KNOT INVARIANTS FROM A KOHNO-KONTSEVICH INTEGRAL FOR BRAIDS

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Abstract

An invariant of knots is constructed from an integral for geometric braids due to Kohno and Kontsevich. It takes values in a quotient by a certain ideal of the algebra generated by chord diagrams over the circle.

1. Preliminaries

In this introductory section we present some basic notions concerning braids and knots, as well as the definition and a couple of properties of an integral for braids due to Kohno and Kontsevich. For more detailed reviews of these subjects, containing further background material, the reader is referred to articles by Birman [1] and Bar-Natan [2].

Let \(0 < x_1 < \cdots < x_n\) be a set of \(n\) numbers lying on the positive real axis of the complex plane \(\mathbb{C}\). Identify \(\mathbb{R}^3\) with \(\mathbb{C} \times \mathbb{R}\), coordinatized by \((z, t)\) with \(z \in \mathbb{C}\) and \(t \in \mathbb{R}\). An \(n\)-strand braid, or simply braid, \(b\), is a one-dimensional piecewise-smooth oriented submanifold of \(\mathbb{R}^3\) lying between the planes \(t = 0\) and \(t = 1\), which is topologically the disjoint union of \(n\) intervals called the strands of \(b\), whose boundary is \(\{(x_i, j) : i = 1, \ldots, n, j = 0, 1\}\), and whose strands point upwards everywhere in the sense that the \(t\) component of the orientation vector is everywhere positive. Frequently one makes the “natural” choice of endpoints \(x_i = i\) for \(i = 1, \ldots, n\), but for reasons which will become clear we wish to allow greater flexibility for the endpoints.

Two \(n\)-strand braids \(b_1\) and \(b_2\) are said to be equivalent if \(b_1\) is carried into \(b_2\) under a diffeomorphism of \(\mathbb{C} \times [0, 1]\) which preserves each horizontal plane, as well as the real axes of the top and bottom planes, and is identity-connected when restricted to the real axes of the top and bottom planes. In the special case of the diffeomorphism restricting to the identity on the top and bottom planes, we will speak of restricted equivalence of braids. The unrestricted equivalence allows the endpoints to move within the positive real axes without changing their order.

Two \(n\)-strand braids \(b_1\) and \(b_2\) sharing the same endpoints may be multiplied: \(b_1b_2\) is the braid obtained by shrinking both braids by a factor \(1/2\) in the \(t\) direction and putting \(b_2\) on top of \(b_1\). Given a braid \(b\), its inverse \(b^{-1}\) is the mirror reflection of \(b\) in the bottom plane, lifted in the positive \(t\) direction to lie again between the \(t = 0\) and \(t = 1\) planes. The restricted equivalence classes of braids sharing a fixed set of endpoints form a group under these operations, with the unit being the class of the trivial braid whose strands point vertically upwards everywhere.
Braids may be used as a tool to study knots and links through the closure construction. Since $b$ is compact it is contained in $U \times [0,1]$ for some open neighbourhood $U \subseteq \mathbb{C}$. The closure of $b$ is the knot or link obtained by connecting the corresponding points $(x_i,0)$ and $(x_i,1)$ for $i = 1, \ldots, n$ by $n$ parallel curves contained in the complement of $U \times ]0,1]$. Since knots and links are identified up to ambient isotopy the closure is well-defined. Any knot or link can be obtained as the closure of some braid by a theorem of Alexander [4]. Furthermore the closure of two equivalent braids gives the same knot or link. When two braid equivalence classes have the same closure they are called Markov equivalent. Markov’s theorem [4][5] states that Markov equivalence is generated by two types of move: (Markov 1) $b_1b_2 \Leftrightarrow b_2b_1$ and (Markov 2) $b \Leftrightarrow (b \bowtie)\sigma_n^{\pm 1}$, where $b$ is an $n$-strand braid contained in $U \times [0,1]$ for some open neighbourhood $U \subseteq \mathbb{C}$, $b \bowtie$ is an $(n+1)$-strand braid consisting of $b$ together with an $(n+1)$th strand of the form $\{(x_{n+1},t) : 0 \leq t \leq 1\}$ contained in the complement of $U \times [0,1]$, and $\sigma_n^{\pm 1}$ is the elementary $(n+1)$-strand braid where the $n$th and $(n+1)$th strands cross over (+) or under (−) each other, whilst the other strands are vertical. Two strands crossing over (under) is taken to mean that they circle halfway round each other in an anticlockwise (clockwise) direction as $t$ goes from 0 to 1.

The notion of chord diagrams for one-dimensional manifolds has been used by a number of authors, in particular in the context of Vassiliev invariants [3][4][5][6][7][10]. Let $X$ be an oriented one-dimensional manifold. A chord diagram over $X$ is a finite set of disjoint pairs of distinct points belonging to the interior of $X$, represented pictorially by dashed lines connecting the points of each pair. Chord diagrams are identified up to orientation and component-preserving diffeomorphisms of $X$. The space of chord diagrams over $X$, denoted $\mathcal{A}(X)$, is the vector space generated over $\mathbb{C}$ by the chord diagrams over $X$, quotiented by the four-term (4T) relations. The 4T relations hold between any four diagrams which are identical apart from in three disjoint open intervals contained in $X$, where they are as in Figure 1.

![4T Relations](image)

Figure 1: The 4T relations

The vector space $\mathcal{A}(X)$ is naturally graded over $\mathbb{N}$ by the number of chords. Denote by $\mathcal{A}_m(X)$ the subspace of $\mathcal{A}(X)$ generated by chord diagrams with precisely $m$ chords, modulo the 4T relations. We will in fact be considering the completion of $\mathcal{A}(X)$ by this grading, also denoted $\hat{\mathcal{A}}(X)$. For the case $X = S^1$ we write $\mathcal{A}$ instead of $\mathcal{A}(S^1)$. Two chord diagrams over $S^1$ can be multiplied by forming the connected sum of the two circles, making the connection in regions free of chord endpoints. Because of the 4T relations this product is well-defined (independent of how the connected sum is formed) and, extending the product to the whole of $\mathcal{A}$ by linearity, $\mathcal{A}$ becomes a commutative algebra with unit (namely the 0-chord diagram over $S^1$ denoted $\bigcirc$). In the case where $b_1$ and $b_2$ are $n$-strand braids there is a natural product operation $\mathcal{A}(b_1) \times \mathcal{A}(b_2) \rightarrow \mathcal{A}(b_1b_2)$ given, for two individual chord diagrams, by putting the chord diagram over $b_2$ on top of the chord diagram over $b_1$, and extended by linearity to general elements. Finally for two manifolds $X_1$, $X_2$, there is a product $\mathcal{A}(X_1) \times \mathcal{A}(X_2) \rightarrow \mathcal{A}(X_1 \amalg X_2) (c_1,c_2) \mapsto c_1 \otimes c_2$, where $\amalg$ is the disjoint union, induced by the natural inclusion maps from the manifolds to their disjoint union.
The following integral associated with a braid $b$ and taking values in $\mathcal{A}(b)$ was introduced by Kontsevich $\mathbb{[8]}$ in the context of Vassiliev invariants and developed further in $\mathbb{[2]} \| \mathbb{[11]}$. It is closely related to an integral introduced earlier by Kohno $\mathbb{[12]} \| \mathbb{[13]}$ taking values in a matrix algebra and reduces to the Kohno integral in a matrix representation of $\mathcal{A}(b)$. Take $m$ values of $t$, $0 < t_1 < \cdots < t_m < 1$ and for each $t_i$ choose a pair of distinct points $(z_i(t_i), t_i)$ and $(z'_i(t_i), t_i)$ belonging to $b$. This is called a pairing, $P$, and there is an obvious chord diagram $D_P \in \mathcal{A}(b)$ associated with it. Now we define:

$$Z(b) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{0<t_1<\cdots<t_m<1} \prod_{P} \frac{d(z_i - z'_i)}{z_i - z'_i} D_P. \quad (1)$$

$Z(b)$ is best understood geometrically as a path-ordered exponential representing parallel transport in the configuration space of $n$ particles in the plane $\mathbb{C}^n \setminus \Delta$ where $\Delta = \{(z_1, \ldots, z_n) : z_i = z_j \text{ for } i \neq j\}$, with respect to the flat Knizhnik-Zamolodchikov $\mathbb{[14]}$ connection $\omega_{KZ} = (1/2\pi i) \sum_{i<j=1}^n d(z_i - z_j)/(z_i - z_j) H_{ij}$ where $H_{ij} \in \mathcal{A}(b)$ is the chord diagram with a single chord connecting strands $i$ and $j$. Due to the flatness of $\omega_{KZ}$, $Z(b)$ has the property:

**Theorem 1** $Z(b)$ respects the restricted equivalence of braids.

Furthermore it has the multiplicative property $\mathbb{[2]} \| \mathbb{[3]}$:

**Theorem 2** $Z(b_1 b_2) = Z(b_1) Z(b_2)$.

2. **Construction of a knot invariant**

Let us for simplicity restrict attention from now on to those braids whose closure is a knot, i.e. a one-component link. The construction which follows may be easily adapted to the general case. For such braids $b$ there is a natural map $p : \mathcal{A}(b) \rightarrow \mathcal{A}$ given by the inclusion of $b$ in its closure, topologically a circle. $pZ$ enjoys the following two properties:

**Theorem 3** $pZ(b_1 b_2) = pZ(b_2 b_1)$.

Proof: This follows from Thm.2 and the obvious property $p(c_1 c_2) = p(c_2 c_1)$ for $c_1 \in \mathcal{A}(b_1)$, $c_2 \in \mathcal{A}(b_2)$. \(\square\)

**Theorem 4** $pZ$ respects the equivalence of braids.

Proof: The Kontsevich integral is defined also for braid-like objects having $n$ upward-pointing strands but whose endpoints in the top and bottom planes, both identified with $\mathbb{C}$, are not the same. Let $b_1$ and $b_2$ be equivalent braids such that the endpoints of $b_1$ and $b_2$ are $\{(x_i, k) : i = 1, \ldots, n, k = 0,1\}$ and $\{(x'_i, k) : i = 1, \ldots, n, k = 0,1\}$ respectively. Consider the braid-like object $\tilde{b}$ which has bottom endpoints $\{(x_i, 0) : i = 1, \ldots, n\}$ and top endpoints $\{(x'_i, 1) : i = 1, \ldots, n\}$, and whose strands are straight lines connecting corresponding endpoints. Let $b^{-1}$ be its inverse in the sense of the mirror construction used for braids. The equivalence of $b_1$ and $b_2$ implies that $b_1$ and $\tilde{b} b^{-1} \tilde{b}$ are equivalent in the restricted sense. Adapting Thm. 3 to braid-like objects one has $pZ(b_1) = pZ(\tilde{b} b^{-1} \tilde{b}) = pZ(b_2)$, where the last equality uses Thm. 1. \(\square\)

Now we calculate $pZ$ for the simplest non-trivial 2-strand braid $\sigma_1$ and its inverse:

$$pZ(\sigma_1^\pm 1) = \bigcirc + \sum_{m=1}^{\infty} (\pm 1/2)^m / m! \bigcirc_m \quad (2)$$
where \( \mathcal{O}_m \), for \( m = 1, \ldots, \infty \), denotes the \( m \)-chord diagram over \( S^1 \) such that the endpoints of each chord are diametrically opposite points of the circle. This is because the coefficient of \( \mathcal{O}_m \) coming from equation (1) reduces to a path-ordered integral along a single contour \( C \) given by \( z(t) = \exp(\pi it) \) for \( 0 \leq t \leq 1 \), namely \((1/2\pi i)^m \mathcal{P} \int_C dz_1/z_1 \cdots dz_m/z_m\) where \( \mathcal{P} \) denotes path-ordering, which is given by:

\[
(1/2)^m \int_{t_m=0}^1 \int_{t_{m-1}=0}^{t_m} \cdots \int_{t_1=0}^{t_2} dt_1 dt_2 \cdots dt_m = (1/2)^m 1/m!.
\]

Now we define the \( \mathcal{A} \) elements

\[
r_{\pm} = pZ(\sigma_1^{\pm}) - \mathcal{O}.
\]

\( r_\pm \) are non-invertible and thus the ideal \( \mathcal{I} = r_+ \mathcal{A} + r_- \mathcal{A} \) is non-trivial. Let \( \tilde{\mathcal{A}} \) denote the quotient of \( \mathcal{A} \) by this ideal and let \( k : \mathcal{A} \to \tilde{\mathcal{A}} \) be the canonical projection.

The following main theorem employs a limit technique similar to one used in Le and Murakami [9], Lemma 2.3.4.

**Theorem 5** Let a knot \( K \) be presented as the closure of a braid \( b \). Then \( Y(K) = kpZ(b) \) is well-defined and thus gives rise to an \( \tilde{\mathcal{A}} \)-valued knot invariant.

**Proof:** Since \( pZ \) respects braid equivalence and the first Markov move (Thms. 3 and 4) it remains to show that \( kpZ \) respects the second Markov move. Let \( b \) be an \( n \)-strand braid and \( (b \Pi) \sigma_n^{\pm 1} \) be the corresponding \((n+1)\)-strand braid in the move. Translate the final strand and the over/undercrossing \( \sigma_n^{\pm 1} \) out to the right by a distance \( R - 1 \) as in Figure 2. Call this new braid \( b_n(R) \). It factorizes as

\[
b_n(R) = (b \Pi_R |) \tilde{b}_R(|^{n-1} \Pi_R \sigma_1^{\pm 1}) \tilde{b}_R^{-1}
\]

where \( b \Pi_R \) is the braid \( b \Pi \) with a distance \( R \) separating \( b \) and the final strand, \( \tilde{b}_R \) is a braid-like object with all strands pointing vertically upwards except the \( n \)th one, \(|^{n-1} \Pi_R \sigma_1^{\pm 1} \) is the elementary braid \( \sigma_n^{\pm 1} \) where the final two strands have been moved out to the right by a distance \( R \) and finally \( \tilde{b}_R^{-1} \) is the inverse of \( \tilde{b}_R \) (see Figure 2). In the corresponding factorization of \( Z(b_n(R)) \) we split each factor into an \( R \)-independent and an \( R \)-dependent part:

\[
Z((b \Pi_R |)) = Z(b) \otimes Z_0(|) + O(1/R)
\]
The second terms on the right hand sides of (6) and (7) are $O(1/R)$ as $R \to \infty$ since the corresponding chords are multiplied by coefficient integrals of order $O(1/R)$. In (8) the $R$-independent term is $Z_0(\tilde{b}_R^{\pm 1})$ (note that $A_0(\tilde{b}_R^{\pm 1})$ is $R$-independent) since for $R = 1$, $\tilde{b}_R^{\pm 1}$ is the trivial braid and thus $\sum_{i \geq 1} Z_i(\tilde{b}_R^{\pm 1}) = 0$. The terms of chord number $i$ in this remainder are $O(\ln(R)^i)$ as $R \to \infty$. Now $pZ(b_n(R))$ is $R$-independent by Thm. 4. Since the only convergent limits which can be obtained by multiplying terms of order $O(1/R)$ and terms of order $O((\ln R)^i)$ are 0, and since $pZ(b_n(R))$ as a whole is convergent, we conclude that $pZ((b \downarrow |)\sigma_n^{\pm 1})$ is given by

$$pZ((b \downarrow |)\sigma_n^{\pm 1}) = p((Z(b) \otimes Z_0(|))Z_0(\tilde{b}_R)(Z_0(\lvert^{n-1}) \otimes Z(\sigma_1^{\pm 1}))Z_0(\tilde{b}_R))$$

When $k$ is applied to both sides of this equation, on the right-hand-side only the 0-chord term remains in the third factor and we thus obtain $Y((b \downarrow |)\sigma_n^{\pm 1}) = Y(b)$. □

We finally prove a small non-triviality result.

**Theorem 6** The invariant $Y$ distinguishes the trefoil knot and its mirror image.

**Proof:** The trefoil may be presented as the closure of $\sigma_1^3$, whereas its inequivalent mirror image is the closure of $\sigma_1^{-3}$. Now we expand $Y(\sigma_1^3)$ and $Y(\sigma_1^{-3})$ up to 3-chords:

$$Y(\sigma_1^3) = \bigcirc + (3/2) \bigcirc_1 + (3/2)^2 1/2! \bigcirc_2 + (3/2)^3 1/3! \bigcirc_3 + \cdots$$

$$= \bigcirc + 6r_+ + 3r_- + (1/2) \bigcirc_3 + \cdots$$

and by a similar calculation

$$Y(\sigma_1^{-3}) = \bigcirc + 3r_+ + 6r_- - (1/2) \bigcirc_3 + \cdots$$

where the dots represent terms with higher chord diagram number. An arbitrary linear combination of the form $\bigcirc_3 + \cdots$ does not belong to the ideal $I$, as may be easily established by studying 4T relations for 3-chord diagrams in $A$. Thus the coefficient of $\bigcirc_3$, being the first non-trivial coefficient in the $Y$ invariant, differs for the trefoil and its mirror image. □

We conclude with a few remarks.

Previous approaches to using the Kontsevich integral for obtaining knot and link invariants [2][6][7] led to severe computational problems. The integral [4], adapted to tangles, has to be multiplied by a certain element of $A$ before it becomes a knot invariant. This element is rich in structure, being related to Euler sums/Zagier zeta functions [8][14][15], but complicated to use. Whilst the integral [4] for braids remains far from trivial, one may hope that the fairly rigid structure of braids will enable efficient calculational methods to be found.

The invariant takes values in $\tilde{A} = A/I$, a satisfyingly large space. The ideal $I$ is, loosely speaking, only twice as large as the ideal $P_A$ of separated chord diagrams, which occurs in the Bar-Natan approach to Vassiliev invariants [2]. It should be pointed out that in our construction there is no need to quotient out by this ideal.

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