Integrability and Conformal Symmetry in the BCS model

Germán Sierra
Instituto de Física Teórica, CSIC-UAM, Spain

Abstract

The exactly solvable BCS Hamiltonian of superconductivity is considered from several viewpoints: Richardson’s ansatz, conformal field theory, integrable inhomogenous vertex models and Chern-Simons theory.

1 Introduction

The exact solution of a simplified version of the BCS model, obtained by Richardson in 1963, has received in the last few years growing attention due to its physical applications to Condensed Matter and Nuclear Physics, and its connection with integrable models, conformal field theory and Chern-Simons theory. In this article we briefly review the subject from a historical perspective, focusing on the relationships between various approaches.

2 The BCS model of Superconductivity (1957)

In 1957 Bardeen, Cooper and Schrieffer proposed a model to describe the superconducting properties of some metals at low temperatures [1]. In its simplest form the BCS Hamiltonian is given by

\[ H_{BCS} = \sum_{j,\sigma=\pm} \varepsilon_j \sigma c_{j\sigma}^\dagger c_{j\sigma} - gd \sum_{j,j'} c_{j+}^\dagger c_{j-}^\dagger c_{j'-} c_{j'+} \]  

(1)

where \( c_{j,\pm} \) (resp. \( c_{j,\pm}^\dagger \)) is an electron destruction (resp. creation) operator in the time-reversed states \(|j, \pm\rangle\) with energies \( \varepsilon_j \), \( d \) is the mean level spacing and \( g \) is the BCS dimensionless coupling constant. The sums in (1) run over a set of \( \Omega \) doubly degenerate energy
levels $\varepsilon_j (j = 1, \ldots, \Omega)$. One assumes that the energy levels are all distinct, i.e. $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$. The Hamiltonian (1) is a simplified version of the reduced BCS Hamiltonian where all couplings have been set equal to a single one, namely $g$, and it describes the pair creation and annihilation between electrons belonging to different energy levels.

In their historical paper BCS gave an ansatz in the Grand Canonical ensemble (g.c.) for the ground state of this Hamiltonian which reads

$$|BCS\rangle \propto \exp \left[ \sum_j \frac{v_j}{u_j} c_{j,+} c_{j,-} \right] |0\rangle$$  \hspace{1cm} (2)

where $u_j$ and $v_j$ are the variational parameters given by the formulae

$$u_j^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_j - \mu}{E_j} \right), \quad v_j^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_j - \mu}{E_j} \right)$$  \hspace{1cm} (3)

where $\mu$ is the chemical potential, $E_j = [(\varepsilon_j - \mu)^2 + \Delta^2]^{1/2}$ is the quasiparticle energy and $\Delta$ is the superconducting gap. The g.c. state (2) is asymptotically exact in the limit when the number of electrons goes to infinity.

3 The Projected BCS ansatz (50’s-60’s)

Soon after the BCS work, the pairing model was applied in Nuclear Physics, but it became clear that a canonical version of the BCS state should be more appropriate to describe nuclei with small number of nucleons [2]. This gave rise to the so called Projected BCS ansatz (PBCS) which is given by

$$|PBCS\rangle \propto \left( \sum_j \frac{v_j}{u_j} c_{j,+} c_{j,-} \right)^N |0\rangle$$  \hspace{1cm} (4)

which has exactly $N$ fermion pairs. Again, $u_j$ and $v_j$ are variational parameters but the equations fixing them are much more complicated than those of the g.c. case (3). Of course, in the limit when $N \to \infty$ the PBCS results agree with those obtained with the BCS state. The PBCS state was applied in the 90’s to study ultrasmall metallic grains. A peculiarity of the PBCS ansatz is that, having a fixed number of fermions, the superconducting (SC) order parameter vanishes identically, i.e. $\langle c_{j,+} c_{j,-} \rangle = 0$, which thus obscures the nature of the superconducting correlations. There are however alternative definitions of SC order parameters in the canonical ensemble, which converge to the g.c. one for large systems.
4 Anderson’s conjecture (1959)

In 1959 Anderson made the conjecture that superconductivity must disappear for metallic grains where the mean level spacing $d$, which is inversely proportional to the volume, is of the order of the SC gap in the bulk $\Delta$ $[3]$. A simple argument supporting this conjecture is that the ratio $\Delta/d$ measures the number of electronic levels involved in the formation of Cooper pairs, and thus when $\Delta/d \leq 1$ there are no energy levels correlated by the pairing interaction. Apart from some theoretical studies this conjecture remained largely unexplored until the fabrication of ultrasmall metallic grains.

5 Ultrasmall metallic grains (1996-97)

In the years 96-97 Ralph, Black and Tinkham (RBT), in a series of experiments, studied the superconducting properties of ultrasmall Aluminium grains at the nanoscale $[4]$. These grains have radius $\sim 4$-5 nm, mean level spacing $d \sim 0.45$ mev, Debye energy $\omega_D \sim 34$ mev and charging energy $E_C \sim 46$ mev. Since the bulk gap of Al is $\Delta \sim 0.38$ mev one meets the Anderson’s condition for the possible non existence of superconductivity, namely $d \geq \Delta$. Moreover the large charging energy $E_C$ implies that these grains have a fix number of electrons, while the Debye frequency gives an estimation of the number of energy levels involved in the pairing, namely $\Omega = 2\omega_D/d \sim 150$, which is rather small. Among another things RBT found an interesting parity effect, similar to the one happening in Nuclear Physics, where grains with an even number of electrons display properties associated with a SC gap, while the odd grains showed gapless behaviour.

These experimental findings produced a burst of theoretical activity focused on the study of the reduced BCS Hamiltonian ($[\Omega]$) with equally spaced levels, i.e. $\varepsilon_j = jd$. Some of the approaches used to study this model have been: i) g.c. BCS ansatz projected in parity, ii) the PBCS ansatz, iii) Lanczos method up to $\Omega = 23$ energy levels, iv) Perturbative RG methods, v) Density Matrix Renormalization Group (DMRG) up to $\Omega = 400$ levels, etc. (for a review on this topic consult $[5]$).

6 Richardson’s exact solution (1963)

Surprisingly enough the aforementioned theoretical works were done in the ignorance of the existence of an exact solution of the reduced BCS model, obtained in 1963 by Richardson $[6]$ and his collaborator Sherman (1964) $[7]$, and further developed and generalized to other models in a series of papers in the 60’s and 70’s by Richardson himself. In order to describe Richardson’s solution it is convenient to define the hard-core boson operators $b_j = c_{j, -} c_{j, +}$, $b_j^\dagger = c_{j, +} c_{j, -}^\dagger$, $N_j = b_j b_j^\dagger$, which satisfy the commutation relations, $[b_j, b_j^\dagger] = \delta_{j,j'} (1 - 2N_j)$.

The Hamiltonian $[\Omega]$ can then be written as
Richardson showed that the eigenstates of this Hamiltonian with \( N \) pairs have the (unnormalized) product form \([7]\)

\[
|N\rangle_R = \prod_{\nu=1}^{N} B_\nu |\text{vac}\rangle
\]

where the parameters \( e_\nu (\nu = 1, \ldots, N) \) are, in general, complex solutions of the \( N \) coupled algebraic equations

\[
\frac{1}{gd} + \sum_{\mu=1(\neq \nu)}^{N} \frac{1}{e_\mu - e_\nu} = \sum_{j=1}^{\Omega} \frac{1}{2\varepsilon_j - e_\nu}
\]

The total energy of these states is given by \( E(N) = \sum_{\nu=1}^{N} e_\nu \). The (normalized) states \([8]\) can also be written as

\[
|N\rangle_R = \frac{C}{\sqrt{N!}} \sum_{j_1,\ldots,j_N} \psi^R(j_1,\ldots,j_N) b_{j_1}^\dagger \cdots b_{j_N}^\dagger |\text{vac}\rangle
\]

where the sum excludes double occupancy of pair states and the wave function \( \psi \) takes the form

\[
\psi^R(j_1,\ldots,j_N) = \sum_{\mathcal{P}} \prod_{k=1}^{N} \frac{1}{2\varepsilon_{j_k} - e_{\mathcal{P}k}}
\]

The sum in \([8]\) runs over all the permutations, \( \mathcal{P} \), of \( 1,\ldots,N \). The constant \( C \) in \([8]\) guarantees the normalization of the state \([8]\). The BCS Hamiltonian can be given a spin representation in terms of the operators \( t_0^j = 1/2 - N_j, t_+^j = b_j \) and \( t_-^j = b_j^\dagger \) which provide a spin 1/2 representation of the \( SU(2) \) group associated to each level \( j \). The Hamiltonian \([8]\) can then be written as

\[
H_{\text{BCS}} = -\sum_j 2\varepsilon_j t_0^j + \frac{gd}{2} (T^+ T^- + T^- T^+) + \text{ctes}
\]

where the matrices \( T^a = \sum_{j=1}^{\Omega} t_0^a \) \( (a = 0, +, -) \) satisfy the standard \( SU(2) \) algebra, with Casimir \( \mathbf{T} \cdot \mathbf{T} = T^0 T^0 + \frac{1}{2} (T^+ T^- + T^- T^+). \)
7 Integrability of BCS (1997)

The integrability of the reduced BCS model was established by Cambiaggio, Rivas and Saraceno in 1997 [9]. These authors found a set of operators

\[ R_i = -t_i^0 - gd \sum_{j(\neq i)} \frac{t_i \cdot t_j}{\varepsilon_i - \varepsilon_j}, \quad (i = 1, \ldots, \Omega) \] (11)

which commute among themselves and with the BCS Hamiltonian \( H_{BCS} \), which in fact can be expressed in terms of these operators as \( H_{BCS} = \sum_j 2\varepsilon_j R_j + \text{ctes} \). The commutativity condition of the \( R_i \) operators follows from: i) the classical Yang-Baxter equation

\[ [r_{i,j}, r_{j,k}] + [r_{i,j}, r_{i,k}] + [r_{i,k}, r_{j,k}] = 0 \] (12)

where \( r_{i,j} = \frac{t_i \cdot t_j}{\varepsilon_i - \varepsilon_j} \) is the classical Yang-Baxter \( r \)-matrix, and ii) the equation \( [t_i^0 + t_j^0, r_{i,j}] = 0 \). Unfortunately CRS, unaware of the Richardson’s exact solution, gave not the eigenvalues \( r_i \) of the conserved quantities \( R_i \), which were found in reference [10] using CFT methods,

\[ r_i = -\frac{1}{2} + gd \left( \sum_{\nu=1}^{N} \frac{1}{2\varepsilon_i - e_{\nu}} - \frac{1}{4} \sum_{j=1(\neq i)}^{\Omega} \frac{1}{\varepsilon_i - \varepsilon_j} \right) \] (13)

8 Gaudin’s model (1976)

Inspired in part by Richardson’s work on BCS, Gaudin proposed in 1976 a class of spin models based on a set of commuting Hamiltonians \( H_i \)

\[ H_i = \sum_{j(\neq i)}^{\Omega} \frac{t_i \cdot t_j}{\varepsilon_i - \varepsilon_j}, \quad (i = 1, \ldots, \Omega) \] (14)

which can be diagonalized by an ansatz similar to (6) with the parameters \( e_{\nu} \) satisfying the equations

\[ \sum_{\mu=1(\neq \nu)}^{N} \frac{2}{e_{\mu} - e_{\nu}} = \sum_{j=1}^{\Omega} \frac{1}{2\varepsilon_j - e_{\nu}}, \] (15)

and eigenvalues
\[ h_i = -\sum_{\nu=1}^{N} \frac{1}{2\varepsilon_i - \varepsilon_\nu} + \frac{1}{4} \sum_{j=1(\neq i)}^{\Omega} \frac{1}{\varepsilon_i - \varepsilon_j} \]  

Comparing eqs. (14) and (11) it is obvious that \( h_i = -\lim_{g \to \infty} R_i/gd \), and similarly \( h_i = -\lim_{g \to \infty} r_i/gd \). On the other hand, in limit \( g \to \infty \), Richardson’s eqs.(8) become Gaudin’s eqs.(13). The model defined by the Hamiltonians (14) is SU(2) invariant and it is known as the rational case. Gaudin also proposed a trigonometric model, which breaks SU(2) down to U(1) and an elliptic model which breaks this U(1) to \( Z_2 \). Gaudin’s trigonometric version have been generalized, to include a g-term à la BCS, in references [12, 13]. There are also bosonic pairing Hamiltonians satisfying the same type of equations as in the fermionic case [14, 15]. Generalizations of the SU(2) BCS-like models to other Lie groups G [16] and supergroups [17] have been worked out.

9 Conformal Field Theory Picture (2000)

In reference [10] it was proposed a CFT interpretation and derivation of the exact’s solution of the BCS model. This was based on several observations: i) analytic structure of the Richardson wave function similar to the one that arises in the computation of conformal blocks, ii) common origin of the integrability of the BCS model and the Knizhnik-Zamolodchikov (KZ) equations [18], namely the classical YB equations, and iii) similarity between the electrostatic analogue model of the Richardson’s eqs. and the Coulomb gas representation of the Wess-Zumino-Witten (WZW) model [19].

The electrostatic picture was already noticed by Gaudin [20] Richardson [21], who observed that the equations (7) and (15) are nothing but the equilibrium conditions for a set of N point-like charges, with charge \( Q = 2 \), located at the positions \( e_\nu \) in the complex plane, subject to their mutual repulsion and the attraction of \( \Omega \) charges, with charge \( Q = -1 \), located at the positions \( \Xi_i = 2\varepsilon_i \), plus a constant electric field generated by a linear charge at infinity with density \( \rho_L = -1/(\pi gd) \). In the Gaudin’s model the latter term is absent. The holomorphic piece, \( U \), of the total 2D-electrostatic potential is given by

\[ U = \sum_{i<j}^{\Omega} \ln(z_i - z_j) - 2 \sum_{\nu<\mu}^{N} \ln(u_\nu - u_\mu) + \frac{1}{4g}(\sum_{i=1}^{\Omega} z_i + 2 \sum_{\nu=1}^{N} u_\nu) \]  

It is easy to verify that \( (\partial U/\partial u_\nu)_{u_\mu=e_\mu} = 0 \) reproduces eqs.(7) and (15). Moreover the eigenvalues \( r_i \) and \( h_i \) are proportional to the forces \( (\partial U/\partial z_i)_{u_\mu=e_\mu} \) exerted on the \( \Omega \) fixed charges \( z_i \). This analogue model was in fact used by Gaudin and Richardson to derive the standard BCS equations for the SC gap, chemical potential, total energy and occupation numbers of the energy levels, in the asymptotic limit when \( N \to \infty \) and \( d \sim 1/N \to 0 \) with \( N/\Omega \), kept fixed [20, 21].
From a CFT viewpoint one can recognized $U$ as arising from the following chiral correlator $^{10}$

$$e^{-\alpha_0^2 U(z,u)} = \langle W_g \prod_{i=1}^\Omega V_{-\alpha_0}(z_i) \prod_{\nu=1}^N V_{2\alpha_0}(u_\nu) \rangle$$

(18)

where $V_\alpha(z)$ and $W_g$ are the following vertex operators

$$V_\alpha(z) = e^{i\alpha \phi(z)}$$

$$W_g = \exp \left( i \alpha_0 g \oint dz \phi(z) \right)$$

constructed from a chiral boson $\phi(z)$ with background charge $2\alpha_0$. In (18) we have neglected an operator at infinity needed to neutralize the overall background charge. We see from (18) that the Coulomb gas charges are equal to $\alpha = -\alpha_0$ for the $z_i$'s, and $\alpha = 2\alpha_0$ for the $u_\nu$'s, in agreement with the Gaudin-Richardson electrostatic model, up to the overall $\alpha_0$ factor. The line of charge at infinity is represented by the vertex operator $W_g$.

The other ingredient of the CFT construction is the so called $\beta - \gamma$ system formed by two boson fields $\beta(z)$ and $\gamma(z)$ which have a correlator $\langle \beta(z)\gamma(w) \rangle = 1/(z - w)$ $^{19}$. Using this formula and the Wick theorem one can write the Richardson’s wave function (9) as follows

$$\psi_R^{m_1,\ldots,m_\Omega}(z,e) = \langle \prod_{\nu=1}^\Omega \gamma_{-m_i}(z_i) \prod_{\nu=1}^N \beta(e_\nu) \rangle$$

(20)

where $m_i = 1/2$ ( resp. $-1/2$) if the corresponding energy level $z_i$ is empty ( resp. occupied) by an electron pair. We have also neglected in (24) some extra $\beta - \gamma$ fields placed at infinity. This suggest to associate every energy level to a primary field $V_{j,m}(z)$ of the $SU(2)$ WZW model with total spin $j$ and third component $m$, which have the Coulomb gas realization $\Phi^j_m(z) = \gamma^{j-m}(z) \exp(-2j\alpha_0\phi(z))$. The total spin $j$ is given by half the maximum number of electron pairs that can occupy a single energy level $z_i$, and so, in the actual case it is given by $j = 1/2$. Based on the eqs. (18) and (20) it is natural to consider the following perturbed WZW conformal block (PWZW)

$$\psi_{m_1,\ldots,m_\Omega}^{ZW}(z) = \langle \prod_{\nu=1}^\Omega \gamma^{j_\nu-m_i}(z_i) \prod_{\nu=1}^N \beta(e_\nu) \rangle$$

where $S(z) = \beta(z)\exp(2i\alpha_0\phi(z))$ are the screening operators needed to balance the charge and $C_\nu$ are their contours of integration. Using the results explained above, one can write the PWZW conformal block as

$$\psi_{m}^{PWZW}(z) = \oint_{C_1} du_1 \ldots \oint_{C_N} du_N e^{-\alpha_0^2 U(z,u)} \psi_{m}^{R}(z,u)$$

(22)

Hence, in the limit $\alpha_0 \to \infty$ these integrals are dominated by the saddle point configurations, i.e. $(\partial U/\partial u_\nu)_{u_\nu=e_\nu} = 0$, where the positions $u_\nu = e_\nu$ satisfy the Richardson eqs.,
and $\psi_m^{PWZ} \propto \psi_m^R$. Curiously enough, the proportionality factor $C$ in eq. (8) follows from the gaussian integration around the saddle points, while the $1/\sqrt{N!}$ factor arises naturally in the Coulomb gas approach. Given the relation $k + 2 = 1/(2\alpha_0^2)$, where $k$ is the level of the Kac-Moody algebra $SU(2)_k$, it follows that the limit $\alpha_0 \to \infty$ corresponds to a singular limit in the representation theory of $SU(2)_k$. Finally the PWZW conformal blocks (22) satisfy the perturbed KZ eqs.

$$\left(\frac{k+2}{2} \frac{\partial}{\partial z_i} - \frac{t_i^0}{2gd} - \sum_{j \neq i}^{\Omega} \frac{\mathbf{t}_i \cdot \mathbf{t}_j}{z_i - z_j} \right) \psi_m^{PWZ}(z) = 0 \tag{23}$$

In the limit when $k + 2 \to 0$ one can easily derive from (23) that $\psi_R$ is indeed an eigenstate of the operators $R_i$ with eigenvalues $r_i = \frac{1}{2} \frac{t_i^0}{gd} \frac{\partial U}{\partial z_i}$ (eq. (13)). In this manner the Richardson’s solution and the integrals of motion of BCS get unified in the framework of perturbed CFT.

10 Gaudin’s, BCS and Integrable Vertex models (1993-2001)

The CFT interpretation of the BCS model, explained in the previous section, turns out to be closely related to the works of Babujian [22], and Babujian, Flume [23] who in 1993 rederived Gaudin’s exact solution using the so called off-shell algebraic Bethe ansatz (OSBA). These authors also pointed out that Gaudin’s eigenstates can be used to build the conformal blocks of the WZW models, along the same lines as was shown in the previous section. The work in reference [10] was done with no knowledge of the papers [22, 23], which may then be regarded as another example of lack of information that these topics have unfortunately suffered in the past.

The similarity between the works [10] and [22, 23] suggested that the Richardson’s solution of the reduced BCS model should also be derivable using the OSBA method. This was done by Amico, Falci and Fazio [24], and later on clarified in references [25, 26, 27], where the BCS coupling constant parametrizes a boundary operator that appears in the transfer matrix of the inhomogenous vertex model, whose semi-classical limit gives rise to the CRS conserved quantities.

The OSBA approach starts from an inhomogenous vertex model whose transfer matrix is given by

$$T(\lambda; z_1, \ldots, z_\Omega) = tr_0 \left( K_0 R^{0\Omega}(\lambda - z_\Omega) \ldots R^{01}(\lambda - z_1) \right) \tag{24}$$

where

$$R^{0j}(\lambda - z_j) = I_0 \otimes I_k + \frac{2\eta}{\eta - 2(\lambda - z_k)} \sigma_0 \otimes S_k \tag{25}$$
is the $R$-matrix acting on the space $0 \otimes k$ ( $S_k$ are the spin matrices associated to the spin $s_j$ irrep of $SU(2)$ ) and

$$K_0 = \begin{pmatrix} e^{-\frac{\eta}{2g^d}} & 0 \\ 0 & e^{\frac{\eta}{2g^d}} \end{pmatrix}$$ (26)

is a boundary matrix acting on the auxiliary space $0$. Defining the Bethe ansatz state

$$|N\rangle = B(\lambda_1) \ldots B(\lambda_N)|\uparrow, \ldots, \uparrow\rangle$$ (27)

one can follow two approaches:

i) **On shell approach:** Impose that $|N\rangle$ is an eigenstate of the transfer matrix $T$ (24), which is guaranteed by the Bethe ansatz eqs.

$$e^{\frac{\eta}{2g^d}} \prod_{i=1}^{\Omega} \frac{\lambda_\alpha - z_i - \eta/2 + \eta s_i}{\lambda_\alpha - z_i - \eta/2 - \eta s_i} = \prod_{\beta \neq \alpha}^{N} \frac{\lambda_\alpha - \lambda_\beta + \eta}{\lambda_\alpha - \lambda_\beta - \eta}$$ (28)

which in the limit $\eta \to 0$ yield the Richardson eqs.

$$\frac{1}{2g^d} + \sum_{i=1}^{\Omega} \frac{s_i}{\lambda_\alpha - z_i} = \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta}$$ (29)

The operators $R_i$ (11) appear at order $\eta^2$ in the power expansion of the transfer matrix (24).

ii) **Off Shell approach:** Let the state $|N\rangle$ to be not an eigenstate of $T$ but instead use it for finding solutions to the PKZ eq.(23). Then in the limit when $k \to -2$ one gets the Richardson’s eqs. as saddle points conditions, as was shown in the previous section. For the Gaudin’s model, this observation was made by Reshetikhin and Varchenko [28].

The derivation of Richardson’s solution from an integrable vertex model has also allowed the computation of correlators and form factors [29, 25] using the “determinantal" techniques developed in [30, 31]. This works generalize the old results by Richardson and Gaudin’s, concerning the norm of the eigenstates and the occupation numbers [8, 20].

From a more mathematical viewpoint the CFT approach to the Gaudin model has a counterpart in the representation theory of Kac-Moody algebras at the critical level, as was shown by Feigin, Frenkel and Reshetikhin [32]. These authors remark that their construction fits into the program of geometric Langlands correspondence proposed by Beilinson and Drinfeld. There are also connections with the Gaudin-Calogero model and the Hitchin’s systems [33, 34]. An interesting mathematical problem is to investigate the meaning of the Richardson eqs. in connection with the representation theory of Kac-Moody algebras at the critical level.
11 BCS and Chern-Simons theory (2001)

In this section we shall briefly mention the last step in the understanding of the integrability properties of the BCS model. In the Gaudin’s case we can draw the following chain of relations: Gaudin’s magnets → Integrable vertex models → WZW → Chern-Simons theory. The last step refers to the well known connection between the 3D-CS theory and the 2D-WZW model \[35\]. Hence we may ask what is the origin of the BCS model in the CS theory?

In reference \[16\] it is shown that the field theoretical origin of BCS can be traced back to a $SU(2)$ CS theory interaction with a one-dimensional distribution of colored matter which breaks both gauge and conformal invariance. This connection is quite remarkable because Chern-Simons theory has been advocated to be mainly connected with effective descriptions of fractional quantum Hall effect and high $T_c$ superconductivity, but never with standard superconductivity. The Chern-Simons theory is not defined in the physical space, which might be two or three-dimensional, but rather in the complex energy plane which is always two-dimensional. This explains why this field theoretical connection of BCS theory remained unveiled for so long time. The connection of Chern-Simons theory with BCS model can be understood in a more general framework when one considers a scaling limit of the twisted Chern-Simons theory defined on a torus, that is, the twisted elliptic Chern-Simons theory. On a torus the KZ equations \[18\] are replaced by the Knizhnik-Zamolodchikov-Bernard (KZB) equations \[36\], which depend on the coordinates $z_i$ of the punctures, the moduli of the torus $\tau$ and a set of parameters $u_j$ characterizing the toroidal flat gauge connections. The later parameters $u_j$ define the twisted boundary conditions for the WZW fields on the torus.

The main result of \[16\] is to show that, for a generic simple simply connected, compact Lie group $G$, the Richardson equations, the CRS conserved quantities and their eigenvalues arise from the KZB connection and their associated horizontal sections. This is done in a limit where the torus degenerates into the cylinder and then into the complex plane. In this limiting procedure the generalized BCS coupling constants appear as conjugate variables of the parameters $u_j$, when this parameters go to infinity. This gives the $G$-based BCS models a suggestive geometrical and group theoretical meaning.

In analogy with Gaudin’s model we can complete the chain of relations in the BCS case: Richardson solution → Integrable vertex models with boundary operators → perturbed WZW → twisted elliptic Chern-Simons. These connections suggest further generalizations both of the CS theory and the CFT, which deserve further investigation.

12 Summary

In this article we have summarized some of the works concerning the integrability and conformal properties of the exactly solvable reduced BCS model and the closely related Gaudin’s model.
Acknowledgments

I would like to thank conversations with L. Amico, M. Asorey, A. Belavin, F. Braun, J. Dukelsky, F. Falceto, G. Falci, E.H. Kim, J. Links, A. Mastellone, R.W. Richardson, J.M. Roman and J. von Delft. I am also grateful to G. Mussardo and A. Cappelli for the invitation to participate in the NATO Advanced Research Workshop on “Statistical Field Theories”, Como 18-23 June 2001. This work has been supported by the grant MCyT BFM2000-1320-C02-01.

References

[1] J. Bardeen, L.N. Cooper and J.R. Schrieffer, Phys. Rev. 108, 1175 (1957).
[2] P. Ring and P. Schuck, The Nuclear Many Body Problem, Springer-Verlag (1980).
[3] P.W. Anderson, J. Phys. Chem. Solids, 11, 28 (1959).
[4] D.C. Ralph, C.T. Black and M. Tinkham, Phys. Rev. Lett. 76, 688 (1996); 78, 4087 (1997).
[5] Jan von Delft and D. C. Ralph, Physics Reports, 345, 61 (2001).
[6] R.W. Richardson, Phys. Lett. 3, 277 (1963).
[7] R.W. Richardson and N. Sherman, Nucl. Phys. B 52, (1964) 221.
[8] R.W. Richardson, J. Math. Phys. 6, 1034 (1965).
[9] M.C. Cambiaggio, A.M.F. Rivas and M. Saraceno, Nucl. Phys. A 624, 157 (1997).
[10] G. Sierra, Nucl. Phys. B 572 [FS](2000) 517.
[11] M. Gaudin, J. Physique 37, 1087 (1976).
[12] L. Amico, A. Di Lorenzo and A. Osterloh, Phys.Rev.Lett. 86 (2001) 5759.
[13] J. Dukelsky, C. Esebbag and P. Schuck, Phys.Rev.Lett. 87 (2001) 66403.
[14] R.W. Richardson, J. Math. Phys. 9, 1327 (1968).
[15] J. Dukelsky and P. Schuck, Phys.Rev.Lett. 86 (2001) 4207.
[16] M. Asorey, F. Falceto and G. Sierra, hep-th/0110266.
[17] P. P. Kulish, N. Manojlovic, nlin.SI/0103010.
[18] V. Knizhnik, A. B. Zamolodchikov, Nucl. Phys B 247, 83-103 (1984).
[19] V.S. Dotsenko, Nucl. Phys. B 338 (1990) 747.
[20] M. Gaudin, "Étude d’un modèle à une dimension pour un système de fermions en interaction," Thèse, Univ. Paris 1967.

M. Gaudin, "États propes et valeurs propes de l’Hamiltonien d’appariement," in Travaux de Michel Gaudin, Modèles exactament résolus, Les Éditions de Physique, France, 1995.

[21] R.W. Richardson, J. Math. Phys. 18, 1802 (1977).

[22] H.M. Babujian, J. Phys. A 26, 6981 (1993).

[23] H.M. Babujian and R. Flume, Mod. Phys. Lett. 9, 2029 (1994).

[24] L. Amico, G. Falci and R. Fazio, J. Phys. A 34 (2001) 6425-6434

[25] H.-Q. Zhou, J. Links, R.H. McKenzie and M.D. Gould, cond-mat/0106390.

[26] Jan von Delft and R. Poghossian, cond-mat/0106405.

[27] G. Sierra, lectures on the BCS model, Univ. of Zaragoza ( May 2001) (unpublished work).

[28] N. Reshetikhin and A. Varchenko, hep-th/9402126.

[29] L. Amico and A. Osterloh, cond-mat/0105141.

[30] V.E. Korepin, Comm. Math. Phys. 86, 391 (1982).

[31] E.K. Sklyanin, Lett. Math. Phys. 47, 275 (1999).

[32] B. Feigin, E. Frenkel and N. Reshetikhin, Commun.Math.Phys. 166 (1994) 27.

[33] B. Enriquez and V. Rubtsov, Math. Res. Lett. 3 343 (1996).

[34] N. Nekrasov, Commun.Math.Phys. 180 (1996) 587.

[35] E. Witten, Commun. Math. Phys. 121, 351 (1989).

[36] D. Bernard, Nucl. Phys. B 303, 77-93 (1988).