Abstract. A finite-time singularity of 2D harmonic map flow will be called “strictly type-II” if the outer energy scale satisfies
\[ \lambda(t) = O(T - t)^{1/2}. \]
We prove that the body map at a strict type-II blowup is Hölder continuous. This is relevant to a conjecture of Topping.

1. Introduction

Let \( M \) and \( N \) be compact Riemannian manifolds. For differentiable maps \( u : M \to N \), we may define the Dirichlet energy:
\[ \frac{1}{2} \int_M |du|^2 \, dV. \]
Its downward gradient flow is given by
\[ \frac{\partial u}{\partial t} = \mathcal{T}(u). \]
Here, \( \mathcal{T}(u) \) is the tension field of \( u \), a generalization of the Laplace-Beltrami operator to maps between manifolds. The evolution equation (1.1), known as harmonic map flow, was introduced in 1964 by Eells and Sampson [5] and has been studied since then almost without interruption.

When \( M \) has dimension two, the Dirichlet functional is conformally invariant; we shall be concerned exclusively with this case. Struwe [11] constructed a global weak solution \( u(t) \) of (1.1) starting from any initial map of Sobolev class \( W^{1,2}(M,N) \). The solution is smooth away from finitely many singular times, when isolated singularities (“bubbles”) may form. For a singular time \( T < \infty \), the limit
\[ u(T) = \lim_{t \to T} u(t) \]
exists weakly in \( W^{1,2} \) and smoothly away from the bubbling set, and is referred to as the body map.

Note that Struwe’s construction leaves open the possibility that the body map will be discontinuous. Topping [13] demonstrated that for certain target manifolds and initial data, \( u(T) \) can indeed have an essential singularity at a bubble point. At the same time, he conjectured that for well-behaved (specifically, real-analytic) metrics on the target, the body map always extends continuously across the bubbling set. Topping’s conjecture is the sine qua non for future geometric applications of harmonic map flow.

In previous joint work with C. Song [10], we established Hölder continuity of the body map when \( N \) is compact Kähler with nonnegative holomorphic bisectional curvature and...
the energy of the initial map is near the holomorphic energy. The argument relied partly on establishing a bound on the outer energy scale of the form
\begin{equation}
\lambda(t) = O(T - t)^\frac{1 + \alpha}{2},
\end{equation}
where $0 < \alpha \leq 1$. We refer to a singularity satisfying (1.2) as a strict type-II blowup. Note that (1.2) is a refinement of the ordinary type-II blowup estimate for 2D harmonic map flow:
\begin{equation}
\lambda(t) = o(T - t)^\frac{1}{2}.
\end{equation}
For a proof of (1.3), see [13, Theorem 1.6v].

The strict type-II bound (with $\alpha = 1$) is most familiar from the rotationally symmetric setting. Angenent, Hulshof, and Matano [1] proved that the finite-time blowup first constructed by Chang, Ding, and Ye [3] occurs with rate $\lambda(t) = o(T - t)$. Raphael and Schweyer [9] determined a large set of rotationally symmetric initial data that blows up under (1.1) with the precise rate
\begin{equation}
\lambda(t) \sim \kappa \frac{T - t}{\ln(T - t)^2}.
\end{equation}
They also proved in this context that the body map is $W^{2,2}$, hence $C^\beta$ for each $\beta < 1$.\footnote{This also follows from the main theorem of [10].}

Davila, Del Pino, and Wei [4] produced a larger set of examples with blowup rate (1.4), whose body maps are continuous by construction.

In another direction, Topping [12] proved continuity of the body map if the Dirichlet energy is Hölder continuous as a function of time. One step in the proof was to establish that the strict type-II bound (1.2) follows from this assumption (see [12, Lemma 2.2]). Hence, the strict type-II bound with arbitrary exponent has appeared in previous work, although a much stronger assumption was required to obtain continuity of the body map.

Accordingly, the known examples of strict type-II blowup all have continuous body maps. On the other hand, the only known example with discontinuous body map, due to Topping, fails to be strictly type-II—see [13, Theorem 1.14e]. Our main theorem confirms the implication, as follows.

**Theorem 1.1.** For any strict type-II blowup of harmonic map flow in dimension two, with $0 < \alpha \leq 1$ in (1.2), the body map is $C^{\frac{\alpha}{2}}$.

Our main technical result is Theorem 3.3 below, and Corollary 3.4 gives the formal statement of Theorem 1.1.

The proof depends on obtaining decay estimates for the differential of $u$ in the “neck region,” i.e., the area near the singularity but outside the (vanishing) energy scale. The required estimate on the angular component of $du$ is already known [10, Lemma 5.4], so it remains only to estimate the radial component. We obtain a bound on the difference between the radial and angular components from a well-known identity (2.6), giving an integral bound on the radial component under the flow. Using a specialized parabolic estimate (Proposition 2.3) and a bootstrap argument, we are able to promote this integral bound to pointwise decay throughout the neck region.
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2. Technical results

For an introduction to harmonic map flow, we refer the reader to [10, §2] or to the textbook of Lin and Wang [7].

Let \((M, g)\) and \((N, h)\) be Riemannian manifolds. Given any smooth map \(u : M \to N\), we denote the pullback of \(h\) to \(u^*TN\) by \(\langle \cdot, \cdot \rangle\), which we combine with \(g\) on tensors. The differential \(\text{du}\) is a section of \(T^*M \otimes u^*TN\), with norm squared
\[
|\text{du}|^2 = g^{ij} \langle \partial_i u, \partial_j u \rangle.
\]
The tension field is given by
\[
\mathcal{T}(u) = \text{tr}_g \nabla \text{du}.
\]
Here, \(\nabla\) is the Levi-Civita connection on \(T^*M\) coupled with the pullback to \(u^*TN\) of the Levi-Civita connection on \(N\).

Suppose that \(\dim(M) = 2\). We write
\[
g = \xi^2 (dr^2 + r^2 d\theta^2)
\]
for the given metric in conformal coordinates, where \(\xi\) is a smooth function. By adjusting the conformal chart to second order, we may assume \(d\xi(0) = 0\). We may further assume
\[
|\xi - 1| + r|d\xi| + r^2 |\nabla d\xi| \leq \xi_0 r^2
\]
for a constant \(0 \leq \xi_0 \leq \frac{1}{2}\), after rescaling \(g\) by a constant. This implies
\[
|r - \text{dist}_g(x_0, \cdot)| \leq C \xi_0 r^2.
\]
In view of these bounds, the difference between conformal and geodesic coordinates will be of no significance for our results; we use conformal coordinates only for convenience.

Let
\[
S_{ij} = \langle \partial_i u, \partial_j u \rangle - \frac{1}{2} g_{ij} |\text{du}|^2
\]
be the stress-energy tensor of \(u\). This is a symmetric 2-tensor on \(M\), which satisfies
\[
\nabla^i S_{ij} = \langle \mathcal{T}(u), \partial_j u \rangle.
\]
For a derivation of (2.4), see [10, §2.2].

The radial vector field \(X^i\) in conformal coordinates is conformally Killing, i.e.
\[
\nabla^i X^j + \nabla^j X^i = \lambda g^{ij}
\]
for a scalar function \(\lambda\). Contracting with \(X^i\) in (2.4), we have
\[
\nabla^i (X^j S_{ij}) = \langle \mathcal{T}(u), X^j \partial_j u \rangle + \frac{1}{2} \left( \nabla^i X^j + \nabla^j X^i \right) S_{ij}
\]
\[
= \langle \mathcal{T}(u), X^j \partial_j u \rangle,
\]
since \(g^{ij} S_{ij} = 0\) in dimension two. Integrating (2.5) over a disk \(D_r\) in the conformal chart, and applying the divergence theorem, we obtain
\[
\int_{S_{1r}^2} X^i X^j S_{ij} d\theta = \int_{D_r} \langle \mathcal{T}(u), X^i \partial_j u \rangle dV.
\]
This identity is well known from the theory of approximate harmonic maps, and will be used crucially below.

Next, we need the following parabolic estimates. For \(0 \leq \nu \leq 1\) and \(\mu > 1\), let
\[
\Box_\nu = \partial_t - \left( \frac{\partial^2}{r^2} + \frac{1}{r} \partial_r - \frac{\nu^2}{r^2} \right),
\]
\[
\Delta_\mu = \partial^2_r + \frac{(\mu - 1)}{r} \partial_r.
\]
We have
\[(2.7)\]
\[
\Box_\nu (r^\nu y) = r^\nu \left( \partial_t - \Delta_{2(\nu+1)} \right) y.
\]
Also notice that
\[(2.8)\]
\[
\Box_\nu r^\beta = \left( \nu^2 - \beta^2 \right) r^{\beta-2}.
\]

**Lemma 2.1.** Let \(u\) be a solution of (1.1) with respect to a metric \(g\) of the form (2.1-2.2). Suppose that for some \(0 < \eta^2 < \varepsilon_0\), we have
\[(2.9)\]
\[
|du| + r^2|\nabla du| + r^3|\nabla^2 du| + r^4|\nabla^3 du| \leq \eta.
\]
Then the angular energy
\[
f = f(u; r, t) := \sqrt{\int_{S^1_r} |u_\theta(r, \theta, t)|^2 d\theta + \int_{S^1_r} |\nabla_\theta u_\theta(r, \theta, t)|^2 d\theta}
\]
satisfies a differential inequality
\[(2.10)\]
\[
\Box_\nu f \leq C\xi_0 \eta,
\]
where \(\nu = \sqrt{1 - C\eta}\). The radial energy
\[(2.11)\]
\[
g = g(u; r, t) := \sqrt{\int_{S^1_r} r^2|u_r(r, \theta, t)|^2 d\theta}
\]
satisfies
\[(2.12)\]
\[
\Box_\nu \left( \frac{g}{r} \right) \leq \frac{6f}{r^3} + \frac{C\xi_0 \eta}{r}.
\]
Here, \(\varepsilon_0\) depends on the geometry of \(N\), and \(\xi_0\) is the constant of (2.2).

**Proof.** The proof is an elementary extension of prior calculations; see Appendix A. \(\square\)

**Proposition 2.2.** Let \(-\nu \leq \beta_i \leq \nu \leq 1\), for \(i = 0, 1\). Suppose that \(f(r, t)\) is continuous on \([\rho, 1] \times [\tau, T]\) and satisfies
\[(2.13)\]
\[
\Box_\nu f \leq A,
\]
with
\[(2.14)\]
\[
|f(r, \tau)| \leq A \left( \frac{\rho}{r} \right)^{\beta_0} + r^{\beta_1}, \quad f(\rho, t) \leq A, \quad f(1, t) \leq A.
\]
Then, given $0 < \kappa \leq 1/2$, for
\[ \frac{\rho}{\kappa} \leq r \leq \kappa \text{ and } \tau + \frac{r^2}{\kappa^2} \leq t < T, \]
we have
\[ |f(r,t)| \leq C_{2.2} A \left( \kappa^{\nu+\beta_0} \left( \frac{\rho}{r} \right)^{\beta_0} + \kappa^{\nu-\beta_1} r^{\beta_1} \right). \]
Here $C_{2.2}$ depends on $\beta_0, \beta_1$, and $\nu$.

**Proof.** Let $\mu = 2\nu + 2$. Using the results of Appendix B.1 and (2.7-2.8), we can construct a supersolution for (2.13) of the form
\[ \bar{v}(r,t) = r^\nu v_0(r,t) + 2A \left( \left( \frac{\rho}{r} \right)^\nu + r^\nu \right) - \frac{A}{4 - \nu^2} r^2, \]
where
\[ \left( \partial_t - \Delta_{\mu} \right) v_0 = 0, \]
\[ v_0(r,0) = r^{-\nu} f(r,0), \quad v_0(\rho,t) = 0 = v_0(1,t). \]
Applying the comparison principle to (2.13), we have
\[ g(r,t) \leq \bar{v}(r,t), \]
so it suffices to check (2.15) for $\bar{v}$.

By Proposition B.2 and (2.14), we have
\[ v_0(r,t) \leq CA \left( \frac{\rho^{\beta_0}}{r^{3\beta_0+\nu}} w^{\beta_0+\nu}(r,t-\tau) + r^{\beta_1-\nu} w^{\nu-\beta_1}(r,t-\tau) \right), \]
where $w^a(r,t)$ is defined by
\[ w^a(r,t) = \left( \frac{r^2}{r^2 + t} \right)^{a/2}. \]
Overall, from (2.16), we obtain
\[ |f(r,t)| \leq C A \left( \left( \frac{\rho}{r} \right)^{\beta_0} w^{\beta_0+\nu}(r,t-\tau) + r^{\beta_1-\nu} w^{\nu-\beta_1}(r,t-\tau) + \left( \frac{\rho}{r} \right)^\nu + r^\nu + r^2 \right). \]
For $r \geq \rho/\kappa$, we have
\[ \left( \frac{\rho}{r} \right)^\nu \leq \kappa^{\nu-\beta_0} \left( \frac{\rho}{r} \right)^{\beta_0}. \]
For $r \leq \kappa$ and $t - \tau \geq r^2/\kappa^2$, we have
\[ w^{\beta_0+\nu}(r,t-\tau) \leq \kappa^{\beta_1+\nu}, \quad w^{\nu-\beta_1}(r,t-\tau) \leq \kappa^{\nu-\beta_1}, \quad r^\nu \leq \kappa^{\nu-\beta_1} r^{\beta_1}. \]
Substituting into (2.17), we obtain (2.15). \(\square\)

**Proposition 2.3.** Let $-\nu \leq \gamma_i \leq \nu \leq 1$ and $\beta_i$ with $|1 \pm \beta_i| \neq \nu$, for $i = 0, 1$. Given $0 < \kappa \leq 1/2$, let $\rho \leq \rho_1 \leq \kappa$.

Suppose that $g(r,t)$ is continuous on $[\rho_1, 1] \times [\tau, T)$ and satisfies
\[ \square_{\nu} \left( \frac{g}{r} \right) \leq A \frac{r^\beta_0}{r^{3\beta_0+\nu}} \left( \frac{\rho}{r} \right)^{\beta_0} + r^{\beta_1}, \]
with
\[ |g(r, \tau)| \leq A \left( \left( \frac{\rho}{r} \right)^{\gamma_0} + r^{\gamma_1} \right), \]
(2.19)
\[ \int_\tau^T g(\rho_1, t)^2 \, dt \leq B^2, \quad |g(1, t)| \leq A. \]

Then, for
\[ 2\rho_1 \leq r \leq \kappa \quad \text{and} \quad \tau + \frac{r^2}{\kappa^2} \leq t < T, \]
(2.20)
we have
\[ |g(r, t)| \leq C_{2.3} \left( B_{\rho_1}^{\nu-1} + A \left( \frac{\rho_1^{\beta_0}}{\rho_1^{\beta_0+1}} + r^{\beta_1-1} \right) + \frac{\rho_1^{\nu-1}}{r^{\nu-1}} \right). \]
(2.21)
Here, \( C_{2.3} \) depends on \( \gamma_0, \gamma_1, \beta_0, \beta_1, \) and \( \nu. \)

**Proof.** As in the previous proof, we construct a supersolution for (2.18) of the form
\[ \bar{v}(r, t) = r^\nu \left( v_1(r, t) + v_2(r, t) \right) + CA \left( \frac{\rho_1^{\beta_0}}{\rho_1^{\beta_0+1}} + r^{\beta_1-1} \right), \]
where
\[ (\partial_t - \Delta_\mu) v_i = 0, \quad i = 1, 2, \]
with
\[ v_1(r, 0) = r^{-\nu-1}g(r, 0), \quad v_1(\rho, t) = 0 = v_1(1, t), \]
and
\[ v_2(\rho_1, t) = \rho_1^{-\nu-1}g(\rho_1, t), \quad v_2(r, 0) = 0 = v_2(1, t). \]

Applying the comparison principle to (2.18), we have
\[ |g(r, t)| \leq r\bar{v}(r, t). \]
(2.22)

By Proposition B.4 and (2.19), we have
\[ v_1(r, t) \leq CA \left( \frac{\rho_1^{\gamma_0}}{r^{\gamma_0+\nu+1}} w^{\gamma_0+\nu+1}(r, t) + r^{\gamma_1-\nu-1}w^{\nu-\gamma_1+1}(r, t) \right). \]

By Proposition B.4 and (2.19), since \( \mu - 2 = 2\nu, \) we have
\[ v_2(r, t) \leq C B_{\rho_1}^{\nu-1} \left( \frac{\rho_1^{2\nu}}{r^{2\nu+1}} \right) \leq C B_{\rho_1}^{\nu-1} \frac{\rho_1^{\nu-1}}{r^{2\nu+1}}. \]

Overall, from (2.22), we obtain
\[ |g(r, t)| \leq C \left( B_{\rho_1}^{\nu-1} + A \left( \left( \frac{\rho_1}{r} \right)^{\gamma_0} w^{\gamma_0+\nu+1}(r, t) + r^{\gamma_1} w^{\nu-\gamma_1+1}(r, t) + \frac{\rho_1^{\beta_0}}{\rho_1^{\beta_0+1}} + r^{\beta_1} \right) \right). \]

As in the previous proof, under the assumptions (2.20) on \( r \) and \( t, \) this implies (2.21). \( \square \)
3. Main Theorem

Fix \( x_0 \in M \) and let \( r_g(x) = \text{dist}_g(x_0, x) \). We shall denote the geodesic ball of radius \( R \) centered at \( x_0 \) by \( B_R(x_0) \), or more typically by \( B_R \). An annulus will be denoted by \( U_\rho^R = B_R \setminus \bar{B}_\rho \).

**Definition 3.1** ([10], Definition 4.2). Given a \( W^{1,2} \) map \( u : B_R(x_0) \to N \), the outer energy scale \( \lambda_{\varepsilon, R, x_0}(u) \) is the smallest nonnegative number \( \lambda \) such that

\[
\sup_{\lambda < \rho \leq R} \int_{U^\rho_R(x_0)} |du|^2 dV < \varepsilon.
\]

Note that \( \lambda = R \) satisfies (3.1) vacuously, so \( 0 \leq \lambda_{\varepsilon, R, x_0}(u) \leq R \) by definition.

We first establish the following “baby case” of the main theorem.

**Lemma 3.2.** Given \( E, \lambda > 0 \) and \( 0 < \varepsilon < \varepsilon_0 \), there exists \( \delta > 0 \) as follows. Suppose that \( u \) is a smooth solution of (1.1) on \( B_R \times (-R^2, T) \), with \( 0 < R < R_0 \) and \( T > 0 \), which satisfies

\[
\sup_{-R^2 < t < T} \int_{B_R} |u_t|^2 dV \leq E,
\]

\[
\int_{-R^2}^T \int_{B_R} |\mathcal{T}|^2 dV dt < \delta^2,
\]

and

\[
\sup_{-R^2 < t < T} \lambda_{\varepsilon, R, x_0}(u(t)) \leq R \lambda.
\]

Then for \( 2R\lambda \leq r_g(x) \leq R/2 \) and each integer \( k \geq 0 \), we have

\[
r_g(x)^{1+k}|\nabla^{(k)}u(x, 0)| \leq C_k \sqrt{\varepsilon \left( \frac{R\lambda}{r_g(x)} + \frac{r_g(x)}{R} \right)}.
\]

Here, \( R_0 > 0 \) depends on the geometry of \( M \), \( \varepsilon_0 > 0 \) depends on the geometry of \( N \), and \( \delta \) depends on \( E, \varepsilon \), and \( \lambda \).

**Proof.** Assuming that \( R_0 \) is sufficiently small, we may rescale so that \( R = 1 \) and our metric \( g \) takes the form (2.1-2.2), with \( \xi_0 \leq \frac{1}{2} \). By (2.3), it suffices to establish (3.5) with the conformal coordinate \( r \) in place of \( r_g \).

In view of (3.4), the standard \( \varepsilon \)-regularity lemma (see e.g. Theorem 3.4 of [10]) implies

\[
\sup_{\lambda \leq r \leq R} \left( r |du| + r^2 |\nabla du| + r^3 |\nabla^2 du| + r^4 |\nabla^3 du| \right) \leq C \sqrt{\varepsilon}.
\]

Hence, by Lemma 2.1, \( f \) satisfies an evolution equation

\[\Box_\nu f \leq C \sqrt{\varepsilon},\]

where \( \nu = \sqrt{1 - C \sqrt{\varepsilon}} \). We can apply Proposition 2.2 with \( \rho = \lambda, A = C \sqrt{\varepsilon}, \beta_0 = \beta_1 = 0 \), and \( \tau = -\frac{1}{2} \). Since \( w^\alpha(r, s) \leq C r^\alpha \) for \( s \geq \frac{1}{4} \), (2.17) gives

\[
\sup_{-\frac{1}{4} \leq t \leq 0} |f(r, t)| \leq C \sqrt{\varepsilon \left( r^{\nu + r^\beta_1} + \left( \frac{\lambda}{r} \right)^\nu + r^\nu + r^2 \right)}
\]

\[
\leq C \sqrt{\varepsilon \left( \left( \frac{\lambda}{r} \right)^\nu + r^\nu \right)}.
\]
Applying Hölder’s inequality identity to (2.6), we have
\[
\left\| \int_{-\frac{1}{2}}^{0} (g^2(r,t) - f^2(r,t)) \, dt \right\| = \left\| \int_{t_0}^{t_1} \int_{S_{r_1}} X^i X^j S_{ij}(x,t) \, d\theta dt \right\|
\]
(3.8)
\[
\leq \left( \int_{t_0}^{t_1} \int_{D_{r_1}} |T(u)|^2 \, dV dt \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_{D_{r_1}} r^2 |du|^2 \, dV, dt \right)^{\frac{1}{2}}
\]
\[
\leq C\delta \sqrt{E}.
\]
Since \(|du|^2 \leq f^2 + g^2\), for \(\delta\) sufficiently small, (3.7-3.8) imply
\[
\int_{-\frac{1}{2}}^{0} \int_{r/2}^{r} |du|^2 \, dV dt \leq C\varepsilon \left( \left( \frac{\lambda}{r} \right)^{\nu} + r^{\nu} \right)^{2}.
\]
Applying \(\varepsilon\)-regularity (Theorem 3.4 of [10]) again, we obtain
\[
r^{1+k} |\nabla^{(k)} \, du(x,0)| \leq C_k \sqrt{\varepsilon} \left( \left( \frac{\lambda}{r} \right)^{\nu} + r^{\nu} \right).
\]
(3.9)
To obtain the same estimate with \(\nu = 1\), one can apply a supersolution argument as in Propositions 2.2-2.3, with (3.9) in place of (3.6). Since the sharp result will not be used below, we omit the proof.

\section*{Theorem 3.3.}
Given \(E > 0\), \(0 < \varepsilon < \varepsilon_0\), and \(0 < \alpha \leq 1\), there exists \(\delta_0 > 0\) as follows. Suppose that \(u : B_R \times (-R^2, T) \to N\) is a smooth solution of (1.1), with \(0 < R < R_0\) and \(T > 0\), which satisfies
\[
\sup_{-R^2 < t < T} \int_{B_R} |du(t)|^2 \, dV \leq E
\]
(3.10)
and
\[
\int_{-R^2}^{T} \int_{B_R} |T|^2 \, dV dt < \delta_0^2.
\]
(3.11)
Suppose further that for some \(0 < \lambda_1 \leq \frac{1}{2}\) and \(0 < t_1 < T\), we have
\[
\lambda_{\varepsilon, R, x_0}(u(t)) \leq R \left( \lambda_1 + \left( \frac{t_1 - t}{R^2} \right)^{\frac{1}{2}} \right)
\]
(3.12)
for all \(-R^2 < t < t_1\). Then for \(2R\lambda_1 \leq r_g(x) \leq R/2\) and each integer \(k \geq 0\), we have
\[
r_g(x)^{1+k} |\nabla^{(k)} \, du(x, t_1)| \leq C_{k, \alpha} \sqrt{\varepsilon} \left( \frac{R\lambda_1}{r_g(x)} + \left( \frac{r_g(x)}{R} \right)^{\alpha} \right)^{\frac{1}{2}}.
\]
(3.13)
Here, \(R_0 > 0\) depends on the geometry of \(M\), \(\varepsilon_0 > 0\) depends on the geometry of \(N\), and \(\delta_0\) depends on \(E, \varepsilon\), and \(\alpha\).

\textbf{Proof.}
For convenience, we replace (3.12) by
\[
\lambda_{\varepsilon, R, x_0}(u(t)) \leq R \left( \lambda_1 + \kappa^{\alpha} \left( \frac{t_1 - t}{R^2} \right)^{\frac{12\alpha}{2}} \right),
\]
(3.14)
where $0 < \kappa \leq \frac{1}{2}$ is a constant to be determined by the argument below (depending only on $\alpha$ and $\nu$), and $a$ is sufficiently large, for instance
\begin{equation}
(3.15) \quad a = \frac{72(1 + \alpha)}{\alpha}.
\end{equation}
The assumptions (3.14) and (3.12) are equivalent after rescaling and redefining constants.

We will also prove the two estimates
\begin{equation}
(3.16) \quad f(u; r, t_1) \leq C_0 \sqrt{\varepsilon} \max \left[ \frac{\lambda_1}{r}, r^\alpha \right]^{2\nu - 1},
\end{equation}
\begin{equation}
(3.17) \quad g(u; r, t_1) \leq C_0 \sqrt{\varepsilon} \max \left[ \frac{\lambda_1}{r}, r^\alpha \right]^{\frac{1}{2}},
\end{equation}
which clearly imply (3.13). Here, $\nu$ is any number with
\begin{equation}
(3.18) \quad \frac{26}{27} \leq \nu \leq \sqrt{1 - C\sqrt{\varepsilon}}.
\end{equation}

It suffices to prove the theorem for $\lambda_1$ of the form
\begin{equation*}
\lambda_1 = 2^{-n} \kappa^a,
\end{equation*}
so we may proceed by induction. For $\lambda_1 = \kappa^a$, Lemma 3.2 gives $\delta_0 = \delta > 0$ such that (3.16-3.17) hold, with $C_0 > 1$ universal. This establishes the base case. Note that we are free to assume $\kappa$ is arbitrarily small.

For the induction step, suppose that (3.16-3.17) hold for all $\lambda_1 \geq 2\bar{\lambda}_1$, where $0 < \bar{\lambda}_1 \leq \kappa^a$; i.e., the conclusion of the theorem holds for all such $\lambda_1$ and solutions $u$ satisfying the hypotheses. We must establish the Theorem for $\lambda_1 = \bar{\lambda}_1$.

Let $u(t)$ be a solution satisfying the hypotheses, with $\lambda_1 = \bar{\lambda}_1$. By rescaling, it suffices to assume $R = 1$.

In view of (3.12), the standard $\varepsilon$-regularity lemma (see e.g. Theorem 3.4 of [10]) implies
\begin{equation*}
\sup_{\lambda_1(t) \leq r \leq R} \left( r|du| + r^2|\nabla du| + r^3|\nabla^2 du| + r^4|\nabla^3 du| \right) \leq C\sqrt{\varepsilon}.
\end{equation*}

Hence, by Lemma 2.1, $f$ and $g$ satisfy evolution equations
\begin{align*}
\Box_\nu f &\leq C\sqrt{\varepsilon} \\
\Box_\nu \left( \frac{g}{r} \right) &\leq \frac{6f}{r^3} + C\frac{\sqrt{\varepsilon}}{r},
\end{align*}
for $\lambda_1 + \kappa^a (t_1 - t) \frac{1 + \alpha}{2} \leq r \leq 1$. Let
\begin{equation*}
\rho = 2\bar{\lambda}_1, \quad \zeta = \rho \frac{1}{1 + \alpha}.
\end{equation*}
Since $\bar{\lambda}_1 \leq \kappa^a$ (by the base case), we have
\begin{equation}
(3.19) \quad \frac{\rho}{\kappa} \leq \zeta \leq 2\kappa \frac{\alpha}{1 + \alpha}.
\end{equation}
Also notice that $\rho = \zeta^{1 + \alpha}$, so
\begin{equation}
(3.20) \quad \frac{\rho}{\zeta} = \zeta^\alpha.
\end{equation}
We now apply the induction hypotheses to $u$ with $R = 1/2$, $\lambda_1 = \rho$, and $R\rho = \bar{\lambda}_1$. In view of (3.20), we clearly have (3.16-3.17) for $r \leq \zeta/2$. Applying the induction hypothesis again, with $R = 1$ and $\lambda_1 = \rho$, we also obtain (3.16-3.17) for all $r \geq 2\zeta$. Hence, it remains only to establish (3.16-3.17) for

$$\frac{1}{2} \zeta \leq r \leq 2\zeta. \tag{3.21}$$

In other words, for $r$ as in (3.21), we must show

$$f(r, t_1) \leq \frac{C_0}{2} \sqrt{\varepsilon} \zeta^\alpha (2\nu - 1) \tag{3.22}$$

and

$$g(r, t_1) \leq \frac{C_0}{2} \sqrt{\varepsilon} \zeta^{\frac{2}{3}}. \tag{3.23}$$

Let

$$t_0 = t_1 - \frac{\zeta^2}{\kappa^2}.$$

From (3.14), we have $\lambda(t) \leq \rho$ for all $t_0 \leq t \leq t_1$. By the induction hypothesis, we have

$$f(r, t) \leq C \sqrt{\varepsilon} \left( \frac{\rho}{r} + r^\alpha \right)^{2\nu - 1} \tag{3.24}$$

and

$$g(r, t) \leq C \sqrt{\varepsilon} \left( \frac{\rho}{r} + r^\alpha \right)^{\frac{1}{3}} \tag{3.25}$$

for all $t_0 \leq t \leq t_1$, where $C$ is a multiple of $C_0$. Combining these, we also have

$$r|du(x, t)| \leq C \sqrt{\varepsilon} \left( \frac{\rho}{r} + r^\alpha \right)^{\frac{1}{2}} \tag{3.26}$$

for all $t_0 \leq t \leq t_1$.

To obtain the estimate (3.22) on $f$, we apply Proposition 2.2. From (3.24-3.25), we obtain

$$\sup_{\zeta/2 \leq r \leq 2\zeta} f(r, t_1) \leq CC_2 \sqrt{\varepsilon} \left( \kappa^{1-\nu} \left( \frac{\rho}{\zeta} \right)^{2\nu - 1} + \kappa^{\nu-\alpha(2\nu - 1)} \zeta^\alpha (2\nu - 1) \right)$$

$$\leq C \sqrt{\varepsilon} \left( \kappa^{1-\nu} + \kappa^{\nu-\alpha(2\nu - 1)} \right) \zeta^{2\nu - 1},$$

where we have used (3.19-3.20). Assuming that $\kappa$ is small enough that

$$C \sqrt{\varepsilon} \left( \kappa^{1-\nu} + \kappa^{\nu-\alpha(2\nu - 1)} \right) \leq \frac{C_0}{2},$$

this establishes the desired estimate (3.22) on $f$.\(^2\)

Next, to obtain the estimate on $g$, let

$$\rho_1 = \zeta^{1+\frac{\alpha}{2\nu}}.$$

By the induction hypothesis, we have

$$\sup_{t_0 \leq t \leq t_1} f(\rho_1, t) \leq C \sqrt{\varepsilon} \left( \frac{\rho}{\rho_1} \right)^{2\nu - 1} \leq C \sqrt{\varepsilon} \zeta^{\frac{7}{16} \alpha} \leq C \sqrt{\varepsilon} \zeta^{\frac{3}{8} \alpha}.$$

\(^2\)The decay estimate on $f$ can also be obtained directly from [10], Lemma 5.4. We have re-proven it here (by a different method) for the sake of exposition.
Using (3.18), we obtain
\[
\int_{t_0}^{t_1} f(\rho_1, t)^2 \, dt \leq \frac{C\varepsilon}{\kappa^2} \zeta^{2 + \frac{4}{3}\alpha}.
\]

We now integrate (2.6) in time and apply Hölder’s inequality:
\[
\left| \int_{t_0}^{t_1} (g^2(\rho_1, t) - f^2(\rho_1, t)) \, dt \right| = \left| \int_{t_0}^{t_1} \int_{\mathcal{S}^1_{\rho_1}} X^i X^j S_{ij}(x, t) \, d\theta dt \right|
\leq \left( \int_{t_0}^{t_1} \int_{\mathcal{D}_{\rho_1}} |T(u)|^2 \, dV dt \right)^{\frac{1}{2}} \left( \int_{t_0}^{t_1} \int_{\mathcal{D}_{\rho_1}} \rho^2 |u|^2 \, dV_r dt \right)^{\frac{1}{2}}
\leq C\delta_0 \left( (t_1 - t_0) \left( \rho^2 E + C\varepsilon \rho_1^2 \left( \frac{\rho}{\rho_1} \right)^{\frac{2}{3}} \right) \right)^{\frac{1}{2}}.
\]

Here we have used the assumptions (3.10-3.11) and (3.26). We have
\[
(t_1 - t_0) \rho^2 E = \frac{\zeta^{4 + 2\alpha}}{\kappa^2} E
\]
and
\[
(t_1 - t_0) \rho_1^2 \left( \frac{\rho}{\rho_1} \right)^{\frac{2}{3} \alpha} = \frac{\zeta^2}{\kappa^2} \zeta^{2 + \frac{2}{3} \alpha} \zeta^{\frac{3}{2} \alpha} = \frac{\zeta^{4 + \frac{12}{5} \alpha}}{\kappa^2}.
\]

Hence, for \(\delta_0\) sufficiently small (independently of \(\bar{\lambda}_1\)), (3.28) reduces to
\[
\left| \int_{t_0}^{t_1} (g^2(\rho_1, t) - f^2(\rho_1, t)) \, dt \right| \leq \kappa \varepsilon \zeta^{2 + \frac{12}{5} \alpha}.
\]

Combining this with (3.27), we obtain
\[
\int_{t_0}^{t_1} g(\rho_1, t)^2 \, dt \leq \kappa \varepsilon \zeta^{2 + \frac{12}{5} \alpha} + \frac{C\varepsilon}{\kappa^2} \zeta^{2 + \frac{4}{3}\alpha}
\leq C \varepsilon \left( \kappa + C \kappa^{\frac{11}{17} + \frac{\alpha}{3}} \right) \zeta^{2 + \frac{12}{5} \alpha}
\leq C \kappa \varepsilon \zeta^{2 + \frac{12}{5} \alpha},
\]
where we have used (3.15) and (3.19).

We may now apply Proposition 2.3, to obtain
\[
\sup_{\zeta/2 \leq r \leq 2\zeta} g(r, t_1) \leq CC_{2.3} \left( \frac{\sqrt{\kappa \varepsilon \zeta^{1 + \frac{12}{5} \alpha}}}{\zeta^{1 + \frac{12}{5} \alpha}} \right)^{\nu} \varepsilon \left( \kappa^{\nu + \frac{1}{3} + \frac{1}{2} + 1} \left( \frac{\zeta^{1 + \alpha}}{\zeta} \right)^{\frac{1}{3}} + \kappa^{\nu + \frac{2}{3} + 1} \zeta^{\frac{2}{3}} \right)
\leq C \left( \sqrt{\kappa \varepsilon \zeta^{\frac{1}{3}} (17 + 27(\nu - 1))} + \sqrt{\kappa \varepsilon \zeta^{\frac{2}{3} \alpha}} + \varepsilon \zeta^{\frac{2\alpha}{3}} \right)
\leq C \varepsilon \left( \sqrt{\kappa + \kappa^{\alpha} \left( \frac{2\alpha}{2 + (1 + \alpha) \nu} + \frac{1}{2} \right)} \right) \zeta^{\frac{2}{3}},
\]
where we have used (3.18) and (3.19). For \(\kappa\) sufficiently small, this implies the desired bound (3.23), completing the induction.
The tension field with respect to \( g \) with respect to the metric \( g \)
Letting \( \lambda \) denote the flat cylindrical metric, we have
\[
\nabla \text{denotes the pullback connection on } u^*TN, \text{ as above. The heat-flow equation (1.1) with respect to the metric } g \text{ becomes}
\]
\[
(A.1) \quad u_t = \mathcal{T}(u) = \xi^{-2}e^{-2s}\mathcal{T}_0(u).
\]
We start from the identity
\[
(A.2) \quad \frac{1}{2} \partial_s^2 f_1^2 = \int_{S^1} (|\nabla_s \nabla_{\theta} u_\theta|^2 + (\nabla_s^2 \nabla_{\theta} u_\theta, \nabla_{\theta} u_\theta)).
\]
We have
\[(A.3)\]
\[\nabla_s u_s + \nabla_\theta u_\theta = \mathcal{T}_0(u) = \xi^2 e^{2s} u_t.\]
Applying \(\nabla_\theta\), we obtain
\[\nabla_\theta \nabla_s u_s + \nabla_\theta^2 u_\theta = 2\xi \nabla_\theta \xi e^{2s} u_t + \xi^2 e^{2s} \nabla_\theta u_t + \xi^2 e^{2s} \nabla_t u_\theta,\]
and
\[\nabla_\theta^2 u_s = -\nabla_\theta^3 u_\theta + \left(2(\nabla_\theta \xi)^2 + 2\xi \nabla_\theta^2 \xi\right) e^{2s} u_t + 2\xi \nabla_\theta \xi e^{2s} \nabla_t u_\theta + \xi^2 e^{2s} \nabla_\theta \nabla_t u_\theta + I,\]
where
\[\[A.4\]
\[I = (\xi^2 - 1) e^{2s} \nabla_\theta^2 u_t + e^{2s} R(\theta, u_t) u_\theta + \left(2(\nabla_\theta \xi)^2 + 2\xi \nabla_\theta^2 \xi\right) e^{2s} u_t + 2\xi \nabla_\theta \xi e^{2s} \nabla_\theta u_t.\]
We may also commute derivatives to obtain
\[\nabla_s^2 \nabla_\theta u_\theta = \nabla_s \left(\nabla_\theta \nabla_s u_\theta + R(u_s, u_\theta) u_\theta\right)\]
\[= \nabla_\theta \nabla_s^2 u_\theta + \nabla_s \left(R(u_s, u_\theta) u_\theta\right)\]
\[= \nabla_\theta \nabla_s (\nabla_\theta u_s) + \nabla_s \left(R(u_s, u_\theta) u_\theta\right)\]
\[= \nabla_\theta^2 \nabla_s u_s + II,\]
where
\[II = \nabla_\theta \left(R(u_s, u_\theta) u_s\right) + \nabla_s \left(R(u_s, u_\theta) u_\theta\right)\]
\[= \nabla R(u_\theta, u_s, u_\theta) u_s + R(\nabla_\theta u_s, u_\theta) u_s + R(u_s, \nabla_\theta u_\theta) u_s + R(u_s, u_\theta) \nabla_\theta u_s\]
\[+ \nabla R(u_s, u_s, u_\theta) u_\theta + R(\nabla_s u_s, u_\theta) u_\theta + R(u_s, \nabla_s u_\theta) u_\theta + R(u_s, u_\theta) \nabla_s u_\theta.\]
Inserting (A.4) and (A.6) into (A.2), integrating by parts, and rearranging, we obtain
\[\[A.8\]
\[\frac{1}{2} \left( e^{2s} \partial_t f_1^2 - \partial_s^2 f_1^2 \right) = -\int_{S^1} \left( |\nabla_s \nabla_\theta u_\theta|^2 + |\nabla_\theta^2 u_\theta|^2\right) - \int_{S^1} \left( I + II, \nabla_\theta u_\theta\right).\]
We need the following simple estimates. Since \(|u_\theta| \leq \eta\) is small, we may assume that the image of the curve \(u(s, \theta, t) : S^1 \rightarrow N\) lies in a coordinate chart of \(N\) where the Christoffel symbol \(\Gamma\) is bounded by \(C_N\). Then, in local coordinates, we have
\[|\partial_\theta u_\theta| = |\nabla_\theta u_\theta - \Gamma(u_\theta, u_\theta)| \leq |\nabla_\theta u_\theta| + |\Gamma(u_\theta, \nabla_\theta u_\theta)| \leq \eta + C_N \eta^2 \leq 2\eta,\]
assuming that \(\eta\) is sufficiently small (depending on \(N\)). Thus
\[|\nabla_\theta u_\theta|^2 \geq |\partial_\theta u_\theta + \Gamma(u_\theta, u_\theta)|^2\]
\[\geq |\partial_\theta u_\theta|^2 - 2|\partial_\theta u_\theta| \Gamma(u_\theta, u_\theta) + |\Gamma(u_\theta, u_\theta)|^2\]
\[\geq |\partial_\theta u_\theta|^2 - C\eta |u_\theta|^2.\]
Then the ordinary Poincaré inequality on \(S^1\) yields
\[\[A.9\]
\[f_1^2 = \int_{S^1} |\nabla_\theta u_\theta|^2 \geq \int_{S^1} |u_\theta|^2 - C\eta \int_{S^1} |u_\theta|^2 = (1 - C\eta) f_0^2.\]
A similar argument, applied to $\nabla^2_{\theta} u_\theta$, gives

\[(A.10) \quad \int_{S^1} |\nabla^2_{\theta} u_\theta|^2 \geq (1 - C\eta) f_1^2. \]

We first apply (A.10) and Hölder’s inequality to (A.8), to obtain

\[(A.11) \quad \frac{1}{2} \left(e^{2s} \partial_t f_1^2 - \partial^2_s f_1^2 + (1 - C\eta) f_1^2\right) \leq \int_{S^1} (\nabla_s \nabla_\theta u_\theta)^2 - \int_{S^1} (I + II, \nabla_\theta u_\theta)
\leq \int_{S^1} |\nabla_s \nabla_\theta u_\theta|^2 + (\|I\|_{L^2(S^1)} + \|II\|_{L^2(S^1)}) f_1. \]

Note that

\[\frac{1}{2} \partial_s f_1^2 = f_1 \partial_s f_1 = \int_{S^1} (\nabla_s \nabla_\theta u_\theta, \nabla_\theta u_\theta) \leq \left(\int_{S^1} |\nabla_s \nabla_\theta u_\theta|^2\right)^{1/2} f_1, \]

so we have

\[|\partial_s f_1|^2 \leq \int |\nabla_s \nabla_\theta u_\theta|^2. \]

On the other hand,

\[\frac{1}{2} \partial^2_s (f_1^2) = f_1 \cdot \partial^2_s f_1 + |\partial_s f_1|^2. \]

Hence, “dividing out” by $f_1$ in (A.11) (which is justified in the distribution sense), we obtain

\[(A.12) \quad e^{2s} \partial_t f_1 - \partial^2_s f_1 + (1 - C\eta) f_1 \leq 2 \left(\|I\|_{L^2(S^1)} + \|II\|_{L^2(S^1)}\right). \]

It remains to estimate the terms on the RHS of (A.12). By (2.9), we have

\[e^{2s} (|u_t| + |\nabla_\theta u_t| + |\nabla^2_\theta u_t|) \leq C\eta. \]

Combining this with (2.2), from (A.5), we obtain

\[\|I\|_{L^2(S^1)} \leq C\xi_0 \eta e^{2s} + \eta^2 f_0 \leq C\xi_0 \eta e^{2s} + \eta f_1, \]

where we have used (A.9). From (A.7), since each term has at least one factor of $u_\theta$ or $\nabla_\theta u_\theta$, we also obtain

\[\|II\|_{L^2(S^1)} \leq C_N \eta^2 f_1 \leq \eta f_1, \]

for $\eta$ sufficiently small. Inserting these estimates into (A.12), and absorbing the $C\eta f_1$ terms in the LHS, we obtain

\[(A.13) \quad e^{2s} \partial_t f_1 - \partial^2_s f_1 + (1 - C\eta) f_1 \leq C\xi_0 \eta e^{2s}. \]

Translating the above equation back to polar coordinates, we get (2.10).

To estimate the radial energy

\[g = \sqrt{\int_{\{e^s\} \times S^1} |u_s|^2 d\theta}, \]

we start from the identity

\[(A.14) \quad \frac{1}{2} \partial_s^2 g^2 = \int_{S^1} \left(|\nabla_s u_s|^2 + \langle \nabla^2_s u_s, u_s \rangle\right). \]
Applying $\nabla_s$ to (A.3), we obtain
\[
\nabla_s^2 u_s + \nabla_s \nabla_t u_{\theta} = (2\xi \partial_s \xi + 2\xi^2) e^{2s}u_t + \xi^2 e^{2s} \nabla_s u_t
\]
\[
= 2\left(\xi^{-1} \partial_s \xi + 1\right) T_0(u) + \xi^2 e^{2s} \nabla_t u_s
\]
\[
= 2(\nabla_s u_s + \nabla_\theta u_{\theta}) + e^{2s} \nabla_t u_s + 2\xi^{-1} \partial_s \xi (\nabla_s u_s + \nabla_\theta u_{\theta}) + (\xi^2 - 1) e^{2s} \nabla_s u_t.
\]
We also have
\[
\nabla_s \nabla_\theta u_{\theta} = \nabla_\theta^2 u_s + R(u_s, u_{\theta}) u_{\theta}.
\]
Returning to (A.14), we have
\[
\frac{1}{2} \partial_s^2 g^2 = \int |\nabla_s u_s|^2 + \int \left( e^{2s} \nabla_t u_s + 2\nabla_s u_s + 2\nabla_\theta u_{\theta}, u_s \right)
\]
\[
+ \int |\nabla_\theta u_{\theta}|^2 + \int \left( 2\xi^{-1} \partial_s \xi (\nabla_s u_s + \nabla_\theta u_{\theta}) + (\xi^2 - 1) e^{2s} \nabla_s u_t, u_s \right) - \int R(u_s, u_{\theta}) u_{\theta},
\]
where we have integrated by parts once. Applying Hölder’s inequality, rearranging, and using (2.2), we obtain
\[
\frac{1}{2} \left( e^{2s} \partial_t - \partial_s^2 + 2\partial_s \right) g^2 \leq -\int |\nabla_s u_s|^2 + 2f_1 g + C\xi_0 \eta e^{2s} g
\]
\[
- \int |\nabla_\theta u_{\theta}|^2 + C_N \eta f_0 g.
\]
We choose $\eta$ small enough that $C_N \eta \leq 1$, and discard the $-\int |\nabla_\theta u_{\theta}|^2$ term. After “dividing out” by $g$ as above, we obtain
\[
(A.15) \quad \left( e^{2s} \partial_t - \partial_s^2 + 2\partial_s \right) g \leq 6f_1 + C\xi_0 \eta e^{2s}.
\]
Changing back to polar coordinates and dividing by $r$, we get the desired evolution equation.

**Appendix B. Radial heat kernel**

In this appendix, we extract several results from the appendix of [14], replacing the integer dimension by a real number $\mu > 1$. The proofs of Propositions 2.2-2.3 are based on these results.

Let
\[
\Delta_{\mu} = \partial_r^2 + \frac{\mu - 1}{r} \partial_r.
\]
In the case that $\mu$ is an integer, the spherical average of the Euclidean heat kernel is given by
\[
(B.1) \quad H(r, s, t) = \frac{c_\mu e^{-\frac{(r^2 + s^2)}{4t}}}{t^{\mu/2}} I \left(\frac{r s}{2t}\right),
\]
where $c_\mu$ is an appropriate constant, and \(^3\)
\[
(B.2) \quad I(x) = \int_0^\pi e^{x \cos \theta} \sin^{\mu-2} \theta \, d\theta.
\]
\(^3\)Since $I$ satisfies the ODE (B.4), we in fact have
\[
I(x) = x^{1-\frac{\mu}{2}} I_{\frac{\mu}{2}-1}(x),
\]
where $I_{\frac{\mu}{2}-1}$ is the modified Bessel function. This recovers formula (3.3) of Bragg [2].
Lemma B.1. For any real $\mu > 1$, the above function $H$ satisfies
\[
(\partial_t - \Delta_\mu) H(\cdot, s, t) = 0, \quad t > 0,
\]
\[
H(r, s, t) > 0 \text{ for } 0 < r, s, t < \infty,
\]
\[
H(r, s, t) \to \frac{1}{s^{\mu-1}} \delta(r - s) \quad t \to 0,
\]
and
\[
\frac{C e^{-x^2/4t}}{t^{\mu/2}} \leq H(r, s, t) \leq \frac{C e^{-x^2/4t}}{t^{\mu/2}}.
\]

Proof. We calculate
\[
(\partial_t - \Delta_\mu) H(r, s, t) = \frac{C e^{-x^2/4t}}{t^{\mu/2}} \left( I''(x) + \frac{\mu - 1}{x} I'(x) - I(x) \right).
\]

For $I(x)$ given by (B.2), we have
\[
I''(x) + \frac{\mu - 1}{x} I'(x) - I(x) = \int_0^\pi e^x \cos \theta \left( \cos^2 \theta + \frac{\mu - 1}{x} \cos \theta - 1 \right) \sin^{\mu-2} \theta \, d\theta
\]
\[
= \int_0^\pi e^x \cos \theta \left( -\sin^2 \theta + \frac{\mu - 1}{x} \cos \theta \right) \sin^{\mu-2} \theta \, d\theta
\]
\[
= 0,
\]
after integrating by parts. Hence, $H$ solves the PDE as required for any real $\mu$.

Borrowing a factor of $e^{x^2/4t}$ in (B.2), we have
\[
H(r, s, t) = \frac{C e^{-x^2/4t}}{t^{\mu/2}} I_1 \left( \frac{rs}{2t} \right),
\]
where
\[
I_1(x) = \int_0^\pi e^{x \cos \theta - 1} \sin^{x^2/2} \theta \, d\theta.
\]

Then $I_1(x)$ clearly tends to a positive constant as $x \to 0$. By the substitution $u = \sqrt{x(1 - \cos \theta)}$, it follows that the integral is bounded by a constant times $x^{-\mu/2}$. Hence
\[
I_1(x) \leq \frac{C}{(1 + x)^{\mu/2}} \leq \left( \frac{t}{rs + t} \right)^{\mu/2}.
\]
Substituting into (B.5), we obtain the desired bound. \qed

B.1. Initial data. Let $H_{[\rho, R]}(r, s, t)$ be the Dirichlet kernel for the operator $\partial_t - \Delta_\mu$ on the interval $[\rho, R]$, satisfying
\[
(\partial_t - \Delta_\mu) H_{[\rho, R]}(\cdot, s, t) = 0, \quad t > 0,
\]
\[
H_{[\rho, R]}(\rho, s, t) = 0 = H_{[\rho, R]}(R, s, t) \quad \rho \leq s \leq R, \quad t > 0,
\]
\[
H_{[\rho, R]}(r, s, t) \to \frac{1}{s^{\mu-1}} \delta(r - s) \quad t \to 0.
\]
By the maximum principle, we have $0 \leq H_{[\rho,R]}(r,s,t) \leq H(r,s,t)$, and so

$$0 \leq H_{[\rho,R]}(r,s,t) \leq \frac{C_\mu e^{-\frac{(r-s)^2}{4t}}}{t^{1/2}(rs + t)^{\mu/2}}.$$ \hfill (B.6)

Given an initial function $\varphi(r)$ on $[\rho,R]$, the solution of the initial-value problem is given by

$$v_0(r,t) = \int_\rho^R H_{[\rho,R]}(r,s,t)\varphi(s)s^{\mu-1} \, ds.$$ \hfill (B.7)

Let

$$w_0(r,t) = \left(\frac{r^2}{r^2 + t}\right)^{a/2}.$$

**Proposition B.2.** For $0 \leq k \leq \mu - 1$, assuming that $|\varphi(r)| \leq Ar^{-k}$, we have

$$|v_0(r,t)| \leq C_\mu Ar^{-k}w^k(r,t)w^{\mu-k}(R,t).$$

**Proof.** From (B.6) and (B.7), we have

$$|v_0(r,t)| \leq CA \int_\rho^R e^{-\frac{(r-s)^2}{4t}} \frac{s^{-k+\mu-1}}{(rs + t)^{\mu/2}} \frac{ds}{t^{1/2}} \leq CA
R^{-k} \int_\rho^R e^{-\frac{(r-s)^2}{4t}} \frac{r^k s^{-k+\mu-1}}{(rs + t)^{\mu/2}} \frac{ds}{t^{1/2}}.$$ By Lemma A.1a of [14], applied with $a = k, b = \mu - k - 1, c = d = 0$, we have

$$\int_\rho^R e^{-\frac{(r-s)^2}{4t}} \frac{r^k s^{\mu-k-1}}{(rs + t)^{\mu/2}} \frac{ds}{t^{1/2}} \leq C \frac{R - \rho}{R - \rho + \sqrt{t}} w^k(r,t)w^{\mu-k-1}(R,t) \leq C w^k(r,t)w^{\mu-k}(R,t).$$

The result follows. \hfill \square

### B.2. Boundary data.

To construct a kernel for the boundary data at the inner radius $\rho = 1$, we follow the argument of [14], Appendix A.3. Suppose $R > 1$, and let

$$h(r) = \frac{r^{2-\mu} - R^{2-\mu}}{1 - R^{2-\mu}}.$$  

Let

$$y_1(r,t) = h(r) - \int_1^R H_{[1,R]}(r,s,t)h(s) s^{\mu-1} \, ds.$$  

This satisfies

$$(\partial_t - \Delta_\mu)y_1 = 0$$

$$y_1(r,0) = 0, \quad 1 < r < R$$

$$y_1(1,t) = 1, \quad y_1(R,t) = 0, \quad t > 0.$$  

The function

$$G_{[1,R]}(r,t) = \partial_t y_1(r,t)$$

satisfies

$$\lim_{r \searrow 1} G_{[1,R]}(r,t) = \delta(t).$$
Lemma B.3. We have
\[ 0 \leq G_{[1,R]}(r,t) \leq \frac{C_{\mu}e^{-\frac{(r-1)^2}{5t}}}{t^{(t+1)^{\frac{\mu}{2}-1}}} \begin{cases} \min\left[\frac{(r-1)}{\sqrt{t}}, 1\right] & (t \leq 1) \\ \min\left[r-1, 1\right] & (t \geq 1). \end{cases} \]

Proof. Replacing $n$ by $\mu$, the bound is identical to that of Lemma A.4b of [14], and the proof there carries over.

To obtain an inner boundary kernel for $[\rho,R]$, we let
\[ G_{[\rho,R]}(r,t) = \frac{1}{\rho^2} G_{[1,R/\rho]}(r/\rho, t/\rho^2). \]

By Lemma B.3, it satisfies
\begin{equation}
(G.8) \quad G_{[\rho,R]}(r,t) \leq \frac{C_{\mu}e^{-\frac{(r-\rho)^2}{5t}}\rho^{\mu-2}}{t^{(t+\rho^2)^{\frac{\mu}{2}-1}}} \begin{cases} \min\left[\frac{(r-\rho)}{\sqrt{t}}, 1\right] & (t \leq 1) \\ \min\left[r-\rho, 1\right] & (t \geq 1). \end{cases} \tag{B.8}
\end{equation}

The solution of the boundary problem with data $\psi(t)$ at $r = \rho$ is given by
\begin{equation}
(v.9) \quad v_1(r,t) = \int_0^t \psi(\tau) G_{[\rho,R]}(r,t-\tau) \, d\tau. \tag{B.9}
\end{equation}

Proposition B.4. For $2\rho \leq r \leq R$ and $t \geq 0$, we have
\[ |v_1(r,t)| \leq C_{\mu} e^{-\frac{(r-\rho)^2}{6t}} \left( \frac{\rho^{\mu-2}}{r^{\mu-1}} \right) \sqrt{\int_0^t \psi^2(\tau) \, d\tau}. \]

Proof. We apply Hölder’s inequality as in the proof of Proposition A.5b of [14]. From (B.8-B.9), we have:
\begin{align*}
\rho^{-1}|v_1(r,t)| & \leq C_{\mu} \rho^{\mu-2} \int_0^t |\psi(\tau)| \frac{\rho^{\mu-1} e^{-\frac{(r-\rho)^2}{3(t-\tau)}}}{(t-\tau)(t-\tau+\rho^2)^{\frac{\mu}{2}-1}} \, d\tau \\
& \leq C_{\mu} \rho^{\mu-2} \sqrt{\int_0^t \psi^2(\tau) \, d\tau} \\
& \quad \cdot \sqrt{\int_0^t \frac{\rho^{2\mu-2}(r-\rho)^2}{(t-\tau)^2(t-\tau+\rho^2)^{\mu-2} e^{-\frac{2(r-\rho)^2}{3(t-\tau)}}}} \frac{e^{-\frac{2(r-\rho)^2}{3(t-\tau)}}}{(r-\rho)^2}. \tag{B.10}
\end{align*}

The result follows by changing variables $u = \frac{\tau}{(r-\rho)^2}$.

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University of Wisconsin, Madison

Email address: waldron@math.wisc.edu