Parameterized Algorithms for Zero Extension and Metric Labelling Problems

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Abstract

We consider the problems Zero Extension and Metric Labelling under the paradigm of parameterized complexity. These are natural, well-studied problems with important applications, but have previously not received much attention from parameterized complexity.

Depending on the chosen cost function \(\mu\), we find that different algorithmic approaches can be applied to design FPT-algorithms: for arbitrary \(\mu\) we parameterized by the number of edges that cross the cut (not the cost) and show how to solve Zero Extension in time \(O(|D|^{O(k^2)}n^4\log n)\) using randomized contractions. We improve this running time with respect to both parameter and input size to \(O(|D|^{O(k)}m)\) in the case where \(\mu\) is a metric. We further show that the problem admits a polynomial sparsifier, that is, a kernel of size \(O(k|D|+1)\) that is independent of the metric \(\mu\).

With the stronger condition that \(\mu\) is described by the distances of leaves in a tree, we parameterize by a gap parameter \((q-p)\) between the cost of a true solution \(q\) and a ‘discrete relaxation’ \(p\) and achieve a running time of \(O(|D|^{q-p}T|m+T|\phi(n,m))\) where \(T\) is the size of the tree over which \(\mu\) is defined and \(\phi(n,m)\) is the running time of a max-flow computation. We achieve a similar running for the more general Metric Labelling, while also allowing \(\mu\) to be the distance metric between an arbitrary subset of nodes in a tree using tools from the theory of VCSPs. We expect the methods used in the latter result to have further applications.

1 Introduction

The task of extending a partial labelling of a few data points to a full data set while minimizing some error function is a natural computational step for scientific and engineering tasks. For the particular case of data imposed with a (binary) relationship, we find that the problems Zero Extension and Metric Labelling are well-suited for optimization in image processing\(^1\), Markov Random Fields\(^3\), social network classification\(^2\), or sentiment analysis in natural language processing\(^2\).

The problem settings are as follow. For Zero Extension, we are given a graph \(G\) and a partial labelling \(\tau: S \to D\), for some set of terminals \(S \subseteq V(G)\), alongside a cost function \(\mu: D \times D \to \mathbb{R}^+\). Our task is to compute a labelling \(\lambda: V(G) \to D\) which agrees with \(\tau\) on \(S\), subject to the following cost: for each edge \(uv \in G\) we pay the cost \(\mu(\lambda(u), \lambda(v))\).

In Metric Labelling, we are given a graph \(G\), a cost function \(\mu\) as above, and a labelling cost \(\sigma: V(G) \times D \to \mathbb{R}^+\). Again we are asked to compute a labelling \(\lambda\) and in addition to the above edge-costs we now also pay \(\sigma(v, \lambda(v))\) for each vertex. Note that this model allows us to emulate terminals, by making the cost \(\sigma(v, \lambda(v))\) prohibitive for all but the required label \(\lambda(v)\). Both problems are generalizations of the Multiway Cut problem (we simply let \(\mu\) be identically one for all distinct pairs), which has garnered considerable

\(^1\)The recent advent of convolutional neural networks seems to have lessened the importance here, but Metric Labelling is still used (for a recent example see e.g.\(^2\))
attention from the FPT community in the past and formed a crystallization nucleus for the
very fruitful research of cut-based problems (see e.g. [23, 19, 4, 13, 25]).

We will find application for most of these tools in our results listed below, but we wish to
particularly highlight the use of tools and relaxations from Valued CSPs (VCSPs) for design-
ing FPT algorithms under gap parameters. VCSPs are a general framework for expressing
optimisation problems, via the specification of a set $\Gamma$ of cost functions (also referred to
as a constraint language). Many important problems correspond to VCSP for a specific
language $\Gamma$, including every choice of a specific metric for the problems above. Thapper and
Živný [33] characterized the languages $\Gamma$ for which the resulting VCSP is tractable.

The use of a tractable VCSP as a discrete relaxation of an NP-hard optimisation problem
has previously been used for the design of surprisingly powerful FPT algorithms [13] (see
also related improvements [14, 35]). In this paper, we advance this research in two ways.
First, previous algorithms of this type have required the relaxation to have a persistence
property, which allows an optimum to be found by sequentially fixing variables. In this
paper, we relax this condition to a weaker domain consistency property. Second, we use a
folklore result from VCSP research to restrict the behaviour of optimal solutions to a VSCP
instance in order to facilitate the proof that the domain consistency property holds for the
relevant VCSPs. See Section 5 for details.

Related work. So far, Zero Extension and Metric Labelling have been re-
searched primarily from the perspective of efficient and approximation algorithms (for a
more complete overview and hardness results we refer to the paper by Manokaran, Naor,
Raghavendra and Schwartz [22]). Kleinberg and Tardos [32] introduced Metric Labelling
and provided a $O(\log |S| \log \log |S|)$ approximation. A result by Fakcharoenphol, Rao, and
Talwar regarding embedding general metrics into tree metrics [8] improves the ratio of this
algorithm to $O(\log |S|)$ and a lower bound of $O((\log |S|)^{2/3})$ was proved by Chuzhoy and
Naor [5]. Karzanov [16] introduced Zero Extension with the specific case of $\mu$ being a
graph metric, that is, equal to the distance metric of some graph $H$. The central question of
his work—for which graphs $H$ the problem is tractable—was just recently fully answered by
Hirai [11]. Picard and Ratliff much earlier showed that an equivalent problem is tractable
on trees [30]. Fakcharoenphol, Harrelson, Rao, and Talwar showed that the problem can be
approximation to within a factor of $O(\log |S|/ \log \log |S|)$ [7]. Karloff, Khot, Mehta, Rabani
used the approach by Chuzhoy and Naor to show that no factor of $O((\log |S|)^{1/4})$ for
any $\varepsilon > 0$ is possible unless $\text{NP} \subseteq \text{QP}$ [15]. More recently, Hirai and Pap [10] [12]
studied the problem from a more structural angle and we make use of their duality result in the
following.

Our results. In this paper we study both problems from the perspective of parameterized
complexity. As the choice of metric has a strong effect on the complexity of the problem, we
give a range of results, ranging from the more generally applicable to the algorithmically
stronger, both in terms of running time and parameterization. In the most general setting,
when $\mu$ is a general cost function or a metric, we will parametrize not by the cost of a
solution but by the number of crossing edges which are precisely the bichromatic edges under
a labelling $\lambda$. This in particular allows us to include the case of zero cost pairs under $\mu$.
For general cost functions, we employ the technique of randomized contractions [31] and prove
the following:

**Theorem 1.1.** Zero Extension can be solved in time $O(D^{O(k^2)} n^4 \log n)$ where $k$ is a
given upper bound on the number of crossing edges in the solution.

When $\mu$ is a metric, we are able to give a linear-time FPT algorithm, while also improving
the dependency on the parameter, using important separators [24].

**Theorem 1.2.** Zero Extension with metric cost functions can be solved in time $O(|D|^{O(k)} m)$
where $k$ is a given upper bound on the number of crossing edges in the solution.

For the general metric setting, we also have our most surprising result, demonstrating that
Zero Extension admits a sparsifier; that is, we prove that it admits a polynomial kernel
independent of the metric $\mu$. This result crucially builds on the technique of representative
sets [19, 24, 21]. The exact formulation of the result is somewhat technical and we defer
it to Section 4.3 but roughly, we obtain a kernel of size $O(k^{|S|^{k+1}})$, independent of $\mu$, where $k$ is again the number of crossing edges. This result is a direct, seemingly far-reaching generalization of the polynomial kernel for $s$-Multiway Cut \cite{fomin2012kernelization}.

Next, we consider the case when $\mu: D \to \mathbb{Z}^+$ is induced by the distance in a tree $T$ with $D \subseteq V(T)$. Here, relaxing the problem to allow all labels $V(T)$ as vertex values defines a tractable discrete relaxation, in the sense discussed above. Using techniques from V CSP, we design a gap-parameter algorithm:

**Theorem 1.3.** Let $I = (G, \tau, \mu, q)$ be an instance of Zero Extension where $\mu$ is an induced tree metric on a set of labels $D$ in a tree $T$, and let $\hat{I} = (G, \hat{\tau}, \mu, q)$ be the relaxed instance. Let $p = \text{cost}(\hat{I})$. Then we can solve $I$ in time $O(|D|^{|\hat{\mu}|}T|D|m)$.

For the further restriction when $\mu$ corresponds to the distances of the leaves $D$ of a tree $T$, we obtain an algorithm with a slightly better polynomial dependence. Moreover, it uses only elementary operations like computing cuts and flows:

**Theorem 1.4.** Let $I = (G, \tau, \mu, q)$ be an instance of Zero Extension where $\mu$ is a leaf metric on a set of labels $D$ in a tree $T$, and let $\hat{I} = (G, \hat{\tau}, \mu, q)$ be the relaxed instance. Let $p = \text{cost}(\hat{I})$. Then we can solve $I$ in time $O(|D|^{|\hat{\mu}|}Tm + |T|\phi(n, m))$, where $\phi$ is the time needed to run a max-flow algorithm.

Finally, we apply the V CSP toolkit to Metric Labelling and obtain a similar gap algorithm (see Section 5\textsuperscript{5} for undefined terms).

**Theorem 1.5.** Let $I = (G, \sigma, \mu, q)$ be an instance of Metric Labelling where $\mu$ is an induced tree metric for a tree $T$ and a set of nodes $D \subseteq V(T)$, and where every unary cost $\sigma(v, \cdot)$ admits an interpolation on $T$. Let $\hat{I} = (\hat{G}, \hat{\sigma}, \hat{\mu}, q)$ be the relaxed instance, and let $\rho = \text{cost}(\hat{I})$. Then the instance $I$ can be solved in time $O^*(|D|^{\mu-\rho})$. In particular, this applies for any $\sigma$ if $D$ is the set of leaves of $T$.

## 2 Preliminaries

For a graph $G = (V, E)$ we will use $n_G = |V|$ and $m_G = |E|$ to denote the number of vertices and edges, respectively. We write $d_G$ for the distance-metric induced by $G$, that is, $d_G(u, v)$ is the length of a shortest path between vertices $u, v \in V(G)$. We denote by $N_G(v)$ and $N_G[v]$ the open and closed neighbourhood of a vertex. For a vertex set $S \subseteq V(G)$ we write $\delta(S)$ to denote the set of edges with exactly one endpoint in $S$. We omit the subscript $G$ if clear from the context all these notations.

For a tree $T$ we call a sequence of nodes $x_1x_2\ldots x_p$ a monotone sequence if $x_1 \preceq_P x_2 \preceq_P \ldots \preceq_P x_p$ where $P$ is a path in $T$ and $\preceq_P$ is the linear order induced by $P$. Note that $x_i = x_{i+1}$ is explicitly allowed. For two nodes $x, y \in T$ we will denote the unique $x$-$y$-path in $T$ by $T[x, y]$. For a vertex set $S$, an $S$-path packing is a collection of edge-disjoint paths $\mathcal{P}$ that connect pairs of vertices in $S$. We will also consider half-integral path packings, here every edge of the graph is allowed to be used by up to two paths.

Let $D$ be a set of labels. For a graph $G$ we call a function $\tau: S \to D$ for $S \subseteq V(G)$ a partial labelling and a function $\lambda: V(G) \to D$ a labelling. The labelling $\lambda$ is an extension of $\tau$ if $\lambda$ and $\tau$ agree on $S$, that is, for every vertex $u \in S$ we have that $\lambda(u) = \tau(u)$. Given a graph $G$ and a labelling $\lambda$ we call an edge $uv \in E(G)$ crossing if $\lambda(u) \neq \lambda(v)$. A $\tau$-path packing is a collection $\mathcal{P}$ of edge-disjoint paths such that every path $P \in \mathcal{P}$ connects to vertices that receive distinct labels under $\tau$ (and both are labelled).

### 2.1 Cost functions, metrics, and extensions

A cost function over $D$ is a symmetric positive function $\mu: D \times D \to \mathbb{R}^+$. We call it simple if $\mu(x, x) = 0$. A cost function is a a metric if further it obeys the triangle inequality and it is a tree metric if it corresponds to the distance metric of a tree. We derive an induced tree metric from a tree metric by restricting its domain to a subset $D$ of the nodes of the underlying tree. A leaf metric is an induced tree metric where $D$ is the set of leaves of the
tree. Given a cost function $\mu$, we define the cost of a labeling $\lambda$ of a graph $G$ under a cost function $\mu$ as
\[
\text{cost}_\mu(\lambda, G) = \sum_{u,v \in G} \mu(\lambda(u), \lambda(v)).
\]
With these definitions in place, we can now define the problem in question:

**Zero Extension**

**Input:** A graph $G$ with a partial labeling $\tau$ over a finite domain $D$, a simple cost function $\mu$ over $D$ and an integer $q$.

**Problem:** Does $G$ admit an extension $\lambda$ of $\tau$ such that $\text{cost}_\mu(\lambda, G) \leq q$?

For cost functions $\mu$ that are uniform on all non-diagonal values we recover (up to some constant scaling of the parameter) the problem Multiway Cut. Picard and Ratliff proved that for tree metrics, the problem is solvable in polynomial time [30]. We will call the special case in which the distance function is restricted to a leaf metric Zero Leaf Extension.

### 3 Cost functions: Randomized Contractions

We will apply the framework by Chitnis et al. [4] to show that the most general case of Zero Extension is in FPT when parameterized by the number of crossing edges. Note that in this setting crossing edges could incur an arbitrary cost, including zero. However, the stronger parameterization of only counting the number of crossing edges at non-zero cost makes for an intractable problem: With such zero-cost edges, we can express the problem $H$-Retraction for reflexive graphs $H$, which asks us to compute a retraction of a graph $G$ into the fixed graph $H$. This problem is already NP-complete for $H$ being the reflexive 4-cycle [M4] and thus Zero Extension is paraNP-complete for parameter $k = 0$ if parameterized by the number of non-zero-cost crossing edges (or indeed if parameterized by the total cost).

A $(\sigma, \kappa)$-good separation is a partition $(L, R)$ of $V(G)$ such that $|L|, |R| > \sigma$, $|E(L, R)| \leq \kappa$, and both $G[L]$ and $G[R]$ are connected. There exists an algorithm that finds a $(\sigma, \kappa)$-good separation in time $O((\sigma + \kappa)^{O(\min(\sigma, \kappa))} n^4 \log n)$ (Lemma 2.2 in [4]) or concludes that the graph is $(\sigma, \kappa)$-connected, that is, no such separation exists. The following lemma is a slight reformulation of Lemma 1.1 in [4] which in turn is based on splitters as defined by Noar et al. [24].

**Lemma 3.1** (Edge splitter). Given a set $E$ of size $m$ and integers $0 \leq a, b \leq m$ one can in time $O((a + b)^{O(\min(a, b))} m \log m)$ construct a set family $F$ over $E$ of size at most $O((a + b)^{O(\min(a, b))} \log m)$ with the following property: for any disjoint sets $A, B \subseteq E$ with $|A| \leq a$ and $|B| \leq b$ there exists a set $H \in F$ with $A \subseteq H$ and $B \cap H = \emptyset$.

Let us first demonstrate how Zero Extension can be solved on such highly connected instances and then apply the ‘recursive understanding’ framework to handle graphs with good separations. In the following, let $I = (G, \tau, \mu, q)$ be the input instance with $\mu$ being a cost function over the domain $D$.

**Lemma 3.2.** Let $G$ be $(\sigma, k)$-connected for some $\sigma > k$. Then we can find an optimal solution in time $O((|D| + 2\sigma k + k)^{O(k)} (n + m) \log n)$.

**Proof.** Let $\lambda \in \text{opt}(I)$ be an optimal solution and let $E_\lambda$ be the crossing edges. We write $V(E_\lambda)$ to denote the endpoints of these edges. Let $C_0, C_1, \ldots, C_\ell$ be the connected components of $G - E_\lambda$ with $C_0$ being the largest one. Since the sets $C_i$ have only one label each under $\lambda$, each one contains at most one terminal.

Since $G$ is $(\sigma, k)$-connected, we know that $\ell \leq k$ and that all components $C_1, \ldots, C_\ell$ have size at most $\sigma$ (cf. Lemma 3.6 [4]). We will assume in the following that $C_0$ contains more than $\sigma$ vertices, otherwise $G$ contains less than $\sigma k$ vertices and we find the set $E_\lambda$ in time $O((\sigma k)^{2k})$ by brute-force.

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2They prove it for a variant of the Facility Location problem.
Otherwise, we proceed by colouring the edges of \( G \) using an edge splitter (details below). Such a colouring is successful if

1. the crossing edges \( E_\lambda \) are red;
2. each component \( C_i, 1 \leq i \leq \ell \), contains a blue spanning tree; and
3. each vertex \( u \in C_0 \cap V(E_\lambda) \) is contained in a blue tree of size at least \( \sigma + 1 \).

By fixing a collection of (arbitrary) spanning trees for the components \( C_i, 1 \leq i \leq \ell \) and a collection of trees in \( C_0 \) with \( \sigma + 1 \) vertices that contain the \( \leq k \) boundary vertices \( C_0 \cap V(E_\lambda) \) we can see that we need to correctly colour a set \( B \subseteq E(G) \) of at most \( (\sigma - 1)\ell + \sigma k \leq 2\sigma k \) edges blue while colouring a set \( R \subseteq E(G) \) of at most \( k \) edges red. We apply Lemma 3.1 with \( a = 2\sigma k \) and \( b = k \) to construct an edge splitter \( F \) of size at most \( O((2\sigma k + k)^{O(k)} \log m) \) in time \( O((2\sigma k + k)^{O(k)} m \log m) \) for which we are guaranteed that at least one member \( H \in F \) will contain \( B \) while avoiding \( R \).

It is left to show that we can easily find a solution in successfully coloured graph. Let \( G_B \) be the graph induced by the blue edges. We will call a component of \( G_B \) small if it contains \( q \) or less vertices and big otherwise. Our task is to recover the solution-induced components \( C_0, C_1, \ldots, C_\ell \). First notice that every \( C_i, i \geq 1 \) must be a small component in \( G_B \) and further that all components reachable from \( C_i \) via red edges must either be another component \( C_j, j \geq 1 \), or be a big component in \( G_B \). Thus, we ‘discover’ the sets \( C_1, \ldots, C_\ell \) by the following marking algorithm:

1. Guess which terminal \( x \) lies in the big component \( C_0 \)
2. Mark all small components of \( G_B \) that contain terminals other than \( x \)
3. Repeat exhaustively: mark all small components of \( G_B \) which have a red edge into an already marked component.

If our initial guess of \( x \) is correct this procedure will exactly mark the components \( C_1, \ldots, C_\ell \). Indeed, any small component \( C_i \) not marked by this process would be a small component containing no terminal and with all edges of \( E_\lambda \) connecting to the big component, in which case there is a solution with at most the same cost which merges \( C_i \) with \( C_0 \). The same holds for any collection of small components with no red edge to a marked component. From this we can deduce \( C_0 \), the crossing edges \( E_\lambda \), and \( \lambda \) itself. In case the colouring step was unsuccessful or our guess of \( x \) was wrong, the above procedure will produce some set of edges \( E_\lambda \) of non-minimal cost.

This verification step, given a colouring, is possible in time \( O((n + m)|S|) \). Since \( G \) is connected, we can assume that \( |S| \leq k \) and the total running time to identify \( E_\lambda \) is \( (2\sigma k + k)^{O(k)} (n + m) \log n \). Given \( E_\lambda \), the final step is to find an optimal assignment. While the assignment for components \( C_i \) containing terminals is fixed, we need to try all possible assignments for the remaining components in time \( O(|D|^k) = O(|D|^k k) \). Taken both steps together yields the claimed running time.

With the well-connected cases handled we can now proceed to solve the general problem.

**Theorem 1.1.** 
**Zero Extension** can be solved in time \( O(|D|^{O(k^2)} n^4 \log n) \) where \( k \) is a given upper bound on the number of crossing edges in the solution.

**Proof.** We will assume that \( G \) is connected (otherwise we compute an optimal solution for every connected component), therefore the number of terminals \( |S| \) is bounded by \( k + 1 \). Let \( \sigma := |D|^k + 1 \). We run the algorithm of Chitnis et al. [4] to find a \((\sigma, k)\)-good separation. Assume for now that such a good separation \((V_1, V_2)\) is found with at most \( k \) edges \( E(V_1, V_2) \) crossing it.

Let \( S_1 := E(V_1, V_2) \cap V_1 \) be the \( \leq k \) vertices in \( S_1 \) on the border of the separation. We iterate through all \( |D|^{O(k)} \leq |D|^k \) ways the vertices \( S_1 \) could receive colours by a solution. For each such colouring \( \tau \), we construct a sub-instance \( G_\tau \) by identifying all vertices that have the same terminal-labelling and recursively solve the instance. For each solution
of $G_\tau$ with at most $k$ crossing edges we collect said crossing edges in $E^\ast$. That is, the set $E^\ast$ contains all edges that are crossing for an optimal solution of some labelling of $S_1$. Note that $|E^\ast| < \sigma$ by our choice of $\sigma$. Since $G_1$ is connected, there is at least one edge in $\hat{E} = E(G_1) \setminus E^\ast$. Now, every solution $\lambda$ of $G$ that has at most $k$ crossing edge can be modified to have only edges of $E^\ast$ and thus no edges of $\hat{E}$ crossing it; we simply observe what labelling $\tau$ the solution $\lambda$ applies to $S_1$ and replace the colour $\lambda$ applies to $V_1$ for the colours that the optimal solution of $G_\tau$ applies to it.

Consequently, we can assume that the optimal solutions we are interested in are not crossed by $\hat{E}$, therefore it is safe to contract $\hat{E}$ and therefore reduce the size of $V_1$ to $\sigma$ (recall that $G[V_1]$ is connected and the contraction of course preserves that property). We repeat this procedure until the resulting graph is $(\sigma, k)$-connected (alternatively, any of the recursive calls could tell us that no solution with at most $k$ crossing edges exists in which case we return that the instance has no solution). Then by Lemma 13.2, we can decide the problem in time $O((|D| + 2\sigma k + k)^{O(k)}(n + m) \log n) = O(|D|^{O(k^2)}n^2 \log n)$.

Let us now analyse the total running time $T(n)$: note that $k$ and $\sigma$ do not change with each recursive call. First, finding a good separation takes time $O((\sigma + k)^{O(k)}n \log n) = O(|D|^{O(k^2)}n^3 \log n)$ and constructing the instance $G_\tau$ at total of $O(|D|^kn + m)$, which is dominated by the former running time. The recursive call on $V_1$ with $n_1 := |V_1|$ costs us $T(n_1)$, after which we are left to work on an instance of size at most $n - n_1 + \sigma$ which will cost us $T(n - n_1 + \sigma)$. Note that, by the properties of $(\sigma, k)$-good separations, it holds that $\sigma + 1 \leq n_1 \leq n - \sigma - 1$. We therefore need to resolve the recurrence

$$T(n) \leq \max_{\sigma + 1 \leq n_1 \leq n - \sigma - 1} \left( |D|^{O(k^2)}n^3 \log n + T(n_1) + T(n - n_1 + \sigma) \right).$$

As noted in 13.4, the maximum in this expression is attained at the extreme values for $n_1$ and that the claimed running time is a bound on $T(n)$.

4 General metrics: Pushing separators

We now consider the more restricted, but reasonable case that $\mu$ is a metric, observing the triangle inequality. We find that this allows a ‘greedy’ operation of pushing in a solution $\lambda$, which allows both the design of a faster algorithm (Section 4.2) and the computation of a metric sparsifier (Section 4.3).

4.1 The pushing lemma

Let in the following $I = (G = (V, E), \tau, \mu, g)$ be an instance of Zero Extension for an arbitrary metric $\mu$. Let $S \subseteq V$ be the range of $\tau$, and let $D$ be the set of labels. We assume that the following reductions have been performed on $G$: For every label $\ell$ used by $\tau$ there is a terminal $t_\ell$, and every vertex $v$ such that $\tau(v) = \ell$ has been identified with this terminal $t_\ell$.

Let $\lambda: V \to D$ be an extension of $\tau$, and let $U = \lambda^{-1}(\ell)$ for some $\ell \in D$. By pushing from $\ell$ in $\lambda$ we refer to the operation of relabelling vertices to grow the set $U$ “as large as possible”, without increasing the number of crossing edges. Formally, this refers to the following operation: Let $C$ be the furthest min-cut between vertex sets $U$ and $S - t_\ell$ (respectively $S$ if there is no terminal $t_\ell$), let $U'$ be the vertices reachable from $U$ in $G - C$, and let $\lambda'$ be the labelling where $\lambda'(v) = \ell$ for $v \in U'$ and $\lambda'(v) = \lambda(v)$ otherwise. Clearly, $\lambda'$ is an extension of $\tau$. The purpose of this section is to show that as long as $\mu$ is a metric (in particular, observes the triangle inequality), this operation does not increase the cost of the solution.

Lemma 4.1 (Pushing Lemma). For any $\tau$-extension $\lambda$ and every label $\ell \in D$, pushing from $\ell$ in $\lambda$ yields a $\tau$-extension $\lambda'$ with $\text{cost}_\mu(\lambda', G) \leq \text{cost}_\mu(\lambda, G)$.

Proof. Write $V_\ell = \lambda^{-1}(\ell)$, and for a set of vertices $U$ let $\delta(U)$ denote the edges with one endpoint in $U$. Let $P$ be a max-flow from $V_\ell$ to the set $S - t_\ell$, let $C$ be a furthest corresponding min-cut, and let $V_\ell^+ \supseteq V_\ell$ be the set of vertices reachable from $t_\ell$ in $G' - C$. 

As shown in [4], the maximum in this expression is attained at the extreme values for $n_1$ and that the claimed running time is a bound on $T(n)$. 

\end{proof}

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By Menger’s theorem, $\mathcal{P}$ partitions the set of edges $\delta(V_i^+)$. Finally, let $\mathcal{P}^−$ consist of the prefixes of the paths $P \in \mathcal{P}$ up until and including the edges of $C$. For $P \in \mathcal{P}^−$, let $\lambda(P)$ be the label of its final edge. Let $\lambda'$ be the assignment resulting from letting $\lambda'(v) = \ell$ for every $v \in V_i^+$ and $\lambda'(v) = \lambda(v)$ otherwise. We make two quick observations.

**Claim.** The cost incurred by $\lambda'$ on every path $P \in \mathcal{P}^−$ is precisely $\mu(\ell, \lambda(P))$, whereas for every edge $uv$ not on any such path the cost is at most as high as for $\lambda$, i.e., $\mu(\lambda(u), \lambda(v)) \leq \mu(\lambda(0), \lambda(v))$.

**Proof.** For every path $P \in \mathcal{P}^−$ only the final vertex $v$ has a label $\lambda'(v) \neq \ell$, and the final edge has cost $\mu(\ell, \lambda(v))$ where $\lambda(v) = \lambda(P)$. For the second part, only edges with at least one end point in $V_i^+$ have changed cost from $\lambda$ to $\lambda'$, and among such edges only the edges of $C$ have non-zero cost. Since $C$ is a min-cut, all such edges are covered by paths $P \in \mathcal{P}^−$.

**Claim.** The cost incurred by $\lambda$ on a path $P \in \mathcal{P}^−$ is at least $\mu(\ell, \lambda(P))$.

**Proof.** Let $P = t_1v_1 \ldots v_r$ where $v_r$ may be a terminal. This describes a walk $\lambda(t_1) = \ell$, $\ldots$, $\lambda(v_r) = \lambda(P)$ from $\ell$ to $\lambda(P)$, and every edge $uv$ of $P$ where $\lambda(u) \neq \lambda(v)$ incurs the corresponding cost. By the triangle inequality for $\mu$, the sum of these costs is at least $\mu(\ell, \lambda(P))$.

The result follows immediately from the above two claims.

An immediate consequence of the above lemma is the following reduction rule.

**Corollary 4.2.** We can reduce to the case where for every label $\ell$ used by $\tau$, there is a single terminal $t_\ell$ of value $\tau(t_\ell) = \ell$ such that $\delta(t_\ell)$ is the unique isolating min-cut for $t_\ell$.

**Proof.** The first part of the reduction (to terminals $t$) is trivial. Let $S$ be the set of terminals. Let $t_\ell$ be a terminal with label $\ell$, let $C_0$ be a furthest isolating min-cut between $t_\ell$ and $S - t_\ell$, and let $V^0_\ell$ be the vertices reachable from $t_\ell$ in $G - C_0$.

Now let $\lambda: V \rightarrow D$ be a $\tau$-extension and let $V_\ell = \lambda^{-1}(\ell)$. Let $V^0_\ell \supseteq V_\ell$ be produced by pushing from $\ell$ in $\lambda$. By Lemma 4.1, replacing $V_\ell$ by $V^0_\ell$ does not incur a larger cost. To see that our reduction is valid, let $f: 2^V \rightarrow \mathbb{Z}$ be the edge-cut function for $G$. Then, by submodularity,

$$f(V^0_\ell) + f(V^+_\ell) \geq f(V^0_\ell \cap V^+_\ell) + f(V^0_\ell \cup V^+_\ell) \geq f(V^0_\ell) + f(V^0_\ell \cup V^+_\ell),$$

where the last step follows since $\delta(V^0_\ell)$ is a minimum isolating cut. Thus $V^0_\ell \cup V^+_\ell$ is a cut between $t_\ell$ and $S - t_\ell$ of cost no more than $V^+_\ell$, which implies that they are equal and $V^0_\ell \subseteq V^+_\ell$. Hence any solution $\lambda$ can be modified to let $\lambda(V^0_\ell) = \ell$, and we may contract all vertices $V^0_\ell$ into $t_\ell$. Repeating this for all vertices yields an instance as described.

### 4.2 An FPT algorithm

Let $I = (\mathcal{G}, \tau, \mu, q)$ be an instance of ZERO EXTENSION where $\mu$ is a metric, e.g. a simple cost function observing the triangle inequality. Let $D$ be the domain of $\mu$, and let $S$ be the terminals in $G$. We will show that the problem is FPT parameterized by $k + |D|$ where $k$ is a bound on the number of crossing edges in an optimum $\lambda$.

The algorithm uses the pushing lemma (Lemma 4.1) to guess a solution $\lambda$ using the technique of important separators. This is a classical ingredient for FPT algorithms for cut problems, pioneered by Marx [24]. We focus on the edge version. Let $G = (V, E)$ be a graph with disjoint vertex sets $S$ and $T$, and let $C \subseteq E$ be a minimal $(S, T)$-cut which is not necessarily minimum. Let $U$ be the set of vertices reachable from $S$ in $G - C$. Then $C$ is an important separator if for every set $U' \supseteq U$ with $T \cap U' = \emptyset$ we have $|\delta(U')| > |C|$. In other words, $C$ represents a greedy “furthest cut” from $S$ for its size. The important realization is that for every bound $k$ on $|C|$, there are at most $f(k) = 4^k$ important $(S, T)$-separators in $G$.

The algorithm we will use is inspired by the classical algorithm for MULTICUT of Chen, Liu and Lu [3], improving on Marx [24]. This algorithm works by repeatedly selecting
a terminal \( t \), guessing an important separator around \( t \) (against the other terminals \( S - t \)), then deleting the chosen separator and proceeding with the next terminal with a decreased budget \( k \). The important aspect to us is the process of enumerating important separators, which for edge-cuts can be described as follows.

1. Assume that we are enumerating important separators between sets \( S \) and \( T \), which may be singleton sets (the process is the same regardless). Assume that \( k \) is our budget for the maximum separator size.

2. Compute a furthest min-cut \( C \) between \( S \) and \( T \). If \( |C| > k \), abort. Otherwise, contract edges so that \( \delta(S) = C \). Initialise all edges as unmarked.

3. Select an unmarked edge \( uv \in \delta(S) \) and branch recursively on two cases:
   
   (a) Contract \( uv \) into \( S \), thereby increasing the max-flow
   
   (b) Mark \( uv \) as part of the final separator

   Abort a branch whenever the resulting max-flow is more than \( k \).

4. Once all edges of \( \delta(S) \) are marked, output \( \delta(S) \) as an important separator.

For the running time, we may analyse this in terms of the gap between \( |C| \) and \( k \). More precisely, consider an alternative lower bound where every marked edge of \( C \) counts for 1 point, but unmarked edges of \( C \) are worth 1/2 point. Then it is clear that the lower bound increase by 1/2 in both branches of the recursive calls, and at most \( 2^k \) important separators are generated.

Compared to \textsc{Edge Multiway Cut}, there are two complications to an algorithm for \textsc{Zero Extension}. First, if \( C \) are the crossing edges on an optimal solution \( \lambda \), it may be that not every connected component of \( G - C \) contains a terminal; therefore the algorithm may need to branch on non-terminal vertices as well. This is not an obstacle in itself, but it allows for only a weaker running time analysis. Second, even after \( C \) has been identified, it remains to find an assignment \( \lambda \) that minimizes cost. The complexity of this may vary depending on the particular metric \( \mu \).

For this reason, we describe the algorithm below as being composed of stages, where the first stage identifies all crossing edges reachable from a terminal, the second stage identifies the remaining crossing edges, and the third stage finds an assignment \( \lambda \). For specific metrics \( \mu \), it may be possible to skip or speed up the second and third stages; e.g., for a leaf metric it can be checked that the algorithm is finished after the first stage.

In summary, we show the following.

\textbf{Theorem 1.2.} \textsc{Zero Extension} with metric cost functions can be solved in time \( O(|D|^{O(k)}, m) \) where \( k \) is a given upper bound on the number of crossing edges in the solution.

We begin by proving the running time for the first stage. This is analysed in terms of a lower bound \( p \) on the crossing number of any labelling \( \lambda \). This bound is computed as follows. Let \( S \) be the set of terminals. Observe that for any \( \tau \)-extension \( \lambda \), the crossing edges in \( \lambda \) form a multiway cut for \( S \) in \( G \). This can be lower-bounded as follows. For every \( t \in S \), let \( f_t \) be the value of a max-flow between \( t \) and \( S - t \), and let \( p = \sum_{t \in S} f_t/2 \). Then there is a half-integral packing of \( p \) terminal-terminal paths in \( G \) (i.e., a \( \tau \)-path-packing for the tree which is a star with \( S \) as leaves, as in Section 5.1 [31]), hence there is no solution \( \lambda \) with crossing number less than \( p \). Then the first phase can be computed in time parameterized by the gap \( k - p \). (We note that this is a weaker lower bound, and hence a weaker gap parameter, compared to the relaxation lower bound \( \rho \) used later in this paper. However, it of course applies to any metric.)

\textbf{Lemma 4.3.} Let \( p \) be the lower bound as above In \( O(4^{k-p} km) \) time and \( 4^{k-p} \) guesses, we can reduce to the case where every edge of \( \delta(t) \) is a crossing edge in the optimal solution for every \( t \in S \).
Proof. This phase works almost exactly as Edge Multiway Cut. By Corollary 4.2 we assume that for every \( t \in S \), \( \delta(t) \) is the unique \((t, S - t)\)-min cut. Since our parameter is a cardinality parameter, we implement this step by a standard augmenting path algorithm, aborting whenever more than \( k \) paths have been found. More specifically, we proceed as follows. Let \( f \) trace the number of paths found in total across all terminals; initially \( f = 0 \). Then for each terminal \( t \in S \) in turn, we pack \( t - (S - t) \)-paths in \( G \), increasing \( f \) at each point, and aborting if \( f > 2k \) is reached. Assuming this does not happen, we can in \( O(m) \) time compute the furthest \((t, S - t)\)-min cut \( C \), and the set \( V_t \) of vertices reachable from \( t \) in \( G - C \), and finally the new graph \( G \) resulting from contracting \( V_t \) into \( t \). In total, this process takes \( O(km) \) time and either aborts or produces a reduced graph. We assume in the rest of the proof that the process has succeeded and proceed as follows.

Initialize all edges \( t t' \) for \( t, t' \in S \) as marked, all other edges as unmarked. Compute a lower bound \( p \) on \( k \) by counting 1 for every marked edge, and \( 1/2 \) for every unmarked edge of \( \delta(S) \). If \( p > k \), reject the instance. Otherwise, proceed as in the enumeration of important separators, i.e. select an unmarked edge \( tv \) incident with some \( t \in S \) and branch on two cases: Either contract \( tv \) into \( t \), and recompute the max-flow and value of \( p \); or mark \( tv \) as part of the solution and select another edge. Whenever \( p > k \), abort the branch; whenever every edge of \( \delta(S) \) is marked, return the instance.

For the correctness, the value \( p \) is a lower bound on the crossing number of \( \lambda \), as argued above. Hence if \( p > k \), the current instance has no solution. Otherwise, the branching is exhaustive, marking \( tv \) as either crossing or non-crossing, and in the latter case we are allowed to contract out to the new furthest min-cut by the pushing lemma (Lemma 4.1). Hence, assuming the instance has a solution at all, in at least one output instance the marked edges correspond to the crossing edges of some optimal labelling that are reachable from a terminal.

Regarding running time, Corollary 4.2 can be applied in \( O(S|km) \) time by an augmenting path approach. For every further branching step, recomputing a new max-flow can be done in \( O(km) \) time. Throughout the branching process, the value of \( k - p \) decreases by at least \( 1/2 \) in every branch (in particular, the max-flow number increases in the contraction branch). Hence the branching process produces at most \( 2^{2(k-p)} \) outputs.

Next, we show a similar branching algorithm (without a lower bound) to mark the remaining crossing edges of a solution.

Lemma 4.4. Given an input from stage 1, with \( p \) edges already marked, in \( O(4^{2k-p}m) \) time and \( 4^{2k-p} \) guesses we can reduce to the case where every edge of \( G \) is crossing in the optimal solution.

Proof. We assume that the input from stage 1 has the property that an edge is marked if and only if it is incident with a terminal. We proceed with a branching process as follows. Let \( v \) be an arbitrary non-terminal vertex incident with at least one non-marked edge; if none exists, simply output the instance. Compute a furthest min-cut \( C \) between \( v \) and \( S \) in \( G \), aborting if \( |C| > k \). Let \( U \) be the vertices reachable from \( v \) in \( G - C \). If \( U \) contains any marked edge, abort the branch as being the result of inconsistent choices; otherwise contract \( U \) into \( v \). If every edge of \( \delta(v) \) is marked, proceed with a different starting vertex \( v \). Otherwise, as above select one unmarked edge from \( \delta(v) \) and branch on either marking it or contracting it and recomputing the max-flow and min-cut. If at any point more than \( k \) edges have become marked, abort the branch. Once only marked edges remain, output the instance.

The correctness argument is similar as in stage 1. Assume that we are currently working with a non-terminal vertex \( v \), and that \( \lambda(v) = i \) in an optimal solution to the current instance; by Lemma 4.1 we may assume that pushing from \( i \) in \( \lambda \) has no effect. Let \( V_i = \lambda^{-1}(i) \) and let \( V_i \) be the vertices of the connected component of \( G[V_i] \) that contains \( v \). Then \( \delta(V_i) \) is an important separator, and it follows that the same holds for \( \delta(V_v) \) (as otherwise pushing would produce a bigger set \( V_v \)). Thus contracting it to a furthest min-cut is allowed. After this, the branching process is exhaustive as above.

For the running time, we view edges as being double-marked, receiving one mark from each endpoint. Edges marked in stage 1 are viewed as having received a mark from each
terminal side, i.e. at least \( p \) marks have already been placed at the start of the algorithm. We then view the branching around a vertex \( v \) as placing one such mark on the \( v \)-side of the solution edges connecting to \( V_v \) in \( V_v^c \). Hence, both the packing of a path from \( v \) and the final marking of an edge represents the placing of half a mark, and in total at most \( 2k - p \) such marks will be placed (ignoring the final leaf of an aborted branch where \( |C| \) grows too large). Hence the total number of branches in this stage is at most \( 2^{2(k-p)} \). Finally, for the polynomial part of the running time, we consider the amount of work done in a single node, before branching. As noted, every additional path added in the max-flow computation takes \( O(m) \) time to find and counts as a half-mark against our budget, hence at most \( O(k) \) paths are found before aborting. Additionally, every contraction step (out to a furthest min-cut) takes \( O(m) \) time in total, and by the same argument only \( O(k) \) contraction steps are performed. Hence the local work per node is bounded as \( O(km) \).

After stage 2, the remaining graph contains at most \( k \) edges, hence at most \( O(k) \) vertices, and it only remains to find the min-cost labelling of the non-terminal vertices. In the absence of any stronger structural properties of the metric \( \mu \), this last phase can be completed in \( |D|^{O(k)}O(m) \) time. Theorem \ref{thm:terminal} follows.

### 4.3 A kernel for any metric

We show that \textsc{Zero Extension} has a kernel of \( O(k^{s+1}) \) vertices for any metric \( \mu \), where \( k \) is a bound on the crossing number of a solution and \( s \) is the number of labels of \( \mu \). In fact, stronger than this, we show that such a kernel can be computed without access to \( \mu \). We can find a set \( Z \) of \( O(k^{s+1}) \) edges such that for every instance \( I = (G, \tau, \mu, q) \) with terminal set \( S \), if \( I \) admits any solution with at most \( k \) crossing edges, then \( F \) contains all crossing edges of at least one optimal solution of this type for \( I \). By contracting all edges not in \( Z \) this allows us to construct a graph \( G' \) with \( O(k^{s+1}) \) edges such that \( (G', S) \) has the `same behaviour' as \( (G, S) \), up to the values of \( k \) and \( s \). We refer to this as a metric sparsifier. Let us make this more precise.

**Definition 4.5.** For an instance \( I = (G, \tau, \mu, q) \) of \textsc{Zero Extension} and an integer \( k \), the \( k \)-bounded cost \( \text{cost}(I, k) \) of \( I \) is the minimum cost of a \( \tau \)-extension \( \lambda \) with crossing number at most \( k \), or \( \infty \) if no such \( \tau \)-extension exists. Let \( G = (V, E) \) be a graph with a set of terminals \( S \subseteq V \), and let \( k \) and \( s \) be integers, \( s \geq |S| \). A \( k \)-bounded metric sparsifier for \((G, S)\) (for metrics with up to \( s \) labels) is a graph \( G' = (V', E') \) with \( S \subseteq V' \), such that for any metric \( \mu \) on a set \( D \) of at most \( s \) labels, and for any injective labelling \( \tau: S \rightarrow D \), we have \( \text{cost}((G, \tau, \mu, q), k) = \text{cost}((G', \tau, \mu, q), k) \).

We show the following result, which implies the existence of a small metric sparsifier.

**Theorem 4.6.** Let \( s \geq 3 \) be a constant. For every graph \( G = (V, E) \) with a set \( S \) of terminals, \(|S| \leq s \), and integer \( k \), there is a randomized polynomial-time computable set \( Z \subseteq E \) with \( |Z| = O(k^{s+1}) \) such that for any instance \( I = (G, \tau, \mu, q) \) of \textsc{Zero Extension} with \( S \) being the set of terminals in \( I \) and \( \mu \) having at most \( s \) labels, if \( \text{cost}(I, k) < \infty \) then there is a \( \tau \)-extension \( \lambda \) with crossing number at most \( k \) and cost \( \text{cost}(I, k) \) such that every crossing edge of \( \lambda \) is contained in \( Z \).

The kernel is an adaptation of the irrelevant vertex strategy used in the kernel of \textsc{Multiway Cut} for terminals of Kratsch and Wahlström \cite{KratschWahlstrom2008}. In that paper, a kernel is produced by first computing a set \( Z_0 \) of \( O(k^{s+1}) \) vertices which contains all vertices \( v \) which are contained in every minimum solution of size at most \( k \). Then one irrelevant vertex \( v \notin Z_0 \) is chosen and removed from the graph via a bypassing operation, and the process is repeated until the computed set \( Z_0 \) covers all non-terminal vertices, at which point \( Z_0 \) is the desired final set \( Z \). The set \( Z_0 \) is computed using a tool from matroid-theory called representative sets. These same components will yield a kernel for our present problem.

We give only a brief sketch of the technical background, and focus on the kernelization result itself. For a complete description of the technical tools involved, see Kratsch and Wahlström \cite{KratschWahlstrom2008}.
A matroid is an “independence system” $M = (E, \mathcal{I})$, $\mathcal{I} \subseteq 2^E$, subject to certain axioms. The sets $S \in \mathcal{I}$ are the independent sets of $M$. Matroids have broad applications in combinatorics in general; see Oxley [28] and Schrijver [31]. A representation of a matroid is a matrix $A$ with columns labelled by $E$ such that for every $S \subseteq V$, $S \in \mathcal{I}$ if and only if the corresponding columns of $A$ are linearly independent. A gammoid is a matroid corresponding to flows in a graph. Given a directed graph $G = (V, E)$ with a set of source vertices $S \subseteq V$, and a set $U \subseteq V$, the gammoid defined from $G$, $S$ and $U$ is a matroid $M = (U, \mathcal{I})$ where $T \subseteq U$ is independent if and only if the fully vertex-disjoint max-flow from $S$ to $T$ is of size $|T|$; equivalently, there is no $(S, T)$-cut in $G$ of fewer than $|T|$ vertices. A representation of a gammoid can be computed in randomized polynomial time [28]. In our application, we need the gammoid to represent edge cuts instead of vertex cuts; clearly this can be done by subdividing edges of $E$ and multiplying every vertex $v$ of $G$ into $d(v)$ copies. Undirected edges of $G$ can be implemented as a pair of directed edges in opposite directions. Let $G'$ be the directed graph resulting from applying these modifications to $G$.

The main tool in our kernel is the following result. (See also Fomin et al. [9] for a faster algorithm.)

**Lemma 4.7 (21, 23).** Let $M = (E, \mathcal{I})$ be a linear matroid represented by a matrix $A$ of rank $r+s$, and let $\mathcal{Y}$ be a collection of independent sets of $M$, each of size $s$. Assume that $s$ is a constant. Then in polynomial time we can compute a set $\mathcal{Y}^* \subseteq \mathcal{Y}$ of size at most $(r+s)$ such that for every set $X \subseteq E$, there is a set $Y \in \mathcal{Y}$ such that $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$ if and only if there is such a set $Y' \in \mathcal{Y}^*$.

As shorthand, for an independent set $X$ we say that $Y$ extends $X$ if $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$. The following is a useful characterization of this notion in gammoids.

**Proposition 4.8 (Prop. 1 of [19]).** Let $X$ be a minimum $(S, X)$-vertex cut closest to $S$ in $G$, which may overlap $S$. Then for a vertex $v \in U$, the set $\{v\}$ extends $X$ in the gammoid if and only if $v$ is reachable from $S$ in $G - X$.

For further terms from matroid theory used below, see [19], alternatively Oxley [28] and Marx [24].

We are now ready to prove Theorem 4.6.

**Proof.** Let a graph $G = (V, E)$ with a terminal set $S \subseteq V$, $|S| \leq s$, and an integer $k$ be given. We apply Corollary 4.2 to $G$, then create a digraph $G'$ from $G$ as above. Additionally, for every edge $e = uv \in E$ we introduce a sink-copy $e'$ of $e$ in $G'$, with in-arcs from all copies of vertices $u$ and $v$ in $G'$, and no further in- or out-arcs. Let $E_S = \bigcup_{t \in S} \delta(t)$. We first note that if $|E_S| > 2k$ then we may reject the input.

**Claim.** If $\sum_{t \in S} d(t, G) > 2k$, then every labelling $\lambda$ that is injective on $S$ has more than $k$ crossing edges.

**Proof.** The crossing edges of any such $\lambda$ form a multiway cut of $(G, S)$. It is known that the cardinality of a multiway cut is at least $\sum_{t \in S} d(t, G)/2$, as noted previously in Section 4.2.

Hence if $|E_S| > 2k$, then we reject the input (alternatively, simply produce $Z = \emptyset$). Otherwise proceed as follows. Let an instance over $(G, S)$ be an instance $I = (\tau, \mu, q) = (G, \tau, \mu, q)$ of Zero Extension where $\mu$ is a metric over a set of labels $D$ with $|D| \leq s$ and $\tau$ is an injective labelling $\tau : S \to D$. We first observe that the existence of a $\tau$-extension $\lambda$ with crossing number at most $k$ for any instance over $(G, S)$ is a property purely of $(G, S)$ and $k$, hence independent of the metric; explicitly, it exists if and only if $(G, S)$ has a multiway cut of at most $k$ edges. Since the theorem is vacuous otherwise, we assume that such a multiway cut exists, hence that $\text{cost}(I, k) < \infty$ for every instance $I$ over $(G, S)$. Say that an edge $e$ is essential in $G$ if there is some instance $I = I(\tau, \mu, q)$ over $(G, S)$ such that there is at least one $\tau$-extension $\lambda$ with crossing number at most $k$ and cost at most $q$, and the edge $e$ is crossing in every such $\lambda$. We compute a set $Z_0$ that contains all essential edges;
any edge of $G$ not contained in $Z_0 \cup E_S$ is then irrelevant. The computation of $Z_0$ makes up the major part of this proof.

Number the vertices of $S$ as $S = \{t_1, \ldots, t_r\}$, $r = |S|$. We define a matroid $M$ as the disjoint union of the following matroids: Matroids $M_1$ through $M_s$ are disjoint copies of the gammoid defined from $G'$ with source set $E_S = \bigcup_{e \in S} \delta(t)$. Note that $E_S$ is a vertex set in $G'$. Finally, let $M_0$ be the uniform matroid $U_{n,k}$ of rank $k$ on ground set $E(G)$. Since $M_0$ has a representation over every sufficiently large field, we can compute a representation of $M$ in randomized polynomial time with exponentially small failure probability $\frac{1}{\text{poly}(k)}$. We refer to $M_1, \ldots, M_s$ and $M_0$ as the layers of $M$. For an independent set $X$ of $M$, a set $Y$ extends $X$ in $M$ if and only if the restriction of $Y$ to layer $i$ extends the restriction of $Y$ to layer $i$ for every layer $i$. Since the rank of each gammoid is $|E_S| \leq 2k$, the rank of $M$ is at most $(2s + 1)k = O(k)$ since $s$ is a constant. For an edge $e \in E(G)$, let $e_i$ (respectively $e_i'$) refer to the copy of $e$ (of $e'$) in $M_i$, $i \in [s]$, and $e_0$ refers to the copy of $e$ in $M_0$.

We now define the collection $\mathcal{Y}$ of sets of size $s + 1$. For an edge $e \in E(G)$, let $Y(e) = (e_1', \ldots, e_r', e_0)$, and define $\mathcal{Y} = \{Y(e) \mid e \in E(G) \setminus E_S\}$. Compute a representative set $\mathcal{Y}^* \subseteq \mathcal{Y}$ in $M$, and let $Z_0 \subseteq E = \{e \in E \mid Y(e) \in \mathcal{Y}^*\}$. Then $|Z_0| = O(k^{s+1})$ by Lemma 4.7.

We show that $Z_0 \cup E_S$ contains every edge that is essential in $G$.

Claim. If $e$ is essential in $G$ for some instance $I = I(\tau, \mu, q)$ over $(G, S)$, then $e \in Z_0 \cup E_S$.

Proof. Let $D$ be the set of labels of $\mu$. Let $\lambda$ be an extension of $\tau$ with at most $k$ crossing edges, and of cost $\text{cost}(I, k)$. Assume that among all such extensions, $\lambda$ has the minimum crossing number. Let $C$ be the set of crossing edges in $\lambda$, and for $i \in D$ let $V_i = \lambda^{-1}(i)$. Hence $C = \bigcup_{e \in D} \delta(V_i)$. Define a set $X$ in $M$ as follows: In layers $i = 1, \ldots, r$, $X$ contains the copy of edges $\delta(V_i) \cup \delta(t_i)$; in layers $i = r + 1, \ldots, s$, $X$ contains the copy of edges $\delta(V_i)$; and in the final layer (representing $M_0$), $X$ contains the copy of $C - e$. It is clear that $X$ is independent in $M$, as otherwise there is some label $i$ such that (by Lemma 4.7) the pushing lemma pushing from some label $i$ would yield a labelling $\lambda'$ with smaller crossing number and at most the same cost. The claim is now that $Y(f)$ extends $X$ if and only if $f = e$.

In the one direction, assume that $Y(f)$ extends $X$. If $f \in C$, then $f = e$ by $M_0$. Otherwise, $f$ lies in some set $V_i$, $i \in D$. But then the restriction of $X$ to layer $i$ separates $f$ from $E_S$, so $f'$ cannot possibly extend $X$ in this layer. Hence $f \in C$, and $f = e$.

In the other direction, we show that $Y(e)$ indeed extends $X$. This is clear in $M_0$, so we focus on a layer $i \in [s]$. Let $\lambda_i$ be the result of pushing from $i$ in $\lambda$ and let $C_i$ be the set of crossing edges in $\lambda_i$. Let $V'_i = \lambda_i^{-1}(i)$. By Lemma 4.7, $\lambda_i$ has a cost at most as high as $\lambda$, and clearly has at most $k$ crossing edges, hence $\lambda_i$ is an optimal solution to $I$, and by assumption $e \in C_i$. Let $v$ be an endpoint of $e$ such that $\lambda_i(v) \neq i$; by assumption, $v$ exists. It follows from the pushing operation that $v$ is reachable from $S - t_i$ (respectively from $S$ if $i > r$) avoiding $\delta(V'_i)$. Thus, by Prop. 4.8, $e'$ extends $\delta(V'_i) \cup \delta(t_i)$ in $M_i$ (respectively $\delta(V_i)$ if $i > r$), which is the restriction of $X$ to layer $i$. Thus $Y(e)$ extends $X$.

Since $\mathcal{Y}^*$ contains at least one set $Y(f)$ that extends $X$, we conclude $e \in Z_0$. 

It only remains to show that sequentially contracting irrelevant edges (one at a time, while recomputing $Z_0$ at every step) yields a final set $Z$ that contains crossing edges of optimal solutions for all metrics as described. This should be clear. The effect of contracting an edge $uv$ is precisely to restrict the solution space to labellings $\lambda$ where $uv$ is non-crossing. All other edges of the graph remain identifiable, and for every instance $I$ over $(G, S)$ there still exist some optimal solution $\lambda$ after the contraction. That is, at every stage, for every instance $I$ over $(G, S)$ there exists some solution $\lambda$ with crossing number at most $k$ and of cost $\text{cost}(I, k)$, such that the edge that is about to be contracted is non-crossing in $\lambda$. Thus the final graph, where $Z \cup E_S$ covers the entire edge set, also contains such a solution. 

Corollary 4.9. For every graph $G$ with terminal set $S$, and integers $k, s$, there is a $k$-bounded metric sparsifier $G'$ for $(G, S)$, for metrics with up to $s$ labels, with $O(k^{s+1})$ edges, which can be computed in randomized polynomial time. Furthermore, for every metric $\mu$ on $s$ labels, ZERO EXTENSION admits a randomized polynomial kernel with $O(q^{s+1})$ edges.
inner odd-join

We now give more powerful algorithms parameterized by the gap parameter for problems
where \( \lambda \) is precisely a zero-extension of the terminal-set \( S \) and the sets \( F \subseteq E \) are so-called
inner odd-join, that is, a set of edges whose deletion leaves every non-terminal vertex with an
even degree. It follows that the maximum value of a half-integral \( \tau \)-path packing is just the
minimum cost of a \( \tau \)-extension \( \lambda \), since a half-integral path-packing is just a path-packing
in the graph where every edge of \( G \) has been duplicated, and such a graph has no vertices
of odd degree.

Let in the following \( I = (G, \tau, \mu, q) \) be an instance of Zero Leaf Extension, where \( \mu \)
is a leaf metric over a tree \( T \) with leaves \( D \). Let \( \hat{\mu} = d_T \) be the underlying tree metric. We
define the relaxed instance \( \hat{I} = (G, \tau, \hat{\mu}, q) \). Let \( \text{opt}(I) \), \( \text{opt}(\hat{I}) \) denote the set of optimal
solutions for the integral and the relaxed instance, respectively. As mentioned in Section \ref{sec:alg}
it is known that the relaxed instance can be solved optimally in polynomial time \[30\]. For
convenience, we say that a vertex \( u \) is integral with respect to a solution \( \lambda \) if \( \lambda(u) \in D \) and
we say that an edge \( uv \in G \) is integral with respect to \( \lambda \) if both endpoints are integral.

Using the above notation, we can summarize the duality between an minimum relaxed
labelling and a path packing as follows: Given a relaxed instance \( \hat{I} \), there exists a half-integral
\( \tau \)-path-packing \( \mathcal{P} \) of cost precisely \( \text{cost}(\hat{I}) \). We will not explicitly compute \( \mathcal{P} \) in the final
algorithm, instead we use its existence to derive useful properties of the problem. In the
following we will use \( S \) to denote the terminals of the instance, i.e. the vertices labelled by \( \tau \).
By the usual identification argument, we can assume that \( \tau \) is a bijection and a \( \tau \)-path
packing is equivalent to an \( S \)-path packing.
Lemma 5.1. Let $\mathcal{P}$ be an half-integral $\tau$-path packing that satisfies

$$
\frac{1}{2} \sum_{P \in \mathcal{P}} \mu(\tau(s_P), \tau(t_P)) = \text{cost}(\hat{I}).
$$

Let $\lambda \in \text{opt}(\hat{I})$ be a relaxed optimum and let $P \in \mathcal{P}$ with endpoints $s, t$. Then

$$\text{cost}_\mu(\lambda, P) = \mu(\tau(s), \tau(t)).$$

Proof. First, consider any $s$-$t$-path $P'$ in $G$. Then

$$\text{cost}_\mu(\lambda, P) = \sum_{uv \in P'} \bar{\mu}(\lambda(u), \lambda(v)) \geq \mu(\tau(s), \tau(t)).$$

We therefore find that the inequality

$$\sum_{P \in \mathcal{P}} \bar{\mu}(\tau(s_P), \tau(t_P)) \leq \sum_{P \in \mathcal{P}} \text{cost}_\mu(\lambda, P) \leq 2 \text{cost}_\mu(\lambda, G)$$

holds. But according to our assumption, the left-hand side and right-hand side are equal and we conclude that

$$\sum_{P \in \mathcal{P}} \bar{\mu}(\tau(s_P), \tau(t_P)) = \sum_{P \in \mathcal{P}} \text{cost}_\mu(\lambda, P).$$

and therefore that for every $P \in \mathcal{P}$, $\text{cost}_\mu(\lambda, P) = \mu(\tau(s_P), \tau(t_P))$, as claimed. \qed

A direct consequence is that if we trace an $s$-$t$-path $P \in \mathcal{P}$, then the labels assigned by any relaxed optimum $\lambda$ to $P$ induce a monotone sequence from $s$ to $t$ in $T$. That is, not only will we only encounter those labels that lie on $T[s, t]$, we also will encounter them ‘in order’. We further can conclude the following:

Corollary 5.2. Let $e \in G$ be an edge that is not part of the path-packing $\mathcal{P}$. Then under every relaxed optimum $\lambda \in \text{opt}(\hat{I})$ the edge $e$ has cost zero.

Consider an edge $xy \in E(T)$. Then, as a consequence of Lemma 5.1, the set of edges $C_{xy}(\lambda) = \{uv \in E(G) \mid \lambda(u) \in T_x, \lambda(v) \in T_y\}$ between the vertex sets with labels in $T_x$ and $T_y$, respectively, must be saturated by paths of the packing $\mathcal{P}$. For cuts right above leaves of $T$, this implies the following.

Lemma 5.3. Let $S$ be the vertices labelled by $\tau$ in $G$ and assume that $\tau$ is a bijection. Let $C$ be any minimum $(x, S - x)$-cut for some terminal $x \in S$. Then every optimal, half-integral $S$-path-packing in $G$ will saturate $C$.

Proof. Let us first consider the closest minimal cut $C_{\min}(x)$ and the furthest minimal cut $C_{\max}(x)$. Let $\mathcal{P}$ be a max-value $\tau$-path-packing, and let $\lambda$ be the corresponding min-cost extension of $S$. Let $y$ be the ancestor of $x$ in $T$ and consider $C_{yz}(\lambda)$. By the above, $C_{yz}(\lambda)$ is saturated by $\mathcal{P}$. Since every path of $\mathcal{P}$ induces a monotone sequence in $T$ under $\lambda$, every path $P \in \mathcal{P}$ crossing $yz$ in $T$ must have $x$ as an endpoint. But the total weight of such paths $P_x \subseteq \mathcal{P}$ is at most $C_{\min}(x)$. Since $C$ is a $(x, S - x)$-cut, we must have $C_{yz}(\lambda) = C_{\min}(x)$.

Now, since $C_{\max}(x)$ is a minimal $(x, S - x)$-cut as well, $\mathcal{P}_x$ saturates $C_{\max}(x)$ as well. This in particular means that every path in $\mathcal{P}$ that intersects $C_{\max}(x)$ must end in $x$, e.g. those paths are exactly $\mathcal{P}_x$. Consequently, every min-cut around $x$ is saturated by $\mathcal{P}_x$, proving the statement. \qed

Lemma 5.4. Let $\lambda$ be a not necessarily optimal solution for $\hat{I}$. Then

$$\text{cost}_\mu(\lambda, G) = \sum_{xy \in T} |C_{xy}(\lambda)|.$$
Proof. The claim is equivalent to proving that
\[ \sum_{uv \in G} \mu(\lambda(u), \lambda(v)) = \sum_{xy \in T} |C_{xy}(\lambda)|. \]

We proof the above equality by double-counting. Consider an edge uv, then \( \mu(\lambda(u), \lambda(v)) \) is by definition \( |T[\lambda(u), \lambda(v)]| \). Note that uv appears in exactly those cuts \( C_{xy} \) with \( xy \in T[\lambda(u), \lambda(v)] \), hence we can charge the cost of \( \mu(\lambda(u), \lambda(v)) \) on those cuts and the equation follows.

Lemma 5.5. Let \( u \in G \) be such that \( u \notin C_{\text{max}}(x) \) for all terminals \( x \in S \). Then \( u \) does not receive any integral value by any relaxed optimum.

Proof. Assume otherwise: let \( \lambda \in \text{opt}(\hat{I}) \) be a relaxed optimum that assigns \( u \) some integral value \( v \in T \). Let \( y \) be the parent of \( x \) in \( T \). Then we conclude that \( C_{xy}(\lambda) \) cannot be a minimum cut, since \( u \) is part of \( \lambda^{-1}(x) \), but \( u \) is not in the furthest min-cut \( C_{\text{max}}(x) \). But then \( \sum_{xy \in T} |C_{xy}(\lambda)| \) cannot be minimal, and by Lemma 5.4 therefore \( \text{cost}_\mu(\lambda, G) \) is not either, contradicting our assumption.

Lemma 5.6. For every triple \( x, y, z \in S \) of distinct terminals it holds that \( C_{\text{max}}(y) \cap C_{\text{max}}(z) = \emptyset \).

Proof. Assume towards a contradiction that there exists \( u \in C_{\text{max}}(x) \cap C_{\text{max}}(y) \cap C_{\text{max}}(z) \) and let us choose \( u \) such that it is incident to edges that cross the cut \( C_{\text{max}}(x) \).

Let \( P \) be an optimal half-integral S-path packing. By Lemma 5.3 all paths of \( P \) that enter \( C_{\text{max}}(c) \) for \( c \in S \) must have a endpoint in \( c \); moreover, they saturated the cut \( C_{\text{max}}(c) \). Since \( x \) is incident to edges that cross the cut \( C_{\text{max}}(x) \) it must therefore lie on at least one path \( P \) that ends in \( x \). Now, since \( u \) also lies in \( C_{\text{max}}(y) \), the path \( P \) crosses \( C_{\text{max}}(y) \) and must therefore end in \( y \). However, the same argument holds for \( z \) and we arrive at a contradiction. We conclude that the intersection of the three cuts must indeed be empty.

Theorem 1.4. Let \( I = (G, \tau, \mu, q) \) be an instance of Zero Extension where \( \mu \) is a leaf metric on a set of labels \( D \) in a tree \( T \), and let \( \hat{I} = (G, \tau, \hat{\mu}, q) \) be the relaxed instance. Let \( p = \text{cost}(\hat{I}) \). Then we can solve \( I \) in time \( O(|D|^{q+1}|T|m + |T|\phi(n, m)) \), where \( \phi \) is the time needed to run a max-flow algorithm.

Proof. Given the input graph \( G \) we first construct for every edge \( ij \in T \) a flow network \( H_{ij} \) from \( G \) as follows: let \( D_i \) be those leaves that lie in the same component as \( i \) in \( T - ij \) and \( D_j \) all others. Then \( H_{ij} \) is obtained from \( G \) by identifying all terminals \( \tau^{-1}(D_i) \) into a source \( s \) and all terminals \( \tau^{-1}(D_j) \) into a sink \( t \). For each networks \( H_{ij} \) we compute a maximum flow \( f_{ij} \) in time \( \phi(n, m) \). By Lemma 5.4 we have that for every \( \nu \in \text{opt}(\hat{I}) \) it holds that
\[ \sum_{xy \in T} |f_{ij}| = \sum_{xy \in T} |C_{xy}(\lambda)| = \text{cost}_\mu(\hat{I}). \]

Note that we can also, in linear time, find the closest cuts \( C_{\text{min}}(x) \) and furthest cuts \( C_{\text{max}}(x) \) for terminals \( x \in S \) using the residual network of \( (H_{ij}, f_{ij}) \) with \( i = \tau(x) \) and \( j \) the parent of \( i \) in \( T \).

Thus, in linear time, we can identify whether \( G \) contains a vertex that is not part of any furthest min-cut \( C_{\text{max}}(x) \) for all \( x \in S \). By Lemma 5.5 such a vertex cannot take an integral value in any relaxed optimum. We therefore branch on the \( |D| \) possible integral values it could take: for \( x \in S \) with \( \tau(x) \in D \) being the chosen integral value, we update the flow networks \( (H_{ij}, f_{ij}) \) by adding an edge of infinite capacity from \( x \) to \( u \) and then augment the flow \( f_{ij} \) until it is maximum again. Note that the number of augmentations necessary are at most \( k - p \), since each augmentation witnesses the increase of \( p \) and thus the decrease of the parameter by one. In our analysis we can therefore charge each augmentation to a level of the search tree (treating each augmentation like a descent to the next node) and thus spend only \( O(m) \) time per flow \( f_{ij} \), for a total of \( O(|T|m) \).

Otherwise, we find that every vertex of the current graph \( G \) is contained in at least one furthest min-cut. By Lemma 5.6 the intersection of three or more such cuts is empty.
and we can partition the vertices of $G$ into sets $\{V_x\}_{x \in S} \cup \{U_{xy}\}_{x,y \in S}$ where $V_x$ contains all vertices that are only contained in $C_{\max}(x)$ while $U_{xy}$ contains those that live in the intersection $C_{\max}(x) \cap C_{\max}(y)$. By Lemma 5.3 we have that $U_{xy}$ only contains edges towards $V_x$ and $V_y$ (since these edges are exactly saturated by a half-integral path-packing and the paths saturating these edges have endpoints $x$ and $y$). Therefore we construct an integral solution $\lambda$ as follows: every set $V_x$, $x \in S$ is coloured $\tau(x)$ and for every non-empty set $U_{xy}$, $x, y \in S$ we choose colour $\tau(x)$ or $\tau(y)$ arbitrarily. By Lemma 5.4, the cost of $\lambda$ is precisely

$$\text{cost}_\mu(\lambda, G) = \sum_{xy \in T} |C_{xy}(\lambda)| = \sum_{xy \in T} |f_{ij}| = \text{opt}(\hat{I})$$

and we conclude that $\lambda$ is an integral solution that matches the relaxed optimum of the current instance. In this case, we return $\lambda$ as a solution to the original instance. The claimed running time follows if we prune every branch of the search tree in which the parameter drops below zero.

\[ \square \]

### 5.2 VCSP toolkit

We now review some required terminology and tools for the proof of the algebraic properties of distance problems on trees.

Given a set of cost functions $\Gamma$ over a domain $D$, an instance $I$ of VCSP($\Gamma$) is defined by a set of variables $V$ and a sum of valued constraints $f_i(\bar{v}_i)$, where for each $i$, $f_i \in \Gamma$ and $\bar{v}_i$ is a tuple of variables over $V$. We write $f_i(\bar{v}) \in I$ to signify that $f_i(\bar{v})$ is a valued constraint in $I$.

It is known that the tractability of a VCSP is characterized by certain algebraic properties of the set of cost functions. In full generality, such conditions are known as \textit{fractional polymorphisms} for the finite-valued case and more general \textit{weighted polymorphisms} in the general-valued case. Dichotomies are known in these terms both for the finite-valued [33] and general case of VCSP [15], i.e., characterizations of each VCSP as being either in P or \textsc{NP}-hard. We will only need a less general term.

A \textit{binary multimorphism} $\langle \circ, \bullet \rangle$ of a language $\Gamma$ over a domain $D$ is a pair of binary operators that satisfy

$$f(\bar{x}) + f(\bar{y}) \geq f(\bar{x} \circ \bar{y}) + f(\bar{x} \bullet \bar{y}) \quad \forall f \in \Gamma, \bar{x}, \bar{y} \in D^{\text{ar}(f)},$$

where $\text{ar}(f)$ is the arity of $f$ and where we extend the binary operators to vectors by applying them coordinate-wise. An operator $\circ$ is \textit{idempotent} if $x \circ x = x$ for every $x \in D$, and \textit{commutative} if $x \circ y = y \circ x$. A (finite, finite-valued) language $\Gamma$ with a binary multimorphism where both operators are idempotent and commutative is solvable in polynomial time via an LP-relaxation [33]. The most basic example is the Boolean domain $D = \{0, 1\}$, in which case the multimorphism $\langle \land, \lor \rangle$ corresponds to the well-known class of \textit{submodular functions}, which is a tractable class that generalizes cut functions in graphs.

The following is folklore, but will be important to our investigations. Again, the corresponding statements apply for arbitrary fractional polymorphisms, but we only give the version we need in the present paper.

**Definition 5.7** (Preserved under equality). Let $f$ be a function that admits a multimorphism $\langle \circ, \bullet \rangle$. We say that two tuples $\bar{x}, \bar{y} \in D^{\text{ar}(f)}$ are preserved under equality if

$$f(\bar{x}) + f(\bar{y}) = f(\bar{x} \circ \bar{y}) + f(\bar{x} \bullet \bar{y}).$$

For a relation $R \subseteq D^{\text{ar}(r)}$, we say that $f$ is preserved under equality in $R$ if every pair of tuples $\bar{x}, \bar{y} \in R$ is preserved under equality and $\bar{x} \circ \bar{y}, \bar{x} \bullet \bar{y} \in R$.

**Lemma 5.8.** Let $\Gamma$ be a language of cost functions that admit a multimorphism $\langle \circ, \bullet \rangle$ and let $\lambda_1, \lambda_2 \in \text{opt}(I)$ for some instance $I$ of VSIP($\Gamma$). Then for every valued constraint $f(\bar{v}) \in I$ it holds that

$$f(\lambda_1(\bar{v})) + f(\lambda_2(\bar{v})) = f((\lambda_1 \circ \lambda_2)(\bar{v})) + f((\lambda_1 \bullet \lambda_2)(\bar{v})).$$
where \( f(\lambda(\bar{v})) = f(\lambda(v_1), \ldots, \lambda(v_r)) \) for \( \bar{v} = v_1, \ldots, v_r \) is the value of \( f(\bar{v}) \) under \( \lambda \). In other words, every valued constraint \( f(\bar{v}) \in I \) is preserved under equality in \( \text{opt}(I) \).

Proof. Let \( \text{cost}_f(\lambda) \) be the sum of all valued constraints \( f(\bar{v}) \in I \) under \( \lambda \). By the multimorphism, we have that
\[
\text{cost}_f(\lambda_1) + \text{cost}_f(\lambda_2) \geq \text{cost}_f(\lambda_1 \circ \lambda_2) + \text{cost}_f(\lambda_1 \bullet \lambda_2)
\]
and since \( \lambda_1 \) and \( \lambda_2 \) are optimal we obtain that
\[
\text{cost}_f(\lambda_1) + \text{cost}_f(\lambda_2) = 2 \text{opt}(\hat{I}) = \text{cost}_f(\lambda_1 \circ \lambda_2) + \text{cost}_f(\lambda_1 \bullet \lambda_2)
\]
and therefore that \( \text{cost}_f(\lambda_1 \circ \lambda_2) = \text{cost}_f(\lambda_1 \bullet \lambda_2) = \text{opt}(\hat{I}) \). For two variables \( u, v \) that appear together in a valued constraint \( f \) let us define
\[
\Delta_f(\bar{v}) := f(\lambda_1(\bar{v})) + f(\lambda_2(\bar{v})) - f((\lambda_1 \circ \lambda_2)(\bar{v})) - f((\lambda_1 \bullet \lambda_2)(\bar{v}))
\]
then by the multimorphism property it follows that \( \Delta_f(\bar{v}) \geq 0 \). Since, by definition,
\[
\text{cost}_f(\lambda_1) + \text{cost}_f(\lambda_2) - \text{cost}_f(\lambda_1 \circ \lambda_2) - \text{cost}_f(\lambda_1 \bullet \lambda_2) = \sum_{f(\bar{v}) \in I} \Delta_f(\bar{v})
\]
and the left-hand side evaluates to zero, we conclude that \( \sum_{u,v \in G} \Delta_f(u,v) = 0 \) and therefore that \( \Delta_f(u,v) = 0 \) for every constraint \( f(u,v) \in I \).

To illustrate, let us return again to the case of graph cut functions and submodularity over the Boolean domain. Let \( G = (V,E) \) be an undirected graph, and define the cut function \( f_G : 2^V \rightarrow \mathbb{Z} \) as \( f_G(S) = |\delta(S)| \). Then \( f_G \) is the sum over binary valued constraints \( f(u,v) = [u \neq v] \) over all edges \( uv \in E \), in Iverson bracket notation. Since a single valued constraint \( f(u,v) \) is submodular, the same holds for the cut function as a whole. Then Lemma 5.8 specialises into the statement that for two sets \( A,B \subseteq V \) such that \( \delta(A), \delta(B) \) are minimum \( s,t \)-cuts in \( G \) for some \( s,t \in V \), there is no edge between \( A \setminus B \) and \( B \setminus A \). This kind of observation is a common tool in, e.g., graph theory and approximation algorithms.

The above lemma will be very useful when reasoning about the structure of \( \text{opt}(I) \) subject to more complex multimorphisms, as we will define next.

5.3 Submodularity on trees

Let \( \preceq_T \) denote the ancestor relationship in a rooted tree \( T \). For a path \( P[x,y] \subseteq T \), let \( z_1, z_2 \) be the middle vertices of \( P[x,y] \) (allowing \( z_1 = z_2 \) in case \( P[x,y] \) has odd length) such that \( z_1 \preceq_T z_2 \). Define the commutative operators \( \land, \lor \) as returning exactly those two mid vertices, e.g. \( x \land y = y \land x = z_1 \) and \( x \lor y = y \lor x = z_2 \). Languages admitting the multimorphism \( \langle \land, \lor \rangle \) are called strongly tree-submodular.

Define the commutative operator \( \uparrow \) to return the common ancestor of two nodes \( x, y \) in a rooted tree \( T \). Define \( x \uparrow y \) to be the vertex \( z \) on \( P[x,y] \) which satisfies \( d_T(x,z) = d_T(y,x \uparrow y) \). In other words, to find \( z = x \uparrow y \), we measure the distance from \( y \) to the common ancestor of \( x \) and \( y \) and walk the same distance from \( x \) along \( P[x,y] \). Languages that admit \( \langle \uparrow, \lor \rangle \) as a multimorphism are called weakly tree-submodular. In particular, all strongly tree-submodular languages are weakly tree-submodular [17]. Tree-metric are, not very surprisingly, strongly tree-submodular:

Lemma 5.9. Every tree-metric is strongly tree-submodular for every rooted version of the tree.

Proof. Let \( T \) be a rooted tree and let \( a, b, x, y \in T \) not necessarily distinct nodes. We let \( d \) be the distance-metric on \( T \). We need to show that
\[
d(a,b) + d(x,y) \geq d(a \land x, b \land y) + d(a \lor x, b \lor y).
\]
First, consider the case that $P[a, b] \cap P[x, y] = P[m_1, m_2]$ is a non-empty path. Assume that $P[a, x]$ and $P[b, y]$ are disjoint. Then the left-hand side of (1) is equal to

$$d(a, b) + d(x, y) = |P[a, b]| + |P[x, y]| = 2|P[m_1, m_2]| + |P[a, x]| + |P[b, y]|.$$  \hspace{1cm} (2)

Since the nodes $a \cap x$, $a \cap b$ both lie on $P[a, x]$ and the nodes $b \cap y$, $b \cap b$ on $P[a, y]$, the right-hand side of (1) cannot be larger than the right-hand side of (2), thus (1) holds in this case. In the alternative case where $P[a, x]$ and $P[b, y]$ are non-disjoint, we instead use the paths $P[a, y]$, $P[b, x]$. In this case, the nodes $a \cap x$, $a \cap b$, $b \cap y$, $b \cap y$ could now also lie on $P[m_1, m_2]$ but the argument remains the same.

Thus consider the second case: $P[a, b]$ and $P[x, y]$ do not intersect. Let now $P[c, z]$ be the unique path connecting $P[a, b]$ and $P[x, y]$ with $c \in P[a, b]$ and $z \in P[x, y]$. First, we simplify our lives by observing that the right-hand side of (1) can be replaced using

$$d(a \cap x, b \cap y) + d(a \cap x, b \cap y) = 2d(m_{ax}, m_{by})$$

where we allow the mid-points $m_{ax}$ of $P[a, x]$ and $m_{by}$ of $P[b, y]$ to lie in the middle of an edge (by some abuse of notation we extend $d$ to such mid-points of edges and allow it to take half-integral values). Consider the following re-writing of (1):

$$d(a, b) + d(x, y) \geq 2d(m_{ax}, m_{by}).$$  \hspace{1cm} (3)

Clearly, it holds in the degenerate case of $a = b = c$ and $x = y = z$. We prove the remainder by induction through the insertion of an arbitrary edge. First, assume that an edge is inserted into $P[a, c]$. This increases the left-hand side of (3) by one and moves $m_{ax}$ by half a unit, thus at most increasing the right-hand side by one as well. The same holds, by symmetry, for any edge inserted into $P[b, c]$, $P[x, z]$, and $P[y, z]$. It remains to consider edges inserted into $P[c, z]$ whose addition does not contribute to the left-hand side. If such an edge additionally lies on the path between $m_{ax}$ and $m_{by}$, the distance between these two mid-points decreases by one; otherwise both midpoints are shifted in such a way that they remain equidistant. In neither scenario does the right-hand side increase, proving the claim. Finally, since $d$ does not depend on the choice of root in $T$, the result also holds for every root.

\[\square\]

**Corollary 5.10.** Every tree-metric is weakly tree-submodular for every rooted version of the tree.

We will need the following characterization of which value-pairs are preserved under equality by strong tree submodularity for tree distance functions.

**Lemma 5.11.** Two tuples $(a, b), (x, y) \in V(T) \times V(T)$ are preserved under equality by $d_T$ with multimorphism $(\cap, \cap)$ iff all four nodes lie on a single path $P$ in $T$ and either $a, b \leq_P x, y$ or $a, x \leq_P b, y$.

Below is a complete enumeration of all orders in which the nodes $a, b, x, y$ might appear on a path, where we removed all cases in which $b$ appears before $a$ (to break mirror symmetry) and all cases derivable by exchanging $a$ with $x$ and $b$ with $y$, or applying both operations. The cases on the right side show how the nodes might all appear on a single path and yet not be preserved under equality by $(\cap, \cap)$ (if they were, the sum of the magenta lines would match the sum of the cyan lines).
Proof. Let $m_{ax}$ and $m_{bx}$ denote the mid-points of $P[a, x]$ and $P[b, y]$, as in the proof of Lemma 5.9 we allow these points to lie in the middle of an edge in case these paths are of even length. We drop the subscript of $d_T$ in the following.

For the one direction, assume that $a, b, x, y$ all lie on some common path $P$. We now need to show that

$$d(a, b) + d(x, y) = 2d(m_{ax}, m_{by}). \tag{4}$$

It will be helpful to identify $P$ with the interval $I = [0, |P|]$. We distinguish several cases depending on the order imposed on these nodes by $P$, the two principal cases are depicted below. Since $a, b$ are exchangeable, we will assume that $a \leq_P b$ in all cases.

![Diagram showing two cases](image)

In both of these pictures we can easily verify that $m_{ax}$ lies at position $d(a, x)/2$ on $I$ and $m_{by}$ at $d(a, b) + d(b, y)/2$, hence

$$2d(m_{ax}, m_{by}) = \left| \frac{d(a, x)}{2} - \left( d(a, b) + \frac{d(b, y)}{2} \right) \right| = \left| d(a, x) - d(b, y) - 2d(a, b) \right| = d(a, b) + d(x, y).$$

Note that the case $axyb$ is equivalent to the case $axby$, since exchanging $y$ and $b$ does not affect $m_{by}$ and thus none of the relevant distances. This proves the first direction.

In the other direction, first assume that the nodes do all lie on a path $P$ but do no fulfil the second property. After removing symmetries we are left with the four cases $aybx, ayxb, yaxb$. Consider $aybx$ first. We have that

$$2d(m_{ax}, m_{by}) = \left| \frac{d(a, x)}{2} - \left( d(a, y) + \frac{d(y, b)}{2} \right) \right| = \left| d(a, x) - d(y, b) - 2d(a, y) \right|$$

which is smaller than either $d(a, b)$ or $d(x, y)$. The calculation for $yaxb$ is essentially the same, which leaves us with $ayxb$. In that case (skipping to the part where the computation diverges) we obtain that

$$2d(m_{ax}, m_{by}) = \left| d(a, x) - d(y, b) - 2d(a, y) \right|$$

which is smaller than $d(a, b)$ unless $x = y$.

This concludes the case in which the nodes all lie on a path, hence we are left with cases in which the nodes $a, b, x, y$ do not lie on a single path in $T$. First, consider the case $aybx$ (as in the above figure) and imagine introducing edges to take $y$ and $b$ away from the path $P[a, x]$. Every edge introduced in this manner will contribute exactly one the the left-hand side of $[\overline{1}]$ and at most one to the right-hand side (since $m_{by}$ moves by half a unit). Hence no tree derivable from $aybx$ can ever achieve equality. The same argument holds for $ayxb$ (where we introduced edges to remove $y$ and $x$ from $P[a, b]$) and $yaxb$ (were we remove $a$ and $x$ from $P[y, b]$).

In the case of $axby$, removing $x$ or $b$ from $P[a, y]$ by introducing an edge will contribute one to the left-hand side of $[\overline{1}]$ and decrease the right-hand side (since $m_{ax}$ would move away from $a$ and $m_{by}$ away from $y$), hence equality is broken. A similar argument works for $axyb$ and $abxy$, in both cases the midpoints move in a way that decreases the right-hand side of $[\overline{4}]$. This proves the claim. \( \square \)
Corollary 5.12. Let $d_T$ be preserved under equality in $R$ for some $R \subseteq V_T \times V_T$, with at least one pair $(a, b) \in R$ with $a \neq b$. Then there is a path $P$ in $T$ which can be oriented as a directed path such that for every pair $(a, b) \in R$ the nodes $a$ and $b$ lie on $P$ with $a \preceq_P b$.

Proof. We first show that it holds for all pairs $(a, b) \in R$ with $a \neq b$. Let $ij$ be an edge of $T$ and let $T_i, T_j$ be the trees of $T - ij$. By the above lemma, there are no two pairs $(a, b), (x, y) \in R$ such that $a, y \in T_i$ and $b, x \in T_j$. Hence we can define an oriented subforest $T'$ of $T$ by including a directed edge $ij$ whenever there is a pair $(a, b) \in R$ with $a \in T_i, b \in T_j$. Then again by the above lemma, there is no path $P$ in $T$ such that $T'$ contains edges of $P$ oriented in conflicting directions. This implies that $T'$ is a subgraph of a directed path in $T$.

Next, let $P$ be a minimal directed path as above, i.e., $P = T[s, t]$ for some $s, t$. Let $ij$ be an edge of $T$ and let $T_i, T_j$ be the trees of $T - ij$. By the above lemma, there are no two pairs $(a, b), (x, y) \in R$ such that $a, y \in T_i$ and $b, x \in T_j$. Hence we can define an oriented subforest $T'$ of $T$ by including a directed edge $ij$ whenever there is a pair $(a, b) \in R$ with $a \in T_i, b \in T_j$. Then again by the above lemma, there is no path $P$ in $T$ such that $T'$ contains edges of $P$ oriented in conflicting directions. This implies that $T'$ is a subgraph of a directed path in $T$.

5.4 The domain consistency property

Consider a problem VCSP($\Gamma$) over a domain $D_T$ and a discrete relaxation VCSP($\Gamma'$) of VCSP($\Gamma$) over a domain $D \supseteq D_T$. We say that the relaxation has the domain consistency property if the following holds: for any instance $I$ of VCSP($\Gamma'$), if for every variable $v$ there is an optimal solution to $I$ where $v$ takes a value in $D_T$, then there is an optimal solution where all variables take values in $D_T$, i.e. an optimal solution to the original problem of the same cost. We will use the results of the previous section to show that the relaxations we are using for ZERO EXTENSION and METRIC LABELLING on induced tree metrics have the domain consistency property (in the latter case with a suitable restriction on the unary costs), allowing for FPT algorithms under the gap parameter via simple branching algorithms.

The result will follow from a careful investigation of the binary constraints that $\text{opt}(I)$ can induce on a pair of vertices $u, v \in V$, also for cases when there is no edge $uv$ in $G$. The result builds on Corollary 5.12.

For the rest of the section, let us fix a relaxed instance $I = (G = (V, E), \tau, \mu, q)$ of ZERO EXTENSION where $\mu$ is a tree metric defined by a tree $T$, and the original (non-relaxed) metric is the restriction of $\mu$ to a set of nodes $D_T$. Note that $I$ can be expressed as a VCSP instance using assignments and the valued constraint $\mu$. Let $\text{opt}$ be the set of optimal labellings. For a vertex $v \in V$, let $D(v)$ denote the set $\{\lambda(u) \mid \lambda \in \text{opt}\}$, and let $D_T(v) = D_T \cap D(v)$. Furthermore, for a pair of vertices $u, v \in V$, let $R(u, v) = (\lambda(u), \lambda(v)) \mid \lambda \in \text{opt}$ be the projection of $\text{opt}$ onto $(u, v)$, and $R_T(u, v) = R(u, v) \cap (D_T \times D_T)$ the integral part of this projection. Let $F \subseteq E$ be the set of edges that are crossing in at least one $\lambda \in \text{opt}$ and let $E_0 = E \setminus F$.

We begin by observing that the “path property” of Corollary 5.12 applies to all vertices and edges in $\text{opt}$.

Lemma 5.13. For every vertex $v$ that lies in a connected component of $G$ containing at least one terminal, $D(v)$ is a path in $T$. Furthermore, for every edge $uv \in E$, $R(u, v)$ embeds into the transitive closure of a directed path in $T$.

Proof. First note that $v$ has the same domain as any vertex $u$ reachable from $v$ via edges of $E_0$; therefore we can focus on the case that $v$ is incident with an edge $uv \in F$. In this case, the valued constraint $d_T(u, v)$ is present in $I$, and by Lemma 5.8 is preserved under equality in $R(u, v)$. Since $D(v)$ is just the projection of $R(u, v)$ to $v$, it follows from
Corollary 5.12 that $D(v)$ is contained in a path. Furthermore, $D(v)$ itself is closed under the $\cap$, $\sqcap$ operations, which implies that $D(v)$ covers the whole path. For the second part, if $uv \in F$ then the claim is Corollary 5.12. Otherwise, $R(u, v)$ is a collection of pairs $(x, x)$ for all $x \in D(v)$, which is equal to a path in $T$.

Next, we show the main result of this section: if $u$ and $v$ is a pair of variables, then whether or not there is an edge $uv$ in $E$, the constraint $R(u, v)$ induced on $u$ and $v$ by opt is only non-trivial on values in $D(u) \cap D(v)$. Note that the proof only uses the algebraic properties of weak tree submodularity, hence the only assumption that is specific to Zero Extension is that $D(u)$ and $D(v)$ form paths in $T$.

Lemma 5.14. Let $u$ and $v$ be a pair of variables and $a \in D(u)$, $b \in D(v)$ a pair of values. If $(a, b) \notin R(u, v)$, then $a, b \in D(u) \cap D(v)$ and $a \neq b$.

Proof. Refer to the values of $D(u) \cap D(v)$ as shared values, and other values of $D(u)$ and $D(v)$ as non-shared values. Since $D(u)$ and $D(v)$ is each a path in $T$, $R(u, v)$ is contained in the product of these paths, and the shared values (if any) induce a common path. Let $P_u$ be the path on vertices $D(u)$ and $P_v$ the path on vertices $D(v)$. Let $\omega_d$ denote the $d$-rooted weak tree submodularity multimorphism. Using the operation $\uparrow$ rooted in $d$, we find for any $d$ that $R(u, v)$ contains a pair $(a, b)$ where $a \in D(u)$ and $b \in D(v)$ are both chosen as the point closest to $d$. We refer to this as the $d$-closest pair. We first handle the case that there are no shared values.

Claim. If there are no shared values, then $R(u, v) = D(u) \times D(v)$.

Proof. Let $p \in P_u$ and $q \in P_v$ be the points on the respective path closest to the other path. By considering the $p$-closest pair in $R(u, v)$, we find $(p, q) \in R(u, v)$. Now let $a \in D(u)$ and $b \in D(v)$ be arbitrary. By considering the $a$-closest and $b$-closest pairs in $R(u, v)$, we find $(a, q), (p, b) \in R(u, v)$. But then $\omega_p$ on this pair produces $(a, b) \in R(u, v)$.

Next, assume that there are shared values, and let $P_1$ be the path induced on these values. Note that $(r, r) \in R(u, v)$ for every $r \in V(P_1)$, by considering the $r$-closest pair. We also have the following.

Claim. Every vertex $r \in D(u) \cap D(v)$ is compatible with every vertex of $D(u) \triangle D(v)$, i.e., $(a, r) \in R(u, v)$ for every $a \in D(u) \setminus D(v)$, and $(r, b) \in R(u, v)$ for every $b \in D(v) \setminus D(u)$.

Proof. Let $r_0$ and $r_n$ be the endpoints of $P_1$. The vertices of $D(u) \triangle D(v)$ are split into those closest to $r_0$ and those closest to $r_n$. We first note that every vertex $a \in D(u) \triangle D(v)$ is compatible with that endpoint it is closest to, by considering the $a$-closest pair in $R(u, v)$. Next, let $a \in D(u) \setminus D(v)$ be a point closer to $r_0$ than $r_n$, and write $P_1 = r_0r_1 \ldots r_n$. We claim by induction that $(a, r_i) \in R(u, v)$ for every $i = 0, \ldots, n$. As a base case, we already have $(a, r_0) \in R(u, v)$. The inductive step then goes as follows:

1. Assume $(a, r_i) \in R(u, v)$ for some $i < n$. Then $(a, r_i)$ and $(r_{i+1}, r_{i+1})$ via $\omega_{r_i}$ produces $(a', r_{i+1})$ where $a'$ is the vertex following $a$ on $P_u$ and may be $a' = r_0$

2. $(a, r_i), (a', r_{i+1})$ via $\omega_{r_i}$ produces $(a, r_{i+1})$

By induction we then get $(a, r) \in R(u, v)$ for every $r \in V(P_1)$, and the rest of the cases of the claim follow by symmetry.

The rest of the proof is now easy. Let $a \in D(u) \setminus D(v)$ and $b \in D(v)$; the other case is symmetric. If $b$ is a shared value, then $(a, b) \in R(u, v)$ by the above. Otherwise, let $r$ be an arbitrary shared value; then $(a, r), (r, b) \in R(u, v)$ and $\omega_r$ produces $(a, b) \in R(u, v)$.

Hence the only excluded pairs are on shared values. Finally, $(r, r) \in R(u, v)$ for every shared value $r$ by considering the $r$-closest pair.

We also need the following standard result.
Lemma 5.15. Let $\lambda_0$ be a partial labelling on a set of vertices $U \subseteq V$. Then there exists a labelling $\lambda \in \text{opt}$ that extends $\lambda_0$ if and only if $\langle \lambda_0(u), \lambda_0(v) \rangle \in \mathcal{R}(u, v)$ for every pair $u, v \in U$.

Proof. Define the operation $m(a, b, c) = ((a \uparrow b) \downarrow (a \uparrow c)) \downarrow (b \uparrow c)$ (for an arbitrary choice of root). Then $m$ preserves $\text{opt}$, and it is readily verified that $m$ is a majority operation, i.e., $m(x, x, y) = m(x, y, x) = m(y, y, x)$. It follows that $\text{opt}$ is characterized by its binary projections. 

This gives us the following algorithmic consequence.

Lemma 5.16. There is a labelling $\lambda \in \text{opt}$ such that for every variable $v$ with $D_T(v)$ non-empty, we have $\lambda(v) \in D_T$.

Proof. Let $U = \{v \in V \mid D_T(v) \neq \emptyset\}$ be the set of vertices with at least one integral value in the support. By Lemma 5.15 it suffices to produce a partial labelling $\lambda_0: U \rightarrow D_T$ such that every binary projection of $\lambda_0$ is supported by $\text{opt}$, i.e., for every pair of distinct vertices $u, v \in U$ we have $\langle \lambda_0(u), \lambda_0(v) \rangle \in \mathcal{R}(u, v)$. For this, define an arbitrary total order on $D_T$, and define $\lambda_0$ by selecting for every $v \in U$ the value of $D_T(v)$ that is earliest according to this order. Then for every pair $u, v \in U$ either $\lambda_0(u) = \lambda_0(v)$ or one of the values $\lambda_0(u), \lambda_0(v)$ is not shared in $\mathcal{R}(u, v)$. In both cases, by Lemma 5.14 we have $\langle \lambda_0(u), \lambda_0(v) \rangle \in \mathcal{R}(u, v)$.

Let us for reusability spell out the explicit assumptions and requirements made until now.

Theorem 5.17. Let $I = (G = (V, E), \tau, \mu, q)$ be an instance of Zero Extension with no isolated vertices and where every connected component of $G$ contains at least two terminals, and where $\mu$ is an induced tree metric for some tree $T$ and integral nodes $D_T \subseteq T$. Additionally, assume a collection of cost functions $F = \{f_i(v_i)\}_{i=1}^n$ has been given, where for every $f_i$ the scope is contained in $V$ and where $f_i$ is weakly tree submodular for every rooted version of $T$. Let $I'$ be the VCSP instance created from the sum of the cost functions of $I$ and $F$. Then $I'$ has the domain consistency property, i.e., there is an integral relaxed optimum if and only if every vertex $v$ is integral in at least one relaxed optimum of $I'$.

Proof. As noted in Lemma 5.13 due to the instance $I$ every vertex $v$ is either incident with at least one edge of $E$ that has non-zero length in at least one optimum, or it holds that $u = v$ in every optimal assignment, where $u$ is such a vertex. Hence $D(v)$ forms a path in $T$, and for every pair of variables $u, v \in V$ the conclusion of Lemma 5.14 applies, even for the projection $\mathcal{R}(u, v)$ of $\text{opt}(I')$ (as opposed to just $\text{opt}(I)$). The result follows as in Lemma 5.16.

5.5 Gap algorithms for general induced tree metrics

We now use the results of Theorem 5.17 to provide FPT algorithms parameterized by the gap parameter $k - \rho$.

Theorem 1.3. Let $I = (G = (V, E), \tau, \mu, q)$ be an instance of Zero Extension where $\mu$ is an induced tree metric on a set of labels $D$ in a tree $T$, and let $\hat{I} = (G, \tau, \hat{\mu}, q)$ be the relaxed instance. Let $p = \text{cost}(\hat{I})$. Then we can solve $I$ in time $O(|D|^{q-k}|T||D|^{nm})$.

Proof. This algorithm is similar to the algorithm for a leaf metric, except that we are not as easily able to test whether every variable has an integral value in $\text{opt}$. By the results of Section 5.1 the value of $\text{opt}$ is witnessed by the collection of min-cuts for edges in $T$; we will use this as a value oracle for $I$. We initially compute a max-flow across every edge of $T$, then for every assignment made we can compute the new value of $\text{opt}$ using $O(|T|)$ calls to augmenting path algorithms. This allows us to test for optimality of an assignment in $O(|T|^{nm})$ time. The branching step then in general iterates over at most $n$ variables, testing at most $|D|$ assigned values for each, and testing for optimality each time. Hence the local work in a single node of the branching tree is $O(|T|\cdot|D|^{nm})$. This either produces a variable for branching on or (by Theorem 5.17) produces an integral assignment, and in each branching step the value of $\rho$ increases but $q$ does not. The time for the initial max-flow computation is eaten by the factor $|T|^{nm}$. The result follows.
For **Metric Labelling**, we first need to restrict the unary cost functions to be weakly tree-submodular.

**Lemma 5.18.** Let \( f : V(T) \to \mathbb{R} \) be a unary function on a tree \( T \). Then \( f \) is weakly tree submodular on \( T \) for every choice of root \( r \in V_T \) if and only if it observes the following interpolation property: for any nodes \( u, v \in V(T) \), at distance \( d_T(u, v) = d \), and every \( i \in [d - 1] \), let \( w_i \) be the node on \( T[u, v] \) satisfying \( d_T(u, w_i) = i \). Then for any such choice of \( u, v \) and \( i \), it holds that \( f(w_i) \leq ((d - i)/d)f(u) + (i/d)f(v) \).

**Proof.** On the one hand, assume that \( f \) has the gradient property, and let \( u, v \in V(T) \) and let \( r \in V(T) \) be a choice of root. We need to show that \( f \) is preserved by the \( r \)-rooted weakly tree submodular multimorphism. If \( r \) does not lie on \( T[u, v] \), or \( r \in \{u, v\} \), then this is vacuous. Otherwise, let \( ^\uparrow(u, v) \) be the \( i \)th node of \( T[u, v] \), \( i \in [d - 1] \). Then we need to show

\[
 f(u) + f(v) \geq f(w_i) + f(w_{d-i}),
\]

which clearly holds. On the other hand, let \( u, v \in V(T) \) and \( i \in [d_T(u, v) - 1] \) be such that the interpolation property does not hold. Let \( w_a \), \( a < i \) be the last vertex before \( w_i \) on \( T[u, v] \) such that the interpolation equality holds for \( u, v \) and \( a \), and let \( w_b \), \( b > i \) be the first vertex after \( w_i \) for which it holds. These vertices clearly exist, possibly with choices \( w_a = u \) and \( w_b = v \). Then \( w_a \) and \( w_b \) are a witness that the \( w_i \)-rooted weak tree submodularity is not a multimorphism of \( f \).

In particular, let \( f_0 : U \to \mathbb{Z}^+ \) be a non-negative function defined on a subset \( U \) of the nodes of a tree \( T \), and say that \( f_0 \) admits an interpolation on \( T \) if there is an extension \( f : V(T) \to \mathbb{Z}^+ \) with the interpolation property such that \( f(v) = f_0(v) \) for every \( v \in U \). In particular, if \( U \) is the set of leaves of \( T \), then every function \( f_0 \) admits an interpolation by simply padding with zero values (although stronger interpolations are in general both possible and desirable).

**Theorem 1.5.** Let \( I = (G, \sigma, \mu, q) \) be an instance of **Metric Labelling** where \( \mu \) is an induced tree metric for a tree \( T \) and a set of nodes \( D \subseteq V(T) \), and where every unary cost \( \sigma(v, \cdot) \) admits an interpolation on \( T \). Let \( I = (G, \bar{\sigma}, \bar{\mu}, q) \) be the relaxed instance, and let \( \rho = \text{cost}(\hat{I}) \). Then the instance \( I \) can be solved in time \( O^*(|D|q^{n-\rho}) \). In particular, this applies for any \( \sigma \) if \( D \) is the set of leaves of \( T \).

**Proof.** Assume that \( G \) is connected, or else repeat the below for every connected component of \( G \). Select two arbitrary vertices \( u, v \in V \) and exhaustively guess their labels; in the case that you guess them to have the same label, identify the vertices in \( I \) (adding up their costs in \( \sigma \)) and select a new pair to guess on. Note that this takes at most \( O(|D|^2n) \) time, terminating whenever you have guessed more than one label in a branch or when you have guessed that all vertices are to be identical. This guessing phase can only increase the value of \( \rho \). We may now treat \( u \) and \( v \) as terminals, and the instance \( I \) as the sum of a **Zero Extension** instance on those two terminals and a collection of additional unary cost functions \( \sigma(v', \cdot) \), as in **Theorem 5.17**. Note that the resulting VCSP is tractable, i.e., the value of an optimal solution can be computed in polynomial time. The running time from this point on consists of iterating through all variables verifying whether each one has an integral value in some optimal assignment, and branching exhaustively on its value if not.

In particular, as noted, for a leaf metric \( \mu \) the algorithm applies without any assumptions on \( \sigma \) (and without \( T \) being explicitly provided).

### 6 Conclusions

We have given a range of algorithmic results for the **Zero Extension** and **Metric Labelling** problems from a perspective of parameterized complexity.

Most generally, we showed that **Zero Extension** is FPT parameterized by the number of **crossing edges** of an optimal solution, i.e. the number of edges whose endpoints receive
distinct labels, for a very general class of cost functions \( \mu \) that need not even be metrics. This is a relatively straight-forward application of the technique of recursive understanding [4].

For the more reasonable case that \( \mu \) is a metric, i.e. observing the triangle inequality, we gave two stronger results for the same parameter. First, we showed a linear-time FPT algorithm, with a better parameter dependency, using an important separators-based algorithm. Second, and highly surprisingly, we show that every graph \( G \) with a terminal set \( S \) admits a polynomial-time computable, polynomial-sized metric sparsifier \( G' \), with \( O(k+1) \) edges, such that \( (G', S) \) mimics the behaviour of \( (G, S) \) over any metric on at most \( s \) labels (up to solutions with crossing number \( k \)). This is a direct and seemingly far-reaching generalization of the polynomial kernel for \( s \)-Multiway Cut [19], which corresponds to the special case of the uniform metric.

Finally, we further developed the toolkit of discrete relaxations to design FPT algorithms under a gap parameter for Zero Extension and Metric Labelling where the metric is an induced tree metric (i.e. a restriction of a tree metric to a subset of the values). This in particular involves a more general FPT algorithm approach, supported by an applicability condition of domain consistency, relaxing the previously used persistence condition.

Let us highlight some questions. First, is there a lower bound on the size of a metric sparsifier for \( s \) labels for Zero Extension? This is particularly relevant since the existence of a polynomial kernel for \( s \)-Multiway Cut whose degree does not scale with \( s \) is an important open problem, and since the metric sparsifier is a more general result.

Second, can the FPT algorithms for induced tree metrics parameterized by the relaxation gap be generalised to restrictions of other tractable metrics, such as graph metrics for median graphs or the most general tractable class of orientable modular graphs [11]? Complementing this, what are the strongest possible gap parameters that allow FPT algorithms for metrics that are either arbitrary, or do not explicitly provide their relaxation?

More broadly, we also ask how far the method of discrete relaxations stretches in general. Let Planning VCSP be the class of problems defined by a (presumably tractable) integer-valued language \( \Gamma \) on a domain \( D \), and a subset \( D_I \subseteq D \) of integral values in the domain, where the problem is equivalent to the VCSP on \( \Gamma \) restricted to the domain \( D_I \), but where we ask for an FPT algorithm parameterized by the corresponding relaxation gap. Can it be characterized for which \( \Gamma \) and \( D_I \) this problem is FPT, and/or for which cases the domain consistency property holds?

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