Optimal Learning for Multi-pass Stochastic Gradient Methods

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Abstract

We analyze the learning properties of the stochastic gradient method when multiple passes over the data and mini-batches are allowed. In particular, we consider the square loss and show that for a universal step-size choice, the number of passes acts as a regularization parameter, and optimal finite sample bounds can be achieved by early-stopping. Moreover, we show that larger step-sizes are allowed when considering mini-batches. Our analysis is based on a unifying approach, encompassing both batch and stochastic gradient methods as special cases.

1 Introduction

Modern machine learning applications require computational approaches that are at the same time statistically accurate and numerically efficient [1]. This has motivated a recent interest in stochastic gradient methods (SGM), since on the one hand they enjoy good practical performances, especially in large scale scenarios, and on the other hand they are amenable to theoretical studies. In particular, unlike other learning approaches, such as empirical risk minimization or Tikhonov regularization, theoretical results on SGM naturally integrate statistical and computational aspects.

Most generalization studies on SGM consider the case where only one pass over the data is allowed and the step-size is appropriately chosen, [2][3][4][5][6][7] (possibly considering averaging [8]). In particular, recent works show how the step-size can be seen to play the role of a regularization parameter whose choice controls the bias and variance properties of the obtained solution [4][5][6]. These latter works show that balancing these contributions, it is possible to derive a step-size choice leading to optimal learning bounds. Such a choice typically depends on some unknown properties of the data generating distributions and in practice can be chosen by cross-validation.

While processing each data point only once is natural in streaming/online scenarios, in practice SGM is often used as a tool for processing large data-sets and multiple passes over the data are typically considered. In this case, the number of passes over the data, as well as the step-size, need then to be determined. While the role of multiple passes is well understood if the goal is empirical risk minimization [9], its effect with respect to generalization is less clear and a few recent works have recently started to tackle this question. In particular, results in this direction have been derived in [10] and [11]. The former work considers a general stochastic optimization
setting and studies stability properties of SGM allowing to derive convergence results as well as finite sample bounds. The latter work, restricted to supervised learning, further develops these results to compare the respective roles of step-size and number of passes, and show how different parameter settings can lead to optimal error bounds. In particular, it shows that there are two extreme cases: one between the step-size or the number of passes is fixed a priori, while the other one acts as a regularization parameter and needs to be chosen adaptively. The main shortcoming of these latter results is that they are in the worst case, in the sense that they do not consider the possible effect of capacity assumptions \cite{12,13} shown to lead to faster rates for other learning approaches such as Tikhonov regularization. Further, these results do not consider the possible effect of mini-batches, rather than a single point in each gradient step \cite{14,15,16,17}. This latter strategy is often considered especially for parallel implementation of SGM.

The study in this paper, fills in these gaps in the case where the loss function is the least squares loss. We consider a variant of SGM for least squares, where gradients are sampled uniformly at random and mini-batches are allowed. The number of passes, the step-size and the mini-batch size are then parameters to be determined. Our main results highlight the respective roles of these parameters and show how can they be chosen so that the corresponding solutions achieve optimal learning errors. In particular, we show for the first time that multi-pass SGM with early stopping and a universal step-size choice can achieve optimal learning bounds, matching those of ridge regression \cite{18,13}. Further, our analysis shows how the mini-batch size and the step-size choice are tightly related. Indeed, larger mini-batch sizes allow to consider larger step-sizes while keeping the optimal learning bounds. This result could give an insight on how to exploit mini-batches for parallel computations while preserving optimal statistical accuracy. Finally we note that a recent work \cite{19} is tightly related to the analysis in the paper. The generalization properties of a multi-pass incremental gradient are analyzed in \cite{19}, for a cyclic, rather than a stochastic, choice of the gradients and with no mini-batches. The analysis in this latter case appears to be harder and results in \cite{19} give good learning bounds only in restricted setting and considering iterates rather than the excess risk. Compared to \cite{19} our results show how stochasticity can be exploited to get faster capacity dependent rates and analyze the role of mini-batches.

The rest of this paper is organized as follows. Section 2 introduces the learning setting and the SGM algorithm. Main results with discussions and proof sketches are presented in Section 3. Finally, simple numerical simulations are given in Section 4 to complement our theoretical results.

\textbf{Notation} For any $a, b \in \mathbb{R}$, $a \vee b$ denotes the maximum of $a$ and $b$. $\mathbb{N}$ is the set of all positive integers. For any $T \in \mathbb{N}$, $[T]$ denotes the set $\{1, \cdots, T\}$. For any two positive sequences $\{a_t\}_{t \in [T]}$ and $\{b_t\}_{t \in [T]}$, the notation $a_t \lesssim b_t$ for all $t \in [T]$ means that there exists a positive constant $C \geq 0$ such that $C$ is independent of $t$ and that $a_t \leq C b_t$ for all $t \in [T]$.

2 Learning with SGM

We begin by introducing the learning setting we consider, and then describe the SGM learning algorithm. Following \cite{19}, the formulation we consider is close to the setting of functional regression, and covers the reproducing kernel Hilbert space (RKHS) setting as special cases. In
particular, it reduces to standard linear regression for finite dimensions.

### 2.1 Learning Problems

Let $H$ be a separable Hilbert space, with inner product and induced norm denoted by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$, respectively. Let the input space $X \subseteq H$ and the output space $Y \subseteq \mathbb{R}$. Let $\rho$ be an unknown probability measure on $Z = X \times Y$, $\rho_X(\cdot)$ the induced marginal measure on $X$, and $\rho(\cdot|x)$ the conditional probability measure on $Y$ with respect to $x \in X$ and $\rho$.

Considering the square loss function, the problem under study is the minimization of the risk,

$$
\inf_{\omega \in H} \mathcal{E}(\omega), \quad \mathcal{E}(\omega) = \int_{X \times Y} (\langle \omega, x \rangle_H - y)^2 d\rho(x, y),
$$

when the measure $\rho$ is known only through a sample $z = \{z_i = (x_i, y_i)\}_{i=1}^m$ of size $m \in \mathbb{N}$, independently and identically distributed (i.i.d.) according to $\rho$. In the following, we measure the quality of an approximate solution $\hat{\omega} \in H$ (an estimator) considering the excess risk, i.e.,

$$
\mathcal{E}(\hat{\omega}) - \inf_{\omega \in H} \mathcal{E}(\omega).
$$

Throughout this paper, we assume that there exists a constant $\kappa \in [1, \infty[$, such that

$$
\langle x, x' \rangle_H \leq \kappa^2, \quad \forall x, x' \in X.
$$

### 2.2 Stochastic Gradient Method

We study the following SGM (with mini-batches).

**Algorithm 1.** Let $b \in [m]$. Given any sample $z$, the $b$-minibatch stochastic gradient method is defined by $\omega_1 = 0$ and

$$
\omega_{t+1} = \omega_t - \eta_t \frac{1}{b} \sum_{i=b(t-1)+1}^{bt} (\langle \omega_t, x_{j_i} \rangle_H - y_{j_i})x_{j_i}, \quad t = 1, \ldots, T,
$$

where $\{\eta_t > 0\}$ is a step-size sequence. Here, $j_1, j_2, \cdots, j_{bT}$ are independent and identically distributed (i.i.d.) random variables from the uniform distribution on $[m]$.

Different choices for the (mini-)batch size $b$ can lead to different algorithms. In particular, for $b = 1$, the above algorithm corresponds to a simple SGM, while for $b = m$, it is a stochastic version of the batch gradient descent. The aim of this paper is to derive excess risk bounds for the above algorithm under appropriate assumptions. Throughout this paper, we assume that $\{\eta_t\}$ is non-increasing, and $T \in \mathbb{N}$ with $T \geq 3$. We denote by $J_t$ the set $\{j_l : l = b(t-1) + 1, \cdots, bt\}$ and by $J$ the set $\{j_l : l = 1, \cdots, bT\}$.

### 3 Main Results with Discussions

In this section, we first state some basic assumptions. Then, we present and discuss our main results.
3.1 Assumptions

We first make the following assumption.

**Assumption 1.** There exists constants $M \in ]0, \infty[$ and $v \in ]1, \infty[$ such that
\[
\int_{Y} y^{2v} d\rho(y|x) \leq !M^{v}, \quad \forall \in \mathbb{N}, \tag{5}
\]
\(\rho_{X}\)-almost surely.

Assumption (5) is related to a moment hypothesis on \(|y|^{2}\). It is weaker than the often considered bounded output assumption, and trivially verified in binary classification problems where \(Y = \{-1, 1\}\).

To present our next assumption, we introduce the operator \(\mathcal{L} : L^{2}(H, \rho_{X}) \rightarrow L^{2}(H, \rho_{X})\), defined by \(\mathcal{L}(f) = \int_{X} (x, \cdot)_{H} f(x) \rho_{X}(x)\). Under Assumption (3), \(\mathcal{L}\) can be proved to be positive trace class operators, and hence \(\mathcal{L}^{\zeta}\) with \(\zeta \in \mathbb{R}\) can be defined by using the spectrum theory [20].

The Hilbert space of square integral functions from \(H\) to \(\mathbb{R}\) with respect to \(\rho_{X}\), with induced norm given by \(\|f\|_{\rho} = (\int_{X} |f(x)|^{2} \rho_{X}(x))^{1/2}\), is denoted by \((L^{2}(H, \rho_{X}), \|\cdot\|_{\rho})\). It is well known that the function minimizing \(\int_{X} (f(x) - y)^{2} \rho_{X}(z)\) over all measurable functions \(f : H \rightarrow \mathbb{R}\) is the regression function, which is given by
\[
f_{\rho}(x) = \int_{X} y \rho(x|y), \quad x \in X. \tag{6}
\]

Define another Hilbert space \(H_{\rho} = \{f : X \rightarrow \mathbb{R} | \exists \omega \in H \text{ with } f(x) = (\omega, x)_{H, \rho_{X}} \}-\text{almost surely}\). Under Assumption (3) it is easy to see that \(H_{\rho}\) is a subspace of \((L^{2}(H, \rho_{X}), \|\cdot\|_{\rho})\). Let \(f_{H}\) be the projection of the regression function \(f_{\rho}\) onto the closure of \(H_{\rho}\) in \((L^{2}(H, \rho_{X}), \|\cdot\|_{\rho})\). It is easy to see that the search for a solution of Problem (1) is equivalent to the search of a linear function from \(H_{\rho}\) to approximate \(f_{H}\). From this point of view, bounds on the excess risk of a learning algorithm naturally depend on the following assumption, which quantifies how well, the target function \(f_{H}\) can be approximated by \(H_{\rho}\).

**Assumption 2.** There exist \(\zeta > 0\) and \(R > 0\), such that \(\|\mathcal{L}^{-\zeta} f_{H}\|_{\rho} \leq R\).

The above assumption is fairly standard [20, 19] in non-parametric regression. The bigger \(\zeta\) is, the more stringent the assumption is, since \(\mathcal{L}^{\zeta_{1}} (L^{2}(H, \rho_{X})) \subseteq \mathcal{L}^{\zeta_{2}} (L^{2}(H, \rho_{X}))\) when \(\zeta_{1} \geq \zeta_{2}\).

In particular, for \(\zeta = 0\), we are assuming \(\|f_{H}\|_{\rho} < \infty\), while for \(\zeta = 1/2\), we are requiring \(f_{H} \in H_{\rho}\), since [21, 19]
\[
H_{\rho} = \mathcal{L}^{1/2}(L^{2}(H, \rho_{X})).
\]

Finally, the last assumption relates to the capacity of the hypothesis space.

**Assumption 3.** For some \(\gamma \in ]0, 1]\) and \(c_{\gamma} > 0\), \(\mathcal{L}\) satisfies
\[
\text{tr}(\mathcal{L}(\mathcal{L} + \lambda I)^{-1}) \leq c_{\gamma} \lambda^{-\gamma}, \quad \text{for all } \lambda > 0. \tag{7}
\]

The left-hand side of (7) is the so-called effective dimension, or the degrees of freedom [12, 13]. It can be related to covering/entropy number conditions, see [21] for further details. Assumption (3) is always true for \(\gamma = 1\) and \(c_{\gamma} = \kappa^{2}\), since \(\mathcal{L}\) is a trace class operator which implies the eigenvalues of \(\mathcal{L}\), denoted as \(\sigma_{i}\), satisfy \(\text{tr}(\mathcal{L}) = \sum_{i} \sigma_{i} \leq \kappa^{2}\). This is referred as the capacity independent setting. Assumption (3) with \(\gamma \in ]0, 1]\) allows to derive better error rates. It is satisfied, for example, if the eigenvalues of \(\mathcal{L}\) satisfy a polynomial decaying condition \(\sigma_{i} \sim i^{-\gamma}\), or with \(\gamma = 0\) if \(\mathcal{L}\) is finite rank.
3.2 Main Results

We start with the following corollary, which is a simplified version of our main results stated next.

Corollary 3.1. Under Assumptions 2 and 3, let $\zeta \geq 1/2$ and $|y| \leq M$ $\rho_X$-almost surely for some $M > 0$. Consider the SGM with

1) $p_\ast = \lceil m^{1/2} \rceil$, $b = 1$, $\eta_t \simeq \frac{1}{m}$ for all $t \in \lbrack (p_\ast, m) \rbrack$, and $\omega_{p_\ast} = \omega_{p_\ast, m+1}$.

If $m$ is large enough, with high probability\(^2\) there holds

$$
\mathbb{E}_J[\mathcal{E}(\omega_{p_\ast})] - \inf_{\omega \in H} \mathcal{E} \lesssim m^{-\frac{2\zeta}{1 - \zeta}}.
$$

Furthermore, the above also holds for the SGM with\(^3\)

2) or $p_\ast = \lceil m^{1/2} \rceil$, $b = \sqrt{m}$, $\eta_t \simeq \frac{1}{\sqrt{m}}$ for all $t \in \lbrack (p_\ast, \sqrt{m}) \rbrack$, and $\omega_{p_\ast} = \omega_{p_\ast, \sqrt{m}+1}$.

In the above, $p_\ast$ is the number of ‘passes’ over the data, which is defined as $\lceil \frac{m}{2} \rceil$ at $t$ iterations.

The above result asserts that, at $p_\ast$ passes over the data, the simple SGM with fixed step-size achieves optimal learning error bounds, matching those of ridge regression\([13]\). Furthermore, using mini-batch allows to use a larger step-size while achieving the same optimal error bounds.

Our main theorem of this paper is stated next, and provides error bounds for the studied algorithm. For the sake of readability, we only consider the case $\zeta \geq 1/2$ in a fixed step-size setting. General results in a more general setting ($\eta_t = \eta t^{-\theta}$ with $0 \leq \theta < 1$, and/or the case $\zeta \in (0, 1/2)$) can be found in the appendix.

Theorem 3.2. Under Assumptions 2, 3, and 4 let $\zeta \geq 1/2$, $\delta \in [0, 1]$, $\eta_t = \eta t^{-2}$ for all $t \in [T]$, with $\eta \leq \frac{1}{8(\log T + 1)}$. If $m \geq m_\delta$, then the following holds with probability at least $1 - \delta$ for all $t \in [T]$,

$$
\mathbb{E}_J[\mathcal{E}(\omega_{t+1})] - \inf_{\omega \in H} \mathcal{E} \leq q_1 (\eta t)^{-2\zeta} + q_2 m^{-\frac{2\zeta}{1 - \zeta}} \left(1 + m^{-\frac{2\zeta}{1 - \zeta}} \eta t\right)^2 \log^2 T \log \frac{1}{\delta} + q_3 \eta b^{-1} (1 \vee m^{-\frac{2\zeta}{1 - \zeta}} \eta t) \log T.
$$

Here, $m_\delta, q_1, q_2$ and $q_3$ are positive constants depending on $\kappa^2, \|T\|, M, v, \zeta, R, c_\gamma, \gamma$, and $m_\delta$ also on $\delta$ (which will be given explicitly in the proof).

There are three terms in the upper bounds of (8). The first term depends on the regularity of the target function and it arises from bounding the bias, while the last two terms result from estimating the sample variance and the computational variance (due to the random choices of the points), respectively. To derive optimal rates, it is necessary to balance these three terms. Solving this trade-off problem leads to different choices on $\eta, T$, and $b$, corresponding to different regularization strategies, as shown in subsequent corollaries.

The first corollary gives generalization error bounds for SGM, with a universal step-size depending on the number of sample points.

Corollary 3.3. Under Assumptions 2, 3, and 4 let $\zeta \geq 1/2$, $b = 1$ and $\eta_t \simeq \frac{1}{m}$ for all $t \in [T]$, where $T \leq m^2$. If $m \geq m_0$, then with probability at least $1 - 1/m$, there holds

$$
\mathbb{E}_J[\mathcal{E}(\omega_{t+1})] - \inf_{\omega \in H} \mathcal{E} \lesssim \left\{ \left( \frac{m}{t} \right)^{2\zeta} \right\} + m^{-\frac{2\zeta + 2}{1 - \zeta}} \left( \frac{1}{m} \right)^{2} \log^4 m, \quad \forall t \in [T],
$$

\(^2\)Here, ‘high probability' refers to the sample $z$.

\(^3\)Here, we assume that $\sqrt{m}$ is an integer.
and in particular,
\[ \mathbb{E}_J[\mathcal{E}(\omega_{T^*+1})] - \inf_{\omega \in H} \mathcal{E} \lesssim m^{-\frac{2\zeta}{2\zeta+1}} \log^4 m, \] (10)
where \( T^* = \lceil m^{\frac{2\zeta+1}{2\zeta}} \rceil \). Here, \( m_0 \) is a positive integer depending only on \( \kappa, \|T\|, \zeta \) and \( \gamma \), and will be given explicitly in the proof.

**Remark 3.4.** Ignoring the logarithmic term and letting \( t = pm \), Eq. \((9)\) becomes
\[ \mathbb{E}_J[\mathcal{E}(\omega_{pm+1})] - \inf_{\omega \in H} \mathcal{E} \lesssim p^{-2\zeta} + m^{-\frac{2\zeta+2}{2\zeta+1}} p^2. \]
A smaller \( p \) may lead to a larger bias, while a larger \( p \) may lead to a larger sample error. From this point of view, \( p \) has a regularization effect.

The second corollary provides error bounds for SGM with a fixed mini-batch size and a fixed step-size (which depend on the number of sample points).

**Corollary 3.5.** Under Assumptions 1, 2 and 3, let \( \zeta \geq 1/2, b = \lceil \sqrt{m} \rceil \) and \( \eta_t \approx \frac{1}{\sqrt{m}} \) for all \( t \in [T] \), where \( T \leq m^2 \). If \( m \geq m_0 \), then with probability at least \( 1 - 1/m \), there holds
\[ \mathbb{E}_J[\mathcal{E}(\omega_{t+1})] - \inf_{\omega \in H} \mathcal{E} \lesssim \left\{ \left( \frac{\sqrt{m}}{t} \right)^{2\zeta} + m^{-\frac{2\zeta+2}{2\zeta+1}} \left( \frac{t}{\sqrt{m}} \right)^2 \right\} \log^4 m, \forall t \in [T], \] (11)
and particularly,
\[ \mathbb{E}_J[\mathcal{E}(\omega_{T^*+1})] - \inf_{\omega \in H} \mathcal{E} \lesssim m^{-\frac{2\zeta}{2\zeta+1}} \log^4 m, \] (12)
where \( T^* = \lceil m^{\frac{1}{2\zeta+1}} \rceil \).

The above two corollaries follow from Theorem 3.2 with the simple observation that the dominating terms in \((8)\) are the terms related to the bias and the sample variance, when a small step-size is chosen. The only free parameter in \((9)\) and \((11)\) is the number of iterations/passes. The ideal stopping rule is achieved by balancing the two terms related to the bias and the sample variance, showing the regularization effect of the number of passes. Since the ideal stopping rule depends on the unknown parameters \( \zeta \) and \( \gamma \), a hold-out cross-validation procedure is often used to tune the stopping rule in practice. Using an argument similar to that in Chapter 6 from [21], it is possible to show that this procedure can achieve the same convergence rate.

We give some further remarks. First, the upper bound in \((10)\) is optimal up to a logarithmic factor, in the sense that it matches the minimax lower rate in \((13)\). Second, according to Corollaries 3.3 and 3.5, \( \frac{b_T^\gamma}{m} \approx m^{\frac{1}{2\zeta+1}} \) passes over the data are needed to obtain optimal rates in both cases. Finally, in comparing the simple SGM and the mini-batch SGM, Corollaries 3.3 and 3.5 show that a larger step-size is allowed to use for the latter.

In the next result, both the step-size and the stopping rule are tuned to obtain optimal rates for simple SGM with multiple passes. In this case, the step-size and the number of iterations are the regularization parameters.

**Corollary 3.6.** Under Assumptions 1, 2 and 3, let \( \zeta \geq 1/2, b = 1 \) and \( \eta_t \approx m^{-\frac{2\zeta}{2\zeta+1}} \) for all \( t \in [T] \), where \( T \leq m^2 \). If \( m \geq m_0 \), and \( T^* = \lceil m^{\frac{1}{2\zeta+1}} \rceil \), then \((10)\) holds with probability at least \( 1 - 1/m \).

**Remark 3.7.** If we make no assumption on the capacity, i.e., \( \gamma = 1 \), Corollary 3.6 recovers the result in \([3]\) for one pass SGM.
The next corollary shows that for some suitable mini-batch sizes, optimal rates can be achieved with a constant step-size (which is nearly independent of the number of sample points) by early stopping.

**Corollary 3.8.** Under Assumptions 1, 2 and 3, let $\zeta \geq 1/2$, $b = \lceil m^{\frac{2\zeta}{\zeta + 1}} \rceil$ and $\eta_t \simeq \frac{1}{\log m}$ for all $t \in [T]$, where $T \leq m^2$. If $m \geq m_0$, and $T^* = \lceil m^{\frac{2\zeta}{\zeta + 1}} \rceil$, then (10) holds with probability at least $1 - \frac{1}{m}$.

According to Corollaries 3.6 and 3.8, around $m^{\frac{\zeta}{\zeta + 1}}$ passes over the data are needed to achieve the best performance in the above two strategies. In comparisons with Corollaries 3.3 and 3.5 where around $m^{\frac{\zeta}{\zeta + 1}}$ passes are required, the latter seems to require fewer passes over the data. However, in this case, one might have to run the algorithms multiple times to tune the step-size, or the mini-batch size.

Finally, the last result gives generalization error bounds for ‘batch’ SGM with a constant step-size (nearly independent of the number of sample points).

**Corollary 3.9.** Under Assumptions 1, 2 and 3, let $\zeta \geq 1/2$, $b = m$ and $\eta_t \simeq \frac{1}{\log m}$ for all $t \in [T]$, where $T \leq m^2$. If $m \geq m_0$, and $T^* = \lceil m^{\frac{2\zeta}{\zeta + 1}} \rceil$, then (10) holds with probability at least $1 - \frac{1}{m}$.

As will be seen in the proof from the appendix, the above result also holds when replacing the sequence $\{\omega_t\}$ by the sequence $\{\nu_t\}$ generated from real batch GM in (14). In this sense, we study the gradient-based learning algorithms simultaneously.

### 3.3 Discussions

We compare our results with previous works. For non-parametric regression with the square loss, one pass SGM has been studied in, e.g., [4, 22, 5, 6]. In particular, [4] proved capacity independent rate of order $O(m^{\frac{-\zeta}{\zeta + 1}} \log m)$ with a fixed step-size $\eta \simeq m^{\frac{-\zeta}{\zeta + 1}}$, and [6] derived capacity dependent error bounds of order $O(m^{\frac{-\zeta}{\zeta + 1}} \min(\zeta, 1) + \gamma)$ (when $2\zeta + \gamma > 1$) for the average. Note also that a regularized version of SGM has been studied in [5], where the derived convergence rate is of order $O(m^{\frac{-\zeta}{\zeta + 1}})$ assuming that $\zeta \in [\frac{1}{2}, 1]$. In comparison with these existing convergence rates, our rates from (10) are comparable, either involving the capacity condition, or allowing a broader regularity parameter $\zeta$ (which thus improves the rates).

More recently, [19] studied multiple passes SGM with a fixed ordering at each pass, also called incremental gradient method. Making no assumption on the capacity, rates of order $O(m^{\frac{-\zeta}{\zeta + 1}})$ in $L^2(H, \rho_X)$-norm with a universal step-size $\eta \simeq \frac{1}{m}$ are derived. In comparisons, Corollary 3.3 achieves better rates, while considering the capacity assumption. Note also that [19] proved sharp rate in $H$-norm for $\zeta \geq 1/2$ in the capacity independent case. In fact, we can extend our analysis to the $H$-norm for Algorithm 4. We postpone this extension to a longer version of this paper.

The idea of using mini-batches (and parallel implements) to speed up SGM in a general stochastic optimization setting can be found, e.g., in [14, 15, 16, 17]. Our theoretical findings, especially the interplay between the mini-batch size and the step-size, can give further insights on parallelization learning. Besides, it has been shown in [23, 15] that for one pass mini-batch SGM with a fixed step-size $\eta \simeq b/\sqrt{m}$ and a smooth loss function, assuming the existence of at least one solution in the hypothesis space for the expected risk minimization, the convergence
rate is of order $O(\sqrt{1/m} + b/m)$ by considering an averaging scheme. When adapting to the learning setting we consider, this reads as that if $f_H \in H_\rho$, i.e., $\zeta = 1/2$, the convergence rate for the average is $O(\sqrt{1/m} + b/m)$. Note that, $f_H$ does not necessarily belongs to $H_\rho$ in general. Also, our derived convergence rate from Corollary \[3.5\] is better, when the regularity parameter $\zeta$ is greater than $1/2$, or $\gamma$ is smaller than $1$.

### 3.4 Error Decomposition

The key to our proof is a novel error decomposition, which may be also used in analysing other learning algorithms. We first introduce two sequences. The population iteration is defined by $\mu_1 = 0$ and

$$
\mu_{t+1} = \mu_t - \eta_t \int_X ((\mu_t, x)_H - f_{\rho}(x))xd\rho_X(x), \quad t = 1, \ldots, T. \tag{13}
$$

The above iterated procedure is ideal and can not be implemented in practice, since the distribution $\rho_X$ is unknown in general. Replacing $\rho_X$ by the empirical measure and $f_{\rho}(x_i)$ by $y_i$, we derive the sample iteration (associated with the sample $z$), i.e., $\nu_t = 0$ and

$$
\nu_{t+1} = \nu_t - \eta_t \frac{1}{m} \sum_{i=1}^{m} ((\nu_t, x_i)_H - y_i)x_i, \quad t = 1, \ldots, T. \tag{14}
$$

Clearly, $\mu_t$ is deterministic and $\nu_t$ is a $H$-valued random variable depending on $z$. Given the sample $z$, the sequence $\{\nu_t\}_t$ has a natural relationship with the learning sequence $\{\omega_t\}_t$, since

$$
E_J[\omega_t] = \nu_t. \tag{15}
$$

Indeed, taking the expectation with respect to $J_t$ on both sides of \[14\], and noting that $\omega_t$ depends only on $J_1, \ldots, J_{t-1}$ (given any $z$), one has

$$
E_J[\omega_{t+1}] = E_J[\omega_t] - \eta_t \frac{1}{m} \sum_{i=1}^{m} ((E_J[\omega_t], x_i)_H - y_i)x_i,
$$

and thus,

$$
E[\omega_{t+1}] = E[\omega_t] - \eta_t \frac{1}{m} \sum_{i=1}^{m} ((E[\omega_t], x_i)_H - y_i)x_i, \quad t = 1, \ldots, T,
$$

which satisfies the iterative relationship given in \[14\]. By an induction argument, \[15\] can then be proved.

Let $S_\rho : H \to L^2(H, \rho_X)$ be the linear map defined by $(S_\rho \omega)(x) = \langle \omega, x \rangle_H, \forall \omega, x \in H$. We have the following error decomposition.

**Proposition 3.10.** We have

$$
E_J[\mathcal{E}(\omega_t)] - \inf_{f \in H} \mathcal{E}(f) \leq 2\|S_\rho \mu_t - f_H\|_\rho^2 + 2\|S_\rho \nu_t - S_\rho \mu_t\|_\rho^2 + E_J[\|S_\rho \omega_t - S_\rho \nu_t\|_\rho^2]. \tag{16}
$$

**Proof.** For any $\omega \in H$, we have \[21\] \[19\]

$$
\mathcal{E}(\omega) - \inf_{f \in H} \mathcal{E}(f) = \|S_\rho \omega - f_H\|_\rho^2. \tag{17}
$$

Thus, $\mathcal{E}(\omega_t) - \inf_{f \in H} \mathcal{E}(f) = \|S_\rho \omega_t - f_H\|_\rho^2$, and

$$
E_J[\|S_\rho \omega_t - f_H\|_\rho^2] = E_J[\|S_\rho \omega_t - S_\rho \nu_t + S_\rho \nu_t - f_H\|_\rho^2]
$$

$$
= E_J[\|S_\rho \omega_t - S_\rho \nu_t\|_\rho^2 + \|S_\rho \nu_t - f_H\|_\rho^2] + 2E_J(S_\rho \omega_t - S_\rho \nu_t, S_\rho \nu_t - f_H)_\rho.
$$

8
Using (15) to the above, we get \( \mathbb{E}_\lambda[\|S_\rho \omega_t - f_H\|_\rho^2] = \mathbb{E}_\lambda[\|S_\rho \omega_t - S_\rho \nu_t\|_\rho^2 + \|S_\rho \nu_t - f_H\|_\rho^2] \). Now the proof can be finished by considering
\[
\|S_\rho \nu_t - f_H\|_\rho^2 = \|S_\rho \nu_t - S_\rho \mu_t + S_\rho \mu_t - f_H\|_\rho^2 \leq 2\|S_\rho \nu_t - S_\rho \mu_t\|_\rho^2 + 2\|S_\rho \mu_t - S_\rho f_H\|_\rho^2.
\]

There are three terms in the upper bound of the error decomposition (16). We refer to the deterministic term \( \|S_\rho \mu_t - f_H\|_\rho^2 \) as the bias, the term \( \|S_\rho \nu_t - S_\rho \mu_t\|_\rho^2 \) depending on \( z \) as the sample variance, and \( \mathbb{E}_\lambda[\|S_\rho \omega_t - S_\rho \mu_t\|_\rho^2] \) as the computational variance. These three terms will be estimated in the appendix, see Lemma B.2, Theorem C.6 and Theorem D.9. The bound in Theorem 3.2 thus follows plugging these estimations in the error decomposition.

4 Numerical Simulations

![Error decompositions for gradient-based learning algorithms on synthesis data](image1)

Figure 1: Error decompositions for gradient-based learning algorithms on synthesis data, where \( m = 100 \).

In order to illustrate our theoretical results and the error decomposition, we first performed some simulations on a simple problem. We constructed \( m = 100 \) i.i.d. training examples of the form \( y = f_\rho(x_i) + \omega_i \). Here, the regression function is \( f_\rho(x) = |x - 1/2| - 1/2 \), the input point \( x_i \) is uniformly distributed in \([0, 1]\), and \( \omega_i \) is a Gaussian noise with zero mean and standard deviation 1, for each \( i \in [m] \). We perform three experiments with the same \( H \), a RKHS associated with a Gaussian kernel \( K(x, x') = \exp(-(x - x')^2/(2\sigma^2)) \) where \( \sigma = 0.2 \). In the first experiment, we run mini-batch SGM, where the mini-batch size \( b = \sqrt{m} \), and the step-size \( \eta_t = 1/(8\sqrt{m}) \). In the second experiment, we run simple SGM where the step-size is fixed as \( \eta_t = 1/(8m) \), while in the third experiment, we run batch GM using the fixed step-size \( \eta_t = 1/8 \). For each experiment,

![Classification Errors of Minibatch SGM](image2)

Figure 2: Misclassification Errors for gradient-based learning algorithms on BreastCancer dataset.
we run the algorithm 50 times. For mini-batch SGM and SGM, the total error $\|S_\rho \omega_t - f_\rho\|_{L_2}^2$, the bias $\|S_\rho \hat{\mu}_t - f_\rho\|_{L_2}^2$, the sample variance $\|S_\rho \nu_t - S_\rho \hat{\mu}_t\|_{L_2}^2$ and the computational variance $\|S_\rho \omega_t - S_\rho \nu_t\|_{L_2}^2$, averaged over 50 trials, are depicted in Figures 1a and 1b, respectively. For batch GM, the total error $\|S_\rho \nu_t - f_\rho\|_{L_2}^2$, the bias $\|S_\rho \hat{\mu}_t - f_\rho\|_{L_2}^2$ and the sample variance $\|S_\rho \nu_t - \hat{\mu}_t\|_{L_2}^2$, averaged over 50 trials are depicted in Figure 1c. Here, we replace the unknown marginal distribution $\rho_X$ by an empirical measure $\hat{\rho} = \frac{1}{2000} \sum_{i=1}^{2000} \delta_{\hat{x}_i}$, where each $\hat{x}_i$ is uniformly distributed in $[0, 1]$. From Figure 1a or 1b we see that as the number of passes increases the bias decreases, while the sample error increases. Furthermore, we see that in comparisons with the bias and the sample error, the computational error is negligible. In all these experiments, the minimal total error is achieved when the bias and the sample error are balanced. These empirical results show the effects of the three terms from the error decomposition, and complement the derived bound (8), as well as the regularization effect of the number of passes over the data. Finally, we tested the simple SGM, mini-batch SGM, and batch GM, using similar step-sizes as those in the first simulation, on the BreastCancer data-set. The classification errors on the training set and the testing set of these three algorithms are depicted in Figure 2. We see that all of these algorithms perform similarly, which complement the bounds in Corollaries 3.3, 3.5 and 3.9.

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4Note that the terminology ‘running the algorithm with $p$ passes’ means ‘running the algorithm with $\lceil mp/b \rceil$ iterations’, where $b$ is the mini-batch size.
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A Preliminary

A.1 Notation

We first introduce some notations. For $t \in \mathbb{N}$, $\Pi_{t+1}^T(L) = \prod_{k=t+1}^T (I - \eta_k L)$ for $t \in [T-1]$ and $\Pi_{t+1}^T(L) = I$, for any operator $L : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space and $I$ denotes the identity operator on $\mathcal{H}$. $E[x]$ denotes the expectation of a random variable $x$. For a given bounded operator $L : L^2(H, \rho_X) \to H$, $\|L\|$ denotes the operator norm of $L$, i.e., $\|L\| = \sup_{f \in L^2(H, \rho_X), \|f\|_2 = 1} \|Lf\|_H$. We will use the conventional notations on summation and production: $\prod_{i=t+1}^T 1 = 1$ and $\sum_{i=t+1}^T 1 = 0$.

We next introduce some auxiliary operators. Let $S_\rho : H \to L^2(H, \rho_X)$ be the linear map $\omega \to \langle \omega, \cdot \rangle_H$, which is bounded by $\kappa$ under Assumption (5). Furthermore, we consider the adjoint operator $S_\rho^* : L^2(H, \rho_X) \to H$, the covariance operator $T : H \to H$ given by $T = S_\rho^* S_\rho$, and the operator $L : L^2(H, \rho_X) \to L^2(H, \rho_X)$ given by $S_\rho S_\rho^*$. It can be easily proved that $S_\rho^* g = \int_X x g(x) d\rho_X(x)$ and $T = \int_X \langle \cdot, x \rangle_H d\rho_X(x)$. The operators $T$ and $L$ can be proved to be positive trace class operators (and hence compact). For any $\omega \in H$, it is easy to prove the following isometry property (21)

$$\|S_\rho \omega\|_\rho = \|\sqrt{T} \omega\|_H. \quad (18)$$

We define the sampling operator $S_\xi : H \to \mathbb{R}^m$ by $(S_\xi \omega)_i = \langle \omega, x_i \rangle_H$, $i \in [m]$, where the norm $\|\cdot\|_\mathbb{R}^m$ in $\mathbb{R}^m$ is the Euclidean norm times $1/m$. Its adjoint operator $S_\xi^* : \mathbb{R}^m \to H$, defined by $\langle S_\xi^* \nu, \omega \rangle_H = \langle y, S_\xi \omega \rangle_{\mathbb{R}^m}$ for $y \in \mathbb{R}^m$ is thus given by $S_\xi^* \nu = \frac{1}{m} \sum_{i=1}^m y_i x_i$. Moreover, we can define the empirical covariance operator $T_\xi : H \to H$ such that $T_\xi = S_\xi^* S_\xi$. Obviously,

$$T_\xi = \frac{1}{m} \sum_{i=1}^m \langle \cdot, x_i \rangle_H x_i. \quad (13)$$

With these notations, (13) and (14) can be rewritten as

$$\mu_{t+1} = \mu_t - \eta_t (T \mu_t - S_\rho^* f_\rho), \quad t = 1, \ldots, T, \quad (19)$$

and

$$\nu_{t+1} = \nu_t - \eta_t (T_\xi \nu_t - S_\xi^* \nu), \quad t = 1, \ldots, T, \quad (20)$$

respectively.

Using the projection theorem, one can prove that

$$S_\rho^* f_\rho = S_\rho^* f_H. \quad (21)$$

Indeed, since $f_H$ is the projection of the regression function $f_\rho$ onto the closer of $H_\rho$ in $L^2(H, \rho_X)$, according to the projection theorem, one has

$$\langle f_H - f_\rho, S_\rho \omega \rangle_\rho = 0, \quad \forall \omega \in H,$$

which can be written as

$$\langle S_\rho^* f_H - S_\rho^* f_\rho, \omega \rangle_H = 0, \quad \forall \omega \in H,$$

and thus leads to (21).
A.2 Concentration Inequality

We need the following concentration result for Hilbert space valued random variable used in Caponnetto and De Vito [13] and based on the results in Pinelis and Sakhanenko [25].

Lemma A.1. Let $w_1, \ldots, w_m$ be i.i.d random variables in a Hilbert space with norm $\| \cdot \|$. Suppose that there are two positive constants $B$ and $\sigma^2$ such that

$$\mathbb{E}[\|w_1 - \mathbb{E}[w_1]\|^l] \leq \frac{1}{2} l! B^{l-2} \sigma^2, \quad \forall l \geq 2. \tag{22}$$

Then for any $0 < \delta < 1$, the following holds with probability at least $1 - \delta$,

$$\left\| \frac{1}{m} \sum_{k=1}^{m} w_m - \mathbb{E}[w_1] \right\| \leq 2 \left( \frac{B}{m} + \frac{\sigma}{\sqrt{m}} \right) \log \frac{2}{\delta}.$$  

In particular, (22) holds if

$$\|w_1\| \leq \frac{B}{2} \text{ a.s.}, \quad \text{and} \quad \mathbb{E}[\|w_1\|^2] \leq \sigma^2. \tag{23}$$

A.3 Basic Estimates

Lemma A.2. Let $\theta \in [0,1]$, and $t \in \mathbb{N}$. Then

$$\frac{t^{1-\theta}}{2} \leq \sum_{k=1}^{t} k^{-\theta} \leq \frac{t^{1-\theta}}{1-\theta}.$$  

Proof. Note that

$$\sum_{k=1}^{t} k^{-\theta} \leq 1 + \sum_{k=2}^{t} \int_{k-1}^{k} u^{-\theta} \, du = 1 + \int_{1}^{t} u^{-\theta} \, du = \frac{t^{1-\theta} - 1}{1-\theta},$$

which leads to the first part of the desired result. Similarly,

$$\sum_{k=1}^{t} k^{-\theta} \geq \sum_{k=1}^{t} \int_{k}^{k+1} u^{-\theta} \, du = \int_{1}^{t+1} u^{-\theta} \, du = \frac{(t+1)^{1-\theta} - 1}{1-\theta},$$

and by mean value theorem, $(t+1)^{1-\theta} - 1 \geq (1-\theta)t(t+1)^{-\theta} \geq (1-\theta)t^{1-\theta}/2$. This proves the second part of the desired result. The proof is complete.

Lemma A.3. Let $\theta \in \mathbb{R}$ and $t \in \mathbb{N}$. Then

$$\sum_{k=1}^{t} k^{-\theta} \leq t^{\max(1-\theta,0)} (1 + \log t).$$  

Proof. Note that

$$\sum_{k=1}^{t} k^{-\theta} = \sum_{k=1}^{t} k^{-1} k^{1-\theta} \leq t^{\max(1-\theta,0)} \sum_{k=1}^{t} k^{-1},$$

and

$$\sum_{k=1}^{t} k^{-1} \leq 1 + \sum_{k=2}^{t} \int_{k-1}^{k} u^{-1} \, du = 1 + \log t.$$  

\hfill \Box
Lemma A.4. Let $q \in \mathbb{R}$ and $t \in \mathbb{N}$ with $t \geq 3$. Then
\[
\sum_{k=1}^{t-1} \frac{1}{t-k} k^{-q} \leq 2 t^{-\min(q,1)} (1 + \log t).
\]

Proof. Note that
\[
\sum_{k=1}^{t-1} \frac{1}{t-k} k^{-q} = \sum_{k=1}^{t-1} \left( \frac{1}{t-k} + 1 \right) = \frac{2}{t} \sum_{k=1}^{t-1} \frac{1}{k} \leq 2 \left( 1 + \log t \right).
\]

B Bias

In this section, we develop upper bounds for the bias, i.e., $\|S_{\rho \mu t} - f_{\tilde{h}}\|_p^2$. Towards this end, we introduce the following lemma, whose proof borrows idea from [4, 5].

Lemma B.1. Let $L$ be a compact self-adjoint operator on a separable Hilbert space $H$. Assume that $\eta_1 \|L\| \leq 1$. Then for $t \in \mathbb{N}$ and any non-negative integer $k \leq t - 1$,
\[
\| \Pi_{k+1} L \xi \| \leq \left( \frac{\zeta}{e^\sum_{j=k+1}^t \eta_j} \right)^\zeta. \tag{24}
\]

Proof. Let $\{\sigma_i\}$ be the sequence of eigenvalues of $L$. We have
\[
\| \Pi_{k+1} L \xi \| = \sup_i \prod_{l=k+1}^t (1 - \eta_i \sigma_i) \sigma_i^\zeta.
\]

Using the basic inequality
\[
1 + x \leq e^x \quad \text{for all } x \geq -1,
\]
with $\eta_1 \|L\| \leq 1$, we get
\[
\| \Pi_{k+1} L \xi \| \leq \sup_i \exp \left\{ -\sigma_i \sum_{l=k+1}^t \eta_l \right\} \sigma_i^\zeta \leq \sup_{x \geq 0} \exp \left\{ -x \sum_{l=k+1}^t \eta_l \right\} x^\zeta.
\]

The maximum of the function $g(x) = e^{-cx} x^\zeta$ (with $c > 0$) over $\mathbb{R}_+$ is achieved at $x_{\max} = \zeta/c$, and thus
\[
\sup_{x \geq 0} e^{-cx} x^\zeta = \left( \frac{\zeta}{ce} \right)^\zeta. \tag{26}
\]

Using this inequality, one can get the desired result (24). \qed

With the above lemma and Lemma A.2 from the appendix, we can derive the following result for the bias.
Proposition B.2. Under Assumption \( \underline{2} \), let \( \eta_1 \kappa^2 \leq 1 \). Then, for any \( t \in \mathbb{N} \),
\[
\| S_{\rho_\mu t+1} - f_H \|_\rho \leq R \left( \frac{\zeta}{2 \sum_{j=1}^t \eta_j} \right) \zeta.
\]  
(27)

In particular, if \( \eta = \eta t^{-\theta} \) for all \( t \in \mathbb{N} \), with \( \eta \in [0, \kappa^{-2}] \) and \( \theta \in [0, 1] \), then
\[
\| S_{\rho_\mu t+1} - f_H \|_\rho \leq R \zeta \eta^{-\zeta t(\theta-1)\zeta}.
\]  
(28)

Proof. The result is essentially proved in \cite{26}, see also \cite{19}. For the sake of completeness, we provide a proof here. Since \( \mu_{t+1} \) is given by (19), introducing with (21),
\[
\mu_{t+1} = \mu_t - \eta_t (T \mu_t - S^*_\rho f_H).
\]  
(29)

Thus,
\[
S_{\rho_\mu t+1} = S_{\rho_\mu t} - \eta_t S_{\rho_\mu} (T \mu_t - S^*_\rho f_H) = S_{\rho_\mu t} - \eta_t L (S_{\rho_\mu t} - f_H).
\]  
(30)

Subtracting both sides by \( f_H \),
\[
S_{\rho_\mu t+1} - f_H = (I - \eta_t L) (S_{\rho_\mu t} - f_H).
\]

Using this equality iteratively, with \( \mu_1 = 0 \),
\[
S_{\rho_\mu t+1} - f_H = -\Pi_1 (L) f_H.
\]

Taking the \( L^2(H, \rho X) \)-norm, by Assumption \( \underline{2} \)
\[
\| S_{\rho_\mu t+1} - f_H \|_\rho = \| \Pi_1 (L) f_H \|_\rho \leq \| \Pi_1 (L) \| R.
\]

By applying Lemma B.1 we get \( \underline{27} \). Combining \( \underline{27} \) with Lemma A.2 we get \( \underline{28} \). The proof is complete.

The following lemma gives upper bounds for the sequence \( \{ \mu_t \}_{t \in \mathbb{N}} \) in \( H \)-norm. It will be used for the estimation on the sample variance in the next section.

Lemma B.3. Under Assumption \( \underline{2} \), the following holds for all \( t \in \mathbb{N} \):
1) If \( \zeta \geq 1/2 \),
\[
\| \mu_t \|_H \leq R \kappa^{2\zeta - 1}.
\]  
(31)

2) If \( \zeta \in [0, 1/2] \),
\[
\| \mu_t \|_H \leq \kappa^{2\zeta - 1} \vee \left( \sum_{k=1}^t \eta_k \right)^{\frac{1}{2} - \zeta}.
\]  
(32)

Proof. The proof for the fixed step-size can be found in \cite{19}. Following from \( \underline{29} \), we have
\[
\mu_{t+1} = (I - \eta_t T) \mu_t + \eta_t S^*_\rho f_H.
\]

Applying this relationship iteratively, and introducing with \( \mu_1 = 0 \), we get
\[
\mu_{t+1} = \sum_{k=1}^t \eta_k \Pi_{k+1} (T) S^*_\rho f_H = \sum_{k=1}^t \eta_k S^*_\rho \Pi_{k+1} (L) f_H.
\]

Therefore, using Assumption \( \underline{2} \) and the spectrum theory,
\[
\| \mu_{t+1} \|_H \leq \sum_{k=1}^t \eta_k S^*_\rho \Pi_{k+1} (L) \| \leq R \max_{\sigma \in [0, \kappa^2]} \sigma^{1/2 + \zeta} \sum_{k=1}^t \eta_k \Pi_{k+1} (\sigma).
\]
If $\zeta \geq 1/2$, for any $\sigma \in [0, \kappa^2]$, 
\[
\sigma^{1/2+\zeta} \sum_{k=1}^{t} \eta_k \Pi_{k+1}^\epsilon(\sigma) \leq \kappa^{2\zeta-1} \sigma \sum_{k=1}^{t} \eta_k \Pi_{k+1}^\epsilon(\sigma) \leq \kappa^{2\zeta-1},
\]
where for the last inequality, we used
\[
\sum_{k=1}^{t} \eta_k \sigma \Pi_{k+1}^\epsilon(\sigma) = \sum_{k=1}^{t} (1 - (1 - \eta_k) \sigma) \Pi_{k+1}^\epsilon(\sigma) = \sum_{k=1}^{t} \Pi_{k+1}^\epsilon(\sigma) - \sum_{k=1}^{t} \Pi_{k}^\epsilon(\sigma) = 1 - \Pi_{1}^\epsilon(\sigma).
\]
Thus,
\[
\|\mu_{t+1}\|_H \leq R \kappa^{2\zeta-1}.
\]
The case for $\zeta \geq 1/2$ is similar to that in [19]. We omit it. The proof is complete. \qed

\section{Sample Variance}

In this section, we aim to estimate the sample variance, i.e., $\mathbb{E}[\|S_\rho \mu_t - S_\rho \nu_t\|^2_\rho]$. Towards this end, we need some preliminary analysis. We first introduce the following key inequality, which provides the hinge idea on estimating $\mathbb{E}[\|S_\rho \mu_t - S_\rho \nu_t\|^2_\rho]$.

\textbf{Lemma C.1.} For all $t \in [T]$, we have
\[
\|S_\rho \nu_{t+1} - S_\rho \mu_{t+1}\|_\rho \leq \sum_{k=1}^{t} \eta_k \left\| T^{\frac{1}{2}} \Pi_{k+1}^\epsilon(T_k) N_k \right\|_H,
\]
where
\[
N_k = (T \mu_k - S_\rho^* f_\rho) - (T \mu_k - S'_\rho y), \quad \forall k \in [T].
\]

\textbf{Proof.} Since $\nu_{t+1}$ and $\mu_{t+1}$ are given by (20) and (19), respectively,
\[
\nu_{t+1} - \mu_{t+1} = \nu_t - \mu_t + \eta_t \{ (T \mu - S_\rho^* f_\rho) - (T_k \nu_t - S'_\rho y) \}
\]
\[
= (I - \eta_t T_k)(\nu_t - \mu_t) + \eta_t \{ (T \mu - S_\rho^* f_\rho) - (T_k \mu_t - S'_\rho y) \},
\]
which is exactly
\[
\nu_{t+1} - \mu_{t+1} = (I - \eta_t T_k)(\nu_t - \mu_t) + \eta_t N_t.
\]
Applying this relationship iteratively, with $\nu_1 = \mu_1 = 0$,
\[
\nu_{t+1} - \mu_{t+1} = \Pi_1^\epsilon(T_k)(\nu_t - \mu_t) + \sum_{k=1}^{t} \eta_k \Pi_{k+1}^\epsilon(T_k) N_k = \sum_{k=1}^{t} \eta_k \Pi_{k+1}^\epsilon(T_k) N_k.
\]
By [18], we have
\[
\|S_\rho \nu_{t+1} - S_\rho \mu_{t+1}\|_\rho = \left\| \sum_{k=1}^{t} \eta_k T^{\frac{1}{2}} \Pi_{k+1}^\epsilon(T_k) N_k \right\|_H,
\]
which leads to the desired result [33]. The proof is complete. \qed

The above lemma demonstrates that in order to upper bound $\mathbb{E}[\|S_\rho \mu_t - S_\rho \nu_t\|^2_\rho]$, one may only need to bound $\left\| T^{\frac{1}{2}} \Pi_{k+1}^\epsilon(T_k) N_k \right\|_H$. A detailed look at this latter term indicates that one may analysis the terms $T^{\frac{1}{2}} \Pi_{k+1}^\epsilon(T_k)$ and $N_k$ separately, since $\mathbb{E}[N_k] = 0$ and the properties of the deterministic sequence $\{\mu_k\}_k$ are well developed in Section 3.

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Lemma C.2. **Under Assumptions** $\Box$ and $\Box$, let $\zeta \geq 1/2$. Then for any fixed $\lambda > 0$, with probability at least $1 - \delta_1$, the following holds for all $k \in \mathbb{N}$:

1) If $\zeta \geq 1/2$,

$$
\| (T + \lambda)^{-\frac{1}{2}} N_k \|_H \leq 4(R\kappa^{2\zeta} + \sqrt{M}) \left( \frac{\kappa}{m\sqrt{\lambda}} + \frac{\sqrt{2\sqrt{\kappa}c_\gamma}}{\sqrt{m\lambda^2}} \right) \log \frac{4}{\delta_1}.
$$

(35)

2) If $\zeta \in [0,1/2]$,

$$
\| (T + \lambda)^{-\frac{1}{2}} N_k \|_H \leq 4 \left( \kappa \left( \kappa^{2\zeta - 1} \left( \sum_{i=1}^{k} \eta_i \right)^{\frac{1}{2} - \zeta} \right) + \sqrt{M} \right) \left( \frac{\kappa}{m\sqrt{\lambda}} + \frac{\sqrt{2\sqrt{\kappa}c_\gamma}}{\sqrt{m\lambda^2}} \right) \log \frac{4}{\delta_1}.
$$

(36)

**Proof.** We will apply Bernstein inequality from Lemma $\Box$ to prove the result.

Bounding $\| (T + \lambda)^{-\frac{1}{2}} (S^*_\rho f_\rho - S^*_y y) \|_H$

For all $i \in [m]$, let $w_i = y_i (T + \lambda I)^{-\frac{1}{2}} x_i$. Obviously, from the definitions of $f_\rho$ (see (6)) and $S_\rho$,

$$
E[w_1] = E_{x_1}[f_\rho(x_1)(T + \lambda I)^{-\frac{1}{2}} x_1] = (T + \lambda I)^{-\frac{1}{2}} S^*_\rho f_\rho.
$$

Thus,

$$(T + \lambda)^{-\frac{1}{2}} (S^*_\rho f_\rho - S^*_y y) = \frac{1}{m} \sum_{i=1}^{m} (E[w_i] - w_i).$$

We next estimate the constants $B$ and $\sigma^2(w_1)$ in $\Box$. Note that for any $l \geq 2$,

$$
E[\| w_1 - E[w_1] \|^l_H] \leq E[(\| w_1 \|_H + E[\| w_1 \|_H])^l].
$$

By using Hölder’s inequality twice,

$$
E[\| w_1 - E[w_1] \|^l_H] \leq 2^{l-1} E[\| w_1 \|^l_H + (E[\| w_1 \|_H])^l] \leq 2^{l-1} E[\| w_1 \|^l_H + E[\| w_1 \|^l_H]].
$$

The right-hand side is exactly $2^l E[\| w_1 \|^l_H]$. Therefore, by recalling the definition of $w_1$ and expanding the integration,

$$
E[\| w_1 - E[w_1] \|^l_H] \leq 2^l \int_Y y^l d\rho(y|x) \int_X \| (T + \lambda I)^{-\frac{1}{2}} x \|_H d\rho_X(x).
$$

(37)

Note that by using Hölder’s inequality,

$$
\int_Y y^l d\rho(y|x) \int_X \leq \left( \int_Y |y|^2 d\rho(y|x) \right)^{\frac{l}{2}}.
$$

Using Assumption $\Box$ to the above,

$$
\int_Y y^l d\rho(y|x) \int_X \leq \sqrt{l! M^l v} \leq l! (\sqrt{M})^l \sqrt{v}.
$$

Plugging the above into (37), we reach

$$
E[\| w_1 - E[w_1] \|^l_H] \leq l! (2\sqrt{M})^l \sqrt{v} \int_X \| (T + \lambda I)^{-\frac{1}{2}} x \|^l_H d\rho_X(x).
$$

Using Assumption $\Box$ which implies

$$
\| (T + \lambda I)^{-\frac{1}{2}} x \|_H \leq \frac{\| x \|_H}{\sqrt{\lambda}} \leq \frac{\kappa}{\sqrt{\lambda}},
$$

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we get that
\[ \mathbb{E}[\|w_1 - \mathbb{E}[w_1]\|_H^2] \leq t!(2\sqrt{M})^l \sqrt{\nu} \left( \frac{\kappa}{\sqrt{\lambda}} \right)^{l-2} \int_X \| (T + \lambda I)^{-\frac{1}{2}} x \|_H^2 d\rho_x(x). \]

Using the fact that \( \mathbb{E}[\|\xi\|^2_H] = \mathbb{E}[\text{tr}(\xi \otimes \xi)] = \text{tr}(\mathbb{E}[\xi \otimes \xi]) \) and \( \mathbb{E}[\xi \otimes x] = T \), we know that
\[ \int_X \| (T + \lambda I)^{-\frac{1}{2}} x \|_H^2 d\rho_x(x) = \text{tr}((T + \lambda I)^{-\frac{1}{2}} T (T + \lambda I)^{-\frac{1}{2}}) = \text{tr}((T + \lambda I)^{-1} T), \]
and as a result of the above and Assumption 3,
\[ \int_X \| (T + \lambda I)^{-\frac{1}{2}} x \|_H^2 d\rho_x(x) \leq c_\gamma \lambda^{-\gamma}. \]

Therefore,
\[ \mathbb{E}[\|w_1 - \mathbb{E}[w_1]\|_H^2] \leq t!(2\sqrt{M})^l \sqrt{\nu} \left( \frac{\kappa}{\sqrt{\lambda}} \right)^{l-2} c_\gamma \lambda^{-\gamma} = \frac{1}{2} t!(\frac{2\kappa^2\sqrt{M}}{\sqrt{\lambda}})^{l-2} 8M \sqrt{v} c_\gamma \lambda^{-\gamma}. \]

Applying Berstein inequality with \( B = \frac{2\kappa\sqrt{M}}{\sqrt{\lambda}} \) and \( \sigma = \sqrt{8M \sqrt{v} c_\gamma \lambda^{-\gamma}} \), we get that with probability at least \( 1 - \frac{4}{\delta_1} \), there holds
\[ \left\| (T + \lambda I)^{-\frac{1}{2}} (S^*_\rho f_d - S^*_\xi y) \right\|_H \leq \frac{1}{m} \sum_{i=1}^m \left( \mathbb{E}[w_i] - w_i \right) \leq 4\sqrt{M} \left( \frac{\kappa}{m\sqrt{\lambda}} + \frac{\sqrt{2v}\sqrt{c_\gamma}}{\sqrt{m\lambda^3}} \right) \log \frac{4}{\delta_1}. \]  

**Bounding \( \| (T + \lambda I)^{-\frac{1}{2}} (T - T_k) \| \)**

Let \( \xi_i = (T + \lambda I)^{-\frac{1}{2}} x_i \otimes x_i \), for all \( i \in [m] \). It is easy to see that \( \mathbb{E}[\xi_i] = (T + \lambda I)^{-\frac{1}{2}} T \), and that \( (T + \lambda I)^{-\frac{1}{2}} (T - T_k) = \frac{1}{m} \sum_{i=1}^m (\mathbb{E}[\xi_i] - \xi_i) \). Denote the Hilbert-Schmidt norm of a bounded operator from \( H \) to \( H \) by \( \| \cdot \|_H \). Note that
\[ \left\| \xi_i \right\|_H^2 = \left\| x_i \right\|_H^2 \text{Trace}((T + \lambda I)^{-1/2} x_1 \otimes x_1 (T + \lambda I)^{-1/2}) = \left\| x_i \right\|_H^2 \text{Trace}((T + \lambda I)^{-1} x_1 \otimes x_1). \]

By Assumption 3,
\[ \left\| \xi_i \right\|_H \leq \sqrt{\kappa^2 \text{Trace}((T + \lambda I)^{-1} x_1 \otimes x_1)} \leq \sqrt{\kappa^2 \text{Trace}(x_1 \otimes x_1)/\lambda} \leq \kappa^2 / \sqrt{\lambda}, \]
and furthermore, by Assumption 3,
\[ \mathbb{E}[\|\xi_i\|_H^2] \leq \kappa^2 \mathbb{E}[\text{Trace}((T + \lambda I)^{-1} x_1 \otimes x_1)] = \kappa^2 \text{Trace}((T + \lambda I)^{-1} T) \leq \kappa^2 c_\gamma \lambda^{-\gamma}. \]

According to Lemma A.1, we get that with probability at least \( 1 - \frac{4}{\delta_1} \), there holds
\[ \left\| (T + \lambda I)^{-\frac{1}{2}} (T - T_k) \right\|_H \leq 2\kappa \left( \frac{2\kappa}{m\sqrt{\lambda}} + \frac{\sqrt{2v}\sqrt{c_\gamma}}{\sqrt{m\lambda^3}} \right) \log \frac{4}{\delta_1}. \]  

Finally, using the triangle inequality, we have,
\[ \left\| (T + \lambda I)^{-\frac{1}{2}} N_k \right\|_H \leq \left\| (T + \lambda I)^{-\frac{1}{2}} (T - T_k) \right\|_H \|\mu_k\|_H + \left\| (T + \lambda I)^{-\frac{1}{2}} (S^*_\rho f_d - S^*_\xi y) \right\|_H. \]

Applying Lemma B.3 to the above, introducing with 38 and 39, and then noting that \( \kappa \geq 1 \) and \( v \geq 1 \), one can prove the desired results. \( \square \)

The next lemma is borrowed from [27], derived by applying a recent Bernstein inequality from [28][29] for a sum of random operators.
Lemma C.3. Let $\delta_2 \in (0, 1)$ and $\frac{2\kappa^2}{m} \log \frac{n}{\delta_2} \leq \lambda \leq \|T\|$. Then the following holds with probability at least $1 - \delta_2$,

$$\|(T_x + \lambda I)^{-\frac{1}{2}} T^{\frac{1}{2}}\| \leq \|(T_x + \lambda)^{-\frac{1}{2}} (T + \lambda)^{\frac{1}{2}}\| \leq 2. \quad (40)$$

Now we are in a position to estimate the sample variance.

**Proposition C.4.** Let $\eta k^2 \leq 1$ and $\|T\| \leq 2$ for all $k \in [T]$. Assume that $\|T\| \leq 2$ holds. Then the following holds for all $t \in [T]$:

1) If $\zeta \geq 1/2$,

$$\|S_{\rho t+1} - S_{\rho t+1}\|_p \leq 4(R\kappa^2 + \sqrt{M}) \left( \frac{k}{m\sqrt{\lambda}} + \sqrt{\frac{2\sqrt{\nu_c}}{m\lambda^2}} \right) \left( \sum_{k=1}^{t-1} \frac{\eta_k / 2}{\sum_{i=k+1}^t \eta_i} + \lambda \sum_{k=1}^{t-1} \eta_k + \sqrt{2\kappa^2 \eta_t} \right) \log \frac{4}{\delta_1}. \quad (41)$$

2) If $\zeta \leq 1/2$,

$$\|S_{\rho t+1} - S_{\rho t+1}\|_p \leq 4 \left( \kappa \left( \kappa^{2t-1} \sqrt{\sum_{i=1}^t \eta_i} \right)^{-\frac{1}{2} - \zeta} \right) + \sqrt{M} \left( \frac{k}{m\sqrt{\lambda}} + \sqrt{\frac{2\sqrt{\nu_c}}{m\lambda^2}} \right) \log \frac{4}{\delta_1}. \quad (42)$$

**Proof.** For notational simplicity, we let $T_x = T + \lambda I$ and $T_{x,\lambda} = T_x + \lambda I$. Note that by Lemma C.1, we have $\|T\| \leq 2$. When $k \in [t-1]$, by rewriting $T^{\frac{1}{2}} \Pi_{k+1}^t(T_x) N_k$ as

$$T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} N_k,$$

we can upper bound $\|T^{\frac{1}{2}} \Pi_{k+1}^t(T_x) N_k\|_H$ as

$$\|T^{\frac{1}{2}} \Pi_{k+1}^t(T_x) N_k\|_H \leq \|T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} N_k\|_H.$$

Applying (40), the above can be relaxed as

$$\|T^{\frac{1}{2}} \Pi_{k+1}^t(T_x) N_k\|_H \leq 4 \|T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} N_k\|_H,$$

which is equivalent to

$$\|T^{\frac{1}{2}} \Pi_{k+1}^t(T_x) N_k\|_H \leq 4 \|T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} N_k\|_H.$$

Thus, following from $\eta_k k^2 \leq 1$ which implies $\eta_k \|T_x\| \leq 1$,

$$\|T_{x,\lambda} \Pi_{k+1}^t(T_x)\| \leq \|T_{x,\lambda} T_{k+1}^t(T_x)\| + \|\lambda \Pi_{k+1}^t(T_x)\| \leq \|T_{x,\lambda} T_{k+1}^t(T_x)\| + \lambda.$$

Applying Lemma B.1 with $\zeta = 1$ to bound $\|T_{x,\lambda} \Pi_{k+1}^t(T_x)\|$, we get

$$\|T_{x,\lambda} \Pi_{k+1}^t(T_x)\| \leq \frac{1}{e \sum_{j=k+1}^t \eta_j} + \lambda.$$

When $k = t$,

$$\|T^{\frac{1}{2}} \Pi_{k+1}^t(T_x) N_k\|_H = \|T^{\frac{1}{2}} N_k\|_H \leq \|T^{\frac{1}{2}}\| \|T^{\frac{1}{2}}\| \|T^{\frac{1}{2}} N_k\|_H \leq \|T\| \|T\| \|T^{\frac{1}{2}} N_k\|_H.$$
Since $\lambda \leq \|T\| \leq \text{tr}(T) \leq \kappa^2$, we derive

$$\|T^{\frac{1}{2}} \Pi_{k+1}^c(T) N_t\|_H \leq \sqrt{2} \kappa^2 \|T^{-\frac{1}{2}} N_t\|_H.$$  

From the above analysis, we conclude that $\sum_{k=1}^t \eta_k \left\|T^{\frac{1}{2}} \Pi_{k+1}^c(T) N_k\right\|_H$ can be upper bounded by

$$\leq \sup_{k \in [t]} \|T^\frac{1}{2}_\lambda N_k\|_H \left( \sum_{k=1}^{t-1} \frac{\eta_k}{\sum_{i=k+1}^t \eta_i} + \lambda \sum_{k=1}^t \eta_k + \sqrt{2} \kappa^2 \eta_t \right).$$

Plugging (35) (or (36)) into the above, and then combining with (33), we get the desired bound (41) (or (42)). The proof is complete. \qed

Setting $\eta_t = \eta_1 t^{-\theta}$ in the above proposition, with some basic estimates from Appendix A, we get the following explicit bounds for the sample variance.

**Proposition C.5.** Let $\eta_t = \eta_1 t^{-\theta}$ and (35) for all $t \in [T]$, with $\eta_1 \in [0, \kappa^{-2}]$ and $\theta \in [0, 1]$. Assume that (40) holds. Then the following holds for all $t \in [T]$:

1) If $\zeta \geq 1/2$,

$$\|S_{\rho \mu t+1} - S_{\rho \mu t+1}\|_\rho \leq 4 (R \kappa^{2\zeta} + \sqrt{M}) \left( \frac{2 \lambda \eta_1 t^{1-\theta}}{1 - \theta} + \log t + 1 + \sqrt{2} \eta_2 \kappa^2 \right) \left( \frac{\kappa}{m \sqrt{\lambda}} + \sqrt{\frac{2 \sqrt{\nu c}}{\sqrt{m \lambda}} \zeta} \right) \log \frac{4}{\delta_1}. \quad (43)$$

2) If $\zeta \leq 1/2$,

$$\|S_{\rho \mu t+1} - S_{\rho \mu t+1}\|_\rho \leq 4 \left( \kappa \left( \kappa^{2(1-\zeta)} + \frac{2 \lambda \eta_1 t^{1-\theta}}{1 - \theta} \right) + \sqrt{M} \right) \times \left( \frac{2 \lambda \eta_1 t^{1-\theta}}{1 - \theta} + \log t + 1 + \sqrt{2} \eta_2 \kappa^2 \right) \left( \frac{\kappa}{m \sqrt{\lambda}} + \sqrt{\frac{2 \sqrt{\nu c}}{\sqrt{m \lambda}} \zeta} \right) \log \frac{4}{\delta_1}. \quad (44)$$

**Proof.** By Proposition C.4 we have (41). Note that

$$\sum_{k=1}^{t-1} \eta_k \sum_{i=k+1}^t \eta_i \leq \sum_{k=1}^{t-1} \sum_{i=k+1}^t \eta_i = \sum_{k=1}^{t-1} \sum_{i=k+1}^t i^{-\theta} \leq \sum_{k=1}^{t-1} \frac{k^{-\theta}}{t^{-\theta}}.$$

Applying Lemma A.4 we get

$$\sum_{k=1}^{t-1} \frac{\eta_k}{\sum_{i=k+1}^t \eta_i} \leq 2 + 2 \log t,$$

and by Lemma A.2

$$\sum_{k=1}^{t-1} \eta_k = \eta_1 \sum_{k=1}^{t-1} k^{-\theta} \leq \frac{2 \eta_1 t^{1-\theta}}{1 - \theta}.$$  

Introducing the last two estimates into (41) and (43), one can get the desired results. The proof is complete. \qed

In conclusion, we get the following result for the sample variance.

**Theorem C.6.** Under Assumptions 1, 2 and 3, let $\delta_1, \delta_2 \in [0, 1]$ and $\frac{m^2}{\nu c} \log \frac{\eta_1}{\delta_2} \leq \lambda \leq \|T\|$. Let $\eta_t = \eta_1 t^{-\theta}$ for all $t \in [T]$, with $\eta_1 \in [0, \kappa^{-2}]$ and $\theta \in [0, 1]$. Then with probability at least $1 - \delta_1 - \delta_2$, the following holds for all $t \in [T]$:

1) if $\zeta \geq 1/2$, we have (43).

2) if $\zeta < 1/2$, we have (44).
In this section, we estimate the computational variance, $E[\|S_\rho \omega_t - S_\rho \omega_t \|^2_\rho]$. For this, a series of lemmas is necessarily introduced.

### D.1 Bounding the Empirical Risk

This subsection is devoted to upper bounding $E_\mathcal{F}[\mathcal{E}_\mathcal{F}(\omega_t)]$. The process relies on some tools from convex analysis and a decomposition related to the weighted averages and the last iterates from [22, 30]. We begin by introducing the following lemma, a fact based on the square loss’ special properties.

**Lemma D.1.** Given any sample $z$, and $l \in \mathbb{N}$, let $\omega \in H$ be independent from $J_1$, then

$$\eta (\mathcal{E}_\mathcal{F}(\omega_t) - \mathcal{E}_\mathcal{F}(\omega)) \leq \|\omega_t - \omega\|^2_H - E_{\mathcal{F}} \|\omega_{t+1} - \omega\|^2_H + \eta^2 \kappa^2 \mathcal{E}_\mathcal{F}(\omega_t). \tag{45}$$

**Proof.** Since $\omega_{t+1}$ is given by [4], subtracting both sides of [4] by $\omega$, taking the square $H$-norm, and expanding the inner product,

$$\|\omega_{t+1} - \omega\|^2_H = \|\omega_t - \omega\|^2_H + \frac{\eta^2}{b^2} \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j) x_j \| H^2 + \frac{2\eta}{b} \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j)(\omega - \omega_t, x_j)_H.$$

By Assumption [3], $\|x_j\|_H \leq \kappa$, and thus

$$\left\| \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j)x_j \right\|_H^2 \leq \left( \sum_{i=b(l-1)+1}^{bl} |\langle \omega_i, x_j \rangle_H - y_j| \right)^2 \leq \kappa^2 b \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j)^2,$$

where for the last inequality, we used Cauchy-Schwarz inequality. Thus,

$$\|\omega_{t+1} - \omega\|^2_H \leq \|\omega_t - \omega\|^2_H + \frac{\eta^2 \kappa^2}{b} \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j)^2$$

$$+ \frac{2\eta}{b} \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j)(\langle \omega_i, x_j \rangle_H - \langle \omega, x_j \rangle_H).$$

Using the basic inequality $a(b - a) \leq (b^2 - a^2)/2, \forall a, b \in \mathbb{R}$,

$$\|\omega_{t+1} - \omega\|^2_H \leq \|\omega_t - \omega\|^2_H + \frac{\eta^2 \kappa^2}{b} \sum_{i=b(l-1)+1}^{bl} (\langle \omega_i, x_j \rangle_H - y_j)^2$$

$$+ \frac{\eta}{b} \sum_{i=b(l-1)+1}^{bl} \left( (\langle \omega_i, x_j \rangle_H - y_j)^2 - (\langle \omega, x_j \rangle_H - y_j)^2 \right).$$

Noting that $\omega_t$ and $\omega$ are independent from $J_1$, and taking the expectation on both sides with respect to $J_1$,

$$E_{J_1} \|\omega_{t+1} - \omega\|^2_H \leq E_{J_1} \|\omega_t - \omega\|^2_H + \eta^2 \kappa^2 \mathcal{E}_\mathcal{F}(\omega_t) + \eta_i (\mathcal{E}_\mathcal{F}(\omega) - \mathcal{E}_\mathcal{F}(\omega_t)),$$

which leads to the desired result by rearranging terms. The proof is complete. \qed
Using the above lemma and a decomposition related to the weighted averages and the last iterates from [22, 30], we can prove the following relationship.

**Lemma D.2.** Let \( \eta \kappa^2 \leq 1/2 \) for all \( t \in \mathbb{N} \). Then

\[
\eta t E_\lambda[\mathcal{E}_\lambda(\omega_t)] \leq 4E_\lambda(0) \frac{1}{t} \sum_{i=1}^{t} \eta_t + 2\kappa^2 \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^{t-1} \eta_t^2 E_\lambda[\mathcal{E}_\lambda(\omega_i)]. \quad (46)
\]

**Proof.** For \( k = 1, \ldots, t-1 \),

\[
\frac{1}{k(k+1)} \sum_{i=t-k}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] - \frac{1}{k+1} \sum_{i=t-k}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] \]

\[
= \frac{1}{k(k+1)} \left\{ (k+1) \sum_{i=t-k}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] - k \sum_{i=t-k}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] \right\} \]

\[
= \frac{1}{k(k+1)} \sum_{i=t-k+1}^{t} (\eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] - \eta_{i-1} E_\lambda[\mathcal{E}_\lambda(\omega_{i-1})]).
\]

Summing over \( k = 1, \ldots, t-1 \), and rearranging terms, we get [30]

\[
\eta_t E_\lambda[\mathcal{E}_\lambda(\omega_t)] \leq \frac{1}{t} \sum_{i=1}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] + \frac{1}{k+1} \sum_{i=t-k+1}^{t} (\eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] - \eta_{i-1} E_\lambda[\mathcal{E}_\lambda(\omega_{i-1})]).
\]

Since \( \{\eta_t\} \) is decreasing and \( E_\lambda[\mathcal{E}_\lambda(\omega_{i-1})] \) is non-negative, the above can be relaxed as

\[
\eta_t E_\lambda[\mathcal{E}_\lambda(\omega_t)] \leq \frac{1}{t} \sum_{i=1}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] + \frac{1}{k+1} \sum_{i=t-k+1}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i) - E_\lambda(\omega_{i-1})]. \quad (47)
\]

In the rest of the proof, we will upper bound the last two terms of the above.

To bound the first term of the right side of (47), we apply Lemma D.1 with \( \omega = 0 \) to get

\[
\eta_t E_\lambda(\mathcal{E}_\lambda(\omega_t) - E_\lambda(0)) \leq E_\lambda[\|\omega_t\|_H^2 - \|\omega_{t+1}\|_H^2] + \eta_t^2 E_\lambda[\mathcal{E}_\lambda(\omega_t)].
\]

Rearranging terms,

\[
\eta_t (1 - \eta_t \kappa^2) E_\lambda[\mathcal{E}_\lambda(\omega_t)] \leq E_\lambda[\|\omega_t\|_H^2 - \|\omega_{t+1}\|_H^2] + \eta_t E_\lambda(0).
\]

It thus follows from the above and \( \eta_t \kappa^2 \leq 1/2 \) that

\[
\eta_t E_\lambda[\mathcal{E}_\lambda(\omega_t)]/2 \leq E_\lambda[\|\omega_t\|_H^2 - \|\omega_{t+1}\|_H^2] + \eta_t E_\lambda(0).
\]

Summing up over \( t = 1, \ldots, t \),

\[
\sum_{i=1}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)]/2 \leq E_\lambda[\|\omega_1\|_H^2 - \|\omega_{t+1}\|_H^2] + E_\lambda(0) \sum_{i=1}^{t} \eta_i.
\]

Introducing with \( \omega_1 = 0, \|\omega_{t+1}\|_H^2 \geq 0 \), and then multiplying both sides by \( 2/t \), we get

\[
\frac{1}{t} \sum_{i=1}^{t} \eta_i E_\lambda[\mathcal{E}_\lambda(\omega_i)] \leq 2E_\lambda(0) \frac{1}{t} \sum_{i=1}^{t} \eta_i. \quad (48)
\]

It remains to bound the last term of (47). Let \( k \in \{t-1\} \) and \( i \in \{t-k, \ldots, t\} \). Note that given the sample \( z, \omega_i \) is depending only on \( J_1, \ldots, J_{i-1} \) when \( i > 1 \) and \( \omega_1 = 0 \). Thus, we can apply Lemma D.1 with \( \omega = \omega_{t-k} \) to derive

\[
\eta_t (E_\lambda(\omega_t) - E_\lambda(\omega_{t-k})) \leq \|\omega_i - \omega_{t-k}\|_H^2 - E_\lambda[\|\omega_{i+1} - \omega_{t-k}\|_H^2 + \eta_t^2 E_\lambda(\omega_i)].
\]

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Therefore,

\[ \eta_i \mathbb{E}_J [ \mathcal{E}_u (\omega_i) - \mathcal{E}_u (\omega_{t-k}) ] \leq \mathbb{E}_J [ ||\omega_i - \omega_{t-k}||^2_J - ||\omega_i + 1 - \omega_{t-k}||^2_J ] + \eta_i^2 \kappa^2 \mathbb{E}_J [ \mathcal{E}_u (\omega_i) ] \cdot \]

Summing up over \( i = t - k, \ldots, t, \)

\[ \sum_{i=t-k}^t \eta_i \mathbb{E}_J [ \mathcal{E}_u (\omega_i) - \mathcal{E}_u (\omega_{t-k}) ] \leq \kappa^2 \sum_{i=t-k}^t \eta_i^2 \mathbb{E}_J [ \mathcal{E}_u (\omega_i) ] \cdot \]

Note that the left hand side is exactly \( \sum_{i=t-k+1}^t \eta_i \mathbb{E}_J [ \mathcal{E}_u (\omega_i) - \mathcal{E}_u (\omega_{t-k}) ] \). We thus know that the last term of (47) can be upper bounded by

\[ \kappa^2 \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^t \eta_i^2 \mathbb{E}_J [ \mathcal{E}_u (\omega_i) ] \]

Using the fact that

\[ \sum_{k=1}^{t-1} \frac{1}{k(k+1)} = \sum_{k=1}^{t-1} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{t} \leq 1, \]

and \( \kappa^2 \eta_t \leq 1/2, \) we get that the last term of (47) can be bounded as

\[ \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k+1}^t \eta_i (\mathbb{E}_J [ \mathcal{E}_u (\omega_i) ] - \mathbb{E}_J [ \mathcal{E}_u (\omega_{t-k}) ] ) \]

\[ \leq \kappa^2 \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^t \eta_i^2 \mathbb{E}_J [ \mathcal{E}_u (\omega_i) ] + \eta_t \mathbb{E}_J [ \mathcal{E}_u (\omega_t) ] / 2. \]

Plugging the above and (48) into the decomposition (47), and rearranging terms

\[ \eta_t \mathbb{E}_J [ \mathcal{E}_u (\omega_t) ] / 2 \leq 2M^2 \frac{1}{t} \sum_{t=1}^t \eta_t + \kappa^2 \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^t \eta_i^2 \mathbb{E}_J [ \mathcal{E}_u (\omega_i) ] \]

which leads to the desired result by multiplying both sides by 2. The proof is complete. \( \square \)

We also need to the following lemma, whose proof can be done by using an induction argument.

**Lemma D.3.** Let \( \{ u_t \}_{t=1}^T, \{ A_t \}_{t=1}^T \) and \( \{ B_t \}_{t=1}^T \) be three sequences of non-negative numbers such that \( u_1 \leq A_1 \) and

\[ u_t \leq A_t + B_t \sup_{i \in \{ t-1 \}} u_i, \quad \forall t \in \{ 2, 3, \ldots, T \}. \] (49)

Let \( \sup_{t \in [T]} B_t \leq B < 1 \). Then for all \( t \in [T], \)

\[ \sup_{k \in [t]} u_t \leq \frac{1}{1-B} \sup_{k \in [t]} A_k. \] (50)

**Proof.** When \( t = 1, \) (50) holds trivially since \( u_1 \leq A_1 \) and \( B < 1 \). Now assume for some \( t \in \mathbb{N} \) with \( 2 \leq t \leq T, \)

\[ \sup_{i \in \{ t-1 \}} u_i \leq \frac{1}{1-B} \sup_{i \in \{ t-1 \}} A_i. \]
Then, by (49), the above hypothesis, and $B_t \leq B$, we have
\[ u_t \leq A_t + B_t \sup_{i \in [t-1]} u_i \leq A_t + \frac{B_t}{1-B} \sup_{i \in [t-1]} A_i \leq \sup_{i \in [t]} A_i \left(1 + \frac{B_t}{1-B} \right) \leq \sup_{i \in [t]} A_i \frac{1}{1-B}. \]
Consequently,
\[ \sup_{k \in [t]} u_t \leq \frac{1}{1-B} \sup_{k \in [t]} A_k, \]
thereby showing that indeed (50) holds for $t$. By mathematical induction, (50) holds for every $t \in [T]$. The proof is complete.

Now we can bound $\mathbb{E}_t[E_\omega f_k]$ as follows.

**Lemma D.4.** Let $\eta_1 \kappa^2 \leq 1/2$ and for all $t \in [T]$ with $t \geq 2$,
\[ \frac{1}{\eta t} \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^{t-1} \eta_i^2 \leq \frac{1}{4\kappa^2}. \] (51)
Then for all $t \in [T]$,
\[ \sup_{k \in [t]} \mathbb{E}_t[E_\omega f_k] \leq 8E_\omega(0) \sup_{k \in [t]} \left\{ \frac{1}{\eta k} \sum_{i=1}^{k} \eta_i \right\}. \] (52)

**Proof.** By Lemma D.2 we have (49). Dividing both sides by $\eta_t$, we can relax the inequality as
\[ \mathbb{E}_t[E_\omega(\omega_t)] \leq 4E_\omega(0) \frac{1}{\eta t} \sum_{i=1}^{t} \eta_i + 2\kappa^2 \frac{1}{\eta t} \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^{t-1} \eta_i^2 \sup_{i \in [t-1]} \mathbb{E}_i[E_\omega(\omega_i)]. \]
In Lemma D.3, we let $u_t = \mathbb{E}_t[E_\omega(\omega_t)]$, $A_t = 4E_\omega(0) \frac{1}{\eta t} \sum_{i=1}^{t} \eta_i$ and
\[ B_t = 2\kappa^2 \frac{1}{\eta t} \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=t-k}^{t-1} \eta_i^2. \]
Condition (51) guarantees that $\sup_{t \in [T]} B_t \leq 1/2$. Thus, (50) holds, and the desired result follows by plugging with $B = 1/2$. The proof is complete.

Finally, we need the following lemma to bound $E_\omega(0)$, whose proof follows from applying the Bernstein Inequality from Lemma A.1.

**Lemma D.5.** Under Assumption 7 with probability at least $1 - \delta_3$ ($\delta_3 \in [0,1]$), there holds
\[ E_\omega(0) \leq Mv + 2M \left( \frac{1}{m} + \frac{\sqrt{2v}}{\sqrt{m}} \right) \log \frac{2}{\delta_3}. \]
In particular, if $m \geq 32 \log^2 \frac{2}{\delta_3}$, then
\[ E_\omega(0) \leq 2Mv. \] (53)

**Proof.** Following from (5),
\[ \int_T y^2 d\rho \leq \frac{1}{2} \Phi(M^{-2} \cdot (2M^2 v)), \quad \forall \nu \in \mathbb{N}. \]
Applying Lemma A.1 with $\omega_i = y_i^2$ for all $i \in [m]$, $B = M$ and $\sigma = M\sqrt{2v}$, we know that with probability at least $1 - \delta_3$, there holds
\[ \frac{1}{m} \sum_{i=1}^{m} y_i^2 - \int_T y^2 d\rho \leq 2M \left( \frac{1}{m} + \frac{\sqrt{2v}}{\sqrt{m}} \right) \log \frac{2}{\delta_3}. \]
By setting \( l = 1 \) in (5),
\[
\int_Z y^2 d\rho \leq Mv.
\]
It thus follows that
\[
\frac{1}{m} \sum_{i=1}^{m} y_i^2 \leq \int_Z y^2 d\rho + 2M \left( \frac{1}{m} + \frac{\sqrt{2\nu}}{\sqrt{m}} \right) \log \frac{2}{\delta_3} \leq Mv + 2M \left( \frac{1}{m} + \frac{\sqrt{2\nu}}{\sqrt{m}} \right) \log \frac{2}{\delta_3},
\]
which leads to the desired results by noting that the left-hand side is exactly \( \mathcal{E}_Z(0) \) and \( \nu \geq 1 \).

The proof is complete. \( \square \)

D.2 Bounding \( \| T^\frac{1}{2} \Pi_{k+1}^t (T_x) \| \)

**Lemma D.6.** Assume (40) holds for some \( \lambda > 0 \) and \( \eta_{1\kappa^2} \leq 1 \). Then
\[
\| T^\frac{1}{2} \Pi_{k+1}^t (T_x) \| \leq \| T^\frac{1}{2} (T_x + \lambda I)^{-\frac{1}{2}} \Pi_{k+1}^t (T_x) \|.
\]

**Proof.** Note that we have
\[
\| T^\frac{1}{2} \Pi_{k+1}^t (T_x) \| \leq \| T^\frac{1}{2} (T_x + \lambda I)^{-\frac{1}{2}} \Pi_{k+1}^t (T_x) \|.
\]

Using (40), we can relax the above as
\[
\| T^\frac{1}{2} \Pi_{k+1}^t (T_x) \| \leq 2 \| (T_x + \lambda I)^\frac{1}{2} \Pi_{k+1}^t (T_x) \|,
\]
which leads to
\[
\| T^\frac{1}{2} \Pi_{k+1}^t (T_x) \|^2 \leq 4 \| (T_x + \lambda I)^\frac{1}{2} \Pi_{k+1}^t (T_x) \|^2.
\]

Since
\[
\| (T_x + \lambda I)^\frac{1}{2} \Pi_{k+1}^t (T_x) \|^2 = \| (T_x + \lambda I) \Pi_{k+1}^t (T_x) \Pi_{k+1}^t (T_x) \|
\]
\[
\leq \| T_x \Pi_{k+1}^t (T_x) \Pi_{k+1}^t (T_x) \| + \lambda
\]
\[
= \| T_x^\frac{1}{2} \Pi_{k+1}^t (T_x) \|^2 + \lambda,
\]
and with \( \eta_{1\kappa^2} \leq 1, \| T_x \| \leq \text{tr}(T_x) \leq \kappa^2, \) by Lemma \( \text{B.1} \)
\[
\| T_x^\frac{1}{2} \Pi_{k+1}^t (T_x) \|^2 \leq \frac{1}{2e \sum_{i=k+1}^{t} \eta_i} \leq \frac{1}{4 \sum_{i=k+1}^{t} \eta_i},
\]
we thus derive the desired result. The proof is complete. \( \square \)

D.3 Deriving Error Bounds

With Lemmas \( \text{D.4} \) and \( \text{D.6} \) we are ready to estimate the computational variance \( E_J \| f_t - g_t \|^2 \rho \), as follows.

**Proposition D.7.** Assume (40) holds for some \( \lambda > 0 \), \( \eta_{1\kappa^2} \leq 1/2 \), (51) and (53). Then, we have for all \( t \in [T] \),
\[
E_J \| S^\rho \omega_{t+1} - S^\rho \nu_{t+1} \|^2 \rho \leq \frac{16Mv\kappa^2}{b} \sup_{k \in [t]} \left\{ \frac{1}{\eta_k} \sum_{i=1}^{k} \eta_i \right\} \left( \sum_{k=1}^{t-1} \frac{\eta_k^2}{\sum_{i=k+1}^{t} \eta_i} + 4\lambda \sum_{k=1}^{t-1} \frac{\eta_k^2}{\sum_{i=k+1}^{t} \eta_i} + \eta_1^2 \kappa^2 \right).
\]

(54)
Proof. Since \( \omega_{t+1} \) and \( \nu_{t+1} \) are given by (14) and (20), respectively,

\[
\omega_{t+1} - \nu_{t+1} = (\omega_t - \nu_t) + \eta_t \left\{ (T_k \omega_t - S_k^* \gamma) - \frac{1}{b} \sum_{i=b(t-1)+1}^{bt} \langle (\omega_t, x_j), H - y_j, x_j \rangle \right\}
\]

\[
= (I - \eta_t T_k) (\omega_t - \nu_t) + \eta_t \sum_{i=b(t-1)+1}^{bt} \left\{ (T_k \omega_t - S_k^* \gamma) - \langle (\omega_t, x_j), H - y_j, x_j \rangle \right\}.
\]

Applying this relationship iteratively,

\[
\omega_{t+1} - \nu_{t+1} = \Pi_{\omega_k}^t (T_k) (\omega_t - \nu_t) + \frac{1}{b} \sum_{k=1}^t \sum_{i=b(k-1)+1}^{bk} \eta_t \Pi_{\omega_k}^{i+1} (T_k) M_{k,i},
\]

where we denote

\[
M_{k,i} = (T_k \omega_k - S_k^* \gamma) - \langle (\omega_k, x_j), H - y_j, x_j \rangle.
\]

Introducing with \( \omega_1 = \nu_1 = 0, \)

\[
\omega_{t+1} - \nu_{t+1} = \frac{1}{b} \sum_{k=1}^t \sum_{i=b(k-1)+1}^{bk} \eta_t \Pi_{\omega_k}^{i+1} (T_k) M_{k,i}.
\]

Therefore,

\[
E_\mathcal{J} \left\| S_\rho \omega_{t+1} - S_\rho \nu_{t+1} \right\|_\rho^2 = \frac{1}{b^2} E_\mathcal{J} \left\| \sum_{k=1}^t \sum_{i=b(k-1)+1}^{bk} \eta_t \Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \right\|_\rho^2
\]

\[
= \frac{1}{b^2} \sum_{k=1}^t \sum_{i=b(k-1)+1}^{bk} \eta_t^2 \left\| \Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \right\|_\rho^2,
\]

where for the last equality, we use the fact that if \( k \neq k', \) or \( k = k' \) but \( i \neq i' \) then

\[
E_\mathcal{J} (\Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \Pi_{\omega_{k'}}^{i'+1} (T_k) M_{k',i'})_\rho = 0.
\]

Indeed, if \( k \neq k' \), without loss of generality, we consider the case \( k < k' \). Recalling that \( M_{k,i} \) is given by (55) and that given any \( z, f_k \) is depending only on \( J_1, \ldots, J_{k-1} \), we thus have

\[
E_\mathcal{J} (\Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \Pi_{\omega_{k'}}^{i'+1} (T_k) M_{k',i'})_\rho = E_{\mathcal{J}_1, \ldots, J_{k'-1}} (\Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \Pi_{\omega_{k'}}^{i'+1} (T_k) E_{\mathcal{J}_{k'}} [M_{k',i'}])_\rho = 0.
\]

If \( k = k' \) but \( i \neq i' \), without loss of generality, we assume \( i < i' \). By noting that \( \omega_k \) is depending only on \( J_1, \ldots, J_{k-1} \) and \( M_{k,i} \) is depending only on \( \omega_k \) and \( z_{ji} \) (given any sample \( z \)),

\[
E_\mathcal{J} (\Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \Pi_{\omega_{k'}}^{i'+1} (T_k) M_{k',i'})_\rho = E_{\mathcal{J}_1, \ldots, J_{k'-1}} (\Pi_{\omega_k}^{i+1} (T_k) E_{\mathcal{J}_{j,i}} [M_{k,i}] \Pi_{\omega_{k'}}^{i'+1} (T_k) E_{\mathcal{J}_{j,i}} [M_{k',i'}])_\rho = 0.
\]

Using the isometry property (18) to (56),

\[
E_\mathcal{J} \left\| \Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \right\|_\rho^2 = E_\mathcal{J} \left\| \mathcal{T} \frac{1}{b} \Pi_{\omega_k}^{i+1} (T_k) M_{k,i} \right\|_H^2 \leq \left\| \mathcal{T} \frac{1}{b} \Pi_{\omega_k}^{i+1} (T_k) \right\|_H^2 \left\| M_{k,i} \right\|_H^2,
\]

and by applying the inequality \( E[\|\xi - E[\xi]\|_H^2] \leq E[\|\xi\|_H^2] \),

\[
E_\mathcal{J} \left\| M_{k,i} \right\|_H^2 \leq E_\mathcal{J} \left\| (\omega_k, x_j), H - y_j, x_j \right\|_H^2 \leq \kappa^2 E_\mathcal{J} [((\omega_k, x_j), H - y_j)^2] = \kappa^2 E_\mathcal{J} [\mathcal{E}_k (\omega_k)],
\]

This is possible only when \( b \geq 2 \).
where for the last inequality we use \( \eta \). Therefore,

\[
\mathbb{E}_\mathcal{A}\|\mathcal{S}_\rho \nu_{t+1} - \mathcal{S}_\rho \nu_t\|_\rho^2 \leq \frac{8}{b} \sum_{k=1}^t \frac{\eta_k^2}{\sum_{i=k+1}^t \eta_i^2} \left\| T_k \hat{\Pi}^k_{t+1}(\mathcal{T}_k) \right\|^2 \mathbb{E}_\mathcal{A}[\mathcal{E}_\mathcal{A}(\omega_k)].
\]

According to Lemma D.7, we have \( \eta \). It thus follows that

\[
\mathbb{E}_\mathcal{A}\|\mathcal{S}_\rho \nu_{t+1} - \mathcal{S}_\rho \nu_{t+1}\|_\rho^2 \leq \frac{8\mathbb{E}_\mathcal{A}(0)\kappa^2}{b} \sup_{k \in [\rho]} \left\{ \frac{1}{\eta_k} \sum_{i=k}^t \eta_i \right\} \sum_{k=1}^t \frac{\eta_k^2}{\eta_i} \left\| T_k \hat{\Pi}^k_{t+1}(\mathcal{T}_k) \right\|^2.
\]

Now the proof can be finished by applying Lemma D.8 which tells us that

\[
\sum_{k=1}^t \frac{\eta_k^2}{\eta_i} \left\| T_k \hat{\Pi}^k_{t+1}(\mathcal{T}_k) \right\|^2 = \sum_{k=1}^{t-1} \frac{\eta_k^2}{\eta_i} \left\| T_k \hat{\Pi}^k_{t+1}(\mathcal{T}_k) \right\|^2 + \eta_i^2 \left\| T_0 \hat{\Pi}^0_{t+1}(\mathcal{T}_0) \right\|^2.
\]

and \( \eta \) to the above. The proof is complete.

Setting \( \eta_t = \eta_t t^{-\theta} \) for some appropriate \( \eta_t \) and \( \theta \) in the above proposition, we get the following explicitly upper bounds for \( \mathbb{E}_\mathcal{A}\|\mathcal{S}_\rho \nu_{t+1} - \mathcal{S}_\rho \nu_{t+1}\|_\rho^2 \).

**Proposition D.8.** Assume \( \lambda \) holds for some \( \lambda > 0 \) and \( \lambda \). Let \( \eta_t = \eta_t t^{-\theta} \) for all \( t \in [T] \), with \( \theta \in [0, 1] \) and

\[
0 < \eta_t \leq \frac{\xi}{8\kappa^2(\log t + 1)}, \quad \forall t \in [T].
\]

Then, for all \( t \in [T] \),

\[
\mathbb{E}_\mathcal{A}\|\omega_{t+1} - \nu_{t+1}\|_\rho^2 \leq \frac{16M\kappa^2}{b(1-\theta)} \left( 5\eta_t t^{-\min(\theta,1-\theta)} + 8\lambda \eta_t^2 t^{(1-2\theta)} + 1 \right) (1 \vee \log t). \tag{58}
\]

**Proof.** We will use Proposition D.7 to prove the result. Thus, we need to verify the condition \( \eta \). Note that

\[
\sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=1}^{t-1} \frac{1}{k(k+1)} = \sum_{i=1}^{t-1} \eta_i \left( \frac{1}{t-i} - \frac{1}{t} \right) \leq \sum_{i=1}^{t-1} \eta_i \frac{1}{t-i}.
\]

Substituting with \( \eta_t = \eta_t t^{-\theta} \), and by Lemma A.4

\[
\sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=1}^{t-1} \eta_i^2 \frac{1}{t-i} \leq \sum_{i=1}^{t-1} \frac{t^{-2\theta}}{t-i} \leq 2\eta_t^2 t^{-\min(2\theta,1)}(\log t + 1).
\]

Dividing both sides by \( \eta_t \) \( (= \eta t^{-\theta}) \), and then using \( \eta_t \)

\[
\frac{1}{\eta_t} \sum_{k=1}^{t-1} \frac{1}{k(k+1)} \sum_{i=1}^{t-1} \frac{1}{t-i} \eta_i^2 \leq 2\eta_t t^{-\min(\theta,1-\theta)}(\log t + 1) \leq \frac{1}{4\kappa^2}.
\]

This verifies \( \eta \). Note also that by taking \( t = 1 \) in \( \eta \), for all \( t \in [T] \),

\[
\eta_t \kappa^2 \leq \eta \kappa^2 \leq \frac{1}{8\kappa^2} \leq \frac{1}{2}.
\]

We thus can apply Proposition D.7 to derive \( \eta \). What remains is to control the right hand side of \( \eta \). Since

\[
\sum_{k=1}^{t-1} \frac{\eta_k^2}{\sum_{i=k+1}^{t-1} \eta_i} = \eta_k \sum_{k=1}^{t-1} \frac{k^{-2\theta}}{\sum_{i=k+1}^{t-1} t^{-\theta}} \leq \eta_k \sum_{k=1}^{t-1} \frac{k^{-2\theta}}{(t-k)t^{-\theta}}.
\]

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Introducing the last three estimates into (54) and using that
≥ \eta t^{-(0,1-\theta)} (\log t + 1).

Also, by Lemma A.2
\[ \frac{1}{\eta_k} \sum_{i=k+1}^{l} \eta_i = \frac{1}{\kappa} \sum_{i=1}^{k} t^{-\theta} \leq \frac{1}{1 - \theta}, \]
and by Lemma A.3
\[ \sum_{k=1}^{t-1} \eta_k = \sum_{k=1}^{t-1} \kappa^{-2\theta} = \eta_1 t^{\max(1-2\theta,0)} (\log t + 1). \]

Introducing the last three estimates into (54) and using that \( \eta t^{2\theta} \leq \eta t^{-\theta} \) by (57), we get the desired result. The proof is complete.

Collect some of the above analysis, we get the following result for the computational variance.

**Theorem D.9.** Under Assumptions 1 and 3, let \( \eta_2 \in [0,1], \frac{98\eta^2}{m} \log \frac{m}{\eta} \leq \lambda \leq ||T||, \delta_3 \in [0,1], m \geq 32 \log^2 \frac{\lambda}{\delta}, \) and \( \eta = \eta t^{-\theta} \) for all \( t \in [T], \) with \( \theta \in [0,1] \) and \( \eta \) such that (57). Then, with probability at least \( 1 - \delta_2 - \delta_3, \) (58) holds for all \( t \in [T]. \)

### E \ Deriving Total Error Bounds

The purpose of this section is to derive total error bounds.

#### E.1 \ Attainable Case

We have the following general theorem for \( \zeta \geq 1/2, \) with which we prove our main results stated in Section 3.

**Theorem E.1.** Under Assumptions 1, 2 and 3, let \( \zeta \geq 1/2, T \in \mathbb{N} \) with \( T \geq 3, \delta \in [0,1], \eta_t = \eta t^{-\theta} \) for all \( t \in [T], \) with \( \theta \in [0,1] \) and \( \eta \) such that
\begin{equation}
0 < \eta \leq \frac{t^{\min(0,1-\theta)}}{8(\log t + 1)}, \quad \forall t \in [T].
\end{equation}

If for some \( \epsilon \in [0,1], \)
\begin{equation}
m \geq \frac{18\kappa^2}{\epsilon ||T||^{\log \left( \frac{27\kappa^2}{\epsilon ||T||^2} \right)^{1/\epsilon}},
\end{equation}
then the following holds with probability at least \( 1 - \delta: \) for all \( t \in [T], \)
\begin{equation}
\mathbb{E}[\mathcal{E}(\omega_{t+1})] - \inf_{\omega \in H} \mathcal{E}(\omega) \leq q_1 (\eta t^{1-\theta})^{-2\kappa} + q_2 m^{\gamma(1-\epsilon)-1}(1 \lor \eta^2 m^{2-2\theta}) (\log T)^2 \log^2 \frac{12}{\delta} + q_3 m^{\gamma(1-\theta) - 1} (\log T).
\end{equation}

Here, \( q_1 = 2R^2 \epsilon^{-2\kappa}, q_2 = \frac{800(R\kappa^2 + \sqrt{M})}{(1-\theta)^2} + \frac{2\sqrt{2\epsilon \mathbb{E}_{\omega} ||T||^2}}{(1-\theta)^2}, \) and \( q_3 = \frac{208 M V}{1-\theta}. \)

**Proof.** Let \( \lambda = ||T||m^{\epsilon-1}. \) Clearly, \( \lambda \leq ||T||. \) For any \( A \geq 0 \) and \( B \geq 1, \) by applying (26) with \( \zeta = 1, x = (Bm)^{\epsilon} \) and \( c = \frac{1}{2AB}. \)
\begin{equation}
A \log(Bm) = \frac{A}{e} \log((Bm)^{\epsilon}) \leq \frac{A e}{e} \log \left( \frac{2AB^\epsilon}{e} \right) + \frac{1}{2} m^{\epsilon} \leq \frac{A e}{e} \log \left( \frac{AB}{e} \right) + \frac{1}{2} m^{\epsilon}.
\end{equation}
Using the above inequality with $A = \frac{9\kappa^2}{m}$ and $B = \frac{1}{\kappa^2}$, one can prove that the condition (60) ensures that $\frac{9\kappa^2}{m} \log \frac{2\kappa}{\delta} \leq \lambda$ is satisfied with $\delta_2 = \frac{\delta}{2}$. Therefore, by Lemma C.3, (40) holds with probability at least $1 - \delta_2$. Similarly the condition (60) implies that $m \geq \frac{32\log^2 \frac{2\kappa}{\delta}}{\delta}$ is satisfied with $\delta_3 = \frac{\delta}{4}$, and thus by Lemma D.3, (53) holds with probability at least $1 - \delta_3$. Combining with Lemma C.2, by taking the union bound, we know that with probability at least $1 - \delta_1 - \delta_2 - \delta_3$, (40), (53) and (55) hold for all $k \in [T]$. Now, we can apply Propositions C.5 and D.8 to get (43) and (58). Noting that by (57), $\sqrt{2\eta} \leq 1$, and by a simple calculation, we derive from (43) that

\[
\|S_{\rho_{t+1}} - S_{\rho_{t+1}}\|^2 \leq \frac{400(R\kappa^2 + \sqrt{M})(\kappa/\|T\| + 2\sqrt{\nu\gamma/\|T\|^2})^2 m^{(1-\epsilon)}(1 \lor \lambda^2 \eta^2 t^{2-2\theta} \lor \log^2 t) \log^2 \frac{4}{\delta_1}.
\]

where for the last inequality, we used $\|T\| \leq \kappa^2$. Similarly, by a simple calculation, we get from (58) that

\[
\mathbb{E}_j\|S_{\rho_{t+1}} - S_{\rho_{t+1}}\|^2 \leq \frac{208Mv}{\delta(1 - \theta)} (\eta^2 \min(0,1-\theta) \lor \lambda^2 \eta^2 t^{2-2\theta} - t^{2-2\theta}) (1 \lor \log t)
\]

Letting $\delta_1 = \frac{\delta}{3}$, and introducing the above estimates and (28) into (16), we get (61). The proof is complete.

**Proof of Theorem 7.3.** By choosing $\epsilon = 1 - \frac{1}{2\kappa + 1}$ and $\theta = 0$ in Theorem E.1, then the condition (60) reduces to $m \geq m_\delta$, where

\[
m_\delta = \frac{18\kappa^2 p}{\|T\|} \log \left( \frac{27\kappa^2 p}{\|T\|^2} \right)^p, \quad p = \frac{2\zeta + \gamma}{2\zeta + \gamma - 1}.
\]

The desired result thus follows by applying Theorem E.1.

**Proof of Corollary 3.3.** We use Theorem 3.2 to prove the result. We first prove that the conditions of Theorem 3.2 are true. When $\delta = 1/m$, the condition $m \geq m_\delta$ with $m_\delta$ given by (63), is equivalent to

\[
m^{1/p} \geq \frac{18\kappa^2 p}{\|T\|} \log \left( \frac{27\kappa^2 p}{\|T\|^2} \right).
\]

Using (62) with $\epsilon = 1/p$, $A = \frac{18\kappa^2 p}{\|T\|}$ and $B = \frac{27\kappa^2 p}{\|T\|^2}$, we know that

\[
\frac{18\kappa^2 p}{\|T\|} \log \left( \frac{27\kappa^2 p}{\|T\|^2} \right) \leq \frac{18\kappa^2 p}{\|T\|} \log \left( \frac{486\kappa^4 p^3}{\|T\|^2} \right) + \frac{1}{2} m^{1/p},
\]

and consequently we know that the condition $m \geq m_\delta$ is met if $m \geq m_0$, where

\[
m_0 = \left( \frac{36\kappa^2 p^2}{\|T\|} \log \left( \frac{486\kappa^4 p^3}{\|T\|^2} \right) \right)^{1/p}.
\]

Letting $\eta_t = \eta^{-2}$ with $\eta = \frac{1}{16m}$ (which is $\eta_t \simeq \frac{1}{m}$), we know that $\eta \leq \frac{1}{8(\log T + 17)}$, since $3 \leq T \leq m^2$ and $\log m \leq m - 1$. We thus verified the conditions of Theorem 3.2. We thus have (6). By introducing with $\delta = 1/m$, $\eta = \frac{1}{16m}$ and $T \leq m^2$,

\[
\mathbb{E}_j\|S_{\rho_{t+1}} - f_H\|_{\rho} \leq \left( m^{-1}t \right)^{-2\kappa} + m^{-2\kappa+1}(1 + m^{-\frac{2\kappa}{2\kappa+1}} - 1)^2 \log^6 m + m^{-1}(1 \lor m^{-\frac{2\kappa}{2\kappa+1}} - 1) \log m.
\]

Using $m^{-1} \leq m^{-\frac{2\kappa}{2\kappa+1}} \leq \left( m^{-1}t \right)^{-2\kappa} + m^{-2\kappa+1}(m^{-\frac{2\kappa}{2\kappa+1}} - 1)^2$, one can prove the desired result.

\[\text{[B] is always greater than 1 since } \kappa^2 \geq \|T\| \text{ and } p \geq 1.\]
The proofs for the other corollaries parallelize to the above. We thus omit.

E.2 Non Attainable Case

Theorem E.2. Under Assumptions 1, 2 and 3, let \( \zeta \leq 1/2 \), \( T \in \mathbb{N} \) with \( T \geq 3 \), \( \delta \in [0, 1] \), \( \eta_t = \eta \kappa^{-2t-\theta} \) for all \( t \in [T] \), with \( \theta \in [0, 1] \) and \( \eta \) such that \([59]\) and for some \( \epsilon \in [0, 1] \), \([61]\) holds. Then the following holds with probability at least \( 1 - \delta \): for all \( t \in [T] \),

\[
E_{\delta} [E(\omega_{t+1})] - \inf_{\omega \in H} E(\omega) \lesssim (\eta t^{1-\theta})^{-2\zeta} + m \gamma (1-\epsilon)^{-1} (1 \vee \eta^2 m^{2\epsilon-2t-2\theta}) (1 \vee \eta t^{1-\theta})^{1-2\zeta} \log^2 t \log^2 \frac{4}{\delta_1} \\
+ \eta b^{-1} (t - \min(\theta, 1-\theta) \vee m^{\epsilon-1} \eta t^{1-2\theta}) \log T.
\]

Proof. The proof is similar to that for Theorem E.1. We include the sketch only. Similar to the proof of Theorem E.1, one can prove that with probability at least \( 1 - \delta_1 - \delta_2 - \delta_3 \), \([40]\), \([53]\), and \([56]\) hold for all \( k \in [T] \). Now, we can apply Propositions C.5 and D.8 to get \([44]\) and \([58]\). Noting that by \([57]\), \( \sqrt{2} \eta \leq 1 \), and by a simple calculation, we derive from \([44]\) that

\[
\|S_{\rho_{t+1}} - S_{\rho_t+t+1}\|_{\rho}^2 \leq \frac{400 \left( \kappa^2 \left( 1 \vee \frac{2m^{1-\epsilon}}{1-\theta} \right) ^{1-\zeta} + \sqrt{M} \right)^2 (\kappa / \sqrt{T})^2 + \sqrt{2} \sqrt{\epsilon c_{\gamma} / \|T\| \gamma}^2}{(1-\theta)^2} \times m^\gamma (1-\epsilon)^{-1} (1 \vee \lambda^2 \eta^2 \kappa^{-4t^2-2\theta} \vee \log^2 t) \log^2 \frac{4}{\delta_1}.
\]

The rest of the proof parallelizes to that for Theorem E.1.

Remark E.3. Letting \( \theta = 0 \) in the above theorem, and ignoring the logarithmic terms, the bound \([64]\) reads as

\[
E_{\delta} [E(\omega_{t+1})] - \inf_{\omega \in H} E(\omega) \lesssim (\eta t)^{-2\zeta} + m \gamma (1-\epsilon)^{-1} (1 \vee m^{\epsilon-1} \eta t)^2 (1 \vee \eta t)^{1-2\zeta} + \eta b^{-1} (1 \vee m^{\epsilon-1} \eta t).
\]

Remark E.4. Better bounds for the case \( \zeta \leq 1/2 \) will be proved in the longer version of this paper.