Gravitational perturbations about a Kerr black hole in the Newman-Penrose formalism are concisely described by the Teukolsky equation. New numerical methods for studying the evolution of such perturbations require not only the construction of appropriate initial data to describe the collision of two orbiting black holes, but also to know how such new data must be imposed into the Teukolsky equation. In this paper we show how Cauchy data can be incorporated explicitly into the Teukolsky equation for non-rotating black holes. The Teukolsky function $\Psi$ and its first time derivative $\partial_t \Psi$ can be written in terms of only the 3-geometry and the extrinsic curvature in a gauge invariant way. Taking a Laplace transform of the Teukolsky equation incorporates initial data as a source term. We show that for astrophysical data the straightforward Green function method leads to divergent integrals that can be regularized like for the case of a source generated by a particle coming from infinity.

I. INTRODUCTION AND OVERVIEW

Black holes coalescence is considered to be one of the strongest astrophysical sources of gravitational radiation and a primary candidate to be detected by gravitational wave observatories now under construction. For this reason, several groups of researchers are now attacking the problem of solving Einstein’s equations numerically, with supercomputers \cite{1}. Many difficulties must still be solved in this approach, such as the presence of instabilities in the numerical evolution due to the nonlinearity of the Einstein’s equations and that of finding a better qualitative prescription for truly astrophysical initial data representing black holes collisions.

Meanwhile, the perturbation theory represents a valid and complementary approach to full numerical simulations, with some clear advantages with respect the supercomputer project: It is a considerably more economical approach and it is semianalytical. This latter aspect is a very important one since it allows to study some fundamental and conceptual problems. Among the new theoretical results in perturbation theory stands out the close limit approximation \cite{2} which approximates the collision of two black holes by a single perturbed one. The subsequent extension to second order perturbations \cite{3}, and to moving (head-on) holes \cite{4}, confirmed the success of this approximation. Also recently \cite{5,6,7}, the collision of a small and a big black hole by perturbative methods have been revisited in order to incorporate non-vanishing initial data. However, only perturbations about Schwarzschild black holes have been studied so far, by use of the Zerilli-Moncrief \cite{8,9} equation (a gauge invariant description). The success of these techniques encourages now to extend them to the more realistic rotating background. In particular, for the most plausible astrophysical scenario of a final single rotating hole, the first order “curvature” perturbations are compactly described by the Teukolsky equation \cite{10,11}, which in the non-rotating case reduces to the so called Bardeen-Press equation \cite{12}.

The Teukolsky equation has been studied already in the early seventies in the frequency domain. In order to avoid the important problem of the imposition of the initial data into that equation, the computation of the gravitational radiation have been carried out in the case of unbounded particle trajectories (or circular motion) \cite{13}. The divergent integrals encountered when one wants to calculate the gravitational radiation generated by a test mass falling into a black hole from infinity using the standard Green function, have been made finite by systematically discarding infinite surface terms \cite{14}. In a different way to tackle this problem, Nakamura and Sasaki \cite{15} transformed, in the frequency domain, the Teukolsky equation into other “more regular” equation. Only very recently, Poisson \cite{16} (in the non-rotating case) and Campanelli and Lousto \cite{17} (for the rotating hole) showed that there is nothing intrinsically wrong
with the radial Teukolsky equation when dealing with unbounded source terms, and that the divergent integrals can be regularized in a natural way. In the Appendix, we will show how those results about regularization also apply when dealing with initial data for a *bounded* problem (like for the close limit case).

The problem of imposition of initial data in the Teukolsky equation is not, of course, restricted to the frequency domain form of the equation. Very recently an evolution code to integrate the Teukolsky equation in its time-domain form have been developed \[13\]. However, until now the evolution of initial data have been restricted to the simple case of a bell-shaped burst, which cannot represent accurately realistic astrophysical initial data for the late stages of binary black hole coalescence. The imposition of the initial data into the Teukolsky equation, to our knowledge, seems not to have a clear framework yet. Consequently, in this paper we address to the important issue of the imposition of the initial data into the Teukolsky equation. In the non-rotating case, the gauge invariant Teukolsky functions \( \psi_0 \) (ingoing) and \( \psi_4 \) (outgoing), and their first time derivatives can be completely expressed in terms of the Moncrief waveform (and its time derivative), and then, in terms of the 3-geometry and the extrinsic curvature. By evaluating these functions on a constant time hypersurface for any specific astrophysical initial data, one is in principle able to numerically evolve the Teukolsky equation and to study very interesting astrophysical scenarios, like the close limit approximation \[19\]. The imposition of initial data into the Bardeen-Press equation \[12\] can suggest what features need to be generalized in the rotating case.

The paper is organized as follows: Section II is devoted to the problem of expressing the Teukolsky functions \( \psi_0 \) and \( \psi_4 \) on a given hypersurface in terms of the 3-geometry and the extrinsic curvature of that hypersurface. The computations are specifically performed in the Regge-Wheeler gauge \[20\] for both even and odd parity waves and then reexpressed in a gauge invariant way by use of the Moncrief functions. In Sec. III we discuss the applicability of these relations and the possibility of generalization of these results to the rotating case. In the Appendix we briefly review the Teukolsky equation in its frequency domain and use the Laplace transform to mathematically incorporate initial data into the radial equation as an effective source term. We explicitly replace Brill-Lindquist initial data into the Teukolsky equation in its frequency domain and use the Laplace transform to mathematically incorporate initial data into the radial Teukolsky equation. The source term is shown to lead to a divergent integral as the domain form of the equation. Very recently an evolution code to integrate the Teukolsky equation in its time-domain extends to infinity. The regularization procedure already used for unbounded sources is employed successfully in the present case as described in the Appendix.

The notation and conventions employed in this paper are those of Misner, Thorne and Wheeler \[21\]. That is, the metric signature is (− + + +); the Riemann tensor is defined by the equation \( R^{\alpha\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\gamma\delta} - \Gamma^{\alpha}_{\beta\delta\gamma} + \Gamma^{\alpha}_{\gamma\delta\beta} - \Gamma^{\alpha}_{\gamma\beta\delta} \), and the Ricci tensor by equation \( R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \). Throughout the paper an overdot indicates differentiation with respect to the time variable \( t \), a prime differentiation with respect to the radial coordinate \( r \), an overbar represents complex conjugation, and units are such that \( G = c = 1 \). The conventions for the NP quantities, i.e. spin coefficients, operators (denoted by an overhat and the Kinnersley tetrad), employed in this paper are those given in the table of Ref. \[17\].

II. INITIAL DATA FOR THE TEUKOLSKY EQUATION

In the Newman–Penrose formalism, gravitational perturbations of the Kerr metric can be decoupled and described by the Teukolsky \[11\] equation. In the Boyer-Lindquist coordinates \( (t, r, \theta, \varphi) \) this equation takes the following form

\[
\begin{align*}
\left\{ \left[ a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right] \partial_t - \frac{4M ar}{\Delta} \partial_\varphi - 2s \left[ (r + ia \cos \theta) - \frac{Mr^2 - a^2}{\Delta} \right] \partial_t \\
+ \Delta^{-s} \partial_r \left( \Delta^{s+1} \partial_r \right) + \frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta \right) + \left[ \frac{1}{\sin^2 \theta} \frac{a^2}{\Delta} \right] \partial_\varphi \\
+ 2s \left[ \frac{a(r - M)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right] \partial_\varphi - s \left( s \cot^2 \theta - 1 \right) \right\} \Psi = 4\pi \Sigma T,
\end{align*}
\]

where \( M \) is the mass of the black hole, \( a \) its angular momentum per unit mass, \( \Sigma \equiv r^2 + a^2 \cos^2 \theta \), \( \Delta \equiv r^2 - 2Mr + a^2 \), and \( s \) is the spin parameter characterizing the perturbation.

The field \( \Psi \) represents either the outgoing radiative part of the perturbed Weyl tensor, \( \psi^{(1)}_4 \) \((s = -2)\), or the ingoing radiative part, \( \psi^{(1)}_0 \) \((s = +2)\). The corresponding unperturbed quantities \( \psi^{(0)}_4 \) and \( \psi^{(0)}_0 \) vanish identically in the Kerr spacetime.

\[
\Psi(t, r, \theta, \varphi) = \begin{cases} 
\rho^{-4} \psi^{(1)}_4 & \text{for } s = -2 \\
\rho^{-4} \psi^{(1)}_0 & \text{for } s = +2
\end{cases}
\]

(2.2)
The knowledge of only one of these two first order complex NP components, formed by projecting the perturbed Weyl tensor along the Kinnersley tetrad, is sufficient \([2,4]\) to uniquely and completely specify the gravitational perturbation (up to changes of the Kerr parameters \(a\) and \(M\)). In fact, \(\psi^{(1)}_4\) and \(\psi^{(1)}_0\) are gauge invariant under both coordinate transformations and infinitesimal tetrad (Lorentz) rotations. These two scalars carry information, in their real and imaginary parts, about the two dynamical degrees of freedom of the perturbed field. More precisely, in the nonrotating case, the real part of \(\Psi\) contains information about the even parity (or polar) perturbations of the metric (which are invariant under the transformation \(\varphi \leftrightarrow -\varphi\)) while its imaginary part contains information of odd parity (or axial) perturbations of the metric (that change sign under a sign reversal of \(\varphi\)). The source term \(T\) in Eq. \((2.1)\) is constructed out of the \(T_{\mu\nu}^{(1)}\) and its expression can be found in Refs. \([1,4,7]\).

In order to study perturbations around a Kerr hole one has to supplement the Teukolsky equation with initial data that allow to start the integration. So far the Teukolsky equation has been studied (in the frequency domain) for perturbations generated by particles coming from infinity, i.e. with vanishing initial data. There are, though, astrophysically relevant situations where initial data contribution are important. This is the case of the very successful close limit approximation \([2]\). We can divide the initial value problem for the Teukolsky equation into two subproblems: First, one must find what Cauchy data can better represent astrophysical scenarios for colliding black holes. Slices of constant Boyer-Linquist time of the Kerr geometry give a non-conformally flat three-geometry, making it incompatible with all initial data studied so far. It is not known wheter a slice of the Kerr geometry exists such that its three-geometry is conformally flat. It seems that, on the other hand, it is simpler to find solutions to the initial value problem compatible with Kerr perturbations on the usual hypersurfaces \([23,24]\). In the present paper we will be more concerned with the question of how to impose initial data to the Teukolsky equation.

Cauchy data in General Relativity consists of \((g_{ik}, K_{ik})\) and a three-dimensional spacelike hypersurface \(\Sigma_t\), where \(g_{ik}\) is a Riemannian three-metric on \(\Sigma_t\), and \([21]\)

\[K_{ik} = \frac{1}{2N}(N_{i,k} + N_{k,i} - \partial_t g_{ik} - 2\Gamma^p_{ik} N_p)\]  \((2.3)\)

is the extrinsic curvature which describes the “embedding” of \(\Sigma_t\) in the spacetime, written in terms of the lapse \(N = (-g^{tt})^{-1/2}\) and of the shift \(N_t = g_{tt}\). In perturbation theory, to first order in the metric perturbation about the background metric \(g^{(0)}_{\mu\nu}\), the Cauchy data for Einstein’s equations, \(\delta G_{\mu\nu}^{(1)} = 8\pi \delta T_{\mu\nu}^{(1)}\), are given in terms of the first order perturbations of the metric, \(h_{ik}^{(1)}\), and in terms of the first order extrinsic curvature, \(K_{ik}^{(1)}\), on a constant time hypersurface. In the case of the Teukolsky equation, the first order initial value problem must be translated into terms of \(\Psi|_{t=0} = 0\) and \(\partial_t \Psi|_{t=0}\). As we will show in this Section relating \((h_{ik}^{(1)}, K_{ik}^{(1)})\) at the initial hypersurface \(t = 0\) with \((\Psi|_{t=0}, \partial_t \Psi|_{t=0})\) is not straightforward.

The starting point is given by Chrzanowski \([21]\), who related the Teukolsky functions to metric perturbations

\[\psi^{(1)}_4 = \frac{1}{2} \left\{ (\ddot{\delta} - 3\alpha - \dot{\beta} - \pi + \tau)(\ddot{\delta} - 2\alpha - 2\dot{\beta} + \pi + \tau)h_{nn} + (\ddot{\Delta} - \mu - \dot{\mu} - 3\gamma + \tau)(\ddot{\Delta} + \mu - \dot{\mu} - 2\gamma + 2\tau)h_{m\bar{m}} \\
- 2 \left[ (\ddot{\Delta} - \mu - \dot{\mu} - 3\gamma + \tau)(\dot{\pi} + \tau) + (\ddot{\Delta} - 3\alpha - \dot{\beta} - \pi + \tau)(\ddot{\Delta} - 2\dot{\mu} - 2\gamma) \right] h_{n\bar{m}} \right\} \]  \((2.4)\)

and

\[\psi^{(1)}_0 = \frac{1}{2} \left\{ (\ddot{\sigma} + \dot{\alpha} + 3\beta - \ddot{\pi} + \tau)(\ddot{\sigma} + 2\alpha + 2\beta - \ddot{\pi} - \tau)h_{ll} + (\ddot{D} + \rho + \dot{\rho} + 3\epsilon - \dot{\epsilon})(\ddot{D} + \rho - \dot{\rho} + 2\epsilon - 2\dot{\epsilon})h_{mm} \\
- 2 \left[ (\ddot{D} + \rho + \dot{\rho} + 3\epsilon - \dot{\epsilon})(\dot{\tau} + \ddot{\pi}) + (\ddot{\sigma} + \dot{\alpha} + 3\beta - \ddot{\pi} + \tau)(\ddot{D} + \dot{\rho} + 2\epsilon) \right] h_{l\bar{m}} \right\} \]  \((2.5)\)

where \(h_{nn} = n^\mu n^\nu h_{\mu\nu}\), \(h_{l\bar{m}} = l^\mu m^\nu h_{\mu\nu}\), etc., contain not only three-metric perturbations but all the components of the first order metric perturbations (here \(l, n, m\) and \(\bar{m}\) are the null vectors of the Kinnersley tetrad). This represents a practical problem at the moment of imposing initial data, since one is normally given only the 3-geometry and the extrinsic curvature, not the 4-geometry and its time derivative (nor we should need it). The point is that \(\psi_4\) (and \(\psi_0\)) are gauge invariant quantities, but we do not have them explicitly expressed in terms of purely hypersurface quantities.

Many simplifications in the analysis are possible when the background has spherical symmetry. In the Schwarzschild black hole case expressions \([2,4,5-2.3]\) reduce to
parity perturbations are given by the real part (Re) of $\psi$ and because the radiated energy can be described in terms of only the derivative are given by $h_{\theta \theta} - h_{\varphi \varphi}/\sin^2 \theta - 2i h_{\varphi \theta}/\sin \theta$. Below we shall decompose all metric perturbations in multipoles with (implicit) index $\ell m$ when, for the sake of simplicity, we have chosen to work in the Regge-Wheeler gauge. To do so, we will use a result of Ref. [26] where the metric perturbations in the RW gauge have been expressed in terms of $H_{\ell m}$ and its time derivative are given by

$$h_{\theta \theta} = 1 - 2M/r$$ and $f' = 2M/r^2$.

The imposition of spherical symmetry carry also the following computational (but not necessarily crucial) advantage: the multipole decomposition of the metric perturbations in terms of spin-weighted harmonics $-2Y^m_\ell(\theta)$ can be performed [27], and even and odd parity perturbations decouple so they can be considered independently. Below we shall decompose all metric perturbations in multipoles with (implicit) index $\ell m$ (not to be confused with the tetrad vectors).

### A. Even or polar parity waves

From now on we will consider only $\psi_4$ because of its better behavior for numerical integration compared to $\psi_0$ and because the radiated energy can be described in terms of only $\psi_4$. Similar equations will hold for $\psi_0$. Even parity perturbations are given by the real part (Re) of $\psi_4$. In the Regge-Wheeler (RW) gauge, $h_{\theta \theta} = h_{\varphi \varphi} = h_{\varphi \theta} = h_{r \theta} = 0, h_{\theta \varphi} = h_{\varphi \theta}/\sin^2 \theta$, we have $h_{\ell m} = h_{nm} = h_{mm} = h_{lm} = 0$, then Eq. (2.6) and its time derivative are given by

$$\text{Re} \psi_4^{(1)} = -\frac{(1 - 2M)}{16r^2} \sum_{\ell m} \sqrt{(\ell - 1)/(\ell + 1)}(\ell + 2)\{H_0 - 2H_1 + H_2\} Y^m_\ell$$

$$\text{Re} \partial_\ell \psi_4^{(1)} = -\frac{(1 - 2M)}{16r^2} \sum_{\ell m} \sqrt{(\ell - 1)/(\ell + 1)}(\ell + 2)(\partial_\ell H_0 - 2\partial_\ell H_1 + \partial_\ell H_2) Y^m_\ell$$

Using Einstein’s equations in the RW gauge, Zerilli’s [8], Eq. (C7e) gives,

$$H_0 = H_2 + \frac{32\pi r^2}{\sqrt{2(\ell - 1)/(\ell + 1)(\ell + 2)}} F_{\ell m},$$

where $F_{\ell m}$ is a source term given in Table III of Ref. [8]. The above expressions allow us to write Eqs. (2.8) in terms of $H_2, H_1,$ and its time derivatives. Note that here and below there is an implicit index $\ell m$ coming from the multipole decomposition of the metric perturbations. The following step is to restore the gauge invariance, lost in principle when, for the sake of simplicity, we have chosen to work in the Regge-Wheeler gauge. To do so, we will use a result of Ref. [27] where the metric perturbations in the RW gauge have been expressed in terms of $\phi_M$, the Moncrief invariant function

$$\phi_M = \frac{r}{\lambda + 1} \left[ K + \frac{r - 2M}{\lambda r + 3M} (H_2 - r \partial_\ell K) + \frac{(r - 2M)}{\lambda r + 3M} (r^2 \partial_\ell G - 2h_1) \right], \quad \lambda = \frac{1}{2}(\ell - 1)(\ell + 2),$$

which is expressed entirely in terms of the 3-geometry and is gauge invariant under changes of coordinates in the spacetime. The other piece of initial data, $\partial_\ell \phi_M$, is expressed in terms of the extrinsic curvature $K_{ij}$ as in Ref. [27]
\[ \partial_t \phi_M = \frac{-2r}{\lambda + 1} \left[ \sqrt{1 - \frac{2M}{r} K_{(K)}} + \frac{r - 2M}{\lambda r + 3M} \left( \frac{1 - \frac{2M}{r}}{K_{rr}} - r \partial_r \left( \sqrt{1 - \frac{2M}{r} K_{(K)}} \right) \right) \right] + \frac{(r - 2M)}{\lambda r + 3M} \left( r^2 \partial_r \left( \sqrt{1 - \frac{2M}{r} K_{(G)}} \right) - 2 \sqrt{1 - \frac{2M}{r} K_{r}^{\text{even}} \right) \right) \] (2.12)

where \( K_{(K)} \) and \( K_{(G)} \) are respectively the "K" and "G" parts of the extrinsic curvature, in analogy to the Regge-Wheeler \([20]\) decomposition of the metric tensor (see also a similar decomposition for \(\tilde{K}\) in Eqs. (2.32)-(2.35) of Ref. \([6]\)). Also, here and below, extrinsic curvature components (of even and odd parity) have indices \( \ell m \) implicit. In Ref. \([26]\) it was found by use of the definition (2.11) and the Hamiltonian constraint (see for instance Zerilli’s equation \((7a)\) in Ref. \([8]\))

\[ H_2 = -\frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2 (\lambda + 1)r^3}{r^2(\lambda r + 3M)^2} \phi_M + \frac{3M^2 - \lambda M r + \lambda r^2}{r(\lambda r + 3M)} \phi_M + (r - 2M) \phi_M'' \] (2.13)

here and below, primes denote \( \partial_r \).

From the momentum constraint, equation (C7b) in Ref. \([8]\)

\[ H_1 = r \partial_t \phi_M' + \frac{\lambda r^2 - 3\lambda M r - 3M^2}{(r - 2M)(\lambda r + 3M)} \partial_t \phi_M. \] (2.14)

The time derivatives of these metric coefficients in terms of \( \phi_M \) have also been found in Ref. \([26]\)

\[ \partial_t H_2 = (r - 2M) \partial_t \phi_M'' + \frac{3M^2 - \lambda M r + \lambda r^2}{(\lambda r + 3M)r} \partial_t \phi_M' \\
- \frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2 (\lambda + 1)r^3}{(\lambda r + 3M)^2r^2} \partial_t \phi_M \] (2.15)

and from the time derivative of Eq. (2.14), \( \partial_t H_1 = r \partial_t^2 \phi_M' + (\lambda r^2 - 3\lambda M r - 3M^2)/(r - 2M)/(\lambda r + 3M) \partial_t^2 \phi_M \), and the Zerilli \([8]\) wave equation in vacuum, \( \partial_t^2 \phi_M = \partial_t^2 \phi_M - V_\ell(r) \phi_M \), we obtain

\[ \partial_t H_2 = \frac{(r - 2M)^2}{r} \phi_M'' + \frac{(r - 2M)(\lambda r^2 + 3\lambda M r + 15M^2)}{r^2(\lambda r + 3M)} \phi_M' \\
+ \left\{ \frac{2M \left[ -\lambda r^2 + 3(\lambda - 2) M r + 15M^2 \right]}{r^3(\lambda r + 3M)} - r V_\ell(r) \right\} \phi_M \\
- \left\{ \frac{\lambda r^2 - 3\lambda M r - 3M^2}{(r - 2M)(\lambda r + 3M)} V_\ell(r) + r V_\ell'(r) \right\} \phi_M \] (2.16)

where \( V_\ell(r) \) is the Zerilli potential \([8]\). Here it is evident that in order to rewrite \( \psi_4 \) and \( \partial_t \psi_4 \) in terms of hypersurface data we had to use the evolution equations, not only the constraints.

For the sake of simplicity we have written Eqs. (2.13)-(2.16) in the sourceless case, but source terms can be straightforwardly included \([24]\). The above relations together with (2.11) provide the expressions that allow us to write \( \psi_4 \) and \( \partial_t \psi_4 \) in terms of \( \phi_M \) and \( \partial_t \phi_M \) (and radial derivatives of them). This relation holds for all \( r \) and \( t \) and is gauge invariant since all involved final expressions are (in spite of having used the RW gauge as an intermediate step), in particular, this relation holds at the hypersurface where we want to give initial data. Since \( \phi_M \) can be written only in terms of the perturbed 3-geometry and \( \partial_t \phi_M \) in terms of only the extrinsic curvature (cfr. Eqs. (2.11) and (2.12)) we will have, at \( \Sigma_t \)

\[ \psi_4 = \psi_4(h_{ij}, K_{ij}), \quad \partial_t \psi_4 = \partial_t \psi_4(h_{ij}, K_{ij}) \] (2.17)

our desired relations. Note that, in general, even for stationary, time-symmetric initial data neither \( \psi_4 \) nor \( \partial_t \psi_4 \) vanish. This is in contrast with what happens with the corresponding initial data for the Moncrief wave form, where \( \partial_t \psi_M = 0 \), and with the initial data taken in Ref. \([8]\).
B. Odd or axial parity waves

For odd parity perturbations we shall parallel the procedure we just followed in the even parity case. Let us recall that odd parity is described by the imaginary part (Im) of $\psi_4$. In the RW gauge ($h_2 = 0$) we have

$$\text{Im} \psi^{(1)}_4 = -\frac{1}{8r^2} \left[ \partial_t - \left( 1 - \frac{2M}{r} \right) \partial_r + \frac{2M}{r^2} \right] \sum_{l,m} \sqrt{(l-1)(l+1)(l+2)} \left( h_0 - \left( 1 - \frac{2M}{r} \right) h_1 \right) -2Y^{lm}_\ell (2.18)$$

and

$$\text{Im} \partial_t \psi^{(1)}_4 = -\frac{1}{8r^2} \left[ \partial_t - \left( 1 - \frac{2M}{r} \right) \partial_r + \frac{2M}{r^2} \right] \sum_{l,m} \sqrt{(l-1)(l+1)(l+2)} \left( \partial_t h_0 - \left( 1 - \frac{2M}{r} \right) \partial_t h_1 \right) -2Y^{lm}_\ell (2.19)$$

here $h_1$ and $h_0$ are the odd parity metric perturbations (not to confuse with the same symbols used for the even parity perturbations, that in the RW gauge are taken to vanish).

The corresponding gauge invariant waveform for odd parity given by Moncrief [5] is

$$Q = \frac{1}{r} \left( 1 - \frac{2M}{r} \right) \left[ h_1 + \frac{1}{2} \left( \partial_t h_2 - \frac{2}{r} h_2 \right) \right]$$

and

$$\partial_t Q = \frac{1}{r} \left( 1 - \frac{2M}{r} \right) \left[ \sqrt{1 - \frac{2M}{r} K^{\text{odd}}_{\varphi\varphi}} + \frac{1}{2} \left( \partial_r \left( \sqrt{1 - \frac{2M}{r} K^{\text{odd}}_{\varphi\varphi}} \right) - \frac{2}{r} \sqrt{1 - \frac{2M}{r} K^{\text{odd}}_{\varphi\varphi}} \right) \right].$$

Again, in the RW gauge ($h_2 = 0$) we can write (from Eq. (2.20))

$$h_1 = \frac{r}{1 - \frac{2M}{r}} Q, \quad \partial_t h_1 = \frac{r}{1 - \frac{2M}{r}} \partial_t Q,$$

(2.22)

from Zerilli’s [5] Eq. (C6c), one obtains

$$\partial_t h_0 = \left( 1 - \frac{2M}{r} \right) \left[ \partial_t (rQ) + \frac{8\pi i r^2 D_{lm}}{\sqrt{(\ell - 1)(\ell + 1)(\ell + 2)/2}} \right]$$

(2.23)

where $D_{lm}$ is a source term given in Table III of Ref. [8], and finally $h_0$ can be obtained from the definition of extrinsic curvature, i.e. Eq. (2.3), $\partial_t h_2 = 2 \left( \sqrt{1 - \frac{2M}{r} K_{\varphi\varphi}} - h_0 \right)$, then in the RW gauge

$$h_0 = \sqrt{1 - \frac{2M}{r} K_{\varphi\varphi}}$$

(2.24)

In spite of having constructed Eqs. (2.22)-(2.24) in the RW gauge, when we replace their right hand side in the expressions for Im $\psi_4$ and Im $\partial_t \psi_4$, given by the Eqs. (2.18) and (2.19), they will be gauge invariant if we consider the general form of $Q$ and $K_{\varphi\varphi}$ . We have then succeeded in our objective of expressing also the odd parity perturbations (the imaginary part of $\psi_4$) in terms only of the 3-geometry and the extrinsic curvature.

III. DISCUSSION

In this paper we have faced (and solved for the $a = 0$ case) the problem of giving initial data to the equation that describe in a gauge invariant way perturbations around the Kerr geometry, i.e. the Teukolsky equation in the time and frequency domains. We first observed that the connection between metric perturbations and gauge invariant objects that fully describe gravitational perturbations $\psi_4$ (or $\psi_0$) as given by Chrzanowski [2], is not explicitly written only in terms of the 3-geometry and the extrinsic curvature. This poses a difficulty when one wants to use this relation to impose initial data, since, normally one is given only the 3-geometry and the extrinsic curvature on a given hypersurface (and that should be all we need). In the nonrotating ($a = 0$) case (Schwarzschild background), we have been able to reexpress $\psi_4$ (and $\psi_0$) and its time derivative in terms only of the 3-geometry and the extrinsic
curvature. To do so, we have momentarily gone to the Regge-Wheeler gauge and then restored gauge invariance by writing the metric perturbations in terms of $\phi_M$, the Moncrief’s invariant waveform and its time derivative (and radial derivatives on the hypersurface). As a by-product we obtained relations between the waveforms of Teukolsky and Moncrief (of course, they are not independent of each other) on any given hypersurface. These relations (for both the even and odd parity cases) represent the generalization of the Chandrasekhar transformations in the time domain when source terms are present. It is worth stressing here that to obtain the above described relations we used the evolution equations in order to eliminate the higher than the first time derivatives of the metric perturbations in the RW gauge.

An interesting application of these relations is to compute the “close limit approximation” for black hole collisions by directly integrating the Teukolsky equation \[1\]. This constitutes a test for both, our relations on each hypersurface and the numerical accuracy of the code \[3\] for integrating the Teukolsky equation.

All this could be accomplished for the nonrotating case, which provides the $a = 0$ limit to be recovered by the more complicated (and realistic) case of $a \neq 0$, for which many steps should be generalized. In fact, in the rotating case we do not have the multipole decomposition of the metric perturbations, nor a Regge-Wheeler gauge that can be reexpressed explicitly in terms of other invariants of the 3-geometry and the extrinsic curvature only, neither the generalization of $\phi_M$ for that case (that would be an interesting result in itself). For these reasons a more geometrical approach (making use of the Gauss-Codazzi relations) should be followed in order to rewrite $\psi_4$ in the Kerr background \[28\].

A second aspect we have been able to study is the question of the regularization of the Green function solution to the Teukolsky equation in the frequency domain when nonvanishing initial data are present (see Appendix). The appropriate method to incorporate initial data in this case is to Laplace transform the Teukolsky equation. In this way, initial data appear as an additional source term and can be easily manipulated. We have explicitly shown in the nonrotating case and using Brill-Lindquist initial data that $\ell = 2$ and $\ell = 3$ modes need to be regularized. This has to be a general feature, valid also for the rotating background and for any other kind of astrophysically reasonable initial data for black hole collisions, and has to do with the fact that realistic initial data are not of compact support (or do not decay fast enough at spatial infinity).

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APPENDIX A: REGULARIZATION OF THE INITIAL DATA

In this appendix we shall consider again the problem of divergent integrals that appear when in the frequency domain one tries to solve the Teukolsky equation with sources that extend to infinity using the standard Green function technics. It was first proven by Poisson \[16\] that, for a source generated by a particle released at rest from infinity (which has vanishing initial data), the divergent integrals appearing in the formal solution can be regularized in the nonrotating case. Soon after, the present authors \[17\] extended to the rotating (Kerr) background Poisson’s approach. Here, we show how the same problem appears when one consider bounded sources, but nonvanishing initial data, like Misner or Brill-Lindquist data \[29,30\] which are relevant to study black hole head-on collisions, in the close limit approximation. We thus apply the same regularization technics developed in references \[16,17\] to show that the Green function formulation can be regularized. We shall refer to the above two references for further details of the problem and computations, and below only sketch how things go for the case of nonvanishing initial data.

It is well known that the Teukolsky equation \[27\] can be separated by mode decomposition of the field and source into a complex radial wave equation and an angular equation satisfied by the so-called spin-weighted oblate spheroidal harmonics $sS^{\ell m} (\theta, a \omega) \[31\] , that in the case $a = 0$ reduce \[32\] to the spin-weighted spherical harmonics $Y^{\ell m} (\theta)$. By use of this separation of variables, the Teukolsky equation, in its frequency domain, was widely studied in the early seventies \[14,13\] providing extraordinary results like the proof of the stability of rotating black hole and the computation of the gravitational radiation for unbounded particle trajectories around rotating black holes.

A Fourier analysis in the complex plane can be used in the case one want to study the problem of a particle falling into the black hole from any finite distance. In this case one must consider the initial value problem, with $\Psi$ and $\partial_t \Psi$ specified on an initial hypersurface (at $t = 0$). To this purpose it is very useful to define the Laplace transform $\Psi$ of $\Psi$ to be
\[ \Psi(\omega, r, \theta, \varphi) \equiv \int_0^\infty e^{i\omega t} \Psi(t, r, \theta, \varphi) \, dt . \]  

(A1)

We take \( \Psi \) to vanish for \( t < 0 \), which means that \( \Psi \) must be analytic in the upper half of the complex \( \omega \) plane. Applying the Laplace transform to Eq. (2.1), the Teukolsky equation may be written in the following form

\[
\left\{ \begin{array}{l}
\Delta^{-1} \partial_r \left( \Delta^{\ell+1} \partial_r \right) - \omega^2 \left[ a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right] + 2i\omega \left[ (r + ia \cos \theta) - \frac{M(r^2 - a^2)}{\Delta} \right] \\
+ \frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta \right) + \left[ \frac{1}{\sin^2 \theta} \frac{a^2}{\Delta} \right] \partial_\varphi + 2s \left[ \frac{a}{\Delta} (r - M + \frac{2i\omega Mr}{s}) + \frac{i \cos \theta}{\sin^2 \theta} \partial_\varphi \right] \\
- s \left( s \cot^2 \theta - 1 \right) \end{array} \right\} \Psi = T + S_{ID} ,
\]

(A2)

where

\[
S_{ID} = \left[ a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right] \left( i \omega \right) \Psi|_{t=0} - \left[ a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right] \left( i \omega \right) \Psi|_{t=0} .
\]

(A3)

and

\[
T(\omega, r, \theta, \varphi) \equiv \int_0^\infty e^{i\omega t} 4\pi \Sigma T(t, r, \theta, \varphi) \, dt .
\]

(A4)

In this way Cauchy data are explicitly incorporated into the Teukolsky equation as part of the source term. Eq. (A2) can be separated into a radial and an angular part and applied to study particular problems involving finite radii infall.

Let us consider the source term given by Eq. (A3) in the nonrotating case

\[
\frac{S_{ID}}{\Delta^2} = \frac{1}{r^4 (1 - 2M)^3} \left[ (i \omega r^2 + 4r - 12M) \psi_4(t = 0, r) - r^2 \partial_r \psi_4(t = 0, r) \right] \equiv \tilde{G} g
\]

(A5)

where \( \tilde{G} = \sqrt{(\ell - 1)\ell(\ell + 1)(\ell + 2)(2\ell + 1)/(4\pi)} \), and \( g(r) \) takes the following form for the Brill-Lindquist initial data

\[
g(r) = \frac{\left( i \omega r^2 + 4r - 10M \right)}{(r - 2M)^2} \left( \frac{\tau}{r} \right)^{1/2} \sum_{\ell} \left( \frac{M}{\tau} \right)^{\ell+1}
\]

(A6)

where \( \tau \) is the coordinate in the conformal space, that we can take as the isotropic radial coordinate, related to \( r \) as \( \tau = (\sqrt{r} + \sqrt{r - 2M})^2/4 \).

This expression holds for the case of two equal masses black holes initially at rest, and in the perturbative regime (close limit \( a \rightarrow 0 \)), but a very similar expression holds for the case of unequal masses (included the particle limit). In particular, the conclusions about regularization will hold the same also for different initial data such as the Misner ones (symmetrization through the throats).

As noted by Poisson [16] the divergent integral generated by the Green method, \( I_{\text{div}} \), can be regularized by introducing a function \( h(r) \) and integrating by parts i.e. writing it in the following form

\[
I_{\text{div}} = \int \left[ \tilde{G} g(r) \mathcal{G}_{\text{div}} e^{i\omega t} \mathcal{L} X^{H,\infty} + \frac{d}{dr} \left( h e^{i\omega t} \mathcal{L} X^{H,\infty} \right) \right] dr - h e^{i\omega t} \mathcal{L} X^{H,\infty} \bigg|_{\text{boundaries}}
\]

(A7)

where \( \mathcal{G}_{\text{div}} = 2i\omega^2 (1 - 3M/r + i\omega r) \), \( \mathcal{L} = d/dr^* + i\omega \) (with \( r^* = r + 2M \ln (r/2M - 1) \)), and \( X^{H,\infty} \) are solutions to the Regge-Wheeler equation (in the frequency domain) with purely ingoing and purely outgoing behavior in the horizon and at infinity respectively. The function \( h(r) \) is determined by
\[ h = -\hat{G}e^{-i\omega r'} \int e^{i\omega r'} g\text{div}dr. \]

By direct power counting in the first addend of Eq. (A7) we observe that the more important modes for computing gravitational radiation, modes \( \ell = 2 \) and \( \ell = 3 \) (in the unequal masses case) need to be regularized since the integral diverges linearly and logarithmically, respectively.

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