POSITIVITY OF PETERSON SCHUBERT CALCULUS

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Abstract. The Peterson variety is a subvariety of the flag manifold $G/B$ equipped with an action of a one-dimensional torus, and a torus invariant paving by affine cells, called Peterson cells. We prove that the equivariant pull-backs of Schubert classes indexed by arbitrary Coxeter elements are dual (up to an intersection multiplicity) to the fundamental classes of Peterson cell closures. Dividing these classes by the intersection multiplicities yields a $\mathbb{Z}$-basis for the equivariant cohomology of the Peterson variety. We prove several properties of this basis, including a Graham positivity property for its structure constants, and stability with respect to inclusion in a larger Peterson variety. We also find formulae for intersection multiplicities with Peterson classes. This explains geometrically, in arbitrary Lie type, recent positivity statements proved in type A by Goldin and Gorbutt.

1. Introduction

Among the most important properties of the cohomology rings of complex flag manifolds is that they have a distinguished basis: the Schubert basis, $\{\sigma_v : v \in W\}$, indexed by the Weyl group $W$. The Schubert structure constants for the product in singular cohomology in this basis are positive, which means that all nonzero coefficients in the product expansion

$$\sigma_u \sigma_v = \sum_{w \in W} c_{uv}^w \sigma_w$$

are positive. This is a consequence of transversality properties, as the coefficients $c_{uv}^w$ count the number of points in the intersection of three Schubert varieties translated in general (transversal) position. The study of positivity properties of these coefficients across a wide range of cohomology theories has spurred a large body of literature in algebraic geometry, representation theory, and combinatorics.

In the case of torus-equivariant cohomology, a conjecture by Peterson [Pet97], proved by Graham [Gra01], states that the Schubert structure constants for the torus equivariant cohomology rings of flag manifolds are polynomials in the simple roots with positive coefficients. Graham’s proof relies on refined transversality techniques (see also [AGM11]), which are expected to be useful in analyzing the equivariant cohomology ring of varieties related to flag manifolds.

Hessenberg varieties form a remarkable family of subvarieties of flag manifolds, and appear across multiple disciplines within mathematics; see [AH20] for a recent survey. In this paper we focus on a particular class of regular nilpotent Hessenberg varieties, namely Peterson varieties.
Peterson varieties may also be realized as a flat degeneration of certain regular semisimple Hessenberg varieties such as the permutohedral variety; see e.g., [Kly85, Kly95, AFZ20]. Peterson varieties share many properties with flag varieties, and provide a fertile ground for exploration. They initially appeared in the study of the quantum cohomology ring of (generalized) flag manifolds $G/B$ in [Kos96, Pet97].

The purpose of this paper is to prove a positivity property of the equivariant cohomology ring of Peterson varieties, similar to that of flag varieties. This may be seen as a step towards investigating positivity and transversality properties of the (equivariant) cohomology ring of the larger family of Hessenberg varieties, little of which is known. We give next a more precise account of our results.

Let $G$ be a complex, semisimple Lie group, let $B, B^- \subset G$ be opposite Borel subgroups, and let $T := B \cap B^-$ be the associated maximal torus. Let $\Delta$ be the set of positive simple roots corresponding to the choice of $B$, and let $e$ be a principal nilpotent element contained in $\bigoplus_{\alpha \in \Delta} g_\alpha$, where $g_\alpha$ is the root space in $g := \text{Lie}(G)$ corresponding to the root $\alpha$. Denote by $\mathbf{W}$ the Weyl group associated to $(G, T)$ with length function $\ell: \mathbf{W} \to \mathbb{N}$, and denote by $w_0 \in \mathbf{W}$ the longest element. Let $G^e$ be the centralizer of $e$ in $G$.

The Peterson variety

$$(1) \quad P := \overline{G^e.w_0 B} \hookrightarrow G/B,$$

is the closure of the $G^e$-orbit of $w_0 B$ inside the flag manifold $G/B$. It admits an action of a one-dimensional torus $S \subset T$ with finitely many fixed points. We denote by $\iota: P \hookrightarrow G/B$ the inclusion.

In this manuscript we investigate $H^*_S(P) := H^*_S(P; \mathbb{Z})$, the integral $S$-equivariant cohomology ring of the Peterson variety. A presentation of $H^*_S(P; \mathbb{Q})$ by generators and relations was given by Harada, Horiguchi and Masuda in [HHM15].

Let $\sigma_{v_I} \in H^*_S(G/B)$ be a Schubert class indexed by some Coxeter element $v_I \in W$ for $I \subset \Delta$, let $p_I := \iota^*(\sigma_{v_I})$ be the pullback of $\sigma_{v_I}$ along the inclusion $\iota: P \to G/B$, and let $m(v_I)$ be the intersection multiplicity from Theorem [1.1] below. We show that $\left\{ p_I / m(v_I) \right\}_{I \subset \Delta}$ is a $H^*_S(pt)$-basis of $H^*_S(P)$, and that this basis is dual to the equivariant Borel-Moore homology basis constituted by the fundamental classes of Peterson varieties (see Section 2). In particular, while the class $p_I$ depends on the choice of $v_I$, the class $p_I / m(v_I)$ is independent of this choice.

In our main result we prove that the structure constants of multiplication are positive with respect to the basis $\left\{ p_I / m(v_I) \right\}$ in the sense of Graham [Gra01]. This generalizes recent results in Lie type A by Goldin and Gorbutt [GG22], who found manifestly positive combinatorial formulae for the structure constants in question. Special cases of such formulae were found earlier by Harada and Tymoczko [HT11], and by Drellich [Dre15].

We now present a more precise version of our results. For $I \subset \Delta$, let $w_I$ be the maximal element of the Weyl group of $I$, and let $P^\circ_I := P \cap Bw_IB/B$. Tymoczko [Tym07] and Bălibanu [Băl17] proved that $P^\circ_I \cong \mathbb{C}^{|I|}$, and the
cells $P_I$ (called Peterson cells) form an affine paving of the Peterson variety. Consequently, the fundamental classes $[P_I]_S \in H^S_{2|I}|(P)$, where $P_I = P^\setminus I$, form a basis of $H^S_0(P)$ over $H^S_0(pt)$. For $v \in W$, let $X^v := B^-vB/B$ denote the (opposite) Schubert variety in $G/B$, and let $\sigma_v \in H^{2\ell(v)}_S(G/B)$ be the corresponding Poincaré dual class, satisfying the equality $\langle \sigma_v \cap [G/B]|_S = [X^v]_S \rangle$. Consider the pairing $\langle \cdot, \cdot \rangle : H^S_0(P) \otimes H^S_0(P) \to H^S_0(pt)$ of equivariant cohomology and equivariant homology defined by $\langle a, b \rangle = \int_P a \cap b$; see [2]. Our first result is the following (cf. Theorem 1.3 below):

**Theorem 1.1** (Duality Theorem). Let $I,J$ be subsets of the set of simple roots $\Delta$ and let $v_I \in W$ be any Coxeter element for $I$. Then

$$\langle \iota^*\sigma_{v_I}, [P_J]_S \rangle = m(v_I)\delta_{I,J},$$

where $m(v_I) \in \mathbb{Z}_{>0}$ is the multiplicity of the (unique) intersection point of $X^{v_I} \cap P_I$.

This follows because the varieties $X^{v_I}$ and $P_I$ intersect at a unique point, namely $w_I$, the longest element in the subgroup $W_I$ determined by $I$. The remaining part of the proof exploits the poset structure of the affine paving by Peterson cells, along with the duality of Schubert classes in $G/B$.

A non-equivariant version of this theorem has appeared in the literature in type A in a recent preprint [AHKZ24], and may be deduced in general Lie type from [IT16, Ins15], where the intersection $X^{v_I} \cap P(I)$ was analyzed. Besides working non-equivariantly, a key restriction in all these papers is that the Coxeter elements $v_I$ are not arbitrary, but depend on a certain ordering of the simple roots. Our approach removes this restriction. In Section 3.2, we give algorithms to calculate the aforementioned multiplicities based on equivariant localization, and closed formulae for a particular basis; see also Theorem 1.3 below and the related discussion. A formula for $m(v_I)$ for any Coxeter element $v_I$ was recently obtained in [GS21]; see Equation (19) below.

The duality theorem has several consequences. For each Coxeter element $v_I$ for $I$, recall that $p_I := \iota^*\sigma_{v_I} \in H^{2|I}|_S(P)$. Then the classes $\left\{ \frac{p_I}{m(v_I)} \right\}_{I \subseteq \Delta}$ form a $H^S_0(pt)$-basis of $H^S_0(P)$; see Corollary 1.4. By the duality theorem, the equivariant push forward $t_+: H^S_0(P) \to H^S_0(G/B)$ is injective. Non-equivariantly, the injectivity was proved in [IT16].

The cocharacter $h$ of $T$ satisfying $\alpha(h) = 2$ for all $\alpha \in \Delta$ determines a one dimensional subtorus $S \subset T$, satisfying $\alpha(S) = \alpha'(S)$ for any $\alpha, \alpha' \in \Delta$; see Section 3.2. Consequently there is a well defined element $t \in H^S_0(pt)$ given by $t := \alpha|S$ for $\alpha \in \Delta$.

**Theorem 1.2** (Positivity). Let $I,J,K$ be subsets of $\Delta$. The structure constants of multiplication, $c^K_{I,J} \in H^S_0(pt)$, given by

$$p_I \cdot p_J = \sum_K c^K_{I,J} p_K$$

are polynomials in $t$ with non-negative coefficients.
Theorem 1.2 (Theorem 5.3 below) generalizes several positivity statements to arbitrary Lie type, while providing a uniform proof in all cases. In the case where \(|I| = 1\), i.e., \(p_I\) is a divisor class, a positive Monk-Chevalley formula for the structure constants \(c^K_{I,J}\) was obtained by Harada and Tymoczko [HT11] in Lie type A, and in arbitrary Lie type by Drellich [Dre15].

For general \(c^K_{I,J}\), and in Lie type A, Goldin and Gorbutt [GG22] found a manifestly positive combinatorial formula for all equivariant coefficients \(c^K_{I,J}\) in the expansion (2). A different combinatorial model computing these coefficients in non-equivariant cohomology was recently obtained in [AHKZ24].

The proof of Theorem 1.2 relies on the duality theorem, and on positivity statements proved by Graham [Gra01]. The structure constants of the multiplication \(\sigma_u \cup \sigma_v \in H^*_S(G/B)\) are positive in the sense of Theorem 1.2. Hence it suffices to show that the coefficients \(b^I_w \in H^*_S(pt)\) of the restricted classes \(c^* \sigma_w = \sum_J b^I_w p_J\) (\(w \in W\) arbitrary) satisfy the same positivity. By the duality theorem, the positivity of the coefficients \(b^I_w\) is equivalent to the positivity (in a suitable sense) of the coefficients \(c^I_v\) in the Schubert expansion

\[
\tau_* [P_I]_S = \sum_{v \in W} c^I_v [X_v]_S \in H^*_S(G/B),
\]

where \([X_v]_S\) denotes the homology class of the Schubert variety \(X_v := BvB/B \) in \(G/B\). In the non-equivariant case, this is clear. Indeed, by Kleiman transversality [Kle74], the Schubert classes are a basis for the Chow group of \(G/B\), and form a set of primitive generators for the cone of effective algebraic cycles. Since the Chow group is equal to \(H_*(G/B)\) (cf. [Fu98 Ex. 19.1.11]), the claim follows. Equivariantly, we deduce the positivity of \(c^I_v\) from a general positivity result of Graham [Gra01] for expansions of fundamental classes of torus invariant varieties; cf. Theorem 5.2. A different geometric approach to positivity (in the non-equivariant setting) was pursued in [AHKZ24]; see Remark 5.4 below.

Using equivariant localization, we obtain formulae for the multiplicities \(m(v_I)\) in the duality theorem, and an effective algorithm to find the Schubert expansion from Equation (3); see Section 1.2. Our algorithms for the coefficients \(c^I_v\), and for the multiplicities \(m(v_I)\), rely on the restriction of the Schubert classes \(\sigma_v\) to the fixed points \(w_I \in G/B\) (see Section 2). These equivariant localizations may be calculated using formulae developed by Andersen, Jantzen, and Soergel [AJS94] and Billey [Bil99].

**Theorem 1.3.** (a) Let \(I\) be a connected Dynkin diagram with the standard labelling, see [Bou02], and set \(v_I = s_1 s_2 \cdots s_n\). Then,

\[
m(v_I) = \begin{cases} 
1 & \text{if } I = A_n, \\
2^{n-1} & \text{if } I = B_n, C_n, \\
2^{n-2} & \text{if } I = D_n, \\
72 = 2^3 \cdot 3^2 & \text{if } I = E_6, \\
864 = 2^5 \cdot 3^3 & \text{if } I = E_7, \\
51840 = 2^7 \cdot 3^4 \cdot 5 & \text{if } I = E_8, \\
48 = 2^4 \cdot 3 & \text{if } I = F_4, \\
6 = 2 \cdot 3 & \text{if } I = G_2.
\end{cases}
\]

(b) Let \(I_1, \ldots, I_k\) be the connected components of a Dynkin diagram \(I\), and let \(v_{I_1}, \ldots, v_{I_k}\) be any Coxeter elements for \(I_1, \ldots, I_k\) respectively. Then \(v := v_{I_1} \cdots v_{I_k}\) is a Coxeter element for \(I\), and \(m(v) = \prod m(v_{I_j})\).
See Theorem 7.6 below. The factorization is related to the exponents of the Lie algebra of $G$, see §7 below, and also the recent paper [CS21]. Theorem 1.3 generalizes a result of Insko [Ins15], who showed that when $I = A_n$, $m(s_1 \cdots s_n) = 1$. More generally, the theorem addresses [IT16, Question 1] by providing an explicit formula for these multiplicities; in particular, it disproves the conjecture by Insko and Tymoczko that the multiplicities are always 1 or 2 in classical Lie types. The proof of part (a) of Theorem 1.3 utilizes equivariant localization, while the proof of part (b) utilizes the stability property of Peterson classes explained below. Using parts (a) and (b) concurrently allows us to compute $m(v_I)$ for some Coxeter element $v_I$ in each Dynkin diagram $I$, and hence allows us to construct the dual class $ι^∗ σ_vI m(v_I)$ of any Peterson subvariety $P_I ⊂ P$.

Consider $I ⊂ Δ$, a subset of the Dynkin diagram, and let $G_I$ be a semisimple group with Dynkin diagram $I$. Let $G_I/B_I$ and $P(I) = P$ denote the flag variety and the Peterson variety of $G_I$ respectively, and let $S_I$ be the one-dimensional subtorus defined analogously to $S$, and acting on $P(I)$. There is a natural closed embedding $i : G_I/B_I \hookrightarrow G/B$, but unfortunately there may not be a morphism $S_I \rightarrow S$ which is compatible with this embedding. This leads to some technical subtleties explained in Section 6.2. The upshot is that there is an algebra isomorphism $H^*_S(\text{pt}; \mathbb{Q}) \cong H^*_S(I; \mathbb{Q})$, and induced maps $H^*_S(G/B; \mathbb{Q}) \xrightarrow{i^*} H^*_S(G_I/B_I; \mathbb{Q})$ and $H^*_S(G_I/B_I; \mathbb{Q}) \xrightarrow{i^*} H^*_S(G/I/B; \mathbb{Q})$. The stability theorem, proved in Proposition 6.5 and Theorem 6.6, is the following.

**Theorem 1.4 (Stability).** (a) $i(P(I)) = P \cap i(G_I/B_I) = P_I$, as subsets of $G/B$.

(b) For $J \subset I$, we have $i_*(P(J)|_{S_I}) = [P_J]_{S_I}$, as classes in $H^*_S(P; \mathbb{Q})$.

(c) Let $j : P(I) \hookrightarrow P$ be the restriction of $i : G_I/B_I \rightarrow G/B$. For $K \subset Δ$, we have

$$j^*(p_K) = \begin{cases} p_K & \text{if } K \subset I, \\ 0 & \text{otherwise}, \end{cases}$$

as classes in $H^*_S(P)(I; \mathbb{Q})$.

Furthermore, in the non-equivariant case, the statements in (b) and (c) hold over $\mathbb{Z}$.

The proof of the stability theorem utilizes a common alternate description of the Peterson variety, namely

$$P = \left\{ gB \in G/B \left| \text{Ad}(g^{-1})e \in b \oplus \bigoplus_{\alpha \in Δ} g_{-\alpha} \right. \right\},$$

where $b = \text{Lie}(B)$. In Appendix A, we take the opportunity to present a proof that the definitions (1) and (4) are equivalent, a matter of folklore implied by, and implicit in, Kostant’s original work [Kos96].

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Conventions. We work over the field of complex numbers. By a variety
we mean a reduced, irreducible scheme of finite type. Schemes defined as
algebraic group orbits, and closures of group orbits, are always equipped
with the induced reduced scheme structure; see, e.g., [Sta24 tag 01IZ].

2. Equivariant (Co)Homology

Let $X$ be a complex algebraic variety equipped with a left action of a torus
$T$. We recall aspects of the $T$-equivariant homology and cohomology of $X$.
We will use the Borel model of equivariant cohomology, and equivariant
Borel-Moore homology, following the setup in Graham’s paper [Gra01]. We
refer to [Ful98 Ch 19], [Ful97 Appendix B], [CG97 §2.6] for more details
about cohomology and Borel-Moore homology.

Since we are working with algebraic varieties, our statements and proofs
could have been written using the language of equivariant Chow groups
[EG98]. For full results, this requires some additional properties of the
operational Chow ring of linear varieties proved by Totaro [Tot14]. Aware of
this technicality, the reader may use the equivariant cycle map from [EG98]
to freely swap between the Borel-Moore and Chow theories.

Fix an identification $T \cong (\mathbb{C}^*)^r$ and let $ET = (\mathbb{C}^\infty \setminus 0)^r$ be the universal
$T$-bundle with classifying space $BT = (\mathbb{P}^\infty)^r$. The product $ET \times X$ has
a right $T$-action given by $(e, y).t := (et, t^{-1}y)$. The action is free, and the
orbit space $X_T := (ET \times X)/T$ is called the Borel mixing space of $X$. The
universal $T$-bundle $ET \to BT$ admits finite dimensional approximations
$ET_n \to BT_n$, where $ET_n = (\mathbb{C}^{n+1} \setminus 0)^r$ and $BT_n := (\mathbb{P}^n)^r$. These induce
finite dimensional approximations of the Borel mixing space $X_{T,n} := (ET_n \times
X)/T$, and inclusions $X_{T,n_1} \subset X_{T,n_2}$ for $n_1 < n_2$.

We define the equivariant cohomology ring by $H^*_T(X) := H^*(X_T)$; note
that we have $H^*_T(X) = H^i(X_{T,n})$ for sufficiently large $n$. The equivariant
Borel-Moore homology groups are defined via a limiting property,

$$H^T_i(X) := H^RM_{i+2nr}(X_{T,n}), \quad \text{for } n \gg 0$$
where the right hand side is the ordinary Borel-Moore homology. If $V \subset X$ is a closed $T$-stable subvariety of $X$ of complex dimension $d$, its fundamental class $[V]_T$ is an element in $H^*_{2d}(X)$. The cap product gives the (total) equivariant homology $H^*_T(X) = \bigoplus_n H_n^T(X)$ a graded module structure over the equivariant cohomology ring $H^*_T(X)$.

If $X = pt$, then $H^*_T(pt) = H^*(BT)$ is naturally identified with the symmetric algebra $\text{Sym}(\mathfrak{X}(T))$ of the character group $\mathfrak{X}(T) := \text{Hom}(T, \mathbb{C}^*)$ of $T$ (written additively). For any map $S \to T$ of tori, we have a natural map of algebras $H^*_T(X) \to H^*_S(X)$, compatible with the algebra map $H^*_T(pt) \to H^*_S(pt)$ induced by $\mathfrak{X}(T) \to \mathfrak{X}(S)$. Taking $S$ to be the trivial subgroup in $T$, we obtain a ring homomorphism $H^*_T(X) \to H^*(X)$. (One can show that this map is surjective for spaces with affine pavings in the sense of Lemma 2.1 below; we will not need this fact.)

The morphism $X_T \to BT$ that projects onto the first factor gives the equivariant cohomology $H^*_T(X)$ the structure of a graded algebra over $H^*_T(pt)$. In addition, the cap product $\cap$ endows the equivariant homology $H^*_T(X)$ with a graded module structure over $H^*_T(X)$. Equivalently, there is a compatibility of cap and cup products given by $(a \cup b) \cap c = a \cap (b \cap c)$, for $a, b, c \in H^*_T(X)$.

Each irreducible, $T$-stable, closed subvariety $Z \subset X$ of complex dimension $k$ has a fundamental class $[Z]_T \in H^*_{2k}(X)$. If $X$ is smooth and irreducible, then there exists a unique (Poincaré dual) class $\eta_Z \in H^{2(\dim X - k)}(X)$ such that $\eta_Z \cap [X]_T = [Z]_T$.

Any $T$-equivariant morphism of $T$-varieties $f : X \to Y$ induces a degree preserving pull-back morphism of $H^*_T(pt)$-algebras $f^* : H^*_T(Y) \to H^*_T(X)$. For a point $x \in X$ fixed by the $T$ action, the inclusion $\iota_x : \{x\} \to X$ induces a localization map $\iota^*_x : H^*_T(X) \to H^*_T(\{x\}) = H^*_T(pt)$.

If $f$ is proper then there is a push-forward $f_* : H^*_T(X) \to H^*_T(Y)$, defined as follows. Let $Z \subset X$ be closed, irreducible and $T$-stable. Then $f_*[Z]_T = d_Z[f(Z)]_T$ if $\dim f(Z) = \dim Z$, where $d_Z$ is the generic degree of the restriction $f : Z \to f(Z)$, and $f_*[Z]_T = 0$ if $\dim f(Z) < \dim Z$. The push-forward and pull-back are related by the usual projection formula $f_* (f^*(a) \cap c) = a \cap f_* (c)$.

An important particular case is when $X$ is complete, thus $f : X \to pt$ is proper. For a homology class $c \in H^*_T(X)$, we denote by $\int_X c$ the class $f_* (c) \in H^*_T(pt)$.

Recall that the equivariant homology $H^*_T(pt)$ of a point is a free $H^*_T(pt)$-module with basis $[pt]_T$. Therefore we identify $H^*_T(pt) = H^*_T(pt)$ via the map $a \mapsto a \cap [pt]_T$. Then we may define a pairing,

$$\langle \cdot, \cdot \rangle : H^*_T(X) \otimes_{H^*_T(pt)} H^*_T(X) \to H^*_T(pt); \quad \langle a, c \rangle := \int_X a \cap c. \quad (5)$$

We often abuse notation and for a cohomology class $a \in H^*_T(X)$ we write $\int_X a$ to mean $\int_X (a \cap [X]_T)$.

\[1\text{We note that } f_* (c) \text{ agrees with "integration over the fiber" when } X \text{ is smooth, justifying the notation.} \]
Following [Ful98 Ex 1.9.1] (see also [Gra01]) we say that a $T$-variety $X$ admits a $T$-stable affine paving if it admits a filtration $X := X_n \supseteq X_{n-1} \supseteq \ldots$ by closed $T$-stable subvarieties such that each $X_i \setminus X_{i-1}$ is a finite disjoint union of $T$-invariant varieties $U_{i,j}$ isomorphic to affine spaces $\mathbb{A}^i$. The following has been proved by Graham; see [Gra01 Prop 2.1].

**Lemma 2.1.** Assume $X$ admits a $T$-stable affine paving, with cells $U_{i,j}$. 

(a) The equivariant homology $H^T_*(X)$ is a free $H_T^*(pt)$-module with basis $\{[U_{i,j}]_T\}$.

(b) If $X$ is complete, the pairing from Equation (5) is perfect, and so we may identify $H_T^*(X) = \text{Hom}_{H_T^*(pt)}(H^T_*(X), H_T^*(pt))$.

### 3. Flag manifolds and Peterson Varieties

In this section we recall some basic definitions about flag manifolds, Schubert varieties, and Peterson varieties. We mostly follow the setup in [Tym07] and [Bål17], from which we will need several important results.

#### 3.1. Flag manifolds and Schubert varieties

Fix a complex semisimple Lie group $G$, a Borel subgroup $B \subseteq G$, $B^- \subseteq G$ an opposite Borel subgroup, and let $T := B \cap B^-$ be a maximal torus. Denote by $\Delta$ the system of simple positive roots associated to $(G, B, T)$ and by $\Phi^+_\Delta \subseteq \Phi_{\Delta}$ the set of positive roots included in the set of all roots. The Weyl group $W := N_G(T)/T$ is generated by simple reflections $s_i := s_{\alpha_i}$ where $\alpha_i \in \Delta$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function and $w_0$ the longest element in $W$. Then $B^- = w_0Bw_0$.

Any subset $I \subseteq \Delta$ determines a Weyl subgroup $W_I := \langle s_i : \alpha_i \in I \rangle$ and a corresponding standard parabolic subgroup $P_I$. We denote by $w_I$ the longest element of $W_I$. The flag manifold $G/B$ is a smooth algebraic variety of complex dimension $\ell(w_0)$ with a transitive action of $G$ given by left multiplication. The flag manifold has a stratification into finitely many $B$-orbits, respectively $B^-$-orbits, called the Schubert cells. $X^w := BwB/B \cong \mathbb{C}^{\ell(w)}$ and $X^{w,\circ} := B^-wB/B \cong \mathbb{C}^{\ell(w_0w)}$; we have

$$G/B = \bigsqcup_{w \in W} X^w_\circ = \bigsqcup_{w \in W} X^{w,\circ}.$$  

The closures $X_w := \overline{X_w}$ and $X^w := \overline{X^{w,\circ}}$ are called Schubert varieties and opposite Schubert varieties, respectively. The Bruhat order is a partial order on $W$ characterized by inclusions of Schubert varieties and opposite Schubert varieties. In particular, $X_v \subseteq X_w$ if and only if $v \leq w$, and $X^w \subseteq X^v$ if and only if $v \leq w$. Following Lemma 2.1, the homology classes $\{[X_v]_T \mid v \leq w\}$ form a basis of $H^T_*(X_w)$, while $\{[X^v]_T \mid w \leq v\}$ form a basis of $H^T_*(X^w)$.

The cohomology classes $\sigma_v \in H^T_*(X)$ Poincaré dual to the $[X^v]_T$, i.e. characterized by the equation $\sigma_v \cap [G/B]_T = [X^v]_T$, are called Schubert classes.

Note that Lemma 2.1 also implies $\{\sigma_v \mid v \in W\}$ is a basis of $H^T_*(G/B)$ as a module over $H_T^*(pt)$. Under the pairing in Equation (5), the basis $\{\sigma_v \mid v \in W\}$ is dual to the basis $\{[X_v]_T \mid v \in W\}$, i.e, we have $\langle \sigma_v, [X_w]_T \rangle = \delta_{v,w}$.
3.2. **The Peterson variety and Peterson cells.** The Peterson variety appeared in the unpublished work of Peterson [Pet97], in relation to the quantum cohomology of $G/B$; we refer the reader to [Kos96, Rie03] for details.

We recall the definition of the Peterson variety. Let $g := \text{Lie}(G)$, $\mathfrak{h} := \text{Lie}(T)$, and consider the Cartan decomposition

$$g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$ 

For each simple root $\alpha \in \Delta$, choose a root vector $e_\alpha \in \mathfrak{g}_\alpha$, and let

$$e := \sum_{\alpha \in \Delta} e_\alpha.$$ 

The element $e$ is a regular nilpotent element in the Lie algebra $\mathfrak{h}$ of $B$; see [Kos59] or [CM93] Thm 4.1.6. Denote by $G^e \subset G$ the stabilizer of $e$ for the adjoint action of $G$ on $g$. We have $G^e = (G^e)^0 \times Z(G)$, where $(G^e)^0$ is the identity component of $G^e$, and $Z(G)$ the center of $G$. The identity component $(G^e)^0$ is a subgroup of the unipotent radical $U$ of $B$, isomorphic to the affine variety $\mathbb{C}^n$, where $n := |\Delta|$ is the number of simple roots, i.e., the rank of $G$, cf. [Kos59] Cor 5.3. For instance, if $G = \text{SL}_n(\mathbb{C})$, then $(G^e)^0$ is the subgroup of upper triangular unipotent matrices with equal entries along each superdiagonal. The Peterson variety is defined by

$$P := \overline{G^e.w_0B} \subset G/B.$$

This is an irreducible subvariety of $G/B$ of dimension $|\Delta|$, singular in general.

For any $\omega \in \mathfrak{h}$ contained in the coroot lattice, the map $\varphi_\omega : \mathbb{C} \to \mathfrak{h}$ defined by $\varphi_\omega(z) = z\omega$ lifts to a cocharacter $exp(\varphi_\omega) : \mathbb{C}^* \to T$. (Here the differential of $\exp(\varphi_\omega)$ is equal to $\varphi_\omega$. In complex differential geometry, the map $\exp(\varphi_\omega)$ intertwines with the (non-algebraic) exponential maps $\exp : \mathbb{C} \to \mathbb{C}^*$ and $\exp : \mathfrak{h} \to T$; the cocharacter $\exp(\varphi_\omega)$ is itself an algebraic map.) This identifies the coroot lattice of $\mathfrak{h}$ with a subset of the cocharacters of $T$. See, e.g., [GOV97, Ch. 3, Prop. 1.15] (in the algebraic setting), or [PH91] p. 373-4] (in the manifold setting).

We take $h = \sum_{\alpha \in \Phi_+} \alpha^\vee$ to be the sum of the positive coroots, and denote by $S \subset T$ the image of the cocharacter corresponding to $h$. Following [Bou02] Ch 6, Prop 29], we have $\alpha(h) = 2$ for all $\alpha \in \Delta$, because $h$ is equal to twice the sum of the fundamental coweights. In particular, it follows that $\alpha|S = \alpha'|S$ for any $\alpha, \alpha' \in \Delta$. We set $t := \alpha|S \in X(S) \subset H^*_S(pt)$.

**Example 3.1.** Consider $G = \text{SL}_n$, and let $T \subset G$ be the set of diagonal matrices:

$$T = \left\{ \left( \begin{array}{ccc}
 z_1 & 0 & 0 \\
 0 & \ddots & 0 \\
 0 & 0 & z_n
\end{array} \right) \mid z_1 \cdots z_n = 1 \right\}.$$ 

The $\alpha_i, 1 \leq i \leq n - 1$, given by $\alpha_i(diag(z_1, \cdots, z_n)) \mapsto z_i/z_{i+1}$, form a set of simple roots. The coroot $h$ corresponds to the one-dimensional subtorus
\[ S = \left\{ \begin{pmatrix} z^{n-1} & 0 & 0 & 0 \\ 0 & z^{n-3} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & z^{-n+1} \end{pmatrix} \right\} \quad z \in \mathbb{C}^* \]

The character \( t \) of \( S \) is the map given by \( \text{diag}(z^{n-1}, z^{n-3}, \ldots, z^{-n+1}) \mapsto z^2 \).

**Remark 3.2.** The element \( t \) need not be a generator of the ring \( H^*_S(pt) \). For example, if \( G = SL_2 \), we have

\[ S = \left\{ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \right\}, \quad t : \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \mapsto z^2. \]

The character group \( \mathcal{X}(S) \) is generated by \( t/2 \), which is the map

\[ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \mapsto z. \]

However, we always have either \( H^*_S(pt) = \mathbb{Z}[t] \), or \( H^*_S(pt) = \mathbb{Z}[t/2] \).

Since \([h,e_\alpha] = 2e_\alpha\) for each simple root \( \alpha \), we have \([h,e] = 2e\), from which we observe that \( S \) normalizes \( G^e \), cf. [Kos96, Theorem 10], resulting in an action of the semidirect product \( S \ltimes G^e \) on the Peterson variety.

The following was proved in classical types by Tymoczko [Tym07, Thm 4.3] and generalized to all Lie types by Precup [Pre18].

**Proposition 3.3.** For \( I \subset \Delta \), let \( w_I \) denote the longest element in the Weyl subgroup \( W_I \).

(a) The intersection \( P \cap Bw_B/B \) is nonempty if and only if \( w = w_I \) for some subset \( I \subset \Delta \).

(b) The set theoretic intersection \( P^I := P \cap Bw_IB/B \) is an affine space of dimension \(|I|\). In particular, its closure \( P_I \) is an irreducible subvariety of \( X_{w_I} \).

Some of the details proving part (b) are implicit in [Bål17]. We take the opportunity to make these details explicit in Proposition A.1 below. We will refer to \( P^I \) as a Peterson cell; its closure \( P_I \subset X_{w_I} \) is an irreducible variety, and the Schubert cell decomposition of Schubert varieties yields an affine paving

\[ P_I = \bigcup_{J \subset I} P_J^I. \]

Following [Gra01, Prop 2.1(a)], the classes \( \{[P_I]_S | I \subset \Delta \} \) form a basis of \( H^S_*(P) \). Observe that \( S \subset T \) is a regular subtorus, thus the fixed point loci for \( S \) and \( T \) in \( G/B \) coincide, i.e., \( (G/B)^T = (G/B)^S \); see e.g. [Hum75, §24.2, §24.3]. It follows that

\[ P^S = (G/B)^S \cap P = (G/B)^T \cap P = \{w_I : I \subset \Delta\}, \]

where we utilize the usual identification \( (G/B)^T = W \).

For \( I \subset \Delta \), an element \( v \in W \) is called a Coxeter element for \( I \) if \( v = s_{\alpha_1} \cdots s_{\alpha_k} \) for some enumeration \( \alpha_1, \ldots, \alpha_k \) of \( I \). Recall the following result, cf. [TT16, Lemma 7]:

\[ \]
**Proposition 3.4.** Let $v_I$ be a Coxeter element for some subset $I \subset \Delta$. Then the intersection $X^{v_I} \cap P_I$ is the single (possibly non-reduced) point $w_I$.

**Proof.** The intersection $Y := X^{v_I} \cap P_I$ is proper and $S$-stable. Any fixed point in $Y_S \subset P_I^S$ is of the form $w_J$, for some $J \subset I$. On the other hand, since $w_J \in X^{v_I}$, we have $w_J \geq v_I$. Since $v_I$ is a Coxeter element for $W_I$, $I \subset J$, and so $I = J$. Thus $Y$ contains a unique $S$-fixed point; hence by [Bor91, Prop 13.5], we have $Y = \{w_I\}$. □

**Corollary 3.5.** Let $\eta_I \in H^*_S(G/B)$ be the Poincaré dual of $[P_I]_S \in H^*_S(G/B)$, $v_I$ a Coxeter element for $I$, and $\tau_{w_I}$ the Poincaré dual of the point class $[w_I]_S$. Then

$$\sigma_{v_I} \cup \eta_I = m(v_I)\tau_{w_I} \quad \text{and} \quad \int_{G/B} \sigma_{v_I} \cup \eta_I = m(v_I),$$

where $m(v_I)$ is the multiplicity of $w_I$ in the intersection $X^{v_I} \cap P_I$.

**Proof.** Observe from Lemma 2.1 that $H^*_S(G/B)$ is torsion-free, and hence the localization map $H^*_S(G/B) \to \bigoplus_{w \in W} H^*_S(w)$ is injective (over $\mathbb{Z}$); see [GKM98, Cor 1.3.2, Thm 1.6.2] and [Hsi75, Thm 3.1]. By Proposition 3.4 the only potentially non-zero localization of $\sigma_{v_I} \cup \eta_I$ is at $w_I$, and therefore $\sigma_{v_I} \cup \eta_I = m(v_I)\tau_{w_I}$ for some integer $m(v_I)$. Under the specialization $H^*_S(G/B) \to H^*(G/B)$, the class $\tau_{w_I}$ maps to $1 \in H^*(G/B)$. It now follows from [Ful97, Eq (31)] that $m(v_I)$ is the multiplicity of the intersection $X^{v_I} \cap P_I$. □

In Section 4 we provide a formula for $m(v_I)$ based on equivariant localization, and compute the value of $m(v_I)$ for certain Coxeter elements $v_I$.

### 4. Poincaré duality and consequences

Let $G$ be a complex semisimple group, and $\iota : P \to G/B$ the corresponding Peterson variety, as in Section 3. In Theorem 4.3, we construct a basis $\{p_I\}_{I \subset \Delta}$ of $H^*_S(P)$ dual (up to scaling) to the basis $\{[P_I]_S\}_{I \subset \Delta}$ of $H^*_S(P)$. Theorem 4.3 relates the Schubert expansion of a Peterson class $[P_I]_S$ to the expansion in the $\{p_I\}$ basis of the pull-backs $\iota^*\sigma_w$; the latter can be computed using equivariant localization and Gaussian elimination. We sketch an example in Section 4.2.

#### 4.1. Peterson classes and duality.

**Lemma 4.1.** Let $I \subset \Delta$, and consider the expansion

$$\iota_*[P_I]_S = \sum_{w \in W} c^I_v[X_v]_S \in H^*_S(G/B).$$

Then $c^I_v = 0$ unless $v \leq w_I$.

**Proof.** By Lemma 2.1 the equivariant homology $H^*_S(X_{w_I})$ has a $H^*_S(pt)$-basis given by the fundamental classes $[X_v]_S$, where $v \leq w_I$. Since $P_I$ is a subvariety of $X_{w_I}$, we have $\iota_*[P_I]_S = \sum_{v \leq w_I} c^I_v[X_v]_S$, for some $c^I_v \in H^*_S(pt)$. □
Lemma 4.2. Let $I \subset \Delta$, and consider the expansion
\[ \iota_\ast [P_I]_S = \sum_{v \in W} c^v_I [X_v]_S \in H^*_S(G/B). \]
If $v$ is a Coxeter element for $J \neq I$, then $c^v_I = 0$.

Proof. Suppose $v$ is a Coxeter element for some subset $J \subset \Delta$ for which $c^v_J \neq 0$. Following Lemma 4.1, we have $v \leq w_I$, hence $J \subset I$. On the other hand, since the expansion is homogeneous, we have $|J| = \ell(v) \geq \dim P_I = |I|$, and hence $J = I$. \qed

Theorem 4.3 (Duality Theorem). Let $I, J$ be subsets of the set of simple roots $\Delta$, and let $v_I$ be a Coxeter element for $I$. We have
\[ \langle \iota_\ast \sigma_{v_I}, [P_J]_S \rangle = m(v_I) \delta_{I,J}, \]
where $m(v_I)$ is the multiplicity of the (unique) intersection point of $X^{v_I} \cap P_I$. In particular, $m(v_I)$ is a positive integer.

Proof. Consider the Schubert expansion $\iota_\ast [P_J]_S = \sum c^v_J [X_v]_S$. Then
\[ \langle \iota_\ast \sigma_{v_I}, [P_J]_S \rangle = \langle \sigma_{v_I}, \iota_\ast [P_J]_S \rangle = c^v_I, \]
since the set $\{ \sigma_v \}_{v \in W}$ forms a dual basis to the fundamental classes $\{ [X_v]_S \}_{v \in W}$. It follows from from Lemma 4.2 that $c^v_J = 0$ for $I \neq J$. For $J = I$, Corollary 4.4 implies
\[ c^v_I = \langle \sigma_{v_I}, \iota_\ast [P_I]_S \rangle = \int_X \sigma_{v_I} \cup \eta_I = m(v_I) > 0. \]
Finally, $m(v_I) \in \mathbb{Z}_+$ because the pairing (5) has values in integral cohomology. \qed

We record the following consequence of the Duality theorem.

Corollary 4.4. For each $I \subset \Delta$, fix a Coxeter element $v_I$, and set $p_I := \iota_\ast \sigma_{v_I} \in H^*_S(P)$.

(a) The classes $\left\{ \frac{p_I}{m(v_I)} \in H^*_S(P) \right\}_{I \subset \Delta}$ form a $H^*_S(pt)$-basis of $H^*_S(P)$.

(b) The map $\iota_\ast : H^*_S(P) \to H^*_S(G/B)$ is injective.

Proof. By Theorem 4.3, the classes $\frac{p_I}{m(v_I)}$ are dual to the classes $[P_I]_S$, and part (a) follows from Lemma 2.1. For part (b), observe that the pairing
\[ \langle \sigma_{v_J}, \iota_\ast [P_I]_S \rangle = m(v_I) \delta_{I,J}, \]
along with the linear independence of the $\sigma_{v_J}$ in $H^*_S(G/B)$, implies that the $\iota_\ast [P_I]_S$ are linearly independent. It follows that the map $\iota_\ast : H^*_S(P) \to H^*_S(G/B)$ is injective. \qed

Remark 4.5. Part (a) of Corollary 4.4 was proved in various generalities, and for particular choices of Coxeter elements $v_I$, in [Dre15] [IT16] [AHKZ24]. The non-equivariant version of part (b) was proved in [IT16 Thm 2].

We also record the following immediate corollary, which will be utilized in the proof of the positivity statement Theorem 5.2.
Corollary 4.6. For each $I \subset \Delta$, fix a Coxeter element $v_I$, and set $p_I := \iota^* \sigma_{v_I} \in H^*_S(P)$. Consider the expansions

$$
\iota^* \sigma_w = \sum_{J \subset \Delta} b^I_w p_J, \quad \iota^* [P_I]_S = \sum_{u \in W} c^I_u [X_u]_S.
$$

Then $c^I_u = m(v_I) b^I_u$ for all $u$, where $m(v_I) > 0$ is the coefficient from the Duality Theorem 4.3.

Proof. Using Theorem 4.3 and the equality $\langle \sigma_v, [X_u]_S \rangle_{G/B} = \delta_{u,v}$, we calculate,

$$
c^I_u = \langle \sigma_u, \iota^*[P_I]_S \rangle_{G/B} = \langle \iota^* \sigma_u, [P_I]_S \rangle_P = m(v_I) b^I_u.
$$

Here the first equality follows from the definition of $c^I_u$, the second from the projection formula, and the third from Theorem 4.3 together with the definition of $b^I_u$. \hfill \square

4.2. Schubert Expansion of the Peterson Classes. In their study of certain regular Hessenberg varieties, Abe, Fujita and Zeng [AFZ20] found a beautiful closed formula for the non-equivariant Schubert expansions of the fundamental classes of these varieties. For the Peterson varieties discussed here, their formula states that

$$
\iota^*[P] = \prod_{\alpha \in \Phi^+_\Delta \setminus \Delta} c_1(G \times_B C_{-\alpha}) \cap [G/B] \in H_*(G/B);
$$

see Cor. 3.9 in loc. cit. However, since the line bundles $G \times B C_{-\alpha}$ are not globally generated, this formula involves cancellations. A manifestly positive formula was recently found by Nadeau and Tewari [NT23], and further investigated by Horiguchi [Hor24], in relation to mixed Eulerian numbers. The origins of this approach lie in the realization of the Peterson variety as a flat degeneration of a smooth projective toric variety, called the (generalized) permutohedral variety; see [AFZ20] NT23. The permutohedral variety is a regular semisimple Hessenberg variety; its cohomology ring has been classically studied e.g. by Klyachko [Kly85] Kly95. In this section we present a different algorithm, which calculates the equivariant Schubert expansion of $\iota^*[P]_S$. The algorithm is based on Corollary 4.6 and it depends on the multiplicities $m(v_I)$ for some choice of Coxeter elements $v_I$, $I \subset \Delta$. The values $m(v_I)$ for a particular such choice are computed in Theorem 7.6. It would be interesting to utilize this algorithm to extend the formulae from [AFZ20] NT23 to the equivariant setting; this will be left for elsewhere.

Proposition 4.7. Fix Coxeter elements $v_I$ for each subset $I \subset \Delta$, and consider the matrices,

$$
A_{u,I} = \iota^* \sigma_u, \quad C_{I,J} = \iota^* \sigma_{v_I}, \quad M_{I,J} = m(v_I) \delta_{I,J}.
$$

Here $A$ is a $|W| \times 2^{|\Delta|}$ matrix, and $C$ and $M$ are $2^{|\Delta|} \times 2^{|\Delta|}$ matrices. The fundamental classes $[P_I]_S$ and $[X_u]_S$ are related by the matrix equation,

$$
([P_I]_S)_{I \subset \Delta} = (AC^{-1}M)^T ([X_u]_S)_{u \in W}.
$$

(8)
Proof. Consider the commutative diagram,

$$
\begin{array}{ccc}
H^*_S(G/B) & \xrightarrow{\oplus u^*_S} & \bigoplus_{u \in W} H^*_S(u) \\
\downarrow^{i^*} & & \downarrow \\
H^*_S(P) & \xrightarrow{\oplus \iota_{uI}^*} & \bigoplus_{I \subseteq \Delta} H^*_S(w_I).
\end{array}
$$

Let $Q$ be the fraction field of the integral domain $H^*_S(pt)$, and let $R_Q := R \otimes_{H^*_S(pt)} Q$ for any $H^*_S(pt)$-module $R$. The map $H^*_S(P) \to \bigoplus_{I \subseteq \Delta} H^*_S(w_I)$ induces an isomorphism $H^*_S(P)_Q \xrightarrow{\sim} \bigoplus_{I \subseteq \Delta} H^*_S(w_I)_Q$; see [GKM98, Cor 1.3.2, Thm 1.6.2, Thm 6.3]. Observe that $H^*_S(P)$ is torsion-free, and is naturally identified as a lattice in the $Q$-vector space $H^*_S(P)_Q$. Let $\tau_I$ denote a generator of $H^*_S(w_I)_Q$, and consider the column vectors,

$$
\sigma = (\iota^* \sigma_u)_{u \in W}, \quad \tau = (\tau_I)_{I \subseteq \Delta}, \quad p = (p_I)_{I \subseteq \Delta}, \quad q = \left(\frac{p_I}{m(v_I)}\right)_{I \subseteq \Delta}.
$$

We have the following equalities in $H^*_S(P)_Q$:

$$
(10) \quad p = Mq, \quad \sigma = A\tau, \quad p = C\tau.
$$

The matrix $C$ is invertible since both $\{p_I\}_{I \subseteq \Delta}$ and $\{\tau_I\}_{I \subseteq \Delta}$ are bases for $H^*_S(P)_Q$. We deduce that $\sigma = AC^{-1}Mq$. Equation (8) now follows from Corollary 4.6. $\square$

Remark 4.8. The coefficients $A_{w,I}$ and $C_{I,J}$ in Proposition 4.7 can be computed by composing the localization formula for the $T$-equivariant Schubert classes (cf. [AJS94, Bil99]) with the restriction map $X(T) \to X(S)$ defined by $\lambda \mapsto \lambda|S$.

Remark 4.9. The invertibility of the matrix $C$ in Proposition 4.7 can be directly deduced from the observation that $\iota_{wJ}^* \sigma_{vI} \neq 0$ if and only if $I \subseteq J$, and hence $C$ is upper triangular with respect to the partial order $I \leq J \iff I \subseteq J$.

Example 4.10. We use Proposition 4.7 to compute the Schubert expansion of $[P]_S$ in the case $\Delta = B_2$, with $v_\Delta = s_1s_2$. Set

$$
p_\phi = \iota^* \sigma_{id}, \quad p_{\{1\}} = \iota^* \sigma_1, \quad p_{\{2\}} = \iota^* \sigma_2, \quad p_{\{1,2\}} = \iota^* \sigma_{12}.
$$
Composing the localization formula for Schubert classes (cf. [AJS94, Bil99]) with the restriction map \( X(T) \to X(S) \), we obtain the \( S \)-equivariant localizations of the Schubert classes:

\[
\begin{pmatrix}
\iota^*\sigma_{id} \\
\iota^*\sigma_1 \\
\iota^*\sigma_2 \\
\iota^*\sigma_{12} \\
\iota^*\sigma_{21} \\
\iota^*\sigma_{121} \\
\iota^*\sigma_{1212}
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & t & 0 & 4t \\
0 & 0 & t & 3t \\
0 & 0 & 0 & 6t^2 \\
0 & 0 & 0 & 6t^2 \\
0 & 0 & 0 & 6t^3 \\
0 & 0 & 0 & 6t^4
\end{pmatrix} \begin{pmatrix}
\tau_0 \\
\tau_1 \\
\tau_2 \\
\tau_{12}
\end{pmatrix}.
\]

The \( 8 \times 4 \) matrix in Equation (11) corresponds to the matrix \( A \) in Equation (10), and the matrix \( C \) is precisely its top \( 4 \times 4 \) submatrix. The multiplicities \( m(v_I) \) are computed in Theorem 7.6; we have \( m(v_I) = 1 \) for all \( I \subset B_2 \) and \( m(v_{\Delta}) = 2 \), i.e.,

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

Applying Equation (5), we obtain

\[
\begin{pmatrix}
\iota_*[P_{\phi}]_S \\
\iota_*[P_{\{1\}}]_S \\
\iota_*[P_{\{2\}}]_S \\
\iota_*[P]_S
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix} \begin{pmatrix}
\iota_*[X_{id}]_S \\
\iota_*[X_1]_S \\
\iota_*[X_2]_S \\
\iota_*[X_{\Delta}]_S
\end{pmatrix} = \begin{pmatrix}
\iota_*[X_{id}]_S \\
\iota_*[X_1]_S \\
\iota_*[X_2]_S \\
\iota_*[X_{\Delta}]_S
\end{pmatrix}.
\]

In particular, we have \( \iota_*[P]_S = 2[X_{12}]_S + 2[X_{21}]_S + 2t[X_{121}]_S + 2t[X_{212}]_S + 2t^2[X_{1212}]_S \).

5. Positivity

We recall a theorem of Graham [Gra01, Thm. 3.2], which plays a key role in the proof of our positivity results, Theorems 5.2 and 5.3.

**Theorem 5.1.** Let \( B' \) be a connected solvable group with unipotent radical \( N' \), and let \( T' \subset B' \) be a maximal torus, so that \( B' = T'N' \). Let \( \alpha_1, \ldots, \alpha_d \) be the weights of \( T' \) acting on \( \text{Lie}(N') \). Let \( X \) be a scheme with a \( B' \)-action, and \( Y \) a \( T' \)-stable subvariety of \( X \). Then there exist \( B' \)-stable subvarieties \( D_1, \ldots, D_k \) of \( X \) such that in the equivariant homology \( H^*_T(X) \),

\[
[Y]_{T'} = \sum f_i [D_i]_{T'},
\]

where each \( f_i \in H^*_T(pt) \) is a linear combination of monomials in \( \alpha_1, \ldots, \alpha_d \) with non-negative integer coefficients.
Theorem 5.2. Let $I$ be a subset of $\Delta$, let $\iota : P \hookrightarrow G/B$ be the inclusion, and consider the Schubert expansion,
\[ \iota_*[P_I] \cdot s = \sum_{v \in W} c^v_I[X_v] \cdot s. \]
Then $c^v_I \in H^*_S(pt)$ is a polynomial in $t$ with non-negative coefficients.

Proof. We apply Graham’s positivity theorem to the following situation: $Y = P_I \subset X = G/B$, $T' = S$, and $B' = SU$, where $U$ is the unipotent radical of $B$. We have $U \subset B' \subset B$, and since the $U$-orbits and $B$-orbits in $G/B$ coincide, the $B'$-orbits in $G/B$ are precisely the Schubert cells $X_v^\circ$.

Observe that the restriction map $X(T) \to X(S)$ is given by $\alpha \mapsto ht(\alpha)t$ for $\alpha \in \Phi^+_\Delta$, where $ht(\alpha)$ is the height of $\alpha$. It follows that the weights for the $S$-action on $\text{Lie}(U)$ are positive integer multiples of $t$. It follows from Theorem 5.1 that each $c^v_I \in H^*_S(pt)$ is a polynomial in $t$ with non-negative coefficients. \end{proof}

Theorem 5.3. Let $p_I := \iota^*\sigma_{v_I} \in H^*_S(P)$ for some Coxeter element $v_I$, and consider the multiplication in $H^*_S(P)$,
\[ p_I \cdot p_J = \sum_{K \subset \Delta} c^K_{I,J} p_K. \]
The structure constants $c^K_{I,J} \in H^*_S(pt)$ are polynomials in $t$ with non-negative coefficients.

Proof. By Graham’s equivariant positivity theorem [Gra01, Prop 2.2, Thm 3.2], the structure constants $c^K_{u,v}$ in the expansion
\[ \sigma_u \cdot \sigma_v = \sum c^K_{u,v} \sigma_w \in H^*_T(G/B) \]
are polynomials in the $T$-weights of $\text{Lie}(U)$ with non-negative coefficients. Then
\[ p_I \cdot p_J = \iota^*\sigma_{v_I} \cdot \iota^*\sigma_{v_J} = \sum d^K_{u,v} \iota^*\sigma_w, \]
where $d^K_{u,v}$ is the image of $c^K_{u,v}$ under the restriction map $X(T) \to X(S)$; in particular, $d^K_{u,v}$ is a polynomial in $t$ with non-negative coefficients. The result now follows from Theorem 5.2 and Corollary 4.6 since the classes $\iota^*\sigma_w$ expand into the classes $p_K$ with coefficients having the same positivity property as the $d^K_{u,v}$. \end{proof}

Remark 5.4. In the recent preprint [GG22], Goldin and Gorbutt found a manifestly positive formula for the coefficients $c^K_{I,J}$, in Lie type $A$, and for a particular choice of the Coxeter elements $v_I$. While this paper was in preparation, a different combinatorial model, in the non-equivariant cohomology, appeared in the preprint [AHK24] by Abe, Horiguchi, Kuwata and Zeng. They also provide a geometric proof of positivity (cf. Prop. 4.15 in loc. cit.), which utilizes a ‘Giambelli formula’, writing the classes $p_I$ in terms of products of pull-backs of the (effective) line bundles $GL_n \times^B C_{-\omega_i}$ associated to the fundamental weights $\omega_i$. This argument should extend to arbitrary Lie type if one utilizes instead the more general equivariant Giambelli formulae obtained by Drellich [Dre15], specialized to ordinary cohomology.
6. Stability Properties

In this section, we utilize a common alternate construction of the Peterson variety in order to prove a stability property of Peterson varieties. For each finite-type Dynkin diagram \( \Delta \), we construct a variety \( P(\Delta) \) inside the flag manifold \( F(\Delta) \) which is isomorphic to the Peterson variety \( P \) corresponding to any group \( G \) whose Dynkin diagram is \( \Delta \). The equality \( P(\Delta) = P \) is well-known to experts; in Appendix \( \text{A} \) we present a proof following Kostant [Kos96].

For \( I \subset \Delta \), we show that there is a natural inclusion \( P(I) \hookrightarrow P(\Delta) \) identifying \( P(I) \) with the Peterson cell closure \( P_I \). This implies that the fundamental classes \( [P_K] \) and the cohomology classes \( p_K \) are stable for the inclusion \( P(I) \hookrightarrow P(\Delta) \), and that the Peterson Schubert varieties of [IT16] are simply Peterson varieties corresponding to smaller groups.

6.1. The Flag Manifold of a Dynkin diagram. Let \( \Phi_\Delta \) (resp. \( \Phi_\Delta^+ \), \( W_\Delta \)) denote the root system (resp. positive roots, Weyl group) corresponding to a finite-type Dynkin diagram \( \Delta \). Following [Tit66], let \( g_\Delta \) denote the root system (resp. positive roots, Weyl group) corresponding to \( g \). We fix a connected Lie group \( G \) whose Dynkin diagram is \( \Delta \). Let \( \Phi_\Delta \) (resp. \( \Phi_\Delta^+ \), \( W_\Delta \)) denote the root system (resp. positive roots, Weyl group) corresponding to \( \Delta \). We show that there is a natural inclusion \( P(I) \hookrightarrow P(\Delta) \) identifying \( P(I) \) with the Peterson cell closure \( P_I \). This implies that the fundamental classes \( [P_K] \) and the cohomology classes \( p_K \) are stable for the inclusion \( P(I) \hookrightarrow P(\Delta) \), and that the Peterson Schubert varieties of [IT16] are simply Peterson varieties corresponding to smaller groups.

Equation (12).

The stabilizer of \( b_\Delta \) in \( G \) is the Borel subgroup \( B \subset G \) satisfying \( b_\Delta = Lie(B) \), hence we have the usual \( G \)-equivariant identification,

\[
\varphi : G/B \simeq F(\Delta).
\]

For \( I \subset \Delta \), the subalgebra of \( g_\Delta \) spanned by \( \{e_\alpha, h_\alpha\}_{\alpha \in \Phi_I} \) is precisely the Lie algebra \( g_I \) associated to the Dynkin diagram \( I \). We have \( h_I = h \cap g_I \) and \( b_I = b_\Delta \cap g_I \). Let \( T_I, B_I \), and \( G_I \) be the connected subgroups of \( G \) corresponding to \( h_I, b_I \) and \( g_I \) respectively. The induced map \( G_I/B_I \to G/B \) corresponds to an embedding \( F(I) \to F(\Delta) \) via Equation (12). In Equation (14), we give a characterization of this embedding in terms of Equation (12).

**Lemma 6.1.** If \( u \subset g_\Delta \) is a \( |\Phi_\Delta^+| \)-dimensional subalgebra containing only nilpotent elements, then its normalizer \( N(u) = \{ x \in g \mid ad(x)u \subset u \} \) is a Borel subalgebra of \( g_\Delta \).

**Proof.** Following [Bou05] p. 162, Cor 2], every subalgebra \( u \) containing only nilpotent elements is contained in some Borel subalgebra \( b \), and further, \( u \subset [b, b] \) [Bou05] p. 91, Prop 5(b)]. Comparing dimensions, we deduce that \( u = [b, b] \), and hence \( N(u) = b \). □
Let $b'_I$ be any Borel subalgebra of $g_I$. Observe that $v_I = \bigoplus_{\alpha \in \Phi^+_\Delta} g_{\alpha}$ is $g_I$-stable, and hence it is an ideal in the $|\Phi^+_\Delta|$-dimensional subalgebra $[b'_I, b'_I] \oplus v_I$. By [Bor98, p. 71, Lemma 1], we see that $[b'_I, b'_I] \oplus v_I$ is a $|\Phi^+_\Delta|$-dimensional subalgebra of $g_\Delta$ containing only nilpotent elements. Following Lemma 6.1, we see that $N([b'_I, b'_I] \oplus v_I)$ is a Borel subalgebra of $g_\Delta$. Hence we have an embedding,

$$\text{(14)} \quad i : F(I) \to F(\Delta), \quad b'_I \mapsto N([b'_I, b'_I] \oplus v_I).$$

The embedding $i : F(I) \to F(\Delta)$ is $G_I$-equivariant, and sends $b_I$ to $b_\Delta$. It follows that under the identifications $F(I) = G_I/B_I$ and $F(\Delta) = G/B$ of Equation (13), the map $i$ is precisely the map $G_I/B_I \to G/B$ induced by the inclusion $G_I \hookrightarrow G$; observe that $B_I = B \cap G_I$ follows from, e.g., [Bor91, §11.2, Corollary and Thm. 11.16].

We will say that a map of Lie groups $F : G_1 \to G_2$ lifts a Lie algebra map $f : g_1 \to g_2$ if $\operatorname{Lie}(G_i) = g_i$ for $i = 1, 2$, and $f$ is the differential of $F$ at the identity.

**Remark 6.2.** The inclusion $i : F(I) \to F(\Delta)$ is $f$-equivariant for any map $f : G'_1 \to G$ lifting the inclusion $g_I \to g_\Delta$.

**Lemma 6.3.** Fix $w \in W_I$, and let $b'_w = h_I \oplus \bigoplus_{\alpha \in \Phi^+_\Delta} g_{\alpha} \cap \mathfrak{g}_w$, and $b_w = h_\Delta \oplus \bigoplus_{\alpha \in \Phi^+_\Delta} g_{\alpha}$. Consider the Schubert varieties

$$X^I_w = \text{Ad}(B_I)b'_w \subset F(I) \quad \text{and} \quad X_w = \text{Ad}(B)b_w \subset F(\Delta).$$

Then $i(b'_w) = b_w$ and $i(X^I_w) = X_w$. We view the $X^I_w$ as $B$-varieties via this identification. Consider the Schubert classes $\sigma_w \in H^*_T(F(I))$ and $\sigma'_w \in H^*_T(F(I))$. We have

$$i_*[X^I_w]_T = [X_w]_T, \quad i^*\sigma_w = \sigma'_w.$$

**Proof.** Since $w \in W_I$, we have $w(\Phi^+_\Delta \setminus \Phi^+_I) = \Phi^+_\Delta \setminus \Phi^+_I$, and hence

$$\bigoplus_{\alpha \in \Phi^+_\Delta} g_{\alpha} = v_I \oplus \bigoplus_{\alpha \in \Phi^+_I} g_{\alpha} = [b'_I, b'_I] \oplus v_I.$$ 

It follows that $i(b'_w) = b_w$. Next, since $B_I \subset B$, we have $i(X^I_w) \subset X_w$. Further, both varieties are irreducible of dimension $l(w)$, hence they are equal. Consequently, we have $i_*[X^I_w]_T = [X_w]_T$; since the Schubert classes $\sigma_w$ (resp. $\sigma'_w$) are dual to the fundamental classes $[X_w]_T$ (resp. $[X^I_w]_T$), we further obtain $i^*\sigma_w = \sigma'_w$. \hfill \square

### 6.2. The Peterson Variety

Given a Borel subalgebra $b \subset g_\Delta$, let $h$ be a Cartan subalgebra of $b$, let $\Phi_h$ denote the root system of $(g_\Delta, h)$, and let $\Delta_h \subset \Phi_h$ be the set of simple roots for which $b$ is the Borel subalgebra corresponding to the positive roots. We define

$$\text{(15)} \quad \mathcal{H}(b) = b \oplus \bigoplus_{\alpha \in \Delta_h} g_{-\alpha, h}$$

where $g_{\alpha, h}$ is the root space corresponding to $\alpha \in h^*$. 
Observe that the subspace $H(b)$ is independent of the choice of $h$. Indeed, any two Cartan subgroups $h$ and $h'$ of $b$ are conjugate via an inner automorphism of $b$ [Bou05, Ch 7, §3, Prop 5]. Since $H(b)$ is stable under the adjoint action of $b$, the automorphism preserves $H(b)$. (Alternatively, $H(b) = [u, u]^\perp$, where $u$ is the nilpotent radical of $b$, and $\perp$ is taken with respect to the Killing form.)

**Definition 6.4.** Let $e := \sum_{\alpha \in \Delta} e_\alpha$. The Peterson variety $P(\Delta)$ is defined by

$$P(\Delta) := \{ b \in Fl(\Delta) \mid e \in H(b) \}.$$ 

We recall that $e$ is a regular nilpotent element of $g_\Delta$. Under the $G$-equivariant isomorphism $G/B \sim Fl(\Delta)$ from Equation (13) we have

$$P(\Delta) = \left\{ gB \in G/B \mid e \in H(Ad(b) \Delta) \right\}$$

(16) $$= \left\{ gB \in G/B \mid Ad(g^{-1}) e \in H(b) = b_\Delta \oplus \bigoplus_{\alpha \in \Delta} C e_{-\alpha} \right\}.$$ 

Let $G, G_I$, and $\phi : G_I \to G$ be as in Section 6.1 and let $S_I \subset T_I$ be the one-dimensional torus corresponding to $h_I = \sum_{\alpha \in \Phi_I^+} \alpha \vee$.

**Proposition 6.5.** Consider the map $i : Fl(I) \hookrightarrow Fl(\Delta)$ from Equation (14). Then $i(P(I)) = P_I$, as algebraic varieties. Furthermore, $P_I$ is also equal to the set theoretic intersection $P(\Delta) \cap Fl(I)$.

**Proof.** Let $e_I = \sum_{\alpha \in I} e_\alpha$ and $e_T = \sum_{\alpha \in \Delta \backslash I} e_\alpha$, so that $e = e_I + e_T$. Recall that

$$P(I) = \{ b_I' \in Fl(I) \mid e_I \in H(b_I') \}.$$ 

Consider $b_I' \in Fl(I)$, and set $i(b_I') = b'$. We see from Equations (14) and (15) that

$$H(b_I') \oplus v_I \subset H(b').$$

Suppose $b_I' \in P(I)$. We have $e_I \in v_I$, and hence

$$e_I \in H(b_I') \implies e = e_I + e_T \in H(b_I') \oplus v_I \subset H(b') \implies b' \in P(\Delta).$$

We deduce that $i(P(I)) \subset P(\Delta)$.

Using the natural basis $\{e_\alpha, h_\alpha\}_{\alpha \in \Phi_\Delta}$ for $g_\Delta$, and its sub-basis of $g_I$, consider the $g_I$-equivariant projection $pr : g_\Delta \to g_I$ defined by:

$$pr(e_\alpha) = \begin{cases} e_\alpha & \text{if } \alpha \in \Phi_I, \\ 0 & \text{otherwise.} \end{cases}$$

$$pr(h_\alpha) = \begin{cases} h_\alpha & \text{if } \alpha \in \Phi_I, \\ 0 & \text{otherwise.} \end{cases}$$

Now, suppose $b' \in P(\Delta)$. Then $e \in H(b')$, and hence

$$e_I = pr(e) \in pr(H(b')) = H(b_I').$$

It follows that $Fl(I) \cap P(\Delta) = P(I)$. The equality $P(I) = P_I$ is a consequence of the observation that $Fl(I) = X_{w_I}$, and the irreducibility of $P(I)$; see Lemma A.3.\]
We will denote by \( j : P(I) \rightarrow P(\Delta) \) the inclusion induced by restricting \( i \) to \( P(I) \). In order to discuss stability for Peterson classes, we first need to construct algebra homomorphisms \( H^*_S(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H^*_S(\mathcal{F}l(I); \mathbb{Q}) \), compatible with restrictions to Peterson subvarieties. To this end, we replace \( S_I \) and \( S \) by a \( \mathbb{C}^* \) ‘parametrizing’ (not necessarily injectively) these tori via the the defining cocharacters \( h_I : \mathbb{C}^* \rightarrow S_I \) and \( h : \mathbb{C}^* \rightarrow S \). This \( \mathbb{C}^* \) acts on \( \mathcal{F}l(I) \), respectively on \( \mathcal{F}l(\Delta) \), via its image \( S_I \subset T_I \) and \( S \subset T \). The embedding \( g_I \rightarrow g \) is \( \mathbb{C}^* \)-equivariant, and hence so is the embedding \( \mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta) \) described in (14). These facts are summarized in the diagram below. The question marks signify that a map may not exist; see Remark 6.7 below.

\[
\begin{array}{ccc}
\mathbb{C}^* & \xrightarrow{h_I} & S_I \\
\downarrow & & \downarrow \\
\mathcal{F}l(I) & \xrightarrow{\ ? \ } & S \\
\end{array}
\]

The cocharacter \( h \) induces an isomorphism \( \text{Lie}(\mathbb{C}^*) \rightarrow \text{Lie}(S) \), and hence a ring isomorphism \( H^*_{\mathbb{C}^*}(pt; \mathbb{Q}) \rightarrow H^*_{\mathbb{C}^*}(pt; \mathbb{Q}) \). (In general the corresponding map over integer coefficients, \( H^*_{\mathbb{C}^*}(pt; \mathbb{Z}) \rightarrow H^*_{\mathbb{C}^*}(pt; \mathbb{Z}) \), may not be an isomorphism.) The identity map \( \mathcal{F}l(\Delta) \rightarrow \mathcal{F}l(\Delta) \) is equivariant with respect to the cocharacter \( h : \mathbb{C}^* \rightarrow S \), therefore by functoriality we have induced isomorphisms \( H^*_S(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H^*_S(\mathcal{F}l(\Delta); \mathbb{Q}) \) and \( H^*_C(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H^*_S(\mathcal{F}l(\Delta); \mathbb{Q}) \). Further, since \( P(\Delta) \) is \( S \)-stable, it inherits a \( \mathbb{C}^* \)-action through \( h \), giving isomorphisms

\[
H^*_S(P(\Delta); \mathbb{Q}) \sim H^*_C(P(\Delta); \mathbb{Q}) \quad \text{and} \quad H^*_C(P(\Delta); \mathbb{Q}) \sim H^*_S(P(\Delta); \mathbb{Q}).
\]

All these isomorphisms are natural with respect to the closed embedding \( P(\Delta) \subset \mathcal{F}l(\Delta) \). A similar discussion for the cocharacter \( h_I \) yields isomorphisms

\[
H^*_S(P(I); \mathbb{Q}) \sim H^*_C(P(I); \mathbb{Q}) \quad \text{and} \quad H^*_C(P(I); \mathbb{Q}) \sim H^*_S(P(I); \mathbb{Q}),
\]

natural with respect to \( P(I) \subset \mathcal{F}l(I) \). Consequently, the \( \mathbb{C}^* \)-equivariant inclusion \( j : P(I) \rightarrow P(\Delta) \) yields a pullback map,

\[
H^*_S(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H^*_S(\mathcal{F}l(I); \mathbb{Q})
\]

compatible with the algebra isomorphism \( H^*_{\mathbb{C}^*}(pt; \mathbb{Q}) \rightarrow H^*_{\mathbb{C}^*}(pt; \mathbb{Q}) \), and we obtain a commutative diagram,

\[
H^*_S(P(\Delta); \mathbb{Q}) \xrightarrow{j^*} H^*_S(P(I); \mathbb{Q}) \quad \text{(17)}
\]

In a similar fashion, we also obtain a pushforward \( j_* : H^*_S(P(I); \mathbb{Q}) \rightarrow H^*_S(P(\Delta); \mathbb{Q}) \).

The following is the main result of this section.
Theorem 6.6. Consider the map $i : \mathcal{F}l(I) \hookrightarrow \mathcal{F}l(\Delta)$ from Equation (14).

(a) For $J \subset I$, we have $i_*[P_J]_{S_I} = [P_J]_{S_I}$ in $H^S(\mathcal{F}l(\Delta); \mathbb{Q})$.

(b) Let $j^* : H^S_S(\mathcal{P}(\Delta); \mathbb{Q}) \to H^S_I(\mathcal{P}(I); \mathbb{Q})$ denote the pullback induced from the inclusion $\mathcal{P}(I) \hookrightarrow \mathcal{P}(\Delta)$. For $K \subset \Delta$, we have

$$j^*p_K = \begin{cases} p_K & \text{if } K \subset I, \\ 0 & \text{otherwise.} \end{cases}$$

In the non-equivariant case, the equalities in (a) and (b) hold with integral coefficients.

Proof. For $J \subset I \subset \Delta$, the inclusions $\mathcal{F}l(J) \hookrightarrow \mathcal{F}l(I) \hookrightarrow \mathcal{F}l(\Delta)$ are $\mathbb{C}^*$-equivariant for the action given by the cocharacters $h_J, h_I$ and $h$, respectively. By Proposition 6.3, we have $i'(\mathcal{P}(J)) = P_J \subset \mathcal{P}(I)$ and $i'(\mathcal{P}(J)) = P_J \subset \mathcal{P}(\Delta)$, and consequently $[P_J]_{\mathbb{C}^*} = i_*([P_J])_{\mathbb{C}^*} = i_*([P_J])_{\mathbb{C}^*}$ in $H^S_C(\mathcal{F}l(\Delta))$. Then part (a) follows because the $\mathbb{C}^*$-equivariance may be replaced by the $S_I$, respectively $S$-equivariance, as explained above. Part (b) follows from Lemma 6.5 and the commutativity of the diagram,

$$\begin{array}{ccc}
P(I) & \xrightarrow{j} & P(\Delta) \\
\downarrow{i} & & \downarrow{i} \\
\mathcal{F}l(I) & \xrightarrow{i} & \mathcal{F}l(\Delta)
\end{array}$$

utilizing again that all maps are $\mathbb{C}^*$-equivariant.

In the non-equivariant case, all (co)homology morphisms are defined over $\mathbb{Z}$, and the classes $[P_I]$ and $p_I$ are integral, by their definition. This finishes the proof. \[\square\]

Remark 6.7. The reader may wonder whether an algebra map $H^S_S(\mathcal{F}l(\Delta)) \to H^S_I(\mathcal{F}l(I))$ may be directly constructed from the inclusion $i : \mathcal{F}l(I) \to \mathcal{F}l(\Delta)$, equivariant with respect to a map $\varphi_I : S_I \to S$. The requirement that $i$ is $\varphi_I$-equivariant implies that the differential $d \varphi_I : \text{Lie}(S_I) \to \text{Lie}(S)$ must send $h_I \mapsto h$. (Note that this is not the restriction of the natural map $\text{Lie}(T_I) \to \text{Lie}(T)$. The existence of a lift $S_I \to S$ of this Lie algebra map cannot be guaranteed. For instance, consider the inclusion $G_I := SL_3 \subset G := SL_4$ given by the natural embedding of Dynkin diagrams $A_2 \subset A_3$. The tori $S_I$ and $S$ are the images of cocharacters

$$h_I(z) = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} \quad \text{and} \quad h(z) = \begin{pmatrix} z^3 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-3} \end{pmatrix},$$

respectively. In this case there is no group homomorphism $\varphi_I : S_I \to S$ satisfying $\varphi_I(h_I(z)) = h(z)$.

Corollary 6.8. Let $I \subset \Delta$ and assume that the map $\text{Lie}(S_I) \to \text{Lie}(S)$ sending $h_I \mapsto h$ lifts to a map $\varphi_I : S_I \to S$. Then the push-forward and pull-back maps

$$j^* : H^S_S(\mathcal{P}(\Delta)) \to H^S_{S_I}(\mathcal{P}(I)) \quad \text{and} \quad j_* : H^S_{S_I}(\mathcal{P}(I)) \to H^S_S(\mathcal{P}(\Delta))$$
may be defined with \( \mathbb{Z} \) coefficients. In particular, the statements in Theorem 6.6 also hold over \( \mathbb{Z} \).

**Proof.** The claim follows because the \( G_I \)-equivariant map \( i : \mathcal{F}(I) \to \mathcal{F}(\Delta) \) from (14) restricts to the \( \varphi_I \)-equivariant map \( j : \mathcal{P}(I) \to \mathcal{P}(\Delta) \). Then \( j_* \) and \( j_*' \) may be defined over \( \mathbb{Z} \).

**Remark 6.9.** The results of this section can be extended to the case of reductive groups \( G \) and a one-dimensional torus \( S \subset T \) satisfying \( \alpha \mid S = \beta \mid S \) for all simple roots \( \alpha, \beta \). For \( G \) semisimple, there is a unique \( S \subset T \) satisfying this condition. For an arbitrary reductive group \( G \), this may not determine \( S \) uniquely.

It is common in the literature on type A Peterson varieties to use the group \( G = GL_n \) and the one-dimensional torus \( S = diag(z^n, z^{n-1}, \ldots, z) \). In this case, we have an identification between the one dimensional subtori of \( GL_n \) and \( GL_{n+1} \) given by \( diag(z^n, z^{n-1}, \ldots, z) \mapsto diag(z^{n+1}, z^n, \ldots, z) \). Then the diagram in (17), and hence the statements in Theorem 6.6 hold over \( \mathbb{Z} \).

### 7. Intersection multiplicities

Different choices of Coxeter elements \( v_I \) lead to different bases \( \{p_I = \iota^*\sigma_{v_I}\} \) for \( H_S^*(\mathcal{P}; \mathbb{Q}) \). By Theorem 4.3, the transition matrix between two such bases \( \{p_I\} \) and \( \{p'_I\} \) is diagonal, with entries given by ratios

\[
\frac{m(v_I)}{m(v'_I)} = \frac{\langle p_I, P_I | s \rangle}{\langle p'_I, P_I | s \rangle}.
\]

It is natural to ask whether there are choices for the Coxeter elements \( v_I \) for which \( m(v_I) = 1 \), and more generally, to ask for formulæ for the \( m(v_I) \). In Proposition 7.3, we give a formula for \( m(v_I) \) in terms of the localization of the Schubert variety \( X^{v_I} \) at the point \( w_I \), and in Theorem 7.6, we use this formulæ to compute \( m(v_I) \) for certain Coxeter elements \( v_I \). Theorem 7.6 settles Question 1 of [IT16] for all classical types. As a further application of Proposition 7.3(b), we show in Example 7.4 that not all choices of \( v_I \) lead to \( m(v_I) = 1 \) in type A, and in Example 7.5 that for \( I \in \{B_2, C_2\} \), there is no Coxeter element \( v_I \) for which \( m(v_I) = 1 \).

#### 7.1. The Exponents of a Dynkin diagram

Let \( \Delta \) be a Dynkin diagram with \( n \) nodes. The exponents \( m_1, \ldots, m_n \) of \( \Delta \) are fundamental invariants, appearing in many contexts. We will utilize the following two characterizations found in [Car72], Ch. 10; see also [Kos59]:

1. Let \( \mathfrak{g} \) be the Lie algebra with Dynkin diagram \( \Delta \), and let \( \{e, f, h\} \) be an \( \mathfrak{s}\mathfrak{l}_2 \)-triple in \( \mathfrak{g} \), such that \( e \) is a regular nilpotent element in \( \mathfrak{g} \); see [Mor42] [CM93]. The \( \mathfrak{s}\mathfrak{l}_2 \)-decomposition of \( \mathfrak{g} \) is precisely \( \oplus V(\text{dim} k) \), where \( V(k) \) denotes the irreducible finite dimensional \( \mathfrak{s}\mathfrak{l}_2 \)-representation with highest weight \( k \).

2. Let \( a_i \) be the number of roots of height \( i \) in \( \Phi^+_\Delta \). Then \( (a_1, \ldots, a_k) \) is a partition, and the conjugate partition is precisely \( (m_1, \ldots, m_n) \).

Throughout this section, we will denote by \( m_1, \ldots, m_n \), the exponents of \( \Delta \).
Lemma 7.1. The weights for the $S$-action on $\text{Lie}(G^e)$ are precisely $m_1 t, \cdots, m_n t$.

Proof. Recall that $S \subset T$ corresponds to the cocharacter $h$ satisfying $\alpha(h) = 2$ for all $\alpha \in \Delta$, and that $|h, e| = 2e$. Identifying $X(S)$ as a lattice in $\text{Lie}(S)^*$, we view $t$ as an element of $\text{Lie}(S)^*$. Let $\varpi \in \text{Lie}(S)^*$ be the fundamental weight dual to $h$, i.e., given by $\varpi(h) = 1$. Comparing the weights of the $h$-action and $S$-action on $e$, we deduce that $t = 2\varpi$.

Consider now an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, with $e$ (resp. $h$) as the nilpositive (resp. neutral) element. Since $e$ is a principal nilpotent element of $\mathfrak{g}$, the decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2$-representation is given by $\mathfrak{g} = \oplus V(2m_i \varpi) = \oplus V(m_i t)$.

Now, simply observe that

$$\text{Lie}(G^e) = \{ x \in \text{Lie}(U) \mid [e, x] = 0 \} = \ker(ad(e))$$

is spanned by the highest weight vectors in $\mathfrak{g}$, whose weights are precisely $m_1 t, \cdots, m_n t$. \hfill \qed

Lemma 7.2. The $S$-equivariant Euler class of the tangent space $T_{w_I} \text{Fl}(I)$ is $(\prod m_i!) t_N$, where $N = \dim \mathcal{Fl}(I)$.

Proof. Observe that the map $X(T) \rightarrow X(S)$ is given by $\alpha \mapsto t$, for all $\alpha \in \Delta$. Consequently, the $T$-weight space $\mathfrak{g}_\alpha$, for $\alpha \in \Phi^+_I$, is an $S$-weight space of weight $ht(\alpha)t$. The tangent space at $w_I$ admits a $T$-decomposition,

$$T_{w_I}(G/B) = \bigoplus_{\alpha \in \Phi^+_I} \mathfrak{g}_\alpha;$$

hence the $S$-equivariant Euler class of $T_{w_I}(G/B)$ is $t^{a_1}(2t)^{a_2} \cdots (kt)^{a_k}$, where $a_i$ is the number of roots of height $i$ in $\Phi^+_I$. Following Section 7.1, the partition $(a_1, \ldots, a_k)$ is conjugate to $(m_1, \ldots, m_n)$; consequently, the $S$-equivariant Euler class of $T_{w_I}(G/B)$ is precisely $m_1! m_2! \cdots m_n! t_N$. \hfill \qed

We are now ready to calculate the multiplicities $m(v_I)$ using the map in cohomology obtained by restricting to the fixed point set.

Proposition 7.3. Let $v w : H^*_S(G/B) \rightarrow H^*_S(w)$ be the map induced by the inclusion $wB/B \hookrightarrow G/B$. Define $b \in \mathbb{Z}$ by $v w \sigma v_I = bt^n$.

(a) We have $m(v_I) = \frac{b}{m_1 \cdots m_n}$.

(b) Suppose $X^{v_I}$ is smooth at $w_I$. Let $\{ \beta_1, \cdots, \beta_n \} = \{ \alpha \in \Phi^+_I \mid s_\alpha \not\leq v_I w_I \}$. Then

$$m(v_I) = \frac{ht(\beta_1) \cdots ht(\beta_n)}{m_1 \cdots m_n}.$$
Proof. Recall from Corollary 3.5 that
\[ (18) \quad \sigma_{v_I} \cup \eta_I = m(v_I)\tau_{w_I}, \]
where \( \eta_I \) and \( \tau_{w_I} \) are Poincaré dual to \([P_I]_S\) and \([w_I]_S\), respectively, in \( H^*_S(G/B)\).

We restrict both sides to \( w_I \) under the map \( \iota^*_w : H^*_S(G/B) \to H^*_S(w_I) \). By Theorem 6.6, we may assume \( \Delta = I \), so that the tangent space \( T_{w_I}P_I = \text{Lie}(G^e) \) has \( S \)-weights \( m_1, \ldots, m_n \) as described in Lemma 7.1. Following Proposition 4.1 and Lemma 7.2, we see that the \( S \)-equivariant Euler class at \( w_I \) of the normal bundle of \( P \) is \( \frac{m_1!m_2! \cdots m_n!t^N}{m_1m_2 \cdots m_n t^n} \). Applying \( \iota^*_w \) to both sides of Equation (18) yields
\[ \frac{m_1!m_2! \cdots m_n!t^N}{m_1m_2 \cdots m_n t^n} \iota^*_w \sigma_{v_I} = m(v_I) \iota^*_w \tau_{w_I}. \]
Using Lemma 7.2, we have \( \iota^*_w \tau_{w_I} = m_1!m_2! \cdots m_n!t^N \), and part (a) follows.

For part (b), since \( X^{v_I} \) is smooth at \( w_I \), the normal space of \( X^{v_I} \) at \( w_I \) is spanned by
\[ \{ g_\alpha \mid \alpha \in \Phi^+_I, s_\alpha \neq v_I \} = \{ g_\alpha \mid \alpha \in \Phi^+_I, s_\alpha \neq v_I w_I \}; \]
see [Kum02 Cor 12.1.10]. Part (b) now follows from (a), along with the observation that the map \( \mathcal{X}(T) \to \mathcal{X}(S) \) is given by \( \beta \mapsto \text{ht}(\beta) t \).

Example 7.4. Let \( I = A_3 \), and \( v_I = s_1s_3s_2 \). Then \( m(v_I) = 2 \).

Example 7.5. For \( I \in \{ B_2, C_2 \} \), we have \( m(v_I) = 2 \) for every Coxeter element \( v_I \).

In [IT16, Question 1], Insko and Tymoczko conjecture that \( m(v_I) = 1 \) for certain Coxeter elements, when \( I \) is contained in some sub-diagram of type \( A \), and that \( m(v_I) = 2 \) otherwise. As an application of Proposition 7.3, we compute \( m(v_I) \) for one Coxeter element in each Dynkin diagram; this formula proves their conjecture in type \( A \), and disproves it in other cases.

Theorem 7.6. (a) Let \( I \) be a connected Dynkin diagram with the standard labelling (see [Bou02]), and set \( v_I = s_1s_2 \cdots s_n \). Then,
\[ m(v_I) = \begin{cases} 1 & \text{if } I = A_n, \\ 2^{n-1} & \text{if } I = B_n, C_n, \\ 2^{n-2} & \text{if } I = D_n, \end{cases} \]
\[ = \begin{cases} 72 = 2^3 \cdot 3^2 & \text{if } I = E_6, \\ 864 = 2^5 \cdot 3^3 & \text{if } I = E_7, \\ 51840 = 2^7 \cdot 3^4 \cdot 5 & \text{if } I = E_8, \end{cases} \]
\[ = \begin{cases} 48 = 2^4 \cdot 3 & \text{if } I = F_4, \\ 6 \cdot 2 = 3 & \text{if } I = G_2. \end{cases} \]

(b) Let \( I_1, \ldots, I_k \) be the connected components of a Dynkin diagram \( I \), and let \( v_{I_1}, \ldots, v_{I_k} \) be Coxeter elements for \( I_1, \ldots, I_k \) respectively. Then \( v := v_{I_1} \cdots v_{I_k} \) is a Coxeter element for \( I \), and \( m(v) = \prod_{j=1}^k m(v_{I_j}) \).

Proof of Theorem 7.6. If \( I \) is a diagram of classical type, the variety \( X^{v_I} \) is smooth at \( w_I \), cf. [IT16 Thm 3]. Consequently, we can use Proposition 7.3(b) to compute \( m(v_I) \). We show the details of the calculations in Appendix B. For the exceptional cases, a computer calculation suffices: we
use the localization formula (cf. [AJS94, Bil99]) to compute $i^*_w \sigma_v$, and apply Proposition 7.3(a).

Following Theorem 6.6 we may assume $\Delta = I$. The integer $m(v)$ is the multiplicity of the intersection of $X^v$ with $P$. We have

$$F(I) = \prod F(I_j), \quad X^v = \prod X^{v_j}, \quad P(I) = \prod P(I_j),$$

and hence the multiplicity $m(v)$ is the product of the multiplicities $m(v_j)$.

\[\square\]

Remark 7.7. We conjecture for all Coxeter elements $v_I$ a type-independent formula for the intersection multiplicity, namely:

\begin{equation}
  m(v_I) = \left| \mathcal{R}(v_I) \right||W_I| = \mathcal{R}(v_I) \prod_{\alpha \in I} a_\alpha.
\end{equation}

Here $\mathcal{R}(v_I)$ is the set of reduced expressions for $v_I$, $W_I$ is the Weyl group of $I$, $C_I$ is the Cartan matrix of the Dynkin diagram $I$, and the integers $a_\alpha$ are the coefficients of the highest root $\theta_I = \sum_{\alpha \in I} a_\alpha \alpha$ of $I$. The second equality follows from [Bou02, p. 297]. For the Coxeter elements in Theorem 7.6, we have verified this formula with type-by-type calculations. For a type-independent proof, see [GS21].

Remark 7.8. The formula $m(v_I) = 1$ for $I = A_n$ was first obtained by Insko in [Ins15], who proved that the scheme-theoretic intersection $X^{v_I} \cap P_I$ is reduced.

Corollary 7.9. Suppose $I$ is contained in some sub-diagram $J$ of type $A$, and let $v$ be the Coxeter element of $I$ obtained by multiplying the simple reflections in increasing order (for the standard type $A$ labelling of nodes in $J$). Then $m(v) = 1$.

Proof. Observe that each connected component $I_j \subset I$ is of type $A$. Let $v_j$ be the Coxeter element of $I_j$ obtained by multiplying the simple reflections in increasing order (for the standard type $A$ labelling of nodes in $I_j$), so that $v = \prod v_j$. Following Theorem 7.6 we have $m(v_j) = 1$, and $m(v) = \prod m(v_j) = 1$. \[\square\]

APPENDIX A. TWO DEFINITIONS OF THE PETERSON VARIETY

In this section we recall the affine paving of the Peterson variety (Appendix A.1), and we show in Proposition A.3 that the two definitions of the Peterson variety,

$$P := G \cdot w_0 B,$$

$$P(\Delta) := \left\{ gB \in G/B \bigg| Ad(g^{-1})e \in \text{Lie}(B) \oplus \bigoplus_{\alpha \in \Delta} C_{e-\alpha} \right\},$$

agree. These results are well-known to experts, but either some statements are only implicitly present in the literature, or we present slightly different proofs. A key point is the irreducibility of $P(\Delta)$, which we prove utilizing results of Kostant [Kos96]. We also present in Remark A.5 an alternate
proof following [AT10, Pre18, AFZ20], as explained to us by Băilăbanu. Their arguments extend to the wider setting of regular Hessenberg varieties.

**A.1. Paving by Affines.** For $I \subset \Delta$, let $P(I)^0 = P(I) \setminus \bigcup_{J \nsubseteq I} P(J)$. Let $U_I$ be the unipotent Lie group corresponding to the Dynkin diagram $I$, and let $A_I$ be the centralizer of $e_I$ in $U_I$; see [Tit66].

The following proposition was proved in various cases by Tymoçko [Tym07, Theorem 4.3] and Băilăbanu [Băl17, Section 6]. Following the exposition in [Băl17], we recall the main steps in the proof.

**Proposition A.1.**

(a) The group $A_I$ acts transitively and faithfully on $P(I)^0$.

(b) $P(\Delta) = \bigsqcup_{I \subset \Delta} P(I)^0$ is a paving by affines.

(c) The intersection $P(\Delta) \cap X^0_w$ is nonempty if and only if $w = w_I$ for some subset $I \subset \Delta$.

**Proof.** Following [Băl17, Prop 6.3], we have $P(I)^0 = A_I w_I B / B$, i.e., $A_I$ acts transitively on $P(I)^0$. Further, $U_I$ acts faithfully on the Schubert cell $X^0_{w_I}$, and hence the action of $A_I \subset U_I$ is faithful at the point $w_I$. Next, observe that $P(I)^0$ is a principal space for $A_I$, hence is an affine space. Finally, the observation $P(I)^0 \subset X^0_{w_I}$, along with part (b) implies that $P(\Delta) \cap X^0_w$ is empty unless $w = w_I$ for some $I$. □

**A.2. Equivalence of two definitions of the Peterson variety.**

**Lemma A.2.** ([Kos96]) The variety $P(\Delta)$ is locally irreducible at the point $1B$.

**Proof.** Let $U^-$ be the unipotent radical of the opposite Borel subgroup $B^-$, and let $\mathcal{N}$ be the variety of nilpotent elements in $g$. Consider the map $\eta : U^- \to g$ given by $u \mapsto \text{Ad}(u^{-1})e$. Following [Kos96, Thm. 17], the map $\eta$ induces an isomorphism,

$$\eta : U^- \cong (e + b^-) \cap \mathcal{N},$$

where $b^- = \text{Lie}(B^-)$. Recall that the $U^-$-orbit of $1B$ is an open set (namely the opposite Schubert cell) in $G/B$. Hence $Z := P(\Delta) \cap U^- B / B$ is an open neighborhood of $1B$ in $P(\Delta)$, and it suffices to show that $Z$ is irreducible. Let $\mathfrak{f} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}$, and let $\mathcal{N}^{reg}$ denote the set of regular nilpotent elements in $g$. Since the intersection $U^- \cap B$ is trivial, it follows that $Z = \eta^{-1}(b \oplus \mathfrak{f})$. Then

$$Z = \eta^{-1}(b \oplus \mathfrak{f}) \cong (b \oplus \mathfrak{f}) \cap (e + b^-) \cap \mathcal{N}$$

$$= (e + \mathfrak{h} + \mathfrak{f}) \cap \mathcal{N} = (e + \mathfrak{h} + \mathfrak{f}) \cap \mathcal{N}^{reg},$$

where the last equality is from [Kos96, §3.2]. The result now follows from the irreducibility of $(e + \mathfrak{h} + \mathfrak{f}) \cap \mathcal{N}^{reg}$, cf. [Kos96, Thm. 6]. □

**Lemma A.3.** The variety $P(\Delta)$ is irreducible.
Proof. Let $Y$ be an irreducible component of $P(\Delta)$. Recall from Appendix A.1 that the (connected) group $A_\Delta = Stab_Y(e)$ acts on $P(\Delta)$, hence it acts on $Y$. Since $Y$ is a closed (hence projective) subvariety of $G/B$ and since $A_\Delta$ is solvable, $Y$ admits an $A_\Delta$-fixed point by [Bor91, Thm.10.4]. This point must necessarily be $1B$, as this is the unique $A_\Delta$-fixed point in $G/B$. In other words, every irreducible component of $P(\Delta)$ contains $1B$; the irreducibility of $P(\Delta)$ now follows from the local irreducibility of $P(\Delta)$ at $1B$; see Lemma A.2.

Proposition A.4. The two definitions of the Peterson variety in Equation (20) agree, i.e., $P = P(\Delta)$.

Proof. Observe that $G^e = A_\Delta \times Z(G)$, where $Z(G)$ is the center of $G$, cf. [Kos96, p. 9]. Since $Z(G) \subset B$, we have an equality $A_\Delta w_0 B = G^e w_0 B \subset P \cap P(\Delta)$. It follows from Proposition A.1 that $G^e w_0 B$ is an open subset of $P(\Delta)$, and since $P(\Delta)$ is irreducible by Lemma A.3, $P = P(\Delta)$.

Remark A.5. We recall an alternate proof of the irreducibility of the variety $P(\Delta)$, following Precup [Pre18, Cor 14] and [AT10, Lemma 7.1], as explained to us by Bălibanu.

Let $H = b \oplus (\oplus_{\alpha \in \Delta} g_{-\alpha})$, and consider the variety $Z = G \times^B H$, equipped with the map $Z \to g$ given by $(g, x) \mapsto Ad(g)x$. For $x$ a regular semisimple element of $g$, the fiber $Z_x$ has dimension $|\Delta|$; see [Pre18, Cor 3]. Since regular semisimple elements are dense in $g$, it follows from [Mum88, Ch. 1, §8, Thms. 2, 3] that each irreducible component of the fiber $Z_e = P(\Delta)$ has dimension greater than or equal to $|\Delta|$, and hence $P(\Delta)$ is pure-dimensional. Following [Ful98, §1.5], the fundamental classes of the irreducible components of $P(\Delta)$ freely generate the top Chow group of $P(\Delta)$. Since there is a unique top-dimensional cell in the affine paving of Proposition A.1(b), it follows that $P(\Delta)$ has a unique irreducible component.

APPENDIX B. INTERSECTION MULTIPLEITIES FOR CLASSICAL DIAGRAMS

We present here the details of our calculation in Theorem 7.6 of the intersection multiplicities $m(v_I)$ for classical Dynkin diagrams.

B.1. Type A. Let $V$ be a vector space with orthonormal basis $\epsilon_1, \cdots, \epsilon_n$. The vectors $\{\epsilon_i - \epsilon_j\}$ form a root system with Dynkin diagram $A_{n-1}$. A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i < n$, and the Weyl group is naturally identified with the symmetric group on $\{\epsilon_1, \cdots, \epsilon_n\}$. We calculate $v_I w_I = [1, n, \cdots, 2]$, and $\{\alpha \in \Phi_I^+ \mid s_{\alpha} \not\in v_I w_I\} = \{\epsilon_i - \epsilon_i \mid 2 \leq i \leq n\}$. Now, $ht(\epsilon_1 - \epsilon_i) = i - 1$. Consequently Proposition 7.3 and Table 1 yield $m(v_I) = 1$.

B.2. Type B. Let $V$ be a vector space with orthonormal basis $\epsilon_1, \cdots, \epsilon_n$. The vectors $\{\pm \epsilon_i \pm \epsilon_j\} \cup \{\pm \epsilon_i\}$ form a root system with Dynkin diagram $B_n$. A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$, and $\alpha_n = \epsilon_n$.

Let $S_{2n}$ be the symmetric group on the letters $\{1, \cdots, n, \overline{n}, \cdots, \overline{1}\}$, and let $r_{ij} \in S_{2n}$ be the transposition switching the letters $i$ and $j$. The Weyl
group $W$ can be viewed as the subgroup of $S_{2n}$ generated by the reflections, 
\[ s_{\epsilon_i - \epsilon_j} = r_{ij} r_{ij}^{-1}, \quad s_{\epsilon_i + \epsilon_j} = r_{ij}^{-1} r_{ij}, \quad s_{\epsilon_i} = r_{i i}, \quad 1 \leq i < j \leq n. \]

Given $v, w \in W$, if $v \leq w$ in the Bruhat order on $W$, then $v \leq w$ in the Bruhat order on $S_{2n}$; see [LR08, §6.1.1]. We calculate 
\[ v_I w_I = [2, \ldots, n, 1, n, 2, \ldots, 1] \]

and \[ \{ \alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I \} = \{ \epsilon_1 + \epsilon_i \mid 2 \leq i \leq n \} \cup \{ \epsilon_1 \}. \]
Now, $ht(\epsilon_1) = n$, and $ht(\epsilon_1 + \epsilon_i) = 2n + 1 - i$. Following Proposition 7.3 and Table 1, we have 
\[ m(v_I) = \frac{n(n + 1) \cdots (2n - 1)}{(1)(3)(2n - 1)} = 2^{n - 1}. \]

### B.3. Type C.

Let $V$ be a vector space with orthonormal basis $\epsilon_1, \ldots, \epsilon_n$. The set of vectors \( \{ \pm \epsilon_i \pm \epsilon_j \} \cup \{ \pm 2\epsilon_i \} \) forms a root system with Dynkin diagram $C_n$. A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$, and $\alpha_n = 2\epsilon_n$. The Weyl group of $C_n$ is isomorphic to the Weyl group of $B_n$. We calculate \[ \{ \alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I \} = \{ \epsilon_1 + \epsilon_i \mid 2 \leq i \leq n \} \cup \{ 2\epsilon_1 \}. \]
Now, $ht(2\epsilon_1) = 2n - 1$, and $ht(\epsilon_1 + \epsilon_i) = 2n - 1 - i$, for $2 \leq i \leq n$. Following Proposition 7.3 and Table 1, we have 
\[ m(v_I) = \frac{n(n + 1) \cdots (2n - 1)}{(1)(3)(2n - 1)} = 2^{n - 1}. \]

### B.4. Type D.

Let $V$ be a vector space with orthonormal basis $\epsilon_1, \ldots, \epsilon_n$. The set of vectors \( \{ \pm \epsilon_i \pm \epsilon_j \} \cup \{ \pm 2\epsilon_i \} \) forms a root system with Dynkin diagram $D_n$. A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$, and $\alpha_n = \epsilon_{n-1} + \epsilon_{n+1}$.

Let $S_{2n}$ be the symmetric group on the letters \( \{ 1, \ldots, n, n, \ldots, 1 \} \), and let $r_{i,j} \in S_{2n}$, be the transposition switching the letters $i$ and $j$. The Weyl group $W$ can be viewed as the subgroup of $S_{2n}$ generated by the reflections, 
\[ s_{\epsilon_i - \epsilon_j} = r_{ij}, \quad s_{\epsilon_i + \epsilon_j} = r_{ij}^{-1}, \quad 1 \leq i < j \leq n. \]

Given $v, w \in W$, if $v \leq w$ in the Bruhat order on $W$, then $v \leq w$ in the Bruhat order on $S_{2n}$; see [LR08, §7.1.1]. A simple computation yields 
\[ v_I w_I = \begin{cases} 2, \ldots, n - 1, 1, n & \text{if } n \text{ is even}, \\ 2, \ldots, n - 1, 1, \overline{n} & \text{if } n \text{ is odd}. \end{cases} \]

Observe that \[ \{ \alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I \} = \{ \epsilon_1 + \epsilon_i \mid 2 \leq i \leq n \} \cup \{ \epsilon_1 - \epsilon_n \}. \]
Now $ht(\epsilon_1 - \epsilon_n) = n - 1$, and $ht(\epsilon_1 + \epsilon_i) = 2n - 1 - i$, for $2 \leq i \leq n$. Consequently, we deduce from Proposition 7.3 and Table 1 that 
\[ m(v_I) = \frac{(n - 1)(n - 1)n \cdots (2n - 3)}{(1)(3) \cdots (2n - 1)(n - 1)} = \frac{(n - 1) \cdots (2n - 3)}{(1)(3) \cdots (2n - 3)} = 2^{n - 2}. \]

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