Q-ball Formation in Affleck-Dine Baryogenesis with Gravity-mediated SUSY Breaking

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Abstract

To date, the properties of Q-balls arising from an Affleck-Dine condensate in gravity-mediated SUSY breaking have been obtained primarily through numerical simulations. In this work, we will derive the expected charge of the Q-balls formed in such a scenario through an analytical treatment. We will also examine the numerically observed difference between Q-ball formation in weakly charged condensates and formation in strongly charged condensates.

1 Introduction

A Q-ball is a non-perturbative solution of the equation of motion for a scalar field which is charged under a continuous U(1) symmetry [1]. The MSSM requires several such scalars, and so it is expected that Q-balls could be formed in a supersymmetric universe [2]. An obvious context in which to look for Q-balls is the (baryon charged) scalar condensate that is required for Affleck-Dine (AD) baryogenesis [3]. This possibility has been explored both analytically using linear perturbation theory and numerically.

On the analytical side, two distinct methods of formation have been discussed in the literature. A strongly charged condensate might fracture into Q-balls. This possibility is discussed in [4]. It has also been noted that the characteristic AD scalar potential in gravity mediated SUSY breaking can give rise to negative pressure that causes perturbation growth [5].

Numerically, exhaustive simulations have been performed by Kawasaki and Kasuya [7,8]. They observe Q-ball formation for a wide range of initial AD field configurations, including both strongly and weakly charged condensates (we will quantify the terms “strongly” and “weakly” later in the paper). These simulations reveal that the mechanism of Q-ball formation is not the same in
all condensates. For strongly charged condensates, a proportionality between the maximum Q-ball charge and the condensate charge density is observed. There is, however, a distinct change in this relationship for weakly charged condensates. This change in scaling behavior is accompanied by (or results from) the appearance of Q-balls carrying a charge opposite in sign to that initially carried by the condensate in addition to those of the same sign. Thus, for weakly charged condensates, numerical simulations imply that Q-balls can be effectively pair-produced.

In this work, we will briefly revisit the analytical work that has been done on Q-ball formation. We will show that the two different approaches to Q-ball formation can help to explain why weakly charged and strongly charged condensates form Q-balls differently. We will attempt to explain the observed relationship between the charge density of the condensate and Q-ball charge for strongly charged condensates.

2 Review of Q-balls in the AD Scenario

2.1 Flat Directions

AD baryogenesis will require a scalar field that carries baryon number and has a very flat potential so that it can be given a large expectation value [3]. There are numerous flat directions in the renormalizable MSSM without soft SUSY breaking terms. These have been cataloged [9]. High energy ($\geq M_{\text{GUT}}$) physics and SUSY breaking, however, ensure that no direction will be completely flat. In modern formulations of the AD scenario [10,11] non-renormalizable, soft-breaking, and inflaton-coupling terms are added to the renormalizable MSSM potential. Such terms lift a flat direction $\phi$ by generating a potential of the form:

$$U(\phi) = (m_\phi^2 - c_H H^2)|\phi|^2 + \frac{m_{3/2}^3}{M^{n-3}}(a_m\phi^n + h.c.) + \frac{|\lambda|^2}{M^{2n-6}}|\phi|^{2n-2}$$  \hspace{1cm} (1)

where $c_H \sim 1$, $|a_m m_{3/2}| \approx m_\phi \approx 1 \text{ TeV}$ (assuming gravity-mediated breaking), and $M$ is a large mass scale.

The phase dependent term is necessary for AD baryogenesis, as it will allow dynamical charge creation in the condensate. For the purpose of this discussion, however, it is irrelevant. We will simply assume charge creation has occurred. We are instead most interested in the negative mass term proportional to the Hubble constant. If we have a large enough $H$ during inflation, the $\phi$ field will spontaneously move out to a finite expectation value. (Note
that the negative sign is chosen. It is equally likely to be positive [10], but this is an uninteresting case.)

Once $H$ drops below the value $H_{osc} = m_\phi c_H^{-1/2}$, however, we expect the positive mass term to begin to dominate, and $\phi$ will begin coherent oscillations about zero. We can find the value of $\phi$ when the oscillations begin. For $H \sim H_{osc}$, minimizing the potential (1) yields a $\phi$ vacuum value of:

$$|\phi_{osc}| \approx \left( \frac{m_\phi M^{n-3}}{\sqrt{|\lambda|^2(n-1)}} \right)^{1/(n-2)}.$$ (2)

This gives a characteristic initial condition for the scalar condensate. Q-ball formation, however, has to wait until the phase-dependent term has become irrelevant, otherwise charge is not conserved. This will require the field value to drop by a factor of order 10 from the value of equation (2). Examining the form of the potential (1) tells us that for such small $\phi$ values, we can use the effective baryon-conserving potential:

$$U(\phi) = m_\phi^2(\phi)|\phi|^2$$ (3)

where the $\phi$ dependence of $m_\phi$ is essential to Q-ball stability. We will discuss this point in the next section.

2.2 Radiative Corrections and the Q-ball Condition

The next step in forming Q-balls is to show that the potential (1) is compatible with the Q-ball condition [1]:

$$\min_{\phi} \left[ \frac{U}{|\phi|^2} \right] < \frac{1}{2} U''(|\phi| = 0).$$ (4)

This condition can be realized when radiative corrections are taken into account [5]. The gauge interactions associated with the components of the (composite) $\phi$ field add a scale-dependence to the mass, so that:

$$m_\phi^2(\phi) \approx m_\phi^2(\mu) \left( 1 + K \log \left[ \frac{|\phi|^2}{\mu^2} \right] \right)$$ (5)

where $K$ is negative, with a magnitude in the range 0.01 to 0.1 [5,6]. The negative value of $K$ is important, as it implies that equation (4) is satisfied.
The potential relevant for Q-ball formation, then, is:

\[ U(|\phi|) = m_\phi^2(\mu)|\phi|^2 \left( 1 + K \log \left[ \frac{|\phi|^2}{\mu^2} \right] \right) + \frac{|\lambda|^2}{M^{2n-6}}|\phi|^{2n-2}. \]  

(6)

Given the form of this potential, the solution for thin-walled Q-balls has a characteristic \( \phi \) value [1,5]:

\[ |\phi_0| = \left( \frac{|K|}{|\lambda|^2(n-2) m_\phi M^{n-3}} \right)^{1/(n-2)}. \]  

(7)

Note that this is almost the same expression as our expected initial value of \( \phi \) when the condensate begins oscillations (equation (2)). This is something of a problem for Q-ball formation. Recall that Q-balls cannot form until the phase dependent term has run its course. We do not expect the AD field to be large enough at that time to form thin-walled Q-balls.

### 2.3 Thick-Walled Q-balls

Since AD baryogenesis will have to form thick-walled Q-balls if any, we must consider the properties of such constructions. These were examined in [6]. There it was shown that we can assume a Gaussian profile for thick-walled Q-balls. That is, we can take:

\[ \phi(r, t) = \phi(r)e^{i\omega t} \equiv |\phi_c|e^{-r^2/R^2}e^{i\omega t} \]  

(8)

where \( \phi_c \) and \( R \) are constants describing the radial profile of our Q-ball.

To see why this works, we must consider the equation of motion for the radial part of \( \phi \) [1]:

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} = -\omega^2 \phi + \frac{\partial U}{\partial \phi}. \]  

(9)

For a radial profile of the form (8) and a potential of the form (6) this yields:

\[ \frac{4r^2}{R^4} - \frac{6}{R^2} = -\omega^2 + m_\phi^2 \left[ 1 + \log \left( \frac{\phi_c^2}{\phi_0^2} \right) \right] + |K|m_\phi^2 \frac{2r^2}{R^2} \]  

(10)

where we have ignored the contribution of the non-renormalizable term on the right hand side, because for \( \phi_c < \phi_0 \) it will be negligible.
Comparing the sides of the above expression term by term then tells us:

\[ R^2 = \frac{2}{|K|m_\phi^2} \]

(11)

while

\[ \omega^2 = m_\phi^2 + 2|K|m_\phi^2 \left[ 1 + \log \left( \frac{\phi_0}{\phi_c} \right) \right] \]

(12)

We can also determine the charge of a Gaussian Q-ball. The Noether current for our complex scalar is:

\[ n_B = \beta i (\dot{\phi}^* \phi - \phi^* \dot{\phi}) \]

(13)

where \( \beta \) is the baryon charge per meson of the field \( \phi \). Now we can use equations (8) and (13) to see that the total charge can be expressed [6]:

\[ Q = 8\pi \beta \omega \phi_c^2 \int_0^\infty e^{-2r^2/R^2} r^2 dr = 2 \left( \frac{\pi}{2} \right)^{3/2} \beta \omega \phi_c^2 R^3. \]

(14)

3 Strongly Charged Condensates

3.1 Background

Having studied the properties of Q-balls, we now investigate whether they are likely to be formed in an AD baryogenesis scenario. For a strongly charged condensate, we anticipate that the uniform scalar field could simply fracture into stable Q-balls. In the last section, we showed that the Q-balls will have a characteristic size. Therefore, the process of breakup should yield Q-balls with a charge \( Q \) proportional to the initial charge density of the condensate \( q_0 \) [4] where \( q_0 \) in the simulations of [7,8] is defined as the charge density when \( H \approx m_\phi \).

The exact proportionality will depend on the time at which breakup occurs. This is because in an expanding universe the baryon charge density will fall with time. For a matter-dominated universe (Affleck-Dine baryogenesis occurs during the epoch of inflationary reheating when inflaton matter dominates the energy density of the universe), the charge density decreases as \( q \propto t^{-2} \propto H^2 \).
3.2 Q-ball Stability

We first show that a strongly charged condensate is energetically capable of breaking into stable Q-balls. Recall that Coleman’s condition for Q-ball stability is that the energy per charge in the Q-ball be less than the energy per charge in free particles of the scalar field in question. The energy of free scalars, however, is simply the mass of the field: \( m_\phi = \frac{U''(0)}{2} \). This line of reasoning leads directly to equation (4) \([1]\). That equation tells us that the binding energy per charge for a thin-walled Q-ball will be:

\[
\frac{1}{\beta} \left( \frac{1}{2} U''(0) \right)^{1/2} - \frac{1}{\beta} \left( \frac{U(|\phi_c|)}{\phi_c^2} \right)^{1/2}
\]

where \( \phi_c \) is the field value within the Q-ball. Note that this binding energy can be maximized in gravity mediated scenarios at the specific value \( \phi_0 \) defined in equation (7).

A problem arises when we try to calculate the value of the binding energy, however. It is a well-known difficulty (see e.g. \([12]\)) that the running mass of our scalar field has an infrared divergence associated with \( \phi = 0 \). Even worse, however, the one-loop running becomes untrustworthy for large values of the log term, since the next loop order will bring in higher powers of the log. This implies that we cannot actually calculate the mass of a free \( \phi \) meson by one-loop running from the value of the mass at large \( \phi \) and, more importantly, the qualitative behavior becomes suspect at large values of the log term.

This could be a very important issue for thick-walled Q-balls, which necessitate \( \phi_c << \phi_0 \) so that we are already stretching the one-loop running to the limits of applicability. If we want to have a truer idea of the Q-ball binding energy, we should try to make use of the renormalization group, or at the very least move to two-loop running. This has not been attempted in the numerical simulations to date, however. Therefore, let us consider the consequences of assuming simple one-loop running holds over a large range of \( \phi \).

With this assumption, we see that although \( \phi_0 \) maximizes the binding energy per charge, the binding energy will remain positive (formally infinite) for any value \( \phi < \phi_0 \). This is very important for us, since we have already shown that the AD scenario will only generate thick-walled Q-balls. We can now see that they are potentially stable. Unfortunately, our discussion so far is not technically valid for thick-walled Q-balls since they will require the addition of a gradient term to the expression for the Q-ball energy.
The energy of a complex scalar field is given by [1]:

\[ E = \int d^3x \left[ \dot{\phi}^* \dot{\phi} + \nabla \phi \cdot \nabla \phi^* + U(|\phi|) \right]. \]  

(16)

Note that if we do have gradient energy, we can treat it as an effective contribution to the scalar potential. That is, we can define an effective potential for thick-walled Q-balls:

\[ \tilde{U} = U + \nabla \phi \cdot \nabla \phi^*. \]  

(17)

We can estimate the contribution of the gradient term to \( \tilde{U} \) by using the Gaussian approximation. Recall that thick-walled Q-balls have the profile:

\[ \phi(r, t) = |\phi_c|e^{-|K|m^2r^2/2}e^{i\omega t} \]  

(18)

which means:

\[ \nabla \phi \cdot \nabla \phi^* = |K|^2m_\phi^4r^2e^{-|K|m^2r^2}|\phi_c|^2. \]  

(19)

Looking at this equation, we see that the gradient term will make a contribution to the potential which is at maximum of order \( |K|^{1/2}m_\phi \). Since the energy per charge has the form:

\[ E_Q = \frac{1}{\beta} \sqrt{\frac{U(|\phi_c|)}{|\phi_c|^2}} \]  

(20)

we expect the gradient energy to yield a contribution of order \( |K|^{1/2}m_\phi \). We would like to know if this is significant on the scale of the expected binding energy per charge.

Even though the binding energy is formally infinite, we can still compare the characteristic scale of the binding energy with the characteristic scale of the gradient energy. In the thick-walled case, we expect that the potential will be dominated by the mass term. Thus, we can estimate the binding energy per baryon as:

\[ \frac{1}{\beta} \sqrt{m_\phi^2(0)} - \frac{1}{\beta} \sqrt{m_\phi^2(\phi_c)} \approx \frac{1}{\beta} |K|^{1/2}m_\phi \log(\phi_c/0). \]  

(21)

This equation shows that although the binding energy is formally infinite, it has the characteristic magnitude \( |K|^{1/2}m_\phi \) (we will assume that the log running is somehow regulated by higher-order effects). This implies that we
can expect a binding energy per charge of at least a few times $|K|^{1/2}m_\phi$, and so the gradient energy of thick-walled Q-balls is not too large to admit bound Q-balls. These conclusions are borne out in numerical simulations. In [7,8] Q-balls are shown to form over a wide range of initial $\phi$ values (many decades).

3.3 Partially Charged Q-balls

So far we have only considered Q-balls which follow a circular path in phase space ($\omega$ is constant). As we shall see, this corresponds to a scalar field that is completely charge asymmetric (all baryons or all anti-baryons). This is not a necessary condition for a strongly charged condensate, however. In [7] it is seen that even condensates with considerable ellipticity in their phase space rotation will form Q-balls that obey the same scaling relations as for completely charged condensates. This implies that elliptically rotating condensates can fracture into bound Q-balls as well. We would like to examine the energetics of this process.

The first step is to parameterize an elliptical condensate. Recall our effective potential after the phase dependent terms have become unimportant is:

$$U(|\phi|) = m_\phi |\phi|^2 \left(1 - |K| \log \left( \frac{\phi^2}{\phi^2_1} \right) \right)$$

(22)

where $\phi_1$ is basically arbitrary. As long as $\phi$ does not change more than an order of magnitude or so, the log corrections are essentially unimportant assuming small $|K|$ (as we shall see, this is true for a condensate with significant charge). Therefore we effectively have a $|\phi|^2$ potential. Such a potential allows for closed orbits in the complex $\phi$ plane [15]. We can parameterize such an orbit in the form:

$$\phi = A \cos(m_\phi t) + iB \sin(m_\phi t).$$

(23)

Note that in keeping with our orbit analogy the analog of angular momentum in this case is $\theta_\phi |\phi|^2$ which, by the Noether current, is equal to $n_B/2\beta$ where $n_B$ is the number density of excess baryons. Conservation of angular momentum is guaranteed by the central force character of our potential (even with logarithmic correction). Plugging our parameterization into the Noether current yields:

$$n_B = 2\beta m_\phi AB.$$ 

(24)
We can now compare this number to the total number of $\phi$ particles that our condensate contains. This is approximated by taking the total energy density in the condensate, as given by equation (16), and dividing by $m_\phi$:

$$n_\phi \approx m_\phi A^2 + m_\phi B^2. \quad (25)$$

It is now apparent that if we have a circular “orbit” the condensate is entirely composed of baryons (or antibaryons). For elliptical orbits, higher eccentricity means less net baryon fraction.

Now let us evaluate the energy per charge of an elliptical Q-ball. We can make the analysis easier by picking a particular point along the trajectory. The simplest to use is when $\phi = A$. Using our small $\phi$ approximation for the potential and ignoring gradient terms for simplicity, we obtain:

$$E = (m_\phi^2 B^2 + m_\phi^2 A^2)V \quad (26)$$

where $V$ is the volume occupied by the Q-ball. The charge of the Q-ball will be $n_B V$ where $n_B$ is given in equation (24). Thus, we have an energy per charge:

$$\frac{E}{Q} = \frac{1}{2\beta} \left( m_\phi \frac{B}{A} + m_\phi \frac{A}{B} \right). \quad (27)$$

Note that throughout this analysis we have ignored the running of the mass with $\phi$. Now we can see that this was justified. The characteristic contribution of the ellipticity to the energy per charge is of order $m_\phi$, while the binding energy is characteristically of order $|K|^{1/2}m_\phi$.

This is a serious concern. By comparing equation (27) to the result of equation (21) we can see that once $A$ is a few times greater than $B$ it would require the one-loop mass running to be valid over several orders of magnitude to yield a bound Q-ball. In contrast, numerical simulations show that elliptical condensates can certainly form Q-balls up to ratios $A/B \sim 10$, and possibly beyond [7].

We can reconcile this apparent discrepancy by noting that the cause of the instability of Q-balls with a large energy per net baryon is quantum mechanical decay. Such decay is not treated in numerical simulations of the classical field. Thus, our estimates would lead us to conclude that unless some way of damping out the ellipticity of a partially-charged AD condensate is present, we must question whether such condensates will truly form Q-balls.

One possibility for such damping is scalar self-interaction in a nonhomogeneous condensate. It is possible that this has already been observed in the numerical
work of [7,8] and simply not published. Perhaps a more interesting possibility, however, is that in this case the problem is also the cure.

A rotating scalar condensate need not decay only to free scalar particles. We anticipate couplings of the AD field to many other particle species. In this case, we would anticipate that a dense condensate could experience annihilations that remove the charge-neutral part while leaving behind the baryon asymmetry. This would effectively remove the excess energy per baryon.

We have calculated the annihilations that would be mediated by an interaction term of the form $|\phi|^2|\chi|^2$ where $\chi$ is another scalar field. Such an interaction would generically be present for a flat direction composed of squarks. Our calculations show that the annihilations mediated by this interaction would reduce the ellipticity of the condensate dramatically on a time scale that is considerably less than the time scale for Q-ball formation [16].

3.4 Breakup

So far in this section, we have summarized the evidence that a uniformly charged AD condensate is in a state that meets the criteria to be considered a bound Q-ball immediately after beginning coherent oscillations about $\phi = 0$. On the strength of these arguments, we expect that by the time the universe is of the age $t \sim \mathcal{O}(1) \times m_\phi^{-1}$ the condensate is energetically ready to form Q-balls. Numerical simulations, however, show that the time of Q-ball formation is more nearly of the order $t \sim \mathcal{O}(1000) \times m_\phi^{-1}$ for $|K| = 0.01$ [7].

This discrepancy is a result of the finite age of the universe (or, equivalently, the expansion of the universe). In a matter dominated universe, the maximum radius of a causally connected patch is

$$r_{cc} = 3t = 2H^{-1}. \quad (28)$$

Thus, even if the scalar field would energetically prefer to form Q-balls, the Q-ball sized patches cannot form a coherent object until $r_{cc} \approx |K|^{-1/2}m_\phi^{-1}$.

For $|K| = 0.01$ Q-ball formation would then be causally allowed by $t \sim \mathcal{O}(10) \times m_\phi^{-1}$. Thus, we still have a large factor to make up. As we shall see, this is because the forces responsible for Q-ball formation propagate at a characteristic sound speed rather than the speed of light.

Q-ball formation is the result of interactions in the fluid of squarks represented by the scalar condensate. Such fluid interactions will have a sound speed associated with them. The sound speed in a completely charge-asymmetric scalar
condensate is given in [1] as:

\[ v_s^2 = \frac{U''(|\phi|)|\phi| - U'(|\phi|)}{U''(|\phi|)|\phi| + 3U'(|\phi|)}. \] (29)

Evaluating this using the effective potential of equation (3) gives the result that \( v_s^2 \approx K/2 \). It is important to note that because \( K \) is negative, \( v_s \) is technically imaginary. As we shall see, this will lead to the amplification of small perturbations.

To see this amplification, consider the plane-wave solution to the wave equation:

\[ \psi(\vec{r}, t) = \psi_0 \exp \left( i\vec{k} \cdot \vec{r} \pm i\omega_s t \right) \] (30)

where for sound waves \((\omega_s/|k|)^2 = v_s^2\).

Now, consider the situation when \( v_s^2 \) is negative. We can construct a very similar solution, except that now we must take \( \omega_s \) to be imaginary. Thus, we anticipate solutions of the form:

\[ \psi(\vec{r}, t) = \psi_0 \exp \left( i\vec{k} \cdot \vec{r} \right) \exp \left( \pm |\omega_s| t \right). \] (31)

This solution yields growing modes which increase exponentially with e-folding time equal to:

\[ t_{e-fold} = |\omega_s|^{-1} = \frac{1}{|v_s||k|}. \] (32)

Since a sound wave is constructed of small density fluctuations in the medium, this tells us that such fluctuations will grow exponentially. Further, since the density of the medium is proportional to \(|\phi|^2\), we anticipate that small fluctuations in the field behave as \( \delta\rho \sim |\phi|\delta\phi \), so that \( \phi \) fluctuations will grow on basically the same time scale as density perturbations.

Consider what this means for our condensate. We expect to form Q-balls with characteristic diameter \( 2^{3/2}|K|^{-1/2}m_\phi^{-1} \). Thus, to decide how fast perturbations on this scale will grow, we can use this diameter as half the wavelength in determining \(|k|\) in the above equation. We have already calculated \( v_s \). Combining these results with the results above gives:

\[ t_{e-fold} \approx \frac{2\sqrt{2}}{\pi|K|m_\phi} \approx \frac{1}{|K|m_\phi}. \] (33)
For $|K| = 0.01$, this evaluates to a characteristic time scale for the growth of Q-ball-sized perturbations:

$$t_{e-fold} \approx 100m_{\phi}^{-1}. \quad (34)$$

Further, since the numerical simulations of interest start off with perturbations suppressed by a factor of $10^7$ [7], we anticipate Q-ball formation will require about 16 e-folding times. This yields exactly the relation found in the numerical simulations:

$$t_{Q-ball} \approx 16t_{e-fold} \approx 3000m_{\phi}^{-1}. \quad (35)$$

Calculating the total charge contained in the Q-ball sized perturbation at this time, we find that:

$$Q \approx \frac{4\pi}{3}R_Q^3 q(0)t_{Q-ball}^{-2}m_{\phi}^{-2} \approx 0.0057m_{\phi}^{-3}q(0) \quad (36)$$

which compares very well with the results obtained on three-dimensional lattices in [8].

Before we leave this section, we should comment that this analysis is only valid for imaginary values of the sound speed in Q-matter. Looking back at equation (29), this means that Q-ball formation will require:

$$U''(|\phi|) - \frac{U'(|\phi|)}{|\phi|} < 0. \quad (37)$$

This condition can be restated using the equation of motion for $|\phi|$. From this, we can see that the phase rotation rate $\omega$ for a completely charge-asymmetric condensate is given by [1]:

$$\omega^2 = U'(|\phi|)/|\phi|. \quad (38)$$

Thus, we can rewrite our condition for Q-balls:

$$U''(|\phi|) - \omega^2 < 0 \quad (39)$$

which is precisely the condition for Q-ball formation in a completely charge-asymmetric condensate arrived at in [4] using the equations of motion for $|\phi|$ and $\omega$. 

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4 Weakly Charged Condensates

4.1 Background

For condensates with little or no net charge, we expect that breakup into Q-balls must follow a different path than for strongly charged condensates. The only way to form Q-balls is to somehow separate the neutral condensate into positive and negatively charged regions. This is observed to happen in numerical simulations [7]. We wish to try to understand this process analytically.

4.2 Negative Pressure and Growth of Perturbations

In one of the early papers on Q-balls in gravity-mediated AD scenarios, it was assumed that Q-ball formation was due to negative pressure in the AD condensate [5]. For the case of a real (non-complex) scalar field, negative pressure condensates result from the negative value of $K$ in the one-loop corrected scalar potential (see equation (6)) [13,14]. We wish to explicitly consider the pressure arising for a charged (complex) scalar.

It is important to note that in this section we will have to be careful to retain terms of order $|K|$ in the expressions we are about to derive. These small corrections will be responsible for any pressure that we find. The first such correction to take note of is that when we parameterize our rotating condensate in the form of an ellipse, we must use the more general form:

$$\phi = A \cos(\omega t) + iB \sin(\omega t)$$

where to first order in $|K|$, $\omega$ is given by:

$$\omega = m_\phi \sqrt{1 - |K|}.$$  \hspace{1cm} (41)

(This result follows from the equation of motion.) Note that for a treatment of “weakly charged” condensates we are interested in $A/B > 10$.

With this form in hand, we can move on to explicitly calculate the pressure. In [13] it was shown that for a rapidly (relative to the Hubble time) oscillating condensate, the average pressure can be written in terms of the energy density through the use of the parameter:

$$\gamma = \frac{1}{T} \int_0^{T} \frac{p + \rho}{\rho} dt$$

(42)
where $p$ is the pressure, $\rho$ is the energy density, and $T$ is the period of the oscillation. In terms of $\gamma$, we would have the relationship between average pressure and energy density:

$$\bar{p} = (\gamma - 1)\rho.$$  

(43)

To evaluate this expression for our charged condensate we must use the energy-momentum tensor for a complex scalar:

$$T^\mu_\nu = \frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi + \frac{\partial L}{\partial \partial_\mu \phi^*} \partial_\nu \phi^* - L \delta^\mu_\nu.$$  

(44)

which tells us:

$$\rho = \dot{\phi}\dot{\phi}^* + U$$  

(45)

and:

$$p = \dot{\phi}\dot{\phi}^* - U.$$  

(46)

These, along with the generic elliptical parameterization of equation (40), yield the expressions:

$$\rho + p = 2\dot{\phi}\dot{\phi}^* = 2(\rho - U)$$  

(47)

and:

$$\rho = m_\phi^2 (A^2 + B^2) - |K|m_\phi^2 B^2.$$  

(48)

Now, assuming that $\rho$ changes very little over the course of one oscillation, we can write:

$$\gamma = \frac{\omega}{\pi} \int_0^{2\pi/\omega} \left( 1 - \frac{U}{m_\phi^2 A^2 + \omega^2 B^2} \right) dt.$$  

(49)

This expression can be numerically integrated. Doing so for various values of $B/A$ and $K$ yields the curve shown in Figure 1. The feature of current interest is the intercept at $B = 0$, which is consistently equal to $-0.39|K|$.

With this result we see that the analysis of [5] is essentially correct. The pressure of a weakly charged AD condensate is expected to be negative, with
Fig. 1. Plot of $p/(|K|\rho)$ for condensates with varying ellipticities. The solid line is $K = -0.1$, while the dotted line is $K = -0.01$.

$\gamma - 1 \approx K/2$. Note, however, that we have made something of an approximation in our parameterization of the AD condensate as a perfect ellipse. A study of “true” AD condensates obtained via numerical integration of the classical equations of motion shows that the numerical treatment for the idealized elliptical trajectory is accurate to within a factor of 2. Therefore, we have settled on $K/2$ as a reasonable first approximation to the pressure. Note also that this is the result obtained using the polynomial approximation of [14].

We can now follow [5] to see that perturbations in even a perfectly charge-symmetric AD condensate will grow exponentially. In fact, since the factor $\gamma - 1$ is identical to the sound speed $v_s$ found for a strongly charged condensate, the exponential growth will occur with the same timescale as the growth of perturbations in a strongly charge-asymmetric condensate.

4.3 Charge Separation

The negative pressure of the condensate can serve to give us growing Q-ball sized perturbations. The next question, however, is how these regions can acquire a charge asymmetry. The dynamics of such a process are beyond a simple perturbative analysis. We can, however, do some back of the envelope calculations to guess at the charges we could expect.

Consider two overdense Q-ball-sized perturbations that have evolved “next to” one another. We will consider the epoch when these perturbations have
just gone nonlinear, so $\delta \equiv \delta \rho / \bar{\rho} \approx 1$ where $\bar{\rho}$ is the average energy density in the scalar field and $\delta \rho$ is the peak energy density in the perturbation. The perturbations will be separated by an underdense region of approximately Q-ball size.

We must assume that they have interacted in such a way as to impart on one another a very weak rotation. By conservation of baryon number we know one will have a rotation corresponding to positive baryon asymmetry and the other will carry negative baryon asymmetry. We now wish to show that this weak rotation is enough to cause the positive perturbation to preferentially accrete baryons as it grows, while the negative perturbation will preferentially accrete anti-baryons. Thus the charge asymmetries will be enhanced, eventually yielding a Q-ball/anti-Q-ball pair.

Consider a tiny box (side length $L$) filled with baryons that lies on the line joining the two perturbations. Imagine that it is closer to the center of the negative perturbation. We can calculate work required to remove this box from the negative perturbation. The work will be:

$$W_{rm} = \int_{r_0}^{\infty} \Delta p L^2 dr \quad (50)$$

where $r_0$ is the distance between the center of the negative perturbation and the small box to begin with, and $\Delta p$ is the pressure difference between the leading edge and the trailing edge of the box, given approximately by

$$\Delta p \approx -\frac{dp}{dr} L. \quad (51)$$

So, using the results of the previous section:

$$W_{rm} = \int_{r_0}^{\infty} L^3 |K| \frac{d\rho}{dr} = L^3 \frac{|K|}{2} \rho(r_0). \quad (52)$$

We want to compare this work with the energy difference that we would expect to see if the box of baryons left the negative perturbation and joined the positive perturbation. We can estimate this by considering the perturbations to be highly elliptical Q-balls. Thus, we expect that the perturbations contain net baryon number $N_B$ equal to (see equation (25)):

$$N_B = \pm 2 \beta m_\phi ABV_Q \quad (53)$$

where the minus denotes the negative perturbation and $V_Q$ is the volume of
the perturbation. Similarly, each perturbation has an energy (see equation (26)):

\[
E = (m_{\phi}^2 A^2 + m_{\phi}^2 B^2) V_Q
\]

(54)

where in each of these formulas, we expect \( A \gg B \).

Adding the box of baryons to either perturbation implies a change in both the total baryon asymmetry and the energy. For the net baryon number, we have:

\[
\delta N_B = \beta \frac{\rho_b}{m_{\phi}} L^3 = \pm 2 \beta m_{\phi} (A\delta B + B\delta A) V_Q
\]

(55)

where \( \rho_b \) is the energy density within the small box, and we have differentiated \( N_B \) given above. The energy relation is:

\[
\delta E = \rho_b L^3 = 2m_{\phi}^2 (A\delta A + B\delta B) V_Q.
\]

(56)

We can solve these equations simultaneously to obtain:

\[
\delta A_+ = \delta B_+ = \frac{\rho_b L^3}{2m_{\phi}^2 V_Q} \frac{A - B}{A^2 - B^2}
\]

(57)

for the positive perturbation, while for the negative perturbation we would have:

\[
\delta A_- = -\delta B_- = \frac{\rho_b L^3}{2m_{\phi}^2 V_Q} \frac{A + B}{A^2 - B^2}.
\]

(58)

We have solved for these by explicitly assuming the energy contribution of the box of baryons is the same for each perturbation. To zeroth order in \( K \) we expect this to be the case, as there is no attractive force present. Once we “turn on” \( K \), however, we do expect to see an attraction between baryons. We can estimate this attraction by using our zeroth order solutions for \( \delta A \) and \( \delta B \) in the first order expression for energy. From equation (48) we can see:

\[
E_{unlike} - E_{like} = 2|K|\rho_b L^3 \frac{BA}{A^2 - B^2} \approx 2|K|\rho_b L^3 \frac{B}{A}.
\]

(59)

where “like” denotes the energy of a positive perturbation with the positive box of baryons or of a negative perturbation with a negative box, and “unlike” represents the other two possibilities. Thus we see that at first order in \( K \) the energy of the box is lower when it is within the positive Q-ball than when it is
within the negative Q-ball. This energy difference can provide the work needed to separate the negative charges from the growing positive perturbation and vice-versa.

4.4 Q-ball Charge

We will now make a very rough estimate of the amount of charge separation that will occur in the formation of a perturbation from a weakly charged condensate.

In the simplest picture, we are forming spherical perturbations with radius $R$ (approximately $|K|^{-1/2}m^{-1}$) which are separated from one another by voids of about $2R$ in width. Thus, each perturbation has a sphere of influence that extends a radius $2R$ from its center. Theoretically it will gather up all the matter in this sphere to create the final perturbation.

We will now assume that we have a pair of oppositely charged perturbations next to one another. Now in the region between these, the baryons will be preferentially attracted to the positive perturbation, while the antibaryons will be preferentially attracted to the negative one. Thus, we will somewhat arbitrarily define an extended sphere of influence for these charged Q-balls of radius $3R$. In the overlap region of these extended spheres, we will assume that the charges separate completely and collect in their energetically preferred Q-ball.

To determine the final charge, then, we are interested in the ratio of this overlap volume to that of the complete $2R$ sphere of influence. A quick calculation shows that the ratio in question is approximately 1/10. Thus, we expect a final charge fraction in the Q-ball of 1/10 the value that a Q-ball formed in a completely charge asymmetric condensate of the same initial $|\phi|$ value would have. This is in reasonable agreement with numerical simulation [7,8].

5 Summary

We anticipate that strongly charged condensates can easily break up into bound Q-balls. It is clear from a simple analysis, however, that any significant ellipticity in the phase rotation of the condensate might compromise this binding by making the energy per charge larger than that in a free scalar plasma. This concern leads us to believe that damping of this ellipticity is important for Q-ball formation. This has been considered in a separate work [16]. (Note added: After publication of this paper, I was informed that the delayed decay
of Affleck-Dine condensates has also been considered in [17].

Assuming that strongly charged condensates can reach a nearly circular rotational state, we recover the known result [4] that breakup will occur. By considering the sound speed in the scalar condensate, we have shown that the time scales for breakup observed in numerical simulations [7,8] are reproduced in a simple analytical treatment. Given the results of [7,8] we expect these “strongly charged” results to be useful for condensates with $A/B$ as high as $O(10)$.

For condensates with even larger $A/B$ ratios, we have shown that the negative pressure of the condensate [5,14] can still allow for the growth of seed perturbations. This, coupled with the weak attractive force between baryons can lead to Q-ball/anti-Q-ball pairs.

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