Two-scale homogenization of piezoelectric perforated structures

Houari Mechkour

Abstract

We are interested in the homogenization of elastic-electric coupling equation, with rapidly oscillating coefficients, in periodically perforated piezoelectric body. We justify the two first terms in the usual asymptotic development of the problem solution. For the main convergence results of this paper, we use the notion of two-scale convergence. A two-scale homogenized system is obtained as the limit of the periodic problem. While in the static limit the method provides homogenized electroelastic coefficients which coincide with those deduced from other homogenization techniques (asymptotic homogenization [5], Γ-convergence [16]).

Key words. Homogenization; Piezoelectricity; Perforations

1 Introduction

Composites and perforated (lattice) materials are widely used in many practical applications, such as aircraft, civil engineering, electrotechnics, and many others. These materials are with a large number of heterogeneities (inclusions or holes), and in strong contrast to continuum materials, their behavior is definitively influenced by micromechanical events.

The first goal of this work we study the homogenization of the equation of the elastic-electric coupling with rapidly oscillating coefficients in a periodically perforated domain. The homogenized of this problem for a fixed domain has already been studied, by the author (Feng and Wu [9], Castillero and all. [5], Ruan and all. [14]). But in this work we give new convergence results concerning the same model by using homogenization technique of “two-scale convergence”, which permits us to conclude the limit problem, the approximation of final state is altered by a constant named as the volum fraction which depends on the proportion of material in the perforated domain and is equal to 1 when there are no holes.

*Centre de Mathématiques Appliquées (UMR 7641) École Polytechnique, 91128 Palaiseau, France. (mechkour@cmap.polytechnique.fr).
The second goal of this paper is to apply the technique of a formal asymptotic homogenization, to determine the effective elastic, piezoelectric and dielectric moduluses of periodic medium. The final formulae for the effective parameters are given in a relatively simpler closed form.

The third goal of this paper is to also establish a corrector-type theorem, which permits to replace the sequence by its “two-scale” limit using the result on the strong convergence, and permits to justify the two first terms in the usual asymptotic expansion of the solution. In the last section we treat of the energy aspect of our problem.

2 Homogenization problem

Throughout this paper $L^2(\Omega)$ in the Sobolev space of real-valued functions that are measurable and square summable in $\Omega$ with respect of the Lebesgue measure. We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^3$ that are periodic of $Y$. Then, $L^2_\#(Y)$ (respectively, $H^1_\#(Y)$) is the completion for the norm of $L^2(Y)$ (respectively, $H^1(Y)$) of $C^\infty_\#(Y)$.

2.1 Geometric of the medium

Let $\Omega \subset \mathbb{R}^3$ be a bounded three dimensional domain with the boundary $\Gamma = \partial \Omega$. We denote $x$ the macroscopic variable and by $y = \frac{x}{\varepsilon}$ the microscopic variable. Let us define $\Omega_\varepsilon$ of periodically perforated subdomains of a bounded open set $\Omega$. The period of $\Omega_\varepsilon$ is $\varepsilon Y^*$, where $Y^*$ is a subset of the unit cube $Y = (0,1)^3$, which represents the solid or material domain, $S^*$ obtained by $Y$-periodicity from $Y^*$, is a smooth connected (the material is in one piece) open set in $\mathbb{R}^3$. Denoting by $\chi(y)$ the characteristic function of $S^*$ ($Y$-periodic), in $\Omega_\varepsilon$ well be defined analytically by

$$\Omega_\varepsilon = \{ x \in \Omega, \chi(y) = 1 \}$$

2.2 Model problem

We adopt the convention of Einstein for the summation of repeated indices, we use Latin indices, understood from 1 to 3, we note by $u^\varepsilon$ the fields of displacement in elastic, and by $\varphi^\varepsilon$ of electric potential. The equations of equilibrium and Gauss’s law of electrostatics in the absence of free charges, written as

$$\begin{cases}
-\text{div} \sigma^\varepsilon(u^\varepsilon, \varphi^\varepsilon) = f \text{ in } \Omega_\varepsilon, \\
-\text{div} D^\varepsilon(u^\varepsilon, \varphi^\varepsilon) = 0 \text{ in } \Omega_\varepsilon,
\end{cases}$$

we complete the boundary conditions,

$$\begin{cases}
(u^\varepsilon, \varphi^\varepsilon) = (0,0) \text{ on } \partial \Omega, \\
\sigma^\varepsilon(u^\varepsilon, \varphi^\varepsilon).n^\varepsilon = 0 \text{ on the boundary of holes } \partial \Omega_\varepsilon - \partial \Omega, \\
D^\varepsilon(u^\varepsilon, \varphi^\varepsilon).n^\varepsilon = 0 \text{ on the boundary of holes } \partial \Omega_\varepsilon - \partial \Omega,
\end{cases}$$

2
where \( f \in L^2(\Omega_\varepsilon) \). The second-order stress tensor \( \sigma^\varepsilon = (\sigma^\varepsilon_{ij}) \), and the electric displacement vector \( \mathbf{D}^\varepsilon = (D^\varepsilon_i) \), are linearly related to the second-order strain tensor \( s_{kl}(\mathbf{u}) = \frac{1}{2}(\partial_k \mathbf{u}_l + \partial_l \mathbf{u}_k) \) and the electric field vector \( \partial_k \varphi^\varepsilon \) by the constitutive law

\[
\begin{align*}
\sigma^\varepsilon_{ij}(\mathbf{u}^\varepsilon, \varphi^\varepsilon) &= e^\varepsilon_{ijkl} s_{kl}(\mathbf{u}^\varepsilon) + e^\varepsilon_{ij} \partial_k \varphi^\varepsilon \quad \text{in } \Omega_\varepsilon, \\
D^\varepsilon_i(\mathbf{u}^\varepsilon, \varphi^\varepsilon) &= -e^\varepsilon_{kl} s_{kl}(\mathbf{u}^\varepsilon) + d^\varepsilon_{ij} \partial_j \varphi^\varepsilon \quad \text{in } \Omega_\varepsilon.
\end{align*}
\]

(3)

where \( (\text{div } \sigma^\varepsilon)^j = \partial_i \sigma^\varepsilon_{ij}, \text{ div } \mathbf{D}^\varepsilon = \partial_i D^\varepsilon_i, \partial_i = \frac{\partial}{\partial x_i}, x = (x_i) \in \Omega \). And the material properties are given by the fourth-order stiffness tensor \( c^\varepsilon_{ijkl} \) measured at constant electric field, the elastic coefficients satisfy the following symmetries and ellipticity uniformly in \( \varepsilon \),

\[
\begin{cases}
c^\varepsilon_{ijkl}(x) = c_{ijkl}(x, \frac{x}{\varepsilon}), \\
c^\varepsilon_{ijkl} = c^\varepsilon_{ikjl} = c^\varepsilon_{jikl} = c^\varepsilon_{ijlk}, \\
c^\varepsilon_{ijkl}(x, y) \in L^\infty(\Omega; C^1(\mathbb{Y})), \\
\exists \alpha_\varepsilon \neq \alpha_0(\varepsilon) > 0 : c^\varepsilon_{ijkl} X_{ij} X_{kl} \geq \alpha_\varepsilon X_{ij} X_{ij}, \quad \forall X_{ij} = X_{ji} \in \mathbb{R}.
\end{cases}
\]

(4)

The third-order piezoelectric tensor \( e^\varepsilon_{ijk} \) (the coupled tensor), verify the following symmetry,

\[
\begin{cases}
e^\varepsilon_{ijk} = e_{ijk}(x, \frac{x}{\varepsilon}), \\
e^\varepsilon_{ijk} = e^\varepsilon_{ik,j}, \\
e_{ijk}(x, y) \in L^\infty(\Omega; C^1(\mathbb{Y})).
\end{cases}
\]

(5)

The second-order electric tensor \( d^\varepsilon_{ij} \) (dielectric permittivity), measured at constant strain, verify the conditions of symmetric and ellipticity uniformly by \( \varepsilon \),

\[
\begin{cases}
d^\varepsilon_{ij} = d_{ij}(x, \frac{x}{\varepsilon}), \\
d^\varepsilon_{ij} = d^\varepsilon_{ji}, \\
d_{ij}(x, y) \in L^\infty(\Omega; C^1(\mathbb{Y})), \\
\exists \alpha_d \neq \alpha_0(\varepsilon) > 0 : d^\varepsilon_{ij} X_i X_j \geq \alpha_d X_i X_i, \quad \forall X_i \in \mathbb{R}.
\end{cases}
\]

(6)

### 2.3 Variational problem

Introducing the two Hilbert spaces

\[
\begin{align*}
\mathbf{V}_\varepsilon(\Omega_\varepsilon) &= \{ \mathbf{v} \in H^1(\Omega_\varepsilon), \mathbf{v} = 0 \text{ on } \partial\Omega \}, \\
W_\varepsilon(\Omega_\varepsilon) &= \{ \psi \in H^1(\Omega_\varepsilon), \psi = 0 \text{ on } \partial\Omega \}.
\end{align*}
\]

With two norms : \( \| \cdot \|_{\mathbf{V}_\varepsilon(\Omega_\varepsilon)} = \| \cdot \|_{H^1(\Omega_\varepsilon)}, \| \cdot \|_{W_\varepsilon(\Omega_\varepsilon)} = \| \cdot \|_{H^1(\Omega_\varepsilon)}. \) The variational problem is defined by:

\[
\begin{cases}
\text{Find } (\mathbf{u}^\varepsilon, \varphi^\varepsilon) \in \mathbf{V}_\varepsilon(\Omega_\varepsilon) \times W_\varepsilon(\Omega_\varepsilon), \text{ such that } \\
a_\varepsilon((\mathbf{u}^\varepsilon, \varphi^\varepsilon), (\mathbf{v}, \psi)) = L_\varepsilon(\mathbf{v}, \psi) \quad \forall (\mathbf{v}, \psi) \in \mathbf{V}_\varepsilon(\Omega_\varepsilon) \times W_\varepsilon(\Omega_\varepsilon),
\end{cases}
\]

(7)
where

\[
\begin{aligned}
o_{\varepsilon}(\{u^\varepsilon, \varphi^\varepsilon\}, \{v, \psi\}) &= \int_{\Omega_\varepsilon} \left\{ [c_{ijkl}^\varepsilon s_{kl}(u^\varepsilon) + e_{ijkl}^\varepsilon \partial_k \varphi^\varepsilon] s_{ij}(v) + [-e_{ijkl}^\varepsilon s_{kl}(u^\varepsilon) + d_{ijkl}^\varepsilon \partial_j \varphi^\varepsilon] \partial_i \psi \right\} \, dx \\
L_{\varepsilon}(v, \psi) &= \int_{\Omega_\varepsilon} f_i v_i \, dx
\end{aligned}
\]

It is pointed out that under assumptions (4)-(5)-(6), the variational problem (7)-(8) have a unique solution \((u^\varepsilon, \varphi^\varepsilon) \in V_{\varepsilon}(\Omega_\varepsilon) \times W_{\varepsilon}(\Omega_\varepsilon)\), corresponding the saddle point of this functional (see [12]) :

\[
(v, \psi) \rightarrow \frac{1}{2} \int_{\Omega_\varepsilon} (c^\varepsilon(v, v) + 2e^\varepsilon(u, \psi) - d^\varepsilon(\psi, \psi)) \, dx - \int_{\Omega_\varepsilon} f \, v \, dx,
\]

where

\[
\begin{aligned}
c^\varepsilon(u, v) &= c_{ijkl}^\varepsilon s_{ij}(u) \, s_{kl}(v) \\
e^\varepsilon(u, \psi) &= e_{ijkl}^\varepsilon s_{kl}(u) \, \partial_i \psi \\
d^\varepsilon(\psi, \psi) &= d_{ijkl}^\varepsilon \, \partial_j \psi \, \partial_i \psi
\end{aligned}
\]

### 2.4 A priori estimates

In order to prove the main convergence results of this paper we use the notion of two-scale convergence which was introduced in [10] and developed further in [1]. The idea of this convergence is based in first step by taking a priori estimates for displacement field and the electric potentiel. The second step we use the relatively compact property with the classical procedure of prolongation (wich is the extension by 0 from \(\Omega_\varepsilon\) to \(\Omega\)). Finally we pass the limit \(\varepsilon \to 0\), in order to obtain the homogenized and the local problems in same time.

**Proposition 1** By using the two equivalent norms of \(V_{\varepsilon}(\Omega_\varepsilon)\), \(W_{\varepsilon}(\Omega_\varepsilon)\), for any sequence of solution \((u^\varepsilon, \varphi^\varepsilon) \in V_{\varepsilon}(\Omega_\varepsilon) \times W_{\varepsilon}(\Omega_\varepsilon)\) of variational problem (7)-(8). Then, this solution is bounded, and we have this a priori estimate uniformly by \(\varepsilon\)

\[
\| u^\varepsilon \|_{H^1(\Omega_\varepsilon)} + \| \varphi^\varepsilon \|_{H^1(\Omega_\varepsilon)} \leq C,
\]

where \(C\) is constant strictement positive and independent by \(\varepsilon\).

**Proof.**

By choosing \(v = u^\varepsilon\) and \(\psi = \varphi^\varepsilon\) in variational formulae (7)-(8), and by using the Korn’s and Poincaré’s inequalities in perforated domains (see Oleinik et al. [13] for the Korn’s inequality and Allaire-Murat [2] for Poincaré’s inequality), we see that \(u^\varepsilon\) and \(\varphi^\varepsilon\) are bounded, by a constant which does not depend on \(\varepsilon\). For other details see [12].
3 Two-scale convergence

We denote by \( \tilde{\cdot} \) the extension by zero in the holes \( \Omega - \Omega_\varepsilon \). The sequence of solution \((u_\varepsilon, \varphi_\varepsilon) \subset V_\varepsilon(\Omega_\varepsilon) \times W_\varepsilon(\Omega_\varepsilon)\) of variational problem (7)-(8) verify (9) and, in this case, by adding the relatively compact property and elementary properties of two-scale convergence, imply

**Lemma 1**

1. There exists \( u(x) \in H^1_0(\Omega) \) and \( \varphi(x) \in H^1(\Omega) \) such that, the two sequences \((\tilde{u}_\varepsilon)_{\varepsilon}, (\tilde{\varphi}_\varepsilon)_{\varepsilon}\) two-scale converge to \( \chi(y)u(x), \chi(y)\varphi(x) \), respectively.

2. There exists \( u_1(x,y) \in L^2[\Omega;H^1_0(Y^*)/\mathbb{R}], \varphi_1(x,y) \in L^2[\Omega;H^1_0(Y^*)/\mathbb{R}] \) such that,

\[
\tilde{\nabla} u_\varepsilon \to \chi(y)[\nabla_x u(x) + \nabla_y u_1(x,y)] \quad \text{in two-scale sense}
\]

\[
\tilde{\nabla} \varphi_\varepsilon \to \chi(y)[\nabla_x \varphi(x) + \nabla_y \varphi_1(x,y)] \quad \text{in two-scale sense}
\]

3. We have

\[
\tilde{s}(u_\varepsilon) \to \chi(y)[s_x(u(x)) + s_y(u_1(x,y))] \quad \text{in two-scale sense}
\]

index \( x \) or \( y \) means that the derivatives are with respect to the variable.

**Proof.** For details see [1], [10], [11] and [12].

**Corollary 1** The sequence \((\tilde{u}_\varepsilon)_{\varepsilon>0}\) (resp. \((\tilde{\varphi}_\varepsilon)_{\varepsilon>0}\)) converge weakly to a limit \( \theta u \) (resp. \( \theta \varphi \)) in \( L^2(\Omega) \) (resp. \( L^2(\Omega) \)).

**Remark 1** Let \( \rho \in L^2(Y) \), define \( \rho^\varepsilon(x) = \rho(\frac{x}{\varepsilon}) \), and \((v^\varepsilon)_{\varepsilon} \subset L^2(\Omega)\) two-scale converge to a limit \( v \in L^2(\Omega \times Y) \). Then \((\rho^\varepsilon v^\varepsilon)_{\varepsilon} \) two-scale converges to a limit \( \rho v \) (see [12]).

From last results, we can state next theorem

**Theorem 1** The sequences \((u_\varepsilon)_{\varepsilon}, (\tilde{s}(u_\varepsilon))_{\varepsilon}, (\tilde{\nabla} u_\varepsilon)_{\varepsilon}, (\tilde{\nabla} \varphi_\varepsilon)_{\varepsilon}\) two-scale converge to \( \chi(y)u(x), \chi(y)[s_x(u) + s_y(u_1)], \chi(y)\varphi(x) \) and \( \chi(y)[\nabla_x \varphi + \nabla_y \varphi_1] \) respectively, where \((u(x), u_1(x,y), \varphi(x), \varphi_1(x,y))\) are the unique solutions in \( H_0^1(\Omega) \times L^2[\Omega;H^1_0(Y^*)/\mathbb{R}] \times H_0^1(\Omega) \times L^2[\Omega;H^1_0(Y^*)/\mathbb{R}] \) of the following two-
scale homogenized system;

\[
\begin{aligned}
\frac{\partial}{\partial x_j} \left[ \int_{Y^*} \left\{ c_{ijkl}(x,y) [s_{kl,x}(u) + s_{kl,y}(u_1)] + e_{ki,j}(x,y) [\partial_{k,x}\varphi + \partial_{k,y}\varphi_1] \right\} dy \right] = \theta f_i(x) \quad \text{in } \Omega, \\
\frac{\partial}{\partial x_i} \left[ \int_{Y^*} \left\{ -e_{ikl}(x,y) [s_{kl,x}(u) + s_{kl,y}(u_1)] + d_{ij}(x,y) [\partial_{j,x}\varphi + \partial_{j,y}\varphi_1] \right\} dy \right] = 0 \quad \text{in } \Omega, \\
\frac{\partial}{\partial y_j} \left\{ c_{ijkl}(x,y) [s_{kl,x}(u) + s_{kl,y}(u_1)] + e_{ki,j}(x,y) [\partial_{k,x}\varphi + \partial_{k,y}\varphi_1] \right\} = 0 \quad \text{in } \Omega \times Y^*, \\
\frac{\partial}{\partial y_i} \left\{ -e_{ikl}(x,y) [s_{kl,x}(u) + s_{kl,y}(u_1)] + d_{ij}(x,y) [\partial_{j,x}\varphi + \partial_{j,y}\varphi_1] \right\} = 0 \quad \text{in } \Omega \times Y^*, 
\end{aligned}
\]

(10)

and we have this boundary conditions

\[
\begin{aligned}
\begin{cases}
  u(x) & = 0 \quad \text{on } \partial \Omega, \\
  \varphi(x) & = 0 \quad \text{on } \partial \Omega, \\
  \{ c_{ijkl}(x,y) [s_{kl,x}(u) + s_{kl,y}(u_1)] + e_{ki,j}(x,y) [\partial_{k,x}\varphi + \partial_{k,y}\varphi_1] \} n_j & = 0 \quad \text{on } \partial Y^* - \partial Y, \\
  \{ -e_{ikl}(x,y) [s_{kl,x}(u) + s_{kl,y}(u_1)] + d_{ij}(x,y) [\partial_{j,x}\varphi + \partial_{j,y}\varphi_1] \} n_i & = 0 \quad \text{on } \partial Y^* - \partial Y.
\end{cases}
\end{aligned}
\]

(11)

\[
\begin{cases}
  y \rightarrow u_1(x,y) & \text{is } Y \text{ - periodic}, \\
  y \rightarrow \varphi_1(x,y) & \text{is } Y \text{ - periodic},
\end{cases}
\]

(12)

where \( \theta \) is the volum fraction of material (i.e. \( \theta = \langle \chi \rangle = \int_Y \chi(y) \, dy = |Y^*| \)), \( |Y| \) denote mesure of \( Y \).

The equations (10) - (11) - (12) are referred to as the two-scale homogenized system.

**Proof.** From the idea of G.Nguetseng [10], the test functions in (7)-(8) are chossed on the form

\[
\begin{aligned}
  v^e(x) &= v(x, \frac{x}{\varepsilon}) = v^0(x) + \varepsilon v^1(x, \frac{x}{\varepsilon}), \\
  \psi^e(x) &= \psi(x, \frac{x}{\varepsilon}) = \psi^0(x) + \varepsilon \psi^1(x, \frac{x}{\varepsilon}),
\end{aligned}
\]

\[6\]
where \( \mathbf{v}^0 \in C^\infty_0(\Omega) \), \( \psi^0 \in C^\infty_0(\Omega) \), \( \mathbf{v}^1 \in C^\infty_0(\Omega; C^\infty_0(Y)) \) and \( \psi^1 \in C^\infty_0(\Omega; C^\infty_0(Y)) \), we obtain

\[
\int_{\Omega^*} \left\{ \left[s_{ijkl}(\mathbf{u}^x) + e_{ijkl} \partial_y \varphi\right][s_{ijkl}(\mathbf{v}^0)(x) + \{s_{ijkl}(\mathbf{v}^1) + \varepsilon s_{ijkl}(\mathbf{v}^1)\}(x, \frac{x}{\varepsilon})] \right. \\
- \left[-\varepsilon_{ijkl} s_{ijkl}(\mathbf{u}^x) + d_{ijkl} \partial_y \varphi\right][\partial_i \psi^0 + \{\partial_i \psi^1 + \varepsilon \partial_i \psi^1\}(x, \frac{x}{\varepsilon})] \} \ dx \\
= \int_{\Omega^*} f_i(x) [\varepsilon \psi^0_i(x) + \varepsilon \psi^1_i(x, \frac{x}{\varepsilon})] \ dx
\]

Under the precedent hypotheses, and passing to the two-scale limit, yields

\[
\int_{\Omega^*} \int_{Y^*} \left[ c_{ijkl}(x, y) \chi(y)(s_{ijkl}(\mathbf{u}) + s_{ijkl}(\mathbf{u}_1)) + e_{ijkl}(x, y) \chi(y)(\partial_y \varphi + \partial_y \varphi_1)\right] \chi(y) \\
[s_{ijkl}(\mathbf{v}^0) + s_{ijkl}(\mathbf{v}^1)] \ dx \ dy \\
- \int_{\Omega^*} \int_{Y^*} \left[-\varepsilon_{ijkl}(x, y) \chi(y)(s_{ijkl}(\mathbf{u}) + s_{ijkl}(\mathbf{u}_1)) + d_{ijkl}(x, y) \chi(y)(\partial_y \varphi + \partial_y \varphi_1)\right] \chi(y) \\
[\partial_i \psi^0 + \partial_i \psi^1] \ dx \ dy \\
= \int_{\Omega^*} \int_{Y^*} f_i(x) \chi(y) \psi^0_i(x) \ dx \ dy.
\]

By definition of \( \chi \), we have

\[
\int_{\Omega^*} \int_{Y^*} \left[ c_{ijkl}(x, y)(s_{ijkl}(\mathbf{u}) + s_{ijkl}(\mathbf{u}_1)) + e_{ijkl}(x, y)(\partial_y \varphi + \partial_y \varphi_1)\right] \\
[s_{ijkl}(\mathbf{v}^0) + s_{ijkl}(\mathbf{v}^1)] \ dx \ dy \\
+ \int_{\Omega^*} \int_{Y^*} \left[-\varepsilon_{ijkl}(x, y)(s_{ijkl}(\mathbf{u}) + s_{ijkl}(\mathbf{u}_1)) + d_{ijkl}(x, y)(\partial_y \varphi + \partial_y \varphi_1)\right] \\
[\partial_i \psi^0 + \partial_i \psi^1] \ dx \ dy \\
= \theta \int_{\Omega^*} f_i(x) \psi_i(x) \ dx,
\]

where \( \theta = \int_Y \chi(y) \ dy \), by density of spaces from which we chose the test functions, the equation (13) holds true for any \( \mathbf{v}^0 \in H^1_0(\Omega) \), \( \psi^0 \in H^1_0(\Omega) \), and for any \( \mathbf{v}^1 \in L^2[\Omega; H^1_0(Y^*)/\mathbb{R}] \), \( \psi^1 \in L^2[\Omega; H^1_0(Y^*)/\mathbb{R}] \).

Integrating by parts, shows that (14) is variational formulation associated to the two-scale homogenized system

\[
\begin{align*}
-\partial_j \left[ \int_{Y^*} \left\{ c_{ijkl}(x, y)[s_{ijkl}(\mathbf{u}) + s_{ijkl}(\mathbf{u}_1)] + e_{ijkl}(x, y)[\partial_y \varphi + \partial_y \varphi_1]\right\} \ dx \right] &= \theta f_i(x) \\
-\partial_i \left[ \int_{Y^*} \left\{ -\varepsilon_{ijkl}(x, y)[s_{ijkl}(\mathbf{u}) + s_{ijkl}(\mathbf{u}_1)] + d_{ijkl}(x, y)[\partial_y \varphi + \partial_y \varphi_1]\right\} \ dx \right] &= 0
\end{align*}
\]

We complete (15) by the boundary conditions (11)-(12). To prove existence and uniqueness in (14), by application of the Lax-Milgram lemma, let focus on
the coercivity in $H^1_0(\Omega) \times L^2(\Omega; H^1_0(\Omega)) \times L^2(\Omega; H^1_0(\Omega))$ of the bilinear form defined by the left-hand side of (14) (For a complete demonstration see [12]).

**Remark 2** It is evident that the two-scale homogenized problem (10)-(11)-(12) is a system of four equation, four unknown $(u, u_1, \varphi, \varphi_1)$, each dependent on both space variables $x$ and $y$ (i.e. the macroscopic and microscopic scales) which are mixed. Although seems to be complicated, it is well-posed system of equations. Also it is clear that the two-scale homogenized problem has the same form as the original equation.

The object of new paragraph is to give another form of theorem which is more suitable for further physical interpretations. Indeed, we shall eliminate the microscopic variable $y$ (one doesn’t want to solve the small scale structure), and decouple the two-scale homogenized problem (10)-(11)-(12) in homogenized and cell equations. However, it is preferable, from a physical or numerical point of view (see [12]).

### 4 Derivation of the homogenized coefficients

Due to the linearity of the original problem, and assuming the regularity in variation of the coefficients, we take

$$u_1(x, y) = s_{mh,x}(u(x))w^{mh}(y) + \frac{\partial \varphi(x)}{\partial x_n} q^n(y), \quad (16)$$

$$\varphi_1(x, y) = s_{mh,x}(u(x))\varphi^{mh}(y) + \frac{\partial \varphi(x)}{\partial x_n} \psi^n(y), \quad (17)$$

where $w^{mh}, \varphi^n, q^{mh}$ and $\psi^n$ are $Y^*$-periodic functions in $y$, independent of $x$, solutions of these two locals problems in $Y^*$

$$\begin{cases} 
- \frac{\partial}{\partial y_j} \left\{ e_{ijkl}(x, y) \left[ \tau_{kl}^{mh} + s_{kl,y}(w^{mh}) \right] + e_{kij}(x, y) \frac{\partial \varphi^{mh}}{\partial y_k} \right\} = 0 \text{ in } Y^*, \\
- \frac{\partial}{\partial y_i} \left\{ -e_{ikl}(x, y) \left[ \tau_{kl}^{mh} + s_{kl,y}(w^{mh}) \right] + d_{ij}(x, y) \frac{\partial \varphi^{mh}}{\partial y_j} \right\} = 0 \text{ in } Y^*, \\
w^{mh}, \varphi^{mh} \quad Y^*-\text{periodics,} 
\end{cases} \quad (18)$$

where

$$\tau_{kl}^{mh} = \frac{1}{2} [\delta_{km} \delta_{lh} + \delta_{kh} \delta_{lm}] \quad 1 \leq k, m, l, h \leq 3.$$
From the relation (20)-(21), after lengthy calculations we arrive at the homogenized (effective) coefficients:

$$
\begin{align*}
- \frac{\partial}{\partial y_j} \left\{ c_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + e_{kj}(x,y) \left[ \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right] \right\} & = 0 \text{ in } Y^*, \\
\frac{\partial}{\partial y_i} \left\{ - e_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + d_{ij}(x,y) \left[ \delta_{jn} + \frac{\partial \psi^n}{\partial y_j} \right] \right\} & = 0 \text{ in } Y^*,
\end{align*}
$$

(19)

Calling \( \tau_{mh} \) the basic of symmetric second order tensors \( \tau_{mh} = \frac{1}{2} \left[ \delta_{kn} \delta_{lh} + \delta_{kl} \delta_{im} \right] \), where \( \delta_{ij} \) is the Kronecker symbol.

Analogously, we substitute the expansions (16) and (17) in this equation

$$
- \frac{\partial}{\partial y_j} \left\{ - e_{ijkl}(x,y)s_{kl,x}(\mathbf{u}) + k_{ijkl}(x,y) \left[ \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right] \right\} = 0 \text{ in } \Omega \times Y^*,
$$

we obtain

$$
\begin{align*}
- \frac{\partial \delta_{mn,xy}(\mathbf{u})}{\partial y_j} \left\{ c_{ijkl}(x,y) \left[ \tau_{mh}^{kl} + s_{kl,y}(\mathbf{w}^m) \right] + e_{kj}(x,y) \left[ \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right] \right\} & \\
+ \frac{\partial \phi}{\partial x_n} \left\{ c_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + e_{kj}(x,y) \left[ \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right] \right\} & = 0.
\end{align*}
$$

(20)

Calling \( \tau_{mh} \) the basic of symmetric second order tensors \( \tau_{mh} = \frac{1}{2} \left[ \delta_{kn} \delta_{lh} + \delta_{kl} \delta_{im} \right] \), where \( \delta_{ij} \) is the Kronecker symbol.

Analogously, we substitute the expansions (16) and (17) in this equation

$$
- \frac{\partial}{\partial y_i} \left\{ - e_{ijkl}(x,y)s_{kl,x}(\mathbf{u}) + d_{ij}(x,y) \left[ \delta_{jn} + \frac{\partial \psi^n}{\partial y_j} \right] \right\} = 0 \text{ in } \Omega \times Y^*,
$$

we obtain

$$
\begin{align*}
- \frac{\partial \delta_{mn,xy}(\mathbf{u})}{\partial y_i} \left\{ - e_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + d_{ij}(x,y) \left[ \delta_{jn} + \frac{\partial \psi^n}{\partial y_j} \right] \right\} & \\
+ \frac{\partial \phi}{\partial x_n} \left\{ - e_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + d_{ij}(x,y) \left[ \delta_{jn} + \frac{\partial \psi^n}{\partial y_j} \right] \right\} & = 0.
\end{align*}
$$

(21)

From the relation (20)-(21), after lengthy calculations we arrive at the homogenized (effective) coefficients:

$$
\begin{align*}
c_{ijkl}^H & = \left\langle c_{ijkl}(x,y) \left[ \tau_{mh}^{kl} + s_{kl,y}(\mathbf{w}^m) \right] + e_{kj}(x,y) \left[ \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right] \right\rangle, \\
e_{ijkl}^H & = \left\langle c_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + e_{kj}(x,y) \left[ \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right] \right\rangle, \\
f_{ijkl}^H & = \left\langle e_{ijkl}(x,y) \left[ \tau_{mh}^{kl} + s_{kl,y}(\mathbf{w}^m) \right] - d_{ij}(x,y) \left[ \delta_{jn} + \frac{\partial \psi^n}{\partial y_j} \right] \right\rangle, \\
d_{ijkl}^H & = \left\langle - e_{ijkl}(x,y)s_{kl,y}(\mathbf{q}^n) + d_{ij}(x,y) \left[ \delta_{jn} + \frac{\partial \psi^n}{\partial y_j} \right] \right\rangle.
\end{align*}
$$

(22)-(25)
where $\langle h \rangle = \int_Y h(y) \, dy$, the measurements on $Y^*$ of function $h$.

Now we give the results concerning some properties of elasticity homogenized tensor.

**Proposition 2** The coefficients of elasticity homogenized tensor $C^H = (c^H_{ijkl})$ defined by (22) satisfy:

a) $c^H_{ijkl} = c^H_{klij} = c^H_{ijlk} = c^H_{jilk}$, $\forall 1 \leq i, j, k, l \leq 3$,

b) There exists $\alpha^H_c > 0$, such that for all $\xi$, symmetric tensor ($\xi_{ij} = \xi_{ji}$),

$$c^H_{ijkl} \xi_{ij} \xi_{kl} \geq \alpha^H_c \xi_{ij} \xi_{ij}$$

**Proof.**

The part of the symmetry of these coefficients is evident $c^H_{ijmh} = c^H_{mijn} = c^H_{mijn}$.

Following the ideas, we transform the above expression to obtain a symmetric form.

We defined the tensor of second order $\Sigma$, by $\Sigma_{kl} = \frac{1}{2}(y_k \epsilon_l + y_l \epsilon_k)$, we define $3 \times 3$ $H^k$ matrix by $H^k = s_y(\Sigma^k)$, it is evident the coefficients of this matrix its defined by

$$[H^k]_{mh} = s^k_{mh} = \frac{1}{2} \left[ \delta_{km} \delta_{lh} + \delta_{kh} \delta_{lm} \right] \quad 1 \leq k, l, m, h \leq 3.$$ 

If we use this new notation, we can rewrite the problem (18), as the form given under

$$\begin{align*}
- \frac{\partial}{\partial y_j} \left\{ c_{ijkl}(x,y) s_{kl,y} (\Sigma^m + w^m) + e_{kij}(x,y) \frac{\partial \varphi^m}{\partial y_k} \right\} &= 0 \text{ in } Y^*, \\
- \frac{\partial}{\partial y_i} \left\{ - e_{ikl}(x,y) s_{kl,y} (\Sigma^m + w^m) + d_{ij}(x,y) \frac{\partial \varphi^m}{\partial y_j} \right\} &= 0 \text{ in } Y^*, \\
w^m, \varphi^m &\quad Y^* - \text{periodics.}
\end{align*}$$

We introduce the problem functions $(w^{ij}, q^{ij})$, solutions of the problem

$$\begin{align*}
- \frac{\partial}{\partial y_j} \left\{ c_{k\alpha\beta}(x,y) s_{\alpha\beta,y} (\Sigma^{ij} + w^{ij}) + e_{kij}(x,y) \frac{\partial \varphi^{ij}}{\partial y_k} \right\} &= 0 \text{ in } Y^*, \\
- \frac{\partial}{\partial y_k} \left\{ - e_{k\alpha\beta}(x,y) s_{\alpha\beta,y} (\Sigma^{ij} + w^{ij}) + d_{kj}(x,y) \frac{\partial \varphi^{ij}}{\partial y_j} \right\} &= 0 \text{ in } Y^*, \\
w^{ij}, \varphi^{ij} &\quad Y^* - \text{periodics.}
\end{align*}$$
The coefficient of elasticity tensor can be rewritten as

\[ c_{Hijm} = \int_{Y'} c_{ijkl}(x,y) s_{kl,y} \left( \Sigma_{mh} + w_{mh} \right) dy + \int_{Y'} e_{kij}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} dy. \]  

(28)

The second integral of the right-hand side of precedent expression, is evaluated as follows

\[ \int_{Y'} e_{kij}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} dy = \int_{Y'} e_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} \delta_{alpha} \delta_{beta} dy, \]

\[ = \frac{1}{2} \int_{Y'} e_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} \left( \delta_{alpha} \delta_{beta} + \delta_{alpha} \delta_{beta} \right) dy, \]

\[ = - \int_{Y'} e_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} s_{alpha,y}(w_{ij}) dy \]

\[ + \int_{Y'} e_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} s_{alpha,y}(\Sigma_{ij} + w_{ij}) dy. \]  

(29)

We use the variationnal formulation of the first equation of problem \(26\), and taking the test function \( v = w_{ij} \), we obtain

\[ \int_{Y'} e_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} s_{alpha,y}(w_{ij}) dy = \int_{Y'} e_{kalpha}(x,y) s_{alpha,y}(\Sigma_{ij} + w_{ij}) dy. \]  

(30)

Multiplying the second equation by \( \varphi_{mh} \), and integrating by parts, we have

\[ \int_{Y'} e_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} s_{alpha,y}(\Sigma_{ij} + w_{ij}) dy = \int_{Y'} d_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} \frac{\partial \varphi_{ij}}{\partial y_alpha} dy. \]  

(31)

Regrouping these results, and using the definition \(28\), we derive

\[ c_{ijm} = \int_{Y'} c_{ijkl}(x,y) s_{kl,y} \left( \Sigma_{mh} + w_{mh} \right) dy + \int_{Y'} e_{kij}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} dy, \]

\[ = \int_{Y'} c_{ijkl}(x,y) s_{kl,y} \left( \Sigma_{mh} + w_{mh} \right) dy \]

\[ + \int_{Y'} c_{alpha}(x,y) s_{alpha,y}(\Sigma_{ij} + w_{ij}) dy, \]

\[ = \int_{Y'} c_{alpha}(x,y) s_{alpha,y}(\Sigma_{ij} + w_{ij}) dy \]

\[ + \int_{Y'} d_{kalpha}(x,y) \frac{\partial \varphi_{mh}}{\partial y_k} \frac{\partial \varphi_{ij}}{\partial y_alpha} dy. \]  

(32)
It is immediate from above, that the coefficients of elasticity tensor satisfies
\[ c_{ijmh}^H = c_{mlij}^H \]

This is the end of the proof of the first section of proposition i.e of symmetry.

We now study the ellipticity of the coefficients of the elasticity tensor, we recall \( c_{ijkl}^H \) is elliptic, if for all the second order tensor \( X_{ij} \) symmetric \((X_{ij} = X_{ji})\), we have
\[ \exists \alpha_c^H > 0, \quad c_{ijkl}^H X_{ij} X_{kl} \geq \alpha_c^H X_{ij} X_{ij} \]

Consider expression (22) of the tensor \( c_{ijmh}^H \), we have
\[
c_{ijmh}^H X_{ij} X_{mh} = \int_{Y^*} c_{ijmh}(x,y) X_{ij} X_{mh} \, dy + \int_{Y^*} c_{ijkl}(x,y)s_{kl,y}(w_{mh}) X_{ij} X_{mh} \, dy \\
+ \int_{Y^*} e_{kj}(x,y) \frac{\partial\varphi_{mh}}{\partial y_k} X_{ij} X_{mh} \, dy,
\]
where \( w = w_{mh}X_{mh}, \ \zeta = \varphi_{mh}X_{mh} \) and \( P_{ij} = \tau_{mlij}X_{mh} = X_{ij} \). Therfore the couple \((w, \zeta)\) is a saddle point of the functional \( J \) defined by
\[
J(v, \psi) = \frac{1}{2} \int_{Y^*} \left\{ c_{ijkl}(s_{ij,y}(v) + P_{ij})(s_{kl,y}(v) + P_{kl}) + 2e_{kj}(x,y) (s_{ij,y}(v) + P_{ij}) - d_{ij} \frac{\partial \psi}{\partial y_i} \frac{\partial \psi}{\partial y_j} \right\} dy,
\]
By definition of the saddle point, we have
\[
J(w, \zeta) \leq J(w, \psi) \leq J(v, \zeta) \quad \forall (v, \psi) \text{ periodic functions.}
\]
Or
\[
J(w, 0) = \frac{1}{2} \int_{Y^*} c_{ijkl}(s_{ij,y}(w) + P_{ij})(s_{kl,y}(w) + P_{kl}) \, dy.
\]

But, if we use the first equation of system (13), we obtain
\[
c_{ijmh}^H X_{ij} X_{mh} = 2J(w, \zeta), \geq J(w, 0), \geq \frac{1}{2} \int_{Y^*} c_{ijkl}(s_{ij,y}(w) + P_{ij})(s_{kl,y}(w) + P_{kl}) \, dy, \geq 0,
\]
We have the homogenized elasticity tensor \( C^H = (c_{ijkl}^H) \), which is elliptic.
Now we give a results concerning some properties of dielectric homogenized tensor.
Proposition 3  The coefficients of dielectric homogenized tensor $D^H = (d_{in}^H)$ defined by (23) satisfy:

\begin{enumerate}
\item $d_{in}^H = d_{ni}^H$, $\forall 1 \leq i, n \leq 3$,
\item There exists $\alpha_d^H > 0$, such that for all vector $\xi$,
\[ d_{in}^H \xi_i \xi_n \geq \alpha_d^H \xi_i \xi_i \]
\end{enumerate}

Proof.

By analogy, we transform these coefficients to obtain a symmetric form

\[ d_{in}^H = d_{ni}^H \]

The problem (19), can be rewritten as

\begin{equation}
\begin{cases}
- \frac{\partial}{\partial y_j} \left\{ c_{ijkl}(x,y) s_{kl,y}(q^n) + e_{kij}(x,y) \frac{\partial(y_n + \psi^n)}{\partial y_k} \right\} = 0 \text{ in } Y^*, \\
- \frac{\partial}{\partial y_i} \left\{ -e_{ikl}(x,y) s_{kl,y}(q^n) + d_{ij}(x,y) \frac{\partial(y_n + \psi^n)}{\partial y_j} \right\} = 0 \text{ in } Y^*,
\end{cases}
\end{equation}

(33)

Introducing $(q^i, \psi^i)$, in the solution of this local problem

\begin{equation}
\begin{cases}
- \frac{\partial}{\partial y_j} \left\{ c_{ijkl}(x,y) s_{kl,y}(q^i) + e_{kij}(x,y) \frac{\partial(y_i + \psi^i)}{\partial y_k} \right\} = 0 \text{ in } Y^*, \\
- \frac{\partial}{\partial y_i} \left\{ -e_{ikl}(x,y) s_{kl,y}(q^i) + d_{ij}(x,y) \frac{\partial(y_i + \psi^i)}{\partial y_j} \right\} = 0 \text{ in } Y^*,
\end{cases}
\end{equation}

(34)

We can rewrite these coefficients of electric tensor in form given as

\[ d_{in}^H = \int_{Y^*} -e_{ikl}(x,y) s_{kl,y}(q^n) \, dy + \int_{Y^*} d_{ij}(x,y) \frac{\partial(y_n + \psi^n)}{\partial y_j} \, dy. \]

(35)

The first term of the second integral of precedent expression, is evaluated as follows

\[ -e_{ikl}(x,y) s_{kl,y}(q^n) \, dy = \int_{Y^*} -e_{akl}(x,y) s_{kl,y}(q^n) \delta_{ni} \, dy, \]

\[ = \int_{Y^*} -e_{akl}(x,y) s_{kl,y}(q^n) \frac{\partial y_i}{\partial y_a} \, dy, \]

\[ = \int_{Y^*} -e_{akl}(x,y) s_{kl,y}(q^n) \frac{\partial \psi^i}{\partial y_a} \, dy, \]

\[ - \int_{Y^*} -e_{akl}(x,y) s_{kl,y}(q^n) \frac{\partial(y_i + \psi^i)}{\partial y_a} \, dy. \]

(36)
Using the variationnal formulation of the second equation of system (33), and choosing a test function \( \varphi = \psi^i \), we obtain

\[
\int_{Y^*} -e_{ijkl}(x,y)s_{kl,y}(q^n) \frac{\partial \psi^i}{\partial y^a} dy = \int_{Y^*} d_{ij}(x,y) \frac{\partial(y^n + \psi^a)}{\partial y^a} \frac{\partial \psi^i}{\partial y^a} dy.
\]

Let us now consider the second integral in (36). Multiplying the first equation of system (34) by \( \phi^n \), and integrating by parts, we have

\[
\int_{Y^*} -e_{ijkl}(x,y)s_{kl,y}(q^n) \frac{\partial(y_i + \psi^i)}{\partial y^a} dy = -\int_{Y^*} c_{klo\beta}(x,y)s_{\alpha\beta,y}(q^i)s_{kl,y}(\rho^n) dy.
\]

Finally, we regroup these last results, and using the definition (35), we obtain

\[ d_{in}^H = \int_{Y^*} -e_{ikl}(x,y)s_{kl,y}(q^n) dy + \int_{Y^*} d_{ij}(x,y) \frac{\partial(y^n + \psi^a)}{\partial y^a} dy,
\]

\[ = \int_{Y^*} d_{ij}(x,y) \frac{\partial(y^n + \psi^a)}{\partial y^a} dy - \int_{Y^*} c_{klo\beta}(x,y)s_{\alpha\beta,y}(q^i)s_{kl,y}(\rho^n) dy.
\]

It is clear from that the coefficients of electric tensor is symmetric.

Now we are interested in the ellipticity of this tensor, recall \( d_{in}^H \) is elliptic, if for all vector \( X_i \), we have

\[ \exists \alpha_d^H > 0, \quad d_{in}^H X_i X_n \geq \alpha_d^H X_i X_i.
\]

We consider the expression (25) of tensor \( d_{in}^H \), we derive

\[ d_{in}^H X_i X_n = \int_{Y^*} d_{in}(x,y) X_i X_n dy - \int_{Y^*} e_{ikl}(x,y)s_{kl,y}(q^n) X_i X_n dy
\]

\[ + \int_{Y^*} d_{ij}(x,y) \frac{\partial \psi^a}{\partial y^a} X_i X_n dy,
\]

\[ = \int_{Y^*} d_{in}(x,y) X_i X_n dy - \int_{Y^*} e_{ikl}(x,y)s_{kl,y}(q^n X_n) X_i dy
\]

\[ + \int_{Y^*} d_{ij}(x,y) \frac{\partial \psi^a}{\partial y^a} X_i dy,
\]

\[ = \int_{Y^*} d_{ij}(x,y) (Q_j + \frac{\partial \xi}{\partial y^j}) Q_i dy - \int_{Y^*} e_{ikl}(x,y)s_{kl,y}(\xi)Q_i dy,
\]

\[ + \int_{Y^*} d_{ij}(x,y) \frac{\partial \psi^a}{\partial y^a} X_i dy.
\]
where $\xi = \psi^n X_n$, $\varsigma = \varphi^n X_n$ and $Q_i = \delta_{in} X_n = X_i$. Or the couple $(\xi, \varsigma)$ is a saddle point of the functional $G$ defined by:

$$(v, \psi) \rightarrow G(v, \psi)$$

$G(v, \psi) = \frac{1}{2} \int_{Y^*} \left\{ c_{ijkl} s_{ij,y}(v) s_{kl,y}(v) + 2 e_{ijkl} s_{ij,y}(v) \left( Q_k + \frac{\partial \psi}{\partial y_k} \right) - d_{ij} (Q_i + \frac{\partial \psi}{\partial y_i}) \left( Q_j + \frac{\partial \psi}{\partial y_j} \right) \right\} \, dy.$

By definition of the saddle point, we have

$$G(\xi, \psi) \leq G(\xi, \varsigma) \leq G(v, \varsigma) \quad \forall (v, \psi) \text{ periodic functions.}$$

Or

$$G(0, \varsigma) = -\frac{1}{2} \int_{Y^*} d_{ij}(x,y) (Q_i + \frac{\partial \varsigma}{\partial y_i}) (Q_j + \frac{\partial \varsigma}{\partial y_j}) \, dy.$$

But, if we use the second equation of system (19), we have

$$G(0, \varsigma) = -\frac{1}{2} \int_{Y^*} d_{ij}(x,y) (Q_j + \frac{\partial \varsigma}{\partial y_j}) Q_i \, dy + \frac{1}{2} \int_{Y^*} e_{ijkl} s_{kl,y}(\varsigma) X_i \, dy,$$

$$= -\frac{1}{2} d_{in}^H X_i X_n, \quad < 0.$$

We have the dielectric homogenized tensor which is elliptic.

Now we give the results concerning some properties of piezoelectric homogenized tensor.

**Proposition 4** The coefficients of piezoelectric homogenized tensor $\mathcal{E}^H = (e_{nij}^H)$ defined by (23) satisfy:

$$e_{nij}^H = e_{nji}^H$$

Moreover, we have the identity

$$e_{nij}^H = f_{nij}^H$$

**Proof.**

By definition of coefficients $e_{nij}^H$ (using the fact that $c_{ijkl}(x,y) = c_{jikl}(x,y)$, $e_{kji}(x,y) = e_{kij}(x,y)$), we have

$$e_{nij}^H = \int_{Y^*} \left\{ c_{ijkl}(x,y) s_{kl,y}(q^n) + e_{kij}(x,y) \left( \delta_{kn} + \frac{\partial q^n}{\partial y_k} \right) \right\} \, dy,$$

$$= \int_{Y^*} \left\{ c_{ijkl}(x,y) s_{kl,y}(q^n) + e_{kji}(x,y) \left( \delta_{kn} + \frac{\partial q^n}{\partial y_k} \right) \right\} \, dy,$$

$$= e_{nji}^H.$$

(38)
We can rewrite the coefficients $e_{nij}^H$, as given as under

$$
e_{nij}^H = \int_{\mathcal{Y}^*} \left\{ c_{ijkl}(x,y)s_{kl,y}(q^n) + e_{kij}(x,y)\left( \delta_{kn} + \frac{\partial \psi^n}{\partial y_k} \right) \right\} dy,$$

$$= \int_{\mathcal{Y}^*} \left\{ e_{nij}(x,y) + e_{kij}(x,y)\frac{\partial \psi^n}{\partial y_k} + c_{ijkl}(x,y)s_{kl,y}(q^n) \right\} dy.$$  (39)

Same, we can rewrite the coefficients $f_{nij}^H$, as form givens under

$$f_{nij}^H = \int_{\mathcal{Y}^*} \left\{ e_{nkl}(x,y)\left( r_{ij}^{kl} + s_{kl,y}(w^i) \right) - d_{nt}(x,y)\frac{\partial \phi^i}{\partial y_t} \right\} dy,$$

$$= \int_{\mathcal{Y}^*} \left\{ e_{nij}(x,y) + e_{nkl}(x,y)s_{kl,y}(w^i) - d_{nt}(x,y)\frac{\partial \phi^i}{\partial y_t} \right\} dy.$$  (40)

Using the two variations formulations corresponding of problems (18) and (19), and choosing the appropriate test functions, we can directly prove as $e_{nij}^H = f_{nij}^H$.

Finally, using the three last propositions, we can purpose the alternative form of the principal convergence theorem

**Theorem-Bis (the alternative form)**

Set $(u, \varphi)$ solution of the two-scale homogenized problem (10)-(11)-(12), then $(u, \varphi)$ is defined by that the solution of this homogenized problem

$$\begin{cases}
-\text{div} \sigma^H(u, \varphi) = \theta f & \text{in } \Omega,

-\text{div} D^H(u, \varphi) = 0 & \text{in } \Omega,
\end{cases}$$

(41)

where the boundary conditions

$$\begin{cases}
u(x) = 0 & \text{on } \partial \Omega,

\varphi(x) = 0 & \text{on } \partial \Omega,
\end{cases}$$

(42)

$\sigma_{ij}^H$ and $D_{ij}^H$ are defined by the homogenized constitutive law

$$\begin{cases}
\sigma_{ij}^H(u, \varphi) = c_{ijkl}^H s_{ml,x}(u) + e_{nij}^H \frac{\partial \varphi}{\partial x_n},

D_{ij}^H(u, \varphi) = -e_{inm}^H s_{ml,x}(u) + d_{nij}^H \frac{\partial \varphi}{\partial x_n},
\end{cases}$$

(43)

the homogenized coefficients $c_{ijkl}^H$, $e_{nij}^H$ and $d_{nij}^H$ are defined respectively by (22), (23) and (24).

### 5 Correctors result

The corrector results are obtained easily by the two-scale convergence method.

The objective of the next theorem justify rigorously the two first terms in the
usual asymptotic expansion of the solution.
Following the idea of Allaire \(\text{[1]}\), we introduce the following definition

**Definition 1** We call \(\psi(x, y)\) an admissible test function, if it is \(Y\)-periodic, and satisfies the following relation

\[
\lim_{\varepsilon \to 0} \int_\Omega \psi(x, \frac{x}{\varepsilon})^2 dx = \int_\Omega \int_Y \psi(x, y)^2 dx \, dy
\] (44)

Here we recall the Allaire's lemma

**Lemma 2 (Allaire \(\text{[1]}\))**: Let the function \(\psi(x, y) \in L^2(\Omega; C^\#(Y))\), then \(\psi(x, y)\) is an admissible test function in the sense of Definition 1.

Using this lemma, we obtain the following proposition.

**Proposition 5** The two functions \(s_{ij,y}(u_1(x, y))\) and \(\partial_{i,y}\varphi_1(x, y)\) are admissible test functions in the sense of Definition 1.

**Proof.** By definition we have

\[
u_1(x, y) = s_{mh,x}(u(x))w^{mh}(y) + \frac{\partial \varphi(x)}{\partial x_n}q^n(y),
\]

\[\varphi_1(x, y) = s_{mh,x}(u(x))\varphi^{mh}(y) + \frac{\partial \varphi(x)}{\partial x_n}q^n(y),\]

we obtain

\[
s_{ij,y}(u_1(x, y)) = s_{mh,x}(u(x))s_{ij,y}(w^{mh}(y)) + \frac{\partial \varphi(x)}{\partial x_n}s_{ij,y}(q^n(y)),
\]

\[\partial_{i,y}\varphi_1(x, y) = s_{mh,x}(u(x))\partial_{i,y}\varphi^{mh}(y) + \frac{\partial \varphi(x)}{\partial x_n}\partial_{i,y}q^n(y).
\]

Using Lemma 2, \(s_{ij,y}(u_1(x, y))\) and \(\partial_{i,y}\varphi_1(x, y)\) are the admissible test functions in the sense of Definition 1.

**Theorem 2** We have these two strong convergence results

\[
\begin{cases}
\tilde{u}^\varepsilon(x) - \chi(\frac{x}{\varepsilon})[u(x) + u_1(x, \frac{x}{\varepsilon})] \to 0 \text{ strongly in } H^1_0(\Omega) \\
\tilde{\varphi}^\varepsilon(x) - \chi(\frac{x}{\varepsilon})[\varphi(x) + \varphi_1(x, \frac{x}{\varepsilon})] \to 0 \text{ strongly in } H^1_0(\Omega)
\end{cases}
\]
Proof. We consider the variational formulation under the following form

\[
\int_{\Omega} \{ [c_{ijkl} s_{kl}(u^\varepsilon) + e_{kij} \partial_k \varphi^\varepsilon] s_{ij}(v) + [-c_{ijkl} s_{kl}(u^\varepsilon) + d_{ij} \partial_j \varphi^\varepsilon] \partial_i \psi \} \, dx = \int_{\Omega} f_i(x) v_i(x) \, dx
\]

(45)

Chosing \( v = u^\varepsilon \) and \( \psi = \varphi^\varepsilon \) in (45), we obtain

\[
\int_{\Omega} \{ [c_{ijkl} s_{kl}(u^\varepsilon) + e_{kij} \partial_k \varphi^\varepsilon] s_{ij}(u^\varepsilon) + [-c_{ijkl} s_{kl}(u^\varepsilon) + d_{ij} \partial_j \varphi^\varepsilon] \partial_i \psi \} \, dx = \int_{\Omega} f_i(x) u_i^\varepsilon(x) \, dx
\]

By simplification, we get

\[
\int_{\Omega} \{ c_{ijkl} (x) s_{ij}(u^\varepsilon) (x) s_{kl}(u^\varepsilon)(x) + d_{ij} (x) \partial_j \varphi^\varepsilon(x) \partial_i \varphi^\varepsilon(x) \} \, dx = \int_{\Omega} f_i(x) u_i^\varepsilon(x) \, dx
\]

(46)

By applying (46), we can write

\[
\int_{\Omega} c_{ijkl}(x) \left\{ s_{ij}(u^\varepsilon) - \chi(x, \varepsilon) \left[ s_{ij,x}(u) + s_{ij,y}(u_1) \right] \right\} \left\{ s_{kl}(u^\varepsilon) - \chi(x, \varepsilon) \left[ s_{kl,x}(u) + s_{kl,y}(u_1) \right] \right\} \, dx
\]

\[
+ \int_{\Omega} d_{ij}(x) \left\{ \partial_i \varphi^\varepsilon(x) - \chi(x, \varepsilon) \left[ \partial_i x \varphi + \partial_i y \varphi_1 \right] \right\} \left\{ \partial_j \varphi^\varepsilon(x) - \chi(x, \varepsilon) \left[ \partial_j x \varphi + \partial_j y \varphi_1 \right] \right\} \, dx
\]

\[
= \int_{\Omega} f_i(x) u_i^\varepsilon(x) \, dx
\]

\[
+ \int_{\Omega} c_{ijkl}(x) \chi(x, \varepsilon) \left[ s_{ij,x}(u(x)) + s_{ij,y}(u_1(x, \frac{x}{\varepsilon})) \right] \left[ s_{kl,x}(u(x)) + s_{kl,y}(u_1(x, \frac{x}{\varepsilon})) \right] \, dx
\]

\[
+ \int_{\Omega} d_{ij}(x) \chi(x, \varepsilon) \left[ \partial_i x \varphi(x) + \partial_i y \varphi_1(x, \frac{x}{\varepsilon}) \right] \left[ \partial_j x \varphi(x) + \partial_j y \varphi_1(x, \frac{x}{\varepsilon}) \right] \, dx
\]

\[
- 2 \int_{\Omega} c_{ijkl}(x) \chi(x, \varepsilon) \left[ \partial_i \varphi^\varepsilon(x) \left[ \partial_j x \varphi(x) + \partial_j y \varphi_1(x, \frac{x}{\varepsilon}) \right] \right] \, dx
\]

\[
- 2 \int_{\Omega} d_{ij}(x) \chi(x, \varepsilon) \left[ \partial_i \varphi^\varepsilon(x) \left[ \partial_j x \varphi(x) + \partial_j y \varphi_1(x, \frac{x}{\varepsilon}) \right] \right] \, dx.
\]
Using the ellipticity property of the elastic \((c_{ijkl})\) and dielectric \((d^{ij}_{kl})\) tensors, we get
\[
\alpha_c \|\tilde{s}_{ij}(\mathbf{u}^\varepsilon) - \chi(\frac{x}{\varepsilon}) s_{ij,x}(\mathbf{u}(x)) - \chi(\frac{x}{\varepsilon}) s_{ij,y}(\mathbf{u}_1(x, \frac{x}{\varepsilon})) \|_{L^2(\Omega)}^2
\]
\[\quad + \quad \alpha_d \|\tilde{\partial}_i \varphi^\varepsilon(x) - \chi(\frac{x}{\varepsilon}) \partial_{i,x} \varphi(x) - \chi(\frac{x}{\varepsilon}) \partial_{i,y} \varphi_1(x, \frac{x}{\varepsilon}) \|_{L^2(\Omega)}^2 \]
\[
\leq \int_{\Omega} f_i(x) u_i(x) \, dx
\]
\[\quad + \quad \int_{\Omega} \tilde{c}_{ijkl}(x) \chi(\frac{x}{\varepsilon}) \left[ s_{ij,x}(\mathbf{u}(x)) + s_{ij,y}(\mathbf{u}_1(x, \frac{x}{\varepsilon})) \right] \left[ s_{kl,x}(\mathbf{u}(x)) + s_{kl,y}(\mathbf{u}_1(x, \frac{x}{\varepsilon})) \right] \, dx
\]
\[\quad + \quad \int_{\Omega} \tilde{d}_{ij}(x) \chi(\frac{x}{\varepsilon}) \left[ \partial_{i,x} \varphi(x) + \partial_{i,y} \varphi_1(x, \frac{x}{\varepsilon}) \right] \left[ \partial_{j,x} \varphi(x) + \partial_{j,y} \varphi_1(x, \frac{x}{\varepsilon}) \right] \, dx
\]
\[\quad - \quad 2 \int_{\Omega} \tilde{c}_{ijkl}(x) \chi(\frac{x}{\varepsilon}) \tilde{s}_{ij}(\mathbf{u}^\varepsilon) \left[ s_{kl,x}(\mathbf{u}(x)) + s_{kl,y}(\mathbf{u}_1(x, \frac{x}{\varepsilon})) \right] \, dx
\]
\[\quad - \quad 2 \int_{\Omega} \tilde{d}_{ij}(x) \chi(\frac{x}{\varepsilon}) \tilde{\partial}_i \varphi^\varepsilon(x) \left[ \partial_{j,x} \varphi(x) + \partial_{j,y} \varphi_1(x, \frac{x}{\varepsilon}) \right] \, dx.
\]

Using the fact that \(s_{ij,y}(\mathbf{u}_1(x, y))\) and \(\partial_{i,y} \varphi_1(x, y)\) are the admissible test functions and taking the limit in the sense of the two-scale convergence in the second right-hand side of the inequality, we obtain
\[
\alpha_c \lim_{\varepsilon \to 0} \|\tilde{s}_{ij}(\mathbf{u}^\varepsilon) - \chi(\frac{x}{\varepsilon}) \left[ s_{ij,x}(\mathbf{u}(x)) - s_{ij,y}(\mathbf{u}_1(x, \frac{x}{\varepsilon})) \right] \|_{L^2(\Omega)}^2
\]
\[\quad + \quad \alpha_d \lim_{\varepsilon \to 0} \|\tilde{\partial}_i \varphi^\varepsilon(x) - \chi(\frac{x}{\varepsilon}) \left[ \partial_{i,x} \varphi(x) - \partial_{i,y} \varphi_1(x, \frac{x}{\varepsilon}) \right] \|_{L^2(\Omega)}^2 \]
\[
\leq \int_{\Omega} \int_{Y^*} f_i(x) u_i(x) \, dx \, dy
\]
\[\quad - \quad \int_{\Omega} \int_{Y^*} \tilde{c}_{ijkl}(x,y) \left[ s_{ij,x}(\mathbf{u}(x)) + s_{ij,y}(\mathbf{u}_1(x, y)) \right] \left[ s_{kl,x}(\mathbf{u}(x)) + s_{kl,y}(\mathbf{u}_1(x, y)) \right] \, dx \, dy
\]
\[\quad - \quad \int_{\Omega} \int_{Y^*} \tilde{d}_{ij}(x,y) \left[ \partial_{i,x} \varphi(x) + \partial_{i,y} \varphi_1(x, y) \right] \left[ \partial_{j,x} \varphi(x) + \partial_{j,y} \varphi_1(x, y) \right] \, dx \, dy \quad (47)
\]

Recalling the form of the two-scale homogenized problem \([10]-[11],[12]\), we observe that the right-hand side of the inequality \((47)\) vanishes, so that, we obtain
\[
\lim_{\varepsilon \to 0} \|\tilde{s}_{ij}(\mathbf{u}^\varepsilon) - \chi(\frac{x}{\varepsilon}) \left[ s_{ij,x}(\mathbf{u}(x)) - s_{ij,y}(\mathbf{u}_1(x, \frac{x}{\varepsilon})) \right] \|_{L^2(\Omega)} = 0
\]
and
\[
\lim_{\varepsilon \to 0} \|\tilde{\partial}_i \varphi^\varepsilon(x) - \chi(\frac{x}{\varepsilon}) \left[ \partial_{i,x} \varphi(x) - \partial_{i,y} \varphi_1(x, \frac{x}{\varepsilon}) \right] \|_{L^2(\Omega)} = 0.
\]
6 Conclusion

In this work, we have given the new convergence results, and the explicite forms of the elastic, piezoelectric and dielectric homogenized coefficients. The two-scale convergence is applied to our problem yields the strong convergence result on the correctors. This technique of two-scale convergence can handle also other homogenization problems, in medium which has periodic structure for example the laminated piezocomposite materials or fiber materials (see [5] [9] [11] [14] [12]). Numerical implementation for perforated, laminated and fiber structures, will be presented in forthcoming publications (see [12]).

Acknowledgment. This work has been supported in part by the Ministry for higher education and scientific research of Algeria (University of Oran, Departement of Mathematics). The author is grateful to Professor Bernadette Miara for helpful discussions.

References

[1] Allaire G. Homogenization and two scale-convergence, SIAM J. Math. Anal., 23 (26), (1992) 1482-1518.
[2] Allaire G., Murat F. Homogenization of Neumann problem with non-isolated holes, Asymptotic. Anal. 7, (1993) 81-95.
[3] Berger H., Gabbet U., Köppe H., Rodriguez-Ramos R., Bravo-Castillero J., Guinovart-Diaz R., Otero J.A., Maugin G.A. Finite element and asymptotic homogenization methods applied to smart composite materials. Comp. Mech. 33, (2003) 61-67.
[4] Bensoussan A., Lions J.L., Papanicolaou G. Asymptotic Analysis for Periodic Structures, North Holland, Amsterdam (1978).
[5] Castillero J.B., Otero J.A., Ramos R.R., Bourgeat A. Asymptotic homogenization of laminated piezocomposite materials, Int. J. Solids Structures. 35 (1998) 527-541.
[6] Cioranescu D., Damlamian A., Griso G. Periodic unfolding and homogenization, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 99-104.
[7] Ghergu M., Griso G., Mechkour H., Miara B. Homogénéisation de coques minces piézoélectriques perforées. C. R. Acad. Sci. Paris, Ser. II : Mécanique 333 (2005) 249-255.
[8] Ghergu M., Griso G., Labat B., Mechkour H., Miara B., Rohan E., Zidi M. Homogénéisation et piézoélectricité. Aide à la conception d’un biomatériaux. Annals of University of Craiova. Math. Comp. Sci. Ser. 32 (2005): 9-15.
[9] Feng M.L., Wu C.C. *A study of three-dimensional four-step braided piezoceramic composites by the homogenization method*, Comp Scien Tech, **61** (2001) 1889-1898.

[10] Nguetseng G. *A general convergence result for a functionnal related to the theory of homogenization*, SIAM J. Math. Anal., **20**(3), (1989) 608-623.

[11] Mechkour H., Miara B. *Modelling and control of piezoelectric perforated structures*, Proceedings of The Third World Conference On Structural Control. John Wiley, Chichester. F. Casciati : Editor. Vol 3, (2003) 329-336.

[12] Mechkour H *Homogénéisation et simulation numérique de structures piézoélectriques perforées et laminées*. PhD thesis, University of Marne-La-Vallée 2004 (in french).

[13] Oleinik O.A., Shamaev G.A., Yosifian G.A. *Mathematical problems in elasticity and homogenization*, North Holland, Amsterdam (1992).

[14] Ruan X., Safari A., Chou T.W. *Effective elastic, piezoelectric and dielectric properties of braided fabric composites*, Comp Part A **30** (1999)1435-1444.

[15] Pastor J *Homogenization of linear piezoelectric media*. Mech. Res. Comm. **24**(2), (1997) 145-150.

[16] Telega J.J. *Piezoelectricity and homogenization. Application to biomechanic*, In: Maugin, G.A.(Ed.), Continum Models and Discrete Systems, Vol. 2. Longman, Harlow, Essex, (1991) 220-229.