Phase space theory for open quantum systems with local and collective dissipative processes

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Abstract

In this article we investigate driven dissipative quantum dynamics of an ensemble of two-level systems given by a Markovian master equation with collective and local dissipators. Exploiting the permutation symmetry in our model, we employ a phase space approach for the solution of this equation in terms of a diagonal representation with respect to certain generalized spin coherent states. Remarkably, this allows to interpolate between mean field theory and finite system size in a formalism independent of Hilbert-space dimension. Moreover, in certain parameter regimes, the evolution equation for the corresponding quasiprobability distribution resembles a Fokker–Planck equation, which can be efficiently solved by stochastic calculus. Then, the dynamics can be seen as classical in the sense that no entanglement between the two-level systems is generated. Our results expose, utilize and promote techniques pioneered in the context of laser theory, which can be powerful tools to investigate problems of current theoretical and experimental interest.

Keywords: open quantum systems, spin coherent state, quantum decoherence

(Some figures may appear in colour only in the online journal)
1. Introduction

The subtle interplay of quantum systems with their environment is a key factor for understanding nature and has important implications on various research fields, e.g. for precision experiments in metrology [1, 2], quantum-optical simulations for condensed matter phenomena [3–5], superconducting circuits [6, 7] and quantum-information science [8–10]. A central objective for the theoretical community is to investigate quantum optical models, such as the Dicke model, where an ensemble of identical two-level atoms is coupled to a cavity mode, which have been among the first systems where emergent collective behavior such as super-radiance has been investigated [11]. The cooperative effects can be described in terms of collective atomic operators in such a way that the ensemble of $N$ two-level atoms is treated as a large spin of length $j = \frac{N}{2}$. Interestingly, such a cooperative state can still possess non-trivial quantum correlations [11]. The proposal on how to implement the Dicke model in an optical cavity QED system [12] and the experimental realization using a super-fluid gas trapped inside an optical cavity [13] are important milestones, which have inspired more studies on the role of collective effects in quantum phase transitions, such as [14, 15]. Recently, there have been major developments in the experimental realization of out of equilibrium quantum many body systems, with a variety of different platforms such as Rydberg atoms in optical lattices [16, 17] as well as superconducting qubits and circuit quantum electrodynamics, which are promising setups to investigate and exploit strong light–matter interactions even further [6–8]. Some of the experiments could already show validity of theoretical predictions [18, 19]. Dissipative effects play an important role in these setups and have been investigated both theoretically and experimentally [20–23].

Purely collective dynamics, as in the Dicke model, cannot always be achieved due to experimental conditions, for example when the ensemble of two level systems is inhomogeneous [24] or when the individual two level systems couple to different reservoirs [25]. In these cases, the dimensionality of the relevant Hilbert space in general grows exponentially with the system size, which drastically restricts theoretical investigations. An interesting alternative regime exists when in addition to collective processes only local processes occur which preserve permutation invariance. This holds if all local processes are identical for each constituent in the ensemble. In this regime, permutation invariance can be utilized to reduce the effective dimension of the problem [22, 26]. Formally, this can be understood as follows. In an ensemble of $N$ two-level systems, collective processes confine the dynamics on a subspace $H_j \subset \mathbb{C}^{2N}$ spanned by the Dicke states $|j, m\rangle$, where $-j \leq m \leq j$, whereas the permutation invariant local processes couple the different subspaces $H_j$ spanned by Dicke ladders labeled by differing $j$. The reduction in the effective dimension emerges due to high degeneracy of the Dicke states [27].

When local and collective processes coexist, a host of interesting phenomena can be studied. For example, incoherent pumping on a super-radiant ensemble of two level systems can lead to robust steady state super-radiance [28]. Competing collective phenomena, such as driving and dissipation can lead to non-equilibrium phase transitions [29, 30] and recently the robustness of such transitions against local dephasing has been investigated [31]. Recent work also highlights the importance of finite size effects that may lead to a large deviation from mean field theory despite the fact that the models are fully-connected [22, 32]. These studies are mostly based on numerical investigations. To gain a better understanding of the underlying physical mechanisms a theory which allows further analytical treatment is highly desirable. The phase space formalism that we developed here can be a starting point for this and even allows a mapping of quantum dynamics to classical stochastic systems, albeit only in restricted cases.
In this article we will investigate a paradigmatic driven, dissipative open system consisting of \( N \) two level atoms which is affected by both collective and permutation invariant local dissipative processes. We describe the dynamics of such a system with the following Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) type master equation \[ \frac{\partial}{\partial t} \rho = -i[H, \rho] + L_c[\rho] + L_l[\rho]. \] (1)

In this article we consider a linear Hamiltonian \[ H = \sum_{\lambda=1}^{N} \frac{1}{2} \vec{B} \cdot \vec{\sigma}_\lambda \equiv \vec{B} \cdot \vec{J}, \] (2)

with \( \vec{B} \in \mathbb{R}^3, \sigma_i^\lambda = \sigma_i \) for \( i = z, \pm \) are Pauli spin matrices \( \sigma_z, \sigma_\pm = \sigma_x \pm i \sigma_y \) acting on system \( \lambda \), and \( \vec{J} \) is the collective spin operator with components \( J_z \) and \( J_\pm = J_x \pm i J_y \). We stress that non-linear terms, such as spin squeezing terms, could also be included straightforwardly in the phase space theory that we develop. The collective dissipation is given by

\[ L_c[\rho] = \sum_{k=z,\pm} L_{k}^c[\rho] = \sum_{k=z,\pm} 2\kappa_k \left( J_k \rho J_k^\dagger - \frac{1}{2} \left\{ J_k^\dagger J_k, \rho \right\} \right), \] (3)

whereas the permutation invariant local dissipation is given by

\[ L_l[\rho] = \sum_{k=z,\pm} \sum_{\lambda=1}^{N} \gamma_k \left( \sigma_k^\lambda \rho \sigma_k^{\dagger \lambda} - \frac{1}{2} \left\{ \sigma_k^{\dagger \lambda}, \sigma_k^\lambda, \rho \right\} \right). \] (4)

We assume that all of the local and collective rates, \( \gamma_k \) and \( \kappa_k \) respectively, are non-negative, so that the dynamics generated by the master equation is completely positive. Note that the local term does not commute with the total angular momentum \( \vec{J}^2 \). It therefore couples different eigenspaces of \( \vec{J}^2 \) in contrast to the collective terms.

Instead of using the basis of Dicke states directly, we will employ a phase space approach by representing the state of the open system in terms of generalized spin coherent states and the associated \( P \)-function \[35, 36\]. This type of approach to collective phenomena has been already discussed in the quantum optics community during the 70s and 80s of the previous century, in the context of cooperative fluorescence \[37–40\]. The generalized coherent state approach that we take has been pioneered in \[41–43\] and used also more recently in \[44\]. Here we show that in certain parameter regions this method can be used to map the solution of master equation (1) to a Fokker–Planck equation for any system size. Then the dynamics is classical in the sense that no quantum correlations between the two-level systems build up. In the method the system size is merely a parameter determining the strength of diffusion in phase space. Therefore, it allows to analytically interpolate between finite size and mean field theory.

The outline of the article is the following. First we introduce a generalization of spin coherent states in section 2. In section 3 we derive the equation of motion for the associated \( P \)-function. The phase space approach provides immediately a consistent mean field theory, which we explore in section 4. In section 5 we focus on the exact semi-classical regime of our model, where the equations of motion can be efficiently solved. At last, we conclude with discussion and outlook in section 6.
2. Generalized spin coherent states

The main technical tool of this paper is to expand the state of all identical two level systems in terms of a generalized P-representation. More precisely, consider the density operator $\alpha(\vec{r})$ of the simplest product state where all two level systems are identical

$$\alpha(\vec{r}) = \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}).$$

This state is positive iff the Bloch vector $\vec{r}$ has a length smaller or equal to 1. It is pure iff $|\vec{r}|^2 = 1$. We show later on that in this case the definition (5) reduces to the standard spin coherent states in the symmetric Dicke subspace with $j = \frac{N}{2}$. Due to this property, we call these states the generalized coherent states. However, we stress that these are not pure states and do not have the same group theoretical interpretation as regular coherent states [45]. We will later express the state in terms of a diagonal representation using a $P$-function associated with these generalized coherent states. Firstly, let us discuss how certain operators act on these states. A direct calculation shows that

$$\sigma_\lambda (1 + \vec{\sigma} \cdot \vec{r}) = [r_k + (\delta_{ki} - i e_{ijk} r_j - r_k r_j) \partial_i] (1 + \vec{\sigma} \cdot \vec{r}),$$

$$(1 + \vec{\sigma} \cdot \vec{r}) \sigma_k = [r_k + (\delta_{ki} + i e_{ijk} r_j - r_k r_j) \partial_i] (1 + \vec{\sigma} \cdot \vec{r}),$$

where the derivatives are with respect to $\vec{r}$, i.e. $\partial_i = \partial / \partial r_i$, and summation over repeated indices is implied, see appendix A. With these relations we can replace the action of operators acting on generalized coherent states by differential operators. One example is the total angular momentum operator $J$

$$J_\lambda \alpha(\vec{r}) = \frac{1}{2} \sum_{\lambda=1}^{N} \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}) \otimes \cdots \otimes \sigma_\lambda \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}) \otimes \cdots \otimes \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r})_{\lambda^{th \ position}}$$

$$= (\text{use product rule})$$

$$= \frac{N}{2} r_\lambda \alpha(\vec{r}) + \frac{1}{2} (\delta_{ki} - i e_{ijk} r_j - r_k r_j) \partial_i \alpha(\vec{r}).$$

This gives us the algebra of spin coherent states. It is analogous to well known relations such as $a|z\rangle = z|z\rangle$ in the case of bosons, where $|z\rangle$ is a bosonic coherent state. With these rules we can express the action of the Hamiltonian part, as well as the collective dissipative part of the Lindbladian (1), onto a coherent state as a differential operator with respect to the coherent state label $\vec{r}$. This also works for non-linear Hamiltonians which are not explicitly considered here. Non-linear terms will generate higher order differential operators, leading to a more complicated evolution equation.

Due to our more general definition of spin coherent states, this differential representation also works for the local dissipators. Consider for instance identical local dephasing (in the $\sigma_z$ basis with rate $\gamma_c/2$) of all two level systems

$$L_k [\alpha(\vec{r})] = \frac{\gamma_c}{2} \sum_{\lambda} L_{kj} \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}) \otimes \cdots \otimes \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r})_{\lambda^{th \ position}}$$

$$= \frac{\gamma_c}{2} \sum_{\lambda=1}^{N} \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}) \otimes \cdots \otimes \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r})_{\lambda^{th \ position}}.$$
Now, the action of any superoperator on a two level system can always be written as a first order differential operator with respect to the Bloch vector. This is obvious because the state itself is linear in $\vec{r}$. In the example of dephasing we find

$$L^\gamma_j \left[ \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}) \right] = \sigma^x \frac{1}{2} \left( \frac{1}{2} \right) (1 + \vec{\sigma} \cdot \vec{r}) - \left\{ \sigma^x, \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}) \right\} = (-2x\partial_x - 2y\partial_y) \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{r}).$$

(10)

with canonical notation $r_1 = x, r_2 = y, r_3 = z$. Inserting this in the above expression and using the product rule one finds for the coherent state

$$L^\gamma_j [\alpha(\vec{r})] = (-\gamma_r x\partial_x - \gamma_s y\partial_y)\alpha(\vec{r}).$$

(11)

Of course, the same calculations can be done for local decay and pump channels. In summary, we can now express the action of the entire Lindblad superoperator on a generalized spin coherent state as a differential operator with respect to the state label, that is the vector $\vec{r}$. Let us briefly make the connection to the standard spin coherent states. These are usually defined in an eigenspace of $\hat{J}^2$ with eigenvalue $j = \frac{1}{2} N$ (symmetric Dicke states). They are rotations of the ‘fully polarized’ state $\alpha(\vec{e}_z)$. We recover these states from our generalized coherent states by setting $|\vec{r}| = r = 1$. For better illustration, let us parametrize the vector with spherical coordinates

$$\vec{r} = \left( \begin{array}{c} r \sqrt{1 - \eta^2} \cos \phi \\ r \sqrt{1 - \eta^2} \sin \phi \end{array} \right), \quad \phi \in [0, 2\pi), \ \eta \in [-1, 1], \ r \in [0, 1],$$

(12)

where one may identify $\eta = \cos \theta$. The differential representation of the dissipators in these coordinates are provided in appendix C. The main insight is that for $r = 1$ in the collective operators, no derivatives with respect to $r$ appear. The radius is preserved because the dynamics does not lead out of the symmetric $j = N/2$ subspace. This is different in the case of local dissipation. The local dissipators are not confined to this subspace which is reflected in the occurrence of derivatives with respect to $r$. If the dynamics remains in the Dicke subspace however, the equations of motions which we derive in the next section will automatically conserve $r = 1$ so that both cases are covered in the formalism. Now as an ansatz we express the state with a diagonal representation in terms of the generalized coherent states

$$\rho = \int_{r \leq 1} d^3r \ P(\vec{r}) \alpha(\vec{r}),$$

(13)

with a generalized $P$-function $P(\vec{r})$, which is a quasiprobability distribution on the unit ball in $\mathbb{R}^3$ centered at the origin. As $\alpha(\vec{r})$ is a polynomial in $\vec{r}$ of order $N$, it is obvious that $P$ is not unique. If $\rho$ is a normalized state, the distribution is normalized as well

$$1 = \text{tr} \{ \rho \} = \int_{r \leq 1} d^3r \ P(\vec{r}) \text{tr} \{ \alpha(\vec{r}) \} = \int_{r \leq 1} d^3r \ P(\vec{r}).$$

(14)

If $P(\vec{r})$ is a positive function, the state is also positive by construction. However, $P(\vec{r})$ does not need to be positive. In fact, if $P(\vec{r})$ is positive for all $\vec{r}$, then the state is separable by definition, as it is a classical mixture of separable states. An entangled state necessarily has
a non-positive $P$ function. Once the $P$ function of the state is known, all observables can be computed easily

$$\text{tr}\{\rho O\} = \int_{r \leq 1} d^3r P(\vec{r}) \text{tr}\{\alpha(\vec{r})O\}.$$ (15)

For example, in the case of the angular momentum operator one finds $\text{tr}\{\alpha(\vec{r})\vec{J}\} = N/2 \vec{r}$ so that

$$\text{tr}\{\rho \vec{J}\} = \frac{N}{2} \int_{r \leq 1} d^3r P(\vec{r}) \vec{r}.$$ (16)

### 3. Equation of motion for the $P$-function

With the differential form of operators at hand, it is a straightforward task to derive an equation of motion for the $P$-function of a state evolving under the GKSL equation (1). One simply plugs in the expression (13) in the equation of motion

$$\partial_t \rho = \int_{r \leq 1} d^3r \partial_t P(\vec{r}) \alpha(\vec{r}) = \mathcal{L}[\rho] = \int_{r \leq 1} d^3r P(\vec{r}) \mathcal{L}[\alpha(\vec{r})].$$ (17)

Using the differential representation, we can express the action of the GKSL-generator as a second order differential operator

$$\mathcal{L}[\alpha(\vec{r})] = \left( \partial_i a_i(\vec{r}) \partial_i + \frac{1}{2N} D_{ij}(\vec{r}) \partial_i \partial_j \right) \alpha(\vec{r}).$$ (18)

To find the ‘forward in time’ equation of motion for $P$ one performs partial integration under the integral, neglecting boundary terms. Comparison of the integrands on the left and right-hand side gives the following partial differential equation

$$\partial_t P(\vec{r}) = \left( -\partial_i a_i(\vec{r}) + \frac{1}{2N} \partial_i \partial_j D_{ij}(\vec{r}) \right) P(\vec{r}).$$ (19)

General expressions for $a(\vec{r})$ and $D(\vec{r})$ are provided in appendix B. This exact equation has the form of a second order Kramers–Moyal expansion. However, we point out that the matrix $D(\vec{r})$ is in general not positive for all $\vec{r}$. When $D(\vec{r})$ is positive semidefinite for all $\vec{r}$, the equation corresponds to a Fokker–Planck equation. Since the Fokker–Planck equation always preserves the positivity, it is clear that the diffusion can be positive only if the dynamics does not generate entanglement between the two level systems. Indeed, we do find certain parameter regimes in our model where this holds true. Then the partial differential equation is well behaved and can easily be integrated numerically using stochastic differential equations. We give a detailed description of this in section 5.

### 4. Mean field theory

From equation (19) follows directly a ‘mean field’ approximation by neglecting the diffusion term. In general, this approximation is exact in the thermodynamical limit $N \rightarrow \infty$ because
Figure 1. Solutions of the mean field equation (22) depicted on the unit ball. As collective parameters we choose \( \vec{B} = B / \sqrt{2(\vec{e}_z - \vec{e}_x)} \), \( \kappa_+ = 0.8B \) and \( \kappa_- = 0 \). (a) Only collective processes are present \( \gamma_i = 0 \) so that the trajectories stay on the surface of the ball. (b) Due to the local decay, the stable fixed point lies inside the ball. (c) If local dephasing is present, all trajectories approach the fully mixed state \( r = 0 \).

then the diffusion vanishes exactly due to the prefactor \( \frac{1}{N} \). The remaining equation of motion is first order and can be solved by the method of characteristics

\[
\partial_t P(\vec{r}) = -\partial_a \alpha(\vec{r}) P(\vec{r}). \tag{20}
\]

The solution is given by the time evolution of the initial distribution along the mean field trajectories

\[
P_t(\vec{r}) = \int \delta^3 \vec{r}_0 P_0(\vec{r}_0) \delta(\vec{r} - \vec{r}(t)), \quad \vec{r}(0) = \vec{r}_0, \quad \dot{\vec{r}}(t) = \vec{a}(\vec{r}(t)). \tag{21}
\]

If the system starts in a coherent state, i.e. \( \rho_0 = \alpha(\vec{r}_0) \) or \( P_0(\vec{r}) = \delta(\vec{r} - \vec{r}_0) \), then the state remains a coherent state under mean field evolution and \( \rho_t = \alpha(\vec{r}(t)) \) or \( P_t(\vec{r}) = \delta(\vec{r} - \vec{r}(t)) \).

For our master equation the mean field equations of motion are explicitly

\[
\begin{align*}
\dot{x} &= B_y z - B_z y + x(\gamma_- - \gamma_+ + \kappa_- z - \kappa_+ z), \\
\dot{y} &= -B_z z + B_x x + y(\gamma_- - \gamma_+ + \kappa_- z - \kappa_+ z), \\
\dot{z} &= B_x y - B_y x + (\kappa_- - \kappa_+) \left( x^2 + y^2 \right) + 2 \gamma_-(z - 1) - 2 \gamma_+(z + 1),
\end{align*} \tag{22}
\]

where we have neglected subleading terms of order \( 1/N \) consistent with neglecting of the diffusion. Note that collective dephasing does not influence the mean field dynamics. Except for some fine tuned cases [30, 32], mean field theory characterizes the steady state phases of the model [46]. Phase diagrams can be derived by finding the fixed points of (22) and analyzing their stability. We show a few examples of solutions of the mean field equation in figure 1. The images give an intuitive picture of the influence of the different dissipators onto the mean field solutions.

5. Exact semi-classical regime

Going beyond mean field one must include the diffusion term in the \( P \)-function evolution equation. This term is generated by collective dissipation in the master equation (1). Using
spherical coordinates, the diffusion matrix reads

\[
\begin{pmatrix}
\frac{2(1-\eta^2)(r\kappa_+ + \kappa_-)}{r} & 0 & -\eta (\kappa_+ - \kappa_-) \\
0 & 2\kappa \pm 2\eta (\kappa_+ (\eta r + 1) + \kappa_- (\eta r - 1)) / (1-\eta^2) r & 0 \\
-\eta^2 (\kappa_+ - \kappa_-) & 0 & 0 \\
\end{pmatrix}
\]

(23)

If the diffusion matrix is not positive semidefinite, the partial differential equation is no longer parabolic and numerical solution with, for example, finite element methods, is unstable. There exist parameter regimes where this matrix has only positive eigenvalues. In this case the dynamics is described by a Fokker–Planck equation and can be solved with stochastic methods. An obvious example is the case of dephasing only, i.e. \( \kappa_- = \kappa_+ = 0 \). Dephasing can be realized by a stochastic unitary evolution \([47–49]\). The corresponding stochastic Hamiltonian is non-interacting, so that the full dynamics can be mapped to a classical stochastic process. This is reflected by the positivity of the diffusion matrix. More generally, whenever the loss and the pump rates are identical \( \kappa_+ = \kappa_- \), the diffusion is positive.

Let us now focus on the purely collective case with no local decay processes. Then the Fokker–Planck equation preserves the radial direction \( r \), so that we can set \( r = 1 \) and recover the canonical spin coherent state \( P \)-function. Then, the matrix (23) has two non-zero eigenvalues given by

\[
\lambda_1 = 2(1-\eta^2)((1+\eta)\kappa_- + (1-\eta)\kappa_+),
\]

(24)

\[
\lambda_2 = 2\kappa_+ - 2\eta / (1-\eta) - \kappa_+ + 2\eta / (1+\eta).
\]

(25)

We see that equation (19) is a proper Fokker–Planck equation if both \( \lambda_1, \lambda_2 \) are positive. \( \lambda_1 \) is always positive since \(-1 \leq \eta \leq 1\) and \( \lambda_2 \) is positive for all \( \eta \) and \( \phi \) if

\[
\kappa_+ + \kappa_- \leq 2\kappa_+.
\]

(26)

Satisfying this condition, the open quantum system evolution described by equation (1) can be mapped to a classical diffusion process on the sphere. Nevertheless the state lies in a finite-dimensional Hilbert-space and contains quantum fluctuations due to the non-zero overlap of coherent states. Relation (26) can be understood in the sense that one simply has to add enough dephasing to compensate the positivity violation of the diffusion matrix due to collective losses. As noted above, the dephasing can be understood as classical noise in the Hamiltonian. This noise can destroy quantum features such as entanglement thus leading to a positive \( P \)-function. We point out, however, that the dephasing channel vanishes in the mean field theory and thus does not influence the different phases of the model.

The standard way of solving the Fokker–Planck equation is to consider the set of stochastic differential equations

\[
d\eta = a_\eta \, dt + \sqrt{\frac{\lambda_1}{N}} \, d\xi_\eta,
\]

(27)

\[
d\phi = a_\phi \, dt + \sqrt{\frac{\lambda_2}{N}} \, d\xi_\phi,
\]

(28)

where \( d\xi_i \) is a real Ito increment with properties \( d\xi_i \, d\xi_j = \delta_{ij} \, dt \) and the drift terms are given in appendix C. In the absence of the noise terms we obtain again the mean field theory, as in
Figure 2. Numerical solution of the master equation (1) with the classical stochastic trajectories obeying equation (27). The light and bold curves are averaged over 100 and 10000 sampled trajectories, respectively. Dashed lines display the predictions from mean field theory. The initial state is a spin coherent state. In both figures we choose the parameters $N = 40$, $\kappa_z = \kappa_+ + \kappa_-$ and $B_z = B_x = 0$. There are no local processes $\gamma_{\pm z} = 0$.
quantum state, which allows to investigate correlations, such as the variances in figure 2, for arbitrary system sizes.

6. Discussion

In this article we have further developed a particular phase space picture for driven dissipative spin systems first introduced in the context of laser theory [43]. Our approach incorporates both collective and permutation invariant local decay processes into a unified framework, which allows for a very simple derivation of evolution equations for the corresponding P-function.

As is well known, the coefficient matrix for the second order terms in the equation of motion for phase space quasiprobability distributions is typically not positive semidefinite. We have examined conditions under which initially positive phase space densities remain positive for all times. Then the dynamics can be considered classical in the sense that no entanglement is generated between the two-level systems. In particular, in the case of purely collective dynamics, presence of sufficiently large dephasing ensures the positivity. Another interesting case is the combination $\kappa_+ = \kappa_-$, which results in positive diffusion even in presence of local processes. This corresponds to a Markovian environment at infinite temperature, where the loss $\kappa_+ = (\bar{n} + 1)\kappa_0$ and pump $\kappa_- = \bar{n}\kappa_0$ rates become identical as $\bar{n} \gg 1$. In case the diffusion matrix is not positive, solving the problem using phase space techniques is challenging as this leads to regions where $P$ is negative and the evolution equation is no longer parabolic. This scenario typically occurs when non-linear Hamiltonians are considered.

The phase space approach provides immediately a consistent mean field picture just by neglecting the subleading second order terms in the evolution equation. By analyzing the stability of the fixed points of the mean field equations, phase diagrams for non-equilibrium phases can be obtained. However, scenarios are known where finite-size corrections become relevant even for large system sizes [30]. In the case of positive diffusion, such finite-size corrections can be incorporated exactly by adding noise to the mean field equation. This remarkable feature allows to efficiently solve the problem with stochastic methods, where the numerical effort is independent of the dimension of the underlying Hilbert space. Nevertheless, we can access the full quantum state, which allows to access difficult non-local observables. An important
example are overlaps, such as Loschmidt amplitudes, which are currently of great interest in the field of dynamical phase transitions [50–52]. There, one is typically interested in the large $N$ limit which can in some cases be mimicked with semi-classical techniques [52]. Our theory gives an analytically exact starting point for studying the large $N$ limit including collective and local dissipation.

Generally, for fully connected models it is often desirable to develop a Gaussian theory which captures fluctuations of localized steady state states. In the case of $\vec{J}^2$ conserving models this can be achieved by the Holstein–Primakov transformation [53, 54]. A similar theory capturing local processes could be obtained upon expanding the equation of motion for the $P$-function around a stable mean field fixed point.

The formalism presented in this paper can be straightforwardly generalized for SU($n$)-type models [55,56], by using generalized Bloch-representations, for instance for three-level systems [57], and defining the coherent states accordingly.

We believe that the presentation of the phase space methods in this article is easily accessible and can be applied effortlessly to problems of current experimental and theoretical interest.

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Appendix A. Differential operator relations

Here, we derive the relations (6) and (7) by direct calculation. Using the product rule for two Pauli matrices $\sigma_k \sigma_i = \delta_{ki} \mathbb{1} + i \varepsilon_{kji} \sigma_j$ gives

$$\sigma_k (1 + \vec{\sigma} \cdot \vec{r}) = \delta_{ki} \sigma_i + \delta_{ki} r_j \mathbb{1} - i \varepsilon_{kji} \sigma_j. \quad \text{(A1)}$$

To find a differential representation of this operation consider

$$\left( v_k(\vec{r}) + w_k(\vec{r}) \partial_j \right) (1 + \vec{\sigma} \cdot \vec{r}) = v_k(\vec{r})(1 + r_i \sigma_i) + w_k(\vec{r}) \sigma_i. \quad \text{(A2)}$$

Comparing both expressions yields $v_k(\vec{r}) = r_k$ and $w_k(\vec{r}) = \delta_{ki} - i \varepsilon_{kij} r_j - r_k r_i$.

Appendix B. Drift and diffusion in $xyz$ parametrization

In this appendix we provide the full expressions for the diffusion matrix and the drift term. These can be derived by expressing the action of the generator of master equation (1) onto a spin coherent state

$$\mathcal{L} \alpha(\vec{r}) = \left( \partial_i a_i(\vec{r}) + \frac{1}{2N} \partial_i \partial_j D_{ij}(\vec{r}) \right) \alpha(\vec{r}). \quad \text{(B1)}$$

Expressing the state via (13), this results in an evolution equation for the $P$-function of model (1)

$$\partial_t P(\vec{r}) = \left( -\partial_i a_i(\vec{r}) + \frac{1}{2N} \partial_i \partial_j D_{ij}(\vec{r}) \right) P(\vec{r}). \quad \text{(B2)}$$
The drift term \( \tilde{a} \) reads

\[
\begin{align*}
\tilde{a}_x &= B_z B_y x + \frac{x (\kappa_- + \kappa_+ + (N-1) (\kappa_- - \kappa_+)) - \kappa_- x}{N} + x (\gamma_- - \gamma_+ - \gamma_c), \\
\tilde{a}_y &= -B_z B_y x + \frac{y (\kappa_- + \kappa_+ + (N-1) (\kappa_- - \kappa_+)) - \kappa_- y}{N} + y (\gamma_- - \gamma_+ - \gamma_c), \\
\tilde{a}_z &= B_y B_z x + \frac{2 \kappa_- + 2 \kappa_+ + \kappa_- (- (N-1) (x^2 + y^2) - 2z) + \kappa_+ ((N-1) (x^2 + y^2) - 2z)}{N} \\
&+ 2 \gamma_- (z - 1) - 2 \gamma_+ (z + 1), \quad (B3)
\end{align*}
\]

and the symmetric diffusion matrix \( D = D^T \) is given as

\[
\begin{align*}
D_{xx} &= 2 (\kappa_- z (x^2 + z + 1) + \kappa_+ z (x^2 + z - 1) + \kappa_- y^2), \\
D_{xy} &= -2 xy (\kappa_- + z (\kappa_- - \kappa_+)), \\
D_{xz} &= x (\kappa_- (x^2 + y^2) - (z + 1)^2) - \kappa_+ (x^2 + y^2) - (z - 1)^2), \\
D_{yy} &= 2 (\kappa_- x^2 + \kappa_- z (y^2 + z + 1) + \kappa_+ z (y^2 + z - 1)), \\
D_{yz} &= y (\kappa_- (x^2 + y^2) - (z + 1)^2) - \kappa_+ (x^2 + y^2) - (z - 1)^2), \\
D_{zz} &= 2 (x^2 + y^2) (\kappa_- + \kappa_+ + \kappa_- z - \kappa_+ z). \quad (B4)
\end{align*}
\]

**Appendix C. Drift and diffusion in spherical parametrization**

Parametrize the spin coherent state with spherical coordinates as in (12), we can transform the derivatives in (B1) accordingly. Expressing the state as

\[
\rho = \int d\eta \int d\phi \int dr \alpha(\tilde{r}) P(\eta, \phi, r), \quad (C1)
\]

the \( P \)-function obeys an evolution equation of the same form as (19). In particular

\[
\frac{\partial}{\partial \tau} P = \left( - \sum_{\beta=\eta,\phi,r} \partial_\beta a_\beta + \sum_{\beta,\beta'=\eta,\phi,r} \frac{1}{2N} \partial_\beta \partial_{\beta'} D_{\beta\beta'} \right) P, \quad (C2)
\]

with drift

\[
\begin{align*}
a_\eta &= \sqrt{1 - \eta^2} (B_x \cos(\phi) - B_y \sin(\phi)) + 2 \gamma_- (\eta \tau - 1) - 2 \gamma_+ (\eta \tau + 1) \\
&- \frac{1}{2} (\eta^2 - 1) r (\kappa_- - \kappa_+) + \frac{2 (\kappa_- - \kappa_+) + 2 \eta (\kappa_- - \kappa_+) + 2 r (\kappa_- + \kappa_+)}{2N r}, \\
a_\phi &= -\sqrt{1 - \eta^2} \eta (B_x \cos(\phi) + B_y \sin(\phi)) - B_z \eta^2 + B_z, \\
a_r &= -2 \eta (\gamma_- + \gamma_+) + (\eta^2 + 1) r (\gamma_- - \gamma_+) + \gamma_+ (\eta^2 - 1) r, \quad (C3)
\end{align*}
\]
and diffusion [cf equation (23)]

\[
D_{\eta,\eta} = -\frac{2 (\eta^2 - 1) (\eta (\kappa_- - \kappa_+) + r (\kappa_- + \kappa_+))}{r},
\]

\[
D_{\phi,\phi} = -\frac{2 (\eta^2 - \kappa_+ + \eta^2 r (\kappa_- + \kappa_+ - \kappa_z) + \kappa_z r)}{(\eta^2 - 1) r},
\]

\[
D_{\eta,\phi} = (\eta^2 - 1) (- (r^2 - 1)) (\kappa_- - \kappa_+),
\]

\[
D_{\phi,\eta} = D_{\phi,\phi} = D_{\phi,\phi} = 0.
\]

\[\text{(C4)}\]

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**References**

[1] Kominis I K, Kornack T W, Allred J C and Romalis M V 2003 *Nature* **422** 596
[2] Itano W M, Bergquist J C, Bollinger J J, Gilligan J M, Heinzen D J, Moore F L, Raizen M G and Wineland D J 1993 *Phys. Rev. A* **47** 3554
[3] Morrison S and Parkins A S 2008 *Phys. Rev. Lett.* **100** 040403
[4] Greiner M, Mandel O, Esslinger T, H"ansch T W and Bloch I 2002 *Nature* **415** 39
[5] You J Q and Nori F 2011 *Nature* **474** 589
[6] Baumann K, Guerlin C, Brennecke F and Esslinger T 2010 *Nature* **464** 1301
[7] Huybrechts D, Minganti F, Nori F, Wouters M and Shammah N 2020 *Phys. Rev. B* **101** 214302
[8] Lalumi`ere K, Sanders B C, van Loo A F, Fedorov A, Wallraff A and Blais A 2013 *Phys. Rev. A* **88** 043806
[9] Galve F, Mandarino A, Paris M G A, Benedetti C and Zambrini R 2017 *Sci. Rep.* **7** 42050
[10] Shammah N, Ahmed S, Lambet N, De Liberato S and Nori F 2018 *Phys. Rev. A* **98** 063815
[27] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press).

[28] Meiser D and Holland M J 2010 *Phys. Rev. A* 81 033847

[29] Iemini F, Russomanno A, Keeling J, Schirò M, Dalmonte M and Fazio R 2018 *Phys. Rev. Lett.* 121 035301

[30] Link V, Luoma K and Strunz W T 2019 *Phys. Rev. A* 99 062120

[31] Tucker K, Zhu B, Lewis-Swan R J, Marino J, Jimenez F, Restrepo J G and Rey A M 2018 *New J. Phys.* 20 123003

[32] Hannukainen J and Larson J 2018 *Phys. Rev. A* 98 042113

[33] Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* 17 821

[34] Lindblad G 1976 *Commun. Math. Phys.* 48 119

[35] Gisin N and Cibils M B 1992 *J. Phys. A: Math. Gen.* 25 5165

[36] Puri R R and Lawande S V 1980 *Physica A* 101 599

[37] Walls D F, Drummond P D, Hassan S S and Carmichael H J 1978 *Prog. Theor. Phys. Suppl.* 64 307

[38] Walls D F 1980 *J. Phys. B: At. Mol. Phys.* 13 2001

[39] Drummond P D and Carmichael H J 1978 *Opt. Commun.* 27 160

[40] Carmichael H J 1980 *J. Phys. B: At. Mol. Phys.* 13 3551

[41] Weidlich W, Risken H and Haken H 1967 *Z. Phys.* 201 396

[42] Weidlich W, Risken H and Haken H 1967 *Z. Phys.* 204 223

[43] Haken H, Risken H and Weidlich W 1967 *Z. Phys.* 206 355

[44] Patra A, Altshuler B L and Yuzbashyan E A 2019 *Phys. Rev. A* 99 033802

[45] Zhang W-M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* 62 867

[46] Ferreira J S and Ribeiro P 2019 *Phys. Rev. B* 100 184422

[47] Strunz W T 2005 *Open Syst. Inf. Dyn.* 12 65

[48] Grotz T, Heaney L and Strunz W T 2006 *Phys. Rev. A* 74 022102

[49] Müller K, Luoma K and Strunz W T 2019 *Geometric phase gates in dissipative quantum dynamics* (arXiv:1907.08033 [quant-ph])

[50] Lang J, Frank B and Halimeh J C 2018 *Phys. Rev. Lett.* 121 130603

[51] Mumford J, Turner E, Sprung D W L and O’Dell D H J 2019 *Phys. Rev. Lett.* 122 170402

[52] Halimeh J C and Zauner-Stauber V 2017 *Phys. Rev. B* 96 134427

[53] Lamberti N, Emary C and Brandes T 2004 *Phys. Rev. Lett.* 92 073602

[54] Hayn M, Emary C and Brandes T 2011 *Phys. Rev. A* 84 053856

[55] Barry D W and Drummond P D 2008 *Phys. Rev. A* 78 052108

[56] Grass T, Juliá-Díaz B, Ku M and Lewenstein M 2013 *Phys. Rev. Lett.* 111 090404

[57] Goyal S K, Simon B N, Singh R and Simon S 2016 *J. Phys. A: Math. Theor.* 49 165203