Instability of large solitary water waves

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Abstract

We consider the linearized instability of 2D irrotational solitary water waves. The maxima of energy and the travel speed of solitary waves are not obtained at the highest wave, which has a 120 degree angle at the crest. Under the assumption of non-existence of secondary bifurcation which is confirmed numerically, we prove linear instability of solitary waves which are higher than the wave of maximal energy and lower than the wave of maximal travel speed. It is also shown that there exist unstable solitary waves approaching the highest wave. The unstable waves are of large amplitude and therefore this type of instability can not be captured by the approximate models derived under small amplitude assumptions. For the proof, we introduce a family of nonlocal dispersion operators to relate the linear instability problem with the elliptic nature of solitary waves. A continuity argument with a moving kernel formula is used to study these dispersion operators to yield the instability criterion.

1 Introduction

Preliminaries. The water-wave problem in its simplest form concerns two-dimensional motion of an incompressible inviscid liquid with a free surface, acted on only by gravity. Suppose, for definiteness, that in the \((x, y)\)-Cartesian coordinates gravity acts in the negative \(y\)-direction and that the liquid at time \(t\) occupies the region bounded from above by the free surface \(y = \eta(t; x)\) and from below by the flat bottom \(y = -h\), where \(h > 0\) is the water depth. In the fluid region \(\{(x, y) : -h < y < \eta(t; x)\}\), the velocity field \((u(t; x, y), v(t; x, y))\) satisfies the incompressibility condition

\[
\partial_x u + \partial_y v = 0 \tag{1.1}
\]

and the Euler equation

\[
\begin{aligned}
\partial_t u + u \partial_x u + v \partial_y u &= -\partial_x P \\
\partial_t v + u \partial_x v + v \partial_y v &= -\partial_y P - g,
\end{aligned} \tag{1.2}
\]

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where $P(t; x, y)$ is the pressure and $g > 0$ denotes the gravitational constant of acceleration. The kinematic and dynamic boundary conditions at the free surface $\{y = \eta(t; x)\}$

$$v = \partial_t \eta + u \partial_x \eta \quad \text{and} \quad P = P_{\text{atm}}$$

express, respectively, that the boundary moves with the velocity of the fluid particles at the boundary and that the pressure at the surface equals the constant atmospheric pressure $P_{\text{atm}}$. The impermeability condition at the flat bottom states that

$$v = 0 \quad \text{at} \quad \{y = -h\}. \quad (1.4)$$

In this paper we consider the irrotational case with $\text{curl} \, v = 0$, for which the Euler equation (1.1)–(1.4) is reduced to the Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g \eta = c(t)$$

where $\phi$ is the vector potential such that $(u, v) = \nabla \phi$. The local well-posedness of the full water wave problem was proved by Wu (68) for deep water and by Lannes for water of finite depth (54).

We consider a traveling solitary wave solution of (1.1)–(1.4), that is, a solution for which the velocity field, the wave profile and the pressure have space-time dependence $(x + ct, y)$, where $c > 0$ is the speed of wave propagation. With respect to a frame of reference moving with the speed $c$, the wave profile appears to be stationary and the flow is steady. It is traditional in the traveling-wave problem to define the relative stream function $\psi(x, y)$ and vector potential $\phi(x, y)$ such that:

$$\psi_x = -v, \quad \psi_y = u + c \quad (1.5)$$

and

$$\phi_x = u + c, \quad \phi_y = v. \quad (1.6)$$

The boundary conditions at infinity are

$$(u, v) \to (0, 0), \quad \eta(x) \to 0, \text{ as } |x| \to +\infty.$$ 

The solitary wave problem for (1.1)–(1.4) is then reduced to an elliptic problem with the free boundary $\{y = \eta(x)\}$ (11):

Find $\eta(x)$ and $\psi(x, y)$, in $\{(x, y) : -\infty < x < +\infty, \ -h < y < \eta(x)\}$, such that

$$\Delta \psi = 0 \quad \text{in} \quad -h < y < \eta(x), \quad (1.7a)$$

$$\psi = 0 \quad \text{on} \quad y = \eta(x), \quad (1.7b)$$

$$|\nabla \psi|^2 + 2gy = c^2 \quad \text{on} \quad y = \eta(x), \quad (1.7c)$$

$$\psi = -ch \quad \text{on} \quad y = -h, \quad (1.7d)$$

with

$$\nabla \psi \to (0, c), \quad \eta(x) \to 0, \text{ as } |x| \to +\infty.$$
First we give a summary of the existence theory of solitary water waves. Lavrentiev ([35]) got the first proof of the existence of small solitary waves by studying the long wave limit. A direct construction of small solitary waves was given by Fridrichs and Hyers ([26]), and their proof was readdressed by Beale ([8]) via the Nash-Moser method. The existence of large amplitude solitary waves was shown by Amick and Toland ([4]). The highest wave was also shown to exist by Toland ([66]), and its angle at the crest was shown to be 120 degree (Stokes’s Conjecture in 1880) by Amick, Toland and Fraenkel ([6]). The symmetry of solitary waves was studied by Craig and Sternberg ([23]). Plotnikov ([58]) studied the secondary bifurcation and showed that solitary waves are not unique for certain travelling speed. The particle trajectory for solitary waves was studied by Constantin and Escher ([21]). We list some properties of the solitary waves which will be used in the study of their stability. Denote the Froude number by

\[ F = \frac{c}{\sqrt{gh}} \]

and the Nekrasov parameter by

\[ \mu = \frac{6gch}{\pi q_c^3} \]

where \( q_c \) is the (relative) speed of the flow at the crest. We note that \( \mu \) is the bifurcation parameter used in [4]. The highest wave corresponds to \( \mu = +\infty \) since \( q_c = 0 \). The following properties of solitary waves are proved:

(P1) ([4]) There exists a curve of solitary waves that are symmetric, positive \((\eta > 0)\) and monotonically decay on either side of the crest, with the parameter \( \mu \in (\frac{6}{\pi}, +\infty) \). When \( \mu \nearrow +\infty \), the solitary waves tend to the highest wave with the 120 degree angle at the crest. When \( \mu \searrow \frac{6}{\pi} \), the solitary waves tend to the small waves constructed in [26] and [8]. Moreover, we have \( \nabla \psi \to (0, c) \), \( \eta (x) \to 0 \) exponentially as \( |x| \to +\infty \). Below, we call this solitary wave curve the primary branch.

(P2) ([61], [4], [53]) Any positive and symmetric solitary wave which decays monotonically on either side of its crest is supercritical, that is, \( F > 1 \) or equivalently \( c > \sqrt{gh} \). The limit of small waves corresponds to \( F \searrow 1 \) ([4], [26]).

(P3) ([23]) Any supercritical solitary wave \((F > 1)\) is symmetric, positive and decays monotonically on either side of its crest. Moreover, any nontrivial solitary wave curve connected to the primary branch must have \( F > 1 \).

(P4) ([58]) For small amplitudes waves with \( \mu \approx \frac{6}{\pi} \), there is no secondary bifurcation on the primary branch. When the highest wave is approached, that is, when \( \mu \to +\infty \), there are infinitely many points on the primary branch which are either a secondary bifurcation point or a turning point where \( c' (\mu) = 0 \).

The property (P4) is essentially what was proved in [58], though our statement above adapts the explanation in [15], p. 245. Moreover, numerical evidences ([17], [23]) indicate that the following assumption holds true:

(H1) There are no secondary bifurcation points on the primary branch.
Under the assumption (H1), above property (P4) implies that there are infinitely many turning points where \( c'(\mu) = 0 \). So the travel speed \( c \) does not always increase with the wave amplitude, and this differs greatly from KDV and other approximate models for which the higher waves travel faster. More precisely, for full solitary water waves the travel speed obtains its maximum before the highest wave and then it becomes highly oscillatory near the highest wave. This fact was first observed from numerical computations ([7], [48]), then confirmed by the asymptotic analysis ([49], [51]). Indeed, almost all physical quantities (i.e. energy and momentum) do not achieve their maxima at the highest wave, and are highly oscillatory around the highest wave (see above references). This fact turns out to imply the instability of large solitary waves, which was first discovered from numerical computations ([64]) and is rigorously proved in this paper.

**Main results.** Denote by \( \mu_1 \) the first turning point where \( c(\mu) \) obtains its global maximum and, by \( \tilde{\mu}_1 \) the first and also the global maximum point of \( E(\mu) \), where

\[
E(\mu) = \int_{-h}^{\eta(x)} \frac{1}{2} (u^2 + v^2) \ dydx + \int \frac{1}{2} g\eta^2 \ dx. \quad (1.8)
\]

is the energy of the solitary wave with the parameter \( \mu \). Numerical computations ([48], [64], [52]) indicate that \( \mu_1 > \tilde{\mu}_1 \), and \( \tilde{\mu}_1 \) is the only critical point of \( E(\mu) \) in \( (\frac{\pi}{4}, \mu_1) \). We state it as another hypothesis:

(H2) The energy maximum is achieved on the primary branch before the wave of the maximal travel speed (the first turning point).

**Theorem 1** Under the assumptions (H1) and (H2), the solitary wave is linearly unstable when \( \mu \in (\tilde{\mu}_1, \mu_1) \), where \( \mu_1 \) and \( \tilde{\mu}_1 \) are the maxima points of the travel speed and energy, respectively. The linear instability is in the sense that there exists a growing mode solution \( e^{\lambda t} [\eta(x), \psi(x,y)] (\lambda > 0) \) to the linearized problem (2.2), where \( \eta(x), \psi(x,y) \in C^\infty \cap H^k \) for any \( k > 0 \).

Our next theorem shows that there exist unstable solitary waves approaching the highest wave.

**Theorem 2** Under the assumption (H1), there exists infinitely many intervals \( I_i \subset (\mu_1, +\infty), (i = 1, 2, \cdots) \) with \( \lim_{n \to \infty} \max \{\mu | \mu \in I_n\} = +\infty \), such that solitary waves with the parameter \( \mu \in I_i \) are linearly unstable in the sense of Theorem 7.

Theorem 2 suggests that the highest wave \( (\mu = +\infty) \) constructed in [6] is unstable. This contrasts with the stability of peaked solitary waves in some shallow water wave models ([22], [46]). Numerical evidences ([64], [52]) suggest that solitary waves are spectrally stable when \( \mu \in (\frac{\pi}{4}, \tilde{\mu}_1) \), and linearly unstable when \( \mu > \tilde{\mu}_1 \), at least before the first few turning points where the computations are reliable. We note that the amplitude of the maximal energy wave with the parameter \( \tilde{\mu}_1 \) is already close to the maximal height ([64]). So the unstable waves
proved in Theorems 1 and 2 are of large amplitude, and therefore this type of instability can not appear in approximate models which are derived under the small amplitude assumptions, such as KDV equation or Boussinesq systems. Numerical evidences ([65]) also suggest that this large amplitude instability can lead to wave breaking. Such wave breaking induced by large unstable waves had also been used to explain the breaking of waves approaching beaches ([25],[56],[57]). More discussions of these issues are found in Remarks 1 and 2 (Section 5).

The proof of Theorems 1 and 2 also has some implications for the spectral stability of solitary waves with $\mu < \tilde{\mu}_1$. We note that the traveling waves of full water waves even with vorticity are shown ([12],[31]) to be always highly indefinite energy saddles under the constraints of constant momentum, mass etc. Therefore, their stability cannot be studied by the traditional method of proving (constrained) energy minimizers as in many model equations such as the KDV type equations ([9],[13]). So far there are few effective methods for proving nonlinear stability of energy saddles. So naturally, the first step is to study their spectral stability, namely, to show that there does not exist an exponentially growing solution to the linearized problem. The following theorem might be useful for this purpose.

**Theorem 3** Assume the hypothesis (H1). Suppose that there is a sequence of purely growing modes $e^{\lambda_n t} [\eta_n(x), \psi_n(x, y)]$ $(\lambda_n > 0)$ to the linearized problem for solitary waves with parameters $\{\mu_n\}$, with $\lambda_n \to 0^+$ and $\mu_n \to \mu_0$ where $\mu_0$ is not a turning point, then we must have \( \frac{\partial E}{\partial \mu} (\mu_0) = 0 \).

By the above theorem, if an oscillatory instability can be excluded, that is, any growing mode is shown to be purely growing, then the transition of instability can only happen at the energy extrema or turning points. Numerical results in [64], [52] justify that the growing modes found are indeed purely growing for solitary waves before the first few turning points. If additionally the spectral stability of small solitary waves can be proved, then it follows that the solitary waves are spectrally stable up to the wave of maximal energy.

**Comments and ideas of the proof.** First, we comments on related results in the literature. In [59], Saffman considered the spectral stability of periodic waves in deep water (Stokes waves), under perturbations of the same period (so called superharmonic perturbations). The picture of superharmonic instability of Stokes waves ([62]) is similar to that of the instability of solitary waves. The approach of [59] is to take the finite mode truncation of the linearized Hamiltonian formulation of Zakharov ([69]) and study the eigenvalue problem for the matrix obtained. By assuming the existence of a sequence of purely growing modes with the growth rate $\lambda_n \to 0^+$ and parameters $\mu_n \to \mu_0$, the solvability conditions are checked to the second order to show that $\mu_0$ must be an energy extremum. That is, an analogue of Theorem 3 was established in [59] for Stokes waves. However, the analysis in [59] is at a rather formal level. First, Zakharov’s Hamiltonian formulation has a highly indefinite quadratic form that is unbounded from both below and above. This is due to the indefiniteness of the
energy functional of the pure gravity water wave problem ([12]) as mentioned before. So it is unclear how to pass the finite truncation results in [59] to the original water wave problem. Secondly, an implicit assumption in [59] is that the truncated matrix has the zero eigenvalue of geometric multiplicity 1. It is unclear how to check and relate this assumption to the properties of steady waves. For solitary waves, the truncation approach of [59] seems difficult to apply because of the unbounded domain. Recently, in [32], Kataoka recovered Saffman’s formal result (or analogues of Theorem 3) for periodic waters in water of finite depth and for interfacial solitary waves in a different way. The analysis of [32], [33] is again formal and of similar nature as [59]. That is, by assuming the existence of purely growing modes with vanishing growth rates, the first two solvability conditions were checked to show that the limiting parameter is an energy extremum. We note that in the above papers of Kataoka and Saffman, the existence of a sequence of purely growing modes with vanishing growth rates was only assumed but never proved. Moreover, their analysis are perturbative, only for travelling waves near energy extrema. In this paper, we rigorously prove the linear instability of large solitary waves and our method is non-perturbative, which can apply to solitary waves far from the energy extrema.

Below, we briefly discuss main ideas in the proof of Theorems 1 and 2. To avoid the issue of indefiniteness of energy functional, we do not adapt Zakharov’s Hamiltonian formulation in terms of the vector potential \( \phi \) on the free surface and the wave profile \( \eta \). We use the linearized system derived in [31], in terms of the infinitesimal perturbations of the wave profile \( \eta \) and the stream function \( \psi \) restricted on the steady surface \( S_e \). Then we further reduce this system to get a family of operator equations \( \mathcal{A}_\lambda \psi|_{S_e} = 0 \), where \( \lambda > 0 \) is the unstable eigenvalue to be found. The operator \( \mathcal{A}_\lambda \) is the sum of the Dirichlet-Neumann operator and a bounded but nonlocal operator. The idea of above reduction is to relate the eigenvalue problems to the elliptic type problems for steady waves. The hodograph transformation is then used to get equivalent operators \( \mathcal{A}_\lambda \) defined on the whole line. The existence of a purely growing mode is equivalent to find some \( \lambda > 0 \) such that the operator \( \mathcal{A}_\lambda \) has a nontrivial kernel. This is achieved by using a continuity argument to exploit the difference of the spectra of \( \mathcal{A}_\lambda \) near infinity and zero.

The idea of introducing nonlocal dispersion operators with a continuity argument to get instability criteria originates from our previous works ([40], [39], [38]) on 2D ideal fluid and 1D electrostatic plasma, which have also been extended to galaxy dynamics [28] and 3D electromagnetic plasmas [11], [12]. The new issue in the current case is the influence of the symmetry of the problem. More specifically, we need to understand the movement of the kernel of \( \mathcal{A}_0 \) that is due to the translation symmetry, under the perturbation of \( \mathcal{A}_0 \) to \( \mathcal{A}_\lambda \) for small \( \lambda \). This is obtained in a moving kernel formula (Lemma 5.1). The convergence of \( \mathcal{A}_\lambda \) to \( \mathcal{A}_0 \) is very weak, so the usual perturbation theories do not apply and the asymptotic perturbation theory by Vock and Hunziker ([67]) has to be used to study perturbations of the eigenvalues of \( \mathcal{A}_0 \). An important technical part in our proof is to use the supercritical property \( F > 1 \) and the decay of solitary waves to obtain a priori estimates and gain certain compactness. In particular,
$F > 1$ implies that the essential spectra of the operators $A^\lambda$ lie in the right half complex plane. The techniques developed in this paper have been recently extended to show instability of large Stokes waves \([43]\) and get instability criteria for periodic and solitary waves of rather general dispersive wave equations \([44], [45]\).

In Lemma \([4.1]\) we prove that the zero-limiting operator $A^0$ is exactly the same operator used in \([58]\) for studying the secondary bifurcation of solitary waves. This link is interesting and a little unexpected since our derivation of the operator $A^0$ is totally unrelated to the formulation used in \([58]\). We note that the bifurcation results in \([58]\) have no implications for instability of solitary waves. Indeed, for water wave problems, there seems to be no definite relations between the stability and bifurcation of travelling waves. For example, it was shown in \([31]\, Remark 4.13\) that the bifurcation of nontrivial traveling water waves are unrelated to the exchange of stability of trivial flows. From numerical works \([64], [17], [33]\), the exchange of instability at energy extrema for solitary waves does not imply any secondary bifurcation there.

This paper is organized as follows. In Section 2, we give the formulation of the linearized problem and derive the nonlocal dispersion operators $A^\lambda$. Section 3 is devoted to study properties of the operators $A^\lambda$, in particular, their essential spectrum. In Section 4, we apply the asymptotic perturbation theory to study the eigenvalues of $A^\lambda$ for $\lambda$ near 0. In Section 5, we derive a moving kernel formula and prove the main theorems. Some important formulae are proved in Appendix.

### 2 Formulation for linear instability

In this Section, a solitary wave solution of \((1.7)\) is held fixed, as such it serves as the undisturbed state about which the system \((1.1)\)–\((1.4)\) is linearized. The derivation is performed in the moving frame of references, in which the wave profile appears to be stationary and the flow is steady. Let us denote the undisturbed wave profile and relative stream function by $\eta_e(x)$ and $\psi_e(x, y)$, respectively, which satisfy the system \((1.7)\). The steady relative velocity field is

$$(u_e(x, y) + c, v_e(x, y)) = (\psi_{ey}(x, y), -\psi_{ex}(x, y)),$$

and the steady pressure $P_e(x, y)$ is determined through

$$P_e(x, y) = \frac{1}{2}c^2 - \frac{1}{2} |\nabla \psi_e(x, y)|^2 - gy.$$  \hspace{1cm} (2.1)

Let

$$D_e = \{(x, y) : -\infty < x < +\infty, -h < y < \eta_e(x)\}$$

and

$$S_e = \{(x, \eta_e(x)) : -\infty < x < +\infty\}$$

denote, respectively, the undisturbed fluid domain and the steady wave profile.
Let us denote

\[(\eta(t; x), u(t; x, y), v(t; x, y), P(t; x, y))\]

to be the infinitesimal perturbations of the wave profile, the velocity field and the pressure respectively. The stream function perturbation is \(\psi(t; x, y)\), such that \((u, v) = (\psi_y, -\psi_x)\).

The linearized water-wave problem was derived in [31], and it takes the following form in the irrotational case:

\[\Delta \psi = 0 \quad \text{in} \quad \mathcal{D}_e,\]

\[\partial_t \eta + \partial_x (\psi_{xy} \eta) + \partial_y \psi = 0 \quad \text{on} \quad \mathcal{S}_e;\]

\[P + P_{xy} \eta = 0 \quad \text{on} \quad \mathcal{S}_e;\]

\[\partial_t \partial_n \psi + \partial_x (\psi_{xy} \partial_n \psi) + \partial_r P = 0 \quad \text{on} \quad \mathcal{S}_e;\]

\[\partial_z \psi = 0 \quad \text{on} \quad \{y = -h\},\]

where

\[\partial_r f = \partial_x f + \eta_{ex} \partial_y f \quad \text{and} \quad \partial_n f = \partial_y f - \eta_{ex} \partial_x f\]
denote the tangential and normal derivatives of a function \(f(x, y)\) on the curve \(\{y = \eta_e(x)\}\). Alternatively, \(\partial_r f(x) = \frac{d}{dx} f(x, \eta_e(x))\). Note that the above linearized system may be viewed as one for \(\psi(t; x, y)\) and \(\eta(t; x)\). Indeed, \(P(t; x, \eta_e(x))\) is determined through \((2.2c)\) in terms of \(\eta(t; x)\) and other physical quantities are similarly determined in terms of \(\psi(t; x, y)\) and \(\eta(t; x)\).

A growing mode refers to a solution to the linearized water-wave problem \((2.2a)-(2.2e)\) of the form

\[(\eta(t; x), \psi(t; x, y)) = (e^{\lambda t} \eta_e(x), e^{\lambda t} \psi(x, y))\]

and \(P(t; x, \eta_e(x)) = e^{\lambda t} P(x, \eta_e(x))\) with \(\text{Re} \lambda > 0\). For a growing mode, the linearized system \((2.2)\) becomes

\[\Delta \psi = 0 \quad \text{in} \quad \mathcal{D}_e\]

and the following boundary conditions on \(\mathcal{S}_e\),

\[\lambda \eta(x) + \frac{d}{dx} (\psi_{xy}(x, \eta_e(x)) \eta(x)) = -\frac{d}{dx} \psi(x, \eta_e(x)),\]

\[P(x, \eta_e(x)) + P_{xy}(x, \eta_e(x)) \eta(x) = 0,\]

\[\lambda \psi_n(x) + \frac{d}{dx} (\psi_{xy}(x, \eta_e(x)) \psi_n(x)) = -\frac{d}{dx} P(x, \eta_e(x)).\]

We impose the following boundary condition on the flat bottom

\[\psi(x, -h) = 0,\]
from which (2.2e) follows. In summary, the growing-mode problem for a solitary
water-wave is to find a nontrivial solution of (2.3)-(2.7) with Re $\lambda > 0$. Below,
we look for purely growing modes with $\lambda > 0$ and reduce the system (2.3)-(2.7)
to one single equation for $\psi|_{S_e}$. For simplicity, here and in the sequel we identify
$\psi_{ey}(x)$ with $\psi_{ey}(x, \eta(x))$ and $\phi(x)$ with $\phi(x, \eta(x))$, etc. First, we introduce
the following operator

$$C^\lambda = \left( \lambda + \frac{d}{dx} (\psi_{ey}(x) \cdot ) \right)^{-1} \frac{d}{dx}. \tag{2.8}$$

Note that $\psi_{ey} > 0$ in $D_e$ by the maximum principle and Hopf’s principle (23),
and the fact that $\psi_{ey} = u_e + c \rightarrow c$ as $|x| \rightarrow \infty$. Thus

$$c_0 \leq \psi_{ey}(x, \eta(x)) = u_e + c \leq c_1, \tag{2.9}$$

for some constant $c_0, c_1 > 0$. Defining the operator

$$D \phi = \frac{d}{dx} (\psi_{ey}(x) \phi(x)), \tag{2.10}$$

we can write $C^\lambda$ as

$$C^\lambda = \frac{D}{\lambda + D} \frac{1}{\psi_{ey}(x)} = \left( 1 - \frac{\lambda}{\lambda + D} \right) \frac{1}{\psi_{ey}(x)}. \tag{2.10}$$

Denote $L^2_{\psi_{ey}}(S_e)$ to be the $\psi_{ey}$-weighted $L^2$ space on $S_e$. Because of the bound
(2.9), $L^2_{\psi_{ey}}(S_e)$ and $L^2(S_e)$ are norm equivalent. Note that the operator $D$
is anti-symmetric on $L^2_{\psi_{ey}}(S_e)$.

**Lemma 2.1** For $\lambda > 0$, define the operator $\mathcal{E}^{\lambda, \pm} : L^2_{\psi_{ey}}(S_e) \rightarrow L^2_{\psi_{ey}}(S_e)$ by

$$\mathcal{E}^{\lambda, \pm} = \frac{\lambda}{\lambda \pm D}. \tag{2.11}$$

Then,

(a) The operator $\mathcal{E}^{\lambda, \pm}$ is continuous in $\lambda$,

$$\|\mathcal{E}^{\lambda, \pm}\|_{L^2_{\psi_{ey}}(S_e) \rightarrow L^2_{\psi_{ey}}(S_e)} \leq 1, \tag{2.11}$$

and

$$\|1 - \mathcal{E}^{\lambda, \pm}\|_{L^2_{\psi_{ey}}(S_e) \rightarrow L^2_{\psi_{ey}}(S_e)} \leq 1. \tag{2.12}$$

(b) When $\lambda \rightarrow 0^+$, $\mathcal{E}^{\lambda, \pm}$ converges to 0 strongly in $L^2_{\psi_{ey}}(S_e)$.

(c) When $\lambda \rightarrow +\infty$, $\mathcal{E}^{\lambda, \pm}$ converges to 1 strongly in $L^2_{\psi_{ey}}(S_e)$.

**Proof.** Denote $\{M_\alpha; \alpha \in \mathbb{R}^1\}$ to be the spectral measure of the self-adjoint
operator $\mathcal{R} = -iD$ on $L^2_{\psi_{ey}}(S_e)$. Then

$$\|\mathcal{E}^{\lambda, \pm} \phi\|^2_{L^2_{\psi_{ey}}} = \int_\mathbb{R} \frac{\lambda}{\lambda \pm i\alpha}^2 d\|M_\alpha \phi\|^2_{L^2_{\psi_{ey}}} \leq \int_\mathbb{R} d\|M_\alpha \phi\|^2_{L^2_{\psi_{ey}}} = \|\phi\|^2_{L^2_{\psi_{ey}}}$

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and \((2.11)\) follows. Similarly, we get the estimate \((2.12)\). To prove (b), we take any \(\phi \in L^2_{\psi_{ey}}(S_e)\) and denote the function \(\xi(\alpha)\) to be such that \(\xi(\alpha) = 0\) for \(\alpha \neq 0\) and \(\xi(0) = 1\). Then by the dominant convergence theorem, when \(\lambda \to 0^+\),

\[
\|\mathcal{E}^{\lambda, \pm} \phi\|^2_{L^2_{\psi_{ey}}} = \int_{\mathbb{R}} \left| \frac{\lambda}{\lambda \pm i\alpha} \right|^2 \|M_\alpha \phi\|^2_{L^2_{\psi_{ey}}} \to \int_{\mathbb{R}} \|M_0 \phi\|^2_{L^2_{\psi_{ey}}}
\]

Note that \(M_{\{0\}}\) is the projector of \(L^2_{\psi_{ey}}\) to \(\ker \mathcal{D} = \{0\}\). So \(M_{\{0\}} = 0\) and \(\mathcal{E}^{\lambda, \pm} \phi \to 0\) in \(L^2_{\psi_{ey}}\). The proof of (c) is similar to that of (b) and we skip it.

By the above lemma and the bound \((2.9)\) on \(\psi_{ey}\), we have

**Lemma 2.2** For \(\lambda > 0\), the operator \(C^\lambda : L^2(S_e) \to L^2(S_e)\) defined by \((2.8)\) has the following properties:

(a) 
\[
\|C^\lambda\|_{L^2(S_e) \to L^2(S_e)} \leq C,
\]

for some constant \(C\) independent of \(\lambda\).

(b) When \(\lambda \to 0^+\), \(C^\lambda\) converges to \(\frac{1}{\psi_{ey}(x)}\) strongly in \(L^2(S_e)\).

(c) When \(\lambda \to +\infty\), \(C^\lambda\) converges to \(0\) strongly in \(L^2(S_e)\).

By using the operator \(C^\lambda\), the growing mode system \((2.3)-(2.7)\) is reduced to

\[
\psi_n(x) + C^\lambda P_{ey}(x) C^\lambda \psi = 0, \quad \text{on } S_e,
\]

\[
\Delta \psi = 0 \quad \text{in } D_e,
\]

\[
\psi(x, -h) = 0.
\]

We define the following Dirichlet-Neumann operator \(N_e : H^1(S_e) \to L^2(S_e)\) by

\[
N_e f = \partial_n \psi_f = (\partial_y \psi_f - \eta_x \partial_x \psi_f) (x, \eta_e(x)),
\]

where \(\psi_f\) is the unique solution of the following Dirichlet problem for \(f \in H^1(S_e)\)

\[
\Delta \psi_f = 0 \quad \text{in } D_e,
\]

\[
\psi_f|_{S_e} = f,
\]

\[
\psi_f(x, -h) = 0.
\]

Then the existence of a purely growing mode is reduced to find some \(\lambda > 0\) such that the operator \(A_e^\lambda\) defined by

\[
A_e^\lambda = N_e + C^\lambda P_{ey}(x) C^\lambda : H^1(S_e) \to L^2(S_e)
\]

has a nontrivial kernel. Note that if we denote by \(\tilde{\phi}_f\) the holomorphic conjugate of \(\psi_f\) in \(D_e\), then \(\partial_n \psi_f = \frac{\partial}{\partial x} \tilde{\phi}_f\). This motivates us to define an analogue of the Hilbert transformation as in \([58]\), by

\[
(C_e f) (x) = \int_0^x N_e f \, dx.
\]
Then the operator \( N_e \) can be written as \( N_e = \frac{d}{dx} C_e \). From the definition, \( C_e f + i f \) and \( f - iC_e f \) are the boundary values on \( S_e \) of some analytic functions in \( D_e \). Below, we further reduce the operator \( A^\lambda \) to one defined on the real line. First, we define the holomorphic mapping \( F : D_e \rightarrow \mathbb{R} \times (-ch, 0) \) by \( F(x, y) = (\phi_e(x, y), \psi_e(x, y)) \). We denote 
\[
(\xi, \varsigma) = \frac{1}{c} (\phi_e, \psi_e) \in D_0 = \mathbb{R} \times (-h, 0)
\]
and define the mapping \( G : D_0 \rightarrow D_e \) by 
\[
G(\xi, \varsigma) = (x(\xi, \varsigma), y(\xi, \varsigma)) = F^{-1} (c\xi, c\varsigma)
\]
The flat Dirichlet-Neumann operator \( N : H^1 (\mathbb{R}) \rightarrow L^2 (\mathbb{R}) \) is defined by 
\[
N f = \partial_\varsigma \psi_f |_{\varsigma = 0},
\]
where \( \psi_f \) is the solution of the following Dirichlet problem for \( f \in H^1 (\mathbb{R}) \)
\[
\Delta \psi_f = 0 \quad \text{in} \quad D_0,
\]
\[
\psi_f |_{\varsigma = 0} = f,
\]
\[
\psi_f |_{\varsigma = -h} = 0.
\]
Similarly, we define the operator \( C \) by \( C f = \int_0^x N f \, dx \). Then \( N = \frac{d}{d\xi} C \) and \( C f + i f \) or \( f - iC f \) are the boundary values on \( \{\varsigma = 0\} \) of analytic functions in \( D_0 \). Moreover, \( N \) is a Fourier multiplier operator with the symbol (2.15)
\[
n(k) = \frac{k}{\tanh(kh)}.
\]
To separate the uniform flow \((c, 0)\), we rewrite
\[
(x(\xi, \varsigma), y(\xi, \varsigma)) = (\xi, \varsigma) + (x_1(\xi, \varsigma), y_1(\xi, \varsigma))
\]
Denote
\[
w(\xi) = y_1(\xi, 0)
\]
Then we can set \( x_1(\xi, 0) = Cw \) by adding a proper constant to the vector potential \( \phi_e \). The mapping \( G \) restricted on \( \{\varsigma = 0\} \) induces a mapping \( B : \mathbb{R} \rightarrow S_e \) defined by \( B(\xi) = (\xi + Cw, w) \). Denote \( z = x + iy \) and \( p = \xi + i\varsigma \), then 
\[
u_e + c - iv_e = \frac{d(\phi_e + i\psi_e)}{dz} = c \frac{dp}{dz} = c \frac{1}{1 + \partial_\xi x_1 + i\partial_\varsigma y_1}.
\]
So on \( \{\varsigma = 0\} \), we get 
\[
u_e + c = c \frac{1 + N w}{|W|^2}, \quad v_e = c \frac{w'}{|W|^2}
\]
where
\[ W = (1 + \partial_\xi x_1 + i \partial_\xi y_1) |_{\xi = 0} = 1 + Cw' + iw' = 1 + Nw + iw' \] (2.18)
and \('\) denotes the \( \xi \)-derivative. From (2.17),
\[
1 + Nw = \frac{(ue + c)c}{(ue + c)^2 + \nu^2},
\]
and thus by (2.9), there exists \( c_2, c_3 > 0 \) such that
\[
c_2 < 1 + Nw < c_3. \tag{2.19}
\]
We define the operator \( B : L^2(\mathcal{S}_e) \to L^2(\mathbb{R}) \) by
\[
(Bf)(\xi) = f(B(\xi)) = f(\xi + Cw(\xi), w(\xi)), \text{ for any } f \in L^2(\mathcal{S}_e).
\]
Since \( BC_e = CB \) and \( \frac{d}{d\xi}B = (1 + Nw)B \frac{d}{dx} \), we have
\[
BN_e B^{-1} = B \frac{d}{dx} C_e B^{-1} = \frac{1}{1 + Nw} \frac{d}{d\xi} C = \frac{1}{1 + Nw} N,
\]
and
\[
BA^\lambda B^{-1} = BN_e B^{-1} + (BC^\lambda B^{-1}) BP_e B^{-1} (BC^\lambda B^{-1})
\]
\[
= \frac{1}{1 + Nw} N + \tilde{C}^\lambda P_{ey}(\xi) \tilde{C}^\lambda.
\]
Here,
\[
\tilde{C}^\lambda = BC^\lambda B^{-1} = \left( \lambda + \frac{1}{1 + Nw} \frac{d}{d\xi} (\psi_{ey}(\xi)) \right)^{-1} \left( \frac{1}{1 + Nw} \frac{d}{d\xi} \right)
\]
and we use \( \psi_{ey}(\xi), P_{ey}(\xi) \) to denote \( \psi_{ey}(B(\xi)), P_{ey}(B(\xi)) \) etc. For \( \lambda > 0 \), we define the operator \( A^\lambda : H^1(\mathbb{R}) \to L^2(\mathbb{R}) \) by
\[
A^\lambda = N + (1 + Nw) \tilde{C}^\lambda P_{ey}(\xi) \tilde{C}^\lambda.
\]
Then the existence of a purely growing mode is equivalent to finding some \( \lambda > 0 \) such that the operator \( A^\lambda \) has a nontrivial kernel.

## 3 Properties of the operator \( A^\lambda \)

In this section, we study the spectral properties of the operator \( A^\lambda \). First, we have the following estimate for the Dirichlet-Neumann operator \( \mathcal{N} \).
Lemma 3.1 There exists $C_0 > 0$, such that for any $\delta \in (0, 1)$ and $f \in H^{1/2} (\mathbb{R})$, we have

$$(N f, f) \geq (1 - \delta) \frac{1}{h} \|f\|_{L^2}^2 + C_0 \delta \|f\|_{H^{1/2}}^2,$$

where

$$\|f\|_{H^{1/2}}^2 = \int (1 + |k|) \left| \hat{f}(k) \right|^2 dk$$

and $\hat{f}(k)$ is the Fourier transformation of $f$.

Proof. By the definition (2.15),

$$(N f, f) = \int \frac{k}{\tanh (kh)} \left| \hat{f}(k) \right|^2 dk = \int \frac{|k|}{\tanh (|k| h)} \left| \hat{f}(k) \right|^2 dk.$$

It is easy to check that the function

$$h(x) = \frac{x}{\tanh (xh)}, \quad x \geq 0$$

satisfies

$$h(x) - \frac{1}{h} \geq 0 \text{ and } \lim_{x \to \infty} \frac{h(x) - \frac{1}{h}}{x} = 1.$$

So there exists $K > 0$, such that $h(x) - \frac{1}{h} \geq \frac{1}{2} x$, when $x > K$. Thus

$$(N f, f) \geq \frac{1}{h} \int \left| \hat{f}(k) \right|^2 dk + \frac{1}{2} \int_{|k| \geq K} |k| \left| \hat{f}(k) \right|^2 dk$$

$$\geq (1 - \delta) \frac{1}{h} \|f\|_{L^2}^2 + \frac{1}{2} \int_{|k| \geq K} |k| \left| \hat{f}(k) \right|^2 dk$$

$$\geq (1 - \delta) \frac{1}{h} \|f\|_{L^2}^2 + \min \left\{ \frac{\delta}{2h}, \frac{\delta}{2Kh}, \frac{1}{2} \right\} \int (1 + |k|) \left| \hat{f}(k) \right|^2 dk$$

This proves the Lemma with $C_0 = \min \left\{ \frac{\delta}{2h}, \frac{\delta}{2Kh}, \frac{1}{2} \right\}$. $lacksquare$

We have the following properties for the operator $\hat{C}^\lambda$.

Lemma 3.2 For $\lambda > 0$, the operator $\hat{C}^\lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by (2.8) satisfies:

(a) $\left\| \hat{C}^\lambda \right\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq C$,

for some constant $C$ independent of $\lambda$.

(b) When $\lambda \to 0+$, $\hat{C}^\lambda$ converges to $\frac{1}{\psi \circ (\xi)}$ strongly in $L^2(\mathbb{R})$.

(c) When $\lambda \to +\infty$, $\hat{C}^\lambda$ converges to $0$ strongly in $L^2(\mathbb{R})$.

Proof. By (2.19), the operator $B$ and $B^{-1}$ are bounded. Since $\hat{C}^\lambda = B\hat{C}^\lambda B^{-1}$, the above lemma follows directly from Lemma 2.2. $lacksquare$
To simply notations, we denote $b(\xi) = 1 + Nw$ and define the operators

$$\tilde{D} = \frac{1}{b(\xi)} \frac{d}{d\xi} (\psi_{ey}(\xi) \cdot) \quad \text{and} \quad \tilde{E}_{\lambda, \pm} \phi(x) = \frac{\lambda}{\lambda \pm D}.$$ 

The operator $\tilde{D}$ is anti-symmetric in the $b\psi_{ey}$-weighted space $L^2_{b\psi_{ey}}(\mathbb{R})$. Similar to the proof of Lemma 2.1, we have

**Lemma 3.3**  
(a) For any $\lambda > 0$,

$$\left\| \tilde{E}_{\lambda, \pm} \right\|_{L^2_{b\psi_{ey}} \rightarrow L^2_{b\psi_{ey}}} \leq 1,$$ 

and

$$\left\| 1 - \tilde{E}_{\lambda, \pm} \right\|_{L^2_{b\psi_{ey}} \rightarrow L^2_{b\psi_{ey}}} \leq 1.$$ 

(b) When $\lambda \to 0^+$, $\tilde{E}_{\lambda, \pm}$ converges to 0 strongly in $L^2_{b\psi_{ey}}$.

(c) When $\lambda \to +\infty$, $\tilde{E}_{\lambda, \pm}$ converges to 1 strongly in $L^2_{b\psi_{ey}}$.

The operator $\tilde{C}_{\lambda}$ can be written as

$$\tilde{C}_{\lambda} = \left(1 - \frac{\lambda}{\lambda + D}\right) \frac{1}{\psi_{ey}(\xi)} = \left(1 - \tilde{E}_{\lambda, +}\right) \frac{1}{\psi_{ey}(\xi)}.$$ 

**Proposition 1**  
For any $\lambda > 0$, we have

$$\sigma_{ess}(A_{\lambda}) \subset \left\{ z \mid \text{Re} \lambda \geq \frac{1}{2} \left( \frac{1}{h} - \frac{g}{c^2} \right) \right\}.$$ 

We note that

$$\delta_0 := \frac{1}{h} - \frac{g}{c^2} > 0$$

by Property (P2), so the above Proposition shows that the essential spectrum of $A_{\lambda}$ lies on the right half plane and is away from the imaginary axis. To prove Proposition 1 we need the following lemmas.

**Lemma 3.4**  
For any $u \in H^\frac{1}{2}(\mathbb{R})$, we have

(i) For any $\lambda > 0$,

$$\left\| \tilde{E}_{\lambda, \pm} u \right\|_{H^\frac{1}{2}} \leq C \left\| u \right\|_{H^\frac{1}{2}},$$

for some constant $C$ independent of $\lambda$.

Below, let $F(\xi)$ be a fixed bounded function that decays at infinity. Then

(ii) Given $\lambda > 0$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$\left\| F \tilde{E}_{\lambda, \pm} u \right\|_{L^2} \leq \varepsilon \left\| u \right\|_{H^\frac{1}{2}} + C_\varepsilon \left\| \frac{u}{1 + \xi^2} \right\|_{L^2}.$$
(iii) For any \( \varepsilon > 0 \), there exists \( \lambda_\varepsilon > 0 \), such that when \( 0 < \lambda < \lambda_\varepsilon \),
\[
\left\| F \tilde{\xi}^{\lambda, \pm} u \right\|_{L^2} \leq \varepsilon \left\| u \right\|_{H^{\frac{1}{2}}}. 
\]

(iv) For any \( \varepsilon > 0 \), there exists \( \Lambda_\varepsilon > 0 \), such that when \( \lambda > \Lambda_\varepsilon \),
\[
\left\| F \left( 1 - \tilde{\xi}^{\lambda, \pm} \right) u \right\|_{L^2} \leq \varepsilon \left\| u \right\|_{H^{\frac{1}{2}}}.
\]

**Proof.** Proof of (i): Denote \( \{ \tilde{M}_\alpha; \alpha \in \mathbb{R}^1 \} \) to be the spectral measure of the self-adjoint operator \( \tilde{R} = -i\tilde{D} \) on \( L^2_{b \psi, y} \). For \( s \geq 0 \), we define the space
\[
\tilde{H}^s = \left\{ u \in L^2_{b \psi, y} \mid \left\| \tilde{R}^s u \right\|_{L^2_{b \psi, y}} \right\}
\]
with the norm
\[
\left\| u \right\|_{\tilde{H}^s} = \left\| u \right\|_{L^2_{b \psi, y}} + \left\| \tilde{R}^s u \right\|_{L^2_{b \psi, y}} = \left\| u \right\|_{L^2_{b \psi, y}} + \left( \int_{\mathbb{R}} \left| \alpha \right|^{2s} d\tilde{M}_\alpha \left\| u \right\|_{L^2_{b \psi, y}}^2 \right)^{\frac{1}{2}},
\]
where \( \left\| \tilde{R}^s \right\| \) is the positive self-adjoint operator defined by \( \int |\alpha|^s d\tilde{M}_\alpha \). We claim that the norm \( \left\| \cdot \right\|_{\tilde{H}^s} \) is equivalent to the norm \( \left\| \cdot \right\|_{H^s} \), for \( 0 \leq s \leq 1 \). When \( s = 0 \), \( \tilde{H}^0 = L^2_{b \psi, y} \) and \( H^0 = L^2 \). Since \( b \) and \( \psi_{cy} \) are bounded with positive lower bounds, \( \left\| \cdot \right\|_{L^2_{b \psi, y}} \) and \( \left\| \cdot \right\|_{L^2} \) are equivalent. When \( s = 1 \), we have
\[
\left\| u \right\|_{\tilde{H}^1} = \left\| u \right\|_{L^2_{b \psi, y}} + \left( \int_{\mathbb{R}} \frac{1}{b} d\xi \left( \psi_{cy} u \right)^2 b \psi_{cy} \right)^{\frac{1}{2}},
\]
which is clearly equivalent to \( \left\| u \right\|_{H^1}^2 \), again due to the bounds of \( b \) and \( \psi_{cy} \). When \( 0 < s < 1 \), the spaces \( \tilde{H}^s \) \((H^s)\) are the interpolation spaces of \( \tilde{H}^0(H^0) \) and \( \tilde{H}^1 \) \((H^1)\). So by the general interpolation theory (11), we get the equivalence of the norms \( \left\| \cdot \right\|_{\tilde{H}^s} \) and \( \left\| \cdot \right\|_{H^s} \). Thus, there exists \( C_1, C_2 > 0 \), such that
\[
C_1 \left\| u \right\|_{\tilde{H}^\frac{s}{2}} \leq \left\| u \right\|_{H^\frac{s}{2}} \leq C_2 \left\| u \right\|_{\tilde{H}^\frac{s}{2}}. \tag{3.6}
\]
Since \( \tilde{R} \) and \( \tilde{\xi}^{\lambda, \pm} \) are commutable, we have
\[
\left\| \tilde{\xi}^{\lambda, \pm} u \right\|_{\tilde{H}^\frac{s}{2}} = \left\| \tilde{\xi}^{\lambda, \pm} u \right\|_{L^2_{b \psi, y}} + \left\| \tilde{\xi}^{\lambda, \pm} \left( \left\| \tilde{R}^{\frac{s}{2}} u \right\|_{L^2_{b \psi, y}} \right) \right\|_{L^2_{b \psi, y}} \leq \left\| \left\| u \right\|_{L^2_{b \psi, y}} + \left\| \tilde{R}^{\frac{s}{2}} u \right\|_{L^2_{b \psi, y}} \right\|_{\tilde{H}^\frac{s}{2}} = \left\| u \right\|_{\tilde{H}^\frac{s}{2}}.
\]
The estimate (3.5) follows from above and (3.6).
Proof of (ii): Suppose otherwise, then there exists $\varepsilon_0 > 0$ and a sequence $\{u_n\} \in H^{1,2}_0(\mathbb{R})$ such that

$$\left\| F\tilde{\mathcal{E}}^{\lambda, \pm} u_n \right\|_{L^2} \geq \varepsilon_0 \left\| u_n \right\|_{H^{1,2}} + n \frac{\left\| u_n \right\|_{1 + \lambda^2}}{1 + \lambda^2}.$$  

We normalize $u_n$ by setting $\left\| F\tilde{\mathcal{E}}^{\lambda, \pm} u_n \right\|_{L^2} = 1$. Then

$$\left\| u_n \right\|_{H^{1,2}} \leq \frac{1}{\varepsilon_0}, \quad \left\| \frac{u_n}{1 + \lambda^2} \right\|_{L^2} \leq \frac{1}{n}.$$  

So there exists $u_\infty \in H^{1,2}$, such that $u_n \rightharpoonup u_\infty$ weakly in $H^{1,2}$. Since $\frac{u_n}{1 + \lambda^2} \to 0$ strongly in $L^2$, we have $u_\infty = 0$. Thus $v_n = \tilde{\mathcal{E}}^{\lambda, \pm} u_n$ converges to 0 weakly in $L^2$. By (i),

$$\left\| v_n \right\|_{H^{1,2}} \leq C \left\| u_n \right\|_{H^{1,2}} \leq C \varepsilon_0.$$

Let $\chi_R \in C^\infty_0$ be a cut-off function for $\{\xi \leq R\}$. We write

$$F = F\chi_R + F(1 - \chi_R) = F_1 + F_2.$$  

Then

$$\left\| F_2 v_n \right\|_{L^2} \leq C \max_{\left| \xi \right| \geq R} \left| F(\xi) \right| \left\| u_n \right\|_{L^2} \leq C \frac{1}{\varepsilon_0} \max_{\left| \xi \right| \geq R} \left| F(\xi) \right| \leq \frac{1}{2},$$

when $R$ is chosen to be big enough. Since $F_1$ has a compact support and $H^{1,2} \hookrightarrow L^2$ is locally compact, so $F_1 v_n \to 0$ strongly in $L^2$. Thus, when $n$ is large enough,

$$\left\| F v_n \right\|_{L^2} \leq \left\| F_1 v_n \right\|_{L^2} + \left\| F_2 v_n \right\|_{L^2} \leq \frac{3}{4}.$$  

This is a contradiction to the fact that $\left\| F v_n \right\|_{L^2} = \left\| F\tilde{\mathcal{E}}^{\lambda, \pm} u_n \right\|_{L^2} = 1$.

Proof of (iii): Suppose otherwise, then there exists $\varepsilon_0 > 0$ and a sequence $\{u_n\} \in H^{1,2}_0(\mathbb{R})$, $\lambda_n \to 0+$, such that

$$\left\| F\tilde{\mathcal{E}}^{\lambda_n, \pm} u_n \right\|_{L^2} \geq \varepsilon_0 \left\| u_n \right\|_{H^{1,2}}.$$  

Normalize $u_n$ by $\left\| F\tilde{\mathcal{E}}^{\lambda_n, \pm} u_n \right\|_{L^2} = 1$. Then $\left\| u_n \right\|_{H^{1,2}} \leq \frac{1}{\varepsilon_0}$. Let $u_n \rightharpoonup u_\infty$ weakly in $H^{1,2}$. Then for any $v \in L^2$, we have

$$\left( \tilde{\mathcal{E}}^{\lambda_n, \pm} u_n, v \right) = \left( u_n, b\psi_{cy} \tilde{\mathcal{E}}^{\lambda_n, \pm} \left( \frac{v}{b\psi_{cy}} \right) \right) \to 0,$$

because by Lemma $b\psi_{cy} \tilde{\mathcal{E}}^{\lambda_n, \pm} \left( \frac{v}{b\psi_{cy}} \right) \to 0$ strongly in $L^2$ when $\lambda_n \to 0+$.

So $\tilde{\mathcal{E}}^{\lambda_n, \pm} u_n \to 0$ weakly in $L^2$, and this leads to a contradiction as in the proof of (ii).

Proof of (iv) is the same as that of (iii), except that we use the strong convergence $1 - \tilde{\mathcal{E}}^{\lambda_n, \pm} \to 0$ when $\lambda_n \to \infty.$
Lemma 3.5 Consider any sequence 
\[ \{u_n\} \in H^2(R), \quad \|u_n\|_2 = 1, \quad \text{supp } u_n \subset \{\xi \mid |\xi| \geq n\}. \]
Then for any complex number \( z \) with \( \text{Re } z < \frac{1}{2}\delta_0 \), we have 
\[ \text{Re } \left( (A - z) u_n, u_n \right) \geq \frac{1}{4}\delta_0, \]
when \( n \) is large enough. Here, \( \delta_0 \) is defined by (3.4). 

Proof. We have 
\[ \text{Re } \left( (A - z) u_n, u_n \right) = (N u_n, u_n) - \text{Re } z + \text{Re } \left( b\tilde{C}^\lambda P_{ey}(\xi) \tilde{C}^\lambda u_n, u_n \right). \] (3.7)

For \( 0 < \delta < 1 \) (to be fixed later), by Lemma 3.1
\[ (N u_n, u_n) \geq (1 - \delta) \frac{1}{h} + C_0\delta \|u_n\|^2_{H^2}. \] (3.8)

Note that by (2.1)
\[ P_{ey}(\xi) = -g + \psi_{ey}(\xi) \tilde{a}(\xi), \quad \tilde{a}(\xi) = \frac{d}{dx}(\psi_{ex})(\xi). \] (3.9)

Then \( \tilde{a}(\xi) \) decays exponentially when \( |\xi| \to \infty \). We have 
\[ \left( b\tilde{C}^\lambda P_{ey}(\xi) \tilde{C}^\lambda u_n, u_n \right) \]
\[ = g \left( b \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n, u_n \right) \]
\[ + \left( b \left(1 - \tilde{C}^{\lambda, +} \right) \tilde{a} \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n, u_n \right) \]
\[ = T_1 + T_2. \]

Denote 
\[ \tilde{b}(\xi) = b - 1, \quad \tilde{c}(\xi) = \frac{1}{\psi_{ey}} - \frac{1}{c}. \] (3.10)

Then \( \tilde{b}, \tilde{c} \) tends to zero exponentially when \( |\xi| \to \infty \). The first term can be written as 
\[ T_1 = -g \left( b \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n, \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n \right) \]
\[ = -g \left( b\psi_{ey} \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n, \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n \right) \]
\[ - g \left( b\psi_{ey} \left[ \frac{1}{\psi_{ey}}, 1 - \tilde{C}^{\lambda, +} \right] \frac{1}{\psi_{ey}} u_n, \left(1 - \tilde{C}^{\lambda, +} \right) \frac{1}{\psi_{ey}} u_n \right) \]
\[ = T_1^1 + T_1^2, \]
where in the above we use the fact that the operator $\hat{D}$ is anti-symmetric in the space $L^2_{\psi_{\text{ey}}}$. In the rest of this paper, we use $C$ to denote a generic constant in the estimates. By Lemma 3.2 and the assumption that $\text{supp} u_n \subset \{\xi \mid |\xi| \geq n\}$, we have

$$|T_1^1| \leq g \left\| \left(1 - \hat{\mathcal{E}}^{\lambda,+}\right) \frac{1}{\psi_{\text{ey}}} u_n \right\|_{L^2_{\psi_{\text{ey}}}} \left\| \left(1 - \hat{\mathcal{E}}^{\lambda,-}\right) \frac{1}{\psi_{\text{ey}}} u_n \right\|_{L^2_{\psi_{\text{ey}}}}$$

$$\leq g \left\| \frac{1}{\psi_{\text{ey}}} u_n \right\|_{L^2_{\psi_{\text{ey}}}} \left\| \frac{1}{\psi_{\text{ey}}} u_n \right\|_{L^2_{\psi_{\text{ey}}}}$$

$$= g \left(\int b \frac{1}{\psi_{\text{ey}}} |u_n|^2 \, d\xi\right)^\frac{1}{2} \left(\int b \frac{1}{\psi_{\text{ey}}} |u_n|^2 \, d\xi\right)^\frac{1}{2}$$

$$\leq g \left(\frac{1}{c^3} + C \max_{|\xi| \geq n} \left|\hat{b}(\xi)\hat{c}(\xi)\right| + \left|\hat{b}(\xi)\right| + \left|\hat{c}(\xi)\right|\right)^\frac{1}{2} \cdot \left(\frac{1}{c} + C \max_{|\xi| \geq n} \left|\hat{b}(\xi)\hat{c}(\xi)\right| + \left|\hat{b}(\xi)\right| + \left|\hat{c}(\xi)\right|\right)^\frac{1}{2} \|u_n\|_{L^2}^2$$

$$= \frac{g}{c^2} + O\left(\frac{1}{n}\right).$$

Since

$$\left[\frac{1}{\psi_{\text{ey}}}, 1 - \hat{\mathcal{E}}^{\lambda,+}\right] = [\hat{c}, \hat{\mathcal{E}}^{\lambda,+}] = \hat{\mathcal{E}}^{\lambda,+} - \hat{\mathcal{E}}^{\lambda,+}\hat{c},$$

we have

$$|T_1^2| \leq \left\| b\psi_{\text{ey}} \left[\frac{1}{\psi_{\text{ey}}}, 1 - \hat{\mathcal{E}}^{\lambda,+}\right] \frac{1}{\psi_{\text{ey}}} u_n \right\|_{L^2} \left\| \left(1 - \hat{\mathcal{E}}^{\lambda,-}\right) \frac{1}{\psi_{\text{ey}}} u_n \right\|_{L^2}$$

$$\leq C \left(\left\| \hat{\mathcal{E}}^{\lambda,+} \left(\frac{1}{\psi_{\text{ey}}} u_n\right) \right\|_{L^2} + \left\| \hat{\mathcal{E}}^{\lambda,+} \left(\hat{c} \frac{1}{\psi_{\text{ey}}} u_n\right) \right\|_{L^2}\right).$$

Since $\hat{c}(\xi)$ decays at infinity, by Lemma 3.4 (ii), for $\varepsilon > 0$ (to be fixed later), there exists $C_{\varepsilon}$ such that

$$\left\| \hat{\mathcal{E}}^{\lambda,+} u_n \right\|_{L^2} \leq \varepsilon \|u_n\|_{H^{\frac{1}{2}}} + C_{\varepsilon} \left\| \frac{u_n}{1 + \xi^2} \right\|_{L^2} \leq \varepsilon \|u_n\|_{H^{\frac{1}{2}}} + \frac{C_{\varepsilon}}{n^2}.$$
Since
\[ \|\tilde{\mathcal{E}} \psi + \left(\frac{1}{\psi} \tilde{u}_n\right)\|_{L^2} \leq C \left\| \frac{1}{\psi} \tilde{u}_n\right\|_{L^2} = O\left(\frac{1}{n}\right), \]
so
\[ |T_1^2| \leq C \left( \varepsilon \|u_n\|_{H^1_n} + \frac{C_\varepsilon}{n^2} + \frac{1}{n} \right) \]
and thus
\[ |T_1| \leq \frac{g}{c^2} + C \left( \varepsilon \|u_n\|_{H^1_n} + \frac{C_\varepsilon}{n^2} + \frac{1}{n} \right). \]
The term \(T_2\) can be written as
\[
T_2 = \left( b \left(1 - \tilde{\mathcal{E}} \lambda^+\right) \tilde{a} \left(1 - \tilde{\mathcal{E}} \lambda^+\right) \frac{1}{\psi} \tilde{u}_n, u_n \right)
\]
\[
= \left( b \psi \tilde{a} \left(1 - \tilde{\mathcal{E}} \lambda^+\right) \frac{1}{\psi} \tilde{u}_n, \left(1 - \tilde{\mathcal{E}} \lambda^+\right) \frac{1}{\psi} \tilde{u}_n \right)
\]
\[
= \left( b \psi \tilde{a} \left(1 - \tilde{\mathcal{E}} \lambda^+\right) \frac{1}{\psi} \tilde{u}_n, \left(1 - \tilde{\mathcal{E}} \lambda^-\right) \frac{1}{\psi} \tilde{u}_n \right)
\]
\[ + \left( b \psi \tilde{a} \left(1 - \tilde{\mathcal{E}} \lambda^+\right) \frac{1}{\psi} \tilde{u}_n, \left(1 - \tilde{\mathcal{E}} \lambda^-\right) \frac{1}{\psi} \tilde{u}_n \right)
\]
\[ = T_1^1 + T_2^2. \]

Similar to the estimates for \(T_1\), we have
\[ |T_2^1| \leq \left\| \tilde{a} \frac{1}{\psi} \tilde{u}_n \right\|_{L^2_{\psi \tilde{a} \psi}} \left\| \frac{1}{\psi} \tilde{u}_n \right\|_{L^2_{\psi \tilde{a} \psi}} \leq C \max |\tilde{a} (\xi)| = O \left(\frac{1}{n}\right) \]
and
\[ |T_2^2| \leq C \left( \|\tilde{a} \tilde{\mathcal{E}} \lambda^+ \left(\frac{1}{\psi} \tilde{u}_n\right)\|_{L^2} + \|\tilde{\mathcal{E}} \lambda^+ \left(\tilde{a} \frac{1}{\psi} \tilde{u}_n\right)\|_{L^2} \right)
\]
\[ \leq C \left( \varepsilon \|u_n\|_{H^1_n} + \frac{C'_\varepsilon}{n^2} + \frac{1}{n} \right). \]
So
\[ |T_2| \leq C \left( \varepsilon \|u_n\|_{H^1_n} + \frac{C'_\varepsilon}{n^2} + \frac{1}{n} \right) \]
Thus
\[ |\text{Re} \left( t\tilde{\mathcal{E}} \lambda P_{\psi} (\xi) \tilde{\mathcal{E}} \lambda u_n, u_n \right) | \leq |T_1| + |T_2|
\]
\[ \leq \frac{g}{c^2} + C \left( \varepsilon \|u_n\|_{H^1_n} + \frac{C_\varepsilon + C'_\varepsilon}{n^2} + \frac{1}{n} \right). \]
Combining with (3.8), we have
\[
\text{Re} \left( (A^\lambda - z) u_n, u_n \right) \\
\geq (1 - \delta) \frac{1}{h} + C_0 \delta \| u_n \|_{H^\beta}^2 - \frac{1}{2} \delta_0 - \frac{g}{c^2} - C \left( \varepsilon \| u_n \|_{H^\beta} + \frac{C_\varepsilon + C_\nu}{n^2} + \frac{1}{n} \right) \\
= \frac{1}{2} \delta_0 - \frac{\delta}{h} + (C_0 \delta - C \varepsilon) \| u_n \|_{H^\beta} - C \left( \frac{C_\varepsilon + C_\nu}{n^2} + \frac{1}{n} \right) \\
\geq \frac{1}{4} \delta_0, \text{ when } n \text{ is large enough,}
\]
by choosing \( \varepsilon > 0 \) and \( \delta \in (0, 1) \) such that \( \varepsilon \leq \frac{C_\varepsilon}{C} \delta \) and \( \delta \leq \frac{1}{8} \delta_0 h \). This finishes the proof of the lemma.

To study the essential spectrum of \( A^\lambda \), we first look at the Zhislin Spectrum \( Z(A^\lambda) \) \([29]\). A Zhislin sequence for \( A^\lambda \) and \( z \in \mathbb{C} \) is a sequence \( \{ u_n \} \in H^1 \), \( \| u_n \|_2 = 1 \), \( \text{supp} \ u_n \subset \{ \xi \mid |\xi| \geq n \} \) and \( \| (A^\lambda - z) u_n \|_2 \to 0 \) as \( n \to \infty \). The set of all \( z \) such that a Zhislin sequence exists for \( A^\lambda \) and \( z \) is denoted by \( Z(A^\lambda) \). From the above definition and Lemma 3.5 we readily have
\[
Z(A^\lambda) \subset \left\{ z \in \mathbb{C} \mid \text{Re} \ z \geq \frac{1}{2} \delta_0 \right\}. \tag{3.11}
\]

Another related spectrum is the Weyl spectrum \( W(A^\lambda) \) \([29]\). A Weyl sequence for \( A^\lambda \) and \( z \in \mathbb{C} \) is a sequence \( \{ u_n \} \in H^1 \), \( \| u_n \|_2 = 1 \), \( u_n \to 0 \) weakly in \( L^2 \) and \( \| (A^\lambda - z) u_n \|_2 \to 0 \) as \( n \to \infty \). The set \( W(A^\lambda) \) is all \( z \) such that a Weyl sequence exists for \( A^\lambda \) and \( z \). By \([29\) Theorem 10.10\]), \( W(A^\lambda) \subset \sigma_{\text{ess}}(A^\lambda) \) and the boundary of \( \sigma_{\text{ess}}(A^\lambda) \) is contained in \( W(A^\lambda) \). So to prove Proposition 11 it suffices to show that \( W(A^\lambda) = Z(A^\lambda) \). Since if this is true, then (3.8) follows from (3.11). By \([29\) Theorem 10.12\]), the proof of \( W(A^\lambda) = Z(A^\lambda) \) can be reduced to proving the following lemma.

Lemma 3.6 Given \( \lambda > 0 \). Let \( \chi \in C_0^\infty(\mathbb{R}) \) be a cut-off function such that \( \chi|_{\{|\xi| \leq R_0\}} = 1 \) for some \( R_0 > 0 \). Define \( \chi_d = \chi(\xi/d) \), \( d > 0 \). Then for each \( d \), \( \chi_d(A^\lambda - z)^{-1} \) is compact for some \( z \in \rho(A^\lambda) \), and that there exists \( \varepsilon(d) \to 0 \) as \( d \to \infty \) such that for any \( u \in C_0^\infty(\mathbb{R}) \),
\[
\| [A^\lambda, \chi_d] u \|_2 \leq \varepsilon(d) \left( \| A^\lambda u \|_2 + \| u \|_2 \right). \tag{3.12}
\]

**Proof.** Since \( A^\lambda = N + \mathcal{K}^\lambda \), where \( N \) is positive and
\[
\mathcal{K}^\lambda = b \tilde{C}^\lambda P_{ey} \tilde{C}^\lambda : L^2 \to L^2 \tag{3.13}
\]
is bounded, so if \( z = -k \) with \( k > 0 \) sufficiently large, then \( z \in \rho(A^\lambda) \). The compactness of \( \chi_d(A^\lambda + k)^{-1} \) follows from the local compactness of \( H^1 \hookrightarrow L^2 \). To show (3.12), we note that the graph norm of \( A^\lambda \) is equivalent to \( \| \cdot \|_{H^1} \). First, we write
\[
[\mathcal{K}^\lambda, \chi_d] = b \left[ \tilde{C}^\lambda, \chi_d \right] P_{ey} \tilde{C}^\lambda + b \tilde{C}^\lambda P_{ey} \left[ \tilde{C}^\lambda, \chi_d \right].
\]
We have

\[
\left[ \tilde{C}^\lambda, \chi_d \right] = \left[ \left( 1 - \frac{\lambda}{\lambda + D} \right) \frac{1}{\psi_{ey}}, \chi_d \right]
\]

\[
= -\left[ \frac{\lambda}{\lambda + D}, \chi_d \right] \frac{1}{\psi_{ey}} = -\frac{\lambda}{\lambda + D} \left[ \chi_d, \tilde{D} \right] \frac{1}{\lambda + D} \frac{1}{\psi_{ey}}
\]

\[
= \frac{1}{\lambda d} \tilde{\varepsilon}^{\lambda,+} \left( \frac{1}{b} \chi' (\xi/d) \psi_{ey} \right) \tilde{\varepsilon}^{\lambda,+} \frac{1}{\psi_{ey}}.
\]

Since \( \| \tilde{\varepsilon}^{\lambda,+} \|_{L^2 \to L^2} \) is bounded, so

\[
\| \left[ \tilde{C}^\lambda, \chi_d \right] \|_{L^2 \to L^2} \leq \frac{C}{\lambda d}
\]

and therefore

\[
\| \left[ [K^\lambda, \chi_d] \right] u \|_2 \leq \frac{C'}{\lambda d} \| u \|_2.
\] (3.14)

Denote \( \mathcal{N}_1 = 1 + \frac{d}{d\xi} \) and \( \mathcal{N}_2 \) is the Fourier multiplier operator with the symbol

\[
n_2 (k) = \frac{k}{\tanh (kh) (1 + ik)}.
\] (3.15)

Then \( \mathcal{N} = \mathcal{N}_2 \mathcal{N}_1 \) and thus

\[
[\mathcal{N}, \chi_d] = \mathcal{N}_2 [\mathcal{N}_1, \chi_d] + [\mathcal{N}_2, \chi_d] \mathcal{N}_1.
\]

Since \( [\mathcal{N}_1, \chi_d] = \frac{1}{\xi} \chi' (\xi/d) \) and \( \| \mathcal{N}_2 \|_{L^2 \to L^2} \) is bounded, we have

\[
\| \mathcal{N}_2 [\mathcal{N}_1, \chi_d] u \|_2 \leq \frac{C}{d} \| u \|_2.
\]

To estimate \( [\mathcal{N}_2, \chi_d] \), for \( v \in C^\infty_0 (\mathbb{R}) \), we follow [15] p.127-128] to write

\[
[\mathcal{N}_2, \chi_d] v = -(2\pi)^{-\frac{1}{2}} \int \n_2 (\xi - y) (\chi_d (\xi) - \chi_d (y)) v (y) \, dy
\]

\[
= -\int_0^1 \left( (2\pi)^{-\frac{1}{2}} (x - y) \n_2 (\xi - y) \right) \chi'_d (\rho (\xi - y) + y) v (y) \, dy \, d\rho
\]

\[
= \int_0^1 A_\rho v \, d\rho,
\]

where \( A_\rho \) is the integral operator with the kernel function

\[
K_\rho (\xi, y) = -(2\pi)^{-\frac{1}{2}} (\xi - y) \n_2 (\xi - y) \chi'_d (\rho (\xi - y) + y).
\]
Note that \( \alpha (\xi) = \xi \tilde{n}_2 (\xi) \) is the inverse Fourier transformation of \( in_2' (k) \) and obviously \( n_2' (k) \in L^2 \), so \( \alpha (\xi) \in L^2 \). Thus
\[
\int \int |K_\rho (\xi, y)|^2 \, d\xi dy = 2\pi \int \int |\alpha|^2 (\xi - y) |\chi_d'|^2 (\rho (\xi - y) + y) \, d\xi dy = 2\pi \int \int |\alpha|^2 (\xi) |\chi_d'|^2 (y) \, d\xi dy = 2\pi \|\alpha\|_{L^2}^2 \|\chi_d'|_{L^2}^2 = \frac{2\pi}{d} \|\alpha\|_{L^2}^2 \|\chi'\|_{L^2}^2.
\]
So
\[
\|[N_2, \chi_d]\|_{L^2 \to L^2} \leq \frac{C}{d^2}
\]
and
\[
\|[N_2, \chi_d]N_1 u\|_{L^2} \leq \frac{C}{d^2} \|N_1 u\|_{L^2} \leq \frac{C}{d^2} \|u\|_{H^1}.
\]
Thus
\[
\| [N, \chi_d] u \|_{L^2} \leq C \left( \frac{1}{d^2} + \frac{1}{d} \right) \| u \|_{H^1}.
\]
Combining above with (3.14), we get the estimate (3.12). This finishes the proof of the lemma and thus Proposition 1. \[\square\]

Recall that to find growing modes, we need to find \( \lambda > 0 \) such that \( A^\lambda \) has a nontrivial kernel. We use a continuity argument, by comparing the spectra of \( A^\lambda \) for \( \lambda \) near 0 and infinity. First, we study the case near infinity.

**Lemma 3.7** There exists \( \Lambda > 0 \), such that when \( \lambda > \Lambda \), \( A^\lambda \) has no eigenvalues in \( \{ z \mid \text{Re } z \leq 0 \} \).

**Proof.** Suppose otherwise, then there exists a sequence \( \{ \lambda_n \} \to \infty \), and \( \{ k_n \} \in \mathbb{C}, \{ u_n \} \in H^1 (\mathbb{R}) \), such that \( \text{Re } k_n \leq 0 \) and \( (A^\lambda_n - k_n) u_n = 0 \). Since \( \| A^\lambda - N \| = \| K^\lambda \| \leq M \) for some constant \( M \) independent of \( \lambda \) and \( N \) is a self-adjoint positive operator, all discrete eigenvalues of \( A^\lambda \) lie in
\[
D_M = \{ z \mid \text{Re } z \geq -M \text{ and } |\text{Im } z| \leq M \}.
\]
Therefore, \( k_n \to k_\infty \in D_M \) with \( \text{Re } k_\infty \leq 0 \). Denote
\[
e (\xi) = \max \left\{ |\tilde{a} (\xi)|, |\tilde{b} (\xi)|, |\tilde{c} (\xi)| \right\}, \qquad (3.16)
\]
where \( \tilde{a} (\xi), \tilde{b} (\xi), \tilde{c} (\xi) \) are defined in (3.9) and (3.11). Then \( e (\xi) \to 0 \) as \( |\xi| \to \infty \). Define the \( e (\xi) \)-weighted \( L^2 \) space \( L^2_e \) with the norm
\[
\| u \|_{L^2_e} = \left( \int e (\xi) |u|^2 \, d\xi \right)^{\frac{1}{2}}. \quad (3.17)
\]
We normalize \( u_n \) by setting \( \| u_n \|_{L^2_e} = 1 \). We claim that
\[
\| u_n \|_{H^\frac{1}{2}} \leq C, \text{ for a constant } C \text{ independent of } n. \quad (3.18)
\]
Assuming (3.18), we have $u_n \to u_\infty$ weakly in $H^{1/2}$. Moreover, $u_\infty \neq 0$. To show this, we choose $R > 0$ large enough such that $\max_{|\xi| \leq R} e(\xi) \leq \frac{1}{2C}$. Then

$$\int_{|\xi| \geq R} e(\xi) |u_n|^2 \, d\xi \leq \frac{1}{2C} \|u_n\|_{L^2} \leq \frac{1}{2}.$$  

Since $u_n \to u_\infty$ strongly in $L^2(\{|\xi| \leq R\})$, we have

$$\int_{|\xi| \leq R} e(\xi) |u_n|^2 \, d\xi = \lim_{n \to \infty} \int_{|\xi| \leq R} e(\xi) |u_n|^2 \, d\xi \geq \frac{1}{2}$$

and thus $u_\infty \neq 0$. By Lemma 3.2, $A^{\lambda_n} \to \mathcal{N}$ strongly in $L^2$, therefore $A^{\lambda_n} u_n \to \mathcal{N} u_\infty$ weakly. Thus $\mathcal{N} u_\infty = k_\infty u_\infty$. Since $\Re k_\infty \leq 0$, this is a contradiction to that $\mathcal{N} > 0$. It remains to show (3.18). The proof is quite similar to that of Lemma 3.5, so we only sketch it. From $(A^{\lambda_n} - k_n) u_n = 0$, we have

$$(\mathcal{N} u_n, u_n) + \Re \left( b^{\tilde{\lambda}_n} P_{\psi_0} (\xi) \tilde{C}^{\lambda_n} u_n, u_n \right) = \Re k_n \|u_n\|_{L^2}^2 \leq 0. \quad (3.19)$$

By Lemma 3.4

$$(\mathcal{N} u_n, u_n) \geq (1 - \delta) \frac{1}{h} \|u_n\|_{L^2}^2 + C_0 \delta \|u_n\|_{H^{1/2}}^2.$$  

Following the proof of Lemma 3.5, we write

$$(b^{\tilde{\lambda}_n} P_{\psi_0} (\xi) \tilde{C}^{\lambda_n} u_n, u_n)$$

$$= -g \left( b_{\psi_0} \left( 1 - \tilde{\xi}^{\lambda_n, +} \right) \frac{1}{\psi_0^2} u_n, \left( 1 - \tilde{\xi}^{\lambda_n, -} \right) \frac{1}{\psi_0} u_n \right)$$

$$- g \left( b_{\psi_0} \left[ \tilde{c}, 1 - \tilde{\xi}^{\lambda_n, +} \right] \frac{1}{\psi_0} u_n, \left( 1 - \tilde{\xi}^{\lambda_n, -} \right) \frac{1}{\psi_0} u_n \right)$$

$$+ \left( b_{\psi_0} \left( 1 - \tilde{\xi}^{\lambda_n, +} \right) \frac{1}{\psi_0} u_n, \left( 1 - \tilde{\xi}^{\lambda_n, -} \right) \frac{1}{\psi_0} u_n \right)$$

$$= T_1^1 + T_1^2 + T_2^1 + T_2^2.$$  

The first term is estimated as

$$|T_1^1| \leq g \left( \int b \frac{1}{\psi_0^3} |u_n|^2 \, dx \right)^{1/2} \left( \int b \frac{1}{\psi_0} |u_n|^2 \, dx \right)^{1/2}$$

$$\leq g \left( \frac{1}{c^3} \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2 \right)^{1/2} \left( \frac{1}{c} \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2 \right)^{1/2}$$

$$\leq g \left( \frac{1}{c^2} \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2 \right) \left( \frac{1}{c} \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2 \right)$$

$$\leq \frac{g}{C} \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2 \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2$$

$$\leq \frac{g}{C} \|u_n\|_{L^2}^2 + \epsilon \|u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2.$$  

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where in the second inequality, we use the fact that

\[ |b - 1|, \left| \frac{1}{\psi c y^3} - \frac{1}{c^3} \right|, \left| \frac{1}{\psi c y} - \frac{1}{c} \right| \leq C e(\xi). \]

The second term is controlled by

\[ |T_1| \leq C \left( \left\| \left( 1 - \tilde{\xi} \right) u_n \right\|_{L^2} + \| u_n \|_{L^2} \right) \| u_n \|_{L^2}
\leq C \left( \varepsilon \| u_n \|_{H^{1/2}} + \| u_n \|_{L^2} \right) \| u_n \|_{L^2} \leq C \varepsilon \| u_n \|_{H^{1/2}} + C \| u_n \|_{L^2}^2, \]

where in the second inequality we use Lemma 3.4 (iv). The third term is

\[ |T_2| \leq C \| u_n \|_{L^2} \| u_n \|_{L^2} \leq \varepsilon \| u_n \|_{L^2}^2 + C \| u_n \|_{L^2}^2. \]

By the same estimate as that of $T_1$, we have

\[ |T_2| \leq C \varepsilon \| u_n \|_{H^{1/2}} + C \| u_n \|_{L^2}^2. \]

Plugging all of the above estimates into (3.19), we have

\[ 0 \geq \left( 1 - \delta \right) \frac{1}{h^2} \| u_n \|_{L^2}^2 + \left( C_0 \delta - C \varepsilon \right) \| u_n \|_{H^{1/2}}^2 - C \| u_n \|_{L^2}^2
\geq \frac{1}{2} \delta_0 \| u_n \|_{L^2}^2 + \frac{1}{2} C_0 \delta \| u_n \|_{H^{1/2}}^2 - C \| u_n \|_{L^2}^2, \]

by choosing $\delta, \varepsilon$ such that

\[ \delta = \frac{1}{2} h \delta_0, \quad \varepsilon = \frac{1}{2} C_0 \delta. \]

Then (3.18) follows.

4 Asymptotic perturbations near zero

In this Section, we study the eigenvalues of operator $A^\lambda$ when $\lambda$ is very small. By Lemma 3.2, when $\lambda \to 0^+$, $A^\lambda \to A^0$ strongly, where

\[ A^0 = \mathcal{N} + \frac{b P_{ey}}{\psi c y^2} (\xi). \]

The related operator in the physical space is $A^0_e : H^1(\mathcal{S}_c) \to L^2(\mathcal{S}_c)$ defined by

\[ A^0_e = \mathcal{N}_e + \frac{P_{ey}}{\psi c y} (x) = B^{-1} \left( \frac{1}{b} A^0 \right) B, \]

which is the strong limit of $A^\lambda$ when $\lambda \to 0^+$. We have the following properties of $A^0$. We use $A^0(\mu)$ to denote the dependence on the solitary wave parameter $\mu$. 

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Lemma 4.1  (i) The operator $A^0 : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint and

$$
\sigma_{\text{ess}}(A^0) = \left[ \frac{1}{h} - \frac{g}{c^2}, +\infty \right].
$$

(ii) $\psi_{ex}(\xi) \in \ker A^0$ and $A^0$ has at least one negative eigenvalue that is simple.

(iii) Under the hypothesis (H1) of no secondary bifurcation, $\ker A^0(\mu) = \{\psi_{ex}(\xi)\}$ when $\mu$ is not a turning point and $\ker A^0(\mu) = \{\psi_{ex}(\xi), \frac{\partial \psi_{ex}(\mu)}{\partial \mu}\}$ when $\mu$ is a turning point. For any $\mu > \frac{\pi}{6}$, $\psi_{ex}(\xi)$ is the only odd kernel of $A^0(\mu)$.

(iv) When $\mu - \frac{\pi}{6}$ is small enough, $A^0(\mu)$ has exactly one negative eigenvalue and $\ker A^0(\mu) = \{\psi_{ex}(\xi)\}$. Under hypothesis (H1), the same is true for $A^0(\mu)$ with $\mu \in (\frac{\pi}{6}, \mu_1)$, where $\mu_1$ is the first turning point.

(v) When $\mu \to \infty$, the number of negative eigenvalues of $A^0(\mu)$ increases without bound.

Proof. (i) The essential spectrum bound follows from the observations that $\sigma_{\text{ess}}(N_{1,2}) = \left[ \frac{1}{h}, +\infty \right)$ and $bP_{ey}/\psi_{ey}^2 \to -\frac{g}{c^2}$ when $|\xi| \to \infty$.

Proof of (ii): To show $\psi_{ex}(\xi) \in \ker A^0$, it is equivalent to show that $\psi_{ex}(x) = \psi_{ex}(x, \eta_c(x)) \in \ker A^0_e$. On $S_e$, we have

$$
\psi_{ex}(x) + \eta_c \psi_{ey}(x) = 0, \quad P_{ex}(x) + \eta_c P_{ey}(x) = 0 \quad (4.1)
$$

and

$$
P_{ex}(x) = -(\psi_{ex} \psi_{exx} + \psi_{ey} \psi_{eyx}) = -\psi_{ey} (\eta_c \psi_{eyy} + \psi_{eyx})\quad (4.2)
$$

$$
= -\psi_{ey} \frac{d}{dx} (\psi_{ey}) = -\psi_{ey} \frac{d}{dx} (\phi_{ex}).
$$

So

$$
P_{ey} = \frac{\psi_{ex}}{\psi_{ey}} P_{ex}(x) = \frac{\psi_{ey}}{\psi_{ey}} \eta_c = \frac{P_{ex}(x)}{\psi_{ey}} = -\frac{d}{dx} (\phi_{ex}) = -N_e (\psi_{ex}(x)),
$$

and thus $A^0_{ey} \psi_{ex}(x) = 0$. Now we show that $A^0$ has a negative eigenvalue. We note that the Fourier multiplier operator $N_{1,2}$ has the same symbol as in the Intermediate Long Wave equation (IIW), for which it was shown in [2] that for $K > 0$ large, the operator $(N_{1,2} + K)^{-1}$ is positivity preserving. Thus, by the spectrum theory for positivity preserving operators ([2]), the lowest eigenvalue of $A^0$ is simple with the corresponding eigenfunction of one sign. Since $\psi_{ex}(x)$ is odd, $\psi_{ex}(\xi)$ has a zero at $\xi = 0$. So 0 is not the lowest eigenvalue of $A^0$ and $A^0$ has at least one simple negative eigenvalue.

To prove (iii)-(iv), first we show that the operator $A^0(\mu)$ is exactly the operator $A(\lambda)$ introduced by Plotnikov ([58, p. 349]) in the study of the bifurcation
of solitary waves. In \[58\], \( h \) is set to 1 and the parameter \( \lambda = \frac{1}{F(\mu)} \) is the inverse square of the Froude number, then the operator \( A(\lambda) \) is defined by

\[
A(\lambda) = N - a, \quad a(\xi) = \lambda \exp(3\tau) \cos \theta + \theta'(\xi)
\]

In the above, \( \exp(\tau + i\theta) = W \) where \( W \) is defined by \( (2.18) \), as can be seen from \( \[58\] (4.2), p. 348\) with \( u = w \). To show that \( A^0(\mu) = A(\lambda(\mu)) \), it suffices to prove that

\[
A(\lambda) \psi_{\text{ex}}(\xi) = 0. \tag{4.3}
\]

Since this implies that

\[
0 = (A(\lambda) - A^0(\mu)) \psi_{\text{ex}}(\xi) = (-a - bP_{eg}/\psi_{eg}^2) \psi_{\text{ex}}(\xi)
\]

and thus \( bP_{eg}/\psi_{eg}^2 = -a \). We prove \( (4.3) \) below. In \[58\], solitary waves are shown to be critical points of the functional

\[
J(\lambda, w) = \frac{1}{2} \int_{\mathbb{R}} \{ wN'w - \lambda w^2 (1 + N'w) \} d\xi. \tag{4.4}
\]

Let the self-adjoint operator \( A_0(\lambda) \) to be the second derivative of \( J(\lambda, w) \) at a solitary wave solution. In \[58\] p. 349], the operator \( A(\lambda) \) is defined via

\[
A(\lambda) = M^* A_0(\lambda) M.
\]

Here, the operator \( M : L^2 \to L^2 \) is defined by

\[
Mf = f (1 + Cw') + w'\mathcal{C}f = \text{Re} \{ W\mathcal{R}f \}, \tag{4.5}
\]

where \( \mathcal{C} \) is defined in Section 2 such that \( \mathcal{R}f = f - i\mathcal{C}f \) is the boundary value on \( \{ \varsigma = 0 \} \) of an analytic function on \( D_0 \). Our definition \( (4.3) \) above adapts the notations in \[14\] p. 228, which studies the bifurcation of Stokes waves by using a similar variational setting as \[58\]. Taking \( d/d\xi \) of the equation \( \nabla_w J(\lambda, w) = 0 \) for a solitary wave solution \( w \), we have \( A_0(\lambda) w' = 0 \). Since

\[
M^{-1}w' = \text{Re} \left\{ \frac{\mathcal{R}w'}{W} \right\} = \text{Re} \left\{ \frac{w' - i\mathcal{C}w'}{1 + \mathcal{C}w' + iw'} \right\} = \text{Re} \left\{ \frac{(w' - i\mathcal{C}w')(1 + \mathcal{C}w' - iw')}{|W|^2} \right\} = \frac{w'}{|W|^2} = \frac{1}{c} v_e = -\frac{1}{c} \psi_{\text{ex}}(\xi),
\]

we have \( M\psi_{\text{ex}}(\xi) = -cw' \) and thus

\[
A(\lambda) \psi_{\text{ex}}(\xi) = -cM^* A_0(\lambda) w' = 0.
\]

This finishes the proof that \( A^0(\mu) = A(\lambda) \).
Proof of (iii): By applying the analytic bifurcation theory in \cite{14}, \cite{15} to the variational setting (4.4) for the solitary waves, one can relate the secondary bifurcation of solitary waves with the null space of $A^0$ (equivalently $\nabla^2_{ww}J$). Under the hypothesis (H1), there is no secondary bifurcation and therefore the kernel of $A^0$ is either due to the trivial translation symmetry ($\psi_{ex}$) or due to the loss of monotonicity of $\lambda(\mu)$ at a turning point which generates an additional kernel $\partial_\mu \psi_e$. In the Appendix, we prove that at a turning point $\mu_0$, $A^0 \partial_\mu \psi_e = 0$. By \cite{23} there is no asymmetric bifurcation for solitary waves with $F > 1$, so $\psi_{ex}(\xi)$ is the only odd kernel of $A^0(\mu)$.

Proof of (iv): Let $\lambda(\rho) = \exp(-3\rho^2)$, then $\mu \approx \frac{\pi}{6}$ is equivalent to $\lambda \approx 1$ and thus $\rho$ is a small parameter. Consider an eigenvalue $\nu$ of $A^0(\mu(\lambda))$, let $\nu = \rho^2(3 - \alpha(\rho))$. By using the KDV scaling, it was shown in \cite{58, p. 353} that when $\rho \to 0$, the limit $\alpha(0)$ is an eigenvalue of the operator $B = \frac{1}{3} \frac{d^2}{dx^2} + 9 \text{sech}^2 \left( \frac{3}{2} x \right)$ which has three eigenvalues $\frac{3}{4}, 3$ and $\frac{27}{4}$. Therefore, when $\rho$ is small, $A^0$ has three eigenvalues $-\frac{9}{4} \rho^2 + o(\rho^2)$, $\alpha(\rho^2)$ and $\frac{9}{4} \rho^2 + o(\rho^2)$. Since $0$ is an eigenvalue of $A^0$, the middle one must be zero and the rest two eigenvalues are one positive and one negative. Under the hypothesis (H1), when $\mu < \mu_1$, that is, before the first turning point, we have $\ker A^0(\mu) = \{ \psi_{ex}(\xi) \}$. Then for all $\mu \in (\frac{\pi}{6}, \mu_1)$, the operator $A^0(\mu)$ always has only one negative eigenvalue. Suppose otherwise, then when $\mu$ increases from $\frac{\pi}{6}$ to $\mu_1$, the eigenvalues of $A^0(\mu)$ must go across zero at some $\mu = \mu^* \in (\frac{\pi}{6}, \mu_1)$. This implies that $\dim \ker A^0(\mu^*) \geq 2$, a contradiction to (H1).

Property (v) is Theorem 4.3 in \cite{58}. $\blacksquare$

We note that by Lemma 4.1 (iv), there is no secondary bifurcation for small solitary waves. Although this fact was not stated explicitly in \cite{58}, it comes as a corollary of results there.

Next, we study the eigenvalues of $A^\lambda$ for small $\lambda$. Since the convergence of $A^\lambda \to A^0$ is rather weak, we cannot use the regular perturbation theory. We use the asymptotic perturbation theory developed by Vock and Hunziker (\cite{67}), see also \cite{29}, \cite{30}. First, we establish some preliminary lemmas.

**Lemma 4.2** Given $F \in C_0^\infty(\mathbb{R})$. Consider any sequence $\lambda_n \to 0+$ and $\{u_n\} \in H^1(\mathbb{R})$ satisfying

$$\|A^{\lambda_n} u_n\|_2 + \|u_n\|_2 \leq M_1 < \infty$$

for some constant $M_1$. Then if $w - \lim_{n \to \infty} u_n = 0$, we have

$$\lim_{n \to \infty} \|Fu_n\|_2 = 0$$

and

$$\lim_{n \to \infty} \|[A^{\lambda_n}, F] u_n\|_2 = 0.$$ 

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Proof. Since (4.6) implies that \( \|u_n\|_{H^1} \leq C \), (4.7) follows from the local compactness of \( H^1 \to L^2 \). For the proof of (4.8), we use the same notations as in the proof of Lemma 3.6. We write \( A^\lambda_n = N + K^\lambda_n \). Then

\[
[N,F] = [N_2N_1,F] = N_2[N_1,F] + [N_2,F]N_1,
\]

where \( N_1 = 1 + \frac{d}{dk} \) and \( N_2 \) has the symbol \( n_2(k) \) defined by (3.15). We have \( \|[N_1,F]u_n\|_2 \to 0 \), again by the local compactness. Since \( \frac{d}{dk}n_2(k) \to 0 \) when \( |k| \to \infty \), by [18] Theorem C] the commutator \([N_2,F]: L^2 \to L^2 \) is compact. This can also be seen from the proof of Lemma 3.6 since \([N_2,F] = \int_0^1 A_\rho \, d\rho \) where \( A_\rho \) is an integral operator with a \( L^2(\mathbb{R} \times \mathbb{R}) \) kernel. Since \( \|N_1u_n\|_2 \leq \|u_n\|_{H^1} \leq C \) and \( u_n \to 0 \) weakly in \( L^2 \), we have \( v_n = N_1u_n \to 0 \) weakly in \( L^2 \). Therefore, \( \|[N_2,F]N_1u_n\|_2 = \|[N_2,F]v_n\|_2 \to 0 \).

Thus, \( \|[N,F]u_n\|_2 \to 0 \). Since

\[
[K^\lambda_n,F]u_n = \left[ b \left( 1 - \tilde{\lambda}^{\lambda_n,+} \right) \frac{P_{cy}}{\psi_{cy}} \left( 1 - \tilde{\lambda}^{\lambda_n,+} \right) \frac{1}{\psi_{cy}}, F \right]
\]

\[
= x \left[ F, \tilde{\lambda}^{\lambda_n,+} \right] \frac{P_{cy}}{\psi_{cy}} \left( 1 - \tilde{\lambda}^{\lambda_n,+} \right) \frac{1}{\psi_{cy}}u_n + x \left( 1 - \tilde{\lambda}^{\lambda_n,+} \right) \frac{P_{cy}}{\psi_{cy}} \left[ F, \tilde{\lambda}^{\lambda_n,+} \right] u_n
\]

\[
= p_n + q_n.
\]

For any \( \varepsilon > 0 \), by Lemma 3.3 (iii), when \( n \) is large we have

\[
\left\| F \tilde{\lambda}^{\lambda_n,+} u_n \right\|_{L^2} \leq \varepsilon \|u_n\|_{H^1} \leq \varepsilon \|u_n\|_{H^1} \leq \varepsilon M_1.
\]

So

\[
\|q_n\|_2 \leq C \left( \|F \tilde{\lambda}^{\lambda_n,+} u_n\|_2 + \|F u_n\|_2 \right) \leq \varepsilon C M_1 + C \|F u_n\|_2 \leq 2 \varepsilon C M_1,
\]

when \( n \) is large. By the same proof as that of (3.3), for any \( \lambda > 0 \), we have the estimate

\[
\left\| \tilde{\lambda}^{\lambda,+} \right\|_{H^1 \to H^1} \leq C \quad \text{(independent of } \lambda)\n\]

Denote

\[
r_n = \frac{P_{cy}}{\psi_{cy}} \left( 1 - \tilde{\lambda}^{\lambda_n,+} \right) \frac{1}{\psi_{cy}} u_n,
\]

then \( \|r_n\|_{H^1} \leq C \|u_n\|_{H^1} \leq C M_1 \). Since \( w - \lim_{n \to \infty} u_n = 0 \), we have \( w - \lim_{n \to \infty} r_n = 0 \) as in the proof of Lemma 3.4 (iii). Then similar to the estimate of \( \|g_n\|_2 \), we have

\[
\|p_n\|_2 \leq C \left( \|F \tilde{\lambda}^{\lambda_n,+} r_n\|_2 + \|F r_n\|_2 \right) \leq 2 \varepsilon C M_1,
\]

when \( n \) is large. Therefore, \( \left\| [K^\lambda_n,F] u_n \right\|_2 \leq 4 \varepsilon C M_1 \) when \( n \) is large enough. Since \( \varepsilon \) is arbitrary, we have \( \left\| [K^\lambda_n,F] u_n \right\|_2 \to 0 \), when \( n \to \infty \). This finishes the proof of (4.8).
Lemma 4.3  Let \( z \in \mathbb{C} \) with \( \text{Re} \, z \leq \frac{1}{2} \delta_0 \), then for some \( n > 0 \) and all \( u \in C^\infty(|x| \geq n) \), we have

\[
\| (A^\lambda - z) u \|_2 \geq \frac{1}{4} \delta_0 \| u \|_2 ,
\]

when \( \lambda \) is sufficiently small. Here \( \delta_0 > 0 \) is defined by (3.4).

Proof. The estimate (4.9) follows from

\[
\text{Re} \, ((A^\lambda - z) u, u) \geq \frac{1}{4} \delta_0 \| u \|_2^2 .
\]

The proof of (4.10) is almost the same as that of Lemma 3.5, except that Lemma 3.4 (iii) is used in the estimates. So we skip it.

With above two lemmas, we can use the asymptotic perturbation theory ([29], [30]) to get the following result on eigenvalue perturbations of \( A^\lambda \).

Proposition 2  Each discrete eigenvalue \( k_0 \) of \( A^0 \) with \( k_0 \leq \frac{1}{2} \delta_0 \) is stable with respect to the family \( A^\lambda \) in the following sense: there exists \( \lambda_1, \delta > 0 \), such that for \( 0 < \lambda < \lambda_1 \), we have

(i) \( B(k_0; \delta) = \{ z \mid 0 < |z - k_0| < \delta \} \subset P(A^\lambda) \),

where

\[
P(A^\lambda) = \left\{ z \mid R^\lambda(z) = (A^\lambda - z)^{-1} \text{ exists and is uniformly bounded for } \lambda \in (0, \lambda_1) \right\} .
\]

(ii) Denote

\[
P_\lambda = \oint_{|z-k_0|=\delta} R^\lambda(z) \, dz \quad \text{and} \quad P_0 = \oint_{|z-k_0|=\delta} R^0(z) \, dz
\]
to be the perturbed and unperturbed spectral projection. Then \( \dim P_\lambda = \dim P_0 \) and \( \lim_{\lambda \to 0} \| P_\lambda - P_0 \| = 0 \).

It follows from above that for \( \lambda \) small, the operators \( A^\lambda \) have discrete eigenvalues inside \( B(k_0; \delta) \) with the total algebraic multiplicity equal to that of \( k_0 \).

5  Moving kernel and proof of main results

To study growing modes, we need to understand how the zero eigenvalue of \( A^0 \) is perturbed, in particular its moving direction. In this Section, we derive a moving kernel formula and use it to prove the main results. We assume hypothesis (H1) and that \( \mu \) is not a turning point. Then by Lemma 4.1 (iii), \( \ker A^0(\mu) = \{ \psi_{\mu \xi}(\xi) \} \). Let \( \lambda_1, \delta > 0 \) be as given in Proposition 2 for \( k_0 = 0 \). Since \( \dim P_\lambda = \dim P_0 = 1 \), when \( \lambda < \lambda_1 \) there is only one real eigenvalue of \( A^\lambda \) inside \( B(0; \delta) \), which we denote by \( k_\lambda \in \mathbb{R} \). The following lemma determines the sign of \( k_\lambda \) when \( \lambda \) is sufficiently small.
Lemma 5.1 Assume hypothesis (H1) and that $\mu$ is not a turning point. For $\lambda > 0$ small enough, let $k_\lambda \in \mathbb{R}$ to be the eigenvalue of $A^\lambda$ near zero. Then

$$\lim_{\lambda \to 0^+} \frac{k_\lambda}{\lambda^2} = -\frac{1}{c} \frac{dE}{dc} / \|\psi_{\text{ex}}\|_{L^2}^2,$$

(5.1)

where $E(\mu)$ is the total energy defined in (L5).

The following a priori estimate is used in the proof.

Lemma 5.2 For $\lambda > 0$ small enough, consider $u \in H^1(\mathbb{R})$ satisfying the equation $(A^\lambda - z)u = v$, where $z \in \mathbb{C}$ with $\text{Re} z \leq \frac{1}{2} \delta_0$ and $v \in L^2$. Then we have

$$\|u\|_{H^1} \leq C \left( \|u\|_{L^2} + \|v\|_{L^2} \right),$$

(5.2)

for some constant $C$ independent of $\lambda$. Here, the norm $\|\cdot\|_{L^2}$ is defined in (7.17) with the weight $e(\xi)$ defined by (5.16).

Proof. The proof is almost the same as that of the estimate (3.18) in the proof of Lemma 3.7. So we only sketch it. We have

$$(\mathcal{N}u, u) + \text{Re} \left( b(\lambda) p \psi_{\text{ex}}(\xi) \tilde{C}^\lambda u, u \right) - \text{Re} z \|u\|^2 = \text{Re} (u, v).$$

By the same estimates as in proving (3.18), except that Lemma 3.4(iii) is used, we have

$$\left[ (1 - \delta) \frac{1}{h} - \frac{g}{c^2} - \frac{1}{2} \delta_0 \right] \|u\|_{L^2}^2 + (C_0 \delta - C \varepsilon) \|u\|_{H^1}^2 - C \varepsilon \|u\|_{L^2}^2$$

$$\leq \varepsilon \|u\|_{L^2}^2 + \frac{1}{\varepsilon} \|v\|_{L^2}^2.$$ 

Then the estimate (5.2) follows by choosing $\varepsilon > 0$ and $\delta \in (0, 1)$ properly. 

Assuming Lemma 5.1 we prove Theorem 1.

Proof of Theorem 1. We fixed $\mu \in (\mu_1, \mu_1)$. Under the assumption (H1), it follows from Lemma 4.1 that $A^0(\mu)$ has only one negative eigenvalue $k_0^-$, and $\text{ker} A^0(\mu) = \{\psi_{\text{ex}}(\xi)\}$. By Proposition 2 and Lemma 5.1, there exists $\lambda_1, \delta > 0$ small enough, such that for $0 < \lambda < \lambda_1$, $A^\lambda$ has one negative eigenvalue $k^\lambda_ \lambda$ in $B(k_0^-; \delta)$ with multiplicity 1 and one positive eigenvalue $k^\lambda_ \lambda$ in $B(0; \delta)$ because $\frac{dE}{d\nu} = E'(\mu) / c'(\nu) < 0$ for $\mu \in (\mu_1, \mu_1)$. Consider the region

$$\Omega = \{ z \mid 0 > \text{Re} z > -2M \text{ and } |\text{Im} z| < 2M \},$$

where $M$ is the uniform bound of $\|A^\lambda - \mathcal{N}\|$. We claim that: for $\lambda$ small enough, $A^\lambda$ has exactly 2 eigenvalues (counting multiplicity) in

$$\Omega_\delta = \{ z \mid 2\delta > \text{Re} z > -2M \text{ and } |\text{Im} z| < 2M \}.$$

That is, all eigenvalues of $A^\lambda$ with real parts no greater than $2\delta$ lie in $B(k_0^-; \delta) \cup B(0; \delta)$. Suppose otherwise, there exists a sequence $\lambda_n \to 0^+$ and

$$\{u_n\} \in H^1(\mathbb{R}), z_n \in \Omega \setminus (B(k_0^-; \delta) \cup B(0; \delta))$$
such that \((A^n - z_n) u_n = 0\). We normalize \(u_n\) by setting \(\|u_n\|_{L^2} = 1\). Then by Lemma 3.2, we have \(\|u_n\|_{H^\frac{n}{2}} \leq C\). By the same argument as in the proof of Lemma 3.7, \(u \to u_\infty \neq 0\) weakly in \(H^\frac{n}{2}\). Let

\[
 z_n \to z_\infty \in \overline{\Omega}/ \left( B(0^{-}; \delta) \cup B(0; \delta) \right),
\]

then \(A^n u_\infty = z_\infty u_\infty\) which is a contradiction. This proves the claim. Thus for \(\lambda\) small enough, \(A^\lambda\) has exactly one eigenvalue in \(\Omega\).

Suppose the conclusion of Theorem 1 does not hold, then \(A^\lambda (\mu)\) has no kernel for any \(\lambda > 0\). Define \(n_\Omega(\lambda)\) to be the number of eigenvalues (counting multiplicity) of \(A^\lambda\) in \(\Omega\). By (3.3), the region \(\Omega\) is away from the essential spectrum of \(A^\lambda\), so \(n_\Omega(\lambda)\) is a finite integer. For \(\lambda\) small enough, we have proved that \(n_\Omega(\lambda) = 1\). By Lemma 3.7, \(n_\Omega(\lambda) = 0\) for \(\lambda > \Lambda\). Define the two sets

\[
 S_{\text{odd}} = \{ \lambda > 0 \mid n_\Omega(\lambda) \text{ is odd} \} \quad \text{and} \quad S_{\text{even}} = \{ \lambda > 0 \mid n_\Omega(\lambda) \text{ is even} \}.
\]

Then both sets are non-empty. Below, we show that both \(S_{\text{odd}}\) and \(S_{\text{even}}\) are open. Let \(\lambda_0 \in S_{\text{odd}}\) and denote \(k_1, \ldots, k_l \ (l \leq n_\Omega(\lambda_0))\) to be all distinct eigenvalues of \(A^{\lambda_0}\) in \(\Omega\). Denote \(ih_1, \ldots, ih_m\) to be all eigenvalues of \(A^{\lambda_0}\) on the imaginary axis. Then \(|h_j| \leq M, 1 \leq j \leq m\). Choose \(\delta > 0\) sufficiently small such that the disks \(B(k_i; \delta) \ (1 \leq i \leq l)\) and \(B(ih_j; \delta) \ (1 \leq j \leq m)\) are disjoint, \(B(k_i; \delta) \subset \Omega\) and \(B(ih_j; \delta)\) does not contain \(0\). Note that \(A^\lambda\) depends on \(\lambda\) analytically in \((0, +\infty)\). By the analytic perturbation theory (29), if \(|\lambda - \lambda_0|\) is sufficiently small, any eigenvalue of \(A^\lambda\) in \(\Omega\) lies in one of the disks \(B(k_i; \delta)\) or \(B(ih_j; \delta)\). So \(n_\Omega(\lambda)\) is \(n_\Omega(\lambda_0)\) plus the number of eigenvalues in \(\bigcup_{i=1}^{m} B(ih_j; \delta)\) with the negative real part. The later number must be even, since the complex eigenvalues of \(A^\lambda\) appears in conjugate pairs. Thus, \(n_\Omega(\lambda)\) is odd for \(|\lambda - \lambda_0|\) small enough. This shows that \(S_{\text{odd}}\) is open. For the same reason, \(S_{\text{even}}\) is open. Thus, \((0, +\infty)\) is the union of two non-empty, disjoint open sets \(S_{\text{odd}}\) and \(S_{\text{even}}\). A contradiction.

So there exists \(\lambda > 0\) and \(0 \neq u \in H^1(\mathbb{R})\) such that \(A^\lambda u = 0\). Let \(f = B^{-1} u \in H^1(S_c)\), then \(A^\lambda f = 0\). Define \(\eta(x) = e^{\lambda x} f, P(x) = -P_{xy} \eta(x)\) and \(\psi(x, y)\) to be the solution of the Dirichlet problem

\[
 \Delta \psi = 0 \quad \text{in} \quad D_c, \quad \psi|_{S_c} = f, \quad \psi(x, -h) = 0.
\]

Then \((\eta(x), \psi(x, y))\) satisfies the system (2.3)-(2.7), thus \(e^{\lambda t} [\eta(x), \psi(x, y)]\) is a growing mode solution to the linearized problem (2.2). Below we prove the regularity of \([\eta(x), \psi(x, y)]\). Since the operator \(C^\lambda\) is regularity preserving, from \(A^\lambda f = 0\) we have

\[
 \psi_n(x) = -C^\lambda P_{xy} C^\lambda f \in H^1(S_c).
\]

By the elliptic regularity of Neumann problems (11), we have \(\psi(x, y) \in H^{5/2}(D_c)\). So by the trace theorem, \(f = \psi|_{S_c} \in H^2(S_c)\). Repeating this process, we get \(\psi_n(x) \in H^2(S_c)\) and \(\psi(x, y) \in H^{7/2}(D_c)\). Since the irrotational solitary wave profile and the boundary \(S_c\) are analytic (36), we can repeat the above process to show \(\psi(x, y) \in H^k(D_c)\) for any \(k > 0\). Therefore \(\eta(x) = C^\lambda (\psi|_{S_c}) \in\)
\( H^k (\mathcal{S}_x), \) for any \( k > 0. \) By Sobolev embedding, \([\eta(x), \psi(x, y)] \in C^\infty.\) This finishes the proof of Theorem 1. 

**Proof of Theorem 2.** Let \( \mu_1 < \mu_2, \cdots < \mu_n < \cdots \) be all the turning points. Then \( \mu_n \rightarrow +\infty. \) Under the assumption (H1), \( \ker A^0 (\mu) = \{ \psi_{xx} (\xi) \}, \) for \( \mu \in (\mu_i, \mu_{i+1}), i \geq 1. \) Denote by \( n^- (\mu) \) the number of negative eigenvalues of \( A^0 (\mu). \) Then \( n^- (\mu) \) is a constant in \( (\mu_i, \mu_{i+1}), \) by the same argument in the proof of Lemma 4.1 (iv). Denote \( \tilde{\mu}_1 < \tilde{\mu}_2, \cdots < \tilde{\mu}_n < \cdots \) to be all the critical points of \( E(\mu). \) Each \( \tilde{\mu}_k \) lies in some interval \( (\mu_i, \mu_{i+1}). \) Then the sign of \( \frac{dE}{dc} = \frac{E'(\mu)}{c'(\mu)} \) changes at \( \tilde{\mu}_k \) in \( (\mu_i, \mu_{i+1}). \) So we can find an interval \( I_k \subset (\mu_i, \mu_{i+1}) \) such that the number

\[ \tilde{n}^- (\mu) = n^- (\mu) + (1 + \text{sign} (E'(\mu)/c'(\mu))) / 2 \]

is odd for \( \mu \in I_k. \) Note that \( \tilde{n}^- (\mu) \) is the number of eigenvalues of \( A^0 (\mu) \) in the left half plane, for \( \lambda \) sufficiently small. So by the same proof as that of Theorem 1 we get a purely growing mode for solitary waves with \( \mu \in I_k. \) Since \( \tilde{\mu}_n \rightarrow \infty, \) the intervals \( I_k \) goes to infinity.

We make two remarks about the unstable solitary waves proved above.

**Remark 1** In terms of the parameter \( \omega = 1 - F^2 q^2, \) it was found (\cite{64}) from numerical computations that the energy maximum is achieved at \( \omega \approx 0.88, \) which corresponds to the amplitude-to-depth ratio \( \alpha = \eta_e (0) / h = 0.7824, \) (\cite{52}). The highest wave has the parameters \( \omega = 1, \alpha = 0.8332 \) and the maximal travel speed is achieved at \( \omega = 0.917, \alpha = 0.790, \) (\cite{64}, [28]). So the unstable waves proved in Theorem 1 and 2 are of large amplitude, and their height is comparable to the water depth. Therefore, this type of instability cannot be captured in the approximate models based on small amplitude assumptions. Indeed, although some approximate models share certain features of the full water wave model, no unstable solitary waves have been found. For example, the Green-Naghdi model is proposed to model water waves of larger amplitude and it also has an indefinite energy functional, but the numerical computation (\cite{57}, P. 529) indicates that all the G-N solitary waves are spectrally stable. For Camassa-Holm and Degasperis-Procesi equations, there exist cornered solitary waves (peakons). However, these peakons are shown to be nonlinearly stable (\cite{22}, \cite{46}). So the instability of large solitary waves seems to be a particular feature of the full water wave model.

**Remark 2** The linear instability suggests that the solitary wave cannot preserve its shape for all the time. The long time evolution around an unstable wave was studied numerically in (\cite{56}). It was found that small perturbations with the same amplitude but opposite signs can lead to totally different long time behaviors. For one sign, the perturbed wave breaks quickly and for the other sign, the perturbed wave never breaks and it finally approaches a slightly lower stable solitary wave with almost the same energy. Note that in the breaking case, the initial perturbed profile has a rather negative slope (\cite{56}). The wave breaking for shallow water waves models such as Camassa-Holm (\cite{21}) and Whitham equations (\cite{60}, [54])
is due to the initial large negative slope. It would be interesting to clarify whether or not the wave breaking found in \[65\] has the same mechanism.

The wave breaking due to the instability of large solitary waves had been used to explain the breaking waves approaching beaches \([25\), \([56\), \([57\)]\. When a wave approaches the beach, the amplitude-to-depth ratio can increase to be near the critical ratio \((\approx 0.7824)\) for instability and consequently the wave breaking can occur.

It remains to prove the moving kernel formula \((5.1)\).

**Proof of Lemma 5.1** As described at the beginning of this Section, for \(\lambda > 0\) small enough, there exists \(u_\lambda \in H^1(\mathbb{R})\), such that \((A^\lambda - k_\lambda) u_\lambda = 0\) with \(k_\lambda \in \mathbb{R}\) and \(\lim_{\lambda \to 0^+} k_\lambda = 0\). We normalize \(u_\lambda\) by \(\|u_\lambda\|_{L^2} = 1\). Then by Lemma \((5.2)\) we have \(\|u_\lambda\|_{H^\frac{1}{2}} \leq C\) and as in the proof of Lemma \((3.7)\) \(u_\lambda \to u_0 \neq 0\) weakly in \(H^\frac{1}{2}\). Since \(A^0 u_0 = 0\) and \(\ker A^0 (\mu) = \{\psi_{ex}(\xi)\}\), we have \(u_0 = c_0 \psi_{ex}(\xi)\) for some \(c_0 \neq 0\). Moreover, we have \(\|u_\lambda - u_0\|_{H^\frac{1}{2}} = 0\). To show this, first we note that \(\|u_\lambda - u_0\|_{L^2} \to 0\), since

\[
\|u_\lambda - u_0\|_{L^2}^2 \leq \int_{|\xi| \leq R} e(\xi)|u_\lambda - u_0|^2 \, d\xi + \max_{|\xi| \geq R} e(\xi) \|u_\lambda - u_0\|_{L^2}^2,
\]

and the second term is arbitrarily small for large \(R\) while the first term tends to zero by the local compactness. Since

\[
(A^\lambda - k_\lambda)(u_\lambda - u_0) = k_\lambda u_\lambda + (A^0 - A^\lambda) u_0,
\]

by Lemma \((5.2)\) we have

\[
\|u_\lambda - u_0\|_{H^\frac{1}{2}} \leq C \left(\|u_\lambda - u_0\|_{L^2}^2 + |k_\lambda| \|u_\lambda\|_{L^2}^2 + \|(A^0 - A^\lambda) u_0\|_{L^2}^2\right) \to 0,
\]

when \(\lambda \to 0^+\). We can set \(c_0 = 1\) by renormalizing the sequence.

Next, we show that \(\lim_{\lambda \to 0^+} \frac{k_\lambda}{\lambda} = 0\). From \((A^\lambda - k_\lambda) u_\lambda = 0\), we have

\[
\frac{A^0 u_\lambda}{\lambda} + \frac{A^\lambda - A^0}{\lambda} u_\lambda = \frac{k_\lambda}{\lambda} u_\lambda.
\]

(5.3)

Taking the inner product of above with \(\psi_{ex}(\xi)\), we have

\[
\frac{k_\lambda}{\lambda} (u_\lambda, \psi_{ex}(\xi)) = \left(\frac{A^\lambda - A^0}{\lambda} u_\lambda, \psi_{ex}(\xi)\right) := m(\lambda).
\]

We compute the integral \(m(\lambda)\) in the physical space, by the change of variable \(\xi \to x\). Denote by \((\cdot, \cdot)\) the inner product in \(L^2(\mathcal{S}_e)\). Noting that \(dx = b(\xi) \, d\xi\),

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we have

\[
m(\lambda) = \left\langle \frac{A_e^\lambda - A_0^0}{\lambda} u_\lambda(x), \psi_{ex}(x) \right\rangle
\]

\[
= - \left\langle \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_{ey}} u_\lambda, \psi_{ex} \right\rangle - \left\langle \frac{1}{\psi_{ey}} \frac{1}{\lambda + D} P_{ey} u_\lambda, \psi_{ex} \right\rangle
\]

\[
+ \left\langle \frac{1}{\lambda + D} \frac{P_{ey} \mathcal{E}_\lambda + 1}{\psi_{ey}} u_\lambda, \psi_{ex} \right\rangle
\]

\[
= m_1 + m_1 + m_3,
\]

where we use

\[
\frac{A_e^\lambda - A_0^0}{\lambda} = \frac{1}{\lambda} \left( \mathcal{L}_{P_{ey}}(x) \mathcal{L} - \frac{P_{ey}}{\psi_{ey}} \right)
\]

\[
= \frac{1}{\lambda} \left( \left( 1 - \frac{\lambda}{\lambda + D} \right) \frac{1}{\psi_{ey}} P_{ey}(x) \left( 1 - \frac{\lambda}{\lambda + D} \right) \frac{1}{\psi_{ey}} - \frac{P_{ey}}{\psi_{ey}^2} \right)
\]

\[
= - \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_{ey}^2} - \frac{P_{ey}}{\psi_{ey}} \frac{1}{\lambda + D} \frac{1}{\psi_{ey}} + \frac{1}{\lambda + D} \frac{P_{ey} \mathcal{E}_\lambda + 1}{\psi_{ey}}.
\]

We compute each term separately. For the first term, when \( \lambda \to 0^+ \),

\[
m_1(\lambda) = \left\langle \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_{ey}} u_\lambda, \psi_{ey} \eta_{ex} \right\rangle = \left\langle \frac{P_{ey}}{\psi_{ey}} u_\lambda, \frac{1}{\lambda - D} \eta_{ex} \right\rangle
\]

\[
= \left\langle \frac{P_{ey}}{\psi_{ey}} u_\lambda, \left( \mathcal{E}_\lambda - 1 \right) \frac{1}{\psi_{ey}} \eta_e \right\rangle \to \left\langle \frac{P_{ey}}{\psi_{ey}} \psi_{ex}, \frac{1}{\psi_{ey}} \eta_e \right\rangle = 0,
\]

where we use Lemma 2.1 (b) in the above and the resultant integral is zero because \( P_{ey}, \psi_{ey}, \eta_e \) are even and \( \psi_{ex} \) is odd in \( x \). The second term is

\[
m_2(\lambda) = \left\langle \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_{ey}} u_\lambda, \psi_{ey} \psi_{ex} \right\rangle = \left\langle \frac{1}{\lambda + D} \frac{1}{\psi_{ey}} u_\lambda, \psi_{ey} \right\rangle \left\langle \frac{1}{\lambda - D} \psi_{ey} d \psi_{ex} \right\rangle
\]

\[
= \left\langle u_\lambda, \frac{1}{\lambda - D} \psi_{ey} \psi_{ex} \right\rangle = \left\langle u_\lambda, \left( \mathcal{E}_\lambda - 1 \right) \frac{1}{\psi_{ey}} \psi_{ex} \right\rangle
\]

\[
\to - \left\langle \psi_{ex}, \frac{1}{\psi_{ey}} u_e \right\rangle = 0,
\]

where the relations (4.1) and (4.2) are used in the above computation. For the last term,

\[
m_3(\lambda) = - \left\langle \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_{ey}} \mathcal{E}_\lambda + \frac{1}{\psi_{ey}} u_\lambda, \psi_{ey} \eta_{ex} \right\rangle = - \left\langle \frac{P_{ey} \mathcal{E}_\lambda + 1}{\psi_{ey}} u_\lambda, \frac{1}{\lambda - D} \eta_{ex} \right\rangle
\]

\[
= \left\langle P_{ey} \mathcal{E}_\lambda + \frac{1}{\psi_{ey}} u_\lambda, \left( \mathcal{E}_\lambda - 1 \right) \frac{1}{\psi_{ey}} \eta_e \right\rangle \to 0,
\]

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because $E^{\lambda, +} \frac{1}{\psi_{ey}} u_{\lambda} \to 0$ weakly in $L^2$, when $\lambda \to 0$. So $m(\lambda) \to 0$, and thus

$$\lim_{\lambda \to 0^+} \frac{k_{\lambda}}{\lambda} = \lim_{\lambda \to 0^+} \frac{m(\lambda)}{(u_{\lambda}, \psi_{ex}(\xi))} = 0.$$  

Now we write $u_{\lambda} = c_{\lambda} \psi_{ex} + \lambda v_{\lambda}$, with $c_{\lambda} = (u_{\lambda}, \psi_{ex}) / (\psi_{ex}, \psi_{ex})$. Then $(v_{\lambda}, \psi_{ex}) = 0$ and $c_{\lambda} \to 1$ as $\lambda \to 0^+$. We claim that: $|v_{\lambda}|_{L^2} \leq C$ (independent of $\lambda$). Suppose otherwise, there exists a sequence $\lambda_n \to 0^+$ such that $|v_{\lambda_n}|_{L^2} \geq n$. Denote $\tilde{v}_{\lambda_n} = v_{\lambda_n}/|v_{\lambda_n}|_{L^2}$. Then $|\tilde{v}_{\lambda_n}|_{L^2} = 1$ and $\tilde{v}_{\lambda_n}$ satisfies

$$A^{\lambda_n} \tilde{v}_{\lambda_n} = \frac{1}{|\tilde{v}_{\lambda_n}|_{L^2}^2} \left( \frac{k_{\lambda_n}}{\lambda_n} u_{\lambda_n} - c_{\lambda_n} A^{\lambda_n} - A^0 \psi_{ex}(\xi) \right).$$  

Denote

$$g_n(\xi) = \frac{A^{\lambda_n} - A^0}{\lambda_n} \psi_{ex}(\xi) = b(\xi) w_{\lambda_n}(x(\xi)),$$

where

$$w_{\lambda}(x) = \frac{A^{\lambda} - A^0}{\lambda} \psi_{ex}(x) = - \left( \frac{1}{\lambda + D} \psi_{ey} \frac{P_{ey}}{\psi_{ey}} \frac{1}{\lambda + D} \psi_{ey} - \frac{1}{\lambda + D} \psi_{ey} \right) \psi_{ey} \eta_{ex}$$

$$= \frac{1}{\lambda + D} \frac{d}{dx}(u_{c}) + \frac{P_{ey}}{\psi_{ey}} \frac{1}{\lambda + D} \eta_{ex} - E^{\lambda,+} \frac{1}{\psi_{ey}} \psi_{ey} \eta_{ex}$$

$$= (1 - E^{\lambda,+}) \frac{u_{c}}{\psi_{ey}} + \frac{P_{ey}}{\psi_{ey}} (1 - E^{\lambda,+}) \frac{\eta_{c}}{\psi_{ey}} - E^{\lambda,+} \frac{1}{\psi_{ey}} \psi_{ey} \eta_{ex}.$$  

By Lemma 2.1, $w_{\lambda} \in L^2(S_\alpha)$, and moreover

$$w_{\lambda}(x) \to u_{c} \frac{\psi_{ey}}{\psi_{ey}} + \frac{P_{ey} \eta_{c}}{\psi_{ey}^2}$$

strongly in $L^2(S_\alpha)$, when $\lambda \to 0^+$. (5.7)

So $|g_{\lambda}|_{L^2} \leq C$ and thus by applying the estimate (5.2) to (5.3), we have $|\tilde{v}_{\lambda_n}|_{H^\frac{1}{2}} \leq C$. Therefore, as before, $\tilde{v}_{\lambda_n} \to \tilde{v}_0$ weakly in $H^\frac{1}{2}$. Since $k_{\lambda_n}/|\tilde{v}_{\lambda_n}|_{L^2} \to 0$, we have $A^0 \tilde{v}_0 = 0$. So $\tilde{v}_0 = c_1 \psi_{ex}(\xi)$ for some $c_1 \neq 0$. But $(\tilde{v}_{\lambda_n}, \psi_{ex}(\xi)) = 0$ implies $(\tilde{v}_0, \psi_{ex}(\xi)) = 0$, a contradiction. This proves that $|v_{\lambda_n}|_{L^2} \leq C$. The equation satisfied by $v_{\lambda}$ is

$$A^{\lambda} v_{\lambda} = \frac{k_{\lambda}}{\lambda_n} u_{\lambda} - c_{\lambda} A^{\lambda} - A^0 \psi_{ex}(\xi) = \frac{k_{\lambda}}{\lambda_n} u_{\lambda} - c_{\lambda} b(\xi) w_{\lambda}(x(\xi)).$$

Applying Lemma 5.2 to the above equation, we have $|v_{\lambda}|_{H^\frac{1}{2}} \leq C$ and thus $v_{\lambda} \to v_0$ weakly in $H^\frac{1}{2}$. By (5.7), $v_0$ satisfies

$$A^0 v_0 = -b(\xi) \left( \frac{u_{c}}{\psi_{ey}} + \frac{P_{ey} \eta_{c}}{\psi_{ey}^2} \right)(\xi).$$

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It is shown in the Appendix that

\[ A^0_c \partial_c \bar{\psi}_e (x) = - \left( \frac{u_e}{\psi_{cy}} + \frac{P_{cy} \eta_e}{\psi_{cy}^2} \right) (x), \tag{5.8} \]

and equivalently,

\[ A^0_c \partial_c \bar{\psi}_e (\xi) = - b (\xi) \left( \frac{u_e}{\psi_{cy}} + \frac{P_{cy} \eta_e}{\psi_{cy}^2} \right) (\xi), \]

where

\[ \bar{\psi}_e (x, y) = \psi_e (x, y) - cy. \]

So \( A^0 \left( v_0 - \partial_c \bar{\psi}_e (\xi) \right) = 0. \) Since \((v_0, \psi_{ex} (\xi)) = \lim_{\lambda \to 0^+} (v_\lambda, \psi_{ex} (\xi)) = 0, \) we have

\[ v_0 = \partial_c \bar{\psi}_e (\xi) + d_0 \psi_{ex} (\xi), \quad d_0 = - (\partial_c \bar{\psi}_e, \psi_{ex}) / \| \psi_{ex} \|_{L^2}^2. \]

By the same argument as in the proof of \( \| u_\lambda - u_0 \|_{H^\frac{1}{2}} \to 0, \) we have \( \| v_\lambda - v_0 \|_{H^\frac{1}{2}} \to 0. \) We rewrite

\[ u_\lambda = c_\lambda \psi_{ex} + \lambda v_\lambda = \bar{c}_\lambda \psi_{ex} + \lambda \bar{v}_\lambda, \]

where \( \bar{c}_\lambda = c_\lambda + \lambda d_0, \) \( \bar{v}_\lambda = v_\lambda - d_0 \psi_{ex}. \) Then \( \bar{c}_\lambda \to 1, \) \( \bar{v}_\lambda \to \partial_c \bar{\psi}_e (\xi) \) when \( \lambda \to 0^+. \)

Now we compute \( \lim_{\lambda \to 0^+} \frac{k_\lambda}{\lambda^2}. \) From (5.3), we have

\[ A^0 \frac{u_\lambda}{\lambda^2} + \frac{A^\lambda - A^0}{\lambda} \left( \frac{\bar{c}_\lambda}{\lambda} \psi_{ex} + \bar{v}_\lambda \right) = \frac{k_\lambda}{\lambda^2} u_\lambda. \]

Taking the inner product of above with \( \psi_{ex} (\xi), \) we have

\[ \frac{k_\lambda}{\lambda^2} (u_\lambda, \psi_{ex} (\xi)) = \bar{c}_\lambda \left( \frac{A^\lambda - A^0}{\lambda^2} \psi_{ex}, \psi_{ex} \right) + \left( \frac{A^\lambda - A^0}{\lambda} \bar{v}_\lambda, \psi_{ex} \right) \]

\[ = \bar{c}_\lambda I_1 + I_2. \]

Again, we do the computations in the physical space. For the first term, we use (5.6) to get

\[ I_1 = \left( \frac{A^\lambda - A^0}{\lambda^2} \psi_{ex} (x), \psi_{ex} (x) \right) = \left( \frac{w_\lambda (x)}{\lambda}, \psi_{ex} (x) \right) \]

\[ = \left( \frac{\mathcal{D}}{(\lambda + D) \lambda \psi_{cy}} \psi_{ex} (x) \right) + \left( \frac{P_{cy} \eta_e}{\psi_{cy} (\lambda + D) \lambda \psi_{cy}} \right) \psi_{ex} (x) \]

\[ - \left( \frac{1}{\lambda + D} \psi_{cy} \left( 1 - \mathcal{E}^{\lambda^+} \right) \frac{\eta_e}{\psi_{cy}} \right) \psi_{ex} (x) \]

\[ = I^1_1 + I^1_2 + I^1_3. \]
We have

\[ I_1 = - \left< \left( \frac{1}{\lambda} - \frac{1}{\lambda + D} \right) \frac{u_e}{\psi_y}, \psi_y \eta \right> \]

\[ = - \frac{1}{\lambda} \left< u_e, \eta \right> + \left< \frac{1}{\lambda + D} \frac{u_e}{\psi_y}, \psi_y \eta \right> = \left< \frac{1}{\lambda + D} \frac{u_e}{\psi_y}, \psi_y \eta \right> \]

\[ = \left< u_e, \frac{1}{\lambda - D} \eta \right> = \left< u_e, (\mathcal{E}^{\lambda,-} - 1) \frac{\eta}{\psi_y} \right> \]

\[ \to - \left< u_e, \frac{\eta}{\psi_y} \right>, \]

and

\[ I_2 = - \left< P_{ey} \left( \frac{1}{\lambda} - \frac{1}{\lambda + D} \right) \frac{\eta}{\psi_y}, \psi_y \eta \right> = \left< P_{ey} \left( 1 - \mathcal{E}^{\lambda,+} \right) \frac{\eta}{\psi_y}, \psi_y \eta \right> \]

\[ = \left< P_{ey} \left( 1 - \mathcal{E}^{\lambda,+} \right), \eta \right> \frac{\eta}{\psi_y} \rightarrow \left< \frac{1}{\lambda - D} \eta, \psi_y \eta \right> \]

\[ = I_1^2 + I_1^3. \]

So

\[ \lim_{\lambda \to 0^+} I_1 (\lambda) = - \left< \frac{2u_e}{\psi_y} + \frac{P_{ey} \eta}{\psi_y^2} \right>. \]

To compute \( I_2 \), we write

\[ I_2 = \left< \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_y} \tilde{v}_\lambda, \psi_y \eta \right> + \left< \frac{1}{\lambda + D} \frac{P_{ey}}{\psi_y} \tilde{v}_\lambda, \psi_y \eta \right> \]

\[ - \left< \frac{1}{\lambda - D} \frac{P_{ey}}{\psi_y} \mathcal{E}^{\lambda,+} \frac{1}{\psi_y} \tilde{v}_\lambda, \psi_y \eta \right> \]

\[ = I_2^1 + I_2^2 + I_2^3. \]

We have

\[ I_2^1 = \left< P_{ey} \frac{1}{\psi_y} \tilde{v}_\lambda, \frac{1}{\lambda - D} \eta \right> = \left< P_{ey} \frac{1}{\psi_y} \tilde{v}_\lambda, (\mathcal{E}^{\lambda,-} - 1) \frac{\eta}{\psi_y} \right> \]

\[ \to - \left< \frac{P_{ey} \eta}{\psi_y^2}, \frac{\eta}{\psi_y} \right>, \]

\[ I_2^2 = \left< \frac{1}{\lambda + D} \frac{1}{\psi_y} \tilde{v}_\lambda, \frac{d}{dx} (u_e) \right> = \left< \tilde{v}_\lambda, \frac{1}{\lambda - D} \frac{d}{dx} (u_e) \right> \]

\[ = \left< \tilde{v}_\lambda, (1 + \mathcal{E}^{\lambda,-}) \frac{u_e}{\psi_y} \right> \to - \left< \partial_x \tilde{v}_\lambda, \frac{u_e}{\psi_y} \right>. \]
and

\[ I_2^2 = - \left\langle \frac{1}{\lambda + \mathcal{D}} \psi_{cy} \mathcal{E}^\lambda + \frac{1}{\psi_{cy}} \psi_{cy} \eta_{ex} \right\rangle = - \left\langle \psi_{cy} \mathcal{E}^\lambda + \frac{1}{\psi_{cy}} \psi_{cy} \frac{1}{\lambda - \mathcal{D}} \eta_{ex} \right\rangle \]

\[ = - \left\langle \psi_{cy} \mathcal{E}^\lambda + \frac{1}{\psi_{cy}} \psi_{cy} \left( -1 + \mathcal{E}^\lambda \right) \frac{\eta_{e}}{\psi_{cy}} \right\rangle \to 0. \]

So

\[ \lim_{\lambda \to 0^+} I_2(\lambda) = - \left\langle \partial_e \tilde{\psi}_e, \frac{u_e}{\psi_{cy}} + \psi_{cy} \frac{\eta_{e}}{\psi_{cy}} \right\rangle. \]

Thus

\[ \lim_{\lambda \to 0^+} \frac{k_\lambda}{\lambda^2} = \lim_{\lambda \to 0^+} \frac{\tilde{c}_\lambda I_1 + I_2}{\left( u_\lambda, \psi_{cy} \right)} = - \left\langle \eta_{e}, \frac{2u_e}{\psi_{cy}} + \frac{\psi_{cy}}{\eta_{e}} \frac{\eta_{e}}{\psi_{cy}} \right\rangle - \left\langle \partial_e \tilde{\psi}_e, \frac{u_e}{\psi_{cy}} + \psi_{cy} \frac{\eta_{e}}{\psi_{cy}} \right\rangle. \]

It is shown in the Appendix that

\[ \frac{dP}{dc} = - \left\langle \eta_{e}, \frac{2u_e}{\psi_{cy}} + \frac{\psi_{cy}}{\eta_{e}} \frac{\eta_{e}}{\psi_{cy}} \right\rangle - \left\langle \partial_e \tilde{\psi}_e, \frac{u_e}{\psi_{cy}} + \psi_{cy} \frac{\eta_{e}}{\psi_{cy}} \right\rangle, \tag{5.9} \]

where the momentum \( P \) is defined by

\[ P = \int_{S_e} \eta_e \frac{d}{dx} \left( \tilde{c}_e (x, \eta_e (x)) \right) dx, \]

with \( \tilde{c}_e = \phi_e - cx \). It is shown in [10] (see also [47]) that for a solitary wave solution

\[ \frac{dE}{dc} = - \frac{dP}{dc}. \]

where we note that the travel direction considered in this paper is opposite to the one in the above references. A combination of above results yields the formula [5.1].

As a corollary of the above proof, we show Theorem 3.

**Proof of Theorem 3**

The proof is very similar to that of Lemma 5.1, so we only sketch it. The main difference is that the computations depend on the parameter \( \mu_n \). We use \( \eta_{e,n}, \psi_{cy,n} \) etc. to denote the dependence on \( \mu_n \), and \( \eta_{e}, \psi_{cy} \) etc. for quantities depending on \( \mu_0 \). Denote \( \eta_{e} (\xi) = B_n (\psi_{n}, \lambda_{n}) \). Then \( \mathcal{A}^\mu_n u_n = 0 \) and we normalize \( u_n \) by \( \| u_n \|_{L^2} = 1 \). First, we show that \( \tilde{\kappa}_{\lambda_n^2} \to 0 \) strongly in \( L^2 (\mathbb{R}) \). Indeed, for any \( v \in L^2 (\mathbb{R}) \), since the norms \( \| \cdot \|_{L^2} \) and \( \| \cdot \|_{L^2, \psi_{cy,n}} \) are equivalent, we have

\[ \| \tilde{\kappa}_{\lambda_n^2} \|_{L^2} \leq C \| \tilde{\kappa}_{\lambda_n^2} \|_{L^2, \psi_{cy,n}}^2 = C \int \frac{\lambda_n^2}{\lambda_n^2 + \alpha^2} d \| \tilde{M}_\alpha v \|_{L^2}^2 \leq C \frac{\lambda_n^2}{\delta^2} \| v \|_{L^2, \psi_{cy,n}}^2 + C \int d \| \tilde{M}_\alpha v \|_{L^2, \psi_{cy,n}}^2 = A_1 + A_2, \]

where

\[ A_1 = C \int \frac{\lambda_n^2}{\lambda_n^2 + \alpha^2} d \| \tilde{M}_\alpha v \|_{L^2}^2 \]

and

\[ A_2 = C \int d \| \tilde{M}_\alpha v \|_{L^2}^2. \]
where \( \delta > 0 \) is arbitrary and \( \{ \tilde{M}_{\alpha,n}; \alpha \in \mathbb{R}^1 \} \) is the spectral measure of the self-adjoint operator \( \tilde{R}_n = -i \tilde{D}_n \) on \( L^2_{b_n \psi_{e,y,n}} \). For any \( \varepsilon > 0 \), since \( M_{(0)} = 0 \) we can choose \( \delta \) small enough such that

\[
\int_{|\alpha| \leq \delta} d\| \tilde{M}_n v \|_{L^2_{b_n \psi_{e,y}}}^2 \leq \frac{\varepsilon}{2}.
\]

Since the measure \( d\| \tilde{M}_n v \|_{L^2_{b_n \psi_{e,y}}}^2 \) converges to \( d\| \tilde{M}_n v \|_{L^2_{b_n \psi_{e,y}}}^2 \), we have \( A_2 \leq \varepsilon \) when \( n \) is big enough. The term \( A_1 \) tends to zero when \( n \to \infty \). Because \( \varepsilon \) can be arbitrarily small, we have

\[
\lim_{n \to \infty} \| \tilde{E}^\lambda_{n,v} \|_{L^2}^2 = 0.
\]

So \( A_n^\lambda \to A^0 \) strongly and similar to the proof of Lemma 5.1, we have \( u_n \to \psi_{ex}(\xi) \) in \( H^\frac{1}{2}(\mathbb{R}) \) by a renormalization. We write \( u_n = c_n \psi_{ex,n}(\xi) + \lambda_n v_n \), with \( c_n = (u_n, \psi_{ex,n}) / (\psi_{ex,n}, \psi_{ex,n}) \). Then \( c_n \to 1 \) and \( (v_n, \psi_{ex,n}) = 0 \). As before, we have \( v_n \to v_0 \) in \( H^\frac{1}{2}(\mathbb{R}) \), where

\[
v_0 = \partial_c \tilde{\psi}_e(\xi) + d_0 \psi_{ex}(\xi), \quad d_0 = -\left( \partial_c \tilde{\psi}_e, \psi_{ex} \right) / \left( \psi_{ex}, \psi_{ex} \right).
\]

We rewrite

\[
u_n = c_n \psi_{ex,n}(\xi) + \lambda_n v_n = \tilde{c}_n \psi_{ex,n} + \lambda_n \tilde{v}_n,
\]

where \( \tilde{c}_n = c_n + \lambda_n d_0 \) and \( \tilde{v}_n = v_n - d_0 \psi_{ex,n} \). Then \( \tilde{c}_n \to 1 \), \( \tilde{v}_n \to \partial_c \tilde{\psi}_e(\xi) \) when \( n \to \infty \). Similarly, we have

\[
0 = \tilde{c}_n \left( \frac{A_n^\lambda - A_0}{\lambda_n} \psi_{ex,n}, \psi_{ex,n} \right) + \left( \frac{A_n^\lambda - A_0}{\lambda_n} \tilde{v}_n, \psi_{ex,n} \right) = \tilde{c}_n I_1 + I_2.
\]

As before, the calculations of \( I_1, I_2 \) are first done in the physical space \( S_{e,n} \) with the inner product \( \langle ., . \rangle_n \). We use the same notations as in the proof of Lemma 5.1. By the same computations, we have

\[
I_1^1 = \left\langle u_{e,n}, (-1 + \tilde{E}^\lambda_{n,\cdot}) \frac{\eta_{e,n}}{\psi_{e,y,n}} \right\rangle_n = \left\langle b_n u_{e,n}(\xi), \left( -1 + \tilde{E}^\lambda_{n,\cdot} \right) \frac{\eta_{e,n}}{\psi_{e,y,n}}(\xi) \right\rangle

\]

\[
- \left\langle b_n(\xi), \frac{\eta_{e}}{\psi_{e,y}}(\xi) \right\rangle = - \left\langle u_e, \frac{\eta_{e}}{\psi_{e,y}} \right\rangle.
\]

The other terms are handled similarly. So finally we have

\[
0 = \lim_{n \to \infty} \tilde{c}_n I_1 + I_2 = -\left\langle \eta_{e}, 2 \frac{u_e}{\psi_{e,y}} + \frac{P_{e,y}}{\psi_{e,y}^2} \eta_{e} \right\rangle - \left\langle \partial_c \tilde{\psi}_e, \frac{u_e}{\psi_{e,y}} + \frac{P_{e,y}}{\psi_{e,y}^2} \eta_{e} \right\rangle

\]

\[
= -\frac{1}{c} \frac{dE}{dc} u_0.
\]

So \( E'(\mu_0) = 0 \). This finishes the proof of Theorem 4. 

\[39\]
6 Appendix

In this Appendix, we prove (5.8) and (5.9).

Proof of (5.8). We derive (5.8) from the linearized system (2.2) to avoid working on the parameter dependent fluid domains and wave profiles. Note that (2.2) describes the evolution of the first order variations of the wave profile and the stream function in the travelling frame of the basic wave. The basic wave is \((\eta_e(x; c), \psi_e(x, y; c))\) in its travelling frame \((x + ct, y, t)\). Here, the stream function \(\psi_e\) and the relative stream function \(\bar{\psi}_e\) are related by \(\bar{\psi}_e = \psi_e - cy\), and thus \(\bar{\psi}_e \rightarrow 0\) when \(|x| \rightarrow \infty\). As an example to illustrate the ideas, we first consider a perturbed solution with a trivial translation \((\eta_e(x + \delta x; c), \bar{\psi}_e(x + \delta x, y; c))\).

The first order variations \(\delta x (\eta_{ex, \psi_{ex}})\) and \(\delta x \bar{P}_{ex}\) satisfy the linearized system (2.2), so we have

\[
\Delta \bar{\psi}_{ex} = 0 \quad \text{in} \quad D_e,
\]

\[
\frac{d}{dx}(\psi_{ey\eta_{ex}}) + \frac{d}{dx}\bar{\psi}_{ex} = 0 \quad \text{on} \quad S_e;
\]

\[
\bar{P}_{ex} + P_{ey\eta_{ex}} = 0 \quad \text{on} \quad S_e;
\]

\[
\frac{d}{dx}(\psi_{ey \partial_n \bar{\psi}_{ex}}) + \frac{d}{dx} \bar{P}_{ex} = 0 \quad \text{on} \quad S_e;
\]

\[
\bar{\psi}_{ex} = 0 \quad \text{on} \quad \{y = -h\},
\]

from which we get \(\mathcal{A}_e \bar{\psi}_{ex} = 0\). Since \(\bar{\psi}_{ex} = \psi_{ex}\), this recovers \(\mathcal{A}_e \psi_{ex} = 0\) which is proved in Lemma 4.1.1. The solitary wave with the speed \(c + \delta c\) is given by

\[
(\eta_e(x + \delta ct; c + \delta c), \bar{\psi}_e(x + \delta ct, y; c + \delta c))
\]
in the \((x + ct, y, t)\) frame. The first order variation are

\[
(\eta_e(x + \delta ct; c + \delta c), \bar{\psi}_e(x + \delta ct, y; c + \delta c)) - (\eta_e(x; c), \bar{\psi}_e(x, y; c)) = \delta c \left(\eta_{ex}(x; c) t + \partial_x \eta_e(x; c), \bar{\psi}_{ex}(x, y; c) t + \partial_x \bar{\psi}_e(x, y; c)\right).
\]

So

\[
(\eta_{ex}(x; c) t + \partial_x \eta_e(x; c), \bar{\psi}_{ex}(x, y; c) t + \partial_x \bar{\psi}_e(x, y; c))
\]

and \(\bar{P}_{ex}(x, y; c) t + \partial_x \bar{P}_{ex}(x, y; c)\) satisfy the linearized system (2.2). By using the above linear system for \((\eta_{ex}, \bar{\psi}_{ex})\), we get

\[
\Delta \partial_x \bar{\psi}_e = 0 \quad \text{in} \quad D_e,
\]

\[
\eta_{ex} + \frac{d}{dx}(\psi_{ey \partial_c \eta_e}) + \frac{d}{dx}(\partial_x \bar{\psi}_e) = 0 \quad \text{on} \quad S_e; \quad (6.2)
\]

\[
\partial_c \bar{P}_e + P_{ey \partial_c \eta_e} = 0 \quad \text{on} \quad S_e; \quad (6.3)
\]

\[
\partial_n \bar{\psi}_{ex} + \frac{d}{dx}(\psi_{ey \partial_n (\partial_c \bar{\psi}_e)}) + \frac{d}{dx}(\partial_c \bar{P}_e) = 0 \quad \text{on} \quad S_e; \quad (6.4)
\]

\[
\partial_c \bar{\psi}_e = 0 \quad \text{on} \quad \{y = -h\}.
\]
From \(6.2\),

\[
\partial_c \eta_c = -\frac{1}{\psi_{ey}} (\partial_c \psi_c + \eta_c). \tag{6.5}
\]

Since \(\partial_n \tilde{\psi}_{ex} = \frac{d}{dx} (\tilde{\phi}_{ex}) = \frac{d}{dx} (u_e)\) and \(u_e \to 0\) when \(|x| \to \infty\), combining \(6.3\) with \(6.4\) and \(6.3\), we have

\[
\partial_n (\partial_c \tilde{\psi}_c) = -\frac{1}{\psi_{ey}} (u_e - P_{ey} \partial_c \eta_c) = -\frac{u_e}{\psi_{ey}} - \frac{P_{ey} \eta_c}{\psi_{ey}^2} - \frac{P_{ey} \partial_c \psi_c}{\psi_{ey}}. \tag{6.6}
\]

So

\[
\partial_n (\partial_c \tilde{\psi}_c) + \frac{P_{ey} \partial_c \psi_c}{\psi_{ey}} \partial_c \tilde{\psi}_c = -\frac{u_e}{\psi_{ey}} - \frac{P_{ey} \eta_c}{\psi_{ey}^2},
\]

that is,

\[
\mathcal{A}_e^0 \partial_c \tilde{\psi}_c = -\frac{u_e}{\psi_{ey}} - \frac{P_{ey} \eta_c}{\psi_{ey}^2}.
\]

Lastly, we show that at a turning point \(\mu_0\), \(\mathcal{A}_e^0 \partial_\mu \tilde{\psi}_c = 0\). Indeed, the first order variation of

\[
(\eta_c(x + \delta ct; \mu_0 + \delta \mu), \tilde{\psi}_c(x + \delta ct, y; \mu_0 + \delta \mu)) \rightarrow \eta_c(x; \mu_0), \tilde{\psi}_c(x, y; \mu_0)
\]

is \(\delta \mu (\partial_\mu \eta_c, \partial_\mu \tilde{\psi}_c)\), since \(\delta c = O(\delta \mu^2)\) is of higher order. So \((\partial_\mu \eta_c, \partial_\mu \tilde{\psi}_c)\) satisfies the same system for \((\eta_{ex}, \tilde{\psi}_{ex})\), which yields \(\mathcal{A}_e^0 \partial_\mu \tilde{\psi}_c = 0\).

**Proof of (5.9).** We have

\[
P (c) = \int_{S_e} \eta_c \frac{d}{dx} \tilde{\phi}_c \, dx = \int \eta_c (x; c) \partial_n \tilde{\psi}_c ((x, \eta_c (x; c); c)) \, dx
\]

Since

\[
\frac{d}{dc} \left( \partial_n \tilde{\psi}_c (x, \eta_c (x; c); c) \right)
\]

\[
= \frac{d}{dc} \left( \tilde{\psi}_{ey} (x, \eta_c (x; c); c) - \eta_{ex} (x; c) \tilde{\psi}_{ex} (x, \eta_c (x; c); c) \right)
\]

\[
= \partial_y \left( \partial_c \tilde{\psi}_c \right) + \tilde{\psi}_{ey} \partial_c \eta_c - \partial_x \left( \partial_c \eta_c \right) \tilde{\psi}_{ex} - \eta_{ex} \left( \partial_x \left( \partial_c \tilde{\psi}_c \right) + \tilde{\psi}_{ex} \partial_c \eta_c \right)
\]

\[
= \left( \partial_y \left( \partial_c \tilde{\psi}_c \right) - \partial_x \left( \partial_c \tilde{\psi}_c \right) \right) - \partial_x \left( \tilde{\psi}_{ex} + \eta_{ex} \tilde{\psi}_{ex} \right) - \partial_x \left( \partial_c \eta_c \right) \tilde{\psi}_{ex}
\]

\[
= \partial_n \left( \partial_c \tilde{\psi}_c \right) - \partial_c \eta_c \frac{d}{dx} \tilde{\psi}_{ex} - \frac{d}{dx} \left( \partial_c \eta_c \right) \tilde{\psi}_{ex} = \partial_n \left( \partial_c \tilde{\psi}_c \right) - \frac{d}{dx} \left( \partial_c \eta_c \tilde{\psi}_{ex} \right),
\]
we have

\[
\frac{dP}{dc} = \int \left\{ \partial_c \eta \partial_n \tilde{\psi} + \eta \frac{d}{dc} \left( \partial_n \tilde{\psi} \right) \right\} \, dx
\]

\[
= \int \left\{ \partial_c \eta \partial_n \tilde{\psi} e + \eta \left( \partial_n \left( \partial_c \tilde{\psi} e \right) - \frac{d}{dx} \left( \partial_c \eta \tilde{\psi} ex \right) \right) \right\} \, dx
\]

\[
= \int \left\{ \eta \left( u e - \eta ex \tilde{\psi} ex \right) + \eta \partial_n \left( \partial_c \tilde{\psi} e \right) + \eta ex \partial_c \eta \tilde{\psi} ex \right\} \, dx
\]

\[
= \int \left\{ \eta \left( \frac{1}{\psi ey} (\partial_c \tilde{\psi} e + \eta) \right) + \left( \eta, - \frac{u e}{\psi ey} - \frac{P ey \eta e}{\psi^2 ey} \right) + \left( \eta, \frac{P ey \eta e}{\psi^2 ey} \right) \right\} \, dx
\]

(by equations (6.5) and (6.6))

\[
= - \left( \eta, \frac{2 u e}{\psi ey} + \frac{P ey \eta e}{\psi^2 ey} \right) - \left( \partial_c \tilde{\psi} e, \frac{u e}{\psi ey} + \frac{P ey \eta e}{\psi^2 ey} \right).
\]

\[\Box\]

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References

[1] Agranovich, M. S., *Elliptic boundary problems*, Partial Differential equations IX pp. 1–132,

[2] Albert, J. P.; Bona, J. L.; Henry, D. B. *Sufficient conditions for stability of solitary-wave solutions of model equations for long waves*. Phys. D 24 (1987), no. 1-3, 343–366.

[3] Albert, J. P.; Bona, J. L. *Total positivity and the stability of internal waves in stratified fluids of finite depth*. IMA J. Appl. Math. 46 (1991), no. 1-2, 1–19.

[4] Amick, C. J.; Toland, J. F. *On solitary water-waves of finite amplitude*. Arch. Rational Mech. Anal. 76 (1981), no. 1, 9–95.

[5] Amick, C. J.; Toland, J. F. *On Periodic Water-Waves and their Convergence to Solitary Waves in the Long-Wave Limit*, Proc. Roy. Soc. (London) Ser. A, 303, (1981), 633-669.

[6] Amick, C. J.; Fraenkel, L. E.; Toland, J. F. *On the Stokes conjecture for the wave of extreme form*. Acta Math. 148 (1982), 193–214.
[7] Byatt-Smith, J. G. B.; Longuet-Higgins, M. S. On the speed and profile of solitary waves. Proc. Roy. Soc. London Ser. A 350 (1976), no. 1661, 175–189.

[8] Beale, J. Thomas The existence of solitary water waves. Comm. Pure Appl. Math. 30 (1977), no. 4, 373–389.

[9] Benjamin, T. B. The stability of solitary waves. Proc. Roy. Soc. (London) Ser. A 328 (1972), 153–183.

[10] Benjamin, T. B., Lectures on nonlinear wave motion, In ”Nonlinear Wave Motion,” (ed. A. C. Newell) American Math. Soc: Providence, R. I. Lecture Notes in Applied Mathematics 15 (1974), 3–47.

[11] Bergh, Jörn; Löfström, Jörgen, Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[12] Bona, J. L.; Sachs, R. L. The existence of internal solitary waves in a two-fluid system near the KdV limit. Geophys. Astrophys. Fluid Dynamics 48 (1989), no. 1-3, 25–51.

[13] Bona, J. L.; Souganidis, P. E.; Strauss, W. A. Stability and instability of solitary waves of Korteweg-de Vries type. Proc. Roy. Soc. London Ser. A 411 (1987), no. 1841, 395–412.

[14] Buffoni, B.; Dancer, E. N.; Toland, J. F. The regularity and local bifurcation of steady periodic water waves. Arch. Ration. Mech. Anal. 152 (2000), no. 3, 207–240.

[15] Buffoni, B.; Dancer, E. N.; Toland, J. F. The sub-harmonic bifurcation of Stokes waves. Arch. Ration. Mech. Anal. 152 (2000), no. 3, 241–271.

[16] Buffoni, Boris; Toland, John Analytic theory of global bifurcation. An introduction. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2003.

[17] Chen, B.; Saffman, P. G. Numerical evidence for the existence of new types of gravity waves of permanent form on deep water. Stud. Appl. Math. 62 (1980), no. 1, 1–21.

[18] Cordes, H. O., On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators. J. Funct. Anal. 18 (1975), 115–131.

[19] A. Constantin, The trajectories of particle in Stokes waves, Invent. Math., 166, pp. 523–535 (2006).

[20] Constantin, Adrian; Escher, Joachim Particle trajectories in solitary water waves, Bull. Amer. Math. Soc. (N.S.) 44 (2007), no. 3, 423–431.
[21] Constantin, Adrian; Escher, Joachim *Wave breaking for nonlinear nonlocal shallow water equations*. Acta Math. **181** (1998), no. 2, 229–243.

[22] Constantin, Adrian; Strauss, Walter A. *Stability of peakons*. Comm. Pure Appl. Math. **53** (2000), no. 5, 603–610.

[23] Craig, W. and Sternberg, P., *Symmetry of solitary waves*, Comm. PDE **13** (1988), 603–633.

[24] Craig, W., *Nonexistence of solitary water waves in three dimensions*, Phil. Trans. Royal Soc. London A **360** (2002), 1–9.

[25] Duncan, J. H., *Spilling breakers*, Annu. Rev. Fluid Mech., **33** (2001), 519-547.

[26] Friedrichs, K. O. and Hyers, D. H., *The existence of solitary waves*, Comm. Pure Appl. Math. **7** (1954), 517–550.

[27] Grillakis, M., Shatah, J. and Strauss, W., *Stability theory of solitary waves in the presence of symmetry. I*, Journal of Functional Analysis, **74**, no. 1, (1987), 160–197.

[28] Guo, Yan and Lin, Zhiwu, *Unstable and Stable Galaxy Models*, to appear in Comm. Math. Phys.

[29] Hislop, P. D., and Sigal, I. M., *Introduction to Spectral theory. With applications to Schrödinger operators*, Springer-Verlag, New York, 1996.

[30] Hunziker, W. *Notes on asymptotic perturbation theory for Schrödinger eigenvalue problems*. Helv. Phys. Acta **61** (1988), no. 3, 257–304.

[31] Hur, Vera and Lin, Zhiwu, *Unstable surface waves in running water*, to appear in Comm. Math. Phys.

[32] Kataoka, T. *On the superharmonic instability of surface gravity waves on fluid of finite depth*. J. Fluid Mech. **547** (2006), 175–184.

[33] Kataoka, Takeshi *The stability of finite-amplitude interfacial solitary waves*. Fluid Dynam. Res. **38** (2006), no. **12**, 831–867.

[34] Lannes, D., *Well-posedness of the water-waves equations*, J. Amer. Math. Soc., **18**, (2005) 605–654.

[35] Lavrentiev, M. A. I. *On the theory of long waves. II. A contribution to the theory of long waves*. Amer. Math. Soc. Translation 1954, (1954). no. **102**.

[36] Lewy, H., *A note on harmonic functions and a hydrodynamic application*, Proc. Amer. Math. Soc. **3** (1952), 111–113.

[37] Li, Yi A. *Linear stability of solitary waves of the Green-Naghdi equations*. Comm. Pure Appl. Math. **54** (2001), no. 5, 501–536.
[38] Lin, Zhiwu *Instability of periodic BGK waves.* Math. Res. Lett. 8 (2001), no. 4, 521–534.

[39] Lin, Zhiwu *Some stability and instability criteria for ideal plane flows.* Comm. Math. Phys. 246 (2004), no. 1, 87–112.

[40] Lin, Zhiwu *Instability of some ideal plane flows.* SIAM J. Math. Anal. 35 (2003), no. 2, 318–356.

[41] Lin, Zhiwu; Strauss, Walter A. *Linear stability and instability of relativistic Vlasov-Maxwell systems.* Comm. Pure Appl. Math. 60 (2007), no. 5, 724–787.

[42] Lin, Zhiwu and Strauss, Walter A., *A sharp stability criterion for Vlasov-Maxwell systems,* to appear in Invent. Math.

[43] Lin, Zhiwu, *Instability of large Stokes waves,* in preparation.

[44] Lin, Zhiwu, *Instability of nonlinear dispersive solitary waves,* Submitted.

[45] Lin, Zhiwu, *Instability of nonlinear dispersive periodic waves,* in preparation.

[46] Lin, Zhiwu and Liu, Yue, *Stability of peakons for the Degasperis-Procesi equation,* to appear in Comm. Pure Appl. Math.

[47] Longuet-Higgins, M. S. *On the mass, momentum, energy and circulation of a solitary wave.* Proc. Roy. Soc. (London) Ser. A 337 (1974), 1–13.

[48] Longuet-Higgins, M. S.; Fenton, J. D. *On the mass, momentum, energy and circulation of a solitary wave. II.* Proc. Roy. Soc. (London) Ser. A 340 (1974), 471–493.

[49] Longuet-Higgins, M. S.; Fox, M. J. H. *Theory of the almost-highest wave: the inner solution.* J. Fluid Mech. 80 (1977), no. 4, 721–741.

[50] Longuet-Higgins, M. S.; Fox, M. J. H. *Theory of the almost-highest wave. II. Matching and analytic extension.* J. Fluid Mech. 85 (1978), no. 4, 769–786.

[51] Longuet-Higgins, M. S.; Fox, M. J. H. *Asymptotic theory for the almost-highest solitary wave.* J. Fluid Mech. 317 (1996), 1–19.

[52] Longuet-Higgins, Michael; Tanaka, Mitsuhiro *On the crest instabilities of steep surface waves.* J. Fluid Mech. 336 (1997), 51–68.

[53] McLeod, J. B. *The Froude number for solitary waves.* Proc. Roy. Soc. Edinburgh Sect. A 97 (1984), 193–197.

[54] Naumkin, P. & Shishmarev, I., *Nonlinear Nonlocal Equations in the Theory of Waves.* Transl. Math. Monographs, 133. Amer. Math. Soc., Providence, RI, 1994.
[55] Pego, Robert L.; Weinstein, Michael I. Eigenvalues, and instabilities of solitary waves. Philos. Trans. Roy. Soc. London Ser. A 340 (1992), no. 1656, 47–94.

[56] Peregrine, D. H., Breaking waves on beaches, Annu. Rev. Fluid Mech., 15 (1983), 149-178.

[57] Peregrine, D. H., Water waves and their development in space and time, Proc. Roy. Soc. (London) Ser. A, 400 (1985), 1-18.

[58] Plotnikov, P. I. Nonuniqueness of solutions of a problem on solitary waves, and bifurcations of critical points of smooth functionals. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), no. 2, 339–366; translation in Math. USSR-Izv. 38 (1992), no. 2, 333–357

[59] Saffman, P. G. The superharmonic instability of finite-amplitude water waves. J. Fluid Mech. 159 (1985), 169–174.

[60] Seliger, R., A note on the breaking of waves. Proc. Roy. Soc. Ser. A, 303 (1968), 493–496.

[61] Starr, Victor P. Momentum and energy integrals for gravity waves of finite height. J. Marine Research 6 (1947), 175–193.

[62] Tanaka, M., The stability of steep gravity waves, J. Phys. Soc. Japan 52 (1983), pp. 3047–3055.

[63] Tanaka, M., The stability of steep gravity waves. II. J. Fluid Mech. 156 (1985), 281–289.

[64] Tanaka, M., The stability of solitary waves. Phys. Fluids 29 (1986), no. 3, 650–655.

[65] Tanaka, M., Dold, J.W., Lewy, M. and Peregrine. D.H. (1987) Instability and breaking of a solitary wave, J. Fluid Mech. 185, .235-248.

[66] Toland, J. F. On the existence of a wave of greatest height and Stokes’s conjecture. Proc. Roy. Soc. London Ser. A 363 (1978), no. 1715, 469–485.

[67] Vock, E.; Hunziker, W. Stability of Schrödinger eigenvalue problems. Comm. Math. Phys. 83 (1982), no. 2, 281–302.

[68] Wu, S. Well-posedness in Sobolev spaces of the full water wave problem in 2-D, Invent. Math., 130 (1997), 39–72.

[69] Zakharov, V. Stability of periodic waves of finite amplitude on the surface of a deep fluid. J. Appl. Mech. Tech. Phys. 9 (1968), 190–194.