ON DIRECTIONAL DERIVATIVES FOR CONE-CONVEX FUNCTIONS

KRZYSZTOF LEŚNIEWSKI

Abstract. We investigate the relationship between the existence of directional derivatives for cone-convex functions with values in a Banach space $Y$ and isomorphisms between $Y$ and $c_0$.

1. Introduction

Cone convexity of vector-valued functions plays a similar role as convexity for real-valued functions (e.g. local Pareto minima are global etc.). Directional derivatives of cone-convex functions are used in constructing descent methods for vector-valued mappings (see, e.g. [8]). However, in contrast to the real-valued case the existence of directional derivative for cone-convex vector-valued functions is not automatically guaranteed.

If we assume that cone $K \subset Y$ is normal and $Y$ is weakly sequentially complete Banach space, then directional derivatives for $K$-convex functions $F : X \to Y$ always exists (see [19]). The question is what can we say about converse implication.

In Theorem 5.2 of [3] it was proved that the existence of directional derivatives of a cone-convex mapping $F : X \to Y$ defined on linear space $X$ with values in a real Banach space $Y$ implies the weak sequential completeness of the space $Y$.

In the proof of Theorem 5.2 of [3] we construct a cone $K$ and a $K$-convex function such that the directional derivative does not exist at 0 for some $h \in X \setminus \{0\}$. However in [3], cone $K$ is not normal.

In the present paper we relate the existence of directional derivatives for $K$-convex functions $F : X \to Y$, where cone $K$ is normal to the existence of isomorphisms between image space $Y$ and $c_0$.

Isomorphisms of a Banach space $Y$ and the space $c_0$ have been investigated e.g. in [4, 7, 17]. A sequence $\{b_j\}$ in a Banach space $Y$ is strongly summing (s.s.) if $\{b_j\}$ is a weak Cauchy basic sequence and for any scalars...
\{c_j\} satisfying \( \sup_n \| \sum_{j=1}^n c_j b_j \| < +\infty \), the series \( \sum c_j \) converges. In \[17\] Rosenthal proved the following theorem.

**Theorem 1.1.** (Theorem 1.1. of \[17\]) A Banach space \( Y \) contains no subspace isomorphic to \( c_0 \) if and only if every non-trivial weak-Cauchy sequence in \( Y \) has a \((s.s.)\)-subsequence.

In \[4\] Bessaga and Pełczyński proved Bessaga-Pełczyński \( c_0 \)-theorem.

**Theorem 1.2** (Theorem 5 of \[4\]). A Banach space \( Y \) does not contain a subspace isomorphic to \( c_0 \) if and only if every series \( \sum_{k=1}^\infty x_k \) such that \( \sum_{k=1}^\infty |\langle x_k, x^* \rangle| < \infty \) \( \forall x^* \in Y^* \) is unconditionally convergent.

In \[7, 11\] we can find interesting result for Banach lattices.

**Theorem 1.3.** A Banach lattice does not contain a subspace isomorphic to \( c_0 \) if and only if it is a weakly sequentially complete.

The organization of the paper is as follows. In Section 2 we present basic notions and facts about cone-convex functions. Section 3 is devoted to normal cones in Banach spaces. In Section 4 we present basic construction of cone-convex functions (cf. \[3\]) which is used in our main result. Section 5 contains the main result.

### 2. Notations and preliminaries

Let \( X \) be a linear space over reals. Let \( Y \) be a normed space over reals and let \( Y^* \) be the norm dual of \( Y \).

**Definition 2.1.** Let \( A \subset X \). The function \( F : A \rightarrow Y \) is **directionally differentiable** at \( x_0 \in A \) in the direction \( h \neq 0 \) such that \( x_0 + th \in A \) for all \( t \) sufficiently small if the limit

\[
F'(x_0; h) := \lim_{t \downarrow 0} \frac{F(x_0 + th) - F(x_0)}{t}
\]

exists. The element \( F'(x_0; h) \) is called the **directional derivative** of \( F \) at \( x_0 \) in the direction \( h \).

A nonempty subset \( K \) of \( Y \) is called a **cone** if \( \lambda K \subset K \) for every \( \lambda \geq 0 \) and \( K + K \subset K \). The relation \( x \preceq_K y \ (y \succeq_K x) \) is defined as follows

\[
x \preceq_K y \ (y \succeq_K x) \iff y - x \in K.
\]
The dual cone \( K^* \) of a cone \( K \) is defined as
\[
K^* = \{ y^* \in Y^* : y^*(y) \geq 0 \ \text{for all} \ y \in K \}.
\]
In the space \( c_0 := \{ x = (x_1, x_2, \ldots), x_i \in \mathbb{R}, \lim_{i \to \infty} x_i = 0 \} \), the cone \( c_0^+ \),
\[
c_0^+ := \{ x = (x_1, x_2, \ldots) \in c_0 : x_i \geq 0, \ i = 1, 2, \ldots \},
\]
is a closed convex pointed (i.e. \( K \cap (-K) = \{0\} \)). Cone \( c_0^+ \) is generating in \( c_0 \) i.e. \( c_0 = c_0^+ - c_0^+ \).

**Example 2.2.** Let us show that \( (c_0^+)^* = l_1^+ := \{ g = (g_1, g_2, \ldots) \in l_1 : g_i \geq 0, \ i = 1, 2 \ldots \} \). Since \( c_0^* = l_1 \) for any \( y^* = (g_1, g_2, \ldots) \in (c_0^+)^* \), we have \( \sum_{i=1}^{\infty} |g_i| < \infty \). Since every \( e_i = (0, 0, \ldots, 1_i, 0, \ldots) \), \( i = 1, 2, \ldots \) is an element of \( c_0^+ \) we have
\[
y^*(e_i) = g_i, \ i = 1, 2, \ldots.
\]
By the definition of dual cone (2.2), we get \( g_i \geq 0 \) for all \( i = 1, 2, \ldots \). On the other hand, for any \( x \in (c_0^+)^+ \) and any \( y^* = (g_1, g_2, \ldots) \in l_1^+ \) we have \( y^*(x) \geq 0 \).

**Definition 2.3.** Let \( K \subset Y \) be a cone. Let \( A \subset X \) be a convex set. We say that a function \( F : X \rightarrow Y \) is \( K \)-convex on \( A \) if \( \forall x, y \in A \) and \( \forall \lambda \in [0, 1] \)
\[
\lambda F(x) + (1 - \lambda) F(y) - F(\lambda x + (1 - \lambda)y) \in K.
\]
Some properties of \( K \)-convex functions can be found in e.g. [3, 8, 12]. The following characterization is given in [12] for finite dimensional case.

**Lemma 2.4** (Lemma 3.3 of [3]). Let \( A \subset X \) be a convex subset of \( X \). Let \( K \subset Y \) be a closed convex cone and let \( F : X \rightarrow Y \) be a function. The following conditions are equivalent.

1. The function \( F \) is \( K \)-convex on \( A \).
2. For any \( u^* \in K^* \), the composite function \( u^*(F) : A \rightarrow \mathbb{R} \) is convex.

**3. Normal Cones**

In a normed space \( Y \) a cone \( K \) is normal (see [16]) if there is a number \( C > 0 \) such that
\[
0 \leq_K x, y \Rightarrow \|x\| \leq C\|y\|.
\]
Some useful characterizations of normal cones are given in the following lemmas.

**Lemma 3.1.** [14] Let \( Y \) be an ordered topological vector space with positive cone \( K \). The following assertions are equivalent.
• $K \subset Y$ is normal.
• For any two nets $\{x_\beta : \beta \in I\}$ and $\{y_\beta : \beta \in I\}$, if $0 \leq_K x_\beta \leq_K y_\beta$ for all $\beta \in I$ and $\{y_\beta\}$ converges to 0, then $\{x_\beta\}$ converges to 0.

For lattice cones in Riesz spaces defined as in [2] we get the following Lemma.

**Lemma 3.2** (Lemma 2.39 of [2]). Every lattice cone in Riesz space is normal closed and generating.

In some infinite dimensional spaces there are pointed generating cones which are not normal.

**Example 3.3** (Example 2.41 of [2]). Let $Y = C^1[0,1]$ be the real vector space of all continuously differentiable functions on $[0,1]$ and let cone $K$ be defined as

$$K := \{x \in C^1[0,1] : x(t) \geq 0 \text{ for all } t \in [0,1]\}.$$ 

Let us consider the norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty,$$

where $x'$ denotes the derivative of $x \in Y$. Cone $K$ is closed and generating but it is not normal. Let $x_n := t^n$ and $y_n := 1$, we have $0 \leq_K x_n \leq_K 1$. There is no constant $c > 0$ such that the inequality $\|x_n\| = \|t^n\|_\infty + \|nt^{n-1}\|_\infty = n + 1 \leq c = c\|y\|$ holds for all $n$.

Some interesting results (see e.g. [10, 13, 15]) for closed convex cones are using the concept of a basis of a space.

**Definition 3.4** (Definition 1.1.1 of [1]). A sequence $\{x_n\} \subset Y$ in an infinite-dimensional Banach space $Y$ is said to be a basis of $Y$ if for each $x \in Y$ there is a unique sequence of scalars $\{a_n\}_{n \in \mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

For basis $\{x_n\}$ we can define the cone associated to the basis $\{x_n\}$.

**Definition 3.5** (Definition 10.2 of [13]). Let $\{x_n\}$ be a basis of a Banach space $Y$. The set

$$K_{\{x_n\}} := \{y \in Y : y = \sum_{i=1}^{\infty} \alpha_i x_i \in Y : \alpha_i \geq 0, i = 1, 2, \ldots \}$$

is called the cone associated to the basis $\{x_n\}$.
Cone $K\{x_n\}$ is a closed and convex and coincides with the cone generated by $\{x_n\}$ i.e. it is the smallest cone containing $\{x_n\}$.

For a basis $\{x_n\}$ of a Banach space $Y$ functionals $\{x_n^*\}$ are called biorthogonal functionals if $x_n^*(x_j) = 1$ if $k = j$, and $x_n^*(x_j) = 0$ otherwise, for any $k, j \in \mathbb{N}$ and $x = \sum_{i=1}^{\infty} x_i^*(x)x_i$ for each $x \in X$. The sequence $\{x_n, x_n^*\}$ is called the biorthogonal system.

It is easy to see that $\{e_i\}$, where $e_i = (0, \ldots, 1, \ldots)$, $i = 1, 2, \ldots$ is a basis for $c_0$ and $\{e_i^*\}$, $e_i^* := e_i$ are biorthogonal functionals.

We also have $c_0^+=K\{e_i\}$ and $(c_0^+)^* = l_1^+ = K\{e_i^*\}$.

**Definition 3.6** (Definition 3.1.1 of [1]). A basis $\{x_n\}$ of a Banach space $Y$ is called unconditional if for each $x \in Y$ the series $\sum_{n=1}^{\infty} x_n^*(x)x_n$ converges unconditionally.

A basis $\{x_n\}$ is conditional if it is not unconditional. A sequence $\{x_n\} \subset X$ is complete (see [18]) if span$\{x_n\} = X$.

**Theorem 3.7** (Proposition 3.1.3 of [1]). Let $\{x_n\}$ be a complete sequence in a Banach space $Y$ such that $x_n \neq 0$ for every $n$. Then the following statements are equivalent.

(1) $\{x_n\}$ is an unconditional basis for $Y$.
(2) $\exists C_1 \geq 1 \forall N \geq 1 \forall c_1, \ldots, c_N \forall \varepsilon_1, \ldots, \varepsilon_N = \pm 1$,

(3.1) $\|\sum_{n=1}^{N} \varepsilon_n c_n x_n\| \leq C_1 \|\sum_{n=1}^{N} c_n x_n\|.$

First example of conditional basis for $c_0$ was given by Gelbaum [9].

**Example 3.8.** Basis $\{x_n\}$ defined as $x_n := (1, 1, \ldots, 1, 0, \ldots) = \sum_{i=1}^{n} e_i, \{e_i\}$ is the canonical basis for $c_0$, $n = 1, 2, \ldots$ is a conditional basis for $c_0$. All calculations can be found in Example 14.1 p.424 [13].

**Example 3.9.** Let us show that $\{b_i\} \subset c_0$ defined as

(3.2) $b_i = \frac{1}{i} e_i, \ i = 1, 2, \ldots$,

$\{e_i\}$ is the canonical basis for $c_0$ is an unconditional basis.
Since \(\|x\| = \sup_i |x^i|\) for \(x = (x^1, x^2, \ldots) \in c_0\), inequality (3.1) is satisfied with \(C_1 = 1\)
\[
\|\varepsilon_1 c_1(1, 0, \ldots) + \cdots + \varepsilon_N c_N(0, 0, \ldots, 1_N, 0, \ldots)\| = \\
\|\varepsilon_1 c_1, \ldots, \varepsilon_N c_N(\frac{1}{N}, 0, \ldots)\| = \|c_1, \ldots, c_N(\frac{1}{N}, 0, \ldots)\|.
\]

In [10, 13] we can find a characterization of normal cones in terms of unconditional bases.

**Theorem 3.10** (Theorem 16.3 of [13]). Let \(\{x_n, f_n\}\) be a complete biorthogonal system in a Banach space \(Y\). The following are equivalent.

1. \(\{x_n\}\) is an unconditional basis for \(Y\).
2. \(K_{\{x_n\}}\) is normal and generating.

**Example 3.11.** Cone \(K_{\{x_n\}} \subset c_0\), where
\[
x_n = (0, 0, \ldots, 0, -1, 1, 0, 0, \ldots), n = 1, 2, \ldots
\]
is generating pointed and not normal in \(c_0\).

Now let us present some facts about cone isomorphisms.

**Definition 3.12** ([5, 15]). Let \(X\) and \(Y\) be normed spaces ordered by cones \(P \subset X\) and \(K \subset Y\), respectively. We say that \(P\) is conically isomorphic to \(K\) if there exists an additive, positively homogeneous, one-to-one map \(i\) of \(P\) onto \(K\) such that \(i\) and \(i^{-1}\) are continuous in the induced topologies. Then we also say that \(i\) is a conical isomorphism of \(P\) onto \(K\).

**Proposition 3.13.** Let \(X\) be a linear space and let \(Y, Z\) be Banach spaces. Let \(P\) and \(K\) be convex cones in \(Z\) and \(Y\), respectively. Let function \(F : X \to P\) be a \(P\)-convex. If there exists a conical isomorphism \(i : P \to K\), where cone \(P\) is generating in \(Z\), then the function \(\bar{F} : X \to K\), where
\[
\bar{F} := i \circ F
\]
is a \(K\)-convex function.

**Proof.** By Theorem 4.4 of [5], in view of the fact that \(P\) is a generating cone in \(Z\), the conical isomorphism \(i : P \to K\) can be extended to the function \(G : P \to K - K\) defined as
\[
G(x) = i(x^1) - i(x^2), \text{ where } x = x^1 - x^2, x^1, x^2 \in P.
\]
Function \(G : Z \to K - K\) is linear. Indeed, let us take \(x, y \in Z\), since \(P\) is generating \(x = x^1 - x^2, y = y^1 - y^2\), where \(x^1, x^2, y^1, y^2 \in P\).
\[
G(x + y) = G(x^1 - x^2 + y^1 - y^2) = i(x^1 + y^1) - i(x^2 + y^2) = \\
i(x^1) - i(x^2) + i(y^1) - i(y^2) = G(x) + G(y).
\]
Let us take $\lambda < 0$. We have
\[ G(\lambda x) = G(\lambda x^1 - \lambda x^2) = i(-\lambda x^2) - (-\lambda x^1) = -\lambda i(x^2) + \lambda i(x^1) = \lambda G(x). \]

For $\lambda \geq 0$ the calculations are analogous.

Now let us take $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. By Definition 2.3 and the linearity of $G$,
\[ F(\lambda x_1 + (1 - \lambda)x_2) \leq_p \lambda F(x_1) + (1 - \lambda)F(x_2) \quad \text{i.e.} \]
\[ \lambda F(x_1) + (1 - \lambda)F(x_2) - F(\lambda x_1 + (1 - \lambda)x_2) \in P. \]

By the definition of $G$,
\[ G(\lambda F(x_1) + (1 - \lambda)F(x_2) - F(\lambda x_1 + (1 - \lambda)x_2)) = i(\lambda F(x_1) + (1 - \lambda)F(x_2) - F(\lambda x_1 + (1 - \lambda)x_2)) \in K. \]

Furthermore,
\[ G(\lambda F(x_1) + (1 - \lambda)F(x_2) - F(\lambda x_1 + (1 - \lambda)x_2)) = \lambda G(F(x_1)) + (1 - \lambda)G(F(x_2)) - G(F(\lambda x_1 + (1 - \lambda)x_2)) = \lambda i(F(x_1)) + (1 - \lambda)i(F(x_2)) - i(F(\lambda x_1 + (1 - \lambda)x_2)) \geq_K 0. \]

The latter inequality is equivalent to
\[ \bar{F}(\lambda x_1 + (1 - \lambda)x_2) \leq_K \lambda \bar{F}(x_1) + (1 - \lambda)\bar{F}(x_2) \]
which completes the proof. $\square$

**Proposition 3.14.** Let $K \subset Y$ be closed convex and normal cone in a Banach space $Y$. If $i : Y \to Z$ is an isomorphism, then $i(K)$ is a closed convex and normal cone in $Z$.

**Proof.** Let us take
\[ (3.3) \quad 0 \leq_{i(K)} z^1_n \leq_{i(K)} z^2_n \quad n = 1, 2, \ldots \]
for some sequences $z^1_n, z^2_n \in Z$ and let $\lim_{n \to \infty} z^2_n \to 0$. We want to show that $\lim_{n \to \infty} z^1_n \to 0$. By (3.3), we have $z^2_n - z^1_n \in i(K)$. By assumption $i^{-1} : Z \to Y$ is continuous linear and onto i.e. we get,
\[ K \ni i^{-1}(z^2_n - z^1_n) = i^{-1}(z^2_n) - i^{-1}(z^1_n). \]
It means that $i^{-1}(z^2_n) \geq_K i^{-1}(z^1_n)$. Since $z_2 \to 0$ we get $i^{-1}(z^2_n) \to 0$. From the fact that $K$ is normal $\lim_{n \to 0} i^{-1}(z^1_n) = 0 \equiv \lim_{n \to 0} z^1_n = 0$. By Banach Open Mapping Theorem, cone $i(K)$ is closed. $\square$

From Proposition 3.14 and Example 3.11 we get the following corollary.

**Corollary 3.15.** Every Banach space isomorphic to $c_0$ contains a cone which is closed convex and generating but not normal.
Let $J$ be a James space i.e. $J := \{ x = (x_n)_{n \in \mathbb{N}} : \lim_{n \to 0} x_n = 0 \}$ with the norm $\|x\| = \sup\{ \sum_{i=1}^{n} (x_{m_{2i-1}} - x_{m_{2i}})^2 : 0 = m_0 < m_1 < \cdots < m_{n+1} \} < \infty$. An interesting result is the fact that James space does not contain an isomorphic copy of $c_0$ or $l_1$.

In [4] Pełczyński and Bessaga proved the following theorem.

**Theorem 3.16** (Theorem 6.4 of [10, 4]). A separable Banach space having the space $J$ of James as a subspace (e.g. $C[0,1]$) does not have an unconditional basis.

It is easy to find not normal cones in infinite dimensional Banach spaces (see e.g. [13]).

### 4. Useful Constructions of convex functions

In this section we recall a constructions of some convex functions introduced in [3]. Let us start with the following lemma.

**Lemma 4.1** (Lemma 4.2 of [3]). Let $\{a_m\}, \{t_m\} \subset \mathbb{R}$ be sequences with $\{t_m\}$ decreasing. Function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$g(r) := \sup_m f_m(r),$$

where

$$f_m(x) := a_m + \frac{x - t_m}{t_{m+1} - t_m} (a_{m+1} - a_m) \quad \text{for } x \in \mathbb{R}.$$ is convex and

$$g(t_m) = a_m.$$

if and only if

$$\frac{a_{m+1} - a_m}{t_{m+1} - t_m} \geq \frac{a_{m+2} - a_{m+1}}{t_{m+2} - t_{m+1}} \quad \text{for } m \in \mathbb{N}. \quad (4.1)$$

**Proof.** $\Leftarrow$ this proof has beed presented in [3].

$\Rightarrow$ Let us assume that

$$\frac{a_{m+1} - a_m}{t_{m+1} - t_m} < \frac{a_{m+2} - a_{m+1}}{t_{m+2} - t_{m+1}} \quad \text{for some } m \in \mathbb{N}.$$

We get

$$(a_{m+1} - a_m) \left(1 - \frac{x - t_m}{t_{m+1} - t_m}\right) + \frac{x - t_{m+1}}{t_{m+2} - t_{m+1}} (a_{m+2} - a_{m+1}) > 0$$

and $f_{m+1}(x) > f_m(x)$ for $x \geq t_{m+1}$. Let us take $x = t_m \geq t_{m+1}$ we get

$$f_{m+1}(t_m) > f_m(t_m) = a_m,$$

which is contrary to $g(t_m) = a_m$. \qed
Let \( Y \) be a Banach space and \( \{y_i\} \) be an arbitrary sequence of elements of \( Y \). Let \( \{t_i\} \) be a sequence of positive reals tending to zero.

Let \( \bar{F} : X_h := \{x \in X : x = \beta h, \beta \geq 0\} \rightarrow Y \) be a function defined as in [33], i.e. for \( r > 0 \)

\[
(4.2) \quad \bar{F}(rh) := \sum_{i=1}^{\infty} \bar{F}_i(rh),
\]

where \( \bar{F}_i : \{x \in X : x = \beta h, \beta \geq 0\} \rightarrow Y \) is defined as

\[
\bar{F}_1(rh) := \begin{cases} 
y_1t_1 + \frac{r-t_1}{t_2-t_1}(y_2t_2 - y_1t_1) & t_2 < r \\
0 & r < t_2
\end{cases}
\]

and for \( i \geq 2 \)

\[
\bar{F}_i(rh) := \begin{cases} 
y_i t_i + \frac{r-t_i}{t_{i+1}-t_i}(y_{i+1}t_{i+1} - y_it_i) & t_{i+1} < r \leq t_i \\
0 & r \notin (t_{i+1}, t_i]
\end{cases}
\]

Observe that for \( r = t_k \) we have \( \bar{F}(t_k h) = \bar{F}_k(t_k h) = y_k t_k \).

The following proposition is a simple consequence of Lemma 4.1.

**Proposition 4.2.** Let \( K \subset Y \) be a closed convex cone with the dual \( K^* \subset Y^* \). Let \( \{y_i\} \subset Y \) be a sequence in \( Y \). The function \( \bar{F} \) defined by \((4.2)\) with \( t_k = \frac{1}{k}, k = 1, 2, \ldots \), is \( K \)-convex on \( X_h \) if and only if

\[
(4.3) \quad y^*(2y_{k+1} - y_k - y_{k+2}) \leq 0 \quad \text{for all } y^* \in K^*, \; k = 1, 2, \ldots.
\]

**Proof.** By Lemma 4.1 the function \( \bar{F} \) is \( K \)-convex on \( X_h \) if and only if inequality \((4.1)\) holds, i.e.

\[
\frac{a_{k+1} - a_k}{t_{k+1} - t_k} \geq \frac{a_{k+2} - a_{k+1}}{t_{k+2} - t_{k+1}},
\]

where \( t_k = \frac{1}{k}, a_k := y^*(\bar{F}(t_k h)) \) and \( y^* \in K^* \). We get

\[
\frac{1}{k+2} y^*(y_{k+2}) - \frac{1}{k+1} y^*(y_{k+1}) \leq \frac{1}{k+1} y^*(y_{k+1}) - \frac{1}{k} y^*(y_k) \equiv \frac{1}{k+2} y^*(y_{k+2}) - \frac{1}{k+1} y^*(y_{k+1}) \leq (k+1)y^*(y_{k+2}) - (k+2)y^*(y_{k+1}) \equiv (k+1)y^*(2y_{k+1} - y_k - y_{k+2}) \leq 0, \; k = 1, 2, \ldots \equiv y^*(2y_{k+1} - y_k - y_{k+2}) \leq 0, \; k = 1, 2, \ldots.
\]

\[\Box\]

**Proposition 4.3.** Let \( \{y_i\} \subset Y \) be a sequence in a Banach space \( Y \). If \( \{b_k\} \subset Y \) defined as

\[
2y_{k+1} - y_k - y_{k+2} =: b_k, \; k = 1, 2, \ldots,
\]
forms an unconditional basis in $Y$, the function $\bar{F} : X \to Y$ defined by (4.2) with $t_k = \frac{1}{k}, k = 1, 2, \ldots$, is $(-K_{(b_k)})$-convex, where

$$K_{(b_k)} := \{ y \in Y : y = \sum_{i=1}^{\infty} a_i b_i, a_i \geq 0, i = 1, 2, \ldots \}$$

is a closed generating and normal cone in $Y$.

Proof. In view of Proposition 3.7, the cone

$$K_{(b_k)} = \{ y \in Y : y = \sum_{i=1}^{\infty} \alpha_i b_i \in Y : \alpha_i \geq 0, i = 1, 2, \ldots \}$$

is normal and generating in $Y$. Let us observe that dual cone is defined as

$$K^* := -K_{(b_k^*)},$$

where $\{b_i, b_i^*\}$ is the biorthogonal system. Inequality (4.3) is satisfied because

$$y^*(b_k) = -\alpha_k \leq 0$$

for some $\alpha_k \in \mathbb{R}$. By Proposition 4.2, the function $\bar{F}$ is $(-K_{(b_k)})$-convex. □

5. Main result

Now we are ready to formulate our main result.

**Theorem 5.1.** Let $X$ be a linear space and $Y$ be a Banach space. If for every closed convex and normal cone $K \subset Y$ and for every $K$-convex function $F : X \to Y$ there exist directional derivatives for every $h \in X$ at $x_0 = 0$, then there is no subspace in $Y$ isomorphic to $c_0$.

Proof. We proceed by contradiction. We assume that there exists a subspace $Z \subset Y$ isomorphic to $c_0$, i.e. there is a continuous linear mapping $i : c_0 \to Z$. Our aim is to construct a cone-convex function $\bar{F} : X \to Y$ which does not possess the directional derivative at 0 for some $h \in X, h \neq 0$.

Let us take $h \in X, h \neq 0$ and $t_k := \frac{1}{k}, k = 1, 2, \ldots$. Let $\{y_k\} \subset c_0$ be a sequence in $c_0$.

The proof will be in two steps.

Step 1. By using (4.2), let us construct a cone-convex function $\bar{F} : X \to c_0$ such that

$$\bar{F}(\frac{1}{k}h) = y_k \frac{1}{k}; k = 1, 2, \ldots .$$

Let $\bar{F} : \{ x \in X : x = \beta h, \beta \geq 0 \} \to c_0$ be defined as follows. For $r > 0$

$$\bar{F}(rh) := \sum_{i=1}^{\infty} \bar{F}_i(x),$$
ON DIRECTIONAL DERIVATIVES FOR CONE-CONVEX FUNCTIONS

where $\tilde{F}_i : \{x \in X : x = \beta h, \beta \geq 0\} \to c_0$ is defined as
\[
\tilde{F}_1(x) := \begin{cases} 
 y_1 t_1 + \frac{r - t_1}{t_2 - t_1} (y_2 t_2 - y_1 t_1) & t_2 < r \\
 0 & r < t_2
\end{cases}
\]
and for $i \geq 2$
\[
\tilde{F}_i(x) := \begin{cases} 
 y_i t_i + \frac{r - t_i}{i+1-t_i} (y_{i+1} t_{i+1} - y_i t_i) & t_{i+1} < r \leq t_i \\
 0 & r \notin (t_{i+1}, t_i]
\end{cases}
\]

In view of Proposition 4.3 we need to find a sequence $\{y_k\}$ such that
1. the sequence $\{b_k\}$ defined as
\[
(5.1) \quad 2y_{k+1} - y_k - y_{k+2} = b_k, \quad k = 1, 2, \ldots
\]
is an unconditional basis of $c_0$,
2. $\{y_k\}$ is not weakly convergent.

Let us prove by induction that
\[
(5.2) \quad y_k = (k - 1)y_2 - (k - 2)y_1 - \sum_{i=1}^{k-2} (k - i - 1)b_i.
\]

Let us assume that equality (5.2) holds for all $n \leq k$. By (5.1) we have
\[
y_{k+1} = 2y_k - y_{k-1} - e_{k-1} = 2[(k - 1)y_2 - (k - 2)y_1 - \sum_{i=1}^{k-2} (k - i - 1)b_i] - y_{k-1} - b_{k-1} = 2[(k - 1)y_2 - (k - 2)y_1 - \sum_{i=1}^{k-2} (k - i - 1)b_i] - \sum_{i=1}^{k-3} (k - i - 2)b_i - b_{k-1} = y_{k-1} - \sum_{i=1}^{k-1} (k - i) b_i
\]
which proves (5.2).

Let $\{b_i\} \subset c_0$ be the unconditional basis defined in Example 3.11, i.e. $b_i = (0, 0, \ldots, \frac{1}{i}, 0, \ldots), i = 1, 2, \ldots$.

Let $K := -K_{\{b_i\}}$. By Proposition 4.3 cone $K$ is normal and generating in $c_0$.

Let us take $y_1 = (0, 0, \ldots)$ and $y_2 = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. 

By (5.2), we get
\[ y_3 = (1, 1, \frac{2}{5}, \frac{2}{3}, \frac{2}{6}, \ldots), \]
\[ \ldots \]
\[ y_k = (1, 1, \ldots, \frac{k - 1}{k}, \frac{k - 1}{k + 1}, \ldots). \]

Moreover, observe that \( \{y_k\} \) is weak Cauchy. Let \( y^* = (f_1, f_2, \ldots) \in l_1 \).

We have
\[ y^*(y_{k+1} - y_k) = f_{k+1} \frac{2}{k + 1} + f_{k+2} \frac{1}{k + 2} + f_{k+3} \frac{1}{k + 3} + \ldots \xrightarrow{k \to \infty} 0. \]

Since \( y_k \to (1, 1, \ldots) \notin c_0 \) sequence \( \{y_k\} \) is not weakly convergent.

Step 2. By assumption, there exists an isomorphism \( i \) between \( c_0 \) and the subspace \( Z \subset Y \).

Let us define \( F : X \to Y \) by the formula \( F := i \circ \bar{F} \). From Proposition 3.13 the function \( F \) is \( i(K) \)-convex.

The directional derivative for the function \( F \) at \( x_0 = 0 \) is equal
\[ \lim_{k \to \infty} \frac{F(t_kh)}{t_k} = \lim_{k \to \infty} \frac{i(\bar{F}(t_kh))}{t_k} = \lim_{k \to \infty} i \left( \frac{\bar{F}(t_kh)}{t_k} \right) = \lim_{k \to \infty} i(y_k). \]

By the fact that \( i \) is an isomorphism and by Proposition 3.14 cone \( i(K) \) is closed normal and generating in \( Z \). Since \( \{y_k\} \) is not weakly convergent, the function \( F \) is \( i(K) \)-convex and is not directionally differentiable at \( x_0 = 0 \) in the direction \( 0 \neq h \in X \).

Since \( \{b_i\} \) is an unconditional basis for \( c_0 \), from Theorem 3.10 we get the following corollary.

**Corollary 5.2.** Let \( X \) be a linear space and \( Y \) be a Banach space. If for every closed convex normal and generating cone \( K \subset Y \) and for every \( K \)-convex function \( F : X \to Y \) there exist directional derivative for every \( h \in X \) at \( x_0 = 0 \), then \( Y \) is not isomorphic to \( c_0 \).

If \( Y \) is weakly sequentially complete Banach space, then \( K \)-convex function, where \( K \subset Y \) is closed convex and normal has directional derivative for every \( x_0 \in X, h \in X \setminus \{0\} \) (see [19]). From Theorem 1.3 and Corollary 5.2 we get the characterization of weakly sequentially complete Banach spaces in terms of existence of directional derivative for \( K \)-convex functions.

**Theorem 5.3.** Let \( Y \) be a Banach lattice. Space \( Y \) is weakly sequentially complete if and only if, for every closed convex normal cone \( K \subset Y \) and every \( K \)-convex function \( f : X \to Y \) the directional derivative exist for all \( x_0 \in X, h \in X \setminus \{0\} \).
References

[1] F. Albiac, N. J. Kalton, *Topics in Banach Spaces*, Springer-Verlag, 2000
[2] Ch. D. Aliprantis, R. Tourky, *Cones and Duality*, Graduate Studies in Mathematics Vol 84, 2007
[3] E. Bednarczuk, K. Leśniewski, *On weakly sequentially complete Banach spaces*, Journal of Convex Analysis 24 (2017), No. 4
[4] Cz. Bessaga, A. Pelczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), pp. 151-164
[5] E. Casini, E. Miglierina, *Cones with bounded and unbounded bases and reflexivity*, Nonlinear Analysis, 72 (2010), pp. 2356-2366
[6] J. Jahn, *Mathematical Vector Optimization in Partially Ordered Linear Spaces*, Verlag Peter Lang, Frankfurt am Main 1986
[7] G. Ya. Lozanowski, Banach structures and bases, Funct. Anal. Appl. 1, (1967), pp. 249. (rus.)
[8] L. M. Graña Drummond, F. M.P. Raupp, B.F. Svaiter, A quadratically convergent Newton method for vector optimization, Optimization, 63 No 5, (2014), pp. 661-677
[9] Gelbaum, *Expansions in Banach spaces*, Duke Math. J. 17 (1950), pp. 187196.
[10] C.W. McArthur, Developments in Schauder basis theory, Bulletin of The American Mathematical Society, Vol. 78 (1972) No. 6, pp. 877-908
[11] P. Meyer-Nieberg, Zur schwachen Kompaktkheit in Banachverba nden, Math. Z. 134, (1973), pp. 303315.
[12] T. Pennanen, *Graph-convex mappings and K-convex functions*, Journal of Convex Analysis, Vol. 6 (1999) No.2, pp. 235–266
[13] I. Singer, *Bases of Banach Spaces II*, Springer, 1970
[14] A. L. Peressini, *Ordered Topological Vector Spaces*, Harper & Row, New York, Evanston, and London, 1967
[15] I. A. Polyak, *Cones Locally isomorphic to the Positive cone of l_1(\Gamma)*, J. Math. Anal. Appl. 338 (2008), pp. 695–704
[16] I. A. Polyak, *Cone characterization of reflexive Banach lattices*, Glasgow Mathematical Journal, Vol. 37, Issue 1, (1995), pp. 65-67
[17] H. Rosenthal, *A characterization of Banach spaces containing c_0*, Journal of American Mathematical Society, 7 No 3, (1994), pp. 707-748
[18] P. Terenzi, *Every separable Banach space has a bounded strong norming biorthogonal sequence which is also Steintz basis*, Studia Mathematica, 111, No 3, (1994)
[19] M. Valadier , *Sous-Différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné*, Math. Sand. 30 (1972), pp. 65–74

Faculty of Mathematics and Information Science, Warsaw University of Technology, 00-662 Warszawa, Poland, ul. Koszykowa 75
E-mail address: k.lesniewski@mini.pw.edu.pl