Superfield Generating Equation of Field-Antifield Formalism

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Abstract—A simple quantum superfield generating equation of the field-antifield formalism is proposed. The Schroedinger equation with the Hamiltonian having $\Delta$-exact form is derived. An symmetric extension to the main construction, with specific features caused by the principal fact that all basic equations become $Sp(2)$ vector-valued ones, is presented. A principal role of quantum antibrackets in formulation of the Heisenberg equations of motion is shown.

DOI: 10.1134/S1063779618050052

1. INTRODUCTION

The field-antifield formalism or Batalin–Vilkovisky method is a unique closed approach to the problem of covariant quantization of general gauge theories [1, 2]. The quantum master equation formulated in terms of the odd nilpotent Laplacian operator ($\Delta$-operator) for a quantum action plays a fundamental role in deriving all basic properties of the formalism (gauge independence of $S$-matrix, Ward identity, gauge invariant renormalization and so on).

In the present paper we present a new sight on the field-antifield formalism at the quantum superfield level. Firstly, within a superfield approach [3–5], we formulate a simple quantum generating equation of the field-antifield formalism as having its configuration space identified with the antisymplectic phase space of fields and antifields. The latter generating equation is presented in terms of a superfield covariant derivative with respect to the two-dimensional super-time whose Boson component is the “ordinary” time, purely formal in its origin, while its Fermion component is identified naturally with the BRST parameter. The covariant derivative squared is just the “ordinary” time derivative. Then we derive the standard Schroedinger equation by applying again the covariant derivative to the generating superfield equation. We provide effectively for the Hamiltonian commuting with the $\Delta$-operator and being a $\Delta$-exact one. Choosing a special form of the wave function in this Schroedinger equation we reproduce the quantum master equation of the field-antifield formalism. We present an $Sp(2)$ symmetric extension to the main construction, with specific features caused by the principal fact that all basic equations become $Sp(2)$ vector-valued ones.

2. PHASE SPACE

Our starting point is the phase space for which the co-ordinate operators $Z^\alpha$,

$$Z^\alpha = (\Phi^\alpha;\Phi^\alpha_\ast), \quad \epsilon(\Phi^\alpha) = \epsilon(\Phi^\ast_\alpha) + 1, \quad (1)$$

are identified with the standard full set of the field-antifield variables, and $P_\alpha$,

$$P_\alpha = -i\hbar\partial_\alpha(-1)^{\epsilon_\alpha} \Rightarrow [Z^\alpha, P_\beta] = i\hbar\delta^\alpha_\beta. \quad (2)$$

are their respective canonically-conjugate momenta operators.

3. SUPERFIELD

We introduce a superfield $\Psi$,

$$\Psi = \Psi(Z; t, \tau), \quad \epsilon(t) = 0, \quad \epsilon(\tau) = 1, \quad (3)$$

as a function depending on the co-ordinates of phase space and the “time” $t$ and its superpartner $\tau$.

4. SUPERFIELD DYNAMICS

We note that the generating equation of the field-antifield formalism takes the very simple form of a superfield Schroedinger equation,

$$(i\hbar D - Q)\Psi = 0, \quad (4)$$

where $D$ is a covariant super-time derivative,

$$D = \partial_\tau + \tau\partial_t, \quad \epsilon(D) = 1, \quad [D, D] = 2\partial_t, \quad (5)$$

$^1$The article is published in the original.
Q is a super-charge
\[ Q = \Delta - F, \quad \varepsilon(Q) = \varepsilon(\Delta) = \varepsilon(F) = 1, \]
whose kinetic part is the odd Laplacian, \( \Delta \),
\[ \Delta = \frac{1}{2} P_A E^{AB} P_B (\text{const}), \quad E^{AB} = \text{const}, \]
with \( E^{AB} \) being antisymplectic structure,
\[ \varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1, \]
\[ E^{AB} = -E^{BA}(-1)^{\varepsilon_A + (\varepsilon_B + 1)} \]
and \( F \) is a super-potential,
\[ F = F(Z) \Rightarrow [F, F] = 0, \]

It follows from (4) that the standard Schroedinger equation holds,
\[ (i\hbar \partial_t - H) \Psi_0 = 0, \]
with the Hamiltonian \( H \),
\[ H = \frac{-1}{2} (i\hbar)^{-1} [Q, Q] = (i\hbar)^{-1} [\Delta, F], \]
and the decomposition of superfield \( \Psi_0 \),
\[ \Psi(Z; t, \tau) = \exp\{\tau(i\hbar)^{-1}Q\} \Psi_0(Z; t). \]

Due to the nilpotency of \( \Delta \)-operator and the definition (12), we have
\[ [\Delta, H] = 0. \]

From (14) we conclude that if \( \Psi_0 \) satisfies the Schr"{o}dinger equation (11) then \( \Delta \Psi_0 \) is a solution to this equation as well,
\[ (i\hbar \partial_t - H) \Delta \Psi_0 = 0. \]

In its turn, it implies the following statement
\[ \Delta \Psi_0 \big|_{t=0} = 0 \Rightarrow \Delta \Psi_0 \big|_{t=0} = 0. \]

Then, by choosing \( \Psi_0(Z; t) = \exp\{\frac{i}{\hbar} W(Z)\} \), we arrive at the quantum master equation of the field-antifield formalism [1, 2] for quantum action \( W = W(Z) \), when \( t = 1 \),
\[ \Delta \exp\{\frac{i}{\hbar} W\} = 0. \]

5. \( Sp(2) \) SYMMETRIC VERSION

In the case of \( Sp(2) \)-covariant quantization scheme [6–9] co-ordinate part \( Z^A \) of the phase space,
\[ Z^A = (\Phi^a, \Phi^{ab}, \Phi^{ab}_a, \Phi^{ab}_a), \]
are identified with the set of all field and antifield variables. The generating equation of \( Sp(2) \) formalism takes the form of \( Sp(2) \) vector valued superfield Schroedinger equation
\[ (i\hbar D^a - Q^a) \Psi = 0, \quad D^a = \partial_{\tau^a} + g^{ab} \tau_b \partial_t, \]
where the following conventions hold for the required \( Sp(2) \) vector valued operators
\[ [D^a, D^b] = 2g^{ab} \partial_t, \quad g^{ab} = g^{ba} = \text{const}, \]
\[ Q^a = \Delta^a - F^a, \]
\[ \Delta^a = \Delta^a + \frac{i}{\hbar} V^a, \]
\[ V^a = -i\hbar \epsilon^{ab} \Phi^{ab}_a \partial_t (\text{const}), \]
\[ F^a = g^{ab} \epsilon_{bc} (i\hbar)^{-1} [\Delta^c, B]. \]

Choosing \( B \) as a Boson functional depending on fields \( \{ \Phi^{a} \} \) only,
\[ B = B(\Phi), \]
we arrive at the implication
\[ (B, B)^a = 0 \Rightarrow [F^a, F^b] = 0, \]
where the notation \( (G, H)^a \) for extended antibrackets is used [10]. Using the component form of superfield \( \Psi = \Psi(Z; t, \tau_a) \)
\[ \Psi = \exp\{\tau_5(i\hbar)^{-1}Q^5\} \Psi_0(Z; t), \]
we find that \( \Psi_0 \) satisfies the Schrödinger equation
\[ (i\hbar \partial_t - H) \Psi_0 = 0, \]

with the Hamiltonian \( H \),
\[ H = \frac{-1}{4} g_{ab} (i\hbar)^{-1} [Q^a, Q^b] \]
\[ = \frac{1}{2} (i\hbar)^{-2} [\Delta^a, \epsilon_{ab} [\Delta^b, B]], \]
where \( g_{ab} \) is inverse to \( g^{ab} \). Due to the anti-commutativity of the operators \( \Delta^a \),
\[ [\Delta^a, \Delta^b] = 0, \]
the Hamiltonian \( H \) (29) commutes with \( \Delta^a \),
\[ [\Delta^a, H] = 0, \]
and, as a consequence, the \( \Delta^a \Psi_0 \) satisfies the Schrödinger equation,
\[ (i\hbar \partial_t - H) \Delta^a \Psi_0 = 0, \]
if the $\Psi_0$ does satisfy. It follows from (32) that the implication holds of
\[ \Delta^a \Psi_{0\mid t=0} = 0 \Rightarrow \Delta^a \Psi_{0\text{any}t} = 0. \] (33)

By choosing $\Psi_0(Z; t) = \exp \left\{ \frac{i}{\hbar} W(Z) \right\}$, we arrive at the quantum master equation of the $Sp(2)$-covariant quantization [6–8],
\[ \Delta^a \exp \left\{ \frac{i}{\hbar} W \right\} = 0, \] (34)
as written for the quantum action $W = W(Z)$, when $t = 1$.

6. HEISENBERG EQUATIONS OF MOTION

In this Section, we are going to emphasize a principal role of the quantum antibrackets [11, 12] in formulation of the Heisenberg equations of motion both in the $Sp(l)$ [1, 2] and the $Sp(2)$ [6–8] cases.

Denote with $\Gamma$ the full set of the Schroedinger canonical variable operators,
\[ \Gamma = (Z^A; P_A), \] (35)
and let $\tilde{\Gamma}(t, \tau)$ be the respective superfield Heisenberg canonical variable operators,
\[ \tilde{\Gamma} = \tilde{\Gamma}(t, \tau). \] (36)

In the $Sp(l)$ case, the superfield Heisenberg equations of motion have the form,
\[ i\hbar \frac{\partial}{\partial t} \tilde{\Gamma} = [\tilde{Q}, \tilde{\Gamma}], \quad i\hbar \frac{\partial}{\partial \tau} \tilde{Q} = [\tilde{Q}, \tilde{\Gamma}]. \] (37)

It follows from these equations [12],
\[ (i\hbar)^2 \frac{\partial}{\partial t} \tilde{\Gamma} = -\frac{1}{2} [\tilde{Q}, [\tilde{Q}, \tilde{\Gamma}]] = -\frac{2}{3} [\tilde{Q}, \tilde{\Gamma} \tilde{Q}], \] (38)
where the quantum 2—antibracket, $(A, B)_Q$, is defined by
\[ (A, B)_Q = \frac{1}{2} (\llbracket A, [Q, B] \rrbracket - (A \leftrightarrow B)(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}). \] (39)

In the $Sp(2)$ case, the respective superfield Heisenberg equations of motion have the form,
\[ i\hbar D^a \tilde{\Gamma} = [\tilde{Q}^a, \tilde{\Gamma}], \quad i\hbar D^b \tilde{Q} = [\tilde{Q}^b, \tilde{\Gamma}]. \] (40)

It follows from these equations,
\[ (i\hbar)^2 \frac{\partial}{\partial t} \tilde{\Gamma}^a = -\frac{1}{4} g_{ab} [\tilde{Q}^b, [\tilde{Q}^a, \tilde{\Gamma}^a]] = -\frac{1}{3} g_{ab} [\tilde{Q}^b, \tilde{\Gamma}^a \tilde{Q}^a], \] (41)
where the $Sp(2)$ vector valued quantum 2-antibracket, $(A, B)_Q^a$ [13], is defined by
\[ (A, B)_Q^a = \frac{1}{2} \llbracket A, [Q^a, B] \rrbracket - (A \leftrightarrow B)(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}). \] (42)

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