Local scaling limits of Lévy driven fractional random fields

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We obtain a complete description of local anisotropic scaling limits for a class of fractional random fields $X$ on $\mathbb{R}^2$ written as stochastic integral with respect to infinitely divisible random measure. The scaling procedure involves increments of $X$ over points the distance between which in the horizontal and vertical directions shrinks as $O(\lambda)$ and $O(\lambda^\gamma)$ respectively as $\lambda \downarrow 0$, for some $\gamma > 0$. We consider two types of increments of $X$: usual increment and rectangular increment, leading to the respective concepts of $\gamma$-tangent and $\gamma$-rectangent random fields. We prove that for above $X$ both types of local scaling limits exist for any $\gamma > 0$ and undergo a transition, being independent of $\gamma > \gamma_0$ and $\gamma < \gamma_0$, for some $\gamma_0 > 0$; moreover, the ‘unbalanced’ scaling limits ($\gamma \neq \gamma_0$) are $(H_1,H_2)$-multi self-similar with one of $H_i$, $i = 1,2$, equal to 0 or 1. The paper extends Pilipauskaite and Surgailis (2017) and Surgailis (2020) on large-scale anisotropic scaling of random fields on $\mathbb{Z}^2$ and Benassi et al. (2004) on 1-tangent limits of isotropic fractional Lévy random fields.

Keywords: Fractional random field; local anisotropic scaling limit; rectangular increment; Lévy random measure; scaling transition; multi self-similar random field

1. Introduction

The present paper discusses local anisotropic scaling behavior of a class of fractional type infinitely divisible random fields (RFs) on $\mathbb{R}^2$. This behavior is characterized by limits of shrinking increments of RF under anisotropic scaling, where anisotropy is due to the fact that the ‘horizontal’ and ‘vertical dimensions’ of the increment tend to 0 with $\lambda \downarrow 0$ at different rates $\lambda$ and $\lambda^\gamma$ for any given $\gamma > 0$.

Given a RF $X = \{X(t), t \in \mathbb{R}^2\}$, the two basic types of its increment at a point $t_0 = (t_{01}, t_{02}) \in \mathbb{R}^2$ are (ordinary) increment $X(t_0 + t) - X(t_0)$ and rectangular increment

$$X((t_0, t_0 + t)) := X(t_{01} + t_1, t_{02} + t_2) - X(t_{01}, t_{02} + t_2) - X(t_{01} + t_1, t_{02}) + X(t_{01}, t_{02}) \quad (1.1)$$

for $t = (t_1, t_2) \in \mathbb{R}^2_+$. These two notions of increment give rise to different notions of RF with stationary increments and stationary rectangular increments [5]. For $\gamma > 0$, $\lambda > 0$, let $\Gamma = \text{diag}(1, \gamma)$, $\lambda^\Gamma = \text{diag}(1, \lambda^\gamma)$. With $\lambda^\Gamma t = (\lambda t_1, \lambda^\gamma t_2)$, (anisotropic) local scaling limits of RF $X$ at $t_0$ can be defined as the limits (in the sense of weak convergence of finite-dimensional distributions) as $\lambda \downarrow 0$:

$$d_{\lambda,\gamma}^{-1}(X(t_0 + \lambda^\Gamma t) - X(t_0)) \xrightarrow{\text{fdd}} T_\gamma(t), \quad (1.2)$$

$$d_{\lambda,\gamma}^{-1}X((t_0, t_0 + \lambda^\Gamma t)) \xrightarrow{\text{fdd}} V_\gamma(t), \quad (1.3)$$

where $d_{\lambda,\gamma}^{-1}$ is a normalization, the latter being generally different for (1.2) and (1.3). Here and below, we suppress the dependence of these scaling limits on $t_0$ also because in our theorems they actually do
not depend on it. In the case of isotropic scaling $\gamma = 1$ the limit $T_1$ in (1.2) (called the tangent RF) was introduced in [15] with \( \lim_{\lambda \downarrow 0} \) replaced by a stronger functional convergence. The existence of (non-trivial) tangent RF is also termed local asymptotic self-similarity [8, 13, 14] since the tangent RF is self-similar [15]. In fact, following [15], [37, Prop. 2.1], one can prove that under mild additional conditions all scaling limits in (1.2)–(1.3) satisfy the $(H, \gamma)$-SS (self-similarity) property:

\[ U(\lambda^r t) \overset{\text{fdd}}{=} \lambda^H U(t), \quad \forall \lambda > 0, \tag{1.4} \]

with some $H = H(\gamma) > 0$; moreover, the normalization $d_{\lambda, \gamma}$ is regularly varying with exponent $H$ as $\lambda \downarrow 0$. Superseding this terminology we call $T_\gamma$ in (1.2) the $\gamma$-tangent RF and $V_\gamma$ in (1.3) the $\gamma$-rectangent RF (‘rectangent’ as the abridge for ‘rectangular tangent’). Note that for $\gamma = 1$ (1.4) yields the (usual) SS property for RF indexed by $\mathbb{R}^2$ or $\mathbb{R}_+^2$, see [24, 15, 41], for general $\gamma > 0$ (1.4) is a particular case of operator scaling property discussed in [10].

The RFs $X$ for which $\gamma$-tangent and $\gamma$-rectangent RF ($\gamma > 0$ arbitrary) are identified in this paper are written as stochastic integrals

\[ X(t) := \int_{\mathbb{R}^2} \{g(t-u) - g^0_1((t_1,0) - u) - g^0_2((0,t_2) - u) + g^0_{12}(-u)\} M(du), \quad t = (t_1, t_2) \in \mathbb{R}_+^2, \tag{1.5} \]

where $M$ is an infinitely divisible random measure (also called a Lévy basis) on $\mathbb{R}^2$ and $g, g^0_1, g^0_2, g^0_{12}$ are deterministic functions satisfying some conditions guaranteeing the existence of (1.5). For reasons explained below, we call $X$ in (1.5) a Lévy driven fractional RF. It follows from (1.5) that RF $X$ has stationary rectangular increments which do not depend on the ‘initial’ functions $g^0_1, g^0_2, g^0_{12}$, viz.,

\[ X((0,t)) = \int_{\mathbb{R}^2} g((-u, t - u)) M(du), \quad t \in \mathbb{R}_+^2, \tag{1.6} \]

where $g((-u, t - u)) := g(t_1 - u_1, t_2 - u_2) - g(-u_1, t_2 - u_2) - g(t_1 - u_1, -u_2) + g(-u_1, -u_2)$ in accordance with the notation in (1.1). Similarly, if $g^0_1 = g^0_2 = 0$ then $X$ in (1.5) has stationary (ordinary) increments. In both cases, the scaling limits in (1.2)–(1.3) (provided they exist) do not depend on $t_0$ and depend on scaling properties of kernel $g$ and the Lévy basis $M$ specified in Assumptions (G)$_\alpha$ and (M)$_\alpha$ below (roughly, the last assumption means that the small-scale behavior of $M$ is $\alpha$-stable with $0 < \alpha \leq 2$). The class of RFs in (1.5) is quite large and contains many fractional RFs studied in [8, 13, 44, 22, 20] and elsewhere. (1.5) also constitute a natural spatial generalization of Lévy driven moving average processes with one-dimensional time studied in [6].

The main results of this paper can be summarised as follows. We prove that for a class of fractional Lévy driven RFs in (1.5) the $\gamma$-rectangent limits $V_\gamma$ exist for any $\gamma > 0$ and are $\alpha$-stable RFs; moreover, the limit family $\{V_\gamma, \gamma > 0\}$ exhibits a ‘scaling transition’ in the sense that there exists $\gamma_0 > 0$ such that $V_\gamma = V_+ (\text{respectively, } V_\gamma = V_-)$ do not depend on $\gamma > \gamma_0 (\text{respectively, on } \gamma < \gamma_0)$ and $V_+ \neq cV_-$ for any $c > 0$; moreover, the ‘unbalanced’ $\gamma$-rectangent limits $V_\pm$ are $(H_1, H_2)$-multi self-similar (MSS) RFs (see (1.7)) with one of the self-similarity parameters $H_i, i = 1, 2$, equal 1 or 0. We also prove somewhat similar although more straightforward results, including a ‘scaling transition’, about $\gamma$-tangent limits $T_\gamma$ for a related class of fractional Lévy driven RFs in (1.5).

Related trichotomy of the scaling behavior was reported in large-scale anisotropic scaling for several classes of long-range dependent (LRD) planar RF models, with rectangular increment replaced by a sum or integral of the values on large rectangle with sides increasing at different rates $\lambda$ and $\lambda^\gamma$ as $\lambda \to \infty$ for any given $\gamma > 0$; see [36, 37, 32, 33, 34, 43]. See also [11]. In the above works, this trichotomy was termed the scaling transition, with $V_+$ the unbalanced and $V_{\gamma_0}$ the well-balanced scaling limits. The
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present paper can be regarded as a continuation of the above research and we use the same terminology in the case of the ‘small-scale’ limits in (1.2)–(1.3). As noted above the unbalanced limits in (1.2)–(1.3) have a very particular dependence structure being \((H_1, H_2)\)-MSS RFs with one of \(H_i, i = 1, 2\), equal 0 or 1. Following [17] we call a RF \(V = \{V(t), t \in \mathbb{R}_+^2\}\) \((H_1, H_2)\)-MSS with parameters \(H_i \geq 0, i = 1, 2\), if
\[
V(\lambda_1 t_1, \lambda_2 t_2)^{\text{fdd}} \overset{\text{def}}{=} \lambda_1^{H_1} \lambda_2^{H_2} V(t), \quad \forall \lambda_1 > 0, \forall \lambda_2 > 0.
\]
(We note that while [17] assume \(q_i, i = 1, 2\), relation (1.7) makes sense when \(H_1 \wedge H_2 = 0\) as well.) The ‘classical’ case of \((H_1, H_2)\)-MSS RF is Fractional Brownian Sheet (FBS) \(B_{H_1, H_2}\) defined as a Gaussian process on \(\mathbb{R}_+^2\) with zero mean and the covariance function:
\[
E B_{H_1, H_2}(t) B_{H_1, H_2}(s) = \frac{1}{4} \sum_{i=1}^{2} (t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}),
\]
for \(t = (t_1, t_2) \in \mathbb{R}_+^2, s = (s_1, s_2) \in \mathbb{R}_+^2\). The parameters \(H_i, i = 1, 2\), of FBS usually take values in the interval \((0, 1]\) or \((0, 1)\), see [2]; the extension to \(H_i \in (0, 1]\) was defined in [43] from (1.8) by continuity as \(H_i \downarrow 0, i = 1, 2\), see Definition 1. FBS with \(H_1 \wedge H_2 = 0\) are very unusual and extremely singular objects and their appearance in limit theorems is surprising [43]. We note the FBS with \((H_1, H_2) = (0, \frac{1}{2})\) or \((\frac{1}{2}, 0)\) as anisotropic partial sums limits of linear RFs on \(\mathbb{Z}^2\) with negative dependence and edge effects were obtained in [43]; in the case of LRD RFs on \(\mathbb{Z}^2\) unbalanced Gaussian limits were proved to be FBS with at least one of \(H_i, i = 1, 2\), equal or less than 1 [33, 34].

Let us describe the results of the present paper informally in more detail. Rigorous formulations are given in Sec. 5 and 6. We assume that \(g\) in (1.5) and (1.6) features a power-law behavior at the origin with possibly different exponents along the coordinate axes, more precisely, as \(t \to 0\),
\[
g(t) \sim g_0(t) := \rho(t)^\chi L(t), \quad \text{where } \rho(t) := |t_1|^{q_1} + |t_2|^{q_2},
\]
\(q_1 > 0, q_2 > 0, Q := \frac{1}{q_1} + \frac{1}{q_2}, \chi \neq 0\) are parameters, and \(L(t), t \in \mathbb{R}_+^2\), is a (generalized invariant) function satisfying some boundedness and regularity conditions. For \(L(t) \equiv 1\), the kernel \(g_0(t) = \rho(t)^\chi\) vanishes or explodes at the origin \(t = 0\) depending on the sign of the parameter \(\chi\) suggesting a different scaling behavior in (1.3) in the cases \(\chi > 0\) and \(\chi < 0\). Actually, the limit results essentially depend on the two parameters
\[
p_i := q_i(Q - \chi) > 0, \quad i = 1, 2,
\]
alone, making the parametrization \(p_i, i = 1, 2\), in (1.10) more convenient than \(\chi, q_i, i = 1, 2\), in (1.9). The parameters (1.10) satisfy
\[
\frac{c_1}{1 + \alpha} < P < \alpha, \quad \text{where } P := \frac{1}{p_1} + \frac{1}{p_2}.
\]
Note for \(p_i, i = 1, 2\), in (1.10) satisfying (1.11) \(\chi > 0\) is equivalent to \(P > 1\). For \(c_i > 0, i = 1, 2\), denote
\[
P_{c_1, c_2} := \frac{c_1}{p_1} + \frac{c_2}{p_2}
\]
so that \(P = P_{1,1}\). The main results of this paper are represented in Table 1 and Figure 1 showing four sets \(R_{11}, R_{12}, R_{21}, R_{22}\) in the parameter region (1.11) determined by segments \(P_{\frac{1}{2}, \frac{1}{2}} \cdot \frac{1}{1 + \alpha} = 1, P_{\frac{1}{2}, \frac{1}{2}} \cdot \frac{1}{1 + \alpha} = 1\) with different unbalanced rectangent limits \(V_{\pm}\). The critical or the scaling transition point in all four regions \(R_{ij}, i, j = 1, 2\), is the same, namely
\[
\gamma_0 := \frac{p_1}{p_2} = \frac{q_1}{q_2}.
\]
The four RFs $\tilde{Y}_{\alpha,i}$, $Y_{\alpha,i}$, $i = 1, 2$, in Table 1 are defined in Sec. 4 as integrals with respect to (w.r.t.) $\alpha$-stable random measure on $\mathbb{R}^2$ of non-random integrands determined by certain increments of $g_0$ in (1.9) or its partial derivatives. All four RFs have $\alpha$-stable finite-dimensional distributions and are $(H_1, H_2)$-MSS with indices shown in Table 1. In the Gaussian case $\alpha = 2$ the RFs in Table 1 agree with FBS with the corresponding parameters $(H_1, H_2)$.

Let us briefly describe our results concerning $\gamma$-tangent limits in (1.2). As mentioned above we consider the case of (1.5) with $g_1^0 = g_2^0 = 0$ so that

$$X(t) = \int_{\mathbb{R}^2} (g(t - u) - g(-u)) M(du), \quad t \in \mathbb{R}^2,$$

provided $g_{12}^0 = -g$. When $g(t) = (|t_1|^2 + |t_2|^2)^{H-\frac{1}{2}}$, $t = (t_1, t_2) \in \mathbb{R}^2$, is an isotropic homogeneous function, $0 < H < 1$, and $M$ has finite variance, the RF in (1.14) is called Lévy fractional RF; see Sec. 3. Tangent (1-tangent) RFs of Lévy fractional RF with truncated $\alpha$-stable $M$ were studied in [8, 13, 14]. There it was shown that the tangent RF has a similar representation as in (1.14) with $M$ replaced by $\alpha$-stable $W_\alpha$. In our paper we extend these results to anisotropic kernels $g$ satisfying (1.9) with $\chi < 0$ and more general $M$ and show that they correspond to the well-balanced limit $T_{\gamma_0}$ in (1.2) at $\gamma = \gamma_0$ given in (1.13). We also prove the existence of the unbalanced limits $T_+ = T_{\gamma} (\gamma > \gamma_0)$ and $T_- = T_{\gamma}$
(γ < γ₀) given by \( T_+(t) := T_{γ₀}(t₁) \) and \( T_-(t) := T_{γ₀}(t₂) \), \( t = (t₁, t₂) ∈ \mathbb{R}^2_+ \), which depend on only one coordinate in contrast to rectangent limits in Table 1.

The rest of the paper is organized as follows. Sec. 2 provides rigorous assumptions about \( g \) and \( M \) and some preliminary facts needed to prove our results. Sec. 3 presents some examples of fractionally integrated RFs satisfying the assumptions in Sec. 2. The \( α \)-stable MSS RFs in Table 1 are defined in Sec. 4. The main results (Theorem 1) pertaining to Table 1 are given in Sec. 5. Sec. 6 discusses \( \gamma \)-tangent limits. Some concluding remarks are given in Sec. 7.

**Notation.** In what follows, \( C \) denote generic positive constants which may be different at different locations. We write \( d, d, \frac{d}{d} \neq (\frac{d}{d}, \frac{d}{d}, \frac{d}{d}) \) for the weak convergence, equality, and inequality of (finite-dimensional) distributions. \( 1 := (1, 1), \ 0 := (0, 0), \ \mathbb{R}^2_0 := \mathbb{R}^2 \setminus \{0\}, \ \mathbb{R}_0 := \mathbb{R} \setminus \{0\}, \mathbb{R}^2_+ := \{\mathbf{x} = (x₁, x₂) ∈ \mathbb{R}^2 : x₁ > 0, i = 1, 2\}, \ \mathbb{R}^+ := (0, ∞), \ \mathbb{R} := (0, 1), \ \mathbf{x} := (0, x₁) × (0, x₂)\), \( x = (x₁, x₂) ∈ \mathbb{R}^2_+, \ e₁ = (1, 0), \ e₂ = (0, 1), |x| = |x₁| + |x₂|, \ |x| := (x₁² + x₂²)₁², \ x · y = x₁y₁ + x₂y₂, \ ||f||_α := (∫_{\mathbb{R}^2} |f(u)|^α du)₁^α, \ α > 0. \text{I}(A) \) stands for indicator function of a set \( A \).

**2. Assumptions and preliminaries**

Given a function \( f : \mathbb{R}^2 → \mathbb{R} \) we use the following notation for partial derivatives at \( t = (t₁, t₂) ∈ \mathbb{R}^2_+ \):

\[
∂_i f(t) = ∂(f(t))/∂t_i, \quad i = 1, 2, \quad ∂_12 f(t) := ∂^2 f(t)/∂t_1∂t_2.
\]

Following [34], we say that a measurable function \( f : \mathbb{R}^2_0 → \mathbb{R} \) is **generalized homogeneous** (respectively, **generalized invariant**) if there exist some positive \( q₁, q₂ \) such that \( λf(λ^{1/q₁}t₁, λ^{1/q₂}t₂) = f(t) \) holds for all \( λ > 0, t ∈ \mathbb{R}^2_0 \) (respectively, \( f(λ^{1/q₁}t₁, λ^{1/q₂}t₂) \) does not depend on \( λ > 0 \) for any \( t ∈ \mathbb{R}^2_0 \)). Every generalized homogeneous function \( f(t) \) \( t ∈ \mathbb{R}^2_0 \), can be represented as \( f(t) = ρ(t)^{−1}ℓ(t) \) with \( ρ(t) = |t₁|^{q₁} + |t₂|^{q₂} \) and a generalized invariant function \( ℓ(t) = ℓ(t₁/ρ(t)^{1/q₁}, t₂/ρ(t)^{1/q₂}) \), where \( ℓ \) is a restriction of \( f \) to \( t ∈ \mathbb{R}^2_0 : ρ(t) = 1 \), see [34]. We also note that if \( f > 0 \) is a generalized homogeneous function and \( χ ∈ \mathbb{R}_0 \), then the function \( f^χ \) and its partial derivatives \( ∂_i f^χ, i = 1, 2 \) (provided they exist) satisfy the following scaling relations: for all \( λ > 0, t ∈ \mathbb{R}^2_0 \),

\[
f^χ(λ^{1/q₁}t₁, λ^{1/q₂}t₂) = λ^{−κ}f^χ(t), \quad ∂_i f^χ(λ^{1/q₁}t₁, λ^{1/q₂}t₂) = λ^{−κ−1/q_i}∂_i f^χ(t), \quad i = 1, 2.
\]

We shall also need some properties of the above function \( ρ \) from [33, 34, 42]. Note the elementary inequality: for any \( ν > 0 \),

\[
C₁ρ(t) ≤ ρ(|t₁|^ν, |t₂|^ν)^{1/ν} ≤ C₂ρ(t), \quad t ∈ \mathbb{R}^2_0,
\]

with \( C_i > 0, i = 1, 2 \), independent of \( t \), see [42, (2.16)]. From (2.3) and [33, Prop. 5.1] we obtain for any \( δ, ν > 0 \) with \( Q = \frac{1}{q₁} + \frac{1}{q₂} \),

\[
∫_{\mathbb{R}^2} ρ(t)^{−ν}I(ρ(t) < δ)dt < ∞ ⇐⇒ Q > ν, \quad ∫_{\mathbb{R}^2} ρ(t)^{−ν}I(ρ(t) ≥ δ)dt < ∞ ⇐⇒ Q < ν.
\]

Moreover, with \( q = \max\{q₁, q₂, 1\}, \)

\[
ρ(t + s)^{1/ν} ≤ ρ(t)^{1/ν} + ρ(s)^{1/ν}, \quad t, s ∈ \mathbb{R}^2_0,
\]
see [33, (7.1)]. We shall work on the following assumptions, where \(0 < \alpha \leq 2\) is the stability parameter of limit RFs discussed in the Introduction.

**Assumption (G)\(_\alpha\)**. The functions \(g(t)\) and \(g_0(t) = \rho(t)^\chi L(t)\) are as in (1.9), where \(L(t)\) is a generalized invariant function, the parameters \(\chi \in \mathbb{R}_0, q_i > 0, i = 1, 2\), with \(Q = \frac{1}{q_1} + \frac{1}{q_2}\) are such that

\[
-\frac{1}{\alpha} Q < \chi < (1 - \frac{1}{\alpha}) Q,
\]

moreover, \(g(t)\) and \(g_0(t)\) have partial derivatives in (2.1) for \(t \in \mathbb{R}^2\_0\). They satisfy as \(|t| \to 0,\)

\[
g(t) = g_0(t) + o(\rho(t)^\chi), \quad \partial_i g(t) = \partial_i g_0(t) + o(\rho(t)^{\chi - \frac{1}{q_i}}), \quad i = 1, 2,
\]

\[
\partial_{12} g(t) = \partial_{12} g_0(t) + o(\rho(t)^{\chi - Q}),
\]

and for all \(t \in \mathbb{R}^2\_0,\)

\[
|g_0(t)| \leq C \rho(t)^\chi, \quad |\partial_i g_0(t)| \leq C \rho(t)^{\chi - \frac{1}{q_i}}, \quad i = 1, 2, \quad |\partial_{12} g_0(t)| \leq C \rho(t)^{\chi - Q}. \tag{2.8}
\]

**Remark 1.** In view of (2.7) the bounds in (2.8) extend to \(g\) and its derivatives \(\partial_i g, \partial_{ij} g, \partial_{12} g\) in the neighborhood of the origin. We note that the function \(g_0(t) = \rho(t)^\chi = (|t_1|^{q_1} + |t_2|^{q_2})^\chi\), corresponding to \(L(t) \equiv 1\) satisfies the bounds in (2.8) for all \(t \in \mathbb{R}^2_0\) provided \(q_i \geq 1, i = 1, 2\). The last fact follows from the expressions of partial derivatives

\[
\partial_i \rho(t)^\chi = \rho(t)^{\chi - \frac{1}{q_i}} \ell_i(t), \quad i = 1, 2, \quad \partial_{12} \rho(t)^\chi = \rho(t)^{\chi - Q} \ell_{12}(t)
\]

with bounded

\[
\ell_i(t) := \chi q_i \text{sgn}(t_i) \left(\frac{|t_i|}{\rho(t)^{\frac{1}{q_i}}}\right)^{q_i-1}, \quad i = 1, 2, \quad \ell_{12}(t) := \prod_{i=1}^{2} (\chi + 1 - i) q_i \text{sgn}(t_i) \left(\frac{|t_i|}{\rho(t)^{\frac{1}{q_i}}}\right)^{q_i-1}.
\]

**Remark 2.** Let \(\rho_0(t) := |t_1|^{p_1} + |t_2|^{p_2}, t \in \mathbb{R}^2,\) for \(P = \frac{1}{p_1} + \frac{1}{p_2}\) with \(p_i, i = 1, 2,\) of (1.10). Using (2.3), the bounds in (2.8) can be respectively replaced by

\[
|g_0(t)| \leq C \rho_0(t)^{P-1}, \quad |\partial_i g_0(t)| \leq C \rho_0(t)^{P - \frac{1}{q_i}-1}, \quad i = 1, 2, \quad |\partial_{12} g_0(t)| \leq C \rho_0(t)^{-1}. \tag{2.9}
\]

Assumption (G)\(_\alpha\) pertains to the behavior of \(g\) alone. It is complemented by Assumption (G)\(^0\)\(_\alpha\) guaranteeing the existence of the associated RF \(X\) in (1.5), where infinitely divisible random measure satisfies (M)\(_\alpha\).

**Assumption (G)\(^0\)\(_\alpha\)**. For any \(t = (t_1, t_2) \in \mathbb{R}^2, \delta > 0,\) the functions \(g, g_1^0, g_2^0, g_{12}^0 : \mathbb{R}^2 \to \mathbb{R}\) satisfy

\[
\int_{\mathbb{R}^2} |g(t - u) - g(t)(t_1 e_1 - u) - \frac{1}{2} |t_2 e_2 - u||^\alpha \, du < \infty \quad (0 < \alpha \leq 2) \tag{2.10}
\]

and

\[
\int_{|u| > \delta} \left(\sum_{i=1}^{2} |\partial_i g(u)|^\alpha + |\partial_{12} g(u)|^\alpha\right) \, du < \infty \quad (1 \leq \alpha \leq 2). \tag{2.11}
\]
Moreover, if $0 < \alpha < 1$, then there exist $\delta_0 > 0$ and functions $\bar{g}_i(u), \bar{g}_{12}(u)$, $u = (u_1, u_2) \in \mathbb{R}^2_+$, monotone decreasing in each $u_j > 0$, $j = 1, 2$, and satisfying $|\partial_i g(u)| \leq \bar{g}_i(|u_1|, |u_2|)$, $|\partial_{12} g(u)| \leq \bar{g}_{12}(|u_1|, |u_2|)$, $|u| > \delta_0$, such that (2.11) holds with $\partial_i g, \partial_{12} g$ replaced by $\bar{g}_i, \bar{g}_{12}$, $i = 1, 2$.

Next we make assumptions about infinitely divisible random measure $M = \{M(A), A \in B_0(\mathbb{R}^2)\}$, where $B_0(\mathbb{R}^2)$ denotes the family of all bounded Borel sets. We recall that the infinitely divisible measure $M$ is such a random process that, for every sequence $\{A_i, i \in \mathbb{N}\}$ of pairwise disjoint sets in $B_0(\mathbb{R}^2)$, $M(A_i)$, $i \in \mathbb{N}$, are independent random variables (r.v.s), and, if $\cup_{i=1}^{\infty} A_i \in B_0(\mathbb{R}^2)$, then we also have $M(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} M(A_i)$ a.s. In addition, for every $A \in B_0(\mathbb{R}^2)$, $M(A)$ is an infinitely divisible r.v. We assume that for every $A \in B_0(\mathbb{R}^2)$, the characteristic function of $M(A)$ has the form

$$Ee^{i\theta M(A)} = \exp \left\{ \text{Leb}(\mathbb{R}) \left( -\frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta \tau_\alpha(y)) \nu(dy) \right) \right\}, \quad \theta \in \mathbb{R}, \quad (2.12)$$

where $\sigma^2 \geq 0, \nu$ is a Lévy measure on $\mathbb{R}$ satisfying Assumption (M)$_\alpha$ below and for $y \in \mathbb{R}$,

$$\tau_\alpha(y) := \begin{cases} y, & 1 < \alpha \leq 2, \\ y 1(|y| \leq 1), & \alpha = 1, \\ 0, & 0 < \alpha < 1. \end{cases} \quad (2.13)$$

Particularly, when $1 < \alpha \leq 2$ we have that $\int_{\mathbb{R}} |y| \wedge |y|^2 \nu(dy) < \infty$ and $E|M(A)| < \infty$, $EM(A) = 0$.

Assumption (M)$_\alpha$. The characteristics $\sigma, \nu$ in (2.12) satisfy the following: either

(i) $\alpha = 2, \sigma > 0$ and $\int_{\mathbb{R}} y^2 \nu(dy) < \infty$, or

(ii) $0 < \alpha < 2, \sigma = 0$ and there exist $\lim_{y \downarrow 0} y^\alpha \nu([y, \infty)) = c_+, \lim_{y \downarrow 0} y^\alpha \nu((-\infty, -y]) = c_-$ for some $c_+ \geq 0, c_+ + c_- > 0, \sup_{y > 0} y^\alpha \nu(\{u \in \mathbb{R} : |u| > y\}) < \infty$. Moreover, if $\alpha = 1$, then $\nu$ is symmetric, i.e. $\nu(dy) = \nu(-dy), y > 0$.

The above assumption is rather general. For comparison, [8, Prop. 4.1] consider the truncated stable case $\nu(dy) = I(|y| \leq 1)|y|^{-1-\alpha}dy$ only, which obviously satisfies (M)$_\alpha$ part (ii) with $c_+ = c_- = \frac{1}{\alpha}$. Assumption (M)$_\alpha$ implies that $M$ belongs to the domain of local attraction of $\alpha$-stable random measure $W_\alpha$ on $\mathbb{R}^2$ with characteristic function

$$Ee^{i\theta W_\alpha(A)} = e^{-\text{Leb}(\mathbb{R})/\theta^\alpha \omega_\alpha(\theta)}, \quad \theta \in \mathbb{R}, \quad A \in B_0(\mathbb{R}^2), \quad (2.14)$$

where

$$\omega_\alpha(\theta) := \begin{cases} \frac{1}{\pi} \sigma^2, & \alpha = 2, \\ \frac{1}{\pi} \frac{2-\alpha}{1-\alpha} ((c_+ + c_-) \cos(\frac{\pi \alpha}{2}) - i(c_+ - c_-) \text{sgn}(\theta) \sin(\frac{\pi \alpha}{2})), & 0 < \alpha < 2, \alpha \neq 1, \\ (c_+ + c_-)^2, & \alpha = 1, \ c_+ = c_. \end{cases} \quad (2.15)$$

The last fact can be formulated in terms of local scaling limits of the associated Lévy sheet $\{M(t) := \int_{(0,t]} M(du), t \in \mathbb{R}^2_+\}$. Namely, Assumption (M)$_\alpha$ implies that

$$\left(\lambda_1 \lambda_2\right)^{-\frac{1}{\alpha}} M(\lambda_1 t_1, \lambda_2 t_2) \xrightarrow{\text{f.d.d.}} W_\alpha(t), \quad \lambda_i \downarrow 0, \quad i = 1, 2, \quad (2.16)$$

where $\{W_\alpha(t) := \int_{(0,t]} W_\alpha(du), t \in \mathbb{R}^2_+\}$ is $\alpha$-stable Lévy sheet defined by (2.14). The fact in (2.16) seems to be well-known, see e.g. [3, Prop. 5.1] and also follows from Proposition 1 providing a general
Assumption (M) also recall that stochastic integral is well-defined for any \( f \) for some \( \alpha \) as \( C \) independent. Accordingly, \( \inf \) infinitely divisible random measure with characteristics \( \mu \) towards that w.r.t. criterion for weak convergence of stochastic integrals w.r.t. infinitely divisible random measure. It suffices to prove the convergence of characteristic functions:

\[
E \exp \left\{ i \theta \int_{\mathbb{R}^2} f(u) W_\alpha(du) \right\} = \exp \left\{ - |\theta|^\alpha \int_{\mathbb{R}^2} |f(u)|^\alpha \omega_\alpha(\theta f(u)) du \right\}, \quad \theta \in \mathbb{R}. \tag{2.17}
\]

Also recall that stochastic integral \( \int_{\mathbb{R}^2} f(u)M(du) \) w.r.t. infinitely divisible random measure satisfying Assumption (M) is well-defined for any \( f \in L_\alpha(\mathbb{R}^2) \) and has an infinitely divisible distribution with characteristic function

\[
E \exp \left\{ i \theta \int_{\mathbb{R}^2} f(u) M(du) \right\} = \exp \left\{ \int_{\mathbb{R}^2} \left( - \frac{1}{2} \sigma^2 \theta^2 f(u)^2 + \int_{\mathbb{R}} (e^{i \theta f(u)y} - 1 - i \theta f(u) \tau_\alpha(y)) \nu(dy) \right) du \right\}. \tag{2.18}
\]

To prove the (weak) convergence of stochastic integrals in (2.18) towards stable integral in (2.17), we use the following proposition. For any \( 0 < \alpha \leq 2, \mu_i > 0, i = 1, 2 \) and any function \( f = f_\lambda : \mathbb{R}^2 \to \mathbb{R} \)

possibly depending on \( \lambda > 0 \), define the re-scaled function \( f_\lambda^1 : \mathbb{R}^2 \to \mathbb{R} \) by

\[
f_\lambda^1(u) := \lambda^{-\alpha/2} f_\lambda(\lambda^{\mu_1} u_1, \lambda^{\mu_2} u_2), \quad u \in \mathbb{R}^2. \tag{2.19}
\]

**Proposition 1.** Assume that the infinitely divisible random measure \( M \) satisfies (M) for some \( 0 < \alpha \leq 2 \). Let \( f_\lambda \in L_\alpha(\mathbb{R}^2), \lambda > 0 \). If there exists \( h \in L_\alpha(\mathbb{R}^2) \) such that

\[
\lim_{\lambda \downarrow 0} \| f_\lambda^1 - h \|_\alpha = 0, \tag{2.20}
\]

for some \( \mu_i > 0, i = 1, 2 \), then

\[
\int_{\mathbb{R}^2} f_\lambda(u) M(du) \mathop{\rightarrow}^{\text{d}} \int_{\mathbb{R}^2} h(u) W_\alpha(du), \quad \lambda \downarrow 0. \tag{2.21}
\]

**Proof.** It suffices to prove the convergence of characteristic functions: \( E \exp \{i \theta \int_{\mathbb{R}^2} f_\lambda(u) M(du)\} =: C_\lambda(\theta) \to C_0(\theta) := E \exp \{i \theta \int_{\mathbb{R}^2} h(u) W_\alpha(du)\}, \forall \theta \in \mathbb{R} \), as given in (2.18), (2.17). W.l.g., assume that \( f_\lambda^1(u) \to h(u) \) for a.e. \( u \in \mathbb{R}^2 \). Split \( M(du) = M_1(du) + M_2(du) \), where \( M_1 \) (respectively, \( M_2 \)) is infinitely divisible random measure with characteristics \( (0, \nu) \) (respectively, \( (\sigma, 0) \)) and \( M_1, M_2 \) are independent. Accordingly, \( C_\lambda(\theta) = C_{\lambda, 1}(\theta) C_{\lambda, 2}(\theta) \), where \( C_{\lambda, i}(\theta) := E \exp \{i \theta \int_{\mathbb{R}^2} f_\lambda^1(u) M_i(du)\}, i = 1, 2 \). The subsequent proof is split into four parts depending on the value of \( \alpha \). Let \( \lambda := \lambda^{\mu_1 + \mu_2}, \quad \Psi_\alpha(y) := e^{iy} - 1 - i \tau_\alpha(y) \) for \( 0 < \alpha \leq 2, y \in \mathbb{R} \).

Case \( \alpha = 2 \). Then \( \sigma > 0 \) and \( C_{\lambda, 2}(\theta) \to C_0(\theta) \) is immediate from (2.20), hence, (2.21) follows from \( C_{\lambda, 1}(\theta) \to 1 \). We have \( C_{\lambda, 1}(\theta) = e^{I_\lambda} \), where \( I_\lambda = \int_{\mathbb{R}^2 \times \mathbb{R}} \Psi_2(\theta f_\lambda(u)) d\mu(dy) = \hat{\lambda} \int_{\mathbb{R}^2 \times \mathbb{R}} \Psi_2(\theta^{1/2} f_\lambda(u)) d\mu(dy) \). By Pratt's lemma [35], \( I_\lambda = o(1) \). Indeed, using \( |\Psi_2(y)| \leq \min\{2|y|, \frac{1}{2} |y|^2\} \) gives \( \hat{\lambda} |\Psi_2(\theta^{1/2} f_\lambda(u))| \leq C \lambda^{1/2} |f_\lambda(u)| = o(1) \) for \( y \in \mathbb{R} \), a.e. \( u \in \mathbb{R}^2 \). For \( \alpha < 2 \) such that \( f_\lambda^1(u) \to h(u) \), moreover, \( \hat{\lambda} |\Psi_2(\theta^{1/2} f_\lambda^1(u))| \leq C |f_\lambda^1(u)|^2 \to \int_{\mathbb{R}^2 \times \mathbb{R}} |h(u)|^2 d\mu(dy) \), proving (2.21) for \( \alpha = 2 \).
Case $1 < \alpha < 2$. In this case $\sigma = 0$. Let $\hat{\nu}_+(y) := \nu((y, \infty))$, $\hat{\nu}_-(y) := \nu((\infty, -y])$, $y > 0$. Then

$$C_\lambda(\theta) = \exp\{\lambda \int_{\mathbb{R}^2 \times \mathbb{R}} \Phi_\lambda(\theta f^\dagger_\lambda(u), y) d\nu(dy)\} =: C_\lambda,$$

where $I_\lambda = \int_{\mathbb{R}^2 \times \mathbb{R}} \Phi_\lambda(\theta f^\dagger_\lambda(u), y) dudy$ follows integrating by parts with

$$\Phi_\lambda(u, y) := \hat{\nu}_{\text{sgn}(y)} \left(\frac{\hat{\lambda} \theta |y|}{|u|}\right) (e^{i \text{sgn}(u)y} - 1) \text{sgn}(uy) \rightarrow |u|^{\alpha} \frac{\text{sgn}(u)}{|y|} (e^{i \text{sgn}(u)y} - 1) \text{sgn}(uy) := \Phi_0(u, y),$$

hence, $\Phi_\lambda(\theta f^\dagger_\lambda(u), y) \rightarrow \Phi_0(\theta h(u), y)$ for any $(u, y) \in \mathbb{R}^2 \times \mathbb{R}$ such that $f^\dagger_\lambda(u) \rightarrow h(u)$. Therefore we can expect that

$$I_\lambda \rightarrow \int_{\mathbb{R}^2 \times \mathbb{R}} \Phi_0(\theta h(u), y) dudy = -|\theta|^\alpha \int_{\mathbb{R}^2} |h(u)|^\alpha \omega_\alpha(\theta h(u)) du,$$

(2.22)

where the last equality follows from [21, proof of Thm. 2.2.2]. To justify convergence of the integrals in (2.22), by $|\hat{\nu}_\pm(y)| \leq Cy^{-\alpha}$, $y > 0$, we have $|\Phi_\lambda(u, y)| \leq C|u|^\alpha |y|^{-\alpha} \min\{|y|, 1\} =: \Phi(u, y)$, implying $\Phi_\lambda(\theta f^\dagger_\lambda(u), y) \leq \Phi(\theta f^\dagger_\lambda(u), y)$ where $\Phi(\theta f^\dagger_\lambda(u), y) \rightarrow \Phi(\theta h(u), y)$ for a.e. $(u, y) \in \mathbb{R}^2 \times \mathbb{R}$ and $\int_{\mathbb{R}^2 \times \mathbb{R}} \Phi(\theta f^\dagger_\lambda(u), y) dudy \rightarrow \int_{\mathbb{R}^2 \times \mathbb{R}} \Phi(\theta h(u), y) dudy < \infty$ so that (2.22) follows by Pratt’s lemma as in the case $\alpha = 2$ above. Hence, $C_\lambda(\theta) \rightarrow C_0(\theta)$ ($\forall \theta \in \mathbb{R}$).

Case $0 < \alpha < 1$. Then $C_\lambda(\theta) = \exp\{\int_{\mathbb{R}^2} J_\lambda(\theta f^\dagger_\lambda(u)) du\}$, where for a large $K > 0$ we split $J_\lambda(u) = J_{\lambda,K,0}(u) + J_{\lambda,K,1}(u)$ with

$$J_{\lambda,K,0}(u) := \hat{\lambda} \int_{|y| \leq \frac{1}{\lambda K}} (e^{\hat{\lambda} \frac{1}{2} u y} - 1) \nu(dy), \quad J_{\lambda,K,1}(u) := \hat{\lambda} \int_{|y| > \frac{1}{\lambda K}} (e^{\hat{\lambda} \frac{1}{2} u y} - 1) \nu(dy).$$

Integrating by parts we rewrite $J_{\lambda,K,0}(u) = \Phi_\lambda(u, K) - \Phi_\lambda(u, -K) - \int_{|y| \leq K} \phi_\lambda(u, y) dy$ with

$$\Phi_\lambda(u, y) := -\hat{\lambda} \nu_{\text{sgn}(y)} \left(\frac{\hat{\lambda} \frac{1}{2} u y}{|u|}\right) \text{sgn}(y)(e^{i \text{sgn}(u)y} - 1),$$

$$\phi_\lambda(u, y) := -\hat{\lambda} \nu_{\text{sgn}(y)} \left(\frac{\hat{\lambda} \frac{1}{2} u y}{|u|}\right) i \text{sgn}(uy)e^{i \text{sgn}(u)y}.$$ 

In the corresponding decomposition of $I_{\lambda,K,0} := \int_{\mathbb{R}^2} J_{\lambda,K,0}(\theta f^\dagger_\lambda(u)) du$ we use $\Phi_\lambda(\theta f^\dagger_\lambda(u), y) \rightarrow \Phi_0(\theta h(u), y)$ and $\phi_\lambda(\theta f^\dagger_\lambda(u), y) \rightarrow \phi_0(\theta h(u), y)$ for any $(u, y) \in \mathbb{R}^2 \times \mathbb{R}$ such that $f^\dagger_\lambda(u) \rightarrow h(u)$, where

$$\Phi_0(u, y) := -|u|^{\alpha} \frac{\text{sgn}(u)}{|y|}|y| \text{sgn}(uy)e^{i \text{sgn}(u)y} - 1), \quad \phi_0(u, y) := -|u|^{\alpha} \frac{\text{sgn}(u)}{|y|}|y| \text{sgn}(uy)e^{i \text{sgn}(u)y} - 1).$$

Arguing as in the proof of (2.22), we obtain $I_{\lambda,K,0} \rightarrow I_{K,0} := \int_{\mathbb{R}^2} J_{K,0}(\theta h(u)) du$ with

$$J_{K,0}(u) := \Phi_0(u, K) - \Phi_0(u, -K) - \int_{|y| \leq K} \phi_0(u, y) dy = |u|^{\alpha} \int_{|y| \leq K} \alpha \frac{\text{sgn}(uy)}{|y|^{\alpha+1}} (e^{i \text{sgn}(u)y} - 1) dy,$$

where $I_{\lambda,K,0} \rightarrow -|\theta|^{\alpha} \int_{\mathbb{R}^2} |h(u)|^\alpha \omega_\alpha(\theta h(u)) du$ as $K \rightarrow \infty$. The fact that $\int_{\mathbb{R}^2} J_{\lambda,K,1}(\theta f^\dagger_\lambda(u)) du$ is negligible follows from $|I_{\lambda,K,1}| \leq 2\hat{\lambda} \int_{\mathbb{R}^2} \nu(|y| > \frac{1}{\lambda K}) du \leq \frac{C}{\lambda K} \int_{\mathbb{R}^2} |\theta f^\dagger_\lambda(u)|^\alpha du \leq \frac{C}{\lambda K}$, by choosing $K$ large enough. This proves $C_\lambda(\theta) \rightarrow C_0(\theta)$ ($\forall \theta \in \mathbb{R}$) in case $0 < \alpha < 1$. 


Case $\alpha = 1$. By symmetry of $\nu$ and $W_1$, $C_\lambda(\theta) = e^{2I_\lambda}$, $C_0(\theta) = e^{2I_0}$, where

$$I_\lambda := \tilde{\lambda} \int_{\mathbb{R}^2 \times \mathbb{R}^+} (\cos(\theta f_\lambda(u) \tilde{\lambda}^{-1} y) - 1) d\nu(dy),$$

$$I_0 := c_+ \int_{\mathbb{R}^2 \times \mathbb{R}^+} (\cos(\theta h(u) y) - 1) y^{-2} d\nu(dy) = -c_+ |\theta| \|h\|_1 J,$$

where $J := \int_0^\infty (1 - \cos y) y^{-2} dy = \frac{\pi}{2}$. Similarly as in the case $0 < \alpha < 1$, for a large $K > 0$ split $I_\lambda = I_{\lambda,K,0} + I_{\lambda,K,1}$, where

$$I_{\lambda,K,0} := \tilde{\lambda} \int_{\mathbb{R}^2} \{0 < y \leq \frac{\lambda^K}{|\nu f_\lambda(u)|}\} \cdots \quad I_{\lambda,K,1} := \tilde{\lambda} \int_{\mathbb{R}^2} \{y > \frac{\lambda^K}{|\nu f_\lambda(u)|}\} \cdots$$

Then integrating by parts and using (M), Pratt’s lemma similarly as in the case $0 < \alpha < 1$ we obtain

$$I_{\lambda,K,0} \to c_+ \int_{\mathbb{R}^2} |\theta h(u)| du \int_0^K (\cos y - 1) y^{-2} dy =: I_{K,0},$$

where $I_{K,0} \to I_0$ ($K \to \infty$) while $I_{\lambda,K,1}$ can be made arbitrarily small uniformly in $\lambda > 0$ by choosing $K > 0$ large enough. Proposition 1 is proved. \qed

Remark 3. We have $\lambda^{-\frac{1}{\alpha}(1+\gamma)} M(\lambda t_1, \lambda^\gamma t_2) = \int_{\mathbb{R}^2} f_\lambda(u) M(du)$ with $f_\lambda(u) := \lambda^{-\frac{1}{\alpha}(1+\gamma)} I(u \in (0, \lambda t_1] \times (0, \lambda^\gamma t_2])$. Then if $\mu_1 = 1, \mu_2 = \gamma$ the corresponding re-scaled function in (2.19) $f_{\lambda i}^i(u) = I(u \in (0, t_i])$ does not depend on $\lambda$ and trivially satisfies (2.20) with $h(u) = I(u \in (0, t_i])$. Accordingly by Proposition 1 the convergence in (2.16) holds when $\lambda_i, i = 1, 2$, tend to zero as $\lambda_1 = \lambda, \lambda_2 = \lambda^\gamma$. Actually Proposition 1 extends to more general scaling by $\lambda_i \downarrow 0, i = 1, 2$, as in (2.16) but for our purposes the scaling in (2.19) suffices.

3. Examples of fractional Lévy driven RFs

In this section we discuss three examples of Lévy driven fractional RFs related to fractional powers of classical partial differential operators. Fractional operators and equations naturally appear in the study of fractional RFs with LRD or negative dependence, see [23, 26] and references therein. In each of these examples Assumptions (G) and (G) are verified and Theorems 1 and 2 apply in respective parameter regions.

3.1 Fractional Lévy RF. Let Assumption (M) hold and $g(t) = \|t\|^{H-\frac{3}{2}}, g_0^i(t) = 0, i = 1, 2, g_0^0(t) = -\|t\|^{H-\frac{3}{2}}$ for all $t \in \mathbb{R}^2$ and some $H \in (0, 1), \alpha \in (0, 2]$. The corresponding Lévy driven RF $X$ takes the form

$$X(t) := \int_{\mathbb{R}^2} \{\|t - u\|^{H-\frac{3}{2}} - \|u\|^{H-\frac{3}{2}}\} M(du), \quad t \in \mathbb{R}^2. \quad (3.1)$$

If $EM(du)^2 = \sigma^2_2 d\nu$ and $\alpha = 2$ in the integrand, then the covariance function of $X$ in (3.1) is given by

$$EX(t)X(s) = E[X(e_1)^2] \frac{1}{2} \{\|t\|^{2H} + \|s\|^{2H} - \|t - s\|^{2H}\}, \quad t, s \in \mathbb{R}^2, \quad (3.2)$$
where \( E[X(e_1)^2] = \sigma_M^2 \int_{\mathbb{R}^2} (\|e_1 + u\|^{H-1} - \|u\|^{H-1})^2 \, du < \infty \). If \( M = W_\alpha \) is \( \alpha \)-stable random measure, then RF \( X \) in (3.1) is called a moving average fractional stable RF (a fractional Brownian RF in the case \( \alpha = 2 \)), see [8, 13, 44]. Fractional Lévy RF can be defined on arbitrary \( \mathbb{R}^d \), \( d \geq 1 \). See the review paper [27] and the numerous references therein on various mathematical and probabilistic aspects of the fractional Brownian RF, including extension to arbitrary \( H \in \mathbb{R} \) (as a generalized RF or random tempered distribution) and relation to fractional powers of the Laplace operator. Remark 1 shows that the above \( g \) verifies Assumption (G) with \( g_0 = g \), \( q_1 = q_2 = 2 \), \( Q = 1 \), \( \chi = \frac{H}{2} - \frac{1}{\alpha} \in (-\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha}) \). Moreover, parameters in (1.10) of the limiting kernel satisfy: \( p_1 = p_2 = 2 - H + \frac{2}{\alpha} \) with \( \frac{\alpha}{1+\alpha} < P < \frac{2\alpha}{2+\alpha} \). \( P \frac{1+\alpha}{\alpha} = P \frac{1}{\alpha} < 1 \).

### 3.2 Isotropic fractional Laplace or Matérn RF

Let

\[
 g(t) := \frac{2^{2+\chi}}{c^{2+\chi}(-\chi)} \{(c\|t\|)^\chi K_\chi(c\|t\|) - I(\chi > 0)\Gamma(\chi)2^{\chi-1}\}, \quad t \in \mathbb{R}_0^2, \tag{3.3}
\]

where \( c > 0 \), \( \chi \in (-\frac{1}{\alpha}, 1-\frac{1}{\alpha}) \backslash \{0\} \) and \( K_\chi \) denotes the modified Bessel function of the second kind. Using \( K_\chi(t) = K_{-\chi}(t) \sim \Gamma(\chi)2^{-\chi}t^{-\chi} \) as \( t \downarrow 0 \) \((\chi > 0)\), see [1, 9.6.9, p.375], we see that \( g(t) \sim \|t\|^{2\chi} \) \((\chi < 0)\), \( g(t) \to 0 \) \((\chi > 0)\) as \( \|t\| \to 0 \); moreover, for \( 2\chi = H - \frac{2}{\alpha} < 0 \) we have \( \lim_{c \downarrow 0} g(t) = \|t\|^{H - \frac{2}{\alpha}} \) hence \( g(t) \) of Example 3.1 can be regarded as the limiting case of (3.3) when \( c \downarrow 0 \). Let Assumption (M) hold for \( \alpha \in (0, 2) \) and

\[
 X(t) := \int_{\mathbb{R}^2} \{g(t - u) - I(\chi > 0)g(-u)\} M(du), \quad t \in \mathbb{R}^2. \tag{3.4}
\]

Clearly (3.4) is a particular case of (1.5) corresponding to \( g_0^0(t) = g_1^0(t) = g_2^0(t) = 0 \) \((\chi < 0)\) and \( g_0^1(t) = g_1^0(t) = 0, g_1^1(t) = -g_0(t) \) \((\chi > 0)\).

**Proposition 2.** For any \( \alpha \in (0, 2) \) the kernel \( g(t) \) in (3.3) satisfies Assumption (G) with

\[
 g_0(t) := \|t\|^{2\chi}, \quad t \in \mathbb{R}_0^2, \tag{3.5}
\]

\( q_1 = q_2 = 2, Q = 1 \), and \( \chi \) as in (3.3), or \( p_1 = p_2 = 2(1 - \chi) \), \( P \in (\frac{\alpha}{1+\alpha}, \alpha) \), \( P \neq 1 \). Moreover, the integrand in (3.4) satisfies Assumption (G)_0. As a consequence, the RF \( X \) in (3.4) is well-defined for any \( M \) satisfying Assumption (M)\(_\alpha\).

**Proof.** Using the relation \( K_\chi(t) = K_{-\chi}(t) = \frac{1}{\Gamma(\chi)}\Gamma(1-\chi)(I_{-\chi}(t) - I_{\chi}(t)) \) (c.f. [1, 9.6.2, 9.6.6, 9.6.10, p.375]), the modified Bessel function of the first kind can be expressed as \( I_{\nu}(t) = \frac{1}{\Gamma(1+\nu)}t^\nu + \frac{1}{\nu\Gamma(2+\nu)}t^{2+\nu} + o(t^{2+\nu}) \) \((t \downarrow 0)\), (c.f. [1, 9.6.10, p.375]), the asymptotics \( g(t) = g_0(t)(1 + o(1)) \) with \( g_0(t) \) as in (3.5) follows since \( |\chi| < 1 \) in case \( 0 < \chi < 1 - \frac{1}{\alpha} \), and as shown before in case \( \chi < 0 \). Note the derivatives

\[
 \partial_i g_0(t) = 2\chi\|t\|^{2\chi-1}\partial_i\|t\|, \quad i = 1, 2, \quad \partial_1^2 g_0(t) = 4\chi(\chi - 1)\|t\|^{2\chi-2}\sum_{i=1}^2 \partial_i\|t\|. \]

It suffices to verify (2.7) for derivatives \( \partial_i \tilde{g}(t), \partial_i \tilde{g}(t), i = 1, 2 \), where \( \tilde{g}(t) := \frac{2^{1+\chi}}{\Gamma(-\chi)}\|t\|^\chi K_\chi(\|t\|) \).

Using the recurrence relation \( K'_\nu(t) = -K_{\nu-1}(t) - \nu t^{-1}K_\nu(t) \) (c.f. [1, 9.6.26, p.376]) and \( \partial_i\|t\| =
\[ \|t\|^{-1}t_i, \text{ we find} \]

\[ \partial_1 g(t) = \frac{2^{1+\chi}}{\Gamma(-\chi)}(\chi \|t\|^{\chi-1}K_\chi(\|t\|) + \|t\|^{\chi}K'_\chi(t))\partial_1 \|t\| = -\frac{2^{1+\chi}}{\Gamma(-\chi)}\|t\|^{\chi}K_{\chi-1}(\|t\|)\partial_1 \|t\|, \quad i = 1, 2, \]

and

\[ \partial_2 g(t) = -\frac{2^{1+\chi}}{\Gamma(-\chi)}((\chi - 1)\|t\|^{\chi-2}K_{\chi-1}(\|t\|) + \|t\|^{\chi-1}K'_{\chi-1}(\|t\|)t_1 \partial_2 \|t\|) \]

\[ = \frac{2^{1+\chi}}{\Gamma(-\chi)}\|t\|^{\chi}K_{\chi-2}(\|t\|) \prod_{i=1}^2 \partial_i \|t\|. \]

Thus (2.7) follows using \( K_{-\nu}(t) = K_\nu(t) \sim \Gamma(\nu)2^{\nu-1}t^{-\nu}, t \downarrow 0 (\nu > 0). \) The remaining facts of Assumption (G)\(_\alpha\) follow from the definition in (3.3). Finally, Assumption (G)\(_\alpha\) is guaranteed since \( K_\nu(t) \) decays exponentially as \( t \to \infty \) for any \( \nu > 0 \). \( \Box \)

**Remark 4.** The stationary Matérn RF on \( \mathbb{R}^2 \) is defined by

\[ Y(t) := \int_{\mathbb{R}^2} h(t - u)M(du), \quad t \in \mathbb{R}^2, \quad (3.6) \]

where

\[ h(t) := \frac{2^{1+\chi}}{\Gamma(1+\chi)}\|t\|^{\chi}K_\chi(c\|t\|), \quad \chi > -\frac{1}{\alpha}, \]

agrees with \( g(t) \) in (3.3) for \( \chi < 0 \); for \( \chi > 0 \) we have \( g(t) = h(t) - \lim_{\|s\| \to 0} h(s) \). Therefore, for \( \chi > 0 \) the RF \( X(t) \) in (3.4) is the increment RF \( X(t) = Y(t) - Y(0), \quad t \in \mathbb{R}^2, \) of the Matérn RF \( Y \) in (3.6). Clearly, RF \( X \) and \( Y \) have identical ordinary and rectangular increments and tangent limits. Finite-variance Matérn RFs and their covariance functions are widely used in spatial applications, see [19, 22, 20, 12, 45] and the references therein. If \( E(M(du))^2 = \sigma_M^2 du \) and \( \chi > -\frac{1}{2}, \) then \( E|Y(0)|^2 < \infty \) and the covariance function of RF \( Y \) is given by

\[ R(t) := EY(0)Y(t) = E|Y(0)|^2 \frac{|c|^{1+\chi}K_{1+\chi}(c\|t\|)}{\Gamma(1+\chi)2^{\chi+2}}, \quad t \in \mathbb{R}^2. \]

Whence, \( E(X(t)X(s)) = R(t - s) - R(s) - R(t) + R(0) \) and \( E(X(t) - X(s))^2 = 2(R(0) - R(t - s)) + R(t) \), \( t, s \in \mathbb{R}^2, \) for \( \chi \in (0, \frac{1}{2}) \). We note that the finite-variance RF \( Y \) in (3.6) can be regarded as a stationary solution of the fractional Helmholtz equation

\[ (c^2 - \Delta)^{1+\chi}Y(t) = \tilde{M}(t), \quad t \in \mathbb{R}^2, \]

where \( \Delta := \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} \) is the Laplace operator and \( \tilde{M}(t) = \partial_1 M(t) \) is the Lévy white noise (the generalized random process), see [46, 12]. Non-Gaussian Matérn RF with \( M \) belonging to some parametric class are discussed in [12, 45].

**3.3 Anisotropic fractional heat operator RF.** Let

\[ g(t) = \frac{t^\chi}{2^\chi(2\pi)^{\frac{\chi}{2}}c_1 \Gamma(\chi+\frac{\chi}{2})} \exp\left(-c_1 t_1 - \frac{t_2^2}{4c_2 t_1}\right) I(t_1 > 0), \quad t \in \mathbb{R}^2, \quad (3.7) \]
where \( \chi > -\frac{3}{2\alpha}, c_1 > 0, c_2 > 0 \) are parameters. In the case \( \alpha = 2 \), the kernel in (3.7) is related to the fractional heat operator \((c_1 + \Delta_{12})^{\chi + \frac{3}{2}}, \Delta_{12} := \frac{\partial}{\partial t} - c_2^2 \frac{\partial^2}{\partial x^2}\), as explained below. (For \( \chi = -\frac{3}{2} \) it solves the equation \((c_1 + \Delta_{12})g(t) = 0, t_1 > 0, \) as expected.) The stationary solution of the corresponding stochastic equation

\[
(c_1 + \Delta_{12})^{\chi + \frac{3}{2}} X(t) = \dot{M}(t), \quad t \in \mathbb{R}^2, 
\]

with Gaussian white noise \( \dot{M} \) is defined in [23, (3.2)] as a moving average RF

\[
X(t) = \int_{\mathbb{R}^2} g(t - u) M(du) \tag{3.9}
\]

with the spectral density (given by the squared Fourier transform of \( g \)) of the form

\[
f(x) = \frac{\sigma^2}{(2\pi)^2} |\tilde{g}(x)|^2 = \frac{\sigma^2}{(2\pi)^2} \frac{1}{\left( x^2 + c_1^2 + c_2^2 \right)^{\chi + \frac{3}{2}}}, \quad x \in \mathbb{R}^2. \tag{3.10}
\]

We claim that the corresponding \( g \in L_2(\mathbb{R}^2) \) is given by (3.7) for \( \chi > -\frac{3}{4} \). Indeed, its Fourier transform can be found from [18, 3.944.5–6]:

\[
\int_{\mathbb{R}^2} g(t)e^{it\cdot x} dt = \frac{1}{2\pi^\chi} \int_0^\infty t_1^\chi e^{-c_1 t_1} \frac{1}{\left( 4\pi c_1^2 t_1 \right)^{\frac{\chi}{2}}} e^{-\frac{x_1^2}{4c_1^2 t_1} + iz_2^2} d\sigma_2 
\]

\[
= \frac{1}{2\pi^\chi} \int_0^\infty \frac{1}{\left( x_1^2 + c_2^2 \right)^{\chi + \frac{3}{2}}} \exp\{i(\chi + \frac{3}{2}) \arctan(\frac{x_1}{c_2})\} \tag{3.11}
\]

and hence \( g \) in (3.7) satisfies (3.10). We note that the representation of \( g \) in [23, (3.7)] is not explicit; the expression in (3.7) was suggested by the derivation of the asymptotics of the fundamental solution of the fractional heat equation on \( \mathbb{Z}^2 \) in [33, proof of Prop. 4.1]. For \( t \in \mathbb{R}^2 \), let \( g_0(t) := \Lambda(t)^\chi \ell(t) \) with

\[
\rho(t) := |t_1| + |t_2|, \quad q_1 := 1, q_2 := 2, Q = \frac{3}{2}, \tag{3.12}
\]

\[
\ell(t) := \frac{2^\chi}{2^\chi(2\pi)^{\frac{3}{2}}} \frac{e^{-c_2^2 \Gamma(\chi + \frac{3}{2})}}{\left( \chi + \frac{3}{2} \right)^{\chi}} \exp\left\{ -\frac{1}{4c_2^2} \left( \frac{1}{2} - 1 \right) \right\} I(t_1 > 0), \quad \text{where } z := \frac{t_1}{\rho(t)} \in (0, 1).
\]

Note \( \ell \) in (3.12) is a bounded general invariant function for any \( \chi > -\frac{3}{2} \); particularly, \( \ell(t) \to 0 \) as \( z \downarrow 0 \). We have

\[
g(t) = \rho(t)^\chi \ell(t)(1 + o(1)) = g_0(t)(1 + o(1)), \quad |t| \to 0. \tag{3.13}
\]

The form of \( g_0 \) and the asymptotics in (3.13) are similar to [33, (4.8)] and [43, (4.11)] in the lattice case.

**Proposition 3.** The kernel \( g \) in (3.7) satisfies Assumptions \((G)_{\alpha_i}, (G)_{\alpha_i}' \) with \( g_0, \ell, i = 1, 2 \), as in (3.12) for any

\[
-\frac{3}{2\alpha} < \chi < \frac{3}{2}(1 - \frac{1}{\alpha}), \quad \alpha \neq 0, \tag{3.14}
\]

(equivalently, \( P = \frac{3}{2\alpha - 2} \in (\frac{\alpha}{1 + \alpha}, \alpha), P \neq 1 \) and \( \alpha \in (0, 2] \). As a consequence, the RF \( X \) is (3.9) is well-defined for any \( \dot{M} \) satisfying Assumption \((M)_{\alpha_i} \).
Proof. It suffices to prove the proposition for \( \tilde{g}(t) := t_1^\chi \exp\{-t_1 - \frac{t_2}{t_1^2}\} I(t_1 > 0) \) and \( \tilde{g}_0(t) := t_1^\chi \exp\{-\frac{t_2}{t_1}\} I(t_1 > 0), t \in \mathbb{R}_0^2 \). Let us verify (2.8) for \( t = (t_1, t_2) \in \mathbb{R}_0^2 \) such that \( t_1 > 0 \). It is convenient to change the variables as \( u := t_1 > 0, z := \frac{t_2}{t_1} > 0 \). For any \( \chi \in \mathbb{R} \), there exists \( C > 0 \) such that \( e^{-z} \leq C(1 + z)\chi \) for all \( z > 0 \), hence \( \tilde{g}_0(t) = u^\chi e^{-z} \leq Cu^\chi (1 + z)^\chi = C\rho(t)^\chi \). In a similar way,

\[
|\partial_1 \tilde{g}_0(t)| = u^{\chi-1}|\chi + z|e^{-z} \leq C(u(1 + z))^{\chi-1} = C\rho(t)^{\chi-\frac{1}{\gamma}},
\]

\[
|\partial_2 \tilde{g}_0(t)| = 2u^{\chi-1}z^{\frac{1}{2}}|e^{-z} \leq C(u(1 + z))^{\chi-\frac{1}{2}} = C\rho(t)^{\chi-\frac{1}{2}},
\]

\[
|\partial_1 \tilde{g}_0(t)| = 2u^{\chi-\frac{1}{2}}(1 - \chi - z)\frac{1}{2}e^{-z} \leq C(u(1 + z))^{\chi-\frac{3}{2}} = C\rho(t)^{\chi-Q},
\]

proving (2.8). Relations (2.7) follow from (2.8) and \( \tilde{g}(t) = \tilde{g}_0(t)e^{-t_1} \) since \( e^{-t_1} = 1 + O(t_1) \), \( t_1 \downarrow 0 \), together with its derivatives. Finally, let us verify Assumption (G)\( _0 \). After the above-given change of variables, \( \int_{\mathbb{R}^2} |\tilde{g}(t)|^a \, dt = \int_0^\infty u^\alpha \frac{1}{2} e^{-\alpha u} \, du \int_0^\infty z \frac{1}{2} e^{-\alpha z} \, dz < \infty \) since \( \chi > -\frac{3}{2\gamma} \).

Moreover, \( \int_{\mathbb{R}^2} |\partial_1 \tilde{g}_0(t)|^a e^{-t_1} I(t_1 > 0) \, dt = \int_0^\infty J(u) u^\alpha \chi + \frac{1}{2} \frac{1}{2} e^{-\alpha u} \, du < \infty \) since \( J(u) := \int_0^\infty \chi |\chi + z|\frac{1}{2} e^{-\alpha z} \, dz \leq C(u^{\frac{1}{2}} - \alpha \frac{1}{2} z e^{-\alpha z}) I(u \leq 1) + I(u > 1)) \). In a similar way, we can show \( \int_{\mathbb{R}^2} (|\partial_2 \tilde{g}(t)|^a + |\partial_1 \tilde{g}(t)|^a) \, dt < \infty \).

\[\square\]

4. A class of \( \alpha \)-stable MSS RFs with one of the self-similarity parameters equal 0 or 1

In this section we define the \( \alpha \)-stable RFs \( \mathcal{Y}_{\alpha,i}, \tilde{\mathcal{Y}}_{\alpha,i}, i = 1, 2 \), of Table 1 as integrals w.r.t. \( \alpha \)-stable random measure on \( \mathbb{R}^2 \). In the Gaussian case \( \alpha = 2 \) these RFs up to a scale factor coincide with standard FBS defined via the covariance function in (1.8) for \( H \in (0, 1]^2 \). The following definition extends the last covariance to \( H \in [0, 1]^2 \).

**Definition 1** ([43]). Standard FBS \( B_H = \{B_H(t), t \in \mathbb{R}^2_+\} \) with \( H = (H_1, H_2) \in (0, 1]^2, H_1 \land H_2 = 0 \), is defined as a Gaussian process with zero-mean and covariance function \( EB_H(t)B_H(s) = \prod_{i=1}^2 R_{H_i}(t_i, s_i), t, s \in \mathbb{R}^2_+ \), where for \( t, s \in \mathbb{R}_+ \),

\[
R_{H_i}(t, s) = \frac{1}{2}(2^{2H} + s^{2H} - |t - s|^{2H}), \quad 0 < H \leq 1,
\]

\[
R_0(t, s) = \lim_{H \downarrow 0} R_{H}(t, s) = 1 - \frac{1}{2} I(t \neq s).
\]

**Remark 5.** The covariance in (1.8) implies that the restriction of FBS \( B_H \) to horizontal/vertical line agrees with fractional Brownian motion (FBM) \( B_H = \{B_H(t), t \in \mathbb{R}_+\} \) with the corresponding Hurst parameter \( H = H_i \in (0, 1], i = 1, 2 \). Following Definition 1 we may define FBM \( B_H \) with \( H = 0 \) as a Gaussian process on \( \mathbb{R}_+ \), with zero-mean and the covariance function \( EB_0(t)B_0(s) = 1 - \frac{1}{2} I(t \neq s), t, s \in \mathbb{R}_+ \). The last process is \( H \)-SS SI with \( H = 0 \) and satisfies the strange property that \( E(B_0(t) - B_0(s))^2 = 1 = \lim_{H \downarrow 0} \frac{1}{2}(W(t) - W(0)) \), where \( W(t), t \in [0, \infty) \), is (uncountable) family of independent \( N(0, 1) \) r.v.s. See [40, Examples 1.3.1, 8.2.3]. We note that the above \( B_0 \) is different from the ‘regularized’ FBM with \( H = 0 \) defined in [16, p.2985], which is not \( 0 \)-self-similar and has a.s. continuous paths. Non-constant
The document discusses the local scaling limits of Lévy-driven fractional RFs. It describes the process $Y$ in detail and mentions the $Y$-stationary RFs $\tilde{Y}$ of a singular RF $\tilde{Y}$. It also presents a proposition about the bounds for the RFs, focusing on the conditions for $\alpha$ and $P$. The proposition is stated and proved, with a focus on the bounds for the RFs under specific conditions.
Let

\[ H_{\alpha,1} := \frac{1+\alpha}{\alpha} (1 + \frac{p_1}{p_2}) - p_1, \quad \tilde{H}_{\alpha,1} := \frac{1+\alpha}{\alpha} + \frac{p_1}{\alpha p_2} - p_1, \tag{4.3} \]

\[ H_{\alpha,2} := \frac{1+\alpha}{\alpha} (1 + \frac{p_2}{p_1}) - p_2, \quad \tilde{H}_{\alpha,2} := \frac{1+\alpha}{\alpha} + \frac{p_2}{\alpha p_1} - p_2. \]

Note the equivalencies:

\[ P > \frac{\alpha}{1+\alpha} \land P_1 \frac{1+\alpha}{\alpha} < 1 \iff 0 < H_{\alpha,1} < 1, \]

\[ P < \alpha \land P_1 \frac{1+\alpha}{\alpha} > 1 \iff 0 < \tilde{H}_{\alpha,2} < 1, \]

\[ P_1 \frac{1+\alpha}{\alpha} = 1 \iff H_{\alpha,1} = 1 \land \tilde{H}_{\alpha,2} = 0. \]

Similar equivalencies hold for \( \tilde{H}_{\alpha,1} \) and \( H_{\alpha,2} \) by symmetry. Also note that \( \tilde{H}_{\alpha,1} = \tilde{H}_{\alpha,2} = 1 \) when \( P = \alpha \) and \( H_{\alpha,1} = H_{\alpha,2} = 0 \) when \( P = \frac{\alpha}{1+\alpha} \).

**Corollary 1.** Let \( g_0, p_i, i = 1, 2 \), satisfy the conditions of Proposition 4. Then the random processes in (4.1) are well-defined, H-SS SI and have \( \alpha \)-stable finite dimensional distributions in the parameter regions indicated in Table 2 below:

| Parameter region | \( Y_{\alpha,1} \) | \( \tilde{Y}_{\alpha,2} \) | \( Y_{\alpha,2} \) | \( \tilde{Y}_{\alpha,1} \) |
|------------------|--------------------|--------------------|--------------------|--------------------|
| \( H_{\alpha,1} \) | \( P_1 \frac{1+\alpha}{\alpha} < 1 \) | \( P_1 \frac{1+\alpha}{\alpha} > 1 \) | \( P_1 \frac{1+\alpha}{\alpha} < 1 \) | \( P_1 \frac{1+\alpha}{\alpha} > 1 \) |
| \( H_{\alpha,2} \) | \( H_{\alpha,2} \) | \( H_{\alpha,2} \) | \( H_{\alpha,2} \) | \( H_{\alpha,2} \) |

Table 2. Parameter regions and self-similarity indices of \( \alpha \)-stable random processes \( Y_{\alpha,i}, \tilde{Y}_{\alpha,i}, i = 1, 2, \) in (4.1).

**Particularly, in the Gaussian case** \( \alpha = 2 \) the processes \( Y_{2,i}, \tilde{Y}_{2,i}, i = 1, 2, \) agree with corresponding FBM, viz.,

\[ Y_{2,i} \overset{\text{fdd}}{=} \sigma_i B_{H_{2,i}}, \quad \tilde{Y}_{2,i} \overset{\text{fdd}}{=} \tilde{\sigma}_i B_{\tilde{H}_{2,i}}, \quad i = 1, 2, \tag{4.4} \]

where \( \sigma_i^2 := \|h_i(1;\cdot)\|_2^2, \tilde{\sigma}_i^2 := \|\tilde{h}_i(1;\cdot)\|_2^2, i = 1, 2. \)

**Proof.** The fact that the processes in (4.1) are well-defined and have \( \alpha \)-stable distribution follow from Proposition 4 and general properties of stochastic integrals w.r.t. \( \alpha \)-stable random measure [41]. The stationarity of increments property is a consequence of the form of the integrands in (4.2) and the invariance of \( W_\alpha \) w.r.t. to shifts in \( \mathbb{R}^2 \). Similarly, the H-SS property follows from change of variables in the stochastic integral; particularly, \( \{Y_{\alpha,1}(\lambda t), t \in \mathbb{R}_+\} \overset{\text{fdd}}{=} \{\lambda H_{\alpha,1}Y_{\alpha,1}(t), t \in \mathbb{R}_+\} \) from

\[
Y_{\alpha,1}(\lambda t) = \int_{\mathbb{R}^2} (g_0(\lambda t - u_1, -u_2) - g_0(-u_1, -u_2)) W_\alpha(du)
= \int_{\mathbb{R}^2} (g_0(\lambda t - \lambda u_1, -\lambda u_2) - g_0(-\lambda u_1, -\lambda u_2)) W_\alpha(d\lambda u_1, d\lambda u_2),
\]

where \( \lambda > 0 \).
by using \( g_0(\lambda t_1, \lambda^{\frac{1}{2}} t_2) = \lambda^{\chi_1} g_0(t), \chi_2 = p_1(P - 1), \frac{\partial}{\partial t_2} = \frac{\partial}{\partial t}, \) see (2.2), (1.10), and \( W_\alpha(d\lambda u_1, d\lambda^{\frac{1}{2}} u_2) \text{ fdd } \lambda^{-\left(1 + \frac{2}{p_1}\right)}W_\alpha(du). \) The self-similarity \( \{\tilde{Y}_{\alpha,1}(t), t \in \mathbb{R}_+\} = \{\lambda^{H_{\alpha,1}}\tilde{Y}_{\alpha,1}(t), t \in \mathbb{R}_+\} \) follows analogously from

\[
\tilde{Y}_{\alpha,1}(t) = \int_{\mathbb{R}^2} (\partial_2 g_0(\lambda t - \lambda u_1, -\lambda^{\frac{1}{2}} u_2) - \partial_2 g_0(-\lambda u_1, -\lambda^{\frac{1}{2}} u_2))W_\alpha(d\lambda u_1, d\lambda^{\frac{1}{2}} u_2).
\]

Finally, (4.4) is a consequence of the well-known characterization of Gaussian \( H\text{-SS} \) SI processes [41].

Next we define two classes of RFs on \( \mathbb{R}_+^2 \) following the definitions in (4.1). The first class is defined as

\[
\tilde{Y}_{\alpha,1}(t) := t_2 \tilde{Y}_{\alpha,1}(t_1), \quad \tilde{Y}_{\alpha,2}(t) := t_1 \tilde{Y}_{\alpha,2}(t_2), \quad t \in \mathbb{R}_+^2.
\] (4.5)

Thus, (4.5) are nothing but simple line extensions of the processes \( \tilde{Y}_{\alpha,i}, i = 1, 2; \) for any fixed \( t_1, \tilde{Y}_{\alpha,1}(t) \) is a half-line in \( t_2 \) with random slope \( \tilde{Y}_{\alpha,1}(t_1), \) the definition of \( \tilde{Y}_{\alpha,2}(t) \) being analogous.

The definition of the second class of RFs (corresponding to \( Y_{\alpha,i}, i = 1, 2 \)) is more involved. For \( m \in \mathbb{N} \), let \( Y_{\alpha,1,j}, j = 0, 1, \ldots, m, \) be independent copies of \( Y_{\alpha,1} \) in (4.1). Pick \( (t_{j,1}, t_{j}) \in \mathbb{R}_+^2, j = 1, \ldots, m, \) with different ordinates \( t_1 < \cdots < t_m \). Define finite dimensional distribution of a RF \( \tilde{Y}_{\alpha,1} \):

\[
\{Y_{\alpha,1}(t_{j,1}, t_{j}), j = 1, \ldots, m\} \overset{d}{=} \{Y_{\alpha,1}^{(j)}(t_{j,1}) - Y_{\alpha,1}^{(0)}(t_{j,1}), j = 1, \ldots, m\}.
\] (4.6)

This definition extends to arbitrary finite collection of points in \( \mathbb{R}_+^2 \). Namely, to extend (4.6) let \( (t_{j,k}, t_{j}) \in \mathbb{R}_+^2, k = 1, \ldots, n_j, j = 1, \ldots, m, \) be given with \( t_1 < \cdots < t_m \). Then

\[
\{Y_{\alpha,1}(t_{j,k}, t_{j}), k = 1, \ldots, n_j, j = 1, \ldots, m\} \overset{d}{=} \{Y_{\alpha,1}^{(j)}(t_{j,k}) - Y_{\alpha,1}^{(0)}(t_{j,k}), k = 1, \ldots, n_j, j = 1, \ldots, m\}.
\]

Similarly, we define a RF \( \{Y_{\alpha,2}(t), t \in \mathbb{R}_+^2\} \) such that for each finite collection of points \( (t_j, t_{j,k}) \in \mathbb{R}_+^2, k = 1, \ldots, n_j, j = 1, \ldots, m, \) with abscissas \( t_1 < \cdots < t_m \):

\[
\{Y_{\alpha,2}(t_j, t_{j,k}), k = 1, \ldots, n_j, j = 1, \ldots, m\} \overset{d}{=} \{Y_{\alpha,2}^{(j)}(t_j, k) - Y_{\alpha,2}^{(0)}(t_j, k), k = 1, \ldots, n_j, j = 1, \ldots, m\},
\] (4.7)

where \( Y_{\alpha,2}^{(j)}, j = 0, 1, \ldots, m, \) are independent copies of \( Y_{\alpha,2} \) in (4.1). Finally, for \( t \in \mathbb{R}_+^2 \), set

\[
T_{\alpha,0}(t) := \int_{\mathbb{R}_+^2} h_0(t; u)W_\alpha(du).
\] (4.8)

The following corollary summarises the properties of the introduced RFs and details their MSS indices mentioned in the beginning of this section.

**Corollary 2.** Let \( g_0, \alpha, p_i, i = 1, 2, \) satisfy the conditions in Proposition 4. Then:

(i) RFs \( \tilde{Y}_{\alpha,0} \) in (4.8) is well-defined;
(ii) RFs \( Y_{\alpha,i}, \tilde{Y}_{\alpha,i} \) are well-defined in the parameter regions shown in Table 2 for \( Y_{\alpha,i}, \tilde{Y}_{\alpha,i} \) respectively, \( i = 1, 2; \)
(iii) RFs \( Y_{\alpha,0}, \tilde{Y}_{\alpha,0}, Y_{\alpha,i}, \tilde{Y}_{\alpha,i}, i = 1, 2, \) have \( \alpha \)-stable finite dimensional distributions and stationary rectangular increments;
(iv) \( \tilde{Y}_{\alpha,i} \) is MSS RF with index \( H = (\tilde{H}_{\alpha,1}, 1) \) if \( i = 1 \), and \( H = (1, \tilde{H}_{\alpha,2}) \) if \( i = 2 \);
(v) \( \bar{Y}_{\alpha,i} \) is MSS RF with index \( H = (H_{\alpha,1}, 0) \) if \( i = 1 \), and \( H = (0, H_{\alpha,2}) \) if \( i = 2 \);
(vi) In the Gaussian case \( \alpha = 2 \) RFs \( \tilde{Y}_{2,i}, \tilde{Y}_{2,i}, i = 1, 2 \), agree with FBS, viz.,

\[
\begin{align*}
\tilde{Y}_{2,1} & \equiv \sigma_1 B_{(H_{2,1}, 0)}, \\
\tilde{Y}_{2,2} & \equiv \sigma_2 B_{(0, H_{2,2})}, \\
\tilde{Y}_{2,1} & \equiv \sigma_1 B_{(H_{2,1}, 1)}, \\
\tilde{Y}_{2,2} & \equiv \sigma_2 B_{(1, H_{2,2})},
\end{align*}
\]

where \( \sigma_1^2, \sigma_2^2, i = 1, 2 \), are given in (4.4).

**Proof.** Most facts in Corollary 2 follow from Proposition 4 and the definitions of the introduced RFs. Let us check the MSS property of \( \bar{Y}_{\alpha,i} \), viz.,

\[
\{ \bar{Y}_{\alpha,i}(\lambda_1 t_1, \ldots, \lambda_2 t_2), \ t \in \mathbb{R}_+^2 \} \overset{d}{=} \{ \lambda_i^{H_{\alpha,i}} \bar{Y}_{\alpha,i}(t), \ t \in \mathbb{R}_+^2 \}, \quad \forall \lambda_1 \in \mathbb{R}_+, \lambda_2 \in \mathbb{R}_+,
\]

for \( i = 1, 2 \). Let \( \{ (t_{j,k}, t_{j}) \in \mathbb{R}_+^2, k = 1, \ldots, n_j, j = 1, \ldots, m \} \) be an arbitrary collection of points with \( t_1 < t_2 < \cdots < t_m \). Then the rescaled collection \( \{ (\lambda_1 t_{j,k}, \lambda_2 t_{j}) \in \mathbb{R}_+^2, k = 1, \ldots, n_j, j = 1, \ldots, m \} \) satisfies the same property: \( \lambda_2 t_1 < \lambda_2 t_2 < \cdots < \lambda_2 t_m \). Therefore, with \( Y_{i,j} \), \( j = 0, 1, \ldots, n \), independent copies of \( H_{\alpha,1} \)-SS process \( Y_{\alpha,1} \) in (4.1),

\[
\begin{align*}
\{ \bar{Y}_{\alpha,1}(\lambda_1 t_{j,k}, \lambda_2 t_{j}), k = 1, \ldots, n_j, j = 1, \ldots, m \} \\
\overset{d}{=} \{ Y_{\alpha,1}^{(j)}(\lambda_1 t_{j,k}) - Y_{\alpha,1}^{(0)}(\lambda_1 t_{j,k}), k = 1, \ldots, n_j, j = 1, \ldots, m \} \\
\overset{d}{=} \{ \lambda_1^{H_{\alpha,1}} Y_{\alpha,1}^{(j)}(t_{j,k}) - Y_{\alpha,1}^{(0)}(t_{j,k}), k = 1, \ldots, n_j, j = 1, \ldots, m \} \\
\overset{d}{=} \{ \lambda_1^{H_{\alpha,1}} Y_{\alpha,1}(t_{j,k}, t_{j}), k = 1, \ldots, n_j, j = 1, \ldots, m \},
\end{align*}
\]

proving (4.10) for \( i = 1 \). \( \square \)

### 5. Rectangent limits of Lévy driven fractional RF

The following Theorem 1 is the main result of our paper.

**Theorem 1.** Let Lévy driven fractional RF \( X \) in (1.5) satisfy Assumptions \((G)_{\alpha'}, (G)_{\alpha} \) and \((M)_\alpha; 0 < \alpha \leq 2, \frac{\alpha}{1 + \alpha} < P < \alpha, P \neq 1, P_{\frac{\alpha}{1 + \alpha}} \neq 1, P_{\frac{\alpha}{1 + \alpha}} \neq 1 \). Then the \( \gamma \)-rectangent RF in (1.3) exists for any \( \gamma > 0 \), \( t_0 \in \mathbb{R}_+^2 \) and satisfies the trichotomy

\[
V_{\gamma} = \begin{cases} 
V_+, & \gamma > \gamma_0, \\
V_-, & \gamma < \gamma_0, \\
V_0, & \gamma = \gamma_0.
\end{cases}
\]

with \( \gamma_0 = \frac{q_1}{q_2} = \frac{P_2}{P_1} \), \( V_0 := Y_{\alpha,0} \) defined in (4.8) and

\[
\begin{align*}
V_- := \begin{cases} 
\hat{Y}_{\alpha,2}, & P_{\frac{\alpha}{1 + \alpha}}^{1 + \alpha} > 1, \\
\bar{Y}_{\alpha,1}, & P_{\frac{\alpha}{1 + \alpha}}^{1 + \alpha} < 1.
\end{cases} \\
V_+ := \begin{cases} 
\hat{Y}_{\alpha,1}, & P_{\frac{\alpha}{1 + \alpha}}^{1 + \alpha} > 1, \\
\bar{Y}_{\alpha,2}, & P_{\frac{\alpha}{1 + \alpha}}^{1 + \alpha} < 1.
\end{cases}
\end{align*}
\]

\( \gamma \)-rectangent RF in (1.3) is defined in the proof of Theorem 1.
\textbf{Proof.} Case $\gamma = \gamma_0$. Let $H(\gamma_0) = (1 + \gamma_0)\frac{1+\alpha}{\alpha} - p_1 = (1 + \gamma_0)\frac{1}{\alpha} + q_1 \chi$. Then $\lambda^{-H(\gamma_0)} X((0, \lambda^\Gamma t)) = \int_{\mathbb{R}^2} f_\lambda(t;u) M(du)$, where $f_\lambda(t;u) = \lambda^{-H(\gamma_0)} g((-u, \lambda^\Gamma t - u))$, $t \in \mathbb{R}_+^2$, $u \in \mathbb{R}^2$. It suffices to show that for any $\theta_i \in \mathbb{R}$, $t_i \in \mathbb{R}_+^2$, $i = 1, \ldots, m$, $m \geq 1$,

$$\sum_{i=1}^{m} \theta_i \int_{\mathbb{R}^2} f_\lambda(t_i;u) M(du) \overset{d}{\to} \sum_{i=1}^{m} \theta_i \int_{\mathbb{R}^2} h_0(t_i;u) W(du), \tag{5.3}$$

where $h_0(t;u) = g_0((-u, t - u))$, $t \in \mathbb{R}_+^2$, $u \in \mathbb{R}^2$, as in (4.2). Using Proposition 1 this follows from

$$\left\| \sum_{i=1}^{m} \theta_i (f_\lambda(t_i;\cdot) - h_0(t_i;\cdot)) \right\|_\alpha \to 0. \tag{5.4}$$

Obviously, it suffices to prove (5.4) for $m = \theta_1 = 1, t_1 = t$. Letting $\mu_1 = 1, \mu_2 = \gamma_0$, we have $f_\lambda^1(t;u) = \lambda^{-\chi\gamma_0} g_0((-\lambda^\Gamma u, \lambda^\Gamma u - t)) + o(1) = h_0(t;u) + o(1)$ for any $u \neq 0, (0, t_2), (t_1, 0), t$, by (2.7). Consequently, we have (5.4) by the DCT using (2.7), (2.8) similarly as in the proof of Proposition 4 (i). Indeed, the above relations imply the existence of $\epsilon > 0$ such that for all $0 < |u| < \epsilon$,

$$|g(u)| \leq C\rho(u)^{\chi}, \quad |\partial_1 g(u)| \leq C\rho(u)^{\chi-\frac{1}{\gamma_0}}, \quad i = 1, 2, \quad |\partial_2 g(u)| \leq C\rho(u)^{\chi-Q}. \tag{5.5}$$

Letting $t = 1$ and using (5.5) we have that for all $\lambda > 0$ small enough, $|u| < \epsilon$,

$$|f_\lambda^1(1;u)| \leq C\lambda^{-\chi\gamma_0} (\rho(\lambda^\Gamma (1 - u))^\chi + \rho(\lambda^\Gamma (e_2 - u))^\chi + \rho(\lambda^\Gamma (e_1 - u))^\chi + \rho(-\lambda^\Gamma u)^\chi)$$

$$= C(\rho(1 - u)^\chi + \rho(e_2 - u)^\chi + \rho(e_1 - u)^\chi + \rho(-u)^\chi),$$

where the dominating function is $\alpha$-integrable on $|u| < \epsilon$, see (2.4). Similarly, for all $|\lambda^\Gamma u| < \epsilon \leq |u|$,

$$|f_\lambda^1(1;u)| = |\lambda^{-\chi\gamma_0} \int_{[0,1]} \partial_1 g(\lambda^\Gamma (s - u))ds| \leq C \int_{[0,1]} \rho(s - u)^{\chi-Q} ds,$$

where the dominating function is $\alpha$-integrable on $|u| \geq \epsilon$. Finally, we have $\int_{|\lambda^\Gamma u| \geq \epsilon} |f_\lambda^1(1;u)|^\alpha du = \lambda^\alpha(1 + \gamma_0 - H(\gamma_0)) I_{\lambda} = o(1)$, because $1 + \gamma_0 - H(\gamma_0) = q_1 (Q(1 - \frac{1}{\alpha}) - \chi) = \frac{p_1}{\alpha} (\alpha - P) > 0$ and

$$I_{\lambda} = \int_{|u| \geq \epsilon} \int_{[0,1]} |\partial_1 g(\lambda^\Gamma s - u)|^\alpha ds \overset{d}{\to} \int_{|u| \geq \epsilon} |\partial_1 g(u)|^\alpha du < \infty$$

for $1 \leq \alpha \leq 2$ follows by Jensen’s inequality from Assumption (G)\textsuperscript{0}; for $0 < \alpha < 1$ relation $I_{\lambda} = O(1)$ is a consequence of the monotonicity of the dominating function $g_1$ in (2.11), Assumption (G)\textsuperscript{0}. This completes the proof of (5.4).

Case $\gamma < \gamma_0, P_{\frac{1}{2}, \frac{1+\alpha}{\alpha}} < 1$. Let $H(\gamma) = H(\alpha, 1)$. It suffices to prove the convergence

$$\sum_{i=1}^{n} \theta_i \lambda^{-H(\alpha, 1)} X((0, \lambda^\Gamma t_i)) \overset{d}{\to} \sum_{i=1}^{n} \theta_i Y_{\alpha,1}(t_i), \tag{5.6}$$
for any $\theta_i \in \mathbb{R}, t_i \in \mathbb{R}^2_+, i = 1, \ldots, n, n \geq 1$. Write the collection $\{t_i, i = 1, \ldots, n\} = \{(t_{k_1}^{(j)}, t_{k_2}^{(j)})\}, k = 1, \ldots, n^{\langle j\rangle}, j = 1, \ldots, m$ as a union of $m$ sub-collections of points in $\mathbb{R}^2_+$ with different ordinates, viz., $t_{k_2}^{(1)} < t_{k_2}^{(2)} < \cdots < t_{k_2}^{(m)}$ and $\sum_{j=1}^{m} n^{\langle j\rangle} = n$. Set $t_0^{\langle 0\rangle} := 0$ and first let $m = 1$. For $t = (t_1, t_2) \in \mathbb{R}^2_+$ such that $t_2 = t_2^{(1)}$, split $\lambda^{-H_{0,1}} X((0, \lambda^t t)] = X_{\lambda}^{(1)}(t_1) - X_{\lambda}^{(0)}(t_1)$, where

$$\hat{X}_{\lambda}^{(j)}(t_1) = \int_{\mathbb{R}^2} f_{\lambda}(t_1; u - \lambda^{-\gamma} t_{k_2}^{(j)} e_2) M(du), \quad j = 0, 1,$$

(5.7)

with $f_{\lambda}(t_1; u) = \lambda^{-H_{0,1}}(g(\lambda t_1 e_1 - u) - g(-u)), u \in \mathbb{R}^2$. Note the two processes $\{X_{\lambda}^{(j)}(t_1), t_1 \in \mathbb{R}_+, j = 0, 1\}$, are identically distributed and asymptotically independent as it follows from the argument below. Let us prove that as $\lambda \downarrow 0$,

$$(X_{\lambda}^{(0)}(t_1), X_{\lambda}^{(1)}(t_1)) \overset{\text{fdd}}{\rightarrow} (Y_{\alpha,1}^{(0)}(t_1), Y_{\alpha,1}^{(1)}(t_1)),$$

(5.8)

where $Y_{\alpha,1}^{(j)}(t_1), j = 0, 1$, are independent copies of $Y_{\alpha,1}$ in (4.1). For this purpose, write $f_{\lambda}(t_1; u)$ as a sum of

$$f_{\lambda,0}(t_1; u) = f_{\lambda}(t_1; u) I(|u_2| < \epsilon \lambda^\gamma, |u| < \epsilon),$$

$$f_{\lambda,1}(t_1; u) = f_{\lambda}(t_1; u) I(|u_2| \geq \epsilon \lambda^\gamma, |u| < \epsilon),$$

$$f_{\lambda,2}(t_1; u) = f_{\lambda}(t_1; u) I(|u| \geq \epsilon)$$

for all $u \in \mathbb{R}^2$ and some small enough $\epsilon > 0$. Then, using the fact that the processes $\{f_{\lambda,0}(t_1; u - \lambda^{-\gamma} t_{k_2}^{(j)} e_2) M(du), t_1 \in \mathbb{R}_+, j = 0, 1\}$, are independent and have the same distribution, relation (5.8) follows if we show that with $h_1$ defined by (4.2),

$$\int_{\mathbb{R}^2} f_{\lambda,0}(t_1; u) M(du) \overset{\text{fdd}}{\rightarrow} \int_{\mathbb{R}^2} h_1(t_1; u) W_{\alpha}(du) = Y_{\alpha,1}(t_1) \quad \text{and} \quad \|f_{\lambda,i}(t_1; \cdot)\|_{\alpha} = o(1), \quad i = 1, 2,$$

(5.10)

The first relation in (5.10) follows by Proposition 1 with $\mu_1 = 1, \mu_2 = \gamma_0$ from $\|f_{\lambda,0}^{\uparrow}(t_1; \cdot) - h_1(t_1; \cdot)\|_{\alpha} = o(1)$, which in turn follows by the DCT. Indeed, since $\gamma < \gamma_0$ we have $f_{\lambda,0}^{\uparrow}(t_1; u) = f_{\lambda,0}(t_1; u) I(|u_2| < \epsilon \lambda^{-\gamma} \lambda^\gamma_0 |u| < \epsilon) \sim f_{\lambda}^{\uparrow}(t_1; u)$, where

$$f_{\lambda}^{\uparrow}(t_1; u) = \lambda^{-q_1 \chi}(g(\chi_{\lambda}^0 (t_1 e_1 - u)) - g(-\chi_{\lambda}^0 u)) \sim g_0(t_1 e_1 - u) - g_0(-u) = h_1(t_1; u)$$

using (2.7) and $\lambda^{-q_1 \chi} o(\rho(\chi_{\lambda}^0 u) \chi) = o(1)$ for any $u \neq 0, t_1 e_1$. Moreover,

$$|f_{\lambda,0}^{\uparrow}(t_1; u)| \leq |f_{\lambda}^{\uparrow}(t_1; u)| I(|\chi_{\lambda}^0 u| < \epsilon) \leq C \left((\rho_0(t_1 e_1 - u)^{P-1} + \rho_0(-u)^{P-1}) I(|u| < 1) + \int_0^{t_1} \rho_0(s e_1 - u)^{\frac{1}{P^2} - 1} ds I(|u| \geq 1)\right),$$

because $|\chi_{\lambda}^0 u| < \epsilon$ implies $|\chi_{\lambda}^0 (t_1 e_1 - u)| < 2 \epsilon$, and consequently we can use (2.7), (2.9) and (2.8) to bound the terms of $f_{\lambda}^{\uparrow}(t_1; u)$ and integrand of rewritten $f_{\lambda}^{\uparrow}(t_1; u) = \lambda^{-q_1 \chi} f_{\lambda}^{\uparrow} \partial g_1(\chi_{\lambda}^0 (s e_1 - u)) ds.$
The above function, which dominates \( f^\dagger_{\lambda,0}(t_1; \cdot) \), belongs to \( L_0(\mathbb{R}^2) \), see the proof of Proposition 4 (ii). The above domination also holds for \( f^\dagger_{\lambda,1}(t_1; \cdot) \), except that now we have

\[
f^\dagger_{\lambda,1}(t_1; u) = f^\dagger_{\lambda}(t_1; u)I(|u_2| \geq \epsilon \lambda^{\gamma-\gamma_0}, |\lambda^{\Gamma_0} u| < \epsilon) = o(1)
\]

point-wise (since \( \lambda^{\gamma-\gamma_0} \rightarrow \infty \)), resulting in \( \|f^\dagger_{\lambda,1}(t_1; \cdot)\|_\alpha = \|f_{\lambda,1}(t_1; \cdot)\|_\alpha = o(1) \). Finally, rewriting \( \int_{\mathbb{R}^2} |f_{\lambda,2}(t_1; u)|^\alpha du = \int_{|u| \geq \epsilon} |f_{\lambda}(t_1; u)|^\alpha du \) with \( f_{\lambda}(t_1; u) = \lambda^{1-\gamma} f_{\lambda,1}(t_1; u) \), we get

\[
\int_{|u| \geq \epsilon} |f_{\lambda,2}(t_1; \cdot)|_\alpha = o(1)
\]

for arbitrary \( t = (t_1, t_2) \in \mathbb{R}^2 \) such that \( t_2 = t^{(j)}_2 \). Consider a similar decomposition as in (5.7), viz.,

\[
\lambda^{-H_{\alpha,1}} X((0, \lambda^\Gamma t)) = \tilde{X}^{(j)}(t_1) - \tilde{X}^{(j)}(0)(t_1),
\]

where

\[
\tilde{X}^{(j)}(t_1) = \int_{\mathbb{R}^2} f_{\lambda}(t_1; u - \lambda^{\gamma} t^{(j)}_2 e_2)M(du), \quad i = 0, j,
\]

with \( f_{\lambda}(t_1; u) = \sum_{i=0}^{\infty} f_{\lambda,i}(t_1; u) \) for all \( u \in \mathbb{R}^2 \) and some \( \epsilon > 0 \) as in (5.9). If \( \epsilon > 0 \) is small enough, then \( \int_{\mathbb{R}^2} f_{\lambda,0}(t_1; u - \lambda^{\gamma} t^{(j)}_2 e_2)M(du), t_1 \in \mathbb{R}^+ \), \( j = 0, \ldots, m \), are independent, moreover,

\[
\int_{\mathbb{R}^2} f_{\lambda,0}(t_1; u - \lambda^{\gamma} t^{(j)}_2 e_2)M(du) \overset{\text{fdd}}{\rightarrow} Y^{(j)}_{\alpha,1}(t_1), \quad j = 0, \ldots, m,
\]

whereas other terms are negligible as in (5.10), which proves (5.6).

**Case \( \gamma < \gamma_0, P_1 > 1+\alpha > 1 \):** Let \( H(\gamma) = 1 + \gamma H_{\alpha,1} \). It suffices to prove one-dimensional convergence

\[
\lambda^{-1-\gamma} \lambda_{\alpha,1} X((0, \lambda^\Gamma t)) \overset{d}{\rightarrow} t_1 \tilde{Y}_{\alpha,2}(t_2) = t_1 \int_{\mathbb{R}^2} \tilde{h}_2(t_2; u)W_\alpha(du)
\]

for arbitrary \( t = (t_1, t_2) \in \mathbb{R}^2 \) with \( \tilde{h}_2(t; u) = \partial_1 g_0(\lambda t_2 e_2 - u) - \partial_1 g_0(-u), t \in \mathbb{R}^+, u \in \mathbb{R}^2 \) in (4.2). Since the l.h.s. of (5.11) writes as \( \int_{\mathbb{R}^2} f_{\lambda}(t; u)M(du) \) then using Proposition 1 with \( \mu_1 := \frac{\gamma}{\alpha}, \mu_2 := \gamma \) it suffices to prove

\[
\|f^\dagger_{\lambda}(t; \cdot) - t_1 \tilde{h}_2(t_2; \cdot)\|_\alpha = o(1),
\]

where with \( \Gamma^0_0 = \text{diag}(\frac{\gamma}{\alpha}, \gamma), \lambda' = \lambda^{1-\gamma} \).

\[
f^\dagger_{\lambda}(t; u) = \lambda^{\frac{\gamma}{\alpha} + \gamma - 1} \gamma \tilde{h}_{\alpha,2}(\lambda^{-\gamma} g((\lambda^{-\gamma} u, \lambda^\Gamma t - \lambda^{\Gamma_0} u)) \int_0^{t_1} \{ \partial_1 g(\lambda^{\Gamma_0}(\lambda' s e_1 + t_2 e_2 - u)) - \partial_1 g(\lambda^{\Gamma_0}(\lambda' s e_1 - u)) \} ds
\]

satisfies

\[
f^\dagger_{\lambda}(t; u) \sim \lambda^{\frac{\gamma}{\alpha} - \gamma} g(0) \int_0^{t_1} \{ \partial_1 g_0(\lambda^{\Gamma_0}(\lambda' s e_1 + t_2 e_2 - u)) - \partial_1 g_0(\lambda^{\Gamma_0}(\lambda' s e_1 - u)) \} ds
\]
\[
\int_0^{t_1} \{ \partial_1 g_0(\lambda' s e_1 + t_2 e_2 - u) - \partial_1 g_0(\lambda' s e_1 - u) \} \, ds \sim \int_1 h_2(t_2; u)
\]
for almost all \( u \in \mathbb{R}^2 \). To get the dominated function it is convenient to use \( \chi q_2 = p_2(P - 1) \) and \( |\partial_1 g_0(u)| \leq C \rho_0(u) \frac{1}{P^2 + 1} \), see (2.9). Whence setting \( t = 1 \), for \( 2^p > \rho_0(u) \geq 2^p \lambda^p \), we get

\[
|f^1_\lambda(1; u)| \leq C \int_0^1 (\rho_0(\lambda' s e_1 + e_2 - u) \frac{1}{P^2 - 1} + \rho_0(\lambda' s e_1 - u) \frac{1}{P^2 - 1}) \, ds,
\]
where \( \rho_0(\lambda' s e_1 - u) \frac{1}{P^2} \geq \frac{1}{2} \rho_0(u) \frac{1}{P^2} \) with \( \int_{\rho_0(u) < 2^p} \rho_0(u)^{\alpha \frac{1}{P^2} - 1} \, du < \infty \), see the proof of Proposition 4 (iii), and similar domination holds for the first term on the r.h.s. of (5.13). For \( \rho_0(u) \geq 2^p > \rho_0(\lambda^p u) \) we write

\[
|f^1_\lambda(1; u)| = \lambda^{\gamma P^2} \left| \int_{[0,1]} \partial_1 g_0(\lambda_0(\lambda' s_1 e_1 + s_2 e_2 - u)) \, ds \right|
\]

\[
\leq C \int_{[0,1]} \rho_0(\lambda' s_1 e_1 + s_2 e_2 - u)^{-1} \, ds \leq C \rho_0(u)^{-1}
\]
with \( \int_{\rho_0(u) < 2^p} \rho_0(u)^{-\alpha} \, du < \infty \), see the proof of Proposition 4 (iii). For \( \rho_0(u) < 2^p \lambda^p \), we get

\[
|f^1_\lambda(1; u)| \leq C(\lambda)^{-1} (\rho_0(\lambda e_1 + e_2 - u)^{P-1} + \rho_0(e_2 - u)^{P-1})
\]
\[+ \rho_0(\lambda e_1 - u)^{P-1} + \rho_0(-u)^{P-1}, \]
where \( \int_{\rho_0(u) < 2^p \lambda^p} \rho_0(u)^{\alpha(P-1)} \, du = o(\lambda^{\alpha P}) \) since \( 1 + \frac{p_1}{P^2} + p_1 \alpha(P - 1) > \alpha \) or \( P^2 \frac{1}{\alpha} + \frac{p_1}{\alpha} > 1 \), hence, \( \int_{\rho_0(u) < 2^p \lambda^p} |f^1(1; u)|^{\alpha} \, du = o(1) \). In the same manner, \( \int_{\rho_0(e_2 - u) < 2^p \lambda^p} |f^1(1; u)|^{\alpha} \, du = o(1) \). Finally, \( \int_{\rho_0(\lambda^p u) < 2^p} \int_{[0,1]} \partial_1 g_0(\lambda^p s - u) \, ds \, d\mu < C \) follows from Assumption (G) as in the two previous cases of \( \gamma \). This proves (5.12) and completes the proof of Theorem 1. \( \square \)

Extending Theorem 1 we may ask the following two questions: q1) when the (anisotropic) \( \gamma \)-rectangular limits in (1.3) agree with the Lévy sheet \( \mathcal{W}_o \) in (2.16)? (Obviously, the limits \( V_0 \) in Theorem 1 never agree with \( \mathcal{W}_o \).) Concerning q1), recall Table 1, where in the parameter region \( R_{11} \) the unbalanced limits \( V_0 \) of Theorem 1 are MSS RFs with indices \( (H_1, H_2) \rightarrow (1, 1) \) as \( P \rightarrow \alpha \), the last fact an easy observation from formulas in (4.3). One may expect that for \( P > \alpha \) all scaling limits are \( (1, 1) \)-MSS RFs of the form \( \{ t_1 t_2 V; (t_1, t_2) \in \mathbb{R}_+^2 \} \), where \( V \) is a r.v. See [40, Prop. 8.2.10] for related fact in the case of one-parameter processes. As shown in Proposition 5 this is true indeed under weaker conditions on the infinitely divisible random measure \( M \) which do not imply asymptotic stability.

We recall some facts about stochastic integrals w.r.t. \( M \) whose characteristics \( (\sigma, \nu) \) satisfy either Assumption (M) for \( \alpha = 2, \sigma = 0, \sup_{y > 0} y^{\alpha} \nu(\{ u \in \mathbb{R} : |u| > y \}) < \infty \) for some \( 0 < \alpha < 2 \), moreover, \( \nu \) is symmetric for \( \alpha = 1 \). Then \( M \) can be represented as \( M(d\mu) = \sigma W_2(d\mu) + \int_R y\tilde{N}(d\mu, dy) \) with \( \tilde{N} = (N - n) I(1 < \alpha \leq 2) + N I(0 < \alpha < 1) + (N' - N'') I(\alpha = 1) \), where \( W_2 \) is a Gaussian random measure on \( \mathbb{R}^2 \) with intensity \( d\mu \), \( N \) is a Poisson random measure on \( \mathbb{R}^2 \times \mathbb{R} \) with intensity \( n(d\mu, dy) = d\mu \nu(dy) \), \( N', N'' \) are Poisson random measures on \( \mathbb{R}^2 \times \mathbb{R} \) with the same intensity.
\( \frac{1}{2}, n, W_2, N, N', N'' \) are mutually independent. Thus, stochastic integral w.r.t. \( M \) can be defined as
\[
\int_{\mathbb{R}^2} f(u) M(du) = \sigma \int_{\mathbb{R}} f(u) W_2(du) + \int_{\mathbb{R}^2 \times \mathbb{R}} f(u) \gamma N(du, dy),
\]
where the last integral exists for any \( f : \mathbb{R}^2 \to \mathbb{R} \) satisfying \( \int_{\mathbb{R}^2} V(f(u)) du < \infty \), where
\[
V(u) = \int_{\mathbb{R}} ((|uy| \wedge |uy|^2) I(1 < \alpha \leq 2) + (1 \wedge |uy|) I(0 < \alpha < 1) + (1 \wedge |uy|^2) I(\alpha = 1)) \nu(dy)
\]
for \( u \in \mathbb{R} \), see [38], [31, Sec. 3]. Integrating the last expression by parts w.r.t. \( y \) we obtain that it does not exceed \( C|u|^\alpha \) hence the integral \( \int_{\mathbb{R}^2} f(u) M(du) \) is well-defined for any \( f \in L_\alpha(\mathbb{R}^2) \). Moreover, \( f_\lambda \to f \) in \( L_\alpha(\mathbb{R}^2) \) implies \( \int_{\mathbb{R}^2} f_\lambda(u) M(du) \overset{d}{\to} \int_{\mathbb{R}^2} f(u) M(du) \).

**Proposition 5.** Let \( X \) be a Lévy driven RF given by (1.5), where the Lévy characteristics \((\sigma, \nu)\) of \( M \) satisfy either Assumption \((M)\), for \( \alpha = 2 \) or \( \sigma = 0 \), \( \sup_{y>0} y^\alpha \nu(\{u \in \mathbb{R} : |u| > y\}) < \infty \) for some \( 0 < \alpha < 2 \), moreover, \( \nu \) is symmetric for \( \alpha = 1 \). Let (2.10) hold, where the kernel \( g \) admits the \( L_\alpha(\mathbb{R}^2) \)-derivative \( \partial_{12}g \) in the sense that
\[
\lim_{t_1, t_2 \downarrow 0, i=1,2} \int_{\mathbb{R}^2} |(t_1 t_2)^{-1} g((u, u + t)) - \partial_{12}g(u)|^\alpha du = 0.
\]
Then for any \( \gamma > 0 \), as \( \lambda \downarrow 0 \),
\[
\lambda^{-1-\gamma} X((0, \lambda^{\Gamma} t)) \overset{fdd}{\rightarrow} t_1 t_2 V,
\]
where \( V := \int_{\mathbb{R}^2} \partial_{12}g(u) M(du) \).

**Proof.** Follows from (1.6), (5.14) and the criterion for the convergence of stochastic integrals before the proposition.

Clearly, (5.14) is not directly related to the asymptotic form of \( g(t) \) at 0 as specified in Assumption \((G)_\alpha \). On the other hand if \( g \) satisfies the latter assumption then from (2.7), (2.8), (2.9) and (2.4) we see that \( \|\partial_{12}g\|_\alpha < \infty \) and (5.14) hold when \( P > \alpha \), relating Proposition 5 to question q1). Concerning question q2), in order that the rectangent limits are given by a Lévy Sheet, we replace the power law behavior of \( g \) in Assumption \((G)_\alpha \) by a condition that \( g \) is bounded but discontinuous at 0. Let \( \mathbb{R}_{ij}^2 := \{t \in \mathbb{R}^2 : \text{sgn}(t_1) = i, \text{sgn}(t_2) = j\}, i, j = \pm 1 \), denote the 4 open quadrants in \( \mathbb{R}^2 \).

**Assumption \((G)'_\alpha \).** The function \( g \) has finite limits \( g_{ij} = \lim_{|t| \to 0, t \in \mathbb{R}_{ij}^2} g(t), i, j \in \{1, -1\} \), on each quadrant such that
\[
g[0] := \sum_{i,j \in \{1,-1\}} ig_{ij} \neq 0.
\]
Moreover,
\[
\int_{\mathbb{R}^2} |g((-u, t-u)) - g[0]I(u \in (0,t))|^\alpha du = o(t_1 t_2), \quad t_i \downarrow 0, i = 1, 2.
\]

Note that condition (5.15) and the existence of the ‘limites quadrantales’ \( g_{ij} \) is typical for distribution functions in \( \mathbb{R}^2 \) having an atom at 0 or more generally, functions from the Skorohod space \( D(\mathbb{R}^2) \),
which are discontinuous at the origin, see [9]. The most simple example of \( g \) satisfying (5.15) and (5.16) is the function taking constant values on each quadrant, viz.,

\[
g(t) = \sum_{i,j \in \{1,-1\}} g_{ij} I(t \in \mathbb{R}^2_{ij}), \quad t \in \mathbb{R}^2. \tag{5.17}
\]

For \( g \) in (5.17), \( g((-u, t - u)) = g(0) I(u \in (0, t)) \) implying

\[
\lambda^{-\frac{1}{\alpha}(1+\gamma)} X((0, \lambda^\Gamma t)) = g(0) \lambda^{-\frac{1}{\alpha}(1+\gamma)} M((0, \lambda^\Gamma t)) \xrightarrow{\text{fdd}} g(0) W_\alpha(t) \tag{5.18}
\]
as in Remark 3. A similar result holds for more general \( g \) satisfying Assumption (G)'a.

**Proposition 6.** Let \( X \) be a Lévy driven RF in (1.5) satisfying (2.10), Assumptions (G)'a and (M)_a for some \( 0 < \alpha \leq 2 \). Then for any \( \gamma > 0 \),

\[
\lambda^{-\frac{1}{\alpha}(1+\gamma)} X((0, \lambda^\Gamma t)) \xrightarrow{\text{fdd}} g(0) W_\alpha(t). \tag{5.19}
\]

**Proof.** Split \( X((0, t]) = X'(0, t]) + X''((0, t]) \), where \( X'(0, t]) := g(0) M((0, t]) \), \( X''((0, t]) := \int_{\mathbb{R}^2} [g((-u, t - u)) - g(0) I(u \in (0, t])] d M(u) \). Then \( \lambda^{-\frac{1}{\alpha}(1+\gamma)} X'(0, \lambda^\Gamma t) \xrightarrow{\text{fdd}} g(0) W_\alpha(t) \) according to (5.18), while \( \lambda^{-\frac{1}{\alpha}(1+\gamma)} X''((0, \lambda^\Gamma t)) \xrightarrow{\text{fdd}} 0 \) in view of condition (5.16). \( \Box \)

### 6. Tangent limits of Lévy driven fractional RF

In this section we identify \( \gamma \)-tangent limits in (1.2) of Lévy driven fractional RF in (1.14), viz.,

\[
X(t) = \int_{\mathbb{R}^2} \{g(t-u) - g_{12}(-u)\} M(du), \quad t \in \mathbb{R}^2, \tag{6.1}
\]

with stationary increments whose distribution depends on \( g \) and the infinitely divisible random measure \( M \).

**Theorem 2.** Let \( X \) be a Lévy driven fractional RF in (6.1), where \( M \) satisfies Assumption (M)_a, \( 0 < \alpha \leq 2 \), and \( \int_{\mathbb{R}^2} |g(t-u) - g_{12}(-u)|^q dM < \infty \) for all \( t \in \mathbb{R}^2 \). Let \( g(t) = g_0(t)(1 + o(1)) \) as \( |t| \to 0 \), where \( g, g_0 \) have partial derivatives \( \partial_i, i = 1, 2 \), in (2.1) on \( \mathbb{R}^2_0 \) such that

\[
|g(t)| + |g_0(t)| \leq C \rho(t)^{\chi}, \quad |\partial_1 g(t)| + |\partial_1 g_0(t)| \leq C \rho(t)^{-\frac{1}{\alpha}}, \quad i = 1, 2, \tag{6.2}
\]

for all \( t \in \mathbb{R}^2_0 \). Moreover, \( g_0(t) = \rho(t)^{\chi} L(t), \quad t \in \mathbb{R}^2_0 \), as in (1.9), where \( L \) is a generalized invariant function and the parameters \( \chi < 0, \ q_i > 0, \ i = 1, 2, \) with \( Q = \frac{1}{q_1} + \frac{1}{q_2} \) satisfy

\[
-\frac{1}{\alpha} Q < \chi < -\frac{1}{q_1 \sqrt{q_2}} - \frac{1}{\alpha} Q. \tag{6.3}
\]

Then the \( \gamma \)-tangent limits \( T_\gamma \) in (1.2) of \( X \) exist for any \( \gamma > 0 \) under normalization \( d_{\chi, \gamma} = \lambda^{H(\gamma)} \), \( H(\gamma) := (1 - \frac{\gamma}{q_0}) H(\gamma_0), \ H(\gamma_0) := \frac{1 + \gamma_0}{\alpha} + \chi q_1 \) and are given by

\[
T_\gamma = \begin{cases} T_+, & \gamma > \gamma_0; \\ T_-, & \gamma < \gamma_0; \\ T_0, & \gamma = \gamma_0. \end{cases} \tag{6.4}
\]
where $\gamma_0 = \frac{q_1}{q_2}$, and

$$T_0(t) := \int_{\mathbb{R}^2} \{ g_0(t - u) - g_0(-u) \} W_\alpha(du), \quad t \in \mathbb{R}^2_+,$$

(6.5)

and $W_\alpha$ is $\alpha$-stable random measure as in Theorem 1, and the ‘unbalanced’ $\gamma$-tangent RFs

$$T_+(t) := T_0(t_1, 0), \quad T_-(t) := T_0(0, t_2), \quad t = (t_1, t_2) \in \mathbb{R}^2_+,$$

depend on only one coordinate on the plane.

Remark 6. We can rewrite the condition (6.3) in terms of $p_i, i = 1, 2$, in (1.10) as $P \geq \frac{1}{1 + \alpha} < 1,$ $P \geq \frac{1}{1 + \alpha} < 1,$ $P > \frac{\alpha}{1 + \alpha},$ which corresponds to parameter region $R_{22}$ in Figure 1.

Remark 7. For $\alpha = 2$, in Example 3.1 we have $g(t) = g_0(t) = |t|^{H-1}$ with $q_1 = q_2 = 2$, $Q = 1,$ $\chi = \frac{H-1}{2}$ and conditions (6.3) are equivalent $\frac{1-H}{2} < \frac{1}{2} - \frac{H}{2} \text{ or } 0 < H < 1$. Condition (6.2) in this example is also satisfied for all $t \in \mathbb{R}^2_0$ since $\partial_i |t|^{H-1} = (H-1)|t|^{H-2} \partial_i |t|$ with $\partial_i |t| = \frac{t_i}{|t|}.$

Proof. Let us check that the stochastic integral in (6.5) is well-defined for any $t \in \mathbb{R}^2$ or $\|h_0(t; \cdot)\|_\alpha < \infty,$ where $h_0(t; u) := g_0(t - u) - g_0(-u)$. Let $q := \max\{q_1, q_2, 1\}$. Then using (6.2), we get $\int_{\rho(u)<2\rho(t)} |h_0(t; u)|^{\alpha} du \leq C \int_{\rho(u)<2\rho(t)} \rho(u)^{\alpha} du < \infty \text{ as } Q > -\chi,$ see (2.4). Next, for $\rho(u) \geq 2\rho(t)$ we rewrite $h_0(t; u) = \int_0^{\rho(u)} \partial_0 g_0(s - u_1, t_2 - u_2) ds + \int_0^{\rho(u)} \partial_2 g_0(-u_1, s - u_2) ds,$ where $\partial_1 g_0(s - u_1, t_2 - u_2) \leq C \rho(u)^{\gamma} \frac{1}{s}$, $\partial_2 g_0(-u_1, s - u_2) \leq C \rho(u)^{-\frac{1}{\gamma}}$ by (6.2), (2.5). We get $\int_{\rho(u)\geq 2\rho(t)} |h_0(t; u)|^{\alpha} du \leq C \int_{\rho(u)\geq 2\rho(t)} \rho(u)^{\alpha} \left(\frac{\gamma}{\rho(u)}\right) du < \infty.$ Hence, $\|h_0(t; \cdot)\|_\alpha < \infty.$

To prove the convergence in (1.2) we use Proposition 1. It suffices to consider the case $t_0 = 0$. Let first $\gamma = \gamma_0$ with $H(\gamma_0) = \frac{1}{\alpha} (1 + \gamma_0) + \frac{\gamma_0}{Q}$. Then $\lambda^{-H(\gamma_0)} X(\lambda^{\Gamma_0} t) = \int_{\mathbb{R}^2} f_\lambda(t; u) M(du),$ where $f_\lambda(t; u) := \lambda^{-H(\gamma_0)} (g(\lambda^{\Gamma_0} t - u) - g(-u))$ satisfies $f_\lambda(t; u) = \lambda^{-\mu_1} (g(\lambda^{\Gamma_0} t - u) - g(-\lambda^{\Gamma_0} u)) \rightarrow h_0(t; u)$ for all $u \neq 0, t$ according to $g(t) = g_0(t) (1 + o(1)), |t| \rightarrow 0.$ The domination argument implying $f_\lambda(t; \cdot) \rightarrow h_0(t; \cdot)$ in $L_\alpha(\mathbb{R}^2)$ uses (6.2) and follows as in the proof $\|h_0(t; \cdot)\|_\alpha < \infty$ above.

Next, let $\gamma < \gamma_0$ with $H(\gamma) = \frac{\gamma}{\gamma_0} H(\gamma_0) = \frac{1}{\alpha} (\gamma + \frac{\gamma}{Q}) + \frac{\gamma_0}{Q}$. Then we have $\lambda^{-H(\gamma)} X(\lambda^{\Gamma} t) = \int_{\mathbb{R}^2} f_\lambda(t, u) M(du),$ where using Proposition 1 with $\mu_1 = \frac{\gamma}{\gamma_0}, \mu_2 = \gamma$, we see that $f_\lambda(t, u) = \lambda^{-\mu_1} g(\lambda t - \lambda^{\mu_1} u_1, \lambda t_2 - \lambda^{\mu_1} u_2) \rightarrow h_0(0, t_2; u)$ for all $u \neq 0, (0, t_2).$ The $L_\alpha(\mathbb{R}^2)$-convergence $f_\lambda(t, \cdot) \rightarrow h_0(0, t_2; \cdot)$ in the case $\gamma < \gamma_0$ and the proof of (1.2) for $\gamma > \gamma_0$ follows analogously and we omit the details. Theorem 2 is proved. 

7. Concluding comments

1. Unbalanced scaling ($\gamma \neq \gamma_0$) of fractional RF on $\mathbb{R}^2$ may lead to degenerate dependence in one direction (either horizontal or vertical). The critical $\gamma_0$ coincides with the ‘intrinsic local dependence ratio’ $\frac{q_1}{q_2} = \frac{p_1}{p_2}$ of the RF, defined in terms the exponents $q_i, i = 1, 2,$ in the asymptotic form (1.9) of the moving-average kernel $g(t)$ at $t = 0$. The degeneration is apparent in $(H_1, H_2)$-MSS property of rectangential limits with one of $H_i, i = 1, 2,$ equal 1 or 0, indicating either extreme positive $(H_i = 1)$ or
extreme negative \((H_i = 0)\) dependence of the limit RF in direction \(e_i, i = 1, 2\). The above facts are very important for statistical estimation of \(p_i, i = 1, 2\), using power variations of rectangular increments on a dense grid (see Comment 5 below).

2. We believe that our results can be extended to fractional type infinitely divisible RFs \(X\) on \(\mathbb{R}^d, d \geq 3\). However, the results in [11, 42] on large-scale anisotropic scaling suggest that the class of the limit rectangent RFs in higher dimensions \(d \geq 3\) is more complex and its complete description can be difficult. On the other side, it would be very interesting to extend the class of two-parameter Lévy driven fractional RFs in (1.5) by including a random volatility RF (independent of \(M\)) as in the case of ambit RF [4].

3. The \(\gamma\)-tangent and \(\gamma\)-rectangent limits in our paper are defined through finite-dimensional convergence only. This raises the question of a functional convergence in these limits and also about the path properties of the limit tangent and rectangent RFs. While in some cases these limits are a.s. continuous and classical, in other cases they are extremely singular, see Remark 5, suggesting that a functional convergence is not feasible in such cases.

4. As noted by a referee, the scaling in (1.2), (1.3) with diagonal matrix \(\Gamma\) is quite particular, raising the question about these limits for general \(2 \times 2\) matrix \(\Gamma\) with positive real parts of the eigenvalues, c.f. [10]. This question is interesting and open. We expect that a possible answer depends on \(\Gamma\) or on the way the point \(\lambda^\Gamma t\) tends to \(0\) as \(\lambda \to 0\). If the trajectory \(\{\lambda^\Gamma t, 0 < \lambda < 1\}\) winds up around \(0\) infinitely many times, the limit in (1.3) under the premises of Theorem 1 probably do not exist. On the other hand, if the trajectory of \(\lambda^\Gamma t\) approaches \(0\) along some ‘oblique’ direction, the limit in (1.3) is more likely to exist since this case resembles the scaling of RF with ‘oblique’ dependence axis [34].

5. We expect that our results can be applied for statistical estimation and identification of fractional parameters \(p_i, i = 1, 2\), based on observations of \(X\) in (1.5) on a dense rectangular grid of \([0, 1]^2\).

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