)\mathcal{G}(K + Q_1)\mathcal{G}(K + Q_1 + Q_2)\tau^{1,3}] \\
\times \left( k \cdot q_1 + \frac{q_1^2}{2} \right) \left( k \cdot q_2 + q_1 \cdot q_2 + \frac{q_2^2}{2} \right) \frac{1}{m^2} + (Q_1 \leftrightarrow Q_2). \quad (D4)
$$

For our purposes, it is sufficient to obtain the $O(q^4)$ contribution to $\Gamma_{1,3}^{(3)}$ at zero frequencies. A lengthy computation yields

$$
\Gamma_3^{(3)} = \int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0 k^2}{12m^2E_k^3} \left[ q_1 \cdot q_2(q_1^2 + q_2^2) + 2q_1^2q_2^2 \right] \\
\Gamma_1^{(3)} = \int \frac{d^3k}{(2\pi)^3} \frac{\Delta_0 k^2}{12m^2E_k^3} \left[ q_1 \cdot q_2(q_1^2 + q_2^2) + 2q_1^2q_2^2 \right] + \frac{q_1^2q_2^2}{6m^2E_k^3} (q_1 \cdot q_2)^2 + O(q^5). \quad (D5)
$$

We observe that the coefficients of $(q_1^2 + q_2^2) + 2q_1^2q_2^2$ in $\Gamma_3^{(3)}$ and $\Gamma_1^{(3)}$ are given respectively by $-m^{-1}\partial \chi_{jn}/\partial \omega$ and $-m^{-1}\partial \chi_{jl}/\partial \omega$ (see Eqs. (C9) and (C10)). This agrees with Eq. (D3), after the replacement $\sigma \rightarrow g\chi_{11}Y/(2 + g\chi_{11})$ — from the gap equation — is made in the term of the effective action proportional to $\Gamma_1^{(3)}\sigma \theta \theta$. Simple inspection of the coefficients of $(q_1 \cdot q_2)^2$ and $q_1^2q_2^2$ shows that $g_3(\mu)/g_2(\mu) = 2$, and therefore $g_3(X)/g_2(X) = 2$, after dropping an irrelevant integration constant.

The result $g_3(X)/g_2(X) = 2$ is strikingly simple, and one might wonder whether this is just a peculiarity of the one-loop approximation used here. But, unlike the ratio $f_3/f_2 = -3$ found in Section III, the value $g_3(X)/g_2(X) = 2$ does not seem to be the consequence of any obvious symmetry.

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