ERGODIC PROPERTIES OF
RANDOM HOLOMORPHIC ENDOMORPHISMS OF \( \mathbb{P}^k \)

TURGAY BAYRAKTAR

Abstract. We study ergodic properties of evaluation processes generated by independent applications of holomorphic endomorphisms of the complex projective space chosen at random according to some probability distribution. Along the way, we construct positive closed currents which have good invariance and convergence properties. We provide a sufficient condition for these currents to have Hölder continuous quasi-potentials. We also prove central limit theorem for d.s.h and Hölder continuous observables.

1. Introduction

Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a holomorphic map of algebraic degree \( d \geq 2 \) and \( \omega_{FS} \) denote the Fubini-Study form on \( \mathbb{P}^k \) normalized by \( \int \omega_{FS}^k = 1 \). Dynamical Green current \( T_f \) of \( f \) is defined to be the weak limit of the sequence of smooth forms \( \{ d^{-n}(f^n)^*\omega_{FS} \} \) ([Bro65, HP94, FS95]). Green currents play an important role in the dynamical study of holomorphic endomorphisms of the projective space [FS95, Sib99]. The current \( T_f \) has Hölder continuous quasi-potentials, hence by Bedford-Taylor theory the exterior products

\[ T_f^j = T_f \wedge \cdots \wedge T_f \]

are also well-defined and dynamically interesting currents. In particular, the top degree intersection \( \mu_f = T_f^k \) yields the unique \( f \)-invariant measure of maximal entropy ([Lju83, Sib99, BD01]).

Recall that the set of rational endomorphisms \( f : \mathbb{P}^k \to \mathbb{P}^k \) with fixed algebraic degree \( d \) can be identified with \( \mathbb{P}^N \) where \( N = (k + 1)(d+1) - 1 \). We denote the set of holomorphic parameters in \( \mathbb{P}^N \) by \( \mathcal{H}_d(\mathbb{P}^k) \). It is well know that the complement of this set \( \mathcal{M} := \mathbb{P}^N \setminus \mathcal{H}_d(\mathbb{P}^k) \) is an irreducible hypersurface [GKZ94]. We consider \( \mathbb{P}^N \) as a metric space furnished with the Fubini-Study metric. We let \( m \) denote a Borel probability measure on \( (\mathbb{P}^N, \mathcal{B}) \) and assume throughout the paper that a rational endomorphism \( f \in \mathbb{P}^N \) is holomorphic with probability one. By a random holomorphic endomorphism we mean a \( \mathbb{P}^N \)-valued random variable with distribution \( m \).

In this paper, we consider the following canonical construction (see for instance [Kif86]): Let \( \Omega = \prod_{i=1}^{\infty} \mathbb{P}^N \) is the product of copies of \( \mathbb{P}^N \) endowed with the product \( \sigma \)-algebra \( \mathcal{B} \) and the probability measure \( \mathbb{P} \) which is the product measure generated by finite dimensional probabilities. To avoid measurability problems, throughout this paper we assume that all probability spaces are complete and with some abuse of notation we call the completed Borel \( \sigma \)-algebra still as Borel algebra.

In the sequel we assume that \( d \geq 2 \). Then the elements \( \lambda \in \Omega \) are sequences of rational maps

\[ \lambda = (f_0, f_1, \ldots) \text{ with } \lambda(n) = f_n : \mathbb{P}^k \to \mathbb{P}^k. \]

of degree \( d \geq 2 \). We let \( X_n : \Omega \to \mathbb{P}^N \) denote the projection onto the \( n \)th coordinate that is

\[ X_n(\lambda) = \lambda(n). \]
Note that $X_n$’s are identically distributed independent $\mathbb{P}^N$-valued random variables with distribution $m$. We also define the unilateral shift operator

$$\theta : \Omega \to \Omega$$

$$(\theta \lambda)(n) = \lambda(n + 1)$$

for all $n \geq 0$.

The measure $\mathbb{P}$ is $\theta$-invariant and ergodic. Next, we consider the natural skew product on $X := \Omega \times \mathbb{P}^k$ defined by

$$\tau : X \to X$$

$$(\lambda, x) \to (\theta(\lambda), X_0(\lambda)x)$$

then we have

$$\tau^n(\lambda, x) = (\theta^n(\lambda), F_{\lambda,n}(x))$$

where $F_{\lambda,n} := f_{n-1} \circ \cdots \circ f_1 \circ f_0 : \mathbb{P}^k \to \mathbb{P}^k$ is a rational map of algebraic degree $\leq d^n$. We remark that the results in this paper do not depend on the specific choice of the random variables (1.1) but their distribution $m$. Note that if $m$ is a Dirac mass supported at $f \in \mathcal{H}_d(\mathbb{P}^k)$ then the deterministic case emerges.

Our first result indicates that the sequence of pull-backs of a smooth form by a random sequence of rational maps is equidistributed with a positive closed current.

**Theorem 1.1.** There exists a set $\mathcal{A} \subset \Omega$ of probability one and $\theta$-invariant such that for $1 \leq p \leq k$ and for every $\lambda \in \mathcal{A}$ the sequence \(\{d^{-m}F_{\lambda,n}^* \omega_{FS}\}\) converges in the sense of currents to a positive closed bidegree $(p, p)$ current $T_p(\lambda)$ such that

$$f_0^n(T_p(\theta(\lambda))) = d^n T_p(\lambda).$$

Furthermore, if

$$\log \text{dist}(\cdot, \mathcal{M}) \in L^1_m(\mathbb{P}^N)$$

then with probability one the current $T_p(\lambda)$ has Hölder continuous super-potentials.

Random iteration of perturbation of holomorphic maps was studied in [FW00] (see also [DS03, Pet05, DS06b] for the non-autonomous setting). A local version of Theorem 1.1 was proved in [FW00] when $m$ is the Lebesgue measure. More recently, dynamics of fibered rational maps has been studied in [Jon99, Jon00, Sum00, dT12a, dT12b]. We provide a new construction of “random Green currents” $T_p(\lambda)$. Namely, we use the super-potentials of Dinh and Sibony and quantitative estimates for resolution of $\partial \overline{\partial}$-equations [DS09, GS90]. We remark that similar results for the existence of random Green currents in the setting of fibered rational maps were obtained in [dT12a] under the assumption that the function log $\text{dist}(\cdot, \mathcal{M}) \in L^1_m(\mathbb{P}^N)$. We do not require integrability of log $\text{dist}(\cdot, \mathcal{M})$ for the existence however, the later assumption provides Hölder continuity of super-potentials of $T_p(\lambda)$ with probability one. Theorem 1.1 and its consequences can be extended to the setting of random dynamical systems of holomorphic endomorphisms (cf. [Jon00, dT12a]).

Next, we give an application of Theorem 1.1 to the value distribution theory. We let $P$-almost every $\lambda \in \Omega$ there exists a pluripolar set $E_\lambda \subset G(p, k)$ such that

$$\frac{1}{d^{pn}}(F_{\lambda,n})^*[W] \to T_p(\lambda)$$

in the sense of currents as $n \to \infty$ for every $W \in G(p, k) \backslash E_\lambda$.

For $p = k$ and $\lambda \in \mathcal{A}$ each $\mu_\lambda := T_k(\lambda)$ is a Borel probability measure on $\mathbb{P}^k$ and we can define a probability measure $\mu$ whose action on a continuous function $\phi : \Omega \times \mathbb{P}^k \to \mathbb{R}$ is given by

$$(\mu, \phi) := \int_{\Omega} (T_k(\lambda), \phi(\lambda, \cdot)) dP(\lambda).$$
It follows from Theorem 1.1 that the measure $\mu$ is well-defined and $\tau$-invariant. In the sequel, we consider some ergodic properties of the dynamical system $(X, \mathcal{B}, \tau, \mu)$. We prove that $(X, \mathcal{B}, \tau, \mu)$ exponential decay of correlations for d.s.h and Hölder continuous observables under the assumption that $\log \text{dist}(\cdot, \mathcal{M}) \in L^\infty_m(\mathbb{P}^k)$.

Recall that a quasi-plurisubharmonic (qpsh for short) function is an $L^1(\mathbb{P}^k)$ function which can be locally written as difference of a plurisubharmonic function and a smooth function. A d.s.h function is equal to difference of two qpsh functions outside of a pluripolar set. In particular, smooth functions are dsh. The class of dsh functions were introduced by Dinh and Sibony; they are useful for the study of equidistribution problems in complex dynamics (see [DS06b] for instance). One can define a norm on the set of dsh functions $\text{DSH}(\mathbb{P}^k)$ (see section 4.3 for details). For a dsh function $\psi \in \text{DSH}(\mathbb{P}^k)$ we denote $\hat{\psi} = \psi \circ \pi$ where $\pi : X \to \mathbb{P}^k$ is the canonical projection.

**Theorem 1.3.** If $\log \text{dist}(\cdot, \mathcal{M}) \in L^\infty_m(\mathbb{P}^N)$ then there exists $C > 0$ such that

$$\sum_{n=0}^{N-1} |(\varphi \circ \tau^n)^n \hat{\psi} d\mu - \int_X \varphi d\mu| \leq C d^{-n}||\varphi||_{L^p(X)} ||\psi||_{\text{DSH}(\mathbb{P}^k)}$$

for $n \geq 0$, $\varphi \in L^p(X)$ with $p > 1$ and $\psi \in \text{DSH}(\mathbb{P}^k)$.

In the special case, $m = \delta_f$ for $f \in H_0(\mathbb{P}^k)$ we recover the corresponding result of [DNS10]. Next, we focus on some stochastic properties of the invariant measure $\mu$. We say that a function $\psi : \mathbb{P}^k \to \mathbb{R}$ is a coboundary if $\psi = h \circ \tau - h$ for some $h \in L^2_{\mu}(X)$. We prove central limit theorem (CLT for short) for dsh and Hölder continuous observables.

**Theorem 1.4.** Assume that $\log \text{dist}(\cdot, \mathcal{M}) \in L^\infty_m(\mathbb{P}^N)$. If $\psi \in \text{DSH}(\mathbb{P}^k)$ (respectively Hölder continuous) which is not a coboundary such that $\langle \mu, \psi \circ \pi \rangle = 0$ then $\hat{\psi} = \psi \circ \pi$ satisfies CLT. That is for every interval $I \subset \mathbb{R}$

$$\lim_{N \to \infty} \mu\left\{ (\lambda, x) : \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \psi(F_{\lambda, n}(x)) \in I \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_I \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

for some $\sigma > 0$.

In the deterministic case, CLT was obtained for Hölder observables in [CLB05, DS06a, Dup10, DNS10] and for dsh observables in [DNS10]. However, the published version of [CLB05] contains an error. Here, we follow the strategy developed by [DNS10]; namely, we use the strong mixing property (1.3) and apply Gordin’s method [Gor69] to derive CLT.

Finally, we consider a time homogenous Markov chain associated with the pre-images of random holomorphic maps. We define the transition probability by

$$P : \mathbb{P}^k \times \mathcal{B} \to [0, 1]$$

$$P(x, G) := \int_{\mathbb{P}^N} \Lambda_f(\chi_G)(x) dm(f)$$

where $\mathcal{B}$ denote the Borel algebra on $\mathbb{P}^k$, $\chi_G$ denotes the indicator of $G$ and

$$\Lambda_f \chi_G(x) := d^{-k} \sum_{y \in \Gamma^{-1}(x)} \chi_G \circ f(y).$$

Let $Y$ denote the infinite product space $Y := \prod_{n=1}^{\infty} \mathbb{P}^k$ endowed with the product algebra $\mathcal{B}^\otimes$ and $\theta : Y \to Y$ be the unilateral shift operator. Given an initial distribution $\alpha$ on the state space $(\mathbb{P}^k, \mathcal{B})$, we define $\mathbb{P}_\alpha$ to be the product measure on $Y$ generated by $\alpha$. We let $Z_0$ be a $\mathbb{P}^k$-valued random variable whose distribution is $\alpha$ that is

$$\mathbb{P}_\alpha[Z_0 \in G] = \alpha(G)$$

for every Borel set $G \subset \mathbb{P}^k$. Then we define the random variables

$$Z_n : Y \to \mathbb{P}^k$$

for every interval $I \subset \mathbb{R}$.
\[ Z_n(y) := Z_0 \circ \partial^n(y). \]

The sequence \((Z_n)_{n \geq 0}\) induces a time homogenous Markov chain with state space \((\mathbb{P}^k, \mathcal{B})\) with transition probability is \(P\) such that its law \(P_\alpha\) satisfies
\[
P_\alpha[Z_{n+1} \in G \mid Z_n = x] = P(x, G) \text{ and } P_\alpha[Z_0 \in G] = \alpha(G).
\]

Now, we denote \(\nu := \pi_\mu\) where \(\pi : X \to \mathbb{P}^k\) is the canonical projection. It follows that the probability measure \(\nu\) is \(P\)-invariant and ergodic (see Proposition 6.1), hence, \((Z_n)_{n \geq 0}\) is stationary under \(P_\nu\). We say that \(\psi \in L^2_0(\mathbb{P}^k)\) is a coboundary for the Markov chain \((Z_n)_{n \geq 0}\) if \(\int_{\mathbb{P}^k} \psi^2 - (P\psi)^2 d\nu = 0\). We prove CLT for the Markov chain \((Z_n)_{n \geq 0}\) with initial distribution \(\nu\) for dsh and Hölder continuous observables.

**Theorem 1.5.** If \(\log \text{dist}(\cdot, \mathcal{M}) \in L^\infty_m(\mathbb{P}^N)\) then every \(\psi \in DSH(\mathbb{P}^k)\) (respectively Hölder continuous) which is not a coboundary such that \(\langle \nu, \psi \rangle = 0\) satisfies CLT for the Markov chain \((Z_n)_{n \geq 0}\). That is for every interval \(I \subset \mathbb{R}\)
\[
\lim_{N \to \infty} \mathbb{P}_\nu\{y \in Y : \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \psi(Z_n(y)) \in I\} = \frac{1}{2\pi} \int_I \exp\left(-\frac{x^2}{2\sigma^2}\right) dx
\]
where \(\sigma^2 = \int_{\mathbb{P}^k} \psi^2 - (P\psi)^2 d\nu\).

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## 2. Background

### 2.1. Super-potentials of positive closed currents.

Let \(\mathbb{P}^k\) denote the complex projective space and \(\omega\) be the Fubini-Study form normalized by
\[
\int_{\mathbb{P}^k} \omega^k = 1.
\]
We denote the set of smooth \((p, q)\) forms on \(\mathbb{P}^k\) by \(D_{p,q}(\mathbb{P}^k)\) and let \(D^{p,q}(\mathbb{P}^k) = (D_{k-p, k-q}(\mathbb{P}^k))'\) denote the set of bidegree \((p, q)\) currents. We say that a \((p, p)\) form \(\Phi\) is (strongly) positive if at every point it can be written as a linear combination of forms of type \(ia_1 \land \pi_1 \land \cdots \land ia_p \land \pi_p\) where each \(a_i \in D_{1,0}\). In particular, a positive \((k, k)\) form is a product of a volume form and positive function. We say that a \((p, p)\) form \(\varphi\) is weakly positive if \(\Phi \land \varphi\) is a volume form for every positive form \(\Phi \in D_{k-p, k-p}\). We say that \(\Phi\) is a negative \((p, p)\) form if \(-\Phi\) is positive.

A \((p, p)\) current \(T\) is called (strongly) positive if \(T \land \varphi\) is a positive measure for every weakly positive form \(\varphi \in D_{k-p, k-p}\). A \((p, p)\) current \(T\) is said to be negative if \(-T\) is positive. We say that \(T\) is closed if \(dT = 0\) in the sense of distributions. The mass of a positive closed \((p, p)\) current \(T\) is defined by \(\|T\| := \int_{\mathbb{P}^k} T \land \omega^{k-p}\). We denote the set of all positive closed bidegree \((p, p)\) currents of mass one by \(\mathcal{E}_p\) endowed with the weak topology of currents. This is a compact convex set. We refer the reader to the manuscript [Dem09] for basic properties of positive closed currents.

Let \(T \in \mathcal{E}_p\) where \(p \geq 1\), a \((p - 1, -1)\) current \(U\) is called a quasi-potential of \(T\) if it satisfies the equation
\[
(2.1) \quad T = \omega^p + dd^c U
\]
where \(d = \partial + \overline{\partial}\) and \(d^c := \frac{1}{2\pi i} (\overline{\partial} - \partial)\). In particular, if \(p = 1\) a quasi-potential is nothing but a qpsh function. Note that two qpsh functions satisfying (2.1) differ by a constant. When \(p > 1\) the quasi-potentials differ by \(dd^c\)-closed currents. The quantity \(\langle U, \omega^{k-p+1}\rangle\) is called the mean of \(U\). The following result provides solutions to (2.1) with quantitative estimates.
Theorem 2.1. [DS09] Let \( T \in \mathcal{C}_p \). Then there exists a negative quasi-potential \( U \) of \( T \) which depends linearly on \( T \) such that for every \( 1 \leq r \leq \frac{k}{2kn} \) and \( 1 \leq s \leq \frac{2k}{2kn} \) we have
\[
\|U\|_{\mathcal{L}_r} \leq c_r \quad \text{and} \quad |dU|_{\mathcal{L}_r} \leq c_s
\]
where \( c_r, c_s \) are positive constants independent of \( T \).

Note that an immediate corollary of Theorem 2.1 is that the mean of \( U \) is bounded by a constant which is independent of \( T \in \mathcal{C}_p \).

Super-potentials of positive closed currents were introduced by Dinh and Sibony [DS09] which extends the notion of quasi-potential defined for the positive closed bidegree \((1,1)\) currents. For \( T \) a smooth form in \( \mathcal{C}_p \), super-potential of \( T \) of mean \( m \) is defined by
\[
\mathcal{U}_T : \mathcal{C}_{k-p+1} \to \mathbb{R} \cup \{-\infty\}
\]
(2.2)
\[
\mathcal{U}_T(R) = \langle U_T, R \rangle
\]
where \( U_T \) is a quasi-potential of \( T \) of mean \( m \). Then it follows that (see [DS09, Lemma 3.1.1])
\[
\mathcal{U}_T(R) = \langle T, U_R \rangle
\]
where \( U_R \) is a quasi-potential of \( R \) of mean \( m \). In particular, the definition of \( \mathcal{U}_T \) in (2.2) is independent of the choice of \( U_T \) of mean \( m \). Note that super-potential of \( T \) of mean \( m' \) is given by \( \mathcal{U}_T + m' - m \). The definition of super-potentials can be extended to arbitrary positive closed currents by approximation (see [DS09, Proposition 3.1.6]). Moreover, super-potentials determine the currents. We refer the reader to [DS09] for basic properties of super-potentials.

3. Random Green currents

3.1. Random Green currents. In this section we prove Theorem 1.1 in a slightly more general context:

Theorem 3.1. Let \( \{T_n\}_{n \geq 0} \) be a sequence of positive closed bidegree \((p,p)\) currents such that \( \|\mathcal{U}_{T_n}\|_{\infty} = o(d^n) \). Then the sequence \( \{d^{-mp}(F_{\lambda,n})^*T_n\} \) almost surely converges weakly to a positive closed bidegree \((p,p)\) current \( T_p(\lambda) \) such that
\[
f_{\theta}^*(T_p(\theta(\lambda))) = d^p T_p(\lambda).\tag{3.1}
\]

The current \( T_p(\lambda) \) is called as random Green current associated with the distribution \( m \).

Proof. Let \( \mathcal{A} \) be the set of \( \lambda \in \Omega \) such that \( d^{-mp}(F_{\lambda,n})^*T_n \) is well-defined for every \( n \geq 0 \) and converges weakly to a positive closed bidegree \((p,p)\) current. We also denote
\[
\mathcal{H}_d := \{ \lambda \in \Omega : f_i := \lambda(i) \in \mathcal{H}_d(\mathbb{P}^k) \text{ for every } i \geq 0 \}.
\]
Note that \( \mathcal{H}_d \) has probability one and invariant under the shift \( \theta \). We will prove that \( \mathcal{H}_d \subseteq \mathcal{A} \). Assuming \( \mathcal{H}_d \subseteq \mathcal{A} \) for the moment, since \( (\Omega, \mathcal{B}, \mathbb{P}) \) is a complete probability space this implies that \( \mathcal{A} \) is measurable and has probability one. Then we define
\[
\mathscr{A} := \cap_{n=0}^{\infty} \theta^n(\mathcal{A})
\]
which is clearly measurable and invariant under \( \theta \). Moreover, \( \mathscr{A} \) contains \( \mathcal{H}_d \) hence, \( \mathscr{A} \) has probability one.

Next, we prove that \( \mathcal{H}_d \subseteq \mathcal{A} \). In the rest of the proof, we denote \( f_i := \lambda(i) \) where \( \lambda \in \mathcal{H}_d \) and denote the pull-back and push-forward operators by
\[
L_j := \frac{1}{d^p} f_j^* : \mathcal{C}_p \to \mathcal{C}_p
\]
and
\[
\Lambda_j := \frac{1}{d^{p-1}} (f_j)_* : \mathcal{C}_{k-p+1} \to \mathcal{C}_{k-p+1}.
\]
Since $f_i$'s are holomorphic both operators are well defined and continuous (see for instance [Moe96, DS07, DS09]). We also define
\[ \Lambda^j := \Lambda_{j-1} \circ \Lambda_{j-2} \circ \cdots \circ \Lambda_0 \]
for $j \geq 1$ with the convention that $\Lambda^0 = id$. By Theorem 2.1 there exists smooth negative $(p-1, p-1)$ currents $U_{L_j(\omega^p)}$ such that
\[ dd^c U_{L_j(\omega^p)} := \frac{1}{dp} f_j^* \omega^p - \omega^p. \]
Moreover, there exists $C < 0$ such that
\[ C \leq m_j := \langle U_{L_j(\omega^p)}, \omega^{k-p+1} \rangle \leq 0 \]
for every $j \geq 1$. Let $\mathcal{U}_{L_j(\omega^p)}$ be the super-potential of $L_j(\omega^p)$ of mean $m_j$. It follows from [DS09, Lemma 5.3.8] that
\[ \mathcal{U}_n(\lambda) := \frac{1}{dn} \mathcal{U}_{T_n} \circ \Lambda^n + \sum_{j=0}^{n-1} \frac{1}{dj} \mathcal{U}_{L_j(\omega^p)} \circ \Lambda^j \]
is a super-potential of $d^{-p}(F_{\lambda,n})^* \omega^p$ on the smooth forms in $\mathcal{C}_{k-p+1}$. The first term converges to zero by assumption. Hence, $\mathcal{U}_n$ decreases to
\[ \mathcal{U}_p(\lambda) := \sum_{j=0}^{\infty} \frac{1}{dj} \mathcal{U}_{L_j(\omega^p)} \circ \Lambda^j. \]
on the smooth forms in $\mathcal{C}_{k-p+1}$. By [DS09, Corollary 3.2.7], it is enough to show that $\mathcal{U}_p(\lambda)$ is not identically $-\infty$. To this end, it is sufficient to prove that the sequence of means $\mathcal{U}_n(\omega^{k-p+1})$ is bounded from below.

Now, since $\Lambda_j(\omega^{k-p+1})$ is a positive closed bidegree $(k-p+1, k-p+1)$ current of mass one, we may write it as
\[ \Lambda_j(\omega^{k-p+1}) = \omega^{k-p+1} + dd^c R_j \]
where $R_j$ is a negative bidegree $(k-p, k-p)$ current given by Theorem 2.1 and its mean $M \leq c_j := \langle R_j, \omega^p \rangle \leq 0$ for some constant $M < 0$ independent of $j$.

Note that for $f \in H_d(\mathbb{P}^k)$ the operator $\Lambda_f$ can be continuously extended to set of negative bidegree $(k-p, k-p)$ currents $R$ such that $dd^c R \geq -\omega^{k-p+1}$. Moreover,
\[ 0 \geq \langle \Lambda_f(R), \omega^p \rangle = \langle R, \frac{1}{dp+1} f^* \omega^p \rangle = \langle R, \omega^p \rangle + \langle R, dd^c U_f \rangle = \langle R, \omega^p \rangle + \langle dd^c R, U_f \rangle \geq \langle R, \omega^p \rangle - \langle \omega^{k-p+1}, U_f \rangle \geq \langle R, \omega^p \rangle \]
where $U_f$ is a smooth bidegree $(p-1, p-1)$ negative form which is a quasi-potential of $L_f(\omega^p)$. This implies that
\[ \Lambda^n(\omega^{k-p+1}) = \omega^{k-p+1} + dd^c S_n \]
where $S_n = \sum_{i=0}^{n-2} \Lambda_{n-1} \circ \cdots \circ \Lambda_{i+1}(R_i) + R_{n-1}$ which is a decreasing sequence of negative bidegree $(k-p, k-p)$ currents such that $\frac{1}{dn} \langle S_n, \omega^p \rangle$ is bounded.

Now,
\[ \mathcal{U}_{L_j(\omega^p)} \circ \Lambda^j(\omega^{k-p+1}) = \mathcal{U}_{L_j(\omega^p)}(\omega^{k-p+1} + dd^c S_j) = m_j + \langle U_{L_j(\omega^p)}, dd^c S_j \rangle \]
and since $U_{L_j(\omega^p)}$ is smooth we have
\[ \langle U_{L_j(\omega^p)}, dd^c S_j \rangle = \langle dd^c U_{L_j(\omega^p)}, S_j \rangle \]
(3.2)
\[ = \langle \frac{1}{dp} (f_j)^* \omega^p - \omega^p, S_j \rangle \]
(3.3)
Note that
\[(3.4) \quad \langle \frac{1}{d} f_j^* \omega^p, S_j \rangle = \frac{1}{d} \langle \omega^p, S_j + 1 - R_j \rangle \]
Since \( R_j \) is a negative current combining (3.2) and (3.4) we get
\[ \mathcal{W}_{L_j(\omega^p)} \circ \Lambda^j(\omega^{k-p+1}) = m_j - \frac{1}{d} \langle R_j, \omega^p \rangle + \langle \frac{1}{d} S_j + 1 - S_j, \omega^p \rangle \geq m_j + \langle \frac{1}{d} S_j + 1 - S_j, \omega^p \rangle \]
Hence,
\[ \mathcal{W}_n(\omega^{k-p+1}) \geq \sum_{j=0}^{n-1} m_j + \frac{1}{d^n} \langle S_n, \omega^p \rangle \]
from \( 0 \geq m_j \geq C \) we deduce that
\[ \mathcal{W}_n(\omega^{k-p+1}) \geq C \frac{d}{d-1} + \frac{1}{d^n} \langle S_n, \omega^p \rangle. \]
Since the last term is bounded the first assertion follows.

Note that the super-potential \( \mathcal{W}_n(\theta(\lambda)) \) of \( \{d^{-p^m}(F_{\theta(\lambda), n})^* \omega^p\} \) satisfies
\[(3.5) \quad \frac{1}{d} \mathcal{W}_n(\theta(\lambda)) \circ \Lambda_0 + \mathcal{W}_{L_0(\omega^p)} = \mathcal{W}_{n+1}(\lambda) \]
on smooth forms in \( \mathcal{C}_{k-p+1} \) this implies that
\[ (f_0)^*(d^{-p^m}(F_{\theta(\lambda), n})^* \omega^p) = d^{-p^m}(F_{\lambda, n+1})^* \omega^p. \]
Since both sequences are convergent and \( L_0 = \frac{1}{d} f_0 \) is continuous on \( \mathcal{C}_p \) passing to the limit we see that
\[ f_0(T_p(\theta(\lambda))) = d^p T_p(\lambda). \]

Following, [DS09] for \( \alpha > 0 \) we define a distance function on \( \mathcal{C}_p \) by
\[ \text{dist}_a := \sup_{\| \Phi \|_{\infty} \leq 1} |\langle R - R', \Phi \rangle| \]
where \( \Phi \) is a smooth \((k-p, k-p)\) form on \( \mathbb{P}^k \). It follows from interpolation theory between Banach spaces that
\[ \text{dist}_{\beta} \leq \text{dist}_a \leq C_{\alpha \beta} \text{dist}_{\beta} \| \Phi \|_{\infty}^\beta \]
for \( 0 < \beta \leq \alpha \) (see [DS09] for the proof). Moreover, for \( \alpha > 1 \)
\[ \text{dist}_a(\delta_a, \delta_b) \approx \| a - b \| \]
where \( \delta_a \) denotes the Dirac mass at \( a \) and \( \| a - b \| \) denotes the distance on \( \mathbb{P}^k \) induced by the Fubini-Study metric. Continuity of quasi-potentials of random Green currents were observed in the setting of [FW00, dT12a]. In the sequel, we will show that super potentials of \( T_p(\lambda) \) are Hölder continuous with respect to the \( \text{dist}_a \) for some (equivalently for all) \( \alpha > 0 \) under the assumption that \( \log \text{dist}(\cdot, \mathcal{M}) \in L^1_m(\mathbb{P}^k) \).

**Theorem 3.2.** Assume that \( \log \text{dist}(\cdot, \mathcal{M}) \in L^1_m(\mathbb{P}^k) \). Then with probability one random Green current \( T_p(\lambda) \) has Hölder continuous super-potentials.

We use the same notation as in the proof of Theorem 3.1. We prove several lemmas which will be useful in the proof of Theorem 3.2.

First, we show that with probability one the maps in the of tail of \( \lambda \) do not get too close to the complement of the set of holomorphic maps in \( \mathbb{P}^N \). More precisely,
Lemma 3.3. If $\log \text{dist}(\cdot, \mathcal{M}) \in L^1_m(\mathbb{P}^k)$ then the set

$$A := \{ \lambda \in \Omega : \frac{1}{n} \sum_{i=0}^{n-1} \log \text{dist}(f_n, \mathcal{M}) \to \int_{\mathbb{P}^N} \log \text{dist}(f, \mathcal{M})dm(f) \}$$

has probability one. Furthermore for every $\epsilon > 0$ and $\lambda \in A$ there exists $N_\epsilon(\lambda)$ such that

$$\text{dist}(f_n, \mathcal{M}) \geq \exp(-n\epsilon)$$

for every $n \geq N_\epsilon$.

Proof. We define $\mathbb{P}^N$-valued random variables

$$X_n(\lambda) := \log \text{dist}(f_n, \mathcal{M}).$$

Note that $X_n$'s are independent, identically distributed sequence random variables with finite mean. The fist assertion is by independence of $X_n(\lambda) = f_n$'s and the second one follows from the assumption. Thus, it follows from strong law of large numbers (see [Bil12, Theorem 22.1]) that with probability one, $\frac{1}{n} \sum_{i=0}^{n-1} X_i$ converges to the mean of $X_1$ namely, $E(X_1) = \int_{\mathbb{P}^N} \log \text{dist}(f, \mathcal{M})dm(f)$. This proves the first assertion.

To prove the second assertion, let $\epsilon > 0$ be small and define

$$B_n := \{ \lambda \in A : \text{dist}(f_n, \mathcal{M}) < e^{-n\epsilon} \}$$

Note that $B_n$'s are independent events. Since $\log \text{dist}(\cdot, \mathcal{M}) \in L^1_m(\mathbb{P}^N)$ and $f_n$'s are i.i.d. for every $\epsilon > 0$ the sum $\sum_{n=1}^{\infty} \mathbb{P}(B_n)$ converges. Indeed, since $\mathbb{P}$ is the product measure we have

$$\int_{\mathbb{P}^N} -\log \text{dist}(f, \mathcal{M}) dm = \int_0^\infty m\{ \log \text{dist}(f, \mathcal{M}) < -t \} dt$$

$$\geq \sum_{n=0}^{\infty} m\{ \log \text{dist}(f, \mathcal{M}) < -n\epsilon \}$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\text{dist}(f_n, \mathcal{M}) < e^{-n\epsilon})$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(B_n).$$

Thus, by Borel-Cantelli Lemma [Bil12] we have

$$\mathbb{P}(\limsup_{n \to \infty} B_n) = 0$$

where

$$\limsup_{n \to \infty} B_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} B_k.$$ 

Note that the set $A$ is invariant under the shift $\theta : \Omega \to \Omega$. Moreover, if $\lambda \in A$ then $f_n \in \mathcal{H}_d(\mathbb{P}^k)$ for every $n \geq 0$.

Next we observe that $\Lambda_j$ is Lipschitz with respect to the distance $\text{dist}_\alpha$ (cf. [DS09, Lemma 5.4.3]):

Lemma 3.4. Let $f \in \mathcal{H}_d(\mathbb{P}^k)$ and $\alpha > 0$ then there exists constants $K > 0, q \in \mathbb{N}$ such that

$$\text{dist}_\alpha(\Lambda_j(R), \Lambda_j(R')) \leq K \text{dist}(f, \mathcal{M})^{-q} \text{dist}_\alpha(R, R')$$

Proof. First, we prove that if $\Phi$ is a $(p-1, p-1)$ smooth from with $\|\Phi\|_{\mathcal{C}^\alpha} \leq 1$ then there exists constants $\rho_\alpha > 0$ and $q \in \mathbb{N}$ such that

$$\|f^*\Phi\|_{\mathcal{C}^\alpha} \leq \rho_\alpha \text{dist}(f, \mathcal{M})^{-q} \|\Phi\|_{\mathcal{C}^\alpha}.$$
Indeed, we consider the meromorphic map
\[ \Psi : \mathbb{P}^N \times \mathbb{P}^k \to \mathbb{P}^k \]
\[ \Psi(g, x) = g(x). \]
Since \( f \in \mathcal{H}_d(\mathbb{P}^k) \) by [DD04, Lemma 2.1] there exists \( C > 0 \) and \( q_1 \in \mathbb{N}^* \) such that
\[ \|D_x f\| \leq \|D(f, x)\| \leq C \text{dist}(f, \mathcal{M})^{-q_1} \]
for every \( x \in \mathbb{P}^k \).
Note that \( \|\Phi\|_{\mathcal{C}^\alpha} \) is the \( \mathcal{C}^\alpha \) norms of the coefficients in a fixed atlas. In local coordinates we may write \( \Phi = \sum \phi_{IJ} dz_I \wedge d\bar{z}_J \). Then
\[ f^* \Phi = \sum \phi_{IJ} \circ f \ df_I \wedge d\bar{f}_J \]
and we infer that
\[ \|f^* \Phi\|_{\mathcal{C}^\alpha} \leq \rho_{\alpha} \text{dist}(f, \mathcal{M})^{-q}\|\Phi\|_{\mathcal{C}^\alpha} \]
for some \( q \in \mathbb{N}^* \). Now,
\[ |\langle \lambda f(R - R'), \Phi \rangle| = d^{1-q}|\langle R - R', f^* \Phi \rangle| \]
Thus, the assertion follows. \( \square \)

Note that the same reasoning as in Lemma 3.4 implies that \( \mathcal{U}_{L_j(\omega^p)} \) is \( \beta \)-Hölder continuous for \( 0 < \beta \leq 1 \) with respect to the distance \( \text{dist}_{\alpha} \). More precisely:

**Lemma 3.5.** For \( f_j \in \mathcal{H}_d(\mathbb{P}^k) \) and every \( 0 < \beta \leq 1 \) there exists constants \( r_{\alpha} > 0 \) and \( q \in \mathbb{N}^* \) such that
\[ |\mathcal{U}_{L_j(\omega^p)}(R) - \mathcal{U}_{L_j(\omega^p)}(R')| \leq r_{\alpha} \text{dist}(f_j, \mathcal{M})^{-q} \text{dist}_{\alpha}(R, R')^\beta \]
for \( R, R' \in \mathcal{C}_{k-p+1} \).

**Proof.** It follows from Theorem 2.1 that there exists a negative smooth \((p - 1, p - 1)\) form \( U_{L_j(\omega^p)} \) of mean \( m_j \) such that
\[ L_j(\omega^p) = \omega^p + dd^c U_{L_j(\omega^p)}. \]
the form \( U_{L_j(\omega^p)} \) is defined by
\[ U_{L_j(\omega^p)}(z) = \int_{\zeta \neq z} L_j \omega^p \wedge K(z, \zeta) \]
where \( K(z, \zeta) \) is a negative kernel on \( \mathbb{P}^k \times \mathbb{P}^k \) (see [DS09, Theorem 2.3.1]). Applying the same argument as in Lemma 3.4 we see that \( \|U_{L_j(\omega^p)}\|_{\mathcal{C}^\alpha} \) is bounded by \( r_{\alpha} \text{dist}(f_j, \mathcal{M})^{-q} \) for some constant \( r_{\alpha} > 0 \) and \( q \in \mathbb{N}^* \). Now, since \( \mathcal{U}_{L_j(\omega^p)} \) denotes the super-potential of \( L_j(\omega^p) \) of mean \( m_j \) we have
\[ |\mathcal{U}_{L_j(\omega^p)}(R) - \mathcal{U}_{L_j(\omega^p)}(R')| = |\langle U_{L_j(\omega^p)}, R - R' \rangle| \]
and the result follows. \( \square \)

In the sequel we replace \( A \) by \( \mathcal{A} \cap A \) and still denote by \( \mathcal{A} \) which is clearly \( \theta \)-invariant and has probability one. Now, we fix \( \lambda \in \mathcal{A} \) and we will show that \( T^\lambda(\lambda) \) has Hölder continuous superpotential with respect to the \( \text{dist}_{\alpha} \). Our approach is similar to that of [DS09].

**Proof of Theorem 3.2.** Let \( \epsilon > 0 \) small, by Lemma 3.3 there exists \( N \) such that \( \text{dist}(f_j, \mathcal{M}) \geq e^{-\epsilon j} \) for \( j \geq N \) and
\[ \prod_{i=0}^{j-1} \text{dist}(f_j, \mathcal{M})^{-q} \leq M^j \]
where $M := e^{-q(m,\log\text{dist}(\cdot,\mathcal{M}))}$. Let $\beta < \frac{\log d - eq}{\max(1,\log(KM))}$ then by Lemma 3.5 and Lemma 3.4 we have
\[
|\mathcal{U}_\mu(\lambda)(R) - \mathcal{U}_\mu(\lambda)(R')| \lesssim \sum_{j=0}^\infty d^{-j} \text{dist}(f_j, \mathcal{M})^{-\alpha} \text{dist}_\alpha(A_j(R), A_j(R'))^\beta
\]
\[
\lesssim \text{dist}_\alpha(R, R')^\beta \left( \sum_{j=0}^N d^{-j} \text{dist}(f_j, \mathcal{M})(KM)^\beta + \sum_{j=N+1}^\infty \frac{e^{eq}(KM)^\beta}{d^j} \right)
\]
\[
\lesssim (C_{\lambda,N} + \frac{e^{eq}(KM)^\beta}{d})^{-N} \text{dist}_\alpha(R, R')^\beta
\]
Hence, we conclude that super-potential $\mathcal{U}_{\lambda,p}$ is $\beta$-Hölder continuous with $\beta < \frac{\log d}{\max(1,\log(KM))}$. □

**Remark 3.6.** Note that for $\lambda \in \mathcal{A}$ fixed, the proof Theorem 3.2 and implies that super-potentials of random Green currents satisfy a uniform Hölder estimate in the following sense
\[
|\mathcal{U}_{\theta^n(\lambda),p}(R) - \mathcal{U}_{\theta^n(\lambda),p}(R')| \leq C_\lambda \text{dist}_\alpha(R, R')^\beta
\]
for $n \geq 0$ where $C_\lambda$ does not depend on $n$. Furthermore, if $\log \text{dist}(\cdot, \mathcal{M}) \in L^\infty_m(\mathbb{P}^N)$ then one can obtain the same estimate where $C_\lambda$ replaced with $C > 0$ which does not depend on $\lambda \in \mathcal{A}$.

Following [DS03, DNS10], we say that a positive closed bidegree $(p, p)$ current is *moderate* if for every compact family of qpsh functions $\mathcal{K}$ there exists constants $c > 0$ and $\alpha > 0$ such that
\[
\int p e^{-\alpha \varphi} T \wedge \omega^{k-p} \leq c
\]
for every $\varphi \in \mathcal{K}$. It follows that every positive closed bidegree $(p, p)$ current with Hölder continuous super-potentials is moderate (see [DNS10, DN12]). As a consequence we have the following uniform estimate which will be useful in the sequel.

**Corollary 3.7.** With probability one the current $T_p(\lambda)$ is moderate for each $1 \leq p \leq k$. In particular, for $p = k$ and $\lambda \in \mathcal{A}$ there exists constants $\beta, C > 0$ such that
\[
\int p e^{-\beta \varphi} d\mu_{\theta^n(\lambda)} \leq C
\]
for every $n \in \mathbb{N}$ and $\omega$-psh function $\varphi \in \mathcal{K}$ where $\mathcal{K}$ is a compact family of qpsh functions.

**4. Ergodic Properties of $(X, \mathcal{B}, \mu, \tau)$**

Let $X := \Omega \times \mathbb{P}^k$ and $\tau : X \to X$ is the skew product as previously defined. In the previous section we proved that there exists a Borel set $\mathcal{A} \subset \mathcal{H}_d$ which is invariant under the shift operator and of probability one such that for every $\lambda \in \mathcal{A}$ the sequence $d^{-k n} F^*_{\lambda,n} \omega^k$ converges weakly to a probability measure which we denote by $\mu_\lambda$. The measure $\mu_\lambda$ has Hölder continuous super-potentials. Moreover, by the invariance property (3.1) we have
\[
f_0^* \mu_{\theta(\lambda)} = d^{-k} \mu_\lambda
\]
for every $\lambda \in \mathcal{A}$. Furthermore, since $d^{-k} f_* f^* = \text{id}$ for every $f \in \mathcal{H}_d(\mathbb{P}^k)$ we infer that
\[
(f_0)_*(\mu_\lambda) = \mu_{\theta(\lambda)}.
\]
We denote $\mu_{\lambda,n} := \mu_{\theta^n(\lambda)}$.

Note that for every $(k - p, k - p)$ test form $\Phi$ on $\mathbb{P}^k$ the map
\[
(4.2) \quad \lambda \to \chi_{\mathcal{A}}(\lambda)(T_p(\lambda), \Phi)
\]
is measurable. Indeed, for each $n$ we consider the map
\[
\lambda \to \chi_{\mathcal{A}}(\lambda)(d^{-p n} F^*_{\lambda,n} \omega^p, \Phi)
\]
which is measurable since for each $\lambda \in \mathcal{A}$, the form $F^*_{\lambda, \omega} \omega^p$ is smooth and its coefficients depend continuously on $\lambda$ as $\lambda$ varies in $\mathcal{H}_d(\mathbb{P}^k)$. Now, being limit of measurable maps (4.2) defines a measurable map. Therefore,

$$T_p := \int_{\Omega} \langle T_p(\lambda), \cdot \rangle d\mathbb{P}(\lambda)$$

defines a positive closed bidegree $(p, p)$ current on $\mathbb{P}^k$. Let

$$L_p : C_p \to C_p$$

$$L_p(S) := d^{-p} \int_{\mathbb{P}^k} f^* S \, dm(f)$$

It follows from (3.1) and $\theta$-invariance of $\mathbb{P}$ that the current $T_p$ is invariant under the operator $L_p$. That is

$$L_p(T_p) = T_p.$$  

We denote the top degree current by $\nu := T_k$ which is a Borel probability measure on $\mathbb{P}_k$.

Now, by above reasoning we may also define a probability measure on $X$ by

$$\langle \mu, \varphi \rangle = \int_{\Omega} \langle \mu_\lambda, \varphi \rangle d\mathbb{P}(\lambda)$$

where $\varphi$ is a continuous function on $X$. Note that $\pi_* \mu = \nu$ where $\pi : X \to \mathbb{P}^k$ is the canonical projection. Indeed, by (4.1) and $\theta$-invariance of $\mathbb{P}$ we have

$$\langle \pi_* \mu, \varphi \rangle = \langle \mu, \varphi \circ \pi \rangle = \int_{\Omega} \langle \mu_{\lambda_\theta}(dx), \varphi(\theta(\lambda), f_0(x)) \rangle d\mathbb{P}(\lambda)$$

$$= \int_{\Omega} \langle \mu_{\theta}(\lambda)(dx), \varphi(\theta(\lambda), x) \rangle d\mathbb{P}(\lambda)$$

$$= \int_{\Omega} \langle \mu_\lambda, \varphi \rangle d\mathbb{P}(\lambda)$$

$$= \langle \mu, \varphi \rangle$$

Furthermore, since $(\Omega, \mathbb{P}, \theta)$ is mixing, it is classical that $(X, \mu, \tau)$ is mixing (see [Jon00, Proposition 4.1]). In section 4.2 we will show that the dynamical system $(X, \mu, \tau)$ has strong mixing properties for dsh and Hölder continuous observables.

**DSH Functions:** A function $\psi \in L^1(\mathbb{P}^k)$ is called dsh if outside of a pluripolar set $\psi = \varphi_1 - \varphi_2$ where $\varphi_i$ are negative qpsh functions. This implies that

$$dd^c \psi = T^+ - T^-$$

for some positive closed $(1, 1)$ currents $T^\pm$. Two dsh functions are identified if they coincide outside of a pluripolar set; we denote the set of all dsh functions by $DSH(\mathbb{P}^k)$. Note that dsh functions are stable under pull-back and push-forward operators induced by meromorphic maps of $\mathbb{P}^k$ and have good compactness properties inherited from those of qpsh functions. Following [DS06b] one can define a norm on $DSH(\mathbb{P}^k)$ as follows:

$$\|\psi\|_{DSH} := \|\psi\|_{L^1(\mathbb{P}^k)} + \inf \|T^\pm\|$$

where $dd^c \psi = T^+ - T^-$ and the infimum is taken over all such representations.

If $\mu$ is a probability measure on $\mathbb{P}^k$ such that all qpsh functions are $\mu$-integrable then one can define

$$\|\psi\|_{DSH}^\mu := \langle \mu, \psi \rangle + \inf \|T^\pm\|$$

where $T^\pm$ as above. The following proposition is proved in [DS05] we state it here for convenience of the reader:
Proposition 4.1. Let $\psi \in DSH(\mathbb{P}^k)$ then there exists negative qpsh functions $\varphi_1, \varphi_2$ such that $\psi = \varphi_1 - \varphi_2$ and $dd^c \varphi_i \geq -c||\psi||_{DSH}$ where $c > 0$ independent of $\psi$ and $\varphi_i$’s. Moreover, $|\psi|$ is also a dsh function and $|||\psi|||_{DSH} \leq c||\psi||_{DSH}$. Furthermore, if $\psi$ is a bounded dsh function and $\chi$ is a convex function then $\chi(\psi)$ is also dsh.

4.1. Fiberwise Mixing. In this section, we explore the speed of mixing over the “fibers” of $\tau$. For fixed $\lambda \in \mathcal{A}$ each $f_n := \lambda(n) \in \mathcal{H}_d(\mathbb{P}^k)$ induces a unitary operator

$$U_n : L^2_{\mu,\lambda,n+1}(\mathbb{P}^k) \to L^2_{\mu,\lambda,n}(\mathbb{P}^k)$$

$$\varphi \mapsto \varphi \circ f_n.$$ 

We denote the adjoint of this operator by

$$\Lambda_n : L^2_{\mu,\lambda,n}(\mathbb{P}^k) \to L^2_{\mu,\lambda,n+1}(\mathbb{P}^k)$$

$$\Lambda_n(\psi)(x) = d^{-k} \sum_{f_n(y)=x} \psi(y).$$

Proposition 4.2. Let $\lambda \in \mathcal{A}$ be fixed. If $\varphi \in L^p(\mu_{\lambda,n})$ and $\psi \in DSH(\mathbb{P}^k)$ then

$$||\langle \mu_{\lambda}, (\varphi \circ f_{n-1} \cdots \circ f_0) \psi \rangle - \langle \mu_{\lambda,n}, \varphi \rangle \mu_{\lambda,n} \psi \rangle \leq C_\lambda d^{-n} ||\varphi||_{L^p(\mu_{\lambda,n})} ||\psi||^{\mu_{\lambda,n}}_{DSH}$$

where $C_\lambda > 0$ depends only on $\lambda$ and $p$.

We need several preliminary lemmas to prove Proposition 4.2. The next lemma is an improved version of [dT12a, Proposition 7] and it will be helpful in the sequel.

Lemma 4.3. For $\lambda \in \mathcal{A}$ there exists $C_\lambda > 0$ such that

$$||\psi||^{\mu_{\lambda,n}}_{DSH} \leq C_\lambda ||\psi||_{DSH}$$

for every $n \in \mathbb{N}$ and $\psi \in DSH(\mathbb{P}^k)$.

Proof. Let $\psi \in DSH(\mathbb{P}^k)$ then by Proposition 4.1 there exists qpsh functions $\phi_i$ such that $\psi = \phi_1 - \phi_2$ and $dd^c \phi_i \geq -c||\psi||_{DSH}$ where $c > 0$ is independent of $\psi$ and $\phi_i$. Since random Green currents have H"older continuous super potentials with H"older exponent $0 < \beta \leq 1$, by Remark 3.6 and [DN12, Lemma 3.5] we obtain

$$||\langle \mu_{\lambda,n}, \psi \rangle || \leq C_\lambda \max(||\psi||_{L^1}, c^{1-\beta} ||\psi||^{1-\beta}_{DSH} ||\psi||^\beta_{L^1})$$

If $||\psi||_{L^1} \geq c^{1-\beta} ||\psi||^{1-\beta}_{DSH} ||\psi||^\beta_{L^1}$ we’re done. If not then $||\psi||_{L^1} < c||\psi||_{DSH}$ and this implies that

$$||\langle \mu_{\lambda,n}, \psi \rangle || \leq cC_\lambda ||\psi||_{DSH}$$

□

Remark 4.4. By using a similar technique and using Lemma 4.3 one can also show that for $\lambda \in \mathcal{A}$ there exists a constant $C_\lambda > 0$ such that

$$||\psi||_{DSH} \leq C_\lambda \max ||\psi||^{\mu_{\lambda,n}}_{DSH}.$$ 

for every $n \geq 0$ (cf. [dT12a, Proposition 8]).

The following lemma is essentially due to [DNS10], however, we need to make some modifications to adapt it in our setting.

Lemma 4.5. Let $\lambda \in \mathcal{A}$ and $\psi \in DSH(\mathbb{P}^k)$. If $\langle \mu_{\lambda}, \psi \rangle = 0$ then there exists $C_\lambda > 0$ such that for every $q \geq 1$

$$|||A_{n-1} \cdots A_1 \circ A_0(\psi)|||_{L^q(\mu_{\lambda,n})} \leq qC_\lambda d^{-n} ||\psi||^{\mu_{\lambda,n}}_{DSH}$$

for $n \geq 1$.  

(4.3)
Proof. Note that $(f_{n-1})^*\mu_{\lambda,n} = d^k\mu_{\lambda,n-1}$ for $n \geq 1$. This implies that

$$(\mu_{\lambda,n}, \Lambda_{n-1}(A_{n-2} \circ \cdots \circ A_0\psi)) = 0$$

and

$$\|\Lambda_{n-1}(A_{n-2} \circ \cdots \circ \Lambda_0\psi)\|_{DSH}^{\mu_{\lambda,n}} \leq d^{1-1} \|\Lambda_{n-2} \circ \cdots \circ \Lambda_0\psi\|_{DSH}^{\mu_{\lambda,n-1}}$$

Indeed, we may write

$$dd^c(\Lambda_{n-2} \circ \cdots \circ \Lambda_0\psi) = R_{n-2}^+ - R_{n-2}^-$$

where $R_{n-2}^\pm$ are some positive closed $(1,1)$ currents then

$$\|\Lambda_{n-1}(A_{n-2} \circ \cdots \circ \Lambda_0\psi)\|_{DSH}^{\mu_{\lambda,n}} \leq \|\Lambda_{n}(R_{n-2}^\pm)\| = d^{-1}\|R_{n-2}^\pm\|.$$

Now by Proposition 4.1, Remark 4.4 and Lemma 4.3 we obtain

$$\|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{DSH} \leq C\|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{DSH} \leq C_1\|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{DSH} \leq C_2\|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{DSH} \leq C_3\|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{DSH}$$

where $C_3 > 0$ depends on $\lambda$ but does not depend on $n$ nor $\psi$. Thus, by above estimate and Lemma 4.3 it is enough to prove the case $\|\psi\|_{DSH} > 0$. Since

$$\frac{d^n}{\|\psi\|_{DSH}}|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)|$$

is a bounded sequence in $DSH(\mathbb{P}^k)$ by Corollary 3.7 there exists $\beta > 0$ and $C_{\lambda} > 0$ such that

$$(\mu_{\lambda,n}, \exp(\beta \frac{d^n}{\|\psi\|_{DSH}}|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)|)) \leq C_{\lambda}.$$

Now, by using the inequality $\frac{x^n}{\beta^n} \leq e^x$ for $x \geq 0$ we conclude that

$$\|\Lambda_{\lambda-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{L^\beta} \leq \frac{q}{\beta} C_{\lambda}d^{-n}\|\psi\|_{DSH}$$

□

Proof of Proposition 4.2. Let $\lambda \in \mathscr{A}$ be fixed. If $\psi$ is constant then the assertion follows from the invariance properties

$$(f_j)^*\mu_{\lambda,j} = \mu_{\lambda,j+1}.$$}

Thus, replacing $\psi$ by $\psi - (\psi, \mu_\lambda)$ we may assume that $\langle \psi, \mu_\lambda \rangle = 0$. Then by Hölder’s inequality and Lemma 4.5 we obtain

$$\|\langle \mu_\lambda, (\varphi \circ f_{n-1} \circ \cdots \circ f_0)\psi \rangle\| = d^{-kn}\|\langle F_{n-1}^*\mu_{\lambda,n}, (\varphi \circ f_{n-1} \circ \cdots \circ f_0)\psi \rangle\| \leq \|\mu_{\lambda,n}, \varphi A_{n-1} \circ \cdots \circ A_1 \circ A_0(\psi)\| \leq \|\varphi\|_{L^p(\mu_{\lambda,n})}\|\Lambda_{n-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{L^q(\mu_{\lambda,n})} \leq qC_{\lambda}d^{-n}\|\varphi\|_{L^p(\mu_{\lambda,n})}\|\psi\|_{DSH}$$

for some $c > 0$ independent of $\psi$ and for all $n \geq 0$. □

In the deterministic case, as a consequence of interpolation theory between the Banach spaces $\mathcal{C}^0$ and $\mathcal{C}^2$, it was observed in [DNS10] that a holomorphic map $f \in \mathcal{H}_d(\mathbb{P}^k)$ posses strong mixing property for $\beta$-Hölder continuous functions with $0 < \beta \leq 1$ (see [DNS10, Proposition 3.5]). Adapting their argument to our setting, we obtain the succeeding lemma. We omit the proof as it is very similar to the one given above and to that of [DNS10, Proposition 3.5].

Lemma 4.6. Let $\lambda \in \mathscr{A}$ be fixed and $0 < \beta \leq 1$. If $\psi : \mathbb{P}^k \rightarrow \mathbb{R}$ be a $\beta$-Hölder continuous function such that $\langle \mu_\lambda, \psi \rangle = 0$ then there exists a constant $C > 0$ such that

$$\|\Lambda_{n-1} \circ \cdots \circ A_1 \circ A_0(\psi)\|_{L^\beta(\mu_{\lambda,n})} \leq C_{\lambda,\beta}d^{-\frac{n\beta}{2}}\|\psi\|_{\mathcal{C}^\beta}$$
4.2. Exponential Mixing. Let \((X, \mathcal{B}, \tau, \mu)\) be as above. In this section we prove that the dynamical system \((X, \mathcal{B}, \tau, \mu)\) is exponentially mixing for dsh and Hölder continuous observables. Denote by \(\pi : \Omega \times \mathbb{P}^k \to \mathbb{P}^k\) the canonical projection. For a measurable function \(\varphi : X \to \mathbb{R}\) we denote \(\varphi_\lambda(x) := \varphi(\lambda, x)\) and for a measurable function \(\psi : \mathbb{P}^k \to \mathbb{R}\) we define \(\tilde{\psi} := \psi \circ \pi\). Note that

\[
\|\tilde{\psi}\|_{L^2_\mu(X)}^2 = \int_\Omega \langle \mu_\lambda, |\psi|^2 \rangle \, d\mathbb{P}(\lambda) = \int_\Omega \|\psi\|_{L^2_{\mu, \lambda}}^2 \, d\mathbb{P}(\lambda)
\]

Let us denote the unitary operator induced by \(\tau\)

\[
U_\tau : L^2(X) \to L^2(X)
\]

\[
\varphi \to \varphi \circ \tau
\]

and \(P_\tau = U_\tau^*\) is the adjoint operator.

**Proposition 4.7.** If \(\psi \in L^2(\mathbb{P}^k)\) then

\[
P_\tau \tilde{\psi}(\lambda, x) = \int_{\mathbb{P}^N} \Lambda_f \psi(x) \, df(\lambda)
\]

**Proof.** Let \(\varphi \in L^2_\mu(X)\) and assume for the moment that \(\psi\) is real-valued, the general case follows from linearity. By (3.1) and using the fact that \(\mathbb{P}\) is the product measure we obtain

\[
\langle U_\tau \varphi, \tilde{\psi} \rangle = \int_\Omega \langle \mu_\lambda(dx), \varphi_0(\lambda)(f_0x)\psi(x) \rangle \, d\mathbb{P}(\lambda)
\]

\[
= \int_\Omega \langle d^{-k} f_0^* \mu_\theta(\lambda)(dx), \varphi_0(\lambda)(f_0x)\psi(x) \rangle \, d\mathbb{P}(\lambda)
\]

\[
= \int_\Omega \int_{\mathbb{P}^N} \langle \mu_{\lambda'}(dx), \varphi_{\lambda'}(f_0x) \rangle \, df(\lambda') \, d\mathbb{P}(\lambda)
\]

\[
= \int_\Omega \langle \mu_{\lambda'}(\varphi_{\lambda'}), \int_{\mathbb{P}^N} \Lambda_f \psi dm(\lambda') \rangle \, d\mathbb{P}(\lambda)
\]

where the third line follows from the invariance of \(\mathbb{P}\) under \(\theta\) and \(\lambda = (f, \lambda')\). \(\Box\)

For \(\varphi, \psi \in L^2_\mu(X)\) we define the correlation function by

\[
C_n(\varphi, \psi) := \langle \mu, \varphi \circ \tau^n \psi \rangle.
\]

Note that the dynamical system \((X, \mu, \tau)\) is mixing if \(C_n(\varphi, \psi) \to 0\) as \(n \to \infty\) for every \(\varphi, \psi \in L^2_\mu(X)\). Next, we prove that \(C_n\) decays exponentially fast for dsh observables.

**Proof of Theorem 1.3.** Again by the invariance properties (3.1) without lost of generality we may assume that \(\langle \mu, \tilde{\psi} \rangle = \int_\Omega \langle \mu_\lambda, \psi \rangle \, d\mathbb{P}(\lambda) = 0\). Since \(\tilde{\psi}\) is real valued we have

\[
C_n(\varphi, \tilde{\psi}) = \langle \mu, \varphi P^n_\tau(\tilde{\psi}) \rangle
\]

hence, we need to bound the quantity \(\|\langle \mu, \varphi P^n_\tau(\tilde{\psi}) \rangle\|\).

Now, by a straightforward calculation and \(\langle \mu, \tilde{\psi} \rangle = 0\) we see that

\[
P^n_\tau \tilde{\psi}(\lambda, x) = \int_\Omega \Lambda_{f_{n-1}} \circ \cdots \circ \Lambda_{f_0} \psi(x) \, d\mathbb{P}(\lambda)
\]

\[
= \int_\Omega (\Lambda_{f_{n-1}} \circ \cdots \circ \Lambda_{f_0} \psi(x) - \langle \mu_\lambda, \psi \rangle) \, d\mathbb{P}(\lambda)
\]

On the other hand, since \(\log \text{dist}(\cdot, \mathcal{M}) \in L^\infty(\mathbb{P}^N)\) by Remark 3.6, Corollary 3.7 and Lemma 4.5 for every \(\lambda \in \mathcal{B}\) we have

\[
\|\Lambda_{f_{n-1}} \circ \cdots \circ \Lambda_{f_0} \psi - \langle \mu_\lambda, \psi \rangle\|_{L^2_{\mu, \lambda}} \leq Cd^{-n}\|\psi\|_{DSH}
\]
where $C > 0$ does not depend on $\lambda$ or $n$. Then by Hölder's inequality, by (4.4) and from above argument we infer that

$$|C_n(\varphi, \tilde{\psi})| = |\langle \mu, \varphi P^n(\tilde{\psi}) \rangle|$$

$$\leq \|\varphi\|_{L^2}(\int_{\Omega} \|P^n(\tilde{\psi})\|_{L^2_{\mu}}^2 d\mu(\lambda))^\frac{1}{2}$$

$$= \|\varphi\|_{L^2}(\int_{\Omega} \|P^n(\tilde{\psi})\|_{L^2_{\mu,\lambda}}^2 d\mu(\lambda))^\frac{1}{2}$$

$$\leq Cd^{-n}\|\varphi\|_{L^2}\|\tilde{\psi}\|_{DSH}$$

where the forth equality follows from $\theta^*P = \mu$ and $P^n\tilde{\psi}$ does not depend on $\lambda$.

□

**Remark 4.8.** Note that we can also obtain strong mixing properties for Hölder continuous functions by using Lemma 4.6 and applying the above argument.

5. Stochastic Properties of $(X, B, \mu, \tau)$

5.1. **Central Limit Theorem.** In this section we prove a Central Limit theorem (CLT) for d.s.h and Hölder continuous observables. Our proof relies on verifying Gordin's condition.

**Gordin’s Method:** Let $(X, F, T, \alpha)$ be an ergodic dynamical system. We let $U : L^2_{\alpha}(X) \to L^2_{\alpha}(X)$ denote the unitary operator induced by $T$ and let $P := U^*$ be its adjoint operator. We denote the $\sigma$-algebra $F_n := T^{-n}(F)$ and let $E(\cdot | F_n)$ be the associated conditional expectation. Recall that $E(\phi| F_n)$ is the orthogonal projection of $\phi \in L^2_{\alpha}(X)$ onto closed subspace of $F_n$ measurable functions in $L^2_{\alpha}(X)$. Then it follows from an easy calculation that for $n \geq 0$

$$\|E(\phi| F_n)\|_{L^2_{\alpha}} = \|P^n\phi\|_{L^2_{\alpha}} \text{ and } E(\phi| F_n) = U^n P^n \phi$$

almost everywhere with respect to $\alpha$ restricted to $F_n$. We say that $\psi \in L^2_{\alpha}(X)$ is a coboundary if $\psi = u \circ T - u$ for some $u \in L^2_{\alpha}(X)$.

**Theorem 5.1.** [Gor69]

Let $\phi \in L^2_{\alpha}(X)$ be such that $\langle \alpha, \phi \rangle = 0$. Assume that

$$\sum_{n \geq 0} \|P^n\phi\|_{L^2_{\alpha}} < \infty$$

then the non-negative real number $\sigma$ defined by

$$\sigma^2 = \lim_{N \to \infty} \frac{1}{N} \int_X \left( \sum_{n=0}^{N-1} \phi \circ T^n \right)^2 d\alpha$$

is a finite number. Moreover, $\sigma > 0$ if and only if $\phi$ is not a coboundary. In this case,

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \phi \circ T^n \Rightarrow N(0, \sigma).$$

as $N \to \infty$.

**Proof of Theorem 1.4.** We will verify condition (5.1). To this end, it is enough to show that

$$\sum_{n \geq 0} \|P^n \tilde{\psi}\|_{L^2_{\mu}} < \infty$$
where $P_{\tau}$ is defined previously and by Proposition 4.7 we have
\[
P_{\tau}^n \tilde{\psi}(x) = \int_{\Omega} \Lambda_{f_{n-1}} \circ \cdots \circ \Lambda_{f_0} \psi(x) d\mathbb{P}(\lambda)
\]
Now, since $\langle \mu, \tilde{\psi} \rangle = 0$ by the argument in the proof of Theorem 1.3 we have
\[
\|P_{\tau}^n \tilde{\psi}\|_{L^2_\mu} \leq C d^{-n} \|\psi\|_{DSH}
\]
thus, the assertion follows. \qed

\textbf{Remark 5.2.} Note that applying the same reasoning and using Lemma 4.6, one can obtain CLT for H"older continuous functions.

6. A Markov chain associated with random pre-images

In this section we introduce a Markov chain associated with random pre-images of random holomorphic maps. We use the same notation as in previous sections. We consider $(\mathbb{P}^k, \mathcal{B}, \nu)$ as a probability space where $\mathcal{B}$ denotes the Borel algebra, $\nu := \pi_* \mu$ and $\pi: \Omega \times \mathbb{P}^k \to \mathbb{P}^k$ is the canonical projection. We let $\Lambda_f$ denote the Peron-Frobenius operator associated with $f \in \mathcal{H}_d(\mathbb{P}^k)$, precisely
\[
\Lambda_f(\phi)(x) = d^{-k} \sum_{\{y : f(y) = x\}} \phi(y)
\]
for $\phi \in L^2_\nu(\mathbb{P}^k)$. We define the transition probability by
\[
P: \mathbb{P}^k \times \mathcal{B} \to [0, 1]
\]
\[
P(x, G) := \int_{\mathbb{P}^k} \Lambda_f(\chi_G)(x) dm(f)
\]
\[
= \int_{\mathbb{P}^k} d^{-k} \sum_{y \in f^{-1}(x)} \delta_y(G) dm(f)
\]
where $\delta_y$ denotes the Dirac mass at $y$ and $\chi_G$ denotes the indicator function of $G$. First, we observe that $P(x, G)$ is well-defined. To this end it is enough to show that the map
\[
\mathcal{H}_d(\mathbb{P}^k) \to [0, 1]
\]
\[
f \to \Lambda_f(\chi_G)(x)
\]
is measurable. This follows from noting that for fixed $x \in \mathbb{P}^k$ as $f$ varies in $\mathcal{H}_d(\mathbb{P}^k)$ the solutions $y \in \mathbb{P}^k$ such that $f(y) = x$ vary continuously. The same reasoning shows that $x \to P(x, G)$ is a measurable map for every Borel set $G$. Moreover, $G \to P(x, G)$ defines a probability on $\mathbb{P}^k$. Thus, we may define the Markov operator on non-negative measurable functions by
\[
P\phi(x) := \int_{\mathbb{P}^k} \phi(y) P(x, dy)
\]
\[
= \int_{\mathbb{P}^k} \Lambda_f \phi(x) dm(f)
\]
which is again a non-negative measurable function. The following is a direct consequence of Theorem 3.1 and Theorem 1.3:

\textbf{Proposition 6.1.} \textit{The measure $\nu$ is an $P$-invariant ergodic measure.}

\textbf{Proof.} To prove invariance, we need to show that for every bounded measurable function $\phi$ on $\mathbb{P}^k$ we have
\[
\langle \nu, P\phi \rangle = \langle \nu, \phi \rangle.
\]
By definition of $\nu$ and Fubini's theorem we have
\[
\langle \nu, P\phi \rangle = \langle \mu, P\phi \circ \pi \rangle = \int_\Omega \int_{\mathbb{P}^k} \langle \mu_\lambda, \Lambda_f \phi \rangle dm(f) d\mathbb{P}(\lambda) = \int_\Omega \langle \mu_\theta(\lambda'), \phi \rangle d\mathbb{P}(\lambda') = \langle \mu, \phi \circ \pi \rangle = \langle \nu, \phi \rangle
\]
where $\lambda' = (f, \lambda)$. To prove ergodicity of $\nu$ we need to show that for every bounded measurable function $\phi$ on $\mathbb{P}^k$, $P\phi = \phi$ implies that $\phi \circ \pi$ is constant $\mu$-a.e. This follows from the strong mixing property proved in Theorem 1.3. \hfill \Box

Let $Y, \mathbb{P}_\nu, (Z_n)_{n \geq 0}$ and $\vartheta : Y \to Y$ be as previously defined. It follows from Proposition 6.1 that $\mathbb{P}_\nu$ is invariant and ergodic with respect to the shift $\vartheta$ hence, $(Z_n)_{n \geq 0}$ is stationary under $\mathbb{P}_\nu$. Thus, by Birkhoff’s ergodic theorem, for every $\phi \in L^1(\mathbb{P}_\nu, \nu)$ the series
\[
\frac{1}{N} \sum_{n=0}^{N-1} \phi(Z_n) = \frac{1}{N} \sum_{n=0}^{N-1} \phi(Z_0 \circ \vartheta^n)
\]
converges to $\langle \nu, \phi \rangle$, for $\mathbb{P}_\nu$-a.e. $y \in Y$. We say that $\phi$ satisfies Central Limit Theorem (CLT) for the Markov chain $(Z_n)_{n \geq 0}$ if $\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \phi(Z_n)$ converges in law under the invariant measure $\mathbb{P}_\nu$ to the normal distribution $N(0, \sigma^2)$ for some $\sigma > 0$. The following result is a consequence of [GL78]:

**Theorem 6.2.** If $\phi = g - Pg$ for some $g \in L^2(\mathbb{P}_\nu)$ then
\[
\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \phi(Z_n) \Rightarrow N(0, \sigma^2)
\]
where $\sigma^2 = \int_{\mathbb{P}_\nu} (g^2 - (Pg)^2) d\nu$. Moreover, if $\sigma = 0$ then the partial sum converges to Dirac mass at 0.

**Proof of Theorem 1.5.** Note that the condition in the Theorem 6.2 is satisfied if
\[
\sum_{j \geq 0} \|P^j \psi\|_{L^2_\nu} < \infty.
\]
Indeed, if $g := \sum_{j \geq 0} P^j \psi$ converges in $L^2(\mathbb{P}_\nu)$ then $\psi = g - Pg$.

Now, by $\langle \nu, \psi \rangle = 0$ we have
\[
P^j \psi(x) = \int_{\Omega} \Lambda_j \cdots \Lambda_1 \Lambda_0(\psi)(x) d\mathbb{P}(\lambda) = \int_{\Omega} \Lambda_j \cdots \Lambda_1 \Lambda_0(\psi)(x) - \langle \mu_\lambda, \psi \rangle d\mathbb{P}(\lambda)
\]
Thus, by Remark 3.6, Corollary 3.7 and Lemma 4.5 we have
\[
\|\Lambda_{n-1} \cdots \Lambda_1 \Lambda_0(\psi) - \langle \mu_\lambda, \psi \rangle\|_{L^2(\mu_{\lambda,n})} \leq C d^{-n} \|\psi\|_{DSH}^{\mu_{\lambda}}
\]
On the other hand by invariance property $\theta^* \mathbb{P} = \mathbb{P}$.
\[ \|P_j \psi\|^2_{L^2} = \int_{\Omega} \|P_j \psi\|^2_{L^2_{\mu,\lambda}} d\mathbb{P}(\lambda) \]
\[ = \int_{\Omega} \|P_j \psi\|^2_{L^2_{\mu,\lambda,n}} d\mathbb{P}(\lambda) \]
\[ \leq Cd^{-n} \|\psi\|_{DSH} \]
where the second line follows from \( \theta_* \mathbb{P} = \mathbb{P} \) and \( P_j \psi \) does not depend on \( \lambda \). \( \square \)

**Remark 6.3.** Applying the same argument and using Lemma 4.6 one can obtain CLT for Hölder continuous observables.

**References**

[BD01] Jean-Yves Briend and Julien Duval, *Deux caractérisations de la mesure d’équilibre d’un endomorphisme de \( P^k(C) \)*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 93, 145–159. MR 1863737 (2002k:32027)

[Bil12] P. Billingsley, *Probability and measure*, vol. 939, Wiley, 2012.

[Bro65] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. 6 (1965), 103–144 (1965). MR 0194595 (33 #2805)

[CLB05] Serge Cantat and Stéphane Le Borgne, *Théorème limite central pour les endomorphismes holomorphes et les correspondances modulaires*, Int. Math. Res. Not. (2005), no. 56, 3479–3510. MR 2200586 (2007c:60024)

[DD04] T.-C. Dinh and C. Dupont, *Dimension de la mesure d’équilibre d’applications méromorphes*, J. Geom. Anal. 14 (2004), no. 4, 613–627. MR 2111420 (2006k:37117)

[Dem09] J.-P. Demailly, *Complex analytic and differential geometry*, http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf, 2009.

[DN12] T.C. Dinh and V.A. Nguyen, *Characterization of monge-ampere measures with holder continuous potentials*, arXiv preprint arXiv:1204.4883 (2012).

[DNS10] T.C. Dinh, V.A. Nguyê’n, and N. Sibony, *Exponential estimates for plurisubharmonic functions*, Journal of Differential Geometry 84 (2010), no. 3, 465–488.

[DS03] T.-C. Dinh and N. Sibony, *Dynamique des applications d’allure polynomiale*, J. Math. Pures Appl. (9) 82 (2003), no. 4, 367–423. MR 1992375 (2004e:37063)

[DS05] Tien-Cuong Dinh and Nessim Sibony, *Equidistribution for meromorphic transforms and the \( dd^c \)-method*, Science in China Series A: Mathematics 48 (2005), 180–194.

[DS06a] T.-C. Dinh and N. Sibony, *Decay of correlations and the central limit theorem for meromorphic maps*, Comm. Pure Appl. Math. LIX (2006), no. 0754–0768, 754–768.

[DS06b] T.-C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. 81 (2006), no. 1, 221–258. MR 2208805 (2007i:32017)

[DS07] Tien-Cuong Dinh and Nessim Sibony, *Pull-back of currents by holomorphic maps*, manuscripta mathematica 123 (2007), no. 3, 357–371.

[DS09] T.-C. Dinh and N. Sibony, *Super-potentials of positive closed currents, intersection theory and dynamics*, Acta Math. 203 (2009), no. 1, 1–82. MR 2545825 (2011b:32052)

[dT12a] H. de Thelin, *Endomorphismes aleatoires dans les espaces projectifs i*, arXiv preprint arXiv:1205.1601 (2012).

[dT12b] , *Endomorphismes aleatoires dans les espaces projectifs ii*, arXiv preprint arXiv:1209.3597 (2012).

[Dup10] Christophe Dupont, *Bernoulli coding map and almost sure invariance principle for endomorphisms of \( P^k \)*, Probab. Theory Related Fields 146 (2010), no. 3-4, 337–359. MR 2574731 (2010k:37088)

[FS95] J. E. Fornaess and N. Sibony, *Complex dynamics in higher dimension. II*, Modern methods in complex analysis (Princeton, NJ, 1992), Ann. of Math. Stud., vol. 137, Princeton Univ. Press, Princeton, NJ, 1995, pp. 135–182. MR 1369137 (97g:32033)

[FW00] John Erik Fornæss and Brendan Weickert, *Random iteration in \( P^k \)*, Ergodic Theory Dynam. Systems 20 (2000), no. 4, 1091–1109. MR 1779395 (2001j:32016)

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994. MR 1264417 (95e:14045)

[GL78] M. I. Gordin and B. A. Lifšic, *Central limit theorem for stationary Markov processes*, Dokl. Akad. Nauk SSSR 239 (1978), no. 4, 766–767. MR 0501277 (58 #18672)

[Gor69] M. I. Gordin, *The central limit theorem for stationary processes*, Dokl. Akad. Nauk SSSR 188 (1969), 739–741. MR 0251785 (40 #5012)

[GS90] Henri Gillet and Christophe Soulé, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. (1990), no. 72, 93–174 (1991). MR 1087394 (92d:14016)
[HP94] J. H. Hubbard and P. Papadopol, *Superattractive fixed points in $\mathbb{C}^n$*, Indiana Univ. Math. J. **43** (1994), no. 1, 321–365. MR 1275463 (95e:32025)

[Jon99] Mattias Jonsson, *Dynamics of polynomial skew products on $\mathbb{C}^2$*, Math. Ann. **314** (1999), no. 3, 403–447. MR 1704543 (2000e:32025)

[Jon00] Mattias Jonsson, *Ergodic properties of fibered rational maps*, Ark. Mat. **38** (2000), no. 2, 281–317. MR 1785403 (2002k:37073)

[Kif86] Yuri Kifer, *Ergodic theory of random transformations*, Progress in Probability and Statistics, vol. 10, Birkhäuser Boston Inc., Boston, MA, 1986. MR 884892 (89c:58069)

[Lju83] M. Ju. Ljubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 351–385. MR 741393 (85k:58049)

[Meo96] M. Meo, *Image inverse d’un courant positif fermé par une application analytique surjective*, C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 12, 1141–1144. MR 1396655 (97d:32013)

[Pet05] Han Peters, *Non-autonomous dynamics in $\mathbb{P}^k$*, Ergodic Theory Dynam. Systems **25** (2005), no. 4, 1295–1304. MR 2158406 (2006b:37085)

[RS97] A. Russakovskii and B. Shiffman, *Value distribution for sequences of rational mappings and complex dynamics*, Indiana Univ. Math. J. **46** (1997), no. 3, 897–932. MR 1488341 (98h:32046)

[Sib99] N. Sibony, *Dynamique des applications rationnelles de $\mathbb{P}^k$*, Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, pp. ix–x, xi–xii, 97–185. MR 1760844 (2001c:32026)

[Sum00] Hiroki Sumi, *Skew product maps related to finitely generated rational semigroups*, Nonlinearity **13** (2000), no. 4, 995–1019. MR 1767945 (2001g:37060)

Mathematics Department, Johns Hopkins University 21218 Maryland, USA

*E-mail address*: bayraktar@jhu.edu