Comparing scalar-tensor gravity and $f(R)$-gravity in the Newtonian limit

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Recently, a strong debate has been pursued about the Newtonian limit (i.e. small velocity and weak field) of fourth order gravity models. According to some authors, the Newtonian limit of $f(R)$-gravity is equivalent to the one of Brans-Dicke gravity with $\omega_{BD} = 0$, so that the PPN parameters of these models turn out to be ill defined. In this paper, we carefully discuss this point considering that fourth order gravity models are dynamically equivalent to the O’Hanlon Lagrangian. This is a special case of scalar-tensor gravity characterized only by self-interaction potential and that, in the Newtonian limit, this implies a non-standard behavior that cannot be compared with the usual PPN limit of General Relativity. The result turns out to be completely different from the one of Brans-Dicke theory and in particular suggests that it is misleading to consider the PPN parameters of this theory with $\omega_{BD} = 0$ in order to characterize the homologous quantities of $f(R)$-gravity. Finally the solutions at Newtonian level, obtained in the Jordan frame for a $f(R)$-gravity, reinterpreted as a scalar-tensor theory, are linked to those in the Einstein frame.

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I. INTRODUCTION

Recently, several authors claimed that higher order theories of gravity, in particular $f(R)$-gravity [1], are characterized by an ill-defined behavior in the Newtonian regime. In a series of papers [2], it is discussed that higher order theories violate experimental constraints of General Relativity (GR) since a direct analogy between $f(R)$-gravity and Brans-Dicke gravity [3] gives the Brans-Dicke characteristic parameter, in metric formalism, $\omega_{BD} \rightarrow \infty$ while it should be $\omega_{BD} \rightarrow 0$ to recover the standard GR. Actually despite the calculation of the Newtonian limit of $f(R)$, directly performed in the Jordan frame, have showed that this is not the case [4, 5], it remains to clarify why the analogy with Brans-Dicke gravity seems to fail its predictions also if one is assuming $f(R) \approx R^{1+\epsilon}$ with $\epsilon \rightarrow 0$. The shortcoming could be overcome once the correct analogy between $f(R)$-gravity and the scalar-tensor framework is taken into account.

The action of the Brans-Dicke gravity, in the Jordan frame, reads:

$$A_{JF}^{BD} = \int d^4x\sqrt{-g}\left[\phi R + \omega_{BD}\frac{\partial\phi}{\partial\phi} + \mathcal{L}_m\right], \quad (1)$$

where there is a generalized kinetic term and no potential is present. On the other hand, considering a generic function $f(R)$ of the Ricci scalar $R$, one has:

$$A_{JF}^{f(R)} = \int d^4x\sqrt{-g}\left[f(R) + \mathcal{L}_m\right]. \quad (2)$$

In both cases, $\mathcal{X} = \frac{8\pi G}{c^4}$, is the standard Newton coupling, $\mathcal{L}_m$ is the perfect fluid matter Lagrangian and $g$ is the determinant of the metric.

As said above, $f(R)$-gravity can be re-interpreted as a scalar-tensor theory by introducing a suitable scalar field $\phi$ which non-minimally couples with the gravity sector. It is important to remark that such an analogy holds in a

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formalism in which the scalar field displays no kinetic term but is characterized by means of a self-interaction potential which determines the dynamics (O’Hanlon Lagrangian) \[6\]. This consideration, therefore, implies that the scalar field Lagrangian, equivalent to the purely geometrical \(f(R)\) one, turns out to be different with respect to the above ordinary Brans-Dicke definition \[11\]. This point represents a crucial aspect of our analysis. In fact, as we will see below, such a difference will imply completely different results in the Newtonian limit of the two models and, consequently, the impossibility to compare predictions coming from the PPN approximation of Brans-Dicke models to those coming from \(f(R)\)-gravity.

The layout of paper is the following. In Sec. II, we discuss the solutions in the Newtonian limit of \(f(R)\)-gravity by using the analogies with the O’Hanlon theory. Sec. III is devoted to the analysis of the solutions in the limit \(f(R) \to R\) and the interpretation of PPN parameters \(\gamma, \beta\). Conformal transformations and the solutions in the Newtonian limit approximation are considered in Sec. IV. Concluding remarks are drawn in Sec. V.

II. THE NEWTONIAN LIMIT OF \(f(R)\)-GRAVITY BY O’HANLON THEORY

Before starting with our analysis, let us remind that the field equations in metric formalism, coming from \(f(R)\)-gravity, are

\[
H_{\mu\nu} = f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - f'(R) g_{\mu\nu} + g_{\mu\nu} \Box f' = \mathcal{X} T_{\mu\nu}
\]

which have to be solved together to the trace equation

\[
\Box f'(R) + \frac{f'(R) R - 2 f(R)}{3} = \frac{\mathcal{X}}{3} T.
\]

Let us notice that this last expression assigns the evolution of the Ricci scalar as a dynamical quantity. Here, \(T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}\) is the the energy-momentum tensor of matter, while \(T = T^\sigma_{\sigma}\) is the trace, \(f'(R) = \frac{d f(R)}{dR}\). The conventions for Ricci’s tensor is \(R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}\) while for the Riemann tensor is \(R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta
u,\mu} + \ldots\). The affine connections are the Christoffel symbols of the metric: \(\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^\mu_{\sigma} (g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})\). The adopted signature is \((+ - - -)\).

On the other hand, the so-called O’Hanlon Lagrangian \[6\] can be written as

\[
\mathcal{A}_{\text{O}H}^{JF} = \int d^4 x \sqrt{-g} \left[ \phi R - V(\phi) + \mathcal{X} L_m \right],
\]

where \(V(\phi)\) is the self-interaction potential. Field equations are obtained by varying Eq. \[5\] with respect both \(g_{\mu\nu}\) and \(\phi\) which now represent the dynamical variables. Thus, one obtains

\[
\phi G_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} - \phi g_{\mu\nu} + g_{\mu\nu} \Box \phi = \mathcal{X} T_{\mu\nu},
\]

\[
R - \frac{d V(\phi)}{d\phi} = 0,
\]

\[
\Box \phi - \frac{1}{3} \left[ \phi \frac{d V(\phi)}{d\phi} - 2 V(\phi) \right] = \frac{\mathcal{X}}{3} T,
\]

where we have displayed the field equation for \(\phi\). Eq. \[8\] is a combination of the trace of \[6\] and \[7\]. \(f(R)\)-gravity and O’Hanlon gravity can be mapped one into the other considering the following equivalences

\[
\phi = f'(R)
\]
\[ V(\phi) = f'(R)R - f(R) \]  

and supposing that the Jacobian of the transformation $\phi = f'(R)$ is non-vanishing. Henceforth we can consider, instead of Eqs. (3)-(4), a new set of field equations determined by the equivalence between the O'Hanlon gravity and the $f(R)$-gravity:

\[ \phi R_{\mu\nu} - \frac{1}{6} \left[ V(\phi) + \phi \frac{dV(\phi)}{d\phi} \right] g_{\mu\nu} - \phi_{,\mu\nu} = \chi \Sigma_{\mu\nu} \]  

\[ \Box \phi - \frac{1}{3} \left[ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right] = \frac{\chi}{3} T \]  

where $\Sigma_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu}$.

Let us, now, calculate the Newtonian limit of Eqs. (12)-(13). To perform this calculation, the metric tensor $g_{\mu\nu}$ and the scalar field $\phi$ have to be perturbed with respect to the background. After, one has to search for solutions at the $(v/c)^2$ order in term of the metric and the scalar field entries. It is

\[ g_{\mu\nu} \simeq \begin{pmatrix} 1 + g^{(2)}_{00} & 0 \\ 0 & -\delta_{ij} + g^{(2)}_{ij} \end{pmatrix} \]  

\[ \phi \sim \phi^{(0)} + \phi^{(2)} \]  

The differential operators turn out to be approximated as

\[ \Box \approx \partial_0^2 - \Delta \quad \text{and} \quad \partial_{\mu\nu} \approx \partial_\mu^2 \]  

since time derivatives increase the degree of perturbation, they can be discarded. From a physical point of view, this position holds since Newtonian limit implies also the slow motion.

Actually in order to simplify calculations, we can exploit the gauge freedom that is intrinsic in the metric definition. In particular, we can choose the harmonic gauge $g^{\rho\sigma} T^\rho_{\mu\nu} = 0$ so that the components of Ricci tensor reduces to

\[ \begin{cases} R^{(2)}_{00} = \frac{1}{2} \Delta g^{(2)}_{00} \\ R^{(3)}_{0i} = 0 \\ R^{(2)}_{ij} = \frac{1}{2} \Delta g^{(2)}_{ij} \end{cases} \]  

Accordingly, we develop the self-interaction potential at second order. In particular, the quantities in Eqs. (12) and (13) read:

\[ V(\phi) + \phi \frac{dV(\phi)}{d\phi} \simeq V(\phi^{(0)}) + \phi^{(0)} \frac{dV(\phi^{(0)})}{d\phi} + \left[ \phi^{(0)} \frac{d^2V(\phi^{(0)})}{d\phi^2} + 2 \frac{dV(\phi^{(0)})}{d\phi} \right] \phi^{(2)} \]  

\[ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \simeq \phi^{(0)} \frac{dV(\phi^{(0)})}{d\phi} - 2V(\phi^{(0)}) + \left[ \phi^{(0)} \frac{d^2V(\phi^{(0)})}{d\phi^2} - \frac{dV(\phi^{(0)})}{d\phi} \right] \phi^{(2)} \]
Field Eqs. (12) - (13), solved at 0-th order of approximation, provide the two solutions

\[ V(\phi(0)) = 0 \quad \text{and} \quad \frac{dV(\phi(0))}{d\phi} = 0 \]  

which fix the 0-th order terms of the self-interaction potential; therefore we have

\[ \phi \frac{dV(\phi)}{d\phi} \approx \phi(0) \frac{d^2V(\phi(0))}{d\phi^2} \phi(2) \approx 3m^2 \phi(2), \]

where the constant factor \( m^2 \) can be easily interpreted as a mass term as will become clearer in the following analysis (see also [9]). Now, taking into account the above simplifications, we can rewrite the field equations at the \((v/c)^2\) order in the form:

\[ \triangle g^{(2)}_{00} = \frac{2\lambda}{\phi(0)} \Sigma_{00}^{(0)} - m^2 \phi(2) \phi(0), \]

\[ \triangle g^{(2)}_{ij} = \frac{2\lambda}{\phi(0)} \Sigma_{ij}^{(0)} + m^2 \phi(2) \phi(0) \delta_{ij} + 2 \frac{\phi^{(2)}_{ij}}{\phi(0)}, \]

\[ \triangle \phi(2) - m^2 \phi(2) = -\frac{\lambda}{3} T^{(0)}. \]

The scalar field solution can be easily obtained from Eq. (24) as:

\[ \phi^{(2)}(x) = \frac{-\lambda}{3} \int d^3x' G(x, x') T^{(0)}(x') \]

where \( G(x, x') \) is the Green function of the operator \( \triangle - m^2 \), while, for \( g^{(2)}_{00} \) and \( g^{(2)}_{ij} \), we have

\[ g^{(2)}_{00}(x) = -\frac{\lambda}{2\pi\phi(0)} \int d^3x' \frac{\Sigma_{00}^{(0)}(x')}{|x - x'|} + \frac{m^2}{4\pi\phi(0)} \int d^3x' \frac{\phi^{(2)}(x')}{|x - x'|}, \]

\[ g^{(2)}_{ij}(x) = -\frac{\lambda}{2\pi\phi(0)} \int d^3x' \frac{\Sigma_{ij}^{(0)}(x')}{|x - x'|} - \frac{m^2\delta_{ij}}{4\pi\phi(0)} \int d^3x' \frac{\phi^{(2)}(x')}{|x - x'|} + \frac{2}{\phi(0)} \left[ \frac{x_i x_j}{|x|^3} \phi^{(2)}(x) + \left( \delta_{ij} - \frac{3x_i x_j}{|x|^2} \right) \frac{1}{|x|^3} \int_0^{|x|} d|x'| |x'|^2 \phi^{(2)}(x') \right]. \]

The above three solutions are a completely general result [7]. An example can make clearer the discussion. We can consider a fourth order gravity Lagrangian\(^1\) of the form \( f(R) = aR + bR^2 \) so that the “dummy” scalar field reads \( \phi = a + 2bR \). The relation between \( \phi \) and \( R \) is \( R = \frac{\phi - a}{2b} \) while the self-interaction potential turns out the be \( V(\phi) = \frac{(\phi - a)^2}{4b} \) satisfying the conditions \( V(a) = 0 \) and \( V'(a) = 0 \). In relation to the definition of the scalar field, we can opportunistically identify \( a \) with a constant value \( \phi(0) = a \). Furthermore, the scalar field mass can be expressed in term

\(^1\) It is important to stress that, in the Newtonian limit of any analytic \( f(R) \)-gravity model, we need to consider only the first two derivatives of \( f(R) \) [4].
of the Lagrangian parameters as \( m^2 = -\frac{1}{3} \phi^{(0)} \frac{d^2 V(\phi^{(0)})}{d\phi^2} = \frac{a}{6b} \). Since the Ricci scalar at lowest order (Newtonian limit) reads

\[
R \simeq R^{(2)}(x) = \frac{\phi^{(2)}(x)}{2b} = -\frac{\chi}{6b} \int d^3x' G(x, x') T^{(0)}(x'),
\]

(28)

if we consider a point-like mass \( M \), the energy-momentum tensor components become respectively \( T_{00} = \rho, T \sim \rho \) while \( \rho = M \delta(x) \), therefore we obtain

\[
R^{(2)} = -\frac{(2\pi)^{1/2} r_g m^2 e^{-m|x|}}{a} |x|,
\]

(29)

where \( r_g \) is the Schwarzschild radius. The immediate consequence is that the solution for the scalar field \( \phi \), up to the second order of perturbation, is given by

\[
\phi = a + \frac{(2\pi)^{1/2} r_g e^{-m|x|}}{3a} |x|.
\]

(30)

In the same way, one can deduce the expressions for \( g^{(2)}_{00} \) and \( g^{(2)}_{ij} \), where \( \Sigma_{00}^{(0)} = \frac{2}{3} \rho c^2 \) and \( \Sigma_{ij}^{(0)} = \frac{1}{3} \rho c^2 \delta_{ij} = \frac{1}{3} \Sigma_{00}^{(0)} \delta_{ij} \). As matter of fact, the metric solutions at the second order of perturbation are

\[
g_{00} = 1 - \frac{4}{3a} \frac{r_g}{|x|} - \frac{(2\pi)^{1/2} r_g e^{-m|x|}}{3a} \left( \frac{1}{|x|} - \frac{2}{m^2 |x|^3} \right) e^{-m|x|} - \frac{2}{m^2 |x|^3} |x|,
\]

(31)

\[
g_{ij} = - \left\{ \frac{1}{3a} \frac{r_g}{|x|} \left( \frac{1}{|x|} - \frac{2}{m^2 |x|^3} \right) e^{-m|x|} - \frac{2}{m^2 |x|^3} |x| \right\} \delta_{ij}
\]

\[+ \frac{2(2\pi)^{1/2} r_g}{3a} \left( \frac{1}{|x|} + \frac{3}{m^2 |x|^2} + \frac{3}{m^2 |x|^3} \right) e^{-m|x|} - \frac{3}{m^2 |x|^3} |x| \delta_{ij} x_i x_j \].

(32)

These quantities show that the gravitational potential coming from the O’Hanlon theory of gravity is non-Newtonian. The corrections have the meaning of scale parameters defining characteristic sizes and masses \([4, 9]\).

III. THE BEHAVIOR OF SOLUTIONS FOR \( f(R) \rightarrow R \) AND THE INTERPRETATION OF PPN PARAMETERS \( \gamma, \beta \)

The results \([31] - [32]\) are equivalent to those obtained in \( f(R) \)-gravity \([4]\). This point is very important since such a behaviour prevents from obtaining the standard definition of the PPN parameters as corrections to the Newtonian potential. As matter of fact, at the Newtonian level, it is indeed not true that a generic \( f(R) \)-gravity model corresponds to a Brans-Dicke model with \( \omega_{BD} = 0 \). In particular, in such a limit, it is not correct to consider the PPN parameter \( \gamma = \frac{1}{2} + \frac{\omega_{BD}}{2 + \omega_{BD}} \) (see \([3]\) of Brans-Dicke gravity and evaluating this at \( \omega_{BD} = 0 \). In this case, one obtains \( \gamma = 1/2 \) as suggested in \([2]\) and the standard Newton potential \( \gamma = 1/2 \) could never be recovered. Differently, because of the presence of the self-interaction potential \( V(\phi) \) in the O’Hanlon theory, a Yukawa-like correction appears and it contributes in a completely different way to the post-Newtonian limit. As matter of fact one obtains a different gravitational potential with respect to the ordinary Newtonian one and the fourth order corrections in term of the \( \nu/c \) ratio (PPN level) have to be evaluated in a different way. In other words, considering Brans-Dicke and O’Hanlon theories, despite of their similar structure, will imply completely different predictions in Newtonian limit. Such an achievement represents a significant argument against the claim that fourth order gravity models can be ruled out only on the bases of the analogy with Brans-Dicke PPN parameters.

Another important point has to be considered. The PPN-parameters \( \gamma \) and \( \beta \), in the GR context, are intended to parameterize the deviations from the Newtonian behaviour of the gravitational potentials. They are defined according to the standard Eddington metric
\[ g_{00} = 1 - \frac{r_g}{|x|} + \frac{\beta}{2} \frac{r_g^2}{|x|^2}, \] (33)

\[ g_{ij} = -\left(1 + \frac{\gamma r_g}{|x|}\right) \delta_{ij}. \] (34)

In particular, the PPN parameter \( \gamma \) is related with the second order correction to the gravitational potential while \( \beta \) is linked with the fourth order perturbation in \( v/c \). Actually, if we consider the limit \( f(R) \to R \) from Eqs. (31) and (32), we have

\[ g_{00} = 1 - \frac{4}{3a} \frac{r_g}{|x|}, \] (35)

\[ g_{ij} = -\left(1 + \frac{2}{3a} \frac{r_g}{|x|}\right) \delta_{ij}. \] (36)

Since \( a \) is an arbitrary constant, in order to match the Newtonian gravitational potential of GR, we should fix \( a = 4/3 \). This assumption implies

\[ g_{00} = 1 - \frac{r_g}{|x|}, \] (37)

\[ g_{ij} = -\left(1 + \frac{1}{2} \frac{r_g}{|x|}\right) \delta_{ij}. \] (38)

which suggest that the PPN parameter \( \gamma \), in this limit, is 1/2, that is in striking contrast with GR predictions. Such a result is not surprising. In fact, the GR limit of the O’Hanlon theory requires \( \phi \sim const \) \( V(\phi) \to 0 \) but such approximations induce mathematical inconsistencies in the field equations of \( f(R) \)-gravity, once these have been obtained by a general O’Hanlon Lagrangian. Actually, this is a general feature of the O’Hanlon theory. In fact it can be easily demonstrated that the field Eqs. (12) and (13) do not reduce to the standard GR ones since we have:

\[ R_{\mu\nu} = \frac{\mathcal{X}}{a} S_{\mu\nu}, \] (39)

\[ \frac{\mathcal{X}}{3} T = 0 \] (40)

but \( S_{\mu\nu} \) components read \( S_{00} = \frac{4}{3} \rho \) and \( S_{ij} = \frac{1}{2} \rho \delta_{ij} = \frac{1}{2} S_{00} \delta_{ij} \) instead of \( S_{00} = \frac{1}{2} \rho \) and \( S_{ij} = \frac{1}{2} \rho \delta_{ij} = S_{00} \delta_{ij} \) as usual. Of course, \( S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \) with \( T_{\mu\nu} \) the energy-momentum tensor of matter and Einstein equations are written in the form

\[ R_{\mu\nu} = \mathcal{X} S_{\mu\nu}. \] (41)

Such a pathology is in order even when the GR limit is recovered from the Brans-Dicke theory. In such a case, in order to match the Hilbert-Einstein Lagrangian, one needs \( \phi \sim const \) and \( \omega = 0 \), the immediate consequence is that the PPN parameter \( \gamma \) turns out to be 1/2, while it is well known that Brans-Dicke model fulfils the low energy limit prescriptions of GR in the limit \( \omega \to \infty \). Even in this case, the problem, with respect to the GR prediction, is that the GR limit of the model introduces inconsistencies in the field equations. In other words, it is not possible to impose the same transformation which leads BD theory into GR at the Lagrangian level on the solutions and the observables obtained by solving field equations descending from the general Lagrangian. The relevant aspect of this discussion is that considering a \( f(R) \)-model, in analogy with the O’Hanlon theory and then supposing that the self-interaction potential is negligible, introduces a pathological behaviour on the solutions leading to the PPN parameter \( \gamma = 1/2 \). This is what happens when an effective approximation scheme is introduced in the field equations in order to calculate the weak field limit of fourth order gravity by means of Brans-Dicke model. Such a result seems, from another point of view, to enforce the claim that fourth order gravity models have to be carefully investigated in this limit and their analogy with scalar-tensor gravity should be considered accordingly.
IV. THE CONFORMAL TRANSFORMATIONS AND THE NEWTONIAN LIMIT

Along the paper we have discussed the Newtonian limit of $f(R)$-gravity in term of scalar-tensor gravity rigorously remaining in the Jordan frame. In this section, we discuss the Newtonian limit when a conformal transformation is applied to the O’Hanlon theory. In other words we discuss fourth order gravity models in the Einstein frame in place of the Jordan one when a redefinition of the metric in the conformal sense is performed. A scalar-tensor gravity theory is, in some sense, a generalization of both the Brans-Dicke and the O’Hanlon theories, that is

$$\mathcal{A}_{ST}^{JF} = \int d^4x \sqrt{-g} \left[ F(\phi) R + \omega(\phi) \phi \phi^{\alpha} - V(\phi) + \lambda \mathcal{L}_m \right].$$

(42)

Such a theory can be transformed by means of a conformal transformation $\tilde{g}_{\mu\nu} = A(x) g_{\mu\nu}$, with $A(x) > 0$ satisfying the condition $F(\phi) A(x)^{-1} = \Lambda \in \mathbb{R}$, as

$$\mathcal{A}_{ST}^{EF} = \int d^4x \sqrt{-\tilde{g}} \left[ \Lambda \tilde{R} + \Omega(\varphi) \varphi, \varphi^{\alpha} - W(\varphi) + \lambda \tilde{\mathcal{L}}_m \right].$$

(43)

The relations between the quantities in the two frames are

$$\begin{align*}
\Omega(\varphi) d\varphi^2 &= \Lambda \left[ \frac{\omega(\phi)}{F(\phi)} - \frac{1}{2} \left( \frac{d \ln F(\phi)}{d \phi} \right)^2 \right] d\phi^2 \\
W(\varphi) &= \frac{\Lambda^2}{F(\phi)^2} V(\phi(\varphi)) \\
\tilde{\mathcal{L}}_m &= \frac{\Lambda^2}{F(\phi)^2} \mathcal{L}_m \left( \frac{\Lambda \phi}{F(\phi)} \right)
\end{align*}$$

(44)

In the case of the O’Hanlon Lagrangian, (5), i.e. $F(\phi) = \phi$, $\omega(\phi) = 0$, the previous Lagrangian turns out to be simplified and the transformation rule between the two scalar fields reads

$$\Omega(\varphi) d\varphi^2 = -\frac{3 \Lambda}{2} \frac{d \phi^2}{\phi^2} \quad \Rightarrow \quad \phi = k e^{\pm \int \sqrt{-\frac{2 \Omega(\varphi)}{3 \Lambda}} d\varphi},$$

(45)

where $k$ is a integration constant. If we suppose $\Omega(\varphi) = -\Omega_0 < 0$, we have

$$\phi = k e^{\pm Y \varphi},$$

(46)

where $Y = \sqrt{\frac{2 \Omega_0}{3 \Lambda}}$. The transformed action (5) in the Einstein frame is

$$\mathcal{A}_{OH}^{EF} = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{\Lambda} \tilde{R} - \Omega_0 \varphi, \varphi^{\alpha} - \frac{\Lambda^2}{k^2} e^{2Y \varphi} V(k e^{\pm Y \varphi}) + \frac{\lambda \Lambda^2}{k^2} e^{2Y \varphi} \mathcal{L}_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\mu\nu} \right) \right].$$

(47)

The field equations are now

$$\begin{cases}
\Lambda \tilde{G}_{\mu\nu} + \frac{1}{2} \frac{\Lambda^2}{k^2} e^{2Y \varphi} V(k e^{\pm Y \varphi}) \tilde{g}_{\mu\nu} - \Omega_0 \left( \varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} \varphi, \varphi^{\alpha} \tilde{g}_{\mu\nu} \right) = \lambda \tilde{T}_{\mu\nu} \\
2 \Omega_0 \tilde{\Box} \varphi - \frac{\Lambda^2}{k^2} e^{2Y \varphi} \left[ \frac{\lambda}{k} (k e^{\pm Y \varphi} \mp 2 Y V(k e^{\pm Y \varphi})) + \lambda \frac{\tilde{\mathcal{L}}_m}{\tilde{\varphi}} \right] = 0 \\
\tilde{R} = -\frac{\lambda}{k} \tilde{T}_{\varphi} + \frac{\Omega_0}{\Lambda} \varphi, \varphi^{\alpha} + \frac{2 \Lambda}{k^2} e^{2Y \varphi} V(k e^{\pm Y \varphi})
\end{cases}$$

(48)

where the matter tensor, which now coupled with the scalar field $\varphi$, in the Einstein frame [14] reads
\[
\tilde{T}^\varphi_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-\tilde{g}} \tilde{L}_m)}{\delta \tilde{g}^{\mu\nu}} = \frac{\Lambda^2}{k^2} e^{\mp 2Y \varphi} \left[ L_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho\sigma} \right) \tilde{g}_{\mu\nu} - 2 \frac{\delta}{\delta \tilde{g}^{\mu\nu}} L_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho\sigma} \right) \right],
\]

and

\[
\frac{\partial \tilde{L}_m}{\partial \varphi} = \frac{\Lambda^2}{k^2} e^{\mp 2Y \varphi} \left[ 2 L_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho\sigma} \right) + \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho\sigma} \frac{\delta L_m}{\delta \tilde{g}_{\rho\sigma}} \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho\sigma} \right) \right].
\]

Actually, in order to calculate the Newtonian limit of the model in the Einstein frame, we can develop the two scalar fields at the second order \( \phi \sim \phi^{(0)} + \phi^{(2)} \) and \( \varphi \sim \varphi^{(0)} + \varphi^{(2)} \) with respect to a background value. This choice gives the relation:

\[
\begin{align*}
\phi^{(0)} &= \pm Y^{-1} \ln \frac{\phi^{(0)}}{k} \\
\varphi^{(2)} &= \pm Y^{-1} \varphi^{(2)}
\end{align*}
\]

Let us consider the conformal transformation \( \tilde{g}_{\mu\nu} = \frac{\phi}{\Lambda} g_{\mu\nu} \). From this relation and considering the (46) one obtains, if \( \phi^{(0)} = \Lambda \), that

\[
\begin{align*}
\tilde{g}^{(2)}_{00} &= g^{(2)}_{00} + \frac{\phi^{(2)}}{\phi^{(0)}} \\
\tilde{g}^{(2)}_{ij} &= g^{(2)}_{ij} - \frac{\phi^{(2)}}{\phi^{(0)}} \delta_{ij}
\end{align*}
\]

As matter of fact, since \( g^{(2)}_{00} = 2 \Phi^{JF}, g^{(2)}_{ij} = 2 \Psi^{JF} \delta_{ij} \) and \( \tilde{g}^{(2)}_{00} = 2 \Phi^{EF}, \tilde{g}^{(2)}_{ij} = 2 \Psi^{EF} \delta_{ij} \) from (49) it descends a relevant relation which links the gravitational potentials of Jordan and Einstein frame:

\[
\begin{align*}
\Phi^{EF} &= \Phi^{JF} + \frac{\phi^{(2)}}{2 \phi^{(0)}} = \Phi^{JF} \pm \frac{Y}{2} \varphi^{(2)} \\
\Psi^{EF} &= \Psi^{JF} - \frac{\phi^{(2)}}{2 \phi^{(0)}} = \Psi^{JF} \mp \frac{Y}{2} \varphi^{(2)}
\end{align*}
\]

If we introduce the variations of two potentials: \( \Delta \Phi = \Phi^{JF} - \Phi^{EF} \) and \( \Delta \Psi = \Psi^{JF} - \Psi^{EF} \) we obtain the most relevant result of this section:

\[
\Delta \Phi = - \Delta \Psi = - \frac{\phi^{(2)}}{2 \phi^{(0)}} = \mp \frac{Y}{2} \varphi^{(2)} \propto b \propto f''(R = 0).
\]

From the above expressions, one can notice that there is an evident difference between the behavior of the two gravitational potentials in the two frames. Such achievement suggests that, at the Newtonian level, it is possible to discriminate between the two frames. Specifically, once the gravitational potential is calculated in the Jordan frame and the dynamical evolution of \( \phi \) is taken into account at the suitable perturbation level, these can be substituted in the first of Eqs. (53) to obtain its evolution in the Einstein frame. The final step is that the two potentials have to be matched with experimental data in order to investigate what the physical solution is.

V. CONCLUSIONS

In this paper, we have used the analogy between the \( f(R) \)-gravity and the O’Hanlon theory to discuss, in the Jordan frame, the Newtonian limit of the theory. The main result is that it is not possible to consider the analogy between \( f(R) \)-gravity and the Brans-Dicke theory to achieve the correct PPN limit, as done several times in literature, since the result \( \omega_{BD} = 0 \) implying \( \gamma = 1/2 \) is a pathology of the theory (both \( f(R) \) and Brans-Dicke). This means that the PPN-parameters have to be redefined accordingly in the Jordan frame of \( f(R) \)-gravity without transforming the theory.
When we perform the GR limit of \( f(R) \)-gravity, the correspondence is needed at any level (Lagrangian, field equations and solutions) between the GR and the \( f(R) \)-gravity, in particular as soon as \( f(R) \sim R^{1+\epsilon} \) with \( \epsilon \to 0 \). This means that the statement \( \gamma = 1/2 \) for any \( f(R) \) \[2\] is not correct since the transformation in terms of Brans-Dicke theory does not work. Supposing to modify the Hilbert-Einstein Lagrangian, the correction to the solutions can not produce the same displacement from Schwarzschild solution for any \( f(R) \)-gravity and such displacement could not be independent from the analytical form of \( f(R) \). In fact when \( f(R) \to R \) the solutions of field equations are not the solutions of GR. Furthermore the Eddington parameterization \[10\] is based on the hypothesis that metric has to match second order differential field equations. This means that gravitational potential admits the same Green function of the Newtonian theory \[11\]). In \( f(R) \)-gravity case, field equations are fourth-order in metric approach. The field equations, in Newtonian limit, admit Yukawa-like corrections and the standard Eddington parameterization cannot work \[3, 12, 13\]. On the other hand, in order to compare results in Einstein frame and Jordan frame, one can perform the Newtonian limit in both frames and then compare the solutions. Immediately it emerges that results are different. As final remark, it is worth saying that \( f(R) \)-gravity can fully evade Solar system tests but results have to be carefully analyzed in the right frame. Forthcoming experiments could clearly give indications in this sense \[15\].

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