DYNAMICAL CANONICAL HEIGHTS FOR JORDAN BLOCKS, ARITHMETIC DEGREES OF ORBITS, AND NEF CANONICAL HEIGHTS ON ABELIAN VARIETIES

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Abstract. Let $f : X \to X$ be an endomorphism of a normal projective variety defined over a global field $K$. We prove that for every $x \in X(\bar{K})$, the arithmetic degree $\alpha_f(x) = \lim_{n \to \infty} h_X(f^n(x))^{1/n}$ of $x$ exists, is an algebraic integer, and takes on only finitely many values as $x$ varies over $X(\bar{K})$. Further, if $X$ is an abelian variety defined over a number field, $f$ is an isogeny, and $x \in X(\bar{K})$ is a point whose $f$-orbit is Zariski dense in $X$, then $\alpha_f(x)$ is equal to the dynamical degree of $f$. The proofs rely on two results of independent interest.

First, if $D_0, D_1, \ldots \in \text{Div}(X) \otimes \mathbb{C}$ form a Jordan block with eigenvalue $\lambda$ for the action of $f^\ast$ on $\text{Pic}(X) \otimes \mathbb{C}$, then we construct associated canonical height functions $\hat{h}_{D_k}$ satisfying Jordan transformation formulas $\hat{h}_{D_k} \circ f = \lambda \hat{h}_{D_k} + \hat{h}_{D_{k-1}}$. Second, if $A/\mathbb{Q}$ is an abelian variety and $\hat{h}_D$ is the canonical height on $A$ associated to a nonzero nef divisor $D$, then there is a unique abelian subvariety $B_D \subseteq A$ such that $\hat{h}_D(P) = 0$ if and only if $P \in B_D(\mathbb{Q}) + A(\mathbb{Q})_{\text{tors}}$.

Introduction

Let $A/\mathbb{Q}$ be an abelian variety defined over a number field, and let $D$ be a symmetric divisor on $A$. The canonical height associated to $D$ is a quadratic form

$$\hat{h}_D : A(\mathbb{Q}) \to \mathbb{R}$$

that is of fundamental importance in studying the arithmetic geometry of $A$. Among its useful properties is the fact that if $D$ is ample, then

$$\hat{h}_D(P) = 0 \iff P \in A(\mathbb{Q})_{\text{tors}}.$$ 

This is the analogue for abelian varieties of Kronecker’s theorem that $\alpha \in \mathbb{Q}^\ast$ is a root of unity if and only if $h(\alpha) = 0$.

Our first main result studies what happens if we relax the ampleness condition on the divisor $D$. We recall that $D \in \text{Div}(A) \otimes \mathbb{R}$ is said to be numerically effective (nef) if $D \cdot C \geq 0$ for all curves $C \subseteq A$. Alternatively, one may define $D$ to be nef if its numerical equivalence class is in the closure of the cone of ample divisors.
Theorem 1. Let $\mathbb{A}/\overline{\mathbb{Q}}$ be an abelian variety, let $D \in \text{Div}(\mathbb{A}) \otimes \mathbb{R}$ be a nonzero symmetric nef divisor, and let $\hat{h}_{A,D}$ be the the canonical height on $A$ with respect to $D$. Then there is a unique abelian subvariety $B_D \subseteq A$ such that

$$\{x \in \mathbb{A}(\overline{\mathbb{Q}}) : \hat{h}_{A,D}(x) = 0\} = B_D(\overline{\mathbb{Q}}) + A(\overline{\mathbb{Q}})_{\text{tors}}.$$

Remark 2. The utility of Theorem 1 lies in the following. Let $f : A \to A$ be an isogeny. In general, the map

$$f^* : \text{NS}(A) \otimes \mathbb{C} \to \text{NS}(A) \otimes \mathbb{C}$$

that $f$ induces on the Néron–Severi group of $A$ need not have any ample eigendivisor classes, but there is always a nontrivial symmetric nef eigendivisor class $[D_f] \in \text{NS}(A) \otimes \mathbb{R}$. (See Lemma 31 for a general statement.) More precisely,

$$(0.1) \quad f^* D_f \equiv \delta_f D_f \quad \text{in } \text{NS}(A) \otimes \mathbb{R},$$

where in general the dynamical degree $\delta_f$ of a morphism $f : X \to X$ of projective varieties is the spectral radius of the linear transformation

$$f^* : \text{NS}(X) \otimes \mathbb{R} \to \text{NS}(X) \otimes \mathbb{R}.$$ (More generally, for a dominant rational map $f : X \dashrightarrow X$, the dynamical degree $\delta_f$ is $\delta_f = \lim\rho((f^n)^*)^{1/n}$, where $\rho$ denotes the spectral radius of the indicated map, and we note that it is often the case that $(f^n)^* \neq (f^*)^n$.)

The equivalence (0.1) implies that the canonical height associated to $D_f$ satisfies

$$\hat{h}_{D_f,A}(f^n(P)) = \delta_f^n \hat{h}_{D_f,A}(P) \quad \text{for all } n \geq 0.$$

Applying Theorem 1 we see that if $\hat{h}_{D_f,A}(P) = 0$, then the entire $f$-orbit

$$\mathcal{O}_f(P) := \{f^n(P) : n \geq 0\}$$

of $P$ is contained in $B_{D_f} + A_{\text{tors}}$. But $\mathcal{O}_f(P)$ is contained in $A(K)$ for some number field $K$, from which it is not hard to deduce that $\mathcal{O}_f(P)$ is contained in a finite union of translates of $B_{D_f}$. In particular, we conclude that if $\hat{h}_{D_f,A}(P) = 0$, then $\mathcal{O}_f(P)$ is not Zariski dense in $A$. See the proof of Theorem 29(c) and Corollary 32 for details.

The other main aim of this paper is to study more generally the arithmetic behavior of algebraic points in orbits. Let $K$ be a global field (as defined in Section 1), let $X/K$ be a normal projective variety, let $h_X : X(\overline{K}) \to \mathbb{R}$ be a Weil height associated to an ample divisor, and let $h_X^\vee = \max\{h_X, 1\}$. We are interested in the arithmetic properties of the orbit of a point $x \in X(\overline{K})$ under iteration of a $K$-endomorphism $f : X \to X$. We recall [6,11] that the $f$-arithmetic degree of $x$ is the quantity

$$(0.2) \quad \alpha_f(x) = \lim_{n \to \infty} h_X(v(f^n(x)))^{1/n},$$

if the limit exists.
The arithmetic degree of \(x\) provides a rough, but useful, measure of the arithmetic complexity of the \(f\)-orbit of \(x\). For example, it was observed in [6] that the existence of \(\alpha_f(x)\) determines the asymptotic growth of the orbit counting function via the formula

\[
\lim_{T \to \infty} \frac{\# \{ n \geq 0 : h_X(f^n(x)) \leq T \}}{\log T} = \frac{1}{\log \alpha_f(x)},
\]

where the limit is understood to be \(\infty\) if \(\alpha_f(x) = 1\).

We also recall from [6] that the arithmetic degree, if it exists, satisfies the inequality

\[
\alpha_f(x) \leq \delta_f.
\]

We prove the following results, which were conjectured in [6] to hold more generally for dominant rational maps \(f : X \to X\). (See also [7,11].)

**Theorem 3.** Let \(f : X \to X\) be a \(K\)-endomorphism of a normal projective variety as described above.

(a) For every \(x \in X(\bar{K})\), the limit defining \(\alpha_f(x)\) converges.
(b) The arithmetic degree \(\alpha_f(x)\) is an algebraic integer.
(c) The collection of arithmetic degrees

\[
\{ \alpha_f(x) : x \in X(\bar{K}) \}
\]

is a finite set.

**Theorem 4.** Let \(X/\bar{Q}\) be an abelian variety, let \(f : X \to X\) be an isogeny, and let \(x \in X(\bar{K})\) be a point whose \(f\)-orbit is Zariski dense in \(X\). Then \(\alpha_f(x)\) is equal to \(\delta_f\), the dynamical degree of \(f\).

As the reader will have surmised, the proof of Theorem 4 relies on the nef canonical height theorem (Theorem 1). Indeed, it was this application that led us to prove Theorem 1.

The proof of Theorem 3 uses an extension of the classical theory of dynamical canonical heights [4] to Jordan blocks, a result that is of independent interest. We recall that if a divisor \(D \in \text{Div}(X) \otimes \mathbb{C}\) satisfies a linear equivalence \(f^* D \sim \lambda D\) for some \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\), then the classical theory [4] says that for every \(x \in X(\bar{K})\), the Tate limit

\[
\hat{h}_{f,D}(x) = \lim_{n \to \infty} \lambda^{-n} h_D(f^n(x))
\]

exists, and the resulting function \(\hat{h}_{f,D} : X(\bar{K}) \to \mathbb{C}\) satisfies the following functional equation and normalization condition:

\[
\hat{h}_{f,D} \circ f = \lambda \hat{h}_{f,D} \quad \text{and} \quad \hat{h}_{f,D} = h_D + O(1).
\]

We generalize this construction by allowing a sequence of divisors that form a Jordan block for the action of \(f^*\) on \(\text{Pic}(X) \otimes \mathbb{C}\).

**Theorem 5.** Let \(X/K\) be a normal projective variety, let \(f : X \to X\) be a \(K\)-morphism, let \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\), and let

\[
D_0, D_1, D_2, \ldots \in \text{Div}(X) \otimes \mathbb{C}
\]
be divisors satisfying linear equivalences in Jordan block form,
\[
\begin{align*}
    f^*D_0 & \sim \lambda D_0 \\
    f^*D_1 & \sim D_0 + \lambda D_1 \\
    f^*D_2 & \sim D_1 + \lambda D_2 \\
    \vdots & \vdots
\end{align*}
\]
Further, for each \(k\), let \(h_{D_k}\) be a Weil height function associated to the divisor \(D_k\).

(a) There are unique functions \(\hat{h}_{D_0}, \hat{h}_{D_1}, \hat{h}_{D_2}, \ldots : X(\bar{K}) \to \mathbb{C}\)
satisfying the normalization conditions
\[
\hat{h}_{D_k} = h_{D_k} + O(1)
\]
and the functional equations
\[
\hat{h}_{D_k} \circ f = \lambda \hat{h}_{D_k} + \hat{h}_{D_{k-1}},
\]
where by convention we set \(\hat{h}_{D_{-1}} = 0\).

(b) The canonical height functions described in (a) satisfy the recursively defined limit formulas
\[
\hat{h}_{D_k}(x) = \lim_{n \to \infty} \left( \lambda^{-n} h_{D_k}(f^n(x)) - \sum_{i=1}^{k} \binom{n}{i} \lambda^{-i} \hat{h}_{D_{k-i}}(x) \right).
\]

We make a number of remarks, after which we give a brief section-by-section summary of the contents of the paper.

Remark 6. A standard construction \[10\] page 60 associates to each divisor \(D \in \text{Div}(A)\) a morphism \(\varphi_D : A \to \hat{A}\) from \(A\) to its dual \(\hat{A}\). If the nef divisor \(D\) in Theorem 1 is in \(\text{Div}(A)\), then the associated abelian subvariety \(B_D\) is simply the kernel of the morphism \(\varphi_D : A \to A\). However, in general the divisor \(D\) is in \(\text{Div}(A) \otimes \mathbb{R}\), so it only induces a map on \(A \otimes \mathbb{R}\). In order to describe \(B_D\), one first writes \(D\) as a linear combination \(D = \sum_{i=1}^{r} c_i D_i\) with \(D_i \in \text{Div}(A)\) and \(c_i \in \mathbb{R}\) such that \(r\) is minimal. This forces \(c_1, \ldots, c_r\) to be linearly independent over \(\mathbb{Q}\); cf. the proof of Lemma 30. Then \(B_D\) may be described geometrically as
\[
B_D = \bigcap_{i=1}^{r} \ker(\varphi_{D_i}).
\]

Remark 7. Iterating (0.4) gives a general functional equation
\[
\hat{h}_{D_k} \circ f^n = \sum_{i=0}^{k} \binom{n}{i} \lambda^{-i} \hat{h}_{D_{k-i}},
\]
valid for all \(n \geq 0\); see Remark 16.

Remark 8. An elaboration of the proof of Theorem 13 can be used to construct local canonical height functions (also sometimes called Green functions) that are suitably normalized and satisfy analogous local functional equations. We refer the reader to [4] for a detailed description of the case of a single eigendivisor \(f^*D \sim \lambda D\). But since the focus of this paper is on global results, we have restricted attention to that case.
Remark 9. If the Jordan block linear equivalences in Theorem 5 are replaced by algebraic equivalences, then a modification of the proof of Theorem 5 gives canonical heights for points whose (upper) arithmetic degrees are smaller than $|\lambda|^2$. These heights satisfy the functional equation (0.4) and a weak form of the normalization condition (0.3). We briefly describe these “algebraic equivalence canonical heights” and their properties, without proof, in Section 5.

Question 10. In general, if $f : X \to X$ is an endomorphism of an arbitrary normal variety with dynamical degree $\delta_f > 1$, then there is always a divisor $D \in \operatorname{Div}(X) \otimes \mathbb{R}$ satisfying $f^* D \equiv \delta_f D$ in $\operatorname{NS}(X) \otimes \mathbb{R}$. Let $\hat{h}_{X,D,f}(x) = \lim_{n \to \infty} \delta_f^{-n} h_{X,D}(f^n(x))$ be the usual canonical height associated to $D$ and $f$; see Remark 22 and [6, Theorem 5]. If we further assume that $D$ is ample, then it is a standard elementary fact that $\hat{h}_{X,D,f}(x) = 0$ if and only if $x$ has finite $f$-orbit. It seems to be a very interesting question to give a geometric description of the set $\{x \in X(\overline{K}) : \hat{h}_{X,D}(x) = 0\}$ when $D \neq 0$ is nef, but not ample.

Remark 11. Theorem 4 gives a criterion for $\alpha_f(x) = \delta_f$ when $f$ is an isogeny of an abelian variety. In an earlier paper [11], the second author proved an analogous result for endomorphisms of the torus $\mathbb{G}_m^N$. (Note that these monomial maps do not extend to self-morphisms of $\mathbb{P}^N$.) The proofs have some features in common, especially a final linear algebra step in which one analyzes the kernel of a linear transformation defined over a field that is larger than the field of definition of the underlying vector space. On the other hand, the two cases use somewhat different tools to reduce to the linear algebra step. For the toric case, one uses local canonical height functions and Baker’s theorem on linear forms in logarithms, while for the abelian variety case, one uses the classification of nef divisors and two fundamental global height formulas (Propositions 27 and 28). It would be interesting to combine these two techniques to prove the analogous result for semi-abelian varieties.

We conclude this introduction with a summary of the contents of the paper. Section 1 begins by setting notation and proving an elementary estimate for powers of a Jordan matrix. In Section 2 we prove Theorem 5 which we restate slightly more generally as Theorem 13. This gives the existence of canonical heights for Jordan blocks. Section 4 contains the proof of Theorem 3. In Section 5 we state, without proof, a version of Theorem 5 in which the linear equivalences are replaced by algebraic equivalences. In Section 6 we study nef divisors, canonical heights, and arithmetic degrees on abelian varieties defined over number fields. After some preliminaries, and the proof of two canonical height formulas (Propositions 27 and 28) that should be of independent interest, we prove Theorem 1 and then use it as the primary tool to prove Theorem 4. Our general theory is illustrated in Sections 3 and 7. The former has an explicit description of the 3-dimensional Jordan block canonical heights associated to certain isogenies $f : E^2 \to E^2$, where $E$ is a non-CM elliptic curve, while the latter examines the proof of Theorem 1 for abelian varieties $A$ whose endomorphism algebra $\operatorname{End}(A) \otimes \mathbb{Q}$ is a real quadratic field.

1. Definitions and notation

In this section we set notation and give definitions that will be used throughout the remainder of this paper (except for Section 6 where we will take $K$ to be a
number field). We begin by fixing:

- $K$ a global field, which for the purposes of this paper will mean a field $K$ of characteristic 0, a fixed algebraic closure $\bar{K}$, and a collection of absolute values on $\bar{K}$ such that there is a well-defined theory of Weil height functions, as explained for example in [8, Chapters 1–4].
- $X/K$ a normal projective variety.
- $f$ a $K$-morphism $f : X \to X$.
- $h_X$ a Weil height on $X(\bar{K})$ relative to an ample divisor.
- $h_X^+$ $= \max\{h_X, 1\}$.

To ease notation, we use subscripts $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ on abelian groups to indicate tensoring over $\mathbb{Z}$ with the indicated field. So for example, if $X$ is a variety and $A$ is an abelian variety, then

\[
\operatorname{Div}(X)_\mathbb{C} = \operatorname{Div}(X) \otimes \mathbb{C}, \quad \operatorname{NS}(X)_\mathbb{R} = \operatorname{NS}(X) \otimes \mathbb{R}, \quad A(\mathbb{Q})_\mathbb{Q} = A(\mathbb{Q}) \otimes \mathbb{Q}.
\]

**Definition.** Let $x \in X(\bar{K})$. The arithmetic degree of $x$ is the limit

\[
\alpha_f(x) = \lim_{n \to \infty} h_X^+(f^n(x))^{1/n},
\]

if the limit exists. In any case, the upper and lower arithmetic degrees of $x$ are the quantities

(1.1) $\underline{\alpha}_f(x) = \liminf_{n \to \infty} h_X^+(f^n(x))^{1/n}, \quad \overline{\alpha}_f(x) = \limsup_{n \to \infty} h_X^+(f^n(x))^{1/n}.$

**Definition.** For a given $\ell \geq 0$ and $\lambda \in \mathbb{C}$, we write $\Lambda$ for the $(\ell + 1)$-dimensional lower Jordan block matrix

$\Lambda = \begin{pmatrix} \lambda & 0 & \ldots & 0 \\ 1 & \lambda & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & \lambda \end{pmatrix}.$

Since we will be studying relationships between height functions associated to many different divisors, it is convenient to use vector-valued height functions; cf. [2].

**Definition.** For divisors $D_0, \ldots, D_\ell \in \operatorname{Div}(X)_\mathbb{C}$ and associated height functions $h_{D_0}, \ldots, h_{D_\ell}$, we define a (column) vector-valued height function $h_D : X(\bar{K}) \to \mathbb{C}^{\ell+1}, \quad h_D(x) = (h_{D_0}(x), \ldots, h_{D_\ell}(x)).$ If the divisors and heights have been fixed, to ease notation we write $h_k = h_{D_k}$ for $0 \leq k \leq \ell$.

**Definition.** We use $\| \cdot \|$ to denote the sup norm of a vector or a matrix, i.e., for vectors $a = (a_i)$ and matrices $B = (b_{ij})$ with complex coordinates,

$\|a\| = \max |a_i|$ and $\|B\| = \max |b_{ij}|.$

We will frequently use the elementary triangle inequality estimate

$\|Ba\| \leq \dim(B) \cdot \|B\| \cdot \|a\|.$

We prove two elementary estimates about the powers of a Jordan matrix.
Lemma 12. (a) For all \( n \geq 1 \),
\[
\|\Lambda^n\| \leq n^\ell \max\{|\lambda|, 1\}^n.
\]
(b) If \( \mathbf{v} \in \mathbb{C}^{\ell+1} \) is a nonzero vector, then
\[
\lim_{n \to \infty} \|\Lambda^n \mathbf{v}\|^{1/n} = |\lambda|.
\]
Proof. We write \( \Lambda = \lambda I + N \) with \( N \) nilpotent. Then \( N^{\ell+1} = 0 \), so \( \Lambda^n = (\lambda I + N)^n = \sum_{i=0}^\ell \binom{n}{i} \lambda^{n-i} N^i \).

(a) Using the nilpotent form of \( N \), we see that the \( ij \)th-entry of \( \Lambda^n \) is
\[
\Lambda^n_{ij} = \binom{n}{i-j} \lambda^{n-(i-j)},
\]
where we set \( \binom{n}{k} = 0 \) if \( k < 0 \), and of course \( \binom{n}{k} = 0 \) if \( k > n \). Hence
\[
\|\Lambda^n\| = \max_{0 \leq k \leq \min\{\ell, n\}} \binom{n}{k} |\lambda|^{n-k} \leq n^\ell \max\{|\lambda|, 1\}^n.
\]
(b) If \( \lambda = 0 \), then \( \Lambda^{\ell+1} = 0 \), so the result is trivially true. We assume now that \( \lambda \neq 0 \). We write \( \mathbf{v} = \ell(v_0, \ldots, v_\ell) \). Then the \( i \)th-coordinate of \( \Lambda^n \mathbf{v} \) satisfies
\[
(A^n \mathbf{v})_i = \sum_{j=0}^\ell \Lambda^n_{ij} v_j = \sum_{j=0}^\ell \binom{n}{i-j} \lambda^{n-(i-j)} v_j = O(n^\ell \lambda^n).
\]
This holds for all \( i \), so we have
\[
(1.2) \quad \limsup_{n \to \infty} \|\Lambda^n \mathbf{v}\|^{1/n} \leq \limsup_{n \to \infty} \|O(n^\ell \lambda^n)\|^{1/n} \leq |\lambda|.
\]
Next, since \( \mathbf{v} \neq \mathbf{0} \), there is an index \( 0 \leq k \leq \ell \) such that
\[
v_0 = \cdots = v_{k-1} = 0 \quad \text{and} \quad v_k \neq 0.
\]
Hence the \( k \)th-coordinate of \( \Lambda^n \mathbf{v} \) has the form
\[
(A^n \mathbf{v})_k = \sum_{j=1}^\ell \Lambda^n_{kj} v_j = \sum_{j=0}^\ell \binom{n}{k-j} \lambda^{n-(k-j)} v_j = \lambda^n v_k,
\]
where the last equality follows from the facts that \( v_j = 0 \) for \( j < k \) and \( \binom{n}{k-j} = 0 \) for \( j > k \). This gives
\[
(1.3) \quad \liminf_{n \to \infty} \|\Lambda^n \mathbf{v}\|^{1/n} \geq \liminf_{n \to \infty} |(A^n \mathbf{v})_k|^{1/n} = \liminf_{n \to \infty} |\lambda^n v_k|^{1/n} = |\lambda|,
\]
where the last equality follows from the fact that \( v_k \neq 0 \). Combining (1.2) and (1.3) gives
\[
\lim_{n \to \infty} \|\Lambda^n \mathbf{v}\|^{1/n} = |\lambda|,
\]
which completes the proof of Lemma 12. \( \square \)
2. Jordan Block Canonical Heights

In this section we construct canonical heights associated to Jordan blocks in \( \text{Pic}(X)_\mathbb{C} \). These exist for eigenvalues satisfying \( |\lambda| > 1 \). We also give an estimate that holds for all \( \lambda \).

**Theorem 13.** Let \( \lambda \in \mathbb{C} \), and let \( D_0, D_1, \ldots, D_\ell \in \text{Div}(X)_\mathbb{C} \) be divisors that form a Jordan block in \( \text{Pic}(X)_\mathbb{C} \).

\[
(2.1) \quad f^*D_0 \sim \lambda D_0, \quad f^*D_1 \sim D_0 + \lambda D_1, \quad \ldots, \quad f^*D_\ell \sim D_{\ell-1} + \lambda D_\ell.
\]

(a) There is a constant \( C = C(D_0, \ldots, D_\ell, \lambda) \) such that
\[
\|h_D(f^n(x))\| \leq Cn^\ell \max\{|\lambda|, 1\}^n \cdot (\|h_D(x)\| + 1)
\]
for all \( x \in X(\bar{K}) \) and all \( n \geq 0 \).

(b) If \( |\lambda| > 1 \), then there is a unique function \( \hat{h}_D : X(\bar{K}) \to \mathbb{C}^{\ell+1} \) satisfying the functional equation
\[
(2.2) \quad \hat{h}_D \circ f = \Lambda \hat{h}_D
\]
and the normalization condition
\[
(2.3) \quad \hat{h}_D = h_D + O(1).
\]

(c) The coordinate functions of the canonical height function
\[
\hat{h}_D = \ell(\hat{h}_0, \hat{h}_1, \ldots, \hat{h}_\ell)
\]
described in (b) satisfy the limit recursion relations
\[
(2.4) \quad \hat{h}_k(x) = \lim_{n \to \infty} \left( \lambda^{-n} f^n(x) \right) - \sum_{i=1}^{\ell} \binom{n}{i} \lambda^{-i} \hat{h}_{k-i}(x).
\]

**Remark 14.** We note that in Theorem 13 if \( D_0, \ldots, D_\ell \in \text{Div}(X)_\mathbb{R} \) and \( \lambda \in \mathbb{R} \), then the associated canonical height functions take values in \( \mathbb{R} \), i.e., \( \hat{h}_D : X(\bar{K}) \to \mathbb{R}^{\ell+1} \).

**Remark 15.** We also note that the proof of Theorem 13 does not use the assumption that the global field \( K \) has characteristic 0, so it in fact holds for any field on which there is a theory of Weil height functions.

**Remark 16.** The functional equation for the individual \( \hat{h}_k \circ f^n \) mentioned in Remark 7 follows immediately from the functional equation (2.2). To see this, we iterate (2.2) to obtain \( \hat{h}_D \circ f^n = \Lambda^n \hat{h}_D \), expand \( \Lambda^n = (\lambda I + N)^n \) using the binomial theorem, and apply the fact (already used in the proof of Lemma 12(a)) that \( \Lambda_{ij}^n = \binom{n}{i-j} \lambda^{-n(i-j)} \).

**Proof of Theorem 13.** We define a vector-valued “error function”
\[
E_D : X(\bar{K}) \to \mathbb{C}^{\ell+1}, \quad E_D = h_D \circ f - \Lambda h_D.
\]
The assumption (2.1) that \( D_0, \ldots, D_\ell \) form a Jordan block says that each coordinate function of \( E_D \) is a Weil height function with respect to a divisor that is linearly equivalent to zero. A standard property of Weil height functions [5, Theorem B.3.2(d)] then implies that there is a constant \( C_1 \) such that
\[
(2.5) \quad \|E_D(x)\| \leq C_1 \quad \text{for all } x \in X(\bar{K}).
\]
We now begin the proof of (a). For \( N \geq 1 \) we consider the telescoping sum

\[
h_D \circ f^N = \Lambda^N h_D + \sum_{n=0}^{N-1} \Lambda^{N-n-1}(h_D \circ f^{n+1} - \Lambda h_D \circ f^n)
\]

\[
= \Lambda^N h_D + \sum_{n=0}^{N-1} \Lambda^{N-n-1} E_D \circ f^n.
\]

To ease notation, we let

\[
|\lambda^+| = \max\{|\lambda|, 1\},
\]

so Lemma 12(a) says that \( \|\Lambda^n\| \leq n^\ell |\lambda^+|^n \). For \( x \in X(\bar{K}) \) we compute

\[
\|h_D(f^N(x))\| \\
\leq \|\Lambda^N h_D(x)\| + \sum_{n=0}^{N-1} \|\Lambda^{N-n-1} E_D(f^n(x))\|
\]

\[
\leq (\ell + 1)\|\Lambda^N\| \cdot \|h_D(x)\| + \sum_{n=0}^{N-1} (\ell + 1)\|\Lambda^{N-n-1}\| \cdot \|E_D(f^n(x))\|
\]

\[
\leq (\ell + 1)N^\ell |\lambda^+|^N \cdot \|h_D(x)\| \\
+ (\ell + 1) \sum_{n=0}^{N-1} (N - n - 1)^\ell |\lambda^+|^{N-n-1} \cdot \|E_D(f^n(x))\| \quad \text{from Lemma 12(a)}
\]

\[
\leq (\ell + 1)N^\ell |\lambda^+|^N \cdot \|h_D(x)\| \\
+ (\ell + 1) \sum_{n=0}^{N-1} (N - n - 1)^\ell |\lambda^+|^{N-n-1} C_1 \quad \text{from 2.5}
\]

\[
\leq (\ell + 1)N^\ell |\lambda^+|^N \cdot \|h_D(x)\| + C_1 (\ell + 1)N^\ell |\lambda^+|^N.
\]

This completes the proof of (a).

We next show that there is at most one vector-valued height function \( \hat{h}_D \) satisfying the conditions given in (b). So we suppose that \( \hat{h}_D' \) is another such function, we let \( \tilde{g}_D = \hat{h}_D' - \hat{h}_D \), and we prove by contradiction that \( \tilde{g}_D = 0 \). So we suppose that there is a point \( x \in X(\bar{K}) \) such that \( \tilde{g}_D(x) \neq 0 \).

Taking the difference of the functional equations for \( \hat{h}_D \) and \( \hat{h}_D' \) gives a functional equation

\[
(2.6) \quad \tilde{g}_D \circ f = \Lambda \tilde{g}_D
\]

for \( \tilde{g}_D \). This allows us to compute

\[
|\lambda| = \limsup_{n \to \infty} \|\Lambda^n \tilde{g}_D(x)\|^{1/n} \quad \text{from Lemma 12(b), since } \tilde{g}_D(x) \neq 0
\]

\[
= \limsup_{n \to \infty} \|\tilde{g}_D(f^n(x))\|^{1/n} \quad \text{from 2.6}
\]

\[
= \limsup_{n \to \infty} \|\hat{h}_D(f^n(x)) - \hat{h}_D'(f^n(x))\|^{1/n} \quad \text{definition of } \tilde{g}_D
\]
\[ \limsup_{n \to \infty} \left( \| \hat{h}_D(f^n(x)) - h_D(f^n(x)) \| + \| \hat{h}'_D(f^n(x)) - h'_D(f^n(x)) \| \right)^{1/n} \leq 1 \text{ from (2.3).} \]

Hence \(|\lambda| \leq 1\), which contradicts the assumption in (b) that \(|\lambda| > 1\). This contradiction completes the proof that \( \hat{g}_D = 0 \), which shows that \( \hat{h}_D \) is uniquely determined by (2.2) and (2.3).

The assumption that \(|\lambda| > 1\) allows us to estimate the norm of the negative powers of \( \Lambda \) as follows:

\[
\| \Lambda^{-n} \| = \| (\lambda I + N)^{-n} \| = \left\| \sum_{i=0}^{\ell} \binom{-n}{i} \lambda^{-n-i} N^i \right\| \text{ since } N^{\ell+1} = 0 \\
\leq |\lambda|^{-n} \cdot (\ell + 1) \cdot \max_{0 \leq i \leq \ell} \left| \binom{-n}{i} \right| \\
= C_2 n^{\ell+1} |\lambda|^{-n},
\]

where the constant \( C_2 \) depends on \( \ell \) and \( \lambda \), but does not depend on \( n \).

**Claim 17.** For all \( x \in X(\bar{K}) \), the vector-valued series

\[
\hat{h}_D(x) := h_D(x) + \sum_{n=0}^{\infty} \Lambda^{-n-1} E_D(f^n(x))
\]

(2.8)

is absolutely convergent and defines a vector-valued height function

\( \hat{h}_D : X(\bar{K}) \to \mathbb{C}^{\ell+1} \) satisfying \( \| \hat{h}_D(x) - h_D(x) \| \leq C \) for a constant \( C \) that is independent of \( x \).

**Proof.** We compute

\[
\sum_{n=0}^{\infty} \| \Lambda^{-n-1} E_D(f^n(x)) \| \leq (\ell + 1) \sum_{n=0}^{\infty} \| \Lambda^{-n-1} \| : \| E_D(f^n(x)) \| \\
\leq C_3 \sum_{n=0}^{\infty} n^{\ell} |\lambda|^{-n} \| E_D(f^n(x)) \| \text{ from (2.7)} \\
\leq C_3 C_1 \sum_{n=0}^{\infty} n^{\ell} |\lambda|^{-n} \text{ from (2.5)} \\
\leq C_4 \text{ since } |\lambda| > 1 \text{ by assumption.}
\]

This shows that the series appearing in (2.8) is absolutely convergent, while simultaneously giving the desired upper bound for \( \| \hat{h}_D(x) - h_D(x) \| \), which completes the proof of the claim. \( \square \)

Claim 17 gives us a well-defined function \( \hat{h}_D : X(\bar{K}) \to \mathbb{C}^{\ell+1} \) that satisfies the normalization condition (2.4). It remains to prove that \( \hat{h}_D \) satisfies the functional
equation (2.2). Since we know that the series defining \( \hat{h}_D \) is absolutely convergent, the proof is a formal calculation using the definitions of \( \hat{h}_D \) and \( E_D \). Thus

\[
\hat{h}_D \circ f = h_D \circ f + \sum_{n=0}^{\infty} \Lambda^{-n-1} E_D \circ f^{n+1}
\]

\[
= h_D \circ f + \sum_{n=1}^{\infty} \Lambda^{-n} E_D \circ f^n
\]

\[
= h_D \circ f - E_D + \sum_{n=0}^{\infty} \Lambda^{-n} E_D \circ f^n
\]

\[
= \Lambda h_D + \sum_{n=0}^{\infty} \Lambda^{-n-1} E_D \circ f^n
\]

\[
= \Lambda \hat{h}_D.
\]

(c) We have already proven that the canonical height function \( \hat{h}_D \) exists and satisfies a normalization condition and a functional equation. In particular, the normalization condition implies that

\[
\hat{h}_D \circ f^n = h_D \circ f^n + O(1).
\]

We compute

\[
\hat{h}_D = \hat{h}_D + \lambda^{-n}(\hat{h}_D \circ f^n - \Lambda^n \hat{h}_D)
\]

from the functional equation

\[
= \lambda^{-n} h_D \circ f^n - ((\lambda^{-1} \Lambda - I) \hat{h}_D + \lambda^{-n}(\hat{h}_D \circ f^n - h_D \circ f^n))
\]

\[
= \lambda^{-n} h_D \circ f^n - ((\lambda^{-1} \Lambda - I) \hat{h}_D + O(\lambda^{-n})
\]

writing \( \Lambda = \lambda I + N \)

\[
= \lambda^{-n} h_D \circ f^n - \ell \sum_{i=1}^{\ell} \binom{n}{i} \lambda^{-i} N^i \hat{h}_D + O(\lambda^{-n})
\]

since \( N^{\ell+1} = 0 \).

Evaluating at \( x \) and letting \( n \to \infty \) yields

\[
\hat{h}_D(x) = \lim_{n \to \infty} \left( \lambda^{-n} h_D(f^n(x)) - \sum_{i=1}^{\ell} \binom{n}{i} \lambda^{-i} N^i \hat{h}_D(x) \right)
\]

Multiplication by the matrix \( N \) is the right-shift operator

\[
t(a_0, \ldots, a_{\ell}) \mapsto t(0, a_0, \ldots, a_{\ell-1}),
\]

so the \( k \)-th-coordinate of \( N^i \hat{h}_D(x) \) is \( \hat{h}_{k-i} \). (By convention we set \( \hat{h}_j = 0 \) if \( j < 0 \).) This shows that the vector formula (2.10) is a succinct way of writing the formulas (2.4) that we are trying to prove.

\[
\square
\]

3. AN EXAMPLE OF A JORDAN BLOCK CANONICAL HEIGHT

In this section we illustrate Theorem 13 for \( X = E^2 \), where \( E/K \) is a non-CM elliptic curve. In this case, the Néron–Severi group of \( X \) is generated by the three divisors

\[
H_1 = (O) \times E, \quad H_2 = E \times (O), \quad \text{and} \quad \Delta = \{(x,x) : x \in E\}.
\]
It is more convenient to take as our basis for \( \text{NS}(X)_\mathbb{Q} \) the three divisors 
\[ H_1, \ H_2, \ \text{and} \ H_3 = H_1 + H_2 - \Delta. \]

We consider an endomorphism of the form 
\[ f : X \rightarrow X, \quad f(x,y) = (ax + by, ay) \]
with \( a, b \in \mathbb{Z} \) satisfying \( |a| \geq 2 \) and \( b \neq 0 \). An elementary intersection theory calculation yields the formulas 
\[ f^* H_1 = a^2 H_1 + b^2 H_2 + ab H_3, \]
\[ f^* H_2 = a^2 H_2, \]
\[ f^* H_3 = 2ab H_2 + a^2 H_3, \]
from which it is a linear algebra exercise to construct divisors 
\[ D_0 = 4a^3b^2 H_2, \quad D_1 = 2a^2b H_3, \quad D_2 = 2a H_1 - b H_3, \]
satisfying 
\[ f^* D_0 = a^2 D_0, \quad f^* D_1 = D_0 + a^2 D_1, \quad f^* D_2 = D_1 + a^2 D_2. \]

Thus \( D_0, D_1, D_2 \) form a Jordan block basis for \( \text{NS}(X)_\mathbb{R} \), and their associated Jordan block canonical heights satisfy 
\[ \hat{h}_{D_0} \circ f = a^2 \hat{h}_{D_0}, \quad \hat{h}_{D_1} \circ f = \hat{h}_{D_0} + a^2 \hat{h}_{D_1}, \quad \hat{h}_{D_2} \circ f = \hat{h}_{D_1} + a^2 \hat{h}_{D_2}. \]

A short calculation shows that \( \hat{h}_{D_0}, \hat{h}_{D_1}, \hat{h}_{D_2} \) may be written in terms of the canonical height pairing on \( E \) as follows:
\[ \hat{h}_{D_0}(x,y) = 4a^3b^2 \langle y, y \rangle_E, \]
\[ \hat{h}_{D_1}(x,y) = 2a^2b \langle x, y \rangle_E, \]
\[ \hat{h}_{D_2}(x,y) = \langle x, 2ax - by \rangle_E. \]

4. An application to arithmetic degrees

In this section we prove Theorem 3 which says that for morphisms \( f : X \rightarrow X \), the arithmetic degree \( \alpha_f(x) \) of a point \( x \in X(\bar{K}) \) exists, is an algebraic integer, and takes on only finitely many values as \( x \) ranges over \( X(\bar{K}) \).

We recall that the lower and upper arithmetic degrees (1.1) are defined, respectively, to be the liminf and limsup of \( h^+_X(f^n(x))^{1/n} \). It is proven in [6, Proposition 12] that the values of \( \alpha_f(x) \) and \( \overline{\alpha}_f(x) \) do not depend on the choice of the ample height function \( h_X \).

Before beginning the proof of Theorem 3, we note that the equivalence class of Weil height functions associated to a divisor \( D \in \text{Div}(X)_{\mathbb{C}} \) consists of complex valued functions 
\[ h_D : X(\bar{K}) \rightarrow \mathbb{C} \]
obtained by writing \( D \) as a linear combination 
\[ D = c_1 D_1 + \cdots + c_t D_t \]
with \( c_i \in \mathbb{C} \) and \( D_i \in \text{Div}(X) \) and setting 
\[ h_D = c_1 h_{D_1} + \cdots + c_t h_{D_t} + O(1). \]

On the other hand, we note that the concept of ample divisor only makes sense in \( \text{Div}(X)_{\mathbb{R}} \), not in \( \text{Div}(X)_{\mathbb{C}} \). Despite this potential problem, the following helpful
Lemma 18. Let \( f : X \to X \) be a morphism, let \( D \in \text{Div}(X)_\mathbb{C} \) be any divisor, and let \( x \in X(\bar{K}) \). Then

\[
α_f(x) ≥ \liminf_{n \to \infty} |h_D(f^n(x))|^{1/n}.
\]  

Proof. If
\[
\lim_{n \to \infty} |h_D(f^n(x))| \neq \infty,
\]
then the right-hand side of (4.1) is less than or equal to 1, so (4.1) is automatically true, since the definition of \( α_f(x) \) ensures that \( α_f(x) ≥ 1 \). We may thus assume that \( |h_D(f^n(x))| \to \infty \).

Any two nontrivial norms on \( \mathbb{C} \) are related by \(|z|_1 \propto |z|_2\), so the right-hand side of (4.1) is independent of the chosen norm on \( \mathbb{C} \). We will use the norm \(|a + bi| = \max\{|a|, |b|\} \). We write
\[
D = D_1 + iD_2 \quad \text{with} \quad D_1, D_2 \in \text{Div}(X)_\mathbb{R}.
\]
We fix an ample divisor \( H \in \text{Div}(X) \). Then there is an \( ε > 0 \) so that the four divisors
\[
H + εD_1, \quad H - εD_1, \quad H + εD_2, \quad H - εD_2
\]
are all in the ample cone. Since \( H \) is ample, we may choose a height function \( h_H \) for \( H \) satisfying \( h_H ≥ 0 \).

Using standard properties of height functions, we estimate
\[
\max\{h_{H+εD_1}, h_{H-εD_1}\} = h_H + ε \max\{h_{D_1}, -h_{D_1}\} + O(1)
\]
\[
= h_H + ε|h_{D_1}| + O(1)
\]
\[
≥ ε|h_{D_1}| + O(1),
\]
where the last line follows because \( h_H ≥ 0 \). Replacing \( D_1 \) with \( D_2 \) gives an analogous inequality, so we find that
\[
\max\{h_{H+εD_1}, h_{H-εD_1}, h_{H+εD_2}, h_{H-εD_2}\} ≥ ε \max\{|h_{D_1}|, |h_{D_2}|\} + O(1)
\]
\[
= ε|h_{D_1}| + O(1).
\]

We now evaluate (4.2) at \( f^n(x) \), take the \( n \)th-root, and take the liminf as \( n \to \infty \). Since the divisors \( H ± εD_i \) are all ample, we can use the fact that \( α_f(x) \) may be computed using any ample divisor [6, Proposition 12] to deduce that the left-hand side of (4.2) goes to \( α_f(x) \). (We remark that the main theorems in [6] assume that \( X \) is smooth and \( f \) is dominant, but neither of these assumptions is used in the proof of [6, Proposition 12].) On the other hand, since \( ε > 0 \) and \(|h_D(f^n(x))| \to \infty \), we see that the right-hand side of (4.2) is equal to \( \lim inf|h_D(f^n(x))|^{1/n} \).

Lemma 19. Let \( f : X \to X \) be a morphism. Then there is a monic integral polynomial \( P_f(t) \in \mathbb{Z}[t] \) with the property that \( P_f(f^n) \) annihilates \( \text{Pic}(X) \).

Proof. The Picard group of \( X \) fits into an exact sequence
\[
0 \to A \to \text{Pic}(X) \to \text{NS}(X) \to 0,
\]
where \( A = \text{Pic}^0(X) \) is an abelian variety, and where the Néron–Severi group \( \text{NS}(X) \) is a finitely generated abelian group by the theorem of the base [8] Chapter 6,
Theorem 6.1. In particular, since $\text{NS}(X)$ is finitely generated, there is a monic $Q_f(t) \in \mathbb{Z}[t]$ such that $Q_f(f^*)$ annihilates $\text{NS}(X)$.

The map $f^* : \text{Pic}(X) \to \text{Pic}(X)$ maps $\text{Pic}^0(X)$ to itself, and the resulting map is an endomorphism, which we denote by $\varphi_f : A \to A$. Let $R_f(t) \in \mathbb{Z}[t]$ be the characteristic polynomial of $\varphi_f$ acting on the Tate module $T_t(A)$; cf. [10, Theorem 3 (p.176)]. Then $R_f(\varphi_f) \in \text{End}(A)$ annihilates all of the $\ell$-power torsion of $A$, so $R_f(\varphi_f) = 0$. Setting $P_f(t) = R_f(t)Q_f(t)$ then gives a monic integral polynomial satisfying $P_f(f^*)(D) \sim 0$ for all $D \in \text{Pic}(X)$. \hfill $\square$

**Proof of Theorem 6.3.** Let $P_f(t) \in \mathbb{Z}[t]$ be the monic polynomial from Lemma 19 having the property that $P_f(f^*)(D) \sim 0$ for all $D \in \text{Pic}(X)$, and let $d = \deg(P_f)$. We fix an ample divisor $H \in \text{Div}(X)$, and we let

$$V = \text{Span}_\mathbb{Q}(H, f^*H, (f^*)^2H, \ldots, (f^*)^{d-1}H) \subset \text{Pic}(X)_{\mathbb{Q}}.$$ 

Then the fact that $P_f(f^*)(H) \sim 0$ implies that $V$ is an $f^*$-invariant subspace of $\text{Pic}(X)_{\mathbb{Q}}$. We let $\rho = \dim(V)$.

Extending scalars to $\mathbb{C}$, we choose divisors $E_1, \ldots, E_\rho \in \text{Div}(X)_{\mathbb{C}}$ whose divisor classes in $\text{Pic}(X)_{\mathbb{C}}$ form a $\mathbb{C}$-basis for $V_{\mathbb{C}}$ such that the associated matrix of $f^*|_V$ is in Jordan normal form. Thus for each $1 \leq i \leq \rho$, we have either

$$f^*E_i \sim \lambda_i E_i \quad \text{or} \quad f^*E_i \sim \lambda_i E_i + E_{i-1},$$

where by convention we set $E_0 = 0$.

Relabeling the divisors, we may assume that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_\sigma| > 1 \geq |\lambda_{\sigma+1}| \geq \cdots \geq |\lambda_\rho|. \tag{4.3}$$

Theorem 13(b) and Remark 7 tell us that for each $1 \leq i \leq \sigma$, there is a canonical height function $\hat{h}_{E_i}$ having various useful properties, including

$$\hat{h}_{E_i} = h_{E_i} + O(1) \tag{4.4}$$

and

$$\hat{h}_{E_i}(f^n(x)) = \sum_{j=0}^{\ell(i)} \binom{n}{j} \lambda_i^{-j} \hat{h}_{E_{i-j}}(x), \tag{4.5}$$

where $\ell(i)$ is chosen so that $E_i, E_{i-1}, \ldots, E_{i-\ell(i)}$ is the appropriate piece of the Jordan block that contains $E_i$.

On the other hand, for $\sigma < i \leq \rho$, taking the $n$th-root of Theorem 13(a) and using the fact that $|\lambda_i| \leq 1$ for these $i$, we find that

$$\limsup_{n \to \infty} |\hat{h}_{E_i}(f^n(x))|^{1/n} \leq 1 \quad \text{for all } \sigma < i \leq \rho. \tag{4.6}$$

Now take a point $x \in X(\bar{K})$. We first consider the case that $\hat{h}_{E_i}(x) \neq 0$ for some $1 \leq i \leq \sigma$. We let $k$ be the smallest such index, so

$$\hat{h}_{E_k}(x) \neq 0 \quad \text{and} \quad \hat{h}_{E_{k-1}}(x) = \hat{h}_{E_{k-2}}(x) = \cdots = \hat{h}_{E_1}(x) = 0. \tag{4.7}$$

Then

$$\hat{h}_{E_k}(f^n(x)) = \sum_{j=0}^{\ell(k)} \binom{n}{j} \lambda_k^{-j} \hat{h}_{E_{k-j}}(x) \quad \text{from (4.5)} \tag{4.8}$$

$$= \lambda_k^{n/k} \hat{h}_{E_k}(x) \quad \text{from (4.7)}. \tag{4.8}$$

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This allows us to estimate
\[
\alpha_f(x) \geq \liminf_{n \to \infty} |h_{E_k}(f^n(x))|^{1/n}
\]
from Lemma 18
\[
\geq \liminf_{n \to \infty} \left( |\hat{h}_{E_k}(f^n(x))| - O(1) \right)^{1/n}
\]
from (4.4)
\[
= \liminf_{n \to \infty} \left( |\lambda_k^n \hat{h}_{E_k}(x)| - O(1) \right)^{1/n}
\]
from (4.8)
\[
(4.9)
= |\lambda_k| \quad \text{since } |\lambda_k| > 1 \text{ and } \hat{h}_{E_k}(x) \neq 0.
\]

In order to find a complementary upper bound, we recall that we fixed an ample divisor \( H \in \text{Div}(X) \) and used it to define \( V \). So we can write the divisor class of \( H \) in terms of our \( \mathbb{C} \)-basis for \( V \),
\[
H \sim c_1 E_1 + \cdots + c_\rho E_\rho \quad \text{with } c_1, \ldots, c_\rho \in \mathbb{C}.
\]

Since \( |\lambda_k| > 1 \), we can fix an \( \epsilon \) satisfying \( 0 < \epsilon < |\lambda_k| - 1 \) and compute as follows, where the big-\( O \) constants are independent of \( n \):
\[
h_H(f^n(x)) = \sum_{i=1}^\rho c_i h_{E_i}(f^n(x)) + O(1)
\]
\[
= \sum_{i=1}^\sigma c_i \hat{h}_{E_i}(f^n(x)) + O(1) + \sum_{i=\sigma+1}^\rho c_i h_{E_i}(f^n(x)) \quad \text{from (4.4)}
\]
\[
(4.10)
= \sum_{i=1}^\sigma c_i \hat{h}_{E_i}(f^n(x)) + O((1+\epsilon)^n) \quad \text{from (4.6)}
\]
\[
= \sum_{i=\sigma}^\rho c_i \hat{h}_{E_i}(f^n(x)) + O((1+\epsilon)^n) \quad \text{from (4.6)}
\]
\[
\leq O \left( \max_{1 \leq i \leq \sigma} n^\rho |\lambda_i^n| \right) + O((1+\epsilon)^n) \quad \text{from (4.5)}
\]
\[
\leq O(n^\rho |\lambda_k|^n) + O((1+\epsilon)^n) \quad \text{from (4.3)}
\]
\[
\leq O(n^\rho |\lambda_k|^n) \quad \text{since } \epsilon < |\lambda_k| - 1.
\]

Hence
\[
(4.11)
\overline{\alpha}_f(x) = \limsup_{n \to \infty} h_H(f^n(x))^{1/n} \leq \limsup_{n \to \infty} O(n^\rho |\lambda_k|^n)^{1/n} = |\lambda_k|.
\]

Combining (4.9) and (4.11) gives
\[
|\lambda_k| \leq \alpha_f(x) \leq \overline{\alpha}_f(x) \leq |\lambda_k|,
\]
which completes the proof that if \( \hat{h}_{E_k}(x) \neq 0 \), then
\[
\alpha_f(x) = \lim_{n \to \infty} h_H(f^n(x))^{1/n} = |\lambda_k|.
\]

It remains to deal with the case that
\[
\hat{h}_{E_1}(x) = \cdots = \hat{h}_{E_\rho}(x) = 0,
\]
which using (4.5) implies that
\[
(4.12)
\hat{h}_{E_1}(f^n(x)) = \cdots = \hat{h}_{E_\rho}(f^n(x)) = 0 \quad \text{for all } n \geq 0.
\]
Substituting (4.12) into the earlier calculation of \( h_H(f^n(x)) \), more specifically into the line labeled (4.10), we find that
\[
h_H(f^n(x)) = O((1 + \epsilon)^n).
\]
So taking \( n^{th} \)-roots and letting \( n \to \infty \) gives
\[
\overline{\alpha}_f(x) = \limsup_{n \to \infty} h_H(f^n(x))^{1/n} \leq 1 + \epsilon,
\]
and since this holds for all \( \epsilon > 0 \), we find that
\[
\overline{\alpha}_f(x) \leq 1.
\]
But it is clear from the definition that \( \overline{\alpha}_f(x) \geq 1 \), so we conclude in this case that the limit defining \( \alpha_f(x) \) exists and is equal to 1.

This completes the proof that the limit defining the arithmetic degree exists. Further, we have shown that either \( \alpha_f(x) = 1 \) or else \( \alpha_f(x) \) is equal to the absolute value of one of the eigenvalues of \( f^* \) acting on the finite-dimensional vector space \( V \subset \text{Pic}(X)_\mathbb{Q} \). But we also know that \( P_f(f^*) \) annihilates \( \text{Pic}(X)_\mathbb{Q} \), so the minimal polynomial of \( f^*|_V \) must divide \( P_f(t) \), and in any case, the eigenvalues of \( f^*|_V \) are roots of \( P_f(t) \). Hence
\[
\{ \alpha_f(x) : x \in X(\bar{K}) \} \subset \{ 1 \} \cup \{ |\lambda| : \lambda \in \mathbb{C} \text{ is a root of } P_f(t) \},
\]
which simultaneously shows that \( \alpha_f(x) \) is an algebraic integer and that \( \alpha_f(x) \) takes on only finitely many distinct values as \( x \) ranges over \( X(\bar{K}) \).

This completes the proof of Theorem 3 but we also remark that with somewhat more work, one can show that one need only consider eigenvalues of \( f^* : \text{NS}(X)_\mathbb{Q} \to \text{NS}(X)_\mathbb{Q} \); see Remark 23 for further details. \( \square \)

5. Canonical heights for algebraic equivalence polarizations

In this section we state an analogue of Theorem 5 in which the linear equivalences are replaced by algebraic equivalences. We omit the proof which, \emph{mutatis mutandis}, follows the proof of Theorem 5.

**Theorem 20.** Let \( X/K \) be a normal projective variety, let \( f : X \to X \) be a \( K \)-morphism, let \( \lambda \in \mathbb{C} \), and let
\[
X^{(\lambda)}(\bar{K}) = \{ x \in X(\bar{K}) : \alpha_f(x) < |\lambda|^2 \}.
\]
Let \( D_0, D_1, \ldots, D_\ell \in \text{Div}(X)_\mathbb{C} \) be divisors that form a Jordan block with eigenvalue \( \lambda \) for the linear transformation \( f^* : \text{NS}(X)_\mathbb{C} \to \text{NS}(X)_\mathbb{C} \), i.e.,
\[
f^*D_0 \equiv \lambda D_0, \quad f^*D_1 \equiv D_0 + \lambda D_1, \ldots, \quad f^*D_\ell \equiv D_{\ell-1} + \lambda D_\ell,
\]
where \( \equiv \) denotes algebraic equivalence. Then there is a unique function \( \hat{h}_D : X^{(\lambda)}(\bar{K}) \to \mathbb{C}^{\ell+1} \) satisfying the functional equation
\[
\hat{h}_D \circ f = \Lambda \hat{h}_D
\]
and the normalization condition
\[
\limsup_{n \to \infty} \left\| \hat{h}_D(f^n(x)) - h_D(f^n(x)) \right\|^{1/n} \leq \alpha_f(x)^{1/2}.
\]
The coordinate functions of \( \hat{h}_D \) satisfy the recursion relations (4.13) stated in Theorem 5.
Remark 21. If a divisor $D$ is algebraically equivalent to 0, i.e., $D \equiv 0$, then a classical height estimate [8, Chapter 4, Corollary 3.4] says that $h_D = o(h_X^+)$.

This estimate is not strong enough to prove Theorem 20. Instead one uses the stronger estimate $h_D = O \left( (h_X^+)^{1/2} \right)$, which follows from the Néron–Tate theory of canonical heights on abelian varieties; see [5, Theorem B.5.9].

Remark 22. Continuing with algebraic equivalence relations as in Theorem 20, if we further assume that $f$ is dominant and that the eigenvalue is sufficiently large, then we can obtain a stronger result. More precisely, if the eigenvalue $\lambda$ satisfies $|\lambda| > \delta_f^{1/2} > 1$, where $\delta_f$ is the dynamical degree of $f$, then [6] tells us that $X(\lambda)(\overline{K}) = X(\overline{K})$, and the weak normalization condition (5.2) in Theorem 20 may be replaced by the stronger condition

$$\hat{h}_D = h_D + O \left( (h_X^+)^{1/2} \right).$$

The proof, which we omit, again follows the lines of the proof of Theorem 20, but uses a key height inequality proven in [6]. (The reason that we require $f$ to be dominant here is because this is assumed in [6].)

Remark 23. Our proof of Theorem 3 shows that either $\alpha_f(x) = 1$ or else $\alpha_f(x)$ is equal to the absolute value of an eigenvalue of the linear transformation $f^* : \text{Pic}(X)_\mathbb{Q} \rightarrow \text{Pic}(X)_\mathbb{Q}$.

Lemma 19 implies that this set of eigenvalues is finite. However, using Theorem 20 and suitably modifying the proof of Theorem 3, one can show that in fact $\alpha_f(x)$ is either 1 or the absolute value of an eigenvalue of $f^* : \text{NS}(X)_\mathbb{Q} \rightarrow \text{NS}(X)_\mathbb{Q}$, so there is in fact no need to consider the eigenvalues coming from the action of $f^*$ on $\text{Pic}^0(X)_\mathbb{Q}$.

6. Nef heights and arithmetic degrees for endomorphisms of abelian varieties

Our primary goal in this section is to prove the nef canonical height theorem (Theorem 1) on abelian varieties, and then to use it prove that if the $f$-orbit of a point is Zariski dense, then its arithmetic degree is maximal (Theorem 4). Along the way we prove two useful height formulas (Propositions 27 and 28). We will make extensive use of the geometry of abelian varieties and their Néron–Severi groups and endomorphism algebras as described in Mumford’s classic book [10, Sections 19–21].

For this section we set the following notation:

- $A/\overline{\mathbb{Q}}$: An abelian variety defined over $\overline{\mathbb{Q}}$.
- $\hat{A}$: The dual of the abelian variety $A$, i.e., $\hat{A} = \text{Pic}^0(A)$.
- $\delta_f$: The dynamical degree of an endomorphism $f \in \text{End}(A)$, which by definition is the spectral radius of the induced map $f^*$ on $\text{NS}(A)_\mathbb{Q}$.
- $H$: An ample divisor on $A$.
- $H_{\alpha, \beta}$: For $\alpha, \beta \in \text{End}(A)_\mathbb{Q}$, the divisor

  $$H_{\alpha, \beta} = (\alpha \pi_1 + \beta \pi_2)^*H - (\alpha \pi_1)^*H - (\beta \pi_2)^*H \in \text{Div}(A^2)_\mathbb{Q},$$

  where $\pi_1, \pi_2 : A \times A \rightarrow A$ are the projection maps.

- $\varphi_D$: For a divisor class $[D] \in \text{NS}(A)_\mathbb{Q}$, the map

  $$\varphi_D : A \rightarrow \hat{A} = \text{Pic}^0(A), \quad \varphi_D(x) = [T_x^*D - D],$$

  where $T_x : A \rightarrow A$ is the translation-by-$x$ map; see [10, page 60].
The inclusion
\[ \Phi : \text{NS}(A) \hookrightarrow \text{End}(A), \quad \Phi_D = \varphi_H^{-1} \circ \varphi_D, \]
induced by the ample divisor $H$; see \cite[pages 190, 208]{10}.

The quadratic part of the canonical height on $A$ relative to the divisor $D$, defined by $\hat{q}_{A,D}(x) = \lim n^{-2} h_{A,D}(nx)$.

The associated height pairing on $A$, defined by $\langle x, y \rangle_{A,D} = \hat{q}_{A,D}(P + Q) - \hat{q}_{A,D}(P) - \hat{q}_{A,D}(Q)$.

We extend the pairing $\mathbb{R}$-linearly to $A$.

The induced map $\hat{\alpha} : \hat{B} \to \hat{A}$ for a homomorphism $\alpha : A \to B$ of abelian varieties.

The Rosati involution on $A$ associated to $H$, defined by
\[ \alpha' = \varphi_H^{-1} \circ \hat{\alpha} \circ \varphi_H, \quad \text{where } \alpha, \alpha' \in \text{End}(A); \]
see \cite[page 189]{10}.

We begin with a number of geometric results that will be used as input to the height machinery.

**Lemma 24.** Let $A$ and $B$ be abelian varieties, let $f : B \to A$ be a homomorphism, and let $D \in \text{NS}(B)$. Then
\[ \varphi_{f^*D} = \hat{f} \circ \varphi_D \circ f. \]

If further $A = B$ and we fix an ample divisor $H$ to define the inclusion (6.1) and the Rosati involution (6.2), then
\[ \Phi_{\alpha^*D} = \alpha' \circ \Phi_D \circ \alpha \quad \text{for all } \alpha \in \text{End}(A). \]

**Proof.** For $x \in B$ we compute
\[ \varphi_{f^*D}(x) = [T^*_x(f^*D) - f^*D] \]
\[ = [f^*(T^*_{f(x)}D) - f^*D] \]
\[ = f^*[T^*_{f(x)}D - D] \]
\[ = \hat{f}(\varphi_D(f(x))). \]

Hence $\varphi_{f^*D} = \hat{f} \circ \varphi_D \circ f$, which proves (6.3).

In the case that $A = B$, we have
\[ \Phi_{\alpha^*D} = \varphi_H^{-1} \circ \varphi_{\alpha^*D} \quad \text{definition of } \Phi; \text{ see (6.1)} \]
\[ = \varphi_H^{-1} \circ \hat{\alpha} \circ \varphi_D \circ \alpha \quad \text{from (6.3)} \]
\[ = \varphi_H^{-1} \circ \hat{\alpha} \circ \varphi_H \circ \varphi_H^{-1} \circ \varphi_D \circ \alpha \]
\[ = \alpha' \circ \Phi_D \circ \alpha \quad \text{definitions (6.1) and (6.2) of } \Phi \text{ and Rosati,} \]
which completes the proof of (6.4). \qed

**Lemma 25.** Let $\alpha \in \text{End}(A)$. Then
\[ H_{\alpha,1} \equiv H_{1,\alpha'} \quad \text{in } \text{NS}(A^2). \]
Proof. It is a standard fact that
\[
\alpha \rightarrow \hat{\alpha} \quad \text{is a ring homomorphism} \quad \text{End}(A) \rightarrow \text{End}(\hat{A}).
\]

To see this, we start with [5 Proposition A.7.3.2], which says that \((\pi_1 + \pi_2)^* = \pi_1^* + \pi_2^*\) on \(\text{Pic}^0(A)\). Hence
\[
(\alpha + \beta)^* = ((\pi_1 + \pi_2) \circ (\alpha \times \beta))^* = (\pi_1^* + \pi_2^*) \circ (\alpha \times \beta)^*
\]
\[
= (\alpha \times \beta)^* \circ (\pi_1^* + \pi_2^*) = (\pi_1 \circ (\alpha \times \beta))^* + (\pi_2 \circ (\alpha \times \beta))^* = \alpha^* + \beta^*.
\]

We let \(\alpha, \beta \in \text{End}(A)_{\mathbb{R}}\) and compute more generally
\[
\varphi_{H_{\alpha, \beta}} = \varphi_{(\alpha \pi_1 + \beta \pi_2)^* H} - \varphi_{(\alpha \pi_1)^* H} - \varphi_{(\beta \pi_2)^* H} \quad \text{definition of} \ \ H_{\alpha, \beta}
\]
\[
= (\alpha \pi_1 + \beta \pi_2) \circ \varphi \circ (\alpha \pi_1 + \beta \pi_2) - (\alpha \pi_1^*) \circ \varphi \circ (\alpha \pi_1)
\]
\[
- (\beta \pi_2) \circ \varphi \circ (\beta \pi_2) \quad \text{from Lemma 24} \quad \text{equation (6.3)}
\]
\[
= (\alpha \pi_1 + \beta \pi_2) \circ \varphi \circ (\alpha \pi_1 + \beta \pi_2) - (\alpha \pi_1^*) \circ \varphi \circ (\alpha \pi_1)
\]
\[
- (\beta \pi_2) \circ \varphi \circ (\beta \pi_2) \quad \text{from (6.5)}
\]
\[
= \hat{\alpha} \circ \pi_1 \circ \varphi_{H} \circ \beta \circ \pi_2 + \hat{\beta} \circ \pi_2 \circ \varphi_{H} \circ \alpha \circ \pi_1
\]
\[
= \hat{\pi}_1 \circ \hat{\alpha} \circ \varphi_{H} \circ \beta \circ \pi_2 + \hat{\beta} \circ \varphi_{H} \circ \alpha \circ \pi_1
\]
\[
= \hat{\pi}_1 \circ \varphi_{H} \circ \alpha' \circ \beta \circ \pi_2 + \hat{\pi}_2 \circ \varphi_{H} \circ \beta' \circ \alpha \circ \pi_1
\]
\[
\quad \text{definition of the Rosati involution.}
\]

Hence
\[
\varphi_{H_{\alpha, 1}} = \hat{\pi}_1 \circ \varphi_{H} \circ \alpha' \circ \pi_2 + \hat{\pi}_2 \circ \varphi_{H} \circ \alpha \circ \pi_1
\]
and
\[
\varphi_{H_{1, \alpha'}} = \hat{\pi}_1 \circ \varphi_{H} \circ \alpha' \circ \pi_2 + \hat{\pi}_2 \circ \varphi_{H} \circ \alpha'' \circ \pi_1.
\]

Since \(\alpha'' = \alpha\), this shows that \(\varphi_{H_{\alpha, 1}} = \varphi_{H_{1, \alpha'}}\). To complete the proof that \(H_{\alpha, 1} \equiv H_{1, \alpha'}\), it suffices to note that the map
\[
\text{NS}(X)_{\mathbb{R}} \rightarrow \text{End}(X)_{\mathbb{R}}, \quad D \mapsto \varphi_D,
\]
is injective [10 page 208]. □

Proposition 26. Let \(D \in \text{NS}(A)_{\mathbb{R}}\) be a nef divisor. Then there is an endomorphism \(\alpha \in \text{End}(A)_{\mathbb{R}}\) satisfying
\[
\Phi_D = \alpha' \circ \alpha \quad \text{and} \quad \alpha' = \alpha.
\]

Proof. The \(\mathbb{R}\)-algebra \(\text{End}(A)_{\mathbb{R}}\) is isomorphic to a product of matrix algebras of the form \(\mathcal{M}_n(\mathbb{R})\), \(\mathcal{M}_n(\mathbb{C})\), and \(\mathcal{M}_n(\mathbb{H})\), and the isomorphism may be chosen so that the Rosati involution on \(\text{End}(A)_{\mathbb{R}}\) corresponds to the standard involution \(T \rightarrow \hat{T}\) on the matrix algebras; cf. [10] pages 208–209. (Here \(t \rightarrow \hat{t}\) is the identity on \(\mathbb{R}\), complex conjugation on \(\mathbb{C}\), and quaternionic conjugation on \(\mathbb{H}\).) The map \(\Phi\) gives an isomorphism [10 page 208],
\[
\Phi : \text{NS}(A)_{\mathbb{R}} \xrightarrow{\sim} \{\alpha \in \text{End}(A)_{\mathbb{R}} : \alpha' = \alpha\},
\]
so \(\text{NS}(A)_{\mathbb{R}}\) is identified with a product of Jordan algebras of the form \(\mathcal{H}_n(\mathbb{R})\), \(\mathcal{H}_n(\mathbb{C})\), and \(\mathcal{H}_n(\mathbb{H})\), where
\[
\mathcal{H}_n(\mathbb{K}) = \{T \in \mathcal{M}_n(\mathbb{K}) : \hat{T} = T\}
\]
denotes the set of Hermitian matrices for \(\mathbb{K} = \mathbb{R}, \mathbb{C}, \text{or} \mathbb{H}\) [10 Theorem 6, page 208].
The matrices in $H_n(\mathbb{K})$ have real eigenvalues, since they are self-adjoint. It is proven in [10] page 210 that a divisor $D$ is ample if and only if the eigenvalues associated to $\Phi_D$ are all strictly positive. Since the nef cone is the closure of the ample cone in $\text{NS}(A)_{\mathbb{R}}$, we see that $D$ is nef if and only if all of the eigenvalues associated to $\Phi_D$ are nonnegative. Equivalently, $D$ is nef if and only if the matrices associated to $\Phi_D$ are self-adjoint and positive semi-definite.

A standard result in linear algebra says that a self-adjoint positive semi-definite matrix $T \in H_n(\mathbb{K})$ can be written in the form $T = ^tSS$ for some $S \in H_n(\mathbb{K})$. See, e.g., [11, Theorem 7.27] for the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, and [9, Corollary 2.6] for $\mathbb{K} = \mathbb{H}$. (More precisely, [9, Corollary 2.6] says that $T$ is unitarily equivalent to a diagonal matrix in $\mathcal{M}_n(\mathbb{R})$. So $T = ^t\bar{U}\Delta U$ with $^t\bar{U} = U^{-1}$ and $\Delta$ diagonal and real. The positive semi-definiteness of $T$ implies that $\Delta$ has nonnegative entries, so $\Delta$ has a square root in $\mathcal{M}_n(\mathbb{R})$, say $\Delta = \Gamma^2$. Since $^t\bar{\Gamma} = \Gamma$, it follows that $T = ^t\bar{S}S$ with $S = ^t\bar{U}\Delta U$ satisfying $^t\bar{S} = S$.) Hence with the indicated identifications, we can find an $\alpha \in \text{End}(A)_{\mathbb{R}}$ satisfying $\Phi_D = \alpha' \circ \alpha$ and $\alpha' = \alpha$.

We now turn to some arithmetic consequences of these geometric facts.

**Proposition 27.** Let $\alpha \in \text{End}(A)_{\mathbb{R}}$ and let $x, y \in A(\bar{\mathbb{Q}})$. Then
\[
\langle \alpha(x), y \rangle_{A,H} = \langle x, \alpha'(y) \rangle_{A,H}.
\]

**Proof.** We first compute
\[
\hat{q}_{A^2,H_{\alpha,\beta}}(x,y) = \hat{q}_{A^2,\alpha_1+\beta_2}H(x,y) - \hat{q}_{A^2,\alpha_1}H(x,y)
- \hat{q}_{A^2,\beta_2}H(x,y)
\quad \text{by definition of } H_{\alpha,\beta}
\]
\[
= \hat{q}_{A,H}(\alpha(x) + \beta(y)) - \hat{q}_{A,H}(\alpha(x)) - \hat{q}_{A,H}(\beta(y))
\quad \text{(6.6)}
\]
Hence
\[
\langle \alpha(x), y \rangle_{A,H} = \hat{q}_{A^2,H_{\alpha}}(x,y)
\quad \text{and } \langle x, \alpha'(y) \rangle_{A,H} = \hat{q}_{A^2,H_{1,\alpha'}}(x,y).
\]

But Lemma 25 says that $H_{\alpha,1} \equiv H_{1,\alpha'}$, and the (quadratic part of the) canonical height on an abelian variety depends on only the algebraic equivalence class of the divisor, which completes the proof of Proposition 27.

**Proposition 28.** Let $D \in \text{Div}(A)_{\mathbb{R}}$ and $x, y \in A(\bar{\mathbb{Q}})_{\mathbb{R}}$. Then
\[
\langle x, y \rangle_{A,D} = \langle x, \Phi_D(y) \rangle_{A,H}.
\]

**Proof.** For $E \in \text{Div}(A)$ and $x, z \in A$, we compute
\[
\hat{q}_{A,\varphi_E}(z)(x) = \hat{q}_{A,T_zE-E}(x) \quad \text{definition of } \varphi_E
= \hat{q}_{A,T_zE}(x) - \hat{q}_{A,E}(x) \quad \text{linearity}
= \hat{q}_{A,E}(T_z(x)) - \hat{q}_{A,E}(T_z(0)) - \hat{q}_{A,E}(x)
\quad \text{from [5, Theorem B.5.6(d)]}
= \hat{q}_{A,E}(x + z) - \hat{q}_{A,E}(z) - \hat{q}_{A,E}(x)
\quad \text{(6.7)}
= \langle x, z \rangle_{A,E}.
\]
Applying (6.7) twice, we find that

\[
\langle x, \Phi_D(y) \rangle_{A,H} = \hat{q}_{A,D}(x) \from (6.7) \text{ with } z = \Phi_D(y) \text{ and } E = H
\]

\[
= \hat{q}_{A,\varphi_D(y)}(x) \text{ since } \Phi_D = \varphi_H^{-1} \circ \varphi_D
\]

\[
= \langle x, E \rangle_{A,D} \from (6.7) \text{ with } z = y \text{ and } E = D.
\]

This completes the proof of Proposition 28. \qed

We now have the tools needed to prove Theorem 1, which we restate as the first part of the following theorem.

**Theorem 29.** Let \( A/\overline{\mathbb{Q}} \) be an abelian variety defined over \( \overline{\mathbb{Q}} \), let \( D \in \text{Div}(A)_R \) be a nonzero nef divisor, and let \( \hat{q}_{A,D} \) be the quadratic part of the canonical height on \( A \) with respect to \( D \).

(a) There is a unique abelian subvariety \( B_D \subset A \) such that

\[
\{ x \in A(\overline{\mathbb{Q}}) : \hat{q}_{A,D}(x) = 0 \} = B_D(\overline{\mathbb{Q}}) + A(\overline{\mathbb{Q}})_{\text{tors}}.
\]

(b) Let \( f \in \text{End}(A) \) and suppose that \( f^*D \equiv \lambda D \) in \( \text{NS}(A)_R \). Then the abelian subvariety \( B_D \) from (a) is \( f \)-invariant, i.e., \( f(B_D) \subseteq B_D \).

(c) Let \( K/\mathbb{Q} \) be a number field over which \( A \) and \( D \) are defined. Then

\[
\{ x \in A(K) : \hat{q}_{A,D}(x) = 0 \}
\]

is not Zariski dense in \( A \).

**Proof of Theorem 1.** Since \( D \) is nef, we can use Proposition 26 to find an \( \alpha \in \text{End}(A)_R \) (depending on \( D \)) satisfying

\[
\Phi_D = \alpha' \circ \alpha \quad \text{and} \quad \alpha' = \alpha.
\]

We compute

\[
\hat{q}_{A,D}(x) = \frac{1}{2} \langle x, x \rangle_{A,D}
\]

\[
= \frac{1}{2} \langle x, \Phi_D(x) \rangle_{A,H} \from \text{Proposition 28}
\]

\[
= \frac{1}{2} \langle x, \alpha' \circ \alpha(x) \rangle_{A,H} \from (6.8)
\]

\[
= \frac{1}{2} \langle \alpha(x), \alpha(x) \rangle_{A,H} \from \text{Proposition 27} \text{ with } y = \alpha(x)
\]

\[
= \hat{q}_{A,H}(\alpha(x)).
\]

But \( \hat{q}_{A,H} \) is the (quadratic part of the) canonical height on \( A \) relative to an ample divisor, so it is a positive definite quadratic form on \( A(\overline{\mathbb{Q}}) \otimes \mathbb{R} \); see [5, Proposition B.5.3]. (Recall that the canonical height pairing on \( A(\overline{\mathbb{Q}}) \) is extended \( \mathbb{R} \)-linearly to \( A(\overline{\mathbb{Q}})_R \).) Hence

\[
\hat{q}_{A,D}(x) = 0 \iff \alpha(x) = 0 \in A(\overline{\mathbb{Q}})_R.
\]

To complete the proof of Theorem 1 we use the following elementary linear algebra result.
Lemma 30. Let $V$ be a $\mathbb{Q}$-vector space, and fix a $\mathbb{Q}$-vector subspace $D \subset \text{End}(V)$ of the ring of $\mathbb{Q}$-linear endomorphisms of $V$. Let $\alpha \in D \otimes \mathbb{R}$ be an $\mathbb{R}$-linear transformation of $V \otimes \mathbb{R}$. Then there is a finite collection of endomorphisms $\beta_1, \ldots, \beta_r \in D$ with the property that for $v \in V$,

$$\alpha(v) = 0 \text{ in } V \otimes \mathbb{R} \iff \beta_1(v) = \cdots = \beta_r(v) = 0 \text{ in } V.$$ 

Proof. We write $\alpha \in D \otimes \mathbb{R}$ as a sum

$$\alpha = \sum_{i=1}^{r} c_i \beta_i$$

where we choose $r$ to be minimal. We claim that this implies that $c_1, \ldots, c_r \in \mathbb{R}$ are $\mathbb{Q}$-linearly independent.

Suppose not. Then after relabeling, we can write $c_1 = \sum_{i=2}^{r} b_i c_i$ with $b_i \in \mathbb{Q}$. But then $\alpha = \sum_{i=1}^{r} c_i (b_i \beta_1 + \beta_i)$ with $b_i \beta_1 + \beta_i \in D$, contradicting the minimality of $r$.

We next claim that for $v_1, \ldots, v_r \in V$ we have

$$\sum_{i=1}^{r} c_i v_i = 0 \quad \text{in } V \otimes \mathbb{R} \iff v_1 = \cdots = v_r = 0.$$ 

To prove this claim, we let $\{e_j\}_{j \in J}$ be a $\mathbb{Q}$-basis for $V$, so $\{e_j\}_{j \in J}$ is also automatically an $\mathbb{R}$-basis for $V \otimes \mathbb{R}$. For each $1 \leq i \leq r$, write $v_i = \sum_{j \in J} b_{ij} e_j$ with all $b_{ij} \in \mathbb{Q}$ and almost all $b_{ij} = 0$. Then $0 = \sum_i c_i v_i = \sum_{j \in J} (\sum_i c_i b_{ij}) e_j$, so the assumptions that the $e_j$ are $\mathbb{R}$-linearly independent and the $c_i$ are $\mathbb{Q}$-linearly independent implies that $b_{ij} = 0$ for all $i, j$.

We now see that for $v \in V$ we have

$$\alpha(v) = 0 \text{ in } V \otimes \mathbb{R} \iff \sum_{i=1}^{r} c_i \beta_i(v) = 0 \text{ in } V \otimes \mathbb{R} \iff \beta_1(v) = \cdots = \beta_r(v) = 0 \text{ in } V,$$

where for the last implication we have used (6.10) and the fact that $\beta_1(v), \ldots, \beta_i(v)$ are in $V$. \hfill \square

We now resume the proof of Theorem 29. We let $\alpha \in \text{End}(A) \mathbb{R}$ be the nonzero endomorphism appearing in (6.8). We apply Lemma 30 to the $\mathbb{Q}$-vector space $V = A(\mathbb{Q})_Q$ and the $\mathbb{Q}$-subspace

$$D = \text{End}(A) \mathbb{Q} \subset \text{End}(V).$$

(Here $\text{End}(A)$ denotes the ring of algebraic maps $A \to A$, while $\text{End}(V)$ denotes the ring of $\mathbb{Q}$-linear maps $V \to V$.) Lemma 30 says that we can find endomorphisms $\beta_1, \ldots, \beta_r \in \text{End}(A) \mathbb{Q}$ so that for $x \in A(\mathbb{Q})_Q$,

$$\alpha(x) = 0 \iff \beta_1(x) = \cdots = \beta_r(x) = 0.$$ 

Combining (6.9) and (6.11) yields

$$\hat{q}_{A,D}(x) = 0 \iff \beta_1(x) = \cdots = \beta_r(x) = 0 \in A(\mathbb{Q})_Q.$$ 

Replacing each of the finitely many $\beta_i \in \text{End}(A) \mathbb{Q}$ by an appropriate nonzero integral multiple $m_i \beta_i$, we may assume that the $\beta_i$ all lie in $\text{End}(A)$.

We let

$$B = \bigcap_{i=1}^{r} \ker(\beta_i).$$
We note that $B$ is a (not necessarily connected) algebraic subgroup of $A$, and further, $B \neq A$, since $\alpha \neq 0$, so at least one $\beta_i \neq 0$. (This is where we use the assumption that $D \neq 0$, which ensures that $\Phi_D \neq 0$, so $\alpha \neq 0$.) The definition of $B$ and (6.12) imply that

$$\{ x \in A(\bar{\mathbb{Q}}) : \hat{q}_{A,D}(x) = 0 \} = B(\bar{\mathbb{Q}})_{\mathbb{Q}}.$$ 

It follows that

$$\{ x \in A(\bar{\mathbb{Q}}) : \hat{q}_{A,D}(x) = 0 \} = B(\bar{\mathbb{Q}})^{\text{div}},$$

where

$$B(\bar{\mathbb{Q}})^{\text{div}} = \{ y \in A(\bar{\mathbb{Q}}) : my \in B(\bar{\mathbb{Q}}) \text{ for some } m \geq 1 \}$$

is the divisible hull of $B$ in $A$. Without loss of generality, we may replace $B$ by its connected component, since that won’t change the group $B(\bar{\mathbb{Q}})^{\text{div}}$. But if $B$ is connected, then we claim that

$$B(\bar{\mathbb{Q}})^{\text{div}} = B(\bar{\mathbb{Q}}) + A(\bar{\mathbb{Q}})_{\text{tors}}.$$ 

One inclusion is clear. For the other, let $x \in B(\bar{\mathbb{Q}})^{\text{div}}$. Then $mx \in B$ for some $m \geq 1$. Since $B$ is connected, there is a $y \in B(\bar{\mathbb{Q}})$ with $my = mx$. Then $x = y + (x - y)$ with $y \in B(\bar{\mathbb{Q}})$ and $x - y \in A(\bar{\mathbb{Q}})_{\text{tors}}$.

In order to complete the proof of Theorem 24(a), it remains only to show that $B$ is uniquely determined by $D$. So suppose that $B_1 \subset A$ and $B_2 \subset A$ are abelian subvarieties of $A$ satisfying $B_1 + A_{\text{tors}} = B_2 + A_{\text{tors}}$. We may view $B_2 + A_{\text{tors}}$ as a scheme over $\text{Spec}(\bar{\mathbb{Q}})$, and our assumption implies that $B_1$ is a $\mathbb{Q}$-subscheme of $B_2 + A_{\text{tors}}$. But $B_1$ is a scheme of finite type over $\text{Spec}(\mathbb{Q})$, so it is contained in a subscheme of $B_2 + A_{\text{tors}}$ of finite type. Hence there is an integer $m \geq 1$ such that $B_1 \subset B_2 + A[m]$. But $B_1$ and $B_2$ are irreducible, so $B_1 \subset B_2 + t$ for some $t \in A[m]$. Finally, the fact that $B_1$ and $B_2$ contain 0 implies that $B_1 \subset B_2$. Reversing the roles of $B_1$ and $B_2$ gives the opposite inclusion, so $B_1 = B_2$.

(b) We have

$$\hat{q}_{A,D} \circ f = \hat{q}_{A,f^*D} = \hat{q}_{A,\lambda D} = \lambda \hat{q}_{A,D},$$

where the middle equality follows from the fact that the quadratic part of the canonical height depends only on the algebraic equivalence class of the divisor. Hence applying (a) twice, we find that

$$f(B_D) \subset f(B_D + A_{\text{tors}}) = f(\{ x \in A : \hat{q}_{A,D}(x) = 0 \}) \subset \{ x \in A : \hat{q}_{A,D}(x) = 0 \} = B_D + A_{\text{tors}}.$$ 

But as in the proof of (a), the image $f(B_D)$ is a subscheme of $B_D + A_{\text{tors}}$ that is of finite type over $\text{Spec}(\mathbb{Q})$, so $f(B_D)$ is contained in $B_D + A[m]$ for some $m \geq 1$. But $f(B_D)$ is irreducible, so it is contained in $B_D + t$ for some $t \in A[m]$, and finally the fact that 0 is in both $B_D$ and $f(B_D)$ implies that $f(B_D) \subset B_D$.

(c) Let

$$\Gamma = A(K) \cap (B_D(\bar{\mathbb{Q}}) + A(\bar{\mathbb{Q}})_{\text{tors}}).$$

It follows (a) that

$$\{ x \in A(K) : \hat{q}_{A,D}(x) = 0 \} \subset \Gamma,$$

so in order to prove (c), it suffices to show that $\Gamma$ is not Zariski dense in $A$. 

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The group $A(K)$ is finitely generated [3] Theorem C.0.1], so its subgroup $\Gamma$ is also finitely generated. Let $y_1, \ldots, y_l$ be generators for $\Gamma$. Each $y_i$ is in $B + A_{\text{tors}}$, so we can find an integer $m \geq 1$ such that $my_1, \ldots, my_l \in B_D$. It follows that $m\Gamma \subset B_D$, and hence $\Gamma \subset B_D + A[m]$ is not Zariski dense in $A$. □

**Lemma 31.** Let $\alpha : X \to X$ be a morphism of a normal projective variety. Then there exists a nonzero nef eigendivisor $F \in \text{NS}(X)_R$ satisfying $f^*F \equiv \delta_f F$.

**Proof.** The existence of $F$ follows from an elementary Perron–Frobenius-type result of Birkhoff [3] applied to the vector space $\text{NS}(X)_R$, the linear transformation $f^*$, and the nef cone in $\text{NS}(X)_R$. See [6] Remark 29 and Proposition 30 for details. □

**Corollary 32.** Let $f \in \text{End}(A)$, let $F \in \text{Div}(A)_R$ be a divisor satisfying $f^*F \equiv \delta_f F$ as described in Lemma 31, let $x \in A(\mathbb{Q})$, and let $\overline{O_f(x)}$ denote the Zariski closure of $x$ in $A$ of the $f$-orbit of $x$. Then the following implications hold:

$$\overline{O_f(x)} = A \implies \hat{q}_{A,F}(x) > 0 \implies \alpha_f(x) = \delta_f.$$

**Proof.** Applying (6.13) with $\lambda = \delta_f$, we have

$$\hat{q}_{A,F} \circ f = \delta_f \hat{q}_{A,F}.$$

Further, if $D$ is symmetric, then we have $\hat{q}_{A,D} = h_{A,D} + O(1)$, so

$$\hat{q}_{A,F} = \frac{1}{2} \hat{q}_{A,F} + [-1] \cdot F = \frac{1}{2} h_{A,F} + [-1] \cdot F + O(1).$$

We also note that the canonical height associated to the nef divisor $F$ is nonnegative, because $F + \epsilon H$ is ample for any $\epsilon > 0$ and any ample divisor $H$, so

$$\hat{q}_{A,F} = \hat{q}_{A,F} + \epsilon H - \hat{q}_{A,F} \geq -\epsilon \hat{q}_{A,H}.$$ 

Since $\hat{q}_{A,H} \geq 0$ and $\epsilon$ is arbitrary, we see that $\hat{q}_{A,F} \geq 0$.

We now suppose that $\hat{q}_{A,F}(x) > 0$ and compute

$$\alpha_f(x) = \alpha_f(x) \quad \text{from Theorem 3}
\geq \liminf_{n \to \infty} [h_{A,F} + [-1] \cdot F(f^n(x))]^{1/n} \quad \text{from Lemma 18}
= \liminf_{n \to \infty} \hat{q}_{A,F}(f^n(x))^{1/n} \quad \text{from (6.14)}
= \liminf_{n \to \infty} (\delta_f^n \hat{q}_{A,F}(x))^{1/n} \quad \text{from (6.13)}
= \delta_f \quad \text{since } \hat{q}_{A,F}(x) > 0.$$

For the other direction, we note that [5] says that $\alpha_f(x) \leq \delta_f$ for any dominant rational map $f : X \dashrightarrow X$ of smooth projective varieties. We have not assumed that our map $f : A \to A$ is dominant, but since $f^n(A)$ is a sequence of abelian subvarieties of $A$ of nonincreasing dimension, it eventually stabilizes, say $B = f^n(A)$ with $f : B \to B$ an isogeny. And of course, the abelian variety $B$ is smooth, so [6] gives $\alpha_{f|B}(f^n(x)) \leq \delta_{f|B}$. Then one easily checks that $\alpha_f(x) = \alpha_{f|B}(f^n(x))$ and $\delta_f = \delta_{f|B}$, which completes the proof of the implication

$$\hat{q}_{A,F}(x) > 0 \implies \alpha_f(x) = \delta_f.$$

Next we let $K/\mathbb{Q}$ be a number field such that $A$, $D$, and $f$ are defined over $K$ and such that $x \in A(K)$. Then $O_f(x) \subset A(K)$. Now suppose that $\hat{q}_{A,F}(x) = 0$. Then (6.14) tells us that $\hat{q}_{A,F}(f^n(x)) = \delta_f^n \hat{q}_{A,F}(x) = 0$ for all $n \geq 0$, so

$$O_f(x) \subset \{ y \in A(K) : \hat{q}_{A,F}(y) = 0 \}.$$
Theorem (29c) tells us that the set on the right is not Zariski dense in \( A \), so the same is true of \( \mathcal{O}_f(x) \). This completes the proof of the implication
\[
\hat{q}_{A,F}(x) = 0 \implies \overline{\mathcal{O}_f(x)} \neq A,
\]
which combined with the fact that \( \hat{q}_{A,F}(x) \geq 0 \) gives the other desired implication
\[
\overline{\mathcal{O}_f(x)} = A \implies \hat{q}_{A,F}(x) > 0.
\]
This completes the proof of Corollary 32. \( \square \)

Remark 33. We give examples to show that neither of the implications in Corollary 32 is true in the opposite direction. Let \( E \) be an elliptic curve, and take
\[
A = E^2, \quad H = \pi_1^1(O) + \pi_2^2(O), \quad \text{and} \quad f(P, Q) = (2P, 2Q).
\]
Then \( H \) is ample and symmetric (so \( \hat{h}_{A,H} = \hat{h}_{A,H} \)) and satisfies \( f^* H \sim 4H \), so in particular \( \delta_f = 4 \). Further,
\[
\hat{h}_{A,H}(P, Q) = \hat{h}_{E,(O)}(P) + \hat{h}_{E,(O)}(Q).
\]
Then for any nontorsion point \( P \in E \) we have
\[
\hat{h}_{A,H}(P, O) = \hat{h}_{E,(O)}(P) + \hat{h}_{E,(O)}(O) = \hat{h}_{E,(O)}(P) > 0
\]
and
\[
\overline{\mathcal{O}_f(P, O)} = E \times \{ O \},
\]
which shows that the implication \( \overline{\mathcal{O}_f(x)} = A \implies \hat{q}_{A,F}(x) > 0 \) cannot be reversed. Continuing with the assumption that \( P \notin E_{\text{tors}} \), we compute
\[
\hat{h}_{A,H}(f^n(P, O)) = \hat{h}_{A,H}(f^n(P, O)) + O(1)
\]
\[
= 4^n \hat{h}_{A,H}(P, O) + O(1)
\]
\[
= 4^n \hat{h}_{E,(O)}(P) + O(1),
\]
so the fact that \( \hat{h}_{E,(O)}(P) > 0 \) and the definition of arithmetic degree tell us that \( \alpha_f(P, O) = 4 = \delta_f \). On the other hand, consider the nonzero nef divisor \( F = \pi_2^*(O) \). It satisfies
\[
f^* F \sim 4F \quad \text{and} \quad \hat{h}_{A,F}(P, O) = \hat{h}_{E,(O)}(O) = 0.
\]
Thus \( \alpha_f(P, O) = \delta_f \) and \( \hat{h}_{A,F}(P, O) = 0 \), which shows that the implication
\[
\hat{q}_{A,F}(x) > 0 \implies \alpha_f(x) = \delta_f
\]
cannot be reversed.

7. An example of nef heights on certain CM abelian varieties

In this section we illustrate Theorem 11 and Corollary 32 by working out the details for a nontrivial example, specifically for an abelian variety \( A \) whose endomorphism algebra \( \operatorname{End}(A)_\mathbb{Q} \) is isomorphic to a real quadratic field \( \mathbb{Q}(\sigma) \), where \( \sigma^2 = m \) is a positive nonsquare integer. In this case the Rosati involution is the identity map, so \( \Phi : \text{NS}(A)_\mathbb{Q} \to \operatorname{End}(A)_\mathbb{Q} \) is an isomorphism. We always have \( \Phi_H = 1_A \), and we choose a divisor \( F \in \text{NS}(A)_\mathbb{Q} \) such that \( \Phi_F = \sigma \). Then
\[
\operatorname{End}(A)_\mathbb{R} \cong \mathbb{R}(\sigma) \xrightarrow{i} \mathbb{R} \times \mathbb{R}, \quad i(a + b\sigma) = (a + b\sqrt{m}, a - b\sqrt{m}),
\]
and \( D \in \text{NS}(A)_\mathbb{R} \) is nef if and only if both coordinates of \( i(\Phi_D) \) are nonnegative; cf. [10, page 210].
The divisor
\[ D = \sqrt{m}H + F \in \text{NS}(A)_{\mathbb{R}} \]
satisfies
\[ i(\Phi_D) = i(\sqrt{m}\Phi_H + \Phi_F) = i(\sqrt{m} + \sigma) = (2\sqrt{m}, 0), \]
so \( D \) is nef. Proposition \[26\] says that \( \Phi_D \) can be written in the form \( \alpha' \circ \alpha \) for
some \( \alpha \in \text{End}(A)_{\mathbb{R}} \). In fact, we explicitly have
\[ \Phi_D^2 = (\sqrt{m} + \sigma)^2 = 2m + 2\sqrt{m}\sigma = 2\sqrt{m}\Phi_D, \]
so
\[ \Phi_D = \alpha' \circ \alpha = \alpha^2 \quad \text{with} \quad \alpha = (4m)^{-1/4}\Phi_D. \]
Then \( \hat{h}_{A,D}(x) = \hat{h}_{A,H}(\alpha(x)) \) (cf. the computation in the proof of Theorem \[11\]), so
\[ \hat{h}_{A,D}(x) = 0 \iff \alpha(x) = 0 \quad \text{in} \ A(\bar{\mathbb{Q}})_{\mathbb{R}}. \]
But
\[ \alpha(x) = (4m)^{-1/4}\Phi_D(x) = (4m)^{-1/4}(\sqrt{m}x + \sigma(x)), \]
so we see that
\[ (7.1) \quad \hat{h}_{A,D}(x) = 0 \iff \sqrt{m}x + \sigma(x) = 0 \quad \text{in} \ A(\bar{\mathbb{Q}})_{\mathbb{R}}. \]
Consider the linear transformation
\[ T : A(\bar{\mathbb{Q}})_{\mathbb{R}} \rightarrow A(\bar{\mathbb{Q}})_{\mathbb{R}}, \quad x \mapsto (\sqrt{m} + \sigma)(x) = \sqrt{m}x + \sigma(x). \]
It follows from \((7.1)\) that \( \hat{h}_{A,D}(x) = 0 \) if and only if \( x \in \ker(T) \). If we also assume that \( x \in A(\bar{\mathbb{Q}})_{\mathbb{Q}} \), then choosing a basis \( \{v_i\} \) for \( A(\bar{\mathbb{Q}})_{\mathbb{Q}} \) and writing \( x = \sum a_i v_i \) with \( a_i \in \mathbb{Q} \), we have (all sums have finitely many nonzero terms)
\[
0 = T(x) = \sum_i a_i T(v_i) \\
= \sum_i a_i \sqrt{m}v_i + a_i \sigma(v_i) \\
= \sum_i a_i \sqrt{m}v_i + a_i \sum_j b_{ij}v_j \quad \text{for some} \ b_{ij} \in \mathbb{Q} \\
= \sum_i \left( a_i \sqrt{m} + \sum_j a_j b_{ji} \right) v_i.
\]
Since \( \{v_j\} \) is a \( \mathbb{Q} \)-basis for \( A(\bar{\mathbb{Q}})_{\mathbb{Q}} \), it is a fortiori an \( \mathbb{R} \)-basis for \( A(\bar{\mathbb{Q}})_{\mathbb{R}} \), so we see that \( a_i \sqrt{m} + \sum_j a_j b_{ji} = 0 \) for all \( i \). But \( a_i, b_{ij} \in \mathbb{Q} \), while \( \sqrt{m} \notin \mathbb{Q} \), so we conclude that \( a_i = 0 \) for all \( i \). Hence \( x = 0 \) in \( A(\bar{\mathbb{Q}})_{\mathbb{Q}} \), which is equivalent to \( x \) being a torsion point. We note that this is the appropriate conclusion from Theorem \[11\] since \( A \) is simple, so the abelian subvariety \( B \subseteq A \) must be \( B = 0 \), and thus \( B^{\text{div}} = A_{\text{tors}} \). Finally, we remark that an easy calculation shows that \( D \) is an eigendivisor for every endomorphism \( f \in \text{End}(A) \), and more precisely, that \( f^* D = \delta_f D \). Hence if \( x \notin A_{\text{tors}} \), then we have proven that \( \hat{h}_{A,D}(x) > 0 \), from which we conclude (as in the proof of Corollary \[22\]) that
\[
\alpha_f(x) \geq \liminf \hat{h}_{A,D}(f^n(x))^{1/n} = \liminf(\delta_f^n \hat{h}_{A,D}(x))^{1/n} = \delta_f.
\]
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