On BF-type higher-spin actions in two dimensions

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ABSTRACT: We propose an interacting higher-spin theory in two dimensions for an infinite multiplet of massive scalar fields coupled to infinitely many topological higher-spin gauge fields. The action functional is of BF type and generalizes the known higher-spin Jackiw-Teitelboim gravity via an extension of the standard higher-spin algebra.
1 Introduction

The problem of constructing interacting theories of higher-spin (HS) gauge fields is notoriously difficult, especially at the level of the action (see e.g. [1, 2] for introductory reviews). In fact, in dimensions four and higher the examples of fully nonlinear actions compatible with the minimal coupling to the spin-two subsector are pretty scarce although such cubic interaction vertices are known since a long time [3, 4]. On the one hand, for conformal HS gravity there exists a perturbatively local action [5, 6] (see also [7]) in any even dimension, whose low-spin truncation gives Maxwell and Weyl actions. Unfortunately, this action expanded around conformally flat background is higher-derivative and thereby clashes with pertubative unitary. On the other hand, nonlinear equations [8] of four-dimensional HS (super)gravity are known since several decades (and their higher-dimensional bosonic analogue [9] since more than a decade) but it was only recently that action functionals were proposed [10, 11] (see also the review [12]) as an off-shell formulation of minimal bosonic four-dimensional HS gravity. The action principles from [10, 11] share the unusual property of being formulated in terms of differential forms on a base space of higher dimension than the spacetime manifold itself. Another example of complete HS action is given by four-dimensional chiral HS gravity (i.e. the HS extension of self-dual Yang-Mills and self-dual gravity) in the light-cone formulation, both in flat [13, 14] and anti de Sitter [15] spacetimes. However, note that this action is real only in Euclidean signature.

In dimensions three and two, the situation simplifies drastically because HS gauge fields become topological. In the frame-like formulation, HS gravity theories without matter are the smooth generalizations of their spin-two counterparts. In the absence of matter, HS gauge field are described, on-shell, by a flat connection taking values in the HS algebra and, off-shell,
by either a Chern-Simons (CS) action in three dimensions or by a BF action in two dimensions. More precisely, the HS extension of CS gravity with a negative cosmological constant \([16, 17]\) (respectively, of CS conformal gravity \([18]\)) was provided in \([19, 20]\) (respectively, in \([21, 22]\) for the conformal case) while the HS extension of Jackiw-Teitelboim gravity \([23–25]\) was proposed in \([26–28]\) (see \([29, 30]\) for the (super)conformal case).

Let us stress that, in three dimensions, the inclusion of matter (in the fundamental representation of the HS algebra) is known to reinstate the same level of intricacy as in four dimensions. However, in two dimensions the same degree of simplicity happens to be preserved if one includes a specific type of matter (motivated by holography): infinitely many scalar fields with fine-tuned masses spanning the “twisted-adjoint” representation of the HS algebra.

The HS symmetry in two (or three) dimensions is identified with the Lie algebra \(\mathfrak{hs}[\lambda]\), originally introduced in \([31, 32]\) (or, respectively, two copies thereof), where \(\lambda\) is a non-negative real parameter. For integer values \(\lambda = N \in \mathbb{N}\), this HS algebra can be truncated to the finite-dimensional algebra \(\mathfrak{sl}(N, \mathbb{R})\) and the corresponding spectrum is spanned by gauge fields of spin 2, 3, ..., \(N\). However, the holographic duals of such type-N higher-spin gravity theories appear to be non-unitary conformal field theories (CFT).

In any dimension, the HS algebra completely defines the kinematics of HS gravity through several of its representations: singleton, adjoint, and twisted-adjoint. In the two-dimensional case at hand, the singleton module encodes the boundary CFT\(_1\) fundamental degrees of freedom while the adjoint and twisted-adjoint modules lead, in the two-dimensional bulk, to topological HS fields and massive scalar fields respectively \([33]\). Extending the original HS algebra \(\mathfrak{hs}[\lambda]\), via a product with the group \(Z_2\) generated by an involutive automorphism (called “twist”), allows one to unify the gravity and matter sector into a single framework. Moreover, the BF-type action associated to this extended HS algebra provides a natural extension of the HS Jackiw-Teitelboim gravity, the linearization of which reproduces the correct equations of motion for topological and local modes dictated by symmetries. This is our main result. Note that, contrary to standard BF theories, this BF-type higher-spin theory is not purely topological but includes propagating matter fields. This is possible because the gauge algebra is a subtle smash product of an infinite-dimensional Lie algebra with a finite group, thereby producing fields in the twisted-adjoint representation of \(\mathfrak{hs}[\lambda]\). The latter representation decomposes into infinite-dimensional irreducible representations of AdS\(_2\) isometry group corresponding to matter fields with local degrees of freedom.

The paper is organized as follows. In Section 2, we review the HS symmetry algebra in two dimensions and its representations. In Section 3, we discuss HS-invariant equations of motion for gauge and matters fields. In Section 4 the HS extension of Jackiw-Teitelboim gravity is reviewed. Section 5 defines the extended HS algebra and considers the corresponding BF-type theory. Concluding remarks are given in Section 6.
Consider the universal enveloping algebra $\mathcal{U}(\mathfrak{so}(2,1))$ of the isometry algebra $\mathfrak{so}(2,1)$ of two-dimensional anti-de Sitter spacetime AdS$_2$, and its ideal

$$\mathcal{I} = \left( C_2 - \frac{1}{4}(\lambda^2 - 1) \right) \mathcal{U}(\mathfrak{so}(2,1)) \quad (2.1)$$

generated by the eigenvalue $\frac{1}{4}(\lambda^2 - 1)$ (for $\lambda \in \mathbb{R}$) of the quadratic Casimir element $C_2 \in \mathcal{U}(\mathfrak{so}(2,1))$. The quotient

$$\text{Mat}[\lambda] = \mathcal{U}(\mathfrak{so}(2,1))/\mathcal{I} \quad (2.2)$$

is an associative algebra which, for generic $\lambda \in \mathbb{R}$, is an infinite-dimensional analogue of the finite-dimensional associative algebra $\text{Mat}(N, \mathbb{R})$ of $N \times N$ matrices, as emphasized by our choice of notation. Moreover, for integer $\lambda = N \in \mathbb{N}$, the algebra (2.2) contains an infinite-dimensional ideal $\mathcal{J}_N$ to be factored out, and $\text{Mat}[N]/\mathcal{J}_N \cong \text{Mat}(N, \mathbb{R})$. The space $\mathcal{U}(\mathfrak{so}(2,1))/\mathcal{I}$ endowed with the commutator as Lie bracket, is a reductive Lie algebra, which is often denoted $\mathfrak{gl}[\lambda]$ because, for generic $\lambda \in \mathbb{R}$, it is an infinite-dimensional analogue of the general linear algebra $\mathfrak{gl}(N, \mathbb{R})$ [31]. Note that the enveloping algebra of the so-called “Wigner deformed oscillator algebra” provides a useful realization of $\text{Mat}[\lambda]$ [32].

**Higher-spin algebra.** The centre of $\mathcal{U}(\mathfrak{so}(2,1))$ is spanned by the polynomials in the quadratic Casimir element $C_2$. Accordingly, the centre of $\text{Mat}[\lambda]$ is the one-dimensional subalgebra $Z \cong \mathbb{R}$, which is what remains of the centre of $\mathcal{U}(\mathfrak{so}(2,1))$ after quotienting the ideal (2.1). Its Lie algebra counterpart forms a $\mathfrak{u}(1)$ ideal of $\mathfrak{gl}[\lambda]$. The Lie algebras of HS

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### Table 1. Relevant algebras

| Algebra (parameter) | Infinite-dim. $(\lambda \in \mathbb{R}_{>0} \cap \mathbb{N})$ | Finite-dim. $(N \in \mathbb{N})$ | Centre | Trace $(\lambda \in \mathbb{R}_{>0})$ |
|---------------------|-------------------------------------------------|---------------------------------|--------|----------------------------------|
| associative         | $\text{Mat}[\lambda]$                          | $\text{Mat}(N, \mathbb{R})$    | $Z \cong \mathbb{R}$         | projection on $Z$ |
| reductive Lie       | $\mathfrak{gl}[\lambda]$                       | $\mathfrak{gl}(N, \mathbb{R})$ | $\mathfrak{u}(1)$            | projection on $\mathfrak{u}(1)$ |
| simple Lie          | $\mathfrak{hs}[\lambda]$                       | $\mathfrak{sl}(N, \mathbb{R})$ | 0                               | traceless |
| associative         | $\text{Mat}[\lambda] \rtimes \mathbb{Z}_2$    | $\text{Mat}(N, \mathbb{R}) \rtimes \mathbb{Z}_2$ | $Z \cong \mathbb{R}$         | projection on $Z$ |
| quadratic Lie       | $\mathfrak{gl}[\lambda] \rtimes \mathbb{Z}_2$ | $\mathfrak{gl}(N, \mathbb{R}) \rtimes \mathbb{Z}_2$ | $\mathfrak{u}(1)$            | projection on $\mathfrak{u}(1)$ |
| centerless Lie      | $\mathfrak{ch}_s[\lambda]$                     | $\mathfrak{esl}(N, \mathbb{R})$ | 0                               | traceless |

1 A Lie algebra is said reductive if it is a direct sum of an abelian ideal and a semisimple subalgebra.
symmetries in two dimensions are traditionally defined by subtracting the one-dimensional Abelian ideal,

$$\mathfrak{gl}[\lambda] = \mathfrak{u}(1) \oplus \mathfrak{hs}[\lambda],$$

so that $\mathfrak{hs}[\lambda]$ is an infinite-dimensional analogue of $\mathfrak{sl}(N, \mathbb{R})$. The structure of $\mathfrak{hs}[\lambda]$ was described in the papers [31, 32], from which one may extract the following relevant facts: The Lie algebra $\mathfrak{hs}[\lambda]$ always contains $\mathfrak{so}(2, 1)$ as a subalgebra. Moreover, $\mathfrak{hs}[\lambda]$ is simple if and only if $\lambda \notin \mathbb{N}$. The Lie algebra $\mathfrak{hs}[N]$ contains an infinite-dimensional ideal $J_N$ to be factored out and the corresponding quotient is finite-dimensional, $\mathfrak{hs}[N]/J_N \cong \mathfrak{sl}(N, \mathbb{R})$ [31, 32]. Consequently, the family of Lie algebras of HS symmetries in two dimensions that are simple and that allow for unitary representations, is $\mathfrak{hs}[\lambda]$ for $\lambda \notin \mathbb{N}$. The other algebras in the upper half of Table 1 are useful auxiliary tools (e.g. the associative algebras) or illustrative toy models (e.g. the finite-dimensional algebras) but the kinematics of pure HS gravity theories in two dimensions is determined by the one-parameter family of Lie algebras $\mathfrak{hs}[\lambda]$ and representations thereof.

**Twist automorphism.** Let us consider basis elements $T_A = (P_a, L)$ of the Lie algebra $\mathfrak{so}(2, 1)$, where $A = 0, 1, 2$ and $a = 0, 1$. They have been split into transvection generators $P_a$ and Lorentz generator $L$. One can introduce the involutive automorphism $\tau$ of $\mathfrak{so}(2, 1)$ acting as $\tau(P_a) = -P_a$ and $\tau(L) = L$. This automorphism can be promoted to the whole algebra $\mathcal{U}(\mathfrak{so}(2, 1))$ by the associativity and by setting $\tau(1) = 1$. The Casimir element $C_2 = \frac{1}{2}T_AT^A = P^aP_a + L^2$ is left invariant by $\tau$, therefore the automorphism $\tau$ of $\mathcal{U}(\mathfrak{so}(2, 1))$ descends to an automorphism of both $\mathfrak{Mat}[\lambda]$, $\mathfrak{gl}[\lambda]$ and $\mathfrak{hs}[\lambda]$, in which cases it is called “twist” (see e.g. [33, 34] for reviews). Moreover, the ideals $J_N$ mentioned above are also $\tau$-invariant so that the twist consistently descend to the finite-dimensional algebras $\mathfrak{Mat}(N, \mathbb{R})$, $\mathfrak{gl}(N, \mathbb{R})$ and $\mathfrak{sl}(N, \mathbb{R})$ as well.

**Adjoint vs twisted-adjoint representations.** Let us review two important representations of $\mathfrak{gl}[\lambda]$ on itself. Firstly, as any Lie algebra $\mathfrak{gl}[\lambda]$ acts on itself via the adjoint action,

$$^*\text{ad}_y(a) = [y, a]_* := y \ast a - a \ast y, \quad \forall y, a \in \mathfrak{gl}[\lambda],$$

where $\ast$ stands for the associative product in $\mathfrak{Mat}[\lambda]$. The same holds for its subalgebra $\mathfrak{hs}[\lambda]$. Secondly, the twisted-adjoint action of $\mathfrak{gl}[\lambda]$ on itself is defined as

$$^\tau\text{ad}_y(a) = [y, a]_{\tau} := y \ast a - a \ast \tau(y), \quad \forall y, a \in \mathfrak{gl}[\lambda].$$

Specifying $y \in \mathfrak{hs}[\lambda]$ defines the twisted-adjoint action of the higher-spin algebra $\mathfrak{hs}[\lambda]$ on the linear space of $\mathfrak{gl}[\lambda]$.

Restricting these two actions to elements $y \in \mathfrak{so}(2, 1) \subset \mathfrak{gl}[\lambda]$, we obtain the adjoint and twisted-adjoint actions of $\mathfrak{so}(2, 1)$ on $\mathfrak{gl}[\lambda]$, denoted respectively as $\mathcal{T} := ^*\text{ad}_T$ and $\mathcal{I} := ^\tau\text{ad}_T$. The two corresponding $\mathfrak{so}(2, 1)$-modules are infinite-dimensional and reducible. They can be decomposed into irreducible submodules of $\mathfrak{so}(2, 1)$ which are finite-dimensional ("Killing")
modules for the adjoint action and infinite-dimensional (“Weyl”) modules for the twisted-
adjoint action. The latter modules are in fact Verma modules of $\mathfrak{so}(2,1)$ with running weights expressed in terms of $\lambda$ (see [33] for details).

3 Linearized higher-spin equations in two dimensions

Let $M_2$ be a two-dimensional spacetime manifold. The fields are differential $p$-forms ($p = 0,1,2$) taking values in the vector space $\mathfrak{gl}[\lambda]$. These differential $p$-forms will be denoted accordingly as $X_{[p]} \in \Omega^p(M_2) \otimes \mathfrak{gl}[\lambda]$. A differential 1-form $X_{[1]} \in \Omega^1(M_2) \otimes \mathfrak{gl}[\lambda]$ will be called a (Cartan) connection 1-form if its $\mathfrak{so}(2,1)$ piece $X_{[1]}^A T_A = e^A_{[1]} P^a + \omega^A_{[1]} L$ is such that the components $e^a_{[1]}$ along the transvection generators define a non-degenerate zweibein.

In particular, let $W_{[1]} \in \Omega^1(M_2) \otimes \mathfrak{so}(2,1)$ be an $\mathfrak{so}(2,1)$-valued connection 1-form. The respective (twisted-)adjoint covariant derivatives read

$$\nabla = d + W_{[1]}^A T_A , \quad \tilde{\nabla} = d + W_{[1]}^A \bar{T}_A ,$$

where $d$ is the de Rham differential, while $T_A$ and $\bar{T}_A$ are basis elements of $\mathfrak{so}(2,1)$ in the (twisted-)adjoint representations (2.4) and (2.5).

Both squared covariant derivatives yield the curvature 2-form $R_{[2]} \in \Omega^2(M_2) \otimes \mathfrak{so}(2,1)$ as follows

$$\nabla^2 = R_{[2]}^A T_A , \quad \tilde{\nabla}^2 = R_{[2]}^A \bar{T}_A ,$$

where

$$R_{[2]}^A = dW_{[1]}^A + \frac{1}{2} \epsilon^{ABC} W_{[1]}^B \wedge W_{[1]}^C ,$$

and $\epsilon^{ABC}$ stands for the $\mathfrak{so}(2,1)$ Levi-Civita tensor.

From now on, we will assume that the connection 1-form $W_{[1]}$ solves the zero-curvature condition $R_{[2]}^A = 0$, thereby defining AdS$_2$ spacetime (locally). Then, we can introduce the following covariant constancy equations

$$\nabla \Omega_{[1]} \equiv (d + W_{[1]}^A T_A) \wedge \Omega_{[1]} = 0 ,$$

for the adjoint-valued 1-form field $\Omega_{[1]} \in \Omega^1(M_2) \otimes \mathfrak{gl}[\lambda]$, and

$$\tilde{\nabla} C_{[0]} \equiv (d + W_{[1]}^A \bar{T}_A) C_{[0]} = 0 ,$$

for the twisted-adjoint-valued 0-form field $C_{[0]} \in \Omega^0(M_2) \otimes \mathfrak{gl}[\lambda]$.

The first equation, (3.4), describes free topological HS fields that are pure gauge and thus do not carry local degrees of freedom. The second equation, (3.5), describes an infinite tower of free massive scalar fields with ascending masses $[33]$

$$\left( \Box_{\text{AdS}} + m^2_n \right) \varphi_n = 0 , \quad m^2_n = \frac{(n-\lambda)(n-\lambda+1)}{R^2_{\text{AdS}}} , \quad n = 0,1,2,\ldots ,$$

(3.6)
where $\Box_{AdS_2}$ is the wave operator on the AdS$_2$ spacetime of curvature radius $R_{AdS}$\footnote{A single massive scalar on constant curvature spaces is known to be described by such twisted-adjoint equations (also known as unfolded equations), see e.g. \cite{35}.}. The space of states of each massive scalar field spans a Verma module of $so(2,1)$ with lowest energy $\Delta_n$ such that $m_n^2 = \Delta_n(\Delta_n - 1)$, or, equivalently, spans the particular irreducible module under the twisted-adjoint action of $so(2,1)$ discussed above.

### 4 Higher-spin Jackiw-Teitelboim gravity

The definition of a BF action requires an invariant symmetric bilinear form on the Lie algebra of symmetries. In other words, the latter algebra must be quadratic.\footnote{A symmetric bilinear form $\langle \ , \ \rangle : g \otimes g \to \mathbb{R}$ over a Lie algebra $g$ is (adjoint-)invariant if $\langle [a_1, a_2], a_3 \rangle = \langle a_1, [a_2, a_3] \rangle$, $\forall a_1, a_2, a_3 \in g$.} Fortunately, there exists a trace over $Mat[\lambda]$ realized either via deformed oscillators \cite{32} or via the quotient algebra construction \cite{28}. By definition, it is a linear form $Tr : Mat[\lambda] \to \mathbb{R}$ obeying the cyclicity property

$$Tr[ [a_1, a_2], ] = 0, \quad \forall a_1, a_2 \in Mat[\lambda].$$

Equivalently, the corresponding linear form $Tr : gl[\lambda] \to \mathbb{R}$ on the associated Lie algebra must be degenerate on the derived Lie algebra $gl[\lambda]'$ spanned by Lie brackets.\footnote{Lie algebras endowed with a non-degenerate invariant symmetric bilinear form are called (regular) quadratic algebras. All finite-dimensional reductive Lie algebras are quadratic. The number of independent such bilinear forms is equal to the number of abelian and simple algebras summands, e.g. two for $gl(N, \mathbb{R}) = u(1) \oplus sl(N, \mathbb{R})$. It is remarkable that the infinite-dimensional reductive Lie algebra $gl[\lambda]$ is quadratic (for generic $\lambda$).} Note that $u(1)' = 0$ (since it is Abelian) and $hs[\lambda]' = hs[\lambda]$ for generic $\lambda$ (since it is simple), hence $gl[\lambda]' = hs[\lambda]$.

Therefore, the only possibility (up to a multiplicative constant) is that the trace $Tr : gl[\lambda] \to \mathbb{R}$ identifies with the projector on the $u(1)$ ideal, i.e. $Tr[a] = 0$ if $a \in hs[\lambda]$ and $Tr[a] = a$ if $a \in u(1)$ where the centre $u(1)$ is identified with $\mathbb{R}$ (cf. Table 1). Accordingly, the linear form $Tr : Mat[\lambda] \to \mathbb{R}$ identifies with the projector on the center $Z \cong \mathbb{R}$. An important property follows: the trace is twist-invariant,

$$Tr[ \tau(a) ] = Tr[a], \quad \forall a \in Mat[\lambda],$$

since $\tau(a) = a$ for any $a \in \mathbb{R}$.

Note that the trace on $Mat[\lambda]$ automatically defines an invariant symmetric bilinear form on $gl[\lambda]$ (and on its subalgebra $hs[\lambda]$ as well)

$$\langle a_1, a_2 \rangle_{gl[\lambda]} := Tr[a_1 \ast a_2],$$

which is non-degenerate for non-integer $\lambda$ (see below for the case of integer $\lambda$). This parallels the relation between the Killing form on $sl(N, \mathbb{R})$ and the trace on $Mat(N, \mathbb{R})$. Note that $u(1)$
and $\mathfrak{h}s[\lambda]$ are orthogonal to each other with respect to (4.4) since $\text{Tr}[a_1 \ast a_2] = a_1 \text{Tr}[a_2] = 0$ for $a_1 \in \mathbb{R}$ and $a_2 \in \mathfrak{h}s[\lambda]$.

The HS extension [26, 28] of Jackiw-Teitelboim gravity is described in the frame-like formulation by the BF action

$$S_{HS, JT}[A, B] = \int_{\mathcal{M}_2} \text{Tr}[B_{[0]} \ast F_{[2]}] ,$$

where:

- $B_{[0]}$ is an adjoint-valued 0-form, i.e. $B_{[0]} \in \Omega^0(\mathcal{M}_2) \otimes \mathfrak{g}[\lambda]$ and $\delta_\epsilon B_{[0]} = [\epsilon_{[0]}, B_{[0]}]_*$ is a gauge transformation with 0-form gauge parameter $\epsilon_{[0]} \in \Omega^0(\mathcal{M}_2) \otimes \mathfrak{g}[\lambda]$.

- $D = d + A_{[1]}$ is the covariant derivative of the adjoint-valued connection 1-form $A_{[1]} \in \Omega^1(\mathcal{M}_2) \otimes \mathfrak{g}[\lambda]$, such that $\delta_\epsilon A_{[1]} = D\epsilon_{[0]}$ under a gauge transformation.

- $F_{[2]} = dA_{[1]} + A_{[1]} \wedge \ast A_{[1]}$ is the curvature 2-form of the connection 1-form $A_{[1]}$, thus $F_{[2]} \in \Omega^2(\mathcal{M}_2) \otimes \mathfrak{g}[\lambda]$. Under gauge transformations, it transforms homogeneously: $\delta_\epsilon F_{[2]} = [\epsilon_{[0]}, F_{[2]}]_*$. 

The equations of motion following from the action (4.5),

$$F_{[2]} = 0 , \quad DB_{[0]} = 0 ,$$

impose flatness and covariant constancy conditions, respectively. It is clear that AdS$_2$ space-time, i.e. (3.3) with $R^A_{[2]} = 0$, provides a solution of (4.6) by setting $A_{[1]} = W_{[1]} \in \Omega^1(\mathcal{M}_2) \otimes \mathfrak{so}(2, 1)$ and $B_{[0]} = 0$. Let us linearize the above equations around AdS$_2$ solution by setting

$$A_{[1]} = W_{[1]} + \Omega_{[1]} ,$$

i.e. $\Omega_{[1]}$ is seen as the fluctuation of the connection 1-form $A_{[1]}$ over the AdS$_2$ background $W_{[1]}$. The linearization of (4.6) yields, respectively,

$$\nabla \Omega_{[1]} = 0 , \quad \nabla B_{[0]} = 0 ,$$

where the covariant derivative $\nabla$ is defined in (3.1) (for more details see [28]). The first equation in (4.8) reproduces (3.4) while the second one determines the $\mathfrak{g}[\lambda]$-valued scalar $B_{[0]}(x)$ everywhere on the base manifold $\mathcal{M}_2$ in terms of its value $B_{[0]}(x_0)$ at any given point $x_0$ (through parallel transport by the flat $\mathfrak{so}(2, 1)$-connection 1-form $W_{[1]}$). These global degrees of freedom are the HS generalization of the dilaton solutions in Jackiw-Teitelboim gravity and they are in one-to-one correspondence with the Killing tensor fields of the AdS$_2$ background.

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5 More generally, in any quadratic Lie algebra $\mathfrak{g}$ the centre $\mathfrak{z} \subset \mathfrak{g}$ and the derived algebra $\mathfrak{g}' \subset \mathfrak{g}$ are the orthogonal complements of each other, as follows from the invariance condition (4.1). Therefore, any quadratic Lie algebra decomposes into the direct sum $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ of its center and its derived algebra (see e.g. (2.3)).
Two comments are in order. Firstly, the above construction of the BF action applies exactly in the same way if one replaces $\mathfrak{gl}[\lambda]$ by $\mathfrak{hs}[\lambda]$ according to (2.3) everywhere in this section. Since $\mathfrak{u}(1)$ and $\mathfrak{hs}[\lambda]$ are orthogonal, the only effect is in subtracting the BF action of the $\mathfrak{u}(1)$ subsector: this $\mathfrak{u}(1)$ term describes on-shell a constant scalar field together with a topological spin-one gauge field. Secondly, for $\lambda \in \mathbb{N}$ the algebra $\mathfrak{hs}[\lambda]$ is not simple and contains an infinite-dimensional ideal $\mathcal{J}_{\lambda}$ so that $\mathfrak{sl}(N, \mathbb{R}) = \mathfrak{hs}[\lambda]/\mathcal{J}_{\lambda}$. The bilinear form (4.4) is then degenerate [32] so that the fields taking values in the ideal $\mathcal{J}_{\lambda}$ do not contribute to the action. The only non-vanishing contributions are identified with $\mathfrak{sl}(N, \mathbb{R})$-valued differential forms. It follows that the resulting higher-spin BF action (4.5) then reduces to $\mathfrak{sl}(N, \mathbb{R})$ BF action [26]. At $N = 2$ we reproduce the original Jackiw-Teitelboim theory in the BF form [25].

5 Extended higher-spin BF-type theory

There exists an extension of the previous HS Jackiw-Teitelboim gravity where the higher-spin algebra $\mathfrak{hs}[\lambda]$ is replaced with an extended HS algebra, denoted $\mathfrak{ehs}[\lambda]$, based on the trick of replacing $\mathfrak{gl}[\lambda]$ by two copies of itself endowed with a subtle product between them. More precisely, the corresponding extension of $\text{Mat}[\lambda]$ is defined via a smash product of the original associative algebra and a finite group $\mathbb{Z}_2$ of its automorphisms.\(^7\)

Let $\mathbb{Z}_2 = \{1, \tau\}$ be the group of automorphisms generated by the twist $\tau$. Then, the associative algebra denoted $\text{Mat}[\lambda] \rtimes \mathbb{Z}_2$ is the vector space spanned by elements that can be written as $a \mathbf{1} + b \mathbf{\tau}$, where $a, b \in \text{Mat}[\lambda]$, endowed with the smash product $\star$ defined as follows

$$(a_1 \mathbf{1} + b_1 \mathbf{\tau}) \star (a_2 \mathbf{1} + b_2 \mathbf{\tau}) = \left( a_1 \star a_2 + b_1 \star \tau(b_2) \right) \mathbf{1} + \left( a_1 \star b_2 + b_1 \star \tau(a_2) \right) \mathbf{\tau}, \quad (5.1)$$

where $\star$ denotes the product in $\text{Mat}[\lambda]$.

The associated Lie algebra will be denoted $\mathfrak{gl}[\lambda] \rtimes \mathbb{Z}_2$. It is spanned by elements $a \mathbf{1} + b \mathbf{\tau}$, where $a, b \in \mathfrak{gl}[\lambda]$, endowed with the $\star$-commutator

$$[a_1 \mathbf{1} + b_1 \mathbf{\tau}, a_2 \mathbf{1} + b_2 \mathbf{\tau}]_\star =$$

$$= \left( [a_1, a_2]_\star + (b_1 \star \tau(b_2) - b_2 \star \tau(b_1)) \right) \mathbf{1} + \left( [a_1, b_2]_\tau - [a_2, b_1]_\tau \right) \mathbf{\tau}, \quad (5.2)$$

\(^6\)Note that a similar trick was also used in [35] for a distinct proposal of two-dimensional HS gravity. The matter spectrum in [35] is made of a single massive scalar with a fixed mass, so this proposal appears very different.

\(^7\)The smash product can be defined as follows (see e.g. the section 3.9 of [36]). Let $H$ be a group. Consider an $H$-module algebra $A$ and let $\pi$ denote the corresponding action of $H$ on $A$. The so-called skew group ring of $H$ over $A$ is denoted as $A\#H$ and consists of pairs $(a, h)$, where $a \in A$ and $h \in H$, endowed with the smash product $(a_1, h_1)\#(a_2, h_2) = (a_1 \pi_{h_1}(a_2), h_1 h_2)$, where $\pi_h$ denotes the action of an element $h \in H$ on $A$. In our case, $A = \text{Mat}[\lambda]$ and $H = \mathbb{Z}_2$, and the smash product is realized on $\text{Mat}[\lambda] \# \mathbb{Z}_2$ as in (5.1). With a slight abuse of the standard mathematical notation, we will denote the smash product algebra as $\text{Mat}[\lambda] \rtimes \mathbb{Z}_2$. In the higher-spin theory, skew group rings of various finite groups (sometimes called outer Kleinians) over associative algebras were extensively used in constructing non-linear equations of motion, see e.g. [37, 38] for earlier literature and [39, 40] for recent studies.
where \([a, b]_\ast = a \ast b - b \ast a\) is the \(*\) commutator and \([a, b]_\tau = a \ast b - b \ast \tau(a)\) is the twisted \(*\) commutator. This implies that the \(*\) adjoint action of \(gl[\lambda]\) on its extension \(gl[\lambda] \ltimes Z_2\) unifies the adjoint and twisted-adjoint actions of \(gl[\lambda]\) on itself,

\[ *ad_y (a 1 + b \tau) := [y 1, a 1 + b \tau]_\ast = *ad_y (a) 1 + *ad_y (b) \tau , \quad \forall y, a, b \in gl[\lambda], \tag{5.3} \]

where we used (2.4)-(2.5). One can show that the center \(Z\) of the associative algebra \(Mat[\lambda] \ltimes Z_2\) is the one-dimensional subalgebra \(Z = \mathbb{R} 1 := \{ c 1 | \forall c \in \mathbb{R} \}\).

The projection to the center \(Z\) leads to a natural trace over \(Mat[\lambda] \ltimes Z_2\). Let us denote by \(\mathbb{T}\) the trace over \(Mat[\lambda] \ltimes Z_2\), defined as the restriction to the trace \(\text{Tr}\) over \(Mat[\lambda]\), that is to say

\[ \mathbb{T}[a 1 + b \tau] = \text{Tr}[a] , \quad \forall a, b \in Mat[\lambda]. \tag{5.4} \]

The cyclicity property,

\[ \mathbb{T}[a_1 1 + b_1 \tau, a_2 1 + b_2 \tau] = \text{Tr}[a_1 1 + b_1 + \tau(b_2) - b_2 + \tau(b_1)] = 0 , \tag{5.5} \]

follows from the commutation relation (5.2), the definition (5.4), the involution and automorphism properties \(\tau^2 = 1\) and \(\tau(a \ast b) = \tau(a) \ast \tau(b)\) of the twist, together with the properties (4.2), (4.3). Note that although the extended trace \(\mathbb{T}\) is degenerate along the direction of the extra generator \(\tau\), the symmetric bilinear form that it defines,

\[ \langle a_1 1 + b_1 \tau, a_2 1 + b_2 \tau \rangle_{gl[\lambda] \ltimes Z_2} := \]

\[ := \mathbb{T}[(a_1 1 + b_1 \tau) \ast (a_2 1 + b_2 \tau)] = \text{Tr}[a_1 a_2 + b_1 + \tau(b_2)] , \tag{5.6} \]

is non-degenerate (we used (5.1) and (4.4) to obtain the last two lines). The cyclicity of the extended trace and the Jacobi identity of the \(*\) commutator ensures that this symmetric bilinear form is also invariant with respect to the \(*\) adjoint action of \(gl[\lambda] \ltimes Z_2\) on itself. In other words, the infinite-dimensional algebra \(gl[\lambda] \ltimes Z_2\) is a quadratic Lie algebras. And these two properties (non-degeneracy and adjoint-invariance) are enough to define a proper BF action. Moreover, it also implies that this Lie algebra decomposes into the direct sum (cf footnote 5):

\[ gl[\lambda] \ltimes Z_2 = u(1) \oplus chs[\lambda] , \tag{5.7} \]

where the centre is the Abelian ideal \(u(1) = \mathbb{R} 1\) and the derived algebra, denoted \(chs[\lambda] = (gl[\lambda] \ltimes Z_2)’\), is the extension of the \(hss[\lambda]\) algebra (2.3) (see Table 1) spanned by elements \(a 1 + b \tau\) where \(a \in hss[\lambda]\) and \(b \in gl[\lambda]\). This extended HS algebra \(chs[\lambda]\) is an ideal of \(gl[\lambda] \ltimes Z_2\) and, in particular, the \(*\) adjoint action (5.3) can be consistently restricted to \(chs[\lambda]\).

A smash-product extension of the higher-spin BF-type action (4.5) is

\[ S_{ehs,jt}[\mathcal{A}, \mathcal{B}] = \int_{M_2} \mathbb{T}[\mathcal{B}[0] \ast \mathcal{F}[2]] , \tag{5.8} \]

where
\[ B_0 = B_0 \mathbf{1} + C_0 \tau \] is a 0-form valued in \( \mathfrak{gl}[\lambda] \rtimes \mathbb{Z}_2 \), with \( \delta_\varepsilon B_0 = [\varepsilon_0, B_0]_\star \) as a gauge transformation.

\[ D = d + A_1 ] \] is the extended HS covariant derivative of the connection 1-form \( A_1 = A_1 \mathbf{1} + Z_1 \tau \) valued in the \( \star \)-adjoint representation, with \( \delta_\varepsilon A_1 = D\varepsilon_0 \) as a gauge transformation.

\[ F_2 = dA_1 + A_1 \wedge \star A_1 \] is the extended curvature \( \mathfrak{gl}[\lambda] \rtimes \mathbb{Z}_2 \)-valued 2-form with \( \delta_\varepsilon F_2 = [\varepsilon_0, F_2]_\star \) as a gauge transformation.

The proposed BF-type theory (5.8) is natural for various reasons. Firstly, it is clear that truncating all fields to \( \mathfrak{gl}[\lambda] \), i.e.

\[ A_1 = A_1 \mathbf{1} , \quad B_0 = B_0 \mathbf{1} , \quad (5.9) \]

the extended HS action (5.8) is reduced to the HS Jackiw-Teitelboim action (4.5).

Secondly, the equations of motion following from the BF-type action (5.8) impose the on-shell flatness and covariant constancy conditions

\[ F_2 = 0 , \quad DB_0 = 0 , \quad (5.10) \]

which are natural extensions of (4.6).

Thirdly, the flatness condition \( F_2 = 0 \) (5.10) implies that, locally, the 1-form connection describes AdS$_2$, i.e. \( A_1 = W_1 \mathbf{1} \) in a suitable gauge. Note that this is true except if one imposes by hand a non-degeneracy condition on \( Z_1 \) similar to the one on \( A_1 \). However, we do not see presently any clear motivation for introducing such a second zweibein-like field. Then, due to the property (5.3), the covariant constancy condition \( DB_0 = 0 \) decomposes as

\[ \nabla B_0 = 0 , \quad \bar{\nabla} C_0 = 0 . \quad (5.11) \]

In this sense, the equations of motion (5.8) of the extended BF-type theory can be thought of as a higher-spin covariantization of the twisted-adjoint equation (3.5) on the extra 0-form \( C_0 \), together with the adjoint equation (4.8) on the 0-form \( B_0 \) already present in the HS Jackiw-Teitelboim theory.

Fourthly, a consistent truncation of the equations (5.10) is

\[ A_1 = A_1 \mathbf{1} , \quad B_0 = C_0 \tau \quad (5.12) \]

leading to the system

\[ F_2 = 0 , \quad \bar{D} C_0 = 0 , \quad (5.13) \]

where \( \bar{D} \) is the twisted-adjoint covariant derivative with respect to the on-shell flat connection \( A_1 \). The linearization of these equations around AdS$_2$ solution \( W_1 \) (3.3) via the decomposition (4.7) of the connection 0-form \( A_1 \) are respectively

\[ \nabla \Omega_1 = 0 , \quad \bar{\nabla} C_0 = 0 , \quad (5.14) \]
which perfectly reproduces the linear equations (3.4) and (3.5).

Finally, let us make two comments similar to the ones at the end of the previous section. First, to construct the BF-type action one may consider differential forms taking values in the subalgebra \( \mathfrak{e}_h(\lambda) \) (instead of \( \mathfrak{g}(\lambda) \times \mathbb{Z}_2 \)) in order to get rid of the \( \mathfrak{u}(1) \) subsector. More explicitly, this means that \( A^{[1]} \) and \( B^{[0]} \) are restricted to \( \mathfrak{h}_s(\lambda) \), as in HS Jackiw-Teitelboim gravity, while \( Z^{[1]} \) and \( C^{[0]} \) still take values in the whole algebra \( \mathfrak{g}(\lambda) \). For instance, in the truncation (5.12) the \( \mathfrak{u}(1) \)-connection decouples from the adjoint sector, while scalar components are still present in the twisted-adjoint sector that keeps the equations (5.14) consistent. Second, we note that at integer values of the parameter (i.e. \( \lambda = N \in \mathbb{N} \)), the extended HS algebra \( \mathfrak{g}(N) \times \mathbb{Z}_2 \) contains an infinite-dimensional ideal denoted \( \mathcal{J}_N \times \mathbb{Z}_2 \), where \( \mathcal{J}_N \) are infinite-dimensional ideals in \( \mathfrak{g}(N) \) (see Section 4). The resulting quotient

\[
\frac{\mathfrak{g}(N) \times \mathbb{Z}_2}{\mathcal{J}_N \times \mathbb{Z}_2} = \{ a\mathbf{1} + b\tau \mid \forall a, b \in \mathfrak{g}(N, \mathbb{R}) \}
\]

(5.15)

is a finite-dimensional Lie algebra which will be denoted as \( \mathfrak{g}(N, \mathbb{R}) \times \mathbb{Z}_2 \). Its centerless part is defined from the decomposition (see Table 1):

\[
\mathfrak{g}(N, \mathbb{R}) \times \mathbb{Z}_2 = \mathfrak{u}(1) \oplus \mathfrak{c}(N, \mathbb{R}).
\]

(5.16)

One may consider a BF theory with the gauge algebra \( \mathfrak{g}(N, \mathbb{R}) \times \mathbb{Z}_2 \) which yields a topological system of equations of motion which are (twisted-)adjoint covariant constancy conditions on finite-dimensional field spaces. In particular, for (5.12) the equations of motion are reduced to the standard \( \mathfrak{g}(N, \mathbb{R}) \) BF equations along with new topological equations in the sector of 0-forms. The latter are analogous to those discussed in three-dimensional HS theory [38, 41] as a topological subsystem decoupled from the original dynamical twisted-adjoint equations at integer \( \lambda = N \).

6 Concluding remarks

To summarize, the non-Abelian BF-type action (5.8) provides a natural extension of the action (4.5) of HS Jackiw-Teitelboim gravity via the addition of a matter multiplet. The corresponding equations of motion describe an infinite tower of scalar fields with fine-tuned increasing masses (3.6), interacting with the topological gauge fields of HS Jackiw-Teitelboim gravity. In particular, their linearization around AdS_2 background reproduces the correct equations fixed by the HS symmetries in two dimensions. Let us stress that our construction of the extended HS gravity action (5.8) with the above properties relies only on the existence of a twist-invariant trace on the HS algebra, not on its explicit form (though the latter would become important to write down the expression of the action in components).

The existence of the BF-type action (5.8) is remarkable and contrasts with the situation in three-dimensional HS gravity where the inclusion of matter in the fundamental representation

\footnote{An explicit realization of the twist on \( \mathfrak{g}(N, \mathbb{R}) \) and respective equations will be considered elsewhere.}
of the HS algebra in three dimensions (hence in the “bifundamental” representation of $\mathfrak{hs}[^2\lambda]$, since the latter algebra comes in two copies) is known to reinstate the same level of intricacy as in dimension four.\footnote{For instance, the two action principles \cite{42,43} which have been proposed for the nonlinear equations of motion in \cite{38} are not usual CS actions (in particular, the action in \cite{42} has no base space while the base space of the action in \cite{43} is of higher dimension than the spacetime manifold).}

The nonlinear equations (5.10) are the two-dimensional analogues of the equations considered in the approach of \cite{40} (and references therein) to the Noether procedure in the unfolded formulations of higher-dimensional HS gravity theories. More precisely, following the same logic as \cite{40}, applied to two dimensions, one should consider a deformation of the extended HS algebra considered above.\footnote{We thank A. Sharapov and E. Skvortsov for discussions on this point.} Note that if this deformed extended HS algebra admits a trace, then the present BF-type construction would generalize to this deformed case as well.

BF-type HS theories in two dimensions obviously require further study. In particular, the physical content of the untruncated spectrum in the above model should be analyzed further for several reasons. For instance, there seems to be no genuine non-linearity in the matter fields (the field equations are linear in the 0-forms) nor backreaction on the gauge fields from the presence of matter (the 0-form sector does not source the 1-form sector). This feature might be improved by making use of the interactions generated from a deformation of the extended HS algebra. Let us point out that, since any BF-type action takes the form of a topological-like\footnote{Note that here the target space is the dual of the extended HS algebra. The latter involves a smash product of an infinite-dimensional algebra. This is the reason why this BF-type action can (and does) describe local degrees of freedom.} Poisson Sigma model, a reasonable expectation is that the fully interacting action still takes the form of a Poisson Sigma model, whose Poisson bivector field is a nonlinear deformation of the undeformed linear one. Last but not least, for holography one should add some right boundary terms and check whether the corresponding total action may capture correlators of single-trace operators in some suitable CFT$_1$.

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