A WEAK CHEVALLEY-WARNING THEOREM FOR QUASI-FINITE FIELDS

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Abstract. There exists a function $f: \mathbb{N} \to \mathbb{N}$ such that for every positive integer $d$, every quasi-finite field $K$ and every projective hypersurface $X$ of degree $d$ and dimension $\geq f(d)$, the set $X(K)$ is non-empty. This is a special case of a more general result about intersections of hypersurfaces of fixed degree in projective spaces of sufficiently high dimension over fields with finitely generated Galois groups.

1. Introduction

The Chevalley-Warning theorem [7, I, Th. 3] asserts that every finite field $K$ is $C_1$, i.e., for every $n \in \mathbb{N}$ and every degree $n$ hypersurface $X$ of dimension $\geq n$, the set $X(K)$ is non-empty. One might ask whether the Chevalley-Warning theorem extends to all quasi-finite fields (i.e., perfect fields with Galois group isomorphic to $\hat{\mathbb{Z}}$). Ax [1], answering a question of Serre [6, II, §3], gave an example of a quasi-finite field which is not $C_n$ for any $n$.

In this note, we prove a weak version of Chevalley-Warning, as follows:

Theorem 1. There exists a function $f: \mathbb{N} \to \mathbb{N}$ such that for every positive integer $d$, every quasi-finite field $K$, and every projective hypersurface $X$ of degree $d$ and dimension $\geq f(d)$, the set $X(K)$ is non-empty.

The proof can in principle be used to produce a function $f$, but it has very rapid growth. Of course, we have already seen that $f(d)$ cannot have polynomial growth.

Theorem 1 is a special case of the following more general result.

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Theorem 2. There exists a function \( h : \mathbb{N}^4 \to \mathbb{N} \) such that given:

- any perfect field \( K \), not formally real, such that \( G_K \) has a generating set with \( g \) elements,
- any positive integers \( d, e, k \),
- any sequence \( d_1, \ldots, d_e \) of positive integers \( \leq d \),
- any vector space \( V \) over \( K \) of dimension \( n + 1 > h(d, e, g, k) \),
- any sequence of forms \( F_i \in \text{Sym}^{d_1} V^*, \ldots, F_e \in \text{Sym}^{d_e} V^* \),

there exists a subspace \( W \subset V \) of dimension \( k \) on which all the forms \( F_i \) are identically zero.

Specializing to \( k = 1 \), we get the following:

Theorem 3. There exists a function \( h : \mathbb{N}^3 \to \mathbb{N} \) such that if \( K \) is any perfect field, not formally real, such that \( G_K \) has a generating set with \( g \) elements; \( d \) and \( e \) are positive integers; and \( X \) is an intersection of \( e \) degree \( \leq d \) hypersurfaces in \( \mathbb{P}^n \) for \( n \geq h(d, e, g) \), then \( X(K) \) is non-empty.

We remark that there are interesting examples of fields \( K \) for which \( G_K \) is finitely generated but not abelian. For instance, it is known [4, Th. 7.5.10] that every \( p \)-adic field has a finitely generated Galois group. For \( K = \mathbb{Q}_p, p > 2 \), we can take \( g = 4 \). For this family of fields, Theorem 3 is due to Schmidt [5].

Specializing Theorem 3 to the case \( e = g = 1 \), we obtain Theorem 1, since by definition, quasi-finite fields are perfect, and a quasi-finite field cannot be formally real. Indeed, the Brauer group of a quasi-finite field is trivial [8, XIII, Prop. 5]. Therefore, the Severi-Brauer curve \( x^2 + y^2 + z^2 = 0 \) ([8, X, §7, Ex. (e)]) has a rational point over \( K \), which means that \(-1\) is a sum of two squares in \( K \). Thus \( K \) satisfies the hypotheses of Theorem 3.

The idea of the proof of Theorem 2 is to use a polarization argument due to Brauer [2] to reduce to the case of diagonal forms. By a Galois cohomology argument, we can further reduce to the case of Fermat hypersurfaces. These can be treated using identities introduced by Hilbert in his work on Waring’s problem.

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We begin with Fermat hypersurfaces

\[ F_{d,n} : x_0^d + x_1^d + \cdots + x_n^d = 0. \]

Theorem 4. If \( d \) is a positive integer and \( K \) is a field, not formally real, such that \( K_d := K^{\times}/(K^{\times})^d \) is finite, then \( F_{d,n}(K) \) is non-empty whenever \( n \geq |K_d| \).
Proof. Let
\[ \Sigma_{d,r} := \left( (K^\times)^d + \cdots + (K^\times)^d \right) \cup \{0\} \]
Then \( \Sigma_{d,r} \) is stable by multiplication by \((K^\times)^d\) and is therefore characterized by the image in \( K_d \) of its non-zero elements. As \( \Sigma_{d,r} \subset \Sigma_{d,r+1} \), it follows that the sequence must stabilize, and every sum of \( d \)th powers of elements of \( K \) can be expressed as a sum of at most \(|K_d|\) such powers.

It remains to show that for every positive integer \( d \), \(-1\) is a sum of \( d \)th powers. If the characteristic of \( K \) is positive, this is obvious, so we assume that \( \text{char } K = 0 \). By hypothesis, \(-1\) is a sum of squares in \( K \). This implies that \( \Sigma_{2,r} = K \) for \( r \gg 0 \). Indeed, the condition \( a_1^2 + \cdots + a_n^2 = -1 \) implies
\[ \left( \frac{c+1}{2} \right)^2 + \sum_{i=1}^n \left( a_i \left( \frac{c-1}{2} \right) \right)^2 = c. \]
For odd \( d \), it is obvious that \(-1\) is a sum of \( d \)th powers. If for some \( d \) and some \( r \), \(-1 \in \Sigma_{d,r} \), then we can write
\[ -1 = \sum_{i=1}^r \left( \sum_{j=1}^{\left| K_d \right|} a_{i,j}^2 \right)^d. \]
By Hilbert’s identity [3 Th. 3.4], this implies that \(-1\) is a sum of \( 2d \)th powers of certain \( \mathbb{Q} \)-linear combinations of the \( a_{i,j} \). The theorem follows by induction on the largest power of 2 dividing \( d \).

Next we consider diagonal hypersurfaces in general.

Lemma 5. Let \( G \) be a profinite group and \( H < G \) an open subgroup. If \( G \) can be topologically generated by \( g \) elements, then \( H \) can be topologically generated by \( 1 + [G : H](g - 1) \) elements.

Proof. By compactness, it suffices to prove this when \( G \) and \( H \) are finite groups. Let \( G' \) denote a free group on \( g \) elements and \( \pi : G' \to G \) a surjective homomorphism. Let \( H' := \pi^{-1}(H) \). Thus \( H' \) is a subgroup of \( G' \) of index \([G : H]\). Identifying \( G' \) with the fundamental group of a join \( X \) of \( g \) circles, the quotient of the universal cover of \( X \) by \( H' \) is a covering space of \( X \) of degree \([G' : H']\) and therefore a finite, connected, 1-dimensional CW-complex \( Y \). Thus \( H' \) is free, and the number of its generators is the rank of \( H_1(Y, \mathbb{Z}) \). We have
\[ \chi(Y) = [G' : H'] \chi(X) = [G : H] \chi(X) = [G : H](g - 1), \]
where \( \chi \) denotes the Euler characteristic. Thus, \( H' \) has \( 1 + [G : H](g-1) \) generators, and the same is true of its quotient \( H \). \( \square \)

**Proposition 6.** If \( K \) is a perfect field such that \( G_K \) can be generated by \( g \) elements and \( d \) is a positive integer, then

\[
|K_d| \leq d^{dg+1}.
\]

**Proof.** Let \( K_d \) denote the extension of \( K \) generated by \( \mu_d \), the group of solutions of \( x^d = 1 \) in \( \bar{K} \). We have Kummer isomorphisms

\[
K^\times / (K^\times)^d \cong H^1(K, \mu_d)
\]

and

\[
K_d^\times / (K_d^\times)^d \cong H^1(K_d, \mu_d) \cong \text{Hom}(G_{Kd}, \mu_d)
\]

Now, \( G_{Kd} \) is the kernel of the homomorphism \( G_K \to \text{Gal}(K_d/K) \) whose image has order \( \leq d-1 \). Applying Lemma 5 to \( G_{Kd} \subset G_K \), we see that

\[
|\text{Hom}(G_{Kd}, \mu_d)| \leq d^{d-1(g-1)+1}.
\]

The order of the cohomology group \( H^1(\text{Gal}(K_d/K), \mu_d) \) is bounded above by the number of 1-cochains. By the inflation-restriction sequence,

\[
|K^\times / (K^\times)^d| \leq |H^1(\text{Gal}(K_d/K), \mu_d)| \cdot |H^1(K_d, \mu_d)| \leq d^{dg+1}.
\]

The proposition follows. \( \square \)

We can now prove Theorem 2.

**Proof.** By [2, Th. C], it suffices to find a bound \( N \), depending only on \( d \) and \( g \), such that every diagonal homogeneous form of degree \( d \) in \( K \) in more than \( N \) variables has a solution in \( K \). Let \( n = d^{dg+1} \), \( N = n^2 \), and

\[
F := a_0 x_0^d + \cdots + a_N x_N^d
\]

a diagonal form. If any coefficients \( a_i \) is zero, then \( F \) has an obvious non-trivial solution. Otherwise at least \( n + 1 \) of the \( a_i \) must belong to a single coset \( a(K^\times)^d \). Assuming without loss of generality that \( a_0, \ldots, a_n \in a(K^\times)^d \) and letting \( (b_0, \ldots, b_n) \) denote a non-trivial solution of \( x_0^d + \cdots + x_n^d = 0 \) in \( K \), the non-zero vector

\[
(b_0, \ldots, b_n, 0, \ldots, 0)
\]

is a solution of \( F \). \( \square \)

We conclude with two examples, to show that the hypotheses on \( K \) are really needed.

If \( K = \mathbb{R} \) and \( F := x_1^2 + x_2^2 + \cdots + x_n^2 \), there are no non-trivial solutions regardless of how large \( n \) may be.
If $K$ is a separable closure of the infinite-dimensional function field $\mathbb{F}_p(t_1, t_2, \ldots)$, then
\[
F := t_1 x_1^p + t_2 x_2^p + \cdots + t_n x_n^p = 0
\]
has no solutions regardless of how large $n$ may be since the $t_i$ are linearly independent over $\mathbb{F}_p(t_1^p, t_2^p, \ldots)$.

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