A Numerical study of the unsteady flow of two immiscible micro polar and Newtonian fluids through a horizontal channel using DQM with B-Spline basis function

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Abstract. In this paper the differential quadrature method, using cubic B-spline basis function is applied to get numerical solution for the unsteady flow of two immiscible micro polar and Newtonian fluids through a horizontal channel. This numerical technique is not only subsequently easy to enforce but also inexpensive in phrases of data complexity. The two immiscible micro-polar and Newtonian fluids were taken into account as combined partial differential equations. For fluid velocity and micro rotation regarding different fluid parameter sets, empirical outcomes are obtained. The analysis of results is discussed in three situations, with constant, periodic and decreasing pressure gradient. The effects of admissible fluid parameters i.e. Reynolds number, ratio of viscosities, micro polarity parameter and time on fluid velocities, micro rotation and volume flow rate have been illustrated through graphs. The outcomes in terms of volume flow rate across the channel for fluid velocities with different fluid parameters are addressed.

Keywords: Newtonian fluid, micro-polar fluid, immiscible fluid, Unsteady flow, Cubic B- spline, Modified cubic B- spline, Differential quadrature method (DQM)

1. Introduction
Many of the fluids that are relevant to the engineering industry are not like Newtonian fluids. Many researchers have presented different models to study the behavior of such fluids. The micro-polar fluid model is one of the popular non-Newtonian fluids that attracted many researchers around the globe. Eringen [1],[2] presented and explained the theory of micro-polar fluids. Detailed thermodynamic limitations are examined and density, velocity and micro-rotational field equations have been achieved. Peddieson [17] has obtained numerical solutions for the flat or axis symmetric stagnant point flow of a micro-polar fluid with the characteristics of a turbulent shear flux. Lubrication flow studies [12], [6], [21] and [27] demonstrated that the carriage of a micro-polar fluid is better than a Newtonian fluid. Devakar and Iyenger [26] used the state space approach to address the unstable flow of a micro-polar fluid between parallel plates. Analytical results are developed for Fluid velocity and micro-rotation and the effect of various material parameters is discussed on flow changes. A micro-polar fluid model was studied by Mekheimer and Kot [22] for stenosis tapered artery. The homotopy analysis approach was used by Srinivasacharya and Shiferaw [10] to find solutions for micro-polar liquid fluid flow between the Soret and Dufour parallel sheets. Wang et al. [23] explored the movement of micro-polar fluids via a micro parallel stream. Some other work can be included in [31-32, 24] on micro-polar fluid flow.

There are many engineering problems, including desalination processes and the system in which oil pipelines are lubricated with water film admitted the immiscible fluid flow. The above developments have resulted in significant studies into immiscible liquid flows. Precise solutions were obtained in [29] for laminar flow between parallel plates of two immiscible Newtonian fluids. Some additional studies are available in [16,18–4] about the movement of
immiscible liquids between parallel plates for Newtonian, non-Newtonian liquids. In a vertical canal the analytical investigation of fully developed free-convective flow for the micro-polar and viscous fluids was done in [13]. In [14] Srinivas and Ramana Murthy discuss the flow of two immiscible couple strain fluids in two homogeneous permeable beds. They also examined the effect of the entropy level in [15] for the flow of two micro-polar fluids in a horizontal stream. The magneto convection reverse flow of two Newtonian immiscible fluids in a vertical stream was investigated in Borrelli et al [3]. Nevertheless, all natural and artificial flows have circumstances in which the dimension of time evolution is highly desirable. It means all the real flow circumstances are approximately unstable. Consequently, the study, for their application understanding of time dependent flows of immiscible fluids is more important in biomechanics. Researchers performed a significant number of studies on the steady flow of immiscible Newtonian and micro-polar fluids by various geometries with many consequences. Apart from Devakar et al[25] the unsteady flow of two immiscible micro-polar and Newtonian fluids has not yet been observed. Here the fluid velocities and micro-rotation were measured by the Crank-Nicolson finite difference numerical approach. The low order finite difference, finite element and finite volume approaches can accomplish many computational simulations of engineering problems. To reach a reasonable level of precision, still many grid points are used to obtain precise solutions at these specified points in the low order methods. In the pursuit of an effective technique of discretization to obtain precise numerical solutions with substantially limited grid levels, the differential quadrature method has been introduced by Bellman et al.[30] Where the function’s partial derivative with respect to a coordinate direction are represented at all mesh points along that direction as a linear weighed sum of all functional values. Quan and Chang [20], [19] were further strengthened to overcome weighting coefficients. For the determination of weighting coefficients, different test functions, including spline functions, Lagrange interpolation polynomials, sink function etc., are employed in [7],[8],[9].To solve advection-diffusion equation the cubic B-spline function was used with DQM Korkmaz and Dag [5].With the modulated B-spline collocation process, Mittal and Jain [28] have solved the Burgers equation. Arora and Joshi [33],[34] solved nonlinear Schrodinger and Cable equations numerically using DQM. The numerical solutions for an incompressible magnetohydrodynamic Jeffrey fluid between two parallel plates through a porous medium was discussed by Katta Ramesh and Varun Joshi [35].Burgers and one dimensional hyperbolic telegraph equations are solved numerically using DQM with trigonometric and modified trigonometric basis function[36],[37] .The updated cubic-B-spline foundation functions were used in DQM in order to determine weighting coefficients by Arora and Singh [11] and it was observed that the matrix size and complexity were reduced when applying and less grid points are needed compared with the methods given above. This approach is thus simple to use and economically effective in terms of data sophistication, which leads to less error and MCB-DQM can easily be applied and the low storage is retained as a gain of that system. Hence in this paper the differential quadrature method with cubic B- spline basis function is used to evaluate the velocity and micro-rotation of unsteady flow of two immiscible micro polar and Newtonian fluids through a horizontal channel. The analysis of results for fluid velocities and microrotation profile is explained along with three cases for applied pressure gradient. The outcomes in terms of volume flow rate across the channel for fluid velocities with different fluid parameters are addressed and also analyzed with Devakar et al [25].

2. Basic equation and Formulation

Field equations governing micro-polar fluid flow are provided by [1]

\[ \chi_t + \nabla (\chi \nu) = 0 \]  \hspace{1cm} (1)

\[ \chi \mu - \nabla \rho + \delta \nabla \times \omega - (\alpha + \delta) \nabla \times \nu + (\gamma_1 + 2 \alpha + \delta) \nabla (\nabla \cdot \nu) - \chi \frac{d \nu}{dt} = 0 \]  \hspace{1cm} (2)
\[ \chi \xi - 2\delta \omega + \delta \nabla \times v - \lambda_1 \nabla \times \nabla \times \omega + (\lambda_1 + \lambda_2 + \lambda_3)\nabla(\nabla \cdot \omega) \cdot \chi \frac{d\alpha}{dt} = 0 \] (3)

Here \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are gyro-viscosity coefficients, the material constants \( \gamma_1, \alpha \) and \( \delta \) are viscosity coefficients, \( v \) is the velocity, \( \omega \) is the micro-rotation, \( \mu \) is the body force per unit mass, \( \xi \) is the body couple per unit mass, \( \chi \) is the density co-efficient, \( \tau \) is the gyration co-efficient, \( \rho \) is the fluid pressure at any point. Equation (2) can be reduced to following equation if there is no micro rotational effect means \( \omega = 0, \delta \to 0 \)

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\[ \chi \frac{du}{dt} = \chi \mu - \nabla \rho - \alpha \nabla \times \nabla \times v + (\gamma_1 + 2\alpha)\nabla(\nabla \cdot v) \] (4)

The equation (4) is known as the Navier-Stokes equation of motion for the classical Newtonian fluid model. Consider the flow is as unsteady, laminar and unidirectional of two micro-polar and Newtonian fluids and the following assumptions are being made:

The fluid is incompressible and immiscible and it is flowing in between two horizontal parallel plates and the distance between these plates is 2k both the plates existing in X and Z direction shown in figure 1. Micro-polar fluid occurs in lower region-I (\( -k \leq y \leq 0 \)) having the fluid velocity \( u_1 \), density \( \chi_1 \), viscosity \( \alpha_1 \), micro-rotation vector \( \omega \), and vortex viscosity \( \delta \). Newtonian fluid occurs in upper region-II (\( 0 \leq y \leq k \)) and having the fluid velocity \( u_2 \), density \( \chi_2 \), viscosity \( \alpha_2 \), micro-rotation vector \( \omega \) and vortex viscosity \( \delta \).

In the both regions the transportation properties are constants. In the direction of X-axis the common pressure gradients is applied hence the fluid flows in both the regions. The equations (1), (2) and (3) governs the fluid flow in the region –I and the equations (1) and (4) governs the fluid flow in the region –II.

The fluid flow is governed by following equations under above mentioned assumptions region-I for Micro-polar fluid (\( -k \leq y \leq 0 \)).

\[ \nabla \cdot (\chi u_1) = 0 \] (5)

\[ \chi_1 \frac{du_1}{dt} = -\nabla \rho + \delta \nabla \times \omega + (\alpha_1 + \delta) \nabla \times \nabla \times u_1 \] (6)

\[ \chi_1 \frac{d\omega}{dt} = -2\delta \omega + \delta \nabla \times u_1 - \lambda_1 \nabla \times \nabla \times \omega + (\lambda_1 + \lambda_2 + \lambda_3)\nabla(\nabla \cdot \omega) \] (7)

![Fig. 1. Geometrical configuration of the problem.](image-url)
Region-II for Newtonian fluid \((0 \leq y \leq k)\):

\[ \nabla \cdot (\chi \nu_2) = 0 \]  
\[ \chi \frac{d\nu_2}{dt} = -\nabla \rho - \alpha_2 \nabla \times \nabla \times \nu_2 \] (8)

The fluid velocity \((v_i(y, t))\) and micro-rotation vector in both regions\((i=I, II)\) are presumed as \((v_i = (v_i(y, t), 0, 0))\) and \((\omega = (0, 0, C_\nu(y, t)))\) respectively because fluids flow is unstable, incompressible and powered by a pressure gradient in the \(X\) direction also the character of fluid velocity is unidirectional. Hence the equations (6), (7) and (9) can be streamlined accordingly as

Region-I for Micro-polar fluid \((-k \leq y \leq 0)\)

\[ \chi_1 v_1_t = -\rho_x + 3C_\nu y + (\alpha_1 + \delta) v_{1yy} \] (10)

\[ \chi_1 \tau C_\nu_t = \lambda_1 C_{\nu yy} - \delta(2C_\nu + v_{1y}) \] (11)

Region-II for Newtonian fluid \((0 \leq y \leq k)\):

\[ \chi_2 \nu_2_t = \alpha_2 v_{2yy} - \rho_x \] (12)

The fluid layers are connected mechanically through the device by exchanging momentum. Impetus transition benefits from rate and shear pressure stability across the interface. But we presume the flow rates and the shear pressure is constant at the liquid-liquid interface. This can be written as follows in numerical terms.

Initial conditions: At \(t \leq 0\),

\[ v_1(y, t) = 0 \text{ for } -k \leq y \leq 0 \] (13)

\[ v_2(y, t) = 0 \text{ for } 0 \leq y \leq k \] (14)

\[ C_\nu(y, t) = 0 \text{ for } -k \leq y \leq 0 \] (15)

Boundary and interface conditions: At \(t > 0\),

\[ v_1(-k, t) = 0 \] (16)

\[ v_2(k, t) = 0 \] (17)

\[ C_\nu(-k, t) = 0 \] (18)

\[ v_1(0, t) = v_2(0, t) \] (19)

\[ C_\nu(0, t) = -\frac{1}{2} v_{1y} \] (20)

\[ \alpha_2 v_{2y} = (\alpha + \delta)v_{1y} + \delta C_\nu \text{ at } y = 0 \] (21)

Introducing the non-dimensional parameters

\[ x = \frac{x}{k}, y = \frac{y}{k}, \bar{v}_1 = \frac{v_1}{V}, \bar{v}_2 = \frac{v_2}{V}, \bar{C}_\nu = \frac{kC_\nu}{V}, \bar{\rho} = \frac{\rho}{\chi_1 V^2}, \bar{t} = \frac{tV}{k}, \]

After dropping the bars and following these assumptions \(\lambda_1 = (\alpha_1 + \delta/2)\tau\) with \(\tau = k^2\) and \(n_1 = \frac{\delta}{a_1}\)

\[ Re = \frac{\chi_1 V k}{a_1}, m_1 = \frac{\alpha_2}{a_1}, m_2 = \frac{\chi_2}{\chi_1} \]

the governing equations (10)–(12) take following form.

Region-I \((-k \leq y \leq 0)\) (Micro-polar fluid region):
\[ u_{1t} = -\frac{\rho_x}{Re x k} + \frac{n_1 C_y y}{Re x k} + (n_1 + 1) \frac{u_{1yy}}{Re x k} \]  
(22)

\[ C_{\tau t} = \frac{\frac{n_1}{Re x k}}{Re x k\ y_2} - \frac{n_1 (2C_x + u_y y)}{Re x v k^2} \]  
(23)

Region-II for Newtonian fluid (0 ≤ y ≤ k):

\[ u_{2t} = \frac{m_1 v k}{Re m_2} u_{2yy} - \frac{\rho_x}{Re m_2} \]  
(24)

Where \( V k = 1, Re = 1 \) and \( n_1 \) is the micro-polarity parameter, \( Re \) is the Reynolds number, \( V \) is the maximum velocity in the channel, \( m_1 \) is the ratio of viscosities and \( m_2 \) is the ratio of densities, \( -\rho_x = Ge(t) \) is the time dependent pressure gradient with \( t > 0 \). Three different cases for \( Ge(t) \) are considered to get numerical results.

Case-I: \( Ge(t) = Ge \) (constant pressure gradient)

Case-II: \( Ge(t) = Ge * Sin(wt) \) (periodic pressure gradient with pulsating parameter \( w \))

Case-III: \( Ge(t) = Ge * e^{-kt} \) (decaying pressure gradient with decaying parameter). The non-dimensional conditions, after dropping bars, become

Initial conditions: At \( t \leq 0 \),
\[ v_1(y, t) = 0 \]  
(25)
\[ v_2(y, t) = 0 \]  
(26)
\[ C_s(y, t) = 0 \]  
(27)

Boundary and interface conditions: At \( t > 0 \),
\[ v_1(-1, t) = 0 \]  
(28)
\[ v_2(1, t) = 0 \]  
(29)
\[ C_s(-1, t) = 0 \]  
(30)
\[ v_1(0, t) = v_2(0, t) \]  
(31)
\[ C_s(0, t) = -\frac{1}{2} u_{1y} \]  
(32)
\[ m_1 v_{2y} = (1 + n_1) v_{1y} + n_1 C_s \]  
(33)

3. Numerical solution

We first divide the domain [-1 1] into [-1 0] for micro-polar fluid (region I) and [0 1] for Newtonian fluid (region II) respectively. For phase length \( h \) in the y-direction and \( k \) in the time scales, both the domains are equally discretized. The nodes are expected to be spread equally.

\[ a = y_1 < y_2 < \cdots < y < x_n = b \], such that \( y_{i+1} - y_i = h \) on the real axis.  
(34)

After that the first and second order derivatives of \( v_1(y, t) \), \( v_2(y, t) \) and \( C_s(y, t) \) are obtained at any time on the nodes \( x_i \).

For \( 1, 2, 3, \ldots, n \).
\[ v_{1y}(y, t) = \sum_{j=1}^{N} a^i j v_1(y, t) \], for \( j = 1, 2, \ldots, N \)  
(35)
\[ v_{1yy}(y, t) = \sum_{j=1}^{N} b^i j v_1(y, t) \], for \( j = 1, 2, \ldots, N \)  
(36)
In this case, and measure first and second order derivatives coefficients with respect to \( y \), and obtained using the modified cubic B-spline functions. The cubic B-spline functions at the knots are described below.

Where \( \{ \} \) forms a basis over the region \([a, b]\).

The updated cubic B-spline functions are described in the nodes as follows.

| \( y_j \) | \( y_{j-1} \) | \( y_j \) | \( y_{j+1} \) | \( y_{j+2} \) |
|---|---|---|---|---|
| \( \vartheta_j(y) \) | 0 | 1 | 4 | 1 | 0 |
| \( \vartheta_j'(y) \) | 0 | \( \frac{3}{h} \) | 0 | \( -\frac{3}{h} \) | 0 |
| \( \vartheta_j''(y) \) | 0 | \( \frac{6}{h^2} \) | \( -\frac{12}{h^2} \) | \( -\frac{6}{h^2} \) | 0 |

Table 1: Value of spline functions at different knots.

Using the modified cubic B-spline function the weighting coefficients are calculated as follow.
The first order derivative approximation is given by
\[ \psi'_k(y_i) = \sum_{j=1}^{N} a^*_{ij} \theta_k(y_j) \quad \text{for } i = 1, 2, \ldots, N \]
\[ k = 1, \ldots, N \]  
(47)

The estimate can be provided for the first knot point \( y_1 \).
\[ \psi'_k(y_1) = \sum_{j=1}^{N} a^*_{1j} \theta_k(y_j) \quad \text{for } i = 1, 2, \ldots, N \]
\[ k = 1, \ldots, N \]  
(48)

Then the tri-diagonal system of equations is formed as
\[
\begin{bmatrix}
6 & 1 & 0 & 0 \\
0 & 4 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 4 & 1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 1 & 4 & 1 \\
0 & 0 & \cdots & 0 & 1 & 4 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 6 \\
\end{bmatrix}
\begin{bmatrix}
a^*_{11} \\
a^*_{12} \\
a^*_{13} \\
\vdots \\
a^*_{N-3} \\
a^*_{N-2} \\
a^*_{N-1} \\
a^*_{N} \\
\end{bmatrix}
= \begin{bmatrix}
-6/h \\
6/h \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Similarly for the point, \( y_2 \) we have
\[ \psi'_k(y_2) = \sum_{j=1}^{N} a^*_{2j} \theta_k(y_j) \quad \text{for } i = 1, 2, \ldots, N \]
\[ k = 1, \ldots, N \]  
(49)

Then again the tri-diagonal system of equations is formed as
\[
\begin{bmatrix}
6 & 1 & 0 & 0 \\
0 & 4 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 4 & 1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 1 & 4 & 1 \\
0 & 0 & \cdots & 0 & 1 & 4 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 6 \\
\end{bmatrix}
\begin{bmatrix}
a^*_{21} \\
a^*_{22} \\
a^*_{23} \\
\vdots \\
a^*_{2N-3} \\
a^*_{2N-2} \\
a^*_{2N-1} \\
a^*_{2N} \\
\end{bmatrix}
= \begin{bmatrix}
-3/h \\
0 \\
3/h \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

We continue to find the tri-diagonal systems for the remaining \( y_i \)'s and the system for last \( y_N \) is given by
\[
\begin{bmatrix}
6 & 1 & 0 & 0 \\
0 & 4 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 4 & 1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 1 & 4 & 1 \\
0 & 0 & \cdots & 0 & 1 & 4 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 6 \\
\end{bmatrix}
\begin{bmatrix}
a^*_{N1} \\
a^*_{N2} \\
a^*_{N3} \\
\vdots \\
a^*_{NN-3} \\
a^*_{NN-2} \\
a^*_{NN-1} \\
a^*_{NN} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
-6/h \\
\end{bmatrix}
\]
\]
The solution of the above systems provides the co-efficient $a \_{11}, a \_{12}, \ldots, a \_{1N}$, $a \_{21}, a \_{22}, \ldots, a \_{2N}$, $a \_{N1}, a \_{N2}, \ldots, a \_{NN}$. Then the value of $b \_{ij}$ for $i = 1, 2, 3 \ldots N, j = 1, 2, 3 \ldots N$ is calculated as follows:

$$b \_{ij} = \begin{cases} 2a \_{ij} \left( a \_{ij} - \frac{1}{y_i - y_j} \right) & \text{for } i \neq j \\ - \sum_{i=1, i \neq j}^{N} b \_{ij} & \text{for } i = j \end{cases}$$

(50)

Replacing the approximation of the spatial components of the first and second order obtained by using MCB-DQM, the Equations 22-24 could be updated as:

Region-I ($-k \leq y \leq 0$) (Micro-polar fluid region):

$$u_{1t} = Ge(t) + \frac{n_1}{Re} \left( \sum_{j=1}^{N} a \_{ij} C \_{r}(y_j, t) \right) + \frac{(n_1+1)}{Re} \left( \sum_{j=1}^{N} b \_{ij} u_1(y_j, t) \right)$$

(51)

$$C \_{s} = \frac{(n_1+2)}{2Re} \left( \sum_{j=1}^{N} b \_{ij} C \_{r}(y_j, t) \right) - \frac{n_1}{Re} \left( 2C \_{s} + \sum_{j=1}^{N} a \_{ij} u_1(y_j, t) \right)$$

(52)

Region-II for Newtonian fluid ($0 \leq y \leq k$):

$$u_{2t} = \frac{m_1}{Re m_2} \left( \sum_{j=1}^{N} b \_{ij} u_2(y_j, t) \right) + \frac{Ge(t)}{m_2}$$

(53)

Thus, Eq. 51, 52 and 53 are reduced into a system of ordinary differential equations in time, that is, for $i=1, 2, 3 \ldots, N$ and the system is solved by four stage order three SSP RK43 scheme. The velocities in both regions and micro-rotation are obtained as:

At first stage for $i=1, 2, 3 \ldots n$

Region-I ($-k \leq y \leq 0$) (Micro-polar fluid region):

$$u_{1t} = u_{10} + \frac{\Delta t}{2} \left( Ge(t) + \frac{n_1}{Re} \left( \sum_{j=1}^{N} a \_{ij} C \_{r}(y_j, t) \right) + \frac{(n_1+1)}{Re} \left( \sum_{j=1}^{N} b \_{ij} u_{10}(y_j, t) \right) \right)$$

(54)

$$C \_{s} = C \_{s0} + \frac{\Delta t}{2} \left( \frac{(n_1+2)}{2Re} \left( \sum_{j=1}^{N} b \_{ij} C \_{r}(y_j, t) \right) - \frac{n_1}{Re} \left( 2C \_{s0} + \sum_{j=1}^{N} a \_{ij} u_{10}(y_j, t) \right) \right)$$

(55)

Region-II for Newtonian fluid ($0 \leq y \leq k$):

$$u_{2t} = u_{20} + \frac{\Delta t}{2} \left( \frac{m_1}{Re m_2} \left( \sum_{j=1}^{N} b \_{ij} u_{20}(y_j, t) \right) + \frac{Ge(t)}{m_2} \right)$$

(56)

At first stage of the scheme, the initial and boundary conditions (equation 25-33) are considered accordingly.

At second stage for $i=1, 2, 3 \ldots n$

Region-I ($-k \leq y \leq 0$) (Micro-polar fluid region):

$$u_{1t} = u_{11} + \frac{\Delta t}{2} \left( Ge(t) + \frac{n_1}{Re} \left( \sum_{j=1}^{N} a \_{ij} C \_{r}(y_j, t) \right) + \frac{(n_1+1)}{Re} \left( \sum_{j=1}^{N} b \_{ij} u_{11}(y_j, t) \right) \right)$$

(57)

$$C \_{s} = C \_{s1} + \frac{\Delta t}{2} \left( \frac{(n_1+2)}{2Re} \left( \sum_{j=1}^{N} b \_{ij} C \_{r}(y_j, t) \right) - \frac{n_1}{Re} \left( 2C \_{s1} + \sum_{j=1}^{N} a \_{ij} u_{11}(y_j, t) \right) \right)$$

(58)

Region-II for Newtonian fluid ($0 \leq y \leq k$):
At second stage of the scheme, the initial and boundary conditions (equation 25-33) are considered accordingly.

At third stage for \( i=1,2,3\ldots,n \)

Region-I \((-k \leq y \leq 0)\) (Micro-polar fluid region):

\[
v_{13} = \frac{2u_{10}}{3} + \frac{v_{12}}{3} + \frac{\Delta t}{6} \left( Ge(t) + \frac{n_1}{Re} \left( \sum_{j=1}^{N} b_{*ij}v_{21}(y_j, t) \right) + \frac{(n_1+1)}{Re} \left( \sum_{j=1}^{N} b_{*ij}v_{11}(y_j, t) \right) \right)
\]  

(60)

\[
C_{*3} = \frac{2c_{*2}}{3} + \frac{c_{*2}}{3} + \frac{\Delta t}{6} \left( \frac{(n_1+2)}{Re} \left( \sum_{j=1}^{N} b_{*ij}C_{*2}(y_j, t) \right) - \frac{n_1}{Re} \left( 2C_{*2} + \sum_{j=1}^{N} a_{*ij}v_{12}(y_j, t) \right) \right)
\]  

(61)

Region-II for Newtonian fluid \((0 \leq y \leq k)\):

\[
v_{23} = \frac{2v_{20}}{3} + \frac{v_{22}}{3} + \frac{\Delta t}{6} \left( \frac{m_1}{Re \cdot m_2} \left( \sum_{j=1}^{N} b_{*ij}v_{21}(y_j, t) \right) + \frac{Ge(t)}{m_2} \right)
\]  

(62)

At third stage of the scheme, the initial and boundary conditions (equation 25-33) are considered accordingly.

At forth stage for \( i=1,2,3\ldots,n \)

Region-I \((-k \leq y \leq 0)\) (Micro-polar fluid region):

\[
v_{1} = v_{13} + \frac{\Delta t}{2} \left( Ge(t) + \frac{n_1}{Re} \left( \sum_{j=1}^{N} a_{*ij}C_{*3}(y_j, t) \right) + \frac{(n_1+1)}{Re} \left( \sum_{j=1}^{N} b_{*ij}v_{13}(y_j, t) \right) \right)
\]  

(63)

\[
C_{*} = C_{*3} + \frac{\Delta t}{2} \left( \frac{(n_1+2)}{2Re} \left( \sum_{j=1}^{N} b_{*ij}C_{*2}(y_j, t) \right) - \frac{n_1}{Re} \left( 2C_{*2} + \sum_{j=1}^{N} a_{*ij}v_{13}(y_j, t) \right) \right)
\]  

(64)

Region-II for Newtonian fluid \((0 \leq y \leq k)\):

\[
v_{2} = v_{23} + \frac{\Delta t}{2} \left( \frac{m_1}{Re \cdot m_2} \left( \sum_{j=1}^{N} b_{*ij}v_{22}(y_j, t) \right) + \frac{Ge(t)}{m_2} \right)
\]  

(65)

At forth stage of the scheme, the initial and boundary conditions (equation 25-33) are considered accordingly.

In non-dimensional form, for both the fluid velocities, the volume flow rate across the canal is calculated as

\[
Q^* = \int_{-1}^{0} v_1(y_j, t) \, dy + \int_{0}^{1} v_2(y_j, t) \, dy
\]  

(66)

4. Results and analysis

4.1. Case 1: Constant pressure gradient. The fluid velocity profiles for the constant pressure gradient at different times and different fluid parameters are demonstrated in figures 2-6, and it is found to be in parabolic nature. It is spotted from figure 2 and 3 that fluid velocity rise in both regions with increasing time and Ge(pressure gradients) value furthermore; variations in the Newtonian region of fluid are higher in term of magnitude than in the micro-polar region of fluid. The
behaviour of fluid velocity can be seen in figure 4 and also it is noticed that the velocities in both the regions increases the Reynolds number increases and this attainment is reduced near the interface. Figure 5 with varying parameter $n_1$ shows that, initially in the region-I, there is a very little decrease in the velocity and then it rises a bit near to interface. A slight increase in the velocity can be seen in the region-II as the micro-polar parameter $m_1$ grows and velocity reaches the steady state at greater value of $n_1$. The figure 6 shows that the fluid velocity declines significantly in region-II and declines slightly in region-I with an increase in the $m_1$. In region-II, the gain in velocity is reduced near the interface as the viscosity ratio increases hence variations in the Newtonian region of fluid are higher in term of magnitude than in the micro-polar region of fluid.

Similar kind of behaviour is observed in the figures 7, 8 and 11 for the micro rotation profile for micro-polar fluid with varying time and parameters $Ge$ and $m_1$. It increases as $Ge$, $m_1$, and time augments reach the steady state at greater values. Figure 9 with varying Reynolds number ($Re$) shows that, initially there is a little decrease in the micro rotation for micro-polar fluid and then it rise significantly as the parameter $Re$ grows. Figure 10 demonstrated the micro rotation profile for micro-polar fluid with varying parameter $n_1$ and it can be seen that the micro-rotation decreases as the $n_1$ augments.

![Figure 2: Velocity profile for constant pressure gradient Ge=10 with varying time when $Re = 2, n_1 = 0.5$ and $m_1 = 0.5$](image-link)
Figure 3: Velocity profile for constant pressure gradient with varying Ge when \( t = 0.2 \), \( Re2, n_1 = 0.5 \) and \( m_1 = 0.5 \).

Figure 4: Velocity profile for constant pressure gradient Ge=10 with varying Reynolds number \( Re \) when, \( t = 0.2, n_1 = 0.5 \) and \( m_1 = 0.5 \).
Figure 5: Velocity profile for constant pressure gradient $Ge=10$ with varying micro polar parameter $n_1$ When, $t = 0.2$, $Re = 2$ and $m_1 = 0.5$

Figure 6: Velocity profile for constant pressure gradient $Ge=10$ with varying ratio of viscosity $m_1$ When, $t = 0.2$, $n_1 = 0.5$ and $Re = 2$
Figure 7: Microrotation profile for constant pressure gradient Ge=10 with varying time

When $Re = 2, n_1 = 0.5$ and $m_1 = 0.5$

Figure 8: Microrotation profile for constant pressure gradient with varying Ge

When $t = 0.2, Re = 2, n_1 = 0.5$ and $m_1 = 0.5$
Figure 09: Microrotation profile for constant pressure gradient Ge=10 with varying Reynolds number Re When, $t = 0.2, n_1 = 0.5$ and $m_1 = 0.5$

Figure 10: Microrotation profile for constant pressure gradient Ge=10 with varying micro polar parameter $n_1$ When, $t = 0.2, Re = 2$ and $m_1 = 0.5$
4.2. Case II: Periodic pressure gradient. Figure 12 and 13 shows the velocity and micro-rotation profile when the periodic pressure gradient $Ge(t) = Ge \times \sin(\omega t)$ is applied with differing time. It is observed that the fluid velocity is oscillated in both the regions like it increases as $0 < \omega t \leq \pi$ and the velocity profile decreases after $\omega t > \pi$. Similarly the micro-rotation also swings with variation in time. The fluid velocity and the micro-rotation profiles for the periodic pressure gradient with different fluid parameters like Reynolds number $\text{Re}$ viscosity ratio $m_1$ and the micro-polar parameter $n_1$ are also examined. It is found that in both regions the obtained results with varying values of $\text{Re}$, $m_1$ and $n_1$ are exactly same as the velocity and micro-rotation profiles for constant pressure gradient case can be seen in the figure 4, 5, 6, 9, 10 and 11.

4.3. Case III: Decaying pressure gradient. The fluid velocity and micro rotation profiles are obtained for decaying pressure gradient $Ge(t) = Ge \times e^{-\lambda t}$ with varying time and different fluid parameters like Reynolds number $\text{Re}$ viscosity ratio $m_1$ and the micro-polar parameter $n_1$. It is observed that all the results with varying values of $\text{Re}$, $m_1$ and $n_1$ are exactly same as the velocity and micro-rotation profiles for constant pressure gradient case can be seen in the figure 4, 5, 6, 9, 10 and 11. It can be noticed from figure 14 that initially the fluid velocity increases in both the regions as $0 < t \leq 1$ and then it shows the significant declines after $t > 1$. The fluid velocity approaches to zero in both regions for a high value of time. Similarly the micro-rotation profile for decaying pressure gradient in region I can be seen in figure 15. It can be noticed that initially the micro-rotation increases in region-I as $0 < t \leq 1$ and then it shows the significant declines after $t > 1$. 

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Figure 11: Microrotation profile for constant pressure gradient $Ge=10$ with varying ratio of viscosity $m_1$ when, $t = 0.2, n_1 = 0.5$ and $Re = 2$
Figure 12: Velocity profile for periodic pressure gradient Ge=10 with varying time

When $Re = 2$, $n_1 = 0.5$ and $m_1 = 0.5$

Figure 13: Microrotation profile for periodic pressure gradient Ge=10 with varying time when $Re = 2$, $n_1 = 0.5$ and $m_1 = 0.5$
Figure 14: Velocity profile for decaying pressure gradient Ge=10 with varying time when, $Re = 2, n_1 = 0.5$ and $m_1 = 0.5$

Figure 15: Microrotation profile for constant decaying gradient Ge=10 with varying time when $Re = 2, n_1 = 0.5$ and $m_1 = 0.5$
To study the volume flow rate for different fluid parameters like pressure gradient $Ge$, Reynolds number $Re$, viscosity ratio $m_1$ and the micro-polar parameter $n_1$ the fluid velocities $v_1(y_j,t)$, $v_2(y_j,t)$ for both regions are computed numerically with DQM using B-Spline basis function and then numerical integration using equation 66 is obtained.

| GB-DQM  | Crank-  | CB-DQM  | Crank-  | CB-DQM  | Crank-  | CB-DQM  | Crank-  | CB-DQM  | Crank-  |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\Delta t = 10^3$ | Nicolson approach[25] | $\Delta t = 10^3$ | Nicolson approach[25] | $\Delta t = 10^3$ | Nicolson approach[25] | $\Delta t = 10^3$ | Nicolson approach[25] | $\Delta t = 10^3$ | Nicolson approach[25] | $\Delta t = 10^3$ |
| $Ge$ | $Q^*$ | $Ge$ | $Q^*$ | $Re$ | $Q^*$ | $Re$ | $Q^*$ | $m_1$ | $Q^*$ | $m_1$ | $Q^*$ | $n_1$ | $Q^*$ | $n_1$ | $Q^*$ |
| 5 | 2.124 | 2 | 2.46044 | 1 | 3.424 | 8 | 0.15 | 4.76089 | 4 | 0.3 | 7.079 | 7 | 0.1 | 4.15083 | 6 | 0.3 | 6.114 | 6 |
| 10 | 4.248 | 117 | 10 | 5.905 | 3 | 4.57228 | 7 | 2 | 5.905 | 12 | 0.30 | 4.49609 | 7 | 0.5 | 5.995 | 2 | 0.2 | 4.17971 | 81 |
| 15 | 6.372 | 626 | 15 | 8.857 | 4 | 4.76381 | 7 | 3 | 7.447 | 2 | 0.49 | 4.30279 | 3 | 0.7 | 5.197 | 0 | 0.3 | 4.20540 | 03 |
| 20 | 8.496 | 835 | 20 | 11.81 | 5 | 4.89505 | 1 | 4 | 8.463 | 7 | 0.66 | 4.15143 | 4 | 0.9 | 4.740 | 1 | 0.4 | 4.22815 | 54 |
| 0.5 | 4.2484 | 17 |

Table 2. Volume flow rate for constant pressure gradient for various values of $Ge$,

$Re$, $m_1$ and $n_1$ by cubic-B spline DQM and Crank-Nicolson approach

5. Conclusion

This paper provides numerical analysis of the unstable flow of two micro-polar and Newtonian fluids via a horizontal channel, using the differential quadrature method with cubic B-shaping basis function. In DQM, the four stage order three SSP RK43 scheme is used to get fluid velocity and micro-rotation. The fluid velocity and shear stress are believed to be continuous throughout the interface. The core effectuations on fluid velocity, micro-rotation and volume flow have been discussed by the corresponding fluid parameters. The behaviour of fluid velocity in both the regions and micro-rotation in region-I is sighted with respect to different applied pressure gradient. Hence it concluded that the fluid velocities in both regions and micro-rotation for constant pressure gradient are raised with enhancement of time. The impact of micro-polar parameter on fluid velocity in both regions, have seen while the micro polar fluid is only present in the lower region. The fluctuating behaviour is sighted of the velocities and micro-rotation with respect to increase in time for periodic pressure gradient. For the decaying pressure gradient, the declined nature is noted for fluid velocity with the intensified time. Enlarging the ratio of viscosity for all three applied pressure gradient, decline the velocities in both regions and let up the micro-rotation. The fluid velocities in both regions and micro-rotation in region-I enhanced with raising the Reynolds number and the pressure gradient.

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