A Wave-Mechanical Approach to Cosmic Structure Formation

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ABSTRACT
The dynamical equations describing the evolution of a self-gravitating fluid can be rewritten in the form of a Schrödinger equation coupled to a Poisson equation determining the gravitational potential. This wave-mechanical representation allows an approach to cosmological gravitational instability that has numerous advantages over standard fluid-based methods. We explore the usefulness of the Schrödinger approach by applying it to a number of simple examples of self-gravitating systems in the weakly non-linear regime. We show that consistent description of a cold self-gravitating fluid requires an extra “quantum pressure” term to be added to the usual Schrödinger equation and we give examples of the effect of this term on the development of gravitational instability. We also show how the simple wave equation can be modified by the addition of a non-linear term to incorporate the effects of gas pressure described by a polytropic equation-of-state.

Key words: Cosmology: theory – galaxies: clustering – large-scale structure of the Universe.

1 INTRODUCTION
The local Universe displays a rich hierarchical pattern of galaxy clustering that encompasses a large range of length scales, culminating in rich clusters and superclusters; the early Universe, however, was almost smooth, with only slight ripples seen in the cosmic microwave background radiation (e.g. Smoot et al. 1992). Models of the evolution of structure link these observations by appealing to the effect of gravitational instability, e.g. Lahav et al. (2002). Low-amplitude primordial perturbations of wave-like character within a largely homogeneous universe become amplified by their own self-gravity, first linearly which then, as their amplitudes build up, result in non-linear collapse. The linear theory of perturbation growth for cosmological density fluctuations is well-established, and is founded on tried-and-tested representation of matter on cosmological scales as a cold fluid (e.g. Peebles 1980). The non-linear regime is much more complicated and generally not amenable to analytic solution. Instead, numerical N-body simulations using particles rather than waves to trace cosmic structure, have led the way towards an understanding of strongly developed clustering. Analytic approaches based on particles, chiefly the Zel’dovich (1970) approximation, have also proved useful. Although these agree to an extent with fully numerical calculations (Coles, Melott & Shandarin 1993), they break down when particle trajectories cross to form singularities known as caustics.

In this paper we adopt an approach to cosmic structure that has advantages over both fluid and particle descriptions. Following Widrow & Kaiser (1993) and Coles (2002), we construct a formalism in which the dynamics of gravitational is couched in the language of quantum mechanics and governed by a Schrödinger wave equation. The advantages of this approach will be made apparent as we explore its behaviour in simple examples which can be solved exactly and, within the Schrödinger approach, to various levels of approximation. Our aim is to construct a useful tool for evolving density fluctuations into the mildly non-linear regime, beyond the breakdown of linear perturbation theory.

The importance of the search for analytic understanding of structure formation is often overlooked in the light of the tremendous advances that have been made recently in computational astrophysics. One motivation for a wider range of theoretical techniques is practical. High-resolution N-body experiments are CPU-intensive and it is difficult to find sufficient resources to use them to explore a large range of choices of initial fluctuations, different cosmological parameters, and so on. A neat analytic approximation can do this job more efficiently, at least to the point of narrowing down the field of contenders for fuller study. The other motivation is methodological: the complex “cosmic web” of large-scale structure requires explanation rather than reproduction, and explanation rests on using analytic approaches to identify the important physics as much as possible.
The layout of this paper is as follows. In Section 2 we run through the standard basics of gravitational instability in an expanding universe, including perturbation theory and the Zel’dovich approximation. We then, in Section 3, present the details of the Schrödinger approach and derive a spherically-symmetric solution. These are the solutions, and the other two cases are space sections of self-gravitating fluid. These are the equations governing the behaviour of a dynamical field equations which can be written

\begin{equation}
\Omega = \frac{8\pi G}{3} \rho, \quad \frac{\dot{a}}{a} = \frac{3\pi G}{3} \left( \frac{\rho + \frac{p}{c^2}}{a^2} \right) + \frac{\Lambda a^2}{3}, \quad \dot{\rho} = -3\frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right).
\end{equation}

These equations determine the time evolution of the cosmic scale factor \(a(t)\) (the dots denote derivatives with respect to cosmological proper time) and \(\Omega\) (the Hubble parameter).

### 2.1 The Background Cosmology

Space-times consistent with the Cosmological Principle can be described by the Robertson–Walker metric

\begin{equation}
d^2s^2 = c^2 dt^2 - a^2(t) \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),
\end{equation}

where \(\kappa\) is the spatial curvature, scaled so as to take the values 0 or \(\pm1\). The case \(\kappa = 0\) represents flat space sections, and the other two cases are space sections of constant positive or negative curvature, respectively. The time coordinate \(t\) is cosmological proper time and \(a(t)\) is the cosmic scale factor. The dynamics of a Friedmann-Robertson-Walker universe are determined by the Einstein gravitational field equations which can be written

\begin{equation}
3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G \rho - \frac{3\kappa c^2}{a^2} + \Lambda c^2,
\end{equation}

\begin{equation}
\frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{p}{c^2} \right) + \frac{\Lambda a^2}{3},
\end{equation}

\begin{equation}
\dot{\rho} = -3\frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right).
\end{equation}

These equations determine the time evolution of the cosmic scale factor \(a(t)\) (the dots denote derivatives with respect to cosmological proper time) and therefore describe the global expansion or contraction of the universe. The behaviour of these models can further be parametrized in terms of the Hubble parameter \(H = \dot{a}/a\) and the density parameter \(\Omega = 8\pi G \rho/3H^2\), a suffix 0 representing the values of these quantities at the present epoch when \(t = t_0\).

### 2.2 Fluid Treatment

In the standard treatment of the Jeans Instability one begins with the dynamical equations governing the behaviour of a self-gravitating fluid. These are the Euler equation

\begin{equation}
\frac{\partial (\rho v)}{\partial t} + (v \cdot \nabla)v + \frac{1}{\rho} \nabla p + \nabla \phi = 0;
\end{equation}

the continuity equation

\begin{equation}
\frac{\partial \rho}{\partial t} + \nabla (\rho v) = 0,
\end{equation}

expressing the conservation of matter; and the Poisson equation

\begin{equation}
\nabla^2 \phi = 4\pi G \rho,
\end{equation}

describing Newtonian gravity. If the length scale of the perturbations is smaller than the effective cosmological horizon \(d_H = c/H\), a Newtonian treatment of cosmic structure formation is still expected to be valid in expanding world models. In an expanding cosmological background, the Newtonian equations governing the motion of gravitating particles can be written in terms of

\begin{equation}
x \equiv r/a(t)
\end{equation}

(the comoving spatial coordinate, which is fixed for observers moving with the Hubble expansion),

\begin{equation}
v \equiv \dot{r} - Hr = a\dot{x}
\end{equation}

(the peculiar velocity field, representing departures of the matter motion from pure Hubble expansion), \(\rho(x, t)\) (the matter density), and \(\phi(x, t)\) (the peculiar Newtonian gravitational potential, i.e. the fluctuations in potential with respect to the homogeneous background) determined by the Poisson equation in the form

\begin{equation}
\nabla_x^2 \phi = 4\pi G a^2 \left( \rho - \rho_0 \right) = 4\pi G a^2 \rho_0 \delta.
\end{equation}

In this equation and the following the suffix on \(\nabla_x\) indicates derivatives with respect to the new comoving coordinates. Here \(\rho_0\) is the mean background density, and

\begin{equation}
\delta \equiv \frac{\rho - \rho_0}{\rho_0}
\end{equation}

is the density contrast. Using these variables the Euler equation becomes

\begin{equation}
\frac{\partial (\rho v)}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho} \nabla_p p - \nabla_x \phi.
\end{equation}

The first term on the right-hand-side of equation \(\mathbf{12}\) arises from pressure gradients, and is neglected in dust-dominated cosmologies. Pressure effects may nevertheless be important in the (collisional) baryonic component of the mass distribution when nonlinear structures eventually form. The second term on the right-hand side of equation \(\mathbf{12}\) is the peculiar gravitational force, which can be written in terms of \(g = -\nabla_x \phi/a\), the peculiar gravitational acceleration of the fluid element. If the velocity flow is irrotational, \(v\) can be rewritten in terms of a velocity potential \(\phi_v\):

\begin{equation}
v = -\nabla_x \phi_v/a.
\end{equation}

This is expected to be the case in the cosmological setting because (a) there are no sources of vorticity in these equations and (b) vortical perturbation modes decay with the expansion. We also have a revised form of the continuity equation:

\begin{equation}
\frac{\partial \rho}{\partial t} + 3H \rho + \frac{1}{a} \nabla_x (\rho v) = 0.
\end{equation}

### 2.3 Linear Perturbation Theory

The procedure for handling perturbations is to linearise the Euler, continuity and Poisson equations by perturbing physical quantities defined as functions of Eulerian coordinates, i.e. relative to an unperturbed coordinate system. Expanding \(\rho, v\) and \(\phi\) perturbatively and keeping only the first-order terms in equations \(\mathbf{12}\) and \(\mathbf{13}\) gives the linearised continuity equation:

\begin{equation}
\frac{\partial \delta \rho}{\partial t} + 3H \delta \rho + \frac{1}{a} \nabla_x (\rho \delta v) = 0.
\end{equation}
where can be inverted, with a suitable choice of boundary conditions, to yield
\[ \delta = -\frac{1}{aHf} (\nabla \cdot \mathbf{v}) . \] (16)

The function \( f \approx \Omega_0^6 \); this is simply a fitting formula to the full solution (Peebles 1980). The linearised Euler and Poisson equations are
\[ \frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} \mathbf{v} = -\frac{1}{\rho a} \nabla \rho + \frac{1}{a} \nabla \phi , \] (17)
\[ \nabla \phi = 4\pi G a^2 \rho_0 \delta ; \] (18)

\[ |v|, |\phi|, |\delta| \ll 1 \) in equations (16), (17) & (18). From these equations, and if one ignores pressure forces, it is easy to obtain an equation for the evolution of \( \delta \):
\[ \ddot{\delta} + 2H \dot{\delta} - \frac{3}{2} \Omega H^2 \delta = 0 . \] (19)

For a spatially flat universe dominated by pressureless matter, \( \rho_0(t) = 1/6\pi G \Omega t^2 \) and equation (19) admits two linearly independent power law solutions \( \delta(x,t) = D_+(t)\delta(x) \), where \( D_+(t) \propto a(t) \propto t^{2/3} \) is the growing mode and \( D_-(t) \propto t^{-1} \) is the decaying mode.

The above considerations apply to the evolution of a single Fourier mode of the density field \( \delta(x,t) = D_+(t)\delta(x) \). What is more likely to be relevant, however, is the case of a superposition of waves, resulting from some kind of stochastic process in which the density field consists of a superposition of such modes with different amplitudes. A statistical description of the initial perturbations is therefore required, and any comparison between theory and observations will also have to be statistical. Many versions of the inflationary scenario for the very early universe (Guth 1981; Guth & Pi 1982; Linde 1982; Albrecht & Steinhardt 1982) predict the initial density fluctuations to take the form of a Gaussian random field in which the initial Fourier modes of the perturbation field have random phases.

The linearised equations of motion provide an excellent description of gravitational instability at very early times when density fluctuations are still small \( (\delta \ll 1) \). The linear regime of gravitational instability breaks down when \( \delta \) becomes comparable to unity, marking the commencement of the quasi-linear (or weakly non-linear) regime. During this regime the density contrast may remain small \( (\delta < 1) \), but the phases of the Fourier components \( \delta_k \) become substantially different from their initial values resulting in the gradual development of a non-Gaussian distribution function even if the primordial density field is Gaussian owing to mode–mode interactions.

Perturbation theory fails when non-linearity develops, but the fluid treatment is intrinsically approximate anyway. A fuller treatment of the problem requires a solution of the Boltzmann equation for the full phase-space distribution of the system \( f(x,v,t) \) coupled to the Poisson equation (7) that determines the gravitational potential. In cases where the matter component is collisionless, the Boltzmann equation takes the form of a Vlasov equation:
\[ \frac{\partial f}{\partial t} = \sum_{i=1}^3 \left( \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} - v_i \frac{\partial f}{\partial x_i} \right) . \] (20)

The fluid approach outline above can only describe cold material where the velocity dispersion of particles is negligible. But even if the dark matter is cold, there may be hot components of baryonic material whose behaviour needs also to be understood. Moreover, the fluid approach assumes the existence of a single fluid velocity at every spatial position. It therefore fails when orbits cross and multi-streaming generates a range of particle velocities through a given point.

### 2.4 The Zel'dovich Approximation

Fortunately the formation of the main elements of large-scale structure can nevertheless be understood using relatively simple tools, such as the Zel’dovich approximation (Zel’dovich 1970). Let the initial (i.e. Lagrangian) coordinate of a particle in this unperturbed distribution be \( \mathbf{q} \). Now each particle is subjected to a displacement corresponding to a density perturbation. In the Zel’dovich approximation the Eulerian coordinate of the particle at time \( t \) is
\[ \mathbf{r}(t, \mathbf{q}) = a(t)[\mathbf{q} - b(t) \nabla \Phi_0(\mathbf{q})] , \] (21)
where \( \mathbf{r} = a(t)\mathbf{x} \), with \( \mathbf{x} \) a comoving coordinate, and we have made \( a(t) \) dimensionless by dividing throughout by \( a(t_i) \), where \( t_i \) is some reference time which we take to be the initial time. Notice that particles in the Zel’dovich approximation execute a kind of inertial motion on straight line trajectories. The derivative on the right hand side is taken with respect to the Lagrangian coordinates. The dimensionless function \( b(t) \) describes the evolution of a perturbation in the linear regime, with the condition \( b(t_i) = 0 \), and therefore solves the equation
\[ \ddot{b} + 2\left( \frac{\dot{a}}{a} \right) \dot{b} - 4\pi G \rho b = 0 ; \] (22)

which means that the velocity field is irrotational. The quantity \( \Phi_0(\mathbf{q}) \) is related to the density perturbation in the linear regime by the relation
\[ \delta = \frac{b
abla^2 \Phi_0}{A} , \] (24)

which is a simple consequence of Poisson’s equation.

The Zel’dovich approximation, though simple, has a number of interesting properties. First, it is exact for the case of one dimensional perturbations up to the moment of shell crossing. As we have mentioned above, it also incorporates irrotational motion, which is required to be the case if it is generated only by the action of gravity (due to the Kelvin circulation theorem). For small displacements between \( \mathbf{r} \) and \( a(t)\mathbf{q} \), one recovers the usual (Eulerian) linear regime: in fact, equation (22) defines a unique mapping between the coordinates \( \mathbf{q} \) and \( \mathbf{r} \) (as long as trajectories do not cross); this means that \( \rho(\mathbf{r}) d^3r = \langle \rho(t_i) \rangle d^3q \) or
\[ \rho(\mathbf{r}, t) = \frac{\langle \rho(t) \rangle}{|J(r,t)|} , \] (25)

where \( |J(r,t)| \) is the determinant of the Jacobian of the mapping between \( \mathbf{q} \) and \( \mathbf{r} \). Since the flow is irrotational
the matrix \( J \) is symmetric and can therefore be locally diagonalised. Hence
\[
\rho(\mathbf{r}, t) = \langle \rho(t) \rangle \prod_{i=1}^{3} [1 + b(t) \alpha_i(\mathbf{q})]^{-1} : \tag{26}
\]
the quantities \( 1 + b(t) \alpha_i \) are the eigenvalues of the matrix \( J \) (the \( \alpha_i \) are the eigenvalues of the deformation tensor). For times close to \( t_i \), when \( |b(t) \alpha_i| \ll 1 \), equation (26) yields
\[
\delta \simeq - (\alpha_1 + \alpha_2 + \alpha_3) b(t), \tag{27}
\]
which is just the law of perturbation growth obtained by solving equation (14).

At some time \( t_{sc} \), when \( b(t_{sc}) = -1/\alpha_j \), a singularity appears and the density becomes formally infinite in a region where at least one of the eigenvalues (in this case \( \alpha_j \)) is negative. This condition corresponds to the situation where two points with different Lagrangian coordinates end up at the same Eulerian coordinate. In other words, particle trajectories have crossed and the mapping between Lagrangian and Eulerian space is no longer unique. A region where the shell-crossing occurs is called a caustic. For a fluid element to be collapsing, at least one of the \( \alpha_i \) must be negative. If more than one is negative, then collapse will occur first along the axis corresponding to the most negative eigenvalue. If there is no special symmetry, one therefore expects collapse to be generically one-dimensional, from three dimensions to two. We shall make use of the fact that in one-dimensional cases, the Zel’dovich solution is exact until the moment of shell-crossing.

The Zel’dovich approximation matches very well the evolution of density perturbations in full \( N \)-body calculations until the point where shell crossing occurs (Coles, Melott & Shandarin 1993). After this, the approximation breaks down completely. Particles continue to move through the caustic in the same direction as they did before. Particles entering a pancake from either side merely sail through it and pass out the opposite side. The pancake therefore appears only instantaneously and is rapidly smeared out. In reality, the matter in the caustic would feel the strong gravity there and be pulled back towards it before it could escape through the other side. Since the Zel’dovich approximation is only kinematic it does not account for these close-range forces and the behaviour in the strongly non-linear regime is therefore described very poorly. Furthermore, this approximation cannot describe the formation of shocks and phenomena associated with pressure. Attempts to understand properties of non-linear structure using a fluid description therefore resort to further approximations (Saichev & Coles 1995) to extend the approach beyond shell-crossing. One method involves the so-called adhesion model (Gurbatov, Saichev & Shandarin 1989). In this model one assumes that the particles stick to each other when they enter a caustic region because of an artificial viscosity which is intended to simulate the action of strong gravitational effects inside the overdensity forming there. This ‘sticking’ results in a cancellation of the component of the velocity of the particle perpendicular to the caustic. If the caustic is two-dimensional, the particles will move in its plane until they reach a one-dimensional interface between two such planes. This would then form a filament. Motion perpendicular to the filament would be cancelled, and the particles will flow along it until a point where two or more filaments intersect, thus forming a node. The smaller is the filaments intersect, the thinner will be the sheets and filaments, and the more point-like will be the nodes. Outside these structures, the Zel’dovich approximation is still valid to high accuracy. See Shandarin & Zel’dovich (1989) for further discussion.

2.5 The Strongly Nonlinear Regime

Further into the non-linear regime, bound structures form. The baryonic content of these objects may then become important dynamically: hydrodynamical effects (e.g. shocks), star formation and heating and cooling of gas all come into play. The spatial distribution of galaxies may therefore be very different from the distribution of the (dark) matter, even on large scales. Attempts are only just being made to model some of these processes with cosmological hydrodynamics, such as those based on Smoothed Particle Hydrodynamics (SPH; Monaghan 1992), but it is some measure of the difficulty of understanding the formation of galaxies and clusters that most studies have only just begun to attempt to include modelling the detailed physics of galaxy formation. In the front rank of theoretical efforts in this area are the so-called semi-analytical models which encode simple rules for the formation of stars within a framework of merger trees that allows the hierarchical nature of gravitational instability to be explicitly taken into account.

The spatial distribution of particles obtained using the adhesion approximation represents a sort of “skeleton” of the real structure: non-linear evolution generically leads to the formation of a quasi-cellular structure, which is a kind of “tessellation” of irregular polyhedra having pancakes for faces, filaments for edges and nodes at the vertices. This skeleton, however, evolves continuously as structures merge and disrupt each other through tidal forces; gradually, as evolution proceeds, the characteristic scale of the structures increases. In order to interpret the observations we have already described, one can think of the giant “voids” as being the regions internal to the cells, while the cell nodes correspond to giant clusters of galaxies. While analytical methods, such as the adhesion model, are useful for mapping out the skeleton of structure formed during the non-linear phase, they are not adequate for describing the highly non-linear evolution within the densest clusters and superclusters. In particular, the adhesion model cannot be used to treat the process of merging and fragmentation of pancakes and filaments due to their own (local) gravitational instabilities, which must be done using full numerical computations.

3 WAVE MECHANICS AND STRUCTURE FORMATION

3.1 Introduction

The combination of Eulerian fluid-based perturbation theory, Lagrangian approaches like the Zel’dovich approximation and brute-force \( N \)-body numerics has been successful at establishing a standard framework within which the basic features of large-scale structure can be described and understood. There remain significant obstacles to fuller analytic...
where the density becomes infinite.

To begin with, consider the continuity equation and approximation and its variations break down at shell-crossing, where the density is positive.

(i) Perturbation Theory. Standard perturbation methods do not guarantee a density field that is everywhere positive. This basic problem is obvious in the standard case where there is a Gaussian distribution of initial fluctuations. When the variance of the distribution of \( \delta \) is of order unity (its mean is, by definition, always unity) then a Gaussian distribution assigns a non-zero probability to regions with \( \delta < -1 \), i.e. with \( \rho < 0 \).

(ii) Shell-crossing. Methods such as the Zel’dovich approximation and its variations break down at shell-crossing, where the density is positive.

(iii) Gas Pressure. Analytic techniques for modelling the effects of gas pressure are scarce, even in relatively simple systems such as the Lyman-\( \alpha \) absorbing systems (Matarrese & Mohayee 2002). Fully numerical techniques, such as SPH, are generally expected to be the way forward but even they are not yet able to incorporate all relevant gas physics.

\[ \nabla^2 \Phi = 4\pi G \rho^*, \quad (33) \]

taking \( V \) to be \( \Phi \).

This system looks very similar to a Schrödinger equation, except for the extra "operator" \( \rho \). The role of this term becomes clearer if one leaves it out of equation \( 34 \) and works backwards. The result is an extra term in the Bernoulli equation that resembles a pressure gradient. This is often called the "quantum pressure" that arises when one tries to understand a quantum system in terms of a classical fluid behaviour. Leaving it out to model a collisionless fluid can be justified only if \( \alpha \) varies only slowly on the scales of interest. On the other hand one can model situations in which one wishes to model genuine effects of pressure by adjusting (or omitting) this term in the wave equation. Widrow & Kaiser (1993) advocated simply writing

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\phi(x)\psi, \quad (34) \]
i.e. simply ignoring the quantum pressure term. In this spirit, the constant \( \hbar \) is taken to be an adjustable parameter that controls the spatial resolution \( \lambda \) through a de Broglie relation \( \lambda = \hbar/mv \). In terms of the parameter \( \nu \) used above, we have \( \nu = \hbar/m \), giving the correct dimensions for Planck’s constant. Note that the wavefunction \( \psi \) encodes the velocity part of phase space in its argument through the ansatz

\[ \psi(x) = \sqrt{\rho(x)} \exp[i\theta(x)/\hbar], \quad (35) \]

where \( \nabla \theta(x) = p(x) \), the local ‘momentum field’. This formalism thus yields an elegant description of both the density and velocity fields in a single function.

It should be clear how this approach bypasses immediately the first two items of the list given in Section 3.1. First, the construction \( 35 \) requires that \( \rho = |\psi|^2 \) is positive. Secondly, since particles are not treated as point-like entities with definite trajectories, shell-crossing does not have the catastrophic from in this approach that it does in the Zel’dovich approximation. Note that no singularities occur in the wavefunction at any time. Although the simplest ansatz \( 35 \) does assume a unique velocity at every fluid location, it is possible to construct more complex representations that allow for multi-streaming (Widrow & Kaiser 1993) although we shall not use them in this paper.

### 3.2 The Widrow–Kaiser Approach

A novel approach to the study of collisionless matter, with applications to structure formation, was suggested by Widrow & Kaiser (1993). It involves re-writing of the fluid equations given in Section 2 in the form of a non-linear Schrödinger equation. The equivalence between this and the fluid approach has been known for some time; see Spiegel (1980) for historical comments. Originally the interest was to find a fluid interpretation of quantum mechanical effects, but in this context we shall use it to describe an entirely classical system.

To begin with, consider the continuity equation and Euler equation for a curl-free flow in which \( \nu = \nabla \phi \), in response to some general potential \( V \). In this case the continuity equation can be written

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \phi) = 0. \quad (28) \]

It is convenient to take the first integral of the Euler equation, giving

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 = -V, \quad (29) \]

which is usually known as the Bernoulli equation. The trick then is to make a transformation of the form

\[ \psi = \alpha \exp(i\phi/\nu), \quad (30) \]

where \( \rho = \psi^* \psi = |\psi|^2 = \alpha^2 \); the wavefunction \( \psi(x,t) \) evidently complex. Notice that the dimensions of \( \nu \) are the same as \( \phi \), namely \( [L^2T^{-1}] \). After some algebra it emerges that the two equations above can be written in one equation of the form

\[ i\nu \frac{\partial \psi}{\partial t} = -\frac{\nu^2}{2} \nabla^2 \psi + V \psi + P \psi, \quad (31) \]

where

\[ P = \frac{\nu^2 \nabla^2 \alpha}{\alpha}. \quad (32) \]

To accommodate gravity we need to couple equation \( 31 \) to the Poisson equation in the form

\[ \nabla^2 \Phi = 4\pi G \rho^*, \quad (33) \]
but we are looking for a static solution so obviously the fluid flow velocity should be zero. We therefore need to set $\phi$ to be constant in the above transformation so that $\nabla \phi$ can be zero everywhere. Without loss of generality we can exploit the global $U(1)$ symmetry of quantum mechanics to set the phase to zero. Hence we can set $\psi = R$ and consequently $\nabla^2 \psi = \nabla^2 R$. The quantum pressure term has the form

$$P = \frac{\nu^2 \nabla^2 R}{2},$$

(38)

so that this now cancels the first term on the right-hand-side of equation (36). In this event, and requiring a static solution for $\psi$ we get

$$\Phi \psi + \kappa^2 |\psi|^2 \psi = 0$$

(39)

to be solved simultaneous with Poisson’s equation. We therefore obtain a simple Helmholtz equation for $\Phi$:

$$\nabla^2 \Phi + \left(\frac{4\pi G}{\kappa^2}\right) \Phi = 0.$$  

(40)

The solution for spherically symmetry is straightforward:

$$\Phi = \Phi_c \frac{\sin \lambda r}{\lambda r}.$$  

(41)

where

$$\lambda^2 = \frac{4\pi G}{\kappa^2}$$  

(42)

and $\Phi_c$ is constant. There is a particularly neat relation for $\rho$ using equation (40) yielding

$$4\pi G \rho + \frac{4\pi G}{\lambda^2} \Phi = 0,$$  

(43)

which gives

$$\rho(r) = \rho_c \frac{\sin \lambda r}{\lambda r}.$$  

(44)

The resulting density profile will be recognized as that of an $n = 1$ polytrope, i.e. a fluid in which the gas pressure is related to the density via $p = K \rho^2$, with $K$ a constant.

The important point emerging from this analysis is that one can only obtain a static solution in the presence of the quantum pressure term. A self-consistent solution requires this term to be included and its effect is to stabilize the system. In order to apply the wave mechanical approach to structure formation, therefore, one has to be more careful with this term than Widrow & Kaiser (1993) in their equation (40). We investigate this issue further in the following Section.

4 ONE-DIMENSIONAL COLLAPSE

In this section we look at some simple one-dimensional examples of gravitational instability using the Schrödinger equation. We do this by comparing the solutions we obtain with the Zel’dovich approximation, which gives an exact solution up to the point of shell-crossing (Shandarin & Zel’dovich 1989). We also want to consider the effects of varying the parameter $\nu$ in the Schrödinger equation and how this changes the spatial and velocity resolution of the solution, as well as elucidating more clearly the effects the “quantum pressure” term.

4.1 The “Free-Particle” Solution to the Schrödinger Equation

The equation for the trajectory of a particle using the Zel’dovich approximation is identical in form to that of inertial (or ballistic) motion, with the displace replaced by the comoving co-ordinate $x$ and the time replaced by $b$ as defined in equation (22). The most obviously analogous approach to take with the Schrödinger equation is obtained by setting the potential $V$ equal to zero, replacing $t$ with $b$ and taking spatial derivatives with respect to the comoving co-ordinate $x$. Equation (31) then becomes

$$\frac{\nu}{b} \frac{\partial \psi}{\partial b} = -\frac{\nu^2}{2} \nabla_x \psi,$$

(45)

or, in one-dimension,

$$\frac{\nu}{b} \frac{\partial \psi}{\partial b} = -\frac{\nu^2}{2} \frac{\partial^2 \psi}{\partial x^2}.$$  

(46)

This is the “free-particle” Schrödinger equation and the solutions are given by

$$\psi(x, t) = \sum_k A_k e^{ikx},$$

(47)

with $\omega = k^2 \nu/2$. The boundary conditions are given by the initial value of the wave function at $b = 0$:

$$\psi(x, 0) = \sum_k A_k e^{ikx}.$$  

(48)

The values of $A_k$ can be calculated using the Fast Fourier Transform of the initial wavefunction, and the value of the wavefunction at any future time can be calculated directly using equation (47).

In order to show the equivalence between the Zel’dovich approximation and Schrödinger method we consider the two solutions when the initial velocity potential is given by a single sine wave. In the Schrödinger equation we start with an initial wave function given by

$$\psi(x, 0) = \sqrt{\rho_0} \exp(-i\cos x/\nu).$$  

(49)

This is equivalent to setting the initial density contrast $\delta = 0$ and the velocity $v = \sin x$. In one dimension, the Zel’dovich approximation for the position of a particle at time $t$ is given by

$$x(q, t) = q - b \frac{d\Phi_0(q)}{dq},$$

(50)

cf. equation (21). Conservation of mass is enforced by

$$\rho_0 dq = \rho dx,$$

(51)

and therefore the density contrast is given by

$$\delta = \left(\frac{\partial x}{\partial q}\right)^{-1} - 1 = \left[1 - b \frac{d^2 \Phi_0(q)}{dq^2}\right]^{-1} - 1.$$  

(52)

If the velocity potential $\Phi_0(q) = \cos q$ then the density contrast at time $t$ is then given by

$$\delta = (1 + b \cos x)^{-1} - 1,$$

(53)

and shell-crossing occurs when $b = 1$ when the density contrast becomes infinite. Figure 1 shows the density contrast obtained using both of these methods at three different times and for three different values of $\nu$. When $b = 0.3$ density
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Figure 1. Comparison of the Zel’dovich approximation to the “free-particle” Schrödinger equation. The Zel’dovich approximation is given by the dashed line and the solution to the Schrödinger equation by the solid line. At early times the two methods are almost identical. Close to shell crossing the results are similar, but with the Schrödinger result behaving more smoothly than the Zel’dovich solution. The effect of varying $\nu$ can be most clearly seen after shell-crossing.

growth is still approximately linear and the density contrast given by the Schrödinger and Zel’dovich methods are almost identical. When $b = 0.9$, just before shell-crossing, the density growth has become highly non-linear. The very high density peaks seen in the Zel’dovich approximation are not seen in the Shrödinger solution.

The value of $\nu$ corresponds to a “de Broglie wavelength” which gives the spatial resolution of the solution. The effects of this can be seen most clearly when $b = 2.0$ after shell-crossing has occurred. Although the Schrödinger and Zel’dovich methods show qualitatively different behaviour after shell-crossing, this is not physical, and the overall behaviour of the results is in any case similar, particularly for small values of $\nu$.

4.2 The Schrödinger equation with time-independent potential

Setting the potential $V(x) = 0$ in the Schrödinger equation gives a good agreement with the Zel’dovich approximation. However, in both methods the particles continue on their trajectories after shell-crossing with no reference to the self-gravity of the collapsing system. Here we look at how the inclusion of a potential term in the Schrödinger equation might give a more realistic solution after shell-crossing. This approach is motivated by the realisation that, at least during linear perturbation growth, the gravitational potential corresponding to a density fluctuation barely changes. In a different context this has motivated the so-called “frozen-potential” approximation (Matarrese et al. 1992; Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994), which is similar in spirit to the approach we adopt here.

The simplest way to include the potential is to include its initial value, but not to allow this to vary with time. For this example we again ignore the quantum pressure term. We therefore need to solve the equation

$$i\nu \frac{\partial \psi}{\partial t} = -\frac{\nu^2}{2} \frac{\partial^2 \psi}{\partial x^2} + \cos(x)\psi.$$  \hspace{1cm} (54)

It should be noted that in this example no account is taken of the expansion of the universe. This means that a direct comparison with this Zel’dovich approximation is not as straightforward as in the previous example and in particular the time coordinate is not the same in the two methods. However a comparison may be made by considering the growth at early times in the linear regime. In the Zel’dovich approximation the density contrast is given by
\[ \delta \simeq b \frac{\partial^2 \phi}{\partial x^2} \simeq -b \cos x, \]  
\[ (55) \]

and the velocity in comoving coordinates is given by
\[ v = \frac{dx}{dt} \simeq -b \frac{d\phi_0}{dx} \simeq \dot{b} \sin x. \]  
\[ (56) \]

In a non-expanding universe with a constant potential given by \( V(x) = V_0 \cos x \) the density contrast at early times is
\[ \delta \simeq -\frac{V_0}{2} t^2 \cos x. \]  
\[ (57) \]

We can therefore compare the Zel’dovich approximation at time \( b \) with the Schrödinger equation at time \( t = \sqrt{2b/V_0} \). Starting with the initial wavefunction given by
\[ \psi(x, 0) = \sqrt{\rho_0}, \]  
\[ (58) \]

and evolving the wavefunction according to equation (54) using Cayley’s method (Goldberg, Schey & Schwartz 1967) leads to the results shown in the first column of Figure 2. These are for \( \nu = 0.1 \) at times corresponding to the values of \( b = 0.3, 0.9 \) and 2.0 used before. Comparison with Figure 1 shows that the extensive smearing of density peaks which occurs after shell-crossing in both the “free-particle” and Zel’dovich solution is no longer seen in the case where the initial potential is included.

**4.3 The Schrödinger equation with time-varying potential**

In the second column of Figure 2 we show the solution to the Schrödinger equation with the full potential calculated using the Poisson equation (33). We start by considering the equation without the quantum pressure term. In this case we need to solve the coupled Schrödinger and Poisson equations:
\[ i \nu \frac{\partial \psi}{\partial t} = -\nu^2 \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi, \]  
\[ (59) \]
\[ \frac{\partial^2 V}{\partial x^2} = 4\pi G \psi \psi^*, \]  
\[ (60) \]

To set up the initial wavefunction we choose a single sine wave for the density profile,
\[ \delta = -\delta_0 \cos x. \]  
\[ (61) \]

In the linear regime the density contrast at time \( t \) is given by
\[ \delta \simeq -\delta_0 \exp(\sqrt{4\pi G t}) \cos x, \]  
\[ (62) \]

and the velocity is
\[ v = \sqrt{4\pi G \delta_0} \exp(\sqrt{4\pi G t}) \sin x. \]  
\[ (63) \]
Accordingly we compare the Zel’dovich approximation at time $b$ with the Schrödinger equation at time $t = 1/\sqrt{4\pi G \ln(b/b_0)}$. We start with the initial wavefunction, using equation (35), given by,

$$
\psi(x, 0) = \sqrt{\rho_0(1 - \delta_0 \cos x)} \exp \left(-i \sqrt{4\pi G \delta_0} \cos x / \nu \right), \tag{64}
$$

and evolve the wavefunction according to equation (59). In this case the potential $V(x)$ is calculated by solving the Poisson equation numerically at each time step. The results are shown in the second column of Figure 2 for $\nu = 0.1$ at times corresponding to $b = 0.3, 0.9$ and 2.0 used before.

The third column of Figure 2 shows the results for the same initial wavefunction, but this time including the quantum pressure term, as in equation (31). The inclusion of this term makes a significant difference to the results at later times, particularly after shell-crossing, but is slight earlier on during the evolution. This means that useful approximations for the mildly non-linear regime can still be obtained even if this term is ignored.

### 4.4 Solutions in an expanding background

Although we do not make further use of it in this paper, it is useful to end this section with a few comments about what arises when one places the system in an expanding background. To see what happens, let us define a scaled density $\chi = \rho / \rho_0 = (1 + \delta)$ and take $\Omega = 1$. The continuity equation then becomes

$$
\frac{\partial \chi}{\partial a} + \nabla \cdot (\chi \nabla \phi) = 0, \tag{65}
$$

where the velocity potential $\phi$ is now such that $u = dx / dt = \nabla \phi$ and $a$ is the scale factor. It is convenient to take the first integral of the Euler equation, giving

$$
\frac{\partial \phi}{\partial a} + \frac{1}{2} (\nabla \phi)^2 = -\frac{3}{2a} (\phi + \theta), \tag{66}
$$

where $\theta = 2\Phi / 3a^3 H^2$ and $\Phi$ is the gravitational potential. After some more algebra the system again becomes a Schrödinger-like wave equation, but in $a$ rather than in $t$ and using $\psi^2 = \chi$, such that

$$
\frac{\partial \psi}{\partial a} = -\frac{\nu}{2} \nabla^2 \psi + V \psi + P \psi, \tag{67}
$$

with $V = \phi + \theta$ and $P$ as before. It is relatively straightforward to incorporate the extra terms.

### 5 DISCUSSION

In this paper we have set out an approach to the formation of cosmic structure from evolving cosmological density fluctuations that relies on a transformation of the evolution equations into a Schrödinger-Poisson system. One advantage of this new formalism was pointed out by Coles (2002): it yields a rather more convincing approach to understanding the origin of spatial intermittency in the galaxy distribution than that offered by Jones (1999). It also makes a connection in the underlying physics with other systems that display similar phenomena. In this paper we have argued further that it provides an analytical approach that offers benefits over standard approaches of both Lagrangian and Eulerian types. On the one hand, it always guarantees a positive density and on the other it does not produce singularities at shell-crossing events. It is clearly important to investigate both of these issues further. Szapudi & Kaiser (2003) have already completed a study of perturbation theory and its statistical ramifications within this approximation. Dealing fully with post-shell-crossing phenomena is beyond the scope of this paper as it requires a more sophisticated representation of the wave-function in velocity space; see Widrow & Kaiser (1993) for discussion.

In this paper we showed how this approach works in practice by applying it to a static spherically-symmetric case and to some examples of one-dimensional collapse. In doing this we looked at a “free-particle” solution of the system that resembles the Zel’dovich approximation in some respects. Incorporating a fixed gravitational potential yields a solution that has some similarities to the “frozen-potential” approximation (Matarrese et al. 1992). We think the Schrödinger formalism provides a useful tool for modelling fluctuations in the mildly non-linear regime. In future work we will look at its use in modelling the intergalactic medium in order to understand properties of Lyman-α absorption systems.

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