ON TACHYON KINKS FROM THE DBI ACTION

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We consider solitonic solutions of the DBI tachyon effective action for a non-BPS brane in the presence of an electric field. We find that for a constant electric field \( \tilde{E} \leq 1 \), regular solitons compactified on a circle admit a singular and decompactified limit corresponding to Sen’s proposal provided the tachyon potential satisfies some restrictions. On the other hand for the critical electric field \( \tilde{E} = 1 \), regular and finite energy solitons are constructed without any restriction on the potential.

1 Introduction

In addition to the stable D-branes, type II superstrings admit non-supersymmetric D-branes. The instability of these non-BPS branes is signalled by the presence of a tachyon on their worldvolume, and their decay is described by the dynamics of the tachyon. The BPS branes can be viewed as tachyon kinks on the non-BPS branes with one dimension higher. The dynamics of the decaying tachyon can be captured using the Dirac-Born-Infeld (DBI):

\[
S = - \int d^p x dt V(T) \sqrt{(1 + \partial_{\mu} T \partial^{\mu} T \eta^{\mu\nu})}. \tag{1}
\]

Here \( V(T) \) is the tachyon potential which is even and vanishes at infinity where it reaches its minimum. Near the global maximum at \( T = 0 \), \( V(T) = T_p (1 - \beta^2 T^2 / 2) + \ldots \) where \( T_p \) is the tension of the non-BPS \( p \)-brane and the potential encodes the mass of the tachyon near the perturbative vacuum \( T = 0 \). Finally \( \eta^{\mu\nu} = (-1, +1, \ldots, +1) \) is the \( p + 1 \) Minkowski metric.

It happens that finite energy solitons of the DBI action are singular. One would like to regularise the behaviour of such solitons. This can be achieved by compactifying one spatial dimension on a circle.

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singular solitons can be then obtained in the decompactified limit provided some restrictions on the tachyon potential are imposed. On the other hand, there seems to be an intrinsic ambiguity in the regularization process of the DBI action leading to singular solutions. In particular, adding an extra kinetic term for the tachyon and then taking the limit where this term vanishes\(^a\) leads to singular kinks with no restriction on the shape of the potential. Such an ambiguity can be lifted for kinks in an electric field background. In particular for a critical value \(E = 1\) of the electric field, one always finds regular and finite energy solutions of the DBI action.

The plan of this paper is as follows. In Section 2 we revisit Derrick’s theorem which gives necessary conditions for the existence of finite energy solitons for actions of the form (1). In Section 3, we compactify one spatial coordinate on a circle of radius \(R\) in order to get regular kinks. In fact, we obtain regular solitons describing \(n\) pairs of kinks and anti-kinks. Then we determine the conditions on the potential for which the decompactifying limit exists. The stability of the kink-antikink configurations is also described. Finally we consider the case of charged kinks\(^b\) and focus on the critical case where regular and finite energy solitons on the infinite line can be found.

\section{Uncharged solitons and Derrick’s theorem}

We are interested in kink-like solutions of (1). These kink solutions should represent the stable BPS \(p-1\) branes into which the non-BPS \(p\)-brane decays. Before doing this explicitly, it is worthwhile recalling Derrick’s theorem\(^c\); in the case of the usual Klein-Gordon action for scalar fields, it tells us that finite energy static solitonic solutions on an infinite space are only possible in (1+1)-dimensions. Here will draw similar conclusions starting from (1).

Let us consider a slightly more general action:

\[ S = - \int d^p x \, dt \, V(T) \left( 1 + \partial_\mu T \partial_\nu T \eta^{\mu \nu} \right)^q, \]  

where the real scalar field \(T\) is dimensionless (we set \(\alpha' = 1\) throughout). For \(q = 1/2\) this is just (1). Let \(T_1(x)\) be a static solution of the equations of motion, and hence an extremum, \(\delta E = 0\), of the finite static energy functional

\[ E[T] = \int d^p x V(T) \left( 1 + \partial_i T \partial^i T \right)^q. \]  

\(^a\)Equivalently one can work with the action (2) below with \(q \neq 1/2\) and then take the limit \(q \to 1/2^+\) at the end of the calculation.
Now consider a family $T_\lambda(x) = T_1(\lambda x)$, using $\frac{d}{d\lambda} E[T_\lambda]|_{\lambda=1} = 0$ we find
\[
\int d^p x V(T) \left( 1 + \partial_i T \partial^i T \right)^{q-1} \left[ p + (p - 2q) \partial_i T \partial^i T \right] = 0. \tag{4}
\]

When $q = 1/2$, the case we focus on here, the square bracket in (4) is $p + (p - 1) \partial_i T \partial^i T$ which can never vanish for $p \geq 1$. Thus no finite energy static solutions seem to be permitted. However, there is formally a way around this: when $p = 1$ Eq. (4) becomes
\[
0 = \int d^p x \frac{V(T)}{\sqrt{1 + T'^2}} \tag{5}
\]
which can vanish if
\[T' \to \pm\infty \quad \text{or} \quad T \to \pm\infty
\]
with $E[T]$ remaining finite.

Let us suppose that the equations of motion admit a solution with $T' \to \pm\infty$ describing a single kink on the infinite line (below we will find the conditions on $V(T)$ such that this is the case),
\[
T(x) = \lim_{C \to 0} \frac{x}{C} = \begin{cases} 
\infty & x > 0 \\
0 & x = 0 \\
-\infty & x < 0
\end{cases} \tag{6}
\]
This is a typical case of the solutions discussed by Sen it describes an infinitely thin topological kink interpolating between the two vacua. Notice that $V = 0$ everywhere apart from when $T = 0 = x$. Substitution of (6) into the energy functional (3) (taking carefully the $C \to 0$ limit) gives the the energy of this singular solution to be
\[
E_{\text{Sen}} = \int_{-\infty}^{\infty} V(x) dx. \tag{7}
\]
Sen has argued that such singular kinks are stable, as the BPS brane should be, and furthermore that their effective action is exactly the required DBI action. Hence such solutions are of great interest, and would suggest that BPS branes are infinitely thin. Note that the parameters of $V(T)$ should be tuned such that $E_{\text{Sen}} = T_{p-1}$.

In this note, we will construct such singular solitons as limits of regularised kink solutions. Sen’s singular limit can then be approached in the decompactification limit. We are looking for static kink-like solutions in which $T$ has a non-trivial dependence on only one spatial coordinate; $T = T(x)$. Let us
assume that the kink is centered at the origin, $T(0) = 0$. The equations of motion coming from (2) with $q = 1/2$ have a first integral,

$$\frac{V(T)}{\sqrt{1 + (T')^2}} = V_0, \quad (8)$$

where $V_0 \geq 0$ is a constant. The energy of this solution is given by

$$E = \int dx V(T) \sqrt{1 + (T')^2} = \frac{1}{V_0} \int dx V^2(T). \quad (9)$$

Since $T'^2$ is positive, solutions of equation (8) exist in the region $V(T) \geq V_0$. Furthermore, the solutions $T(x)$ are periodic with a $V_0$-dependent amplitude which, from (8), diverges as $V_0 \to 0$. Note that within one period there must be both a kink and an anti-kink (corresponding to the two points for which $T(x) = 0$). Also, the energy density of the kinks becomes more and more localised as $V_0 \to 0$. Below we will study in detail the dependence of both the period and $E$ on $V_0$.

In order for $E$ to be finite, one must have $|T| \to \infty$ as $|x| \to \infty$. This immediately implies from (8) that $V_0 = 0$ — a topological kink. Sen’s solution is such a singular solution in which $T$ vanishes for only one value of $x$: in other words Sen’s solution is periodic with a divergent wavelength. Furthermore in that case $E = 2E_{Sen}$.

We would like to approach the singular ($V_0 \to 0$) kink(s) as the limit of regular solutions with $V_0 \neq 0$. In order to get regular and finite energy solutions we shall suppose that $x$ is a compact direction of length $2\pi R$, so that $V_0$ can now be non-zero. Our aim is to see whether the limit $R \to \infty$ with $E$ being finite exists. Note that if this limit exists we expect to get twice (7) as the energy of the resulting solution. In fact, BPS branes have a RR charge and on a circle the sum of the charges must vanish so all we can get are pairs of branes and anti-branes. Sen’s solution corresponds to the brane infinitely distant from the anti-brane.

The period of the solutions of equation (8) is $4\zeta$ where

$$\zeta(V_0) = \int_0^{T_0} \frac{dT}{\sqrt{\left(\frac{V(T)}{V_0}\right)^2 - 1}}, \quad (10)$$

and $T_0$ is defined by $V(T_0) = V_0$. The radius is thus given by

$$2\pi R = 4n\zeta, \quad (11)$$

\[b\] Notice that this condition is identical to (5).
where \( n = 1, 2, \ldots \). The separation between the brane and the anti-brane is \( 2\zeta \). The \( V_0 \)-dependent energy of this solution can be written as

\[
E(V_0) = 4n \int_0^{T_0} dT \frac{V}{\sqrt{1 - \left(\frac{V_0}{V}\right)^2}} \quad (12)
\]
or equivalently \( 4n\mathcal{E}(V_0) \).

The behaviour of \( \zeta(V_0) \) as \( V_0 \to 0 \) depends critically on the form \( V(T) \) for large \( T \). Since the potential is assumed positive, let us write it in the form \( V = e^{-\sigma(T)} \), then the behavior of \( \zeta(V_0) \) depends on \( \sigma'(T) \) at large \( T \)[13] \( \zeta(V_0) \propto [\sigma'(T_0)]^{-1} \). Let us examine the limit where the radius goes to infinity.

There are three possibilities

1. \( \zeta \to \infty \) as \( V_0 \to 0 \). This happens when \( \sigma' \to 0 \). The kink anti-kink separation tends to infinity, and as \( R \to \infty \), equation [11] can be satisfied with \( n = 1 \) (a single kink and anti-kink). The energy of this solution will, however, depend on the behaviour of \( \zeta V_0 \). Indeed, the energy goes to \( 2 \int_{-\infty}^{\infty} V(T) dT \) if and only if \( \zeta V_0 \to 0 \) [13]. This latter case is the one which reproduces a single singular solution in the non-compact limit.

2. \( \zeta \to \text{const} \neq 0 \) as \( V_0 \to 0 \). Now the kink anti-kink distance tends to a constant and one can never get an isolated kink. As \( R \to \infty \), equation [11] can only be satisfied if \( n \to \infty \). In fact the realistic potential \( V = v / \cosh(\beta T) \) falls in this category.

3. \( \zeta \to 0 \) as \( V_0 \to 0 \). The kink and anti-kink separation tends to zero and it is clearly not possible to obtain a single kink and anti-kink in the non-compact limit. This behaviour occurs for many potentials considered in the literature; for example \( V(T) = ve^{-T^2} \).

We now ask which conditions \( V(T) \) must satisfy so that as \( V_0 \) and \( R \to \infty \) the energy \( E \) in [12] with \( n = 1 \) reduces to \( 2E_{\text{Sen}} \) in [7]. We find that \( \mathcal{E}(V_0) \) will be finite if and only if, as \( T \to \infty \),

\[
\frac{|V'|}{V^2} \to \infty.
\]

In this case, \( \mathcal{E} \to \int_0^\infty V(T)dT \). Thus from [13] we conclude that for exponential potentials

\[
V(T \to \infty) \sim \exp(-T^a) \quad \Rightarrow \quad \mathcal{E} \text{ finite } \forall \ a > 0 \quad (14)
\]

whereas for power-law potentials

\[
V(T \to \infty) \sim \frac{1}{T^{1/\alpha}} \quad \Rightarrow \quad \mathcal{E} \text{ finite } \iff \alpha < 1. \quad (15)
\]
For any $R$ and non-zero $V_0$ the kink anti-kink array is unstable\cite{18}. On the other hand the singular kink is always stable.

3 Charged Solitons

Now consider solitons in an electric field background\cite{19} $A_0(x)$ in the gauge where $A_1 = 0$ and all the other components of the gauge field are set to zero. The DBI action depends on the tachyon and the field strength $F_{\mu\nu}$:

$$S = - \int d^{p+1}x V(T) \sqrt{-\gamma}$$

where $\gamma$ is the determinant of $\gamma_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu T \partial_\nu T + F_{\mu\nu}$ and the only non-zero component of $F_{\mu\nu}$ is $F_{01} = -\partial_1 A_0(x) \equiv \tilde{E}$. The dynamics are determined by the Lagrangian density

$$\mathcal{L} = V(T) \sqrt{1 + T'^2 - \tilde{E}^2}$$

which now has two second integrals $V_0$ and $e$ where

$$V_0 = \frac{V(T)}{\sqrt{1 + T'^2 - \tilde{E}^2}}$$

and from the Euler-Lagrange equations for $A_\mu$

$$e = \frac{\tilde{E} V(T)}{\sqrt{1 + T'^2 - \tilde{E}^2}}$$

Since $V_0$ and $e$ are constant, the electric field

$$\tilde{E} = \frac{e}{V_0}$$

is also constant. The tachyonic dynamics can be deduced from

$$\dot{V}_0 = \frac{V(T)}{\sqrt{1 + (\frac{dT}{dy})^2}}$$

where $dy = \sqrt{1 - \tilde{E}^2} dx$ and $\dot{V}_0 = \sqrt{1 - \tilde{E}^2} V_0$. Notice that there is a maximal value for the electric field $\tilde{E}_0 = 1$. As soon as $\tilde{E} < \tilde{E}_0$, one can use the previous analysis on the shape of the potential $V(T)$ with the same restrictions as in the neutral case.

More interestingly, the critical case $\tilde{E} = 1$ leads to non-singular solitons on the real line. In that case the dynamics are governed by

$$T'^2 = \frac{V^2}{V_0^2}.$$
Let us define \( T_{V_0}(x) = T_1 \left( \frac{x}{V_0} \right) \) then \( T_1 \) satisfies the universal profile
\[
|T'_1| = V(T). \tag{23}
\]
The solution
\[
x = \int_0^T \frac{du}{V(u)} \tag{24}
\]
defines a soliton interpolating between the \( T = -\infty \) at \( x = -\infty \) and \( T = \infty \) at \( x = \infty \), i.e. between two vacua of the theory. The width of the soliton is of order \( O(V_0) \) implying that one gets an infinitely thin soliton in the \( V_0 \to 0 \) limit. For any other value of \( V_0 \) the soliton is of finite width.

Finally notice that for any value of \( V_0 \), the energy of the solitons when \( \tilde{E} = 1 \) is simply
\[
E = \int_{-\infty}^{\infty} V(T)dT, \tag{25}
\]
i.e. Sen's value for the tension of the BPS brane. This value is independent of the width \( V_0 \) of the soliton. Therefore the zero-width limit does not modify the energy of the soliton.

We have thus found that introducing a constant electric field allows one to define regular solitons of the non-BPS DBI action on the infinite line. Moreover infinitely thin solitons can also be obtained by taking the limit where the width of the regular solitons goes to zero.

**4 Conclusion**

We have performed an analysis of the regular solitons on the circle of the non-BPS DBI action, both in the absence and in the presence of a constant electric field in the soliton direction. For an electric field below the critical value \( \tilde{E} = 1 \) and with the regularisation scheme presented here, one finds that singular solitons on the infinite line can only be obtained for restricted tachyon potentials. On the other hand when \( \tilde{E} = 1 \), regular solitons of finite energy exist in noncompact space.

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