HIGHER SYMPLECTIC STRUCTURE ON TORSIONLESS LIE RINEHART PAIRS

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ABSTRACT. We define an n-plectic structure as a commutative and torsionless Lie Rinehart pair, together with a distinguished cocycle from its Chevalley-Eilenberg complex.

This 'n-plectic cocycle' gives rise to an extension of the Chevalley-Eilenberg complex by so called symplectic tensors. The cohomology of this extension generalizes Hamiltonian functions and vector fields to tensors and cotensors in a range of degrees, up to certain coboundaries and has the structure of a Lie $\infty$-algebra.

Finally we show, that momentum maps appear in this context just as weak Lie $\infty$-morphisms.

1. INTRODUCTION

We define n-plectic structures as commutative and torsionless Lie Rinehart pairs, together with a distinguished $(n+1)$-cocycle from their Chevalley-Eilenberg complexes.

By torsionless we mean, that the Lie algebra partner of the pair is a torsionless module with respect to its commutative partner. The natural pairing between tensors and cotensors is then non-degenerate and we can define general Cartan calculus.

In contrast to symplectic geometry, where the symplectic form has to be non-degenerate on vectors, we do not impose additional properties on the n-plectic cotensor, other than being a cocycle. Since we deal with tensors of arbitrary degree almost all n-plectic cocycles will be degenerate to some extend on higher tensors and therefore we drop any distinction between degenerate and non-degenerate right from the beginning.

The theory in this paper appears as a refinement of the work from Rogers on n-plectic manifolds as given in [9] and the Lie $\infty$-algebra of Lie Rinehart pairs as given by me in [7]. In fact the papers [3] and [1] have been an important inspiration.

The basic generalization however is, that we do not restrict to Hamiltonian (or symplectic) vectors, but consider appropriate tensors of arbitrary degree instead. This is possible, since the n-plectic structure behaves well with respect to the Lie $\infty$-algebra of exterior tensors and we can define higher Poisson brackets as some kind of Lie $\infty$-algebra extension.

A main concerns of this work is to convince the reader, that exterior (co)tensors of higher degrees are a central part of the theory, not only as a need from physics but for pure mathematical reasons. In fact this idea has a long history, as it appears...
first in the work of Kanatchikov (see [4], [5]) and was then pushed further by Forger, Paufler and Römer as in [2] and the references therein.

In addition we show, that n-plectic geometry can be defined verbatim on arbitrary torsionless Lie Rinehart pairs without much additional afford.

2. Lie Rinehart pairs

We start with a short introduction to Lie Rinehart pairs, which have the additional property, that the natural map from the Lie partner into its double dual module is injective. We call them torsionless or semi-reflexive. In analogy to the prototypical Lie Rinehart pair of smooth functions and vector fields, these pairs allow for a general definition of exterior derivative, contraction and Lie derivative.

In what follows \( g \) will always be a real Lie algebra, that is a \( \mathbb{R} \)-vector space together with an antisymmetric, bilinear map, 
\[
\left[\cdot, \cdot\right] : g \times g \to g
\]
called Lie bracket, such that for any three vector \( x_1, x_2, x_3 \in g \) the Jacobi identity
\[
\left[\left[ x_1, x_2 \right], x_3 \right] + \left[\left[ x_2, x_3 \right], x_1 \right] + \left[\left[ x_3, x_1 \right], x_2 \right] = 0
\]
is satisfied.

In addition \( A \) will always be a real associative and commutative algebra with unit, that is a \( \mathbb{R} \)-vector space together with an associative and commutative, bilinear map
\[
\cdot : A \times A \to A
\]
called multiplication and a unit \( 1_A \in A \). According to a better readable text, we frequently suppress the symbol of the multiplication in \( A \) and just write \( ab \) instead of \( a \cdot b \).

Moreover \( \text{Der}(A) \) will be the Lie algebra of derivations of \( A \), that is the vector space of linear endomorphisms of \( A \), with \( D(ab) = D(a)b + aD(b) \) and Lie bracket 
\[
[D, D'](a) := D(D'(a)) - D'(D(a))
\]
for any \( a, b \in A \) and \( D, D' \in \text{Der}(A) \).

Before we get to Lie Rinehart pairs, it is handy to define Lie algebra modules first:

**Definition 2.1** (Lie algebra module). Let \( g \) be a real Lie algebra, \( A \) an \( \mathbb{R} \)-algebra and \( D : g \to \text{Der}(A) \) a Lie algebra morphism. Then \( A \) is called a Lie algebra module (or just \( g \)-module) and \( D \) is called the \( g \)-scalar multiplication.

Now a Lie Rinehart pair is nothing but a Lie algebra and an associative algebra, each of them being a module with respect to the other, such that a particular compatibility equation of their products is satisfied:

**Definition 2.2** (Lie Rinehart Pair). Let \( A \) be an associative and commutative algebra with unit, \( \mathfrak{g} \) a Lie algebra and \( \cdot_A : A \times \mathfrak{g} \to \mathfrak{g} \) as well as \( D : \mathfrak{g} \to \text{Der}(A) ; x \mapsto D_x \) maps, such that \( A \) is a \( \mathfrak{g} \)-module with \( \mathfrak{g} \)-scalar multiplication \( D \), the vector space \( \mathfrak{g} \) is an \( A \)-module with \( A \)-scalar multiplication \( \cdot_A \) and the Leibniz rule
\[
[x, a \cdot_A y] = D_x(a) \cdot_A y + a \cdot_A [x, y]
\]
is satisfied for any \( x, y \in \mathfrak{g} \) and \( a \in A \). Then \( (A, \mathfrak{g}) \) is called a Lie Rinehart pair.

Maybe the most prominent example is provided by smooth functions and vector fields on a differentiable manifold:
Example 1. Let $M$ be a differentiable manifold, $C^\infty(M)$ the algebra of smooth, real valued functions and $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$. $\mathfrak{X}(M)$ is a $C^\infty(M)$-module and vector fields acts as derivations on smooth functions, that is the map

$$ D : \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M) ; (X, f) \mapsto D_X(f) := X(f) $$

satisfies the equation $D_X(fg) = D_X(f)g + fD_X(g)$. Moreover the Leibniz rule $[X, fY] = D_X(f)Y + f[X, Y]$ holds and it follows that $(C^\infty(M), \mathfrak{X}(M))$ is a Lie Rinehart pair.

Morphisms of Lie Rinehart pairs are pairs of appropriate algebra maps, which interact properly with respect to the additional module structures [8]:

**Definition 2.3 (Lie Rinehart Morphism).** Let $(A, \mathfrak{g})$ and $(B, \mathfrak{h})$ be two Lie Rinehart pairs. A **morphism of Lie Rinehart pairs** is a pair of maps $(f, g)$, such that $f : A \to B$ is a morphism of associative and commutative, real algebras with unit, $g : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras and the equations

$$ g(a \cdot_A x) = f(a) \cdot_B g(x) \quad \text{and} \quad f(D_x(a)) = D_{g(x)}(f(a)) \quad (4) $$

are satisfied for any $a \in A$ and $x \in \mathfrak{g}$.

The Lie structure of $\mathfrak{g}$ can be extended into a graded\(^1\) Lie algebra on the direct sum of the partners, concentrated in degree zero and one. This differs from [8], where no grading was considered.

**Definition 2.4 (Associated Lie algebra).** Let $(A, \mathfrak{g})$ be a Lie Rinehart pair. Its **associated** (graded) Lie algebra is the graded direct sum $A \oplus \mathfrak{g}$, where $A$ is concentrated in degree zero, $\mathfrak{g}$ is concentrated in degree one and the Lie bracket is defined by

$$ [\cdot, \cdot] : A \oplus \mathfrak{g} \times A \oplus \mathfrak{g} \to A \oplus \mathfrak{g} $$

$$ ((a, x), (b, y)) \mapsto (D_x(a) + D_y(b), [x, y]). \quad (5) $$

In particular this means, that we can talk about Lie $\infty$-morphisms in the context of Lie Rinehart pairs.

To define torsionless Lie Rinehart pairs, let $\mathfrak{g}^\vee_A := \text{Hom}_A(\mathfrak{g}, A)$ be the dual of the $A$-module $\mathfrak{g}$ and $\mathfrak{g}^{\vee\vee}$ its appropriate double dual.

**Definition 2.5 (Torsionless Lie Rinehart pair).** Let $(A, \mathfrak{g})$ be a Lie Rinehart pair, such that the natural map $\mathfrak{g} \to \mathfrak{g}^{\vee\vee}$ : $x \mapsto (\mathfrak{g}^{\vee} \to A ; f \mapsto f(x))$ is injective. Then $(A, \mathfrak{g})$ is called **torsionless** or **semi-reflexive**.

Torsionless Lie Rinehart pairs gives rise to a non degenerate pairing between vectors and covectors, which in turn allows to consider Cartan calculus. To see that we define $\otimes^0_A \mathfrak{g} := A$ as well as $\otimes^0_A \mathfrak{g}^{\vee} := A$ and write $\otimes^n_A \mathfrak{g}$ as well as $\otimes^n_A \mathfrak{g}^\vee$ for the appropriate $n$-fold $A$-tensor products of the $A$-modules $\mathfrak{g}$ and $\mathfrak{g}^{\vee}$. Since $A$ is commutative, $\otimes^n_A \mathfrak{g}$ and $\otimes^n_A \mathfrak{g}^{\vee}$ are $A$-modules.

\(^1\)The term 'graded' will always mean $\mathbb{Z}$-graded. See the appendix for an introduction to $\mathbb{Z}$-graded vector spaces.
Proposition 2.8. Let \((A, g)\) be a Lie Rinehart pair. Then the natural pairing is non-degenerate.

Proof. This follows since the natural map \(g \to g^{\vee \vee}\) is injective.

With a non-degenerate pairing the contraction of a cotensor along a tensor can be defined consistently.
Definition 2.9 (Contraction). Let \((A, g)\) be a torsionless Lie Rinehart pair and \(x \in E^k(A, g)\) a tensor homogeneous of degree \(k\). Then the map
\[
i_x : E_{-n}(A, g^\vee) \to E_{k-n}(A, g^\vee)
\]
uniquely defined for any \(n \in \mathbb{N}_0\) and homogeneous cotensor \(f \in E_{-n}(A, g^\vee)\) by the equation
\[
\langle i_x f, y \rangle = \langle f, x \wedge y \rangle
\]
for all \(y \in E^{n-k}(A, g)\) and then extended to all of \(E(A, g^\vee)\) by additivity, is called the contraction along \(x\).

Since the natural pairing is non-degenerate, the contraction is uniquely defined by this equation.

The exterior tensor power of a Lie Rinehart pair has the structure of a Lie \(\infty\)-algebra \([7]\), with higher operators defined in terms of the exterior product and the Schouten-Nijenhuis bracket.

Definition 2.10 (Higher Lie Brackets). Let \((A, g)\) be a Lie Rinehart pair with exterior algebra \(E(A, g)\) and \([\cdot, \cdot]_S\) the antisymmetric Schouten-Nijenhuis bracket. Then the tensor Lie \(k\)-bracket
\[
[\cdot, \cdot, \ldots, \cdot]_k : E(A, g) \times \cdots \times E(A, g) \to E(A, g)
\]
is defined for any integer \(k \geq 2\) and homogeneous tensors \(x_1, \ldots, x_k \in E(A, g)\) by
\[
[x_1, \ldots, x_k]_k := \sum_{s \in S_k(2, k-2)} e(s; x_1, \ldots, x_k) e(x_{s(1)}) x_{s(k)} \wedge \cdots \wedge x_{s(3)} \wedge [x_{s(2)}, x_{s(1)}]_S
\]
and is then extended to all of \(E(A, g)\) by \(A\)-additivity.

These higher tensor brackets are graded symmetric and homogeneous of degree \(-1\) with respect to the tensor grading. The unary bracket \([\cdot]_1\) has to be the zero operator in general \([7]\) and \((E(A, g), [\cdot, \cdot]_S)\) is a Lie \(\infty\)-algebra, concentrated in non-negative (tensor) degrees.

In what follows we will always write \([\cdot, \ldots, \cdot]_k\) for the graded symmetric higher tensor brackets as well as \([\cdot, \cdot]_S\) for the common Schouten-Nijenhuis bracket.

The natural inclusion \(A \oplus g \hookrightarrow E(A, g)\) : \((a, x) \mapsto (a, x)\) is not a morphism of Lie \(\infty\)-algebras, since such a strict morphism has to commute with all higher brackets, but these brackets are zero on \(A \oplus g\). Instead the inclusion now comes as a weak morphism of Lie \(\infty\)-algebras:

Definition 2.11. Let \((A, g)\) be a Lie Rinehart pair with associated graded Lie algebra \(A \oplus g\), exterior tensor algebra \(E(A, g)\) and
\[
i_k : \ A \oplus g \times \cdots \times A \oplus g \to E(A, g)
\]
for any \(k \in \mathbb{N}\). Then the sequence \(i_\infty := (i_k)_{k \in \mathbb{N}}\) is called the natural inclusion of the Lie Rinehart pair into its exterior tensor Lie \(\infty\)-algebra.

On the other side, the exterior cotensor algebra has the structure of a differential graded algebra, at least if the Lie Rinehart pair is torsionless. This generalizes the well-known De Rham Complex of differential forms to the exterior cotensor algebra of such Lie Rinehart pairs:
Definition 2.12 (Exterior derivative). Let \((A, g)\) be a torsionless Lie Rinehart pair with exterior cotensor algebra \(E(A, g)\). Then the **exterior derivative**
\[
d : E(A, g) \to E(A, g)
\]
is defined for a homogeneous cotensor \(f \in E_{-k}(A, g)\) and any \(x_0, \ldots, x_k \in g\) by the equation
\[
d f(x_0, \ldots, x_k) = \sum_j (-1)^j D_{x_j}(f(x_0, \ldots, \hat{x}_j, \ldots, x_k))
\]
and is then extended to all of \(E(A, g)\) by \(A\)-linearity.

Since the natural pairing is non degenerate, the exterior derivative is uniquely defined by this equation and is moreover a graded map, homogeneous of degree \(-1\), with \(d^2 = 0\). In particular the Chevalley-Eilenberg complex is a cochain complex with respect to the tensor grading.

With the contraction along tensors and the exterior derivative, we can now define Lie derivatives along tensors in terms of the graded Cartan formula:

Definition 2.13. Let \((A, g)\) be a torsionless Lie Rinehart pair and \(x \in E(A, g)\) a homogeneous tensor. Then the map
\[
L_x : E(A, g) \to E(A, g) ; f \mapsto df - \underbrace{(-1)^{|x|}i_x \circ df}_{(12)}
\]
is called the **Lie derivative** along \(x\).

The following proposition summarizes basic computation rules as we need them in what follows:

Proposition 2.14. Let \((A, g)\) be a torsionless Lie Rinehart pair, \([\cdot, \cdot]\)s the graded antisymmetric Schouten-Nijenhuis bracket, \(x, y \in E(A, g)\) homogeneous tensors and \(f \in E(A, g)\) a cotensor. Then
\[
\begin{align*}
   dL_x f &= (-1)^{|x|-1}L_x df \\
i_{[x, y]} f &= (-1)^{|x|-1}|y|L_x i_y f - i_y L_x f \\
L_{[x, y]} f &= (-1)^{|x|-1}(|y|-1)L_x L_y f - L_y L_x f \\
L_x \wedge y f &= (-1)^{|y|}i_y L_x f + L_y i_x f
\end{align*}
\]

Proof. A proof is given in [2] in the context of multivector fields and differential forms, but can be carried over verbatim into our general setting.

3. Higher Symplectic Geometry

3.1. **N-plectic structures.** We define higher symplectic structures as torsionless Lie Rinehart pairs together with a distinguished cocycle from their Chevalley-Eilenberg complex. We look at the ‘infinitesimal invariants’ of such a cocycle and show that they are a Lie \(\infty\)-algebra with respect to the higher tensor brackets.

Definition 3.1. Let \((A, g)\) be a torsionless (semi-reflexive) Lie Rinehart pair and \(\omega \in E_{-(n+1)}(A, g)\) a cocycle of tensor degree \(-(n+1)\) for some \(n \in \mathbb{N}\). Then \((A, g, \omega)\) is called an **\(n\)-symplectic (or just \(n\)-plectic) structure** and \(\omega\) is called its **\(n\)-plectic cocycle**.
We do not distinguish between $n$-plectic cocycles that are degenerate on vectors and those that are not. The fundamental pairing between functions and vector fields generalizes to tensors and cotensors in a range of degrees and almost all $n$-plectic cocycles have a non-trivial kernel on higher tensors, whether they are degenerate on vectors or not. Unique pairings are an exception in general $n$-plectic structures.

**Example 3.** Any symplectic manifold $(M, \omega)$ is in particular a 1-plectic structure $(C^\infty(M), \mathfrak{X}(M), \omega)$ on the Lie Rinehart pair of smooth functions and vector fields.

**Example 4 (Trivial $n$-plectic structure).** Let $(A, g)$ be a torsionless Lie Rinehart pair and $\omega \in E_{-(n+1)}(A, g^\vee)$ the zero cotensor. Then $(A, g, \omega)$ is called the trivial $n$-plectic structure.

If the Lie derivation of a symplectic form along a vector vanishes, the vector is called symplectic. In a general $n$-plectic setting however, there is no reason to restrict to vectors, but to ask for all infinitesimal invariants of an $n$-plectic cocycle:

**Definition 3.2 (Symplectic and Hamiltonian tensors).** Let $(A, g, \omega)$ be an $n$-plectic structure and $x \in E(A, g)$ an exterior tensor (of arbitrary tensor degree). We say that $x$ is a **symplectic tensor** if the contraction of $\omega$ along $x$ is again a cocycle, that is if

$$di_x \omega = 0.$$  \hfill (14)

In addition we say that $x$ is a **Hamiltonian tensor**, if the contraction of $\omega$ along $x$ is a coboundary, i.e. if there is an exterior cotensor $f \in E(A, g^\vee)$ such that

$$df = i_x \omega.$$  \hfill (15)

We write $\text{Sym}(A, g, \omega)$ for the set of all symplectic tensors. Since $d\omega = 0$, an exterior tensor $x$ is symplectic, if and only if $L_x \omega = 0$.

**Remark.** Any Hamiltonian tensor is clearly a symplectic tensor, but the converse depends on the de-Rham cohomology of the Chevalley-Eilenberg complex $E(A, g^\vee)$. However in sharp contrast to symplectic geometry, the first cohomology group isn’t sufficient anymore, since $i_x \omega$ now can have a tensor degree other than $-1$.

The following proposition is one of the central observations in this work and a crucial technical detail. It links the Lie $\infty$-algebra of exterior tensors to the cochain complex of exterior cotensors.

**Proposition 3.3.** Let $(A, g, \omega)$ be an $n$-plectic structure and $[\cdot, \ldots, \cdot]_k$ the tensor Lie $k$-bracket on $E(A, g)$. Then

$$i_{[x_1, \ldots, x_k]} \omega = di_{x_k} \wedge \cdots \wedge di_{x_1} \omega$$  \hfill (16)

for any $k \in \mathbb{N}$ with $k \geq 2$ and symplectic tensors $x_1, \ldots, x_k \in \text{Sym}(A, g, \omega)$.

**Proof.** We proof this by induction on $k$. For $k = 2$ it is the well known link between the Poisson bracket of functions and the Lie bracket of symplectic vector field, but generalized to symplectic tensors of arbitrary degree:

$$i_{[x_1, x_2]} \omega = e(x_1)i_{[x_2, x_1]} \omega = e(x_1, x_2)L_{x_2} i_{x_1} \omega = e(x_1, x_2) di_{x_2} i_{x_1} \omega = di_{x_2} \wedge i_{x_1}.$$
Then we apply (13) to the right side of the last expression. After simplification and reindexing this leads to

\[
\frac{1}{2(k-1)!} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) L_{x_{s(1)}} i_{x_{s(k+1)} \wedge \ldots \wedge x_{s(2)}} \omega \\
- \frac{1}{2(k-1)!} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_{s(1)}) i_{x_{s(k+1)} \wedge \ldots \wedge x_{s(2)}} \omega \\
+ \frac{1}{2(k-1)!} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_{s(1)}) e(x_{s(2)}) i_{x_{s(2)} \wedge x_{s(1)}} \omega.
\]

Now we rewrite the Lie derivation operator in the previous expression using Cartan’s formula (12), which gives

\[
\frac{k(k+1)}{2} i_{x_{k+1} \wedge \ldots \wedge x_1} \omega \\
- \frac{1}{k-1} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_{s(1)}) i_{x_{s(k+1)} \wedge \ldots \wedge x_{s(2)}} \omega \\
+ \frac{1}{2(k-1)!} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_{s(1)}) e(x_{s(2)}) i_{x_{s(2)} \wedge x_{s(1)}} \omega
\]

and apply the induction hypothesis on this expression. Recall that if \(k = 2\), the last sum vanishes, so this is well defined:

\[
\frac{k(k+1)}{2} i_{x_{k+1} \wedge \ldots \wedge x_1} \omega \\
- \frac{1}{k-1} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_{s(1)}) i_{x_{s(k+1)} \wedge \ldots \wedge x_{s(1)}} \omega \\
+ \frac{1}{2(k-1)!} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_{s(1)}) e(x_{s(2)}) i_{x_{s(2)} \wedge x_{s(1)}} \omega
\]

In the next step, we substitute the higher tensor brackets again by their definition (10). (According to a better readable text we write \(Sh(p, q)\) for the set of \((p, q)\)-shuffle permutations defined explicit on the finite set \(S\).) This transforms the previous expression into

\[
\frac{k(k+1)}{2} i_{x_{k+1} \wedge \ldots \wedge x_1} \omega \\
- \frac{1}{k-1} \sum_{s \in S_{k+1}} \sum_{t \in Sh((t_2, \ldots, t_{k-2}))} (2, k-2) e(s; x_1, \ldots, x_{k+1}) \cdot e(t; x_{s(2)}, \ldots, x_{s(k+1)}) e(x_{s(1)}) e(x_{t(1)}) i_{x_{s(1)} i_{x_{s(k+1)} \wedge \ldots \wedge x_{s(4)} \wedge x_{t(2)}}) \omega
\]
Now observe, that for any \( s \in S_{k+1} \) and shuffle \( t \in Sh_{a(2), \ldots, a(k+1)}(2, k-2) \), the permutation \( (s(1), ts(2), \ldots, ts(k+1)) \) is again an element of \( S_{k+1} \) and since there are precisely \( \frac{k!}{2(k-2)!} \) many shuffles in \( Sh_{a(2), \ldots, a(k+1)}(2, k-2) \) we can just 'absorb' the first sum over shuffles in the previous expression into the sum over general permutation. A similar argument holds for the second sum. After simplification this gives:

\[
\frac{(k+1)!}{2} d_{x_{k+1}^+ \cdots x_1^+} \omega - \frac{1}{(k-1)!} \frac{k!}{2(k-2)!} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_s(1)) i_{x_{k+1}^+ \cdots x_1^+} \sum_{s \in S_{k+1}} e(s; x_1, \ldots, x_{k+1}) e(x_s(1)) i_{x_{k+1}^+ \cdots x_1^+} \omega
\]

Now we can transform this sum over arbitrary permutations back into a sum over shuffle permutations, such that we can apply the definition of the Lie \((n+1)\)-bracket another time.

\[
\frac{(k+1)!}{2} i_{x_{k+1}^+ \cdots x_1^+} \omega = \frac{(k+1)!}{2} d_{x_{k+1}^+ \cdots x_1^+} \omega - k(k-1) i_{x_{k+1}^+ \cdots x_1^+} \omega + \frac{(k-1)(k-2)}{2} i_{x_{k+1}^+ \cdots x_1^+} \omega
\]

Again recall, that the last sum is omitted for \( k = 2 \). Summarizing this long computation we have shown that under the induction hypothesis, the equation

\[
\frac{(k+1)!}{2} d_{x_{k+1}^+ \cdots x_1^+} \omega - \frac{1}{(k-1)!} \frac{k!}{2(k-2)!} \sum_{s \in Sh(2,k-1)} e(s; x_1, \ldots, x_{k+1}) e(x_s(1)) i_{x_{k+1}^+ \cdots x_1^+} \sum_{s \in Sh(2,k-1)} e(s; x_1, \ldots, x_{k+1}) e(x_s(1)) i_{x_{k+1}^+ \cdots x_1^+} \omega
\]

is satisfied for any \( k \geq 2 \). This in turn leads to the equation

\[
(1 - \frac{(k-1)(k-2)}{2} + k(k-1)) i_{x_{k+1}^+ \cdots x_1^+} \omega = \frac{(k+1)!}{2} d_{x_{k+1}^+ \cdots x_1^+} \omega
\]

which completes the proof on homogeneous symplectic tensors and hence on arbitrary symplectic tensors.

An immediate consequence of equation (16) is that the set of symplectic tensors is closed under the operation of all higher tensor brackets:

**Corollary 3.4.** \( Sym(A, g, \omega) \) is a sub Lie \( \infty \)-algebra of \( (E(A, g), [\cdot, \cdot, \cdot], k \in \mathbb{N}) \).

**Proof.** Since \( L_x \omega = 0 \) is the defining equation of a symplectic tensor and Lie derivation is \( \mathbb{R} \)-linear in both arguments, \( Sym(A, g, \omega) \) is a graded vector subspace of \( E(A, g) \). To see that it is moreover closed under the operation of all brackets, we use (16) to compute \( L_{[x_1, \ldots, x_k]} \omega = d_i[x_1, \ldots, x_k] \omega = 0 \) for any \( x_1, \ldots, x_k \in Sym(A, g, \omega) \).

**Remark.** \( Sym(A, g, \omega) \) is in general not a sub \( A \)-module of \( E(A, g) \). In fact since \( L_a \omega = da \wedge \iota_x \omega \), it is a sub \( A \)-module, if and only if \( da = 0 \) for any \( a \in A \). This is well known from symplectic geometry.
Corollary 3.5. The set of Hamiltonian tensors is an ideal in the Lie $\infty$-algebra $\overline{\text{Sym}}(A, g, \omega)$.

Proof. This is an immediate consequence of equation (16), because the higher tensor brackets of all symplectic tensors are Hamiltonian. 

Since we consider symplectic tensors of arbitrary degree and potentially very high degenerated n-plectic cocycles, the kernel of $\omega$ can be very large. However it is always an ideal in Hamiltonian and hence in symplectic tensors, as the following corollary shows:

Corollary 3.6. The kernel $\ker(\omega)$ of the n-plectic cocycle is an ideal in the Lie $\infty$-algebra of Hamiltonian tensors.

Proof. We have to show $[x_1, \ldots, x_k] \in \ker(\omega)$ for all $k \in \mathbb{N}$, if at least one argument $x_i$ is element of the kernel of $\omega$, but this follows from (16), since $i[x_1, \ldots, x_k] \omega = d(x_1 \wedge \cdots \wedge x_k) \omega = 0$, for some $x_i \in \ker(\omega)$.

In what follows we will consider symplectic tensors only up to elements of the kernel of $\omega$. Since the kernel is an ideal, we can divide it out and write

$$\Sym(A, g, \omega) := \overline{\text{Sym}}(A, g, \omega) / \ker(\omega)$$

for the Lie $\infty$-algebra of symplectic tensors modulo elements of the kernel of $\omega$. According to a better readable text we just write $x$ instead of $[x]$ for an appropriate equivalence class. Contraction and Lie derivative of $\omega$ along such a tensor, do not depend on a particular representative.

3.2. N-plectic extensions of symplectic tensors. We show that any choice of an n-plectic cocycle gives rise to an extension of the Lie $\infty$-algebra of symplectic tensors by the Chevalley-Eilenberg complex of cotensors.

Usually this cochain complex of cotensors is not seen as a Lie $\infty$-algebra, but as a differential graded algebra. However with respect to the tensor grading we can see it as an abelian Lie $\infty$-algebra, where all brackets except the unary are zero.

To define this extension we need the shift $A_{[n]}$ of a (co)chain complex $A$, which is again a (co)chain complex given by $A^k_{[n]} = A^{k+n}$ and $d^k_{A_{[n]}} := (-1)^n d^k_{A^n}$.

Definition 3.7 (The n-plectic extension). Let $(A, g, \omega)$ be an n-plectic structure, $\Sym(A, g, \omega)$ the Lie $\infty$-algebra of symplectic tensors up to $\ker(\omega)$ and $E(A, g^\vee)_{[n]}$ the abelian Lie $\infty$-algebra of cotensors, but with the tensor degree shifted by $n$. Then the map

$$d_\omega : E(A, g^\vee)_{[n]} \oplus \Sym(A, g, \omega) \to E(A, g^\vee)_{[n]} \oplus \Sym(A, g, \omega)$$

(18)

defined by $d_\omega(f_{[n]}, x) = ((i_x \omega - df)_{[n]}, 0)$ is called the $\omega$-extension of the exterior derivative.

Moreover, let $B_j \in \mathbb{N}_0$ be the sequence of Bell numbers and $k \in \mathbb{N}$ with $k \geq 2$. Then the map

$$\{1, \ldots, k\}_\omega : \chi^k(E(A, g^\vee)_{[n]} \oplus \Sym(A, g, \omega)) \to E(A, g^\vee)_{[n]} \oplus \Sym(A, g, \omega)$$

(19)

defined by

$$\{(f_{[n]}, x_1), \ldots, (f^k_{[n]}, x_k)\}_\omega = (B_{k-1} \cdot i_{x_1} \wedge \cdots \wedge x_k \omega_{[n]}, [x_1, \ldots, x_k]_\omega)$$

(20)

is called the $\omega$-extension of the $k$-th tensor bracket.
Recall from definition 2.6 that we consider cotensors as graded but concentrated in non positive degrees. This implies that the $n$-shifted cotensor $f_{[n]}$ of any homogeneous $f \in E(A,\mathfrak{g}^\vee)$ has tensor degree $(n-|f|)$.

The following theorem shows, that this gives indeed a Lie $\infty$-algebra, which can be seen as an extension of the tensor Lie $\infty$-algebra by the Chevalley-Eilenberg cochain complex along the n-plectic cocycle:

**Theorem 3.8.** The graded direct sum $E(A,\mathfrak{g}^\vee)_{[n]} \oplus \text{Sym}(A,\mathfrak{g},\omega)$ of graded vector spaces is a Lie $\infty$-algebra with respect to the n-plectic extensions of the exterior derivative and the higher tensor brackets.

**Proof.** Since the contraction and all tensor brackets are graded linear, so are their extensions. To see that they are homogeneous of tensor degree $-1$, suppose that $(f_{[n]}^1, x_1), \ldots, (f_{[n]}^k, x_k) \in E(A,\mathfrak{g}^\vee)_{[n]} \oplus \text{Sym}(A,\mathfrak{g},\omega)$ are homogeneous. Then $|x_i| = |f_{[n]}^i| = n - |f^i|$ and from the homogeneity of the higher tensor bracket follows

$$|x_1| + \ldots + |x_k| - 1 = |[x_1, \ldots, x_k]|_k = n + (-n + 1) + (|x_1| + \ldots + |x_k|) = n + (|\omega| + (|x_1| + \ldots + |x_k|)) = n + |i_{x_1} \wedge \ldots \wedge x_k \omega| = (i_{x_1} \wedge \ldots \wedge x_k, \omega)_{[n]}.$$

Therefore the $k$-ary extended bracket is homogeneous of tensor degree $-1$. Similar we compute $|(i_{\bar{x}} \omega - df)_{[n]}| = n + |i_{\bar{x}} \omega - df| = n + |i_{\bar{x}} \omega| = n + (|x| - (n + 1)) = |x| - 1$ for $|x| = f_{[n]}$ which means, that $d_{\omega}$ is homogeneous of degree $-1$, too.

Graded symmetry follows from the graded symmetry of the higher tensor brackets and the exterior product.

To proof the first Jacobi equation, we have to show $d_{\omega}^2 = 0$. Therefore recall that $di_{x} \omega = 0$ on symplectic tensors and compute

$$d_{\omega}(f_{[n]}, x))^2 = d_{\omega}((i_{x} \omega - df)_{[n]}, 0) = ((-d_{i_{x}} \omega + d^2 f)_{[n]}, 0) = (0, 0).$$

To calculate the other Jacobi equations, observe $\{d_{\omega}(f^1_{[n]}, x_1), \ldots, (f^k_{[n]}, x_k)\}_k = (0, 0)$, for any $k \geq 2$, since the extended bracket vanish if at least one argument contains the zero symplectic tensor up to elements of $\ker(\omega)$.

Using this we can simplify the second Jacobi equation according to

$$d_{\omega}((f^1_{[n]}, x_1), (f^2_{[n]}, x_2))^2 + d_{\omega}(f^1_{[n]}, x_1), (f^2_{[n]}, x_2))^2_{\omega} + d_{\omega}(f^1_{[n]}, x_1), (f^2_{[n]}, x_2))^2_{\omega} + d_{\omega}((f^1_{[n]}, x_1), (f^2_{[n]}, x_2))^2 = d_{\omega}((f^1_{[n]}, x_1), (f^2_{[n]}, x_2))^2_{\omega}).$$

To see that the single term on the right vanishes too, we use equation (16) and $B_1 = 1$ to compute

$$d_{\omega}((f^1_{[n]}, x_1), (f^2_{[n]}, x_2))^2_{\omega}) = d_{\omega}((B_1 i_{x_1} \wedge x_2, \omega)_{[n]}, [x_1, x_2]_{2}) = (i_{[x_1, x_2]} \omega - B_1 d_{i_{x_1} \wedge x_2} \omega)_{[n]}, 0) = (0, 0).$$
It remains to show that the general weak Jacobi equation vanishes for any \( k \geq 3 \), that is
\[
\sum_{p+q=k+1} \sum_{s \in \text{Sh}(p,q-1)} e(s; x_1, \ldots, x_k) \cdot \\
\cdot \left\{ \left( f_{[1]}^p, x_{1} \right), \ldots, \left( f_{[p]}^p, x_{p} \right) \right\}_p, \left\{ \left( f_{[1]}^{p+1}, x_{p+1} \right), \ldots, \left( f_{[1]}^{k}, x_{k} \right) \right\}_q = (0, 0)
\]
is satisfied. To see that, recall that all terms for \( p = 1 \) vanishes, since the extended brackets are zero, if at least one argument contains the zero symplectic tensor. In addition the term for \( q = 1 \) becomes
\[
((i_{x_1,\ldots,x_k})\omega - B_{k-1} dx_{k} \wedge \ldots \wedge x_1 \omega)_{[n]}, 0).
\]
Using the definition of the n-plectic extended brackets, the remaining sum of the weak Jacobi expression can be written as
\[
\sum_{p+q \geq 2} \sum_{p+q=k+1} \sum_{s \in \text{Sh}(p,q-1)} e(s; x_1, \ldots, x_k) \cdot \\
\cdot \left( B_{q-1} x_{s(k)} \wedge \ldots \wedge x_{s(p+1)} \wedge [x_{s(1)}, \ldots, x_{s(p)}] \omega \right)_{[n]}, \left[ x_1, \ldots, x_p, x_{p+1}, \ldots, x_k \right]_q
\]
and taken the weak Jacobi identity of the higher tensor brackets into account, this can be expressed as
\[
\sum_{p+q \geq 2} \sum_{p+q=k+1} \sum_{s \in \text{Sh}(p,q-1)} e(s; x_1, \ldots, x_k) \cdot \\
\cdot \left( B_{q-1} x_{s(k)} \wedge \ldots \wedge x_{s(p+1)} \wedge [x_{s(1)}, \ldots, x_{s(p)}] \omega \right)_{[n]}, 0)
\]
It follows, that the Jacobi identity holds in dimension \( k \), if and only if the equation
\[
B_{k-1} dx_{k} \wedge \ldots \wedge x_1 \omega - i_{x_1,\ldots,x_k} \omega = \\
\sum_{p+q \geq 2} \sum_{p+q=k+1} \sum_{s \in \text{Sh}(p,q-1)} e(s; x_1, \ldots, x_k) B_{q-1} x_{s(k)} \wedge \ldots \wedge x_{s(p+1)} \wedge [x_{s(1)}, \ldots, x_{s(p)}] \omega
\]
is satisfied.

To simplify this equation we use (16), and the graded symmetry of the exterior product and the higher tensor brackets, to rewrite it into
\[
B_{k-1} dx_{k} \wedge \ldots \wedge x_1 \omega - dx_{k} \wedge \ldots \wedge x_1 \omega = \\
\sum_{p+q \geq 2} \sum_{p+q=k+1} \frac{1}{p!(q-1)!} B_{q-1} \sum_{s \in S_k} e(s; x_1, \ldots, x_k) i_{x_1,\ldots,x_k} \omega
\]
Then we substitute the tensor \( p \)-bracket by its definition. (According to a better readable text we write \( Sh_{S(p,q)} \) for the set of \( (p,q) \)-shuffle permutations defined explicit on the finite set \( S \).) This transforms the equation into
\[
(B_{k-1} - 1) dx_{k} \wedge \ldots \wedge x_1 \omega = \\
\sum_{p+q \geq 2} \sum_{p+q=k+1} \frac{1}{p!(q-1)!} B_{q-1} \sum_{s \in S_k} \sum_{t \in Sh_{\{s(1),\ldots,s(p)\}}(2,p-2)} e(t; x_1, \ldots, x_k) \cdot \\
\cdot e(t; x_{s(1)}, \ldots, x_{s(p)}) e(t x_{s(1)}) i_{x_{s(k)} \wedge \ldots \wedge x_{s(p+1)} \wedge x_{t_1} \wedge \ldots \wedge x_{t_3} \wedge [x_{t_2}, x_{t_3}]) \omega
\]
Now observe, that for any \( s \in S_k \) and shuffle \( t \in Sh_{\{s(1),\ldots,s(p)\}}(2,p-2) \), the permutation \( (ts(1), \ldots, ts(p), s(p+1), \ldots, s(k)) \) is again an element of \( S_k \). Since there are precisely \( \frac{p!}{2(p-2)!} \) many shuffles in \( Sh_{\{s(1),\ldots,s(p)\}}(2,p-2) \) we can just
satisfied and the n-plectic extension is a Lie

but this equation holds for all

and after using the definition of the

Considering the Chevalley-Eilenberg complex of exterior c otensors

Corollary 3.9. is a short exact sequence of Lie

\[
\sum_{p+q=k+1} \frac{B_{p+1}}{2(q-1)!(p-2)!} \sum_{s \in S_k} e(s; x_1, \ldots, x_k) e(x_s(1)) e(x_s(2), x_s(3)) e(x_s(4)) \left[ x_{s(1)} \wedge x_{s(2)} \wedge x_{s(3)} \wedge x_{s(4)} \right] \wedge \omega.
\]

Then we transform the summation over arbitrary permutations back into a sum over (2, k-2)-shuffles, such that we can apply the definition of the tensor k-bracket again. This gives

\[
(B_{k-1} - B_0) di_{x_1 \wedge \ldots \wedge x_k} \omega = \sum_{p+q=k+1} \frac{(k-2)!}{(q-1)!(p-2)!} B_{q-1} \cdot \sum_{s \in Sh(2, k-2)} e(s; x_1, \ldots, x_k) e(x_s(1)) e(x_s(2), x_s(3)) e(x_s(4)) \left[ x_{s(1)} \wedge x_{s(2)} \wedge x_{s(3)} \wedge x_{s(4)} \right] \wedge \omega
\]

and after using the definition of the k-ary tensor bracket as well as (16), this rewrites into

\[
(B_{k-1} - B_0) di_{x_1 \wedge \ldots \wedge x_k} \omega = \sum_{p+q=k+1} \frac{(k-2)!}{(q-1)!(p-2)!} B_{q-1} i_{x_1, \ldots, x_k} \wedge \omega = \sum_{p+q=k+1} \frac{(k-2)!}{(q-1)!(p-2)!} B_{q-1} di_{x_1 \wedge \ldots \wedge x_k} \omega.
\]

Summarizing this computation, the weak Jacobi equation in dimension k is satisfied, if and only if

\[
\sum_{q=2}^{k-1} \frac{(k-2)!}{(q-1)!(k-1-q)!} B_{q-1} = B_{k-1} - B_0
\]

but this equation holds for all k \geq 3, as we can see from the recurrence relation

\[
B_{k+1} = \sum_{p=0}^{k} \binom{k}{p} B_p
\]

of the Bell numbers. Therefore all weak Jacobi equations are satisfied and the n-plectic extension is a Lie \(\infty\)-algebra.

\[\square\]

**Corollary 3.9.** Considering the Chevalley-Eilenberg complex of exterior cotensors as a Lie \(\infty\)-algebra with only non vanishing unary bracket, the diagram

\[
E(A, g^\vee)[n] \xrightarrow{\delta} E(A, g^\vee)[n] \oplus \text{Sym}(A, g, \omega) \xrightarrow{\delta} \text{Sym}(A, g, \omega)
\]

is a short exact sequence of Lie \(\infty\)-algebras, where the morphisms are the natural inclusion and projection, respectively.

**3.3. The Hamiltonian Cohomology.** On a connected symplectic manifold any two Hamiltonian functions associated to the same Hamiltonian vector field have equal exterior derivatives and hence differ by a constant only.

From another perspective, we could say that Hamiltonian functions are only defined up to locally constant functions, or that Hamiltonian functions associated to the same vector field differ by closed 0-forms only.

We generalize this idea to Hamiltonian tensors of possible higher tensor degrees, by passing to the cohomology of \(E(A, g^\vee)[n] \oplus \text{Sym}(A, g, \omega)\) with respect to the coboundary map \(\delta_\omega\). The cocycle property of \(\delta_\omega\) then provides the fundamental pairing between Hamiltonian tensors and cotensors and passing to the cohomology takes care of the previously mentioned ambiguity inherent in this pairing.

The central part of this work is to show that there is a Lie \(\infty\)-structure on this cohomology, which generalizes the usual *Poisson Lie bracket* of symplectic functions to the general higher context.

To start we first look at the cocycles and coboundaries of the n-plectic extended derivation \(d_\omega\). As the following proposition shows the cocycle property is then nothing but the fundamental pairing well known from symplectic geometry:
Proposition 3.10. Let $(A, g, \omega)$ be an $n$-plectic structure with $n$-plectic extension $E(A, g^\vee)[n] \oplus \text{Sym}(A, g, \omega)$. Then a pair $(f[n], x) \in E(A, g^\vee)[n] \oplus \text{Sym}(A, g, \omega)$ is a cocycle with respect to $d_\omega$, if and only if
\[ i_x \omega = df. \] (22)

It is moreover a coboundary, if and only if $x \in \ker(\omega)$ and there is a symplectic tensor $y \in \text{Sym}(A, g, \omega)$ as well as a cotensor $h \in E(A, g^\vee)$, such that $f = i_y \omega - dh$. Two cocycles $(f[n], x_1), (f[n], x_2)$ are cohomologous, precisely if $x_1 - x_2 \in \ker(\omega)$ and $f_1 - f_2 = i_y \omega + dh$.

Proof. Apply the definition of $d_\omega$ and recall that $0 \in \text{Sym}(A, g, \omega)$ is the kernel of $\omega$. □

Now passing to the cohomology, restricts the $n$-plectic extension of symplectic tensors precisely to the Hamiltonian tensors, together with particular cotensors, connected by the fundamental pairing, up to closed forms.

Definition 3.11 (Hamiltonian Cohomology). Let $(A, g, \omega)$ be an $n$-symplectic structure, $E(A, g^\vee)[n] \oplus \text{Sym}(A, g, \omega)$ the $n$-plectic extension of symplectic tensors and $d_\omega$ the $n$-plectic extension of the exterior derivative. Then the graded $\mathbb{R}$-vector space
\[ H(A, g, \omega) := \bigoplus_{k \in \mathbb{Z}} \ker(d^k_\omega) / \text{im}(d_{k+1}^k) \]
is called the Hamiltonian cohomology of the $n$-plectic structure. Cohomology classes $[f[n], x] \in H(A, g, \omega)$ are called (pairs of) Hamiltonian tensors and cotensors and the defining equation of a cocycle
\[ i_x \omega = df \] (23)
is called the fundamental pairing of Hamiltonian tensors and cotensors.

Elements of a cohomology class $[f[n], x] \in H^k(A, g, \omega)$ are pairs of Hamiltonian tensors, homogeneous of tensor degree $k$ and cotensors, homogeneous of tensor degree $(n - k)$, linked by the equation $i_x \omega = df$. All representative tensors are equal up to elements of the kernel of $\omega$ and all such cotensors are equal up to (certain) closed forms.

The next proposition shows, the Hamiltonian cohomology complex is bounded:

Proposition 3.12. Let $(A, g, \omega)$ be an $n$-plectic structure. Then the Hamiltonian cohomology is an $\mathbb{N}_0$-graded complex, where in particular the $k$-th cohomology $H^k(A, g, \omega)$ is trivial for all $k > (n + 1)$.

Proof. To see $H^{-k}(A, g, \omega) = \{0\}$ for all $k \in \mathbb{N}$, observe that any representative Hamiltonian tensor $x$ of a cohomology class $[f[n], x] \in H^{-k}(A, g, \omega)$ has tensor degree $-k$ and is therefore zero. From the cocycle condition $i_x \omega = df$ then follows that any representative Hamiltonian cotensor $f$ has to satisfy $df = 0$, which means, that it is a coboundary with respect to $d_\omega$. Hence the class is the zero class.

To see the other bound, suppose $[f[n], x] \in H^k(A, g, \omega)$ for some $k > (n + 1)$. Then the tensor degree of a any representative tensor $x$ is $|x| = k > (n + 1)$ and hence $x \in \ker(\omega)$. From the cocycle condition $i_x \omega = df$ then follows $df = 0$ and again this means, that $f$ is a coboundary with respect to $d_\omega$ and that the class is the zero class. □
As the central part of this work, we now generalize the usual Poisson Lie bracket of Hamiltonian functions to a sequence of 'higher brackets' on the Hamiltonian cohomology of any n-plectic structure and show that this arranges into a Lie \( \infty \)-algebra.

**Definition 3.13** (Poisson brackets). Let \((A, g, \omega)\) be an n-plectic structure, with Hamiltonian cohomology \(H(A, g, \omega), \{ \cdot, \cdot \}_k\) the sequence of higher tensor brackets \((10)\). Then the map

\[
\{\cdot, \cdot\}_k : H(A, g, \omega) \times \cdots \times H(A, g, \omega) \rightarrow H(A, g, \omega)
\]

defined for any \(k \in \mathbb{N}\) and cohomology classes \([f^1_{[n]}], x_1, \ldots, [f^k_{[n]}], x_k] \in H(A, g, \omega)\) by the equation

\[
\{[f^1_{[n]}], x_1, \ldots, [f^k_{[n]}], x_k]\}_k = [i_x \omega, [x, \omega]_1, \ldots, [x, \omega]_k]_k,
\]

is called the **Poisson Lie k-bracket** (or just \(k\)-ary Poisson Lie bracket) of the Hamiltonian cohomology.

As the following theorem show, these brackets are well defined and arrange the Hamiltonian cohomology into a Lie \( \infty \)-algebra.

**Theorem 3.14.** The Hamiltonian cohomology \(H(A, g, \omega)\) is a Lie \( \infty \)-algebra, with respect to the sequence \(\{\cdot, \cdot\}_k\) of Poisson Lie k-brackets.

**Proof.** Since the Hamiltonian tensors \(x\) of all representatives \([f_{[n]}], x\) of a cohomology class \([f_{[n]}], x] \in H(A, g, \omega)\) are equal up to elements of the kernel of \(\omega\), the bracket does not depend on the particular chosen representative.

To see that the unary bracket \(\{\cdot\}_1\) is the zero operator, recall from, that the tensor bracket \(\cdot\) is zero and that \(i_x \omega\) is closed. Then

\[
\{[f_{[n]}], x\}_1 = [i_x \omega, [x, \omega]_1] = [i_x \omega, 0] = [0, 0].
\]

Since \(\{\cdot\}_1\) is zero, it only remains to show that the weak Jacobi equations

\[
\sum_{p+q=k+1} \sum_{s \in Sh(p, q-1)} e(s; x_1, \ldots, x_k) \cdot \left\{ \{f^1_{[n]}, x_1\}, \ldots, \{f^p_{[n]}, x_p\}_p, \{f^{p+1}_{[n]}, x_{p+1}\}, \ldots, \{f^k_{[n]}, x_k\}_k \right\}_q = [0, 0]
\]

are satisfied for all \(k \geq 3\) as well as \([f^1_{[n]}], x_1, \ldots, [f^k_{[n]}], x_k] \in H(A, g, \omega)\) and arguing similar to the proof of theorem 3.8 we need to show that

\[
\sum_{p+q=k+1} \sum_{s \in Sh(p, q-1)} e(s; x_1, \ldots, x_k)(i_{x_1} \omega, \ldots, i_{x_{(p)}}(x_1, \ldots, x_{(q)}) \omega)\omega^n
\]

is closed, since \([0, 0]\) is the equivalence class of elements of the kernel of \(\omega\) and closed forms. Again we know from the proof of theorem 3.8 that this expression can be rewritten into

\[
\sum_{p+q=k+1} \frac{(k-2)!}{(q-1)(p-2)!} d i_{x_1} \wedge \cdots \wedge x_1 \omega
\]

which then completes the proof.

**Remark.** Since the Poisson Lie 1-bracket \(\{\cdot\}_1\) is zero on the Hamiltonian cohomology, the general Jacobi equation \((\Lambda.2)\) simplifies for \(n = 3\) into the usual (strict) Jacobi equation of a graded Lie algebra. This is an important fact as it allows Lie algebra representations and in particular momentum maps to be handled similar to the common symplectic case.
However the higher brackets don’t vanish and so more general (weak) representations can be expected which are not yet part of symplectic geometry.

3.4. Momentum maps. In order to study Lie group actions on symplectic manifolds, the momentum map of such an action was originally defined as a certain map from the manifold into the dual of the appropriate Lie algebra. However it became clear that momentum maps can equivalently be seen as certain morphisms $J : \mathfrak{g} \to C^\infty(M)$ of Lie algebras, which fit into otherwise exact, commutative diagrams

\[
\begin{array}{ccccccc}
0 & \rightarrow & H^0(M) & \rightarrow & C^\infty(M) & \rightarrow & Sym(M) & \rightarrow & H^1(M) & \rightarrow & 0 \\
& & \downarrow J & & & & & & \end{array}
\]

in the category of Lie algebras, where $Sym(M)$ is the Lie algebra of symplectic vector fields, $C^\infty(M)$ a Lie algebra with respect to the symplectic Poisson Lie bracket and the de-Rham cohomologies $H^i(M)$ are abelian Lie algebras.

In $[3]$ the authors generalizes this to an appropriate diagram in Lie $\infty$-algebras, where the function Poisson-algebra is replaced by a certain Lie $\infty$-algebra, that can be seen as a sub structure of our n-plectic tensor extension. We continue their generalization to symplectic tensors and cotensors in a wider range of tensor degrees.

In the symplectic setting a momentum map (if it exists) is in general only a morphism up to $H^0(M)$. Translated to higher symplectic geometry, this should be a morphism into the Hamiltonian cohomology, which then is a morphism up to closed forms only.

Moreover since the Hamiltonian cohomology is internal to Lie $\infty$-algebras, this kind of momentum map makes sense for more general Lie $\infty$-algebras, not just for Lie algebras associated to a particular Lie group action.

Putting this together, we are able to give a conceptual very simple definition of a momentum map:

**Definition 3.15.** Let $(A, g, \omega)$ be an n-plectic structure, with Hamiltonian cohomology $H(A, g, \omega)$ and $(L, D_{k \in \mathbb{N}})$ a Lie $\infty$-algebra. Then an n-plectic momentum map is a weak morphism of Lie $\infty$-algebras

$$J_\infty : L \rightarrow H(A, g, \omega).$$

The structure equations for general weak Lie $\infty$-morphisms (see Appendix) take care of the properties one would expect from a momentum map. The interested reader is encouraged to show that this resamples precisely the common definition of a momentum map, in case $L$ is the Lie algebra of a symplectic action on an appropriate manifold.

4. Conclusion and Outlook

We developed a general definition of higher symplectic structures and defined a reasonable momentum map, to study Lie $\infty$-algebra representations in this context. However a well understood obstruction theory for momentum maps as well as a good notion of ‘higher symplectic morphisms’ has still to be found. In $[3]$ the authors started to look at the first of these problems.
Appendix A. Lie $\infty$-algebras

We recall the most basic stuff about Lie $\infty$-algebras. There are many incarnations of them \cite{6}, but we will only look at their graded symmetric, ‘many brackets’ version, since that picture fits nicely into the Schouten calculus and is moreover useful when it comes to actual computations.

Lie $\infty$-algebras are defined on $\mathbb{Z}$-graded vector spaces and consequently we recall them first:

A.1. Graded Vector Spaces. In what follows $\mathbb{K}$ will always be a field and $\mathbb{Z}$ the Abelian group of integers with respect to addition. A $\mathbb{Z}$-graded $\mathbb{K}$-vector space $V$ is the direct sum $\bigoplus_{n \in \mathbb{Z}} V_n$ of $\mathbb{K}$-vector spaces $V_n$. Since this is a coproducit, there are natural injections $i_n : V_n \to V$ and a vector is called homogeneous of degree $n$ if it is in the image of the injection $i_n$. In that case we write $\deg(v)$ or $|v|$ for its degree.

According to a better readable text we just write graded vector space as a shortcut for $\mathbb{Z}$-graded $\mathbb{K}$-vector space.

A morphism $f : V \to W$ of graded vector spaces, homogeneous of degree $r$, is a sequence of linear maps $f_n : V_n \to W_{n+r}$ for any $n \in \mathbb{Z}$ and the integer $r \in \mathbb{Z}$ is called the degree of $f$, denoted by $\deg(f)$ (or $|f|$).

For any $n \in \mathbb{N}$, an $n$-multilinear map $f : V_1 \times \cdots \times V_n \to W$, homogeneous of degree $r$ is a sequence of $n$-multilinear maps $f_k : (V_1)_{n_1} \times \cdots \times (V_k)_{n_k} \to W_{\sum n_i+r}$ for all $j_i \in \mathbb{Z}$ with $\sum j_i = k$.

The $\mathbb{Z}$-graded tensor product $V \otimes W$ of two graded vector spaces $V$ and $W$ is given by

$$(V \otimes W)_n := \bigoplus_{i+j=n} (V_i \otimes W_j)$$

and the Koszul commutativity constraint $\tau : V \otimes W \to W \otimes V$ is on homogeneous elements $v \otimes w \in V \otimes W$ defined by

$$\tau(v \otimes w) := (-1)^{\deg(v)\deg(w)} w \otimes v$$

and then extended to $V \otimes W$ by linearity.

Remark. We define the symbols $e(v) := (-1)^{\deg(v)}$, $e(v, w) := (-1)^{\deg(v)\deg(w)}$. The Koszul sign $e(s; v_1, \ldots, v_k) \in \{-1, +1\}$ is defined for any permutation $s \in S_k$ and any homogeneous vectors $v_1, \ldots, v_k \in V$ by

$$v_1 \otimes \cdots \otimes v_k = e(s; v_1, \ldots, v_k)v_{s(1)} \otimes \cdots \otimes v_{s(k)}. \quad (27)$$

In an actual computation it can be determined by the following rules: When a permutation $s \in S_k$ is a transposition $j \leftrightarrow j + 1$ of consecutive neighbors, then $e(s; v_1, \ldots, v_k) = (-1)^{\deg(v_j)\deg(v_{j+1})}$ and if $t \in S_k$ is another permutation, then $e(ts; v_1, \ldots, v_k) = e(t; v_{s(1)}, \ldots, v_{s(k)})e(s; v_1, \ldots, v_k)$.

A graded $k$-linear morphism $f : \bigwedge^k V \to W$ is called graded symmetric if

$$f(v_1, \ldots, v_k) = e(s; v_1, \ldots, v_k)f(v_{s(1)}, \ldots, v_{s(k)})$$

for all $s \in S_k$. 
A.2. **Shuffle Permutation.** Let $S_k$ be the symmetric group, i.e. the group of all bijective maps of the ordinal $[k]$.

**Definition A.1** (Shuffle Permutation). For any $p, q \in \mathbb{N}$ a $(p, q)$-shuffle is a permutation $s \in S_{p+q}$ with $s(1) < \ldots < s(p)$ and $s(p+1) < \ldots < s(p+q)$. We write $Sh(p, q)$ for the set of all $(p, q)$-shuffles.

More generally for any $p_1, \ldots, p_n \in \mathbb{N}$ a $(p_1, \ldots, p_n)$-shuffle is a permutation $s \in S_{p_1+\ldots+p_n}$ with $s(p_{j-1}+1) < \ldots < s(p_{j-1}+p_{j})$. We write $Sh(p_1, \ldots, p_n)$ for the set of all $(p_1, \ldots, p_n)$-shuffles.

A.3. **Lie $\infty$-algebras.** On the structure level Lie $\infty$-algebras generalize (differential graded) Lie-algebras to a setting where the Jacobi identity isn’t satisfied anymore, but holds up to particular higher brackets. This can be defined in many different ways [6], but the one that works best for us is its ‘graded symmetric, many bracket’ version.

**Definition A.2.** A Lie $\infty$-algebra $(V, (D_k)_{k \in \mathbb{N}})$ is a $\mathbb{Z}$-graded $\mathbb{R}$-vector space $V$, together with a sequence $(D_k)_{k \in \mathbb{N}}$ of graded symmetric, $k$-multilinear maps $D_k : \times^k V \to V$, homogeneous of of degree $-1$, such that the weak Jacobi equations

$$\sum_{i+j=n+1} \left( \sum_{s \in Sh(j,n-j)} e(s; v_1, \ldots, v_n) D_j \left( v_{s_1}, \ldots, v_{s_j} \right), v_{s_{j+1}}, \ldots, v_{s_n} \right) = 0$$

are satisfied for any integer $n \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in V$.

In particular Lie $\infty$-algebras generalizes ordinary Lie algebras, if the grading is chosen right:

**Example 5** (Lie Algebra). Every Lie algebra $(V, [\cdot, \cdot])$ is a Lie $\infty$-algebra if we consider $V$ as concentrated in degree one and define $D_k = 0$ for any $k \neq 2$ as well as $D_2(\cdot, \cdot) := [\cdot, \cdot]$.

Very different from common Lie theory is, that a morphism of Lie $\infty$-algebras is not necessarily just a single map. In fact such a morphism is a sequence of maps, satisfying a particular structure equation.

**Definition A.3.** For any two Lie $\infty$-algebras $(V, (D_k)_{k \in \mathbb{N}})$ and $(W, (L_k)_{k \in \mathbb{N}})$ a morphism of Lie $\infty$-algebras is a sequence $(f_k)_{k \in \mathbb{N}}$ of graded symmetric, $k$-multilinear maps

$$f_k : V \times \cdots \times V \to W$$

homogeneous of degree 0, such that the structure equations

$$\sum_{p+q=n+1} \left( \sum_{s \in Sh(q,p-1)} e(s) f_p \left( D_q (v_{s(1)}, \ldots, v_{s(q)}) \right), v_{s(q+1)}, \ldots, v_{s(n)} \right)$$

$$= \sum_{p \geq 1} \frac{1}{p!} \sum_{s \in Sh(k_1, \ldots, k_p)} e(s) P \left( f_{k_1} \left( v_{s(1)}, \ldots, v_{s(k_1)} \right), \ldots, f_{k_p} \left( v_{s(n-k_p+1)}, \ldots, v_{s(n)} \right) \right)$$

are satisfied for any $n \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in V$.

The morphism is called strict, if in addition $f_k = 0$ for all $k \geq 2$, that is, if the morphism is a single map, that commutes with all brackets.

**References**

[1] DOMENICO FIORENZA, CHRISTOPHER L. ROGERS, URS SCHREIBER: L-infinity algebras of local observables from higher prequantum bundles, arXiv:1304.6292

[2] M. FORGER, C. PAUFLE: The Poisson Bracket for Poisson Forms in Multi-symplectic Field Theory, Rev. Math. Phys. 15 (2003) 705-744; math-ph/0202043
[3] Yael Fregier, Christopher L. Rogers, Marco Zambon: Homotopy moment maps, arXiv:1304.2051
[4] I.V. Kanatchikov: Canonical Structure of Classical Field Theory in the Polymomentum Phase Space, DOI: 10.1016/S0034-4877(98)80182-1
[5] I.V. Kanatchikov: On Field Theoretic Generalizations of a Poisson Algebra, DOI: 10.1016/S0034-4877(97)85919-8
[6] Jean-Louis Loday, Bruno Vallette: Algebraic Operads, DOI: 10.1016/S0034-4877(97)85919-8
[7] Mirco Richter: Lie infinity algebras from Lie Rinehart pairs, arxiv:1311.2228
[8] G. Rinehart: Differential forms for general commutative algebras, Trans. Amer. Math. Soc. 108 (1963), 195222.
[9] C.L. Rogers: Higher Symplectic Geometry, Ph.D. thesis, Univ. of California, http://arxiv.org/abs/1106.4068