Research of numerical methods for solving ordinary differential equations in MS Excel

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Abstract. Methods of Euler and Runge-Kutta are the most famous ones among the numerical methods for solving ordinary differential equations. Euler's method has slow convergence; therefore, methods of a higher order of accuracy are often used. There are various modifications of the Euler method to increase its accuracy. Euler methods are mostly common corrected and modified. The corrected and modified Euler methods are second-order methods; their accuracy is proportional to $h^2$. Based on a comparative analysis of the obtained results of the study of numerical methods for solving ordinary differential equations in MS Excel, it was found that the modified Euler's method provides the highest accuracy.

It is known that only a small number of types of first-order differential equations allow integration by quadratures (they say that a differential equation is solved in quadratures if its general solution is expressed in terms of one or more integrals). Even less often it is possible to obtain a solution in elementary functions. The more important are the numerical methods that make it possible to obtain an approximate solution of an ordinary differential equation (ODE) of the first order.

Let it be required to find an approximate solution to the differential equation:

$$\frac{dy}{dx} = y' = f(x, y),$$

satisfying the initial condition $y(x_0) = y_0$.

This problem is called the Cauchy problem. Numerical solution of the Cauchy problem consists in calculating a table of approximate values $y_1, y_2, ..., y_n$ in points $x_1, x_2, ..., x_n$.

Generally, $x_i = x_0 + ih$, где $i = 1, 2, ..., n$.

The points $x_i$ are called grid nodes, and $h$ is the grid step, and $0 < h < 1$. The construction of the discrete Cauchy problem is based on one way or another of replacing a differential equation with its discrete analogue.

Among the numerical methods for solving ordinary differential equations, the most famous are the methods of Euler, Runge-Kutta $[1-4]$.

These methods belong to the group of one-step methods, in which to calculate the point $y_{i+1} = y(x_{i+1})$, information is required only about the last calculated point $y_i$.

Euler's formula for calculating any $y$ is:

$$y_{i+1} = y_i + h \times f(x_i, y_i), \text{где } i = 0, 1, 2, ..., n-1.$$
Since the exact solution of the Cauchy problem is often unknown, the Runge rule or the double recalculation rule is used to estimate the error of the method: the calculation is repeated with a step \( h/2 \) and the absolute difference is calculated:

\[
\left| y^h_i - y^{h/2}_i \right| \over 2^p - 1
\]

Where: \( y^h_i \) – the value of the function at the point \( x_i \) at the step \( h \); \( y^{h/2}_i \) – the value of the function at the point \( x_i \) at the step \( h/2 \); \( p \) – the order of the method (Euler’s method is a first-order method, since its accuracy increases linearly with decreasing step \( h \)).

The error of the method is estimated using the following formula:

\[
\max \left| y^h_i - y^{h/2}_i \right| \over 2^p - 1 \quad \text{for } i = 0, 1, 2, ..., n.
\]

Euler’s method has slow convergence; therefore, methods of a higher order of accuracy are often used. There are various modifications of the Euler method to increase its accuracy. The most widespread are the corrected and modified Euler methods [1-4].

We denote \( K = f(x_i, y_i) \). Then both methods are described by the formula:

\[
y_{i+1} = y_i + h \times \Phi(x_i, y_i),
\]

where: \( \Phi(x_i, y_i) = a_1 K + a_2 f(x_i + hb_1, y_i + hb_2 K) \).

For the corrected method \( a_1 = a_2 = 0.5 \) and \( b_1 = b_2 = 1 \).

For the modified method \( a_1 = 0, a_2 = 1 \) and \( b_1 = b_2 = 0.5 \).

The corrected and modified Euler methods are second-order methods, their accuracy is proportional to \( h^2 \).

Consider the numerical solution of the Cauchy problem \( y' = 0.25 y^2 + x^2, y(0) = -1 \) on the interval \([0; 0.5]\) in 0.1 increments. To solve the problem, use the usual, corrected and modified Euler methods.

The estimation of the errors of the methods will be carried out on the basis of the Runge rule.

Let us present the algorithm of actions for solving the Cauchy problem by numerical methods on the MS Excel worksheet [5-8]:

1. Start MS Excel and save the workbook file with the name "ODE Solution.xlsx". Design the template as shown in figure 1.

   Comment. For the convenience of understanding the ODE solution, it is recommended to develop the proposed templates in the ranges of cells indicated in the figures.

2. In the range A4: A14, set the values of the variable \( x \) on the interval \([0; 0.5]\) in 0.05 steps. The choice of a step of 0.05, and not 0.1 (as specified in the example) is due to the fact that according to the task it is necessary to estimate the error of the method using Runge’s rule. This rule involves recalculation with a step of \( h/2 \). In cell B4, enter the variable \( x \) and the initial condition \( y(0) = -1 \).

3. Enter formulas in the template. The entered formulas correspond to the expression \( y_{i+1} = y_i + h \times f(x_i, y_i) \). After entering all the formulas, the results of solving the problem will be displayed in the cells with yellow filling (figure 2).

4. Estimate the solution error using Runge’s rule. To do this, follow these steps:

   - In cell C4 enter the initial condition \( y(0) = -1 \);
   - in the range C5: C14 enter the formulas that match the expression \( y_{i+1} = y_i + h/2 \times f(x_i, y_i) \), where \( h/2 = 0.05 \);
   - in the range D4:D14 enter the formulas that match the expression \( \left| y^h_i - y^{h/2}_i \right| \) (Euler’s method is a first order method, so \( 2^p - 1 = 2^1 - 1 = 1 \));
   - in cell D15 enter the formula corresponding to the expression \( \max \left| y^h_i - y^{h/2}_i \right| \).
The results of evaluating the error in solving the ODE by the Euler method are shown in figure 3.

Figure 1. Template for solving ODEs by Euler's method.

Figure 2. Formulas for solving ODEs by Euler's method.

5. Develop a template for solving ODEs with corrected and modified Euler methods.
6. In the range A21: A31, enter the values of the variable $x$ in the interval $[0; 0.5]$ in 0.05 steps. Enter the initial condition in cells B21 and E21 $y(0) = -1$. In cells F18, F19, H18 and H19 enter the values of the coefficients $a_1$, $a_2$, $b_1$, $b_2$ for the corrected Euler method (figure 4).

Figure 3. Estimation of the error in solving ODE by Euler's method.
7. Enter formulas in the template as appropriate.

The formulas entered in the column $y$ for $h$ correspond to the expression $y_{i+1} = y_i + h \times \Phi(x_i, y_i)$.

The formulas entered in column $K$ at $h$ correspond to the expression $K = f(x_i, y_i)$.

The formulas entered into the column $\Phi$ at $h$ correspond to the expression $\Phi(x_i, y_i) = a_1 K + a_2 f(x_i + h b_1, y_i + h b_2 K)$.

After entering all the formulas, the results of solving the problem will be displayed in the cells with yellow filling.

8. Estimate the solution error using Runge’s rule. To do this, follow these steps:

- In the range E22: E31 enter the formulas corresponding to the expression $y_{i+1} = y_i + h/2 \times \Phi(x_i, y_i)$, where $h/2 = 0.05$;
- In the range F21: E31 enter the formulas corresponding to the expression $K = f(x_i, y_i)$;
- in the range G21: G31 enter the formulas corresponding to the expression $\Phi(x_i, y_i) = a_1 K + a_2 f(x_i + h/2 b_1, y_i + h/2 b_2 K)$, where $h/2 = 0.05$;
- in the range H21: H31 enter the formulas corresponding to the expression $\left| \frac{y_{i+1} - y_{i+1/2}^{h/2}}{3} \right|$ (the corrected and modified Euler methods are second order methods, so $2^n - 1 = 2^2 - 1 = 3$);
- in cell H32 enter the formula corresponding to the expression $\max \left| \frac{y_{i+1} - y_{i+1/2}^{h/2}}{3} \right|$.

The results of estimating the error in solving the ODE by the corrected Euler method are shown in figure 4.

9. To solve the ODE by the modified Euler method, change the values of the coefficients $a_1, a_2, b_1$, and $b_2$.

The results are shown in figure 5.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
|---|---|---|---|---|---|
| 18 | Corrected method of Euler              | a1 | 0.5 | b1 | 1  |
| 19 | a2 | 0.5 | b2 | 1  |
| 20 | x | y by h | K by h | \Phi by h | y by h/2 | K by h/2 | \Phi by h/2 | d |
| 21 | 0.00 | -1.00 | 0.2500 | 0.2483 | -1.00 | 0.2500 | 0.2484 | 0.00000 |
| 22 | 0.05 | -0.98759 | 0.24633 | 0.24706 |
| 23 | 0.10 | -0.97524 | 0.24777 | 0.25102 | 0.00004 |
| 24 | 0.15 | -0.96269 | 0.25419 | 0.25990 |
| 25 | 0.20 | -0.94969 | 0.26458 | 0.27360 | 0.00008 |
| 26 | 0.25 | -0.93601 | 0.28153 | 0.29201 |
| 27 | 0.30 | -0.92141 | 0.30235 | 0.31505 | 0.00013 |
| 28 | 0.35 | -0.90566 | 0.32756 | 0.34263 |
| 29 | 0.40 | -0.88853 | 0.35737 | 0.37469 | 0.00017 |
| 30 | 0.45 | -0.86979 | 0.39164 | 0.41118 |
| 31 | 0.50 | -0.84857 | 0.40022 |
| 32 |   |   |   |   |   | 0.00022 |

Figure 4. Estimation of the error in solving the ODE by the corrected Euler method.
Figure 5. Estimation of the error in solving the ODE by the modified Euler method.

10. Compare the errors in solving the ODE by the usual, corrected and modified Euler methods (table 1).

Table 1. Comparison of errors in solving ODEs according to Runge’s rule.

| Method                | Method error |
|-----------------------|--------------|
| Euler’s method        | 0.00382      |
| Corrected Euler method| 0.00022      |
| Modified Euler method | 0.00007      |

It can be seen from table 1, the modified Euler method provides the highest accuracy.

In practice, the most common method for solving ordinary differential equations is the fourth order Runge-Kutta method. This method is more accurate than Euler’s method (first-order method) and then the revised and modified Euler’s methods (second-order methods).

To estimate the value of the derivative in the Runge-Kutta method, four auxiliary function evaluations are used, but the method is still one-step.

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