Recent developments in affine Toda quantum field theory

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Invited lectures at the CRM-CAP Summer School ‘Particles and Fields 94’
August 16-24, 1994
Banff, Alberta, Canada.

December 1994
1. Introduction

It is not intended in these four talks to give a detailed review of all the recent activities in the area of Toda field theory, but rather to highlight some of the interesting developments, and to point out some of the currently outstanding problems. The list of references is by no means exhaustive.

Affine Toda field theory \([1,2]\) is a theory of \(r\) scalar fields in two-dimensional Minkowski space-time, where \(r\) is the rank of a compact semi-simple Lie algebra \(g\). The classical field theory is determined by the lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi)
\]  

(1.1)

where

\[
V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i \epsilon \beta \alpha_i \cdot \phi.
\]  

(1.2)

In (1.2), \(m\) and \(\beta\) are real (classically unimportant) constants, \(\alpha_i, \ i = 1, \ldots, r\) are the simple roots of the Lie algebra \(g\), and \(\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i\) is a linear combination of the simple roots; it corresponds to the extra spot on an extended Dynkin diagram for \(g\), at least in so far as representing its inner products with the simple roots is concerned. For notational reasons, \(n_0 = 1\) in (1.2), but the other integers \(n_i\) are characteristic for each type of theory. They are tabulated in many places, for example in Kac’ book \([3]\). The quantity \(h = \sum_{i=0}^{r} n_i\) is called the Coxeter number. For most purposes, in the present context \(\alpha_0\) will not represent a simple root of the affine algebra \(\hat{g}\).

If the term \(i = 0\) is omitted from (1.2) in the lagrangian (1.1), then the theory, both classically and after quantisation is conformal, and will be referred to as conformal Toda field theory or, simply, as Toda field theory. With the term \(i = 0\), the conformal symmetry is broken but the theory remains classically integrable, in the sense that there are infinitely many independent conserved charges in involution. The recent renewal of interest in Toda field theories was stimulated by Zamolodchikov’s ideas concerning perturbations of conformal field theories \([4,5]\). The possible root systems which can be used in the lagrangian (1.1), maintaining the classical integrability are in one to one correspondence with the untwisted and twisted affine Dynkin-Kac diagrams \([2]\). However, in what follows, it is useful to distinguish those which are unchanged (apart from a possible flip) under the transformation

\[
\alpha_i \rightarrow 2\alpha_i/|\alpha_i|^2,
\]  

(1.3)
and those which are ‘dual’ pairs under this transformation. The self dual set are $a_n^{(1)}$, $d_n^{(1)}$, $e_n^{(1)}$ whose roots are all of equal length (conventionally, the longest root satisfies $|\alpha_i|^2=2$) and $a_{2n}^{(2)}$, the dual pairs are $(b_n^{(1)}, a_{2n-1}^{(2)})$, $(c_n^{(1)}, d_{n+1}^{(2)})$, $(g_2^{(1)}, d_4^{(3)})$, $(f_4^{(1)}, e_6^{(2)})$.

Each of the members of the self-dual set are untwisted with roots of equal length, except for $a_{2n}^{(2)}$ which is twisted and contains roots of three different lengths. The affine Toda theory corresponding to the simplest case $a_1^{(1)}$ is recognised to be the sinh-Gordon theory (for real coupling), or the sine-Gordon theory (for imaginary coupling, or real coupling and imaginary fields).

Each of the non simply-laced or twisted root systems can be obtained by ‘folding’ one of the simply-laced Dynkin or affine Kac-Dynkin diagrams, respectively. A pair of examples will suffice to illustrate this. Consider the Dynkin diagram for $d_4$ (first diagram).

![Diag: $d_4$]

It has a symmetry $\alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_1$, under which $\beta_1 = \alpha_4$ and $\beta_2 = (\alpha_1 + \alpha_2 + \alpha_3)/3$ are clearly invariant. The two roots $\beta_1, \beta_2$ are simple roots for $g_2$ (that is, the second diagram, where the shorter root $\beta_2$ corresponds to the black spot). The extra root $\alpha_0 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)$ for $d_4$ is also invariant and becomes the extra root $\beta_0 = -(2\beta_1 + 3\beta_2)$ for $g_2^{(1)}$. On the other hand, the dual of $g_2^{(1)}$, $d_4^{(3)}$ is obtained using the threefold symmetry of the extended Kac-Dynkin diagram of $e_6^{(1)}$ (first diagram).

![Diag: $e_6$]

In this case, $\beta_1 = \alpha_3$, $\beta_2 = (\alpha_2 + \alpha_4 + \alpha_6)/3$ and $\beta_0 = (\alpha_0 + \alpha_1 + \alpha_5)/3 = -\beta_1 - 2\beta_2$ are the invariant combinations under the symmetry and provide the root system for $d_4^{(3)}$ (the second diagram, in which the additional root attached to the $g_2$ Dynkin diagram is short).

The two types of $g_2$ extension lead to quite different classical field theories.
2. Classical integrability and classical data

Toda field theory is classically integrable and indeed conformal [6]. Affine Toda field theory is classically integrable and also, in a generalised version, conformal [7]. Consider a conformal transformation in light-cone variables:

$$x_\pm = (x^0 \pm x^1)/2 \rightarrow \bar{x}_\pm(x_\pm).$$

(2.1)

Clearly, since the second derivative of the scalar fields transforms via

$$\partial_+ \partial_- \phi \rightarrow \bar{\partial}_+ \bar{\partial}_- \phi = \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \partial_+ \partial_- \phi,$$

the equations of motion are invariant provided the potential term also scales in a suitable manner:

$$\sum_{i=1}^r n_i \alpha_i e^{\beta \alpha_i \phi} \rightarrow \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \sum_{i=1}^r n_i \alpha_i e^{\beta \alpha_i \phi}.$$

The latter requires the fields themselves to transform according to

$$\phi(x_+, x_-) \rightarrow \bar{\phi}(\bar{x}_+, \bar{x}_-) = \phi(x_+, x_-) + \frac{\rho}{\beta} \ln \left( \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \right),$$

(2.2)

where the vector $\rho$ enjoys the property

$$\rho \cdot \alpha_i = 1, \quad i = 1, 2, 3 \ldots, r.$$

(2.3)

Since the fundamental weights satisfy

$$2\lambda_i \cdot \alpha_j / |\alpha_j|^2 = \delta_{ij},$$

$\rho$ may be expressed in terms of the fundamental weights:

$$\rho = \sum_{i=1}^r \frac{2}{|\alpha_i|^2} \lambda_i.$$

It is immediately clear, since

$$\rho \cdot \alpha_0 = - \sum_{i=1}^r n_i = 1 - h,$$

that adding the extra term in the lagrangian (proportional to $n_0$) breaks the conformal symmetry.
On the other hand, suppose the extra affine term is included, and further suppose
that the set of roots is actually taken to be the set of simple roots for the affine algebra
itself. Then, the set $\hat{\alpha}_i$, $i = 0, 1, 2 \ldots, r$ are independent, lying in a Minkowski space of
signature $(r + 1, 1)$ and, once again, a vector $\hat{\rho}$ can be found for which

$$\hat{\rho} \cdot \hat{\alpha}_i = 1 \quad i = 0, 1, 2, \ldots, r.$$ 

Using this, the argument of the last paragraph may be repeated to conclude the theory is
conformal even with the affine term included [7]. The penalty being paid for this is that
the scalar fields $\phi$ no longer take values in a Euclidean space and the energy is no longer
a positive definite functional of the field components. Restricting the fields to a Euclidean
space breaks the conformal invariance and, effectively, introduces a mass scale.

This situation is reminiscent of string theory which, in its most basic form, contrives to
describe families of massive states starting from a conformally invariant lagrangian whose
fields take values in space-time.

Once conformal Toda field theory is quantised, it provides a coupling dependent rep-
resentation of the Virasoro algebra whose central charge (\textit{ade series}) is given by [5]:

$$c(\beta) = r + 48\pi |\rho|^2 \left( \frac{\beta}{4\pi} + \frac{1}{\beta} \right)^2. \quad (2.4)$$

This central charge is clearly symmetric under the transformation $\beta \rightarrow 4\pi/\beta$, revealing
that the quantum conformal field theory enjoys a weak-strong coupling symmetry not
apparent in the original lagrangian.

Throughout these notes it will be assumed the fields take values in an $r$-dimensional
Euclidean space, spanned by the simple roots of the Lie algebra $g$.

The classical integrability of the affine Toda field theories relies on the existence of a
Lax pair from which the conserved quantities may be established. The details of this is a
story in itself [2,8] but from our present perspective it is enough to be aware of some of the
main results. First of all, it is relatively straightforward to check the equivalence between
the zero curvature property

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = 0,$$
and the affine Toda field equations provided the two components of the two-dimensional vector potential $A_\mu$ are given by:

\begin{align}
A_0 &= H \cdot \partial_1 \phi / 2 + \sum_{i=0}^{r} m_i (\lambda E_{\alpha_i} - 1 / \lambda E_{-\alpha_i}) e^{\alpha_i \cdot \phi / 2} \\
A_1 &= H \cdot \partial_0 \phi / 2 + \sum_{i=0}^{r} m_i (\lambda E_{\alpha_i} + 1 / \lambda E_{-\alpha_i}) e^{\alpha_i \cdot \phi / 2},
\end{align}

(2.5)

where $H, E_{\alpha_i}$ and $E_{-\alpha_i}$ are the Cartan subalgebra and the generators corresponding to the simple roots and the extra root, respectively, of $g$. Thus, in particular,

\[
[H, E_{\alpha_i}] = \alpha_i E_{\alpha_i},
\]

\[
[E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} \frac{2\alpha_j \cdot H}{|\alpha_j|^2}.
\]

The spectral parameter is $\lambda$, and the coefficients $m_i$ are chosen to satisfy

\[m_i^2 = n_i \alpha_i^2 / 8.\]

For convenience, the classically unimportant constants $m$ and $\beta$ have been scaled away.

Since the path ordered integral of $A_1$,

\[T(a, b; \lambda) = P \exp \int_a^b dx^1 A_1\]

satisfies

\[\frac{d}{dt} T = T A_0(b) - A_0(a) T ,\]

then

\[Q(\lambda) = tr T(-\infty, \infty; \lambda)\]

(2.6)

is time independent when $\partial_1 \phi \to 0$ as $x^1 \to \pm \infty$ and $\phi(\infty) = \phi(-\infty) + 2\kappa$, where $\kappa \cdot \alpha_i$ is an integer.

An important fact about the Lax pair is the possibility of performing a gauge transformation after which the potentials lie in a Cartan subalgebra $h_i$ of $g$, two members of which are

\[E_{\pm 1} = \sum_{i=0}^{r} m_i E_{\pm \alpha_i}.\]
Once this gauge transformation has been done, the potential $A_1$ takes the form

$$ a_1 = \lambda E_1 + \sum_{s \geq 1} \lambda^{-s} h_s I_0^{(s)}, $$

where the sum on the right hand side runs over the exponents of the algebra $g$ (another characteristic set of integers which will be met again below), modulo $h$, the Coxeter number. The elements of the Cartan subalgebra are conveniently labelled by the $r$ exponents, and $h_{s+n} = h_s$. The zero curvature condition reads

$$ \partial_0 a_1 = \partial_1 a_0, $$

and therefore the integral of $a_1$ over the whole line is conserved. Since $\lambda$ is arbitrary, there are infinitely many conserved quantities

$$ Q_s = \int_{-\infty}^{\infty} dx^1 I_0^{(s)}. $$

Adding or subtracting the equations (2.5) reveals that $\lambda$ scales under a Lorentz transformation ($\lambda \rightarrow l \lambda$) in order to guarantee the correct transformation of the light-cone components of the vector potential. Consequently, the conserved quantities $Q_s$ must scale under the transformation by a factor $l^s$. (There is an alternative abelianisation for which there is a similar expression for $a_1$ after the gauge transformation expressed as a series of positive powers in $\lambda$. From this, a matching set of conserved quantities of the opposite spin is obtained.)

It is possible to demonstrate the involutary nature of the charges by first demonstrating the existence of a classical $r$-matrix for which

$$ \{T(\lambda), \otimes T(\mu)\} = [r(\lambda/\mu), T(\lambda) \otimes T(\mu)], \quad T(\lambda) \equiv T(-\infty, \infty; \lambda), $$

follows from the canonical equal-time Poisson bracket between the fields and their conjugate momenta. Indeed, Olive and Turok [8] give $r$ in the form:

$$ r(\lambda/\mu) = \frac{\mu^h + \lambda^h}{\mu^h - \lambda^h} \sum_{i=1}^{r} H_i \otimes H_i + \frac{2}{\mu^h - \lambda^h} \sum_{\alpha > 0} |\alpha|^2 \left( \lambda^{l(\alpha)} \mu^{h-l(\alpha)} E_{\alpha} \otimes E_{-\alpha} + \lambda^{h-l(\alpha)} \mu^{l(\alpha)} E_{-\alpha} \otimes E_{\alpha} \right), $$

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where the sum is over all positive roots of \( g \) (ie all those roots expressible as combinations of simple roots with positive integer coefficients), and \( l(\alpha) \) is the length of a root (ie the sum of the integers in its expansion in terms of simple roots).

As mentioned above, the classically conserved charges are two dimensional Lorentz tensors, labelled by their ‘spin’ in light-cone coordinates, the possible spins being the exponents of the algebra repeated modulo its Coxeter number \( h \). In other words, the conserved charges may be denoted \( Q_{s+kh} \), where \( s \) is an exponent and \( k \) is an integer. The quantities \( Q_{\pm 1} \) correspond to the light-cone components of the energy-momentum. If the quantised field theory retains the integrability property, it is expected that the conserved quantities will survive as mutually commuting quantum operators whose eigenstates are the particles of the theory. Thus, for single-particle states,

\[
Q_p|a> = q_p^a e^{p\theta_a}|a> \quad p = s + kh,
\]

where \( \theta_a \) is the rapidity of the particle labelled \( a \):

\[
p_a \equiv m_a (\cosh \theta_a, \sinh \theta_a),
\]

and \( m_a \) is its mass.

Taking the classical lagrangian as the starting point for the definition of a quantum field theory, the classical masses can be computed by expanding the potential (1.2) as far as the quadratic term. Thus the mass matrix is

\[
(M^2)_{ab} = m^2 \sum_{i=0}^r n_i \alpha_i^a \alpha_i^b.
\]

For most cases, the mass matrix was diagonalised some time ago [2]. However, more recently, it was noticed [3,4] and then proved Lie algebraically by Freeman and others [10], that except for the twisted cases the eigenvalues of the mass matrix \( m_a^2 \) were themselves the squares of the components of the lowest eigenvalue eigenvector of the Cartan matrix corresponding to \( g \). In other words, it is possible to choose an ordering of the masses so that \( m = (m_1, m_2, \ldots, m_r) \) and

\[
Cm = 4 \sin^2 \frac{\pi}{2h} m.
\]

This is quite a remarkable result since it allows the particles to be assigned unambiguously (up to mass degeneracies), to the Dynkin diagram for \( g \). Curiously, the gravitational ordering once this assignment is made follows the ‘weight’ ordering in terms of the dimension
of the fundamental representations also assigned to the spots on the Dynkin diagram. For example, the $a_n^{(1)}$ masses ($m_a = 2m \sin \frac{\pi n}{N}$) increase from the ends of the Dynkin diagram working in, and are doubly degenerate corresponding to the folding symmetry of the diagram; for $e_8^{(1)}$ the masses are assigned as follows:

\[ m_2 \quad m_6 \quad m_8 \quad m_7 \quad m_5 \quad m_3 \quad m_1 \]

Even more remarkably, for the ade series of simply-laced algebras (and for one of the twisted cases $a_{(2)}^{(2)}$), the classical mass ratios are preserved in perturbative field theory at least to one-loop order [11,12], suggesting in turn that the set of eigenvalues $q_i^g$ in (2.7) is an eigenvector of the Cartan matrix for $g$. In a while, a generalisation of this result will be discussed. The fact that the radiative corrections to the classical masses in most of the non simply-laced cases are not universal is the first hint that these theories will be rather different as quantum field theories.

Again at the classical level, it is interesting to examine the cubic term in the expansion of (1.2) since this defines the classical three-point couplings, needed to carry out for example the one-loop check mentioned above. Once the mass eigenstates are known, it is possible to compute the couplings, $c^{abc} = \sum_n n_i \alpha_i^a \alpha_i^b \alpha_i^c$. For many triples, the coupling vanishes. However, when the coupling is not zero it is proportional always [11,12] to the area of a triangle whose sides have lengths equal to the masses of the three participating particles $a, b, c$. One consequence of this is that the coupling defines a set of angles (the angles in the triangle), by for example

\[ m_c^2 = m_b^2 + m_b^2 - 2m_a m_b \cos \bar{\theta}_{ab}, \tag{2.11} \]

where

\[ \bar{\theta} = \pi - \theta. \tag{2.12} \]

Just which couplings are non-zero will be explained further below once some of the geometry of root systems has been explored.

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1 There is a convention in the literature that the outside angles of the triangle are denoted by $\theta_{ab}^c$, etc.
It is tempting to suppose eq(2.10) generalises (at least for the simply-laced cases) and the other conserved quantities have values constituting the components of the remaining eigenvectors of the Cartan matrix [13]:

\[ Cq^{(s)} = 4 \sin^2 \frac{5\pi}{2h} q^{(s)} \]  

(2.13)

This is true in the quantum theory, in the sense that it is consistent with other known facts. Again, a fuller discussion is deferred.

**Geometry associated with the Coxeter element**

There is some very useful geometry associated with roots and weights which is less familiar than facts about representation theory. For that reason it will be reviewed briefly here—further details may be found in several books, for example, Bourbaki or Humphreys [14].

A simple Weyl reflection \( w_i \) corresponds to a linear transformation on the root lattice representing a reflection in a plane orthogonal to the simple root \( \alpha_i \) given by

\[ w_i : x \rightarrow x - 2 \frac{\alpha_i \cdot x}{\alpha_i^2} \alpha_i \]  

(2.14)

A Coxeter element of the Weyl group is a product over the simple roots of the simple Weyl reflections. Clearly, once a set of simple roots have been chosen (ie a set of \( r \) independent roots such that any other root is either a linear combination of them with positive coefficients, or a linear combination with negative coefficients), this product could be taken with different orderings of the individual simple reflections. However, different orderings lead to conjugate Coxeter elements. Alternative choices of simple roots also lead to conjugate Coxeter elements. For present purposes, there is a special ordering which is extremely useful and which relies on the fact that Dynkin diagrams have no closed loops. The latter fact allows the simple roots to be divided into two sets such that the roots within each set are mutually orthogonal (ie members of the same set are not joined by a line in the Dynkin diagram). The members of the two sets are distinguished by assigning a colour to them, either black or white. Thus, for example, the \( e_8 \) diagram can be coloured in this way as follows:

The same is true of any other Dynkin diagram, as you can easily check\(^2\). Obviously, the

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\(^2\) It is not true of all extended Dynkin diagrams, however; think of \( a^{(1)}_{\text{even}} \)
product of Weyl reflections corresponding to simple roots within one of these special sets no longer matters since the Weyl reflections commute. With this choice, the Coxeter element is only ambiguous up to the relative black-white ordering, and for definiteness, the Coxeter element will be chosen for the rest of these talks to be

$$ w = w_\bullet w_\circ \equiv \prod_{k \in \bullet} w_k \prod_{k \in \circ} w_k. \quad (2.15) $$

Notice, that each of the factors $w_\bullet$ and $w_\circ$ separately squares to unity. Notice, too, that there is a close relationship between the two factors of the Coxeter element and the Cartan matrix of $g$:

$$(w_\bullet + w_\circ)\alpha_i = \sum_j (2\delta_{ij} - C_{ij})\alpha_j. \quad (2.16)$$

This is easily checked on the black and white roots separately. On the other hand,

$$(w_\bullet + w_\circ)^2 = 2 + w + w^{-1},$$

and therefore

$$(2 + w + w^{-1}) \sum_i x_i \alpha_i = \sum_i x_i (2 - C)^2_{ij} \alpha_j, \quad (2.17)$$

revealing a close relationship between the eigenvectors of the Coxeter element and the Cartan matrix. Indeed, the eigenvalues of the Cartan matrix have been mentioned already in connection with the classical data and, using them it is straightforward to deduce the eigenvalues of the Coxeter element. The eigenvalues of the Cartan matrix are given in (2.13), therefore the eigenvalues of the Coxeter element are also labelled by the spins $s$ and are computed from (2.17) to be

$$ e^{2i\pi s/h}. $$

Hence, the order of the Coxeter element is $h$.

To understand how the Coxeter element affects the roots, it is convenient to define certain linear combinations of the simple roots whose coefficients are the eigenvectors of the Cartan matrix. Consider, for each spin $s$,

$$ a_\bullet^{(s)} = \sum_{i \in \bullet} q_i^{(s)} \alpha_i \quad l_\bullet^{(s)} = \sum_{i \in \bullet} q_i^{(s)} \lambda_i $$

$$ a_\circ^{(s)} = \sum_{i \in \circ} q_i^{(s)} \alpha_i \quad l_\circ^{(s)} = \sum_{i \in \circ} q_i^{(s)} \lambda_i \quad (2.18) $$

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where the $\lambda_i$ are fundamental weights. Then,

$$w_{\bullet}a_{\bullet}^{(s)} = -a_{\bullet}^{(s)}, \quad w_{\bullet}a_{\circ}^{(s)} = a_{\circ}^{(s)} + 2\cos \theta_s a_{\bullet}^{(s)}, \quad \theta_s = \frac{s\pi}{h}. \quad (2.19)$$

The first of (2.19) follows directly from the definition of $w_{\bullet}$ and the mutual orthogonality of the black roots; the second is less straightforward and requires a sequence of steps. Since the white roots have inner products with the black roots represented by the entries of the Cartan matrix,

$$w_{\bullet}a_{\circ}^{(s)} = a_{\circ}^{(s)} - \sum_{i \in \bullet} q_i^{(s)} C_{ij} \alpha_j$$

$$= a_{\circ}^{(s)} - \sum_{i \in \bullet} q_i^{(s)} C_{ij} \alpha_j + \sum_{i \in \bullet} q_i^{(s)} C_{ij} \alpha_j$$

$$= a_{\circ}^{(s)} - \lambda^{(s)} a_{\bullet}^{(s)} + 2a_{\bullet}^{(s)},$$

and the last line is the second relation in (2.19). Thus,

$$|w_{\bullet}a_{\circ}^{(s)}|^2 = |a_{\circ}^{(s)}|^2 + 4\cos \theta_s |a_{\circ}^{(s)}||a_{\bullet}^{(s)}|^2$$

$$= |a_{\circ}^{(s)}|^2,$$

and there is a similar relation with black and white interchanged. Comparing the two leads to

$$|a_{\circ}^{(s)}|^2 = |a_{\bullet}^{(s)}|^2 \quad \text{and} \quad a_{\circ}^{(s)} \cdot a_{\bullet}^{(s)} = -\cos \theta_s |a_{\circ}^{(s)}||a_{\bullet}^{(s)}|. \quad (2.20)$$

Using the fact relating simple roots to fundamental weights, one also has

$$a_{\bullet}^{(s)} = \sum_{i \in \bullet} q_i^{(s)} C_{ij} \lambda_j$$

$$= 2(l_{\bullet}^{(s)} - \cos \theta_s l_{\circ}^{(s)}),$$

with a similar expression for $a_{\circ}^{(s)}$. Hence,

$$l_{\circ}^{(s)} = \frac{a_{\circ}^{(s)} + \cos \theta_s a_{\bullet}^{(s)}}{2 \sin^2 \theta_s}, \quad l_{\bullet}^{(s)} = \frac{a_{\bullet}^{(s)} + \cos \theta_s a_{\circ}^{(s)}}{2 \sin^2 \theta_s}, \quad (2.21)$$

from which it is easily seen that

$$l_{\bullet}^{(s)} \cdot a_{\circ}^{(s)} = 0 = l_{\circ}^{(s)} \cdot a_{\bullet}^{(s)} \quad \text{and} \quad l_{\bullet}^{(s)} \cdot l_{\circ}^{(s)} = \frac{\cos \theta_s}{4 \sin^2 \theta_s} |a_{\bullet}^{(s)}|^2$$

$$|l_{\bullet}^{(s)}|^2 = |l_{\circ}^{(s)}|^2 = \frac{1}{4 \sin^2 \theta_s} |a_{\bullet}^{(s)}|^2. \quad (2.22)$$
Clearly, all four vectors lie in a plane, for each choice of $s$. The Coxeter element acts as a clockwise rotation in this plane through an angle $2\theta_s$. Notice, that although it might happen that $a_\circ^{(s)}$ and $a_\circ^{(s)}$ lie on the same Coxeter orbit, this will never be the case for $-a_\bullet^{(s)}$ and $a_\circ^{(s)}$ (nor indeed for $l_\bullet^{(s)}$ and $l_\circ^{(s)}$). The various vectors are illustrated in the diagram below.

Vectors in the $s$ plane

The sets $w^p\alpha_i$, $i \in \circ$ and $-w^p\alpha_i$, $i \in \bullet$, for $p = 1, \ldots, h$, of images of simple roots do not intersect for distinct simple roots. They provide rank $r$ orbits each of $h$ elements, together providing the full set of roots for the algebra $g$. Since it is possible to normalise the eigenvectors of the Cartan matrix so that

$$\sum_s q_i^{(s)} q_j^{(s)} = \delta_{ij},$$

the relationship between $a_\circ^{(s)}$ and $a_\bullet^{(s)}$ can be inverted to find that each simple root has a component in the spin $s$ plane, either along $a_\bullet^{(s)}$ or along $a_\circ^{(s)}$, according to its colour. Moreover, the images of each simple root $\alpha_i$ under repeated application of the Coxeter element each have a component in this plane of the same magnitude, equal to $q_i^{(s)}$. In particular, for $s = 1$, and according to the earlier observation (2.10), each orbit has a classical mass associated with it and therefore the whole orbit may be assigned to a particular particle.

Consider three roots which make a triangle. The projection of this triangle onto the $s = 1$ Coxeter plane provides another triangle the sides of which have lengths equal to the masses of the particles associated with the three orbits to which the three roots belong.

More interestingly, it is now possible to give a characterisation of the three-point couplings:
The coupling $c^{abc}$ between three particles $a$, $b$ and $c$ is non-zero if and only if there are three vectors, one from each of the orbits representing the particles, which sum to zero.

This was first proved on a case by case basis by Dorey [15] and then deduced from the classical lagrangian by Fring, Liao and Olive by extending the ideas of Freeman [10]. For all cases, the couplings actually correspond to a subset of the Clebsch-Gordan series in the sense that if a coupling $abc$ is non-zero then the tensor product of the representations assigned to the particles according to their assignment to the Dynkin diagram contains the trivial representation, ie $a \otimes b \otimes c \supset 1$. Except for the cases corresponding to $a_n^{(1)}$, $d_4^{(1)}$ the converse is not true. The relationship between the couplings and the Clebsch-Gordan series, and other matters, has been further elucidated by Braden [16].

There is another way to label the Coxeter orbits [17] which will turn out to be useful in the next section, and which naturally incorporates the minus sign. Let the elementary Weyl reflections and the roots be labelled so that

$$w = w_\circ w_\bullet = w_1 \ldots w_b w_{b+1} \ldots w_r,$$

then set

$$\phi_k = w_r w_{r-1} \ldots w_{k+1} \alpha_k$$

$$= \begin{cases} 
\alpha_k & \text{for } k \in \circ \\
-w^{-1} \alpha_k & \text{for } k \in \bullet 
\end{cases}$$

(2.23)

$$= (1 - w^{-1}) \lambda_k.$$

The last two lines of (2.23) follow directly from the definition in the first. One consequence of the second fact is that the set of images of distinct $\phi_k$ never overlap and, therefore, these vectors may be used equally well to label the orbit which has been associated with a particle. A curious property of these vectors, which turns out to have a use in the next section, is the following. The image of each of them under the inverse Coxeter element is a positive root and successive images remain positive until the middle of the orbit, after which they all change sign, remaining negative subsequently for the rest of the orbit.

As an illustration, consider $d_4$ labelled as before, with the outer spots coloured black and the centre spot white. Then

$$\phi_k = \alpha_k + \alpha_4 \text{ for } k = 1, 2, 3 \quad \phi_4 = \alpha_4,$$
and the orbits of the inverse Coxeter element are

1: \( \alpha_1 + \alpha_4; \alpha_2 + \alpha_3 + \alpha_4; \alpha_1; -\alpha_4 - \alpha_1; \ldots \)

2: \( \alpha_2 + \alpha_4; \alpha_1 + \alpha_3 + \alpha_4; \alpha_2; -\alpha_4 - \alpha_2; \ldots \)

3: \( \alpha_3 + \alpha_4; \alpha_2 + \alpha_1 + \alpha_4; \alpha_3; -\alpha_4 - \alpha_3; \ldots \)

4: \( \alpha_4; \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4; \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4; -\alpha_4; \ldots \)

Clearly, these orbits provide the full set of roots as promised.

3. Aspects of the quantum field theory

In this lecture, the intention is to provide certain basic facts and formulae which have proved to be remarkably universal. Most of the time, the \( ade \) sequence of theories will be considered. The affine diagrams for these are invariant under the transformation \([1.3]\). For background, and further references on S-matrix theory in two dimensions, the review article by Zamolodchikov and Zamolodchikov \([18]\) is strongly recommended.

It will be supposed, as a working hypothesis, that (1) after quantisation the conserved charges remain conserved and in involution—ie commute with one another, and (2) the particle spectrum of any one of these theories is as simple as possible—in other words, the particles are exactly \( r \) in number, stable and distinguishable, if not by their masses then by one or other of the conserved charges. With this in mind, it will be assumed that there is a set of one particle states which are eigenstates of the quantum conserved charge operators (which have not been properly constructed yet), ie \((2.7)\):

\[
Q_p|a> = q_a^p e^{p \theta_a}|a> \quad p = s + kh, \tag{3.1}
\]

where \( \theta_a \) is the rapidity of particle \( a \). In addition two-particle states are also assumed to be eigenstates of the conserved charge operators, ie:

\[
Q_p|a, b> = (q_a^p e^{p \theta_a} + q_b^p e^{p \theta_b})|a, b>, \tag{3.2}
\]

and so on.

---

3 The sine-Gordon model is not as simple as this. There are two ‘soliton’ states which are only distinguished by a zero spin charge. Such a distinction is already too relaxed for the present purposes; it permits the mixing of soliton and anti-soliton in a scattering process, leading, in turn, to a greatly enriched spectrum of bound states.
There is no elementary definition of these particle states, although they may be approximated perturbatively. If it is further supposed that two-particle states, which are functions of a pair of rapidities, one for each particle, may under certain circumstances be dominated by a single particle state, then

\[ q_a^p e^{p \theta_a} + q_b^p e^{p \theta_b} = q_{\bar{c}}^p e^{p \theta_{\bar{c}}}, \]  

(3.3)

where the particle \( \bar{c} \) must itself be part of the conjectured spectrum. If the particle \( c \) is to be stable then this situation cannot occur for real rapidity difference \( \Theta_{ab} = \theta_a - \theta_b \). Rather, considering the spin \( \pm 1 \) charges (the light-cone components of energy-momentum, \( q_{k}^{\pm 1} = m_k \)), the situation may arise only when the rapidity difference satisfies

\[ m_{\bar{c}}^2 = m_a^2 + m_b^2 + m_a m_b \cosh \Theta_{ab} = m_a^2 + m_b^2 + m_a m_b \cos U_{ab}^c, \]  

(3.4)

and the masses \( m_a, m_b \) and \( m_c \) are the sides of a triangle with internal angles \( U_{ab}^c, U_{ac}^b, U_{bc}^a \). The same triangle equally well permits a description of the energy-momentum conservation for the virtual processes \( ac \to \bar{b} \) and \( bc \to \bar{a} \). For these special rapidity differences, the rapidities themselves may be written conveniently as

\[ \theta_a = \theta_{\bar{c}} - i U_{ac}^b \quad \theta_b = \theta_{\bar{c}} + i U_{bc}^a. \]  

(3.5)

One might expect that for a certain rapidity difference the vacuum state may dominate a particle- anti-particle state. For this, energy momentum requires \( \Theta_{a\bar{a}} = i \pi \) and therefore,

\[ q_a^p e^{p \theta_a} + q_{\bar{a}}^p e^{p(\theta_a - i \pi)} = 0 \quad \text{ie} \quad q_{\bar{a}}^p = (-)^{p+1} q_a^p. \]  

(3.6)

One consequence of this is immediate. Particles and anti-particles are distinguished only by even spin charges. Affine Toda theories for which the exponents are odd must contain self-conjugate particles only (this includes \( e_7^{(1)} \) and \( e_8^{(1)} \) which have no mass-degenerate states, and \( d_{even}^{(1)} \) which has mass degenerate states corresponding to the prongs of the fork in the Dynkin diagram).

More generally, using (3.6), the \( ab \to \bar{c} \) conserved charge relation may be rewritten

\[ q_a^p e^{p(U_{ac}^b + U_{bc}^a)} + q_c^p e^{pU_{bc}^a} + q_b^p = 0, \]  

(3.7)

which represents a series of ‘triangular ’ relations, one for each \( p \).
At this stage, it is tempting to identify the set of triangle relations (3.7) with the projections of the root triangles which represent the classical couplings described in the last section [15]. This would require the masses of the particles in the quantum spectrum to be essentially the same (upto an overall scale) as the mass parameters in the classical lagrangian; the coupling angles $U^c_{ab}$ would also be the same as those for the classical mass triangles (2.11). The eigenvalues of the conserved quantities $q^p_a$ would repeat modulo $\hbar$, and the first $r$ of them, labelled by the exponents of the algebra, would be the components of the corresponding Cartan eigenvector (2.13). These consequences of the initial hypotheses are very strong and would need to be verified by direct calculation. In fact, if one examines the members of the self-dual affine Toda theories perturbatively, all infinities may be removed by normal-ordering and a calculation of the ‘bubble’ diagrams which contribute to mass corrections at lowest order reveals that the identification of the classical masses with the quantum masses is natural in the sense that the mass corrections are independent of particle type. This is quite definitely not the case for those theories which have a different dual partner. For them, the mass corrections are type-dependent and it would seem unnatural to insist on the quantum masses being the same as the classical ones. In the next lecture an alternative and more attractive approach to these will be presented.

Given the large number of conserved charges and the set of distinguishable particles, the two particle scattering of affine Toda particles is expected to be simple in the sense that the character of the particles is unchanged, there is no production, and the initial and final momenta are the same [18]. Indeed, the ‘in’ and ‘out’ states may differ only by a phase which may at most (because of Lorentz invariance) be a function of the rapidity difference of the two particles and of the coupling $\beta^2$ (or $\bar{\hbar}$). Ie

$$|a, b >_{out} = S_{ab}(\Theta_{ab}; \beta)|a, b >_{in}. \quad (3.8)$$

For each pair of particles there will be such a phase factor. The set of factors will be called the two-particle S-matrix although there is no real scattering going on. The phase factors regarded as functions of complex rapidity difference are far from trivial, however. Indeed, they are analytic functions of the rapidity difference$^4$, with an intricate set of zeroes and poles characteristic of each specific theory. The ‘physical strip’ consists of the region $0 < Im(\Theta) < i\pi$, the boundary $Im(\Theta) = 0$ being the physical $s-$channel, the boundary

$^4$ Using the rapidity variable effectively removes the $s, t$ threshold cuts and there are no others because there is no production.
$\text{Im}(\Theta) = i\pi$ being the physical $t$-channel. The region $0 > \text{Im}(\Theta) > -i\pi$ is the second, unphysical sheet from the point of view of the Mandelstam variables. The continuation of the unitarity and crossing relations away from $\text{Im}(\Theta) = 0$ requires

$$S_{ab}^{-1}(\Theta) = S_{ab}(-\Theta) \quad \text{and} \quad S_{\bar{a}b}(\Theta) = S_{ab}(i\pi - \Theta), \quad (3.9)$$

respectively. Taken together, these imply the S-matrix elements are $2\pi i$ periodic.

If it is further assumed that the scattering is factorisable (one more feature which will need substantiating ultimately), then the three-particle S-matrix elements may be regarded as products of two-particle S-matrix elements. The ordering ambiguity (normally resolved by the Yang-Baxter equation) is absent here because of the special nature of the two-particle scattering (there is no reflection). Thus, the Yang-Baxter equation itself plays no rôle. On the other hand, a two-particle state for complex rapidity difference may share the quantum numbers of a single particle state. The signal for this is a direct channel pole in the physical strip at a purely imaginary rapidity difference. The fusing idea allows a set of ‘bootstrap’ relations to be formulated which relates the scattering of particle $d$, say, with $a$ and $b$, to the scattering of $d$ with $\bar{c}$. Ie, pictorially,

and algebraically:

$$S_{d\bar{c}}(\Theta) = S_{da}(\Theta - i\bar{U}^b_{ac})S_{db}(\Theta + iU^a_{bc}). \quad (3.10)$$

The latter, in the case of the two particle state $a, \bar{a}$ at a relative rapidity of $i\pi$, is in agreement with the crossing relation.

Using (3.9), (3.10) can be rearranged to

$$S_{da}(\Theta + iU^b_{ac} + iU^a_{bc})S_{dc}(\Theta + iU^a_{bc})S_{db}(\Theta) = 1, \quad (3.11)$$

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which is a ‘product’ version of the sum rule (3.7).

The equation (3.10) is extremely useful but it does not fix the S-matrix elements uniquely. What it does do is supply a set of consistency conditions which must be supplemented by other data or prejudices. A natural idea, given the hypothesis concerning the masses of the particles, is to suppose that the possible fusings for which the bootstrap works are to be given precisely by the classical couplings $c^{abc}$ and their associated angles [1,5,11,12]. Before checking this, however, it is also necessary to make some remarks concerning the coupling dependence of the S-matrix elements.

Clearly, when $\beta = 0$ the S-matrix elements ought to be unity since the particles are free. When $\beta \neq 0$, the poles indicating the fusings are at fixed positions and these must be the only poles on the physical strip since it has been presumed that the classical spectrum is complete. Therefore, the S-matrix elements must contain travelling zeroes on the physical strip which coincide with the fixed poles to cancel them when $\beta = 0$. Because of unitarity, each zero has an accompanying pole which must be situated on the unphysical strip for any choice of the coupling in the range $0 \leq \beta \leq \infty$. For the simply-laced conformal Toda theories, it was pointed out (eq(2.4)) that there is a symmetry between strong and weak coupling in the sense that the Virasoro central charge is actually invariant under $\beta \rightarrow 4\pi/\beta$. There is an elegant solution to the bootstrap which also enjoys this symmetry and which neatly parametrises the coupling dependence to ensure the other desirable properties.

It is useful to have a convenient notation for the basic ratio of functions satisfying the periodicity and unitarity relations [11]. Set

$$ (x)_{\Theta} = \frac{\sinh \left( \frac{\Theta}{2} + \frac{i \pi x}{2h} \right)}{\sinh \left( \frac{\Theta}{2} - \frac{i \pi x}{2h} \right)}, \quad (3.12) $$

bearing in mind that the fusing angles are always multiples of $\pi/h$. Often, this will be referred to merely as $(x)$. The fixed poles will be represented by terms of this kind. The coupling dependence may be incorporated by assembling blocks of this type as follows:

$$ \{x\}_{\Theta} = \frac{(x - 1)(x + 1)}{(x - 1 + B)(x + 1 - B)}, \quad (3.13) $$

where $B$ is coupling dependent and, in fact, universal,

$$ B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + 4\pi/\beta^2}. \quad (3.14) $$

Clearly, for small $\beta$, $\{x\}_{\Theta}$ approaches unity and, because $B(\beta) = 2 - B(4\pi/\beta)$, for large $\beta$, exactly the same is true; the pole-cancelling zeroes cross over and cancel the opposite
pole, as $\beta$ runs from 0 to $\infty$. In principle, other functions of $\beta$ might be acceptable under these constraints but this is the one originally suggested by Arinshtein, Fateev and Zamolodchikov \[1\] for the $a_n^{(1)}$ series, on the basis of a comparison with the sin/sinh-Gordon model \[15\].

Rather than simply writing down the conjectures for the S-matrices, it is instructive to build one up, watching the bootstrap in action. An interesting case is $d_4^{(1)}$ labelled as in fig(1.4). There are four distinguished particles 1,2,3 with mass $\sqrt{2}m$ and 4 with mass $\sqrt{6}m$, and possible couplings

$$c^{123}, c^{aa4} (a = 1, 2, 3) \quad \text{and} \quad c^{444}.$$  

In this case, the Coxeter number $h = 6$. The particles are self-conjugate and therefore the S-matrix elements are crossing symmetric.

To begin with, make the simplest compatible conjecture for $S_{12}$, say. It ought to have a pole at $\Theta = 2i\pi/3$ with a positive residue, and a crossed partner at $\Theta = i\pi/3$ with negative residue, and no other fixed poles. In the above notation, a reasonable guess would be

$$S_{12}(\Theta) = \{3\} \equiv \frac{(2)(4)}{(2 + B)(4 - B)} \sim \frac{i}{\Theta - 2i\pi/3} \frac{\pi B}{6} \quad (= S_{13} = S_{23}) \quad . \quad (3.15)$$

Using this pole together with the bootstrap (3.10), leads to

$$S_{33}(\Theta) = S_{13}(\Theta - i\pi/3) S_{13}(\Theta + i\pi/3)$$

$$= \{1\}\{5\} \equiv \frac{(0)(2)(4)(6)}{(B)(2 - B)(4 + B)(6 - B)} \quad (= S_{11} = S_{22}) \quad , \quad (3.16)$$

which is again crossing symmetric. However, because $6 = -1$, it is the pole at $\Theta = i\pi/3$ which has the positive residue this time, indicating the non-zero coupling $c^{334}$ (or $c^{114}$, $c^{224}$). Using the latter with the bootstrap yields

$$S_{14}(\Theta) = S_{13}(\Theta - i\pi/6) S_{13}(\Theta + i\pi/6)$$

$$= \{2\}\{4\} \equiv \frac{(1)(3)^2(5)}{(1 + B)(3 - B)(3 + B)(5 - B)} \quad (= S_{24} = S_{34}) \quad , \quad (3.17)$$

and

$$S_{44}(\Theta) = S_{14}(\Theta - i\pi/6) S_{14}(\Theta + i\pi/6)$$

$$= \{1\}\{3\}\{5\} \equiv -\frac{(2)^3(4)^3}{(B)(2 - B)(2 + B)^2(4 - B)^2(4 + B)(6 - B)} \quad . \quad (3.18)$$

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All other bootstrap relations are verified by direct checking.

The S-matrix elements may be represented conveniently in the following diagram.

In the diagram, the physical strip is marked at intervals of $\pi/6$ and the boxes represent the basic block factors in the S-matrix elements whose labels appear on the right. The vertical lines represent the fixed pole positions on the physical strip.

Now, return to the Coxeter element orbits provided in (2.24). There is a striking correlation between the coefficient of $\alpha_i$ in the expressions for the vectors in the positive part of the orbit of $\phi_j$ (ie the vectors $w^{-p}\phi_j$ for $p=0,1,2$), and the boxes which appear in the diagram representing $S_{ij}$. Indeed, the boxes are labelled by $2p + 1 + \epsilon_{ij}$, where $\epsilon_{ij}$ depends only on the colour of the pair $i,j$:

$$
\epsilon_{\bullet\bullet} = \epsilon_{\circ\circ} \quad \epsilon_{\circ\bullet} = -\epsilon_{\bullet\circ} = 1.
$$

(3.19)
This observation suggests there is a formula for the S-matrix elements in terms of roots weights and Coxeter orbits [13]:

\[ S_{ab}(\Theta) = \prod_{p=1}^{h} \left( 2p + 1 + \epsilon_{ab} \right)^{\lambda_a \cdot w^{-p} \phi_b}. \]  

(3.20)

(The + subscript indicates that because the blocks are all accounted for by traversing the positive part of the orbit of \( \phi_b \) only, it is necessary, when extending the product over the whole Coxeter orbit, to realise that the numerators of the blocks are reconstructed by the positive part of the orbit and the denominators by the negative part.) Because the formula depends only on the roots/weights it promises to be universal. Actually, that is indeed the case. The S-matrix formula once postulated can be shown to be symmetrical under \( a \leftrightarrow b \), to be unitary, to satisfy crossing and to satisfy the bootstrap relation [15, 20, 21]. This is a beautiful result, which applies only to the ade series of cases, and it is a pity there is no direct derivation of it from the field theory.

The observant will have noticed that among the poles in the S-matrix elements (3.17) and (3.18) there are some of order two and three. These are clearly required by the bootstrap and are, in a sense, fortuitously useful. The point is that there is little hope of computing directly the S-matrix elements perturbatively, at least for arbitrary rapidity, but there is some hope of calculating the coefficients of higher order poles. This is because the poles appear as Landau singularities in Feynman diagrams and there is a well developed calculus for dealing with them. For example, there is no time to go into details, but the double poles all arise from singularities of box diagrams and it has been checked that the coefficients of the poles to order \( \beta^4 \) agree with the predictions of the S-matrix elements (not just for \( d_4^{(1)} \), but in all cases). This is quite important because the observant will also have noticed that there was no attempt to check the bootstrap on the double pole in (3.17). The fact that the poles are an artefact of the perturbation expansion and there is no order \( \beta^2 \) tree-graph with a simple pole sharing the same pole position strongly suggests this is a correct interpretation of the bootstrap rules. On the other hand, third order poles (and in general odd-order poles) appear as dressings of tree processes and one would expect that their existence does really signal a bound state which ought to participate in the bootstrap. It has been found that there are a number of diagrams of different types contributing to the third order poles (all two-loop diagrams, since the leading contribution to the third order pole is order \( \beta^6 \)), but never more than twenty-six (!) as one ranges over the ade series. The sum of the contributions from these diagrams agrees exactly with the
prediction from the conjectured S-matrix elements whenever the cubic poles occur in the ade series. The number of diagrams to be computed is prohibitively large for the fourth and higher poles (up to order twelve in the $e_8^{(1)}$ S-matrix elements), and for these a direct check is not possible. The type of checking advocated here is complicated and makes it abundantly clear how inefficient the perturbation series is from a computational point of view. These poles have been checked to order three in [22], and other perturbative matters have been investigated in [23].

4. Dual pairs

The theories based on non simply-laced algebras work in a very different way which will be partially explained by reference to a particular example. Further details are obtainable in the recent literature [24-29] although there is much yet to do before the final version of the story can be told.

For definiteness, consider the pair of classical theories based on the extended Dynkin diagrams $g_2^{(1)}$ and $d_4^{(3)}$:

In each case, the black spots denote short roots and it is clear that to obtain one diagram from the other involves an inversion of roots (1.3).

In each case, there are two particles labelled 1 and 2 but their mass ratios are different [11,12]. In the first case, the classical mass parameters are simply those of $d_4^{(1)}$ without the mass degeneracy, since this has been removed by the folding corresponding to the threefold symmetry of the $d_4$ Dynkin diagram. In the second case, the root system is obtained by applying the folding procedure to the extended Dynkin diagram $e_6^{(1)}$, which also has a threefold symmetry; hence, in this case the masses are a subset of those to be found in the $e_6^{(1)}$ theory. In summary, the two mass ratios are

$$\frac{m_1}{m_2} \bigg|_{g_2^{(1)}} = \frac{\sin(\pi/6)}{\sin(2\pi/6)} \quad \frac{m_1}{m_2} \bigg|_{d_4^{(3)}} = \frac{\sin(\pi/12)}{\sin(2\pi/12)}. \quad (4.1)$$

Moreover, the non-zero three-point couplings for the two cases are:

$$g_2^{(1)} : 111, 112, 222, \quad d_4^{(3)} : 111, 112, 222, 221. \quad (4.2)$$
Hence, from a classical point of view these two theories are very different.

Some time ago, it was also noted that guesses for the S-matrix for $g_2$ were problematical if based on maintaining poles at the positions of the classical masses [1]. There were always extra singularities whose origin could not be traced in perturbation theory. It was also found that radiative mass corrections, which worked very well for the simply-laced theories, did not preserve the classical mass ratios, suggesting either that cases such as $g_2$ were in a sense anomalous and therefore not quantum integrable or, that they were quantum integrable but that the relationship with the classical theory was much less clear cut. The principle step in suggesting a resolution of these difficulties has been provided by Delius, Grisaru and Zanon [24]. They have noted how the bootstrap might be satisfied, even in a situation where there are particles with coupling dependent masses, in such a manner that the small coupling approximation is provided by the $g_2^{(1)}$ theory and the large coupling limit is provided by the $d_4^{(3)}$ theory. In other words, there is a quantum field theory corresponding to the pair together rather than either classical theory separately, and the transformation

$$\beta \rightarrow 4\pi/\beta,$$

effectively implements the inversion [1.3] which interchanges the two extended Dynkin diagrams. A similar mechanism is working for all the non simply-laced algebras which come in the pairs listed previously and related by [1.3]. The exceptions to this are the members of the $a_{2n}^{(2)}$ sequence which are ‘self-dual’.

The first thing to note is that the masses may be parametrised conveniently by setting

$$\left. \frac{m_1}{m_2} \right|_\beta = \frac{\sin(\pi/H(\beta))}{\sin(2\pi/H(\beta))} \quad \text{with} \quad 6 \leq H(\beta) \leq 12 \quad \text{for} \quad 0 \leq \beta \leq \infty, \quad (4.3)$$

where the functional dependence of $H$ on $\beta$ is really a matter of informed conjecture. A few words will be said about it at the end of the section. That the masses do depend on the coupling has been confirmed by Watts and Weston [27] who have investigated the coupling dependence in a Monte-Carlo lattice simulation of the model. Their results leave little doubt that the masses do indeed flow with the coupling although the numerical accuracy of the simulation is not yet sufficient to pin down the actual dependence on $\beta$.

The couplings (4.2) are more problematical since the two theories have different numbers of three-point couplings. However, the two self-couplings are clearly permitted whatever the masses might be and always correspond to a coupling angle of $2\pi/3$ in the notation
introduced before (eq(2.11)). Also, the 112 coupling is quite natural with a coupling angle of $2i\pi/H$, since

$$\sin^2 \left( \frac{2\pi}{H} \right) \equiv 2\sin^2 \left( \frac{\pi}{H} \right) + 2\sin^2 \left( \frac{\pi}{H} \right) \cos \left( \frac{2\pi}{H} \right),$$

whatever the value of $H$ might be, whereas the coupling 221 is quite unnatural. As far as an $ab$ S-matrix element is concerned, one would expect the $abc$ couplings to emerge as poles (or possibly multiple poles) in the physical strip with a positive coefficient (times $i$). That was certainly what happened in the simply-laced sequences of models. However, in this and other similar cases, the mere positivity of the pole coefficient is not enough and it appears to be necessary to strengthen the requirement to **positivity over the whole range of $\beta$**. Once this is done it is found that there is a consistent set of bootstrap conditions satisfied by a subset of the classical couplings, but not all of them.

To examine the S-matrix, it is helpful to use a diagrammatic representation (see below) which displays the poles on the physical strip (solid lines) and compensating zeroes (dashed lines), as they travel from their positions at $\beta = 0$ to their positions at $\beta = \infty$. The filled circles represent points on the physical strip at intervals of $\pi/h$ or $\pi/h^\vee$. Thus, the upper row represents the physical strip marked at intervals of $\pi/6$ for $g_{2}^{(1)}$ while the lower row represents the physical strip marked at intervals of $\pi/12$ for $d_{4}^{(3)}$. The dashed lines always meet solid lines at the top and bottom of the diagram indicating that the poles and zeroes precisely cancel there, as they ought because the S-matrix elements are unity at $\beta = 0$ or $\infty$. The first of the diagrams represents $S_{11}(\Theta)$

\[ S_{11}(\Theta) \]

for which the algebraic expression is

$$\frac{(0) (2)}{(H/3 - 2)(4 - H/3)} \frac{(H/3) (2H/3)}{(4)(H - 4)} \frac{(H - 2) (H)}{(2 + 2H/3)(4H/3 - 4)},$$

(4.4)

25
where the bracket notation has been adjusted to represent

\[
(x) = \frac{\sinh \left( \frac{\Theta}{2} + \frac{x\pi}{2H} \right)}{\sinh \left( \frac{\Theta}{2} - \frac{x\pi}{2H} \right)}.
\]

It is clear from the diagram that there are two physical simple poles and their crossed partners—the third vertical line represents the self coupling 11 → 1, and the oblique line next to it represents the 11 → 2 coupling. Using the bootstrap relation on the 11 → 2 coupling leads directly to the S-matrix element \( S_{12}(\Theta) \) for which the algebraic expression is

\[
\frac{(1) \left( \frac{2H}{3} - 1 \right)}{(H - 5)(5 - H/3)} \frac{(H/3 + 1)(H - 1)}{(4H/3 - 5)(5)}.
\]

and which is represented diagrammatically by

In this case, the rightmost solid line represents the physical pole for the expected bound state in the channel 12 → 2 and it looks as if there is another pole at \( \Theta = (2/3 - 1/H)\pi i \) which meets its crossed partner at the top of the diagram but is separated from it at the bottom of the diagram. However, moving down the diagram, the coefficient of this pole has the wrong sign to be interpreted as a bound state until it is crossed by a zero represented by a dotted line. There the coefficient changes sign and remains positive up to the bottom of the diagram. A reasonable interpretation of this is that near the \( d_4^{(3)} \) theory this pole looks like the one appropriate for the 12 → 2 coupling but, far away it does not. A reasonable hypothesis amends the bootstrap principle to include just those poles which never change sign over the whole interval. At first sight this seems strange. However, changing \( \beta \) is actually equivalent to adjusting \( \bar{\hbar} \) (remember, there is no classical coupling really), and therefore in a sense it is merely being suggested that the structure of the quantum field theory should be independent of a particular scale choice for \( \bar{\hbar} \). It would be difficult to check this statement in perturbative field theory because the zero in a pole coefficient is hard to find. Neverthess, Delius, Grisaru and Zanon do give preliminary
perturbative arguments for the ‘floating’ masses [24]. On the other hand, there is another argument, based on the Coleman-Thun mechanism [30], which suggests that a pole with an indefinite coefficient might be best thought of as a double pole with a compensating zero. There is no time to pursue this argument here but it is described in some detail in ref[28].

The bootstrap principle applied to the coupling $11 \rightarrow 2$ also yields the third S-matrix element $S_{22}(\Theta)$ whose diagram is

![Diagram](image)

with the corresponding algebraic form

$$S_{22}(\Theta) = \frac{(0) (2H/3 - 2)}{(H - 6)(4 - H/3)} \frac{(2) (2H/3)}{(H - 4)(6 - H/3)} \times \frac{(H/3) (H - 2)}{(4H/3 - 6)(4)} \frac{(H/3 + 2) (H)}{(H/3 - 4)(6)}$$

(4.6)

This matrix element is quite fascinating. There are a number of poles but all but two of them have coefficients which change sign. The poles (at $\Theta = 2\pi i/3$, corresponding to the self-coupling $22 \rightarrow 2$) which do not change sign fail to do so because two zeroes happen to collide. All the other poles can be accommodated within the extended Coleman-Thun scheme.

The other non simply-laced cases have all been listed elsewhere [4,24,28] and will not be mentioned explicitly here. In every case, the generalised bootstrap principle alluded to above is consistent and all poles not included in the bootstrap have a plausible explanation within the extended Coleman-Thun scheme.

One intriguing question in all of this is, what (if anything) replaces the beautiful structure surrounding the Coxeter element which plays such a unifying rôle in the simply-laced cases? What replaces the formula (3.20)? Presumably, whatever the structure is, it transcends the root lattices of the pair but offers a geometrical setting for both of them (see [21]).
Finally, it is worth mentioning that there is a convenient way to parametrise the coupling angles, and the floating masses [29]. Define first, generalising the earlier notation,

\[ B_g(\beta) \equiv B(\beta; g, g^\vee) = \frac{2\beta^2}{\beta^2 + 4\pi(h/h^\vee)}, \quad (4.7) \]

where \( g \) and \( g^\vee \) have extended Dynkin diagrams related by (1.3), and associated Coxeter numbers \( h \) and \( h^\vee \), respectively. Then, the following identity is true

\[ B(\beta; g, g^\vee) = 2 - B(4\pi/\beta; g^\vee, g). \quad (4.8) \]

The duality of the coupling angles for the positive definite poles is then rendered transparent by setting

\[ \Theta_{ab}(\beta) = \frac{2 - B_g}{2} \Theta_{ab}(0) + \frac{B_g}{2} \Theta_{ab}(\infty). \quad (4.9) \]

A straightforward comparison with the coupling angles for the \( g_2^{(1)} - d_4^{(3)} \) case yields the consistent choice

\[ \frac{1}{H(\beta)} = \frac{1}{12} \left( 2 - \frac{B_{g_2^{(1)}}(\beta)}{2} \right). \]

5. A word on solitons

If complex solutions to the affine Toda field equations are permitted then there is a whole extra dimension to the Toda activity. At first sight, the idea of allowing the Toda field to be complex is unattractive since the classical hamiltonian will not be positive definite and it is not immediately clear how such solutions ought to be interpreted, or what their rôle in the quantum Toda theory might be. On the other hand, it has been pointed out by Hollowood [31] that the soliton solutions, although complex, actually have real energy and momentum associated with them, despite the fact that their energy-momentum density is complex. In addition, the masses associated with the solitons are closely related to the particle masses in the real theory and their couplings, in the sense of a fusing rule, are also identical to the couplings of the real particles, at least for the \( ade \) sequence of possibilities. The static solitons are labelled by ‘topological charges’ corresponding to weights of the fundamental representations of the Lie algebra underlying the Toda theory. This is quite easy to check but there is something of a mystery associated with the topological charges in the sense that complete sets of weights are only rarely found (in the \( a_n^{(1)} \) sequence of
theories, and even then only in the smallest dimension representations). This puzzle will be mentioned again at the end of this section.

To see that the possibility of soliton solutions exists is not difficult. It is enough to note that the Toda potential has local stationary points whenever the field is constant and taken to be

\[ \phi = \frac{2i\pi \lambda}{\beta} \text{ with } \lambda \cdot \alpha_k = \text{integer}, \ k = 0, 1, ..., r . \] (5.1)

At each of these values of the field, the potential vanishes. Soliton solutions to the equations of motion interpolate pairs of these ‘vacua’ with, typically,

\[ \phi(-\infty, t) = 0 \quad \phi(\infty, t) = \frac{2i\pi \lambda}{\beta}. \]

The topological charge is defined to be

\[ \lambda = \frac{\beta}{2i\pi} (\phi(\infty, t) - \phi(-\infty, t)). \] (5.2)

At each of these values of the field, the potential vanishes. Notice that this set up generalises the situation to be found in the sine-Gordon theory which may be regarded as a purely imaginary version of the \( a_1^{(1)} \) affine Toda theory. Note, too, that the sine-Gordon theory supplies the only example for which all the soliton solutions are effectively real.

Each of the affine Toda theories contains soliton solutions and many of the solutions have been catalogued elsewhere [32]. For definiteness and ease of computation, the \( a_n^{(1)} \) types will be illustrated here using the so-called Hirota method as it was originally adopted by Hollowood. This relies on the ansatz

\[ \phi = -\frac{1}{\beta} \sum_0^r \alpha_k \ln \tau_k \] (5.3)

for which the (time-independent) Toda field equations reduce to:

\[ \sum_0^r \alpha_k \left( \frac{\tau_k''}{\tau_k} - \frac{\tau_k' \tau_k'}{\tau_k^2} + \prod_l \frac{\tau_l^{-\alpha_k \cdot \alpha_l}}{\tau_l^{\alpha_k \cdot \alpha_l}} \right) = 0 . \] (5.4)

These are the relevant equations for static solitons. Using the explicit form of the \( a_r^{(1)} \) Cartan matrix (5.4) may be rewritten

\[ \sum_0^r \alpha_k \left( \frac{\tau_k''}{\tau_k} - \frac{\tau_k' \tau_k'}{\tau_k^2} + \frac{\tau_{k-1} \tau_{k+1}}{\tau_k^2} \right) = 0 , \]
and solved by setting

$$\frac{\tau_k''}{\tau_k} - \frac{\tau_k'\tau_k'}{\tau_k^2} + \frac{\tau_{k-1}\tau_{k+1}}{\tau_k^2} = 1 \quad \text{for } k = 0, 1, 2, \ldots, r,$$

(5.5)

where

$$\tau_k = 1 + \Omega_k e^{\sigma x + x_0},$$

provided

$$\sigma^2 \Omega_k - 2\Omega_k + \Omega_{k-1} + \Omega_{k+1} = 0$$

$$\Omega_{k-1} \Omega_{k+1} - \Omega_k^2 = 0$$

$$\Omega_{k+r+1} = \Omega_k.$$ (5.6)

The last pair of eqs (5.6) are solved by taking

$$\Omega_k^{(a)} = e^{2 \pi ak/r+1} = \omega^{ak} \quad \text{for each choice } a = 1, 2, \ldots, r,$$

where $\omega$ is the primitive $r+1$st root of unity. The first of the eqs (5.6) then imply a corresponding constraint on $\sigma$ leading (for each choice of $a$) to

$$\sigma^{(a)} = 2 \sin \frac{\pi a}{r+1}.$$ (5.7)

The replacement $x \rightarrow -x$ gives another solution, and $x_0$ is an arbitrary constant. Assembling all these pieces, there is a solution for each $a$ of the form (5.3):

$$\phi^{(a)} = -\frac{1}{\beta} \sum_0^r \alpha_k \ln \left(1 + \omega^{ak} e^{\sigma^{(a)} x + x_0^{(a)}}\right) = -\frac{1}{\beta} \sum_1^r \alpha_k \ln \left(\frac{1 + \omega^{ak} e^{\sigma^{(a)} x + x_0^{(a)}}}{1 + e^{\sigma^{(a)} x + x_0^{(a)}}}\right).$$ (5.8)

These solutions are generally complex. The same solutions may be obtained via a more sophisticated and general algebraic method given by Olive, Turok and Underwood based on the work of Leznov and Saveliev who pioneered a general approach to Toda wave equations some years ago [33-35]. Moreover, it appears there are no other single soliton solutions to be found using the more general techniques. This is perhaps surprising given the special nature of the ansatz (5.3) and the particular choice of solution within it, eq (5.3).

To calculate the energy of these solutions, it is extremely convenient to use a formula for the energy-momentum tensor established in the article by Olive, Turok and Underwood [34], using arguments rooted in conformal Toda field theory. They found

$$T_{\mu\nu} = (\eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu}) C,$$ (5.9)
where the function $C$ for solitons is given by
\[ C = -\frac{2}{\beta^2} \sum_0^r \ln \tau_k. \]

Using this, the energy of a static soliton (i.e., its mass) can be calculated
\[ M^{(a)} = \int_{-\infty}^{\infty} dx T_{00} = \left. \frac{\partial C}{\partial x} \right|_{-\infty}^{\infty} = \frac{2}{\beta^2} \sum_0^r \frac{\tau_k'}{\tau_k} \left|_{-\infty}^{\infty} = \frac{2(r + 1)}{\beta^2} \sigma^{(a)}. \tag{5.10} \]

It is worthy of note that the mass is real despite the fact that the energy density is complex and, moreover, each mass is proportional to the mass of a corresponding elementary scalar particle in the real coupling Toda theory provided the label $a$ is suitably interpreted.

To provide the interpretation, first recall from lecture (1) that the scalar particles of the real coupling Toda theory are associated with the fundamental representations of the Lie algebra $a_n$, the lightest particles corresponding to the smallest $(n + 1)$ dimensional representation of the algebra or its conjugate, the next lightest to the representations of dimension $n(n + 1)/2$, and so on. The solitons, on the other hand, are labelled naturally by their topological charges which may be calculated from the explicit solutions using (5.2). The calculation of the topological charges is slightly tricky and must be performed with some care. First of all note that the argument of the logarithm in eq(5.8) must never vanish or diverge for any choice of $x$, otherwise the solution will be singular. This requires that the constant $x_0^{(a)}$ (which may be complex) has an imaginary part which is not entirely arbitrary; it is confined to regions in the range $[0, 2\pi]$ whose boundaries correspond to those choices of $\text{Im}x_0$ for which at least one of the logarithmic arguments will vanish or diverge. Hence, the number of such boundary points provides the maximum number of possibly different topological charges which might be described by the solution (5.8). Provided the boundary points are avoided, the arguments of the logarithms change continuously, the logarithm cannot jump its branch and the topological charge is defined uniquely. McGhee [36] has calculated all the topological charges that are possible given (5.8), and has confirmed that the topological charges of the solution whose mass corresponds to that of the classical particle $a$ do indeed lie among the weights of the associated representation. However, he has also noted that the total number of possible topological charges typically falls far short of the number required to fill up the whole representation. Indeed there is a neat formula for the total number of topological charges obtainable:
\[ \text{number of charges of type } a = \frac{n + 1}{\gcd(a, n + 1)}. \]
Indeed, the only representation with its full complement of topological weights is the representation of dimension $n + 1$, or its conjugate.

McGhee has also examined a number of other theories and has found that the Hirota solution in all other cases always fails to fill the associated weight set and often the discrepancy is huge \[37\]. A selection of the results are given below, the numbers below the Dynkin diagram points denoting the number of possible topological charges obtainable via the Hirota ansatz.

For the $e_8$ case the topological charges present are a small fraction of the conjectured total. If it is really the case that the approach of Olive, Turok and Underwood captures all the solitons but is effectively equivalent to the Hirota ansatz, then one must take seriously the ‘gaps’ in the topological spectrum.

It has been suggested that there might be a consistent quantum field theory corresponding to the complex Toda theories (a generalisation of the sine-Gordon situation), in which the particle spectrum consists of multiplets corresponding to the representations of a quantised affine Lie algebra (as is the case for the sine-Gordon theory in which the soliton and anti-soliton are a doublet of $su_q(2)$). This would appear to be natural but at the same time very mysterious without a detailed mechanism to explain the enormous gaps in the classical soliton spectrum. Presumably, the classical spectrum must be enhanced in the quantum theory. Hollowood \[38\], and Bernard and LeClair \[39\] have presented arguments
suggesting that the quantum theory ought to have such an enlarged spectrum. Certainly, it is possible to obtain solutions to the Yang-Baxter equations, based on quantum group ideas. There is no time to consider these arguments here but one ought to bear in mind the intriguing behaviour of the real coupling non simply-laced theories which would appear to be difficult to mirror in the complex theory, since it is not at all clear which quantum group one ought to be choosing. There is evidence that the quantum corrections to the soliton masses preserve the classical mass ratios for the ade cases but fail to do so for the others. Hollowood \cite{40}, using old ideas of Dashen, Hasslacher and Neveu, has calculated the lowest order quantum mass corrections to the soliton mass spectrum for $a_n^{(1)}$. Remarkably, the mass corrections do not spoil the mass ratios. Very recently, Mackay and Watts, and Delius and Grisaru \cite{41} have performed similar calculations for non simply-laced solitons and the classical mass ratios are not preserved. One would imagine that something akin to the duality going on in the real coupling theories should persist provided the quantum field theory of the complex theories really makes sense. It is conceivable a truncated spectrum will be necessary in order to permit the S-matrix elements to enjoy floating bound-state poles. That some truncation of the spectrum might be needed is also indicated by a need for unitarity in a theory with a non-hermitean Hamiltonian; an apparently serious fault which might be alleviated by removing parts of the spectrum to leave a unitary core.

It remains to be seen how this extremely interesting story will unfold.

6. Other matters

There are several interesting developments which cannot be described here. For example, a full understanding of the quantum field theory would require much more than the S-matrix/conserved quantity considerations presented here. Indeed, there is a sizeable literature concerning the calculation of form factors for Toda theory and related topics (for example, see \cite{42}).

What happens if affine Toda field theory is restricted to a segment of the real line, or to a half-line? The general question of integrability in the presence of boundary conditions has its own literature but some recent articles dealing specifically with Toda theories are given in refs\cite{43}. Surprisingly, there are strong constraints on the possible form of the boundary condition maintaining classical integrability, but it is not clear how these will affect the quantum theory—another question to be resolved in the future.
7. Acknowledgements

I am grateful to the organisers of the school for the opportunity to talk about Toda field theories, and to many colleagues and students for stimulating interactions. In particular, I would like to thank Harry Braden, Patrick Dorey, Richard Hall, Tim Hollowood, Niall Mackay, William McGhee, Rachel Rietdijk, Ryu Sasaki, Gérard Watts, and Robert Weston for enjoyable discussions and collaborations. I am indebted to Patrick Dorey for various pictorial representations of the singularities of S-matrix elements some of which have been used in these lectures to illustrate the dual pairs of non simply-laced models.
References

[1] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, Phys. Lett. B87 (1979) 389.

[2] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, Comm. Math. Phys. 79 (1981) 473;
G. Wilson, Ergod. Th. and Dynam. Sys. 1 (1981) 361;
D. I. Olive and N. Turok, Nucl. Phys. B215 (1983) 470.

[3] V. Kac, Infinite Dimensional Lie Algebras (Birkhauser Verlag 1983).

[4] A. B. Zamolodchikov, Int. J. Mod. Phys. A3 (1988) 743;
T. J. Hollowood and P. Mansfield, Phys. Lett. B226 (1989) 73;
T. Eguchi and S-K Yang, Phys. Lett. B224 (1989) 373;
V. A. Fateev and A. B. Zamolodchikov, Int. J. Mod. Phys. A5 (1990) 1025;
P. Christe, Proceedings of the NATO Conference on Differential Geometric Methods in Theoretical Physics, Lake Tahoe, USA 2-8 July 1989 (Plenum 1990);
G. Mussardo, Proceedings of the NATO Conference on Differential Geometric Methods in Theoretical Physics, Lake Tahoe, USA 2-8 July 1989 (Plenum 1990).

[5] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, Proceedings of the NATO Conference on Differential Geometric Methods in Theoretical Physics, Lake Tahoe, USA 2-8 July 1989 (Plenum 1990).

[6] J-L. Gervais and A. Neveu, Nucl. Phys. B224 (1983) 329;
P. Mansfield, Nucl. Phys. B222 (1983) 419;
E. Braaten, T. Curtright, G. Ghandour and C. Thorn, Phys. Lett. B125 (1983) 301.

[7] O. Babelon and L. Bonora, Phys. Lett. B267 (1991) 71;
L. Bonora, Int. J Mod. Phys. B6 (1992) 2015.

[8] D. Olive and N. Turok, Nucl. Phys. B257 (1985) 277.

[9] P. G. O. Freund, T. Klassen and E. Melzer, Phys. Lett. B229 (1989) 243.

[10] M. D. Freeman, Phys. Lett. B261 (1991) 57;
A. Fring, H.C. Liao and D.I. Olive, Phys. Lett. 266B (1991) 82.

[11] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, Nucl. Phys. B338 (1990) 689.

[12] P. Christe and G. Mussardo, Nucl. Phys. B330 (1990) 465;
P. Christe and G. Mussardo, Int. J. Mod. Phys. A5 (1990) 4581.

[13] T. R. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485.

[14] N. Bourbaki, Groupes et algèbres de Lie IV, V, VI, (Hermann, Paris 1968);
J. E. Humphreys, Reflection Groups and Coxeter Groups, (Cambridge University Press 1990).

[15] P. E. Dorey, Nucl. Phys. B358 (1991) 654;
P. E. Dorey, Nucl. Phys. B374 (1992) 741.

[16] H. W. Braden, J. Phys. A25 (1992) L15.

[17] B. Kostant, Am. J. Math. 81 (1959) 973.
[18] A.B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys.* **120** (1979) 253.

[19] M. Karowski, *Nucl. Phys.* **B153** (1979) 244.

[20] A. Fring and D.I. Olive, *Nucl. Phys.* **B379** (1992) 429.

[21] P.E. Dorey, In the proceedings of the conference “Integrable Quantum Field Theories”, Como, Italy, 13-19 September 1992 (Plenum, 1993).

[22] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B356** (1991) 469.

[23] H.W. Braden and R. Sasaki, *Phys. Lett.* **B255** (1991) 343;
   H.W. Braden and R. Sasaki, *Nucl. Phys.* **B379** (1992) 377;
   H.W. Braden, H.S. Cho, J.D. Kim, I.G. Koh and R. Sasaki, *Prog. Theor. Phys.* **88** (1992) 1205;
   R. Sasaki and F.P. Zen, *Int. J. Mod. Phys.* **A8** (1993) 115.

[24] G.W. Delius, M.T. Grisaru and D. Zanon, *Nucl. Phys.* **B382** (1992) 365.

[25] H.S. Cho, I.G. Koh and J.D. Kim, *Phys. Rev.* **D47** (1993) 2625.

[26] H. Kausch and G.M.T. Watts, *Nucl. Phys.* **B386** (1992) 166.

[27] G.M.T. Watts, R. A. Weston, *Phys. Lett.* **B289** (1992) 61.

[28] E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B408** (1993) 579.

[29] P.E. Dorey, *Phys. Lett.* **B312** (1993) 291.

[30] S. Coleman and H. Thun, *Commun. Math. Phys.* **61** (1978) 31.

[31] T.J. Hollowood, *Nucl. Phys.* **384** (1992) 523.

[32] H. Aratyn, C.P. Constantinidis, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, *Nucl. Phys.* **B406** (1993) 727;
   N.J. Mackay and W.A. McGhee, *Inter. J. Mod. Phys.* **A8** (1993) 2791;
   Z. Zhu and D.G. Caldi, *Multi-soliton solutions of affine Toda models*, SUNY, hep-th/9307175.

[33] A. N. Leznov and M. V. Saveliev, *Group-theoretical methods for the integration of nonlinear dynamical systems* (Birkhäuser Verlag 1992).

[34] D.I. Olive, N. Turok and J.W.R. Underwood, *Nucl. Phys.* **B401** (1993) 663.

[35] D.I. Olive, N. Turok and J.W.R. Underwood, *Nucl. Phys.* **B409** (1993) 509.

[36] W.A. McGhee, *Int. J. Mod. Phys.* **A9** (1994) 2666.

[37] W.A. McGhee, Durham PhD Thesis, 1994.

[38] T.J. Hollowood, *Int. J. Mod. Phys.* **A8** (1993) 947.

[39] D. Bernard and A. LeClair, *Commun. Math. Phys.* **142** (1991) 99;
   D. Bernard and A. LeClair, *Nucl. Phys.* **B399** (1993) 709.

[40] T.J. Hollowood, *Phys. Lett.* **B300** (1993) 73.

[41] G. Watts, *Phys. Lett.* **B338** (1994) 40;
   N.J. Mackay and G.M.T. Watts, *Quantum mass corrections for affine Toda solitons*, DAMTP-94-36, hep-th/9411169;
   G.W. Delius and M.T. Grisaru, *Toda soliton mass corrections and the particle-soliton duality conjecture*, King’s College preprint, hep-th/9411176.
[42] F.A. Smirnov, *Form factors in completely integrable models of quantum field theory*, (World Scientific 1992);
A. Fring, G. Mussardo and P. Simonetti, *Phys. Lett.* B307 (1993) 83;
G. Delfino and G. Mussardo, *Phys. Lett.* B324 (1994) 40;
A. Koubek, *Nucl. Phys.* B428 (1994) 655;
F.A. Smirnov, *Int. J. Mod. Phys.* A9 (1994) 5121.

[43] S. Ghoshal and A. B. Zamolodchikov, *Int. J. Mod. Phys.* A9 (1994) 3841;
S. Ghoshal, *Int. J. Mod. Phys.* A9 (1994) 4801;
A. Fring and R. Köberle, *Nucl. Phys.* B421 (1994) 159;
A. Fring and R. Köberle, *Nucl. Phys.* B419 (1994) 647;
R. Sasaki, in Proceedings of the conference “Interface between physics and mathematics”, Hangzhou, China, 6-17 September 1993, (World Scientific 1994);
E. Corrigan, P.E. Dorey, R. Rietdijk and R. Sasaki, *Phys. Lett.* B333 (1994) 83;
E. Corrigan, P.E. Dorey and R. Rietdijk, *Aspects of affine Toda field theory on a half line*, Durham preprint DTP-94/29, [hep-th/9407148](http://arxiv.org/abs/hep-th/9407148).
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