Abstract. We prove a stronger version of the Kontsevich Formality Theorem for orientable manifolds, relating the Batalin-Vilkovisky (BV) algebra of multivector fields and the homotopy BV algebra of multidifferential operators of the manifold.

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1. Introduction

Given a manifold $M$, the space of multidifferential operators of $M$, $D_{\text{poly}}(M)$ is a smooth version of the Hochschild complex of the functions on $M$. Both $D_{\text{poly}}(M)$ and the space $T_{\text{poly}}(M)$ of multivector fields of $M$ are (shifted) differential graded Lie algebras. These two objects are related by the Hochschild-Kostant-Rosenberg Theorem that provides us with a quasi-isomorphism $T_{\text{poly}}(M) \to D_{\text{poly}}(M)$. However, this map not compatible with the Lie structure.

Searching for a canonical formal quantization of Poisson manifolds, in [KO] M. Kontsevich establishes the existence of a homotopy Lie quasi-isomorphism $T_{\text{poly}}(M) \to D_{\text{poly}}(M)$ extending the Hochschild-Kostant-Rosenberg map. This map, nowadays called Kontsevich's Formality morphism, has a very explicit description involving integrals over configuration spaces of points when $M = \mathbb{R}^d$.

Taking the wedge product into consideration $T_{\text{poly}}$ is a Gerstenhaber algebra, and even if $D_{\text{poly}}$ is not a Gerstenhaber algebra, its homology is in a standard way. It is natural to ask whether one can put a homotopy Gerstenhaber algebra structure on $D_{\text{poly}}$ that induces the usual Gerstenhaber algebra in the cohomology (Deligne’s conjecture) and find a Formality morphism satisfying the Gerstenhaber structure up to homotopy. This question has been answered affirmatively by D. Tamarkin [Ta, Hi].
In [Wi], T. Willwacher uses a different model for the Gerstenhaber operad, the Braces operad, that acts naturally on \( D_{\text{poly}} \) given the nature of the formulas. Willwacher proves in loc. cit. a homotopy Braces version of the Formality morphism.

In this paper we intend to take the final step on this chain of results by showing a BV version of the Formality Theorem(s). As described in Section 2 we can endow both \( T_{\text{poly}}(\mathbb{R}^d) \) and the cohomology of \( D_{\text{poly}}(\mathbb{R}^d) \) with a degree -1 operator, extending the previous Gerstenhaber structures to BV algebra structures.

The cyclic structure of \( D_{\text{poly}}(\mathbb{R}^d) \) leads to the construction of \( \mathbb{CBr} \), the Cyclic Braces operad which is a refinement of the Braces operad. We show that the operad \( \mathbb{CBr} \) is quasi-isomorphic to \( \mathbb{BV} \), the operad governing BV algebras, and the action of \( \mathbb{CBr} \) on \( D_{\text{poly}}(\mathbb{R}^d) \) descends to the canonical BV algebra structure on \( D_{\text{poly}}(\mathbb{R}^d) \).

In section 5 we show that the BV action on \( T_{\text{poly}}(\mathbb{R}^d) \) can be lifted to an action of \( \mathbb{CBr}_\infty \), a resolution of \( \mathbb{CBr} \) and we show the first main Theorem.

**Theorem 1.** There exists a \( \mathbb{CBr}_\infty \) quasi-isomorphism \( T_{\text{poly}}(\mathbb{R}^d) \to D_{\text{poly}}(\mathbb{R}^d) \).

The components of this morphism are defined through integrals similarly to Kontsevich’s case.

The formality of the Cyclic Braces operad implies that in the previous Theorem \( \mathbb{CBr}_\infty \) can be replaced any other cofibrant resolution of \( \mathbb{BV} \), namely its minimal model or the Koszul resolution of \( \mathbb{BV} \).

If we require orientability of the manifold \( M \), the spaces \( T_{\text{poly}}(M) \) and \( H(D_{\text{poly}}(M)) \) still have natural BV structures. Using formal geometry techniques, together with the formalism of twisting of bimodules, in Section 6 we show a global version Theorem 1.

**Theorem 2.** Let \( M \) be an oriented manifold. There exists a \( \mathbb{CBr}_\infty \) quasi-isomorphism \( T_{\text{poly}}(M) \to D_{\text{poly}}(M) \) extending Kontsevich’s Formality morphism.

Applications of this theorem to string topology are expected.

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1.2. **Notation and conventions.** In this paper we work over the field \( \mathbb{R} \) of real numbers, even though the “algebraic” results hold in any field of characteristic zero.

All algebraic objects objects are differential graded, or dg for short, unless otherwise stated so.

If \( \mathcal{P} \) is a 2-colored operad, we denote the space of operations with \( m \) inputs in color 1, \( n \) inputs in color 2 and output in color \( i \) by \( \mathcal{P}^i(m, n) \) and we might denote \( \mathcal{P} \) by \( (\mathcal{P}^1, \mathcal{P}^2) \).

2. **Preliminaries**

2.1. **BV algebras.** Let us recall the definition of a BV algebra and also fix degree conventions.
Definition 1. A Batalin-Vilkovisky algebra or BV-algebra is a quadruplet $(A,\cdot,[\ ,\ ],\Delta)$, such that:

- $(A,\cdot)$ is a (differential graded) commutative associative algebra,
- $(A,[\ ,\ ])\text{ is a 1-shifted Lie algebra (i.e., the bracket has degree -1),}$
- $(A,\cdot,[\ ,\ ])\text{ is a Gerstenhaber algebra, i.e., for all }a\in A\text{ of degree }|a|,\text{ the}$
  \begin{equation*}
  [a,-]\text{ is a derivation of degree }|a|-1.
  \end{equation*}
- $\Delta: A \rightarrow A\text{ is a unary linear operator of degree }-1\text{ such that }\Delta\text{ is a derivation}$
  \begin{equation*}
  \text{of the bracket},
  \end{equation*}
- The bracket is the failure of $\Delta$ being a derivation for the product, i.e.,
  \begin{equation*}
  [-,-] = \Delta \circ (-\cdot-) - (\Delta (-) \cdot -) - (- \cdot \Delta (-)).
  \end{equation*}

We denote by BV, the operad governing BV algebras.

2.2. Hochschild cochain complex. In this section we recall the basics of Hochschild cohomology. For a more detailed introduction, along with the missing proofs, see [Lo].

Let $A$ be a non-graded associative algebra.

For $f:A^m\rightarrow A$ and $g:A^n\rightarrow A$, we define $f\circ_i g: A^{m+n-1}\rightarrow A$, for $i = 1,\ldots, m$, to be the insertion of $g$ at the $i$-th slot of $f$,

\begin{equation*}
  f \circ_i g(a_1,\ldots,a_{m+n-1}) = f(a_1,\ldots,a_{i-1},g(a_i,\ldots,a_{i+n-1}),\ldots,a_{m+n-1}).
\end{equation*}

Lemma 3. Let $f:A^m\rightarrow A$ and $g:A^n\rightarrow A$. The operation $f \circ g: A^{m+n-1}\rightarrow A$ given by

\begin{equation*}
  f \circ g = \sum_{i=1}^{m} (-1)^{i-1} f \circ_i g,
\end{equation*}

defines a pre-Lie product (of degree -1).

This defines a $-1$ shifted graded Lie algebra structure on $\prod_{n\geq 0} \text{Hom}(A^n, A)$. Let $\mu: A^{\otimes 2} \rightarrow A$ be the multiplication of the algebra.

Since $A$ is an associative algebra, we have

\begin{equation*}
  [\mu, \mu](a_1, a_2, a_3) = 2\mu(\mu(a_1, a_2), a_3) - 2\mu(\mu(a_1, a_2), a_3) = 0,
\end{equation*}

i.e., $\mu$ is a Maurer-Cartan element of the Lie algebra $\prod_{n\geq 0} \text{Hom}(A^n, A)$.

Definition 2. Let $A$ be an associative algebra. The Hochschild cochain complex of $A$, $(C^\ast(A), d)$ is defined by

\begin{equation*}
  C^n(A) = \text{Hom}(A^\otimes n, A); \quad d = [\mu, -].
\end{equation*}

Explicitly, for $f \in C^n(A)$ and $a_i \in A$, the differential is given by $df(a_1,\ldots,a_{n+1}) = a_1f(a_2,\ldots,a_{n+1}) + \sum_{i=1}^{n-1} (-1)^{i-1} f(a_1,\ldots,a_ia_{i+1},\ldots,a_n) + (-1)^n f(a_1,\ldots,a_n)a_{n+1}$.

Definition 3. The Hochschild cohomology of an associative algebra $A$ is the cohomology of the complex $C^\ast(A)$ and is denoted by $HH^\ast(A)$.

Definition 4. Let $f \in C^n(A)$ and $g \in C^n(A)$. The cup product on Hochschild cochains $f \cup g \in C^{m+n}(A)$ is defined by

\begin{equation*}
  f \cup g(a_1,\ldots,a_{m+n}) = f(a_1,\ldots,a_m) \cdot g(a_{m+1},\ldots,a_{n+m}),
\end{equation*}

The cup product is trivially associative but, in general, non-commutative and it
does not satisfy the desired compatibility with the Lie bracket. However, as the
following proposition tells us, this is rectified at the cohomological level.

**Proposition 4.** The cup product and the Lie bracket above defined, induce a Ger-
stenhaber algebra structure on $HH^*(A)$.

### 2.3. Multidifferential operators

Let $M$ be an oriented manifold. One of the central objects of this paper is the space of multidifferential operators of $M$, which are a smooth analog of the Hochschild cochain complex.

**Definition 5.** Let $A = C^\infty(M)$, the algebra of smooth functions of $M$. The space of multidifferential operators $D^\bullet_{poly}(M)$ or just $D_{poly}$ if there is no ambiguity, is a subspace of $C^\bullet(A)$, given by

$$D^n_{poly} = \left\{ D: C^\infty(M)^\otimes^n \to C^\infty(M) \mid D \text{ is locally } \sum f \frac{\partial}{\partial x_{I_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{I_n}} \right\},$$

where the $I_j$ are finite sequences of indices between 1 and $\dim(M)$ and $\frac{\partial}{\partial x_{I_j}}$ is the multi-index notation representing the composition of partial derivatives.

We will now describe an action of the group $C_{n+1} = \langle \sigma_n | \sigma_n^{n+1} = e \rangle$ on $D^n_{poly}$.

Since every multidifferential operator is uniquely determined by evaluation on the compactly supported functions $C^\infty_c(M)$, then, $D^n_{poly}$, for $n \geq 1$ can be seen as a subspace of $\text{Hom}(C^\infty_c(M)^\otimes n, C^\infty_c(M))$. One can equally see $D_{poly}$ as a subspace of $\text{Hom}(C^\infty_c(M)^\otimes n+1, \mathbb{R})$ in the following way:

Let us denote by $vol$ the given volume form $M$. We identify $D \in D^n_{poly} \subset \text{Hom}(C^\infty_c(M)^\otimes n, C^\infty_c(M))$, with

$$\left[ f_1 \otimes \cdots \otimes f_{n+1} \mapsto \int_M f_1 D(f_2, \ldots, f_{n+1})\ vol \right] \in \text{Hom}(C^\infty_c(M)^\otimes n+1, \mathbb{R}).$$

The reverse identification can be obtained by integrating by parts in order to remove differential operators from $f_1$.

From now on we drop the $M$ as the domain of integration and the $\text{vol}$ to make the notation lighter.

There is an action of $C_{n+1}$ on $D^n_{poly} \subset \text{Hom}(C^\infty_c(M)^\otimes n+1, \mathbb{R})$ is given by the cyclic permutation of the inputs.

$$\int f_1 D(f_2, \ldots, f_{n+1}) = \int f_2 D(f_3, \ldots, f_{n+1}, f_1).$$

**Definition 6.** The Connes $B$ operator on $D_{poly}$, is the map $B: D^\bullet_{poly} \to D^{\bullet-1}_{poly}$ defined for all $D \in D^n_{poly}$ by

$$B(D)(f_1, \ldots, f_{n-1}) = \sum_{k=0}^{n} (-1)^k D^k(f_1, \ldots, f_{n-1}), \quad \forall f_i \in C^\infty(M).$$

**Proposition 5.** The $B$ operator induces a well defined map in the cohomology of $D_{poly}$ and together with the Lie bracket and cup product defined in the previous section induces a BV-algebra structure in $H^*(D_{poly})$.

The proposition can be proved “by hand”, but also will also follow from the result that the operad $CBr$, whose homology is the $BV$ operad, acts on $D_{poly}$.
2.4. Multivector fields.

**Definition 7.** Let $M$ be an oriented manifold. The graded vector space $T_{\text{poly}}(M)$ or just $T_{\text{poly}}$ of multivector fields on $M$ is

$$T_{\text{poly}}^\bullet = \Gamma(M, \bigwedge^\bullet T_M),$$

where $T_M$ is the tangent bundle of $M$.

$T_{\text{poly}}$ has a natural Gerstenhaber algebra structure by taking as product the wedge product of multivector fields and as bracket, the Schouten-Nijenhuis bracket, i.e., the unique $\mathbb{R}$-linear bracket satisfying

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(|X|-1)(|Y|-1)} Y \wedge [X, Z], \quad \forall X, Y, Z \in T_{\text{poly}}^\bullet$$

that restricts to the usual Lie bracket of vector fields.

We can define a map $f: T_{\text{poly}}^\bullet(M) \to \Omega^n_{dR}(M)$ that sends a multivector field to its contraction with the volume form of $M$.

This map is easily checked to be an isomorphism of vector spaces. We define the divergence operator $\text{div}$ to be the pullback of the de Rham differential via $f$, i.e.

$$\text{div} = f^{-1} \circ d_{dR} \circ f.$$

A series of straightforward calculations prove the following:

**Proposition 6.** The space $T_{\text{poly}}^\bullet(M)$, with the wedge product, the Schouten-Nijenhuis bracket and the divergence operator forms a BV-algebra.

3. Cyclic Swiss Cheese type operads

3.1. Cyclic operads. The standard notion of an operad is used in order to describe operations on a certain vector space with a given number of inputs and one output. A symmetric operad is used when one wants to take into consideration the symmetries on the inputs. The notion of a cyclic operad [GK, LV], introduced by Getzler and Kapranov, arises when one considers the output as an additional input that can be cyclically permuted along with the remaining inputs. This can arise naturally in many situations, for example, when one is given a finite dimensional vector space $V$ equipped with a non-degenerate symmetric bilinear form, the space $\text{Hom}(V \otimes^n V, V)$ can be identified with $\text{Hom}(V \otimes^{n+1} V, \mathbb{R})$.

**Definition 8.** A cyclic operad on a symmetric monoidal category $(\mathcal{C}, \otimes, I, s)$ is the data of a non-symmetric operad $\mathcal{P}$ and a right action of $C_{n+1} = \langle \sigma_n | \sigma_n^{n+1} = e \rangle$, the symmetric group of order $n+1$ on $\mathcal{P}(n)$ satisfying the following axioms:

a) The cyclic action on the unit in $\mathcal{P}(1)$ is trivial.

b) For every $m, n \geq 1$, the diagram

$$\begin{array}{cccc}
\mathcal{P}(m) \otimes \mathcal{P}(n) & \xrightarrow{s_1} & \mathcal{P}(m + n - 1) \\
\downarrow \sigma_m \otimes \sigma_n & & \downarrow \sigma_{m+n-1} \\
\mathcal{P}(m) \otimes \mathcal{P}(n) & \xrightarrow{s} & \mathcal{P}(n) \otimes \mathcal{P}(m) & \xrightarrow{s_n} & \mathcal{P}(m + n - 1)
\end{array}$$

commutes.
c) For every \( m, n \geq 1 \) and \( 2 \leq i \leq m \), the following diagram commutes:

\[
P(m) \otimes P(n) \xrightarrow{\sigma_i} P(m + n - 1) \\
\xrightarrow{\sigma_m \otimes \text{id}_{P(n)}} \\
P(m) \otimes P(n) \xrightarrow{\sigma_{i-1}} P(m + n - 1).
\]

3.2. Operad of Cyclic Swiss Cheese type.

**Definition 9.** Let \( P \) be a 2-colored operad that is non-symmetric in color 2. We say that \( P \) is of Swiss Cheese type if \( P^1(m, n) = 0 \) if \( n > 0 \).

Furthermore, \( P \) is said to be of Cyclic Swiss Cheese type if these two following additional conditions hold:

- The cyclic group of order \( n+1 \), \( C_{n+1} \), acts on the right on \( P^2(m, n) \) satisfying the same axioms as the axioms of a cyclic operad,
- The cyclic action is \( P^1 \), equivariant,
- There is a distinguished element \( 1_P \in P^2(0, 0) \).

For simplicity of notation we denote \( P^1(m, 0) \) by \( P^1(m) \). Using the distinguished element \( 1_P \) we define the “forgetful” map \( \text{Forget}_{\infty}; P^2(m, n) \to P^2(m, n - 1) \) by \( \text{Forget}_{\infty}(p) = p^0(\text{id}_{P^1}, \ldots, \text{id}_{P^1}; \text{id}_{P^2}, \ldots, \text{id}_{P^2}, 1_P) \).

A morphism \( P \to Q \) of Cyclic Swiss Cheese type operads is a colored operad morphism that is equivariant with respect to the cyclic action and sends \( 1_P \) to \( 1_Q \).

3.3. Examples.

3.3.1. Multidifferential operators as an operad. Let \( M \) be an oriented manifold. The operad of multidifferential operators \( \hat{D}_{\text{poly}}(M) \), or just \( \hat{D}_{\text{poly}} \), is a differential graded operad concentrated in degree zero with zero differential given by

\[
\hat{D}_{\text{poly}}^n := \hat{D}_{\text{poly}}(n) = \left\{ D: C^\infty(M)^{\otimes n} \to C^\infty(M) \bigg| D \begin{colonequals} \sum f \frac{\partial}{\partial x_1} \otimes \cdots \otimes \frac{\partial}{\partial x_n} \right\} \mathbb{R}
\]

The operadic structure is the one induced by the endomorphisms operad of \( C^\infty(M) \), i.e., given by composition of operators. As any other operad, \( \hat{D}_{\text{poly}} \) can be seen as a 2-colored operad simply by declaring that there are no operations with inputs or outputs in color 1. To endow \( \hat{D}_{\text{poly}} \) with a Cyclic Swiss Cheese Operad type structure we use the cyclic action defined in Section 2.3 and the distinguished element \( 1 \in \hat{D}_{\text{poly}}^0 = C^\infty(M) \) is defined to be the constant function 1.

For every \( D \in \hat{D}_{\text{poly}}^n \) \( \in \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d)) \) we have

\[
\text{Forget}_{\infty}(D) = \int D(\cdot) \text{vol} \in \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n}, \mathbb{R}).
\]

3.3.2. Configurations of framed points. The Fulton-MacPherson topological operad \( \text{FM}_2 \), introduced by Getzler and Jones [GJ] is constructed in such a way that the \( n \)-ary space \( \text{FM}_2(n) \) is a compactification of the configuration space of points labeled \( 1, \ldots, n \) in \( \mathbb{R}^2 \), modulo scaling and translation. The spaces \( \text{FM}_2(n) \) are manifolds with corners with each boundary stratum representing a set of points that got infinitely close.

\[\text{This is almost the object introduced in Section 2.3.} \]

The tilde is a reminder that there is no grading or differential.
Figure 1. Composition of an element of $\mathbb{F}\mathbb{F}_{M}^{2}$ with an element in $\mathbb{F}\mathbb{H}$.

The first few terms are
- $\mathbb{F}\mathbb{M}_{2}(0) = \emptyset$,
- $\mathbb{F}\mathbb{M}_{2}(1) = \{\ast\}$,
- $\mathbb{F}\mathbb{M}_{2}(2) = S^{1}$.

The operadic composition $\circ_{i}$ is given by inserting a configuration at the boundary stratum at the point labeled by $i$. For details on this construction see also [CKTB, Part IV] or [Ko].

**Definition 10.** Let $\mathcal{P}$ be a topological operad such that there is an action of topological group $G$ on every space $\mathcal{P}(n)$ and the operadic compositions are $G$-equivariant. The semi-direct product $G \ltimes \mathcal{P}$ is a topological operad with $n$-spaces

$$(G \ltimes \mathcal{P})(n) = G^{n} \times \mathcal{P}(n),$$

and composition given by

$$(\overline{g}, p) \circ_{i} (\overline{g'}, p') = (g_{1}, \ldots, g_{i-1}, g_{i} g'_{1}, \ldots, g_{i} g'_{m}, g_{i+1}, \ldots, g_{n}, p \circ_{i} (g_{i} \cdot p')), $$

where $\overline{g} = (g_{1}, \ldots, g_{n})$ and $\overline{g'} = (g'_{1}, \ldots, g'_{m})$.

The topological group $S^{1}$ acts on $\mathbb{F}\mathbb{M}$ by rotations. We define the Framed Fulton-MacPherson topological operad $\mathbb{F}\mathbb{F}_{M}^{2}$ to be the semi-direct product $S^{1} \ltimes \mathbb{F}\mathbb{M}_{2}$. Equivalently, $\mathbb{F}\mathbb{F}_{M}^{2}(n)$ is the compactification of the configuration space of points modulo scaling and translation such that at every point we assign a frame, i.e., an element of $S^{1}$. When the operadic composition is performed, the configuration inserted rotates according to the frame on the point of insertion.

We denote by $\mathbb{F}\mathbb{H}_{m,n}$, the space of configurations of $m$ points in the upper half plane labeled by $1, \ldots, m$ and $n$ points at the boundary, labeled by $\overline{1}, \ldots, \overline{n}$, modulo scaling and horizontal translations, with a similar compactification. Similarly, $\mathbb{F}\mathbb{H}_{m,n}$ shall be the compactification of the space of configurations of $m$ framed points in the upper half plane and $n$ non-framed points at the boundary. These spaces are considered unital in the sense that $\mathbb{F}\mathbb{H}_{0,0}$ is topologically a point, instead of the empty space.

Together they form a Swiss Cheese type topological operad $\mathcal{P}$, with $\mathcal{P}^{1} = \mathbb{F}\mathbb{F}_{M}^{2}$ and $\mathcal{P}^{2} = \mathbb{F}\mathbb{H}$ with composition of color 2 being insertion of the correspondent configuration in the boundary stratum and composition of color 1 on the vertex labeled by $i$ being the insertion at the boundary stratum at the point $i$ after applying the corresponding rotation given by the frame of $i$. We shall consider that a framing pointing upwards represents the identity of $S^{1}$, see Figure 1.

In fact they can be endowed with a Cyclic Swiss Cheese type operad structure.

The open upper half plane is isomorphic to the Poincaré disk via a conformal (angle preserving) map. This isomorphism sends the boundary of the plane to the boundary of the disk except one point, that we label by $\infty$. We define the cyclic
action of $C_{n+1}$ in $\text{FFM}_2(m,n)$ by cyclic permutation of the point labeled by infinity with the other points at the boundary.

\[ \begin{array}{c}
\sigma \cdot \infty \\
\sigma \\
\end{array} \]

The element $I$ is defined to be the unique point in $\text{FHM}_{0,0}$. Insertion of this element represents forgetting a certain point at the boundary.

The forgetful map is defined by forgetting the point at infinity and labeling the first point as the new $\infty$ point and the previous $\infty$ becomes the new $n-1$.

\[ \begin{array}{c}
\text{Forget}_\infty \\
\end{array} \]

3.3.3. Two kinds of graphs. A directed graph $\Gamma$ is the data of a finite set of vertices, $V(\Gamma)$ and a set of edges $E(\Gamma)$ consisting of ordered pairs of vertices, that is, a subset of $V(\Gamma) \times V(\Gamma)$. Notice that tadpoles (edges connecting a vertex to itself) are allowed.

Let $\text{BVKGra}'(m,n)$ be the graded vector space spanned by directed graphs with $m$ vertices of type I labeled with the numbers $\{1, \ldots, m\}$, $n$ labeled with the numbers $\{1, \ldots, n\}$ of type II and edges labeled with the numbers $\{1, \ldots, \#\text{edges}\}$, such that there are no edges starting on a vertex of type II. The degree of a graph is $-\#\text{edges}$, i.e., every edge has degree $-1$. For every non-negative integer $d$, there is an action of $S_d$ on $\text{CPT}_d^r(n)$ by permutation of the labels of the edges.

We define the space $\text{BVKGra}$ of BV Kontsevich Graphs by

\[ \text{BVKGra}(m,n) := \bigoplus_d \text{BVKGra}'_d(m,n) \otimes_{S_d} \text{sgn}_d, \]

where $\text{sgn}_d$ is the sign representation.

We define the space of BV Graphs, $\text{BVGr}a(n) := \text{BVKGra}(n,0)$. There is a natural $S_n$ action by permutation of the labels and we define a symmetric operad structure in $\text{BVGr}a$ by setting the composition $\Gamma_1 \circ \Gamma_2$ to be the insertion of $\Gamma_2$ in the $i$-th vertex of $\Gamma_1$ and sum over all possible ways of connecting the edges incident to $i$ to $\Gamma_2$.

We can form a Swiss Cheese type operad by setting $\text{BVGr}a$ to be the operations in color 1 and $\text{BVKGra}$ to be the operations in color 2, considering the symmetric action permuting the labels of type I vertices and ignoring the symmetric action of...
type II vertices. The partial compositions are given as in BVGra, i.e., by insertion on the corresponding vertex and connecting in all possible ways.

The type II vertices in BVKGra will be later seen as boundary vertices when we relate BVKGra with FH, and since we wish to distinguish between BVGra(⋅) and BVKGra(⋅, 0), we draw the latter with a line passing by the type II vertices.

The space BVKGra(m, n) forms a graded commutative algebra with product of two graphs defined by superposing the vertices and taking the union of the edges. This algebra generated by

\[ \Gamma_i^j = \begin{array}{cccccc} i & \cdots & j & \cdots & n \end{array}, \text{with } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \text{ and } \]

\[ \Gamma^{i,j} = \begin{array}{cccccc} i & \cdots & \pi \end{array}, \text{with } 1 \leq i, j \leq m. \] For simplicity, the dependance of m and n is dropped from the notation.

Let \( \Gamma_i^j \in BVKGra(m, n) \). The action of the generator \( \sigma \) of \( C_{n+1} \) on \( \Gamma_i^j \) is \( \sigma(\Gamma_i^j) = \Gamma_{j-i}^j \) if \( j \neq 1 \) and \( \sigma(\Gamma_i^j) = -\sum_{k=1}^n \Gamma_k^j - \sum_{k=1}^m \Gamma_k^i \). The action of \( \sigma \) on \( \Gamma^{i,j} \in BVKGra(m, n) \) is \( \sigma(\Gamma^{i,j}) = \Gamma^{i,j} \), for \( 1 \leq i, j \leq m \).

We define the cyclic action of \( C_{n+1} = \langle \sigma^{n+1}, e \rangle \) on one-edge graphs of BVKGra(m, n) by \( \sigma(\Gamma_i^j) = \Gamma_{j-1}^j \) if \( j \neq 1 \) and \( \sigma(\Gamma_i^j) = -\sum_{k=1}^n \Gamma_k^j - \sum_{k=1}^m \Gamma_k^i \). The action of \( \sigma \) on \( \Gamma^{i,j} \in BVKGra(m, n) \) is defined by \( \sigma(\Gamma^{i,j}) = \Gamma^{i,j} \), for \( 1 \leq i, j \leq m \).

Since \( \sigma^2(\Gamma_i^j) = \Gamma_{m+1}^m \), we have that \( \sigma^{n+1} \) acts as the identity in every one-edge graph, therefore the action of \( C_{n+1} \) is well defined.

We extend this action to BVKGra(m, n) by declaring that the action distributes over a product of graphs (i.e., making the cyclic action a morphism of unital algebras).

The element \( 1 \in BVKGra(0, 0) \) is the empty graph, the unique graph with no vertices. The insertion \( \Gamma \circ \alpha \) is zero if there is any edge incident to the vertex labeled by \( j \) or, if there are no such edges, it forgets the vertex labeled by \( j \).

4. The Cyclic Braces operad

In [KS], Kontsevich and Soibelman introduced an operad that they call minimal operad that acts naturally on the Hochschild cochain complex of \( A_\infty \) algebras.
They show that this operad is quasi-isomorphic to $\mathbf{Ger}$, the operad governing Gerstenhaber algebras (see also [MS]). In this paper we call this operad $\mathbf{Br}$, standing for Braces. In this section we introduce the Cyclic Braces operad, which is a refinement of the Braces operad that is meant to take into account the a unit and a cyclic action. A similar operad was constructed by Ward in [Wa].

4.1. The Cyclic Planar Trees operad. Let $\mathbf{CPT}'(n)$ be the graded vector space spanned by rooted planar trees with vertices labeled with the numbers in $\{1, \ldots, n\}$ with the additional feature that every vertex can have additional edges connecting to a symbol $\mathbb{1}$ and every vertex has a marked edge, that can be one of the additional edges $\mathbb{2}$. The non-root edges (including the ones connecting to $\mathbb{1}$) are labeled by the numbers $\{1, \ldots, \#\text{edges}\}$. The degree of a rooted planar tree is $-\#\text{edges}$. For every non-negative integer $d$, there is an action of $\mathbb{S}_d$ on $\mathbf{CPT}'_d(n)$ by permuting the labels of the edges.

We define the operad $\mathbf{CPT}$ of Cyclic Planar Trees by

$$\mathbf{CPT}(n) := \bigoplus_d \mathbf{CPT}''_d(n) \otimes_{\mathbb{S}_d} \text{sgn}_d,$$

where $\text{sgn}_d$ is the sign representation and $\mathbf{CPT}''$ is the quotient of $\mathbf{CPT}'$ by trees in which there is a vertex is connected to an element $\mathbb{1}$ whose mark is not pointing towards $\mathbb{1}$.

The operadic composition $T_1 \circ_j T_2$ is given by inserting the tree $T_2$ in the vertex labeled $j$ of the tree $T_1$, orienting the root of $T_2$ with the marking at the vertex $j$ of $T_1$, forgetting both the root and the mark at the vertex $j$ and reconnecting all incident edges in all planar possible ways.

Since it it unambiguous, for simplicity of the drawing we draw only a mark between two edges when some vertex is connected to $\mathbb{1}$.

Example 1. Examples of insertion:

The operad is generated by $T^n_i$, $i = 1, \ldots, n$ and $T^{n,i+1}_n$, $i = 1, \ldots, n$, see Figure 2.

---

2In fact, we want at most one edge connecting to $\mathbb{1}$ per vertex and for vertices having an edge connecting to $\mathbb{1}$ we want to force the marked edge to be that one, but imposing this condition directly would not be stable by the composition that we define below. This is resolved by considering a quotient of $\mathbf{CPT}'$, rather than a subspace.
4.2. Algebras over CPT. The operad CPT acts naturally on spaces with cyclic structure.

**Proposition 7.** Let \( \mathcal{P} \) be an operad of Cyclic Swiss Cheese type. Its total space, \( \prod_n \mathcal{P}^2(n)[n] \), forms a \( \text{CPT} \)-bimodule.

**Proof.** To describe the left action of \( \text{CPT} \) we use the following multi-insertion notation:

For \( p_1, p_2, \ldots, p_n \in \mathcal{P}^2 \), \( p_1 \) in arity \( N \), we say that \( I \) is a planar insertion of \( p_2, \ldots, p_n \) in \( p_1 \) if \( I \) is an \( N \)-uple containing each \( p_2, \ldots, p_n \) exactly once, in that order and the other entries are filled with \( \text{id}_{\mathcal{P}^2} \). For \( i = 1, \ldots, n \), we define \( i(I) \) as the position of \( p_i \) in \( I \). By \( p_1(I) \), we mean the operadic composition given by \( I \) (ignoring insertions in color 1).

The action of \( T_n \in \text{CPT} \) is given by braces operations, i.e., \( T_n(p_1, p_2, \ldots, p_n) = p_1\{p_2, \ldots, p_n\} \). The action of \( T_n^i \in \text{CPT} \), for \( i = 1, \ldots, n \) is given by a composition of the braces operation and a permutation of \( C_{N+1} \) “turning the mark in the direction of the root”. Explicitly, if \( \sigma \) is the generator of the cyclic group, \( T_n^i(p_1, p_2, \ldots, p_n) = \sum_I p_1^{\sigma^{-i}(I)}(I) \), where the sum runs over all possible planar insertions \( I \) of \( p_2, \ldots, p_n \) in \( p_1 \).

The action of \( T_n^{i,i+1} \) is given by \( T_n^{i,i+1}(p_1, p_2, \ldots, p_n) = \sum_I (\sum_{k=0}^{(i+1)(I)} \text{Forget}_{\infty}(p_1^{\sigma^{-k}}))(I) \), where the first sum runs over all possible planar insertions \( I \) of \( p_2, \ldots, p_n \) in \( p_1 \). This corresponds to the insertion of the element \( 1_\mathcal{P} \) in the marked space and the permutation sending the mark back to the direction of the root.

**Lemma 8.** A morphism of Cyclic Swiss Cheese type operads induces a morphism of bimodules.

**Proof.** Since a morphism of Cyclic Swiss Cheese type operads is in particular a morphism of colored operads, the induced map on the total space is a morphism of right bimodules. Since the definition of the action of \( \text{CPT} \) uses only the cyclic action and \( \text{Forget}_{\infty} \) and by hypothesis a morphism of Cyclic Swiss Cheese type operads commutes with these maps, the induced map on the total spaces is a left module morphism.

4.3. The operad CBr. We now finish the construction of the Cyclic Braces operad via operadic twisting. There is a map \( F: \text{Lie}(1) \to \text{CPT} \) sending the Lie bracket to
Using $F$ we consider the (dg) operad given by the operadic twisting of $\text{CPT}$, $\text{TwCPT}$ (see the Appendix for details).

The space $\text{TwCPT}(n) = \left( \prod_k \text{CPT}(n + k) \otimes \mathbb{K}[-2]^{\otimes k} \right) S_n$ is made out of trees, similar to the ones in $\text{CPT}$ but with vertices of two different kinds. There are $n$ external vertices, labeled from 1 to $n$ and $k$ internal unlabeled vertices, that we draw as a full black vertex. The degree of each edge or marked space is $-1$, the degree of an external vertex is 0 and the degree of an internal vertex is 2.

This operad is generated by elements as in Figure 2 together with $T^n_{0i}$ and $T^n_{0i+1}$, $i = 0, \ldots, n$:

The differential has two pieces, the first is computed by taking the operadic Lie bracket with $\begin{tikzpicture} \node (a) at (0,0) {1}; \node (b) at (1,0) {2}; \node (c) at (1.5,0) {}; \node (d) at (2,0) {3}; \node (e) at (2.5,0) {}; \node (f) at (3,0) {n}; \node (g) at (1.75,0.5) {}; \node (h) at (1.75,-0.5) {}; \node (i) at (2.25,0.5) {}; \node (j) at (2.25,-0.5) {}; \draw (a) -- (b); \draw (b) -- (c); \draw (c) -- (d); \draw (d) -- (e); \draw (e) -- (f); \draw (h) -- (g); \draw (j) -- (i); \end{tikzpicture}$, which amounts to split an internal vertex at every external vertex, but subtracting some combinations with one 1-valent or 2-valent internal vertex. The second piece just splits an internal vertex out of every internal vertex.

Lemma 9. The subspace $(\text{TwCPT})' \subset \text{TwCPT}$ spanned by trees whose internal vertices are at least 3-valent is a suboperad of $\text{TwCPT}$.

Proof. The composition of trees in $(\text{TwCPT})'$ cannot create internal vertices with valence 1 or 2.

The differential, however can create both kinds of vertices, so we must check that these contributions are canceled.

1-valent internal vertices can be created at every internal vertex by splitting it and reconnecting all edges incident edges to one of the internal vertices. Similarly, 1-valent internal vertices can be created at an external vertex when inserting $T^1_1 + T^1_2 \circ_1 T^0_0$ at that vertex and then reconnect to the external vertex. These contributions are all canceled by the remaining term of the differential consisting in inserting the tree in $T^1_1 + T^1_2 \circ_1 T^0_0$.

To see that 2-valent internal vertices contributions are canceled, it is enough to notice that every time such a vertex is created, it will be canceled by a similar contribution on the other adjacent vertex. \qed
**Definition 11.** We define the Cyclic Braces operad as $\text{CBr} := (T \circ \text{CPT})' / J$, where $J$ is the operadic ideal generated by $T^{i} - T^{i-1}$, $i = 0, \ldots, n$ and $(1)$.

**Remark 10.** The $T^{i} - T^{i-1}$ in $J$ mean that in $\text{CBr}$ the marks at internal vertices are irrelevant. We will therefore not draw them in pictures and we will denote the image of $T^{i}$ in $\text{CBr}$ just by $T^{i}$.

**Convention 11.** Since $J$ is not homogeneous with respect to the number of (internal) vertices, the number of (internal and therefore the total number of) vertices of a cyclic braces tree is a priori ill defined. We shall consider that whenever we have subsection of a tree like this that there is only one edge and no vertices.

**4.4. The homology of the Cyclic Braces operad.** In this subsection we show that the homology of $\text{CBr}$ is the BV operad. For this, we make use of the operad $\text{Br}$ whose homology, as mentioned in the beginning of this section, is the operad $\text{Ger}$.

**Definition 12.** The operad $\text{Br}$ is defined as the suboperad of $\text{CBr}$ generated by $T^{1}_{n}$ and $T^{i}_{n}$, or equivalently, the suboperad spanned by trees whose marks at every vertex are pointing towards the root. In $\text{Br}$ we “forget” that there are marks at vertices, therefore when referring to this operad we use the notation $T$ instead of $T^{1}_{n}$ and we do not draw the marks in the pictures.

Two trees in $\text{CBr}$ are said to have the same shape if when one forgets about the marks at vertices and connections to $\mathbb{B}$, they are the same. For example, $T^{i}_{n}$ and $T^{i+1}_{n}$ have the same shape.

Let us consider the map $f = \otimes f: \text{Br}(n) \otimes (K \oplus K[1])^\otimes n \to \text{CBr}(n)$ sending $T \otimes \epsilon$, where $T$ is braces tree and $\epsilon = \epsilon_{1} \otimes \cdots \otimes \epsilon_{n} \in (K \oplus K[1])^\otimes n$, to the a sum of cyclic braces trees of the same shape, according to the following rules:

- If the $\epsilon_{i} = (1, 0)$, the vertex labeled by $i$ is sent to the same vertex with the marking pointing in the direction of the root.
- If the $\epsilon_{i} = (0, 1)$, the vertex labeled by $i$ is sent to a sum over all possible ways of inserting an edge connecting to $\mathbb{B}$.

**Lemma 12.** $f$ is a quasi-isomorphism of chain complexes.

**Proof.** Since marked spaces have degree $-1$, $f$ preserves the degree. Since the differential acts by derivations, it is enough to check that $f$ commutes with the differentials on every vertex $i$ and this is clearly the case if $\epsilon_{i} = (1, 0)$.

Let us consider the case of $T_{n} = \otimes \epsilon \in \text{Br}$ with $\epsilon_{1} = (0, 1)$.
where the sum runs over all planar possible ways of connecting the incident edges such that the internal vertex is at least trivalent.

We have \( f(T_n) = \sum_{i=1}^{\nu_n} T_{n, i+1} \), following the notation in Figure 2. If we compute \( df(T_n) \), the part of the differential given by the insertion of \( \ast \) on every \( T_{n, i+1} \) is canceled over all the sum.

Therefore \( df(T_n) = \sum_{i} \sum_{\text{ways of connecting}} T_{n, i+1} \). To see that \( df(T_n) = f(dT_n) \), we note that there are two possibilities. Every summand of \( df(T_n) \) has either the internal vertex connected to the root vertex or the vertex labeled by 1 connected to the root vertex. If the root is connected to the internal vertex, we find that same summand on the image by \( f \) of the second type of trees on equation (2), and similarly if the root is connected to the vertex 1.

Conversely, all trees that we get when we compute \( f(dT_n) \) appear only once (due to the planar ordering of edges and marks around a vertex) and can be obtained as a summand in \( df(T_n) \).

To show that \( f \) is a quasi-isomorphism, we filter \( \mathbb{CBr} \) and \( \mathbb{Br} \) by the number of internal vertices (see Remark 10). The map \( f \) is compatible with these filtrations and on the zeroth page of the corresponding spectral sequence in \( \mathbb{CBr} \) one obtains the only piece of the differential that does not increase the number of internal vertices. Explicitly \( d_0(T_{n, i+1}) = T_{n, i+1} - T_n \) and \( d_0(T_{n, j}^j) = 0 \). On the correspondent spectral sequence in \( \mathbb{Br} \) one obtains the zero differential.

The differential \( d_0 \) respects the shape of the tree. Therefore the complex \( (\mathbb{CBr}, d_0) \) splits as

\[
\mathbb{CBr}(n) = \bigoplus_{\text{Shape } S} V_S,
\]

where the sum runs over all possible shapes \( S \) of trees with \( n \) external vertices and \( V_S \) is the subcomplex spanned by all trees with the shape \( S \).

\[\text{(2)} \quad dT_n = \sum_{\text{ways of connecting}} T_{n, i+1} + \sum_{\text{ways of connecting}} T_{n, j}^j,\]
The differential acts on the tree by acting on every vertex by means of the Leibniz rule, therefore if \( V_S^i \) represents the space of the \( i \)-th vertices of the trees with the given shape, then each \( V_S \) splits as a complex as \( V_S = \bigotimes_{i=1}^n V_S^i \) (up to some degree shift).

But \( (V_S^i, d_0) \) is isomorphic to the simplicial complex of the \( k \)-gon, where \( k \) is the valence of the vertex \( i \) (again, up to some degree shift).

Therefore \( H(CBr, d_0) = \bigoplus_{\text{Shape } S} \left( \bigotimes_{i=1}^n H(V_S^i) \right)[k_S] = \bigoplus_{\text{Shape } S} \left( \bigotimes_{i=1}^n (K \oplus K[1]) \right)[k_S] \), where \( k_S \) is a degree shift dependent only on the shape of the tree.

Then, at the level of the homology of the zeroth pages of the spectral sequences we get an induced map \( Br(n) \otimes (K \oplus K[1])^{\otimes n} \to \bigoplus_{\text{Shape } S} (K \oplus K[1])^{\otimes n}[k_S] \).

Since clearly every possible shape of Cyclic Braces trees has a unique representative that is a Braces tree, this induced map is an isomorphism. Therefore \( f \) induces a quasi-isomorphism on the zeroth page of the spectral sequence, which implies that \( f \) is a quasi-isomorphism between the original complexes.

\[ \square \]

**Corollary 13.** The homology of \( CBr \) is \( BV \), the operad governing \( BV \) algebras.

**Proof.** As a consequence of Lemma 12 we have \( H(CBr(n)) = H(Br(n) \otimes (K \oplus K[1])^{\otimes n}) = H(Br(n)) \otimes (K \oplus K[1])^{\otimes n} \cong \text{Ger}(n) \otimes (K \oplus K[1])^{\otimes n} \cong BV(n) \).

Keeping track of signs, it is easy to check that \( \circ \), \( [ \ , \ ] \) and \( \Delta \), the generators of \( BV \).

For example, the equality

\[
\begin{align*}
1 \circ_1 2 &= 1 + 2 + 1 - 2 + 1 + d 1 + d 2 \\
1 \circ_2 2 &= 2 + 1 - 2 + 1 + d 1 + d 2 
\end{align*}
\]

corresponds in homology to the equation \( \Delta \circ_1 = [ \ , \ ] + \circ_1 \Delta + \circ_2 \Delta \).

Therefore, since the dimensions in every arity are the same (and finite), the operad \( H(CBr) \) is canonically isomorphic to \( BV \). \[ \square \]

5. **Operadic bimodule maps**

Given an operad \( P \) and a resolution \( P_{\infty} \subset P_{\text{bimod}} \subset P_{\infty} \) of the canonical bimodule \( P \subset P \subset P \), an infinity morphism of \( P_{\infty} \) algebras \( A \) and \( B \), can be expressed as the following bimodule map:

\[
\begin{array}{ccc}
P_{\infty} & \subset & P_{\text{bimod}} \\
\Downarrow & & \Downarrow \\
\text{End } B & \subset & \text{Hom}(A^{\otimes}, B) \\
\end{array}
\]

\[
\begin{array}{ccc}
P_{\infty} & \subset & P_{\text{bimod}} \\
\Downarrow & & \Downarrow \\
\text{End } A, & \subset & \text{End } A,
\end{array}
\]
where by \( \text{End} A \) we mean the operadic endomorphisms \( \text{End} A(n) = \text{Hom}(A^\otimes n, A) \) and the bimodule structure on \( \text{Hom}(A^\otimes B, B) \) is the natural one using composition of maps. In this section we prove Theorem 1 by expressing it in terms of a morphism of bimodules.

5.1. \( \text{Chains}(F_{\mathbb{H}_{m,n}}) \rightarrow \text{BVKGra} \). The topological operad of Cyclic Swiss Cheese type \( (F_{\mathbb{M}}^2, F_{\mathbb{H}_{m,n}}) \) introduced in 3.3.2 is in fact an operad on the category of semi-algebraic manifolds\(^{\text{HLTV, LV}} \). We consider the functor \( \text{Chains} \) of semi-algebraic chains. This functor is monoidal so it induces a functor from semi-algebraic Cyclic Swiss Cheese type operads to dg Cyclic Swiss Cheese type operads.

In this section we define a morphism of Cyclic Swiss Cheese type operads

\[
\text{Chains}(F_{\mathbb{M}}^2), \text{Chains}(F_{\mathbb{H}_{m,n}}) \rightarrow (\text{BVGr}, \text{BVKGra})
\]

We start by defining a map \( f_2: \text{BVKGra}^* \rightarrow \Omega(F_{\mathbb{H}_{m,n}}) \), where \( \Omega \) is the functor sending a semi-algebraic manifold to its algebra of semi-algebraic forms.

Notice that \( F_{\mathbb{H}_{m,n}} \) is a quotient of the configuration space of \( m \) points in the upper half plane and \( n \) points at the boundary by a group of conformal maps. The identification of \( \mathbb{H} \) with the Poincaré Disk necessary for the definition of the cyclic action and the forgetful map is also conformal. Therefore, given a point \( p \) in the upper half plane and a point \( q \) either in the upper half plane or at the boundary the angle between the hyperbolic line passing by the point at \( \infty \) and \( p \) and the hyperbolic line passing by the points \( p \) and \( q \) is well defined (up to a multiple of \( 2\pi \)).

We define \( d\phi^i_j \in \Omega^1(F_{\mathbb{H}_{m,n}}) \), for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) as the 1-form given by the angle made by the hyperbolic line defined by the point at \( \infty \) and the point labeled by \( i \) and the hyperbolic line defined by the point labeled by \( i \) and the point labeled by \( j \).

Similarly, \( 1 \leq i \neq j \leq m \), we define \( d\phi^{i,j} \in \Omega^1(F_{\mathbb{H}_{m,n}}) \) as the 1-form given by the angle defined by the line passing by \( \infty \) and \( i \) and the line passing by \( i \) and \( j \).

Finally, we define \( d\phi^{i,i} \in \Omega^1(F_{\mathbb{H}_{m,n}}) \) as the 1-form corresponding to the angle between the line passing by \( \infty \) and \( i \) and the frame at \( i \).

\[
\begin{align*}
\text{Figure 3.} \quad \text{The hyperbolic angle } \phi^{i,j} \\
\end{align*}
\]

There is a canonical basis of \( \text{BVKGra}(m, n) \) given by the graphs and, by abuse of notation, we denote by the same graph the dual basis of \( \text{BVKGra}^*(m, n) \).

Following the notation in 3.3.3 we define \( f_2(\Gamma_i^j) := \frac{d\phi^i_j}{2\pi} \) for \( 1 \leq i \leq m \), \( 1 \leq j \leq n \) and \( f_2(\Gamma^{i,j}) := \frac{d\phi^{i,j}}{2\pi} \) for \( i \neq j \) between 1 and \( m \).
BVKGra*(m, n) admits a similar algebra structure by defining the product of two graphs as the superposition of edges. We extend the map \( f_2 \) to BVKGr^* by requiring it to be a morphism of unital algebras.

A \( C_{n+1} \) action on \( \text{BVKGra}^*(m, n) \) can be defined via the pullback of the cyclic action on \( \text{BVKGra}(m, n) \). Notice that this is not the standard definition of an action of a group on the dual space (one normally uses the pullback via the inverse of the map), but since \( C_{n+1} \) is abelian no problems arise from this.

\( \Omega(F\text{H}_{m,n}) \) inherits a \( C_{n+1} \) cyclic action from the cyclic action in \( F\text{H}_{m,n} \) (also by pullback).

**Lemma 14.** The map \( f_2: \text{BVKGra}^*(m, n) \to \Omega(F\text{H}_{m,n}) \) is \( C_{n+1} \) equivariant.

**Proof.** Notice actually that the algebra structure on \( \text{BVKGra}(m, n) \) is in fact the exterior algebra \( \Lambda V \), where \( V \) is the (finite dimensional) vector space concentrated in degree -1 spanned by all graphs with exactly one edge.

We had defined the cyclic action on \( V \), extended this action to \( \Lambda V \) by requiring the action to commute with the product and defined an action on \( (\Lambda V)^* = \text{BVKGra}(m, n) \). Alternatively, the cyclic action on \( V \) induces a cyclic action on \( V^* \) which induces a cyclic action on \( \Lambda V^* \). Under the identification \( \Lambda V^* = (\Lambda V)^* \) these two actions are the same. This is an immediate consequence of the fact that if \( e_1, \ldots, e_n \) are part of a basis of \( V \) and \( e_1^*, \ldots, e_n^* \) are the corresponding parts of the dual basis, then \( e_1^* \wedge \cdots \wedge e_n^* \) is dual to \( e_1^* \wedge \cdots \wedge e_n^* \).

This allows us to conclude that the cyclic action on \( \text{BVKGra}^*(m, n) \) commutes with the product of graphs.

It is therefore enough to show that \( f_2 \) is equivariant with respect to one-edge graphs.

The cyclic action of \( C_{n+1} \) on one-edge graphs in \( \text{BVKGra}^*(m, n) \) is given by \( (\Gamma^{i,j})^\sigma = \Gamma^{i,j} - \Gamma^i_1 \) and \( (\Gamma^i_1)^\sigma = \Gamma^{j+1}_1 - \Gamma^i_1 \) with the convention that \( \Gamma^{n+1}_n = 0 \).

Since the cyclic action on \( F\text{H}_{m,n} \) is by rotation of the \( n \) points at the boundary with the point \( \infty \), we have \( (d\phi^{i,j})^\sigma = d(\phi^{i,j} - \phi^i_1) \) and similarly \( (d\phi^{i,j})^\sigma = (d\phi^{i,j} - d\phi^i_1) \), therefore \( f_2 \) commutes with the action. \( \Box \)

Analogously, a map \( f_1: \text{BVGr}^*(n) \to \Omega(F\text{FM}_2)(n) \) can be defined on one-edge graphs by considering the angle with the vertical and extending as a morphism of algebras.

**Remark 15.** It is easy to check on generators that these maps produce a map of colored cooperads.

\[ (f_1, f_2): (\text{BVGr}^*, \text{BVKGra}^*) \to (\Omega(F\text{FM}_2), \Omega(F\text{H}_{m,n})) \]

Let us sketch the verification for the case of \( \Gamma^{1,2} \in \text{BVKGra}^*(2, 0) \).

The composition map in \( (F\text{FM}_2, F\text{H}) \) is done by insertion at the boundary stratum with an appropriate rotation given by the framing. Since the cocomposition map is given by the pullback of the composition map, the part of the cocomposition given by \( \Omega(F\text{H}) \to \Omega(F\text{H}) \otimes \Omega(F\text{FM}_2) \) sends \( d\phi^{1,2} \in F\text{H}(2,0) \) to \( d\phi^{1,2} \otimes 1 + 1 \otimes d\phi^{1,2} \in \Omega(F\text{H}(1,0)) \otimes \Omega(F\text{FM}_2(2)) \) (recall Figure [4]).

The corresponding cocomposition in BVKGr^* sends \( \Gamma^{1,2} \) to

---

[3] Strictly speaking, the right hand side is not a cooperad, but this does not affect what follows.
\[
\left( \Gamma^{1,1} \otimes 1 \otimes 2 \right) + \left( 1 \otimes 1 \otimes 2 \right) \in \text{BVKGra}^\ast(2,0) \otimes \text{BVGra}^\ast(2), \text{ therefore the diagrams commute. The general case for } \Gamma^{i,j} \in \text{BVKGra}^\ast(m,n) \text{ is similar and all the remaining cases are as simple or even simpler to check.}
\]

We define a map \( g_1 : \text{Chains}(\text{FFM}_2) \to \Omega^\ast(\text{FFM}_2) \) that maps every elementary semi-algebraic chain \( c \in \text{Chains}(\text{FFM}_2) \) to the linear form \( \omega \mapsto \int_c \omega \). Similarly we define \( g_2 : \text{Chains}(\text{FHM}) \to \Omega^\ast(\text{FHM}) \) sending a chain to integration over that chain.

Clearly \( \text{BVKGra}(m,n) \) is finite dimensional for a fixed degree, therefore its double dual of \( \text{BVKGra}(m,n) \) can be identified with the original space.

Finally, the map of Cyclic Swiss Cheese type operads \( \langle \rangle \) that we were searching is defined as the composition

\[
(\text{Chains}(\text{FFM}_2), \text{Chains}(\text{FHM})) \xrightarrow{(g_1,g_2)} (\Omega^\ast(\text{FFM}_2), \Omega^\ast(\text{FHM})) \xrightarrow{(f_1,f_2)} (\text{BVGra,BVKGra}).
\]

This is a colored operad map as a consequence of Remark \( \langle \rangle \) it commutes with the cyclic action as a consequence of Lemma \( \langle \rangle \) and by hand one checks that \( \mathcal{I}_{\text{Chains}(\text{FHM})} \) is sent to \( \mathcal{I}_{\text{BVKGra}} \).

Explicitly, given a chain \( c \in \text{Chains}(\text{FFM}_2) \), we have \( f_1^* \circ g_1(c) = \sum_{\Gamma} \Gamma \int_c f_1(\Gamma) \), where \( \Gamma \) runs through all the graphs in \( \text{BVGra} \). This sum is finite because the integral is zero every time the degree of \( \Gamma \) differs from the degree of the chain \( c \).

Recall section \( \langle \rangle \) where we saw that given a Cyclic Swiss Cheese type operad \( \mathcal{P} \) one can endow the total space \( \Pi_n \mathcal{P}^2(\cdot,n)[n] \) with a a \( \mathcal{CPT} - \mathcal{P} \)-bimodule structure. Moreover, morphism of Cyclic Swiss Cheese type operads induce morphisms of bimodules. Therefore we obtain a bimodule map

\[
\begin{align*}
\text{CPT} & \circ \prod_n \text{Chains}(\text{FHM},n)[-n] \circ \text{Chains}(\text{FFM}_2) \\
\downarrow \text{id} & \downarrow & \downarrow \\
\text{CPT} & \circ \prod_n \text{BVKGra}(\cdot,n)[-n] \circ \text{BVGra}.
\end{align*}
\]

We choose a Maurer Cartan element \( \mu \in (\prod_n \text{Chains}(\text{FHM}_0,n)[-n])_2 \) to be \( \theta = \prod_{n \geq 2} c_n \), where \( c_n \) is the fundamental chain of the space \( \text{FHM}_0,n \).

It is easy to see that the image of \( c_n \) is zero for \( n > 2 \) and for \( n = 2 \) is the single graph in \( \text{BVKGra}(0,2)[-2] \) with no edges.

By twisting both \( \prod_n \text{Chains}(\text{FHM},n)[-n] \) and \( \prod_n \text{BVKGra}(\cdot,n)[-n] \) with respect to \( \mu \) and its image, we get a map of \( T\text{wCPT} \)-modules \( \prod_n \text{Chains}^\mu(\text{FHM},n)[-n] \to \prod_n \text{BVKGra}^\mu(\cdot,n)[-n] \) where the superscript \( \mu \) indicates that there is a changed differential induced by the Maurer-Cartan elements. Since the ideal generated by \( \langle \rangle \) acts as zero, we can restrict our action to the subquotient \( \mathcal{CBr} \), of \( T\text{wCPT} \), thus obtaining a morphism of left \( \mathcal{CBr} \)-modules.

Since the right action of \( \text{Chains}(\text{FFM}_2) \) on \( \text{Chains}(\text{FHM}) \) is on the non-boundary points, and analogously, the action of \( \text{BVGra} \) on \( \text{BVKGra} \) is on the type II vertices, it is clear that the morphism commutes with the right action. We obtain then the following bimodule map:
The projection map $p_{m,n} : FHH_{m,n} \to FHH_{m,0}$ that forgets the points at the boundary induces a strongly continuous chain \[ p_{m,n} : FHH_{m,0} \to Chains(FHH_{m,n}). \] Intuitively the image of a configuration of points in $FHH_{m,0}$ is the same configuration of points but with $n$ points at the real line that are freely allowed to move. If we consider the complex $Chains(FHH_{\bullet,0}) = \bigoplus_{m \geq 1} Chains(FHH_{m,0})$, this induces a degree preserving map \[ p^{-1} : Chains(FHH_{\bullet,0}) \to \prod_{n \geq 0} Chains^\mu(FHH_{\bullet,n})[-n]. \]

**Lemma 16.** $p^{-1}$ is a morphism of right $Chains(FFM_2)$-modules and its image is a $CBr - Chains(FFM_2)$-bimodule.

**Proof.** The morphism clearly commutes with the right action. Let us check that $p^{-1}$ commutes with the differentials.

Let $c \in Chains(FHH_{m,0})$.

The boundary term $\partial p_{m,n}^{-1}(c)$ has two kind of components. When at least two points at the upper half plane get infinitely close, giving us the term $p_{m,n}^{-1}(\partial c)$, and when points at the real line get infinitely close, giving us $p_{m,n}^{f\partial}(c)$, where the $f\partial$ superscript represents that we are considering the boundary at every fiber.

Then, we have $p^{-1}(\partial c) = \prod_{n \geq 0} p_{m,n}^{-1}(\partial c) = \prod_{n \geq 0} \partial p_{m,n}^{-1}(c) \pm p_{m,n}^{f\partial}(c)$. The first summand corresponds to the normal differential in $Chains(FHH_{m,n})$ and the second summand is precisely the extra piece of the differential induced by the twisting.

It remains to check the stability under the left $CBr$ action. It is enough to check the stability under the action of the generators $T^i_n, T^{i+1}_n, T'_n$ and $T^{n,i+1}_n$.

Let $c_1, \ldots, c_n \in Chains(FHH_{\bullet,0})$ of arbitrary degree. It is not hard to see that \[ p^{-1} \circ p(T^i_n(p^{-1}(c_1), \ldots, p^{-1}(c_n))) = T^1_n(p^{-1}(c_1), \ldots, p^{-1}(c_n)). \]

This follows essentially from the fact that on the right hand side the projection in $Chains^\mu(FHH_{\bullet,k})[-k]$ is the sum over all the possibilities of distributing $k$ points on the boundary stratum of $c_i$, for $i = 2, \ldots, n$ and $k_1$ boundary points not infinitely close to any of these chains, with $k_1 + \ldots + k_n = k$, whereas the left hand is taking all of these possibilities into account at once.

For the remaining $T^{n,i+1}_n$, the stability follows from the remark that if a chain is in the image of $p^{-1}$, then any cyclic permutation of it is still in the image of $p^{-1}$. Since forgetting one of the boundary points of a chain in the image of $p^{-1}$ leaves it in the image of $p^{-1}$, we get stability under the action of $T^{n,i+1}_n$.

The other generators follow from similar arguments. \[ \square \]

$p^{-1}$ is right inverse to the projection map, therefore it is an embedding of right $Chains(FFM_2)$-modules. We can therefore transport back the left $CBr$ action on its image, making $p^{-1}$ a morphism of $CBr - Chains(FFM_2)$-bimodules.

By composition with the map \ref{4}, we obtain the following bimodule map:
5.2. A representation on the colored vector space $D_{\text{poly}} \oplus T_{\text{poly}}$. In this section we drop the $\mathbb{R}^d$ from the notation $T_{\text{poly}}$, $D_{\text{poly}}$ and $D_{\text{poly}}$, for simplicity. In Section 6 we globalize the results obtained here.

Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{R}^n$ and let $\xi_1, \ldots, \xi_n$ be the corresponding basis of vector fields. We define an action of BVGra on the graded algebra of multivector fields $T_{\text{poly}}$ in $\mathbb{R}^d$ by setting

$$\Gamma(x_1, \ldots, x_k) = \left( \prod_{(i,j) \in \Gamma} \frac{\partial}{\partial x_i} \right)(x_1 \wedge \cdots \wedge x_k),$$

where $\Gamma \in \text{BVGra}(k)$. $x_1, \ldots, x_k$ are multivector fields, the product runs over all edges of $\Gamma$ in the order given by the numbering of edges and the superscripts $(i)$ and $(j)$ mean that the partial derivative is being taken on the $i$-th and $j$-th component of $x_1, \ldots, x_k$. This is equivalent to an operad morphism $\text{BVGra} \to \text{End}_{\text{poly}}$.

Seeing $\Gamma$ as an element of $\text{BVGra}(m+n)$ and, using the action of $\text{BVGra}$ in $T_{\text{poly}}$, together with the fact that $C^\infty$ functions are degree zero multivector fields we define a map $g: \text{BVGra}(m,n) \to \text{Hom}(T_{\text{poly}}^n \otimes C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d))$ by

$$g(\Gamma)(x_1, \ldots, x_m)(f_1, \ldots, f_n) = \Gamma(x_1, \ldots, x_m, f_1, \ldots, f_n)\text{.}$$

These two maps form a colored operad morphism from (BVGra, BVKgra) to the Swiss Cheese type operad $\left( \text{End}_{\text{poly}}, \text{Hom}(T_{\text{poly}}^n \otimes C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d)) \right)$, a sub-operad of the colored operad $\text{End}(T_{\text{poly}} \otimes C_c^\infty(\mathbb{R}^d))$.

The Tensor-Hom adjunction allows us to rewrite $\text{Hom}(T_{\text{poly}}^\otimes m \otimes C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d))$ as $\text{Hom}(T_{\text{poly}}^\otimes m, \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d)))$ and the bilinear form $f: C_c^\infty(\mathbb{R}^d)^{\otimes n} \otimes C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ induces a map

$$\text{Hom}(T_{\text{poly}}^\otimes m, \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d))) \to \text{Hom}(T_{\text{poly}}^\otimes m, \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R})).$$

There is a natural $C_{n+1}$ action on $\text{Hom}(T_{\text{poly}}^\otimes m, \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R}))$ given by the action on $C_c^\infty(\mathbb{R}^d)^{\otimes n+1}$ and also a distinguished element $1$ map given by the insertion of the constant function $\equiv 1$ on the first input of $\text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R})$.

Lemma 17. With the above described map and cyclic action, the composition of the maps (5) and (7) induces a morphism of Cyclic Swiss Cheese type operads

$$(\text{BVGra}, \text{BVKgra}) \to \left( \text{End}_{\text{poly}}, \text{Hom}(T_{\text{poly}}^\otimes, \text{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R})) \right).$$

4We set all $\xi_i = 0$. 
Proof: It is clear that the map is a morphism of colored operads and it sends one distinguished element to the other. It is enough to check the compatibility with the cyclic action.

Notice that the image of a graph under the morphism

$$\text{BVKGra}(m, n) \rightarrow \text{Hom}(\mathcal{T}^\otimes_{\text{poly}}, \text{Hom}(C^\infty_c (\mathbb{R}^d)^{\otimes n+1}, \mathbb{R}))$$

actually lands inside of \(\text{Hom}(\mathcal{T}^\otimes_{\text{poly}}, D_{\text{poly}}(n))\) and this space is an algebra with product given by the product of functions.

It is clear by the definition of this morphism that it commutes with products, therefore to check the compatibility with the cyclic action it is enough to check it on graphs with just one edge.

Let \(\Gamma^i_j \in \text{BVKGra}(m, n)\). Recall that the action of the generator \(\sigma\) of \(C_{n+1}\) on \(\Gamma^i_j\) is \(\sigma(\Gamma^i_j) = \Gamma^i_{j-1}\) if \(j \neq 1\) and \(\sigma(\Gamma^i_1) = -\sum_{k=1}^n \Gamma^i_k - \sum_{k=1}^m \Gamma^{i,k}\). The action of \(\sigma\) on \(\Gamma^{i,j} \in \text{BVKGra}(m, n)\) is \(\sigma(\Gamma^{i,j}) = \Gamma^{i,j},\) for \(1 \leq i, j \leq m\).

Let \(X_1, \ldots, X_m \in T_{\text{poly}}\) and let \(f_0, \ldots, f_n \in C^\infty(\mathbb{R}^d)\).

Notice that \(g(\Gamma^i_1)(X_1, \ldots, X_m)\) can only be non-zero if all the \(X_j\), for \(j \neq i\) are in \(T_{\text{poly}}\) \(= C^\infty(\mathbb{R}^d)\) and \(X_i \in T_{\text{poly}}^0 = \Gamma(\mathbb{R}^d, T_{\text{poly}}^0)\).

The operator \((g(\Gamma^i_1)(X_1, \ldots, X_m))^\sigma\) is defined by

$$\int f_0 g(\Gamma^i_1)(X_1, \ldots, X_m)(f_1, \ldots, f_n) = \int f_1 (g(\Gamma^i_1)(X_1, \ldots, X_m))^\sigma (f_2, \ldots, f_n, f_0),$$

i.e., by “taking the derivatives from \(f_1\)”.

Let us write \(X_i = \sum_{k=1}^d \psi_k \frac{\partial}{\partial x_k}\). Expanding the first integral we have

$$\int f_0 g(\Gamma^i_1)(X_1, \ldots, X_m)(f_1, \ldots, f_n) =$$

$$\sum_{k=1}^d \int \frac{\partial f_1}{\partial x_k} \psi_k X_1 \ldots \hat{X}_i \ldots X_m f_2 \ldots f_n f_0 =$$

$$- \sum_{k=1}^d \int f_1 \frac{\partial \psi_k}{\partial x_k} X_1 \ldots \hat{X}_i \ldots X_m f_0 f_2 \ldots f_n + f_1 \psi_k \frac{\partial X_1}{\partial x_k} X_2 \ldots \hat{X}_i \ldots X_m f_2 \ldots f_n f_0 +$$

$$+ \cdots + f_1 \psi_k X_1 \ldots \hat{X}_i \ldots X_m f_2 \ldots f_n \frac{\partial f_0}{\partial x_k}.$$  

Therefore

$$(g(\Gamma^i_1)(X_1, \ldots, X_m))^\sigma (a_1, \ldots, a_n) =$$

$$- \Gamma^{i,i} (X_1, \ldots, X_m, a_1, \ldots, a_n) - \sum_{k=1, k \neq i}^m \Gamma^{i,k} (X_1, \ldots, X_m, a_1, \ldots, a_n) - \sum_{k=1}^d \Gamma^i_k (X_1, \ldots, X_m, a_1, \ldots, a_n) =$$

$$g(- \sum_{k=1}^m \Gamma^{i,k} - \sum_{k=1}^n \Gamma^i_k)(X_1, \ldots, X_m)(a_1, \ldots, a_n) =$$

$$g(\Gamma^i_1, \sigma)(X_1, \ldots, X_m)(a_1, \ldots, a_n).$$

The verification for the case \(\Gamma^{i,j}\) is trivial and the case \(\Gamma^i_j\) with \(j \neq 1\) is also immediate because there is only permutation of variables involved. \(\square\)

We obtain then a bimodule map
The image of the Maurer-Cartan element $μ \in BVKGra(0, 2)[−2]$ is the element induced by the multiplication map

$$\mu : C^\infty_\cdot(\mathbb{R}^d)^{\otimes 2} \to C^\infty_\cdot(\mathbb{R})$$

By twisting with respect to these Maurer-Cartan elements we obtain a map of $Tw\text{CPT}$ from $\prod_n BVKGra(\cdot, n)[−n]$ to $Hom_\cdot(T^\otimes_\cdot, Hom(C^\infty_\cdot(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R}))[−n]$. Notice that in this last space, the differential coming from the twisting is the same as the one induced by the Hochschild differential and the degrees also agree with the Hochschild complex. In fact, the image of the map (8) lands in $Hom_\cdot(T^\otimes_\cdot, D_\text{poly})$.

Since $\epsilon \in Tw\text{CPT}$ acts trivially on both spaces, this induces an action of its subquotient $\text{CBr}$, therefore we obtain the following maps of bimodules:

$$\text{CBr} \odot \prod_n BVKGra(\cdot, n)[−n] \odot \text{BVGr}$$

Also, the $\text{CBr}$ action on $Hom(T^\otimes_\cdot, D_\text{poly})$ comes from the action of $\text{CBr}$ on $D_\text{poly}$ (as seen in (3.3.1)), which translates into an operadic morphism $\text{CBr} \to End D_\text{poly}$. Thus, by composition with the map (5) we obtain

$$End D_\text{poly} \odot \text{Hom}(T^\otimes_\cdot, D_\text{poly}) \odot End T^\otimes_\cdot$$

5.3. A zig-zag of quasi-torsors. Let us recall the definition of an operadic quasi-torsor from [CW]:

**Definition 13.** Let $\mathcal{P}$ and $\mathcal{Q}$ be two differential graded operads and let $M$ be a $\mathcal{P} - \mathcal{Q}$ operadic differential graded bimodule, i.e., there are compatible actions

$$\mathcal{P} \odot M \odot \mathcal{Q}.$$
We say that $M$ is a $P$-$Q$ quasi-torsor if there is an element $1 \in M^0(1)$ such that the canonical maps

$$\begin{align*}
l & : P \to M \\
r & : Q \to M \\
p & \mapsto p \circ (1, \ldots, 1) \\
q & \mapsto 1 \circ q
\end{align*}$$

are quasi-isomorphisms.

**Lemma 18.** $\text{Chains}(\mathbb{H}_{*,0})$ is a $\text{CBr} - \text{Chains}(\text{FFM}_2)$ quasi-torsor.

**Proof.** Let us consider the element $1 \in \text{Chains}_0(\mathbb{H}_{1,0})$ corresponding to a single point on the upper half plane with frame is pointing upwards.

Let $i : \text{FFM}_2 \to \mathbb{H}_{*,0}$ be the map that sends a configuration in $c \in \text{FFM}_2$ to the configuration in $\mathbb{H}_{*,0}$ given by one boundary stratum on the upper half plane with $c$ on it. It is clear that $i$ is a homotopy equivalence (with homotopy inverse being the map that “forgets” the boundary of the upper half plane). The map $r : \text{Chains}(\text{FFM}_2) \to \text{Chains}(\mathbb{H}_{*,0})$, as in Definition 10 is the image of $i$ via the functor $\text{Chains}$. Since $i$ is a homotopy equivalence, $r$ is a quasi-isomorphism.

It was shown in [Ge] that $H(\text{FFM}_2) = \text{BV}$.

The map $l$ sends $1 \in \text{CBr}_{-1}(1)$ to the fundamental chain of the circle. It sends

$1$ to the zero chain consisting of two horizontally aligned points in the upper half plane with frames pointing upwards. And it sends $1$ to the 1-chain corresponding to two points rotating around each other.

Since the homologies of $\text{CBr}$ and of $\mathbb{H}_{*,0}$ are both $\text{BV}$ and $l$ sends (representatives of) generators to (representatives of) generators, $l$ is a quasi-isomorphism. □

The main Theorem of [CW] states that if the $P$-$Q$-bimodule $M$ is an operadic quasi-torsor, then there is a zig-zag of quasi-isomorphisms connecting $P \circ M \circ Q$ to the canonical bimodule $P \circ P \circ P$.

It follows then from Lemma 18 that there is a zig-zag of bimodules

$$\begin{align*}
\text{CBr} & \circ \quad \text{CBr} & \circ \quad \text{CBr} \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\uparrow & & \uparrow \\
\text{CBr} & \circ \quad \text{Chains}(\mathbb{H}_{*,0}) & \circ \quad \text{Chains}(\text{FFM}_2).
\end{align*}$$

Let $\text{CBr}_{\text{bimod}}^\infty$ be a cofibrant resolution of the canonical bimodule $\text{CBr}$. $\text{CBr}_{\text{bimod}}^\infty$ is a $\text{CBr}_\infty - \text{CBr}_\infty$-bimodule, where $\text{CBr}_\infty$ is a cofibrant resolution of the operad $\text{CBr}$. 
Finally, the zig-zag can be lifted up to homotopy to a bimodule map

\[
\begin{array}{c}
\text{CBr}_\infty \\
\Downarrow
\end{array} \quad \circ \quad \\
\begin{array}{c}
\text{CBr}^\text{bimod}_\infty \\
\Downarrow
\end{array} \quad \circ \quad \\
\begin{array}{c}
\text{End} D\text{poly} \\
\Downarrow
\end{array} \quad \circ \quad \\
\begin{array}{c}
\text{Hom}(T^\bullet_{\text{poly}}, D_{\text{poly}}) \\
\Downarrow
\end{array} \quad \circ \quad \\
\begin{array}{c}
\text{End} T\text{poly}.
\end{array}
\]

giving us the desired quasi-isomorphism and thus proving Theorem 1.

It also follows from Lemma 18 and [CW] that CBr is quasi-isomorphic to Chains(FFM₂). Due to the formality of FFM₂ [GS], it follows that we can replace CBr∞ in Theorem 1 by any cofibrant replacement of the operad BV.

6. Globalization

Let M be a d-dimensional oriented manifold. In this section we globalize the BV∞ quasi-isomorphism \( T_{\text{poly}}(\mathbb{R}^d) \to D_{\text{poly}}(\mathbb{R}^d) \) from Theorem 1 to a quasi-isomorphism \( T_{\text{poly}}(M) \to D_{\text{poly}}(M) \), thus proving Theorem 2. To do this we use standard formal geometry techniques.

6.1. The idea: We refer the reader to the paper [Do], from which we borrow the notation.

Theorem 1 is valid if we replace \( \mathbb{R}^d \) by \( \mathbb{R}^d_{\text{formal}} \), its formal completion at the origin, i.e., the space whose ring of functions is given by formal power series on the coordinates \( x_1, \ldots, x_d \).

We consider \( T_{\text{poly}} \) (resp. \( D_{\text{poly}} \)), the vector bundle on M of fiberwise formal multivector fields (resp. multidifferential operators) tangent to the fibers. We can then construct the vector bundles \( \Omega(T_{\text{poly}}, M) \) of forms valued in \( T_{\text{poly}} \) and \( \Omega(D_{\text{poly}}, M) \) of forms valued in \( D_{\text{poly}} \) with appropriate differentials.

The fibers of the bundles \( T_{\text{poly}} \) and \( D_{\text{poly}} \) are isomorphic to \( T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and \( D_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \), respectively. Therefore, the formal version of the formality map can be used to find a vector bundle CBr∞ quasi-isomorphism

\[
U^f: \Omega(T_{\text{poly}}, M) \to \Omega(D_{\text{poly}}, M)
\]

These two vector bundles can be related with \( T_{\text{poly}}(M) \) and \( D_{\text{poly}}(M) \). In fact, with an appropriate change of differential that comes from a choice of a flat connection, \( \Omega(T_{\text{poly}}, M) \) becomes a resolution of \( T_{\text{poly}}(M) \) and \( \Omega(D_{\text{poly}}, M) \) becomes a resolution of \( D_{\text{poly}}(M) \). This change of differential can be seen locally as a twist via a Maurer-Cartan element \( B \) in \( \Omega^1(T_{\text{poly}}^1, U) = \Omega^1(D_{\text{poly}}^1, U) \). However, the linear part of \( B \) (in the fiber coordinates) is not globally well defined.

\[\text{(11)}\]

\[U^f: \Omega(T_{\text{poly}}, M) \to \Omega(D_{\text{poly}}, M)\]

Using the fact that the formality morphism is invariant by linear transformation of coordinates.

\[\text{(11)}\]
6.2. **An extension of Kontsevich’s \( L_\infty \) morphism.** In this section we show that the BV\(_\infty \) formality morphism from Theorem 1 can be obtained in such a way that it extends Kontsevich’s original \( L_\infty \) morphism [Ko].

We have the following chain of maps:

\[
\begin{array}{cccc}
\text{hoLie}_1 & \circlearrowleft & \text{hoLie}_1^{\mathrm{bimod}} & \circlearrowleft & \text{hoLie}_1 \\
\downarrow & & \downarrow & & \downarrow \\
\text{CB}_\infty & \circlearrowleft & \text{CB}_\infty^{\mathrm{bimod}} & \circlearrowleft & \text{CB}_\infty \\
\downarrow & & \downarrow & & \downarrow \\
\text{CB} & \circlearrowleft & \text{Chains}(F\mathbb{H}_{*,0}) & \circlearrowleft & \text{Chains}(FFM_2) \\
\downarrow & & \downarrow & & \downarrow \\
\text{CB} & \circlearrowleft & \prod_n BVKGra^{\mu}(\cdot,n)[-n] & \circlearrowleft & \text{BVGr} \\
\downarrow & & \downarrow & & \downarrow \\
\text{End} D_{\text{poly}} & \circlearrowleft & \text{Hom}(T_{\text{poly}}^{\otimes}, D_{\text{poly}}) & \circlearrowleft & \text{End} T_{\text{poly}}.
\end{array}
\]

where \( \text{hoLie}_1 = \Omega(\text{Lie}\{1\}^\vee) \), the first downwards maps are induced by the inclusion \( \text{Lie} \to \text{CB} \) and the other maps follow from the proof of Theorem 1. Showing that our morphism extends Kontsevich’s formality morphism amounts to showing that the full composition of the maps in (12) gives Kontsevich’s map. This is clear for the left column. For the other two columns the argument is similar so we will only prove it for the right column given that the notation is simpler. Let us call \( \mu_n \) the generator of \( \text{Lie}\{1\}^\vee(n) \).

We recall that in [Ko] the construction of \( U_n \), the \( L_\infty \) components of the formality morphism are constructed by sending \( \mu_n \) to the fundamental chain of \( H_{n,0} \). We wish then to show the commutativity of the following diagram, where the upper horizontal maps represent Kontsevich’s approach and \( \text{Gr} \) is the suboperad of \( \text{BVGr} \) in which tadpoles are not admitted.

\[
\begin{array}{cccc}
\text{hoLie}_1 & \longrightarrow & \text{Chains}(FM_2) & \longrightarrow & \text{Gr} & \longrightarrow & \text{End}(T_{\text{poly}}). \\
\downarrow & & & & \downarrow & & \\
\text{CB}_\infty & \longrightarrow & \text{Chains}(FFM_2) & \longrightarrow & \text{BVGr}
\end{array}
\]

As semi-algebraic manifolds, \( FFM_2(n) = FM_2(n) \times (S^1)^n \), therefore there exists an inclusion map \( \varepsilon : FM_2 \to FFM_2 \) that is the identity on the \( FM_2 \) component and constant equal to the vertical direction in the \( S^1 \) components.

Naming the relevant maps, diagram (13) becomes
It is clear that the right triangle diagram and the adjacent square diagram are commutative. To conclude the commutativity of the exterior diagram it is enough to show that the left square is commutative but this need not be the case. Fortunately this can be rectified if one is careful when constructing the map $g$ as a lift over quasi-isomorphisms. We sketch here the argument that is nothing but an adapted version or the argument of Lemmas 12 and 13 in [CW].

The fact that $\text{Lie}\{1\}$ can be seen embedded in $CBr$ via the map $F$ in section 4.3 implies that the generators $\mu_n$ of $\text{hoLie}_1$ can be seen as part of the generators of $CBr_\infty$ (via the map $i_L$) and the map $f$ sends $\mu_n$ to the fundamental chain of $\text{FM}_2(n)$.

To construct $g$ one starts with a filtration $0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset CBr_\infty$ such that when differentiating the generators we fall in the previous degree of the filtration and then we construct the map recursively using the following diagram:

$$
\text{CBr}_\infty \xrightarrow{g} \text{Chains}(\text{FFM}_2) \xrightarrow{BVGra} \text{BVGra}
$$

where all maps are quasi-isomorphisms, $E$ is the operad through which the zig-zag connecting $CBr$ and $\text{Chains}(\text{FFM}_2)$ goes and $F$ is the operad resulting from the “surjective trick”, i.e., an operad that surjects both onto $E$ and $\text{Chains}(\text{FFM}_2)$ such that the depicted triangle commutes up to homotopy. At every stage we wish to map $\mu_n$ to a pre-image of the fundamental chain of $\text{FM}_2(n)$ and essentially one has to check that $dg'(\mu_n) = g'(d\mu_n)$, but this follows from the fact that the boundary of the fundamental chain of $\text{FM}_2(n)$ is computed the same way as the cocomposition of $\mu_n$ in $\text{Lie}\{1\}^\vee$.

6.3. The bimodule BVKGraphs. Notice that due to the chain of morphisms there is a morphism of bimodules

$$
\text{hoLie}_1 \overset{f}{\longrightarrow} \text{Chains}(\text{FM}_2) \xrightarrow{i_*} \text{Gra} \xrightarrow{\text{End}(T_{\text{poly}})}.
$$

$$
\text{CBr}_\infty \xrightarrow{g} \text{Chains}(\text{FFM}_2) \xrightarrow{BVGra} \text{BVGra}
$$

Using the formalism of twisting of bimodules described in the Appendix we can perform the bimodule twisting with respect to this morphism, thus obtaining the operadic bimodule $TwCBr \overset{\text{bimod}}{\otimes} \prod_n \text{BVKGra}^a(\cdot,n)[-n] \otimes \text{BVGra}$. 

The elements in $TwBVGra(n)$ can be seen as linear combinations of directed graphs with at least $n$ vertices, where from these, $n$ of them are labeled by numbers from 1 to $n$ and the remaining ones are indistinguishable. The labeled vertices are called external vertices and the unlabeled ones are called internal vertices. In a similar way, the elements of $Tw \prod_n BVKGr^k(m,n)[-n]$ consist of the same kind of graphs, but where now the type I vertices come in two flavors, the indistinguishable internal vertices and the $m$ labeled external vertices.

**Proposition/Definition 19.** The operad $TwBVGra$ has a suboperad that we call $BVGraphs$ spanned by graphs satisfying the following properties:

1. There are no 1-valent internal vertices or 2-valent internal vertices with exactly one incoming and one outgoing edges;
2. There are no tadpoles on internal vertices.

**Proof.** It is clear that the operadic composition preserves each of the conditions imposed in $BVGraphs$, therefore we only need to check that $BVGraphs$ is preserved under the action of the differential.

The differential $d$ in $TwBVGra$ has the form $d = d_1 + d_2$, where $d_1$ is defined by $d_1 \Gamma = ( \xrightarrow{\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Let \( c \in \text{Chains}(\text{FFM}_2)(n) \). To define \( \iota(c) \) it is enough to define its projection in \( \text{Chains}(\text{FFM}_2)(n+k)^3_k \). We define this projection to be \( \pi_{k,n} \).

To see that this is an operad map, we need to check that \( \iota(c \circ c') = \iota(c) \circ \iota(c') \). This equality follows from the observation that fixed a boundary stratum of a configuration of points, having \( k \) points varying freely is the same as \( i \) points inside that boundary stratum and \( k-i \) outside, for \( i = 0, \ldots, k \). □

The operad morphism \( \text{Chains}(\text{FFM}_2) \to \text{BVGr} \) and the functoriality of \( Tw \) and the canonical projections \( TwP \to P \) give us the following commutative square

\[
\begin{array}{ccc}
Tw \text{Chains}(\text{FFM}_2) & \longrightarrow & Tw\text{BVGr} \\
\downarrow & & \downarrow \\
\text{Chains}(\text{FFM}_2) & \longrightarrow & \text{BVGr}
\end{array}
\]

As a corollary of the previous Lemma, the operad morphism \( \text{Chains}(\text{FFM}_2) \to \text{BVGr} \) factors as \( \text{Chains}(\text{FFM}_2) \to Tw\text{BVGr} \to \text{BVGr} \). Explicitly, the first map is given by

\[
(15) \quad c \in \text{Chains}(\text{FFM}_2)(n) \mapsto \sum_{\Gamma} \int_{\pi_{\Gamma}^{-1}(c)} f_1(\Gamma),
\]

where \( f_1(\Gamma) \) is the form associated to the graph \( \Gamma \), as defined in Section 5.1 and for \( \Gamma \) a graph with \( n \) external and \( m \) internal vertices, \( \pi_{\Gamma}^{-1}(c) \) is the chain in \( \text{Chains}(\text{FFM}_2)(n+m) \) in which the \( m \) points corresponding to the internal vertices vary freely in \( \mathbb{R}^d \) while their frame is constantly pointing upwards.

**Proposition 21.** The operad morphism \( \text{Chains}(\text{FFM}_2) \to \text{BVGr} \) defined above factors through \( \text{BVGraphs} \).

**Proof.** It is enough to check that the morphism (15) lands in \( \text{BVGraphs} \) and for this one must check that the coefficient of the graphs that are “forbidden” in \( \text{BVGraphs} \) is zero. This is clear if the graph contains a 1-valent internal vertex, since the computation of the coefficient involves an integral of a 1-form (corresponding to the incident edge) over a 2 dimensional space.

Suppose the graph \( \Gamma \) contains an internal vertex with exactly one incoming and one outcoming edges. Let us call this vertex \( i \) and let us also call \( a \) and \( b \) the vertices to which these two edges connect.

\[\text{Figure 4. An internal vertex connected to two (internal or external) vertices.}\]

By Fubini’s Theorem for fibrations, the integral \( \int_{\pi_{\Gamma}^{-1}(c)} f_1(\Gamma) \) can be rewritten as
where $X_{x_a,z_b}$ is the space of configurations in which the points labeled by $a$ and $b$ are in positions $z_a$ and $z_b$, and the point labeled by $i$ moves freely. It suffices therefore to show that the integral

$$(16) \int_{X_{x_a,z_b}} d\phi_{ai} d\phi_{ib}$$

vanishes. To check this, notice that by (the fibration integral version of) Stokes' Theorem, we have

$$d \int_{Y_{x_a,z_b}} d\phi_{ai} d\phi_{ij} d\phi_{jb} = \int_{\partial Y_{x_a,z_b}} d\phi_{ai} d\phi_{ij} d\phi_{jb},$$

where $Y_{x_a,z_b}$ is the configuration space of four points $(i,j,a,b)$ where $a$ and $b$ are fixed at $z_a$ and $z_b$ and the points labeled by $i$ and $j$ are free. The integral on the left hand side vanishes by degree reasons. The boundary terms on the right hand side vanish except on the following cases:

- The boundary stratum in which $a$ and $i$ are infinitely close,
- The boundary stratum in which $i$ and $j$ are infinitely close,
- The boundary stratum in which $j$ and $b$ are infinitely close.

In each of these cases, the result is an integral of the form of integral (16) (possibly with different signs), therefore it is zero.

If a graph $\Gamma$ contains an internal vertex with a tadpole, the form $\text{f}_1(\Gamma)$ includes a term of the form $d\phi$, where $\phi$ is the angle between the vertical direction and the frame at the corresponding point. However $\pi^{-1}_G(c)$ is a chain in which the frame of that point does not vary, therefore the integral $\int_{\pi^{-1}_G(c)} \text{f}_1(\Gamma)$ vanishes. $\square$

As a consequence of Lemma 26, the canonical projections $Tw\text{CBr} \to \text{CBr}$ and $Tw\text{BVGraphs} \to \text{BVGraphs}$ admit right inverses. This defines a $\text{CBr} - \text{BVGraphs}$ bimodule structure on $Tw \prod_{\pi} \text{BVKGra}_\mu(\cdot,n)[-n]$. Elements of this bimodule are (sequences of) graphs with type I and type II vertices as before, but now there are two kinds of type I vertices. Using the same designations as in $\text{CBr}$ we refer to the labeled type I vertices as external vertices and the indistinguishable type I vertices as internal vertices.

**Proposition/Definition 22.** The $\text{CBr} - \text{BVGraphs}$ bimodule $Tw \prod_{\pi} \text{BVKGra}_\mu(\cdot,n)[-n]$ has a subbimodule that we call $\text{BVKGraphs}$ that is spanned by the graphs with the following properties:

1. There is at least one type I external vertex,
2. There are no 0-valent type I internal vertices
3. There are no 1-valent type I internal vertices with an outgoing edge,
4. There are no 2-valent type I internal vertices with one incoming and one outgoing edge (in particular there are no internal vertices with one tadpole and no other incident edges).
Proof. We must check that BVKGraphs is preserved by the differential, the left CBr and right BVGraphs actions. This is clear for the right BVGraphs action.

To check that BVKGraphs is closed under the action on CBr we start by considering the action of the generator $T^n_1$. Let $\Gamma_1, \ldots, \Gamma_n$ be graphs in BVKGraphs. The element $T^n_1(\Gamma_1, \ldots, \Gamma_n)$ is determined by inserting $\Gamma_2, \ldots, \Gamma_n$ at the type II vertices of $\Gamma_1$, therefore every type I vertex in $T^n_1(\Gamma_1, \ldots, \Gamma_n)$ can be identified as coming from one of the $\Gamma_i$. Since there are only incoming edges at type II vertices, the action of $T^n_1$ can increase or maintain the number of incoming edges at a type I vertex but it can only maintain the number of outgoing edges at every type I vertex, thus proving that properties (2), (3) and (4) are preserved. Property (1) is clearly preserved.

The action of $T^n_j$ is given by insertions of the $\Gamma_i$ in the type II vertices on cyclic permutations of $\Gamma_1$, using the cyclic action of BVKGra described in section 3.3.3. Since the cyclic action preserves properties (1)- (4), BVKGraphs is closed under the action of $T^n_j$.

To show that BVKGraphs is closed under the action of $T^n_j$, it is enough to check that summands of the Maurer-Cartan element by which $\prod BVKGra^\mu(\cdot,n)[{-n}]$ was twisted (image of the generators of hoLie\textsubscript{bimod}) satisfy the following two properties:

(a) The only graph containing a 1-valent type I internal vertex is the 2 vertex graph \[ \begin{array}{c} \uparrow \\ \downarrow \end{array} \], with coefficient 1.

(b) There are no graphs with vertices like the ones in property (4).

To verify these properties we recall that the map $hoLie\textsubscript{bimod} \to \prod BVKGra^\mu(\cdot,n)[{-n}]$ involves at some step the integration of differential forms over $FH_{n,0}$. Then, property (a) follows from degree reasons and property (b) has a proof similar to Proposition 21.

It remains to check that the differential preserves BVKGraphs. The differential is composed of the following pieces:

- The original splitting of type II vertices,
- Insertion of $\begin{array}{c} \uparrow \\ \downarrow \end{array}$ at type I external vertices,
- Insertion of $\begin{array}{c} \uparrow \\ \downarrow \end{array}$ at type I internal vertices,
- Bracket with the image of the generators of hoLie\textsubscript{bimod}.

The first piece of the differential clearly preserves BVKGraphs. Properties (1) and (2) are trivially preserved by all pieces of the differential. It remains to check properties (3) and (4). The remaining pieces of the differential can produce vertices like (3) and (4), so we must verify that these graphs cancel. There are 3 possibilities to obtain a vertex of the kind (3) with the differential:

Using the second piece of the differential on a graph $\Gamma \in$ BVKGraphs, at every external vertex we get a forbidden 1-valent vertex connecting to it, corresponding to inserting $\begin{array}{c} \uparrow \\ \downarrow \end{array}$ and reconnecting all the originally incident edges to the external vertex. Similarly, for every internal vertex of $\Gamma$, the second piece of the differential produces one 1-valent internal vertex with one outgoing edge connecting to it.
Due to property (a), the only “problematic” graphs that may arise from the fourth piece of the differential are coming from bracket with $\Gamma \pm \sum_i \Gamma_\circ i$. The bracket with this element gives $\Gamma_\circ i$ where on the first summand we connect the internal vertex to every possible (type I or II) vertex of $\Gamma$ and on the second summand the $\circ_i$ represents an insertion at the vertices of $\Gamma$ of type II. In $\Gamma$, the edges connecting to type I vertices in $\Gamma$ are all canceled out with the second and third pieces of the differential as described above. The edges connecting to type II vertices are canceled by the terms in $\sum_i \Gamma_\circ i$ in which all the incident edges to $i$ are reconnected to the type II vertex after the insertion.

To check that the differential preserves property (4), one can see that every time an internal vertex having property (4) is created due to type I internal or external vertex splitting, this term is canceled by a splitting on the other adjacent vertex to the 2-valent vertex that was created. This also holds for splitting of vertices adjacent to type II vertices, but in that case the cancellation is done with a term coming from $\sum_i \Gamma_\circ i$. Due to property (b), no more forbidden graphs are produced by the fourth piece of the differential. □

Lemma 23. $\text{Chains}(H_{\bullet,0})$ is natively twistable.

The construction of the map $\text{Chains}(H_{\bullet,0}) \to \text{Tw Chains}(H_{\bullet,0})$ is identical to Lemma 20 and the compatibility with the left and right actions is immediate.

As a consequence, the bimodule morphism $\text{Chains}(H_{\bullet,0}) \to \text{BVKGra}$ factors through $\text{Tw} \prod_n \text{BVKGra}^d(\cdot,n)[-n]$. The explicit formula is given by

\begin{equation}
(17) \quad c \in \text{Chains}(H)(n) \mapsto \sum_i \int_{\pi^{-1}_i(c)} f_2(\Gamma),
\end{equation}

where $f_2(\Gamma)$ is the form associated to the graph $\Gamma$, as defined in Section 5.1 and if $\Gamma$ is a graph with $n$ external and $m$ internal type I vertices and $k$ type II vertices, $\pi^{-1}_i(c)$ is the chain in $\text{Chains}(H_{n+m,k})$ in which the $m$ points corresponding to the internal vertices vary freely in the upper half plane while their frame is constantly pointing upwards.

Proposition 24. The bimodule morphism $\text{Chains}(H_{\bullet,0}) \to \text{BVKGra}$ factors through $\text{BVKGraphs}$.

The proof is essentially the same as the one of Proposition 21.

6.4. The Twist. As a consequence of the previous section we have the following map of bimodules representing the last layer of the formality morphism:
As described in section 6.1, the fibers of the vector bundles $T_{\text{poly}}$ and $D_{\text{poly}}$ are isomorphic to $T_{\text{poly}}(R^d_{\text{formal}})$ and $D_{\text{poly}}(R^d_{\text{formal}})$, therefore this induces the following map of bimodules:

$$\begin{align*}
\text{CBr} & \circlearrowleft \text{BVGraphs} & \circlearrowright \text{BVGraphs} \\
\text{End} D_{\text{poly}}(R^d_{\text{formal}}) & \circlearrowleft \text{Hom}(T_{\text{poly}}(R^d_{\text{formal}}), D_{\text{poly}}(R^d_{\text{formal}})) & \circlearrowright \text{End} T_{\text{poly}}(R^d_{\text{formal}})
\end{align*}$$

Since the $\text{CBr}_{\infty}$ formality morphism from Theorem 1 is an extension of Kontsevich’s $L_\infty$ formality morphism (see section 6.2), its $L_\infty$ part satisfies properties P1)-P5) from section 7 in [Ko]. In particular, property P4) implies that for $n \geq 2$, $U_n(B, \ldots, B) = 0$ and thus $B^B = \sum_{n=1}^{\infty} \frac{1}{n!} U_n(B, \ldots, B) = U_1(B) = B$, under the identification $\Omega^1(T_{\text{poly}}^1) = \Omega^1(D_{\text{poly}}^1)$.

On the other hand, the bimodule $\text{BVGraphs}$ is obtained from a twist therefore it is natively twistable.

Therefore, following the Appendix, we obtain a map of bimodules:

$$\begin{align*}
\text{CBr} & \circlearrowleft \text{BVGraphs} & \circlearrowright \text{BVGraphs} \\
\text{End} \Omega(D_{\text{poly}}) & \circlearrowleft \text{Hom}(\Omega(T_{\text{poly}}) \otimes \Omega(D_{\text{poly}})) & \circlearrowright \text{End} \Omega(T_{\text{poly}})
\end{align*}$$

where the superscript $B$ indicates that we are considering the twisted differential. For this twist it is important that $\text{BVGraphs}$ forbids 1-valent internal vertices with an outgoing edge and 2-valent internal vertices with one incoming and one outgoing edges, since the linear part of $B$ is not globally well-defined.

Composing with this map with bimodule maps $\text{CBr}_{\infty}^{\text{bimod}} \to \text{Chains}(H_{\bullet,0}) \to \text{BVGraphs}$, we obtain the desired global $\text{CBr}_{\infty}$ quasi-isomorphism.

**APPENDIX A. Twisting**

In this Appendix we give an overview on the theory of operadic twisting following [DW] that we need for this paper and we define a notion of twisting of operadic bimodules, which is not more than an adaptation of the same theory. We advise the reader to read the third section of loc. cit. if they are not familiar with twisting of operads.
We make the standard assumptions used in the context of twisting with respect to Maurer-Cartan elements. Namely, all algebras $g$ (over $\text{hoLie}_1$ or another operad) are equipped with complete decreasing filtrations $g = F_0 g \supset F_1 g \supset \ldots$, such that the operations are compatible with the filtration and $g = \lim_i g/F_i g$. These assumptions are made so that infinite sums (going deeper in the filtration) are allowed.

Let $g$ be a $\text{hoLie}_1$ algebra, an element $\mu \in F_1 g$ of degree 2 is said to be Maurer-Cartan element of $g$ if it satisfies the Maurer-Cartan equation:

$$d\mu + \sum_{n=2}^{\infty} \frac{1}{n!} l_n (\mu, \ldots, \mu) = 0,$$

where $l_n$ are the generating operations in $\text{hoLie}_1$.

Given such a Maurer-Cartan element, one can construct the twisted $\text{hoLie}_1$ algebra $g^\mu$, that is as a graded vector space just $g$, but with a changed (called twisted) differential, denoted by $d^\mu$, that is defined by $d^\mu(x) = dx + \sum_{n=1}^{\infty} \frac{1}{n!} l_{n+1} (\mu, \ldots, \mu, x)$, and new brackets given by $l^\mu_n(x_1, \ldots, x_k) = \sum_{n=1}^{\infty} \frac{1}{n!} l_{n+k} (\mu, \ldots, \mu, x_1, \ldots, x_k)$.

A1. Twisting of operads. Let $P$ be an operad and let us assume the existence of an operad morphism $F: \text{hoLie}_1 \to P$. If $g$ is a $P$ algebra, thanks to $F$. Therefore it makes sense to talk about Maurer-Cartan elements of $g$. If $\mu$ be a Maurer-Cartan element of $g$, the twisted algebra $g^\mu$ is no longer necessarily a $P$ algebra. It is, however, an algebra over the operad $TwP$, whose construction depends on the map $F$.

As an $S$-module, we have

$$TwP(p) = \prod_{r \geq 0} (P(r + p) \otimes \mathbb{K}[-2r])^{S_r},$$

where $S_r$ here is the subgroup of $S_{r+p}$ fixing the last $p$ entries. The $r$ non-symmetric inputs should be thought as representing the insertion of $r$ Maurer-Cartan elements. The composition is defined using the original composition in $P$, but summing over sufles to ensure that it lands in the invariants over the action of $S_{r+p}$.

To describe the differential we need an auxiliary dg Lie algebra:

$$\mathcal{L}_P := \text{Conv}(\text{Lie}(1)^\vee, P) = \prod_{n \geq 1} P(n)^{S_n} [2 - 2n].$$

The Lie algebra $\mathcal{L}_P$ acts on $TwP$, by composition on the non-symmetric inputs. $TwP(1)$ acts on $TwP$ by inner derivations.

There is an obvious degree zero map $\kappa: \mathcal{L}_P \to TwP(1)$.

The map $F$ induces a Maurer-Cartan element $\tilde{F}$, and the final differential is $d_{Tw} = d_P + d_\kappa + d_{\kappa(F)}$, where the first piece is induced by the original differential in $P$, the second one comes from the $L_P$ action and the third one comes from the $TwP(1)$ action.

The fact that this is a differential is essentially a consequence of the following Proposition:

**Proposition 25.** [DW Prop. 3.3] The map

$$\mathcal{L}_P \to \mathcal{L}_P \times TwP(1)$$

$$v \mapsto v + \kappa(v)$$

is a morphism of Lie algebras.
Lemma 26. [Wj Lemma 16] Let $\mathcal{P}$ be an operad (with an implicit map $\text{hoLie}_1 \to \mathcal{P}$). $Tw\mathcal{P}$ is natively twistable.

Notice that $TwTw\mathcal{P}(n) = \prod_{r_1, r_2 \geq 0} \left( (\mathcal{P}((n+r_1)+r_2) \otimes \mathbb{K}[-2r_1] \otimes \mathbb{K}[-2r_2])^{S_{r_1}} S_{r_2} \right) = \prod_{r_1, r_2 \geq 0} (\mathcal{P}(n+r_1+r_2) \otimes \mathbb{K}[-2(r_1+r_2)])^{S_{r_1} \times S_{r_2}} = \prod_{r_1, r_2 \geq 0} (\mathcal{P}(n+r) \otimes \mathbb{K}[-2r])^{S_{r_1} \times S_{r_2}}.$

For $p \in T\mathcal{P}(n)$, the map $Tw\mathcal{P} \to T\mathcal{P}$ is defined by the inclusion of $p_r \in (\mathcal{P}(n+r) \otimes \mathbb{K}[-2r])^{S_{r}}$ in the factors of $T\mathcal{P}(n)$ in which $r_1 + r_2 = r$ and zero if $r_1 + r_2 \neq r$.

A.2. Twisting of bimodules. Let $\mathfrak{g}$ and $\mathfrak{h}$ be $\text{hoLie}_1$ algebras. Given an infinity morphism from $\mathfrak{g}$ to $\mathfrak{h}$, we define a Maurer-Cartan element of this morphism to be a pair $(\mu, \mu')$, where $\mu$ is a Maurer-Cartan element of $\mathfrak{g}$ and $\mu'$ is a Maurer-Cartan element of $\mathfrak{h}$ such that the $\text{hoLie}_1$ morphism sends $\mu$ to $\mu'$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be (dg) operads and $\mathcal{M}$ be a $\mathcal{P} - \mathcal{Q}$ operadic bimodule, that we assume to come with an implicit bimodule morphism $F: \text{hoLie}_1^\text{bimod} \to \mathcal{M}$.

$$
\begin{array}{cccc}
\text{hoLie}_1 & \circ & \text{hoLie}_1^\text{bimod} & \circ & \text{hoLie}_1 \\
\downarrow F_p & & \downarrow F & & \downarrow F_Q \\
\mathcal{P} & \circ & \mathcal{M} & \circ & \mathcal{Q}
\end{array}
$$

Let $\mathfrak{g}$ be a $\mathcal{P}$ algebra and let $\mathfrak{h}$ be a $\mathcal{Q}$ algebra. Due to the map $F$, a morphism of bimodules

$$
\begin{array}{cccc}
\mathcal{P} & \circ & \mathcal{M} & \circ & \mathcal{Q} \\
\downarrow \text{End} \mathfrak{h} & & \downarrow \text{Hom}(\mathfrak{g}^*, \mathfrak{h}) & & \downarrow \text{End} \mathfrak{g}
\end{array}
$$

\[\text{Lemma 26. [Wj Lemma 16] Let } \mathcal{P} \text{ be an operad (with an implicit map } \text{hoLie}_1 \to \mathcal{P}. \text{ } \mathcal{P} \text{ is natively twistable.}
\]

The action of $Tw\mathcal{P}$ on $\mathfrak{g}^n$ is given by inserting Maurer-Cartan elements in the non-symmetric slots. Explicitly, let $p \in T\mathcal{P}(n)$ and let $x_1, \ldots, x_n \in \mathfrak{g}$.

$$
p(x_1, \ldots, x_n) := \sum_{r \geq 0} \frac{1}{r!} p_r(x_1, \ldots, x_n),
$$

where $p_r$ is the projection of $p$ in the factor $(\mathcal{P}(r+n) \otimes \mathbb{K}[-2r])^{S_r}$.

There is a natural operad projection map $Tw\mathcal{P} \to \mathcal{P}$, sending $\prod_{r \geq 1} (\mathcal{P}(r+n) \otimes \mathbb{K}[-2r])$ to zero. At the algebra level this tells us that not only twisted $\mathfrak{g}^n$ but the original $\mathfrak{g}$ are naturally $Tw\mathcal{P}$ algebras.

On the other direction, an operad $\mathcal{P}$ is said to be natively twistable if there exists an operad morphism $\mathcal{P} \to Tw\mathcal{P}$ such that $\mathcal{P} \to Tw\mathcal{P} \to \mathcal{P}$ is the identity. In this case, the twist of a $\mathcal{P}$-algebra is still a $\mathcal{P}$-algebra.

6Evidently for a fixed $\text{hoLie}_1$ infinity morphism, $\mu$ determines a unique $\mu'$. 
determines a hoLie\textsubscript{1} infinity morphism from $\mathfrak{g}$ to $\mathfrak{h}$. We wish to construct a $Tw\mathcal{P} - Tw\mathcal{Q}$ bimodule $\mathcal{M}$ such that for every $(\mu, \mu')$, Maurer-Cartan element of this morphism, there is a natural map of bimodules

$$
\begin{array}{ccc}
Tw\mathcal{P} & \widehat{\odot} & Tw\mathcal{M} & \widehat{\odot} & Tw\mathcal{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\text{End}\mu' & \odot & \text{Hom}(\mathfrak{g}^{\mu}\otimes\mathfrak{h}^{\mu'}, \mathfrak{h}^{\mu'}) & \odot & \text{End}\mathfrak{g}^\mu
\end{array}
$$

We start by giving the description of $Tw\mathcal{M}$ as an $S$-module.

**Definition 14.** The $Tw\mathcal{P} - Tw\mathcal{Q}$ bimodule $Tw\mathcal{M}$ is the space

$$
Tw\mathcal{M}(n) = \prod_{r \geq 0} (\mathcal{M}(r + n) \otimes \mathbb{K}[-2r])^{S_r},
$$

with differential $d_{Tw}$, where $S_r$ here is the subgroup of $S_{r+n}$ fixing the last $n$ entries.

We need now to clarify the left and right actions, as well as the differential.

Let $m \in Tw\mathcal{M}(n) = \prod_{r \geq 0} (\mathcal{M}(p + r) \otimes \mathbb{K}[-2r])^{S_r}$. We denote by $m_r$ it’s projection in $(\mathcal{M}(p + r) \otimes \mathbb{K}[-2r])^{S_r}$. We use a similar notation $p_r, q_r$.

The right $Tw\mathcal{Q}$ action on $\mathcal{M}$ is defined in the following way: Let $m \in Tw\mathcal{M}(n)$ and $q \in Tw\mathcal{Q}(l)$.

$$(m \circ q)_r := \sum_{p=0}^{r} \sum_{\sigma \in Sh_{p,r-p}} \gamma_{i,\sigma}(m_p, q_{r-p}),$$

where $Sh_{p,r-p} \subset S_r$ are the $(p, r - p)$ shuffles $\gamma_{i,\sigma}$ is the composition given by the following tree

\[ \sigma(1) \cdots \sigma(p) r + 1 \cdots r+i-1 \cdots r+n+1 \]

\[ \sigma(p+1) \cdots \sigma(r) r + 1 \cdots r+i-1 \cdots r+n+1 \]

We write $d_{Tw} = d_{\mathcal{M}} + d_R + d_L$, where $d_{\mathcal{M}}$ is the differential induced by the differential in $\mathcal{M}$.

The Lie Algebra $L_{\mathcal{Q}}$ acts on $(Tw\mathcal{M}, d_{\mathcal{M}})$ by operadic derivations. The proof of this is the same as [DW, Proposition 3.2].

The Lie Algebra $Tw\mathcal{Q}(1)$ acts on the right on $Tw\mathcal{M}$ by

$$m \cdot q = \sum_{i=1}^{n} m \circ_i q,$$

where $m \in Tw\mathcal{M}(n)$ and $q \in Tw\mathcal{Q}$.

Multiplying by a minus sign, the previous right action becomes a left action, thus inducing a dg Lie algebra action $L_{\mathcal{Q}} \ltimes Tw\mathcal{Q}(1) \quad (Tw\mathcal{M}, d_{\mathcal{M}})$. 

The map \( F : \text{hoLie}_1 \rightarrow \mathcal{Q} \) gives us a Maurer-Cartan element in \( \mathcal{L}_Q \). Due to Lemma 25, we can twist \((Tw\mathcal{M}, d_M)\) with respect to this Maurer-Cartan element, giving us the module \((Tw\mathcal{M}, d_M + d_R)\).

There is an obvious left \( \mathcal{P} \) action on \((Tw\mathcal{M}, d_M)\), using the original \( \mathcal{P} \) action on \( \mathcal{M} \). It is easy to see that \( \mathcal{P} \) also acts on \((Tw\mathcal{M}, d_M + d_R)\).

Indeed, the equation of compatibility with the differential

\[
(d_M + d_R)(p \circ m) = d_{\mathcal{P}} p \circ m + (-1)^{|p|} p \circ (d_M + d_R)m
\]

is equivalent to \(d_R(p \circ m) = (-1)^{|p|} p \circ d_Rm\), and the associativity axiom involving the left and right actions of an operadic bimodule, together with the fact that \(d_R\) uses right compositions ensures that this equality holds for all \(p \in Tw\mathcal{P}\) and \(m \in Tw\mathcal{M}\).

The map \( F : \text{hoLie}_1 \rightarrow \mathcal{M} \) gives us a Maurer-Cartan element in \(\prod_r \text{Hom}_{\mathcal{E}}(\mathbb{K}[2r], \mathcal{M}(r)) = \prod_r (\mathcal{M}(r) \otimes \mathbb{K}[-2r])^{\text{gtr}} = Tw\mathcal{M}(0)\). Twisting with respect to this Maurer-Cartan element we obtain a left action of \(Tw\mathcal{P}\) on \((Tw\mathcal{M}, (d_M + d_R) + d_L)\).

Using a similar argument of compatibility with the differential, we see that \(Tw\mathcal{Q}\) acts on the right on \((Tw\mathcal{M}, d_M + d_R + d_L) = (Tw\mathcal{M}, d_{Tw}), \) and the associativity of the left \(Tw\mathcal{P}\) and right \(Tw\mathcal{Q}\) actions is clear and so we finished the construction of the bimodule \(Tw\mathcal{M}\).

\[\text{A.2.1. The action on } \text{Hom}(g^\mu \otimes \bullet, h^{\mu'})\]

As described in the beginning of the section, we wish now to construct a map of bimodules

\[
\begin{array}{ccc}
Tw\mathcal{P} & \circ & Tw\mathcal{M} \\
\downarrow & & \downarrow \\
\text{End} h^{\mu'} & \circ & \text{Hom}(g^\mu \otimes \bullet, h^{\mu'}) \\
\downarrow & & \downarrow \\
\text{End} g^\mu & & \text{Hom}(g^\mu \otimes \bullet, h^{\mu'})
\end{array}
\]

(19)

The two outer maps are the maps induced by the usual twisting of operads. For the main map, informally we do the usual procedure of inserting the Maurer-Cartan element on the non-symmetric slots. Formally, if \(m \in Tw\mathcal{M}(n)\),

\[
m(x_1, \ldots, x_n) = \sum_{r=0}^{\infty} \frac{1}{r!} m_r(\mu, \ldots, \mu, x_1, \ldots, x_n), \quad x_i \in g,
\]

where we identify an element of \(\mathcal{M}\) (resp. \(Tw\mathcal{M}\)) with its image in \(\text{Hom}(g, h)\) (resp. \(\text{Hom}(g^\mu, h^{\mu'})\)).

The only thing that remains to be checked is the commutativity of the left and right squares, as well as the compatibility with the differential of the central vertical map. Let as call \(l_r^P\) the image of the \(r\)-ary generator of \(\text{hoLie}_1\) in \(\mathcal{P}\), and we define similarly \(l_r^Q\) and \(l_r^M\).

Due to the original bimodule morphism \(IS\), the right square is trivially commutative, and the commutativity of the left square is a simple consequence \(IS\) together with the hypothesis \(\sum_r \frac{1}{r!} l_r^M(\mu, \ldots, \mu) = \mu'.\) Also, thanks to this equation, when we evaluate \(d_L m\) in \(\text{Hom}(g^\mu, h^{\mu'}), \) \(mu'\) will replace the Maurer-Cartan element of \(Tw\mathcal{M}\).
We wish to show that for $m \in Tw\mathcal{M}(n)$ and $x_1, \ldots, x_n \in g^\mu$,
\[
d^\mu(m(x_1, \ldots, x_n)) = (d_M m + d_L m + d_R m)(x_1, \ldots, x_n) + \sum_{i=1}^{n} (-1)^{|m|+|x_1|+\cdots+|x_{i-1}|} m(x_1, \ldots, d^\mu x_i, \ldots, x_n).
\]

Keep in mind in the following computations that $m_r$ has $r$ non-symmetric inputs and $n$ symmetric inputs, whereas $l^Q_k$ will be of arity $r$ but will have $r-1$ non-symmetric inputs. Expanding the right hand side we get
\[
\sum_{r \geq 0} \frac{1}{r!} d_M m_r(\mu, \ldots, \mu, x_1, \ldots, x_n) + \sum_{k \geq 2, r \geq 0} \frac{1}{(k-1)!} l^Q_k(\mu', \ldots, \mu', m_r(\mu, \ldots, \mu, x_1, \ldots, x_n)) +
\]
\[
- \sum_{r \geq 0, k \geq 2} \frac{r}{r! k!} m_r(l^Q_k(\mu, \ldots, \mu, x_1, \ldots, x_n)) +
\]
\[
- \sum_{r \geq 0, k \geq 2} \frac{(-1)^{|x_1|+\cdots+|x_{k-1}|}}{r!(k-1)!} m_r(\mu, \ldots, \mu, x_1, \ldots, l^Q_k(\mu, \ldots, \mu, x_1, \ldots, x_n)) +
\]
\[
- \sum_{i=1}^{n} \sum_{r \geq 0, k \geq 2} \frac{(-1)^{|x_1|+\cdots+|x_{k-1}|}}{r!(k-1)!} m_r(\mu, \ldots, \mu, x_1, \ldots, d x_i, \ldots, x_n).
\]

Using the Maurer-Cartan equation, the third summand simplifies to

\[
\sum_{r \geq 0} \frac{r}{r!} m_r(d\mu, \mu, \ldots, \mu, x_1, \ldots, x_n),
\]

therefore, the first, third and fifth summands add up to $d(m(x_1, \ldots, x_n))$, while the fourth and sixth summands cancel out, leaving us precisely with $d^\mu(m(x_1, \ldots, x_n))$.

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