Foundation of The Two dimensional Quantum Theory of Gravity

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Abstract
The two dimensional substructure of general relativity and gravity, and the two dimensional geometry of quantum effect by black hole are disclosed. Then the canonical quantization of the two dimensional theory of gravity is performed. It is shown that the resulting uncertainty relations can explain black hole quantum effects. A quantum gravitational length is also derived which can clarify the origin of Planck length.
1. Introduction

In this report we will point out the main role played by the two dimensional substructures of the four dimensional Riemannian space time manifold, i.e. the curvature two form and the two dimensional subspace of motion or orbits, in the classical- as well as in the quantum gravity.

We show in the following that not only the gravitational effects are two dimensional effects, but also the ”four dimensional” theory of gravity has a two dimensional origin. Accordingly we will conjecture a two dimensional model for gravity which is defined on this subspace of motion. In other words we describe the motion or dynamics of a gravitational system just in the motion space or the dynamical manifold which is constituted from dynamical degrees of freedom of the system. Note that the space of motion is embeded in the space-time manifold, but is not identical with this. Thus we consider only the actual or dynamical degrees of freedom of the object under gravitation field, which is equal to the two dimensions of the surface where the motion of such an object takes place. If one of such two dynamical degrees of freedom is considered as time dependent, then one may speak of one space- and one time coordinate or degree of freedom. Nevertheless in accord with the phase space philosophy the time is not considered in our model as a new independent variable. Therefore in general the space of motion in our model is the surface of motion which is described like any surface by two variables, this surface is the dual object to the gravitational curvature which causes the gravitational motion under consideration (see also below).

We will show that, since all gravitational effects including the quantum effect by black holes [1] are two dimensional effects, i.e. they depend on maximally two independent variables, therefore a two dimensional theory is adequate to describe the gravity. Thus such a two dimensional theory has the advantage that it is quantizable, despite of the non-renormalizable four dimensional general relativistic theory (GRT) of gravity.

The main point about the geometrical description of gravity with respect to the curvature of space around a gravitational source, which becomes clarified later in detail, is the following:

In view of the fact that in accord with the general definition of curvature by Gauss, the total curvature or the surface integral of curvature is given as the excess of the sum of the angles of a triangle, on the surface, from 2\pi, i.e.: \( \phi = \delta \Phi = \phi - 2\pi = \int_{(surface)} R_{ij} dx^i \wedge dx^j \). Therefore since the surface and triangle are two dimensional concepts, hence the curvature of a manifold is always a two dimensional quantity,
irrespective of the dimension of that manifold. Thus the curvature tensor of any manifold has dimension $L^{-2}$. In this sense all effects of gravity, i.e. the effects of the curvature of the manifold around the gravitational source, are of two dimensional origin, irrespective of dimension of that manifold.

Note that it is the inherent and invariant two dimensional property, i.e. the $L^{-2}$ character of the curvature tensor $R_{[ij]}$, which results in the mentioned two dimensional features of gravity: Since the dimension of the tensor components of a curvature two-form remains always $L^{-2}$ independent of the dimensions of the manifold where it is defined. This means but that only two independent variables or components are needed to define such a curvature, thus also only two independent variables are needed to define the dual surface. Thus irrespective of dimensions of the underlying manifold where the curvature form and its action-invariant are defined, the resulting equations of motion for the curvature components are constituted so that these invariant character of two dimensionality $\sim L^{-2}$ remains unchanged. In other words in view of the fact that the action invariant respects the invariant properties of its constituents, i.e. of its Lagrangian content and of its integration manifold. Therefore the rest components of a curvature tensor on a higher dimensional manifold, which is involved in an action invariant on such a manifold, are related by the equations of motion in such a manner that only two independent components remain. This is the same what one confirmed by the two independent components of the metric field in the gravitational waves, although one started there from the four dimensional Einstein theory.

Thus in the same way that gravitational effects like the planetary motion take place on the two dimensional projective surfaces, also one needs to describe them only by two degrees of freedom on such surfaces, since these are described also by only two independent variables. Thus degrees of freedom of a dynamical system are equivalent to the independent variables of the system.

In other words to describe dynamical gravitational effects one needs to describe the motion orbits of the object under gravitational force. However in accord with the $L^{-2}$ dimensionality of the gravitational curvature or in accord with the $L^{-1}$ dimensionality of the gravitational potential, these orbits are always two dimensional curves like ellipse or circle which can be described by only two independent variables (see Kepler problem or perihelion shift, respectively). In this respect these two variables and the two degrees of freedom of the moving object under gravitational force are synonyms, since the geometrical concept of dimension is related with the dynamical concept of degrees of freedom in accord with the
geometrical interpretation of gravitational force by the curvature.

In this sense the two dimensional theory should be formulated on a two dimensional curved submanifold of the three dimensional space, where all the gravitational effects including the planetary motion, take place. Thus in accord with the geometrical equivalence between the gravitational force and the curvature which are both of dimension $L^{-2}$ in geometric units, the mentioned two dimensional curved submanifold of gravitational effects can be considered as the curved surface around the gravitational source. Accordingly, this curved surface ($\sim L^2$) and the gravitational curvature ($\sim L^{-2}$) can be considered as "dual" aspects of the same gravity. Thus the integral of the two dimensional curvature ($\sim L^{-2}$) over this two dimensional surface ($\sim L^2$) which is the topological invariant of the surface manifold, is the action invariant of our two dimensional model.

The paper is organized as follows: In the second section we disclose the fundamental two dimensional substructure of classical and quantum gravitational effects as well as of the four dimensional theory of gravitation. In sect. 3 we introduce the two dimensional classical and quantum theory of gravity. The forth section contains the conclusion on the classical limit and the quantum structure of space-time. In the appendix we apply the Hirzebruch-Riemann-Roch theorem to the mentioned two dimensional theory of gravity in order to demonstrate the relevance of its geometric quantization.

2. The two dimensional substructure of gravity

There are various classical- and quantum hints about the main role played by the underlying two dimensional spatial substructure (2D) of the space-time four manifold in the gravitational effects, in Einstein-Hilbert action and in the original approach to quantum effects by black holes (BH). However, the dominance of four dimensional relativistic point of view in classical physics and the hierarchy of local point of view in quantum physics, i.e. the hierarchy of differential equations, overrun these hints.

Note that all the following hints for the two dimensional background of gravity can be considered also as the advantages of a two dimensional model of gravity.

The main physical hint about the two dimensional character of gravity is the inherent and exclusive two dimensional structure of all measurable gravitational effects and the fact that only certain two body problems are exactly solvable. Thus such a two body problem can be reduced to a "single body" problem with two degrees of freedom or to the relative motion of a single body with the reduced mass.
Note that as the planetary motion in the Newtonian theory, also all ”general relativistic” gravitational
effects, i. e. the perihelion shift, the bending of light, the precession of gyroscope and the redshift, are
two dimensional effects, or a (1+1)-dimensional effect which is equivalent two a two dimensional one:
The first three are describable as the curvature or ”angle”-effects on the surface of motion of test body
in a gravitational field, where the angle which manifests any of these effects, i. e. $\Delta \phi \sim \frac{M}{r}$, is given by
the $surface$ integral of the curvature two form: $\phi \propto \int_{(surface)} \bar{R}$ \[4\]. The redshift is a (1+1)-dimensional
effect, since in this effect only the frequence ($\sim (T)^{-1}$) and the radial coordinate of Schwarzschild metric
($\sim r$) are relevant. Note that a two dimensional effect is an effect which is of order $1 \sim L^{-1}$ or $1 \sim L^{-2}$
in geometric units. Thus it can be described within a two dimensional system which is described by the
action $W_G = \phi \propto \int_{(surface)} \bar{R} = \oint_{(contour)} \Gamma$ in accord with Stokes theorem, where $\Gamma$ is the gravitational
connection form. Note also that, in view of $M \sim L$ dimension in general relativistic units, the
usual measurable value $\Delta \phi \propto \frac{M}{L} \sim \frac{L}{L}$ is in accord with $\phi \propto L \int_{(surface)} \bar{R}$, from which one obtains:
$\Delta \phi \propto \bar{R} \cdot \Delta A \sim L^{-2} \cdot L^2$, where $A$ is the area of the involved surface.
The second hint is already obvious from the structure of Einstein-Hilbert (E-H) action $\kappa^{-1} \int d^4x \sqrt{-g} \bar{R}$, where in view of $L^{-2}$ dimensionality of curvature scalar $\bar{R}$ and $L^4$ dimensionality of four volume,
($\kappa \sim G$) is forced to be of dimension $L^2(\sim l_P^2)$. Whereby it is just this circumstance which leads
to the non-renormalizability of quantized (E-H) action \[3\]. Thus the four dimensionality of volume is
retained here only with the help of $L^2$ dimensionality of $\kappa \sim G$ which prevents however a renormalizable
quantization of action. From this hint one can conclude (carefully) that a renormalizable quantization
is in some sense related with a $2D$ volume, since the $L^{-2}$ dimension of disturbing term compensates
just the $L^2$ part of four volume. Further note that the scalar $\bar{R}$ is constructed by mixed traces between
the tangential and curved indices of the curvature tensor. Therefore also this action is based on the
curvature two form which can be defined, as a two form, entirely on a two dimensional manifold.
The third hint for the two dimensionality of gravity comes from the theoretical analysis of the gravita-
tional waves. It is known that these waves, despite of their four dimensional background, are transversal
and possess only two degrees of freedom which gives the dynamical degrees of freedom of the metric field
\[6\]. In view of the fact that the linearization of theory, which enables one to derive these waves, does not
affect the number of degrees of freedom of the theory, also the assumed full ”non-linear” gravitational
waves or the dynamical gravitational metric field possess only two degrees of freedom. This is not only in accord with the related concept of square of length or area which is a two dimensional invariant, but it is also in accord with the number of degrees of freedom of the curvature form which can be considered as the second derivative of the Riemannian metric. Thus in view of the fact that the number of independent degrees of freedom of the metric field determines the number of independent components of this field and this depends on the dimension of the underlying manifold, hence the dimension of the underlying manifold of such a metric should be also two. This is the two dimensional submanifold of the four dimensional spac-time manifold which is involved in the gravitational dynamics that can be described either with respect to the metric tensor or with respect to the related curvature form.

The forth hint comes from the original approach where the BH entropy is given by

\[ S = (\text{constant}) \cdot \frac{1}{4} A_{(3D)} \]

with \( A \) the area of BH event horizon(s). Now the area of \( S^2 \) which is the surface of general topological 3D object, i.e. the sphere, is \( 4\pi r^2 \). One quarter of which is just the area of its cross section, a disc, which is the general topological 2D object. Therefore, the entropy \( S \), as a quantum theoretical quantity, should be considered as to be proportional to the area of the 2D submanifold and not to the area of a classical 3D object, i.e. it should be \( S := (\text{constant}) \cdot A_{(2D \subset 3D)} \) (see below). Thus the proportionality between entropy and area even in the four dimensional approach in view of the fact that the area is a two dimensional invariant, shows the fundamental role of 2D structure within the quantum structures on four manifolds.

The fifth hint for the two dimensional structure of the GRT of gravity and of black holes comes directly from the analysis of singularity in GRT which was lead to the concept of black holes: Although the BH is considered as the singularity of the four dimensional Schwarzschild solution of Einstein equations, nevertheless the singularity analysis shows that such a four dimensional solution is not suitable to determine the true singularities.

First note that the "four dimensional" Schwarzschild solution in the spherical coordinates is in fact a three dimensional solution, since the relation \( x^2 + y^2 + z^2 = r^2 \) in spherical coordinates reduces the number of independent spatial variables from three to two. Thus \( r \) is here not a new variable, but a function of \( (x, y, z) \) and the \( x^2 + y^2 + z^2 = r^2 \) is the equation of a spherical surface which can be described also by only two independent variables. Moreover in the so called extended solutions, i.e.
in Lemaitre, Eddington-Finkelstein and Kruskal metrics, one reduced these three variables by further linear combination to only two independent variables. Thus in the phase space description of Einstein equations, any spherical, i. e. any three dimensional solution is in fact a two dimensional solution, since the time is not a true independent variable in the phase space, but a parameter. In other words even the so called "four dimensional" solutions of Einstein equations are in fact two dimensional ones. This is not surprising, since in accord with the above discussion the variable part of the E-H action comes from a two dimensional curvature tensor ($L^{-2}$).

Thus it is known that just the extended two dimensional Kruskal metric is the metric which delivers the true singularity in $r = 0$, whereas the "four dimensional" Schwarzschild metric contain false singularity in $r = 2M$ which is related with coordinate effects. Note that this fact is a hint about the two dimensional background of BH solutions of Einstein equations. Again in view of the fundamental relation between the dimensions of metric and of the underlying Riemannian manifold, the underlying manifold with such a two dimensional metric is also of dimension two. Therefore the responsible theory of gravity for such a "two dimensional" singularity, which is a dynamical theory of the underlying manifold, should be two dimensional theory. In other words black hole as singularity is a two dimensional concept and in this sense it is also a concept of two dimensional theories. Thus in four dimensional theories with four dimensional metrics one needs a reduction to the two dimensional metric to come close to the true singularities. It is also important to mention that in view of the fact that singularities belong to the invariant properties of a theory, therefore if one will replace a theory, then the new theory should contain the same singularities as the old one. Hence even in this sense the two dimensional theory of gravity is the best candidate to replace the four dimensional theory.

As the last classical hint in favour of a two dimensional approach let us mention the constraint disadvantage of four dimensional theories, e. g. the GRT. Such constraints arise from the superficiality of the time component of dynamical variable of the theory, which however can not be a dynamical component in view of the canonical or symplectic structure of phase space of theory. Note that the time is not a true variable in the phase space of a physical system. Thus in the so called extended phase space of a one dimensional motion, where momentum and position are considered as functions of time parameter, time plays the role of "canonical conjugate" parameter to the Hamiltonian which is itself a function
of momentum and position variables. Hence we have to do even in the extended phase space with two interdependent pair of independent canonical conjugate variables, i.e. momentum, position or time and Hamiltonian, from which only one pair are independent physical variables. Related with this disadvantage is the problem of a covariant definition of energy-momentum complex in the four dimensional theories, which becomes more critical with respect to the quantization. Thus in the GRT one can not define a covariantly conserved quantity for the energy of gravitational field.

Note however that the discussed two dimensional substructure of the four dimensional space-time does not affect the general relativity between space and time "coordinates”. Thus all measurable consequences of such a relativity can be described with respect to one space and the time coordinates, which is equivalent to a two dimensional effect in view of the theory of relativity where $T \sim L$.

These hints disclose the two dimensional substructure of the four dimensional theory of gravity. This is done in the sense that we showed that although Einstein theory and its solutions for gravity are formulated in a four dimensional fashion, however since they are based on the two dimensional concept of curvature, therefore they possess only two truly independent dimensions or two degrees of freedom. A fact which is in accord with the two dimensional nature of gravitational effects. The rest two dimensions of this four dimensional theory are frozen in the dimensional coupling constant of the theory $\kappa$, so that they play no dynamical role.

Now we discuss a quantum hint which comes from the quantum effect by black holes. A geometrical analysis of the original approach to this effect shows that also the underlying model is a two dimensional model with respect to the quantum effect of particle creation by BH. Since the responsible quantum effect in BH is a "Aharonov-Bohm" like gravitational effect, whereby the gravitational curvature field plays the same role as the magnetic field in Aharonov-Bohm effect. Note that even the Aharonov-Bohm effect is a two dimensional effect in the sense that the electron moves on a two dimensional surface which contains a cross section of the solenoid. Thus the phase change of electron is given by a contour integral of the electromagnetic potential which surround this surface. Moreover, it is a typical global quantum phase effect which rises in the original approach just by a global comparison between quantum operations in two spatially separated regions which are asymptotically flat. Thus this asymptotic flatness of outside region of BH, with respect to the curvature of space-time, is comparable with the vanishing of
magnetic field outside of solenoid in Bohm-Aharonov effect. Hence from quantum mechanical point of view the Hermitian scalar quantum field $\Phi$ receives a phase surplus, i.e. a change of frequency, which is caused in the original approach by the comparison of anihilation operations in the two asymptotically flat regions in view of the imposibility of an invariant definition of positive and negative frequencies under the influence of the curved BH metric [1]. Then this global comparison can be considered, as we will show, as a looping around the strong gravitational field of BH, where the looping takes place in view of comparision of "operations" in the two asymptotically flat regions around the BH in the following way:

First recall that in the original approach where $\Phi = \sum_i \{ f_i a_i + \tilde{f}_i a_i^\dagger \}$ the operators $a_{1i}$ and the quantum state $|0_1>$ are defined in the asymptotically flat region (1), whereas the operators $a_{3i}$ and $|0_3>$ are defined in the asymptotically flat region (3) with the BH region (2) between them. Therefore, the operation $a_{3i}|0_1>$ can be considered as if $|0_1>$ is moved to the region (3), where $a_{3i}$ is defined, to experience this operation and the subsequent comparison of $a_{3i}|0_1>$ with $a_{1i}|0_1>$ which is defined in region (1) can be considered as if $|0_1>$ is moved back from region (3) to its original region (1). The quantum effect rises then by this global comparison between operations of anihilation operators on vacuum state in two distant regions (1) and (3). In this sense one can consider the above comparision of operations on the vacuum state $|0_1>$ as a looping of state $|0_1>$ around the strong gravitional region (2) from region (1) to region (3) and back to the region (1). Obviously one can consider instead of that also the loop of operator $a_{3i}$ from region (3) to region (1) for operation $a_{3i}|0_1>$ and back to the region (3) where the quantum effect rises by comparison of $a_{3i}|0_1>$ $\neq 0$ with $a_{3i}|0_3> = 0$. Hereby the quantized flux of gravitional curvature field in region (2) plays the similar role as the role played by the flux of magnetic field in Bohm-Aharonov effect. Therfore even the four dimensional original model to describe quantum effects by BH is properly a two dimensional model on the loop surface of vacuum state of $\Phi$ which is similar to the two dimensional Bohm-Aharonov model on the motion surface of electrons around the cross section of solenoid. In this sense the discussed quantum effect by BH is again a "phase" or "angle" effect which is given by the surface integral of the curvature two form.

Thus one can conclude that not only the classical gravitational effects but also the quantum gravitational effect by black hole are two dimensional effects which are caused by the gravitational curvature tensor in accord with the action $W_G = \phi \propto \int (\text{surface}) R_{mn}dx^m \wedge dx^n = \oint (\text{contour}) \Gamma_m dx^m$ ; $m, n = 1, 2$. Therefore we
conjecture in next section that the gravitational action functional should be given, up to a *dimensionless* coupling constant $\tilde{G}$, by this action. Such an action describes the gravitational flux through the surface which is bounded by the contour orbit of a moving object under the gravitational influence. This is in the same manner as the magnetic flux perceived by the electron moving around the magnetic field.

Note that there is an intrinsic merit of the two dimensional model with respect to the four dimensional theory, since the length or distance $x_m$ is a canonical variable in the two dimensional model in accord with the fact that the canonical conjugate variables of this action are $\{\Gamma_m, x^m\}$. Therefore the two dimensional quantized model possess a quantum of length which is given in accord with the quantization of the action in canonical manner: $W_G = \tilde{G} \int_{(surface)} R_{mn} dx^m \wedge dx^n = \oint_{(contour)} \Gamma_m dx^m = Nh; \; N \in \mathbb{Z}$ (see next section). Such a quantum of length is given in our model by: $l_G^2 = \frac{\hbar}{G R}$ similar to the magnetic length in magnetic quantization. Here $R := \epsilon_{mn} R_{mn}$ is the constant gravitational curvature around the source, since in two dimensional case the curvature is constant. Thus we can define and speak of a quantum of length like the Planck length in the two dimensional model. Whereas since the length is no canonical conjugate variables in the four dimensional GRT and GRT is not quantizable, one can not define a quantum of length like Planck length in GRT or even in any hypothetical quantization of GRT.

Summarizing the above discussions the advantages of two dimensional model with respect to the four dimensional GRT of gravity are manifolds: Among them: The two dimensional model avoids the coordinate effects which appear in the four dimensional theory with respect to the singularities. Despite of constraints in the four dimensional theory which results in ambiguities in quantization of this theory, the two dimensional model possess no constraint. Thus the two dimensional model is canonically quantizable, whereas the four dimensional theory is not quantizable in view of its dimensional constant $\kappa$, its metrical structure and constraints. The two dimensional model is more appropriate for gravity in view of the two dimensional structure of gravitational effects including quantum effects by black holes. Thus the two dimensional model is formulated directly in the motion space where the gravitational effects take place.

3. The two dimensional classical and quantum theory of gravity

The main reason for the emergence of quantum effects in two dimensions is that quantum effects are invariant (global) effects which are based principielly on the existence of a quantum structure on the phase
space of physical system which itself is based on the existence of a flat complex line bundle ($\sim U(1)_{flat}$) over the phase space. Thus the reason that such a two dimensional quantization can be responsible for quantum effects, which are originally formulated for a four dimensional theory, is that the quantization of a system takes place on the phase space of system and not on the space-time where the system is usually defined. In other words the number of dimensions of space-time has no influence on the possibility that a system becomes a quantum system, but only on the structure of its quantization \[1\]. Whereas the possibility that a system becomes a quantum system depends on the value of its action $W$, i. e. if $W \sim \hbar$ \[2\].

Moreover the mentioned flat principal $U(1)$ bundle is closely related with the symplectic structure of phase space of system: $\omega := d\pi_m \wedge dq^m$ \[3\]. Therefore we give first the symplectic structure of the two dimensional model:

The curvature two form $\bar{R} = R_{mn}dx^m \wedge dx^n$ admits a symplectic structure on the two dimensional surface $M$, i. e. $\bar{R} \sim \omega = d\pi_m \wedge dq^m$, in view of its closedness $d\bar{R} = 0$ and its non-degeneracy $R_{mn}^{(surface)} := constant \neq 0$. Hence $q^m \sim x^m$ and $\pi_m \sim \Gamma_m = R_{mn} \cdot x^n$. In this sense the two dimensional classical action of our two dimensional model for gravity: $W_G = \oint_{(contour)} \Gamma_m dx^m = \int_{(surface)} R_{mn}dx^m \wedge dx^n$ is equivalent to the symplectic action function $\int_{\text{phase space}} d\pi_m \wedge dq^m$ on the phase space of this two dimensional gravitational system with its canonical conjugate variables $\{R_{mn}, x^m\}$. This is again equivalent by $\int_{\text{phase space}} d\pi_m \wedge dq^m = \oint_{\text{phase space}} \pi_m dq^m$ to the integral $\oint_{\text{phase space}} \pi_m dq^m$ on the same phase space with the equivalent canonical conjugate variables $\{\Gamma_m, x^m\}$ (see also below).

From now on we consider the curved surface of integration $(M; \bar{R})$ as a symplectic manifold with respect to the curvature structure $\bar{R}$. Hence in this sense the symplectic manifold $(M; \bar{R})$ can be considered as the phase space of two dimensional gravitational system.

This identification of symplectic structure on $(M; \bar{R})$ is in principle enough to consider a canonical quantization on $(M; \bar{R})$ by postulating $W_G = Nh$ as usual. Thus for physically reasonable situations, e. g. for strong gravitational curvature where $W_G \sim \hbar$, it can be expected that we meet the quantization of such a symplectic structure.

One obtains the two dimensional equations of motion for $\Gamma_m$ from the variation of this action with respect to the variation of $x^n$. Here, i. e. on the surface, $\Gamma_m$ depends also on $x^n$ so that one should consider
the variation of $W_G$ only with respect to the variation of $x^n$ whereby $dx^m = \frac{\partial x^m}{\partial x^n} dx^n$. Then the Euler-Lagrange equation $\frac{\partial L}{\partial x^m} = \frac{\partial}{\partial x^m} \frac{\partial L}{\partial \dot{x}^m}$ results in the equations of motion $\partial_n \partial^m \Gamma_m = (d^l d + dd^l) \Gamma = 0$ which is the usual Laplace equation of motion for $\Gamma_m$ in two dimensions \[13\]. Note also that in the two dimensional case under consideration, where $\bar{R}$ is a constant two form, the equations $d\bar{R} = 0$ and $d^l \bar{R} = 0$ which are equivalent with the above equations of motion \[13\], are identities. In this sense the equations of motion in the two dimensional case are purely geometrical statements.

Note that, in view of $W_G = \int_{\text{contour}} \Gamma_m \dot{x}^m dt$, the canonical conjugate variables of the phase space of this two dimensional gravitational action are given in accord with the Legendre formula $\frac{\partial L}{\partial \dot{x}^m}$ by: $\{ \Gamma_m, x^m \}$ variables. Hence we can postulate the canonical quantization of the related system directly by $W_G = \hbar N$. Nevertheless we show in the appendix that such a canonical quantization is even in accord with the general structures of geometric quantization.

To perform the canonical quantization of two dimensional gravitational system recall that the canonical quantization of action $W := \int_{\text{phase space}} \omega = \int_{\text{phase space}} d\pi_a \wedge dq^a$ can be described by the equivalent postulates: $W = \hbar N$, $N \in \mathbb{Z}$ or $[\hat{\pi}_a, \hat{q}^b] = -i\hbar \delta^b_a$. Therefore, in view of the fact that the symplectic manifold $(M; \bar{R})$ plays the role of phase space of our two dimensional gravitational system, one can expect that it can be canonically quantized, if the value of gravitational action is of order $\hbar$ \[12\]. In this case the two dimensional gravitational system can be considered as a quantum system on the compact orientable two dimensional Riemannian manifold $M$ without boundary and its quantization can be postulated by

$$W^Q_G = \int_{\text{surface}} R_{mn} dx^m \wedge dx^n = \oint_{\text{contour}} \Gamma_m dx^m = \hbar N \quad (1)$$

Note also that we have to do here with a non-simply connected region which contains of inside curved region of BH (surface) $M$ and the outside asymptotically flat (contour) region, where curvature $\bar{R}$ and connection $\Gamma$ have different values. Thus $R_{mn}^{(\text{surface})} = \text{constant}$ and $R_{mn}^{(\text{contour})} = 0$,

whereas $\Gamma_m^{(\text{surface})} = R_{mn}^{(\text{surface})} \cdot x^n$ and $\Gamma_m^{(\text{contour})} = \Gamma_m^{(\text{flat})}$ with $d\Gamma_m^{(\text{flat})} \equiv 0$. Note that although $\Gamma_m^{(\text{contour})} = \Gamma_m^{(\text{flat})}$ can be locally gauged away, nevertheless $\oint_{\text{contour}} \Gamma_m dx^m \neq 0$ is an invariant of the system which is equal to the value of action functional of system $W_G = \int_{\text{surface}} R_{mn} dx^m \wedge dx^n$ \[14\].
Therefore the canonical quantization of this two dimensional gravitational action can be performed either on the phase space of contour region with canonical conjugate variables \( \{ \Gamma_m, x_m \} \) in accord with \([\hat{\Gamma}_m, \hat{x}_m] = -i\hbar \) or it can be performed on the equivalent phase space of the surface region with canonical conjugate variables \( \{ R_{mn}, x_m \} \) in accord with \( R[\hat{x}_m, \hat{x}_n] = -i\hbar \epsilon_{mn} \). The equivalent uncertainty relations are then given by, respectively:

\[
\Delta \Gamma_m \cdot \Delta x_m \geq \hbar , \quad R \cdot \Delta A = R \cdot |\epsilon_{mn}| \Delta x_m \cdot \Delta x_n \geq \hbar ,
\]

where \( \Delta A \) is the area uncertainty.

Thus the quantum commutator postulate \([\hat{\Gamma}_m, \hat{x}_m] = -i\hbar \) and the equivalent quantization

\[
\oint_{\text{contour}} \Gamma_m dx^m = N\hbar
\]

are comparable, respectively, with the canonical quantization postulates:

\[
\oint_{\text{phase space}} \pi_m dq^m = N\hbar.
\]

Moreover the operators \( \hat{\Gamma}_m \) and \( \hat{x}_m \) are given in the \( \Psi_G(x_m, t) \)- or \( \Psi_G(\Gamma_m, t) \) representation of the wave function of quantized gravitational system, respectively, by \( \{ \hat{\Gamma}_m := -i\hbar \frac{\partial}{\partial x_m} \text{ and } \hat{x}_m := x_m \} \) or by \( \{ \hat{\Gamma}_m := \Gamma_m \text{ and } \hat{x}_m := i\hbar \frac{\partial}{\partial \Gamma_m} \} \). Thus in both representations the commutator postulate:

\[
[\hat{\Gamma}_m, \hat{x}_n] = -i\hbar \delta_{mn} ,
\]

is fulfilled.

However the relative weakness \( 10^{-40} \) of gravitational interaction with respect to electromagnetic interaction sets limits on the values of curvature and area of a BH in a quantum state. Thus in view of such a relation a dimensionless gravitational coupling constant \( \tilde{G} \) can be introduced by \( W_G := \tilde{G} \oint_{\text{surface}} R_{mn} dx^m \wedge dx^n = \tilde{G} \oint_{\text{contour}} \Gamma_m dx^m = N\hbar \) in analogy with the electric charge \( e \). Obviously such a coupling constant should match with the Newtonian constant in the classical limit of our quantized gravitational system. Nevertheless, if not necessary, we set in the rest \( \tilde{G} = 1 \).

Analysing the geometrical structure of this two dimensional quantum gravity in accord with

\( \Delta \Gamma_m \cdot \Delta x_m \geq \hbar \) it becomes obvious that in view of \( \Delta x_m > 0 \) the two dimensional quantum gravitational system or the quantized curved surface \((M, \tilde{R})_Q\) possesses no one dimensional boundary in the classical sense, i. e. \( \partial(M, \tilde{R})_Q = \emptyset \). Since the contour region of \( M \) has in the quantized case a width of
\[ \Delta x > 0 \text{ which can not be undercut under quantum conditions. Thus the quantum contour area with the undetermined width } \Delta x \text{ can be considered as belonging not to the BH but to the outside region of BH.} \]

Furthermore, in view of uncertainty relation \( R \cdot \Delta A \geq \hbar > 0 \) the area uncertainty \( \Delta A \) is always positive and the BH area is increasing. Thus the entropy uncertainty of BH system which is related with gravitional quantum effects is given by \( \Delta S_G \sim \Delta W_G = R \cdot \Delta A \geq \hbar > 0 \). Hence the entropy of BH system increases permanently as like as its area in accord with quantum gravitional processes.

Moreover, in analogy with the two dimensional quantization of electromagnetic systems (see the appendix), one should expect that in two dimensional quantum gravitational case there exists a quantum of length \( (\Delta x_m)_{\text{minimum}} := l_G \), which is defined in analogy with the magnetic length \( (\sim l_B^2 := \frac{\hbar}{eB}) \) by:

\[
\frac{1}{2} \frac{l_G^2}{l_G^2} := \frac{\hbar}{G R} \tag{4}
\]

It is the most minimal length which is quantum mechanically obtainable under the action of constant curvature \( \bar{R} \). In other words one can derive a quantum gravitational length \( l_G \) from the obtained uncertainty relation by the "uncertainty equation": \( R \cdot l_G^2 = \hbar \) or \( \bar{G} R \cdot l_G^2 = \hbar \), since \( \Delta A \geq l_G^2 \). In this sense if a strong gravitational source is in a quantum state, then all length and areas around it can be quantized in the units \( l_B \) and \( l_B^2 \).

Therefore the quantization of action \( W_G := \tilde{G} \int_M R_{mn} dx^m \wedge dx^n = Nh \) in view of constance of \( R_{mn} \) is due to the quantization of area \( A_{BH} = N \cdot 2\pi l_G^2 \). Thus the quantum entropy should be related also with the quantization of area by \( S_G \sim W_G = \tilde{G} R \cdot N \cdot 2\pi l_G^2 \). Thus it is a major merit of the two dimensional model that by this model one can quantize lengths and areas which is at the heart of a quantization of geometrical gravity.

It is also possible to relates this gravitational length with the Planck length \( l_{Pl}^2 := \frac{\hbar G}{c^3} \cdot \tilde{G} = \frac{c^3}{G R} \). In this sense the canonically defined quantum of length should replace the Planck length. Note however that only, since in the two dimensional model \( \Gamma \) and \( x_m \sim (\text{length}) \) are the canonical conjugate variables, therefore the length is quantized by the canonical quantization of the model. Whereas in the four dimensional GRT the length is no canonical variable and therefore the length can not be quantized in GRT. Therefore to speak of a quantum of length or Planck length even in a hypothetically quantized GRT is an assumption without any quantum theoretical foundation. Thus the canonical quantizability of two
dimensional model, which results in the quantization of length and areas, confirms the consistency of this model.

As it is mentioned above in view of the uncertainty relations there are amount of momentum $\Delta \Gamma \geq \bar{\hbar}$ or $\Delta \Gamma_m \geq \bar{\hbar} \frac{c}{G l_G}$ which can be considered as radiated, i. e. as belonging to the region outside of BH. This particle creation is related with the indeterminacy of contour area as it is discussed above. Of course one can relate this particle creation also with the energy time uncertainty relation which is equivalent to momentum position uncertainty relation or to the area uncertainty relation as discussed above.

To compare our time independent approach with the original one note that as in any time dependent system we can also use the extended phase space $\mathbb{R}$ where the phase space variables, e. g. $\{\Gamma_m, x_m\}$ depend on time parameter. In accord with the extended canonical action: $\tilde{W} = \oint \pi_m dq^m - \int H dt$ the extended two dimensional gravitational action of BH is rewritten by: $\tilde{W}_G = \oint \Gamma_m dx^m - \int H_G dt$ where $H_G$ is the Hamiltonian of the two dimensional BH system. Hence in accord with the canonical quantization $\tilde{W} = Nh$ where the uncertainty relations: $\Delta \pi_m \cdot \Delta q_m \sim \Delta E \cdot \Delta t \geq \bar{\hbar}$ rise, then also in the canonical quantized gravitational system $\tilde{W}_G = Nh$ the following uncertainty relations should rise: $\Delta \Gamma_m \cdot \Delta x_m \sim \Delta E_G \cdot \Delta t \geq \bar{\hbar}$. In this sense the "free" amount of gravitational energy $\Delta E_G \geq \bar{\hbar} \frac{c}{N}$ can be transformed within the time $\Delta t$ into the pair production energy. Recall however that the equivalent connection ($\sim$ momentum)- position uncertainty, which also can be responsible for the pair production by BH, was already derived by the time independent canonical quantization of gravitational system.

Furthermore the following temperature uncertainty $\Delta T$ can be calculated from the relation between energy and entropy by $\Delta E_G = \bar{\hbar} \cdot \Delta S \cdot \Delta T$ where $\Delta S = (\hbar)^{-1} \cdot \Delta W_G$. In this manner the increase of entropy and area of BH are related together and with the particle production in accord with $\Delta \Gamma_m \cdot \Delta x_m \sim \Delta E_G \cdot \Delta t \sim \Delta S \cdot \Delta T > 0$. Therefore, the thermodynamics of BH can be obtained also from the two dimensional approach. For example to get a zero temperature in the quantum mechanical sense, i. e. to arrive $\Delta T = 0$ one needs infinite amount of operations, i. e. infinite amount of $\Delta S$. Recall that, as it is mentioned above, $\Delta S \sim \Delta A > 0$ are always positive and $\Delta S \sim R_{(constant)} \cdot \Delta A$. Moreover since $\Delta A_{(minimum)} = l^2_G \sim l^2_{Pl}$ and the area is quantized in units of $l^2_G \sim l^2_{Pl}$, hence in view of $\Delta S_G \sim R \cdot \Delta A$ and $S_G \sim N \cdot \Delta S_G$ the entropy can be considered as given by the quantum number: $S_G \sim N$. 

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Thus all quantum aspects of BH gravitations can be deduced also from this type of two dimensional quantum gravities which can be formulated also as a time dependent theory.

Furthermore for time dependent variables \( \{ \Gamma_m(t), x_m(t) \} \) of an extended phase space, one obtains also a conventional equation of motion for a test particle moving in a gravitational field:

Considering the action:

\[
S_G = \frac{1}{2} \left( \oint P_m dx^m + \tilde{G} \oint \Gamma_m dx^m \right),
\]

for such a test particle with momentum and position coordinates \( P_m \) and \( x_m \), the equations of motion should be given by:

\[
\dot{P}_m = -\tilde{G}(\dot{\Gamma}_m + \dot{x}_n \cdot R_{mn}),
\]

which is of Lorentz force type.

Note that the canonical quantization of action (6), which posses the canonical conjugate variables: \( \{ \frac{1}{2}(P_m + \tilde{G} \Gamma_m), x^m \} \), results beyond the test particle commutator \( [\hat{P}_m, \hat{x}_m] = -i\hbar \), also in the above obtained commutator (4) with a constant factor \( \tilde{G} \). Thus one can obtain the above quantization of (1) or (2) also from the quantization of (6).

Note also that there is a relation to a possible \((2 + 1)\) dimensional Chern-Simons theory of gravitation which can be given by the action \( \int (2+1) \Gamma \wedge \tilde{R} \). Since in our model, in accord with \( \Gamma_m = R_{mn} \cdot x^n \), the commutators \( R \cdot \epsilon_{mn}[\hat{x}_m, \hat{x}_n] = [\hat{\Gamma}_m, \hat{x}_m] = -i\hbar \) are equivalent to the commutator \( [\hat{\Gamma}_m, \hat{\Gamma}_n] = -i\hbar R \cdot \epsilon_{mn} \) which is the commutator postulate for the quantization of the gauged Chern-Simons functional \( \int (2+1) \Gamma_m d\Gamma_n \epsilon_{mn} \) in the \( \Gamma_0 = 0 \) gauge. Thus our gauge free two dimensional quantum gravity model is related with a gauged \((2 + 1)\) dimensional quantized Chern-Simons gravity. Note however that in our model the curvature is in view of the two dimensionality a constant one, whereas in the usual Chern-Simons model the curvature is constrained to be zero.

Conclusion: The classical limit and the quantum structure of "space-time"
The conclusion is that in view of our discussion of classical gravitational effects and of the two dimensional structure of original approach to quantum effect by BH, all these effects can be described without lose of generality as two dimensional effects. Thus, just in view of general relativistic relation between time and space "variables" in the extended configuration space any (1+1) dimensional effect is equivalent to a two dimensional effect. Hence in this sense the classical limit of our two dimensional model is straightforward, since all classical gravitational effects including the planetary motion can be considered as two dimensional effects.

In the same manner the flat space-time appears as the large scale limit where the curvature vanishes. Since by vanishing of curvature and connection, the motion of test particle which is given by (7) becomes rectilinear. Thus even the quantum effect by BH can be well described with the above canonically quantized two dimensional model.

Furthermore it seems that the event horizon of BH is comparable with the contour region in our approach where the integral \[ \oint_{\Gamma} \] takes place and which has in quantized case \[ \oint_{\Gamma} dx^m = Nh \] a width of \[ \Delta x_m \geq l_G > 0 \]. The effect of strong curvature \( R \) is to bind particles to move in this contour region as like as in a potential well which is caused by the strong field of curvature. In quantum gravitational case one can expect that the absorbed particles by BH gravitation are bound in the contour region of BH flowing permanently in this region so that they can not scape from this region as long as the gravitational field is stronger than possible repulsive fields acting on these particles.

As a last remark let us mention that this model has consequences for the possible quantum structure of "space-time" which will be discussed in more details elsewhere. Nevertheless the first lesson from this model for quantum spaces is that such quantum structures are possible in two spatial dimensions. In other words the position coordinates of a particle moving in a very strong gravitational curvature have to be considered in a quantum theory, in accord with the above discussion, as quantum operators which fulfil the commutator postulate: \[ R \cdot \epsilon_{mn}[\hat{x}_m, \hat{x}_n] = -i\hbar \]. It is in view of the cyclotron motion of particles in a curvature field, which is neccessary for the periodic quantization, or in view of the two dimensionality of the curvature field (\( \sim L^{-2} \)), that as usual only the surface coordinates of the particle becomes quantized. Nevertheless in view of the fact that the time is not a true variable in the phase space which is the space of quantization, the time "coordinate" of a particle can not be quantized directly. On the other hand, if the
position and momentum coordinates in the phase space of a system which should be quantized become functions of a time parameter, then it is possible to define a Hamiltonian as a function of the phase space variables and to quantize the energy integral of the system or equivalently to obtain uncertainty relations for the product of time and energy. This is an indirect canonical quantization of time which requires however, in principle, to introduce a quantum time ”coordinate” operator of the moving particle which should be in general proportional to an energy derivation in the energy representation of the wave function of the system of a single particle in the gravitational curvature: $\Psi \propto \exp(i \int \bar{R})$.

Recall that although there are various models of quantum space-times, but none of them gives a complete canonical quantization of space-time with respect to a possible phase space- or action functional quantization \[16\]. Thus any correct quantum commutator postulate of space-time variable operators or any uncertainty relation for them should be equivalent to the action functional quantization. Moreover, in view of the dimensional discussion at the beginning of this paper and in accord with the mentioned two dimensional structure within the original four dimensional approach, it seems that any quantum theory of space-time or of quantum space-time effects should be equivalent to some two dimensional quantum structure.

The main lesson is that in accord with $\hbar \sim L^0$ in geometrical units: In view of the restriction of all known physical quantities to those up to two dimensional ones, e. g.: \{invariants $\sim L^0$ : (action, charge)\}, \{connections $\sim L^{-1}$ : (momentums, energies, potentials)\} and \{curvatures $\sim L^{-2}$ : (forces, field strengths)\}. A reasonable quantum relation such as the quantum action postulate, the quantum commutator postulate or the uncertainty relation should be dimensionless within this two dimensional structure.
Appendix

First note that any 2-form which is defined on a two dimensional manifold like \( M \) is by definition a closed 2-form, since there is no 3-form on \( M \). In this sense all such 2-forms may represent spatially constant field strengths. Note also that \( R_{mn} \) can be considered as a constant almost complex structure on the two dimensional manifold under consideration. Nevertheless this constance of almost complex structure results then in vanishing of Nijenhuis tensor \([17]\) on the considered two dimensional (surface) manifold, so that \((M; \bar{R})\) admit a complex structure which simulates a \( U(1) \) bundle over the real surface manifold. In other words one can consider such a surface either as a real manifold with a \( U(1) \) bundle structure over it, or one may consider it as a complex manifold with a constant almost complex structure.

As a reasonable basis for the geometric quantization of our system, which defines a geometric quantization by a polarization of the underlying symplectic manifold of phase space \([8]\), our arguments are as follows: The reason that a ”spatial” quantization on \( M \) is possible is that in accord with Hirzebruch-Riemann-Roch theorem on a compact complex manifold without boundary \( M^C \), the Euler characteristic is given by \( \chi := \int_{M^C} \bar{R}(T M) = \int_{M^C} ch_1(T M^+) \) where \( ch_1(F) = c_1(F) = (-2\pi i)^{-1} F \). Here we consider only the case where \( M \) is the above discussed two dimensional symplectic manifold and \( F \) is a \( U(1) \) valued Yang-Mills curvaure. Thus the surface integral of curvature two form on the complexified manifold \( M \) is equal to the integral of first Chern class of a flat \( U(1) \) bundle over the complexified \( M \), since \( ch_1(T M^+) = ch_1(T M; U(1)_{flat}) \). Note that the geometric quantization of this system requires by the integrality condition that just the integral of \( ch_1(T M^+) \) on \( M \) should be an integer multiple of Planck constant.

Recall also that the polarization, which is required by the geometric quantization \([8]\), is here due to the holomorphic polarization on \( TM \) which is given by considering the Chern class on \( TM^+ \).

Moreover the required complexness of the underlying manifold by the Hirzebruch-Riemann-Roch theorem is given here by the above mentioned property that the two dimensional manifold \( M \) is a real manifold with vanishing Nijenhuis tensor. Hence it admits a global complex structure which simulates the global \( U(1) \) bundle structure which is required to define the first Chern class: \( c_1(F) \). Therefore all condition which are needed for a geometric quantization can be fulfilled in the above discussed case of \((M; \bar{R})\).
In other words, in view of the fact that Euler characteristic on a two dimensional orientable manifold without boundary $M$ is always given by $\chi := \int_M \bar{R}(TM)$ and in accord with the above theorem that $\chi := \int_M \bar{R}(TM) = \int_M ch_1(TM; U(1)_{flat})$. Then the quantization which is achieved by the integrality of first Chern class of a flat $U(1)$ bundle over this two dimensional manifold, i.e. by $\int_M ch_1(TM; U(1)_{flat}) = Nh$, can be expressed by the integrality of the equivalent Euler class, i.e. by $\int_M \bar{R}(TM) = \int_{\text{surface}} R_{mn} dx^m \wedge dx^n = Nh$. Also since the curvature tensor $R_{mn}^{(\text{surface})} = \frac{1}{2} (\partial_m \Gamma_n - \partial_n \Gamma_m)$ on the surface $(M; \bar{R})$ can be considered as the $U(1)$ curvature, i.e. as the field strength $F_{mn} = \frac{1}{2} (\partial_m A_n - \partial_n A_m)$ on the simulated $U(1)$ bundle over $M$. This underlines the application of the above introduced theorem in our case with respect to the relation between the Euler class ($\sim \bar{R} = R_{mn} dx^m \wedge dx^n$) and Chern class ($\sim F = F_{mn} dx^m \wedge dx^n$).

Thus for relevant physical conditions where the two dimensional gravitational $W_G$ system has to be considered as a quantum system, the canonical quantization of this system can be described in accord with the geometric quantization by the quantum postulate $W_G = \int_M \bar{R} = Nh$.

A physical example of such a quantization on a two dimensional configuration space is the cyclotron motion of electron in a magnetic field \cite{18} \cite{19}. In this two dimensional case the canonical quantization is given similar to the experimentally verified flux quantization by postulating: $\int_{\text{surface}} e \tilde{F}_{mn} dx^m \wedge dx^n = \oint_{\text{contour}} e \tilde{A}_m dx^m = Nh$ where $\tilde{A}$ and $\tilde{F}$ are the $U(1)$ valued electromagnetic 1 and 2 forms. Thus if one considers the flux quantization as the canonical quantization of a two dimensional electromagnetic system of electrons \cite{19}, then the equivalent quantum commutator postulate is given by $eB[\hat{x}_m, \hat{x}_n] = -i\hbar \epsilon_{mn}$ which is known, phenomenologically, as the non-commutativity of the relative coordinate operators $\hat{x}_m$ in the cyclotron motion of electrons \cite{18}, where $B$ is the applied spatially constant magnetic field. There is also an uncertainty relation $eB \Delta x_m \cdot \Delta x_n |_{\epsilon_{mn}} = e \Delta A_m \cdot \Delta x_m \geq \hbar$ \cite{19}.

Obviously this two dimensional electromagnetic quantization is comparable with the above quantization of two dimensional gravity. Thus the magneto-quantization on the two dimensional system is due to the applied spatially constant strong magnetic field $F$, whereas the gravito-quantization is due to the spatially constant strong gravitational field $R$ of BH.

A geometrical analysis of this quantum structure shows that, as it is mentioned above, the curvature ten-
sors $F_{mn}$ or $R_{mn}$ act here as constant almost complex structures on respective two dimensional manifolds which provide the underlying two dimensional manifolds with the flat $U(1)$ structure that is necessary for the geometric quantization of respective action functionals. Hence it is the values of these field strengths and of areas of respective interactions regions that determine the value of action functional which decides about the dominance of quantum or classical level of interaction. Recall that in accord with [12] in the case where $W \sim \hbar$ the quantum modes are dominant.

However, whereas the two dimensional electromagnetic quantization is verifiable in strong magnetic fields by cyclotron motion or flux quantization, the two dimensional gravitational quantization is, in view of the $10^{-40}$ weakness of gravitational interaction with respect to electromagnetic interaction, only verifiable in very strong gravitational field of BH.

Footnotes and references

References

[1] S. W. Hawking, Commun. Math. Phys., 43, 199-220 (1975);

See also the metric approach to the original model, e.g.: R. M. Wald, ”General Relativity” (The University of Chicago Press 1984).

[2] The invariant or inherent dimensions of curvature- or field strength two-form can be determined in accord with the following fact that every r-form is dimensionless. Hence in view of the fact that $dx^i \sim x^i \sim L$, then $dx^i \wedge dx^j \sim L^2$ and the $R_{ij}$ tensor components of the two-form $R_{ij}dx^i \wedge dx^j$ are of dimension $L^{-2}$. In the same manner a potential component of a connection one-form is of invariant dimension $L^{-1}$. Note that this invariant dimension is different than the usual dimension which vary with the dimension of the underlying manifold where the forms are defined.

[3] V. I. Arnol’d: ”Mathematical methods in classical mechanics”, (graduate text in mathematics, Springer-Verlag 1978). Note that only the cases with $n = 2$, or those which can be reduced to them, have a satisfactory solution.
The precession of gyroscope is, like the Lense-Thiring effect, given by the angular velocity $\dot{\phi}$ which is in accord to the definition of $\phi$ also a two dimensional effect, since time should be considered in the phase space of system, not as a variable, but only as a parameter.

E. Witten, hep-th/9206069.

Recall also that even the electromagnetic waves ($\sim$ photon) possess only two degrees of freedom.

Note that the $\theta$ and $\phi$ "coordinates" in this metric are spherical coordinates in subspaces with spherical symmetry, so that they can not be considered as true independent coordinates and can be suppressed with respect to the analysis of singularities.

N. Woodhouse, "Geometric Quantization", (Oxford University, Clarendon Press, 1980, 1990); N. Hitchin, Commun. Math. Phys., 131, 347-380 (1990).

Note with respect to the quantization that the question of metaplectic or corrected geometric quantization which founds the usual $\frac{1}{2}$ term in the energy quantization postulate ($\sim W = \frac{E}{\nu} := (N + \frac{1}{2})h$) is here omitted (see the above cited Book).

Typical global or topological quantum effects are Aharonov-Bohm effect, Flux quantization and quantum Hall effect, which are caused by the flux of magnetic field through a surface. The electrons moving on the contour of such a surface perceive a phase change which is given in accord with the Stokes' theorem by:

$$e \oint_{\text{contour}} F_{mn} dx^m \wedge dx^n = e \oint_{\text{contour}} A_m dx^m$$

where $e$, $F$ and $A$ are the electric charge, magnetic field and the electromagnetic potential. Obviously these results are two dimensional topological invariants in the sense that the flux surface is a two dimensional manifold.

The magnetic length is defined by:

$$L_B^2 = \frac{h}{eB}$$

where $B$ is the applied constant magnetic field which is the curvature of the $U(1)$ bundle. For phenomenological definition of magnetic length see text books in solid state physics.

Recall that the quantization structure for the $(3 + 1)$ dimensional $U(1)$ Yang-Mills theory:

$$\int_{(3+1)} F \wedge *F$$

is postulated by $[\hat{A}_i, \hat{E}_j] \propto \hbar \delta_{ij}$ where $E_i := \dot{A}_i$ and $i, j = 1, 2, 3$, whereas the quantization of the $(2 + 1)$ dimensional $U(1)$ Chern-Simons theory:

$$\int_{(2+1)} A \wedge F$$

is postulated in accord with $A_0 = 0$ gauge fixing by $[\hat{A}_m, \hat{A}_n] \propto \hbar \epsilon_{mn}$ where $m, n = 1, 2$ (see also Ref. [3]).
Of course one can obtain the same equations of motion also from the equivalent action functional on the contour region \( W_G = \oint_{\text{contour}} \Gamma_m dx^m \). However in this case, i.e. on the contour region, one should consider that the equation of motion \( \partial_m \Gamma_m = 0 \) or \( d^\dagger \Gamma = 0 \) should be fulfilled by flat connection \( \Gamma_{\text{contour}} := \Gamma_{\text{flat}} \) for which \( \epsilon_{mn} \partial_n \Gamma_m = 0 \) or \( d^\dagger \Gamma = 0 \). These two group of equations, i.e. \( d^\dagger \Gamma = 0 \) and \( d\Gamma = 0 \) together are equivalent to the same Laplace equation \( (d^\dagger d + dd^\dagger)\Gamma = \partial_n \partial^n \Gamma_m = 0 \).

This is a typical situation in topological quantum effects, as e.g. in Bohm-Aharonov effect, where just the contour integral of the locally vanishing flat connection, which is equal to the topological surface integral of non-vanishing curvature, causes the observable quantum phase.

According to geometric quantization the classical vector fields related to the canonical conjugate variables \( \{\pi_m, q^m\} \) on the phase space are given by:

\[
X_{\pi_m} = \frac{\partial \pi_m}{\partial q^n} \frac{\partial}{\partial q^n} - \frac{\partial \pi_m}{\partial q^n} \frac{\partial}{\partial \pi_n}, \quad X_{q^m} = \frac{\partial q^m}{\partial \pi_n} \frac{\partial}{\partial \pi_n} - \frac{\partial q^m}{\partial q^n} \frac{\partial}{\partial q^n},
\]

Furthermore, the inner product of any globally hamiltonian vector field \( X_f \) of a function \( f \) on the phase space of system with the symplectic 2-form \( \omega = d\pi_m \wedge dq^m \), should result in: \( \langle X_f, \omega \rangle = df \).

In other words \( X_{\pi_m} = \frac{\partial}{\partial q^n} \) and \( X_{q^m} = -\frac{\partial}{\partial \pi_n} \).

Moreover in the polarized (quantum) phase space, i.e. in a certain representations as in \( \Psi(\pi_m, t) \)- or \( \Psi(q^m, t) \) representation, the quantum operators for \( \{\pi_m, q^m\} \) variables are given, respectively, by: \( \hat{\pi}_m = \pi_m, \hat{q}^m = -i\hbar X_{\pi_m} = i\hbar \frac{\partial}{\partial \pi_n} \) or \( \hat{\pi}_m = -i\hbar X_{\pi_m} = -i\hbar \frac{\partial}{\partial q^n}, \hat{q}^m = q^m \).

For a four dimensional model of ”quantum space time” and the discussion of other four dimensional models see: S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys., 188-220 (1995). Note that the central question of space-time quantization, i.e. the physical meaning of the commutator postulate in this model, is not answered.

M. Nakahara: ”Geometry, Topology And Physics” (Adam Hilger, 1990); C. Nash: ”Differential Topology and Quantum Field Theory”, (Academic Press 1991).
For a general discussion of canonical quantization of two dimensional electrodynamics see:

F. Ghaboussi: (cond-mat/9710092), (quant-ph/9702054), (cond-math/9701128), (cond-math/9703080).