Sigma models in (4,4) harmonic superspace

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Abstract

We define basics of (4,4) 2D harmonic superspace with two independent sets of $SU(2)$ harmonic variables and apply it to construct new superfield actions of (4,4) supersymmetric two-dimensional sigma models with torsion and mutually commuting left and right complex structures, as well as of their massive deformations. We show that the generic off-shell sigma model action is the general action of constrained analytic superfields $q^{(1,1)}$ representing twisted $N = 4$ multiplets in (4,4) harmonic superspace. The massive term of $q^{(1,1)}$ is shown to be unique; it generates a scalar potential the form of which is determined by the metric on the target bosonic manifold. We discuss in detail (4,4) supersymmetric group manifold $SU(2) \times U(1)$ WZNW sigma model and its Liouville deformation. A deep analogy of the relevant superconformally invariant analytic superfield action to that of the improved tensor $N = 2$ 4D multiplet is found. We define (4,4) duality transformation and find new off-shell dual representations of the previously constructed actions via \textit{unconstrained} analytic (4,4) superfields. The main peculiarities of the (4,4) duality transformation are: (i) it preserves manifest (4,4) supersymmetry; (ii) dual actions reveal a gauge invariance needed for the on-shell equivalence to the original description; (iii) in the actions dual to the massive ones 2D supersymmetry is modified off shell by $SU(2)$ tensor central charges. The dual representation suggests some hints of how to describe (4,4) models with non-commuting complex structures in the harmonic superspace.
1 Introduction

Two-dimensional $(4,4)$ supersymmetric sigma models with torsion [1 - 5] have a number of interesting applications, e.g. in the string theory (see, e.g., [6]) and the $2D$ black hole business [7, 8]. An important subclass of these models is provided by those on group manifolds, that is, by $(4,4)$ supersymmetric WZNW sigma models [1, 3, 4, 5]. A superconformally invariant deformation of the simplest model of that kind, with $SU(2) \times U(1)$ as the bosonic target, is the $N=4$ $SU(2)$ super Liouville theory [1, 3, 9]. It proved to be the first example of integrable $N=4$ $2D$ system and it is expected to play a crucial role in $N=4$ $2D$ induced supergravity [10, 11]. Presumably, $N=4$ superextensions of other integrable bosonic systems, such as the Toda and sine-Gordon ones, can also be obtained as some deformations of the appropriate $(4,4)$ WZNW sigma models. These theories could encode a rich set of invariances and, by analogy with $N=4$ super Liouville model, be related to $N=4$ superextensions of induced $W$ gravities.

To explore all these exciting issues in a full generality and to clearly understand their underlying geometric aspects, one needs a convenient off-shell superfield description of the $(4,4)$ sigma models in question.

Until now such superfield formulations were given in the $(2,2)$ and projective $(4,4)$ superspaces [2, 7, 8, 12, 13], as well as in the conventional $(4,4)$ superspace [9]. Basically, the simplest case of sigma models with mutually commuting left and right complex structures was treated. As for the more general class of models with non-commuting structures, at present only some proposals exist how to describe them in the superfield terms [12, 13]. On the other hand, an adequate framework for theories with extended supersymmetry ($N \geq 2$ in $4D$, $N \geq 3$ in $2D$) is provided by the harmonic superspace approach [14]. Though up to now its main applications concerned supersymmetric theories in four dimensions, it is quite natural to expect that this approach is applicable with equal efficiency to theories with two-dimensional extended supersymmetry, including the aforementioned $(4,4)$ sigma models. We believe that the most appropriate arena to handle the problems mentioned in the beginning is just the $(4,4)$ harmonic superspace.

In the present paper we define basic elements of the $(4,4)$ harmonic superspace calculus in two dimensions and apply it to construct new off-shell superfield formulations of $(4,4)$ $2D$ sigma models with torsion as well as of their massive deformations. We limit our consideration to the simplest case of mutually commuting left and right complex structures, the more general class of models with non-commuting structures will be attacked in further publications. Here (in the end of Sect.6) we make only some comments on possible ways of describing these models in the harmonic superspace.

We start in Sect.2 by giving generalities of $(4,4)$ harmonic superspace and its analytic subspace which has twice as few Grassmann coordinates. The most characteristic feature of this formalism is the presence of two sets of harmonic variables which are associated with the $SU(2)$ automorphism groups of two $2D$ light-cone copies of $N=4$ supersymmetry. Among other things, we show that there exist two different $N=4$ $SU(2)$ superconformal groups preserving harmonic $(4,4)$ analyticity, their closure being $N=4$ $SO(4) \times U(1)$ (“large”) superconformal group.

In Sect.3 the general form of the $(4,4)$ sigma model action with two commuting sets of complex structures is given as an integral over an analytic subspace of $(4,4)$ harmonic
superspace. This action is the general action of the analytic superfield \( q^{(1,1)} \) which has a constrained harmonic dependence and represents the twisted \((4,4)\) supermultiplet in the harmonic superspace. The relevant component action is shown to be completely specified by the metric on the physical bosons manifold. The general action is always invariant under one of two \( N = 4 \) SU(2) superconformal groups defined in Sect.2, with the SU(2) Kac-Moody subgroup acting only on fermions.

In Sect.4 we show that the requirement of invariance under another superconformal group the SU(2) Kac-Moody subgroup of which is realized both on the bosonic and fermionic fields, uniquely fixes the action to be that of \((4,4)\) superextension of the group manifold \( SU(2) \times U(1) \) WZNW sigma model. The action has an unexpectedly simple form and bears a deep analogy with the \( N = 2 \) 4D harmonic superspace action of the improved tensor \( N = 2 \) multiplet \([15]\). We discuss some unusual properties of the action constructed and explain in detail the process of descending to components.

Sect.5 is devoted to constructing massive deformations of the superfield actions presented in the previous Sections. We find that, without allowing for central charges, the massive term of superfields \( q^{(1,1)} \) is defined uniquely. It preserves one of the superconformal groups mentioned above, namely the one with SU(2) acting both on bosons and fermions. Being added to the \( SU(2) \times U(1) \) action, it produces the \( N = 4 \) SU(2) WZNW - Liouville model of ref. \([4,5,6]\). In general it adds to the physical component action the potential terms strictly specified by the form of the \((4,4)\) sigma model action one started with.

In Sect.6 we define the \((4,4)\) duality transformation: insert the harmonic constraints the superfields \( q^{(1,1)} \) satisfy into the action and then find a dual form of the action in terms of the relevant unconstrained analytic Lagrange multiplier superfields. This duality transformation, in contrast to the one used in ref. \([4,5,6]\), manifestly preserves \((4,4)\) supersymmetry. The crucially new feature of the dual action is the presence of an infinite number of auxiliary fields. While in the original action the correct physical field content is ensured by the harmonic constraints, in the dual action this is achieved due to an appropriate gauge invariance. We explicitly give the dual form of the \( SU(2) \times U(1) \) action with and without Liouville term. An interesting peculiarity of the dual form of massive actions is the spontaneous generation of “semi-central” charges breaking the commutativity of the left and right light-cone \((4,4)\) supertranslations and possessing nontrivial tensor properties with respect to the automorphism SU(2)’s.

2 Generalities of \((4,4)\) harmonic superspace

2.1 Conventional \((4,4)\) superspace. We start with the basic relations of the algebra of \((4,4)\) 2D covariant spinor derivatives\(^1\)

\[
\{D_+, i\}, \{D_-, i\} = 2i\delta^j_i \partial^{++}, \quad \{D_+, a\}, \{D_-, a\} = 2i\delta^a_i \partial^{--}, \quad \{D_+, i, D_+, a\} = \{D_+, i, D_- a\} = 0. \quad (2.1)
\]

\(^1\)We use the standard conventions: SU(2) doublet indices are raised and lowered with the help of totally antisymmetric tensors \( \epsilon_{ik}, \epsilon_{ab}, \epsilon^{ik}, \epsilon^{ab}, (\epsilon_{12} = -\epsilon^{12} = 1) \), etc.
Here

\[ D_{+i} = \frac{\partial}{\partial \theta^{+i}} + i\bar{\theta}^{+i} \partial_{++}, \quad \overline{D}_{+}^{i} = -\frac{\partial}{\partial \theta^{+i}} - i\bar{\theta}^{+i} \partial_{++} \]

\[ D_{-a} = \frac{\partial}{\partial \theta^{-a}} + i\bar{\theta}^{-a} \partial_{--}, \quad \overline{D}_{-}^{a} = -\frac{\partial}{\partial \theta^{-a}} - i\bar{\theta}^{-a} \partial_{--} \]

\[ \partial_{\pm \pm} = \frac{\partial}{\partial x_{\pm \pm}}. \quad (2.2) \]

The 2D Lorentz indices \"+\", \"-\" mark quantities related to the left and right light-cone sectors of the (4, 4) 2D superspace,

\[ S^{(1,1|4,4)} \equiv L^{(1|4)} \otimes R^{(1|4)}, \quad (2.3) \]

\[ L^{(1|4)} = \{x^{++}, \theta^{+i}, \bar{\theta}^{+\bar{j}}\} \equiv \{Z_L\}, \quad R^{(1|4)} = \{x^{--}, \theta^{-a}, \bar{\theta}^{-\bar{b}}\} \equiv \{Z_R\}, \quad (2.4) \]

the isodoublet indices \( i, j = 1, 2 \) and \( a, b = 1, 2 \) are associated with two independent automorphism \( SU(2) \) groups acting in the left and right sectors.

The left (4, 4) 2D Poincaré supertranslations are realized on the coordinates of \( L^{(1|4)} \) by

\[ x^{++'} = x^{++} + i(\theta^{+i} \epsilon^{+i} - \epsilon^{-i} \bar{\theta}^{+\bar{i}}), \quad \theta^{+i'} = \theta^{+i} + \epsilon^{+i}, \quad \bar{\theta}^{+\bar{i'}} = \bar{\theta}^{+\bar{i}} + \epsilon^{-\bar{i}}, \quad (2.5) \]

where \( \epsilon^{+i} \) is the related constant parameter. The realization of the right supertranslations on the coordinates of \( R^{(1|4)} \) has the same form, up to the replacements \( + \rightarrow -, i \rightarrow a \).

For our further purposes it will be of crucial importance that (4, 4) 2D supersymmetry possesses two commuting automorphism groups \( SU(2)_L \) and \( SU(2)_R \) acting, respectively, on the doublet indices \( i, j \) and \( a, b \). Note that, besides these explicit \( SU(2) \)'s, the relations (2.1) reveal covariance with respect to another two \( SU(2) \) groups which rotate spinor quantities through their conjugates and commute with each other and with the explicit \( SU(2) \)'s. All these \( SU(2) \) automorphisms (forming the group \( SO(4)_L \times SO(4)_R \)) become manifest in the quartet notation (we suppress the light-cone indices)

\[ (\theta^i, \bar{\theta}^{\bar{j}}) \equiv \theta^{i\underline{k}}, \quad (D_i, \overline{D}_{\underline{i}}) \equiv D_{i\underline{k}}, \quad \underline{k} = 1, 2 \]

\[ (\theta^a, \bar{\theta}^{\bar{b}}) \equiv \theta^{a\underline{b}}, \quad (D_a, \overline{D}_{\underline{a}}) \equiv D_{a\underline{b}}, \quad \underline{b} = 1, 2 \]

\[ (\theta^{\underline{k}})^\dagger = \epsilon_{\underline{k}\underline{m}} \theta^{\underline{m}}, \quad (D_{i\underline{k}})^\dagger = -\epsilon^{i\underline{k}\underline{l}} \epsilon_{\underline{l}\underline{m}} D_{\underline{k}\underline{m}}, \quad \text{etc.} \quad (2.6) \]

For instance, in this notation the algebra of spinor derivatives belonging to the left world can be summarized as the single relation

\[ \{D_{+i\underline{k}}, D_{+j\underline{l}}\} = -2i\epsilon_{ij} \epsilon_{\underline{k}\underline{l}} \bar{\theta}^{+\underline{++}}. \quad (2.7) \]

Actually, the harmonic extension of (4, 4) superspace we will deal with in the present paper uses the harmonic variables on the automorphism groups \( SU(2)_L \) and \( SU(2)_R \) which are explicit in the doublet notation (2.1), (2.2). For our purposes it will be of no need to harmonize two other \( SU(2) \)'s, they will be treated as additional automorphism groups of the (4, 4) harmonic superspace and an analytic subspace of the latter.
2.2 Harmonic (4,4) superspace. In constructing a harmonic extension of the (4,4) 2D superspace $S^{(1,1|4,4)}$ we will closely follow the lines of ref. [14]. The main new feature of the present case is the possibility to introduce two independent sets of harmonic variables associated with the mutually commuting automorphism groups $SU(2)_L$, $SU(2)_R$.

Thus we define the harmonic (4,4) superspace $HS^{(1+2,1+2|4,4)}$ as

$$
HS^{(1+2,1+2|4,4)} = HL^{(1+2|4)} \otimes HR^{(1+2|4)},
$$

(2.8)

$$
HL^{(1+2|4)} = L^{(1|4)} \otimes S_L^2 \equiv \{Z_L, u^{(\pm 1)} i\}, \quad HR^{(1+2|4)} = R^{(1|4)} \otimes S_R^2 \equiv \{Z_R, v^{(\pm 1)} a\}.
$$

(2.9)

Here the harmonic coordinates $u^{(1)} i, u^{(-1)} i$ and $v^{(1)} a, v^{(-1)} a$ parametrize two-dimensional spheres $S_L^2, S_R^2$:

$$
S_L^2 \sim SU(2)_L/U(1)_L = \{u^{(1)} i, u^{(-1)} j\}, \quad S_R^2 \sim SU(2)_R/U(1)_R = \{v^{(1)} a, v^{(-1)} b\}
$$

(2.10)

$$
u^{(1)} i u^{(-1)} i = 1, \quad v^{(1)} a v^{(-1)} a = 1.
$$

(2.11)

Actually, each set of harmonics brings just two independent parameters in agreement with the dimension of the cosets $S_L^2, S_R^2$. One of three parameters remaining after employing the unitarity conditions (2.11) does not contribute because we require the strict preservation of the relevant $U(1)$ charge (such a requirement is standard for the harmonic superspace approach [14]).

One may define, for each set of the harmonic variables, three derivatives compatible with the conditions (2.11)

$$
D^{(\pm 2,0)} = u^{(\pm 1)} i \frac{\partial}{\partial u^{(\mp 1)} i}, \quad D_u^{(0,0)} = u^{(1)} i \frac{\partial}{\partial u^{(1)} i} - u^{(-1)} i \frac{\partial}{\partial u^{(-1)} i},
$$

$$
D^{(0,\pm 2)} = v^{(\mp 1)} a \frac{\partial}{\partial v^{(1)} a}, \quad D_v^{(0,0)} = v^{(1)} a \frac{\partial}{\partial v^{(1)} a} - v^{(-1)} a \frac{\partial}{\partial v^{(-1)} a}.
$$

(2.12)

They form two commuting algebras $su(2)$

$$
[D^{(2,0)}, D^{(-2,0)}] = D_u^{(0,0)}, \quad [D_u^{(0,0)}, D^{(\pm 2,0)}] = \pm 2 D^{(\pm 2,0)} ,
$$

(2.13)

$$
[D^{(0,2)}, D^{(0,-2)}] = D_v^{(0,0)}, \quad [D_v^{(0,0)}, D^{(0,\pm 2)}] = \pm 2 D^{(0,\pm 2)} .
$$

(2.14)

In general, the superfields given on the superspace (2.9), harmonic (4,4) superfields, are characterized by two $U(1)$ charges which are eigenvalues of the operators $D_u^{(0,0)}, D_v^{(0,0)}$

$$
\Phi^{(q,p)} = \Phi^{(q,p)}(Z_L, Z_R, u, v), \quad D_u^{(0,0)} \Phi^{(q,p)} = q \Phi^{(q,p)}, \quad D_v^{(0,0)} \Phi^{(q,p)} = p \Phi^{(q,p)}.
$$

(2.15)

These superfields in general contain infinite numbers of components coming from two independent harmonic expansions on the two-spheres $S_L^2$ and $S_R^2$, i.e. with respect to the harmonics $u$ and $v$. For instance

$$
\Phi^{(1,1)}(Z_L, Z_R; u, v) = \Phi^{ia}(Z_L, Z_R) u_i^{(1)} u_a^{(1)} + ...
$$

(2.16)

2.3 (4,4) harmonic analyticity. Let us pass to another (analytic) basis in the left and right harmonic superspaces (2.9)

$$
HL^{(1+2|4)} = \{z^{++}, \theta^{+ \pm (1,0)}, \bar{\theta}^{+ \pm (1,0)}, u^{(\pm 1) i}\},
$$

$$
HR^{(1+2|4)} = \{z^{--}, \theta^{- \pm (0,1)}, \bar{\theta}^{- \pm (0,1)}, v^{(\pm 1) a}\},
$$

(2.17)
where
\[ z^{\pm \pm} = x^{\pm \pm} + i(\theta^{\pm (1,0)} \bar{\theta}^{\pm (-1,0)} + \theta^{\pm (-1,0)} \bar{\theta}^{\pm (1,0)}) \]
\[ \theta^+ (\pm 1, 0) = \theta^+ i u_i^{(\pm 1)} , \quad \bar{\theta}^+ (\pm 1, 0) = \bar{\theta}^+ i u_i^{(\pm 1)} , \]
\[ \theta^- (0, \pm 1) = \theta^- a_{\nu_a}^{\pm (1)} , \quad \bar{\theta}^- (0, \pm 1) = \bar{\theta}^- a_{\nu_a}^{\pm (1)} . \] (2.18)

Henceforth, for brevity, we will omit the light-cone Lorentz indices of spinor coordinates in the analytic basis.

It is easy to check that the coordinate sets
\[ \mathbf{AL}^{(1+2|2)} = \{ z^{++}, \theta^{(1,0)}, \bar{\theta}^{(1,0)}, u^{(\pm 1)} \} \equiv \{ \zeta_L, u \} , \]
\[ \mathbf{AR}^{(1+2|2)} = \{ z^{--}, \theta^{(0,1)}, \bar{\theta}^{(0,1)}, v^{(\pm 1)} \} \equiv \{ \zeta_R, v \} \] (2.19)
are closed under the $(4,4)$ supersymmetry transformations (2.23) and so form invariant analytic subspaces in the above harmonic superspaces. Their product is the analytic harmonic $(4,4)$ superspace \footnote{To avoid a confusion, we point out that the symbol $(4,4)$ indicates the numbers of left and right supersymmetries, but not the Grassmann dimension of superspaces where these supersymmetries are realized.}
\[ \mathbf{AS}^{(1+2,1+2|2,2)} = \mathbf{AL}^{(1+2|2)} \otimes \mathbf{AR}^{(1+2|2)} = \{ \zeta_L, \zeta_R, u, v \} \equiv \{ \zeta, u, v \} . \] (2.20)

The existence of this analytic subspace matches with the form of covariant spinor derivatives in the analytic basis
\[ D^{(\pm 1,0)} \equiv D^i u_i^{(\pm 1)} , \quad \bar{D}^{(\pm 1,0)} \equiv \bar{D}^i u_i^{(\pm 1)} , \]
\[ D^{(1,0)} = -\frac{\partial}{\partial \theta^{(1,0)}} , \quad \bar{D}^{(1,0)} = -\frac{\partial}{\partial \bar{\theta}^{(1,0)}} , \]
\[ D^{(-1,0)} = \frac{\partial}{\partial \theta^{(1,0)}} + 2i \bar{\theta}^{(-1,0)} \partial_{++} , \quad \bar{D}^{(-1,0)} = \frac{\partial}{\partial \bar{\theta}^{(1,0)}} - 2i \theta^{(-1,0)} \partial_{++} \] (2.21)
\[ (D^{0,\pm 1} \equiv D^a_{\nu_a}^{\pm (1)} \text{ and } \bar{D}^{0,\pm 1} \equiv \bar{D}^a_{\nu_a}^{\pm (1)} \text{ are given by analogous formulas). Harmonic superfields obeying the (4,4) harmonic Grassmann analyticity conditions} \]
\[ D^{(1,0)} \Psi^{(q,p)} = \bar{D}^{(1,0)} \bar{\Psi}^{(q,p)} = D^{(0,1)} \Psi^{(q,p)} = \bar{D}^{(0,1)} \bar{\Psi}^{(q,p)} = 0 \] (2.22)
are called analytic $(4,4)$ superfields. In the basis (2.18) the spinor derivatives entering (2.22) are reduced to the partial ones, so the conditions (2.22) mean that the analytic superfields do not depend on the coordinates \( \theta^{(-1,0)}, \bar{\theta}^{(-1,0)}, \theta^{(0,-1)}, \bar{\theta}^{(0,-1)} \) in this basis
\[ \Psi^{(q,p)} = \Psi^{(q,p)}(\zeta_L, \zeta_R, u, v) . \] (2.23)

In the sequel we will need the expressions for the harmonic derivatives \( D^{(2,0)}, D^{(0,2)} \) in the basis (2.18) and in the realization on analytic superfields
\[ D^{(2,0)} = \partial^{(2,0)} + 2i \theta^{(1,0)} \bar{\theta}^{(1,0)} \partial_{++} , \quad D^{(0,2)} = \partial^{(0,2)} + 2i \theta^{(0,1)} \bar{\theta}^{(0,1)} \partial_{--} , \] (2.24)
where partial harmonic derivatives are given by the expressions (2.12). We see that these harmonic derivatives (as well as $D_u^{(0,0)}$, $D_v^{(0,0)}$) preserve (4, 4) analyticity: the result of their action on an analytic superfield is again an analytic superfield. For completeness, we also present the analytic superspace form of the $U(1)$ charge operators

$$
D_u^{(0,0)} = \partial_u^{(0,0)} + \theta^{(1,0)} \frac{\partial}{\partial \theta^{(1,0)}} + \bar{\theta}^{(1,0)} \frac{\partial}{\partial \bar{\theta}^{(1,0)}}, \\
D_v^{(0,0)} = \partial_v^{(0,0)} + \theta^{(0,1)} \frac{\partial}{\partial \theta^{(0,1)}} + \bar{\theta}^{(0,1)} \frac{\partial}{\partial \bar{\theta}^{(0,1)}}.
$$

(2.25)

Finally, we note that, like in the $N = 2$ 4D case [14], the analytic subspace $A_{L(1+2|2)}$ ($AR^{(1+2|2)}$) is real with respect to the generalized involution “∼” which is the product of ordinary complex conjugation and an antipodal map of the sphere $S^2_L$ ($S^2_R$)

$$(\theta^{(1,0)}) = -\bar{\theta}^{(1,0)} , \quad (\bar{\theta}^{(1,0)}) = \theta^{(1,0)} , \quad (u^{(\pm 1)}_i) = -u^{(\pm 1)}_i , \quad (v^{(\pm 1)}_a) = -v^{(\pm 1)}_a,$$

(2.26)

(and similarly for $\theta^{(0,1)}$, $\bar{\theta}^{(0,1)}$, $v^{(\pm 1)}_a$). The analytic superfields $\Psi^{(p,q)}(\zeta_L, \zeta_R, u, v)$ can be chosen real with respect to this involution, provided $|p + q| = 2n$

$$(\tilde{\Psi}^{(p,q)}) = \Psi^{(p,q)} , \quad |p + q| = 2n .$$

(2.27)

Of course, for the component fields in the $\theta$ and $u, v$ expansion of $\Psi^{(p,q)}$ one obtains ordinary reality conditions.

This is an appropriate place to comment on the relation to the projective (4, 4) 2D superspace formalism [2 14 2].

Bearing some formal resemblances to the latter, the harmonic superspace approach differs in a number of important aspects. This mainly regards the treatment of extra bosonic variables which are present in both approaches. In the projective 2D superspace formalism they form two sets of complex variables with respect to which one takes contour integrals, while in the harmonic superspace formalism they are $SU_L(2)/U_L(1)$ and $SU_R(2)/U_R(1)$ spinor harmonics $u^{(\pm 1)}_i$, $v^{(\pm 1)}_a$, and all the involved fields are assumed to be decomposable into the harmonic series in these variables. The harmonic variables $u, v$ represent the left and right spheres $SU(2)_{L,R}/U_{L,R}(1)$ in a parametrization-independent way, while the complex coordinates of the projective superspace can be viewed as particular parametrizations of the same spheres. In the projective superspace approach, when doing contour integration, there arises an uneasy problem of how to choose the relevant integration contours. There is no such problem in the harmonic superspace approach where the integral over additional variables is understood as the double harmonic integral on the product $SU_L(2)/U_L(1) \otimes SU(2)_R/U_R(1)$. It can be defined by the rules [14]

$$
\int du \ 1 = 1 , \quad \int du \ v^{(1)}_{i_1} u^{(1)}_{i_2} u^{(-1)}_{i_3} \ldots = 0
$$

(2.28)

(for $v$ integration the rules are the same). The basic feature of the harmonic superspace approach is that the harmonic variables are treated on equal footing with other superspace coordinates: with respect to them one not only integrates, but also differentiates, they essentially enter into the formulas relating central and analytic bases, they are responsible
for the presence of an infinite number of auxiliary fields in unconstrained analytic (4, 4) superfields, etc.

2.4 Two N=4 SU(2) superconformal groups. As the last topic of this Section we will discuss realizations of two \( N = 4 \) SU(2) superconformal groups in the analytic (4, 4) superspace \([20]\). To know them will be very important for our further purposes.

Both these \( N = 4 \) SU(2) superconformal groups consist of two commuting branches independently acting in the left and right analytic subspaces, so it will be sufficient to consider their action, say, in AL\(^{(1+2|2)}\).

The existence of two different \( N = 4 \) SU(2) superconformal groups in AL\(^{(1+2|2)}\) is related to the fact that the most general superconformal group which can be defined in this superspace is the so called “large” \( N = 4 \) superconformal group with \( SO(4) \times U(1) \sim SU(2) \times SU(2) \times U(1) \) affine Kac-Moody subgroup in its bosonic sector [16, 17, 5, 18]. The \( N = 4 \) SU(2) superconformal groups in question are two different subgroups of this “large” superconformal group, each including one of two \( SU(2) \) factors of the \( SO(4) \) just mentioned. Transformations of the “large” supergroup on the coordinates of AL\(^{(1+2|2)}\) have been already given in [20]. We present the coordinate realizations of its two \( N = 4 \) SU(2) subgroups separately, as they have essentially different implications in the sigma models we are going to discuss. Our way of deducing these realizations and parametrizing them slightly differ from the one adopted in [20].

First of these superconformal groups acts on all coordinates of AL\(^{(1+2|2)}\) including the harmonic ones (for brevity we omit the light-cone indices):

\[
\begin{align*}
\delta_I z &= \Lambda_I(\zeta_L, u) , \quad \delta_I \theta^{(1,0)} = \Lambda^{(1,0)}_I(\zeta_L, u) , \quad \delta_I \bar{\theta}^{(1,0)} = \bar{\Lambda}^{(1,0)}_I(\zeta_L, u) , \\
\delta_I u^{(1)}_i &= \Lambda^{(2,0)}_I u^{(-1)}_i , \quad \delta_I u^{(-1)}_i = 0
\end{align*}
\]

and is fully determined by the requirement that the harmonic derivative \( D^{(2,0)} \) transforms as

\[
\delta_I D^{(2,0)} = -\Lambda^{(2,0)}_I D^{(0,0)}_u .
\]

From this condition one obtains the relations [20]

\[
\begin{align*}
D^{(2,0)} \Lambda_I &= 2i (\bar{\theta}^{(1,0)} \Lambda_I^{(1,0)} - \theta^{(1,0)} \bar{\Lambda}_I^{(1,0)}) , \\
D^{(2,0)} \Lambda_I^{(1,0)} &= \Lambda^{(2,0)} I \theta^{(1,0)} , \quad D^{(2,0)} \bar{\Lambda}_I^{(1,0)} = \Lambda^{(2,0)}_I \bar{\theta}^{(1,0)} , \quad D^{(2,0)} \Lambda_I^{(2,0)} = 0
\end{align*}
\]

which have a simple general solution via the constrained analytic function \( a(\zeta_L, u) \)

\[
\begin{align*}
\Lambda_I &= a - \frac{1}{2} \partial^{(2,0)} D^{(2,0)} a , \quad \Lambda_I^{(1,0)} = \frac{i}{4 \partial \theta^{(1,0)}} D^{(2,0)} a , \\
\bar{\Lambda}_I^{(1,0)} &= -\frac{i}{4 \partial \bar{\theta}^{(1,0)}} D^{(2,0)} a , \quad \Lambda_I^{(2,0)} = -\frac{1}{2} D^{(2,0)} \partial a \equiv D^{(2,0)} \Lambda_I^{(0,0)} ,
\end{align*}
\]

\[
(D^{(2,0)})^2 a(\zeta_L, u) = 0 .
\]

The explicit form of \( a(\zeta_L, u) \) is as follows

\[
a(\zeta_L, u) = a_0(z) + a_0^{(ij)}(z) u^{(1)}_i u^{(-1)}_j + \theta^{(1,0)} \xi^{(z)}(z) u^{(-1)}_i + \bar{\theta}^{(1,0)} \bar{\xi}^{(z)}(z) u^{(-1)}_i \\
-2i \theta^{(1,0)} \bar{\theta}^{(1,0)} \partial z a_0^{(ij)}(z) u^{(1)}_i u^{(-1)}_j .
\]
Here, \(a_0(z), \xi^i(z), \bar{\xi}^i(z), \partial_2 a_0^{(ij)}(z)\) are, respectively, parameters of conformal, supersymmetry and \(SU(2)\) Kac-Moody transformations forming the \(N = 4\) \(SU(2)\) superconformal group. Notice that the Kac-Moody transformation parameter enters \(a\) via its “prepotential” \(a^{(ij)}\). It is an easy exercise to check that the transformations \((2.29), (2.32)\) preserve the \(\text{AL}(1^{+2}/2)\) integration measure

\[
\mu^{(-2,0)} = \left[ dz \, d\theta^{(1,0)} \, d\bar{\theta}^{(1,0)} \right] du,
\]

\[
\delta_I \mu^{(-2,0)} = \left( \partial_z \Lambda_I + \bar{\theta}^{(-2,0)} \Lambda^{(2,0)}_I - \frac{\partial \Lambda^{(1,0)}_I}{\partial \theta^{(1,0)}} - \frac{\partial \bar{\Lambda}^{(1,0)}_I}{\partial \bar{\theta}^{(1,0)}} \right) \mu^{(-2,0)} = 0. \tag{2.35}
\]

Note that the above analytic superspace realization of \(N = 4\) \(SU(2)\) superconformal group is basically of the same form as that of \(N = 2\) 4D superconformal group in the corresponding harmonic analytic superspace \([13]\).

The second \(N = 4\) \(SU(2)\) superconformal group has no direct analog in \(N = 2\) 4D case. It does not affect harmonic variables

\[
\delta_{II} \theta^{(1,0)}(\zeta_L, u), \delta_{II} \bar{\theta}^{(1,0)}(\zeta_L, u), \delta_{II} \zeta^{(1,0)}(\zeta_L, u), \delta_{II} u^{(1)}_i = 0 \tag{2.36}
\]

and is fully determined by requiring \(D^{(2,0)}\) to be invariant

\[
\delta_{II} D^{(2,0)} = 0. \tag{2.37}
\]

The latter equation implies

\[
D^{(2,0)} \Lambda^{(1,0)}_{II} = D^{(2,0)} \bar{\Lambda}^{(1,0)}_{II} = 0,
\]

\[
D^{(2,0)} \Lambda_{II} = 2i \left( \bar{\theta}^{(1,0)} \Lambda^{(1,0)}_{II} - \theta^{(1,0)} \bar{\Lambda}^{(1,0)}_{II} \right). \tag{2.38}
\]

Combining \(\theta^{(1,0)}, \bar{\theta}^{(1,0)}\) into a doublet of an extra \(SU(2)\) (recall \((2.6)\)),

\[
\{ \theta^{(1,0)}, \bar{\theta}^{(1,0)} \} \equiv \{ \theta^{(1,0)} \}, \tag{2.39}
\]

one can write the general solution to \((2.38)\) as

\[
\Lambda^{(1,0)}_{II}(\zeta_L, u) = \lambda^{(1)}_{k} u^{(1)}_k + \theta^{(1,0)} \bar{\lambda}^{(1)}_{k} \zeta_L - \delta^{(1)}_{i} \partial_2 \lambda(z)) - 2i \theta^{(1,0)} \bar{\lambda}^{(1,0)} \zeta_L (\lambda^{(1)}_{k} u^{(-1)}_k),
\]

\[
\Lambda_{II}(\zeta_L, u) = \lambda(z) + 2i \theta^{(1,0)} \bar{\lambda}^{(1,0)} (\lambda^{(1)}_{k} u^{(-1)}). \tag{2.40}
\]

Here, \(\lambda(z), \lambda^{(1)}_{k}(z), \lambda^{(1)}_{k}(z)\) are, respectively, the parameters of conformal, second \(SU(2)\) Kac-Moody and supersymmetry transformations. The analytic superspace integration measure is also preserved by this \(N = 4\) \(SU(2)\) superconformal group

\[
\delta_{II} \mu^{(-2,0)} = 0. \tag{2.41}
\]

Finally, we wish to mention once more that these two superconformal groups do not commute; their closure is the “large” \(N = 4\) superconformal group. We will not discuss here the detailed structure of this closure.
3 (4,4) sigma models in harmonic superspace

The main reason why we applied to the (4,4) harmonic superspace formalism was the hope to construct, within its framework, an off-shell formulation of general (4,4) supersymmetric sigma models with torsion and, as a special subclass of the latter, (4,4) superextensions of the group manifold WZNW sigma models. As a first step in approaching our ultimate aim, in this Section we rewrite in the harmonic superspace the general (4,4) sigma models with two mutually commuting sets of complex structures.

3.1 Harmonic superspace description of (4,4) twisted multiplet. In (2,2) superspace the (4,4) sigma models with mutually commuting left and right complex structures are described by an action of paired chiral and twisted chiral superfields [2, 7, 8]. These pairs comprise the (4,4) twisted [2] (or analytic [1]) multiplet, so the (4,4) superspace form of the sigma model action in question should coincide with a general action of (4,4) superfields representing the (4,4) twisted multiplets. It turns out that this action admits a natural formulation in the analytic harmonic superspace AS(1+2|1+2,2)(2.20).

We start by recalling how this multiplet is described in conventional (4,4) superspace. It is represented by a real quartet superfield \( q^{ia}(Z_L, Z_R) \), \( (q^{ia})^\dagger = \epsilon_{ik} \epsilon_{ab} q^{kb} \), subject to the following irreducibility conditions [1, 2, 3, 21]

\[
D^a (j^a + q^{ia}) = 0 , \quad D_\bar{a} (b^{\bar{a}} - q_{\bar{a}a}) = 0 .
\]

These constraints leave in \( q^{ia} \) 8 + 8 independent field components that is just the off-shell field content of the (4,4) twisted multiplet (it reduces to 4 + 4 on shell).

Let us now convert the \( SU(2) \) indices of \( q^{ia} \) and spinor derivatives in (3.1) with the harmonics \( u^{(1)}_i, v^{(1)}_a \) in order to rewrite (3.1) in the following equivalent form

\[
D^{(1,0)} (q^{(1,1)}) = D^{(1,0)} (q^{(1,1)}) = 0 , \quad D^{(0,1)} (q^{(1,1)}) = D^{(0,1)} (q^{(1,1)}) = 0 ,
\]

where the involved projections of the spinor derivatives are defined in eq. (2.21) and

\[
q^{(1,1)}(Z_L, Z_R, u, v) \equiv q^{ia}(Z_L, Z_R) u^{(1)}_i v^{(1)}_a , \quad q^{(1,1)} = q^{(1,1)} .
\]

The homogeneity property (3.3) can be equivalently reexpressed as the harmonic constraints [14]

\[
D^{(2,0)} (q^{(1,1)}) = D^{(0,2)} (q^{(1,1)}) = 0 ,
\]

after which, comparing (3.2) with eqs. (2.22), one concludes that the (4,4) twisted multiplet is represented in the harmonic superspace by a real analytic superfield

\[
q^{(1,1)} = q^{(1,1)}(\zeta_L, \zeta_R, u, v)
\]

which obeys the harmonic constraints (3.4).

---

3The constraints in this form have been given for the first time in [1].
where “bar” on the fields means ordinary complex conjugation and some numerical factors have been inserted for further convenience. We see that the fields $q^{ia}_i, \psi^a_+ , \bar{\psi}^a_+ , \chi^i_+ , \bar{\chi}^-_i$ and $F, \bar{F}, L, \bar{L}$ have appropriate dimensions to represent, respectively, physical and auxiliary degrees of freedom. Note a formal similarity of the (4,4) harmonic constraints (3.4) to the constraints defining the $N=2$ tensor multiplet in the harmonic $\mathcal{N}=2$ 4D superspace \(\mathcal{I}\). The crucial difference between the two types of constraints is that the latter implies a differential condition for a vector component of the relevant superfield, requiring it to be divergenceless, while this is not the case for the constraints (3.4).

As a last topic of this Subsection we discuss the transformation properties of $q^{(1,1)}$ under two $N=4$ $SU(2)$ superconformal groups defined in Subsect.2.4. These transformation laws are uniquely fixed by the requirement of preserving the harmonic constraints (3.4) and turn out to be very simple (we again omit the light-cone indices)

$$
\delta_I q^{(1,1)}(\zeta_L, \zeta_R, u, v) \simeq q^{(1,1)'}(\zeta_L', \zeta_R', u', v') - q^{(1,1)}(\zeta_L, \zeta_R, u, v) = \Lambda^{(0,0)}(\zeta_L, \zeta_R, u, v)
= -\frac{1}{2} \left( \partial_2 a(\zeta_L, u) \right) q^{(1,1)}(\zeta_L, \zeta_R, u, v) ,
$$

$$
\delta_{II} q^{(1,1)}(\zeta_L, \zeta_R, u, v) \simeq q^{(1,1)'}(\zeta_L', \zeta_R', u, v') - q^{(1,1)}(\zeta_L, \zeta_R, u, v) = 0 .
$$

The transformation rules with respect to the right light-cone branches of these superconformal groups are given by similar formulas.

The basic difference between the realizations I and II lies in the action of the $SU(2)$ affine Kac-Moody subgroup: in the case I it acts both on the physical bosonic and fermionic fields as rotations of their indices $i, a$ while in the case II it does not affect the physical bosons $q^{ia}_i$ at all and acts only on fermions, mixing $\psi$ with $\bar{\psi}$ and $\chi$ with $\bar{\chi}$. The auxiliary fields are scalars with respect to the $SU(2)$ Kac-Moody subgroup of the realization I and split into a singlet and triplet with respect to an analogous subgroup of the realization II. All these properties become manifest in the “quartet” notation (2.35).
For instance, four auxiliary fields terms in (3.5) are combined into the single term

$$2\theta^{(1,0)} i\theta^{(0,1)} a F^{ia} , \quad F^{ia} = \begin{pmatrix} F & L \\ \bar{L} & -F^{i} \end{pmatrix} . \tag{3.8}$$

As a useful example of the component transformation properties we explicitly give the transformation rule of the bosonic field \(q^{ia}\) under the \(SU(2)\) Kac-Moody subgroup

$$\delta_{SU(2)} q^{ia}(z^{++}, z^{--}) = \frac{1}{2} \left( \partial a^{(ik)}_{0} (z^{++}) \right) q^{a}_{k}(z^{++}, z^{--}) . \tag{3.9}$$

Note that the superconformal transformations of the constrained superfield \(q^{ia}(Z_{L}, Z_{R})\) representing the \((4,4)\) twisted multiplet in the conventional \((4,4)\) superspace were given in [9]. These look much more complicated compared to the \((4,4)\) analytic superfield ones (3.6), (3.7).

### 3.2 General action of the superfields \(q^{(1,1)}\)

By the dimensionality reasoning and keeping in mind the requirement of conservation of the \(U(1)\) charges, it is straightforward to write the most general action of self-interacting superfields \(q^{(1,1)} M (M = 1, 2, \ldots)\)

$$S_{q} = \int \mu^{(-2,-2)} \mathcal{L}^{(2,2)}(q^{(1,1)} M (\zeta_{L}, \zeta_{R}, u, v), u, v) . \tag{3.10}$$

Here

$$\mu^{(-2,-2)} \equiv \mu^{(-2,0)} \mu^{(0,-2)} = d^{2}z d^{2}\theta^{(1,0)} d^{2}\theta^{(0,1)} du dv$$

is the measure of integration over the analytic \((4,4)\) superspace. The dimensionless analytic superfield Lagrangian \(\mathcal{L}^{(2,2)}(q^{(1,1)} M , u^{(\pm1)}, v^{(\pm1)})\) bears in general an arbitrary dependence on its arguments, the only restriction being a compatibility with the external \(U(1)\) charges \((2, 2)\) of the Lagrangian. The free action of \(q^{(1,1)} M\) is given by

$$S_{q}^{free} \sim \int \mu^{(-2,-2)} q^{(1,1)} M q^{(1,1) M} , \tag{3.11}$$

so for consistency we are led to assume

$$\det \left( \frac{\partial^{2} \mathcal{L}^{(2,2)}}{\partial q^{(1,1) M} \partial q^{(1,1) N}} \right) |_{q^{(1,1)}=0} \neq 0 . \tag{3.12}$$

For completeness, we also add the constraints on \(q^{(1,1)} M (\zeta_{L}, \zeta_{R}, u, v)\)

$$D^{(2,0)} q^{(1,1) M} = D^{(0,2)} q^{(1,1) M} = 0 . \tag{3.13}$$

It is straightforward to substitute the component expansion of \(q^{(1,1)}\), (3.5), into (3.10), to integrate over \(\theta^{'s}\) and to obtain the component form of the action. It is instructive to give here its physical and auxiliary bosons parts, with all fermions omitted. These pieces can be written as follows

$$S_{phb} = 2 \int d^{2}z \left\{ G_{Mia Njb} (q) \partial_{++} q^{ia M} \partial_{--} q^{jb N} + B_{Mia Njb} (q) \partial_{++} q^{ia M} \partial_{--} q^{jb N} \right\} , \tag{3.14}$$
\[ S_{auxb} = 4 \int d^2 z \, G_{MN}(q) \left( F^M F^N + L^M \tilde{L}^N \right), \]  

(3.15)

where

\[ G_{Mia Njb}(q) = \int dudv \, g_{MN}(q_0^{(1,1)}, u, v) \epsilon_{ij} \epsilon_{ab}, \]  

(3.16)

\[ B_{Mia Njb}(q) = \int dudv \, g_{MN}(q_0^{(1,1)}, u, v) [\epsilon_{ij} v^{(1)}(a) v^{(-1)}_b - \epsilon_{ab} u^{(1)}(i) u^{(-1)}_j], \]  

(3.17)

\[ G_{MN}(q) = \int dudv \, g_{MN}(q_0^{(1,1)}, u, v), \]  

(3.18)

\[ g_{MN}(q_0^{(1,1)}, u, v) = \frac{\partial^2 \mathcal{L}^{(2,2)}}{\partial q^{(1,1)} M \partial q^{(1,1)} N} \bigg|_{\theta=0}, \]  

(3.19)

where \( q_0^{(1,1)} \equiv q^{(1,1)}|_{\theta=0} \). The objects \( G_{Mia Njb}(q) \), \( B_{Mia Njb}(q) \) are, respectively, symmetric and antisymmetric under the simultaneous permutation of the indices \( M \leftrightarrow N, i \leftrightarrow j, a \leftrightarrow b \) and so they can be identified with the metric and torsion potential on the target space.

Sometimes it is advantageous to represent the second term in (3.14) through the torsion field strength. It is introduced by the standard expression

\[ H_{Mia Njb T kd} = \frac{\partial B_{Njb T kd}}{\partial q^{ia} M} + \frac{\partial B_{Mia Njb}}{\partial q^{kd} T} + \frac{\partial B_{T kd Mia}}{\partial q^{jb} N}, \]  

(3.20)

and is totally antisymmetric with respect to permutations of the triples \( Mia, Njb, Tkd \). Letting \( q^{ia} M \) depend on an extra parameter \( t \), with \( q^{ia} M(t, z)|_{t=1} \equiv q^{ia} M(z), q^{ia} M(t, z)|_{t=0} = \epsilon^{ia} \), one can locally rewrite the torsion term as

\[ B_{Mia Njb} \partial_+ q^{ia} M \partial_- q^{jb} N = \int_0^1 dt \, H_{Mia Njb T kd} \partial_+ q^{ia} M \partial_- q^{jb} N \partial_- q^{kd} T. \]  

(3.21)

For \( B_{Mia Njb} \) given by eq. (3.17), \( H_{Mia Njb T kd} \) is reduced to

\[ H_{Mia Njb T kd}(q) = \partial_{(Mid} G_{NT)}(q) \epsilon_{ab} \epsilon_{jk} + \partial_{(Mka} G_{NT)}(q) \epsilon_{db} \epsilon_{ij}, \]  

(3.22)

where \( \partial_{Mid} \equiv \partial/\partial q^{id} M \) and symmetrization is with respect to indices \( M, N, T \). Note that all the fermionic terms in the action (3.10) are also expressed through the function \( G_{MN}(q) \) and its derivatives.

Thus we see that in the harmonic superspace formalism all the target geometry objects associated with the off-shell sigma model action (3.10) are expressed in terms of the metric \( G_{MN}(q) \) which is given by a double harmonic integral of the second derivative of the single function, the analytic superspace Lagrangian \( \mathcal{L}^{(2,2)}(q^{(1,1)}, u, v) \). A similar representation for these geometric objects has been obtained earlier in the projective superspace approach [2], with contour integrals instead of the harmonic ones. The quantity \( \mathcal{L}^{(2,2)} \) is an analog of the hyper-Kähler potential [22]. It would be tempting to find out the appropriate geometric setting within which the representation (3.16) - (3.19), (3.22) would follow from certain first principles, like this has been done, e.g., for the hyper-Kähler geometry in [22], and for the geometries of sigma models with heterotic supersymmetry in [23]. There, by solving the defining constraints on the curvature and torsion, expressions for all geometric quantities through a few unconstrained potentials have been obtained.
As the last remark we note that the action (3.10) with arbitrary \( L(2,2) \) respects invariance under the second of two \( N = 4 \) SU(2) superconformal groups realized in the analytic superspace (eqs. (2.30) - (2.41), (3.7)). As for the superconformal group defined by eqs. (2.29) - (2.35), (3.6), in general it does not constitute an invariance group of the action. Even in the free case (3.11) this symmetry is broken. An action possessing this superconformal symmetry will be presented in the next Section.

4 \( SU(2) \times U(1) \) WZNW sigma model in harmonic superspace

In this Section we show that the requirement of invariance under the \( N = 4 \) SU(2) superconformal group (2.29) - (2.35), (3.6) uniquely fixes the \( q^{(1,1)} \) sigma model action to be that of \( N = 4 \) SU(2) \( \times U(1) \) WZNW sigma model \([3, 7, 9]\).

4.1 \( N = 4 \) SU(2) superconformally invariant \( q^{(1,1)} \) action. Let us specialize to a single superfield \( q^{(1,1)} \) and construct for it an action invariant under the superconformal group defined by eqs. (2.29) - (2.35), (3.6).

As was already mentioned, the free action (3.11) does not respect this superconformal invariance (even its rigid scale and SU(2) subsymmetries), so the invariant action should necessarily include self-interaction terms. To find its precise form, we apply the procedure which has been employed earlier in \([15]\) for constructing the action of improved \( N = 2 \) 4D tensor multiplet in the analytic harmonic \( N = 2 \) 4D superspace. Namely, we split \( q^{(1,1)} \) as

\[
q^{(1,1)} = \hat{q}^{(1,1)} + c^{(1,1)} , \quad c^{(1,1)} \equiv c^{ia} u^{(1)}_i v^{(1)}_a , \quad (4.1)
\]

\[
D^{(2,0)} q^{(1,1)} = D^{(0,2)} \hat{q}^{(1,1)} = 0 , \quad (4.2)
\]

with \( c^{ia} \) being a quartet of arbitrary constants, and represent the sought analytic superspace action as a series in \( \hat{q}^{(1,1)} \)

\[
S_{sc} = \int \mu^{(-2,-2)} \sum_{n=2}^{\infty} b_n (\hat{q}^{(1,1)})^n (c^{(-1,-1)})^n = 2 . \quad (4.3)
\]

Here the appropriate degrees of \( c^{(-1,-1)} = c^{ia} u^{(-1)}_i v^{(-1)}_a \) have been inserted for the balance of \( U(1) \) charges. The newly introduced analytic superfield \( \hat{q}^{(1,1)} \) transforms inhomogeneously under the superconformal transformations (2.29) - (2.35), (3.6) (it will be sufficient to consider only the left light-cone branch of the whole superconformal group)

\[
\delta_I \hat{q}^{(1,1)} = \Lambda_I^{(0,0)} \hat{q}^{(1,1)} + \Lambda_I^{(0,0)} c^{(1,1)} - \Lambda_I^{(2,0)} c^{(-1,1)} , \quad c^{(-1,1)} = c^{ia} u^{(-1)}_i v^{(1)}_a , \quad (4.4)
\]

so there arises an opportunity to achieve the invariance of (4.3) by requiring that the variations of the terms of different order in \( \hat{q}^{(1,1)} \) cancel each other up to full harmonic derivatives. Namely, we take into account the invariance of the integration measure and then demand the homogeneous part of the variation of the second order term to be cancelled by the inhomogeneous part of the variation of the third order term, etc (the
inhomogeneous part of the variation of the second order term is a full harmonic derivative in its own right upon using the constraints \((4.2)\). In the process, one exploits the defining constraints \((4.2)\) and the identities

\[
c^{(1,-1)}c^{(-1,1)} = c^{(-1,-1)}c^{(1,1)} - \frac{1}{2}c^2, \quad c^2 \equiv c^ia_i c_a,
\]

\[
D^{(2,0)}D^{(0,2)}(c^{(-1,1)})^n = n^2(c^{(-1,-1)})^{n-1} c^{(1,1)} - \frac{1}{2}c^2n(n-1)(c^{(-1,-1)})^{n-2},
\]

which follow from the completeness relations

\[
u_i(1)u_j(-1) - u_i(-1)u_j(1) = \epsilon_{ij}, \quad v_a^{(1)}v_b^{(-1)} - v_a^{(-1)}v_b^{(1)} = \epsilon_{ab}
\]

and the definition of \(D^{(2,0)}, D^{(0,2)}\). Doing in this way, one finally proves that the action \((4.3)\) is invariant provided the following recurrence relations between the coefficients \(b_n\) hold

\[
b_{n+1} = -\frac{2}{c^2} \left( \frac{n^2}{n^2-1} \right) b_n,
\]

whence one finds

\[
b_n = 2 \left( \frac{2}{c^2} \right)^{n-2} n^2 b_2.
\]

Now, introducing

\[
X \equiv 2 \left( \frac{c^{(-1,-1)} \hat{q}^{(1,1)}}{c^2} \right),
\]

it is straightforward to show that the series in \((4.3)\) is summed up to the expression

\[
S_{sc} \equiv -2b_2 \int \mu^{(-2,-2)}\mathcal{L}_{sc}^{(2,2)}(\hat{q}^{(1,1)}, u, v) = -2b_2 \int \mu^{(-2,-2)} \hat{q}^{(1,1)} \hat{q}^{(1,1)} \left( \frac{\ln(1 + X)}{X} \right) \prime
\]

\[
= -2b_2 \int \mu^{(-2,-2)} \hat{q}^{(1,1)} \hat{q}^{(1,1)} \left( \frac{1}{(1 + X)X} - \frac{\ln(1 + X)}{X^2} \right).
\]

The action \((4.9)\) is the sought superconformally invariant \(q^{(1,1)}\) action. We will prove later (in Subsect.4.2) that it is just the off-shell action of the \(N = 4\) \(SU(2) \times U(1)\) WZNW sigma model (actually this could be figured out already from the fact that the \(SU(2)\) Kac-Moody subgroup of the superconformal group in question acts on the physical bosonic fields \(q^{ia}(z)\) in the way just specific for the realizations of Kac-Moody symmetries in WZNW models, see eq. \((3.9)\)). Now we dwell on some peculiarities of this action.

Let us first mention that its superconformal invariance can be checked directly, without expanding it in a series in \(\hat{q}^{(1,1)}\). After a straightforward computation with making use of the following simple formula for the variation

\[
\delta S_{sc} = 2b_2 \int \mu^{(-2,-2)} \hat{q}^{(1,1)} \delta \hat{q}^{(1,1)} \frac{1}{(1 + X)^2},
\]

one finds

\[
\delta I \mathcal{L}_{sc}^{(2,2)} = D^{(2,0)}(c^{(-1,1)} \hat{q}^{(1,1)} \frac{1}{(1 + X)^2} \Lambda_I^{(0,0)}) - D^{(0,2)}(c^{(1,-1)} \hat{q}^{(1,1)} \frac{2 + X}{(1 + X)^2} \Lambda_I^{(0,0)})
\]

\[
+ 2D^{(0,2)} \hat{q}^{(1,1)} c^{(-1,1)} \frac{1}{(1 + X)^3} \Lambda_I^{(0,0)} - D^{(2,0)} \hat{q}^{(1,1)} c^{(-1,1)} \frac{1 - X}{(1 + X)^3} \Lambda_I^{(0,0)} \quad (4.11)
\]
By virtue of the constraints (4.2) this variation reduces to full harmonic derivatives, thus ensuring the invariance of the action $S_{sc}$ (4.9). In the same way one checks the invariance of (4.9) under the right 2D light-cone branch of the first $N = 4$ $SU(2)$ superconformal group. Of course, being a particular case of the action (3.10), (4.9) is manifestly invariant under the second type $N = 4$ $SU(2)$ superconformal transformations (2.36) - (2.41), (3.7).

As a next comment we point out that the action (4.9) can be uniquely restored (the relations (4.6), (4.7) can be deduced) merely by requiring it to be invariant under some special subgroups of the left (or right) $N = 4$ $SU(2)$ superconformal group: either under rigid scale transformations with $\Lambda^{(0,0)} = \lambda$, $\partial z \lambda = 0$, or under rigid $SU(2)_c$ transformations with $\Lambda^{(0,0)} = \lambda^{(ij)} u^{(1)}_i u^{(-1)}_j$, $\Lambda^{(2,0)} = \lambda^{(ij)} u^{(1)}_i u^{(1)}_j$, $\partial z \lambda^{(ij)} = 0$; considering the whole set of $z$ dependent left and right superconformal transformations brings nothing new in the proof of invariance.

Last two remarks concern the already mentioned analogy with the improved $N = 2$ 4D tensor multiplet.

The original Ansatz for the action (4.3) and the transformation law (4.4) look very similar to those used in ref. [15] (leaving aside the fact that the $N = 2$ 4D superconformal group is finite-dimensional while its 2D counterpart is infinite-dimensional). However, the relevant recurrence relations between the coefficients $b_n$ and the final expressions for the superconformally invariant action radically differ. The origin of this difference lies in different superconformal properties of the integration measure of the $N = 2$ 4D and (4,4) 2D analytic harmonic superspaces: in the former case it has the dimension $[cm^2]$ and is transformed by the relevant superconformal group, while in the latter case it is dimensionless and superconformally invariant. As a result, the superconformal invariance of the action is achieved under different conditions on the coefficients $b_n$.

A close analogy with the $N = 2$ 4D tensor multiplet action is retained in what concerns the constant $c^{ia}$. In both cases the presence of such a constant (an isotriplet one in the $N = 2$ 4D case and an isoquartet one in the (4,4) 2D case) is inevitable in the analytic superfield Lagrangian density, while the invariant action, being rewritten via the unshifted superfield, i.e. $q^{(1,1)}$ in the case at hand, does not depend on the specific choice of this constant. This latter property is directly related to the invariance of the action (4.9) under the rigid scale and $SU(2)_c$ subgroups of $N = 4$ $SU(2)$ superconformal group.

To demonstrate this, let us put

$$b_2 = \frac{1}{4\sqrt{2}} \frac{1}{c \kappa^2}, \quad c \equiv \sqrt{c^{ia} c_{ia}},$$

and rewrite the action (4.9) through $q^{(1,1)}$

$$S_{sc} \equiv S_{sc}(q^{(1,1)}, c^{ia}).$$

Next, let us consider infinitesimal deformations of (4.13) under some rigid dilatations and $SU(2)$ rotations of the constant $c^{ia}$

$$\delta S_{sc} \simeq S_{sc}(q, c + \delta c) - S_{sc}(q, c),$$

(a) $\delta_1 c^{ia} = \alpha c^{ia}$; (b) $\delta_2 c^{ia} = \alpha^{(ik)} c^a_k$.

$$\delta S_{sc} \simeq S_{sc}(q, c + \delta c) - S_{sc}(q, c),$$

(a) $\delta_1 c^{ia} = \alpha c^{ia}$; (b) $\delta_2 c^{ia} = \alpha^{(ik)} c^a_k$. (4.14)
Keeping in mind eq. (4.12), it is an easy exercise to check that these deformations are reduced, up to full harmonic derivatives in the variation of the analytic superfield Lagrangian $L_{sc}^{(2,2)}$, to particular $N = 4$ SU(2) superconformal transformations of $S_{sc}$ with the parameters

$$(a) \Lambda_j^{(0,0)} = -\alpha; \quad (b) \Lambda_j^{(0,0)} = \alpha^{(ik)} u_i^{(1)} u_k^{(-1)}, \quad \Lambda^{(2,0)} = \alpha^{(ik)} u_i^{(1)} u_k^{(1)}.$$

Hence, because of superconformal invariance of the action,

$$(a) \delta_1 S_{sc} = 0; \quad (b) \delta_2 S_{sc} = 0. \quad \text{(4.15)}$$

From first of these relations one concludes that the action does not depend on the norm of the four-vector $c^{ik}$; from now on, we choose

$$c^2 = 2 \Rightarrow b_2 = \frac{1}{8\kappa^2}, \quad X = c^{(-1,-1)} \hat{q}^{(1,1)}, \quad \text{(4.16)}$$

From the second relation and an analogous relation which comes from considering an SU(2) rotation of $c^{ia}$ in the index $a$ it follows that the action does not depend on the angular part of the four-vector $c^{ia}$ as well. So one can put

$$c^{ia} = \epsilon^{ia}. \quad \text{(4.17)}$$

Though $c^{ia}$ drops out from the action, its presence in $L_{sc}^{(2,2)}$ (even written through $q^{(1,1)}$) is unavoidable. As was noticed in [15], the presence of an arbitrary isotriplet constant in the analytic superfield Lagrangian of the improved $N = 2$ 4D tensor multiplet has a deep topological meaning: this constant parametrizes a Dirac-like string of singularities appearing in the component action when the latter is written through the field strength of notoph [24]. A meaning of the constant $c^{ia}$ is somewhat more obscure. It seems to reflect an ambiguity in resolving the torsion field strength 3-form (which is closed but not exact in the case at hand) through the 2-form potential. Just the latter enters into the component Lagrangian directly following from $L_{sc}^{(2,2)}$. This interpretation is supported by the fact that the explicit $c^{ik}$ dependence in the torsion term of the Lagrangian disappears if one writes this term through the torsion field strength which is well defined globally, rather than through the torsion potential (see the next Subsection).

Keeping in mind eq. (4.16), the final form of the action (4.9) is as follows

$$S_{sc} = -\frac{1}{4\kappa^2} \int \mu^{(-2,-2)} L_{sc}^{(2,2)} \quad \text{(4.18)}$$

4.2 Passing to components. In order to demonstrate that the action (4.18) indeed describes the $N = 4$ SU(2) × U(1) WZNW model, we give here its component form and show that it precisely coincides with the component $N = 4$ SU(2) × U(1) WZNW action [3, 8].
Let us begin with the bosonic part of the action. It is given by a sum of the physical and auxiliary bosonic fields terms defined in eqs. (3.14) - (3.19). In the present case:

\[ S_{sc}^{bos} = \frac{1}{2\kappa^2} \int d^2 z \{ G(\hat{q}) \partial_{++} q^{ia} \partial_{--} q_{ia} + B_{ia \ jb}(\hat{q}) \partial_{++} q^{ia} \partial_{--} q^{jb} + 2G(\hat{q}) (FF + LL) \} \]  

(4.19)

\[ G(\hat{q}) = - \int dudv \frac{\partial^2 L_{sc}^{(2,2)}}{\partial q^{(1,1)} \partial q^{(1,1)}} \big|_{\theta=0} = \int dudv \frac{1 - X}{(1 + X)^3} \]  

(4.20)

\[ B_{ia \ jb}(\hat{q}) = \int dudv \frac{1 - X}{(1 + X)^3} [\epsilon_{ij} v_{(a}^{(1)} v_{b)}^{(-1)} - \epsilon_{ab} u_{(i}^{(1)} u_{j)}^{(-1)}] . \]  

(4.21)

It turns out that all the target geometry quantities present in the Lagrangian (including its fermionic part) are eventually expressed through the single object \( G(\hat{q}) \) (4.20). It has been computed in Appendix. It is a function of the original field \( q^{ia}(z) \) and it contains no explicit dependence on \( c^{ia} \)

\[ G(q) = 2(q^{ia} q_{ia})^{-2} \equiv 2\rho^{-2} . \]  

(4.22)

Parametrizing the \( 4 \times 4 \) physical bosons matrix \( q^{ia}(z) \) as

\[ q^{ia} = \epsilon^{u(z)} \tilde{q}^{ia}(z) , \]  

(4.23)

where \( \tilde{q}^{ia}(z) \) is an unitary \( SU(2) \) matrix,

\[ \tilde{q}^{ia} \tilde{q}^j_a = \epsilon^{ji} , \tilde{q}^{ia} \tilde{q}^b_i = \epsilon^{ba} , \]  

(4.24)

one finds that

\[ G(q) = e^{-2u} . \]  

(4.25)

So, the metric term in (4.19) is reduced to a sum of the free Lagrangian of the field \( u(z) \) and the Lagrangian of the \( SU(2) \) principal sigma model

\[ G(q) \partial_{++} q^{ia} \partial_{--} q_{ia} = 2\partial_{++} u \partial_{--} u + \partial_{++} \tilde{q}^{ia} \partial_{--} \tilde{q}_{ia} \]  

The last, torsion term in (4.19) needs a bit more careful treatment. It is difficult to directly integrate there over harmonic variables, the reason is that the method of doing such integration which we applied in [15] and in the Appendix requires a manifest \( SU(2) \) covariance of the relevant harmonic integral, while the \( SU(2) \) variation of the integrand in the torsion term vanishes only modulo full \( z \) derivatives. To get round this difficulty, one can do as in Subsect.4.2 and rewrite the torsion term via the field strength of the potential \( B_{ia \ jb} \). The totally antisymmetric (with respect to permutations of pairs of the indices \( ia , jb , ... \) torsion field strength \( H_{ia \ jb \ kd} \) defined by the general formula (3.20) in the given specific case is reduced to the simple expression

\[ H_{ia \ jb \ kd} = \frac{\partial G(q)}{\partial q^{kd}} \epsilon_{ab} \epsilon_{ik} - \frac{\partial G(q)}{\partial q^{kb}} \epsilon_{ad} \epsilon_{ij} , \]  

(4.26)

or, with taking account of eq. (4.22),

\[ H_{ia \ jb \ kd} = 4\rho^{-4} (q_{kb} \epsilon_{ad} \epsilon_{ij} - q_{jd} \epsilon_{ab} \epsilon_{ik}) . \]  

(4.27)
After substituting this expression into the torsion term
\[ B_{ia,jb} \partial_{++} q^{ia} \partial_{--} q^{jb} = \int_0^1 dt \ H_{ia,jb,kd} \ \partial_t q^{ia} \partial_{++} q^{jb} \partial_{--} q^{kd} \] (4.28)
and passing to the parametrization \([4.23]\), the r.h.s. of \([4.28]\) takes the form
\[ \int_0^1 dt \ \partial_t \tilde{q}_{ia} \ \tilde{q}_{jb} \ \partial_{++} \tilde{q}^{ib} \partial_{--} \tilde{q}^{ja} - \partial_{++} \tilde{q}^{ja} \partial_{--} \tilde{q}^{ib} \] (4.29)
which is the standard SU(2) WZNW term.

Note that in this form WZNW term is well-defined globally, it is also immediately seen from the representation \([4.26]\), \([4.27]\) that the field strength \( H_{ia,jb,kd} \) transforms as a tensor under the rigid SU(2)\(_c\). As is explained in Appendix, this is the main reason why this object depends only on unshifted \( q^{ja} \) and reveals no dependence on the constant \( c^{ja} \). On the other hand, the torsion potential \( B_{ia,jb} \) directly appearing in the component action is defined only locally and it possesses no tensor properties under SU(2)\(_c\). By this reason it cannot be expressed only in terms of \( q^{(1,1)} \); one can check that the potential inevitably includes a dependence on \( c^{ja} \) which disappears only after performing \( z \) integration. So the presence of this constant in \( B_{ia,jb} \) reflects the global uncertainty in resolving \( H_{ia,jb,kd} \) through \( B_{ia,jb}\).

Summing up the above contributions, one may write the final expression for the bosonic part of the action \([4.18]\)
\[ S_{bos}^{sc} = \frac{1}{k^2} \int d^2 z \ \{ \partial_{++} u \partial_{--} u + \frac{1}{2} \partial_{++} \tilde{q}^{ia} \partial_{--} \tilde{q}_{ia} + \frac{1}{2} \int_0^1 dt \ \partial_t \tilde{q}_{ia} \ \tilde{q}_{jb} \ \partial_{++} \tilde{q}^{ib} \partial_{--} \tilde{q}^{ja} - \partial_{++} \tilde{q}^{ja} \partial_{--} \tilde{q}^{ib} \} + e^{-2u} (FF + LL) \] (4.30)

Let us now apply to the fermionic sector. The fermionic part of the component action consists of three pieces
\[ S_{ferm}^{sc} = S_{sf} + S_{auxf} + S_{kinf} \]
which correspond, respectively, to the term quartic in fermionic fields, a term involving auxiliary fields and the kinetic term of the component Lagrangian. Explicitly, these are as follows
\[ S_{sf} = \frac{4}{k^2} \int d^2 z \ \frac{\partial^2 G(q)}{\partial q^{ja} \partial q^{jb}} \ \psi^{(a}_{\bar{\psi}^b}_{\bar{\psi}^b} (i-j) \] (4.31)
\[ S_{auxf} = \frac{2}{k^2} \int d^2 z \ \frac{\partial G(q)}{\partial q^{ja}} \ \left\{ F\bar{\psi}^a_{\bar{\psi}^b} \chi^i_{\chi^i} - \bar{F}\psi^a_{\psi^b} \chi^i_{\chi^i} + \bar{L}\psi^a_{\psi^b} \chi^i_{\chi^i} + L\bar{\psi}^a_{\bar{\psi}^b} \chi^i_{\chi^i} \right\} \] (4.32)
\[ S_{kinf} = \frac{1}{k^2} \int d^2 z \ \{ iG(q) \ \partial_{++} \chi^i_{\chi^i} - \partial_{--} \chi^i_{\chi^i} + \partial_{--} \bar{\psi}^a_{\bar{\psi}^b} \psi^a_{\psi^b} + \partial_{++} \psi^a_{\psi^b} \} \] (4.33)

\[ \text{The off-shell action of } N = 4 \ SU(2) \times U(1) \text{ WZNW model in the conventional } (4,4) \ 2D \text{ superspace from the beginning contains an integral over extra } t \ [4.3], \text{ so after passing to components it yields just the } H \text{ form of the torsion term. On the other hand, while the same model is formulated in terms of } (2,2) \ 2D \text{ superfields (chiral and twisted chiral ones) } [4.3], \text{ the torsion term appears in its } B \text{ form, like in the } (4,4) \text{ harmonic superspace formulation.} \]
Using the explicit expressions (4.22), (4.25) for $G(q)$, one observes:

(i). After the field redefinition

\[ F = F' - 2e^{-u} \tilde{q}_{ia} \psi^i_a \chi_-, \]
\[ L = L' + 2e^{-u} \tilde{q}_{ia} \psi^i_a \tilde{\chi}_- \]  \hspace{1cm} (4.34)

the sum of $S_4$ and $S_{aux}$ is entirely cancelled by the contribution coming from $S_{bos}^{aux}$. Thus the off-shell superconformally invariant $q^{(1,1)}$ action does not contain 4-fermionic term which in general is present in the generic action (3.10). The full auxiliary fields part of the action takes the simple form

\[ S_{aux}^{sc} = \frac{1}{\kappa^2} \int d^2 z \ e^{-2u} (F' F' + L' L') . \]  \hspace{1cm} (4.35)

(ii). Being written through redefined fermionic fields

\[ \chi^a_- = e^{-u} \tilde{q}^a_i \chi^i_-, \ \tilde{\chi}^a_- = e^{-u} \tilde{q}^a_i \tilde{\chi}^i_-, \ 
\psi^i_a = e^{-u} \tilde{q}^a_i \psi^i_+, \ \tilde{\psi}^i_+ = e^{-u} \tilde{q}^a_i \tilde{\psi}^i_+, \]  \hspace{1cm} (4.36)

$S_{kinf}$ is reduced to a sum of the free fermionic terms

\[ S_{kinf} = \frac{1}{\kappa^2} \int d^2 z \ i \{ \partial_{++} \chi^- \chi^- - \partial_{++} \tilde{\chi}^- \tilde{\chi}^- + \partial_{--} \psi^i_+ \tilde{\psi}^i_+ - \partial_{--} \tilde{\psi}^i_+ \psi^i_+ ) . \]  \hspace{1cm} (4.37)

Thus, on shell $S_{sc}$ is reduced to a sum of the free action for the scalar field $u(z)$, $SU(2)$ WZNW sigma model action for the field $\tilde{q}^a_i$ and the free actions for fermionic fields $\chi^a_-$ and $\psi^i_+$. This is just the action of $N = 4 \ SU(2) \times U(1)$ WZNW sigma model [3, 4, 19]. The off-shell component action (with the redefinitions (4.34), (4.36)) is also identical to the one given in [4].

Finally, it is worth clarifying the term “$SU(2) \times U(1)$ WZNW sigma model” in the present context. The field $\tilde{q}^a_i$ contains just three independent parameters-fields and so parametrizes the coset $SU(2)_{cl} \times SU(2)_{cr}/SU(2)_{diag}$, where $SU(2)_{cl}$ and $SU(2)_{cr}$ are rigid $SU(2)$ subgroups of two commuting left and right light-cone branches of the full $N = 4 \ SU(2)$ superconformal group. What is the group-theoretical meaning of the scalar field $u(z)$? Let us remind that the entire symmetry of the considered model is the “large” $N = 4 \ SO(4) \times U(1)$ superconformal symmetry which is a closure of two $N = 4 \ SU(2)$ ones defined in Subsect.2.4. Two local supersymmetries present in $N = 4 \ SU(2)$ superconformal groups I and II contain in their commutator $U(1)$ Kac-Moody transformations which are realized in the given model as shifts of the field $u(z)$ by a sum of two arbitrary holomorphic functions of, respectively, $z^{++}$ and $z^{--}$. So, one is led to identify $u(z)$ with a parameter of the coset $U(1)_L \times U(1)_R/U(1)_{diag}$, and this explains the term “$SU(2) \times U(1)$ WZNW model”. Let us stress that in the present case the internal symmetry does not commute with supersymmetry, in contrast, e.g., to $N = 1$ supersymmetric WZNW models [25]. Instead, it constitutes a nontrivial part of the underlying superconformal symmetry.
5 Massive deformations of (4,4) sigma models

A standard way to generate potential (in particular, mass) terms in \( N = 4 \) \( 2D \) \( (N = 2 \ 4D) \) sigma model actions is to modify the supersymmetry algebra by central charges and to identify the latter with some isometries of the original action [26, 27]. As was noticed in [3] and, recently, in [28], in the \( (4,4) \) sigma models there exists a possibility to add such terms without changing the supersymmetry algebra. The explicitly elaborated example is the \( N = 4 \) \( SU(2) \) WZNW - Liouville system of refs. [1, 3, 9] which is a superconformally invariant deformation of the \( N = 4 \) \( SU(2) \times U(1) \) WZNW model discussed in the previous Section. Here we reproduce this example within the harmonic superspace formalism and comment on a more general situation when the generic action (3.10) is subjected to a deformation of this kind. It turns out that the deforming term in the superfield \( q^{(1,1)} \) action is defined in a unique way. As a result, potential terms in the component action are almost completely specified by the original target space bosonic metric.

5.1 Deformations of the general \( q^{(1,1)} \) action. Keeping in mind that the superfield \( q^{(1,1)} \) and the integration measure \( \mu^{(-2,-2)} \) are dimensionless, the only way to construct a manifestly analytic massive term for \( q^{(1,1)} \) is to allow for explicit \( \theta \)'s in the action.

The simplest term of this kind reads

\[
S_m = m \int \mu^{(-2,-2)} \theta^{(1,0)} \mathbb{J} q^{(1,1)} M ; \quad [m] = cm^{-1} ,
\]

where \( C_M^{\underline{1}\underline{2}} \) are arbitrary constants (subject to the appropriate reality conditions), and we once again resorted to the quartet notation (see eqs. (2.39)). It immediately follows that, despite the presence of explicit \( \theta \)'s, (5.1) is invariant under rigid \( (4,4) \) supersymmetry: one represents the supertranslation of, say, \( \theta^{(1,0)} \) as

\[
\delta_{\text{SUSY}} \theta^{(1,0)} = \epsilon^{\underline{k}\underline{l}} u^{(1)}_{\underline{k}} = D^{(2,0)} \epsilon^{\underline{k}\underline{l}} u^{(-1)}_{\underline{k}} ,
\]

integrate by part with respect to \( D^{(2,0)} \) and make use of the defining constraints (3.13).

It is easy to argue that this linear term is the only possible supersymmetric massive term of \( q^{(1,1)} M \). Indeed, adding of any higher degree monomial of \( q^{(1,1)} M \) to (5.1) would require inserting explicit harmonics \( u^{(-1)}_{\underline{i}} , v^{(-1)}_{\underline{a}} \) to ensure the balance of \( U(1) \) charges. After taking off the harmonic derivatives \( D^{(2,0)} , D^{(0,2)} \) from the supervariations of the analytic \( \theta \)'s and integrating by part, these derivatives would hit not only the superfields \( q^{(1,1)} M \), but also the harmonics \( u^{(-1)}_{\underline{i}} , v^{(-1)}_{\underline{a}} \), in the latter case with a non-vanishing result.

It is also straightforward to check that (5.1) respects invariance not only under rigid \( (4,4) \) supersymmetry, but also under the whole \( N = 4 \) \( SU(2) \) superconformal group defined by eqs. (2.29) - (2.35), (3.6). At the same time, it breaks invariance under the second \( N = 4 \) \( SU(2) \) superconformal group and, hence, the “large” \( N = 4 \) \( SO(4) \times U(1) \) superconformal group. The only additional manifest invariance one can achieve is the diagonal \( SU(2) \) in the product of two rigid \( SU(2) \)'s acting on the indices \( \underline{i} \) and \( \underline{b} \) of Grassmann coordinates in (5.1), provided \( C_{\underline{i}\underline{b} M} \sim \epsilon_{\underline{i}\underline{b}} \).

Note that (5.1) can be rewritten in the standard \( (4,4) \) superspace as a Fayet-Iliopoulos term of the unconstrained prepotential solving the irreducibility conditions (3.1) [4]. In
such a form the mass term does not involve explicit $\theta$’s and so is manifestly supersymmetric.

Let us examine how the adding of (5.1) to the generic action (3.10) influences the component structure of the latter. After integrating over Grassmann and harmonic variables, (5.1) reads

$$S_m = -2m \int d^2 z \ F_{ab}^M C_{ab}^M ,$$

where we made use of the matrix notation (3.8) for auxiliary fields. After eliminating the auxiliary fields in the sum $S_q + S_m$, the physical component action acquires new terms: certain Yukawa type couplings between fermions and the field $q^{iaM}$ as well as a potential term of the latter. All these new terms are expressed through the “metric” $G_{MN}(q)$ defined by eq. (3.16) and its inverse $G^{MN}(q)$ ($G^{MN}G_{NK} = \delta^M_K$). We give here explicitly only the potential term of $q^{(1,1)M}$

$$S_{pot}^q = \frac{m^2}{2} \int d^2 z \ G^{MN}(q) \ (C_{\underline{ab}} C_{\underline{ab}}^M) .$$

5.2 $N = 4$ SU(2) WZNW - Liouville model. As an instructive example we will discuss a massive deformation of the superconformal action (1.3).

As was already noticed, the mass term (5.1) preserves the $N = 4$ SU(2) superconformal symmetry I, so the model described by the action

$$S_{sc}^m = -\frac{1}{4\kappa^2} \int \mu^{(-2,-2)} \{ \hat{q}^{(1,1)} \hat{q}^{(1,1)} \left( \frac{1}{(1+X)X} - \frac{\ln(1+X)}{X^2} \right) + 2m \theta^{(1,0)} \hat{q}^{(0,1)} \hat{q}^{(1,1)} \}$$

is a superconformally invariant deformation of $N = 4$ SU(2) $\times$ U(1) WZNW model. Actually, the mass parameter in (5.4) is inessential as it can be fixed at any non-zero value by rescaling $q^{(1,1)}$ as

$$q^{(1,1)} \Rightarrow \gamma q^{(1,1)} .$$

The sigma model part of the action (5.4) is invariant under this rescaling because the relevant variation looks precisely the same as the rigid dilatation one. Also, using the invariance of the sigma model part under the group $SO(4) \sim SU(2) \times SU(2)$ acting on the underlined indices (these $SU(2)$’s enter into the left and right branches of superconformal group II) and absorbing the norm of the four-vector $C_{\underline{ab}}$ into a renormalization of the parameter $m$, one may rotate this constant vector into the form

$$C_{\underline{ab}} = \epsilon_{\underline{ab}} ,$$

which explicitly shows that (5.4) possesses an extra symmetry with respect to the diagonal $SU(2)$ from the $SO(4)$ just mentioned.

With taking account of the last remark, in the component language the mass term in (5.4) reads

$$2m \int \mu^{(-2,-2)} \theta^{(1,0)} \hat{q}^{(0,1)} \hat{q}^{(1,1)} = 4m \int d^2 z \ F_{\underline{ab}} \epsilon_{\underline{ab}} .$$
After eliminating auxiliary fields it gives rise to the following physical component action of the deformed $N = 4 \; SU(2) \times U(1)$ WZNW sigma model

$$
S_{sc(m)} = S_{sc}^{bos}(F = L = 0) + S_{kinf} + S_m ,
$$

(5.6)

where $S_{sc}^{bos}$ and $S_{kinf}$ are given by eqs. (4.30) and (4.37) and

$$
S_m = \frac{1}{\kappa^2} \int d^2 z \left\{ m^2 e^{2u} + 2m e^{-u} \tilde{q}_{ia} \left( \tilde{\psi}^a_i \chi_i - \psi^a_i \bar{\chi}_i \right) \right\} .
$$

(5.7)

The action (5.6) is just the on-shell action of $N = 4 \; SU(2)$ WZNW - Liouville system [1, 3, 9].

6 (4,4) duality transformation

Up to now we dealt with the constrained analytic superfields $q^{(1,1)}$. However, for several reasons it would be advantageous to have unconstrained superfield formulations of the models presented in the previous Sections. Firstly, this seems necessary for the complete understanding of the target space geometry hidden in the sigma model actions (3.10), (3.13). Secondly, such formulations would allow to straightforwardly deduce the relevant superfield equations of motion. To know such equations is important, e.g., while analyzing the integrability properties of given model (the existence of a zero curvature representation, an infinite number of superfield conserved quantities, etc.). Thirdly, unconstrained formulations could prompt how to extend the above formalism to accommodate more general class of $(4,4)$ sigma models with non-commuting left and right complex structures.

One way to achieve an unconstrained superfield formulation is to express the action through prepotentials solving the harmonic conditions (3.13) [21, 9]. Unfortunately, when doing so, one loses the privilege of manifest harmonic analyticity as the relevant prepotential is a general $(4,4)$ superfield. Besides, it is dimensionful and therefore can hardly be identified with any object of the target space geometry (e.g., a coordinate on the target manifold). Therefore, like in the $N = 2 \; 4D$ case [29], we prefer another way which is based on implementing (3.13) in the action with the help of unconstrained analytic Lagrange multipliers. After eliminating the original $q^{(1,1)}$ by its algebraic equation of motion one gets a new, dual representation of the action via the Lagrange multipliers. Below we present dual forms of the previously constructed $q^{(1,1)}$ actions and discuss some peculiarities of them.

6.1 Transforming the general $q^{(1,1)}$ action. Let us consider the following modification of the $q^{(1,1)}$ action (3.10)

$$
S_{q,\omega} = \int \mu^{-2,-2} \left\{ \mathcal{L}^{(2,2)}(q^{(1,1)} M, u, v) + \omega^{(-1,1)} M D^{(2,0)} q^{(1,1)} M + \omega^{(1,-1)} M D^{(0,2)} q^{(1,1)} M \right\} .
$$

(6.1)

The analytic superfields $q^{(1,1)} M$, $\omega^{(1,-1)} M$, $\omega^{(-1,1)} M$ are unconstrained and one can vary them to get the superfield equations of motion. Varying $\omega^{(1,-1)} M$, $\omega^{(-1,1)} M$ yields the constraints (3.13) and we recover the original action (3.10). Alternatively, one can vary
So one may wonder how the equivalence to the original $\omega^{(1,1)}$ in the harmonic expansions of superfields. In particular, there are two sets of the bosonic fields appearing as first terms component level.

[\text{unconstrained} \text{commuting left and right complex structures via auxiliary fields} \cite{14}]. Thus, in the case at hand the physical component action for $4^n$ bosons and $4^n$ fermions is restored only after eliminating an infinite tower of auxiliary fields coming from the double harmonic expansion of superfields $\omega^{(1,-1)}(\zeta, u, v), \omega^{(-1,1)}(\zeta, u, v)$. We postpone the discussion of the component content of the dual (4,4) sigma model actions to further publications. Here we will briefly mention only some salient features of the dual formulation.

The action (6.4) provides a new off-shell formulation of (4,4) sigma models with commuting left and right complex structures via \textit{unconstrained} analytic (4,4) superfields. The most characteristic feature of such formulations is the presence of infinite number of auxiliary fields \cite{14}. Thus, in the case at hand the physical component action for $4^n$ bosons and $4^n$ fermions is restored only after eliminating an infinite tower of auxiliary fields coming from the double harmonic expansion of superfields $\omega^{(1,-1)}(\zeta, u, v), \omega^{(-1,1)}(\zeta, u, v).

We postpone the discussion of the component content of the dual (4,4) sigma model actions to further publications. Here we will briefly mention only some salient features of the dual formulation.

The action (6.4) has a clear analog in the $N = 2$ 4D harmonic analytic superspace: it is the $\omega$ hypermultiplet action dual to the action of tensor $N = 2$ 4D multiplet \cite{15}. An essentially new feature of the (4,4) 2D action is the presence of two independent sets of Lagrange multipliers $\omega^{(1,-1)}M, \omega^{(-1,1)}M$. An inspection of their field content shows that they contain twice as many physical fields as compared with the original constrained $q^{(1,1)}$ superfields. In particular, there are two sets of the bosonic fields appearing as first terms in the harmonic expansions of $\omega^{(1,-1)}M, \omega^{(-1,1)}M$

$$\omega^{(1,-1)}M(\zeta, u, v) = \omega^{ia}_0 u^{(1)}_i v^{(-1)}_a + \ldots, \omega^{(-1,1)}M(\zeta, u, v) = \omega^{[ia}_0 u^{(-1)}_i v^{(1)}_a + \ldots.$$  \hspace{1cm} (6.5)

So one may wonder how the equivalence to the original $q^{(1,1)}$ action is achieved at the component level.

The answer proves very simple: the actions (6.1) and (6.4) are invariant under the gauge transformations

$$\delta \omega^{(1,-1)}M = D^{(2,0)}\sigma^{(-1,1)}M, \delta \omega^{(-1,1)}M = -D^{(0,2)}\sigma^{(-1,1)}M,$$

with $\sigma^{(-1,1)}M = \sigma^{(-1,1)}M(\zeta, u, v)$ being arbitrary analytic functions, and this gauge freedom takes away just half of the lowest superisospin multiplets in the superfields $\omega^{(1,-1)}M, \omega^{(-1,1)}M$, thus restoring the correct physical field content. For instance, the first components of these superfields are transformed as

$$\delta \omega^{(1,-1)}M(z) = \partial^{(2,0)}\sigma^{(-1,1)}M(z), \delta \omega^{(-1,1)}M(z) = -\partial^{(0,2)}\sigma^{(-1,1)}M(z),$$  \hspace{1cm} (6.7)
and one may fix the gauge so as to entirely remove one set of these fields (other gauge choices are possible as well). Thus, in contrast to the \( q^{(1,1)} \) superfield formulation, where the necessary set of the physical fields is ensured by imposing the harmonic constraints on \( q^{(1,1)} \), the same goal in the dual formulation is achieved thanks to the gauge freedom (6.6) (and after eliminating an infinite set of auxiliary fields).

Note that (6.6) is a generalization of the abelian shift isometries of the \( \omega \) hypermultiplet Lagrange multipliers of the dual \( N = 2 \) 4D actions \[29\]. The origin of these isometries lies in the fact that the scalar components undergoing constant shifts are dual to the notoph field strengths which are present in the initial formulation via the tensor multiplet superfields, and therefore these components always enter the action through their \( x \) derivatives. However, such an interpretation seems not possible for the transformations (6.6), because there is no analog of the notoph field strength in the \( \theta \) expansion of \( q^{(1,1)} \) (3.5). Both in the \( q \) and \( \omega \) languages the bosonic degrees of freedom in the considered case are represented by the canonical scalar fields. So it remains a mystery what is the geometric origin of the gauge transformations (6.6) (see, however, the next comment).

One more interesting feature of the dual formulation is revealed while applying the (4, 4) duality transformation to the mass term deformed action

\[ S_q + S_m, \]

where \( S_m \) is given by eq. (5.4). The algebraic equation (6.2) is now modified by \( \theta \) dependent terms

\[ \frac{\partial \mathcal{L}^{(2,2)}}{\partial q^{(1,1)}} M = A^{(1,1)} M - m \theta^{(1,0)} \theta^{(0,1)} b \mathcal{C}_M. \]  

(6.8)

Accordingly, \( \theta \) terms appear in the corresponding dual action. It is seen from (6.8) (or directly from considering (6.1) with the mass term added) that in this dual formulation (4, 4) 2D supersymmetry is realized in a non-standard way

\[ \delta_{\text{SUSY}} \omega^{(1,-1)} M = -m \epsilon^{a b}_R v_a^{(-1)} \theta^{(1,0)} b \mathcal{C}_M, \]

\[ \delta_{\text{SUSY}} \omega^{(-1,1)} M = m \epsilon^{i j}_L u_i^{(-1)} \theta^{(0,1)} b \mathcal{C}_M. \]  

(6.9)

Here \( \epsilon^{i j}_L, \epsilon^{a b}_R \) are the constant parameters of the left and right supertranslations in the quartet notation. To reveal the meaning of this modification, let us consider the Lie bracket of the left and right supertranslations. While in the original \( q \) formulation they commute irrespective of the presence or absence of the mass term in the action, in the realization on the superfields \( \omega \) their bracket surprisingly turns out non-vanishing. Namely, in the obvious notation,

\[ [\delta_L, \delta_R] \omega^{(1,-1)} = m \epsilon^{i j}_L \epsilon^{a b}_R \mathcal{C}_M u_i^{(-1)} v_a^{(-1)} \]

(6.10)

But this is just a subclass of transformations (6.6) with \( z \) independent parameters homogeneous in harmonics

\[ \sigma_0^{(-1,-1)} M = m \epsilon^{i j}_L \epsilon^{a b}_R \mathcal{C}_M u_i^{(-1)} v_b^{(-1)}. \]  

(6.11)

Thus the realization (6.9) corresponds to the following extension of the standard 2D (4, 4) SUSY algebra

\[ \{ Q^{ii}_{+}, Q^{ja}_{-} \} \sim m \mathcal{Z}_{-}^{i a}, \]  

(6.12)
where the “semi-central” charge generator $Z^{ii\alpha\beta}$ is realized by shifts (6.6), (6.11), with $\sigma^{ij}_{\alpha\beta}$ being the associated transformation parameters.

Note that the effect of activating central charge operators in the SUSY algebras by duality transformations is well known (see, e.g., ref. [30] and, in context of $N=2$ 4D harmonic superspace, ref. [29]). For instance, one may introduce a mass term similar to (5.1) in the general analytic superfield action of tensor $N=2$ 4D multiplets [29] with preserving the original realization of 4D supersymmetry. After passing to the dual $\omega$ hypermultiplet action one then finds that on $\omega$ superfields the supersymmetry algebra is realized with a non-zero central charge proportional to an abelian shift isometry generator.

An unusual feature of the case under consideration is that the operator $Z^{ii\alpha\beta}$ transforms in general as a non-trivial tensor of the automorphism group (hence the term “semi-central”), while in the $N=2$ 4D example just mentioned a similar operator is a pure singlet of the automorphism $SU(2)$. Note that the appearance of “quaternionic” central charges in certain $(4,4)$ supersymmetric models with non-zero scalar fields potentials has been already observed in [31].

The relationship with central charges could clarify the geometric meaning of the strange gauge invariance (6.6): perhaps, it can be viewed as a gauging of the transformations (6.11).

In the rest of this Section we will present the dual actions for $N=4$ $SU(2) \times U(1)$ WZNW sigma model and its Liouville extension.

### 6.2 Dual form of N=4 WZNW and WZNW - Liouville actions.

The extended action (6.1) specialized to $N=4$ $SU(2) \times U(1)$ WZNW model can be chosen as

$$S_{sc}(q,\omega) = -\frac{1}{4\kappa^2} \int \mu^{(-2,-2)} \left\{ L^{(2,2)}_{sc} + \frac{1}{2} \omega^{(-1,1)} D^{(2,0)} q^{(1,1)} + \frac{1}{2} \omega^{(1,-1)} D^{(0,2)} q^{(1,1)} \right\} .$$

(6.13)

By varying (6.1) with respect to the involved superfields we obtain the equations of motion of $N=4$ $SU(2) \times U(1)$ WZNW model in the analytic superspace

$$q^{(1,1)} = \frac{1}{(1+X)^2}, \quad D^{(2,0)} q^{(1,1)} = D^{(0,2)} q^{(1,1)} = 0 .$$

(6.14)

First of them is algebraic and it serves to express $q^{(1,1)}$ via $A^{(1,1)}$

$$q^{(1,1)} = -2 \frac{A^{(1,1)}}{(1 + \sqrt{1 + 2A})^2}, \quad A \equiv e^{(-1,-1)} A^{(1,1)} .$$

(6.15)

The second pair of equations in (6.14), being kinematical constraints in the original formulation, becomes the dynamical equations in the dual description; substitution of the expression (6.15) into them yields a closed set of the equations of motion for the superfields $\omega^{(1,-1)}, \omega^{(-1,1)}$. They can be independently deduced from the action obtained by substituting (6.13) into (6.13)

$$S_{sc}(\omega) = -\frac{1}{4\kappa^2} \int \mu^{(-2,-2)} \left( \frac{\tilde{A}^{(1,1)}}{A} \right)^2 \left( \tilde{A}^2 - \tilde{A} + \ln(1 + \tilde{A}) \right) ,$$

(6.16)
with
\[ \tilde{A}^{(1,1)} = \frac{A^{(1,1)}}{1 + \sqrt{1 + 2A}}, \quad \tilde{A} \equiv c^{(-1,-1)} A^{(1,1)}. \] (6.17)

The action (5.16) provides a dual description of $N = 4$ $SU(2) \times U(1)$ WZNW sigma model in terms of unconstrained analytic $(4,4)$ superfields $\omega^{(1,-1)}, \omega^{(-1,1)}$. Recall that they are subjected to the gauge freedom (5.6), which ensures the correct physical fields content of the action.

With making use of the explicit form of the superconformal variation of $L^{(2,2)}_{sc}$ (4.11), it is easy to see that the action (6.13) is still invariant under the transformations (4.4), provided Lagrange multipliers transform as
\[ \delta_I \omega^{(1,-1)} = -\Lambda_I^{(0,0)} \left( \omega^{(1,-1)} + 2c^{(1,-1)} \frac{1}{(1 + X)^3} \right), \]
\[ \delta_I \omega^{(-1,1)} = -\Lambda_I^{(0,0)} \left( \omega^{(-1,1)} - 2c^{(-1,1)} \frac{1 - X}{(1 + X)^3} \right). \] (6.18)

Transformations from the right light-cone branch are given by the same formulas, with the $U(1)$ charges interchanged as $1 \leftrightarrow -1$. Substituting in (6.18) the expression for $q^{(1,1)}$ (6.15), one may rewrite (6.18) entirely in terms of the $\omega$ fields. The resulting transformations realize the superconformal group I in the dual $\omega$ formulation of $N = 4$ $SU(2) \times U(1)$ WZNW model. Of course, the above actions are invariant under the superconformal group II, with respect to which $\omega^{(1,-1)}, \omega^{(-1,1)}$ are scalars like $q^{(1,1)}$.

Finally, let us obtain the dual form of WZNW - Liouville action (5.4). The relevant extended action is given by a sum of (5.4) and the same Lagrange multipliers term as in (6.13). Then, passing to the dual formulation is accomplished by the following replacement in the above equations
\[ \tilde{A}^{(1,1)} \Rightarrow A^{(1,1)} - 4m (\theta^{(1,0)} \bar{\theta}^{(0,1)} + \theta^{(0,1)} \bar{\theta}^{(1,0)}). \] (6.19)

The modified expression for $q^{(1,1)}$ reads
\[ \hat{q}^{(1,1)} = -2 \frac{A^{(1,1)}}{(1 + \sqrt{1 + 2A})^2} + 4m \frac{1}{\sqrt{1 + 2A}} (\theta^{(1,0)} \bar{\theta}^{(0,1)} + \theta^{(0,1)} \bar{\theta}^{(1,0)}) \]
\[ -16m^2 \frac{c^{(-1,-1)}}{(1 + 2A)^{3/2}} \theta^{(1,0)} \bar{\theta}^{(0,1)} \theta^{(0,1)} \bar{\theta}^{(0,1)}. \] (6.20)

Accordingly, the dual $N = 4$ WZNW - Liouville action is given by
\[ S_{(sc)m}^{\text{dual}} = -\frac{1}{4\kappa^2} \int \mu^{-2} \left\{ \left( \frac{A^{(1,1)}}{\bar{A}^{(1,1)}} \right)^2 \left( \tilde{A}^2 - \tilde{A} + \ln(1 + \tilde{A}) \right) \right. \]
\[ +2m \frac{1}{1 + \tilde{A}} \tilde{A}^{(1,1)} (\theta^{(1,0)} \bar{\theta}^{(0,1)} + \theta^{(0,1)} \bar{\theta}^{(1,0)}) \]
\[ -8m^2 \frac{1}{(1 + 2\tilde{A})(1 + \tilde{A})} \theta^{(1,0)} \bar{\theta}^{(0,1)} \theta^{(0,1)} \bar{\theta}^{(0,1)} \} \]. (6.21)
This action is invariant under the appropriate semi-central charge modification of superconformal transformations \(7.18\)

\[
\delta_{I}^{\text{mod}} \omega^{(1,-1)} = \delta_{I} \omega^{(1,-1)}, \quad \delta_{I}^{\text{mod}} \omega^{(-1,1)} = \delta_{I} \omega^{(-1,1)} + i m(\frac{\partial a}{\partial \theta^{(1,0)}} \bar{\theta}^{(0,1)} + \frac{\partial a}{\partial \theta^{(1,0)}} \theta^{(0,1)}) \quad (6.22)
\]

(and analogously for transformations from the right light-cone branch). Note in this connection that the substitution (6.19) can be interpreted as the following central-charge modification of the harmonic derivatives \(D^{(2,0)}, D^{(0,2)}\) in the \(\omega\) superfield strength \(A^{(1,1)}\)

\[

\begin{align*}
D^{(2,0)} & \Rightarrow D^{(2,0)} - 2m(\theta^{(1,0)} \bar{\theta}^{(0,1)} + \theta^{(0,1)} \bar{\theta}^{(1,0)}) Z^{(1,-1)} \\
D^{(0,2)} & \Rightarrow D^{(0,2)} + 2m(\theta^{(1,0)} \bar{\theta}^{(0,1)} + \theta^{(0,1)} \bar{\theta}^{(1,0)}) Z^{(-1,1)}, \quad (6.23)
\end{align*}
\]

where the generators \(Z^{(1,-1)}, Z^{(-1,1)}\) act on the \(\omega\) superfields as shifts

\[
Z^{(1,-1)} \omega^{(-1,1)} = 1, \quad Z^{(-1,1)} \omega^{(-1,1)} = -1, \quad (6.24)
\]

and are the appropriate harmonic projections of the operator appearing in (6.12)

\[
Z^{(\pm 1, \mp 1)} \sim Z^{ia_{\pm} A_{i a_{\pm}}^{(\pm 1)} b_{i a_{\mp}}^{\mp 1}}.
\]

We end with two comments.

First of all, we stress that the \((4,4)\) duality transformation discussed in this Section is a natural generalization of the analogous transformation proposed in [29] in the framework of \(N = 2\) 4D harmonic superspace.

The latter transformation has been used to give a constructive proof of the statement that all self-interactions of the constrained matter \(N = 2\) multiplets are equivalent to some particular classes of the general self-interaction of the universal unconstrained \(N = 2\) multiplet, the analytic \(q_i^+\) hypermultiplet [14, 29, 13]. Representing \(q_i^+\) as a pair of analytic superfields, \(q_i^+ \propto \{L^{++}, \omega\}\), and eliminating \(L^{++}\) by its algebraic equation of motion, the general \(q^+\) action can also be written as a general action of self-interacting \(\omega\) hypermultiplets [24]. These general actions are characterized by lacking of any isometries, while this is not the case for their particular cases corresponding to the constrained matter \(N = 2\) multiplets.

In our case the direct analog of the \(N = 2\) \(\omega\) hypermultiplet is the pair of unconstrained analytic superfields \(\omega^{(1,-1)}, \omega^{(-1,1)}\) and one of the characteristic features of the dual action (3.4) is the gauge invariance (3.6) which serves to remove the doubling of physical degrees of freedom. This invariance substitutes the isometries of the dual \(N = 2\) 4D \(\omega\) hypermultiplet actions. Then, by analogy, one might conjecture that the most general class of \((4,4)\) 2D sigma models (including those with non-commuting left and right complex structures) is described by the general \(\omega^{(1,-1)}, \omega^{(-1,1)}\) action respecting no gauge invariance (3.6). Such action definitely cannot be reformulated in terms of constrained \(q^{(1,1)}\) superfields and so cannot be written through pairs of chiral and twisted

\[5\text{It seems that in the present case the notion of } q^+ \text{ hypermultiplet is not so useful as in the } N = 2 \ 4D \text{ case because in order to accommodate, e.g., the triple of superfields } q^{(1,1)}, \omega^{(1,-1)}, \omega^{(-1,1)} \text{ (the } (4,4) \text{ analog of the pair } L^{++}, \omega) \text{ one needs two such hypermultiplets related by some algebraic constraint.}\]
chiral $(2,2)$ superfields. An alternative, perhaps more attractive option could be to somehow non-abelize the gauge freedom, still avoiding the doubling of physical degrees of freedom. We hope to analyze these possibilities in further publications.

Second remark concerns the relationship with the duality transformation elaborated in ref. for $N = 4$ $SU(2) \times U(1)$ WZNW model in the formulation through chiral and twisted chiral $(2,2)$ superfields. This kind of duality transformation replaces the twisted multiplet by a chiral one and so brings the model into the torsionless form of $(2,2)$ Kähler sigma model for two chiral superfields with a specific Kähler potential. Though we do not know as yet the precise relation of this transformation to ours, the basic difference between either seems to lie in the fact that the former breaks manifest $(4,4)$ supersymmetry while the latter respects it at any stage. We postpone the full analysis of the component structure of the dual $N = 4$ $SU(2) \times U(1)$ action to the future. However, some preliminary tests show that after fixing a proper gauge with respect to (6.10) and eliminating an infinite tower of auxiliary fields, the action of physical bosonic fields proves the same as in the original $q^{(1,1)}$ formulation, thus indicating that the geometry of bosonic manifold is not affected by the $(4,4)$ duality transformation in contrast to the aforementioned $(2,2)$ one. This point deserves a further study.

7 Conclusions

In this paper we have introduced the basic concepts of $(4,4)$ $2D$ harmonic superspace with two independent sets of $SU(2)$ harmonics in the left and right light-cone sectors, and constructed in its framework off-shell superfield actions which describe $(4,4)$ supersymmetric sigma models with commuting left and right complex structures and provide massive deformations of these models. We discussed both the case of general bosonic target manifolds of this kind and the special case of the $SU(2) \times U(1)$ group manifold WZNW sigma model. The analytic superfield action of the latter has been shown to unambiguously follow from the requirement of $N = 4$ $SU(2)$ superconformal invariance, quite similarly to the action of improved tensor $N = 2$ multiplet in harmonic $N = 2$ $4D$ superspace. Besides the formulation in terms of constrained analytic $q^{(1,1)}$ superfields which is basically equivalent to the formulations via constrained superfields in the projective or conventional $(4,4)$ superspaces, we have found new formulations of these models via unconstrained analytic superfields $\omega^{(1,-1)}, \omega^{(-1,1)}$ with infinite sets of auxiliary fields and with a gauge invariance which serves to remove the doubling of propagating degrees of freedom. We achieved this with the help of $(4,4)$ duality transformation which directly generalizes the duality transformation defined earlier for sigma models in harmonic $N = 2$ $4D$ superspace. Some interesting features of the dual actions have been found, in particular the appearance, after passing to the dual formulation of massive $q^{(1,1)}$ models, of the $SU(2)$ tensor “semi-central” charges in the anticommutators of the left and right $2D$ super Poincaré generators.

We conclude by listing some further problems we hope to solve with the help of the $(4,4)$ harmonic superspace formalism.

(A). Constructing off-shell formulations of $(4,4)$ sigma models with non-commuting left
and right complex structures. Some conceivable ways of approaching this difficult problem within the \((4,4)\) harmonic superspace were already indicated in the end of previous Section. The dual formulations seem especially promising in this respect.

(B). Extending the integrability concepts to the harmonic superspace. In Sect. 5 we have shown the uniqueness of massive deformation of general \(q^{(1,1)}\) action (without modifying \((4,4)\) supersymmetry): the relevant component potential terms are almost entirely (up to a constant matrix) specified by the metric of the original bosonic manifold. This effect is quite new, e.g., in comparison with the \((2,2)\) models where the superpotential terms can be introduced independently of the sigma model term as arbitrary holomorphic functions of chiral (or twisted chiral) superfields. The \(N = 4\) \(SU(2) \times U(1)\) WZNW model generates in this way the \(N = 4\) WZNW - Liouville system. The superfield equations of the latter (as well as those of the initial sigma model), while written in conventional \((4,4)\) superspace, are integrable in the sense that they amount to the vanishing of some supercurvature \([1]\). So, this system has actually proved the first example of integrable \((4,4)\) supersymmetric model. It is interesting to see how the same integrability properties manifest themselves when the \(N = 4\) WZNW - Liouville system is represented by the equations \((6.14)\) in the analytic harmonic superspace. Do these equations admit a zero-curvature interpretation? To answer this question seems especially important, having in perspective to construct \((4,4)\) extensions of the Toda systems, both conformal and affine (including the sine-Gordon model). We conjecture that these systems are described by the deformed actions of the type \(S_q + S_m\), with \(S_m\) defined in \((5.1)\) and appropriately chosen metric functions \(G(q)_{MN}\). It is highly desirable to have convenient manifestly supersymmetric criterions of integrability of the corresponding analytic superfield equations of motion.

(C). Coupling to \((4,4)\) world-sheet conformal supergravity. To know the \((4,4)\) sigma models - supergravity couplings is important for constructing self-consistent superstring and higher \(p\) branes models with the sigma model target manifolds as a background. We hope that the conformal supergravity in the analytic \((4,4)\) 2D superspace can be constructed by analogy with that in the \(N = 2\) 4D analytic superspace \([32]\). On the other hand, it has been argued in \([14, 4, 11]\) that, e.g., the component action of \(N = 4\) \(SU(2)\) WZNW - Liouville model itself can be interpreted as a result of fixing appropriate gauges in the action of the Polyakov type \(N = 4\) 2D supergravity. It would be interesting to understand how this occurs in the framework of the analytic superfield description.

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Appendix

Here we compute the double harmonic integral

\[ G(q) = \int \frac{1 - X}{(1 + X)^3}, \quad X = c^{(-1,-1)}q^{ia}(z)u_i^{(1)}v_a^{(1)}, \quad (A.1) \]

which is the basic object of the component \( N = 4 \ SU(2) \times U(1) \) WZNW action.

To simplify our task, we will closely follow ref. \[15\] where a heave use of different \( SU(2) \)'s realized on the fields and harmonics has been made to compute analogous integrals.

First of all, we exploit the invariance of the integration measure in (A.1) under two independent \( SU(2) \) rotations of harmonics in indices \( i \) and \( a \). Using this freedom, one may bring \( c^{ia} \) into the form

\[ c^{ia} = \epsilon^{ia}. \quad (A.2) \]

Secondly, it is straightforward to check that (A.1) is invariant under the transformations of rigid \( SU(2) \) subgroups of the left and right branches of \( N = 4 \ SU(2) \) superconformal group I. Recall that these transformations, e.g. from the left light-cone sector, are given by

\[ \delta_{SU(2)_L} q_0^{(1,1)} = \lambda^{(ij)} u_i^{(1)} u_j^{(-1)} (\hat{q}^{(1,1)} + c^{(1,1)}) - \lambda^{(ij)} u_i^{(1)} u_j^{(-1)} c^{(-1,1)} \]
\[ \delta_{SU(2)_L} u_i^{(1)} = \lambda^{(kj)} u_k^{(1)} u_j^{(-1)}, \quad \delta_{SU(2)_L} u_i^{(-1)} = 0. \quad (A.3) \]

Under these transformations and their right counterparts the field \( q^{ia}(z) = \hat{q}^{ia} + c^{ia} \) undergoes independent \( SU(2) \) rotations in indices \( i \) and \( a \), so one may choose the frame where

\[ \hat{q}^{ia} = \epsilon^{ia} \rho(z), \quad \rho^2 = \frac{1}{2} q^{ia} q_{ia}. \quad (A.4) \]

With the choice (A.2), (A.4):

\[ X = (\rho - 1) (u_2^{(-1)} v_1^{(-1)} - u_1^{(-1)} v_2^{(-1)})(u_2^{(1)} v_1^{(1)} - u_1^{(1)} v_2^{(1)}). \quad (A.5) \]

Next simplifying step is to represent the integrand in (A.1) as

\[ \frac{1 - X}{(1 + X)^3} = \left( 1 + 3 \frac{\partial}{\partial \alpha} + \alpha \frac{\partial^2}{\partial \alpha^2} \right) \frac{1}{1 + \alpha X} |_{\alpha = 1}, \quad (A.6) \]

thereby reducing the problem to the computation of the harmonic integral

\[ I_\alpha(q) \equiv \int \frac{1}{1 + \alpha X}. \quad (A.7) \]

Choosing the Euler angle parametrization for the harmonics

\[ u_i^{(1)} = i \sin \theta/2 e^{-i\phi/2}, \quad u_i^{(-1)} = \cos \theta/2 e^{-i\phi/2}, \]
\[ u_2^{(1)} = \cos \theta/2 e^{i\phi/2}, \quad u_2^{(-1)} = i \sin \theta/2 e^{i\phi/2}, \]
\[ v_1^{(1)} = i \sin \omega/2 e^{-i\gamma/2}, \quad v_1^{(-1)} = \cos \omega/2 e^{-i\gamma/2}, \]
\[ v_2^{(1)} = \cos \omega/2 e^{i\gamma/2}, \quad v_2^{(-1)} = i \sin \omega/2 e^{i\gamma/2}. \quad (A.8) \]
one rewrites (A.7) as
\[ I_\alpha(q) = \int dudv \{ 1 + \frac{\alpha}{2}(\rho - 1)[1 - \cos \theta \cos \omega - \sin \theta \sin \omega \cos (\phi - \gamma)]\}^{-1}, \]
\[ \int dudv = \frac{1}{(4\pi)^2} \int_0^\pi \sin \theta d\theta \int_0^\pi \sin \omega d\omega \int_0^{2\pi} d\phi \int_0^{2\pi} d\gamma. \] (A.9)

A straightforward calculation yields
\[ I_\alpha(q) = \ln(1 + \alpha(\rho - 1)) \]
whence, using (A.7),
\[ G(q) = \rho^{-2}. \] (A.10)

Note that (A.7) could be computed without resorting to the explicit parametrization of harmonics, by expanding (A.7) in a series in \( \alpha \) and applying the formal rules of integration over harmonic variables (2.28).

Finally, we notice that the above method is not directly applicable for computing the harmonic integrals in the torsion potential (4.21). The latter is not a tensor under the action of groups \( SU(2)_{cL}, \, SU(2)_{cR} \): it is shifted by full \( z \) derivatives and its \( SU(2) \) variations vanish only after performing \( z \) integration. As a result, one cannot choose \( q^i \) in the simple form (A.4), while performing the \( u,v \) integration. As we know, this difficulty can be got over by passing to the torsion field strength \( H_{ia,jb,kd} \) which have tensor properties with respect to superconformal \( SU(2) \)'s.

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