Energy partition in low energy fission

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The intrinsic excitation energy of fission fragments is dynamically evaluated in terms of the time dependent pairing equations. These equations are corroborated with two conditions. One of them fixes the number of particles and the another separates the pairing active spaces associated to the two fragments in the vicinity of the scission configuration. The fission path is obtained in the frame of the macroscopic-microscopic model. The single particle level schemes are obtained within the two center Woods-Saxon shell model. It is shown that the available intrinsic dissipated energy is not shared proportionally to the masses of the two fission fragments. If the heavy fragment possesses nucleon numbers close to the magic ones, the accumulated intrinsic excitation energy is lower than that of the light fragment.

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I. INTRODUCTION

Under the action of a mutual Coulomb repulsion, at scission the fission fragments are accelerated in opposite directions. These fragments are highly excited, as underlined in many review papers [1,2]. The maximal kinetic energy issued in the process amounts to the Q-value in the case of cold fission. The fragments decay on their ground state mainly by evaporation of neutrons and by radiation emission. It is known that the motion of any physical system is governed by conservative forces and by frictional ones that give rise to dissipation. Therefore, the excitation energy of the fragments must depend on the dynamics of the nuclear system in its path to scission.

The fission process offers a possibility to investigate how two nuclei in contact share their excitation energy. In the analyzes of experimental data, the authors of Ref. [3] evidenced an energy sorting mechanism based on statistical arguments. Considering a postulated difference between the temperatures of the two nascent fragments in conjunction with the condition of maximum entropy, they emphasized a flow of energy from one fragment to another. This flow of energy depends on the available states in the fragments in the scission configuration. In this context, an explanation for the violation of the constant temperature hypothesis [4] that involves a proportionality between the intrinsic excitation energy division and the mass ratio of the fragment is offered.

Experimental direct indications about the excitation energies of the fragments are obtained by measuring their evaporated neutrons [5]. Despite a similar temperature of the neutron velocity distributions, the experiment revealed that a larger excitation energy characterizes the light mass distribution in comparison with the heavy one. The shifting in the sawtoothlike behavior of the neutron multiplicity as function of the parent excitation were attributed mainly to the deformation energy, and not to the intrinsic heat. The thermal neutron induced fission of 231U [6] analyzed in Ref. [7] evidenced a small neutron multiplicity in the A=132 region. By increasing the excitation energy of the compound nucleus, it was observed that in this mass region the kinetic energy decreases by about 2 MeV, that is a large value. The interpretation ascribed a similar temperature of fragments as scission and it was speculated that this increment in excitation energy is due to a modification of the shape sequences during fission leading to a deformed heavy fragment. Furthermore, the multiplicity obtained for two neutron induced 237Np fission energies [8] revealed a modification of the heat of only the fragments in the heavier mass distribution. It is an experimental indication for a sorting mechanism in the intrinsic excitation energy.

Motivated by these aspects, in this work, the intrinsic excitation energy of the fragments are evaluated dynamically in terms of the time dependent pairing equations in the cold fission regime. The macroscopic-microscopic is employed to obtain the fission path using the minimal action principle. The method is briefly described in the next section. The basic ingredients for the time dependent pairing equations are the single-particle diagrams that must be computed from the initial state of the fission nucleus up the configuration given by two separated fragments. The Woods-Saxon two center shell model [9] used to determine realistic level scheme along the fission path is presented in section III. In the section IV the formalism concerning the time dependent pairing equations is introduced and its relevance in calculating dissipation energy is emphasized. In section V, the formalism is extended for two separated nuclei. In Section VI the results concerning the 234U fission are reported. The last section is devoted to conclusions.

II. THE FISSION TRAJECTORY

In order to calculate the energy levels diagrams for the fissioning system, the first step is the determination of a fission path that satisfies the minimal action criteria [10]. The sequence of shapes that follow a nucleus when it passes from the ground state to the scission point depend principally on the potential energy surface and the inertia.
In the macroscopic-microscopic method, the whole system is characterized by some collective coordinates that determine approximately the behavior of many other intrinsic variables. The basic ingredient in such an analysis is the shape parametrization that depends on several macroscopic degrees of freedom. The generalized coordinates associated with these degrees of freedom vary in time leading to a split of the nuclear system in two separated fragments. The macroscopic deformation energy is calculated within the liquid drop model. A microscopic potential must be constructed to be consistent with this nuclear shape parametrization. A microscopic correction is then evaluated using the Strutinsky procedure.

First of all, in our description, it is required to define a nuclear shape parametrization. In the following, an axial symmetric nuclear shape is obtained by smoothly joining two spheroids of semi-axis $a_i$ and $b_i$ $(i=1,2)$ with a neck surface generated by the rotation of a circle around the axis of symmetry. By imposing the condition of volume conservation we are left by five independent generalized coordinates $\{b_i\} (i=1,5)$ that can be associated to five degrees of freedom: the elongation $R$ given by the distance between the centers of the spheroids; the necking parameter $C = S/R_3$ related to the curvature of the neck, the eccentricities $\epsilon_i$ associated with the deformations of the nascent fragments and the mass asymmetry parameter $\eta = a_1/a_2$. The notations that describe this parametrization can be identified by inspecting the Fig. 1. Due to the axial symmetry, the surface equation is given in cylindrical coordinates for the three regions involved:

$$
\rho(z) = \begin{cases} 
    b_1 \sqrt{1-(z-z_1)^2/a_1^2}, & z \leq z_{c1}; \\
    \rho_3 - S \sqrt{R_3^2 - (z-z_3)^2}, & z_{c1} < z < z_{c2}; \\
    b_2 \sqrt{1-(z-z_2)^2/a_2^2}, & z \geq z_{c2}.
\end{cases} \quad (1)
$$

It is known that a nuclear shape is well adapted for the fission process is the following conditions are satisfied [11]: (i) The three most important degrees of freedom, that is, elongation, necking and mass-asymmetry are taken into account; (ii) A single sphere and two separated fragments are allowed configurations; (iii) The flatness of the neck is an independent variable. All these conditions are fulfilled by the above parametrization. If $S=1$, the shapes are necked in the median surface characterizing scission shapes and for $S=-1$ the shapes are swollen addressing the ground state and the saddle configurations.

If we consider that the elongation $q_1 = R$ is the main coordinate, the dependencies of the other generalized coordinates $q_i = f_i(R)$ $(i = 2,5)$ must be obtained. As specified in Ref. [10], such trajectories emerge by minimizing the action functional.

$$
P = -\frac{2}{\hbar} \int_{R_i}^{R_f} \sqrt{2M(q_i, \partial q_i/\partial R)V(q_i)}dR \quad (2)
$$

where $M(q_i, \partial q_i/\partial R)$ is the inertia along the trajectory and $V(q_i)$ is the deformation energy. $R_i$ and $R_f$ stand for the elongation associated to the ground state and the exit from the barrier, respectively. In our calculation the reference of the deformation energy is always taken as the energy in the ground state. So the next condition is fulfilled $V(R_i) = V(R_f) = 0$. As it can be seen in formula (2), as the fissioning nucleus passes from its ground state to the scission configuration, the sequences of shapes depends mainly on the deformation energy and the inertia. The deformation energy is obtained in the frame of the macroscopic-microscopic model [12] while the inertia is computed within the cranking approximation [10, 13, 14]. The deformation energy was obtained by summing the liquid drop energy $E_{LDM}$ with the shell and the pairing corrections $\delta E$.

$$
V = E_{LDM} + \delta E \quad (3)
$$

The macroscopic energy $E_{LDM}$ is obtained in the framework of the Yukawa - plus - exponential model [16] extended for binary systems with different charge densities as detailed in Ref. [17]:

$$
E_{LDM} = E_m + E_C + E_V \quad (4)
$$

![FIG. 1: Nuclear shape parametrization. Two ellipsoids of different eccentricities are smoothly joined with a third surface. Two cases are obtained: (a) the curvature of the neck is positive $S=1$ and (b) the curvature is negative $S=-1.$](image)
where
\[ E_n = -\frac{a_2}{8\pi^2 r_0^4 a^4} \int_{v_1} \int_{v_2} \left( \frac{r_{12}}{a} - 2 \right) \exp \left( -\frac{r_{12}}{a} \right) d^3 r_1 d^3 r_2 \]
is the nuclear term,
\[ E_C = \frac{1}{2} \int_{v_1} \int_{v_2} \rho_\nu(\vec{r_1}) \rho_\nu(\vec{r_2}) r_{12} d^3 r_1 d^3 r_2 \]
is the Coulomb energy, and \( E_V \) is the volume energy. In the previous definitions \( \rho_\nu \) are charge densities and \( r_{12} = |r_1 - r_2| \). The numerical values of the parameters \( a_2, r_0, a \) are taken from Ref. [18].

The shell effects \( \delta E \) are obtained as a sum between the shell and the pairing microscopic corrections. In this context, the Strutinsky procedure [10] was used. These corrections represent the varying parts of the total binding energy caused by the shell structure. The single particle level diagrams are computed within the Woods-Saxon supersymmetric two-center shell model.

The effective mass is computed within the cranking adiabatic approximation [10, 13, 14]. In a multidimensional deformation space, where the nuclear shape is described by the set of \( n \) independent generalized coordinates \( q_i \), the inertia tensor \( M_{ij} \) is defined by the equation of the kinetic energy \( T \):
\[ T = \frac{1}{2} \sum_{i,j=1}^{n} M_{ij}(q_1, ..., q_n) \frac{\partial q_i}{\partial t} \frac{\partial q_j}{\partial t} \]  

In the adiabatic description of the collective behavior of a nucleus, the nucleons are assumed to move in an average deformed potential. Using a Hamiltonian \( H(q_1, ..., q_n) \) that includes pairing interactions, introducing the collective parameters \( q_i \) by means of the Lagrange multipliers, it is possible to obtain the response of the nuclear system for slow changes of the shape within the cranking model formula
\[ M_{ij}(q_1, ..., q_n) = \frac{\hbar^2}{2} \sum_{\nu, \mu} \langle \nu | \frac{\hbar^2}{2m} | \nu \rangle \langle \nu | a_{\mu}^+ a_{\mu} | \nu \rangle \times (u_{\mu} v_{\nu} + u_{\nu} v_{\mu})^2 + P_{ij} \]

where \( |\nu > \) and \( |\mu > \) are single particle wave functions, \( E_{\nu}, u_{\nu} \) and \( v_{\nu} \) are the quasiparticle energy, the vacancy and occupation amplitudes of the state \( \nu \), respectively, in the BCS approximation, and \( P_{ij} \) is a correction that depends on the variation of the pairing gap and the Fermi energy as function of the deformation coordinates. Recently, the formula [8] was generalized by taking into account the intrinsic excitation produced during the fission process itself [15]. The inertia \( M \) along a trajectory in the configuration space spanned by the generalized coordinates \( q_i \) \((i=1,5)\) can be obtained within the formula
\[ M = 5 \sum_{i=1}^{5} \sum_{j=1}^{5} M_{ij} \frac{\partial q_i}{\partial R} \frac{\partial q_j}{\partial R} \]

The total inertia is the sum of the contributions that correspond to the proton and to the neutron level schemes. Usually, the matrix elements of the derivatives of the Hamiltonian in Ref. [8] are replaced by the matrix elements of the derivatives of the mean field potential alone.

### III. SINGLE PARTICLE ENERGIES

A microscopic potential must be constructed to be consistent within our nuclear shape parametrization. The simplest way it to use a semi-phenomenological Woods-Saxon potential. In order to take into account nuclear deformations going over to separate shapes and obtain two separated fragments, a two-center shell model with a Woods-Saxon potential was developed recently [9]. Other recipes that allows to treat strongly deformed nuclei are presented in Ref. [10, 20]. The mean field potential is defined in the frame of the Woods-Saxon model:
\[ V_0(\rho, z) = -\frac{V_c}{1 + \exp \left[ \frac{\Delta(\rho, z)}{a} \right]} \]

where \( \Delta(\rho, z) \) represents the distance between a point \((\rho, z)\) and the nuclear surface. This distance is measured only along the normal direction on the surface and it is negative if the point \((\rho, z)\) is located in the interior of the nucleus. \( V_c \) is the depth of the potential while \( a \) is the diffuseness parameter. In our work, the depth is \( V_c = V_{0c}[1 + \kappa(N_0 - Z_0)/(N_0 + Z_0)] \) with plus sign for protons and minus sign for neutrons, \( V_{0c} = 51 \text{ MeV}, a = 0.67 \text{ fm}, \kappa = 0.67 \). Here \( A_0, N_0 \) and \( Z_0 \) represent the mass number, the neutron number and the charge number of the parent, respectively. This parametrization, referred as the Blomqvist-Walhborn one in Ref. [21], is adopted because it provides the same radius constant \( r_0 \) for the mean field and the pairing field. That ensures a consistency of the shapes of the two fields at hyperdeformations, i.e., two tangent ellipsoids. The Hamiltonian is obtained by adding the spin-orbit and the Coulomb terms to the Woods-Saxon potential. The eigenvalues are obtained by diagonalization of the Hamiltonian in the semi-symmetric harmonic two center basis. In this work, the major quantum number used is \( N_{\text{max}} = 12 \). The two center Woods-Saxon model will be used to computer shell and pairing corrections together with inertia in this work. The two center shell model represents a valuable instrument to investigate the role of individual orbitals for the treatment of a wide variety of nuclear processes like cold fission [22], formation of superheavy elements [23] or supersymmetric disintegration processes, pertaining to cluster- and alpha-decays [27, 29].

### IV. TIME DEPENDENTPAIRING EQUATIONS

In the actual formalism, the starting point is a many-body Hamiltonian with pairing residual interactions.
This Hamiltonian depends on some time-dependent collective parameters \( q(t) = \{ q_i(t) \} \) \((i = 1, \ldots, n)\), such as the inter-nuclear distances between atoms or nuclei:

\[
H(t) = \sum_{k>0} \epsilon_k [q(t)] (a_k^+ a_k + a_k^+ a_k^*) - G \sum_{k,i>0} a_k^+ a_k^* a_i a_i^*.
\]

(11)

Here, \( \epsilon_k \) are single-particle energies of the molecular potential, \( a_k^+ \) and \( a_k \) denote operators for creating and destroying a particle in the state \( k \), respectively. The state characterized by a bar signifies the time-reversed partner of a pair. The pairing correlation arises from the short range interaction between fermions moving in time-reversed orbits. The essential feature of the pairing correction can be described in terms of a constant pairing interaction \( G \) acting between a given number of particles.

In this paper, the sum over pairs generally runs over the index \( k \). Because the pairing equations diverge for an infinite number of levels, a limited number of levels are used in the calculation, that is \( N \) levels above and below the Fermi energy \( E_F \).

In order to obtain the equations of motion, we shall start from the variational principle taking the following energy functional

\[
\mathcal{L} = \langle \varphi \mid H - i\hbar \frac{\partial}{\partial t} \mid \varphi \rangle
\]

by assuming the many-body state as time dependent BCS seniority-zero wave function

\[
| \varphi(t) \rangle = \prod_k (u_k(t) + v_k(t) a_k^+ a_k^*)
\]

(13)

To minimize this functional, the expression (12) is derived with respect the independent variables \( u_k \), together with their complex conjugates, and the resulting equations are set to zero. Eventually, the next coupled-channel equations are obtained:

\[
\begin{align*}
\dot{\rho}_k &= \kappa_k \Delta^* - \kappa_k^* \Delta \\
\dot{\kappa}_k &= (2\rho_k - 1)\Delta - 2\kappa_k \epsilon_k
\end{align*}
\]

(14)

where \( \rho_k = | v_k |^2 \) are occupation probabilities, \( \kappa_k = u_k^* v_k \) are pairing moment components, and \( \Delta = G \sum_k \kappa_k \) is the pairing gap. \( u_k \) and \( v_k \) are the complex BCS occupation and vacancy amplitudes. The variations of single-particle densities \( \rho_k \) can be evaluated for different values of the generalized velocities by solving the previous system of coupled equations as done in Ref [30]. Eqs. (14) are also generically known as the time dependent Hartree-Fock-Bogoliubov equations [31, 32]. As mentioned in Ref. [32], a connection with the Landau-Zener effect is included in these equations. Levels undergo Landau-Zener transitions on virtual levels with coupling strengths given by the magnitude of the gap \( \Delta \). Recently, these equations were generalized to take into account the Landau-Zener effect in seniority one systems [3, 33] and the pair breaking mechanism [34].

These time-dependent pairing equations can offer a measure of the dissipated energy. The difference between the total energy value \( E \) obtained within the TDHFB equations and \( E_0 \) given by the static BCS-equations represents an approximate measure for the dissipation \( E_D \):

\[
E_D = E - E_0.
\]

(15)

\( E \) is expressed simply in terms of \( \rho_k \) and \( \kappa_k \)

\[
E = 2 \sum_k \epsilon_k \rho_k - G \sum_k \kappa_k |\Delta|^2 - G \sum_k \rho_k^2.
\]

(16)

\( E_0 \) corresponds to \( \rho_0^2 \) and \( \kappa_0^2 \) associated to the lower energy state, that is, obtained from BCS equations.

The sum of single particle densities derivatives of Eqs. (14) is

\[
i\hbar \sum_k \dot{\rho}_k = \sum_k (\kappa_k \Delta^* - \kappa_k^* \Delta)
\]

\[
= \frac{\hbar}{2} \left[ |\Delta|^2 - | \Delta_i |^2 \right] = 0
\]

(17)

This result shows that the sum of the single particle occupation probabilities is a constant quantity as the system evolves in time according to Eqs. (14), that is, the average number of particles in the pairing active space is a constant of the motion.

V. NUMBER OF PARTICLES

After scission, the levels from the pairing active space will be shared between the two fission fragments. The levels of the core are sorted accordingly. The sum of occupation probabilities of the levels located in each fragment must be equal to the number of nucleons. This is a problem that can be solved by appealing to two properties of the nuclear system.

First of all, the sorting is produced in a continuous manner: the wave function associated to one single-particle energy level is transferred gradually to one of the two potential wells obtained asymptotically, after the scission. For example, the Woods-Saxon wave function of the lowest single particle energy is displayed in Fig. 2 for different values of the distance between the centers of the fragments, that is, for different shapes of the potential. When the two fragments are completely separated, the wave function is located in one of the two wells. Making use of this property, it is possible to identify the final localization of the single particle level even when the system behave as one single nucleus if the final mass-asymmetry is known. For this purpose we calculate two quantities \( Q_{1k} = \langle \varphi_k(z) \mid \Theta(-z) \mid \varphi_k(z) \rangle \) and \( Q_{2k} = \langle \varphi_k(z) \mid \Theta(z) \mid \varphi_k(z) \rangle \) where \( \Theta \) is the Heaviside function and \( \varphi_k \) is the Woods-Saxon single particle wave function of the level \( k \) along the axis \( z \). If \( Q_{1k} > Q_{2k} \), the wave function is located in the well 1. It is worth to note that this procedure fails only in the avoided level crossing regions.

Secondly, the matrix element of the pairing interaction \( G \) is in principle dependent of the overlap of the wave functions of the pairs [35, 36]. As long as a single nucleus is involved, the monopole approximation is
FIG. 2: Right panel. For the left axis, the lowest energy Woods-Saxon wave function for the two center model $\varphi_0(z)$ as function of $z$ for different values of the distance between the centers is displayed with a thin line. The right axis corresponds to a plot with a thick curve of a section in the middle of the neutron Woods-Saxon potential $V_0(z)$ at the same values of the distance between centers. In the left panel, a representation of the Woods-Saxon potential $V_0$ in the cylindrical $\rho$ and $z$ coordinates is made. The distances between centers $R$ are 0, 6, 12, 18 and 21 fm from top to bottom. The configurations displayed correspond to the minimal $^{234}$U fission trajectory.

considered to perform well [37] and all the values of the matrix elements are considered to be equal in the active level space. However, after the scission, the matrix elements of the pairing interaction between wave function belonging to different fragments must be zero. If the pairing matrix elements between pairs located in different fragments at scission is zero ($G_{12}=0$), then the energy given by Rel. (16) becomes

$$E = 2 \sum_k \epsilon_k \rho_k - G \left| \sum_k \kappa_k \right|^2 - G \sum_k \rho_k^2$$

$$\Rightarrow 2 \sum_{k_1} \epsilon_{k_1} \rho_{k_1} - G_1 \left| \sum_{k_1} \kappa_{k_1} \right|^2 - G_1 \sum_{k_1} \rho_{k_1}^2 + \sum_{k_2} \epsilon_{k_2} \rho_{k_2} - G_2 \sum_{k_2} \kappa_{k_2}^2 - G_2 \sum_{k_2} \rho_{k_2}^2$$

$$= E_1 + E_2$$

(18)

where the sum over $k_1$ and $k_2$ means that the levels belong to fragments 1 and 2, respectively. We used an arrow in the previous relation to indicate that the constant
value $G$ of the parent nucleus can be also transformed in the two values $G_1$ and $G_2$ associated to the two fragments. On other words, the monopole approximation is considered valid separately in each fragment by considering constant values of $G_1$ and $G_2$. The relation \[ \text{(18)} \] shows that the total energy $E$ is decomposed in two fragment total energies $E_1$ and $E_2$ given by relations of the type \[ \text{(16)} \] if the values of the matrix element of the pairing interaction addressing different fragments $G_{12}=0$.

Within these properties, a simple recipe to fix the average number of particles at scission can be elaborated. Taking as example the proton level scheme, the conditions that the sum of occupation probabilities of levels in the two wells must be equal with the number of nucleons of the fragments can be written as:

\[ 2Z_p \sum_k \rho_{k_1} = 2Z_{p_1} \sum_k \rho_{k_2} \]
\[ Z_{p_1} + Z_{c_1} = Z_1 \]
\[ Z_{p_2} + Z_{c_2} = Z_2 \]

(19)

where $Z_i$ ($i=1,2$) are the number of protons in the two fragments, $Z_c$, and $Z_p$, stand for the number of protons in the core and the number of protons in the pairing active space, respectively. For an initial number of pairs $N$ considered in the pairing active space, the values of $Z_c$ and $Z_p$ are simply obtained by counting the levels given by the two center model. The occupations probabilities $\rho_k$, must be obtained from the time dependent pairing equations \[ \text{(14)}. \]

Exploiting the two previous properties, the problem to fix the final average number of particle within equations \[ \text{(14)} \] is now trivial. After the passage of the external saddle point, in the descent to scission, we insert the condition \[ \text{(19)} \] in the functional \[ \text{(12)} \] and we continue to solve the equations of motion. When the good numbers of particles are obtained, we impose the condition that $G_{12}$ is zero between the wave functions belonging to separated fragments. Both conditions \[ \text{(19)} \] is introduced after the passage of the external saddle point.

In the operator notation the condition \[ \text{(19)} \] becomes

\[ Z_{p_1} \hat{Z}_{p_1} = Z_{p_1} \hat{Z}_{p_2} \]
\[ \hat{Z}_{p_1} = \sum_k \left( a_{k_1} a_{k_1}^+ + \kappa_{k_1} a_{k_1}^+ \right) \]
\[ \hat{Z}_{p_2} = \sum_k \left( a_{k_2} a_{k_2}^+ + \kappa_{k_2} a_{k_2}^+ \right) \]

(20)

This condition is introduced in the energy functional \[ \text{(12)} \]

\[ \mathcal{L} = \langle \varphi | H - ih \frac{\partial}{\partial t} - \lambda (Z_{p_2} \hat{Z}_{p_1} - Z_{p_1} \hat{Z}_{p_2}) | \varphi \rangle \]

(21)

using a Lagrange multiplier $\lambda$. Imposing also the condition that the interaction matrix element $G$ between pairs of the same fragment is not the same than those of different fragments the new time dependent equations read, eventually:

\[ ih \dot{\rho}_{k_1} = \kappa_{k_1} \Delta_{k_1} - \kappa_{k_1} \Delta_1 \]
\[ ih \dot{\rho}_{k_2} = \kappa_{k_2} \Delta_{k_2} - \kappa_{k_2} \Delta_2 \]
\[ ih \kappa_{k_1} = (2\rho_{k_1} - 1) \Delta_1 - 2\kappa_{k_1} (\epsilon_{k_1} - \lambda Z_{p_2}) \]
\[ ih \kappa_{k_2} = (2\rho_{k_2} - 1) \Delta_2 - 2\kappa_{k_2} (\epsilon_{k_2} + \lambda Z_{p_1}) \]

(22)

where $\Delta_1 = G_1 \sum_k \kappa_{k_1} + G_{12} \sum_k \kappa_{k_2}$ and $\Delta_2 = G_{12} \sum_k \kappa_{k_1} + G_2 \sum_k \kappa_{k_2}$. When $G_{12}=0$, it can be easily verified the average number of particles in the two fragments are conserved according to conditions of the type \[ \text{(17)} \] applied separately on the two working spaces. That means, $Z_1$ and $Z_2$ behave as constants. The previous recipe represents the simplest dynamical method to project the average number of particles onto two separate nuclei.

VI. RESULTS

To obtain the fission trajectory, the action integral \[ \text{(4)} \] must be minimized in our five-dimensional space. The first turning point $R_i$ is obtained by determining the ground state configuration while the second one $R_f$ lies on the equipotential surface that characterize the exit from the outer barrier. That means, $R_f$ is defined by the multidimensional function $V(R, C, \epsilon_1, \epsilon_2, \eta)=0$. Different methods are currently envisaged to obtain the heights of the barriers. In static calculations \[ \text{(8)} \] the immersion procedure is extensively used while for dynamical paths \[ \text{(39-41)} \] the Ritz method is applicable. To minimize the action integral we used a numerical method initiated in Ref. \[ \text{(42)} \] and used for fission processes in a large range of mass asymmetries \[ \text{(34-43)} \] The dependencies $C(R)$, $\epsilon_1(R)$, $\epsilon_2(R)$, and $\eta(R)$ were considered as spline functions of $n$ variables $C_k$, $\epsilon_{1k}$, $\epsilon_{2k}$ and $\eta_k$ ($k = 1, n$) that are associated to the fixed mesh points $R_k$ located in the interval $(R_i, R_f)$. The integral action is transformed in a numerical function that depends on the $4n$ variables and it is minimized numerically.

Determination of potential energies $V$ and of effective masses $M_{ij}$ are very expensive in computing time. For the numerical minimization procedure, a large number of iterations is required and it is not possible to calculate the values of $V$ and $M_{ij}$ for each iteration. An interpolation of calculated values of the energy and of the effective masses will be used. Therefore, to make the problem tractable, first of all, a grid of deformation values was fixed in the five-dimensional configuration space: 25 values of $R$ between 0 fm and 24 fm, 7 values of the eccentricities $\eta_i$ between 0 and 0.6, 7 values of the ratio $\eta$ in the interval 1 and 1.6, and 23 values of $C$ between −0.115 fm$^{-1}$ and 0.105 fm$^{-1}$. The pertinent region of deformations for the possible trajectories between the ground state and the exit point from the fission barrier was spanned. In each point of the configuration space, the deformation energy and the tensor of inertia was computed. During the minimization process, interpolated values of the deformation energy and the inertia were used. Different initial values of the generalized coordinates were used as input parameters for the minimization. Therefore different local minimum were obtained. The best value was selected. Moreover, additional calculation of the action integral were performed by slowly varying the generalized coordinates around the best trajectory previously.
FIG. 3: $^{234}\text{U}$ fission barrier $V$ for a final partition $^{102}\text{Zr}+^{132}\text{Te}$ determined along the minimal action trajectory. Some particular shapes related to the ground state, the extremes of the barrier, the exit point and the scission point are inserted in the plot. The distances for the elongation $R$ that characterizes the shapes are $4.17$, $7.7$, $10.43$, $12.64$, $15.53$, $17.53$ and $20.2$ fm.

obtained from the numerical procedure. Among all results, the best final trajectory for the least action was retained. Such a procedure was used in determining a theoretical systematic of fission barrier heights in Ref. [46]. In the present work, the trajectory in the configuration space was modified after the saddle of the outer barrier. The generalized coordinates were changed to obtain at scission a required final configuration as explained below.

The minimization leads to a definite path in the sub-barrier region. Its extrapolation to the scission point is not unique [47]. We will choose the scission configuration by searching the best candidates that have deformations close to those obtained in the exit point of the barrier. We found that the eccentricities and mass-asymmetry parameters in the exit point of the outer barrier are consistent with the formation of a pair given by the isotopes $^{102}\text{Zr}$ and $^{132}\text{Te}$. The ground state configurations of the fragments were calculated for this purpose. By introducing the ground states of the fragments in the scission configuration, the excitation due to the deformations energy is minimized and we are left only with the dissipation. As evidenced from evaluations [48], the partition $^{102}\text{Zr}+^{132}\text{Te}$ is one of the larger enough probable partitions. Moreover, this final configuration is also interesting due to the fact that the numbers of nucleons in the heavy fragments are close to the magic ones. This configuration allows also a simple comparison with the hypothesis made in Ref. [3]. The fission barrier together with some particular shapes are plotted in Fig. 3.

The neutron and proton single particle diagrams are calculated along the minimal action trajectory, from the ground state of the parent up to the formation of two

FIG. 4: Neutron level diagram for the $^{234}\text{U}$ fission with respect the elongation $R$.

FIG. 5: Same as Fig. 4 for the proton diagram.
separated fragments. These level schemes are plotted in Figs. 4 and 5. In Fig. 4, at scission, the energy slope for the light fragment is larger than that for the heavy one. The energy slope for the light fragment becomes spherical after scission, so its levels are not degenerate due to its deformation. In the proton diagram of Fig. 5, it can be observed a smooth decrease of the single particle energies after the scission due to the Coulomb mutual polarization. The energy slope for the light fragment is larger than that for the heavy one.

It is known that if the projection on a given particle number changes the pairing coupling constant $G$. However, for our exploratory investigation, the same $G$ will be kept for the parent nucleus and for the fragments. After scission, we calculate the matrix elements for each fragment separately. The values of $G$, $G_1$, $G_2$ are obtained from the renormalization procedure [16] by taking into account the number of levels in the pairing active space.

To solve the Eqs. (14) we need the velocity of the generalized coordinate. Our calculations will pertain to the cold fission regime and are characterized by small values of the collective kinetic energy and of the excitations energies. These properties can be obtained by selecting an appropriate value for the collective velocity. Therefore, different constant values of the inter-nuclear velocity $\dot{R}$ ranging from $10^4$ to $10^6$ m/s were tested. These values can be translated in a time to penetrate the barrier ranging in the interval $[10^{-18}, 10^{-20}]$ s. The time for the descent between saddle to scission reaches about $4 \times 10^{-21}$ s for a velocity of $10^6$ m/s, a value considered a typical time for scission [2]. Within the semi-adiabatic cranking model [13], the inertia in the ground state and in the asymptotic region of two separated fragments are 1.26 and 1.208 $\hbar^2$/MeV$^2$/fm$^2$, respectively. Within this estimation of the mass, the velocities can also be translated in a macroscopic kinetic energy that amounts up to 0.3 MeV, of the order of magnitude of the zero point vibration energy. The system (14) is solved numerically for the selected velocities. The initial conditions for $\rho_k$ and $\kappa_k$ were given by the BCS solutions for the parent ground state, a configuration characterized by $R \approx 4$ fm in Fig. 5. In Fig. 6 the dependencies of the dissipated energy $E^*$ for the different internuclear velocities investigated are displayed as function of the distance between the centers of fragments $R$. No conditions for fixing the number of particles are used. The dissipated energy for the proton level scheme is lower that that of the neutron one. The dissipated energy for the lower collective velocity is negligible, while for the higher one the total dissipation reaches about 10.5 MeV. This value is larger than in previous estimations [15] because of the imposed modification of the fission trajectory that gives a particular configuration at scission. In connection with the shape of the barrier displayed in Fig. 3 it can be deduced that the larger part of the excitation is formed during the penetration of the second barrier and, in the descend between saddle to fission. This result is not in line with the observation that the excitation energy does not increase after the passage of the saddle point [19].

It will be interesting to compare the dissipated energy and the number of nucleons attributed to the two nuclei in the case of the treatment without constraints and the treatment with projection of the average number of particle. In Fig. 7(a) the unconstrained dissipation energy for the neutron workspace is plotted with a thick curve as function of $R$ at $dR/dt=10^6$ m/s. The dissipation obtained within the scheme proposed for projecting the average number of particles is displayed with a thin line. Up to the the external saddle, the Eqs. (14) are solved. After $R \approx 16$ fm, we imposed the condition $\kappa_k$ [20]. It can be observed that the time dependent pairing equations with constraint give a larger dissipation. In Fig. 7(b) the sums $N_{p_1} = \sum_k k^2 \rho^2_k$ and $N_{p_2} = \sum_k k^2 \rho^2_k$ for the unconstrained equations are plotted with a thick line. $k_1$ and $k_2$ are levels in the active pairing space attributed to the fragment 1 and the fragment 2, respectively. The total

![Graph](image-url)
number of pairs is conserved $N_1 + N_2 = N = 30$. After
imposing the condition (20), the number of pairs located
on the two level schemes of the two fragment reach the
correct values, that is $N_1 = 15$ and $N_2 = 15$, that define the
partition analyzed.

It was remarked in Ref. [31] that the maxima of the
dissipated energy arise because of the character of the
ground state solution is changing rapidly that the sys-

tem cannot adjust itself. So, the system appears excited
not because of any changes in the occupation amplitudes
but because the changes in the ground state to which the
dynamic state is studied. In this respect, in Fig. 7 we
analyzed also the behavior of the system by replacing the
ground state. Instead of the ground state of the parent,
we used as $E_0$ the sum between the lower energy states
calculated for the two level schemes of the two nascent
fragments. The result is plotted with a dashed line. At
scission we obtain approximately 8 MeV excitation en-
ergy for both fragments.

In Fig. 8 the dissipation obtained for the two frag-
ments are displayed separately. It is found that the dissi-
pated energy for the heavy fragment is much lower than






related on the distribution of pairing occupation proba-


ties of the levels at scission. These probabilities can
be obtained by solving an appropriate set of equations of
motion.

FIG. 7: (a) The dissipated energy without constraints is
plotted with a full thick line as function of the elongation
$R$. The dissipated energy after imposing the condition for
projecting the number of particles is plotted with a thin line.
The dissipated energy obtained by replacing the ground state
of the parent nucleus within the ground states of the two
nascent fragments. (b) The number of pairs $N_{p1}$ and $N_{p2}$
located on the levels that belongs to the first and second well
are plotted with thick lines. The number of particles after
imposing the conditions for projecting the number of particles
are plotted with thin lines. $N_{p2}$ corresponds to the heavy
fragment and it is always larger than $N_{p1}$.

VII. CONCLUSIONS

Dynamical estimations of the excitation energies in
cold fission are evaluated within time dependent pair-
ing equations. Using conditions that fix the number of
particles in each fragment it was possible to obtain for
the first time the excitation energy of each nucleus issued
in the process. A recent hypothesis [3] that claims the
excitation energy is not equilibrated between fragments
was confirmed in the cold fission regime. If the heavy
fragment is close to magic number, its excitation energy
is smaller than that of the light one. It was found that

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FIG. 8: Excitations energies for $^{102}$Zr (dashed line) and $^{132}$Te (full line) as function of the internuclear velocity $dR/dt$. In panels (a) and (b) the final excitation are displayed for the neutron and proton schemes, respectively. In panel (c), the sum between the excitations energies obtained for protons and neutrons are presented.

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