String amplitudes in arbitrary dimensions

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Abstract

We calculate gravitational dressed tachyon correlators in non critical dimensions. The 2D gravity part of the theory is constrained to constant curvature. Then scaling dimensions of gravitational dressed vertex operators are equal to their bare conformal dimensions. Considering the model as $d+2$ dimensional critical string we calculate poles of generalized Shapiro–Virasoro amplitudes.

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1 Introduction

It is the idea of string theory that all elementary particles and their resonances can be considered as excitations of strings moving through space time. The main problem of such a picture is that the space time dimension $d$ has to be 26 (bosonic case) or 10 (super symmetric case) in order to get results not depending on the metric of the world sheet swept out by the string.

Then it was proposed in [1] to integrate over all world sheet metrics which can be considered as an inclusion of 2D quantum gravity. Unfortunately that approach provides complex scaling dimensions in the interesting region of space time dimensions between one and 25 [3, 4, 5, 6].

A natural way to overcome that problem is to weight the integration over all metrics by a factor

$$\int \mathcal{D}g \to \int \mathcal{D}g e^{-S_{pg}},$$

where $S_{pg}$ is an action for pure 2D gravity. The Einstein–Hilbert action does not solve the problem because it is a topological density in two dimensions. However, there is another natural candidate for $S_{pg}$ proposed in [7] and rederived in a topological framework in [8],

$$S_{pg} = \frac{i}{\pi} \int d^2z \sqrt{g} \phi (R + \Lambda),$$

(1.1)

where $\phi$ is a two dimensional dilaton which has to be taken into account as a quantum field. $R = R(g)$ is the two dimensional curvature and $\Lambda$ is a cosmological constant. (We use conventions of [9]). The dilaton enters the action like a Lagrange multiplier and hence (1.1) produces the constraint of constant curvature. There are also possible generalizations of (1.1) discussed in [10].

In [11] it has been shown that the inclusion of (1.1) provides a real string susceptibility (scaling dimension of the partition function) in arbitrary space time dimensions $d$. Later on it was shown in [12] (perturbative approach) and in [13] (non perturbative approach) that scaling dimensions of vertex operators are equal to their bare conformal dimensions and hence real. (The perturbative approach is also discussed in [14]).

The mass spectrum of the model was obtained in [14] via a sigma model interpretation, i.e. linear combinations of the Liouville field $\sigma$ (see below) and the dilaton $\phi$ are considered as new string coordinates. In the following we will call that point of view $d + 2$ dimensional critical string.

Our paper is organized as follows. In the second section we calculate correlators in the gravity part of the theory. That will be a preparation of the third section where correlators of gravitational dressed tachyons are written down and their gravitational dressed dimensions are calculated. In the fourth section we consider the $2 + d$ dimensional critical string and give an expression in the form of Shapiro–Virasoro amplitudes. A rough argumentation provides the poles of single scattering channels. In $d = 12$ dimensions the four point function will be considered more detailed. We get poles in S,T and U channels and also leg poles which are created by scattering with background tachyons of fixed momentum.
2 Correlators in 2D gravity

We will confine ourselves to simply connected world sheets and use the conformal gauge

\[ g_{\alpha\beta} = e^{2\sigma(z)} \delta_{\alpha\beta}. \] (2.1)

The 2D gravity action contains besides the pure gravity action (1.1) the matter induced Liouville action [1],

\[ S_g = S_{pg} + (26 - d)S_L, \] (2.2)

\[ S_L = \frac{1}{12\pi} \int d^2 z \left( \frac{1}{2} \partial_\alpha \sigma \partial_\alpha \sigma + \mu e^{2\sigma} \right), \] (2.3)

where \( \mu \) is the Liouville mass. We define fixed area expectation values as follows,

\[ Z(A) = \int Dg \phi Dg \sigma d\Lambda \delta \left( 1 - \frac{1}{A} \int d^2 z \sqrt{g} \right) \delta (\Lambda A + 4\pi) e^{-S_g} \]

\[ \langle \cdots \rangle_A = \frac{1}{Z(A)} \int Dg \phi Dg \sigma d\Lambda \delta \left( 1 - \frac{1}{A} \int d^2 z \sqrt{g} \right) \delta (\Lambda A + 4\pi) e^{-S_g} \langle \cdots \rangle. \] (2.4)

The second delta function comes from the constraint of constant curvature and the Gauß–Bonnet theorem. The first delta function is redundant, because the constraint of constant curvature implies a constant area. However, when we use translation invariant measures the term \( \phi \sqrt{g} \) in (1.1) will be renormalized and the constraint of constant curvature is no longer manifest. The arguments of delta functions are arranged in such a way that \( Z = \int dAZ(A) \) is dimensionless. Now we consider the correlator of \( N \) 2D gravitons and dilatons,

\[ \langle \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2i\gamma_i \phi(z_i)} \rangle |_{A}. \]

In order to be able to calculate this correlator we have to move to translation invariant measures [3]. Therefore we split \( \sigma \) into a classical part \( \hat{\sigma} \) and into a quantum part \( \sigma \) and turn to measures with \( g \) replaced by \( \hat{g} \),

\[ Dg \rightarrow D\hat{g}, \quad \hat{g}_{\alpha\beta} = e^{2\hat{\sigma}} \delta_{\alpha\beta}. \]

The Jacobian can be calculated by methods given in [4] or [16]. We get

\[ \langle \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2i\gamma_i \phi(z_i)} \rangle |_{A} = \]

\[ \frac{1}{Z(A)} e^{-(26-d)S_{\hat{g}}[\hat{\theta}]} \prod_{i=1}^{N} e^{2\beta_i \hat{\sigma}(z_i)} \]

\[ \int D\hat{g} \hat{\sigma} D\hat{g} \phi e^{-12\alpha \hat{S}_L - \hat{S}_{\hat{g}}} \int d\Lambda \delta \left( 1 - \frac{1}{A} \int d^2 z \sqrt{\hat{g}(e^{2\hat{\sigma}})_\text{ren}} \right) \]

\[ \delta (\Lambda A + 4\pi) \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2i\gamma_i \phi(z_i)}, \] (2.5)
where

\[ a = \frac{24 - d}{12} \]

\[ \hat{S}_L = \frac{1}{12\pi} \int d^2 z \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{\alpha\beta} \partial_{\alpha} \sigma \partial_{\beta} \sigma + \hat{R} \sigma + \mu (e^{2\sigma})_{\text{ren}} \right) \]

\[ \hat{S}_{pg} = \frac{i}{\pi} \int d^2 z \sqrt{\hat{g}} \left[ \phi (\hat{\Delta} \sigma + \hat{R}) + \Lambda (\phi e^{2\sigma})_{\text{ren}} \right]. \]  \tag{2.6}\]

It is useful to introduce

\[ \psi = \sigma + \frac{i}{a} \phi \]  \tag{2.7}\]

instead of \( \sigma \). Then the massless part of the action becomes diagonal,

\[ 12a\hat{S}_L |_{\mu=0} + \hat{S}_{pg} |_{\Lambda=0} = \]

\[ \frac{a}{\pi} \int d^2 z \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{\alpha\beta} \partial_{\alpha} \psi \partial_{\beta} \psi + \hat{R} \psi \right) + \]

\[ \frac{1}{a\pi} \int d^2 z \sqrt{\hat{g}} \frac{1}{2} \hat{g}^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi. \]  \tag{2.8}\]

(2.8) is a conformal field theory with central charge

\[ c = 12a + 1 + 1 = 26 - d \]

which cancels the central charge of the matter part of the theory (compare next section). In (2.5) we have admitted a renormalization of \( e^{2\sigma} \) and \( \phi e^{2\sigma} \). The operators \( (e^{2\sigma})_{\text{ren}} \) and \( (\phi e^{2\sigma})_{\text{ren}} \) must not destroy the conformal invariance and therefore we require them to be primary fields of dimension (1,1). (Otherwise we would get \( \hat{\sigma} \) dependent expressions).

The conformal dimension of a general composite vertex operator can be calculated via an operator product expansion. That provides

\[ \Delta \left( e^{2\rho \sigma} e^{2i(\omega - \hat{\sigma}) \phi} \right) = \Delta \left( e^{2\rho \nu} \right) + \Delta \left( e^{2i(\omega - \hat{\sigma}) \phi} \right) = \rho - \frac{\rho^2}{2a} + \frac{a}{2} \left( \omega - \frac{\rho}{a} \right)^2. \]  \tag{2.9}\]

The requirement \( \Delta = 1 \) yields one equation for two parameters and hence we have infinite many operators of dimension (1,1), (and there are even more of them \([15]\)). A natural additional requirement is that renormalized and unrenormalized operators should coincide in the semi classical limit \( (a \rightarrow \infty) \). That leads to

\[ (e^{2\sigma})_{\text{ren}} = e^{2\sigma} \]  \tag{2.10}\]

\[ (\phi e^{2\sigma})_{\text{ren}} = -\frac{ia}{4} e^{2\sigma} (e^{4i\phi} - 1) = \]

\[ = \phi e^{2\sigma} + o \left( \frac{1}{a} \right). \]  \tag{2.11}
Performing zero mode, and $\Lambda$ integration and neglecting uninteresting factors provides
\[
\langle \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2i\gamma_i \phi(z_i)} \rangle_{\Lambda} = \frac{A^{-t-s-1}}{Z(A)} e^{-\mu A} \Gamma(-t) \prod_{i=1}^{N} e^{2\beta_i \hat{\sigma}(z_i)} e^{-(26-d)\hat{S}_L[\sigma]} \int D\hat{g}_\perp D_{\hat{g}_\perp} \phi e^{-\hat{S}_{\hat{g}_\perp}[\Lambda=0]-\hat{S}_{\hat{L}}[\mu=0]} \left( \int d^2 z \sqrt{\hat{g}} e^{2\alpha e^{4i\phi}} \right)^t \left( \int d^2 z \sqrt{\hat{g}} e^{2\alpha} \right)^s \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2i\gamma_i \phi(z_i)},
\]
(2.12)
where
\[
t = a \left( 1 - \frac{1}{2} \sum_{i=1}^{N} \gamma_i \right) \quad \text{and} \quad s = 2a - t - \sum_{i=1}^{N} \beta_i = a + \sum_{i=1}^{N} \left( \frac{a}{2} \gamma_i - \beta_i \right),
\]
(2.13)
The right angle indicates that zero modes are integrated out. We observe that the partition function behaves like
\[
Z(A) \sim A^{-2a-1} e^{-\mu A} = A^{\frac{d-d_{\text{flat}}}{d+1}} e^{-\mu A}
\]
(2.14)
which coincides with the calculation with non translation invariant measures performed in [11]. The calculation of the partition function is rather simple with non translation invariant measures. In the case of general correlators it turns out to be more difficult. For non vanishing $\gamma_i$’s one has to solve the Liouville equation with additional delta function like sources as it was stated in [15]. But also for vanishing $\gamma_i$’s the problem is involved, because one has to integrate over all solutions of the Liouville equation. If one is only interested in the partition function the integrand will not depend on the special form of the solution of the Liouville equation and the integral over all solutions gives an uninteresting factor. The situation is much more complicated for correlation functions where the integrand depends on the special form of the solution. However, if one was able to solve that problem one could check whether our assumptions (2.10) and (2.11) are correct at the level of higher point functions, too.

Now we return to (2.12) and define the zero modes in a covariant way [6]
\[
\phi_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \hat{R} \phi \quad \text{and} \quad \sigma_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \hat{R} \sigma.
\]
(2.15)
Then we use the field redefinition (2.7) and get, up to a factor, for $\psi$ and $\phi$ the same $\hat{\sigma}$–dependent propagator
\[
G(z, z'; \hat{\sigma}) = -\log(M // z - z') - \frac{\hat{\sigma}(z)}{2} - \frac{\hat{\sigma}(z')}{2} + 3S_L[\hat{\sigma}].
\]
(2.16)
As usual (compare e.g. [17]) we take \( t \) and \( s \) as positive integers during the calculation and assume that final results could be continued to real values. That leads to

\[
\langle \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2\gamma_i \phi(z_i)} \rangle|_A = \\
\Gamma(-t) \frac{\Gamma(2a-t-s)}{\Gamma(-a)} e^{-12aS_L[\theta]} \prod_{i=1}^{N} e^{2\beta_i \hat{\sigma}(z_i)}
\]

\[
\int \left( \prod_{i=N+1}^{N+t+s} d^2 z_i e^{2\beta_i \hat{\sigma}(z_i)} \right) \prod_{i=1}^{N+t+s} \prod_{i,j} e^{(2\gamma_i \beta_j - a \gamma_i \gamma_j)G(z_i, z_j|\hat{\sigma})},
\]

where

\[
\gamma_{N+1} = \cdots = \gamma_{N+t} = \frac{2}{a} \\
\gamma_{N+t+1} = \cdots = \gamma_{N+t+s} = 0 \tag{2.18}
\]

\[
\beta_{N+1} = \cdots = \beta_{N+t+s} = 1
\]

which implies

\[
\sum_{i=1}^{N+t+s} \gamma_i = 2 \quad \text{and} \quad \sum_{i=1}^{N+t+s} \beta_i = 2a. \tag{2.19}
\]

It is easy to see that the \( \hat{\sigma} \) dependence drops out which represents a nontrivial check of our calculation.

We regularize \( \log(M0) \) by \( \log(M\epsilon/\eta) \), \( \eta \) is a renormalization scale and \( \epsilon \) is a UV cut off. Finally we get

\[
\langle \prod_{i=1}^{N} e^{2\beta_i \sigma(z_i) + 2\gamma_i \phi(z_i)} \rangle|_A = \\
A^{2a-t-s} \frac{\Gamma(-t)}{\Gamma(-a)} \left( \frac{M\epsilon}{\eta} \right)^{-\sum_{i=1}^{N}(2\gamma_i \beta_i - \gamma_i^2 a)}
\]

\[
\int \prod_{i=1}^{t} \prod_{l=1}^{s} d^2 u_i d^2 w_l \prod_{j=1}^{N} \prod_{i=1}^{t} (M | z_j - u_i |)^{-\frac{4}{a}\beta_i + 2\gamma_i} \prod_{j=1}^{N} \prod_{l=1}^{s} (M | z_j - w_l |)^{-2\gamma_j} \sum_{l=1}^{N} \prod_{i=1}^{t} (M | u_i - w_l |)^{-\frac{4}{a} \sum_{i \neq j}^{N} (M | z_i - z_j |)^{\gamma_i \gamma_j a - 2\gamma_i \beta_j}}. \tag{2.20}
\]

### 3 Matter coupled to 2D gravity

In the conformal gauge (2.1) the string action is given by

\[
S_M = \frac{1}{4\pi} \int d^2 z \frac{1}{2} \partial_\alpha X^\mu(z) \partial_\alpha X_\mu(z), \quad \mu = 1, \cdots, d. \tag{3.1}
\]
Like the authors of [6] we define a normal ordered tachyon operator by multiplying it with a conformal cut off and a scalar density. We allow that 2D dilatons take part in the scalar density,  
\[ e_{ik_j \cdot X(z_j)} := e^{ik_j \cdot X(z_j)} B_j(z_j) \]  
(3.2)

where  
\[ B_i(z_i) = (e^{2e^{2\sigma(z_i)}})^{\beta_i-1} e^{2\gamma_i \phi(z_i)} \]  
(3.3)

The N-point tachyon correlator is then given by  
\[ \langle \prod_{i=1}^N : e^{ik_i \cdot X(z_i)} : \rangle = \frac{1}{Z} \int Dg \sigma Dg \phi Dg(ghost) e^{-S_{pg}-S_M-S_{ghost}} \prod_{i=1}^N : e^{ik_i \cdot X(z_i)} : \]  
(3.4)

where the ghost integral arises from conformal gauge fixing. Using the result (2.20) from the previous section we get  
\[ \langle \prod_{i=1}^N : e^{ik_i \cdot X(z_i)} : \rangle |_{A} = \]

\[ A^{2a-t-s} \frac{\Gamma(-t)}{\Gamma(-a)} \left( \sum_{i=1}^N k_i \right) \left( \frac{\epsilon M}{\eta} \right) \sum_{i=1}^N (k_i^2 - 2\gamma_i \beta_i + \gamma_i^2 a) \epsilon^2 \sum_{i=1}^N (\beta_i-1) \]

\[ \int \prod_{i=1}^t d^2 u_i \prod_{l=1}^s d^2 w_l \prod_{j=1}^N \prod_{i=1}^t (M \mid z_j - u_i \mid)^{-\frac{4}{3} \beta_j + 2\gamma_j} \prod_{j=1}^t \prod_{l=1}^s (M \mid z_j - w_l \mid)^{-2\gamma_j} \]

\[ \prod_{i=1}^t \prod_{j=1}^s \prod_{l=1}^t (M \mid u_i - w_l \mid)^{-\frac{4}{3} \left( k_i \cdot k_j + \gamma_i \gamma_j a - 2\gamma_i \beta_j \right)} \]  
(3.5)

Requiring independence on the UV cut off \( \epsilon \) yields  
\[ \beta_i - \gamma_i \beta_i + \frac{\gamma_i^2 a}{2} = 1 - \frac{k_i^2}{2} \equiv 1 - \Delta_i^{(0)} \]  
(3.6)

where \( \Delta_i^{(0)} \) is the bare conformal dimension, i.e. the conformal dimension with respect to the matter part only. With (2.9) and (3.6) follows  
\[ \Delta (B_i(z_i)) = 1 - \Delta_i^{(0)}. \]  
(3.7)

Thus our definition of gravitational dressing is equivalent to the definition used in [5]. The gravitational dressed dimensions are defined via the scaling behavior of (3.5),  
\[ \langle \prod_{i=1}^N : e^{k_i \cdot X(z_i)} : \rangle |_{A} \sim \prod_{i=1}^N A^{1-\Delta_i}. \]  
(3.8)
That leads to
\[ \Delta_i = 1 - \beta_i \]  
and is not unique due to the presence of the \( \gamma_i \). The most natural restriction is
\[ \gamma_i = 0, \]  
i.e. gravitational dressing is carried by gravitons only. It ensures also that the unit operator has scaling dimension zero. Restriction (3.10) leads to a trivial KPZ relation
\[ \Delta_i = \Delta_i^{(0)}. \]  
(3.11)
(3.11) coincides with the perturbative result [12].

4 \( d + 2 \) dimensional critical string

We consider the integrated N point functions
\[ A_{N}(k_1, \ldots, k_N) = \frac{1}{Vol(SL(2, R))} \int \prod_{i=1}^{N} d^2 z_i \langle \prod_{i=1}^{N} : e^{ik_i \cdot X(z_i)} : \rangle \]  
(4.1)
and rewrite them in a suggestive form, (we neglect pre factors),
\[ A_{N}(k_1, \ldots, k_N) = \frac{1}{Vol(SL(2, R))} \int \prod_{j=1}^{N} d^2 z_j \prod_{t=1}^{t} d^2 u_t \prod_{J=1}^{s} d^2 w_J \]
\[ \prod_{i<j} | z_i - z_j |^{2K_i \cdot K_j} \prod_{j, \alpha} | z_j - u_{\alpha} |^{2iK_j \cdot K} \]
\[ \prod_{j, J} | z_j - w_J |^{2iK_j \cdot K} \prod_{\alpha, J} | u_{\alpha} - w_J |^{-2K_j \cdot K}, \]  
(4.2)
where we have introduced the index conventions
\[ j = 1, \ldots, N; \ \alpha = 1, \ldots, t; \ \ J = 1, \ldots, s; \]
and the \( K_i \) are \( d + 2 \) dimensional vectors. Comparison of (4.2) with (3.5) provides
\[ K_j = \left( k_{j1}, \ldots, k_{jd}, \frac{i \beta_i}{\sqrt{a}}, \sqrt{a} \gamma_j - \frac{\beta_j}{\sqrt{a}} \right), \]
(4.3)
\[ i\tilde{K} = \left( 0, \ldots, 0, \frac{i}{\sqrt{a}} \right), \]
(4.4)
\[ i\tilde{K} = \left( 0, \ldots, 0, \frac{i}{\sqrt{a}}, -\frac{1}{\sqrt{a}} \right). \]
(4.5)
Equation (3.6) can be written in the form

\[-m_i^2 \equiv K_i^2 + n \cdot K_i = 2,\] (4.6)

with

\[n = (0, \ldots, 0, -2i\sqrt{a}, 0)\] (4.7)

(4.6) is the mass shell condition of [15]. The scalar product in (4.4) is Euclidean and the time like coordinate is the pure imaginary one. (The case \(a = 0\) is the ordinary 26 dimensional critical string and is not considered here). However, (4.6) could as well be written in the form

\[-\tilde{m}_i^2 \equiv (K_i + \frac{n}{2})^2 = 2 + \frac{n^2}{4} = 2 - a.\]

Then the ‘tachyon’ would become massless in \(0 + 2\) dimensions which is common for two dimensional critical strings [18]. In the following we will use the mass definition (4.6).

One can extract the poles of a single channel by splitting the integration intervals and considering the region where \(|z_i - z_j|\) is smaller than all other distances [19]. Than one expands the integrand in a Taylor series around \(|z_i - z_j| = 0\) and integrates out that relative distance. That leads to the following poles,

\[S_{kl} \equiv (K_k + K_l)^2 + n \cdot (K_k + K_l) = 2 - 2j\] (4.8)

\[S_{kI} \equiv (K_k + i\bar{K})^2 + n \cdot (K_k + i\bar{K}) = 2 - 2j\] (4.9)

\[S_{ko} \equiv (K_k + iK)^2 + n \cdot (K_k + iK) = 2 - 2j\] (4.10)

\[S_{I\alpha} \equiv (iK + i\bar{K})^2 + n \cdot (iK + i\bar{K}) = 2 - 2j\] (4.11)

where \(j\) is a non negative integer. From (4.8) we get the same mass spectrum as the author of [15]. Insertion of (4.4) and (4.5) provides

\[S_{kl} = 4 - \frac{4\beta_k}{a} + 2\gamma_k\] (4.12)

\[S_{ko} = 4 - 2\gamma_k\] (4.13)

\[S_{I\alpha} = 4 - \frac{4}{a}\] (4.14)

Due to the presence of background tachyons with fixed momenta we have leg poles (4.12) and (4.13). For

\[a = \frac{4}{2j + 2},\] (4.15)

\((j = \text{non negative integer})\), we expect divergent expressions because we are then exactly at poles of \((I, \alpha)\)-channels. These divergencies can be regularized by a cut off \(|u_\alpha - w_J| > \lambda\).

Now we want to consider the four point function in more detail. Because we have not succeeded in the continuation to real \(t\) and \(s\) until now we restrict ourself to the case \(a = 1\) (i.e. \(d = 12\)). Here we remark that in the super symmetric case one gets very similar
formulas with 2D super space integrals and \( a = (8 - d)/4 \). Then \( a = 1 \) corresponds to the physically interesting case \( d = 4 \). A detailed discussion of the super symmetric case will be given in [20]. Furthermore we set

\[
\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0, \tag{4.16}
\]

i.e. gravitational dressing is carried by gravitons only. Thus we have \( t = 1 \) and \( s = 1 - \sum \beta_i \) and do not need the continuation to real values of \( t \). We use M"obius invariance to set

\[
z_1 = 0, z_2 = 1, z_3 = \infty, z_4 = z \tag{4.17}
\]

and get

\[
A_4(k_1, k_2, k_3, k_4) = \int d^2z \int d^2w \int \prod_{\alpha=1}^8 d^2u_\alpha \left| z \right|^{2k_1 k_4} \left| 1 - z \right|^{2k_2 k_4} \left| z - w \right|^{-4\beta_4}
\]

\[
\left| w \right|^{-4\beta_1} \left| 1 - w \right|^{-4\beta_2} \prod_{\alpha=1}^8 \left| u_\alpha - w \right|^{-4}. \tag{4.18}
\]

The calculation of the \( z \)-integral is given in [21]. There the two dimensional integral is expressed in terms of contour integrals which provide the hypergeometric function. Before giving the result we introduce suitable Mandelstam variables,

\[
S = (k_3 + k_4)^2 + 2(\beta_3 + \beta_4)
\]

\[
T = (k_1 + k_4)^2 + 2(\beta_1 + \beta_4)
\]

\[
U = (k_2 + k_4)^2 + 2(\beta_2 + \beta_4). \tag{4.19}
\]

Adding the Mandelstam variables provides

\[
S + T + U = 8 + 4\beta_4 = -4m^2 + 4\beta_4, \tag{4.20}
\]

where \( m^2 = -2 \) is the tachyon mass. The sum (4.20) is not constant because background tachyons take part in scattering. With the help of

\[
K_5 = (0, \ldots, 0, i, 1) = i\vec{K} \tag{4.21}
\]

we define a further Mandelstam variable \( (\beta_5 = 1) \)

\[
V = (K_4 + K_5)^2 + 2(\beta_4 + \beta_5) \tag{4.22}
\]

and get

\[
S + T + U + V = 12 = -6m^2. \tag{4.23}
\]
We note that (4.16) implies that there is no interaction between external tachyons and the s background tachyons

\[ \left( \int d^2 z e^{2\sigma} \right)^s. \]

In terms of S, T, U, and V (4.17) becomes

\[ A_4 = \pi \int d^2 w \left\{ \frac{\Delta(\frac{1}{2}T - 1)\Delta(\frac{1}{2}V - 1)}{\Delta(\frac{1}{2}(T + V) - 2)} | w |^{T+V-4} | F(2 - \frac{1}{2}U, \frac{1}{2}T - 1, \frac{1}{2}(T + V) - 2; w) |^2 + \right. \]

\[ \left. \frac{\Delta(\frac{1}{2}S - 1)\Delta(\frac{1}{2}U - 1)}{\Delta(\frac{1}{2}(U + S) - 2)} | F(2 - \frac{1}{2}V, \frac{1}{2}S - 1, \frac{1}{2}(U + S) - 2; w) |^2 \right\} | w |^{-4\beta_1} | 1 - w |^{-4\beta_2} \int \prod_{\alpha=1}^s d^2 u_\alpha | u_\alpha - w |^{-4}. \]  

(4.24)

\( F \) is the hypergeometric function and

\[ \Delta(x) = \frac{\Gamma(x)}{\Gamma(1 - x)}. \]

In the S-channel for example we observe the poles

\[ S = 2 - 2j \]  

(4.25)

which confirms (4.8). In fact poles in V are leg poles because

\[ V = 4 - 4\beta_4. \]  

(4.26)

These leg poles occur because background tachyons with fixed momenta take part in scattering. The \( u \)-integrals are divergent because \( a = 1 \) is contained in (4.15). The divergence can be regularized by a cut off

\[ | u_\alpha - w | > \lambda \]

and renormalized via

\[ \left( \int d^2 z e^{2\sigma} \right)^s \to \left( \lambda^2 \int d^2 z e^{2\sigma} \right)^s. \]

Then one can solve the \( u \)-integrals. Expanding the \( w \)-integrand in a series around \( | w | = 0 \) and \( | w | = \infty \) one can convince himself that there are no additional poles in S, T, and U channels.

One could regard (4.24) also as an off shell 12 dimensional non critical string amplitude. That would be closer to the original approach of sections 1, 2, and 3. Then poles in scattering amplitudes occur when the intermediate particles are gravitationally dressed according to equation (3.6). Unfortunately we do not know how to get a mass spectrum in such a picture. However, we can guess a mass spectrum by consistency requirements.
Suppose we are given a ground state or ‘tachyon’ mass $m_0$. Then the on shell four point function should have a pole at

$$(k_3 + k_4)^2 = -m_0^2$$

(4.27)

Together with equation (4.25) that leads to the restriction

$$m_0^2 = -2j_0 - 2$$

(4.28)

with $j_0$ a non negative integer. The on shell value of $\beta_i$ is then

$$\beta_i = 1 + \frac{m_0^2}{2} = -j_0.$$}

Hence there are no additional divergences of the on shell amplitude due to leg poles. We obtain the full mass spectrum via (4.25)

$$2 - 2j = -M_j^2 + 4 + 2m_0^2$$

$$M_j^2 = 2j - 2 - 4j_0, \quad j = 0, 1, 2, \ldots$$

(4.29)

Now we require the ground state to be the lightest one,

$$M_j^2 \geq m_0^2, \quad \forall j.$$ 

(4.30)

That leads to

$$j_0 = 0$$

(4.31)

and we obtain the same mass spectrum as in 26 dimensional critical string theory.

### 5 Conclusions

We have calculated the N point tachyon amplitude in a model where the gravitational part is trivialized by the constraint of constant curvature. Reasonable assumptions provided the correct string susceptibility and a trivial KPZ relation.

Although we were not able to give a closed formula for arbitrary dimensions and external momenta we got an expression for the four point function where poles in different scattering channels are manifest. Interpreting the model as $d+2$ dimensional critical string we obtained, like the author of [15], the same mass spectrum as in 26 dimensional critical string theory. Furthermore we have obtained hints that also a $d$ dimensional non critical string picture provides the same mass spectrum.

The most serious open problem is the continuation to real values of $t$ and $s$. The upper barrier $d = 1$ of the model with induced gravity only is in the model considered here a lower barrier at $d = 0$ ($a = 2$). We have doubts whether one can use our calculation directly for $d = 0$. In the gravity part of our theory Möbius invariance is not manifest, e.g. the two point function is not zero for different conformal weights. We can hope that the zero mode integration of the matter part selects configurations where Möbius
invariance survives. But for $d = 0$ there is no matter part. Therefore we expect that the lower barrier, $d = 0$, should be considered as a limit $c_{\text{matter}} \rightarrow 0$. In order to be able to perform that limit one needs the continuation to real values of $t$ and $s$ urgently.

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**References**

[1] A.M. Polyakov, Phys. Lett. **B113**(1981)207

[2] A.M. Polyakov, Mod. Phys. Lett. **A2**(1987)899

[3] V.G. Knizhnik, A.M. Polyakov, A.A. Zamolodchikov, Mod. Phys. Lett. **A3**(1988)819

[4] F. David, Mod. Phys. Lett. **A3**(1988)1651

[5] J. Distler, H. Kawai, Nucl. Phys. **B321**(1989)509

[6] H. Dorn, H.-J. Otto, Phys. Lett. **B232**(1989)327; CERN-TH. 6285/91

[7] R. Jackiw, Quantum theory of gravity (ed. S. Christiensen, Adam Hilger, Bristol 1984); p. 403
   C. Teitelboim, Quantum theory of gravity; p. 327

[8] A.H. Chamseddine, D. Wyler, Nucl. Phys. **B340**(1990)595

[9] E. D’Hoker, D.H. Phong, Rev. Mod. Phys. **60**(1988)987

[10] I.M. Lichtzier, S.D. Odintsov, Mod. Phys. Lett. **A6**(1991)1953

[11] A.H. Chamseddine, Phys. Lett. **256B**(1991)379; **258B**(1991)97

[12] F.D. Mazzitelli, N. Mohammed, Phys. Lett. **B268**(1991)12

[13] S. Förste, Proceedings XXV Int. Symp. Ahrenschoop (Sept. 1991)

[14] S.D. Odintsov, I.L. Shapiro, Madrid preprint(1991), FTUAM-91-33

[15] A.H. Chamseddine, Nucl. Phys. **B368**(1992)98

[16] N. Mavromatos, J. Miramontes, Mod. Phys. Lett. **A4**(1989)1849; E. D’Hoker, P.S. Kurzepa, Mod. Phys. Lett. **A5**(1990)1411
[17] M. Goulian, M. Li, Phys. Rev. Lett. 66(1991)2051

[18] D. Kutasov: Some properties of (non)critical strings, PUPT-1277

[19] J.A. Shapiro, Phys. Lett. 33B(1970)361

[20] S. Förste, in preparation

[21] Vl. S. Dotsenko: Lectures on Conformal Field Theory (Kyoto Univ., Oct. 1986)