Non-integrability of restricted double pendula

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Abstract

We consider two special types of double pendula, with the motion of masses restricted to various surfaces. In order to get quick insight into the dynamics of the considered systems the Poincaré cross sections as well as bifurcation diagrams have been used. The numerical computations show that both models are chaotic which suggest that they are not integrable. We give an analytic proof of this fact checking the properties of the differential Galois group of the system’s variational equations along a particular non-equilibrium solution.

Keywords: non-integrability; double pendulum; chaotic Hamiltonian systems; variational equations; differential Galois group; Morales-Ramis theory.

1. Introduction

The complicated behaviour of the simple double pendulum is well known but still fascinating—because it is probably the simplest mechanical system exhibiting chaos. We can find many interesting information about its behaviour on the web [9], in books [1, 4] as well as in the scientific articles [5, 2, 3]. Since such a system depends on parameters, like masses and lengths of arms, its integrability analysis is very difficult. Until now, there is no closed mathematical proof confirming its non-integrability. However, in the nineties of the XX century, Morales-Ruiz and Ramis showed that integrability in the Liouville sense imposes a very restrictive condition for the identity component of the differential Galois group of variational equations obtained by the linearization of the equations of motion along a particular non-equilibrium solution. The main theorem of this theory states that if the system is integrable in the Liouville sense, then the identity component of differential Galois group of normal variational equations is Abelian. Note that this is a necessary condition but not a sufficient one. For the precise definition of Morales–Ramis theory and differential Galois group see e.g. [8, 10, 12].

The weak point of the Morales–Ramis theory is that it requires the knowledge of a particular solution that is not an equilibrium position. Unfortunately, for the ordinary planar double pendulum we cannot find it, while for the fully three-dimensional double pendulum, the solutions are too simple to provide any restrictions. Thus, in order to solve this inconvenience we will consider some special types of double pendula restricted to certain surfaces, for which the particular non-equilibrium solutions are known. Thanks to this delicate modification we are able to proceed with the whole integrability analysis of these models, so that we can move closer to the proof of the non-integrability of the double pendulum than ever before.

2. Model 1: simple broken pendulum

The first model under consideration consists of two simple pendula such that the first one with length $l_1$ and mass $m_1$ is attached to a pivot point and moves in the $(x, y)$ plane. The second pendulum with $l_2$ and $m_2$ is restricted to the plane containing $m_1$ and parallel to the $(x, z)$ plane. Looking at the Figure[1] we can immediately write the Lagrange function...
In order to have invertible Legendre transformation we assume that the determinant of the kinetic energy matrix does not vanish, which is valid provided that

\[ l_1 l_2 m_2 \neq 0. \]

Then, the Hamiltonian function is the following

\[
H = \frac{1}{2} \left( l_1^2 (m_1 + m_2) \dot{\phi}_1^2 + l_2^2 m_2 \dot{\phi}_2^2 + 2 l_1 l_2 m_2 \dot{\phi}_2 \phi_1 \sin \phi_1 \sin \phi_2 \right) + g l_1 m_1 \cos \phi_1 + g m_2 \left( l_1 \cos \phi_1 + l_2 \cos \phi_2 \right) .
\]

(2.1)

and its corresponding canonical equations are given by

\[
\dot{\phi}_1 = \frac{l_2 \dot{p}_1 - l_1 \dot{p}_2 \sin \phi_1 \sin \phi_2}{l_1^2 l_2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 (\phi_2) \right) + m_1 \right)} , \]

\[
\dot{\phi}_2 = \frac{l_2 m_2 \dot{p}_1 \sin \phi_1 \sin \phi_2 - l_1 \left( m_1 + m_2 \right) \dot{p}_2}{l_1 l_2^2 \left( m_2 \left( \sin^2 \phi_1 \sin^2 \phi_2 - 1 \right) - m_1 \right)} ,
\]

\[
\dot{p}_1 = -\frac{m_2 \dot{p}_2^2 \sin \phi_1 \sin^2 \phi_2 \cos \phi_1}{l_1^2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 \phi_2 \right) + m_1 \right)} - \frac{(m_1 + m_2) \dot{p}_2^2 \sin (2 \phi_1) \sin^2 \phi_2}{2 l_2^2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 \phi_2 \right) + m_1 \right)}
\]

(2.2)

\[
+ \frac{p_1 p_2 \sin \phi_1 \sin \phi_2 \cos \phi_1 \left( m_2 \left( \sin^2 \phi_1 \sin^2 \phi_2 + 1 \right) + m_1 \right)}{l_1 l_2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 \phi_2 \right) + m_1 \right)} - gl_1 \left( m_1 + m_2 \right) \sin \phi_1,
\]

\[
\dot{p}_2 = -\frac{m_2 \dot{p}_1^2 \sin \phi_1 \sin^2 \phi_2 \cos \phi_2}{l_1^2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 \phi_2 \right) + m_1 \right)} - \frac{(m_1 + m_2) \dot{p}_1^2 \sin^2 \phi_1 \sin \phi_2 \cos \phi_2}{2 l_2^2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 \phi_2 \right) + m_1 \right)}
\]

\[
+ \frac{p_1 p_2 \sin \phi_1 \cos \phi_2 \left( m_2 \left( \sin^2 \phi_1 \sin^2 \phi_2 + 1 \right) + m_1 \right)}{l_1 l_2 \left( m_2 \left( 1 - \sin^2 \phi_1 \sin^2 \phi_2 \right) + m_1 \right)} - gl_2 m_2 \sin \phi_2.
\]

(2.3)

2.1. Numerical analysis

As the evolution of our system takes place in a four dimensional phase space, it is convenient to use the so-called Poincaré sections. These are simply the intersections of orbits with a suitably chosen surface. This and all the other numerical results were obtained using the Bulirsch-Stoer modified midpoint method with Richardson extrapolation. Figures 2 and 3 present such sections for increasing values of energy. They were constructed for the following constant parameters

\[
m_1 = 2, \quad m_2 = 1, \quad l_1 = 2, \quad l_2 = 3, \quad g = 1,
\]

(2.4)
Figure 2: The Poincaré cross sections of the first system on the surface $\phi_1 = 0$.

(a) $E = -8.9$

(b) $E = -6$

Figure 3: The Poincaré cross sections of the first system on the surface $\phi_1 = 0$.

(a) $E = -4$

(b) $E = -3$

with the surface $\phi_1 = 0$ and $p_1 > 0$, restricted to the plane $(\phi_2, p_2)$. It is easy to verify that the global energy minimum corresponding to the state of rest in the bottom position of the pendula has the value $E_0 = -9$.

As expected, at the energy level $E = -8.9$ the pendula oscillate near the equilibrium point as shown in Figure 2(a). The image is very regular, in the center we detect the stable particular periodic solution that is surrounded by invariant tori. The situation becomes more complex when we increase energy to the value of $E = -6$. The invariant tori located at the center become visibly deformed. Some of them decay and we observe the bifurcations that lead to emergence of stable periodic solutions which are enclosed by a separatrix. Thus, we may expect the first sign of chaotic behaviour exactly in this region. Our suspicion is confirmed in Figure 3. We can notice at first sight that the chaotic behaviour appears in the region were the separatrix was located. For a non-integrable Hamiltonian system, these random-looking points correspond to the fact that the trajectories can freely wander over larger regions of the phase space. The trajectories (in contrast to the integrable system) are no longer confined to the surfaces of a set of nested tori, but they begin to move outside the tori. Loosely speaking, the tori are destroyed. The last Poincaré section shows the highly chaotic stage of the system.

For the energy $E = -3$ almost all of the regular orbits merge into a global chaotic region. Figure 4 shows the bifurcation diagram representing the relationship between oscillations in $\phi_2$
and energy $E$. For a given initial condition $(\phi_2, p_2) = (0, 4.9)$ we effectively construct the Poincaré cross sections with cross plane $\phi_1 = 0$ with gradually increasing energy. The periodic motion diverges as the energy increases and finally becomes chaotic. We can also detect the stable “windows” between completely chaotic regions.

Clearly, the numerical analysis demonstrates that the system is chaotic, and the main goal of this article is to also prove that the system is not integrable for a wide range of parameters (almost everywhere).

2.2. The non-integrability proof

Below we formulate the main theorem of this subsection

**Theorem 2.1.** The system governed by Hamiltonian (2.2) is not integrable in the Liouville sense in the class of meromorphic functions of coordinates and momenta, except possibly for energy $E \in \{-g[l_1(m_1 + m_2) + l_2m_2], -g[l_1(m_1 + m_2) - l_2m_2]\}$.

**Proof.** The system (2.3) has two known invariant manifolds

$$\mathcal{N}_1 = \left\{ (\phi_1, \phi_2, p_1, p_2) \in \mathbb{C}^4 \mid \phi_1 = p_1 = 0 \right\}, \quad \mathcal{N}_2 = \left\{ (\phi_1, \phi_2, p_1, p_2) \in \mathbb{C}^4 \mid \phi_2 = p_2 = 0 \right\}. $$

For further analysis we chose $\mathcal{N}_1$, due to considerably simpler characteristic exponents, which will become apparent in a moment. Restricting the right hand sides of (2.3) to $\mathcal{N}_1$, we obtain

$$\dot{\phi}_1 = 0, \quad \dot{\phi}_2 = \frac{p_2}{l_2^2m_2}, \quad \dot{p}_1 = 0, \quad \dot{p}_2 = -gl_2m_2 \sin \phi_2. \quad (2.5)$$

Hence, we have a one-parameter family of particular solutions lying on $H = E$

$$\phi_2^2 = \frac{2[E + gl_2m_2 \cos \phi_2 + gl_1(m_1 + m_2)]}{l_2^2m_2}. \quad (2.6)$$

Solving equations (2.5) and taking into account (2.6), we obtain our particular solution $\phi(t) = (0, \phi_2(t), 0, p_2(t))$, where by a slight abuse of notation we use the variable symbols for particular functions of time. Let $\nu = (\Phi_1, \Phi_2, \bar{P}_1, \bar{P}_2)^T$ denote the variation of $(\phi_1, \phi_2, p_1, p_2)^T$, then the variational equations along
\( \varphi(t) \) take the form
\[
\frac{dv}{dt} = Lv,
\]
\[
L = \begin{pmatrix}
-\frac{p_2 \sin \phi_2}{l_1 l_2 m_3} & 0 & \frac{1}{l_1 l_2 m_3} & 0 \\
0 & 0 & 0 & \frac{1}{l_1 l_2 m_2} \\
-\frac{p_2 \sin \phi_2}{l_2 m_3} & -gl_1 m_3 & 0 & \frac{p_2 \sin \phi_2}{l_1 l_2 m_3} \\
0 & -gl_2 m_2 \cos \phi_2 & 0 & 0
\end{pmatrix},
\]
where \( m_3 = m_1 + m_2 \). Notice that equations for \( \Phi_1 \) and \( P_1 \) form a closed subsystem, called the normal variational equations, that can be rewritten as one second order differential equation
\[
\Phi'' + \frac{p_2^2 \cos \phi_2 + gl_2^3 m_2 (m_2 \cos^2 \phi_2 + m_1)}{l_1 l_2^3 m_2 (m_1 + m_2)} \Phi = 0, \quad \Phi = \Phi_1.
\]

Next, by means of the change of the independent variable
\[
t \rightarrow z = \cos \phi_2(t),
\]
and transformation of derivatives
\[
\frac{d}{dt} = z \frac{dz}{d z'}, \quad \frac{d^2}{dt^2} = z^2 \frac{d^2 z}{d z'^2} + \frac{dz}{dz'},
\]
we can rewrite equation (2.8) as
\[
\Phi'' + p(z)\Phi' + q(z) \Phi = 0, \quad ' \equiv \frac{d}{dz'}
\]
with rational coefficients
\[
p = \frac{z}{z^2 - 1} + \frac{1}{2(z + \gamma)}, \quad q = -\frac{\beta + 3z^2 + 2\gamma z}{2a(\beta + 1) (z^2 - 1) (z + \gamma)},
\]
where in the last steep we used (2.6). The explicit forms of the parameters \((\alpha, \beta, \gamma)\) are the following
\[
\alpha = \frac{l_1}{l_2}, \quad \beta = \frac{m_1}{m_2}, \quad \gamma = \alpha + \alpha \beta + \frac{E}{l_2 m_2 g}.
\]

Next, let us apply the classical change of the dependent variable
\[
\Phi = w \exp \left[ -\frac{1}{2} \int_{z_0}^{z} p(s) ds \right],
\]
which transforms (2.10) into its reduced form
\[
w'' = r(z)w, \quad r(z) = -q(z) + \frac{1}{2} p'(z) + \frac{1}{4} p(z)^2,
\]
where the coefficient \( r(z) \) is given by
\[
r(z) = -\frac{3}{16} \left[ \frac{1}{(z - 1)^2} + \frac{1}{(z + 1)^2} + \frac{1}{(z + \gamma)^2} \right] + \frac{4\beta + \gamma (a \beta + a + 8z) + 3z (a \beta + a + 4z)}{8a(z^2 - 1)(z + \gamma)(\beta + 1)}.
\]

In order to avoid the confluence of the singularities we assume that \( \gamma \neq \pm 1 \). This corresponds to the fact that certain values of energy should be excluded, namely
\[
\{E_1, E_2\} = \{-g[l_1(m_1 + m_2) + l_2 m_2], -g[l_1(m_1 + m_2) - l_2 m_2]\}.
\]

Notice that these two energies are the stationary point energies of the system (2.2). The first one, \( E_1 \), is the global energy minimum corresponding to the equilibrium solution when both pendula are at
rest. The second energy, \( E_2 \), corresponds to the case when the first pendulum is pointing down and the second one up, both at rest, although the equilibrium position is not the only solution with this energy.

If there existed an additional first integral, it would not depend on the energy value, so in particular it would exist for all generic values of energy other than \( E_1 \) and \( E_2 \). In order to exclude such an integral we can thus safely assume that (2.14) is always satisfied.

However, it could happen that there is a first integral, which is conserved only on those special hypersurfaces. We cannot preclude its existence from the below analysis, but note that such an isolated constant of motion would be of much less practical value. Also, the physical solution corresponding to \( E_1 \) is just the equilibrium position; while the Poincaré section in Figure 3(b) shows heavy chaos for energy \( E_2 \) indicating that not even a local first integral exists.

It is important to note that transformations (2.9) and (2.12) change the differential Galois group of equation (2.8). However, the key is that they do not change the identity component of this group, see [8]. Hence, in order to make the non-integrability proof it is enough to show that the identity component of the differential Galois group of (2.13) is not Abelian. For equation (2.13) the differential Galois group \( G \) is an algebraic subgroup of \( \text{SL}(2, \mathbb{C}) \). The following lemma describes all possible types of \( G \) and relates them to the forms of a solution of (2.13), see [8, 9].

**Lemma 2.1.** Let \( G \) be the differential Galois group of equation (2.13). Then one of the four cases can occur.

1. \( G \) is conjugate to a subgroup of triangular group

\[
T = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\},
\]

and equation (2.13) has an exponential solution \( w = P \exp[\int \omega], P \in \mathbb{C}[z], \omega \in \mathbb{C}(z) \).

2. \( G \) is conjugated with a subgroup of

\[
D^* = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \mid c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \mid c \in \mathbb{C}^* \right\};
\]

in this case (2.13) has a solution of the form \( w = \exp[\int \omega] \), where \( \omega \) is algebraic function of degree 2.

3. \( G \) is finite and all solutions of equation (2.13) are algebraic.

4. \( G = \text{SL}(2, \mathbb{C}) \) and equation (2.13) has no Liouvillian solution.

**Remark 2.1.** Let us write \( r(z) \in \mathbb{C} \) in the form

\[
r(z) = \frac{s(z)}{t(z)}, \quad s(z), t(z) \in \mathbb{C}[z].
\]

The roots of \( t(z) \) are the poles of \( r(z) \). The order of the pole is the multiplicity of the zero of \( t \), and the order of \( r(z) \) at \( \infty \) is \( \deg(t) - \deg(s) \).

**Lemma 2.2.** The following conditions are necessary for the respective cases given in Lemma 2.1

1. Every pole of \( r \) must have even order or else have order 1. The order of \( r \) at \( \infty \) must be even or else greater than 2.

2. \( r \) must have at least one pole that either has odd order greater than 2 or else has order 2.

3. The order of a pole of \( r \) cannot exceed 2 and the order of \( r \) at \( \infty \) must be at least 2. If the partial fraction expansion of \( r \) is

\[
r(z) = \sum_i \frac{a_i}{(z-z_i)^2} + \sum_j \frac{b_j}{z-z_j},
\]

then \( \Delta_i = \sqrt{1 + 4a_i} \in \mathbb{Q} \) for each \( i \), \( \sum_j b_j = 0 \) and if

\[
g = \sum_i a_i + \sum_j b_j d_j,
\]

then \( \sqrt{1 + 4g} \in \mathbb{Q} \).
Equation (2.13) has four singularities

\[ z_{1,2} = \pm 1, \quad z_3 = -\gamma, \quad z_{\infty} = \infty. \]

For \( \gamma \neq \pm 1 \) the singularities \{\( z_{1,2}, z_3, z_{\infty} \)\} are poles of the second order, and the degree of infinity is 1. Taking into account the character of singularities we deduce, according to the Lemma 2.3, that the differential Galois group of reduced equation (2.13) cannot be reducible or finite. Differential Galois group can be only dihedral or SL(2, \( \mathbb{C} \)). In order to check the first possibility we apply the second case of the so-called Kovacic algorithm. Here and below we will use the original formulation of this algorithm given in [6] as well as the notation introduced in this paper.

**Lemma 2.3.** The differential Galois group of equation (2.13) is SL(2, \( \mathbb{C} \)).

**Proof.** First, for singular points \( z_i \), we calculate the auxiliary sets

\[ E_i = \{2 + k\Delta_i \mid k = 0, \pm 2\} \cap \mathbb{Z}, \quad i = 1, 2, 3, \]

where \( \Delta_i \) are the difference of exponents defined by \( \Delta_i = \sqrt{1 + 4a_i} \), and \( a_i \) are the coefficients appearing in expansion of \( r(z) \)

\[ r = \sum_{i=1}^{3} \frac{a_i}{(z - z_i)^2}. \]

In our case \( a_i = -3/16 \). Since the order of infinity is one, the auxiliary sets are

\[ E_1 = E_2 = E_3 = \{1, 2, 3\}, \quad E_{\infty} = \{1\}. \quad (2.15) \]

Next, following the algorithm we look for elements \( e = (e_{\infty}, e_1, e_2, e_3) \) of Cartesian product \( E = E_{\infty} \times \prod_{i=1}^{3} E_i \) such that

\[ d(e) := \frac{1}{2} \left( e_{\infty} - \sum_{i=1}^{3} e_i \right) \in \mathbb{N}_0, \quad (2.16) \]

where \( \mathbb{N}_0 \) is a set of non-negative integers with 0. In is not difficult to note that there are no elements of \( E \) satisfying this condition. Thus, according to the algorithm the second case cannot occur, which implies that only the fourth case is possible, i.e., \( G = \text{SL}(2, \mathbb{C}) \), and equation (2.13) has no Liouvillian solution.

From our short analysis we can formulate the following conclusion. Since the differential Galois group of reduced equation (2.13) is SL(2, \( \mathbb{C} \)), the identity component of differential Galois group of variational equations (2.17) is not Abelian. This implies that the model of two pendula restricted to \( (x, y) \) and \( (x, z) \) planes respectively, is not integrable in the class of function meromorphic in coordinates and momenta.

3. Model 2: toroidal pendulum

Let us consider the the second model presented in Figure 5. Like in the previous case, the first pendulum with mass \( m_1 \) and length \( l_1 \) is attached to the fixed point and moves in the \( (x, y) \) plane. However, the second one with parameters \( m_2 \) and \( l_2 \) is restricted to a plane containing the first pendulum and perpendicular to the \( (x, y) \) plane. In other words it moves on a torus. The positions of the masses \( m_1 \) and \( m_2 \), can be defined parametrically in the following way

\[ I_1 = l_1[\cos \phi_1, \sin \phi_1, 0]^T, \quad I_2 = [(l_1 + l_2 \cos \phi_2) \cos \phi_1, (l_1 + l_2 \cos \phi_2) \sin \phi_1, l_2 \sin \phi_2]^T. \]

From this, we can write immediately the Lagrange function

\[ L = \frac{1}{2} \left( [l_1^2m_1 + m_2(l_1 + l_2 \cos \phi_2)^2]\dot{\phi}_1^2 + l_2m_2\dot{\phi}_2^2 \right) + g[l_1(m_1 + m_2) + l_2m_2 \cos \phi_2] \cos \phi_1. \quad (3.1) \]

This form is quadratic in velocities and is non-singular provided \( l_2m_2 \neq 0 \) and then the Legendre transformation can be carried out. From now on we assume that this condition is always fulfilled. The Hamiltonian function is given by

\[ H = \frac{1}{2} \left( \frac{p_1^2}{l_1^2m_1 + m_2(l_1 + l_2 \cos \phi_2)^2} + \frac{p_2^2}{l_2^2m_2} \right) - g[l_1(m_1 + m_2) + l_2m_2 \cos \phi_2] \cos \phi_1, \quad (3.2) \]
and the equations of motion are as follows
\[
\begin{align*}
\dot{\phi}_1 &= \frac{p_1}{l_1^2 m_1 + m_2 (l_1 + l_2 \cos \phi_2)^2}, \\
\dot{\phi}_2 &= \frac{p_2}{l_2^2 m_2}, \\
\dot{p}_1 &= -g[l_1 (m_1 + m_2) + l_2 m_2 \cos \phi_2] \sin \phi_1, \\
\dot{p}_2 &= -l_2 m_2 \left( \frac{(l_1 + l_2 \cos \phi_2)^2}{l_2^2 m_1 + m_2 (l_1 + l_2 \cos \phi_2)^2} + g \cos \phi_1 \right) \sin \phi_2.
\end{align*}
\] (3.3)

3.1. Numerical analysis

We analyse the dynamics of this model with the same values of parameters as in the first one, see (2.4). Figures 6-7 present the Poincaré cross sections on the \((\phi_2, p_2)\) plane with \(\phi_1 = 0\) and \(p_1 > 0\). As previously, the energy minimum corresponding to equilibria of both pendula is \(E_0 = -9\). We note that for the energy level \(E = -8.9\) the first section presented in 6(a) is strictly similar to the one (see Fig. 2(a)) given in our first model. The whole figure is filled with quasi-periodic orbits surrounding the stable periodic particular solution. Looking at Figure 6(a) we can also detect the region of stable periodic solution related to high order resonance. The situation becomes more complex when we increase energy to the value of \(E = -5\) and the invariant tori deform strongly. Some of them decay giving rise to stable periodic solutions enclosed by separatrices. Thus, as expected for higher values of energies the invariant tori become more and more deformed and the region corresponding to chaotic motion appears, see Figure 7. Figure 8 shows the bifurcation diagram for the coordinate \(\phi_2\) as a function of \(E\). The successive sections were constructed by choosing the cross plane \(\phi_1 = 0\) with the initial condition \((\phi_2, p_2) = (0.61, 0)\).
3.2. The non-integrability proof

Let us formulate the main theorem of this subsection.

**Theorem 3.1.** The system governed by Hamiltonian (3.2) has no additional meromorphic first integral for

\[ E \notin \left\{ -gl_1(m_1 + m_2) \pm il_2m_2, -gl_1(m_1 \pm i \sqrt{m_1/m_2}) \right\}. \] (3.4)

In other words, it is not integrable in the Liouville sense in the class of meromorphic functions of coordinates and momenta.

**Proof.** The system of equations (3.3) has two known invariant manifolds defined by

\[ N_1 = \left\{ (\phi_1, \phi_2, p_1, p_2) \in \mathbb{C}^4 \mid \phi_1 = p_1 = 0 \right\}, \quad N_2 = \left\{ (\phi_1, \phi_2, p_1, p_2) \in \mathbb{C}^4 \mid \phi_2 = p_2 = 0 \right\}. \]

Indeed, restricting the equations (3.3) to \( N_1 \), we obtain

\[ \dot{\phi}_1 = 0, \quad \dot{\phi}_2 = \frac{p_2}{l_2^2m_2}, \quad \dot{p}_1 = 0, \quad \dot{p}_2 = -gl_2m_2 \sin \phi_2. \] (3.5)
Thus, we have a one-parameter family of particular solutions lying on $H = E$

$$
\Phi^2 = \frac{2[E + gl_2 m_2 \cos \phi_2 + g l_1 (m_1 + m_2)]}{l_2^2 m_2},
$$

(3.6)

If we denote by $[\Phi_1, \Phi_2, P_1, P_2]^T$ the variations of $[\phi_1, \phi_2, p_1, p_2]^T$, then the variational equations along this particular solutions are the following

$$
\begin{pmatrix}
\Phi_1 \\
\Phi_2 \\
P_1 \\
P_2
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & a_{13} & 0 \\
0 & 0 & 0 & a_{24} \\
a_{31} & 0 & 0 & 0 \\
0 & a_{42} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\Phi_2 \\
P_2 \\
P_1 \\
P_2
\end{pmatrix},
$$

(3.7)

with

$$
a_{13} = \frac{1}{l_1^2 l_2^2 m_1 + m_2 (l_1 + l_2 \cos \phi_2)^2}, \quad a_{24} = \frac{1}{l_2^2 m_2},
$$

(3.8)

$$
a_{31} = -g (l_1 (m_1 + m_2) + l_2 m_2 \cos \phi_2), \quad a_{42} = -g l_2 m_2 \cos \phi_2.
$$

The system for $\Phi_1$, and $P_1$ forms normal variational equations that can be rewritten as a one second-order differential equation

$$
\Phi + a \Phi + b \Phi = 0, \quad \Phi \equiv \Phi_1,
$$

(3.9)

with coefficients

$$
a = -\frac{a_{13}}{a_{13}} = -\frac{2 (l_1 + l_2 \cos \phi_2) p_2 \sin \phi_2}{l_2 (l_1^2 m_1 + m_2 (l_1 + l_2 \cos \phi_2)^2)}, \quad b = -a_{13} a_{31} = \frac{g [l_1 (m_1 + m_2) + l_2 m_2 \cos \phi_2]}{l_1^2 m_1 + m_2 (l_1 + l_2 \cos \phi_2)^2}.
$$

In order to transform this equation into one with rational coefficients we make the following change of the independent variable

$$
t \rightarrow z = \cos \phi_2(t).
$$

(3.10)

After this transformation the variational equation (3.9) converts into

$$
\Phi'' + p(z) \Phi' + q(z) \Phi = 0, \quad ' \equiv \frac{d}{dz},
$$

(3.11)

with coefficients

$$
p = \frac{z}{z^2 - 1} + \frac{2(z + \alpha)}{(z + \alpha)^2 + \alpha^2 \beta}, \quad q = \frac{1}{2(z + \gamma)}, \quad \alpha = \frac{l_1}{l_2}, \quad \beta = \frac{m_1}{m_2}, \quad \gamma = \frac{E + gl_1 (m_1 + m_2)}{gl_2 m_2},
$$

(3.12)

where

$$
\alpha = \frac{l_1}{l_2} \quad \beta = \frac{m_1}{m_2}, \quad \gamma = \frac{E + gl_1 (m_1 + m_2)}{gl_2 m_2},
$$

(3.13)

have been introduced. Now we apply the standard change of dependent variable

$$
\Phi = w \exp \left[ -\frac{1}{2} \int_{z_0}^z p(s) ds \right],
$$

(3.14)

which transforms (2.10) into its reduced form

$$
w'' = r(z) w, \quad r(z) = -q(z) + \frac{1}{2} p'(z) + \frac{1}{4} p(z)^2,
$$

(3.15)

where the coefficient $r(z)$ is given by

$$
r(z) = -\frac{3}{16} \left( \frac{1}{(z + \gamma)^2} + \frac{1}{(z + 1)^2} + \frac{1}{(z - 1)^2} \right) + \frac{\alpha^2 \beta}{(z + \alpha - i\alpha \sqrt{\beta})^2 (z + \alpha + i\alpha \sqrt{\beta})^2}
$$

$$
+ \frac{\alpha (\alpha + i\alpha \sqrt{\beta}) (\alpha - i\alpha \sqrt{\beta}) (z + \alpha + i\alpha \sqrt{\beta})}{8 (z^2 - 1) (z + \gamma) (z + \alpha - i\alpha \sqrt{\beta}) (z + \alpha + i\alpha \sqrt{\beta})}.
$$

(3.15)
Equation (3.14) has six regular singular points
\[ z_{1,2} = \pm 1, \quad z_{3,4} = -\alpha \pm i\alpha \sqrt{\beta}, \quad z_5 = -\gamma, \quad z_6 = \infty, \]  
(3.16)
where the singularities \( z_1, \ldots, z_5 \) are the poles of the second order, and degree of infinity is 2. The respective differences of exponents at singularities are following
\[ \Delta_1 = \Delta_2 = \frac{1}{2}, \quad \Delta_3 = \Delta_4 = 0, \quad \Delta_5 = \frac{1}{2}, \quad \Delta_6 = \frac{5}{2} \]  
(3.17)
In order to avoid their confluences the conditions
\[ \gamma \notin \{ \pm 1, \alpha \pm i\alpha \sqrt{\beta} \}, \quad \beta \neq -\frac{(\alpha + 1)^2}{\alpha^2} \]  
(3.18)
must be satisfied. Since \( \beta \) and \( \alpha \) are real positive parameters only the first condition remains valid. From this, we have the following exclusions for the energy
\[ E \notin \left\{ -g(l_1(m_1 + m_2) \pm l_2m_2), -gl_1(m_1 \pm i\sqrt{m_1m_2}) \right\}. \]  
(3.19)
Now, we can prove the following

**Lemma 3.1.** The differential Galois group of equation (3.14) satisfying (3.18) is \( SL(2, \mathbb{C}) \).

**Proof.** As \( \Delta_{3,4} = 0 \), local solutions in a vicinity of \( z_* = z_3 \) and \( z_* = z_4 \) contain a logarithmic term. Two linearly independent solutions \( w_1 \) and \( w_2 \) of (3.14) have the following forms
\[ w_1(z) = (z - z_*)^\rho f(z), \quad w_2(z) = w_1(z) \ln(z - z_*) + (z - z_*)^\rho h(z), \]  
(3.20)
where \( f(z) \) and \( h(z) \) are holomorphic at \( z_* \), and \( f(z_*) \neq 0 \). The local monodromy matrix corresponding to continuation of the matrix of fundamental solutions along a small loop encircling \( z_* \) counterclock-wise has the following form
\[ M_* = \begin{pmatrix} -1 & -2\pi i \\ 0 & -1 \end{pmatrix}, \]
for details consult [7]. A subgroup of \( SL(2, \mathbb{C}) \) generated by a non-diagonal triangular matrix cannot be finite, and thus also differential Galois group cannot be finite. Moreover, this matrix is non-diagonalisable. Thus also differential Galois group \( G \) of this equation cannot be a subgroup of dihedral group that correspond to the second case of the Kovacic algorithm. Thus the only possibilities are that \( G \) is the full triangular group or \( SL(2, \mathbb{C}) \).

In order to check the first possibility we apply the first case of the Kovacic algorithm. If it is satisfied, then equation (3.14) has an exponential solution. First, for singular points \( z_i \) we calculate sets of exponents \( E_i \) of local solutions
\[ E_i := \{(1 \pm \Delta_i)/2\}, \quad \text{for } i = 1, \ldots, 6. \]
Thus, we have
\[ E_1 = E_2 = \left\{ \frac{3}{4}, \frac{1}{4} \right\}, \quad E_3 = E_4 = \left\{ \frac{1}{2}, \frac{1}{2} \right\}, \quad E_5 = \left\{ \frac{3}{4}, \frac{1}{4} \right\}, \quad E_6 = \left\{ \frac{7}{4}, -\frac{3}{4} \right\}. \]  
(3.21)
Next, according to the algorithm we look for elements \( e = (e_1, e_2, e_3, e_4, e_5, e_6) \) of Cartesian product \( E = \prod_{i=1}^6 E_i \) such that
\[ d(e) := e_6 - \sum_{i=1}^5 e_i \in \mathbb{N}_0. \]  
(3.22)
In our case there exists only one element of \( E \) satisfying this condition, namely
\[ e = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right\}. \]
for which \( d(e) = 0 \). Now we pass to the third step of the Kovacic algorithm. We look for a polynomial \( P \neq 0 \) of degree \( d(e) \), such that it is a solution of the following differential equation

\[
P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0, \tag{3.23}
\]

where

\[
\omega = \sum_{i=1}^{5} \frac{e_i}{z - z_i},
\]

and \( r \) is given in (3.15). In the considered case, we have \( P = 1 \), so equation (3.23) gives equality

\[
z + \alpha(1 + \beta) = 0,
\]

which cannot be satisfied for arbitrary \( z \).

4. Conclusions

Although the obstructions to integrability obtained with the Morales–Ramis theory are one of the strongest known, the frequent obstacle in its application is finding a particular solution for a given dynamical system. The classical double pendulum, both three- and two-dimensional, has no obvious solutions which could be used to explicitly linearize the equations of motion and allow for determination of the differential Galois group. Thus, despite the numerical evidence \([1, 5]\) and theoretical \([2, 3]\) work, a proof of Liouvillian non-integrability still eludes us.

The present work is a step towards proving the conjecture that the three-dimensional double pendulum has no additional first integrals. Otherwise one would expect them to be present also in some restrictions of the original system. Note also, that the finite set of energies for which our non-integrability result does not hold does not leave any hope for practical solving of the equations of motion, as that requires a global first integral, independent of energy.

A peculiar feature of both models, which allows for such a concise analysis, is the fact that the characteristic exponents of the variational equations do not depend on the physical parameters. Since the differential Galois group depends on the exponents, this is a huge simplification. Usually a mechanical system admits first integrals for special values of parameters because the group depends on them essentially. In this sense, the models considered here are quite exceptional.

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