THE INDEX OF BIHARMONIC MAPS IN SPHERES

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Abstract. Biharmonic maps are the critical points of the bienergy functional and generalise harmonic maps. We investigate the index of a class of biharmonic maps, derived from minimal Riemannian immersions into spheres. This study is motivated by three families of examples: the totally geodesic inclusion of spheres, the Veronese map and the Clifford torus.

1. Introduction

In their original 1964 paper [7], Eells and Sampson proposed an infinite dimensional Morse theory on the manifold of smooth maps between Riemannian manifolds. Though their main results concern harmonic maps, they also suggested other functionals. The interest encountered by harmonic maps, and to a lesser account by $p$-harmonic maps, has overshadowed the study of other possibilities, e.g. exponential harmonicity [5]. While the examples cited so far, are all first order functionals, one can investigate problems involving higher derivatives. A prime example of these, is the bienergy, not only for the role it plays in elasticity and hydrodynamics, but also because it can be seen as the next stage, should the theory of harmonic maps fail. Witness the case of the two-torus $\mathbb{T}^2$ and the two-sphere $\mathbb{S}^2$: Eells and Wood showed in [3], that there exists no harmonic map from $\mathbb{T}^2$ to $\mathbb{S}^2$ (whatever the metrics chosen) in the homotopy classes of Brower degrees $\pm 1$. But the situation is drastically different for biharmonicity. By Palais-Smale, in each and every homotopy class from a surface (or a three-manifold) to a compact manifold, there exists a biharmonic map (cf. [4]). While this biharmonic map could very well turn out to be also harmonic, this cannot happen to maps of degrees $\pm 1$ between $\mathbb{T}^2$ and $\mathbb{S}^2$.

Let $\phi : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds. We define its tension field to be: $\tau(\phi) = \text{trace} \nabla d\phi$, and, for a compact domain $K \subseteq M$, its bienergy:

$$E^2(\phi) = \frac{1}{2} \int_K |\tau(\phi)|^2 v_g.$$
The critical points of $E^2$, with respect to continuous deformations, are called \textit{biharmonic maps}. The Euler-Lagrange operator associated to $E^2$ is:

$$\tau^2(\phi) = -\Delta^\phi(\tau(\phi)) - \text{trace } R^N(d\phi, \tau(\phi))d\phi,$$

where

$$\Delta^\phi = -\text{trace}_g(\nabla^\phi)^2 = -\text{trace}_g \left( \nabla^\phi \nabla^\phi - \nabla^\phi_{\nabla^\phi} \right)$$

is the Laplacian on sections of the pull-back bundle $\phi^{-1}TN$ and $R^N$ is the curvature operator:

$$R^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z, \quad \forall X, Y, Z \in C(TN).$$

Hence, a map is biharmonic if and only if $\tau^2(\phi) = 0$.

It is interesting to observe that $\tau^2(\phi) = -J^\phi(\tau(\phi))$, where the Jacobi operator $J^\phi$ plays an important role in the theory of harmonic maps, since it gives the second variation of the energy functional $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$ at its critical points, the well-known \textit{harmonic maps}. Moreover, the Euler-Lagrange operator related to $E$ is precisely the tension field $\tau(\phi)$.

The Jacobi operator being linear, harmonic maps are trivially biharmonic, in fact, global minima of $E^2$. Besides, by means of a Bochner technique, Jiang established a partial converse:

\textbf{Theorem 1.1.} \cite{10} A biharmonic map from a compact domain into a non-positively curved manifold, is harmonic.

One can easily construct a non-harmonic biharmonic map, by choosing a third order polynomial mapping between Euclidean spaces, since, in this situation, the biharmonic operator is nothing but the Laplacian composed with itself.

However, starting from a compact manifold, one must choose positively curved targets, e.g. the sphere. Remembering that, on the one hand, mapping $S^n$ into $S^{n+1}$ as the equator yields a harmonic map, i.e. a biharmonic map of vanishing bienergy, and that, on the other hand, sending the whole of $S^n$ onto the north pole of $S^{n+1}$ also has zero bienergy, one can infer the existence of a non-harmonic biharmonic map from $S^n$ into $S^{n+1}$.

A more precise argument (cf. \cite{1}) shows that it actually is the embedding of $S^n(s_{\frac{1}{\sqrt{2}}})$ as the tropic, cancer or capricorn, of $S^{n+1}$.

In \cite{3}, Caddeo, Montaldo and Piu show that a biharmonic curve on a surface of non-positive Gaussian curvature is a geodesic, i.e. is harmonic, and give examples of non-harmonic biharmonic curves on spheres, ellipses, unduloids and nodoids. On the three-sphere, the only non-harmonic biharmonic curves are certain circles and helices, while $S^2(s_{\frac{1}{\sqrt{2}}})$ is the unique non-harmonic, that is non-minimal, biharmonic surface (\cite{11}). More generally, for the sphere $S^n$, only circles and helices are non-harmonic biharmonic curves (\cite{2}).

Since biharmonicity derives from a variational principle, one is naturally tempted to take the second variation of the functional $E^2$. While this was
done in general by Jiang [10], in light of the restrictions given by Theorem 1.1 on the curvature of the target space, we will restrict ourselves to maps into the unit sphere. In this configuration, the second variation formula for the bienergy was obtained by the second author:

**Theorem 1.2.** [13] Assume that \((M,g)\) is a compact manifold and \(\phi : (M,g) \to S^n\) is a biharmonic map. Take \(\{\phi_{s,t}\}_{s,t \in \mathbb{R}}\) a two parameter variation of \(\phi\) and set \(\frac{\partial \phi}{\partial s} \bigg|_{s=0} = V\) and \(\frac{\partial \phi}{\partial t} \bigg|_{t=0} = W\), then

\[
\frac{\partial^2 E^2(\phi_{s,t})}{\partial s \partial t} \bigg|_{(s,t) = (0,0)} = (I(V), W) = \int_M <I(V), W> v_g,
\]

where

\[
I(V) = \Delta^\phi(\Delta^\phi V) + \Delta^\phi(\text{trace } V, d\phi \cdot d\phi - |d\phi|^2 V)
+ 2 <d\tau(\phi), d\phi > V + |\tau(\phi)|^2 V
- 2 \text{trace } < V, d\tau(\phi) \cdot d\phi \cdot > - 2 \text{trace } < \tau(\phi), dV \cdot > d\phi - 
- \text{trace } < d\phi \cdot, \text{trace } < V, d\phi \cdot > d\phi \cdot > d\phi \cdot
- 2 |d\phi|^2 \text{trace } < d\phi \cdot, V > d\phi \cdot
+ 2 < dV, d\phi > \tau(\phi) - |d\phi|^2 \Delta^\phi V + |d\phi|^4 V.
\]

For this aspect of the theory, the two most important quantities to determine are the nullity and the index:

**Definition 1.1.** Let \(\phi : (M,g) \to S^n\) be a biharmonic map. The dimension of the vector space \(\{V \in C^0(\phi^{-1\text{T}S^n}) : I(V) = 0\}\) is called the **nullity** of \(\phi\).

The first case to look at, for nullity, is the identity map \(1 : S^n \to S^n\), for which the operator \(I(V)\) becomes:

\[
I(V) = \Delta(V) - 2(n-1)\Delta V + (n-1)^2 V,
\]

and the nullity is the dimension of the vector space \(\{V \in C^0(\text{T}S^n) : \Delta V = (n-1)V\}\).

The Hodge decomposition, the eigenvalues of the sphere and the dimension of the space of Killing vector fields, show:

**Theorem 1.3.** [13] The identity \(1 : S^n \to S^n\) is a biharmonic map of nullity equal to 6, if \(n = 2\), and \(\frac{n(n+1)}{2}\), when \(n \geq 3\).

**Definition 1.2.** Let \(\phi : (M,g) \to S^n\) be a biharmonic map. The **index** of \(\phi\) is the dimension of the largest vector space \(\{V \in C^0(\text{T}S^n) : (I(V), V) < 0\}\).

A map of index zero is called **stable**, otherwise, it is said to be **unstable**.

**Remark 1.1.** i) Any harmonic is clearly a stable biharmonic map.

ii) Equation (1) shows that \(I\) is a linear elliptic self-adjoint differential operator, and the Hilbert space of sections \(L^2(\phi^{-1\text{T}S^n})\) is the orthogonal sum of the finite dimensional eigenspaces of \(I\). Its spectrum consists of real numbers, bounded from below. Therefore the nullity and the index are finite.
The purpose of this article is the computation of the index of biharmonic maps constructed in the following manner:

**Theorem 1.4.** Let $M$ be a compact manifold, $\psi : M \to \mathbb{S}^n(1/\sqrt{2}) \times \{1/\sqrt{2}\}$ a nonconstant harmonic map and $i : \mathbb{S}^n(1/\sqrt{2}) \times \{1/\sqrt{2}\} \to \mathbb{S}^{n+1}$ the inclusion map.

Then $\phi = i \circ \psi : M \to \mathbb{S}^{n+1}$ is nonharmonic biharmonic if and only if $e(\psi)$ is constant.

**Proof.** Let $p \in M$ be an arbitrary point, $\{X_i\}_{i=1,\ldots,m}$ a geodesic frame at $p$, and $\eta$ the unit section of the normal bundle. Using the chain rule for the tension field:

$$\tau(\phi) = di(\tau(\psi)) + \text{trace} \, d^2 i(\psi, \psi)$$

$$= - \sum_{i=1}^m <d\psi(X_i), d\psi(X_i) - \eta$$

$$= -2e(\psi)\eta,$$

so $\phi$ is not harmonic. Here the second fundamental form of the inclusion is $\nabla di(X,Y) = B(X,Y) = -<X,Y> \eta$.

By a straightforward computation, we obtain, at the point $p$:

$$\frac{1}{2} \nabla^\phi_{X_i} \nabla^\phi_{X_i} \tau(\phi) = -(X_i X_i e(\psi))\eta + e(\psi) <d\psi(X_i), d\psi(X_i) - \eta$$

$$-2d\phi[(X_i e(\psi)) X_i] - e(\psi) \nabla d\psi(X_i, X_i),$$

and

$$-R^{\mathbb{S}^{n+1}}(d\phi(X_i), \tau(\phi))d\phi(X_i) = <d\psi(X_i), d\psi(X_i) - \tau(\phi).$$

Thus

$$\frac{1}{2} \tau^2(\phi) = (\Delta e(\psi))\eta - 2d\phi(\text{grad } e(\psi)).$$

Now, the map $\phi$ is biharmonic if and only if

$$\Delta e(\psi) = 0$$

and $d\phi(\text{grad } e(\psi)) = 0$,

and the theorem follows from the compactness of $M$. \hfill \Box

In the general case we choose to study, we consider a minimal (i.e. harmonic) isometric immersion of a compact Riemannian manifold into the unit sphere and adapt its radius, to obtain $\psi : (\mathbb{M}^m, g) \to \mathbb{S}^n(1/\sqrt{2})$, of constant energy density $\frac{m}{2}$, which is, then, composed with the tropical inclusion into $\mathbb{S}^{n+1}$, in order to attain, according to Theorem 1.4, a non-harmonic biharmonic map $\phi : (\mathbb{M}^m, g) \to \mathbb{S}^{n+1}$, of which we will analyse the stability.

This framework will remain throughout the article.

Our general modus operandi will be to look at the pull-back bundle $\phi^{-1} T\mathbb{S}^{n+1}$ as three cases:

1. The one-dimensional normal sub-bundle, spanned by the vector field $\eta$ tangent to $\mathbb{S}^{n+1}$ but normal to the tropic $\mathbb{S}^n(1/\sqrt{2}) \times \{1/\sqrt{2}\}$.
(2) The image of the tangent space of the domain, i.e. \( d\phi(TM) \), named the tangent sub-bundle.

(3) The sub-space consisting of vector fields tangent to the tropic but orthogonal to the image of the map and referred to as the vertical sub-bundle.

2. The normal sub-bundle

As the image of \( \phi \) actually lies in \( S^n \left( \sqrt{\frac{1}{2}} \right) \times \{ \sqrt{\frac{1}{2}} \} \), we start by examining vector fields of \( C(\phi^{-1}TS^{n+1}) \) of the form \( V = f\eta \), \( f \) being a function on \( M \) and \( \eta(p) = (\psi(p), -\sqrt{\frac{1}{2}}) \), \( \forall p \in M \), i.e. normal to the tropical hypersphere.

**Proposition 2.1.** Take \( V = f\eta \), \( f \in C^\infty(M) \), then:

\[
(I(V), V) = \int_M \left( |\Delta^\phi V - mV|^2 - 4m^2|V|^2 \right) v_g.
\]

Furthermore, if \( \Delta f = \lambda f \):

\[
(2) \quad (I(V), V) = (\lambda^2 + 4\lambda - 4m^2) \int_M f^2 v_g.
\]

**Proof.** When \( V = f\eta \), the various elements of Formula (1) become:

\[
\tau(\phi) = -m\eta, \quad \text{trace } <V, d\phi \cdot > d\phi \cdot = 0, \quad |d\phi|^2 = m,
\]

\[
< d\tau(\phi), d\phi > = -m^2, \quad |\tau(\phi)|^2 = m^2, \quad \text{trace } <V, d\tau(\phi) \cdot > d\phi \cdot = 0,
\]

\[
\text{trace } <\tau(\phi), dV \cdot > d\phi \cdot = -md\phi(\text{grad } f), \quad < \tau(\phi), V > \tau(\phi) = m^2V,
\]

\[
\text{trace } <d\phi \cdot , \Delta^\phi V > d\phi \cdot = (\Delta^\phi V)^T, \quad < dV, d\phi > \tau(\phi) = -m^2V,
\]

and:

\[
I(V) = \Delta^\phi (\Delta^\phi V) - 2m\Delta^\phi V - 3m^2V + 2md\phi(\text{grad } f) + (\Delta^\phi V)^T.
\]

As \( V \) is normal to \( S^n \left( \sqrt{\frac{1}{2}} \right) \times \{ \sqrt{\frac{1}{2}} \} \):

\[
< I(V), V >/= < \Delta^\phi (\Delta^\phi V), V >/= -2m < \Delta^\phi V, V >/= -3m^2|V|^2,
\]

and integrating by parts yields:

\[
(I(V), V) = \int_M \left( |\Delta^\phi V - mV|^2 - 4m^2|V|^2 \right) v_g.
\]

To conclude, we observe that:

\[
\Delta^\phi V = \Delta^\phi (f\eta) = (\Delta f)\eta - 2d\phi(\text{grad } f) + m f\eta.
\]

\( \square \)

**Remark 2.1.** A crucial observation, for this section, is that if \( V = f\eta \) and \( W = g\eta \), where \( f \) and \( g \) are eigenfunctions orthogonal one to the other, then \( (I(V), W) = 0 \). So counting the eigenfunctions, for which \( (I(f\eta), f\eta) \) is negative, together with their multiplicities, computes the index on the normal sub-bundle.
The principal significance of Proposition 2.1 is to reveal the influence of the small eigenvalues of $\Delta$ on the index of $\phi$:

**Corollary 2.1.** Let $V = f\eta$ and $\Delta f = \lambda f$, then $(I(V), V)$ is negative if and only if $\lambda \in [0, 2(\sqrt{m^2 + 1} - 1)]$.

**Example 2.1** (Generalised Veronese map). The well-known Veronese map $\mathbb{S}^2(\sqrt{2}) \to \mathbb{S}^4$ is generalised by the immersion of $\mathbb{S}^m(\sqrt{\frac{2m+1}{m}})$ into $\mathbb{S}^{m+p}$, $p = \frac{(m-1)(m+2)}{2}$ (cf. [1]).

We modify the radius, compose with the tropical inclusion and obtain a non-harmonic biharmonic map $\phi : \mathbb{S}^m(\sqrt{\frac{m+1}{m}}) \to \mathbb{S}^{m+p+1}$.

As $(\mathbb{S}^m(\sqrt{\frac{m+1}{m}}), g_{can})$ is isometric to $(\mathbb{S}^m, \frac{m+1}{m}g_{can})$, its eigenvalues are \[ \lambda_k = \frac{m}{m+1}k(m + k - 1) : k \in \mathbb{N} \], with multiplicity $\frac{(2k + m - 1)(k + m - 2)!}{k!(m-1)!}$ (2), and, whatever the dimension, $\lambda_1 < 2(\sqrt{m^2 + 1} - 1)$ while $\lambda_k > 2(\sqrt{m^2 + 1} - 1)$, $\forall k \geq 2$.

So, for normal vector fields, contributions to the index of $\phi$, come only from the first two eigenvalues, $\lambda_0$ and $\lambda_1$, making it greater or equal to $1 + (m + 1) = m + 2$.

Furthermore, the generalised Veronese map being quadratic, it also defines a biharmonic map from $\mathbb{R}P^m$, equipped with the metric $\frac{m+1}{m}g_{can}$, into $\mathbb{S}^{m+p+1}$. But the spectrum, in this case, is reduced to $\{ \tilde{\lambda}_k = \frac{m}{m+1}2k(m + 2k - 1) : k \in \mathbb{N} \}$ and, as $\tilde{\lambda}_1$ is larger than $2(\sqrt{m^2 + 1} - 1)$, only $\eta$ adds to the index.

**Example 2.2** (Generalised Clifford torus).

An extension to higher dimensions of the Clifford torus, is the minimal Riemannian immersion $\mathbb{S}^l(\frac{1}{2}) \times \mathbb{S}^l(\frac{1}{2}) \to \mathbb{S}^{m+1}(\frac{1}{2})$, $(m = 2l)$, which gives rise to a non-harmonic biharmonic map $\phi : \mathbb{S}^l(\frac{1}{2}) \times \mathbb{S}^l(\frac{1}{2}) \to \mathbb{S}^{m+2}$.

The eigenvalues of a product being the sum of the eigenvalues of each factor, the spectrum of the Laplacian on $\mathbb{S}^l(\frac{1}{2}) \times \mathbb{S}^l(\frac{1}{2})$ is $\{ \lambda_k = 4(p(l + p - 1) + q(l + q - 1)) : p, q \in \mathbb{N}, p + q = k \}$. So $\lambda_1 = 4l = 2m$ is greater than $2(\sqrt{m^2 + 1} - 1)$, and, among normal vector fields, only $\eta$ is counted in the index.

Since the spectrum of the Laplacian invariably contains the eigenvalue 0:

**Theorem 2.1.** Let $(M^m, g)$ be a compact Riemannian manifold.

A biharmonic map $\phi : (M^m, g) \to \mathbb{S}^{n+1}$, obtained as the composition of a minimal isometric immersion $\psi : (M^m, g) \to \mathbb{S}^n(\frac{1}{\sqrt{2}})$ and the tropical inclusion $\mathbb{S}^n(\frac{1}{\sqrt{2}}) \times \{ \frac{1}{\sqrt{2}} \} \to \mathbb{S}^{n+1}$, has a strictly positive index, i.e. is unstable.

**Remark 2.2.** This result is clearly reminiscent of the instability of harmonic maps into the sphere $[11]$.

To further refine Corollary 2.1 one can attempt to control the value of the first non-zero eigenvalue, $\lambda_1$, through geometrical constraints. Before anything else, note the upper bound $2m$ on $\lambda_1$, imposed by the hypotheses...
on \( \psi \). Indeed, immersing \( S^n(\frac{1}{\sqrt{2}}) \xrightarrow{j} \mathbb{R}^{n+1} \) makes each component of the map \( j \circ \psi : M \rightarrow \mathbb{R}^{n+1} \) an eigenfunction of the Laplacian on \((M,g)\), of eigenvalue \(2m\), hence the bound on \( \lambda_1 \).

Given that many examples are built from spheres, a helpful condition is inspired by Einstein manifolds:

**Proposition 2.2.** Let \( \psi : (M,g) \rightarrow S^n(\frac{1}{\sqrt{2}}) \) be a minimal Riemannian immersion and \( \phi : (M,g) \rightarrow S^{n+1} \) the biharmonic map constructed above. Assume that \( \text{Ricci}^M \geq \kappa g, \) \( (\kappa > 0) \) and let \( \kappa(m) = \frac{2(m-1)}{m} \left( \sqrt{m^2 + 1} - 1 \right) \).

If \( \kappa \geq \kappa(m) \), then the contribution of normal vector fields to the index of \( \phi \) is 1.

**Proof.** All hinges on a theorem of Lichnerowicz ([12]), stating that \( \text{Ricci}^M \geq \kappa g \) implies \( \lambda_1 \geq \frac{m}{m-1} \kappa \).

By Corollary 2.1 we only need to ensure that \( \frac{m}{m-1} \kappa \geq 2(\sqrt{m^2 + 1} - 1) \). \( \Box \)

**Example 2.3** (Totally geodesic inclusion).

A very simple example of a minimal isometric immersion into the sphere, is the totally geodesic inclusion of a lower dimension sphere in the equator, \( S^m(\frac{1}{\sqrt{2}}) \rightarrow S^n(\frac{1}{\sqrt{2}}) \), \((m \leq n)\), producing a non-harmonic biharmonic map \( \phi : S^m(\frac{1}{\sqrt{2}}) \rightarrow S^{n+1} \).

The sphere \( S^m(\frac{1}{\sqrt{2}}) \) being isometric to \( (S^m, \frac{1}{2}g_{can}) \), we can apply the above proposition for \( \kappa = 2(m-1) \), and deduce that solely the vector field \( \eta \) will contribute to the index of the biharmonic map \( \phi \) constructed from this totally geodesic inclusion.

**Remark 2.3.** Although the projective space of Example 2.1 and the product of spheres of Example 2.2 are Einstein manifolds, and therefore admit a constant \( \kappa \) satisfying \( \text{Ricci} \geq \kappa g \), in both instances \( \kappa < \kappa(m) \).

### 3. The Tangent Sub-Bundle

Vector fields of the tangent sub-bundle are in the image of \( TM \) by \( d\phi \) and can be written \( V = d\phi(X), X \in C(TM) \).

**Proposition 3.1.** Let \( V = d\phi(X) \) be a tangential vector field. Then:

\[
(I(V), V) = \int_M \left( |\Delta \phi V + (1 - m)V|^2 - m^2|V|^2 \right) v_g
\]

**Proof.** As \( \phi \) is an isometric immersion and \( V \) is in \( d\phi(TM) \), and therefore, normal to \( \eta \) and \( \tau(\phi) \), the components of Equation (II) are:

\[
\tau(\phi) = -m\eta, \quad \text{trace} \ < V, d\phi > d\phi = V, \quad |d\phi|^2 = m,
\]

\[
< d\tau(\phi), d\phi > = -m^2, \quad |\tau(\phi)|^2 = m^2, \quad \text{trace} \ < V, d\tau(\phi) > d\phi = -mV,
\]

\[
\text{trace} \ < \tau(\phi), dV > d\phi = mV, \quad < \tau(\phi), V > \tau(\phi) = 0, \quad \text{trace} \ < d\phi, \Delta \phi V > d\phi = (\Delta \phi V)^T,
\]

\[
\text{trace} \ < d\phi, \text{trace} < V, d\phi > d\phi > d\phi = V, \quad < dV, d\phi > = \text{div} X,
\]
consequently:
\[ I(V) = \Delta^\phi(\Delta^\phi V) + (1 - 2m)\Delta^\phi V + (\Delta^\phi V)^T + (1 - 2m)V - 2m(\text{div } X)\eta.\]
Since \( V \) is normal to \( \eta \) and tangent to the image of \( M \):
\[ <I(V), V> = <\Delta^\phi(\Delta^\phi V), V> + 2(1 - m) <\Delta^\phi V, V> + (1 - 2m)|V|^2,\]
and after an integration by parts:
\[ (I(V), V) = \int_M \left( |\Delta^\phi V| - (1 - m)|V|^2 - m^2|V|^2 \right) v_g. \]

Exploiting Formula (3) requires a kind of Bochner formula linking the Laplacian of the pull-back bundle to the Hodge-de Rham Laplacian on vector fields tangent to \( M \).

**Proposition 3.2.** Let \( X \) be a vector field tangent to \( M \) and \( V = d\phi(X) \) a vector field of \( \phi^{-1}TS^{m+1} \). Then:
\[ (I(V), V) = \int_M \left( 4|\text{trace } \nabla d\psi(\nabla X, .)|^2 + 4(\text{div } X)^2 \right. \]
\[ \left. + |\Delta H(X) - 2\text{Ricci}(X) + mX|^2 - m^2|X|^2 \right) v_g\]

**Proof.** Choosing a geodesic frame \( \{X_i\}_{i=1,...,m} \) around \( p \in M \), means that, at \( p \), \( \nabla X_i X_j = 0 \), \( \forall i, j = 1, \ldots, m \), and:
\[ \Delta^\phi V = -\sum_{i=1}^m \nabla^\phi_{X_i} \nabla^\phi_{X_i} V. \]
Moreover, the second fundamental form of the tropical injection \( \iota \) is \( B(X, Y) = -<X, Y> \eta \), so:
\[ \nabla^\phi_{X_i} V = \nabla^\psi_{X_i} d\psi(X) - <X, X> \eta. \]
Composing the preceding expressions implies:
\[ \nabla^\phi_{X_i} \nabla^\phi_{X_i} V = \nabla^\phi_{X_i} \nabla^\psi_{X_i} d\psi(X) - <X, \nabla X, X > \eta - <X, X> d\phi(X_i) \]
\[ = \nabla d\iota(d\psi(X_i), \nabla^\psi_{X_i} d\psi(X)) + \nabla^\psi_{X_i} \nabla^\psi_{X_i} d\psi(X) \]
\[ - <X, \nabla X, X> \eta <X, X> d\phi(X_i) \]
\[ = \nabla^\psi_{X_i} \nabla^\psi_{X_i} d\psi(X) - <d\psi(X_i), \nabla X_i X> d\psi(X_i) + d\psi(\nabla X_i X) > \eta \]
\[ - <X, \nabla X, X> \eta <X, X> d\phi(X_i) \]
\[ = \nabla^\psi_{X_i} \nabla^\psi_{X_i} d\psi(X) - 2<X, \nabla X, X> \eta <X, X> d\phi(X_i) \]
We conclude that:
\[ \Delta^\phi V = \Delta^\psi V + 2(\text{div } X)\eta + V. \]
The second half of the proof consists of putting to use the harmonicity of $\psi$ and the curvature tensor of $M$, to express $\Delta^\psi V$ in terms of $\nabla d\psi$:

$$\Delta^\psi V = -\sum_{i=1}^{m} \nabla_{X_i}^\psi \nabla_{X_i}^\psi V$$

$$= -\sum_{i=1}^{m} \left( \nabla_{X_i}^\psi [\nabla d\psi(X_i, X)] + \nabla d\psi(X_i, \nabla_X X) + d\psi(\nabla_X \nabla_X X) \right),$$

because $\nabla_{X_i}^\psi V = \nabla d\psi(X_i, X) + d\psi(\nabla_X X)$.

But:

$$\nabla_{X_i}^\psi [\nabla d\psi(X_i, X)] = \nabla_{X_i}^\psi [\nabla d\psi(X_i, X_i)]$$

$$= \nabla_{X_i}^\psi \nabla_{X_i}^\psi d\psi(X_i) - \nabla_{X_i}^\psi d\psi(\nabla_X X_i)$$

$$= \nabla_{X_i}^\psi \nabla_{X_i}^\psi d\psi(X_i) + \nabla_{[X_i, X_i]}^\psi d\psi(X_i) + R^\psi(X_i, X) d\psi(X_i)$$

$$= \nabla_{X_i}^\psi \nabla_{X_i}^\psi d\psi(X_i) + \nabla_{[X_i, X_i]}^\psi d\psi(X_i) - d\psi(\nabla_X \nabla_X X_i) - d\psi(\nabla_{[X_i, X_i]} X_i) + R(X_i, X) X_i$$

$$= \nabla_{X_i}^\psi \nabla_X^\psi d\psi(X_i) + \nabla_{[X_i, X_i]}^\psi d\psi(X_i) + 2\{<X_i, X > d\psi(X_i) - d\psi(X)\}$$

$$- \nabla_{X_i}^\psi d\psi(\nabla_X X_i) + \nabla d\psi(X, \nabla_X X_i) - d\psi(\nabla_{[X_i, X_i]} X_i) - d\psi(R(X_i, X) X_i)$$

$$= \nabla_{X_i}^\psi \nabla d\psi(X_i, X_i) + \nabla d\psi([X_i, X], X_i) + 2 \left( <X_i, X > d\psi(X_i) - d\psi(X) \right) - d\psi(R(X_i, X) X_i).$$

Since, $[X_i, X] = \nabla_{X_i} X$ at $p$:

$$\Delta^\psi V = -2 \sum_{i=1}^{m} \nabla d\psi(\nabla_X X_i, X_i) + d\psi \left( 2(m - 1) X - \text{trace} \nabla^2 X - \text{Ricci}(X) \right)$$

$$= -2 \sum_{i=1}^{m} \nabla d\psi(\nabla_X X_i, X_i) + d\psi \left( 2(m - 1) X + \Delta_H(X) - 2 \text{Ricci}(X) \right).$$

(5)

The proposition, then, follows from Equations (3), (11) and (5). □

**Theorem 3.1.** Let $(M^n, g)$ $(m \geq 2)$ be a compact Riemannian manifold and $\phi = i \circ \psi : M \to S^{n+1}$ a biharmonic map, forged as the composition of a minimal isometric immersion $\psi$ and the tropical inclusion $i$.

Let $V$ be a vector field of the pull-back bundle $\phi^{-1}TS^{n+1}$, such that $V = d\phi(X), X \in C(TM)$.

If $\dim M \leq 4$ or $\text{div}(X) = 0$, then $(I(V), V) \geq 0$.

**Proof.** Let $V = d\phi(X), X \in C(TM)$, via the Yano formula (13):

$$\int_M (\text{div} X)^2 - <\text{trace} \nabla^2 X + \text{Ricci} (X), X > v_g = \int_M \frac{1}{2} |L_X g|^2 v_g,$$
we know that:
\[(6)\]
\[(I(V), V) \geq \int_M \left( \left( \text{trace } \nabla^2 X + \text{Ricci}(X) \right)^2 + m |L_X g|^2 + 2(2-m)(\text{div } X)^2 \right) v_g.\]

Recall that for \(X\) tangent to \((M, g)\):
\[|L_X g|^2 \geq \frac{4}{m}(\text{div } X)^2,\]
since for a local orthonormal frame field \(\{X_i\}_{i=1, \ldots, m}\) on \(M\):
\[
|L_X g|^2 = \sum_{i,j=1}^m ((L_X g)(X_i, X_j))^2 = \sum_{i,j=1}^m \left( < \nabla_{X_i} X, X_j > + < \nabla_{X_j} X, X_i > \right)^2 \\
\geq 4 \sum_{i=1}^m ( < \nabla_{X_i} X, X_i > )^2,
\]
and by the Cauchy inequality:
\[|L_X g|^2 \geq \frac{4}{m} \left( \sum_{i=1}^m < \nabla_{X_i} X, X_i > \right)^2 = \frac{4}{m}(\text{div } X)^2.\]

Substituting in (6), we reach:
\[(I(V), V) \geq 2(4-m) \int_M (\text{div } X)^2 v_g.\]

\[\square\]

**Remark 3.1.** Using the fact that for a Killing vector field \(X\):
\[
\sum_{i=1}^m \nabla d\psi(\nabla_{X_i} X, X_i) = 0,
\]
and trace \(\nabla^2 X + \text{Ricci } X = 0\), one can also show that \(I(V) = 0\). Moreover, the converse also holds, if \(m \leq 4\) or \(\text{div } X = 0\), since, in these cases, \(I(V) = 0\) implies \(L_X g = 0\), by Equation (6), i.e. \(X\) is a Killing vector field.

**Corollary 3.1.** The nullity of the biharmonic map \(\phi\) is bounded from below by the dimension of \(\text{Isom}(M, g)\).

We can complement Theorem 3.1 when the first eigenvalue of the Laplacian is large enough:

**Corollary 3.2.** Let \(\phi : M^m \to S^{n+1}\) be a biharmonic map, fashioned as before, and \(\lambda_1\) the first non-zero eigenvalue of the Laplacian on \((M, g)\).
If \(\lambda_1 \geq \frac{m^2}{4}\) then \((I(V), V) \geq 0\) for all vector fields \(V \in C(\phi^{-1} T S^{n+1})\) such that \(V = d\phi(\text{grad } f)\), \(\Delta f = \lambda f\).
Proof. Let \( V = d\phi(X) \) and \( X = \text{grad} \, f \), where \( \Delta f = \lambda_1 f \).

The proposition comes from the inequalities:

\[
(I(V), V) = \int_M \left( 4 \left| \operatorname{trace} \nabla d\psi(\nabla, X, \cdot) \right|^2 + 4(\text{div} X)^2 
+ |\Delta_H(X) - 2\text{Ricci}(X) + mX|^2 - m^2|X|^2 \right) v_g 
\geq \int_M \left( 4(\text{div} X)^2 - m^2|X|^2 \right) v_g 
= \int_M \left( 4(\Delta f)^2 - m^2|\text{grad} f|^2 \right) v_g 
= \lambda_1(4\lambda_1 - m^2) \int_M f^2 v_g. 
\]

\[\square\]

Example 3.1 (Generalised Clifford torus).

The first eigenvalue of \( S^l(\frac{1}{\sqrt{2}}) \times S^l(\frac{1}{\sqrt{2}}) \) is \( 4l \), so, for the biharmonic map \( \phi : S^l(\frac{1}{\sqrt{2}}) \times S^l(\frac{1}{\sqrt{2}}) \to S^{m+2} \) \( (m = 2l) \), \( (I(V), V) \) will be positive for vector fields \( V = d\phi(\text{grad} f) \), \( f \) eigenfunction of the Laplacian, when \( 2m \geq \frac{m^2}{4} \), that is \( m \leq 8 \).

Owing to the nature of our construction, we can link the stability of the biharmonic map \( \phi : M \to S^{n+1} \) to the one of the identity.

**Theorem 3.2.** Let \( \psi \) be a minimal isometric immersion from a compact Riemannian manifold \( (M, g) \) into \( S^n(\frac{1}{\sqrt{2}}) \), \( \phi : M \to S^{n+1} \) the associated biharmonic map and \( I \) its second variation operator.

If the identity \( 1 : (M, g) \to (M, g) \) is stable, as a harmonic map, then \( (I(V), V) \) is positive for any vector field \( V \) of \( \phi^{-1}TS^{n+1} \) tangent to \( \phi(M) \).

**Proof.** Writing a vector field \( V \in C(\phi^{-1}TS^{n+1}) \), as \( d\phi(X), X \in C(TM) \), yields:

\[
(I(V), V) = \int_M \left( |\Delta \phi V + (1 - m)V|^2 - m^2|V|^2 \right) v_g. 
\]

with \( \Delta \phi V = \Delta \psi V + 2(\text{div} X)\eta + V \).

On the other hand, the Jacobi operator of \( \psi \) is:

\[
J\psi(V) = \Delta \psi V + \operatorname{trace} R^{S^n(\frac{1}{\sqrt{2}})}(d\psi, V)d\psi, 
\]

with

\[
\operatorname{trace} R^{S^n(\frac{1}{\sqrt{2}})}(d\psi, V)d\psi = \sum_{i=1}^m R^{S^n(\frac{1}{\sqrt{2}})}(d\psi(X_i), V)d\psi(X_i) 
= 2 \sum_{i=1}^m (X_i, X - d\psi(X_i) - X_i, X_i, X) 
= 2(1 - m)V, 
\]
so that:
\[ J^\psi(V) = \Delta^\psi V + 2(1 - m)V. \]
Therefore:
\[ \Delta^\phi V = J^\psi(V) + 2(\text{div } X)\eta + (2m - 1)V, \]
and
\[ \Delta^\phi V + (1 - m)V = J^\psi(V) + 2(\text{div } X)\eta + mV. \]
Integrating shows that:
\[
(I(V), V) = \int_M \left( |J^\psi(V) + mV|^2 + 4(\text{div } X)^2 - m^2|V|^2 \right) v_g
= \int_M \left( |J^\psi(V)|^2 + 4(\text{div } X)^2 + 2m < J^\psi(V), V > \right) v_g
\geq 2m \int_M < J^\psi(V), V > \ v_g = 2m \int_M < J^1(V), V > \ v_g.
\]
\[ \square \]

**Remark 3.2.** The stability of the identity of \((M, g)\) is, in fact, equivalent to \((J^\psi(V), V)\) positive for any section \(V\) of \(\psi^{-1}T^S\mathbb{S}^{n}(\frac{1}{\sqrt{2}})\) and tangent to \(\phi(M)\). In this respect, \(\phi\) inherits part of the stability of \(\psi\).

**Example 3.2 (Generalised Clifford torus—Revisited).**
The manifold \(S^{l}(\frac{1}{4}) \times S^l(\frac{1}{4})\) being Einstein of constant \(\kappa = 4(l - 1)\), its identity map will be stable if and only if \(2\kappa \leq \lambda_1\) \((14)\), i.e. provided \(l\) equals 1 or 2. For these values of \(l\), the index of the biharmonic map \(\phi : S^{l}(\frac{1}{4}) \times S^l(\frac{1}{4}) \rightarrow S^{2l+2}\), as we already knew, has no contribution from vector fields of the tangent sub-bundle.

When \(l > 2\), we need a finer analysis of the sign of \((I(V), V)\). Staying with a section \(V\) of \(\phi^{-1}TS^{2l+2}\), such that \(V = d\phi(X), X = \text{grad } f\) with \(\Delta f = \lambda f\), we know that:
\[
(I(V), V) \geq \int_{S^l(\frac{1}{4}) \times S^l(\frac{1}{4})} \left( 4(\text{div } X)^2 + |\Delta_H(X) - 2\text{Ricci}(X) + mX|^2 - m^2|X|^2 \right) v_g
= \int_{S^l(\frac{1}{4}) \times S^l(\frac{1}{4})} \left( 4\lambda^2 f^2 + (\lambda - 8(l - 1) + m)^2\lambda f^2 - m^2\lambda f^2 \right) v_g
= \lambda \left( \lambda^2 + 2\lambda(10 - 3m) + 8m^2 - 48m + 64 \right) \int_{S^l(\frac{1}{4}) \times S^l(\frac{1}{4})} f^2 v_g,
\]
where \(m = 2l\).
Therefore, when \(P(\lambda) = \lambda^2 + 2\lambda(10 - 3m) + 8m^2 - 48m + 64\) is positive, so is \((I(V), V)\), and, the first non-zero eigenvalue \(\lambda_1\) of the Laplace-Beltrami operator on \(S^l(\frac{1}{4}) \times S^l(\frac{1}{4})\) being \(4l = 2m\), \(P(\lambda_1) \geq 0\) if and only if \(m \leq 8\).
The polynomial \(P\) is increasing on the interval \([2m, +\infty)\) when \(m \leq 10\), so \(P(\lambda_k) \geq P(\lambda_1) \geq 0, \forall k \geq 1\) as long as \(m \leq 8\) (this recoups Example 3.1).
For dimensions superior to 8, \(P(\lambda_1)\) is negative but \(P(\lambda_2)\), and therefore \(P(\lambda_k)\) for \(k \geq 2\), is positive, since \(\lambda_k = 4p(l + p - 1) + 4q(l + q - 1) (p + q = k)\).
In conclusion, if where $q$ is the projection on the second factor.

Nonetheless for $\lambda_1$, we can push this investigation a little further, by explicitly computing the term overlooked in Inequality (7), i.e. $4 \left| \text{trace} \, \nabla d\psi(\nabla X, .) \right|^2$.

Let $f \in C^\infty(S^l(\frac{1}{2}))$ such that $\Delta^{S^l(\frac{1}{2})} f = 2mf$ and $\bar{f} = f \circ p$, $p$ being the first projection $p : S^l(\frac{1}{2}) \times S^l(\frac{1}{2}) \to S^l(\frac{1}{2})$. Then $\Delta^{S^l(\frac{1}{2})} \times S^l(\frac{1}{2}) \bar{f} = 2m\bar{f}$, $X = \text{grad} \, \bar{f} = \text{grad} \, f$ and, $f$ being the first eigenvalue of the Laplacian on a sphere, $\nabla X_a X = \nabla^{S^l(\frac{1}{2})} f \text{grad} = -4f X_a$, where $\{X_a\}_{a=1,...,l}$ is an orthonormal frame of $S^l(\frac{1}{2})$.

Then, if $\{X_a, X_b\}_{a,b=1,...,l}=\{X_i\}_{i=1,\ldots,2l}$ is an orthonormal frame of $S^l(\frac{1}{2}) \times S^l(\frac{1}{2})$:

$$4 \left| \text{trace} \, \nabla d\psi(\nabla X, .) \right|^2 = 4 \left| \sum_{i=1}^{2l} \nabla d\psi(\nabla X, X_i) \right|^2$$

$$= 4 \left| \sum_{a=1}^{l} \nabla d\psi(\nabla X_a X, X_a) + \sum_{b=1}^{l} \nabla d\psi(\nabla X_b X, X_b) \right|^2$$

$$= 4 \left| -4f \sum_{a=1}^{l} \nabla d\psi(X_a, X_a) + \sum_{b=1}^{l} \nabla d\psi(0, X_b) \right|^2$$

$$= 64f^2 \left| \sum_{a=1}^{l} \sqrt{2} \xi \right|^2$$

$$= 32m^2 f^2,$$

and:

$$(I(V), V) = (2m(-8m + 64) + 32m^2) \int_{S^l(\frac{1}{2}) \times S^l(\frac{1}{2})} f^2 v_g > 0.$$

A similar computation yields the same result for $\bar{f} = f \circ q$ and $\bar{f} = f \circ p + f \circ q$, where $q$ is the projection on the second factor.

In conclusion, if $V = d\phi(\text{grad} \, f)$ with $\Delta f = \lambda f$, then $(I(V), V) > 0$, whatever the dimension.

**Remark 3.3.** When our examples have dimension less than four, calling on Theorem 3.1 is a far easier option.

As in the previous section, when the domain is an Einstein manifold, conditions can be imposed on the Einstein constant to ensure the positivity of $(I(V), V)$.

**Proposition 3.3.** Let $(M^m, g)$ be an Einstein manifold of constant $\kappa \in \mathbb{R}$, i.e. $\text{Ricci} = \kappa g$ and $\phi : M \to S^{n+1}$ a biharmonic map constructed as previously.

If $\kappa \geq \frac{(m+2)^2}{8}$ then $(I(V), V) \geq 0$, for $V = d\phi(\text{grad} \, f)$, $\Delta f = \lambda f$.

**Proof.** Let $X \in C(TM)$ and $V = d\phi(X)$.

Assume that $X = \text{grad} \, f$, $f \in C^\infty(M)$ and $\Delta f = \lambda f$. 


By Proposition \[ \text{Proposition 3.2} \]

\[
(I(V), V) \geq \int_{M} \left(4(\text{div} \, X)^2 + |\Delta_H(X) - 2 \text{Ricci}(X) + mX|^2 - m^2|X|^2 \right) v_g,
\]

but \( \Delta_H X = \Delta_H(\text{grad} \, f) = \text{grad} \, \Delta f = \lambda X \), \( \text{Ricci} \, X = \kappa X \) and:

\[
\int_{M} |X|^2 v_g = \int_{M} |\text{grad} \, f|^2 v_g = \int_{M} \lambda f^2 v_g
\]

\[
\int_{M} (\text{div} \, X)^2 v_g = \int_{M} (\Delta f)^2 v_g = \lambda \int_{M} f^2 v_g.
\]

Therefore:

\[
(I(V), V) \geq \lambda \left( \lambda^2 + 2(m - 2 - 2\kappa)\lambda + 4\kappa^2 - 4\kappa m \right) \int_{M} f^2 v_g,
\]

and, if \( \kappa \geq \frac{(m+2)^2}{8} \), \( \lambda^2 + 2(m - 2 - 2\kappa)\lambda + 4\kappa^2 - 4\kappa m \geq 0 \), so \((I(V), V)\) is positive. \(\square\)

**Example 3.3** (Totally geodesic inclusion).

Seeing that the sphere \( S^m(\frac{1}{\sqrt{2}}) \) has \( \kappa = 2(m - 1) \) for Einstein constant, we can apply the above proposition to show that the biharmonic map \( \phi : S^m(\frac{1}{\sqrt{2}}) \rightarrow S^{n+1} \) is stable in the direction of vector fields \( d\phi(\text{grad} \, f) \) \( (\Delta f = \lambda f) \), as soon as \( 2(m - 1) \geq \frac{(m+2)^2}{8} \), i.e. \( m \in [2, 9] \).

If \( m = 1 \), then \( \kappa = 0 \) and \((I(V), V)\) is always positive for \( V = d\phi(\text{grad} \, f) \).

When \( m \geq 10 \) and \( V = d\phi(\text{grad} \, f) \) \( (\Delta f = \lambda f) \):

\[
(I(V), V) = \int_{S^m(\frac{1}{\sqrt{2}})} \left(4(\text{div} \, X)^2 + |\Delta_H(X) - 2 \kappa X|^2 + 2m < \Delta_H(X) - 2 \kappa X, X > \right) v_g
\]

\[
= \lambda \left( \lambda^2 + 6(2 - m)\lambda + 8(m - 1)(m - 2) \right) \int_{S^m(\frac{1}{\sqrt{2}}}) f^2 v_g,
\]

which is always positive as the spectrum of the Laplacian on \( S^m(\frac{1}{\sqrt{2}}) \) is \( \{ \lambda_k = 2k(m + k - 1) : k \in \mathbb{N} \} \).

**Example 3.4** (Generalised Veronese map).

To the minimal Riemannian immersion of \( S^m(\sqrt{\frac{2m+1}{m}}) \) into \( S^{m+p} \), \( p = \frac{(m-1)(m+2)}{2} \), Theorem \[ \text{Theorem 1.4} \] associates a non-harmonic biharmonic map from \( \phi : S^m(\sqrt{\frac{m+1}{m}}) \rightarrow S^{m+p+1} \).

Since \( S^m(\sqrt{\frac{m+1}{m}}) \), equipped, as usual, with the canonical metric, is an Einstein manifold of constant \( \kappa = \frac{m(m-1)}{m+1} \), we deduce that if \( V = d\phi(X) \) for \( X = \text{grad} \, f, \ f \in C^\infty(S^m(\sqrt{\frac{m+1}{m}})) \) with \( \Delta f = \lambda f \), as we saw in the proof of Proposition \[ \text{Proposition 3.3} \] \((I(V), V)\) is positive as soon as \( P(\lambda) = \lambda^2 + 2\lambda(m + 2 - 2\kappa) + 4\kappa^2 - 4\kappa m \kappa \) is positive.

Recall that the eigenvalues of the Laplacian on \( S^m(\sqrt{\frac{m+1}{m}}) \) are \( \{ \lambda_k = \frac{m}{m+1}k(m + k - 1) : k \in \mathbb{N} \} \).

The roots of \( P \) are \( x_1 = \frac{m^2-5m-2}{m+1} \), \( x_2 = \frac{m^2-5m-2}{m+1} + \)
\[
\sqrt{\frac{m^3-3m^2+16m+4}{m+1}}, \text{ and, } \lambda_k \geq x_2, \forall m \geq 2 \text{ and } \forall k \geq 2, \text{ while } \lambda_1 \geq x_2 \text{ only if } m = 2, 3 \text{ or } 4 \text{ (as predicted by Theorem 3.1).}
\]

If \( m \geq 5 \), then \( \lambda_1 \in [x_1, x_2] \) and we need to study in details the term \( |\text{trace} \nabla d\psi(\nabla X,.)|^2 \), for \( X = \text{grad} f \) with \( \Delta f = \lambda_1 f \). This is, again, made possible by the fact that, on a sphere, the first eigenfunction of the Laplacian takes the form \( f(x) = \langle u, x \rangle \) with \( u \in \mathbb{R}^{m+1} \), and satisfies \( \nabla X \text{grad} f = -\frac{m}{m+1} fX \). So, for an orthonormal frame field \( \{X_i\}_{i=1,\ldots,m} \):

\[
\sum_{i=1}^{m} \nabla d\psi(\nabla X_i X_i) = -\frac{m}{m+1} f \sum_{i=1}^{m} \nabla d\psi(X_i, X_i) = 0,
\]

by the harmonicity of \( \psi \).

In conclusion, for the generalised Veronese map, \((I(V), V)\) is positive, when \( V = d\phi(\text{grad} f) \) and \( \Delta f = \lambda f \), except for \( \lambda = \lambda_1 \).

4. The vertical sub-bundle

The third case of vector fields are sections of the pull-back bundle \( \phi^{-1}T\mathbb{S}^{n+1} \), tangent to the tropical sphere \( \mathbb{S}^n(\sqrt{2}) \) but orthogonal to the image of the map.

For such a vector field \( V \), the second variation operator is easily worked out:

**Proposition 4.1.** Let \( V \in C(\phi^{-1}T\mathbb{S}^{n+1}) \) be orthogonal to \( \eta \) and \( d\phi(TM) \), then:

\[
(I(V), V) = \int_M |\Delta^2 V|^2 - 2m < \Delta^2 V, V > v_g
\]

**Proof.** The basic properties of \( V \), i.e. \( < V, \eta = < V, d\phi(X) = 0 \), \( \forall X \in TM \), imply that the non-vanishing constituents of Equation (11) are:

\[
I(V) = \Delta^2 V - 2 \Delta^2 V - 2 \nabla^2 d\phi - 2 \nabla^2 d\phi - |d\phi|^4 V.
\]

Taking the inner-product with \( V \), produces:

\[
< I(V), V > = < \Delta^2 V - m \Delta^2 V - m^2 V - m^2 V, V > = < \Delta^2 V, V >,
\]

and integrating by parts yields:

\[
(I(V), V) = \int_M |\Delta^2 V|^2 - 2m < \Delta^2 V, V > v_g
\]

\( \square \)

Though Equation (8) is generally unworkable, assuming that \( m = n - 1 \) and that the normal sub-bundle of \( \psi^{-1}T\mathbb{S}^n \) is parallelizable enables the expression of \( \Delta^2 V \) in function of the shape operator.
Proposition 4.2. Let $\psi : M^m \to S^n(\frac{1}{\sqrt{2}})$ be a minimal isometric immersion and $\phi : M^m \to S^{n+1}$ the coupled biharmonic map. Assume that $m = n - 1$ and that there exists a unit section $\xi$ of the normal bundle of $M$ in $\psi^{-1}TS^n(\frac{1}{\sqrt{2}})$.

Let $V = f\xi, f \in C^\infty(M)$ and $A_\xi$ the shape operator of $\xi$, then:

\[
< \Delta^\psi V, V > = (\Delta f) f + f^2 |\nabla d\psi|^2,
\]

\[
|\Delta^\psi V|^2 = (\Delta f + |\nabla d\psi|^2 f)^2 + |2A_\xi (d\psi(\text{grad } f)) + f \text{ trace } (\nabla A_\xi)(d\psi(\cdot), d\psi(\cdot))|^2.
\]

Proof. Let $V = f\xi$ with $f \in C^\infty(M)$, so that $< V, \eta > = < V, d\phi(X) > = 0, \forall X \subset C(TM)$.

The first thing to observe is that for $X \subset C(TM)$:

\[
\nabla^\phi_X V = \nabla^S_{d\phi(X)} V = \nabla^S_{d\psi(X)} V = \nabla^S_{d\psi(X)} V - < d\phi(X), V > \eta
\]

\[
= \nabla^S_{d\psi(X)} V = \nabla^\psi_X V.
\]

Furthermore:

\[
\nabla^\phi_Y \nabla^\phi_X V = \nabla^\phi_Y \nabla^\psi_X V
\]

\[
= \nabla^\phi_Y \nabla^\psi_X V - < d\psi(Y), \nabla^\psi_X V > \eta
\]

\[
= \nabla^\phi_Y \nabla^\psi_X V + < \nabla^\phi_Y d\psi(Y), V > \eta
\]

\[
= \nabla^\phi_Y \nabla^\psi_X V + < \nabla d\psi(X, Y) + d\psi(\nabla_X Y), V > \eta
\]

\[
= \nabla^\phi_Y \nabla^\psi_X V + < \nabla d\psi(X, Y), V > \eta,
\]

and $\Delta^\phi V = \Delta^\psi V$, as $\psi$ is harmonic.

Now

\[
\nabla^\psi_X (f\xi) = (Xf)\xi + f\nabla^\psi_X \xi
\]

\[
= (Xf)\xi + f \left( \nabla^S_{d\phi(X)} \xi - A_\xi(d\phi(X)) \right).
\]

The second order of differentiation is:

\[
\nabla^\psi_Y \nabla^\psi_X (f\xi) = \nabla^\psi_Y \left( (Xf)\xi + f \left( \nabla^S_{d\phi(X)} \xi - A_\xi(d\phi(X)) \right) \right)
\]

\[
= Y(Xf)\xi + (Xf)(\nabla^S_{d\phi(Y)} \xi - A_\xi(d\phi(Y))) + \nabla^\psi_Y \left( f(\nabla^S_{d\phi(X)} \xi - A_\xi(d\phi(X))) \right)
\]

\[
= Y(Xf)\xi + (Xf)\nabla^S_{d\phi(Y)} \xi - A_\xi(d\phi(Y))(Yf) \left( \nabla^S_{d\phi(X)} \xi - A_\xi(d\phi(X)) \right)
\]

\[
+ f \left( \nabla^S_{d\phi(Y)} \nabla^S_{d\phi(X)} \xi - A_\xi(d\phi(Y)) \right) - \nabla^S_{d\phi(Y)}(A_\xi(d\phi(X))) - B(d\psi(Y), A_\xi(d\phi(X))).
\]
\[\Delta^\psi V = (\Delta f)\xi - 2\nabla^\perp_{d\psi(\nabla f)}\xi + 2A_\xi(d\psi(\nabla f))
\]
\[+ f \left( \Delta^\perp\xi + |\nabla d\psi|^2\xi + \sum_{i=1}^m \left( A_{\nabla^\perp_{d\psi(X_i)}}\xi d\psi(X_i) + \nabla_{d\psi(X_i)}A_\xi(d\psi(X_i)) \right) \right)\]
\[=(\Delta f)\xi - 2A_\xi(d\psi(\nabla f)) + f|\nabla d\psi|^2\xi + f \sum_{i=1}^m \nabla_{d\psi(X_i)}(A_\xi(d\psi(X_i))),\]

since \(\nabla^\perp\xi = 0\).

As a result:
\[<\Delta^\psi V, V> = (\Delta f)f + f^2|\nabla d\psi|^2\]

and:
\[|\Delta^\psi V|^2 = (\Delta f + |\nabla d\psi|^2 f)^2 + |2A_\xi(d\psi(\nabla f)) + f \sum_{i=1}^m \nabla_{d\psi(X_i)}A_\xi(d\psi(X_i))|^2.\]

Example 4.1 (Generalised Clifford torus).
The minimal isometric immersion of the generalised Clifford torus \(S^l(\frac{1}{2}) \times S^l(\frac{1}{2}) \rightarrow S^{2l+1}(\frac{1}{2})\) has codimension one and a parallelizable normal bundle, the vector field \(\xi\) being defined by \(\xi(p,p') = \sqrt{2}(p, -p')\) for \((p, p') \in S^l(\frac{1}{2}) \times S^l(\frac{1}{2})\). One can readily verify that \(|\xi| = 1\) and \(<\xi, X> = 0\), \(\forall X \in TS^l \times TS^l\).

Furthermore, the shape operator at \(\xi\) is:
\[A_\xi = \sqrt{2} \begin{pmatrix} -\text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix},\]

so, for a function \(f\) on \(S^l(\frac{1}{2}) \times S^l(\frac{1}{2})\) and an orthonormal frame \(\{X_i\}_{i=1,\ldots,2l}\):
\[A_\xi(d\psi(\nabla f)) = \sum_{i=1}^{2l} A_\xi(d\psi(X_i(f)X_i))\]
\[= -\sqrt{2} \sum_{i=1}^l d\psi(X_i(f)X_i) + \sqrt{2} \sum_{i=l+1}^{2l} d\psi(X_i(f)X_i),\]

and
\[|A_\xi(d\psi(\nabla f))|^2 = 2|\nabla f|^2.\]

Moreover:
\[\text{trace} \left( \nabla A_\xi \right)(d\psi(\cdot), d\psi(\cdot)) = \sum_{i=1}^{2l} (\nabla A_\xi)(d\psi(X_i), d\psi(X_i))\]
\[= \sum_{i=1}^{2l} (\nabla d\psi(X_i))A_\xi(d\psi(X_i)) - A_\xi(\nabla d\psi(X_i)d\psi(X_i))\]
\[= 0.\]
since $A_\xi$ acts as the identity (up to a constant) on each of the factors of $S^{l}(\frac{1}{2}) \times S^{l}(\frac{1}{2})$.

The rest follows easily, for a section $V = f\xi$:

\[
(I(V), V) = \int_{S^{l}(\frac{1}{2}) \times S^{l}(\frac{1}{2})} |\Delta V|^2 - 2m < \Delta V, V > v_g
\]

\[
= \int_{S^{l}(\frac{1}{2}) \times S^{l}(\frac{1}{2})} (\Delta f + 4lf)^2 + 8|\text{grad } f|^2 - 4l(f(\Delta f) + 4lf^2) v_g
\]

\[
= \int_{S^{l}(\frac{1}{2}) \times S^{l}(\frac{1}{2})} (\Delta f)^2 + 8lf\Delta f + 16l^2 f^2 + 8|\text{grad } f|^2 - 4l f(\Delta f) - 16l^2 f^2 v_g
\]

\[
= \int_{S^{l}(\frac{1}{2}) \times S^{l}(\frac{1}{2})} (\Delta f)^2 + 4lf(\Delta f) + 8|\text{grad } f|^2 v_g
\]

\[
= \int_{S^{l}(\frac{1}{2}) \times S^{l}(\frac{1}{2})} (\Delta f)^2 + 4(l + 2)|\text{grad } f|^2 v_g.
\]

So, for the generalised Clifford torus, $(I(V), V)$ is always positive on the vertical sub-bundle.

**Example 4.2** (Totally geodesic inclusion).

Note that, though the normal bundle of $S^m(\sqrt{2})$ in $S^n(\sqrt{2})$ is not of codimension one, it is parallelized by $\{e_{m+2}, \ldots, e_{n+1}\}$ and we can compute its vertical index.

Let $V = f e_{m+i}$, $(2 \leq i \leq n-m)$, $f \in C^\infty(S^m(\sqrt{2}))$, then $<V, d\phi(X)> = 0$, $\forall X \in C(TM)$ and $<V, \eta> = 0$. A short computation shows that $\Delta V = (\Delta f) e_{m+i}$.

Assuming that $\Delta f = \lambda f$, we have:

\[
(I(V), V) = \int_{S^m(\sqrt{2})} \left( <\Delta V, \Delta V > - 2m <\Delta V, V > \right) v_g
\]

\[
= \lambda(\lambda - 2m) \int_{S^m(\sqrt{2})} f^2 v_g
\]

\[
\geq 0.
\]

Thus, vector fields $f e_{m+i}$ ($\Delta f = \lambda f$) do not contribute to the index of $\phi : S^m(\sqrt{2}) \to S^{n+1}$.

5. **The index**

For some examples, the study of the three different sub-bundles of the pull-back leads to an estimation of the index:

**Proposition 5.1.** The index of the non-harmonic biharmonic map $S^m(\sqrt{2}) \to S^{n+1}$, $(m \leq n)$, obtained from the totally geodesic inclusion $S^m(\sqrt{2}) \to S^n(\sqrt{2})$, is 1.

In particular, the inclusion map $S^n(\sqrt{2}) \to S^{n+1}$ has index 1.
Remark 5.1. It is interesting to compare Proposition 5.1 with the index of the totally geodesic embedding $S^n(\frac{1}{2}) \to S^n(\frac{1}{2})$ as a harmonic map, which is $n + 1$ if $m \geq 3$ and $n - 2$ for $m = 2$.

Proof. One can easily verify that $(I(V), W) = 0$ when $V$ and $W$ are orthogonal vector fields of any of the three cases, except for $V = d\phi(X)$ and $W = d\phi(Y)$, with $\text{div} \, X = \text{div} \, Y = 0$, and when $V = f\eta$ and $W = d\phi(\text{grad} \, f)$ ($\Delta f = \lambda f$, as usual).

In the first situation, a linear combination of $V$ and $W$ can be written as $d\phi(Z)$ with $\text{div} \, Z = 0$, and therefore span $\{V, W\}$ does not influence the index.

In the second case, a simple computation leads to:

$$(I(V), W) = -4\lambda(\lambda + 2 - 2m) \int_{S^n(\frac{1}{2})} f^2 v_9$$

and, since $(I(V), V)$ and $(I(W), W)$ are already known (and positive), we have:

$$(I(V), W)^2 \leq (I(V), V)(I(W), W),$$

for any eigenvalue $\lambda$ and dimension $m$, so that the second variation operator is positive on the span of $V$ and $W$.

The index of $S^n(\frac{1}{2}) \to S^{n+1}$ is therefore the sum of the different contributions examined in Examples 2.3, 3.3 and 4.2.

Proposition 5.2. The nullity of the non-harmonic biharmonic map $S^n(\frac{1}{2}) \to S^{n+1}$, $(m \leq n)$ is $\frac{1}{2}(m + 1)(m + 2) + (m + 2)(n - m)$.

Proof. A direct use of the previous computations shows that the operator $I$ preserves the following sub-spaces:

1. $S_1 = \left\{ f\eta : f \in C^\infty(S^n(\frac{1}{2})) \right\} \oplus \left\{d\phi(\text{grad} \, g) : g \in C^\infty(S^n(\frac{1}{2})) \right\}$
2. $S_2 = \left\{d\phi(X) : \text{div} \, X = 0 \right\}$
3. $S_3 = \left\{ f_1 e_{m+2} + \cdots + f_{n-m} e_{n+1} : f_1, \ldots, f_{n-m} \in C^\infty(S^n(\frac{1}{2})) \right\}$

For the first space, one can easily check that, if $f$ is an eigenfunction, $I(2f\eta + d\phi(\text{grad} \, f)) = 0$ if and only if $f$ corresponds to the first non-zero eigenvalue $\lambda_1 = 2m$ (which has multiplicity $m + 1$). Moreover, any vector field of $S_1$ can be written $V = \sum_{i \in \mathbb{N}} 2\alpha_i f_i \eta + \beta_i d\phi(\text{grad} \, f_i)$, with $f_i$ eigenfunction corresponding to $\lambda_i$.

If $I(V) = 0$, then, by previous remarks, $I(2\alpha_i f_i \eta + \beta_i d\phi(\text{grad} \, f_i)) = 0 \quad \forall i \in \mathbb{N}$, and, necessarily, $V = \alpha_1 (2f_1 \eta + d\phi(\text{grad} \, f_1))$, where $\Delta f_1 = \lambda_1 f_1$. The contribution of $S_1$ to the nullity is $m + 1$.

Remark 5.1 shows that the space of Killing vector fields on $S^n(\frac{1}{2})$, which has dimension $\frac{m(m+1)}{2}$, is the kernel of $I$ restricted to $S_2$.

Finally, for a vertical vector field $V = f_1 e_{m+2} + \cdots + f_{n-m} e_{n+1}$, since $\Delta V = \Delta(f_1)e_{m+2} + \cdots + \Delta(f_{n-m}) e_{n+1}$, we see that $I(V) = 0$ implies $\Delta \Delta f_1 -$
2mΔfi = 0 \quad \forall i = 1, \ldots, n - m, so that each fi is a linear combination of a
constant function and eigenfunctions of the first eigenvalue. Consequently,
on the vertical sub-bundle, the nullity is (m + 2)(n - m).

\[ \Delta f_i = 0 \quad \forall i = 1, \ldots, n - m, \]

However, in general, one cannot hope to have a parallelizable normal sub-
bundle and, having to forgo some of the sections, our results can only be
lower bounds:

**Proposition 5.3.** The biharmonic map derived from the generalised Veronese
map \( S^m(\sqrt{\frac{m+1}{m}}) \rightarrow S^{m+p} \) (\( p = \frac{(m-1)(m+2)}{2} \)) has index at least equal to 2m + 3.

**Proof.** From the study of Example 2.1, we know that the index of the gen-
eralised Veronese map is at least \( m + 2 \). As we know that \((I(V), V)\) is also
negative when \( V = d\phi(\text{grad } f) \), \( \Delta f = \lambda_1 f \) (and no other eigenvalue),
a better lower bound of the index will be obtained if these vector fields com-
bine to span an even larger space on which \((I(V), V)\) is negative definite.
This requires computing \((I(V), W)\) in three different cases: first when \( V = d\phi(\text{grad } f) \) and
\( W = d\phi(\text{grad } g) \), \( \Delta f = \lambda_1 f \) and \( \Delta g = \lambda_1 g \) (\( f \neq g \));
then \( V = f\eta, W = d\phi(\text{grad } g) \), \( \Delta f = \lambda f \) (\( \lambda = \lambda_0 \) or \( \lambda_1 \)) and \( \Delta g = \lambda_1 g \), and,
finally, \( V = f\eta \) and \( W = d\phi(\text{grad } f) \), \( \Delta f = \lambda_1 f \).
If \( V = d\phi(\text{grad } f) \) and \( W = d\phi(\text{grad } g) \), where \( f \) and \( g \) are two differ-
ent eigenfunctions of the Laplacian corresponding to the first eigenvalue \( \lambda_1 \)
such that \( \int_{S^m(\sqrt{\frac{m+1}{m}})} fg \, v_g = 0 \), then:

\[ (I(V), W) = 0. \]

On the other hand, if \( V = f\eta \) and \( W = d\phi(\text{grad } g) \), \( \Delta f = \lambda f \), \( \lambda = \lambda_0 \) or \( \lambda_1 \),
and \( \Delta g = \lambda_1 g \) (\( f \neq g \)), then:

\[ (I(V), W) = 0, \]

since \( \int_{S^m(\sqrt{\frac{m+1}{m}})} fg \, v_g = 0. \)

If \( f = g \), then \((I(V), W) = \frac{(2m)^3}{(m+1)^2} \int_{S^m(\sqrt{\frac{m+1}{m}})} f^2 \, v_g \) and \((I(V), W)^2 <
(I(V), V)(I(W), W)\), so the bilinear form \((I(V), V)\) is definite negative on
the space spanned by \( f\eta \) and \( d\phi(\text{grad } f) \).
Together, the vector fields \( \{\eta, f\eta, d\phi(\text{grad } g), \Delta f = \lambda_1 f, \Delta g = \lambda_1 g\} \) (the mul-
tiplicity of \( \lambda_1 \) being \( m + 1 \)) span a \((2m+3)\)-dimensional space on which
\((I(V), V)\) is negative definite.

Various attempts to find vector fields, other than \( \eta \), for which \((I(V), V)\)
is negative has led us to:

**Conjecture 5.1.** The biharmonic map from \( S^l(\frac{1}{2}) \times S^l(\frac{1}{2}) \) into \( S^{2l+2} \),
constructed from the generalised Clifford torus, has index 1.
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