GROUP INVARIANTS OF CERTAIN BURN LOOP CLASSES

GÁBOR P. NAGY

1. Introduction

To any loop \((L, \cdot)\), one can associate several groups, for example its multiplication groups \(G_{\text{left}}(L)\) and \(G_{\text{right}}(L)\) and \(M(L) = \langle G_{\text{left}}(L), G_{\text{right}}(L) \rangle\), the groups of (left or right) pseudo-automorphisms, group of automorphisms, or the group of collineations of the associated 3-net. Groups, which are isotope invariants are of special interest. For example, the groups \(G_{\text{left}}(L)\), \(G_{\text{right}}(L)\) and \(M(L)\) are isotope invariant for any loop \(L\). These groups contain many information about the loop \(L\), the standard references on this field are [2], [4], [11].

For some special loop classes, other isotope invariant groups can be defined. For Bol loops, M. Funk and P.T. Nagy [7] investigated the collineation group generated by the Bol reflections. The notion of the core was first studied by R.H. Bruck [4] for Moufang loops and by V.D. Belousov [3] for Bol loops. Recently, this concept was intensively used in P.T. Nagy and K. Strambach [10].

In the papers [5, 6], R.P. Burn defined the infinite classes \(B_{4n}, n \leq 2\) and \(C_{4n}, n \leq 2, n\) even of Bol loops. These examples satisfy the left conjugacy closed property, that is, their section \(S(L) = \{\lambda_x : x \in L\}\) is invariant under conjugation with elements of the group \(G_{\text{left}}(L) = \langle \lambda_x | x \in L \rangle\) generated by the (left) translations \(\lambda_x : y \mapsto xy\).

In this paper, we determine the collineation groups generated by the Bol reflections, the core, the automorphism groups and the full direction preserving collineation groups of the loops \(B_{4n}\) and \(C_{4n}\) given by R.P. Burn. We also prove some lemmas and use new methods in order to simplify the calculations in these groups.

2. Basic concepts

A loop \(L\) is said to be a Bol loop, if

\[ x \cdot (y \cdot xz) = (x \cdot yx) \cdot z \]

holds for all \(x, y, z \in L\). This is equivalent with \(\lambda_x \lambda_y \lambda_x \in S(L)\) for all \(x, y \in L\). In any Bol loop, the group

\[ N = \langle (\lambda_x^{-1}, \rho_x, \lambda_x) | x \in L \rangle \]

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belongs to the loop $L$ is a normal subgroup of the directions preserving collineation group of the 3-net belonging to the loop $L$, cf. [7], [8]. Actually, the fact that $(\lambda_x^{-1} \rho_x^{-1}, \lambda_x)$ is a direction preserving collineation for all $x \in L$ is equivalent with the Bol property for the coordinate loop. As in [7], we define the endomorphism $\Phi$ by

$$\Phi : \begin{cases} N \to G(L) \\ (\lambda_x^{-1} \rho_x^{-1}, \lambda_x) \mapsto \lambda_x. \end{cases}$$

This map $\Phi$ will help us to determine the group $N$, which acts transitively on the set of horizontal lines, and so, plays an important role in the description of the full collineation group of the 3-net. In general, about the kernel of $\Phi$ one can only know, that it is isomorphic to a subgroup of the left nucleus of $L$ full collineation group of the 3-net. In particular, it is non-Moufang, see [5, 6].

The groupoid $(L, +)$ is the groupoid $(L, +)$, where the binary operation “+” is defined by

$$x + y = x \cdot y^{-1}x, \quad x, y \in L.$$ 

This groupoid satisfies the following identities:

\begin{align*}
  x + x &= x \\
  x + (x + y) &= y \\
  x + (y + z) &= (x + y) + (x + z)
\end{align*}

An alternative way to define the core is via the action of the Bol reflections on the set of vertical lines of the associated 3-net. In this way, the core turns out to be strongly related to the group $N$.

We say that the loop $L$ is left conjugacy closed, if $S(L)$ is invariant under the conjugation with the elements of $G(L)$. This concept was introduced in the paper [9] by P.T. Nagy and K. Strambach. They also defined the notion of Burn loop, which is a left conjugacy closed Bol loop. Examples for such loops are the following constructions due to R.P. Burn [5, 6].

The section $S(L)$ of a loop $L$ is a sharply transitive set of permutations, for any $x \in L$, there is a uniquely defined $\lambda_x$ mapping the unit element 1 to $x$. Thus, by $x \cdot y = y^{\lambda_x}$, the multiplication of $L$ is given by the set $S(L)$ and the unit elements 1. Theorem 1.7 in [5, 6] says that if the set $S(L)$ is invariant under conjugation with its own elements, different choices of the unit element still give isomorphic loops, hence a Burn loop is completely determined by its section $S(L)$ (up to isomorphism).

The loops $B_{4n}$, $n \geq 2$: Let the group $G_{8n}$ be generated by the elements $\alpha, \beta, \gamma$ with the relations $\alpha^{2n} = \beta^2 = \gamma^2 = (\alpha \beta)^2 = id$, $\alpha \gamma = \gamma \alpha$ and $\beta \gamma = \gamma \beta$.

The set $S(B_{4n})$ will be

$$S(B_{4n}) = \{\alpha^{2^i} \beta^{2^j+1} \beta, \alpha^k \beta \gamma : i, j \in \{1, \ldots, n\}, k \in \{1, \ldots, 2n\}\}.$$ 

Then, with the action of $G_{8n}$ on the right cosets of $\langle \beta \rangle$, $B_{4n}$ is a Burn loop, for all $n \geq 2$. Moreover, it is non-Moufang, see [5, 6].

The loops $C_{4n}$, $n \geq 2$, $n$ even: Let the group $H_{8n}$ be

$$H_{8n} = \langle \alpha, \beta, \gamma : \alpha^{2n} = \beta^2 = \gamma^2 = (\alpha \beta)^2 = id, \alpha \gamma = \gamma \alpha, \beta \gamma = \gamma \beta \alpha^n \rangle.$$ 

The set $S(C_{4n})$ will be

$$S(C_{4n}) = \{\alpha^{2^i} \beta^{2^j+1} \beta, \alpha^k \beta \gamma : i, j \in \{1, \ldots, n\}, k \in \{1, \ldots, 2n\}\}.$$ 

Choosing again the action of $H_{8n}$ on the right cosets of $\langle \beta \rangle$, the section $C_{4n}$ becomes a Burn loop, for all $n \geq 2$, $n$ even. It is non-Moufang, see [5, 6].
In \[9\], the authors showed that the square of any element of a Burn loop belongs to the intersection of the left and middle nuclei. In any Bol loop, these two nuclei coincide (cf. \[3\], Proposition 2.1) and form a normal subgroup of the loop (see Lemma 2.1). Thus, if \(L\) denotes a (left) Bol loop, one can speak of the factor loop \(L/N_{\lambda}\).

**Lemma 2.1.** Let \((L, \cdot)\) be a (left) Bol loop. Then its left nucleus \(N_{\lambda}\) is a normal subgroup of \(L\).

**Proof.** Let \((L, \cdot)\) be a left Bol loop. Let us denote by \(G_{\text{left}}(L)\) and \(G_{\text{right}}(L)\) the groups generated by the left and right translations of \(L\), respectively. Let \(M(L)\) denote the group generated by \(G_{\text{left}}(L)\) and \(G_{\text{right}}(L)\). The Bol identity \(x \cdot (y \cdot xz) = (x \cdot yx) \cdot z\) can also be expressed by \(\rho_x \lambda_x = \rho_x \lambda_x \rho_z\), or equivalently, \(\lambda_x \rho_z \lambda_x^{-1} = \rho_x \rho_z^{-1} \in G_{\text{right}}(L)\). This means that \(G_{\text{right}}(L)\) is a normal subgroup of \(M(L)\).

Let now \(u\) be a permutation of \(L\) with \(1^u = n\) and let us suppose that \(u\) centralizes the group \(G_{\text{right}}(L)\). Then for any \(x \in L\)

\[
x^u = 1^{\rho_x} = 1^{u \rho_z} = n x,
\]

that is, \(u = \lambda_n\). Moreover, \(\lambda_n \rho_z = \rho_z \lambda_n\) for all \(x \in L\) means exactly that \(n\) is an element of the left nucleus \(N_{\lambda}(L)\) of \(L\). Hence, \(U = \{\lambda_n : n \in N_{\lambda}(L)\}\) is the centralizer of the normal subgroup \(G_{\text{right}}(L)\) in \(M(L)\), it should also be normal. This implies that \(N_{\lambda}(L) = 1^U\) is a normal subgroup of \(L\), see \[3\], Theorem 3. □

**Remark.** Clearly, if \(L\) is a Burn loop, the factor loop \(L/N_{\lambda}\) is Burn as well. This means that in the quotient loop \(L/N_{\lambda}\) of a Burn loop \(L\) every element has order 2.

### 3. The Kernel of the Map \(\Phi\) in Burn Loops

In this chapter, the kernel of the map \(\Phi\) will be determined, for the case that the loop is of Burn type. The elements of ker \(\Phi\) are of the form \((\lambda, \text{id})\), with \(\lambda \in G(L)\); thus ker \(\Phi\) is isomorphic to a subgroup of \(G(L)\), let us denote this subgroup by \(K\). (By Theorem 3.1 of \[7\], even \(K \leq S(N_{\lambda})\) holds.)

If \(a_1, \ldots, a_k\) are elements of a group, then \([a_1, \ldots, a_k]\) denotes the commutator \(a_1^{-1} \cdots a_k^{-1} a_1 \cdots a_k\). Let \(L\) be a Burn loop. For \(k \geq 2\), we define the following subgroup \(H_k\) of \(G(L)\):

\[
H_k = \langle [\lambda_{x_1}, \ldots, \lambda_{x_k}] | x_1, \ldots, x_k \in L, \lambda_{x_1} \cdots \lambda_{x_k} \in S(L) \rangle.
\]

**Lemma 3.1.** In any Bol loop, \(K = \cup_k H_k\). If the loop is of Burn type, we have ker \(\Phi \leq G(L)\).

**Proof.** An element of ker \(\Phi\) is of the form \((\rho_{x_0} \lambda_{x_0} \cdots \rho_{x_k} \lambda_{x_k}, \lambda_{x_0}^{-1} \cdots \lambda_{x_k}^{-1})\), where \(\lambda_{x_0}^{-1} \cdots \lambda_{x_k}^{-1} = \text{id}, \lambda_{x_0} = \lambda_{x_1}^{-1} \cdots \lambda_{x_k}^{-1}\). Thus

\[
x_0 \cdot \cdots \cdot (x_{k-2} \cdot x_{k-1} x_k)\ldots = 1.
\]

The Bol property immediately implies that \(\rho_x \lambda_x \rho_y = \rho_y \lambda_x\) for all \(x, y \in L\). Then

\[
\rho_{x_0} \lambda_{x_0} \cdots \rho_{x_k} \lambda_{x_k} = \rho_{x_0} (\cdots (x_{k-2} \cdot x_{k-1} x_k)\ldots) \lambda_{x_0} \cdots \lambda_{x_k} = \lambda_{x_0} \cdots \lambda_{x_k} = \lambda_{x_1}^{-1} \cdots \lambda_{x_k}^{-1} \lambda_{x_0} \cdots \lambda_{x_k} = [\lambda_{x_1}, \ldots, \lambda_{x_k}].
\]
By the left inverse property, there exists an \( x_0 \in L \) such that \( \lambda x_0 \cdots \lambda x_k = id \) if and only if \( \lambda x_1 \cdots \lambda x_k \in S(L) \). So we have

\[
\ker \Phi = \{(\lambda x_0, \ldots, \lambda x_k) | x_0, \ldots, x_k \in L, \lambda x_0 \cdots \lambda x_k \in S(L) \} = \bigcup_k H_k.
\]

Since in a Burn loop, the set \( S(L) \) is invariant under the conjugation with elements \( \lambda y \), we have \( \ker \Phi \triangleleft G(L) \). □

As the square of any element of the Burn loop \( L \) is in \( N_\lambda \), for all \( n \in N_\lambda, x, y \in L \), the commutators \([\lambda_n, \lambda_x]\) and \([\lambda_x^2, \lambda_y]\) belong to \( H_2 \). Using this we prove the following lemma.

**Lemma 3.2.** Let \( \alpha_1, \ldots, \alpha_k \in S(L) \) and \( \bar{\alpha}_i \in S(N_\lambda) \).

(i) \([\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k] \equiv [\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_k] \pmod{H_2} \);

(ii) \([\alpha_1, \ldots, (\alpha_i \bar{\alpha}_i), \ldots, \alpha_k] \equiv [\alpha_1, \ldots, \bar{\alpha}_i, \alpha_i, \ldots, \alpha_k] \pmod{H_2} \);

(iii) \([\alpha_1 \cdots \alpha_k, \bar{\alpha}_i] \in H_2 \);

(iv) \([\alpha_1, \ldots, \alpha_i, \bar{\alpha}_i, \ldots, \alpha_k] \equiv [\alpha_1, \ldots, \alpha_k] \pmod{H_2} \).

(v) If the element on the right side of the equivalence (i), (ii) or (iv) is in \( H_k \), then the element on the left side is in \( H_k \), as well.

**Proof.**

(i) We have \( \alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k = \alpha_1 \cdots \alpha_{i+1} \bar{\alpha}_i \cdot \cdots \cdot \alpha_k \). On the other hand,

\[
\alpha_1^{-1} \cdots \alpha_i^{-1} \alpha_{i+1}^{-1} \cdots \alpha_k^{-1} = \alpha_1^{-1} \cdots \alpha_{i+1}^{-1} (\alpha_i^{-1})^{\alpha_{i+1}} [\alpha_i^{-1} (\alpha_i^{-1})^{\alpha_{i+1}}]^{-1} \alpha_{i+2}^{-1} \cdots \alpha_k^{-1} = \alpha_1^{-1} \cdots \alpha_{i+1}^{-1} (\alpha_i^{-1})^{\alpha_{i+1}} \cdots \alpha_k^{-1} (\alpha_i^{-1})^{\alpha_{i+1}} \beta,
\]

where \( \beta = \alpha_{i+2}^{-1} \cdots \alpha_k^{-1} \in S(L) \). Now, it is sufficient to show that the expression in the square bracket is an element of \( H_2 \): \( \alpha_i^{\alpha_{i+1}} (\alpha_i^{-1})^{\alpha_{i+1}} = [\alpha_i^{2} \alpha_i^{-2}, \alpha_i^{-1}] \in H_2 \).

(ii) By some similar calculation one can show that

\[
[\alpha_1, \ldots, (\alpha_i \bar{\alpha}_i), \ldots, \alpha_k] = [\alpha_1, \ldots, \bar{\alpha}_i, \alpha_i, \ldots, \alpha_k] [\alpha_i, \bar{\alpha}_i]^{\alpha_{i+1} \cdots \alpha_k},
\]

and because of \( \bar{\alpha}_i \in S(N_\lambda) \), the last factor is an element of \( H_2 \).

(iii) \([\alpha_1 \cdots \alpha_k, \bar{\alpha}_i] = [\alpha_2 \cdots \alpha_k, \bar{\alpha}_i^{\alpha_1}] [\alpha_1, \bar{\alpha}_i] \equiv [\alpha_2 \cdots \alpha_k, \bar{\alpha}_i^{\alpha_1 \cdots \alpha_k}] \equiv id \pmod{H_2} \).

(iv) \([\alpha_1, \ldots, \alpha_i, \bar{\alpha}_i, \ldots, \alpha_k] = [\alpha_1, \ldots, \alpha_k, \bar{\alpha}_i^{\alpha_{i+1} \cdots \alpha_k}] \equiv [\alpha_1, \ldots, \alpha_k] [\alpha_1 \cdots \alpha_k, \bar{\alpha}_i^{\alpha_{i+1} \cdots \alpha_k}] \pmod{H_2} \).

(v) This follows from \( H_2 \triangleleft H_k \triangleleft G(L) \). □

**Proposition 3.3.** Let \( L \) be a Burn loop and \( \Phi \) and \( H_k \) \((k \geq 2)\) be defined as in the beginning of this section and let \( s = |L : N_\lambda| \). Then \( \ker \Phi = H_{s-1} \) if \( s \geq 3 \), and \( \ker \Phi = H_2 \) if \( s = 1 \) or \( 2 \).

**Proof.** Let \( B \) be a set of representatives from the cosets of \( N_\lambda \) in \( L \) such that \( 1 \in B \). Then any element of \( L \) can be written in a unique way as the product \( n b \), with \( n \in N_\lambda, b \in B \). Let us choose elements \( x_1, \ldots, x_k \), \( x_i = n_i b_i \), from \( L \) such that \( \lambda x_1 \cdots \lambda x_k \in S(L) \). By (4.2) and (iv), \([\lambda x_1, \ldots, \lambda x_k] \equiv [\lambda b_1, \ldots, \lambda b_k] \pmod{H_2} \). Applying (4.2) and \( b_i^2 \in N_\lambda \) several times, one gets \([\lambda x_1, \ldots, \lambda x_k] \equiv
Thus for any trivial, we still have to show \((i) \implies \lambda \) holds in the quotient loop \(L/N\). Then the followings are equivalent.

\begin{align*}
(i) & \quad \lambda_b \lambda_d \lambda_b \in S(L), \\
(ii) & \quad \lambda_b \lambda_d \lambda_b \in S(L) \text{ with } \{i, j, k\} = \{1, 2, 3\}, \\
(iii) & \quad \lambda_b \lambda_d \in S(L), \\
(iv) & \quad \lambda_b, \lambda_d \in S(L) \text{ for all } i, j \in \{1, 2, 3\}.
\end{align*}

Proof. (i) \(\Rightarrow\) (iii). From \(b_3N_\lambda \cdot (b_2N_\lambda \cdot b_3N_\lambda) = N_\lambda\) we get \(\lambda_b, \lambda_d \lambda_b = \lambda_n, n \in N_\lambda\). Hence \(\lambda_b \lambda_d = \lambda_{b_3^{-1}n} \in S(L)\).

(iii) \(\Rightarrow\) (i). The quotient is a Burn loop, thus \(b_3N_\lambda = b_2N_\lambda \cdot b_1N_\lambda, b_2b_1 = b_3n, \lambda_b \lambda_d = \lambda_n, \lambda_b \lambda_d = \lambda_{b_3^{-1}n} \in S(L)\).

The equivalence (ii) \(\iff\) (iv) can be shown in the same manner. (iv) \(\Rightarrow\) (iii) being trivial, we still have to show (i) \(\Rightarrow\) (ii). Supposing (i), we have

\[ \lambda_{b_2} \lambda_d \lambda_n \in S(L) \]

and

\[ S(L) \ni \lambda_{b_3}^{-1} \lambda_{b_2}^{-1} \lambda_{b_1}^{-1} = \lambda_{b_2} \lambda_{b_3} \lambda_{b_2} \lambda_{b_3} \lambda_{b_1} \lambda_n = \lambda_{b_3} \lambda_{b_2} \lambda_{b_1} \lambda_n, \]

with \(n_1, n_2, n_3, n \in N_\lambda\), and so \(\lambda_{b_3} \lambda_{b_2} \lambda_{b_1} \in S(L)\). This is enough to imply (ii). \(\square\)

Proposition 3.6. If \(s = |L : N_\lambda| \leq 7\), then \(s \in \{1, 2, 4\}\) and

\[ \ker \Phi = [S(N_\lambda), G(L)] = [(\lambda, \lambda x)|n \in N_\lambda, x \in L]. \]

Proof. The quotient \(L/N_\lambda\) is a Bol loop of order \(s \leq 7\), and so a group (cf. 3.5). In \(L\), the square of any element is in \(N_\lambda\), since \(L/N_\lambda\) is an elementary abelian 2-group, \(s \in \{1, 2, 4\}\). For \(s = 1\) or 2 the statement follows directly from 3.5. Let us suppose that \(s = 4\). If \(b_1N_\lambda, b_2N_\lambda, b_3N_\lambda\) are different nontrivial elements of \(L/N_\lambda\), then \(b_3N_\lambda \cdot b_2N_\lambda \cdot b_1N_\lambda = N_\lambda\). Suppose that \(\lambda_{b_1} \lambda_{b_2}\) or \(\lambda_{b_1} \lambda_{b_3}\) is an element of \(S(L)\). Then, by \(\lambda_{b_3}\) for all \(i, j \in \{1, 2, 3\}\), one has \(\lambda_{b_i} \lambda_{b_j} \in S(L)\). This means that for any \(x_i, x_j \in L\), \(x_i, x_j = b_i b_j n_{i,j}\) with \(n_{i,j} \in N_\lambda\),

\[ \lambda_{x_i} \lambda_{x_j} = \lambda_{x_i} \lambda_{b_i} \lambda_{x_j} \lambda_{b_j} = \lambda_{x_i} \lambda_{x_j} \lambda_{b_i} \lambda_{b_j} \in S(L), \]

thus \(L\) is a group, which contradicts \(s = 4\).

This shows that \(\ker \Phi = H_3 = [S(N_\lambda), G(L)]\). \(\square\)

4. The groups generated by the Bol reflections and the cores of the loops \(B_{4n}\) and \(C_{4n}\)

Let us denote by \(\sigma_m\) the Bol reflection with axis \(x = m\) (see 4), by \(N^+\) the collineation group generated by these reflections and by \(N\) the subgroup generated by products of even length of reflections. Since a Bol reflection interchanges the horizontal and transversal directions, \(N^+\) does not preserve the directions, but the group \(N\) does.

Clearly, \(N\) is a normal subgroup of index 2 of \(N^+\) and by the geometric properties of Bol reflections, the set \(\Sigma = \{\sigma_x|x \in L\}\) is invariant in \(N^+\). Thus, the elements
σ_xσ_1 generate N. Using the coordinate system, we get the form σ_xσ_1 = (p_x, λ_x) for these generators, where p_x = λ_x^{-1}ρ_x^{-1}, see [3].

The following lemma will help us to determine the orbit of the y-axis under N.

**Lemma 4.1.** Let (L, ·) be a Burn loop and let us define the groups

\[ F = \langle p_x | x \in L \rangle, \quad U = \langle \lambda_x^2 | x \in L \rangle. \]

Then, the orbits \( 1^F \) and \( 1^U \) coincide.

**Proof.** Using that L is left conjugacy closed, we have

\[ 1^F \subseteq 1^U, \]

which means \( 1^F \subseteq 1^U \). On the other hand,

\[ 1^F \subseteq 1^U \]

shows that \( 1^F \) is invariant under \( U \). Thus, \( 1^F = 1^U \). \( \square \)

**Lemma 4.2.** Let (L, ·) be a Burn loop and U \( \subseteq G(L) \) be an Abelian group containing the left translations \{λ_m : m \in N_λ\}. Then the group \( Φ^{-1}(U) \) of collineations is Abelian, too.

**Proof.** The action of an arbitrary collineation \((u, v)\) on the set of transversal lines is \( vλ_a, \) where \( a = 1^a, \) see [2]. If \((u, v) \in Φ^{-1}(U),\) then by Lemma 4.1 \( a \in N_λ, \) hence \( λ_a \in U \) and \( vλ_a \in U. \) And since U is Abelian, this means that the commutator elements of \( Φ^{-1}(U) \) act trivially on the set of horizontal and vertical lines, thus on the whole point set. \( \square \)

**Theorem 4.3.** Let (L, ·) be one of the loops \( B_{4n} \) or \( C_{4n}, \) \( n \geq 2. \)

1. The group N is equal to \( ker Φ \times \bar{G}, \) where \( Φ \) induces an isomorphism from the subgroup \( \bar{G} \) to \( G(L). \) Let us denote the respective generators of \( \bar{G} \) by \( \bar{α}, \) \( \bar{β} \) and \( \bar{γ}, \) and by \( δ \) the generator of \( ker Φ. \) Then, \( \bar{α} \) and \( \bar{γ} \) act trivially on \( ker Φ, \) and \( \bar{β}δ\bar{β} = δ^{-1}. \)

2. The reflection \( σ_1 \) is an automorphism of N, which inverts the generators \( (p_x, λ_x). \) It always leaves \( \bar{α} \) and \( \bar{β} \) invariant and acts on \( \bar{γ} \) and \( δ \) in the following way.

\[ σ_1 : \{ \begin{array}{l}
\bar{γ} \mapsto \bar{γ}, \quad δ \mapsto \bar{α}^{-4}\bar{δ}^{-1} \quad \text{if} \quad L = B_{4n}, n \geq 2 \text{ or } C_{4n}, n \equiv 0 \pmod{4}; \\
\bar{γ} \mapsto \bar{α}^n\bar{γ}, \quad δ \mapsto \bar{α}^{-4}\bar{δ}^{-1} \quad \text{if} \quad L = C_{4n}, n \equiv 2 \pmod{4};
\end{array} \]

3. The group \( G_{core} \) generated by the core is isomorphic to \( N^+/Z(N^+) \) where

\[ Z(N^+) = \begin{cases}
\langle \bar{α}^n, \bar{γ}, σ_1 \rangle & \text{if} \ L = B_8; \\
\langle \bar{α}^n, \bar{γ} \rangle & \text{if} \ L = B_{4n}, n \not\equiv 0 \pmod{4}, n > 2; \\
\langle \bar{α}^n, \bar{γ}, δ\bar{δ} \rangle & \text{if} \ L = B_{4n}, n \equiv 0 \pmod{4}; \\
\langle \bar{γ}\bar{α}^n, δ\bar{δ} \rangle & \text{if} \ L = C_{4n}, n \equiv 0 \pmod{4}; \\
\langle \bar{α}^n \rangle & \text{if} \ L = C_{4n}, n \equiv 2 \pmod{4}.
\end{cases} \]

**Proof.** If \( L \) is either \( B_{4n}, n \geq 2 \text{ or } C_{4n}, n \equiv 0 \pmod{4}, \) then by Table 1 \( ker Φ \) acts regularly on the orbit \((y-axis)^N. \) Hence, in these cases, \( \bar{G} = N_{y-axis} \) is a good choice.

Let us suppose \( L = C_{4n}, n \equiv 2 \pmod{4}. \) Let \( m \) be \( 1^α^2. \) Then \( m \) has order \( n \) in \( L, \) it is a generator of the cyclic group \( N_λ, \) and the generating element \( δ \) of \( ker Φ \) can be assumed to be in the form \( (λ_2^{-n}, id). \) Let \( X \) be the set of vertical
lines of equation $x = 1$ or $x = m\frac{k}{2}$. Let us define the subgroup $G$ as the setwise stabilizer of $X$ in $N$. To the left translation $\lambda_x = \beta\gamma$ the $N$-generator $(p_x, \lambda_x)$ is associated; since $1p_x = 1(\beta\gamma)^{x} = 1a^n = m\frac{k}{2}$, this generator interchanges the lines in $X$. Therefore $[G : Ny\mbox{-axe}] = 2$ and $|N : G| = n/2$. Clearly, $G \cap \ker \Phi = \{id\}$, and so, $G$ is a transversal to $\ker \Phi$.

To complete the proof of point 1, we consider the action of $G$ on $\ker \phi$. Applying Lemma 4.2 to $U = \langle \alpha, \gamma \rangle$ we see that $\tilde{\alpha}$ and $\tilde{\gamma}$ commute with $\ker \Phi$. Furthermore, since in each cases of $L, \tilde{\beta} \in Ny\mbox{-axe},$ hence $\tilde{\beta} = (\beta, \beta) \in N_{(1,1)}$ and $\delta^{\tilde{\beta}} = \delta^{-1}$.

To determine the action of $\sigma_1$ on the elements $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and $\delta$, we have to express the generators $(p_x, \lambda_x)$ of $N$ by these elements. We claim that this is done in Table 2. We therefore use the fact that two collineations $(u, v)$ and $(u', v')$ coincide if $v = v'$ and $1u = 1u'$, see 2. Moreover, if $(u, v)$ is a generator element for $N$, then we have $1u = 1v^{-2}$.

Again, the cases $L = B_{4n}, n \geq 2$ or $C_{4n}, n \equiv 0 \pmod{4}$ are trivial, since then $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ stabilize the $y$-axe and $\delta$ acts on it in a well known way. Let us suppose $L = C_{4n}, n \equiv 2 \pmod{4}$ and denote the $N$-generator associated to $\alpha^k\beta\gamma$ by $(u, \alpha^k\beta\gamma)$. Then one has $1u = 1(\alpha^k\beta\gamma)^2 = 1a^n = m\frac{k}{2}$, and so, $(u, \alpha^k\beta\gamma) \in G$. This gives $(u, \alpha^k\beta\gamma) = \tilde{\alpha}^k\tilde{\beta}\tilde{\gamma}$. The results of Table 2 and point 2 of the theorem follow.

The core of the Bol loop $(L, \cdot)$ is the grupoid $(L, +)$ with $x + y = x \cdot y^{-1}x$. Isomorphic versions of the grupoid can be defined in the following ways.

\begin{align*}
(S(L), \oplus), & \quad \lambda_x \oplus \lambda_y = \lambda_x \lambda_y^{-1}\lambda_x; \\
(\Sigma, \otimes), & \quad \Sigma = \{ \sigma_x : x \in L \}, \quad \sigma_x \otimes \sigma_y = \sigma_x \sigma_y \sigma_x.
\end{align*}

The isomorphism $(L, +) \cong (S(L), \oplus)$ is trivial, and $(S(L), \oplus) \cong (\Sigma, \otimes)$ can be shown using $\sigma_x \sigma_1 = (p_x, \lambda_x)$. Hence, the permutation group generated by the core acts on $L$ like $N^+$ acts on $\Sigma$ by conjugation and this action equals to the action of $N^+$ on the set of vertical lines. And since $\Sigma$ generates $N^+$, the group $G_{core}$ generated by the core is isomorphic to $N^+/Z(N^+)$. Thus, we only have to compute the centres $Z(N^+)$. 

| $B_{4n}$, $n$ odd | $B_{4n}$, $n$ even | $C_{4n}$, $n \equiv 2 \pmod{4}$ | $C_{4n}$, $n \equiv 0 \pmod{4}$ |
|------------------|------------------|------------------|------------------|
| $\ker \Phi$     | $C_n$            | $C_n$            | $C_n$            |
| $|\{(y\mbox{-axe})\}|$ | $n$              | $\frac{n}{2}$   | $n$              | $\frac{n}{2}$   |

Table 1. The kernel of $\Phi$ and the orbit of the $y$-axe under $N$.
If $L = B_8$, then $\sigma_1$ acts trivially on $N$. In any other cases, $\sigma_1$ is a non-trivial outer automorphism and we have $Z(N^+) = C_{Z(N)}(\sigma_1)$, which is very easy to calculate.

5. Automorphisms of Burn loops of type $B_{4n}$ and $C_{4n}$

Let $(L, \cdot)$ be a loop and let $u$ denote an automorphism of $L$. Then, by conjugation, $u$ induces an automorphism of the group $G(L)$. Moreover $u$ leaves the section $S(L)$ and the stabilizer $G(L)_1$ invariant. Conversely, let $u$ be an automorphism of $G(L)$, normalizing the subgroup $G(L)_1$ and the set $S(L)$. Then $u$ induces a permutation on the cosets of $G(L)_1$, hence on $L$. The induced permutation will fix 1 and normalize $S(L)$, thus $u^{-1} \lambda_x u = \lambda_y$ for all $x \in L$. Applying this to 1, one gets $y = x^n$, hence $\lambda_x^n = \lambda_{x^n}$ for all $x \in L$. This means $u \in \text{Aut}(L)$.

In the case of the given loops the stabilizer of 1 consists of $\{id, \beta\}$. First we calculate its normalizer in the automorphism groups of the left translation groups, that is, the groups $C_{\text{Aut}(G)}(\beta)$, where $G$ is $G_{8n}$ or $H_{8n}$.

**Lemma 5.1.** Let $G$ denote the group $G_{8n}$, $n$ odd. Then $C_{\text{Aut}(G)}(\beta) \cong Z_{2n}^* \times S_3$, and the elements of $C_{\text{Aut}(G)}(\beta)$ normalize $S(B_{4n})$.

**Proof.** Let us define the subgroups $A = \langle \alpha^2 \rangle$ and $B = \langle \alpha^n, \beta, \gamma \rangle$ of $G$. As $|A| = n$ is odd, $A$ is a characteristic subgroup of $G = A \times B$. Moreover, $B = Z(G) \langle \beta \rangle$ is invariant in $C_{\text{Aut}(G)}(\beta)$, as well. Hence, $C_{\text{Aut}(G)}(\beta) = \text{Aut}(A) \times C_{\text{Aut}(B)}(\beta) \cong Z_{2n}^* \times S_3$.

On the other hand, $S(L) = A \{id, \alpha^n \beta, \beta \gamma, \alpha^n \beta \gamma\}$. Since the set

$$\{id, \alpha^n \beta, \beta \gamma, \alpha^n \beta \gamma\}$$

is invariant under $C_{\text{Aut}(B)}(\beta)$, the statement follows. \hfill $\Box$

**Lemma 5.2.** Let $G$ denote the group $G_{8n}$, $n$ even. Then $C_{\text{Aut}(G)}(\beta) \cong Z_n^* \times D_8$, and the elements of $C_{\text{Aut}(G)}(\beta)$ normalize $S(B_{4n})$.

**Proof.** It is enough to consider the possible images of $\alpha$ and $\gamma$, let us write them as $\hat{\alpha} = \alpha^i \gamma k \beta^j$ and $\hat{\gamma} = \alpha^p \gamma q \beta^s$, respectively. Clearly, $\hat{\beta} = \beta$.

If $j = 1$ then $\hat{\alpha} = id$, which is impossible. The order of $\hat{\alpha}$ must be $2n$, thus $i \in Z_{2n}$. The elements $\hat{\alpha}$ and $\hat{\gamma}$ must commute, $s$ cannot be 1. Also the elements $\hat{\beta}$ and $\hat{\gamma}$ commute, we must have $p = ln$ with $l \in Z_2$.

Let us now suppose that $q = 0$. Then $l = 0$ implies $\hat{\gamma} = id$ and $k = 0$ implies $\gamma \not\in \langle \hat{\alpha}, \hat{\beta}, \hat{\gamma} \rangle$, hence we have $l = k = 1$. This means $\hat{\alpha}^n = \alpha^i = \alpha^n = \hat{\gamma}$, a contradiction.

Let us denote by $u(i, k, l)$ the automorphism induced by

$$\alpha \mapsto \alpha^i \gamma k \beta^j, \quad \beta \mapsto \beta, \quad \gamma \mapsto \alpha^l \gamma,$$

with $i \in Z_{2n}$, $k, l \in Z_2$. It is easy to check that this is really an element of $C_{\text{Aut}(G)}(\beta)$. Moreover,

$$u(i, j, k)u(i', j', k') = u(ik + lk' + k + k', l + l'),$$

where one calculates modulo $2n$ in the first and modulo 2 in the second and third position.

Let us decompose $Z_{2n}^*$ into $Z_n^* \times Z_2$ by $i = i_0 + i_1 n$, $i_0 \in Z_n^*$, $i_1 \in Z_2$. Then the group $C_{\text{Aut}(G)}(\beta)$ decomposes into the direct factors

$$\{u(i_0, 0, 0) : i_0 \in Z_n^*\} \text{ and } \{u(i_1 n, k, l) : i_1, k, l \in Z_2\}.$$
An easy calculation is to show that the second factor is isomorphic to the dihedral group $D_8$ of 8 elements.

Since we gave explicitly the elements of $C_{\text{Aut}(G)}(\beta)$, it can be checked directly that they leave $S(L)$ invariant. \hfill \Box

**Lemma 5.3.** Let $G$ denote the group $H_{8n}$, $n > 2$ even. Then $C_{\text{Aut}(G)}(\beta) \cong Z_{2n}^* \times Z_2$, and the elements of $C_{\text{Aut}(G)}(\beta)$ normalize $S(4n)$.

**Proof.** As in the preceding proof, we consider the images $\hat{\alpha} = \alpha^{i}\gamma^{k}\beta^l$, $\hat{\gamma} = \alpha^p\gamma^q\beta^s$ of $\alpha$ and $\gamma$.

If $j = 1$, then $\hat{\alpha}^2 = \alpha^{i}\gamma^{k}\beta^l\alpha^{i}\gamma^{k}\beta^l = (\gamma^k\beta^l)^2 = \alpha^{kn}$, $\hat{\alpha}^4 = id$, which is not possible because of $n > 2$. If $k = 1$, then $(\hat{\alpha}\hat{\beta})^2 = (\gamma\beta)^2 = \alpha^n \neq id$, hence $k = 0$ and $\hat{\alpha} = \alpha^i$, with $i \in Z_{2n}$.

As before, $\hat{\alpha}\hat{\gamma} = \hat{\gamma}\hat{\alpha}$ implies $s = 0$ and $\gamma \in \langle \hat{\alpha}, \hat{\beta}, \hat{\gamma} \rangle$ implies $q \neq 0$. Finally, $p \in \{0, n\}$ since $\hat{\gamma} = (\alpha^p\gamma^q) = \alpha^{2p} = id$.

Thus, any element of $C_{\text{Aut}(G)}(\beta)$ is induced by

$$\alpha \mapsto \alpha^i, \quad \beta \mapsto \beta, \quad \gamma \mapsto \alpha^n\gamma,$$

and it leaves $S(L)$ invariant. \hfill \Box

**Theorem 5.4.** Let $(L, \cdot)$ be one of the loops $B_{4n}$ or $C_{4n}$ defined at the beginning of this section. Then

$$\text{Aut}(L) \cong \begin{cases} Z_n^* \times S_4 & \text{if } L = B_{4n}, \text{ } n \text{ odd} \\ Z_n^* \times D_8 & \text{if } L = B_{4n}, \text{ } n \text{ even} \\ Z_{2n} \times Z_2 & \text{if } L = C_{4n}, \text{ } n > 2, \text{ } n \text{ even} \\ D_8 & \text{if } L = C_8 \end{cases}$$

Moreover, in any of these loops, each left pseudo-automorphism is an automorphism.

**Proof.** The case $L = C_8$ is handled in [3], the others in Lemmas 4.1, 4.2 and 4.3. We only have to prove the second statement. Therefore, let us suppose that $u$ is a left pseudo-automorphism of $L$ with companion $c$, that is, for all $x, y \in L$,

$$(c \cdot x^n) \cdot y^n = c \cdot (xy)^n.$$ 

This can be expressed by $u\lambda cx^n = \lambda xu\lambda$, which implies $S(L)^u = S(L)\lambda c^{-1}$.

The following results are to find in [3]: If $L = B_{4n}$, then the principal isotopes of $L$ have the four representation $S(L)$, $\alpha\beta S(L)$, $\alpha\beta\gamma S(L)$, and $\beta\gamma S(L)$. If $n$ is even, then these sections contain $3n + 1$, $n + 3$, $n + 3$ and $n + 1$ elements of order 2. If $n$ is odd, $S(L)$ contains $3n$ elements of order 2 and the others contain $n + 2$ elements of order 2, $n > 2$. That means that $c$ is a left companion element of $L$ if and only if $S(L)\lambda c = S(L)$, it is, $c \in N_\lambda$ and $u$ is an automorphism.

Let now $L$ be equal to $C_{4n}$. Again the principal isotopes are $S(C_{4n})$, $\alpha\beta S(C_{4n})$, $\alpha\beta\gamma S(C_{4n})$, and $\beta\gamma S(C_{4n})$, they contain $n + 1$, $n + 3$, 3 and 1 involutions, respectively. If $n > 2$, then one sees with the above argument that $c \in N_\lambda$ and $u$ is an automorphism. \hfill \Box

### 6. Collineation groups of the given 3-nets

In this chapter, we determine the full collineation group $\Gamma$ of the 3-nets belonging to $B_{4n}$, $n \geq 3$, and $C_{4n}$, $n \geq 4$, $n$ even. The cases $B_8$ and $C_8$ are completely described in [3].
Denote by $P$ the orbit $(1,1)^T$ of the origin under $\Gamma$. As we know by Corollary 2.8 of \[3\], for any Burn loop, $P$ is a union of vertical lines and its intersection with the $x$-axe constitute of the points belonging to the left companion elements. In our cases, these are the elements of $N_\lambda$, see Theorem \[5\]. Hence $|P| = 4n^2$.

Let $\Lambda_0$ be the subgroup $\langle \alpha, \gamma \rangle$ of $G(L)$. The centralizer element $\alpha^j\beta\gamma^j \notin \Lambda_0$ in $\Lambda_0$ has order 4, that is, any Abelian subgroup not contained in $\Lambda_0$ has order at most 8. This means that if $n > 2$ then $\Lambda_0$ is the only Abelian subgroup of index 2 in $G(L)$, it must therefore be characteristic in $G(L)$.

Now, we define the following subgroups of $\Gamma$.

$$
T = \{ (\lambda_m, id) : m \in N_\lambda \}, \quad \Lambda = \Phi^{-1}(\Lambda_0),
$$

$$
A = \{ (\sigma, \sigma) : \sigma \in \text{Aut}(L) \}, \quad M = TA.
$$

** Lemma 6.1.** The subgroup $M$ is an Abelian normal subgroup of $\Gamma$. Moreover, it is isomorphic to the direct product $N_\lambda \times \Lambda_0$ and acts regularly on the orbit $P$ of the origin.

**Proof.** First we show that $M$ is Abelian. By Lemma 4.2, one sees that the permutation action of the elements of $\Lambda$ are all in $\langle \alpha, \gamma \rangle$; the same can be said about the elements of $T$. These actions commute, and so, all the elements must commute.

Clearly, $T$ is normal in $\Gamma$. The subgroup $\Lambda$ is invariant in $\Gamma$ as well, for it is the homomorphic preimage of a characteristic subgroup.

Suppose that $(u, v)$ is an element of $M_{(1,1)}$. Then $v = id$, since $v = \beta$ is not possible. This implies $u = \lambda_m$, $m \in N_\lambda$; from which $u = id$ follows. Furthermore, on the one hand, by $\Lambda \cap T = \ker \Phi$, we have $M_{y\text{-axe}} \cong M/T \cong \Lambda_0$. On the other hand, $T \subset M_{x\text{-axe}}$ acts transitively on $P \cap y\text{-axe}$. This means that $M$ acts transitively on $P$, thus, regularly. Finally, $M = T \times M_{y\text{-axe}} \cong N_\lambda \times \Lambda_0$. $\square$

** Theorem 6.2.** Let $\Gamma$ be the full collineation group of the 3-net, coordinatized by the loop $L$, with $L = B_{4n}$ or $C_{4n}$, $n > 2$. Then, $\Gamma$ can be written as the semidirect product $M \rtimes \text{Aut}(L)$, where $M$ is defined as above and the action of $\text{Aut}(L)$ on $M$ is defined by $(u,v)^\sigma = (u^\sigma, v^\sigma)$.

**Proof.** Obviously, $A$ is isomorphic to $\text{Aut}(L)$. By Theorem 10.1 of \[2\], $A$ is equal to the stabilizer $\Gamma_{(1,1)}$ of the origin $(1,1)$ in $\Gamma$. By Lemma 6.1, $M$ is a normal subgroup of $\Gamma$, acting regularly on the orbit $P = (1,1)^T$. Then, $\Gamma$ can be written as the semidirect product $M \rtimes A \cong M \rtimes \text{Aut}(L)$. $\square$

**Remark.** Note that there is an interesting analogy with the case of group 3-nets: then one has $\Gamma \cong (G \times G) \rtimes \text{Aut}(G)$ (cf. \[2\], Theorem 10.1).

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JATE Bolyai Institute, Aradi v´ertan´uk ter 1. H-6720 Szeged (Hungary)

E-mail address: nagyg@math.u-szeged.hu