Stability of an additive-quadratic-quartic functional equation

Abstract: In this paper, we investigate the stability of an additive-quadratic-quartic functional equation

\[ f(x + 2y) + f(x - 2y) - 2f(x + y) - 2f(x - y) - 2f(x - y) - 2f(y - x) + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0 \]

by the direct method in the sense of Găvruta.

Keywords: Hyers-Ulam stability, hyperstability, quadratic functional equation, fixed point theorem

MSC: 39B82, 39B52

1 Introduction

A. K. Hassan et al. [1], M. Mohamadi et al. [2] and C. Park et al. [3] investigated the stability of the AQQ (additive-quadratic-quartic) functional equation

\[ f(x + 2y) + f(x - 2y) - 2f(x + y) - 2f(x - y) - 2f(y - x) + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0 \] (1.1)

in various spaces. For the terminology “AQQ (additive-quadratic-quartic) functional equation”, refer to the papers [1–3]. The second author [4] also studied different type of the additive-quadratic-quartic functional equation

\[ f(x + ky) + f(x - ky) - k^2f(x + y) - k^2f(x - y) + 2(k^2 - 1)f(x) + (k^2 + k)f(y) + (k^2 - k)f(-y) - 2f(ky) = 0, \]

where \( k \) is a fixed real constant with \( k \neq 0, \pm 1 \).

In this paper, let \( V \) and \( W \) be real vector spaces and \( Y \) be a real Banach space. For a given mapping \( f : V \rightarrow W \), we use the following abbreviations

\[ f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}, \]

\[ Df(x, y) = f(x + 2y) + f(x - 2y) - 2f(x + y) - 2f(x - y) - 2f(x - y) - 2f(x - y) + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) \]

for all \( x, y \in V \). In this paper, we will prove the stability of the functional equation (1.1) in the sense of Găvruta [5] (See also [6, 7]). In other words, from the given mapping \( f \) that approximately satisfies the functional...
equation (1.1), we will show that the mapping \( F \), which is the solution of the functional equation (1.1), can be constructed using the formula

\[
F(x) = \lim_{n \to \infty} \left( \frac{f_0(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n} \right)
\]

or

\[
F(x) = \lim_{n \to \infty} \left( 2^n f_0 \left( \frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n f_e \left( \frac{x}{2^n} \right)}{3} - \frac{16^{n+1} - 4^{n+2} f_e \left( \frac{x}{2^{n+1}} \right)}{3} \right)
\]

and we will prove that the mapping \( F \) is the unique solution mapping of functional equation (1.1) near the mapping \( f \).

2 Main results

**Lemma 1.** If a mapping \( f : V \to W \) satisfies \( Df(x, y) = 0 \) for all \( x, y \in V \), then the equalities

\[
f(x) = \frac{f_0(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n}, \tag{2.1}
\]

\[
f(x) = 2^n f_0 \left( \frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n f_e \left( \frac{x}{2^n} \right)}{3} - \frac{16^{n+1} - 4^{n+2} f_e \left( \frac{x}{2^{n+1}} \right)}{3} \tag{2.2}
\]

hold for all \( x \in V \) and all \( n \in \mathbb{N} \cup \{0\} \).

**Proof.** If a mapping \( f : V \to W \) satisfies \( Df(x, y) = 0 \) for all \( x, y \in V \), then the equality (2.1) can be derived from the equalities

\[
f(x) = \frac{f_0(2^n x)}{2^n} - \frac{16f_e(2^n x) + f_e(2^{n+1} x)}{12 \cdot 4^n} + \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n}
\]

\[
= \sum_{i=0}^{n-1} \frac{2f_0(2^i x) - f_e(2^{i+1} x)}{2^i 12 \cdot 4^i} + \sum_{i=0}^{n-1} \frac{64f_e(2^i x) - 20f_e(2^{i+1} x) + f_e(2^{i+2} x)}{12 \cdot 4^i}
\]

\[
+ \sum_{i=0}^{n-1} \frac{-64f_e(2^i x) - 20f_e(2^{i+1} x) + f_e(2^{i+2} x)}{12 \cdot 16^{i+1}}
\]

\[
= \sum_{i=0}^{n-1} \frac{-Df_0(2^i x, 2^{i-1} x)}{2^{i+1}} + \sum_{i=0}^{n-1} \frac{Df_e(2^i x, 2^{i+1} x) + 4Df_e(2^i x, 2^i x)}{12 \cdot 4^{i+1}} + \sum_{i=0}^{n-1} \frac{-Df_e(2^{i+1} x, 2^{i+2} x) - 4Df_e(2^i x, 2^{i+1} x)}{12 \cdot 16^{i+1}}
\]

for all \( x \in V \) and \( n \in \mathbb{N} \cup \{0\} \). The equality (2.2) can be easily obtained in a similar way.

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (1.1) in the sense of Găvruţa.

**Theorem 1.** Let \( f : V \to Y \) be a mapping for which there exists a function \( \varphi : (V \setminus \{0\})^2 \to [0, \infty) \) such that the inequality

\[
\|Df(x, y)\| \leq \varphi(x, y) \tag{2.3}
\]

holds for all \( x, y \in V \setminus \{0\} \) and let \( f(0) = 0 \). If \( \varphi \) has the property

\[
\sum_{n=0}^{\infty} \varphi(2^n x, 2^n y) < \infty \tag{2.4}
\]

for all \( x, y \in V \), then there exists a unique solution mapping \( F : V \to Y \) of the functional equation (1.1) satisfying the inequality

\[
\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left( \frac{\varphi(2^i x, 2^{i-1} x) + \varphi(2^{i+1} x, 2^i x) + 4\varphi(2^i x, 2^{i+1} x)}{12 \cdot 4^{i+1}} \right) \tag{2.5}
\]
for all \( x \in V \setminus \{0\} \), where \( \varphi_x \) is the function defined by \( \varphi_x(x, y) := \frac{\varphi(x) + \varphi(-x, y)}{2} \). In particular, \( F \) is represented by

\[
F(x) = \lim_{n \to \infty} \left( \frac{f_0(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n} \right)
\]  

(2.6)

for all \( x \in V \).

**Proof.** First, we define a set \( A := \{ f : V \to Y \mid f(0) = 0 \} \) and a mapping \( J_n : A \to A \) by

\[
J_n f(x) := \frac{f_0(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n}
\]

for \( x \in V \) and \( n \in \mathbb{N} \cup \{0\} \). Notice that

\[
\| J_n f(x) - J_{n+1} f(x) \| = \left\| \frac{f_0(2^n x)}{2^n} - \frac{f_0(2^{n+1} x)}{2^{n+1}} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{16f_e(2^{n+1} x) - f_e(2^{n+2} x)}{12 \cdot 4^{n+1}} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n} + \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^{n+1}} \right\|
\]

\[
= \left\| \frac{-Df_0(2^n x, 2^{n-1} x)}{2^{n+1}} + \frac{Df_e(2^{n+1} x, 2^n x) + 4Df_e(2^n x, 2^n x)}{12 \cdot 4^{n+1}} - \frac{Df_e(2^{n+1} x, 2^n x - 4Df_e(2^n x, 2^n x)}{12 \cdot 16^{n+1}} \right\|
\]

\[
\leq \frac{\varphi_e(2^n x, 2^{n-1} x)}{2^{n+1}} + \frac{\varphi_e(2^n x, 2^{n-1} x)}{12 \cdot 4^{n+1}} + 4\varphi_e(2^n x, 2^n x)
\]

(2.7)

for all \( x \in V \setminus \{0\} \). It follows from (2.7) that

\[
\| J_n f(x) - J_{n+m} f(x) \| \leq \sum_{i=n}^{n+m-1} \| J_i f(x) - J_{i+1} f(x) \|
\]

\[
\leq \sum_{i=n}^{n+m-1} \left( \frac{\varphi_e(2^n x, 2^{n-1} x)}{2^{n+1}} + \frac{\varphi_e(2^{n+1} x, 2^n x) + 4\varphi_e(2^n x, 2^n x)}{12 \cdot 4^{n+1}} \right)
\]

(2.8)

for all \( x \in V \setminus \{0\} \). In view of (2.4) and (2.8), the sequence \( \{ J_n f(x) \} \) is a Cauchy sequence for all \( x \in V \setminus \{0\} \). Since \( Y \) is complete and \( f(0) = 0 \), the sequence \( \{ J_n f(x) \} \) converges for all \( x \in V \). Hence, we can define a mapping \( F : V \to Y \) by

\[
F(x) := \lim_{n \to \infty} \left( \frac{f_0(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n} \right)
\]

for all \( x \in V \). Moreover, letting \( n = 0 \) and passing the limit \( m \to \infty \) in (2.8) we get the inequality (2.5). With the definition of \( F \), we easily get the equality \( DF(x, y) = 0 \) from the relations

\[
\| DF(x, y) \| = \lim_{n \to \infty} \left\| \frac{Df_0(2^n x, 2^n y)}{2^n} + \frac{16Df_e(2^n x, 2^n y) - Df_e(2^{n+1} x, 2^{n+1} y)}{12 \cdot 4^n} - \frac{4Df_e(2^n x, 2^n y) - Df_e(2^{n+1} x, 2^{n+1} y)}{12 \cdot 16^n} \right\|
\]

\[
\leq \lim_{n \to \infty} \left( \frac{\varphi_e(2^n x, 2^n y)}{2^n} + \frac{16\varphi_e(2^n x, 2^n y)}{12 \cdot 4^n} + 4\varphi_e(2^n x, 2^n y) \right)
\]

\[
= 0
\]

for all \( x, y \in V \setminus \{0\} \). From the equality \( DF(x, y) = 0 \) for all \( x, y \in V \setminus \{0\} \) and \( f(0) = 0 \), it is easy to see that \( DF(x, y) = 0 \) for all \( x, y \in V \). To prove the uniqueness of \( F \), let \( F' : V \to Y \) be another solution mapping of the functional equation (1.1) satisfying the inequality (2.5). Instead of the condition (2.5), it
is sufficient to show that there is a unique mapping $F$ satisfying the simpler condition $\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{\phi_e(2^{i+1}x, 2^{i+1}x) + \phi_e(2^{i+1}x, 2^{i+1}x) + 4\phi_e(2^{i+1}x, 2^{i+1}x)}{2^{i+1}}$. By (2.1), the equality $F'(x) = J_n F'(x)$ holds for all $n \in \mathbb{N}$. Therefore, we have

$$\|J_n f(x) - F'(x)\| = \|J_n f(x) - J_n F'(x)\| \leq \frac{\phi_e(2^n x) + 16\phi_e(2^{n+1}x) - 4\phi_e(2^n x) - 4\phi_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{\phi_e(2^n x) - 4\phi_e(2^n x)}{12 \cdot 4^n} \leq \frac{\phi_e(2^n x) - 4\phi_e(2^n x)}{12 \cdot 4^n} + \left(\frac{16}{12 \cdot 4^n} - \frac{4}{12 \cdot 16^n}\right)\|f_e(2^n x) - F' e(2^n x)\| \leq \sum_{i=0}^{\infty} \frac{\phi_e(2^{i+1}x, 2^{i+1}x) + \phi_e(2^{i+1}x, 2^{i+1}x) + 4\phi_e(2^{i+1}x, 2^{i+1}x)}{2^{i+1}} \leq 4 \sum_{i=0}^{\infty} \phi_e(2x, 2^{i+1}x) + \phi_e(2x, 2^{i+1}x) + 4\phi_e(2x, 2^{i+1}x)$$

for all $x \in V \backslash \{0\}$ and all $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \to \infty$, we conclude that $F'(x) = \lim_{n \to \infty} J_n f(x)$ for all $x \in V \backslash \{0\}$. Because $F(0) = 0 = F'(0)$, this means that the equality $F(x) = F'(x)$ holds for all $x \in V$.

In the following corollary, we obtain the hyperstability of the functional equation (1.1).

**Corollary 1.** Let $p < 0$ be a real number and $X$ be a real normed space. If $f : X \to Y$ is a mapping such that

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

(2.9)

for all $x, y \in X \backslash \{0\}$ and $f(0) = 0$, then $f : X \to Y$ satisfies the equality $Df(x, y) = 0$ for all $x, y \in X$.

**Proof.** According to Theorem 1, there is a unique solution mapping $F$ of the functional equation $DF(x, y) = 0$ such that

$$\|f(x) - F(x)\| \leq \left(1 + \frac{2^p}{2p|2 - 2p|} + \frac{2^p + 9}{12 |4 - 2p|}\right)\|x\|^p$$

for all $x \in X \backslash \{0\}$. From the equality

$$Df((2n + 1)x, nx) = Df((2n + 1)x, nx) - DF((2n + 1)x, nx) = (f - F)((3n + 1)x) + (f - F)(2f - F)((3n + 1)x) - 2(f - F)((3n + 1)x) - 2(f - F)((3n + 1)x) - 2(f - F)((3n + 1)x) + 4(f - F)((2n + 1)x) + 4(f - F)((2n + 1)x) - (f - F)(2nx) - (f - F)(2nx) + 4(f - F)(nx) + 4(f - F)(nx)$$

for all $x \in X \backslash \{0\}$ and $n \in \mathbb{N}$, we have the inequality

$$\|f(x) - F(x)\| \leq \|Df((2n + 1)x, nx)\| + \|(f - F)((3n + 1)x)\| + 2\|(f - F)((3n + 1)x)\|$$
Proof the inequality $\forall x \in V \setminus \{0\}$ and $n \in \mathbb{N}$. Since $(4n + 1)^p, (3n + 1)^p, (2n + 1)^p, (n + 1)^p + (2n)^p, n^p$ tend to 0 as $n \to \infty$ and $f(0) = F(0)$, we get $f(x) = F(x)$ for all $x \in X$. Therefore, the equality $Df(x, y) = DF(x, y) = 0$ holds for all $x, y \in X$.

\textbf{Theorem 2.} Let $f : V \to Y$ be a mapping for which there exists a function $\varphi : V^2 \to [0, \infty)$ such that the inequality

$$\|Df(x, y)\| \leq \varphi(x, y)$$

(2.10)

holds for all $x, y \in V$ and let $f(0) = 0$. If $\varphi$ has the property

$$\sum_{n=0}^{\infty} 16^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) < \infty$$

(2.11)

for all $x, y \in V$, then there exists a unique solution mapping $F : V \to Y$ of the functional equation (1.1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \sum_{n=0}^{\infty} \left( \frac{16^{n+1}}{3} \varphi e \left( \frac{x}{2^n}, \frac{x}{2^{n+2}} \right) + \left( \frac{4 \cdot 16^n}{3} + 2^n \right) \varphi e \left( \frac{x}{2^{n+1}}, \frac{x}{2^{n+2}} \right) \right)$$

(2.12)

for all $x \in V \setminus \{0\}$. In particular, $F$ is represented by

$$F(x) = \lim_{n \to \infty} \left( 2^n f_0 \left( \frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_0 \left( \frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_0 \left( \frac{x}{2^{n+1}} \right) \right)$$

(2.13)

for all $x \in V$.

\textbf{Proof.} First, we define $A := \{ f : V \to Y \mid f(0) = 0 \}$ and a mapping $I_n : A \to A$ by

$$I_n f(x) := 2^n f_0 \left( \frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_0 \left( \frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_0 \left( \frac{x}{2^{n+1}} \right)$$

for $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. Notice that

$$\|I_n f(x) - I_{n+1} f(x)\| \leq \left\| \frac{4 \cdot 16^n - 4^n}{3} f_0 \left( \frac{x}{2^n} \right) - 20 f_0 \left( \frac{x}{2^{n+1}} \right) + 64 f_0 \left( \frac{x}{2^{n+2}} \right) \right\|$$

(2.14)

$$+ 2^n \left( f_0 \left( \frac{x}{2^n} \right) - 2 f_0 \left( \frac{x}{2^{n+1}} \right) \right)$$

$$\leq \frac{4 \cdot 16^n}{3} \left\| Df_0 \left( \frac{x}{2^n}, \frac{x}{2^{n+2}} \right) + 4 Df_0 \left( \frac{x}{2^{n+1}}, \frac{x}{2^{n+2}} \right) \right\| + 2^n \left\| Df_0 \left( \frac{x}{2^n}, \frac{x}{2^{n+2}} \right) \right\|$$

$$\leq 2^n \varphi e \left( \frac{x}{2^n}, \frac{x}{2^{n+2}} \right) + \frac{4 \cdot 16^n}{3} \left\| \varphi e \left( \frac{x}{2^{n+1}}, \frac{x}{2^{n+2}} \right) + 4 \varphi e \left( \frac{x}{2^{n+2}}, \frac{x}{2^{n+2}} \right) \right\|$$

for all $x \in V \setminus \{0\}$. It follows from (2.14) that

$$\|I_n f(x) - I_{n+m} f(x)\| \leq \sum_{i=n}^{n+m-1} \left( \frac{16^{i+1}}{3} \varphi e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) + \left( \frac{4 \cdot 16^i}{3} + 2^i \right) \varphi e \left( \frac{x}{2^{i+1}}, \frac{x}{2^{i+2}} \right) \right)$$

(2.15)

for all $x \in V \setminus \{0\}$.
In view of (2.11) and (2.15), the sequence \( \{J_n f(x)\} \) is a Cauchy sequence for all \( x \in V \setminus \{0\} \). Since \( Y \) is complete and \( f(0) = 0 \), the sequence \( \{J_n f(x)\} \) converges for all \( x \in V \). Hence, we can define a mapping \( F : V \to Y \) by

\[
F(x) := \lim_{n \to \infty} \left( 2^n f_0 \left( \frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_e \left( \frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e \left( \frac{x}{2^{n+1}} \right) \right)
\]

for all \( x \in V \). Moreover, letting \( n = 0 \) and passing the limit \( n \to \infty \) in (2.15) we get the inequality (2.12). From the definition of \( F \), we easily get

\[
\|DF(x, y)\| = \lim_{n \to \infty} \|2^n Df_0 \left( \frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} Df_e \left( \frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} Df_e \left( \frac{x}{2^{n+1}} \right) \|
\]

\[
\leq \lim_{n \to \infty} \left( \|2^n Df_0 \left( \frac{x}{2^n} \right)\| + \left\| \frac{4 \cdot 16^n - 4^n}{3} Df_e \left( \frac{x}{2^n} \right) \right\| + \left\| \frac{16^{n+1} - 4^{n+2}}{3} Df_e \left( \frac{x}{2^{n+1}} \right) \right\| \right)
\]

\[
= 0
\]

for all \( x, y \in V \setminus \{0\} \), which means that \( DF(x, y) = 0 \) for all \( x, y \in V \) from the same reason in Theorem 1.

To prove the uniqueness of \( F \), let \( F^* : V \to Y \) be another solution of the functional equation (1.1) satisfying (2.12). Instead of the condition (2.12), it is sufficient to show that there is a unique mapping satisfying the simpler condition

\[
\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 16^{i+1} \left( \varphi_e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) + \varphi_e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) \right)
\]

for all \( x \in V \). By (2.2), the equality \( F^*(x) = J_n F^*(x) \) holds for all \( x \in V \) and all \( n \in \mathbb{N} \). Therefore, we have

\[
\|J_n f(x) - F^*(x)\|=\|J_n f(x) - J_n F^*(x)\|
\]

\[
\leq \sum_{i=0}^{\infty} 16^{i+1} \left( \varphi_e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) + \varphi_e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) \right)
\]

\[
+ \sum_{i=0}^{\infty} 16^{i+2} \left( \varphi_e \left( \frac{x}{2^{i+3}}, \frac{x}{2^{i+3}} \right) + \varphi_e \left( \frac{x}{2^{i+3}}, \frac{x}{2^{i+3}} \right) \right)
\]

\[
+ 16 \sum_{i=0}^{\infty} 16^{i+1} \left( \varphi_e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) + \varphi_e \left( \frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) \right)
\]

\[
\leq 17 \sum_{i=0}^{\infty} 16^i \left( \varphi_e \left( \frac{x}{2^{i+1}}, \frac{x}{2^{i+1}} \right) + \varphi_e \left( \frac{x}{2^{i+1}}, \frac{x}{2^{i+1}} \right) \right)
\]

for all \( x \in V \) and all \( n \in \mathbb{N} \). Taking the limit in the above inequality as \( n \to \infty \), we can conclude that

\[
F'(x) = \lim_{n \to \infty} J_n f(x) \quad \text{for all} \quad x \in V.
\]

This means that \( F(x) = F'(x) \) for all \( x \in V \).
3 Conclusions

We have proved the stability of an additive-quadratic-quartic functional equation

\[ f(x + 2y) + f(x - 2y) - 2f(x + y) - 2f(-x - y) - 2f(x - y) - 2f(y - x) \\
+ 4f(-x) + 2f(x) + f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0 \]

by the direct method in the sense of Găvruta.

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