CONVEX-CYCLIC WEIGHTED TRANSLATIONS ON
LOCALLY COMPACT GROUPS

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Abstract. A bounded linear operator $T$ on a Banach space $X$ is called a convex-cyclic operator if there exists a vector $x \in X$ such that the convex hull of $\text{Orb}(T, x)$ is dense in $X$. In this paper, for given an aperiodic element $g$ in a locally compact group $G$, we give some sufficient conditions for a weighted translation operator $T_{g, w} : f \mapsto w \cdot f \ast \delta_g$ on $L^p(G)$ to be convex-cyclic. A necessary condition is also studied. At the end, to explain the obtained results, some examples are given.

1. Introduction

A bounded linear operator $T$ on a separable infinite-dimensional Banach space $X$ over the field $\mathbb{C}$ is called hypercyclic and supercyclic if there exists a vector $x \in X$ such that $\text{Orb}(T, x) = X$ and $\mathbb{C} \text{Orb}(T, x) = X$, respectively, where $\text{Orb}(T, x) = \{T^n x; \ n \in \mathbb{N}\}$. If $\text{span}(\text{Orb}(T, x))$ is dense in $X$, then $T$ is called a cyclic operator. Recall that, the notion of hypercyclicity was already studied by Birkhoff [7] when he introduced the notion of the topological transitivity. To be precise, an operator $T$ is topologically transitive, if for every pair of nonempty open subsets $U, V$ of $X$, there exists a non-negative integer $n$ such that $T^n(U) \cap V \neq \emptyset$. It is not difficult to observe that

$$\text{Transitivity} \iff \text{Hypercyclicity} \implies \text{Supercyclicity} \implies \text{Cyclicity}.$$  

Similar to the definition of the transitivity, if there exists a non-negative integer $N$ such that $T^n(U) \cap V \neq \emptyset$ for all integers $n \geq N$, then $T$ is called topologically mixing and it is clear that

$$\text{Mixing} \implies \text{Transitivity}.$$  

Like supercyclicity, another well known concept can be appeared between cyclicity and hypercyclicity that is when the convex hull generated by an orbit $\text{Orb}(T, x)$ is dense in $X$. In this case, $x$ is called

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a convex-cyclic vector for $T$ and also, $T$ is called a convex-cyclic operator. Note that, if $\mathcal{C}_p$ denotes the set of all convex polynomials, then $co(\text{Orb}(T, x)) = \{P(T)x : P \in \mathcal{C}_p\}$. Similar to the definition of the transitivity, a bounded linear operator $T$ on $X$ is called a convex-transitive operator if for every nonempty open subsets $V$ and $U$ of $X$, the intersection $P(T)(V) \cap U$ is nonempty for some convex polynomial $P \in \mathcal{C}_p$. The relations between convex-cyclicity and convex-transitivity is exhibited in the following diagram.

$\text{Convex – transitivity } \Rightarrow \text{Convex – cyclicity.}$

The converse of the above diagram is correct when the point spectrum of the adjoint of the associated operator is empty [14, Theorem 3.9]. Initially, the notion of the convex-cyclicity has been studied by H. Rezaei in [14]. After that, other authors studied this notion in [8], [16] and [18]. For example, the Hahn-Banach characterization for convex-cyclicity theorem is an important theorem which we will use it in the next section. Therefore we state it below and of course, its proof can be found in [8]:

**Hahn-Banach Characterization for Convex-Cyclicity:** Let $X$ be a Banach space over the complex numbers, $T$ be a bounded linear operator on $X$ and $x \in X$. Then the following are equivalent:

i) $x$ is a convex-cyclic vector for $T$.

ii) $\sup_{n \geq 0} \text{Re}(\Lambda(T^n x)) = +\infty$ for every linear functional $\Lambda \in X^* \setminus \{0\}$.

Let us mention that for an arbitrary bounded sequence $w = \{w_j > 0\}_{j \in \mathbb{N}}$ and the canonical basis $\{e_j\}_{j \in \mathbb{N}}$ of $\ell_p(\mathbb{N})$ when $p \in [1, +\infty)$, the operator $B_w$ on $\ell_p(\mathbb{N})$, which is defined by

$$B_w(e_j) = \begin{cases} w_je_{j-1}, & j \geq 2 \\ 0 & j = 1 \end{cases},$$

is called a unilateral weighted backward shift. First of all, Rolewicz [15] showed that $\lambda B$ on $\ell_2(\mathbb{N})$ is hypercyclic for any complex number $|\lambda| > 1$ while the backward shift operator $B$ is not hypercyclic. Then for $p \in [1, +\infty)$ and an arbitrary bounded sequence $w = \{w_j > 0\}_{j \in \mathbb{Z}}$ and the standard base $\{e_j\}_{j \in \mathbb{Z}}$ for $\ell_p(\mathbb{Z})$ the hypercyclic bilateral weighted shift operators on $\ell_p(\mathbb{Z})$, which is defined by $B_w(e_j) = w_je_{j-1}$, have been characterized by Salas [17] in terms of their weights.

In [14], Rezaei characterized a convex-cyclic weighted backward shift on $\ell_p(\mathbb{N})$. Indeed, he proved that a weighted backward shift on $\ell_p(\mathbb{N})$ is hypercyclic if and only if it is convex-cyclic. Then he claimed that a certain condition is sufficient for a bilateral weighted shift on $\ell_p(\mathbb{Z})$ to be convex-cyclic [14, Theorem 4.2], but the proof of the associate
theorem is correct only in a special case. So in this paper, we will give a full generalization of this result to weighted translation operators on the Lebesgue spaces.

In [9] and [10] the hypercyclic weighted translation operators on the Lebesgue space $L^p(G)$ in terms of the weight were characterized in discrete and non-discrete groups. Therefore, it is natural to raise the following question:

**Question 1.1.** Is there a description of the convex-transitive weighted translation operators on the Lebesgue space $L^p(G)$?

For deep understanding of the hypercyclicity, we would like to mention two excellent books [6] and [11]. Also, dynamics of the weighted translations in different settings have been studied in [1, 2, 3, 4, 5].

### 2. Main results

The operators that we want to study in this paper, are in the complex Lebesgue space $L^p(G)$, when $G$ is a locally compact group with a right Haar measure $\vartheta$. Note that, the Lebesgue space $L^p(G)$ with recent properties is separable whenever $G$ is second countable. We remind that a torsion element in a locally compact group $G$ is an element of finite order, and an element $g \in G$ is called compact [13] (or periodic [12]) if the closed subgroup $G(g)$ generated by $g$ is compact. Also, an element in $G$ is called aperiodic if it is not periodic. Observe that in discrete groups, a periodic element is a torsion element and conversely.

Given an element $g \in G$, the unit point mass function at $g$ is denoted by $\delta_g$, and a weight on $G$ is considered as a real bounded function $w : G \to (0, +\infty)$. Then a weighted translation operator $T_{g,w}$ is defined on $L^p(G)$ by

$$T_{g,w}(f)(x) := w(x)(f*\delta_g)(x) = w(x) \int_G f(xh^{-1})d\delta_g(h) = w(x)f(xg^{-1}).$$

It is not difficult to observe that,

$$T_{g,w}^n(f) = \left( \prod_{i=0}^{n-1} w * \delta_{g^i} \right)(f * \delta_{g^n}). \quad (2.1)$$

Chen and Chu [10] were succeeded in showing how topologically mixing of $T_{g,w}$ depends on the behavior of successive translations of $w$ by $g$. More precisely, if $T_{g,w} : L^p(G) \to L^p(G)$ is a weighted translation, then the following statements are equivalent ([10, Theorem 2.2]).

**i)** $T_{g,w}$ is a topologically mixing weighted translation.

**ii)** If $K$ is an arbitrary compact subset of $G$ with $\vartheta(K) > 0$, then there
exists a sequence of Borel sets \( \{B_n\} \subset K \) such that \( \lim_{n \to \infty} \vartheta(B_n) = \vartheta(K) \) and both sequences \( \{w_n\} \) and \( \{\tilde{w}_n\} \) which
\[
w_n := \prod_{i=1}^{n} w \ast \delta_{g^{-i}} \quad \text{and} \quad \tilde{w}_n := \frac{1}{\prod_{i=0}^{n-1} w \ast \delta_i}
\]
satisfy
\[
\lim_{n \to \infty} \|w_n|_{B_n}\|_{\infty} = \lim_{n \to \infty} \|\tilde{w}_n|_{B_n}\|_{\infty} = 0.
\]

We are now ready to show how the convex-cyclicity of the weighted translation \( T_{g,w} \) depends on the behavior of successive translations of the weight \( w \) by an aperiodic element \( g \). Since we will use the convex-transitive criterion, so it is stated \([14]\).

**Convex-transitive Criterion Theorem.** Let \( X \) be a separable Banach space and \( T \) be a bounded linear operator on \( X \). If there exist dense subsets \( Y \) and \( Z \) of \( X \) such that for every vectors \( y, z \) in \( Y \) and \( Z \), respectively, there exist a sequence \( \{P_k\}_{k>1} \subseteq \mathcal{P} \) and functions \( S_k : Z \to X \) such that
\[
i) \quad P_k(T)y \to 0,
\[ii) \quad S_k z \to 0 \quad \text{and} \quad P_k(T)S_k z \to z,
\]
then \( T \) is convex-transitive and consequently is convex-cyclic.

In the following theorem, we are going to state the sufficient conditions for the weighted translation operator \( T_{g,w} \) to be convex-cyclic. This result not only extends \([14, \text{Theorem } 4.2]\) but also improves its proof.

In what follows, to prevent the text from prolongation, we use the following symbols.
\[
i) \quad \mathcal{AP}(G) \text{ is instead of all aperiodic elements of } G.
\[ii) \quad \text{The set of all bounded positive functions } w : G \to (0, +\infty) \text{ will be denoted by } \Psi(G).
\]

**Theorem 2.1.** Let \( g \in \mathcal{AP}(G) \) and \( w \in \Psi(G) \). Then the weighted translation operator \( T_{g,w} : \mathcal{L}^p(G) \to \mathcal{L}^p(G) \) is convex-cyclic whenever there exists a scalar \( \beta \geq 1 \) such that
\[
i) \liminf_{k \to \infty} \|\beta^{-k} \prod_{i=1}^{k} w \ast \delta_{g^{-i}}\|_{\infty} = 0, \quad \liminf_{k \to \infty} \|\beta^{k} \prod_{i=0}^{k-1} w \ast \delta_{g^i}\|_{\infty}^{-1} = 0
\]
whenever \( \beta > 1 \) and
\[
(ii) \liminf_{k \to \infty} \|k^{-1} \prod_{i=1}^{k} w \ast \delta_{g^{-i}}\|_{\infty} = 0, \quad \liminf_{k \to \infty} \|k^{k-1} \prod_{i=0}^{k-1} w \ast \delta_{g^i}\|_{\infty}^{-1} = 0
\]
whenever \( \beta = 1 \).

**Proof.** The condition \( (i) \) implies that \( T_{g,\beta^{-1}w} = \beta^{-1}T_{g,w} \) is hypercyclic, \([10, \text{Theorem } 2.3]\). Hence the set \( \mathcal{H} \) consisting of all hypercyclic vectors
for $T_{g,\beta^{-1}w}$ are dense in $\mathfrak{L}^p(G)$. Also, the condition $(i)$ implies that $\ker(\beta I - T_{g,w}) = \{0\}$ and hence $(\beta I - T_{g,w})$ has dense range.

Now we want to show that $T_{g,w}$ satisfies the convex-cyclic criterion. For this goal, we put $Y = (\beta I - T_{g,w})(\mathcal{H})$ and $Z = C_c(G)$, the set of all continuous $f \in \mathfrak{L}^p(G)$ with compact support. Thus $Y$ and $Z$ are dense subsets of $\mathfrak{L}^p(G)$.

Now we consider a convex polynomial

$$P_k(t) = \frac{\beta - 1}{\beta^k - 1}(\beta^{k-1} + \beta^{k-2}t + \cdots + t^{k-1}), \quad k \in \mathbb{N},$$

and if we define

$$S_{g,w}(f) = \left(\frac{1}{w}f\right) * \delta_{g^{-1}}, \quad f \in \mathfrak{L}^p(G),$$

then set

$$S_k := \frac{\beta^k - 1}{1 - \beta}(\beta I - T_{g,w})S_{g,w}^k. \quad (2.2)$$

For an arbitrary $f_0 \in \mathcal{H}$, we have $(\beta I - T_{g,w})f_0 \in Y$ and hence,

$$P_k(T_{g,w})(\beta I - T_{g,w})(f_0) = \frac{\beta - 1}{\beta^k - 1}(\beta^k f_0 - T_{g,w}^k(f_0))$$

$$= \frac{\beta - 1}{\beta^k - 1} \left(\beta^k f_0 - \left(\prod_{i=0}^{k-1} w * \delta_{g^i}\right) f_0 * \delta_{g^k}\right)$$

$$= \frac{\beta^k(\beta - 1)}{\beta^k - 1} \left(f_0 - (\beta^{-k} \prod_{i=0}^{k-1} w * \delta_{g^i}) f_0 * \delta_{g^k}\right)$$

$$= \frac{\beta^k(\beta - 1)}{\beta^k - 1} \left(f_0 - T_{g,\beta^{-1}w}^k(f_0)\right).$$

Now, above equality implies that

$$\|P_k(T_{g,w})(\beta I - T_{g,w})(f_0)\|_p = \|\frac{\beta^k(\beta - 1)}{\beta^k - 1} \left(f_0 - T_{g,\beta^{-1}w}^k(f_0)\right)\|_p \to 0$$

as $k \to \infty$. On the other side, the definition of $S_{g,w}$ implies that

$$S_{g,w}^k(h) = \left(\prod_{i=1}^{k} w * \delta_{g^{-i}}\right)^{-1} (h * \delta_{g^{-k}}), \quad h \in C_c(G).$$

Now, from (2.2) we get that

$$\|S_k(h)\|_p \leq \frac{1}{\beta - 1}(\beta + \|T_{g,w}\|)\|\beta^k(\prod_{i=0}^{k-1} w * \delta_{g^i})^{-1}\|_\infty \|h\|_p$$,
so \( \|S_k(h)\|_p \to 0 \) as \( k \to \infty \). Moreover, for every \( h \in C_c(G) \) we have

\[
\|P_k(T_{g,w})S_k(h) - h\|_p = \|\beta^k S_{g,w}^k(h)\|_p \leq \|\beta^k (\prod_{i=0}^{k-1} w \ast \delta_{g_i})^{-1}\|_\infty \|h\|_p.
\]

Hence, the condition (i) and the above inequality implies \( P_k(T_{g,w})S_k(h) \) tends to \( h \) in \( \| \cdot \|_p \)-norm, whenever \( k \to \infty \). Therefore \( T_{g,w} \) satisfies the convex-transitive criterion and consequently it is convex-cyclic.

When the condition (ii) holds, then we may consider the convex polynomial

\[
P_k(t) = k^{-1}(1 + t + t^2 + \cdots + t^{k-1}),
\]

and

\[
S_k := k(I - T_{g,w})S_{g,w}^k
\]

for any \( k \in \mathbb{N} \). The condition (ii) implies that \( \ker(I - T_{g,w}) = \{0\} \) and hence the operator \( (I - T_{g,w}) \) has dense range. Indeed, if there exists a non-zero \( h \in L^p(G) \) such that \( T_{g,w}(h) = h \), then since \( T_{g,w} \) is invertible and \( T_{g,w}^{-1} = T_{g^{-1},(\frac{1}{w} \ast \delta_{g^{-1}})} \), we obtain

\[
h = T_{g^{-1},(\frac{1}{w} \ast \delta_{g^{-1}})}^k(h) = (\prod_{i=1}^{k} w \ast \delta_{g^{-i}})^{-1}(h \ast \delta_{g^{-k}}), \quad k \in \mathbb{N}.
\]

Now, the inequality

\[
\|h\|_p^p = \int_G |(\prod_{i=1}^{k} w \ast \delta_{g^{-i}})^{-1}(x)|^p |(h \ast \delta_{g^{-k}})(x)|^p d\vartheta(x)
\]

\[
= \int_G |(\prod_{i=0}^{k-1} w \ast \delta_{g_i})^{-1}(x)|^p |h(x)|^p d\vartheta(x)
\]

\[
\leq \|k(\prod_{i=0}^{k-1} w \ast \delta_{g_i})^{-1}\|_\infty \|h\|_p^p
\]

implies that \( \ker(I - T_{g,w}) = \{0\} \).

Now set \( Y = Z = C_c(G) \). Then for each \( f \in C_c(G) \) we have

\[
\|P_k(T_{g,w})(I - T_{g,w})(f)\|_p = \frac{1}{k} f - \frac{1}{k} (\prod_{i=0}^{k-1} w \ast \delta_{g_i}) f \ast \delta_{g^k}\|_p
\]

\[
= \frac{1}{k} f \ast \delta_{g^{-k}} - \frac{1}{k} (\prod_{i=1}^{k} w \ast \delta_{g^{-i}}) f\|_p
\]

\[
\leq \frac{1}{k} \|f \ast \delta_{g^{-k}}\|_p + \frac{1}{k} \|\prod_{i=1}^{k} w \ast \delta_{g^{-i}}\|_\infty \|f\|_p \to 0
\]
as $k \to \infty$. Similarly

$$\|S_k(f)\|_p \leq (1 + \|T_{g,w}\|)k(\prod_{i=0}^{k-1} w \ast \delta_{g^i})^{-1}\|f\|_p$$

and

$$\|P_k(T_{g,w})S_k(f) - f\|_p = \|S_k^k(f)\|_p \leq k(\prod_{i=0}^{k-1} w \ast \delta_{g^i})^{-1}\|f\|_p,$$

that yields $S_k(f)$ and $P_k(T_{g,w})S_k(f)$ approaches to 0 and $f$, respectively, as $k \to \infty$. Therefore, $T_{g,w}$ is again convex-cyclic in this case. □

We would like to state the following lemma because we will use it in the proof of the next Theorem.

**Lemma 2.2.** Let $x_0 \in X$ be a convex-cyclic vector for an operator $T$ on $X$, $\varepsilon > 0$ and $N_0 \in \mathbb{N}$. Then there is a convex polynomial $P_k(t) := a_0 + a_1 t + \cdots + a_k t^k$, $a_k > 0$ such that $k > N_0$ and $\|P_k(T)(x_0) - x\| < \varepsilon$ for every $x \in X$.

**Proof.** Since $x_0 \in X$ is a convex-cyclic vector for $T$, so there is a convex polynomial $P_n(t) := \sum_{i=0}^{n} a_i t^i$, $a_n > 0$ such that $\|P_n(T)(x_0)\| < \frac{\varepsilon}{\|T\|^{N_0}}$ and consequently

$$\|T^{N_0}(a_0 x_0 + a_1 T(x_0) + \cdots + a_n T^n(x_0))\| < \varepsilon. \quad (2.3)$$

Now for an arbitrary vector $x \in X$, there is a convex polynomial $Q_m(t) := \sum_{i=0}^{m} b_i t^i$, $b_m > 0$ such that $\|Q_m(T)(x_0) - 2x\| < \varepsilon$. Thus

$$\|\frac{1}{2}Q_m(T)(x_0) - x\| < \frac{\varepsilon}{2}. \quad (2.4)$$

From (2.3) and (2.4) we get that

$$\|\frac{1}{2}(T^{N_0}P_n(T)(x_0) + Q_m(T)(x_0)) - x\| < \varepsilon.$$

Note that the polynomial $\frac{1}{2}(T^{N_0}P_n(T)x + Q_m(T)x)$ is a convex polynomial such that its degree is equal to $l := \max\{n + N_0, m\}$. Therefore $l > N_0$ and the proof is completed. □

**Theorem 2.3.** Let $g \in \mathfrak{M}(G)$, $w \in \Psi(G)$ and also, let the weighted translation operator $T_{g,w}$ on $\mathfrak{L}^p(G)$ be convex-cyclic. If $\sigma_p(T_{g,w}^*) = \emptyset$, then for each compact subset $K \subset G$ with positive measure, there exist...
\( N_0 \in \mathbb{N}, \) a sequence of Borel subsets \( (E_n) \subseteq K \) and a sequence \( (a_n) \subseteq [0,1] \) with \( \sum_{i=0}^{\infty} a_i = 1 \) such that

\[
1/\lim \inf_{n \to \infty} \| (a_0 \prod_{i=0}^{N_0-1} w \ast \delta_{g^i} + \cdots + a_n \prod_{i=0}^{N_0+n-1} w \ast \delta_{g^i})|_{E_n} \|_\infty = 0.
\]

**Proof.** Let \( \varepsilon > 0 \) and \( K \subset G \) be an arbitrary compact set with \( \vartheta(K) > 0 \). Then there exists a positive integer \( N_0 \) such that for every \( n > N_0 \), \( K \cap Kg^n = \emptyset \), because \( g \) is aperiodic [10, Lemma 2.1]. Note that, if \( \sigma_p(T_{g,w}^*) = \emptyset \), then the set of all convex-cyclic vectors for \( T_{g,w} \) is dense in \( \mathfrak{L}^p(G) \) [14, Theorem 3.5]. Consider \( \chi_K \in \mathfrak{L}^p(G) \) as the characteristic function of \( K \) and choose \( \theta \in (0, \frac{\varepsilon}{2 + \varepsilon}) \), then there is a convex-cyclic vector \( h \in \mathfrak{L}^p(G) \) such that \( \|h - \chi_K\|_p < \theta^2 \). Also there exists a convex polynomial \( P_n \in \mathfrak{C}_p \) with the real coefficients of the finitely supported sequence \( (a_n) \subseteq [0,1] \) such that \( a_0 + a_1 + \cdots + a_n = 1 \) and

\[
\|P_n(T_{g,w})(h) - \chi_K\|_p < \theta^2.
\]

To attain the desired result, by Lemma 2.2, the power of the polynomial \( P_n \) is somehow chosen greater than \( N_0 \) and sufficiently large as well. Furthermore, by [8, Corollary 2.2], it does not make an ambiguity if we start the first term of \( P_n \) of order \( N_0 \) i.e., zero is the root of the order \( N_0 \). Indeed, \( P_n(t) = a_0 t^{N_0} + a_1 t^{N_0+1} + \cdots + a_n t^{N_0+n} \). Now, consider the following Borel subsets of \( G \).

\[
B_\theta = \{ x \in G \setminus K : \ |h(x)| \geq \theta \},
\]

\[
C_\theta = \{ x \in K : \ |(a_0( \prod_{i=0}^{N_0-1} w \ast \delta_{g^i})h \ast \delta_{g^{N_0}} + a_1( \prod_{i=0}^{N_0} w \ast \delta_{g^i})h \ast \delta_{g^{N_0+1}} + \cdots + a_n( \prod_{i=0}^{N_0+n-1} w \ast \delta_{g^i})h \ast \delta_{g^{N_0+n}})(x) - 1 | \geq \theta \}.
\]

Note that

\[
\theta^{2p} > \|h - \chi_K\|_p^p > \int_{G \setminus K} |h - \chi_K|^p d\vartheta \\
\geq \int_{B_\theta} |h|^p d\vartheta \geq \theta^p \vartheta(B_\theta).
\]
Moreover observe that,
\[ \theta^{2p} > \| P_n(T_{g,w})(h) - \chi_K \|^p \]
\[ = \int_G |a_0(\prod_{i=0}^{N_0-1} w * \delta_{g^i})h * \delta_{g^{N_0}} + a_1(\prod_{i=0}^{N_0} w * \delta_{g^i})h * \delta_{g^{N_0+1}} + \cdots + a_n(\prod_{i=0}^{N_0+n-1} w * \delta_{g^i})h * \delta_{g^{N_0+n}} - \chi_K|^p d\vartheta \]
\[ \geq \int_K |a_0(\prod_{i=0}^{N_0-1} w * \delta_{g^i})h * \delta_{g^{N_0}} + a_1(\prod_{i=0}^{N_0} w * \delta_{g^i})h * \delta_{g^{N_0+1}} + \cdots + a_n(\prod_{i=0}^{N_0+n-1} w * \delta_{g^i})h * \delta_{g^{N_0+n}} - 1|^p d\vartheta \]
\[ \geq \theta^p \partial(C_\theta), \]
which implies that \( \vartheta(C_\theta) \leq \theta^p \). Furthermore, on the subset \( K \setminus C_\theta \) we have
\[ 1 - \theta < |a_0(\prod_{i=0}^{N_0-1} w * \delta_{g^i})h * \delta_{g^{N_0}} + a_1(\prod_{i=0}^{N_0} w * \delta_{g^i})h * \delta_{g^{N_0+1}} + \cdots + a_n(\prod_{i=0}^{N_0+n-1} w * \delta_{g^i})h * \delta_{g^{N_0+n}}| \]
\[ \leq (a_0 \prod_{i=0}^{N_0-1} w * \delta_{g^i} + a_1 \prod_{i=0}^{N_0} w * \delta_{g^i} + \cdots + a_n \prod_{i=0}^{N_0+n-1} w * \delta_{g^i}) \max_{N_0 \leq i \leq N_0+n-1} |h * \delta_{g^i}|. \]
On the other hand, according the fact that \( K \cap K^g = \emptyset \), then on the subset \( E_n := K \setminus (B_{\theta g^{N_0}} \cup B_{\theta g^{N_0+1}} \cup \cdots \cup B_{\theta g^{N_0+n}} \cup C_\theta) \) we obtain
\[ 1/(a_0 \prod_{i=0}^{N_0-1} w * \delta_{g^i} + a_1 \prod_{i=0}^{N_0} w * \delta_{g^i} + \cdots + a_n \prod_{i=0}^{N_0+n-1} w * \delta_{g^i}) \]
\[ < \frac{1}{1 - \theta} \max_{N_0 \leq i \leq N_0+n} |h * \delta_{g^i}| \]
\[ < \frac{\theta}{1 - \theta} \]
\[ < \varepsilon, \]
which completes the proof. \( \square \)
Remark 2.4. If one sets \( w = 1 \) in the statement of Theorem 2.3, it is pointed out that the translation operator \( T_g \) can not be convex-cyclic itself.

Example 2.5. Let \( G = \mathbb{R} \), be the group of the real numbers equipped with the Lebesgue measure. Fix a non-zero negative \( g \in \mathbb{R} \) and consider the convolution \((f \ast \delta_g)(x) = f(x - g), x \in \mathbb{R} \) and \( f \in \mathcal{L}^p(\mathbb{R}) \). Choose arbitrary real numbers \( s, t \) as \( 1 < s < t \), then define the weight function \( w \) on \( \mathbb{R} \) by

\[
    w(x) = \begin{cases} 
    t, & 1 \leq x, \\
    -\frac{x}{2} + 1, & -1 < x < 1, \\
    s, & x \leq -1.
    \end{cases}
\]

By taking any \( \beta \in \mathbb{R} \) with \( s < \beta < t \), Theorem 2.1 ensures that \( T_{g,w} \) is convex-cyclic on \( \mathcal{L}^p(\mathbb{R}) \) while is not hypercyclic by [10, Theorem 2.2].

Example 2.6. Observe that by the second case in Theorem 2.1, the weighted translation operator \( \tilde{T} := T_{g,(w \ast \delta_g)} \) on \( \mathcal{L}^p(\mathbb{Z}) \) is a convex-cyclic operator whenever \( g = -1 \) and

\[
    w = \{ w_i \} = \begin{cases} 
    2, & i \geq 1 \\
    1, & i \leq 0.
    \end{cases}
\]

We show that the point spectrum of the adjoint of \( \tilde{T} \) is empty. Indeed, if \( \lambda \in \sigma_p(T^*) \), then there is a non-zero \( f \in \mathcal{L}^q(\mathbb{Z}) \) such that \( T^*f = \lambda f \).

Since \((T^*)^n f = \lambda^n f\) for every \( n \in \mathbb{N} \), so \( \prod_{i=0}^{n-1} w(m-i) f(m-n) = \lambda^n f(m) \) for every \( m \in \mathbb{Z} \). If \( m = 0 \), then \( f(-n) = \lambda^n f(0) \). This implies that \( |\lambda| < 1 \) because \( |f(-n)| \to 0 \) as \( n \to \infty \). But if \( m > 0 \), then \( 2^m f(0) = \lambda^m f(m) \). Thus \( 2 < |\lambda| \) because \( |f(m)| \to 0 \) as \( n \to \infty \). This contradiction shows that \( \sigma_p(T^*) = \phi \).

Now we consider \( P_n(x) = a_0 x^{N_0} + a_1 x^{N_0+1} + \cdots + a_n x^{N_0+n} \) as a convex polynomial when \( N_0 \in \mathbb{N} \) such that \( N_0 = \vartheta(K_{N_0}) \) for an arbitrary finite subset \( K_{N_0} = \{ m+1, m+2, \cdots, m+N_0 \} \) of \( \mathbb{Z} \). It is not difficult to see that \( K_{N_0} \cap K_{g \pm N_0} = \phi \). For simplicity assume that all elements in \( K_{N_0} \) are negative. If \( 0 < \theta < 1 \), then there exist an \( h \in \mathcal{L}^p(\mathbb{Z}) \) and a convex polynomial \( P_n(x) \) similar above such that

\[
    \| h - \chi_{K_{N_0}} \|_p \leq \theta^2 \quad \text{and} \quad \| P_n(T)(h) - \chi_{K_{N_0}} \|_p \leq \theta^2.
\]
Let

\[ A = \{ x \in K_{N_0} : |h(x) - 1| \geq \theta \}, \]
\[ B = \{ x \in \mathbb{Z} \setminus K_{N_0} : |h(x)| \geq \theta \}, \]
\[ C = \{ x \in K_{N_0} : |a_0 \prod_{i=1}^{N_0} w(x + i)h(x + N_0) + \]
\[ a_1 \prod_{i=1}^{N_0+1} w(x + i)h(x + N_0 + 1) + \cdots + \]
\[ a_n \prod_{i=1}^{N_0+n} w(x + i)h(x + N_0 + n) - 1 | \geq \theta \} \]
\[ \text{and} \]
\[ D = \{ x \in K_{N_0} : |a_0 \prod_{i=1}^{N_0} w(x + i - n)h(x + N_0 - n) + \]
\[ a_1 \prod_{i=1}^{N_0+1} w(x + i - n)h(x + N_0 + 1) + \cdots + \]
\[ a_n \prod_{i=1}^{N_0+n} w(x + i - n)h(x + N_0) | \geq \theta \}. \]

Then \( \vartheta(A) < \theta^p \), \( \vartheta(B) < \theta^p \), \( \vartheta(C) < \theta^p \) and \( \vartheta(D) < \theta^p \) imply that \( \vartheta(A) = \vartheta(B) = \vartheta(C) = \vartheta(D) = 0 \). Now, for an \( x \in K_{N_0} \) we have that

\[ a_0 \prod_{i=1}^{N_0} w(x + i) + a_1 \prod_{i=1}^{N_0+1} w(x + i) + \cdots + a_n \prod_{i=1}^{N_0+n} w(x + i) \to \infty \]

as \( n \to \infty \), but

\[ a_0 \prod_{i=1}^{N_0} w(x + i - n) + a_1 \prod_{i=1}^{N_0+1} w(x + i - n) + \cdots + a_n \prod_{i=1}^{N_0+n} w(x + i - n) \]

\[ \to \sum_{j=0}^{n} a_j = 1 \text{ as } n \to \infty. \]

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