On some tensor inequalities based on the t-product

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Abstract

In this work, we investigate the tensor inequalities in the tensor t-product formalism. The inequalities involving tensor power are proved to hold similarly as standard matrix scenarios. We then focus on the tensor norm inequalities. The well-known arithmetic-geometric mean inequality, Hölder inequality, and Minkowski inequality are generalized to tensors. Furthermore, we obtain some t-eigenvalue inequalities.

Key words: t-positive semidefiniteness; tensor power; tensor norm inequality; t-eigenvalue

1 Introduction

Tensors are higher-order extensions of matrices. An order $m$ tensor can be regarded as a multidimensional array, which takes the form

$$\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}.$$ 

In this paper, we mainly focus on tensors of order three. The well-known representations of tensors are the CANDECOMP/PARAFAC [4] and Tucker model [20]. In the last ten years, the t-product [3,10] has been introduced as a generalization of matrix multiplication for tensors. It is shown that a discrete Fourier transform can be performed along the tube fibers of each tensor to compute the t-product efficiently. The tensor operation of the t-product has been proved to be a useful tool in many areas, finding applications in image processing [15,19], signal processing [12,18], and tensor compression [22], to name but a few.

Based on the t-product framework, Lund [14] posed the definition for the tensor t-function. Later, Miao et al. [16] defined the generalized tensor t-function by the tensor singular value decomposition. After that, the authors in [23] presented the definition of the t-positive (semi)definiteness of third-order symmetric tensors. In order to motivate further the development of the tensor analysis, one topic we are

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interested in is the results specific to particular t-functions. We will first concentrate on the tensor power including the square root of the symmetric t-positive semidefinite tensors. The other topic of interest is the tensor t-eigenvalue. A recent study of t-eigenvalues introduced some t-eigenvalue inequalities for Hermitian tensor such as Weyl’s theorem and Cauchy’s interlacing theorem [11].

As an important research field of scientific computing, matrix inequalities reflect the quantitative aspect of matrix analysis. Much work has been carried out to the development of matrix equalities [7,21]. So it is natural to talk about the tensor inequalities. In fact, Chang has built some inequalities in the aspects regarding trace function, Golden-Thompson inequality, Jenson’s inequality and Klein’s inequality, et al., for the t-product tensors [5,6]. In this paper, we will generalize some inequalities of matrix power, matrix norm inequalities and several classical eigenvalue inequalities to tensors.

This paper is organized as follows. In section 2, we review basic definitions and notations. Section 3 details the inequalities of tensor power. By the properties of the tensor spectral and Frobenius norm, results for the norm inequalities of tensor functions are presented in section 4. Section 5 studies the tensor t-eigenvalue inequalities. We give a conclusion in section 6.

2 Preliminaries

In this section, we review the t-product introduced by Kilmer et al. [9,10] and give some needed notations. Throughout this paper, third-order tensors denoted by calligraphic script letters are considered. Capital letters refer to matrices, and lower case letters to vectors. The ith frontal slice of tensor \( A \) will be denoted by \( A^{(i)} \). For Hermitian matrices \( G, H \) and symmetric tensors \( A \) and \( B \), we write \( G \leq H \) and \( A \leq B \) to mean that \( H - G \) is Hermitian positive semidefinite and \( B - A \) is a symmetric t-positive semidefinite tensor. In particular, \( H > (\geq)0 \) and \( A > (\geq)0 \) indicate that \( H \) is Hermitian positive (semi)definite and \( A \) is a symmetric t-positive (semi)definite tensor respectively.

For \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), define \( \text{bcirc} \) as a block circulant matrix of size \( n_1 n_3 \times n_2 n_3 \)

\[
\text{bcirc}(A) = \begin{bmatrix}
A^{(1)} & A^{(n_3)} & \cdots & A^{(2)} \\
A^{(2)} & A^{(1)} & \cdots & A^{(3)} \\
\vdots & \vdots & \ddots & \vdots \\
A^{(n_3)} & A^{(n_3-1)} & \cdots & A^{(1)}
\end{bmatrix}.
\]

The command \texttt{unfold} reshapes a tensor \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) into an \( n_1 n_3 \times n_2 \) block-column vector (the first block-column of \( \text{bcirc}(A) \)), while \texttt{fold} is the inverse, i.e., \( \text{fold}(\text{unfold}(A)) = A \).

**Definition 2.1.** (t-product) [9] Let \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) and \( B \in \mathbb{R}^{n_2 \times n_4 \times n_3} \). The t-product \( A \ast B \) is the tensor \( C \in \mathbb{R}^{n_1 \times n_4 \times n_3} \) defined by

\[ C = \text{fold}(\text{bcirc}(A) \cdot \text{unfold}(B)). \]
Note that the t-product reduces to the standard matrix multiplication when \( n_3 = 1 \). The Discrete Fourier Transformation (DFT) plays a core role in tensor-tensor product. The DFT on \( v \in \mathbb{R}^n \), denoted as \( \tilde{v} = F_n v \in \mathbb{C}^n \). Here \( F_n \) is the DFT matrix

\[
F_n = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{bmatrix},
\]

where \( \omega = e^{\frac{-2\pi i}{n}} \) is a primitive \( n \)th root of unity with \( i = \sqrt{-1} \) and \( F_n \) satisfies \( F_n^* F_n = F_n F_n^* = n I_n \). The block circulant matrix can be block diagonalized by the DFT, i.e.,

\[
(F_{n_3} \otimes I_{n_1}) \cdot \text{bcirc}(A) \cdot (F_{n_3}^{-1} \otimes I_{n_2}) = \tilde{A},
\]

where \( \otimes \) denotes the Kronecker product and \( \tilde{A} = \text{diag}(\tilde{A}^{(1)}, \tilde{A}^{(2)}, \ldots, \tilde{A}^{(n_3)}) \). By taking the Fast Fourier Transform (FFT) along each tubal scalar of \( A \), \( \tilde{A} = \text{fold}(A) = \text{fft}(A, [], 3) \) and \( A = \text{ifft}(\tilde{A}, [], 3) \). Then we have the following lemma.

**Lemma 2.1.** \([9]\) For \( A, B \) and \( C \) of appropriate size, the following statements hold:

\[
C = A \ast B \iff \tilde{C} = \tilde{A} \cdot \tilde{B},
\]

\[
C = A + B \iff \tilde{C} = \tilde{A} + \tilde{B}.
\]

**Definition 2.2.** (identity tensor) \([9]\) The identity tensor \( I \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) is the tensor with \( I^{(1)} \) being the \( n \times n \) identity matrix, and other frontal slices being zeros.

**Definition 2.3.** (tensor transpose) \([9]\) If \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), then \( A^T \) is the \( n_2 \times n_1 \times n_3 \) tensor obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through \( n_3 \).

**Definition 2.4.** (1-diagonal tensor) \([10]\) A tensor is called 1-diagonal if each of its frontal slices is a diagonal matrix.

**Definition 2.5.** (inverse tensor) \([9]\) An \( n \times n \times n_3 \) tensor \( A \) has an inverse \( B \), provided that \( A \ast B = I_{n_3} \) and \( B \ast A = I_{n_3} \).

**Definition 2.6.** \([9, 13]\) For \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), The Frobenius norm and the spectral norm of \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) are defined as

\[
\|A\|_F = \sqrt{\sum_{ijk} |a_{ijk}|^2}, \quad \|A\|_2 = \|\text{bcirc}(A)\|_2.
\]

We first summarize some basic properties about tensor norms.

**Lemma 2.2.** \([13]\) Let \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \). Then

\[
\|A\|_2 = \|\tilde{A}\|_2, \quad \|A\|_F = \frac{1}{\sqrt{n_3}}\|\tilde{A}\|_F.
\]
Definition 2.7. \([23]\) Let \(X, Y \in \mathbb{R}^{n_1 \times n_2 \times n_3}\). The inner product between \(X\) and \(Y\) is defined as
\[
\langle X, Y \rangle = \sum_{i,j,k} a_{ijk} b_{ijk}.
\]

Definition 2.8. (orthogonal tensor) \([9]\) An \(n \times n \times n_3\) real-valued tensor \(Q\) is orthogonal if \(Q^T \ast Q = Q \ast Q^T = I\).

Definition 2.9. (symmetric t-positive (semi)definite tensor) \([23]\) Let \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\). We say \(A\) is a symmetric t-positive (semi)definite tensor, if and only if \(A\) is a symmetric tensor and
\[
\langle X, A \ast X \rangle > (\geq) 0
\]
holds for any \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3} \setminus \{0\}\).

Note that the definition of symmetric t-positive (semi)definiteness given above is consistent with that in \([9]\).

Lemma 2.3. \([23]\) Suppose that \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) can be block diagonalized as (2.1). Then \(A\) is a symmetric t-positive (semi)definite tensor if and only if all the matrices \(\bar{A}^{(i)}\) (\(i = 1, \ldots, n_3\)) are Hermitian positive (semi)definite.

In view of the tensor t-function \([14]\), for any \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), we know that \((A^T \ast A)^{1/2}\) is a symmetric t-positive semidefinite tensor. Like the matrix case, we denote it as \(|A|\), i.e. \(|A| \equiv (A^T \ast A)^{1/2}\). The following result is a straightforward corollary of Lemma 2.3.

Corollary 2.1. Let \(A, B \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), then \(A \geq B\) if and only if \(\bar{A} \geq \bar{B}\).

Definition 2.10. \([11]\) Let \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\). Suppose that \(X \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) and \(X \neq 0\). If
\[
A \ast X = \lambda X, \quad \lambda \in \mathbb{C},
\]
then \(\lambda\) is called a t-eigenvalue of \(A\) and \(X\) is a t-eigenvector of \(A\) associated to \(\lambda\).

The authors in \([11]\) pointed out all the t-eigenvalues of \(A\) are actually the eigenvalues of the matrix \(\text{bcirc}(A)\), and vice versa. Hence the definition of the t-eigenvalues is equivalent to that given in \([17]\).

Definition 2.11. For \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), we say \(A\) a normal tensor if and only if
\[
A^T \ast A = A \ast A^T.
\]

### 3 Tensor power inequalities

We consider the generalization of several inequalities involving matrix power to the tensor scenarios.
### 3.1 Tensor Löwner-Heinz inequality

The celebrated Löwner-Heinz inequality can be generalized to tensors as follows.

**Theorem 3.1.** Let \( \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n \times n} \), \( \mathbf{A} \geq \mathbf{B} \geq 0 \), \( 0 \leq r \leq 1 \), then
\[
\mathbf{A}^r \geq \mathbf{B}^r. \tag{3.1}
\]

**Proof.** From Corollary 2.1 and the assumption, we know that \( \bar{\mathbf{A}}^{(i)} \geq \bar{\mathbf{B}}^{(i)} \geq 0 \). Due to \( [21, \text{Theorem 1.1}] \), \( (\bar{\mathbf{A}}^{(i)})^r \geq (\bar{\mathbf{B}}^{(i)})^r \), which means that \( (\bar{\mathbf{A}}^{(i)})^r - (\bar{\mathbf{B}}^{(i)})^r \) is positive semidefinite. The proof is complete as Lemma 2.3. \( \square \)

An illustrative example below shows that this result does not hold for \( r > 1 \).

**Example 3.1.** Assume \( \mathbf{A}, \mathbf{B} \in \mathbb{R}^{2 \times 2 \times 2} \), with
\[
\mathbf{A}^{(1)} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B}^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]
and the other frontal slices of \( \mathbf{A} \) and \( \mathbf{B} \) are zeros. Notice that \( \mathbf{A} \geq \mathbf{B} \geq 0 \), however, the first frontal slice of \( (\mathbf{A}^2 - \mathbf{B}^2) \) is \( \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \), and the second frontal slice is a zero matrix. Therefore, \( \mathbf{A}^2 \geq \mathbf{B}^2 \) does not hold.

Likewise, for the tensor power, we also have the following inequalities.

**Theorem 3.2.** Let \( \mathbf{Q} \) be orthogonal and \( \mathbf{X} \geq 0 \). Then
\[
\mathbf{Q} \ast \mathbf{X}^r \ast \mathbf{Q} \leq (\mathbf{Q} \ast \mathbf{X} \ast \mathbf{Q})^r, \text{ if } 0 < 1 \leq 1, \tag{3.2}
\]
\[
\mathbf{Q} \ast \mathbf{X}^r \ast \mathbf{Q} \geq (\mathbf{Q} \ast \mathbf{X} \ast \mathbf{Q})^r, \text{ if } 1 \leq r \leq 2. \tag{3.3}
\]

**Proof.** From the definition of orthogonality and Lemma 2.1 we know that \( \bar{\mathbf{Q}} \bar{\mathbf{X}}^r \bar{\mathbf{Q}} \leq (\bar{\mathbf{Q}} \bar{\mathbf{X}} \bar{\mathbf{Q}})^r, 0 < r \leq 1 \).

Applying Corollary 2.1 yields (3.2). (3.3) can be proved similarly. \( \square \)

**Theorem 3.3.** If \( \mathbf{A} \geq \mathbf{B} \geq 0 \), then
\[
(\mathbf{B}^r \ast \mathbf{A}^p \ast \mathbf{B}^r)^{1/q} \geq \mathbf{B}^{(p+2r)/q} \tag{3.4}
\]
and
\[
\mathbf{A}^{(p+2r)/q} \geq (\mathbf{A}^r \ast \mathbf{B}^p \ast \mathbf{A}^r)^{1/q} \tag{3.5}
\]
for \( r \geq 0, p \geq 0, q \geq 1 \) with \( (1 + 2r)q \geq p + 2r \).

**Proof.** The inequality (3.4) is a direct consequence of Corollary 2.1 and
\[
(\bar{\mathbf{B}}^r \cdot \bar{\mathbf{A}}^p \cdot \bar{\mathbf{B}}^r) \geq \bar{\mathbf{B}}^{(p+2r)/q},
\]
which follows from \( [21, \text{Theorem 1.16}] \). We can also obtain (3.5) in the same way. \( \square \)
Notice that the case \( p = q \geq 1 \) of Theorem 3.3 is the following.

**Corollary 3.1.** Suppose \( \mathcal{A} \geq \mathcal{B} \geq 0 \). We obtain that
\[
(B^r * \mathcal{A}^p * B^r)^{1/p} \geq B^{(p+2r)/p} \quad \text{and} \quad \mathcal{A}^{(p+2r)/p} \geq (\mathcal{A}^r * \mathcal{B}^r * \mathcal{A}^r)^{1/p}
\]
hold for all \( r \geq 0 \) and \( p \geq 1 \). Especially, if \( r = 1, p = 2 \), then
\[
(B^2 * \mathcal{A}^2 * B)^{1/2} \geq B^2 \quad \text{and} \quad \mathcal{A}^2 \geq (\mathcal{A} * \mathcal{B}^2 * \mathcal{A})^{1/2}.
\]

### 3.2 Tensor Young inequality

The gist of the most important case of the Young inequality is that if \( 1/p + 1/q = 1 \), with \( p, q > 1 \), then \( |ab| \leq |a|^p/p + |b|^q/q \) for \( a, b \in \mathbb{C} \). Ando in [1] pointed out that if \( \mathcal{A}, \mathcal{B} \) is a commuting pair and \( \mathcal{A} \mathcal{B} \geq 0 \), then it is clear that
\[
\mathcal{A} \mathcal{B} \leq \frac{\mathcal{A}^p}{p} + \frac{\mathcal{B}^q}{q}.
\]
Now we extend this classical inequality to tensors.

**Theorem 3.4.** If \( \mathcal{A}, \mathcal{B} \geq 0 \) is a commuting pair, i.e., \( \mathcal{A} \mathcal{B} = \mathcal{B} \mathcal{A} \) and \( \mathcal{A} \mathcal{B} \geq 0 \), then for \( p, q > 1 \),
\[
\mathcal{A} \mathcal{B} \leq \frac{1}{p} \mathcal{A}^p + \frac{1}{q} \mathcal{B}^q.
\]

**Proof.** Begin with the fact that \( \text{bcirc}(\mathcal{A} + \mathcal{B}) = \text{bcirc}(\mathcal{A}) + \text{bcirc}(\mathcal{B}) \). Then we have
\[
(F_{n_3} \otimes I_{n_1}) \cdot \text{bcirc}(\mathcal{A} \mathcal{B}) \cdot (F_{n_3}^{-1} \otimes I_{n_1}) = \bar{\mathcal{A}} \bar{\mathcal{B}} \leq \frac{1}{p} \bar{\mathcal{A}}^p + \frac{1}{q} \bar{\mathcal{B}}^q,
\]
\[
= (F_{n_3} \otimes I_{n_1}) \cdot \text{bcirc}(\frac{1}{p} \mathcal{A}^p + \frac{1}{q} \mathcal{B}^q) \cdot (F_{n_3}^{-1} \otimes I_{n_1}).
\]
Applying Corollary 2.1 again gives the conclusion. \( \Box \)

A generalized matrix Young inequality was given in [1].

**Lemma 3.1.** [1] Let \( p, q > 0 \) be mutually conjugate exponents, that is, \( 1/p + 1/q = 1 \). Then for any pair \( \mathcal{A}, \mathcal{B} \) of \( n \times n \) complex matrices, there is a unitary matrix \( U \) depending on \( \mathcal{A}, \mathcal{B} \) such that
\[
U^H |\mathcal{A} \mathcal{B}| U \leq \frac{|\mathcal{A}|^p}{p} + \frac{|\mathcal{B}|^q}{q}.
\]

It turns out that the generalized tensor Young inequality can be stated below.

**Theorem 3.5.** Let \( p, q > 0, 1/p + 1/q = 1 \). Then for any pair \( \mathcal{A}, \mathcal{B} \in \mathbb{R}^{n \times n \times n_3} \), there exists an orthogonal tensor \( \mathcal{U} \) depending on \( \mathcal{A} \) and \( \mathcal{B} \), such that
\[
\mathcal{U}^T * |\mathcal{A} * \mathcal{B}^T| * \mathcal{U} \leq \frac{1}{p} |\mathcal{A}|^p + \frac{1}{q} |\mathcal{B}|^q.
\]
Proof. First, we apply (2.1) to \( A, B \) and then we get \( \tilde{A} \) and \( \tilde{B} \), which are block diagonal matrices. According to Lemma 3.1, for any pair \( \tilde{A}(i), \tilde{B}(i) \), there is a unitary matrix \( U(i) \) such that

\[
(U(i))^H |\tilde{A}(i)(\tilde{B}(i))^H| U(i) \leq |\tilde{A}(i)|^p/p + |\tilde{B}(i)|^q/q.
\]

Set

\[
\tilde{U} = \text{fold} \left( \begin{bmatrix} U(1) \\ U(2) \\ \vdots \\ U(n) \end{bmatrix} \right), \quad \mathcal{U} = \text{ifft} (\tilde{U}, [\ ], 3),
\]

and

\[
\tilde{C} = \text{fold} \left( \begin{bmatrix} |\tilde{A}(1)(\tilde{B}(1))^H| \\ |\tilde{A}(2)(\tilde{B}(2))^H| \\ \vdots \\ |\tilde{A}(n)(\tilde{B}(n))^H| \end{bmatrix} \right), \quad \mathcal{C} = \text{ifft} (\tilde{C}, [\ ], 3).
\]

Obviously, \( \mathcal{U} \) is an orthogonal tensor and \( \mathcal{C} = |A \ast B|^T \). Analogously, we can get \( |A|^p \) and \( |B|^q \).

In particular, when we take \( p = q = 2 \), we get a corollary which is a tensor generalization of the result obtained by Bhatia and Kittaneh in [2].

**Corollary 3.2.** For any pair \( A, B \in \mathbb{R}^{n \times n \times n} \), there is an orthogonal tensor \( \mathcal{U} \) depending on \( A \) and \( B \), such that

\[
\mathcal{U}^T \ast |A \ast B|^T \ast \mathcal{U} \leq \frac{1}{2} |A|^2 + \frac{1}{2} |B|^2.
\]

4 **Tensor norm inequalities**

In this section, we derive some tensor norm inequalities based on the Frobenius norm and the spectral norm. Particularly, we will prove some norm inequalities related to the tensor power.

Notice that any tensor \( T \in \mathbb{C}^{n \times n \times n} \) can be written as \( T = A + iB \), where \( A, B \in \mathbb{R}^{n \times n \times n} \), and \((iB)(i, j, k) = iB(i, j, k)\). Then we have

\[
(F_{n_3} \otimes I_n) \cdot \text{bcirc}(T) \cdot (F_{n_3}^{-1} \otimes I_n)
= (F_{n_3} \otimes I_n) \cdot \text{bcirc}(A) \cdot (F_{n_3}^{-1} \otimes I_n) + (F_{n_3} \otimes I_n) \cdot \text{bcirc}(iB) \cdot (F_{n_3}^{-1} \otimes I_n)
= \tilde{A} + i(F_{n_3} \otimes I_n) \cdot \text{bcirc}(B) \cdot (F_{n_3}^{-1} \otimes I_n)
= \tilde{A} + i\tilde{B}.
\]

Before we move on to several classical inequalities, we mention some inequalities for the complex tensors.

**Theorem 4.1.** Let \( T = A + iB, A, B \in \mathbb{R}^{n \times n \times n} \).

(a) if \( A \) and \( B \) are symmetric, then

\[
(\|A\|_2^2 + \|B\|_2^2) \leq \|T\|_2^2 \leq 2(\|A\|_2^2 + \|B\|_2^2),
\]

where \( \| \cdot \|_2 \) is the Frobenius norm.
\[4(\|A\|_F^2 + \|B\|_F^2) \geq \|T\|_F^2 \geq (\|A\|_F^2 + \|B\|_F^2),\]

\[\|(A^2 + B^2)^{1/2}\|_2 \leq \|T\|_2 \leq \sqrt{2}||(A^2 + B^2)^{1/2}\|_2,\]

\[\|(A^2 + B^2)^{1/2}\|_F = \|T\|_F.\]

(b) if \(A\) is symmetric t-positive semidefinite and \(B\) is symmetric, then
\[\|T\|_2^2 \leq \|A\|_2^2 + 2\|B\|_2^2, \quad \|T\|_F^2 \geq \|A\|_F^2 + 2\|B\|_F^2.\]

(c) if \(A\) and \(B\) are symmetric t-positive semidefinite, then
\[\|T\|_2^2 \leq \|A\|_2^2 + \|B\|_2^2, \quad \|T\|_F^2 \leq \|A\|_F^2 + \|B\|_F^2.\]

**Proof.** The proof makes use of [21, Theorem 3.21] and Lemma 2.2. Indeed,
\[
4(\|A\|_F^2 + \|B\|_F^2) = \frac{4}{n^3}(\|\bar{A}\|_F^2 + \|\bar{B}\|_F^2) \\
\geq \frac{1}{n^3} \|\bar{T}\|_F^2 = \|T\|_F^2 \geq \frac{1}{n^3}(\|\bar{A}\|_F^2 + \|\bar{B}\|_F^2) = (\|A\|_F^2 + \|B\|_F^2),
\]

and
\[
\|(A^2 + B^2)^{1/2}\|_2 = \|(\bar{A}^2 + \bar{B}^2)^{1/2}\|_2 \\
\leq \|\bar{T}\|_2 = \|T\|_2 \leq \sqrt{2}||(\bar{A}^2 + \bar{B}^2)^{1/2}\|_2 = \sqrt{2}||(A^2 + B^2)^{1/2}\|_2.
\]

A similar procedure can be used for (b) and (c). \(\square\)

We are now set to state three classical inequalities. The arithmetic-geometric mean inequality for complex numbers \(a, b\) is \(|ab| \leq (|a|^2 + |b|^2)/2\). One tensor version of this inequality is the following result.

**Theorem 4.2.** For any real tensors \(A, \mathcal{X}\) and \(B\) of appropriate size, we have
\[\|A \ast \mathcal{X} \ast B^T\| \leq \frac{1}{2}\|A^T \ast \mathcal{X} + \mathcal{X} \ast B^T \ast B\|\]
for the Frobenius norm and the spectral norm.

**Proof.** Upon consideration of [21, Theorem 4.19] and Lemma 2.2 we see that
\[\|A \ast \mathcal{X} \ast B^T\|_F = \frac{1}{\sqrt{n^3}} \|\bar{A}\bar{X}\bar{B}^H\|_F \\
\leq \frac{1}{2\sqrt{n^3}} \|\bar{A}^H\bar{A}\bar{X} + \bar{X}\bar{B}^H\bar{B}\|_F = \frac{1}{2}\|A^T \ast \mathcal{X} + \mathcal{X} \ast B^T \ast B\|_F.
\]

The proof of the spectral norm case is entirely analogous. \(\square\)

If, additionally, the tensors \(A, \mathcal{X}\) and \(B\) are symmetric t-positive definite, then the following theorem arises by combining [21, Theorems 4.24 and 4.25] and Lemma 2.2. \(\square\)
Theorem 4.3. Let $A, X, B \in \mathbb{R}^{n \times n \times 3}$ with $A$ and $B$ symmetric t-positive semidefinite. Then for the Frobenius norm and the spectral norm, the following two inequalities hold.

(1) For any real numbers $r, t$ satisfying $1 \leq 2r \leq 3, -2 < t \leq 2$,
\[
(2 + t)\|A^r * X * B^{2-r} + A^{2-r} * X * B^r\| \leq 2\|A^2 * X + tA * X * B + X * B^2\|.
\]
(2)
\[
4\|A * B\| \leq \|(A + B)^2\|.
\]

Now we turn to the tensor Hölder inequality.

Theorem 4.4. Let $A, X, B \in \mathbb{R}^{n \times n \times 3}$ with $A$ and $B$ being symmetric t-positive semidefinite. Then
\[
\| |A * X * B| \|_F \leq n^{\frac{1}{2p} + \frac{1}{2q} - \frac{1}{3}} \| |A^p * X^q| \|_F^{1/p} \cdot \| |B^q| \|_F^{1/q},
\]
and
\[
\| |A * X * B| \|_2 \leq \| |A^p * X^q| \|_2^{1/p} \cdot \| |X * B^q| \|_2^{1/q},
\]
for all positive real numbers $r, p, q$ with $1/p + 1/q = 1$.

Proof. By [21, Theorem 4.29], it is easy to verify that
\[
\| |A * X * B| \|_F = \frac{1}{\sqrt{n^3}} \| |A^p * X^q| \|_F
\]
\[
\leq \frac{1}{\sqrt{n^3}} \left( \| |A^p * X^q| \|_F^{1/p} \cdot \| |X * B^q| \|_F^{1/q} \right)
\]
\[
= n^{\frac{1}{2p} + \frac{1}{2q} - \frac{1}{3}} \| |A^p * X^q| \|_F^{1/p} \cdot \| |X * B^q| \|_F^{1/q}.
\]
The second inequality follows by mimicking the above argument. \qed

Corollary 4.1. For $A, B \in \mathbb{R}^{n \times n \times n_3}$, then
\[
\| |A * B| \|_F \leq n^{\frac{1}{2p} + \frac{1}{2q} - \frac{1}{3}} \| |A^p| \|_F \cdot \| |B^q| \|_F^{1/q},
\]
and
\[
\| |A * B| \|_2 \leq \| |A^p| \|_2 \cdot \| |B^q| \|_2^{1/q},
\]
where $r, p, q$ are positive real numbers with $1/p + 1/q = 1$.

The subsequent result is another tensor Hölder inequality.

Theorem 4.5. Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. Then for all $A, B, C, D \in \mathbb{R}^{n \times n \times n_3}$, we have
\[
2^{-\frac{1}{2p} - \frac{1}{2q}} \| C^T * A + D^T * B \|_F \leq n^{\frac{1}{2p} + \frac{1}{2q} - \frac{1}{3}} \| |A^p + |B^p| \|_F^{1/p} \cdot \| |C^q + |D^q| \|_F^{1/q},
\]
and
\[
2^{-\frac{1}{2p} - \frac{1}{2q}} \| C^T * A + D^T * B \|_2 \leq \| |A^p + |B^p| \|_2^{1/p} \cdot \| |C^q + |D^q| \|_2^{1/q}.
\]
Proof. We only consider the Frobenius norm case. It is straightforward to show that
\[
2^{-\frac{1}{p} - \frac{1}{q}} \| C^T * A + D^T * B \|_F = \frac{1}{\sqrt{n_3}} 2^{-\frac{1}{p} - \frac{1}{q}} \| \bar{C}^H A + \bar{D}^H \bar{B} \|_F
\]
\[
\leq \frac{1}{\sqrt{n_3}} \| |\bar{A}|^p + |\bar{B}|^p \|_F^{1/p} \cdot \| |\bar{C}|^q + |\bar{D}|^q \|_F^{1/q}
\]
\[
= n_3^{\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} \| |A|^p + |B|^p \|_F^{1/p} \cdot \| |C|^q + |D|^q \|_F^{1/q},
\]
where we utilize [21, Theorem 4.34].

We continue in this section by discussing the tensor Minkowski inequality.

**Theorem 4.6.** Let \( 1 \leq p \leq \infty \). For \( A_i \) and \( B_i \in \mathbb{R}^{n_1 \times n_1 \times n_3} (i = 1, 2) \),
\[
2^{-\frac{1}{p} - \frac{1}{q}} \| |A_1 + A_2|^p + |B_1 + B_2|^p \|_F^{1/p} \leq \| |A_1|^p + |B_1|^p \|_F^{1/p} + \| |A_2|^p + |B_2|^p \|_F^{1/p},
\]
holds for both the Frobenius norm and the spectral norm.

**Proof.** It follows from [21, Theorem 4.35] that
\[
2^{-\frac{1}{p} - \frac{1}{q}} \| |A_1 + A_2|^p + |B_1 + B_2|^p \|_F^{1/p}
\]
\[
= n_3^{\frac{1}{p}} 2^{-\frac{1}{p} - \frac{1}{q}} \| |\bar{A}_1 + \bar{A}_2|^p + |\bar{B}_1 + \bar{B}_2|^p \|_F^{1/p}
\]
\[
\leq n_3^{\frac{1}{p}} \| |\bar{A}_1|^p + |\bar{B}_1|^p \|_F^{1/p} + \| |\bar{A}_2|^p + |\bar{B}_2|^p \|_F^{1/p}
\]
\[
= \| |A_1|^p + |B_1|^p \|_F^{1/p} + \| |A_2|^p + |B_2|^p \|_F^{1/p}.
\]
The spectral norm inequality arises analogously.

\[\square\]

## 5 Tensor t-eigenvalue inequalities

Let \( A \) be an \( n \times n \) matrix, with eigenvalues \( \lambda_1, \ldots, \lambda_n \). The Schur inequality [8] says that
\[
\sum_{i=1}^n |\lambda_i|^2 \leq \| A \|_F^2.
\]
Based on the concepts of the tensor t-eigenvalue and Lemma 2.2, we can easily extend (5.1) to tensors.

**Theorem 5.1.** For \( A \in \mathbb{R}^{n \times n \times n_3} \) with t-eigenvalues \( \lambda_i, \ i = 1, 2, \ldots, nn_3 \), we have
\[
\sum_{i=1}^{nn_3} |\lambda_i|^2 \leq n_3 \| A \|_F^2.
\]

Suppose \( A \) is an \( n \times n \) matrix. The Gershgorin circle theorem makes the observation: discs \( G(A) = \bigcup_{i=1}^n \{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \} \) that are centered at the points \( a_{ii} \) are guaranteed to contain the eigenvalues of \( A \). The upcoming theorem shows that we still have similar results in tensors.
Theorem 5.2. Let $A = [a_{ijk}] \in \mathbb{C}^{n \times n \times n^3}$ and denote the $n$ Gershgorin discs

$$G_i(A) = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j,k \neq i} |a_{ijk}| \right\}.$$  

Then the $t$-eigenvalues of $A$ are in the union of Gershgorin discs

$$G(A) = \bigcup_{i=1}^{n} G_i(A).$$

Furthermore, if the union of $k$ of the $n$ discs that comprise $G(A)$ forms a set that is disjoint from the remaining $n - k$ discs, then this set contains exactly $k$ $t$-eigenvalues of $A$, counted according to their algebraic multiplicities.

Proof. Suppose that $\text{bcirc}(A) = [b_{ij}]_{n^3 \times n^3}$. Since all the $t$-eigenvalues of $A$ can be regarded as those of matrix $\text{bcirc}(A)$, they are contained in

$$G_i(\text{bcirc}(A)) = \left\{ z \in \mathbb{C} : |z - b_{ii}| \leq \sum_{j \neq i} |b_{ij}| \right\}, \quad i = 1, 2, \ldots, n^3,$$

which can also be characterized below by noticing the block circulant structure of $\text{bcirc}(A)$,

$$G_i(\text{bcirc}(A)) = \left\{ z \in \mathbb{C} : |z - a_{ii1}| \leq \sum_{j,k \neq i} |a_{ijk}| \right\} = G_i(A), \quad i = 1, 2, \ldots n.$$  

The conclusion follows directly by the matrix Gershgorin circle theorem. \hfill \Box

The next theorem is the tensor Bauer-Fike theorem.

Theorem 5.3. Let $A, B \in \mathbb{C}^{n \times n \times n^3}$ be diagonalizable, and suppose that $A = Q^{-1} \ast S \ast Q$, in which $S$ is $t$-diagonal and $Q$ is invertible. If $\lambda$ is a $t$-eigenvalue of $A$, there always exists a $\mu$ being a $t$-eigenvalue of $B$, such that

$$|\lambda - \mu| \leq \left\| Q^{-1} \right\|_2 \cdot \left\| Q \right\|_2 \cdot \left\| A - B \right\|_2.$$  

Proof. From $A = Q^{-1} \ast S \ast Q$, we have

$$\text{bcirc}(A) = \text{bcirc}(Q^{-1} \ast S \ast Q) = (\text{bcirc}(Q))^{-1} \cdot \text{bcirc}(S) \cdot \text{bcirc}(Q),$$

where $\text{bcirc}(S)$ is a diagonal matrix. According to the Bauer-Fike theorem for matrices, for any eigenvalue $\lambda$ of $\text{bcirc}(A)$, there exists an eigenvalue $\mu$ of $\text{bcirc}(B)$, such that

$$|\lambda - \mu| \leq \left\| \text{bcirc}(Q)^{-1} \right\|_2 \cdot \left\| \text{bcirc}(Q) \right\|_2 \cdot \left\| \text{bcirc}(A - B) \right\|_2$$

$$= \left\| Q^{-1} \right\|_2 \cdot \left\| Q \right\|_2 \cdot \left\| A - B \right\|_2,$$

completing the proof of the theorem. \hfill \Box
Next we derive the tensor Hoffman-Wielandt theorem.

**Theorem 5.4.** Let $A$ and $B \in \mathbb{C}^{n \times n \times n_3}$ be both normal, with $\lambda_1, \ldots, \lambda_{nn_3}$ being the $t$-eigenvalues of $A$, and assume that $\mu_1, \ldots, \mu_{nn_3}$ are the $t$-eigenvalues of $B$. There is a permutation $\pi(\cdot)$ of the integers $1, \ldots, nn_3$ such that

$$\left( \sum_{i=1}^{nn_3} |\mu_{\pi(i)} - \lambda_i|^2 \right)^{\frac{1}{2}} \leq n_3 \|B - A\|_F.$$

**Proof.** It is clear that $\text{bcirc}(A)$ and $\text{bcirc}(B)$ are both normal matrices. Consequently, there is a permutation $\pi(\cdot)$ of the integers $1, \ldots, nn_3$ such that

$$\left( \sum_{i=1}^{nn_3} |\mu_{\pi(i)} - \lambda_i|^2 \right)^{\frac{1}{2}} \leq \|\text{bcirc}(B) - \text{bcirc}(A)\|_F$$

$$= \|\text{bcirc}(B - A)\|_F = n_3 \|B - A\|_F. \quad \Box$$

The tensor Hoffman-Wielandt theorem leads to the following corollary.

**Corollary 5.1.** Suppose that $A$ and $B \in \mathbb{R}^{n \times n \times n_3}$ are symmetric, where $\lambda_1 \leq \ldots \leq \lambda_{nn_3}$ are the $t$-eigenvalues of $A$, and $\mu_1 \leq \ldots \leq \mu_{nn_3}$ are the $t$-eigenvalues of $B$. Then

$$\left( \sum_{i=1}^{nn_3} |\mu_i - \lambda_i|^2 \right)^{\frac{1}{2}} \leq n_3 \|B - A\|_F.$$

Similar treatment yields the last theorem.

**Theorem 5.5.** Let $T = A + iB \in \mathbb{C}^{n \times n \times n_3}$ with real symmetric tensors $A$ and $B$. Let $\alpha_i$ and $\beta_i$ be the $t$-eigenvalues of $A$ and $B$ respectively, which are ordered such that $|\alpha_1| \geq |\alpha_2| \geq |\alpha_{nn_3}|$ and $|\beta_1| \geq |\beta_2| \geq |\beta_{nn_3}|$. Then

$$\frac{1}{n_3} \|\text{diag}(\alpha_1 + i\beta_1, \ldots, \alpha_{nn_3} + i\beta_{nn_3})\|_F \leq \sqrt{2} \|T\|_F$$

and

$$\|\text{diag}(\alpha_1 + i\beta_1, \ldots, \alpha_{nn_3} + i\beta_{nn_3})\| \leq \sqrt{2} \|T\|_2.$$

6 Conclusion

We have provided some inequalities for a variety of tensor topics. Most of the results and proofs presented here are derived through the technique of unfolding tensors into block circulant matrices. Several directions can be pursued to expand the results throughout this paper.
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