STOKES MATRICES FOR THE QUANTUM COHOMOLOGIES OF
ORBIFOLD PROJECTIVE LINES

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Abstract. We prove the Dubrovin’s conjecture for the Stokes matrices for the quantum
cohomology of orbifold projective lines. The conjecture states that the Stokes matrix of
the first structure connection of the Frobenius manifold constructed from the Gromov-
Witten theory coincides with the Euler matrix of a full exceptional collection of the
bounded derived category of the coherent sheaves. Our proof is based on the homological
mirror symmetry, primitive forms of affine cusp polynomials and the Picard-Lefschetz
theory.

1. Introduction

For a smooth projective variety $X$ over $\mathbb{C}$, the quantum cohomology ring of $X$ is
defined as a generalization of usual cohomology ring. The quantum cohomology ring
coincide with the usual cohomology ring as a vector space, but the product structure is
“quantum corrected” from the usual cup product, by counting the number of holomorphic
curves in $X$ hitting the cycles Poincaré dual to the cohomology classes. Such counting
numbers are called the Gromov-Witten invariants of $X$. This idea comes from physics and
attracted the interests of mathematicians because of spectacular predictions for classical
enumeration problems in algebraic geometry through the mirror symmetry. Now the
theory of quantum cohomology and the Gromov-Witten invariants are extended to cases
when $X$ is an orbifold [1, 5].

The theory of Frobenius manifold formulated by Dubrovin [6], which first appeared
as the flat structure in K. Saito’s study of the deformation space of an isolated hypersurface
singularity, enable us to treat quantum cohomology systematically. A Frobenius manifold
is a complex (formal) manifold whose tangent space at any point has a bilinear form
and an associative commutative product with certain compatibility conditions. From
these compatibility conditions, it turns out to be that, the structure constants of the
product structure are given by third derivatives of a function on the Frobenius manifold.
The function is called Frobenius potential. The quantum cohomology of $X$ satisfies the
axioms of Frobenius manifolds, and its Frobenius potential is the generating function of
the genus zero Gromov-Witten invariants of $X$. 

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The above compatibility conditions give a property of integrable systems to Frobenius manifolds. Namely, for any Frobenius manifold $M$, we can construct a flat connection on the tangent bundle of $M \times \mathbb{P}^1$, called the first structure connection of $M$. The differential equation satisfied by the flat sections of the first structure connection can be regarded as an isomonodromic family of a meromorphic ordinary differential equation on $\mathbb{P}^1$, parametrized by points of $M$. The equation has two singular points on $\mathbb{P}^1$; one is a regular singular point and the other one is an irregular singular point. If a point on $M$ is semi-simple, that is, if there are no nilpotent elements in the product structure on the tangent space at the point, then one can define the monodromy data of the differential equation at the point, consisting of the monodromy matrix at the regular singular point, the Stokes matrix at the irregular singular point, and the connection matrix between these two singular points. These data do not depend on the choice of a semi-simple point due to the isomonodromy property. Moreover, as is shown in [8], from these data we can reconstruct the Frobenius structure by the Riemann-Hilbert correspondence.

After the Zaslow’s work [24], Dubrovin formulated a conjecture in [7] about a close relationship between the monodormy data of the Frobenius manifold constructed from the Gromov-Witten theory of $X$ and the structure of the bounded derived category of coherent sheaves on $X$. To state his conjecture, we recall some notions here.

**Definition 1.1.** Let $\mathcal{T}$ be a $\mathbb{C}$-linear triangulated category $\mathcal{T}$ with a translation functor $T$.

(i) An object $E$ in $\mathcal{T}$ is called an exceptiona object (or is called exceptional) if $\mathcal{T}(E, E) = \mathbb{C} \cdot \text{id}_E$ and $\mathcal{T}(E, T^pE) = 0$ when $p \neq 0$.

(ii) An exceptional collection in $\mathcal{T}$ is an ordered set $(E_1, \ldots, E_\mu)$ of exceptional objects satisfying the condition $\mathcal{T}(E_i, T^pE_j) = 0$ for all $p \in \mathbb{Z}$ and $i > j$.

(iii) An exceptional collection $(E_1, \ldots, E_\mu)$ in $\mathcal{T}$ is called full if the smallest full triangulated subcategory of $\mathcal{T}$ containing $E_1, \ldots, E_\mu$ is equivalent to $\mathcal{T}$.

**Definition 1.2.** Let $\mathcal{T}$ be a $\mathbb{C}$-linear triangulated category $\mathcal{T}$ with a translation functor $T$. Assume that $\mathcal{T}$ is finite, namely, for all objects $E, E' \in \mathcal{T}$ one has

\[ \sum_{p \in \mathbb{Z}} \dim_{\mathbb{C}} \mathcal{T}(E, T^pE') < \infty. \tag{1.1} \]

(i) Let $K_0(\mathcal{T})$ be the Grothendieck group of $\mathcal{T}$. The pairing $\chi : K_0(\mathcal{T}) \times K_0(\mathcal{T}) \to \mathbb{Z}$ defined by

\[ \chi ([E], [E']) := \sum_{p \in \mathbb{Z}} (-1)^p \dim_{\mathbb{C}} \mathcal{T}(E, T^pE') \tag{1.2} \]

is called the Euler pairing.
(ii) Suppose that $\mathcal{T}$ is generated by a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_\mu)$. We shall call the $\mu \times \mu$-matrix $\chi = (\chi_{ij})$, $\chi_{ij} = \chi([\mathcal{E}_i], [\mathcal{E}_j])$ the Euler matrix of the exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_\mu)$.

Then, (a part of) the Dubrovin’s conjecture is formulated as follows:

**Conjecture 1.3** (7). Let $X$ be a smooth projective variety over $\mathbb{C}$ and let $M$ be the Frobenius manifold constructed from the Gromov–Witten theory for $X$

(i) The Frobenius manifold $M$ is semi-simple if and only if the bounded derived category $D^b\text{coh}(X)$ of coherent sheaves on $X$ admits a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_\mu)$ where $\mu := \sum_{p \in \mathbb{Z}} \dim \mathbb{C}H^{p,p}(X)$.

(ii) Fix a semi-simple point $t \in M$ such that the values of the canonical coordinates at the point $t$ are pairwise distinct. Then the Stokes matrix of the first structure connection of the Frobenius structure defined at the point $t$ is identified with the Euler matrix of an exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_\mu)$ in $D^b\text{coh}(X)$.

The conjecture is proved for some examples of smooth projective varieties (cf. [8, 11, 22, 23]).

In this paper, we shall consider a generalization of Dubrovin’s conjecture to an orbifold projective line $\mathbb{P}^1_A := \mathbb{P}^1(a_1,a_2,a_3)$, an orbifold $\mathbb{P}^1$ with at most three isotropic points of orders $a_1, a_2, a_3$ satisfying the condition $\chi_A := 1/a_1 + 1/a_2 + 1/a_3 - 1 > 0$. The existence of a full exceptional collection in $D^b\text{coh}(\mathbb{P}^1_A)$ is already shown by Geigle–Lenzing (Proposition 4.1 in [10]). The semi-simplicity of the quantum cohomology ring of $\mathbb{P}^1_A$ is a corollary (see Corollary 3.12 below) of the classical mirror symmetry, the isomorphism of Frobenius manifolds between the one constructed from Gromov–Witten theory for $\mathbb{P}^1_A$ and the one constructed from the theory of primitive forms for the polynomial $f_A := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - x_1x_2x_3$.

The following is our main theorem:

**Theorem 1.4.** Let $M_{\mathbb{P}^1_A}$ be the Frobenius manifold constructed from the Gromov–Witten theory for $\mathbb{P}^1_A$. Fix a semi-simple point $t \in M_{\mathbb{P}^1_A}$ such that the values of the canonical coordinates at the point $t$ are pairwise distinct. Then the Stokes matrix of the first structure connection of the Frobenius structure defined at the point $t$ is identified with the Euler matrix of a full exceptional collection in $D^b\text{coh}(\mathbb{P}^1_A)$.

The proof is based on the homological mirror symmetry; an triangulated equivalence between $D^b\text{coh}(\mathbb{P}^1_A)$ and $D^b\text{Fuk}^\rightarrow(f_A)$. The latter is the bounded derived category of the directed Fukaya category $\text{Fuk}^\rightarrow(f_A)$ of $f_A$. A full exceptional collection of the $D^b\text{Fuk}^\rightarrow(f_A)$
is given by the vanishing cycles in the fiber of a suitable deformation of $f_A$, and the Euler forms of them can be written in terms of intersection numbers of these vanishing cycles multiplied by $-1$. On the other hand, due to the classical mirror symmetry, the oscillatory integral of the primitive form for $f_A$ along a certain cycle called Lefschetz thimble, gives a flat sections of the first structure connection of the Frobenius manifold constructed from $\mathbb{P}^1_A$. Using the idea of [23] and the Picard-Lefschetz theory, we can compute the Stokes matrix of the oscillatory integrals by the intersection numbers of vanishing cycles. Thus we obtain the main theorem.

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2. Preliminary

2.1. Definition of the Frobenius manifold. In this section, we recall the definition and some basic properties of the Frobenius manifold [6]. The definition below is taken from Saito-Takahashi [16].

**Definition 2.1.** Let $M = (M, \mathcal{O}_M)$ be a connected complex manifold or a formal manifold over $\mathbb{C}$ of dimension $\mu$ whose holomorphic tangent sheaf and cotangent sheaf are denoted by $\mathcal{T}_M, \mathcal{O}^1_M$, respectively and $d$ be a complex number.

A Frobenius structure of rank $\mu$ and dimension $d$ on $M$ is a tuple $(\eta, \circ, e, E)$, where $\eta$ is a non-degenerate $\mathcal{O}_M$-symmetric bilinear form on $\mathcal{T}_M$, $\circ$ is $\mathcal{O}_M$-bilinear product on $\mathcal{T}_M$, defining an associative and commutative $\mathcal{O}_M$-algebra structure with the unit $e$, and $E$ is a holomorphic vector field on $M$, called the Euler vector field, which are subject to the following axioms:

(i) The product $\circ$ is self-ajoint with respect to $\eta$: that is,

$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M.$$

(ii) The Levi–Civita connection $\nabla : \mathcal{T}_M \otimes \mathcal{O}_M \mathcal{T}_M \to \mathcal{T}_M$ with respect to $\eta$ is flat: that is,

$$[\nabla_\delta, \nabla_{\delta'}] = \nabla_{[\delta, \delta']}, \quad \delta, \delta' \in \mathcal{T}_M.$$

(iii) The tensor $C : \mathcal{T}_M \otimes \mathcal{O}_M \mathcal{T}_M \to \mathcal{T}_M$ defined by $C_\delta \delta' := \delta \circ \delta'$, $(\delta, \delta' \in \mathcal{T}_M)$ is flat: that is,

$$\nabla C = 0.$$
(iv) The unit element $e$ of the $\circ$-algebra is a $\nabla$/flat homolophic vector field: that is, $\nabla/e = 0$.

(v) The metric $\eta$ and the product $\circ$ are homogeneous of degree $2 - d$ ($d \in \mathbb{C}$), 1 respectively with respect to Lie derivative $\text{Lie}_E$ of Euler vector field $E$: that is, $\text{Lie}_E(\eta) = (2 - d)\eta$, $\text{Lie}_E(\circ) = \circ$.

A manifold $M$ equipped with a Frobenius structure $(\eta, \circ, e, E)$ is called a Frobenius manifold.

From now on in this section, we shall always denote by $M$ a Frobenius manifold. We expose some basic properties of Frobenius manifolds without their proofs.

Let us consider the space of horizontal sections of the connection $\nabla$/:

$$\mathcal{T}_M^f := \{ \delta \in \mathcal{T}_M \mid \nabla_/\delta' = 0 \text{ for all } \delta' \in \mathcal{T}_M \}$$

which is a local system of rank $\mu$ on $M$ such that the metric $\eta$ takes constant value on $\mathcal{T}_M^f$. Namely, we have

$$\eta(\delta, \delta') \in \mathbb{C}, \quad \delta, \delta' \in \mathcal{T}_M^f. \quad (2.1)$$

**Proposition 2.2.** At each point of $M$, there exist a local coordinate $(t_1, \ldots, t_\mu)$, called flat coordinates, such that $e = \partial_1$, $\mathcal{T}_M^f$ is spanned by $\partial_1, \ldots, \partial_\mu$ and $\eta(\partial_i, \partial_j) \in \mathbb{C}$ for all $i, j = 1, \ldots, \mu$, where we denote $\partial/\partial t_i$ by $\partial_i$.

The axiom $\nabla/C = 0$, implies the following:

**Proposition 2.3.** At each point of $M$, there exist the local holomorphic function $F$, called Frobenius potential, satisfying

$$\eta(\partial_i \circ \partial_j, \partial_k) = \eta(\partial_i, \partial_j \circ \partial_k) = \partial_i \partial_j \partial_k F, \quad i, j, k = 1, \ldots, \mu,$$

for any system of flat coordinates. In particular, one has

$$\eta_{ij} := \eta(\partial_i, \partial_j) = \partial_i \partial_j F.$$

The product structure on $\mathcal{T}_M$ is described locally by $F$ as

$$\partial_i \circ \partial_j = \sum_{k=1}^{\mu} c_{ij}^k \partial_k \quad i, j = 1, \ldots, \mu, \quad (2.2)$$

$$c_{ij}^k := \sum_{l=1}^{\mu} \eta^{kl} \partial_i \partial_j \partial_l F, \quad (\eta^{ij}) = (\eta_{ij})^{-1}, \quad i, j, k = 1, \ldots, \mu. \quad (2.3)$$
2.2. First structure connection of the Frobenius manifold. For a Frobenius manifold $M$, one can associate a connection $\hat{\nabla}$ on $T_{\mathbb{P}^1 \times M}$ [6]:

\[
\hat{\nabla}_{\delta'} \delta = \nabla_{\delta'} \delta - \frac{1}{u} \delta \circ \delta', \quad \delta, \delta' \in \mathcal{T}_M, \quad (2.4a)
\]

\[
\hat{\nabla}_{u \frac{d}{du}} \delta = \frac{1}{u} \mathcal{U}(\delta) - \mathcal{V}(\delta), \quad \delta \in \mathcal{T}_M, \quad (2.4b)
\]

\[
\hat{\nabla}_{\delta} \frac{d}{du} = \hat{\nabla}_{\frac{d}{du}} \frac{d}{du} = 0, \quad \delta \in \mathcal{T}_M. \quad (2.4c)
\]

Here $u$ is the coordinate of $\mathbb{P}^1$, and $\mathcal{U}$, $\mathcal{V}$ are following $\mathcal{O}_M$-linear operators acting on $\mathcal{T}_M$:

\[
\mathcal{U}(\delta) = E \circ \delta, \quad \mathcal{V}(\delta) = \nabla_{\delta} E - \frac{2 - d}{2} \delta. \quad (2.5)
\]

**Proposition 2.4** (Proposition 3.1 in [6]). The connection $\hat{\nabla}$ is flat.

Note that the parameter $z$ in [6] is $1/u$ in this paper. The flat connection $\hat{\nabla}$ is called the first structure connection of the Frobenius manifold $M$.

Let $\varphi = \varphi(t, u)$ be a function on an open subset in $M \times \mathbb{P}^1$. We say that $\varphi$ is $\hat{\nabla}$-flat if it satisfies

\[
\hat{\nabla} \left( \sum_{i=1}^{\mu} u \frac{\partial \varphi}{\partial t_i} dt_i \right) = 0
\]

under the identification $\mathcal{T}_M \cong \Omega^1_M$ by the non-degenerate bilinear form $\eta$. That is, the gradient $\Phi = \nabla (u \partial_1 \varphi, \ldots, u \partial_\mu \varphi)$ of a $\hat{\nabla}$-flat function $\varphi$ satisfies

\[
\frac{\partial}{\partial t_i} \Phi = \frac{1}{u} C_i \Phi, \quad i = 1, \ldots, \mu, \quad (2.6a)
\]

\[
u \frac{d}{du} \Phi + \frac{1}{u} \mathcal{U} \Phi - \mathcal{V} \Phi = 0. \quad (2.6b)
\]

Here $(C_i)_{jk} = c_{ij}^k$, $\mathcal{U}$ and $\mathcal{V}$ are the matrices representing the $\mathcal{O}_M$-linear operators $\mathcal{U}$ and $\mathcal{V}$ respectively in the above identification:

\[
\mathcal{U}(dt_i) = \sum_{k=1}^{\mu} U_{ki} dt_k, \quad \mathcal{V}(dt_i) = \sum_{k=1}^{\mu} V_{ki} dt_k, \quad i = 1, \ldots, \mu. \quad (2.7)
\]

The equation (2.6b) can be considered as a family of meromorphic differential equations on $\mathbb{P}^1$ parametrized by points on $M$, and Proposition 2.4 implies that this family is isomonodromic. The equation has a regular singular point at $u = \infty$ and an irregular singular point of Poincaré rank one at $u = 0$. 
2.3. **Semi-simple Frobenius manifold and the canonical coordinate.** In this section we recall the notion of semi-simple Frobenius manifolds.

**Definition 2.5.** Let \( M \) be a Frobenius manifold.

(i) A point on \( t \in M \) is called **semi-simple** if there are no nilpotent elements in the product \( \circ \) on the tangent space \( T_t M \).

(ii) A Frobenius manifold is called **semi-simple** if general points are semi-simple.

(iii) The set \( K := \{ t \in M \mid t \text{ is not semi-simple} \} \) is called the set of **caustics** of the Frobenius manifold \( M \).

The semi-simplicity is an open property, namely, the set \( M \setminus K \) is an open subset of \( M \).

**Proposition 2.6** (Theorem 3.1 in [8]). Near a semi-simple point the eigenvalues \( \{ w_a \}_{a=1}^\mu \) of the matrix \( U \) give local coordinates of \( M \setminus K \). They satisfy the following:

\[
\frac{\partial}{\partial w_a} \circ \frac{\partial}{\partial w_b} = \delta_{ab} \frac{\partial}{\partial w_a}, \quad a, b = 1, \ldots, \mu, \tag{2.8a}
\]

\[
e = \sum_{a=1}^\mu \frac{\partial}{\partial w_a}, \tag{2.8b}
\]

\[
E = \sum_{a=1}^\mu w_a \frac{\partial}{\partial w_a}. \tag{2.8c}
\]

**Definition 2.7.** The local coordinates \( (w_1, \ldots, w_\mu) \) of \( M \setminus K \) are called the **canonical coordinates** of the Frobenius manifold \( M \).

We recall the following important fact:

**Proposition 2.8** (Corollary 3.1 in [8]). All the points \( t \in M \) where the eigenvalues \( \{ w_a \}_{a=1}^\mu \) of the matrix \( U \) are pairwise distinct are semi-simple.

Set \( \mathcal{B} := \{ t \in M \mid \text{some of eigenvalues of the matrix } U \text{ coincide} \} \) and call \( \mathcal{B} \) the **bifurcation set** of the Frobenius manifold \( M \). It follows from the above Proposition that \( (M \setminus \mathcal{B}) \subset (M \setminus K) \).

2.4. **Stokes matrix of the first structure connection.** Fix a point on \( M \setminus \mathcal{B} \), where the values of canonical coordinates are pairwise distinct. Define the \( \mu \times \mu \) matrix \( \Psi = (\Psi_{ai})_{a,i=1,\ldots,\mu} \) by

\[
\Psi_{ai} := \frac{\partial w_a}{\partial t_i} \cdot \eta \left( \frac{\partial}{\partial w_a}, \frac{\partial}{\partial w_a} \right)^{\frac{1}{2}}, \quad a, i = 1, \ldots, \mu. \tag{2.9}
\]

It follows from the equation (2.8c) that the matrix \( \Psi \) diagonalizes the matrix \( U \):

\[
\Psi^{-1}U \Psi = \text{diag}(w_1, \ldots, w_\mu). \tag{2.10}
\]
We can construct a formal solution of \((2.6b)\) near \(u = 0\) at the point as follows.

**Proposition 2.9** (Lemma 4.3 in [8]). There exists a formal matrix fundamental solution of the differential equation \((2.6b)\) in the form

\[
\Phi_{\text{formal}}(u) = \Psi G(u) \exp(W/u).
\]

Here \(\Psi\) is given by \((2.9)\), \(W = \text{diag}(w_1, \ldots, w_\mu)\) and \(G(u) = 1 + G_1 u + G_2 u^2 + \cdots\) is a \(\mu \times \mu\) matrix-valued formal power series satisfying

\[
G^T(-u)G(u) = 1.
\]

Such \(G(u)\) is unique.

Since \(u = 0\) is an irregular singular point of Poincaré rank one, the formal power series \(G(u)\) is a divergent power series in general.

**Definition 2.10.** For \(0 \leq \phi < \pi\), a line \(\ell = \{u \in \mathbb{C}^\times | \arg u = \phi, \phi - \pi\}\) is called admissible if the line through \(w_a\) and \(w_b\) is never orthogonal to \(\ell\) for any \(a, b \in \{1, \ldots, \mu\}\) with \(a \neq b\).

Fix such a line \(\ell\), and chose a small number \(\varepsilon > 0\) so that any line passing through the origin with angle between \(\phi - \varepsilon\) and \(\phi + \varepsilon\) is admissible. Define sectors in \(u\)-plane by

\[
D_{\text{right}} = \{u \in \mathbb{C}^\times | \phi - \pi - \varepsilon < \arg u < \phi + \varepsilon\},
\]

\[
D_{\text{left}} = \{u \in \mathbb{C}^\times | \phi - \varepsilon < \arg u < \phi + \pi + \varepsilon\}.
\]

The following statement is a consequence of the general theory for ordinary differential equations.

**Proposition 2.11** (e.g., Theorem A in [4]). There exists unique solutions \(\Phi_{\text{right/left}}\) of \((2.6b)\) analytic in \(u\) in the sectors \(D_{\text{right/left}}\) having the following asymptotic properties:

\[
\Phi_{\text{right}}(u) \sim \Phi_{\text{formal}}(u) \quad \text{as} \quad u \to 0, \quad u \in D_{\text{right}},
\]

\[
\Phi_{\text{left}}(u) \sim \Phi_{\text{formal}}(u) \quad \text{as} \quad u \to 0, \quad u \in D_{\text{left}}.
\]

In the sector

\[
D_+ = \{u \in \mathbb{C}^\times | \phi - \varepsilon < \arg u < \phi + \varepsilon\} \subset D_{\text{right}} \cap D_{\text{left}}
\]

we have two analytic solutions \(\Phi_{\text{right}}\) and \(\Phi_{\text{left}}\). They must be related as

\[
\Phi_{\text{left}}(u) = \Phi_{\text{right}}(u)S, \quad u \in D_+
\]

with a matrix \(S\) independent of \(u\). The following proposition is a consequence of the isomonodromic property of the differential equation \((2.6b)\).
Proposition 2.12 (Theorem 4.4 in [8]). The matrix $S$ is locally constant as a function on $M \setminus B$.

Definition 2.13. The matrix $S$ is called the Stokes matrix of the first structure connection of $M$ (for the admissible line $\ell$).

3. Mirror Isomorphism of Frobenius manifolds

Let $A := (a_1, a_2, a_3)$ be a triple of positive integers. Set
\[
\mu_A = 2 + \sum_{k=1}^{3} (a_k - 1),
\]
(3.1)
\[
\chi_A := 2 + \sum_{k=1}^{3} \left( \frac{1}{a_k} - 1 \right).
\]
(3.2)

We shall only consider $A$ satisfying the condition $\chi_A > 0$ in this paper.

3.1. Gromov–Witten theory for $\mathbb{P}^1_A$. Following Geigle–Lenzing (cf. Section 1.1 in [10]), we shall introduce orbifold projective lines. First, we prepare some notations.

Definition 3.1. Define a ring $R_A$ by
\[
R_A := \mathbb{C}[X_1, X_2, X_3] / I ,
\]
(3.3a)
where $I$ is an ideal generated by the homogeneous polynomial
\[
X_3^{a_3} - X_2^{a_2} + X_1^{a_1}.
\]
(3.3b)

Denote by $L_A$ an abelian group generated by three letters $\bar{X}_i$, $i = 1, 2, 3$ defined as the quotient
\[
L_A := \bigoplus_{i=1}^{3} \mathbb{Z}\bar{X}_i / M_A ,
\]
(3.4a)
where $M_A$ is the subgroup generated by the elements
\[
a_i\bar{X}_i - a_j\bar{X}_j, \quad 1 \leq i < j \leq 3.
\]
(3.4b)

We then consider the following quotient stack:

Definition 3.2. Define a stack $\mathbb{P}^1_A$ by
\[
\mathbb{P}^1_A := \left( \text{Spec}(R_A) \setminus \{0\} \right) / \text{Spec}(\mathbb{C} L_A) ,
\]
(3.5)
which is called the orbifold projective line of type $A$. 
An orbifold projective line of type $A$ is a Deligne–Mumford stack whose coarse moduli space is a smooth projective line $\mathbb{P}^1$.

For $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(\mathbb{P}^1_A, \mathbb{Z})$, the moduli space (stack) $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_A, \beta)$ of orbifold (twisted) stable maps of genus $g$ and call it the genus $g$ moduli space is a smooth projective line $\mathbb{P}^1$. We also consider the generating function $P_{\text{vir}}$ and call it the genus $g$ divisor axiom and we assume that there exists a virtual fundamental class $[\mathcal{M}_{g,n}(\mathbb{P}^1_A, \beta)]_{\text{vir}}$ and Gromov–Witten invariants of genus $g$ with $n$-marked points of degree $\beta$ are defined as usual except for that we have to use the orbifold cohomology group $H^*_\text{orb}(\mathbb{P}^1_A, \mathbb{Q})$:

$$\langle \Delta_1, \ldots, \Delta_n \rangle_{g,n,\beta}^{\mathbb{P}^1_A} : = \int_{[\mathcal{M}_{g,n}(\mathbb{P}^1_A, \beta)]_{\text{vir}}} ev^*_1 \Delta_1 \wedge \cdots \wedge ev^*_n \Delta_n, \quad \Delta_1, \ldots, \Delta_n \in H^*_\text{orb}(\mathbb{P}^1_A, \mathbb{Q}),$$

where $ev^*_i : H^*_\text{orb}(\mathbb{P}^1_A, \mathbb{Q}) \to H^*(\mathcal{M}_{g,n}(\mathbb{P}^1_A, \beta), \mathbb{Q})$ denotes the induced homomorphism by the evaluation map. We also consider the generating function

$$F_g^{\mathbb{P}^1_A} := \sum_{n,\beta} \frac{1}{n!} \langle \Delta_1, \ldots, \Delta_n \rangle_{g,n,\beta}^{\mathbb{P}^1_A}, \quad \Delta = \sum_{i=1}^{\mu_A} t_i \Delta_i$$

and call it the genus $g$ potential where $\{\Delta_1 = 1, \Delta_2, \ldots, \Delta_{\mu_A}\}$ denotes a $\mathbb{Q}$-basis of $H^*_\text{orb}(\mathbb{P}^1_A, \mathbb{Q})$. Note that $F_g^{\mathbb{P}^1_A}|_{t_1=0}$ is a polynomial in $t_2, \ldots, t_{\mu_A-1}, e^{t_{\mu_A}}$ since we have the divisor axiom and we assume that $\chi_A > 0$.

The Gromov–Witten theory for orbifolds developed by Abramovich–Graber–Vistoli [1] and Chen–Ruan [5] gives us the following.

**Proposition 3.3.** The genus zero potential $F_0^{\mathbb{P}^1_A}$ satisfies the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations. In particular, there exists a structure of a Frobenius manifold of rank $\mu_A$ and dimension one on $M := \mathbb{C}^{\mu_A-1} \times (\mathbb{C} \setminus \{0\})$ whose non-degenerate symmetric $\mathcal{O}_M$-bilinear form $\eta$ on $\mathcal{T}_M$ is given by the Poincaré pairing.

**Proof.** See Theorem 6.2.1 in [1] and Theorem 3.4.3 in [5]. \hfill \Box

For simplicity, we shall denote by $M_{\mathbb{P}^1_A}$ the complex manifold $M$ together with the Frobenius structure on $M$ obtained in Proposition 3.3 and call it the Frobenius manifold constructed from the Gromov–Witten theory for $\mathbb{P}^1_A$.

### 3.2. Theory of primitive forms for $f_A$.

Consider an cusp polynomial of type $A$, namely, a polynomial $f_A(x) \in \mathbb{C}[x_1, x_2, x_3]$ given as

$$f_A(x) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1} \cdot x_1 x_2 x_3 \quad (3.6)$$

for some $q \in \mathbb{C} \setminus \{0\}$. One can easily show that the $\mathbb{C}$-vector space

$$\mathbb{C}[x_1, x_2, x_3] / \left( \frac{\partial f_A}{\partial x_1}, \frac{\partial f_A}{\partial x_2}, \frac{\partial f_A}{\partial x_3} \right)$$
is of dimension $\mu_A$ since we assume that $\chi_A > 0$. We can consider the universal unfolding of $f_A$, a deformation $F_A$ of $f_A$ defined on $\mathbb{C}^3 \times M$, $M := \mathbb{C}^{\mu_A-1} \times (\mathbb{C} \setminus \{0\})$ over a $\mu_A$-dimensional parameters $(s, s_{\mu_A}) \in M$ given as follows:

**Definition 3.4.** Define a function $F_A(x; s, s_{\mu_A})$ defined on $\mathbb{C}^3 \times M$ as follows;

$$F_A(x; s, s_{\mu_A}) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - s_{\mu_A}^{-1} \cdot x_1x_2x_3 + s_1 \cdot 1 + \sum_{i=1}^{a_i-1} \sum_{j=1}^{a_i} s_{i,j} \cdot x_i^j.$$  \hfill (3.7)

Denote by

$$p : \mathbb{C}^3 \times M \rightarrow M, \quad (x; s, s_{\mu_A}) \mapsto (s, s_{\mu_A})$$

the projection map from the total space to the deformation space. Set

$$p_\ast \mathcal{O}_C := \mathcal{O}_M[x_1, x_2, x_3] / \left( \left( \frac{\partial F_A}{\partial x_1}, \frac{\partial F_A}{\partial x_2}, \frac{\partial F_A}{\partial x_3} \right) \right).$$  \hfill (3.8)

$p_\ast \mathcal{O}_C$ can be thought of as the direct image of the sheaf of relative algebraic functions on the relative critical set $C$ of $F_A$ with respect to the projection $p : \mathbb{C}^3 \times M \rightarrow M$.

**Proposition 3.5** (Proposition 2.4 in [13]). The function $F_A(x; s, s_{\mu_A})$ satisfies the following conditions:

(i) $F_A(x; 0, q) = f_A(x)$.

(ii) The $\mathcal{O}_M$-homomorphism $\rho$ called the Kodaira–Spencer map defined as

$$\rho : T_M \rightarrow p_\ast \mathcal{O}_C, \quad \delta \mapsto \delta F_A,$$  \hfill (3.9)

is an isomorphism.

**Definition 3.6.** We shall denote by $\circ$ the induced product structure on $T_M$ by the $\mathcal{O}_M$-isomorphism $3.9$. Namely, for $\delta, \delta' \in T_M$, we have

$$(\delta \circ \delta')F_A = \delta F_A \cdot \delta' F_A \text{ in } p_\ast \mathcal{O}_C.$$  \hfill (3.10)

**Definition 3.7.** The vector field $e$ and $E$ on $M$ corresponding to the unit 1 and $F$ by the $\mathcal{O}_M$-isomorphism $3.9$ is called the primitive vector field and the Euler vector field, respectively. That is,

$$eF_A = 1 \text{ and } EF_A = F_A \text{ in } p_\ast \mathcal{O}_C.$$  \hfill (3.11)

Note that the primitive vector field $e$ and the Euler vector field $E$ on $M$ are given by

$$e = \frac{\partial}{\partial s_1}, \quad E = s_1 \frac{\partial}{\partial s_1} + \sum_{i=1}^{a_i-1} \sum_{j=1}^{a_i} \frac{a_i - j}{a_i} s_{i,j} \frac{\partial}{\partial s_{i,j}} + \chi_A s_{\mu_A} \frac{\partial}{\partial s_{\mu_A}}.$$  \hfill (3.12)

One can then construct the filtered de Rham cohomology group $\mathcal{H}_{F_A}$ (whose increasing filtration is denoted by $\mathcal{H}_{F_A}^{(p)} (p \in \mathbb{Z})$), the Gauss–Manin connection $\nabla$ on $\mathcal{H}_{F_A}$ and the
higher residue pairings $K_{F_A}$ on $\mathcal{H}_{F_A}$, which are necessary to define a notion of a primitive form. In this paper, we omit the details about these objects and refer the interested reader to [13, 16].

A primitive form is obtained by Ishibashi–Shiraishi–Takahashi in [13]:

**Theorem 3.8 (Theorem 3.1 in [13]).** The element

$$\zeta_A := [s_{\mu_A}^{-1} dx_1 \wedge dx_2 \wedge dx_3] \in \mathcal{H}_{F_A}^{(0)}$$

(3.13)

is a primitive form for the tuple $(\mathcal{H}_{F_A}^{(0)}, \nabla, K_{F_A})$ with the minimal exponent $r = 1$.

Once we have a primitive form $\zeta_A$, we obtain a Frobenius structure on $M$ by the general theory developed by K. Saito.

**Corollary 3.9 (Corollary 3.2 in [13]).** The primitive form $\zeta_A$ determines a Frobenius structure of rank $\mu_A$ and dimension one on the deformation space $M$ of the universal unfolding of $f_A$. More precisely, the non-degenerate symmetric bilinear form $\eta$ on $\mathcal{T}_M$ defined by

$$\eta(\delta, \delta') := u^{-3} K_{F_A}(u \nabla_\delta \zeta_A, u \nabla_\delta' \zeta_A), \quad \delta, \delta' \in \mathcal{T}_M,$$

(3.14)

together with the product $\circ$ on $\mathcal{T}_M$, the primitive vector field $e \in \Gamma(M, \mathcal{T}_M)$ and the Euler vector field $E \in \Gamma(M, \mathcal{T}_M)$ define a Frobenius structure on $M$ of rank $\mu_A$ and dimension one.

For simplicity, we shall denote by $M_{f_A, \zeta_A}$ the deformation space $M$ together with the Frobenius structure on $M$ obtained in Corollary 3.9 and call it the *Frobenius manifold constructed from the pair $(f_A, \zeta_A)$*.

3.3. Mirror isomorphism.

**Theorem 3.10 (Corollary 4.5 in [13]).** There exists an isomorphism of Frobenius manifolds between $M_{P_{\mathbb{F}_A}^1}$ and $M_{f_A, \zeta_A}$.

**Remark 3.11.** A part of the statement is given by Milanov–Tseng [14] for the case $a_1 = 1$ and by Rossi [15] for the case $\chi_A > 0$.

**Corollary 3.12.** The quantum cohomology ring of the orbifold $\mathbb{P}_A^1$ is semi-simple.

**Proof.** The statement easily follows from the fact that the Frobenius structure constructed from the pair $(f_A, \zeta_A)$ is semi-simple whose canonical coordinates are given by the critical values. \qed
Let \( t = (t_1, \ldots, t_{\mu_A}) \) be the flat coordinate of \( M_{\mathrm{P}^1_A} \) such that 
\[
E = \sum_{i=1}^{\mu_A} E_i \frac{\partial}{\partial t_i} = \sum_{i=1}^{\mu_A} (1-q_i)t_i \frac{\partial}{\partial t_i} + \chi_A \frac{\partial}{\partial t_{\mu_A}}
\]
where \( q_1 = 0 \). In the above flat coordinate of \( M_{\mathrm{P}^1_A} \) the matrices in (2.7) are given by
\[
U_{ij} = \sum_{k=1}^{\mu_A} E_k c_{ik}^j, \quad V_{ij} = \left( q_i - \frac{1}{2} \right) \delta_{ij}, \quad i, j = 1, \ldots, \mu_A.
\]  
(3.15)
See Section 2 in [8] for detail. Another important corollary of Theorem 3.8 is the following:

**Corollary 3.13.** Let \( \hat{\nabla} \) be the first structure connection of the Frobenius manifold \( M_{\mathrm{P}^1_A} \). On a neighborhood of a semi-simple point of \( M_{\mathrm{P}^1_A} \), the oscillatory integral
\[
I(t, u) = (2\pi u)^{-\frac{3}{2}} \int_{\Gamma(u)} e^{F_A(x; t, e^{t_{\mu_A}})/u} \zeta_A
\]  
(3.16)
gives a \( \hat{\nabla} \)-flat function, where \( t = (t_1, \ldots, t_{\mu_A-1}) \). Here \( \Gamma(u) \in H_3(\mathbb{C}^3; \operatorname{Re}(F_A(\bullet; t, e^{t_{\mu_A}})/u) \ll 0; \mathbb{Z}) \) is a Lefschetz thimble defined in Section 5.

**Proof.** Due to the definition (P4) of the primitive form \( \zeta_A \) in Definition 2.38 in [13] and Lemma 3.4 in [13], we can show that the oscillatory integral (3.16) satisfies the following system of differential equations:
\[
\begin{align*}
&u \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} I = \left( \frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} \right) I, \quad i, j = 1, \ldots, \mu_A. \tag{3.17a}

&\left( u \frac{d}{du} + E \right) \left( u \frac{\partial}{\partial t_i} I \right) = \left( q_i - \frac{1}{2} \right) \left( u \frac{\partial}{\partial t_i} I \right), \quad i = 1, \ldots, \mu_A. \tag{3.17b}
\end{align*}
\]
The differential equation (3.17a) and (2.2) imply that the gradient of \( I \) satisfies (2.6a). Since it follows from (3.17a) that
\[
E \left( u \frac{\partial}{\partial t_i} I \right) = \left( E \circ \frac{\partial}{\partial t_i} \right) I, \quad i = 1, \ldots, \mu_A,
\]
the equation (2.6b) follows from (3.17b) and (3.15). \( \square \)

4. Homological Mirror Symmetry

4.1. Derived directed fukaya category \( D^b \text{Fuk}^+ (f_A) \) for \( f_A \). We regard \( F_A \) as a globally defined tame polynomial on \( \mathbb{C}^3 \times M \) and then consider the derived category \( D^b \text{Fuk}^+ (X_{w; t, e^{t_{\mu_A}}} ) \) of the directed Fukaya category \( \text{Fuk}^+ (X_{w; t, e^{t_{\mu_A}}} ) \). Here, \( \text{Fuk}^+ (X_{w; t, e^{t_{\mu_A}}} ) \) is a directed \( A_\infty \)-category which can be thought of as a “categorification” of a distinguished basis of vanishing cycles in the affine variety \( X_{w; t, e^{t_{\mu_A}}} \) at a point \( (w; t, e^{t_{\mu_A}}) \in (\mathbb{C} \times M) \setminus \mathcal{D} \) defined as
\[
X_{w; t, e^{t_{\mu_A}}} := \{ x \in \mathbb{C}^3 \mid F_A(x; t, e^{t_{\mu_A}}) = w \},
\]  
(4.1)
Definition 4.1. The directed Fukaya category \( \text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}}) \) is a strictly unital \( A_\infty \)-category consists of

- \( \mu_A \) vanishing graded Lagrangian submanifolds \( \mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A} \) in the affine variety \( \mathcal{X}_{w; t, e^{t\mu A}} \) together with an ordering of these objects as \( (\mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A}) \) such that

\[
\text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}})(\mathcal{L}_i, \mathcal{L}_j) = \begin{cases} 
0 & \text{if } i > j, \\
\mathbb{C} \cdot \text{id}_{\mathcal{L}_i} & \text{if } i = j, \\
\bigoplus_{p \in \mathcal{L}_i \cap \mathcal{L}_j} \mathbb{C}[^\text{deg}(p)] & \text{if } i < j,
\end{cases}
\]

(4.3)

where \([-\cdot\cdot\cdot]_{\mathcal{L}_i} \) denotes the translation of the complex, \( \text{deg}(p) \) is defined by the gradings \( \text{gr}_{\mathcal{L}_i} : \mathcal{L}_i \to \mathbb{R} \) and \( \text{gr}_{\mathcal{L}_j} : \mathcal{L}_j \to \mathbb{R} \) as the largest integer less than or equal to \( \text{gr}(\mathcal{L}_j)|_p - \text{gr}(\mathcal{L}_i)|_p \),

- the (non-trivial) composition maps

\[
m^n_A : \text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}})(\mathcal{L}_{i_1}, \ldots, \mathcal{L}_{i_n}) \otimes_{\mathbb{C}^*} \cdots \otimes_{\mathbb{C}^*} \text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}})(\mathcal{L}_{i_1}, \mathcal{L}_{i_2})
\to \text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}})(\mathcal{L}_{i_1}, \mathcal{L}_{i_n})[2 - n], \quad i_1 < \cdots < i_n,
\]

(4.4)

defined by the “numbers of pseudo-holomorphic polygons” with boundaries on \( \mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A} \) and corners on intersection points.

The \( A_\infty \)-category \( \text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}}) \) depends on many choices other than the initial data \( f_A \), especially, on the choice of the point \( (w; t, e^{t\mu A}) \in (\mathbb{C} \times M) \setminus \hat{\mathcal{D}} \). However, it turns out that the derived category \( D^b\text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}}) \) of \( \text{Fuk}^\rightarrow(\mathcal{X}_{w; t, e^{t\mu A}}) \) becomes an invariant of the polynomial \( f_A \) as a triangulated category, which we shall usually denote by \( D^b\text{Fuk}^\rightarrow(f_A) \) for simplicity.

Note also that the ordered set of objects \( (\mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A}) \) forms a full exceptional collection in \( D^b\text{Fuk}^\rightarrow(f_A) \) by definition.

4.2. Mirror equivalence. The following theorem is proven by Auroux-Katzarkov-Orlov [3], Seidel [17] and van Straten [19] for the case \( A = (1, p, q), p, q \geq 1 \) and by the second named author [21] for the cases \( A = (2, 2, r), r \geq 2, A = (2, 3, 3), A = (2, 3, 4) \) and \( A = (2, 3, 5) \).

Theorem 4.2. There exists a triangulated equivalence

\[
\sigma : D^b\text{Fuk}^\rightarrow(f_A) \cong D^b\text{coh}(\mathbb{P}^1_A).
\]

(4.5)
In particular, \((\sigma(L_1), \ldots, \sigma(L_{\mu_A}))\) forms a full exceptional collection in \(D^b_{\text{coh}}(\mathbb{P}_A^1)\) and
\[
\chi(\sigma(L_i), \sigma(L_j)) + \chi(\sigma(L_j), \sigma(L_i)) = -I(L_i, L_j), \quad i, j = 1, \ldots, \mu_A,
\]
where \(I\) is the intersection form on the middle homology group \(H_2(X_{w; t, e^{t\mu_A}}; \mathbb{Z})\) of the affine variety \(X_{w; t, e^{t\mu_A}}\).

5. Lefschetz thimbles and vanishing cycles

In what follows, we identify \(M_{\mathbb{P}_A^1}\) and \(M_{f_{\mathbb{A}}}\) by the mirror isomorphism of Theorem 3.10 and denote it by \(M\). Fix a point \((t, e^{t\mu_A}) \in M \setminus \mathcal{B}\) and \(u \in \mathbb{C}^\times\) such that \(e^{\pi \sqrt{-1}/2} u\) lies on an admissible line in the sense of Definition 2.10. Then, for each critical point \(p_{\text{crit}} \in \mathbb{C}^3\) of \(F_A(x; t, e^{t\mu_A})\), we can define a relative 3-cycle
\[
\Gamma(u) \in H_3(\mathbb{C}^3, \text{Re}(F_A(\bullet; t, e^{t\mu_A})/u) \ll 0; \mathbb{Z}),
\]
called the Lefschetz thimble for \(p_{\text{crit}}\) as follows. The image of \(\Gamma(u)\) by \(F_A(x; t, e^{t\mu_A})\) is a half-line starting from \(w_{\text{crit}} = F_A(p_{\text{crit}}; t, e^{t\mu_A})\) in the direction of \((\pi + \arg u)\), and the fiber above a point \(w\) on this half-line is the 2-cycle in \(X_{w; t, e^{t\mu_A}}\) which vanishes at \(p_{\text{crit}}\) by the parallel transport along this half-line (see Figure 1). Note that all critical points of \(F_A(x; t, e^{t\mu_A})\) are non-degenerate since we take a semi-simple point.

![Figure 1. Lefschetz thimble.](image)

**Proposition 5.1.** For any point \((w; t, e^{t\mu_A}) \in (\mathbb{C} \times M) \setminus \mathcal{D}\), we have an isomorphism
\[
\partial : H_3(\mathbb{C}^3, X_{w; t, e^{t\mu_A}}; \mathbb{Z}) \cong H_2(X_{w; t, e^{t\mu_A}}; \mathbb{Z}).
\]

**Proof.** The relative homology long exact sequence yields the statement. \(\square\)
For a point \((t, e^r_A) \in M \setminus B\) and an admissible \(u\), take a regular value \(w_0 \in C\) with \(\text{Re}(w_0/u)\) is small enough so that

\[
H_3(\mathbb{C}^3, \text{Re}(F_A(\bullet; t, e^r_A))/u) \ll 0; \mathbb{Z}) \cong H_3(\mathbb{C}^3, X_{w_0:t, e^r_A}; \mathbb{Z})
\]

holds. Then, the homology class represented by Lefschetz thimbles are uniquely characterized by the vanishing cycles in the affine variety \(X_{w_0:t, e^r_A}\) by Proposition 5.1.

6. Main Theorem

In this section we discuss on a neighborhood of a point on \(M \setminus B\). Fix an admissible line \(\ell = e^{\sqrt{-1}\phi}\mathbb{R} \setminus \{0\}\) \((0 \leq \phi < \pi)\). Let \(p_1, \ldots, p_{\mu_A}\) be the critical points of \(F_A(x; t, e^r_A)\), \(\Gamma_1(u), \ldots, \Gamma_{\mu_A}(u)\) be the Lefschetz thimbles for these critical points when \(e^{\pi\sqrt{-1}/2}u \in \ell\), and define

\[
\mathcal{I}_i(t, u) = (2\pi u)^{-\frac{3}{2}} \int_{\Gamma_i(u)} \frac{e^{F_A(x; t, e^r_A)}}{u} \zeta_A, \quad i = 1, \ldots, \mu_A.
\]

(6.1)

By the saddle-point method, the oscillatory integral has the asymptotic expansion

\[
\mathcal{I}_i(t, u) = e^{-t_{\mu_A}e^{w_i(t)/u}}(1 + O(u)), \quad i = 1, \ldots, \mu_A.
\]

(6.2)

as \(u \to 0\) with the fixed argument since \(s_{\mu_A} = e^{r_{\mu_A}}\) (see sentences after Lemma 4.3 in [13]). Here \(w_i(t) = F_A(p_i; t, e^r_A)\) is the \(i\)-th critical value, and \(\Delta_i(t)\) is the Hessian at \(p_i\)

\[
\Delta_i(t) := \det \left( \frac{\partial^2 F_A}{\partial x_k \partial x_l}(p_i; t, e^r_A) \right)_{k,l=1,2,3}, \quad i = 1, \ldots, \mu_A.
\]

(6.3)

Since \(\partial/\partial w_1, \ldots, \partial/\partial w_{\mu_A}\) are basic idempotents, it follows that

\[
\frac{\partial F_A}{\partial w_i}(p_i; t, e^r_A) = 1.
\]

(6.4)

Therefore, by Definition 2.29 and Definition 2.33 (iv) in [13], we have

\[
\eta \left( \frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j} \right) = -\frac{e^{-2t_{\mu_A}}}{\Delta_i(t)}, \quad i = 1, \ldots, \mu_A,
\]

(6.5)

and hence

\[
\Psi_{1i} = \frac{e^{-t_{\mu_A}}}{\sqrt{-\Delta_i(t)}}, \quad i = 1, \ldots, \mu_A.
\]

(6.6)

The equality (6.6) and Corollary 3.13 imply that the asymptotic expansion of the gradient \(T(u\partial_1 \mathcal{I}_i, \ldots, u\partial_{\mu_A} \mathcal{I}_i)\) coincides with the \(i\)-th column of the formal matrix solution \(\Phi_{\text{formal}}\) if a suitable branch of the square root is chosen.

The integration cycles \(\Gamma_i(u)\) undergoes a discontinuous change when \(u\) cross a line such that the half-line starting from the critical value \(w_i\) in the direction of \((\pi + \arg u)\) pass through another critical value \(w_j\). These discontinuities cause Stokes phenomena for the oscillatory integrals. In order to determine the Stokes matrix \(S\) of the Frobenius
manifold $M$ for the admissible line $\ell$, we establish the correspondence between the analytic solutions $\Phi_{\text{right/left}}$ of (2.6b) and $I_i$.

![Figure 2](image)

**Figure 2.** The image of $\Gamma_{i,\text{right/left}}(u = +\sqrt{-1})$ by $F_A$.

In what follows, without loss of generality, we assume that the positive imaginary axis $\ell = \sqrt{-1} \mathbb{R} \setminus \{0\}$ is admissible, and discuss the Stokes matrix for $\ell$. Take a small positive number $\varepsilon$ such that all lines passing through the origin with angle between $\pi/2 - \varepsilon$ to $\pi/2 + \varepsilon$ are admissible. Order critical values $\{w_i\}_{i=1}^{\mu_A}$ so that

$$|e^{w_1/u}| \ll |e^{w_2/u}| \ll \cdots \ll |e^{w_{\mu_A}/u}|$$

holds as $u \to 0$ along the line $\text{arg} u = \pi/2$. Consider the local system on $\mathbb{C}^\times$ whose fiber on $u \in \mathbb{C}^\times$ is the relative homology group $H_3(\mathbb{C}^3, \text{Re}(F_A(\bullet; t, e^{\nu_A}))/u) \ll 0; \mathbb{Z})$, and let $\Gamma_{i,\text{right/left}}$ be a section of the local system on $D_{\text{right/left}}$ satisfying the following condition:

(A) for $u \in \{u \in \mathbb{C}^\times \mid -\varepsilon < \text{arg} u < \varepsilon\} \subset D_{\text{right}}$ (resp., $u \in \{u \in \mathbb{C}^\times \mid \pi - \varepsilon < \text{arg} u < \pi + \varepsilon\} \subset D_{\text{left}}$), $\Gamma_{i,\text{right}}(u)$ (resp., $\Gamma_{i,\text{left}}(u)$) coincides with the relative homology class represented by the Lefschetz thimble for the $i$-th critical point.

Since $D_{\text{right/left}}$ is simply-connected, the condition (A) determines the section $\Gamma_{i,\text{right/left}}$ uniquely. Figure 2 describes the projection of cycles in $H_3(\mathbb{C}^3, \text{Re}(F_A(\bullet; t, e^{\nu_A}))/u) \ll 0; \mathbb{Z})$ representing these homology classes when $u = +\sqrt{-1} \in D_+$ in the case $\mu_A = 3$. Solid curves (resp., dotted curves) in Figure 2 express the image of cycles representing
The integrand is single-valued. That is, for $u$ desired Stokes matrix $F$ of $D$ sector holds. Fix such a point formal is a fundamental solution of (2.6) which is asymptotic to $\Phi$ \textit{See Figure 3 for an example of such paths. Denote by $L$}\{(•, t, e^{\mu A} )\} with $Re(w_0/u) < 0$ is small enough such that $H_\beta(\mathbb{C}^3, Re(F_A(•, t, e^{\mu A} )/u) \ll 0; \mathbb{Z}) \cong H_2(X_{w_0,t,e^{\mu A}}; \mathbb{Z})$ (6.11) holds. Fix such a point $w_0$ and take paths $c_i$ between $w_0$ and $w_i$ for $i = 1, \ldots, \mu_A$ satisfying
\begin{itemize}
  \item for $i \neq j$, $c_i$ and $c_j$ have a unique common point $w_0$,
  \item $c_1, \ldots, c_{\mu_A}$ are ordered such that the following holds; for $i < j$, $w \in c_i$ and $w' \in c_j$,
\end{itemize}
$Re w < Re w'$ holds if $Im w = Im w'$. See Figure 3 for an example of such paths. Denote by $\mathcal{L}_i$ the cycle in $X_{w_0, t, e^{\mu A}}$ which vanishes at $w_i$ by the parallel transport along the path $c_i$ ($i = 1, \ldots, \mu_A$). Then, the ordered set $(\mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A})$ of cycles form a \textit{distinguished basis of vanishing cycles} in the sense in [2].

According to the Picard-Lefschetz theory, the relationship between the cycles $\{\Gamma_{i, \text{right}}\}_{i=1}^{\mu_A}$ and $\{\Gamma_{i, \text{left}}\}_{i=1}^{\mu_A}$ are expressed by the intersection numbers of these vanishing cycles. Here we recall the Picard-Lefschetz formula.

**Proposition 6.1** (e.g., Section 2 in [2]). For $i = 1, \ldots, \mu_A$, let $h_i \in \text{Aut}(H_2(X_{w_0, t, e^{\mu A}}; \mathbb{Z}))$ be the monodromy operator along the loop $\tau _i \in \pi_1(\mathbb{C}\setminus\{w_1, \ldots, w_{\mu_A}\}, w_0)$, which goes along the path $c_i$ from $w_0$ to $w_i$, turns around $w_i$ in the positive direction (anti-clockwise) and returns to $w_0$ along $c_i$. For any cycle $\mathcal{L} \in H_2(X_{w_0, t, e^{\mu A}}; \mathbb{Z})$, we have

$$h_i(\mathcal{L}) = \mathcal{L} + I(\mathcal{L}, \mathcal{L}_i)\mathcal{L}_i, \quad i = 1, \ldots, \mu_A.$$ (6.12)
Using the Picard-Lefschetz formula, we obtain the following.

**Proposition 6.2.** The following equality holds:

$$
\Gamma_{j, \text{left}}(u) = \Gamma_{j, \text{right}}(u) - \sum_{i=1}^{j-1} I(L_i, L_j) \Gamma_{i, \text{right}}(u), \quad j = 1, \ldots, \mu_A \quad (6.13)
$$

for $u \in D_+$. Here $I$ is the intersection form on $H_2(X_{w_0;}; \mathbb{Z})$.

**Proof.** For $u \in D_+$ and $i = 1, \ldots, \mu_A$, let $L_{i, \text{right/left}} \in H_2(X_{w_0;}; \mathbb{Z})$ be the vanishing cycles corresponding to the homology class $\Gamma_{i, \text{right/left}}(u)$ by the isomorphism $(6.11)$. That is, $L_{i, \text{right/left}}$ is the cycle which vanishes at $w_i$ along the path $c_{i, \text{right/left}}$ in Figure 4 and hence $L_i = L_{i, \text{left}}$ since $c_i$ and $c_{i, \text{left}}$ are homotopic as paths on $\mathbb{C} \setminus \{w_1, \ldots, w_{\mu_A}\}$. In view of Figure 4,

$$
L_{1, \text{left}} = L_{1, \text{right}} \quad (6.14)
$$

holds obviously. For $L_{2, \text{right/left}}$, since we can deform the path $c_{2, \text{left}}$ cycles homotopically as in Figure 5 we have $L_{2, \text{right}} = h_1(L_{2, \text{left}})$. It follows from $(6.12)$ and $(6.14)$ that

$$
L_{2, \text{left}} = L_{2, \text{right}} - I(L_1, L_2) L_{1, \text{right}}. \quad (6.15)
$$

Similarly, $L_{i, \text{right}} = (h_1 \circ h_2 \circ \cdots \circ h_{i-1})(L_{i, \text{left}})$ holds for general $i$. Using the Picard-Lefschetz formula $(6.12)$ iteratively, we obtain

$$
L_{j, \text{left}} = L_{j, \text{right}} - \sum_{i=1}^{j-1} I(L_i, L_j) L_{j, \text{right}}, \quad j = 1, \ldots, \mu_A. \quad (6.16)
$$

The equality $(6.13)$ follows from $(6.16)$ and the isomorphism $(6.11)$. \hfill \Box
By Proposition 6.2 we have
\[
S_{ij} = \begin{cases} 
0 & \text{if } i > j, \\
1 & \text{if } i = j, \\
-I(L_i, L_j) & \text{if } i < j.
\end{cases} \tag{6.17}
\]
That is, the Stokes matrix $S$ and the intersection matrix $I = (I(L_i, L_j))_{i,j=1,\ldots,\mu_A}$ are related as
\[
S + TS = -I. \tag{6.18}
\]
On the other hand, as a consequence of the homological mirror symmetry of Section 4, the intersection matrix $I$ and the Euler matrix $\chi = (\sigma(L_i), \sigma(L_j))_{i,j=1,\ldots,\mu_A}$ for the full exceptional collection $(\sigma(L_1), \ldots, \sigma(L_{\mu_A}))$ of $D^b\text{coh}(\mathbb{P}^1_A)$ also satisfy
\[
\chi + T\chi = -I. \tag{6.19}
\]
Since both the Stokes matrix $S$ and the Euler matrix $\chi$ are upper triangular matrices with all diagonal entries one, the relation (6.18) and (6.19) implies that $S = \chi$. That is, we have the following statement, which shows the Dubrovin’s conjecture (7) for $\mathbb{P}^1_A$ with $\chi_A > 0$.

**Theorem 6.3.** For any point on $M \setminus B$ and any admissible line, there exists a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_{\mu_A})$ of $D^b\text{coh}(\mathbb{P}^1_A)$ such that the Stokes matrix $S = (S_{ij})_{i,j=1,\ldots,\mu_A}$
of the first structure connection of the Frobenius manifold $M_{\mathbb{P}^1_A}$ coincides with the Euler matrix of them:

$$S_{ij} = \chi([\mathcal{E}_i], [\mathcal{E}_j]), \quad i, j = 1, \ldots, \mu_A. \quad (6.20)$$

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