NONLINEAR DISCRETE SYSTEMS WITH NONANALYTIC DISPERSION RELATIONS

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A discrete system of coupled waves (with nonanalytic dispersion relation) is derived in the context of the spectral transform theory for the Ablowitz Ladik spectral problem (discrete version of the Zakharov-Shabat system). This 3-wave evolution problem is a discrete version of the stimulated Raman scattering equations, and it is shown to be solvable for arbitrary boundary value of the two radiation fields and initial value of the medium state. The spectral transform is constructed on the basis of the $\mathcal{D}$-approach.

I. INTRODUCTION

This paper relates the study of the following discrete coupled system for the three fields $A_1(\theta, n, t)$, $A_2(\theta, n, t)$ and $q(n, t)$

$$
A_1(\theta, n, t) - A_1(\theta, n-1, t) = e^{-in\theta}q(n, t)A_2(\theta, n, t)
$$

$$
A_2(\theta, n, t) - A_2(\theta, n-1, t) = -e^{in\theta}q(n, t)A_1(\theta, n, t)
$$

$$
q_t(n, t) = \frac{\rho(n, t)}{2\pi} \int_{-\pi}^{+\pi} d\theta e^{in\theta}(A_1 \ast A_2)(\theta, n, t)
$$

where $\theta \in [-\pi, +\pi]$, $n \in \mathbb{Z}$ and $t \in \mathbb{R}^+$. The interaction term here above is defined as the coupling factor

$$
(A_1 \ast A_2)(\theta, n, t) = g(\theta, t)A_1(\theta, n-1, t)A_2(\theta, n, t) + \overline{g(\theta, t)}A_1(\theta, n, t)\overline{A_2}(\theta, n-1, t)
$$

where $g(\theta, t)$ is an arbitrary function in $L^2([-\pi, +\pi])$ (which could also be time dependent), and where the energy ratio $\rho(n, t)$ at the site $n$ (it will be shown that this quantity is actually $\theta$-independent) is defined as

$$
\rho(n, t) = \frac{|I_1(\theta, t)|^2 + |I_2(\theta, t)|^2}{|A_1(\theta, n, t)|^2 + |A_2(\theta, n, t)|^2}
$$

for the following definition of the boundary values $I_1(\theta, t)$ and $I_2(\theta, t)$ (input data)

$$
I_1 = \lim_{n \to +\infty} A_1(\theta, n, t), \quad I_2 = \lim_{n \to +\infty} A_2(\theta, n, t).
$$

One of the main results is the proof that the system (1) with data (initial-boundary value problem)

$$
q(n, 0), \quad I_1(\theta, t), \quad I_2(\theta, t)
$$

is integrable. In particular this work provides the first instance of an integrable nonlinear discrete system with nonanalytic dispersion relation.

An interesting limit of the above equation arises when the arbitrary distribution $g(\theta)$ goes to a Dirac delta function, for instance $\delta(\theta)$. Then the system reads (now $A_j(n, t)$ denotes $A_j(\theta, n, t)|_{\theta=0}$)

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\begin{align}
A_1(n, t) - A_1(n-1, t) &= q(n, t)A_2(n, t) \\
A_2(n, t) - A_2(n-1, t) &= -\overline{q}(n, t)A_1(n, t) \\
q(n, t) &= \rho(n, t)[A_1(n-1, t)\overline{A}_2(n, t) + A_1(n, t)\overline{A}_2(n-1, t)]
\end{align}

and it is called the \textit{sharp line limit of} \([\mathfrak{I}].\) Although a definite physical application of such an equation does not exist by now, it can still be understood in the following way. \(A_1\) and \(A_2\) are the two envelopes of some high frequency (HF) oscillations (say at frequency \(\omega_1\) and \(\omega_2\)) which interact resonantly on each site \(n\) with a medium constituted of oscillators of envelope \(q(n)\) and frequency \(\Omega = \omega_1 - \omega_2,\) with a coupling intensity proportional to the relative amount \(\rho(n)\) of the total HF energy which has reached the site \(n.\) Then the physical data are the input values \(A_{1,2}(\theta, n, t)\) at \(n = \infty\) of the HF external excitations, and the initial state of the medium oscillators.

The method used to build and solve (1) is the inverse spectral (or scattering) transform (IST) well known also as the nonlinear Fourier transform \([\mathfrak{I}].\) Indeed the method, in its principle, works like a Fourier transform. It associates to the field, solution of a nonlinear evolution equation, its spectral transform which evolves linearly. Then the field at time \(t\) is reconstructed from the spectral transform at time \(t\) by solving the inverse spectral problem. This method has been widely studied and extended to various interesting nonlinear evolution problems. We are particularly interested in three types of extension which will be used all three together.

The first extension involved here is the use of \textit{discrete} spectral problems to solve discrete (in space) nonlinear evolution equations. Famous instances of integrable discrete systems are the Toda lattice \([\mathfrak{I}^4]\) and a special discrete version of the nonlinear Schrödinger equation which has been proposed and integrated by Ablowitz and Ladik by using a discrete version of the Zakharov–Shabat spectral problem \([\mathfrak{I}^3].\) This so called Ablowitz–Ladik spectral problem and the related nonlinear difference–difference equations with polynomial dispersion relations have been extensively studied (see \([\mathfrak{I}^4] – [\mathfrak{I}^17].\) Recently a different version of the discrete Zakharov-Shabat system has been proposed in order to keep the canonical Poisson structure of the continuous case \([\mathfrak{I}^18].\)

The second domain considered here concerns the extension of the spectral transform to the case of \textit{nonanalytic dispersion relations} \([\mathfrak{I}^14, \mathfrak{I}^24].\) The first instance of such an integrable system is the self-induced transparency (SIT) equations of McCall–Hahn \([\mathfrak{I}^21]\) which was shown to possess a Lax pair in \([\mathfrak{I}^22]\), was given a N-soliton solution in \([\mathfrak{I}^23]\) and later studied and completely solved in \([\mathfrak{I}^24].\) These systems generally describe wave-wave interactions for which some boundary value are prescribed. These boundary values are strongly dependent on the physical problem under consideration. For instance the problem of superfluorescence in two-level media results in the same equation as SIT but with different boundary values and consequently quite different generic properties resulting mainly from a linear but \textit{non homogeneous} evolution of the spectral transform \([\mathfrak{I}^23].\)

The third extension used is the generalization of the solution of an evolution equation with a nonanalytic dispersion relation to the case of \textit{arbitrary boundary values} \([\mathfrak{I}^26].\) In this case, the evolution of the spectral transform can be not only non homogeneous but also \textit{nonlinear} and still has interesting physical application. In particular the problem of stimulated Raman scattering of a high energy long Laser pulse in a gas has been solved by this technique \([\mathfrak{I}^27, \mathfrak{I}^28].\)

The paper is organized as follows. In sect. 2 we summarize the method of solution of the system \([\mathfrak{I}^1]\) and provide there only the resulting formulae.

In sect. 3 the principal Lax operator or, more precisely, the associated spectral problem (a special reduction of the Ablowitz–Ladik spectral problem) is used to define the spectral data (or nonlinear Fourier transform). This is done by selecting the basic set of Jost solutions and then proving that they obey a Riemann-Hilbert problem in the spectral parameter.

The inverse spectral problem is then solved in sect. 4 by means of the \(\partial\)-formulation of the spectral problem, which means that the \textit{potentials} are reconstructed from the \textit{spectral data}. There the compatible reductions are also considered, which will allow in particular to obtain simpler integrable equations with an easier interpretation.

The sect. 5 is devoted to the formulation of the inverse spectral transform on the basis of the \(\partial\)-problem. More precisely, having previously shown that the spectral problem (for the principal Lax operator) leads naturally to a \(\partial\)-problem, we prove here the reverse statement. This is useful in the following for considering the \(\partial\)-problem itself as the starting tool.

The general discrete integrable systems with nonanalytic dispersion relations is then constructed by requiring a time evolution of the spectral transform with a \textit{non-analytic} dispersion law and a \textit{non-homogeneous} term.

These results are used in sect. 6 to prove that indeed the system \([\mathfrak{I}^1]\) with the arbitrary boundary values \([\mathfrak{I}^4]\) is solvable. That means that we obtain the time evolution of the spectral data in terms of the boundary values and the spectral transform of the initial datum \(q(n, 0)\). An interesting case corresponds to the growth of the field \(q(n, t)\) on an initial medium at rest, that is for \(q(n, 0) \equiv 0\). The method furnishes in such a case the explicit output values of the HF fields \(A_1(\theta, n, t)\) and \(A_2(\theta, n, t)\) for \(n = -\infty.\)
II. SOLUTION OF THE SYSTEM. A SUMMARY

The general method to generate solutions of (3) is sketched hereafter. The starting point is the spectral transform of the initial datum \( q(n,0) \), namely the set of two scalar functions \( \alpha \) and \( \beta \) defined on the unit circle, a sequence of \( N \) discrete points \( k_j \) outside the unit disc to each of which are associated \( N \) complex constants \( C_j \). This set is given at \( t = 0 \) as

\[
\alpha(\zeta, 0), \quad \beta(\zeta, 0), \quad \zeta = e^{i\theta}; \quad C_j(0), \quad k_j, \quad |k_j| > 1, \quad j = 1, \ldots, N.
\] (7)

In the language of the scattering theory, \( \alpha \) is called the reflection coefficient, \( \beta \) the transmission coefficient, \( N \) the number of bound states \( k_j \) and \( C_j \) the related normalization coefficients. The effective construction of these data from \( q(n,0) \) is displayed in sect. 3, but here we just consider that the set (7) is given and we show how to build from it a solution of (3). It is worth mentioning that in the linear limit case of \( q(x,0) \), \( \alpha(\zeta,0) \) becomes the Fourier transform of \( q \) (with parameter \( 2\zeta \)), \( \beta(\zeta,0) \) become 1 and all the \( C_j \)'s vanish (no discrete spectrum, or else no solitons in the linear limit).

The first step is to construct the spectral transform at time \( t \) by solving

\[
\begin{align*}
\partial_t \alpha &= \alpha g + \frac{\bar{g}}{2} (|I_1(\theta,t)|^2 - |I_2(\theta,t)|^2) - (g + \bar{g})I_1I_2 \\
-\alpha &\int \frac{d\zeta'}{2\pi i} P \int_C \frac{d\zeta'}{\zeta' - \zeta} (g + \bar{g})(|I_1|^2 - |I_2|^2) \\
+\alpha &\int \frac{d\zeta'}{2\pi i} \frac{d\zeta'}{\zeta'} (g|I_1|^2 - \bar{g}|I_2|^2), \\
\partial_t k_j &= 0, \\
\partial_t C_j(t) &= -C_j(t) \int \frac{d\zeta'}{2\pi i} P \int_C \frac{d\zeta'}{\zeta' - k_j} (g + \bar{g})(|I_1|^2 - |I_2|^2) \\
+ C_j(t) &\int \frac{d\zeta'}{2\pi i} \frac{d\zeta'}{\zeta'} (g|I_1|^2 - \bar{g}|I_2|^2).
\end{align*}
\] (8-13)

Unlike in the continuum case, the transmission coefficient \( \beta(\zeta,t) \) cannot be computed directly from \( \alpha(\zeta,t) \) and it becomes necessary to solve the equation

\[
\beta(\zeta,t)^{-1} \partial_t \beta(\zeta,t) = -\frac{1}{2} \frac{\partial_t |\alpha|^2}{|\alpha|^2} + \frac{1}{2\pi} \int_C \frac{d\zeta'}{\zeta' - \zeta} \frac{\partial_t |\alpha|^2}{|\alpha|^2}.
\] (14)

In the integrals here above, \( P \) denotes the Cauchy principal value and \( C \) the unit circle in the complex plane.

Although not elementary, in the case when \( I_1 \) and \( I_2 \) are given independently of \( \alpha \) and \( \beta \) the above system of equations is linear and can in principle be explicitly solved as soon as the initial data (3) and the boundary values (7) are known. Before going further, it is already worth remarking that if the quantity \( I_1I_2 \) does not vanish, then the evolution for the reflection coefficient \( \alpha(\zeta,t) \) has a non-homogeneous term. Consequently the solution can grow on the initial vacuum \( q(n,0) = 0 \) which has the spectral transform

\[
\alpha(\zeta, 0) = 0, \quad \beta(\zeta, 0) = 1; \quad N = 0.
\] (15)

The second step consists in solving the following system of linear integral equation for the unknowns \( \phi_n(k,n,t) \) for \( |k| < 1 \)

\[
\begin{align*}
\left( \begin{array}{c}
\phi_1(k) \\
\phi_2(k)
\end{array} \right) &= \left( \begin{array}{c}
1 \\
0
\end{array} \right) + \frac{1}{2\pi} \int_C \frac{d\zeta'}{\zeta' - k} \left( \zeta' \right)^{-n} \alpha(\zeta') \left( -\overline{\phi_2(\zeta')} \overline{\phi_1(\zeta')} \right) + \\
&+ \sum_{j=1}^N \frac{C_j}{k_j - k} \left( \zeta' \right)^{-n} \left( -\overline{\phi_2(1/k_1)} \overline{\phi_1(1/k_j)} \right), \quad n = 1, 2, \ldots, \infty.
\end{align*}
\] (16)

The solution of (3) then reads (last step) for \( \zeta = e^{i\theta} \)

\[
\begin{align*}
\left( \begin{array}{c}
A_1 \\
A_2
\end{array} \right) &= I_1 \left( \phi_1(\zeta) \zeta^n \phi_2(\zeta) \right) + I_2 \left( -\zeta^{-n} \overline{\phi_2(\zeta)} \overline{\phi_1(\zeta)} \right) \\
q(n+1,t) &= -\phi_2^{(-1)}(n,t).
\end{align*}
\] (17) (18)
with $\phi_2^{(-1)}$ the coefficient of $k^{-1}$ in the Laurent expansion of the solution $\phi_2(k, n, t)$. This achieves the solution of the nonlinear system \([\text{(4)}]\) with the arbitrary boundary values \([\text{(4)}]\) as a sequence of linear operations.

An interesting information here is the output values (vs. the input values) of the fields $A_j$ (values for $n \to -\infty$) which will be proved to be

$$\left( \begin{array}{l} A_1 \\ A_2 \end{array} \right) \underset{n \to -\infty}{\longrightarrow} \frac{1}{1 + |\alpha|^2} \left( \begin{array}{c} I_1 \beta + I_2 \bar{\beta} \\ -I_1 \alpha \beta + I_2 \beta \end{array} \right)$$

for the input

$$\left( \begin{array}{l} A_1 \\ A_2 \end{array} \right) \underset{n \to +\infty}{\longrightarrow} \left( \begin{array}{c} I_1 \\ I_2 \end{array} \right)$$

(note the necessity to compute not only the reflection coefficient $\alpha$ at time $t$ but also the transmission coefficient $\beta$). This result, besides having a physical interest, has the nice property of being explicit. Indeed it does not require solving the integral equations \([\text{(16)}]\). Actually, when the system \([\text{(4)}]\) is viewed as describing the interaction of radiation components $A_j$ with matter, the relevant (measured) physical information is the output values of the radiation components.

Now we can compute the ratio $\rho(-\infty, t)$ of transmitted photon number, defined in \([\text{(3)}]\), as

$$\rho(n, t) \underset{n \to -\infty}{\longrightarrow} \frac{1 + |\alpha|^2}{|\beta|^2},$$

while we have obviously

$$\rho(n, t) \underset{n \to +\infty}{\longrightarrow} 1.$$

This is unlike in the continuous case for which we would find $\rho(-\infty, t) = 1$, and results effectively from the discrete nature of \([\text{(4)}]\). Indeed a direct calculation leads to the following total photon number non-conservation

$$|A_1(\zeta, n-1)|^2 + |A_2(\zeta, n-1)|^2 = (1 + |g(n)|^2)(|A_1(\zeta, n)|^2 + |A_2(\zeta, n)|^2).$$

As a consequence we obtain from \([\text{(3)}]\) and the above relation

$$\rho^{-1}(n, t) = \prod_{i=n+1}^{\infty} (1 + |q(i, t)|^2)$$

which proves in particular that the energy ratio $\rho(n, t)$ indeed does not depend on the variable $\theta$.

### III. THE SPECTRAL PROBLEM

In the case of the discrete variable, a spectral problem is understood as a difference equation for some unknown $\mu(n)$, which involves explicitly an external parameter, the spectral parameter, and a set of given $n$-dependent coefficients, the potentials. Solving a spectral problem results in defining the set of spectral data (functions of the parameter $k$) in such a way that they are in bijection with the set of potentials (in some given class of functions). We shall work here in the space of $2 \times 2$ matrices and adopt an equivalent form of the reduced Ablowitz-Ladik spectral problem used in \([\text{(3)}]\) to integrate the discrete nonlinear Schrödinger equation. In our case we are able to write the spectral transform as a $\mathcal{D}$-problem for the matrix $\mu(k, n)$, which results in a simple formulation of the inverse problem together with a very convenient tool for building and solving nonlinear evolutions, in particular those with nonanalytic dispersion relations and boundary value data.

#### A. Equation and Jost solutions

Let us consider the discrete spectral problem

$$\mu(k, n+1) - \Lambda^{-1} \mu(k, n) \Lambda = Q(n+1) \mu(k, n+1),$$

(25)
with the following definitions

\[
\Lambda(k) = \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}, \quad z^2 = k,
\quad Q(n) = \begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix}
\]

(26)

where \( k \) is the spectral parameter which belongs to the domain \( D = \mathbb{C} - \{0, \infty\} \). Up to a change of \( n \to -n \), \( k \to 1/k \) and a rescaling by convenient powers of \( z \) of the matrix elements of \( \mu \) it is just the special reduction of the Ablowitz–Ladik spectral problem associated to the integrable discrete nonlinear Schrödinger equation. The solution of (25) possesses the property

\[
\det\{\mu(k, n - 1)\} = \det\{\mu(k, n)\}[1 - r(n)q(n)].
\]

(27)

The solution of this spectral problem goes through the construction of some well chosen solutions (the Jost solutions) out of some particular asymptotic behaviors. These solutions are denoted by \( \mu^\pm \) and are defined by the following discrete integral equations

\[
\begin{pmatrix} \mu_{11}^+(k, n) \\ \mu_{21}^+(k, n) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\sum_{i=-n+1}^{+\infty} q(i)\mu_{21}^+(k, i) \\ \sum_{i=-n}^{+\infty} k^{i-n}r(i)\mu_{11}^+(k, i) \end{pmatrix}
\]

(28)

\[
\begin{pmatrix} \mu_{11}^-(k, n) \\ \mu_{21}^-(k, n) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\sum_{i=-n+1}^{+\infty} q(i)\mu_{21}^-(k, i) \\ \sum_{i=-n}^{+\infty} k^{i-n}r(i)\mu_{11}^-(k, i) \end{pmatrix}
\]

(29)

\[
\begin{pmatrix} \mu_{12}^-(k, n) \\ \mu_{22}^-(k, n) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\sum_{i=-n+1}^{+\infty} k^{n+i}q(i)\mu_{22}^-(k, i) \\ -\sum_{i=-n+1}^{+\infty} r(i)\mu_{12}^-(k, i) \end{pmatrix}
\]

(30)

\[
\begin{pmatrix} \mu_{12}^+(k, n) \\ \mu_{22}^+(k, n) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \sum_{i=-n}^{+\infty} k^{n-i}q(i)\mu_{22}^+(k, i) \\ -\sum_{i=-n+1}^{+\infty} r(i)\mu_{12}^+(k, i) \end{pmatrix}
\]

(31)

We will make use also of the notation

\[
\mu_1^\pm = \begin{pmatrix} \mu_{11}^\pm \\ \mu_{21}^\pm \end{pmatrix}, \quad \mu_2^\pm = \begin{pmatrix} \mu_{12}^\pm \\ \mu_{22}^\pm \end{pmatrix}.
\]

(32)

The above integral equations allow then to obtain, for some given class of potentials, the analytical properties of the solutions in the domain \( D \) of the complex \( k \)-plane, for all \( n \). The function \( \mu_1^+(k, n) \) is holomorphic inside the unit circle, the function \( \mu_2^-(k, n) \) is holomorphic outside the unit circle, the function \( \mu_1^-(k, n) \) is meromorphic outside the unit circle where it has a finite number \( N^- \) of simple poles \( k_j^- \), the function \( \mu_2^+(k, n) \) is meromorphic inside the unit circle where it has a finite number \( N^+ \) of simple poles \( k_j^+ \). Moreover the two solutions \( \mu^\pm \) are continuous on the unit circle.

### B. Riemann-Hilbert problem and spectral data

The method to obtain from the integral equations defining \( \mu^\pm \) the related Riemann-Hilbert problem is standard. We proceed through direct computation of the difference of the two column vectors \( \mu_1^+ \) and \( \mu_1^- \) and obtain the integral equation for this difference. It obeys the same equation as the quantity \( -\alpha^-(k)k^{-n}\mu_2^-(k, n) \) (the quantity \( \alpha^- \) is defined below) and, based on the uniqueness of the solution of such equations, we conclude...
\[
\mu_1^+(k, n) - \mu_1^-(k, n) = -\alpha^-(k)k^{-n}\mu_2^-(k, n), \quad |k| = 1. \tag{33}
\]

The same approach is applied to \(\mu_2^+\) and we get
\[
\mu_2^+(k, n) - \mu_2^-(k, n) = \alpha^+(k)k^n\mu_1^+(k, n), \quad |k| = 1
\tag{34}
\]
where the reflection coefficients \(\alpha^-(k)\) and \(\alpha^+(k)\) are defined (still for \(|k| = 1\)) as
\[
\alpha^-(k) = \sum_{i=-\infty}^{+\infty} k^i r(i)\mu_{11}(k, i), \quad \alpha^+(k) = \sum_{i=-\infty}^{+\infty} k^{-i} q(i)\mu_{22}^+(k, i). \tag{35}\]

For future use we define also
\[
\beta^-(k) = 1 - \sum_{i=-\infty}^{+\infty} q(i)\mu_{21}^-(k, i), \quad \beta^+(k) = 1 - \sum_{i=-\infty}^{+\infty} r(i)\mu_{12}^+(k, i) \tag{36}\]

which are called the transmission coefficients. Note that, due to the analytical properties of \(\mu_{21}^-\) (resp. \(\mu_{12}^+\)), \(\beta^-(k)\) can be continued analytically in \(|k| \geq 1\) (resp. \(\beta^+(k)\) in \(|k| \leq 1\)). Actually the vectors \(\mu_1^-\) and \(\mu_2^+\) have poles where the transmission coefficients \(\beta^{\pm}(k)\) have poles and we derive from the integral equations (after multiplication by \(k - k_j^\pm\) and limit \(k \to k_j^\pm\))

\[
\text{Res}_{k_j^-} \{\mu_1^-(k, n)\} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_j^-)^{-n} C_j^- + \begin{pmatrix} -\sum_{i=n+1}^{+\infty} q(i)\text{Res}_{k_j^-} \{\mu_{21}(k, i)\} \\ -\sum_{i=n+1}^{+\infty} (k_j^-)^{-i} r(i)\text{Res}_{k_j^-} \{\mu_{11}(k, i)\} \end{pmatrix} \tag{37}\]
\[
\text{Res}_{k_j^+} \{\mu_2^+(k, n)\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (k_j^+)^{n} C_j^+ + \begin{pmatrix} -\sum_{i=n+1}^{+\infty} (k_j^+)^{-i} q(i)\text{Res}_{k_j^+} \{\mu_{22}^+(k, i)\} \\ -\sum_{i=n+1}^{+\infty} r(i)\text{Res}_{k_j^+} \{\mu_{12}(k, i)\} \end{pmatrix} \tag{38}\]

with the following definitions of the \(C_j^{\pm}\)s
\[
C_j^- = \sum_{i=-\infty}^{+\infty} (k_j^-)^{i} r(i)\text{Res}_{k_j^-} \{\mu_{11}(k, i)\} \tag{39}\]
\[
C_j^+ = \sum_{i=-\infty}^{+\infty} (k_j^+)^{-i} q(i)\text{Res}_{k_j^+} \{\mu_{22}(k, i)\} \tag{40}\]

which are called the normalization coefficients.

Since the vectors \(\mu_1^+\) and \(\mu_2^-\) are holomorphic, we can write down their integral equations evaluated respectively in \(k_j^+\) and \(k_j^-\) and compare them to the above integral equations for the residues. We obtain
\[
\text{Res}_{k_j^-} \{\mu_1^-(k, n)\} = (k_j^-)^{-n} C_j^- \mu_2^-(k_j^-, n) \tag{41}\]
\[
\text{Res}_{k_j^+} \{\mu_2^+(k, n)\} = (k_j^+)^{n} C_j^+ \mu_1^+(k_j^+, n). \tag{42}\]

Finally the Riemann-Hilbert problem \([33] [34]\) is completed by the behaviors of the solution \(\mu^\pm\) on the boundaries \(|k| = 0\) and \(|k| = \infty\) of \(\mathcal{D}\), which read
\[
\mu_1^+(k, n) \xrightarrow[k \to 0]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2^-(k, n) \xrightarrow[k \to \infty]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{43}\]
The vectorial Riemann-Hilbert problem (33), (34) with singular points given by (41), (42) and the boundary behaviors (43) constitutes a closed problem which will be solved in the next section.

The behaviors of \( \mu^{-1}(k, n) \) at large \( k \) and of \( \mu^{+2}(k, n) \) at small \( k \) will be useful for the following and we define

\[
\mu_{-11}(k, n) \xrightarrow{k \to \infty} f(n). \tag{44}
\]

Then

\[
\mu_{-21}(k, n) \xrightarrow{k \to \infty} r(n)f(n) \tag{45}
\]

and one easily gets that \( f(n) \) satisfies the integral equation

\[
f(n) = 1 - \sum_{n+1}^{+\infty} q(i)r(i)f(i), \tag{46}
\]

the solution of which is

\[
f(n) = \prod_{n+1}^{+\infty} [1 - q(i)r(i)]. \tag{47}
\]

The same computation holds for \( \mu^{+2}(k, n) \) and we have finally

\[
\mu^{-}(k, n) \xrightarrow{k \to \infty} \begin{pmatrix} f(n) & 0 \\ r(n)f(n) & 1 \end{pmatrix} \tag{48}
\]

\[
\mu^{+}(k, n) \xrightarrow{k \to 0} \begin{pmatrix} 1 & q(n)f(n) \\ 0 & f(n) \end{pmatrix}. \tag{49}
\]

It can be shown easily by using (33) and (34) that the determinant of the matrix \( \mu(k, n) \) is analytic in the whole domain \( \mathcal{D} \). Hence it follows from the Liouville theorem and the boundary values (48), (49) that

\[
\text{det}\{\mu(k, n)\} = f(n). \tag{50}
\]

Note that, within the reduction \( r = -q \), \( f(n) = \rho(n)^{-1} \) as given by (24).

C. Asymptotic behaviors and unitarity relation

By taking the limit at large \( n \) directly on the integral equations, the functions \( \mu^{\pm} \) obey for \( |k| = 1 \)

\[
\mu^{-}(k, n) \xrightarrow{n \to +\infty} \begin{pmatrix} 1 & 0 \\ k^{-n}\hat{\alpha}^{-}(k) & 1 \end{pmatrix} \tag{51}
\]

\[
\mu^{+}(k, n) \xrightarrow{n \to -\infty} \begin{pmatrix} \beta^{+}(k) & 0 \\ k^{n}\hat{\beta}^{-}(k) & \beta^{-}(k) \end{pmatrix} \tag{52}
\]

\[
\mu^{+}(k, n) \xrightarrow{n \to +\infty} \begin{pmatrix} 1 & k^{n}\hat{\alpha}^{+}(k) \\ 0 & 1 \end{pmatrix} \tag{53}
\]

\[
\mu^{-}(k, n) \xrightarrow{n \to -\infty} \begin{pmatrix} \hat{\beta}^{-}(k) & 0 \\ -k^{-n}\hat{\alpha}^{+}(k) & \beta^{+}(k) \end{pmatrix} \tag{54}
\]

where the following alternative scattering data are defined as

\[
\hat{\alpha}^{-}(k) = \sum_{-\infty}^{+\infty} k^{-i}q(i)\mu_{-22}(k,i), \quad \hat{\alpha}^{+}(k) = \sum_{-\infty}^{+\infty} k^{i}r(i)\mu_{11}^{+}(k,i) \tag{55}
\]

\[
\hat{\beta}^{-}(k) = 1 - \sum_{-\infty}^{+\infty} r(i)\mu_{22}^{+}(k,i), \quad \hat{\beta}^{+}(k) = 1 - \sum_{-\infty}^{+\infty} q(i)\mu_{11}^{-}(k,i). \tag{56}
\]
The quantities $\hat{\alpha}^\pm$ are also called the reflection coefficients to the left (referring to the limit $n \to -\infty$) when $\alpha^\pm$ are the reflection coefficients to the right.

It is easy to prove the following relations

$$
\hat{\alpha}^- = \frac{\alpha^+ \beta^-}{1 - \alpha^- \alpha^+}, \quad \hat{\alpha}^+ = \frac{\alpha^- \beta^+}{1 - \alpha^- \alpha^+},
$$

(57)

$$
\hat{\beta}^- = \frac{\beta^+}{1 - \alpha^- \alpha^+}, \quad \hat{\beta}^+ = \frac{\beta^-}{1 - \alpha^- \alpha^+}.
$$

(58)

Indeed, by using the Riemann-Hilbert problems (33) and (34) for $\mu$ (still for $|k| = 1$) we have

$$
\hat{\alpha}^- = \sum_{-\infty}^{+\infty} k^{-i} q(i) \mu_{22}^+(k, i) - \alpha^+ \sum_{-\infty}^{+\infty} q(i) \mu_{21}^+(k, i)
$$

$$
= \alpha^+ \left( 1 - \sum_{-\infty}^{+\infty} q(i) \mu_{21}^+(k, i) \right) = \alpha^+ \hat{\beta}^+, \quad \text{(59)}
$$

$$
\hat{\beta}^+ = 1 - \sum_{-\infty}^{+\infty} q(i) \mu_{21}^-(k, i) + \alpha^- \sum_{-\infty}^{+\infty} k^{-i} q(i) \mu_{22}^-(k, i)
$$

$$
= \beta^- + \alpha^- \hat{\alpha}^-, \quad \text{(60)}
$$

and so on for the other relations.

Now from (27) the determinant of $\mu(k, n)$ as $n \to -\infty$ can be computed and, by using the behaviors of $\mu(k, n)$, it leads to the relation

$$
\beta^+ \beta^- = (1 - \alpha^- \alpha^+) \prod_{-\infty}^{+\infty} [1 - r(i) q(i)] \quad \text{(61)}
$$

which is called the unitarity relation.

D. Reduction

A reduction denotes a simple (possibly algebraic) relation between the potentials (here $r(n)$ and $q(n)$) for which one can derive the counterpart relations for the spectral data. In other word a reduction is a relation which conserves the bijection between potentials and spectral data.

In the case

$$
r(n) = -\overline{q}(n) \leftrightarrow \overline{Q}(n) = \sigma_2 Q(n) \sigma_2, \quad \text{(62)}
$$

it is easy to check that the function

$$
\nu(k, n) = \sigma_2 \overline{\mu}(1/\overline{k}, n) \sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{(63)}
$$

obeys the same equation as $\mu(k, n)$. To compare them it is enough to consider their behaviors as $n \to \pm \infty$. Since

$$
\sigma_2 \overline{\mu}^-(1/\overline{k}, n) \sigma_2 \quad \text{as} \quad n \to \pm \infty \quad \rightarrow \quad \begin{pmatrix} 1 & -k^p \sigma_2^- (1/\overline{k}) \\ 0 & 1 \end{pmatrix}, \quad \text{(64)}
$$

we conclude that

$$
\sigma_2 \overline{\mu}^-(1/\overline{k}, n) \sigma_2 = \mu^+(k, n), \quad \text{(65)}
$$

$$
\sigma^- (1/\overline{k}) = -\alpha^+(k). \quad \text{(66)}
$$

The same calculation at $n \to -\infty$ gives that

$$
\overline{\beta}^- (1/\overline{k}) = \beta^+(k), \quad \text{(67)}
$$
and also that the alternative scattering data obey similar relations.

For the discrete spectrum, the relation (65) implies

$$\text{Res}_{k_j^+} \{ \mu_2^+(k, i) \} = \text{Res}_{k_j^+} \left( \frac{-\pi_{21}(1/k, i)}{\pi_{11}(1/k, i)} \right).$$

(68)

Using the basic relations

$$\text{Res}_{k_0} \{ f(1/k) \} = -(k_0)^2 \text{Res}_{1/k_0} \{ f(k) \}, \quad \text{Res}_{k_0} \{ \overline{g}(k) \} = \overline{\text{Res}}_{k_0} \{ g(k) \},$$

(69)

the equation (68) becomes

$$\text{Res}_{k_j^+} \{ \mu_2^+(k, i) \} = -(k_j^+)^2 \text{Res}_{1/k_j^+} \left( \frac{-\pi_{21}(k, i)}{\pi_{11}(k, i)} \right).$$

(70)

This last equation holds if

$$N^+ = N^-, \quad k_j^+ = \frac{1}{k_j^-}$$

(71)

for which it reads

$$\text{Res}_{k_j^+} \{ \mu_2^+(k, i) \} = -(k_j^+)^2 \text{Res}_{k_j^-} \left( \frac{-\pi_{21}(k, i)}{\pi_{11}(k, i)} \right).$$

(72)

The above relation together with (65) and (71) implies then

$$\frac{C_j^+}{k_j^+} = \frac{\overline{C_j^-}}{k_j^-}.$$  

(73)

In the case of the reduction (62) we will use the following simplified notations (already used in sect. 2)

$$\alpha = \alpha^-, \quad \beta = \beta^-, \quad C_j = C_j^-, \quad k_j = k_j^-, \quad N = N^+ = N^-.$$  

(74)

In order to not over complicate this paper, we do not consider the other reduction \( r = \bar{q} \) for which similar results can be easily obtained, but which corresponds to a spectral problem without discrete spectrum.

Last, it is useful for the following to rewrite the asymptotic boundary behaviors (51)-(54) within the reduction (and with the above notations) and for \(|\zeta| = 1\) as:

$$\begin{pmatrix} \beta^\nu \alpha \gamma \\ 0 \end{pmatrix} \begin{pmatrix} \gamma^\nu \alpha \gamma \\ 0 \end{pmatrix} \xleftarrow{n \to -\infty} \mu^-(\zeta, n) \xrightarrow{n \to +\infty} \begin{pmatrix} 1 \nu \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

(75)

$$\begin{pmatrix} \gamma^\nu \alpha \gamma \\ 0 \end{pmatrix} \begin{pmatrix} \beta^\nu \alpha \gamma \\ 0 \end{pmatrix} \xleftarrow{n \to -\infty} \mu^+(\zeta, n) \xrightarrow{n \to +\infty} \begin{pmatrix} 1 \nu \alpha & 0 \\ 0 & 1 \end{pmatrix},$$

(76)

$$\gamma = \frac{\beta}{1 + |\alpha|^2}.$$  

(77)

Similarly, the unitarity relation (61) together with the definition (47) reads here

$$|\beta|^2 = (1 + |\alpha|^2) f(-\infty).$$

(78)
IV. THE INVERSE SPECTRAL PROBLEM

The inverse spectral problem consists in reconstructing the potentials \( q(n) \) and \( r(n) \) from the spectral data

\[
\alpha^+(k), \beta^+(k), \quad |k| = 1;
\]
\[
C_j^-, k_j^-, |k_j^-| < 1, \quad j = 1, \ldots, N^+, \quad |k_j^-| > 1, \quad j = 1, \ldots, N^-.
\]  

(79)

A simple way of doing this is to reformulate the analytical properties of the matrix \( \mu(k,n) \) in the domain \( D \) as a \( \overline{\partial} \)-problem.

A. Inverse problem as a \( \overline{\partial} \)-problem

Indeed, the set of fundamental relations (83), (84), (81) and (82), which contain all the information about the analytical properties of \( \mu(k,n) \), can be summarized in the formula

\[
\frac{\partial}{\partial R} \mu(k,n) = \mu(k,n) R(k,n), \quad k \in D,
\]  

(80)

where the spectral transform contains all the information and reads

\[
R(k,n) = \begin{pmatrix} 0 & \alpha^+(k) \delta^+(k,1) \\ -\alpha^-(k) \delta^-(k,1) & 0 \end{pmatrix} \begin{pmatrix} k_n & 0 \\ 0 & k^n \end{pmatrix} - 2i\pi \begin{pmatrix} 0 & \sum_{j=1}^{N^+} C_j^+ \delta(k-k_j^+) \\ \sum_{j=1}^{N^-} C_j^- \delta(k-k_j^-) & 0 \end{pmatrix} \begin{pmatrix} k_n & 0 \\ 0 & k^n \end{pmatrix}.
\]  

(81)

The distributions \( \delta^\pm(k,1) \) are defined in the appendix and the distribution \( \delta(k) \) is normalized by requiring that \( \iint dk \wedge d\bar{k} \delta(k) = 1 \). Using the method and tools described in the appendix, and for the behaviors

\[
\mu_1(k,n) \xrightarrow{k \to 0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2(k,n) \xrightarrow{k \to \infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]  

we have the following solution of this \( \overline{\partial} \)-problem

\[
\mu(k,n) = 1 + \frac{1}{2i\pi} \iint_D \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} \mu(\lambda,n) R(\lambda,n) \begin{pmatrix} k/\lambda & 0 \\ 0 & 1 \end{pmatrix}.
\]  

(83)

Due to the particular structure (81) of \( R(k) \), the above matrix valued equation has actually to be understood as two vectorial integral equations, for \( \mu_1^+(k,n) \) and for \( \mu_2^-(k,n) \). As will be seen hereafter, the knowledge of these two vectors is sufficient for solving completely the problem.

B. Reconstruction of the potentials

The potentials are obtained from the asymptotic expansion of the Jost solutions \( \mu_2^- \) and \( \mu_1^+ \) of (83) via the formulae

\[
q(n+1) = -\mu_2^-(n), \quad r(n+1) = -\mu_2^+(n),
\]  

(84)

where \( \mu_2^-(n) \) is the coefficient of \( k^{-1} \) in the Laurent expansion for \( k \to \infty \) of \( \mu_2^-(k,n) \), and \( \mu_2^+(n) \) the coefficient of \( k \) in the Taylor expansion for \( k \to 0 \) of \( \mu_2^+(k,n) \). In particular from (83) we get

\[
r(n+1) = \frac{1}{2i\pi} \oint_C d\zeta \alpha^-(\zeta) \zeta^{-n-2} \mu_2^-(\zeta,n) - \sum_{j=1}^{N^-} C_j^- (k_j^-)^{-n-2} \mu_2^-(k_j^-,n)
\]  

(85)

\[
q(n+1) = \frac{1}{2i\pi} \oint_C d\zeta \alpha^+(\zeta) \zeta^n \mu_1^+(\zeta,n) + \sum_{j=1}^{N^+} C_j^+ (k_j^+)^n \mu_1^+(k_j^+,n)
\]  

(86)
One could easily check that the potentials given by (86) and (85) do obey the reduction (62) when \( \mu(k,n) \) obeys (63), \( \alpha(k) \) obeys (66), \( C_\pm \) obey (73) and \( k_\pm^2 \) obey (71).

Finally the transmission coefficients \( \beta^-(k) \) and \( \beta^+(k) \) are computed from their definitions (36), where the entries \( \mu_{21}^-(k,n) \) and \( \mu_{12}^+(k,n) \) are obtained from the solution \( \mu_1^+ \) and \( \mu_2^- \) by using the explicit relations (33), (34). Equivalently one can use the relations (58) and the behaviors (52) and (54) to get (\(|\zeta|=1\))

\[
\beta^- (\zeta) = \left[ 1 - \alpha^+ \alpha^- \right] \lim_{n \to -\infty} \mu_{22}^-(\zeta, n) \tag{87}
\]

\[
\beta^+ (\zeta) = \left[ 1 - \alpha^+ \alpha^- \right] \lim_{n \to -\infty} \mu_{11}^+(\zeta, n) \tag{88}
\]

Remark. From the other components of \( \mu_1^+ \) and \( \mu_2^- \) we obtain in (25) the following relations

\[
\mu_{22}^{-(1)}(n) = \sum_{n+1}^{\infty} r(i)q(i+1), \quad \mu_{11}^{+(1)}(n) = \sum_{n+1}^{\infty} r(i+1)q(i). \tag{89}
\]

V. THE METHOD OF THE \( \partial \)-PROBLEM

We have shown in the preceding sections that the spectral problem (25) can be mapped to the \( \partial \)-problem (80), namely

\[
\frac{\partial}{\partial k} \mu(k) = \mu(k) R(k), \quad k \in D, \tag{90}
\]

with the boundary behaviors (82). The solution of such a boundary value problem in the complex plane solves the Cauchy-Green integral equation

\[
\mu(k) = 1 + \frac{1}{2\pi i} \int_D \frac{d\lambda \wedge d\overline{\lambda}}{\lambda - k} \mu(\lambda) R(\lambda) \begin{pmatrix} k/\lambda & 0 \\ 0 & 1 \end{pmatrix}. \tag{91}
\]

The purpose of the following is to show that the above integral equation, for the unknown \( \mu \) and the datum \( R \), can be taken as the starting tool. More precisely we shall show how a parametric dependence of \( R \) (on an integer \( n \) and on a real \( t \)) induces the spectral problem (25) and a nonlinear evolution equation.

A. The principal spectral problem

We restrict this study to off-diagonal matrices \( R(k) \) and consider the integral equation (91) as the given tool. If \( R(k) \) depends now on an external integer \( n \), the solution \( \mu(k,n) \) solves then the \( \partial \)-problem (90) with the behaviors

\[
\mu(k,n) \xrightarrow{k \to 0} \begin{pmatrix} 1 & g(n) \\ 0 & f(n) \end{pmatrix} + k\mu^{(1)}(n) + \ldots \tag{92}
\]

\[
\mu(k,n) \xrightarrow{k \to \infty} \begin{pmatrix} f'(n) & 0 \\ h(n) & 1 \end{pmatrix} + \frac{1}{k}\mu^{(-1)}(n) + \ldots \tag{93}
\]

where the functions \( f, f', g \) and \( h \) have to be evaluated. The determinant of \( \mu(k,n) \) is analytic in \( D \) as indeed the off-diagonal structure of \( R(k,n) \) implies

\[
\frac{\partial}{\partial k} \det \{ \mu(k,n) \} = 0, \tag{94}
\]

and from the above behavior the Liouville theorem implies

\[
\det \{ \mu(k,n) \} = f(n) = f'(n). \tag{95}
\]

We chose now the following explicit dependence of \( R(k,n) \) on the discrete variable \( n \)

\[
R(k,n+1) = \Lambda(k)^{-1} R(k,n) \Lambda(k), \tag{96}
\]
with $\Lambda(k)$ defined in (26). The basic fundamental property which allows us to derive from the choice (96) a difference equation for $\mu$ in the variable $n$ is the following

$$\frac{\partial}{\partial k} H(k, n) = 0, \quad H(k, n) = \mu(k, n+1)\Lambda(k)^{-1}\mu(k, n)^{-1}\Lambda(k). \quad (97)$$

The above function $H(k, n)$ can then be reconstructed from its behaviors on the boundary of $\mathcal{D}$ ($k = \infty$ and $k = 0$) which reads from (92) and (93)

$$H(k, n) \rightarrow_{k \to 0} \frac{1}{f(n)} \begin{pmatrix} f(n) - g(n+1)\mu_{21}^{(1)}(n) & g(n+1) \\ -f(n+1)\mu_{21}^{(1)}(n) & f(n+1) \end{pmatrix} \quad (98)$$

$$H(k, n) \rightarrow_{k \to \infty} \frac{1}{f(n)} \begin{pmatrix} f(n+1) & -f(n+1)\mu_{12}^{(-1)}(n) \\ h(n+1) & f(n) - h(n+1)\mu_{12}^{(-1)}(n) \end{pmatrix} \quad (99)$$

Since $H(k, n)$ is analytic, these two behaviors are equal, which implies the following four equations

$$f(n+1) = f(n) - g(n+1)\mu_{21}^{(1)}(n)$$
$$f(n+1) = f(n) - h(n+1)\mu_{12}^{(-1)}(n)$$
$$g(n+1) = -f(n+1)\mu_{12}^{(-1)}(n)$$
$$h(n+1) = -f(n+1)\mu_{21}^{(1)}(n), \quad (100)$$

which are solved by first defining the potentials as

$$q(n+1) = -\mu_{12}^{(-1)}(n), \quad r(n+1) = -\mu_{21}^{(1)}(n), \quad (101)$$

and hence

$$g(n) = q(n)f(n), \quad h(n) = r(n)f(n), \quad (102)$$

with the recursion relation for $f(n)$

$$f(n) = f(n+1)[1 - r(n+1)q(n+1)] \quad (103)$$

of which the solution is indeed given by (47).

Finally, the solution of the $\partial$-problem (97) reads

$$H(k, n) = \frac{1}{1 - r(n+1)q(n+1)} \begin{pmatrix} 1 & q(n+1) \\ r(n+1) & 1 \end{pmatrix} \quad (104)$$

which can be written with (97) as the discrete spectral problem

$$\mu(k, n+1)\Lambda(k)^{-1}\mu(k, n)^{-1}\Lambda(k) = [1 - Q(n+1)]^{-1} \quad (105)$$

where

$$Q(n) = \begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix}. \quad (106)$$

It is convenient for the following to define the quantity

$$U(n) = [1 - Q(n+1)]^{-1} \quad (107)$$

such that the equation for $\mu(k, n)$ reads

$$\mu(k, n+1) = U(n)\Lambda(k)^{-1}\mu(k, n)\Lambda(k) \quad (108)$$

which is precisely the spectral problem (25).
B. Nonanalytic Dispersion Relations. A Theorem

We consider now that $R(k, n)$ depends also on an external real $t$ and address the problem of computing the expression of the time dependence of the solution $\mu$ of (12). The result can be stated as a theorem.

**Theorem.** When the spectral transform $R(k, n, t)$ evolves according to

$$R_t(k, n, t) = [R(k, n, t), \Omega(k, t)] + M(k, n, t)$$

(109)

where

$$M(k, n+1, t) = \Lambda(k)^{-1} M(k, n, t) \Lambda(k), \quad [\Lambda(k), \Omega(k, t)] = 0,$$

(110)

and where $\Omega(k, t)$ is the nonanalytic dispersion relation

$$\Omega(k, t) = \frac{1}{2\pi} \int\int_D d\lambda \wedge d\lambda' \frac{\partial \Omega(\lambda, t)}{\partial \lambda} \left( \frac{k}{\lambda} \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right)$$

(111)

the potential $Q$ obeys the following evolution

$$Q_t(n+1, t) = \left[ \sigma_3, \frac{1}{2\pi} \int\int_D d\lambda \wedge d\lambda' \frac{\partial \Omega(\lambda, t)}{\partial \lambda} T(\lambda, n, t) \right],$$

(112)

where

$$T(k, n, t) = \Lambda(k)^{-1} \mu(k, n, t) \{M(k, n, t) - \frac{\partial \Omega(k, t)}{\partial \lambda} \} \Lambda(k) \mu^{-1}(k, n+1, t).$$

(113)

This theorem is proved hereafter.

**The auxiliary spectral problem.**

Let us define the matrix

$$V(k, n, t) = \{\mu_t(k, n, t) - \mu(k, n, t)\Omega(k, t) \} \mu^{-1}(k, n, t),$$

(114)

and compute its $\partial$-derivative which, from (80), (109) and (110), obeys

$$\frac{\partial V(k, n, t)}{\partial k} = \mu(k, n, t) \{M(k, n, t) - \frac{\partial \Omega(k, t)}{\partial \lambda} \} \Lambda(k) \mu^{-1}(k, n, t).$$

(115)

To solve the above $\partial$-problem we need the behaviors of $V_1$ as $k \to 0$ and $V_2$ as $k \to \infty$. Since

$$\mu(k, n, t) \underset{k \to 0}{\longrightarrow} \begin{pmatrix} 1 & q(n, t)f(n, t) \\ 0 & f(n, t) \end{pmatrix} + k \mu^{(1)}(n, t) + \ldots$$

(116)

$$\mu(k, n, t) \underset{k \to \infty}{\longrightarrow} \begin{pmatrix} f(n, t) \\ r(n, t) \end{pmatrix} + \frac{1}{k} \mu^{(-1)}(n, t) + \ldots$$

(117)

it is easy to obtain, thanks also to the choice (111) (it would not be so in the case of a regular dispersion relation)

$$V_1(k, n, t) \underset{k \to 0}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad V_2(k, n, t) \underset{k \to \infty}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. $$

(118)

Consequently the solution reads

$$V(k, n, t) = \frac{1}{2\pi} \int\int_D d\lambda \wedge d\lambda' \frac{\partial \Omega(\lambda, n, t)}{\partial \lambda} S(\lambda, n, t) \left( \frac{k}{\lambda} \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right).$$

(119)

where we have defined

$$S(k, n, t) = \mu(k, n, t) \{M(k, n, t) - \frac{\partial \Omega(k, t)}{\partial \lambda} \} \mu^{-1}(k, n, t).$$

(120)

Now, with the above value of $V$, the definition (114) can be written as the auxiliary spectral problem
\[ \mu_k(k, n, t) = V(k, n, t) \mu(k, n, t) + \mu(k, n, t) \Omega(k, t), \]  

Fundamental property of \( V(k, n, t) \)

For simplicity of notations, we omit from now on the variable \( t \). By direct computation the matrix \( S \) obeys the equation

\[ S(k, n+1)U(n) = U(n) \Lambda(k)^{-1} S(k, n) \Lambda(k). \]  

The next step consists in seeking an analogous property for \( V(k, n) \), by computing the quantities \( U(n)^{-1} V(k, n+1) \) on one side and \( \Lambda(k)^{-1} V(k, n) \Lambda(k) U(n)^{-1} \) on the other side. Using

\[
\Lambda(\lambda)^{-1} S(\lambda, n) \Lambda(\lambda) U(n)^{-1} \begin{pmatrix} k/\lambda & 0 \\ 0 & 1 \end{pmatrix} - \Lambda(k)^{-1} S(\lambda, n) \begin{pmatrix} k/\lambda & 0 \\ 0 & 1 \end{pmatrix} \Lambda(k) U(n)^{-1} = \frac{\lambda - k}{\lambda} \begin{pmatrix} r(n+1) s_{21}(\lambda, n) - (1/\lambda) s_{22}(\lambda, n) & -q(n+1) s_{11}(\lambda, n) + \lambda s_{12}(\lambda, n) \\ 0 & 0 \end{pmatrix}
\]

we obtain finally the required property of \( V(k, n) \)

\[ U(n)^{-1} V(k, n+1) - \Lambda(k)^{-1} V(k, n) \Lambda(k) U(n)^{-1} = P(n) \]  

\[ P(n) = \frac{1}{2\pi} \int \int d\lambda \int d\tilde{\lambda} \left[ \sigma_3, T(\lambda, n) \right] \]  

\[ T(k, n) = \Lambda(k)^{-1} S(k, n) \Lambda(k) U(n)^{-1} = U(n)^{-1} S(k, n+1). \]  

We have used here above \( S(k, n) \) to rewrite \( T \) in a more convenient form, and finally from the definition \( \Omega(k, n) \) and the spectral problem \( \Omega(k, n) \) it reads

\[ T(k, n) = \Lambda(k)^{-1} \mu(k, n) \{ M(k, n) - \frac{\partial \Omega(k)}{\partial k} \} \Lambda(k) \mu^{-1}(k, n+1). \]  

Note that the matrix \( T(k, n) \) obeys a property similar to \( S(k, n) \) since it can be checked directly that

\[ U(n-1) T(k, n-1) = \Lambda(k) T(k, n) U(n) \Lambda(k)^{-1}. \]  

The evolution equation

The nonlinear evolution of \( Q(n, t) \) is now obtained in the usual way by requiring the compatibility between \( \mu \) and \( \Omega \) which reads

\[
\frac{\partial}{\partial t} \mu(k, n+1, t) = \frac{\partial}{\partial t} \left\{ U(n) \Lambda(k)^{-1} \mu(k, n) \Lambda(k) \right\} \\
= V(k, n+1, t) \mu(k, n+1, t) + \mu(k, n+1, t) \Omega(k, t).
\]

By means of \( \Lambda(k) \) it is then easy to obtain the equation

\[ U_1(n) = U(n) P(n) U(n), \]  

which readily gives the evolution \( U_1 \) since \( U(n) = [1 - Q(n+1)]^{-1} \). This ends the proof of the theorem.

C. Reduction

If we consider the reduction \( r(n) = -7(n) \) the matrices

\[ \Omega(k, t) = \begin{pmatrix} \omega_1(k, t) & 0 \\ 0 & \omega_2(k, t) \end{pmatrix} \]  

\[ M(k, n, t) = \begin{pmatrix} 0 & m_2(k, t) \\ m_1(k, t) & 0 \end{pmatrix} \begin{pmatrix} k^{-n} & 0 \\ 0 & k^n \end{pmatrix}. \]  

must be compatible with the preservation of the structure of \( R(n) \) in the time evolution equation \( \mu \). We choose, therefore, \( \omega_1 \) and \( \omega_2 \) analytic inside and outside the unite circle with limit values on the two sides of the circle satisfying the symmetry properties \( \zeta = e^{i \theta} \)
\[ \omega_1^-(\zeta, t) = \overline{\omega_2^+(\zeta, t)} \]
\[ \omega_2^-(\zeta, t) = \overline{\omega_1^+(\zeta, t)} \]

and we choose \( m_1 \) and \( m_2 \) as
\[ m_1(k, t) = m^-_1(k, t)\delta^-(k, 1) \equiv m(\zeta, t)\delta^-(k, 1) \]
\[ m_2(k, t) = m^+_2(k, t)\delta^+(k, 1) \equiv m(\zeta, t)\delta^+(k, 1) \]

where
\[ \zeta = \frac{k}{|k|} \]

and \( m(\zeta, t) \) is a given function defined on the circle \( |k| = 1 \). Note that the discontinuity of \( \omega_1 \)
\[ p(\zeta, t) = \omega_1^+(\zeta, t) - \omega_1^-(\zeta, t) \]
is related to the discontinuity of \( \omega_2 \) by the formula
\[ \omega_2^+(\zeta, t) - \omega_2^-(\zeta, t) = -\overline{p(\zeta, t)} \]
The analytic properties of \( \omega_1 \) and \( \omega_2 \) are summarized by the formulae (\( \zeta = k/|k| \))
\[ \frac{\partial \omega_1}{\partial k} = p(\zeta)\delta(k, 1) \]
\[ \frac{\partial \omega_2}{\partial k} = -\overline{p(\zeta)}\delta(k, 1) \]

where the distribution \( \delta(k, 1) \) is defined in the appendix. Requiring that \( \omega_1 \to 0 \) for \( k \to 0 \) and \( \omega_2 \to 0 \) for \( k \to \infty \), \( \Omega(k) \) is defined by the following Cauchy-Green formula
\[ \Omega(k) = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta - k} \begin{pmatrix} p(\zeta, t) & 0 \\ 0 & -\overline{p(\zeta, t)} \end{pmatrix} \begin{pmatrix} k/|k| \\ 0 \end{pmatrix} \cdot \]

It results that
\[ \omega_1(k) = \overline{\omega_2(1/|k|)} \]
in agreement with the conditions on the boundaries (132).

It can be shown that with the choices indicated in (132) and (133) the evolution equation (112) is compatible with
\[ q_1(n+1, t) = \frac{1}{2\pi f(n+1, t)} \int_{-\pi}^{+\pi} d\gamma(\theta, n, t), \]

where
\[ \gamma(\theta, n) = -\omega_1^-(\zeta)\mu_1^1(\zeta, n)\mu_2^2(\zeta, n+1) + \omega_1^+(\zeta)\mu_1^1(\zeta, n)\mu_1^2(\zeta, n+1) \]
+ \( \omega_2^-(\zeta)\mu_2^2(\zeta, n)\mu_1^1(\zeta, n+1) - \omega_2^+(\zeta)\mu_1^2(\zeta, n)\mu_1^1(\zeta, n+1) \)
+ \( \alpha(\omega_1^-(\zeta) - \omega_1^+(\zeta))\zeta^{-n}\mu_2^2(\zeta, n)\mu_1^2(\zeta, n+1) \)
+ \( \overline{\mu}(\omega_1^+(\zeta) - \omega_1^-(\zeta))\zeta^{n+1}\mu_1^1(\zeta, n)\mu_1^2(\zeta, n+1) \)
- \( m(\zeta)\zeta^{-n}\mu_1^2(\zeta, n)\mu_2^2(\zeta, n+1) + \overline{\mu}(\zeta)\zeta^{n+1}\mu_1^1(\zeta, n)\mu_1^1(\zeta, n+1) \)

with \( \zeta = e^{i\theta} \) (remember that \( \alpha \equiv \alpha^- \) and \( \alpha^+(\zeta) = -\overline{\mu}(\zeta) \)). Note that in computing \( T(\lambda, n) \) in (112) the term containing \( \Omega(k, t) \) must be written as follows
\[ \Lambda^{-1}\mu(n)\frac{\partial \Omega}{\partial k}\Lambda\mu^{-1}(n+1) = \]
\[ \frac{\partial}{\partial k} \left( \Lambda^{-1}\mu(n)\Omega\Lambda\mu^{-1}(n+1) \right) - \Lambda^{-1}\mu(n) [R(n), \Omega] \Lambda\mu^{-1}(n+1) \]

which is a well defined local formulation of a \( \overline{\partial} \)-problem for a sectionally holomorphic function.

It is convenient, by using equations (133) and (134), to rewrite (141) in terms of \( \mu_1^+ \) and \( \mu_2^- \)
\[ \gamma(\theta, n) = p(\zeta)\mu_1^1(\zeta, n)\mu_2^2(\zeta, n+1) + \overline{\mu}(\zeta)\mu_1^2(\zeta, n)\mu_1^1(\zeta, n+1) \]
- \( m(\zeta)\zeta^{-n}\mu_1^2(\zeta, n)\mu_2^2(\zeta, n+1) + \overline{\mu}(\zeta)\zeta^{n+1}\mu_1^1(\zeta, n)\mu_1^1(\zeta, n+1) \)

which shows explicitly that the evolution equation depends only on the discontinuity of \( \Omega(k) \) on the unit circle.
VI. INTEGRABLE DISCRETE INITIAL-BOUNDARY VALUE PROBLEM

By using the tools previously developed we prove now that the nonlinear system \( [\text{I}] \) is integrable when it is related to the initial boundary value \( [\text{I}] \). We rewrite hereafter this system in the variable \( \zeta = e^{i\theta} \) and with the relation \( [\text{I}] \) as

\[
g_t(n, t) \prod_{i=n+1}^{\infty} (1 + |q(i, t)|^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{in\theta} (A_1 \ast A_2)(\zeta, n, t) \tag{144}
\]

\[
A_1(\zeta, n, t) - A_1(\zeta, n-1, t) = \zeta^{-n} q(n, t) A_2(\zeta, n, t) \tag{145}
\]

\[
A_2(\zeta, n, t) - A_2(\zeta, n-1, t) = -\zeta^n \overline{\theta}(n, t) A_1(\zeta, n, t) \tag{146}
\]

where the interaction term is defined as

\[
(A_1 \ast A_2)(\zeta, n, t) = g(\theta, t) A_1(\zeta, n-1, t) \overline{A}_2(\zeta, n, t) + \overline{g}(\theta, t) A_1(\zeta, n, t) \overline{A}_2(\zeta, n-1, t). \tag{147}
\]

**Theorem.** With the datum of the initial value \( q(n, 0) \) and the following arbitrary boundary values as \( n \to +\infty \)

\[
A_1(\zeta, n, t) \to I_1(\zeta, t), \quad A_2(\zeta, n, t) \to I_2(\zeta, t). \tag{148}
\]

The above system is solvable by the spectral transform method.

A. Proof of integrability

The proof is performed by showing that the evolution \( [\text{I}] \) can actually be written under the form \( [\text{I}] \). This then gives a unique definition of the functions \( m(k) \) and \( \omega(k) \) for which the two equations \( [\text{I}] \) and \( [\text{I}] \) are identical. Hence the evolution \( [\text{I}] \) of the spectral transform \( R(k, t) \) is uniquely given via \( [\text{I}] \) and \( [\text{I}] \).

The first useful property is that the following 5 vectors

\[
\begin{pmatrix}
A_1(\zeta, n) \\
\zeta^{-n} A_2(\zeta, n)
\end{pmatrix}, \quad \begin{pmatrix}
\mu^+_{11}(\zeta, n) \\
\mu^+_{22}(\zeta, n)
\end{pmatrix}, \quad \zeta^{-n} \begin{pmatrix}
\mu^+_{12}(\zeta, n) \\
\mu^+_{21}(\zeta, n)
\end{pmatrix}, \tag{149}
\]

solve the equation \( [\text{I}] \). Then, by comparison of their asymptotic behaviors as \( n \to +\infty \), given in \( [\text{I}] \) and in \( [\text{I}] \), we get

\[
\begin{pmatrix}
A_1(\zeta, n) \\
\zeta^{-n} A_2(\zeta, n)
\end{pmatrix} = I_1(\zeta) \begin{pmatrix}
\mu^+_{11}(\zeta, n) \\
\mu^+_{22}(\zeta, n)
\end{pmatrix} + I_2(\zeta) \zeta^{-n} \begin{pmatrix}
\mu^+_{12}(\zeta, n) \\
\mu^+_{21}(\zeta, n)
\end{pmatrix}. \tag{150}
\]

Next, to compute the product \( (A_1 \ast A_2)(\zeta, n) \) we make use of the Riemann-Hilbert relations

\[
\mu^+_{11}(\zeta, n) - \mu^+_{12}(\zeta, n) = \alpha^{-}(\zeta) \zeta^{-n} \mu^-_{22}(\zeta, n) \tag{151}
\]

\[
\mu^+_{22}(\zeta, n) - \mu^+_{21}(\zeta, n) = -\alpha^{+}(\zeta) \zeta^n \mu^+_{11}(\zeta, n) \tag{152}
\]

and rewrite it in terms only of \( \mu^+_{11} \) and \( \mu^+_{22} \)

\[
\zeta^{n+1} (A_1 \ast A_2)(\zeta, n+1) = -\left( g I_1^2 - \overline{\theta} I_2^2 \right) \mu^+_{11}(n) \mu^+_{12}(n+1) - \frac{1}{|I_1|} (g I_1^2 - |I_2|^2) \zeta \mu^+_{12}(n) \mu^+_{11}(n+1)
\]

\[
- (g + \overline{\theta} I_1 I_2) \zeta^{-n} \mu^+_{12}(n) \mu^+_{12}(n+1) + (g + \overline{\theta} I_1 I_2) \zeta^{n+1} \mu^+_{11}(n) \mu^+_{11}(n+1) \tag{153}
\]

Then, thanks to the expression \( [\text{I}] \), the two equations \( [\text{I}] \) and \( [\text{I}] \) are identical if and only if

\[
p(\zeta, t) = -g(\theta, t)|I_1(\theta, t)|^2 + \overline{\theta}(\theta, t)|I_2(\theta, t)|^2 \tag{154}
\]

\[
m(\zeta, t) = (g(\theta, t) + \overline{\theta}(\theta, t)) I_1(\theta, t) I_2(\theta, t). \tag{155}
\]

Finally the theorem is proved and it remains to compute the evolution of the spectral transform.
B. Evolution of the spectral transform

Evolution of \( \alpha(\zeta, t) \)

The time evolution of \( R(k, n, t) \) is given by \([108]\) with \( M \) and \( \Omega \) defined in \([133]\) and in \([138]\). Taking into account the structure \([51]\) of \( R(k, n, t) \) we have

\[
\begin{align*}
\partial_t \alpha(\zeta, t) &= [\omega_1(\zeta, t) - \omega_2(\zeta, t)] \alpha(\zeta, t) - m(\zeta, t) \\
\partial_t k_j &= 0, \quad \partial_t C_j(t) = [\omega_1(k_j, t) - \omega_2(k_j, t)] C_j(t)
\end{align*}
\]

where from \([138]\) and the Sokhotski–Plemelj formula we have

\[
\begin{align*}
\omega_1(\zeta, t) - \omega_2(\zeta, t) &= \frac{1}{2} p(\zeta, t) - \frac{1}{2} p(\zeta, t) \\
&+ \frac{1}{2\pi i} P \oint_C \frac{d\zeta'}{\zeta' - \zeta} p(\zeta', t) \frac{\zeta}{\zeta'} \\
&+ \frac{1}{2\pi i} P \oint_C \frac{d\zeta'}{\zeta' - \zeta} \bar{p}(\zeta', t)
\end{align*}
\]

\[
\begin{align*}
\omega_1(k_j, t) - \omega_2(k_j, t) &= \frac{1}{2} \int_C \frac{d\zeta'}{\zeta' - k_j} p(\zeta', t) \frac{k_j}{\zeta'} \\
&+ \frac{1}{2\pi i} P \oint_C \frac{d\zeta'}{\zeta' - k_j} \bar{p}(\zeta', t).
\end{align*}
\]

and the functions \( p(\zeta, t) \) and \( m(\zeta, t) \) are given in \([154]\) and \([157]\).

As a result the evolution equation of \( \alpha \) and \( C_j \) can be written

\[
\begin{align*}
\partial_t \alpha &= \alpha \frac{g + \gamma}{2} ([I_1(\theta, t)]^2 - [I_2(\theta, t)]^2) - (g + \gamma) I_1 I_2 \\
&- \alpha \frac{1}{2\pi i} P \oint_C \frac{d\zeta'}{\zeta' - \zeta} (g + \gamma)([I_1]^2 - [I_2]^2) \\
&+ \alpha \frac{1}{2\pi i} \oint_C \frac{d\zeta'}{\zeta'} (g[I_1]^2 - \gamma I_2)^2, \\
\partial_t C_j(t) &= -C_j(t) \frac{1}{2\pi i} P \oint_C \frac{d\zeta'}{\zeta' - k_j} (g + \gamma)([I_1]^2 - [I_2]^2) \\
&+ C_j(t) \frac{1}{2\pi i} \oint_C \frac{d\zeta'}{\zeta'} (g[I_1]^2 - \gamma I_2)^2.
\end{align*}
\]

Evolution of \( \beta(\zeta, t) \)

The definition \([58]\) allows to obtain readily

\[
\frac{\hat{\beta}_+^t}{\beta_+} - \beta = -\frac{(|\alpha|^2)}{1 + |\alpha|^2},
\]

which actually can be understood as a Riemann-Hilbert problem on the unit circle. Its solution reads

\[
\begin{align*}
|k| > 1 : \quad \frac{\hat{\beta}_+^t}{\beta_+} &= \frac{\partial}{\partial t} \left( \frac{|\beta|^2}{1 + |\alpha|^2} \right) 1 + |\alpha|^2 \\
&- \frac{1}{2\pi i} \oint_C \frac{d\zeta'}{\zeta' - k} \left( \frac{|\alpha(\zeta')|^2}{1 + |\alpha(\zeta')|^2} \right)
\end{align*}
\]

\[
|k| < 1 : \quad \frac{\beta_+^t}{\beta} = -\frac{1}{2\pi i} \oint_C \frac{d\zeta'}{\zeta' - k} \frac{k}{\zeta'} \left( |\alpha(\zeta')|^2 \right).
\]

Hence, writing the above equation for \( k = \zeta(1 - 0) \), we get the evolution \([14]\).
C. Evolution of the spectral transform from the Lax pair

For completeness, we rederive hereafter the preceding formula (evolution of $\alpha$ and $\beta$, in the absence of bound states for simplicity), by using the traditional approach for which the starting tool is the Lax pair (121). The method consists simply in evaluating the asymptotic boundary values as $n \to \pm \infty$ on the auxiliary spectral problem (119), in which (forget for a while the $(n,t)$-dependence)

$$V(k) = \frac{1}{2i\pi} \int \frac{d\lambda \wedge d\overline{\lambda}}{\lambda - k} \mu(\lambda) \left( M(\lambda) - \frac{\partial \Omega(\lambda)}{\partial \lambda} \right) \mu^{-1}(\lambda) \left( \begin{array}{cc} k/\lambda & 0 \\ 0 & 1 \end{array} \right).$$  \hspace{1cm} (170)

By using the identity (142), the equation (121) can be more conveniently written as

$$\mu(k,n,t) = X(k,n,t)\mu(k,n,t),$$  \hspace{1cm} (171)

where

$$X(k) = \frac{1}{2i\pi} \int \frac{d\lambda \wedge d\overline{\lambda}}{\lambda - k} \mu(\lambda,n,t) \left( M(\lambda) + [R(\lambda),\Omega(\lambda)] \right) \mu^{-1}(\lambda) \left( \begin{array}{cc} k/\lambda & 0 \\ 0 & 1 \end{array} \right).$$  \hspace{1cm} (172)

By inserting in the above equation the explicit forms of $R(k,n,t)$ given in (81) with (66) and no bound states, of $\Omega(k,t)$ given in (138), and of $M(k,n,t)$ given in (131) with (133), we get finally

$$X(k,n,t) = \frac{1}{2i\pi} \int \frac{d\zeta'}{\zeta' - k} \frac{1}{f(n)} \chi(\zeta',n,t) \left( \begin{array}{cc} k/\zeta' & 0 \\ 0 & 1 \end{array} \right),$$  \hspace{1cm} (173)

with the following definition

$$\chi(\zeta) = \zeta^n[\bar{m} + (\omega_1^+ - \omega_2^+)\alpha]\left( \begin{array}{cc} -\mu_{11}^+\mu_{21}^+ & (\mu_{11}^+)^2 \\ -\mu_{12}^+\mu_{22}^+ & (\mu_{12}^+)^2 \end{array} \right) + \zeta^{-n}[m - (\omega_1^- - \omega_2^-)\alpha]\left( \begin{array}{cc} \mu_{12}\mu_{22}^+ & -\mu_{12}^+\mu_{22} \\ \mu_{12}^+\mu_{22} & -\mu_{12}\mu_{22}^+ \end{array} \right).$$  \hspace{1cm} (174)

The main tool is now the asymptotic boundary behaviors (75) and (76) of $\mu^\pm$ which allows to obtain, by taking the limit as $n \to +\infty$ of $\zeta^n\partial_t\mu_{21}^\pm(\zeta,n,t)$, the relations

$$\alpha_t(k) = -\frac{1}{2} \left[ m - (\omega_1^- - \omega_2^-)\alpha \right](k) + \lim_{n \to \infty} \frac{1}{2\pi i} P \int \frac{d\zeta}{\zeta - k} \left( \begin{array}{c} k \\ \zeta \end{array} \right)^{n+1} \left[ m - (\omega_1^- - \omega_2^-)\alpha \right](\zeta)$$

$$0 = -\frac{1}{2} \left[ \bar{m} + (\omega_1^+ - \omega_2^+)\bar{\alpha} \right](k) + \lim_{n \to \infty} \frac{1}{2\pi i} P \int \frac{d\zeta}{\zeta - k} \left( \begin{array}{c} \zeta \\ k \end{array} \right)^{n} \left[ \bar{m} + (\omega_1^+ - \omega_2^+)\bar{\alpha} \right](\zeta).$$

Consequently, with the formula (see appendix)

$$\lim_{n \to \infty} \frac{1}{2\pi i} P \int \frac{d\zeta}{\zeta - k} \left( \begin{array}{c} k \\ \zeta \end{array} \right)^{n+1} \Phi(\zeta) = \frac{1}{2}\Phi(k), \quad |k| = 1, \quad (175)$$

the preceding relations result precisely in the required evolution (156).

In the same way, by taking the limit as $n \to -\infty$ of $\partial_t\mu_{11}^\pm(\zeta,n,t)$, we obtain readily the required evolution (4) of the transmission coefficient $\beta(\zeta,t)$ (note that there, one should use also the unitarity relation (78)).

D. Time evolution of $f(n)$

It could be useful to have also explicitly the time evolution of the quantity $f(n)$. From (50) we have
\[ f_i(n) = f(n) \text{ tr} \left\{ \mu_i(k,n) \mu^{-1}_i(k,n) \right\} \]  

(176)

and then using the auxiliary spectral problem (171)

\[ f_i(n) = f(n) \text{ tr} \{ X(k,n) \} . \]  

(177)

From the expression (173) we obtain that the trace of \( X(k,n) \) is \( k \)-independent and reads

\[ \text{tr} \{ X(k,n) \} = \frac{1}{f(n)} \int \frac{d\zeta}{\bar{\zeta}} \left\{ \zeta^n \bar{\mu}_{11}^+ \mu_{21}^+ - \zeta^{-n} m \mu_{12} \mu_{22}^- + (p + \bar{p}) \mu_{12}^+ \mu_{21}^+ \right. \]
\[ \left. - (\omega_1^+ - \omega_2^+) \mu_{12}^+ \mu_{21}^+ + (\omega_1^- - \omega_2^-) \mu_{12}^- \mu_{21}^- \right\} . \]  

(178)

Due to the analyticity of the function

\[ \frac{1}{k} (\omega_1(k) - \omega_2(k)) \mu_{12}(k) \mu_{21}(k) \]

inside and outside of the circle the last two terms in the r.h.s. vanish and we obtain for the evolution equation of \( f(n) \)

\[ f_i(n) = \int_{-\pi}^{+\pi} d\theta \left\{ \zeta^n \bar{\mu}_{11}^+ \mu_{21}^+ - \zeta^{-n} m \mu_{12} \mu_{22}^- + (p + \bar{p}) \mu_{12}^+ \mu_{21}^+ \right\} . \]  

(179)

This result can also be expressed in terms of the physical quantities \( A_j \) and \( I_j \) by inverting (150) to get on the unit circle

\[ \mu_{11}^+ = \bar{\mu}_{22} = \frac{A_1 \bar{T}_1 + A_2 \bar{I}_2}{|I_1|^2 + |I_1|^2} \]
\[ \mu_{21}^+ = -\bar{\mu}_{12} = -\zeta^{-n} \frac{\bar{A}_1 \bar{I}_2 - A_2 \bar{T}_1}{|I_1|^2 + |I_1|^2} \]

and then by inserting these formulae and those for \( p + \bar{p} \) and \( m \) in (154) and (155) into (173). We obtain finally

\[ f_i(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( g + \bar{g} \right) \frac{|I_1|^2 |A_2|^2 - |I_2|^2 |A_1|^2}{|I_1|^2 + |I_2|^2} . \]  

(180)

Note that \( f(n) \) is conserved if \( g \) is pure imaginary.

**APPENDIX A: MATHEMATICAL TOOLS**

1. Basic distributions

The distributions \( \delta^\pm(\lambda,1) \) and \( \delta(\lambda,1) \) have support on the circle \( C \) of radius 1 in the complex \( \lambda \)-plane and are defined by the following formulae

\[ \int_D d\lambda \wedge d\bar{\lambda} \delta^\pm(\lambda,1)f(\lambda) = \oint_C d\zeta f((1 \mp 0)\zeta) \]  

(A1)

\[ \int_D d\lambda \wedge d\bar{\lambda} \delta(\lambda,1)f(\lambda) = \oint_C d\zeta f(\zeta) \]  

(A2)

or, equivalently, by

\[ \int_D d\lambda \wedge d\bar{\lambda} \delta^\pm(\lambda,1)f(\lambda) = i \int_{-\pi}^{\pi} d\theta e^{i\theta} f((1 \mp 0)e^{i\theta}) \]  

(A3)

\[ \int_D d\lambda \wedge d\bar{\lambda} \delta(\lambda,1)f(\lambda) = i \int_{-\pi}^{\pi} d\theta e^{i\theta} f(e^{i\theta}) . \]  

(A4)

Note that the distributions \( \delta^\pm(\lambda,1) \) can operate on functions which have defined left or right limit on \( C \), while the distribution \( \delta(\lambda,1) \) can operate on functions continuous on \( C \) or with support on \( C \).
By complex conjugation of the equation (A3) and by the change of variable \( \theta \rightarrow -\theta \), we obtain
\[
\iint_D d\lambda \wedge d\lambda \delta^\pm(\lambda, 1) f(\lambda) = i \int_{-\pi}^{\pi} d\theta e^{i\theta} f((1 \mp 0)e^{-i\theta}).
\] (A5)

Next, by means of the change of variable \( \lambda \rightarrow 1/\lambda \) (remember that the domain \( D \) does not contain the point \( \lambda = 0 \)) and \( \theta \rightarrow -\theta \), we obtain
\[
\iint_D d\lambda \wedge d\lambda \delta^\pm(\frac{1}{\lambda}, 1) f(\lambda) = i \int_{-\pi}^{\pi} d\theta e^{i\theta} f((1 \pm 0)e^{-i\theta}),
\] (A6)
and consequently
\[
\delta^\pm(\frac{1}{\lambda}, 1) = \delta^\mp(\lambda, 1).
\] (A7)

Now, through the change of variable \( \lambda \rightarrow \overline{\lambda} \), we obtain
\[
\iint_D d\lambda \wedge d\lambda \delta^\pm(\overline{\lambda}, 1) f(\lambda) = i \int_{-\pi}^{\pi} d\theta e^{i\theta} f((1 \pm 0)e^{-i\theta}),
\] (A8)
which implies the second symmetry property
\[
\delta^\pm(\overline{\lambda}, 1) = \delta^\pm(\lambda, 1).
\] (A9)
These two relation naturally leads to
\[
\delta^+(\lambda, 1) = \delta^-(\frac{1}{\lambda}, 1).
\] (A10)
Similar symmetry properties can be obtained for \( \delta(\lambda, 1) \).

2. The generalized \( \overline{\partial} \)-formula

Let \( F^+ \in C^1(D^+) \) where \( D^+ \) is the open disk of radius 1 centered in the origin and \( F^- \in C^1(CD^+) \), let \( F^+ \) and \( F^- \) satisfy the Hölder condition on the circle \( \mathcal{C} \) of radius 1 and let \( F^- \) vanish at large \( z \). Then by noting \( F(z) \) the function defined as \( F^+(z) \) for \( z \in D^+ \) and as \( F^-(z) \) for \( z \in D^- \equiv \overline{CD^+} \) we have
\[
F(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{F^+(\zeta) - F^-(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \iint_{D^+} \frac{\partial F/\partial \overline{\lambda}}{\lambda - z} d\lambda \wedge d\lambda + \frac{1}{2\pi i} \iint_{D^-} \frac{\partial F/\partial \overline{\lambda}}{\lambda - z} d\lambda \wedge d\lambda
\] (A11)
where the circle \( \mathcal{C} \) is anti clockwise oriented.

If we define the \( \overline{\partial} \)-derivative of a function \( F(z) \) discontinuous on \( \mathcal{C} \) as follows
\[
\frac{\partial F}{\partial \overline{\sigma}} = (F^+(\zeta) - F^-(\zeta)) \delta(z, 1) + \phi_{D^+}(z) \frac{\partial F}{\partial \overline{\sigma}} + \phi_{D^-}(z) \frac{\partial F}{\partial \overline{\sigma}}, \quad \zeta = \frac{z}{|z|}
\] (A12)
where \( \phi_A(z) = 1 \) for \( z \in A \) and \( \phi_A(z) = 0 \) for \( z \notin A \) the generalized \( \overline{\partial} \)-formula (A11) can be rewritten as
\[
F(z) = \frac{1}{2\pi i} \iint_{D^+ \cup D^-} \frac{\partial F/\partial \overline{\lambda}}{\lambda - z} d\lambda \wedge d\lambda.
\] (A13)
Formula (A12) can be considered as the local formulation of the generalized Cauchy–Green formula (A11).

Subtracting formula (A11) at \( z = a \) we obtain
\[ F(z) = F(a) + \frac{1}{2\pi i} \int_{C} \frac{F^+(\zeta) - F^-(\zeta)}{\zeta - z} \, d\zeta + \]
\[ \frac{1}{2\pi i} \int_{D^+_{a,\epsilon}} \frac{\partial F/\partial \lambda}{\lambda - z} \, d\lambda \wedge d\lambda + \frac{1}{2\pi i} \int_{D^-_{a,\epsilon}} \frac{\partial F/\partial \lambda}{\lambda - z} \, d\lambda \wedge d\lambda \]  
\[ (A14) \]

if \( a \in D^+ \) and an analogous formula if \( a \in D^- \). The second integral on the right side is obtained first by computing it on the set \( D_{a,\epsilon} = \{ \lambda : \lambda \in D, |\lambda - a| > \epsilon \} \) and then by taking the limit \( \epsilon \to 0 \). Note that the formula remains valid also if \( F(z) \) is going to a constant different from 0 for \( z \to \infty \).

If for \( z \to \infty \) \( F(z) \to F(\infty) \) we can apply (A11) to \( F(z) - F(\infty) \) getting
\[ F(z) = F(\infty) + \frac{1}{2\pi i} \int_{C} \frac{F^+(\zeta) - F^-(\zeta)}{\zeta - z} \, d\zeta + \]
\[ \frac{1}{2\pi i} \int_{D^+} \frac{\partial F/\partial \lambda}{\lambda - z} \, d\lambda \wedge d\lambda + \frac{1}{2\pi i} \int_{D^-} \frac{\partial F/\partial \lambda}{\lambda - z} \, d\lambda \wedge d\lambda. \]  
\[ (A15) \]

Finally let us note that the Sokhotski-Plemelj formula on the circle reads
\[ \oint_{C} \frac{d\zeta'}{\zeta' - (1 \mp 0)} f(\zeta') = \pm i\pi f(\zeta), \quad |\zeta'| = 1, \]  
\[ (A16) \]
where \( \text{P} \oint \) denote the Cauchy principal value integral.

## 3. Limits of integrals

Let us proof that
\[ \lim_{n \to \infty} \frac{1}{2\pi i} P \oint_{C} \frac{d\zeta}{\zeta - k} \left( \frac{k}{\zeta} \right)^n \Phi(\zeta) = -\frac{1}{2} \Phi(k), \quad |k| = 1. \]  
\[ (A17) \]

Under the following successive changes of variables
\[ \zeta = e^{i\theta}, \quad k = e^{i\varphi}, \quad \alpha = \vartheta - \varphi, \quad x = n\alpha \]  
\[ (A18) \]
we derive
\[ \frac{1}{2\pi i} P \oint_{C} \frac{d\zeta}{\zeta - k} \left( \frac{k}{\zeta} \right)^n \Phi(\zeta) = \frac{1}{4\pi i} P \int_{-\pi+\varphi}^{\pi-\varphi} \, dx \frac{e^{-ix}}{n \sin(x/2n)} e^{ix/2n} \Phi(e^{i(x/2n)+\varphi}) \]  
\[ (A19) \]
and taking the limit, for \(-\pi < \varphi < \pi,\)
\[ \lim_{n \to \infty} \frac{1}{2\pi i} P \oint_{C} \frac{d\zeta}{\zeta - k} \left( \frac{k}{\zeta} \right)^n \Phi(\zeta) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \, dx \frac{e^{-ix}}{x} \Phi(e^{i\varphi}) \]
\[ = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \, dx \frac{\sin x}{x} \Phi(e^{i\varphi}) = -\frac{1}{2} \Phi(e^{i\varphi}). \]

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