LAX MATRICES FROM ANTIDOMINANTLY SHIFTED YANGIANS AND QUANTUM AFFINE ALGEBRAS

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Abstract. We construct a family of $GL_n$ rational and trigonometric Lax matrices $T_D(z)$ parametrized by $\Lambda^+\text{-valued divisors}$ $D$ on $\mathbb{P}^1$. To this end, we study the shifted Drinfeld Yangians $Y_\mu(gl_n)$ and quantum affine algebras $U_{\mu^+,\mu^-}(Lgl_n)$, which slightly generalize their $sl_n$-counterparts of [BFN, FT1]. Our key observation is that both algebras admit the RTT type realization when $\mu$ (resp. $\mu^+$ and $\mu^-$) are antidominant coweights. We prove that $T_D(z)$ are polynomial in $z$ (up to a rational factor) and obtain explicit simple formulas for those linear in $z$. This generalizes the recent construction by the first two authors of linear rational Lax matrices [FP] in both trigonometric and higher $z$-degree directions. Furthermore, we show that all $T_D(z)$ are normalized limits of those parametrized by $D$ supported away of $\{\infty\}$ (in the rational case) or $\{0, \infty\}$ (in the trigonometric case). The RTT approach provides conceptual and elementary proofs for the construction of the coproduct homomorphisms on shifted Yangians and quantum affine algebras of $\mathfrak{sl}_n$, previously established via long computations in [FKPRW, FT1]. Finally, we establish a close relation between a certain collection of explicit linear Lax matrices and the well-known parabolic Gelfand-Tsetlin formulas.

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1. Introduction

1.1. Summary.

Let $G$ be a complex reductive group and let $(C,dz)$ be a complex projective line $\mathbb{P}^1$ with a marked point $z = \infty$, also equipped with a section $dz$ of the canonical line bundle $\mathcal{K}_C$ whose only singularity is a second order pole at $z = \infty$. Let $\langle \cdot , \cdot \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$ of $G$.

To the data $(G, C, \langle \cdot , \cdot \rangle, dz)$ one can associate in the standard way an (infinite-dimensional) Poisson-Lie group $G_1(C)$ of $G$-valued rational functions on $C$ with fixed value 1 at $\infty$. By the formal series expansion at $z = \infty$ there is a natural inclusion $G_1(C) \hookrightarrow G_1[[z^{-1}]]$, where $G_1[[z^{-1}]]$ are $G$-valued power series in $z^{-1}$ with the constant term 1. The group $G_1[[z^{-1}]]$ is the Poisson-Lie group whose Poisson structure is constructed in the standard way from the Lie bialgebra defined by the Manin triple $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], z^{-1}\mathfrak{g}[[z^{-1}]])$ and the residue pairing $\oint_{\infty} \langle \cdot , \cdot \rangle dz$. The quantization of the Poisson-Lie group $G_1[[z^{-1}]]$ produces the Hopf algebra called the Drinfeld Yangian $Y(\mathfrak{g})$.

Let $\Lambda^+$ be a cone of dominant coweights in the coweight lattice $\Lambda$ of $G$. A formal linear combination of points of $C$ with coefficients in $\Lambda^+$ will be called a $\Lambda^+$-valued divisor $D$ on $C$.

The symplectic leaves $\mathfrak{M}_D$ in the Poisson-Lie group $G_1(C)$ are classified by $\Lambda^+$-valued divisors $D = \sum_{x \in \mathbb{F}_1} \lambda_x[x]$ trivial at infinity [8, EP], i.e. with $\lambda_\infty = 0$. Namely, for a given $D$, the symplectic leaf $\mathfrak{M}_D \subset G_1(C)$ consists of those elements in $G_1(C)$ that are regular away from the support of $D$ while having a singularity of the form $G[[z_x]]z_x^{-\lambda_x}G[[z_x]]$ in a neighborhood of each $x \in \text{supp}D$, where $z_x$ is a local coordinate near $x$ vanishing at $x$ and $\lambda_x \in \Lambda^+$ is the coefficient of $D$ at $x$.

The symplectic leaves $\mathfrak{M}_D$ of $G_1(C)$ are interesting in many aspects. A symplectic leaf $\mathfrak{M}_D$ can be identified with:

1. a moduli space of $G$-multiplicative Higgs fields trivially framed at $z = \infty$ [EP]
2. a moduli space of $G_c$-monopoles on $C \times S^1$ regular at infinity and with Dirac singularities whose projection on $C$ is encoded by the $\Lambda^+$-valued divisor $D$, where $G_c$ is the compact group associated to the complex reductive group $G$ [CK, CH]
3. a Coulomb branch of $N = 2$ (ultraviolet fixed point) UV conformal quiver gauge theory on $\mathbb{R}^3 \times S^1$ if $G$ is of ADE type and the ADE quiver is the Dynkin diagram of $\mathfrak{g}$ [CK]
4. a phase space of an algebraic integrable system known in the quantum field theory literature as the Seiberg-Witten integrable system of $N = 2$ ADE UV conformal quiver gauge theory [NP]
5. a classical limit of the GKLO-modules of $Y(\mathfrak{g})$ constructed by Gerasimov, Kharchev, Lebedev and Oblezin [GKLO]

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Let $\mu \equiv \lambda_\infty \equiv D|_\infty$ denote the coefficient of the divisor $D$ at infinity. In the constructions of the above list it was assumed that $\mu$ vanishes. In the constructions (1) and (2), the restriction $\mu = 0$ translates to the regularity either of the Higgs field at $\infty \in \mathbb{P}^1$ or to the regularity of the monopole configuration on the infinity of $\mathbb{R}^2 \times S^1$. In the points (3) and (4), for $G$ of a simple ADE type, $\mu$ encodes the UV $\beta$-function of an $N = 2$ supersymmetric quiver gauge theory, and consequently, the restriction $\mu = 0$ translates to the condition that the UV $\beta$-function of the quiver theory vanishes (cf. [NP]).

It is natural to explore what happens with the constructions listed above when the restriction $\mu = 0$ is lifted. The natural generalizations for not necessarily vanishing $\mu$ are:

1. a moduli space of $G$-multiplicative Higgs fields with the framed singularity $z^\mu$ at $z = \infty$ of the coweight $\mu$
2. a moduli space of $G_c$-monopoles on $C \times S^1$ with a charge $\mu$ at infinity and with Dirac singularities whose projection on $C$ is encoded by the $\Lambda^+$-valued divisor $D$
3. a Coulomb branch of $N = 2$ UV quiver gauge theory on $\mathbb{R}^3 \times S^1$ if $G$ is of ADE type and the ADE quiver is the Dynkin diagram of $\mathfrak{g}$ [NP] with the UV $\beta$-function $-\mu$
4. a phase space of the Seiberg-Witten algebraic integrable system of $N = 2$ supersymmetric ADE quiver gauge theory with the UV $\beta$-function $-\mu$
5. a classical limit of the analogues of the GKLO-modules [GKLO] but for a shifted Yangian $Y_{-\mu}(\mathfrak{g})$ [KWWY, BFNb]

In this paper, we put further details on the construction (5) focusing on $G = GL_n$ and antidominantly shifted Yangians, which in our notations are recorded as $Y_{-\mu}(\mathfrak{gl}_n)$ with $\mu \in \Lambda^+$. Generalizing [D, BK1], we present the isomorphism between the Drinfeld and RTT realizations of $Y_{-\mu}(\mathfrak{gl}_n)$ and both as a consequence and a tool to prove this isomorphism we construct $GL_n$ Lax matrices $T_D(z)$ with prescribed singularities at $D$ for any $\Lambda^+$-valued divisor $D$ (with an additional property that the sum of the coefficients $\sum_{x \in \mathbb{P}^1} \lambda_x$ is in the coroot lattice of $G$).

While in the paper we implicitly assume $h = 1$ (for simplicity of our exposition) and explicitly present only the quantum case, our construction can be naturally generalized to the $\mathbb{C}[h]$-setup: both (antidominantly) shifted Drinfeld and shifted RTT Yangians of $\mathfrak{gl}_n$ become associative algebras over $\mathbb{C}[h]$, $h$ appears in the commutation relations between the canonical coordinates on $\mathfrak{M}_D$ as $[p_{i,r}, e^{h^{g_{i,s}}}] = \delta_{i,j} \delta_{r,s} h e^{g_{i,s}}$, and the rational Lax matrices $T_D(z)$ obviously generalize to keep track of $h$. The classical limit is recovered in the usual way by sending $h \to 0$ and replacing $\frac{1}{h} \{, , \}$ by the Poisson bracket $\{, , \}$.

We conjecture that the classical limit of our construction describes the full family of symplectic leaves in the Poisson-Lie group obtained as the classical limit of the shifted Yangian $Y_{-\mu}(\mathfrak{g})$, and for each $\Lambda^+$-valued divisor $D$ on $C$ we obtain Darboux coordinates on the symplectic leaf $\mathfrak{M}_D$. We leave out for a future work the precise details as well as the details of the construction of the moduli space of multiplicative Higgs fields with a singularity at the framing point and moduli space of singular monopoles on $\mathbb{R}^2 \times S^1$ (cf. [F, Moc] for the relevant constructions of singular monopoles and Kobayashi-Hitchin correspondence in that context).

The Lax matrices $T_D(z)$ can be used to construct explicitly classical commuting Hamiltonians of the corresponding completely integrable systems on $\mathfrak{M}_D$ as well as their quantizations. The classical commuting Hamiltonians are obtained as the coefficients of the spectral curve

$$
det(y - g_\infty T_D(z)) = \sum_{i=0}^n y^{n-i} (-1)^i \text{tr}_{\Lambda^i}(g_\infty T_D(z)). \quad (1.1)$$
Here $g_\infty$ is a regular semi-simple element of $G$ that defines the coupling constants of the respective integrable system or encodes the gauge couplings of the respective quiver gauge theory in case when $\mathfrak{M}_D$ is interpreted as a Coulomb branch [NP]. For a general $G$, the classical complete integrability can be established from the abstract cameral curve construction following [DG].

In the quantum case, using that the homomorphism $\Psi_D$ of Theorem 2.34 factors through the quantized Coulomb branch, see [BFNb, Theorem B.18], the construction of Bethe subalgebras (see [Mol, §1.14] or the original paper [NO]) that uses a quantum version of the spectral curve gives rise to a family of Bethe commutative subalgebras in the quantized Coulomb branches. This answers a question of Boris Feigin posed to one of the authors and M. Finkelberg in March 2017. The pre-quantized Hamiltonians are represented in the algebra of difference operators with rational coefficients on functions of $p_{x,i}$. We do not discuss in this paper the actual quantization (the choice of a polarization, the Hilbert space structure or analytic properties of the wave-functions).

For example, the $i = n$ term in the spectral curve (1.1), the det of the Lax matrix, after a quantization is replaced by the quantum determinant and is given by the formula (2.38):

$$\text{qdet } T_D(z) = \prod_{i=1}^n \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z - x + (i-1)\hbar)^{-\epsilon_i(\lambda_x)}.$$

The Bethe ansatz for these quantum integrable systems was constructed in [NPS].

The origin of the canonical coordinates $(p_x, q_x)$ of the present work goes back to the work of Atiyah-Hitchin on the moduli space of monopoles on $\mathbb{R}^3$, [AH], that identified such moduli space with the moduli space of based rational maps from $C = \mathbb{P}^1$ to the flag variety $G/B$.

For example, for $G = SL_2$ the flag variety $G/B$ is $\mathbb{P}^1$, and the based rational maps from $C$ to $G/B$ are simply rational functions $f(z)$ vanishing at $z = \infty$. Given a coset representative of a based rational map from $C$ to $G/B$ in the form $A(z)/B(z)$, the respective rational function is $f(z) = B(z)/A(z)$. For the divisor $D$ consisting only of a singularity at $\infty \in \mathbb{P}^1$, the coordinates $p_x$ are the locations of zeros of $A(z)$ (i.e. poles of $f(z)$), while the coordinates $e^q$ are the values of $B(z)$ at these zeros. Such canonical coordinates in the space of rational functions also appeared in the work of Sklyanin on separation of variables. Furthermore, Jarvis in his work on monopoles on $\mathbb{R}^3$, [J1, J2], constructed a lift of a based rational map from $C$ to $G/B$ to a rational map from $C$ to $G$. The classical limit of the formulas for the rational Lax matrices $T_D(z)$ presented in this work for $G = GL_n$ could be seen as a canonical realization of Jarvis’s lift of a based rational map from $C$ to $G/B$ to a rational map from $C$ to $G$, equipped with canonical $(p_x, q_x)$-coordinates induced from the Atiyah-Hitchin construction for the based rational maps to $G/B$. We provide some more details in Remark 2.92, while referring the interested reader to [BFNb, 2(xi, xii, xiii)] for a more detailed discussion.

In the second part of the paper we proceed to the trigonometric case by taking $C = \mathbb{P}^1 = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$ equipped with a section $dz/z$ of the canonical bundle $\mathcal{K}_C$ that has order one poles at 0 and $\infty$. Given the Borel decomposition of $\mathfrak{g}$, the section of $\mathcal{K}_C$, and the Killing form on $\mathfrak{g}$, one obtains in the usual way the Lie bialgebra structure on the loop algebra $L\mathfrak{g}$ with the trigonometric $r$-matrix and the corresponding Poisson-Lie loop group. The quantization of this Poisson-Lie group gives rise to the quantum loop algebra $U_q(L\mathfrak{g})$ (also known as the quantum affine algebra with the trivial central charge).
Similar to the rational case, to each $\Lambda^+$-valued divisor $D$ on $C$ we associate a module of $U_q(L\mathfrak{g})$ in a construction analogous to [GKLO]. However, in the trigonometric case there are two special framing points 0 and $\infty$ on $C$. We denote the coefficients of $D$ at these framing points by $\mu^- = \lambda_0 = D|_0$ and $\mu^+ = \lambda_{\infty} = D|_{\infty}$, respectively. Then, for any $\Lambda^+$-valued divisor $D$ on $C$ (with an additional property that the sum of the coefficients $\sum_{x \in \mathbb{P}^1} \lambda_x$ lies in the coroot lattice of $G$), we construct a homomorphism from the shifted quantum affine algebra $U_{-\mu^+,-\mu^-}(L\mathfrak{g})$ to the algebra of $v$-difference operators (see Remark 3.31 and [FT1]), and using an isomorphism between the Drinfeld and the RTT realizations of $U_{-\mu^+,-\mu^-}(L\mathfrak{g}|_{\mathfrak{n}})$, $\mu^+ \in \Lambda^+$, we construct and present explicitly the corresponding $GL_n$ trigonometric Lax matrices $T_D(z)$.

Conjecturally, the classical limit of our construction describes the full family of symplectic leaves in the $(-\mu^+, -\mu^-)$-shifted Poisson-Lie loop group obtained as the classical limit of the shifted quantum affine algebra $U_{-\mu^+,-\mu^-}(L\mathfrak{g})$, where $(\mu^+, \mu^-)$ are the coweights encoding the prescribed singularities at $\infty$ and 0. Conjecturally, each symplectic leaf $\mathcal{M}_D$ is isomorphic as a symplectic variety to the moduli space of multiplicative Higgs bundles on $(\mathbb{P}^1, dz/z)$ with Borel framing at 0 and $\infty$ and with prescribed singularities on $D$. We leave out the precise definitions and details of this construction for a future work.

A subset of $GL_n$ rational Lax matrices constructed in [FP] are known to be the building blocks for the transfer matrices of non-compact spin chains and Baxter $Q$-operators, see [BFLMS, DM] (cf. [T] for a discussion of the trigonometric case). The matrix elements of those Lax matrices are realized as polynomials in the Heisenberg algebra generators in analogy to the free field realization. The Fock vacuum vector serves as the highest weight state and the trace in the transfer matrix construction is taken over the entire Fock space. As discussed in Section 2.7, the realization studied in this paper is closely related to the Gelfand-Tsetlin bases which are the Gelfand-Tsetlin patterns. Consequently, we expect that the transfer matrices can be defined in terms of the Lax matrices presented in this article by introducing the appropriate trace over the Gelfand-Tsetlin oscillator realization. In addition to the construction of transfer matrices from Lax matrices linear in the spectral parameter, this approach should allow for the construction of the commuting family of operators with Lax matrices of higher degree in the spectral parameter. We leave the precise details of this construction as well as generalizations to Lie algebras beyond $A$-type for a future work.

Historically, the shifted Yangians $Y_\nu(\mathfrak{g})$ were first introduced for $\mathfrak{g} = \mathfrak{gl}_n$ and dominant shifts $\nu$ in [BK2], where their certain quotients were identified with type $A$ finite $W$-algebras, the latter being natural quantizations of type $A$ Slodowy slices. This construction was further generalized to any semisimple $\mathfrak{g}$ but still dominant $\nu \in \Lambda^+$ in [KWWY], where it was shown that their GKLO-type quotients (called truncated shifted Yangians) quantize slices in the affine Grassmannians. The generalization to arbitrary shifts $\nu \in \Lambda$ was finally carried out in [BFN1, Appendix B], and it was conjectured that their truncations quantize generalized slices in the affine Grassmannians introduced in loc.cit. In contrast to the original approach, we consider exactly the opposite case, with antidominant shifts, in the current paper (note that any shifted Yangian $Y_\nu(\mathfrak{g})$ may be embedded into the antidominantly shifted one $Y_{-\mu}(\mathfrak{g})$, $\mu \in \Lambda^+$, via the shift homomorphisms of [FKPRW]). The main technical benefit is the RTT realization of those $Y_{-\mu}(\mathfrak{g}|_{\mathfrak{n}})$ (resp. $U_{-\mu^+,-\mu^-}(L\mathfrak{g}|_{\mathfrak{n}})$), and as a result a conceptual explanation of the coproduct homomorphisms of [FKPRW] (resp. of [FT1]). Also, the antidominant case allows us to access interesting algebraic integrable systems that appear on the Coulomb branches of
four-dimensional supersymmetric $N = 2$ ADE quiver gauge theories of the asymptotically free type [NP]; a typical representative of such an integrable system is a closed Toda chain.

1.2. Outline of the paper.

- In Section 2.1, we introduce the *shifted Drinfeld Yangians* of $\mathfrak{gl}_n$, the algebras $Y_\mu(\mathfrak{gl}_n)$, where $\mu \in \Lambda$ is a coweight of $\mathfrak{gl}_n$. These algebras depend only on the associated coweight $\bar{\mu} \in \Lambda$ of $\mathfrak{sl}_n$, up to an isomorphism, see Lemma 2.17. They also contain the shifted Yangians of $\mathfrak{sl}_n$ (introduced in [BFN]) via the natural embedding $\iota_\mu : Y_\mu(\mathfrak{sl}_n) \hookrightarrow Y_\mu(\mathfrak{gl}_n)$ of Proposition 2.19 (generalizing the classical embedding $Y(\mathfrak{sl}_n) \hookrightarrow Y(\mathfrak{gl}_n)$). Moreover, we have the isomorphism $Y_\mu(\mathfrak{gl}_n) \cong ZY_\mu(\mathfrak{gl}_n) \otimes \mathbb{C} Y_\mu(\mathfrak{sl}_n)$ with $ZY_\mu(\mathfrak{gl}_n)$ denoting the center of $Y_\mu(\mathfrak{gl}_n)$, see Corollary 2.23 and Lemma 2.25 (generalizing [Mol, Theorem 1.8.2] in the unshifted case $\mu = 0$).

- In Section 2.2, we introduce the key notion of $\Lambda$-valued divisors on $\mathbb{P}^1$, $\Lambda^+ \text{-valued outside }\{\infty\} \subset \mathbb{P}^1$, see (2.27, 2.28). For each such divisor $D$ satisfying an auxiliary condition (2.29) (which encodes that the sum of all the coefficients of the divisor $D$ lies in the coroot lattice), we construct in Theorem 2.34 an algebra homomorphism $\Psi_D : Y_{-\mu}(\mathfrak{gl}_n) \to A$, where $\mu = D|_\infty$ is the coefficient of $D$ at $\infty$ and the target $A$ is the algebra of difference operators (2.31), see Remark 2.32. This construction generalizes the $A_{n-1}$-case of [BFN, Theorem B.15] as the composition $\Psi_D \circ \iota_{-\mu} : Y_{-\bar{\mu}}(\mathfrak{sl}_n) \to A$ is precisely the homomorphism $\Phi^{\lambda}_{-\mu}$ of loc.cit. (where $\lambda$ is the sum of all coefficients of $D$ outside $\infty$).

- In Section 2.3, we introduce the *(antidominantly) shifted RTT Yangians* of $\mathfrak{gl}_n$, the algebras $Y_{-\bar{\mu}}^{\text{RTT}}(\mathfrak{gl}_n)$ with $\mu \in \Lambda^+$ being a dominant coweight of $\mathfrak{gl}_n$. They are defined via the RTT relation (2.40) and the Gauss decomposition (2.42, 2.43). We construct the epimorphisms $\Upsilon_{-\mu} : Y_{-\mu}(\mathfrak{gl}_n) \twoheadrightarrow Y_{-\bar{\mu}}^{\text{RTT}}(\mathfrak{gl}_n)$ for any $\mu \in \Lambda^+$, see Theorem 2.51. The main result of this section (the proof of which is established in Section 2.4.3), Theorem 2.53, is that $\Upsilon_{-\mu}$ are actually algebra isomorphisms for any $\mu \in \Lambda^+$ (generalizing [D, BK1] in the unshifted case $\mu = 0$ as well as [FT1] in the smallest rank case $n = 2$, see Remark 2.54).

- In Section 2.4, we construct $n \times n$ rational Lax matrices $T_D(z)$ (with coefficients in $A((z^{-1}))$ for each $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (2.29). They are explicitly defined via (2.62, 2.63) combined with (2.57, 2.59, 2.61), while arising naturally as the image of the $n \times n$ matrix $T(z)$ (encoding all the generators of $Y_{-\bar{\mu}}^{\text{RTT}}(\mathfrak{gl}_n)$) under the composition $\Psi_D \circ \Upsilon_{-\bar{\mu}}^{-1} : Y_{-\bar{\mu}}^{\text{RTT}}(\mathfrak{gl}_n) \to A$, assuming Theorem 2.53 has been established, see (2.55, 2.56). As Theorem 2.53 is well-known for $\mu = 0$ and any Lax matrix $T_D(z)$ is a normalized limit of $T_D(z)$ with $D|_\infty = 0$, see Proposition 2.74 and Corollary 2.77, we immediately derive the RTT relation (2.40) for all matrices $T_D(z)$, see Proposition 2.78 (hence, the terminology “rational Lax matrices”). Combining the latter with the key result of [W], see Theorem 2.79, we finally prove Theorem 2.53 in Section 2.4.3. We note that similar arguments may be used to prove the triviality of the centers of shifted Yangians $Y_\mu(\mathfrak{g})$ for any coweight of a semisimple Lie algebra $\mathfrak{g}$, see Remark 2.80. The key property of the rational Lax matrices $T_D(z)$ (their regularity (up to a rational factor (2.65)), see Theorem 2.66 (the proof of the latter is based on a certain cancelation of poles reminiscent to the one appearing in the work on $q$-characters [FP] and $qq$-characters [N], see Remark 2.71). Finally, we derive simplified explicit formulas for all rational Lax matrices $T_D(z)$ which are linear in $z$, see Theorem 2.84. In the smallest rank case $n = 2$, those recover the well-known $2 \times 2$ elementary Lax matrices for the Toda chain, the DST chain, and the Heisenberg magnet, see Remark 2.90. We conclude Section 2.4 with Remark 2.92, which is three-fold: comparing the complete monodromy matrix (2.93) of the Toda chain for $GL_N$ to the degree $N$ rational $2 \times 2$ Lax matrix $T_D(z)$ with $D = N\alpha(\infty)$, identifying the phase spaces of the corresponding classical integrable systems with the $SU(2)$-monopoles of topological charge.
N, and generalizing the latter to SU(2)-monopoles of topological charge N with singularities, thus providing more details to our discussion of Section 1.1.

In Section 2.5, we evaluate explicitly some linear (in z) rational Lax matrices $T_D(z)$ and compare them to the linear rational Lax matrices constructed by the first two authors in [FP] (actually, we treat all the explicit “building blocks” of loc.cit., the fusion of which provides the entire family of the rational Lax matrices $L_{\lambda,\xi,\mu}(z)$ of [FP]).

In Section 2.6, we construct coproduct homomorphisms on antidominantly shifted Yangians. We start by constructing homomorphisms $\Delta_{\rightarrow,\mu_1,\mu_2} : Y_{\rightarrow,\mu_1,\mu_2}(\mathfrak{gl}_n) \to Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \otimes Y_{\rightarrow,\mu_2}(\mathfrak{gl}_n)$ defined via $\Delta_{\rightarrow,\mu_1,\mu_2}(T(z)) = T(z) \otimes T(z)$ for any $\mu_1, \mu_2 \in \Lambda^+$. Evoking the key isomorphism $Y_{\rightarrow,\mu}(\mathfrak{gl}_n) \cong Y_{\rightarrow,\mu}(\mathfrak{gl}_n)$ of Theorem 2.53, this naturally gives rise to homomorphisms $\Delta_{\rightarrow,\mu_1,\mu_2} : Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \to Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \otimes Y_{\rightarrow,\mu_2}(\mathfrak{gl}_n)$, and we compute the images of the finite collection of generators in Proposition 2.134. The latter, in turn, gives rise to homomorphisms $\Delta_{\rightarrow,\mu_1,\mu_2} : Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \to Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \otimes Y_{\rightarrow,\mu_2}(\mathfrak{gl}_n)$ for any dominant $\mathfrak{sl}_n$-coweights $\nu_1, \nu_2 \in \Lambda^+$, see Proposition 2.136, thus providing a conceptual and elementary proof of $A_{n-1}$-case of [FKPRW, Theorem 4.8]. Finally, we note that $\Delta_{\rightarrow,\mu_1,\mu_2}$ with $\mu_1, \mu_2 \in -\Lambda^+$ actually give rise to homomorphisms $\Delta_{\rightarrow,\mu_1,\mu_2} : Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \to Y_{\rightarrow,\mu_1}(\mathfrak{gl}_n) \otimes Y_{\rightarrow,\mu_2}(\mathfrak{gl}_n)$ for any $\nu_1, \nu_2 \in \Lambda$, due to [FKPRW, Theorem 4.12], see Remark 2.140.

In Section 2.7, for any Young diagram $\lambda$ of size $|\lambda| = n$, we show that the homomorphism $Y_{\rightarrow,\mu}(\mathfrak{gl}_n) \to \mathcal{A}$ determined by the rational Lax matrix $T_D(z)$ with $D = \sum_{i=1}^{\lambda} \omega_{n-\lambda}[x_i] - \omega_0[\infty]$ is equal (up to a gauge transformation) to a composition of the evaluation homomorphism $\tilde{ev} : Y_{\rightarrow,\mu}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ (2.163) and the homomorphism $U(\mathfrak{gl}_n) \to \mathcal{A}$ determined by the type $\lambda$ parabolic Gelfand-Tsetlin formulas (which arise naturally via the parabolic Gelfand-Tsetlin formulas (2.156-2.158) for the type $\lambda$ parabolic Verma module over $\mathfrak{gl}_n$), see Proposition 2.165. We note that likewise choosing another standard bases of type $\lambda$ parabolic Verma modules over $\mathfrak{gl}_n$ gives rise to all linear rational Lax matrices of [FP] with $\mu = 0$, see Remark 2.166.

- In Section 3.1, we introduce the shifted Drinfeld quantum affine algebras of $\mathfrak{gl}_n$, the algebras $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, where $\mu^+,\mu^- \in \Lambda$ are coweights of $\mathfrak{gl}_n$. These algebras depend only on the associated coweights $\mu^+,\mu^- \in \Lambda$ of $\mathfrak{sl}_n$, up to an isomorphism, see Lemma 3.13. They also contain the simply-connected versions of the shifted quantum affine algebras of $\mathfrak{sl}_n$ (introduced in [FT]) via the natural embedding $\iota_{\mu^+,\mu^-} : U_{\mu^+,\mu^-}(L\mathfrak{sl}_n) \hookrightarrow U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, while their central extensions $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ of (3.15) contain the adjoint versions of the shifted quantum affine algebras of $\mathfrak{sl}_n$ via $\iota_{\mu^+,\mu^-} : U'_{\mu^+,\mu^-}(L\mathfrak{sl}_n) \hookrightarrow U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, see Proposition 3.16 (generalizing the classical embedding $U_{\mu}(L\mathfrak{sl}_n) \hookrightarrow U_{\mu}(L\mathfrak{gl}_n)$ of quantum loop algebras). Finally, we establish the decomposition $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n) \simeq Z \otimes_{C(\mathfrak{g})} U'_{\mu^+,\mu^-}(L\mathfrak{sl}_n)$, see Lemma 3.22, where $Z \subset U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ is an explicit central subalgebra (which conjecturally coincides with the center of $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, see Remark 3.24).

In Section 3.2, we introduce $\Lambda$-valued divisors on $\mathbb{P}^1$, $\Lambda^+$-valued outside $\{0, \infty\} \in \mathbb{P}^1$, see (3.26, 3.27). For each such $D$ satisfying an auxiliary condition (3.28) (which encodes that the sum of all the coefficients of the divisor $D$ lies in the coroot lattice), we construct in Theorem 3.33 an algebra homomorphism $\Psi_D : U_{-\mu^+,\mu^-}(L\mathfrak{gl}_n) \to \mathcal{A}_{\text{frac}}^\nu$ where $\mu^+ = D|_{\infty}$ and $\mu^- = \mu|_0$ are the coefficients of $D$ at $\infty$ and 0, while the target $\mathcal{A}_{\text{frac}}^\nu$ is the algebra of $\nu$-difference operators (3.30), see Remark 3.31. This construction generalizes the $A_{n-1}$-case of [FT1, Theorem 7.1] as the composition $\Psi_D \circ \iota_{-\mu^+,\mu^-} : U'_{-\mu^+,\mu^-}(L\mathfrak{sl}_n) \to \mathcal{A}_{\text{frac}}^\nu$ essentially.
coincides with the homomorphism \( \overline{\Phi}_\lambda^{\text{rtt}} : U_{\Delta_1} \to U_{\Delta_2} \) of \( \overline{A}_\text{frac} \) of loc. cit. (where \( \lambda \) is the sum of all coefficients of \( D \) outside \([0, \infty)\)), see Remark \( \ref{rem:rtt-deformation} \).

In Section \( \ref{sec:quantum-affine-algebras} \), we introduce the (antidominantly) shifted RTT quantum affine algebras of \( \mathfrak{gl}_n \), the algebra \( U_{\Delta_1} \mathfrak{gl}_n \) with \( \mu^+, \mu^- \in \Lambda^+ \) being dominant coweights of \( \mathfrak{gl}_n \). They are defined via the RTT relation \( \ref{eq:rtt-relation} \), the Gauss decomposition \( \ref{eq:gauss-decomposition}, \ref{eq:gauss-decomposition-2} \), and an additional invertibility condition \( \ref{eq:invertibility} \). We construct the epimorphisms \( \Upsilon_{\mu^+, \mu^-} : U_{\mu^+, \mu^-} \mathfrak{gl}_n \to U_{\mu^+, \mu^-} \mathfrak{gl}_n \) for any \( \mu^+, \mu^- \in \Lambda^+ \), similar to [DF, Main Theorem], see Theorem \( \ref{thm:rtt-invertibility} \). Modulo a trigonometric counterpart of [W, Theorem 12], see Conjecture \( \ref{conj:main-conjecture} \), we prove that \( \Upsilon_{\mu^+, \mu^-} \) are actually algebra isomorphisms for any \( \mu^+, \mu^- \in \Lambda^+ \) (generalizing [DF] in the unshifted case \( \mu^+ = \mu^- = 0 \) as well as [FT1] in the smallest rank case \( n = 2 \), see Remark \( \ref{rem:shifted-case} \).

In Section \( \ref{sec:trig-lax-matrices} \), we construct \( n \times n \) trigonometric Lax matrices \( T_{D(z)} \) (with coefficients in \( \overline{A}_\mathbb{P}(z) \)) for each \( \Lambda^+ \)-valued divisor \( D \) on \( \mathbb{P}^1 \) satisfying \( \ref{eq:divisor-condition} \). They are explicitly defined via \( \ref{eq:shifted-rtt}, \ref{eq:shifted-rtt-2} \) combined with \( \ref{eq:trig-lax-matrices}, \ref{eq:trig-lax-matrices-2}, \ref{eq:trig-lax-matrices-3} \), while arising naturally as the image of the \( n \times n \) matrices \( T^\pm(z) \) (encoding all the generators of \( U_{\Delta_1} \mathfrak{gl}_n \)) under the composition \( \Psi_D \circ \Upsilon_{\mu^+, \mu^-} : U_{\mu^+, \mu^-} \mathfrak{gl}_n \to \overline{A}_\text{frac} \), assuming Theorem \( \ref{thm:rtt-invertibility} \) has been established, see \( \ref{eq:rtt-invertibility-3}, \ref{eq:rtt-invertibility-4} \). As Theorem \( \ref{thm:rtt-invertibility} \) is well-known for \( \mu^+ = \mu^- = 0 \) and any Lax matrix \( T_{D(z)} \) is a normalized limit of \( T_{D(z)} \) with \( D|_\infty = 0 = D|_0 \), see Propositions \( \ref{prop:normalized-limit}, \ref{prop:normalized-limit-2} \) and Corollary \( \ref{cor:normalized-limit} \), we immediately derive the RTT relation \( \ref{eq:rtt-relation} \) for all matrices \( T_{D(z)} \), see Proposition \( \ref{prop:rtt-relation} \) (hence, the terminology “trigonometric Lax matrices”). Combining the latter with the trigonometric generalization of [W, Theorem 12], see Conjecture \( \ref{conj:main-conjecture} \), we finally prove Theorem \( \ref{thm:rtt-invertibility} \) in Section \( \ref{sec:trig-lax-matrices} \). The key property of the trigonometric Lax matrices \( T_{D(z)} \) are their regularity (up to a rational factor \( \ref{eq:regularity} \)), see Theorem \( \ref{thm:regularity} \). Similar to Theorem \( \ref{thm:main-theorem} \), we also derive simplified explicit formulas for all trigonometric Lax matrices \( T_{D(z)} \) which are linear in \( z \), see Theorem \( \ref{thm:explicit-formulas} \). These formulas may be related to the \( \nu \)-deformed parabolic Gelfand-Tsetlin formulas in spirit of Proposition \( \ref{prop:parabolic-gelfand-tsetlin} \), see Remark \( \ref{rem:deformed-parabolic} \).

In Section \( \ref{sec:coproduct-homomorphisms} \), we apply Theorem \( \ref{thm:explicit-formulas} \) to evaluate explicitly all linear trigonometric Lax matrices \( T_{D(z)} \) for \( n = 2 \), thus generalizing the three Lax matrices of [FT1], see Remark \( \ref{rem:generalization} \).

In Section \( \ref{sec:coproduct-homomorphisms} \), we construct coproduct homomorphisms on antidominantly shifted quantum affine algebras. We start by constructing algebra homomorphisms (see Proposition \( \ref{prop:algebra-homomorphisms} \))

\[
\Delta_{\Delta_1}^{\text{rtt}} : U_{\Delta_1} \to U_{\Delta_2} \otimes U_{\Delta_3} \mathfrak{gl}_n \quad \text{defined via} \quad \Delta_{\Delta_1}^{\text{rtt}}(L_{\mu^+}, L_{\mu^-}) = L_{\mu^+} \otimes L_{\mu^-} \mathfrak{gl}_n
\]

for any dominant \( \mathfrak{sl}_n \)-coweights \( \mu^+, \mu^- \in \Lambda^+ \), see Proposition \( \ref{prop:rtt-coproduct} \), thus recovering and providing a more conceptual and simpler proof of [FT1, Theorem 10.22]. These in turn give
rise to homomorphisms $\Delta_{\nu_i^+, \nu_i^-} : U_{\nu_1^+, \nu_2^-, \nu_3^+}^\text{sc} (L\mathfrak{sl}_n) \to U_{\nu_1^+, \nu_2^-, \nu_3^+}^\text{sc} (L\mathfrak{sl}_n) \otimes U_{\nu_1^+, \nu_2^-, \nu_3^+}^\text{sc} (L\mathfrak{sl}_n)$ for any $\mathfrak{sl}_n$-coweights $\nu_1^+, \nu_2^-, \nu_3^+ \in \bar{\Lambda}$, due to [FT1, Theorem 10.26], see Remark 3.111.

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2. Rational Lax matrices

2.1. Shifted Drinfeld Yangians of $\mathfrak{gl}_n$.

Consider the lattice $\Lambda^\vee = \oplus_{j=1}^n \mathbb{Z} \epsilon_i^\vee$ associated with the standard module of $\mathfrak{gl}_n$, so that $\alpha_i^\vee := \epsilon_i^\vee - \epsilon_{i+1}^\vee$ ($1 \leq i < n$) are the standard simple positive roots of $\mathfrak{sl}_n$. Let $\Lambda = \oplus_{j=1}^n \mathbb{Z} \epsilon_j$ be the dual lattice so that $\epsilon_j^\vee (\epsilon_j) = \delta_{i,j}$. We will also need its alternative $\mathbb{Z}$-basis: $\Lambda = \oplus_{i=0}^{n-1} \mathbb{Z} \varpi_i$ with $\varpi_i := -\sum_{j=i+1}^n \epsilon_j$. For $\mu \in \Lambda$, define $\underline{d} = \{d_j\}_{j=1}^n \in \mathbb{Z}^n$, $\underline{b} = \{b_j\}_{j=1}^{n-1} \in \mathbb{Z}^{n-1}$ via

$$d_j := \epsilon_j^\vee (\mu), \quad b_i := \alpha_i^\vee (\mu) = d_i - d_{i+1}. \tag{2.1}$$

Define the \textit{shifted Drinfeld Yangian} of $\mathfrak{gl}_n$, denoted by $Y_{\underline{d}}(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}$-algebra generated by $\{E_i^{(r)} , F_i^{(r)}\}_{r \geq 1 \leq i < n}$ with $\{D_i^{(s)} , \tilde{D}_i^{(s)}\}_{1 \leq i \leq n}$ with the following defining relations (for all admissible $i, j, r, s, t$):

$$\begin{align*}
D_i^{(d_i)} &= 1, \quad \sum_{t=d_i}^{r+d_i} D_i^{(t)} \tilde{D}_i^{(r-t)} = -\delta_{r,0}, \quad [D_i^{(r)} , D_j^{(s)}] = 0, \tag{2.2} \\
[E_i^{(r)} , F_j^{(s)}] &= -\delta_{i,j} \sum_{t=-d_i}^{r+s-1-d_{i+1}} \tilde{D}_i^{(t)} D_{i+1}^{(r+s-t-1)}, \tag{2.3} \\
[D_i^{(r)} , E_j^{(s)}] &= (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=d_i}^{r-1} D_i^{(t)} E_j^{(r+s-t-1)}, \tag{2.4} \\
[D_i^{(r)} , F_j^{(s)}] &= (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=d_i}^{r-1} F_j^{(r+s-t-1)} D_i^{(t)}, \tag{2.5} \\
[E_i^{(r)} , E_j^{(s)}] &= \sum_{t=1}^{r-1} E_i^{(t)} E_i^{(r+s-t-1)} - \sum_{t=1}^{s-1} E_i^{(t)} E_i^{(r+s-t-1)}, \tag{2.6} \\
[F_i^{(r)} , F_i^{(s)}] &= \sum_{t=1}^{s-1} F_i^{(r+s-t-1)} F_i^{(t)} - \sum_{t=1}^{r-1} F_i^{(r+s-t-1)} F_i^{(t)}, \tag{2.7}
\end{align*}$$
\[ [E_i^{(r+1)}, E_i^{(s)}] - [E_i^{(r)}, E_i^{(s+1)}] = -E_i^{(r)}E_i^{(s)}, \quad (2.8) \]
\[ [F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = F_i^{(s)}F_i^{(r)}, \quad (2.9) \]
\[ [E_i^{(r)}, E_j^{(s)}] = 0 \text{ if } |i - j| > 1, \quad (2.10) \]
\[ [F_i^{(r)}, F_j^{(s)}] = 0 \text{ if } |i - j| > 1, \quad (2.11) \]
\[ [E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] = 0 \text{ if } |i - j| = 1, \quad (2.12) \]
\[ [F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] = 0 \text{ if } |i - j| = 1. \quad (2.13) \]

**Remark 2.14.** (a) For \( \mu = 0 \), this definition recovers the Drinfeld Yangian of \( \mathfrak{gl}_n \), see \([D]\) and \([BK1, \text{ Theorem 5.2}]\) (to be more precise, multiplying \( E_i^{(r)}, F_i^{(r)}, D_i^{(r)}, \tilde{D}_i^{(r)} \) by \((-1)^r\) the relations (2.2–2.13) transform into the defining relations (5.7–5.20) of \([BK1]\), cf. Remark 2.50).

(b) Similar to \([BK1, \text{ Remark 5.3}]\), the relations (2.6) and (2.7) are equivalent to the relations
\[ [E_i^{(r+1)}, E_i^{(s)}] - [E_i^{(r)}, E_i^{(s+1)}] = E_i^{(r)}E_i^{(s)} + \tilde{E}_i^{(r)}E_i^{(s)}, \quad (2.15) \]
\[ [F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = -F_i^{(r)}F_i^{(s)} - \tilde{F}_i^{(r)}F_i^{(s)}. \quad (2.16) \]

Let \( \bar{\Lambda} = \bigoplus_{i=1}^{n-1} \mathbb{Z}\omega_i \) be the coweight lattice of \( \mathfrak{sl}_n \), where \( \{\omega_i\}_{i=1}^{n-1} \) are the standard fundamental coweights of \( \mathfrak{sl}_n \). There is a natural \( \mathbb{Z} \)-linear projection \( \Lambda \rightarrow \bar{\Lambda}, \mu \mapsto \bar{\mu} \)

defined via \( \alpha_i^{\bar{\mu}}(\bar{\mu}) = \alpha_i^\mu(\mu) \) for \( 1 \leq i \leq n - 1 \).

Equivalently, we have \( \bar{\omega}_0 = 0 \) and \( \bar{\omega}_i = \omega_i \) for \( 1 \leq i \leq n - 1 \).

The algebra \( Y_\mu(\mathfrak{gl}_n) \) depends only on the associated \( \mathfrak{sl}_n \)-c oweight \( \bar{\mu} \), up to an isomorphism:

**Lemma 2.17.** Let \( \mu_1, \mu_2 \in \Lambda \) be \( \mathfrak{gl}_n \)-coweights such that \( \bar{\mu}_1 = \bar{\mu}_2 \) in \( \bar{\Lambda} \). Then, the assignment
\[ E_i^{(r)} \mapsto E_i^{(r)}, \quad F_i^{(r)} \mapsto F_i^{(r)}, \quad D_i^{(s)} \mapsto D_i^{(s)}(\bar{\mu}_1 - \bar{\mu}_2), \quad \tilde{D}_i^{(s)} \mapsto \tilde{D}_i^{(s)}(\bar{\mu}_1 + \bar{\mu}_2) \]

gives rise to a \( \mathbb{C} \)-algebra isomorphism \( Y_{\mu_1}(\mathfrak{gl}_n) \cong Y_{\mu_2}(\mathfrak{gl}_n) \).

**Proof.** The assignment (2.18) is clearly compatible with the defining relations (2.2–2.13), thus, it gives rise to a \( \mathbb{C} \)-algebra homomorphism \( Y_{\mu_1}(\mathfrak{gl}_n) \rightarrow Y_{\mu_2}(\mathfrak{gl}_n) \). Switching the roles of \( \mu_1 \) and \( \mu_2 \), we obtain the inverse homomorphism \( Y_{\mu_2}(\mathfrak{gl}_n) \rightarrow Y_{\mu_1}(\mathfrak{gl}_n) \). Hence, the result. \( \square \)

We define the generating series of the above generators as follows:
\[ E_i(z) := \sum_{r \geq 1} E_i^{(r)} z^{-r}, \quad F_i(z) := \sum_{r \geq 1} F_i^{(r)} z^{-r}, \]
\[ D_i(z) := \sum_{r \geq d_i} D_i^{(r)} z^{-r}, \quad \tilde{D}_i(z) := \sum_{r \geq -d_i} \tilde{D}_i^{(r)} z^{-r} = -D_i(z)^{-1}. \]

The algebras \( Y_\mu(\mathfrak{g}_n) \) slightly generalize the shifted (Drinfeld) Yangians of \( \mathfrak{sl}_n \), denoted by \( Y_\nu(\mathfrak{sl}_n) \) in \([BFNb, \text{ Definition B.2}]\), where \( \nu \in \bar{\Lambda} \) is an \( \mathfrak{sl}_n \)-c oweight. Recall that the latter is an associative \( \mathbb{C} \)-algebra generated by \( \{E_i^{(r)}, F_i^{(r)}, H_i^{(s)}\}_{1 \leq i \leq n, 1 \leq s} \) with the defining relations of \([BFNb, \text{ Definition B.1}]\) and \( H_i^{(s)\bar{b}_i} = 1 \), where \( b_i := \alpha_i^\nu(\nu) \). We define the generating series
\[ E_i(z) := \sum_{r \geq 1} E_i^{(r)} z^{-r}, \quad F_i(z) := \sum_{r \geq 1} F_i^{(r)} z^{-r}, \quad H_i(z) := \sum_{r \geq -b_i} H_i^{(r)} z^{-r}. \]

The explicit relation between the shifted Drinfeld Yangians of \( \mathfrak{sl}_n \) and \( \mathfrak{g}_n \) is as follows:
Proposition 2.19. For any \( \mu \in \Lambda \), there exists a \( \mathbb{C} \)-algebra embedding

\[
\iota_\mu : Y_\mu(\mathfrak{sl}_n) \hookrightarrow Y_\mu(\mathfrak{gl}_n),
\]

uniquely determined by

\[
E_i(z) \mapsto E_i \left( z + \frac{i}{2} \right), \quad F_i(z) \mapsto F_i \left( z + \frac{i}{2} \right), \quad H_i(z) \mapsto -\bar{D}_i \left( z + \frac{i}{2} \right) D_{i+1} \left( z + \frac{i}{2} \right). \tag{2.21}
\]

Remark 2.22. For \( \mu = 0 \), this recovers the classical embedding \( Y(\mathfrak{sl}_n) \hookrightarrow Y(\mathfrak{gl}_n) \) of Yangians.

Proof of Proposition 2.19. As in the \( \mu = 0 \) case (see Remark 2.22), it is straightforward to see that the assignment (2.21) is compatible with the defining relations of \( Y_\mu(\mathfrak{sl}_n) \), giving rise to a \( \mathbb{C} \)-algebra homomorphism \( \iota_\mu : Y_\mu(\mathfrak{sl}_n) \to Y_\mu(\mathfrak{gl}_n) \). It remains to prove the injectivity of \( \iota_\mu \).

First, we note that the coefficients of the series \( D_1(z)D_2(z+1) \cdots D_n(z+n-1) \) are central elements of \( Y_\mu(\mathfrak{gl}_n) \), due to the defining relations (2.2, 2.4, 2.5), cf. [BK1, Theorem 7.2].

Second, given an abstract polynomial algebra \( \mathcal{B} = \mathbb{C}[\{D_i^{(r)}\}_{1 \leq i \leq n}] \), define the elements \( \{\bar{D}_i^{(s)}\}_{1 \leq i \leq n} \) and \( \{C_s\}_{s \geq d_1+ \cdots + d_n} \) of \( \mathcal{B} \) via

\[
\bar{D}_i(z) := z^{d_i-d_{i+1}} + \sum_{s \geq d_{i+1} - d_i} \bar{D}_i^{(s)} z^{-s} = D_i(z) - D_{i+1}(z),
\]

\[
C(z) := z^{-d_1} \cdots z^{-d_n} + \sum_{s \geq d_1+ \cdots + d_n} C_s z^{-s} = \prod_{i=1}^n D_i(z + i - 1),
\]

where \( D_i(z) := z^{-d_i} + \sum_{r > d_i} D_i^{(r)} z^{-r} \). Clearly, \( \{\bar{D}_i^{(s)}\}_{1 \leq i \leq n} \cup \{C_s\}_{s \geq d_1+ \cdots + d_n} \) is an alternative collection of generators of the polynomial algebra \( \mathcal{B} \). In particular, we have

\[
\mathcal{B} \simeq \mathbb{C}[\{C_s\}_{s \geq d_1+ \cdots + d_n}] \otimes _{\mathbb{C}} \mathbb{C}[\{\bar{D}_i^{(s)}\}_{1 \leq i \leq n}] .
\]

Combining these two observations with the defining relations (2.3–2.5), we immediately get \( Y_\mu(\mathfrak{gl}_n) \simeq Z \otimes \mathbb{C} Y_\mu'(\mathfrak{gl}_n) \), where \( Z \) is the central subalgebra generated by \( \{C_s\}_{s \geq d_1+ \cdots + d_n} \) and \( Y_\mu'(\mathfrak{gl}_n) \) is the \( \mathbb{C} \)-subalgebra of \( Y_\mu(\mathfrak{gl}_n) \) generated by \( \{E_i^{(r)}, F_i^{(r)}, D_i^{(s)}\}_{1 \leq i \leq n} \).

Recalling (2.21), we see that \( \iota_\mu : Y_\mu(\mathfrak{sl}_n) \hookrightarrow Y_\mu'(\mathfrak{gl}_n) \), and hence \( \iota_\mu \) is injective. \( \square \)

The following is a shifted version of the decomposition of [Mol, Theorem 1.8.2]:

**Corollary 2.23.** There is a \( \mathbb{C} \)-algebra isomorphism

\[
Y_\mu(\mathfrak{gl}_n) \simeq \mathbb{C}[\{C_s\}_{s \geq d_1+ \cdots + d_n}] \otimes _{\mathbb{C}} Y_\mu(\mathfrak{sl}_n). \tag{2.24}
\]

In particular, \( Y_\mu(\mathfrak{sl}_n) \) may be realized both as a subalgebra of \( Y_\mu(\mathfrak{gl}_n) \) via (2.20) and as a quotient algebra of \( Y_\mu(\mathfrak{gl}_n) \) by the central ideal \( \{C_s\}_{s \geq d_1+ \cdots + d_n} \).

The following result provides a shifted version of the remaining part of [Mol, Theorem 1.8.2]:

**Lemma 2.25.** (a) The center of the shifted Yangian \( Y_\mu(\mathfrak{sl}_n) \) is trivial for any shift \( \nu \in \bar{\Lambda} \).

(b) The center of the shifted Yangian \( Y_\mu(\mathfrak{gl}_n) \) coincides with \( \mathbb{C}[\{C_s\}_{s \geq d_1+ \cdots + d_n}] \) for any \( \mu \in \Lambda \).

As we will not use Lemma 2.25 in the rest of this paper, we will only sketch the proof of part (a) in Remark 2.80, crucially using the result of [W] discussed below. Part (b) follows immediately from (a), the decomposition (2.24), and the centrality of \( C_s \) established above.
2.2. Homomorphism $\Psi_D$.

In this section, we generalize [BFNb, Theorem B.15] for the type $A_{n-1}$ Dynkin diagram with arrows pointing $i \to i + 1$ for $1 \leq i \leq n - 2$ by replacing $Y_\mu(\mathfrak{sl}_n)$ of loc.cit. with $Y_\mu(\mathfrak{g}_n)$.

Remark 2.26. While similar generalizations exist for all orientations of $A_{n-1}$ Dynkin diagram, for the purposes of this paper it suffices to consider only the above orientation, see Remark 2.72.

A $\mathfrak{g}_n$-coweight $\lambda \in \Lambda$ will be called dominant (denoted by $\lambda \in \Lambda^+$) if the corresponding $\mathfrak{sl}_n$-coweight $\bar{\lambda}$ is dominant ($\bar{\lambda} \in \bar{\Lambda}^+$). Thus, $\sum_{i=0}^{n-1} c_i \varpi_i$ is dominant iff $c_i \in \mathbb{N}$ for $1 \leq i \leq n - 1$.

A $\Lambda$-valued divisor $D$ on $\mathbb{P}^1$, $\Lambda^+$-valued outside $\{\infty\} \in \mathbb{P}^1$, is a formal sum

$$D = \sum_{1 \leq s \leq N} \gamma_s \varpi_i[x_s] + \mu[\infty] \quad (2.27)$$

with $N \in \mathbb{N}$, $0 \leq i_s < n$, $x_s \in \mathbb{C}$, $\gamma_s = \begin{cases} 1, & \text{if } i_s \neq 0 \\ \pm 1, & \text{if } i_s = 0 \end{cases}$, and $\mu \in \Lambda$. We will write $\mu = D|_\infty$. If $\mu \in \Lambda^+$, we call $D$ a $\Lambda^+$-valued divisor on $\mathbb{P}^1$. It will be convenient to present

$$D = \sum_{x \in \mathbb{P}^1 \setminus \{\infty\}} \lambda_x[x] + \mu[\infty] \quad \text{with } \lambda_x \in \Lambda^+ \quad (2.28)$$

related to (2.27) via $\lambda_x := \sum_{1 \leq s \leq N} \gamma_s \varpi_i[x]$.

Set $\lambda := \sum_{s=1}^{N} \gamma_s \varpi_i \in \Lambda^+$. We shall assume that

$$\lambda + \mu = a_1 \alpha_1 + \ldots + a_{n-1} \alpha_{n-1} \quad (2.29)$$

with $a_i \in \mathbb{N}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ($1 \leq i \leq n - 1$) are the simple coroots of $\mathfrak{g}_n$. We also set $a_0 := 0, a_n := 0$.

Remark 2.30. (2.29) is equivalent to $\sum_{j=1}^{n} \epsilon_j^{\gamma} (\lambda + \mu) = 0$ and $\sum_{j=1}^{i} \epsilon_j^{\gamma} (\lambda + \mu) \in \mathbb{N}$ for $1 \leq i < n$.

Consider the associative $\mathbb{C}$-algebra

$$\mathcal{A} = \mathbb{C}\langle p_{i,r}, e^{\pm q_{i,r}}, (p_{i,r} - p_{i,s} + m)^{-1} \rangle_{1 \leq i \leq n, m \in \mathbb{Z}} \quad (2.31)$$

with the defining relations

$$[e^{\pm q_{i,r}}, p_{j,s}] = \mp \delta_{i,j} \delta_{r,s} e^{\pm q_{i,r}}, \quad [p_{i,r}, p_{j,s}] = 0 = [e^{\pm q_{i,r}}, e^{\pm q_{j,s}}], \quad e^{\pm q_{i,r}} e^{\mp q_{i,r}} = 1.$$

Remark 2.32. This algebra $\mathcal{A}$ can be represented in the algebra of difference operators with rational coefficients on functions of $\{p_{i,r} \mid 1 \leq i \leq n\}$ by taking $e^{\mp q_{i,r}}$ to be a difference operator $D_{i,r}^{\pm 1}$ that acts as $(D_{i,r}^{\pm 1} \Psi)(p_{i,1}, \ldots, p_{i,r}, \ldots, p_{n-1,a_{n-1}}) = \Psi(p_{i,1}, \ldots, p_{i,r} \pm 1, \ldots, p_{n-1,a_{n-1}})$.

For $0 \leq i \leq n - 1$ and $1 \leq j \leq n - 1$, we define

$$Z_i(z) := \prod_{1 \leq s \leq N} (z - x_s)^{\gamma_s} = \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z - x)^{\alpha_i^{\gamma}(\lambda_x)}, \quad (2.33)$$

$$P_j(z) := \prod_{r=1}^{a_j} (z - p_{j,r}), \quad P_{j,r}(z) := \prod_{1 \leq s \leq a_j} (z - p_{j,s}),$$

where $\alpha_i^{\gamma} := -\epsilon_i^{\gamma}$. We also define $P_0(z) := 1, P_n(z) := 1$.

The following result generalizes $A_{n-1}$-case of [BFNb, Theorem B.15] stated for semisimple Lie algebras $\mathfrak{g}$ (preceded by [GKLO] for the trivial shift and by [KWWY] for dominant shifts):
Theorem 2.34. Let $D$ be as above and $\mu = D|_{\infty}$. There is a unique $\mathbb{C}$-algebra homomorphism

$$\Psi_D : Y_{-\mu}(\mathfrak{gl}_n) \longrightarrow \mathcal{A}$$

(2.35)
such that

$$E_i(z) \mapsto -\sum_{r=1}^{a_i} \frac{P_{i-1}(pi,r-1)Z_i(pi,r)}{(z-p_i,r)P_{i,r}(p_i,r)} e^{q_i,r},$$

$$F_i(z) \mapsto \sum_{r=1}^{a_i} \frac{P_{i+1}(pi,r+1)}{(z-p_i,r-1)P_{i,r}(p_i,r)} e^{-q_i,r},$$

$$D_i(z) \mapsto \frac{P_i(z)}{P_{i-1}(z-1)} \prod_{0 \leq k < i} Z_k(z) = \frac{P_i(z)}{P_{i-1}(z-1)} \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z-x)^{-\epsilon_i^+}.$$

(2.36)

Remark 2.37. Consider a decomposition $\bar{\lambda} = \sum_{1 \leq i \leq N} z_i \omega_i$ and assign $z_s := x_s - \frac{i_s+1}{2} \in \mathbb{C}$ to the $s$-th summand. Identifying $\mathcal{A}$ with $\mathcal{A}$ of [BFNb, §B(ii)] ($z_i$ of loc.cit. are now specialized to complex numbers) via $p_{i,r} \leftrightarrow w_{i,r} + \frac{i}{2}$ and $e^{2q_i,r} \leftrightarrow w_{i,r}^{-1}$, the (restriction) composition $Y_{-\bar{\mu}}(\mathfrak{sl}_n) \stackrel{\mu}{\longrightarrow} Y_{-\mu}(\mathfrak{gl}_n) \stackrel{\Psi_D}{\longrightarrow} \mathcal{A}$ is just the homomorphism $\Phi_{\bar{\lambda}}$ of [BFNb, Theorem B.15].

Proof of Theorem 2.34. First, we need to verify that under the above assignment (2.36), the image of $D_i(z)$ is of the form $z^d_i + (\text{lower order terms in } z)$. Let $\deg_i$ denote the leading power of $z$ in the image of $D_i(z)$ (clearly the coefficient of $z^{\deg_i}$ equals 1). Then, indeed we have

$$\deg_i = a_i - a_{i-1} - \sum_{x \in \mathbb{P}^1 \setminus \{\infty\}} \epsilon_i^+ - \epsilon_i^- = a_i - a_{i-1} - (a_i - a_{i-1} - \epsilon_i^+ - \epsilon_i^-) = d_i.$$ 

Evoking the decomposition (2.24), it suffices to prove that the restrictions of the assignment (2.36) to the subalgebras $Y_{-\bar{\mu}}(\mathfrak{sl}_n)$ and $\mathbb{C}[\{C_s\}_{s>(d_1+\ldots+d_n)}]$ determine algebra homomorphisms, whose images commute. The former is clear for the restriction to $Y_{-\bar{\mu}}(\mathfrak{sl}_n)$, due to Theorem B.15 of [BFNb] combined with Remark 2.37 above. On the other hand, we have

$$\Psi_D(C(z)) = \prod_{i=1}^{n} \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z+i-1-x)^{-\epsilon_i^+} = \prod_{s=1}^{N} \prod_{k=i_s}^{n-1} (z-x_s+k)^{\gamma_s}.$$ 

(2.38)

Thus, the restriction of $\Psi_D$ to the polynomial algebra $\mathbb{C}[\{C_s\}_{s>(d_1+\ldots+d_n)}]$ defines an algebra homomorphism, whose image is central in $\mathcal{A}$. This completes our proof of Theorem 2.34. □

2.3. Antidominantly shifted RTT Yangians of $\mathfrak{gl}_n$.

Consider the rational $R$-matrix $R_{rat}(z) = z + P$, where $P = \sum_{i,j} E_{ij} \otimes E_{ji} \in (\text{End } \mathbb{C}^n)^{\otimes 2}$ is the permutation operator. It satisfies the Yang-Baxter equation with a spectral parameter:

$$R_{rat;12}(u)R_{rat;13}(u+v)R_{rat;23}(v) = R_{rat;23}(v)R_{rat;13}(u+v)R_{rat;12}(u).$$

(2.39)

Fix $\mu \in \Lambda^+$. Define the (antidominantly) shifted RTT Yangian of $\mathfrak{gl}_n$, denoted by $Y_{-\mu}(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}$-algebra generated by $\{\mu_{ij}^{(r)}\}_{r \in \mathbb{Z}}$ subject to the following two families of relations:

- The first family of relations may be encoded by a single RTT relation

$$R_{rat}(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{rat}(z-w),$$

(2.40)
where $T(z) \in Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)[[z, z^{-1}]] \otimes \mathrm{End} \ C^n$ is defined via
\[
T(z) = \sum_{i,j} t_{ij}(z) \otimes E_{ij} \quad \text{with} \quad t_{ij}(z) := \sum_{r \in \mathbb{Z}} t_{ij}^{(r)} z^{-r}.
\]
(2.41)
Thus, (2.40) is an equality in $Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)[[z, z^{-1}, w, w^{-1}]] \otimes \mathrm{C} \ (\mathrm{End} \ C^n)^{\otimes 2}$.

- The second family of relations encodes the fact that $T(z)$ admits the Gauss decomposition:
\[
T(z) = F(z) \cdot G(z) \cdot E(z),
\]
(2.42)
where $F(z), G(z), E(z) \in Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)((z^{-1})) \otimes \mathrm{End} \ C^n$ are of the form
\[
F(z) = \sum_i E_{ii} + \sum_{i < j} f_{ji}(z) \otimes E_{ji}, \quad G(z) = \sum_i g_i(z) \otimes E_{ii}, \quad E(z) = \sum_i E_{ii} + \sum_{i < j} e_{ij}(z) \otimes E_{ij},
\]
with the matrix coefficients having the following expansions in $z$:
\[
e_{ij}(z) = \sum_{r \geq 1} e_{ij}^{(r)} z^{-r}, \quad f_{ji}(z) = \sum_{r \geq 1} f_{ji}^{(r)} z^{-r}, \quad g_i(z) = z^{d_i} + \sum_{r \geq 1-d_i} g_i^{(r)} z^{-r},
\]
(2.43)
where $\{e_{ij}^{(r)}; f_{ji}^{(r)}; g_i^{(s)}\}_{1 \leq i \leq j \leq n} \subset Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)$.

**Remark 2.44.** (a) For $\mu = 0$, the second family of relations (2.42, 2.43) is equivalent to the relations $t_{ij}^{(r)} = 0$ for $r < 0$ and $t_{ij}^{(0)} = \delta_{i,j}$. Thus, $Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)$ is the RTT Yangian of $\mathfrak{gl}_n$ of [FRT].

(b) Likewise, (2.42) is equivalent to a certain family of algebraic relations on $t_{ij}^{(r)}$. In particular, $T(z) \in Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)((z^{-1})) \otimes \mathrm{End} \ C^n$. For example, (2.43) for $i = 1$ are equivalent to:
\[
t_{11}^{(-d_1)} = 1 \quad \text{and} \quad t_{11}^{(r)} = 0 \quad \text{for} \quad r < -d_1,
\]
\[
t_{1j}^{(r)} = 0 \quad \text{for} \quad r \leq -d_1, 1 < j \leq n.
\]
(c) If $\mu \notin \Lambda^+$, then the above two families of relations are contradictory and thus the algebra $Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)$ is trivial, see Remark 2.49.

**Lemma 2.45.** For any $1 \leq i < j \leq n$ and $r \geq 1$, we have the following identities:
\[
e_{ij}^{(r)} = [e_{ij}^{(1)}, e_{ij}^{(2)}, \cdots, e_{ij}^{(r)}],
\]
\[
f_{ji}^{(r)} = [f_{ji}^{(1)}, f_{ji}^{(2)}, \cdots, f_{ji}^{(r)}],
\]
\[
g_i(z) = z^{d_i} + \sum_{s \geq 1-d_i} g_i^{(s)} z^{-s}.
\]
(2.46)
**Proof.** The proof is analogous to that of [BK1, (5.5)] (see also [FT2, Corollary 2.38]).

**Corollary 2.47.** The algebra $Y_{-\mu}^{\mathrm{RTT}}(\mathfrak{gl}_n)$ is generated by $\{e_{i,i+1}^{(r)}, f_{i+1,i}^{(r)}; g_j^{(s)}\}_{1 \leq i < n, 1 \leq j \leq n}^{r \geq 1, s \geq 1-d_j}$.

The following result is proved completely analogously to [BK1, Lemmas 5.4, 5.5, 5.7]:

**Lemma 2.48.** The following identities hold:
(a) $[g_i(z), g_j(w)] = 0$;
(b) $(z-w) [g_i(z), e_{j,j+1}(w)] = (\delta_{i,j} - \delta_{i,j+1}) g_i(z) (e_{j,j+1}(z) - e_{j,j+1}(w))$;
(c) $(z-w) [g_i(z), f_{j,j+1}(w)] = (\delta_{i,j+1} - \delta_{i,j}) (f_{j,j+1}(z) - f_{j,j+1}(w)) g_i(z)$;
(d) $[e_{i,i+1}(z), f_{j,j+1}(w)] = 0$ if $i \neq j$;
(e) $(z-w) [e_{i,i+1}(z), f_{i+1,i}(w)] = g_i(w)^{-1} g_{i+1}(w) - g_i(z)^{-1} g_{i+1}(z)$;
(f) $(z-w) [e_{i,i+1}(z), e_{i,i+1}(w)] = -(e_{i,i+1}(z) - e_{i,i+1}(w))^2$;
(g) $(z-w) [e_{i,i+1}(z), e_{i+1,i+2}(w)] = -e_{i,i+1}(z) e_{i+1,i+2}(w) + e_{i,i+1}(w) e_{i+1,i+2}(w) - e_{i,i+2}(w) + e_{i,i+2}(z)$;
(h) \([e_{i,i+1}(z), e_{j,j+1}(w)] = 0\) if \(|i-j| > 1\);
(i) \([e_{i,i+1}(z_1), [e_{i,i+1}(z), e_{j,j+1}(w)]] + [e_{i,i+1}(z_2), [e_{i,i+1}(z_1), e_{j,j+1}(w)]] = 0\) if \(|i-j| = 1\);
(j) \((z-w)[f_{i+1,i}(z), f_{i+1,i}(w)] = (f_{i+1,i}(z) - f_{i+1,i}(w))^2\);
(k) \((z-w)[f_{i+1,i}(z), f_{i+2,i+1}(w)] = f_{i+2,i+1}(w)f_{i+1,i+2}(w)f_{i+1,i}(w) + f_{i+2,i}(w) - f_{i+2,i}(z)\);
(l) \([f_{i+1,i}(z), f_{i+1,i}(w)] = 0\) if \(|i-j| > 1\);
(m) \([f_{i+1,i}(z_1), [f_{i+1,i}(z_2), f_{i+1,i}(w)]] + [f_{i+1,i}(z_2), [f_{i+1,i}(z_1), f_{i+1,i}(w)]] = 0\) if \(|i-j| = 1\).

**Remark 2.49.** If \(d_i < d_{i+1}\) for some \(1 < i < n\), then the right-hand side of the identity in Lemma 2.48(e) contains monomials \(z^{d_i+1-d_i}\) and \(w^{d_{i+1}-d_i}\), while all monomials in the left-hand side have negative degrees. Thus, the defining relations of \(Y_{rtt}(\mathfrak{gl}_n)\) are contradictory unless \(\mu\) is dominant (see [FT1, Remark 11.23]) for the trigonometric \(sl_2\)-counterpart of this conclusion.

**Remark 2.50.** The right-hand sides in all identities of Lemma 2.48 have opposite signs to those of [BK1, §5], due to a different choice of the \(R\)-matrix \(R(z) = z - P = -R_{rat}(-z)\) in [BK1].

Comparing the identities of Lemma 2.48 with the defining relations (2.2–2.13) of \(Y_{-\mu}(\mathfrak{gl}_n)\) and recalling Corollary 2.47, we immediately obtain:

**Theorem 2.51.** For any \(\mu \in \Lambda^+\), there is a unique \(\mathbb{C}\)-algebra epimorphism

\[ \Upsilon_{-\mu} : Y_{-\mu}(\mathfrak{gl}_n) \rightarrow Y_{rtt}(\mathfrak{gl}_n) \]

defined by

\[ E_i(z) \mapsto e_{i,i+1}(z), \quad F_i(z) \mapsto f_{i+1,i}(z), \quad D_j(z) \mapsto g_j(z). \]  

(2.52)

Our first main result (the proof of which is postponed till Section 2.4.3) is:

**Theorem 2.53.** \(\Upsilon_{-\mu} : Y_{-\mu}(\mathfrak{gl}_n) \xrightarrow{\sim} Y_{rtt}(\mathfrak{gl}_n)\) is a \(\mathbb{C}\)-algebra isomorphism for any \(\mu \in \Lambda^+\).

**Remark 2.54.** (a) For \(\mu = 0\) and any \(n\), the isomorphism \(\Upsilon_0 : Y(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{rtt}(\mathfrak{gl}_n)\) of Theorem 2.53 was stated in [D], but was properly established only in [BK1, Theorem 5.2].

(b) For \(n = 2\) and \(\mu \in \Lambda^+\), a long straightforward verification shows (see [FT1, Remark 11.26]) that the assignment

\[ t_{11}(z) \mapsto D_1(z), \quad t_{22}(z) \mapsto F_1(z)D_1(z)E_1(z) + D_2(z), \]
\[ t_{12}(z) \mapsto D_1(z)E_1(z), \quad t_{21}(z) \mapsto F_1(z)D_1(z), \]

gives rise to an algebra homomorphism \(Y_{rtt}(\mathfrak{gl}_2) \rightarrow Y_{-\mu}(\mathfrak{gl}_2)\) (the trigonometric \(sl_2\)-counterpart of this result has been properly established in [FT1, Theorem 11.19]), which is clearly the inverse of \(\Upsilon_{-\mu}\). Thus, Theorem 2.53 for \(n = 2\) is essentially due to [FT1].

2.4. Rational Lax matrices via antidominantly shifted Yangians of \(\mathfrak{gl}_n\).

In this section, we construct \(n \times n\) rational Lax matrices \(T_D(z)\) (with coefficients in \(A((z^{-1}))\)) for each \(\Lambda^+\)-valued divisor \(D\) on \(\mathbb{P}^1\) satisfying (2.29). They are explicitly defined via (2.62, 2.63) combined with (2.57, 2.59, 2.61). We note that these long formulas arise naturally as the image of \(T(z) \in Y_{rtt}(\mathfrak{gl}_n)((z^{-1})) \otimes \mathbb{C} \text{End} \mathbb{C}^n\) under the composition \(\Psi_D \circ \Upsilon_{-\mu} : Y_{rtt}(\mathfrak{gl}_n) \rightarrow A\), assuming Theorem 2.53 has been established, see (2.55, 2.56). As the name indicates, \(T_D(z)\) satisfy the RTT relation (2.40), which is derived in Proposition 2.78. Combining the latter with the results of [W], see Theorem 2.79, we finally prove Theorem 2.53 in Section 2.4.3.

We also establish the regularity (up to a rational factor (2.65)) of \(T_D(z)\) in Theorem 2.66, and find simplified explicit formulas for those \(T_D(z)\) which are linear in \(z\) in Theorem 2.84.
2.4.1. Construction of $T_D(z)$ and their regularity.

Consider a $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$, satisfying an extra condition (2.29); note that $\mu = D|_{\infty} \in \Lambda^+$. Composing $\Psi_D: Y_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$ of (2.35) with $\Upsilon_{-\mu}(\mathfrak{gl}_n) \to Y_{-\mu}(\mathfrak{gl}_n)$ (assuming the validity of Theorem 2.53), gives rise to an algebra homomorphism

$$\Theta_D = \Psi_D \circ \Upsilon_{-\mu}: Y_{-\mu}(\mathfrak{gl}_n) \longrightarrow \mathcal{A}. \quad (2.55)$$

Such a homomorphism is uniquely determined by $T_D(z) \in \mathcal{A}((z^{-1})) \otimes \mathbb{C} \text{End} \mathbb{C}^n$ defined via

$$T_D(z) := \Theta_D(T(z)) = \Theta_D(F(z)) \cdot \Theta_D(G(z)) \cdot \Theta_D(E(z)). \quad (2.56)$$

Let us compute explicitly the images of the matrices $F(z), G(z), E(z)$ under $\Theta_D$, which shall provide an explicit formula for the matrix $T_D(z)$ via (2.56).

Combining $\Upsilon_{-\mu}(g_i(z)) = D_i(z)$ with the formula for $\Psi_D(D_i(z))$, we obtain:

$$\Theta_D(g_i(z)) = \frac{P_i(z)}{P_{i-1}(z-1)} \prod_{0 < k < i} Z_k(z) = \frac{P_i(z)}{P_{i-1}(z-1)} \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (x - z)^{-\lambda_i^+(\lambda_x)}. \quad (2.57)$$

Combining $\Upsilon_{-\mu}(e_{i,i+1}(z)) = E_i(z)$ with the formula for $\Psi_D(E_i(z))$, we obtain:

$$\Theta_D(e_{i,i+1}(z)) = - \sum_{r=1}^{a_i} \frac{P_{i-1}(p_{i,r} - 1)Z_i(p_{i,r})}{(z - p_{i,r})P_{i,r}(p_{i,r})} e^{q_{i,r}}. \quad (2.58)$$

As $e_{ij}(z) = [e_{i-1,j-1}^{(1)}, \ldots, e_{i+1,j+1}^{(1)}, e_{i,j+1}(z), \ldots]$ due to (2.46), we obtain (cf. [FT2, (2.66)]):

$$\Theta_D(e_{ij}(z)) = \sum_{1 \leq r_i \leq a_i} \prod_{1 \leq j < a_{i+1}} P_{i+1}(p_{i,r} + 1) \prod_{k=i+1}^{j-1} P_k(p_{i,r}) \frac{P_{i+1}(p_{i,r} + 1)}{(z - p_{i,r} - 1)P_{i,r}(p_{i,r})} e^{-q_{i,r}}. \quad (2.59)$$

Combining $\Upsilon_{-\mu}(f_{i+1,i}(z)) = F_i(z)$ with the formula for $\Psi_D(F_i(z))$, we obtain:

$$\Theta_D(f_{i+1,i}(z)) = \sum_{r=1}^{a_i} \frac{P_{i+1}(p_{i,r} + 1)}{(z - p_{i,r} - 1)P_{i,r}(p_{i,r})} e^{-q_{i,r}}. \quad (2.60)$$

As $f_{ji}(z) = [\cdots [f_{i+1,i}(z), f_{i+2,i+1}(z)], \ldots, f_{j-1,i}^{(1)}]$ due to (2.46), we obtain (cf. [FT2, (2.67)]):

$$\Theta_D(f_{ji}(z)) = \sum_{1 \leq r_j \leq a_j} \prod_{1 \leq j < a_{i+1}} P_{j+1}(p_{j-1,r_j + 1}) \prod_{k=i+1}^{j-1} P_k(p_{j-1,r_j}) \frac{P_{j+1}(p_{j-1,r_j + 1})}{(z - p_{j-1,r_j} - 1)P_{j-1,r_j}(p_{j-1,r_j})} e^{-\sum_{k=i}^{j-1} q_{k,r_k}}. \quad (2.61)$$

While the derivation of the formulas (2.57, 2.59, 2.61) was based on yet unproved Theorem 2.53, we shall use their explicit right-hand sides from now on, without a referral to Theorem 2.53. More precisely, define $\mathcal{A}((z^{-1}))$-valued $n \times n$ diagonal matrix $G_D(z)$, an upper triangular matrix $E_D(z)$, and a lower triangular matrix $F_D(z)$, whose matrix coefficients $g_i^D(z), e_{ij}^D(z), f_{ji}^D(z)$ are given by the right-hand sides of (2.57, 2.59, 2.61) expanded in $z^{-1}$, respectively. From now on, we shall define

$$T_D(z) := F_D(z)G_D(z)E_D(z), \quad (2.62)$$
so that the matrix coefficients of \( T_D(z) \) are given by

\[
T_D(z)_{\alpha,\beta} = \sum_{i=1}^{\min\{\alpha,\beta\}} f_{\alpha,i}^D(z) \cdot g_i^D(z) \cdot e_{i,\beta}^D(z) \quad (2.63)
\]

for any \( 1 \leq \alpha, \beta \leq n \), where the three factors in the right-hand side of (2.63) are determined via (2.61, 2.57, 2.59), respectively, with the conventions \( f_{\alpha,\alpha}^D(z) = 1 = e_{\beta,\beta}^D(z) \).

**Remark 2.64.** We note that \( T_D(z) \) is singular at \( x \in \mathbb{C} \) iff \( \lambda_x \neq 0 \). As \( F_D(z) \) and \( E_D(z) \) are regular in the neighbourhood of \( x \), while \( G_D(z) = (\text{regular part}) \cdot (z - x)^{-\lambda_x} \), we see that in the classical limit \( T_D(z) \) represents a \( GL_n \)-multiplicative Higgs field on \( \mathbb{P}^1 \) with a framing at \( \infty \in \mathbb{P}^1 \) (rational type) and with prescribed singularities on \( D \), cf. [EP].

We shall need normalized rational Lax matrices:

\[
T_D(z) := \frac{T_D(z)}{Z_0(z)},
\]

with the normalization factor determined via (2.33):

\[
\frac{1}{Z_0(z)} = \prod_{1 \leq i \leq N} (z - x_i)^{-\gamma_i} = \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z - x)^{-\alpha_i^0(\lambda_x)}.
\]

The first main result of this section establishes the regularity of the matrix \( T_D(z) \):

**Theorem 2.66.** We have \( T_D(z) \in \mathcal{A}[z] \otimes \text{End} \cdot \mathbb{C}^n \).

**Proof.** In view of (2.63), it suffices to prove for any \( 1 \leq \alpha, \beta \leq n \) that

\[
\frac{1}{Z_0(z)} \sum_{i=1}^{\min\{\alpha,\beta\}} f_{\alpha,i}^D(z) \cdot g_i^D(z) \cdot e_{i,\beta}^D(z) \text{ is polynomial in } z,
\]

where the factors in the right-hand side are determined via (2.61, 2.57, 2.59), respectively. The \( i \)-th summand in (2.67) is explicitly given by

\[
Z_0(z)^{-1} \cdot f_{\alpha,i}^D(z) \cdot g_i^D(z) \cdot e_{i,\beta}^D(z) =
- \sum_{1 \leq r_1 \leq a_1} \cdots \sum_{1 \leq r_{a_1-1} \leq a_{a_1-1}} \frac{P_{i+1,r_{i+1}}(p_{i,r_i} + 1) \cdots P_{\alpha-1,r_{a_1-1}}(p_{\alpha-2,r_{\alpha-2}} + 1)P_{\alpha}(p_{\alpha-1,r_{\alpha-1}} + 1)}{(z - p_{i,r_i} - 1)P_{i,r_i}(p_{i,r_i}) \cdots P_{\alpha-1,r_{\alpha-1}}(p_{\alpha-1,r_{\alpha-1}})}
\times
\]

\[
e^{-q_i}\sum_{1 \leq s_i \leq a_i} \cdots \sum_{1 \leq s_{\beta-1} \leq a_{\beta-1}} P_{i-1}(p_{i,s_i} - 1)P_{i,s_i}(p_{i+1,s_{i+1}} - 1) \cdots P_{\beta-2,s_{\beta-2}}(p_{\beta-1,s_{\beta-1}} - 1) \times
\]

\[
\frac{P_{i}(z)}{P_{i-1}(z - 1)} \cdot Z_1(z) \cdots Z_{i-1}(z) \times
\]

\[
Z_i(p_{i,s_i}) \cdots Z_{\beta-1}(p_{\beta-1,s_{\beta-1}}) \cdot e^{q_i + q_{i+1} + \cdots + q_{\beta-1}}.
\]
Moving \(e^{-q_{i_1}r_1-\ldots-q_{a_1-1}r_{a_1-1}}\) to the rightmost side, we rewrite the right-hand side of (2.68) as

\[
\sum_{1 \leq s_i \leq a_i} e^{-q_{a_1}r_1-\ldots-q_{a_1-1}r_{a_1-1}} \cdot e^{q_{i_1}s_1+\ldots+q_{a_1-1}s_{a_1-1}}.
\]

The coefficient \(Q_{r_1,\ldots,r_{a_1}}^{s_1,\ldots,s_{a_1-1}}(z)\) is a rational function in \(z\) with simple poles at:
- \(\left\{1 + p_{i-1,s}\left| 1 \leq s \leq a_i-1 \right. \right\}\) if \(r_i \neq s_i\);
- \(\left\{1 + p_{i-1,s}\left| 1 \leq s \leq a_i-1 \right. \right\} \cup \left\{1 + p_{i,r}\right\}\) if \(r_i = s_i\).

Thus, the only (at most simple) poles of (2.67) are at \(\{1 + p_{i,r}\mid 1 \leq i < \min\{\alpha, \beta\}, 1 \leq r \leq a_i\}\). The following straightforward result actually shows that the residues at these points vanish:

**Lemma 2.69.** For any \(1 \leq i < \min\{\alpha, \beta\}, 1 \leq r \leq a_i\), and any admissible collection of indices \(r_{i+1}, \ldots, r_{a_1-1}, s_{i+1}, \ldots, s_{a_1-1}\), we have the equality

\[
\text{Res}_{z=1+p_{i,r}} (Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z)dz) + \text{Res}_{z=1+p_{i,r}} (Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z)dz) = 0. \tag{2.70}
\]

**Proof of Lemma 2.69.** Applying the explicit formula (2.68), we find

\[
Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z)e^{-q_{i+1}r_{i+1}} = \frac{A}{z - p_{i+1},r_{i+1} - 1} \cdot \frac{P_{i+1}(z)}{P_i(z)} \cdot Z_i(z) \cdot \frac{P_i(p_{i+1,s_{i+1}} - 1)}{z - p_{i+1,s_{i+1}}}
\]

and

\[
Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z)e^{-q_{i+1}r_{i+1}} = \frac{A \cdot P_{i+1}r_{i+1}(p_{i,r} + 1)}{(z - p_{i,r} - 1)P_{i,r}(p_{i,r})} \cdot \frac{P_i(z)}{P_{i+1}(z)} \cdot \frac{P_{i-1}(p_{i+r})P_i(p_{i+1,s_{i+1}} - 1 + \delta_{r_{i+1},s_{i+1}})Z_i(p_{i,r} + 1)}{(z - p_{i,r} - 1)(z - p_{i,r} + 1)} \cdot e^{q_{i,r}},
\]

where \(A\) is a common \((z, p_{i+1,r_{i+1}})\)-independent factor (its explicit form is irrelevant for us). Hence,

\[
Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z) = A \cdot \frac{P_{i+1,r_{i+1}}(p_{i,r} + 1)}{P_{i,r}(p_{i,r})} \cdot \frac{P_{i,r}(z)}{P_{i-1}(z - 1)} \cdot \frac{P_{i-1}(p_{i+r})P_i(p_{i+1,s_{i+1}} - 1 + \delta_{r_{i+1},s_{i+1}})}{(z - p_{i,r} - 1)} \cdot Z_i(p_{i,r} + 1).
\]

Therefore, the corresponding residues are given by

\[
\text{Res}_{z=1+p_{i,r}} (Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z)dz) = A \cdot \frac{P_{i+1,r_{i+1}}(p_{i,r} + 1)}{P_{i,r}(p_{i,r})} \cdot Z_i(p_{i,r} + 1) \cdot \left(-P_i(p_{i+1,s_{i+1}} - 1 + \delta_{r_{i+1},s_{i+1}})\right),
\]

\[
\text{Res}_{z=1+p_{i,r}} (Q_{r_{i+1},\ldots,r_{a_1-1}}^{s_{i+1},\ldots,s_{a_1-1}}(z)dz) = A \cdot \frac{P_{i+1,r_{i+1}}(p_{i,r} + 1)}{P_{i,r}(p_{i,r})} \cdot \frac{P_{i,r}(p_{i,r} + 1)}{P_{i-1}(p_{i,r})} \cdot \frac{P_{i-1}(p_{i+r})P_i(p_{i+1,s_{i+1}} - 1 + \delta_{r_{i+1},s_{i+1}})}{(z - p_{i,r} + 1)(z - p_{i,r} - 1)} \cdot Z_i(p_{i,r} + 1),
\]

thus summing up to zero and implying (2.70).

\[\square\]

This completes our proof of (2.67) and, hence, of Theorem 2.66.
Remark 2.71. Similar cancelation of poles appeared in the work on $q$-characters [FR] and $qq$-characters [N].

Remark 2.72. Similar to [BFNb, Theorem B.15], one can generalize Theorem 2.34 by constructing the homomorphisms $\Psi_D : Y_{-\mu}(gl_n) \to A$ for any orientation of $A_{n-1}$ Dynkin diagram (so that $\Psi_D \circ \iota_{-\mu} = \Phi_{-\mu}$ as in Remark 2.37, while the images of $D_i(z)$ are given by the same formulas as in (2.36)). However, extending $A$ to its localization $A_{loc}$ by the multiplicative set generated by $\{ p_{i,r} - p_{i+1,s} + m \}_{r \leq \alpha_i, s \leq \alpha_i + 1}^m \in \mathbb{Z}$; these homomorphisms are compositions of the one from (2.35) with algebra automorphisms of $A_{loc}$. Thus, the resulting rational Lax matrices are equivalent to $T_D(z)$ constructed above via algebra automorphisms of $A_{loc}$, cf. Remark 2.26.

2.4.2. Normalized limit description and the RTT relation for $T_D(z)$.

Consider a $\Lambda^+$-valued divisor $D = \sum_{s=1}^N \gamma_s \varpi_{i_s}[x_s] + \mu[\infty]$. As $x_N \to \infty$, we obtain another $\Lambda^+$-valued divisor $D' = \sum_{s=1}^{N-1} \gamma_s \varpi_{i_s}[x_s] + (\mu + \gamma_N \varpi_{i_N})[\infty]$. We will relate $T_{D'}(z)$ to $T_D(z)$.

If $i_N = 0$, then
$$T_{D'}(z) = (z - x_N)^{-\varpi_N} T_D(z), \quad (2.73)$$
due to $F_D(z) = F_{D'}(z)$, $E_D(z) = E_{D'}(z)$, $G_D(z) = (z - x_N)^{\varpi_N} G_{D'}(z)$ and (2.62).

Let us now consider the case $1 \leq i_N \leq n - 1$ (note that $\gamma_N = 1$), so that $(-x_N)^{-\varpi_N} = \text{diag}(1, (-x_N)^{-n-1})$ is the diagonal $n \times n$ matrix with the first $i_N$ diagonal entries equal to 1 and the remaining $n - i_N$ entries equal to $-x_N^{-1}$.

Proposition 2.74. The $x_N \to \infty$ limit of $T_D(z) \cdot (-x_N)^{-\varpi_N}$ equals $T_{D'}(z)$.

Proof. Due to (2.62), $T_D(z) = F_D(z) G_D(z) E_D(z)$ and $T_{D'}(z) = F_{D'}(z) G_{D'}(z) E_{D'}(z)$ with the three factors determined explicitly via (2.61, 2.57, 2.59). Hence, $T_D(z) \cdot (-x_N)^{-\varpi_N}$ has the following Gauss decomposition:
$$T_D(z) \cdot (-x_N)^{-\varpi_N} = F_D(z) \cdot (G_D(z)(-x_N)^{-\varpi_N}) \cdot ((-x_N)^{-\varpi_N} E_D(z)(-x_N)^{-\varpi_N}). \quad (2.75)$$

The leftmost factor in the right-hand side of (2.75) does not depend on $\{ x_s \}_{s=1}^N$ and coincides with $F_D(z)$. As $G_D(z) = (z - x_N)^{-\varpi_N} G_{D'}(z)$ and $\lim_{x_N \to \infty} \frac{z - x_N}{-x_N} = 1$, it is clear that the $x_N \to \infty$ limit of the diagonal factor $G_D(z)(-x_N)^{-\varpi_N}$ of (2.75) coincides with $G_{D'}(z)$. Finally, the matrix coefficients of the upper-triangular factor in (2.75) are $(((-x_N)^{-\varpi_N} E_D(z)(-x_N)^{-\varpi_N})_{\alpha \beta} = e_{D}^{\alpha \beta}_{r}, (-x_N)^{-\varpi_N} E_D(z)(-x_N)^{-\varpi_N})$ and their $x_N \to \infty$ limits exactly coincide with $e_{D'}^{\alpha \beta}(z)$, the matrix coefficients of $E_{D'}(z)$.

This completes our proof of Proposition 2.74. \hfill \Box.

Corollary 2.76. $T_{D'}(z)$ is a normalized limit of $T_D(z)$.

If $\mu \in \Lambda^+$, we can pick a $\Lambda^+$-valued divisor $\bar{D} = \sum_{s=1}^{N+M} \gamma_s \varpi_{i_s}[x_s]$, so that $\{ x_s \}_{s=N+1}^{N+M}$ are some points on $\mathbb{P}^1 \setminus \{ \infty \}$ while $\sum_{s=N+1}^{N+M} \gamma_s \varpi_{i_s} = \mu$. Note that $\infty \not\in \text{supp}(\bar{D})$, i.e. $\bar{D}|_{\infty} = 0$.

Corollary 2.77. For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (2.29), $T_D(z)$ is a normalized limit of $T_D(z)$ with $\infty \not\in \text{supp}(\bar{D})$.

Evoking Remark 2.54(a), we see that the original definition of $T_D(z)$ via (2.55, 2.56) is valid. Hence, $T_D(z)$ defined via (2.62) indeed satisfies the RTT relation (2.40). As a multiplication by diagonal $z$-independent matrices preserves (2.40), we obtain the main result of this section:

Proposition 2.78. For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (2.29), the matrix $T_D(z)$ defined via (2.62) indeed satisfies the RTT relation (2.40).
2.4.4. Linear rational Lax matrices.

Due to Proposition 2.78 and the Gauss decomposition (2.62, 2.63) of \( T_D(z) \) with the factors defined via (2.57, 2.59, 2.61), we see that \( T_D(z) \) indeed gives rise to the algebra homomorphism \( \Theta_D : Y^{\text{unt}}(g\mathfrak{g}_n) \to A \), whose composition with the epimorphism \( \Upsilon_{-\mu} : Y_{-\mu}(g\mathfrak{g}_n) \to Y^{\text{unt}}(g\mathfrak{g}_n) \) of Theorem 2.51 coincides with the homomorphism \( \Psi_D \) of (2.35). Thus, for \( \mu \in \Lambda^+ \) and any \( \Lambda^+ \)-valued divisor \( D \) on \( \mathbb{P}^1 \) (2.27) satisfying (2.29), the homomorphism \( \Psi_D \) factors via \( \Upsilon_{-\mu} \).

The latter immediately implies the injectivity of \( \Upsilon_{-\mu} \), due to the \( g\mathfrak{g}_n \)-counterpart of the following recent result of Alex Weekes:

**Theorem 2.79 ([W]).** For any coweight \( \nu \) of a semisimple Lie algebra \( \mathfrak{g} \), the intersection of kernels of the homomorphisms \( \Phi^*_{-\nu} \) of [BPNb, Theorem B.15] is zero: \( \bigcap_\lambda \ker(\Phi^\lambda_{-\nu}) = 0 \), where \( \lambda \) ranges through all dominant coweights of \( \mathfrak{g} \) such that \( \lambda + \nu = \sum_{i=1}^n a_i \alpha_i \) with \( a_i \in \mathbb{N}, \alpha_i \) being simple coroots of \( \mathfrak{g} \) and points \( \{z_i\} \) of loc.cit. specialized to arbitrary complex parameters.

This completes our proof of Theorem 2.53.

**Remark 2.80.** (A. Weekes) Using similar arguments, one can show that the center of the shifted Yangian \( Y_\nu(g) \) is trivial (cf. Lemma 2.25(a)) for any coweight \( \nu \) of a semisimple Lie algebra \( g \). Indeed, due to Theorem 2.79, it suffices to show that the \( \Phi^\lambda_{-\nu} \)-images have no nonconstant central elements. Assuming \( x \) is central, one can show it is a symmetric rational function in \( p_{x,x} \) (as \( \text{Im}(\Phi^\lambda_{-\nu}) \) contains all symmetric polynomials in \( p_{x,x} \)), and then show that it is actually \( p_{x,x} \)-independent (using the commutativity with the images of \( E_i(z), F_i(z) \)).

2.4.3. Proof of Theorem 2.53.

In this section, we will obtain simplified explicit formulas for all \( T_D(z) \) that are linear in \( z \).

First, let us note that elements of \( \Lambda^+ \) may be encoded by weakly decreasing sequences \( \lambda \) of \( n \) integers \( \lambda_1 \geq \cdots \geq \lambda_n \), which we call pseudo Young diagrams \( \lambda \) (as \( \text{Im}(\Phi^\lambda_{-\nu}) \) contains all symmetric polynomials in \( p_{x,x} \)), and then show that it is actually \( p_{x,x} \)-independent (using the commutativity with the images of \( E_i(z), F_i(z) \)).

Explicitly, such a pseudo Young diagram \( \lambda = (\lambda_1, \cdots, \lambda_n) \) encodes a dominant coweight \( \lambda \in \Lambda^+ \) via

\[
\lambda := - \sum_{1 \leq i \leq n} \lambda_{n-i+1} \epsilon_i = \lambda_n \omega_0 + \sum_{1 \leq i \leq n-1} (\lambda_{n-i} - \lambda_{n-i+1}) \omega_i. \tag{2.81}
\]

We denote \( |\lambda| := \sum_{i=1}^n \lambda_i \). If \( \lambda_n \geq 0 \), then \( \lambda \) is a standard Young diagram of length \( \leq n \).

Fix a pair of pseudo Young diagrams \( \lambda, \mu \). Then, \( \lambda + \mu \) is of the form \( \lambda + \mu = \sum_{i=1}^n a_i \alpha_i \) for some \( a_i \in \mathbb{C} \) iff \( |\lambda| + |\mu| = 0 \). Let us establish the key properties of \( a_i \) in the latter case:

**Lemma 2.82.** (a) \( a_i = - \sum_{j=i+1}^n (\lambda_j + \mu_j) \) for any \( 1 \leq i \leq n - 1 \).
(b) \( a_i \in \mathbb{N} \) for any \( 1 \leq i \leq n - 1 \).
(c) \( a_j - a_{j+1} = -\lambda_{n-j+1} - \mu_{n-j+1} \) for any \( 1 \leq j \leq n \), where we set \( a_0 := 0, a_n := 0 \).

**Proof.** (c) Follows from the equality

\[
\sum_{1 \leq j \leq n} (a_j - a_{j-1}) \epsilon_j = \sum_{1 \leq i \leq n-1} a_i \alpha_i = \lambda + \mu = \sum_{1 \leq j \leq n} (-\lambda_{n-j+1} - \mu_{n-j+1}) \epsilon_j.
\]

(a) Follows by summing the equalities of part (c) for \( j = 1, \ldots, i \).
(b) As \( -\lambda_{n} - \mu_{n} \geq -\lambda_{n-1} - \mu_{n-1} \geq \cdots \geq -\lambda_1 - \mu_1 \), we have an obvious inequality \( \sum_{j=n-i+1}^n (-\lambda_j - \mu_j) \geq \frac{1}{n} \sum_{j=1}^n (-\lambda_j - \mu_j) = 0 \). Thus \( a_i \in \mathbb{N} \), due to part (a).

Thus, \( \Lambda^+ \)-valued divisors on \( \mathbb{P}^1 \) satisfying (2.29) and without summands \( \{-\omega_0[x]\}_{x \in \mathbb{C}} \) may be encoded by pairs \( (\lambda, \mu) \) of a Young diagram \( \lambda \) of length \( \leq n \) and a pseudo Young diagram
\[ \mu \text{ with } n \text{ rows and of total size } |\lambda| + |\mu| = 0, \text{ together with a collection of points } x = \{x_i\}_{i=1}^\lambda \text{ of } \mathbb{C} \text{ (so that } x_i \text{ is assigned to the } i\text{-th column of } \lambda). \text{ Explicitly, given } \lambda, \mu, x \text{ as above, we set } \\
D = D(\lambda, x, \mu) := \sum_{i=1}^\lambda \omega_n - |\lambda_i| [x_i] + \mu[\infty], \text{ where } \lambda_i \text{ is the height of the } i\text{-th column of } \lambda. \]

Due to (2.73), we may further assume that \( D \) does not contain summands \( \{\pm \omega_0[x]\}_{x \in \mathbb{C}} \).

Thus, \( \lambda_n = 0 \) so that \( Z_0(z) = 1 \) and \( T_D(z) = T_D(z) \) is polynomial in \( z \) by Theorem 2.66. Moreover, \( T_D(z)_{11} = g_D(z) \) is a polynomial in \( z \) of degree \( a_1 = -(\lambda_n + \mu_n) = -\mu_n \geq 0 \). We shall also have \( -\mu_n \leq 1 \) for linear Lax matrices \( T_D(z) \). If \( \mu_n = 0 \), then \( \lambda_i = \mu_i = 0 \) for all \( i \), and so \( T_D(z) = T_D(z) = I_n \), the identity matrix. Thus, it remains to treat the case when \( \lambda_n = 0 \) and \( \mu_n = -1 \), which constitutes the key result of this section.

**Remark 2.83.** If \( |\lambda| + |\mu| = 0, \lambda_0 = 0, \mu_n = -1 \), then \( \lambda \) and \( \tilde{\mu} = (\mu_1 + 1, \ldots, \mu_n + 1) \) form a pair of Young diagrams of total size \( |\lambda| + |\tilde{\mu}| = n \). In that setup, an alternative construction of rational Lax matrices \( L_{\lambda, \tilde{\mu}}(z) \) was recently proposed in [FP]. In Section 2.5, we shall compare all explicit Lax matrices \( L_{\lambda, \tilde{\mu}, \tilde{\lambda}}(z) \) of [FP] to the corresponding Lax matrices \( T_D(z) \). However, we do not have an interpretation of the “fusion procedure” of [FP] (used to construct all \( L_{\lambda, \tilde{\mu}, \tilde{\lambda}}(z) \) from the aforementioned explicit “building blocks”) in our current approach.

**Theorem 2.84.** Following the above notations, assume further that \( \lambda_n = 0 \) and \( \mu_n = -1 \). Define \( m := \max \{i | \mu_{n-i+1} = -1\} \) and \( m' := \max \{i | \mu_{n-i+1} \leq 0\} \).

(a) The rational Lax matrix \( T_D(z) \) is explicitly determined as follows:

(I) The matrix coefficients on the main diagonal are:

\[
T_D(z)_{ii} = \begin{cases} 
z + \sum_{r=1}^{a_i} (p_{i-1,r} + 1) - \sum_{r=1}^{a_i} p_{i,r} + \sum_{x \in \mathbb{P}^1 \setminus \{\infty\}} \epsilon^x_i(\lambda x), & \text{if } i \leq m \\
1, & \text{if } m < i \leq m' \\
0, & \text{if } i > m' 
\end{cases} 
\]

(2.85)

(II) The matrix coefficients above the main diagonal are:

\[
T_D(z)_{ij} = \begin{cases} 
- \sum_{1 \leq r_{j-1} \leq a_{j-1}} \frac{P_{i-1}(p_{i,r_{j-1}} - 1) \prod_{k=i}^{j-2} P_{k,r_{k}}(p_{k+1,r_{k+1}} - 1)}{\prod_{k=i}^{j-1} P_{k,r_{k}}(p_{k,r_{k}})} \cdot \prod_{k=i}^{j-1} Z_k(p_{k,r_{k}}) \cdot e^{\sum_{k=i}^{j-1} q_{k,r_{k}}} & \text{if } i \leq m \text{ and } i < j. 
\end{cases} 
\]

(2.87)

(III) The matrix coefficients below the main diagonal are:

\[
T_D(z)_{ji} = \begin{cases} 
0 & \text{if } m < i < j, 
\end{cases} 
\]

(2.88)

(b) \( T_D(z) = T_D(z) \) is polynomial of degree \( 1 \) in \( z \), and the coefficient of \( z \) equals \( \sum_{i=1}^{m} E_{ii} \).

**Proof.** (a) Combining the explicit formulas (2.63, 2.65) for the matrix coefficients \( T_D(z)_{\alpha,\beta} \) with their polynomiality of Theorem 2.66, we may immediately determine all of them explicitly.

The latter is based on the following observations:
The leading power of $z$ in $e_\ell^D_j(z)$ given by the right-hand side of (2.59) expanded in $z^{-1}$ equals $-1$, while the coefficient of $z^{-1}$ is exactly the right-hand side of (2.87) for any $i < j$.

The leading power of $z$ in $f_\ell^j(z)$ given by the right-hand side of (2.61) expanded in $z^{-1}$ equals $-1$, while the coefficient of $z^{-1}$ is exactly the right-hand side of (2.89) for any $i < j$.

The leading power of $z$ in $Z_0(z)^{-1}g_\ell^D(z) = g_\ell^D(z)$ expanded in $z^{-1}$, cf. (2.57), equals

$$a_i - a_{i-1} + (\xi_i^\vee - \xi_i^\vee)(\lambda) = -(\lambda_{n+i-1} - \mu_{n+i-1}) + (\lambda_n + \lambda_{n-i+1}) = -\mu_{n-i+1},$$

due to Lemma 2.82(c) and the assumption $Z$ due to Lemma 2.82(c) and the assumption.

Remark 2.90. Applying Theorem 2.84 for $n = 2$, we obtain three $2 \times 2$ rational Lax matrices

$$\begin{pmatrix} z - p & -e^q \\ e^{-q} & 0 \end{pmatrix}, \begin{pmatrix} z - p & -(p-x_1)e^q \\ e^{-q} & 1 \end{pmatrix}, \begin{pmatrix} z - p & -(p-x_1)(p-x_2)e^q \\ e^{-q} & z + p + 1 - x_1 - x_2 \end{pmatrix},$$

(2.91)

corresponding to $\lambda = (0,0)$ and $\mu = (1,-1)$, $\lambda = (1,0)$ and $\mu = (0,-1)$, $\lambda = (2,0)$ and $\mu = (-1,-1)$, respectively (as $a_1 = 1$, we relabeled $p_1, q_1$ by $p, q$). Those are the well-known $2 \times 2$ elementary Lax matrices for the Toda chain, the DST chain, and the Heisenberg magnet.

Remark 2.92. At this point, it is instructive to discuss higher $z$-degree Lax matrices for $n = 2$. Fix a positive integer $N$ and let $A_N$ denote the algebra $\mathcal{A}$ of (2.31) with $n = 2, a_1 = N$. To simplify our notations, we shall denote the generators $\{p_{1,r}, e^{\pm q_i}, \ldots \}_{i=1}^N$ simply by $\{p_r, e^{\pm q_r} \}_{r=1}^N$.

Let $L_r(z) = \begin{pmatrix} z - p_r & -e^{qr} \\ e^{-qr} & 0 \end{pmatrix}, 1 \leq r \leq N$, be the $2 \times 2$ elementary Lax matrices for the Toda chain, and consider the complete monodromy matrix

$$T_N(z) := L_1(z) \cdots L_N(z) = \begin{pmatrix} A_N(z) & B_N(z) \\ C_N(z) & D_N(z) \end{pmatrix}. \quad (2.93)$$

Note that the matrix coefficients $A_N(z), B_N(z), C_N(z), D_N(z)$ are polynomials in $z$ with coefficients in the algebra $\mathcal{A}^{\otimes N}$ of degrees $N, N-1, N-1, N-2$, respectively. For any $\epsilon \in \mathbb{C}$, the coefficients in powers of $z$ of the linear combination $A_N(z) + \epsilon D_N(z)$ pairwise commute and coincide with Hamiltonians of the quantum closed Toda system of $GL_N$, due to [TF].

Following Remark 2.90 and our construction (2.55, 2.56) of rational Lax matrices $T_\alpha(z)$, we note that local Lax matrices $L_r(z)$ encode the homomorphisms $\Psi_{\alpha}[\infty]: Y_{-\alpha}(\mathfrak{gl}_2) \to \mathcal{A}_1$ of (2.35), where $\alpha := \alpha_1 = -\varpi_0 + 2\varpi_1$ is a simple coroot of $\mathfrak{sl}_2$. Furthermore, evoking the coproduct homomorphisms of Propositions 2.128 and 2.134 below, we see that the complete monodromy matrix $T_N(z)$ of (2.93) encodes the homomorphism $Y_{-\alpha}(\mathfrak{gl}_2) \to \mathcal{A}_1^{\otimes N}$ obtained as a composition of the iterated coproduct homomorphism $Y_{-\alpha}(\mathfrak{gl}_2) \to Y_{-\alpha}(\mathfrak{gl}_2)^{\otimes N}$ and the homomorphism $\Psi_{\alpha}[\infty]: Y_{-\alpha}(\mathfrak{gl}_2)^{\otimes N} \to \mathcal{A}_1^{\otimes N}$.

On the other hand, consider the rational Lax matrix $T_D(z)$ for the $A^+$-valued divisor $D = N\alpha[\infty]$ on $\mathbb{P}^1$. According to Theorem 2.66, the matrix coefficients of $T_D(z)$ are polynomials in $z$ with coefficients in the algebra $\mathcal{A}_N$. Moreover, evoking formulas (2.57, 2.59, 2.61), we find:

$$T_D(z)_{11} = P(z), \quad T_D(z)_{12} = -\sum_{r=1}^N \frac{P_r(z)}{P_r(p_r)} e^{qr}, \quad T_D(z)_{21} = \sum_{r=1}^N \frac{P_r(z)}{P_r(p_r)} e^{-qr},$$
$$T_D(z)_{22} = \frac{1}{P(z) - 1} - \sum_{1 \leq r = 1}^{N} \frac{P_r(z)}{(z - p_r - 1)P_r(p_r)P_r(p_r + 1)} - \sum_{1 \leq r \neq s \leq N} \frac{P_{r,s}(z)}{P_{r,s}(p_r)P_{r,s}(p_s)(p_r - p_s)(p_s - p_r - 1)} e^{q_s - q_r},$$

where

$$P(z) := \prod_{r=1}^{N} (z - p_r), \quad P_r(z) := \prod_{s \neq r} (z - p_s), \quad P_{r,s}(z) := \prod_{1 \leq t \leq N} (z - p_t),$$

cf. (2.33). Due to the RTT relation (2.40) for $T_D(z)$, the coefficients in powers of $z$ of the linear combination $T_D(z)_{11} + \epsilon T_D(z)_{22}$ pairwise commute and define a quantum integrable system. These commuting Hamiltonians can be constructed by applying (1.1) to the Lax matrix $T$.

Notice that the condition $k \leq 2N$ guarantees that the matrix multiplication gives rise to the multiplication homomorphisms $Z_{k,\underline{x}} \times Z_{k',\underline{x}'} \rightarrow Z_{k+k',\underline{x} \cup \underline{x}'}$.
2.5. Examples and comparison to the rational Lax matrices of [FP].

In this section, we consider some examples of the Lax matrices $T_D(z)$ of Theorem 2.84 and compare them to the corresponding Lax matrices $L_{\lambda, z, \vec{\mu}}(z)$ (cf. Remark 2.83) of [FP].

- **Example 1**: $\lambda = (0^n), \mu = (1, 0^{n-2}, -1)$.
  Then $a_1 = \ldots = a_{n-1} = 1$ and $D = (\varpi_1 + \varpi_{n-1} - \varpi_0)[\infty]$. To simplify our notations, we relabel $\{p_i, e^{\pm q_i}\}_{i=1}^{n-1}$ by $\{p_i, e^{\pm \bar{q}_i}\}_{i=1}^{n-1}$. Due to Theorem 2.84, the matrix $T_D(z)$ equals:

$$
T_D(z) = \begin{pmatrix}
  z - p_1 & -e^{q_1} & -e^{q_1+q_2} & \ldots & -e^{q_1+\ldots+q_{n-2}} & -e^{q_1+\ldots+q_{n-1}} \\
  (p_1 + 1 - p_2)e^{-q_1} & 1 & 0 & \ldots & 0 & 0 \\
  (p_2 + 1 - p_3)e^{-q_1-q_2} & 0 & 1 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  (p_{n-2} + 1 - p_{n-1})e^{-q_1-\ldots-q_{n-2}} & 0 & 0 & \ldots & 1 & 0 \\
  e^{-q_1-\ldots-q_{n-1}} & 0 & 0 & \ldots & 0 & 0 
\end{pmatrix}
$$

(2.94)

Let us compare $T_D(z)$ of (2.94) to the Lax matrix $L_{\lambda, z, \vec{\mu}}(z)$ of [FP, (4.11, 4.4–4.7)] with $\vec{\mu} = (2, 1^{n-2}, 0) = \mu + (1^n)$ (cf. Remark 2.83), given by

$$
L_{\lambda, z, \vec{\mu}}(z) = \begin{pmatrix}
  0 & 0 & \ldots & 0 & -e^{-q_{n,n}} \\
  0 & 1 & \ldots & 0 & -p_{2,n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & 1 & -p_{n-1,n} \\
  e^{q_{n,n}} & q_{n,2} & \ldots & q_{n,n-1} & z - p_{n,n} - q_{n,2}p_{2,n} - \ldots - q_{n,n-1}p_{n-1,n} 
\end{pmatrix}.
$$

(2.95)

Conjugating (2.95) by the permutation matrix $\sum_{i=1}^{n} E_{i,n-i+1}$ (which clearly preserves the RTT relation (2.40)), and making the canonical transformation (preserving commutation relations)

$$
q_{n,n-1} = -e^{q_1}, \quad p_{n-i,n} = -p_i e^{-q_i}, \quad e^{q_{n,n}} = -e^{q_{n-1}}, \quad p_{n,n} = -p_{n-1} \quad \text{for } 1 \leq i \leq n-2,
$$

we obtain the following rational Lax matrix:

$$
\tilde{L}_{\lambda, z, \vec{\mu}}(z) = \begin{pmatrix}
  z + p_{n-1} - (p_1 - 1) - \ldots - (p_{n-2} - 1) & -e^{q_1} & \ldots & -e^{q_{n-2}} & -e^{q_{n-1}} \\
  p_1 e^{-q_1} & 1 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  p_{n-2} e^{-q_{n-2}} & 0 & \ldots & 1 & 0 \\
  e^{-q_{n-1}} & 0 & \ldots & 0 & 0 
\end{pmatrix}
$$

(2.96)

Thus $T_D(z)$ of (2.94) and $\tilde{L}_{\lambda, z, \vec{\mu}}(z)$ of (2.96) coincide upon the canonical transformation:

$$
q_i = q_1 + \ldots + q_j, \quad p_i = p_i - p_{i+1} + 1, \quad p_{n-1} = p_{n-1} \quad \text{for } 1 \leq i \leq n-2, 1 \leq j \leq n-1.
$$

- **Example 2**: $\lambda = (0^{2r}), \mu = (1^r, (-1)^r), n = 2r$.
  Then $D = (2\varpi_r - \varpi_0)[\infty]$ and the coefficients $\{a_i\}_{i=1}^{n-1}$ are as follows:

$$
a_1 = 1, \quad a_2 = 2, \quad \ldots, \quad a_{r-1} = r - 1, \quad \alpha_r = r, \quad a_{r+1} = r - 1, \quad \ldots, \quad a_{2r-2} = 2, \quad a_{2r-1} = 1.
$$

Due to Theorem 2.84, $T_D(z)$ is a block matrix of the form

$$
T_D(z) = \begin{pmatrix}
  zI_r - F & K \\
  K & 0 
\end{pmatrix}.
$$

(2.97)
where \( F, K, \bar{K} \) are \( r \times r \) \( z \)-independent matrices, and \( I_r \) is the identity \( r \times r \) matrix.

The first simple property of the matrices \( K, \bar{K} \) is:

**Lemma 2.98.** (a) The matrix elements \( \{K_{ij}\}_{i,j=1}^r \) of the matrix \( K \) pairwise commute.
(b) The matrix elements \( \{\bar{K}_{ij}\}_{i,j=1}^r \) of the matrix \( \bar{K} \) pairwise commute.

**Proof.** This follows immediately from the RTT relation (2.40) for \( T_D(z) \). \( \square \)

A much deeper relation between \( K \) and \( \bar{K} \) is established in the following result:

**Theorem 2.99.** We have \( K \cdot \bar{K} = -I_r \).

**Proof.** Due to (2.87, 2.89), it suffices to prove the following equality:

\[
\sum_{\gamma=1}^r \sum_{1 \leq r_{\gamma-1} \leq r} \sum_{1 \leq r_{\gamma-1} \leq a_{\gamma}} \frac{P_{\alpha-1}(p_{\alpha, r_{\alpha}} - 1) \prod_{k=\alpha}^{r+\gamma-2} P_{k, r_k}(p_{k+1, r_{k+1}} - 1)}{\prod_{k=\alpha}^{r+\gamma-1} P_{k, r_k}(p_{k, r_k})} e^{q_{\gamma, r_{\gamma-1}} + \cdots + q_{r+\gamma-1, r+\gamma-1}} \times \]
\[
\sum_{1 \leq s_{\beta} \leq a_{\beta}} \sum_{1 \leq s_{\gamma-1} \leq a_{\gamma-1}} \frac{P_{\gamma}(p_{\gamma, r_{\gamma-1}} + 1) \prod_{k=\beta}^{\gamma-1} P_{k, s_k}(p_{k-1, s_{k-1}} + 1)}{\prod_{k=\beta}^{r+\gamma-1} P_{k, s_k}(p_{k, s_k})} \times e^{-q_{\gamma, s_{\gamma-1}} - \cdots - q_{r+\gamma-1, s_{r+\gamma-1}}} \delta_{\alpha, \beta} \quad (2.100)
\]

for any \( 1 \leq \alpha, \beta \leq r \).

To evaluate the sum in the left-hand side of (2.100), we first move \( e^{q_{\gamma, r_{\gamma-1}} + \cdots + q_{r+\gamma-1, r+\gamma-1}} \) to the right of \( p_{s, s} \)-terms, then simplify \( e^{q_{i, r_i}} e^{-q_{i, s_i}} \approx 1 \) once \( r_i = s_i \), and finally group together the summands which have the common \( e^{q_{i, r_i}} \)-factor. For each such group, pick the maximal \( k \) (if such exists) such that \( e^{q_{k, s_k}} \) does appear. If \( k \) exists, then \( 1 \leq k \leq 2r - 2 \) as \( a_{2r-1} = 1 \), while \( k \) does not exist if and only if \( \alpha = \beta \) and \( r_i = s_i \) for each \( \alpha \leq i \leq r + \gamma - 1 \).

The equality (2.100) follows from the following result:

**Proposition 2.101.** Pick any of the above groups and consider the associated \( k \) (if it exists).
(a) If \( r \leq k \leq 2r - 2 \), then the sum of terms in the corresponding group is zero.
(b) If \( 1 \leq k < r \), then the sum of terms in the corresponding group is zero.
(c) If \( k \) does not exist, then the sum of terms in the corresponding group equals 1.

**Proof of Proposition 2.101.** (a) Fix any admissible collections \( r_\alpha, \ldots, r_k \) and \( s_\beta, \ldots, s_k \) with \( r_k \neq s_k \). Then, the terms in the corresponding group are parametrized by \( k + 1 - r \leq \gamma \leq r \) and all admissible collections \( r_{k+1} = s_{k+1}, \ldots, r_{r+\gamma-1} = s_{r+\gamma-1} \). Ignoring the common factor, the total sum of terms in this group equals \( \sum_{\gamma=k+1-r}^r S_{\gamma} \), where each summand is given by

\[
S_{\gamma} := \sum_{1 \leq r_{k+1} \leq a_{k+1}} \sum_{1 \leq r_{\gamma-1} \leq a_{\gamma-1}} \frac{P_{k+1, r_{k+1}}(p_{k+1, r_{k+1}} - 1) \cdots P_{r+\gamma-2, r_{r+\gamma-2}}(p_{r+\gamma-2, r_{r+\gamma-2}} - 1) \times \prod_{k=1+1}^{r+\gamma-2} P_{k+1, r_{k+1}}(p_{k+1, r_{k+1}}) \cdots P_{r+\gamma-1, r_{r+\gamma-1}}(p_{r+\gamma-1, r_{r+\gamma-1}})}{P_{k+1, r_{k+1}}(p_{k+1, r_{k+1}} - 1) \cdots P_{r+\gamma-1, r_{r+\gamma-1}}(p_{r+\gamma-1, r_{r+\gamma-1}} - 1)}.
\]

(2.102)

It remains to prove \( \sum_{\gamma=k+1-r}^r S_{\gamma} = 0 \). For the latter, we need the following simple result:
Lemma 2.103. Fix $r < l \leq 2r - 1$ and $1 \leq r_{l-1} \neq s_{l-1} \leq a_{l-1}$. Then, we have

$$1 + \sum_{1 \leq r_{l-1} \leq a_{l-1}} \frac{P_{l-1,r_{l-1}}(p_{l,r_{l}} - 1)}{P_{l,r_{l}}(p_{l,r_{l}})} \cdot \frac{1}{1 + p_{l-1,s_{l-1}} - p_{l,r_{l}}} = 0,$$  \hspace{1cm} (2.104)

$$1 + \sum_{1 \leq r_{l-1} \leq a_{l-1}} \frac{P_{l-1,r_{l-1}}(p_{l,r_{l}} - 1)}{P_{l,r_{l}}(p_{l,r_{l}})} \cdot \frac{1}{p_{l-1,r_{l-1}} - p_{l,r_{l}}} = \frac{P_{l-1,r_{l-1}}(p_{l-1,r_{l-1}} - 1)}{P_{l}(p_{l-1,r_{l-1}})}. \hspace{1cm} (2.105)$$

Proof of Lemma 2.103. Recall that $a_{l-1} = 2r - l$, $a_{l-1} = 2r - l + 1$. Without loss of generality, we may assume that $r_{l-1} = 2r - l + 1$, $s_{l-1} = 2r - l$. To simplify the formulas below, we relabel $\{p_{i,j}\}_{i=1}^{2r-l}$ by $\{c_{i}\}_{i=1}^{2r-l}$ and $\{p_{i-1,j}\}_{i=1}^{2r-l+1}$ by $\{b_{i}\}_{i=1}^{2r-l+1}$, respectively.

Then, the left-hand side of (2.104) becomes

$$1 - \sum_{i=1}^{2r-l} (c_{i} - 1 - b_{1}) \cdots (c_{i} - 1 - b_{2r-l-1}) \cdot \frac{1}{b_{2r-l+1} - c_{i}}.$$

This is a symmetric rational function in $\{c_{i}\}_{i=1}^{2r-l}$ without poles (as symmetric functions may not have simple poles at $c_{i} = c_{j}$ with $i \neq j$), hence, it is polynomial in $\{c_{i}\}_{i=1}^{2r-l}$. However, being of degree $\leq 0$, this polynomial must be a constant (depending on $\{b_{i}\}_{i=1}^{2r-l+1}$). To determine the latter, let $c_{1} \to \infty$, in which case the sum tends to 0. This completes our proof of (2.104).

Likewise, the left-hand side of (2.105) becomes

$$1 + \sum_{i=1}^{2r-l} (c_{i} - 1 - b_{1}) \cdots (c_{i} - 1 - b_{2r-l}) \cdot \frac{1}{b_{2r-l+1} - c_{i}}.$$

This is a symmetric rational function in $\{c_{i}\}_{i=1}^{2r-l}$ with the only poles (which are at most simple) at $c_{i} = b_{2r-l+1}$ ($1 \leq i \leq 2r-l$). Hence, it is of the form $R(\prod_{i=1}^{2r-l+1}(b_{2r-l+1} - c_{i}))$ for some polynomial $R$ of total degree $\text{deg}(R) \leq 2r - l$. Due to (2.104), $R$ is divisible by $\prod_{i=1}^{2r-l+1}(b_{2r-l+1} - 1 - b_{i})$, and thus for degree reasons $R(\{b_{i}\}, \{c_{i}\}) = t \cdot \prod_{i=1}^{2r-l+1}(b_{2r-l+1} - 1 - b_{i})$ with $t \in \mathbb{C}$. Letting $b_{2r-l+1} \to \infty$, we find $t = 1$. This completes our proof of (2.105).

Applying (2.105) to simplify $S_{r-1} + S_{r}$, we find

$$S_{r-1} + S_{r} = \sum_{1 \leq r_{k+1} \leq a_{k+1}} P_{k,r_{k+1}}(p_{k+1,r_{k+1}} - 1) \cdots P_{2r-3,r_{2r-3}}(p_{2r-2,r_{2r-2}} - 1) \cdot \frac{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}} + 1) P_{k+2,r_{k+2}}(p_{k+1,r_{k+1}} + 1) \cdots P_{2r-2,r_{2r-2}}(p_{2r-3,r_{2r-3}})}{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}} - 1) \cdots P_{2r-3,r_{2r-3}}(p_{2r-3,r_{2r-3}} - 1)}.$$

Applying (2.105) again, we may simplify the sum of the above expression and $S_{r-2}$. Proceeding in the same way and applying (2.105) at each step, we eventually get

$$\sum_{k+1 \leq l \leq \gamma \leq r} S_{\gamma} = 1 + \sum_{r_{k+1}} P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}} - 1) \cdot \frac{1}{1 + p_{k,s_{k}} - p_{k+1,r_{k+1}}} = 0,$$

due to (2.104) as $r_{k} \neq s_{k}$. This completes our proof of Proposition 2.101(a).

(b) The proof of Proposition 2.101(b) is completely analogous to the above proof of part (a) and is crucially based both on Lemma 2.103 and its following counterpart:
Lemma 2.106. Fix $1 < l \leq r$ and $1 \leq r_{l-1} \neq s_{l-1} \leq a_{l-1}$. Then, we have
\begin{equation}
\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_l}}{P_{l,r_l}(pt_{r_l})} \left( pt_{r_l} - 1 \right) \frac{1}{1 + pt_{l-1,s_{l-1}} - pt_{r_l}} = 0,
\end{equation}
\begin{equation}
\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_l}}{P_{l,r_l}(pt_{r_l})} \left( pt_{r_l} - 1 \right) \frac{1}{pt_{l-1,r_{l-1}} - pt_{r_l}} = \frac{P_{l-1,r_{l-1}}(pt_{l-1,r_{l-1}} - 1)}{P(l(pt_{l-1,r_{l-1}})).}
\end{equation}

Proof. The proof is similar to that of (2.104, 2.105). We leave details to the interested reader.

(c) The proof of Proposition 2.101(c) is completely analogous to the above proofs of parts (a,b) and is crucially based both on Lemmas 2.103, 2.106 and their following counterpart:

Lemma 2.109. Fix $1 < l \leq r$ and $1 \leq r_{l-1} \leq a_{l-1}$. Then, we have
\begin{equation}
\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_l}}{P_{l,r_l}(pt_{r_l})} (pt_{r_l} - 1) = 0,
\end{equation}
\begin{equation}
\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_l}}{P_{l,r_l}(pt_{r_l})} (pt_{r_l} - 1) = 1.
\end{equation}

Proof. The proof is similar to that of (2.104, 2.105). We leave details to the interested reader.

This completes our proof of Proposition 2.101.

As Proposition 2.101 implies the equality (2.100), the proof of Theorem 2.99 is completed.

It is instructive to compare this Lax matrix $T_D(z)$ to the Lax matrix $L_{\lambda, \tilde{\mu}}(z)$ of [FP, §4.1] with $\tilde{\mu} = (2', 0') = \mu + (1^n)$ (cf. Remark 2.83). Conjugating the latter by the permutation matrix \( \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \), we obtain the following rational Lax matrix
\begin{equation}
\tilde{L}_{\lambda, \tilde{\mu}}(z) = \begin{pmatrix} zI_r - F & K \\ K & 0 \end{pmatrix},
\end{equation}
where $K\tilde{K} = -I_r$ and $K$ encodes all the $q_{s,s}$-variables via [FP, (4.5, 4.6)].

- Example 3: $\lambda = (0^{2r+s}), \mu = (1^r, 0^s, (-1)^t), n = 2r + s$ with $r, s > 0$.

Then $D = (\varpi_r + \varpi_{r+s} - \varpi_0)[\infty]$ and the coefficients \( \{a_i\}_{i=1}^{n-1} \) are as follows:

\[ a_1 = 1, \ldots, a_{r-1} = r - 1, a_r = a_{r+1} = \ldots = a_{r+s} = r, a_{r+s+1} = r - 1, \ldots, a_{2r+s-1} = 1. \]

Due to Theorem 2.84, $T_D(z)$ is a block matrix of the form
\begin{equation}
T_D(z) = \begin{pmatrix} zI_r - F & Q & K \\ -P & I_s & 0 \\ K & 0 & 0 \end{pmatrix},
\end{equation}
where $F = (F_{ij})_{i,j=1}^{r}, K = (K_{ij})_{i,j=1}^{r}, \bar{K} = (\bar{K}_{ij})_{i,j=1}^{r}$ are $r \times r$ matrices, $P = (P_{ij})_{1 \leq i \leq s}^{1 \leq i \leq r}$ is an $s \times r$ matrix, $Q = (Q_{ji})_{1 \leq i \leq s}^{1 \leq i \leq r}$ is an $r \times s$ matrix, and all of them are $z$-independent.

The first simple property of the matrices $P, Q, K, \bar{K}$ is:
**Lemma 2.114.** (a) The matrix elements \( \{K_{ij}\}_{1 \leq i,j \leq s} \) pairwise commute.  
(b) The matrix elements \( \{\bar{K}_{ij}\}_{1 \leq i,j \leq s} \) pairwise commute.  
(c) The commutation relation between the matrix elements of \( P, Q \) is \( [P_{ij}, Q_{j'q'}] = \delta_{i,j'}\delta_{j,j'} \).

**Proof.** This follows immediately from the RTT relation (2.40) for \( T_D(z) \). \( \square \)

Similar to Theorem 2.99, there is also a much deeper relation between \( K \) and \( \bar{K} \):

**Theorem 2.115.** We have \( K \cdot \bar{K} = -I_r \).

**Proof.** The proof of Theorem 2.115 is completely analogous to the above proof of Theorem 2.99. The only extra technical result needed is the following counterpart of Lemma 2.106:

**Lemma 2.116.** For \( r \leq l < r + s \) and \( 1 \leq r_{l-1} \neq s_{l-1} \leq a_{l-1} \), both (2.107, 2.108) hold.

We leave details to the interested reader. \( \square \)

It is instructive to compare the Lax matrix \( T_D(z) \) of (2.113) to the Lax matrix \( L_{\lambda,\bar{\mu}}(z) \) of \( [FP, \S 4.2] \) with \( \bar{\mu} = (2^r, 1^s, 0^r) = \mu + (1^n) \) (cf. Remark 2.83). Conjugating the latter by the permutation matrix \( \begin{pmatrix} 0 & 0 & I_r \\ 0 & I_s & 0 \\ I_r & 0 & 0 \end{pmatrix} \), we obtain the following rational Lax matrix

\[
\bar{L}_{\lambda,\bar{\mu}}(z) = \begin{pmatrix} zI_r - \bar{F} & Q & \bar{K} \\ -P & I_s & 0 \\ K & 0 & 0 \end{pmatrix},
\]

where \( K\bar{K} = -I_r \) and the matrices \( K, Q \) encode all \( q_{*,r} \)-variables via \( [FP, (4.5, 4.6, 4.13)] \).

**Remark 2.118.** We note that \( K \) of (2.117) coincides with \( K \) of (2.112), while \( \bar{K} \) of (2.113) is not the same as \( K \) of (2.97).

- **Example 4:** \( \lambda = (1, 0^{n-1}), \mu = (0^{n-1}, -1) \).
  The corresponding divisor is \( D = \infty_n - x_1 \) with \( x_1 \in \mathbb{C} \). This example is similar to the Example 1 above as the coefficients \( a_i \) are the same: \( a_1 = \ldots = a_{n-1} = 1 \). To simplify our notations, we relabel \( \{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^{n-1} \) by \( \{p_i, e^{\pm q_i}\}_{i=1}^{n-1} \). Due to Theorem 2.84, the matrix \( T_D(z) \) equals:

\[
T_D(z) = \begin{pmatrix} 
z - p_1 & -e^{q_1} & \ldots & -e^{q_1+\ldots+q_{n-2}} & -(p_{n-1} - x_1)e^{q_1+\ldots+q_{n-1}} \\
(p_1 + 1 - p_2)e^{-q_1} & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(p_{n-2} + 1 - p_{n-1})e^{-q_1-\ldots-q_{n-2}} & 0 & \ldots & 1 & 0 \\
e^{-q_1-\ldots-q_{n-1}} & 0 & \ldots & 0 & 1 
\end{pmatrix}.
\]

Let us compare this Lax matrix \( T_D(z) \) to the rational Lax matrix \( L_{\lambda,x_1,\bar{\mu}}(z) \) of \( [FP, (3.2)] \) with \( \bar{\mu} = (1^{n-1}, 0) = \mu + (1^n) \) (cf. Remark 2.83). Conjugating the latter by the permutation...
matrix $\sum_{i=1}^{n} E_{i,n-i+1}$, we obtain the following rational Lax matrix:

$$\tilde{L}_{\lambda,x_1,\tilde{\mu}}(z) = \left( \begin{array}{ccc}
 z - x_1 - q_{n,1}p_{1,n} & \cdots & -q_{n,n-1}p_{n-1,n} & q_{n,n-1} & \cdots & q_{n,2} & q_{n,1} \\
 -p_{n-1,n} & 1 & 0 & 0 & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 -p_{2,n} & 0 & \cdots & 1 & 0 \\
 -p_{1,n} & 0 & \cdots & 0 & 1
\end{array} \right). \quad (2.120)$$

Thus $T_D(z)$ of (2.119) and $\tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)$ of (2.120) coincide upon the canonical transformation:

- **Example 5:** $\lambda = (1^{n-1}, 0), \mu = (0, (1)^{n-1})$.

  The corresponding divisor is $D = \omega[x_1] + (\omega_{n-1} - \omega_0)[\infty]$ with $x_1 \in \mathbb{C}$. This example is similar to Examples 1, 5 above as $a_1 = \cdots = a_{n-1} = 1$. To simplify our notations, we relabel $\{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^{n-1}$ by $\{p_i, e^{\pm q_i}\}_{i=1}^{n-1}$. Due to Theorem 2.84, the matrix coefficients of $T_D(z)$ equal:

$$T_D(z)_{ii} = \begin{cases} 
 z - p_i, & \text{if } i = 1 \\
 z + p_{i-1} - p_i + 1 - x_1, & \text{if } 1 < i < n, \\
 1, & \text{if } i = n
\end{cases} \quad (2.121)$$

The following is straightforward:

**Lemma 2.122.** For any $1 \leq i, j \leq n - 1$, we have $T_D(z)_{ij} = \delta_{i,j}(z - x_1) + T_D(z)_{in}T_D(z)_{nj}$.

Let us compare this Lax matrix $T_D(z)$ to the rational Lax matrix $L_{\lambda,x_1,\tilde{\mu}}(z)$ of [FP, (3.2)] with $\tilde{\mu} = (1, 0^{n-1}) = \mu + (1^n)$ (cf. Remark 2.83). Conjugating the latter by the permutation matrix $E_{12} + \cdots + E_{n-1,n} + E_{n,1}$, we obtain the rational Lax matrix $\tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)$ with the following matrix coefficients:

$$\tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)_{ij} = \delta_{i,j}(z - x_1) - q_{i+1,1}p_{1,j+1} \quad \text{if } 1 \leq i, j < n,$$

$$\tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)_{in} = q_{i+1,1}, \quad \tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)_{ni} = -p_{1,i+1}, \quad \tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)_{nn} = 1 \quad \text{if } 1 \leq i < n. \quad (2.123)$$

Thus $T_D(z)$ of (2.121) and $\tilde{L}_{\lambda,x_1,\tilde{\mu}}(z)$ of (2.123) coincide upon the canonical transformation:

- **Example 6:** $\lambda = (1^r, 0^s), \mu = (0^r, (1)^s)$, $n = r + s$ with $r, s > 0$.

  This example naturally generalizes Example 5 ($r = 1$ case) and Example 6 ($s = 1$ case) above. We have $D = \omega[x_1] + (\omega_r - \omega_0)[\infty]$ with $x_1 \in \mathbb{C}$. Due to Theorem 2.84, $T_D(z)$ is a block matrix of the form

$$T_D(z) = \left( \begin{array}{cc}
 zI_r - F & Q \\
 -P & I_s
\end{array} \right). \quad (2.124)$$
where \( F = (F_{ij})_{i,j=1}^r \) is an \( r \times r \) matrix, \( P = (P_{ij})_{1 \leq i \leq s_1, 1 \leq j \leq r} \) is an \( s \times r \) matrix, \( Q = (Q_{ij})_{1 \leq i \leq s_2, 1 \leq j \leq r} \) is an \( s \times s \) matrix, and all of them are \( z \)-independent.

The first simple property of the matrices \( P, Q \) is:

**Lemma 2.125.** (a) The matrix elements \( \{P_{ij}\}_{1 \leq i \leq \leq s} \) pairwise commute.

(b) The matrix elements \( \{Q_{ij}\}_{1 \leq i \leq \leq s} \) pairwise commute.

(c) The commutation relation between the matrix elements of \( P, Q \) is \( [P_{ij}, Q_{j'i'}] = \delta_{i,i'}\delta_{j,j'} \).

**Proof.** This follows immediately from the RTT relation (2.40) for \( T_D(z) \).

A much deeper relation between \( P, Q \) and \( F \) is established in the following result:

**Theorem 2.126.** We have \( F = x_1I_r + QP \).

**Proof.** The proof of Theorem 2.126 is completely analogous to the above proof of Theorem 2.99. We leave details to the interested reader.

Let us compare this Lax matrix \( T_D(z) \) to the rational Lax matrix \( L_{\mu, x_1, \tilde{\mu}}(z) \) of [FP, (3.2)] with \( \tilde{\mu} = (1^r, 0^s) = \mu + (1^n) \) (cf. Remark 2.83). Conjugating the latter by the permutation matrix \( \sum_{i=1}^s E_{i,s+i} + \sum_{i=1}^s E_{r+i,i} \), we obtain the following rational Lax matrix

\[
\tilde{L}_{\bar{\lambda}, x_1, \tilde{\mu}}(z) = \left( \begin{array}{ccc}
(z - x_1)I_r - QP & Q \\
-P & I_s
\end{array} \right),
\]

(2.127)

where \( \bar{P} = (p_{i,s+j})_{1 \leq i \leq s_1, 1 \leq j \leq s_2} \) and \( \bar{Q} = (q_{s+j,i})_{1 \leq i \leq s_1, 1 \leq j \leq s_2} \) encode all the variables \( p_{s,s}, q_{s,s} \) of [FP].

Thus \( T_D(z) \) of (2.124) and \( \tilde{L}_{\bar{\lambda}, x_1, \tilde{\mu}}(z) \) of (2.127) coincide upon the canonical transformation:

\[
q_{s+j,i} = T_D(z)_{j, r+i}, \quad p_{i,s+j} = -T_D(z)_{r+i, j},
\]

with \( T_D(z)_{j, r+i} \) and \( T_D(z)_{r+i, j} \) evaluated via (2.87) and (2.89), respectively.

2.6. Coproduct homomorphisms for shifted Yangians

One of the crucial benefits of the RTT realization is that it immediately endows the Yangian of \( \mathfrak{gl}_n \) with the Hopf algebra structure, in particular, the coproduct homomorphism

\[
\Delta^{\text{rtt}}: Y^{\text{rtt}}(\mathfrak{gl}_n) \to Y^{\text{rtt}}(\mathfrak{gl}_n) \otimes Y^{\text{rtt}}(\mathfrak{gl}_n)
\]

is defined via \( \Delta^{\text{rtt}}(T(z)) = T(z) \otimes T(z) \).

The main result of this section establishes a shifted version of that:

**Proposition 2.128.** For any \( \mu_1, \mu_2 \in \Lambda^+ \), there is a unique \( C \)-algebra homomorphism

\[
\Delta^{\text{rtt}}_{-\mu_1, -\mu_2}: Y^{\text{rtt}}_{-\mu_1} \otimes Y^{\text{rtt}}_{-\mu_2} \to Y^{\text{rtt}}_{-\mu_1} \otimes Y^{\text{rtt}}_{-\mu_2}
\]

defined by

\[
\Delta^{\text{rtt}}_{-\mu_1, -\mu_2}(T(z)) = T(z) \otimes T(z).
\]

(2.129)

**Proof.** We need to prove that \( T(z) \otimes T(z) \), the \( n \times n \) matrix with values in the algebra \( (Y^{\text{rtt}}_{-\mu_1} \otimes Y^{\text{rtt}}_{-\mu_2})((z^{-1})) \), satisfies the defining relations of \( Y^{\text{rtt}}_{-\mu_1 - \mu_2} \). The first of those, the RTT relation (2.40), follows immediately from the fact that both factors \( T(z) \) satisfy it. Let us now deduce the second relation, the particular form of the Gauss decomposition (2.42, 2.43), from \( \mu_1, \mu_2 \in \Lambda^+ \) and the corresponding relations for both factors \( T(z) \).

We start from the following simple observation. Let \( C \) be an associative algebra and consider a collection of its elements \( \{f_{ji}^{(r)}\}, \{e_{ji}^{(r)}\}_{1 \leq i < j \leq n} \), which are encoded via a lower triangular matrix \( F(z) = \sum E_{ii} + \sum_{i<j} f_{ji}(z) \otimes E_{ji} \) with \( f_{ji}(z) = \sum_{r \geq 1} f_{ji}^{(r)} z^{-r} \) and an upper triangular matrix...
\[ E(z) = \sum_i E_{ii} + \sum_{i<j} e_{ij}(z) \otimes E_{ij} \text{ with } e_{ij}(z) = \sum_{r \geq 1} e_{ij}^{(r)} z^{-r}. \] Then, the product \( E(z) \cdot F(z) \) admits a Gauss decomposition

\[ E(z) \cdot F(z) = \tilde{F}(z) \cdot \tilde{G}(z) \cdot \tilde{E}(z), \] (2.130)

\[ \tilde{F}(z) = \sum_i E_{ii} + \sum_{i<j} \tilde{f}_{ji}(z) \otimes E_{ji}, \quad \tilde{G}(z) = \sum_i \tilde{g}_i(z) \otimes E_{ii}, \quad \tilde{E}(z) = \sum_i E_{ii} + \sum_{i<j} \tilde{e}_{ij}(z) \otimes E_{ij}, \]

with the matrix coefficients having the following expansions in \( z \):

\[ \tilde{e}_{ij}(z) = \sum_{r \geq 1} e_{ij}^{(r)} z^{-r}, \quad \tilde{f}_{ji}(z) = \sum_{r \geq 1} f_{ji}^{(r)} z^{-r}, \quad \tilde{g}_i(z) = 1 + \sum_{r \geq 1} g_i^{(r)} z^{-r} \]

for some elements \( \{ e_{ij}^{(r)}, f_{ji}^{(r)} \}_{1 \leq i < j \leq n} \cup \{ g_i^{(r)} \}_{1 \leq i \leq n} \) of \( \mathbb{C} \).

Moreover, if \( z^d = \text{diag}(z^{d_1}, \ldots, z^{d_n}) \) with \( d_1 \geq \cdots \geq d_n \), then

\[ z^d \tilde{F}(z)(z^d)^{-1} = \sum_i E_{ii} + \sum_{i<j} \tilde{f}_{ji}(z) \otimes E_{ji} \text{ with } \tilde{f}_{ji}(z) = \sum_{r \geq 1} f_{ji}^{(r)} z^{-r} = z^{d_j - d_i} \tilde{f}_{ji}(z) \] (2.131)

and

\[ (z^d)^{-1} \tilde{E}(z) z^d = \sum_i E_{ii} + \sum_{i<j} \tilde{e}_{ij}(z) \otimes E_{ij} \text{ with } \tilde{e}_{ij}(z) = \sum_{r \geq 1} e_{ij}^{(r)} z^{-r} = z^{d_j - d_i} \tilde{e}_{ij}(z) \] (2.132)

for some elements \( \{ f_{ji}^{(r)}, e_{ij}^{(r)} \}_{1 \leq i < j \leq n} \) of \( \mathbb{C} \).

Finally, consider the Gauss decompositions of both factors \( T(z) \):

\[ T(z) \otimes 1 = F^{(1)}(z) G^{(1)}(z) E^{(1)}(z) = F^{(1)}(z) D^{(1)}(z) z^{\mu_1} E^{(1)}(z), \]

\[ 1 \otimes T(z) = F^{(2)}(z) G^{(2)}(z) E^{(2)}(z) = F^{(2)}(z) z^{\mu_2} D^{(2)}(z) E^{(2)}(z), \]

where \( z^{\mu_a} := \text{diag}(z^{d_{1a}}, \ldots, z^{d_{na}}) \), \( D^{(a)}(z) := z^{-\mu_a} G^{(a)}(z) \) with \( d_{ia}^{(a)} := e_{ia}^{(a)}(\mu_a) \) and \( a = 1, 2 \).

To obtain the Gauss decomposition of

\[ T(z) \otimes T(z) = F^{(1)}(z) D^{(1)}(z) z^{\mu_1} E^{(1)}(z) F^{(2)}(z) z^{\mu_2} D^{(2)}(z) E^{(2)}(z), \]

apply the above general observation with \( \mathcal{C} = Y^{\text{rtt}}_{-\mu_1}(\mathfrak{g}_n) \otimes Y^{\text{rtt}}_{-\mu_2}(\mathfrak{g}_n) \) and \( e_{ij}^{(r)} = e_{ij}^{(r)} \otimes 1, f_{ji}^{(r)} = 1 \otimes f_{ji}^{(r)} \) to get the Gauss decomposition of \( E^{(1)}(z) F^{(2)}(z) \) first. As conjugating by \( D^{(a)}(z) \) does not change the leading \( z \)-modes, matrix coefficients appearing in the Gauss decomposition of \( T(z) \otimes T(z) \) have the desired form, due to (2.131, 2.132).

This completes our proof of Proposition 2.128. \( \square \)

The following basic property of \( \Delta^{\text{rtt}}_{\ast_1 \ast_2} \) is straightforward:

**Corollary 2.133.** For any \( \mu_1, \mu_2, \mu_3 \in \Lambda^+ \), the following diagram is commutative:

\[
\begin{array}{ccc}
Y^{\text{rtt}}_{-\mu_1 - \mu_2 - \mu_3}(\mathfrak{g}_n) & \xrightarrow{\Delta^{\text{rtt}}_{-\mu_1 - \mu_2 - \mu_3}} & Y^{\text{rtt}}_{-\mu_1}(\mathfrak{g}_n) \otimes Y^{\text{rtt}}_{-\mu_2 - \mu_3}(\mathfrak{g}_n) \\
\Delta^{\text{rtt}}_{-\mu_1 - \mu_2 - \mu_3} & \downarrow \text{Id} \otimes \Delta^{\text{rtt}}_{-\mu_2 - \mu_3} & \\
Y^{\text{rtt}}_{-\mu_1 - \mu_2}(\mathfrak{g}_n) \otimes Y^{\text{rtt}}_{-\mu_3}(\mathfrak{g}_n) & \xrightarrow{\Delta^{\text{rtt}}_{-\mu_1 - \mu_2} \otimes \text{Id}} & Y^{\text{rtt}}_{-\mu_1}(\mathfrak{g}_n) \otimes Y^{\text{rtt}}_{-\mu_2}(\mathfrak{g}_n) \otimes Y^{\text{rtt}}_{-\mu_3}(\mathfrak{g}_n)
\end{array}
\]

Evoking the isomorphisms \( Y_{-\mu} : Y_{-\mu}(\mathfrak{g}_n) \cong Y^{\text{rtt}}_{-\mu}(\mathfrak{g}_n) \) of Theorem 2.53 for \( \mu = \mu_1, \mu_2, \mu_1 + \mu_2 \), \( \Delta^{\text{rtt}}_{-\mu_1 - \mu_2} \) gives rise to the homomorphism \( \Delta_{-\mu_1, -\mu_2} : Y_{-\mu_1 - \mu_2}(\mathfrak{g}_n) \rightarrow Y_{-\mu_1}(\mathfrak{g}_n) \otimes Y_{-\mu_2}(\mathfrak{g}_n) \).
Proposition 2.134. For any $\mu_1, \mu_2 \in \Lambda^+$, the $\mathbb{C}$-algebra homomorphism

$$
\Delta_{-\mu_1, -\mu_2} : Y_{-\mu_1 - \mu_2}(\mathfrak{gl}_n) \rightarrow Y_{-\mu_1}(\mathfrak{gl}_n) \otimes Y_{-\mu_2}(\mathfrak{gl}_n)
$$

is uniquely determined by the following formulas:

\[
\begin{align*}
F_i^{(r)} & \mapsto F_i^{(r)} \otimes 1 \text{ for } 1 \leq r \leq \alpha_i^\vee(\mu_1), \\
F_i^{(\alpha_i^\vee(\mu_1)+1)} & \mapsto F_i^{(\alpha_i^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes F_i^{(1)}, \\
E_i^{(r)} & \mapsto 1 \otimes E_i^{(r)} \text{ for } 1 \leq r \leq \alpha_i^\vee(\mu_2), \\
E_i^{(\alpha_i^\vee(\mu_2)+1)} & \mapsto 1 \otimes E_i^{(\alpha_i^\vee(\mu_2)+1)} + E_i^{(1)} \otimes 1, \\
D_i^{(-\epsilon_i^\vee(\mu_1+\mu_2)+1)} & \mapsto D_i^{(-\epsilon_i^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes D_i^{(-\epsilon_i^\vee(\mu_2)+1)}, \\
D_i^{(-\epsilon_i^\vee(\mu_1+\mu_2)+2)} & \mapsto D_i^{(-\epsilon_i^\vee(\mu_1)+2)} \otimes 1 + 1 \otimes D_i^{(-\epsilon_i^\vee(\mu_2)+2)} + \\
D_i^{(-\epsilon_i^\vee(\mu_1)+1)} & \otimes D_i^{(-\epsilon_i^\vee(\mu_2)+1)} + 1 \otimes \sum_{\gamma^\vee > 0} \epsilon_i^{(\gamma^\vee)} E_i^{(1)} \otimes F_i^{(1)},
\end{align*}
\]

where the last sum is over all positive roots $\gamma^\vee$ of $\mathfrak{sl}_n$, that is $\gamma^\vee \in \{\alpha_a + \ldots + \alpha_b \}_{1 \leq a < b \leq n}$, and $E_1^{(1)} = [E_1^{(1)}, E_2^{(1)}, \ldots, E_n^{(1)}], F_1^{(1)} = [F_1^{(1)}, F_1^{(1)} + 1, E_1^{(1)}, \ldots, F_n^{(1)} - 1].$

Proof. Since $Y_{-\mu_1 - \mu_2}(\mathfrak{sl}_n)$ is generated by $\{E_i^{(1)}, F_i^{(1)}, D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+1)}, D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+2)}\}_{1 \leq j \leq n}$ as an algebra, it suffices to derive the above formulas (2.135).

Following our notations of the above proof of Proposition 2.128, we note that

\[
\begin{align*}
\overline{f}_{ji}^{(1)} & = \ldots = \overline{f}_{ji}^{(d_i^{(1)}-d_j^{(1)})} = 0, \\
\overline{f}_{ji}^{(d_i^{(1)}-d_j^{(1)})+1} & = \overline{f}_{ji}^{(1)} = f_{ji}^{(1)}, \\
\overline{e}_{ij}^{(1)} & = \ldots = \overline{e}_{ij}^{(d_i^{(2)}-d_j^{(2)})} = 0, \\
\overline{e}_{ij}^{(d_i^{(2)}-d_j^{(2)})+1} & = \overline{e}_{ij}^{(1)} = e_{ij}^{(1)}.
\end{align*}
\]

Thus, following the proof of Proposition 2.128, we immediately get

\[
\begin{align*}
\Delta_{-\mu_1, -\mu_2}^{\text{rtt}}(j_{i+1,i}) & = f_{i+1,i}^{(r)} \otimes 1 \text{ for } 1 \leq r \leq \alpha_i^\vee(\mu_1), \\
\Delta_{-\mu_1, -\mu_2}^{\text{rtt}}(\epsilon_i^{(1)}) & = f_{i+1,i}^{(\alpha_i^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes f_{i+1,i}^{(1)}, \\
\Delta_{-\mu_1, -\mu_2}^{\text{rtt}}(\epsilon_i^{(r)}) & = 1 \otimes e_{i+1,i}^{(r)} \text{ for } 1 \leq r \leq \alpha_i^\vee(\mu_2), \\
\Delta_{-\mu_1, -\mu_2}^{\text{rtt}}(\epsilon_i^{\alpha_i^\vee(\mu_2)+1}) & = 1 \otimes e_{i+1,i}^{(\alpha_i^\vee(\mu_2)+1)} + e_{i+1,i}^{(1)} \otimes 1,
\end{align*}
\]

which give rise to the first four formulas of (2.135) by recalling the construction of $\Upsilon_{-\mu}$.

To deduce the last two formulas of (2.135), it remains to use obvious equalities

\[
\sum_{j > i} e_{ij}^{(1)} \cdot f_{ji}^{(1)} - \sum_{j < i} f_{ji}^{(1)} \cdot e_{ij}^{(1)} = \sum_{1 \leq a < b \leq n} \epsilon_i^{(\alpha_a^\vee + \ldots + \alpha_b^\vee)} e_{ab}^{(1)} \otimes f_{ba}^{(1)}.
\]

This completes our proof of Proposition 2.134. □

Proposition 2.134 provides a conceptual and elementary proof of [FKPRW, Theorem 4.8]:

Proposition 2.136. (a) For any $\nu_1, \nu_2 \in \Lambda^+$, there is a unique $\mathbb{C}$-algebra homomorphism

\[
\Delta_{-\nu_1, -\nu_2} : Y_{-\nu_1 - \nu_2}(\mathfrak{sl}_n) \rightarrow Y_{-\nu_1}(\mathfrak{sl}_n) \otimes Y_{-\nu_2}(\mathfrak{sl}_n)
\]

(2.137)
such that the following diagram is commutative
\[
\begin{array}{c}
Y_{-\bar{\mu}_1-\bar{\mu}_2}(s\mathfrak{l}_n) \xrightarrow{\Delta_{-\bar{\mu}_1-\bar{\mu}_2}\nu} Y_{-\bar{\mu}_1}(s\mathfrak{l}_n) \otimes Y_{-\bar{\mu}_2}(s\mathfrak{l}_n) \\
\downarrow i_{-\mu_1-\mu_2} \hspace{2cm} \downarrow i_{-\mu_1} \otimes i_{-\mu_2}
\end{array}
\]  \hspace{1cm} (2.138)

for any $\mu_1, \mu_2 \in \Lambda^+$. 

(b) The homomorphism $\Delta_{-\nu_1-\nu_2}$ is uniquely determined by the following formulas:

\[
\begin{align*}
F_{i}^{(r)} & \mapsto F_{i}^{(r)} \otimes 1 \text{ for } 1 \leq r \leq \alpha_i^\vee(\nu_1), \\
F_{i}^{(\alpha_i^\vee(\nu_1)+1)} & \mapsto F_{i}^{(\alpha_i^\vee(\nu_1)+1)} \otimes 1 + 1 \otimes F_{i}^{(1)}, \\
E_{i}^{(r)} & \mapsto 1 \otimes E_{i}^{(r)} \text{ for } 1 \leq r \leq \alpha_i^\vee(\nu_2), \\
E_{i}^{(\alpha_i^\vee(\nu_2)+1)} & \mapsto 1 \otimes E_{i}^{(\alpha_i^\vee(\nu_2)+1)} + E_{i}^{(1)} \otimes 1, \\
H_{i}^{(\alpha_i^\vee(\nu_1+\nu_2)+1)} & \mapsto H_{i}^{(\alpha_i^\vee(\nu_1)+1)} \otimes 1 + 1 \otimes H_{i}^{(\alpha_i^\vee(\nu_2)+1)}, \\
H_{i}^{(\alpha_i^\vee(\nu_1+\nu_2)+2)} & \mapsto H_{i}^{(\alpha_i^\vee(\nu_2)+2)} + \sum_{\gamma^\vee > 0} \alpha_i^\vee(\gamma^\vee) E_{i}^{(1)} \otimes F_{i}^{(1)},
\end{align*}
\]  \hspace{1cm} (2.139)

where $E_{\alpha_i^\vee+\ldots+\alpha_{i-1}^\vee}^{(1)} := [E_{\alpha_1}^{(1)}, \ldots, E_{\alpha_i}^{(1)}] \ldots]$ and $F_{\alpha_i^\vee+\ldots+\alpha_{i-1}^\vee}^{(1)} := [\ldots, F_{\alpha_1}^{(1)}, F_{\alpha_i}^{(1)}, \ldots, F_{\alpha_{i-1}}^{(1)}]$. 

Proof. Follows immediately from the formulas (2.135) of Proposition 2.134 combined with the defining formulas (2.21) for the embedding $i_{-\mu}: Y_{-\bar{\mu}}(s\mathfrak{l}_n) \hookrightarrow Y_{-\mu}(s\mathfrak{l}_n)$ of Proposition 2.19. □

Remark 2.140. Due to [FKPRW, Theorem 4.12], $\Delta_{-\nu_1-\nu_2}$ with $\nu_1, \nu_2 \in \bar{\Lambda}^+$ give rise to algebra homomorphisms $\Delta_{\nu_1,\nu_2}: Y_{\nu_1+\nu_2}(s\mathfrak{l}_n) \rightarrow Y_{\nu_1}(s\mathfrak{l}_n) \otimes Y_{\nu_2}(s\mathfrak{l}_n)$ for any $s\mathfrak{l}_n$-coweights $\nu_1, \nu_2 \in \bar{\Lambda}$.

Remark 2.141. We note that [RT, §2.4] contains an attempt to construct the simplest coproduct homomorphism $Y_{-\alpha}(s\mathfrak{l}_2) \rightarrow Y_{-\alpha/2}(s\mathfrak{l}_2) \otimes Y_{-\alpha/2}(s\mathfrak{l}_2)$ from Proposition 2.136.

2.7. Relation to Gelfand-Tsetlin bases of parabolic Verma modules of $\mathfrak{gl}_n$.

Evoking the setup of Section 2.4.4, assume $\mu = ((-1)^n)$ while $\lambda$ is a Young diagram of size $n$ and length $< n$, i.e. $|\lambda| = n$ and $\lambda_n = 0$. Consider the corresponding $\Lambda^+$-valued divisor on $\mathbb{P}^1$: $D = \sum_{k=1}^{\lambda_1} \omega_{i_k}[x_k] - \omega_0[\infty]$ with $x_k \in \mathbb{C}$ (note that $i_k = n - \lambda_k^\vee$). In this section, we show that the homomorphism $\Theta_D: Y^{\text{rtt}}(\mathfrak{gl}_n) \rightarrow A$ of (2.55) may be viewed (up to a gauge transformation) as a composition of the \textit{evaluation} homomorphism $\text{ev}: Y^{\text{rtt}}(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_n)$ and the homomorphism $U(\mathfrak{gl}_n) \rightarrow A$ determined by the \textit{parabolic Gelfand-Tsetlin} formulas.

Recall the explicit formulas for the matrix coefficients $T_D(z)_{i,i}, T_D(z)_{i,i+1}, T_D(z)_{i+1,i}$ of Theorem 2.84 (note that $T_D(z) = \Theta_D(z)$ in the current setup):

\[
T_D(z)_{i,i} = z + \sum_{r=1}^{a_i} (p_{i-1,r} + 1) - \sum_{r=1}^{a_i} p_{i,r} - \sum_{k:i_k \leq i-1} x_k, \hspace{1cm} (2.142)
\]

\[
T_D(z)_{i,i+1} = -\sum_{r=1}^{a_i} \prod_{s=1}^{a_i} (p_{i,r} - 1 - p_{i-1,s}) \prod_{1 \leq s \leq a_i} (p_{i,r} - p_{i,s}) \cdot \prod_{k:i_k = i} (p_{i,r} - x_k) \cdot e^{g_i,r}, \hspace{1cm} (2.143)
\]
\[ T_D(z)_{i+1,i} = \sum_{r=1}^{a_i} \prod_{s=1}^{a_{i+1}} \frac{(p_{i,r} + 1 - p_{i+1,s})}{\prod_{1 \leq s \leq a_i} (p_{i,r} - p_{i,s})} \cdot e^{-q_{i,r}}. \]  

(2.144)

Consider the following factor

\[ S = \prod_{i=1}^{n-2} \prod_{r \leq \alpha_i} \Gamma(p_{i,r} - p_{i+1,s}) \cdot \prod_{i=1}^{a_i} \prod_{r=1}^{a_i} \prod_{k; i,k \leq i-1} \Gamma(p_{i,r} - x_k + 1) \cdot \prod_{i=1}^{r \leq a_i} \Gamma(p_{i,r} - p_{i,r}), \]  

(2.145)

where \( \Gamma(\cdot) \) denotes the classical Gamma function. Then, \( \text{Ad}(S) \) is a well-defined automorphism of \( A \), which shall be referred to as the \textit{gauge transformation with respect to} \( S \). Applying \( \text{Ad}(S) \) to \( T_D(z) \) described by the formulas (2.142, 2.143, 2.144), we obtain

\[ \text{Ad}(S)T_D(z)_{i,i} = z + \sum_{r=1}^{a_i} (p_{i-1,r} + 1) - \sum_{r=1}^{a_i} p_{i,r} - \sum_{k; i,k \leq i-1} x_k, \]  

(2.146)

\[ \text{Ad}(S)T_D(z)_{i,i+1} = \sum_{r=1}^{a_i} (-1)^{a_i+a_{i+1}} \frac{\prod_{s=1}^{a_{i+1}} (p_{i,r} - p_{i+1,s})}{\prod_{1 \leq s \leq a_i} (p_{i,r} - 1 - p_{i,s})} \prod_{k; i,k \leq i} \Gamma(p_{i,r} - x_k + 1) \cdot e^{q_{i,r}}, \]  

(2.147)

\[ \text{Ad}(S)T_D(z)_{i+1,i} = \sum_{r=1}^{a_i} (-1)^{a_i+a_{i-1}+1} \frac{\prod_{s=1}^{a_{i-1}} (p_{i,r} - p_{i-1,s})}{\prod_{1 \leq s \leq a_i} (p_{i,r} + 1 - p_{i,s})} \prod_{k; i,k \leq i-1} \frac{1}{p_{i,r} - x_k + 1} \cdot e^{-q_{i,r}}. \]  

(2.148)

We also consider the factor

\[ U := \prod_{i=1}^{n-1} \prod_{r=1}^{a_i} \left( (-1)^{\lambda_{n-1} p_{i,r}} \cdot e^{-q_{i,r}} \right), \]  

(2.149)

so that \( \text{Ad}(U) \) is a well-defined automorphism of \( A \) which maps

\[ p_{i,r} \mapsto p_{i,r} + i, \quad e^{q_{i,r}} \mapsto (-1)^{\lambda_{n-1} e^{q_{i,r}}}. \]

Applying this automorphism to (2.146, 2.147, 2.148), we obtain

\[ \text{Ad}(US)T_D(z)_{i,i} = z + \sum_{r=1}^{a_i} p_{i-1,r} - \sum_{r=1}^{a_i} p_{i,r} + i(a_{i-1} - a_i) - \sum_{k; i,k \leq i-1} x_k, \]  

(2.150)

\[ \text{Ad}(US)T_D(z)_{i,i+1} = \sum_{r=1}^{a_i} (-1)^{\beta_{i}} \frac{\prod_{s=1}^{a_{i+1}} (p_{i+1,s} - p_{i,r} + 1)}{\prod_{1 \leq s \leq a_i} (p_{i,s} - p_{i,r} + 1)} \prod_{k; i,k \leq i} \Gamma(p_{i,r} - x_k - i) \cdot e^{q_{i,r}}, \]  

(2.151)

\[ \text{Ad}(US)T_D(z)_{i+1,i} = \sum_{r=1}^{a_i} \frac{\prod_{s=1}^{a_{i-1}} (p_{i-1,s} - p_{i,r} - 1)}{\prod_{1 \leq s \leq a_i} (p_{i,s} - p_{i,r} - 1)} \prod_{k; i,k \leq i-1} \frac{1}{x_k - p_{i,r} - i - 1} \cdot e^{-q_{i,r}}, \]  

(2.152)

where \( \beta_{i} := a_{i-1} + a_{i+1} + 1 + \lambda_{n-1} + \lambda_{n-1} \). Evoking \( a_{i-1} - a_{i-1} = 1 - \lambda_{n-1} \), we see that \( \beta_{i} \) is odd. Thus, the formulas (2.150, 2.151, 2.152) may be written as follows:

\[ \text{Ad}(US)T_D(z)_{i,i} = z + \sum_{r=1}^{a_i} p_{i-1,r} - \sum_{r=1}^{a_i} p_{i,r} + i(\lambda_{n-1} - 1) - \sum_{k; i,k \leq i-1} x_k, \]  

(2.153)

\[ \text{Ad}(US)T_D(z)_{i,i+1} = \sum_{r=1}^{a_i} \frac{\prod_{s=1}^{a_{i+1}} (p_{i+1,s} - p_{i,r} + 1)}{\prod_{1 \leq s \leq a_i} (p_{i,s} - p_{i,r} + 1)} \prod_{k; i,k \leq i} \Gamma(p_{i,r} - x_k - i) \cdot e^{q_{i,r}}, \]  

(2.154)
\[
\text{Ad}(US)T_D(z)_{i+1,i} = \sum_{r=1}^{a_i} \prod_{s=1}^{a_i} \left( \frac{p_{i,s} - p_{i,r} - 1}{x_{ki} - p_{i,r} - i - 1} \right) \cdot e^{-q_{i,r}}.
\]

Let us now relate formulas (2.153, 2.154, 2.155) to the parabolic Gelfand-Tsetlin formulas. Let \( p \subseteq gl_n \) be a parabolic subalgebra with the Levi factor \( \Lambda \approx gl_{n_1} \oplus gl_{n_2} \oplus \cdots \oplus gl_{n_t} \) embedded block-diagonally into \( gl_n \). For \( y = (y_1, \ldots, y_{n_2}) \in \mathbb{C}^{\lambda_i} \), let \( \mathbb{C}_y \) be the 1-dimensional \( p \)-module obtained as a pull-back (along the natural projection \( p \rightarrow \mathfrak{h} \)) of the 1-dimensional \( \mathfrak{t} \)-module with \( gl_{\lambda_i} \)-factor acting via \( y \text{tr} \). We also assume that \( y_i \neq y_j \) for \( i \neq j \). Consider the parabolic Verma module \( M_\mathbb{C} := \text{Ind}_p^{gl_n} \mathbb{C}_y \). It has a distinguished basis \( \{\xi_\Lambda\} \), called the Gelfand-Tsetlin basis, parametrized by \( \Lambda \in (\Lambda_{i,j})_{1 \leq j \leq n} \) subject to the following conditions:

(a) \( \Lambda_{n,n} + \ldots + \Lambda_{i-1,i} + a = y_a \) for \( 1 \leq a \leq \lambda_i^\prime \);
(b) \( \Lambda_{i+1,j} - \Lambda_{ij} \in \mathbb{N} \);
(c) if \( \Lambda_{ij} - \Lambda_{i+1,j+1} \in \mathbb{Z} \), then actually \( \Lambda_{ij} - \Lambda_{i+1,j+1} = 1 \in \mathbb{N} \).

Note that conditions (b,c) imply \( \Lambda_{ij} = y_a \) if \( \lambda_i^\prime + \ldots + \lambda_{i-1}^\prime < k \leq \lambda_i^\prime + \ldots + \lambda_j^\prime - (n - i) \). We call such coordinates \((i,k)\) frozen. For \( 1 \leq i \leq n - 1 \), let \( J_i \subseteq \{1, \ldots, i\} \) denote the set of non-frozen coordinates among \( \{(i,*\}) \). It is easy to see that \( |J_i| = a_i \). Set \( l_{i,j} := \Lambda_{ij} - j + 1 \).

Then, the classical Gelfand-Tsetlin formulas \([NT]\) (corresponding to the case \( l \approx gl_{\zeta}^{\text{even}} \)) give rise to the parabolic Gelfand-Tsetlin formulas for the action of \( gl_n \) in the basis \( \xi_\Lambda \) of \( M^\mathbb{C} \):

\[
E_{i,i}(\xi_\Lambda) = \left( \sum_{k \in J_i} l_{i,k} - \sum_{k \in J_i} l_{i-1,k} + \sum_{a: \lambda_a^\prime \geq n - i + 1} (y_a' - i) + (i - 1) \right) \cdot \xi_\Lambda,
\]

\[
E_{i,i+1}(\xi_\Lambda) = -\sum_{k \in J_i} \prod_{m \in J_{i+1} \setminus \{k\}} (l_{i+m} - l_{i,k}) \prod_{a: \lambda_a^\prime \geq n - i} (y_a' - l_{i,k} - i - 1) \cdot \xi_{\Lambda + \delta_{i,k}},
\]

\[
E_{i+1,i}(\xi_\Lambda) = \sum_{k \in J_i} \prod_{m \in J_{i+1} \setminus \{k\}} (l_{i+1,m} - l_{i,k}) \prod_{a: \lambda_a^\prime \geq n - i} \frac{1}{y_a' - l_{i,k} - i} \cdot \xi_{\Lambda - \delta_{i,k}},
\]

where \( y_a' := y_a - (\lambda_1^\prime + \ldots + \lambda_i^\prime) + (n + 1) \) and \( \Lambda \pm \delta_{i,k} \) is obtained from \( \Lambda \) by adding \( \pm 1 \) to its \((i,k)\)-th entry (if \( \Lambda \pm \delta_{i,k} \) does not satisfy (b) or (c), then the corresponding coefficient in front of \( \xi_{\Lambda \pm \delta_{i,k}} \) in (2.157) or (2.158), respectively, is actually zero).

These formulas naturally give rise to the algebra homomorphism \( \rho: U(gl_n) \rightarrow \mathcal{A} \) with

\[
E_{i,i} \mapsto \sum_{k \in J_i} p_{i,k} - \sum_{k \in J_{i-1}} p_{i-1,k} + \sum_{a: \lambda_a^\prime \geq n - i + 1} (y_a' - i) + (i - 1),
\]

\[
E_{i,i+1} \mapsto -\sum_{k \in J_i} e^{q_{i,k}} \prod_{m \in J_{i+1} \setminus \{k\}} (p_{i+1,m} - p_{i,k}) \prod_{a: \lambda_a^\prime \geq n - i} (y_a' - p_{i,k} - i - 1) =
\]

\[
-\sum_{k \in J_i} \prod_{m \in J_{i+1} \setminus \{k\}} (p_{i+1,m} - p_{i,k} + 1) \prod_{a: \lambda_a^\prime \geq n - i} (y_a' - p_{i,k} - i) \cdot e^{q_{i,k}},
\]
\[ E_{i+1,i} \mapsto \sum_{k \in J_i} e^{-q_{i,k}} \prod_{m \in J_{i-1}} (p_{i-1,m} - p_{i,k}) \prod_{m \in J_i \setminus \{k\}} (p_{i,m} - p_{i,k}) \prod_{a : \lambda_i \geq n-i+1} \frac{1}{y_a' - p_{i,k} - i} = \sum_{k \in J_i} \prod_{m \in J_{i-1}} (p_{i-1,m} - p_{i,k} - 1) \prod_{m \in J_i \setminus \{k\}} (p_{i,m} - p_{i,k} - 1) \prod_{a : \lambda_i \geq n-i+1} \frac{1}{y_a' - p_{i,k} - i - 1} \cdot e^{-q_{i,k}}. \] (2.161)

**Remark 2.162.** We note that \( \mathcal{A} \) acts on the bigger space \( \tilde{M}_y \) parametrized by \( \Lambda = (\Lambda_{i,j})_{1 \leq j \leq i \leq n} \) satisfying only the condition (a) via \( p_{i,k} : \xi_{(j)} \mapsto l_{i,k} \xi_{(j)} \), \( e^{\mp q_{i,k}} : \xi_{(j)} \mapsto \xi_{(j) \pm s_i} \). Meanwhile, the same formulas actually define the action of \( \text{Im}(\varrho) \subset \mathcal{A} \) on \( M_y \), composing which with \( \varrho \) recovers the action of \( U(\mathfrak{gl}_n) \) on \( M_y \) defined via (2.156, 2.157, 2.158).

Consider the evaluation homomorphism \( \tilde{e}_\text{ev} : Y^\text{rtt}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n) \) such that
\[
T(z)_{i,j} \mapsto z - (E_{i,i} + 1), \quad T(z)_{i+1,i} \mapsto E_{i+1,i}, \quad T(z)_{j+1,i} \mapsto E_{j+1,i}.
\] (2.163)

**Remark 2.164.** \( \tilde{e}_\text{ev} \) is a composition of the isomorphism \( Y^\text{rtt}(\mathfrak{gl}_n) \overset{\sim}{\to} Y^0_{w_0}(\mathfrak{gl}_n), \) \( T(z) \mapsto zT(z) \) (cf. Lemma 2.17), the evaluation homomorphism \( e_{\text{ev}} : Y_{w_0}^0(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n), \) \( t_{ij}(z) \mapsto \delta_{ij} - E_{ij}z^{-1} \), and the isomorphism \( U(\mathfrak{gl}_n) \overset{\sim}{\to} U(\mathfrak{gl}_n) \) such that \( E_{ii} \mapsto E_{ii} + 1, E_{i,i+1} \mapsto -E_{i,i+1} \).

The key result of this section is:

**Proposition 2.165.** The homomorphism \( \text{Ad}(US) \circ \Theta_D : Y^\text{rtt}(\mathfrak{gl}_n) \to \mathcal{A} \) (the gauge transformation of \( \Theta_D \)) coincides with the composition \( \varrho \circ \tilde{e}_{\text{ev}} : Y^\text{rtt}(\mathfrak{gl}_n) \to \mathcal{A} \), under the identification \( x_k = y'_k \) for \( 1 \leq k \leq \Lambda_1 \).

**Proof.** The proof immediately follows by comparing the formulas (2.153, 2.154, 2.155) with the formulas (2.159, 2.160, 2.161) via (2.163) (as well as recalling that \( i_k = n - \lambda_k' \), hence, for example \( \sum_{k : i_k \leq i - 1} x_k \) of (2.153) coincides with \( \sum_{i : \lambda_i \geq n - i + 1} y'_i \) of (2.159)). \( \square \)

**Remark 2.166.** Choosing a basis of a Lie subalgebra \( \mathfrak{n}_- \subseteq \mathfrak{gl}_n \) such that \( \mathfrak{gl}_n \simeq \mathfrak{p} \oplus \mathfrak{n}_- \), yields another standard basis of \( M_y \) via the vector space isomorphisms \( M_y \simeq U(\mathfrak{n}_-) \simeq S(\mathfrak{n}_-) \), which similar to Proposition 2.165 gives rise to the rational Lax matrices \( L_{\lambda, \mu} : \mu = \theta(z) \) of [FP, §3.2].

### 3. Trigonometric Lax matrices

In this section, we generalize previous results to the trigonometric case.

#### 3.1. Shifted Drinfeld quantum affine algebras of \( \mathfrak{gl}_n \)

For a pair of \( \mathfrak{gl}_n \)-coweights \( \mu^+, \mu^- \in \Lambda \), define \( d^\pm = (d_i^\pm)_{i=1}^n = (\{ d_i^\pm \}_{i=1}^n) \in \mathbb{Z}^n \), \( b^\pm = (b_i^\pm)_{i=1}^{n-1} \in \mathbb{Z}^{n-1} \) via
\[
d_i^\pm := \epsilon_i^\pm(\mu^\pm), \quad b_i^\pm := \alpha_i^\pm(\mu^\pm) = d_i^\pm - d_{i+1}^\pm.
\] (3.1)

Define the **shifted Drinfeld quantum affine algebra of** \( \mathfrak{gl}_n \), denoted by \( U_{\mu^+, \mu^-}(\mathfrak{Lgl}_n) \), to be the associative \( \mathbb{C}(\mathfrak{v}) \)-algebra generated by \( \{ E_{i,r}, F_{i,r} \}_{1 \leq i \leq n} \cup \{ \varphi_{i, \pm s_i}^+, (\varphi_{i, \pm s_i}^+)_{i \neq k}^{-1} \}_{1 \leq i \leq n} \) with the following defining relations (for all admissible \( i,j \) and \( \epsilon, \epsilon' \in \{ \pm \} \)):
\[
[\varphi_i^\epsilon(z), \varphi_j^{\epsilon'}(w)] = 0, \quad \varphi_{i, \pm s_i}^+ \cdot (\varphi_{i, \pm s_i}^+)_{i \neq k}^{-1} = (\varphi_{i, \pm s_i}^+)_{i \neq k}^{-1} \cdot \varphi_{i, \pm s_i}^+ = 1,
\] (3.2)
\[
[E_i(z), F_j(w)] = (\mathfrak{v} - \mathfrak{v}^{-1}) \delta_{ij} \delta \left( \frac{z}{w} \right) ((\varphi_i^+(z))^{-1} \varphi_{i+1}^+(z) - (\varphi_i^-(z))^{-1} \varphi_{i+1}^-(z)),
\] (3.3)
Remark 3.12. For \( \mu^+ = \mu^- = 0 \), we have \( U_{0,0}(\mathfrak{gl}_n)/((\varphi^{\pm}_{r,n})_{r=0} - 1) \simeq U_v(\mathfrak{gl}_n) \) the standard quantum loop (the quantum affine with the trivial central charge) algebra of \( \mathfrak{gl}_n \) as defined in [DF, Definition 3.1]. More precisely, the generating series \( X_i^-(z), X_i^+(z), k_j^\pm(z) \) of loc.cit. correspond to \( E_i(z), F_i(z), \varphi_j^\pm(z) \) of (3.10), respectively.

Similarly to Lemma 2.17, the algebra \( U_{\mu^+, \mu^-}(\mathfrak{gl}_n) \) depends only on the associated \( \mathfrak{sl}_n \)-coweights \( \tilde{\mu}^+, \tilde{\mu}^- \in \Lambda \) up to an isomorphism:

**Lemma 3.13.** Let \( \mu_1^+, \mu_2^+, \mu_1^- \), \( \mu_2^- \in \Lambda \) be \( \mathfrak{sl}_n \)-coweights such that \( \tilde{\mu}^+_1 = \mu^+_2, \tilde{\mu}^-_1 = \mu^-_2 \) in \( \tilde{\Lambda} \). Then, the assignment

\[
\begin{align*}
E_i^{(r)} & \mapsto E_i^{(r)}, \\
F_i^{(r)} & \mapsto F_i^{(r)}, \\
\varphi_i^\pm & \mapsto \varphi_{i, \pm r}^\pm + \epsilon_i'(\mu_i^+ - \mu_i^-)
\end{align*}
\]  

(3.14)

gives rise to a \( \mathbb{C}(v) \)-algebra isomorphism \( U_{\mu^+_1, \mu^-_1}(\mathfrak{gl}_n) \simeq U_{\mu^+_2, \mu^-_2}(\mathfrak{gl}_n) \).

Let \( U'_{\mu^+, \mu^-}(\mathfrak{gl}_n) \) be the associative \( \mathbb{C}(v) \)-algebra obtained from \( U_{\mu^+, \mu^-}(\mathfrak{gl}_n) \) by formally adjoining \( n \)-th roots of its central elements \( \varphi^\pm := \varphi^\pm_{1, \pm d_1^+} \varphi^\pm_{2, \pm d_2^+} \cdots \varphi^\pm_{n, \pm d_n^+} \), that is,

\[
U'_{\mu^+, \mu^-}(\mathfrak{gl}_n) := U_{\mu^+, \mu^-}(\mathfrak{gl}_n)[[(\varphi^+)^{\pm 1/n}, (\varphi^-)^{\pm 1/n}].
\]

(3.15)

The algebras \( U'_{\mu^+, \mu^-}(\mathfrak{sl}_n) \) slightly generalize the shifted (Drinfeld) quantum affine algebras of \( \mathfrak{sl}_n \), denoted by \( U^{\text{sc}, \nu_+, \nu_-}(\mathfrak{sl}_n) \) (the simply-connected version) and \( U^{\text{ad}, \nu_+, \nu_-}(\mathfrak{sl}_n) \) (the adjoint version) in [FT1, §5], where \( \nu^+, \nu^- \in \tilde{\Lambda} \) are \( \mathfrak{sl}_n \)-coweights. Recall that the latter, the algebra
\[ U_{\nu^+, \nu^-}(Lsl_n) \] is an associative \( \mathbb{C}(v) \)-algebra generated by \( \{ e_{i,r}, f_{i,r}, \psi_{i,s}^\pm, (\psi_{i}^\pm)^{1} \}_{r \in \mathbb{Z}, s \geq d_i^\pm} \) with the defining relations \([FT1, (U1-U10)]\), where \( b_i^\pm := \alpha_i^\pm(\nu_i^\pm) \). Define the generating series
\[
e_i(z) := \sum_{r \in \mathbb{Z}} e_{i,r} z^{-r}, \quad f_i(z) := \sum_{r \in \mathbb{Z}} f_{i,r} z^{-r}, \quad \psi_i^\pm(z) := \sum_{r \geq -b_i^\pm} \psi_{i,\pm r} z^{\pm r}.
\]
The explicit relation between the shifted Drinfeld quantum affine algebras of \( sl_n \) and \( gl_n \) is:

**Proposition 3.16.** For any \( \mu^+, \mu^- \in \Lambda \), there exists a \( \mathbb{C}(v) \)-algebra embedding
\[
t_{\mu^+, \mu^-} : U_{\bar{\mu}^+, \bar{\mu}^-}(Lsl_n) \hookrightarrow U_{\mu^+, \mu^-}(Lgl_n),
\]
uniquely determined by
\[
e_i(z) \mapsto \frac{E_i(v^i z)}{v - v^{-1}}, \quad f_i(z) \mapsto \frac{F_i(v^i z)}{v - v^{-1}}, \quad \psi_i^\pm(z) \mapsto (\varphi_i^\pm(v^i z))^{-1} \varphi_{i+1}^\pm(v^i z), \quad \phi_i^\pm \mapsto (\varphi_1^\pm, \varphi_{i+1}^\pm, \ldots, \varphi_n^\pm)^{-1} \cdot (\varphi^\pm)^{i/n}.
\]
Restricting to \( U_{\bar{\mu}^+, \bar{\mu}^-}(Lsl_n) \subset U_{\bar{\mu}^+, \bar{\mu}^-}(Lsl_n) \), gives rise to a \( \mathbb{C}(v) \)-algebra embedding
\[
t_{\mu^+, \mu^-} : U_{\bar{\mu}^+, \bar{\mu}^-}(Lsl_n) \hookrightarrow U_{\mu^+, \mu^-}(Lgl_n),
\]

**Remark 3.20.** For \( \mu^+ = \mu^- = 0 \), this recovers (an extension of) the classical embedding \( U_v(Lsl_n) \hookrightarrow U_v(Lgl_n) \) of quantum loop algebras.

**Proof of Proposition 3.16.** The proof is completely analogous to that of Proposition 2.19. □

Define the generating series
\[
C_i^\pm(z) := \sum_{s \geq d_i^\pm + \ldots + d_n^\pm} C_{i,s}^\pm z^s = \varphi_i^\pm(z) \varphi_{i+1}^\pm(v^2 z) \cdots \varphi_n^\pm(v^{2(n-1)} z).
\]
The coefficients \( C_{i,s}^\pm \) are central elements of both \( U_{\mu^+, \mu^-}(Lgl_n) \) and \( U'_{\mu^+, \mu^-}(Lgl_n) \), due to the defining relations (3.2, 3.4, 3.5). We also note that \( C_{i,s}^\pm + \ldots + d_n^\pm = \varphi_i^\pm \).

The following result provides a trigonometric version of the decomposition (2.24):

**Lemma 3.22.** There is a \( \mathbb{C}(v) \)-algebra isomorphism
\[
U'_{\mu^+, \mu^-}(Lgl_n) \simeq \mathbb{C}[\{ C_{i,s}^\pm, (\varphi^\pm)^{\pm 1/n} \}_{s \geq d_i^\pm + \ldots + d_n^\pm}] \otimes \mathbb{C}(v) U_{\bar{\mu}^+, \bar{\mu}^-}(Lsl_n).
\]
In particular, \( U_{\bar{\mu}^+, \bar{\mu}^-}(Lsl_n) \) may be realized both as a subalgebra of \( U'_{\mu^+, \mu^-}(Lgl_n) \) via (3.17) and as a quotient algebra of \( U'_{\mu^+, \mu^-}(Lgl_n) \) by the central ideal \( \left( C_{i,s}^\pm (\varphi^\pm)^{\pm 1/n - 1} \right)_{s \geq d_i^\pm + \ldots + d_n^\pm} \).

**Remark 3.24.** We expect that the trigonometric version of the key result of [W], see Theorem 2.79 and Conjecture 3.75, holds. Then, the arguments similar to those of Remark 2.80 would yield the triviality of centers of the shifted quantum affine algebras \( U_{\nu^+, \nu^-}(Lg) \) for any coweights \( \nu^+, \nu^- \) of a semisimple Lie algebra \( g \). Combined with (3.23) this would imply that the center of \( U_{\mu^+, \mu^-}(Lgl_n) \) coincides with \( \mathbb{C}[\{ C_{i,s}^\pm, (\varphi^\pm)^{\pm 1/n} \}_{s \geq d_i^\pm + \ldots + d_n^\pm}] \) for any \( \mu \in \Lambda \).
3.2. **Homomorphism \( \Psi_D \).**

In this section, we generalize [FT1, Theorem 7.1] for the type \( A_{n-1} \) Dynkin diagram with arrows pointing \( i \to i+1, 1 \leq i \leq n-2 \), by replacing \( U_{\mu^+,-\mu^-}(L\mathfrak{sl}_n) \) of loc.cit. with \( U_{\mu^+,-\mu^-}(L\mathfrak{g}_n) \).

**Remark 3.25.** While similar generalizations exist for all orientations of \( A_{n-1} \) Dynkin diagram, for the purposes of this paper it suffices to consider only the above equi-oriented case, see Remarks 2.26, 2.72.

A \( \Lambda \)-valued divisor \( D \) on \( \mathbb{P}^1 \), \( \Lambda^+ \)-valued outside \( \{0, \infty\} \in \mathbb{P}^1 \), is a formal sum

\[
D = \sum_{1 \leq s \leq N} \gamma_s w_i(x_s) + \mu^+[\infty] + \mu^-[0] \quad (3.26)
\]

with \( N \in \mathbb{N}, 0 \leq i_s < n, x_s \in \mathbb{C}^x, \gamma_s = \begin{cases} 1, & \text{if } i_s \neq 0 \\ \pm 1, & \text{if } i_s = 0 \end{cases} \), and \( \mu^+, \mu^- \in \Lambda \). We will write \( \mu^+ = D|_{\infty} \) and \( \mu^- = D|_0 \). Note that if \( \mu^+, \mu^- \in \Lambda^+ \), then \( D \) is a \( \Lambda^+ \)-valued divisor on \( \mathbb{P}^1 \). It will be convenient to present

\[
D = \sum_{x \in \mathbb{P}^1 \setminus \{0, \infty\}} \lambda_x[x] + \mu^+[\infty] + \mu^-[0] \quad \text{with } \lambda_x \in \Lambda^+ \quad (3.27)
\]

related to (3.26) via \( \lambda_x := \sum_{s:\Delta x = x} \gamma_s w_i(x_s) \). Set \( \lambda := \sum_{s=1}^N \gamma_s w_i(x_s) \in \Lambda^+ \). We shall assume that

\[
\lambda + \mu^+ + \mu^- = a_1\alpha_1 + \ldots + a_{n-1}\alpha_{n-1} \quad \text{with } a_i \in \mathbb{N} \quad (3.28)
\]

We also set \( a_0 := 0, a_n := 0 \).

Consider the associative \( \mathbb{C}[v, v^{-1}] \)-algebra

\[
\tilde{A}^v = \mathbb{C}\langle D_{i,r}^{\pm 1}, w_{i,s}^{\pm 1/2}, (w_{i,r} - v^m w_{i,s})^{-1}, (1 - v^l)^{-1} \rangle_{1 \leq i \leq n, m \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\}} \quad (3.29)
\]

with the defining relations

\[
D_{i,r} w_{j,s}^{1/2} = v^\delta_{i,j} \delta_{r,s} w_{j,s}^{1/2} D_{i,r}, \quad [D_{i,r}, D_{j,s}] = 0 = [w_{i,r}^{1/2}, w_{j,s}^{1/2}], \quad D_{i,r} D_{i,r}^{\pm 1} = 1 = w_{i,r}^{1/2} w_{i,r}^{1/2} \quad (3.30)
\]

We also define its \( \mathbb{C}(v) \)-counterpart

\[
\tilde{A}^v_{\text{trac}} := \tilde{A}^v \otimes \mathbb{C}(v) \quad (3.31)
\]

**Remark 3.31.** The algebra \( \tilde{A}^v \) can be represented in the algebra of \( v \)-difference operators with rational coefficients on functions of \( \{w_{i,r}\}_{1 \leq i \leq n} \) with the conventions \( w_{i,r}^{1/2} = w_{i,r}^{1/2} \) by taking \( D_{i,r}^{\pm 1} \) to be a \( v \)-difference operator \( D_{i,r}^{\pm 1} \) acting via \( \mathbb{D}_{i,r}^{\pm 1}(\Psi(w_{1,1}, \ldots, w_{1,n}, \ldots, w_{n-1,n}) = \Psi(w_{1,1}, \ldots, v^{1/2}w_{i,r}, \ldots, w_{n-1,n}) \).

For \( 0 \leq i \leq n-1 \) and \( 1 \leq j \leq n-1 \), we define

\[
Z_i(z) := \prod_{1 \leq s \leq N} \left( 1 - \frac{v^{-i} x_s}{z} \right)^{\gamma_s} = \prod_{x \in \mathbb{P}^1 \setminus \{0, \infty\}} \left( 1 - \frac{v^{-i} x}{z} \right)^{\alpha_i^0(\lambda_x)}, \quad (3.32)
\]

\[
W_j(z) := \prod_{r=1}^{a_j} \left( 1 - \frac{w_{j,r}}{z} \right), \quad W_{j,r}(z) := \prod_{1 \leq s \leq a_j} \left( 1 - \frac{w_{j,s}}{z} \right),
\]

where \( \alpha_0^0 = -\epsilon_0^0 \) as before. We also define \( W_0(z) := 1, W_n(z) := 1 \).

The following result generalizes \( A_{n-1} \)-case of [FT1, Theorem 7.1] stated for semisimple Lie algebras \( \mathfrak{g} \):
Theorem 3.33. Let $D$ be as above and $\mu^+ = D|_\infty, \mu^- = D|_0$. There is a unique $\mathbb{C}(v)$-algebra homomorphism

$$\Psi_D : U_{-\mu^+, -\mu^-}(L\mathfrak{g}_s) \to \tilde{A}_v^\text{frac}$$

such that

$$E_i(z) \mapsto z^{-\alpha_i^\vee(\mu^+)} \cdot \prod_{t=1}^{a_i} w_{i,t} \prod_{r=1}^{a_{i-1}} \delta(z^{\psi_i^r} w_{i,r} z) \frac{Z_i(w_{i,r})}{W_i(w_{i,r})} W_{i-1}(v^{-1} w_{i,r}) D_{i,r}^{-1},$$

$$F_i(z) \mapsto -v^{-1} \prod_{t=1}^{a_i+1} w_{i+t,1}^{-1/2} \cdot \sum_{r=1}^{a_i} \delta \left( z^{\psi_i^r} w_{i,r} z \right) \frac{1}{W_{i+1}(v w_{i,r})} W_{i+1}(v w_{i,r}) D_{i,r},$$

$$\varphi_i^\pm(z) \mapsto \prod_{t=1}^{a_i} w_{i,t}^{-1/2} \prod_{t=1}^{a_{i-1}} w_{i-t,1}^{1/2} \cdot \left( z^{\epsilon_i^\vee(\mu^+)} \cdot \prod_{k=0}^{i-1} Z_k(v^{-k} z) \prod_{x \in F_i \setminus \{0, \infty\}} \left( 1 - x/z \right)^{-\epsilon_i^\vee(\lambda_x)} \right)^\pm.$$

We write $\gamma(z)^\pm$ for the expansion of a rational function $\gamma(z)$ in $z^{\pm 1}$, respectively.

Remark 3.36. Let $\tilde{A}_v^\text{frac, ext}$ be the associative $\mathbb{C}(v)$-algebra obtained from $\tilde{A}_v^\text{frac}$ by formally adjoining $n$-th roots of $v, x_s$, and $\Psi_D : U_{-\mu^+, -\mu^-}(L\mathfrak{g}_s) \to \tilde{A}_v^\text{frac, ext}$ be the extended homomorphism. The (restriction) composition $U_{-\mu^+, -\mu^-}^\text{ad}(L\mathfrak{g}_s) \overset{\psi_D}{\to} \tilde{A}_v^\text{frac} \overset{\Phi_D}{\to} \tilde{A}_v^\text{frac, ext}$ coincides with the composition of the natural isomorphism $U_{-\mu^+, -\mu^-}^\text{ad}(L\mathfrak{g}_s) \overset{\gamma}{\to} U_{0, -\mu^+, -\mu^-}^\text{ad}(L\mathfrak{g}_s)$ and the homomorphism $\Phi_D : U_{0, -\mu^+, -\mu^-}^\text{ad}(L\mathfrak{g}_s) \to \tilde{A}_v^\text{frac, ext}$ of [FT1, Theorem 7.1].

Proof of Theorem 3.33. First, we need to verify that under the above assignment (3.35), the images of $\varphi_i^+(z)$ (resp. $\varphi_i^-(z)$) contain only powers of $z$ which are $\leq d_i^+$ (resp. $\geq -d_i^-$), and the corresponding coefficients of $z^d_i$ in $\varphi_i^+(z)$ (resp. of $z^{-d_i}$ in $\varphi_i^-(z)$) are invertible. The claim is clear for $\varphi_i^+(z)$, while its validity for $\varphi_i^-(z)$ follows from the equality

$$-a_i + a_{i-1} + \epsilon_i^\vee(\mu^+) + \epsilon_i^\vee(\lambda) = -\epsilon_i^\vee(\mu^-),$$

due to (3.28).

Evoking the decomposition (3.23), it suffices to prove that the restrictions of the assignment (3.35) to the subalgebras $U_{-\mu^+, -\mu^-}^\text{ad}(L\mathfrak{g}_s)$ and $\mathbb{C}[\{C^\pm_s\}_{s \geq d_1^+ + \ldots + d_n^+}]$ determine algebra homomorphisms, whose images commute. The former is clear for the restriction to $U_{-\mu^+, -\mu^-}^\text{ad}(L\mathfrak{g}_s)$, due to Theorem 7.1 of [FT1] combined with Remark 3.36 above. On the other hand, we have

$$\Psi_D(C^\pm(z)) = A \cdot \prod_{i=1}^{n} \prod_{x \in \mathbb{P}^1 \setminus \{0, \infty\}} \left( 1 - v^{-2(i-1)x} z \right)^{-\epsilon_i^\vee(\lambda_x)} = A \cdot \prod_{s=1}^{N} \prod_{k=1}^{n-1} \left( 1 - v^{-2kx_s} z \right)^{-\epsilon_i^\vee(\mu^+)},$$

where $A := \prod_{i=1}^{n} (v^{2(i-1)x})^\epsilon_i^\vee(\mu^+)$. Thus, the restriction of $\Psi_D$ to the subalgebra $\mathbb{C}[\{C^\pm_s\}_{s \geq d_1^+ + \ldots + d_n^+}]$ defines an algebra homomorphism, whose image is central in $\tilde{A}_v^\text{frac}$. This completes our proof of Theorem 3.33. □
3.3. Antidominantly shifted RTT quantum affine algebras of $\mathfrak{gl}_n$.

Consider the trigonometric $R$-matrix $R_{\text{trig}}(z, w) = R_{\text{trig}}^u(z, w)$ given by

$$R_{\text{trig}}(z, w) := (v - v^{-1}w) \sum_{i=1}^{n} E_{ii} \otimes E_{ij} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (v - v^{-1})z \sum_{i < j} E_{ij} \otimes E_{ji} + (v - v^{-1})w \sum_{i > j} E_{ij} \otimes E_{ji},$$

(3.38)

cf. [DF, (3.7)]. It satisfies the Yang-Baxter equation with a spectral parameter:

$$R_{\text{trig};12}(u, v)R_{\text{trig};13}(u, w)R_{\text{trig};23}(v, w) = R_{\text{trig};23}(v, w)R_{\text{trig};13}(u, w)R_{\text{trig};12}(u, v).$$

(3.39)

Fix $\mu^+, \mu^- \in \Lambda^+$. Define the (antidominantly) shifted RTT quantum affine algebra of $\mathfrak{gl}_n$, denoted by $U_{-\mu^+,-\mu^-}(L\mathfrak{sl}_n)$, to be the associative $\mathbb{C}(v)$-algebra generated by

$$\{t_{ij}^{\pm r}{1 \leq i,j \leq n} \cup \{(g_{i,j}^{\pm})^{-1}{1 \leq i \leq n}\} \text{subject to the following three families of relations:}
$$

- The first family of relations may be encoded by a single RTT relation

$$R_{\text{trig}}(z, w) T^{\epsilon}(z) T^{\epsilon'}(w) = T^{\epsilon'}(w) T^{\epsilon}(z) R_{\text{trig}}(z, w)$$

(3.40)

for any $\epsilon, \epsilon' \in \{+, -\}$, where $T^{\pm}(z) \in U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)[[z, z^{-1}]] \otimes \text{End} \mathbb{C}^n$ are defined via

$$T^{\pm}(z) = \sum_{i,j} t_{ij}^{\pm r}(z) \otimes E_{ij} \text{ with } t_{ij}^{\pm r}(z) := \sum_{r \in \mathbb{Z}} t_{ij}^{\pm r} z^r.$$  

(3.41)

Thus, (3.40) is an equality in $U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)[[z, z^{-1}, w, w^{-1}]] \otimes \text{End} \mathbb{C}^n$ for any $\epsilon, \epsilon'$.

- The second family of relations encodes the fact that $T^{\pm}(z)$ admits the Gauss decomposition:

$$T^{\pm}(z) = F^{\pm}(z) \cdot G^{\pm}(z) \cdot E^{\pm}(z),$$

(3.42)

where $F^{\pm}(z), G^{\pm}(z), E^{\pm}(z) \in U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)((z^{\pm 1})) \otimes \text{End} \mathbb{C}^n$ are of the form

$$F^{\pm}(z) = \sum_{i} E_{ii} + \sum_{i < j} f_{ij}^{\pm}(z) \otimes E_{ji}, \quad G^{\pm}(z) = \sum_{i} g_{i}^{\pm}(z) \otimes E_{ii}, \quad E^{\pm}(z) = \sum_{i} E_{ii} + \sum_{i < j} e_{ij}^{\pm}(z) \otimes E_{ij},$$

with the matrix coefficients having the following expansions in $z$:

$$e_{ij}^{\pm}(z) = \sum_{r \geq 0} e_{ij}^{(r)} z^{-r}, \quad e_{ij}^{-}(z) = \sum_{r < 0} e_{ij}^{(r)} z^{-r},$$

$$f_{ij}^{\pm}(z) = \sum_{r > 0} f_{ij}^{(r)} z^{-r}, \quad f_{ij}^{-}(z) = \sum_{r \leq 0} f_{ij}^{(r)} z^{-r},$$

$$g_{i}^{\pm}(z) = \sum_{r \geq -d_i} g_{i}^{\pm} z^{-r}, \quad g_{i}^{-}(z) = \sum_{r \leq -d_i} g_{i}^{-} z^{-r},$$

(3.43)

where $\{e_{ij}^{(r)} \cup f_{ij}^{(r)}{1 \leq i < j \leq n} \cup \{g_{i}^{\pm} {1 \leq i \leq n}\} \subseteq U_{-\mu^+,-\mu^-}(L\mathfrak{sl}_n)$.

- The third family of relations is just:

$$g_{i}^{\pm} \cdot (g_{i}^{\pm})^{-1} = (g_{i}^{\pm})^{-1} \cdot g_{i}^{\pm} = 1.$$  

(3.44)
Remark 3.45. (a) For $\mu^+ = \mu^- = 0$, the second family of relations (3.42, 3.43) is equivalent to the relations $t_{ij}^r[1] = t_{ij}^{-r}[1] = 0$ for all $i, j$ and $r > 0$ as well as $t_{ij}^r[0] = t_{ij}^{-r}[0] = 0$ for $1 < i < j \leq n$. In this case, adjoining the inverses of $g_{ij}^\pm$, cf. (3.44), is equivalent to adjoining the inverses of $t_{ij}^r[0]$. Thus, $U_{0,0}^{\text{rtt}}(L\mathfrak{g}_n)$ is the RTT quantum loop algebra of $\mathfrak{g}_n$ of [FRT], or more precisely, its extended version $U_\nu^{\text{rtt ext}}(L\mathfrak{g}_n)$ of [GM, (2.15)].

(b) Likewise, (3.43) is equivalent to a certain family of algebraic relations on $t_{ij}^r[1]$. In particular, $T^\pm(z) \in U_{\mu^+, \mu^-}^{-}(\mathbb{C}) \otimes \text{End } \mathbb{C}$. For example, (3.43) for $i = 1$ are equivalent to:

- $t_{11}^r[1] = 0$ for $r < -d_1^+$, $t_{11}^{-r}[1] = 0$ for $r < -d_1^-$,
- $t_{ij}^r[1] = 0$ for $r < -d_i^+$, $j > 1$, $t_{ij}^{-r}[1] = 0$ for $r < -d_i^-$, $j > 1$,
- $t_{11}^r[0] = 0$ for $r < -d_1^+$, $t_{11}^{-r}[0] = 0$ for $r < -d_1^-$, $j > 1$.

Lemma 3.46. For any $1 \leq i < j \leq n$ and $r \in \mathbb{Z}$, we have the following identities:

\begin{align*}
  e_{ij}^{(r)} &= (v - v^{-1})^{j+i-1}[e_{i,j-1}^{(0)}, e_{j-2,j-1}^{(0)}, \ldots, e_{i+1,i+1}^{(0)}, e_{i+1,i+2}^{(0)}, e_{i+2,i+1}^{(0)}]v^{-1}, \\
  f_{ij}^{(r)} &= (v^{-1} - v)^{i-j+1}[[[\cdots [f_{i+1,i}^{(r)}, f_{i+2,i+1}^{(0)}]v, \cdots, f_{j-1,j-2}^{(0)}]v, f_{j,j-1}^{(0)}]v. \\
\end{align*}

\begin{align}
  (3.47)
\end{align}

Proof. The proof is analogous to that of [FT2, Corollary 3.51].

Corollary 3.48. The algebra $U_{\mu^+, \mu^-}(L\mathfrak{g}_n)$ is generated by

\[ \{e_{ij}^{(r)}, f_{ij}^{(r)}, g_{ij}^\pm, (g_{ij}^\pm)_{1,1 < n, 1 < j \leq n} \}. \]

The following result is a shifted version of [DF, Main Theorem] and a trigonometric version of Theorem 2.51:

Theorem 3.49. For any $\mu^+, \mu^- \in \Lambda^+$, there is a unique $\mathbb{C}(v)$-algebra epimorphism

\[ \Upsilon_{\mu^+, \mu^-}: U_{\mu^+, \mu^-}(L\mathfrak{g}_n) \rightarrow U_{\mu^+, \mu^-}(L\mathfrak{g}_n) \]

defined by

\[ E_i^\pm(z) \mapsto e_{i,i+1}^\pm(z), F_i^\pm(z) \mapsto f_{i+1,i}^\pm(z), \varphi_j^\pm(z) \mapsto g_j^\pm(z). \]

(3.50)

Modulo a trigonometric counterpart of [W], see Conjecture 3.75, the following result is proved in Section 3.4.3:

Theorem 3.51. $\Upsilon_{\mu^+, \mu^-}: U_{\mu^+, \mu^-}(L\mathfrak{g}_n) \rightarrow U_{\mu^+, \mu^-}(L\mathfrak{g}_n)$ is a $\mathbb{C}(v)$-algebra isomorphism for any $\mu^+, \mu^- \in \Lambda^+$.

Remark 3.52. (a) For $\mu^+ = \mu^- = 0$ and any $n$, the isomorphism $\Upsilon_{0,0}$ of Theorem 3.51 was established in [DF, Main Theorem] (more precisely, $\Upsilon_{0,0}$ is an isomorphism between the extended versions of both algebras in loc.cit.).

(b) For $n = 2$ and $\mu^+, \mu^- \in \Lambda^+$, a long straightforward verification shows that the assignment

\begin{align*}
  t_{11}^r(z) &\mapsto \varphi_1^r(z), \quad t_{12}^r(z) \mapsto F_2^r(z)\varphi_1^r(z)E_1^r(z) + \varphi_2^r(z), \\
  t_{21}^r(z) &\mapsto \varphi_1^r(z)E_1^r(z), \quad t_{22}^r(z) \mapsto F_1^r(z)\varphi_1^r(z),
\end{align*}

gives rise to a $\mathbb{C}(v)$-algebra homomorphism $U_{\mu^+, \mu^-}(L\mathfrak{g}_2) \rightarrow U_{\mu^+, \mu^-}(L\mathfrak{g}_2)$ (the $\mathfrak{sl}_2$-counterpart of which is due to [FT1, Theorem 11.19]), which is clearly the inverse of $\Upsilon_{-\mu^+, -\mu^-}$. Thus, Theorem 3.51 for $n = 2$ is essentially due to [FT1].
3.4. Trigonometric Lax matrices via antidominantly shifted quantum affine algebras of \( \mathfrak{gl}_n \).

In this section, we construct \( n \times n \) trigonometric Lax matrices \( T_D(z) \) (with coefficients in \( \mathcal{A}_v^p(z) \)) for each \( \Lambda^+ \)-valued divisor \( D \) on \( \mathbb{P}^1 \) satisfying \((3.28)\). They are explicitly defined via \((3.64, 3.65)\) combined with \((3.56, 3.58, 3.60)\). We note that these formulas arise naturally by considering the images of \( T^\pm(z) \in U_{\mu^±,\mu^-}(\mathfrak{gl}_n)((z^{\mp1})) \otimes \text{End} \mathbb{C}^n \) under the composition \( \Psi_D \circ \Upsilon_{\mu^+,\mu^-}^{-1} : U_{\mu^+,\mu^-}(\mathfrak{gl}_n) \rightarrow \mathcal{A}_v^p \), assuming Theorem 3.51 has been established, see \((3.53, 3.54)\) and Proposition 3.63. As the name indicates, \( (T_D(z))_D^\pm \) satisfy the RTT relation \((3.40)\), which is derived in Proposition 3.74. Combining the latter with the conjectured generalization of \([W]\), see Conjecture 3.75, we finally prove Theorem 3.51 in Section 3.4.3.

We also establish the regularity (up to a rational factor \((3.67)\)) of \( T_D(z) \) in Theorem 3.68, and find simplified explicit formulas for those \( T_D(z) \) which are linear in \( z \) in Theorem 3.77. Finally, we show how to degenerate these trigonometric Lax matrices into the rational Lax matrices of Section 2.4.1, see Proposition 3.94.

### 3.4.1. Construction of \( T_D(z) \) and their regularity.

Consider a \( \Lambda^+ \)-valued divisor \( D \) on \( \mathbb{P}^1 \), satisfying an extra condition \((3.28)\); note that \( \mu^+ = D|_\infty \in \Lambda^+ \) and \( \mu^- = D|_0 \in \Lambda^+ \). Composing \( \Psi_D : U_{\mu^+,\mu^-}(\mathfrak{gl}_n) \rightarrow \mathcal{A}_v^p \) of \((3.34)\) with the isomorphism \( \Upsilon_{\mu^+,\mu^-}^{-1} : U_{\mu^+,\mu^-}(\mathfrak{gl}_n) \rightarrow U_{\mu^+,\mu^-}(\mathfrak{gl}_n) \) (assuming the validity of Theorem 3.51), gives rise to an algebra homomorphism

\[
\Theta_D = \Psi_D \circ \Upsilon_{\mu^+,\mu^-}^{-1} : U_{\mu^+,\mu^-}(\mathfrak{gl}_n) \rightarrow \mathcal{A}_v^p.
\]  

(3.53)

Such a homomorphism is uniquely determined by two matrices \( T_D^\pm(z) \in \mathcal{A}_v^p((z^{\mp1})) \otimes \text{End} \mathbb{C}^n \) defined via

\[
T_D^\pm(z) := \Theta_D(T^\pm(z)) = \Theta_D(F^\pm(z)) \cdot \Theta_D(G^\pm(z)) \cdot \Theta_D(E^\pm(z)).
\]  

(3.54)

**Remark 3.55.** Actually \( T_D^\pm(z) \in \mathcal{A}_v^p((z^{\mp1})) \otimes \text{End} \mathbb{C}^n \), due to the formulas \((3.56, 3.58, 3.60)\).

Let us compute explicitly the images of the matrices \( F^\pm(z), G^\pm(z), E^\pm(z) \) under \( \Theta_D \), which shall provide an explicit formula for the matrices \( T_D^\pm(z) \) via \((3.54)\).

Combining \( \Upsilon_{\mu^+,\mu^-}^{-1}(g^\pm_i(z)) = \varphi^\pm_i(z) \) with the formula for \( \Psi_D(\varphi^\pm_i(z)) \), we obtain:

\[
\Theta_D(g^\pm_i(z)) = \prod_{i=1}^{a_i} w_i^{-1/2} \prod_{i=1}^{a_i} w_i^{1/2} \cdot \left( z^{\ell^i(\mu^+)} \frac{W_i(v^{-1}z)}{W_i(v^{-1}z)} \prod_{x \in \mathbb{P}^1 \setminus \{0, \infty\}} (1 - x/z)^{-\ell^i(\lambda)} \right)^\pm.
\]  

(3.56)

Combining \( \Upsilon_{\mu^+,\mu^-}^{-1}(e^\pm_{i,i+1}(z)) = E^\pm_i(z) \) with the formula for \( \Psi_D(E^\pm_i(z)) \), we obtain:

\[
\Theta_D(e^\pm_{i,i+1}(z)) = \prod_{i=1}^{a_i} w_{i,t}^{-1/2} \prod_{i=1}^{a_i} w_{i-1,t}^{1/2} \cdot \prod_{r=1}^{a_i} \frac{\left( v^w w_{i,r} - a^r_i(\mu^+) \right)^\pm}{1 - v^w w_{i,r} / z} \frac{Z_i(w_{i,r}) W_{i-1}(v^{-1}w_{i,r})}{W_{i,r}(w_{i,r})} D_{i,r}.
\]  

(3.57)

As \( e^\pm_{ij}(z) = (v - v^{-1})^{i-j+1} [e^0_{j-1,j}, \ldots, [e^0_{i+1,i+2}, e^\pm_{i,i+1}(z)]_{v^{-1}} \cdots]_{v^{-1}} \) due to \((3.47)\), we obtain (cf. \([FT2, (4.16)]\)): 

\[
\Theta_D(e^\pm_{ij}(z)) = \prod_{j=1}^{a_i} w_{i,t}^{-1/2} \prod_{i=1}^{a_i} w_{i-1,t}^{1/2} \cdot \prod_{r=1}^{a_i} \frac{\left( v^w w_{i,r} - a^r_i(\mu^+) \right)^\pm}{1 - v^w w_{i,r} / z} \frac{Z_i(w_{i,r}) W_{i-1}(v^{-1}w_{i,r})}{W_{i,r}(w_{i,r})} D_{i,r},
\]  

(3.58)

As \( e^\pm_{ij}(z) = (v - v^{-1})^{i-j+1} [e^0_{j-1,j}, \ldots, [e^0_{i+1,i+2}, e^\pm_{i,i+1}(z)]_{v^{-1}} \cdots]_{v^{-1}} \) due to \((3.47)\), we obtain (cf. \([FT2, (4.16)]\)):
Proposition 3.63. The matrix coefficients of the matrices $T_D^\pm(z)$ are the expansions of the same rational functions in $z^{\mp 1}$, respectively.
We will relate the corresponding matrices functions of (3.60, 3.56, 3.58), respectively, with the conventions framing at the classical limit regular in the neighbourhood of for any Remark

First, we claim that Proof.

\[ \Theta_{i,\beta}(z) = f_{\alpha,\beta}^D(z) \]

Theorem 3.68. However, the minimal power of functions in \[ \Lambda \]

3.4.2. Normalized limit description and the RTT relation for \[ T_D(z) \].

Consider a \( \Lambda^+ \)-valued divisor \( D = \sum_{s=1}^{N} \gamma_s \omega_i \sigma_s \) with \( \alpha^+ = \{ \infty \} + \mu | \) and \( \omega_i \). As \( x_N \rightarrow \infty \), we obtain another \( \Lambda^+ \)-valued divisor \( D' = \sum_{s=1}^{N} \gamma_s \omega_i \sigma_s \) with \( (\mu^+ + \gamma_N \omega_i \sigma_s) | \) and \( \mu^- | \) 0, while as \( x_N \rightarrow 0 \), we obtain yet another \( \Lambda^+ \)-valued divisor \( D'' = \sum_{s=1}^{N} \gamma_s \omega_i \sigma_s + \mu^+ | \) and \( (\mu^- + \gamma_N \omega_i \sigma_s) | 0 \). We will relate the corresponding matrices \( T_D(z), T_D'(z) \) to \( T_D(z) \).
If $i_N = 0$, then
\[ T_{D'}(z) = (z - x_N)^{-\gamma N} T_D(z), \quad T_{D''}(z) = (1 - x_N/z)^{-\gamma N} T_D(z), \] (3.69)
due to the defining formula (3.64) and the equalities $F_D(z) = F_{D'}(z) = F_{D''}(z)$, $E_D(z) = E_{D'}(z)$, $G_D(z) = (z - x_N)^{\gamma N} G_{D'}(z) = (1 - x_N/z)^{\gamma N} G_{D''}(z)$.

Let us now consider the case $1 \leq i_N \leq n - 1$ (note that $\gamma_N = 1$).

**Proposition 3.70.** The $x_N \rightarrow 0$ limit of $T_D(z)$ equals $T_{D'}(z)$.

**Proof.** Note that $F_D(z) = F_{D'}(z)$ by (3.60), the $x_N \rightarrow 0$ limit of $G_D(z)$ equals $G_{D'}(z)$ by (3.56), the $x_N \rightarrow 0$ limit of $E_D(z)$ equals $E_{D'}(z)$ by (3.58). This implies the result, due to the defining formulas (3.64, 3.65).

\[ \square \]

To treat the case $x_N \rightarrow \infty$, recall the notation $(-x_N)^{\bar{w}_N} = \text{diag}(i^{i_N}, (-x_N^{-1})^{n-i_N})$.

**Proposition 3.71.** The $x_N \rightarrow \infty$ limit of $T_D(z) \cdot (-x_N)^{\bar{w}_N}$ equals $T_{D'}(z)$.

**Proof.** The proof is completely analogous to our proof of Proposition 2.74.

\[ \square \]

**Corollary 3.72.** (a) $T_{D''}(z)$ is a limit of $T_D(z)$.

(b) $T_{D''}(z)$ is a normalized limit of $T_D(z)$.

If $\mu^+, \mu^- \in \Lambda^+$, we can pick a $\Lambda^+$-valued divisor $\bar{D} = \sum_{s=1}^{N+M} \gamma_s \bar{w}_i [x_s]$, so that $\{x_s\}_{s=N+1}^{N+M}$ are some points on $\mathbb{P}^1 \setminus \{0, \infty\}$ while $\sum_{s=N+1}^{N+M} \gamma_s \bar{w}_i = \mu^+ + \mu^-$. Note that $0, \infty \not\in \text{supp}(\bar{D})$, that is, $\bar{D}|_{0} = 0$ and $\bar{D}|_{\infty} = 0$.

**Corollary 3.73.** For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28), $T_D(z)$ is a normalized limit of $T_D(z)$ with $0, \infty \not\in \text{supp}(\bar{D})$.

Evoking Remark 3.52(a), we see that the original definition of $T_D^{\pm}(z)$ via (3.53, 3.54) is valid. Hence, $T_D^{\pm}(z)$ defined via (3.61) indeed satisfies the RTT relation (3.40), and so is $T_D(z)$. As a multiplication by diagonal $z$-independent matrices preserves (3.40), we obtain the main result of this section:

**Proposition 3.74.** For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28), the matrix $T_D(z)$ defined via (3.64) indeed satisfies the RTT relation (3.40).

3.4.3. **Proof of Theorem 3.51.**

Due to Proposition 3.74 and the Gauss decomposition (3.64, 3.65) of $T_D(z)$ with the factors defined via (3.56, 3.58, 3.60), we see that $T_D(z)$ indeed gives rise to the algebra homomorphism $\Theta_D: U_{-\mu^+, -\mu^-}(Lg_{l_0}) \rightarrow \overline{\mathcal{A}}^v_{\text{frac}}, T^\pm(z) \mapsto (T(z))^\pm$, whose composition with the epimorphism $\Upsilon_{-\mu^+, -\mu^-}: U_{-\mu^+, -\mu^-}(Lg_{l_0}) \twoheadrightarrow U_{-\mu^+, -\mu^-}(Lg_{l_0})$ of Theorem 3.49 coincides with the homomorphism $\Psi_D$ of (3.34). Thus, for $\mu^+, \mu^- \in \Lambda^+$ and any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28) with $D|_{\infty} = \mu^+, D|_{0} = \mu^-$, the homomorphism $\Psi_D$ factors via $\Upsilon_{-\mu^+, -\mu^-}$.

The latter immediately implies the injectivity of $\Upsilon_{-\mu^+, -\mu^-}$ once the following trigonometric counterpart of Theorem 2.79 is established:

**Conjecture 3.75.** For any coweights $\mu^+, \mu^- \in \Lambda$, the intersection of kernels of the homomorphisms $\Psi_D$ of (3.34) is zero: $\bigcap_D \ker(\Psi_D) = 0$, where $D$ ranges through all $\Lambda$-valued divisors on $\mathbb{P}^1$, $\Lambda^+$-valued outside $\{0, \infty\} \subset \mathbb{P}^1$, satisfying (3.28) and such that $D|_{\infty} = \mu^+, D|_{0} = \mu^-$. This completes our proof of Theorem 3.51 modulo Conjecture 3.75.
3.4.4. **Linear trigonometric Lax matrices.**

In this section, we will obtain simplified explicit formulas for all $T_D(z)$ that are linear in $z$.

Following Section 2.4.4, fix a triple of pseudo Young diagrams $\lambda, \mu^+, \mu^-$. They give rise to $\lambda, \mu^+, \mu^- \in \Lambda^+$ via (2.81). Then, $\lambda + \mu^+ + \mu^-$ is of the form $\lambda + \mu^+ + \mu^- = \sum_{i=1}^{n-1} a_i \alpha_i$ for some $a_i \in \mathbb{C}$ if $|\lambda| + |\mu^+| + |\mu^-| = 0$. Moreover, due to Lemma 2.82, we have:

**Lemma 3.76.**

(a) $a_i = -\sum_{j=n-i+1}^{n} (\lambda_j + \mu_j^+ + \mu_j^-)$ for any $1 \leq i \leq n-1$.
(b) $a_i \in \mathbb{N}$ for any $1 \leq i \leq n-1$.
(c) $a_j - a_{j-1} = -\lambda_{n-j+1} + \mu_{n-j+1}^+ - \mu_{n-j+1}^-$ for any $1 \leq j \leq n$, where we set $a_0 := 0, a_n := 0$.

Thus, $\Lambda^+$-valued divisors on $\mathbb{P}^1$ satisfying (3.28) and without summands $\{ -\infty \} \in \mathbb{C}^\times$ may be encoded by triples $(\lambda, \mu^+, \mu^-)$ of a Young diagram $\lambda$ of length $\leq n$ and a pair of pseudo Young diagrams $\mu^+, \mu^-$ with $n$ rows and of total size $|\lambda| + |\mu^+| + |\mu^-| = 0$, together with a collection of points $x = \{ x_i \}_{i=1}^{\lambda_1} \in \mathbb{C}^\times$ (so that $x_i$ is assigned to the $i$-th column of $\lambda$). Explicitly, given $\lambda, \mu^+, \mu^-, x$ as above, we set $D = D(\lambda, x; \mu^+, \mu^-) := \sum_{i=1}^{\lambda_1} (\lambda_i - \lambda_j) + |\mu^+| + |\mu^-| - 0$.

Due to (3.69), we may further assume that $D$ does not contain summands $\{ -\infty \} \in \mathbb{C}^\times$. Thus, $\lambda_n = 0 = \mu_n^-$, so that $Z_0(z) = 1, \epsilon_1^\prime(\lambda + \mu^-) = -\lambda_n - \mu_n^- = 0$, and $T_D(z) = T_D(z)$ is polynomial in $z$ by Theorem 2.66. Moreover, $T_D(z)_{11} = g_D(z)$ is a polynomial in $z$ of degree $\epsilon_1^\prime(\mu^+)$.

**Theorem 3.77.**

Following the above notations, assume further that $\lambda_n = 0, \mu_n = 0, \mu_n^+ = -1$.

(a) The trigonometric Lax matrix $T_D(z)$ is explicitly determined as follows:

(I) The matrix coefficients on the main diagonal are:

$$T_D(z)_{ii} = z \cdot \delta_{\mu_{n-i+1}^+, 1 - i} \cdot \prod_{t=1}^{a_i-1} w_{t,i}^{1/2} \prod_{t=1}^{a_i-1} \prod_{t=1}^{a_i-1} w_{t-1,i}^{1/2} +$$

$$\delta_{\mu_{n-i+1}^-, 0} \cdot \prod_{t=1}^{a_i} w_{t,i}^{1/2} \prod_{t=1}^{a_i-1} w_{t-1,i}^{1/2} \frac{(-v_i)^a_i}{(-v^{i+1})^{a_i-1}} \prod_{1 \leq s \leq \lambda_1} (-x_s),$$

where $i_s := n - \lambda_i^+.$

(II) The matrix coefficients above the main diagonal are:

$$T_D(z)_{ij} = z \cdot \delta_{\mu_{n-i+1}^+, 1 - i} \cdot \prod_{t=1}^{a_i-1} w_{j-t} \prod_{k=t}^{a_k} \prod_{t=1}^{a_i} w_{k,t}^{1/2} \prod_{t=1}^{a_i} w_{t-1,i}^{1/2} \times$$

$$\sum_{1 \leq r_j \leq a_i} \frac{(-v_i w_{i,r_j}) b_i^{1-j} \cdots (v_j w_{j-r_j}) b_j^{j-1} w_{i-1} (v_{i-1} w_{i-1}) \prod_{k=1}^{j-2} w_{k,r_k} (v_{i-1} w_{k+1,r_k+1})}{\prod_{k=1}^{j-1} w_{k,r_k} (w_{k,r_k})} \prod_{k=i}^{j-1} Z_k(w_{k,r_k}) \cdot \frac{w_{i,r_j}}{w_{j-1,r_j}} \prod_{k=i}^{j-1} D_{k,r_k}^{j-1} \text{ for } i < j,$$

where the constants $b_i^j$ are defined via $b_i^j := \mu_{n-i}^+ - \mu_{n-i}^{-1}$. 

(III) The matrix coefficients below the main diagonal are:

\[ T_D(z)_{ji} = \delta_{\mu_{n-i+1,0},(-1)^{i-j+1}v^{i-j+1}} \cdot \prod_{k=-1}^{j-i} \prod_{t=1}^{a_k} w_{k,t}^{-1/2 + \delta_{k,i}} \frac{(-v)^{a_i}}{(-v^{i+1})^{a_{i-1}}} \prod_{1 \leq s \leq A_1} (-x_s) \times \]

\[ \sum_{1 \leq r_j \leq a_i} \prod_{k=i+1}^{j-i} W_{k,r_k}(v w_{k-1,r_{k-1}}) W_{j}(v w_{j-1,r_{j-1}}) \cdot w_{j-1,r_{j-1}}^{-1} \prod_{k=i}^{j-1} D_{k,r_k} \]  

for \( i < j \).

(b) \( T_D(z) = T_D(z) \) is polynomial of degree 1 in \( z \).

**Proof.** (a) Combining the explicit formulas (3.65, 3.67) for the matrix coefficients \( T_D(z)_{\alpha,\beta} \) with their polynomiality of Theorem 3.68, we may immediately determine all of them explicitly. As \( e_{ij}^D(z), f_{ij}^D(z), g_i^D(z) \) are regular at \( z = \infty \) (for the latter, note that \( \epsilon_i^1(\mu^+) - 1 = -\mu_{n-i+1} - 1 \leq 0 \)), each matrix coefficient \( T_D(z)_{\alpha,\beta} \) is a linear polynomial in \( z \), due to Theorem 3.68.

The computation of the coefficients of \( z^1 \) is based on the following observations:

- The \( z \rightarrow \infty \) limit of \( e_{ij}^D(z) \) equals the right-hand side of (3.58) with \( \frac{1}{1-v^{w_{i,r_i}/z}} \) disregarded.

- The \( z \rightarrow \infty \) limit of \( f_{ij}^D(z) \) equals 0.

- The \( z \rightarrow \infty \) limit of \( g_i^D(z) \) equals \( \delta_{\mu_{n-i+1,0},(-1)^{i-j+1}} \cdot \prod_{t=1}^{a_i} w_{t,t}^{-1/2} \prod_{t=1}^{a_{i-1}} w_{t-1,t}^{1/2} \).

The computation of the coefficients of \( z^0 \) is based on the following observations:

- The \( z \rightarrow 0 \) limit of \( e_{ij}^D(z) \) equals 0.

- The \( z \rightarrow 0 \) limit of \( f_{ij}^D(z) \) equals the right-hand side of (3.60) with \( \frac{1}{1-z/v^{w_{i,r_i}/z}} \) disregarded.

- The \( z \rightarrow 0 \) limit of \( g_i^D(z) \) equals \( \delta_{\mu_{n-i+1,0},(-1)^{i-j+1}} \cdot \prod_{t=1}^{a_i} w_{t,t}^{-1/2} \prod_{t=1}^{a_{i-1}} w_{t-1,t}^{-1/2} \frac{(-v)^{a_i}}{(-v^{i+1})^{a_{i-1}}} \prod_{1 \leq s \leq A_1} (-x_s) \).

Part (b) follows immediately from part (a).

**Remark 3.81.** In the particular case when \( \mu^- = (0^n), \mu^+ = ((-1)^n) \), and \( \lambda \) is a Young diagram of size \( n \) and length \( < n \), the Lax matrices \( T_D(z) \) of Theorem 3.77 are closely related to the \( v \)-deformed parabolic Gelfand-Tsetlin formulas (cf. [FT1, Proposition 12.18]), thus providing a \( v \)-deformed version of Section 2.7.

We note that the trigonometric Lax matrices of Theorem 3.77 have the form \( z \cdot T^+ - T^- \). Here, \( T^+ \) is an upper triangular and \( T^- \) is a lower triangular \( z \)-independent matrices, with some of their diagonal entries being zero as prescribed by pseudo Young diagrams \( \mu^\pm \).

We conclude this section by deriving the conditions on a pair of \( n \times n \) matrices \( T^+, T^- \) (with values in an associative algebra \( D \)) which are equivalent to \( T(z) := z \cdot T^+ - T^- \) satisfying the trigonometric RTT relation

\[ R_{\text{trig}}(z,w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{\text{trig}}(z,w). \]  

To this end, recall the (finite) trigonometric \( R \)-matrix \( R = R^v \) given by

\[ R = v^{-1} \sum_{1 \leq i \leq n} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (v^{-1} - v) \sum_{i > j} E_{ij} \otimes E_{ji}. \]

It satisfies the Yang-Baxter equation:

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \]
The final result of this section is:

**Proposition 3.85.** Matrix $T(z) = zT^+ - T^-$ satisfies the trigonometric RTT relation (3.82) if and only if $(T^+, T^-)$ satisfy the following three finite trigonometric RTT relations:

$$RT_1^+ T_2^+ = T_2^+ T_1^+ R, \quad RT_1^- T_2^- = T_2^- T_1^- R, \quad RT_1^+ T_2^- = T_2^+ T_1^- R. \quad (3.86)$$

**Proof.** Recall the following relation between the trigonometric $R$-matrices:

$$R_{\text{trig}}(z, w) = (z - w)R + (v - v^{-1})zP,$$

where $P = \sum_{i,j=1}^n E_{ij} \otimes E_{ji}$ as before. Thus, the relation (3.82) on $T(z)$ may be written as

$$((z - w)R + (v - v^{-1})zP)(zT_1^+ - T_1^-)(wT_2^+ - T_2^-) =$$

$$(wT_2^+ - T_2^-)(zT_1^+ - T_1^-)((z - w)R + (v - v^{-1})zP). \quad (3.87)$$

To prove the “only if” part, compare the coefficients of $z^1w^2, z^0w^1, z^0w^0,$ and $z^2w^0$ in (3.87) to recover the equalities $RT_1^+ T_2^+ = T_2^+ T_1^+ R, RT_1^- T_2^- = T_2^- T_1^- R, RT_1^+ T_2^- = T_2^+ T_1^- R,$ and $\tilde{R}T_1^+ T_2^- = T_2^- T_1^+ \tilde{R},$ respectively, where $\tilde{R} := R + (v - v^{-1})P.$

To prove the “if” part, we note that multiplying the last equality of (3.86) by $R^{-1}$ both on the left and on the right, and conjugating further by the permutation operator $P,$ we get $(PR^{-1}P^{-1})T_1^+ T_2^- = T_2^- T_1^+ (PR^{-1}P^{-1}),$ which together with $PR^{-1}P^{-1} = \tilde{R}$ finally implies

$$RT_1^+ T_2^- = T_2^- T_1^+ \tilde{R}. \quad (3.88)$$

Combining this with (3.86) and $\tilde{R} = R + (v - v^{-1})P,$ the equality (3.87) is equivalent to

$$(v - v^{-1})z^2w(PT_1^+ T_2^- - T_2^+ T_1^+) + (v - v^{-1})z(PT_1^- T_2^- - T_2^- T_1^-) -$$

$$(v - v^{-1})zw(PT_1^+ T_2^- - T_2^+ T_1^- P) + zw(RT_2^+ - T_2^- T_1^+) = 0.$$

The first two summands are clearly zero as $PT_1^+ T_2^- = T_2^+ T_1^+ P$ and $PT_1^- T_2^- = T_2^- T_1^- P,$ while the sum of the latter two equals

$$zw ((R + (v - v^{-1})P)T_1^+ T_2^- - T_2^- T_1^+ (R + (v - v^{-1})P)) = zw \left(\tilde{R}T_1^+ T_2^- - T_2^- T_1^+ \tilde{R}\right) = 0,$$

due to (3.88).

This completes our proof of Proposition 3.85. \hfill \Box

**Remark 3.89.** The above proof is identical to the verification of the fact that the assignment $T^+(z) \mapsto T^+ - z^{-1}T^-,$ $T^-(z) \mapsto T^- - zT^+$ gives rise to the (evaluation) homomorphism $U^\text{trig}_v(Lg_l) \to U^\text{rational}_v(g_l).$ In particular, if it was not for (3.42, 3.43), we would get homomorphisms from shifted quantum affine algebras to the corresponding **contracted algebras** of $[T].$

### 3.4.5. From trigonometric Lax matrices to rational Lax matrices.

In this section, we explain how the trigonometric Lax matrices $T_{\text{trig}}^*(z)$ of Section 3.4.1 may be degenerated into the rational Lax matrices $T_{\text{rat}}^*(z)$ of Section 2.4.1 (superscripts trig, rat are used to distinguish between trigonometric and rational setups). Given a $\Lambda^+$-valued divisor $D = \sum_{s=1}^N \gamma_s x_s [x_s] + \mu^+[\infty] + \mu^-[0]$ on $\mathbb{P}^1$ (with $x_s \in \mathbb{C}^\times$), we consider another $\Lambda^+$-valued divisor $\tilde{D} = \sum_{s=1}^N \gamma_s x_s [x_s] + (\mu^+ + \mu^-)[\infty]$ on $\mathbb{P}^1.$

We make the following change of variables:

$$v \sim v^{1/2}, \quad z \sim e^{x}, \quad x_s \sim e^{x_s}; \quad (3.90)$$

$$w_{i,r} \sim e^{(p_{i,r} - \frac{i}{2})} = e^{w_{i,r}}, \quad \text{where } w_{i,r} := p_{i,r} - i/2 \text{ as in Remark 2.37}; \quad (3.91)$$
We also consider the diagonal \( z \)-independent matrix

\[
\epsilon^{-\mu^+ - \mu^-} := \text{diag}(\epsilon^{-d_1}, \epsilon^{-d_2}, \ldots, \epsilon^{-d_n})
\]
with \( d_i := \epsilon_i^\vee (\mu^+ + \mu^-) = d^+_i + d^-_i \). (3.93)

The main result of this section is:

**Proposition 3.94.** \( \lim_{\epsilon \to 0} \left( T_D^{\text{trig}}(z) \cdot \epsilon^{-\mu^+ - \mu^-} \right) = T_D^{\text{rat}}(x) \).

*Proof.* Recall the Gauss decomposition \( T_D^{\text{trig}}(z) = F_D^{\text{trig}}(z)G_D^{\text{trig}}(z)E_D^{\text{trig}}(z) \) of (3.64) with all three factors determined explicitly in (3.56, 3.58, 3.60). Then, \( T_D^{\text{trig}}(z) \cdot \epsilon^{-\mu^+ + \mu^-} \) has the following Gauss decomposition:

\[
T_D^{\text{trig}}(z) \cdot \epsilon^{-\mu^+ - \mu^-} = F_D^{\text{trig}}(z) \cdot (G_D^{\text{trig}}(z) \epsilon^{-\mu^+ - \mu^-}) \cdot (\epsilon^{\mu^+ + \mu^-} E_D^{\text{trig}}(z) \epsilon^{-\mu^+ - \mu^-}).
\]

(3.95)

On the other hand, recall the Gauss decomposition

\[
T_D^{\text{rat}}(x) = F_D^{\text{rat}}(x) \cdot G_D^{\text{rat}}(x) \cdot E_D^{\text{rat}}(x).
\]

(3.96)

of (2.62) with all three factors determined explicitly in (2.57, 2.59, 2.61).

It remains to note that upon the above change of variables (3.90–3.92), the \( \epsilon \to 0 \) limit of each of the three factors in (3.95) exactly coincides with the corresponding factor in (3.96):

- For the diagonal factors, this immediately follows from

\[
\epsilon^{-a_i} W_i(v^{-i}z) \to P_i(x), \quad \epsilon^{-a_{i-1}} W_{i-1}(v^{-i-1}z) \to P_{i-1}(x-1), \quad \epsilon^{-\alpha_k^\vee(\lambda)} Z_k(v^{-k}z) \to Z_k(x)
\]
as \( \epsilon \to 0 \), combined with the equality

\[
a_i - a_{i-1} + \sum_{k=0}^{i-1} \alpha_k^\vee(\lambda) - d_i = a_i - a_{i-1} - \epsilon_i^\vee(\lambda) - \epsilon_i^\vee(\mu^+ + \mu^-) = a_i - a_{i-1} - \epsilon_i^\vee(\lambda + \mu^+ + \mu^-) = 0;
\]

- For the upper triangular factors, this follows from

\[
\epsilon^{-a_{k+1}} W_{k,r_k}(v^{-1}w_{k+1,r_{k+1}}) \to P_{k,r_k}(p_{k+1,r_{k+1}} - 1), \quad \epsilon^{-a_{k+1}} W_{k,r_k}(w_{k,r_k}) \to P_{k,r_k}(p_{k,r_k}),
\]

\[
\epsilon^{-a_{i-1}} W_{i-1}(v^{-1}w_{i,r_i}) \to P_{i-1}(p_{i,r_i} - 1), \quad \frac{\epsilon}{1 - v^{\alpha_i^\vee(\lambda)}/z} \to \frac{1}{x - p_{i,r_i}}
\]
as \( \epsilon \to 0 \), combined with the equality

\[
a_{i-1} - a_{j-1} + \sum_{k=i}^{j-1} \alpha_k^\vee(\lambda) - \sum_{k=1}^{j-1} s_k + d_i - d_j = a_{i-1} - a_i - a_{j-1} + a_j + (\epsilon_i^\vee - \epsilon_j^\vee)(\lambda + \mu^+ + \mu^-) = 0;
\]

- For the lower triangular factors, this follows from

\[
\epsilon^{-a_{k+1}} W_{k,r_k}(v w_{k-1,r_{k-1}}) \to P_{k,r_k}(p_{k-1,r_{k-1}} + 1), \quad \epsilon^{-a_{k+1}} W_{k,r_k}(w_{k,r_k}) \to P_{k,r_k}(p_{k,r_k}),
\]

\[
\epsilon^{-a_j} W_j(v w_{j-1,r_{j-1}}) \to P_j(p_{j-1,r_{j-1}} + 1), \quad \frac{\epsilon}{1 - z/v^{\alpha_j^\vee(\lambda)}/z} \to \frac{-1}{x - p_{j,r_j} - 1}
\]
as \( \epsilon \to 0 \), combined with the equality

\[
a_j - a_i + \sum_{k=i}^{j-1} s_k = 0.
\]

This completes our proof of Proposition 3.94. \( \Box \)
3.5. Six explicit linear trigonometric Lax matrices for \( n = 2 \).

In this section, we apply Theorem 3.77 to write down explicit linear trigonometric Lax matrices \( T_D(z) \) for \( n = 2 \), corresponding to a triple of pseudo Young diagrams

\[
\lambda = (\lambda_1, 0), \quad \mu^+ = (\mu_1^+, -1), \quad \mu^- = (\mu_1^-, 0)
\]

with \( \lambda_1 \geq 0, \mu_1^+ \geq -1, \mu_1^- \geq 0 \) and \( \lambda_1 + \mu_1^+ + \mu_1^- = 1 \).

We shall also compute their quantum determinant \( q\det T_D(z) \), defined via

\[
q\det T_D(z) := T_D(z)_{11} T_D(v^{-2}z)_{22} - v^{-1}T_D(z)_{12} T_D(v^{-2}z)_{21}.
\]  \((3.97)\)

Note that \( a_1 = -(\lambda_2 + \mu_2^+ + \mu_2^-) = 1 \) manifestly. To simplify the formulas below, we relabel \( D_1^\pm, w_1^\pm \) by \( D^\pm, \tilde{w}^\pm \), respectively.

- **Case** \( \lambda_1 = 0, \mu_1^+ = -1, \mu_1^- = 2 \).
  
  We have
  \[
  T_D(z) = \begin{pmatrix}
  z \cdot \tilde{w}^{-1} - v \tilde{w} & z \cdot \tilde{w} D^{-1} \\
  -v \tilde{w} D & z \cdot \tilde{w}
  \end{pmatrix}
  \]
  and its quantum determinant is \( q\det T_D(z) = v^{-2}z^2 \).

- **Case** \( \lambda_1 = 0, \mu_1^+ = 0, \mu_1^- = 1 \).
  
  We have
  \[
  T_D(z) = \begin{pmatrix}
  z \cdot \tilde{w}^{-1} - v \tilde{w} & z \cdot v^{-1}\tilde{w}^{-1}D^{-1} \\
  -v \tilde{w} D & 0
  \end{pmatrix}
  \]
  and its quantum determinant is \( q\det T_D(z) = v^{-2}z \).

- **Case** \( \lambda_1 = 0, \mu_1^+ = 1, \mu_1^- = 0 \).
  
  We have
  \[
  T_D(z) = \begin{pmatrix}
  z \cdot \tilde{w}^{-1} - v \tilde{w} & z \cdot v^{-2}\tilde{w}^{-3}D^{-1} \\
  -v \tilde{w} D & -v^{-3}\tilde{w}^{-1}
  \end{pmatrix}
  \]
  and its quantum determinant is \( q\det T_D(z) = v^{-2} \).

- **Case** \( \lambda_1 = 1, \mu_1^+ = -1, \mu_1^- = 1 \).
  
  We have
  \[
  T_D(z) = \begin{pmatrix}
  z \cdot \tilde{w}^{-1} - v \tilde{w} & z \cdot \tilde{w} (1 - v^{-1}x_1) \tilde{w}^{-2}D^{-1} \\
  -v \tilde{w} D & z \cdot \tilde{w}
  \end{pmatrix}
  \]
  and its quantum determinant is \( q\det T_D(z) = v^{-2}z(z - x_1) \).

- **Case** \( \lambda_1 = 1, \mu_1^+ = 0, \mu_1^- = 0 \).
  
  We have
  \[
  T_D(z) = \begin{pmatrix}
  z \cdot \tilde{w}^{-1} - v \tilde{w} & z \cdot v^{-1}\tilde{w}^{-1}(1 - v^{-1}x_1) \tilde{w}^{-2}D^{-1} \\
  -v \tilde{w} D & v^{-3}\tilde{w}^{-1}x_1
  \end{pmatrix}
  \]
  and its quantum determinant is \( q\det T_D(z) = v^{-2}(z - x_1) \).

- **Case** \( \lambda_1 = 2, \mu_1^+ = -1, \mu_1^- = 0 \).
  
  We have
  \[
  T_D(z) = \begin{pmatrix}
  z \cdot \tilde{w}^{-1} - v \tilde{w} & z \cdot \tilde{w} (1 - v^{-1}x_1) \tilde{w}^{-2}(1 - v^{-1}x_2) \tilde{w}^{-2}D^{-1} \\
  -v \tilde{w} D & z \cdot \tilde{w} - v^{-3}\tilde{w}^{-1}x_1 x_2
  \end{pmatrix}
  \]
  and its quantum determinant is \( q\det T_D(z) = v^{-2}(z - x_1)(z - x_2) \).

**Remark** 3.104. The first three Lax matrices \((3.98, 3.99, 3.100)\) first appeared in [FT1] (up to a normalization factor, they coincide with those of [FT1, (11.4, 11.9, 11.10)] having \( q\det = 1 \)).
3.6. Coproduct homomorphisms for shifted quantum affine algebras.

A crucial benefit of the RTT realization is that it immediately endows the quantum affine algebra of $\mathfrak{gl}_n$ with the Hopf algebra structure, in particular, the coproduct homomorphism

$$\Delta^{\text{rtt}} : U^{\text{rtt}}_v(L\mathfrak{gl}_n) \rightarrow U^{\text{rtt}}_v(L\mathfrak{gl}_n) \otimes U^{\text{rtt}}_v(L\mathfrak{gl}_n)$$

is defined via $\Delta^{\text{rtt}}(T^\pm(z)) = T^\pm(z) \otimes T^\pm(z)$.

The main result of this section establishes a shifted version of that:

**Proposition 3.105.** For any $\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^- \in \Lambda^+$, there is a unique $\mathbb{C}(v)$-algebra homomorphism

$$\Delta^{\text{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-} : U^{\text{rtt}}_{-\mu_1^+,-\mu_2^+,-\mu_1^-,-\mu_2^-}(L\mathfrak{gl}_n) \rightarrow U^{\text{rtt}}_{-\mu_1^+,-\mu_1^-}(L\mathfrak{gl}_n) \otimes U^{\text{rtt}}_{-\mu_2^+,-\mu_2^-}(L\mathfrak{gl}_n)$$
defined by

$$\Delta^{\text{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}(T^\pm(z)) = T^\pm(z) \otimes T^\pm(z). \quad (3.106)$$

**Proof.** The proof is completely analogous to our proof of Proposition 2.128 with the only minor update of the general observation we used in *loc.cit.* To be more precise, we either add the generators $e_{ij}^{(r)}$ so that $e_{ij}(z) = \sum_{r \geq 0} e_{ij}^{(r)} z^{-r}$ or we add the generators $f_{ji}^{(r)}$ so that $f_{ji}(z) = \sum_{r \geq 0} f_{ji}^{(r)} z^{-r}$. In either case, the product $E(z) \cdot F(z)$ still admits the Gauss decomposition (2.130) with either $\tilde{e}_{ij}(z) = \sum_{r \geq 0} \tilde{e}_{ij}^{(r)} z^{-r}$ and $\tilde{f}_{ji}(z) = \sum_{r \geq 1} \tilde{f}_{ji}^{(r)} z^{-r}$, or $\tilde{e}_{ij}(z) = \sum_{r \geq 1} \tilde{e}_{ij}^{(r)} z^{-r}$ and $\tilde{f}_{ji}(z) = \sum_{r \geq 0} \tilde{f}_{ji}^{(r)} z^{-r}$, respectively. \qed

The following basic property of $\Delta^{\text{rtt}}_{*,*,*,*}$ is straightforward:

**Corollary 3.107.** For any $\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^- \in \Lambda^+$, the following equality holds:

$$(\text{Id} \otimes \Delta^{\text{rtt}}_{-\mu_1^+,-\mu_2^+,-\mu_2^+,-\mu_2^-}) \circ \Delta^{\text{rtt}}_{-\mu_1^+,-\mu_1^-,\mu_2^+,\mu_2^-} = (\Delta^{\text{rtt}}_{-\mu_1^+,-\mu_1^-,\mu_2^+,\mu_2^-} \otimes \text{Id}) \circ \Delta^{\text{rtt}}_{-\mu_1^+,-\mu_1^-,\mu_2^+,\mu_2^-}.$$ 

Evoking the isomorphisms $\Upsilon_{-\mu^+, -\mu^-} : U_{-\mu^+, -\mu^-}(L\mathfrak{gl}_n) \cong U_{-\mu^+, -\mu^-}(L\mathfrak{gl}_n)$ of Theorem 3.51 for $(\mu^+, \mu^-)$ being either of the three pairs $(\mu_1^+, \mu_1^-), (\mu_2^+, \mu_2^-), (\mu_1^+ + \mu_2^+, \mu_1^- + \mu_2^-)$, the algebra homomorphism $\Delta^{\text{rtt}}_{-\mu_1^+,-\mu_1^-,\mu_2^+,\mu_2^-}$ of (3.106) gives rise to the algebra homomorphism

$$\Delta_{-\mu_1^+,-\mu_1^-,\mu_2^+,\mu_2^-} : U_{-\mu_1^+,\mu_1^-}(L\mathfrak{gl}_n) \rightarrow U_{-\mu_1^+,\mu_1^-}(L\mathfrak{gl}_n) \otimes U_{-\mu_2^+,\mu_2^-}(L\mathfrak{gl}_n). \quad (3.108)$$

As the algebra $U_{-\mu_1^+,-\mu_2^+,-\mu_1^-,-\mu_2^-}(L\mathfrak{gl}_n)$ is generated by

$$\{E_{ij}, F_{ij}, \varphi^\pm_{j}, \epsilon^\pm_{j+1}(\mu_1^+ + \mu_2^+), \left(\varphi^\pm_{j}, \epsilon_{j}^{\pm}(\mu_1^+ + \mu_2^+)\right)^{-1}, \varphi^\pm_{j}, \epsilon_{j}^{\pm}(\mu_1^+ + \mu_2^+) \mid 1 \leq j \leq n \},$$

the homomorphism $\Delta_{-\mu_1^+,-\mu_1^-,\mu_2^+,\mu_2^-}$ is uniquely determined by the images of these elements, which were computed explicitly in [FT1, Appendices G, H], cf. [FT1, Theorems G.26, G.36].

Moreover, the homomorphisms (3.108) have natural $\mathfrak{sl}_n$-counterparts:

**Proposition 3.109.** For any $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^- \in \Lambda^+$, there is a unique $\mathbb{C}(v)$-algebra homomorphism

$$\Delta_{-\nu_1^+,\nu_1^-,\nu_2^+,\nu_2^-} : U^{\text{sc}}_{-\nu_1^+,-\nu_2^+,-\nu_1^-,-\nu_2^-}(L\mathfrak{sl}_n) \rightarrow U^{\text{sc}}_{-\nu_1^+,-\nu_1^-}(L\mathfrak{sl}_n) \otimes U^{\text{sc}}_{-\nu_2^+,\nu_2^-}(L\mathfrak{sl}_n)$$

for any $\nu^+, \nu^- \in \Lambda^+$. \qed
such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathfrak{u}^{\text{sc}}_{-\mu_1^+, -\mu_2^+ - \mu_1^- - \mu_2^-} (L \mathfrak{sl}_n) & \xrightarrow{\Delta_{-\mu_1^+, -\mu_2^+ - \mu_1^- - \mu_2^-}} & \mathfrak{u}^{\text{sc}}_{-\mu_1^+, -\mu_2^+} (L \mathfrak{sl}_n) \otimes \mathfrak{u}^{\text{sc}}_{-\mu_1^+, -\mu_2^-} (L \mathfrak{sl}_n) \\
\downarrow & & \downarrow \\
\mathfrak{u}^{\text{sc}}_{-\mu_1^+, -\mu_2^- - \mu_1^- - \mu_2^-} (L \mathfrak{gl}_n) & \xrightarrow{\Delta_{-\mu_1^+, -\mu_2^- - \mu_1^- - \mu_2^-}} & \mathfrak{u}^{\text{sc}}_{-\mu_1^+, -\mu_2^-} (L \mathfrak{gl}_n) \otimes \mathfrak{u}^{\text{sc}}_{-\mu_1^- - \mu_2^-} (L \mathfrak{gl}_n)
\end{array}
\tag{3.110}
\]

for any \( \mu_1^+, \mu_1^-, \mu_2^+, \mu_2^- \in \Lambda^+ \).

Evoking the defining formulas (3.18) for the embedding \( \iota_{-\mu^+, -\mu^-} : U_{-\mu^+, -\mu^-} (L \mathfrak{sl}_n) \hookrightarrow U_{-\mu^+, -\mu^-} (L \mathfrak{gl}_n) \) of Proposition 3.16, one obtains explicit formulas for the \( \Delta_{-\mu_1^+, -\mu_2^-} \) images of the finite generating set, following the proof of [FT1, Theorem 10.14] presented in [FT1, Appendix G]. The resulting formulas coincide with the explicit long formulas of [FT1, Theorem 10.22], thus providing a simpler and more conceptual proof of [FT1, Theorem 10.22].

**Remark 3.111.** Due to [FT1, Theorem 10.26], \( \Delta_{-\nu_1^+, -\nu_2^-} : U_{-\nu_1^+, -\nu_2^-} (L \mathfrak{sl}_n) \hookrightarrow U_{-\nu_1^+, -\nu_2^-} (L \mathfrak{gl}_n) \) with \( \nu_1^+, \nu_2^- \in \Lambda^+ \) give rise to algebra homomorphisms \( \Delta_{\nu_1^+, \nu_2^-} : U_{\nu_1^+, \nu_2^-} (L \mathfrak{sl}_n) \rightarrow U_{\nu_1^+, \nu_2^-} (L \mathfrak{sl}_n) \otimes U_{\nu_1^+, \nu_2^-} (L \mathfrak{sl}_n) \) for any \( \nu_1^+, \nu_2^- \in \Lambda^+ \).

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