Localized bending and longitudinal waves in rods interacting with external nonlinear elastic medium

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Abstract. This paper is concerned with the study of the propagation of bending and longitudinal waves in homogeneous rods interacting with a nonlinear elastic medium. The dynamic behaviour of the rods is determined by refined theories as follows: for a rod fixed on a nonlinear elastic foundation and executing bending vibrations we select the Tymoshenko model while for a rod immersed in a nonlinear elastic medium and performing longitudinal vibrations we choose the Mindlin-Hermann model. It is demonstrated that evolutionary equations are generalizations of the well-known Ostrovsky equation: the modified Ostrovsky equation with a cubic non-linear term and the Ostrovsky equation with an additional quadratic non-linear term. Both generalized equations have exact soliton solutions of the stationary wave class.

1. Introduction

In the dynamics of rods along with engineering (classical) models the so-called refined or non-classical models [1, 2] exist. The said models either consider some additional factors influencing the dynamic process or are free from any hypotheses applicable in engineering theories and limiting the scope of their use.

The classical theory of L. Euler and J. Bernoulli applicable for the description of bending vibrations of a rod is generalized by the Rayleigh model (accounting for the kinetic energy of rotational inertia of a rod element in bending) and the Timoshenko model (accounting also for the potential energy of shear deformation in bending) [1, 2].

The classical theory of D. Bernoulli applicable in the description of longitudinal vibrations of a rod is generalized by the models of Rayleigh-Love (accounting for the kinetic energy of transverse motions of rod particles), Bishop (accounting also for the potential energy of shear deformations), Mindlin-Hermann (free from the hypothesis of uniaxial deformed state of a rod) [1].

Such refined models are generally used in the description of high-frequency wave processes when the wavelength becomes comparable with the rod cross-section diameter and engineering models are conceptually inapplicable.

The problems of the dynamics of the rods interacting with the external medium that possesses elastic or viscoelastic properties are important for practical applications.

This paper investigates the impact of nonlinear elastic properties of the medium on the localization of bending and longitudinal waves propagating in the rods. To describe the propagation of bending waves in the rod we use the Timoshenko model and for longitudinal waves - the Mindlin-Hermann model.
2. Bending Waves

When simulating the processes of interaction between the vehicles and elastic guideways the problems of bending vibrations of a rod are often considered with allowing for shear deformations and rotary inertia of the cross section (the Timoshenko model) lying on the elastic foundation [2–5].

The movement of modern high-speed trains comes with violent vibrations of the railway itself and in the ground around it, which results in the fast wear of the railroad bed and may cause train derailment. Therefore, when constructing long-distance railways, especially on soft soils, the rigidity of the soil is enhanced, which makes it necessary to take into account the nonlinearity of the elastic foundation in calculations.

Let us consider the propagation of bending waves in the Timoshenko-type rod, which lies on a nonlinear-elastic foundation [6]:

\[
\begin{align*}
\rho F \frac{\partial^2 w}{\partial t^2} - \kappa F \frac{\partial^2 w}{\partial x^2} + \mu F \frac{\partial^2 \varphi}{\partial x^2} + h_1 w + h_2 w^3 &= 0 \quad (1) \\
\rho I_y \frac{\partial^2 \varphi}{\partial t^2} - \frac{E I_y}{\kappa \mu} \frac{\partial^2 \varphi}{\partial x^2} + \mu F \left( \varphi - \frac{\partial w}{\partial x} \right) &= 0 \quad (2)
\end{align*}
\]

\(w(x,t)\) is the lateral displacement of particles in the midline of the rod, \(\varphi(x,t)\) is an angle of the rod cross-section deviation from the vertical position, \(\rho\) is the rod material density, \(F\) is the rod cross-sectional area, \(\kappa\) is a correction factor accounting for the deviation from the theory of flat sections, its value depends on the method of determining the average value for the shear angle and the cross-sectional shift distribution pattern (for a rectangular cross-section rod \(\kappa = 5/6\)), \(\mu\) is a shear modulus, \(h_1, h_2\) are coefficients characterizing a nonlinear elastic foundation, \(I_y\) is an axial inertia moment (for a rectangular cross-section \(I_y = a^3 b/12\)), \(E\) is the Young's modulus.

The system of equations (1), (2) is reduced to one equation in the transverse displacement of particles \(w\):

\[
\left( \rho F + \frac{h_1}{\kappa \mu} \frac{\rho I_y}{F} \right) \frac{\partial^2 w}{\partial t^2} - \frac{E h_1 I_y}{\kappa \mu F} \frac{\partial^2 w}{\partial x^2} + \frac{E I_y}{\rho} \frac{\partial^4 w}{\partial x^4} - \rho I_y \left( 1 + \frac{E}{\kappa \mu} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \\
+ \rho^2 I_y \frac{\partial^2 w}{\partial t^4} + h_2 \frac{\rho I_y}{\kappa \mu F} \left( \frac{\partial^2 w}{\partial t^2} - \frac{E \partial^2 w}{\rho \partial x^2} \right) w^3 + h_1 w + h_2 w^3 = 0
\]

In dimensionless variables equation (3) takes on the form:

\[
\left( 1 + a_1 \right) \frac{\partial^2 W}{\partial \tau^2} - a_1 \frac{\partial^2 W}{\partial z^2} + \frac{\partial^4 W}{\partial z^4} \left( 1 + a_3 \right) \frac{\partial^2 W}{\partial z^2 \partial \tau^2} + \\
+ a_3 \frac{\partial^4 W}{\partial t^4} + a_2 \left( \frac{\partial^2 W}{\partial z^2} - \frac{\partial^2 W}{\partial z^2} \right) W^3 + \frac{a_1}{a_3} W + a_2 \frac{a_1}{a_3} W + W^3 = 0
\]

where \(W = w/w_0, z = x/X, \tau = t/T\) are dimensionless values of displacement, coordinates and time, respectively. Characteristic values are assumed to be equal to

\[
X^2 = \frac{I_y}{F}, ~ T^2 = \frac{\rho I_y}{E F} = \frac{I_y}{c_0^2 F^2},
\]

\((c_0 = \sqrt{E/\rho}\) is the propagation velocity of longitudinal waves in the rod), the dimensionless groups of parameters are given by

\[
a_1 = \frac{h_1 I_y}{\kappa \mu F^2}, ~ a_2 = \frac{h_2 I_y}{\kappa \mu F^2 w_0}, ~ a_3 = \frac{E}{\kappa \mu}.
\]
Let us proceed in equation (4) to the moving coordinate system $\xi = z - ct$, $\theta = \epsilon t$, where wave velocity $c$ is unknown in advance, $\epsilon$ is a small parameter ($\epsilon \ll 1$). The choice of variables is explained by the fact that perturbation, propagating with velocity $c$ along the axis $z$, slowly evolves in time because of the nonlinearity, dispersion and dissipation.

Considering the smallness of nonlinear and dispersive terms we find the wave velocity in the zero approximation as follows

$$c^2 = \frac{a_1}{1 + a_1}.$$ 

In the first approximation for a small parameter we have:

$$-2\epsilon(1 + a_1)\frac{\partial^2 W}{\partial \xi \partial \theta} + \left(1 - c^2\right)(1 - a_1 c^2)\frac{\partial^2 W}{\partial \xi^2} - d_2 \left(1 - c^2\right)\frac{\partial^2 W}{\partial \xi^3} \left(W^3\right) + \frac{a_1}{a_3} W + \frac{a_2}{a_3} W^3 = 0. \quad (5)$$

Let us rewrite equation (5) in a more compact form:

$$\frac{\partial}{\partial \xi} \left(\frac{\partial W}{\partial \theta} + d_1 W^2 \frac{\partial W}{\partial \xi} + d_2 \frac{\partial^3 W}{\partial \xi^3}\right) - d_3 W - d_4 W^3 = 0, \quad (6)$$

where coefficients of the equation are equal to

$$d_1 = \frac{3a_2}{2\epsilon(1 + a_1)^2}, \quad d_2 = \frac{a_1 a_2 - (1 + a_1)}{2\epsilon(1 + a_1)^3}, \quad d_3 = \frac{a_1}{2\epsilon(1 + a_1) a_3}, \quad d_4 = \frac{a_2}{2\epsilon(1 + a_1) a_3}.$$

Equation (6) may be classified as the modified Ostrovsky equation [7–10] with an additional nonlinear term. It would be natural to call equation (6) without the last term the “modified Ostrovsky equation” by analogy with the well-known modified Korteweg - de Vries equation (mKdV) and the modified Burgers equation. However, the concept of "modified" for the KdV equation has a slightly different meaning and is associated with the available Miura transformation [11], which converts the solutions of the modified equation to the solutions of the classical equation. For the classical and modified Burgers and Ostrovsky equations no similar transformations have yet been found.

The modified Ostrovsky equation does not have exact solutions but permits qualitative study with the highest derivative to be equal to zero just like the classical Ostrovsky equation [9, 10]. The presence of the additional cubic nonlinear term enables us to find exact solutions of the equation in the form of solitons.

For stationary waves $W(\xi, \theta) = W(\chi)$, $\chi = \xi - v \theta$ equation (6) is written as follows:

$$\left(-v W' + d_1 W^2 W' + d_2 W^3\right)' - d_3 W - d_4 W^3 = 0, \quad (7)$$

where $v$ has the meaning of the nonlinear wave velocity. We consider that the wave propagates in the positive direction of the axis $\xi$.

We shall seek the solution of equation (7) by the simplest equations method [12] using as the simplest equation the Riccati equation with constant coefficients in the form of $Y'(\chi) = -Y(\chi)^2 + B_0$, which has the solution $Y(\chi) = \sqrt{B_0} \tanh(\sqrt{B_0} \chi)$.

The solution of equation (7) has the first-order pole; therefore, we shall search for the solution of the solitary wave type in the form of

$$W(\chi) = b_0 Y(\chi) + b_1. \quad (8)$$

By substituting the solution of (8) into equation (7) and taking into account the Riccati equation we obtain polynomial relative to $Y(\chi)$. By equating to zero the expressions with the same degrees of $Y(\chi)$ we obtain a system of algebraic equations, from which we find values for coefficients:

$$b_0^2 = -\frac{6d_2}{d_1}, \quad b_1 = 0, \quad B_0 = \frac{d_1 d_3}{6 d_2 d_4}.$$

Furthermore, we find the expression for the nonlinear wave velocity:
Taking into account the found coefficient values, solution (8) takes the following form:

\[
W(\chi) = \pm \sqrt{-\frac{d_1}{d_4}} \text{th} \left( \frac{d_3 d_5}{6d_2 d_4} \chi \right). \tag{9}
\]

We will assume that \( W_1 = W_+ \), \( W_2 = W_- \) with the substitution of \( b_{0+} \) and \( b_{0-} \), respectively. Solutions \( W_1, W_2 \) differ only by signs in front of the hyperbolic tangent.

It is obvious that for the existence of real solutions coefficients \( d_1 \) and \( d_2 \) must be of different signs. For the existence of bounded real solutions coefficients in pairs \( d_1, d_2 \) and \( d_3, d_4 \) should be of different signs (Figure 1). With positive radical expressions in (9) the solution \( W_1(\chi) \) like the solution \( W_2(\chi) \) is a solitary wave and has a kink profile - that is a smooth differential between two values of the function (Figure 1a). With negative radical expressions solutions \( W_1(\chi) \) and \( W_2(\chi) \) are periodic unbounded (Figure 1b). Bounded solutions are of greatest interest.

Figure 1. Profiles of bounded (a) and unbounded (b) solutions \( W_1(\chi) \) (solid line) and \( W_2(\chi) \) (long dash line) for different values of \( d_1 - d_4 \).

Kink-like solutions exist on the plane simultaneously, and each of the kinks is symmetric relative to the inflection point which coincides with the origin of coordinates. One of the kinks is monotone increasing, the other is monotone decreasing. Kinks have the same amplitude and width: \( A = 2 \sqrt{-\frac{d_1}{d_4}} \), \( \Delta = \sqrt{\frac{6d_2 d_4}{d_1 d_3}}. \)

We shall return to dimensionless parameters of the problem. The expressions for the velocity, amplitude and width of the wave will take the form:

\[
v = -\frac{1}{2\alpha_1 \alpha_3} \sqrt{\frac{a_1}{1+a_1}}, \quad A = 2 \sqrt{-\frac{a_3}{a_2}}, \quad \Delta = 2 \sqrt{\frac{a_1 a_3(1+a_1)}{2a_1(1+a_1)}}. \tag{10}
\]
From the analysis of dependencies (10) we obtain restrictions on parameters $a_1 > 0$, $a_2 < 0$, $a_3 > 1$. The graphical representation of dependences on parameter $a_1$ and the respective solution profiles $W_1, W_2$ with fixed parameters are shown in Figures 2 and 3.

![Figure 2](image1.png) ![Figure 3](image2.png)

Figure 2. Dependences $A(a_1)$ (long dash line), $\Delta(a_1)$ (solid line), $v(a_1)$ (dash line) with fixed parameters $a_2, a_3$.

Figure 3. Profiles of solutions $W_1$ (solid line), $W_2$ (long dash line) with fixed parameter $a_1$ $(a_1 > \frac{1}{a_3 - 1})$.

As value $a_1$ increases, the amplitude rises, the nonlinear velocity decreases, and the wave propagates in the negative direction of axis $\xi$. The wave width first increases and then decreases. The maximum wave width is reached at point $a_1 = \frac{1 + \sqrt{a_1}}{a_3 - 1}$ and is equal to $\Delta = \sqrt{2\left(\sqrt{a_3} - 1\right)}$, the minimum is at $a_1 \to +\infty$ or $a_1 \to \frac{1}{a_3 - 1}$. With increasing $a_3$ the maximum point in Figure 2 shifts to the left, and the maximum value of the wave width thereat increases.

3. **Longitudinal Waves**

The system of equations covering the propagation of longitudinal waves in the Mindlin-Hermann rod immersed in a nonlinear elastic medium has the form:

$$\frac{\partial^2 u}{\partial t^2} - c_l^2 \frac{\partial^2 u}{\partial x^2} + \frac{2 \kappa_2^2 \lambda}{a \rho} \frac{\partial w}{\partial x} + h_1 u + h_2 u^2 = 0,$$

$$\frac{\partial^2 w}{\partial t^2} - c_l^2 \kappa_2^2 \frac{\partial^2 w}{\partial x^2} + \frac{8 \kappa_2^2 (\lambda + \mu)}{a^2 \rho} w + \frac{4 \kappa_2^2 \lambda}{a^2 \rho} \frac{\partial u}{\partial x} = 0,$$

where $u(x,t), w(x,t)$ are the longitudinal and transverse displacement of the rod particles, $\kappa_1, \kappa_2$ are correction factors to be selected from the condition of proximity of dispersion branches of this model (Mindlin-Hermann) and the exact Pohgammer-Cree solution, $c_l = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$ is the propagation velocity of longitudinal waves in the rod material, $c_l = \sqrt{\frac{\mu}{\rho}}$ is the propagation velocity of shear...
waves \((c_s < c_t)\), \(a\) is the rod cross section radius, \(\rho\) is the rod material density, \(\lambda, \mu\) are Lame coefficients, \(h_1, h_2\) are coefficients characterizing a nonlinear elastic medium.

We shall now reduce the system of equations (11), (12) to one equation relative to the longitudinal displacement of particles \(u\):

\[
\frac{1}{\lambda} \left(4 \left[\frac{4}{a} (\lambda + \mu) a \rho \phi_1 + \frac{\alpha \rho h_1}{2k_0^2} \right] \frac{\partial^2 u}{\partial t^2} - 4 \left[\frac{4}{a} (\lambda + \mu) a \rho \phi_1 + 4k_0^2 \frac{\alpha \rho h_1}{2k_0^2} \right] \frac{\partial^2 u}{\partial x^2} - \frac{\alpha \rho (c_1^2 + c_2^2) a \rho h_1}{2k_0^2} \right) \frac{\partial^4 u}{\partial x^2 \partial t^2} + \\
+ \frac{\alpha \rho}{2k_0^2 \lambda} \frac{\partial^4 u}{\partial t^4} + \left[\frac{\alpha \rho h_2}{2k_0^2 \lambda} \frac{\partial^2}{\partial x^2} - c_1^2 \frac{\partial^2}{\partial t^2} \right] u^2 + \frac{4(\lambda + \mu) h_1}{a \lambda} u + \frac{4(\lambda + \mu) h_2}{a \lambda} u^2 = 0
\]

In dimensionless variables equation (13) takes the form:

\[
\left(1 + a_1 \right) \frac{\partial^2 U}{\partial \tau^2} - \left(1 + a_1 a_4 - a_4 \right) \frac{\partial^2 U}{\partial z^2} + \frac{\partial^4 U}{\partial \tau^4} - \left(1 + a_3 \right) \frac{\partial^4 U}{\partial z^2 \partial \tau^2} + \\
+ a_3 \frac{\partial^4 U}{\partial z^4} + a_1 U + a_2 U^2 + a_4 \left(\frac{\partial^2}{\partial \tau^2} - a_3 \frac{\partial^2}{\partial z^2} \right) U^2 = 0
\]

where \(U = u/u_0\), \(z = x/X\), \(\tau = t/T\) are dimensionless values of the displacement, coordinates and time, respectively. Characteristic values are assumed to be equal to

\[
T^2 = \frac{a^2 \rho}{8c_1^4 (\lambda + \mu)}, \quad X^2 = \frac{a^2 \rho c_1^2}{8c_1^4 (\lambda + \mu)},
\]

and dimensionless groups of parameters have the form

\[
a_1 = \frac{a^2 \rho h_1}{8c_1^4 (\lambda + \mu)}, \quad a_2 = \frac{a^2 \rho h_2 u_0}{8c_1^4 (\lambda + \mu)}, \quad a_3 = \frac{c_2^2}{c_1^2}, \quad a_4 = \frac{\lambda^2}{c_1^2 \rho (\lambda + \mu)}.
\]

As before, we proceed in equation (14) to the moving coordinate system and in the zero-order approximation we obtain the equation

\[
\left(1 + a_1 \right) c^2 + a_4 - 1 - a_1 a_4 \frac{\partial^2 U}{\partial \xi^2} = 0,
\]

from which we obtain wave velocity \(c^2 = \frac{1 + a_1 a_4 - a_4}{1 + a_1}\). In the first approximation for a small parameter we have:

\[
\left( c^4 - (1 + a_1) c^2 + a_3 \frac{\partial^4 U}{\partial \xi^4} - 2\alpha (1 + a_1) \frac{\partial^2 U}{\partial \xi^2 \partial \theta} + a_2 \left(c^2 - a_3 \right) \frac{\partial^2}{\partial \xi^2} (U^2) + a_1 U + a_2 U^2 = 0. \quad (15)
\]

We shall rewrite equation (15) as follows:

\[
\frac{\partial}{\partial \xi} \left( \frac{\partial U}{\partial \xi} + d_1 U \frac{\partial U}{\partial \xi} + d_2 \frac{\partial U}{\partial \xi^3} \right) - d_3 U - d_4 U^2 = 0,
\]

where coefficients are equal to

\[
d_1 = \frac{a_2 (a_3 + a_4 - 1)}{2a (1 + a_1)^2}, \quad d_2 = \frac{(a_3 + a_4 - 1)(a_1 a_4 - a_4)}{2a (1 + a_1)^3}, \quad d_3 = \frac{a_1}{2a (1 + a_1)}, \quad d_4 = \frac{a_2}{2a (1 + a_1)}.
\]

The last equation is the Ostrovsky equation with an additional nonlinear term which has an exact soliton-type solution.

In progressing wave variables equation (16) shall be written as follows:

\[
(-vU' + d_1 UU' + d_2 U^3)' - d_3 U - d_4 U^2 = 0,
\]

where \(v\) is the velocity of a stationary wave propagating in the positive direction of axis \(\xi\), the traveling variable differentiation \(\chi = \xi - v \theta\) is marked by a line bar.
The solution of equation (17) has a second-order pole; therefore, we look for a solitary-wave solution in the form of

$$U(\chi) = b_0Y^2(\chi) + b_1Y(\chi) + b_2,$$  \hspace{1cm} (18)

where $Y(\chi)$ is the solution of the Riccati equation. Values of unknown coefficients and the nonlinear wave velocity have the form:

$$b_0 = -\frac{12d_2}{d_1}, \quad b_1 = 0, \quad b_2 = \frac{d_3}{2d_4} + \frac{8d_2}{d_1} B_0, \quad B_0 = \pm \frac{d_2 d_4}{8d_2 d_4}, \quad \nu = \frac{d_1 d_4}{2d_4} + \frac{2d_2 d_4}{d_1}.$$  

Thus, substituting the found values for coefficients $b_i$ in (18) we obtain two solutions $U_1(\chi)$ and $U_2(\chi)$ for equation (17) corresponding to two values for coefficient $B_0$ ($B_{0+}$ and $B_{0-}$):

$$U_1(\chi) = \frac{d_3}{2d_4} \left(1 - 3\theta^2 \left(\frac{d_1 d_4}{8d_2 d_4} \chi\right)\right),$$ \hspace{1cm} (19)

$$U_2(\chi) = \frac{3d_3}{2d_4} \theta^2 \left(\frac{d_1 d_4}{8d_2 d_4} \chi - 1\right).$$ \hspace{1cm} (20)

For any real values of coefficients $d_i - d_4$ solutions (19), (20) are represented on the real plane (Figure 4). With a positive radical expression both solutions are solitary waves and have a symmetric bell-shaped profile, with a negative one such solutions are periodic unbounded. Bounded solutions are of greatest interest.

![Figure 4. Profiles of bounded and unbounded solutions $U_1(\chi)$ (19) (dash line) and $U_2(\chi)$ (20) (solid line) with different values of $d_i - d_4$.](image)

It is obvious from the analysis of the solutions (19), (20) that both bell-shaped solutions cannot exist simultaneously. The base of solution $U_1(\chi)$ has a nonzero mark equal to $-d_2/d_4$ (soliton with displacement along the vertical axis), the base of solution $U_2(\chi)$ has a zero mark. Depending on the values of $d_i - d_4$ solitons $U_1(\chi)$, $U_2(\chi)$ may be both of a positive and negative polarity. Solitons have the same amplitudes and widths: $A = \frac{3d_3}{2d_4}$, $A = 2\sqrt[4]{\frac{2d_2 d_4}{d_1 d_3}}$. 
Expressions for the wave velocity, amplitude and width in dimensionless parameters of the problem take the form:

\[ v = -\frac{a_4}{2e(1+a_1)} \sqrt{\frac{1+a_1}{1-a_4+a_1a_3}}, \quad A = \frac{3}{2} \frac{a_1}{a_2}, \quad \Delta = 2 \frac{a_1a_3-a_1-a_4}{a_1(1+a_1)}, \tag{21} \]

wherefrom, taking into account the physical meaning of parameters, restrictions follow:

\[ a_1 > 0, \quad 0 < a_3 < 1, \quad 0 < a_4 < 1. \]

The graphical representation of dependences (21) of the soliton amplitude and width on the velocity with fixed parameters \( a_2, a_3, a_4 \) is shown in Figure 5. The appropriate profile of solution \( U_2 \) is shown in Figure 6.

![Figure 5. Dependences \( A(v) \) (long dash line) and \( \Delta(v) \) (solid line) with changing parameter \( a_1 \).](image1.png)

![Figure 6. Profile of solution \( U_2 \) (solid line) and its derivative \( U_2' \) (dash line).](image2.png)

When parameter \( a_1 \) changes from zero to infinity, the wave velocity drops from \( v_* = \frac{a_4}{2e} \sqrt{\frac{1}{1-a_4}} \) to zero, and the wave thereat propagates in the negative direction of axis \( \xi \), the soliton amplitude increases without limit, and its width decreases to zero.

The soliton polarity (Figure 6) is only affected by the sign of parameter \( a_2 \): with parameter positive values (hard nonlinearity) the soliton has a negative polarity, and with parameter negative values (soft nonlinearity) the soliton has a positive polarity.

Solitons with type (19) shift will not propagate in this system as there is always inequality \( a_1(a_3-1) - a_4 < 0 \) obtained with restrictions imposed on parameters.

If the medium rigidity and the rigidity of the rod are of the same order, the longitudinal waves propagate with velocity \( v = \frac{a_4}{4e} \sqrt{\frac{2}{1-a_4+a_3}} \), their amplitude and width are respectively equal to \( A = \frac{3}{2a_2} \) and \( \Delta = \sqrt{2(a_3-1-a_4)} \).

If the medium rigidity is significantly inferior to the rigidity of the rod \( h_1 << c^2/a^2 \) \( (c_1^2 < c_2^2 < c_3^2) \), then \( a_1 \to 0 \) and the wave propagation velocity is bounded above by value \( v_* \). It
means that longitudinal waves in a rigid rod immersed in a soft medium propagate and their velocity is limited, such waves have a small amplitude and wide width.

If the medium rigidity is significantly higher than the rigidity of the rod (\( h_{1} >> c^{2}/a^{2} \)), \( a_{1} \rightarrow \infty \) and the wave propagation velocity tends to zero, i.e. the longitudinal waves in a soft rod immersed in a rigid medium do not propagate.

Thus, the paper shows that accounting for nonlinear elasticity in the foundation and external medium corresponds to the more exact description of bending and longitudinal waves propagating in one-dimensional systems, which contributes to the detection of stationary nonlinear soliton-type waves.

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