On some mathematical models tracking epidemics spreading

Jose M Cerveró
Departamento de Física Fundamental. Universidad de Salamanca, 37008 Salamanca, SPAIN
E-mail: cervero@usal.es

Abstract. A complete integrability of one SEIRD-like dynamical system is presented. Many models like this have been used nowadays in epidemiology and several other descriptions of virological spreading. In this paper we show that one of them is exactly solvable. Only one almost trivial condition for integrability is needed. The statistical perspective is not considered. Our solution is an exact one and the result hereby presented cast some doubts on the interest in this class of deterministic models. One entirely new avenue for tackling the problem of spreading diseases is then proposed. Curiously enough is surprisingly related to Quantum Mechanics in its non-hermitic version also called PT-Quantum Mechanics.

1. The SEIRD-1 system
There is a growing interest nowadays in describing the spread of a pandemic, for obvious reasons. Many models of dynamical systems have been presented but until now, no one of them seems to account for a very rapidly growing expansion of obvious examples of these terrible scourges of the human kind. One of this model has been used frequently: We shall call SEIRD-1 Dynamical System (See SIR-Dynamical Systems and generalizations in the web, and also [1] (specially equation (1) on it), and [2]). Many more have appeared recently ([3], [4] and [5]). Also interest on modelling dynamical systems to account for the present pandemic has fostered an immense body of research as it is easy to understand [6]. In the dynamical system we would like to concentrate, we can easily identify the $S$ with the Susceptible population, $E$ with the Infected Asymptomatic population, $I$ with the Infected Symptomatic population, $R$ with the Recovered population and finally $D$ with the Dead population. The sum of these quantities should be a constant integer number $\mathcal{N}$: The total population.

$$\mathcal{N} = S(\xi) + E(\xi) + I(\xi) + R(\xi) + D(\xi)$$ (1)

The system under scrutiny has the form:

$$\frac{dS(\xi)}{d\xi} = -r_1 S(\xi) E(\xi) - r_2 S(\xi) I(\xi)$$ (2)

$$\frac{dE(\xi)}{d\xi} = r_1 S(\xi) E(\xi) + r_2 S(\xi) I(\xi) - (a_1 + c_1) E(\xi)$$ (3)

$$\frac{dI(\xi)}{d\xi} = c_1 E(\xi) - (a_2 + c_2) I(\xi)$$ (4)
Let us now proceed to manipulate these equations. From (6) it is trivial to see that:

\[
\frac{dR(\xi)}{d\xi} = a_1 E(\xi) + a_2 I(\xi)
\]

(5)

\[
\frac{dD(\xi)}{d\xi} = c_2 I(\xi)
\]

(6)

It is important to remark the fact that this system is consistent with the conservation equation (1). At any rate many virologists [2] have made use of these approaches to analyze their data. Let us now proceed to manipulate these equations. From (6) it is trivial to see that:

\[
I(\xi) = \frac{1}{c_2} \frac{dD(\xi)}{d\xi}
\]

(7)

From (4) for the infected population easily follows that:

\[
E(\xi) = \frac{1}{c_1 c_2} \left[ \frac{d^2D(\xi)}{d\xi^2} + (a_2 + c_2) \frac{dD(\xi)}{d\xi} \right]
\]

(8)

Substitution of (7) and (8) in (5) yields for \( R(\xi) \) the equation

\[
\frac{dR(\xi)}{d\xi} = \frac{1}{c_1 c_2} \left[ a_1 \frac{d^2D(\xi)}{d\xi^2} + (a_1 a_2 + a_1 c_2 + a_2 c_1) \frac{dD(\xi)}{d\xi} \right]
\]

(9)

which easily be integrated as

\[
R(\xi) = \frac{1}{c_1 c_2} \left[ a_1 \frac{dD(\xi)}{d\xi} + (a_1 a_2 + a_1 c_2 + a_2 c_1) D(\xi) \right] + R_0
\]

(10)

A similar substitution performed in (2) provides for \( S(\xi) \) the following differential equation:

\[
\frac{dS(\xi)}{d\xi} = -S(\xi) \frac{1}{c_1 c_2} \left[ \frac{d^2D(\xi)}{d\xi^2} + (r_1 a_2 + r_1 c_2 + c_1 r_2) \frac{dD(\xi)}{d\xi} \right]
\]

(11)

whose trivial integration yields for \( S(\xi) \) the expression:

\[
S(\xi) = S_0 \exp \left\{ -\frac{1}{c_1 c_2} \left[ r_1 \frac{d^2D(\xi)}{d\xi^2} + (r_1 a_2 + r_1 c_2 + c_1 r_2) D(\xi) \right] \right\}
\]

(12)

and the substitution of (7) and (8) in (2) provides:

\[
S(\xi) \left[ r_1 \frac{d^2D(\xi)}{d\xi^2} + (r_1 a_2 + r_1 c_2 + c_1 r_2) \frac{dD(\xi)}{d\xi} \right] =
\]

\[
\left[ \frac{d^2D(\xi)}{d\xi^2} + (a_1 + c_1 + a_2 + c_2) \frac{d^2D(\xi)}{d\xi^2} + (a_1 + c_1)(a_2 + c_2) \frac{dD(\xi)}{d\xi} \right]
\]

(13)

Therefore, the compatibility between (12) and (13) implies

\[
S_0 \exp \left\{ -\frac{1}{c_1 c_2} \left[ r_1 \frac{d^2D(\xi)}{d\xi^2} + (r_1 a_2 + r_1 c_2 + c_1 r_2) D(\xi) \right] \right\} =
\]

\[
\left[ \frac{d^2D(\xi)}{d\xi^2} + (a_1 + c_1 + a_2 + c_2) \frac{d^2D(\xi)}{d\xi^2} + (a_1 + c_1)(a_2 + c_2) \frac{dD(\xi)}{d\xi} \right]
\]

(14)

\[
\left[ r_1 \frac{d^2D(\xi)}{d\xi^2} + (r_1 a_2 + r_1 c_2 + c_1 r_2) \frac{dD(\xi)}{d\xi} \right]
\]

(15)
whose logarithmic derivative yields the following fourth order differential equation for just $D(\xi)$:

$$\left[\frac{d^3D(\xi)}{d\xi^3} - \frac{d^2D(\xi)}{d\xi^3}\right]\left[\frac{d^2D(\xi)}{d\xi^3} + k_3 \frac{dD(\xi)}{d\xi}\right] +$$

$$+ \frac{k_0}{c_1c_2} \left[\frac{d^2D(\xi)}{d\xi^2} + k_3 \frac{dD(\xi)}{d\xi}\right]^2 \left[\frac{d^2D(\xi)}{d\xi^3} + (k_1 + k_2) \frac{d^2D(\xi)}{d\xi^2} + k_1k_2 \frac{dD(\xi)}{d\xi}\right] +$$

$$(k_3(k_1 + k_2) - k_1k_2) \left[\frac{dD(\xi)}{d\xi} \frac{d^3D(\xi)}{d\xi^3} - \left[\frac{d^2D(\xi)}{d\xi^2}\right]^2\right]$$

and the following redefinition of constant has been made:

$$r_1 = k_0$$

$$a_1 + c_1 = k_1$$

$$a_2 + c_2 = k_2$$

$$\frac{1}{r_1}(r_1a_2 + r_1c_2 + c_1r_2) = k_3$$

With this change of variables (7), (8) and (10) read:

$$I(\xi) = \frac{1}{c_2} \frac{dD(\xi)}{d\xi}$$

$$E(\xi) = \frac{1}{c_1c_2} \left[\frac{d^2D(\xi)}{d\xi^2} + k_2 \frac{dD(\xi)}{d\xi}\right]$$

$$R(\xi) = \frac{1}{c_1c_2} \left[ a_1 \frac{dD(\xi)}{d\xi} + (a_1c_2 + a_2c_1)D(\xi) \right] + \mathcal{M}$$

$$S(\xi) = \frac{1}{k_0} \left[ \frac{d^3D(\xi)}{d\xi} + (k_1 + k_2) \frac{d^2D(\xi)}{d\xi^2} + k_1k_2 \frac{dD(\xi)}{d\xi}\right]$$

$$\left[\frac{d^2D(\xi)}{d\xi^2} + k_3 \frac{dD(\xi)}{d\xi}\right]$$

where $\mathcal{M}$ is a constant proportional to the total initial population. Furthermore, the conservation condition (1), can be also used to obtain another equation for $D(\xi)$. The substitution of equations (26) in (2) with the help of (7) give us the following third order differential equation for $D(\xi)$:

$$\frac{c_1c_2}{k_0} \left[\frac{d^3D(\xi)}{d\xi^3} + (k_1 + k_2) \frac{d^2D(\xi)}{d\xi^2} + k_1k_2 \frac{dD(\xi)}{d\xi}\right] +$$

$$+ \left[\frac{d^2D(\xi)}{d\xi^2} + k_3 \frac{dD(\xi)}{d\xi}\right] \left[\frac{d^2D(\xi)}{d\xi^2} + (k_1 + k_2) \frac{dD(\xi)}{d\xi}\right] + k_1k_2D(\xi) + \mathcal{M}c_1c_2 = 0$$

Actually, (27) is a first integral for (18). This property can easily be proved by solving (27) for the third derivative of $D(\xi)$. The substitution of this third order derivative in (18) satisfies identically this equation. Now one can easily see that (27) can also be written as:
\[
c_{1}c_{2} \frac{d}{d\xi} \left[ \frac{d^{2}D(\xi)}{d\xi^{2}} + (k_{1} + k_{2}) \frac{dD(\xi)}{d\xi} + k_{1}k_{2}D(\xi) + MC_{1}c_{2} \right] + \\
+ \left[ \frac{d^{2}D(\xi)}{d\xi^{2}} + k_{3} \frac{dD(\xi)}{d\xi} \right] \left[ \frac{d^{2}D(\xi)}{d\xi^{2}} + (k_{1} + k_{2}) \frac{dD(\xi)}{d\xi} + k_{1}k_{2}D(\xi) + MC_{1}c_{2} \right] = 0 \tag{28}
\]

The equation is identically satisfied if and only if:

\[
\left[ \frac{d^{2}D(\xi)}{d\xi^{2}} + (k_{1} + k_{2}) \frac{dD(\xi)}{d\xi} + k_{1}k_{2}D(\xi) + MC_{1}c_{2} \right] = 0 \tag{29}
\]

This equation is linear and easy to integrate. The general solution with two initial conditions dictated by \( D(0) = K = MC_{1}c_{2} \) and \( \frac{dD}{d\xi} |_{\xi=0} = 0 \):

\[
D(\xi) = \frac{K}{k_{2} - k_{1}} \{ k_{2}e^{-k_{1}\xi} - k_{1}e^{-k_{2}\xi} \} - K \tag{30}
\]

Now in (22) the numerator vanishes. In order to set to zero the denominator as well we insert \( D(\xi) \) in the expression:

\[
\left[ \frac{d^{2}D(\xi)}{d\xi^{2}} + k_{3} \frac{dD(\xi)}{d\xi} \right] = \frac{r_{1}K}{k_{2} - k_{1}} (k_{1}k_{2} - k_{3}) \{ k_{1}e^{-k_{1}\xi} - k_{2}e^{-k_{2}\xi} \} \tag{31}
\]

The only way for this expression to be zero is:

\[
k_{1}k_{2} = k_{3} \implies or \implies r_{1}(a_{1} + c_{1})(a_{2} + c_{2}) = r_{1}a_{2} + r_{1}c_{2} + r_{2}c_{1} \tag{32}
\]

The last relationship among all these constants has to be seen as the **Fundamental Condition for the Integrability of the SEIRD**. The relationship (32) is extremely easy to fulfill and we do not consider such a condition as a very stringent condition for integrability. Note however that the parameters of a real pandemic may or may not be arranged to fulfill equation (32).

As we have shown the SEIRD systems are of very limited use when we try to apply them to the real world. This is the main reason why we should look for other methods (probably not of a purely deterministic nature) that may account for the spreading of epidemics. These methods may combine statistical background with a well defined mathematical foundations. We would like to present here one of these alternatives which combines statistics with a set of mathematical rules.

The main advantage of this method is that there exist as many forms of curves as one wish, given the couple of parameters \( \{A, B\} \). The pair of constants can be adjusted at will. In fact we have generated dozens of curves of different shapes just by changing judiciously the pair \( \{A, B\} \). But all this method and its intrinsic features will be the subject of the next section.

2. Another quite different approach

**Brief Introduction to PT-Quantum Mechanics** Non Hermitian Potentials have been extensively studied, scrutinized and exploited with different physical applications from the beginning of the second millenium. For a recent account see the fundamental reference [7].
Actually the present authors have applied some features of this Non Hermitian Hamiltonians to a set of various kinds of complex potentials and more precisely the band structure of periodic potentials [8]. The diagrams we shall discuss below hasn’t been found by statistical procedures but rather by exactly solving a non Hermitian potential with the conventional Schrödinger equation ([9] and [10]). In this paper we would like to call reader’s attention to one of the most striking properties on one of the Pöschl-Teller potential with complex extension: the Property of Absorption ([11] and [12])

There is no direct relationship between SEIRDS and the PT-symmetric method that will be described below. We have only tried to present an alternative method to systems of deterministic differential equations, using quantum mechanics.

The enormous amount of shapes arising from the solutions provides a manifold of curves that is given by the fact that almost any form of the different rates can be adjusted to empirical evidence by using just two positive constants. The explicit form of the non Hermitic Potential that we shall use, in non-dimensional units is:

\[ V(x) = \frac{V_1}{\cosh^2 x} + i\frac{V_2}{\cosh^2 [x]} \]  \hspace{1cm} (33)

Let us define:

\[ A = \sqrt{V_1 + V_2 - \frac{1}{4}} ; \quad B = \sqrt{V_1 - V_2 - \frac{1}{4}} \]  \hspace{1cm} (34)

As the values of \( V_1 \) and \( V_2 \) are always positive but not necessarily real, the constant \( A \) must be real and positive as well but \( B \) can be real positive and also pure imaginary. Let us now define other set of constants as functions of \( A \) and \( B \):

\[ C_+ = \frac{1}{2} \{ \cosh[2\pi A] + \cosh[2\pi B] \} \]  \hspace{1cm} (35)

\[ C_- = \{ \cosh[\pi A] - \cosh[\pi B] \} \]  \hspace{1cm} (36)

\[ C_\otimes = \{ \cosh[\pi A] \cosh[\pi B] \} \]  \hspace{1cm} (37)

These constants \( C_+ \), \( C_- \) and \( C_\otimes \) enter into the expressions for the Absorption, the Transmission and the Reflection which are given by [11]:

\[ A = \frac{C_- \{ \exp[2\pi x] \cosh[\pi B] - \exp[-2\pi x] \cosh[\pi A] \} + C_-^2}{\cosh^2[2\pi x] + 2C_\otimes \cosh[2\pi x] + C_+} \]  \hspace{1cm} (38)

\[ T = \frac{\{ \sinh[2\pi x] \}^2}{\cosh^2[2\pi x] + 2C_\otimes \cosh[2\pi x] + C_+} \]  \hspace{1cm} (39)

\[ R = \frac{\{ \exp[\pi x] \cosh[\pi B] + \exp[-\pi x] \cosh[\pi A] \}^2}{\cosh^2[2\pi x] + 2C_\otimes \cosh[2\pi x] + C_+} \]  \hspace{1cm} (40)

Notice that as required for conservation of probability:

\[ R + T + A = 1 \]  \hspace{1cm} (41)

The above condition is the counterpart of (1) in SEIRD-like systems.
Figure 1. Red: Deaths, Blue: Susceptible Population, Green: Recovered Population; Value of the Parameters $A = 1$, $B = 0.28i$

Figure 2. Parameters; Left: $A = 1.36$, $B = 1.24$; Right: $A = 1.36$, $B = 1.18$

Figure 1 attach colours to kinds of potential patients. One can easily fix the average statistical data to practically any form of the curves by adequately changing the parameters. The blue curve there shows a deviation from the initial pattern in the direction of increasing the Susceptible people in the middle of the spreading which corresponds to non-immunology effect. In the last pair of drawings one can see a sustained number of Deaths over a long time due to lacking of control measures or useless medication. Actually any intermediate effect in the epidemics can be reproduced by a judicious change of only two constants.

Acknowledgments. This research has been supported in part by MICINN (Grant PID2019-106820RBC22) and Junta de Castilla y León (Grant SA256P18).
Figure 3. Parameters; Left: $A = 7.7$, $B = 1.4$; Right: $A = 6.1$, $B = 0.27i$

References
[1] Karthikeyan R. et al 2020 A fractional model for the novel coronavirus (COVID-19) outbreak *Nonlinear Dynamics* **101** 711-718
[2] Chadi M, Saad-Roy et al 2020 Dynamics in a simple evolutionary-epidemiological model for the evolution of an initial asymptomatic infection stage *Proc. Natl. Acad. Sci. USA* **117** 11541-11550
[3] Saha S, Samanta G P and Nieto J 2020 Epidemic model of COVID-19 outbreak by inducing behavioural response in population *Nonlinear Dynamics* **102** 455–487
[4] Khyar O and Allali K 2020 Global dynamics of a multi-strain SEIR epidemic model with general incidence rates: application to COVID-19 pandemic *Nonlinear Dynamics* **102** 489-509
[5] Kumar Biswas S, Kumar Ghosh J, Susmita Sarkar S and Ghosh U 2020 COVID-19 pandemic in India: a mathematical model study *Nonlinear Dynamics* **102** 537–553
[6] Freeman T 2020 COVID Research and Resources Group brings physicists together *Physics World*
[7] Bender C M 2019 *PT-Symmetry In Quantum And Classical Physics* (World Scientific Publishing Ltd)
[8] Mateo J and Cerveró J M 2003 PT-Symmetry in quantum periodic potentials *Phys. Lett.* A **317** 26-31
[9] Dabrowska J W, Khare A and Sukhatme U P 1988 Explicit wavefunctions for shape-invariant potentials by operator techniques *J. Phys.* A **21** L195-L200
[10] Khare A and Sukhatme U P 1988 Scattering amplitudes for supersymmetric shape-invariant potentials by operator methods *J. Phys.* A **21** L501-L508
[11] Cerveró J M and Rodriguez A 2004 Absorption in atomic wires *Phys. Rev.* A **70** 52705
[12] Rodriguez A and Cerveró J M 2006 One dimensional disordered wires with Poschl-Teller potentials *Phys. Rev.* B **74** 104201