ARTICLE TEMPLATE

On the Leibniz rule and Laplace transform
for fractional derivatives

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ARTICLE HISTORY
Compiled February 1, 2019

ABSTRACT
Taylor series is a useful mathematical tool when describing and constructing a function. With the series representation, some properties of fractional calculus can be revealed clearly. This paper investigates two typical applications: Leibniz rule and Laplace transform. It is analytically shown that the commonly used Leibniz rule cannot be applied for Caputo derivative. Similarly, the well-known Laplace transform of Riemann-Liouville derivative is doubtful for n-th continuously differentiable function. By the aid of this series representation, the exact formula of Caputo Leibniz rule and the explanation of Riemann-Liouville Laplace transform are presented. Finally, three illustrative examples are revisited to confirm the obtained results.

KEYWORDS
Fractional calculus, Taylor series, Leibniz rule, Laplace transform, nonzero initial instant.

1. Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially powerful tools for modelling and controlling those practical plants with history dependent and global correlative properties. As for the recent relevant works, the readers can refer to the excellent papers [1–3] and the references therein.

Fractional derivatives lead to a lot of unusual properties. For example, all the well-known fractional derivatives involving Riemann-Liouville, Caputo and Grünwald-Letnikov definitions violate the usual form of the Leibniz rule $\frac{\text{d}^\alpha}{\text{d}t^\alpha}\{f(t)g(t)\} = f(t)\frac{\text{d}^\alpha}{\text{d}t^\alpha}g(t) + g(t)\frac{\text{d}^\alpha}{\text{d}t^\alpha}f(t)$. Though many references [4–7] confirm this unusual property and present the corresponding Leibniz rule, the questionable rule was still used widely [8,9]. Particularly, the original study of Leibniz rule for fractional derivatives can date back to 1832 when Liouville focused on the issue and developed a correct
Leibniz formula in the form of infinite series summation \([10, 11]\). Generalizations of the Leibniz rule for fractional derivatives are also derived successively by Osler in \([12, 14]\). On this basis, Tremblay improved the fractional Leibniz rule and its integral analogue further \([15]\). These Leibniz rules are especially useful for the evaluation of fractional derivative of a product function, such as, the Lyapunov function. Nonetheless, on the issue of Leibniz rule, only Riemann-Liouville definition was considered.

Under Caputo definition, to the best of our current knowledge, the related Leibniz rule in infinite series form has been adopted in many papers \([16-18]\). In fact, the adopted Leibniz rule in \([16]\) holds for Riemann-Liouville derivative or even Riemann-Liouville integral and it has been highly accepted and widely applied. However, no literature could support the derivation process of such a rule and explain the peculiar items \(C^a \mathcal{D}_{t}^{-k} g (t)\) with \(k > \alpha\). Fortunately, there are several attempts to define a new type of Leibniz rule for Caputo fractional derivative. For instance, Diethelm originated the related discussion and revealed the difference between Caputo and Riemann-Liouville derivative in a pioneering monograph \([19]\). Afterwards, reference \([20]\) revisited the issue in but gave a wrong conclusion. Note that study on such Leibniz rule is still at its early stage. Additionally, the Laplace transform of Riemann-Liouville derivative contains fractional derivatives as initial value which is not practical enough \([10, 21, 22]\). With the adoption of series representation \([23, 24]\), the Caputo Leibniz rule and Riemann–Liouville Laplace transform might be established anew.

Bearing the above discussion in mind, this paper aims at addressing the applications of fractional series representation on Leibniz rule and Laplace transform. To this end, Section 2 presents basic definitions and the so-called series representation of fractional calculus. Section 3 shows the violation of fractional derivative on well-adopted Leibniz rule and Laplace transform and derives the correct counterparts. Section 4 validates the validity of the indicated fact. Section 5 draws the conclusions.

2. Preliminaries

The concept of fractional calculus has been known because of the development of the regular calculus, with the origin probably being associated with the discussion between Leibniz and L'Hôpital in 1695. Today, there are numerous different definitions related to fractional calculus, among which Riemann-Liouville and Caputo definitions are two of the most popular ones which have indeed played a striking role in engineering and science \([25]\).

In 1847, Riemann derived a definition for fractional integral as

\[
\int_{a}^{t} \mathcal{I}_{t}^{\alpha} f (t) \triangleq \frac{1}{\Gamma (\alpha)} \int_{a}^{t} (t - \tau)^{\alpha-1} f (\tau) d \tau,
\]

which is commonly called Riemann–Liouville fractional integral, where \(\alpha > 0\) is the integral order, \(a\) is the constant lower terminal and \(\Gamma (\cdot)\) is the Euler gamma function.

In the light of such a fractional integral in \([1]\), two fractional derivatives were established successively, i.e., Riemann–Liouville case (in 1872)

\[
\int_{a}^{t} \mathcal{D}_{t}^{\alpha} f (t) \triangleq \frac{d^{\alpha}}{dt^{\alpha}} \int_{a}^{t} \mathcal{I}_{t}^{n-\alpha} f (t),
\]

and Caputo case (in 1967)

\[
\int_{a}^{t} \mathcal{D}_{t}^{\alpha} f (t) \triangleq \frac{\int_{a}^{t} \mathcal{I}_{t}^{n-\alpha} \frac{d^{n}}{dt^{n}} f (t),}
\]

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with \( n \in \mathbb{N}_+ \) and \( \alpha \in (n - 1, n) \).

Reference [25] shows that
\[
R_a \mathcal{I}_t^\alpha f(t) = R_a \mathcal{I}_t^{-\alpha} f(t) .
\]

for any \( \alpha > 0 \) and if the mentioned function \( f(t) \) is \((n - 1)\)-times continuously differentiable and \( f^{(n)}(t) \) is integrable for \( t \geq a \), the relationship between the Riemann–Liouville derivative and the Caputo one can be expressed as
\[
R_a \mathcal{I}_t^\alpha f(t) = C_a \mathcal{I}_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+\alpha)} (t-a)^{k-\alpha}.
\]

for \( n \in \mathbb{N}_+ \) and \( \alpha \in (n - 1, n) \).

Based on the aforementioned definitions with \( \alpha \in (n - 1, n) \), \( \forall n \in \mathbb{N}_+ \), the following relations can be found smoothly [24]
\[
\frac{d^n}{dt^n} f(t) = \lim_{\alpha \to n} R_a \mathcal{I}_t^\alpha f(t) = \lim_{\alpha \to n-} C_a \mathcal{I}_t^\alpha f(t) ,
\]

\[
a \mathcal{I}_t^n f(t) = \lim_{\alpha \to n} R_a \mathcal{I}_t^\alpha f(t) ,
\]

which distinctly demonstrate that the fractional calculus could be regarded as the natural generalization of regular calculus. Nevertheless, such fractional derivatives with non-integer order are special integrals in relation to all historical data, which just forms the long memory characteristic of fractional derivatives. Occasionally, this property is also called as nonlocal characteristic or global correlation or history dependence. It is this characteristic that fashion the essential differences between the fractional calculus and the traditional one.

Before moving on, a key lemma will be given first. In light of this lemma, one can use the power functions \((t-a)^k\) and the integer derivatives values to represent the fractional calculus, which brings great convenience in the subsequent theoretical analysis. In this study, these representations are named as fractional Taylor series.

**Lemma 2.1.** *(see [24])* Let \( f(t) \) be a function whose Taylor series expansion exists at \( a \), then
\[
R_a \mathcal{I}_t^\alpha f(t) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{\Gamma(k+\alpha)} (t-a)^{k-\alpha} , \alpha \in (-\infty, +\infty) , \quad (8)
\]

\[
C_a \mathcal{I}_t^\alpha f(t) = \sum_{k=n}^{+\infty} \frac{f^{(k)}(a)}{\Gamma(k+\alpha)} (t-a)^{k-\alpha} , \alpha \in (n - 1, n) , n \in \mathbb{N}_+ , \quad (9)
\]

If \( f(\tau) \), \( \tau \in (a,t) \) can be expressed as a Taylor series at \( t \), then
\[
R_a \mathcal{I}_t^\alpha f(t) = \sum_{k=0}^{+\infty} \left( \frac{\alpha}{k} \right) \frac{f^{(k)}(t)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha} , \alpha \in (-\infty, +\infty) , \quad (10)
\]

\[
C_a \mathcal{I}_t^\alpha f(t) = \sum_{k=n}^{+\infty} \left( \frac{\alpha-n}{k-n} \right) \frac{f^{(k)}(t)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha} , \alpha \in (n - 1, n) , n \in \mathbb{N}_+ . \quad (11)
\]
3. Main Results

This section focuses on the Leibniz rule and the Laplace transform of fractional calculus and develops some interesting observations.

3.1. Leibniz Rule

**Theorem 3.1** (Leibniz rule of Riemann–Liouville calculus). For any constant \( \alpha \in (-\infty, +\infty) \), let \( f(\tau) \) be a function whose Taylor series expansion exists at \( t \), then

\[
\mathcal{D}_t^\alpha \{ f(t) g(t) \} = \sum_{j=0}^{\infty} \binom{\alpha}{j} f^{(j)}(t) \mathcal{D}_t^{\alpha-j} g(t),
\]

and if \( g(\tau) \) can be expressed as a Taylor series at \( t \), one has

\[
\mathcal{D}_t^\alpha \{ f(t) g(t) \} = \sum_{j=0}^{\infty} \binom{\alpha}{j} g^{(j)}(t) \mathcal{D}_t^{\alpha-j} f(t).
\]

**Proof.** With the series expansion of Riemann–Liouville fractional derivative in (10), the following Leibniz rule can be confirmed

\[
\mathcal{D}_t^\alpha \{ f(t) g(t) \} = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(-\alpha)^{k-n}}{\Gamma(k+1-\alpha)} \mathcal{D}_t^k \{ f(t) g(t) \}
\]

in which the facts \( \sum_{k=0}^{\infty} \binom{\alpha}{k} = \sum_{j=0}^{\infty} \binom{\alpha}{j} \), \( (\alpha) \), \( (i+j) \), and \( \binom{\alpha}{j} \) are applied. Notably, considering that the infinite series in (10) is available for Riemann–Liouville definition with \( \alpha \in (-\infty, +\infty) \), Leibniz rule in (12) still holds for \( \alpha \in (-\infty, +\infty) \). During the derivation, \( f^{(k)}(t) \) with \( k \in \mathbb{N}, t \in [a, +\infty) \) are assumed to exist.

With the interchangeability of the product operation, it follows

\[
\mathcal{D}_t^\alpha \{ f(t) g(t) \} = \mathcal{D}_t^\alpha \{ g(t) f(t) \}.
\]

Combining with (12) and (15), (13) can be concluded immediately. The proof is thus completed.

Setting \( g(t) = t^m, m \in \mathbb{N} \), one can obtain the following corollary.

**Corollary 3.2.** For any suitable \( f(t) \)

\[
\mathcal{D}_t^\alpha \{ f(t) t^m \} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} t^{m-k} \mathcal{D}_t^{\alpha+k} f(t),
\]

\[
\mathcal{D}_t^\alpha \{ f(t) t^m \} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)} t^{m-k} \mathcal{D}_t^{\alpha-k} f(t).
\]
Remark 1. Actually, the rule in (12) with \( \alpha > 0 \) has been suggested by Liouville in 1832 as a generalization of the Leibniz rule for regular derivative, so it is convenient to preserve Leibniz’s name also in this case [26,27]. With the help of (6) and \( \Gamma(-k) = \pm \infty, \forall k \in \mathbb{N} \), the \( \alpha = n \in \mathbb{N}_+ \) case follows from (12) and (13) immediately

\[
\frac{d^n}{dt^n} \{ f(t) g(t) \} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(t) g^{(n-j)}(t),
\]

which is just the traditional Leibniz rule.

Similarly, applying the formula (4), a special case, i.e., \( \alpha = 0 \) can be derived from Theorem 3.1 as

\[
\mathcal{R}_a^0 \{ f(t) g(t) \} = f(t) g(t).
\]

When \( \alpha < 0 \), the results in Theorem 3.1 can be simplified as

\[
\mathcal{R}_a \mathcal{I}^{-\alpha} \{ f(t) g(t) \} = \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) \mathcal{R}_a \mathcal{I}^{-\alpha+j} g(t),
\]

and

\[
\mathcal{R}_a \mathcal{I}^{-\alpha} \{ f(t) g(t) \} = \sum_{j=0}^{+\infty} \binom{\alpha}{j} g^{(j)}(t) \mathcal{R}_a \mathcal{I}^{-\alpha+j} f(t),
\]

respectively.

Notably, many published results adopted the Leibniz rule (12) for the Caputo definition, that is to say,

\[
\mathcal{C}_a \mathcal{D}_t^{\alpha} \{ f(t) g(t) \} = \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) \mathcal{C}_a \mathcal{D}_t^{\alpha-j} g(t),
\]

while almost no scholars explored whether it is true or false.

In what follows, we will analytically prove that the Leibniz rule (23) is not established unless the Caputo derivative of \( g(t) \) is equal to the counterpart of Riemann–Liouville case.

Proposition 3.3. Leibniz rule of Caputo derivative (23) does not hold.

Proof. To illustrate the violation of (23), two aspects will be deployed. Likewise, if we put \( \alpha \) in the interval \((n-1,n)\) with positive integer \( n \), then a question will emerge instantly, namely, how to interpret \( \mathcal{C}_a \mathcal{D}_t^{\alpha-j} g(t), \quad j > n \). Recalling Leibniz rule for Riemann–Liouville derivative, one directly assumes that \( \mathcal{C}_a \mathcal{D}_t^{\alpha-j} g(t) = \mathcal{R}_a \mathcal{D}_t^{\alpha-j} g(t) \) for \( j > n \).

\[\bullet\] Conflict with the facts.
Assuming \((23)\) holds, \(g(t) = 1, \ t \geq a\) gives
\[
\begin{align*}
\frac{C}{a} D_t^{\alpha} \{ f(t) g(t) \} &= \frac{C}{a} D_t^{\alpha} \{ f(t) g(t) \} \\
&= \sum_{k=0}^{+\infty} \binom{k}{\alpha} f^{(k)}(t) \sum_{i=0}^{k} \frac{\Gamma(n)}{(k+1)(k-n+1)} (t-a)^{k-i}.
\end{align*}
\]

On one hand, such a formula \((24)\) conflicts with the validated conclusion in \((11)\). For \(n=1\), \(n \in \mathbb{N}_+\), \(h(t)\) calculated by the way of the Riemann–Liouville integral, although there is no reason for this assumption. Apart from this queer assumption, another flaw is that the initial step \(k\) equals to \(n\) not \(0\).

Theorem 3.4 (Leibniz rule of Caputo derivative). For any constant \(\alpha \in (n-1, n), \ n \in \mathbb{N}_+\), if \(f(t)\) can be expressed as a Taylor series, one has
\[
\frac{C}{a} D_t^{\alpha} \{ f(t) g(t) \} = \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)}(t) \sum_{i=0}^{j} \frac{\Gamma(n)}{(j+1)(j-n+1)} (t-a)^{j-i} g(t) + R_1,
\]
and if \( g(t) \) can be expressed as a Taylor series, one has

\[
C^\alpha_a D_t^a \{ f(t) g(t) \} = \sum_{j=0}^{+\infty} \binom{\alpha}{j} g^{(j)}(t) C^\alpha_a D_t^{a-j} f(t) + R_2,
\]

where

\[
R_1 = \sum_{k=0}^{n-1} \sum_{j=0}^k \left( \binom{\alpha}{j} f^{(j)}(t) - \binom{\alpha}{k} f^{(k)}(a) \right) \frac{g^{(k-j)}(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)},
\]

\[
R_2 = \sum_{k=0}^{n-1} \sum_{j=0}^k \left( \binom{\alpha}{j} g^{(j)}(t) - \binom{\alpha}{k} g^{(k)}(a) \right) \frac{f^{(k-j)}(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}.
\]

**Proof.** Applying Theorem 3.1 and (30) yields

\[
C^\alpha_a D_t^a \{ f(t) g(t) \}
= R^\alpha_a D_t^a \{ f(t) g(t) \} - \sum_{k=0}^{n-1} \frac{d^k}{dt^k} \{ f(t) g(t) \}_{t=a}^{(t-a)^{k-\alpha}}
= \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) R^\alpha_a D_t^{a-j} g(t)
- \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{\alpha}{j} f^{(j)}(t) \frac{f^{(k-j)}(a)(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}
= \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) C^\alpha_a D_t^{a-j} g(t)
+ \sum_{j=0}^{n-1} \sum_{k=0}^{k-1} \binom{\alpha}{j} f^{(j)}(t) \frac{g^{(k-j)}(a)(t-a)^{j-\alpha}}{\Gamma(j+1-\alpha)}
- \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{\alpha}{j} f^{(j)}(t) \frac{f^{(k-j)}(a)(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}
= \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) C^\alpha_a D_t^{a-j} g(t)
+ \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{\alpha}{j} f^{(j)}(t) C^\alpha_a D_t^{a-j} g(t)
- \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{\alpha}{j} f^{(j)}(t) C^\alpha_a D_t^{a-j} g(t)
\]

Considering that \( C^\alpha_a D_t^a \{ f(t) g(t) \} = C^\alpha_a D_t^a \{ g(t) f(t) \} \) holds for all possible functions \( f(t), g(t) \) and the order \( n - 1 < \alpha < n \in \mathbb{N}_+ \), (29) can be derived smoothly.

**Remark 2.** Notably, Theorem 3.4 indicates that Leibniz rule in (23) does not always hold, even when \( C^\alpha_a D_t^{a-j} g(t) = R^\alpha_a D_t^{j-\alpha} g(t) \) is assumed for \( j > n \). Two available Leibniz rules for Caputo derivatives are developed via the series representation method and they can also be established with the help of (21), (22) and (30). When the range of \( \alpha \) is set in \((0, 1)\), the proposed formula (28) degenerates into Theorem 3.17 in [19]. From the previous discussion, (28) indicates the following results.

- C^\alpha_a D_t^a \{ f(t) g(t) \} = \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) R^\alpha_a D_t^{a-j} g(t) holds if \( f^{(j)}(a) = 0 \) or \( g^{(j)}(a) = 0 \) for all \( j = 0, 1, \cdots, n-1 \).
- C^\alpha_a D_t^a \{ f(t) g(t) \} = \sum_{j=0}^{+\infty} \binom{\alpha}{j} f^{(j)}(t) C^\alpha_a D_t^{a-j} g(t) holds if \( g^{(j)}(a) = 0 \) for all \( j = 0, 1, \cdots, n-1 \).

**3.2. Laplace Transform**

With the help of the infinite series in Lemma 2.1 some results on Laplace transform can be double checked.
Let us start with the elementary unit $t^\mu$, as

\[
\mathcal{L}\{t^\mu\} = \int_0^{+\infty} e^{-st}t^\mu dt = \frac{1}{s^{\mu+1}} \int_0^{+\infty} e^{-st}(st)^\mu dt (st) = \frac{1}{\Gamma(\mu+1)} \int_0^{+\infty} e^{-\tau\mu} d\tau
\]

in which the definition $\Gamma(z) \triangleq \int_0^{+\infty} x^{z-1}e^{-x} dx$ is adopted. Actually, the corresponding result for $\mu \in \mathbb{N}$ can be obtained via the formula of integration by parts. The range of $\mu$ should be limited as $\mu > -1$ since the integral definition of Gamma function is only suitable for positive argument. From the integrability of $t^\mu$, one has

\[
\mathcal{R} \mathcal{L}^\alpha t^\mu = \int_0^t (t - \tau)^{\alpha-1} \tau^\mu d\tau = \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1}(1 - u)^{\alpha-1}u^\mu du = \frac{\Gamma(\alpha+\mu)}{\Gamma(\alpha+1)} B(\mu + 1, \alpha)
\]

where the integral formula of Beta function holds only for positive arguments, i.e., $\mu + 1 > 0$ and $\alpha > 0$. Without loss of generality, let us suppose $\mu \in (-2, -1)$, then $\mu + 1 > -1$. From the differential property of Laplace transform, it follows

\[
\mathcal{L}\{t^\mu\} = \mathcal{L}\{\frac{d}{dt} t^{\mu+1}\} = s\mathcal{L}\{t^{\mu+1}\} - \frac{\mu+1}{\mu+1} \big|_{t=0} = \frac{s^{\mu+1}}{\Gamma(\mu+1)} - \infty,
\]

which indicates that the Laplace transform of $t^\mu$ with $\mu < -1$ does not exist in the classical sense.

If one defines $F(s) \triangleq \mathcal{L}\{f(t)\}$, by definition

\[
F(s) = \int_0^{+\infty} e^{-st}f(t) dt = \int_0^{+\infty} e^{-st} \sum_{k=0}^{+\infty} \frac{f^{(k)}(t_0)}{k!} t^k dt = \sum_{k=0}^{+\infty} \frac{f^{(k)}(t_0)}{k!} \int_0^{+\infty} e^{-st} t^k dt = \sum_{k=0}^{+\infty} \frac{f^{(k)}(t_0)}{s^{k+1}}
\]

where $f(t)$ can be expressed as a Taylor series at $a$.

From these discussions, the following results can be obtained

\[
\mathcal{R} \mathcal{L}^\alpha f(t) = \int_0^{+\infty} e^{-st} \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{\Gamma(k+1+\alpha)} t^{k+\alpha} dt = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{\Gamma(k+1+\alpha)} \int_0^{+\infty} e^{-st} t^{k+\alpha} dt = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{s^{k+1+\alpha}} = s^{-\alpha} F(s)
\]
for $\alpha > 0$ and

$$\mathcal{L} \left\{ \mathcal{C}_t^\alpha f(t) \right\} = \int_0^{+\infty} e^{-st} \sum_{k=n}^{+\infty} \frac{f^{(k)}(0)}{\Gamma(k+\alpha)} t^{k-\alpha} dt$$

$$= \sum_{k=n}^{+\infty} \frac{f^{(k)}(0)}{s^{k+1-\alpha}}$$

$$= s^n [F(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{s^{k+1-\alpha}}]$$

$$= s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0).$$

(36) for $n - 1 < \alpha < n \in \mathbb{N}_+$. Actually, when the infinitely differentiable condition of $f(t)$ at $t = 0$ is removed, (35) and (36) still hold (see (2.242) and (2.253) of [25]).

Consider the differential property in frequency, one has

$$\frac{d^m}{ds^m} F(s) = \frac{d^m}{ds^m} \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{s^{k+1}}$$

$$= \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{s^{k+m+1}} \Gamma(-k-m)$$

$$= \sum_{k=0}^{+\infty} (-1)^m \frac{f^{(k)}(0) \Gamma(k+m+1)}{s^{k+m+1}} \Gamma(k+1).$$

(37) for any $m \in \mathbb{N}$. From the representation of Riemann–Liouville integral in Lemma 2.1, it follows

$$R_0 \mathcal{I}_t^\alpha \left\{ f(t) (-t)^m \right\} = (-1)^m R_0 \mathcal{I}_t^\alpha \left\{ \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} t^{k+m} \right\}$$

$$= (-1)^m \sum_{k=0}^{+\infty} \frac{f^{(k)}(0) \Gamma(k+m+1)}{k! \Gamma(k+m+\alpha+1)} t^{k+m+\alpha}.$$ (38)

Combing the above discussions, a beautiful result can be reached

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^{\alpha}} F^{(m)}(s) \right\} = R_0 \mathcal{I}_t^\alpha \left\{ f(t) (-t)^m \right\}.$$ (39)

Now, let us consider the general case

$$\mathcal{L} \left\{ (t-a)^\mu \right\} = \int_0^{+\infty} e^{-st} (t-a)^\mu dt$$

$$= e^{-sa} \int_0^{+\infty} e^{-s \tau \xi} d\xi$$

$$= e^{-sa} \int_0^{+\infty} e^{-\tau \mu} d\tau$$

$$= e^{-sa} \Upsilon(\mu + 1, -as),$$

(40)

where $\Upsilon(p, q) \triangleq \int_q^{+\infty} \tau^{p-1} e^{-\tau} d\tau$, $p \in \mathbb{R}_+$, $q \in \mathbb{R}$ and $\Upsilon(p, 0) = \Gamma(p)$.

If $f(t)$ can be expressed as a Taylor series at $a$, then one has

$$\mathcal{L} \left\{ f(t) \right\} = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{s^{k+1}} \frac{e^{-as} \Upsilon(k+1, -as)}{\Gamma(k+1)}.$$ (41)

$$\mathcal{L} \left\{ R_0 \mathcal{I}_t^\alpha f(t) \right\} = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{s^{k+1-\alpha}} \frac{e^{-as} \Upsilon(k+\alpha+1, -as)}{\Gamma(k+\alpha+1)}.$$ (42)

$$\mathcal{L} \left\{ c_0 \mathcal{D}_t^\alpha f(t) \right\} = \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{s^{k+1-\alpha}} \frac{e^{-as} \Upsilon(k-\alpha+1, -as)}{\Gamma(k-\alpha+1)},$$ (43)

where $n - 1 < \alpha < n \in \mathbb{N}_+$. Afterwards, the initial instant $a$ is also assumed to be 0 since the nonzero case leads to an inelegant result.
When \(0 < \alpha < 1, k - \alpha > -1\) hold for all \(k = 0, 1, \cdots\). Then one has

\[
\mathcal{L}\left\{\mathcal{D}_t^\alpha f(t)\right\} = \int_0^{+\infty} e^{-st} \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{\Gamma(k+1-\alpha)} t^{k-\alpha} dt = s^\alpha F(s),
\]

(44)

which matches the result in (2.256) of \[25\].

Supposing \(1 < \alpha < 2, f(0) \neq 0\) and applying (44), yields

\[
\mathcal{L}\left\{\mathcal{D}_t^\alpha f(t)\right\} = \int_0^{+\infty} e^{-st} \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{\Gamma(k+1-\alpha)} t^{k-\alpha} dt = s^\alpha F(s) - \infty,
\]

(45)

which means that in this case the Laplace transform of Riemann–Liouville derivative is singular. Actually, the similar results can be obtained for any \(\alpha > 1\). In other words, the Riemann–Liouville derivative of this function \(f(t)\) does not exist in the classical sense.

Recall the famous Laplace transform of Riemann–Liouville derivative

\[
\mathcal{L}\left\{\mathcal{D}_t^\alpha f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \mathcal{D}_t^{\alpha-k-1} f(t)\big|_{t=0},
\]

(46)

in which \(\mathcal{D}_t^{\alpha-k-1} f(t)\big|_{t=0}\) are the needed initial conditions, including one integral initial value and \(n - 1\) derivative initial values \[10\]. Note that (44) and (45) conflict with (46). Next, we will find out how the differences come into being.

Looking at equations (8) and (10), one singular property can be found on the unit \(t^{-\alpha}\) when \(\alpha > 0\). It cannot be expressed by the bases \(1, t, t^2, \cdots\) with finite weighting, since \(\lim_{t \to 0^+} t^{-\alpha} = +\infty\) and \(\lim_{t \to 0^+} t^k = 0, k \in \mathbb{N}\), which just shows the singularity of Riemann–Liouville derivative. If we consider the behaviour of Riemann–Liouville fractional calculus near the lower terminal, i.e., \(t \to 0^+\), the description

\[
\lim_{t \to 0^+} \mathcal{D}_t^\alpha f(t) = \begin{cases} 0 & \text{if } \alpha < 0 \text{ and } f(0) \neq \infty, \\ f(0) & \text{if } \alpha = 0, \\ \pm\infty & \text{if } \alpha > 0, \alpha \notin \mathbb{N} \text{ and } f(0) \neq 0. \end{cases}
\]

(47)

follows. It clearly illustrates that the initial conditions of fractional derivative are singular in the framework of integer order. If the function \(f(t)\) can be expressed as a Taylor series, the finite \(f^{(k)}(0), k = 0, 1, \cdots, n - 1\) are implied for \(n - 1 < \alpha < n \in \mathbb{N}_+\). For \(\alpha \in (0, 1)\), it follows \(\mathcal{D}_t^{\alpha-1} f(t)\big|_{t=0} = \mathcal{D}_t^{1-\alpha} f(t)\big|_{t=0} = 0\) and then (44) reduces to (44). For \(\alpha \in (1, 2)\), the initial value \(\mathcal{D}_t^{\alpha-2} f(t)\big|_{t=0} = 0\) and the initial value \(\mathcal{D}_t^{\alpha-1} f(t)\big|_{t=0} = \infty\) unless \(f(0) = 0\). As a result, the singular case in (45) appears.

Furthermore, representing the finite initial value \(f^{(k)}(0) = b_k, k = 0, 1, \cdots, n - 1\), then \(F(s)\) can be expressed as

\[
F(s) = \sum_{k=0}^{n-1} \frac{b_k}{s+k} + O\left(\frac{1}{s^{n+1}}\right).
\]

By using the initial value theorem of Laplace transform, nonzero \(b_k\) lead to the
singular results

\[
\lim_{t \to 0^+} \mathcal{D}_t^{\alpha-n} f(t) = 0, \quad (49)
\]

\[
\lim_{t \to 0^+} \mathcal{D}_t^{\alpha-1-k} f(t) = \infty, \quad k = 0, 1, \ldots, n - 2. \quad (50)
\]

**Corollary 3.5.** If the function \( f(t) \) is \((n-1)\)-times continuously differentiable, then the following two conditions are equivalent.

\[
\mathcal{D}_t^{\alpha-1-k} f(t) = 0, \quad k = 0, 1, \ldots, n - 1, \quad (51)
\]

\[
f^{(k)}(t) = 0, \quad k = 0, 1, \ldots, n - 1, \quad (52)
\]

where \( n - 1 < \alpha < n \in \mathbb{N}_+ \).

For all zero \( b_k \), \( \lim_{t \to 0^+} \mathcal{D}_t^{\alpha-1-k} f(t) = 0, k = 0, 1, \ldots, n - 1 \) or even \( \lim_{t \to 0^+} \mathcal{D}_t^{\alpha} f(t) \) equal zero too. In this case, \( \mathcal{D}_t^{\alpha} f(t) = C_t^{\alpha} f(t) \) and their Laplace transform are also equal.

### 4. Simulation Study

In this section, four numerical examples are provided to illustrate the validity of the proposed results. Additionally, \( \mathcal{D}_t^{\alpha-\beta} f(t) \) with \( \beta > 0 \) is calculated by \( \mathcal{D}_t^{\alpha} f(t) \) instead.

**Example 4.1.** Provided that \( 0 < \alpha < 1 \), \( f(t) = t - a \) and \( g(t) = 1 \), it follows

\[
\mathcal{C}_t^{\alpha} (t - a) = \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (53)
\]

The regular Leibniz rule \((23)\) corresponds

\[
\sum_{k=0}^{+\infty} \binom{\alpha}{k} f^{(k)}(t) \mathcal{D}_t^{\alpha-k} g(t) = \binom{\alpha}{0} (t - a) \mathcal{C}_t^{\alpha} g(t) + \binom{\alpha}{1} \mathcal{R}_t^{1-\alpha} g(t) = \alpha \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}, \quad (54)
\]

which differs from \((53)\). This proves the incorrect use of Leibniz rule \((23)\) for Caputo derivative.

From the proposed compensation formula in Theorem 3.4 one has

\[
\mathcal{C}_t^{\alpha} \left\{ f(t) g(t) \right\} = \sum_{k=0}^{+\infty} \binom{\alpha}{k} f^{(k)}(t) \mathcal{C}_t^{\alpha-k} g(t) + \left[ f(t) - f(a) \right] \frac{g(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)} = \alpha \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} (1 - \alpha)\quad (55)
\]

\[
= \mathcal{C}_t^{\alpha} (t - a).
\]
Example 4.2. In a similar way, setting $0 < \alpha < 1$, $f(t) = g(t) = t$, it leads to

$$
\frac{C}{a} D_t^\alpha t^2 = \frac{C}{a} D_t^\alpha \left[ (t-a)^2 + 2a(t-a) + a^2 \right] \\
= \frac{2(t-a)^{2-\alpha}}{\Gamma(3-\alpha)} + 2a \frac{t-a^{1-\alpha}}{\Gamma(2-\alpha)} \\
= \frac{2(t-a)^{1-\alpha}}{\Gamma(3-\alpha)} (t + a - a\alpha). 
$$

By using (23), the result

$$
\sum_{k=0}^{+\infty} \binom{\alpha}{k} f^{(k)}(t) \frac{C}{a} D_t^{\alpha-k} g(t) \\
= \binom{\alpha}{0} t^{(t-a)^{1-\alpha}} + \binom{\alpha}{1} \left( \frac{(t-a)^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{a(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \right) \\
= \frac{(t-a)^{1-\alpha}}{\Gamma(3-\alpha)} (2t + a\alpha - a\alpha^2)
$$

emerges, which is different from (56). After adding the compensation term in Theorem 3.4, the desired result can be obtained

$$
\int_{0}^{t} \frac{C}{a} D_t^{\alpha} f(t) g(t) \\
= \sum_{k=0}^{+\infty} \binom{\alpha}{k} f^{(k)}(t) \frac{C}{a} D_t^{\alpha-k} g(t) + [f(t) - f(a)] \frac{g(a)(t-a)^{\alpha}}{\Gamma(1-\alpha)} \\
= \sum_{k=0}^{+\infty} \binom{\alpha}{k} f^{(k)}(t) \frac{C}{a} D_t^{\alpha-k} g(t) + \frac{(t-a)^{1-\alpha}}{\Gamma(3-\alpha)} (2a - 3a\alpha + a\alpha^2) \\
= \frac{C}{a} D_t^{\alpha} t^2. 
$$

Example 4.3. Defining $f(t) = t^p + k$ with $\alpha > 0$, $p - \alpha > -1$, its Riemann–Liouville derivative follows

$$
\frac{R}{0} D_t^{\alpha} f(t) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} + k \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}. 
$$

By applying the formula (31), one has $\mathcal{L} \{ f(t) \} = \frac{\Gamma(p+1)}{s^{p+1}} + \frac{k}{s}$. Only when $0 < \alpha < 1$, one has

$$
\mathcal{L} \left\{ \frac{R}{0} D_t^{\alpha} f(t) \right\} = \frac{\Gamma(p+1)}{s^{p+\alpha+1}} + \frac{k}{s^{1-\alpha}}, 
$$

and the following relation holds

$$
\mathcal{L} \left\{ \frac{R}{0} D_t^{\alpha} f(t) \right\} = s^\alpha \mathcal{L} \{ f(t) \}. 
$$

Remark 3. This manuscript was first completed in 2016. To facilitate academic exchange, it is attached in arXiv platform. Once it was accepted by some journal, it will be removed here.

5. Conclusions

In this paper, two essential applications of fractional calculus have been studied by introducing the series representation. Specifically, it has proven that the commonly used Leibniz rule is not applicable to Caputo definition and the well-known Laplace transform of Riemann–Liouville is conditional. These concluded points confirm the fact that many recently published results suffer from an incorrect use of such questionable
results. More importantly, the correct form of Leibniz rule and the exact condition of Laplace transform are subsequently deduced. Finally, three examples are presented to illustrate the applicability and efficiency of the presented tools. It is believed that the proposed methods indeed open a new way to solve and analyze the related problems.

Funding

The work described in this paper was supported by the National Natural Science Foundation of China (61601431, 61573332), the Anhui Provincial Natural Science Foundation (1708085QF141), the fund of China Scholarship Council (201806345002), the Fundamental Research Funds for the Central Universities (WK2100100028) and the General Financial Grant from the China Postdoctoral Science Foundation (2016M602032).

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