SEMI-ISOTOPIC KNOTS

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Abstract. A knot is a possibly wild simple closed curve in $S^3$. A knot $J$ is semi-isotopic to a knot $K$ if there is an annulus $A$ in $S^3 \times [0,1]$ such that $A \cap (S^3 \times \{0,1\}) = \partial A = (J \times \{0\}) \cup (K \times \{1\})$ and there is a homeomorphism $e : S^1 \times [0,1] \to A - (K \times \{1\})$ such that $e(S^1 \times \{t\}) \subset S^3 \times \{t\}$ for every $t \in [0,1)$.

Theorem. Every knot is semi-isotopic to an unknot.

1. Introduction

We fix some notation and terminology. Let $I = [0,1]$. A knot is the image of an embedding $S^1 \to S^3$. If the composition of this embedding with some homeomorphism of $S^3$ is a piecewise linear embedding, then the knot is tame. Otherwise, it is wild. An annulus is a space that is homeomorphic to $S^1 \times I$.

Knots $J$ and $K$ are ambiently isotopic if there is a level-preserving homeomorphism $h : S^3 \times I \to S^3 \times I$ such that $h(x,0) = (x,0)$ and $h(J \times \{1\}) = K \times \{1\}$. Note that for such an $h$, $h(J \times \{0\}) = J \times \{0\}$ and $h(J \times \{1\}) = K \times \{1\}$. Of course, classical knot theory is the study of ambient isotopy classes of tame knots in $S^3$.

Knots $J$ and $K$ are (non-ambiently) isotopic if there is a level-preserving embedding $e : J \times I \to S^3 \times I$ such that $e(J \times \{0\}) = J \times \{0\}$ and $e(J \times \{1\}) = K \times \{1\}$. Observe that every knot that pierces a tame disk is isotopic to an unknot. The sequence of pictures in Figure 1 suggests a proof of this observation.

The Bing sling (Figure 2) is a wild knot that pierces no disk. It is not known whether the Bing sling is isotopic to an unknot.

The following conjecture is well-known.

Conjecture. Every knot is isotopic to an unknot.

A knot $J$ is semi-isotopic to a knot $K$ if there is an annulus $A$ in $S^3 \times I$ such that $\partial A = (J \times \{0\}) \cup (K \times \{1\})$ and there is a homeomorphism $e : S^1 \times [0,1] \to A - (K \times \{1\})$ such that $e(S^1 \times \{t\}) \subset S^3 \times \{t\}$ for every $t \in [0,1)$. Note that $e$ may not extend continuously to homeomorphism from $S^1 \times [0,1]$ onto $A$.

The main result of this paper is:

Theorem. Every knot is semi-isotopic to an unknot.

Thus, the Bing sling is semi-isotopic to an unknot.

Knots $J$ and $K$ are (topologically) concordant or $I$-equivalent if there is an annulus $A$ in $S^3 \times I$ such that $A \cap (S^3 \times \{0,1\}) = \partial A = (J \times \{0\}) \cup (K \times \{1\})$. Note that: Isotopic $\Rightarrow$ semi-isotopic $\Rightarrow$ concordant.

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Figure 1. An isotopy of a knot to an unknot

Figure 2. The Bing sling
Example. There is a (wild) two-component link in $S^3$ that is not concordant to any PL link! \[7\]

Observe that Melikhov’s striking example shows that the method of proof of the Theorem won’t extend to two-component links, and that the invariants which Melikhov exploits won’t work for single component knots.

The Theorem is proved by applying the two main results of \[1\]. These results are a reinterpretation and formalization of a technique introduced by the topologist C. H. Giffen in the 1960s called shift-spinning. A consequence of Giffen’s method is that the Mazur 4-manifold \[6\] has a disk pseudo-spine. (A pseudo-spine of a compact manifold $M$ is a compact subset $X$ of $\text{int}(M)$ such that $M - X$ is homeomorphic to $\partial M \times [0,1]$.) After R. D. Edwards’ groundbreaking proof that the double suspension of the boundary of the Mazur 4-manifold is homeomorphic to the 5-sphere in 1975, it was observed that had the existence of a disk pseudo-spine in the Mazur 4-manifold been observed before Edwards’ proof, then a proof of Edwards’ theorem using previously known results could have been given. However, no one made the connection between Giffen’s technique and the existence of a disk pseudo-spine until after Edwards announced his proof. Shift-spinning, the existence of a disk pseudo-spine in the Mazur 4-manifold and why this leads to the conclusion that the double suspension of their boundaries are homeomorphic to $S^5$.

Other topologists have explored Giffen shift-spinning in various contexts. (See pages 404-409 of \[5\] and pages 15-16 of \[4\].)

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2. Mapping swirls

The central concept of \[1\] is the mapping swirl. We recapitulate its definition and state the two main results of \[1\] in forms that are convenient for proving the Theorem.

Let $X$ be a compact metric space. Identify the cone on $X$, $CX$, and the suspension of $X$, $\Sigma X$, as the quotient spaces $CX = ([0,\infty] \times X) / (\{\infty\} \times X)$ and $\Sigma X = ([-\infty,\infty] \times X) / (\{-\infty\} \times X, \{\infty\} \times X)$. Let $(t,x) \mapsto tx$ denote either of the quotient maps $[0,\infty] \times X \to CX$ or $[-\infty,\infty] \times X \to \Sigma X$. For each $t \in [0,\infty]$ or $[-\infty,\infty]$, let $tx$ denote the image of the set $\{t\} \times X$ under the appropriate quotient map. Thus, $\infty X$ denotes the cone point of $CX$, and $(-\infty)X$ and $\infty X$ denote the suspension points of $\Sigma X$.

Let $f : X \to Y$ be a map between compact metric spaces. Observe that, by exploiting the homeomorphism $x \mapsto (x, f(x))$ from $X$ to the graph of $f$, the mapping cylinder of $f$, $Cyl(f)$, can be identified with the subset $\{(tx, f(x)) \in CX \times Y : (t, x) \in [0, \infty) \times X\} \cup (\infty X \times Y)$ of $CX \times Y$. Similarly, the double mapping cylinder of $f$, $DblCyl(f)$, which is obtained by identifying two copies of $Cyl(f)$ along their bases, can be identified
with the subset
\[ \{(tx, f(x)) \in \Sigma X \times Y : (t, x) \in (-\infty, \infty) \times X\} \cup \{(\infty)X, \infty X \times Y\} \]
of \( \Sigma X \times Y \). For each \( t \in (-\infty, \infty) \), the \( t \)-level of \( DblCyl(f) \) is the set
\[ L(f, t) = (tX \times Y) \cap DblCyl(f) = \{(tx, f(x)) : x \in X\}. \]
Furthermore, if \( t \in [0, \infty) \), then \( L(f, t) \) is also called the \( t \)-level of \( Cyl(f) \). Note that for each \( t \in (-\infty, \infty) \), \( x \mapsto (tx, f(x)) : X \rightarrow L(f, t) \) is a homeomorphism. The \( \infty \)-level of \( Cyl(f) \) is the set \( L(f, \infty) = \{\infty X\} \times Y \). The \( (-\infty) \)-level and the \( \infty \)-level of \( DblCyl(f) \) are the sets \( L(f, -\infty) = \{(-\infty)X\} \times Y \) and \( L(f, \infty) = \{(\infty)X\} \times Y \).

Let \( X \) be a compact metric space and let \( f : X \rightarrow S^1 \) be a map. The mapping swirl of \( f \) is the subset
\[ Swl(f) = \{(tx, e^{2\pi it} f(x)) \in CX \times S^1 : (t, x) \in [0, \infty) \times X\} \cup \{\{\infty X\} \times S^1\} \]
of \( CX \times S^1 \). The double mapping swirl of \( f \) is the subset \( DblSwl(f) = \{(tx, e^{2\pi it} f(x)) \in \Sigma X \times S^1 : (t, x) \in (-\infty, \infty) \times X\} \cup \{(\infty)X, \infty X \times S^1\} \)
of \( \Sigma X \times S^1 \). For each \( t \in (-\infty, \infty) \), the \( t \)-level of \( DblSwl(f) \) is the set
\[ L(f, t) = (tX \times S^1) \cap DblSwl(f) = \{(tx, e^{2\pi it} f(x)) : x \in X\}. \]
Furthermore, if \( t \in [0, \infty) \), then \( L(f, t) \) is also called the \( t \)-level of \( Swl(f) \). Note that for each \( t \in (-\infty, \infty) \), \( x \mapsto (tx, e^{2\pi it} f(x)) : X \rightarrow L(f, t) \) is a homeomorphism. The \( \infty \)-level of \( Swl(f) \) is the set \( L(f, \infty) = \{\infty X\} \times S^1 \). The \( (-\infty) \)-level and \( \infty \)-levels of \( DblSwl(f) \) are the sets \( L(f, -\infty) = \{(-\infty)X\} \times S^1 \) and \( L(f, \infty) = \{(\infty)X\} \times S^1 \).

For each \( x \in X \), the \( x \)-fiber of \( Swl(f) \) is the set
\[ \mathcal{F}(f, x) = \{(tx, e^{2\pi it} f(x)) : t \in [0, \infty)\}, \]
and the \( x \)-fiber of \( DblSwl(f) \) is the set
\[ Dbl\mathcal{F}(f, x) = \{(tx, e^{2\pi it} f(x)) : t \in (-\infty, \infty)\}. \]

We now state the two main results of [1].

**Theorem 1.** If \( X \) is a compact metric space and \( f, g : X \rightarrow S^1 \) are homotopic maps, then there is a homeomorphism \( \Omega : Swl(f) \rightarrow Swl(g) \) with the following properties.
1) \( \Omega \) is fiber-preserving: for every \( x \in X \), \( \Omega(\mathcal{F}(f, x)) = \mathcal{F}(g, x) \).
2) \( \Omega \) fixes the \( \infty \)-level: \( \Omega|\mathcal{F}(f, \infty) = \text{id} \).

Compare Theorem 1 to the fact that, in general, there is no homeomorphism between mapping cylinders of homotopic maps.

Note that Theorem 1 holds with \( S^1 \) replaced by any space homeomorphic to \( S^1 \).

Also note that the conclusions of the theorem imply that for any subset \( Y \) of \( X \),
\[ \Omega(Swl(f|Y)) = Swl(g|Y), \Omega(\mathcal{F}(f|Y, 0)) = \mathcal{F}(g|Y, 0) \text{ and } \Omega(\mathcal{F}(f, \infty)) = \mathcal{F}(g, \infty). \]

**Theorem 2.** If \( Y \) is a compact metric space, \( n \) is a non-zero integer and \( f : X \rightarrow S^1 \) is the map satisfying \( f(y, z) = z^n \), then there is a homeomorphism \( \Theta : Cyl(f) \rightarrow Swl(f) \) with the following properties.
1) \( \Theta \) is level-preserving: for every \( t \in [0, \infty] \), \( \Theta(L(f, t)) = \mathcal{L}(f, t) \).
2) \( \Theta \) fixes the \( 0 \)- and \( \infty \)-levels: \( \Theta|L(f, 0) \cup L(f, \infty) = \text{id} \).
Note that Theorem 2 holds for any map \( f : X \to J \) for which there is a commutative diagram:

\[
\begin{array}{ccc}
Y \times S^1 & \xrightarrow{\phi} & S^1 \\
\downarrow & \quad & \downarrow \\
X & \xrightarrow{f} & J
\end{array}
\]

\(\text{homeomorphism}\)

We will need one other elementary fact:

**Proposition 3.** If \( X \) is a compact metric space, \( f : X \to S^1 \) a map and \( \lambda : [0, 1) \to [0, \infty) \) is a homeomorphism, then there is a homeomorphism \( \Lambda : X \times [0, 1) \to Swl(f) - \mathcal{L}(f, \infty) \) such that for every \( t \in [0, 1) \), \( \Lambda(X \times \{t\}) = \mathcal{L}(f, \lambda(t)) \).

Proofs of Theorems 1 and 2 are found in [1]. Because we have modified the statements of these theorems for their use in this paper, we will provide outlines of these proofs as well as a proof of Proposition 3 in section 4 below.

3. THE PROOF OF THE THEOREM

Let \( J \) be a knot. We will prove that \( J \) is semi-isotopic to an unknot.

**Step 1:** There is an unknotted solid torus \( T \) in \( S^3 \) such that \( J \subset int(T) \) and the inclusion \( J \hookrightarrow T \) is a homotopy equivalence.

We can assume \( J \subset S^3 \setminus \{\infty\} \). Let \( p \) and \( q \) be distinct points of \( J \). Let \( \mathcal{Y} \) be an uncountable family of parallel planes in \( \mathbb{R}^3 \) that separate \( p \) from \( q \). Since \( J \) has a countable dense subset, \( J \) does not contain an uncountable pairwise disjoint collection of non-empty open sets. Hence, there is a \( V \in \mathcal{Y} \) such that \( J \cap V \) contains no non-empty open subset of \( J \). It follows that \( J \cap V \) is a totally disconnected subset of \( V \). Let \( J_1 \) and \( J_2 \) be arcs such that \( J_1 \cup J_2 = J \) and \( J_1 \cap J_2 = \partial J_1 = \partial J_2 = \{p, q\} \). Then \( J_1 \cap V \) and \( J_2 \cap V \) are disjoint compact totally disconnected subsets of \( V \). It follows that there is a disk \( D \) in \( V \) such that \( J_1 \cap V \subset int(D) \) and \( (J_2 \cap V) \cap D = \emptyset \). To see this, let \( D' \) be a disk in \( V \), let \( A_1 \) be a subset of \( int(D') \) that is homeomorphic to \( J_1 \cap V \), and let \( A_2 \) be a subset of \( V - D' \) that is homeomorphic to \( J_2 \cap V \). Then according to Theorem 13.7 on pages 93-95 of [8], there is a homeomorphism \( \phi : V \to V \) such that \( \phi(A_i) = J_i \cap V \) for \( i = 1, 2 \). Simply let \( D = \phi(D') \).

We can assume \( D \) is a piecewise linear disk in \( V \). Let \( U \) be a regular neighborhood of \( \partial D \) in \( S^3 \) such that \( U \cap V \) is a regular neighborhood of \( \partial D \) in \( V \) and \( U \cap J = \emptyset \), and let \( T = cl(S^3 - U) \). Then \( U \) and, hence, \( T \) are unknotted solid tori in \( S^3 \), and \( J \subset int(T) \). Let \( V = V \cup \{\infty\} \), and let \( E_1 \) and \( E_2 \) be the components of \( cl(V - U) \) such that \( E_1 \subset int(D) \) and \( \infty \in E_2 \). Then \( E_1 \) and \( E_2 \) are disjoint meridional disks of \( T \) such that \( T \cap V = E_1 \cup E_2 \), \( J_1 \cap V \subset int(E_1) \) and \( J_2 \cap V \subset int(E_2) \). Thus, \( J_1 \cap E_2 = \emptyset = J_2 \cap E_1 \).

Clearly, there is a simple closed curve \( K \subset int(T) \) such that the inclusion \( K \hookrightarrow T \) is a homotopy equivalence, \( p \) and \( q \in K \), \( K_1 \) and \( K_2 \) are arcs such that \( K_1 \cup K_2 = K \), \( K_1 \cap K_2 = \partial K_1 = \partial K_2 = \{p, q\} \), and \( K_1 \cap E_2 = \emptyset = K_2 \cap E_1 \). At this point we will stretch conventional terminology slightly by saying that for subsets \( Z \subset Y \) and \( Z \subset Y' \) of a space \( X \), the inclusions \( Y \hookrightarrow X \) and \( Y' \hookrightarrow X \) are homotopic in \( X \) rel \( Z \) if there is a homotopy \( \xi : Y \times I \to X \) such that \( \xi_0 = id_Y \), \( \xi_1 : Y \to Y' \) is a homeomorphism and \( \xi_t|Z = id_Z \) for every \( t \in I \). Since \( T \) is contractible, then the inclusions \( J_1 \hookrightarrow T - E_2 \) and \( K_1 \hookrightarrow T - E_2 \) are homotopic rel \( \{p, q\} \). Similarly,
since $T - E_1$ is contractible, the inclusions $J_2 \hookrightarrow T - E_1$ and $K_2 \hookrightarrow T - E_1$ are homotopic rel $\{p, q\}$. Therefore, the inclusions $J \hookrightarrow T$ and $K \hookrightarrow T$ are homotopic. It follows that the inclusion $J \hookrightarrow T$ is a homotopy equivalence.

**Step 2:** Consider a homeomorphism $\psi : B^2 \times S^1 \rightarrow T$, let $o \in \text{int}(B^2)$ and let $K = \psi([o] \times S^1)$. Define the homeomorphism $\psi_0 : S^1 \rightarrow K$ by $\psi_0(z) = \psi(o, z)$. Define the map $\tau : B^2 \times S^1 \rightarrow S^1$ by $\tau(y, z) = z$, and define the map $\pi : T \rightarrow K$ by $\pi = \psi_0 \circ \tau \circ \psi^{-1}$. Then we have a commutative diagram in which the vertical arrows are homeomorphisms:

$$
\begin{array}{ccc}
B^2 \times S^1 & \xrightarrow{\tau} & S^1 \\
\downarrow{\psi} & & \downarrow{\psi_0} \\
T & \xrightarrow{\pi} & K
\end{array}
$$

We can now invoke Theorem 2 to obtain a level-preserving homeomorphism $\Theta : Cyl(\pi) \rightarrow Swl(\pi)$ that fixes the 0- and $\infty$-levels. Also observe that since $\tau : B^2 \times S^1 \rightarrow S^1$ is a homotopy equivalence, then so is $\pi : T \rightarrow K$.

**Step 3:** Let $\lambda : I \rightarrow [0, \infty]$ be an order-preserving homeomorphism. Clearly, there is an embedding $j : Cyl(\pi) \rightarrow S^3 \times I$ with the following properties.
1) $j$ maps $L(\pi, 0)$ “identically” onto $T \times \{0\}$; i.e., $j(0x, \pi(x)) = (x, 0)$ for every $x \in T$.
2) For every $t \in I$, $j(L(\pi, \lambda(t))) \subset S^3 \times \{t\}$, and $j(L(\pi, \lambda(t)))$ is a copy of $T$ that is “squeezed” toward $K$.
3) $j$ maps $L(\pi, \infty)$ “identically” onto $K \times \{1\}$; i.e., for every $y \in K$, $j(\{\infty T\} \times \{y\}) = \{(y, 1)\}$.
Thus, $j \circ \Theta^{-1} : Swl(\pi) \rightarrow S^3 \times I$ is an embedding with the following properties.
1) $j \circ \Theta^{-1}$ maps $\mathcal{L}(\pi, 0)$ “identically” onto $T \times \{0\}$; i.e., $j \circ \Theta^{-1}(0x, \pi(x)) = (x, 0)$ for every $x \in T$.
2) For every $t \in I$, $j \circ \Theta^{-1}(\mathcal{L}(\pi, \lambda(t))) \subset S^3 \times \{t\}$.
3) $j \circ \Theta^{-1}$ maps $\mathcal{L}(\pi, \infty)$ “identically” onto $K \times \{1\}$; i.e., for every $y \in K$, $j \circ \Theta^{-1}(\infty T \times \{y\}) = \{(y, 1)\}$.

**Step 4:** Since $\pi|J \rightarrow K$ is the composition of the inclusion $J \hookrightarrow T$ and $\pi : T \rightarrow K$, both of which are homotopy equivalences, then $\pi|J \rightarrow K$ is a homotopy equivalence. Hence, $\pi|J \rightarrow K$ is homotopic to an homeomorphism $\chi : J \rightarrow K$. Therefore, Theorem 1 provides a 0- and $\infty$-level preserving homeomorphism $\omega : Swl(\pi|J) \rightarrow Swl(\chi).$ Thus, $j \circ \Theta^{-1} : Swl(\chi) \rightarrow S^3 \times I$ is an embedding that maps $\mathcal{L}(\chi, 0)$ into $S^3 \times \{0\}$ and maps $\mathcal{L}(\chi, 1)$ onto $K \times \{1\} \subset S^3 \times \{1\}$.

**Step 5:** Since $\chi : J \rightarrow K$ is a homeomorphism, then Theorem 2 provides a level-preserving homeomorphism from $Cyl(\chi)$ to $Swl(\chi)$. Also, since $\chi : J \rightarrow K$ is a homeomorphism, then $Cyl(\chi)$ is an annulus with boundary $L(\chi, 0) \cup L(\chi, 1)$. Therefore, $Swl(\chi)$ is an annulus with boundary $\mathcal{L}(\chi, 0) \cup \mathcal{L}(\chi, 1)$. Since $\omega : Swl(\pi|J) \rightarrow Swl(\chi)$ is a 0- and $\infty$-level preserving homeomorphism, then if follows that $Swl(\pi|J)$ is an annulus with boundary $\mathcal{L}(\pi|J, 0) \cup \mathcal{L}(\pi|J, \infty)$.

Let $A = j \circ \Theta^{-1}(Swl(\pi|J)).$ Then $A$ is an annulus in $S^3 \times I$ with boundary $j \circ \Theta^{-1}(\mathcal{L}(\pi|J, 0) \cup \mathcal{L}(\pi|J, \infty)) = (J \times \{0\}) \cup (K \times \{1\}).$

**Step 6:** Recall that for every $t \in I$, $j \circ \Theta^{-1}(\mathcal{L}(\pi|J, \lambda(t))) \subset S^3 \times \{t\}$. Proposition 3 implies that there is a homeomorphism $\Lambda : J \times [0, 1] \rightarrow Swl(\pi|J) - \mathcal{L}(\pi|J, \infty)$ such that $\Lambda(J \times \{t\}) = \mathcal{L}(\pi|J, \lambda(t))$ for every $t \in [0, 1)$. Therefore, $j \circ \Theta^{-1} \circ \Lambda : J \times [0, 1) \rightarrow A - (K \times \{1\})$ is a homeomorphism such that $j \circ \Theta^{-1} \circ \Lambda(J \times \{t\}) \subset S^3 \times \{t\}$ for every $t \in [0, 1)$. ■
Proof of Theorem 1. Let \( X \) be a compact metric space and let \( f, g : X \to S^1 \) be homotopic maps.

Step 1: There is a homeomorphism \( \Phi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1 \) that carries \( DblSwl(f) \) onto \( DblSwl(g) \).

For each \( x \in X \), observe that \( DblF(f,x) \cup DblF(g,x) \) is a double helix in the cylinder \( (\mathbb{R} x) \times S^1 \subset (\Sigma X) \times S^1 \). We will construct a map \( \sigma : X \to \mathbb{R} \) such that for each \( x \in X \), a first-coordinate shift of \( (\mathbb{R} x) \times S^1 \) through a distance of \(-\sigma(x)\) slides \( DblF(f,x) \) onto \( DblF(g,x) \). (See Figure 3.)

Begin the construction of \( \sigma \) by choosing a homotopy \( h : X \times I \to S^1 \) such that \( h_0 = f \) and \( h_1 = g \). Using complex division in \( S^1 \), define \( k : X \times I \to S^1 \) by \( k(x,t) = h(x,t)/h(x,0) \). Then for every \( x \in X \), \( k(x,0) = 1 \) and \( k(x,1)f(x) = g(x) \).

Let \( e : \mathbb{R} \to S^1 \) be the exponential covering map \( e(t) = e^{2\pi i t} \). The homotopy \( k : X \times I \to S^1 \) lifts to a homotopy \( \tilde{k} : X \times I \to \mathbb{R} \) such that \( e \circ \tilde{k} = k \) and \( \tilde{k}(x,0) = 0 \) for every \( x \in X \). Define \( \sigma : X \to \mathbb{R} \) by \( \sigma(x) = \tilde{k}(x,1) \). Therefore, for every \( x \in X \),

\[
e^{2\pi i \sigma(x)}f(x) = e \circ \sigma(x)f(x) = e \circ \tilde{k}(x,1)f(x) = k(x,1)f(x) = g(x).
\]

Since \( X \) is compact, there is a \( b > 0 \) such that \( \sigma(X) \subset (-b,b) \).

Define \( \Phi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1 \) by

\[
\begin{align*}
\Phi(tx, z) &= ((t - \sigma(x))x, z) \text{ for } (t,x) \in \mathbb{R} \times X \text{ and } z \in S^1, \text{ and} \\
\Phi &= id \text{ on } \{(-\infty)X, \infty X\} \times S^1.
\end{align*}
\]

\( \Phi \) is the first-coordinate shift of \( (\mathbb{R} x) \times S^1 \) through a distance of \(-\sigma(x)\) mentioned above. It remains to show that \( \Phi \) is a homeomorphism of \( (\Sigma X) \times S^1 \) which slides \( DblF(f,x) \) onto \( DblF(g,x) \) for each \( x \in X \).
Φ is continuous at points of \{(-\infty, \infty) \times S^1\} because for every \(x \in X\) and every \(z \in S^1\), \(\Phi((t, \infty]\times \{z\}) \subset ((t - b, \infty] \times \{z\})\) and \(\Phi((-\infty, t + b) \times \{z\}) \subset ((-\infty, t + b) \times \{z\})\).

Φ is a homeomorphism because its inverse \(\overline{\Phi}\) can be defined explicitly by the equations

\[
\begin{align*}
\overline{\Phi}(tx, z) &= ((t + \sigma(x))x, z) \text{ for } (t, x) \in \mathbb{R} \times X \text{ and } z \in S^1, \\
\overline{\Phi} &= \text{id on } \{(-\infty, \infty) \times S^1\}.
\end{align*}
\]

The verification that \(\overline{\Phi} \circ \Phi = \text{id} = \Phi \circ \overline{\Phi}\) is straightforward.

For each \(x \in X\), a typical point of \(Dbl\, F(f, x)\) has the form \((tx, e^{2\pi it} f(x))\), and

\[
\begin{align*}
\Phi((tx, e^{2\pi it} f(x))) &= ((t - \sigma(x))x, e^{2\pi it} f(x)) = \\
((t - \sigma(x))x, e^{2\pi (t - \sigma(x))} e^{2\pi it} f(x)) &= \\
((t - \sigma(x))x, e^{2\pi i(t - \sigma(x))} g(x)) \in Dbl\, F(f, x).
\end{align*}
\]

Hence, \(\Phi(Dbl\, F(f, x)) \subset Dbl\, F(g, x)\). A similar calculation shows

\[\overline{\Phi}(Dbl\, F(g, x)) \subset Dbl\, F(f, x)\]

for every \(x \in X\). Thus,

\[Dbl\, F(g, x) \subset \Phi(Dbl\, F(f, x))\]

for every \(x \in X\). We conclude that

\[\Phi(Dbl\, F(f, x)) = Dbl\, F(g, x)\]

for every \(x \in X\).

We have shown that \(\Phi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1\) is a homeomorphism with the property that \(\Phi|Dbl\, Swl(f) : Dbl\, Swl(f) \to Dbl\, Swl(g)\) is a fiber-preserving homeomorphism that fixes the \((\infty)\)- and \((0)\)-levels. Unfortunately, we can’t conclude that \(\Phi(Swl(f))\) and \(Swl(g)\) are equal subsets of \(Dbl\, Swl(g)\). This completes Step 1.

**Step 2:** There is a homeomorphism \(\Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1\) that maps \(Dbl\, Swl(g)\) onto itself and twists \(\Phi(Swl(f))\) onto \(Swl(g)\).

Observe that for every \(x \in X\), \(\Phi(\mathcal{F}(f, x))\) and \(\mathcal{F}(g, x)\) are contained in the helix \(Dbl\, F(g, x)\) which lies in the cylinder \((\mathbb{R}x) \times S^1\). For each \(x \in X\), \(\Psi\) will map this cylinder to itself with a screw motion that preserves \(Dbl\, F(g, x)\) and twists \(\Phi(\mathcal{F}(f, x))\) onto \(\mathcal{F}(g, x)\).

Recall that \(\sigma(x) \subset (-b, b)\). Hence, there is clearly a map \(\tau : \mathbb{R} \times X \to \mathbb{R}\) such that for each \(x \in X\), \(\tau|\mathbb{R} \times \{x\} : \mathbb{R} \times \{x\} \to \mathbb{R}\) is a homeomorphism such that \(\tau(-\sigma(x)) = 0\) and \(\tau(t, x) = t\) for \(t \in (-\infty, -b] \cup [b, \infty)\).

Define \(\Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1\) by

\[
\begin{align*}
\Psi(tx, z) &= ((\tau(t, x)x, e^{2\pi it(\tau(t, x) - t)} z) \text{ for } (t, x) \in \mathbb{R} \times X \text{ and } z \in S^1, \\
\Psi &= \text{id on } \{(-\infty, 0) \times \infty X} \times S^1\).
\end{align*}
\]

\(\Psi\) is continuous at points of \((-\infty)X, \infty X\) \times S^1\) because \(\Psi = \text{id}\) on \{\(tx : |t| \geq b\) and \(x \in X\)\} \times S^1.

\(\Psi\) is a homeomorphism because its inverse \(\overline{\Psi}\) can be defined explicitly as follows. First observe that there is a map \(\tau : \mathbb{R} \times X \to \mathbb{R}\) so that for every \(x \in X\), \(t \mapsto \tau(t, x) : \mathbb{R} \to \mathbb{R}\) is the inverse of the homeomorphism \(t \mapsto \tau(t, x) : \mathbb{R} \to \mathbb{R}\). Now
define \( \Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1 \) by

\[
\begin{align*}
\Psi(tx, z) &= ((\tau(t, x)x, e^{2\pi i(t, x)}z) \\
\Psi &= id \text{ on } \{(-\infty)X, \infty X\} \times S^1.
\end{align*}
\]

The verification that \( \Psi \circ \Psi = id = \Psi \circ \Psi \) is straightforward.

Let \( x \in X \). To prove that \( \Psi(DblF(g, x)) = DblF(g, x) \), note that a typical point of \( DblF(g, x) \) has the form \((tx, e^{2\pi i t}g(x))\), and

\[
\Psi((tx, e^{2\pi i t}g(x))) = (\tau(t, x)x, e^{2\pi i(t, x)}e^{2\pi i t}g(x)) = (\tau(t, x)x, e^{2\pi i(t, x)}g(x)) \in DblF(g, x).
\]

Therefore, \( \Psi(DblF(g, x)) \subset DblF(g, x) \). A similar calculation shows

\[
\overline{\Psi(DblF(g, x))} \subset DblF(g, x).
\]

Hence,

\[
DblF(g, x) \subset \Psi(DblF(g, x)).
\]

We conclude that

\[
\Psi(DblF(g, x)) = DblF(g, x).
\]

We have shown that \( \Psi : (\Sigma X) \times S^1 \to (\Sigma X) \times S^1 \) is a homeomorphism with the property that \( \Psi DblSwl(g) : DblSwl(g) \to DblSwl(g) \) is a fiber-preserving homeomorphism that fixes the \(-\infty\)- and \( \infty \)-levels.

Again let \( x \in X \). To prove that \( \Psi(\Phi(F(f, x))) = F(g, x) \), note that a typical point of \( \Phi(F(f, x)) \) has the form \((tx, e^{2\pi i t}g(x))\) where \( t \geq -\sigma(x) \), and a typical point of \( F(g, x) \) has the form \((tx, e^{2\pi i t}g(x))\) where \( t \geq 0 \). Also note that \( \tau \) maps \([-\sigma(x), \infty)\) onto \([0, \infty)\). Since

\[
\Psi((tx, e^{2\pi i t}g(x))) = (\tau(t, x)x, e^{2\pi i(t, x)}e^{2\pi i t}g(x)) = (\tau(t, x)x, e^{2\pi i(t, x)}g(x))
\]

then clearly \( \Psi(\Phi(F(f, x))) = F(g, x) \).

It follows that \( \Psi \circ \Phi(Swl(f)) = Swl(g) \), completing Step 2.

We conclude that \( \Psi \circ \Phi(Swl(f)) : Swl(f) \to Swl(g) \) is a fiber-preserving homeomorphism that fixes the \( \infty \)-level.

**Proof of Theorem 2.** Let \( Y \) be a compact metric space, \( n \) a non-zero integer and \( f : Y \times S^1 \to S^1 \) the map satisfying \( f(y, z) = z^n \). We will construct a homeomorphism \( \Theta : C(Y \times S^1) \times S^1 \to C(Y \times S^1) \times S^1 \) that carries \( Cy(f) \) onto \( Swl(f) \).

Define \( \zeta : [0, \infty) \to S^1 \) by \( \zeta(t) = e^{2\pi i t/n} \), and define \( \Theta : C(Y \times S^1) \times S^1 \to C(Y \times S^1) \times S^1 \) by

\[
\begin{align*}
\Theta(t(y, z), w) &= (t(y, \zeta(t)z), w) \text{ for } t \in [0, \infty), (y, z) \in Y \times S^1 \text{ and } w \in S^1, \\
\Theta &= id \text{ on } \{\infty(Y \times S^1)\} \times S^1.
\end{align*}
\]

\( \Theta \) is clearly level-preserving and fixes the \( 0 \)- and \( \infty \)-levels.

We argue that \( \Theta \) is continuous at points of \( \{\infty(Y \times S^1)\} \times S^1 \). For \( a > 0 \), let \( V_a = \bigcup_{t \in [a, \infty]} t(Y \times S^1) \); and for \( w \in S^1 \), let \( M_w \) be a basis for the topology on \( S^1 \) at \( w \). Then for each \( w \in S^1 \), \( \{V_a \times M : a > 0 \} \) and \( M \in M_w \) is a basis for the topology on \( C(Y \times S^1) \times S^1 \) at \( (\infty(Y \times S^1), w) \). Since for each \( a > 0 \) and each \( M \in M_w \), \( \Theta \) carries \( V_a \times M \) into itself, then \( \Theta \) is continuous at \( (\infty(Y \times S^1), w) \).

\( \Theta \) is a homeomorphism because its inverse \( \overline{\Theta} \) can be defined explicitly as follows.

First define the map \( \overline{\zeta} : [0, \infty) \to S^1 \) by \( \overline{\zeta}(t) = e^{2\pi i t/n} \). Then define \( \overline{\Theta} : C(Y \times}
\[ S^1 \times S^1 \to C(Y \times S^1) \times S^1 \] by
\[
\begin{cases}
\Theta(t(y, z), w) = (t(y, \zeta(t)z), w) & \text{for } t \in [0, \infty), (y, z) \in Y \times S^1 \text{ and } w \in S^1, \\
\Theta = id & \text{on } \{\infty \times Y \times S^1\} \times S^1
\end{cases}
\]

The verification that \( \overline{\Theta} \circ \Theta = id = \Theta \circ \overline{\Theta} \) is straightforward.

To prove that \( \Theta(Cyl(f)) = Swl(f) \), note that a typical point of \( Cyl(f) \) has the form \((t(x, z), f(x, z))\), a typical point of \( Swl(f) \) has the form \((t(x, z), e^{2\pi it}f(x, z))\), and
\[
\Theta(t(x, z), f(x, z)) = (t(x, \zeta(t)z), f(x, z)) = (t(x, \zeta(t)z), \zeta(t)) = (t(x, \zeta(t)z), e^{2\pi it}(\zeta(t)z)^n) = (t(x, \zeta(t)z), e^{2\pi it}f(x, \zeta(t)z)) \in Swl(f).
\]

Therefore, \( \Theta(Cyl(f)) \subset Swl(f) \). A similar calculation shows \( \overline{\Theta}(Swl(f)) \subset Cyl(f) \).

Hence, \( Swl(f) \subset \Theta(Cyl(f)) \). We conclude that \( \Theta(Cyl(f)) = Swl(f) \).

It follows that \( \Theta(Cyl(f)) : Cyl(f) \to Swl(f) \) is a level-preserving homeomorphism that fixes the 0- and \( \infty \)-levels.

**Proof of Proposition 3.** A compact metric space \( X \), a map \( f : X \to S^1 \) and a homeomorphism \( \Lambda : [0, 1] \to Swl(f) - \mathcal{L}(f, \infty) \) by \( \Lambda(x, t) = (\lambda(t)x, e^{2\pi it}f(x)) \). We show that \( \Lambda \) is a homeomorphism by exhibiting its inverse. Let \( q : X \times [0, \infty] \to CX \) denote the quotient map \( q(x, t) = tx \). Then \( q(X \times [0, \infty] : X \times [0, \infty] \to CX - \{\infty X\} \) is a homeomorphism; let \( r : CX - \{\infty X\} \to X \times [0, \infty] \) denote its inverse. Let \( p : (CX - \{\infty X\}) \times S^1 \to CX - \{\infty X\} \) denote projection. Define \( \tau : (CX - \{\infty X\}) \times S^1 \to X \times [0, 1] \) by \( \tau = (id_X \times \lambda^{-1}) \circ qr \). It is easily verified that \( \tau \circ \Lambda = id_{X \times [0, 1]} \) and \( \Lambda \circ \tau((Swl(f) - \mathcal{L}(f, \infty))) = id_{Swl(f) - \mathcal{L}(f, \infty)} \). Hence, \( \tau((Swl(f) - \mathcal{L}(f, \infty))) \) is the inverse of \( \Lambda \).  

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