Continuous matter fields in Regge calculus

V.M.Khatsymovskiy

*Budker Institute of Nuclear Physics*

Novosibirsk, 630090, Russia

E-mail address: khatsym@inp.nsk.su

**Abstract**

We find that the continuous matter fields are ill-defined in Regge calculus in the physical 4D theory since the corresponding effective action has infinite terms unremovable by the UV renormalisation procedure. These terms are connected with the singular nature of the curvature distribution in Regge calculus, namely, with the presence in $d > 2$ dimensions of the $(d-3)$-dimensional simplices where the $(d-2)$-dimensional ones carrying different conical singularities are meeting. Possible resolution of this difficulty is discretisation of matter fields in Regge background.
Regge calculus is a (discrete) minisuperspace theory of the continuum spacetime, and correspondingly all other fields living in this spacetime can be taken as either discrete objects or genuine continuous ones. In particular, the most practically important case of the electromagnetic field has been considered on a discrete level in refs. [1, 2]. In ref. [3] the scalar field has been discretised in the 2D Regge calculus. Example of another kind is the field of the coordinate frame deformations $\xi_\mu(x)$ considered in ref. [4]. This field is defined at each point of the spacetime continuum independently, i.e. it is considered as ordinary continuous field.

If continuous matter field is quantised in Regge background, the problem of vacuum stability can arise. Indeed, one can say that gravity field strength is infinite on some subset of points, the conical singularities. On the other hand, this subset is of zero measure. These two competing factors make a priori unclear whether continuous field is well-defined. The routine way to check it is to compute the effective action. This reduces (in the case of free field considered throughout) to calculating some functional determinants. In the 2D case such calculation has been performed on the smooth manifold in ref. [5] for the field of arbitrary spin. In ref. [6] it has been proven to be extendable to the Regge manifold.

In the present paper we show that the continuous matter fields are ill-defined on the Regge manifold in the physical 4D case: their contributions into the effective action suffer from the infinities unremovable by the standard procedure of the UV renormalisation. The source of these are points of intersection of different 2D simplices carrying the conical singularities, i.e. the links.

Consider the conformal trace anomaly $< T_\mu^\mu >$ for a given field. In contrast with the 2D case now we can not restore the effective action from it as in ref. [4], but would like to use it for probing singularities in the action as quantity derivable from this action. Potentially dangerous are the terms in $< T_\mu^\mu >$ squared in curvature requiring a care due to the $\delta$- function nature of the curvature (such the situation does not arise in two dimensions because the trace anomaly is linear in curvature there). First we show that nevertheless the conical singularity alone does not produce unremovable infinities (at least there is no indication to that). Indeed, in some coordinates $x, y, u, v$ conical singularity can be placed in the plane $u = 0, v = 0$, and the curvature tensor made to have the only nonzero component

$$R_{xyxy} \sim \delta(x)\delta(y). \quad (1)$$

The bilinear in curvature invariants $R_{iklm}R^{iklm}, R_{ik}R^{ik}$ and $R^2$ entering $< T_\mu^\mu >$ are proportional to

$$A\delta(x)\delta(y)$$

where $A$ stands for $\delta(0)\delta(0)$ implied in the sense of some regularisation. This is just proportional to $AR$ and can be removed by adding to the action the $R$-term, i.e. by renormalisation of the gravity constant.

Now pass to the intersection of such singularities on a link. Let us coordinatise the neighbourhood of the link by introducing radial distance $r$ in each of the adjacent
4-simplices,
\[ ds^2 = du^2 + dr^2 + r^2 d\Omega^2(\theta, \phi). \]  
(2)

Here \( u \) is a coordinate on the link; \( d\Omega^2(\theta, \phi) \) reduces to a metric on the unit 2-sphere in each 4-simplex, but globally it differs from that one due to the conical singularities. The only nonzero curvature component turns out to be

\[ R_{\theta\phi\theta\phi} = r^2 \left( \frac{1}{2} (2) R - 1 \right) (\gamma_{\theta\theta} \gamma_{\phi\phi} - \gamma_{\theta\phi}^2) \]  
(3)

if

\[ d\Omega^2(\theta, \phi) = \gamma_{\theta\theta} d\theta^2 + 2 \gamma_{\theta\phi} d\theta d\phi + \gamma_{\phi\phi} d\phi^2 \]  
(4)

and \((2)R\) is the scalar curvature of \( \gamma_{ab} \). Thus \( \gamma_{ab} \) is required to ensure \( \frac{1}{2} (2) R - 1 \) being certain combination of \( \delta \)-functions in order to reproduce Regge geometry in the neighbourhood of a link. Explicit solution of this problem is not important for us; knowing eq. (3) allows us to find the curvature invariants,

\[ R = 2r^{-2} \left( \frac{1}{2} (2) R - 1 \right) \]  
(5)

and

\[ R_{iklm} R^{iklm} = 2 R_{ik} R^{ik} = R^2 = \frac{4}{r^4} \left( \frac{1}{2} (2) R - 1 \right)^2. \]  
(6)

Taking into account the \( \delta \)-function nature of \((2)R\), we find that there are the terms in \( < T^\mu_\mu > \) of the type

\[ Ar^{-4} \delta(\theta - \theta_0) \delta(\phi - \phi_0) \]  
(7)

with infinite \( A \). To cancel this the counterterms of dimensionality \((\text{length})^{-4}\) are required. The only such ones in the effective action which give nontrivial contribution to \( < T^\mu_\mu > \) are those bilinear in Riemannian tensor; corresponding contribution to \( < T^\mu_\mu > \) is \( \Delta R \). The expression \( A \Delta R \) contains the term (7) indeed; but it also contains the term

\[ \frac{1}{2} Ar^{-4} \Delta_\gamma \delta(\theta - \theta_0) \delta(\phi - \phi_0) \]  
(8)

which (with reversed sign) thus arises instead of (7) after subtracting \( A \Delta R \). Here \( \Delta_\gamma \) is the 2D covariant Laplacian for the metric \( d\Omega^2 \). Thus, the curvature bilinears (8) are ill-defined in the neighbourhood of the links of the 4D Regge manifold. One might wonder why these bilinears were well-defined (after renormalisation) in the spacetime with an only conical singularity which can be considered as two singularities meeting at a link and being continuation of each other. The matter is in regularisation. The case of the only singularity is the most symmetrical one when the spacetime is decomposed as \( M \times \mathcal{R}^2 \) where \( M \) is 2D spacetime with conical singularity which we regularise independently of other two dimensions \((u, v \) above) in \( \mathcal{R}^2 \). In the case of more singularities sharing a link geometry of the problem only allows to single out the \( u \) and radial coordinate \( r \) and regularise in the angle \( \theta, \phi \)-dependence.
Let us check whether dangerous contributions from different curvature bilinears cancel each other or not. Write out the general expression for the trace anomaly [7],

\[
<T^\mu_\mu> = \frac{1}{2880\pi^2} \left\{ aC^{iklm}C_{iklm} + b \left( R_{ik}R^{ik} - \frac{1}{3}R^2 \right) + c\Delta R + dR^2 \right\},
\]

(9)

and, making use of the values of the curvature invariants (5) and (6), we find for the singular part of \(<T^\mu_\mu>\) that

\[
<T^\mu_\mu>_{\text{sing}} \sim \frac{1}{r^4} \left( \frac{1}{2} (2)R - 1 \right)^2 (2a + b + 6d).
\]

(10)

(the term \(\Delta R\) is well-defined as distribution). It is seen to be nonzero (strictly negative) for all the practically interesting cases: the electromagnetic field with \(a = 13\), \(b = -62\), \(d = 0\); the spinor field with \(a = -7/4\), \(b = -11/2\), \(d = 0\); the scalar field with \(a = -1\), \(b = -1\), \(d = -90 \left( \xi - \frac{1}{6} \right)^2\), \(\xi\) being the curvature-scalar coupling.

Note that (10) gives only anomalous contribution. If the field is not conformally invariant as in the latter case at \(\xi \neq 1/6\), there are also nonanomalous terms. Since these terms depend on the choice of quantum state these can cancel the \(<T^\mu_\mu>_{\text{sing}}\) only at the expense of imposing a kind of a constraint on the state. This ruins the parallel with the continuous counterpart of the theory where such the constraint is absent.

Thus, matter fields generally can not be defined on Regge lattice as the genuine continuous objects. A possible way out of this difficulty may be discretisation of these.

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