LIOUVILLE TYPE THEOREM FOR A NONLINEAR NEUMANN PROBLEM

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Abstract. Consider the following nonlinear Neumann problem

\[
\begin{aligned}
\text{div } (y^a \nabla u(x, y)) &= 0, \quad \text{for } (x, y) \in \mathbb{R}^{n+1}_+ \\
\lim_{y \to 0^+} y^a \frac{\partial u}{\partial y} &= -f(u), \quad \text{on } \partial \mathbb{R}^{n+1}_+ \\
u &\geq 0 \quad \text{in } \mathbb{R}^{n+1}_+
\end{aligned}
\]

where \( a \in (-1, 1) \). A Liouville type theorem is given under suitable conditions on \( f \).

1. Introduction

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space and \( H = \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty) \) be the upper half space, we consider the following nonlinear Neumann problem

\[
\begin{aligned}
\text{div } (y^a \nabla u(x, y)) &= 0, \quad \text{for } (x, y) \in H \\
\lim_{y \to 0^+} y^a \frac{\partial u}{\partial y} &= -f(u), \quad \text{on } \partial H, \\
u &\geq 0 \quad \text{in } \mathbb{R}^{n+1}_+
\end{aligned}
\]

(1.1)

where \( s \in (0, 1), a = 1 - 2s, n > 2s \) are constants. \( f \) is some nonlinear function which will be given later.

Equation (1.1) is closely related to the following nonlinear fractional Laplacian equation

\[
(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n,
\]

(1.2)

which has obtained extensively study in recent years, see, for example [2, 25, 5, 9, 10]. In [2] the authors gave an equivalent characterization to the fractional Laplacian operator \((-\Delta)^s\) in terms of the Dirichlet-Neumann operator which states that for sufficiently regular function \( \phi \) on \( \mathbb{R}^n \), there exists an extension \( \Phi \) from \( \mathbb{R}^n \) to \( H \) such that \( \Phi(x, 0) = \phi(x) \) on \( \partial H \) and solves the equation

\[
\begin{aligned}
\text{div } (y^a \nabla \Phi(x, y)) &= 0, \quad \text{for } (x, y) \in H \\
\lim_{y \to 0^+} y^a \frac{\partial \Phi}{\partial y} &= -d_a(-\Delta)^s \phi, \quad \text{on } \partial H,
\end{aligned}
\]

where \( d_a \) is some positive constant, see [2].

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On the other hand, nonlinear Neumann boundary value problem has its own independent interest to study, especially in the case $s = \frac{1}{2}$ equation (1.1) has been studied with various different nonlinearity on $f$ in the literature, and find applications in the study of heat equations with nonlinear boundary condition and so on, see [13, 9, 24, 18, 17] and the reference therein.

In [13] Theorem 1.2 the author obtained the following Liouville type result

Theorem 1. Let $u$ be a classical solution to the following nonlinear boundary value problem

$$\begin{align*}
\Delta u &= 0, \quad &\text{for } (x, y) \in H \\
\frac{\partial u}{\partial y} &= -u^p, \quad &\text{on } \partial H, \\
u &\geq 0 \quad &\text{in } H,
\end{align*}$$

where $n \geq 2$, $1 < p < \frac{n+1}{n-2}$. Then $u \equiv 0$ on $H$.

See also [14, 18, 24] for different power of $p$ and related results. In [27], an even more general problem was considered, which is worth a generalization in our context.

Our aim in this paper is to obtain Liouville type result of theorem 1 to equation (1.1) which generalizes theorem 1 in two folds. First, we consider the full range of the parameter $s \in (0, 1)$, of which equation (1.1) is singular ($s > 1/2$) or degenerate ($s < 1/2$) in general; second, we study a general nonlinearity $f(u)$ with only continuity assumption together with some monotonicity properties which includes $f(u) = u^p$ as a special case.

To be precise, throughout this paper we assume that $f : [0, \infty) \to [0, \infty)$ is a continuous function such that

1. (H1) $f$ is nondecreasing;
2. (H2) $g(t) = f(t^s)/t^\tau$, $t > 0$ is a nonincreasing and nonconstant function, where
   $$\tau = \frac{n+2s}{n-2s}.$$

Our main result is the following Liouville theorem

Theorem 2. Let $f$ satisfy the condition (H1-H2) and $u$ be a classical nonnegative solution to the nonlinear Neumann problem (1.1), then $u \equiv 0$ on $H$.

As a consequence of [2], we have

Theorem 3. There exists no classical nonnegative solution to the fractional equation (1.2).

Theorem 3 has been studied in many authors’ work, of which we mention the work of Chen, Li and Ou [8]. In this paper, the author introduced the moving plane method for integral equations to solve the nonlinear fractional equation (1.2) with $f(u) = \frac{n+2s}{n-2s}$ (they allow $s$ to range the full interval $(0, n/2)$). Their method has been widely used later in different equations, systems and so on, see [16, 23, 21, 22, 20, 19] for example. Another different approach to Liouville type theorem for equation (1.2) with $f(u) = u^p$ is given in [29].
Our tool is also the famous moving plane method which was invented by the Soviet mathematician Alexandrov in the early 1950s, and later further developed by Serrin [26], Gidas, Ni, Nirenberg [11], Caffarelli, Gidas, and Spruck [3], Li [15], Chen and Li [6, 7], Chang and Yang [4], Chen, Li and Ou [8] and many others. The method has been applied to many different problems, including integral systems and so on. In recent years, this method has been applied to derive radially symmetry for semilinear subelliptic equations on Heisenberg group, see [1, 28]. We use the moving plane method of integral form, the kernel of which is a type of Caccioppoli estimates. With such kind inequality, we can avoid to use too much maximum principles or comparisons, and thus largely simplifies the proof of Theorem 1 in [13, 24].

In section 2, we will give some necessary preliminary results which reveal fine properties of equation (1.1), and in section 3 we will give the proof of theorem 2. Our notations are standard, and positive constants will be denoted by $C, C_\lambda, \ldots$, which may be different from line to line.

2. SOME PROPERTIES OF THE EQUATION (1.1)

There are many properties of equation (1.1-1) which reveal its similarity to the common Laplacian operator in Euclidean space. It is noted that equation (1.1-1) is invariant under the Kelvin transform

$$v(X) = \frac{1}{|X|^{n-2s}} u \left( \frac{X}{|X|^2} \right), X = (x, y) \in \bar{H} \setminus \{0\},$$

so that if $u$ is a solution equation (1.1-1) on $H$, then $v$ is also a solution.

The following comparison principle will be needed in the proof of theorem 2.

**Lemma 4.** (Comparison principle) Let $\Omega \subset H$ be an open set with a part of flat boundary $\Gamma \subset \partial H$ and $u$ be a classical solution to the equation

$$\begin{cases}
\text{div} (y^a \nabla u(x, y)) = 0, & \text{in } \Omega, \\
y^a u_y \leq 0 & \text{on } \Gamma, \\
u \geq 0, u \neq 0 & \text{in } \Omega,
\end{cases}$$

then $u > 0$ on $\Omega \cup \Gamma$.

**Proof.** $u > 0$ in $\Omega$ is a consequence of maximum principle for uniform elliptic operators, see [12] for more information. In fact, since \( \text{div} (y^a \nabla u(x, y)) = 0 \) is equivalent to the nondivergence equation

$$\Delta u(x, y) + \frac{a}{y} u_y = 0,$$

which is a uniformly elliptic equation in any ball $B$ compactly contained in $\Omega$ with bounded smooth lower order coefficients, the strong maximum principle and the Hopf lemma can be applied. Since $B$ is arbitrary, we obtain that $u > 0$ in $\Omega$. 

To show that $u > 0$ on $\Gamma$, let us fix an arbitrary point $X = (x, 0) \in \Gamma$. Suppose that $y^a u_y |_X < 0$, then by continuity, we have $u_y < 0$ for $X$ close to $\Gamma$ enough, which implies that for $y$ small enough $u(x, 0) \geq u(x, y) > 0$ by assumption. In general, let $\phi(y) = y^{1-a}$, then the function $u - \epsilon \phi$ is also a solution of the same equation in $\Omega$ but with boundary condition

$$y^a (u - \epsilon \phi)_y \leq -(1 - a) \epsilon < 0 \quad \text{on } \Gamma,$$

so that $u(x, 0) = (u - \epsilon \phi)(x, 0) > u(x, y) - \epsilon y^{1-a}$ for fixed small $y$. Letting $\epsilon \to 0$ one derives that $u(X) \geq u(x, y) > 0$ by assumption. 

As a corollary we have

**Corollary.** Any nonnegative classical solutions to equation (1.1) must be either positive on $\overline{H}$ or identically equal to zero.

We shall also need the following results related on trace operator. Let $D^{1,a}(H)$ denote the closure of the function space $C_0^\infty(\overline{H})$ with respect to the norm

$$||\phi||^2 = \int_H y^a |\nabla \phi(x, y)|^2 dx dy.$$

It is known that there exists a bounded linear trace operator $T : D^{1,a}(H) \to L_{n-2s}^2(\partial H)$ such that $T \phi = \phi$ if $\phi \in C_0^1(\overline{H})$ and

$$(2.1) \quad \left( \int_{\partial H} |T \phi(x, 0)|^\frac{2n}{n-2s} dx \right)^{\frac{n-2s}{n}} \leq S \int_H y^a |\nabla \phi(x, y)|^2 dx dy$$

for all $\phi \in D^{1,a}(H)$ and for some positive constant $S$, see [10] for details.

### 3. Proof of Theorem 2

As already mentioned in the introduction, the moving plane method is our tool. Some notations are introduced. Let $\lambda \in \mathbb{R}$, $X = (x_1, x_2, \cdots, x_n, y) \in H$, $T_\lambda = \{X \in H; x_1 = \lambda\}$, $\Sigma_\lambda = \{X \in H; x_1 > \lambda\}$, $p_\lambda = (2\lambda, 0, \cdots, 0, 0)$, $X_\lambda = (2\lambda - x_1, x_2, \cdots, x_n, y)$. We divide the proof into several steps.

Step 0. Before starting the procedure of moving plane, we note that if $u$ is a solution, then the Kelvin transform

$$v(X) = \frac{1}{|X|^{n-2s}} u \left( \frac{X}{|X|^2} \right), X = (x, y) \in \overline{H} \setminus \{0\},$$

is a solution to the following problem

$$\begin{align*}
\text{div} \left( y^{1-2s} \nabla v(x, y) \right) &= 0, \quad \text{for } (x, y) \in H \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial v}{\partial y} &= 1 \cdot \frac{y^{1-2s}}{y}, \quad \text{on } \partial H \setminus \{0\}, \\
v &\geq 0 \quad \text{in } H.
\end{align*}$$

Moreover one sees that $v$ has good asymptotic behaviors at infinity:

$$(3.2) \quad \lim_{X \to \infty} |X|^{n-2s} v(X) = u(0).$$
Thus for any \( r > 0, v \in L^{\frac{q}{q-2s}} \cap L^\infty(\Sigma_\lambda) \) for any \( q > n + 1 \) and any \( \lambda > 0 \).

In the following we shall study the function \( v \) instead of \( u \). It is possible that \( v \) has singularity at \( X = 0 \).

Let \( v_\lambda(X) = v(X_\lambda) \), then \( v_\lambda \) solves the equation

\[
\begin{cases}
\text{div} \left( y^{1-2s} \nabla v_\lambda(x,y) \right) = 0, & \text{for } (x,y) \in \mathbb{R}^{n+1}_+ \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial v_\lambda}{\partial y} = -g(|x_\lambda|^{n-2s} v_\lambda(x)) v_\lambda^\tau, & \text{on } \partial \mathbb{R}^{n+1}_+ \setminus \{p_\lambda\}, \\
v_\lambda \geq 0 & \text{in } \mathbb{R}^{n+1}_+.
\end{cases}
\]

Step 1. We show that this procedure can be started for sufficiently large \( \lambda \).

The following integral inequality is our main tool.

**Lemma 5.** For any fixed \( \lambda \), there holds

\[
(3.4) \quad \int_{\Sigma_\lambda} y^\sigma |\nabla (v - v_\lambda)_+|^2 \leq C_\lambda \left( \int_{\partial H \cap \partial A_\lambda} v^{\tau+1} \right)^{\tau-1} \int_{\Sigma_\lambda} y^\sigma |\nabla (v - v_\lambda)_+|^2,
\]

where \( A_\lambda = \{(x,y) \in \Sigma_\lambda; v(x) > v_\lambda(x)\} \), \( C_\lambda (\lambda > 0) \) is a constant bounded away from zero.

This kind of inequality is inspired by the works of [8, 28].

**Proof.** Formula (3.4) is a just a consequence of Caccioppoli’s inequality. Indeed, let \( 0 < \epsilon < \lambda, \eta_\epsilon \in C^\infty_0(\mathbb{R}^{n+1}) \) such that \( 0 \leq \eta_\epsilon \leq 1, \eta_\epsilon \equiv 1 \) for \( 2\epsilon \leq |X - p_\lambda| \leq \epsilon^{-1} \) and \( \eta_\epsilon = 0 \) for \( |X - p_\lambda| \leq \epsilon \) or \( |X - p_\lambda| \geq 2\epsilon^{-1} \), \( |\nabla \eta_\epsilon(X)| \leq C\epsilon^{-1} \) for \( \epsilon \leq |X - p_\lambda| \leq 2\epsilon \) and \( |\nabla \eta_\epsilon(X)| \leq C\epsilon \) for \( \epsilon^{-1} \leq |X - p_\lambda| \leq 2\epsilon^{-1} \). Substituting \( \phi_\epsilon = (v - v_\lambda)_+ \eta_\epsilon^2 \) into the equation of \( v \) and \( v_\lambda \) we have that

\[
\int_{\Sigma_\lambda \cap \{2\epsilon \leq |X - p_\lambda| \leq \frac{\epsilon}{2}\}} y^\sigma |\nabla (v - v_\lambda)_+|^2 \\
\leq \int_{\Sigma_\lambda} y^\sigma |\nabla (\eta_\epsilon (v - v_\lambda)_+)|^2 \\
= \int_{\Sigma_\lambda} y^\sigma \nabla (v - v_\lambda)_+ \nabla \phi_\epsilon + \int_{\Sigma_\lambda} y^\sigma (v - v_\lambda)_+ |\nabla \eta_\epsilon|^2 \\
=: I_1 + I_2.
\]

Let us estimate \( I_1 \) and \( I_2 \) respectively now.

By the divergence theorem and the equations of \( v, v_\lambda \) we have

\[
I_1 = \int_{\Sigma_\lambda} y^\sigma \nabla (v - v_\lambda)_+ \nabla \phi_\epsilon \\
= \int_{\partial\{\Sigma_\lambda \cap \text{supp } \eta_\epsilon\}} \phi_\epsilon y^\sigma \nabla (v - v_\lambda)_+ \cdot \nu \\
= \int_{\{x \in \mathbb{R}^n; x_1 > \lambda_\epsilon \leq |x - p_\lambda| \leq \frac{\epsilon}{2}\}} \left( g(|x|^{n-2s} v_\lambda(x)) v_\lambda^\tau - g(|x_\lambda|^{n-2s} v_\lambda(x)) v_\lambda^\tau \right) \phi_\epsilon.
\]
Since \( |x| \geq |x_\lambda| \) if \( x_1 > \lambda \) and \( v > v_\lambda \) on \( A_\lambda \), and \( g \) is nonincreasing by assumption, it follows that

\[
I_1 \leq \int_{\{x \in \mathbb{R}^n : x_1 > \lambda, \epsilon \leq |x-p_\lambda| \leq \frac{\epsilon}{2}\}} g(|x|^{n-2s} v(x))(v^\tau - v_\lambda^\tau) \phi_\epsilon dx
\]

\[
\leq C_\lambda' \int_{\partial H \cap \partial \Lambda_\lambda} v^{\tau-1}(x)(v-v_\lambda)^2 dx
\]

\[
\leq C_\lambda' \left( \int_{\partial H \cap \partial \Lambda_\lambda} v^{\tau+1}(x) dx \right)^{\frac{\tau-1}{\tau+1}} \left( \int_{\partial H \cap \partial \Sigma_\lambda} (v-v_\lambda)^{\tau+1} dx \right)^{\frac{1}{\tau+1}},
\]

where \( C_\lambda' := \tau \sup_{x_1 > \lambda} g(|x|^{n-2s} v(x)) \). Note that \( |x|^{n-2s} v(x) \to u(0) \) as \( |x| \to \infty \), and so \( C_\lambda' \) is bounded away from zero. By virtue of the trace inequality (2.1), we have that

\[
\left( \int_{\partial H \cap \partial \Sigma_\lambda} (v-v_\lambda)^{\tau+1} dx \right)^{\frac{1}{\tau+1}} \leq S \int_{\Sigma_\lambda} g^a |\nabla (v-v_\lambda)_+|^2.
\]

Therefore,

\[
I_1 \leq C_\lambda \left( \int_{\partial H \cap \partial \Lambda_\lambda} v^{\tau+1}(x) dx \right)^{\frac{\tau-1}{\tau+1}} \int_{\Sigma_\lambda} g^a |\nabla (v-v_\lambda)_+|^2,
\]

where \( C_\lambda \) is a positive constant which is bounded when \( \lambda \) is away from zero.

To estimate \( I_2 \), observe that

\[
I_2 \leq C \epsilon^2 \int_{\Sigma_\lambda \cap R_\epsilon} y^a |(v-v_\lambda)_+|^2 + C \epsilon^2 \int_{\Sigma_\lambda \cap R_1/\epsilon} y^a |(v-v_\lambda)_+|^2
\]

\[
\leq C \epsilon^2 \int_{R_\epsilon} y^a |v|^2 + C \epsilon^2 \int_{R_1/\epsilon} y^a |v|^2,
\]

where \( R_r = \{ y > 0 : r \leq |X-p_\lambda| \leq 2r \} \) for \( r > 0 \). By (3.2), one has that

\[
\epsilon^2 \int_{R_\epsilon} y^a |v|^2 \leq D_\lambda \epsilon^2 \int_{\{X \in H : |X-p_\lambda| \leq 2\epsilon\}} y^a dX = D_\lambda \epsilon^{n-2s}
\]

as \( \epsilon \to 0 \), and

\[
\epsilon^{-2} \int_{\Sigma_\lambda \cap R_1/\epsilon} y^a |v|^2 \leq D_\lambda \epsilon^2 \int_{\{X \in H : 1/\epsilon \leq |X-p_\lambda| \leq 2/\epsilon\}} y^a |X|^{2(2s-n)} dX
\]

\[
\leq D_\lambda \epsilon^{2(n-2s)} \int_{\{X \in H : |X-p_\lambda| \leq 2/\epsilon\}} y^a dX
\]

\[
\leq D_\lambda \epsilon^{n-2s},
\]

for some constant \( D_\lambda > 0 \), hence

\[
I_2 = O(\epsilon^{n-2s}) \quad \text{as } \epsilon \to 0,
\]

which implies that \( \lim_{\epsilon \to 0} I_2 = 0 \). Finally, combining the estimate of \( I_{1,2} \) together we derive the formula (3.4). \( \square \)
As a consequence of (3.4), we see that for \( \lambda > 0 \) large enough,

\[
C_\lambda \left( \int_{\partial H \cap \partial A_\lambda} v^{\tau+1} \right)^{\tau-1} \leq \frac{1}{2}
\]
since \( v(x,0) \in L^{\tau+1}(\partial H \cap \partial \Sigma_\lambda) \), which implies that

\[
\int_{\Sigma_\lambda} y^a |\nabla (v - v_\lambda)|^2 = 0,
\]
and thus \( v \leq v_\lambda \) in \( \Sigma_\lambda \) for \( \lambda \) large enough.

Step 2. Now we can move the plane. Define

\[
\mu = \inf \{ \lambda > 0; v \leq v_\lambda \text{ in } \Sigma_\lambda \}.
\]

Lemma 6. If \( \mu > 0 \), then \( v \equiv v_\mu \) in \( \Sigma_\mu \).

Proof. By continuity, we see that \( v \leq v_\mu \) in \( \Sigma_\mu \). Suppose on the contrary that \( v \not\equiv v_\mu \) in \( \Sigma_\mu \). Then for \( x \in \partial H \cap \partial \Sigma_\mu \), we have that

\[
g(|x|^n v(x))v_\tau(x) = f(|x|^n v(x)) \leq f(|x|^n v_\mu(x)) = g(|x|^n v_\mu(x))v_\tau_\mu(x) \leq g(|x|^n v_\mu(x))v_\tau_\mu(x),
\]
If follows from the comparison principle lemma 4 that

\[
v < v_\mu \text{ for } (x,y) \in \Sigma_\mu \cup \{X \in \partial H, x_1 > \mu\}.
\]
By virtue of the strict inequality, we get that the characteristic function \( \chi_{\partial A_\lambda} \to 0 \) a.e. in \( \mathbb{R}^n \) as \( \lambda \to \mu \), and thus by the Dominated convergence theorem,

\[
\lim_{\lambda \to \mu} C_\lambda \left( \int_{\partial H \cap \partial A_\lambda} v^{\tau+1} \right)^{\tau-1} = 0,
\]
which, together the inequality (3.4), gives rise to a small positive constant \( \delta > 0 \) such that for all \( \lambda \in [\mu - \delta, \mu] \)

\[
v \leq v_\lambda \text{ in } \Sigma_\lambda,
\]
which is against the choice of \( \mu \). This proves the lemma. \( \square \)

Finally, we can show that \( \mu = 0 \). Otherwise \( \mu > 0 \) implies that \( v \equiv v_\mu \) in \( \Sigma_\mu \). But then substituting the equality into equations (3.1)(3.3) leads to the identity

\[
g(|x|^{n-2s}v(x)) = g(|x|^n v_\mu(x)) \text{ for all } x \in \partial \Sigma_\lambda,
\]
which is impossible since \( g \) is not a constant function.

Therefore, we obtain that \( \mu = 0 \), which means that \( v(x_1, x_2, \ldots, x_n, y) \leq v(-x_1, x_2, \ldots, x_n, y) \) for all \( (x_1, x_2, \ldots, x_n, y) \in \{x_1 > 0, y \geq 0\} \). However, note that the function \( v(-x_1, x_2, \ldots, x_n, y) \) is also a solution to equation
(3.1), and one applies the same procedure to obtain the inverse inequality. This shows that \( v \) is symmetric with respect to \( x_1 \). Similarly, one shows that \( v \) is symmetric with respect to any variable \( x_k, k = 1, \ldots, n \), hence \( v \) is radially symmetric with respect to \( x \), which in turn means that \( u \) is radially symmetric with respect to any point on the plane \( \partial H \), which is impossible only if \( u \) depends only on the variable \( y \). But then the equation is explicitly solved by

\[
 u(x, y) = -\frac{f(m)}{1-a}y^{1-a} + m
\]

for some nonnegative constant \( m \), but then \( u \) cannot be nonnegative for \( y \) large enough if \( f(m) > 0 \). However, if \( f(t_0) = 0 \) for some \( t_0 > 0 \), the monotonicity assumptions in (H1)(H2) implies that \( g \equiv 0 \), which is against (H2). Therefore, there is no nonzero nonnegative solutions to the equation (1.1).

This finishes the proof of Theorem 1.

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