Calculating intersection numbers on moduli spaces of pointed curves

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August 14, 2008

Introduction

The purpose of this note is to explain how to calculate intersection numbers on moduli spaces of curves. More specifically, we will discuss computing intersection numbers among tautological classes on the moduli space of stable \(n\)-pointed curves of genus \(g\), denoted \(\overline{M}_{g,n}\). The recipe described below is the cumulation of many results found in various papers [AC, F2, GP, M, W], but is reformulated here with an algorithmic emphasis.

Let \(\overline{M}_{g,n}\) denote the space of Deligne-Mumford stable curves of genus \(g\) with \(n\) marked points, labelled \(1, \ldots, n\). If \(S\) is a finite set, we set \(\overline{M}_{g,S} = \overline{M}_{g,|S|}\) and let \(S\) serves as an index set for the points. The Chow rings \(A^*(\overline{M}_{g,n})\) are often badly behaved, but much geometric information about \(\overline{M}_{g,n}\) can be recovered by restricting our attention to the tautological rings \(R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})\). These are defined as the smallest nonzero system of sub-\(\mathbb{Q}\)-algebras of \(A^*(\overline{M}_{g,n})\), that satisfy the two properties [FP]:

1. the system is closed under pushforwards by the natural forgetful morphism
   \[
   \pi_{n+1} : \overline{M}_{g,n+1} \to \overline{M}_{g,n}
   \]
   which forgets the \((n+1)\)st marked point (and, if necessary, contracts components to stabilize the curve)

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*Research partially supported by an NSF Postdoctoral Fellowship
2. the system is closed under pushforwards by the natural gluing morphisms

\[ t_{\text{irr}} : \overline{M}_{g-1,[n]\cup\{\star,\bullet\}} \rightarrow \overline{M}_{g,n} \]
\[ t_{g_1,S} : \overline{M}_{g_1,S\cup\{\star\}} \times \overline{M}_{g_2,T\cup\{\bullet\}} \rightarrow \overline{M}_{g_1+g_2,n} \]

which identify the points labelled \( \star \) and \( \bullet \). Here, \([n]\) denotes the set \( \{1,\ldots,n\} \), and the subsets \( S,T \subseteq [n] \) form a partition of \([n]\).

More concretely, any tautological class can be expressed in terms of natural classes in the rational Picard group of \( \overline{M}_{g,n} \). Let \( L_i \) denote the line bundle on \( \overline{M}_{g,n} \) whose value at a fixed curve \( C \) is the cotangent space to \( C \) at the \( i \)-th marked point, i.e. \( L_i = \sigma_i^*(\omega_{\pi_{n+1}}) \) where \( \omega \) denotes the relative dualizing sheaf and \( \sigma_i : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n+1} \) is the section of \( \pi_{n+1} \) corresponding to the \( i \)-th marked point. Define divisor classes \( \psi_i \) and codimension \( a \) classes \( \kappa_a \) as follows:

\[ \psi_i = c_1(L_i) \]
\[ \kappa_a = \pi_{1*}(\psi_1^{a+1}) \]

Next, let \( E \) denote the Hodge bundle of \( \overline{M}_{g,n} \), whose value at a pointed curve \( C \) is its \( g \)-dimensional space of differentials, i.e. \( E = \pi_1^*L_1 \). Define codimension \( a \) classes \( \lambda_a \) as follows.

\[ \lambda_a = c_a(E) \]

Computing intersection numbers among tautological classes involves two calculations. First, we must calculate intersection numbers among top intersection products of the \( \kappa \), \( \lambda \), and \( \psi \) classes; in other words, evaluating

\[ \int_{\overline{M}_{g,n,n}} \kappa_1^{a_1} \cdots \kappa_a^{a_a} \lambda_1^{b_1} \cdots \lambda_b^{b_b} \psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \]

This was first implemented by Carel Faber around 1996 in MAPLE (with the limitation \( 3g - 3 + n \leq 20 \)). The second calculation is evaluating these classes as they’re pushed forward by gluing morphisms \( t \). For this, we need the language of decorated stable graphs which is introduced in Section 2.

Let \( \Sigma \) denote the locus of pushforwards of \( \kappa \) and \( \psi \) classes under the natural gluing morphisms. A multiplication formula for classes in \( \Sigma \), first
described in [GP], is given in Section 2.3. It follows from the formula that \( \Sigma \) is closed under multiplication, thus forming a sub-algebra of \( A^*(\mathcal{M}_{g,n}) \). In other words, \( \Sigma \) additively generates \( R^*(\mathcal{M}_{g,n}) \).

We turn to finding relations among these classes. Faber’s conjecture states that \( R^*(\mathcal{M}_{g,n}) \) behaves like the cohomology ring of a smooth projective manifold of dimension \( 3g - 3 + n \) [F3]. In particular, it is Gorenstein with socle in degree \( 3g - 3 + n \), meaning that there is an isomorphism

\[
\phi: R^{3g-3+n} \cong (\mathcal{M}_{g,n})
\]

such that for every \( i \) the bilinear pairing

\[
R^i(\mathcal{M}_{g,n}) \times R^{3g-3+n-i}(\mathcal{M}_{g,n}) \to R^{3g-3+n}(\mathcal{M}_{g,n})
\]

is non-degenerate.

According to Faber’s conjecture, any numerical relation found by calculating intersection numbers among classes of complimentary codimension in \( \Sigma \) must hold in the Chow ring of \( \mathcal{M}_{g,n} \). Thus the tautological ring \( R^*(\mathcal{M}_{g,n}) \) is conjecturally isomorphic to its Gorenstein quotient.

In Section 3 we give relations in genus 3 and 4 that have been found with the help of a computer. Furthermore, we have found a relation \( R^3(\mathcal{M}_{5,0}^{ct}) \), which suggests the existence of a relation in \( R^3(\mathcal{M}_{5,0}) \). In the appendix we display the ranks of intersection pairings for \( \mathcal{M}_{g,n} \), as well as for the related spaces \( \mathcal{M}_{g,n}^{ct} \) the moduli space of curves “with rational tails” (i.e. curves with one component of genus \( g \)), and \( \mathcal{M}_{g,n}^{ct} \), the moduli space of curves of compact type (i.e. curves with compact Jacobians).

## 1 Integrals among \( \psi, \kappa, \) and \( \lambda \) classes

This section contains an overview of how to calculate intersection numbers in \( \kappa, \psi, \) and \( \lambda \) classes. Our starting point is the string and dilaton equations below

\[
\int_{\mathcal{M}_{g,n+1}} \psi_1^{\gamma_1} \psi_2^{\gamma_2} \cdots \psi_n^{\gamma_n} \psi_{n+1} = (2g - 2 + n) \int_{\mathcal{M}_{g,n}} \psi_1^{\gamma_1} \psi_2^{\gamma_2} \cdots \psi_n^{\gamma_n}
\]

\[
\int_{\mathcal{M}_{g,n+1}} \psi_1^{\gamma_1} \psi_2^{\gamma_2} \cdots \psi_n^{\gamma_n} = \sum_{i=1}^{n} \int_{\mathcal{M}_{g,n}} \psi_1^{\gamma_1} \psi_2^{\gamma_2} \cdots \psi_i^{\gamma_i-1} \cdots \psi_n^{\gamma_n},
\]
as well as Witten’s conjecture, or Kontsevich’s theorem. This has many equivalent statements; here is the one we will use. Let \( F \) be the following generating function in the \( \psi \) classes,

\[
F_g := \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1, \ldots, a_n} \left( \int_{\overline{M}_{g,n}} \psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \right) t_{\gamma_1} \cdots t_{\gamma_n}
\]

and set

\[
F = \sum F_g h^{2g-2}. \tag{1}
\]

Witten’s conjecture, or Kontsevich’s theorem, states

\[
(2k+1) \frac{\partial^3}{\partial t_k \partial t_0^2} F = \left( \frac{\partial^2}{\partial t_{k-1} \partial t_0} F \right) \left( \frac{\partial^3}{\partial t_0^3} F \right) + 2 \left( \frac{\partial^3}{\partial t_{k-1} \partial t_0^2} F \right) \left( \frac{\partial^2}{\partial t_0^2} F \right) + \frac{1}{4} \frac{\partial^5}{\partial t_{k-1} \partial t_0^4} F. \tag{2}
\]

These allow us to calculate recursively all top intersection products among \( \psi \) classes.

Integrals involving both \( \psi \) and \( \kappa \) classes can be reduced to the case above using the pullback formula

\[
\kappa_a = \pi_n^* \kappa_a - \psi_n^a.
\]

and the push-pull formula.

\[
\psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \kappa_1^{\alpha_1} \cdots \kappa_a^{\alpha_a} = (\psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \kappa_1^{\alpha_1} \kappa_2^{\alpha_2} \cdots \kappa_a^{\alpha_a})
= \pi_{n+1,*}(\psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \psi_{n+1}^{\alpha_{n+1}} \kappa_2^{\alpha_2} \cdots \kappa_a^{\alpha_a})
= \psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \psi_{n+1}^{\alpha_{n+1}} \pi_{n+1,*}(\kappa_2^{\alpha_2} \cdots \kappa_a^{\alpha_a})
= \psi_1^{\gamma_1} \cdots \psi_n^{\gamma_n} \psi_{n+1}^{\alpha_{n+1}} \prod_{i=2}^{a} (\kappa_i - \psi_{n+1}^{i})^{\alpha_i}
\]

The last expression contains one fewer \( \kappa \) class in its product, and one more marked point; repeated iteration of this equation allows us to eliminate \( \kappa \) classes.

To express \( \lambda \) classes in terms of the other classes we use the Grothendieck-Riemann-Roch formula. Mumford applied this to the morphism \( \overline{M}_{g,1} \rightarrow \)
\( \overline{M}_{g,0} \) to calculate:

\[
\text{ch}(E) = g + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)} \left( \kappa_{2l-1} + \frac{1}{2} \sum_{j \in \{\text{irr},0,\ldots,g\}} \sum_{k=0}^{2l-2} \iota_{h} (-1)^{i} \psi_{i}^{j} \psi_{i}^{2l-2i} \right) \quad (3)
\]

On the left hand side of the above equation, we have the chern character \( \text{ch}(E) \) which is given by the formula

\[
\sum_{j=0}^{\infty} (-t)^{-j} j! \text{ch}_{j} = \left( \sum_{i=0}^{g} i \lambda_{i} t^{i-1} \right) \left( \sum_{i=0}^{g} (-1)^{i} \lambda_{i} t_{i} \right)
\]

On the right hand side, \( \iota_{\text{irr}} \) and \( \iota_{g} \) represent the natural gluing morphisms (with \( n = 0 \)), and \( B_{i} \) is the \( i \)-th Bernoulli number. Pulling equation (3) back to \( \overline{M}_{g,n} \), we obtain

\[
\text{ch}_{a}(E) = \frac{B_{a+1}}{(a+1)!} \left( \kappa_{a} - \sum_{i=1}^{n} \psi_{i}^{a} + \frac{1}{2} \sum_{i=0}^{a-1} \iota_{\text{irr}} \psi_{i}^{a} + \psi_{i}^{a-1-i} \right.
\]

\[
+ \frac{1}{2} \sum_{i=0}^{a-1} \sum_{j \in \{0,\ldots,g\}} (t_{j,S}) S_{\leq |n|} (-1)^{i} \psi_{i}^{a} \psi_{i}^{a-1-i} \right).
\quad (4)
\]

This equation can be used to re-express a product of \( \kappa, \psi, \) and \( \lambda \) classes in terms of only \( \kappa \) and \( \psi \) classes.

2 Boundary classes

2.1 Stable graphs and decorations

By a graph we mean a connected, undirected graph with allowed half-edges, multiple edges, and loops. The set of vertices of a graph \( G \) is denoted \( v(G) \). The incidence function \( n \) assigns to each vertex \( v \) the number of edges and half-edges incident to that vertex. We will also use \( n(G) \) to denote total the number of half edges of a graph \( G \). A stable graph is a graph together with a labelling of the half-edges of the graph (here, the points are always labelled \( 1,\ldots,n(G) \)), and a genus function \( g \) that associates a genus \( g \) to each vertex, such that the stability condition is satisfied: \( 2g(v) - 2 + n(v) > 0 \) for every
vertex $v$ (this is directly analogous to the stability condition for curves). Let $g(G)$ denote the total genus of the graph,

$$g(G) = \sum_{v \in G} g(v) + h^1(G).$$

To any stable pointed curve $C$ we associate a stable graph, or dual graph $G$, which encodes the topological type of $C$. Vertices of $G$ correspond to irreducible components of $C$, edges of $G$ between vertices correspond to nodes of $C$ between respective components, and half-edges correspond to the marked points. Similarly, given any stable graph $G$, we can define $\mathcal{M}_G$ to be the locus of curves in $\overline{\mathcal{M}}_{g(G), n(G)}$ whose dual graph is $G$. This gives a stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type.

Another way to define $\mathcal{M}_G$ is to via the natural morphism of stacks

$$\xi_G : \prod_{v \in v(G)} \overline{\mathcal{M}}_{g(v), n(v)} \rightarrow \overline{\mathcal{M}}_{g,n}.$$

This is the map which identifies families of pointed curves according to how $G$ is formed by identifying two half-edges to an edge. The boundary strata $\mathcal{M}_G$ is the image of the map $\xi_G$.

A decorated stable graph is a stable graph with the additional data of a monomial in $\kappa$, $\lambda$, and $\psi$ classes on each vertex. Choosing such a monomial $\theta_v$ for each vertex $v$, the corresponding class is $\xi_G(\prod \theta_v)$. Since $\psi$ classes depend on the index of the marked points, we denote them by adding arrowheads to the appropriate edge or half-edge incident to the appropriate vertex (see Figure 2.1). If $G$ is a decorated stable graph, let $\overline{G}$ denote the underlying undecorated graph.

Let $G$ be a specialisation of $H$; that is, $G$ is obtained from $H$ by replacing each vertex $v$ with a graph of genus $g(v)$ with $n(v)$ marked points. (This corresponds exactly to specialisation of curves.) We call $H$ a despecialization.
of $G$, and obtain a natural morphism

$$\xi_{G,H} : \prod_{v \in v(G)} \mathcal{M}_{g(v),n(v)} \rightarrow \prod_{v \in v(H)} \mathcal{M}_{g(v),n(v)}.$$

Let $\Sigma$ be the set of tautological classes $\xi_{G^*} \left( \prod_{v \in V(B)} \theta_v \right)$, where $G$ is a graph of genus $G$ with $n$ marked points, and $\theta_v$ is a monomial in $\psi$ and $\kappa$ classes. As explained in the introduction, it is straightforward to check that $\Sigma$ is a sub-algebra of the Chow-ring, closed under push-forwards forgetful and gluing maps, and thus $\Sigma$ spans $R^*(\mathcal{M}_{g,n})$.

2.2 Graph structures

Let $G$ and $A$ be undecorated stable graphs. An $G$-structure on $A$ is an identification of $G$ with a despecialization of $A$. Note that there can be many $A$-structures on the same graph $G$, for example if $G$ and $A$ are the graphs denoted in the pictures below

$$G = \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\infty
\end{array}
\end{array}$$
$$A = \begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\infty
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}$$

then there are four $G$-structures on $A$—the edge of $G$ can be identified with either of the two edges of $A$ in two different ways.

An $(G, H)$-graph is a graph $A$ which has both an $G$-structure and a $H$-structure, called an $(G, H)$-structure. Two $(G, H)$-structures on a graph $A$ are considered isomorphic if the differ by an automorphism of $A$. If $A$ has an $(G, H)$-structure we say that an edge of $A$ is a common $(G, H)$-edge if it is identified with both an edge of $G$ and with an edge of $H$. An $(G, H)$-graph is called called generic if every edge of $A$ is identified with an edge of $G$, an edge of $H$, or both; that is to say. The set of all generic $(G, H)$-structures is denoted $\Gamma(G, H)$.

Continuing the example above, if we define $H = G$, then there are sixteen $(G, H)$-structures on $A$, eight of which are generic. They are isomorphic in pairs; i.e., up to switching the edges of $A$ there are only eight different $(G, H)$-structures, four of which are generic.
2.3 The multiplication formula

If \(G\) and \(H\) are decorated stable graphs, then the intersection of their associated tautological classes is given by the following formula

\[
\mathcal{M}_G \cdot \mathcal{M}_H = \sum_{A \in \Gamma(G, H)} \frac{1}{|\text{aut}(A)|} \xi^*_A (\prod_{v \in G} \theta(v)) \xi^*_A (\prod_{v \in H} \theta(v)) \prod_{e = (e_1, e_2)} (-\psi_{e_1} - \psi_{e_2}) \tag{5}
\]

Here the last product is taken over all common \((G, H)\)-edges of \(G\), and \(e_i\) represents the incidence of a common edge to the appropriate vertex. Alternatively, one may consider only the set of isomorphism classes of generic \((G, H)\)-graphs, and replace the sum above with one representative from each isomorphism class ([GP]). In this case the normalising coefficient \(|\text{aut}(G)|^{-1}\) becomes unnecessary but implementation on a computer becomes more difficult.

The pullbacks of the monomial products \(\prod \theta_v\) to \(A\) are easy to calculate: \(\psi_i\) pulls back as expected, and the pullback of \(\kappa_i\) classes is simply the sum of \(\kappa_i\) on the corresponding vertices [AC].

We give an example of the formula. Let \(G\) and \(H\) be the following decorated stable graphs which represent classes in \(R^3(\overline{M}_{4,0})\) and \(R^6(\overline{M}_{4,0})\).

\[
G = \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} \kappa_1 \quad \quad H = \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\]

There are three generic \((G, H)\)-graphs, namely:

\[
A = \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} \quad B = \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} \quad C = \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\]

There exist four \(G\)-structures and eight \(H\)-structures on \(A\), which gives us a total of 32 \((G, H)\)-structures, each of which has exactly two common \((G, H)\)-edges. The order of the automorphism group of \(A\) is eight.

Graphs \(B\) and \(C\) both have has sixteen generic \((G, H)\)-structure with one, respectively zero, common \((G, H)\)-edges. The order of the automorphism groups of \(B\) and \(C\) are both sixteen.

We start with calculating the summand of formula 5 that corresponds to \(B\). There is only one common \((G, H)\)-edge of \(B\) (one connecting the genus
1 vertex to the rightmost genus zero vertex). The pullback of \(\kappa_1\) from \(A\) to \(B\) is simply the sum of \(\kappa_1\) on the corresponding vertices. Thus we see the summand is the sum over the following eight tautological classes, each of which appears with a coefficient of -1.

\[
\begin{align*}
&\kappa_1, \\
&\kappa_1, \\
&\kappa_1, \\
&\kappa_1,
\end{align*}
\]

All of the above classes are zero except the last one.

A similar calculation for \(A\) and \(C\) yield only one nonzero class on \(C\) that contributes to the final summand with coefficient 1:

\[
\begin{align*}
&\kappa_1.
\end{align*}
\]

Thus the formula becomes:

\[
A \cdot B = -\frac{1}{16} \cdot 16 \left( \int_{\overline{M}_{0,3}} 1 \right)^4 \int_{\overline{M}_{2,1}} \kappa_1 \psi_2 + \frac{1}{16} \cdot 16 \left( \int_{\overline{M}_{0,3}} 1 \right)^5 \int_{\overline{M}_{1,1}} \kappa_1
\]

and the final answer is \(1/8\).

### 2.4 Implementation of the formula

We have implemented an program in Maple which enumerates the set of decorated stable graphs for \(\overline{M}_{g,n}\), and calculates the multiplication formula for any pair of decorated stable graphs \([Y]\). The program works also for \(\overline{M}^{ct}_{g,n}\), the moduli space of \(n\)-pointed genus \(g\) curves of compact type (i.e. curves whose dual graphs are trees), as well as \(\overline{M}^{rt}_{g,n}\), the moduli space of \(n\)-pointed genus \(g\) curves with rational tails (i.e. curves with exactly one component of genus \(g\)).

When enumerating decorated stable graphs, one need not use every possible monomial in the \(\kappa\) and \(\psi\) classes. First, one can disregard monomials of degree \(g\) or more, as well as \(\kappa_{g-1}\), thanks to Ionel’s vanishing theorem \([I]\) and Proposition 2 of \([FP]\). There are also known topological recursion relations due to Getzler and Belorousski-Pandharipande \([G,BP]\). Currently the program works for the pairs \((g,n)\) which are listed in the appendix.
3 Relations in genus 3 and 4

For the relation in $\mathcal{M}_{3,2}$, we let graphs with unassigned half-edges denote the sum of the two graphs that arise from labelling the half-edges both ways:

$$[[\text{graph}]] = [[\text{graph}_1]] + [[\text{graph}_2]]$$

**Proposition 1.** The following relation holds among cycles in $R^2(\overline{\mathcal{M}}_{3,2})$:

$$0 = 42 [[\text{cycle}_1]] - 35 [[\text{cycle}_2]] - \frac{1}{2} \left[ \text{cycle}_3 \right] + \frac{5}{6} \left[ \text{cycle}_4 \right] + 35 \left[ \text{cycle}_5 \right]$$

$$- 175 \left[ \text{cycle}_6 \right] - 12 \left[ \text{cycle}_7 \right] + \frac{80}{3} \left[ \text{cycle}_8 \right] + 30 \left[ \text{cycle}_9 \right]$$

$$- \frac{175}{3} \left[ \text{cycle}_{10} \right] + \frac{200}{3} \left[ \text{cycle}_{11} \right] + \frac{1}{144} \left[ \text{cycle}_{12} \right] - \frac{10}{9} \left[ \text{cycle}_{13} \right]$$

$$+ \frac{14}{3} \left[ \text{cycle}_{14} \right] + \frac{175}{72} \left[ \text{cycle}_{15} \right] - \frac{5}{2} \left[ \text{cycle}_{16} \right] - \frac{65}{6} \left[ \text{cycle}_{17} \right]$$

$$- \frac{25}{9} \left[ \text{cycle}_{18} \right] + \frac{1}{3} \left[ \text{cycle}_{19} \right] - \frac{1}{2} \left[ \text{cycle}_{20} \right] - \frac{2}{3} \left[ \text{cycle}_{21} \right]$$

$$- \frac{1}{6} \left[ \text{cycle}_{22} \right] - 6 \left[ \text{cycle}_{23} \right] + \frac{175}{3} \left[ \text{cycle}_{24} \right]$$

$$- \frac{200}{3} \left[ \text{cycle}_{25} \right] - 25 \left[ \text{cycle}_{26} \right] + 35 \left[ \text{cycle}_{27} \right]$$

$$- \frac{200}{3} \left[ \text{cycle}_{28} \right] + 4 \left[ \text{cycle}_{29} \right] - 16 \left[ \text{cycle}_{30} \right]$$

$$- 4 \left[ \text{cycle}_{31} \right] + 26 \left[ \text{cycle}_{32} \right]$$

**Proof.** The coefficients were found using MAPLE. By Proposition 2 of [FP], the restriction sequence

$$R^* (\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}) \to R^*(\overline{\mathcal{M}}_{g,n}) \to R^*(\mathcal{M}_{g,n}) \to 0$$

is exact in codimension 2, thus the relation holds in $R^*(\overline{\mathcal{M}}_{3,2})$.  

\qed
Proposition 2. The following numerical relation holds among classes in $R^3(M_{4,0})$.

\[
0 = -\frac{7}{12} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 20 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{36} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{7}{24} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]
\[
+ \frac{7}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{5}{6} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{19}{72} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{65}{18} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{73}{36} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
\[
+ \frac{35}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{106}{3} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{19}{34560} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{17}{1728} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]
\[
- \frac{1}{36} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{71}{864} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{37}{864} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{171}{25} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
\[
- \frac{4818}{35} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{7}{360} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{15} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{5}{18} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
\[
- \frac{1}{8} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{7}{36} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{18} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
\[
- \frac{53}{60} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{83}{30} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{373}{120} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
\[
+ \frac{22}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{34}{5} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Finally, we express the class of $\lambda_4 \in R^4(M_{4,0})$ in terms of boundary, $\kappa$, and $\psi$ classes. This is simply a matter of linear algebra; the class $\lambda_4$ intersection with any class in $R^5(M_{4,0})$ is 0 unless the strata is boundary with five edges, in which case the intersection number is 1.

Proposition 3. The following relation holds among cycles in $R^4(M_{4,0})$
\[
\lambda_4 = \frac{2304}{5} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} - \frac{4818}{35} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} - \frac{5778}{5} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} - 
\]
\[
- \frac{828}{5} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} - \frac{1116}{5} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} - \frac{3408}{35} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} - \frac{432}{35} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
- \frac{384}{5} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{2592}{5} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} + \frac{3}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} + \frac{3049}{700} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
- \frac{234}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{99}{5} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} + \frac{99}{10} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} + \frac{468}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{198}{5} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{171}{25} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} + \frac{171}{100} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} + \frac{1362}{35} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{986}{7} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{1132}{5} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} + \frac{2232}{5} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{9426}{175} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} - \frac{2202}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} + \frac{8424}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{5616}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} - \frac{774}{25} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} - \frac{8082}{175} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{2664}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{5616}{25} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} + \frac{7956}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{4374}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{3978}{25} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} + \frac{2376}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{11232}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{1116}{5} \begin{bmatrix} \circ & \circ & \circ \end{bmatrix} - \frac{2208}{5} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} - \frac{1728}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{6624}{175} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} - \frac{144}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} - \frac{1728}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} 
\]
\[
+ \frac{5184}{25} \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix} + \frac{1728}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} + \frac{2448}{25} \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} 
\]
Appendix

The following three tables display the calculated ranks of the intersection pairing on $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}^{st}_{g,n}$, and $\mathcal{M}^{rt}_{g,n}$, respectively.

| Codimension | (g,n) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------------|-------|---|---|---|---|---|---|---|---|---|---|
|             | (1,1) | 1 | 1 |   |   |   |   |   |   |   |   |
|             | (1,2) | 1 | 2 | 1 |   |   |   |   |   |   |   |
|             | (1,3) | 1 | 5 | 5 | 1 |   |   |   |   |   |   |
|             | (1,4) | 1 | 12| 23| 12| 1 |   |   |   |   |   |
|             | (1,5) | 1 | 27| 102| 102| 27| 1 |   |   |   |   |
|             | (2,0) | 1 | 2 | 2 | 1 |   |   |   |   |   |   |
|             | (2,1) | 1 | 3 | 5 | 3 | 1 |   |   |   |   |   |
|             | (2,2) | 1 | 6 | 14| 14| 6 | 1 |   |   |   |   |
|             | (2,3) | 1 | 12| 44| 67| 44| 12| 1 |   |   |   |
|             | (2,4) | 1 | 24| 144| 333| 333| 144| 24| 1 |   |   |
|             | (3,0) | 1 | 3 | 7 | 10| 7 | 3 | 1 |   |   |   |
|             | (3,1) | 1 | 5 | 16| 29| 29| 16| 5 | 1 |   |   |
|             | (3,2) | 1 | 9 | 42| 104| 142| 104| 42| 9 | 1 |   |
|             | (4,0) | 1 | 4 | 13| 32| 50| 50| 32| 13| 4 | 1 |

Figure 2: The rank of the intersection pairing on $\overline{\mathcal{M}}_{g,n}$. 

13
| $(g, n)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----------|---|---|---|---|---|---|---|---|
| $(1, 1)$ | 1 |   |   |   |   |   |   |   |
| $(1, 2)$ | 1 | 1 |   |   |   |   |   |   |
| $(1, 3)$ | 1 | 4 | 1 |   |   |   |   |   |
| $(1, 4)$ | 1 | 11| 11| 1 |   |   |   |   |
| $(1, 5)$ | 1 | 26| 71| 26| 1 |   |   |   |
| $(1, 6)$ | 1 |   |   |   |   |   |   | 1 |
| $(2, 0)$ | 1 | 1 |   |   |   |   |   |   |
| $(2, 1)$ | 1 | 2 | 1 |   |   |   |   |   |
| $(2, 2)$ | 1 | 5 | 5 | 1 |   |   |   |   |
| $(2, 3)$ | 1 | 11| 24| 11| 1 |   |   |   |
| $(2, 4)$ | 1 | 23| 101|101|23| 1 |   |   |
| $(2, 5)$ | 1 | 47| 384|769|384|47| 1 |   |
| $(3, 0)$ | 1 | 2 | 2 | 1 |   |   |   |   |
| $(3, 1)$ | 1 | 4 | 7 | 4 | 1 |   |   |   |
| $(3, 2)$ | 1 | 8 | 24| 24| 8 | 1 |   |   |
| $(3, 3)$ | 1 | 16| 82| 144|82|16| 1 |   |
| $(3, 4)$ | 1 | 32|   |   | 32| 1 |   |   |
| $(4, 0)$ | 1 | 3 | 6 | 6 | 3 | 1 |   |   |
| $(4, 1)$ | 1 | 5 | 17| 27| 17| 5 | 1 |   |
| $(4, 2)$ | 1 | 10| 51| 124|124|51|10| 1 |
| $(5, 0)$ | 1 | 3 | 10| 20|20|10| 3 | 1 |

Figure 3: The rank of the intersection pairing on $\mathcal{M}^{ct}_{g,n}$. 
Figure 4: The rank of the intersection pairing on $\mathcal{M}_{g,n}^{rt}$. 

| Codimension | (g, n) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------|--------|---|---|---|---|---|---|---|
| (2, 0)      |        | 1 |   |   |   |   |   |   |
| (2, 1)      |        | 1 | 1 |   |   |   |   |   |
| (2, 2)      |        | 1 | 3 | 1 |   |   |   |   |
| (2, 3)      |        | 1 | 7 | 7 | 1 |   |   |   |
| (2, 4)      |        | 1 | 15| 35| 15| 1 |   |   |
| (2, 5)      |        | 1 | 31|147|147|31| 1 |   |
| (3, 0)      |        | 1 | 1 |   |   |   |   |   |
| (3, 1)      |        | 1 | 2 | 1 |   |   |   |   |
| (3, 2)      |        | 1 | 4 | 4 | 1 |   |   |   |
| (3, 3)      |        | 1 | 8 |15 | 8 | 1 |   |   |
| (3, 4)      |        | 1 |16 |54 |54|16| 1 |   |
| (3, 5)      |        | 1 |32 |188|333|188|32| 1 |
| (4, 0)      |        | 1 | 1 |   |   |   |   |   |
| (4, 1)      |        | 1 | 2 | 2 | 1 |   |   |   |
| (4, 2)      |        | 1 | 4 | 6 | 4 | 1 |   |   |
| (4, 3)      |        | 1 | 8 |19 |19| 8 | 1 |   |
| (4, 4)      |        | 1 |16 |61 |95|61|16| 1 |
| (5, 0)      |        | 1 | 1 | 1 | 1 |   |   |   |
| (5, 1)      |        | 1 | 2 | 3 | 2 | 1 |   |   |
| (5, 2)      |        | 1 | 4 | 8 | 8 | 4 | 1 |   |
| (5, 3)      |        | 1 | 8 |19 |33|19| 8 | 1 |
| (6, 0)      |        | 1 | 1 | 2 | 1 | 1 |   |   |
| (6, 1)      |        | 1 | 2 | 4 | 4 | 2 | 1 |   |
| (6, 2)      |        | 1 | 4 | 9 |13| 9 | 4 | 1 |
| (7, 0)      |        | 1 | 1 | 2 | 2 | 1 | 1 |   |
| (7, 1)      |        | 1 | 2 | 4 | 5 | 4 | 2 | 1 |
| (8, 0)      |        | 1 | 1 | 2 | 3 | 2 | 1 | 1 |
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