Upper bound on the center-of-mass energy of the collisional Penrose process

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Following the interesting work of Bañados, Silk, and West [Phys. Rev. Lett. 103, 111102 (2009)], it is repeatedly stated in the physics literature that the center-of-mass energy, $E_{\text{c.m.}}$, of two colliding particles in a maximally rotating black-hole spacetime can grow unboundedly. For this extreme scenario to happen, the particles have to collide at the black-hole horizon. In this paper we show that Thorne’s famous hoop conjecture precludes this extreme scenario from occurring in realistic black-hole spacetimes. In particular, it is shown that a new (and larger) horizon is formed before the infalling particles reach the horizon of the original black hole. As a consequence, the center-of-mass energy of the collisional Penrose process is bounded from above by the simple scaling relation $E^\text{max}_{\text{c.m.}}/2\mu \propto (M/\mu)^{1/4}$, where $M$ and $\mu$ are respectively the mass of the central black hole and the proper mass of the colliding particles.

I. INTRODUCTION

The collisional Penrose process, a collision between two particles which takes place within the ergosphere of a spinning Kerr black hole, has attracted the attention of physicists ever since the pioneering work of Piran, Shaham, and Katz more than four decades ago [1, 2]. Interestingly, it was shown [1, 2] that the collision of two particles in the black-hole spacetime can produce two new particles, one of which may escape to infinity while carrying with it some of the black-hole rotational energy [3–5].

The collisional Penrose process has recently gained renewed interest when Bañados, Silk, and West [6] have revealed the interesting fact that the center-of-mass energy of the two colliding particles may diverge if the collision takes place at the horizon of a maximally rotating (extremal) black hole. In particular, it was suggested [6] that these extremely energetic near-horizon collisions may provide a unique probe of the elusive Planck-scale physics [7, 8].

The intriguing discovery of [6], according to which black holes may serve as extremely energetic particle accelerators, has sparked an enormous excitement in the physics community. In particular, following [6] it is repeatedly stated in the physics literature that the center-of-mass energy, $E_{\text{c.m.}}$, of two colliding particles in a maximally rotating black-hole spacetime can grow unboundedly.

It is worth emphasizing again that, for the center-of-mass energy of the colliding particles to diverge, the particles have to collide exactly at the black-hole horizon. The main goal of the present paper is to reveal the fact that Thorne’s famous hoop conjecture [9] precludes this extreme scenario from occurring in realistic black-hole spacetimes. In particular, below we shall show that a new (and larger) horizon is formed before the infalling particles reach the horizon of the original black hole. As a consequence, the center-of-mass energy of the colliding particles in the black-hole spacetime cannot grow unboundedly.

II. THE CENTER-OF-MASS ENERGY OF TWO COLLIDING PARTICLES IN THE BLACK-HOLE SPACETIME

We shall explore the maximally allowed center-of-mass energy associated with a collision of two particles that start falling from rest at infinity towards a maximally rotating (extremal [10]) Kerr black hole of mass $M$ and angular momentum $J = M^2$ [6, 8, 11–13]. The geodesic motions of the particles in the black-hole spacetime are characterized by conserved energies

$$E_1 = E_2 = \mu$$

and conserved angular momenta

$$L_1 = l_1 \cdot M\mu \quad ; \quad L_2 = l_2 \cdot M\mu .$$

Geodesic trajectories that extend all the way from spatial infinity down to the black-hole horizon are characterized by angular momenta in the bounded regime [6, 11]

$$-2(1 + \sqrt{1 + J/M^2}) \leq l \leq 2(1 + \sqrt{1 - J/M^2}) .$$

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In particular, it was shown in [6] that the center-of-mass energy of the collision may diverge if one of the colliding particles (say 1) is characterized by the critical angular momentum (for $J/M^2 = 1$)

$$l_1 = 2.$$  \hfill (4)

The test-particle approximation implies

$$\bar{\mu} \equiv \frac{\mu}{M} \ll 1.$$  \hfill (5)

As shown in [6, 8, 11, 12], the center-of-mass energy of the collision may diverge if the collision takes place at the horizon of the black hole. In particular, defining

$$x_c \equiv r_c - \frac{M}{M}$$  \hfill (6)

as the dimensionless collision radius of the two particles, one finds the leading-order divergent behavior [12]

$$E_{\text{c.m}}^2 = \mu^2 \times \frac{\beta_\pm (2 - l_2)}{x_c} + \mathcal{O}(1)$$  \hfill (7)

for the center-of-mass energy in the near-horizon $x_c \ll 1$ region. As shown in [12], the dimensionless factor $\beta_\pm$ in (7) depends on the sign of the radial momentum of the first (critical [15]) particle (+ for an outgoing particle and − for an ingoing particle), on the value of the Carter constant $Q_1$ [16] which characterizes the polar geodesic motion of that particle in the Kerr black-hole spacetime, and on the polar angle $\theta$ which characterizes the collision point of the two particles in the black-hole spacetime [12]. Specifically, one finds [12]

$$\beta_+ = \frac{2(2 + (2 - \tilde{Q}_1)^{1/2})}{1 + \cos^2 \theta}; \quad \beta_- = \frac{2(2 + \tilde{Q}_1)^{1/2}}{1 + \cos^2 \theta},$$  \hfill (8)

where [12, 17]

$$0 \leq \tilde{Q}_1 = \frac{Q_1}{\mu^2} \leq 2.$$  \hfill (9)

Interestingly, as pointed out in [6, 8, 11, 12], the center-of-mass energy (7) of the colliding particles diverges if the collision takes place at the horizon ($x_c \rightarrow 0$) of the extremal black hole.

III. THE HOOP CONJECTURE AND THE UPPER BOUND ON THE CENTER-OF-MASS ENERGY OF THE COLLISIONAL PENROSE PROCESS

In the previous section we have seen that the center-of-mass energy (7) of the colliding particles can diverge. As emphasized in [6, 8, 11, 12], this intriguing conclusion rests on the assumption that the particles can collide exactly at the horizon of the extremal black hole. In the present section we shall show, however, that Thorne’s famous hoop conjecture [9] implies that, due to the energy carried by the infalling particles, a new (and larger) horizon is formed before the particles reach the horizon of the original black hole. This fact implies, in particular, that the optimal collision [18] between the two particles cannot take place exactly at the horizon of the original black hole.

The hoop conjecture, originally formulated by Thorne more than four decades ago [9], asserts that a gravitating system of total mass (energy) $M$ forms a black hole if its circumference radius $r_c$ is equal to (or smaller than) the corresponding horizon radius $r_{\text{Sch}} = 2M$ of the Schwarzschild black-hole spacetime [14, 21].

In the present study we shall use a weaker version of the hoop conjecture [21]. In particular, we conjecture that: A gravitating system of total mass $M$ and total angular momentum $J$ forms a black hole if its circumference radius $r_c$ is equal to (or smaller than) the corresponding horizon radius $r_{\text{Kerr}} = M + \sqrt{M^2 - (J/M)^2}$ of the Kerr black-hole spacetime. That is, we conjecture that [22]

$$r_c \leq M + \sqrt{M^2 - (J/M)^2} \implies \text{Black-hole horizon exists}.$$  \hfill (10)

In the context of the collisional Penrose process that we consider here, this version of the hoop conjecture implies that a new (and larger) horizon is formed if the energetic particles that fall towards the black hole reach the radial
coordinate $r = r_{\text{hoop}}$, where $r_{\text{hoop}}(E_1, E_2, L_1, L_2)$ is defined by the familiar functional relation of the Kerr black-hole spacetime [see Eq. (10)]

$$r_{\text{hoop}} = M + E_1 + E_2 + \sqrt{(M + E_1 + E_2)^2 - ((J + L_1 + L_2)/(M + E_1 + E_2))^2}.$$  \hfill (11)

Taking cognizance of Eqs. (1), (2), (4), and (11), one finds

$$r_{\text{hoop}} = M + \sqrt{2(2 - l_2)M\bar{\mu} + 2\mu + O(\mu^2/M)}$$  \hfill (12)

for the radius of the new horizon. Assuming that the colliding particles are not engulfed by an horizon, the relation (12) implies [see Eq. (6)]

$$x_{\text{c}}^{\text{min}} = \sqrt{2(2 - l_2)\bar{\mu} + 2\bar{\mu}}$$  \hfill (13)

for the minimally allowed value of the dimensionless collision radius.

Substituting (13) into (7), one finds that, in the collisional Penrose process, the center-of-mass energy of the colliding particles is bounded from above by the relation [24]

$$\mathcal{E}_{\text{c.m.}}^{\text{max}} = \mu \times \sqrt{\frac{(2 - l_2)\beta_\pm}{2(2 - l_2)\bar{\mu} + 2\bar{\mu}}}.$$  \hfill (14)

Assuming the strong inequality $\mu \ll (2 - l_2)M$ [see Eq. (5) and Eq. (19) below], one can approximate (14) by

$$\mathcal{E}_{\text{c.m.}}^{\text{max}} = \mu \times \left(\frac{2 - l_2}{2\bar{\mu}}\right)^{1/4} \beta_\pm^{1/2},$$  \hfill (15)

which yields the simple expression [25]

$$\mathcal{E}_{\text{c.m.}}^{\text{max}} = 2\gamma \cdot M^{1/4} \mu^{3/4}$$  \hfill (16)

for the maximally allowed center-of-mass energy of the colliding particles in the black-hole spacetime, where

$$\gamma \equiv [(2 - l_2)\beta_\pm^2 / 32]^{1/4}.$$  \hfill (17)

What is the maximally allowed value of the dimensionless pre-factor $\gamma$ in (16)? Inspection of Eq. (8) reveals that the factor $\beta_\pm$ is maximized if the collision takes place at the equatorial plane ($\theta = \pi/2$) of the black hole while the first particle is on an outgoing trajectory with $\dot{Q}_1 = 0$ [see Eq. (5)], in which case one finds [see Eq. (8)] [26]

$$\beta_{\text{max}} = 2(2 + \sqrt{2}).$$  \hfill (18)

In addition, taking cognizance of the fact that geodesic trajectories which extend all the way from spatial infinity down to the black-hole horizon are characterized by angular momenta in the bounded regime (3), one finds (for $J/M^2 = 1$)

$$(2 - l_2)_{\text{max}} = 2(2 + \sqrt{2}) .$$  \hfill (19)

Substituting (18) and (19) into (17), one finds [27]

$$\gamma_{\text{max}} = (2 + \sqrt{2})^{3/4} / \sqrt{2}.$$  \hfill (20)

**IV. SUMMARY**

Following the important work of Bañados, Silk, and West [6], it is repeatedly stated in the physics literature that the center-of-mass energy of two colliding particles in an extremal (maximally rotating) black-hole spacetime can diverge. For this extreme scenario to happen, the particles have to collide exactly at the horizon of the black hole.

In this paper we have shown that Thorne’s famous hoop conjecture [3] [and also its weaker version (10)] precludes this infinite-center-of-mass-energy scenario from occurring in realistic black-hole spacetimes. In particular, the hoop conjecture implies that, due to the energy carried by the infalling particles, a new (and larger) horizon is formed before the particles reach the horizon of the original black hole. As a consequence, it was shown that the optimal collision takes place at [see Eq. (12)]

$$r_{\text{c}}^{\text{min}} = M + \sqrt{2(2 - l_2)M\bar{\mu} + 2\mu > M} \quad [28, 31],$$

which implies that the center-of-mass
energy of the colliding particles in the black-hole spacetime is bounded from above by the simple scaling relation [see Eqs. (16) and (20)]

\[
\frac{E_{\text{c.m.}}^{\text{max}}}{2\mu} = \gamma_{\text{max}} \cdot \left(\frac{M}{\mu}\right)^{1/4}.
\]  

(21)

For the case of two protons colliding in a supermassive Kerr black-hole spacetime of \(10^9\) solar masses, the bound (21) implies \(E_{\text{c.m.}}^{\text{max}} \approx 10^{14}\text{TeV}\) for the maximally allowed center-of-mass energy of the collision. Interestingly, this energy, though being finite, is still many orders of magnitude larger than the maximally available center-of-mass energy in the most powerful man-made accelerators [32].

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7. It is important to note that subsequent investigations [3] of the collisional Penrose process have shown that, while the center-of-mass energy of the colliding particles may diverge as claimed in [6], the maximal energy which is radiated to infinity by an escaping particle (after the collision) is only one order of magnitude larger than the initial energy of the colliding particles (this energy gain reflects the extraction of rotational energy from the spinning Kerr black hole).
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10. As shown in [6, 8], assuming the black hole to be a maximally-rotating (extremal) one, allows one to maximize the center-of-mass energy of the colliding particles.
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13. We use natural units in which \(G = c = \hbar = 1\).
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15. It is worth mentioning that the first particle is characterized by the critical angular momentum [4].
16. B. Carter, Phys. Rev. 174, 1559 (1968).
17. It was shown in [12] that a particle infalling from spatial infinity can reach the black-hole horizon provided \(0 \leq Q \leq 2\mu^2\).
18. That is, the collision with the largest possible center-of-mass energy.

[References and acknowledgments follow this point.]

\[x_c \approx x_c^{\text{min}}\]
That is, outside the horizon of the original black hole.

It is worth emphasizing again that, according to the hoop conjecture, it is the presence of the particles themselves within a sufficiently small radius that triggers the formation of a new horizon.

It is important to emphasize that Berti et. al. have shown that the gravitational radiation emitted by the infalling particles is of order $E_{\text{rad}} \sim (\mu^2/M) \cdot \ln(2 - l_1)$. Since $2 - l_1 = O(\mu/M)$ (see [24]), one finds $E_{\text{rad}} \sim - (\mu^2/M) \cdot \ln(\mu/M)$, which implies that the radiated gravitational energy can be neglected (that is, $E_{\text{rad}} \ll \mu$) in the regime $- (\mu/M) \cdot \ln(\mu/M) \ll 1$.

E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius, and U. Sperhake, Phys. Rev. Lett. 103, 239001 (2009).

It is important to emphasize that, had we used the original hoop conjecture [9] instead of its weaker version (10), we would have found that the center-of-mass energy of the colliding particles is bounded from above by a bound which is stronger than our bound (21).

See https://en.wikipedia.org/wiki/List-of-accelerators-in-particle-physics.