Stability analysis for a class of mechanical systems with piecewise constant coefficients

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Abstract. A nonlinear mechanical system described by a vector equation of the Lienard type is considered. It is assumed that the forces acting on the mechanical system are given by functions that are piecewise constant with respect to time. With the aid of a special variable substitution, the equation is reduced to a system that can be considered as an impulsive switched system with infinite numbers of operating modes. The stability problem for the trivial equilibrium position of the obtained system is studied. The stability analysis is carried out using a specially constructed discontinuous Lyapunov function. This function can be understood as a multiple Lyapunov function, consisting of an infinite number of partial Lyapunov functions, each of which is used for a certain time interval. Differentiating the chosen Lyapunov function with respect to solutions of the given system on the corresponding time intervals and applying the theory of differential inequalities, sufficient conditions of the asymptotic stability of the equilibrium position are established.

1. Introduction
The direct Lyapunov method is the most suitable approach for the investigation of stability of nonlinear dynamical systems. However, the main problem of this method is still the construction of a suitable Lyapunov function. The situation becomes more complicated if considered system is a subject to switching or impulse effects [1–5]. Switching changes the structure of the system at certain time points, while maintaining the continuity of the investigated solutions. Impulse effects lead to abrupt changes of solutions of the system. Non-stationary systems with discontinuous coefficients can be considered as a certain class of such hybrid systems. These discontinuities can be understood as a kind of switching. An important feature in this case is that if the total number of discontinuities is infinite, then the system will be considered as a hybrid system with an infinite number of possible operating modes. And the order of changing the operating modes will be hard-coded.

For the stability analysis of impulsive switched systems, the method of multiple Lyapunov functions method was developed [5]. According to this method, for each operating mode of the system, its own Lyapunov function is chosen, and these partial functions are combined into one multiple function. Discontinuous coefficients in the system force the use of discontinuous Lyapunov functions, which can be understood as multiple Lyapunov functions consisting of an infinite set of continuous partial functions. The theory of differential inequalities allows us to estimate the behavior of solutions of the system on the intervals of continuous change of coefficients using partial Lyapunov functions. And it remains only to take into account the
jumps made during the transition from one partial Lyapunov function to another as a result of switching or impulse effects.

In the present paper, the non-stationary vector equation of the Lienard type with piecewise constant time coefficients is investigated. Such equation is widely used for modeling nonlinear oscillatory processes in electromechanical systems (see [6, 7]). Using a class of piecewise constant functions to describe the discontinuous coefficients in an equation on the one hand simplifies the analysis, but on the other hand, many non-stationary changes can be approximated by functions of this kind. For the given equation, a method is proposed for constructing a discontinuous Lyapunov function of a special type, with the help of which sufficient conditions for the asymptotic stability of the trivial equilibrium position are established.

2. Statement of the problem

Let the system
\[ \dot{x} + \frac{\partial G(t, x)}{\partial x} \dot{x} + \frac{\partial P(t, x)}{\partial x} = 0 \] (1)
be given. Here \( t \geq 0, x \in \mathbb{R}^n \); function \( P(t, x) \) and components of vector \( G(t, x) \) are piecewise constant with respect to \( t \) and continuously differentiable with respect to \( x \).

System (1) is the non-stationary vector equation of the Lienard type. It can be used, for instance, for the modelling of mechanical systems, that are under the influence of nonlinear dissipative and potential forces.

Find time instants \( \{\tau_i\}_{i=1,2,\ldots} \), where \( 0 = \tau_0 < \tau_1 < \tau_2 < \ldots \), that are breakpoints with respect to variable \( t \) for \( P(t, x) \) or/and \( G(t, x) \) on the interval \([0, +\infty)\). Without loss of generality, we assume that \( P(t, x) \) and components of vector \( G(t, x) \) are right-continuous functions with respect to variable \( t \) at points \( \{\tau_i\}_{i=1,2,\ldots} \). Excluding the trivial case, we suppose that the total number of breakpoints \( \{\tau_i\}_{i=1,2,\ldots} \) is infinite on the interval \([0, +\infty)\).

So, we investigate continuous solutions \((x(t), \dot{x}(t))^T\) satisfying the equations
\[ \dot{x} + \frac{\partial G(\tau_i, x)}{\partial x} \dot{x} + \frac{\partial P(\tau_i, x)}{\partial x} = 0, \quad t \in [\tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots. \]

This system can be considered as a switched system with infinite number of operating modes. Let us determine sufficient conditions under which the equilibrium position \( x = \dot{x} = 0 \) of the system is asymptotically stable.

In the present paper, we will use the following assumption.

**Assumption 1.** Let function \( P(t, x) \) and vector \( G(t, x) \) satisfy the conditions:

i) for any fixed value of \( t \geq 0 \), function \( P(t, x) \) is a positive definite quadratic form;

ii) for any fixed value of \( t \geq 0 \), components of vector \( G(t, x) \) are homogeneous of the order \( \nu + 1 \) with respect to variable \( x \), \( \nu > 0 \);

iii) for any fixed value of \( t \geq 0 \), function \( \left(\frac{\partial P(t, x)}{\partial x}\right)^T G(t, x) \) is positive definite.

Taking into account Assumption 1, one can find piecewise constant and positive for \( t \geq 0 \) functions \( \bar{p}(t), \bar{q}(t), c(t), g(t) \), such that the estimates
\[ \bar{p}(t)\|x\|^2 \leq P(t, x) \leq \bar{q}(t)\|x\|^2, \quad \left(\frac{\partial P(t, x)}{\partial x}\right)^T G(t, x) \geq g(t)\|x\|^{\nu+2}, \]
\[ \left\|\frac{\partial P(t, x)}{\partial x}\right\| \leq c(t)\|x\|, \quad \|G(t, x)\| \leq g(t)\|x\|^{\nu+1} \]
hold for all \( t \geq 0, x \in \mathbb{R}^n \). Here we use the Euclidean norm of a vector. Note that we allow a situation where functions \( \bar{p}(t), \bar{q}(t), c(t) \) and \( g(t) \) can increase unboundedly on the interval \([0, +\infty)\) or, conversely, approach to zero with time.
Let us rewrite the considered system in the form

\[
\dot{x} = y - G(\tau_i, x), \quad \dot{y} = -\frac{\partial P(\tau_i, x)}{\partial x}, \quad t \in [\tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots
\]  

(2)

It should be noted that since vector \( G(t, x) \) is discontinuous with respect to \( t \), the variable \( y(t) \) will also be discontinuous:

\[
y(\tau_j + 0) = y(\tau_j - 0) + G(\tau_j, x(\tau_j)) - G(\tau_{j-1}, x(\tau_j)), \quad j = 1, 2, \ldots
\]  

(3)

Therefore, system (2), (3) can be understood as an impulsive switched system.

3. Construction of a Lyapunov function

Choose some piecewise constant and positive function \( \gamma(t) \) defined for \( t \geq 0 \) and having discontinuities at time points \( \tau_1, \tau_2, \ldots \).

Construct a Lyapunov function for system (2), (3) in the form

\[
V(t, x, y) = P(t, x) + \frac{1}{2}y^T y - \delta \gamma(t) \| y \| \| y \|^\nu
\]

(4)

where \( \delta \) is a sufficient small positive constant.

Differentiating function (4) with respect to system (2), we obtain equalities

\[
\dot{V} \big|_{(2)} = - \left( \frac{\partial P(\tau_i, x)}{\partial x} \right)^T G(\tau_i, x) - \delta \gamma(\tau_i) \| y \|^\nu + 2
\]

\[
+ \delta \gamma(\tau_i) \| y \|^\nu y^T G(\tau_i, x) + \delta \gamma(\tau_i)x^T \frac{\partial (\| y \|^\nu y)}{\partial y} \frac{\partial P(\tau_i, x)}{\partial x},
\]

that are valid for \( t \in (\tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n \).

Hence, the estimates

\[
\left( \tilde{p}(\tau_i) \| x \|^2 + \frac{1}{2} \| y \|^2 \right) - \delta \gamma(\tau_i) \| x \| \| y \|^{\nu+1} \leq V(t, x, y)
\]

\[
\leq \left( \tilde{p}(\tau_i) \| x \|^2 + \frac{1}{2} \| y \|^2 \right) + \delta \gamma(\tau_i) \| x \| \| y \|^{\nu+1},
\]

\[
\dot{V} \big|_{(2)} \leq - \left( q(\tau_i) \| x \|^{\nu+2} + \delta \gamma(\tau_i) \| y \|^{\nu+2} \right)
\]

\[
+ \beta (\delta \gamma(\tau_i)g(\tau_i) \| x \|^{\nu+1} \| y \|^{\nu+1} + \delta \gamma(\tau_i)c(\tau_i) \| x \|^2 \| y \|^\nu)
\]

hold for \( t \in (\tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n \). Here \( \beta \) is a positive constant.

4. Stability conditions

It is easy to check that if functions

\[
\gamma(t) \left( \tilde{p}(t) \right)^{\frac{\nu-1-\varepsilon_1}{\nu+2}}, \quad g(t) \left( \frac{\gamma(t)}{q(t)} \right)^{\frac{1+\varepsilon_2}{\nu+2}}, \quad c(t) \left( \frac{\gamma(t)}{q(t)} \right)^{\frac{2}{\nu+2}}
\]

(5)

are bounded on the interval \([0, +\infty)\) for some values of

\[
\max\{0; \nu - 1\} \leq \varepsilon_1 \leq \nu, \quad 0 \leq \varepsilon_2 \leq \nu,
\]
then for any \( M_1 \in (0, 1), M_2 > 1 \) and \( M_3 \in (0, 1) \) one can find \( \delta > 0 \) and \( H > 0 \) so that the estimates

\[
M_1 \left( \bar{p}(\tau_i)\|x\|^2 + \frac{1}{2}\|y\|^2 \right) \leq V(t, x, y) \leq M_2 \left( \bar{p}(\tau_i)\|x\|^2 + \frac{1}{2}\|y\|^2 \right),
\]

\[
\dot{V}(t) \leq -M_3 \left( q(\tau_i)\|x\|^{\nu+2} + \delta \gamma(\tau_i)\|y\|^{\nu+2} \right)
\]

hold for \( t \in (\tau_i, \tau_{i+1}), i = 0, 1, \ldots \), if \( \|x^T(t), y^T(t)\| < H \).

Note, that it is always possible to provide the boundedness of functions (5) by appropriate choosing the coefficient \( \gamma(t) \) in Lyapunov function (4) (the restrictions are compatible). Really, let estimates (8), (9) be constructed for impulsive switched system (2), (3). If \( \varphi(t) \to +\infty \) and \( \dot{\varphi}(t) \to +\infty \) as \( t \to +\infty \), then the equilibrium position \( x = y = 0 \) of the system is asymptotically stable.

The proof of Theorem 1 is similar to the proof of Theorem 1 from [8].
5. Conclusion

Methods for the stability analysis of hybrid dynamical systems of various types have been actively developed in recent years. However, there are still many unresolved problems. A special feature of this work is the study of a combination of several factors that complicate the analysis. This includes the non-stationary nature of the system, its non-linearity, switching, and the impulse change of the variables used.

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