Binary self-similar one-dimensional quasilattices: 
Mutual local-derivability classification and substitution rules

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Self-similar binary one-dimensional (1D) quasilattices (QLs) are classified into mutual local-derivability (MLD) classes. It is shown that the MLD classification is closely related to the number-theoretical classification of parameters which specify the self-similar binary 1D QLs. An algorithm to derive an explicit substitution rule, which prescribes the transformation of a QL into another QL in the same MLD class, is presented. An explicit inflation rule, which prescribes the transformation of the self-similar 1D QL into itself, is obtained as a composition of the explicit substitution rules. Symmetric substitution rules and symmetric inflation rules are extensively discussed.

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I. INTRODUCTION

A quasiperiodic structure lacks the periodicity but has a long-range positional order. Quasiperiodic structures have been intensively investigated since the experimental discovery of quasicrystals, which have quasiperiodic structures with non-crystallographic rotational symmetries. In order to discuss the peculiar properties of quasicrystals, many models for the structures of quasicrystals have been proposed. The most successful model is based on a quasilattice (QL), which is a quasiperiodic and discrete set of points.

The mutual local-derivability (MLD) is one of the most important classification schemes for QLs. According to a recent study of a tight-binding model on a one-dimensional (1D) QL, two 1D QLs belonging to different MLD classes belong to different universality classes with respect to their one-electron properties.

Some of the quasiperiodic structures have self-similarities. In particular, all the quasicrystals found experimentally are closely related to self-similar quasiperiodic structures. On the other hand, the self-similarity has been used successfully for a rigorous study of the one-electron properties of 1D QLs because it matches well renormalization methods such as the trace map.

Many studies have focused on self-similar 1D quasiperiodic structures, but some important points have been left. Kaneko and Odagaki have discussed the quasiperiodic sequence of letters (i.e., objects without geometrical length), which are produced by associating different letters to different types of intervals (atomic distance) in QLs. They have shown a procedure to obtain an explicit inflation rule (IR), which prescribes the transformation of a self-similar quasiperiodic structure into itself, for a binary quasiperiodic sequence of letters characterized by a quadratic irrational. However, since the quasiperiodic sequence of letters is not a geometrical object, the geometrical meaning of the explicit IR is obscure. Binary sequences derived from substitution rules (IRs in our terminology) are extensively investigated in Ref. 8. They found that there exist hierarchical series of substitution rules which have good properties with respect to the trace map. They discussed interrelations among different substitution rules. This work also treats the binary sequences as sequences of letters. Yamashita has investigated the relation between the projection method and two series of binary quasiperiodic sequences of letters with inflation symmetries. However, he has not treated the sequences as geometrical objects. As a consequence of considering the quasiperiodic sequence of letters instead of the QL itself, these works have not revealed the geometrical meaning of the self-similarity and the IR.

In order to consider the geometrical meaning of the self-similarity and the IR, it is necessary to investigate the QLs instead of the sequences of letters. In Ref. 13, the self-similarity of 1D QLs has been investigated, and the geometrical meaning of the self-similarity has been revealed. However, a general algorithm to obtain an explicit IR has not been shown. In Ref. 14, the inflation symmetry of binary and ternary 1D QLs has been extensively investigated but this study has been restricted to the cases where the quadratic irrationals characterizing the QLs belong to two infinite series, one of which is the series composed of all the quadratic irrationals of rank one in our terminology to be presented in a later section, while the other is a subseries of quadratic irrationals of rank two.
In the present paper, we restrict our considerations to self-similar binary 1D QLs but we do not impose any condition on the quadratic irrationals characterizing the QLs. The purpose of the present paper is to classify all the self-similar binary 1D QLs into MLD classes and to derive explicit substitution rules (SR), which relate different members of a single MLD class.

In the next section, we make a review of the self-similarity of binary QLs. Here we identify parameters which characterize these QLs. In Sec. II, we classify the QLs into MLD classes on the basis of the number-theoretical properties of the parameters. Also, we give an explicit SR which combines two QLs in a single MLD class. In Sec. III, we discuss composite SRs and IRs. We show several examples of self-similar binary QLs in Sec. IV. In Sec. V, we discuss symmetric SRs and symmetric IRs. Miscellaneous subjects are discussed in Sec. VII. In the last section, we summarize the results, and discuss a few remained subjects.

Finally, a list of some of abbreviated words appearing in the present paper is presented: inflation rule (IR), substitution rule (SR), mutual local-derivability (MLD), unimodular Frobenius matrix (UFM), modular equivalent class (MEC), quasi-permutation-matrix (QPM), reduced quadratic irrational (RQI), and quasi-reduced quadratic irrational (QRQI). The last five words are defined and discussed in Appendix.

II. SELF-SIMILARITY OF BINARY 1D QLS

A. Binary QLs in the projection method

A QL is derived, as shown in Fig. 1, with the projection method from a 2D periodic lattice Λ, which is called a mother lattice. The 2D space embedding Λ is spanned by two orthogonal subspaces E∥ and E⊥, which are called the physical space and the internal space, respectively. We shall use throughout this paper the coordinate system in which the x axis and the y axis are chosen to be E∥ and E⊥, respectively. The physical space is assumed to be in an incommensurate configuration with Λ. One of the two basis vectors, a and b, of Λ is assumed to be in the first quadrant and the other in the fourth quadrant, so that the fundamental parallelogram which is spanned by a and b is cut by E⊥. If all the lattice points within a strip which is parallel to E∥ are projected onto E∥, we obtain a QL as shown in Fig. 1. The QL divides naturally E∥ into intervals, so that we shall identify the QL with the relevant 1D tiling of E∥. In the present paper, we restrict our considerations to binary QLs.

We call the section of the strip cut by E⊥ a window. A binary QL is obtained as shown in Fig. 1 if the size of the window is equal to the vertical width of the fundamental parallelogram. The two types of intervals are denoted as α and β, whose lengths, a and b, are equal to the x-components of the two basis vectors. It is essential in obtaining a binary QL that one of the two basis vectors of Λ is in the first quadrant and the other in the fourth quadrant and that the size of the window is equal to the vertical width of the fundamental parallelogram. Therefore, we shall call such a set of basis vectors a canonical set and the relevant window a canonical window; a canonical window is associated with a canonical set. Since there exist an infinite variety of canonical sets of basis vectors, there exist an infinite variety of canonical windows. This does not mean, however, that there exist an infinite variety of binary QLs with a common mother lattice because different QLs can be (geometrically) similar to one another. It will be shown in a later section that, under a certain condition, there exist only a finite number of binary QLs which are derived with a common mother lattice because different QLs can be (geometrically) similar to one another. A necessary and sufficient condition for two QLs to be mutually LI is that the relevant two windows have a common size. In the present paper, we will not distinguish two QLs if they are mutually LI. We denote Q ≃ Q′ when two QLs, Q and Q′, are mutually LI. A QL derived from a mother lattice is uniquely determined by the window, W, and the QL specified by W is denoted as Q(W). We will not hereafter distinguish the window, W, and its size, |W|.

It is known that a Fourier module, i.e., an additive group formed of reciprocal lattice vectors of a QL, is given by projecting the Fourier module of the mother lattice onto the reciprocal physical space. Therefore different QLs derived from a single mother lattice but with windows of different sizes have a common Fourier module.

The scale of the internal space, E⊥, is irrelevant on the construction of a QL because the projection is made along E⊥ in the projection method. Hence, two mother lattices which only differ in the scale of the internal space are considered to be isomorphic, and are not mutually distinguished. On the other hand, a scale change of the physical
space, $E_{\parallel}$, results in a scaling of a QL, so that to fix the scale is equivalent to fix the lattice constant of the QL. In a usual classification scheme of QLs, two QLs being geometrically similar to each other are assumed to be identical. Then, two mother lattices which differ in the scales of the physical space and/or the internal space are sometimes assumed to be identical.

### B. Self-similar binary QLs

The Fibonacci lattice is a representative self-similar binary QL. The sizes of two types of intervals, $\alpha$ and $\beta$, of the Fibonacci lattice are equal to 1 and $\tau_G$, respectively, where $\tau_G := (1 + \sqrt{5})/2$. If we substitute the intervals $\beta$ and $\alpha\beta$ for $\alpha$ and $\beta$ in the Fibonacci lattice, respectively, we obtain again the Fibonacci lattice. This transformation of the Fibonacci lattice into itself is known as the inflation rule (IR), which is represented as

$$\alpha \rightarrow \beta, \quad \beta \rightarrow \alpha\beta. \tag{2.1}$$

Each interval is scaled up by $\tau_G$ on the inflation.

The self-similarity of a QL is represented generally by an IR which prescribes how the inflated versions of the two intervals are divided into the original intervals. The IR accompanies naturally the inflation matrix $M$, which is a unimodular Frobenius matrix (UFM) as defined in Appendix A.2. For example, the inflation matrix of the Fibonacci lattice is given by

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.2}$$

The golden ratio, $\tau_G$, is not only the Frobenius eigenvalue of $M$ but also the companion quadratic irrational of $M$. Since the explicit form of the inflation matrix $M$ of a self-similar binary QL depends on the order of the two intervals in the defining equation for $M$, we shall fix the order so that the first interval is the shorter of the two. Then, $M$ is a quasi-normal UFM as defined in Appendix A.2. A self-similarity of a QL is derived in the projection method only when the mother lattice has the hyperscaling symmetry: this symmetry is an automorphism of $\Lambda$ and leaves the physical space and the internal space invariant.

Since the inflation matrix $M$ is unimodular, it determines a 2D lattice $\Lambda$ with the hyperscaling symmetry as shown in Appendix A.1. The physical space (or the internal space) is chosen to coincide with the expansive (or contractive) principal axis of the linear transformation $T$ associated with $M$ (see Appendix A.1). More precisely, $T$ enlarges along $E_{\parallel}$ by $\tau$, i.e., the Frobenius eigenvalue of $M$, but shrinks along $E_{\perp}$ by $\tau^{-1} = |\bar{\tau}|$. The left Frobenius eigenvector $(a \ b)$ with $a, b > 0$ satisfies the equation

$$\tau(a \ b) = (a \ b)M, \tag{2.3}$$

while the second left eigenvector, $(a_{\perp} \ b_{\perp})$, satisfies the equation

$$\bar{\tau}(a_{\perp} \ b_{\perp}) = (a_{\perp} \ b_{\perp})M. \tag{2.4}$$

The two basis vectors of $\Lambda$ are given by

$$a = \begin{pmatrix} a \\ a_{\perp} \end{pmatrix}, \quad b = \begin{pmatrix} b \\ b_{\perp} \end{pmatrix}. \tag{2.5}$$

It follows from Eqs. (2.3) and (2.4) that $T$ is an automorphism of $\Lambda$: $T$ yields the hyperscaling symmetry of $\Lambda$. Since the row vector $(a_{\perp} \ b_{\perp})$ is not a Frobenius eigenvector, $a_{\perp}$ and $b_{\perp}$ must have different signs. We can assume that $a_{\perp} < 0$ and $b_{\perp} > 0$. Then, $a$ and $b$ are in the fourth quadrant and the first quadrant, respectively, so that $\{a, b\}$ is a canonical set. The QL $Q(W)$ derived from $\Lambda$ is binary if

$$W = -a_{\perp} + b_{\perp} (= |a_{\perp}| + b_{\perp}), \tag{2.6}$$

and the two types of intervals of the binary QL have the lengths $a$ and $b$. Eq. (2.3) prescribes how the inflated versions of the two intervals are divided into the original intervals.

Let us introduce two positive parameters by

$$\xi := b/a, \quad \zeta := -a_{\perp}/b_{\perp}. \tag{2.7}$$
Then, Eqs. (2.3) and (2.4) are rewritten as
\[ \tau(1, \xi) = (1, \xi)M, \]
(2.8)
\[ \bar{\tau}(-\xi, 1) = (-\xi, 1)M. \]
(2.9)
Note that \( \xi > 1 \) because \( a < b \). Since \( M \) is an integer matrix, the row vector \( (1, \xi) \) must be also a left eigenvector corresponding to the eigenvalue \( \bar{\tau} \) of \( M \). Hence we have \( \xi = \tilde{\xi} \) with \( \tilde{\xi} := -1/\xi \) being the dual to \( \xi \) as defined in Appendix A. It can be shown that the ratio of the frequency of occurrence of the interval \( \alpha \) to that of \( \beta \) is equal to \( \xi \). Although \( \xi \) and \( \zeta \) belong both to the quadratic field \( \mathbb{Q}[\tau] \), their dependences on \( \tau \) are different in general from the case of the Fibonacci lattice for which \( \xi = \zeta = \tau \) (\( = \tau_G \)).

The projections of \( \Lambda \) onto \( E_{\|} \) and \( E_{\perp} \) yield modules, \( \Lambda_{\|} \) and \( \Lambda_{\perp} \), which are dense 1D sets. They are isomorphic to \( \Lambda \), and there are a natural bijection between any pair of the trinity, \( \{ \Lambda, \Lambda_{\|}, \Lambda_{\perp} \} \). A QL is nothing but a discrete subset of \( \Lambda_{\|} \). The projection modules are equal to the \( \mathbb{Z} \)-modules \( \mathbb{Z}[\xi] \) and \( \mathbb{Z}[\zeta] = \mathbb{Z}[\tilde{\xi}] \), respectively, where the physical space and the internal space are scaled so that \( a = b_{\perp} = 1 \), which we will sometimes assume. Remember that \( p + q\xi \in \mathbb{Z}[\xi] \) and \( -p\zeta + q \in \mathbb{Z}[\zeta] \) are combined by the bijection mentioned above.

A well-known formula for a binary QL yields the position of the \( n \)-th site (lattice point) of \( Q(W) \) with \( W = 1 + \xi \):
\[ x_n = n + (\xi - 1) \left\lfloor \frac{n + \phi}{1 + \zeta} \right\rfloor, \]
(2.10)
where \( \phi \) is the phase parameter depending on the position of the window in the internal space.

It is important that the hyperscaling operation \( T \) transforms a canonical set, \( \{ a, b \} \), into another canonical set, \( \{ a', b' \} := \{ Ta, Tb \} \). The new canonical set yields another binary QL, \( Q(W') \), where the new window \( W' \) is related to the old one by \( W' = \tau^{-1}W \). The sizes of the two types of intervals \( \alpha' \) or \( \beta' \) of \( Q(W') \) are given by \( a' := \tau a \) and \( b' := \tau b \). The hyperscaling symmetry of \( \Lambda \) results in that \( \tau Q(W) \cong Q(W') \), which means that \( Q(W) \) is a self-similar QL, whose ratio of the self-similarity is equal to \( \tau \). However, it is not obvious whether the two types of intervals \( \alpha' \) and \( \beta' \) are uniformly decorated by the two types of intervals \( \alpha \) and \( \beta \) because the inflation matrix does not directly prescribe the order of the latter intervals in \( \alpha' \) or \( \beta' \). The uniformity is evident for the case of the Fibonacci lattice as illustrated in Fig. 3. A proof of the uniformity for a generic case will be given in the next section. Note that the ratio of the self-similarity (i.e., the scaling factor \( \tau \)) of a binary QL is restricted to a quadratic irrational of the form \( \frac{a + \sqrt{d}}{1} \) given in Appendix A II.

Since the hyperscaling symmetry of the mother lattice is essential for the self-similarity of the relevant binary QLS, we shall confine our considerations to mother lattice with hyperscaling symmetries. Then, every binary QL obtained by the projection method from such a 2D lattice is self-similar, so that binary QLS appearing in the present paper are all self-similar. Therefore, we may call a self-similar binary QL simply as a binary QL.

It is evident that \( T^n \) with \( n \in \mathbb{Z} \) transforms a canonical set of basis vectors into another. We say that two canonical sets are scaling-equivalent if one of them is transformed by \( T^n \) with \( n \in \mathbb{Z} \) into the other. Then, the relevant two binary QLSs are mutually similar, and we need not distinguish them in a classification of QLSs. The canonical windows of two QLSs which are scaling-equivalent are related to each other by \( W' = \tau^{-n}W \) with \( n \in \mathbb{Z} \).

Let \( \mathcal{W} \) be the set of all the canonical windows. Then, it is a discrete subset of the half line \( \mathbb{R}^+ \). It has the scaling symmetry, \( \tau^{\pm 1} \mathcal{W} = \mathcal{W} \), and is divided into scaling-equivalent classes. Two binary QLSs with windows belonging to different scaling-equivalent classes are not mutually similar.

The parameter \( \xi \) characterizing a binary QL satisfies \( \xi > 1 \) and \( \tilde{\xi} > 0 \), so that is a quasi-reduced quadratic irrational (QRQI) as defined in Refs. [10][11] and discussed in Appendix A. Conversely, the companion matrix of every QRQI is a quasi-normal UFM, which determines a 2D lattice with a hyperscaling symmetry and the relevant binary QL. Thus, there exists a bijection (one-to-one correspondence) between any pair of the three sets: i) the set of all the QRQIs, ii) the set of all the quasi-normal UFMs, and iii) the set of all the binary QLSs. In what follows, a binary QL specified by a QRQI \( \xi \) will be denoted as \( Q\{\xi\} \). Since an inflation matrix is a quasi-normal UFM, it determines uniquely a binary QL (exactly, an LI class of binary QLSs). Note that every quasi-normal UFM can be an inflation matrix of a binary QL.

### III. CLASSIFICATION OF BINARY SELF-SIMILAR QLS INTO MLD CLASSES

If, between two QLSs \( Q \) and \( Q' \), there exists a uniform local-rule for the transformation of \( Q \) into \( Q' \), \( Q' \) is said to be locally derivable from \( Q \); moreover, if \( Q \) is also locally derivable from \( Q' \), we say that \( Q \) and \( Q' \) are mutually locally derivable (MLD) [12].
According to a general theory of MLD relationship developed in Ref. 3, a necessary condition for two QLs to be MLD from each other is that they are derived by the projection method from a common mother lattice, where two mother lattices which differ in the scales of the physical space and the internal space are not distinguished as mentioned at the end of Sec. 1A. Therefore, two binary QLs which are MLD but are not mutually similar are derived by two canonical sets, \( \{ \mathbf{a}, \mathbf{b} \} \) and \( \{ \mathbf{a}', \mathbf{b}' \} \), which are inequivalent with respect to the hyperscaling. Let \( X \) be a unimodular matrix which combines two canonical sets:

\[
(\mathbf{a}' \mathbf{b}') = (\mathbf{a} \mathbf{b})X.
\]  

(3.1)

Then, the argument in Appendix A.4 results in that the ratio of the two intervals, \( \xi' := b'/a' \), is related to the ratio, \( \xi := b/a \), by the modular transformation: \( \xi' = X^{-1}(\xi) \); the two ratios are modular equivalent. This is a necessary condition for two binary QLs, \( Q\{\xi\} \) and \( Q\{\xi'\} \), to be MLD from each other. The next problem is to show that this is also a sufficient condition.

Using Eqs. (2.4) and Eq. (3.1), we can calculate the size of the canonical window \( W' \) as \( W' = |(t - u)\zeta - v + w| \), so that the difference \( W - W' \) with \( W = 1 + \zeta \) belongs to the module \( \mathbb{Z}[\zeta] \), where the relevant UFM \( X \) is assumed to be given by Eqs. (A18). Hence, we can conclude by a general theory of the MLD-relationship among different QLs that \( Q\{\xi\} \) and \( Q\{\xi'\} \) are MLD. This completes a proof of the sufficiency of the condition above. We will show in the next section directly the MLD-relationship between \( Q\{\xi\} \) and \( Q\{\xi'\} \) by deriving an explicit substitution rules (SR) combining the two QLs. At all events, this exists a bijection between the set of all the MLD classes of binary QLs and the set of all the modular equivalent classes (MECs) of QRQIs. Therefore, we can specify an MLD class by the corresponding MEC, \( \langle k_0, k_1, \cdots, k_{n-1} \rangle \), of QRQIs, or, equivalently, by the corresponding MEC of the quasi-normal UFMs, where \( n \) is the rank of the class. Thus, there exist \( N = k_0 + k_1 + \cdots + k_{n-1} \) binary QLs in the relevant MLD class, and the \( N \) binary QLs are: \( Q\{\xi^{(j)}\} \) with \( j = 0, 1, 2, \cdots, n - 1 \) and \( k = 0, 1, 2, \cdots, k_j - 1 \), where the \( \xi^{(j)} \) are QRQIs defined in Appendix A.4. We shall call \( N \) the order of the MLD class. Other important parameters of the MLD class is \( m \) and \( e = (-1)^m \), where \( m \) is the common trace of the relevant UFMs.

It is evident that each MLD class has its own mother lattice. That is, there exists a bijection between the set of all the MLD classes of binary QLs and the set of all the mother lattices with hyperscaling symmetries.

The MLD-relationship between two QLs, \( Q\{\xi\} \) and \( Q\{\xi'\} \), in an MLD class is particularly simple for the case where the modular transformation, \( \xi' = X^{-1}(\xi) \), coincides with either of the two elementary modular transformations presented in Appendix A.3. The matrix \( U \) given by Eq. (A11) is identical to the matrix \( M \) given by Eq. (2.2), and the two canonical sets in this case are related to each other by \( \mathbf{a}' = \mathbf{b} \) and \( \mathbf{b}' = \mathbf{a} + \mathbf{b} \). Then, the SR relating the two sets of the intervals is the same as the SR (2.3) of the Fibonacci lattice:

\[
\alpha' = \beta, \quad \beta' = \alpha \beta.
\]  

(3.2)

This SR is closely related to the matrix \( U \), so that we may call \( U \) the substitution matrix of this SR. On the other hand, the two canonical sets are related, in the case of the matrix \( S \) given by Eq. (A11), to each other by \( \mathbf{a}' = \mathbf{a} \) and \( \mathbf{b}' = \mathbf{a} + \mathbf{b} \). Then, we can show by a similar geometrical argument to the one by which the SR above is derived that the SR for this case is given by

\[
\alpha' = \alpha, \quad \beta' = \beta \alpha,
\]  

(3.3)

which is illustrated in Fig. 3. The substitution matrix for this SR is \( S \).

A simple geometrical rule for the choice between the two SRs (3.2) and (3.3) is given as follows: the rule (3.2) or (3.3) should be chosen according as \( \{ \mathbf{b}, \mathbf{c} \} \) or \( \{ \mathbf{a}, \mathbf{c} \} \) with \( \mathbf{c} := \mathbf{a} + \mathbf{b} \) is a canonical set; \( \mathbf{c} \) is the diagonal vector of the relevant parallelogram. This provides us with an elementary procedure which allows successive generation of canonical sets of basis vectors. This successive procedure can be regarded, alternatively, as a recursive procedure of generating a series of “canonical” basis vectors, \( \mathbf{a}_i, i = 0, 1, 2, \cdots \), with the initial conditions, \( \mathbf{a}_0 = \mathbf{a} \) and \( \mathbf{a}_1 = \mathbf{b} \). We shall call it an additive algorithm. The inverse procedure of it is a subtractive algorithm: one and only one of \( \{ \mathbf{d}, \mathbf{a} \} \) and \( \{ \mathbf{d}, \mathbf{b} \} \) with \( \mathbf{d} := \mathbf{b} - \mathbf{a} \) is a canonical set, where \( \mathbf{d} \) is the second diagonal vector of the relevant parallelogram. The additive algorithm (or the subtractive algorithm) may be called the diagonal (or anti-diagonal) algorithm. At any rate, we can generate a both-infinite set of canonical basis vectors \( \Gamma := \{ \mathbf{a}_i \,| \, i \in \mathbb{Z} \} \), and the \( i \)-th canonical set of basis vectors is given by \( \{ \mathbf{a}_k, \mathbf{a}_i \} \), where \( k \) is the maximum number under the condition that it is smaller than \( i \) and the sign of the second component of \( \mathbf{a}_k \) is opposite to that of \( \mathbf{a}_i \). Hence there is a bijection between \( \Gamma \) and the set of all the canonical sets of basis vectors, and we can identify them. The hyper-scaling symmetry of \( \Lambda \) results in \( \mathbf{a}_{i+N} = \mathbf{T} \mathbf{a}_i \), so that \( \Gamma \) is divided into \( N \) scaling-equivalent classes (series), which form an \( N \)-cycle, i.e., a cyclically ordered set of \( N \) elements.

The properties of the trinity, \( \{ \Lambda, \Lambda_{||}, \Lambda_{\perp} \} \), is naturally succeeded by the trinity, \( \{ \Gamma, \Gamma_{||}, \Gamma_{\perp} \} \), where \( \Gamma_{||} := \{ \mathbf{a}_i \,| \, i \in \mathbb{Z} \} \) and \( \Gamma_{\perp} := \{ \mathbf{a}_i^+ \,| \, i \in \mathbb{Z} \} \) are the projections of \( \Gamma \) onto \( E_{||} \) and \( E_{\perp} \), respectively. In particular, \( \Gamma_{||} \) and \( \Gamma_{\perp} \) have
the scaling symmetry, \( a_{i+N} = \tau a_i \) and \( a_{i+N}^\perp = \bar{\tau} a_i^\perp \), and are divided into \( N \) scaling-equivalent classes, which form \( N \)-cycles. The former of the two is only composed of positive numbers but the latter is not. If the signs of the members are ignored, \( \Gamma \perp \) coincides with \( \mathcal{W} \) because each member of \( \Gamma \) is the “anti-diagonal” of the parallelogram associated with a canonical set of basis vectors. If \( \{ a_k, a_i \} \) is a canonical set above, we obtain \( \xi_i = a_i/a_k \) and \( \zeta_i = -a_k^\perp/a_i^\perp \), which are periodic: \( \xi_{i+N} = \xi_i \) and \( \zeta_{i+N} = \zeta_i \). A 2D matrix is defined with the canonical set by \( A_i := (a_k^\perp, a_i) \).

It satisfies \( A_{i+1} = A_i S \) if \( a_i^\perp \) and \( a_{i+1}^\perp \) have a common sign but \( A_{i+1} = A_i U \) otherwise. It satisfies, furthermore, \( TA_i = A_{i+N} = A_i M \), where \( M \) is the companion UFM of \( \xi \).

We may write \( A_i = (a^\perp, a) X_i \), where

\[
X_i := \begin{pmatrix} t_i & u_i \\ v_i & w_i \end{pmatrix}
\]

is a unimodular matrix. It is a quasi-normal UFM if \( i \geq 2 \) but the reciprocal of a quasi-normal UFM if \( i \leq 0 \), while \( X_1 = I \). It satisfies \( X_{i+N} = X_i M \). Since \( A_i = (a^\perp, a) \), we obtain \( a_i = u_i a + w_i b, \ a_i = u_i + w_i \xi \in \mathbb{Z}[\xi] \), and \( a_i^\perp = -u_i \zeta + w_i \in \mathbb{Z}[\zeta] \). The series of rationals, \( w_i/u_i \) (or \(-u_i/w_i\)), \( i = 1, 2, \ldots \), is just the series of best approximants to \( \xi \) (or \( \zeta \)). The procedure to obtain approximants to an irrational on the basis of the geometry of a lattice has been established in the classical number theory.

It is sometimes convenient to change a single suffix numbering, \( x_i \), into a double suffix numbering, \( x_{i}^{(j)} \), with \( i = j + nN \), where \( j = 1, 2, \ldots, N \) and \( n \in \mathbb{Z} \). Then a relation combining \( x_i \) and \( x_{i+N} \) turns to that combining \( x_{i}^{(j)} \) and \( x_{i+N}^{(j)} \). Using the equality \( X_{i}^{(j)} = X_0^{(j)} M^n \) together with the Cayley-Hamilton theorem, \( M^2 - mM + eI = 0 \), we can conclude that the quantities \( a_{i}^{(j)}, a_{n}^{(j)}, X_{i}^{(j)}, a_{n}^{(j)} \), and \( a_{i}^\perp(\beta) \) satisfy two-term recursion relations which are isomorphic to \( x_{i}^{(j)} = x_{i-N}^{(j)} - \varepsilon x_{i-2}^{(j)} \); the solution of the recursion relation is written as \( x_n = F_n x_1 - eF_{n-1} x_0 \) with \( F_n \) being the generalized Fibonacci numbers associated with the quadratic irrational \( \tau \). Remember, however, that the double suffix numbering must not apply to \( \xi \) and \( \zeta \).

The \( N \) QRQIs in the MEC introduced above form an \( N \)-cycle of modular transformations, and the relation between successive two members of the cycle is given by

\[
\xi_k^{(j)} = S(\xi_{k-1}^{(j)})
\]

if \( k > 0 \) but

\[
\xi_0^{(j)} = U(\xi_{k-1}^{(j)})
\]

with \( k = k_{j-1} - 1 \). It follows that successive two members of the cycle of \( N \) binary QLs are related by the SR \([3.2]\) or \([3.3]\). Moreover, if two members of the \( N \)-cycle are chosen arbitrarily, each of the two is derived from the other by a successive operation of the two types of the elementary SRs. That is, the two QLs are combined by composite SRs.

It is appropriate at this point to define exactly the SR which combines two QLs. We shall distinguish for a while the two types of intervals \( \alpha' \) or \( \beta' \) from two types of sequences \( \sigma_{\alpha} \) or \( \sigma_{\beta} \) composed of the two types of intervals \( \alpha \) or \( \beta \). An SR is represented as \( \alpha' = \sigma_{\alpha}, \ \beta' = \sigma_{\beta} \). Let us denote a QL composed of the two types of intervals \( \alpha \) or \( \beta \) as \( Q[\alpha, \beta] \). If \( Q[\alpha, \beta] \) is represented as a sequence of two types of sequences \( \sigma_{\alpha} \) or \( \sigma_{\beta} \), there exists a new QL \( Q'[\alpha', \beta'] \), so that \( Q[\alpha, \beta] = Q'[\sigma_{\alpha}, \sigma_{\beta}] \). If the SR is used to derive \( Q'[\alpha', \beta'] \) from \( Q[\alpha, \beta] \), we may call it a passive SR. Contrary to it, if it is used to derive the latter from the former, we may call it an active SR. The passive SR and the active one are the inverse procedure of each other.

The substitution matrix of any SR is naturally a quasi-normal UFM. If the QL \( Q[\xi'] \) is transformed by a passive SR into \( Q'[\xi'] \), the relevant two quadratic irrationals is related by the modular transformation as \( \xi = X(\xi') \) with \( X \) being the relevant substitution matrix. If an SR with the substitution matrix \( X \) is operated first, and the second one with \( X' \) is operated subsequently, the substitution matrix of the resulting composite SR is given by \( X X' \) in the passive case but \( X' X \) in the active case. Remember that, for the passive case, the composition of two SRs is anti-isomorphic to the product of the relevant substitution matrices. In the present paper, we will use SRs in the passive meaning unless stated otherwise.

Each of the \( N \) scaling-equivalent classes in \( \mathcal{W} \) has one and only one member in the fundamental interval, \( \bar{F} := [1, \tau] \). Let \( \mathcal{W}_0 := \mathcal{W} \cap \bar{F} \) with \( \bar{F} := [1, \tau] \). Then, it includes \( N + 1 \) windows. The pattern of the distribution of the windows in \( \mathcal{W}_0 \) characterizes completely the relevant MLD class, so that \( \mathcal{W}_0 \) is like a fingerprint of the MLD class. For example, if \( \xi = \{1, 2, 3\} = (1 + \sqrt{10})/3 \), we obtain \( \mathcal{W}_0 = \{1, \xi, 1 + \xi, 1 + 2\xi, 2 + 3\xi\} \), which is shown in Fig. \[4\]. The last member of this “fingerprint” is equal to \( \tau = 3 + \sqrt{10} \), which is scaling-equivalent to the first member.

In general, \( \bar{F} \) is divided into \( N \) subintervals by \( \mathcal{W}_0 \). Every subinterval is limited by two canonical windows, and the two relevant binary QLs have one common interval. If a window is chosen to be an internal point of the subinterval,
the resulting QL is a ternary QL composed of the three types of intervals associated with the two binary QLs. That is, the ternary QL is a hybrid of the two binary QLs.

The canonical windows are vertical widths of the fundamental parallelograms, and we may call their horizontal widths canonical interval (lengths) by a reason to be understood shortly. Let us denote by \( V \) the set of all the canonical intervals. Then it is identical to \( \Gamma \parallel \). The sizes of the two intervals included in a binary QL derived from the relevant mother lattice are given by consecutive two members of \( V \). A new “finger print” \( V_0 \) can be defined with respect to the scaling-invariant set \( V \) in a similar way as the one for \( W_0 \). It is remarkable that the two types of “finger prints” coincide for the MLD classes like \( \langle\langle 1, 2, 1 \rangle\rangle \). The reason will be revealed in a later section.

IV. COMPOSITE SUBSTITUTION RULES AND INFLATION RULES

It is sometimes convenient to employ composite SRs associated with the two types of UFM which are given as blocked forms, \( L = S^l \) and \( K = US^{k-1} \) with \( k \) and \( l \) being natural numbers. The SR for the former case is \( l \) fold repetition of the rule (3.3), so that we obtain

\[
\alpha' = \alpha, \quad \beta' = \beta \alpha^l,
\]

where \( \alpha' \) stands for a concatenation of \( l \) os. This SR is a direct consequence of the equations, \( \alpha' = \mathbf{a} \) and \( \beta' = \mathbf{a} + \mathbf{b} \). The explicit form of the substitution matrix of this SR is given by Eq. (A13) in Appendix A2. On the other hand, the SR for the substitution matrix \( K = US^{k-1} \) is a composition of the rule (4.1) with \( l = k - 1 \) and the rule (3.2), so that we obtain

\[
\alpha' = \beta, \quad \beta' = \alpha \beta^k.
\]

This SR is a direct consequence of the equations, \( \mathbf{a}' = \mathbf{b} \) and \( \mathbf{b}' = \mathbf{a} + \mathbf{k} \). The two block SRs just obtained have simple geometrical meanings, which can also be understood easily. Since a quasi-normal UFM is represented as Eq. (A10), every composite SR can be represented as a composition of a number of block SRs of the types (4.1) and (4.2).

A composition of two SRs as well as a product of the corresponding two substitution matrices is incommutable. For example, the SR with the substitution matrix \( SU \) is given by

\[
\alpha' = \beta \alpha, \quad \beta' = \alpha \beta \alpha,
\]

which is markedly different from the SR with the substitution matrix \( US \); this SR is given by Eq. (1.2) with \( k = 2 \).

Let \( Q \) and \( Q' \) be two QLs which are combined by the SR (1.3). Then, they are combined also by another SR, \( \alpha' = \alpha \beta, \beta' = \alpha \beta^2 \), which is obtained from the former by moving the interval \( \alpha \) at the right ends of the two relevant sequences of the intervals to their left ends. Therefore, these two SRs are equivalent. By similar procedures, we can obtain another two equivalent SRs: \( \alpha' = \beta \alpha, \beta' = \beta \alpha^2 \) and \( \alpha' = \alpha \beta, \beta' = \alpha \beta \). Conversely, by applying repeatedly the inverse procedure to the last SR, we can retrieve other three SRs. These proper and inverse procedures are called cyclic shifts. They yield a linearly ordered finite set of equivalent SRs, and the set is retrieved from any member of the set. The SRs in the set have a common substitution matrix. We will not distinguish equivalent but apparently different SRs. Note that the full set of equivalent SRs has a sort of mirror symmetry; an SR in the ordered set and the one at the mirror site are mutually mirror symmetric because each sequence of the former is the mirror image of the relevant sequence of the latter.

An MLD class of binary QLs has the structure of an \( N \)-cycle as mentioned in the preceding section. Therefore, if the two types of elementary SRs are applied \( N \) times in an appropriate order to an arbitrarily chosen member of the \( N \)-cycle, the chosen member is retrieved. In other words, every binary QL is transformed into itself by a composite SR. Therefore, it is self-similar, and the composite SR is nothing but the IR. Since the inflation matrix of a QL is written as Eq. (A14), the relevant IR is obtained by a composition of the two types of the block SRs (1.1) and (1.2).

V. EXAMPLES

A. MLD classes of rank one

MLD classes of rank one form a series specified by natural numbers, and most of important 1D QLs including the Fibonacci lattice belong to some MLD classes of this series. The MLD class is specified by \( \langle\langle m \rangle\rangle \) in our notation, for which \( N = m \) and \( e = -1 \). It has only one RQI as given by
\[
\tau = \frac{1}{2} \left( m + \sqrt{m^2 + 4} \right),
\]

(i.e., the precious mean, which presents the scale of the self-similarity of the binary QLs in the class. There exist \( m \) QRQIs, \( \xi_k^{(0)} = \tau - k \) with \( k = 0, 1, \cdots, m - 1 \).

The inflation matrix for \( Q\{s_k^{(0)}\} \) is written with Eq. (A16) as

\[
M_k = S^k U S^{m-k-1} = \begin{pmatrix} k & k(m-k) + 1 \\ 1 & m-k \end{pmatrix}.
\]

The IR for \( Q\{\xi_k^{(0)}\} \) is given as a composition of two block SRs corresponding to \( U S^{m-k-1} \) and \( S^k \). Using Eqs. (4.1) and (4.2), we obtain

\[
\alpha' = \beta \alpha^k, \quad \beta' = \alpha (\beta \alpha^k)^{m-k}.
\]

which is equivalent to the IR given in Ref. [14] or given by equation (20) in Ref. [8].

There exists only one MLD class for \( N = 1 \) or \( N = 2 \) but there exist two or more for \( N \geq 3 \). The MLD class \( \langle 1 \rangle \) is only composed of the Fibonacci lattice, while \( \langle 2 \rangle \) is composed of two MLD classes whose ratio of self-similarity is the silver mean, \( \tau = 1 + \sqrt{2} \). The IRs of the two binary QLs in \( \langle 2 \rangle \) are given by the above equations with \( k = 0 \) and \( k = 1 \), and the relevant quadratic irrationals are \( \xi_0^{(0)} = 1 + \sqrt{2} \) and \( \xi_1^{(0)} = \sqrt{2} \), respectively. The two QLs are transformed to each other by the elementary SRs, \( \langle 6 \rangle \) and \( \langle 5 \rangle \).

The ratio of self-similarity, \( \tau \), of a QL must be generally a positive root of the quadratic equation \( z^2 - mz + e = 0 \) with \( m \) being a natural number and \( e = \pm 1 \). There exists at least one MLD class with given \( m \) and \( e \). A representative of such MLD classes is \( \langle m \rangle \) if \( e = -1 \). There exists only one MLD class with given \( m \) and \( e \) if \( m \leq 5 \) and \( e = -1 \). The two MLD classes, \( \langle 6 \rangle \) and \( \langle 1, 2, 1 \rangle \), are commonly characterized by \( m = 6 \) and \( e = -1 \) (namely, \( \tau = 3 + \sqrt{10} \)).

A remarkable feature of binary QLs of rank one is that the relevant parameters satisfy the condition, \( \mathbb{Z}[\xi] = \mathbb{Z}[\tau] \), which results in that the relevant QLs have simple IRs.

### B. MLD classes of rank two

MLD classes of rank two form a 2D “series”, \( \langle k_0, k_1 \rangle \), each of which is specified by a pair of natural numbers, and we obtain \( e = \pm 1 \). We can restrict our considerations to the case \( k_0 > k_1 \). An important subseries is formed of \( \langle m - 2, 1 \rangle \) with \( m \geq 4 \), for which \( N = m - 1 \). This subseries has been investigated by several authors (see, for example, Ref. [7]).

The companion matrix \( M_0 = K_1 K_0 \) of the RQI \( \theta_0 := \langle m - 2, 1 \rangle \) is given by

\[
M_0 = \begin{pmatrix} 1 & m - 2 \\ 1 & m - 1 \end{pmatrix},
\]

whose Frobenius eigenvalue is

\[
\tau = \frac{1}{2} \left( m + \sqrt{m^2 - 4} \right),
\]

(i.e., the “anti-precious mean”. It presents the scale of the self-similarity of the binary QLs in the relevant class, and \( \theta_0 \) is written as \( \theta_0 = \tau - 1 \). The relevant MEC of QRQIs has \( m - 1 \) members, \( \xi_k^{(0)} := \tau - k \) with \( k = 1, 2, \cdots, m - 2 \) and \( \xi_0^{(1)} = \langle 1, m-2 \rangle = (\tau-1)/(m-2) \), where the parameter \( k \) is shifted for a convenience from that following the convention in Sec. [11]. The inflation matrices for the QLs, \( Q\{s_k^{(0)}\} \) and \( Q\{\xi_0^{(1)}\} \), are \( S^{k-1} U^2 S^{m-k-2} \) and \( U S^{m-3} U (= 'M_0) \), respectively. The IR for \( Q\{s_k^{(0)}\} \) is shown to be equivalent to

\[
\alpha' = \alpha^k \beta, \quad \beta' = \alpha^{k-1} \beta (\alpha \beta)^{m-k-1},
\]

which is also equivalent to the IR given in Ref. [14] or given by equation (20) in Ref. [8]. On the other hand, the IR for \( Q\{\xi_0^{(1)}\} \) is shown to be equivalent to

\[
\alpha' = \beta^{m-2} \alpha, \quad \beta' = \beta^{m-1} \alpha.
\]
If the notations for the two types of intervals are exchanged in this IR, the result coincides with the IR (5.6) but with \( k = m - 1 \). Therefore, the formula (5.3) is absorbed in (5.6). Note, however, that the second interval is shorter than the first one after the exchange of the notations. The explicit form of the inflation matrices are:

\[
M_k = S^{k-1}U^2S^{m-k-2} = \begin{pmatrix} k & k(m-k) - 1 \\ m-k & 1 \end{pmatrix}.
\]

(5.8)

The simplest MLD class of the subseries \( \langle m-2, 1 \rangle \) is given by setting \( m = 4 \). It includes three binary QLs whose ratio of self-similarity is \( \tau = 2 + \sqrt{3} \).

The inflation matrices of binary QLs of rank two have the property, \( Z[\xi] = Z[\tau] \) or \( Z[\xi^{-1}] = Z[\tau] \), only if they belong to the subseries above but this property is not possessed by other binary QLs of rank two nor those of higher ranks.

A representative of MLD classes with given \( m \) but with \( e = +1 \) is \( \langle m-2, 1 \rangle \). There exist no other MLD classes if \( m \leq 7 \). The two MLD classes, \( \langle 7, 1 \rangle \) and \( \langle 3, 2 \rangle \), are commonly characterized by \( m = 8 \) and \( e = +1 \) (namely, \( \tau = 4 + \sqrt{15} \)). In fact, the second MLD class is the simplest among rank two MLD classes which do not belong to the subseries above. It has five binary QLs corresponding to five QRQIs:

\[
\xi_0^{(0)} = \frac{3 + \sqrt{15}}{2}, \quad \xi_1^{(0)} = \frac{1 + \sqrt{15}}{2}, \quad \xi_2^{(0)} = \frac{-1 + \sqrt{15}}{2}, \\
\xi_0^{(1)} = 1 + \frac{\sqrt{15}}{3}, \quad \xi_1^{(1)} = \frac{\sqrt{15}}{3}.
\]

The MLD class \( \langle 3, 2 \rangle \) is considered to be the simplest member of the series of rank two MLD classes, \( \langle k, 2 \rangle \) with \( k \geq 3 \). The relevant parameters are: \( m = 2k + 2, e = +1, \) and \( N = k + 2 \). An important QL in the MLD class \( \langle k, 2 \rangle \) is \( Q\{\xi_1^{(1)}\} \), whose inflation matrix is \( S(US^{k-1}U)S^{-1} = SUS^{k-1}U \). Explicitly, the inflation matrix coincides with a quasi-normal UFM defined by

\[
M_k^+ := \begin{pmatrix} k + 1 & k + 2 \\ k & k + 1 \end{pmatrix},
\]

(5.9)

so that \( \theta_0 = (\tau-k-1)/k \). Since \( M \) is factorized into three blocks, \( (S)(US^{k-1})(U) \), the relevant IR is obtained as a composition of the three relevant SRs:

\[
\alpha' = \alpha(\beta\alpha)^k, \quad \beta' = \alpha(\beta\alpha)^{k+1}.
\]

(5.10)

A series of binary quasiperiodic sequences with this type of IRs has been investigated in Ref. [12]. Note, however, that the sequences in Ref. [12] are not geometrical objects but those of letters.

C. MLD classes of rank three

An important series of QLs formed of rank three MLD classes is given by \( \langle 1, k, 1 \rangle \) with \( k \geq 2 \). The relevant parameters are: \( m = 2k + 2, e = -1, \) and \( N = k + 2 \). The companion matrix \( M_0 = UKU = U^2S^{k-1}U \) of the RQI \( \theta_0 := (1, k, 1) \) coincides with the quasi-normal UFM defined by

\[
M_k^- := \begin{pmatrix} k & k + 1 \\ k + 1 & k + 2 \end{pmatrix},
\]

(5.11)

so that \( \theta_0 = (\tau-k)/(k+1) \). The IR for \( Q\{\theta_0\} \), for example, is given by

\[
\alpha' = \beta(\alpha\beta)^k, \quad \beta' = \beta(\alpha\beta)^{k+1}.
\]

(5.12)

A series of binary quasiperiodic sequences with this type of IRs has been also investigated in Ref. [12]. The simplest MLD class in the series is \( \langle 1, 2, 1 \rangle \), which has appeared at the end of the subsection VA. The inflation matrix given by Eq. (5.28) is a special case of the above with \( k = 2 \).
VI. SYMMETRIC SUBSTITUTION RULES AND SYMMETRIC INFLATION RULES

The composite SR (4.2) with \( k = 2 \) is equivalent to

\[
\alpha' = \beta, \quad \beta' = \beta \alpha \beta,
\]

which is explicitly symmetric, and we may say that the SR (4.2) with \( k = 2 \) is symmetric. More generally, the SR (4.2) is symmetric if \( k \) is even but not if \( k \) is odd. A similar argument applies to the SR (4.2). Explicitly, we may write

\[
\alpha' = \alpha, \quad \beta' = \alpha' \beta \alpha',
\]

if \( l = 2l' \) with \( l' \) being a natural number. In particular, if \( l = 2 \), it reduces to

\[
\alpha' = \alpha, \quad \beta' = \alpha \beta.
\]

On the other hand, the two elementary SRs (3.2) and (3.3) are asymmetric. Also the SR (4.3) is asymmetric, while \( \alpha' \) and \( \beta' \) of the symmetric IR (5.12) is \( \alpha' = \alpha' \beta \alpha ' \) and \( \beta' = (\alpha')^{m-k'} \alpha (\alpha')^{m-k'} \) with \( m' = m/2 \) and \( k' = k/2 \). A similar argument applies to the IR (5.3).

It is shown in Appendix A 3 that \( M^2 \), with \( M \) being an inflation matrix is always a QPM if \( m \) is even. Since \( M^2 \) is associated with the double IR, the double IR is equivalent to a symmetric IR. Similarly, the triple IR is equivalent to a symmetric IR if \( m \) is odd. Note that the scaling ratio of the double or triple IR is equal to \( \tau^2 \) or \( \tau^3 \), respectively. For example, if the symmetric IR given by Eq. (6.1) is doubled, we obtain the symmetric IR:

\[
\alpha' = \alpha (\beta \alpha)^2, \quad \beta' = \alpha (\beta \alpha)^3,
\]

whose ratio of inflation-symmetry is \((1 + \sqrt{2})^2\). On the other hand, if the IR of the Fibonacci lattice is tripled, we obtain the symmetric IR: \( \alpha' = \beta \alpha \beta \) and \( \beta' = \beta (\alpha \beta)^2 \), whose ratio of inflation-symmetry is \( \tau_2^3 \).

The central interval in the sequence \( \alpha' \) of the symmetric IR (5.12) is \( \alpha \) and that in \( \beta' \) if \( k \) is odd but these correspondences are reversed if \( k \) is even. The relevant substitution matrix is congruent in modulo 2 with \( I \) for the former case but with \( J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) for the latter. These observations are general properties of symmetric SRs and symmetric IRs.

If one of two QLs is derived from the other by a symmetric SR, and, moreover, the converse is also true, we may say that they are symmetrically MLD (s-MLD). Since the s-MLD is stronger than the conventional MLD, an MLD class can be divided into two or more s-MLD classes. For example, the MLD class \( \langle\langle m \rangle\rangle \) with an even \( m \) necessarily bifurcates because it includes two types of QLs with different scalings with respect to symmetric IRs. Note that a necessary condition for the two QLs to be s-MLD is that the relevant two windows is concentric.

A symmetric SR is called elementary if it cannot be represented as a composition of two symmetric SRs. Such an SR is associated with an elementary QPM defined in Appendix A 3. In particular, the SRs (6.1) and (6.3) are elementary symmetric SRs. Each s-MLD class is a cycle, and its two successive members are combined by an elementary symmetric SR. There are two important series of elementary QPMs, namely, (5.9) and (5.11) with \( k \geq 0 \). This is confirmed by the expressions \( M^k_{\alpha} = SUS^{k-1}U \) and \( M^k_{\beta} = U^2 S^{k-1}U \) for \( k \geq 1 \) but \( M^0_{\alpha} = S^2 = M^0_{\beta} = U \). The corresponding symmetric SRs are identical to the IRs, (5.10) and (5.12), respectively. That is, QLs belonging to the two series of QLs with inflation matrices, \( M^k_{\alpha} \) with \( k \geq 1 \) and \( M^k_{\beta} \) with \( k \geq 0 \), have the property that their inflation matrices are elementary. The two series are the main subjects in Ref. [2].

From here to the end of this section, we use SRs in the active meaning for a convenience; the directions of the arrows to appear should be reversed in the case of the passive meaning. A simplest example is again the case of a rank one MLD class, \( \langle\langle m \rangle\rangle \). For brevity we set \( Q_k := Q\{\xi_k^0\} \), whose inflation matrix is denoted as \( M_k \). If \( m \) is even, the inflation matrix for \( Q_0 \) is factorized into elementary QPMs as \( M_0 = US^{m-1} = (M^+_0)(M^-_0)^{m/2-1} \), which yields the following cycle of symmetric SRs:
\[ Q_0 \to Q_2 \to \cdots \to Q_{m-2} \Rightarrow Q_0, \]  
where the single arrow stands for the SR (6.3) but the double arrow the SR (6.1). The above cycle only includes the QLS \( Q_k \) with even \( k \). For \( Q_k \) with an odd \( k \), the double IR must be employed. The relevant inflation matrix for \( Q_1 \) is written as \( (M_1)^2 = SUS^{m-1}US^{m-2} = (M_0^+)(M_0^-)^{m/2-1} \), so that we obtain the following cycle of odd \( k \) QLS:

\[ Q_1 \to Q_3 \to \cdots \to Q_{m-1} \Rightarrow Q_1, \]  
where the double arrow stands for the symmetric SR which is given by (6.10) with \( k = m \). Thus, the MLD class \( \langle \langle m \rangle \rangle \) with an even \( m \) is divided evenly into the even s-MLD class and the odd s-MLD class. Each of the two s-MLD classes for the case of \( m = 2 \) is composed of a single QL, \( Q_0 \) or \( Q_1 \). The symmetric IR for \( Q_0 \) is given by (6.1), while that for \( Q_1 \) by (6.4).

We shall proceed to the case of the MLD class \( \langle \langle m \rangle \rangle \) with an odd \( m \). Then triple IRs are in order. The relevant inflation matrix for the case of \( Q_0 \) is factorized as

\[ (M_0)^3 = (US^{m-1})^3 = (M_0^-)(M_0^+)^{(m-3)/2}(M_0^+)^{(m-1)/2}, \]  
which reduces to Eq. (A17) if \( m = 5 \). Since the right hand side has \( m \) factors, the \( m \) QLS, \( Q_k \) with \( k = 0, 1, \ldots, m-1 \), are cyclically related by elementary symmetric SRs:

\[ Q_0 \to Q_2 \to \cdots \to Q_{m-1} \Rightarrow Q_1 \to Q_3 \to \cdots \]  
\[ \to Q_{m-2} \Rightarrow Q_0. \]  
That is, this MLD class is simultaneously an s-MLD class.

The above results for the case of a rank one MLD class is readily extended to the case of the subseries \( \langle \langle m-2, 1 \rangle \rangle \) of rank two MLD classes. We only present here the results of factorizations of the relevant inflation matrices. The case with an even \( m \) has two s-MLD classes, and the relevant factorization for the even s-MLD class is \( M_2 = (M_1^+)(M_0^-)^{m/2-2} \) but the one for the odd s-MLD class is \( (M_1)^2 = (M_0^-)(M_0^+)^{m/2-2} \). The relevant factorization for an odd \( m \) is \( (M_1)^3 = (M_{m-2}^-)(M_0^-)(M_0^+)^{(m-3)/2}(M_1^+)^{(m-3)/2} \). Note that \( Q_k \) with \( k = 0 \) is forbidden in the present case, so that the even s-MLD class with even \( m \) is composed of \( m/2 - 1 \) QLS. The situations are more complicated for other MLD classes of rank two or of higher ranks.

A quick derivation of a symmetric IR is possible if we use blocked SRs. The SR (6.2) is associated with the block \( (S^2)^{l'} \). Another two SRs associated with the two important blocks \( (M_k^+)(M_0^-)^p \) and \( (M_k^-)(M_0^+)^p \) are written with \( \dot{\beta} := \alpha^p\beta\alpha^p \) as

\[ \alpha' = \dot{\beta}(\alpha\dot{\beta})^k, \quad \beta' = \dot{\beta}(\alpha\dot{\beta})^{k+1}, \]  
\[ \alpha' = \alpha(\dot{\beta}\alpha)^k, \quad \beta' = \alpha(\dot{\beta}\alpha)^{k+1}. \]  
Although the SR (6.3) is asymmetric, it is equivalent to the symmetric but fractional SR:

\[ \alpha' = (\frac{1}{2}\dot{\beta})\alpha(\frac{1}{2}\dot{\beta}), \quad \beta' = (\frac{1}{2}\dot{\beta})\alpha^2(\frac{1}{2}\dot{\beta}). \]  
It can be shown generally that every asymmetric SR is equivalent to a symmetric but fractional SR. The same is true for asymmetric IRs. Thus, an MLD class is simultaneously an s-MLD class if fractional SRs are allowed.

**VII. MISCELLANEOUS SUBJECTS**

**A. Periodic approximants**

Let \( \alpha' \) and \( \beta' \) be two types of sequences of intervals \( \alpha \) and \( \beta \) of a QL, \( Q \), and assume that the pair of the sequences forms an SR. Then, the periodic structure \( (\alpha')^\infty \) (or \( (\beta')^\infty \)) obtained by an infinite concatenation of \( \alpha' \) (or \( \beta' \)) is called a periodic approximant to \( Q \). It is evident that the period of a periodic approximant is equal to a canonical interval, and different periodic approximants have different periods. Therefore, we shall denote by \( Q_n^{(2)} \) (\( n \geq 0 \)) the
periodic approximant corresponding to the canonical interval \(a_0^{(j)}\). The periodic approximant \(Q^{(j)}_{n+1}\) is changed by an active IR to \(Q^{(j)}_{n+1}\). Hence all the periodic approximants are derived by the successive application of the IR to the prototypes, \(Q^{(j)}_0\) with \(j = 0, 1, 2, \cdots, N-1\). The first three prototypes are \(\alpha, \beta, \) and \(\alpha\beta,\) whose periods are \(1, \xi,\) and \(1 + \xi,\) respectively. The set of the periods of \(N\) prototypes is identical to the “finger print” \(\nu_0.\) There is a bijection between the set of all the periodic approximants and the set of all the best approximants to the relevant QRQI. 

So far we have confined our argument to periodic approximants of a single QL. If an SR is applied to a periodic approximant to the QL, we obtain a periodic approximant to another QL in the same MLD class to which the original QL belongs.

### B. Duality

The properties of a QL are decisively ruled by the relevant quadratic irrational \(\xi.\) A quadratic irrational has the sole algebraic conjugate, and to take the algebraic conjugate of a quadratic irrational in the quadratic field \(Q(\xi)\) is an automorphism of \(Q(\xi).\) This automorphism is a dual relationship, so that the QL may have some properties associated with the duality.

If the mother lattice \(\Lambda\) is a square lattice, it has a canonical set of basis vectors \(\{a, b\}\) satisfying the conditions:

\[
a^2 = b^2, \quad a \cdot b = 0. \tag{7.1}
\]

If \(a\) and \(b\) are represented as Eq. (2.3), this condition leads \(a\alpha b + bb_\perp = 0,\) which is equivalent to \(\xi = \bar{\xi}\) because of Eq. (2.7). Since \(\xi = \bar{\xi},\) this means that \(\xi\) is strongly self-dual. Conversely, if \(\xi\) is strongly self-dual, the condition (7.1) is satisfied provided that the scale of the internal space is chosen appropriately. It follows that the mother lattice of an MLD class of binary QLSs is taken to be a square lattice if and only if the relevant MEC of the normal UFMs is self-dual.

If \(\xi\) is an RQI, so is its dual \(\bar{\xi}.\) We shall denote the mother lattices associated with the two RQIs as \(\Lambda\) and \(\bar{\Lambda},\) respectively; the two are dual to each other. Also, \(\nu_0\) is \(\nu_0\) and \(\nu_0\) are dual to each other. Since the projection modules of the two mother lattices are given by \(\Lambda = \mathbb{Z}[\xi]\) and \(\bar{\Lambda} = \mathbb{Z}[\bar{\xi}],\) the two mother lattices coincide if and only if the relevant MEC of the normal UFMs is self-dual; we may say then that \(\Lambda\) is self-dual. If \(\Lambda\) is self-dual, we obtain \(\nu = \nu_0\) and \(\nu_0 = \nu_0.\) We will discuss below the role of the dual mother lattice \(\Lambda,\) because \((\nu, \nu_0) = (\nu_0, \nu_0)\) because \((\nu, \nu_0) = (\nu_0, \nu_0).\) Note that the area of the rectangle, 

\[
\begin{array}{cc}
2 & 2 \\
0 & 0
\end{array}
\]

with \(a^* := 2\pi/(1 + \xi)\) because \((\nu A^*) = 2\pi I.\) Thus, \(\Lambda\) and \(\Lambda^*\) are dual to each other except a scale factor. The effect of the hyper-scaling on to \(\Lambda^*\) is represented as \(T A^* = A^* T M.\) The structure factor of \(Q\) is closely related to a
1D QL whose mother lattice is $\Lambda^*$. As a consequence, it is approximately self-similar. If $\Lambda$ is self-dual in particular, the structure factor has a similar structure to the QL itself. Note, however, that $\Lambda$ and $\Lambda^*$ are not dual for the case of icosahedral QLs of types $I$ and $F$.

VIII. SUMMARY AND DISCUSSIONS

We have investigated 1D binary QLs with geometrical self-similarities, and found that there exists a bijection between any pair of the four sets: i) the set of all the MLD classes of binary QLs, ii) the set of all the mother lattices with hyperscaling symmetries, iii) the set of all the MECs of quasi-normal UFM, and vi) the set of all the MECs of QRQIs. If a mother lattice with a hyperscaling symmetry is fixed, the following the six $N$-cycles are isomorphic to one another: i) the relevant set of all the MLD classes of binary QLs, ii) the relevant MEC of quasi-normal UFM, iii) the relevant MEC of QRQIs, iv) the set of all the scaling-equivalent class of the canonical basis vectors, v) a similar set of the canonical windows, and vi) a similar set of the canonical intervals. The algorithm which determines the structure of the cycle has been independently given for the three cycles, ii), iii), and iv), although the three algorithms are equivalent. The algorithm which determines the structure of the cycle iii) is identical to the algorithm for the continued-fraction expansion of a QRQI, $\xi$. It is equivalent to the anti-diagonal algorithm which determines the structure of the cycle iv), while the diagonal algorithm is identical to the algorithm for the continued-fraction expansion of a QRQI, $\zeta$. Two successive members of every MLD class of binary QLs are combined by one of the two elementary SRs, and arbitrary chosen two members are combined by composite SRs. The IR of a QL is given as a special composite SR.

Every 1D binary QL has a symmetric IR. A necessary and sufficient condition for an SR to be symmetric is that its substitution matrix is a quasi-permutation-matrix (QPM). A conventional MLD class can be divided into two or more s-MLD classes. There exists a bijection between the set of all the s-MLD classes of binary QLs and the set of all the MECs of QPMs. Each s-MLD class is a cycle, and its two successive members are combined by elementary symmetric SRs.

There exist two quadratic irrationals, $\xi$ and $\tau$, characterizing an MLD class of binary QLs. Of the two, $\xi$ includes complete information on the MLD class but $\tau$ does not in general, so that the former is more fundamental. The latter has complete information only when the case of MLD classes of rank one or of a special series of MLD classes of rank two.

The binary QLs investigated in the present paper have IRs whose inflation matrices are unimodular. Moreover, the IRs are compositions of the elementary SRs of the two types given by Eqs. (3.3) and (3.4). Then, general arguments given in Refs. 8 and 9 result in that the relevant trace maps have the invariant of the standard form: $I \equiv x^2 + y^2 + z^2 - 2xyz - 1$. Therefore, the energy spectra and the one-electron wave functions of the tight-binding models on these structures are fractals which belong to the same classes as those of the Fibonacci lattice. Furthermore, we can conclude that each MLD class of binary QLs has its own universality class with respect to the one-electron properties because two successive members of every MLD class are combined by one of the two elementary SRs.

We have shown that every MLD class is coded by a cycle of natural numbers. However, the MLD class is an $N$-cycle, and is not isomorphic to the cycle coding the MLD class. Therefore, we shall pursue a more convenient coding system. Let us take the MLD class $\langle\langle 3, 2, 4 \rangle\rangle$ as an example, and transform it into the binary code 100010100, which is a reversed concatenation of the three numbers 100, 10, and 1000 with three, two, and four digits, respectively. If the digit 1 in the binary code is replaced by the elementary UFM $U$ and the digit 0 by $S$, we obtain a quasi-normal (exactly normal) UFM which is the inflation matrix of a member of the MLD class. The inflation matrices of other members of the MLD class are obtained similarly from cyclic permutations of the binary code. Therefore, for every binary QL, we shall associate a cyclic binary decimal in the interval $[0, 1]$ so that the first period codes the relevant inflation matrix. Since we can naturally define a cyclic equivalence between two cyclic decimals, we can conclude that i) there exists a bijection between the set of all the binary QLs and the set of all the cyclic binary decimals in the interval $[0, 1]$, ii) there exists a bijection between the set of all the MLD classes and the set of all the cyclic equivalence classes of cyclic binary decimals, and iii) the two cycles related in this bijection is isomorphic. Thus, an MLD class of order $N$ is coded by an irreducible cycle of $N$ binary digits like $\langle\langle 100010100 \rangle\rangle_b$, where the suffix $b$ stands for the symbol being a binary code.
APPENDIX A: MATHEMATICAL GLOSSARY

Most results to be presented in this appendix are found in the literature, e.g., Ref. 25 but several results in appendices A2, A3 and A6 seem new.

1. Unimodular matrix

A matrix $M$ whose matrix elements are all integers is called unimodular if $\det M = e = \pm 1$. Let $\{a, b\}$ a set of basis vectors of a 2D lattice and $\{a', b'\}$ be another set. Then, the two sets are related to each other by a unimodular matrix

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

(A1)
as

$$(a' \ b') = (a \ b)M.$$  

(A2)

Conversely, if $M$ is unimodular, $a'$ and $b'$ form a set of basis vectors of the 2D lattice.

There exists a linear transformation $T$ so that $a' := Ta$ and $b' := Tb$; $T$ is an automorphism of the 2D lattice. Since the signs of basis vectors of a lattice are not very important, we can assume that $m = \text{Tr} M$ is a nonnegative integer. The eigenvalues of $M$ are the roots of the quadratic equation $z^2 - mz + e = 0$. There can be three cases, elliptic, parabolic, and hyperbolic, depending on whether the discriminant $D := m^2 - 4e$ is negative, zero, or positive, respectively. We are interested in the hyperbolic case, to which a general argument to be given hereafter will be confined. Then, $m$ must obey the condition $m \geq 1$ if $e = -1$ but the condition $m \geq 3$ if $e = +1$. Of the two roots of the quadratic equation, the one which is larger than unity is given by the real quadratic irrational:

$$\tau = \frac{1}{2} \left( m + \sqrt{m^2 - 4e} \right).$$  

(A3)

The second root is given by $\bar{\tau}$, the algebraic conjugate of $\tau$, and $\tau$ and $\bar{\tau}$ satisfy

$$\tau \bar{\tau} = e, \quad \tau + \bar{\tau} = m,$$

(A4)

so that $|\tau| < 1$. An important $\mathbb{Z}$-module is generated by $\tau$:

$$\mathbb{Z}[\tau] := \{ t + u\tau \mid t, u \in \mathbb{Z} \}.$$  

(A5)

Remember that the Cayley-Hamilton theorem yields $M^2 - mM + eI = 0$ with $I$ being the unit matrix.

The linear transformation $T$ has two principal axes which we can assume as shown shortly to be mutually orthogonal; $T$ enlarges along one of the axes by $\tau$ but shrinks along the other by $\tau^{-1} = |\tau|$. If the coordinate system matches the principal axes, $T$ is represented by the diagonal matrix

$$T = \begin{pmatrix} \tau & 0 \\ 0 & \bar{\tau} \end{pmatrix}.$$  

(A6)

Let $A := (a \ b)$ be a 2D matrix formed of the two column vectors represented in this coordinate system. Then, Eq. (A2) is equivalent to

$$TA = AM,$$

(A7)

so that $M$ is diagonalized by $A$: $AMA^{-1} = T$, and the two row vectors of $A$ are the left eigenvectors corresponding to the two eigenvalues, $\tau$ and $\bar{\tau}$, respectively. Therefore, the basis vectors $a$ and $b$ are almost determined from $M$ by the orthogonality of the principal axes; we have a freedom of choosing proportionality constants of the two left eigenvectors.

Let $(a \ b)$ be the left eigenvector given by the first row of $A$. Then, it satisfies the equation,

$$\tau(a \ b) = (a \ b)M.$$  

(A8)

Hence the ratio $\xi := b/a$ satisfies $\tau(1 \ \xi) = (1 \ \xi)M$. It follows that
\[ \tau = p + r\xi, \quad \tau\xi = q + s\xi. \quad (A9) \]

Hence \( \xi \) is a real quadratic irrational belonging to the quadratic field \( \mathbb{Q}[\tau] \). It is important that the \( \mathbb{Z} \)-module \( \mathbb{Z}[\xi] \) coincides with \( \mathbb{Z}[\tau] \) only when \( r = 1 \) because \( \mathbb{Z}[\tau] = r\mathbb{Z}[\xi] \). Note that \( \xi \) is a root of the quadratic equation:

\[ r\xi^2 + (p - s)\xi - q = 0. \quad (A10) \]

In this paper, we always mean by a quadratic irrational a real quadratic irrational.

The ratio \( \xi \) is uniquely determined by \( M \) but the converse is not necessarily true because \( M^n \) with \( n \) being any nonzero integer satisfies the condition. A unimodular matrix is called irreducible, if it cannot be written as a power of another unimodular matrix. Every quadratic irrational \( \xi \) has its proper irreducible unimodular matrix; we shall say that \( \xi \) and the relevant matrix are companions of each other.

Another quadratic irrational is defined with \( \xi \) by \( \xi := -\xi^{-1} \). We shall call it the dual to \( \xi \) because the tilde operation is recursive. The companion matrix of \( \xi \) is equal to \( ^tM \), the transposed matrix of \( M \).

The set, \( GL(2, \mathbb{Z}) \), of all the 2D unimodular matrices form an infinite but discrete group. The set, \( GL'(2, \mathbb{Z}) \), of all the hyperbolic and irreducible members of \( GL(2, \mathbb{Z}) \) is, however, not a group because it does not include the unit matrix. We may say generally that two unimodular matrices \( M \) and \( M' \) are modular equivalent if there exists a unimodular matrix \( K \) so that \( M' = K^{-1}MK \). The set \( GL'(2, \mathbb{Z}) \) can be grouped into modular equivalent classes (MECs), which are disjoint. The two eigenvalues are common among different members of an MEC. Also the trace and the determinant are common, so that they are proper numbers of the class. Note, however, that the pair of the two numbers is not sufficient to specify an MEC. For example, the two matrices, \( \left( \begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array} \right) \) and \( \left( \begin{array}{cc} 0 & 1 \\ 2 & 3 \end{array} \right) \) are shown to be not modular equivalent.

There exists a bijection (one-to-one correspondence) between any pair of the three sets: i) \( GL(2, \mathbb{Z}) \), ii) the set of all the sets of basis vectors of a 2D lattice, and iii) the set of all the automorphisms of the 2D lattice. Therefore, we can define primed subsets like \( GL'(2, \mathbb{Z}) \) for the latter two sets as well, and the division of \( GL'(2, \mathbb{Z}) \) into MECs induces similar divisions of other two primed sets.

### 2. Unimodular Frobenius matrices

The case where \( M \in GL'(2, \mathbb{Z}) \) is a Frobenius matrix, whose matrix elements are all nonnegative, is of particular importance. The positive eigenvalue, \( \tau \), of \( M \) is called the Frobenius eigenvalue, and the relevant left or right eigenvector is called the Frobenius eigenvector, which is assumed to have two positive components. It follows that the companion quadratic irrational \( \xi \) is positive.

The set of all the Frobenius matrices in \( GL(2, \mathbb{Z}) \) do not form a group but only do a semigroup because the inverse of a unimodular Frobenius matrix (UFM) has not necessarily the Frobenius property. Let \( M \) and \( M' \) be two UFMs which are modular equivalent to each other. Then, we say that they are strongly modular equivalent to each other if there exist two UFMs \( K \) and \( L \) so that \( M' = K^{-1}MK \) and \( M = L^{-1}M'L \). This condition is equivalent to the one that \( M \) is a product of the two UFMs, \( M = KL \). More generally, if \( M \) is a product of a number of UFMs, it is strongly modular equivalent to another UFM which is obtained from \( M \) by a cyclic permutation of its factors in the product.

A UFM represented as Eq. \((A1)\) is called quasi-normal if it satisfies the two conditions: i) it is irreducible and ii) \( p \leq q \) and \( r \leq s \). It is called normal if it satisfies the condition, iii) \( p \leq r \) and \( q \leq s \), in addition to the former two. A quasi-normal UFM is normal if and only if its transpose is also quasi-normal. In particular, a UFM and its transpose are simultaneously normal. Note that a normal UFM is always hyperbolic but a quasi-normal UFM can be parabolic.

The set of all the quasi-normal (or normal) UFMs in \( GL(2, \mathbb{Z}) \) form a semigroup if the unit matrix is included in it. If an irreducible UFM \( M \neq I \) is not quasi-normal, \( P^{-1}MP \) with \( P := \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right) \) is quasi-normal. That is, a UFM being not quasi-normal is strongly modular equivalent to a quasi-normal UFM. Therefore, properties of the former are investigated from those of the latter. An important property of a quasi-normal UFM is that its companion quadratic irrational is larger than unity, which follows from Eq. \((A9)\).

Simplest but nontrivial quasi-normal UFMs are:

\[ S = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \quad U = \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right). \quad (A11) \]

In fact, the second of the two is normal. A series of normal UFMs is defined by

\[ K := \left( \begin{array}{cc} 0 & 1 \\ 1 & k \end{array} \right). \quad (A12) \]
with \( k \) being natural numbers. We shall call a member of the series an \textit{elementary normal UFM}. The matrix (A12) is factorized as \( K = US^{k-1} \). Similarly, a series of quasi-normal UFMs is defined by

\[
L = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}
\]  
(A13)

with \( l \) being natural numbers. We may write \( L = S^l \), which is a generic form of a parabolic quasi-normal UFM.

If a normal UFM is represented as Eq. (A1), we obtain

\[
MK^{-1} = \begin{pmatrix} q-kp & p \\ s-kr & r \end{pmatrix}.
\]  
(A14)

We shall denote by \( k_0 \) the maximum number \( k \) under the condition that the unimodular matrix \( MK^{-1} \) remains a UFM. Then, \( MK_0^{-1} \) is a normal UFM unless it is equal to the unit matrix. That is, a normal UFM is transformed into another normal UFM with a smaller “norm” provided that the norm is defined appropriately. We can repeat this procedure until we arrive at an elementary normal UFM. Thus, a normal UFM is uniquely factorized into elementary normal UFMs:

\[
M = K_{n-1}K_{n-2} \cdots K_0.
\]  
(A15)

The rank of \( M \) is defined by the total number \( n \) of the factors in the right hand side of this expression. We should mention that the algorithm of this factorization is nothing but a parallel Euclidean algorithm. We can specify \( M \) by the ordered set \( \langle k_0, k_1, \ldots, k_{n-1} \rangle \) of natural numbers, where \( k_i \) is a number specifying \( K_i \); we have adopted the reversed order for a convenience of a later argument. The members of the ordered set are not entirely independent because of the irreducibility condition i) above. For example, \( k_0 \) and \( k_1 \) must be different if \( n = 2 \). If \( M = \langle k_0, k_1, \ldots, k_{n-1} \rangle \), then \( A^M = \langle k_{n-1}, k_{n-2}, \ldots, k_0 \rangle \); \( M \) is symmetric if and only if the relevant ordered set is mirror-symmetric.

As a result of the uniqueness of the factorization above, we can conclude that a normal UFM which is modular equivalent to the normal UFM (A15) has a similar factorization to above but the factors are cyclically permuted. Hence, the rank is common between the two UFMs. An MEC of normal UFMs is composed of \( n \) members, where \( n \) is the common rank of them. The MEC is specified by the cyclically ordered set \( \langle \langle k_0, k_1, \ldots, k_{n-1} \rangle \rangle \), where double angular brackets are used to distinguish the set from the linearly ordered set above. Note that the sign of the determinant of every member of the MEC is equal to \((-1)^{n}\). We will call hereafter a cyclically ordered set simply a cycle.

An MEC is called \textit{self-dual} if it is specified by a cycle with a mirror symmetry. Every member of a self-dual MEC is modular equivalent to its transpose. An MEC is always self-dual if its rank is less than three. An MEC of rank three is self-dual if and only if it takes the form \( \langle \langle k, k', k \rangle \rangle \), while an MEC of rank four is self-dual if and only if its form is one of the two alternatives, \( \langle \langle k, k', k', k \rangle \rangle \) and \( \langle \langle k, k', k''', k' \rangle \rangle \).

An MEC is called \textit{strongly self-dual} if it includes a symmetric member (i.e., a symmetric matrix). A self-dual MEC is strongly self-dual if its rank is odd but not necessary so if its rank is even. For example, an MEC of rank two is never strongly self-dual, while an MEC of rank four is strongly self-dual if and only if its form is the former of the two alternatives above.

3. Quasi-normal UFM and quasi-permutation-matrix

Let \( M' \) be a quasi-normal UFM. Then, we can show by a similar argument to that presented after Eq. (A14) that there exists the maximum number \( l \) under the condition that \( L^{-1}M' \) remains a UFM. In the maximal case, \( L^{-1}M' \) is a normal UFM. Then, we can show that \( M := L^{-1}M'L \) is a normal UFM as well, so that a quasi-normal UFM is strongly modular equivalent to a normal UFM. It follows that a quasi-normal UFM is represented with a normal UFM \( M \) as \( S^lMS^{-l} \) with \( l \) being a nonnegative integer. More precisely, if \( M \) is represented as Eq. (A15), we obtain \( k_0 \) quasi-normal UFMs:

\[
M_k = S^kUS^{k_{n-1}-1}US^{k_{n-2}-1} \cdots US^{k_1-1}US^{k_0-k-1}
\]  
(A16)

with \( k = 0, 1, 2, \ldots, k_0 - 1 \) and \( M_0 := M \), where use has been made of the equations, \( K_i = US^{k_i-1} \). Note that the factorization of \( M_k \) like this is unique. More strongly, there exists an algorithm by which each factor in such a factorization is determined step by step: the right end factor of a quasi-normal UFM \( M \) is \( S \) if \( MS^{-1} \) is quasi-normal but \( U \) otherwise. The two elementary UFMs, \( U \) and \( S \), are two of the four which are given in Ref. 8 as
generators of UFMs. The remaining two are not necessary here because the UFMs under consideration are restricted to quasi-normal ones.

If \((k_0, k_1, \cdots, k_{n-1})\) is a MEC of normal UFMs, we can obtain \(k_i\) quasi-normal UFMs from the \(i\)-th member of the class; the total number of quasi-normal UFMs obtained is equal to \(N := k_0 + k_1 + \cdots + k_{n-1}\). They form an MEC of quasi-normal UFMs, which are strongly modular equivalent to one another. The two MECs, one normal and the other quasi-normal, can be specified by a common cycle of natural numbers although the latter MEC is a cycle composed of \(N\) members. If a member of an MEC of quasi-normal UFMs is factorized as Eq. (A16), \(n\), the number of \(U\), and \(N\), the total number of the factors, are proper numbers of the MEC; \(n\) is nothing but the rank of the MEC. We shall call \(N\) the order of the MEC.

The subject in the remaining part of this subsection concerns only in Sec. VI in the text. There are only two permutation matrices in 2D, i.e., the unit matrix \(I\) and \(J := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\). We shall call a UFM a quasi-permutation-matrix \((QPM)\) if it is congruent in modulo 2 with \(I\) or \(J\). The product of two or more QPMs is also a QPM. If a UFM is not a QPM, it is congruent in modulo 2 with one of the four matrices, \(S, S', U, \) and \(R := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\) because \(\det M = \pm 1\). In what follows, the congruence relation \(x \equiv y \ mod\ 2\) is simply written as \(x \equiv y\). Then, the four matrices just introduced satisfy \(S^2 \equiv (S')^2 \equiv I, U^2 \equiv R, R^2 \equiv U,\) and \(U^3 \equiv R^3 \equiv I\).

Since \(m \equiv 0\) or \(m \equiv 1\) according as \(m := \text{Tr} M\) is even or odd, respectively, a UFM with an odd \(m\) cannot be a QPM. More precisely, \(M \equiv U\) or \(M \equiv R\) for this case, so that \(M^2\) is not a QPM but \(M^3\) is; \(M^3 \equiv I\). On the other hand, a UFM with an even \(m\) satisfies \(M \equiv S\) or \(M \equiv S'\) if \(M\) is not a QPM. Then, \(M^2 \equiv I\).

A QPM is called elementary, if it cannot be represented as a product of two QPMs. Every QPM is uniquely factorized into elementary QPMs. The two QPMs, \(S^2\) and \(US\), are simplest elementary QPMs which are quasi-normal. We present other four types of elementary and quasi-normal QPMs: \(US^kU \equiv I, US^k'U \equiv J, US^lUS^l'U \equiv I,\) and \(US^lUS^l'U \equiv J,\) where \(k\) is an odd integer but \(l\) and \(l'\) are even. There can be more complicated QPMs, e.g., \(U(US^k)^kU\) with \(k\) and \(k'\) being any odd numbers, but we shall not pursue them. A QPM is called irreducible, if it cannot be written as a power of another QPM. If a composite QPM is factorized into elementary QPMs, we obtain another QPM by a cyclic permutation of its factors, and the two QPMs are strongly modular equivalent to each other. Note that an irreducible QPM can be reducible as a usual UFM. For example, if \(M\) is not a QPM, \(M^2\) or \(M^3\) is an irreducible QPM.

Finally, an example of a factorization of a QPM into elementary QPMs is presented:

\[
(U'S^3)^3 = (US)(S^2)(US^4U)(S^2)^2. \tag{A17}
\]

4. Modular transformation

Let us take a unimodular matrix

\[
X = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in GL(2, \mathbb{Z}). \tag{A18}
\]

Then, a transformation from an irrational number \(\xi\) to another irrational \(\xi'\) is uniquely defined by the condition

\[
(1, \ \xi) = \lambda (1, \ \xi')X \text{ or } \lambda (1, \ \xi') = (1, \ \xi)X^{-1}, \tag{A19}
\]

where \(\lambda\) is a parameter determined by this condition. This transformation is called the modular transformation, which is written explicitly as

\[
\xi' = X(\xi) := \frac{tx - u}{-v\xi + w}. \tag{A20}
\]

Our definition of the modular transformation is slightly different from the conventional one but is substantially equivalent to the latter. If \(\xi' = X(\xi)\) and \(\xi'' = X'(\xi')\) with \(X'\) being the second unimodular matrix, we obtain \(\xi'' = X''(\xi)\) with \(X'' := X'X\), which can be readily proved from Eq. (A14). Therefore, the set of all the modular transformations form a group which is homomorphic with \(GL(2, \mathbb{Z})\). More precisely, it is isomorphic with \(GL(2, \mathbb{Z})/\mathbb{Z}_2\), where \(\mathbb{Z}_2 := \{I, -I\}\) is a normal subgroup of \(GL(2, \mathbb{Z})\) because two unimodular matrices \(X\) and \(-X\) yield an identical modular transformation.

A quadratic irrational is changed into another quadratic irrational by a modular transformation. Hence, the set of all the quadratic irrationals can be divided into MECs, which are disjoint.

Let us introduce with the unimodular matrix \(X\) a new set of basis vectors by

\[
\xi'' = X''(\xi) := \frac{tx - u}{-v\xi + w}. \tag{A20}
\]

Our definition of the modular transformation is slightly different from the conventional one but is substantially equivalent to the latter. If \(\xi' = X(\xi)\) and \(\xi'' = X'(\xi')\) with \(X'\) being the second unimodular matrix, we obtain \(\xi'' = X''(\xi)\) with \(X'' := X'X\), which can be readily proved from Eq. (A14). Therefore, the set of all the modular transformations form a group which is homomorphic with \(GL(2, \mathbb{Z})\). More precisely, it is isomorphic with \(GL(2, \mathbb{Z})/\mathbb{Z}_2\), where \(\mathbb{Z}_2 := \{I, -I\}\) is a normal subgroup of \(GL(2, \mathbb{Z})\) because two unimodular matrices \(X\) and \(-X\) yield an identical modular transformation.

A quadratic irrational is changed into another quadratic irrational by a modular transformation. Hence, the set of all the quadratic irrationals can be divided into MECs, which are disjoint.

Let us introduce with the unimodular matrix \(X\) a new set of basis vectors by
are used to specify the corresponding RQI and relevant MEC of RQIs, respectively. The rank and the set of all the RQIs. Then, the symbols used to specify each normal UFM and each MEC of normal UFMs and \( \tilde{\theta} \) is the companion quadratic irrational of \( \theta \). Moreover, we obtain \( \tilde{\theta}' = M'(\tilde{\theta}') \) with \( M' := X^{-1}M \). Thus, there exists a bijection between the set of all the MECs of quadratic irrationals and the set of all the MECs of (hyperbolic) unimodular matrices.

The following proposition provides us with a different view for modular equivalence between two irrationals, \( \xi \) and \( \xi' \): a necessary and sufficient condition for \( \xi \) being larger than unity. Then, we can derive a second irrational \( \xi' \) of quasilattices in any dimensions. A quadratic irrational is called a reduced quadratic irrational if its continued-fraction expansion is periodic. Numbers having periodic continued-fraction expansions are important. Let \( \xi \) be an irrational number being larger than unity. Then, we can derive a second irrational \( \xi' > 1 \) by the equation:

\[
\xi = k + \frac{1}{\xi'}, \quad k := [\xi],
\]

where the symbol \([\star]\) stands for the maximum integer among those which do not exceed the number \( \star \). This equation is written with the use of the modular transformation as

\[
\xi' = K(\xi),
\]

where \( K \) is the normal UFM (A12). Since \( K = US^{k-1} \) with \( S \) and \( U \) being UFMs (A11), this modular transformation is represented as a composition of the two types of elementary procedures:

\[
\xi' = S(\xi) = \xi - 1, \quad \xi' = U(\xi) = \frac{1}{\xi - 1}.
\]

The first of the two procedures is used when \( \xi > 2 \), while the second when \( 1 < \xi < 2 \).

If the recursive procedure (A23) is repeated indefinitely, \( \xi \) is expanded into an infinite continued-fraction. The \( i \)-th step of this recursive procedure is given by

\[
\xi_i = k_i + \frac{1}{\xi_{i+1}}
\]

with \( \xi_0 := \xi \) and \( k_0 := k \), so that

\[
\xi_0 = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \cdots + \frac{1}{k_{i-1} + \frac{1}{\xi_i}}}}.
\]

This equation is equivalent to \( \xi_i = M_i(\xi_0) \) with \( M_i = K_{i-1}K_{i-2} \cdots K_0 \) being a normal UFM, where \( K_j \) is given by Eq. (A12) with \( k = k_j \). If \( \xi_i \) in the continued-fraction (A22) is replaced by an integer \( k \) satisfying \( 1 \leq k \leq k_i \), we obtain a rational number, which is shown to be a “best” approximant to \( \xi_i \).

Numbers having periodic continued-fraction expansions are important. Let \( M \) be the normal UFM derived from the first period of the expansion of such number, \( \xi \). Then \( \xi \) is the companion quadratic irrational of \( M \). Hence, \( \xi \) must be a quadratic irrational. More strongly, \( \xi > 1 \) and \( \xi > 1 \) because \( M \) is normal. However, the continued-fraction expansion of a generic quadratic irrational is not always periodic; exactly, it is periodic except for first several terms.

A quadratic irrational is called a reduced quadratic irrational (RQI) if its continued-fraction expansion is periodic. A necessary and sufficient condition for a quadratic irrational \( \theta \) to be an RQI is that it satisfies the conditions: \( \theta > 1 \) and \( \theta > 1 \), which is equivalent to \(-1 < \theta < 0\). Thus, there exists a bijection between the set of all the normal UFMs and the set of all the RQIs. Then, the symbols used to specify each normal UFM and each MEC of normal UFMs are used to specify the corresponding RQI and relevant MEC of RQIs, respectively. The rank \( n \) of an RQI is nothing
but the shortest period of its continued-fraction expansion, and the number of the members of the relevant MEC of RQIs is equal to $n$.

The subject in the remaining part of this subsection concerns only in Sec. VII D in the text. Let $\theta$ be an RQI specified by the ordered set $\langle k_0, k_1, \cdots, k_{n-1} \rangle$. Then, the dual $\bar{\theta}$ to $\theta$ is specified by the inversely ordered set. An MEC of RQIs is called self-dual or strongly self-dual if the corresponding MEC of normal UFMs is self-dual or strongly self-dual, respectively. A strongly self-dual MEC of RQIs includes an RQI $\theta$ so that $\bar{\theta} = \theta$; the first period of the continued-fraction expansion of $\theta$ is mirror-symmetric. We may say this RQI to be strongly self-dual. If $\xi$ is a member of a self-dual MEC of RQIs, $\bar{\xi}$ is modular equivalent to $\xi$ or, equivalently, $\mathbb{Z}[\xi]$ and $\mathbb{Z}[\bar{\xi}]$ are scaling-equivalent.

A sufficient condition for an MEC of RQIs to be self-dual is that it includes an RQI $\xi$ so that $\xi + \bar{\xi} \in \mathbb{Z}$ or, equivalently, $\xi$ takes the form $\xi := p + q\sqrt{D}$ or $\xi := (p + q\sqrt{D})/2$, where $p, q$, and $D$ are natural numbers. Therefore, the one period of the continued fraction expansion of a quadratic irrational of either of the two forms is mirror symmetric if it is considered to be a cycle.

6. Quasi-reduced quadratic irrational

The companion quadratic irrational $\xi$ of a quasi-normal UFM is a quasi-reduced quadratic irrational (QRQI) which is defined in Refs. 10, 11 because it satisfies the conditions: $\xi > 1$ and $\bar{\xi} > 0$ (or, equivalently, $\bar{\xi} < 0$). A necessary condition for a quadratic irrational being larger than unity to be an QRQI is that its continued-fraction expansion is periodic but for the first term. It can be shown readily that a QRQI $\xi$ is related to an RQI $\theta$ as $\xi = \theta - k$ with $k$ being an integer satisfying the inequality, $0 \leq k \leq k_0 - 1$ with $k_0 := \lceil \theta \rceil$.

The relation $\theta = \bar{\theta} - k$ between a QRQI $\xi$ and an RQI $\theta$ is written in terms of the modular transformation specified by a UFM $S$ given in Eqs. (A11) as $\xi = S_k(\theta)$. It follows from the equation $\theta = M_0(\theta)$ with $M_0$ being the companion matrix of $\theta$ that $\xi$ is the companion quadratic irrational of the quasi-normal UFM, $M_k = S_k M_0 S_k^{-1}$: $\xi = M_k(\xi)$. An explicit form of $M_k$ is given by Eq. (A16), where we have set $M_0 := \langle k_0, k_1, \cdots, k_{n-1} \rangle$. The RQI $\theta$ yields $k_0$ QRQIs, $\xi = \theta - k$ with $k = 0, 1, 2, \cdots, k_0 - 1$, which we shall call companion QRQIs of $\theta$. Thus, there exists a bijection between the set of all the QRQIs and the set of all the quasi-normal UFMs. More strongly, there exists a bijection between the set of all the MECs of QRQIs and that of the quasi-normal UFMs. Both the two types of MECs combined by the bijection have structures of $N$-cycles, and these structures are retained in the bijection. That is, the two $N$-cycles are isomorphic. Therefore, we can specify an MEC of QRQIs by the symbol specifying the corresponding MEC of quasi-normal UFMs: $\langle \langle k_0, k_1, \cdots, k_{n-1} \rangle \rangle$ with $n$ being the rank of the class. Thus, the relevant MEC of QRQIs includes $n$ RQIs, $\theta_j$ with $j = 0, 1, 2, \cdots, n-1$ and $[\theta_j] = k_j$. The companion QRQIs of $\theta_j$ are $\xi^{(j)}_k := \theta_j - k$ with $k = 0, 1, 2, \cdots, k_j - 1$.

As a typical example of the MEC we take $\langle \langle 1, 2, 1 \rangle \rangle$, which is composed of the four QRQIs:

$$
\xi^{(0)}_0 = 1 + \frac{\sqrt{10}}{3}, \quad \xi^{(1)}_0 = 1 + \frac{\sqrt{10}}{2},
\xi^{(1)}_1 = \frac{\sqrt{10}}{2}, \quad \xi^{(2)}_0 = \frac{2 + \sqrt{10}}{3}.
$$

The ratio $\tau$ associated with this MEC is given by $\tau = 3 + \sqrt{10}$. The companion matrix of $\xi^{(0)}_0$, for example, is given by

$$
M = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}.
$$

(A28)

This example can be used to check or understand various results appearing in the present paper.

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FIG. 1. A derivation of a binary 1D QL by the projection method. The mother lattice $\Lambda$ is cut with the strip, and then projected onto $E_\parallel$. 
FIG. 2. A geometrical derivation of the IR of the Fibonacci lattice. The canonical set of basis vectors, \{a, b\}, is transformed by the hyperscaling operation $T$ into another set, \{a', b'\}. The two types of new intervals, $\alpha'$ and $\beta'$, are related to the original intervals, $\alpha$ and $\beta$, by the IR. The window is shrunken by $T$ from $W$ to $W' = \tau^{-1}W$, while the sizes of the two types of intervals are expanded by the factor $\tau$. 
FIG. 3. A geometrical illustration for the SR (3.3). This case is different from the one illustrated in Fig. 2 in that the third basis vector $a + b$ is paired with $a$ but not $b$ to form a canonical set.
FIG. 4. The “finger print” of the MLD class \(\langle(1, 2, 1)\rangle\).