Orthogonal Projections Based on Hyperbolic and Spherical \( n \)-Simplex

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2010 MSC: 51M09, 51M10, 51M20, 51M25, 52A38, 52A55

Key words: Projection, hyperbolic, spherical, simplex, Gram matrix, edge matrix

Abstract

In this paper, orthogonal projection along a geodesic to the chosen \( k \)-plane is introduced using edge and Gram matrix of an \( n \)-simplex in hyperbolic or spherical \( n \)-space. The distance from a point to \( k \)-plane is obtained by the orthogonal projection. It is also given the perpendicular foots from a point to \( k \)-plane of hyperbolic and spherical \( n \)-space.

1 Introduction

One of the fundamental notions in geometry is orthogonal projection and also studies extensively through the long history of mathematics and physics. There are many applications of orthogonal projection. The concept of orthogonal projection plays an important role in the scattering theory, the theory of many-body resonance and different branches of theoretical and mathematical physics.

Let \( R^{n+1}_1 \) be \((n + 1)\)-dimensional vector space equipped with the scalar
product $\langle,\rangle$ which is defined by

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i.$$  

If the restriction of scalar product on a subspace $W$ of $R_1^{n+1}$ is positive definite, then the subspace $W$ is called space-like; if it is positive semi-definite and degenerate, then $W$ is called light-like; if $W$ contains a time-like vector of $R_1^{n+1}$, then $W$ is called time-like.

$S_1^n = \{ x \in R_1^{n+1} | \langle x, x \rangle = 1 \}$ is called de Sitter $n$-space. The $n$-dimensional unit pseudo-hyperbolic space is defined as

$$H_0^n = \{ x \in R_1^{n+1} | \langle x, x \rangle = -1 \},$$

which has two connected components, each of which can be considered as a model for the $n$-dimensional hyperbolic space $H^n$. Throughout this paper we consider hyperbolic $n$-space

$$H^n = \{ x \in H_0^n | x_1 > 0 \}.$$  

Hence, each pair of points $p_i, p_j$ in $H^n$ satisfy $\langle p_i, p_j \rangle < 0$. The hyperbolic distance for $p, q \in H^n$ is defined by $\text{arccosh}(-\langle p, q \rangle)$. Since each $e \in S_1^n$ determines a time-like hyperplane of $R_1^{n+1}$, we have hyperplane $e^+ \cap H^n$ of $H^n$.

Let $R_1^{n+1}$ be $(n+1)$-dimensional vector space equipped with the scalar product $\langle,\rangle_E$ which is defined by

$$\langle x, y \rangle_E = \sum_{i=1}^{n+1} x_i y_i.$$  

The $n$-dimensional unit spherical space $S^n$ is given by

$$S^n = \{ x \in R_1^{n+1} | \langle x, x \rangle_E = 1 \}.$$  

The spherical distance $d_s(p, q)$ between $p$ and $q$ is given by $\text{arccos}(\langle p, q \rangle_E)$.

We consider $W$ is a vector subspace spanned by the vectors $e_1, e_2, \ldots, e_{n-k}$ in $S_1^n$. By using Lemma 27 of [1], one can easily see that $W$ is $(n-k)$-dimensional time-like subspace and $V = e_1^+ \cap e_2^+ \cap \ldots \cap e_{n-k}^+$ is $(k+1)$-dimensional time-like subspace of $R_1^{n+1}$. Consequently, for $i = 1, 2, \ldots, n-k$, the hyperplane $e_i^+ \cap H^n$ intersect at the time-like $k$--plane $V \cap H^n$ of $H^n$. One can define the same tools for spherical $n$--space.
Let \( \triangle \) be a hyperbolic or spherical \( n \)–simplex with vertices \( p_1, \ldots, p_{n+1} \), and \( \triangle_i \) be the face opposite to vertex \( p_i \). Then, according to the first section of [2], we have the edge matrix \( M \) and Gram matrix \( G \) of \( \triangle \). Let \( |M| \) and \( M_{ij} \) be the determinant and \( ij \)th–minor of \( M \), then the unit outer normal vector of \( \triangle_i \) is given by

\[
e_i = \frac{-\epsilon \sum_{j=1}^{n+1} M_{ij} p_j}{\sqrt{M_{ii}|M|}}, \quad i = 1, \ldots, n+1,
\]

where \( \epsilon \) is the curvature of space.

The intersection of \( H^n \) with \((k+1)\)–dimensional time-like subspace is called \( k \)–dimensional plane of \( H^n \) [3]. Similarly, a \( k \)–plane of spherical space is given by the same way.

When a geodesic is drawn orthogonally from a point to a \( k \)–plane, its intersection with the \( k \)–plane is known as perpendicular foot on that \( k \)–plane in \( H^n \) or \( S^n \). The length of a geodesic segment bounded by a point and its perpendicular foot is called the distance between that point and \( k \)–plane. The distance between a vertex and its any opposite \( k \)–face is called \( k \)–face altitude of an \( n \)–simplex.

The orthogonal projection to 2–plane in Euclidean space is well-known (see [4],[5],[6],[7]). The orthogonal projection to \( k \)–plane in Euclidean space is given in [8]. The orthogonal projection taking a point in \( H^n \) and mapping it to its perpendicular foot on a hyperplane are studied in [3] and [9], respectively. The distance between a point and a hyperbolic(spherical) hyperplane is introduced in [10]. The altitude of \((n-1)\)–face of hyperbolic \( n \)–simplex is given in [11].

The orthogonal projection taking a point along a geodesic and mapping to its perpendicular foot, where geodesic meets orthogonally the chosen \( k \)–plane of projection, has not been studied. The aim of this paper is to study such orthogonal projections according to the edge matrix of a simplex in \( H^n \) or \( S^n \).

Let \( m^{k+1} \) be the determinant of sub-matrix \( M(k+1,\ldots,n+1) \) of \( M \) and \( g^{k+1} \) be the determinant of sub-matrix \( G(k+1,\ldots,n+1) \) of \( G \). Suppose that \( m^j_i \) and \( g^j_i \) be the determinant of sub-matrix \( M \begin{pmatrix} 1 & \ldots & k+1 & i \\ 1 & \ldots & k+1 & j \\ \end{pmatrix} \) and \( G \begin{pmatrix} 1 & \ldots & k+1 & i \\ 1 & \ldots & k+1 & j \\ \end{pmatrix} \), respectively.
Lemma 1.1  Let △ be a hyperbolic or spherical \( n \)-simplex with the edge matrix \( M \) and Gram matrix \( G \). Let \( M_{ii} \) and \( G_{ii} \) be \( i \)th minor of \( M \) and \( G \), respectively. Then \( M^{-1} = TGT \) and \( G^{-1} = TMT \) where \( T = \begin{bmatrix} \sqrt{G_{ii}} \delta_{ij} / \| G \| \\ \sqrt{M_{ii}} \delta_{ij} / \| M \| \end{bmatrix} \).

Proof  It can be seen from [12]. □

Let \( M_{11}, M_{12}, M_{22} \) and \( G_{11}, G_{12}, G_{22} \) be \( (k+1) \times (k+1), (k+1) \times (n-k), (n-k) \times (n-k) \) types sub-matrices of \( M \) and \( G \), respectively. Suppose that \( M, G \) and diagonal matrix \( T \) partitioned as \( \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}, \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \) and \( \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \), respectively.

Concerning Lemma 1.1 along with Schur complement of a symmetric matrix, we have the following lemma.

Lemma 1.2  Let \( S_{M_{ii}} \) and \( S_{G_{ii}} \) be Schur complements of the sub-matrices \( M_{ii} \) and \( G_{ii} \). Then
\[
(M_{ii})^{-1} = T_{ii} S_{G_{ii}} T_{ii}, \quad (G_{ii})^{-1} = T_{ii} S_{M_{ii}} T_{ii} \quad i \neq j; \quad i, j = 1, 2.
\]

Proof  It is obvious that \( M, G \) are symmetric and \( M_{ii}, G_{ii} \) are invertible. Since the inverse of Schur complement of \( M_{11} \) in \( M \) is the sub-matrix of \( M^{-1} \), we have
\[
M^{-1} = \begin{bmatrix} (M_{11})^{-1} + (M_{11})^{-1} M_{12} (S_{M_{11}})^{-1} M_{21} (M_{11})^{-1} & -(M_{11})^{-1} M_{12} (S_{M_{11}})^{-1} \\ -(S_{M_{11}})^{-1} M_{21} (M_{11})^{-1} & (S_{M_{11}})^{-1} \end{bmatrix}.
\]

Similarly for the Schur complement of \( M_{22} \), we obtain
\[
M^{-1} = \begin{bmatrix} (S_{M_{22}})^{-1} & -(S_{M_{22}})^{-1} M_{12} (M_{22})^{-1} \\ -(M_{22})^{-1} M_{21} (S_{M_{22}})^{-1} & (M_{22})^{-1} + (M_{22})^{-1} M_{21} (S_{M_{22}})^{-1} M_{12} (M_{22})^{-1} \end{bmatrix}.
\]

Then we have
\[
M^{-1} = \begin{bmatrix} (S_{M_{22}})^{-1} & -(M_{11})^{-1} M_{12} (S_{M_{11}})^{-1} \\ -(M_{22})^{-1} M_{21} (S_{M_{22}})^{-1} & (S_{M_{11}})^{-1} \end{bmatrix},
\]
by the same way, we get
\[
G^{-1} = \begin{bmatrix}
(SG_{22})^{-1} & -(G^{11})^{-1} G^{12} (SG_{11})^{-1} \\
-(G^{22})^{-1} G^{21} (SG_{22})^{-1} & (SG_{11})^{-1}
\end{bmatrix}.
\]
Thus, we obtain the desired results using Lemma 1.1. □

2 Orthogonal Projection to $k$–plane of $H^n$

based on a Hyperbolic $n$–simplex

If $p_1, p_2, ..., p_{n+1}$ are vertices of any hyperbolic $n$–simplex $\triangle$, then $\{p_1, p_2, ..., p_{n+1}\}$ is a basis of $R^{n+1}_1$. Let $W_j$ be a subspace spanned by $\{p_1, p_2, ..., p_j, ..., p_{n+1}\}$, and $e_j$ be the unit outer normal to $W_j$; $j = 1, ..., n+1$. Hence $\{e_1, e_2, ..., e_{n+1}\}$ is another basis of $R^{n+1}_1$.

Let $W^h$ be a $k$–plane which contains a $k$–face with vertices $p_1, p_2, ..., p_{k+1}$ of $\triangle$. Then the set $\{p_1, p_2, ..., p_{k+1}\}$ is a basis of the $(k+1)$–dimensional subspace of $W$ in $R^{n+1}_1$. Since $p_1, p_2, ..., p_{k+1}$ are vertices of $\triangle$, the subset $\{p_1, p_2, ..., p_{k+1}\}$ of $R^{n+1}_1$ can be extended basis $\{p_1, p_2, ..., p_{n+1}\}$ and $\{p_1, ..., p_{k+1}, e_{k+2}, ..., e_{n+1}\}$ of $R^{n+1}_1$. As a consequence, we see that $\{e_{k+2}, ..., e_{n+1}\}$ is a basis of $(n-k)$–dimensional subspace $W^\perp$.

**Theorem 2.1** Let $p$ be a point and $W^h$ be a $k$–plane in $H^n$. Then the orthogonal projection $\sigma(p)$ of $p$ to $W^h$ is given by

\[
\sigma(p) = \frac{p + \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss} M_{tt}} m^*_t (p, e_t) e_s}{|M| m^{k+1}}}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss} M_{tt}} m^*_t (p, e_t) (p, e_s)}{|M| m^{k+1}}}}.
\]

**Proof** For a point $p \in H^n$, by [10, Theorem 3.11], there is a point $\hat{p} \in W$ such that $\hat{p} p \in W^\perp$. Therefore, we can write

\[
\hat{p} p = \sum_{s=k+2}^{n+1} \lambda_s e_s.
\]

Then, we have $-\langle p, e_t \rangle = \sum_{s=k+2}^{n+1} \lambda_s \langle e_s, e_t \rangle$, $t = k+2, ..., n+1$. 
Taking
\[ G^{22} = G \begin{pmatrix} k + 2 & \ldots & n + 1 \\ k + 2 & \ldots & n + 1 \end{pmatrix} = [(e_s, e_t)]_{s,t=k+2,\ldots,n+1}, \quad L = [\lambda_{k+2} \ldots \lambda_{n+1}], \]
we obtain
\[ L = - (G^{22})^{-1} [(p, e_t)]. \quad (2) \]
By Lemma 1.2 and the equation (5) of [13], we see that
\[ (G^{22})^{-1} = \begin{bmatrix} \sqrt{M_{ii}M_{jj}m_i^j} & \vdots \\ \vdots & \ddots \end{bmatrix}_{i,j=k+2,\ldots,n+1}, \]
and this implies
\[ \lambda_s = \sum_{t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m_i^j(p, e_t)}}{|M|m^{k+1}}, \quad s = k + 2, \ldots, n + 1. \quad (3) \]
Substituting (3) into (1), we obtain
\[ \dot{p} = p + \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m_i^j(p, e_t)}}{|M|m^{k+1}} e_s. \quad (4) \]
By [10], there exists a unique \( \sigma(p) \in W^h \) such that \( \sigma(p) = c\dot{p} \). Since \( \dot{p} \) is the orthogonal projection of \( p \) to \( W \), we have
\[ c = \frac{1}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m_i^j(p, e_t)}}{|M|m^{k+1}}}} \]
which completes the proof.

In case of the orthogonal projection to a hyperplane, we obtain \( m_j^j = |M| \) and \( m^{k+1} = M_{jj} \). Substituting these equalities into the statement of Theorem 2.1, we reach the result of [3, Theorem 4.1] and [13, Proposition 2.2], as follows:
\[ \sigma(p) = \frac{p - (p, e_j)e_j}{\sqrt{1 + (p, e_j)^2}}, \quad j = 1, \ldots, n + 1 \]
where \( e_j \) is the unit normal of \( W_j \) in \( R_1^{n+1} \). This result is also a generalization of Theorem 2.1 [3, 13].
Theorem 2.2 Let $p$ be a point and $W^h$ be a $k$–plane in $H^n$. Then,
\[
\cosh \xi(p, W^h) = \sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{M_{ss}M_{tt}m^s_t(p,e_t)p_s(p,e_s)}{|M|m^{k+1}}},
\]
where $\xi(p, W^h)$ is the distance between $p$ and $W^h$.

Proof Since $\langle p, \sigma(p) \rangle = -\cosh \xi(p, W^h)$, the result follows Theorem 2.1 \qed

As an immediate consequence of Theorem 2.2, we obtain the following known result [10, Section 4].

Corollary 2.3 Let $p$ be a point and $W^h_j$ be a hyperplane of $H^n$ determined by $e_j$. Then the distance $\xi(p, W^h_j)$ between $p$ and $W^h_j$ is given by
\[
\cosh \xi(p, W^h_j) = \sqrt{1 + \langle p, e_j \rangle^2}.
\]

By taking $p_j$ instead of $p$ in (4) and using $\langle p_j, e_t \rangle = -\sqrt{|M|} \delta_{jt}$, we obtain
\[
\hat{p}_j = p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} \frac{m^s_j}{m^{k+1}} e_s
\]
and
\[
\langle \hat{p}_j, \hat{p}_j \rangle = -1 - \frac{m^j_j}{m^{k+1}}
\]
where $p_j$ is a vertex of $\triangle$. The proof of following corollary is obvious from Theorem 2.1.

Corollary 2.4 Let $\triangle$ be a hyperbolic simplex with vertices $p_1, \ldots, p_{n+1}$. Then the perpendicular foot from $p_j$ to $k$–face $W^h$ is given by
\[
\sigma(p_j) = \frac{p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} \frac{m^s_j}{m^{k+1}} e_s}{\sqrt{1 + \frac{m^j_j}{m^{k+1}}}}, \quad j = k + 2, \ldots, n + 1,
\]
where $p_1, \ldots, p_k$ are vertices of $k$–face $W^h$.

If we replace $p$ by $p_j$ and use $\langle p_j, e_t \rangle = -\sqrt{|M|} \delta_{jt}$, we see that
\[ \langle \sigma(p_j), p_j \rangle = -\sqrt{1 + \frac{m_j^2}{m^{k+1}}}. \]

If we consider the last equation in the proof of Theorem 2.1, we see that
\[ \cosh \xi(p_j, W^h) = \sqrt{1 + \frac{m_j^2}{m^{k+1}}}, \]
that result is a generalization of \[11, Proposition 4\] to the \(k\)-face \(W^h\) of a hyperbolic \(n\)-simplex. Since \(m_j^2/m^{k+1}\) is the diagonal \(jj\)th-entry of \(S_{M^1} = [a_{ij}]\), the altitude from \(p_j\) to \(k\)-face \(W^h\) with vertices \(p_1, \ldots, p_{k+1}\) is given by
\[ \cosh \xi(p_j, W^h) = \sqrt{1 + a_{jj}} \]
where \(\xi(p_j, W^h)\) is the distance between \(p_j\) and \(k\)-face \(W^h\).

By \(\hat{p}_j = p_j + \sqrt{|M|} \frac{e_j}{M_{jj}}\), for \((n-1)\)-face \(W^h_j\), we have the following corollary.

**Corollary 2.5** Let \(\triangle\) be a hyperbolic simplex with vertices \(p_1, \ldots, p_{n+1}\). Then the perpendicular foot from \(p_j\) to \((n-1)\)-face \(W^h_j\) is given by
\[ \sigma(p_j) = \frac{p_j + \sqrt{|M|} \frac{e_j}{M_{jj}}}{\sqrt{1 + \frac{|M|}{M_{jj}}}}, \quad j = 1, \ldots, n+1 \]
where \(p_1, \ldots, \hat{p}_j, \ldots, p_{n+1}\) are vertices of \(W^h_j\).

Using \(G_{jj} = -\frac{|G|M_{jj}}{|M|}\) for \(j = 1, \ldots, n+1\), we obtain the following known result [11, Proposition 4].

**Corollary 2.6** Let \(\triangle\) be a hyperbolic simplex with vertices \(p_1, \ldots, p_{n+1}\). Then the altitude \(\xi(p_j, W^h_j)\) from \(p_j\) to \((n-1)\)-face \(W^h_j\) is given by
\[ \cosh \xi(p_j, W^h_j) = \sqrt{1 + \frac{|M|}{M_{jj}}}, \quad j = 1, \ldots, n+1 \]
where \(p_1, \ldots, \hat{p}_j, \ldots, p_{n+1}\) are vertices of \(W^h_j\).
3 Orthogonal Projections to a $k$–plane of $S^n$

based on a Spherical $n$–simplex

Let $\triangle$ be with vertices $p_1, \ldots, p_{n+1}$. Then $\{p_1, \ldots, p_{n+1}\}$ is a basis of $R^{n+1}$. If $W_j$ is the subspace spanned by $\{p_1, \ldots, \hat{p}_j, \ldots, p_{n+1}\}$, then $\{e_1, \ldots, e_{n+1}\}$ is another basis of $R^{n+1}$ where $e_j$ is the unit outer normal to $W_j$ for $j = 1, \ldots, n+1$.

Let $W^*$ be a $k$–plane which contains a $k$–face with vertices $p_1, p_2, \ldots, p_{k+1}$. Then the set $\{p_1, p_2, \ldots, p_{k+1}\}$ is a basis of the $(k+1)$–dimensional subspace $W$ in $R^{n+1}$. As a consequence, we have a basis $\{e_{k+2}, \ldots, e_{n+1}\}$ of $(n-k)$–dimensional subspace $W^\perp$.

Theorem 3.1 Let $p$ be a point and $W^*$ be a $k$–plane in $S^n$. Then the orthogonal projection $\sigma(p)$ of $p$ to $W^*$ is given by

$$\sigma(p) = \frac{p - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}} \langle p, e_t \rangle e_s}{|M|m^{k+1}}}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}} \langle p, e_t \rangle \langle p, e_s \rangle}{|M|m^{k+1}}}.$$ 

Proof By [10, Theorem 3.11], for $p \in S^n$, there is a $\hat{p} \in W$ such that $\vec{p}\hat{p} \in W^\perp$. Therefore, we can write

$$\vec{p}\hat{p} = \sum_{s=k+2}^{n+1} \lambda_s e_s$$

Then, we have

$$\langle p, \vec{p}\hat{p} \rangle_E = \sum_{s=k+2}^{n+1} \lambda_s \langle e_s, e_t \rangle_E.$$ 

Taking

$$G^{22} = G \begin{pmatrix} k+2 & \ldots & n+1 \\ k+2 & \ldots & n+1 \end{pmatrix} = [\langle e_s, e_t \rangle_E]_{s,t=k+2,\ldots,n+1}, \quad L = [\lambda_{k+2} \ldots \lambda_{n+1}],$$

we find

$$L = - (G^{22})^{-1} [\langle p, e_t \rangle_E].$$

By Lemma [14.2] and the equation (5) of [14], we see that

$$(G^{22})^{-1} = \left[ \frac{\sqrt{M_{ij}M_{tt}}}{|M|m^{k+1}} \right]_{i,j=k+2,\ldots,n+1}.$$
This implies
\[ \lambda_s = -\sum_{t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m^2_s\langle p, e_t \rangle_E}}{|M|m^{k+1}}, \quad s = k + 2, \ldots, n + 1. \]  
(7)

Substituting (7) into (5), we obtain
\[ \dot{p} = p - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m^2_s\langle p, e_t \rangle_E}}{|M|m^{k+1}}. \]  
(8)

By [10], there exists a unique \( \sigma(p) \in W^* \) such that \( \sigma(p) = c\dot{p} \). Since \( \dot{p} \) is the orthogonal projection of \( p \) to \( W \), we have
\[ c = \frac{1}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m^2_s\langle p, e_t \rangle_E}}{|M|m^{k+1}}}. \] which completes the proof. \( \square \)

By Theorem 3.1, we have
\[ \sigma(p) = \frac{p - \langle p, e_j \rangle_E e_j}{\sqrt{1 - \langle p, e_j \rangle_E^2}} \]
where \( e_j \) is the unit normal of the \( W_j \) in \( R^{n+1} \).

**Theorem 3.2** Let \( p \) be a point and \( W^* \) be a \( k \)-plane in \( S^n \). Then
\[ \cos \theta(p, W^*) = \sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m^2_s\langle p, e_t \rangle_E}}{|M|m^{k+1}}}. \]
where \( \theta(p, W^*) \) is the distance between \( p \) and \( W^* \).

By taking \( p_j \) instead of \( p \) and using \( \langle p_j, e_j \rangle_E = -\sqrt{\frac{|M|}{M_{jj}}} \) in (8), we obtain
\[ \dot{p}_j = p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} m^2_j e_s \]
and
\[ \langle \dot{p}_j, \dot{p}_j \rangle_E = 1 - \frac{m^2_j}{m^{k+1}}, \quad j = k + 2, \ldots, n + 1, \]
where \( p_j \) is a vertex of \( \Delta \). Hence, we have the following corollary.
Corollary 3.3 Let $\triangle$ be a spherical $n-$simplex with vertices $p_1, \ldots, p_{n+1}$, then the perpendicular foot from $p_j$ to $k-$face $W^s$ is given by

$$\sigma(p_j) = \frac{p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|} \frac{m_s}{m_{k+1}}} e_s}{\sqrt{1 - \frac{m_j^2}{m_{k+1}}}}$$

where $p_1, \ldots, p_{k+1}$ are vertices of $W^s$.

Let $\theta(p_j, W^s)$ be the altitude from the vertex $p_j$ to the $k-$face $W^s$ with vertices $p_1, \ldots, p_{k+1}$ for $j = k + 2, \ldots, n + 1$. Then $\theta(p_j, W^s)$ is given by

$$\cos \theta(p_j, W^s) = \sqrt{1 - \frac{m_j^2}{m_{k+1}}}$$

By equality (5) in [14], the $jj$th-entry of the Schur complement $S_{M_{11}} = [b_{ij}]$ satisfy $b_{jj} = \frac{m_j^2}{m_{k+1}}$.

Let $W^s_j$ be the $(n-1)-$face with vertices $p_1, \ldots, \hat{p}_j, \ldots, p_{n+1}$ of $\triangle$. Then, we have

$$\hat{p}_j = p_j + \sqrt{\frac{|M|}{M_{jj}}} e_j,$$

and

$$\langle p_j, e_j \rangle_E = -\sqrt{\frac{|M|}{M_{jj}}}, \quad j = 1, \ldots, n + 1.$$

The proof of the following corollary is obtained by using the above equations.

Corollary 3.4 Let $\triangle$ be a spherical simplex with vertices $p_1, \ldots, p_{n+1}$. Then the perpendicular foot from $p_j$ to $(n-1)-$face $W^s_j$ is given by

$$\sigma(p_j) = \frac{p_j + \sqrt{\frac{|M|}{M_{jj}}} e_j}{\sqrt{1 - \frac{|M|}{M_{jj}}}}$$

where $p_1, \ldots, \hat{p}_j, \ldots, p_{n+1}$ are vertices of $W^s_j$.

Corollary 3.5 Let $\triangle$ be a spherical simplex with vertices $p_1, \ldots, p_{n+1}$. Then the altitude $\theta(p_j, W^s_j)$ from $p_j$ to $(n-1)-$face $W^s_j$ is given by
\[ \cos \theta(p_j, W^*_j) = \sqrt{1 - \frac{|M|}{M_{jj}}}, \quad j = 1, \ldots, n + 1. \]

where \( p_1, \ldots, \hat{p}_j, \ldots, p_{n+1} \) are vertices of \( W^*_j \).

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