A NOTE ON THE FRACTALIZATION OF SADDLE INVARIANT CURVES IN QUASIPERIODIC SYSTEMS

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Abstract. The purpose of this paper is to describe a new mechanism of destruction of saddle invariant curves in quasiperiodically forced systems, in which an invariant curve experiments a process of fractalization, that is, the curve gets increasingly wrinkled until it breaks down. The phenomenon resembles the one described for attracting invariant curves in a number of quasiperiodically forced dissipative systems, and that has received the attention in the literature for its connections with the so-called Strange Non-Chaotic Attractors. We present a general conceptual framework that provides a simple unifying mathematical picture for fractalization routes in dissipative and conservative systems.

1. Introduction. The fractalization route is a mechanism of transition to chaos in a family of quasiperiodically forced dynamical systems: when tuning a control parameter, a smooth invariant curve gets increasingly wrinkled until it breaks down, while no collision with other invariant objects is observed. This mechanism was first described in attracting invariant curves of one-dimensional non-invertible quasiperiodic systems [31] (see also [8, 32, 34]), and later in two-dimensional dissipative quasiperiodic systems [37]. These pioneering papers suggested that the mechanism led to the creation of a strange attractor that is not chaotic [15], a description that has been later refuted partially by more careful numerical computations [23, 29], showing that these attractors were in fact smooth (although very wrinkled) curves. Chaos is supposed to appear in the form of the appearance of a strange chaotic attractor when the control parameter is such that the maximal Lyapunov exponent of the attractor becomes positive. References [18, 21] provide a conceptual framework that describes the phenomenon in terms of spectral and reducibility properties of the linearized dynamics around the invariant curve.

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also [29], in which a mathematical proof of the presence of the fractalization route in a simple quasiperiodic affine system is given.

The previous references discuss fractalizations of attracting invariant curves. As far as we know, there are no descriptions of similar behaviours for saddle invariant curves. In this paper, we describe the fractalization route for saddle invariant curves in quasiperiodic systems. Following the previous lines of thought, the mechanism is associated to loss of reducibility properties of the linearized dynamics around the saddle invariant curve, prior to the fact that one Lyapunov exponent crosses zero. The study of quasiperiodic bifurcations for non-reducible invariant curves is one of the challenging problems stated in [40].

The paper is organized as follows: In Section 2 we give concise definitions and results of quasiperiodic systems, their invariant curves and the fractalization route. In Section 3 we study a numerical model of a 3D volume preserving quasiperiodic system where the fractalization route is observed. Finally, in Section 4 we conclude the paper linking previous results of dissipative fractalization routes with the one presented in this paper, giving a general framework and conjectures comprising both.

2. Quasiperiodic systems, invariant curves and their spectral properties.

Here we give a short account of results of quasiperiodic systems and their invariant objects. More detailed expositions are [11, 20, 38, 39].

2.1. Fiberwise hyperbolic invariant curves of quasiperiodic systems. A quasiperiodically forced system, for short quasiperiodic system, is a skew product over a quasiperiodic rotation that is a smooth bundle map \((F, R_\omega) : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n \times \mathbb{T}\) of the form

\[
\begin{cases}
\bar{z} = F(z, \theta) \\
\bar{\theta} = \theta + \omega \mod 1
\end{cases}
\]

(1)

where \(z \in \mathbb{R}^n, \theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}\) and \(R_\omega(\theta) = \theta + \omega\) is the rotation by an irrational number \(\omega\).

An invariant curve \(K\) of System (1) is the graph of a continuous map \(K : \mathbb{T} \to \mathbb{R}^n\) that satisfies the invariance equation

\[
F(K(\theta), \theta) = K(\theta + \omega).
\]

(2)

The linearized dynamics around the curve \(K\) is given by a quasiperiodically forced linear skew product system, for short quasiperiodic linear system. This is the vector bundle map \((M, R_\omega) : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n \times \mathbb{T}\) of the form

\[
\begin{cases}
\bar{v} = M(\theta)v \\
\bar{\theta} = \theta + \omega \mod 1
\end{cases}
\]

(3)

where \(v \in \mathbb{R}^n\) and \(M(\theta) = DF(K(\theta), \theta)\) is the transfer matrix. We also consider the action of the vector bundle map onto sections of the bundle. This is the transfer operator \(M : C^0(\mathbb{T}, \mathbb{R}^n) \to C^0(\mathbb{T}, \mathbb{R}^n)\), which is defined as the bounded linear operator

\[
M(\Delta)(\theta) = M(\theta - \omega)\Delta(\theta - \omega),
\]

(4)

where \(\Delta : \mathbb{T} \to \mathbb{R}^n\) is continuous. Recall that the spectrum is

\[
\Sigma = \text{Spec}(M, C^0(\mathbb{T}, \mathbb{R}^n)) = \{z \in \mathbb{C} : M - z \text{ is not invertible}\}.
\]

We refer to \(\Sigma\) as the Mather spectrum of the transfer operator.

An invariant curve \(K\) is said to be a Fiberwise Hyperbolic Invariant Torus (FHIT) [11, 12, 14] if the quasiperiodic linear system (3) is uniformly hyperbolic: there is
an invariant continuous Whitney splitting \( \mathbb{R}^n \times T = E^u \oplus E^s \), the unstable and the stable bundles, such that System (3) restricted to \( E^u \) is invertible and there exist constants \( C > 0 \) and \( 0 < \lambda < 1 \) satisfying

- if \( (v, \theta) \in E^u \) then \( |M(\theta + (m - 1)\omega) \cdots M(\theta)v| \leq C\lambda^m|v| \) for all \( m \geq 0 \);
- if \( (v, \theta) \in E^u \) then \( |M(\theta + m\omega)^{-1} \cdots M(\theta - \omega)^{-1}v| \leq C\lambda^{-m}|v| \) for all \( m \leq 0 \).

Notice that uniform hyperbolicity implies that all Lyapunov exponents of System (3) are non zero and the minimum distance between the unstable and stable bundles is positive.

The condition of uniform hyperbolicity on System (3) is equivalent to the fact that the transfer operator (4) is hyperbolic, that is, \( \Sigma \) does not intersect the unit circle of the complex plane. As a byproduct, the implicit function theorem implies the persistence of a FHIT with respect to \( C^1 \) perturbations of the system.

2.2. The annular structure of the Mather spectrum. The set structure of \( \Sigma \) gives information about the growth rate properties of the linearized dynamics around an invariant curve and, in particular, on the hyperbolicity properties. The spectrum \( \Sigma \) is rotationally invariant \([17, 33]\), that is, it is a set of annuli of the complex plane centered at the origin:

\[
\Sigma = \bigcup_{j=1}^{k} A(\rho_j^-, \rho_j^+),
\]

where \( A(\lambda, \mu) = \{ z \in \mathbb{C} \mid \lambda \leq |z| \leq \mu \} \). A spectral annulus can be a circle, a full annulus or, in case \((M, R_\omega)\) is not invertible, a disk. For each spectral annulus \( A(\rho_j^-, \rho_j^+) \) there is a continuous invariant bundle \( E^{\rho_j^- \rho_j^+} \), referred to as spectral bundle, characterized by rates of growth: the Lyapunov multipliers of its vectors are in \([\rho_j^-, \rho_j^+]\). That is:

\[
(v, \theta) \in E^{\rho_j^- \rho_j^+} \iff \rho_j^- \leq \lambda^-(v, \theta), \lambda^+(v, \theta) \leq \rho_j^+,
\]

where

\[
\lambda^-(v, \theta) = \liminf_{m \to -\infty} |M(\theta + m\omega)^{-1} \cdots M(\theta - \omega)^{-1}v|^\frac{1}{m}
\]

and

\[
\lambda^+(v, \theta) = \limsup_{m \to +\infty} |M(\theta + (m - 1)\omega) \cdots M(\theta)v|^\frac{1}{m}
\]

are the backward and forward Lyapunov multipliers, respectively (the corresponding Lyapunov exponents are \( \chi^-(v, \theta) = \log \lambda^-(v, \theta) \) and \( \chi^+(v, \theta) = \log \lambda^+(v, \theta) \)). Moreover, the inner and outer radii of the annulus are Lyapunov multipliers. The spectral bundles split the vector bundle \( \mathbb{R}^n \times T \),

\[
\mathbb{R}^n \times T = \bigoplus_{j=0}^{k} E^{\rho_j^- \rho_j^+}.
\]

In particular, \( k \leq n \). Moreover, for each \( j \) the spectrum of \( \mathcal{M} \) restricted to sections of the the corresponding spectral bundle \( E^{\rho_j^- \rho_j^+} \) is \( A(\rho_j^-, \rho_j^+) \). Bootstrap arguments imply that the spectral bundles inherits the regularity of the vector bundle map \((M, R_\omega)\) \([25, 26]\), as well as the FHIT inherits the regularity of the bundle map \((F, R_\omega)\) \([20, 39]\).

The spectrum of \( \mathcal{M} \) restricted to sections of an invariant bundle \( E \) of rank 1 (i.e., whose fiber has dimension 1) is either a circle or, in case the \((M, R_\omega)\) is not invertible on \( E \), a disk. In the former, under Diophantine properties of \( \omega \), the dynamics on
the rank 1 invariant bundle $E$ can be reduced to a constant (whose absolute value is the radius of the spectral circle). We emphasize that the bundle can be orientable or not. In the latter, one can use double-covering trick for trivialising the bundle. A spectral circle can also be associated to invariant bundles of rank greater than 1. This is the case of a rank 2 invariant bundle $E$ whose dynamics has a fiberwise rotation number [24] that does not resonate with the internal frequency $\omega$, and it can be reduced to a constant linear matrix with complex conjugate eigenvalues [9, 10, 28].

The results presented in this paper can be rephrased by using Sacker-Sell spectral theory [35, 36], instead of using Mather spectral theory [33]. The connections are well-known [5, 6, 25] and can be summarized as follows: $\lambda \in \mathbb{C}$ is in the Mather spectrum of $(M, R_\omega)$ if and only if $(|\lambda|^{-1}M, R_\omega)$ is not uniformly hyperbolic or, equivalently, does not have exponential dichotomy, and this is equivalent to $\log |\lambda|$ being in the Sacker-Sell spectrum. As a consequence, thick annuli in the Mather spectrum correspond to nondegenerate intervals in the Sacker-Sell spectrum, while circles to isolated points. See also [13, 27] for related results.

2.3. Bifurcations and the fractalization route. A subject of fundamental importance is the behaviour of invariant objects when tuning control parameters of a model. The qualitative changes are produced in bifurcations. Local bifurcations have to do with spectral properties of linearized dynamics, while global bifurcations involve interactions with other invariant objects at macroscopic scale (e.g. homo/heteroclinic bifurcations, period-doubling cascades, and creation of strange attractors). In this respect, there is a quasiperiodic bifurcation theory for reducible invariant tori [2, 7], that generalises the well-known bifurcations of fixed points, such as the saddle-node and period-doubling bifurcations. But the development of such a theory and of effective algorithms of computations for non-reducible invariant tori is a challenging problem [40], although there are already some partial results in the literature [4, 14, 18, 21, 29].

Local bifurcations are produced when the spectrum touches the unit circle, leading to a loose of the hyperbolicity property. But the situation is very different if the spectral central component, that collides with the unit circle, is a circle or an annulus, and if the dynamics on the corresponding spectral central bundle is reducible or not. For instance, if the spectral central bundle is one-dimensional, and the corresponding spectral component is a circle, we encounter for instance the quasiperiodic saddle-node and quasiperiodic period-doubling bifurcations. If the spectral central bundle is two-dimensional, and its dynamics is reducible to a constant rotation, then the spectral component is also a circle, and we encounter for instance the quasiperiodic Hopf bifurcation.

The situation in which dynamics is non-reducible is much more difficult. In the so called fractalization route of an attracting invariant curve, in which the curve gradually wrinkles when varying the control parameter up to a value in which the curve is apparently broken down, it has been observed that the central spectral component of the spectrum, the one that gradually approaches the unit circle, is a thick annulus, and the dynamics on the corresponding central bundle is not reducible. This phenomenon has been observed for attracting tori in 1D non-invertible systems [29], in which the spectrum is a disk whose radius approaches 1, and for attracting tori in 2D invertible dissipative systems [18, 21], in which the spectrum is a thick annulus whose outer radius approaches 1. The phenomenon is likely to appear also
in families of quasiperiodic systems that are not homotopic to the identity, since this topological property is an obstruction to reducibility [24].

So far the fractalization route has been observed in attracting invariant curves. We consider here such a mechanism in saddle invariant curves of quasiperiodic invertible and homotopic to the identity systems, in which the central part of the spectrum is a thick annulus that gradually approaches the unit circle up to a critical collision parameter. We emphasise that there are other breakdown mechanisms, in which the slow stable and slow unstable components of the spectrum suddenly grow to a thick annulus containing the unit circle, producing the destruction of the saddle invariant curve [14, 18, 21].

2.4. Saddle invariant curves in 3D quasiperiodic systems. From the previous discussion, the lowest dimension in which a quasiperiodic invertible and homotopic to the identity system can undergo a fractalization route of a saddle invariant torus is 3. For this reason, we consider here the classification of saddle invariant tori in 3D systems, attending their spectral properties. If the stable bundle has rank 2 (resp. 1), we say that the stability index is 2 (resp. 1). We then classify the saddle invariant tori with stability index 2 (the other case is complementary):

- A saddle-node curve: the Mather spectrum consists of three circles of radii \( \rho_1 < \rho_2 < 1 < \rho_3 \). There is a spectral splitting in three continuous invariant bundles: the fast stable bundle \( E^s_1 \) and the slow stable bundle \( E^s_2 \) generate the stable bundle \( E^s \), and the unstable bundle \( E^u = E^3 \) has rank 1. Under regularity properties of the system and Diophantine properties of \( \omega \) the linearized dynamics can be reduced (possible using double covering) to a constant diagonal matrix whose eigenvalues are real and their absolute values are the spectral radii \( \rho_1, \rho_2, \rho_3 \).

- A saddle-focus curve: the Mather spectrum consists of two circles of radii \( \rho < 1 < \rho_3 \). The dynamics on the rank 2 stable bundle has fiberwise rotation number \( \alpha \). Under regularity properties of the system and Diophantine properties of \( \omega \) and \( \alpha \) (the so-called Melnikov conditions), the linearized dynamics on \( E^s \) can be reduced to a focus (a product of a homothety and a rotation).

- A non-reducible saddle curve: the Mather spectrum consists of one annulus of radii \( \rho_1 < \rho_2 < 1 \) and one circle of radius \( \rho_3 > 1 \). There is a splitting in two continuous invariant bundles: the stable bundle \( E^s \) contains the measurable (but non continuous) invariant bundles \( E^1 \) and \( E^2 \) associated to the Lyapunov multipliers \( \rho_1 \) and \( \rho_2 \), respectively, as a result of Oseledec’s theory; the unstable invariant bundle \( E^u = E^3 \) has rank 1.

The previous classification is not exhaustive, and it is object of intensive research (specially in the 2D case). In one-parameter depending families, say, the saddle-node property is an open condition (this is a consequence of upper-semicontinuity of spectrum with respect to parameters), while the other two use to happen in Cantor sets of positive measure in parameter space. The transitions from saddle-node to saddle-focus, that correspond to equality of real eigenvalues, use to occur in zero measure sets in parameter space. More difficult is to understand the transition from saddle-node to non-reducible saddle. This transition is called bundle merging scenario in [18], since two bundles collide non-smoothly. In summary, there can be open intervals in the parameter space in which saddle-node and saddle-focus properties of an invariant curve coexist, to which we refer to as saddle node-focus routes, and others in which saddle-node and non-reducible saddle properties coexist.
to which we refer to as non-reducible saddle routes. Moreover, the different cases are likely to coexist in parameter sets of total measure (a partial result in this direction is [1]).

The case in which a non-reducible saddle route ends in a critical value in which the central radius $\rho_2$ goes to 1 corresponds to the fractalization route mentioned above.

3. The fractalization route in a volume preserving quasiperiodic system.

The quasiperiodic volume preserving system that we focus on in this section is

$$
\begin{align*}
\bar{x} &= \frac{\kappa_1}{2\pi} \sin(2\pi x) + \frac{\kappa_2}{2\pi} \sin(2\pi y) + z - \varepsilon \sin(2\pi \theta) \\
\bar{y} &= x \\
\bar{z} &= y \\
\bar{\theta} &= \theta + \omega \mod 1 
\end{align*}
$$

(5)

where $\omega = \frac{\sqrt{5} - 1}{2}$ is the inverse of the golden mean, $\kappa_1 = 2.1$ and $\kappa_2 = 0.95$, and $\varepsilon$ is the control parameter. As we will see, this model exhibits all the phenomena described in the previous section.

For $\varepsilon = 0$, System (5) has the constant FHIT $K(\theta) = (0, 0, 0)$. It is a saddle-focus: Its Lyapunov exponents are $\chi_1 = \chi_2 \simeq -0.47979481$ and $\chi_3 \simeq 0.95958961$. The 2D stable bundle is associated with a complex pair of eigenvalues (of moduli smaller than 1), and the 1D unstable bundle is associated with a positive real eigenvalue greater than 1.

We have numerically continued this invariant curve with respect to parameter $\varepsilon$ up to values close to breakdown, $\varepsilon = 0.464923$. There are several numerical methods available to do so, such as rational approximation methods [14, 16, 19] and parameterization methods [19, 21]. See [3] for an exposition. Below there is a description of the many transitions suffered by the invariant curve, some of them concerning the topology of its invariant bundles, others concerning its stability and dynamics. To detect these transitions, we use the minimum distance between the invariant bundles and the corresponding Lyapunov exponents, see Figure 1.

The sequence of transitions can be summarized as follows:

![Figure 1](image-url)
1. **Saddle node-focus route.** From $\varepsilon = 0$ to $\varepsilon \simeq 0.452905$ the invariant curve is a saddle with stability index 2, and the stable bundle suffers many node-focus transitions. In each transition from node to focus, the slow and fast stable bundles collide smoothly, and the Mather spectrum changes from being three circles to two circles. In Figure 2 it is shown that the Lyapunov exponents have some open gaps (that correspond to being a saddle-node curve), and in Figure 3 the slow and fast bundles in some of these gaps are displayed. The topology of the bundles changes in different gaps.

2. **Period-doubling bifurcation.** From $\varepsilon \simeq 0.452905$ to $\varepsilon \simeq 0.459160$ the central Lyapunov exponent $\chi_2$ approaches 0, and the slow stable bundle bifurcates smoothly into a slow unstable bundle. Then, at $\varepsilon \simeq 0.459160$ the stability index of the saddle-node torus changes from 2 to 1. Along this bifurcation, the Mather spectrum consists of three circles, and the central one goes from being inside the unit circle to being outside. A careful analysis of this smooth bifurcation reveals that it corresponds to a period-doubling.

3. **Bundle merging bifurcation.** From $\varepsilon \simeq 0.459160$ to $\varepsilon \simeq 0.4598$ the unstable bundle possesses continuous (in fact analytic) slow and fast unstable bundles. As parameter $\varepsilon$ approaches $\varepsilon \simeq 0.4598$, the slow and fast unstable bundles approach each other and finally collide in a non-smooth way, see Figure 4. The Mather spectrum spontaneously grows from three circles to a circle and a thick annulus, implying that the slow and fast unstable bundles are only measurable. The collision of bundles corresponds to the formation of Strange Non-Chaotic Attractors/Repellers in the projective dynamics, see [22, 30] and references therein.

4. **Fractalization route.** From $\varepsilon \simeq 0.4598$ the invariant saddle curve experiences a non-reducible saddle route, that continues up to $\varepsilon \simeq 0.464912$. As the slow unstable Lyapunov exponent approaches 0, see Figure 5, the invariant curve gets more wrinkled, see Figure 6. Figure 7 shows the derivatives of the invariant curves near the breakdown. As $\varepsilon$ increases the derivatives increase. See Figure 8 for a graph in log$_{10}$ scale of the maximum slope of the invariant curve with respect to the parameter $\varepsilon$. It can be observed that the maximum slope increases as the invariant curve fractalizes. In this route, the unstable bundle is not reducible (the Mather spectrum is a circle and a thick annulus) in a set of positive measure, and it is reducible (the Mather spectrum consists of three circles) in tiny gaps of the parameter interval. Moreover, no collision between the unstable and stable bundles is observed, see Figure 9.

4. **Conclusions and conjectures.** From the results and discussions appearing in this paper, we are now ready to propose a general definition of the fractalization route (both for attracting and saddle invariant curves). We say that a family of FHIT $K_\varepsilon$, defined in an interval $[0, \varepsilon_c]$, *fractalizes* at $\varepsilon_c > 0$ if

$$\inf_{\Theta \subset \mathbb{T}} \left( \liminf_{\varepsilon \to \varepsilon_c} \text{diam}(K_\varepsilon(\Theta)) \right) > 0.$$ 

**Remark 1.** In [29] the authors propose an alternative definition of fractalization in the case of 1-D quasiperiodic systems: for any open set $\Theta \subset \mathbb{T}$

$$\lim_{\varepsilon \to \varepsilon_c} \frac{\|K_\varepsilon\|_\Theta}{\|K_\varepsilon\|_\Theta} = \infty,$$
Figure 2. Magnification of the Lyapunov spectra for the parameter values $0.438 \leq \varepsilon \leq 0.454$. The colors are: blue (medium), green (minimal), red (the negative value of the half of the maximal exponent).

Figure 3. Fast (red) and slow (blue) stable bundles of a saddle-node curve for different values of $\varepsilon$ in open gaps. These bundles are plotted as a projection on the 2D stable bundle, using projective coordinates. Notice the different topologies of these bundles.
where $\| \cdot \|_{\Theta}$ is the sup-norm on $\Theta$. This definition is weaker than the one given above and embraces other types of breakdown mechanisms. For example, the folding breakdown mechanism described in [14] satisfies it, although the curve does not fractalizes in the sense discussed in this paper.

We conjecture that a curve fractalizes if, for sufficiently close $\varepsilon$ to $\varepsilon_c$, there is an invariant bundle $E_\varepsilon$ with corresponding spectrum $\sigma_\varepsilon \subset \Sigma_\varepsilon$ satisfying

- $\sigma_\varepsilon$ is either a disk or a full annulus;
- $\lim_{\varepsilon \to \varepsilon_c} d(\sigma_\varepsilon, 1) = 0$. 

**Figure 4.** Invariant curves (left) and fast (red) and slow (blue) unstable bundles (right) for different values of $\varepsilon$. These bundles are plotted as a projection on the 2D unstable bundle, using projective coordinates.
Figure 5. Graphs of the medium Lyapunov exponent for $\varepsilon$ near the breakdown. See text for more details.

Figure 6. Two (smooth) invariant curves in the fractalization route, new breakdown. Note that, as $\varepsilon$ is increased, the curve is more wrinkled. Note that the invariant curves are very wrinkled and, without the magnifications appearing on the right hand side, we cannot distinguish them.

An important question is up to what extent the conjecture is true. In the case that $E_\varepsilon$ is a 1D invariant bundle with $\sigma_\varepsilon$ being a disk inside the unit circle, so that the linearized dynamics is not invertible on $E_\varepsilon$, the fractalization route has been proved in simple cases, such as for an attracting invariant curve of a quasiperiodic
Figure 7. Derivatives of the invariant curves for different values of the $\varepsilon$ parameter near the breakdown. Note that there is a slightly difference between the parameter $\varepsilon$ but the magnitude of the derivatives differs a lot.

Figure 8. Maximum slope function of the invariant curve with respect $\varepsilon$. The $y$ axis is in log scale.

1D non-invertible affine system, but for non-linear models the question is much more complicated and there is only numerical evidence [29].

In quasiperiodic invertible and homotopic to the identity systems, the situation is much more involved because thick spectral components seem to happen in Cantor sets of positive measure in parameter space. So, when the Lyapunov exponent is zero it could happen that the central spectral component is just a circle, not a full annulus. This implies that, to get fractalization, the one parameter family (with parameter $\varepsilon$) must be unfolded to a two parameter family (with parameters $(\varepsilon, \kappa)$). In this case, for an open set of parameter $\kappa$ there is a non-reducible route in which the central Lyapunov exponent goes to zero, and for a Cantor subset of parameter $\kappa$ the route ends with a fractalization of the curve.

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Figure 9. Minimum distance between the invariant unstable and stable bundles near breakdown.

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