A MAXIMAL FUNCTION CHARACTERIZATION OF ABSOLUTELY
CONTINUOUS MEASURES AND SOBOLEV FUNCTIONS

ELIA BRUÈ, QUOC-HUNG NGUYEN, AND GIORGIO STEFANI

Abstract. In this note, we give a new characterization of Sobolev \( W^{1,1} \) functions among \( BV \) functions via Hardy–Littlewood maximal function. Exploiting some ideas coming from the proof of this result, we are also able to give a new characterization of absolutely continuous measures via a weakened version of Hardy–Littlewood maximal function. Finally, we show that the approach adopted in [3, 8] to establish existence and uniqueness of regular Lagrangian flows associated to Sobolev vector fields cannot be further extended to the case of \( BV \) vector fields.

1. Introduction

Let \( \mu \) be a Borel measure in \( \mathbb{R}^d \). We let

\[
M\mu(x) := \sup_{r > 0} \frac{\mu(B(x, r))}{\mathcal{L}^d(B(x, r))} \in [0, +\infty]
\]

be the (Hardy–Littlewood) maximal function of \( \mu \) at \( x \in \mathbb{R}^d \), see [9, Chapter 1]. If the measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure with density \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \), then we can rewrite (1.1) as

\[
Mf(x) := \sup_{r > 0} \int_{B(x, r)} |f(y)| \, dy \in [0, +\infty]
\]

for all \( x \in \mathbb{R}^d \). It is well known that if \( f \in L^p(\mathbb{R}^d) \) for some \( 1 < p \leq +\infty \), then the maximal function in (1.2) satisfies the following strong \( (p, p) \)-type estimate

\[
\|Mf\|_{L^p(\mathbb{R}^d)} \lesssim_d \|f\|_{L^p(\mathbb{R}^d)}.
\]

Here and in the following, given two quantities \( A \) and \( B \), we write \( A \lesssim_d B \) (resp. \( A \gtrsim_d B \)) if there exists a dimensional constant \( C > 0 \) such that \( A \leq CB \) (resp. \( A \geq CB \)). If \( f \in L^1(\mathbb{R}^d) \), then the maximal function in (1.2) satisfies the following weak \((1, 1)\)-type estimate

\[
\sup_{\lambda > 0} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \lesssim_d \|f\|_{L^1(\mathbb{R}^d)}.
\]

Acknowledgements. The authors thank Professors Luigi Ambrosio and Giovanni Alberti for useful conversations about the subject. We also thank Professor Guido De Philippis for his comments on a preliminary version of the manuscript and for having pointed us the reference [6].

Date: December 28, 2021.

2010 Mathematics Subject Classification. 42B25, 46E27, 26A45.

Key words and phrases. Maximal functions, singular measures, \( BV \) and Sobolev functions, regular Lagrangian flows.
For a proof of inequalities (1.3) and (1.4), we refer the interested reader to [9, Theorem 1]. Actually, if \( f \in L^1(\mathbb{R}^d) \), writing \( f = f_\chi \{ |f| > \frac{\lambda}{d} \} + f_\chi \{ |f| \leq \frac{\lambda}{d} \} \) and combining (1.3) and (1.4), we can improve (1.4) as

\[
\lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : Mf(x) > \lambda \}) \lesssim_d \int_{|f| > \frac{\lambda}{d}} |f(y)| \, dy,
\]

so that

\[
\limsup_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : Mf(x) > \lambda \}) = 0.
\]

With a similar reasoning, we also get that

\[
\limsup_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : M\mu(x) > \lambda \}) \lesssim_d \mu^s(\mathbb{R}^d), \tag{1.5}
\]

for all finite Borel measures \( \mu \) (see [7, Section 2] for more details). Here and in the following, \( \mu^s \) denotes the singular part of the measure \( \mu \) with respect to the Lebesgue measure in \( \mathbb{R}^d \).

As remarked in [6, Problem 3.5], it is also possible to establish a reverse version of inequality (1.5), see Proposition 1.1 below.

**Proposition 1.1.** Let \( \mu \) be a finite Borel measure in \( \mathbb{R}^d \). Then

\[
\inf_{\lambda > 0} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : M\mu(x) > \lambda \}) \gtrsim_d |\mu^s(\mathbb{R}^d)|. \tag{1.6}
\]

More in general, given a cube \( Q \subset \mathbb{R}^d \), it holds

\[
\inf \left\{ \lambda \mathcal{L}^d(\{ x \in Q : M\mu(x) > \lambda \}) : \lambda > \frac{\mu(\{ x \in Q : M\mu(x) > \lambda \})}{\mathcal{L}^d(Q)} \right\} \gtrsim_d |\mu^s|(Q). \tag{1.7}
\]

For the reader’s convenience, we give a proof of Proposition 1.1 in Appendix A.

Combining inequalities (1.3) and (1.6), we immediately deduce the following characterization of absolutely continuous measures in \( \mathbb{R}^d \).

**Corollary 1.2.** Let \( \mu \) be a finite Borel measure in \( \mathbb{R}^d \). Then \( \mu \ll \mathcal{L}^d \) if and only if

\[
\lim_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : M\mu(x) > \lambda \}) = 0. \tag{1.8}
\]

Inspired by Corollary 1.2, we give a new characterization of Sobolev \( W^{1,1} \) among \( BV \) functions in term of the behaviour of a suitable maximal function. For \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \), we define

\[
Af(x) := \sup_{r > 0} \frac{1}{\int_{B(x,r)} |f - (f)_{x,r}|} \, dy \in [0, +\infty] \tag{1.9}
\]

for all \( x \in \mathbb{R}^d \), where \( (f)_{x,r} = \frac{1}{\int_{B(x,r)} f} \, dy \). Note that, by Poincaré’s inequality and by inequality (1.5) applied to the measure \( \mu = Df \), we have that

\[
\limsup_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : Af(x) > \lambda \}) \lesssim_d |D^s f|(\mathbb{R}^d).
\]

Our main result is the following, see Section 3 for the proof.

**Theorem 1.3.** Let \( f \in BV(\mathbb{R}^d) \). Then

\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : Af(x) > \lambda \}) \gtrsim_d |D^s f|(\mathbb{R}^d).
\]
In particular, \( f \in W^{1,1}(\mathbb{R}^d) \) if and only if
\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : Af(x) > \lambda \}) = 0.
\]

Exploiting some ideas coming from the proof of the aforementioned Theorem 1.3, we are able to improve Proposition 1.1 as follows. For a finite Borel measure \( \mu \) on \( \mathbb{R}^d \) (possibly with sign), we define
\[
\overline{M}_\mu(x) := \sup_{r > 0} \frac{|\mu(B(x, r))|}{\mathcal{L}^d(B(x, r))} \in [0, +\infty]
\]
for all \( x \in \mathbb{R}^d \). Note that the maximal function defined in (1.10) is weaker than the one recalled in (1.1), in the sense that \( \overline{M}_\mu(x) \leq M_\mu(x) \) for all \( x \in \mathbb{R}^d \). Then the following result holds, see Section 2 for the proof.

**Theorem 1.4.** Let \( \mu \) be a finite Borel measure on \( \mathbb{R}^d \) (possibly with sign). Then
\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : \overline{M}_\mu(x) > \lambda \}) \gtrsim d |\mu|(\mathbb{R}^d).
\]
Hence Corollary 1.2 still holds for any (possibly signed) finite Borel measure \( \mu \) in \( \mathbb{R}^d \) with (1.8) replaced by
\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{ x \in \mathbb{R}^d : \overline{M}_\mu(x) > \lambda \}) = 0.
\]

The last goal of this note — which was our starting motivation for the study of inequality (1.6) — comes from the theory of ordinary differential equations (ODEs) with weakly differentiable vector fields.

The study of this theory was started by DiPerna and Lions in their seminal paper [4], in which they proved existence and uniqueness of solutions of ODEs with Sobolev vector fields. The extension of the results obtained in [4] to vector fields with BV regularity was established by Ambrosio in the groundbreaking paper [1], where the notion of regular Lagrangian flow was introduced as a generalization of the classical definition of flow (see [3, Definition 1.1] for a precise definition). More in detail, the main result of [1] reads as follows. For a time \( T \in (0, +\infty] \), let \( b : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) be a bounded time-dependent vector field such that
\[
b \in L^1((0, T); BV(\mathbb{R}^d; \mathbb{R}^d)), \quad \text{div } b \in L^1((0, T); L^\infty(\mathbb{R}^d, \mathbb{R}^d)).
\]
Consider the associated Cauchy problem
\[
\begin{cases}
\frac{dX_t}{dt}(x) = b_t(X_t(x)), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
X_0(x) = x \in \mathbb{R}^d.
\end{cases}
\]
Then there exists a unique regular Lagrangian flow \( X : [0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) solving (1.13).

Both the results by DiPerna–Lions and Ambrosio rely on the so-called Eulerian approach, meaning that the problem (1.13) is studied indirectly via the closely linked transport equation. A more direct approach, the so-called Lagrangian approach, was proposed by Crippa and De Lellis in [3], where simple a priori estimates were exploited in order to get existence, uniqueness, compactness and even mild regularity properties of the regular Lagrangian flow associated to a Sobolev \( W^{1,p} \) vector field for every \( p > 1 \). Their approach
has been extended to the case $p = 1$ by Jabin in [8], where it was observed that if the quantity
\[
\mathcal{Q}(B; \delta) := |\log \delta|^{-1} \int_0^T \int_B \min\{\delta^{-1}, |Db_t|(x)\} \, dx \, dt
\]
(1.14) satisfies the decay property
\[
\limsup_{\delta \to 0^+} \mathcal{Q}(B; \delta) = 0
\]
(1.15) for all balls $B \subset \mathbb{R}^d$, then there exists a unique regular Lagrangian flow associated to $b$ (for a more detailed exposition of these results, see [7, Section 1]).

Using Proposition 1.1, we can prove that (1.15) holds true if and only if
\[
b \in L^1((0,T); W^{1,1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)),
\]
so that the approach by Crippa–De Lellis and Jabin cannot be further extended to the case of $BV$ vector fields. Our result reads as follows, see Section 4 for the proof.

**Proposition 1.5.** Let $b: (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a vector field satisfying (1.12). Then
\[
\int_0^T |D^s b_t|(B) \, dt \lesssim_d \liminf_{\delta \to 0^+} \mathcal{Q}(B; \delta) \leq \limsup_{\delta \to 0^+} \mathcal{Q}(B; \delta) \lesssim_d \int_0^T |D^s b_t|(B) \, dt
\]
(1.16) for all balls $B \subset \mathbb{R}^d$, where $\mathcal{Q}(B; \delta)$ is as in (1.11). In particular, the decay property (1.15) is satisfied if and only if $b \in L^1((0, T); W^{1,1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d))$.

## 2. PROOF OF THEOREM 1.4

In this section, we prove Theorem 1.4. Write $\mu = \eta|\mu|$, where $\eta: \mathbb{R}^d \to \mathbb{R}$ satisfies $|\eta(x)| = 1$ for $|\mu|$-a.e. $x \in \mathbb{R}^d$. For each $\varepsilon > 0$ let $\eta_\varepsilon \in \text{Lip}(\mathbb{R}^d)$ be a Lipschitz function with Lipschitz constant $C_\varepsilon > 0$ such that
\[
\int_{\mathbb{R}^d} |\eta(x) - \eta_\varepsilon(x)| \, d|\mu|(x) < \varepsilon.
\]
(2.1)

We claim that
\[
\frac{1}{\omega_d r^d} \int_{B(x,r)} \eta_\varepsilon \, d|\mu| + 2C_\varepsilon r |\mu|(x) \geq \frac{1}{\omega_d r^d} \int_{B(x,r)} |\eta_\varepsilon| \, d|\mu|,
\]
(2.2)
for all $x \in \mathbb{R}^d$ and all $r > 0$. Indeed, we can estimate
\[
\left| \int_{B(x,r)} \eta_\varepsilon(y) \, d|\mu|(y) \right| \geq \left| \int_{B(x,r)} \eta_\varepsilon(y) \, d|\mu|(y) - \int_{B(x,r)} |\eta_\varepsilon(y) - \eta_\varepsilon(x)| \, d|\mu|(y) \right|
\geq \int_{B(x,r)} |\eta_\varepsilon(x)| \, d|\mu|(y) - C_\varepsilon r \int_{B(x,r)} d|\mu|(y)
\geq \int_{B(x,r)} |\eta_\varepsilon(y)| \, d|\mu|(y) - \int_{B(x,r)} \|\eta_\varepsilon(y) - |\eta_\varepsilon(x)|\| \, d|\mu|(y) - C_\varepsilon r \int_{B(x,r)} d|\mu|(y)
\geq \int_{B(x,r)} |\eta_\varepsilon(y)| \, d|\mu|(y) - 2C_\varepsilon r \int_{B(x,r)} d|\mu|(y)
\]
for all $r > 0$, from which (2.2) follows.
Let us now consider the measures $\nu_\varepsilon := |\eta - \eta_\varepsilon| |\mu|$ for $\varepsilon > 0$. We claim that
\[
\frac{1}{\omega_d r^d} \left| \int_{B(x,r)} \eta \, d|\mu| \right| + 2C_\varepsilon r \mathcal{M}_\mu(x) + 2\mathcal{M}_\varepsilon(x) \geq \frac{1}{\omega_d r^d} \int_{B(x,r)} |\eta| \, d|\mu| \tag{2.3}
\]
for all $x \in \mathbb{R}^d$ and all $r > 0$. Indeed, assuming $\mathcal{M}_{\mu}(x) < +\infty$ and $\mathcal{M}_{\varepsilon}(x) < +\infty$ without loss of generality, by (2.2) we can estimate
\[
\frac{1}{\omega_d r^d} \left| \int_{B(x,r)} \eta \, d|\mu| \right| \geq \frac{1}{\omega_d r^d} \left| \int_{B(x,r)} \eta_\varepsilon \, d|\mu| \right| - \mathcal{M}_\varepsilon(x)
\]
\[
\geq \frac{1}{\omega_d r^d} \int_{B(x,r)} |\eta| \, d|\mu| - 2C_\varepsilon r \mathcal{M}_\mu(x) - \mathcal{M}_\varepsilon(x)
\]
\[
\geq \frac{1}{\omega_d r^d} \int_{B(x,r)} |\eta| \, d|\mu| - 2C_\varepsilon r \mathcal{M}_\mu(x) - 2\mathcal{M}_\varepsilon(x).
\]

For $\tau > 0$ define
\[
\mathcal{M}_\tau \mu(x) := \sup_{0 < r < \tau} \mathcal{L}^d(B(x,r)) \in [0, +\infty],
\tag{2.4}
\]
and similarly
\[
\mathcal{M}_\tau \mu(x) := \sup_{0 < r < \tau} \mathcal{L}^d(B(x,r)) \in [0, +\infty],
\]
for all $x \in \mathbb{R}^d$. Taking the supremum with respect to $r \in (0, \tau)$ in (2.3), we find
\[
\mathcal{M}_\tau \mu(x) + 2C_\varepsilon \tau \mathcal{M}_\mu(x) + 2\mathcal{M}_\varepsilon(x) \geq \mathcal{M}_\tau \mu(x),
\tag{2.5}
\]
for all $x \in \mathbb{R}^d$.

We claim that
\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\mu(x) > \lambda\}) + C_\varepsilon \tau |\mu^*|[(\mathbb{R}^d)] + \varepsilon \gtrsim_d |\mu^*|(\mathbb{R}^d),
\tag{2.6}
\]
for all $\varepsilon > 0$ and $\tau > 0$. Indeed, since $|\mu|(\mathbb{R}^d) < +\infty$, given $\tau > 0$, for all $\lambda > 0$ sufficiently large it holds
\[
\{x \in \mathbb{R}^d : \mathcal{M}_\tau \mu(x) > \lambda\} = \{x \in \mathbb{R}^d : \mathcal{M}_\mu(x) > \lambda\}.
\tag{2.7}
\]
Thus, on the one hand, by (1.3), we have
\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\tau \mu(x) > \lambda\})
\]
\[
= \liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\mu(x) > \lambda\}) \gtrsim_d |\mu^*|(\mathbb{R}^d).
\]
On the other hand, by (1.3), (2.1) and (2.5) we can estimate
\[
\liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\mu(x) > \lambda\})
\]
\[
\leq \liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\tau \mu(x) + 2C_\varepsilon \tau \mathcal{M}_\mu(x) + 2\mathcal{M}_\varepsilon(x) > \lambda\})
\]
\[
\lesssim_d \liminf_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\mu(x) > \lambda\})
\]
\[
+ C_\varepsilon \tau \limsup_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\mu(x) > \lambda\})
\]
\[
+ \limsup_{\lambda \to +\infty} \lambda \mathcal{L}^d(\{x \in \mathbb{R}^d : \mathcal{M}_\varepsilon(x) > \lambda\})
\]
\[
\leq \lim_{\lambda \to +\infty} \inf \lambda \mathcal{L}^{d}(\{x \in \mathbb{R}^d : \mathcal{M}^\ast \mu(x) > \lambda\}) + C_\varepsilon \tau |\mu^\ast|(\mathbb{R}^d) + \int_{\mathbb{R}^d} |\eta - \eta_\varepsilon| \ d|\mu|
\]
\[
\leq \lim_{\lambda \to +\infty} \inf \lambda \mathcal{L}^{d}(\{x \in \mathbb{R}^d : \mathcal{M}^\ast \mu(x) > \lambda\}) + C_\varepsilon \tau |\mu^\ast|(\mathbb{R}^d) + \varepsilon
\]
\[
= \lim_{\lambda \to +\infty} \inf \lambda \mathcal{L}^{d}(\{x \in \mathbb{R}^d : \mathcal{M} \mu(x) > \lambda\}) + C_\varepsilon \tau |\mu^\ast|(\mathbb{R}^d) + \varepsilon,
\]
for all \( \varepsilon > 0 \) and \( \tau > 0 \). Inequality (2.6) thus follows. Therefore, passing to the limit in (2.6) first as \( \tau \to 0 \) and then as \( \varepsilon \to 0 \), we get (1.11). This concludes the proof.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The idea of the proof is to estimate the quantity \( A_f \) defined in (1.9) from below with the integral average of \( |Df| \), in the spirit of the reverse Poincaré’s inequality. Obviously, it is not possible to get such an estimate for arbitrary \( BV \) functions (with a constant that does not depend on the function itself).

However, it is simple to see that a reverse Poincaré’s inequality is true for one-variable monotone functions, so that one would expect that a sort of reverse Poincaré’s inequality may hold for arbitrary \( BV \) functions if first one specifies a direction \( \nu \in \mathbb{R}^d, |\nu| = 1 \), and then adds a suitable correction term measuring how far is \( f \) from being dependent only on the direction \( \nu \) and monotone.

**Lemma 3.1.** There exist two dimensional constants \( C_1, C_2 > 0 \) such that

\[
\frac{1}{r} \int_{B(x,2r)} |f - (f)_{x,2r}| \ dy + \int_{B(x,2r)} (1 - \langle \nu, \eta \rangle) \ d|Df| \geq C_1 \int_{B(x,r)} |\nabla f| \ dy
\]

for all \( \nu \in \mathbb{R}^d \) with \( |\nu| = 1 \), \( x \in \mathbb{R}^d \), \( r > 0 \) and \( f \in BV(\mathbb{R}^d) \), where \( \eta : \mathbb{R}^d \to \mathbb{R}^d \) satisfies \( Df = \eta |Df| \) and \( |\eta| = 1 \) \( |Df| \)-a.e. in \( \mathbb{R}^d \).

**Proof.** We claim that there exist two dimensional constants \( C_1, C_2 > 0 \) such that

\[
\frac{1}{r} \int_{B(x,2r)} |f - (f)_{x,2r}| \ dy + \int_{B(x,2r)} (|\nabla f| - \langle \nu, \nabla f \rangle) \ dy \geq C_1 \int_{B(x,r)} |\nabla f| \ dy
\]

for all \( \nu \in \mathbb{R}^d \) with \( |\nu| = 1 \), \( x \in \mathbb{R}^d \), \( r > 0 \) and \( f \in C^\infty(\mathbb{R}^d) \). By a standard rescaling argument, we just need to prove that there exists a dimensional constant \( C > 0 \) such that

\[
\int_{2Q} |f - (f)_{2Q}| \ dx + \int_{2Q} (|\nabla f| - \langle e_1, \nabla f \rangle) \ dy \geq C \int_{Q} |\nabla f| \ dy
\]

for all \( f \in C^\infty(\mathbb{R}^d) \), where \( Q = [-1, 1]^d \) and \( (f)_{2Q} := \int_{2Q} f \ dy \). We prove (3.3) by contradiction. For all \( n \in \mathbb{N} \), assume there exists \( f_n \in C^\infty(\mathbb{R}^d) \) such that

\[
\int_{Q} |\nabla f_n| \ dy = 1, \quad (f_n)_{2Q} = 0, \quad \int_{2Q} |f_n| \ dy + \int_{2Q} (|\nabla f_n| - \langle e_1, \nabla f_n \rangle) \ dy < \frac{1}{n}.
\]

Then consider \( g_n : 2I \to \mathbb{R}, I = [-1, 1] \), defined as

\[
g_n(t) := \int_{[-1,1]^{d-1}} g_n(t, y') \ dy', \quad t \in 2I.
\]
For all $s \in [0, 1]$, we have
\[
\frac{1}{n} \geq \int_{2Q} (|\nabla f_n| - \langle e_1, \nabla f_n \rangle) \, dy \\
\geq \int_{1-s}^{1+s} \int_{[-1,1]^{d-1}} |\nabla f_n(y_1, y')| - \langle e_1, \nabla f_n(y_1, y') \rangle \, dy' \, dy_1
\]
\[
\geq 1 - \int_{1-s}^{1+s} g_n(y_1) \, dy_1 = 1 - (g_n(1 + s) - g_n(-1 - s)),
\]
so that
\[
g_n(1 + s) - g_n(-1 - s) \geq 1 - \frac{1}{n}, \quad n \in \mathbb{N}.
\]
This contradicts the fact that $g_n \to 0$ in $L^1(2I)$, since $\|g_n\|_{L^1(2I)} \leq \|f_n\|_{L^1(2Q)} \leq \frac{1}{n}$. This concludes the proof of (3.3) and thus inequality (3.2) follows.

We can now conclude the proof by a standard approximation argument. Given $f \in BV(\mathbb{R}^d)$, by [2, Theorem 3.9 and Proposition 3.13] we can find $f_n \in BV(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $L^1(\mathbb{R}^d)$, $\|\nabla f_n\|_{L^1(\mathbb{R}^d)} \to |Df|(\mathbb{R}^d)$ and $|\nabla f_n| \rightarrow^* |Df|$ in $\mathbb{R}^d$ as $n \to +\infty$. Therefore, inequality (3.1) follows by Reshetnyak’s continuity Theorem, see [2, Theorem 2.39].

**Remark 3.2.** We must have $C_2 > 1$ in Lemma 3.1 as the following example shows. For $n \in \mathbb{N}$, consider $f_n: I \to \mathbb{R}$, $I = [-1, 1]$, defined as
\[
f_n(x) = \begin{cases} 
x x + n - 1 & -1 \leq x < - \left(1 - \frac{1}{n}\right) \\
0 & - \left(1 - \frac{1}{n}\right) \leq x \leq 1 - \frac{1}{n} \\
x x + 1 - n & 1 - \frac{1}{n} < x \leq 1.
\end{cases}
\]
Then $(f_n) = 0$, $\|f_n\|_{L^1(I)} = \frac{1}{n}$ and $\|f'_n\|_{L^1(I)} = 2$, so that inequality (3.6) with $C_2 = 1$, $v = 1$, $x = 0$ and $r = 1$ would imply $C_1 \geq 2n$, a contradiction.

We will not apply Lemma 3.1 directly, but we will use the following easy consequence of it. There exist two dimensional constants $C_1, C_2 > 0$ such that
\[
\frac{1}{r} \int_{B(x,2r)} |f - (f)_{x,C_2r}| \, dy + 2 \int_{B(x,C_2r)} |\eta - v| \, d|Df| \geq C_1 \int_{B(x,r)} |Df| \tag{3.4}
\]
for all $v \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, $r > 0$ and $f \in BV(\mathbb{R}^d)$, where $\eta: \mathbb{R}^d \to \mathbb{R}^d$ is as in Lemma 3.1. The proof of (3.4) is immediate. Indeed, since we can assume $v \neq 0$ without loss of generality, we just need to notice that
\[
2|\eta - v| \geq 1 - \left\langle \frac{v}{|v|}, \eta \right\rangle \quad |Df|-a.e. \text{ in } \mathbb{R}^d,
\]
and apply Lemma 3.1 with $\nu = v/|v|$.

Having inequality (3.4) at our disposal, we are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3** Fix $f \in BV(\mathbb{R}^d)$ and write $Df = \eta \, |Df|$, where $\eta: \mathbb{R}^d \to \mathbb{R}^d$ is as in Lemma 3.1. For each $\varepsilon > 0$, let $\eta_\varepsilon \in \text{Lip}(\mathbb{R}^d)$ be a Lipschitz function with Lipschitz constant $C_3 > 0$ such that the measure $\nu_\varepsilon := |\eta - \eta_\varepsilon| \, |Df|$ satisfies
\[
\nu_\varepsilon(\mathbb{R}^d) = \int_{\mathbb{R}^d} |\eta - \eta_\varepsilon| \, d|Df| < \varepsilon. \tag{3.5}
\]
Now fix $x \in \mathbb{R}^d$. Applying Lemma 3.1 with $\nu = \eta_\epsilon(x)$, we get
\[
\frac{1}{r} \int_{B(x,C_2r)} |f(y) - (f)_x,_{C_2r}| \, dy + 2 \int_{B(x,C_2r)} |\eta(y) - \eta_\epsilon(x)| \, d|Df|(y) \geq C_1 \int_{B(x,r)} d|Df|(y).
\] Note that
\[
\int_{B(x,C_2r)} |\eta(y) - \eta_\epsilon(x)| \, d|Df|(y)
\leq \int_{B(x,C_2r)} |\eta(y) - \eta_\epsilon(x)| \, d|Df|(y) + \int_{B(x,C_2r)} |\eta_\epsilon(y) - \eta_\epsilon(x)| \, d|Df|(y)
\leq \int_{B(x,C_2r)} d\nu_\epsilon(y) + C_2C_\epsilon r \int_{B(x,C_2r)} d|Df|(y).
\] Combining (3.6) and (3.7), we get that
\[
Af(x) + M\nu_\epsilon(x) + \frac{C_\epsilon r}{\omega_d(C_2r)^d} \int_{B(x,C_2r)} d|Df| \geq \frac{1}{\omega_d r^d} \int_{B(x,r)} d|Df|,
\] for all $x \in \mathbb{R}^d$ and $r > 0$, where $Af$ is the function defined in (1.9).

Now fix $\tau > 0$. Taking the supremum for $r \in (0, \tau)$ in (3.8) and recalling the definition in (2.4), we get that
\[
Af(x) + M\nu_\epsilon(x) + C_\epsilon \tau M^{C_2\tau}(Df)(x) \gtrsim_d M^\tau(Df)(x).
\] Thus, by the observation made in (2.7), inequalities (3.5), (1.3) and Proposition 1.1 we conclude that
\[
|D^\epsilon f|(\mathbb{R}^d) \lesssim_d \liminf_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : M(Df)(x) > \lambda\})
= \liminf_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : M^\tau(Df)(x) > \lambda\})
\lesssim_d \liminf_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : Af(x) > \lambda\})
+ \limsup_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : \nu_\epsilon(x) > \lambda\})
+ C_\epsilon \tau \limsup_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : M^{C_2\tau}(Df)(x) > \lambda\})
\lesssim_d \liminf_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : Af(x) > \lambda\}) + \nu_\epsilon(\mathbb{R}^d) + C_\epsilon \tau |D^\epsilon f|(\mathbb{R}^d)
\lesssim_d \liminf_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : Af(x) > \lambda\}) + \epsilon + C_\epsilon \tau |D^\epsilon f|(\mathbb{R}^d),
\] for all $\epsilon > 0$ and $\tau > 0$. Passing to the limit first as $\tau \to 0$ and then as $\epsilon \to 0$, we get
\[
\liminf_{\lambda \to +\infty} \lambda^{d} L_d^d(\{x \in \mathbb{R}^d : Af(x) > \lambda\}) \gtrsim_d |D^\epsilon f|(\mathbb{R}^d).
\] This concludes the proof.
4. Proof of Proposition 1.5

In this section, we prove Proposition 1.5. Let \( b: (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) be a vector field satisfying (1.12). By [7, Remark 10], the quantity
\[
Q(B; \delta) = |\log \delta|^{-1} \int_0^T \int_B \min \{ \delta^{-1}, M|Db_t|(x) \} \, dx \, dt
\]
defined in (1.14) satisfies
\[
\limsup_{\delta \to 0^+} Q(B; \delta) \leq_d \int_0^T |D^s b_t|(B) \, dt
\]
for all balls \( B \subset \mathbb{R}^d \). This proves the second part of (1.16). To prove the first part of (1.16), fix a ball \( B = B_r \subset \mathbb{R}^d \) of radius \( r > 0 \). We claim that
\[
\liminf_{\delta \to 0^+} |\log \delta|^{-1} \int_B \min \{ \delta^{-1}, M|Db_t|(x) \} \, dx \geq_d |D^s b_t|(B_r)
\]
(4.1) holds for a.e. \( t \in (0, T) \), so that the conclusion follows by Fatou's Lemma. Indeed, for any \( \varepsilon \in (0, r/2) \) and for a.e. \( t \in (0, T) \), we can estimate
\[
\int_{B_r} \min \{ \delta^{-1}, M|Db_t|(x) \} \, dx \geq \int_{\delta^{-1/2}}^{\delta^{-1}} \mathcal{L}^d \left( \left\{ x \in B_r : M(1_{B_{r-\varepsilon}}|Db_t|(x) > \lambda) \right\} \right) \, d\lambda.
\]
Note that \( M(1_{B_{r-\varepsilon}}|Db_t|)(x) \leq_d |Db_t|(B_r)\varepsilon^{-d} \) for every \( x \in \mathbb{R}^d \setminus B_r \) and for a.e. \( t \in (0, T) \). Indeed, if \( |x| > r \) then \( B_{r-\varepsilon} \cap B(x, s) = \emptyset \) for all \( s \in [0, \varepsilon] \), so that
\[
M(1_{B_{r-\varepsilon}}|Db_t|)(x) = \sup_{s > 0} \frac{|Db_t|(B_{r-\varepsilon} \cap B(x, s))}{\mathcal{L}^d(B(x, \varepsilon))} \leq \frac{|Db_t|(B_{r-\varepsilon})}{\mathcal{L}^d(B(x, \varepsilon))} \leq_d |Db_t|(B_r)\varepsilon^{-d}.
\]
Now, using the decomposition
\[
\mathcal{L}^d \left( \left\{ x \in B_r : M(1_{B_{r-\varepsilon}}|Db_t|)(x) > \lambda \right\} \right) = \mathcal{L}^d \left( \left\{ x \in \mathbb{R}^d : M(1_{B_{r-\varepsilon}}|Db_t|)(x) > \lambda \right\} \right)
- \mathcal{L}^d \left( \left\{ x \in \mathbb{R}^d \setminus B_r : M(1_{B_{r-\varepsilon}}|Db_t|)(x) > \lambda \right\} \right)
\]
and Proposition 1.1 we get that
\[
\inf_{\lambda \in (\delta^{-1/2}, \delta^{-1})} \lambda \mathcal{L}^d \left( \left\{ x \in B_r : M(1_{B_{r-\varepsilon}}|Db_t|)(x) > \lambda \right\} \right) \gtrsim_d |D^s b_t|(B_{r-\varepsilon})
\]
for all \( \delta \leq_d |Db_t|(B_r)^{-1/2} \varepsilon^{d/2} \) and for a.e. \( t \in (0, T) \). Hence, for \( \delta > 0 \) sufficiently small, we obtain that
\[
|\log \delta|^{-1} \int_{B_r} \min \{ \delta^{-1}, M|Db_t|(x) \} \, dx \gtrsim_d |\log \delta|^{-1} \int_{\delta^{-1/2}}^{\delta^{-1}} |D^s b_t|(B_{r-\varepsilon}) \, d\lambda \gtrsim_d |D^s b_t|(B_{r-\varepsilon}).
\]
Therefore
\[
\liminf_{\delta \to 0^+} |\log \delta|^{-1} \int_B \min \{ \delta^{-1}, M|Db_t|(x) \} \, dx \gtrsim_d |D^s b_t|(B_{r-\varepsilon}),
\]
so that claim (1.14) follows by letting \( \varepsilon \to 0^+ \).
Appendix A. Proof of Proposition 1.1

In this section, we prove Proposition 1.1. The main ingredient of the argument is the following well-known reverse weak \((1,1)\)-type inequality for the maximal function in \((1.2)\). For the proof, which uses Calderon–Zygmund decomposition, we refer to \([9, \text{Chapter 1, Section 5}]\).

Lemma A.1. There exists a dimensional constant \(C > 0\) such that
\[
t \mathcal{L}^d \left( \{ x \in \mathbb{R}^d : Mf(x) > Ct \} \right) \gtrsim_d \int_{\{ f > t \}} f \ d\mathcal{L}^d \tag{A.1}
\]
for all \(t > 0\) and all non-negative \(f \in L^1(\mathbb{R}^d)\). More in general, there exists a dimensional constant \(C > 0\) such that
\[
t \mathcal{L}^d \left( \{ x \in Q : Mf(x) > Ct \} \right) \gtrsim_d \int_{\{ f > t \} \cap Q} f \ d\mathcal{L}^d \tag{A.2}
\]
for all cube \(Q \subset \mathbb{R}^d\), for all \(t > \int_Q f \ d\mathcal{L}^d\) and all non-negative \(f \in L^1_{\text{loc}}(\mathbb{R}^d)\).

We are now ready to prove Proposition 1.1.

Proof of Proposition 1.1. Without loss of generality, we can assume that \(\mu\) is non-negative and singular with respect to \(\mathcal{L}^d\).

For all \(\varepsilon > 0\) define \(f_{\varepsilon} \in L^1(\mathbb{R}^d)\) as
\[
f_{\varepsilon}(x) := \frac{\mu(B(x, \varepsilon))}{\mathcal{L}^d(B(x, \varepsilon))} = \frac{\mu(B(x, \varepsilon))}{\omega_d \varepsilon^d}, \quad x \in \mathbb{R}^d,
\]
where \(\omega_d = \mathcal{L}^d(B(0,1))\).

We claim that \((f_{\varepsilon})_{\varepsilon > 0}\) satisfies the following almost semigroup property: for all \(r > 0\) and \(x \in \mathbb{R}^d\), it holds
\[
(f_{\varepsilon})_{x,r} := \int_{B(x, r)} f_{\varepsilon} \ d y \lesssim_d f_{r+\varepsilon}(x). \tag{A.3}
\]
Indeed, fix \(x \in \mathbb{R}^d\) and \(r > 0\). By Tonelli’s Theorem, we can write
\[
(f_{\varepsilon})_{x,r} = \frac{1}{\omega_d \varepsilon^d \omega_d r^d} \int_{\mathbb{R}^d} \mathcal{L}^d(B(x, r) \cap B(y, \varepsilon)) \ d \mu(y).
\]
Since
\[
\mathcal{L}^d(B(x, r) \cap B(y, \varepsilon)) \leq 1_{B(x, r+\varepsilon)}(y) \min \{ \omega_d \varepsilon^d, \omega_d r^d \},
\]
f or all \(y \in \mathbb{R}^d\), we deduce that
\[
(f_{\varepsilon})_{x,r} \leq \frac{\mu(B(x, r+\varepsilon))}{\omega_d (r+\varepsilon)^d} \min \{ \omega_d \varepsilon^d, \omega_d r^d \} \lesssim_d f_{r+\varepsilon}(x).
\]
This concludes the proof of \((A.3)\).

Thanks to \((A.3)\), we easily get
\[
Mf_{\varepsilon}(x) \lesssim_d M\mu(x),
\]
for all \(x \in \mathbb{R}^d\). Thus, by Lemma A.1, we conclude that
\[
t \mathcal{L}^d \left( \{ x \in \mathbb{R}^d : M\mu(x) > Ct \} \right) \gtrsim_d \int_{\{ f_{\varepsilon} > t \}} f_{\varepsilon} \ d x, \tag{A.4}
\]
for all $t > 0$ and all $\varepsilon > 0$, where $C_d > 0$ is a dimensional constant.

We now claim that
\[
\limsup_{\varepsilon \to 0^+} \int_{\{f_\varepsilon > t\}} f_\varepsilon \, dx \gtrsim_d \mu(\mathbb{R}^d),
\]  
(A.5)
for all $t > 0$, so that (1.6) follows immediately combining (A.4) and (A.5). Indeed, by Tonelli’s Theorem we have
\[
\int_{\{f_\varepsilon > t\}} f_\varepsilon \, dx = \int_{\mathbb{R}^d} \frac{\mathcal{L}^d(\{f_\varepsilon > t\} \cap B(x, \varepsilon))}{\omega_d \varepsilon^d} \, d\mu(x),
\]  
(A.6)
for all $\varepsilon > 0$. Hence, by Fatou’s Lemma, we get that
\[
\limsup_{\varepsilon \to 0^+} \int_{\{f_\varepsilon > t\}} f_\varepsilon \, dx \geq \int_{\mathbb{R}^d} \liminf_{\varepsilon \to 0^+} \frac{\mathcal{L}^d(\{f_\varepsilon > t\} \cap B(x, \varepsilon))}{\omega_d \varepsilon^d} \, d\mu(x).
\]  
We now claim that
\[
\mathcal{L}^d(\{f_\varepsilon > t\} \cap B(x, \varepsilon)) \geq \frac{1}{2^d} \omega_d \varepsilon^d,
\]  
(A.7)
for $\mu$-a.e. $x \in \mathbb{R}^d$ and all $\varepsilon > 0$. To prove (A.7), we need to observe two preliminary facts.

First, notice that, given $\varepsilon > 0$ and $t > 0$, we have
\[
f_{\varepsilon/2}(x) > 2^d t \implies B(x, \varepsilon/2) \subset \{x \in \mathbb{R}^d : f_\varepsilon(x) > t\}.
\]  
(A.8)
Implication (A.8) follows from the trivial inclusion $B(x, \varepsilon/2) \subset B(y, \varepsilon)$ for all $y \in B(x, \varepsilon/2)$.

Second, notice that
\[
\lim_{\varepsilon \to 0^+} f_\varepsilon(x) = +\infty,
\]  
(A.9)
for $\mu$-a.e. $x \in \mathbb{R}^d$. Indeed, we have
\[
\left\{x \in \mathbb{R}^d : \liminf_{\varepsilon \to 0^+} f_\varepsilon(x) < +\infty\right\} \subset \bigcup_{n \in \mathbb{N}} A_n,
\]  
(A.10)
where
\[
A_n := \left\{x \in \mathbb{R}^d : \liminf_{\varepsilon \to 0^+} \frac{\mu(B(x, \varepsilon))}{\omega_d \varepsilon^d} \leq n\right\}.
\]
By a standard covering argument (for instance apply Vitali’s covering Lemma, see [9, Section 1.6]), one can prove that
\[
\mu(E) \leq n \mathcal{L}^d(E) \quad \text{for all Borel sets } E \subset A_n.
\]
Since $\mu$ is singular with respect to $\mathcal{L}^d$, we must have that $\mu(A_n) = 0$ for all $n \in \mathbb{N}$ and thus, by (A.10), we conclude that
\[
\mu \left( \left\{x \in \mathbb{R}^d : \liminf_{\varepsilon \to 0^+} f_\varepsilon(x) < +\infty \right\} \right) = 0.
\]
We can now prove (A.7). Fix $x \in \mathbb{R}^d$ such that (A.8) holds true. Then there exists $\varepsilon_x > 0$ such that $f_{\varepsilon/2}(x) > 2^d t$ for all $\varepsilon < \varepsilon_x$. Hence $B(x, \varepsilon/2) \subset \{f_\varepsilon > t\}$ and so
\[
\mathcal{L}^d(\{\mu_\varepsilon > t\} \cap B(x, \varepsilon)) \geq \frac{1}{2^d} \omega_d \varepsilon^d
\]  
for all $\varepsilon < \varepsilon_x$. Thus (A.7) follows and the proof of (1.6) is complete.
The proof of the local inequality (1.7) similarly follows from (A.2) and is left to the reader. □

REFERENCES

[1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Invent. Math. 158 (2004), no. 2, 227–260.
[2] L. Ambrosio, N. Fusco, and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[3] G. Crippa and C. De Lellis, Estimates and regularity results for the DiPerna-Lions flow, J. Reine Angew. Math. 616 (2008), 15–46.
[4] R. J. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989), no. 3, 511–547.
[5] L. C. Evans, Partial differential equations, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.
[6] C. Muscalu and W. Schlag, Classical and multilinear harmonic analysis. Vol. I, Cambridge Studies in Advanced Mathematics, vol. 137, Cambridge University Press, Cambridge, 2013.
[7] Q. H. Nguyen, Quantitative estimates for regular Lagrangian flows with BV vector fields (2018), preprint, available at https://arxiv.org/abs/1805.01182.
[8] P.-E. Jabin, Differential equations with singular fields, J. Math. Pures Appl. (9) 94 (2010), no. 6, 597–621.
[9] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

Scuola Normale Superiore, Piazza Cavalieri 7, 56126 Pisa, Italy
E-mail address: elia.brue@sns.it

Centro di Ricerca Matematica “Ennio De Giorgi”, Piazza Cavalieri 3, 56126 Pisa, Italy
E-mail address: quochung.nguyen@sns.it

Scuola Normale Superiore, Piazza Cavalieri 7, 56126 Pisa, Italy
E-mail address: giorgio.stefani@sns.it