Wormholes and Black Hole Pair Creation

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Abstract

We analyze the possibility of black holes pair creation induced by three dimensional wormholes. Although this spacetime configuration is nowadays hard to suppose, it can be very important in the early universe, when the wormhole spacetime foam representation can be meaningful. We compare our approach with the no-boundary prescription of Hartle-Hawking.

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I. INTRODUCTION

In recent years, most energies are being focussed on the study of wormholes as significant contributions to quantum gravity, either in the Euclidean or Minkowski formalism. Moreover, it has been argued that wormholes may affect the constant of nature: in particular, they may lead to a vanishing cosmological constant. On the other hand, wormholes can be used as probes for studying the interior of a black hole and finally, an important class termed traversable may well be included to remark the importance of the subject in its various aspects. A classical wormhole is defined by a throat of a given radius, connecting two asymptotically flat three-dimensional spaces, an example can be the Schwarzschild metric,

$$ds^2 = -(1 - \frac{2MG}{r}) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$  \hspace{1cm} (1.1)

describing a four-dimensional wormhole with the horizon located at $2MG$. $M$ is the wormhole mass, $G$ is the gravitational constant and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Another interesting wormhole is described by the Schwarzschild-de Sitter metric

$$ds^2 = -(1 - \frac{2MG}{r} - \frac{\Lambda}{3} r^2) dt^2 + \left(1 - \frac{2MG}{r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega^2.$$  \hspace{1cm} (1.2)

When $\Lambda = 0$, then the metric describes (1.1), while when $M = 0$, we obtain the de Sitter metric. By examining the roots of (1.2), we can distinguish two horizons located at

$$r_+ = \frac{1}{\sqrt{\Lambda}}\cos\left(\frac{\theta}{3}\right), \quad r_{++} = \frac{1}{\sqrt{\Lambda}}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right),$$  \hspace{1cm} (1.3)

where

$$\cos \theta = -3m\sqrt{\Lambda},$$  \hspace{1cm} (1.4)

with the condition

$$0 \leq 9m^2\Lambda \leq 1.$$  \hspace{1cm} (1.5)

Here $r_+$ is the coordinate relative to the black hole horizon and $r_{++}$ is relative to the cosmological horizon. If $9m^2\Lambda = 1$ then $r_{++} = r_+$ and the singularity is eliminated. This
case can be considered as extreme, exactly like the extreme Reissner-Nordström metric. However, since \(9m^2\Lambda = 1\) the cosmological horizon and the black hole horizon have merged. To avoid merging of horizons, we refer to the Nariai metric described in ref. [3]

\[
ds_i^2 = -(1 - \Lambda p^2) dt^2 + (1 - \Lambda p^2)^{-1} dp^2 + \frac{1}{\Lambda} d\Omega^2.
\]

The choice of these particular metrics is not casual. Indeed, when we consider the Euclidean sections of (1.1) and (1.2), we discover negative modes in the saddle point approximation [4,3,11], which means instability with respect to the reference asymptotic background (i.e., flat space and de Sitter space). Moreover, both topology and signature changes are involved together with tunnelling. Then we can see that a rich variety of phenomena appear as a consequence of negative modes. At this point, we can ask ourselves if the same result can appear when the Lorentzian signature is used. Before approaching this problem, an intermediate step has to be investigated and precisely: what happens if we split the time and space in 3+1 dimensions? Concerning this question, the answer comes from an analysis in a Kaluza-Klein type space, where the extra-dimension is represented by a compact manifold.

On this type of product space, we discover instability [14]. The basic strategy is: take a manifold whose topology is \(M^4 \times B\), where \(M^4\) represents the four-dimensional Minkowski spacetime, while \(B\) is a compact manifold of given radius. Then the line element can be written, after an analytic continuation to euclidean space, as

\[
ds^2 = dr^2 + r^2 d\Theta^2 + d\phi^2,
\]

where \(r\) runs from 0 to \(\infty\) and \(d\Theta^2\) is the line element of the three sphere. Comparing this solution with the five-dimensional black hole solution [15]

\[
ds^2 = \frac{dr^2}{1 - \left(\frac{R}{r}\right)^2} + r^2 d\Theta^2 + \left(1 - \left(\frac{R}{r}\right)^2\right) d\phi^2,
\]

we discover that they have the same asymptotic behaviour. Since \(\phi\) is a periodic variable with periodicity \(2\pi R\), we have now a non-singular space which asymptotically approaches the Kaluza-Klein vacuum. Presence of negative action modes in small fluctuations around
showing instability of $M^4 \times S^1$ is guaranteed by the analogous problem treated in one less dimension \[4\]. Motivated by this and by recent results on black hole pair production in inflationary cosmology \[2\], we will investigate the possibility of black hole pair production induced by wormholes, in particular three-dimensional wormholes, which are nothing but “sections” of four dimensional metrics, with $t = \text{const}$. Such sections are holes in space but not in time. Recalling that the probability of decay per unit time per unit volume is given by

$$P \sim |\exp(-I)|^2 \sim |e^{-(\Delta E)(\Delta t)}|^2,$$

it is immediate to see that a possible approach to black hole pair production is by means of the “Energy”. Nevertheless, this choice needs a careful statement because constraint equations both at classical and quantum level impose that dynamics is frozen. Indeed, if we look at the quantum constraint equation \[1\]

$$\mathcal{H}\Psi = 0,$$  \[1.10\]

known as Wheeler-DeWitt equation (WDW), we see that it represents an eigenvalue problem with zero eigenvalue. However eq. \[1.10\], clearly indicates a non vanishing $\Delta E$, otherwise even the action should be meaningless. This means that the problem of computing energy is meaningless for, whatever configuration we fix, we will obtain the same result. Two possibilities of solving this puzzling question come into play. The first one is the resolution of the WDW equation in superspace or in its reduced version of mini-superspace. The second one is the resolution of the algebra of constraints generated by $\mathcal{H}$ and $\mathcal{H}_i$. Although this is a well tested way for probing quantum aspects of gravity it leaves the question of the energy an open issue. Nevertheless, if quantum aspects enter in gravity with some basic principles, they have to be related to the possibility of having a functional Schrödinger equation. Because of \[1.10\], such an equation cannot exist and the associated eigenvalue problem cannot be taken under consideration. However, following ref. \[9\], we shall assume the validity of an “internal” Schrödinger equation associated to each lapse function in such
a way to recover an eigenvalue equation. The plan of the paper is the following: in section II, we will examine the topology of the problem, in section III, we give a simple comparison of the action calculated in a covariant way and in its splitted form of three space and one time; in section IV, we illustrate the results obtained in a saddle point approximation both in four dimensions and in three plus one dimensions. For this purpose we shall use a gaussian variational approach which is very close to the quadratic approximation; in section V, we approach the black hole pair production problem by comparing the results of [2] with our approach; in section VI, we summarize and conclude.

II. THE TOPOLOGY OF THE PROBLEM

In ref. [14], it was shown that, not only the compact Euclidean periodically identified flat four dimensional manifold was unstable, but even its $(3 + 1)$ dimensional version. This suggests the possibility of some basic relations between the three-dimensional space and the four-dimensional one. Here we summarize some aspects of the different topologies associated with metrics (1.1) and (1.2) displaying whenever possible the underlying three-dimensional structure.

a) Let us consider first the static, spherically symmetric metric of a three dimensional wormhole

$$ds_{3}^{2} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

(2.1)

whose topology is $\mathbb{R} \times S^2$. Metric (2.1) possesses an apparent (horizon) singularity at $r = 2M$, that is removable by a suitable coordinate transformation. By defining

$$dx = \pm \frac{dr}{\sqrt{1 - \frac{2M}{r}}}$$

(2.2)

metric (2.1) is transformed into

\[\text{This change of coordinates is present in the mathematics of embedding, i.e., when one wishes to}\]
\[ ds^2 = dx^2 + r^2(x) d\Omega^2, \quad (2.3) \]

where \( x \in [0, \infty) \). However, metric (2.1) can be regarded as a section of a four-dimensional metric. In particular, if we consider constant time sections, i.e. \( t = \text{const} \), we obtain the Schwarzschild wormhole (1.1) and if we consider constant Euclidean time \( \tau = -it \) sections we obtain the Gibbons-Hawking instanton. To avoid conical singularities, the imaginary time axis is periodic with period \( \beta = 8\pi M \). Then two routes for enlarging the topology are possible. The first one is by choosing the Euclidean metric with topology \( \mathbb{R} \times S^2 \times S^1 \) and the second one is by choosing the Lorentzian metric \( \mathbb{R}^2 \times S^2 \). In any case the underlying wormhole topology is preserved and for the Schwarzschild sector, we can summarize in the following diagram the topology enlargement

\[
\begin{array}{c}
\mathbb{R}^2 \times S^2 & \leftrightarrow & \mathbb{R} \times S^2 \times S^1 \\
\text{Lorentzian} & \prec & \text{Euclidean} \\
\end{array}
\]

(2.4)

b) On the other hand the time section of the Schwarzschild-de Sitter metric is

\[
ds_3^2 = \left( 1 - \frac{2M}{r} - \frac{1}{3 \Lambda r^2} \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (2.5)
\]

The Schwarzschild-de Sitter metric has the wormhole topology, but the interesting case is the extremal case (1.6). After eliminating the singularities, by analogy we continue to use the (improper) term of wormhole also for the constant “time” section, whose form metric is

\[
ds_3^2 = (1 - \Lambda p^2)^{-1} dp^2 + \frac{1}{\Lambda} d\Omega^2. \quad (2.6)
\]

construct, in three-dimensional Euclidean space, a two-dimensional with the same geometry at the slice with \( t \) fixed. Formula (2.2), defines the proper radial distance as measured by static observers. The positive sign means that we are performing calculations in the upper universe.
Similarly to the Schwarzschild sector we can consider the diagram

\[
\mathbb{H}^2 \times S^2 \quad \leftrightarrow \quad S^2 \times S^2
\]

representing the possibilities of completing the topology.

### III. COMPUTING THE ACTION ON COMPACT TOPOLOGIES

As a simple application, we calculate here the action contribution on some "test" topologies, comparing the results obtained directly in 4D with the ones obtained in 3+1 dimensions. To obtain finite action contributions, we fix our attention on some compact gravitational instantons, in particular:

- \( S^4 \), corresponding to the compact de Sitter metric (de Sitter),
- \( S^2 \times S^2 \), corresponding to the Nariai metric, that is the extreme Schwarzschild-de Sitter metric,
- \( S^2 \times S^1 \times \mathbb{R} \), corresponding to the Schwarzschild metric.

#### A. The action in 4D

a) \( S^4 \)

The action is represented by

\[
I = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g} (R - 2\Lambda),
\]

where \( \Lambda \) is the positive cosmological constant. By the Einstein equations

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \quad \Rightarrow \quad R = 4\Lambda.
\]
and (3.1) becomes

\[ I = -\frac{2\Lambda}{16\pi G} \int_M d^4x \sqrt{g} = -\frac{2\Lambda}{16\pi G} V_M^{(4)}, \]  

(3.3)

where, \( V_M^{(4)} = \frac{8}{3} \pi^2 \left( \sqrt{\frac{3}{\Lambda}} \right)^4 \) is the four dimensional volume of the compact manifold. Then

\[ I = -\frac{2\Lambda}{16\pi G} \frac{8}{3} \pi^2 \left( \sqrt{\frac{3}{\Lambda}} \right)^4 = -\frac{3\pi}{G\Lambda}. \]  

(3.4)

b) \( S^2 \times S^2 \)

This is the topological product of two spheres of radius \( \sqrt{\frac{3}{\Lambda}} \). The action is represented by eq. (3.3) and \( V_M^{(4)} = 16\pi^2 \left( \sqrt{\frac{1}{\Lambda}} \right)^4 \) and the final value of the action is

\[ I = -\frac{2\Lambda}{16\pi G} 16\pi^2 \left( \sqrt{\frac{1}{\Lambda}} \right)^4 = -\frac{2\pi}{G\Lambda}. \]  

(3.5)

c) \( S^2 \times S^1 \times \mathbb{R} \)

In this case the action needs a boundary term, otherwise the path integral is meaningless. Here \( \Lambda = 0 \) and \( I \) is

\[ -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} [K], \]  

(3.6)

where \([K]\) is the difference in the trace of the second fundamental form of \( \partial M \) in the metric \( g \) and the metric \( \eta \) referred to the flat space. In this case the Einstein equation give \( R_{\mu \nu} = 0 \). This means \( R = 0 \) and the action is given by

\[ I = -\frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} [K] = 4\pi GM^2, \]  

(3.7)

where we have used the fact that the Euclidean “time” is periodic with period \( 8\pi GM \) and the fact that the hypersurface is bounded by the surface \( r = r_0 \).

We will see in next section that all these contributions for the different topologies will be extracted by the same hypersurface term giving the relation between \( 4D \) and \( (3 + 1)D \).
B. The Action in (3 + 1)D

The scalar curvature appearing in the action described in terms of \textit{lapse} and \textit{shift} variables is

\[
(4) R = (3) R - K_{ij}K^{ij} + K^2 - \frac{2}{N} \frac{\partial K}{\partial \tau} + \frac{2}{N} \left[ \nabla^2 N + N^p (K)_{|p} \right],
\]

(3.8)

where \( K \) is the trace of the second fundamental form and \( \tau \) is the Euclidean time related by \( t = -i\tau \). Since we consider only “static spherically symmetric” metrics, the term involving \( N^p \) disappears and \( K_{ij} = 0 \), therefore

\[
(4) R = (3) R + \frac{2}{N} \nabla^2 N,
\]

(3.9)

that is, the contribution for these metrics comes only from a boundary term and from the three scalar curvature. This is a consequence of the fact that we have opened the hypersurfaces, i.e. they do not represent more a compact object. For example, the de Sitter metric can be represented by

\[
d s^2 = d \tau^2 + \cos^2 \tau d \Omega_3^2,
\]

(3.10)

which has a compact \( S^4 \) topology whose spatial section is a closed (compact) \( S^3 \) hypersurface, otherwise we can choose the “\textit{static}” representation

\[
d s^2 = \left( 1 - \frac{\Lambda}{3} r^2 \right) d \tau^2 + \left( 1 - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d \Omega^2,
\]

(3.11)

where for every choice of \( \tau = \text{const} \), the hypersurface covers the region inside the cosmological horizon, giving therefore open topologies\[2, 20\. For this reason, we see immediately that

\[
(4) R = + \frac{2}{N} \nabla^2 N.
\]

(3.12)

\[2\]The same procedure is used in ref. \[19\] to the approach of entropy generation by a quantum tunnelling.
In fact \( (3)^{3}R = 0 \), for the Schwarzschild metric and since we have to consider \( (4)^{R - 2\Lambda} \) for de Sitter and Schwarzschild-de Sitter (Nariai) metrics \( (3)^{R = 2\Lambda} \), we have the result of eq.\((3.12)\). The action is represented by the surface term

\[
I = -\frac{1}{16\pi G} \int dt \int d^3x \sqrt{g} \left[ \frac{2}{N} \frac{\partial}{\partial N} - \left( \frac{\partial}{\partial x} \right) \left( \sqrt{3}gg^{ij}\partial_j N \right) \right]. \tag{3.13}
\]

Now we verify that eq.\((3.13)\) matches with the results of the previous section.

a) de Sitter vs. \( S^4 \) topology.

The line element is

\[
ds^2 = \left( 1 - \frac{\Lambda}{3} r^2 \right) dt^2 + \left( 1 - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.14}\]

and identifying \( N^2 = 1 - \frac{\Lambda}{3} r^2 \), then \( (3.13) \) becomes

\[
I = -\frac{1}{8\pi G} \int dt \int d^3x \left[ \partial_i \left( \sqrt{3}gg^{ij}\partial_j N \right) \right] = -\frac{\Lambda}{6G} \int d\tau r^3 \left| \partial_i \right| = -\frac{\Lambda}{6G} \left( 2\pi \sqrt{\frac{3}{\Lambda}} \right) \left( \frac{3}{\Lambda} \right)^{\frac{3}{2}} \]

\[= -\frac{3\pi}{\Lambda G}. \tag{3.15}\]

b) Nariai vs. \( S^2 \times S^2 \) topology.

The line element is

\[
ds^2 = \left( 1 - \Lambda r^2 \right) dt^2 + \left( 1 - \Lambda r^2 \right)^{-1} dr^2 + \frac{1}{\Lambda} d\Omega^2, \tag{3.16}\]

and performing the same calculation for the a) case we obtain

\[
I = -\frac{1}{8\pi G} \left( \frac{2\pi}{\sqrt{\Lambda}} \right) \left( 4\pi \frac{1}{\Lambda} \right) \int d\tau = -\frac{2\pi}{\Lambda G}. \tag{3.17}\]

c) Schwarzschild vs. \( S^2 \times S^1 \times \mathbb{R} \) topology.

The line element is

\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.18}\]
and the action becomes

\[ I = -\frac{1}{8\pi G} 4\pi \left[ \int dr \partial_r \left( r^2 \sqrt{1 - \frac{2MG}{r}} \partial_r \sqrt{1 - \frac{2MG}{r}} \right) \right] = \frac{1}{8\pi G} \int_{r=r_0} d\tau = -\frac{1}{8\pi G} 4\pi (MG) (8\pi MG) \]

As shown, this gives exactly the term (3.7) with the reversed sign.

IV. NEGATIVE MODES IN 4D AND IN (3 + 1)D

In previous sections we examined the action contribution of some compact topologies from a general point of view. In particular, we have verified that by opening the topologies the exponential background factor agrees with the value obtained performing the calculation on such compact objects. Moreover, the non vanishing surface term that will be related to the decay exponential is deduced from the same term, i.e. the horizon hypersurface. Here, we report, how the action and consequently the partition function modify when a fixed background choice is made to estimate decay ratios. If \( g = \overline{g} + h \), then

\[ I[ g] = I[ \overline{g}] + I_2[ \overline{g}, h] + \ldots \ldots , \]  

where \( I_2 \) is quadratic in the perturbations \( \overline{g} \). In this approximation the decay rate per unit volume is given by

\[ \Gamma = |A| \exp (-I[ \overline{g}]) , \]  

where \( A \) is the prefactor coming from the evaluation of the one loop expansion of the partition function. In particular, it is proportional to its imaginary part.

A. 4D

Limiting us to the transverse-traceless sector (TT or spin 2), eq. (4.2) can be approximated by
\[ \Gamma = |A| \exp \left( -I \left[ M \right] \right) \simeq \left( \det M \right)^{-\frac{1}{2}} \exp \left( -I \left[ M \right] \right), \quad (4.3) \]

where the determinant is represented by the usual gaussian formula

\[ \left( \det M \right)^{-\frac{1}{2}} = \int \mathcal{D}h \exp \left( -\frac{1}{16\pi G} \int d^4x \left[ \frac{1}{4} h^{TT} M^{abcd} h_{cd}^{TT} \right] \right), \quad (4.4) \]

and

\[ M^{abcd} h_{cd} = -\Box h^{ab} - 2R^{acbd} h_{cd} \quad (4.5) \]

The eigenvalues coming from the eigenvalue equation associated with (4.5) are:

i) \( \lambda = -2\Lambda \) for the degenerate Schwarzschild-de Sitter case, i.e., the Nariai metric, \[3,11\]

ii) \( \lambda \simeq -0.19 (GM)^{-2} \) for the Schwarzschild metric\[4\].

**B. (3 + 1)D**

Following ref. \[5\], we expand the scalar curvature around a fixed three dimensional background up to second order, then after having separated the degrees of freedom by means of ultralocal metrics we fix our attention on the resulting traceless transverse sector. The trial wave functional is defined by

\[ \Psi_{\alpha} \left[ h_{ij} \left( \vec{x} \right) \right] = \mathcal{N} \exp \left\{ -\frac{1}{4\ell^2_p} \left( \langle h^{K^{-1} h}_{x,y} \rangle \right) \right\}. \quad (4.6) \]

\( \langle \cdot, \cdot \rangle_{x,y} \) denotes space integration and \( K^{-1} \) is the inverse propagator defined by

\[ K^{-1} (x, x)_{iakl} := \sum_{\mathcal{N}} \frac{h^+_{ia} (x) h^+_{kl} (y)}{2\lambda_{\mathcal{N}} (p)}, \quad (4.7) \]

where \( \lambda_{\mathcal{N}} (p) \) are infinite variational parameters and \( h^+_{ia} (x) \) are the eigenfunctions of

---

\[ ^3 \text{We omit here the discussion on zero modes, since for our purposes this is not relevant.} \]

\[ ^4 \text{However, B. Allen showed that the negative mode can be avoided by restricting the box containing the black hole in such a way to forbid its formation and its growing \[13\].} \]
\[ (-\Delta \delta^a_j + 2R^a_j) h^i_a = -E^2 h^i_j. \]  

(4.8)

\( \Delta \) is the Laplacian in a curved background and \( R^a_j \) is the mixed Ricci tensor. When the cosmological constant is considered, we have

\[ (-\Delta \delta^a_j + 2R^a_j - 2\Lambda\delta^a_j) h^i_a = -E^2 h^i_j. \]  

(4.9)

The “-” sign appears because we are searching for bound states of the two previous equation. Indeed finding bound states is equivalent to searching for negative modes. Solving the previous equations we obtain the following eigenvalues:

i) \( E^2 \simeq -2\Lambda \) for the Nariai “wormhole” case

ii) \( E^2 \simeq -0.24 \left( MG^2 \right)^{-2} \) for the Schwarzschild wormhole.

The numerical discrepancy between the four dimensional case and the three dimensional one is due to different numerical factors coming into play when the Laplacian operator and the connections are computed.

V. BLACK HOLE PAIR CREATION

As mentioned in the introduction and related in section [V], negative modes give evidence of instability. This means that these saddle points are \textit{bounces} instead of being instantons, that is at one loop the approximated functional ceases to be gaussian. Since these backgrounds are periodic in the imaginary time, this periodicity is usually interpreted as finite temperature \( T \). Then the instability of these configurations happens for the hot space. Gross \textit{et al.} interpreted this fact as corresponding to a semiclassical \textit{nonperturbative} instability of hot flat space in a thermal bath due to the nucleation of black holes. The appearance of a single negative eigenvalue is related to the bounce which shifts the energy of the false ground state. Then neglecting the prefactor \([12]\), the approximate value of the probability of nucleating a black hole per unit volume per unit time, which we denote as \( \Gamma \), for the Gibbons-Hawking instanton is
\[ \Gamma \sim \exp -4\pi M^2 G, \]  

(5.1)

while for the Nariai instanton is

\[ \Gamma \sim \exp -\frac{\pi}{\Lambda G}. \]  

(5.2)

In any case the “hot” space cannot describe the ground state, therefore a topology change comes into play and a black hole nucleation can be realized when we consider the hot flat space, while black hole pair creation is the mechanism related to the hot de Sitter space.

It is immediate to recognize that eq. (5.2) represents the decay rate calculated with the no-boundary prescription of Hartle-Hawking. In fact, following ref. [2], we define

\[ \Gamma = \frac{P_{\text{sdS}}}{P_{\text{de Sitter}}} = \exp -\frac{\pi}{G\Lambda}. \]  

(5.3)

On the other hand, in ref. [9] the choice for the wave function solving the Wheeler-DeWitt equation was

\[ \Psi [g_{ij}] = \int dN e^{-N\omega} \Phi [g_{ij}]. \]  

(5.4)

This form was obtained with the Euclidean action defined by

\[ I = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K, \]  

(5.5)

however the three space plus one time integral over decomposition (3.8) represents the same previous action plus the boundary at \( \tau = 0 \): i.e. eq. (2.4) of ref. [2]. Then eq. (5.4) has to be modified with

\[ \Psi [g_{ij}] = \exp \left( \frac{1}{8\pi} \int_{\tau=0} d^3x \sqrt{3g} K \right) \int dN e^{-N\omega} \Phi [g_{ij}]. \]  

(5.6)

\[ ^5 \text{Actually, in ref. [9], the exponential was imaginary because the signature was of the Lontzian type.} \]

\[ ^6 \text{Obviously for the de Sitter and Nariai space, we have to correct the scalar curvature by adding } -2\Lambda. \]
Heuristically speaking, we can conjecture that the only difference with the Hartle-Hawking wave function is that $\Phi [g_{ij}]$ is a trial wave functional of the gaussian type $[5]$. It is quite obvious, at this point, that by repeating the same procedure of cutting half of the instanton, we recover the results that lead to eq. (5.3). The same approach has to be applied for the Gibbons-Hawking instanton which will be cutted in half following the same procedure of ref. [16], where a no-boundary proposal for black holes was presented.

VI. CONCLUSIONS AND OUTLOOKS

Motivated by the recent results on black hole pair creation obtained in the Euclidean sector that we have summarized in section V, we have tried to reproduce the same calculation scheme, but from the Wheeler-DeWitt equation point of view. Although the choice of separating time from space is dense of subtleties and technical problems, a partial reproduction of the above mentioned results has been obtained. Our approach has the same features of a Kaluza Klein space, if the lapse boundary conditions are of periodic type, that it means that the spacetime has topology $\mathbb{R}^3 \times S^1$ namely, flat spacetime is “hot”. However, although we limited our investigation to the TT sector and an improvement including the spin 1 and spin 0 sector is necessary to give robust conclusions, we can realize that our results about instabilities and pair creation seem to suggest a parallel method for semiclassical evaluations, at least for static metrics and consequently for a certain kind of topologies. An interesting feature of this approach is that we can have a better control on the signatures giving therefore hints on the understanding of some evolution or thermal processes. In fact, coming back to the black hole pair creation again, we see that due to boundary conditions, this process is a consequence of a hot background space, namely the cold space is not able in producing black holes or topology changes and therefore this cannot be considered a real vacuum fluctuation. However, our approach seem to be valid without imposing any periodicity and therefore any temperature. This is a direct consequence of choosing a three-dimensional wormhole, whose perturbations reside in the three-space, and since we have investigated a stationary energy
levels problem like in ordinary Quantum Mechanics. An open problem, coming from this observation, could be the possible decay of Minkowski space, namely the zero temperature version of flat space. The positive energy conjecture forbids a classical decay because \( E = 0 \)

if and only if we consider Minkowski space, otherwise \( E \geq 0 \). However, as argued in ref. [17], a possibility is related to pairs of black hole that do not reside in the same universe, but they propagate in universes separated by a wormhole. Actually, the best way to treat this splitting of black holes is by introducing the concept of quasilocal energy [18]. In our case, the Schwarzschild three-dimensional wormhole is nothing but the spatial part of a static Einstein-Rosen bridge (a section or a slice) and the quasilocal energy can be written as

\[
E = E_+ - E_-
\]  

(6.1)

The total energy is zero for boundary conditions symmetric with respect to the bifurcation surface \( S_0 \), which is the case of the asymptotically flat space, where the previous formula becomes

\[
E_{\text{Flat}} = 0 \rightarrow M + (-M) + \text{topology change}.
\]

\( M \) represents the black hole (in the upper universe), while \(-M\) represents the anti-black hole (in the lower universe). This is the same of cutting in half the four dimensional wormhole and consider only one half on a fixed slice. In this situation, we have a non vanishing probability of nucleating black holes, but in different universes [10,17]. Although there is no direct evidence of the existence of such wormholes in present days, they can be considered in the very early universe, where according to ref. [7], pure gravitational wormholes could be existed. This agrees with ref. [2], where neutral black hole pair creation could be relevant in the very early universe, where the cosmological constant could assume the pre-inflationary value \( \Lambda \sim 1 \) (in Planck’s units). To this purpose, an effective cosmological constant is needed and a way to do this is to include scalar matter fields for driving the cosmological constant close to one Planck’s unit [2]. To conclude, we want to remark that the black hole pair creation could be considered a mechanism induced by three-dimensional wormholes.
Nevertheless, our analysis is limited to a single wormhole, while it could be very interesting to consider a multi-wormhole space. Such a collection of wormholes could be viewed as a test for probing spacetime foam as a candidate for a quantum picture of gravity.

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