A model theory of topology

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Abstract. An algebraization of the notion of topology has been proposed more than seventy years ago in a classical paper by McKinsey and Tarski. However, in McKinsey and Tarski’s setting the model theoretical notion of homomorphism does not correspond to the notion of continuity. We notice that the two notions correspond if instead we consider a preorder relation $\subseteq$ defined by $a \subseteq b$ if $a$ is contained in the topological closure of $b$.

A specialization poset is a partially ordered set endowed with a further coarser preorder relation $\subseteq$. We show that every specialization poset can be embedded in the specialization poset naturally associated to some topological space, where the order relation corresponds to set-theoretical inclusion. Specialization semilattices are defined in an analogous way and the corresponding embedding theorem is proved. The interest of these structures arises from the fact that they also occur in many rather disparate settings, even far removed from topology.

1. Introduction

A model theory of topology. As well-known, a topology can be equivalently characterized by means of the corresponding Kuratowski closure operator $K$. The characterization naturally lends itself to an algebraization for the notion of topology. In the seminal paper “The algebra of Topology” [MT] McKinsey and Tarsky introduced closure algebras, which are Boolean algebras endowed with an additional operation $K$ satisfying the formal properties of topological closure for subsets of a topological space.

However, the construction from [MT] is not “functorial”. Algebraically, a homomorphism $\psi$ of closure algebras should satisfy $\psi(Kz) = K\psi(z)$. On the other hand, if $X$, $Y$ are topological spaces, $\varphi$ is a function from $X$ to $Y$ and $\varphi^+$ is the corresponding image function from the power set $P(X)$ to $P(Y)$, then $\varphi$ is continuous if and only if $\varphi^+(Kz) \subseteq K\varphi^+(z)$, for every $z \subseteq X$. The converse inclusion holds if and only if $\varphi$ is a closed map. Not every continuous map is closed.

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Compare also Sikorski [Si] and footnote 1 in Kuratowski [K] Sect. 4.1, p. 20]. Many further references are listed in the quoted sources.
The correspondence between topological spaces and closure algebras is not even contravariant. If $X, Y$ are topological spaces, $\varphi : X \to Y$ is a function and $\varphi^{-\circ}$ denotes the preimage function, then $\varphi$ is continuous if and only if $K\varphi^{-\circ}(z) \subseteq \varphi^{-\circ}(Kz)$, for every $z \subseteq Y$, but equality holds if and only if $\varphi$ is also open. As above, not every continuous function is open.

We first observe that if, instead, we consider a binary relation $z \sqsubseteq w$, to be interpreted as

$$z \sqsubseteq w \quad \text{if} \quad z \text{ is contained in the closure of } w,$$

(1.1)

for $z, w \subseteq X$, then model-theoretical homomorphisms correspond exactly and covariantly to continuous functions; the easy details shall be presented in Proposition 2.4. The above observation is an immediate generalization of the well-known fact that a function $\varphi$ is continuous if and only if $\varphi$ preserves the adherence relation $x \in Kz$ defined between a point $x$ and a subset $z$.

Thus we are led to consider those algebraic properties of topological spaces which are preserved by image functions associated to continuous maps. Since image functions preserve unions but not necessarily intersections or complements, we study join-semilattices with a further preorder $\sqsubseteq$ satisfying some simple compatibility conditions. The preorder $\sqsubseteq$ shall be called a specialization and the above structures specialization semilattices. Their intended interpretation is given by condition (1.1) and an explicit axiomatization will be presented in Definition 3.1. Our main technical result Theorem 4.10 asserts that every specialization semilattice can be embedded into (the structure associated to) some topological space. In particular, our axiomatization captures exactly the universal theory valid for subsets of a topological space in the language $\{\lor, \sqsubseteq\}$. If we consider only those properties holding in the language $\{\leq, \sqsubseteq\}$, where $\leq$ is interpreted as inclusion in the motivating example of topological spaces, we shall speak of specialization posets.

While originally we have pursued only the above-described minimal objective, namely, to detect which structural parts of topological spaces are preserved under model-theoretical homomorphisms, we subsequently realized that the resulting structures turn out to be of independent interest. Before outlining the obvious connections with spaces with a (not necessarily additive, aka, not necessarily topological) closure operator, we first rapidly present just a few examples of “specialization” structures which have appeared in various forms and under different names in very disparate and actually extremely distant settings.

**Examples.** The specialization preorder. Given a topological space $X$, the relation $x \sqsubseteq y$ between points of $X$, defined by $x \in K\{y\}$ is called the specialization preorder. It has various applications, among others, to algebraic geometry [Ha, Ex. 3.17e] and to domain theory [CLD]. In this note we have simply considered the extension of the specialization preorder to all subsets of $X$. To make the example fit with our abstract definition of a specialization
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A tolerance space is a set \( X \) together with a symmetric and reflexive relation \( \tau \) on \( X \). The idea of tolerance spaces originates from Henry Poincaré, while its modern formalization is due to E. C. Zeeman [PN]. If \( X \) is a tolerance space, then, for \( a, b \subseteq X \), let \( a \sqsubseteq \tau b \) if \( \tau(a) \subseteq \tau(b) \), where \( \tau(a) = \{ x \in X \mid y \tau x, \text{ for some } y \in a \} \). In the present terminology, \( \sqcup \) and \( \sqsubseteq \) define the structure of a specialization semilattice on \( P(X) \). The above notions have found applications to image analysis and other information systems [PW].

Causal spaces. Recall that a specialization poset is a set with a partial order \( \leq \) and a coarser pre-order \( \sqsubseteq \). If we further require that \( \sqsubseteq \) is antisymmetric, that is, \( \sqsubseteq \) is assumed to be an order, too, we get a notion equivalent to a causal space, as introduced by E. H. Kronheimer and R. Penrose in [KP] in connection with abstract foundations of general relativity. See Remark 5.2 below for further comments.

Measures. If \( \mu \) is a measure defined on some set \( S \) of subsets of \( X \), let \( a \sqsubseteq \mu b \) if \( \mu(a) \leq \mu(b) \), for \( a, b \in S \). Then subset inclusion and \( \sqsubseteq \) provide \( S \) with the structure of a specialization poset and, if \( \mu \) is 2-valued, of a specialization semilattice. The relation \( \sqsubseteq \) has been widely studied in connection with foundations of probability and with purported economical applications. See Example 5.4 for more details and [Le] for references.

Specializations induced by a quotient. If \( S, T \) are semilattices, \( \varphi : S \rightarrow T \) is a homomorphism and we let \( a \sqsubseteq \varphi b \) in \( S \) if \( \varphi(a) \leq \varphi(b) \) in \( T \), then \( S \) is endowed with the structure of a specialization semilattice. In particular, if \( B \) is a Boolean algebra and \( I \) is an ideal on \( B \), then \( (B, \lor, \sqsubseteq) \) is a specialization semilattice, where \( a \sqsubseteq b \) if \( a/I \leq b/I \) in \( B/I \). Many examples of such structures have been widely studied; we mention just one. If \( B = P(\mathbb{N}) \) and \( I \) is the set of all finite subsets of \( \mathbb{N} \), then \( \sqsubseteq \) is inclusion mod finite. The notion has applications to descriptive and combinatorial set theory [Bla], topology [DH], model theory [MN], among others. Notice that here there is no underlying notion of “closure”: for every subset \( y \) of \( \mathbb{N} \), there are many larger subsets \( x \) such that \( x \sqsubseteq y \), that is \( x \setminus y \) is finite. However, there is no largest such \( x \).

Abstract consequence relations on posets. Abstract consequence relations have been first introduced by A. Tarski (see [T]) in a fashion slightly different from the modern treatment. In the restricted classical sense, an abstract consequence relation is a binary relation \( \vdash \) between sets of formulas and formulas

\[ a \sqsubseteq \varphi b \text{ above, for } \varphi \text{ an arbitrary semilattice homomorphism, is very general; in fact, we shall show in a sequel to the present note that every specialization semilattice can be constructed this way. Moreover, we get an essentially equivalent definition of a specialization semilattice if we consider a semilattice together with a congruence. See Definition 4.6.}
of a formal language. The intended meaning of $\Gamma \vdash \sigma$ is that $\sigma$ is deducible from $\Gamma$ in some deduction system fixed in advance. Consequence relations can be introduced abstractly and provide an equivalent formulation for the notion of a consequence operation or a closure operation \[\text{[Er, GT, Ja, W]}\]. To some consequence relation $\vdash$ one associates the closure operation $C$ sending a set $\Gamma$ of formulas to the set $C(\Gamma)$ of all the formulas deducible from $\Gamma$. The correspondence with topology is patent: a consequence operation is interdefinable with a consequence relation exactly in the same way as topological closure is interdefinable with the adherence relation.

If one abstracts from single formulas “being singletons”, one can obviously introduce a binary relation $\Gamma \vdash \Sigma$ between sets of formulas, whose intended meaning is “everything in $\Sigma$ is deducible from $\Gamma$.” Notice the resemblance with (1.1) (of course, considering the converse relation). This is not love of abstraction for its own sake; the idea provides the possibility of introducing consequence relations in the setting of arbitrary complete lattices, with deep and important applications to algebraic logic, in particular, concerning equivalence and algebraizability \[\text{[GT]}\].

As hinted in \[\text{[GT]}\], a large part of their results and definitions apply to arbitrary partially ordered sets in place of complete lattices. In the setting of partially ordered sets (resp., join semilattices) Conditions (1) - (2) in \[\text{[GT, Subsection 3.1]}\] correspond to the definition of a specialization poset (resp., semilattice) as presented here. If we add Conditions (3) from \[\text{[GT, Subsection 3.1]}\], we get principal specialization semilattices, see Definition 4.2 below.

**Recursive sets of formulas.** As just mentioned, given a deduction system, it is natural to define a relation $\Gamma \vdash \Sigma$ between sets of formulas. In the framework of arbitrary sets of formulas, the definition of $\vdash$ is interchangeable with the definition of the consequence operation $C$ which assigns to a set $\Gamma$ the set $C(\Gamma)$ of all the formulas deducible from $\Gamma$. In fact, from the closure operation $C$ we can recover $\vdash$ by setting $\Gamma \vdash \Sigma$ if $\Sigma \subseteq C(\Gamma)$. Compare again (1.1). Thus, when dealing with arbitrary sets of formulas, it is generally irrelevant whether we deal with $\vdash$ or $C$.

Suppose now that we want to deal only with finite sets of formulas, or, possibly, with recursive sets of formulas. The assumption makes good sense, since only recursive sets of formulas can be effectively described. In this framework the approach using consequence operations is not equivalent to the approach using consequence relations; actually, the former might be not viable, since in general the set $C(\Gamma)$ of the formulas deducible from $\Gamma$ is not recursive, even if $\Gamma$ is. This is an example showing that “specialization” is more apt than “closure” in certain situations.

**Finite pieces of information.** The above example can be immediately reformulated in a setting related to computer science. Suppose that we are dealing with finite pieces of information, such as, e.g., information that can be stored

\[4\text{again, considering the converse of the relation }\vdash.\]
in some database. Suppose that we also have some constraints on our data, so that from some information we may obtain more information. Given a set $\Gamma$ of such information, the set $C(\Gamma)$ of all the informations which can be obtained from $\Gamma$ might be too large to be stored, hence $C(\Gamma)$ might be an object inappropriate in our framework. The relation $\Gamma \vdash \Sigma$, meaning that all the informations in $\Sigma$ can be obtained from the informations in $\Gamma$ is surely more concrete and manageable.

If we consider also the relation of plain containment (union), then the set of the pieces of information under consideration becomes a specialization poset (semilattice) according to our definitions.

Closure spaces. As already hinted above, our results and definitions fit with notions more general than topologies. A closure space is like a topological space, except that the union of two “open” subsets is not assumed to be open. The whole space is not assumed to be open, either. Usually closure spaces are axiomatized in terms of “closed” subsets, that is, complements of “open” subsets. Hence a closure space on some set $X$ is given by a family $\mathcal{F}$ of subsets of $X$ such that $\mathcal{F}$ is preserved under arbitrary intersections. As in the case of topological spaces, closure spaces can be equivalently axiomatized by means of the associate closure operator $K$. See Definition 2.1 for details.

In the present sense, closure does not necessarily correspond to some kind of topological closure, rather, it has the more general flavor of hull, generated by, the smallest object such that…

Closure spaces have found many applications in very disparate settings, with varied and occasionally clashing terminology. Actually, it is quite difficult to collect all instances of applications of closure spaces. An ample discussion is presented in [Er], with illuminating figures and a detailed historical background, highlighting applications, among others, to ordered set, lattice theory, logic, algebra, topology, and connections with category theory. The notion of a closure operation is formally the same as the notion of an abstract (not necessarily finitary) consequence operation, as briefly recalled above. A useful set of references to applications in computer science, notably, in the semantic area, can be found in [R]. For applications to universal algebra, the reader might consult [H].

Model theoretical properties of the algebraic analogue of closure spaces, that is, Boolean algebras with a (not necessarily additive) closure operation are studied in [Sc, Section 8]. Earlier results have been obtained by the italian school, e. g., [Se]. Just like topological spaces and closure algebras furnish an algebraization for the modal system S4, closure spaces and their algebraization are the counterpart of a non-normal “monotonic” modal system. See again [Sc] for further information. Other useful references about closure spaces are, among many others, [Bly, CLD].

A large part of the considerations in this section hold also for closure spaces, not only for topological spaces. Actually, in the case of closure spaces, the notion of specialization seems to capture a significant part of the notion of
“closure”, or perhaps “hull”. In detail, exactly as in the case of topological spaces, to every closure space we associate the structure with the join operation $\cup$ and with a specialization $\sqsubseteq$ given by (1.1). We get the same theory we obtain in the case of topological spaces; this means that, rather unexpectedly, the “universal theories” of topological spaces and of closure spaces are the same. More generally, the same applies to the universal theory of posets with a closure operation.

Conclusions. While the theory of specialization semilattices presented here might prove a bit relevant to foundational studies about topology, it is conceivably too weak to reproduce an important part of topological results. However, the theory appears to be interesting for itself, since it seems to capture significant parts of the notions of closure, hull, generated by..., even in the case when the actual “closure” of some set is too large to be considered “admissible” in the framework under consideration, or, anyway, there are reasons suggesting it should not necessarily be considered.

The fact that many examples of this situation appear in many disparate unrelated fields of mathematics, with applications to other sciences, strongly supports the above point of view. In this sense, our main result asserts that, in each of the above situations, we are always allowed to add “imaginary” elements in such a way that we can pretend to be working in an actual topological space.

Whether or not the above remarks provide some explanation for the success of topology, the present notions seem to deserve some study, even if they are set (or, possibly, just because they are set) in an extremely simpler framework.

Prerequisites. While our original motivation is mainly model-theoretical, we do not assume a specific knowledge of model theory from the reader and only scattered results here mention or use model-theoretical notions. The model-theoretical notions of homomorphism and embedding are an exception, but they are fully explained in detail in Subsection 3.1. We have tried to make the paper as self-contained as possible. Only a minimal mathematical background is assumed, essentially, some basic notions of topology and algebra. A few comments are expressed in logical terminology, but they are not necessary to understand the remaining parts of the paper. When dealing with logic, we always work in the setting of first-order “classical” logic, i.e., finitary, two-valued, with only Boolean connectives, accepting the law of excluded middle...

Summary. The paper is divided as follows. In Section 2 we describe the motivating example in more detail: to every topological space we associate a specialization semilattice and we check that continuous functions between topological spaces correspond exactly to homomorphisms between specialization semilattices. The correspondence works more generally for closure spaces, actually, for partially ordered sets with a closure operation. In Section 3 we
present the actual definitions of specialization semilattices and posets and give a few elementary consequences. In Section 4 we prove our main result Theorem 4.10 to the effect that every specialization semilattice can be embedded into the specialization semilattice associated to some topological space. The proof is divided in various steps. First in Subsection 4.1 we study principal specialization semilattices, roughly, those specialization semilattices in which a notion of closure can be defined. In Subsection 4.2 we prove that every specialization semilattice can be embedded into a principal specialization semilattice, then in Subsection 4.3 we complete the proof of Theorem 4.10 by showing that every principal specialization semilattice can be embedded into the specialization semilattice associated to some topological space. Thus the theory of specialization semilattices is the universal theory of topological spaces in the language of specialization semilattices. In Subsection 4.4 we prove the corresponding results for specialization posets. In Section 5 we present more examples and some counterexamples, while Section 6 is devoted to further comments and problems.

2. The motivating example

As well-known, the topology \( \tau \) of some topological space \((X, \tau)\) can be equivalently described by specifying the family of its closed subsets, equivalently, its associated closure operation \(K_\tau\). The operation \(K_\tau\) sends \(a \subseteq X\) to the intersection of all the closed subsets of \(X\) which contain \(a\). Clearly, \(K_\tau\) is determined by \(\tau\). Conversely, given a function \(K : \mathcal{P}(X) \to \mathcal{P}(X)\) satisfying \(K\emptyset = \emptyset\) (\(K\) preserves \(\emptyset\)), \(Kx \supseteq x\) (extensive), \(KKx = x\) (idempotent) and \(K(x \cup y) = Kx \cup Ky\) (additive, or topological), the family \(\{a \subseteq X \mid Ka = a\}\) is the family of closed sets for some topology. The above constructions are one the inverse of the other, hence they provide equivalent descriptions for a topology. See [K, Sect. 4.1], [En, Proposition 1.2.7]. When no risk of confusion is possible, we shall simply write \(X\) in place of \((X, \tau)\) and \(K\) in place of \(K_\tau\).

Essentially all the arguments in the present note work in a more general context, where the assumptions about the family of closed sets are relaxed. As we mentioned in the introduction, this generalization has applications in many distant mathematical fields.

**Definition 2.1.** A closure space is a set \(X\) together with a family \(\mathcal{F}\) of subsets of \(X\), such that that \(\mathcal{F}\) is preserved under arbitrary intersections. We assume that \(X \in \mathcal{F}\) (this is redundant if one assumes that \(X\) is the intersection of the empty family). Members of \(\mathcal{F}\) are also called closed (sets). Thus we leave out the assumption that \(\mathcal{F}\) is preserved under finite unions, an assumption holding in topological spaces. We are not asking that \(\emptyset\) is closed, either.

As in the case of topologies, closure spaces can be equivalently characterized by the associated closure operator \(K\). In the case of closure spaces \(K\) is assumed to be only extensive, idempotent and isotone; the latter condition
means that $x \subseteq y$ implies $Kx \subseteq Ky$, equivalently, $K(x \cup y) \supseteq Kx \cup Ky$. A closure space defined in terms of a closure operator is a topological space if and only if the operator is additive and preserves $\emptyset$.

As usual, when no risk of confusion might arise, we shall simply say that $X$ is a closure space, with no mention of $F$ or $K$. As we mentioned in the introduction, the terminology about closure spaces is not uniform in the literature and the name of the notion itself greatly vary according to the author or to the field of research. See [Er] for further details, a historical survey, further references and, in particular, [Er, p. 163] for a picture.

If $X$ and $Y$ are closure spaces, then, exactly as in the case of topological spaces, a function $\varphi : X \to Y$ is continuous if and only if $\varphi^{-1}(Ka) \subseteq K\varphi^{-1}(a)$, for every $a \subseteq X$. Equivalently, $\varphi$ is continuous if and only if the preimage of each closed subset of $Y$ is a closed subset of $X$. The equivalence is proved exactly as in the case of topological spaces; for the reader’s convenience, we present explicit details in the following lemma. It should be remarked that in many classic treatises in the topological literature the former condition is presented as the actual definition of continuity.

**Lemma 2.2.** If $X$ and $Y$ are topological spaces, or just closure spaces, and $\varphi$ is function from $X$ to $Y$, then $\varphi$ is continuous if and only if the preimage of each closed subset of $Y$ is a closed subset of $X$.

**Proof.** For topological spaces see, e. g., [K, Sect. 13.IV(1)], [En Proposition 1.4.1]. To prove the result in the general case of closure spaces, suppose that $\varphi$ is continuous and $c$ is closed in $Y$. If, by contradiction, the preimage $\varphi^{-1}(c)$ is not closed in $X$, there exists $x \in K\varphi^{-1}(c)$, such that $x \notin \varphi^{-1}(c)$. Since $\varphi(K\varphi^{-1}(c)) \subseteq K\varphi(\varphi^{-1}(c)) \subseteq K(c) = c$, then $\varphi(x) \in c$, a contradiction.

In the other direction, if preimages of closed are closed, then $\varphi^{-1}(K\varphi^{-1}(b))$ is a closed containing $\varphi^{-1}(\varphi^{-1}(b)) \supseteq b$, hence $\varphi^{-1}(K\varphi^{-1}(b))$ contains $K(b)$. Thus $\varphi(\varphi^{-1}(K\varphi^{-1}(b)))$ contains $\varphi^{-1}(Kb)$ and the conclusion follows from the fact that $\varphi(\varphi^{-1}(K\varphi^{-1}(b))) \subseteq K\varphi^{-1}(b)$. □

Continuity between closure spaces seems to have received less attention than it deserves, since there are many significant examples. Let us mention that non-topological closure spaces and continuity between them naturally arise also in purely topological contexts, see [C] for examples.

As usual, *poset* is an abbreviation for *partially ordered set*.

**Definition 2.3.** Let $X$ be a topological space, or just a closure space.

The *specialization semilattice* $S(X)$ associated to $X$ is the structure $(\mathcal{P}(X), \cup, \subseteq)$, where $\subseteq$ is the binary relation on $\mathcal{P}(X)$ defined by $a \subseteq b$ if $a \subseteq Kb$. Here $a$ and $b$ vary among subsets of $X$ and $K$ is closure.

The *specialization poset* $\mathcal{P}(X)$ associated to $X$ is the structure $(\mathcal{P}(X), \subseteq, \subseteq)$, where $\subseteq$ is defined as above.
When it is necessary to make explicit reference to the topology \( \tau \) on \( X \), we shall write \( S(X, \tau) \) and \( P(X, \tau) \). In the case of a closure space we shall consider \( \tau \) as the family of the complements of members of \( F \).

An explicit definition for an abstract notion of a specialization semilattice and of a specialization poset shall be given in Definition 3.1 below, by means of a few natural conditions. We will show that any structure satisfying the conditions in Definition 3.1 is isomorphic to a substructure of \( S(X) \) or of \( P(X) \), as introduced above, for some topological space \( X \).

Our main interest in the above notions originates from the next easy proposition. Here homomorphisms are always intended in the model-theoretical sense: see Subsection 3.1 below for explicit details in the special cases at hand. Recall that a continuous map \( \iota : X \to Y \) between topological spaces is a \((\text{topological})\) embedding if \( \iota \) induces an homeomorphism from \( X \) to \( \iota^{-1}(X) \), considered as a subspace of \( Y \). In particular, an embedding is an injective function. If \( Y \) is just a closure space and \( Z \subseteq Y \), then, exactly as for topological spaces, \( Z \) inherits the structure of a closure space by taking as closed subsets of \( Z \) the subsets of the form \( Z \cap C \), with \( C \) closed in \( X \). The embeddings of closure spaces are defined as above.

**Proposition 2.4.** Suppose that \( X \) and \( Y \) are topological spaces, or just closure spaces, and \( \varphi : X \to Y \) is a function. Then the following conditions are equivalent.

1. \( \varphi \) is continuous from \( X \) to \( Y \);
2. the image function \( \varphi^{-}\) : \( P(X) \to P(Y) \) is a homomorphism of specialization semilattices from \( S(X) \) to \( S(Y) \);
3. the image function \( \varphi^{-} \) is a homomorphism of specialization posets from \( P(X) \) to \( P(Y) \).

The equivalences still hold if we replace everywhere “continuous” and “homomorphism” with “embedding”.

**Proof.** The image function \( \varphi^{-} \) satisfies \( \varphi^{-}(a \cup b) = \varphi^{-}(a) \cup \varphi^{-}(b) \), for every \( a, b \subseteq X \), with no further special assumption, hence \( \varphi^{-} \) is automatically a \( \cup \)-homomorphism and, similarly, a \( \subseteq \)-homomorphism. It is easy to see that if \( \varphi \) is injective, then \( \varphi^{-} \) is an embedding with respect to both \( \cup \) and \( \subseteq \).

If \( \varphi \) is continuous and \( a \subseteq b \), that is, \( a \subseteq Kb \), we have \( \varphi^{-}(a) \subseteq \varphi^{-}(Kb) \subseteq K \varphi^{-}(b) \), since \( \varphi^{-} \) is a \( \subseteq \)-homomorphism. Hence \( \varphi^{-}(a) \subseteq \varphi^{-}(b) \), thus \( \varphi^{-} \) is a \( \subseteq \)-homomorphism. We have proved (1) \( \Rightarrow \) (2); (2) \( \Rightarrow \) (3) is obvious.

To prove (3) \( \Rightarrow \) (1), suppose that \( \varphi^{-} \) is a \( \subseteq \)-homomorphism, that is, \( a \subseteq b \) implies \( \varphi^{-}(a) \subseteq \varphi^{-}(b) \), for every \( a, b \subseteq X \). In particular, we can take \( a = Kb \) and, since \( Kb \subseteq Kb \), we get \( Kb \subseteq b \), hence \( \varphi^{-}(Kb) \subseteq \varphi^{-}(b) \), thus \( \varphi^{-}(Kb) \subseteq K \varphi^{-}(b) \). Hence \( \varphi \) is continuous.

To prove the last statement, first observe that the following is a chain of equivalent conditions, for \( Z \) a topological or closure space over some subset of \( Y \).
(i) $Z$ is a subspace of $Y$,
(ii) the corresponding closure operations satisfy $K_Z d = Z \cap K_Y d$, for all $d \subseteq Z$.
(iii) $c \subseteq K_Z d$ if and only if $c \subseteq K_Y d$, for all $c, d \subseteq Z$.
(iv) $c \sqsubseteq Z d$ if and only if $c \sqsubseteq Y d$, for all $c, d \subseteq Z$.

Thus if $\varphi : X \to Y$ is an embedding (of topological or closure spaces), $Z = \varphi^{-}(X)$ and $a, b \subseteq X$, then $\varphi^{-}(a) \sqsubseteq Y \varphi^{-}(b)$ if and only if $\varphi^{-}(a) \sqsubseteq Z \varphi^{-}(b)$ if and only if $a \subseteq X b$, since $\varphi$ induces a homeomorphism from $X$ onto $Z = \varphi^{-}(X)$.

Conversely, if $\varphi^{-} : \mathcal{P}(X) \to \mathcal{P}(Y)$ is an embedding of specialization semilattices from $\mathcal{S}(X)$ to $\mathcal{S}(Y)$, then $\varphi : X \to Y$ is injective. Give $Z = \varphi^{-}(X)$ the topology induced by the topology on $X$ through $\varphi$, thus $\varphi^{-}(a) \sqsubseteq Z \varphi^{-}(b)$ if and only if $a \subseteq X b$. This is also equivalent to $\varphi^{-}(a) \sqsubseteq Y \varphi^{-}(b)$, since $\varphi^{-}$ is an embedding of specialization semilattices. Since $\varphi^{-}$ is surjective from $X$ to $Z$, then, for every $c, d \in Z$, there are $a, b \in X$ such that $c = \varphi^{-}(a)$ and $d = \varphi^{-}(b)$. By the equivalence of (iv) and (i), $Z$ is a subspace of $Y$ and this means precisely that $\varphi$ is an embedding.

Remark 2.5. On the other hand, as we mentioned before, if $\varphi$ is continuous, it is not necessarily the case that $\varphi^{-}(K b) \supseteq K \varphi^{-}(b)$ always. In fact, equality holds if and only if $\varphi$ is a closed map [En Exercise 1.4.C]. If $\psi$ is a function between two sets with some unary operation $K$, we shall say that $\psi$ is a homomorphism with respect to $K$ if $\psi K (b) = K \psi (b)$, for all elements $b$ in the domain of $\psi$. In this terminology, $\varphi$ being a continuous function between two topological spaces does not entail $\varphi^{-}$ being a homomorphism with respect to $K$.

Notice that, on the other hand, it is not the case that to every function $\psi : \mathcal{P}(X) \to \mathcal{P}(Y)$ there is an associated function $\varphi : X \to Y$ such that $\psi = \varphi^{-}$, let alone continuity and the notion of homomorphism.

Many of the above definitions, ideas and arguments apply to a general setting in which the underlying structure is just a poset.

Definition 2.6. If $(P, \leq)$ is a poset, a closure operation is an isotone, extensive and idempotent unary operation $K$ on $P$. In the above situation, the triple $(P, \leq, K)$ shall be called a closure poset. If $\leq$ is associated to some semilattice operation $\vee$, we shall say that $(P, \vee, K)$ is a closure semilattice. Again, see [En Section 3.1] for further details and an equivalent characterization in terms of “closed” elements. See also Remark 2.3(b) below.

If $(P, \leq)$ is a poset and $K$ is a closure operation on $P$, the associated specialization poset is the structure $(P, \leq, \sqsubseteq)$, where $\sqsubseteq$ is defined as in [2.3] namely, $a \sqsubseteq b$ if $a \leq Kb$, for $a, b \in P$. If in addition $P$ is a join-semilattice, we shall speak of the associated specialization semilattice $\mathcal{S}(P)$. 
A homomorphism between two closure posets (semilattices) is a poset-(semilattice-) homomorphism which is also a homomorphism with respect to $K$, as defined in Remark 2.5.

Any homomorphism between two closure posets is a homomorphism between the associated specialization semilattices; however, the converse does not necessarily hold, as already seen in the special case of topological spaces. See Remark 2.5.

However, if we introduce a notion of “continuity” between closure posets, a generalization of Proposition 2.4 holds. If $P$ and $Q$ are closure posets (semilattices), we say that a function $\psi : P \to Q$ is continuous if $\psi$ is a morphism of posets (semilattices) and moreover $\psi(K_P a) \leq K_Q \psi(a)$, for every $a \in P$.

The proof of Proposition 2.4 applies in this more general situation and shows the following corollary.

Corollary 2.7. A function $\psi$ between two closure posets (semilattices) is continuous if and only if $\psi$ is a homomorphism between the associated specialization posets (semilattices).

3. Specialization semilattices and posets

Recall that a preorder on some set $P$ is a binary reflexive and transitive relation on $P$. Some authors use the expression quasiorder in place of preorder. A partial order, or simply order, or ordering or order relation is an antisymmetric preorder. A partially ordered set, for short, poset is a set endowed with a partial order.

Recall that (algebraically) a semilattice is a set $S$ together with a binary operation $\lor$ which is commutative, associative and idempotent. Semilattices here will be always intended as join-semilattices, in the sense that a semilattice $(S, \lor)$ induces the partial order $\leq$ on $S$ defined by $a \leq b$ if and only if $a \lor b = b$. Meet semilattices—not considered here—algebraically are defined in the same way, but conventionally are assumed to induce the reverse order. When we mention a partial order in reference to some semilattice, we shall always mean the order $\leq$ introduced above. The symbols $\lor$ and $\land$ shall always be used to denote joins and meets in semilattices or lattices. They should not be confused with the logical propositional operators of conjunction and (inclusive) disjunction, which shall be denoted by “&” and “or”.

Definition 3.1. (a) A specialization poset $S$ is a structure $(S, \leq, \sqsubseteq)$ such that $(S, \leq)$ is a poset and $\sqsubseteq$ is a binary relation on $S$ satisfying

$$a \leq b \Rightarrow a \sqsubseteq b,$$  \hspace{1cm} \text{($\leq$ is finer than $\sqsubseteq$) \hspace{1cm} (S1)}

$$a \sqsubseteq b \land b \sqsubseteq c \Rightarrow a \sqsubseteq c,$$ \hspace{1cm} \text{($\sqsubseteq$ is transitive) \hspace{1cm} (S2)}

for all elements $a, b, c \in S$. 

(b) A specialization semilattice $S$ is a triple $(S, \vee, \sqsubseteq)$ such that $(S, \vee)$ is a semilattice, $S$ satisfies (S1) - (S2), where $\leq$ is the order induced by $\vee$, and

$$a \sqsubseteq b \land a_1 \sqsubseteq b \Rightarrow a \vee a_1 \sqsubseteq b, \quad (\sqsubseteq \text{ respects joins on the 1st comp.}) \quad (S3)$$

for all elements $a, b, a_1 \in S$.

In both cases, the relation $\sqsubseteq$ shall be called a specialization.

In words, a specialization poset is a poset endowed with an additional preorder $\sqsubseteq$ which is coarser (that is, larger) than the poset order. See Remark 3.4(c) below. In the case of specialization semilattices we also ask for the compatibility condition (S3). We will show in Example 5.5 below that (S3) does not follow from the other assumptions.

Occasionally, we shall deal with sets with more structure, thus we might talk of specialization lattices, specialization complete (join) semilattices and so on, however our main interest here is in (finitary) semilattices.

**Remark 3.2.** The structures introduced in Definition 2.3 and, more generally, in Definition 2.6 are easily seen to be specialization semilattices and posets according to the preceding definition.

**Definition 3.3.** Obviously, if $S = (S, \vee, \sqsubseteq)$ is a specialization semilattice, then $R(S) = (S, \leq, \sqsubseteq)$ is a specialization poset, which shall be called the order-specialization-reduct of $S$. Strictly speaking, this is not a reduct of $S$ in the formal sense (it is a reduct of $(S, \vee, \leq, \sqsubseteq)$) but we hope the terminology is sufficiently clear and intuitive.

In particular, everything we shall say about specialization posets will apply to specialization semilattices, too.

**Remarks 3.4.** (a) From (S1) one immediately gets

$$a \sqsubseteq a, \quad (\sqsubseteq \text{ is reflexive}) \quad (S4)$$

while from (S1) - (S2) one gets

$$a \sqsubseteq b \land b \leq c \Rightarrow a \sqsubseteq c, \quad (\sqsubseteq \text{ is weakly transitive on the right}) \quad (S5)$$

$$a \leq b \land b \sqsubseteq c \Rightarrow a \sqsubseteq c, \quad (\sqsubseteq \text{ is weakly transitive on the left}) \quad (S6)$$

for all elements $a, b, c \in S$. Thus every specialization poset satisfies (S4) - (S6).

From $b, b_1 \leq b \lor b_1$, (S3) and (S6) it follows

$$a \sqsubseteq b \land a_1 \sqsubseteq b_1 \Rightarrow a \lor a_1 \sqsubseteq b \lor b_1, \quad (\lor \text{ preserves } \sqsubseteq) \quad (S7)$$

in particular, by (S3) and taking $a_1 = b$ in (S3), respectively, $a_1 = b_1$ in (S7), we get

$$a \sqsubseteq b \Rightarrow a \lor b \sqsubseteq b, \quad (S8)$$

$$a \sqsubseteq b \Rightarrow a \lor a_1 \sqsubseteq b \lor a_1, \quad (S9)$$

thus (S4) - (S9) hold in every specialization semilattice.
(b) From the above remarks we get alternative axiomatizations. For example, a poset with a further relation $\sqsubseteq$ is a specialization poset if and only if (S2), (S4) and (S5) hold (equivalently, (S5) can be replaced by (S6)).

(c) It follows from (S4) and (S2) that if $S$ is a specialization poset (semilattice), then $\sqsubseteq$ is a preorder. However, $\sqsubseteq$ is not required to be antisymmetric, hence it is not necessarily an order. In particular, whenever we speak of meets and joins, these notions shall be always interpreted as relative to $\leq$.

(d) Notice that properties (S1) - (S9) are all expressible as first-order universal sentences, just prefix each formula by an appropriate string of universal quantifiers. Actually (S1) - (S9) are Horn formulas [Ho, Section 9.1] and this implies that they are preserved under direct products, a fact which can also be easily verified directly.

3.1. Homomorphisms and embeddings. For the sake of precision, we give the explicit definitions of homomorphisms and embeddings between specialization posets and semilattices. The notions correspond exactly to the standard model-theoretical notions [Ho, Section 1.2]. Some authors use the expression isomorphic embedding [CK, Section 1.3] for what we call simply an embedding.

If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets and $\varphi : P \to Q$, then $\varphi$ is a homomorphism of posets, or an ordermorphism, or simply a homomorphism when the context is clear, if $a \leq_P b$ implies $\varphi(a) \leq_Q \varphi(b)$, for every $a, b \in P$. An ordermorphism is an order-embedding, or simply an embedding if in addition $\varphi(a) \leq_Q \varphi(b)$ implies $a \leq_P b$, for every $a, b \in P$. Notice that an order-embedding is necessarily injective.

If $(S, \vee_S)$ and $(T, \vee_T)$ are semilattices, a (semilattice) homomorphism is a function $\varphi : S \to T$ such that $\varphi(a \vee_S b) = \varphi(a) \vee_T \varphi(b)$, for every $a, b \in S$. A (semilattice) embedding is an injective homomorphism. Notice that a semilattice homomorphism (embedding) between two semilattices induces an ordermorphism (embedding) for the corresponding “order-reducts”.

If $(S, \leq_S, \sqsubseteq_S)$ and $(T, \leq_T, \sqsubseteq_T)$ are specialization posets, $\varphi$ is a homomorphism (of specialization posets) if $\varphi$ is an ordermorphism from $(S, \leq_S)$ to $(T, \leq_T)$ and moreover

$$a \sqsubseteq_S b \text{ implies } \varphi(a) \sqsubseteq_T \varphi(b), \text{ for every } a, b \in S. \quad \text{(M)}$$

A homomorphism of specialization posets is an embedding if it is an order-embedding and moreover

$$\varphi(a) \sqsubseteq_T \varphi(b) \text{ implies } a \sqsubseteq_S b, \text{ for every } a, b \in S. \quad \text{(E)}$$

If $(S, \vee_S, \sqsubseteq_S)$ and $(T, \vee_T, \sqsubseteq_T)$ are specialization semilattices, a homomorphism (of specialization semilattices) is a semilattice homomorphism satisfying (M). A homomorphism of specialization semilattices is an embedding if it is injective and satisfies (E).
As already mentioned, for structures with a unary operation $K$, e. g., closure posets as introduced in Definition 2.6, a homomorphism is supposed to satisfy $\varphi(K_S a) = K_T \varphi(a)$.

In the above definitions we have distinguished, say, the operation $\lor_S$ on $S$ from the operation $\lor_T$ on $T$ by adding the corresponding subscripts. When no risk of ambiguity might occur, we shall drop the subscripts. To be more accurate, we should have made the distinction between symbols and their interpretations [CK, Ho]. Here we shall not need to make the distinction explicit, hence we shall proceed quite informally.

Notice that in algebra (namely, when dealing with operations) an injective homomorphism is always an embedding. E. g., if $\varphi$ is an injective semilattice homomorphism, then $a \lor b = c$ if and only if $\varphi(a) \lor \varphi(b) = \varphi(c)$. On the other hand, in topology and in model theory when predicates, i.e. relations, are present, injective continuous functions or homomorphisms are not necessarily embeddings. For example, if $X, Y$ are topological spaces, $\varphi : X \to Y$ and $\varphi$ is continuous and injective, it is not necessarily the case that $X$ is homeomorphic to a subspace of $Y$.

4. Embedding theorems

4.1. Principal specialization semilattices. In our motivating example from Definition 2.3 the relation $a \sqsubseteq b$ is defined by $a \subseteq Kb$, where $K$ is the topological closure of some topological space. In particular, in the motivating example, $Kb$ is the largest element of the set $S_b = \{a \in S \mid a \sqsubseteq b\}$. On the other hand, in a general specialization poset, as we have defined it, $S_b$ might not even have a (possibly infinitary) join. See Example 5.1(a) below. Even when $S_b$ has a join, say, $s$, it is not necessarily the case that $s \sqsubseteq b$. See Example 5.1(b). In the following lemma we show that if, for some $b \in S$, the set $S_b$ has a maximum, call it $Kb$, then $Kb$ actually corresponds to some form of “closure”. The lemma works for a single $b$ (possibly, two elements $a$ and $b$) and we do not need the assumption that $S_b$ has a maximum for every $b \in S$. If the latter property actually holds for every $b \in S$, we get a very special class (widely known, in an equivalent formulation) of specialization posets we are going to mention soon.

**Lemma 4.1.** Suppose that $S$ is a specialization poset, $b \in S$ and the set $\{a \in S \mid a \sqsubseteq b\}$ has a maximum (in the sense of $\leq$). Call $Kb$ this maximum. Then, for every $a \in S$, the following conditions are equivalent.

(i) $a \sqsubseteq b$;
(ii) $a \leq Kb$;
(iii) $a \subseteq Kb$.

In particular, $Kb$ is also the maximum of $\{a \in S \mid a \subseteq Kb\}$. Suggestively, if $Kb$ exists, then $KKb$ exists, too, and they are equal.
Suppose further that \( a \in S \) and the set \( \{ c \in S \mid c \sqsubseteq a \} \) has a maximum, call it \( K_a \). Then conditions (i) - (iii) above are also equivalent to

(iv) \( K_a \leq K_b \);
(v) \( K_a \sqsubseteq K_b \).

**Proof.** The fact that (i) implies (ii) is just a restatement of the definition of \( K_b \). From (ii) we immediately get (iii), because of (S1). By the definition of \( K_b \) we get \( K_b \sqsubseteq b \), hence from (iii) and (S2) we obtain (i).

Under the additional assumptions, \( K_a \sqsubseteq a \), hence (iii) implies (v), by (S2). Moreover, (iv) and (v) are equivalent, by applying the equivalence of (ii) and (iii) with \( K_a \) in place of \( a \). Since \( a \leq K_a \), by (S4), we get that (iv) implies (ii). \( \square \)

The above considerations and Lemma 4.1 suggest the following definition.

**Definition 4.2.** (a) We say that a specialization poset (semilattice) \( S \) is principal if the preorder \( \sqsubseteq \) is a principal quasi-order for \( (S, \leq) \). This means that, for every \( b \in S \), the set \( S_b = \{ a \in S \mid a \sqsubseteq b \} \) has a maximum relative to \( \leq \). In other words, \( S \) is principal if and only if every principal \( \sqsubseteq \)-ideal is also \( \leq \)-principal.

(b) Notice that, in the case of a specialization semilattice, \( S_b \) is upward directed, by (S3). In particular, every finite specialization semilattice is principal. However, as we mentioned, when \( S_b \) is infinite it is not necessarily the case that \( S_b \) has a join. Even when some join \( s \) exists, it is not necessarily the case that \( s \sqsubseteq b \). In the definition of a principal quasi-order we not only require that such an \( s \) exists, but we require that actually \( s \sqsubseteq b \).

Notice that, on the other hand, a finite specialization poset is not necessarily principal. See Example 4.3 below.

(c) In particular, if \( S \) is a specialization complete semilattice, then \( S \) is principal if and only if \( S \) satisfies the following infinitary version of condition (S2), for every family \((a_i)_{i \in I}\) of elements of \( S \).

\[
\text{If } a_i \sqsubseteq b, \text{ for every } i \in I, \text{ then } \bigvee_{i \in I} a_i \sqsubseteq b. \tag{S3}_\infty
\]

Principal specialization complete lattices are complete lattices with a symmetric consequence relations in the terminology from [GT, 3.1] (considering the converse of the consequence relation).

(d) If \( S \) is a principal specialization poset (semilattice), then, for every \( b \in S \), let us denote by \( K_b \) the maximum of \( S_b \). We say that a principal specialization semilattice is additive if \( K(a \lor b) = K_a \lor K_b \), for every \( a, b \in S \).

**Remarks 4.3.** (a) The specialization semilattice (poset) associated to a closure space, as in Definition 2.3, is principal. The specialization semilattice associated to some topological space is also additive. Indeed, a closure space is a topological space if and only if its associated specialization semilattice is
additive in the above terminology, and moreover \( K \emptyset = \emptyset \). In particular, not every principal specialization semilattice is additive.

(b) If \( S \) is a principal specialization poset (semilattice), then the function which assigns to \( b \) the maximum \( K b \) of \( S b = \{ a \in S \mid a \sqsubseteq b \} \) is a closure operation. Recall Definition 2.6. Conversely, if \( K \) is a closure operation on some poset (semilattice), then, as noticed in Remark 3.2 we get a principal specialization poset (semilattice) by letting \( a \sqsubseteq b \) if \( a \leq K b \). Notice that the motivating example from Section 2 is a special case of the above construction.

The above constructions provide a bijective correspondence between closure operations and principal specializations on the same poset. Compare also [Er, Proposition 3.9]. We shall not use this correspondence here, except for the next observation. However, let us point out that the notions of homomorphism are distinct in the two settings.

Remark 4.3(b) can be used to show that specializations provide still another equivalent formulation for the notion of a topology, possibly a folklore result in some form or another.

**Observation 4.4.** Fix some set \( X \).

The correspondence assigning to some closure space (topology) \( \tau \) on \( X \) the specialization semilattice \( S(\tau) \) from Definition 2.3 is a bijective correspondence from the set of all closure spaces (topologies) on \( X \) to the set of all the principal (principal and additive) specialization semilattices of the form \( (P(X), \cup, \sqsubseteq) \) (and such that \( a \sqsubseteq \emptyset \) implies \( a = \emptyset \)).

In particular, by Proposition 2.4, the category of topological spaces with continuous functions is equivalent to the category of principal and additive specialization semilattices which can be realized as \( (P(X), \cup, \sqsubseteq) \), for some set \( X \), and such that \( a \sqsubseteq \emptyset \) implies \( a = \emptyset \), with homomorphisms.

**Remarks 4.5.** (a) Notice that a principal specialization semilattice is additive if and only if

\[
K(c \vee d) = c \vee d, \text{ for every } c, d \text{ such that } Kc = c \text{ and } Kd = d. \tag{4.1}
\]

Necessity is obvious. In the other direction, for every \( a \) and \( b \), take \( c = K a \) and \( d = K b \). We have \( Kc = K a \) and \( Kd = K b \) by Lemma 4.1. Then (4.1) provides \( K(Ka \vee Kb) = Ka \vee Kb \), hence \( K(a \vee b) \leq K(Ka \vee Kb) = Ka \vee Kb \), the other inequality being obvious, since \( K \) is isotone, due to (S5).

(b) The condition that some specialization semilattice (poset) is principal can be expressed by a first-order sentence. Indeed, a specialization semilattice (poset) \( S \) is principal if and only if \( S \) satisfies

\[
\forall b \exists c \forall a (a \sqsubseteq b \Leftrightarrow a \leq c). \tag{4.2}
\]
Thus, provided some specialization semilattice $S$ is principal, by (a) we can express the property that $S$ is additive as

$$\forall cd (\forall a (a \subseteq c \Leftrightarrow a \leq c) \& \forall a (a \subseteq d \Leftrightarrow a \leq d) \Rightarrow \forall a (a \subseteq c \lor d \Leftrightarrow a \leq c \lor d)).$$

(4.3)

However, the sentences (4.2) and (4.3) are quite complex; here we are interested in simpler sentences, mainly universal sentences. Of course, if we add further symbols to our language, the statement that a specialization semilattice is principal can be expressed in a simpler way. If we add the unary function $K$, then (4.2) can be expressed as

$$\forall ab (a \subseteq b \Leftrightarrow a \leq Kb).$$

However, $K$ is not definable over every specialization semilattice (poset); in fact, we have $K$ exactly when the specialization semilattice (poset) is principal. If we abbreviate $\forall a (a \subseteq c \Leftrightarrow a \leq c)$ as $C(c)$, then (4.3) simplifies to

$$\forall cd (C(c) \& C(d) \Rightarrow C(c \lor d)).$$

However, as commented above, changing the language changes the notion of homomorphism. In particular, if we add, say, a unary predicate $C$ as defined above, the analogue of Proposition 2.4 fails, since not every continuous function is closed.

4.2. Embedding into principal specialization semilattices. Homomorphic images of specialization semilattices are not necessarily specialization semilattices themselves. See Example 5.6 for an explicit counterexample. Technically, this can be hinted from the fact that (S1) - (S3) are not positive formulas, and shows that the theory of specialization semilattices cannot be axiomatized by positive sentences, by [CK, Theorem 3.2.4]. Henceforth Lemma 4.7 below will be useful. We first recall a classical algebraic definition.

**Definition 4.6.** If $S = (S, \lor)$ is a semilattice, a binary relation $\sim$ is a *congruence* on $S$ if $\sim$ is an equivalence relation on $S$ and furthermore $\sim$ respects $\lor$, that is, $a \sim b$ implies $a \lor c \sim b \lor c$, for every $a, b, c \in S$. This is a special case of a more general algebraic notion [B, Section 1.5].

If $S$ has some further structure, we shall say that $\sim$ is a congruence for the semilattice reduct if the above conditions hold.

**Lemma 4.7.** Suppose that $S = (S, \lor, \sqsubseteq)$ is a specialization semilattice and $\sim$ is an equivalence relation on $S$ such that $\sim$ is a congruence for the semilattice reduct and moreover

$$\text{for every } a, b \in S, \text{ if } a \sim b, \text{ then } a \sqsubseteq b \text{ and } b \sqsubseteq a. \quad (4.4)$$

Then $\bar{S} = (\bar{S}, \lor, \sqsubseteq)$ is a specialization semilattice, where $\bar{S}$ is the set of the $\sim$-equivalence classes, $\lor$ is the standard quotient operation and, for all $a, b \in S$, we let $a \sqsubseteq b$ if $a \sqsubseteq b$. Here we have written, say, $S, a, \ldots$ in place of $S/\sim, a/\sim, \ldots$ in order to improve readability.

Moreover, the projection map $\pi$ which sends $a$ to $\pi(a) = a$ is a homomorphism of specialization semilattices.
Lemma 4.7 in order to get a specialization semilattice $U$ showed that $\sim$, that is, $(\sim)$.

Then, for every $(\sim)$, then $\sim$ is a congruence for the semilattice reduct; moreover, the projection is a semilattice homomorphism.

If $a \leq b$ in $S$, then, by the above paragraph, $(a \lor b)/\sim = a \lor b = b$, that is, $a \lor b \sim b$. By $\sim$, hence $a \leq b$ by $\sim$, thus $a \leq b$, by the definition of $\leq$. We have proved that $(S1)$ holds in $S$.

If $a \leq b$ and $b \leq c$ then $a \leq b$ and $b \leq c$, since the definition of $\leq$ does not depend on the representatives. Hence $a \leq c$ by $(S2)$ and $a \leq c$ by the definition of $\leq$, so that $(S2)$ holds in $S$ as well.

The proof of $(S3)$ is similar, using the already mentioned fact that $(a \lor a_1)/\sim = a \lor a_2 = a_1$. The last statement is trivial.

The assumption that $\sim$ satisfies Condition $(4.4)$ is necessary in Lemma 4.7. See Example 5.6 below.

Theorem 4.8. Every specialization semilattice can be embedded into a principal additive specialization semilattice.

Proof. Suppose that $S = (S, \lor_S, \leq_S)$ is a specialization semilattice. Let $T = \{0, 1\}$ and $T = (T, \max_T, \leq_T)$ be the specialization semilattice such that $x \leq_T y$ for every $x, y \in T$. Let $S \times T$ be defined in the natural way on $S \times T$, by taking the standard semilattice product and letting $a \leq_{S \times T} b$ hold if both $a_1 \leq_S b_1$ and $a_2 \leq_T b_2$, for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $S \times T$ (of course, in the special case at hand, $a \leq_{S \times T} b$ holds if and only if $a_1 \leq_S b_1$ holds). By Remark 3.4(d), $S \times T$ is a specialization semilattice.

The function $\iota$ defined by $\iota(s) = (s, 0)$ is a homomorphism from $S$ to $S \times T$. This fact easily follows from the observation that $\{0\}$ is the universe for a substructure of $T$ and, of course, using $0 \leq_T 0$.

Next, consider the equivalence relation $\sim$ on $S \times T$ defined by $(a_1, a_2) \sim (b_1, b_2)$ if at least one of the following conditions hold

1. $a_1 = b_1$ and $a_2 = b_2$, or
2. $a_2 = b_2 = 1$ and both $a_1 \leq_S b_1$ and $b_1 \leq_S a_1$.

Due to $(S2)$, $\sim$ is transitive, and then obviously an equivalence relation.

We now check that $\sim$ is a congruence for the semilattice reduct. Suppose that $(a_1, 1) \sim (b_1, 1)$, as witnessed by the conditions $a_1 \leq_S b_1$ and $b_1 \leq_S a_1$. Then, for every $(c_1, c_2) \in S \times T$, we have $(a_1, 1) \lor (c_1, c_2) = (a_1 \lor_S c_1, 1)$ and $(b_1, 1) \lor (c_1, c_2) = (b_1 \lor_S c_1, 1)$. From $a_1 \leq_S b_1$ and $(S9)$ we get $a_1 \lor_S c_1 \leq_S b_1 \lor_S c_1$ and, symmetrically, $b_1 \lor_S c_1 \leq_S a_1 \lor_S c_1$, thus $(a_1 \lor_S c_1, 1) \sim (b_1 \lor_S c_1, 1)$, that is, $(a_1, 1) \lor (c_1, c_2) \sim (b_1, 1) \lor (c_1, c_2)$. The other case is trivial. We have showed that $\sim$ is a congruence for the semilattice reduct.

Finally, $\sim$ satisfies Condition $(4.4)$ by construction, hence we can apply Lemma 4.7 in order to get a specialization semilattice $U = (S \times T)/\sim$ and a
homomorphism \( \pi : S \times T \to U \). The composition \( \kappa = \iota \circ \pi \), being a composition of two homomorphisms, is a homomorphism from \( S \) to \( U \).

In order to keep the number of subscripts to a minimum, let us write \([a_1, a_2]\) for \( \pi(a_1, a_2) \), in place of \((a_1, a_2)\) or \((a_1, a_2)/\sim\). The homomorphism \( \kappa \) is injective, since if \( a \neq b \in S \), then, in the above notation, \( \kappa(a) = [a, 0] \) and \( \kappa(b) = [b, 0] \), thus \([a, 0] \neq [b, 0]\), since no two pairs with second component 0 are identified by \( \sim \). By the definition of \( \sqsubseteq \) if \([a, 0] \sqsubseteq [b, 0]\), then \((a, 0) \sqsubseteq_{S \times T} (b, 0)\) and then \( a \sqsubseteq_S b \), by the definition of \( \sqsubseteq_{S \times T} \). Thus \( \kappa \) is an embedding of \( S \) into \( U \).

It remains to show that \( U \) is principal and additive. Taken any element \([a_1, a_2]\) of \( U \), we see that \([a_1, 1] \sqsubseteq [a_1, a_2]\), by the definition of \( \sqsubseteq \) and since \((a_1, 1) \sqsubseteq_{S \times T} (a_1, a_2)\), by (S4) and the definitions of \( \sqsubseteq_T \) and of \( \sqsubseteq_{S \times T} \). Moreover, we claim that \([a_1, 1]\) is the largest element \([c, d] \in U \) such that \([c, d] \sqsubseteq [a_1, a_2]\). Indeed, if \([c, d] \subseteq [a_1, a_2]\), then \([c, d] \sqsubseteq_{S \times T} (a_1, a_2)\), hence \( c \sqsubseteq_S a_1\), by the definitions of \( \sqsubseteq \) and \( \sqsubseteq_{S \times T} \). It follows that \((c, 1) \sqcup_{S \times T} (a_1, 1)\), hence \((c, 1) \sqcup_{S \times T} (a_1, 1) \sqsubseteq_{S \times T} (a_1, 1)\), by (S8). By (S1), \((a_1, 1) \sqsubseteq_{S \times T} (c, 1) \sqcup_{S \times T} (a_1, 1)\), hence \((c, 1) \sqcup_{S \times T} (a_1, 1) \sim (a_1, 1)\), by the definition of \( \sim \). Thus in \( U \)

\[
[c, 1] \cup [a_1, 1] = \pi(c, 1) \cup \pi(a_1, 1) = \pi((c, 1) \cup_{S \times T} (a_1, 1)) = [a_1, 1],
\]

since \( \pi \) is a semilattice homomorphism. It follows that in \( U \) \([c, 1] \leq [a_1, 1]\). Obviously, \([c, d] \leq [c, 1]\), hence \([c, d] \leq [a_1, 1]\).

We have proved that \([a_1, 1]\) is the maximum among those elements \([c, d]\) such that \([c, d] \subseteq [a_1, a_2]\). In the notation from Definition 4.2(d), we have \( K[a_1, a_2] = [a_1, 1]\). Since \([a_1, a_2]\) is arbitrary, the above procedure applies to every element of \( U \), and this means that \( U \) is principal. Moreover \( U \) is additive, since

\[
K([a_1, a_2] \cup [b_1, b_2]) = K(\pi(a_1, a_2) \cup \pi(b_1, b_2)) = K(\pi((a_1, a_2) \cup_{S \times T} (b_1, b_2))) = K(\pi(a_1 \cup_S b_1, a_2 \cup_T b_2)) = K(a_1 \cup_S b_1, a_2 \cup_T b_2) = [a_1 \cup_S b_1, 1] = [a_1, 1] \cup [b_1, 1] = K[a_1, a_2] \cup K[b_1, b_2]. \quad \square
\]

**Remark 4.9.** (a) In practice, the extension \( U \) in the above proof is obtained as follows. Consider the equivalence relation \( \Theta \) on \( S \) defined by \( a \Theta b \) if \( a \sqsubseteq_S b \) and \( b \sqsubseteq_S a \). Since \( \Theta \) satisfies (S4), by (S5), the quotient \( S/\Theta \) naturally becomes a specialization semilattice, call it \( V \). Then \( U \) can be thought of as the “disjoint union” of \( S \) and \( V \), obtained by declaring each class in \( V \) to be larger than every member of the class. While the proof of Theorem 4.8 can be performed along the above lines, the proof becomes harder, since there are a lot of details to be checked by hand. Most of these details automatically follow from the canonical structures on products and quotients, as presented in the given proof of [4.8].

(b) In general, there is not the smallest extension of \( S \) satisfying the conclusions of Theorem 4.8. See Example 5.7.
(c) The embedding $\kappa$ in the proof of 4.8 preserves existing (possibly infini-
tary) meets in $S$. Indeed, in $U$ it never happens that $[a, 1] \preceq [b, 0]$, hence
existing meets are computed as in $S$.

4.3. The universal theory of topological spaces (in the language of
specialization semilattices). Recall from Definition 2.3 that to a topo-
logical space $X$ we have associated the specialization semilattice $S(X) = (\mathcal{P}(X), \sqcup, \sqsubseteq)$ and the specialization poset $P(X) = (\mathcal{P}(X), \subseteq, \sqsubseteq)$, where $a \sqsubseteq b$
if $a \subseteq Kb$, $K$ being the closure induced by the topology on $X$.

We say that a specialization semilattice, resp., poset is topological
if it is isomorphic to $S(X)$, resp., to $P(X)$, for some topological space $X$.

**Theorem 4.10.** Every specialization semilattice can be embedded into a topo-
logical specialization semilattice.

**Proof.** In view of Theorem 4.8, it is enough to show that every principal addi-
tive specialization semilattice can be embedded into a topological specialization
semilattice. The argument is rather standard.

So let $S = (S, \vee, \sqsubseteq)$ be a principal additive specialization semilattice. First
notice that it is no loss of generality to assume that $S$ has a minimum element
0 such that $a \sqsubseteq 0$ if and only if $a = 0$. If not, $S$ can be embedded into a
specialization semilattice $S_0$ with such an element; just add to $S$ a new ⟨-
neutral element 0 and set $0 \sqsubseteq a$, for every $a \in S_0$, and $a \not\sqsubseteq 0$, for every $a \in S$.  
Then the inclusion map is trivially an embedding of $S$ into $S_0$. Notice that
$S_0$ is principal, since $K0 = 0$ and $S$ is assumed to be principal. Of course,
additivity is preserved, too.

Let $\varphi : S \to \mathcal{P}(S)$ be the function defined by $\varphi(a) = \mathcal{F}a = \{b \in S \mid a \not\sqsubseteq b\}$. We shall define a topology $\tau$ on the set $S$ in such a way that $\varphi$ is an embedding from $S$ to
the specialization semilattice associated to $(S, \tau)$. First, $\varphi$ is trivially an injective semilattice homomorphism, hence an embedding from $(S, \vee)$ to $(\mathcal{P}(S), \cup)$, no matter how $\tau$ is defined.

Since $S$ is principal, then, by definition, for every $b \in S$, the set $\{a \in S \mid a \sqsubseteq b\}$ has a maximum $Kb$. In order to define a topology $\tau$ it is enough to
define the family $\mathcal{T}$ of its closed elements. Let the family $\mathcal{F} = \{\mathcal{F}a \mid a \sqsubseteq b\}$
be a closed-base for the topology $\tau$. By this we mean that the members of $\mathcal{T}$
are exactly the intersections of subfamilies of $\mathcal{F}$. The family $\mathcal{F}$ is preserved
under finite unions, since $S$ is additive and $\mathcal{F}$ is a semilattice homomorphism,
thus $\mathcal{F}(Ka \cup Kb) = \mathcal{F}(Ka \cup Kb) = \mathcal{F}(Ka \cup Kb)$. Hence the above definition
actually provides a family of closed sets for a topology, since $\mathcal{T}$ is preserved
under arbitrary intersections, by construction, and, moreover, any finite union
of intersections is an intersection of finite unions. Notice that $S \in \mathcal{T}$, since
$S$ can be considered the intersection of the empty subfamily of $\mathcal{F}$; moreover
$\emptyset \in \mathcal{T}$, since we have assumed that $S$ has a minimum 0 and then $\emptyset = \mathcal{F}0 = \mathcal{F}0$.

Letting $\sqsubseteq_\tau$ be the specialization relation on $\mathcal{P}(S)$ associated to $\tau$, it remains
to show that $\varphi$ is an embedding for the specialization relations, namely, that
a \sqsubseteq b \text{ if and only if } \varphi(a) \sqsubseteq_\tau \varphi(b), \text{ for every } a, b \in S. \text{ Indeed, for } a, b \in S, \text{ the following is a chain of equivalent conditions:}

(i) \ a \sqsubseteq b; 
(ii) \text{ for every } c \in S, \text{ if } b \leq Kc, \text{ then } a \leq Kc; 
(iii) \text{ for every } C \in \mathcal{F}, \text{ if } \varphi(b) \subseteq C, \text{ then } \varphi(a) \subseteq C; 
(iv) \text{ for every closed } C \text{ in } \tau, \text{ if } \varphi(b) \subseteq C, \text{ then } \varphi(a) \subseteq C; 
(v) \ \varphi(a) \subseteq K_\tau \varphi(b); 
(vi) \ \varphi(a) \subseteq_\tau \varphi(b).

We are now going to show that the above conditions are equivalent. If \( a \sqsubseteq b \) and \( b \leq Kc \), then \( a \sqsubseteq Kc \), by (S5), hence \( a \leq Kc \), by Lemma 4.1. Thus (i) \( \Rightarrow \) (ii). To prove the other direction, first observe that \( b \leq Kb \), for every \( b \in S \), by the definition of \( Kb \) and since \( b \sqsubseteq b \), by (S4). Then, taking \( c = b \) in (ii), we get \( a \leq Kb \), that is, \( a \sqsubseteq b \), by the definition of \( Kb \). Hence (i) and (ii) are equivalent.

The equivalence of (ii) and (iii) follows from the definition of \( \mathcal{F} \) and the fact that \( \varphi \) is an order-embedding.

We now deal with (iii) and (iv). Given \( a \) and \( b \), then the property

(*) \text{ if } \varphi(b) \subseteq C, \text{ then } \varphi(a) \subseteq C

holds for every \( C \) belonging to some family \( \mathcal{F} \) if and only if (*) holds for every \( C \) belonging to some intersection of members of \( \mathcal{F} \). Hence, by the definition of \( \tau \), (iii) and (iv) are equivalent.

The equivalence of (iv) and (v) is immediate from the properties of topological closure, and then (v) and (vi) are equivalent by definition. Actually, the equivalence of (iv) and (vi) is a special case of the arguments for the equivalence of (i) and (ii). \( \square \)

Notice that, by Remark 4.3(a), Theorem 4.10 is formally stronger than Theorem 4.8; however, the proof of 4.10 makes use of 4.8.

**Remark 4.11.** In contrast with Remark 4.9(c), the embedding \( \varphi \) in the proof of Theorem 4.10 does not generally preserve existing meets.

In fact, preservation of meets cannot be accomplished, in general. Just consider some lattice \( L \) which is not distributive and let \( K \) be the identity function on \( L \). The associated specialization semilattice is principal and additive, but it cannot be embedded into the specialization semilattice associated to some topological space in such a way that meets are preserved, since every sublattice of a distributive lattice is distributive.

By Remark 3.4(d) the class of specialization semilattices (posets) is axiomatized by a first-order universal theory, which we shall call the *theory of specialization semilattices (posets)*. Recall the definitions of \( S(X) \) and of \( S(P) \) from Definitions 2.3 and 2.6.

**Corollary 4.12.** For every universal first-order sentence \( \varphi \) in the language of specialization semilattices, the following conditions are equivalent.
(1) The sentence $\varphi$ is true in all the structures of the form $S(X)$, where $X$ varies among topological spaces.

(2) The sentence $\varphi$ is true in all the structures of the form $S(X)$, where $X$ varies among closure spaces.

(3) The sentence $\varphi$ is true in all the structures of the form $S(P)$, where $P$ varies among closure semilattices.

(4) The sentence $\varphi$ is a logical consequence of the theory of specialization semilattices.

Proof. (1) $\Rightarrow$ (4) If $\varphi$ is valid in all the structures $S(X)$, with $X$ a topological space, then $\varphi$ is valid in all the specialization semilattices, by Theorem 4.10 and since universal sentences are preserved under taking substructures.

(4) $\Rightarrow$ (3) is obvious by Remark 3.2; (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious. $\Box$

For short, the theory of specialization semilattices is the universal theory of structures associated to topological spaces in the sense of Definition 2.3.

**Problem 4.13.** Characterize those universal-existential sentences valid in all structures $S(X)$.

**Remark 4.14.** The equivalences of (1) - (3) in Corollary 4.12 are quite unexpected. They say that in the language of specialization semilattices the universal sentences valid in all $S(X)$ are the same, no matter whether we let $X$ vary among topological spaces, closure spaces, or even closure semilattices.

Of course, if we consider the closure operation $K$ as part of the language, then topological spaces satisfy $\forall xy K(x \lor y) = Kx \lor Ky$, a sentence which is not necessarily valid in closure spaces (actually a closure space is a topological space if and only if it satisfies this sentence, together with a sentence asserting that the minimal element is a $K$-fixed point).

Coming back to the language of specialization semilattices, we observe that Problem 4.13 has surely different solutions, when $X$ varies on topological or closure spaces. Indeed, the following sentence

$$\forall xyz(x \lor y < z \& z \subseteq x \lor y \Rightarrow \exists w((x < w \& w \subseteq x) \lor (y < w \& w \subseteq y)))$$

(4.5)

is equivalent to a $\forall\exists$ sentence, since the existential quantifier can be moved shortly after the universal quantifier; we have kept it in the present position just to improve readability. Interpreted in topological spaces, the sentence (4.5) asserts that if $x \cup y$ is not closed, then either $x$ or $y$ is not closed; in contrapositive, the union of two closed subsets is closed. Hence if $X$ is a closure space such that $z \subseteq \emptyset$ implies $z = \emptyset$, then (4.5) holds in $S(X)$ if and only if $X$ is a topological space.

A different approach which deals with universal sentences only will be presented in Remark 6.3 below.

**4.4. Embedding specialization posets.** Many arguments from Subsections 4.2 and 4.3 can be extended to specialization posets. We have not
checked exactly which arguments do generalize and leave it as an open problem. Clearly, not everything from the theory of specialization semilattices generalizes: the notion of additivity from Definition 4.2(d) cannot be even expressed, in the absence of a join-semilattice operation.

Rather than reworking the arguments in Subsections 4.2 and 4.3 for specialization posets, we will obtain the corresponding results just noticing that every specialization poset can be “embedded” in a specialization semilattice. Here, of course, embedded should be intended in the sense of the order-specialization-reduct. See Definition 3.3.

**Proposition 4.15.** Suppose that \( P = (P, \leq P, \sqsubseteq P) \) is a specialization poset. Then there is a specialization semilattice \( S = (S, \lor, \sqsubseteq S) \) such that if \( \leq S \) is the order induced by \( \lor \) on \( S \), then there is an embedding \( \iota \) of specialization posets from \( P = (P, \leq P, \sqsubseteq P) \) into \( R(S) = (S, \leq S, \sqsubseteq S) \).

Moreover, some \( S \) as above can be given the structure of a Boolean algebra (with the same join operation \( \lor \)) in such a way that \( \iota \) preserves existing (possibly infinitary) meets in \( P \).

**Proof.** For every \( a \in P \), let \( \downarrow a = \{ b \in P \mid b \leq P a \} \). Let \( S = \mathcal{P}(P) \), thus \( S \) has the structure of a Boolean algebra. In particular, \( (S, \cup) \) is a semilattice and the order induced by \( \cup \) is \( \subseteq \). The function \( \iota \) which assigns to \( a \in P \) the set \( \downarrow a \in S \) is obviously an order-embedding from \( (P, \leq P) \) to \( (S, \subseteq) \) and \( \iota \) trivially preserves existing meets. So far, we have just recalled a classical argument showing that any poset can be order-embedded into a join semilattice (much more can be proved!). We now show that the construction allows the definition of a specialization relation on \( S \) in such a way that \( \iota \) is a specialization embedding.

Define \( \sqsubseteq S \) on \( S \) by \( X \sqsubseteq S Y \) if, for every \( c \in X \), there is \( d \in Y \) such that \( c \leq P d \). Using the corresponding properties of \( \sqsubseteq P \), it is immediate to see that \( (S, \subseteq, \sqsubseteq S) \) satisfies (S1) and (S2). It is also immediate to see that (S3) holds, thus \( (S, \cup, \sqsubseteq S) \) is a specialization semilattice.

It remains to show that \( \iota \) is also an \( \sqsubseteq \)-embedding. We have to show that if \( a, b \in P \), then \( a \sqsubseteq P b \) if and only if \( \downarrow a \sqsubseteq S \downarrow b \). Indeed, if \( a \sqsubseteq P b \) and \( c \in \downarrow a \), then \( c \leq P a \sqsubseteq P b \), thus \( c \sqsubseteq P b \) by (S6). Since \( b \in \downarrow b \), we get \( \downarrow a \sqsubseteq S \downarrow b \). In the other direction, if \( \downarrow a \sqsubseteq S \downarrow b \), then, since \( a \in \downarrow a \), there is \( d \in \downarrow b \) such that \( a \sqsubseteq P d \). Since \( d \in \downarrow b \), we have \( d \leq P b \), hence we get \( a \sqsubseteq P b \) by (S5).

We shall not need the second part of Proposition 4.15 in what follows. In this connection, we notice that in order to prove the first part it is enough to take a much smaller subset of \( \mathcal{P}(P) \), it is enough to take \( S \) equal to the family containing all the finite unions of sets of the form \( \downarrow a \), with \( a \) varying in \( P \) (notice that, by construction, \( S \), as defined here, is closed under finite unions).

Notice also that we cannot use the argument in the proof of Proposition 4.15 in order to prove Theorem 4.10 since \( \iota \) from the proof of 4.15 is generally
not join preserving. However we can put together 4.15 and 4.10 in order to prove the analogue of 4.10 for specialization posets.

Recall that a topological specialization poset is a specialization poset of the form \( P(X) \), for some topological space \( X \), where \( P(X) \) has been introduced in Definition 2.3. Recall also the more general Definition 2.6.

Corollary 4.16. Every specialization poset can be embedded into a topological specialization poset.

Suppose that \( \varphi \) is a universal first-order sentence in the language of specialization posets. Then \( \varphi \) holds in all the structures of the form \( P(X) \) (where \( X \) can equivalently vary among topological spaces, closure spaces, or even closure posets) if and only if \( \varphi \) is a logical consequence of the theory of specialization posets.

**Proof.** If \( P \) is a specialization poset, then, by Proposition 4.15, there is an embedding \( \iota \) from \( P \) into the order-specialization reduct \( R(S) \) of some specialization semilattice \( S \). By Theorem 4.10 there is an embedding \( \varphi \) of \( S \) into some topological specialization semilattice \( T \). Then \( \varphi \) is an embedding of \( R(S) \) into \( R(T) \), and \( R(T) \) is a topological specialization poset. Hence \( \iota \circ \varphi \) is an embedding of \( P \) into a topological specialization poset. \( \square \)

5. More examples

While we intuitively think of \( a \sqsubseteq b \) as “\( a \) is contained in the closure of \( b \)”, we are going to present examples of specialization semilattices which arise in situations in which no recognizable “notion of closure” is present. The next examples help clarifying the distinction between homomorphisms and embeddings in the setting of specialization semilattices.

**Example 5.1.** Let \( \alpha \) be an ordinal, \( \leq \) be the standard order on \( \alpha \) and define \( \beta \sqsubseteq \gamma \) if and only if there is some natural number \( n \) such that \( \beta \leq \gamma + n \).

Then \( P(\alpha) = (\alpha, \leq, \sqsubseteq) \) is a specialization poset. Moreover, if we take \( \sup\{\beta, \gamma\} \) as the join of \( \beta \) and \( \gamma \), then \( S(\alpha) = (\alpha, \sup, \sqsubseteq) \) is a specialization semilattice. In passing, let us notice that every linearly ordered specialization poset becomes a specialization semilattice, if we consider the binary sup as join.

(a) If \( \alpha \) is infinite, then \( P(\alpha) \) and \( S(\alpha) \) are not principal. Recall Definition 4.12. Indeed, \( S_0 = \{ a \in P(\alpha) \mid a \sqsubseteq 0 \} \) has no maximum.

(b) If \( \alpha > \omega \), then \( S_0 \) has a supremum \( \omega \); however, \( \omega \notin S_0 \).

(c) Let \( \alpha > \omega \). We now address the following question.

(\( \Diamond \)) Can we give \( \alpha \) the structure of a principal specialization poset \( P \) in such a way that the identity function is a morphism from \( P(\alpha) \) to \( P \)?

Let us look at the possibilities for \( S_0 = \{ a \in P \mid a \sqsubseteq 0 \} \), as evaluated in some hypothetical such \( P \). Since we assume that \( P \) is principal, then \( S_0 \) has a maximum, call it \( \beta \).
If $\beta$ is not the maximum element of $\alpha$, that is, if $\beta + 1 < \alpha$, then $\beta + 1 \subseteq \beta$ in $P$, since $\beta + 1 \subseteq \beta$ holds in $P(\alpha)$ and we want the inclusion to be a morphism. Since $\beta \in S_0$, then $\beta \subseteq 0$ in $P$, hence $\beta + 1 \subseteq 0$, by $\beta + 1 \subseteq \beta$ and $S_2$. This contradicts the assumption that $\beta$ is the maximum of $S_0$.

Thus the only possibility left is that $\alpha = \beta + 1$ and $\beta$ is the maximum of $S_0$, as computed in $P$. Thus in $P$ we have $\gamma \subseteq \delta$, for every $\gamma, \delta \in \alpha$, by $S_5$ and $S_6$. If $\alpha$ is a successor ordinal, this clearly gives $P$ the structure of a principal specialization semilattice; moreover, the identity function is a morphism from $P(\alpha)$ to $P$, thus $(\diamondsuit)$ has an affirmative answer. The above arguments show that this is the only way to accomplish our goal. Of course, since we have assumed $\alpha > \omega$, the identity function is not an embedding, since $\omega \not\subseteq 0$ in $P(\alpha)$, but $\omega \subseteq 0$ in $P$.

(d) On the other hand, by Corollary 4.16 we can extend $P(\alpha)$ to a principal specialization poset, and similarly by Theorem 4.8 we can extend $S(\alpha)$ to a principal specialization semilattice. The above arguments show that in this case we necessarily should add new elements.

Following Remark 4.9 this can be done by adding to $P(\alpha)$, for every infinite limit ordinal $\gamma \leq \alpha$, a new element, call it $\gamma - 1$, where the ordering on the extended set $P^*(\alpha)$ is defined in the obvious way. We further set $\beta \subseteq \gamma - 1$ when $\gamma$ is the smallest limit ordinal strictly larger than $\beta$, together with the further relations necessary in order to make $P^*(\alpha)$ a specialization poset.

In the notation from the proof of Theorem 4.8 in the case of specialization semilattices, an element $\beta$ of $S(\alpha)$ is identified with $[\beta, 0]$ and $\gamma - 1$ corresponds to $[\beta, 1]$, where, as above, $\gamma$ is the smallest limit ordinal strictly larger than $\beta$.

Remark 5.2. Recall the example of causal spaces from [KP] briefly discussed in the introduction. The order relations considered in [KP] represent causal precedence and chronological precedence on the points—or “events”—of a manifold modeling space-time in the general theory of relativity, or, possibly, in some more abstract generalization.

It might turn out that at the small-scale level events have a composite structure and some corresponding relations are no more antisymmetric at this level. This might suggest the shift from posets to pre-orders. Of course, under the above interpretation, it would be appropriate to consider a theory with two pre-orders, one finer than the other; in other words, in the definition 3.1(a) of a specialization poset, we should weaken the assumption that $\leq$ is an order to a pre-order. The antisymmetric relations from [KP] would emerge back only at the level of events, after we take some quotient turning pre-orders into orders.

Problem 5.3. Study the theory of two preorders, one finer than the other.

Is there any significant difference with the theory of specialization posets as introduced in Definition 3.1(a)?
Example 5.4. Let $X$ be a set and $\mu$ be a measure defined on some subset $S$ of $\mathcal{P}(X)$. Then $(S, \subseteq, \sqsubseteq)$ is a specialization poset, where $\sqsubseteq$ is defined by

$$a \sqsubseteq b \quad \text{if} \quad \mu(a) \leq \mu(b),$$

(5.1)

for $a, b \in S$.

In general, the above definition does not furnish a specialization semilattice, since the properties of a measure are incompatible with $(S3)$. There is a notable exception: if $\mu$ is a 2-valued measure, then $(S, \cup, \sqsubseteq)$ is indeed a specialization semilattice.

There is a vast literature on binary relations representable by the formula (5.1), in various contexts, mostly related with foundational issues about probability and with possible economical applications. See, e. g., [Le] and further references there. The whole line of research seems to have originated from [dF]. See [A].

A poset in which some joins fail to exist cannot be endowed with the structure of a specialization semilattice. Here is the very elementary example of a join semilattice with the structure of a specialization poset, but which is not a specialization semilattice.

Example 5.5. Let $S = \{0, a, b, 1\}$, with the partial order $\leq$ given by $0 < a < 1$ and $0 < b < 1$. Let $1 \sqsubseteq 1$, $x \sqsubseteq y$ and $x \sqsubseteq 1$, for all $x, y \in \{0, a, b\}$ and let no other $\sqsubseteq$-relation hold.

Then $(S, \leq, \sqsubseteq)$ is a specialization poset. Moreover, $\leq$ induces a semilattice operation $\lor$ on $S$, but $(S3)$ fails in $(S, \lor, \sqsubseteq)$, since $a \sqsubseteq 0, b \sqsubseteq 0$ but $a \lor b = 1 \not\sqsubseteq 0$. Henceforth $(S, \lor, \sqsubseteq)$ is not a specialization semilattice.

The same argument implies that $(S, \leq, \sqsubseteq)$ is not principal (Definition 4.2(a)). Indeed, $S_0 = \{x \in S \mid x \sqsubseteq 0\}$ has no maximum.

Example 5.6. Let $S = (S, \lor, \sqsubseteq)$, where $S = \{0, 1, 2, 3\}$, $\lor = \sup$ and the only nontrivially $\sqsubseteq$-related pairs are given by $1 \sqsubseteq 0$ and $3 \sqsubseteq 2$. Of course, we also assume $m \sqsubseteq n$, for $m \leq n \leq 3$. Then $S$ is a specialization semilattice.

If $\sim$ is the equivalence relation whose classes are $\{0\}, \{1, 2\}, \{3\}$, then, in the notations from Lemma 4.7, the standard way to define the quotient structure $\mathcal{S}$ is to set $a \sqsubseteq b$ if and only if $a_1 \sqsubseteq b_1$, for some $a_1, b_1$ with $a_1 \sim a$ and $b_1 \sim b$. This is required if we want the projection to be a homomorphism. However $\sqsubseteq$ is not transitive, since $3 \sqsubseteq 2 = 1 \sqsubseteq 0$, but $3 \sqsubseteq 0$ does not hold. Hence the assumption that $\sim$ satisfies the condition (4.1) in Lemma 4.7 is necessary. More generally, we have showed that a quotient of a specialization semilattice is not necessarily a specialization semilattice.
The point is that, under the assumptions in Lemma 4.7, in the quotient we are always allowed to choose the same representative for the “middle” element $b$ in the implication $\overline{S}_2$, but this is not always true in the general case.

Example 5.7. Let $S = \{a, b, c, 1\}$ with the partial order $\leq$ given by $a < c < 1$ and $b < c < 1$. Let the only nontrivial $\sqsubseteq$-relation be $1 \sqsubseteq c$.

Then $S = (S, \lor, \sqsubseteq)$ is a principal specialization semilattice, with $Ka = a$, $Kb = b$ and $Kc = K1 = 1$. Compare Definition 112. On the other hand, $S$ is not additive, since $K(a \lor b) = Kc = 1 > c = a \lor b = Ka \lor Kb$.

If we add a new element $a_1$ to $S$, prescribing $a < a_1 < 1$ and $a_1 \sqsubseteq a$, then the resulting structure $S_1$ is a principal additive specialization semilattice, since then $K_1a = a_1$. Moreover, the inclusion is an embedding of specialization semilattices (caution! not an embedding, not even a homomorphism, with respect to the operations $K, K_1$).

However, we could perform a symmetric construction by adding some element $b_1 > b$. This shows that we do not necessarily have the smallest extension satisfying the conclusions of Theorem 113.

6. Further remarks

Remark 6.1. If $(S, \leq)$ is a poset, then, among the relations $\sqsubseteq$ making $S$ a specialization poset, there is obviously the coarsest one, namely, $\sqsubseteq = \leq$, and there is the finest relation, namely the universal relation $\sqsubseteq = \mathcal{P}(S \times S)$ such that $a \sqsubseteq b$, for every $a, b \in S$. If in addition $(S, \lor)$ is a semilattice, then the above coarsest and finest relations make $S$ a specialization semilattice.

More generally, given a poset (semilattice) $S$, the set of all the binary relations making $S$ a specialization poset (semilattice) is a complete lattice with maximum and minimum. The maximum and minimum have been described above, and meet is intersection of relations: the meet of $(\sqsubseteq_i)_{i \in I}$ is the relation $\sqsubseteq$ defined by $a \sqsubseteq b$ if and only if $a \sqsubseteq_i b$, for every $i \in I$.

Remark 6.2. Specialization semilattices can be considered the algebraization of the fragment of modal logics consisting of formulas of the type $A \Rightarrow \Diamond B$, where $A$ and $B$ are disjunctions of propositional variables. Similar fragments have been considered in the literature, e. g., [KCS].

A comparison with [KCS] suggests the problem of studying semilattices endowed with more than one specialization.
Remark 6.3. A possible axiomatization using ternary relations. We have seen that, in the language \( \{ \lor, \sqsubseteq \} \) with the interpretation suggested by our motivating example from Definition 2.3, there is no difference between the universal theories of topological spaces and of closure spaces. Recall that the only difference is that in the latter case the empty set, as well as the union of two closed subsets need not be closed; of course, this cannot be expressed in our language by means of universal sentences alone. As we have mentioned in Remark 4.14, a possibility to distinguish between the two cases is to consider the \( \forall \exists \) theory, instead.

However, universal theories are particularly nice from a model-theoretical point of view. If one is not willing to move to existential theories, the following alternative is available. If \( X \) is a topological space, or just a closure space, with closure operator \( K \), consider the following model \( M(X) = (\mathcal{P}(X), \sqcup, R) \), where \( R \) is the ternary relation defined by

\[
R(a; b, c) \text{ if } a \subseteq Kb \sqcup Kc.
\]

Here the semicolon is only for graphical convenience, it has no special meaning. The proof of Proposition 2.4 shows that some function \( \varphi \) is continuous between two topological (or closure) spaces \( X \) and \( Y \) if and only if \( \varphi \to \) is a homomorphism between the corresponding models \( M(X) \) and \( M(Y) \).

Notice that \( a \sqsubseteq b \) from Definition 2.3 is interpretable as \( R(a; b, b) \). Since in a topological space we have \( K(b \sqcup c) = Kb \sqcup Kc \), then, if \( X \) is a topological space, the following sentence is true in \( M(X) \):

\[
R(a; b, c) \iff R(a; b \lor c, b \lor c),
\]

(6.1)

for all \( a, b, c \subseteq X \). On the other hand, (6.1) does not necessarily hold in \( M(X) \), when \( X \) is just assumed to be a closure space. Simply take \( b \) and \( c \) to be two closed whose union \( b \sqcup c \) is not closed and let \( a = K(b \sqcup c) \), thus \( a \supseteq b \sqcup c \). Then \( R(a; b \lor c, b \lor c) \) holds in \( M(X) \), but \( R(a, b, c) \) fails, hence (6.1) fails, as well.

Thus in the language \( \{ \lor, R \} \) the universal theories of topological spaces and of closure spaces are distinct.

Notice that if \( X \) is a topological space, then \( \sqsubseteq \) and \( R \) are interdefinable. Indeed, as we mentioned, \( a \sqsubseteq b \) is interpretable as \( R(a; b, b) \); in the other direction, by (6.1), \( R(a; b, c) \) is the same as \( a \sqsubseteq b \lor c \). In particular, the universal theory of topological spaces in the language \( \{ \lor, R \} \) is given by the universal closure of (6.1) plus the axioms for specialization semilattices (where \( \sqsubseteq \) is considered as a defined symbol as above).

The above argument does not hold for closure spaces. Moreover, the argument cannot be applied when only \( \sqsubseteq \) is considered in place of \( \sqcup \), namely, when we associate to some topological or closure space the model \( M^\ast(X) = (\mathcal{P}(X), \sqsubseteq, R) \). The above remarks suggest the following problems.
**Problem 6.4.** With the above interpretations, provide axioms for the universal theories of closure and topological spaces in the language \{≤, R\}, as well as of closure spaces in the language \{∨, R\}.

Does any qualitative difference arise if we add \(n+1\)-ary relations \(R_n\) whose intended interpretation is \(R_n(a; b_1, b_2, \ldots, b_n) \text{ if } a \leq K b_1 \cup K b_2 \cup \cdots \cup K b_n\)?

**Problem 6.5.** Study the following notions.

A Čech-poset is a structure \((P, \leq, \sqsubseteq)\) such that \((P, \leq)\) is a poset and \((S1), (S5), (S6)\) from Definition 3.1 and Remark 3.4 are satisfied.

A Čech-semilattice is a structure \((P, \lor, \sqsubseteq)\) such that \((P, \lor)\) is a semilattice and \((S1), (S2), (S5)\) and \((S6)\) hold.

Notice that if \(K\) is a monotone and extensive—not necessarily idempotent—operation on some poset \((P, \leq)\) (semilattice \((P, \lor)\)) then setting, as custom by now, \(a \sqsubseteq b\) if \(a \leq Kb\), we get a Čech-poset (semilattice).

**Concluding remark.** In this note we have searched for a theory which could speak of topological spaces and which furthermore satisfies some quite stringent requisites.

(A) The theory is expressible as a first-order theory, possibly using universal sentences. In particular, we do not deal with infinitary operations, or second order properties.

(B) Any notion of homomorphism should correspond to the notion as introduced in its original setting, and

(C) The correspondence between homomorphisms should be covariant, not contravariant.

We have found that the corresponding universal theory is easily described and has independently arisen in various different contexts. Of course, just dropping anyone of requirements (A)-(C), many successful and consolidated approaches are viable; see, among many other possibilities, [BF, CMT, Ed, Er, Jo, MT, MP, Si, V, Z].

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