How degenerate can cosmological neutrinos be?

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Abstract

There are well-known bounds on light neutrino masses from cosmological energy density arguments. These arguments assume the neutrinos to be non-degenerate. We show how these bounds are affected if the neutrinos are degenerate. In this case, we obtain correlated bounds between neutrino mass and degeneracy.

In the standard big bang cosmology, neutrinos play a very important role in the evolution of the universe. As a result, neutrino properties can be significantly constrained from various cosmological parameters. Perhaps the most famous of such constraints is the one on the mass of light neutrinos. Using modern values of the cosmological parameters some of which we will discuss later, this bound comes out to be

$$\sum_i m_i < 46 \text{ eV},$$

where $m_i$ is the mass of the light neutrino $\nu_i$.

The derivation of this bounds assumes that the cosmological neutrinos are non-degenerate, i.e., the number of neutrinos and antineutrinos are equal. If this condition is not satisfied, the bound is modified. For example, if the neutrinos are assumed to be massless and at zero temperature, the degenerate Fermi gas of neutrinos can oversaturate the energy density of the universe unless the chemical potentials $\mu_i$ associated with the neutrinos $\nu_i$ satisfy the constraint

$$\left[ \sum_i \mu_i^4 \right]^{1/4} < 7.4 \times 10^{-3} \text{ eV}.$$

The two cases discussed above are extreme cases — one a bound on mass for vanishing chemical potential, and the other a bound on the chemical potential for vanishing mass. More generally, both mass and chemical potential may be non-zero. One then should obtain correlated bounds on mass and degeneracy of the neutrinos. In view of the importance of the neutrinos in the physics of the early universe, such bounds are potentially important. The purpose of this paper is to derive such bounds. For the sake of simplicity, we assume that only one light neutrino species dominates the energy density of the universe.

To discuss these bounds, it is useful to introduce the parameter $F_\nu$, defined by

$$F_\nu = \frac{(\rho_\nu + \rho_{\bar{\nu}})}{\rho_0},$$

where the numerator is the sum of the present energy densities of the neutrinos and antineutrinos, and $\rho_0$ is the total energy density of the present universe. Obviously, $F_\nu < 1$. Modern theories of structure formation in the universe prefer a value around 0.25 for this parameter. We will present our results for various values of this parameter.

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The total energy density $\rho_0$ is not very well-known. It is usually parametrized in the form

$$\rho_0 = 10^4 h^2 \Omega_0 \text{ eV/cm}^3,$$

where $\Omega_0$ is the density measured in units of the critical density, the latter being dependent on the Hubble parameter, which in turn is parametrized as $100h \text{ km s}^{-1} \text{Mpc}^{-1}$. Combining Eqs. (3) and (4), we can write

$$(\rho_\nu + \rho_{\bar{\nu}})_0 = 10^4 \zeta_\nu \text{ eV/cm}^3,$$

where

$$\zeta_\nu \equiv h^2 \Omega_0 F_\nu.$$ (6)

In the rest of the paper, we will take different values for this parameter $\zeta_\nu$ which measures the importance of the neutrinos in the early universe, and will find the combinations of neutrino mass and degeneracy which can produce the corresponding values for $(\rho_\nu + \rho_{\bar{\nu}})_0$. The bounds quoted in Eqs. (1) and (2) correspond to $\zeta_\nu = 0.5$.

At any epoch, the energy density of neutrinos plus antineutrinos can be written as

$$\rho_\nu + \rho_{\bar{\nu}} = \int \frac{d^3 p}{(2\pi)^3} \sqrt{p^2 + m^2} \left[ f(p) + \bar{f}(p) \right],$$

where $f(p)$ and $\bar{f}(p)$ denote the distribution functions of the neutrinos and the antineutrinos at that epoch, $p$ being the magnitude of the momentum 3-vector.

In the very early universe, the neutrinos were in equilibrium with other particles, so $f(p)$ and $\bar{f}(p)$ were the Fermi-Dirac distribution functions, with appropriate temperature and chemical potential. Due to the expansion of the universe, there reached an epoch when the neutrino interactions were not enough to keep them in thermal equilibrium. This is called the epoch of neutrino decoupling. At this time, let the temperature of the neutrinos be $T_D$ and the chemical potential $\mu_D$. Then, at this epoch, the energy density of neutrinos and antineutrinos was given by Eq. (7), with

$$f_D(p) = \frac{1}{\exp \left( \frac{\sqrt{p^2 + m^2} - \mu_D}{T_D} \right) + 1},$$

and that of the antineutrinos can be obtained by changing the sign of $\mu_D$. After that time, the neutrino momenta did not change at all due to collisions. They only suffered the cosmological red-shift. If the cosmological scale parameter was $a_D$ at the decoupling era and $a_0$ now, all the momenta have decreased by a factor

$$r \equiv \frac{a_0}{a_D}.$$ (9)

The quantity $r$ can be interpreted as the ratio of the scale factors of the era when the neutrinos were last in thermal equilibrium, and the present era. The distribution function of the neutrinos in the present universe can be obtained from Eq. (8) through the relation

$$f_0(p) = f_D(p/r).$$ (10)

If $m = \mu = 0$, this is still a thermal distribution with a redefined temperature $rT_D$. When either the mass or the chemical potential is not zero, the distribution is not thermal. Making a simple change of variable, we can write

$$(\rho_\nu + \rho_{\bar{\nu}})_0 = \frac{r^3}{2\pi^2} \int_0^\infty dp \, p^2 \sqrt{r^2 p^2 + m^2} \left[ f_D(p) + \bar{f}_D(p) \right].$$ (11)
Since the distribution is not thermal as we have already said, it is meaningless to talk about the chemical potential of neutrinos in the present universe. Rather, we can talk about the difference of the number densities of neutrinos and antineutrinos. Using similar argument, it will be given by

\[(n_{\nu} - n_{\bar{\nu}})_0 = \frac{\pi^3}{2\pi^2} \int_0^\infty dp \, p^2 \left[ f_D(p) - \bar{f}_D(p) \right]. \tag{12}\]

In Fig. 1, we have shown for a range of neutrino mass \(m\), the values of the degeneracy parameter

\[\eta_{\nu} \equiv \frac{(n_{\nu} - n_{\bar{\nu}})_0}{n_\gamma}, \tag{13}\]

which give certain preassigned values for the density parameter \(\zeta_{\nu}\). In Eq. (13), \(n_\gamma\) is the number density of photons in the microwave background which has been introduced to obtain a dimensionless parameter for neutrino degeneracy. We will discuss the characteristics of these results later. Before that, we want to discuss different considerations that go into obtaining the results.

From the previous discussion, it seems that if we want to find out the energy density for the neutrinos, we need to know \(m, \mu_D, T_D\) and \(r\). The plot of Fig. 1 is two dimensional, so we need to show that two of these variables are dependent. We take \(m\) and \(\mu_D\) as independent. Given these two, we need to solve for the temperature at which the reaction rate of the neutrinos become equal to the expansion rate of the universe. For this part, we can use \(r = 1\) because, until decoupling took place, the neutrinos were in thermal equilibrium. Once \(T_D\) is obtained this way, we can put

\[r = y_{rh} \frac{T_0}{T_D}. \tag{14}\]

for the calculation of energy and number densities in the present universe, where \(T_0\) is the temperature associated with the microwave background. For \(y_{rh} = 1\), this follows from Eq. (11) and the comment following it, which is valid for a Bose-Einstein distribution as well. The issue of the departure of \(y_{rh}\) from the value of unity will be discussed next.

The value of \(y_{rh}\) should be equal to unity provided the number of photons have not increased due to the annihilation of other species of particles after the neutrinos decoupled. Thus, if the neutrino decoupling temperature is smaller than the electron mass, we should put \(y_{rh} = 1\). If, on the other hand, \(T_D > m_e\), we should take into account the reheating of photons from \(\bar{e}^- - \bar{e}^+\) annihilations. This gives \(y_{rh} = (4/11)^{1/3}\). These considerations have been taken into account in the plot of Fig. 1 as we discuss below.
The decoupling temperature is determined by using the criterion \( \Gamma = H \), where \( \Gamma \) denotes the reaction rate for neutrinos and \( H \) is the Hubble parameter. In the early universe, \( H \) is given by \( T^2/M_P \), apart from numerical factors of order unity. On the other hand, neutrino reaction rates are given by \( \Gamma = n \sigma \), where \( n \) is the number density of the particles that neutrinos interact with, and \( \sigma \) is the scattering cross section. When neutrinos are highly degenerate, they are more abundant than electrons and positrons. Therefore, elastic scattering as well as \( \nu \bar{\nu} \) annihilation are the most efficient mechanisms for redistribution of energy and momentum of neutrinos, which keep them in thermal equilibrium. Hence, in the expression for the reaction rate, one should use \( n = n_\nu + n_\bar{\nu} \). The cross section is given by \( G_\nu^2 E^2 \) where \( E \) is the typical energy of the neutrinos. Since the masses are small in the range under consideration, we can approximate \( E \) by the temperature \( T \). Therefore, the decoupling temperature is determined from the equation

\[
n_\nu + n_\bar{\nu} = \frac{1}{G_\nu^2 M_P^2},
\]

where \( n_\nu \) and \( n_\bar{\nu} \) should be determined using the equilibrium distribution functions.

The plots have been made for three different values for the density of the neutrinos and antineutrinos. First, considerations of the age of the universe dictates \([5]\) that \( n_\nu \leq 1 \times 10^{13} \text{ eV} \), as it should. In fact, the outer vertical line has an intercept on the horizontal axis exactly at 46 eV. Second, for small or vanishing mass, we obtain a horizontal branch which corresponds to the bound of Eq. (2). Here, the intercept on the vertical axis is \( 2.2 \times 10^5 \), which is the number obtained from Eq. (2). Finally, there is an intermediate region which represents the new results obtained in this paper. This part follows roughly a power law behavior, given by

\[
n_\nu \left( \frac{m}{1 \text{ eV}} \right) = 24 \zeta_\nu.
\]

The glitch encountered in this region corresponds to the shift of the value of \( y_{\text{th}} \) from \((4/11)^{1/3}\) to unity. Earlier, we explained how we have calculated the decoupling temperature \( T_D \). If \( T_D \) is less than the electron mass, we have taken \( y_{\text{th}} \) to be unity. Otherwise, it is taken to be \((4/11)^{1/3}\). Because of these sudden approximations, the shift appears as a glitch. In a more detailed calculation of the densities of different particles, involving the Boltzmann equation, there should be a more gradual shift from one value of \( y_{\text{th}} \) to another.

The usefulness of these plots is the following. There are quite a few recent indications that neutrinos are massive. These include the solar neutrino data \([6]\), the atmospheric neutrino data \([7]\) and direct neutrino oscillation experiments \([8]\). Each kind of data points to a different range of neutrino mass, but none of them very close to the value obtained from cosmological dark matter considerations assuming zero degeneracy for neutrinos and taking values of \( h^2 \Omega_0 \) close to 1/2. The point that we make here is that, if we introduce neutrino degeneracy as a free parameter, we can obtain solutions for any energy density given the value of neutrino mass. Such a solution would of course be acceptable if it conforms with other bounds on neutrino degeneracy, e.g., those coming from primordial nucleosynthesis \([9]\). One can also ask whether, given a certain value of mass, the required value of neutrino degeneracy is a plausible one. This can only be addressed within a certain scenario of generation of lepton asymmetry in the universe. This is a separate issue and should be taken up separately.

Note added: After this paper was accepted for publication, we came to know about a paper by Khare and Deo \([10]\) which also found the correlated bounds on neutrino mass and degeneracy from the upper limit on cosmological energy density. However, these authors assumed the neutrino distribution to be thermal even in the present universe. In our analysis, we have taken into account neutrino decoupling and the subsequent deviation of the neutrino distribution from a thermal one.
References

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