SHARP INEQUALITIES FOR DISCRETE AND CONTINUOUS MULTI-TILING, USING THE BOMBIERI-SIEGEL APPROACH.

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ABSTRACT. Given a finite subset $F$ of integer points in $\mathbb{Z}^d$, it is of interest to seek conditions on $F$ that allow it to multi-tile $\mathbb{Z}^d$ by translations. In addition to the continuous multi-tiling results presented here, we also give analogous discrete applications to arithmetic combinatorics. Namely we give a discretized version of the Bombieri-Siegel formula, namely a finite sum of discrete covariograms, taken over any finite set of integer points in $\mathbb{R}^d$. As a consequence, we arrive at a new equivalent condition for multi-tiling $\mathbb{Z}^d$ by translating $F$ with a fixed integer sublattice.

Similar questions related to convex bodies have already been investigated extensively. In order to develop lattice sums of the cross covariogram for any two bounded sets $A, B \subset \mathbb{R}^d$, we prove a refined continuous version of the classical Bombieri-Siegel formula from the geometry of numbers. To achieve this goal, we use a variant of the Poisson Summation formula, adapted for continuous functions of compact support.

As an application of this refined Bombieri-Siegel formula, a new characterization of multi-tilings of Euclidean space by translations of a compact set by using a lattice is given. A further consequence is a spectral formula for the volume of any bounded measurable set.

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1. Introduction

We extend some results of Siegel and of Bombieri from the Geometry of numbers by studying the cross covariogram of any two compact sets \( A, B \subset \mathbb{R}^d \), defined by

\[
\text{g}_{A,B}(x) := \text{vol}(A \cap (B + x)),
\]

defined for all \( x \in \mathbb{R}^d \). An important special case of (1) that comes up here as well is the covariogram of \( A \), denoted by \( \text{g}_A(x) := \text{vol}(A \cap (A + x)) \). We define a body to be a compact subset of \( \mathbb{R}^d \), following the standard conventions of convex geometry. Given any full-rank lattice \( L \subset \mathbb{R}^d \), we study the following sum of the covariogram \( g_A \) over the lattice \( L \):

\[
\sum_{n \in L} g_A(n) := \sum_{n \in L} \text{vol}(A \cap (A + n)),
\]

which is always a finite sum due to the compactness of \( A \). The covariogram \( g_A \) has been studied intensively in recent years and is sometimes also referred to as the set covariance. It follows immediately from basic principles that the covariogram is also equal to the autocorrelation of \( 1_A \):

\[
\text{g}_A(x) := 1_A * 1_A(x) := \int_{\mathbb{R}^d} 1_A(t)1_A(x-t)dt,
\]

where \( 1_A \) is the indicator function of the body \( A \). More generally, we also define the cross-correlation of two bodies \( A, B \) as \( \text{g}_{A,B}(x) := 1_A * 1_B(x) := \int_{\mathbb{R}^d} 1_A(t)1_B(x-t)dt \), and again an elementary computation gives \( \text{g}_{A,B}(x) = \text{vol}(A \cap (B + n)) \).

First, we apply a result of Bombieri [7] to give universal lower bounds for the series \( \sum_{n \in L} g_A(n) \), and completely characterize the compact sets \( A \) that give equality (Corollary 2) in these lower bounds, which turn out to be multi-tiling bodies. We recall the classical formula of Siegel, which in itself is also an extension of Minkowski’s first theorem (see for example [17]).

Theorem (C. L. Siegel, 1935). Let \( \mathcal{P} \subset \mathbb{R}^d \) be a convex body, and \( L \subset \mathbb{R}^d \) a full-rank lattice. If the only lattice point of \( L \) in the interior of \( \frac{1}{2}\mathcal{P} - \frac{1}{2}\mathcal{P} \) is the origin, then we have:

\[
2^d \det L = \text{vol} \mathcal{P} + 4^d \frac{\text{vol} \mathcal{P}}{\text{vol} \mathcal{P}} \sum_{\xi \in L \setminus \{0\}} \left| \hat{1}_{\mathcal{P}}(\xi) \right|^2.
\]

Taking \( Q := \frac{1}{2}\mathcal{P} \), we may equivalently write:

\[
\det L = \text{vol} Q + \frac{1}{\text{vol} Q} \sum_{\xi \in L \setminus \{0\}} \left| \hat{1}_Q(\xi) \right|^2.
\]

\[\square\]

C. L. Siegel’s original proof of (4) used the Parseval identity [19]. Bombieri [7] obtained an extension of Siegel’s formula (again using Parseval’s identity) by relaxing the hypothesis in Siegel’s formula (4), so that \( \mathcal{P} \) is allowed to contain exactly \( N \) interior lattice points of \( L \), with \( N > 1 \).
Theorem (E. Bombieri, 1962). Let $A \subset \mathbb{R}^d$ be a compact set, and $\mathcal{L} \subset \mathbb{R}^d$ any full-rank lattice. Then:

\begin{equation}
\sum_{n \in \mathcal{L}} \text{vol} \left( A \cap (A + x + n) \right) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \left| \hat{1}_A(\xi) \right|^2 e^{2\pi i \langle \xi, x \rangle},
\end{equation}

for each $x \in \mathbb{R}^d$. □

Formula (6) is called the Bombieri-Siegel formula. Here we give an extension of the Bombieri-Siegel formula, to include a more general class of functions, and also to refine the summation index for the series on the left-hand side of (6). As a consequence, we find related inequalities and we classify the cases of equality in terms of multi-tiling bodies.

Our extension of Bombieri’s theorem, namely Theorem 1 below, has more general applications to the interaction between any two distinct convex bodies. We begin with the following known variation of the Poisson summation formula (PSF). This particular variation of Poisson summation doesn’t seem to as well-known as some of the other variations, so we include an independent proof in Appendix A.

Theorem (Poisson summation [16]). Suppose that $g : \mathbb{R}^d \to \mathbb{C}$ is compactly supported, continuous, and $\hat{g} \in L^1(\mathbb{R}^d)$. Then we have:

\begin{equation}
\sum_{n \in \mathcal{L}} g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{g}(m)e^{2\pi i \langle m, x \rangle},
\end{equation}

for any full-rank lattice $\mathcal{L}$, and all $x \in \mathbb{R}^d$. Here the equality holds pointwise, and both series converge absolutely and uniformly to continuous functions. \(\square\)

If identity (7) holds for $g(x)$, then we’ll call $g$ Poisson summation friendly. We include an independent proof of the latter Poisson summation formula (see Appendix), because it is not as well-known as most of the other variants of PSF. Given any measurable sets $A, B \subset \mathbb{R}^d$, if the function

\begin{equation}
g(x) := (1_A * 1_B)(x),
\end{equation}

is Poisson summation friendly, then we say that the sets $A$ and $B$ are Poisson summation friendly. It is very natural to wonder how general the class of such pairs of PSF sets can be, because they arise naturally in the study of cross covariograms. Our first main result is the following slight extension of Bombieri’s Theorem, namely Theorem 1, part (c).

Theorem 1. We fix any two measurable, bounded sets $A, B \subset \mathbb{R}^d$, and we let $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ enjoy the following properties: both functions are bounded, $f$ is compactly supported on $A$, and $g$ is compactly supported on $B$. For any full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$ we have:

(a) $f \ast g$ is continuous on $\mathbb{R}^d$, and is also Poisson summation friendly.

(b) Consequently:

\begin{equation}
\sum_{n \in \mathcal{L}} (f \ast g)(x + n) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{f}(\xi)\hat{g}(\xi)e^{2\pi i \langle \xi, x \rangle},
\end{equation}
for each $x \in \mathbb{R}^d$.

(c) If we choose $f := 1_A$ and $g := 1_B$, then as a special case of part (b), we get:

\[
\sum_{n \in \mathcal{L}} \text{vol} \left( A \cap (B + n) \right) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \hat{A}^\dagger(\xi) \overline{\hat{B}(\xi)} e^{2\pi i \langle \xi, x \rangle},
\]

for each $x \in \mathbb{R}^d$. \hfill (Proof) \hfill \Box

In particular, Theorem 1, part (a) tells us that every bounded, measurable set $B \subset \mathbb{R}^d$ is a Poisson summation friendly set, because in this case the function $f(x) := 1_B * 1_{-B}$ is a Poisson summation friendly function.

The Bombieri-Siegel formula (6) is the case $f = g$, and hence $A = B$. A trivial observation is that if a centrally symmetric body $B$ contains exactly $N$ interior lattice points then $N$ must be odd. The reason is easy: for all nonzero $n \in \text{int}(B) \cap \mathcal{L}$, we have $-n \in \text{int}(B) \cap \mathcal{L}$. Including the origin, we therefore have an odd number of lattice points.

But this brings up another question: can we describe more precisely the lattice sum on the left of (6)? We answer this question in the affirmative, when working with compact sets. Namely, we obtain another refinement of Bombieri’s identity (6), by proving that for compact sets the sum on the left-hand side of (6) may be restricted to precisely the lattice points contained in the interior of $Q - Q$. More generally, we may restrict the lattice sum on the left of (6) to the interior of $A - B$, for any pair of compact sets $A, B$, as the following theorem shows.

Our second main result is the following extension of Bombieri’s Theorem, which gives a precise finite index of summation. Its proof uses the technical Lemma 1 below, and due to this Lemma we will assume henceforth that all of our sets are compact, so that in particular they are also Poisson summation friendly.

**Theorem 2** (Refined Bombieri-Siegel formula). Let $A, B$ be compact sets in $\mathbb{R}^d$, and let $\mathcal{L} \subset \mathbb{R}^d$ be a full-rank lattice.

(a) Then we have

\[
\sum_{n \in \text{int}(A - B) \cap \mathcal{L}} \text{vol} \left( A \cap (B + n) \right) = \frac{1}{\det \mathcal{L}} \text{vol} A \text{vol} B + \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \hat{A}^\dagger(\xi) \overline{\hat{B}(\xi)},
\]

(b) If $P$ is centrally symmetric and convex, then:

\[
\sum_{n \in \text{int}(P) \cap \mathcal{L}} \text{vol} \left( P \cap (P + 2n) \right) = \frac{1}{\det \mathcal{L}} \frac{\text{vol} P}{2^d} + \frac{2^d}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \left| \hat{1}_P^\dagger(\xi) \right|^2.
\]

(Proof) \hfill \Box

We say that a body $P \subset \mathbb{R}^d$ $k$-tiles by translations with a lattice $\mathcal{L} \subset \mathbb{R}^d$ if

\[
\sum_{n \in \mathcal{L}} 1_{P + n}(x) = k,
\]
for all \( x \in \mathbb{R}^d \), except for \( x \in \partial \mathcal{P} + \mathcal{L} \), a measure zero set. When \( k = 1 \), this is the classical definition of tiling by translations, where there are no overlaps between the translated interiors of \( \mathcal{P} \). But when \( k > 1 \), we note that for such a \( k \)-tiling the translates of \( \mathcal{P} \) will always overlap each other. The field of \( k \)-tiling has recently experienced a renaissance, and one of the earliest works relating \( k \)-tiling to Fourier analysis was done by M. Kolountzakis [12].

Next, we discretize our results above, and consider any finite set of integer points \( F \subset \mathbb{Z}^d \), of cardinality \(|F|\). Setting \( \Box := [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d \), we now “thicken” each point \( n \in F \), by replacing it with a cube of sidelength \( \varepsilon \) centered at \( n \). In other words, we replace each integer point \( n \in F \) by the little \( d \)-dimensional cube \( n + [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d := n + \Box \), for a fixed \( 0 < \varepsilon \leq 1 \). Hence we’ve modified the finite set \( F \) into the \( d \)-dimensional compact set

\[
A := \bigcup_{n \in F} (\Box + n),
\]

which we call the \( \varepsilon \)-thickening of \( F \). Moreover, for \( \varepsilon \in (0, 1] \), the \( \varepsilon \)-cubes are centered at integer points and therefore do not overlap in any set of positive measure, allowing us to conclude that \( \text{vol} A = \varepsilon^d |F| \).

**Theorem 3** (Discretized Bombieri-Siegel formula). Let \( F \subset \mathbb{Z}^d \) be any finite set of integer points, and fix a full-rank integer sublattice \( \mathcal{L} \subset \mathbb{Z}^d \). Let \( \mathcal{A} \) be the \( \varepsilon \)-thickening of \( F \), for each fixed \( \varepsilon \in (0, 1] \).

(a) We have:

\[
\sum_{F \cap (F + n)} |F| = \frac{1}{\det \mathcal{L}} |F|^2 \varepsilon^d + \frac{1}{\varepsilon^d \det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} |\hat{1}_A(\xi)|^2.
\]

(b) Equivalently, we have the explicit representation:

\[
\sum_{n \in F} |F \cap (F + n)| = \frac{1}{\det \mathcal{L}} |F|^2 \varepsilon^d + \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \left( \prod_{k=1}^{d} \text{sinc}^2(\pi \varepsilon \xi_k) \left| \sum_{n \in F} e^{2\pi i (\xi, n)} \right|^2 \right),
\]

where \( \text{sinc}(\pi t) := \begin{cases} \frac{\sin(\pi t)}{\pi t} \quad & \text{if } t \neq 0, \\ 1 \quad & \text{if } t = 0. \end{cases} \)

(Proof) \( \square \)

In Theorem 4 below, we will be interested in letting \( \varepsilon = 1 \). Next, we apply Theorem 2, part (a), with \( A = B \), where \( A \) is by definition the \( 1 \)-thickening of a finite set of integer points \( F \subset \mathbb{Z}^d \):

\[
\sum_{n \in \text{int}(A \cap (A + n))} \text{vol}(A \cap (A + n)) = \frac{\text{vol} A}{\det \mathcal{L}} A^2 + \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} |\hat{1}_A(\xi)|^2.
\]

We recall the easy fact that \( F \cap (F + n) \neq \emptyset \iff n \in F - F \). By assumption we have \( F \subset \mathbb{Z}^d \), so that \( F - F \subset \mathbb{Z}^d \) as well.
We now fix a sublattice $\mathcal{L} \subset \mathbb{Z}^d$. From the discussion above, we see that for each $n \in \mathcal{L} \cap (F - F)$, we have the following relation between $F$ and its $\varepsilon$-thickening $A$:

\begin{equation}
\text{vol} \left( A \cap (A + n) \right) = |F \cap (F + n)| \varepsilon^d,
\end{equation}

because any integer translate of $A$ intersects $A$ itself in an integer number of translated copies of the $\varepsilon$-cube $\Box := [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d$. So we can rephrase equation (17) in terms of the finite set $F \subset \mathbb{Z}^d$.

**Corollary 1.** Let $F \subset \mathbb{Z}^d$ be any finite set of integer points, and fix $\mathcal{L} \subset \mathbb{Z}^d$, a full-rank sublattice. Then

\begin{equation}
\sum_{n \in (F - F) \cap \mathcal{L}} |F \cap (F + n)| \geq \frac{1}{\det \mathcal{L}} |F|^2.
\end{equation}

**Proof.** The result follows directly from Corollary 3, part (a), with $\varepsilon = 1$. \hfill \square

Theorem 1 is number-theoretic in the sense that here $\det \mathcal{L}$ and $|F|$ are positive integers. A natural question arises: what are the cases of equality in (19)? There is a particularly interesting answer, which we give in Theorem 4 below. We'll make use of the following trivial but useful observation.

**Observation.** A finite collection of integer points $F \subset \mathbb{R}^d$ multi-tiles $\mathbb{Z}^d$ if and only if its 1-thickening $A$ multi-tiles $\mathbb{R}^d$ with the same multiplicity.

**Theorem 4.** Let $F \subset \mathbb{Z}^d$ be any finite set of integer points, and fix $\mathcal{L} \subset \mathbb{Z}^d$, a full-rank sublattice. Then the following statements are equivalent.

(a) The finite set $F$ multi-tiles $\mathbb{Z}^d$ by translations with $\mathcal{L}$, and with multiplicity $k = \frac{|F|}{\det \mathcal{L}}$.

(b)

\begin{equation}
\sum_{n \in (F - F) \cap \mathcal{L}} |F \cap (F + n)| = k |F|,
\end{equation}

with $k = \frac{|F|}{\det \mathcal{L}}$.

(c) For each nonzero $\xi \in \mathcal{L}^*$ such that none of the coordinates of $\xi$ are integers, the following exponential sum vanishes:

\begin{equation}
\sum_{n \in F} e^{2\pi i \langle \xi, n \rangle} = 0.
\end{equation}

**Proof** \hfill \square

Returning to the continuous context, we may now give several corollaries of Theorem 2, the refined Bombieri-Siegel formula. The following inequality gives us a best-possible lower bound for the sum of the covariogram over a lattice. Interestingly, the lower bound in equation (22) below is achieved by a body $Q$ precisely when $Q$ is a $k$-tiling polytope.
Corollary 2. Let $A \subset \mathbb{R}^d$ be a compact set and let $\mathcal{L} \subset \mathbb{R}^d$ be a full-rank lattice. Then we have:

$\sum_{n \in \mathcal{L} \cap \text{int}(A - A)} \text{vol} \left( A \cap (A + n) \right) \geq \frac{\text{vol}^2 A}{\det \mathcal{L}}$.  \hspace{1cm} (Proof) □

Corollary 3. Let $A \subset \mathbb{R}^d$ be a compact set and let $\mathcal{L} \subset \mathbb{R}^d$ be a full-rank lattice. Then the following are equivalent:

(a) $\sum_{n \in \mathcal{L} \cap \text{int}(A - A)} \text{vol} \left( A \cap (A + n) \right) = \frac{\text{vol}^2 A}{\det \mathcal{L}}$.

(b) The body $A$ $k$-tiles $\mathbb{R}^d$ with the lattice $\mathcal{L}$, where $k = \frac{\text{vol} A}{\det \mathcal{L}}$.

(c) For all compact sets $B \subset \mathbb{R}^d$, we have

$\sum_{n \in \mathcal{L} \cap \text{int}(A - B)} \text{vol} \left( A \cap (B + n) \right) = \frac{1}{\det \mathcal{L}} \text{vol} A \text{vol} B$.  \hspace{1cm} (Proof) □

Another way to conceptualize Corollary 2 is by recalling a well-known identity, as follows. With $f(x) := (1_Q * 1_{-Q})(x) = \text{vol} \left( Q \cap (Q + x) \right)$, we have $\hat{f}(0) = |1_Q(0)|^2 = \text{vol}^2(Q)$. But we also have, via the inverse Fourier transform, $\hat{f}(0) = \int_{\mathbb{R}^d} f(x) dx$, giving us the known identity

$\int_{\mathbb{R}^d} \text{vol} \left( Q \cap (Q + x) \right) dx = \text{vol}^2 Q$.  \hspace{1cm} (25)

Putting this together with the right-hand side of equation (22), we’ve just proved the following reformulation of Corollary 2, which appears to resemble a quadrature formula.

Corollary 4. Let $Q \subset \mathbb{R}^d$ be a compact set and let $\mathcal{L} \subset \mathbb{R}^d$ be a full-rank lattice. Then we have:

$\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol} \left( Q \cap (Q + n) \right) \geq \frac{1}{\det \mathcal{L}} \int_{\mathbb{R}^d} \text{vol} \left( Q \cap (Q + x) \right) dx$.  \hspace{1cm} (26)

Moreover, equality occurs in (26) $\iff$ $Q$ $k$-tiles $\mathbb{R}^d$ with the lattice $\mathcal{L}$. □

We also obtain the following interesting spectral identity for the product of two volumes of any compact sets, provided we pick a sufficiently sparse lattice $\mathcal{L}$.

Corollary 5 (Spectral formula for the product of two volumes). Let $L \subset \mathbb{R}^d$ be a full-rank lattice, and let $A, B \subset \mathbb{R}^d$ be compact sets possessing the following property. We assume that $A$ is disjoint from the nonzero lattice translates $\{ B + n \mid n \in \mathcal{L} \}$. In addition, let $x \in \mathbb{R}^d$ be a nonzero vector with the
property that both \( A \) and \( B + x \) are also mutually disjoint from their nonzero translations by vectors from \( L \). Then

\begin{equation}
\text{vol}(A) \text{vol}(B) = - \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \hat{1}_A(\xi) \overline{\hat{1}_B(\xi)} \cos(2\pi \langle \xi, x \rangle).
\end{equation}

\textbf{(Proof)} □

The special case \( A = B \) of Corollary 5 is interesting in its own right:

\begin{equation}
\text{vol}^2 A = - \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} |\hat{1}_A(\xi)|^2 \cos(2\pi \langle \xi, x \rangle),
\end{equation}

and in particular the latter identity holds for any compact set \( A \) that satisfies the hypothesis of Corollary 5. We remark that given any compact sets \( A, B \subset \mathbb{R}^d \), it’s quite easy to find a lattice \( \mathcal{L} \) that satisfies the hypotheses of Corollary 5.

We also give applications to arithmetic combinatorics, in Section 3, by discretizing the Bombieri-Siegel formula and its extension above, to handle any finite set of integer points in \( \mathbb{Z}^d \). In particular, Corollaries 3, 1, 4 and 6 give us the relevant discretizations.

A related research direction is the question of using a finite set of integer points \( F \) to multi-tile \( \mathbb{Z}^d \) by any possible set \( S \) of integer translations. The \textbf{periodicity conjecture} states that it is always sufficient to let \( S \) be lattice. Bhattacharya [4] recently proved the periodicity conjecture for dimension 2. However, there is a recent breakthrough on the periodicity conjecture by Greenfeld and Tao [10], disproving it for all sufficiently large dimensions \( d \).

Another strong motivation for studying the covariogram is the following conjectural characterization of a convex body \( K \), posed by G. Matheron [15], and studied intensively over the past 30 years by G. Bianchi, as well as other researchers [5] [6].

\textbf{Conjecture 1} (Matheron, 1986). \textit{The covariogram} \( \hat{g}_K \) \textit{determines a convex body} \( K \), \textit{among all convex bodies, up to translations and reflections}. □

We note that knowledge of \( \hat{1}_K(\xi) \), for all \( \xi \in \mathbb{R}^d \), uniquely determines \( K \), for any convex body \( K \); in other words, the Fourier transform is a complete invariant for any body \( K \). Hence Conjecture 1 is equivalent to saying that knowledge of \( |\hat{1}_K| \) determines \( K \) (up to translations and reflections), because \( \hat{g}_K := 1_K * \overline{1_K} = |\hat{1}_K|^2 \).

It is known, for example, that centrally symmetric convex bodies in any dimension are determined by their covariogram, up to translations [6].

\section{2. Preliminaries}

The following somewhat technical result is required for the proof of Theorem 2.

\textbf{Lemma 1}. \textit{Let} \( A, B \in \mathbb{R}^d \) \textit{be compact sets, and let} \( \mathcal{L} \subset \mathbb{R}^d \) \textit{be a full-rank lattice}. 
(a) For any \( n \in L \cap \partial(A - B) \), we have
\[ \text{vol} \left( A \cap (B + n) \right) = 0. \]

(b) Consequently, we also have:
\[ \sum_{n \in L} \text{vol} \left( A \cap (B + n) \right) = \sum_{n \in L \cap \text{int}(A - B)} \text{vol} \left( A \cap (B + n) \right). \]

(Proof) □

We define the function spaces \( L^p(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C} \mid \int_{\mathbb{R}^d} |f(x)|^p dx < \infty \} \), for each \( 1 \leq p < \infty \).

Here we’ll use \( p = 1 \) and \( p = 2 \). Our Fourier transforms are defined by the traditional convention:
\[ \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \]
when the integral converges. A full-rank lattice \( L \subset \mathbb{R}^d \) is a lattice that has a \( d \)-dimensional basis. We begin by mentioning an elementary argument that proves the following fact.

**Lemma 2.** If a body \( Q \) \( k \)-tiles \( \mathbb{R}^d \) we have the identity
\[ \sum_{n \in L} \text{vol} \left( Q \cap (Q + n + x) \right) = k \cdot \text{vol} Q \]
for every fixed \( x \in \mathbb{R}^d \).

**Proof.** For a \( k \)-tiling we must have
\[ \sum_{n \in L} 1_{Q+n}(y) = k \]
for all \( y \in \mathbb{R}^d \), except those points \( y \) that lie on the boundary of \( Q \) or its translates under \( L \). Therefore
\[ \sum_{n \in L} \text{vol} \left( Q \cap (Q + n + x) \right) = \sum_{n \in L} \int_{\mathbb{R}^d} 1_{Q\cap(Q+n+x)}(y) dy \]
\[ = \sum_{n \in L} \int_{\mathbb{R}^d} 1_Q(y) \cdot 1_{Q+n+x}(y) dy \]
\[ = \sum_{n \in L} \int_{\mathbb{R}^d} 1_{Q-x}(y) \cdot 1_{Q+n}(y) dy \]
\[ = \int_{\mathbb{R}^d} 1_{Q-x}(y) \sum_{n \in L} 1_{Q+n}(y) dy \]
\[ = k \cdot \text{vol} Q, \]
where we used the translation-invariance of the integral and exchanged the sum with the integral since the integral is nonzero only for a finite number of \( n \in L \). □

It turns out that the converse of Lemma 2 is also true, and is part of of Corollary 3. A very useful Fourier equivalence for translational tilings is given by the following result, due to M. Kolountzakis [12] (see also [17], Theorem 5.5).
Theorem 5 (Kolountzakis). Let \( \mathcal{P} \) be a body in \( \mathbb{R}^d \), and let \( \mathcal{L} \subset \mathbb{R}^d \) be a full-rank lattice. Then the following conditions are equivalent:

1. \( \hat{1}_\mathcal{P}(\xi) = 0 \), for all nonzero \( \xi \in \mathcal{L}^* \), the dual lattice.
2. \( \mathcal{P} \) \( k \)-tiles \( \mathbb{R}^d \) by translations with \( \mathcal{L} \).

Either of the above conditions already implies that \( k = \frac{\text{vol} \mathcal{P}}{\det \mathcal{L}} \), a positive integer. \( \square \)

Remark 1. We may rewrite equation (6) as

\[
\sum_{\xi \in \mathcal{L}^*} |\hat{1}_\mathcal{Q}(\xi)|^2 \cos(2\pi \langle \xi, x \rangle) + i \sum_{\xi \in \mathcal{L}^*} |\hat{1}_\mathcal{Q}(\xi)|^2 \sin(2\pi \langle \xi, x \rangle),
\]

from which we may conclude that the imaginary part vanishes:

\[
\sum_{\xi \in \mathcal{L}^*} |\hat{1}_\mathcal{Q}(\xi)|^2 \sin(2\pi \langle \xi, x \rangle) = 0,
\]

for all \( x \in \mathbb{R}^d \). It’s also easy to see that this equality (34) follows from first principles. Namely, we first observe that \( |\hat{1}_\mathcal{Q}(\xi)|^2 = \hat{1}_\mathcal{Q}(\xi)\overline{\hat{1}_\mathcal{Q}(\xi)} = \hat{1}_\mathcal{Q}(\xi)\hat{1}_\mathcal{Q}(-\xi) \). Now, we may split \( \mathbb{R}^d \) into 3 regions defined by the hyperplane orthogonal to \( x \), namely \( H := \{ \xi \in \mathbb{R}^d \mid \langle x, \xi \rangle = 0 \} \), and the two half-spaces defined by \( H \). Each \( \xi \in \mathcal{L}^* \) on one side of \( H \) gives a summand that cancels with the corresponding \(-\xi\) on the other side of \( H \), while the \( \xi \)'s that lie on \( H \) yield a vanishing summand.

The above discussion shows that

\[
\sum_{n \in \mathcal{L}} \text{vol} (\mathcal{Q} \cap (\mathcal{Q} + n + x)) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} |\hat{1}_\mathcal{Q}(\xi)|^2 \cos(2\pi \langle \xi, x \rangle),
\]

valid for each \( x \in \mathbb{R}^d \), is equivalent to equation (6) of Theorem 1, a form which Bombieri already had in [7]. \( \square \)

3. Examples of the discretized Bombieri-Siegel formula

Example 1. We let \( F := \{(0,0),(2,1),(1,2),(1,3)\} \), and let \( \mathcal{L} \) be the lattice defined by the basis \( \{(1,1),(-2,2)\} \). It can be verified that \( F \) \( \mathbb{Z}^2 \)-tiles by translations with \( \mathcal{L} \). \( \square \)

As an immediate consequence of Corollary 1 and Theorem 4, we next obtain a discrete version of Minkowski’s formula, as follows.

Corollary 6. Let \( F \subset \mathbb{Z}^d \) be any finite nonempty set of integer points, and fix \( \mathcal{L} \subset \mathbb{Z}^d \), a full-rank sublattice. If the only lattice point of \( \mathcal{L} \) in \( F - F \) is the origin, then

\[
\det \mathcal{L} \geq |F|.
\]

Moreover, equality occurs in (36) if and only if the finite set \( F \) \( \mathbb{Z}^d \)-tiles by translations with \( \mathcal{L} \), and with multiplicity \( k = \frac{|F|}{\det \mathcal{L}} \).
Proof. The inequality of (19) yields
\begin{equation}
|F \cap (F + n)| = \sum_{n=0}^{\infty} |F \cap (F + n)| = |F| \geq \frac{1}{\det L} |F|^2,
\end{equation}
which gives us $\det L \geq |F|$. The equality case follows immediately from Theorem 4.

It is worth noting that Corollary 6 can also be proved in an elementary way by reducing the points of $F$ modulo a fundamental domain of the lattice $L$.

**Example 2.** Consider the set $F := \{1, 3, 4, 6\}$, which happens to tile $\mathbb{Z}$ by the set of translations that belong to the lattice $L := 4\mathbb{Z}$, with multiplicity $k = 1$ (using brute-force). Here
\[ F - F = \{-5, -3, -2, -1, 0, 1, 2, 3, 5\}. \]

To confirm the latter tiling claim by using Theorem 4, we compute
\[ \sum_{n \in (F-F) \cap 4\mathbb{Z}} |F \cap (F + n)| = |F \cap (F + 0)| = 4, \]
whereas $k|F| = 4$ as well, confirming that according to Theorem 4, the finite set $F$ of integers must tile $\mathbb{Z}$, with multiplicity $k = 1$.

**Example 3.** Consider the set of points in $\mathbb{Z}^2$:
\[ F := \{(0,0), (0, 2), (-1, 3), (1, 3), (-1, -3), (0, -2), (1, -3), (1, 1), (-1, -1)\} \]
and the integer sublattice $L$ defined by the basis $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, as shown in Figure 1.

![Figure 1](image_url)  
**Figure 1.** The set $F$ is shown by the Black circles. The integer sublattice $L$ is shown by the blue dots. A basis for $L$ is shown by the two arrows.
Here the difference set \( F - F \) is given by:

\[
F - F = \left\{ (-2, -6), (-2, -4), (-2, -2), (-2, 0), (-2, 2), (-2, 6), (-1, -5), (-1, -3), (-1, -1), (-1, 1), (-1, 3), (-1, 5), (0, -6), (0, -4), (0, 0), (0, 2), (0, 4), (0, 6), (1, -5), (1, -3), (1, -1), (1, 1), (1, 3), (1, 5), (2, -6), (2, -2), (2, 0), (2, 2), (2, 4), (2, 6) \right\}.
\]

We therefore have:

\[
(F - F) \cap L = \{(-2, -2), (-1, -1), (-1, 5), (0, -6), (0, 0), (0, 6), (1, -5), (1, 1), (2, 2)\},
\]

which is the index of summation for the left-hand-side of identity (20), as in Figure 2. Labelling the latter elements by \( n_j \in (F - F) \cap L \), we have:

\[
F \cap (F + n_1) = \{(-1, -1)\};
\]

\[
F \cap (F + n_2) = \{(-1, -3), (-1, -1), (0, 0), (0, 2)\};
\]

\[
F \cap (F + n_3) = \{(-1, 3), (0, 2)\};
\]

\[
F \cap (F + n_4) = \{(-1, -3), (1, -3)\};
\]

\[
F \cap (F + n_5) = \{(-1, -3), (-1, -1), (-1, 3), (0, -2), (0, 0), (0, 2), (1, -3), (1, 1), (1, 3)\};
\]

\[
F \cap (F + n_6) = \{(-1, 3), (1, 3)\};
\]

\[
F \cap (F + n_7) = \{(0, -2), (1, -3)\};
\]

\[
F \cap (F + n_8) = \{(0, -2), (0, 0), (1, 1), (1, 3)\};
\]

\[
F \cap (F + n_9) = \{(1, 1)\}.
\]

Finally, we compute the right-hand side of (20):

\[
\sum_{n \in (F - F) \cap L} |F \cap (F + n)| = |F \cap (F + n_1)| + \cdots + |F \cap (F + n_9)|
\]

\[
= 1 + 4 + 2 + 2 + 9 + 2 + 2 + 4 + 1 = 27.
\]

On the other hand, the definition

\[
k = \frac{|F|}{\det L} = \frac{9}{3 \cdot 1} = 3
\]

gives us the equality

\[
\sum_{n \in (F - F) \cap L} |F \cap (F + n)| = \left| \frac{|F|}{\det L} \cdot |F| \right| = 27.
\]

Hence Corollary (4) part (b) is satisfied for \( F \) and the lattice \( L \), implying that the finite set \( F \)

multi-tiles \( \mathbb{Z}^2 \) with multiplicity \( k = 3 \).
4. A special case: Van der Corput’s inequality

In this short section, we first show that Van der Corput’s classical inequality for compact bodies is an immediate consequence of Corollary 2.

**Theorem 6** (Van der Corput, 1935). *For any body \( Q \) and full-rank lattice \( \mathcal{L} \subset \mathbb{R}^d \), we have*

(a)

\[
\# \{ \text{int}(Q - Q) \cap \mathcal{L} \} \geq \frac{\text{vol} Q}{\det \mathcal{L}}.
\]

(b) *If \( Q \) is also assumed to be convex and centrally symmetric, then we have the inequality*

\[
\# \{ \text{int}(Q) \cap \mathcal{L} \} \geq \frac{1}{2^d} \frac{\text{vol} Q}{\det \mathcal{L}}.
\]

To see how inequality (39) of part (a) follows from Corollary 2, we just note that

\[
\# \{ \text{int}(Q - Q) \cap \mathcal{L} \} \text{vol} Q = \sum_{n \in \text{int}(Q - Q) \cap \mathcal{L}} \text{vol} Q
\]

\[
\geq \sum_{n \in \text{int}(Q - Q) \cap \mathcal{L}} \text{vol} (Q \cap (Q + n))
\]

\[
\geq \frac{1}{\det \mathcal{L}} \text{vol}^2 Q,
\]

where the last inequality is Corollary 2.
To see how inequality (40) of part (b) follows from Theorem 2, eq. (12), we use a similar argument. Here \( Q \) is a convex, centrally symmetric body, so we have

\[
\sum_{n \in \text{int}(Q) \cap L} \text{vol} \left( Q \cap (Q + 2n) \right) = \frac{1}{\det L} \text{vol}^2 Q + 2^d \sum_{\xi \in L^* \setminus \{0\}} \left| \frac{1}{2} Q(\xi) \right|^2.
\]

Now

\[
\# \{\text{int}(Q) \cap L\} \cdot \text{vol} Q = \sum_{n \in \text{int}(Q) \cap L} \text{vol} Q \geq \sum_{n \in \text{int}(Q) \cap L} \text{vol} (Q \cap (Q + 2n)) \geq \frac{1}{\det L} \frac{\text{vol}^2 Q}{2^d},
\]

proving (40). The last inequality followed from (44).

5. A NONCONVEX POLYGON THAT MULTI-TILES WITH MULTIPlicity 2

Example 4. Here we give a nonconvex polygon \( Q \) (see Figure 3) that multi-tiles with multiplicity \( k = 2 \). The main point here is to show how the proof of this 2-tiling follows from Corollary 2.

A nontrivial \( k \)-tiling means that there does not exist an \( m \)-tiling for \( m < k \). So here we also have to show that \( Q \) does not tile with \( k = 1 \) for any lattice. In other words, we have a non-trivial 2-tiling. This phenomenon is in sharp contrast with multi-tiling the plane by using convex polygons, because in that convex context the smallest non-trivial multiplicity is \( k = 5 \) [11].

\[
\text{Figure 3. Left: A body } Q, \text{ and a lattice } L \text{ with } \det L = 2. \\
\text{Right: the difference body } Q - Q, \text{ with its 9 interior lattice points of } L.
\]

The lattice \( L \) we chose for this multi-tiling is defined by the basis \( \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \), and has index 2 in \( \mathbb{Z}^2 \) (also known as the \( D_2 \) lattice, and drawn with green dots in Figure 3). By Corollary 2, to prove
that the body $Q$ gives a 2-tiling of the plane, it suffices to show the following equality:

$$
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) = 2 \text{vol } Q = 8.
$$

A brute-force computation (using Mathematica [23]) reveals that the left-hand side of (48) indeed equals 8, giving us a computational method of verifying that $Q$ indeed 2-tiles the plane with translations by the lattice $\mathcal{L}$.

It is instructive to also see this multi-tiling geometrically, and in Figure 4 we give such a geometric confirmation. We first translate the blue collection of translates of $Q$ by the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, to achieve the green collection. We then translate the same blue collection once again, by the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, to achieve the pink collection. Each of the little squares in the middle of the picture show that their color is the overlap of exactly two translates of the blue collection, and hence every point in their interior gets covered exactly twice. We therefore obtain a 2-tiling.

Finally, to see that $Q$ does not 1-tile with any lattice, we suppose to the contrary that it does. Consulting Figure 5, the translated parallel edge $E_2$, belonging to the purple translate of $Q$, must meet $E_1$, because $E_2$ is the only parallel edge to $E_1$. Similarly the unique edge parallel to $E_3$ must be translated to meet $E_3$, as in the right-hand side of Figure 5. We arrive at a contradiction, because the two translated copies of $Q$, drawn in green and purple, must overlap.

6. A NONCONVEX POLYGON THAT DOES NOT MULTI-TILE

**Example 5.** Here we illustrate Corollary 2 by giving a nonconvex polygon does not multi-tile with the integer lattice. Consider the non-convex body $Q$ drawn on the left-hand side of Figure 6 below.
The difference body of $Q$, namely $Q - Q = \{p - q : p, q \in Q\}$, is shown on the right-hand side of Figure 6, with the red integer points on its boundary.

We will check that here $Q$ does not multi-tile with the integer lattice $\mathbb{Z}^2$, by using Corollary 2. Namely, we’d like to show that

\begin{equation}
\sum_{n \in L \cap \text{int}(Q - Q)} \text{vol} (Q \cap (Q + n)) > \frac{\text{vol}^2 Q}{\det L}.
\end{equation}

For this example, $\text{vol} Q = 5$, and it turns out the the left-hand-side of (49) equals 26 (by a brute force computation that used Mathematica), while the right-hand-side equals $\frac{\text{vol}^2 Q}{\det L} = 25$. So we’ve confirmed that indeed we have $26 > 25$ in our inequality (49), implying by Corollary 2 that $Q$ does not multi-tile with the integer lattice.

□
7. Proof of Theorem 1

Proof. (of Theorem 1) We will apply our Poisson summation formula, namely Theorem 7, to the function $f \ast g$. We first note that because both $f, g \in L^2(\mathbb{R}^d)$, their convolution $f \ast g$ is continuous (for example, [17], Exercise 4.19). Furthermore, because both $f$ and $g$ are compactly supported, so is $f \ast g$. Finally, to satisfy the hypotheses of Theorem 7 we need to check that $f \ast g \in L^1(\mathbb{R}^d)$ and that $\hat{f} \ast \hat{g} \in L^1(\mathbb{R}^d)$. The former condition follows because the convolution of two $L^1$ functions is also in $L^1$ (using Cauchy-Schwarz), and the latter follows because the transform of a compactly supported function (in this case $f \ast g$) is always in $L^1$. So by Theorem 7, we have:

$$\sum_{n \in \mathcal{L}} (f \ast g)(n + x) = \frac{1}{\text{det} \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{f}(m) \hat{g}(m) e^{2\pi i \langle m, x \rangle},$$

proving parts (a) and (b). To prove part (c), we first note that since $f, g \in L^2(\mathbb{R}^d)$, we also have $\hat{f}, \hat{g} \in L^2(\mathbb{R}^d)$. The Cauchy-Schwarz inequality then gives us:

$$\int_{\mathbb{R}^d} |\hat{1}_A \ast \hat{1}_B(\xi)| \, d\xi = \int_{\mathbb{R}^d} |\hat{1}_A(\xi)||\hat{1}_B(\xi)| \, d\xi \leq \left( \int_{\mathbb{R}^d} |\hat{1}_A(\xi)|^2 \, d\xi \right)^{1/2} \left( \int_{\mathbb{R}^d} |\hat{1}_B(\xi)|^2 \, d\xi \right)^{1/2} < \infty,$$

so that $\hat{1}_A \ast \hat{1}_B \in L^1(\mathbb{R}^d)$. We first compute the left hand side of equation (89):
\[
\sum_{n \in \mathcal{L}} (1_A * 1_B)(n + x) = \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_A(y)1_B(n + x - y)dy
\]
\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_A(y)1_B(y - n - x)dy
\]
\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_A(y)1_B(y + n + x)dy
\]
\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_A \cap (B + n + x)(y)dy
\]
\[
= \sum_{n \in \mathcal{L}} \text{vol}(A \cap (B + n + x)).
\]

On the other hand, the right-hand-side of Poisson summation, namely eq. (89), gives us:
\[
\frac{1}{\det L} \sum_{\xi \in \mathcal{L}^*} \hat{1}_A * \hat{1}_B(\xi) e^{2\pi i \langle \xi, x \rangle} = \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^*} \hat{1}_A(\xi) \hat{1}_B(\xi) e^{2\pi i \langle \xi, x \rangle}
\]

Summarizing, Theorem 7 (our variation of Poisson summation) gives us the required identity
\[
\sum_{n \in \mathcal{L}} \text{vol}(A \cap (B + n + x)) = \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^*} \hat{1}_A(\xi) \hat{1}_B(\xi) e^{2\pi i \langle \xi, x \rangle}
\]
\[
= \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^*} \hat{1}_A(\xi) \hat{1}_B(\xi) e^{2\pi i \langle \xi, x \rangle},
\]
for all \(x \in \mathbb{R}^d\). \(\square\)

**Remark 2.** We note that one direction of the known Theorem 5 of Kolountzakis, follows as a Corollary of Bombieri’s Theorem. Namely, suppose that \(Q\) multi-tiles \(\mathbb{R}^d\). Then we know by Lemma 2 that the left hand side of (6) is constant for all \(x\) and therefore by uniqueness of Fourier series, all of the Fourier coefficients on the right-hand side of (6) must vanish, except for \(\xi = 0\). This guarantees that

Theorem 5, part 2 \(\implies\) Theorem 5, part 1, and also that \(k = \frac{\text{vol } Q}{\det L}\).

8. **Proofs of Lemma 1 and Theorem 2**

**Proof.** (of Lemma 1) We observe that the right-hand side of (10) does not depend on boundary points of \(A\) or \(B\), which implies that
\[
\sum_{n \in \mathcal{L}} \text{vol}(A \cap (B + n)) = \sum_{n \in \mathcal{L}} \text{vol}(\text{int } A \cap (\text{int } B + n)).
\]

But
\[
A \cap (B + n) \supset \text{int } A \cap (\text{int } B + n)
\]
\[
\implies \text{vol}(A \cap (B + n)) \geq \text{vol}(\text{int } A \cap (\text{int } B + n)).
\]
Since the left hand side of equation (52) is finite, and all terms of both sides of (52) are non-negative, it follows that

\[
\text{vol} ( A \cap (B + n)) = \text{vol} (\text{int} A \cap (\text{int} B + n)) ,
\]

for each \( n \in \mathcal{L} \). By Lemma 6, we have

\[
\sum_{n \in \mathcal{L}} \text{vol} ( A \cap (B + n)) = \sum_{n \in \mathcal{L} \cap (A - B)} \text{vol} ( A \cap (B + n))
\]

\[
= \sum_{n \in \mathcal{L} \cap \text{int} (A - B)} \text{vol} ( A \cap (B + n)) + \sum_{n \in \mathcal{L} \cap \partial (A - B)} \text{vol} ( A \cap (B + n)) .
\]

where in the last step we have an equality since \( A - B \) is compact (by assumption), hence closed. On the other hand, again by Lemma 6, we have

\[
\sum_{n \in \mathcal{L}} \text{vol} (\text{int} A \cap (\text{int} B + n)) = \sum_{n \in \mathcal{L} \cap (\text{int} A - \text{int} B)} \text{vol} ( A \cap (B + n))
\]

\[
\leq \sum_{n \in \mathcal{L} \cap \text{int} (A - B)} \text{vol} ( A \cap (B + n)) ,
\]

where the last inequality is justified by observing that \( \text{int} A - \text{int} B \subset A - B \); therefore if \( \text{int} A - \text{int} B \) is open, then \( \text{int} A - \text{int} B \subset \text{int} (A - B) \). Combining inequalities (55) and (56), and then using identity (52), we get

\[
\sum_{n \in \mathcal{L} \cap \text{int} (A - B)} \text{vol} ( A \cap (B + n)) \geq \sum_{n \in \mathcal{L} \cap (A - B)} \text{vol} ( A \cap (B + n)) + \sum_{n \in \mathcal{L} \cap \partial (A - B)} \text{vol} ( A \cap (B + n))
\]

Inserting (54) into (57), we obtain

\[
\sum_{n \in \mathcal{L} \cap \partial (A - B)} \text{vol} ( A \cap (B + n)) \leq 0.
\]

Since each volume is non-negative, we must have

\[
\text{vol} ( A \cap (B + n)) = 0,
\]

for each \( n \in \mathcal{L} \cap \partial (A - B) \) as claimed. Part (b) follows as an immediate consequence, using equation (55). \qed
Proof. (of Theorem 2) To prove part (a), we will combine Lemma 1 part (b), with Theorem 1 part (c), using $x = 0$:

$$\sum_{n \in L \cap \text{int}(A - B)} \text{vol} \left( A \cap (B + n) \right) = \sum_{n \in L} \text{vol} \left( A \cap (B + n) \right)$$

$$= \frac{1}{\text{det } L} \sum_{\xi \in L^*} \hat{1}_A(\xi) \hat{1}_B(\xi)$$

$$= \frac{1}{\text{det } L} \text{vol } A \text{ vol } B + \frac{1}{\text{det } L} \sum_{\xi \in L^* \setminus \{0\}} \hat{1}_A(\xi) \overline{\hat{1}_B(\xi)}.$$

To prove part (b), we let $A = B = \frac{1}{2}P$ in (a) above. Because $P$ is now a centrally symmetric and convex body, we have $A - B = \frac{1}{2}P - \frac{1}{2}P = P$. Together with $\text{vol } A = \text{vol } B = \frac{\text{vol } P}{2^d}$, we obtain:

$$\sum_{n \in \text{int}(P) \cap L} \text{vol} \left( \frac{1}{2}P \cap \left( \frac{1}{2}P + n \right) \right) = \frac{1}{\text{det } L} \frac{\text{vol}^2 P}{2^d} + \frac{1}{\text{det } L} \sum_{\xi \in L^* \setminus \{0\}} \left| \hat{1}_P(\xi) \right|^2.$$  

Thus,

$$\sum_{n \in \text{int}(P) \cap L} \text{vol} \left( P \cap (P + 2n) \right) = \frac{1}{\text{det } L} \frac{\text{vol}^2 P}{2^d} + \frac{2^d}{\text{det } L} \sum_{\xi \in L^* \setminus \{0\}} \left| \hat{1}_P(\xi) \right|^2.$$

□

9. PROOFS OF COROLLARIES 2, 3, AND 5

The following proof of Corollary 2 essentially follows by combining Bombieri's Theorem 1, Koloundakis' Theorem 5, and Lemma 1.

Proof. (of Corollary 2) Since $Q \subset \mathbb{R}^d$ is a compact set, equation (6) gives (with $x = 0$):

$$\sum_{n \in L} \text{vol} \left( Q \cap (Q + n) \right) = \frac{1}{\text{det } L} \sum_{\xi \in L^*} \left| \hat{1}_Q(\xi) \right|^2 \geq \frac{1}{\text{det } L} \left| \hat{1}_Q(0) \right|^2 = \frac{\text{vol}^2 Q}{\text{det } L}.$$

We may rewrite the left-hand side of (62), using Lemma 1, part (b), as follows:

$$\sum_{n \in L \cap \text{int}(Q - Q)} \text{vol} \left( Q \cap (Q + n) \right) \geq \frac{\text{vol}^2 Q}{\text{det } L}.$$

Now we'll show that equality holds in (63) if and only if $Q$ multi-tiles, with the lattice $L$. First, let's assume that $Q$ multi-tiles $\mathbb{R}^d$; that is, Theorem 5, part 2 holds. Since Theorem 5, part 2 $\Longrightarrow$ Theorem 5, part 1, we have

$$\sum_{n \in L \cap \text{int}(Q - Q)} \text{vol} \left( Q \cap (Q + n) \right) = \frac{1}{\text{det } L} \left| \hat{1}_Q(0) \right|^2 = \frac{\text{vol}^2(Q)}{\text{det } L}.$$
Conversely, if equality holds in (63), then

\[ \sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) = \frac{\text{vol}^2(Q)}{\det L}, \]

and by (6) with \( x = 0 \) we therefore must have \( |\hat{1}_Q(\xi)|^2 = 0 \) for all \( \xi \in \mathcal{L}^* \), excluding the origin. This means that Theorem 5, part 1 holds, and its equivalence with Theorem 5, part 2 tells us that \( Q \) multi-tiles \( \mathbb{R}^d \) with the lattice \( \mathcal{L} \). \( \square \)

**Proof.** (of Corollary 3) (a) \( \Rightarrow \) (b). Taking \( A = B \) in Theorem 2 part (a) we have

\[ \sum_{n \in \text{int}(A - A) \cap \mathcal{L}} \text{vol} \left( A \cap (A + n) \right) = \frac{1}{\det L} \text{vol} A \text{vol} B + \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \hat{1}_A(\xi) \overline{\hat{1}_B(\xi)} \]

\[ = \frac{1}{\det L} \text{vol}^2 A + \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} |\hat{1}_A(\xi)|^2, \]

implying that

\[ \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} |\hat{1}_A(\xi)|^2 = 0. \]

Hence \( \hat{1}_A(\xi) = 0 \) for all \( \xi \in \mathcal{L}^* \setminus \{0\} \). By Theorem 5 we conclude that \( A \) k-tiles \( \mathbb{R}^d \) by translations with \( \mathcal{L} \).

To prove (b) \( \Rightarrow \) (c), we use Theorem 5. Because \( A \) k-tiles \( \mathbb{R}^d \), \( \hat{1}_A(\xi) = 0 \) for all \( \xi \in \mathcal{L}^* \setminus \{0\} \). Again, by Theorem 2 part (a), we get

\[ \sum_{n \in \text{int}(A - A) \cap \mathcal{L}} \text{vol} \left( A \cap (A + n) \right) = \frac{1}{\det L} \text{vol} A \text{vol} B + \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} \hat{1}_A(\xi) \overline{\hat{1}_B(\xi)} \]

\[ = \frac{1}{\det L} \text{vol} A \text{vol} B + \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^* \setminus \{0\}} 0 \cdot \overline{\hat{1}_B(\xi)} \]

\[ = \frac{1}{\det L} \text{vol} A \text{vol} B, \]

and we are done.

Finally, to prove (c) \( \Rightarrow \) (a), we just take \( A = B \) in part (c) (eq. (24)). \( \square \)

**Proof.** (of Corollary 5) This proof follows from (6) by observing that the hypotheses guarantee that the left-hand side of equation (35) vanishes. Namely, we have:

\[ 0 = \sum_{n \in \mathcal{L}} \text{vol}(Q \cap (Q + n + x)) = \frac{1}{\det L} \sum_{\xi \in \mathcal{L}^*} |\hat{1}_Q(\xi)|^2 \cos (2\pi \langle \xi, x \rangle). \]

Since \( \hat{1}_Q(0) = \text{vol} Q \), we’re done. \( \square \)
Proof. To prove part (a), we simply substitute \( \text{vol}(A \cap (A + n)) = |F \cap (F + n)| \varepsilon^d \) and \( \text{vol} A = \varepsilon^d |F| \) into equation (17), and divide both sides of the ensuing equation by \( \varepsilon^d \).

To prove part (b), we merely have to simplify the expression \( \hat{1}_A(\xi) \). The main point here is that we have the disjoint union

\[
A = \bigcup_{a \in F} (\text{int} \Box + a).
\]

In other words, because \( 0 < \varepsilon \leq 1 \), the interiors of the translated \( \varepsilon \)-cubes are pairwise disjoint. Using this disjoint union, we compute:

\[
\begin{align*}
\hat{1}_A(\xi) &= \sum_{a \in F} \hat{1}_{\Box + a}(\xi) = \sum_{a \in F} \hat{1}_{\Box}(\xi) e^{2\pi i \langle \xi, a \rangle} \\
&= \hat{1}_{\Box}(\xi) \sum_{a \in F} e^{2\pi i \langle \xi, a \rangle} \\
&= \varepsilon^d \prod_{k=1}^d \text{sinc}(\pi \varepsilon \xi_k) \sum_{a \in F} e^{2\pi i \langle \xi, a \rangle},
\end{align*}
\]

where we’ve used the standard sinc formula \( \hat{1}_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) = \prod_{k=1}^d \text{sinc}(\pi \xi_k) \) for the Fourier transform of the unit cube (see for example [17]), as well as the general property \( \hat{1}_{\varepsilon S}(\xi) = \hat{1}_S(\varepsilon \xi) \). \( \Box \)

11. Proof of Theorem 4

Proof. To prove \( (a) \implies (b) \), we note that because the set \( F \) multi-tiles \( \mathbb{Z}^d \), its 1-thickening \( A \) multi-tiles \( \mathbb{R}^d \) so we can apply the implication \( (a) \implies (b) \) of Corollary (3) for the compact set \( A \):

\[
\sum_{n \in L \cap \text{int}(A - A)} \text{vol} \left( A \cap (A + n) \right) = \frac{\text{vol}^2 A}{\det L}.
\]

But, \( \text{vol} \left( A \cap (A + n) \right) = |F \cap (F + n)| \varepsilon^d = |F \cap (F + n)| \), and \( \text{vol} A = \varepsilon^d |F| = |F| \), proving part (b). Similarly, the implication \( (b) \implies (a) \) from Corollary (3) proves that \( (b) \implies (a) \) here.

To prove that \( (b) \implies (c) \), we recall Corollary (3), part (b), with \( \varepsilon = 1 \):

\[
\begin{align*}
\sum_{n \in (F - F) \cap L} |F \cap (F + n)| &= \frac{1}{\det L} |F|^2 + \frac{1}{\det L} \sum_{\xi \in L^* \setminus 0} \left( \prod_{k=1}^d \text{sinc}^2(\pi \xi_k) \left| \sum_{n \in F} e^{2\pi i \langle \xi, n \rangle} \right|^2 \right),
\end{align*}
\]

which holds for any finite set \( F \) of integer points. Therefore, the assumption of (b) is equivalent to the statement:

\[
\sum_{\xi \in L^* \setminus 0} \prod_{k=1}^d \text{sinc}^2(\pi \xi_k) \left| \sum_{n \in F} e^{2\pi i \langle \xi, n \rangle} \right|^2 = 0.
\]

Because all of the summands in the latter sum are nonnegative, equation (72) is equivalent to

\[
\prod_{k=1}^d \text{sinc}^2(\pi \xi_k) \left| \sum_{n \in F} e^{2\pi i \langle \xi, n \rangle} \right|^2 = 0,
\]
for each \( \xi \in \mathcal{L}^* \setminus 0 \). We recall that \( \frac{\sin(\pi \xi_k)}{\pi \xi_k} = 0 \iff \xi_k \in \mathbb{Z} \); so we see that for each nonzero \( \xi \in \mathcal{L}^* \) either one of its coordinates is an integer, or else \( \sum_{n \in F} e^{2\pi i (\xi, n)} = 0 \). This last observation is precisely the assertion of part (e), and hence part (e) is equivalent to part (b).

\[ \square \]

12. More examples

Example 6. While it is tempting to assert that \( \text{int}(Q - Q) = \text{int} Q - \text{int} Q \), for any compact set \( Q \), we give a rather extreme counterexample to this claim. Consider the usual cantor set \( Q \subset [0, 1] \), whose interior is known to be the empty set. It is known \([13]\) that its difference body satisfies the surprising identity \( Q - Q = [-1, 1] \), implying that \( \text{int}(Q - Q) = (-1, 1) \). However, \( \text{int} Q = \emptyset \), so we find that \( \text{int} Q - \text{int} Q = \emptyset \neq \text{int}(Q - Q) \). This example shows why the analysis in Lemma 1 is necessary. \( \square \)

Example 7. Suppose we have a convex, centrally symmetric body \( Q \subset \mathbb{R}^d \) that \( k \)-tiles \( \mathbb{R}^d \) with a lattice \( \mathcal{L} \). Consequently, we know that \( Q - Q = 2Q \). Here we use Theorem 2 to show that if \( Q \) \( k \)-tiles, and enjoys the property that \( Q - Q \) contains exactly 3 lattice points of \( \mathcal{L} \) in its interior, say \( 0, n_0, -n_0 \), then either \( k = 1 \) or \( k = 2 \).

By Theorem 2, we have:

\[
\frac{\text{vol}^2 Q}{\det \mathcal{L}} = \sum_{n \in \text{int}(Q - Q) \cap \mathcal{L}} \text{vol}(Q \cap (Q + n))
= \text{vol} Q + \text{vol}(Q \cap (Q + n_0)) + \text{vol}(Q \cap (Q - n_0))
= \text{vol} Q + 2\text{vol}(Q \cap (Q + n_0)).
\]

By Theorem 5, we also know that \( k = \frac{\text{vol} Q}{\det \mathcal{L}} \), so that

\[
(74) \quad k\text{vol} Q = \text{vol} Q + 2\text{vol}(Q \cap (Q + n_0)).
\]

Since \( Q \cap (Q + n_0) \subset Q \) we must have

\[
k\text{vol} Q = \text{vol} Q + 2\text{vol}(Q \cap (Q + n_0)) < 3\text{vol} Q,
\]

giving us

\[
k \leq 2,
\]
as was claimed. As an aside, together with equation (74), we now also have

\[
(75) \quad k = 1 + 2\frac{\text{vol}(Q \cap (Q + n_0))}{\text{vol} Q} \leq 2.
\]

and therefore

\[
(76) \quad \text{vol}(Q \cap (Q + n_0)) = \frac{1}{2} \text{vol} Q \quad \text{or} \quad \text{vol}(Q \cap (Q + n_0)) = 0.
\]

\[ \square \]
Example 8. The classical identity of Euler, namely $\zeta(2) := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ can be retrieved from Bombieri’s theorem (eq. (6)) by using the integer lattice $\mathcal{L} = \mathbb{Z}^2$ together with the triangle $\mathcal{Q} := \Delta$, defined by the vertices $v_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $v_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, as shown in Figure 8.

On the left-hand side of equation (6), we have (with $x = 0$):

$$\sum_{n \in \mathbb{Z}^2} \text{vol}(\Delta \cap (\Delta + n)) = \text{vol} \Delta = \frac{1}{2}.$$ 

On the right-hand side of (6) we can compute directly the summands $|\hat{1}_\Delta(\xi)|^2$, using the following known formula.

Lemma 3. For the standard $d$-dimensional simplex $\Delta_d := \{ x \in \mathbb{R}^d_{\geq 0} \mid x_1 + \cdots + x_d \leq 1 \} \subset \mathbb{R}^d$, its Fourier transform vanishes for all integer points $\xi$ with the property that all of their coordinates are nonzero and distinct from each other.

Proof. The Fourier transform of $\Delta_d$ is given by:

$$\hat{1}_{\Delta_d}(\xi) := \int_{\Delta_d} e^{-2\pi i \langle \xi, x \rangle} dx = \frac{1}{(2\pi i)^d} \sum_{i=0}^d \prod_{0 \leq j \leq d, j \neq i} (\xi_i - \xi_j),$$

where we let $\xi_i \neq \xi_j$ for every $i \neq j$, and where we’ve defined $\xi_0 := 0$ ([2], or [14], Lemma 21). If we restrict $\xi$ to be an integer point, then for each coordinate $\xi_i \in \mathbb{Z} (i \geq 1)$, we have $e^{-2\pi i \xi_i} = 1$. By the Lagrange interpolation polynomial applied to the points $(\xi_i, 1) \in \mathbb{R}^2$, we have $\hat{1}_{\Delta_d}(\xi) = 0$. The reason for this vanishing is that these points lie on a horizontal line - the constant polynomial. Consequently, all coefficients in the interpolating formula vanish except the constant coefficient, which equals 1 by uniqueness of the polynomial with degree less than or equal to $d + 1$. □

From the discussion above, it is sufficient to restrict attention to only those integer points $\xi \in \mathbb{Z}^d$ which have at least one vanishing coordinate, or at least two equal coordinates. When $d = 2$, the only integer vectors $(\xi_1, \xi_2) \in \mathbb{Z}^2$ for which $\hat{1}_{\Delta_d}(\xi)$ possibly does not vanish are those that are orthogonal to the
sides of $\triangle_2$, namely the integer points belonging to the family $\mathcal{F} = \left\{ \binom{0}{k} \mid k \in \mathbb{Z} \right\} \cup \left\{ \binom{k}{0} \mid k \in \mathbb{Z} \right\} \cup \left\{ \binom{k}{k} \mid k \in \mathbb{Z} \right\}$, as drawn in Figure 8. By symmetry, $\left| \hat{1}_{\triangle_2}(0) \right|^2 = \left| \hat{1}_{\triangle_2}(\frac{k}{k}) \right|^2$, for all $k \in \mathbb{Z}$. We compute, for each $k \in \mathbb{Z} - \{0\}$:

$$\left| \hat{1}_{\triangle_2}(0) \right|^2 = \left| \int_0^1 \int_0^1 e^{-2\pi i k x_1} dx_1 dx_2 \right|^2 = \frac{1}{4 \pi^2 k^2} \left| \int_0^1 (e^{2\pi i k x_2} - 1) dx_2 \right|^2 = \frac{1}{4 \pi^2 k^2}.$$  

A similar computation gives $\left| \hat{1}_{\triangle_2}(\frac{k}{k}) \right|^2 = \frac{1}{4 \pi^2 k^2}$. By Bombieri’s theorem (eq. (6)), we have:

$$\frac{1}{2} = \frac{1}{\det \mathbb{Z}^2} \sum_{\xi \in \mathcal{F}} \left| \hat{1}_{\triangle_2}(\xi) \right|^2 = \left| \hat{1}_{\triangle_2}(0) \right|^2 + 3 \sum_{k \in \mathbb{Z} - \{0\}} \frac{1}{4 \pi^2 k^2} = (\text{vol } \triangle_2)^2 + \frac{3}{4 \pi^2} \sum_{k \in \mathbb{Z} - \{0\}} \frac{1}{k^2} = \frac{1}{4} + \frac{3}{2 \pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

giving the required classical identity $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Figure 9. The Tetrahedron $\triangle_3$, the integer lattice $\mathbb{Z}^3$, and the hyperplanes where $\hat{1}_{\triangle_3}(\xi) \neq 0$. There are four more analogous hyperplanes with this property.
Example 9. Here we show that it is possible to recover the values of the Riemann zeta function at all positive even integers, namely $\zeta(2d)$, by using (6). This approach extends Example 8, and we begin by considering the integer lattice, together with the standard simplex $Q := \triangle_d$. We will compute $\zeta(4) := \sum_{n=1}^{\infty} \frac{1}{n^4}$.

First, we have $\sum_{n \in \mathbb{Z}} \text{vol}(\triangle_3 \cap (\triangle_3 + n)) = \text{vol} \triangle_3 = \frac{1}{6}$, The crucial ingredient on the right-hand side of equation (6) is the transform $|\hat{1}_{\triangle_3}(\xi)|^2 = \left| \int_{\triangle_3} e^{-2\pi i \langle \xi, x \rangle} dx \right|^2$. The only $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{Z}^3$ for which the latter integral does not vanish are those integer points lying on 6 particular hyperplanes, by Lemma 3. More precisely, these are exactly the integer points in the family

$$\mathcal{F} = \{(0, k, k), (k, 0, k), (k, k, 0), (k, k, l), (l, k, k), (l, k, k)\}_{k, l \in \mathbb{Z}}.$$

After some computations we obtain

$$|\hat{1}_{\triangle_3}(0, k, l)|^2 = \frac{1}{16\pi^4} \cdot \frac{1}{k^2l^2} \text{ for } k \neq 0 \text{ and } k \neq l,$$

$$|\hat{1}_{\triangle_3}(0, k, k)|^2 = \frac{1}{16\pi^4} \cdot \frac{4}{k^4} \text{ for } k \neq 0,$$

$$|\hat{1}_{\triangle_3}(k, k, l)|^2 = \frac{1}{16\pi^4} \cdot \frac{1}{k^2(k-l)^2} \text{ for } k, l \neq 0 \text{ and } k \neq l,$$

$$|\hat{1}_{\triangle_3}(0, 0, 0)|^2 = |\hat{1}_{\triangle_3}(k, k, k)|^2 = \frac{1}{16\pi^4} \cdot \left( \frac{1}{k^4} + \frac{\pi^2}{k^2} \right) \text{ for } k \neq 0.$$

By symmetry, all permutations of the coordinates yield the same result for each computation above. By (6) we have

$$\frac{1}{6} = \text{vol} \triangle_3 = \frac{1}{\text{det} \mathbb{Z}^3} \sum_{\xi \in \mathcal{F}} |\hat{1}_{\triangle_3}(\xi)|^2 = |\hat{1}_{\triangle_3}(0, 0, 0)|^2 + 3 \sum_{k, l \neq 0, k \neq l} |\hat{1}_{\triangle_3}(0, k, l)|^2 + 3 \sum_{k, l \neq 0, k \neq l} |\hat{1}_{\triangle_3}(k, k, l)|^2 +$$

$$+ 3 \sum_{k \neq 0} |\hat{1}_{\triangle_3}(0, 0, k)|^2 + \sum_{k \neq 0} |\hat{1}_{\triangle_3}(k, k, k)|^2 + 3 \sum_{k \neq 0} |\hat{1}_{\triangle_3}(0, k, k)|^2$$

$$= (\text{vol} \triangle_3)^2 + \frac{1}{16\pi^4} \left( 3 \sum_{k, l \neq 0, k \neq l} \frac{1}{k^2l^2} + 3 \sum_{k, l \neq 0, k \neq l} \frac{1}{k^2(k-l)^2} + 3 \sum_{k \neq 0} \left( \frac{1}{k^4} + \frac{\pi^2}{k^2} \right) + \sum_{k \neq 0} \left( \frac{1}{k^4} + \frac{\pi^2}{k^2} \right) + 3 \sum_{k \neq 0} \frac{4}{k^4} \right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left( 3 \sum_{k, l \neq 0, k \neq l} \frac{1}{k^2l^2} - 3 \sum_{k \neq 0} \frac{1}{k^4} + 3 \sum_{k \neq 0, k \neq l} \frac{1}{k^2(k-l)^2} - 3 \sum_{k \neq 0} \frac{1}{k^4} + 4 \sum_{k \neq 0} \left( \frac{1}{k^4} + \frac{\pi^2}{k^2} \right) + 3 \sum_{k \neq 0} \frac{4}{k^4} \right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left( 24 \sum_{k, l > 0} \frac{1}{k^2l^2} + 20 \sum_{k > 0} \frac{1}{k^4} + 8 \sum_{k > 0} \frac{\pi^2}{k^2} \right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left( 24 \left( \frac{\pi^2}{6} \right)^2 + 20 \sum_{k > 0} \frac{1}{k^4} + 8\pi^2 \cdot \frac{\pi^2}{6} \right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left( 2\pi^4 + 20 \sum_{k > 0} \frac{1}{k^4} \right)$$

$$= \frac{11}{72} + \frac{5}{4\pi^4} \sum_{k > 0} \frac{1}{k^4}.$$
giving us the well-known identity $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$.

13. Further remarks

Here we mention some open problems, for future research.

**Problem 1.** Extend the notion of Poisson summation friendly sets to include unbounded sets, and classify them.

An answer to this problem would enable a generalization of the current results to functions whose support is unbounded. Regarding multi-tiling bodies, the following simple-sounding question is still open.

**Problem 2.** If $Q$ is a convex body that $k$-tiles $\mathbb{R}^d$ (nontrivially), then $k \neq 2$.

Problem 2 is known to be true in dimensions 2 and 3, due to the recent work of [11], but it is open for $d \geq 4$.

**Problem 3.** Apply the methods and results herein to study Matheron’s Conjecture 1.

**Problem 4.** Using Theorems 1 and 2, with some appropriate Poisson summation friendly set $Q$ (possibly nonconvex), find a geometric interpretation for $\zeta(3), \zeta(5), \text{etc.}$

We note that there are already some fascinating geometric interpretations for the odd special values $\zeta(2n+1)$, following the work of Ed Witten on volumes of certain moduli spaces of flat connections ([22], eq. 4.93). In Example 9, we picked simplices, and we retrieved the even values of $\zeta(s)$ from a spectral representation of their volumes. Perhaps more can be done for Problem 4 by picking more complex sets and computing their volumes in eq. (6), or eq. (10).

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Appendix A. Some useful and known lemmas

For completeness, we include here a new proof of (7), which is a variant of the Poisson summation formula. This variant appears to be not very well known. For a different proof, using locally compact abelian groups, see [16]. We also give proofs of some auxiliary lemmas that we require in the body of the paper, and which are also known. First, we recall the following well-known Plancherel-Polya type inequality (see [18] page 216, Theorem 6, for example).

Lemma 4 (Plancherel-Polya). Let \( g : \mathbb{R}^d \to \mathbb{C} \) be a compactly supported function, with \( \hat{g} \in L^1(\mathbb{R}^d) \). Then there is a positive constant \( M \) such that

\[
\sum_{m \in \mathcal{L}} |\hat{g}(m)| \leq M \int_{\mathbb{R}^d} |\hat{g}(\xi)|. \tag{77}
\]

Proof. Let \( \psi \) be a smooth and compactly supported function such that \( \psi(x) = 1 \) for all \( x \) in the support of \( g \). So we have \( g(x) = \psi(x)g(x) \) for all \( x \in \mathbb{R}^d \). By our assumption that \( \hat{g} \in L^1(\mathbb{R}^d) \) (and clearly \( \hat{\psi} \in L^1(\mathbb{R}^d) \)) we also have \( \hat{g}(\xi) = (\hat{\psi} * \hat{g})(\xi) \). We can now bound the series:

\[
\sum_{m \in \mathcal{L}} |\hat{g}(m)| = \sum_{m \in \mathcal{L}} \left| \int_{\mathbb{R}^d} \hat{\psi}(m - \xi) \hat{g}(\xi) \, d\xi \right| \tag{78}
\]

\[
\leq \sum_{m \in \mathcal{L}} \int_{\mathbb{R}^d} |\hat{\psi}(m - \xi)| |\hat{g}(\xi)| \, d\xi \tag{79}
\]

\[
= \int_{\mathbb{R}^d} \sum_{m \in \mathcal{L}} |\hat{\psi}(m - \xi)||\hat{g}(\xi)| \, d\xi \tag{80}
\]

\[
\leq \sup_{\xi \in \mathbb{R}^d} \left( \sum_{m \in \mathcal{L}} |\hat{\psi}(m - \xi)| \right) \int_{\mathbb{R}^d} |\hat{g}(\xi)| \, d\xi \tag{81}
\]

\[
= M \int_{\mathbb{R}^d} |\hat{g}(\xi)| \, d\xi < \infty. \tag{82}
\]

The integral-sum interchange in (80) is justified by the monotone convergence Theorem. We observe that \( \sum_{m \in \mathcal{L}} |\hat{\psi}(m - \xi)| := h(\xi) \) is a periodic function which is therefore completely determined on \( \mathbb{R}^d/\mathcal{L} \). Due to the rapid decay of \( \hat{\psi} \), \( h(\xi) \) is uniformly convergent on \( \mathbb{R}^d/\mathcal{L} \), and being a series of continuous functions, \( h(\xi) \) is itself continuous. Since \( \mathbb{R}^d/\mathcal{L} \) is compact, \( h \) attains its supremum there, a finite constant \( M > 0 \).

\[
\begin{proof}
\begin{proof}
(A Poisson Summation Formula, equation (7)) The hypothesis that both \( g \) and \( \hat{g} \in L^1(\mathbb{R}^d) \) implies that we can use Fourier inversion (see [8], Theorem 9.36). So \( g(x) = \mathcal{F}(\hat{g})(-x) \) is the image of an \( L^1 \) function under \( \mathcal{F} \), and therefore uniformly continuous. Similarly \( \hat{g} \) is uniformly continuous. Our goal is to prove that:

\[
\sum_{n \in \mathcal{L}} g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{g}(\xi) e^{2\pi i \langle \xi, x \rangle}. \tag{83}
\]
\end{proof}
\end{proof}
\]
To this end, we first pick an infinitely smooth approximate identity \( \varphi_\varepsilon \) that is supported on the unit ball, with \( \varphi_\varepsilon \geq 0 \) and \( \int_{\mathbb{R}^d} \varphi_\varepsilon(x) \, dx = 1 \) for all \( \varepsilon > 0 \) (i.e. \( \varphi_\varepsilon \) is a bump function [20], page 209).

Since \( \varphi_\varepsilon \) vanishes outside a ball of radius \( \varepsilon \) and \( g \) is compactly supported, we have that \( \varphi_\varepsilon \ast g \) is compactly supported on a set that is independent of \( \varepsilon \). Therefore the sum

\[
\sum_{n \in \mathcal{L}} \varphi_\varepsilon \ast g(n + x) := G(x)
\]

has a finite number of terms and is \( \mathcal{L} \)-periodic. Consequently, for \( m \in \mathcal{L}^* \) we have:

\[
\hat{G}(m) := \int_{\mathbb{R}^d / \mathcal{L}} \left( \sum_{n \in \mathcal{L}} \varphi_\varepsilon \ast g(n + x) \right) e^{-2\pi i (m,x)} \, dx
\]

\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d / \mathcal{L}} \varphi_\varepsilon \ast g(n + x) e^{-2\pi i (m,x)} \, dx,
\]

so that we have:

\[
\hat{G}(m) = \sum_{n \in \mathcal{L}} \int_{n+\mathbb{R}^d / \mathcal{L}} \varphi_\varepsilon \ast g(y) e^{-2\pi i (m,y-n)} \, dy
\]

\[
= \sum_{n \in \mathcal{L}} \int_{n+\mathbb{R}^d / \mathcal{L}} \varphi_\varepsilon \ast g(y) e^{-2\pi i (m,y)} \, dy
\]

\[
= \int_{\mathbb{R}^d} \varphi_\varepsilon \ast g(y) e^{-2\pi i (m,y)} \, dy
\]

\[
= \hat{\varphi}_\varepsilon \ast g(y)(m) = \hat{\varphi}_\varepsilon(m) \hat{g}(m) = \hat{\varphi}(\varepsilon m) \hat{g}(m).
\]

We notice that due to the fact that both \( \varphi_\varepsilon \) and \( g \) are compactly supported, and that \( \varphi_\varepsilon \) is infinitely smooth, \( \varphi_\varepsilon \ast g \) belongs to the Schwartz class \( S(\mathbb{R}^d) \). Thus, the basic Poisson summation formula for Schwartz functions holds:

\[
(84) \quad \sum_{n \in \mathcal{L}} \varphi_\varepsilon \ast g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{\varphi}(\varepsilon m) \hat{g}(m) e^{2\pi i (m,x)},
\]

for every \( x \in \mathbb{R}^d \). Because \( g \) is continuous and locally integrable on \( \mathbb{R}^d \), the Lebesgue set of \( g \) is \( \mathbb{R}^d \) and we have

\[
(85) \quad \lim_{\varepsilon \to 0^+} \sum_{n \in \mathcal{L}} \varphi_\varepsilon \ast g(n + x) = \sum_{n \in \mathcal{L}} \lim_{\varepsilon \to 0^+} \varphi_\varepsilon \ast g(n + x) = \sum_{n \in \mathcal{L}} g(n + x).
\]

for every \( x \in \mathbb{R}^d \). Therefore

\[
(86) \quad \sum_{n \in \mathcal{L}} g(n + x) = \lim_{\varepsilon \to 0^+} \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{\varphi}(\varepsilon m) \hat{g}(m) e^{2\pi i (m,x)}.
\]
Our next goal is to allow the interchange of the limit and the latter series in (86), which is a subtle point. First, we have

\[ |\hat{\varphi}(\epsilon m)| = \left| \int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i (x, \epsilon m)} \, dx \right| \leq \int_{\mathbb{R}^d} |\varphi(x)| \, dx = 1. \]

Summarizing, we have \( |\hat{\varphi}(\epsilon x)\hat{g}(x)| \leq |\hat{g}(x)| \), which is an absolutely summable dominating function because

\[ \sum_{m \in \mathcal{L}^*} |\hat{g}(m)| \leq M \int_{\mathbb{R}^d} |\hat{g}(\xi)| < \infty, \]

for some positive constant \( M \), by invoking Lemma 4 (a Plancherel-Polya inequality) with \( \mathcal{L} \) replaced by \( \mathcal{L}^* \). By the Lebesgue dominated convergence theorem (applied to the counting measure on \( \mathcal{L}^* \)), we have

\[ \lim_{\epsilon \to 0^+} \sum_{m \in \mathcal{L}^*} \hat{\varphi}(\epsilon m)\hat{g}(m)e^{2\pi i (m, x)} = \sum_{m \in \mathcal{L}^*} \lim_{\epsilon \to 0^+} \hat{\varphi}(\epsilon m)\hat{g}(m)e^{2\pi i (m, x)} = \sum_{m \in \mathcal{L}^*} \hat{g}(m)e^{2\pi i (m, x)}, \]

where we’ve used the continuity of \( \hat{\varphi} \) in the last equality, and also \( \hat{\varphi}(0) = 1 \). Finally, putting together (86) with the latter computation, we have

\[ \sum_{n \in \mathcal{L}} g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{g}(m)e^{2\pi i (m, x)} \]

for all \( x \in \mathbb{R}^d \). □

We also include the following 3 well-known lemmas, because often the precise relationship between convexity (or lack thereof) of \( Q \) and the difference body \( Q - Q \) can be subtle.

**Lemma 5.** Let \( Q \subset \mathbb{R}^d \) be a centrally symmetric body.

(a) \( Q - Q \supseteq 2Q \).

(b) If we also assume that \( Q \) is convex, then \( Q - Q = 2Q \).

*Proof.* To show (a), let \( x \in 2Q \). Then \( x = 2y \) with \( y \in Q \), implying that \( x = y + y = y - (-y) \in Q - Q \). Here we used the central symmetry of \( Q \) by invoking \(-y \in Q\).

To prove (b), it now suffices to show that \( Q - Q \subseteq 2Q \). So letting \( x \in Q - Q \), \( x = y - z \) with \( y, z \in Q \). We can rewrite \( x = y + (-z) = \left( \frac{1}{2}y + \frac{1}{2}(-z) \right) + \left( \frac{1}{2}y + \frac{1}{2}(-z) \right) \). Since \( Q \) is centrally symmetric we have \(-z \in Q \), and now the convexity of \( Q \) implies that \( \frac{1}{2}y + \frac{1}{2}(-z) \in Q \). Therefore \( x \in 2Q \). □

We observe that convexity is essential in part (b), by considering the following counterexample. Let \( C := [-2, -1] \cup [1, 2] \), a nonconvex set in \( \mathbb{R} \). Here \( C \) is centrally symmetric, yet \( C - C = [-3, 3] \neq [-4, -2] \cup [2, 4] = 2C \).
Lemma 6. Let $A, B \subset \mathbb{R}^d$ be any two sets, and fix any $x \in \mathbb{R}^d$. Then:

\begin{equation}
A \cap (B + x) \neq \emptyset \iff x \in A - B.
\end{equation}

Consequently, if $Q$ is a convex centrally symmetric body, then

\begin{equation}
Q \cap (Q + x) \neq \emptyset \iff x \in 2Q.
\end{equation}

Proof. Let $y \in A \cap (B + x)$. Then $y = z$ and $y = w + x$, where $z \in A$, and $w \in B$. This gives us $z = w + x$, so that $x = y - w \in A - B$, proving (90). The converse follows exactly the same logical steps. To prove condition (91), we first set $A = B = Q$ in (90). Because $Q$ is now also convex and centrally symmetric, Lemma (5) implies that $Q - Q = 2Q$, proving (91).

The following is a standard exercise, but we include it here because it is used in Corollary 3.

Lemma 7. Consider the unit cube $\Box := \left[\frac{1}{2}, -\frac{1}{2}\right]^d$. Then for $\xi \in \mathbb{R}^d$, we have

\begin{equation}
\hat{1}_\Box(\xi) = 0 \iff \text{at least one of the coordinates of } \xi \text{ is a nonzero integer}.
\end{equation}

Proof. We compute:

\begin{equation}
\hat{1}_\Box(\xi) = \int_\Box e^{-2\pi i \langle \xi, x \rangle} dx = \prod_{k=1}^d \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \langle \xi_k x_k \rangle} dx_k.
\end{equation}

The $k$’th term in the latter product (92) is just the sinc function $\text{sinc}(x_k) = \frac{\sin(\pi x_k)}{\pi x_k}$, when $\xi_k \neq 0$, and $\text{sinc}(0) := 1$. So the $k$’th term in the product defined by (92) vanishes if and only if $\xi_k \in \mathbb{Z} \setminus \{0\}$, and we’re done.

In other words, the zero set of the Fourier transform of the unit cube consists precisely of the discrete collection of hyperplanes, defined by $\xi_k = n$, for any nonzero integer $n$, and any index $1 \leq k \leq d$.

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