The minimum number of some resolving sets for the Crystal Cubic Carbon $CCC(n)$ and the Layer Cycle Graph $LCG(n, k)$

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Abstract

The problem of determining resolving sets in graph theory has a long history, as it has many applications in chemistry, robot navigation, combinatorial optimization and utilization of the idea in pattern recognition and processing of images that also provide motivation for founding the theory. Especially, it is well known that these problems are NP hard. In the present work, first we define the structure of the crystal cubic carbon $CCC(n)$ by a new method, and we will find the minimum number of doubly resolving sets and strong resolving sets for the crystal cubic carbon $CCC(n)$. Also, we construct a new class of graphs of order $n + \sum_{p=0}^{k}p^2(n-1)^{p-2}$, is denoted by $LCG(n, k)$ and recall that the layer cycle graph with parameters $n$ and $k$. Moreover, we will compute the minimum number of some resolving sets for the layer cycle graph $LCG(n, k)$.

Keywords: resolving set; doubly resolving; strong resolving.

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1. Introduction and Preliminaries

Throughout this work, we will considering connected simple graphs and we follow the notion and terminology from the book [1]. The structure of a graph in graph theory is often considered as a set of vertices and edges. The problem of determining resolving sets in graph theory has a long history, as it has many applications in chemistry [2,3], robot navigation [4], combinatorial optimization [5], and utilization of the idea in pattern recognition and processing of images that also provide motivation for founding the theory [6]. For example, from a chemical graph theory perspective, a molecular graph is a graph consists of atoms called vertices and the chemical bond between atoms called edges. Especially, if we consider a graph as a chemical compound, then by changing the set of atoms and permuting their positions, a collection of compounds is essentially defined that are characterized by the substructure common to them. In a chemical compound, it is very important for chemists to find the minimum number of atoms so that they can identify other atoms relative to that the smallest set of atoms, so chemists require mathematical forms for a set of chemical compound to give distinct representations to distinct compound structures, and its corresponding in graph theory is to find the minimum number of resolving sets, see [7-11].

Suppose $R = \{r_1, r_2, ..., r_m\}$ is an order subset of vertices belonging to a graph $G$. For each vertex $u$ of $G$, we shall use the notation $r(u|R)$ to denote the representation of $u$ corresponding to $R$ in graph $G$, that is the $m$-tuple $(d(u, r_1), ..., d(u, r_m))$, where $d(u, r_i)$ is the length of geodesic between $u$ and $r_i$, $1 \leq i \leq m$. If the representation of distinct vertices in $V(G) - R$ is distinct, then the order subset $R$ is called a resolving set of graph $G$, see [12]. Therefore, it is important to find the minimum number of resolving sets in graph $G$. The minimum number of resolving sets in graph $G$ is called the metric dimension of $G$ and this minimum number is denoted by $\beta(G)$. The metric dimension and its related parameters has been studied by many researchers over the years, because their remarkable applications in graph theory and other sciences is important. For more specialized topics, see [13-19]. Indeed, the concept and
notation of the metric dimension problem, was first introduced by Slater [6] under the term locating set, and the idea of metric dimension of a graph was individually introduced by Harary and Melter in [20]. One of the more specialized topics related to the metric dimension is a doubly resolving set of graph. Cáceres et al. [21] define the notion of a doubly resolving set. Also, we can verify that an ordered subset of vertices related to the metric dimension is a doubly resolving set of graph. C´ aceres et al. [21] define the notion of a doubly resolving set for $G$, if every two distinct vertices $u$ and $v$ of $G$ are doubly resolved by some two vertices in the set $Z$, that is, for any two vertices $u, v \in V(G)$ we have $r(uZ) = r(vZ) \neq \mu I$, where $\mu$ is an integer, and $I$ denotes the unit $n$-vector $(1, \ldots, 1)$. The minimum number of doubly resolving sets in graph $G$ is denoted by $s(G)$. For more information on the doubly resolving set of graphs, see [22-25]. A vertex $u$ of a graph $G$ is called maximally distant from a vertex $v$ of $G$, if for every $w \in N_G(u)$, we have $d(v, w) \leq d(v, u)$, where $N_G(u)$ to denote the set of neighbors that $u$ has in $G$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then $u$ and $v$ are said to be mutually maximally distant [26]. For vertices $u$ and $v$ of a graph $G$, we use the interval $I_G[u, v]$ to denote as the collection of all vertices that belong to a shortest path between $u$ and $v$. A vertex $w$ strongly resolves two vertices $u$ and $v$ if $v$ belongs to $I_G[u, w]$ or $u$ belongs to $I_G[v, w]$. A set $R = \{r_1, r_2, \ldots, r_k\}$ of vertices of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $R$. The minimum number of strong resolving sets in graph $G$ is denoted by $sdim(G)$, see [5, 27, 28].

In this article we focus on some resolving parameters of two classes of graphs the crystal cubic carbon $\text{CCC}(n)$, and the layer cycle graph will be denoted by $\text{LCG}(n, k)$. We will first describe these classes of graphs that are used in the next section. Note that, these classes of graphs are important because they enable us to obtain many combinatorial problems about resolving sets into chemical compounds.

1). Let $n$ be fixed positive integer and $k$ be an integer so that $2 \leq k \leq n$. The crystal cubic carbon $\text{CCC}(n)$, is defined in [7]. Also, some of the chemical parameters of the crystal cubic carbon $\text{CCC}(n)$ have been calculated by other researchers, further details can be given in [29,30]. In [31] the authors showed that if $n \geq 2$ is an integer, then the minimum number of a resolving set of $\text{CCC}(n)$ is $16 \times 7^{(n-2)}$. In the present paper, we will discuss the minimum number of doubly resolving sets and strong resolving sets for the crystal cubic carbon $\text{CCC}(n)$. For this purpose, first we introduce some notation which is used throughout this article and is related to the crystal cubic carbon $\text{CCC}(n)$, and we define the structure of the crystal cubic carbon $\text{CCC}(n)$ by a new method. Consider two sets $W_1 = \{1,\ldots,4\}$ and $W_2 = \{5,\ldots,8\}$, and let $M$ be a graph with vertex set $\{1,2,\ldots,8\}$. Now suppose that the edge set $E(M)$ is

$$E(M) = \{(ij) \mid i \in W_1, i < j, j-i = 1 \text{or} j-i = 3\} \cup \{(ij) \mid i \in W_2, i < j, j-i = 1 \text{or} j-i = 3\} \cup \{(ij) \mid i \in W_1, j \in W_2, j-i = 4\}.$$  

We can verify that the graph $M$, defined already, is isomorphic to the Cartesian product $C_4 \square P_2$, where $C_4$ and $P_2$ denote the cycle on 4 and the path on 2 vertices, respectively. From now on, for convenience, it can be assumed that $V(C_4 \square P_2) = V(M)$ and $E(C_4 \square P_2) = E(M)$. The cartesian product $C_4 \square P_2$ is depicted in Figure 1.

![Figure 1](attachment://graph.png)

Also, we shall use the notation $Q_n^{(k)}$ to denote a cubic graph of order 8, with vertex set $V(Q_n^{(k)}) = \{(x_1, 1)^{(k)}, \ldots, (x_r, 8)^{(k)}\}$,
and the edge set $E(Q^4_{r_1})$ is

$$E(Q^4_{r_1}) = \{(x_i, i)^{(k)}(x_j, j)^{(k)} | i, j \in W, i < j, j - i = 1 \text{or } j - i = 3\} \cup \{(x_i, i)^{(k)}(x_j, j)^{(k)} | i, j \in W, i < j, j - i = 1 \text{or } j - i = 3\} \cup \{(x_i, i)^{(k)}(x_j, j)^{(k)} | i \in W_1, j \in W_2, j - i = 4\},$$

for $1 \leq r \leq 8$, and $1 \leq s \leq 7^{k-2}$. The graph $Q^4_{r_1}$ is depicted in Figure 2.

![Figure 2](image-url)

We can see that a cubic graph $Q^4_{r_1}$ of order 8 is isomorphic to the cartesian product $C_4 \square P_2$. Now, suppose $H$ is a graph of order $8 + 64 \sum_{k=2}^n 7^{k-2}$ with vertex set

$$V(H) = L_1 \cup L_2 \cup ... \cup L_n,$$

where the sets $L_1, L_2, ..., L_n$ are called the layers of $H$ such that $L_1 = V(C_4 \square P_2) = \{1, 2, ..., 8\}$, and for $k \geq 2$ we have

$$L_k = \{Q^4_{r_1}, Q^4_{r_2}, ..., Q^4_{r_{(n-1)}}, Q^4_{s_1}, Q^4_{s_2}, ..., Q^4_{s_{(n-2)}}\},$$

where $Q^4_{r_1}$ is defined already and every $(x_r, 1)^{(k)} \in Q^4_{r_1}$ is called head vertex of $Q^4_{r_1}$ in the layer $L_1$. Now, let the adjacency relation of graph $H$ given as follows. Suppose that $r$ is an arbitrary vertex in the layer $L_1$, $1 \leq r \leq 8$, and $r$ is adjacent to the head vertex of cubic $Q^4_{r_1}$ in the layer $L_2$ by an edge. Also every vertex in cubic $Q^4_{r_1} \in L_2 (k \geq 2)$, except head vertex $(x_r, 1)^{(k)}$, is adjacent to exactly the head vertex of one cubic in the layer $L_{k+1}$ say $Q^4_{r_1}$ by an edge, then we can see that the resulting graph is isomorphic to the crystal cubic carbon $CCC(n)$. In particular, we say that two cubes are congruous, if both of them lie in the same layer. It is natural to consider its vertex set of crystal cubic carbon $CCC(n)$ as partitioned into $n$ layers. The layers $L_1$ and $L_2$ consisting of the vertices $\{1, 2, ..., 8\}$ and $\{Q^4_{r_1}, Q^4_{r_2}, ..., Q^4_{r_1}\}$, respectively. In particular, each layer $L_k (k \geq 2)$, consisting of the $64 \times 7^{(k-2)}$ vertices. Moreover, each layer $L_k (k \geq 2)$, consisting of the $8 \times 7^{(k-2)}$ cubes. The crystal cubic carbon $CCC(2)$ is depicted in Figure 3.
2). Let $n$ and $k$ be fixed positive integers so that $n \geq 3$, $k \geq 2$ and $[n] = \{1, 2, \ldots, n\}$, also, let $p$ be an integer so that $2 \leq p \leq k$. In this section, we construct a class of graphs of order $n + \sum_{i=2}^{k} n^2(n-1)^{p-2}$, denoted by $LCG(n, k)$ and recall that the layer cycle graph with parameters $n$ and $k$. Moreover, we obtain some resolving parameters for this class of graphs in the next section. For this purpose, first we introduce some notation which is used throughout this paper and is related to the layer cycle graph $LCG(n, k)$ as follows. We shall use the notation $C_r^{(p)}$ to denote a cycle of order $n$, with vertex set

$$V(C_r^{(p)}) = \{(x_r, 1)^{(p)}, (x_r, 2)^{(p)}, \ldots, (x_r, n)^{(p)}\},$$

and the edge set $E(C_r^{(p)})$ is

$$E(C_r^{(p)}) = \{(x_r, i)^{(p)}(x_r, j)^{(p)} | i, j \in [n], i < j, j - i = 1 \text{ or } j - i = n - 1\},$$

for $1 \leq r \leq n$, and $1 \leq s \leq (n - 1)^{p-2}$. We can verify that $C_r^{(p)}$ is isomorphic to the cycle $C_n$, where vertex set of the cycle $C_n$ is $V(C_n) = \{1, 2, \ldots, n\}$ and the edge set $E(C_n)$ is $E(C_n) = \{ij | i, j \in [n], i < j, j - i = 1 \text{ or } j - i = n - 1\}$. Now, suppose $G$ is a graph with vertex set $V(G) = U_1 \cup U_2 \cup \ldots \cup U_k$, where the sets $U_1, U_2, \ldots, U_k$ are called the layers of $G$ such that $U_1 = V(C_n)$, and for $p \geq 2$ we have

$$U_p = \{C_{r_1}^{(p)}, C_{r_2}^{(p)}, \ldots, C_{r_{(n-1)^{p-2}}}^{(p)l}, \ldots, C_{r_2}^{(p)}, \ldots, C_{r_{(n-1)^{p-2}}}^{(p)l}, \ldots, C_{r_1}^{(p)}, C_{r_2}^{(p)}, \ldots, C_{r_{(n-1)^{p-2}}}^{(p)l}\},$$

where $C_{r_i}^{(p)}$ is defined already and every $(x_r, 1)^{(p)} \in C_{r_i}^{(p)}$ is called head vertex of $C_{r_i}^{(p)}$ in the layer $U_p$. Now, let the adjacency relation of graph $G$ given as follows. Suppose that $r$ is an arbitrary vertex in the layer $U_1$, $1 \leq r \leq n$, and $r$ is adjacent to the head vertex of $C_{r_i}^{(p)}$ in the layer $U_2$ by an edge. Also every vertex in cycle $C_{r_i}^{(p)} \in U_p$ ($p \geq 2$), except head vertex $(x_r, 1)^{(p)}$, is adjacent to exactly the head vertex of one cycle in the layer $U_{p+1}$ say $C_{r_{(p+1)}}^{(p)}$ by an edge, then the resulting graph is called the layer cycle graph $LCG(n, k)$ with parameters $n, k$. In particular, we say that two cycles are congruous, if both of them lie in the same layer. It is natural to consider its vertex set of layer cycle graph $LCG(n, k)$ as partitioned into $k$ layers. The layers $U_1$ and $U_2$ consisting of the vertices $\{1, 2, \ldots, n\}$ and $\{C_{1_1}^{(p)}, C_{1_2}^{(p)}, \ldots, C_{1_{(n-1)^{p-2}}}^{(p)l}\}$, respectively. In particular, each layer $U_p$ ($p \geq 2$), consisting of the $n^{(2)(n-1)^{p-2}}$ vertices. The layer cycle graph $LCG(5, 3)$ is depicted in Figure 4.
Figure 4. Layer cycle graph $LCG(5, 3)$

2. Main Results

**Theorem 2.1.** Consider the crystal cubic carbon $CCC(n)$, is defined already. If $n \geq 2$ is an integer, then the minimum number of doubly resolving sets of $CCC(n)$ is $24 \times 7^{n-2}$.

**Proof.** Let $V(CCC(n)) = L_1 \cup L_2 \cup \ldots \cup L_n$, be the vertex set of graph $CCC(n)$, where $L_1, L_2, \ldots, L_n$ are the layers of $CCC(n)$, is defined already. Based on Theorem 1 in [31], we know that the minimum number of resolving sets of $CCC(n)$ is $16 \times 7^{n-2}$. Now, let

$$Z_1 = \{(x_{1,1}, 2)^{(a)}, \ldots, (x_{1_{p-2}, 2})^{(a)}; \ldots; (x_{8, 2})^{(a)}, \ldots, (x_{8_{p-2}, 2})^{(a)}\},$$

be an arranged set of vertices in $CCC(n)$, consisting of exactly one adjacent vertex in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$, and also

$$Z_2 = \{(x_{1, 4})^{(a)}, \ldots, (x_{1_{p-2}, 4})^{(a)}; \ldots; (x_{8, 4})^{(a)}, \ldots, (x_{8_{p-2}, 4})^{(a)}\},$$

be an arranged set of vertices in $CCC(n)$, consisting of exactly one adjacent vertex in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$, then the arranged set $Z_3 = Z_1 \cup Z_2$ of vertices in $CCC(n)$,
consisting of exactly two adjacent vertices in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$ is one of minimal resolving sets in $CCC(n)$. Also it is not hard to see that, for every two vertices $x$ and $y$ so that lie in the layers $L_1 \cup L_2 \cup \ldots \cup L_{n-1}$, we have $r(aZ_n) - r(vZ_n) \neq \mu I$, where $\mu$ is an integer and $I$ denotes the unit $16 \times 7^{n-2}$-vector. In particular, we can show that the set $Z_1$, cannot be doubly resolved all the vertices of each cubic of the layer $L_n$. For this purpose, we consider the cubic $Q_1^{(n)}$ in the layer $L_n$ and suppose $x$ is an arbitrarily element of the set $Z_1$ so that $(x_1, 2, \ldots, x_{n-1}, 4)^{(n)} \neq x$ and the distance between the head vertex $(x_1, 1)^{(n)}$ and $x$ is a positive integer $c$, that is $r((x_1, 1)^{(n)}x) = c$. Now, let $Z = ((x_1, 2, \ldots, x_{n-1}, 4)^{(n)}, x)$ be a subset of the set $Z_1$, we can view that all the vertices in the cubic $Q_1^{(n)}$ cannot be doubly resolved with respect to $Z$. Because, for every $1 \leq i \leq 8$, we have
\[
\begin{aligned}
& r((x_1, 1)^{(n)}Z) = (1, 1, c) \\
& r((x_1, 2)^{(n)}Z) = (0, 2, c + 1) \\
& r((x_1, 3)^{(n)}Z) = (1, 1, c + 2) \\
& r((x_1, 4)^{(n)}Z) = (2, 0, c + 1) \\
& r((x_1, 5)^{(n)}Z) = (2, 2, c + 1) \\
& r((x_1, 6)^{(n)}Z) = (1, 3, c + 2) \\
& r((x_1, 7)^{(n)}Z) = (2, 2, c + 3) \\
& r((x_1, 8)^{(n)}Z) = (3, 1, c + 2),
\end{aligned}
\]
and hence $Z_1$ cannot be doubly resolved all the vertices of each cubic of the layer $L_n$, because $x$ is an arbitrarily element of the set $Z_1$. Now if we consider, the set $Z \cup \{(x_1, 5)^{(n)}\}$ of vertices of $CCC(n)$ then we have,
\[
\begin{aligned}
& r((x_1, 1)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (1, 1, c, 1) \\
& r((x_1, 2)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (0, 2, c + 1, 1, 2) \\
& r((x_1, 3)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (1, 1, c + 2, 3) \\
& r((x_1, 4)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (2, 0, c + 1, 2) \\
& r((x_1, 5)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (2, 2, c + 1, 0) \\
& r((x_1, 6)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (1, 3, c + 2, 1) \\
& r((x_1, 7)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (2, 2, c + 3, 2) \\
& r((x_1, 8)^{(n)}Z \cup \{(x_1, 5)^{(n)}\}) = (3, 1, c + 2, 1),
\end{aligned}
\]
Therefore, every element of $Q_1^{(n)}$ doubly resolved by the set $Z \cup \{(x_1, 5)^{(n)}\}$. Besides, we can view that every minimal resolving set of $CCC(n)$, consisting of exactly two adjacent vertices in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$, and hence $\psi(CCC(n))$ must be greater than $16 \times 7^{(n-2)}$. By the discussion above, we deduce that if
\[
Z_2 = [(x_1, 5)^{(n)}, \ldots, (x_{n-1}, 5)^{(n)}, (x_1, 1)^{(n)}, \ldots, (x_{n-1}, 1)^{(n)}],
\]
is an arranged set of vertices in $CCC(n)$, consisting of exactly one adjacent vertex in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$, then the arranged set $Z_2 = Z_1 \cup Z_3$ of vertices in $CCC(n)$, consisting of exactly three adjacent vertices in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$, is a minimal doubly resolving set for $CCC(n)$. Thus, the minimum number of doubly resolving sets of $CCC(n)$ must be $3 \times 8 \times 7^{(n-2)}$.

**Theorem 2.2.** Consider the crystal cubic carbon $CCC(n)$, is defined already. If $n \geq 2$ is an integer, then the minimum number of strong resolving sets of $CCC(n)$ is $32 \times 7^{(n-2)} - 1$.

**Proof.** Let $V(CCC(n)) = L_1 \cup L_2 \cup \ldots \cup L_n$, be the vertex set of graph $CCC(n)$, where $L_1, L_2, \ldots, L_n$ are the layers of $CCC(n)$, is defined already. Based on Theorem 1 in [31], we know that the minimum number of resolving sets of $CCC(n)$ is $16 \times 7^{(n-2)}$. Besides, the arranged set $Z_2 = Z_1 \cup Z_3$ of vertices in $CCC(n)$, which is defined in the previous Theorem, consisting of exactly two adjacent vertices in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$ is one of minimal resolving sets in $CCC(n)$. Also, every two vertices $u$ and $v$ so that lie in the layers $L_1 \cup L_2 \cup \ldots \cup L_{n-1}$, are strongly resolved by an element of $Z_1$. Without loss of generality, if we consider the cubic $Q_1^{(n)}$ in the layer $L_n$, then every two vertices of the cubic $Q_1^{(n)}$ except two vertices $(x_1, 3)^{(n)}$ and $(x_1, 5)^{(n)}$ are strongly resolved by an element of $Z_1$, and hence if we consider the arranged set $Z_2 = Z_1 \cup Z_3$ of vertices in $CCC(n)$, is defined in the previous Theorem, consisting of exactly three adjacent vertices in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$, then every two vertices so that lie in the one cubic of the layer $L_n$ are strongly resolved by an element of the set $Z_3$, and the number of such vertices is $24 \times 7^{(n-2)}$. Note that, both vertices of $CCC(n)$ so that lie in distinct cubes in the layer $L_n$ and mutually maximally distant, cannot be strongly
resolved by an element of $Z_5$, and hence, from both vertices of distinct cubes so that mutually maximally distant, at least one of them must be belongs to the every minimum strong resolving set of $CCC(n)$. Therefore, in each cube of the layer $L_n$, except one of them, there must be a vertex of that cube that has a maximum distance from the head vertex of that cube in every set of minimum strong resolving set of $CCC(n)$, and hence, the number of such vertices is $8 \times 7^{(n-2)} - 1$. Thus, the minimum number of strong resolving sets of $CCC(n)$ must be $32 \times 7^{(n-2)} - 1$.

**Theorem 2.3.** Consider the layer cycle graph $LCG(n,k)$, is defined already. If $n, k$ are integers so that $n \geq 3$ and $k \geq 2$, then the minimum number of resolving sets of $LCG(n,k)$ is $n(n-1)^{k-2}$.

**Proof.** Let $V(LCG(n,k)) = U_1 \cup U_2 \cup \ldots \cup U_k$ be the vertex set of graph $LCG(n,k)$, where $U_1, U_2, ..., U_k$ are the layers of $LCG(n,k)$, is defined already. If we consider an arranged subset $R_1$ of vertices in the layers $U_1 \cup U_2 \cup \ldots \cup U_{k-1}$, then $R_1$ is not a resolving set for $LCG(n,k)$. In particular, we can express that if $R_2$ is an arranged set of vertices in $LCG(n,k)$, consisting of all the head vertices in the layer $U_k$, then the set $R_2$ is not a resolving set for $LCG(n,k)$, because the degree of each head vertex in the layer $U_k$ is 3, and hence there are two vertices in the cycle $C_1^{(k)}$ of $LCG(n,k)$ so that they are adjacent to the head vertex $(x_1,1)^{(k)}$ in the cycle $C_1^{(k)}$ with the same representations. Now, let $R_3 = \{r_1, r_2, ..., r_i\}$ be a minimal resolving set of $LCG(n,k)$. We claim that there is exactly one vertex of each cycle in the layer $U_k$ belongs to $R_3$. Suppose for a contradiction that none of vertices of each cycle in the layer $U_k$ belong to $R_3$, and hence without loss of generality if we consider the head vertex $(x_1,1)^{(k)}$ in the cycle $C_1^{(k)}$, then we can view that the metric representation of two vertices in the cycle $C_1^{(k)}$ of $LCG(n,k)$ so that they are adjacent to the head vertex $(x_1,1)^{(k)}$ is identical with respect $R_3$. Therefore, we deduce that at least one vertex of each cycle in the layer $U_k$ must be belongs to every minimal resolving set of $LCG(n,k)$. Besides, the layer $U_k$ of graph $LCG(n,k)$ consisting of exactly $n(n-1)^{k-2}$ cycles, and hence we deduce that the minimum number of resolving sets of $LCG(n,k)$ must be equal or greater than $n(n-1)^{k-2}$. Now, suppose that

$$\begin{align*}
R_4 &= \{(x_{11}, n)^{(k)}; \ldots; (x_{1n_1-1,p-2}, n)^{(k)}; \ldots; (x_{n1}, n)^{(k)}; \ldots; (x_{n_n^{(k)}-1,p-2}, n)^{(k)}\},
\end{align*}$$

is an arranged set of vertices in $LCG(n,k)$, consisting of exactly one adjacent vertex in each cycle of the layer $U_k$ with respect to head vertex of each cycle of the layer $U_k$. We claim that the set $R_4$ is a minimal resolving set for $LCG(n,k)$. Since, each vertex in the layer $U_1^p, 1 \leq p \leq k$ is adjacent to exactly one vertex of the layer $U_{p+1}$ say head vertex, it follows that all the vertices in the layer $U_1^p$ have different representations with respect to $R_4$. Therefore, it is necessary to show that all the vertices in layer $U_k$ have different representations with respect to $R_4$. Since the layer $U_k$ consisting of all the cycles so that these cycles are congruous, and the set $R_3$ consisting of exactly one adjacent vertex of each cycle in the layer $U_k$ with respect to head vertex of each cycle of the layer $U_k$, then it is sufficient to show that all the vertices in an arbitrarily cycle of the layer $U_k$ have different representations with respect to $R_4$. For this purpose, we consider the cycle $C_1^{(k)}$ in the layer $U_k$ and suppose $x$ is an arbitrarily element of the set $R_4$ so that $(x_{11}, n)^{(k)} \neq x$, and the distance between the head vertex $(x_{11}, 1)^{(k)} \in C_1^{(k)}$ and $x$ is a positive integer $c$, that is $r((x_{11}, 1)^{(k)}, x) = c$. Now, let $R = \{(x_{11}, n)^{(k)}; x\}$ be a subset of the set $R_4$, we can view that all the vertices in the cycle $C_1^{(k)}$ have different representations with respect to $R$. Because, if $n$ is an even integer then for every $1 \leq i \leq \lceil \frac{n}{2} \rceil$, we have $r((x_{11}, i)^{(k)}; R) = (i, c + i - 1)$, also, if $\lceil \frac{n}{2} \rceil < i \leq n$, then we have $r((x_{11}, i)^{(k)}; R) = (n - i, n + c + 1 - i)$. Note that, if $n$ is an odd integer then there are two vertices in the cycle $C_1^{(k)}$ with maximum distance from the head vertex $(x_{11}, 1)^{(k)}$ and hence for every $1 \leq i \leq \lceil \frac{n}{2} \rceil$, we have $r((x_{11}, i)^{(k)}; R) = (i, c + i - 1)$, also, if $i = \lceil \frac{n}{2} \rceil$, then we have $r((x_{11}, i)^{(k)}; R) = (n - i, n + c + 1 - i)$. Therefore, by the discussion above we deduce that the minimum number of resolving sets of $LCG(n,k)$ is $n(n-1)^{k-2}$.

**Theorem 2.4.** Consider the layer cycle graph $LCG(n,k)$, is defined already. If $n \geq 4$ is an even or odd integer and $k$ is an integer so that $k \geq 2$, then the minimum number of doubly resolving sets of $LCG(n,k)$ is $2n(n-1)^{k-2}$.

**Proof.** From the previous Theorem, we have already seen that the arranged set

$$\begin{align*}
R_4 &= \{(x_{11}, n)^{(k)}; \ldots; (x_{1n_1-1,p-2}, n)^{(k)}; \ldots; (x_{n1}, n)^{(k)}; \ldots; (x_{n_n^{(k)}-1,p-2}, n)^{(k)}\},
\end{align*}$$

of vertices in the layer $U_k$ of $LCG(n,k)$ is one of minimal resolving sets in $LCG(n,k)$. In particular, by the previous Theorem we know that the set $R_4$, cannot be doubly resolved all the vertices of each cycle of the layer $U_k$. Besides,
we can view that every minimal resolving set of \( LC(n,k) \), consisting of exactly one adjacent vertex in each cycle of the layer \( U_k \) with respect to head vertex of each cycle of the layer \( U_k \), and hence \( \psi(LC(n,k)) \) must be greater than \( n(n-1)^{k-2} \). Now, let

\[
R_5 = \{ (x_1, 1)^{(k)}, \ldots, (x_{[\frac{n}{2}]} - 2)^{(k)},\ldots, (x_{n-[\frac{n}{2}]+2}, 1)^{(k)}, \ldots, (x_{n-[\frac{n}{2}]+2}, \frac{n}{2} + 1)^{(k)} \},
\]

be an arranged set of vertices in \( LC(n,k) \), consisting of exactly one vertex in each cycle of the layer \( U_k \) with maximum distance from the head vertex of each cycle of the layer \( U_k \), and \( R_5 = R_4 \cup R_5 \). Then, by applying the same argument is done in proof of Theorem 2.1, we can show that the arranged set \( R_6 \) of vertices in \( LC(n,k) \) is one of minimal doubly resolving sets for \( LC(n,k) \).

\textbf{Theorem 2.5.} Consider the layer cycle graph \( LC(n,k) \), is defined already. If \( n \geq 3 \) is an even or odd integer and \( k \) is an integer so that \( k \geq 2 \), then the minimum number of strong resolving sets of \( LC(n,k) \) is \( \lceil \frac{n}{2} \rceil (n-1)^{k-2} - 1 \).

\textbf{Proof.} Let \( V(LC(n,k)) = U_1 \cup U_2 \cup \ldots \cup U_k \), be the vertex set of graph \( LC(n,k) \), where \( U_1, U_2, \ldots, U_k \) are the layers of \( LC(n,k) \), which is defined already. We can view that, the arranged set

\[
R_7 = \{ (x_1, 2)^{(k)}, \ldots, (x_{[\frac{n}{2}]} - 1)^{(k)}, \ldots, (x_{n-[\frac{n}{2}]+2}, 1)^{(k)}, \ldots, (x_{n-[\frac{n}{2}]+2}, 2)^{(k)} \},
\]

of vertices in \( LC(n,k) \), consisting of exactly one adjacent vertex in each cycle of the layer \( U_k \) with respect to head vertex of each cycle of the layer \( U_k \) is one of minimal resolving sets in \( LC(n,k) \). Also every two vertices \( u \) and \( v \) in the layers \( U_1 \cup U_2 \cup \ldots \cup U_{k-1} \), are strongly resolved by an element of \( R_7 \). In particular, if we consider a cycle and its head vertex in the layer \( U_k \), then each vertex in that cycle that has the maximum distance from the head vertex is strongly resolved by an element of \( R_7 \). Note that, the set \( R_7 \), cannot be strongly resolved other vertices of each cycle of the layer \( U_k \), and hence if we consider the arranged set

\[
R_8 = \{ (x_1, 2)^{(k)}, \ldots, (x_{[\frac{n}{2}]} 1)^{(k)}, \ldots, (x_{n-[\frac{n}{2}]+2}, 2)^{(k)}, \ldots, (x_{n-[\frac{n}{2}]+2}, \frac{n}{2} + 1)^{(k)} \},
\]

of vertices in \( LC(n,k) \), consisting of exactly \( \lceil \frac{n}{2} \rceil - 1 \) elements in each cycle of the layer \( U_k \), then we can see that all the vertices in each cycle of the layer \( U_k \) are strongly resolved by an element of \( R_8 \), and the number of such vertices is \( n(n-1)^{k-2} (-\lceil \frac{n}{2} \rceil) - 1 \). Note that, both vertices of \( LC(n,k) \) so that lie in distinct cycles in the layer \( U_k \) and mutually maximally distant, cannot be strongly resolved by an element of \( R_8 \), and hence, from both vertices of distinct cycles in the layer \( U_k \) so that mutually maximally distant, at least one of them must be belongs to the every minimum strong resolving set of \( LC(n,k) \). Therefore, in each cycle of the layer \( U_k \), except one of them, there must be a vertex of that cycle that has a maximum distance from the head vertex of that cycle in every set of minimum strong resolving set of \( LC(n,k) \), and hence, the number of such vertices is \( n(n-1)^{k-2} - 1 \). Thus, the minimum number of strong resolving sets of \( LC(n,k) \) must be \( \lceil \frac{n}{2} \rceil n(n-1)^{k-2} - 1 \).

3. Conclusion

In the present work, we have constructed the structure of the crystal cubic carbon \( CCC(n) \) by a new method, and we computed the minimum number of doubly resolving sets and strong resolving sets for the crystal cubic carbon \( CCC(n) \). In particular, we have constructed the layer cycle graph \( LC(n,k) \). Moreover, we computed the minimum number of some resolving parameters for the layer cycle graph \( LC(n,k) \), also other researchers can get interesting results on the topological indices for this family of graphs.

\textbf{Data Availability}

No data were used to support this study.

\textbf{Conflicts of Interest}

The authors declare that there are no conflicts of interest.

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