Balance of angular moments in structural mechanics of finite strains: A condition for second spatial derivatives of stress

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Abstract. The conservation laws of linear and angular momentum are among the most fundamental principles of physics. In contemporary science, it is generally postulated that both principles apply to continuum mechanics of deformable solids. Nevertheless, there occasionally occurs the question whether it is necessary to formulate both conservation laws as independent principles. For the case of a statistical theory considering a large ensemble of point masses, the Noether theorem tells that conservation of linear and angular momentum follow automatically from the existence of a Lagrange function for the energy, which is invariant with respect to translation and rotation of the reference system. If one, however, formulates equilibrium of stresses within continuum theory of solids, then it is necessary to treat the balance of angular moments as independent principle. Usually, the symmetry of the Cauchy stress tensor is derived from this postulate. In the present study, it is shown that another condition regarding the second spatial derivatives of stress may be obtained from this principle. It is suggested to use this condition in context with the theory of finite strains.

1. Introduction

Nowadays, the conservation law for angular momentum is generally postulated in mechanical sciences. However, the question whether this principle may be deduced from Newton’s axioms of mechanics is by no means trivial [1]. In this context, a fundamental aspect has been emphasized by Poisson [2], who recognized that a mathematical proof for conservation of angular momentum may be given, if the forces in the system consist of pairwise equilibrated, central forces. In fact, the condition of pairwise central forces is satisfied by electrostatic interaction, and therefore this conservation law is naturally applied to mechanics of solids. Nevertheless, there still remained the question whether it is necessary to consider conservation of angular momentum as independent principle of continuum mechanics. Historically, it has for long been believed that it is sufficient to postulate conservation of linear momentum, until Euler [3] discovered that for rigid bodies conservation of angular momentum must be postulated independently. Thereafter, this conservation law has been extended to all kinds of deformable bodies through the principle of rigidification [4]. If one assumes by hypothesis that the motions of material points in a deformable body are frozen, then the body should behave like a rigid solid.

Next, we come to the question how conservation of linear and angular momentum is considered in the description of stresses. In this context, Euler proposed the principle of a free body diagram in order to visualize the applied forces. A volume element is cut free from the rest of the body, and the forces
along the boundary represent the interaction between the volume element and the remaining part, schematically. Thus, one may integrate over the forces along the boundary line. Consequently, the forces leading to translation of the object must in sum be zero for equilibrium states. An equivalent statement holds for rotational forces. In sum the torques about the centre point of the volume element must vanish. On the basis of this argument, one usually proves the symmetry of the Cauchy stress tensor. However, it has been mentioned by Atanackovic and Guran [5] that one could also derive conditions where stress components depend on higher order gradients of the displacement vector. Unfortunately, they did not work out further details regarding their assessment. Instead, it has just been said that classical elasticity theory is not concerned with such cases. Nevertheless, their statement seems to be an interesting starting point for the investigations of our present study.

Here, it should also be mentioned that similar considerations regarding conservation of angular momentum have been made in the frame of Cosserat theory [6]. Within this approach orientational degrees of freedom are considered in addition to the displacements of material points. Furthermore, related theories of generalized continuum mechanics were established, including micropolar elasticity [7, 8] and couple stress theory [9, 10]. These theories have in common that the elastic response of the described materials deviates significantly from Hooke’s law. But in the present study, we will restrict ourselves to deformable materials, where the elastic response is characterized by the Cauchy stress.

2. Equilibrium states of stress
In this section, the equilibrium conditions of stresses are characterized. At first equations are reemphasized, which represent the modern State of the Art. Thereafter, it is investigated whether further relations may be derived from the postulated balance of angular moments.

2.1. State of the Art
The simplest stress distributions are obtained for homogeneously strained samples consisting of one linear elastic material. In this case, the stresses are homogeneously distributed, as depicted in Fig. 1, schematically. For the moment, the investigation is restricted to 2-dimensional geometries.

![Figure 1](image)

**Figure 1.** Free body diagram of a homogeneously strained plane volume element. The stresses are decomposed into normal stresses (left) and shear stresses (right).

Usually, stresses are decomposed into normal stresses and shear stresses as shown in Figure 1, schematically. For better clarity, the volume element is depicted twice in order to separate the contributions of different components of the stress tensor. It is obvious that the forces acting along the boundaries of the volume element do not introduce translations, because they are in sum compensating each other. However, one could imagine a resulting torque, if the shears \( \tau_{xy} \) and \( \tau_{yx} \) were not equal. Consequently,

\[
\tau_{xy} = \tau_{yx}
\]
In other words, the Cauchy stress tensor is symmetric.

The situation gets more complex, when samples are inhomogeneously strained. Here, we consider the general case where also body forces \( \vec{f} \) might occur. Let us further assume that the first derivatives of stress components are constant across the volume element. The free body diagram of this case is depicted in Figure 2, where again contributions of normal stresses and shear stresses were separated.

![Figure 2. Free body diagram of a volume element showing the first derivatives of stresses. Normal stresses are depicted on the left while shear stresses are depicted on the right side, respectively.](image)

Here, it is necessary to investigate forces which might introduce a translation of the volume element. For every direction of the coordinate system, one equilibrium condition is derived. Thereby, the loads carried by normal and shear stresses must be added. Thus, the equilibrium conditions read as

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0 \tag{2a}
\]

\[
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + f_y = 0 \tag{2b}
\]

This equilibrium condition, in its 3-dimensional representation, was first suggested by Cauchy [11]. While condition (1) was motivated by conservation of angular momentum, condition (2) is due to conservation of linear momentum. In literature, the conditions (1) and (2 a, b) have been reformulated for various representations of continuum mechanics [12]. They are commonly used for elastic and elasto-plastic materials.

2.2. Further examination related to the balance of angular moment

Figure 2 demonstrates the equilibrium of linear moments. But what can be said about the angular moment related to cases, where the first derivatives of stress are considered? Obviously, the first derivatives of normal stresses do not contribute to the torque around the center point. Let us therefore look at the contribution of shear stresses. At first we treat the case where \( \frac{\partial \tau_{xy}}{\partial x} = \text{const} > 0 \), and \( \frac{\partial \tau_{xy}}{\partial y} = 0 \). Further, the shear stress at the point \((x_0, y_0)\) is called \( \tau_0 \). This condition is illustrated in the incomplete free body diagram on the left side of Figure 3. There, we have started to draw a distribution of surface tractions in vertical directions, which are in agreement with our assumptions. Since the Cauchy stress tensor is symmetric, there is only one possibility how to continue the diagram at the corners. Finally, the unique solution for the complete free body diagram is shown in Figure 3 on the right side.
Figure 3. Free body diagram related to $\frac{\partial \tau_{xy}}{\partial x} = \text{const.} > 0$. The first step of the construction is depicted on the left side. The complete diagram is shown on the right side. The sum of torques is zero.

Integration of contributions to the angular moment along the boundary line yields

$$\oint_F \vec{r} \times \vec{F} \, ds = 0,$$

where $\vec{F} \, ds$ is the force increment related to the shear $\tau_{xy}$, and $\vec{r}$ is the vector from a point at the boundary of the volume element to its centre. The sum of torques would not change, if $\frac{\partial \tau_{xy}}{\partial y}$ was also unequal to zero. Hence, the first derivatives of stress do not contribute to the angular moment.

2.3. Conditions derived for higher derivatives of stress

The balance equations including the first derivatives of stress are, however, not sufficient to guarantee balance of angular moments for arbitrary stress distributions. Historically, the application of conservation laws to higher derivatives of stress was not included in the principle of rigidification, because such methods were not needed for the description of rigid bodies either. On the other hand, conditions for the second derivatives of stress were later derived from the compatibility condition of neighbouring volume elements within the linearized theory of elasticity. But this approach uses simplifications, which are not essentially exact. Unfortunately, the linearized strain tensor does not fulfill the requirements of an objective tensor. Therefore, the present study is devoted to the purpose of evaluating conditions, which are in accordance with the principle of material frame indiffERENCE. The results obtained from our new considerations are to be used in combination with the theory of finite strains. A comparison with the simplified conditions of linear elasticity will be given in section 3.

Let us therefore continue with the case, where second derivatives of stress have constant values within the given volume element. First derivatives or constant stress values don’t need to be considered here, since their contributions were already covered in previous steps of our investigation. It is easily found that components of the form $\frac{\partial^2 \sigma_{xx}}{\partial x^2}$ or $\frac{\partial^2 \sigma_{yy}}{\partial y^2}$ do not introduce a torque. However, the derivatives $\frac{\partial^2 \sigma_{xx}}{\partial x \partial y}$ and $\frac{\partial^2 \sigma_{yy}}{\partial x \partial y}$ produce a torque, as illustrated in Figure 4, schematically:
Figure 4. The second derivatives of normal stress components \( \frac{\partial^2 \sigma_{xx}}{\partial x \partial y} \) and \( \frac{\partial^2 \sigma_{yy}}{\partial x \partial y} \) produce a torque, as illustrated in Figure 4 (a) and (b), respectively.

Now, it is necessary to collect all the contributions to the torque caused by the second derivatives of stress in order to require that their sum must be zero. In this context, the equilibrium conditions (2 a, b) must also be considered. In conclusion, one obtains

\[
\frac{\partial^2 \sigma_{xx}}{\partial x \partial y} = \frac{\partial^2 \sigma_{yy}}{\partial x \partial y},
\tag{4}
\]

or equivalently

\[
\frac{\partial^2 \tau_{xy}}{\partial x^2} = \frac{\partial^2 \tau_{xy}}{\partial y^2}.\tag{5}
\]

Here, it should be noticed that equation (4) implies equation (5) and vice versa, if the equations (2 a, b) are valid. Figure 5 demonstrates that the torques caused by \( \frac{\partial^2 \tau_{xy}}{\partial x^2} \) and \( \frac{\partial^2 \tau_{xy}}{\partial y^2} \) point along opposite directions. Consequently, the balance equations (4) and (5) are fully justified.

Figure 5. The torques caused by \( \frac{\partial^2 \tau_{xy}}{\partial x^2} \) and \( \frac{\partial^2 \tau_{xy}}{\partial y^2} \) have opposite sense of direction.
For symmetry reasons, $\partial^2 \tau_{xy} / \partial x \partial y$ cannot introduce a torque, because both coordinate directions are equally loaded. This assessment completes the contributions related to the second derivatives of stress.

Thus, we come to the question whether consideration of higher order derivatives leads to further relations. In principle, higher order derivatives of stress may contribute to a torque. But the equilibrium equations obtained in this way are not independent from those fundamental equations (2 a, b) and (4), which have already been established. Finally, there occurs the question, whether a combination of equations (1) and (4) could lead to a relaxation of the conditions in the sense that only the sum of all torques must vanish. But the condition for the symmetry of the Cauchy stress cannot be ruled out by a condition for second derivatives of stress, because the former condition is derived from a smaller length scale. If the volume element is small enough, then a hypothetical asymmetry of Cauchy stress would outmatch the effect related to second order derivatives. On the other hand, the conditions for second order derivatives of stress are not unnecessary either, because conservation of angular momentum is generally valid for infinitesimal and for finite volumes. In consequence, the equations (1), (2 a, b) and (4) represent an irreducible and complete set of equilibrium conditions.

2.4. Three dimensional representation of balance conditions

For sake of simplicity, we have so far restricted ourselves to the two dimensional case. Hereafter, the results for the three dimensional representation are summarized: The symmetry of the Cauchy stress writes as

$$\tau_{xy} = \tau_{yx}, \quad (6 \ a) \quad \tau_{yz} = \tau_{zy}, \quad (6 \ b) \quad \text{and} \quad \tau_{xz} = \tau_{zx}. \quad (6 \ c)$$

The ordinary equilibrium conditions read as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \quad (7a)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \quad (7b)$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + f_z = 0 \quad (7c)$$

Equivalently, one may express this condition in the form

$$\text{div}(\sigma) + \tilde{f}_b = 0, \quad (8)$$

where $\tilde{f}_b$ is the vector of body force density. Finally, the balance equations for the second order derivatives of the stress tensor are

$$\frac{\partial^2 \sigma_{xx}}{\partial x \partial y} = \frac{\partial^2 \sigma_{yy}}{\partial x \partial y} \quad (9a)$$

$$\frac{\partial^2 \sigma_{yy}}{\partial y \partial z} = \frac{\partial^2 \sigma_{zz}}{\partial y \partial z} \quad (9b)$$

$$\frac{\partial^2 \sigma_{zz}}{\partial z \partial x} = \frac{\partial^2 \sigma_{xx}}{\partial z \partial x} \quad (9c)$$
Further relations may be derived from combinations the equations given here.

3. Comparison with the linearized theory of elasticity

Even though the equations (4 a, b) and (9 a-c) were derived from a fundamental conservation law, one cannot combine these results with existing theories in arbitrary style. In particular, the linearized theory of elasticity incorporates relations for the second derivatives of stress, which do in general not agree with our new results. In order to clarify this issue, the Beltrami-Michell equations [13, 14] are briefly reemphasized here. Linear elasticity is based on the linearized strain tensor $\varepsilon_{ij}$ defined as

$$
\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z},
$$

$$
\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \quad \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)
$$

(10 a-f)

where $u$, $v$ and $w$ are the displacements in $x$-, $y$- and $z$-direction, respectively. Owing to the rules of partial derivatives, the second derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial z}$ of any differentiable function are equivalent.

Consequently, one derives the Saint-Venant compatibility equations, among which we here mention

$$
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial y \partial x}
$$

(11 a)

$$
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}
$$

(11 b)

$$
\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}
$$

(11 c)

Further, Hooke’s law for isotropic linear elastic materials writes as

$$
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{pmatrix}
= E \begin{pmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & 1-2\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 1-2\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 1-2\nu
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{xy} \\
\varepsilon_{yz} \\
\varepsilon_{xz}
\end{pmatrix},
$$

(12)

where $E$ is the Young’s modulus and $\nu$ is the Poisson ratio. Now, combination of (11 a-c) and (12) implies the Beltrami-Michell equations, which write as [14]

$$
\sigma_{ij, kk} + \frac{1}{1+\nu} \sigma_{kk, ij} = 0
$$

(13)

in index notation. Equation (13) represents a set of six elementary equations. According to this notation, summation must be carried out over repeated indices. Obviously, these conditions are not identical with our equations (9 a-c), although they all comprise the second derivatives of stress. For the
moment, it is unclear whether some equivalence between (9) and (13) can be found. Therefore, this fundamental question is here discussed for the case of an elementary example:

In this context, we choose the Kirsch problem [14, 15], which has analytically been solved within linearized elasticity. For this example of a hole in a uniaxial stress field, stress distributions are known, which fulfill the biharmonic equation [14]. Consequently, the Beltrami-Michell equations are also satisfied by this solution. However, a cross-check shows that these classical solutions from literature are not in agreement with the novel equations (4 a, b) proposed in the present article. Therefore, the equations (4) and (9) on the one hand, and equation (13) on the other are not equivalent. In the following section, we give an explanation for this discrepancy.

4. Material frame indifference

Material frame indifference means that a material law should be formulated in such a way where the response of the material is independent of an observer’s position in space and time [16, 17]. In the case of static equilibrium this principle usually reduces to Euclidean objectivity. When material properties are characterized by tensors, then the tensors should transform like objective tensors during rotation of the coordinate system. This requirement is nowadays widely used as prerequisite for the objective formulation of material models. The linearized theory of elasticity is, however, not in perfect accordance with this principle. Unfortunately, the linearized strain tensor defined in equation (10) is not an objective tensor. Instead, objective strain tensors may be derived from the Lagrangian representation of continuum mechanics. This picture of continuum mechanics is based on the deformation gradient tensor

$$F_{ij} = \frac{\partial X_i}{\partial x_j}$$

(14)

where $X_i$ and $x_j$ are the coordinates of the deformed and the undeformed system, respectively. Owing to the theorem of polar decomposition, the deformation gradient

$$F = U \cdot R$$

(15)

may be decomposed into a symmetric stretch tensor $U$ and a matrix $R$ describing rigid body rotation. The stretch tensor may be obtained from the Cauchy-Green tensor

$$C = F^T \cdot F$$

(16)

through the equation

$$U^2 = C$$

(17)

The matrices $U$ and $C$ are both symmetric, and they share their principal axes. Now, there are several possibilities how a finite strain tensor may be related to the stretch tensor $U$. In the present article, we like to choose that strain tensor, which shows the best affinity to the linearized strain. For this purpose, we use the Biot strain tensor [18, 19]

$$E^{\text{Biot}} = U - I$$

(18)

Thus, one may reformulate generalized Hooke’s law.
in material frame indifferent style. This objective version of Hooke’s law may now be combined with the exact stress conditions (9a-c), whereas the formulation of equation (12) should be related to the condition (13).

5. Energy formulation

Instead of using a set of stress conditions, one may formulate the same theory with use of an expression for the internal elastic energy. With respect to the Nöther theorem [20, 21], a system of point masses will automatically preserve momentum and angular momentum, if the system can be described by a Lagrange function of the energy, which is invariant with respect to translation and rotation of the coordinate system. On the other hand, it has in numerous cases been demonstrated that continuum models may be described by discrete systems of material points [22]. In consequence, one may expect that a material model described through an expression for the internal energy will automatically satisfy the fundamental conservation laws of linear and angular momentum, if the required invariance conditions for the energy are satisfied. In this context, equilibrium states are defined as minima of energy. In conclusion, we formulate the internal energy of an elastic material

\[
W = \frac{1}{2} \int_V C^* \left( \mathbf{E}^{\text{Biot}} \right)^2 dV
\]

as quadratic form of an objective strain tensor, where integration is carried out over the deformed volume. Here, \(C^*\) represents the elastic tensor of the material. Such energy formulations are well known in the field of nonlinear elasticity [23]. In the present article we claim that this description automatically satisfies the novel stress conditions (9a-c).

6. Summary and conclusions

The conservation laws of momentum and angular momentum are among the most fundamental principles of physical sciences. However, simplifications of mechanical theories can lead to unexpected inaccuracies. In particular, the linearized theory of elasticity is not in perfect accordance with the principle of material frame indifference, because the linearized strain is not an objective tensor. In consequence, one derives compatibility conditions, which are not essentially exact. Those compatibility conditions imply conditions for second derivatives of stress which show the same inaccuracy. On the other hand, precise conditions for the second spatial derivatives of stress in equilibrium states may be obtained directly from a postulated balance of angular moments. The results for the stress conditions may be combined with material laws satisfying the principle of material objectivity. In the present study, we have reformulated generalized Hooke’s law in material frame indifferent style. Consequently, the novel stress conditions can be used in combination with this material model. However, the results obtained here are applicable in a much wider sense. The conditions leading to balance of angular moments may for instance be applied to elasto-plastic materials. But we have excluded materials like Cosserat solids, which are governed by elastic interactions deviating from the Cauchy stress. Nevertheless, the novel results are valid for a large class of materials. Finally, the energy formulation, which is in some sense related to the Nöther theorem, points the way how the novel stress conditions could easily be implemented by analogy to existing numerical algorithms.
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