In 1935, Einstein-Podolsky-Rosen (EPR) questioned the completeness of quantum mechanics based on locality and realism [1]. Soon after, Schrödinger [2] published a seminal paper defining the notion of entanglement to describe the correlations between two particles. Entanglement, a quantum state which cannot be separated, is indeed the essential entity that evaluates whether a quantum information processing can be accomplished in quantum level. The more entanglement is, the more prowess of the resource has. Various criteria for quantum witnesses of entanglement have been proposed in recent decades. Generally speaking, entanglement measures are mostly as functions of density operator.

EPR steering, like entanglement, was originated from Shrödinger’s reply to the EPR paradox to reflect the inconsistency between quantum mechanics and local realism, and was formalized by Wiseman, Jones, and Doherty [6]. In the steering scenario, for a pure entangled state held by two separated observers Alice and Bob, Bob’s qubit can be “steered” into different ensembles of states although Alice has no access to the qubit. Alice tries to convince Bob that they share two systems in an entangled state. If the systems are actually entangled, quantum mechanics predicts that, by performing different measurements on her system, Alice can remotely prepare different states for Bob’s system. EPR steering is commonly detected by the violation of EPR-steering inequalities in the form of correlations [7, 10]. Although many efforts have been devoted to the investigations of EPR steering, the EPR-steering inequalities in the literature are not effective enough for two-qubit systems. Therefore, it is not possible to observe the EPR steering for some states, especially for mixed states. For EPR steering, entanglement is necessary but not sufficient. By resorting to partial transpose of density operator, entanglement can be certified. This is understandable that the density operator contains all the information of the state. It is hence reasonable to anticipate a criterion based entirely on density matrix for EPR steering witness.

In this work, we propose a criterion to detect EPR steering of an arbitrary two-qubit density matrix $\rho_{AB}$. The criterion can be obtained from the constraints on the eigenvalues of partial transpose matrix $\rho_{AB}^T$. We list some examples to show the utility of our criterion.

Steerability Criterion.—Let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be four eigenvalues of $\rho_{AB}^T$ in the small-to-large order [$\rho_{AB}^T$ and $\rho_{AB}$ share the same eigenvalues]. Then the criterion for EPR steering is given by

$$S = \lambda_1 + \lambda_2 - (\lambda_1 - \lambda_2)^2 < 0,$$

when (1) is satisfied, then EPR steering exists.

Example 1.—The nonmaximal entangled state

$$\rho_1 = \frac{1}{2} \begin{bmatrix} \cos^2 \theta & 0 & 0 & \sin \theta \cos \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin \theta \cos \theta & 0 & 0 & \sin^2 \theta \end{bmatrix}$$

with $\theta \in [0, \pi/4]$ always violates the CHSH inequality as well as steering inequality given in Ref. [9] except $\theta = 0$. In this case, we have $\lambda \in \{ -\sin \theta \cos \theta, \sin^2 \theta, \sin \theta \cos \theta, \cos^2 \theta \}$, and the steerability criterion gives

$$S = -\frac{1}{2} \sin 2\theta(1 + 2 \sin^2 \theta),$$

hence detects all the steering.

Example 2.—The isotropic state

$$\rho_2 = V \rho_0 + (1 - V) \frac{1}{4}$$

$$= \frac{1}{4} \begin{bmatrix} \frac{1+V}{2} & 0 & 0 & \frac{V}{2} \\ 0 & \frac{1-V}{2} & 0 & 0 \\ 0 & 0 & \frac{1-V}{2} & 0 \\ \frac{V}{2} & 0 & 0 & \frac{1+V}{2} \end{bmatrix},$$

where $\rho_0 = \frac{1}{2}(|00\rangle + |11\rangle)(|00\rangle + |11\rangle)$ is the maximal entangled state, and $\mathbf{1}$ is the four-by-four identity matrix. It has been known that the state has the steering in the region $V \in (1/2, 1]$, and no steering in $V \in [0, 1/2]$. In this case, we have $\lambda \in \{ \frac{1-3V}{4}, \frac{1+V}{2}, \frac{1-V}{4}, \frac{1+V}{4} \}$, and the steerability criterion gives

$$S = -\frac{1}{2}(2V - 1)(1 + V),$$
hence detects the critical value $V_{cr} = \frac{1}{3}$.

Example 3.—The Bell-diagonal state

$$\rho_3 = V|\Psi^+\rangle\langle\Psi^+| + (1 - V)|\chi^+\rangle\langle\chi^+|$$

$$= \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & \frac{1-V}{2} & \frac{1-V}{2} & \frac{1-V}{2} \\ 0 & \frac{1-V}{2} & \frac{1-V}{2} & \frac{1-V}{2} \\ \frac{V}{2} & 0 & 0 & \frac{V}{2} \end{pmatrix}$$

(6)

violates the steering inequality in Ref. [9] except $V = \frac{1}{2}$. In this case, we have $\lambda \in \{V - \frac{1}{2}, \frac{1}{2} - V, \frac{1}{2}, \frac{1}{2}\}$, and the steerability criterion gives

$$S = -(1 - 2V)^2,$$

(7)

which recovers the same result.

Example 4.—The nonmaximal entangle state with color noise

$$\rho_4 = \begin{pmatrix} V \cos^2 \theta + \frac{1-V}{2} & 0 & 0 & V \sin \theta \cos \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ V \sin \theta \cos \theta & 0 & 0 & V \sin^2 \theta + \frac{1-V}{2} \end{pmatrix}$$

(8)

with $\theta \in [0, \pi/4]$ always violates CHSH inequality as well as the steering inequality in Ref. [9] except $V = \theta = 0$. In this case, we have $\lambda \in \{-V \sin \theta \cos \theta, V \sin \theta \cos \theta, \frac{1}{2}(1 - V \cos 2\theta), \frac{1}{2}(1 + V \cos 2\theta)\}$, and the steerability criterion gives

$$S = -V^2 \sin^2 2\theta,$$

(9)

which recovers the same result.

Example 5.—The maximally entangled mixed state (MEMS)

$$\rho_5 = \begin{pmatrix} g(\gamma) & 0 & 0 & \gamma/2 \\ 0 & 1 - 2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & g(\gamma) \end{pmatrix},$$

(10)

with $g(\gamma) = 1/3$ for $\gamma \in [0, 2/3]$ and $g(\gamma) = \gamma/2$ for $\gamma \in [2/3, 1]$. It violates the 10-setting steering inequality in Ref. [9] for $\gamma \geq 0.6029$. In the case of $\gamma \in [0, 2/3]$, we have $\lambda \in \{1 - \sqrt{1 + 9\gamma^2}/6, 3 \pm \sqrt{1 + 9\gamma^2}/6\}$, and the steerability criterion gives

$$S = \frac{1}{36}(16 - 9\gamma^2 - 8\sqrt{1 + 9\gamma^2}),$$

(11)

which predicts the critical value

$$\gamma_{cr} = \frac{2}{3} \sqrt{2(6 - \sqrt{33})} \approx 0.4765.$$

Example 6.—The state

$$\rho_6 = \begin{pmatrix} g & 0 & 0 & \gamma/2 \\ 0 & 1/2 - g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & 1/2 \end{pmatrix},$$

(13)

FIG. 1: (Color online) Quantum predictions of steering inequalities. The region above the blue line is steerable detected by the two-setting steering inequality as well as CHSH inequality. The region above the green and red lines are respectively steerable detected by the three- and ten-setting steering inequalities.

with $g = 4/9$ for $\gamma \in [0, 2\sqrt{2}/3]$. It violates the 10-setting steering inequality in Ref. [9] for $\gamma > 0.2564$. In this case, we have $\lambda \in \{\frac{1}{36}(1 - \sqrt{1 + 324\gamma^2}), \frac{1}{36}(1 + \sqrt{1 + 324\gamma^2}), 4/9, 1/2\}$, and the steerability criterion gives

$$S = \frac{17}{324} - \gamma^2,$$

(14)

which predicts the critical value

$$\gamma_{cr} = \frac{\sqrt{17}}{18} \approx 0.2291.$$

(15)

Example 7.—The state

$$\rho_7 = \begin{pmatrix} \frac{1 - \cos \theta}{2} F & 0 & 0 & \sin \theta F \\ 0 & \cos \theta F & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin \theta F & 0 & 0 & 1 - \frac{1 + \cos \theta}{2} F \end{pmatrix},$$

(16)

with $F \in [0, 1]$ and $\theta \in [0, \pi/2]$. The concurrence of the state is given by

$$C = \sqrt{2F(1 - |F(1 + \cos \theta) - 1)|} \sin^2 \frac{\theta}{2},$$

(17)

which vanishes only when $F = 0$ or $\theta = 0$. In this case, $\lambda \in \{-F \sin^2 \frac{\theta}{2}, F \sin^2 \frac{\theta}{2}, F \cos^2 \frac{\theta}{2}, 1 - F \cos^2 \frac{\theta}{2}\}$, we have $\lambda_1 = -\lambda_2 = -F \sin^2 \frac{\theta}{2}$, and the steerability criterion gives

$$S = -4F^2 \sin^4 \frac{\theta}{2},$$

(18)

which predicts that steering always exists. We compare the above result with the violation of the steering inequalities and CHSH inequality (see Fig. 1).

Any two-qubit state can be written in the following form

$$\rho_{AB} = \frac{1}{4}(1 \otimes 1 + \vec{a}^A \cdot \vec{u} \otimes 1 + 1 \otimes \vec{a}^B \cdot \vec{v} + \sum_{i,j=1}^3 \beta_{ij} \sigma_i^A \otimes \sigma_j^B),$$

(19)
where \( \mathbf{u} \) and \( \mathbf{v} \) are Bloch vectors for particles A and B, respectively; \( \beta_{ij} \) are some real numbers. Particularly, we take \( \mathbf{u} = (0, 0, r) \), \( \mathbf{v} = (0, 0, s) \) and \( \beta_{ij} = c_i \delta_{ij} \), then we obtain the five-parameter \( X \)-state as

\[
\rho_{AB} = \frac{1}{4} (1 \otimes 1 + r \sigma_3^A \otimes 1 + 1 \otimes s \sigma_3^B + \sum_{i=1}^{3} c_i \sigma_i^A \otimes \sigma_i^B).
\]

(20)

In Fig. 2 and Fig. 3 we compare the detectable ability of steerability criterion and the ten-setting steering inequality.

In summary, the criterion is proposed due to numerical observation, which works efficiently for detecting EPR steering of two-qubit density matrix. Similar to PPT criterion for detecting entanglement, the steerability criterion may also work as a necessary condition for demonstrating steerability of two qubits. It would be significant to derive the steerability criterion from analytic approach, such as positive maps, and then place it on a firmer foundation.

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