On the Optimum Cyclic Subcode Chains of \(RM(2, m)^*\) for Increasing Message Length

Xiaogang Liu, Yuan Luo and Kenneth W. Shum

Abstract—The distance profiles of linear block codes can be employed to design variational coding scheme for encoding message with variational length and getting lower decoding error probability by large minimum Hamming distance. Considering convenience for encoding, we focus on the distance profiles with respect to cyclic subcode chains (DPCs) of cyclic codes over \(GF(q)\) with length \(n\) such that \(\gcd(n, q) = 1\). In this paper the optimum DPCs and the corresponding optimum cyclic subcode chains are investigated on the punctured second-order Reed-Muller code \(RM(2, m)^*\) for increasing message length, where two standards on the optimums are studied according to the rhythm of increase.

Index Terms—Boolean function, distance profile with respect to cyclic subcode chain (DPC), exponential sum, Reed-Muller code, symplectic matrix.

I. INTRODUCTION

In variational transmission system with linear block code, the changes of the amount of user data will lead to the increase or decrease of the message length, and then lead to the expansion or contraction of linear subcodes. One example is the transport format combination indicator (TFCI) in the 3rd Generation Partnership Project (3GPP) of CDMA, which receives about five hundred patents according to the site of US Patent and Trademark Office (http://patft.uspto.gov/netahtml/PTO/search-adv.htm). Considering convenience for encoding, we focus on the problem of stepwise expansion of cyclic subcodes while keeping the minimum Hamming distances as large as possible, which is a key parameter for evaluating decoding ability. In this paper, the distance profiles with respect to cyclic subcode chains (DPCs) are introduced to deal with this problem on the punctured second-order Reed-Muller code \(RM(2, m)^*\).

The distance profiles and the optimum distance profile (ODP) of a linear block code are about how to select and then include or exclude the basis codewords one by one while keeping the minimum distances of the generated subcodes as large as possible. The concept was introduced by A. J. Han Vinck and Y. Luo in [21], and then investigated for general properties in [19] and for a lower bound on the second-order Reed-Muller codes by Y. Chen and A. J. Han Vinck in [11].

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It can be used to get better error correcting ability in channel coding for informed decoders, see M. van Dijk, S. Baggen, and L. Tolhuizen [23], and to design the TFCI in CDMA system, see H. Holma and A. Toskala [9] and R. Tanner and J. Woodard [21].

One problem is that, for a given linear block code, the algebraic structure of some subcodes may be lost although the properties of the original code may be good, and vice versa. Here we would like to consider cyclic codes and cyclic subcodes, which imply the convenience of encoding at least. In fact, the successive expansion of cyclic subcodes provide a cyclic subcode chain, and the minimum distances of the generated cyclic subcodes form a decreasing distance sequence.

In this paper, we mainly focus on the punctured second-order Reed-Muller code \(RM(2, m)^*\). The basic knowledge is presented in Section II, which includes distance profile, dimension profile, dictionary order, inverse dictionary order, Standard I, Standard II and some counting properties of cyclic subcode chains. In Section III the optimum distance profile with respect to cyclic subcode chains under Standard II, i.e. ODPC-II\(^{inv}\), is studied under one specification that the second selected cyclic subcode is the punctured first-order Reed-Muller code. The result of Section III is suboptimum or a lower bound on ODPC-II\(^{inv}\), but deduces the real ODPC-II\(^{inv}\) of \(RM(2, m)^*\) when \(m\) is even in Section IV. Section V is about some optimum distance profiles under certain requirement for most classes in Standard I, the requirement of which is common. When \(m\) is in the form of a power of 2, we also get a real optimum one in this section. Final conclusion is in Section VI.

II. PRELIMINARIES

There are five subsections in this section, which are about the basic definitions of distance profile of a linear block code (DPB), the optimum distance profile of a linear block code (ODPB), distance profile with respect to cyclic subcode chain of a cyclic code (DPC), and the optimum DPCs under two respective standards (ODPC-I and ODPC-II), etc. In addition, general results about the cyclic subcode chains are presented.

A. Distance Profiles and Subcode Chains of a Linear Block Code

Let \(C\) be an \([n, k]\) linear code over \(GF(q)\) and denote \(C_0 = C\). A sequence of linear subcodes

\[C_0 \supset C_1 \supset \cdots \supset C_{k−1}\]

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\[C_0 \supset C_1 \supset \cdots \supset C_{k−1}\]
is called the subcode chain, where \( \dim[C_i] = k - i \). An increasing sequence

\[
d[C_0] \leq d[C_1] \leq \cdots \leq d[C_{k-1}]
\]

is called the distance profile of the linear block code \( C \) (DPB), where \( d[C_1] \) is the minimum Hamming distance of the subcode \( C_j \). It is easy to see that a distance profile is with respect to a subcode chain.

In the comparison of distance profiles, the inverse dictionary order is for expanding subcodes, i.e., for increasing the message length, which is on the topic of this paper. In details, for any two integer sequences of length \( k \), \( a_0, \ldots, a_{k-1} \) and \( b_0, \ldots, b_{k-1} \), we say that \( a_0, \ldots, a_{k-1} \) is larger than \( b_0, \ldots, b_{k-1} \) in the inverse dictionary order if there is an integer \( t \) such that

\[
a_i = b_i \quad \text{for} \quad k - 1 \geq i \geq t + 1, \quad \text{and} \quad a_t > b_t.
\]

We say that \( a_0, \ldots, a_{k-1} \) is an upper bound on \( b_0, \ldots, b_{k-1} \) in the inverse dictionary order if \( a_0, \ldots, a_{k-1} \) is larger than or equal to \( b_0, \ldots, b_{k-1} \).

A distance profile of an \([n, k]\) linear block code \( C \) is called the optimum distance profile in the inverse dictionary order, which is denoted by ODPI\(_{inv}^*\):

\[
\text{ODP}B[C]^\text{inv} = \text{ODP}B[C]^\text{inv}_0, \text{ODP}B[C]^\text{inv}_{-1}, \ldots, \text{ODP}B[C]^\text{inv}_{k-1},
\]

if it is an upper bound on any distance profile of \( C \) in that order. The ODPI\(_{inv}^*\) will show you how to decrease the minimum distances (a decoding ability) as slowly as possible when expanding the dimensions of the subcodes one by one in a variational transmission system. The existence and uniqueness of the optimum distance profile of a linear block code are obvious. A chain that achieves the optimum distance profile is called an optimum chain in that order.

B. Distance Profiles with Respect to Cyclic Subcode Chains

Although the properties of some applied linear codes may be good, it is known that in many cases few algebraic structures are left in its subcodes, and vice versa. In this paper, we consider the distance profiles with respect to cyclic subcode chains of an \([n, k]\) cyclic code \( C \) over \( GF(q) \), where \( \gcd(n, q) = 1 \).

A cyclic subcode chain of \( C \) is a chain of cyclic subcodes such that:

\[
C_{\tau_0} \supset C_{\tau_1} \supset \cdots \supset C_{\tau_\lambda} \supset \{0^n\},
\]

where \( C_{\tau_0} = C \) and there is no cyclic subcodes between any two neighbors in the chain, i.e., there does not exist a cyclic code \( C^* \) such that \( C_{\tau_\lambda} \supset C^* \supset C_{\tau_{\lambda+1}} \). The increasing sequence

\[
d[C_{\tau_0}] \leq d[C_{\tau_1}] \leq \cdots \leq d[C_{\tau_{\lambda-1}}]
\]

is called the distance profile with respect to the cyclic subcode chain (DPC), where \( \lambda \) is called the length of the profile or the length of the chain. The decreasing sequence

\[
\dim[C_{\tau_0}] > \dim[C_{\tau_1}] > \dim[C_{\tau_2}] > \cdots > \dim[C_{\tau_{\lambda-1}}]
\]

is called the dimension profile with respect to the cyclic subcode chain. In general, math calligraphy \( C_i \) denotes an irreducible cyclic code with primitive idempotent \( \beta_i \) (Subsection III-A), and \( C_{\tau_u} \) denotes a cyclic subcode in a chain.

In the comparison among the DPCs in the inverse dictionary order, according to the dimension profiles or not, two standards are introduced as follows respectively.

1) Standard I: For a given cyclic code \( C \), the lengths of its DPCs are the same, see [17]. In order to compare its DPCs, a classification on the cyclic subcode chains is introduced as follows. Two chains with length \( \lambda \) are set to be in the same class if they have the same dimension profile, i.e.

\[
\dim[C_{\tau_u}] = \dim[C_{\tau_v}] \quad \text{for} \quad 0 \leq u \leq \lambda - 1,
\]

where the superscripts 1 and 2 denote the two chains respectively. In each class, the corresponding DPCs can be compared with each other in the inverse dictionary order, and we are interested in the optimum one denoted by ODPC-I\(_{inv}^*\). The corresponding analysis is said to be under Standard I.

Some counting properties of the classification are presented in Section II-C.

2) Standard II: For a given cyclic code \( C \), the distance profiles of any two cyclic subcode chains can be compared directly in the inverse dictionary order, and the analysis without the condition of same dimension profile is said to be under Standard II. The optimum one is denoted by ODPC-II\(_{inv}^*\). A cyclic subcode chain that achieves the ODPC (I or II) is called an optimum cyclic subcode chain correspondingly.

Standard I considers dimension profile prior to distance profile, and Standard II considers distance profile prior to dimension profile. Since Standard II is without the condition of same dimension profile and Standard I is with the condition, ODPC-II is an upper bound on ODPC-I for each class. As to Standard I, there are different optimum cyclic subcode chains in different classes, and the corresponding ODPC-I's can be different. As to Standard II, there may be more than one optimum cyclic subcode chains, but there exists only one ODPC-II.

C. Key Parameters of Cyclic Subcode Chains

Let \( C \) be an \([n, k]\) cyclic code over \( GF(q) \) such that \( \gcd(n, q) = 1 \). Its generator polynomial \( g(x) \) is a product of some distinct minimal polynomials. Let \( P \) be the set of the minimal polynomials that are factors of \( g(x) \), and \( J(v) \) be the number of the polynomials with degree \( v \) in \( P \). Let \( A \) be the set of all minimal polynomials over \( GF(q) \) that are factors of \( x^n - 1 \).

Let \( m \) be the multiplicative order of \( q \) modulo \( n \), i.e. \( \text{ord}(q, n) \), and the integers modulo \( n \) are considered in \( \{1, 2, \ldots, n\} \). The \( q \)-cyclotomic coset modulo \( n \) which contains \( s \) is \( \{s, sq, sq^2, \ldots, sq^{m_s-1}\} \), where \( m_s \) is the smallest positive integer such that \( s = sq^{m_s} \mod n \), i.e. \( n|s(q^{m_s} - 1) \).

Lemma 1: (Theorem 1, [17]) For the cyclic code \( C \), we have

- The length of its cyclic subcode chains is \( \lambda = |A \setminus P| = \sum_{v \in J} (L(v) - J(v)) \), where \( L(v) \) is the number of \( q \)-cyclotomic cosets modulo \( n \) with size \( v \), i.e. \( L(v) = \sum_{g \in G(v)} \varphi(q^v) \), \( G(v) = \{g : v = \text{ord}(q, n/g), g|n\} \) and \( \varphi(\cdot) \) is the Euler function.
• The number of its cyclic subcode chains is $\lambda!$, i.e. $\lambda$ factorial.
• The number of the chains in each class is $\mu = \prod_{v \in \mathbb{V}} |L(v)|!$.
• The number of classes is $\frac{\lambda!}{\mu}$.

Example 1: Assume that $q = 2, n = 21$, then $m = 6$.
Let $C$ be the cyclic code with generator polynomial $g_1(x) = (1 + x^2 + x^3)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$.
Then
\[
\begin{align*}
J(1) &= J(2) = J(6) = 0, J(3) = 2; \\
L(1) &= 1, L(2) = 1, L(3) = 2, L(6) = 2.
\end{align*}
\]
From Lemma [1], we have $\lambda = 4, \lambda! = 24, \mu = 2$ and $\frac{\lambda!}{\mu} = 12$.

In addition, the set $A$ is
\[
g_1(x), g_1(x)g_2(x), g_1(x)g_2(x)g_3(x), g_1(x)g_2(x)g_3(x)g_4(x)\]
where $g_1(x) = 1 + x^2 + x^3, g_2(x) = 1 + x + x^2 + x^3 + x^6, g_3(x) = 1 + x^2 + x^4 + x^5 + x^6, g_4(x) = 1 + x^2 + x^3 + x^4 + x^5 + x^6,$
and $g_5(x) = 1 + x$. In the investigation of Standard I, for the class of dimension profile 15, 9, 8, 6, there are $\mu = 2$ cyclic subcode chains. One such chain can be obtained from the cyclic subcodes generated by the following polynomials respectively:
\[
g_1(x), g_1(x)g_2(x), g_1(x)g_2(x)g_4(x), g_1(x)g_2(x)g_3(x)g_4(x).
\]
Using Matlab, we find that the corresponding DPC is 2, 6, 6, 8.
In fact, for this class of dimension profile, the two cyclic subcode chains have the same DPC, that is the ODPC-II $^{\text{inv}}$ of $\mathbb{D}_2$.

III. SUBOPTIMUM WITH RESPECT TO ODPC-II $^{\text{inv}}$ OF $\mathcal{R}(2,m)^*$

Let $\mathcal{R}(2,m)$ be the second-order Reed-Muller code. Deleting the first coordinate of each codeword, the well known punctured code $\mathcal{R}(2,m)^*$ is obtained, which is a cyclic code of length $n = 2^m - 1$ and dimension $k = 2^m - 1 + \binom{m}{1} + \binom{m}{2}$. The dual of the Reed-Muller code $\mathcal{R}(r,m)$ is $\mathcal{R}(m-r-1,m)$, which is the extended Hamming code when $r = 1$; $\mathcal{R}(r,m)$ itself is a subcode of the extended BCH code of designed distance $2^m - r - 1$. Reed-Muller codes are widely used, for example, in some communication systems which require fast decoding, in the localization of Malicious Nodes [1], in the Power Control of OFDM Modulation [2], [20], and so on. In [1], linear block codes with optimum distance profiles (ODPB), as defined in [2], were investigated, and the authors provided a lower bound on the optimum distance profile of the second-order Reed-Muller codes, which is proved to be tight for $m \leq 7$.

A. The Case of $m = 2t + 1$ in $\mathcal{R}(2,m)^*$

In this subsection Lemma [2] is cited to show the weight distributions of some subcodes of $\mathcal{R}(2,m)$. Then, by using the symplectic forms derived from the codewords of $\mathcal{R}(2,m) [19]$, we show how to construct the optimum distance profile under certain requirement with respect to ODPC-II $^{\text{inv}}$ in Theorem [1] where the profile is calculated in Lemma [4].

Let $\alpha$ be a primitive $\nu$th root of unity in $GF(q)$, where $q = 2^m$ and $n = 2^m - 1$ is the length of the cyclic code $\mathcal{R}(2,m)^*$. Let $\mathcal{D}_s$ be the cyclotomic coset containing $s$, with primitive idempotent denoted by $\theta_s$, and use $\theta_s^\ast$ to denote the primitive idempotent corresponding to $\mathcal{D}_s$. Easy to see that $\mathcal{D}_s$ and $\mathcal{D}_s$ have the same size. The nonzeros of $\theta_s$ and $\theta_s^\ast$ are $\{\alpha^i : i \in \mathcal{D}_s\}$ and $\{\alpha^i : i \in \mathcal{D}_s\}$ respectively.

Note that, $\theta_0, \theta_1^\ast, \theta_t^\ast (1 \leq i \leq t)$ are all the primitive idempotents contained in $\mathcal{R}(2,m)^*$, which correspond to all the minimal cyclic subcodes of $\mathcal{R}(2,m)^*$. Any cyclic subcode of $\mathcal{R}(2,m)^*$ can be given by idempotent of the form
\[
a_0\theta_0 + a_1\theta_1^\ast + \sum_{j=1}^{t} a_j\theta_{t_j}^\ast, \quad a_j \in \{0, 1\}, -1 \leq j \leq t.
\]

Lemma 2: (Ch.15, [19]) Let $m=2t+1$, and let $h$ be any number in the range $1 \leq h \leq t$. Then there exists a
\[
\left\lfloor \frac{2^m}{m(t+h+1)} \right\rfloor, \left\lfloor \frac{2^m}{m(t+h+1)} + 1 \right\rfloor - 2^m/h - 1
\]
subcode $\mathcal{R}(2t+1)^h$ of $\mathcal{R}(2,m)$.
It is obtained by extending the cyclic subcode of $\mathcal{R}(2,m)^*$ having idempotent
\[
\theta_0 + \theta_1^\ast + \sum_{j=1}^{t} \theta_{t_j}^\ast, \quad l_j = 1 + 2^j.
\]
The code has codewords of weights $2^{m-1}$ and $2^{m-1} - 2^{m-h-1}$ for all $h'$ in the range $h \leq h' \leq t$.

Remark 1: One cyclic subcode chain of $\mathcal{R}(2,m)^*$ can be obtained from Lemma [2]. But there are many other chains. We will compare and select the suboptimum one in Theorem [1].

Lemma 3: (pp. 453, [19]) Let $\Phi_h$ be the set of symplectic forms derived from the codewords of the second-order Reed-Muller code $\mathcal{R}(2,m)$. Suppose that it has the property that the rank of every nonzero form in $\Phi_h$ is at least $2h$, and the rank of the sum of any two distinct forms in $\Phi_h$ is also at least $2h$, here $h$ is some fixed number in the range $1 \leq h \leq |\Phi_h|$, then the maximum size of such a set $\Phi_h$ is $2^{|\Phi_h|}$ if $h = 2t + 1$, and $2^{|\Phi_h|}$ if $h = 2t + 2$.

For each $i (1 \leq i \leq t)$, let $\mathcal{R}(2t+1)^h$ denote the subcode of $\mathcal{R}(2,m)^*$ suggested in Lemma [2] and $\mathcal{R}(2t+1)^h$ denote the cyclic subcode obtained from $\mathcal{R}(2t+1)^h$ by puncturing the first coordinate. Express the one-dimensional cyclic subcode corresponding to $\theta_0$ by $\mathcal{R}(0,m)^*$, and the punctured first-order Reed-Muller code by $\mathcal{R}(1,m)^*$.

Corollary 1: Lemma [2] implies the following cyclic subcode family with a nested structure
\[
\mathcal{R}(0,m)^* \subset \mathcal{R}(1,m)^* \subset \mathcal{R}(2t+1)^h \subset \cdots \subset \mathcal{R}(2t+1)^1.
\]

The following result concerns the minimum distances of the cyclic subcodes in Corollary [1].

Lemma 4: The distance profile of the cyclic subcode chain given in Corollary [1] is
\[
\begin{align*}
d_{r_u} &= 2^{2t} - 2^{2t-u-1} - 1 (0 \leq u \leq t - 1), \\
d_{r_t} &= 2^{m-1} - 1 = 2^{2t} - 1, d_{r_{t+1}} = 2^m - 1 = 2^{2t+1} - 1.
\end{align*}
\]

Proof: Let $c$ be a codeword of the cyclic subcode $\mathcal{R}(2t+1)^i (1 \leq i \leq t)$ in Corollary [1] with symplectic form
of rank \(2d(1 \leq d \leq t)\). From Theorem 4 and Theorem 5 in Chap.15 [19], rewrite the Boolean function \(f\) as \(T(y) = \sum \frac{1}{2}y_2 - y_2 + L(y) + \epsilon,\) here \(L(y)\) and \(\epsilon\) are arbitrary. Choose \(T(y) = \sum \frac{1}{2}y_2 - y_2 + y_1 + y_2 + 1\). As in the proof of Theorem 5, Chap15 [19], the final expression is nonzero for \(2m-1 - 2m-d-1\) coordinates. Since \(T(y) = 1\) when \(y_i = 0, 1 \leq i \leq m,\) deleting the first coordinate, the corresponding weight is \(2m-1 - 2m-d-1\).

**Example 2:** For \(t = 2,\) that is \(m = 5, n = 31,\) Lemma 4 provides a distance profile \(d_{r_0} = 7, d_{r_1} = 11, d_{r_2} = 15, d_{r_3} = 31\) with dimension profile \(16, 11, 6, 1.\) The generated nontrivial cyclic codes [31,11,11] and [31, 6, 15] are optimal [6].

**Theorem 1:** Let \(m = 2t + 1\) where \(t \geq 1.\) Then for the punctured second-order Reed-Muller code \(\mathcal{R} M(2, m)^*\), if we are requiring that, the second selected cyclic subcode is \(\mathcal{R} M(1, m)^*\), the distance profile of the cyclic subcode chain in Corollary 1 is optimum under Standard II.

**Proof:** To get the optimum distance profile under the requirement, we have to add the primitive idempotents one by one accumulatively from \(\theta_0, \theta_1, \theta_1^\ast\) \((1 \leq j \leq t),\) and at the same time try to make the minimum distance of the cyclic subcode generated by the accumulative sum as large as possible according to the following steps.

1) It is obvious that the first cyclic subcode must be \(\mathcal{R} M(0, m)^*\) which has the largest minimum distance \(d_{r_{t+1}} = 2^m - 1 = n\) (code length). That is to say, \(\theta_0\) is selected in this step. Then all the cyclic subcodes of the chain are self-complementary.

2) In the requirement, the second cyclic subcode has idempotent \(\theta_0 + \theta_1^*\) which generates the punctured first-order Reed-Muller code \(\mathcal{R} M(1, m)^*\) satisfying \(d_{r_1} = 2^m - 1.\) Briefly \(\theta_1^*\) is selected here.

3) With decreasing \(h\) from \(t\) to \(1\) and selecting \(l_j,\) idempotents \(\theta_0, \theta_0 + \theta_1, \theta_0 + \theta_1^* + \sum_{s=h} l_j,\) where \(1 \leq j_x \leq t,\) can provide any cyclic subcode chain with \(\mathcal{R} M(0, m)^*\) and \(\mathcal{R} M(1, m)^*.\) In the \((t - h + 3)\) th step, the generated cyclic subcode is denoted by \(C_{r_{t-1}}.\) Note that, in the \((t - h + 3)\) th step of Corollary 1, the cyclic subcode \(\mathcal{R} M(2, 1)^*\) has minimum distance \(d_{r_{t-1}} = 2^m - 1 - 2^m - h - 1\) using Lemma 4.

The set of symplectic forms contained in \(C_{r_{t-1}}\) is with size \(N = 2^{m(t-h+1)}\). According to Lemma 3 the maximum size of the set of symplectic forms satisfying that each element has rank \(\geq 2(h + 1)\) is at most \(2^{2(t+1)}(t-(h+1)+1)\), which is smaller than \(N.\) So there must be some symplectic forms in \(C_{r_{t-1}}\) which have ranks \(2d \leq 2(h + 1).\) According to the proof Lemma 3 \(C_{r_{t-1}}\) has a codeword of weight \(2^m - 1 - 2^m - h - 1 - 1 = d_{r_{t+1}},\)

In one word, the distance profile \(d_{r_0} \leq d_{r_1} \leq \cdots \leq d_{r_{t+1}}\) of the cyclic subcode chain given in Corollary 1 is optimum under the requirement of the theorem.

**B. The Case of \(m = 2t + 2\) in \(\mathcal{R} M(2, m)^*\)**

In this subsection the punctured second-order Reed-Muller code \(\mathcal{R} M(2, m)^*\) is studied for the case of \(m = 2t + 2,\) where \(t \geq 1\) is a positive integer. This case is in parallel with the case of \(m = 2t + 1.\) Corresponding to Lemma 4, Subsection III-A Lemmas 5 is stated for the weight distributions of certain subcodes of \(\mathcal{R} M(2, m).\) Just like Theorem 1 a suboptimum distance profile with respect to ODPC-II\(^{\text{nov}}\) is presented in Theorem 2.

**Lemma 5:** (Theorem 3.6, [11].) Let \(m = 2t + 2,\) and let \(h\) be any number in the range \(1 \leq h \leq t + 1.\) Then there exists a \([2^m, m, t - h - m/2 + 1, 2m - 1 - 2m - h - 1]\) subcode \(\mathcal{R} M(2, m)^*\) of \(\mathcal{R} M(2, m).\) It is obtained by extending the cyclic subcode \(\mathcal{R} M(2, m)^*\) having idempotent \(\theta_0 + \theta_1^* + \sum_{j=h} l_j,\) \(l_j = 1 + 2^j.\)

The code has codewords of weights \(2^m - 1, 2^m - 1 - 2m - h - 1\) for all \(h' \leq h' \leq t + 1.\) Denote \(\mathcal{R} M(0, m)^*, \mathcal{R} M(1, m)^*\) and \(\mathcal{R} M_{2t+2}(1 \leq u \leq t + 1)\) analogously as in Subsection III-A.

**Corollary 2:** Lemma 5 implies the following cyclic subcode family with a nested structure

\[
\mathcal{R} M(0, m)^* \subset \mathcal{R} M(1, m)^* \subset \mathcal{R} M(2, m)^* \subset \cdots \subset \mathcal{R} M_{2t+2}.
\]

Similar to Lemma 4 the following result is for the case of \(m = 2t + 2.\)

**Lemma 6:** The distance profile of the cyclic subcode chain given in Corollary 4 is

\[
d_{r_0} = 2^{2t+1} - 2^{2t-u} - 1(0 \leq u \leq t),
\]

\[
d_{r_{t+1}} = 2^m - 1 = 2^{2t+1} - 1, d_{r_{t+2}} = 2^m - 1 = 2^{2t+2} - 1.
\]

**Example 3:** For \(t = 1,\) that is \(m = 4, n = 15.\) Lemma 6 provides a distance profile \(d_{r_0} = 3, d_{r_1} = 5, d_{r_2} = 7, d_{r_3} = 15\) with dimension profile \(11, 7, 5, 1.\) The generated nontrivial cyclic codes [15,11,3], [15, 7, 5] and [15, 5, 7] are optimal [6].

Using Lemma 5, Theorem 2 can be verified which is the counterpart of Theorem 1.

**Theorem 2:** Let \(m = 2t + 2\) where \(t \geq 1.\) Then for the punctured second-order Reed-Muller code \(\mathcal{R} M(2, m)^*,\) if we are requiring that, the second selected cyclic subcode is \(\mathcal{R} M(1, m)^*,\) the distance profile of the cyclic subcode chain in Corollary 2 is optimum under Standard II.

**IV. THE ODPC-II\(^{\text{nov}}\) OF \(\mathcal{R} M(2, m)^* WHEN m = 2t + 2\)**

In this section we investigate the exact ODPC-II\(^{\text{nov}}\) of the punctured second-order Reed-Muller code \(\mathcal{R} M(2, m)^*\) when \(m = 2t + 2.\) Subsection IV-A is about the basic background on cyclic codes. In Subsection IV-B the weight distributions of cyclic subcodes of \(\mathcal{R} M(2, m)^*\) are studied, and then Theorem 2 is reinvestigated in Theorem 5.

**A. Basic Results on the Weight of Codeword in Cyclic Codes**

Many of the following preliminaries are well referred to [5], [12], [13], [15], [16], [22], [24] which list the properties on weight distributions of cyclic codes, trace functions, exponential sums, quadratic forms and their relations. See also [3], [4], [8] for binary sequences.
Let $q = 2^m$ and $F_q$ be the finite field of order $q$. Let $\pi$ be a primitive element of $F_q$, $Tr:F_{2^m} \rightarrow F_2$ be the trace mapping, and $e(x) = (−1)^{Tr(x)}$ is the canonical additive character on $F_q$. For the binary cyclic code $C$ with length $l = q − 1$ and nonzeros $\pi^{-s_k}x_k, 1 \leq s_k \leq q − 2(1 \leq \lambda \leq u)$, the codewords in $C$ can be expressed by

$$c(\alpha_1, \ldots, \alpha_u) = (c_0, c_1, \ldots, c_{u−1}) \ (\alpha_1, \ldots, \alpha_u \in F_q)$$

where

$$c_i = \sum_{\lambda=1}^{u} Tr(\alpha_i \pi^{s_k}) (0 \leq i \leq l − 1).$$

Therefore the Hamming weight of the codeword $c = c(\alpha_1, \ldots, \alpha_u)$ is

$$w_H(c) = l - \#\{i|0 \leq i \leq l−1, c_i = 0\} = l - l - \frac{1}{2} \sum_{x \in F_q^*} (-1)Tr(f(x)) = 2^{m−1} - \frac{1}{2}S(f, m) \tag{2}$$

where $f(x) = \alpha_1 x^{s_1} + \alpha_2 x^{s_2} + \cdots + \alpha_u x^{s_u} \in F_q[x]$, and $S(f, m) = \sum_{x \in F_q^*} e(f(x))$.

For $f(x) = \alpha x^{2^i + 1} + \beta x^{2^i + 1} + \cdots + \gamma x^{2^i + 1} \in F_q[x]$, we have $S(f, m) = \sum_{x \in F_q^*} (-1)^{x H_{\alpha, \beta, \cdots, \gamma}}$ where $H_{\alpha, \beta, \cdots, \gamma}$ is the matrix of the quadratic form $F_{\alpha, \beta, \cdots, \gamma}(X \in F_q[x])$. $S(f, m)$ is also denoted by $T(\alpha, \beta, \cdots, \gamma)$. For a quadratic form $F$ with corresponding matrix $H$, define $r_F$ to be the rank of the skew-symmetric matrix $H + H^T$. Then $r_F$ is even.

Let $f(x) = \alpha x^{2^i + 1}$ and $Tr(f(x)) = XH_{\alpha, \gamma}^T$ where $\alpha \in F_q^*$. The following lemma can be deduced by using a similar argument as in [5, 10].

Lemma 7: For $\alpha \in F_q^* \setminus \{0\}$, let $r_\alpha$ be the rank of $H_\alpha + H_\alpha^T$. Then $r_\alpha = m − r_{\gcd(2i, m)}$ where $1 \leq i \leq t$.

B. Main Results

Now we focus on the ODPC-II inv of $R, M(2, m)$ when $m$ is even. Lemma 12, Lemma 13 and Corollary 5 investigate the existence of certain one-weight minimal cyclic code. Then in Subsection IV-B1, Corollary 4 gives the optimal cyclic subcode chain for the case of $m = 2^i$. In Subsection IV-B2, Corollary 5 considers the case when $m = 2t + 2$ is not a power of 2. Final results are given by Theorem 5 with an example.

In the subsequent, Lemma 8 is about the greatest common divisor of $2^\alpha + 1$ and $2^\beta − 1$; Lemma 9 is about the size of cyclotomic cosets $D_i$; Lemma 10 is about the exponential sums of quadratic forms. They will be used in Lemma 12 about the existence of certain one-weight minimal cyclic code and then support the determination of the optimum distance profile. In addition, define the 2-adic order function $\nu_2(\ast)$, such that $\nu_2(n) = s$ for $n = 2^sn'$ where $n'$ is odd.

Lemma 8: (Lemma 5.3, 10) Let $\alpha, \beta \geq 1$ be integers. Then

$$\gcd(2^\alpha + 1, 2^\beta − 1) = \begin{cases} 2^\gcd(\alpha, \beta) + 1 & \text{if } \nu_2(\beta) > \nu_2(\alpha), \\ 1 & \text{otherwise}. \end{cases}$$

Lemma 9: (Lemma B.2, 11) If $m = 2t + 1$ is odd, then for $l_i = 1 + 2^i$, the cyclotomic coset $D_i$, has size

$$|D_i| = m, \ 1 \leq i \leq t.$$ 

If $m = 2t + 2$ is even, then for $l_i = 1 + 2^i$, the cyclotomic coset $D_i$, has size

$$|D_i| = \begin{cases} m, & 1 \leq i \leq t/2, \\ m/2, & i = t+1. \end{cases}$$

Lemma 10: (Lemma 1, 13) For the quadratic form $F(X) = XHX^T$ defined as before,

$$S(f, m) = \sum_{x \in F_q^*} (-1)^{Tr(f(x))} = \sum_{x \in F_q^*} (-1)^{F(X)} = \pm 2^{m-2} \text{ or 0}$$

Moreover, if $r_F = m$, then

$$S(f, m) = \sum_{x \in F_q^*} (-1)^{F(X)} = \pm 2^\frac{m}{2}.$$ 

For the irreducible cyclic codes $C_i(\theta_i^t, 1 \leq i \leq t)$, the following lemma will be used in Lemma 12 to characterize their weights.

Lemma 11: (Corollary 3.7 in Ch.3, 15) If $e_1$ and $e_2$ are positive integers, then the greatest common divisor of $x^{e_1} − 1$ and $x^{e_2} − 1$ in $F_q[x]$ is $x^d − 1$, where $d$ is the greatest common divisor of $e_1$ and $e_2$. If $e_2 = q + 1$ and $q = 2^m$, then for any positive integer $e_1$, the number of different solutions of $x^{e_1} − 1 = 0$ in $F_q$ is $d$.

Lemma 12: The irreducible cyclic code $C_i$ is a one-weight cyclic code if and only if $\gcd(2^j + 1, 2^m − 1) = 1$, and in this case the only nonzero weight is $2^{m−1}$.

Proof: From Lemma 9, the minimal cyclic code $C_i$ has dimension $m$. By Lemma 7, the only possible rank of the corresponding skew-symmetric matrix $H_\alpha + H_\alpha^T$ is $m$ or $m - r_{\gcd(2i, m)}$. Applying (3), the weight of the corresponding codeword is $w_H(c) = 2^{m−1} - \frac{1}{2}T(\alpha)$.

Let $2^\prime = \gcd(2i, m)$ since $m$ is even. According to the possible values of $T(\alpha)$ given by Lemma 10 for $\varepsilon = \pm 1$, denote $N_{\varepsilon, i} = \{\alpha \in F_q\setminus\{0\}|T(\alpha) = \varepsilon 2^{\frac{m−1}{2}}\}$, $N_{\varepsilon, i} = |N_{\varepsilon, i}|; N_{\varepsilon, 0} = \{\alpha \in F_q\setminus\{0\}|T(\alpha) = \varepsilon 2^{\frac{m−1}{2}}\}$, $\varepsilon, 0 = |N_{\varepsilon, 0}|; N_0 = \{\alpha \in F_q\setminus\{0\}|T(\alpha) = 0\}; n_0 = |N_0|$.

“Only if” part. Assume $C_i$ has only one nonzero weight. Let $A_1$ and $A_1’ (i = 0, 1, \ldots, 2^{m−1} − 1)$ be the weight distributions of $C_i$ and its dual $C_i^\perp$ respectively. It is easy to see that $A_1’ = 0$. From the MacWilliams identities, see [19 pp.131],

$$\sum_{i=1}^{2^k} \frac{i A_i}{2^k} = \frac{1}{2}(n - A_1’) = \frac{1}{2}n \text{ in our case,} \tag{3}$$

here $n = 2^{m−1}$ is the length of the code, and $k = m$ is the dimension. Equation (3) implies that if there is only one nonzero weight $j, 1 \leq j \leq 2^{m−1} − 1$, then $A_j = 2^{m−1} − 1$ and $j = 2^{m−1}$. We have $n_{1,i} = n_{1,i} = n_{1,i} = 0$, and

$$\sum_{\alpha \in F_q} T(\alpha) = T(0)^2 + (2^{\frac{m}{2}})^2n_{1,0} + (2^{\frac{m}{2}})^2n_{1,0} = \begin{cases} 2^{m} + 2^{m}(n_{1,1} + n_{1,0}) & \text{if } j = 2^{m−1}, \\ 2^{m+2^i}(n_{1,i} + n_{1,-i}) & \text{if } j = 2^{m−1}. \end{cases} \tag{4}$$
where \( T(0) = T(\alpha) = 2^m \).

Now, equation (4) can be calculated in another way

\[
\sum_{\alpha \in \mathbb{F}_q} T(\alpha)^2 = \sum_{\alpha \in \mathbb{F}_q} \sum_{x, y \in \mathbb{F}_q} (-1) \text{Tr}(\alpha(x^{2^i+1} + y^{2^i+1})) = \sum_{x, y \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q} (-1) \text{Tr}(\alpha(x^{2^i+1} + y^{2^i+1})) = \sum_{x^{2^i+1} + y^{2^i+1} = 0} (-1) \text{Tr}(\alpha(x^{2^i+1} + y^{2^i+1})) = 2^m \cdot M_2,
\]

where \( M_2 \) is the number of solutions to the equation \( x^{2^i+1} + y^{2^i+1} = 0 \). From Lemma 11, easy to find that \( M_2 = 1 + (2^m - 1) \text{gcd}(2^i + 1, 2^m - 1) \). Thus \( M_2 = 2^m \), which implies \( \text{gcd}(2^i + 1, 2^m - 1) = 1 \).

"If" part. Assume \( \text{gcd}(2^i + 1, 2^m - 1) = 1 \). Similar to equations (4) and (5), we have

\[
\sum_{\alpha \in \mathbb{F}_q} T(\alpha)^2 = 2^m + 2 \sum n_{1,0} - 2^{-1} n_{-1,0},
\]

\[
\sum_{\alpha \in \mathbb{F}_q} T(\alpha)^3 = 2^{3m} + 2 \sum n_{1,0} - 2 \sum n_{-1,0},
\]

\[
\sum_{\alpha \in \mathbb{F}_q} T(\alpha)^4 = 2^{4m} + 2 \sum n_{1,0} + 2 \sum n_{-1,0},
\]

\[
+ 2 \sum n_{1,0} + 2 \sum n_{-1,0} = 2^{4m}.
\]

Combining with \( n_{1,0} + n_{-1,0} + n_{i,1} + n_{-1,-i} = 2^m - 1 \), it is not difficult to see that \( n_{1,0} = n_{-1,0} = n_{i,1} = n_{-1,-i} = 0 \) and \( n_0 = 2^m - 1 \). That is the only nonzero weight is 2

\[ m-1. \]

**Lemma 13:** The minimal cyclic code \( C_{\mathbb{F}_q} \) with primitive idempotent \( \theta_{\mathbb{F}_q} \) has dimension \( \frac{m}{2} \), and only one nonzero weight \( 2^{m-1} + 2^{-1} \).

**Remark 2:** In Lemma 13 \( \text{gcd}(2^{\frac{m}{2}} + 1, 2^m - 1) = 2^{\frac{m}{2}} + 1 \neq 1 \), but the minimal cyclic code \( C_{\mathbb{F}_q} \) is still a one-weight code.

The following lemma will be used in Corollary 3 for the nonexistence of certain one-weight irreducible cyclic code.

**Lemma 14:** Let \( m = 2t + 2 \) where \( t \geq 1 \). Then \( m \) is not a power of 2 if and only if there exists \( 1 \leq i \leq t \) such that \( \text{gcd}(2^i + 1, 2^m - 1) = 1 \).

**Proof:** “Only if” part. Let \( m = 2^u \cdot m' \) where \( m' \geq 3 \) is odd and \( u \geq 1 \). Set \( i = 2^u \). Then \( t = \frac{m}{2^u} \), and \( i \leq t \). In Lemma 8 \( \nu_2(i) = \nu_2(m) = u \), so \( \text{gcd}(2^i + 1, 2^m - 1) = 1 \).

"If" part. If \( m = 2^u \) is a power of 2 where \( s \geq 2 \), from Lemma 8 for any positive integer \( 1 \leq i \leq t \), we have \( \nu_2(i) < \nu_2(m) = s \). That is \( \text{gcd}(2^i + 1, 2^m - 1) = 2^{\text{gcd}(i,m)} + 1 \geq 3 \).

**Corollary 3:** Let \( m = 2^s = 2t + 2 \) where \( s \geq 2 \). For \( 1 \leq i \leq t \), \( C_i \) is not a one-weight cyclic code, and has weights of the forms \( 2^{m-1} \pm 2^a \) and \( 2^{m-1} - 2^a \) where \( a, a' \) are positive integers.

1) **The Optimum Profile when \( m \) Is a Power of 2:** The following lemma can be derived from Lemma 13 and Corollary 3.

**Lemma 15:** Let \( m = 2^s = 2t + 2 \) where \( s \geq 2 \). For \( 1 \leq i \leq t + 1 \), the cyclic code \( C_i \) with idempotent \( \theta_0 + \theta_1 \), has minimum distance less than \( 2^{m-1} - 1 \).

**Corollary 4:** Let \( m = 2^s \) where \( s \geq 2 \). In the process of selecting the optimum cyclic subcode chain of \( R(M(2, m)^* \) under standard II, \( \theta_0 \) and \( \theta_1^* \) will be the first two selected primitive idempotents. Then \( \{1\} \) is an optimum cyclic subcode chain.

**Proof:** Since the cyclic code \( C_i \) with idempotent \( \theta_0 + \theta_1^* \) has minimum distance \( 2^m - 1 \), from Lemma 15 \( \theta_1^* \) should be the second selected primitive idempotent. And the result follows from Theorem 2.

2) **Other Cases of \( m \):** In this subsection, Corollary 6 investigates the ODPC-II\( m \) of \( R(M(2, m)^* \), where \( m = 2t + 2 \) is not a power of 2. In fact, Corollary 6 is supported by Lemma 17 and Lemma 20 in the investigation of weight distributions and minimum distances. The cyclic code \( C_i \) is with idempotent \( \theta_0^* + \theta_1^* \) and length \( 2^m - 1 \), where \( 1 \leq i \neq j \leq t + 1 \). Assume that at least one of \( i, j \), let’s say \( i \), satisfies \( \text{gcd}(2^i + 1, 2^m - 1) = 1 \).

For the following lemma, we fix some notations. Let \( n_1 \) be an even integer, \( m_1 = n_1/2 \) and \( q_1 = 2^{n_1} \). Let \( k_1 \) be a positive integer, \( 1 \leq k_1 \leq n_1 - 1 \) and \( k_1 \neq m_1 \). Let \( d_i = \text{gcd}(m_1, k_1) \) and \( d_i' = \text{gcd}(m_1 + k_1, 2k_1) \). For \( \alpha \in \mathbb{F}_{2^m}, \beta \in \mathbb{F}_{2^m} \), set

\[ T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} (-1)^{T_i}(\alpha x^{2^m+1} + \beta x^{2^m+1}) \]

\[ T(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} (-1)^{T_i}(\alpha x^{2^m+1} + \beta x^{2^m+1}) \]

\[ \text{is the binary cyclic code of length } l_1 = q_1 - 1 \text{ with nonzero } \pi^{-(2^k+1)} \text{ and } \pi^{-(2^{m+1}+1)}. \]

**Lemma 16:** (Theorem 1, 16) The value distribution of the multi-set \( \{T(\alpha, \beta) | \alpha \in \mathbb{F}_{2^m}, \beta \in \mathbb{F}_{2^m} \} \) and the weight distribution of \( C' \) are shown as following

(i) For the case \( d_i' = d_i \).

(ii) For the case \( d_i' = 2d_i \) (the table at the top of next page).

According to the possible weights of the codewords \( c(\alpha, \beta) \) in Lemma 16 we have

**Lemma 17:** The cyclic code \( C_i \) with idempotent \( \theta_0^* + \theta_1^* \) can not have only three possible nonzero weights \( 2^{m-1}, 2^{m-1} + 2^l \) and \( 2^{m-1} - 2^l \).

**Lemma 18:** There are the following results about the exponential sum \( T(\alpha, \beta) \)
Proof: Exchanging the order of summation

\[
\sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta) = \sum_{\alpha, \beta \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q} (-1) \text{Tr}(\alpha x^{i+1} + \beta x^{j+1})
\]

\[
= \sum_{x \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q} (-1) \text{Tr}(\alpha x^{i+1}) \sum_{\beta \in \mathbb{F}_q} (-1) \text{Tr}(\beta x^{j+1})
\]

\[
= q \cdot \sum_{x=0}^1 \text{Tr}(\alpha x^{i+1}) = 2^{2m};
\]

where \( M_2 \) is the number of solutions to the equation

\[
\begin{align*}
\begin{cases}
   x^{i+1} + y^{j+1} = 0 \\
   x^{j+1} + y^{i+1} = 0. 
\end{cases}
\end{align*}
\]

For any given \( x \in \mathbb{F}_q \), since \( \gcd(2^i + 1, 2^m - 1) = 1 \), there is a unique \( y \in \mathbb{F}_q \) which satisfies the first one of the above equation system \( 6 \), and thus \( y = x \). Therefore \( M_2 = q = 2^m \) and the result is obtained.

Lemma 19: The number of solutions of the following polynomial equation system

\[
\begin{align*}
\begin{cases}
   x^{i+1} + y^{j+1} + z^{i+1} = 0 \\
   x^{j+1} + y^{i+1} + z^{i+1} = 0,
\end{cases}
\end{align*}
\]

is

\[
M_3 = (2^m - 1) + 2 \gcd(i-j, m) + 2 \gcd(i+j, m) - 2 \gcd(|i-j|, i+j, m) + 2^m.
\]

Proof: Here, only the situation \( i > j \) is considered. Divide both sides of the two equations by \( z^{i+1} \) and \( z^{j+1} \) respectively, and then after simplification they become:

\[
\begin{align*}
\begin{cases}
   x^{i+1} + y^{j+1} + 1 = 0 \\
   x^{j+1} + y^{i+1} + 1 = 0.
\end{cases}
\end{align*}
\]

Canceling \( y \) we have \((x^{i+1} + 1)^{2^j+1} = (x^{j+1} + 1)^{2^i+1}\) which is equivalent to

\[
(2^{i+j} + 1)(x^{i+j} + x) = (2^{i+j} + x)^{2^j}(x^{i+j} + x) = 0.
\]

Therefore \( x^{i+j} = x \) or \( x^{i+j} = 0 \), and let’s consider them separately.

Case I: \( x^{i+j} = x \). Set \( k_1 = \gcd(i-j, m) \) and \( q_1 = 2^{k_1} \), then \( F_{q_1} = \mathbb{F}_{2^{k_1}} \). For any \( x \in \mathbb{F}_{q_1} \), since \( \gcd(2^1 + 1, 2^{m-1} - 1) = 1 \), there is a unique element \( y \in \mathbb{F}_q \), such that \( x^{i+j} + y^{i+j} + 1 = 0 \). Shift elements of the last equation, then take the exponential power \( 2^{i+j} \):

\[
(2^{i+j} + 1)^{2^{i+j}} = (x^{i+j} + 1)^{2^{i+j}} + 1 = x^{2^{i+j}} + 1.
\]

In the last step, we have used the fact that \( x \in F_{q_1} \subset F_{2^{i+j}} \), that is \( x^{i+j} = x \). Comparing the left most and right most sides of equation \( 8 \) to the first one of \( 7 \), \( (2^{i+j})^{2^{i+j}} = y^{i+j} \). That is

\[
(2^{i+j})^{2^i+1} = y^{i+j}.
\]

Using again the fact that \( \gcd(2^i + 1, 2^m - 1) = 1 \), equation \( 9 \) implies that \( y^{i+j} = y \), so \( y \in \mathbb{F}_{q_1} \).

For \( x, y \in \mathbb{F}_{q_1} \), take the exponential power \( 2^{i} \) of the second equation of \( 7 \):

\[
(2^{i+j} + 1)^{2^{i}} = (x^{i+j} + 1)^{2^{i}} = x^{2^{i+j}} + y^{2^{i+j}} + 1
\]

\[
= x^{2^i \cdot x^{i+j}} + y^{2^i \cdot y^{i+j}} + 1
\]

\[
= x^{2^i + y^{2^i+j}} + 1,
\]

which implies that the two equations of \( 7 \) are equivalent. So, the number of solutions \( (x, y) \) of \( 7 \) in \( \mathbb{F}_{q_1} \) is \( N_3 = q_1 \).

Case II: \( x^{i+j} = 0 \). Set \( k_2 = \gcd(i+j, m) \), \( q_2 = 2^{k_2} \) and \( F_{q_2} = \mathbb{F}_{2^{k_2}} \). Let \( N_2' \) be the number of \( (x, y) \in \mathbb{F}_{q_2}^2 \) satisfying \( 7 \), then similarly we have \( N_2' = q_2 \).

For the joint of the solution sets of the two cases, set \( k_3 = \gcd(i-j, i+j, m) \), \( q_3 = 2^{k_3} \) and \( F_{q_3} = \mathbb{F}_{2^{k_3}} \). Then the number of \( (x, y) \in \mathbb{F}_{q_3}^2 \) satisfying \( 7 \) is \( N_3 = q_3 \).

Combing above, the number of \( (x, y) \in \mathbb{F}_{q_3}^2 \) satisfying \( 7 \) is \( N' = N_1' + N_2' - N_3' \). Thus \( M_3 = (q-1)N' + M_2 \), and the result of the lemma is obtained.

Corollary 5: There is the following result about the exponential sum

\[
\sum_{\alpha, \beta \in \mathbb{F}_q} T(\alpha, \beta)^{3} = 2^{2m} M_3.
\]

Let \( f(x) = \alpha x^{2^i + 1} + \beta x^{2^j + 1} \), where \( (\alpha, \beta) \in \mathbb{F}_{q}^2 \setminus \{(0, 0)\} \).

According to the relation between the weight of a codeword and corresponding exponential sum \( 2 \), assume that \( T(\alpha, \beta) \) takes only three possible values \( \pm 2^{n/2} \), 0. For \( \varepsilon = \pm 1 \), define
Theorem 3: Let $m = 2t + 2$ where $t \geq 1$. Then for the punctured second-order Reed-Muller code $RM(2,m)^*$, the distance profile of the cyclic subcode chain in Corollary [2] is optimum under Standard II.

Example 4: Set $m = 6 = 2 \cdot 2 + 2$, i.e. $t = 2$. The optimum cyclic subcode chain given in Theorem [3] can be constructed as follows. Note that there are five primitive idempotents here $\theta_0, \theta_1^\dagger, \theta_3^\dagger, \theta_5^\dagger$ and $\theta_6^\dagger$.

- The minimum distance of the cyclic code with idempotent $\theta_0$ is 63, and it is chosen as the first cyclic subcode of the chain.
- The minimum distances of the cyclic subcodes with idempotents $\theta_0 + \theta_1^\dagger, \theta_0 + \theta_3^\dagger, \theta_0 + \theta_5^\dagger$ and $\theta_0 + \theta_6^\dagger$ are 31, 24, 31 and 27 respectively. There are two choices for us: $\theta_0 + \theta_1^\dagger$ or $\theta_0 + \theta_3^\dagger$, which will be suggested later.
- The minimum distances of the cyclic subcodes with idempotents $\theta_0 + \theta_1^\dagger + \theta_3^\dagger, \theta_0 + \theta_1^\dagger + \theta_5^\dagger$ and $\theta_0 + \theta_1^\dagger + \theta_6^\dagger$ are 23, 23, 27 and 24, 23. So, in this step $\theta_0 + \theta_1^\dagger + \theta_6^\dagger$ is selected, and then in last step $\theta_0 + \theta_1^\dagger$ is selected.
- The minimum distances of the cyclic subcodes with idempotents $\theta_0 + \theta_1^\dagger + \theta_3^\dagger + \theta_5^\dagger$ and $\theta_0 + \theta_1^\dagger + \theta_3^\dagger + \theta_6^\dagger$ are 15 and 23 respectively. Select $\theta_0 + \theta_1^\dagger + \theta_3^\dagger + \theta_6^\dagger$ in this step.
- Finally the minimum distance of the cyclic code with idempotent $\theta_0 + \theta_1^\dagger + \theta_3^\dagger + \theta_5^\dagger$, i.e $RM(2,m)^*$, is 15.

Therefore, the ODPC-II$^{inv}$ of the punctured second-order Reed-Muller code $RM(2,6)^*$ is $d_{r_0} = 15, d_{r_1} = 23, d_{r_2} = 27, d_{r_3} = 31, d_{r_4} = 63$.

V. SUBOPTIMUMS WITH RESPECT TO ODPC-I$^{inv}$ OF $RM(2,m)^*$ AND ONE OPTIMUM

In this section, the cyclic subcode chain of the punctured second-order Reed-Muller code $RM(2,m)^*$ is studied under Standard I. Proposition [1] of Subsection V-A gives a suboptimum result with respect to ODPC-I$^{inv}$ for the case $m = 2t + 1$, which considers almost all the subcode chain classes respectively. Proposition [2] of Subsection V-B concerns the case where $m = 2t + 2$ for almost half of the subcode chain classes, and Corollary [7] emphasizes that in fact the optimum result can be obtained when $m$ is a power of 2.

A. The Case of $m = 2t + 1$

From Lemma [1], the length of the cyclic subcode chains is $\lambda = t + 2$. The number of the cyclic subcode chains is $\lambda! = (t + 2)!$. The number of the chains in each class is $\mu = (t + 1)! = (t + 1)!$, the number of the classes is $t + 2$. For the study of the ODPC-I$^{inv}$, consider the dimension profile

$$
(t + 1)m + 1, \ldots, um + 1, \ldots, 2m, m,
$$

where $2 \leq u \leq t$.

Proposition 1: Let $m = 2t + 1$ where $t \geq 2$. For the code $RM(2,m)^*$, consider Standard I with dimension profile (10). If we are requiring that, the cyclic subcode $C_0$ or equivalently the primitive idempotent $\theta_1^\dagger$ is selected first, the cyclic subcode
chain obtained by adding the primitive idempotents one by one in the following order is optimum:
\[
\theta^*_1, \theta^*_i, \ldots, \theta^*_{i-u+2}, \theta_0, \theta^*_{i-u+1}, \ldots, \theta^*_i.
\]
And the distance profile is
\[
\begin{align*}
\sigma_\tau = 2^{2t} - 2^{2t-v-1} - 1 (0 \leq v \leq t - u + 1) \\
\sigma_\tau = 2^{2t} - 2^{2t-v} (t - u + 2 \leq v \leq t) \\
\sigma_{t+1} = 2^{2m-1} = 2^{2t},
\end{align*}
\]
where \(\theta_0\) is selected to be added in the \((u+1)\)th order.

**Example 5:** For \(t = 2\), that is \(m = 5, n = 31, u = 2\).

Proposition 4 provides a distance profile \(d_{\tau_0} = 7, d_{\tau_1} = 11, d_{\tau_2} = 12, d_{\tau_3} = 16\) with dimension profile 16, 11, 10, 5. The generated cyclic codes [31, 11, 11], [31, 10, 12] and [31, 5, 16] are optimal [6].

**B. The Case of** \(m = 2t + 2\)

In this case, the length of the cyclic subcode chains is \(\lambda = t + 3\), and the number of the cyclic subcode chains is \(\lambda! = (t+3)!\). The number of chains in each class is \(\mu = (t+1)! - 1! = (t + 1)\), and the number of the classes is \((t+3)(t+2)\). In Proposition 2, a suboptimum ODPC-\(1^\text{st}\) of \(\mathcal{R}M(2, m)^*\) is presented, with corresponding dimension profile
\[
(t+1)m + \frac{m}{2}, \ldots, (j-1)m + \frac{m}{2} + 1, \ldots, \frac{n}{2}m, n,
\]
where \(2 \leq i < j \leq t + 1\).

**Proposition 2:** Let \(m = 2t + 2\) where \(t \geq 2\). For the code \(\mathcal{R}M(2, m)^*\), consider Standard I with dimension profile \((11)\).

If we are requiring that, the cyclic subcode \(\mathcal{C}_0\) or equivalently the primitive idempotent \(\theta^*_1\) is selected first, the cyclic subcode chain obtained by adding the primitive idempotents one by one in the following order is optimum:
\[
\theta^*_1, \theta^*_i, \ldots, \theta^*_{i-v+2}, \theta_0, \theta^*_{i-v+1}, \ldots, \theta^*_i,
\]
And the distance profile is
\[
\begin{align*}
d_{\tau_0} = 2^{2t+1} - 2^{2t-v} - 1 (0 \leq v \leq t - j + 2) \\
d_{\tau_0} = 2^{2t+1} - 2^{2t-v+1} (j - 3 \leq v \leq t - i + 2) \\
d_{\tau_0} = 2^{2t+1} - 2^{2t-v+2} (t - i + 3 \leq v \leq t + 1) \\
\sigma_{t+1} = 2^{2m-1} = 2^{2t+1},
\end{align*}
\]
where \(\theta^*_i\) is selected in the \((i+1)\)th order, and \(\theta_0\) is selected in the \((j+1)\)th order.

**Example 6:** For \(t = 2\), that is \(m = 6, n = 63, i = 2, j = 3\).

Proposition 2 provides a distance profile \(d_{\tau_0} = 15, d_{\tau_1} = 23, d_{\tau_2} = 24, d_{\tau_3} = 24, d_{\tau_4} = 32\) with dimension profile 22, 16, 15, 12, 6. The generated cyclic codes [63, 16, 23], [63, 15, 24], [63, 12, 24] and [63, 6, 32] are almost optimal [6].

**Corollary 7:** In Proposition 2 if \(m = 2^s\) \((s \geq 2)\), from Corollary 5 we do not require the preassumption that the primitive idempotent \(\theta^*_1\) is the first to be selected, since \(\theta^*_1\) corresponds to the unique nontrivial irreducible cyclic code with minimum distance \(2^{2m-1}\).

VI. CONCLUSION

The optimum distance profile serves as a new research field in coding theory. It has been investigated for the generalized Reed-Solomon code, the Golay code, the first-order Reed-Muller code, the second-order Reed-Muller code, and some other codes in [7], [13] and 11]. Known results on the distance profile of the linear codes can be applied to construct polar codes with good polarizing exponents [14]. Rather than the general linear codes, this paper studies cyclic codes and their cyclic subcode chains because of easy encoding and more algebraic structures.

For the punctured second-order Reed-Muller code \(\mathcal{R}M(2, m)^*\), suboptimum results (optimum under certain requirement) of two standards about the ODPC are presented in Theorem 1, Theorem 2, Proposition 1 and Proposition 2 the requirement of which is common such that only the primitive idempotent \(\theta^*_1\) is fixed early. And the results deduce the ODPCs for the case of \(m = 2t + 2\), see Theorem 3 and Corollary 7.

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