SUMS OF CR FUNCTIONS FROM COMPETING CR STRUCTURES

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Abstract. In this paper we characterize sums of CR functions from competing CR structures in two scenarios. In one scenario the structures are conjugate and we are adding to the theory of pluriharmonic boundary values. In the second scenario the structures are related by projective duality considerations. In both cases we provide explicit vector field-based characterizations for two-dimensional circular domains satisfying natural convexity conditions.

1. Introduction

The Dirichlet problem for pluriharmonic functions is a natural problem in several complex variables with a long history going back at least to Amoroso [Amo], Severi [Sev], Wirtinger [Wir], and others. It was known early on that the problem is not solvable for general boundary data, so we may try to characterize the admissible boundary values with a system of tangential partial differential operators. This was first done for the ball by Bedford in [Bed1]; see §2.1 for details. More precisely, given a bounded domain $\Omega$ with smooth boundary $S$, we seek a system $L$ of partial differential operators tangential to $S$ such that a function $u \in C^\infty(S,\mathbb{C})$ satisfies $Lu = 0$ if and only if there exists $U \in C^\infty(\Omega)$ such that $U|_S = u$ and $\partial\bar{\partial}U = 0$. The problem may also be considered locally.

While natural in its own right, this problem also arises in less direct fashion in many areas of complex analysis and geometry. For instance, this problem plays a fundamental role in Graham’s work on Bergman Laplacian [Gra], Lee’s work on pseudo-Einstein structures [Lee], and Case, Chanillo, and Yang’s work on CR Paneitz operators (see [CCY] and the references therein).

From another point of view, the existence of non-trivial restrictions on pluriharmonic boundary values points to the need to look elsewhere (such as to the Monge-Ampère equations studied in [BeTa]) for Dirichlet problems solvable for general boundary data.

The pluriharmonic boundary value problem is closely related to the problem of characterizing sums of CR functions from different, competing CR structures; indeed, when the competing CR structures are conjugate then these problems coincide (in simply-connected settings); see Theorems 4 and 5 below. Another natural construction leading to competing CR structures arises from the study of projective duality (see §3 or [Bar] for precise definitions).

In each of these two scenarios we precisely characterize sums of CR functions from the two competing CR structures in the setting of two-dimensional circular domains satisfying...
appropriate convexity conditions. For conjugate structures we assume strong pseudoconvexity; our result appears as Theorem 6 below. In the projective duality scenario we assume strong $C$-convexity (see §3.1) and the main result appears as Theorem 23.

Our techniques for these two related problems are interconnected to a surprising extent, and the reader will notice that the projective dual scenario actually turns out to have more structure and symmetry.

This paper is organized as follows. In §2 we focus on the case of conjugate CR structures (the pluriharmonic case). In §3 we study the competing CR structures coming from projective duality. In §4 we prove the main result in the projective setting. In §5 we prove the main result in the case of conjugate structures. Finally in §6 we consider uniqueness questions and further connections with projective duality results in [Bar].

2. Conjugate structures

2.1. Results on the ball. Early work focused on the case of the ball $B^n$ in $\mathbb{C}^n$. In particular, Nirenberg observed that there is no second-order system of differential operators tangent to $S^3$ that exactly characterize pluriharmonic functions (see §6.2 for more details). Third-order characterizations were developed by Bedford in the global case and Audibert in the local case (which requires stronger conditions).

To state these results we define the tangential operators

\begin{align}
L_{kl} &= z_k \frac{\partial}{\partial z_l} - z_l \frac{\partial}{\partial z_k} \\
L_{kl} &= z_k \frac{\partial}{\partial z_l} - z_l \frac{\partial}{\partial z_k}
\end{align}

for $1 \leq k, l \leq n$.

**Theorem 1** ([Bed1]). Let $u$ be smooth on $S^{2n-1}$, then

$$L_{kl}L_{kl}u = 0$$

for $1 \leq k, l \leq n$ if and only if $u$ extends to a pluriharmonic function on $B^n$.

**Theorem 2** ([Aud]). Let $S$ be a relatively open subset of $S^{2n-1}$. and let $u$ be smooth on $S$. Then

$$L_{jk}L_{lm}L_{rs}u = 0 = L_{jk}L_{lm}L_{rs}u$$

for $1 \leq j, k, l, m, r, s \leq n$ if and only if $u$ extends to a pluriharmonic function on a one-sided neighborhood of $S$.

For a treatment of both of these results along with further details and examples, see §18.3 of [Rud].

2.2. Other results. Laville [Lav1, Lav2] also gave a fourth order operator to solve the global problem. In [BeFe] Bedford and Federbush solved the local problem in the more general setting where $b\Omega$ has non-zero Levi form at some point. Later in [Bed2], Bedford used the induced boundary complex $(\partial\bar{\partial})_b$ to solve the local problem in certain settings. In Lee’s work [Lee] on pseudo-Einstein structures, he gives a characterization for abstract CR manifolds using third order pseudohermitian covariant derivatives. Case, Chanillo, and Yang study when the kernel of the CR Paneitz operator characterizes CR-pluriharmonic functions (see [CCY] and the references therein).
2.3. Relation to decomposition on the boundary.

**Remark 3.** Outside of the proof of Theorem 33 below, all forms, functions, and submanifolds will be assumed $C^\infty$-smooth.

**Theorem 4.** Let $S \subset \mathbb{C}^n$ be a compact connected and simply-connected real hypersurface and let $\Omega$ be the bounded domain with boundary $S$. Then for $u: S \to \mathbb{C}$ the following conditions are equivalent.

(2.2a) $u$ extends to a (smooth) function $U$ on $\overline{\Omega}$ that is pluriharmonic on $\Omega$.
(2.2b) $u$ is the sum of a CR function and a conjugate-CR-function.

**Proof.** In the proof that (2.2a) implies (2.2b), the CR term is the restriction to $S$ of an anti-derivative for $\partial U$ on a simply-connected one-sided neighborhood of $S$, and the conjugate-CR term is the restriction to $S$ of an anti-derivative for $\overline{\partial} U$ on a one-sided neighborhood of $S$ (adjusting one term by a constant as needed).

To see that (2.2b) implies (2.2a) we use the global CR extension result [Hör, Thm. 2.3.2] to extend the terms to holomorphic and conjugate-holomorphic functions, respectively; $U$ is then the sum of the extensions. □

**Theorem 5.** Let $S \subset \mathbb{C}^n$ be a simply-connected strongly pseudoconvex real hypersurface. Then for $u: S \to \mathbb{C}$ the following conditions are equivalent.

(2.3a) there is an open subset $W$ of $\mathbb{C}^n$ with $S \subset bW$ (with $W$ lying locally on the pseudoconvex side of $S$) so that $u$ extends to a (smooth) function $U$ on $W \cup S$ that is pluriharmonic on $W$.
(2.3b) $u$ is the sum of a CR function and a conjugate-CR-function.

**Proof.** The proof follows the proof of Theorem 4 above, replacing the global CR extension result by the Hans Lewy local CR extension result as stated in [Bog, Sec. 14.1, Thm. 1]. □

2.4. Circular hypersurfaces in $\mathbb{C}^2$.

**Theorem 6.** Let $S \subset \mathbb{C}^2$ be a strongly pseudoconvex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields $X, Y$ on $S$ satisfying the following conditions.

(2.4a) If $u$ is a function on a relatively open subset of $S$ then $u$ is CR if and only if $Xu = 0$.
(2.4b) If $u$ is a function on a relatively open subset of $S$ then $u$ is CR if and only if $Y\overline{u} = 0$.
(2.4c) If $S$ is compact then a function $u$ on $S$ is a pluriharmonic boundary value (in the sense of Theorem 4) if and only if $XXY u = 0$.
(2.4d) A function $u$ on a relatively open subset of $S$ is a pluriharmonic boundary value (in the sense of Theorem 4) if and only if $XXY u = 0 = XXYu$.

It is not possible in general to have $Y = \overline{X}$.

Theorem 4 is proved in §4 below.

3. Projective dual structures

3.1. Projective dual hypersurfaces. Let $S \subset \mathbb{C}^n$ be an oriented real hypersurface with defining function $\rho$. $S$ is said to be strongly $\mathbb{C}$-convex if $S$ locally equivalent via a projective...
transformation (that is, via an automorphism of projective space) to a strongly convex hypersurface; this condition is equivalent to either of the following two equivalent conditions:

(3.1a) the second fundamental form for $S$ is positive definite on the maximal complex subspace $H_z S$ of each $T_z S$;

(3.1b) the complex tangent (affine) hyperplanes for $S$ lie to one side (the “concave side”) of $S$ near the point of tangency with minimal order of contact.

**Proposition 7.** When $S$ is compact and strongly $\mathbb{C}$-convex the complex tangent hyperplanes for $S$ are in fact disjoint from the domain bounded by $S$.

**Proof.** [APS §2.5].

We note that strongly $\mathbb{C}$-convex hypersurfaces are also strongly pseudoconvex.

A circular hypersurface (that is, a hypersurface invariant under rotations $z \mapsto e^{i\theta}z$) is strongly $\mathbb{C}$-convex if and only if it is strongly convex [Cer, Prop. 3.7].

The proper general context for the notion of strong $\mathbb{C}$-convexity is in the study of real hypersurfaces in complex projective space $\mathbb{C}P^n$ (see for example [Bar] and [APS]).

We specialize now to the two-dimensional case.

**Lemma 8.** Let $S \subset \mathbb{C}^2$ be a compact strongly $\mathbb{C}$-convex hypersurface enclosing the origin. Then there is a uniquely-determined map

$$D: S \to \mathbb{C}^2 \setminus \{0\}$$

$$z \mapsto w(z) = (w_1(z), w_2(z))$$

satisfying

(3.2a) $z_1 w_1 + z_2 w_2 = 1$ on $S$;

(3.2b) the vector field

$$Y \overset{\text{def}}{=} w_2 \frac{\partial}{\partial z_1} - w_1 \frac{\partial}{\partial z_2}$$

is tangent to $S$. Moreover, $Y$ annihilates conjugate-CR functions on any relatively open subset of $S$.

**Proof.** It is easy to check that (3.2a) and (3.2b) force

$$w_1(z) = \frac{\frac{\partial \rho}{\partial z_1}}{z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2}}$$

$$w_2(z) = \frac{\frac{\partial \rho}{\partial z_2}}{z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2}}$$

establishing uniqueness. Existence follows provided that the denominators do not vanish; but the vanishing of the denominators occurs precisely when the complex tangent line for $S$ at $z$ passes through the origin, and Proposition 7 above guarantees that this does not occur under the given hypotheses.
Remark 9. It is clear from the proof that the conclusions of Lemma 8 also hold under the assumption that $S$ is a (not necessarily compact) hypersurface satisfying

\[(3.3)\] no complex tangent line for $S$ passes through the origin.

Remark 10. Any tangential vector field annihilating conjugate-CR functions will be a scalar multiple of $Y$.

Remark 11. The complex line tangent to $S$ at $z$ is given by

\[(3.4)\] \[\{\zeta \in \mathbb{C}^2 : w_1(z)\zeta_1 + w_2(z)\zeta_2 = 1\} \].

Remark 12. The maximal complex subspace $H_z S$ of each $T_z S$ is annihilated by the form $w_1 dz_1 + w_2 dz_2$.

Proposition 13. For $S$ strongly $\mathbb{C}$-convex satisfying \[(3.3)\] the map $D$ is a local diffeomorphism onto an immersed strongly $\mathbb{C}$-convex hypersurface $S^*$, with each maximal complex subspace $H_z S$ of $T_z S$ mapped (non-$\mathbb{C}$-linearly) by $D'$ onto the corresponding maximal complex subspace of $H_{w(z)} S^*$. For $S$ strongly $\mathbb{C}$-convex and compact $S^*$ is an embedded strongly $\mathbb{C}$-convex hypersurface and $D$ is a diffeomorphism.

Proof. \cite{Bar §6], \cite{APS §2.5}. \]

For $S$ strongly $\mathbb{C}$-convex satisfying \[(3.3)\] we may extend $D$ to a smooth map on an open set in $\mathbb{C}^2$; the extended map $D^*$ will be a local diffeomorphism in some neighborhood $U$ of $S$. We may then define vector fields $\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}$ on $U$ by applying $(D^*)^{-1}$ to the corresponding vector fields on $D^*(U)$; these newly-defined vector fields will depend on the choice of the extension $D^*$.

Lemma 14. The non-vanishing vector field

\[V \overset{\text{def}}{=} z_2 \frac{\partial}{\partial w_1} - z_1 \frac{\partial}{\partial w_2}\]

is tangent to $S$ and is independent of the choice of the extension $D^*$.

Proof. From \[(3.3a)\] we have

\[0 = d(z_1 w_1 + z_2 w_2) = z_1 dw_1 + z_2 dw_2 + w_1 dz_1 + w_2 dz_2\]
on $T_z S$.

From Remark 12 we deduce that the null space in $T_z \mathbb{C}^2$ of $z_1 dw_1 + z_2 dw_2$ is precisely the maximal complex subspace $H_z S$ of $T_z S$ (and moreover the null space in $(T_z \mathbb{C}^2) \otimes \mathbb{C}$ of of $z_1 dw_1 + z_2 dw_2$ is precisely $(H_z S) \otimes \mathbb{C}$). If we apply $z_1 dw_1 + z_2 dw_2$ to $V$ we obtain

\[z_1 \cdot V w_1 + z_2 \cdot V w_2 = z_1 \cdot z_2 - z_2 \cdot z_1 = 0\]
showing that $V$ takes values in $(H_z S) \otimes \mathbb{C}$ and is thus tangential.
If an alternate tangential vector field $\tilde{V}$ is constructed with the use of an alternate extension $\tilde{D}^{\ast}$ of $D$ then

$$\tilde{V}w_{j} = \pm z_{3-j} = Vw_{j}$$
$$\tilde{V}\overline{w}_{j} = 0 = V\overline{w}_{j}$$

along $S$, so $\tilde{V} = V$ along $S$. □

**Definition 15.** A function $u$ on relatively open subset of $S$ will be called dual-CR if $Vu = 0$.

**Example 16.** If $S$ is the unit sphere in $\mathbb{C}^2$ then $w(z) = \overline{z}$ and the set of dual-CR functions on $S$ coincides with the set of conjugate-CR functions on $S$.

The set of dual-CR functions will only rarely coincide with the set of conjugate-CR functions as we see from the following two related results.

**Theorem 17.** If $S$ is a compact strongly $\mathbb{C}$-convex hypersurface in $\mathbb{C}^2$ then the set of dual-CR functions on $S$ will coincide with the set of conjugate-CR functions on $S$ if and only if $S$ is a complex-affine image of the unit sphere.

**Theorem 18.** If $S$ is a strongly $\mathbb{C}$-convex hypersurface in $\mathbb{C}^2$ then the set of dual-CR functions on $S$ will coincide with the set of conjugate-CR functions on $S$ if and only if $S$ is locally the image of a relatively open subset of the unit sphere by a projective transformation.

For proofs of these results see [Jen], [DeTr], and [Bol].

**Remark 19.** The constructions of the vector fields $Y$ and $V$ transform naturally under complex-affine mapping of $S$. The construction of the dual-CR structure transforms naturally under projective transformation of $S$. (See for example [Bar, §6].)

**Lemma 20.** Relations of the form

$$V = \chi Y + \sigma \overline{Y}$$
$$Y = \kappa V + \xi \overline{V}$$

hold along $S$ with $\sigma$ and $\xi$ nowhere vanishing.

**Proof.** This follows from the following facts:

- $V, \overline{V}, Y$ and $\overline{Y}$ all take values in the two-dimensional space $(H_z S) \otimes \mathbb{C}$;
- $V$ and $\overline{V}$ are $\mathbb{C}$-linearly independent, as are $Y$ and $\overline{Y}$;
- the non-$\mathbb{C}$-linearity of the map $D'_{z^*}: (H_z S) \otimes \mathbb{C} \to (H_z S^*) \otimes \mathbb{C}$ (see Proposition [13]). □

**Lemma 21.** If $f_1, f_2$ are CR functions and $g_1, g_2$ are dual-CR functions on a connected relatively open subset $W$ of $S$ with $f_1 + g_1 = f_2 + g_2$ then $g_2 - g_1 = f_1 - f_2$ is constant.

**Proof.** From Lemma 20 we deduce that the directional derivatives of $g_2 - g_1 = f_1 - f_2$ vanish in every direction belong to the maximal complex subspace of $TS$. Applying one Lie bracket we find that in fact all directional derivatives along $S$ of $g_2 - g_1 = f_1 - f_2$ vanish. □

**Corollary 22.** If $W$ is a simply-connected relatively open subset of $S$ and $u$ is a function on $W$ that is locally decomposable as the sum of a CR function and a dual-CR function then $u$ is decomposable on all of $W$ as the sum of a CR function and a dual-CR function.
3.2. Circular hypersurfaces in $\mathbb{C}^2$.

**Theorem 23.** Let $S \subset \mathbb{C}^2$ be a strongly ($\mathcal{CR}$-)convex circular hypersurface. Then there exist tangential vector fields $X, T$ on $S$ satisfying the following conditions.

(3.5a) There are scalar functions $\phi$ and $\psi$ so that

$$X = V + \phi \overline{V}, \quad T = Y + \psi \overline{Y}.$$ 

(3.5b) If $u$ is a function on a relatively open subset of $S$ then $f$ is CR if and only if $Xu = 0$; equivalently, $X$ is a non-vanishing scalar multiple $\alpha Y$ of $Y$.

(3.5c) If $u$ is a function on a relatively open subset of $S$ then $g$ is dual-CR if and only if $Tu = 0$; equivalently, $T$ is a non-vanishing scalar multiple $\beta V$ of $V$.

(3.5d) If $S$ is compact then a function $u$ on $S$ is the sum of a CR function and a dual-CR function if and only if $XXTu = 0$.

(3.5e) If $S$ is simply-connected (but not necessarily compact) then a function $u$ on $S$ is the sum of a CR function and a dual-CR function if and only if $XXTu = 0 = TTXu$.

As we shall see the vector field $X$ in Theorem 23 will also work as the vector field $X$ in Theorem 6.

**Example 24.** (Compare [Aud].) The function $\frac{z_1}{w_2}$ satisfies $XXT \frac{z_1}{w_2} = 0$ but is not globally defined. Since $TTX \frac{z_1}{w_2} = 2 \neq 0$ this function is not locally the sum of a CR function and a dual-CR function.

Conditions (3.5a), (3.5b) and (3.5c) uniquely determine $X$ and $T$. See §6.1 for some discussion of what can happen without condition (3.5a).

4. **Proof of Theorem 23**

To prove Theorem 23 we start by consulting Lemma 20 and note that (3.5a), (3.5b) and (3.5c) will hold if we set

$$\alpha = 1/\xi, \quad \beta = 1/\sigma, \quad \phi = \overline{\kappa}/\xi, \quad \psi = \overline{\chi}/\sigma;$$

it remains to check (3.5d) and (3.5e).
We note for future reference and the reader’s convenience that
\begin{align*}
  X w_1 &= z_2 & X w_2 &= -z_1 \\
  \nabla w_1 &= \frac{\bar{\xi}}{2} z_2 & \nabla w_2 &= -\frac{\bar{\xi}}{2} z_1 \\
  X \bar{\xi} &= \phi z_2 & X \bar{\xi} &= -\phi z_1 \\
  X z_1 &= \nabla \bar{z}_1 = 0 & X z_2 &= \nabla \bar{z}_2 = 0 \\
  X \alpha &= \nabla \bar{w}_2 & X \alpha &= -\nabla \bar{w}_1 \\
  T z_1 &= w_2 & T z_2 &= -w_1 \\
  \nabla z_1 &= \sigma w_2 & \nabla z_2 &= -\sigma w_1 \\
  T \bar{\xi} &= \psi \bar{w}_2 & T \bar{\xi} &= -\psi \bar{w}_1 \\
  T w_1 &= \nabla w_1 = 0 & T w_2 &= \nabla w_2 = 0 \\
  T \bar{w}_1 &= \beta \bar{z}_2 & T \bar{w}_2 &= -\beta \bar{z}_1.
\end{align*}
(4.1)

Lemma 25.
\begin{align*}
  [Y, \nabla] &= \bar{\xi} \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) - \xi \left( \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \\
  [V, \nabla] &= \bar{\sigma} \left( w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} \right) - \sigma \left( \bar{w}_1 \frac{\partial}{\partial \bar{w}_1} + \bar{w}_2 \frac{\partial}{\partial \bar{w}_2} \right).
\end{align*}

Proof. The first statement follows from
\begin{equation}
  [Y, \nabla] = (Y \bar{w}_2) \frac{\partial}{\partial \bar{z}_1} - (Y \bar{w}_1) \frac{\partial}{\partial \bar{z}_2} - (\nabla \bar{w}_2) \frac{\partial}{\partial z_1} + (\nabla \bar{w}_1) \frac{\partial}{\partial z_2}
\end{equation}
along with (4.1).

The proof of the second statement is similar. \qed

We note that the assumption that $S$ is circular has not been used so far in this section. We now bring it into play by introducing the real tangential vector field
\[ R \overset{\text{def}}{=} i \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \]
generating the rotations of $z \mapsto e^{i\theta} z$ of $S$.

Lemma 26. The following hold.
\begin{align*}
  \xi &= \xi & \sigma &= \sigma & \alpha &= \alpha & \beta &= \beta \\
  (4.2a) & \bar{\xi} &= \xi & (4.2b) & \bar{\sigma} &= \sigma & (4.2c) & \bar{\alpha} &= \alpha & (4.2d) & \bar{\beta} &= \beta \\
  (4.2e) & R &= -i \left( w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} - \bar{w}_1 \frac{\partial}{\partial \bar{w}_1} - \bar{w}_2 \frac{\partial}{\partial \bar{w}_2} \right) \\
  (4.2f) & [Y, \nabla] &= -i \bar{\xi} R & (4.2g) & [V, \nabla] &= i \sigma R & (4.2h) & [X, Y] &= i R - (Y \alpha) \nabla
\end{align*}
Proof. We start by considering the tangential vector field

\[ [Y, \overline{Y}] + i \xi R = (\xi - \xi) \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right); \]

if (4.2b) fails then \( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \) is a non-vanishing holomorphic tangential vector field on some non-empty relatively open subset of \( S \), contradicting the strong pseudoconvexity of \( S \).

To prove (4.2e) we first note from Lemma 8 that \( w(e^{i\theta} z) = e^{-i\theta} w(z) \); differentiation with respect to \( \theta \) yields (4.2e).

The proof of (4.2a) now may be adapted to prove (4.2b). (4.2c) and (4.2d) follow immediately.

Using Lemma 25 in combination with (4.2a) and (4.2b) we obtain (4.2f) and (4.2g).

From (3.5b) and (4.2f) we obtain (4.2b).

\[ \square\]

Lemma 27. \([X, T] = iR\).

Proof. On the one hand,

\[
[X, T] = [V + \phi \nabla, \beta \nabla] = ((V + \phi \nabla)\beta - \beta(\nabla \phi)) \nabla + i \beta \sigma R = ((V + \phi \nabla)\beta - \beta(\nabla \phi)) \nabla + iR.
\]

On the other hand,

\[
[X, T] = [\alpha \overline{Y}, Y + \psi \overline{Y}] = (\alpha(\overline{Y} \psi) - (Y + \psi \overline{Y})\alpha) \overline{Y} + i \alpha \xi R = (\alpha(\overline{Y} \psi) - (Y + \psi \overline{Y})\alpha) \overline{Y} + iR.
\]

Since \( \nabla \) and \( \overline{Y} \) are linearly independent, it follows that \([X, T] = iR\). \[ \square\]

Lemma 28. The following hold.

\begin{align*}
(4.3a) & \quad [R, Y] = -2iY \\
(4.3b) & \quad [R, \overline{Y}] = 2i\overline{Y} \\
(4.3c) & \quad [R, V] = 2iV \\
(4.3d) & \quad [R, \overline{V}] = -2i\overline{V} \\
(4.3e) & \quad [R, X] = 2iX \\
(4.3f) & \quad [R, \overline{X}] = -2i\overline{X} \\
(4.3g) & \quad [R, T] = -2iT \\
(4.3h) & \quad [R, \overline{T}] = 2iT \\
(4.3i) & \quad R\alpha = 0 \\
(4.3j) & \quad R\beta = 0
\end{align*}

Proof. (4.3a), (4.3b), (4.3c) and (4.3h) follow from direct calculation.

For (4.3g) first note that writing \( T = \beta \nabla \) and using (4.3f) we see that \([R, T]\) is a scalar multiple of \( T \). Then writing

\[
[R, T] = [R, Y + \psi \overline{Y}] = -2iY + (\text{multiple of } \overline{Y})
\]
we conclude using (3.5a) that \([R, T] = -2iT\). The proof of (4.3b) is similar, and (4.3h) and (4.3i) follow by conjugation.

Using (3.5b) along with (4.3b) and (4.3e) we obtain (4.3i); (4.3j) is proved similarly. □

Lemma 29. \(XXf = 0\) if and only if \(f = f_1 w_1 + f_2 w_2\) with \(f_1, f_2\) CR.

Proof. From (3.5b) and (4.1) it is clear that \(XX (f_1 w_1 + f_2 w_2) = 0\) if \(f_1\) and \(f_2\) are CR.

For the other direction, suppose that \(XXf = 0\). Then setting \(f_1 \overset{\text{def}}{=} z_1 f + w_2 Xf\)
and \(f_2 \overset{\text{def}}{=} z_2 f - w_1 Xf\),

it is clear that \(f = f_1 w_1 + f_2 w_2\); with the use of (3.5b) and (4.1) it is also easy to check that \(f_1\) and \(f_2\) are CR. □

Lemma 30. Suppose that \(XXTu = 0\) so that by Lemma 29 we may write \(Tu = f_1 w_1 + f_2 w_2\) with \(f_1, f_2\) CR. Then

\[TTXu = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}.\] (4.4)

In particular, \(TTXu\) is CR.

The non-tangential derivatives appearing in (4.4) may be interpreted using the Hans Lewy local CR extension result previously mentioned in the proof of Theorem 2 or else by rewriting them in terms of tangential derivatives (as in the last step of the proof below).

Proof. We have

\[
TTXu = TXTu + T[T, X]u \\
= TX (f_1 w_1 + f_2 w_2) - iTu \quad \text{(Lemma 27)} \\
= T (f_1 z_2 - f_2 z_1) - iRTu - i[T, R]u \quad \text{(3.5b), (4.1)} \\
= T (f_1 z_2 - f_2 z_1) - iR (f_1 w_1 + f_2 w_2) + 2Tu \quad \text{(4.3b)} \\
= (T f_1) z_2 - f_1 w_1 - (T f_2) z_1 - f_2 w_2 \\
- i (R f_1) w_1 - f_2 w_2 - i (R f_2) w_2 - f_2 w_2 \\
+ 2 (f_1 w_1 + f_2 w_2) \quad \text{(4.1), (4.2e)} \\
= (z_2 T - i w_1 R) f_2 - (z_1 T + i w_2 R) f_2 \\
= (z_2 Y - i w_1 R) f_2 - (z_1 Y + i w_2 R) f_2 \\
= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}. \quad \square
\]

Lemma 31. The following hold.

(4.5a) The operator \(XT\) maps CR functions to CR functions.

(4.5b) The operator \(XY\) maps CR functions to CR functions.

(4.5c) The operator \(TX\) maps dual-CR functions to dual-CR functions.
(4.5d) The operator \(XY\) maps conjugate-CR functions to conjugate-CR functions.

Proof. To prove (4.5a) and (4.5b) note that for \(u\) CR we have \(XTu = XYu = -z_1 \frac{\partial u}{\partial z_1} - z_2 \frac{\partial u}{\partial z_2}\) which is also CR. The other proofs are similar. \(\square\)

Proof of (3.5e). To get the required lower bound on the null spaces, it will suffice to show that \(XX^T\) and \(TTX\) annihilate CR functions and dual-CR functions. This follows from (3.5b) and (3.5c) along with (4.5a) and (4.5c).

For the other direction, if \(XX^Tu = 0 = TTXu\) then from Lemma 30 we have a closed 1-form \(\omega \overset{\text{def}}{=} f_2 dz_1 - f_1 dz_2\) on \(S\) where \(f_1\) and \(f_2\) are CR functions satisfying \(Tu = f_1 w_1 + f_2 w_2\). Since \(S\) is simply-connected we may write \(\omega = df\) with \(f\) CR. Then from (3.5a) we have

\[
Tf = Yf = w_2 f_2 + w_1 f_1 = Tu.
\]

Thus \(u\) is the sum of the CR function \(f\) and the dual-CR function \(u - f\). \(\square\)

To set up the proof of the global result (3.5d) we introduce the form

(4.6) \[\nu \overset{\text{def}}{=} (z_2 dz_1 - z_1 dz_2) \wedge dw_1 \wedge dw_2\]

and the \(\mathbb{C}\)-bilinear pairing

(4.7) \[\langle\langle \mu, \eta \rangle\rangle \overset{\text{def}}{=} \int_S \mu \eta \cdot \nu\]

between functions on \(S\) (but see Technical Remark 35 below).

Lemma 32. \[\langle\langle T\gamma, \eta \rangle\rangle = -\langle\gamma, T\eta \rangle\].

Proof.

\[
\begin{align*}
\langle\langle T\gamma, \eta \rangle\rangle + \langle\gamma, T\eta \rangle &= \int_S T(\gamma \eta) \cdot \nu \\
&= \int_S \iota_T d(\gamma \eta) \cdot \nu \\
&= \int_S d(\gamma \eta) \cdot \iota_T \nu \\
&= \int_S d(\gamma \eta \cdot \iota_T \nu) - \int_S \gamma \eta \cdot d(\iota_T \nu) \\
&= 0 - \int_S \gamma \eta \cdot d(\iota_T ((z_2 dz_1 - z_1 dz_2) \wedge dw_1 \wedge dw_2)) \\
&= -\int_S \gamma \eta \cdot d((z_2 \cdot Tz_1 - z_1 \cdot Tz_2) \cdot dw_1 \wedge dw_2)
\end{align*}
\]
\begin{align*}
&+ \int_{S} \gamma \eta \cdot d((z_2 \, dz_1 - z_1 \, dz_2) \cdot Tw_1 \wedge dw_2) \\
&- \int_{S} \gamma \eta \cdot d((z_2 \, dz_1 - z_1 \, dz_2) \wedge dw_1 \cdot Tw_2) \\
&= - \int_{S} \gamma \eta \cdot d((z_2 w_2 + z_1 w_1) \, dw_1 \wedge dw_2) + 0 - 0 \\
&= - \int_{S} \gamma \eta \cdot dw_1 \wedge dw_2 \\
&= 0.
\end{align*}

Here we have quoted
- the definition (4.7) of the pairing \langle \langle \cdot, \cdot \rangle \rangle;
- the Leibniz rule $\iota_T(\phi_1 \wedge \phi_2) = (\iota_T \phi_1) \wedge \phi_2 + (-1)^{\deg \phi_1} \phi_1 \wedge (\iota_T \phi_2)$ for the interior product $\iota_T$;
- the fact that $S$ is integral for 4-forms;
- Stokes’ theorem;
- the rules (4.1);
- the relation (3.2a).

□

**Theorem 33.** A function $\mu$ on a compact strongly $\mathbb{C}$-convex hypersurface $S$ is CR if and only if $\langle \langle \mu, \eta \rangle \rangle = 0$ for all (smooth) dual-CR $\eta$ on $S$.

**Proof.** [Bar (4.3d) from Theorem 3]. (Note also definition enclosing [Bar (4.2)].) □

**Proof of (3.5d).** Assume that $XXTu = 0$. Noting that $S$ is simply-connected, from (3.5e) it suffices to prove that $TTXu = 0$. From Lemma 30 we know that $TTXu$ is CR. By Theorem 33 it will suffice to show that
\[ \langle \langle TTXu, \eta \rangle \rangle = 0 \]
for dual-CR $\eta$. But from Lemma 32 we have
\[ \langle \langle TTXu, \eta \rangle \rangle = -\langle \langle TXu, T\eta \rangle \rangle \]
\[ = 0 \]
as required. □

**Remark 34.** From symmetry of formulas in Lemma 8 and 14 we have that $X_{S^*} = \mathcal{D}_* T_S, T_{S^*} = \mathcal{D}_* X_S$ and $S^{**} = S$. These facts serve to explain why the formulas throughout this section appear in dual pairs.

**Technical Remark 35.** In [Bar] the pairing (4.7) applies not to functions $\mu, \nu$ but rather to forms $\mu(z) \, (dz_1 \wedge dz_2)^{2/3}, \mu(w) \, (dw_1 \wedge dw_2)^{2/3}$; the additional notation is important in [Bar] for keeping track of invariance properties under projective transformation but is not needed here.
Note also that (4.7) coincides (up to a constant) with the pairing (3.1.8) in [APS] with $s = w_1 dz_1 + w_2 dz_2$.

5. Proof of Theorem 6

Lemma 36. Suppose that $XXY u = 0$ so that by Lemma 27 we may write $Y u = f_1 w_1 + f_2 w_2$ with $f_1, f_2$ CR. Then

\begin{equation}
XXY u = \alpha \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right).
\end{equation}

In particular, $\alpha^{-1} XXY u$ is CR.

Proof. We have

\begin{align*}
XXY u &= XX X u + X[X, Y] u \\
&= XX (\alpha (f_1 w_1 + f_2 w_2)) + X (-i R - (Y \alpha) Y) u \\
&= X (\alpha Y (f_1 w_1 + f_2 w_2)) - i X R u \\
&= X (f_1 z_2 - f_2 z_1) - i R X u - i [X, R] u \\
&= X (f_1 z_2 - f_2 z_1) - i R (\alpha (f_1 w_1 + f_2 w_2)) + 2 X u \\
&= \left( X f_1 \right) \cdot z_2 - f_1 \cdot \alpha w_1 - \left( X f_2 \right) \cdot z_1 - f_2 \cdot \alpha w_2 \\
&\quad - i \alpha ((R f_1) \cdot w_1 - f_1 \cdot (i w_1) + (R f_2) \cdot w_2 - f_2 \cdot (i w_2)) \\
&\quad + 2 \alpha (f_1 w_1 + f_2 w_2) \\
&= \left( X f_1 \right) \cdot z_2 - \left( X f_2 \right) \cdot z_1 - i \alpha ((R f_1) \cdot w_1 + (R f_2) \cdot w_2) \\
&= \alpha (z_2 Y - i w_1 R) f_1 - (z_1 Y + iw_2 R) f_2 \\
&= \alpha \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right).
\end{align*}

□

Proof of (2.4d). To get the required lower bound on the null spaces, it will suffice to show that $XXY$ and $XXY$ annihilate CR functions and conjugate-CR functions. This follows from (2.4c) along with (4.5a) and (4.5c).

For the other direction, if $XXY u = 0 = XXY u$ then from Lemma 30 we have a closed 1-form $\tilde{\omega} \overset{\text{def}}{=} f_2 dz_1 - f_1 dz_2$ on the open subset of $S$ where $f_1$ and $f_2$ are CR functions satisfying $Y u = f_1 w_1 + f_2 w_2$. Restricting our attention to a simply-connected subset, we may write $\omega = df$ with $f$ CR. Then we have

\begin{align*}
Y f &= w_2 f_2 + w_1 f_1 \\
&= Y u.
\end{align*}

Thus $u$ is the sum of the CR function $f$ and the conjugate-CR function $u - f$.

The general case follows by localization. □
Lemma 37. \( \text{div} \, Y \stackrel{\text{def}}{=} \frac{\partial w_2}{\partial z_1} - \frac{\partial w_1}{\partial z_2} \) and \( \text{div} \, \bar{Y} \stackrel{\text{def}}{=} \frac{\partial \bar{w}_2}{\partial \bar{z}_1} - \frac{\partial \bar{w}_1}{\partial \bar{z}_2} \) vanish on \( S \).

Proof. Since \( S \) is circular, any defining function \( \rho \) for \( S \) will satisfy \( \text{Im} \left( z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2} \right) = -\frac{R \rho}{2} = 0 \). Adjusting our choice of defining function we may arrange that \( z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2} \equiv 1 \) in some neighborhood of \( S \). Then from the proof of Lemma 37 we have \( \frac{\partial w_1}{\partial z_1} - \frac{\partial w_2}{\partial z_2} = \frac{\partial^2 \rho}{\partial z_1 \partial z_2} = 0 \).

The remaining statement follows by conjugation.

\[ \square \]

Lemma 38. \[ \int_S (X \gamma) \eta \frac{dS}{\alpha} = - \int_S \gamma (X \eta) \frac{dS}{\alpha} \]

Proof.

\[ \int_S (X \gamma) \eta \frac{dS}{\alpha} = \int_S (Y \gamma) \eta dS \quad (3.3b) \]

\[ = - \int_S \gamma (Y \eta) dS \quad \text{(Lemma 37)} \]

\[ = - \int_S \gamma (X \eta) \frac{dS}{\alpha} \quad (3.3b) \]

(The integration by parts above may be justified by applying the divergence theorem on a tubular neighborhood of \( S \) and passing to a limit.)

\[ \square \]

Proof of (2.4c). Assume that \( X X Y u = 0 \). Noting that \( S \) is simply-connected, from (2.4d) it suffices to prove that \( X X Y u = 0 \). From Lemma 30 we know that \( \alpha^{-1}X X Y u \) is CR. The desired conclusion now follows from

\[ \int_S \left| X X Y u \right|^2 \frac{dS}{\alpha^2} = \int_S \alpha^{-1}X X Y u \cdot X X Y \bar{u} \frac{dS}{\alpha} \]

\[ = - \int_S X (\alpha^{-1}X X Y u) \cdot X Y \bar{u} \frac{dS}{\alpha} \quad \text{(Lemma 38)} \]

\[ = - \int_S 0 \cdot X Y \bar{u} \frac{dS}{\alpha} \]

\[ = 0. \]

\[ \square \]

6. Further comments

6.1. Remarks on uniqueness.
Theorem 39. Suppose that in the setting of Theorem 23 we have vector fields $\tilde{X}, \tilde{T}$ satisfying (suitably-modified) (3.5b) and (3.5c) with the property that $\tilde{X}\tilde{X}\tilde{T}$ annihilates CR functions and dual-CR functions. Then there are CR functions $f_1, f_2$ and $f_3$ so that $f_1w_1 + f_2w_2$ and $f_3$ are non-vanishing and

\begin{align*}
\tilde{X} &= f_3(f_1w_1 + f_2w_2)^2X \\
\tilde{T} &= \frac{1}{f_1w_1 + f_2w_2}T.
\end{align*}

Proof. From (3.5b) and (3.5c) we have $\tilde{X} = \gamma X$, $\tilde{T} = \eta T$ with non-vanishing scalar functions $\gamma$ and $\eta$.

By routine computation we have

$$\tilde{X}\tilde{X}\tilde{T} = \gamma^2\eta XXT + \gamma \left( (2\gamma(X\eta) + \eta(X\gamma)) XT + (X(\gamma(X\eta))T) \right).$$

The operator $\left( (2\gamma(X\eta) + \eta(X\gamma)) XT + (X(\gamma(X\eta))T) \right)f = \left( i(2\gamma(X\eta) + \eta(X\gamma)) R + (X(\gamma(X\eta))T) \right)f$

Since $R$ and $T$ are $\mathbb{C}$-linearly independent and $f$ is arbitrary it follows that we must have

$$X(\gamma\eta^2) = 2\gamma(X\eta) + \eta(X\gamma) = 0$$

$$X(\gamma\eta) = 0.$$

We set $f_3 = \gamma\eta^2$ which is CR and non-vanishing. Then the second equation above yields

$$-f_3 \cdot XX(\eta^{-1}) = X \left( f_3 \eta^{-2}(X\eta) \right)$$

$$= X(\gamma(X\eta))$$

$$= 0$$

and hence $XX(\eta^{-1}) = 0$. From Lemma 29 we have $\eta = \frac{1}{f_1w_1 + f_2w_2}$ with $f_1$ and $f_2$ CR. The result now follows. \qed

Theorem 40. Suppose that in the setting of Theorem 6 we have vector fields $\tilde{X}, \tilde{T}$ satisfying (suitably-modified) (2.4a) and (2.4b) with the property that $\tilde{X}\tilde{X}\tilde{Y}$ annihilates CR functions and conjugate-CR functions. Then there are CR functions $f_1, f_2$ and $f_3$ so that $f_1w_1 + f_2w_2$ and $f_3$ are non-vanishing and

\begin{align*}
\tilde{X} &= f_3(f_1w_1 + f_2w_2)^2X \\
\tilde{Y} &= \frac{1}{f_1w_1 + f_2w_2}Y.
\end{align*}

The proof is similar to that of Theorem 39 using (4.2h) in place of Lemma 29.
6.2. Nirenberg-type result.

Proposition 41. Given a point $p$ on a strongly pseudoconvex hypersurface $S \subset \mathbb{C}^2$, any 2-jet at $p$ of a $\mathbb{C}$-valued function on $S$ is the 2-jet of the restriction to $S$ of a pluriharmonic function on $\mathbb{C}^2$.

Proof. After performing a standard local biholomorphic change of coordinates we may reduce to the case where $p = 0$ and $S$ is described near 0 by an equation of the form

$$y_2 = z_1 \overline{z}_1 + O(\| (z_1, x_2) \|^3).$$

The projection $(z_1, x_2 + iy_2) \mapsto (z_1, x_2)$ induces a bijection between 2-jets at 0 along $S$ and 2-jets at 0 along $\mathbb{C} \times \mathbb{R}$. It suffices now to note that the 2-jet

$$A + Bz_1 + C \overline{z}_1 + Dz_2 + Ez_1^2 + Fz_2^2 + Gz_1 \overline{z}_1 + Hz_1 x_2 + I \overline{z}_1 x_2 + Jx_2^2$$

is induced by the pluriharmonic polynomial

$$A + Bz_1 + C \overline{z}_1 + \frac{D - iG}{2} z_2 + \frac{D + iG}{2} \overline{z}_2 + Ez_1^2 + Fz_2^2 + Hz_1 z_2 + I \overline{z}_1 z_2 + Jz_2^2.$$  

\[\square\]

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