A Dark Energy Quintessence Model of the Universe

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\section*{Abstract}

In this paper, we have presented a model of the FLRW universe filled with matter and dark energy fluids, by assuming an ansatz that the deceleration parameter is a linear function of the Hubble constant. This results in a time-dependent DP having a decelerating-accelerating transition phase of the universe. This is a quintessence model $\omega_{(de)} \geq -1$. The quintessence phase remains for the period ($0 \leq z \leq 0.5806$). The model is shown to satisfy current observational constraints. Various cosmological parameters relating to the history of the universe have been investigated.

\section{Introduction}

We begin with the famous quote by Allan Sandage \cite{1} that "All of observational cosmology is the search for two numbers: Hubble (HP) and deceleration parameters (DP) $H_0$ and $q_0$. The universe is a dynamical system in which its constituents (galaxies) travel like a disciplined march of soldiers and move away from each other with the Hubble rate. The cosmological principle (CP) is the basis of any model describing cosmology. As is well-known \cite{2, 3}, the age and distance problems indicate that the universe is now accelerating, which means that the DP $q_0$ is negative. In the present scenario, higher derivatives of the scale factor such as the jerk $j_0$, snap $s_0$ and the lerk parameters $l_0$ do play a role. It is concluded that dark energy (DE) \cite{4} with negative pressure prevails all over the universe which is responsible for the said acceleration. Many researchers \cite{13} have developed models of the universe in which DE is taken as a perfect fluid with variable equation of state (EoS) parameter $\omega_{de}$ producing negative pressure. These authors have considered different parametric forms of $\omega_{de}$. Off late, many authors \cite{21} (and references therein) also developed DE perfect fluid models.

In this paper, we have presented a model of the FLRW (Friedman-Lemaitre-Robertson-Walker) universe filled with matter and DE fluids by assuming an ansatz that the DP is a linear function of the Hubble constant. This results in a time-dependent DP having a decelerating-accelerating transition phase of universe. This is a quintessence model $\omega_{(de)} \geq -1$. The quintessence phase remains for the period ($0 \leq z \leq 0.5806$). The model is shown to satisfy current observational constraints. Various cosmological parameters relating to the history of the universe have been investigated.

The paper is structured as follows: In Sec. 2, we set up the initial field equations. In Sec. 3, we describe the solution and physical properties of the DE model. Finally, Sec 4 is devoted to conclusions.

\section{Field equations}

The Einstein field equations (EFEs) are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\frac{8\pi G}{c^4}T_{ij},$$

(1)
where $R_{ij}$ is the Ricci tensor, $R$ the scalar curvature, and $T_{ij}$ the stress-energy tensor. It is taken as $T_{ij} = T_{ij}(m) + T_{ij}(de)$, where $T_{ij}(m) = (\rho_m + p_m)u_iu_j - p_mg_{ij}$ and $T_{ij}(de) = (\rho_{de} + p_{de})u_iu_j - p_{de}g_{ij}$. We take \( u^\alpha = 0; \quad \alpha = 1, 2, 3. \)

The FLRW space-time metric (in units $c = 1$) is given by

$$ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 + kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right],$$

where $a(t)$ stands for the scale factor and $k$ is the curvature parameter, $k = +1$ for closed, $k = -1$ for open and $k = 0$ for a spatially flat universe. Solving the EFEs \(1\) for the FLRW metric \(2\), we obtain the following system of equations:

$$\frac{2}{a} \dot{a} + H^2 = -8\pi G (p + \rho_k) \quad (3)$$

$$H^2 = \frac{8\pi G}{3} (\rho + \rho_k) \quad (4)$$

where $H = \dot{a}/a$ is the Hubble constant. Here an overdot means differentiation with respect to proper time $t$.

We define the density and pressure for the curvature energy as $\rho_k = \frac{38\pi G}{8\pi G}, p_k = -\frac{8\pi G}{8\pi G}$. The energy density $\rho$ and pressure $\rho'$ in Eqs. \(3\) and \(4\) are comprised of two types of energies and pressures $\rho = \rho_m + \rho_{de}$ and $p = p_m + p_{de}$. The energy conservation law $T^{ij}_;j = 0$ yields

$$\dot{\rho} + 3H(\rho + \rho_k) = 0, \quad (5)$$

where $\rho = \rho_m + \rho_{de} + \rho_k$ and $p = p_m + p_{de} + p_k$ are the total density and pressure of the universe. As $\dot{\rho}_k + 3H(p_k + \rho_k) = 0$, so $\frac{4}{3}(\rho_m + \rho_{de}) + 3H(p_m + p_{de} + \rho_m + \rho_{de}) = 0$. We assume that both matter and dark energies are minimally coupled so that they are conserved simultaneously, i.e., $\rho_m + 3H(p_m + \rho_m) = 0, \quad \rho_{de} + 3H(p_{de} + \rho_{de}) = 0$. At present, the matter content in the universe is in form of dust for which $p_m = 0$. We assume that the EoS for dark energy is $p_{de} = \omega_{de}\rho_{de}$. Integration of the energy conservation laws yields

$$\rho_m = (\rho_m)_0 (1 + z)^3, \quad \rho_m = (\rho_m)_0 \exp \left( 3 \int_0^z \frac{(1 + \omega_{de})dz}{1 + z} \right), \quad (6)$$

where we have used $\frac{\dot{a}}{a} = 1 + z$. The EoS for the curvature energy is obtained as $p_k = \omega_k\rho_k$ where $\omega_k = -1/3$. This gives $\rho_k \propto a^{-2} = (\rho_k)_0 \left[ \frac{a_0}{a} \right]^2$. We define the density parameters for matter, DE and curvature as $\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_{de} = \frac{\rho_{de}}{\rho_c} \quad & \quad \Omega_k = \frac{\rho_k}{\rho_c}$ where $\rho_c = \frac{3M_H^2}{8\pi G}$, is the critical density. Therefore the FLRW field equations change to

$$H^2(1 - \Omega_{de}) = H_0^2 \left[ (\Omega_m)_0(1 + z)^3 + (\Omega_k)_0(1 + z)^2 \right], \quad (7)$$

and

$$2q = 1 + 3\omega_{de}\Omega_{de} + \frac{3H_0^2}{H^2}\omega_k(\Omega_k)_0(1 + z)^2, \quad (8)$$

where $q$ is the DP defined by $q = -\frac{\ddot{a}}{aH^2}$.

## 3 Results and discussion

In the above, we have found two field equations \(7\) and \(8\) in five unknown variables $a, H, q, \Omega_{de}$ and $\omega_{de}$. Therefore, for a complete solution, we need three more relations involving these variables. As has been discussed in the introduction, in view of the recent observations of Type Ia supernova [01-12], there is a need for a time-dependent DP which describes decelerated expansion in the past ($z \geq 1$) and accelerating expansion at present, so there must be a transition from deceleration to acceleration. The DP must show the change in signature, so we must use an ansatz to determine proper DP. Earlier, Akarsu et al. [27] used a hybrid expansion law [HEL] to express the deceleration to acceleration transition and cosmic history. In this paper we consider $q$ as a linear function of the Hubble parameter, which was earlier used by [28] in $f(R,T)$ gravity.

$$q = \alpha H + \beta \quad (9)$$
Here $\alpha$, and $\beta$ are arbitrary constants. The jerk parameter $j = \frac{\dddot{a}}{aH^2}$ is related to $\dot{q}$ through

$$\dot{q} = (q(2q + 1) - j)H$$ (10)

From Eqs. (9), (10) and $\dot{H} = -(1 + q)H^2$, we get $\alpha = (j - q(2q + 1))/(1 + q)H_{z=0}$ and $\beta = ((-j + q(2 + 3q))/(1 + q))_{z=0}$.

Now, we present followings data regarding values of the cosmological parameters at present due to Hassan and Soroush Amirhashchi [29]: $H_0 = 69.9 \pm 1.7$, $\Omega_0 = 0.279_{-0.014}^{+0.014}$, $\Omega_{0\text{de}} = 0.721_{-0.016}^{+0.016}$, $j_0 = 1.038_{-0.035}^{+0.061}$ and $z_t = 0.707 \pm 0.034$. They have estimated these values using the Pantheon compilation [30] and the JLA data set [31]. This data enables us to determine the constants $\alpha$ and $\beta$. We get $\alpha = 31.619$, $\beta = -2.763$. Eq. (9) provides the following differential equation and its solution:

$$(1 + z)H_{z} = (\delta + \alpha H)H, \quad \delta = 1 + \beta$$ (11)

$$H = (A\delta(1 + z)^\delta)/(1 - A\alpha(1 + z)^\delta),$$ (12)

where $A$ is a constant of integration. It is determined by applying the initial condition that $H_0 = 0.07 \text{ Gyr}^{-1}$ as $A = 7/(100 \delta + 7 \alpha) = 0.155545$. We have also used $\dot{z} = -(1 + z)H$. The transit red shift $z_t$ where DP is zero, is obtained as

$$z_t = \left(\frac{-Ay}{x}\right)^{\frac{1}{1+y}} - 1 = 0.386719 \simeq 9.4699\text{ Gyr}$$ (13)

This means that 4.25 Gyr ago, the universe started accelerating. It is interesting to note that the transit red shift obtained by us is matching with the work referred to in the introduction [13]-[20]. We present the following plots to illustrate our results.
We present the following table which describes observed values of the Hubble parameter along with errors and corresponding theoretical results obtained as per our model in the range \(0 \leq z \leq 1\.)

| \(z\)     | \(H_{ob}\) | \(H_{er}\) | \(H_{th}\) |
|-----------|------------|------------|------------|
| 0.07      | 0.0746     | 0.0302     | 0.0723     |
| 0.1       | 0.0787     | 0.0143     | 0.0734     |
| 0.12      | 0.0908     | 0.0374     | 0.0741     |
| 0.17      | 0.0846     | 0.0085     | 0.0746     |
| 0.179     | 0.0848     | 0.0143     | 0.0765     |
| 0.199     | 0.0971     | 0.0173     | 0.0774     |
| 0.2       | 0.0844     | 0.0079     | 0.0784     |
| 0.27      | 0.1064     | 0.0132     | 0.0808     |
| 0.35      | 0.0899     | 0.0343     | 0.0851     |
| 0.35      | 0.0931     | 0.0409     | 0.0852     |
| 0.4       | 0.1196     | 0.0184     | 0.1136     |
| 0.44      | 0.1636     | 0.0143     | 0.1143     |
| 0.48      | 0.1811     | 0.0235     | 0.1274     |
| 0.57      | 0.1432     | 0.0409     | 0.1452     |
| 0.593     | 0.2066     | 0.0515     | 0.1463     |
| 0.6       | 0.0921     | 0.0184     | 0.1509     |
| 0.781     | 0.1432     | 0.0085     | 0.1942     |
| 0.875     | 0.1636     | 0.0085     | 0.0762     |
| 0.88      | 0.1636     | 0.0085     | 0.0774     |
| 0.9       | 0.1811     | 0.0085     | 0.0808     |
| 0.991     | 0.1432     | 0.0085     | 0.0851     |
| 0.992     | 0.1907     | 0.0085     | 0.0852     |

The observed values and errors are taken from a recent paper by Farook \textit{et al} [32]. The Hubble data is converted into \(Gyr^{-1}\) unit. In order to get quantitative closeness of theory to observation, we obtain \(\chi^2\) from the following formula

\[
\chi^2 = \frac{(H_{ob} - H_{th})^2}{H_{er}^2}
\]

It comes to 19.41469 which is 92.45\% over 21 data sets, which shows a best fit of theory to observation. We present following the following figure to display this closeness.

\(\text{Figure 3: Error bar plot of Hubble parameter } H \text{ showing proximity of theory and observation.}\)

\((\text{ii) DE Parameter } \Omega_{de} \text{ and EoS } \omega_{de}\)

Now from Eqs. [7]-[9], the DE parameter \(\Omega_{de}\) and EoS parameter \(\omega_{de}\) are given by the following equations, and are solved numerically

\[
H^2 \Omega_{de} = H^2 - (\Omega_m)_{0} H_0^2 (1 + z)^3
\]

\[
\omega_{de} = \frac{H^2 (2 \alpha H + 2 \beta - 1)}{3 [H^2 - H_0^2 (\Omega_m)_{0} (1 + z)^3]}
\]

where we have taken \((\Omega_k)_{0} = 0\) for the present spatially flat universe. We solve Eqs. [14] and [15] with the help of Eq. [12] and present the following figures 4 and 5 to illustrate the solution. The figures show that this is a quintessence model \(\omega_{(de)} \geq -1\) which remains for the period \((0 \leq z \leq 0.5806)\). DE favors deceleration at \(z \geq 0.5806\). As per our model, the present value of the DE parameter is 0.7. It decreases over the past, attains a minimum value of \(\Omega_{de} \sim 0.45\) at \(z \sim 0.6\), and then it again increases with red shift. It is interesting to note that the growth of the EoS parameter for DE \(\Omega_{de}\) over redshift during the quintessence regime is matching with the pioneering work by Steinhardt [33] and Johri [34].
(iii) Luminosity Distance:

The redshift-luminosity distance relation [35] is an important observational tool to study the evolution of the universe. The expression for the luminosity distance \( D_L \) is obtained in term of redshift as the light coming out of a distant luminous body gets red shifted due to the expansion of the universe. We determine the flux of a source with the help of the luminosity distance. It is given as \( D_L = a_0 r (1 + z) \), where \( r \) is the radial coordinate of the source. For the FLRW metric Eq. (2), the radial co-ordinate \( r \) is obtained as \( r = \int_0^t \frac{dt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{dz}{h(z)} \), where we have taken \( k = 0 \) and have used \( dt = dz/\dot{z}, \dot{z} = -H(1+z) \) & \( h(z) = \frac{H}{H_0} \). Therefore, we get the luminosity distance as:

\[
\frac{H_0 D_L}{c} = (1 + z) \int_0^z \frac{dz}{h(z)}. \tag{16}
\]

(iv) Distance modulus \( \mu \) and Apparent Magnitude \( m_b \):

The distance modulus \( \mu \) [12] is derived as

\[
\mu = m_b - M = 5 \log_{10} \left( \frac{D_L}{M_{pc}} \right) + 25 = 25 + 5 \log_{10} \left[ \frac{c(1+z)}{H_0} \int_0^z \frac{dz}{h(z)} \right] \tag{17}
\]

The absolute magnitude \( M \) of a supernova [12] is \( M = 16.08 - 25 + 5 \log_{10}(H_0/0.026c) \). Using this, the apparent magnitude \( m_b \) is obtained as

\[
m_b = 16.08 + 5 \log_{10} \left[ \frac{1 + z}{0.026} \int_0^z \frac{dz}{h(z)} \right]. \tag{18}
\]

We solve Eqs (16)−(18) with the help of Eq (12). Following is the value of the integral

\[
\int_0^z \frac{dz}{h(z)} \simeq (1.25 z + 0.09(1 - (1 + z)^{2.76})
\]
(v) $\chi^2$ for Distance Modulus $'\mu'$:

Our theoretical results have been compared with the SNe Ia related 581 data from the Pantheon compilation [29] with possible errors in the range $(0 \leq z \leq 1.3)$ and the derived model was found to be in good agreement with current observational constraints. In order to get quantitative closeness of theory to observation, we obtain $\chi^2$ from the following formula

$$\chi^2 = \frac{(\mu_{ob} - \mu_{th})^2}{\mu_{err}^2}$$

It equates to 562 which is 96.73% over 581 data points, which shows best fit of theory to observation. The following figures 8 depict the closeness of observational and theoretical results, thereby justifying our model.

![Figure 6: Plot of distance modulus ($\mu = M - m_b$) versus redshift ($z$). Crosses are SNe Ia related Pantheon compilation 581 data points with possible corrections](image)

4 Conclusion

In the present paper, we have presented an FLRW universe filled with two fluids (barotropic and dark energy), by assuming that the DP is a linear function of the Hubble constant. This results in a time-dependent DP having a transition from past decelerating to the present accelerating universe. The main findings of our model are itemized point-wise as follows:

- The expansion of the universe is governed by an expansion law $q = \alpha H + \beta$, $H = -\frac{A(\beta+1)(z+1)^{\beta+1}}{0.07(\alpha z(\beta+1)+1-1)}$, where $\alpha = 31.619$, $\beta = -2.76333$ and $A = 0.15$. This describes the transition from deceleration to acceleration.
- The transit red shift is $z_t = 0.386$ and the corresponding time is $T_t = 1.034$ Gyr. The universe is now accelerating and it was decelerating before the time $T_t$.
- This is a quintessence model $\omega_{(de)} \geq -1$ which remains for the period $(0 \leq z \leq 0.5806)$. DE favors deceleration at $z \geq 0.5806$.
- The present value of the DE parameter is 0.7. It decreases over the past, attains a minimum value $\Omega_{de} \sim 0.45$ at $z \sim 0.6$, and then it again increases with red shift.
- The growth of the EoS parameter for DE $\Omega_{de}$ over redshift during the quintessence regime is matching with the pioneering work Steinhardt [33] and Johri [34]
In a nutshell, we believe that the proposed linear law may help in investigations of hidden matter like dark matter, dark energy and black holes.

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