ON A CLASS OF QUOTIENT SPACES OF MOMENT-ANGLE COMPLEXES

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Abstract. For a simplicial complex $K$ and a partition $\alpha$ of the vertex set of $K$, we define a quotient space of the (real) moment-angle complex of $K$ by some (not necessarily free) torus action determined by $\alpha$. We obtain a Hochster type formula to compute the cohomology groups of such spaces. Meanwhile, we can derive the formula from some stable decompositions of these spaces. Moreover, we show that their cohomology rings with $\mathbb{Z}_2$-coefficients are isomorphic as multigraded $\mathbb{Z}_2$-modules (or algebras) to the cohomology of some multigraded differential algebras determined by $K$ and $\alpha$.

1. Introduction

An abstract simplicial complex on a set $[m] = \{v_1, \ldots, v_m\}$ is a collection $\mathcal{K}$ of subsets $\sigma \subseteq [m]$ such that if $\sigma \in \mathcal{K}$, then any subset of $\sigma$ also belongs to $\mathcal{K}$. We always assume that the empty set belongs to $\mathcal{K}$ and refer to $\sigma \in \mathcal{K}$ as an abstract simplex of $\mathcal{K}$. The simplex corresponding to the empty set is denoted by $\emptyset$. In particular, any element of $[m]$ that belongs to $\mathcal{K}$ is called a vertex of $\mathcal{K}$. We call the number of vertices of a simplex $\sigma$ the rank of $\sigma$, denoted by $\text{rank}(\sigma)$. Let $\dim(\sigma)$ denote the dimension of a simplex $\sigma$. So $\text{rank}(\sigma) = \dim(\sigma) + 1$.

Any finite abstract simplicial complex $\mathcal{K}$ admits a geometric realization $|\mathcal{K}|$ in some Euclidean space. If not particularly indicated, we do not distinguish $\mathcal{K}$ and its geometric realization $|\mathcal{K}|$ in this paper.

Given a finite abstract simplicial complex $\mathcal{K}$ on a set $[m]$ and a pair of spaces $(X, A)$ with $A \subset X$, we can construct of a topological space $(X, A)^\mathcal{K}$ by:

$$(X, A)^\mathcal{K} = \bigcup_{\sigma \in \mathcal{K}} (X, A)_{\sigma},$$

where $(X, A)_{\sigma} = \prod_{v_i \in \sigma} X \times \prod_{v_i \notin \sigma} A.$

The symbol $\prod$ here and in the rest of this paper means Cartesian product.

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By definition $(X, A)^K$ is a subspace of the Cartesian product of $m$ copies of $X$. It is called the polyhedral product or the generalized moment-angle complex of $K$ and $(X, A)$. In particular, $Z_K = (D^2, S^1)^K$ and $\mathbb{R}Z_K = (D^1, S^0)^K$ are called the moment-angle complex of $K$ and real moment-angle complex of $K$, respectively (see [5]). Moreover, we can define the polyhedral product $(X, A)^K$ of $K$ with a set of $m$ pairs of spaces $(X, A) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ (see [2] or [6] Sec 4.2).

Originally, $Z_K$ and $\mathbb{R}Z_K$ are introduced by Davis and Januszkiewicz [10] in a completely different way. Let $K'$ denote the barycentric subdivision of $K$. We can consider $K'$ as the set of chains of simplices in $K$ ordered by inclusions. For each simplex $\sigma \in K$, let $F_\sigma$ denote the geometric realization of the poset $K_{\geq \sigma} = \{\tau \in K \mid \sigma \subseteq \tau\}$. Thus, $F_\sigma$ is the subcomplex of $K'$ consisting of all simplices of the form $\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_t$. Let $P_K$ denote the cone on $K'$. If $\sigma$ is a $(k-1)$-simplex, then we say that $F_\sigma \subset P_K$ is a face of codimension $k$ in $P_K$. The polyhedron $P_K$ together with its decomposition into "faces" $\{F_\sigma\}_{\sigma \in K}$ is called a simple polyhedral complex in [10] p.428.

Let $V(K)$ denote the vertex set of $K$. Any map $\lambda: V(K) \to (\mathbb{Z}_2)^r$ is called a $(\mathbb{Z}_2)^r$-coloring of $K$, and any element of $(\mathbb{Z}_2)^r$ is called a color. For any $\sigma \in K$, let $V(\sigma)$ denote the vertex set of $\sigma$ and let $G_\lambda(\sigma)$ denote the subgroup of $(\mathbb{Z}_2)^r$ generated by $\{\lambda(v) \mid v \in V(\sigma)\}$. Define

$$X(K, \lambda) := P_K \times (\mathbb{Z}_2)^r / \sim \quad (2)$$

where $(p, g) \sim (p', g')$ whenever $p' = p \in F_\sigma$ and $g' - g \in G_\lambda(\sigma)$ for some $\sigma \in K$. So when $p$ lies in the relative interior of $F_\sigma$, $(p, g) \sim (p', g')$ if and only if $p' = p$ and $g' - g \in G_\lambda(\sigma)$.

We call $\lambda$ non-degenerate if $\text{rank}_{\mathbb{Z}_2}(G_\lambda(\sigma)) = \text{rank}(\sigma)$ for any simplex $\sigma$ in $K$. In particular, when $r = |V(K)| = m$ and $\{\lambda(v_i) \mid 1 \leq i \leq m\}$ is a basis of $(\mathbb{Z}_2)^m$, the space $X(K, \lambda)$ is homeomorphic to $\mathbb{R}Z_K$. Let $\pi_K: P_K \times (\mathbb{Z}_2)^m \to \mathbb{R}Z_K$ be the corresponding quotient map in [2]. There is a canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}Z_K$ defined by:

$$g' \cdot \pi_K(p, g) = \pi_K(p, g + g')$$

Then any subgroup of $(\mathbb{Z}_2)^m$ acts canonically on $\mathbb{R}Z_K$ through this action.

Let $\{e_1, \cdots, e_m\}$ be a basis of $(\mathbb{Z}_2)^m$. For a map $\lambda: V(K) \to (\mathbb{Z}_2)^r$ whose image spans the whole $(\mathbb{Z}_2)^r$, we have a short exact sequence of groups

$$0 \to (\mathbb{Z}_2)^{m-r} \to (\mathbb{Z}_2)^m \xrightarrow{\tilde{\lambda}} (\mathbb{Z}_2)^r \to 0$$

where $\tilde{\lambda}$ is the group homomorphism defined by $\tilde{\lambda}(e_i) = \lambda(v_i), 1 \leq i \leq m$. The kernel of $\tilde{\lambda}$ is a subgroup of $(\mathbb{Z}_2)^m$, denoted by $H_\lambda$, which is isomorphic to $(\mathbb{Z}_2)^{m-r}$. It is easy to see that $X(K, \lambda)$ is homeomorphic to the quotient space $\mathbb{R}Z_K/H_\lambda$, where $H_\lambda$ acts on $\mathbb{R}Z_K$ via the canonical action.
The cohomology groups of \( \mathbb{R}Z_K \) can be computed via a Hochster type formula as follows (see [3] or 9). For any subset \( J \subset [m] \), let \( K_J \) denote the full subcomplex of \( K \) obtained by restricting to \( J \). Let \( k \) denote a field or \( \mathbb{Z} \) in the rest of the paper. We have

\[
H^q(\mathbb{R}Z_K; k) \cong \bigoplus_{J \subset [m]} \tilde{H}^{q-1}(K_J; k), \quad q \geq 0
\]  

(3)

where \( \tilde{H}^{q-1}(K_J; k) \) is the reduced cohomology groups of \( K_J \). Here and in the rest we adopt the convention \( \tilde{H}^{-1}(K_J; k) = k \).

There is a completely parallel version of the construction (2) for any \( \mathbb{Z}^r \)-coloring \( \Lambda : V(K) \to \mathbb{Z}^r \) (see [10]). Indeed, let \( \hat{G}_\Lambda(\sigma) \) denote the toral subgroup of \( T^r \) corresponding to the subgroup of \( \mathbb{Z}^r \) generated by \( \{\Lambda(v) \mid v \in V(\sigma)\} \). Define

\[
X(K, \Lambda) := P_K \times T^r / \sim
\]  

(4)

where \((p, g) \sim (p', g')\) whenever \( p' = p \in F_\sigma \) and \( g'g^{-1} \in \hat{G}_\Lambda(\sigma) \) for some \( \sigma \in K \).

We call \( \Lambda \) non-degenerate if for any \( \sigma \in K \), the subgroup of \( \mathbb{Z}^r \) generated by \( \{\Lambda(v) \mid v \in V(\sigma)\} \) is a direct summand of \( \mathbb{Z}^r \) with rank equal to \( \text{rank}(\sigma) \). This implies that \( \dim(\hat{G}_\Lambda(\sigma)) = \text{rank}(\sigma) \) for any simplex \( \sigma \) in \( K \). In particular, when \( r = |V(K)| = m \) and \( \{\Lambda(v_i) \mid 1 \leq i \leq m\} \) is a basis of \( \mathbb{Z}^m \), \( X(K, \Lambda) \) is homeomorphic to \( K \). Let \( \tilde{\pi}_K : P_K \times T^m \to Z_K \) be the corresponding quotient map in (3). There is a canonical action of \( T^m \) on \( Z_K \) defined by:

\[
g' \cdot \tilde{\pi}_K(p, g) = \tilde{\pi}_K(p, gg'), \quad p \in P_K, \quad g, g' \in T^m.
\]

Then any subgroup of \( T^m \) acts canonically on \( Z_K \) through this action.

Let \( \{\hat{e}_1, \ldots, \hat{e}_m\} \) be a basis of \( \mathbb{Z}^m \). For a map \( \Lambda : V(K) \to \mathbb{Z}^r \) whose image spans the whole \( \mathbb{Z}^r \), we have a short exact sequence

\[
0 \longrightarrow \mathbb{Z}^{m-r} \longrightarrow \mathbb{Z}^m \quad \Lambda \longrightarrow \mathbb{Z}^r \longrightarrow 0
\]

where \( \Lambda \) is the group homomorphism defined by \( \Lambda(\hat{e}_i) = \Lambda(v_i), 1 \leq i \leq m \). The kernel of \( \Lambda \) is a subgroup of \( \mathbb{Z}^m \) which determines an \( (m-r) \)-dimensional toral subgroup \( S_\Lambda \subset T^m \). Then it is easy to see that \( X(K, \Lambda) \) is homeomorphic to the quotient space \( Z_K/S_\Lambda \), where \( S_\Lambda \) acts on \( Z_K \) via the canonical action.

Similarly, there is a Hochster type formula for the cohomology groups for \( Z_K \) (see [3] or 5).

\[
H^q(Z_K; k) = \bigoplus_{J \subset [m]} \tilde{H}^{q-|J|-1}(K_J; k), \quad q \geq 0.
\]  

(5)

It is shown in [5] that there is a natural bigrading on \( H^*(Z_K; k) \) so that it is isomorphic to \( \text{Tor}_{k[v_1, \ldots, v_m]}(k[K]; k) \) as bigraded algebras where \( k[K] \) is the face ring of \( K \) over \( k \).
Recall that the face ring (or the Stanley-Reisner ring) of a simplicial complex $K$ on the set $[m]$ with $k$-coefficients is the quotient ring

$$k(K) = k[v_1, \cdots, v_m]/\mathcal{I}_K$$

(6)

where $\mathcal{I}_K$ is the ideal generated by the monomials $v_{i_1} \cdots v_{i_s}$ for which $\{v_{i_1}, \cdots, v_{i_s}\}$ does not span a simplex of $K$. The ideal $\mathcal{I}_K$ is called the Stanley-Reisner ideal of $K$. The face ring $k[K]$ can be equivalently defined as the quotient algebra

$$k[K] := k[v_\sigma : \sigma \in K]/\mathcal{I}_K'$$

(7)

where $\mathcal{I}_K'$ is the ideal generated by all the elements of the form

$$v_0 - 1, \ v_\sigma v_\tau - v_{\sigma \wedge \tau} \cdot v_{\sigma \vee \tau}, \ v_\sigma$$

where $\sigma \wedge \tau$ denotes the maximal common face of $\sigma$ and $\tau$, and $\sigma \vee \tau$ denotes the minimal face of $K$ that contains both $\sigma$ and $\tau$. If there is no simplex contains both $\sigma$ and $\tau$, we let $v_{\sigma \vee \tau} = 0$. This way of defining face rings is introduced by R. Stanley in [16] in the context of simplicial posets. The definition (7) of $k[K]$ is related to our previous definition (6) of $k[K]$ by mapping $v_\sigma$ to $\prod_{v_j \in V(\sigma)} v_j$. In particular, we use $v_{(j)}$ to denote the generator corresponding to the vertex $v_j$. We will use either definition of $k[K]$ in our argument depending on which one is more convenient.

The topology of $\mathbb{R}Z_K$ and $Z_K$ for a general finite simplicial complex $K$ has been extensively studied in the recent years. But the study of the quotient spaces of $\mathbb{R}Z_K$ or $Z_K$ by the toral subgroups of $(\mathbb{Z}_2)^m$ or $T^m$ are mainly done for $K$ being a simplicial sphere only.

Let $K^P$ be a simplicial sphere that is dual to an $n$-dimensional simple convex polytope $P$ with $m$ facets. For brevity, we denote $Z_{K^P}$ by $Z_P$ and $\mathbb{R}Z_{K^P}$ by $\mathbb{R}Z_P$. Then $Z_P$ and $\mathbb{R}Z_P$ are $(m+n)$-dimensional and $n$-dimensional closed manifolds, respectively. We call $Z_P$ the moment-angle manifold of $P$ and $\mathbb{R}Z_P$ the real moment-angle manifold of $P$.

- Suppose $H \cong (\mathbb{Z}_2)^{m-n}$ is a subgroup of $(\mathbb{Z}_2)^m$ that acts freely on $\mathbb{R}Z_P$ through the canonical action, the quotient space $\mathbb{R}Z_P/H$ is called a small cover over $P$.

- Suppose $S \cong T^{m-n}$ is a subgroup of $T^m$ that acts freely on $Z_P$ through the canonical action, the quotient space $Z_P/S$ is called a quasitoric manifold over $P$.

The topology of small covers and quasitoric manifolds are thoroughly studied by Davis and Januszkiewicz in [10]. Some other discussions on partial quotients of $Z_P$ can be found in [5, Sec 7.5].

- Let $S \subset T^m$ be a $k$-dimensional subtorus which acts freely $Z_P$ through the canonical action. Then by choosing a basis of $\mathbb{Z}^n$, we can write $S$ in the form:

$$S = \{(e^{2\pi i(a_{11}s_1 + \cdots + a_{1k}s_k)}, \ldots, e^{2\pi i(a_{m1}s_1 + \cdots + a_{mk}s_k)}) \in T^m\},$$
It is clear that there are many properties of $X$ which generalize (3) and (5). These spaces are defined as follows.

$$U \cdot A = 0 \quad \text{(see [5, Proposition 7.34])}.$$ It is shown by [5, Theorem 7.37] that $H^*(Z_p/S; k)$ is isomorphic as a graded algebra to $\text{Tor}_{k[u_1, \ldots, u_{m-k}]}(k[\mathcal{K}^P]; k)$, where the $k[u_1, \ldots, u_{m-k}]$-module structure on $k[\mathcal{K}^P] = k[v_1, \ldots, v_m]/\mathcal{I}_{\mathcal{K}^P}$ is given by

$$k[u_1, \ldots, u_{m-k}] \rightarrow k[v_1, \ldots, v_m]$$

$$u_i \mapsto u_1v_1 + \cdots + u_mv_m.$$ It is easy to see that $Z_p/S$ is homeomorphic to the space $X(\mathcal{K}^P, \Lambda_S)$ (see (1)) where $\Lambda_S(v_j) \in \mathbb{Z}^{m-k}$ is the $j$-th column vector of $U$ for any $1 \leq j \leq m$.

The proof of [5, Theorem 7.37] uses the Eilenberg-Moore spectral sequence of fibrations over simply connected spaces. It is not clear whether the argument can be used to compute the cohomology rings of the free quotient spaces of $Z_p$ or $\mathbb{R}Z_\mathcal{K}$ for a general simplicial complex $\mathcal{K}$. Besides, it is not known whether there exists a Hochster type formula for the cohomology groups $H^*(Z_p/S; k)$ as in [5].

In this paper, we will study a special class of quotient spaces of $\mathbb{R}Z_\mathcal{K}$ and $Z_\mathcal{K}$ by some (not necessarily free) action of toral subgroups of $\mathbb{Z}^m$ and $T^m$. We will show that the cohomology groups of these spaces can be computed by Hochster type formulae which generalize [3] and [6]. These spaces are defined as follows.

Let $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ be a partition of the vertex set $V(\mathcal{K})$ of a simplicial complex $\mathcal{K}$, i.e. $\alpha_i$’s are disjoint subsets of $V(\mathcal{K})$ with $\alpha_1 \cup \cdots \cup \alpha_k = V(\mathcal{K})$.

Let $\{e_1, \ldots, e_k\}$ be a basis of $(\mathbb{Z}_2)^k$ and $\{\hat{e}_1, \ldots, \hat{e}_k\}$ a basis of $\mathbb{Z}^k$. Let $\lambda_\alpha$ be a $(\mathbb{Z}_2)^k$-coloring of $\mathcal{K}$ defined by $\lambda_\alpha(e_i) = e_i$, $1 \leq i \leq k$. Similarly, let $\Lambda_\alpha$ be a $\mathbb{Z}^k$-coloring of $\mathcal{K}$ defined by $\Lambda_\alpha(e_i) = \hat{e}_i$, $1 \leq i \leq k$. Then we obtain two spaces $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ which are essentially determined by $\mathcal{K}$ and $\alpha$. We will see that many properties of $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ are similar to $\mathbb{R}Z_\mathcal{K}$ and $Z_\mathcal{K}$.

For any subset $J \subset [k] = \{1, \ldots, k\}$, let

$$G_J := \text{the subgroup of } (\mathbb{Z}_2)^k \text{ generated by } \{e_j \mid j \in J\}.$$ $$\hat{G}_J := \text{the toral subgroup of } T^k \text{ corresponding to the subgroup of } \mathbb{Z}^k \text{ spanned by } \{\hat{e}_j \mid j \in J\}.$$ For any simplex $\sigma \in \mathcal{K}$, let

$$I_\alpha(\sigma) := \{i \in [k] \mid V(\sigma) \cap \alpha_i \neq \emptyset\} \subset [k],$$ and let $k_\alpha(\sigma) = |I_\alpha(\sigma)|$.

It is clear that $0 \leq k_\alpha(\sigma) \leq \text{rank}(\sigma)$. Notice that by our notations,

$$G_{I_\alpha(\sigma)} = G_{\lambda_\alpha}(\sigma), \quad \hat{G}_{I_\alpha(\sigma)} = \hat{G}_{\Lambda_\alpha}(\sigma).$$
For any subset $L \subset [k] = \{1, \cdots, k\}$, define

$$\mathcal{K}_{\alpha,L} = \text{the subcomplex of } \mathcal{K} \text{ consisting of } \{\sigma \in \mathcal{K} : I_{\alpha}(\sigma) \subset L\}. \quad (8)$$

The main theorems of this paper are the following.

**Theorem 1.1.** Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set of a finite simplicial complex $\mathcal{K}$. Then we have

1. $H^q(X(\mathcal{K}, \lambda_{\alpha}); \mathbb{k}) \cong \bigoplus_{L \subset [k]} \tilde{H}^{q-1}(\mathcal{K}_{\alpha,L}; \mathbb{k})$ for any $q \geq 0$.
2. $H^q(X(\mathcal{K}, \Lambda_{\alpha}); \mathbb{k}) \cong \bigoplus_{L \subset [k]} \tilde{H}^{q-|L|-1}(\mathcal{K}_{\alpha,L}; \mathbb{k})$ for any $q \geq 0$.

Let $\alpha^*$ denote the trivial partition of $V(\mathcal{K})$, i.e. $\alpha^* = (\alpha_1, \cdots, \alpha_m)$ where each $\alpha_j = \{v_j\}$ consists of only one vertex of $\mathcal{K}$. Then by definition $X(\mathcal{K}, \lambda_{\alpha^*}) = \mathbb{R}\mathcal{Z}_\mathcal{K}$, $X(\mathcal{K}, \Lambda_{\alpha^*}) = \mathcal{Z}_\mathcal{K}$, and the formulae in Theorem 1.1 for $\alpha^*$ agree with (3) and (5).

In addition, it is shown in [2, Corollary 2.23] that the Hochster’s formula for the cohomology groups of $\mathcal{Z}_\mathcal{K}$ follows from a stable decomposition of $\mathcal{Z}_\mathcal{K}$. We have completely parallel results for $X(\mathcal{K}, \lambda_{\alpha})$ and $X(\mathcal{K}, \Lambda_{\alpha})$ below.

**Theorem 1.2.** Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set of a finite simplicial complex $\mathcal{K}$. There are homotopy equivalences:

$$\Sigma(X(\mathcal{K}, \lambda_{\alpha})) \cong \bigvee_{L \subset [k]} \Sigma^2(\mathcal{K}_{\alpha,L}), \quad \Sigma(X(\mathcal{K}, \Lambda_{\alpha})) \cong \bigvee_{L \subset [k]} \Sigma^{|L|+2}(\mathcal{K}_{\alpha,L})$$

where the bold $\Sigma$ denotes the suspension.

It is easy to see that Theorem 1.1 follows from the stable decompositions of $X(\mathcal{K}, \lambda_{\alpha})$ and $X(\mathcal{K}, \Lambda_{\alpha})$ in Theorem 1.2. But we can not obtain the cohomology ring structures of $X(\mathcal{K}, \lambda_{\alpha})$ and $X(\mathcal{K}, \Lambda_{\alpha})$ from Theorem 1.2.

We call a partition $\alpha$ of $V(\mathcal{K})$ non-degenerate if for any simplex $\sigma$ of $\mathcal{K}$,

$$|I_{\alpha}(\sigma)| = \text{rank}(\sigma).$$

It is easy to see that the following statements are equivalent.

- $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ is a non-degenerate partition of $V(\mathcal{K})$.
- the two vertices of any 1-simplex of $\mathcal{K}$ belong to different $\alpha_i$.
- The coloring $\lambda_{\alpha}$ or $\Lambda_{\alpha}$ on $\mathcal{K}$ is non-degenerate.

Note that the trivial partition $\alpha^*$ of $V(\mathcal{K})$ is always non-degenerate.

We define a $\mathbb{k}[u_1, \cdots, u_k]$-module structure on $\mathbb{k}[\mathcal{K}] = \mathbb{k}[v_\sigma : \sigma \in \mathcal{K}]/\mathcal{I}_\mathcal{K}$ by

$$\mathbb{k}[u_1, \cdots, u_k] \rightarrow \mathbb{k}[v_\sigma : \sigma \in \mathcal{K}], \quad u_i \mapsto \sum_{v_j \in \alpha_i} v_{\{j\}}. \quad (9)$$
Theorem 1.3. Let $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ be a non-degenerate partition of the vertex set of a finite simplicial complex $K$. Then with respect to some properly defined multi-gradings,

(i) there is an isomorphism of multigraded $\mathbb{Z}_2$-modules from $H^*(X(K, \lambda_\alpha); \mathbb{Z}_2)$ to $\text{Tor}_{\mathbb{Z}_2[u_1, \ldots, u_k]}(\mathbb{Z}_2[K]; \mathbb{Z}_2)$.

(ii) there is an isomorphism of multigraded $\mathbb{Z}_2$-algebras from $H^*(X(K, \Lambda_\alpha); \mathbb{Z}_2)$ to $\text{Tor}_{\mathbb{Z}_2[u_1, \ldots, u_k]}(\mathbb{Z}_2[K]; \mathbb{Z}_2)$.

Remark 1.4. For the trivial partition $\alpha^*$ of $V(K)$, the space $X(K, \lambda_{\alpha^*}) = \mathbb{R}Z_K$ and $X(K, \Lambda_{\alpha^*}) = Z_K$, and the assertions in Theorem 1.3 actually hold for any coefficients $k$ (see [5, Proposition 7.12] and [9, Theorem 4.2]).

If $\alpha$ is a general partition of $V(K)$ which is not necessarily non-degenerate, we can describe $H^*(X(K, \lambda_\alpha); \mathbb{Z}_2)$ and $H^*(X(K, \Lambda_\alpha); \mathbb{Z}_2)$ in terms of some differential graded algebra determined by $K$ and $\alpha$ as follows. Let $\Lambda_k[t_1, \ldots, t_k]$ denote the exterior algebra over $k$ with $k$ generators. For any partition $\alpha$ of $V(K)$, we define a differential $d_\alpha$ on the tensor product $\Lambda_k[t_1, \ldots, t_k] \otimes k[K]$ by:

$$d_\alpha(t_i) := \sum_{v_j \in \alpha_i} v(j), \quad 1 \leq i \leq n, \quad \text{and} \quad d_\alpha(v_\sigma) := \sum_{\sigma \in \omega, \dim(\omega) = \dim(\sigma) + 1} \varepsilon(\sigma, \omega) \cdot v_\omega. \quad (10)$$

where $\varepsilon(\sigma, \omega)$ is the sign of $\sigma$ relative to the boundary of $\omega$ with respect to some chosen orientations of the simplices of $K$ (see [25]). Note that here and in the rest of this paper we use $\otimes$ for $\otimes_k$. It is easy to see the following fact.

Fact: $d_\alpha$ is zero on $k[K]$ if and only if $\alpha$ is non-degenerate.

Theorem 1.5. Let $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ be a partition of the vertex set of a finite simplicial complex $K$. Then with respect to some properly defined multi-gradings,

(i) there is an isomorphism of multigraded $\mathbb{Z}_2$-modules from $H^*(X(K, \lambda_\alpha); \mathbb{Z}_2)$ to $H^*(\Lambda_\mathbb{Z}_2[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[K], d_\alpha)$.

(ii) there is an isomorphism of multigraded $\mathbb{Z}_2$-algebras from $H^*(X(K, \Lambda_\alpha); \mathbb{Z}_2)$ to $H^*(\Lambda_\mathbb{Z}_2[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[K], d_\alpha)$.

Note that when $\alpha$ is non-degenerate, $H^*(\Lambda_k[t_1, \ldots, t_k] \otimes k[K], d_\alpha)$ is isomorphic to $\text{Tor}_{k[u_1, \ldots, u_k]}(k[K]; k)$ with respect to the $k[u_1, \ldots, u_k]$-module structure on $k[K]$ defined in [14]. Indeed, we can compute $\text{Tor}_{k[u_1, \ldots, u_k]}(k[K]; k)$ via the Koszul resolution of $k$ (see [5, Sec 3.4]). Recall that the Koszul resolution of $k$ is:

$$0 \rightarrow \Lambda_k[u_1, \ldots, u_k] \rightarrow \cdots \rightarrow \Lambda_k[t_1, \ldots, t_k] \otimes k[u_1, \ldots, u_k] \rightarrow k[u_1, \ldots, u_k] \rightarrow k \rightarrow 0,$$
where \( d(t_i) = u_i \) and \( d(u_i) = 0 \) for any \( 1 \leq i \leq k \). Then we obtain
\[
\text{Tor}_{k[u_1, \ldots, u_k]}(k[K], k) \cong \text{Tor}_{k[u_1, \ldots, u_k]}(k, k[K]) \\
\cong H^* \left( \left( \Lambda_k[t_1, \ldots, t_k] \otimes k[u_1, \ldots, u_k] \right) \otimes_{k[u_1, \ldots, u_k]} k[K] \right) \\
\cong H^* \left( \Lambda_k[t_1, \ldots, t_k] \otimes k[K] \right)
\]

When \( \alpha \) is non-degenerate, the differential on \( \Lambda_k[t_1, \ldots, t_k] \otimes k[K] \) induced from \( \text{Tor}_{k[u_1, \ldots, u_k]}(k, k[K]) \) is exactly given by \( d_\alpha \) (which is zero on \( k[K] \) in this case). So Theorem 1.3 is actually a corollary of Theorem 1.5.

Our discussion of the cohomology rings of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \) generalizes the discussion of (real) moment-angle complexes in [2, 4, 5, 8, 9, 12, 14] in several aspects. The reader is referred to [3, 6] and [14] for a comprehensive introduction to moment-angle complexes and related topics.

The paper is organized as follows. In section 2, we construct some natural cell decompositions of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \). Then in section 3 we use these cell decompositions to compute the cohomology groups of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \) and the give a proof of Theorem 1.4. In section 4, we use the strategies in [2] to study the stable decompositions of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \), which leads to a proof of Theorem 1.5. In section 5, we build some connections between the differential algebra \( (\Lambda_{\mathbb{Z}_2}[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[K], d_\alpha) \) and the cellular cochain complexes of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \), which imply some multigraded additive isomorphisms from \( H^*(\Lambda_{\mathbb{Z}_2}[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[K], d_\alpha) \) to \( H^*(X(K, \lambda_\alpha); \mathbb{Z}_2) \) and \( H^*(X(K, \Lambda_\alpha); \mathbb{Z}_2) \). In section 6, we study the cohomology ring structures of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \) and finish the proof of Theorem 1.5. In addition, we show that the moment-angle complex of any finite simplicial poset is homotopy equivalent to \( X(K, \Lambda_\alpha) \) for some simplicial complex \( K \) and partition \( \alpha \) of \( V(K) \). In the final section 7, we generalize the main results of this paper to a wider range of spaces.

### 2. Cell decompositions of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \)

For any partition \( \alpha = \{\alpha_1, \ldots, \alpha_k\} \) of the vertex set of a simplicial complex \( K \), we construct some cell decompositions of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \) in this section which will be used for the calculation of their cohomology groups later. Although the cell decompositions of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \) are quite similar in taste, there are still some differences which make it hard to treat them in a uniform way. So we construct the cell decompositions of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \) in two parallel subsections below, which helps the reader to see where the difference lies.

#### 2.1. A cell decomposition of \( X(K, \lambda_\alpha) \).
First of all, let \( \pi_\alpha : P_K \times (\mathbb{Z}_2)^k \to X(\mathcal{K}, \lambda_\alpha) \) be the quotient map in [2]. For any \( p \in P_K \) and \( g \in (\mathbb{Z}_2)^k \), let \([[(p, g)]]\) denote the equivalence class of \((p, g)\) with respect to the \( \sim \) in [2].

By [5, Construction 4.9], \( P_K \) has a natural cubical cell decomposition

\[
P_K = \bigcup_{\sigma \in K} C_\sigma
\]

where \( C_\sigma \) is the cone of the barycentric subdivision of \( \sigma \), and \( \text{dim}(C_\sigma) = \text{rank}(\sigma) \). The boundary of \( C_\sigma \) consists of \( \sigma \) and \( \{C_\tau \mid \tau \preceq \sigma, \text{rank}(\tau) = \text{rank}(\sigma) - 1\} \). In particular, \( C_0 \) be the cone point of \( P_K \). In addition, we let \( \text{Cone}(\mathcal{K}) \) be the cone of \( \mathcal{K} \) as a simplicial complex where the cone point is denoted by \( v_0 \). Then

\[
\text{Cone}(\mathcal{K}) = \bigcup_{\sigma \in \mathcal{K}} \text{Cone}(\sigma),
\]

where \( \text{Cone}(\sigma) \) is the simplex with vertex set \( V(\sigma) \cup \{v_0\} \). It is clear that \( \text{Cone}(\sigma) \) agrees with \( C_\sigma \) as a set, but they have different combinatorial structures.

Let \( b_\sigma \) be the barycenter of \( \sigma \) which lies in the relative interior of \( F_\sigma \). For any \( g \in (\mathbb{Z}_2)^k \), the equivalence class of \([[(b_\sigma, g)]]\) consists of \( 2^{\text{rank}(\sigma)} \) elements in \( P_K \times (\mathbb{Z}_2)^k \). More specifically, \([[(b_\sigma, g)]\) = \{\((b_\sigma, g + h) \mid h \in G_{I_\alpha(\sigma)}\} \subset P_K \times (\mathbb{Z}_2)^k \).

So \( \pi_\alpha(b_\sigma \times (\mathbb{Z}_2)^k) \) consists of exactly \( 2^{\text{rank}(\sigma)} \) points which form an orbit of the canonical \( (\mathbb{Z}_2)^k \)-action on \( X(\mathcal{K}, \lambda_\alpha) \). Indeed, the points in \( \pi_\alpha(b_\sigma \times (\mathbb{Z}_2)^k) \) are indexed by all the elements of \( G_{[k] \cdot I_\alpha(\sigma)} \) where \( g \in G_{[k] \cdot I_\alpha(\sigma)} \) corresponds to

\[
b_{(\sigma, g)} := g \cdot \pi_\alpha(b_\sigma \times \{0\}).
\]

For any \( \sigma \in \mathcal{K} \) and \( g \in G_{[k] \cdot I_\alpha(\sigma)} \), we define a subset \( B_{(\sigma, g)} \subset X(\mathcal{K}, \lambda_\alpha) \) by

\[
B_{(\sigma, g)} = \bigcup_{h \in G_{I_\alpha(\sigma)}} \pi_\alpha(C_\sigma \times \{g + h\}) = \bigcup_{h \in G_{I_\alpha(\sigma)}} g \cdot \pi_\alpha(C_\sigma \times \{h\}). \tag{11}
\]

So \( B_{(\sigma, g)} \) is a union of \( 2^{\text{rank}(\sigma)} \) cubes of dimension \( \text{rank}(\sigma) \).

**Claim:** \( B_{(\sigma, g)} \) is homeomorphic to a closed ball of dimension \( \text{rank}(\sigma) \) which contains \( b_{(\sigma, g)} \) in its interior.

Note that \( B_{(\sigma, g)} = g \cdot B_{(\sigma, 0)} \) for any \( \sigma \in \mathcal{K}, g \in G_{[k] \cdot I_\alpha(\sigma)} \). So we only need to examine \( B_{(\sigma, 0)} \). Note that when \( \sigma = \emptyset, B_{(\emptyset, 0)} \) is just the cone point of \( P_K \times \{0\} \).

Now assume \( V(\sigma) = \{v_1, \ldots, v_n\} \subset V(\mathcal{K}) \). For each \( 1 \leq j \leq n \), let \( f_j \) be the unique cubic facet (codimension-one face) of \( C_\sigma \) containing both \( b_\sigma \) and \( v_j \). The \( (\mathbb{Z}_2)^k \)-coloring \( \lambda_\alpha \) of \( \mathcal{K} \) induces a map

\[
\lambda_\alpha^g : \{f_1, \ldots, f_n\} \longrightarrow G_{I_\alpha(\sigma)} \subset (\mathbb{Z}_2)^k, \text{ where } \lambda_\alpha^g(f_j) = \lambda_\alpha(v_j), 1 \leq j \leq n.
\]

Then similarly to the construction [2], we define a space \( M(C_\sigma, \lambda_\alpha^g) \) by

\[
M(C_\sigma, \lambda_\alpha^g) := C_\sigma \times G_{I_\alpha(\sigma)}/\sim \tag{12}
\]
where \((y, h) \sim (y', h')\) whenever \(y' = y \in f_j\) and \(h' - h = \lambda^\sigma_\alpha(f_j)\) for some \(1 \leq j \leq n\). Note that the facets of \(C_\sigma\) other than \(f_1, \cdots, f_n\) are not glued under the relation \(\sim\) in \((12)\). Let \([(y, h)]\) denote the equivalence class of \((y, h)\) in \((12)\).

There is a natural action of \(G_{I_\alpha(\sigma)}\) on \(M(C_\sigma, \lambda^\sigma_\alpha)\) by:

\[
h' \cdot [(y, h)] = [(y, h + h')], \quad y \in C_\sigma, \ h, h' \in G_{I_\alpha(\sigma)}.
\]

Then we can define a natural equivariant homeomorphism

\[
\Theta_\sigma : M(C_\sigma, \lambda^\sigma_\alpha) \longrightarrow B_{(\sigma, 0)} = \bigcup_{h \in G_{I_\alpha(\sigma)}} \pi_\alpha(C_\sigma \times \{h\})
\]

with respect to the natural actions of \(G_{I_\alpha(\sigma)}\) on \(M(C_\sigma, \lambda^\sigma_\alpha)\) and \(B_{(\sigma, 0)}\) where

\[
\Theta_\sigma([(y, h)]) = \pi_\alpha(y, h), \quad y \in C_\sigma, \ h \in G_{I_\alpha(\sigma)}.
\]

So it amounts to show that \(M(C_\sigma, \lambda^\sigma_\alpha)\) is homeomorphic to a closed ball.

In the rest we will use the following notations.

- Let \(\sigma_{(i)}\) be the face of \(\sigma\) spanned by the vertices \(V(\sigma) \cap \alpha_i\). So we have

\[
\sum_{i \in I_\alpha(\sigma)} \text{rank}(\sigma_{(i)}) = \text{rank}(\sigma).
\]

- For any \(r \geq 0\), let \(\Delta^r\) denote the standard \(r\)-simplex in \(\mathbb{R}^r\).

Now we define

\[
f^\sigma_i = \bigcup_{v_j \in V(\sigma) \cap \alpha_i} f_j, \quad \forall i \in I_\alpha(\sigma).
\]

Notice that all the components in \(f^\sigma_i\) are assigned the same value by \(\lambda^\sigma_\alpha\). We readily think of each \(f^\sigma_i\) as a “big” facet of \(C_\sigma\). It is shown in \([19, \text{Theorem 3.7}]\) that \(C_\sigma\) with these “big” facets \(\{f^\sigma_i; i \in I_\alpha(\sigma)\}\) is still a nice manifold with corners, denoted by \(Q^\sigma_\alpha\), which is combinatorially equivalent to a product of simplices.

More precisely, we have

\[
Q^\sigma_\alpha \cong \prod_{i \in I_\alpha(\sigma)} \text{Cone}(\sigma_{(i)}).
\]

The barycenter \(b_\sigma\) of \(\sigma\) lies in the relative interior of the face \(\bigcap_{i \in I_\alpha(\sigma)} f^\sigma_i\) of \(Q^\sigma_\alpha\). Moreover, the map \(\lambda^\sigma_\alpha\) induces a map

\[
\tilde{\lambda}^\sigma_\alpha : \{f^\sigma_i; i \in I_\alpha(\sigma)\} \longrightarrow G_{I_\alpha(\sigma)} \subset (\mathbb{Z}_2)^k,
\]

where \(\tilde{\lambda}^\sigma_\alpha(f^\sigma_i) = \lambda_\alpha(\alpha_i) = e_i, \ i \in I_\alpha(\sigma)\).

Then we can construct a space \(M(Q^\sigma_\alpha, \tilde{\lambda}^\sigma_\alpha)\) in the same way as \((12)\) by

\[
M(Q^\sigma_\alpha, \tilde{\lambda}^\sigma_\alpha) = Q^\sigma_\alpha \times G_{I_\alpha(\sigma)}/ \sim
\]

\((13)\).
where \((y, h) \sim (y', h')\) whenever \(y' = y + f_i\) and \(y' - y = \lambda_i(f_i)\) for some \(i \in I_\alpha(\sigma)\). Then \(M(Q_\alpha^\sigma, \hat{\lambda}_\alpha^\sigma)\) is obviously homeomorphic to \(M(C_\sigma, \lambda_\alpha^\sigma)\). Moreover, it is easy to see that \(M(Q_\alpha^\sigma, \hat{\lambda}_\alpha^\sigma)\) is homeomorphic to a product of closed balls

\[
M(Q_\alpha^\sigma, \hat{\lambda}_\alpha^\sigma) \cong \prod_{i \in I_\alpha(\sigma)} S^0 \ast \sigma(i),
\]

where \(S^0 \ast \sigma(i)\) is the join of \(S^0\) with \(\sigma(i)\) (i.e., the suspension of \(\sigma(i)\)). Indeed, \(S^0 \ast \sigma(i)\) is obtained by applying the rule in (13) to \(Cone(\sigma(i))\) with \(\sigma(i)\) colored by \(\lambda(\alpha_i)\). For example, Figure 1 shows a 2-simplex \(\sigma\) whose vertices \(v_1, v_2, v_3\) are colored by \(e_1\) and \(e_2\), and we have \(Q_\alpha^\sigma \cong \Delta^1 \times \Delta^2\), \(M(Q_\alpha^\sigma, \hat{\lambda}_\alpha^\sigma) \cong (S^0 \ast \Delta^1) \times (S^0 \ast \Delta^2)\).

Since \(\dim(S^0 \ast \sigma(i)) = \text{rank}(\sigma(i))\), the space \(M(C_\sigma, \lambda_\alpha^\sigma)\) is homeomorphic to a closed ball of dimension \(\text{rank}(\sigma)\). The claim is proved.  

Let \(B_{(\sigma, g)}\) be the relative interior of \(B_{(\sigma, g)}\) in \(X(K, \lambda_\alpha)\). Define

\[
\mathcal{B}_\alpha(K) := \{ B_{(\sigma, g)} \mid \sigma \in K, \ g \in G[k]\setminus I_\alpha(\sigma) \}. \tag{15}
\]

By our construction,

\[
X(K, \lambda_\alpha) = \bigcup_{\sigma \in K \atop g \in G[k]\setminus I_\alpha(\sigma)} B_{(\sigma, g)}, \quad \dim(B_{(\sigma, g)}) = \text{rank}(\sigma).
\]

So \(\mathcal{B}_\alpha(K)\) is a cell decomposition of \(X(K, \lambda_\alpha)\). Let \(C_*(X(K, \lambda_\alpha); k)\) be the cellular chain complex corresponding to \(\mathcal{B}_\alpha(K)\). For any \(J \subset [k]\setminus I_\alpha(\sigma)\), let

\[
B_{(\sigma, J)} := \bigcup_{g \in G[J]} B_{(\sigma, g)}, \quad B_{(\sigma, J)} := \bigcup_{g \in G[J]} B_{(\sigma, g)}. \tag{16}
\]

For any fixed \(\sigma \in K\), \(\{ B_{(\sigma, J)} \mid J \subset [k]\setminus I_\alpha(\sigma) \}\) is a basis of the \(k\)-submodule of \(C_*(X(K, \lambda_\alpha); k)\) spanned by \(\{ B_{(\sigma, g)} \mid g \in G[k]\setminus I_\alpha(\sigma) \}\). In the following sections, we use \(\{ B_{(\sigma, J)} \mid \sigma \in K, \ J \subset [k]\setminus I_\alpha(\sigma) \}\) as the basis of \(C_*(X(K, \lambda_\alpha); k)\).
Notice that $B_{(\sigma, J)} \subset B_{(\sigma, J')} \subset B_{(\sigma, J)}$ for any $J \subset J' \subset [k]\setminus I_\alpha(\sigma)$. So we have the decomposition of $X(\mathcal{K}, \lambda_\alpha)$ (as a set)

$$X(\mathcal{K}, \lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} B_{(\sigma, [k]\setminus I_\alpha(\sigma))} = \bigcup_{\sigma \in \mathcal{K}} B_{(\sigma, [k]\setminus I_\alpha(\sigma))}.$$

### 2.2. A cell decomposition of $X(\mathcal{K}, \Lambda_\alpha)$

We will see that the construction of the cell decomposition of $X(\mathcal{K}, \Lambda_\alpha)$ is very similar to $X(\mathcal{K}, \lambda_\alpha)$ but with an extra ingredient. First of all, we put a standard cell decomposition on $T^k$ for any $k \geq 0$. The circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$ has a natural cell decomposition $\{e^0, e^1\}$ where $e^0 = \{1\} \subset S^1$ and $e^1 = S^1 \setminus e^0$. Then consider $T^k$ as a product of $k$ copies of $S^1$ and equip $T^k$ with the product cell structure (see [11, 3.B]). All the cells in $T^k$ can be indexed by subsets of $[k] = \{1, \cdots, k\}$. Indeed, any $J \subset [k]$ determines a unique cell $U_J$ in $T^k$ where

$$U_J = \prod_{i \in J} e^1_i \times \prod_{i \in [k]\setminus J} e^0_i, \quad \text{dim}(U_J) = |J|.$$ 

Here $\{e^0_i, e^1_i\}$ denote the cells in the $i$-th $S^1$-factor of $T^k$, $1 \leq i \leq k$. Let

$$\hat{\pi}_\alpha : P_{\mathcal{K}} \times T^k \to X(\mathcal{K}, \Lambda_\alpha)$$

be the quotient map in (2). For any simplex $\sigma \in \mathcal{K}$ and $J \subset [k]\setminus I_\alpha(\sigma)$, let

$$\hat{B}_{(\sigma, J)} = \hat{\pi}_\alpha(C_\sigma \times (\hat{G}_{I_\alpha(\sigma)} \times U_J)) \subset X(\mathcal{K}, \Lambda_\alpha).$$ (17)

From the definition of $X(\mathcal{K}, \Lambda_\alpha)$, we can easily see that $\hat{B}_{(\sigma, J)}$ is homeomorphic to $U_J \times \hat{B}_{(\sigma, \varnothing)}$. So to understand $\hat{B}_{(\sigma, J)}$, it amounts to understand $\hat{B}_{(\sigma, \varnothing)}$. Note that when $\sigma = \varnothing$, the space $\hat{B}_{(\varnothing, \varnothing)}$ is just a point. Assume $V(\sigma) = \{v_1, \cdots, v_n\} \subset V(\mathcal{K})$ and use the previous notations for the faces of the cube $C_\sigma$. The $\mathbb{Z}^k$-coloring $\Lambda_\alpha$ of $\mathcal{K}$ induces a map

$$\Lambda_\alpha : \{f_1, \cdots, f_n\} \rightarrow \mathbb{Z}^k, \quad \text{where} \quad \Lambda_\alpha(f_j) = \Lambda_\alpha(v_j), \quad 1 \leq j \leq n.$$ 

Then parallel to the space $M(C_\sigma, \Lambda_\alpha)$, we obtain a space $M(C_\sigma, \Lambda_\alpha)$ by replacing $G_{I_\alpha(\sigma)}$ by $\hat{G}_{I_\alpha(\sigma)}$ and $\lambda_\alpha$ by $\Lambda_\alpha$ in (12). It is clear that $M(C_\sigma, \Lambda_\alpha)$ is homeomorphic to $\hat{B}_{(\sigma, \varnothing)}$. Moreover, on $Q_\alpha$ the map $\Lambda_\alpha$ induces a map

$$\bar{\Lambda}_\alpha : \{f^\sigma_i ; i \in I_\alpha(\sigma)\} \rightarrow \mathbb{Z}^k, \quad \text{where} \quad \bar{\Lambda}_\alpha(f^\sigma_i) = \Lambda_\alpha(\alpha_i) = \hat{e}_i, \quad i \in I_\alpha(\sigma).$$

Then parallel to the space $M(Q_\alpha, \bar{\Lambda}_\alpha)$, we obtain a space $M(Q_\alpha, \bar{\Lambda}_\alpha)$ by replacing $G_{I_\alpha(\sigma)}$ by $\hat{G}_{I_\alpha(\sigma)}$ and $\lambda_\alpha$ by $\bar{\Lambda}_\alpha$ in (13). Similarly, we can show that

$$M(Q_\alpha, \bar{\Lambda}_\alpha) \cong \prod_{i \in I_\alpha(\sigma)} S^1 \ast \sigma(i).$$ (18)
Since \( \dim(S^1 \ast \sigma_{(i)}) = \rank(\sigma_{(i)}) + 1 \), \( \widehat{B}_{(\sigma, \varnothing)} \) is homeomorphic to a closed ball of dimension \( \rank(\sigma) + |I_\alpha(\sigma)| \).

For any subset \( J \subseteq [k] \setminus I_\alpha(\sigma) \), let \( \widehat{B}_{(\sigma, J)} \) be the relative interior of \( \widehat{B}_{(\sigma, \varnothing)} \). So \( \widehat{B}_{(\sigma, J)} \simeq U_J \times \widehat{B}_{(\sigma, \varnothing)} \) is an open ball of dimension \( \rank(\sigma) + |I_\alpha(\sigma)| + |J| \). Define

\[
\widehat{\mathcal{K}}_{\alpha}(\mathcal{K}) := \{ \widehat{B}_{(\sigma, J)} \mid \sigma \in \mathcal{K}, \ J \subseteq [k] \setminus I_\alpha(\sigma) \}
\tag{19}
\]

From the definition of \( \widehat{B}_{(\sigma, J)} \), it is clear that

\[
X(\mathcal{K}, \Lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} \widehat{B}_{(\sigma, J)}. \]

So \( \widehat{\mathcal{K}}_{\alpha}(\mathcal{K}) \) is a cell decomposition of \( X(\mathcal{K}, \Lambda_\alpha) \).

### 3. Cohomology groups of \( X(\mathcal{K}, \lambda_\alpha) \) and \( X(\mathcal{K}, \Lambda_\alpha) \)

For any coefficients \( k \), let \( C^*(X(\mathcal{K}, \lambda_\alpha); k) \) and \( C^*(X(\mathcal{K}, \Lambda_\alpha); k) \) be the cellular cochain complexes corresponding to the cell decompositions \( \mathcal{B}_{\alpha}(\mathcal{K}) \) and \( \widehat{\mathcal{B}}_{\alpha}(\mathcal{K}) \) defined in \([15]\) and \([19]\), respectively. In this section we compute \( H^*(X(\mathcal{K}, \lambda_\alpha); k) \) and \( H^*(X(\mathcal{K}, \Lambda_\alpha); k) \). First, we give a new description of \( X(\mathcal{K}, \lambda_\alpha) \) and \( X(\mathcal{K}, \Lambda_\alpha) \).

Suppose the vertex set of \( \mathcal{K} \) is \([m] = \{v_1, \ldots, v_m\} \). Let \( \Delta^{[m]} \) be the \( m \)-simplex with vertex set \([m]\). For a partition \( \alpha = \{\alpha_1, \ldots, \alpha_k\} \) of \([m]\), let \( \Delta^{\alpha_i} \) denote the face of \( \Delta^{[m]} \) whose vertex set is \( \alpha_i \). Then \( \mathcal{K} \) is a simplicial subcomplex of \( \Delta^{[m]} \). Let \( b_{\Delta^{[m]}} \) be the barycenter of \( \Delta^{[m]} \) and \( C_{\Delta^{[m]}} = P_{\Delta^{[m]}} \) be the cone of the barycentric subdivision of \( \Delta^{[m]} \) which is an \( m \)-dimensional cube. For each \( 1 \leq j \leq m \), let \( f_j \) be the unique cubic facet of \( C_{\Delta^{[m]}} \) containing both \( b_{\Delta^{[m]}} \) and \( v_j \). For any partition \( \alpha = \{\alpha_1, \ldots, \alpha_k\} \) of \([v_1, \ldots, v_m]\),

\[
def \mathbf{f}_i^{\Delta^{[m]}} := \bigcup_{j \in \alpha_i} f_j, \ i \in [k].
\]

Then by the same argument as in the previous section, \( C_{\Delta^{[m]}} \) with these “big” facets \( \{\mathbf{f}_i^{\Delta^{[m]}} \mid i \in [k]\} \) is a nice manifold with corners, denoted by \( Q_{\alpha}^{\Delta^{[m]}} \), which is combinatorially equivalent to a product of simplices \( \prod_{i \in [k]} \Delta^{\alpha_i} \).

Moreover, we can also consider \( \lambda_\alpha \) and \( \Lambda_\alpha \) as a \((\mathbb{Z}_2)^k\)-coloring and \( \mathbb{Z}^k\)-coloring of \( \Delta^{[m]} \), respectively. Then we have the following maps

\[
\lambda^{\Delta^{[m]}}_\alpha : \{f_1, \ldots, f_n\} \longrightarrow (\mathbb{Z}_2)^k, \ \text{where} \ \lambda^{\Delta^{[m]}}_\alpha(f_j) = \lambda_\alpha(v_j), \ 1 \leq j \leq m.
\]

\[
\Lambda^{\Delta^{[m]}}_\alpha : \{f_1, \ldots, f_n\} \longrightarrow \mathbb{Z}^k, \ \text{where} \ \Lambda^{\Delta^{[m]}}_\alpha(f_j) = \Lambda_\alpha(v_j), \ 1 \leq j \leq m.
\]
Then parallel to $M(C_\sigma, \lambda_\sigma^0)$ and $M(C_\sigma, \Lambda_\alpha^0)$, we obtain homeomorphisms:

$$M(C_{[m]}, \lambda_\alpha^{[m]}) \cong \prod_{i \in [k]} S^0 \ast \Delta^{\alpha_i}, \quad M(C_{[m]}, \Lambda_\alpha^{[m]}) \cong \prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i}. \quad (20)$$

By definition, we clearly have homeomorphisms

$$X(\Delta^{[m]}, \lambda_\alpha) \cong M(C_{[m]}, \lambda_\alpha^{[m]}), \quad X(\Delta^{[m]}, \Lambda_\alpha) \cong M(C_{[m]}, \Lambda_\alpha^{[m]}).$$

Note $\dim (X(\Delta^{[m]}, \lambda_\alpha)) = |\alpha_1| + \cdots + |\alpha_k| = m$ and $\dim (X(\Delta^{[m]}, \Lambda_\alpha)) = m + k$.

Since $\mathcal{K}$ is a subcomplex of $\Delta^{[m]}$, $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ are subspaces of $M(C_{\Delta^{[m]}}, \lambda_\alpha^{[m]})$ and $M(C_{\Delta^{[m]}}, \Lambda_\alpha^{[m]})$, respectively. Indeed, the cubical complex $P_\mathcal{K}$ naturally embeds into the cube $P_{\Delta^{[m]}}$ (see [5, 4.2]), which gives the following commutative diagrams

$$X(\mathcal{K}, \lambda_\alpha) \longrightarrow X(\Delta^{[m]}, \lambda_\alpha) \quad X(\mathcal{K}, \Lambda_\alpha) \longrightarrow X(\Delta^{[m]}, \Lambda_\alpha) \quad (21)$$

where the horizontal arrows are embeddings, while the vertical ones are orbit maps for the $(\mathbb{Z}_2)^k$ and $T^k$ actions, respectively.

Note that for any simplex $\sigma \in \mathcal{K}$, $\sigma_{(i)} = \sigma \cap \Delta^{\alpha_i}$. So we can write $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ as:

$$X(\mathcal{K}, \lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_{\alpha}(\sigma)} S^0_{(i)} \ast (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i \in [k] \setminus I_{\alpha}(\sigma)} S^0_{(i)} \right), \quad (22)$$

$$X(\mathcal{K}, \Lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_{\alpha}(\sigma)} S^1_{(i)} \ast (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i \in [k] \setminus I_{\alpha}(\sigma)} S^1_{(i)} \right). \quad (23)$$

Here $S^0_{(i)}$ and $S^1_{(i)}$ denote a copy of $S^0$ and $S^1$ associated to the index $i \in [k]$.

We can easily recover the cell decomposition $\mathcal{B}_\alpha(\mathcal{K})$ of $X(\mathcal{K}, \lambda_\alpha)$ and $\mathcal{B}_\alpha(\mathcal{K})$ of $X(\mathcal{K}, \Lambda_\alpha)$ from (22) and (23), respectively. Indeed, for any simplex $\sigma \in \mathcal{K}$, let

$$D_{\alpha}(\sigma) = \prod_{i \in I_{\alpha}(\sigma)} S^0_{(i)} \ast (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i \in [k] \setminus I_{\alpha}(\sigma)} S^0_{(i)}, \quad (24)$$

$$\hat{D}_{\alpha}(\sigma) = \prod_{i \in I_{\alpha}(\sigma)} S^1_{(i)} \ast (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i \in [k] \setminus I_{\alpha}(\sigma)} S^1_{(i)}. \quad (25)$$

It is clear that $D_{\alpha}(\sigma)$ and $\hat{D}_{\alpha}(\sigma)$ are the closure of $B_{(\sigma, [k] \setminus I_{\alpha}(\sigma))}$ and $\hat{B}_{(\sigma, [k] \setminus I_{\alpha}(\sigma))}$ in $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$, respectively.

Note that $S^0$ can be naturally identified with $\mathbb{Z}_2$. Then the canonical action of $(\mathbb{Z}_2)^k$ (or $T^k$) on $X(\mathcal{K}, \lambda_\alpha)$ (or $X(\mathcal{K}, \Lambda_\alpha)$) can be equivalently defined by: for any
$(g_1, \cdots, g_k) \in (\mathbb{Z}_2)^k$ (or $T^k$), let $g_i \in \mathbb{Z}_2$ (or $S^1$) act on the $S_{(i)}^0$ (or $S_{(i)}^1$) through left translations.

Moreover, a simplex $\sigma \in \mathcal{K}$ with $V(\sigma) = \{v_{i_1}, \cdots, v_{i_s}\}$ can be written as a join

$$\sigma = \bigast_{v \in V(\sigma)} v = v_{i_1} \ast \cdots \ast v_{i_s}.$$  

So we can write $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ equivalently as:

$$X(\mathcal{K}, \lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_\alpha(\sigma)} \left( S_{(i)}^0 \ast \bigast_{v \in V(\sigma) \cap \alpha_i} v \right) \times \prod_{i \in [k] \setminus I_\alpha(\sigma)} S_{(i)}^0 \right),$$

$$X(\mathcal{K}, \Lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_\alpha(\sigma)} \left( S_{(i)}^1 \ast \bigast_{v \in V(\sigma) \cap \alpha_i} v \right) \times \prod_{i \in [k] \setminus I_\alpha(\sigma)} S_{(i)}^1 \right).$$

**Remark 3.1.** The building blocks of $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ are spaces obtained by mixtures of Cartesian products and joins of some simple spaces (points, $S^0$ and $S^1$). The building blocks of polyhedral products $(X, A)^K$, however, only involve Cartesian products of spaces. In addition, we have polyhedral join (see [1]) and polyhedral smash product (see [2]) whose building blocks only involve joins and smash products of spaces. It should be interesting to explore spaces whose building blocks involve mixtures of Cartesian products, joins and smash products.

When we write the boundary maps in the cochain complexes of $X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$, we need to put orientations on their basis elements. To do this, we need to first assign orientations to all the simplicies of $\mathcal{K}$. For convenience, we put a total ordering $<$ on the vertex set $\{v_1, \cdots, v_m\}$ of $\Delta^m$ so that they appear in the increasing order in $\alpha_1$ until $\alpha_k$. In other words, for any $1 \leq i < k$ all the vertices of $\Delta^{\alpha_i}$ have less order than the vertices in $\Delta^{\alpha_i+1}$. Moreover, the vertex-ordering of $\Delta^m$ induces a vertex-ordering of any simplex $\omega \in \Delta^m$, which determines an orientation of $\omega$. Then the boundary of $\omega$ is

$$\partial \omega = \sum_{\sigma \subset \omega, \dim(\sigma) = \dim(\omega) - 1} \varepsilon(\sigma, \omega) \sigma.$$  

Here if $V(\omega) = V(\sigma) \cup \{v\}$, then $\varepsilon(\sigma, \omega)$ is equal to $(-1)^{l(v, \omega)}$ where $l(v, \omega)$ is the number vertices of $\omega$ that are less than $v$ with respect to the vertex-ordering $<$.  

**Definition 3.2** (Simplex with a ghost face). For any $r \geq 0$, the $r$-simplex $\Delta^r$ has an empty face $\hat{0}$ by our convention (note here we do not use bold font since we want to distinguish $\hat{0}$ from the empty simplex $\tilde{0}$ of $K$). In addition, we attach a ghost face $-\hat{1}$ to $\Delta^r$ with the following conventions.

- $\dim(\hat{0}) = \dim(-\hat{1}) = -1$, $\text{rank}(\hat{0}) = \text{rank}(-\hat{1}) = 0$.
- The interiors of $\hat{0}$ and $-\hat{1}$ are themselves.
• The boundary of any vertex of $\Delta^r$ is $\hat{0}$.
• The boundaries of $\hat{0}$ and $-\hat{1}$ are empty.

We denote $\Delta^r$ with the face $\hat{0}$ and $-\hat{1}$ by $\hat{\Delta}^r$. This construction is very useful for us to simplify our argument later.

3.1. Cochain complex of $X(K, \lambda_\alpha)$.

Let $S^0 = \{a^0, a^1\}$ where we assign $-1$ to $a^0$ and $+1$ to $a^1$ as the orientation of $S^0$. Given a total ordering of all the vertices of $\Delta^r$, we can determine an orientation of each face of $\Delta^r$. Let $\sigma^\circ$ denote the relative interior of $\sigma$ in $\Delta^r$. Then we can write a basis of the cellular cochain complex $C^\ast(S^0 \ast \Delta^r; k)$ as

\[
\{a^0\}, \{a^1\}, \{S^0 \ast \sigma^\circ \mid \sigma \text{ is a nonempty simplex in } \Delta^r\},
\]

where the orientation of $S^0 \ast \sigma^\circ$ is canonically determined by the orientations of $S^0$ and $\sigma$. We formally define

\[S^0 \ast \hat{0} := a^1 - a^0, \quad S^0 \ast -\hat{1} := a^0.\]

Then we can write a basis of the cellular chain complex $C_\ast(S^0 \ast \Delta^r; k)$ as

\[
\{S^0 \ast \sigma^\circ \mid \sigma \in \hat{\Delta}^r\}.
\]

Let $\{x^\sigma \mid \sigma \in \hat{\Delta}^r\}$ denote a basis for the cellular cochain complex $C^\ast(S^0 \ast \Delta^r; k)$ where $x^\sigma$ is the dual of $S^0 \ast \sigma^\circ$. So $x^\sigma$ is a cochain in dimension $\text{rank}(\sigma)$.

For a nonempty simplex $\sigma \in \Delta^r$, the differential $d(x^\sigma)$ is:

\[
d(x^\sigma) = \sum_{\substack{\sigma \subset \tau \in \Delta^r \setminus \sigma \subset \Delta^r \setminus \tau \in \Delta^r}} \varepsilon(\sigma, \tau) \cdot x^\tau,
\]

where $\varepsilon(\sigma, \tau) \in \{-1, 1\}$ is defined as in (28). In addition, our convention implies

\[
d(x^{-1}) = 0, \quad d(x^0) = \sum_{v \in V(\Delta^r)} x^v.
\]

Remark 3.3. Let $\{(a^0)^*, (a^1)^*\} \subset C^\ast(S^0; k)$ be the dual basis of $\{a_0, a_1\}$. Then we can write $x^0 = (a^1)^*$ and $x^{-1} = (a^0)^* + (a^1)^*$. The same convention is used in [3] to study the cohomology rings of real moment-angle complexes.

By the homeomorphism in [20], let $X(\Delta^m, \lambda_\alpha) \cong \prod_{i \in [k]} S^0 \ast \Delta^\alpha_i$ be equipped with the product cell structure of each of $S^0 \ast \Delta^\alpha_i$. Then the corresponding cellular cochain complex $C^\ast(X(\Delta^m, \lambda_\alpha); \mathbb{Z}_2)$ has a basis

\[
\{x^\Phi = x^{\sigma_1} \times \cdots \times x^{\sigma_k}; \ \Phi = (\sigma_1, \ldots, \sigma_k), \ \text{where } \sigma_i \in \hat{\Delta}^\alpha_i, 1 \leq i \leq k\}.
\]
Here the “×” should be understood as the cross product of cochains. By the rule of taking differentials of cross products of cochains (see [11, 3.B]), we obtain

\[
d(x^\Phi) := \sum_{1 \leq i \leq k} \iota(\Phi, \sigma_i) x^{\sigma_1} \times \cdots \times dx^{\sigma_i} \times \cdots \times x^{\sigma_k},
\]

where \( \iota(\Phi, \sigma_i) = (-1)^{\sum_{\ell=1}^{i-1} \dim(x^{\sigma_\ell})} \). For a simplex \( \sigma \in \Delta^m \) and \( J \subset [k] \setminus I_\alpha(\sigma) \), let

\[
\Phi^J_\sigma = (\sigma^J_1, \ldots, \sigma^J_k)
\]

where \( \sigma^J_i = \begin{cases} \sigma \cap \Delta^\alpha_i = \sigma_{(i)}, & i \in I_\alpha(\sigma); \\ \hat{0} \in \hat{\Delta}^{\alpha_i}, & i \in J; \\ -\hat{1} \in \hat{\Delta}^{\alpha_i}, & i \in [k] \setminus (I_\alpha(\sigma) \cup J); \end{cases} \)

The sum of those terms \( \sigma^J_i \) where \( \sigma^J_i \) is a cochain of dimension \( \dim(x^{\sigma^J_i}) = \rank(\sigma^J_i) \).

So for \( \Phi = \Phi^J_\sigma \), the equation (30) reads:

\[
d(x^\Phi^J) = \sum_{1 \leq i \leq k} \iota(\Phi^J_\sigma, \sigma^J_i) x^{\sigma^J_1} \times \cdots \times dx^{\sigma^J_i} \times \cdots \times x^{\sigma^J_k}
\]

Note that if \( V(\omega) = V(\sigma) \cup \{v\} \) where \( v \in \alpha_i \), then \( \omega^J_i = \sigma^J_i \cup \{v\} \) and we have

\[
\varepsilon(\sigma, \omega) = \iota(\Phi^J_\sigma, \sigma^J_i) \cdot \varepsilon(\sigma^J_i, \tau).
\]

This follows from our convention on the vertex-ordering of \( \Delta^m \). So we obtain

\[
d(x^\Phi^J) = \sum_{\sigma \subseteq \omega, I_\alpha(\omega) \subseteq I_\alpha(\sigma) \cup J} \varepsilon(\sigma, \omega) x^{\Phi^J_\sigma(\omega) \cup J \setminus I_\alpha(\omega)}.
\]

For the simplicial complex \( K \subset \Delta^m \), let \( C^*(X(K, \lambda_\alpha); k) \) be the cellular cochain complex corresponding to \( B_\alpha(K) \). Then for any \( \sigma \in K \), it is easy to see that \( x^\Phi^J \) is a cochain of dimension \( \rank(\sigma) \) in \( C^*(X(K, \lambda_\alpha); k) \) which is the dual of the chain \( B(\sigma, J) \in C_*(X(K, \lambda_\alpha); k) \) (see (14) and (16)). So we can choose a basis of \( C^*(X(K, \lambda_\alpha); k) \) as

\[
\{ x^\Phi^J_\sigma ; \sigma \in K, J \subset [k] \setminus I_\alpha(\sigma) \}.
\]

Note that for any \( \sigma \in K \), the differential \( d(x^\Phi^J) \) in \( C^*(X(K, \lambda_\alpha); k) \) is only the sum of those terms \( x^{\Phi^J_\sigma(\omega) \cup J \setminus I_\alpha(\omega)} \) on the right side of (32) with \( \omega \in K \).
3.2. Cochain complex of $X(K, \Lambda_\alpha)$.

Let $\{e^0, e^1\}$ be the cell decomposition of $S^1$ where $\dim(e^0) = 0$, $\dim(e^1) = 1$, and $e^0, e^1$ are both oriented. Then given an orientation of each face of $\Delta^r$, we obtain an oriented cell decomposition of $S^1 \ast \Delta^r$ by

$$\{e^0\}, \{e^1\}, \{S^1 \ast \sigma^r \mid \sigma \text{ is a nonempty simplex in } \Delta^r\}.$$  

If we formally define $S^1 \ast \hat{0} = e^1$ and $S^1 \ast -\hat{1} = e^0$, we can write a basis of the cellular chain complex $C_*(S^1 \ast \Delta^r; k)$ as $\{S^1 \ast \sigma^r \mid \sigma \in \hat{\Delta}^r\}$ where the orientation of $S^1 \ast \sigma^r$ is determined canonically by the orientations of $S^1$ and $\sigma$.

Let $\{y^\sigma \mid \sigma \in \hat{\Delta}^r\}$ denote a basis for the cellular cochain complex $C^*(S^1 \ast \Delta^r; \mathbb{Z})$, where $y^\sigma$ is the dual of $S^1 \ast \sigma^r$. For any nonempty simplex $\sigma$ in $\Delta^r$,

$$d(y^\sigma) = \sum_{\sigma \subseteq \tau, \dim(\tau) = \dim(\sigma) + 1} \varepsilon(\sigma, \tau) \cdot y^\tau.$$  

In addition, we have

$$d(y^{-1}) = 0, \quad d(y^0) = \sum_{v \in V(\Delta^r)} y^v.$$  

Note that $y^{-1}$ and $y^0$ are cochains in dimension 0 and 1, respectively.

By the homeomorphism in (20), let $X(\Delta^{[m]}, \Lambda_\alpha) \cong \prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i}$ be equipped with the product cell structure of each $S^1 \ast \Delta^{\alpha_i}$. The corresponding cellular cochain complex $C^*(X(\Delta^{[m]}, \Lambda_\alpha); k)$ has a basis

$$\{y^\Phi = y^{\sigma_1} \times \cdots \times y^{\sigma_k} \mid \Phi = (\sigma_1, \cdots, \sigma_k), \text{ where } \sigma_i \in \hat{\Delta}^{\alpha_i}, 1 \leq i \leq k\}.$$  

We need two more notations for our argument below.

- For any vertex $v$ of $K$ and a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $V(K)$, let $i_\alpha(v) \in [k]$ denote the index so that $v \in \alpha_{i_\alpha(v)}$.
- For any $i \in L \subset [k]$, define $\kappa(i, L) = (-1)^{r(i, L)}$, where $r(i, L)$ is the number of elements in $L$ less than $i$. Moreover, for any simplex $\sigma \in K_{\alpha, L}$ we define

$$\kappa(\sigma, L) := \prod_{v \in V(\sigma)} \kappa(i_\alpha(v), L).$$  

So if $V(\omega) = V(\sigma) \cup \{v\}$, we have $\kappa(\omega, L) = \kappa(\sigma, L) \cdot \kappa(i_\alpha(v), L).$  

The differential of $y^\Phi$ in $C^*(X(\Delta^{[m]}, \Lambda_\alpha); k)$ is given by:

$$d(y^\Phi) := \sum_{1 \leq i \leq k} i(\Phi, \sigma_i) \cdot y^{\sigma_1} \times \cdots \times dy^{\sigma_i} \times \cdots \times y^{\sigma_k},$$  

(37)
where \( i(\Phi, \sigma_i) = (-1)^{\sum_{i=1}^{k} \dim(y_{i1})} \). Notice that when \( \Phi = \Phi^J_{\sigma} = (\sigma^J_1, \ldots, \sigma^J_k) \),

\[
\dim(y_{i1}) = \begin{cases} \text{rank}(\sigma^J_i) + 1, & i \in I_\alpha(\sigma); \\ 1, & i \in J; \\ 0, & i \in [k]\setminus(I_\alpha(\sigma) \cup J). \end{cases}
\]

\[
d(y_{i1}^J) = \sum_{1 \leq i \leq k} i(\Phi^J_{\sigma}, \sigma^J_i) \ y_{i1}^J \times \cdots \times dy_{i1}^J \times \cdots \times y_{i1}^J
\]

\[
= \sum_{1 \leq i \leq k} i(\Phi^J_{\sigma}, \sigma^J_i) \ y_{i1}^J \times \cdots \times \left( \sum_{\sigma^J_i \in \tau \in \Delta^\alpha} \varepsilon(\sigma^J_i, \tau) \ y^\tau \right) \times \cdots \times y_{i1}^J.
\]

Note that if \( V(\omega) = V(\sigma) \cup \{v\} \) where \( v \in \alpha_i \), then \( \omega^J_i = \sigma^J_i \cup \{v\} \) and we have

\[
\kappa(i, I_\alpha(\sigma) \cup J) \cdot \varepsilon(\sigma, \omega) = i(\Phi^J_{\sigma}, \sigma^J_i) \cdot \varepsilon(\sigma^J_i, \tau).
\]

This is because for each \( i \in [k]\setminus(I_\alpha(\sigma) \cup J) \), \( \dim(y_{i1}^J) = 0 \). So we obtain

\[
d(y_{i1}^J) = \sum_{\sigma \subseteq \omega \setminus I_\alpha(\sigma) \cup J \setminus \alpha} \kappa(i_\alpha(\omega \setminus \sigma), I_\alpha(\sigma) \cup J) \cdot \varepsilon(\sigma, \omega) \ y_{\omega^J_{i1}} \quad (38)
\]

where \( \omega \setminus \sigma \) denote the vertex of \( \omega \) that is not in \( \sigma \).

For the simplicial complex \( K \subset \Delta^m \), let \( C^*(X(K, \Lambda_\alpha); k) \) be the cellular cochain complex corresponding to \( \hat{B}_\alpha(K) \). Then for any simplex \( \sigma \in K \), \( y_{i1}^J \) is the cochain of dimension \( \text{rank}(\sigma) + |I_\alpha(\sigma)| + |J| \) in \( C^*(X(K, \Lambda_\alpha); k) \) that is dual to the cell \( \hat{B}_{(\sigma, J)} \) (see (13) and (19)). So \( C^*(X(K, \Lambda_\alpha); Z) \) has a basis

\[
\{ y_{i1}^J ; \sigma \in K, J \subset [k]\setminus I_\alpha(\sigma) \}.
\]

Note that for any \( \sigma \in K \), the differential \( d(y_{i1}^J) \) in \( C^*(X(K, \Lambda_\alpha); k) \) is only the sum of those terms \( y_{i1}^J \) on the right side of (38) with \( \omega \in K \).

**Proof of Theorem 1.1**

For any subset \( L \subset [k] \), let \( C^{*,L}(X(K, \Lambda_\alpha); k) \) denote the linear subspace of \( C^*(X(K, \Lambda_\alpha); k) \) generated by the followings set

\[
\{ B_{(\sigma, J)} | I_\alpha(\sigma) \cup J = L, \sigma \in K, J \subset [k]\setminus I_\alpha(\sigma) \}.
\]

Similarly, let \( C^{*,L}(X(K, \Lambda_\alpha); k) \) denote the submodule of \( C^*(X(K, \Lambda_\alpha); k) \) generated by the set \( \{ B_{(\sigma, J)} | I_\alpha(\sigma) \cup J = L, \sigma \in K, J \subset [k]\setminus I_\alpha(\sigma) \} \).

From the differential of \( C^*(X(K, \Lambda_\alpha); k) \) and \( C^*(X(K, \Lambda_\alpha); k) \) shown in (32) and (33), we see that \( C^{*,L}(X(K, \Lambda_\alpha); k) \) and \( C^{*,L}(X(K, \Lambda_\alpha); k) \) are actually cochain subcomplexes of \( C^*(X(K, \Lambda_\alpha); k) \) and \( C^*(X(K, \Lambda_\alpha); k) \), respectively. We denote
their cohomology groups by $H^*(\mathcal{K}, \lambda_\alpha; k)$ and $H^{*, L}(\mathcal{K}, \Lambda_\alpha; k)$. Then we have the following decompositions:

$$H^*(\mathcal{K}, \lambda_\alpha; k) = \bigoplus_{L \subseteq [k]} H^{*, L}(\mathcal{K}, \lambda_\alpha; k),$$

$$H^*(\mathcal{K}, \Lambda_\alpha; k) = \bigoplus_{L \subseteq [k]} H^{*, L}(\mathcal{K}, \Lambda_\alpha; k).$$

(39) (40)

For any subset $L \subseteq [k]$, let $C^*(\mathcal{K}_{\alpha, L}; k)$ denote the simplicial cochain complex of $\mathcal{K}_{\alpha, L}$. For any simplex $\sigma \in \mathcal{K}$, let $\sigma^*$ be the basis cochain dual to $\sigma$ in $C^*(\mathcal{K}_{\alpha, L}; k)$.

Then we have the following isomorphisms of $k$-modules:

$$\varphi^L_\alpha : C^*(\mathcal{K}_{\alpha, L}; k) \rightarrow C^{*, L}(\mathcal{K}(\mathcal{K}, \lambda_\alpha; k), \varphi^L_\alpha : C^*(\mathcal{K}_{\alpha, L}; k) \rightarrow C^{*, L}(\mathcal{K}(\mathcal{K}, \Lambda_\alpha; k)$$

where $\varphi^L_\alpha(\sigma^*) = x^{\Phi^{L, 1}_{\alpha}(\sigma)}$ and $\hat{\varphi}^L_\alpha(\sigma^*) = \kappa(\sigma, L) y^{\Phi^{L, 1}_{\alpha}(\sigma)}$ for any $\sigma \in \mathcal{K}_{\alpha, L}$. Note

$$\dim(x^{\Phi^{L, 1}_{\alpha}(\sigma)}) = \dim(y^{\Phi^{L, 1}_{\alpha}(\sigma)}) = \dim(\sigma) + |L|.$$

Moreover, by the differential of $C^*(\mathcal{K}(\mathcal{K}, \lambda_\alpha; k))$ and $C^*(\mathcal{K}(\mathcal{K}, \Lambda_\alpha; k)$ shown in (32) and (33), we can prove that $\varphi^L_\alpha$ and $\hat{\varphi}^L_\alpha$ are chain complex isomorphisms. Indeed, the proof for $\varphi^L_\alpha$ we have:

$$d(\hat{\varphi}^L_\alpha(\sigma^*)) = d(\kappa(\sigma, L) y^{\Phi^{L, 1}_{\alpha}(\sigma)})$$

$$= \kappa(\sigma, L) \sum_{\sigma \in \omega \subseteq \mathcal{K}_{\alpha}, \lambda_\alpha(\omega) \subseteq L} \dim(\omega) = \dim(\sigma) + 1$$

$$= \sum_{\sigma \in \omega \subseteq \mathcal{K}_{\alpha}, \lambda_\alpha(\omega) \subseteq L} \dim(\omega) = \dim(\sigma) + 1$$

The third “$=$” uses the relation $\kappa(\omega, L) = \kappa(\sigma, L) \cdot \kappa(i_\alpha(\omega|\sigma), L)$ (see (36)). So we have additive isomorphisms of cohomology groups

$$\tilde{H}^q(\mathcal{K}_{\alpha, L}; k) \cong H^{q+1,L}(\mathcal{K}(\mathcal{K}, \lambda_\alpha; k), \tilde{H}^q(\mathcal{K}_{\alpha, L}; k) \cong H^{q+|L|+1,L}(\mathcal{K}(\mathcal{K}, \Lambda_\alpha; k).$$

Let

$$\varphi_\alpha = \bigoplus_{L \subseteq [k]} \varphi^L_\alpha : \bigoplus_{L \subseteq [k]} \tilde{H}^{q-1}(\mathcal{K}_{\alpha, L}; k) \rightarrow H^q(\mathcal{K}(\mathcal{K}, \lambda_\alpha; k),$$

$$\hat{\varphi}_\alpha = \bigoplus_{L \subseteq [k]} \hat{\varphi}^L_\alpha : \bigoplus_{L \subseteq [k]} \tilde{H}^{q-|L|-1}(\mathcal{K}_{\alpha, L}; k) \rightarrow H^q(\mathcal{K}(\mathcal{K}, \Lambda_\alpha; k).$$

Then $\varphi_\alpha$ and $\hat{\varphi}_\alpha$ are the isomorphisms that satisfy our requirement. \square

In the following, we use Theorem 1.1 to compute the cohomology groups of some examples explicitly.
Example 1. Let $e_1, \ldots, e_5$ be a basis of $(\mathbb{Z}_2)^5$. Let $P$ be a pentagon. Then the real moment-angle complex $M = \mathbb{R}\mathcal{Z}_P$ is an orientable closed surface of genus five. Note that $M$ can be obtained either from $P$ with the $(\mathbb{Z}_2)^5$-coloring shown in the left picture of Figure 2 or from an octagon $P'$ with the $(\mathbb{Z}_2)^3$-coloring shown in the middle picture of Figure 2. Indeed, the octagon $P'$ is obtained from gluing four copies of $P$ along their edges colored by $e_4$ and $e_5$.

The simplicial complex $\mathcal{K}^{P'}$ dual to $P'$ is a simplicial circle with eight vertices that are partitioned into three subsets labelled by $A$, $B$ and $C$. By Theorem 1.1 the cohomology groups of $M$ can be computed from the cohomology groups of the simplicial subcomplexes $\{\mathcal{K}^{P'}_{\alpha,L}; L \subset [3]\}$ which are shown in Figure 3 except $\mathcal{K}^{P'}_{\alpha,\emptyset} = \hat{0}$ and $\mathcal{K}^{P'}_{\alpha,[3]} = \mathcal{K}^{P'}$.

Remark 3.4. The above example implies that for any simplicial complex $\mathcal{K}$, $\mathbb{R}\mathcal{Z}_\mathcal{K}$ (or $\mathcal{Z}_\mathcal{K}$) can be thought of as the quotient space of $\mathbb{R}\mathcal{Z}_{\hat{\mathcal{K}}}$ (or $\mathcal{Z}_{\hat{\mathcal{K}}}$) where $\hat{\mathcal{K}}$ is some larger simplicial complex constructed from $\mathcal{K}$. So we can also compute the cohomology groups of $\mathbb{R}\mathcal{Z}_\mathcal{K}$ (or $\mathcal{Z}_\mathcal{K}$) from some $(\mathbb{Z}_2)^k$-coloring (or $\mathbb{Z}^k$-coloring) on $\hat{\mathcal{K}}$ via Theorem 1.1.
Example 2 (Balanced Simplicial Complex and Pullbacks from the Linear Model).

An \((n - 1)\)-dimensional simplicial complex \(K\) is called balanced if and only if there exists a map \(\phi : V(K) \to [n] = \{1, \ldots, n\}\) such that if \(\{v, v'\}\) is an edge of \(K\), then \(\phi(x) \neq \phi(y)\) (see [17]). We call \(\phi\) an \(n\)-coloring on \(K\). It is easy to see that \(K\) is balanced if and only if \(\phi\) is a non-degenerate simplicial map onto \(\Delta^{n-1}\). In fact, if we identify the vertex set of \(\Delta^{n-1}\) with \([n]\), any \(n\)-coloring \(\phi\) on \(K\) induces a non-degenerate simplicial map from \(K\) to \(\Delta^{n-1}\) which sends a simplex \(\sigma \in K\) with \(V(\sigma) = \{v_1, \ldots, v_i\}\) to the face of \(\Delta^{n-1}\) spanned by \(\{\phi(v_1), \ldots, \phi(v_i)\}\).

Suppose \(\phi : V(K) \to [n]\) is an \(n\)-coloring of \((n - 1)\)-dimensional simplicial complex \(K\). Let \(\{e_1, \ldots, e_n\}\) be a basis of \((\mathbb{Z}_2)^n\). Then \(\phi\) uniquely determines a \((\mathbb{Z}_2)^n\)-coloring \(\lambda^\phi : V(K) \to (\mathbb{Z}_2)^n\) where \(\lambda^\phi(v) = e_{\phi(v)}\). It is clear that \(\lambda^\phi\) is a non-degenerate \((\mathbb{Z}_2)^n\)-coloring on \(K\). The space \(X(K, \lambda^\phi)\) is also called a pullback from the linear model in [10, Example 1.15]. Notice that

\[
\alpha_\phi := \{\phi^{-1}(1), \ldots, \phi^{-1}(n)\}
\]

is a partition of \(V(K)\). So by our notation in section 1, \(X(K, \lambda^\phi) = X(K, \lambda_{\alpha_\phi})\).

Then by Theorem [11] the cohomology groups of \(X(K, \lambda^\phi)\) can be computed by

\[
H^q(X(K, \lambda^\phi); k) = \sum_{L \subseteq [n]} \tilde{H}^{q-1}(K_{\lambda_{\alpha_\phi, L}}; k), \ \forall q \geq 0.
\]

4. Stable Decompositions of \(X(K, \lambda_\alpha)\) and \(X(K, \Lambda_\alpha)\)

It is shown in [2] that the stable homotopy type of a polyhedral product \((X, A)^K\) is a wedge of spaces, which provides corresponding homological decompositions of \((X, A)^K\). In this section, we prove parallel results of for \(X(K, \lambda_\alpha)\) and \(X(K, \Lambda_\alpha)\). Our argument proceeds along the same line as [2].

Let \(\alpha = \{\alpha_1, \ldots, \alpha_k\}\) be a partition of the vertex set \(V(K)\) of a simplicial complex \(K\). For any subset \(L \subseteq [k]\) and any simplex \(\sigma \in K\), define

\[
W^0_{\alpha, L}(\sigma) := \bigwedge_{i \in I_{\alpha}(\sigma) \cap L} S^0_{(i)} \ast (\sigma \cap \Delta^n) \wedge \bigwedge_{i \in L \setminus (I_{\alpha}(\sigma) \cap L)} S^0_{(i)}
\]

\[
W^1_{\alpha, L}(\sigma) := \bigwedge_{i \in I_{\alpha}(\sigma) \cap L} S^1_{(i)} \ast (\sigma \cap \Delta^n) \wedge \bigwedge_{i \in L \setminus (I_{\alpha}(\sigma) \cap L)} S^1_{(i)}
\]

where \(\wedge\) and \(\bigwedge\) denote the smash product. We adopt the convention that the smash product of a space with the empty space is empty. Then the following lemma is immediate from the definitions of \(W^0_{\alpha, L}(\sigma)\) and \(W^1_{\alpha, L}(\sigma)\).

Lemma 4.1. For any subset \(L \subseteq [k]\) and any simplex \(\sigma \in K\),

(i) \(W^0_{\alpha, L}(\sigma) = W^0_{\alpha, L}(\sigma \cap K_{\alpha, L})\), \(W^1_{\alpha, L}(\sigma) = W^1_{\alpha, L}(\sigma \cap K_{\alpha, L})\).


(ii) \( W^{S_0}_{\alpha, L}(\sigma) \) and \( W^{S_1}_{\alpha, L}(\sigma) \) are contractible whenever \( I_\alpha(\sigma) \cap L \neq \emptyset \).

(iii) \( W^{S_0}_{\alpha, L}(\hat{0}) = \bigwedge_{i \in L} S^0_{(i)} \cong S^0 \) while \( W^{S_1}_{\alpha, L}(\hat{0}) = \bigwedge_{i \in L} S^1_{(i)} \cong S^{[L]} \).

Next, we assign a base-point to each of \( S_{(i)} \), which then determines the base-points of \( W^{S_0}_{\alpha, L}(\sigma) \) and \( W^{S_1}_{\alpha, L}(\sigma) \). By \([2\text{, Theorem 2.8}]\), there are natural pointed homotopy equivalences (see \([2, \text{Theorem 4.3}]\)), we obtain homotopy equivalences

\[
\Sigma(D_\alpha(\sigma)) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} W^{S_0}_{\alpha, L}(\sigma) \right), \quad \Sigma(\hat{D}_\alpha(\sigma)) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} W^{S_1}_{\alpha, L}(\sigma) \right)
\]

where \( \Sigma \) is the reduced suspension. Define

\[
E_\alpha(\sigma) := \bigvee_{L \subseteq [k]} W^{S_0}_{\alpha, L}(\sigma), \quad \hat{E}_\alpha(\sigma) := \bigvee_{L \subseteq [k]} W^{S_1}_{\alpha, L}(\sigma).
\]

Let \( \text{Cat}(\mathcal{K}) \) denote the face category of \( \mathcal{K} \) whose objects are simplices \( \sigma \in \mathcal{K} \) and there is a morphism from \( \sigma \) to \( \tau \) whenever \( \sigma \subseteq \tau \). Then we can consider \( D_\alpha \), \( \hat{D}_\alpha \), \( E_\alpha \), \( \hat{E}_\alpha \) as functors from \( \text{Cat}(\mathcal{K}) \) to the category \( \text{CW}_* \) of connected, based \( \text{CW} \)-complexes and based continuous maps. It is clear that

\[
X(\mathcal{K}, \lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} D_\alpha(\sigma) = \text{colim}(D_\alpha(\sigma)), \quad X(\mathcal{K}, \Lambda_\alpha) = \bigcup_{\sigma \in \mathcal{K}} \hat{D}_\alpha(\sigma) = \text{colim}(\hat{D}_\alpha(\sigma)).
\]

For any subset \( L = \{l_1, \ldots, l_\alpha\} \subseteq [k] \), define

\[
X^\wedge(\mathcal{K}_{\alpha, L}, \lambda_\alpha) := \bigcup_{\sigma \in \mathcal{K}} W^{S_0}_{\alpha, L}(\sigma) = \bigcup_{\sigma \in \mathcal{K}_{\alpha, L}} W^{S_0}_{\alpha, L}(\sigma),
\]

\[
X^\wedge(\mathcal{K}_{\alpha, L}, \Lambda_\alpha) := \bigcup_{\sigma \in \mathcal{K}} W^{S_1}_{\alpha, L}(\sigma) = \bigcup_{\sigma \in \mathcal{K}_{\alpha, L}} W^{S_1}_{\alpha, L}(\sigma).
\]

So \( X^\wedge(\mathcal{K}, \lambda_\alpha) \) and \( X^\wedge(\mathcal{K}_{\alpha, L}, \lambda_\alpha) \) are the colimits of \( W^{S_0}_{\alpha, L}(\sigma) \) and \( W^{S_1}_{\alpha, L}(\sigma) \) over face category \( \text{Cat}(\mathcal{K}_{\alpha, L}) \) of \( \mathcal{K}_{\alpha, L} \), respectively. In addition, we have

\[
\text{colim}(E_\alpha(\sigma)) = \bigvee_{L \subseteq [k]} X^\wedge(\mathcal{K}_{\alpha, L}, \lambda_\alpha), \quad \text{colim}(\hat{E}_\alpha(\sigma)) = \bigvee_{L \subseteq [k]} X^\wedge(\mathcal{K}_{\alpha, L}, \Lambda_\alpha).
\]

Since the suspension commutes with the colimits up to homotopy equivalence (see \([2\text{, Theorem 4.3}]\)), we obtain homotopy equivalences

\[
\Sigma(X(\mathcal{K}, \lambda_\alpha)) \simeq \text{colim}(\Sigma(D_\alpha(\sigma))) \simeq \text{colim}(\Sigma(E_\alpha(\sigma))) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} X^\wedge(\mathcal{K}_{\alpha, L}, \lambda_\alpha) \right),
\]

\[
\Sigma(X(\mathcal{K}, \Lambda_\alpha)) \simeq \text{colim}(\Sigma(\hat{D}_\alpha(\sigma))) \simeq \text{colim}(\Sigma(\hat{E}_\alpha(\sigma))) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} X^\wedge(\mathcal{K}_{\alpha, L}, \Lambda_\alpha) \right).
\]
Lemma 4.2. For any $L \subseteq [k]$, there are homotopy equivalences:

$$X^\wedge(K_{\alpha,L}, \lambda_\alpha) \simeq \bigvee_{\sigma \in K_{\alpha,L}} |\Delta((K_{\alpha,L})_{>\sigma})| * W_{\alpha,L}^{S^0}(\sigma),$$

$$X^\wedge(K_{\alpha,L}, \Lambda_\alpha) \simeq \bigvee_{\sigma \in K_{\alpha,L}} |\Delta((K_{\alpha,L})_{>\sigma})| * W_{\alpha,L}^{S^1}(\sigma).$$

where $\Delta((K_{\alpha,L})_{>\sigma})$ is the order complex of the poset $\{\tau \in K_{\alpha,L} \mid \tau \supseteq \sigma\}$ with respect to the inclusion and $|\Delta((K_{\alpha,L})_{>\sigma})|$ is the geometric realization of $\Delta((K_{\alpha,L})_{>\sigma})$.

Proof. Note that the natural inclusion $S^0 \hookrightarrow S^0 * \Delta^r$ is null-homotopic for any $r \geq 0$. Then parallel to the proof of [2] Theorem 2.12, we have a homotopy equivalence $H_L(\sigma) : W_{\alpha,L}^{S^0}(\sigma) \to W_{\alpha,L}^{S^0}(\sigma)$ for each simplex $\sigma \in K_{\alpha,L}$ so that the following diagram commutes for any $\tau \subseteq \sigma \in K_{\alpha,L}$,

$$\begin{array}{ccc}
W_{\alpha,L}^{S^0}(\tau) & \xrightarrow{H_L(\tau)} & W_{\alpha,L}^{S^0}(\tau) \\
\downarrow_{\zeta_{\sigma,\tau}} & & \downarrow_{c_{\sigma,\tau}} \\
W_{\alpha,L}^{S^0}(\sigma) & \xrightarrow{H_L(\sigma)} & W_{\alpha,L}^{S^0}(\sigma)
\end{array}$$

(41)

where $\zeta_{\sigma,\tau}$ is the natural inclusion and $c_{\sigma,\tau}$ is the constant map to the base-point. Then by [2] Theorem 4.1 and [2] Theorem 4.2, there is a homotopy equivalence

$$X^\wedge(K_{\alpha,L}, \lambda_\alpha) = \text{colim}(W_{\alpha,L}^{S^0}(\sigma)) \simeq \bigvee_{\sigma \in K_{\alpha,L}} |\Delta((K_{\alpha,L})_{>\sigma})| * W_{\alpha,L}^{S^0}(\sigma).$$

The argument for $X^\wedge(K, \Lambda_\alpha)$ is completely parallel (by replacing $S^0$ by $S^1$). \qed

From the above discussions, we obtain homotopy equivalences:

$$\Sigma(X(K, \lambda_\alpha)) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} \bigvee_{\sigma \in K_{\alpha,L}} |\Delta((K_{\alpha,L})_{>\sigma})| * W_{\alpha,L}^{S^0}(\sigma) \right),$$

$$\Sigma(X(K, \Lambda_\alpha)) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} \bigvee_{\sigma \in K_{\alpha,L}} |\Delta((K_{\alpha,L})_{>\sigma})| * W_{\alpha,L}^{S^1}(\sigma) \right).$$

Moreover, the above spaces can be simplified by the following facts.

- $W_{\alpha,L}^{S^0}(\sigma)$ and $W_{\alpha,L}^{S^1}(\sigma)$ are contractible whenever $\sigma \neq \emptyset \in K_{\alpha,L}$ (see Lemma 4.2(ii)).
- $\Delta((K_{\alpha,L})_{>\emptyset})$ is isomorphic to the barycentric subdivision $K'_{\alpha,L}$ of $K_{\alpha,L}$ whose geometric realization is homeomorphic to that of $K_{\alpha,L}$.

Then we have the following homotopy equivalences which proves Theorem 1.2

$$\Sigma(X(K, \lambda_\alpha)) \simeq \Sigma\left( \bigvee_{L \subseteq [k]} |K_{\alpha,L}| * S^0 \right) \simeq \bigvee_{L \subseteq [k]} \Sigma^2(|K_{\alpha,L}|),$$
$$\Sigma(X(\mathcal{K}, \Lambda_\alpha)) \simeq \Sigma \left( \bigvee_{L \subset [k]} |\mathcal{K}_{\alpha,L}| \ast S^{\lfloor |L| \rfloor} \right) \simeq \bigvee_{L \subset [k]} \Sigma^{|L|+2}(|\mathcal{K}_{\alpha,L}|).$$

5. Another interpretation of $H^*(X(\mathcal{K}, \lambda_\alpha); \mathbb{Z}_2)$ and $H^*(X(\mathcal{K}, \Lambda_\alpha); \mathbb{Z}_2)$

Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set $V(\mathcal{K})$ of a simplicial complex $\mathcal{K}$. We have defined a differential $d_\alpha$ on $\Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}]$ in (10).

Claim: $d_\alpha \circ d_\alpha = 0$.

Indeed, we have $d_\alpha \circ d_\alpha(t_i) = d_\alpha \left( \sum_{v_j \in \alpha_i} v_{(j)} \right) = \sum_{v_j \in \alpha_i} \sum_{v_j \in \omega, I_\alpha(\omega) = \{i\}} \varepsilon(\omega, v_j) \cdot v_\omega = 0$

since any $v_\omega$ with $\dim(\omega) = 1$ and $I_\alpha(\omega) = \{i\}$ appears twice with opposite signs in the above sum. Similarly, we obtain $d_\alpha \circ d_\alpha(v_\sigma) = 0$ from the face that $\partial \circ \partial = 0$ in $C_*(\mathcal{K}; k)$. The claim is proved. \[\square\]

For any subset $J = \{j_1, \cdots, j_s\} \subset [k]$ with $j_1 < \cdots < j_s$, let

$$t_J := t_{j_1} \cdots t_{j_s} \in \Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}].$$

We introduce a multi-grading $\text{mdeg}_\alpha$ on $\Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}]$ by

$$\text{mdeg}_\alpha(t_J) = (-1, J) \in (\mathbb{Z}, [k]), \quad \text{mdeg}_\alpha(v_\sigma) = (0, I_\alpha(\sigma)) \in (\mathbb{Z}, [k]). \quad (42)$$

Here we define the sum of two elements $(n, J)$, $(n', J') \in (\mathbb{Z}, [k])$ to be

$$(n, J) + (n', J') := (n + n', J \cup J'). \quad (43)$$

So the multi-grading of the product of two monomials in $\Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}]$ is equal to the sum of their multi-gradings.

Next, we identify $\Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}]$ with the cellular cochain complexes of some spaces. These spaces are obtained by replacing each vertex $v$ of $\mathcal{K}$ in (26) and (27) by an infinitely dimensional sphere $S^\infty$, i.e.

$$\mathcal{X}(\mathcal{K}, \alpha, S^0) := \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_\alpha(\sigma)} \left( S^0_{(i)} \ast \ast_{v \in V(\sigma) \cap \alpha_i} S^\infty_v \right) \ast \prod_{i \notin [k]\setminus I_\alpha(\sigma)} S^0_{(i)} \right), \quad (44)$$

$$\mathcal{X}(\mathcal{K}, \alpha, S^1) := \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_\alpha(\sigma)} \left( S^1_{(i)} \ast \ast_{v \in V(\sigma) \cap \alpha_i} S^\infty_v \right) \ast \prod_{i \notin [k]\setminus I_\alpha(\sigma)} S^1_{(i)} \right), \quad (45)$$

where $S^\infty_v$ denotes the copy of $S^\infty$ corresponding to the vertex $v$, and $S^0_{(i)}$ and $S^1_{(i)}$ denote the copy of $S^0$ and $S^1$ corresponding to the index $i \in [k]$, respectively.

Note that $S^\infty$ is a contractible space which has a cell decomposition with exactly one cell in each dimension. The boundary of every $2j$-dimensional cell $f^{2j}$ is the
closure of the \((2j - 1)\)-dimensional cell \(f^{2j-1}\), while the boundary of each \(f^{2j-1}\) is zero. For any vertex \(v\) of \(\mathcal{K}\), the cells in \(S_\v^\infty\) are denoted by \(\{f^m_v\}_{n \geq 0}\).

By identifying each vertex \(v \in V(\mathcal{K})\) as the 0-cell \(e_0^v \in S_\v^\infty\), we can think of \(X(\mathcal{K}, \lambda_\alpha)\) and \(X(\mathcal{K}, \Lambda_\alpha)\) as subspaces of \(\mathcal{X}(\mathcal{K}, \alpha, S^0)\) and \(\mathcal{X}(\mathcal{K}, \alpha, S^1)\), respectively. The deformation retraction of each \(S_\v^\infty\) to \(e_0^v = v\) obviously induces deformation retractions of \(\mathcal{X}(\mathcal{K}, \alpha, S^0)\) to \(X(\mathcal{K}, \lambda_\alpha)\) and \(\mathcal{X}(\mathcal{K}, \alpha, S^1)\) to \(X(\mathcal{K}, \Lambda_\alpha)\), respectively. So we have isomorphisms of cohomology rings

\[
H^*(\mathcal{X}(\mathcal{K}, \alpha, S^0); k) \cong H^*(X(\mathcal{K}, \lambda_\alpha); k). \tag{46}
\]

\[
H^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); k) \cong H^*(X(\mathcal{K}, \Lambda_\alpha); k). \tag{47}
\]

To give explicit descriptions of the cell structures of \(w\):

Definition 5.1 (Weighted Simplex). An \(r\)-simplex \(\Delta^r\) along with a function \(w : V(\Delta^r) \to \mathbb{Z}_{\geq 0}\) is called a weighted \(r\)-simplex, denoted by \((\Delta^r, w)\). We call \(w\) a weight of \(\sigma\). For any face \(\sigma\) of \(\Delta^r\), let \(w_\sigma\) denote the restriction of \(w\) to \(V(\sigma)\), and we call \((\sigma, w_\sigma)\) a weighted face of \((\Delta^r, w)\). Any weight of the empty face \(\hat{0}\) and the ghost face \(-\hat{1}\) of \(\hat{\Delta}^r\) is null. So for brevity, we use \(\hat{0}\) and \(-\hat{1}\) to denote the weighted simplices \((\hat{0}, w_\hat{0})\) and \((-\hat{1}, w_{-\hat{1}})\).

For any \(d \geq 1\), let \(\{e^0, e^d\}\) denote the cell decomposition of the \(d\)-sphere \(S^d\) where \(e^0\) is a 0-cell and \(e^d\) is a \(d\)-cell. When \(S^0 = \{a^0, a^1\}\), we define \(e^0 = a^0\) and \(e^d = a^1 - a^0\) and formally think of \(\{e^0, e^0\}\) as a cell decomposition of \(S^0\).

For any \(d \geq 0\) and a weighted simplex \((\Delta^r, w)\), we define an open ball

\[
S^d \ast (\Delta^r, w) = S^d \ast \bigast_{v \in V(\Delta^r)} f_v^{w(v)} \subset S^d \ast \bigast_{v \in V(\Delta^r)} S_\v^\infty.
\]

We adopt the following conventions in the rest.

- when \(d = 0\), define \(S^0 \ast \hat{0} = \hat{e}^0, S^0 \ast -\hat{1} = e^0\).
- when \(d \geq 1\), define \(S^0 \ast \hat{0} = e^d, S^0 \ast -\hat{1} = e^0\).

Then a cell decomposition of \(S^d \ast \bigast_{v \in V(\Delta^r)} S_\v^\infty\) is given by:

\[
\{S^d \ast (\sigma, w_\sigma) \mid \sigma \in \hat{\Delta}^r, w_\sigma\text{ is an arbitrary weight of }\sigma\}.
\]

Here we write the argument for an arbitrary \(d \geq 0\) because we will generalize our discussion to a wider range of spaces later (see Theorem 7.3 and Theorem 7.4).

Now we can construct cell decompositions of \(\mathcal{X}(\mathcal{K}, \alpha, S^0)\) and \(\mathcal{X}(\mathcal{K}, \alpha, S^1)\) as follows. For any weighted simplex \((\sigma, w_\sigma), \sigma \in \mathcal{K}\) and \(J \subset [k]\) with \(J \cap \Lambda_\alpha(\sigma) = \emptyset\),

\[
B(\sigma, w_\sigma, J) := \prod_{i \in \Lambda_\alpha(\sigma)} S^0_{(i)} \ast (\sigma_{(i)}, w_{\sigma_{(i)}}) \times \prod_{i \in J} S^0_{(i)} \ast \hat{0}_{(i)} \times \prod_{i \in [k] \setminus (\Lambda_\alpha(\sigma) \cup J)} S^0_{(i)} \ast -\hat{1}_{(i)}.
\]
\( \hat{B}(\sigma, w_{\sigma}, J) := \prod_{i \in \lambda(\sigma)} S^1_{(i)} \ast (\sigma_{(i)}, w_{\sigma_{(i)}}) \times \prod_{i \in J} S^1_{(i)} \ast \hat{0}_{(i)} \times \prod_{i \in [k] \setminus \{\lambda(\sigma) \cup J\}} S^1_{(i)} \ast -\hat{1}_{(i)}, \)

where \( \sigma_{(i)} = \sigma \cap \Delta^\alpha \) and \( \hat{0}_{(i)}, -\hat{1}_{(i)} \in \Delta^\alpha \). In particular, when \( \sigma = 0 \in \mathcal{K} \), \( w_0 \) is null and \( 1_{\alpha}(\hat{0}) = \emptyset \). So we have:

\[
B(\hat{0}, w_{\hat{0}}, J) = \prod_{i \in J} S^0 \ast \hat{0}_{(i)} \times \prod_{i \in [k] \setminus J} S^0 \ast -\hat{1}_{(i)}, \quad \forall J \subset [k],
\]

\[
\hat{B}(\hat{0}, w_{\hat{0}}, J) = \prod_{i \in J} S^1 \ast \hat{0}_{(i)} \times \prod_{i \in [k] \setminus J} S^1 \ast -\hat{1}_{(i)}, \quad \forall J \subset [k].
\]

Let \( B(\sigma, w_{\sigma}, J) \) and \( \hat{B}(\sigma, w_{\sigma}, J) \) denote the relative interiors of \( B(\sigma, w_{\sigma}, J) \) and \( \hat{B}(\sigma, w_{\sigma}, J) \), respectively. It is easy to see that the set of all such cells \( \{B(\sigma, w_{\sigma}, J)\} \) and \( \{\hat{B}(\sigma, w_{\sigma}, J)\} \) give us cell decompositions of \( \mathcal{X}(\mathcal{K}, \alpha, S^0) \) and \( \mathcal{X}(\mathcal{K}, \alpha, S^1) \), respectively.

**Remark 5.2.** In fact \( B(\sigma, w_{\sigma}, J) \) is a union of \( 2^{|J|} \) open balls. But it does not hurt to treat \( \{B(\sigma, w_{\sigma}, J)\} \) as a cell decomposition of \( \mathcal{X}(\mathcal{K}, \alpha, S^0) \) for the calculation of the cohomology groups of \( \mathcal{X}(\mathcal{K}, \alpha, S^0) \) because, \( \{B(\sigma, w_{\sigma}, J)\} \) form a basis of the cellular chain complex of an authentic cell decomposition of \( \mathcal{X}(\mathcal{K}, \alpha, S^0) \). A similar construction is used in our discussion of \( \mathcal{X}(\mathcal{K}, \lambda_{\alpha}) \) in section 2.

Let \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^0); k) \) and \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); k) \) denote the cellular cochain complexes corresponding to the above cell decompositions. Let \( x^{(\sigma, w_{\sigma}, J)} \) and \( y^{(\sigma, w_{\sigma}, J)} \) denote the basis of \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^0); k) \) and \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); k) \) that are dual to \( B(\sigma, w_{\sigma}, J) \) and \( \hat{B}(\sigma, w_{\sigma}, J) \), respectively.

For any simplex \( \sigma \) in \( \mathcal{K} \), let \( w_{\sigma}^0 \) denote the weight on \( \sigma \) that assigns 0 to all vertices of \( \sigma \). Then under the inclusions of \( X(\mathcal{K}, \lambda_{\alpha}) \) into \( \mathcal{X}(\mathcal{K}, \alpha, S^0) \) and \( X(\mathcal{K}, \lambda_{\alpha}) \) into \( \mathcal{X}(\mathcal{K}, \alpha, S^1) \), the cell \( x^{\Phi_{\sigma}}_J \) is identified with \( x^{(\sigma, w_{\sigma}, J)} \) and \( y^{\Phi_{\sigma}}_J \) is identified with \( y^{(\sigma, w_{\sigma}, J)} \).

On the other hand, by the definition of \( k[\mathcal{K}] \) in \( \mathcal{K} \), every element of \( k[\mathcal{K}] \) can be uniquely written as a linear combination of monomials of the form \( v_{i_1}^{n_1} \cdots v_{i_s}^{n_s} \) where \( \{v_{i_1}, \ldots, v_{i_s}\} \) span a simplex in \( \mathcal{K} \) and \( n_1, \ldots, n_s > 0 \). The coefficients ring \( k \) is generated by the empty monomial which corresponds to the empty simplex \( \hat{0} \in \mathcal{K} \). So we obtain an additive basis of \( \Lambda_k[t_1, \ldots, t_k] \otimes k[\mathcal{K}] \) over \( k \):

\[
\{t_j v_{i_1}^{n_1} \cdots v_{i_s}^{n_s} \mid J \subset [k], \{v_{i_1}, \ldots, v_{i_s}\} \text{ spans a simplex in } \mathcal{K}, \ n_1, \ldots, n_s \in \mathbb{Z}_+\}.
\]

Suppose \( \{v_{i_1}, \ldots, v_{i_s}\} = V(\sigma) \) for some simplex \( \sigma \in \mathcal{K} \). Then corresponding to \( t_j v_{i_1}^{n_1} \cdots v_{i_s}^{n_s} \), we have the cochains \( x^{(\sigma, w_{\sigma}, J)} \) and \( y^{(\sigma, w_{\sigma}, J)} \) where the weight \( w_{\sigma} \) is:

\[
w_{\sigma}(v_{i_j}) = n_j - 1, \quad 1 \leq j \leq s \quad (\text{if } \sigma \neq \hat{0} \in \mathcal{K}).
\]
In particular, we have the correspondences:

\[ t_i \leftrightarrow x^{(0, w_0(i))} \text{ and } y^{(0, w_0(i))}, \quad \forall i \in [k] \]

\[ v_\sigma \leftrightarrow x^{(\sigma, w_\sigma(\emptyset))} \text{ and } y^{(\sigma, w_\sigma(\emptyset))}, \quad \forall \sigma \in \mathcal{K}. \]

Similarly to the \( x^{p_1}_\alpha \) and \( y^{p_2}_\alpha \) in section 2, we can write \( x^{(\sigma, w_\sigma(\emptyset))} \) and \( y^{(\sigma, w_\sigma(\emptyset))} \) as cross products of cellular cochains in \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^0); k) \) and \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); k) \), and write out their differentials. In particular, we have

\[
d(x^{(0, w_0(i))}) = \sum_{v \in \alpha_i} x^{(v, w_0(\emptyset))}, \quad d(y^{(0, w_0(i))}) = \sum_{v \in \alpha_i} y^{(v, w_0(\emptyset))}
\]

\[
d(x^{(\sigma, w_\sigma(\emptyset))}) = \sum_{\sigma \subset \omega, I_0(\omega) = I_0(\sigma)} \pm x^{(\omega, w_\omega(\emptyset))}, \quad d(y^{(\sigma, w_\sigma(\emptyset))}) = \sum_{\sigma \subset \omega, I_0(\omega) = I_0(\sigma)} \pm y^{(\omega, w_\omega(\emptyset))}.
\]

It is easy to check that when \( k = \mathbb{Z}_2 \), the above correspondences define cochain complex isomorphisms from \( \Lambda_{\mathbb{Z}_2}[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[\mathcal{K}] \) to \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^0); \mathbb{Z}_2) \) and \( C^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); \mathbb{Z}_2) \). The reason why we have to assume the \( \mathbb{Z}_2 \)-coefficients here is: for any \( J \subset [k] \),

\[
d_\alpha(t_J) = \sum_{i \in J} \kappa(i, J) \cdot t_{J \setminus \{i\}} \sum_{v \in \alpha_i} v_{\{j\}}.
\]

But the sign \( \kappa(i, J) \) is not compatible with the sign \( \varepsilon(\sigma, \omega) \) appearing in the boundary of a simplex in \( \mathcal{K} \) unless \( \alpha \) is the trivial partition (also see Remark 6.2).

Combining the above isomorphisms of cochain complexes with the cohomology isomorphisms in (16) and (17), we obtain the following additive isomorphisms.

\[ H^*(\Lambda_{\mathbb{Z}_2}[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[\mathcal{K}], d_\alpha) \cong H^*(\mathcal{X}(\mathcal{K}, \alpha, S^0); \mathbb{Z}_2) \cong H^*(X(\mathcal{K}, \lambda_\alpha); \mathbb{Z}_2) \]

\[ H^*(\Lambda_{\mathbb{Z}_2}[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[\mathcal{K}], d_\alpha) \cong H^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); \mathbb{Z}_2) \cong H^*(X(\mathcal{K}, \Lambda_\alpha); \mathbb{Z}_2) \]

These prove part of the claims in Theorem 1.5

6. COHOMOLOGY RING STRUCTURES OF \( X(\mathcal{K}, \lambda_\alpha) \) AND \( X(\mathcal{K}, \Lambda_\alpha) \)

It is shown in [5] that for any coefficients \( k \), the cohomology ring \( H^*(\mathcal{Z}_\mathcal{K}; k) \) of a moment-angle complex \( \mathcal{Z}_\mathcal{K} \) is isomorphic to the Tor algebra of the face ring \( k[\mathcal{K}] \) of \( \mathcal{K} \). In this section, we obtain parallel results for \( X(\mathcal{K}, \lambda_\alpha) \) under \( \mathbb{Z}_2 \)-coefficients. Similarly to the discussion of \( \mathcal{Z}_\mathcal{K} \) in [5] and [14], we first introduce some differential graded algebras as auxiliary means for our discussion.
6.1. A differential graded algebra $R^*_k(K, \alpha)$.

Suppose $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ is a partition of the vertex set $[m] = \{v_1, \cdots, v_m\}$ of a simplicial complex $K$. Using the definition (7) of $k[K]$, we define a differential (graded) commutative algebra $R^*_k(K, \alpha)$ over $k$ as follows.

$$R^*_k(K, \alpha) := \Lambda_k[t_1, \cdots, t_k] \otimes k[K]/J_{\alpha}$$

where $J_{\alpha}$ is the ideal generated by the following subset of $\Lambda_k[t_1, \cdots, t_k] \otimes k[K]$

$$\{t_i v_\sigma; V(\sigma) \subset \alpha_i, 1 \leq i \leq k \} \cup \{v_\sigma v_\tau; I_{\alpha}(\sigma) \cap I_{\alpha}(\tau) \neq \emptyset\}$$

We denote the quotient map in (48) by

$$\rho_{\alpha} : \Lambda_k[t_1, \cdots, t_k] \otimes k[K] \longrightarrow R^*_k(K, \alpha).$$

Let $\bar{t}_i$ and $\bar{v}_\sigma$ denote the image of $t_i$ and $v_\sigma$ in $R^*_k(K, \alpha)$ under $\rho_{\alpha}$, respectively. $R^*_k(K, \alpha)$ inherits the multi-gradings from $\Lambda_k[t_1, \cdots, t_k] \otimes k[K]$ which gives:

$$\text{mdeg}^\alpha(\bar{t}_i) = \text{mdeg}^\alpha(t_i) = (-1, \{i\}) \in (\mathbb{Z}, [k]),$$

$$\text{mdeg}^\alpha(\bar{v}_\sigma) = \text{mdeg}^\alpha(v_\sigma) = (0, I_{\alpha}(\sigma)) \in (\mathbb{Z}, [k]).$$

Moreover, $R^*_k(K, \alpha)$ inherits a differential $\bar{d}_\alpha$ from $d_{\alpha}$ on $\Lambda_k[t_1, \cdots, t_k] \otimes k[K]$.

$$\bar{d}_\alpha(\bar{t}_i) = \sum_{v_j \in \alpha_i} \bar{v}_j, \ 1 \leq i \leq n, \ \text{and} \ \bar{d}_\alpha(\bar{v}_\sigma) = \sum_{\sigma \subset \omega, I_{\alpha}(\omega) = I_{\alpha}(\sigma), \dim(\omega) = \dim(\sigma) + 1} \varepsilon(\sigma, \omega) \cdot \bar{v}_\omega.$$

It is clear that $\rho_{\alpha}$ is a chain map between $\Lambda_k[t_1, \cdots, t_k] \otimes k[K]$ and $R^*_k(K, \alpha)$.

For any subset $J = \{j_1, \cdots, j_s\} \subset [k]$ with $j_1 < \cdots < j_s$, let

$$\bar{t}_J := \bar{t}_{j_1} \cdots \bar{t}_{j_s} \in R^*_k(K, \alpha).$$

By [13] Lemma 5.4, any element of $k[K]$ can be uniquely written as a linear combination of monomials $v_{\sigma_1}^{t_1}v_{\sigma_2}^{t_2} \cdots v_{\sigma_s}^{t_s}$ corresponding to chains of totally ordered simplices $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_s \subset K \setminus \emptyset$. Then the definition of $J_{\alpha}$ implies that $R^*_k(K, \alpha)$ is generated, as an abelian group, by the set of monomials

$$\{\bar{t}_J \bar{v}_\sigma; J \cap I_{\alpha}(\sigma) = \emptyset, J \subset [k], \sigma \in K\}, \ \text{mdeg}^\alpha(\bar{t}_J \bar{v}_\sigma) = (-|J|, I_{\alpha}(\sigma) \cup J).$$

In particular, $R^*_k(K, \alpha)$ is a free abelian group of finite rank.
For any simplex \( \sigma \in \mathcal{K} \) and \( J \subset [k] \) with \( J \cap I_{\alpha}(\sigma) = \emptyset \), we obtain from (52)
\[
d_{\alpha}(\tilde{t}_J \tilde{v}_\sigma) = \sum_{i \in J} \kappa(i, J) \tilde{t}_{J\setminus\{i\}} \left( \sum_{v_j \in \alpha_i} \tilde{v}_{\sigma} \right) + (-1)^{|J|} \tilde{t}_J \left( \sum_{\sigma \subset \omega, I_{\alpha}(\omega) = I_{\alpha}(\sigma)} \sum_{\dim(\omega) = \dim(\sigma) + 1} \varepsilon(\sigma, \omega) \cdot \tilde{v}_\omega \right)
\]
\[
= \sum_{i \in J} \kappa(i, J) \tilde{t}_{J\setminus\{i\}} \left( \sum_{\sigma \subset \omega, I_{\alpha}(\omega) = I_{\alpha}(\sigma) \cup \{i\}} \tilde{v}_\omega \right) + (-1)^{|J|} \tilde{t}_J \left( \sum_{\sigma \subset \omega, I_{\alpha}(\omega) = I_{\alpha}(\sigma)} \varepsilon(\sigma, \omega) \cdot \tilde{v}_\omega \right). \tag{53}
\]
Note that \( d_{\alpha} \) preserve the \([k]\)-grading of \( R_k^*(\mathcal{K}, \alpha) \). So for any \( L \subset [k] \), let

\[ R_k^{*L}(\mathcal{K}, \alpha) := \text{the } k\text{-submodule of } R_k^*(\mathcal{K}, \alpha) \text{ generated by } \{ \tilde{t}_J \tilde{v}_\sigma \mid J \cap I_{\alpha}(\sigma) = \emptyset, J \cup I_{\alpha}(\sigma) = L \}. \]

Then \( R_k^{*L}(\mathcal{K}, \alpha) \) is a cochain subcomplex of \( R_k^*(\mathcal{K}, \alpha) \) and we have

\[
R_k^*(\mathcal{K}, \alpha) = \bigoplus_{L \subset [k]} R_k^{*L}(\mathcal{K}, \alpha), \quad H^*(R_k^*(\mathcal{K}, \alpha)) = \bigoplus_{L \subset [k]} H^*(R_k^{*L}(\mathcal{K}, \alpha)). \tag{54}
\]

**Lemma 6.1.** Let \( \alpha = \{\alpha_1, \ldots, \alpha_k\} \) be a partition of the vertex set \( V(\mathcal{K}) \) of a simplicial complex \( \mathcal{K} \). There are \( \mathbb{Z}_2 \)-module isomorphisms

\[
H^*(R_{\mathbb{Z}_2}^*(\mathcal{K}, \alpha)) \cong H^*(X(\mathcal{K}, \lambda_{\alpha}); \mathbb{Z}_2), \quad H^*(R_{\mathbb{Z}_2}^{*L}(\mathcal{K}, \alpha)) \cong H^*(X(\mathcal{K}, \Lambda_{\alpha}); \mathbb{Z}_2).
\]

**Proof.** By the decompositions in (54), it is sufficient for us to show that for any subset \( L \subset [k] \) there are cochain complex isomorphisms from \( R_{\mathbb{Z}_2}^{*L}(\mathcal{K}, \alpha) \) to \( C^{*L}(X(\mathcal{K}, \lambda_{\alpha}); \mathbb{Z}_2) \) and \( C^{*L}(X(\mathcal{K}, \Lambda_{\alpha}); \mathbb{Z}_2) \). We define additive isomorphisms

\[
\psi_{\alpha}^L : R_{\mathbb{Z}_2}^{*L}(\mathcal{K}, \alpha) \longrightarrow C^{*L}(X(\mathcal{K}, \lambda_{\alpha}); \mathbb{Z}_2), \quad \hat{\psi}_{\alpha}^L : R_{\mathbb{Z}_2}^{*L}(\mathcal{K}, \alpha) \longrightarrow C^{*L}(X(\mathcal{K}, \Lambda_{\alpha}); \mathbb{Z}_2)
\]

\[
\tilde{t}_J \tilde{v}_\sigma \longmapsto x_{\Phi_\sigma^L}, \quad \tilde{t}_J \tilde{v}_\sigma \longmapsto y_{\Phi_\sigma^L}.
\]

Note that

\[
\dim(x_{\Phi_\sigma^L}) = \text{rank}(\sigma), \quad \dim(y_{\Phi_\sigma^L}) = \text{rank}(\sigma) + |I_{\alpha}(\sigma)| + |J|.
\]

So if we want to require \( \psi_{\alpha}^L \) and \( \hat{\psi}_{\alpha}^L \) to be dimension-preserving, we should define the dimensions of elements in \( R_{\mathbb{Z}_2}^{*L}(\mathcal{K}, \alpha) \) correspondingly.

- For \( \psi_{\alpha}^L \), let \( \dim(\tilde{t}_J) = 0 = \dim(S^0) \) and \( \dim(\tilde{v}_\sigma) = \text{rank}(\sigma) \).
- For \( \hat{\psi}_{\alpha}^L \), let \( \dim(\tilde{t}_J) = 1 = \dim(S^1) \) and \( \dim(\tilde{v}_\sigma) = \text{rank}(\sigma) + |I_{\alpha}(\sigma)| \).

In addition, we define the multi-gradings \( m_{\alpha}(x_{\Phi_\sigma^L}) \) and \( m_{\alpha}(y_{\Phi_\sigma^L}) \) by:

\[
m_{\alpha}(x_{\Phi_\sigma^L}) = m_{\alpha}(y_{\Phi_\sigma^L}) = (-|J|, I_{\alpha}(\sigma) \cup J) \in (\mathbb{Z}, [k]).
\]
Then \( \psi^L_\alpha \) and \( \hat{\psi}^L_\alpha \) also preserve the multi-gradings of the elements in \( R^{**L}_a(K, \alpha) \), \( C^{**L}(X(K, \lambda_\alpha); \mathbb{Z}_2) \) and \( C^{**L}(X(K, \Lambda_\alpha); \mathbb{Z}_2) \).

Now we show that \( \psi^L_\alpha(\tilde{d}_\alpha(\bar{t}_j \bar{v}_\sigma)) = \tilde{d}_\alpha(\psi^L_\alpha(\bar{t}_j \bar{v}_\sigma)) \) for any \( \bar{t}_j \bar{v}_\sigma \in R^{**L}_a(K, \alpha) \). Because of the \( \mathbb{Z}_2 \)-coefficients, we can ignore all the signs in (53).

\[
\psi^L_\alpha(\tilde{d}_\alpha(\bar{t}_j \bar{v}_\sigma)) = \psi^L_\alpha \left( \sum_{i \in J} \bar{t}_{J \setminus \{i\} \setminus I_a(\sigma)} \sum_{\sigma \subset \omega, I_a(\omega) \subset I_a(\sigma)} \bar{v}_\omega \right) + \tilde{d}_\alpha \left( \sum_{\sigma \subset \omega, I_a(\omega) \subset I_a(\sigma)} \bar{v}_\omega \right)
\]

\[
= \psi^L_\alpha \left( \sum_{\sigma \subset \omega, I_a(\omega) \subset I_a(\sigma)} \bar{t}_{I_a(\sigma) \cup J \setminus I_a(\omega)} \bar{v}_\omega \right)
\]

\[
= \sum_{\sigma \subset \omega, I_a(\omega) \subset I_a(\sigma)} \bar{x}^{\Phi_\alpha(\sigma \cup J) \setminus I_a(\omega)} = \tilde{d}_\alpha(\psi^L_\alpha(\bar{t}_j \bar{v}_\sigma)).
\]

A parallel argument shows that \( \hat{\psi}^L_\alpha(\tilde{d}_\alpha(\bar{t}_j \bar{v}_\sigma)) = \tilde{d}_\alpha(\hat{\psi}^L_\alpha(\bar{t}_j \bar{v}_\sigma)) \). So \( \psi^L_\alpha \) and \( \hat{\psi}^L_\alpha \) are cochain complex isomorphisms. Now let

\[
\psi_\alpha = \bigoplus_{L \subset \{k\}} \psi^L_\alpha : R^{**L}_a(K, \alpha) \to C^{**}(X(K, \lambda_\alpha); \mathbb{Z}_2), \quad (55)
\]

\[
\hat{\psi}_\alpha = \bigoplus_{L \subset \{k\}} \hat{\psi}^L_\alpha : R^{**L}_a(K, \alpha) \to C^{**}(X(K, \Lambda_\alpha); \mathbb{Z}_2). \quad (56)
\]

By our construction, \( \psi_\alpha \) and \( \hat{\psi}_\alpha \) are cochain complex isomorphisms which preserve the multi-gradings of the elements in \( R^{**L}_a(K, \alpha) \), \( C^{**}(X(K, \lambda_\alpha); \mathbb{Z}_2) \) and \( C^{**}(X(K, \Lambda_\alpha); \mathbb{Z}_2) \). The theorem is proved. \( \Box \)

**Remark 6.2.** When \( \alpha \) is a nontrivial partition of \( V(K) \) and \( k \neq \mathbb{Z}_2 \), it seems that there is no proper way to define a chain isomorphism between \( R^{**L}_k(K, \alpha) \) and \( C^{**L}(X(K, \lambda_\alpha); k) \) or \( C^{**L}(X(K, \Lambda_\alpha); k) \). This is because when \( \alpha \) is nontrivial, the signs \( \kappa(i, J) \) and \( \varepsilon(\sigma, \omega) \) that appear in the expressions of \( \tilde{d}_\alpha(\bar{t}_j \bar{v}_\sigma), d(\bar{x}^{\Phi_\alpha}) \) and \( d(\bar{y}^{\Phi_\alpha}) \) are not compatible.

### 6.2. Cellular cup products of cochains in \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \).

To understand the cohomology ring structures of \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \), we need to construct explicit cup products of cellular cochains on these spaces. The usual way to do this is to choose cellular approximations of the diagonal maps of these spaces. Indeed, this is what people do in [5] for moment-angle complexes and in [3] for generalized moment-angle complexes. In the following we first construct cellular cup products for some simple spaces, and then write the cellular cup products on \( X(K, \lambda_\alpha) \) and \( X(K, \Lambda_\alpha) \).
For \( S^0 = \{a^0, a^1\} \), let \((a^0)^*\) and \((a^1)^*\) be the basis of \( C^*(S^0; k) \). Then the cellular cup product is given by:
\[
(a^0)^* \cup (a^0)^* = (a^0)^*, \quad (a^1)^* \cup (a^1)^* = (a^1)^*, \quad (a^0)^* \cup (a^1)^* = 0.
\] (57)

Observe that there are no elements in \( C^*(S^0; k) \) with order two.

- For any \( d \geq 1 \), let \( S^d = e^0 \cup e^d \) be a cell decomposition where \( e^d \) is a \( d \)-cell. Let \((e^0)^*, (e^d)^* \in C^*(S^d; k)\) be the dual basis of \( e^0, e^d \). The cellular cup product on \( C^*(S^d; k) \) is given by
\[
(e^0)^* \cup (e^d)^* = (e^d)^*, \quad (e^d)^* \cup (e^d)^* = 0.
\] (58)

- For any \( r \geq 0 \), let \( V(\Delta^r) = \{v_0, \ldots, v_r\} \). We decompose \( \Delta^r \) as its open faces
\[
\Delta^r = \bigcup_{\sigma \in \Delta^r} \sigma^\circ.
\]
Consider \( \{\sigma^\circ \mid \sigma \in \Delta^r\} \) as a basis of the cellular chain complex of \( \Delta^r \) and let \( \sigma^* \in C^*(\Delta^r; k) \) denote the dual basis of \( \sigma^\circ \). Then by the Alexander-Whitney diagonal for \( \Delta^r \) (see [15]), we obtain a cellular cup product on \( \Delta^r \) by
\[
\sigma^* \cup \tau^* = \begin{cases} 
(\sigma \cup \tau)^*, & \sigma \cap \tau = \emptyset \\
0, & \text{else}.
\end{cases}
\]

The above cellular cup products on \( S^d \) and \( \Delta^r \) uniquely determine a cellular cup product on \( S^d \times \Delta^r \times \Delta^1 \) (with the product cell structure) by the rule:
\[
(f_1 \times g_1) \cup (f_2 \times g_2) = (-1)^{|g_1||f_2|} (f_1 \cup f_2) \times (g_1 \cup g_2).
\]
Note that \( S^d \ast \Delta^r \) is a cellular quotient space of \( S^d \times \Delta^r \times \Delta^1 \) by collapsing \( S^d \times \Delta^r \times \{0\} \) to \( S^d \) and \( S^d \times \Delta^r \times \{1\} \) to \( \Delta^r \). Then \( S^d \ast \Delta^r \) inherits the cellular cup product from \( S^d \times \Delta^r \times \Delta^1 \), which can be described explicitly as follows.

Let \( p : S^d \times \Delta^r \times \Delta^1 \to S^d \ast \Delta^r \) be the quotient map. By our argument in Section 3, \( \{S^d \ast \sigma^\circ \mid \sigma \in \hat{\Delta}^r\} \) is a basis for the cellular chain complex of \( S^d \ast \Delta^r \).

- When \( d = 0 \), \( S^0 \ast \hat{0} = a^1 - a^0 \) and \( S^0 \ast \hat{1} = a^0 \), where \( S^0 = \{a^0, a^1\} \);
- When \( d \geq 1 \), \( S^d \ast \hat{0} = e^d \) and \( S^d \ast \hat{1} = e^0 \).

Note that \( p^{-1}(S^d \ast \sigma^\circ) = S^d \times \sigma^\circ \times [0, 1] \) for any nonempty face \( \sigma \) of \( \Delta^r \).

Let \( \{z^\sigma \mid \sigma \in \hat{\Delta}^r\} \) be the basis of \( C^*(S^d \ast \Delta^r; k) \) where \( z^\sigma \) is the dual of \( S^d \ast \sigma^\circ \).

By the notations in Section 3, \( z^\sigma = x^\sigma \) or \( y^\sigma \) when \( d = 0 \) or \( 1 \). The differential of \( z^\sigma \) is parallel to (33) and (34).

By (58), the cellular cup product on \( S^d \ast \Delta^r \) inherited from \( S^d \times \Delta^r \times \Delta^1 \) reads:

- when \( d = 0 \), \( z^0 \hat{0} = (a^1)^* \) and \( z^{-1} \hat{0} = (a^0)^* + (a^1)^* \) (see Remark 3.3). So
\[
z^0 \hat{0} \cup z^0 \hat{0} = z^0 \hat{0}, \quad z^{-1} \cup z^{-1} = z^{-1} \hat{0}, \quad z^{-1} \cup z^{-1} = z^{-1} \hat{0} \quad \text{(by (57))}.
\]
For any nonempty faces $\sigma, \tau$ of $\Delta^r$, we have $z^\sigma \cup z^\tau = \begin{cases} z^{\sigma \wedge \tau}, & \sigma \cap \tau = \emptyset; \\ 0, & \text{else.} \end{cases}$

- when $d \geq 1$, $z^\sigma \cup z^\tau = 0$ for any $\sigma, \tau \in \Delta^r \setminus \hat{1}$, and $z^{-1} \cup z^\sigma = z^\sigma$ for any $\sigma \in \Delta^r$.

- Given any partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $V(\mathcal{K}) = [m]$, we obtain a cellular cup product on the Cartesian product $\prod_{i \in [k]} S^d \ast \Delta^{\alpha_i}$ by:

$$\left(z_1^{\alpha_1} \times \cdots \times z_k^{\alpha_k}\right) \cup \left(z_1^{\tau_1} \times \cdots \times z_k^{\tau_k}\right) = \left(z_1^{\sigma_1} \cup z_1^{\tau_1}\right) \times \cdots \times \left(z_k^{\sigma_k} \cup z_k^{\tau_k}\right).$$

Then $C^\ast(\prod_{i \in [k]} S^d \ast \Delta^{\alpha_i}; k)$ is isomorphic to $\bigotimes_{i \in [k]} C^\ast(S^d \ast \Delta^{\alpha_i}; k)$ as graded algebras with respect to the cup products. Moreover, by the inclusions (see (20))

$$X(\mathcal{K}, \lambda_\alpha) \hookrightarrow \prod_{i \in [k]} S^0 \ast \Delta^{\alpha_i}, \quad X(\mathcal{K}, \Lambda_\alpha) \hookrightarrow \prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i},$$

$X(\mathcal{K}, \lambda_\alpha)$ and $X(\mathcal{K}, \Lambda_\alpha)$ inherit the cellular cup products from $\prod_{i \in [k]} S^0 \ast \Delta^{\alpha_i}$ and $\prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i}$, respectively.

Now we are ready to prove the following lemma which is the key to the proof of Theorem 1.5.

**Lemma 6.3.** Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a partition of the vertex set $V(\mathcal{K})$ of a simplicial complex $\mathcal{K}$. There is a ring isomorphism

$$H^\ast(R_{\mathbb{Z}_2}^\ast(\mathcal{K}, \alpha); \mathbb{Z}_2) \cong H^\ast(X(\mathcal{K}, \Lambda_\alpha); \mathbb{Z}_2).$$

**Proof.** It is sufficient to show that $\hat{\psi}_\alpha : R_{\mathbb{Z}_2}^\ast(\mathcal{K}, \alpha) \longrightarrow C^\ast(X(\mathcal{K}, \Lambda_\alpha); \mathbb{Z}_2)$ defined in (56) is a ring isomorphism with respect to the product structure of $R_{\mathbb{Z}_2}^\ast(\mathcal{K}, \alpha)$ and the cup products of cellular cochains in $X(\mathcal{K}, \Lambda_\alpha)$. In the following we proceed our proof along the same line as [13] Lemma 4.6.

For any coefficients $k$, notice that $R^\ast_k(\Delta^{\lceil m \rceil}, \alpha)$ is isomorphic to the tensor product $\bigotimes_{i \in [k]} R_k^\ast(\Delta^{\alpha_i})$ as differential algebras where

$$R_k^\ast(\Delta^{\alpha_i}) := \Lambda_k[t_i] \otimes k[\Delta^{\alpha_i}]/(t_i v_\sigma = v_\sigma v_\tau = 0, \forall \sigma, \tau \in \Delta^{\alpha_i}).$$

The quotient images of $t_i$ and $v_\sigma$ are denoted by $\bar{t}_i$ and $\bar{v}_\sigma$, respectively. The differential in $R_k^\ast(\Delta^{\alpha_i})$ is defined by (52). It is easy to see that $R_k^\ast(\Delta^{\alpha_i})$ is spanned as a $k$-module by

$$\{1, \bar{t}_i, \bar{v}_\sigma; \sigma \text{ is any nonempty face of } \Delta^{\alpha_i}\}.$$

For any $d \geq 1$, we define a $k$-linear map $\psi^d_{\Delta_i} : R_k^\ast(\Delta^{\alpha_i}) \rightarrow C^\ast(S^d \ast \Delta^{\alpha_i}; k)$ by:

- $\psi^d_{\alpha_i}(1) = z^{-1}$ and $\psi^d_{\alpha_i}(\bar{t}_i) = z^0$;
- $\psi^d_{\alpha_i}(\bar{v}_\sigma) = z^\sigma$ for any nonempty face $\sigma$ of $\Delta^{\alpha_i}$.
It is easy to check that \( \psi_{\alpha_i}^{S^d} \) is a ring isomorphism with respect to the cellular cup product in \( C^*(S^d \ast \Delta^{\alpha_i}; k) \) described above. Moreover, \( \psi_{\alpha_i}^{S^d} \) is a chain map with respect to the differentials in \( R^*_k(\Delta^{\alpha_i}) \) and \( C^*(S^d \ast \Delta^{\alpha_i}; k) \).

Taking the tensor product of \( \psi_{\alpha_i}^{S^d}, 1 \leq i \leq k \), we obtain a ring isomorphism

\[
\Psi^{S^d}_\alpha : R^*_k(\Delta^{[m]}, \alpha) \otimes_{\alpha \in [k]} \psi_{\alpha_i}^{S^d} \to \bigotimes_{i \in [k]} C^*(S^d \ast \Delta^{\alpha_i}; k) \xrightarrow{\sim} C^*\left( \prod_{i \in [k]} S^d \ast \Delta^{\alpha_i}; k \right).
\]

In addition, the inclusion \( K \hookrightarrow \Delta^{[m]} \) induces a ring epimorphism

\[
\eta_\alpha : R^*_k(\Delta^{[m]}, \alpha) \to R^*_k(K, \alpha).
\]

Let \( \xi_\alpha : C^*(\prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i}; k) \to C^*(X(K, \Lambda_\alpha); k) \) be the ring homomorphism induced by the inclusion \( X(K, \Lambda_\alpha) \hookrightarrow \prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i} \). Then from the definitions of \( \widetilde{\psi}_\alpha \) and \( \Phi^{S^1}_\alpha \), we have the following commutative diagram for \( k = \mathbb{Z}_2 \).

\[
\begin{array}{ccc}
R^*_k(\Delta^{[m]}, \alpha) & \xrightarrow{\Psi^{S^1}_\alpha} & C^*(\prod_{i \in [k]} S^1 \ast \Delta^{\alpha_i}; \mathbb{Z}_2) \\
\eta_\alpha \downarrow & & \downarrow \xi_\alpha \\
R^*_k(K, \alpha) & \xrightarrow{\widetilde{\psi}_\alpha} & C^*(X(K, \Lambda_\alpha); \mathbb{Z}_2)
\end{array}
\]

Next we show that \( \widetilde{\psi}_\alpha \) is a ring isomorphism. For any \( \gamma_1, \gamma_2 \in R^*_k(K, \alpha) \), there exist \( \gamma'_1, \gamma'_2 \in R^*_Z(\Delta^{[m]}, \alpha) \) with \( \eta_\alpha(\gamma'_1) = \gamma_1 \) and \( \eta_\alpha(\gamma'_2) = \gamma_2 \) since \( \eta_\alpha \) is surjective. Then since \( \eta_\alpha, \Psi^{S^1}_\alpha \) and \( \xi_\alpha \) are all ring homomorphisms, we obtain

\[
\begin{align*}
\widetilde{\psi}_\alpha(\gamma_1 \gamma_2) &= \widetilde{\psi}_\alpha(\eta_\alpha(\gamma'_1) \eta_\alpha(\gamma'_2)) = \widetilde{\psi}_\alpha(\eta_\alpha(\gamma'_1 \gamma'_2)) \\
&= \xi_\alpha(\Psi^{S^1}_\alpha(\gamma'_1 \gamma'_2)) = \xi_\alpha(\Psi^{S^1}_\alpha(\gamma'_1) \cdot \Psi^{S^1}_\alpha(\gamma'_2)) \\
&= \xi_\alpha(\Psi^{S^1}_\alpha(\gamma'_1)) \cdot \xi_\alpha(\Psi^{S^1}_\alpha(\gamma'_2)) \\
&= \widetilde{\psi}_\alpha(\eta_\alpha(\gamma'_1)) \cdot \widetilde{\psi}_\alpha(\eta_\alpha(\gamma'_2)) = \widetilde{\psi}_\alpha(\gamma_1) \cdot \widetilde{\psi}_\alpha(\gamma_2)
\end{align*}
\]

So \( \widetilde{\psi}_\alpha \) is a ring isomorphism. The lemma is proved. \qed

**Remark 6.4.** The map \( \psi_{\alpha_i}^{S^d} \) is well defined only when \( d \geq 1 \) because when \( d = 0 \), the element \( z^0 \in C^*(S^0 \ast \Delta^{\alpha_i}; k) \) is not of order two. This is why we do not obtain an algebra isomorphism from \( H^*(R^*_k(K, \alpha); \mathbb{Z}_2) \) to \( H^*(X(K, \Lambda_\alpha); \mathbb{Z}_2) \) (also see the remark after [5, Theorem 8.17] and [6, Remark 9]). In addition, it is constructed in [8, Theorem 1.4] a non-commutative differential graded algebra which describes the cellular cochain algebra of a real moment-angle complex \( \mathbb{R}Z_K \).
Remark 6.5. By Lemma 6.1, $C^*(X(K, \Lambda_\alpha); \mathbb{Z}_2)$ is a graded commutative ring with respect to the cellular cup products described above. This is a very special property of $X(K, \Lambda_\alpha)$ since, a general CW-complex may not admit a (graded) commutative cellular cup product of cochains.

Combining the isomorphisms in Lemma 6.3 and Theorem 1.1, we obtain an additive isomorphism

$$f = \hat{\psi}^{-1} \circ \hat{\varphi}_\alpha : \bigoplus_{q \geq 0, L \subseteq [k]} \tilde{H}^{q-1}(K_{\alpha, L}; \mathbb{Z}_2) \longrightarrow \bigoplus_{q \geq 0, L \subseteq [k]} H^{q+|L|}(R_{\alpha, L}^*(K, \alpha)).$$

Since $R_{k}^*(K, \alpha)$ is an algebra, $f$ induces an algebra structure on the left. Consider the products of simplicial cochains of the full subcomplexes $K_{\alpha, L}$ given by:

$$C^{p-1}(K_{\alpha, L}; \mathbb{Z}_2) \otimes C^{q-1}(K_{\alpha, L'}; \mathbb{Z}_2) \longrightarrow C^{p+q-1}(K_{\alpha, L \cup L'}; \mathbb{Z}_2)$$

$$\sigma^* \otimes \tau^* \longmapsto \begin{cases} (\sigma \vee \tau)^*, & \text{if } L \cap L' = \emptyset; \\ 0, & \text{otherwise}. \end{cases} \quad (59)$$

Proposition 6.6. The product in the direct sum $\bigoplus_{q \geq 0, L \subseteq [k]} \tilde{H}^{q-1}(K_{\alpha, L}; \mathbb{Z}_2)$ induced from the isomorphisms in Lemma 6.1 and Theorem 1.1 coincides with the product defined in (59).

Proof. The argument is completely parallel to [6, Proposition 3.2.10]. By the definitions of $\hat{\varphi}_\alpha$ and $\hat{\psi}_\alpha$, for any simplex $\sigma \in K_{\alpha, L}$,

$$f(\sigma^*) = \tilde{t}_{L \setminus I_\alpha(\sigma)} \mathbf{v}_\sigma.$$

Then for any simplices $\sigma \in K_{\alpha, L}$ and $\tau \in K_{\alpha, L'}$, the induced product is given by:

$$\sigma^* \bullet \tau^* := f^{-1}(f(\sigma^*) f(\tau^*)) = f^{-1}(\tilde{t}_{L \setminus I_\alpha(\sigma)} \mathbf{v}_{\sigma} \tilde{t}_{L' \setminus I_\alpha(\tau)} \mathbf{v}_{\tau}).$$

If $L \cap L' \neq \emptyset$, then $\tilde{t}_{L \setminus I_\alpha(\sigma)} \mathbf{v}_{\sigma} \tilde{t}_{L' \setminus I_\alpha(\tau)} \mathbf{v}_{\tau}$ is zero in $R_{\alpha, L \cup L'}^*(K, \alpha)$. Otherwise, we have

$$\tilde{t}_{L \setminus I_\alpha(\sigma)} \mathbf{v}_{\sigma} \tilde{t}_{L' \setminus I_\alpha(\tau)} \mathbf{v}_{\tau} = \tilde{t}_{(L \cup L') \setminus (I_\alpha(\sigma) \cup I_\alpha(\tau))} \mathbf{v}_{\sigma \vee \tau} = \tilde{t}_{(L \cup L') \setminus I_\alpha(\sigma \vee \tau)} \mathbf{v}_{\sigma \vee \tau}.$$

So when $L \cap L' = \emptyset$, the product $\sigma^* \bullet \tau^* = (\sigma \vee \tau)^*$. The claim is proved. \hfill \Box

Lemma 6.7. For any partition $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of the vertex set of a simplicial complex $K$, the homomorphism $\rho_\alpha : \Lambda_{\mathbb{Z}_2}[t_1, \ldots, t_k] \otimes \mathbb{Z}_2[K] \rightarrow R_{\alpha, L}^*(K, \alpha)$ induces a ring isomorphism of cohomology.
Proof. Our argument is parallel to the proof of [12] Theorem 3.6. From our discussion in section 4, we obtain the following commutative diagram

\[
\begin{array}{ccc}
\Lambda_{Z_2}[t_1, \cdots, t_k] \otimes Z_2[\mathcal{K}] & \longrightarrow & C^*(\mathcal{X}(\mathcal{K}, \alpha, S^1); Z_2) \\
\rho_\alpha \downarrow & & \downarrow \psi_\alpha \\
R^*_Z(\mathcal{K}, \alpha) & \longrightarrow & C^*(X(\mathcal{K}, \Lambda_\alpha); Z_2)
\end{array}
\]

where the upper horizontal arrow is an isomorphism of cochain complexes, and the right vertical arrow induces an isomorphism of cohomology rings. By Lemma 6.1, \(\psi_\alpha\) induces isomorphisms of cohomology groups. So \(\rho_\alpha\) induces an isomorphism of cohomology groups. Then since \(\rho_\alpha\) is a ring homomorphism, it induces a cohomology ring isomorphism. \(\square\)

Remark 6.8. It is very likely that Lemma 6.7 holds for arbitrary coefficients \(k\) as well. Specifically, it might be possible to show that \(\rho_\alpha\) is a cochain equivalence using the strategy in the proof of [14] Lemma 4.4. But this approach may involve very cumbersome formulae. For the trivial partition \(\alpha^*\) of \(V(\mathcal{K})\), the proof is given in [14] Lemma 4.4 and [12] Theorem 3.6.

Proof of Theorem 1.5

By Lemma 6.7, we obtain a multi-grading on \(H^*(\Lambda_{Z_2}[t_1, \cdots, t_k] \otimes Z_2[\mathcal{K}], d_\alpha)\) as follows. For any subset \(L \subset [k]\), define

\[
H^{*, L}(\Lambda_{Z_2}[t_1, \cdots, t_k] \otimes Z_2[\mathcal{K}], d_\alpha) = \rho_\alpha^{-1}(H^*(R^*_k(\mathcal{K}, \alpha))).
\]

So by Lemma 6.1, there is an isomorphism from \(H^{*, L}(\Lambda_{Z_2}[t_1, \cdots, t_k] \otimes Z_2[\mathcal{K}], d_\alpha)\) to \(H^{*, L}(X(\mathcal{K}, \lambda_\alpha); Z_2)\) and \(H^{*, L}(X(\mathcal{K}, \Lambda_\alpha); Z_2)\) (as \(Z_2\)-modules).

Moreover, Lemma 6.3 and Lemma 6.7 imply that \(H^*(\Lambda_{Z_2}[t_1, \cdots, t_k] \otimes Z_2[\mathcal{K}], d_\alpha)\) is isomorphic (via \(\hat{\psi}_\alpha \circ \rho_\alpha\)) to \(H^*(X(\mathcal{K}, \Lambda_\alpha); Z_2)\) as multigraded \(Z_2\)-algebras. This finishes the proof of Theorem 1.5. \(\square\)

Suppose \(\alpha\) is a non-degenerate partition of \(V(\mathcal{K})\). In Section 1, we have shown that \(\text{Tor}_{\mathcal{Z}_2[u_1, \cdots, u_k]}[Z_2[\mathcal{K}], Z_2] \cong H^*(\Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}], d_\alpha)\) in this case. Let \(\text{Tor}_{\mathcal{Z}_2[u_1, \cdots, u_k]}^*[k[\mathcal{K}], k]\) denote \(H^*(\Lambda_k[t_1, \cdots, t_k] \otimes k[\mathcal{K}], d_\alpha)\). Then by Lemma 6.1 and Theorem 1.1, we obtain the following group isomorphisms.

\[
\text{Tor}_{\mathcal{Z}_2[u_1, \cdots, u_k]}^*[Z_2[\mathcal{K}], Z_2] \cong H^*(R^*_Z(\mathcal{K}, \alpha)) \cong H^*(X(\mathcal{K}, \Lambda_\alpha); Z_2) \cong \hat{H}^*(\mathcal{K}_{\alpha, \Lambda}; Z_2).
\]

6.3. Connection to Moment-angle Complexes of Simplicial Posets.

A poset (partially ordered set) \(S\) with the order relation \(\leq\) is called simplicial if it has an initial element \(\hat{0}\) and for each \(\sigma \in S\) the lower segment

\[
[\hat{0}, \sigma] = \{\tau \in S : \hat{0} \leq \tau \leq \sigma\}
\]
is the face poset of a simplex. We refer to $\sigma \in S$ as a simplex of dimension $l$ if $[0, \sigma]$ is the face poset of an $l$-simplex.

For each $\sigma \in S$ we assign a geometric simplex $\Delta^\sigma$ whose face poset is $[0, \sigma]$, and glue these geometric simplices together according to the order relation in $S$. We get a cell complex $\Delta^S$ in which the closure of each cell is identified with a simplex preserving the face structure, and all attaching maps are inclusions. We call $\Delta^S$ the geometric realization of $S$. For convenience, we still use $\Delta^\sigma$ to denote the image of each geometric simplex $\Delta^\sigma$ in $\Delta^S$. Then $\Delta^\sigma$ is a maximal simplex of $\Delta^S$ if and only if $\sigma$ is a maximal element of $S$.

The notion of moment-angle complex $Z_S$ associated to a simplicial poset $S$ is introduced and thoroughly studied in [12]. Note that the barycentric subdivision also makes sense for $\Delta^S$. Let $P_S$ denote the cone of the barycentric subdivision of $\Delta^S$. Let the vertex set of $P_S$ be $V(\Delta^S) = \{v_1, \ldots, v_m\}$. Let $\Lambda : V(\Delta^S) \to \mathbb{Z}^m$ be a map so that $\{\Lambda(v_i), 1 \leq i \leq m\}$ is a basis of $\mathbb{Z}^m$. Then we can construct $Z_S$ from $P_S$ and $\Lambda$ via the same rule in (4). So we also denote $Z_S$ by $X(S, \Lambda)$.

**Proposition 6.9.** For any finite simplicial poset $S$, there always exists a finite simplicial complex $K$ and a partition $\alpha$ of $V(K)$ so that $Z_S$ is homotopy equivalent to $X(K, \Lambda_S)$.

**Proof.** We can construct $K$ and $\alpha$ in the following way. Let $p : \Delta^S \times \mathbb{R} \to \Delta^S$ be the projection where $\Delta^S$ is identified with $\Delta^S \times \{0\}$. Then for each maximal simplex $\Delta^\sigma \subset \Delta^S$, we can choose a simplex $\tilde{\Delta}^\sigma \subset \Delta^S \times \{k\}$ for some $0 \leq k \leq n$ where $n$ is a large enough integer, so that

- $p$ maps $\tilde{\Delta}^\sigma$ simplicially isomorphically onto $\Delta^\sigma$.
- The vertices of $\tilde{\Delta}^\sigma$ are in $V(\Delta^S) \times \{0, \ldots, n\}$.
- $\tilde{\Delta}^\sigma \cap \tilde{\Delta}^\tau = \emptyset$ for any maximal elements $\sigma$ and $\tau$ in $S$.

We call $\tilde{\Delta}^\sigma$ a horizontal lifting of $\Delta^\sigma$. We consider the interval $[0, n] \subset \mathbb{R}$ as a 1-dimensional simplicial complex whose vertices are $\{0, \ldots, n\}$, and consider $\Delta^\sigma \times [0, n]$ as the the Cartesian product of $\Delta^\sigma$ and $[0, n]$ as simplicial complexes (see [3] Construction 2.11]). If $\sigma$ and $\tau$ are both maximal, $\Delta^\sigma \cap \Delta^\tau$ is the geometric realization of $\sigma \wedge \tau$. Now define

$$K = \Big( \bigcup_{\sigma \in S \text{ maximal}} \tilde{\Delta}^\sigma \Big) \cup \Big( \bigcup_{\sigma, \tau \in S \text{ maximal}} (\Delta^\sigma \cap \Delta^\tau) \times [0, n] \Big).$$

Then $K$ is clearly a simplicial complex whose vertex set is $V(S) \times \{0, \ldots, n\}$. We call $K$ a stretch of $S$ (see Figure 4 for example). Let $V(\Delta^S) = \{v_1, \ldots, v_m\}$. We define a partition $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ of $V(K)$ by

$$\alpha_i = \{v_j \times \{j\}; 0 \leq j \leq n\}, 1 \leq i \leq m.$$
Observe that the deformation retraction of \((\Delta^\sigma \cap \Delta^\tau) \times [0, n]\) to \((\Delta^\sigma \cap \Delta^\tau) \times \{0\}\) for any maximal simplices \(\sigma, \tau \in S\) gives us a homotopy equivalence between \(K\) and \(\Delta^S\). This induces a homotopy equivalence between \(P_K\) and \(P_S\), which further induces a homotopy equivalence between \(X(K, \Lambda_\alpha)\) and \(X(S, \Lambda) = Z_S\) via the construction \([4]\). □

The notion of the face ring \(k[S]\) of a simplicial poset \(S\) is introduced by Stanley in \([16]\). It is shown in \([12]\) that the integral cohomology ring of \(Z_S\) is isomorphic to \(\text{Tor}_{Z[v_1, \ldots , v_n]}(Z[S], \mathbb{Z})\) as multi-graded algebras. Then Proposition \([6, 9]\) implies that there is a multi-graded algebra isomorphism from \(\text{Tor}_{Z[v_1, \ldots , v_n]}(Z_2[S], \mathbb{Z}_2)\) to \(H^*(\Lambda Z_2[t_1, \ldots , t_m] \otimes \mathbb{Z}_2[K], d_\alpha)\) for some simplicial complex \(K\) and a partition \(\alpha = \{\alpha_1, \ldots , \alpha_m\}\) of \(V(K)\).

7. SOME GENERALIZATIONS

We can generalize our discussion on \(X(K, \lambda_\alpha)\) and \(X(K, \Lambda_\alpha)\) to a wider range of spaces defined below.

(I) We can replace the simplicial complex \(K\) by a simplicial poset \(S\) in the constructions \([2]\) and \([4]\). Then from a geometric realization of \(S\), we can define the simple polyhedral complex \(P_S\) and, construct spaces \(X(S, \lambda_\alpha)\) and \(X(S, \Lambda_\alpha)\) from any partition \(\alpha\) of the vertex set of \(S\). The parallel statements of Theorem \([1.1]\), Theorem \([1.2]\), Theorem \([1.3]\) and Theorem \([1.5]\) (by replacing \(K\) by \(S\)) also hold for these more general spaces. Their proofs are similar to the case of simplicial complexes. In particular, we can consider the simplicial poset \(S\) from a categorical viewpoint and take “colimit” of all the constructions in our proofs (see \([12]\) for the details of this kind of argument).
For a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $V(\mathcal{K})$, we can replace the $S^0$ in (22) and the $S^1$ in (23) by a sequence of spheres $\mathcal{S} = (S^{d_1}, \cdots, S^{d_k})$ and define

$$X(\mathcal{K}, \alpha, \mathcal{S}) = \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_\alpha(\sigma)} S^{d_i} \ast (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i \in [k] \setminus I_\alpha(\sigma)} S^{d_i} \right) \subset \prod_{i \in [k]} S^{d_i} \ast \Delta^{\alpha_i}.$$ 

We obtain the following results for $X(\mathcal{K}, \alpha, \mathcal{S})$ which are parallel to Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.5

**Theorem 7.1.** For any coefficients $k$, there is a $k$-module isomorphism:

$$H^q(X(\mathcal{K}, \alpha, \mathcal{S}); k) \cong \bigoplus_{\mathcal{L} \subset [k]} \tilde{H}^{q-1-\sum_{i \in \mathcal{L}} d_i}(K_{\alpha, \mathcal{L}}; k), \ \forall q \geq 0.$$ 

**Theorem 7.2.** There is a homotopy equivalence

$$\Sigma(X(\mathcal{K}, \alpha, \mathcal{S})) \cong \bigvee_{\mathcal{L} \subset [k]} \Sigma^{\sum_{i \in \mathcal{L}} d_i + 2}(K_{\alpha, \mathcal{L}}).$$

**Theorem 7.3.** Let $\alpha$ be a non-degenerate partition of $V(\mathcal{K})$.

(i) For any family $\mathcal{S} = (S^{d_1}, \cdots, S^{d_k})$, $H^*(X(\mathcal{K}, \alpha, \mathcal{S}); \mathbb{Z}_2)$ is isomorphic to $\text{Tor}_{\mathbb{Z}_2[u_1, \cdots, u_k]}(\mathbb{Z}_2[T], \mathbb{Z}_2)$ as multigraded $\mathbb{Z}_2$-modules.

(ii) For $\mathcal{S} = (S^{d_1}, \cdots, S^{d_k})$ with $d_i \geq 1$, $i = 1, \cdots, k$, $H^*(X(\mathcal{K}, \alpha, \mathcal{S}); \mathbb{Z}_2)$ is isomorphic to $\text{Tor}_{\mathbb{Z}_2[u_1, \cdots, u_k]}(\mathbb{Z}_2[T], \mathbb{Z}_2)$ as multigraded $\mathbb{Z}_2$-algebras.

**Theorem 7.4.** Let $\alpha$ be an arbitrary partition of $V(\mathcal{K})$.

(i) For any family $\mathcal{S} = \{S^{d_1}, \cdots, S^{d_k}\}$, $H^*(\Lambda_{\mathbb{Z}_2}[t_1, \cdots, t_k] \otimes \mathbb{Z}_2[T], \mathbb{Z}_2)$ is isomorphic to $H^*(\Lambda_{\mathbb{Z}_2}[t_1, \cdots, t_k] \otimes \mathbb{Z}_2[T], \mathbb{Z}_2)$ as multigraded $\mathbb{Z}_2$-modules.

(ii) For $\mathcal{S} = (S^{d_1}, \cdots, S^{d_k})$ with $d_i \geq 1$, $i = 1, \cdots, k$, $H^*(X(\mathcal{K}, \alpha, \mathcal{S}); \mathbb{Z}_2)$ is isomorphic to $H^*(\Lambda_{\mathbb{Z}_2}[t_1, \cdots, t_k] \otimes \mathbb{Z}_2[T], \mathbb{Z}_2)$ as multigraded $\mathbb{Z}_2$-algebras.

Moreover, we can make the above isomorphisms in Theorem 7.3 and Theorem 7.4 dimension-preserving by defining

$$\dim(u_i) = d_i, \ 1 \leq i \leq k, \ \text{and} \ \dim(v_\sigma) = \text{rank}(\sigma) + \sum_{i \in I_\alpha(\sigma)} d_i, \ \forall \sigma \in \mathcal{K}.$$ 

The proofs of the above four theorems are left to the reader. Especially, we need to modify our definition of $\kappa(i, \mathcal{L})$ and $\kappa(\sigma, \mathcal{L})$ (see (33)) to adapt to the proof of Theorem 7.1. Indeed, for $X(\mathcal{K}, \alpha, \mathcal{S})$ we should redefine $\kappa(i, \mathcal{L})$ as follows and change $\kappa(\sigma, \mathcal{L})$ accordingly.

$$\kappa(i, \mathcal{L}) = (-1)^{r_{\mathcal{S}}(i, \mathcal{L})}, \ \text{where} \ r_{\mathcal{S}}(i, \mathcal{L}) = \sum_{j \in \mathcal{L}, j < i} d_j, \ \forall i \in \mathcal{L} \subset [k].$$
Remark 7.5. When $\alpha^*$ is the trivial partition of $V(K)$, $X(K, \alpha^*, S)$ is nothing but the polyhedral-product $K^{(D,S)}$ where

$$(D,S) = \{(D^{d_1+1}, S^{d_1}), \ldots, (D^{d_k+1}, S^{d_k})\}.$$  

In these cases, Theorem 7.1 coincides with [9, Theorem 4.2]. Besides, Theorem 7.3 (ii) verifies a claim at the end of [9, Remark 9] for $K^{(D,S)}$.

(III) For a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of the vertex set of a finite simplicial complex $K$ and a sequence of spaces $A = (A_1, \cdots, A_k)$, define

$$X(K, \alpha, A) = \bigcup_{\sigma \in K} \left( \prod_{i \in I_{\alpha}(\sigma)} A_i \ast (\sigma \cap \Delta_{\alpha_i}) \times \prod_{i \in [k] \setminus I_{\alpha}(\sigma)} A_i \right) \subset \prod_{i \in [k]} A_i \ast \Delta_{\alpha_i}.$$  

Similarly to $X(K, \lambda_{\alpha})$ and $X(K, \Lambda_{\alpha})$, we have the following stable decomposition for $X(K, \alpha, A)$:

$$\Sigma(X(K, \alpha, A)) \simeq \Sigma \left( \bigvee_{L \subset [k]} \bigvee_{\sigma \in K_{\alpha,L}} |K_{\alpha,L}| \ast W^A_{\alpha,L}((\sigma)) \right)$$  

(61)

where $W^A_{\alpha,L}((\sigma)) := \bigwedge_{i \in I_{\alpha}(\sigma) \cap L} A_i \ast (\sigma \cap \Delta_{\alpha_i}) \wedge \bigwedge_{i \in L \setminus (I_{\alpha}(\sigma) \cap L)} A_i$.

Since $W^A_{\alpha,L}((\sigma))$ is contractible whenever $\sigma \neq \emptyset \in K_{\alpha,L}$, we can simplify the right side of (61) and obtain the following theorem which is parallel to Theorem 1.2.

Theorem 7.6. There is a homotopy equivalence

$$\Sigma(X(K, \alpha, A)) \simeq \Sigma \left( \bigvee_{L \subset [k]} |K_{\alpha,L}| \ast \bigwedge_{i \in L} A_i \right).$$  

(62)

We can compute the cohomology groups of $X(K, \alpha, A)$ via the above stable decomposition. In particular for the trivial partition $\alpha^*$ of $V(K)$ where $V(K) = [m]$, $X(K, \alpha^*, A)$ is nothing but the polyhedral product $(\text{Cone}(A), A)^K$ where

$$(\text{Cone}(A), A) = \{(\text{Cone}(A_1), A_1), \cdots, (\text{Cone}(A_m), A_m)\}.$$  

The reader is referred to [2] and [18] for the calculations of cohomology groups of polyhedral products in general.
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