LEFT-ORDERABLE FUNDAMENTAL GROUP AND DEHN SURGERY ON TWIST KNOTS

RYOTO HAKAMATA AND MASAKAZU TERAGAITO

Abstract. For any hyperbolic twist knot in the 3-sphere, we show that the resulting manifold by $r$-surgery on the knot has left-orderable fundamental group if the slope $r$ satisfies the inequality $0 \leq r \leq 4$.

1. Introduction

A non-trivial group $G$ is said to be left-orderable if it admits a strict total ordering which is invariant under left-multiplication. Thus, if $g < h$ then $fg < fh$ for any $f, g, h \in G$. Many groups, which arise in topology such as orientable surface groups, knot groups, braid groups, are known to be left-orderable. In 3-manifold topology, it is natural to ask which 3-manifolds have left-orderable fundamental groups. Toward this direction, there is very recent evidence of connections between Heegaard-Floer homology and left-orderability of fundamental groups. More precisely, Boyer, Gordon and Watson [3] conjecture that an irreducible rational homology 3-sphere is an $L$-space if and only if its fundamental group is not left-orderable. An $L$-space is a rational homology 3-sphere $Y$ whose Heegaard–Floer homology group $\hat{HF}(Y)$ has rank equal to $|H_1(Y; \mathbb{Z})|$ ([18]). They confirmed the conjecture for several classes of 3-manifolds including Seifert fibered manifolds, Sol-manifolds. Also, they showed that if $-4 < r < 4$ then $r$-surgery on the figure-eight knot yields a 3-manifold whose fundamental group is left-orderable. Later, Clay, Lidman and Watson [6] added the same conclusion for $r = \pm 4$. Since the figure-eight knot cannot yield $L$-spaces by non-trivial Dehn surgery ([18]), these give supporting evidences of the conjecture.

The purpose of this paper is to push forward them to all hyperbolic twist knots. Any non-trivial twist knot except the trefoil is hyperbolic, and does not admit non-trivial Dehn surgery yielding $L$-spaces ([18]). Hence the following result gives a further supporting evidence of the conjecture mentioned above.

Theorem 1.1. Let $K$ be a hyperbolic twist knot in the 3-sphere $S^3$ as illustrated in Figure 1. If $0 \leq r \leq 4$, then $r$-surgery on $K$ yields a manifold whose fundamental group is left-orderable.

As seen in Figure 1, the clasp is left-handed. The range of the slope in the conclusion of Theorem 1.1 depends on this convention. If the clasp is right-handed, the range would be $[-4, 0]$.

2010 Mathematics Subject Classification. Primary 57M25; Secondary 06F15.

Key words and phrases. left-ordering, Dehn surgery, twist knot.

The second author is partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), 22540088.
For the right-handed trefoil, if \( r \geq 1 \), then \( r \)-surgery yields an \( L \)-space ([13]), and its fundamental group is not left-orderable by [3]. Otherwise, \( r \)-surgery yields a manifold with left-orderable fundamental group ([8]).

In [11], we showed the same conclusion as Theorem 1.1 for the knot \( 5_{2} \), which is the \((-2)\)-twist knot. We will greatly generalize the argument in [11] to handle all hyperbolic twist knots. Our argument works even for the figure-eight knot, and it is much simpler than that in [3], which involves character varieties. We remark that the fact that the figure-eight knot is amphicheiral makes possible to widen the range of slope to \(-4 \leq r \leq 4\).

2. Knot group and representations

Let \( K_{n} \) be the \( n \)-twist knot with diagram illustrated in Figure 1. Our convention is that the twists are right-handed if \( n > 0 \) and left-handed if \( n < 0 \). Thus \( K_{1} \) is the figure-eight knot and \( K_{-1} \) is the right-handed trefoil. If \( n \neq 0, -1 \), then \( K_{n} \) is hyperbolic, and if \( |n| > 1 \), then \( K_{n} \) is not fibered. Throughout the paper, we assume that \( n \neq 0, -1 \). Thus non-trivial Dehn surgery on \( K_{n} \) never yields an \( L \)-space ([18]).

It is easy to see from the diagram that \( K_{n} \) bounds a once-punctured Klein bottle whose boundary slope is 4. (For example, consider a checkerboard coloring of the diagram. Then the bounded surface gives such a once-punctured Klein bottle.) Thus, 4-surgery on \( K_{n} \) yields a non-hyperbolic manifold, which is a toroidal manifold. In [22], we showed that the resulting toroidal manifold by 4-surgery on \( K_{n} \) admits a left-ordering on its fundamental group. Also, 1, 2 and 3-surgeries on \( K_{n} \) are known to yield small Seifert fibered manifolds ([5]), and the resulting manifolds have left-orderable fundamental groups by [3]. However, we do not need the latter fact.

Let \( G = \pi_{1}(S^{3} - K_{n}) \) be the knot group of \( K_{n} \).

**Lemma 2.1.** The knot group \( G \) admits a presentation

\[
G = \langle x, y \mid w^{n}x = yw^{n}, \rangle
\]

where \( x \) and \( y \) are meridians and \( w = xy^{-1}x^{-1}y \).

Furthermore, the longitude \( \mathcal{L} \) is given as \( \mathcal{L} = w^{n}w^{n} \), where \( w_{*} = yx^{-1}y^{-1}x \) is obtained from \( w \) by reversing the order of letters.

This is slightly different from that in [14, Proposition 1], but they are isomorphic.
Proof. We use the surgery diagram of $K_n$ as illustrated in Figure 2, where 1-surgery and $-1/n$-surgery are performed along the second and third components, as indicated by numbers, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{surgery_diagram}
\caption{A surgery diagram of $K_n$}
\end{figure}

Let $\mu_i$ and $\lambda_i$ be the meridian and longitude of the $i$-th component. Then $y = \mu_3^{-1} x \mu_3$, $z = \mu_2^{-1} y \mu_2$, $\lambda_2 = x^{-1} y$ and $\lambda_3 = y z^{-1}$. By 1-surgery on the second component, the relation $\lambda_2 \mu_2 = 1$ arises. Similarly, $-1/n$-surgery on the third component implies $\lambda_3^3 \mu_3^{-1} = 1$.

Hence, $\mu_3^{-1} = \lambda_3^{-n} = (zy^{-1})^n = (x^{-1} y xy^{-1} y)^n$. Let $w = xy^{-1} x y$. Then $x^{-1} y x y^{-1} = x^{-1} y w y^{-1} x$, so $\mu_3^{-1} = x^{-1} y w^n y^{-1} x$.

By substituting this to the remaining relation $y = \mu_3^{-1} x \mu_3$, we obtain $w^n x = y w^n$.

Finally, the longitude $L$ is given by $\mu_3 \mu_2 \mu_3^{-1} \mu_2^{-1} = x^{-1} y w^{-n} y^{-1} x w^n$. Since $x^{-1} y w^{-n} y^{-1} x = (yx^{-1} x y^{-1} x)^n$, we have $L = w^n \ast w$, where $w = y x^{-1} y^{-1} x$.

Let $s > 0$ and $t > 1$ be real numbers. Let $\rho_s : G \rightarrow SL_2(\mathbb{R})$ be the representation defined by the correspondence

$$
\rho_s(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad \rho_s(y) = \begin{pmatrix} t^{-s-1} & 0 \\ \frac{s}{(\sqrt{t} - \frac{1}{\sqrt{t}})^2 - 1} & -s \end{pmatrix}.
$$

For $P = \begin{pmatrix} t^{-1} & 0 \\ \sqrt{t} & \frac{1}{\sqrt{t}} \end{pmatrix}$,

$$
P^{-1} \rho_s(x) P = \begin{pmatrix} \sqrt{t} & \frac{1}{\sqrt{t}} \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad P^{-1} \rho_s(y) P = \begin{pmatrix} \sqrt{t} & 0 \\ -s \sqrt{t} & \frac{1}{\sqrt{t}} \end{pmatrix}.
$$

Hence, (2.1) gives a (non-abelian) representation if $s$ and $t$ satisfy Riley’s equation $z_{1,1} + (1-t) z_{1,2} = 0$, where $z_{i,j}$ is the $(i,j)$-entry of the matrix $P^{-1} \rho_s(w^n) P$ (20).

See also [9]. Then $\phi_n(s, t) = z_{1,1} + (1-t) z_{1,2}$ is called the Riley polynomial of $K_n$.

Since $s$ and $t$ are limited to be positive real numbers in our setting, it is not obvious that there exist solutions for Riley’s equation $\phi_n(s, t) = 0$. However, this will be verified in Proposition 3.2. We temporarily assume that $s$ and $t$ are chosen so that $\phi_n(s, t) = 0$. 


From (2.1), we have

\[ W = \rho_s(w) = \left( 1 + s - st + \frac{s^2}{t-1} \right) \left( \frac{t-1+st}{\sqrt{t}} \frac{\sqrt{t-\frac{w^2}{t}}}{\sqrt{t-\frac{w^2}{t}}} \right). \]

Let

\[ \lambda_{\pm} = \frac{1}{2} \left\{ s^2 - \left( t + \frac{1}{t} - 2 \right) s + 2 \pm \sqrt{\left( s^2 - \left( t + \frac{1}{t} - 2 \right) s + 2 \right)^2 - 4} \right\} \in \mathbb{C}. \]

These are eigenvalues of \( W \), and so \( \lambda_+ + \lambda_- = \text{tr}(W) = s^2 - (t + 1/t - 2)s + 2 \) and \( \lambda_+\lambda_- = 1 \). In Proposition 2.2, we will see that \( s + 2 < t + 1/t < s + 2 + 4/s \). This implies

\[-2 < s^2 - \left( t + \frac{1}{t} - 2 \right) s + 2 < 2,\]

and so \( \lambda_{\pm} = e^{\pm i\theta} \) for some \( \theta \in (0, \pi) \). In particular, we remark that \( \lambda_+ \neq \lambda_- \).

**Proposition 2.2.** The Riley polynomial \( \phi_n(s, t) \) of \( K_n \) is given by

\[ \phi_n(s, t) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-} - \left( t + \frac{1}{t} - 1 - s \right) \frac{\lambda_+^{n} - \lambda_-^{n}}{\lambda_+ - \lambda_-}. \]

**Proof.** The Riley polynomial is explicitly calculated in [16 Proposition 3.1]. Our knot \( K_n \) corresponds to the mirror image of \( J(2, -n) \) in [16]. This gives the conclusion. \( \square \)

By using Lemma 4.4 it is not hard to check directly that \( \rho_s(w^n x) = \rho_s(y u^n) \) holds if and only if \( s \) and \( t \) make the polynomial (2.3) equal to zero.

Set \( T = t + 1/t \) and \( \tau_m = (\lambda_+^{m} - \lambda_-^{m})/(\lambda_+ - \lambda_-) \) for convenience. Then the Riley polynomial of \( K_n \) is expressed simply as \( \phi_n(s, T) = \tau_{n+1} - (T - 1 - s)\tau_n \). Since \( \tau_m \) is symmetric in \( \lambda_+ \) and \( \lambda_- \), it can be expressed as a polynomial of \( \lambda_+ + \lambda_- \), which is \( s^2 - (T - 2)s + 2 \). Also, it is easy to see that a recursive relation

\[ \tau_{m+1} - (\lambda_+ + \lambda_-)\tau_m + \tau_{m-1} = 0 \]

and \( \tau_{-m} = -\tau_m \) hold for any integer \( m \).

**Example 2.3.** Clearly, \( \tau_0 = 0 \) and \( \tau_1 = 1 \). Thus we have \( \tau_2 = s^2 - (T - 2)s + 2 \) and \( \tau_3 = (s^2 - (T - 2)s + 2)^2 - 1 \). From these, the figure-eight knot has the Riley polynomial

\[ \phi_1(s, T) = \tau_2 - (T - 1 - s)\tau_1 = -(s + 1)T + s^2 + 3s + 3. \]

Similarly, the 2-twist knot, \( 6_2 \) in the knot table, has the Riley polynomial

\[ \phi_2(s, T) = \tau_3 - (T - 1 - s)\tau_2 = (s^2 + s)T^2 - (2s^3 + 6s^2 + 7s + 2)T + s^4 + 5s^3 + 11s^2 + 12s + 5. \]

From the recursive relation (2.5), we see that the Riley polynomial \( \phi_n(s, T) \) has degree \( |n| \) in \( T \). Thus we cannot solve the equation \( \phi_n(s, T) = 0 \) for \( T \), in general.
3. Riley polynomials

In this section, we show that Riley’s equation \( \phi_n(s, T) = 0 \) has a pair of solutions \((s, T)\) such as \( s + 2 < T < s + 2 + 4/s \) for any \( s > 0 \). In fact, we can choose \( T \) satisfying \( s + 2 + c/s < T < s + 2 + 4/s \) where \( c \) is a constant depending only on \( n \), unless \( n = 1 \).

Let \( m \) be a positive integer. For \( z = e^{i\theta} \) \((0 \leq \theta \leq \pi)\), set \( T_m(z) = z^{m-1} + z^{m-3} + \cdots + z^3 + z - m \). If \( z \neq \pm 1 \), then \( T_m(z) = (z^m - z^{-m})/(z - z^{-1}) \). Define \( T_0 = 0 \) and \( T_{-m}(z) = -T_m(z) \). Since \( T_m(z) \) is symmetric for \( z \) and \( z^{-1} \), it can be expanded as a polynomial of \( z + z^{-1} \). Furthermore, the recursive relation

\[
T_{m+1}(z) - (z + z^{-1})T_m(z) + T_{m-1}(z) = 0
\]

holds. Also, \( T_m(1) = m \) and \( T_m(-1) = (-1)^{m-1}m \) for any integer \( m \).

**Lemma 3.1.**

1. Let \( m \geq 1 \). Then, \( T_m(e^{\frac{-\pi}{2m+1}i}) = T_{m+1}(e^{\frac{\pi}{2m+1}i}) \), and this value is positive.

2. Let \( m \geq 2 \). Then, \( T_m(e^{\frac{3\pi}{2m+1}i}) = T_{m+1}(e^{\frac{-\pi}{2m+1}i}) \), and this value is negative.

**Proof.**

1. Let \( z = e^{\frac{-\pi}{2m+1}i} \). Then the fact that \( z^{2m+1} = -1 \) immediately implies \( T_m(z) = T_{m+1}(z) \). A direct calculation shows

\[
T_m(z) = \frac{\sin \frac{m\pi}{2m+1}}{\sin \frac{\pi}{2m+1}} > 0.
\]

2. Similarly, set \( z = e^{\frac{3\pi}{2m+1}i} \). Then \( z^{2m+1} = -1 \) holds again. Hence we have \( T_m(z) = T_{m+1}(z) \), and

\[
T_m(z) = \frac{\sin \frac{m\pi}{3m+1}}{\sin \frac{\pi}{3m+1}} < 0.
\]

\[\square\]

Now, fix an \( s > 0 \). We introduce a function \( \Phi_n : [s + 2, s + 2 + 4/s] \rightarrow \mathbb{R} \) by

\[
\Phi_n(T) = T_{n+1}(z) - (T - 1 - s)T_n(z),
\]

where

\[
z = \frac{1}{2} \left\{ s^2 - (T - 2)s + 2 + i\sqrt{4 - (s^2 - (T - 2)s + 2)^2} \right\}.
\]

Since \( s + 2 \leq T \leq s + 2 + 4/s \), we have \(-2 \leq s^2 - (T - 2)s + 2 \leq 2 \). Thus \( z = e^{i\theta} \) for \( \theta \in [0, \pi] \). We will seek a solution \( T \) for \( \Phi_n(T) = 0 \) satisfying \( s + 2 < T < s + 2 + 4/s \), because it gives a pair of solutions \((s, T)\) for Riley’s equation \( \phi_n(s, T) = 0 \).

**Proposition 3.2.**

Riley’s equation \( \phi_n(s, T) = 0 \) has a real solution \( T \) satisfying \( s + 2 < T < s + 2 + 4/s \) for any \( s > 0 \). Moreover, if \( n \neq 1 \), then \( T \) can be chosen so that \( s + 2 + c/s < T < s + 2 + 4/s \), where \( c \) is a constant depending only on \( n \). In particular, \( \phi_n(s, t) = 0 \) has a solution \( t > 1 \) for any \( s > 0 \).

**Proof.**

Suppose \( n > 1 \). By Lemma 3.1,

\[
T_{n+1}(e^{\frac{-\pi}{2n+1}i}) = T_n(e^{\frac{3\pi}{2n+1}i}), \quad T_{n+1}(e^{\frac{3\pi}{2n+1}i}) = T_n(e^{\frac{-\pi}{2n+1}i}).
\]

Let \( c = 2 - 2\cos \frac{\pi}{2n+1} \) and \( c' = 2 - 2\cos \frac{3\pi}{2n+1} \). Then \( c, c' \in (0, 4) \) and \( c < c' \).
Also,
\[ \Phi_n(s + 2 + \frac{c}{s}) = T_{n+1}(e^{\frac{s}{2} \pi i}) - \left(1 + \frac{c}{s}\right) T_n(e^{\frac{s}{2} \pi i}) \]
\[ = \frac{-c}{s} \cdot T_n(e^{\frac{n}{2} \pi i}), \]
\[ \Phi_n(s + 2 + \frac{c'}{s}) = T_{n+1}(e^{\frac{3s}{2} \pi i}) - \left(1 + \frac{c'}{s}\right) T_n(e^{\frac{3s}{2} \pi i}) \]
\[ = \frac{-c'}{s} \cdot T_n(e^{\frac{3s}{2} \pi i}). \]

By Lemma 3.1, these values have distinct signs. We remark that \( \Phi_n(T) \) is a polynomial function of \( T \), so it is continuous. Thus if \( n > 1 \), we have a solution \( T \) for \( \Phi_n(T) = 0 \), satisfying \( s + 2 + c/s < T < s + 2 + c'/s \), from the Intermediate Value Theorem. Since \( T > 2, t + 1/t = T \) has a real solution for \( t \). If we choose \( t = (T + \sqrt{T^2 - 4})/2 \), then \( t > 1 \).

When \( n = 1 \), the Riley polynomial is \( \phi_1(s, T) = -(s + 1)T + s^2 + 3s + 3 \) as shown in Example 2.3. Hence the equation \( \phi_1(s, T) = 0 \) has the unique solution \( T = (s^2 + 3s + 3)/(s + 1) = s + 2 + 1/(s + 1) \) for a given \( s \). This satisfies \( s + 2 < T < s + 2 + 1/s \).

Suppose \( n < 0 \). (Recall that we assume \( n \neq -1 \).) Set \( l = |n| \geq 2 \). If \( l > 2 \), then set \( d = 2 - 2 \cos \frac{\pi}{2l - 1} \) and \( d' = 2 - 2 \cos \frac{3\pi}{2l - 1} \). Then \( d, d' \in (0, 4) \) and \( d < d' \). As before,
\[ T_{l-1}(e^{\frac{s}{2} \pi i}) = T_l(e^{\frac{3s}{2} \pi i}), \quad T_{l-1}(e^{\frac{3s}{2} \pi i}) = T_l(e^{\frac{s}{2} \pi i}). \]
by Lemma 3.4. Thus
\[ \Phi_n(s + 2 + \frac{d}{s}) = - \left(T_{l-1}(e^{\frac{s}{2} \pi i}) - \left(1 + \frac{d}{s}\right) T_l(e^{\frac{s}{2} \pi i})\right) \]
\[ = \frac{d}{s} \cdot T_l(e^{\frac{s}{2} \pi i}), \]
\[ \Phi_n(s + 2 + \frac{d'}{s}) = \frac{d'}{s} \cdot T_l(e^{\frac{s}{2} \pi i}). \]

Since these values have distinct signs, we have a solution \( T \) with \( s + 2 + d/s < T < s + 2 + d'/s \), if \( l > 2 \), as before.

When \( l = 2 \), we have
\[ \Phi_{-2}(s + 2 + \frac{1}{s}) = \frac{1}{s} > 0, \quad \Phi_{-2}(s + 2 + \frac{2}{s}) = -1 < 0. \]
Hence there exists a solution \( T \) with \( s + 2 + 1/s < T < s + 2 + 2/s \). □

4. LONGITUDES

Recall that \( \rho_s : G \to SL_2(\mathbb{R}) \) is the representation defined by (2.1). Two real parameters \( s \) and \( t \) are chosen so that \( \phi_n(s, t) = 0 \). In this section, we examine the image of the longitude \( \mathcal{L} \) of \( G \) under \( \rho_s \). Throughout the section, let
\[ \rho_s(w) = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix}, \quad \rho_s(w^n) = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \]
and \( \sigma = \frac{s(t\sqrt{s^2 - 1} - 1)^s}{(\sqrt{s^2 - 1})^s - s} \).
Lemma 4.1. For $w_n^\sigma$, we have $\rho_s(w_n^\sigma) = \begin{pmatrix} u_{1,1} & \frac{w_{2,1}}{\sigma} \\ u_{1,2} & u_{2,2} \end{pmatrix}$.

Proof. By a direct calculation,

$$\rho_s(xy^{-1}) = \begin{pmatrix} \frac{t-1+st}{t} & \frac{s\sqrt{t}}{t} \\ -s\sqrt{t} & t-1+st \end{pmatrix}, \quad \rho_s(y^{-1}x) = \begin{pmatrix} \frac{t-1+st}{t} & \frac{s\sqrt{t}}{t} \\ -s\sqrt{t} & t-1+st \end{pmatrix}.$$ 

Thus we see that the $(1, 2)$-entry of $\rho_s(y^{-1}x)$ is the $(2, 1)$-entry of $\rho_s(xy^{-1})$ divided by $\sigma$, the $(2, 1)$-entry of $\rho_s(y^{-1}x)$ is the $(1, 2)$-entry of $\rho_s(xy^{-1})$ multiplied by $\sigma$, and the others of $\rho_s(y^{-1}x)$ coincide with those of $\rho_s(xy^{-1})$. The same relation between entries holds for $\rho_s(x^{-1}y)$ and $\rho_s(yx^{-1})$.

In general, such a relation is preserved under the matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix},$$

$$\begin{pmatrix} p & \frac{r}{s} \\ q\sigma & \frac{r}{s} \end{pmatrix} \begin{pmatrix} a & \frac{c}{d} \\ b\sigma & \frac{c}{d} \end{pmatrix} = \begin{pmatrix} ap + br & \frac{brs + dr\sigma}{s} \\ (aq + bs)\sigma & cq + ds \end{pmatrix}.$$ 

Thus we can confirm that the same relation holds for $\rho_s(w^n)$ and $\rho_s(w_n^\sigma)$. \hfill \Box

Proposition 4.2. For the longitude $L$ of $G$, the matrix $\rho_s(L)$ is diagonal, and the $(1, 1)$-entry of $\rho_s(L)$ is a positive real number.

Proof. The first assertion follows from the facts that for a meridian $x$, $\rho_s(x)$ is diagonal but $\rho_s(x) \neq \pm I$ and that $x$ and $L$ commute.

Since $L = w_n^\sigma w^n$ by Lemma 2.1, Lemma 4.1 implies that

$$\rho_s(L) = \rho_s(w_n^\sigma)\rho_s(w^n) = \begin{pmatrix} u_{1,1} & \frac{w_{2,1}}{\sigma} \\ u_{1,2} & u_{2,2} \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} = \begin{pmatrix} u_{1,1}^2 + \frac{w_{2,1}^2}{\sigma} & u_{1,1}u_{2,1} + \frac{w_{2,1}w_{2,2}}{\sigma} \\ u_{1,1}u_{2,1} + \frac{w_{2,1}w_{2,2}}{\sigma} & u_{2,1}^2 + \frac{w_{2,2}^2}{\sigma} \end{pmatrix}.$$ 

Since $\det \rho_s(w^n) = 1$, at least one of $u_{1,1}$ and $u_{2,1}$ is non-zero. Hence the $(1, 1)$-entry is $u_{1,1}^2 + \frac{w_{2,1}^2}{\sigma}$, which is positive, because $s > 0$ and $(\sqrt{t} - 1/\sqrt{t})^2 - s = T - s - 2 > 0$ from Proposition 3.2. \hfill \Box

Remark 4.3. Since $\rho_s(L)$ is diagonal, we also obtain an equation $u_{1,1}u_{1,2}\sigma + u_{2,1}u_{2,2} = 0$. This will be used in the proof of Lemma 4.5.

To diagonalize $W = \rho_s(w)$, let $Q = \begin{pmatrix} w_{1,2} & w_{1,2} \\ \lambda_+ - w_{1,1} & \lambda_- - w_{1,1} \end{pmatrix}$. From (2.2), $w_{1,2} = (t - 1 + st)s/(\sigma\sqrt{t})$. Since $s > 0$, $t > 1$, $\sigma > 0$, we have $w_{1,2} \neq 0$. Also, $\det Q = -w_{1,2}(\lambda_+ - \lambda_-)$. Then a direct calculation shows $Q^{-1}WQ = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$.

Lemma 4.4. The entries of $W^n$ are given as follows.

$$u_{1,1} = w_{1,1}\tau_n - \tau_{n-1}, \quad u_{1,2} = w_{1,2}\tau_n,$$

$$u_{2,1} = w_{2,1}\tau_n, \quad u_{2,2} = \tau_{n+1} - w_{1,1}\tau_n.$$
Proof. This easily follows from $W^n = Q \left( \begin{array}{cc} \lambda^n_+ & 0 \\ 0 & \lambda^n_\cdot \end{array} \right) Q^{-1}$.

For example,

$$w_{2,1} = \frac{1}{\det Q} \left( \lambda^n_+ (\lambda_+ - w_{1,1})(\lambda_- - w_{1,1}) + \lambda^n_- (\lambda_- - w_{1,1})(w_{1,1} - \lambda_+) \right)$$

$$= -\frac{\tau_n}{w_{1,2}} (\lambda_+ - w_{1,1})(\lambda_- - w_{1,1})$$

$$= -\frac{\tau_n}{w_{1,2}} (1 - \text{tr}(W)w_{1,1} + w_{1,1}^2).$$

Since $\text{tr}(W) = w_{1,1} + w_{2,2}$, we have $1 - \text{tr}(W)w_{1,1} + w_{1,1}^2 = 1 - w_{1,1}w_{2,2} = -w_{1,2}w_{2,1}$. Thus $u_{2,1} = -u_{2,1}^\tau_n$.

We omit the others. \(\square\)

**Lemma 4.5.** Let $B_s$ be the $(1,1)$-entry of the matrix $\rho_s(L)$. Then $B_s = -w_{2,1}/(w_{1,2}\sigma)$.

Proof. As noted in Remark 4.4, $u_{1,1}u_{1,2}\sigma + u_{2,1}u_{2,2} = 0$. Since $\det W^n = u_{1,1}u_{2,2} - u_{1,2}u_{2,1} = 1$, we have

$$u_{1,2}B_s = u_{1,1}^1u_{1,2} + \frac{u_{1,2}u_{2,1}^2}{\sigma}$$

$$= u_{1,1}^2u_{1,2} + \frac{u_{2,1}}{\sigma}(u_{1,1}u_{2,2} - 1)$$

$$= u_{1,1}^2u_{1,2} + \frac{u_{1,1}}{\sigma}(-u_{1,1}u_{1,2}\sigma) - \frac{u_{2,1}}{\sigma}$$

$$= -\frac{u_{2,1}}{\sigma}.$$  

By Lemma 4.4, $u_{1,2} = w_{1,2}\tau_n$. As remarked above Lemma 4.4, $w_{1,2} \neq 0$. If $u_{1,2} = 0$, then $\tau_n = 0$. But this implies $\tau_{n+1} = 0$, because $\phi_n(s,t) = \tau_{n+1} - (t + 1/t - 1 - s)\tau_n = 0$. From the recursive relation, this implies $\tau_m = 0$ for all $m$. But this is absurd, because $\tau_1 = 1$. Hence $u_{1,2} \neq 0$, so $B_s = -w_{2,1}/(u_{1,2}\sigma)$. From Lemma 4.4 again, $u_{1,2} = w_{1,2}\tau_n$ and $u_{2,1} = w_{2,1}\tau_n$. Thus $B_s = -w_{2,1}/(w_{1,2}\sigma)$. \(\square\)

5. LIMITS

Let $r = p/q$ be a rational number, and let $M_n(r)$ denote the resulting manifold by $r$-filling on the knot exterior $M_n$ of $K_n$. In other words, $M_n(r)$ is obtained by attaching a solid torus $V$ to $M_n$ along their boundaries so that the loop $x^pL^q$ bounds a meridian disk of $V$, where $x$ and $L$ are a meridian and longitude of $K_n$.

Our representation $\rho_s : G \to SL_2(\mathbb{R})$ induces a homomorphism $\pi_1(M_n(r)) \to SL_2(\mathbb{R})$ if and only if $\rho_s(x)^p\rho_s(L)^q = I$. Since both of $\rho_s(x)$ and $\rho_s(L)$ are diagonal (see (2.1) and Proposition 4.2), this is equivalent to the equation

$$A_s^pB_s^q = 1,$$

where $A_s$ and $B_s$ are the $(1,1)$-entries of $\rho_s(x)$ and $\rho_s(L)$, respectively. We remark that $A_s = \sqrt{7}$ ($> 1$) is a positive real number, so is $B_s$ by Proposition 4.2. Hence the equation (5.1) is furthermore equivalent to the equation

$$-\frac{\log B_s}{\log A_s} = \frac{p}{q}.$$
Let \( g : (0, \infty) \to \mathbb{R} \) be a function defined by
\[
g(s) = -\frac{\log B_s}{\log A_s}.
\]

We will examine the image of \( g \).

**Lemma 5.1.**

(1) If \(|n| > 1\), then \( \lim_{s \to +0} t = \infty \). If \( n = 1 \), then \( \lim_{s \to +0} t = \frac{(3 + \sqrt{5})}{2} \).

(2) \( \lim_{s \to \infty} t = \infty \).

(3) \( \lim_{s \to \infty} (t - s) = 2 \).

(4) \( \lim_{s \to \infty} \frac{t}{s} = 1 \).

**Proof.** (1) If \( n = 1 \), then the equation \( \phi_1(s, T) = 0 \) has the unique solution \( T = (s^2 + 3s + 3)/(s + 1) \) for a given \( s > 0 \), so \( \lim_{s \to +0} T = 3 \). Since \( t = (T + \sqrt{T^2 - 4})/2 \), we have \( \lim_{s \to +0} t = (3 + \sqrt{5})/2 \).

Assume \(|n| > 1\). From Proposition 3.2, we have \( s + 2 + c/s < T \), where \( c \) is a positive constant. Hence \( \lim_{s \to +0} T = \lim_{s \to +0} t = \infty \).

(2) As \( T > s + 2 \), \( \lim_{s \to \infty} T = \lim_{s \to \infty} t = \infty \).

(3) Since \( s + 2 < t + 1/t < s + 2 + 4/s \), (2) implies \( \lim_{s \to \infty} (t - s) = 2 \).

(4) From \( s + 2 < T < s + 2 + 4/s \) again, we have \( \lim_{s \to \infty} T/s = 1 \), which implies \( \lim_{s \to \infty} s/t = 1 \).

**Lemma 5.2.**

(1) \( \lim_{s \to +0} B_s = 1 \).

(2) \( \lim_{s \to \infty} B_s t^2 = 1 \).

**Proof.** (1) By Lemma 4.5,
\[
B_s = \frac{w_{2,1}}{w_{1,2}^\sigma} = \frac{t - s - 1}{-1 + (1 + s)t}.
\]

Lemma 5.1(1) implies \( \lim_{s \to +0} B_s = 1 \).

(2) We decompose \( B_s t^2 \) as
\[
B_s t^2 = (t - s - 1) \cdot \frac{t^2}{-1 + (1 + s)t}.
\]

From Lemma 5.1(3) and (4),
\[
\lim_{s \to \infty} (t - s - 1) = \lim_{s \to \infty} \frac{t^2}{-1 + (1 + s)t} = 1.
\]

Hence \( \lim_{s \to \infty} B_s t^2 = 1 \).

**Proposition 5.3.** The image of \( g \) contains an open interval \((0, 4)\).

**Proof.** By Lemma 5.2(1), \( \lim_{s \to +0} \log B_s = 0 \). Hence
\[
\lim_{s \to +0} g(s) = -\lim_{s \to +0} \frac{\log B_s}{\log A_s} = -\lim_{s \to +0} \frac{\log B_s}{\log \sqrt{t}} = 0.
\]

Also, we have \( \lim_{s \to \infty} (\log B_s + 2 \log t) = 0 \) by Lemma 5.2(2). Thus
\[
\lim_{s \to \infty} g(s) = -\lim_{s \to \infty} \frac{\log B_s}{\log A_s} = -\lim_{s \to \infty} \frac{2 \log B_s}{\log t} = 4.
\]

Hence the image of \( g \) contains an interval \((0, 4)\).
A computer experiment suggests that the image of \(g\) equals to \((0, 4)\), but we do not need this.

6. Universal covering group

We briefly review the description of the universal covering group of \(SL_2(\mathbb{R})\).

Let
\[
SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}
\]
be the special unitary group over \(\mathbb{C}\) of signature \((1, 1)\). It is well known that \(SU(1, 1)\) is conjugate to \(SL_2(\mathbb{R})\) in \(GL_2(\mathbb{C})\). The correspondence is given by \(\psi: SL_2(\mathbb{R}) \rightarrow SU(1, 1)\), sending \(A \mapsto JAJ^{-1}\), where
\[
J = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.
\]
Thus
\[
\psi:\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+d+(b-c)i & a-d-(b+c)i \\ -a+d+(b-c)i & a-d-(b+c)i \end{pmatrix}.
\]

There is a parametrization of \(SU(1, 1)\) by \((\gamma, \omega)\) where \(\gamma = \beta/\alpha\) and \(\omega = \text{arg} \alpha\) defined mod \(2\pi\) (see [1]). Thus \(SU(1, 1) = \{(\gamma, \omega) \mid |\gamma| < 1, -\pi \leq \omega < \pi\}\). Topologically, \(SU(1, 1)\) is an open solid torus \(\Delta \times \mathbb{S}^1\), where \(\Delta = \{\gamma \in \mathbb{C} \mid |\gamma| < 1\}\). The group operation is given by \((\gamma, \omega)(\gamma', \omega') = (\gamma'', \omega'')\), where
\[
(6.1) \quad \gamma'' = \frac{\gamma' + \gamma e^{-2i\omega'}}{1 + \gamma', e^{-2i\omega'}};
\]
\[
(6.2) \quad \omega'' = \omega + \omega' + \frac{1}{2i} \log \frac{1 + \gamma' e^{-2i\omega'}}{1 + \gamma', e^{2i\omega'}}.
\]
These equations come from the matrix operation. Here, the logarithm function is defined by its principal value and \(\omega''\) is defined by mod \(2\pi\). The identity element is \((0, 0)\), and the correspondence between \(\begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix}\) and \((\gamma, \omega)\) gives an isomorphism.

Now, the universal covering group \(\tilde{SL}_2(\mathbb{R})\) of \(SU(1, 1)\) can be described as
\[
\tilde{SL}_2(\mathbb{R}) = \{(\gamma, \omega) \mid |\gamma| < 1, -\infty < \omega < \infty\}.
\]
Thus \(\tilde{SL}_2(\mathbb{R})\) is homeomorphic to \(\Delta \times \mathbb{R}\). The group operation is given by \((6.1)\) and \((6.2)\) again, but \(\omega''\) is not mod \(2\pi\) anymore.

Let \(\chi: \tilde{SL}_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})\) be the covering projection. Then \(\ker \chi = \{(0, 2m\pi) \mid m \in \mathbb{Z}\}\).

**Lemma 6.1.** The subset \((-1, 1) \times \{0\}\) of \(\tilde{SL}_2(\mathbb{R})\) forms a subgroup.

**Proof.** From \((6.1)\) and \((6.2)\), it is straightforward to see that \((-1, 1) \times \{0\}\) is closed under the group operation. For \((\gamma, 0) \in (-1, 1) \times \{0\}\), its inverse is \((-\gamma, 0)\).

For the representation \(\rho_s: G \rightarrow SL_2(\mathbb{R})\) defined by \((2.1)\),
\[
(6.3) \quad \psi(\rho_s(x)) = \frac{1}{2\sqrt{t}} \begin{pmatrix} t+1 & t-1 \\ t-1 & t+1 \end{pmatrix} \in SU(1, 1).
\]
Thus \(\psi(\rho_s(x))\) corresponds to \((\gamma_x, 0)\), where \(\gamma_x = (t-1)/(t+1)\). Since \(t > 1\), \(\gamma_x \in (-1, 1)\).
Also, for the longitude $L$,
\[
\psi(\rho_s(L)) = \frac{1}{2} \left( \frac{B_s + \frac{1}{B_s}}{B_s + \frac{1}{B_s}} \right), \quad B_s > 0
\]
from Proposition 4.2. Thus $\psi(\rho_s(L))$ corresponds to $(\gamma_L, 0)$, where $\gamma_L = (B_s^2 - 1)/(B_s^2 + 1)$. Clearly, $\gamma_L \in (-1, 1)$.

7. Proof of Theorem 1.1

Since the knot exterior $M_n$ of $K_n$ satisfies $H^2(M_n; \mathbb{Z}) = 0$, any $\rho_s : G \to SL_2(\mathbb{R})$ lifts to a representation $\tilde{\rho} : G \to \widetilde{SL_2}(\mathbb{R})$ [10]. Moreover, any two lifts $\tilde{\rho}$ and $\tilde{\rho}'$ are related as follows:
\[
\tilde{\rho}'(g) = h(g)\tilde{\rho}(g),
\]
where $h : G \to ker\chi \subset \widetilde{SL_2}(\mathbb{R})$. Since $ker\chi = \{ (0, 2m\pi) \mid m \in \mathbb{Z} \}$ is isomorphic to $\mathbb{Z}$, the homomorphism $h$ factors through $H_1(M_n)$, so it is determined only by the value $h(x)$ of a meridian $x$ (see [15]).

The following result is the key in [3], which is originally claimed in [15], for the figure eight knot. Our proof most follows that of [3], but it is much simpler, because the values of $\psi(\rho_s(x))$ and $\psi(\rho_s(L))$ are calculated explicitly in Section 6.

**Lemma 7.1.** Let $\tilde{\rho} : G \to \widetilde{SL_2}(\mathbb{R})$ be a lift of $\rho_s$. Then replacing $\tilde{\rho}$ by a representation $\tilde{\rho}' = h \cdot \tilde{\rho}$ for some $h : G \to SL_2(\mathbb{R})$, we can suppose that $\tilde{\rho}(\pi_1(\partial M_n))$ is contained in the subgroup $(-1, 1) \times \{0\}$ of $\widetilde{SL_2}(\mathbb{R})$.

**Proof.** Since $\chi(\tilde{\rho}(L)) = (\gamma_L, 0)$, $\tilde{\rho}(L) = (\gamma_L, 2j\pi)$ for some $j$. On the other hand, $L$ is a commutator, because our knot is genus one. Therefore the inequality (5.5) of [23] implies $-3\pi/2 < 2j\pi < 3\pi/2$. Thus we have $\tilde{\rho}(L) = (\gamma_L, 0)$.

Similarly, $\tilde{\rho}(x) = (\gamma_x, 2l\pi)$ for some $l$. Let us choose $h : G \to \widetilde{SL_2}(\mathbb{R})$ so that $h(x) = (0, -2l\pi)$. Set $\tilde{\rho}' = h \cdot \tilde{\rho}$. Then a direct calculation shows that $\tilde{\rho}'(x) = (\gamma_x, 0)$ and $\tilde{\rho}'(L) = (\gamma_L, 0)$. Since $x$ and $L$ generate the peripheral subgroup $\pi_1(\partial M_n)$, the conclusion follows from these.

**Proof of Theorem 1.1** For $r = 0$, $M_n(0)$ is irreducible and has positive betti number. Hence $\pi_1(M_n(0))$ is left-orderable by [4, Corollary 3.4]. For $r = 4$, [6] and [22] confirmed the conclusion.

Let $r = p/q \in (0, 4)$. By Proposition 5.3 we can fix $s$ so that $g(s) = r$. Choose a lift $\tilde{\rho}$ of $\rho_s$ so that $\tilde{\rho}(\pi_1(\partial M_n)) \subset (-1, 1) \times \{0\}$. Then $\rho_s(x^pL^q) = I$, so $\chi(\tilde{\rho}(x^pL^q)) = I$. This means that $\tilde{\rho}(x^pL^q)$ lies in $ker\chi = \{ (0, 2m\pi) \mid m \in \mathbb{Z} \}$. Hence $\tilde{\rho}(x^pL^q) = (0, 0)$. Then $\tilde{\rho}$ can induce a homomorphism $\pi_1(M_n(r)) \to \widetilde{SL_2}(\mathbb{R})$ with non-abelian image. Recall that $\widetilde{SL_2}(\mathbb{R})$ is left-orderable [21]. Hence any (non-trivial) subgroup of $\widetilde{SL_2}(\mathbb{R})$ is left-orderable. Since $M_n(r)$ is irreducible [12], $\pi_1(M_n(r))$ is left-orderable by [4, Theorem 1.1]. This completes the proof.

**References**

1. V. Bargmann, **Irreducible unitary representations of the Lorentz group**, Ann. of Math. 48 (1947), 568–640.

2. G. Bergman, **Right orderable groups that are not locally indicable**, Pacific J. Math. 147 (1991), 243–248.
3. S. Boyer, C. McA. Gordon and L. Watson, *On L-spaces and left-orderable fundamental groups*, preprint, arXiv:1107.5016.
4. S. Boyer, D. Rolfsen and B. Wiest, *Orderable 3-manifold groups*, Ann. Inst. Fourier (Grenoble) 55 (2005), 243–288.
5. M. Brittenham and Y. Q. Wu, *The classification of exceptional Dehn surgeries on 2-bridge knots*, Comm. Anal. Geom. 9 (2001), 97–113.
6. A. Clay, T. Lidman and L. Watson, *Graph manifolds, left-orderability and amalgamation*, preprint, arXiv:1106.0486.
7. A. Clay and M. Teragaito, *Left-orderability and exceptional Dehn surgery on two-bridge knots*, to appear in the Proceedings of Geometry and Topology Down Under, Contemporary Mathematics Series.
8. A. Clay and L. Watson, *On cabled knots, Dehn surgery, and left-orderable fundamental groups*, Math. Res. Lett. 18 (2011), 1085–1095.
9. J. Dubois, Y. Huyê and Y. Yamaguchi, *Non-abelian Reidemeister torsion for twist knots*, J. Knot Theory Ramifications 18 (2009), 303–341.
10. E. Ghys, *Groups acting on the circle*, Enseign. Math. 47 (2001), 329–407.
11. R. Hakamata and M. Teragaito, *Left-orderable fundamental group and Dehn surgery on the knot 52*, preprint, arXiv:1208.2087.
12. A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. 79 (1985), 225–246.
13. M. Hedden, *On knot Floer homology and cabling. II*, Int. Math. Res. Not. IMRN 2009, 2248–2274.
14. J. Hoste and P. Shanahan, *A formula for the A-polynomial of twist knots*, J. Knot Theory Ramifications 13 (2004), 193–209.
15. V. T. Khoi, *A cut-and-paste method for computing the Seifert volumes*, Math. Ann. 326 (2003), 759–801.
16. T. Morifuji, *Twisted Alexander polynomials of twist knots for nonabelian representations*, Bull. Sci. Math. 132 (2008), 439–453.
17. Y. Ni, *Knot Floer homology detects fibred knots*, Invent. Math. 170 (2007), 577–608.
18. P. Ozsváth and Z. Szabó, *On knot Floer homology and lens space surgeries*, Topology 44 (2005), 1281–1300.
19. R. Riley, *Nonabelian representations of 2-bridge knot groups*, Quart. J. Math. Oxford Ser. (2) 35 (1984), 191–208.
20. D. Rolfsen, *Knots and links*, Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976.
21. H. Schubert, *Knoten mit zwei Brücken*, Math. Z. 65 (1956), 133–170.
22. M. Teragaito, *Left-orderability and exceptional Dehn surgery on twist knots*, to appear in Canad. Math. Bull.
23. J. Wood, *Bundles with totally disconnected structure group*, Comment. Math. Helv. 46 (1971), 257–273.

Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima, Japan 739-8524.

Department of Mathematics and Mathematics Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima, Japan 739-8524.

E-mail address: teragai@hiroshima-u.ac.jp