Curvature Tensor for the Spacetime of General Relativity

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Abstract: In the differential geometry of certain $F$-structures, the role of $W$-curvature tensor is very well known. A detailed study of this tensor has been made on the spacetime of general relativity. The spacetimes satisfying Einstein field equations with vanishing $W$-tensor have been considered and the existence of Killing and conformal Killing vector fields has been established. Perfect fluid spacetimes with vanishing $W$-tensor have also been considered. The divergence of $W$-tensor is studied in detail and it is seen, among other results, that a perfect fluid spacetime with conserved $W$-tensor represents either an Einstein space or a Friedmann-Robertson-Walker cosmological model.

Keywords: $W$-curvature tensor, perfect fluid spacetime, Codazzi tensor, FRW-model.

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1. Introduction

Pokhariyal and Mishra [8] have introduced a new curvature tensor and studied its properties. This (0,4) type tensor, denoted by \( W_2 \), is defined as

\[
W_2(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1}[g(X, Z)Ric(Y, T) - g(Y, Z)Ric(X, T)]
\]

where

\[
R(X, Y, Z, T) = g(R(X, Y, Z), T) \quad \text{and} \quad R(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z
\]

\((D\) being the Riemannian connection) is the Riemann curvature tensor, \( Ric(X, Y) = g(R(X), Y) \) is the (0,2) type Ricci tensor and \( R \) is the scalar curvature.

For the sake of convenience, we shall denote this tensor by \( W \). This curvature tensor, in local coordinates, can be expressed as

\[
W_{abcd} = R_{abcd} + \frac{1}{n-1}[g_{ac}R_{bd} - g_{bc}R_{ad}]
\]

and satisfies the following properties:

\[
W_{abcd} = -W_{bacd}, \quad W_{abcd} \neq -W_{abdc}, \quad W_{abcd} \neq W_{cdab}
\]

\[
W_{abcd} + W_{bca} + W_{cad} = 0
\]

which, in index-free notation, can be expressed as

\[
W(X, Y, Z, T) = -W(Y, X, Z, T)
\]

\[
W(X, Y, Z, T) \neq -W(X, Y, T, Z)
\]

\[
W(X, Y, Z, T) \neq W(Z, T, X, Y)
\]

\[
W(X, Y, Z, T) + W(Y, Z, X, T) + W(Z, X, Y, T) = 0
\]

In the differential geometry of certain \( F \)-structures, this tensor \((W_2\) or \(W\)) has extensively been studied by a number of workers. Thus for example, for a Sasakian manifold this tensor was studied by Pokhriyal [9]; while for a P-Sasakian manifold Matsumoto et al [7] have studied this tensor. On the other hand, in terms of \(W_2\)-tensor, Shaikh et al [10] have introduced the notion of weekly \(W_2\)-symmetric manifolds and studied their properties along with several non-trivial examples. The role of \(W_2\)-tensor in the study of Kenmotsu manifolds has been investigated by Yildiz and De [14] while N(k)-quasi Einstein manifolds satisfying the conditions \(R(\xi, X).W_2 = 0\).
have been considered by Taleshian and Hosseinzadeh [12]. Most recently, Venkatesha et al [13] have studied Lorentzian para-Sasakian manifolds satisfying certain conditions on $W_2$-curvature tensor.

Motivated by the all important role of $W_2$-curvature tensor in the study of certain differential geometric structures, in this paper we have made a detailed study of this tensor on the spacetime of general relativity. In section 2, algebraic properties of $W$-curvature tensor are given, while the spacetimes with vanishing $W$-tensor have been considered in section 3. The existence of Killing and conformal Killing vector fields has been established for such spacetimes. Perfect fluid spacetimes satisfying Einstein field equations for $W$-flat spaces have also been studied. Section 4 deals with a detailed study of divergence of $W$-curvature tensor and perfect fluid spacetimes. A number of results concerning the vanishing of the divergence of $W$-tensor have been proved and it is seen that a perfect fluid spacetime with vanishing divergence of $W$-tensor is either an Einstein space or a Friedmann-Robertson-Walker cosmological model.

2. $W$-Curvature Tensor

It is known that the Bianchi differential identity is given by

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$$

where a semi-colon denotes the covariant differentiation. This equation, in index-free notation can be expressed as

$$(2.1a) \quad (\nabla_U R)(X, Y, Z, T) + (\nabla_Z R)(X, Y, T, U) + (\nabla_T R)(X, Y, U, Z) = 0$$

Let $V_4$ be the 4-dimensional spacetime of general relativity, then equation (1.1) takes the form

$$(2.2) \quad W(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{3}[g(X, Z)Ric(Y, T) - g(Y, Z)Ric(X, T)]$$

From equations (2.1a) and (2.2a) it is easy to see that $W$-curvature tensor satisfies the equation

$$\begin{align*}
(\nabla_X W)(Y, Z, T, U) &+ (\nabla_Y W)(Z, X, T, U) + (\nabla_Z W)(X, Y, T, U) \\
&= \frac{1}{3}[g(Y, T)(\nabla_X Ric)(Z, U) - g(Z, T)(\nabla_X Ric)(Y, U) \\
&+ g(Z, T)(\nabla_Y Ric)(X, U) - g(X, T)(\nabla_Y Ric)(Z, U) \\
&+ g(X, T)(\nabla_Z Ric)(Y, U) - g(Y, T)(\nabla_Z Ric)(X, U)]
\end{align*}$$

(2.3)

which, in local coordinates, can be expressed as

$$\begin{align*}
\nabla_a W_{bcde} + \nabla_b W_{c ade} + \nabla_c W_{abde} \\
&= \frac{1}{3}[g_{bd}(\nabla_a R_{ce} - \nabla_c R_{ae}) + g_{cd}(\nabla_b R_{ae} - \nabla_a R_{be}) + g_{ad}(\nabla_c R_{be} - \nabla_b R_{ce})]
\end{align*}$$

(2.4)
If the Ricci tensor \( R_{ab} \) is of Codazzi type [4], then
\[
(\nabla_X \text{Ric})(Y, Z) = (\nabla_Y \text{Ric})(X, Z) = (\nabla_Z \text{Ric})(X, Y)
\]
or, in local coordinates
\[
(2.5a) \quad \nabla_a R_{ce} = \nabla_c R_{ae} = \nabla_e R_{ac}
\]
(The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been given in [4]. The simplest Codazzi tensors are parallel one.)

From equation (2.5a), equation (2.4) leads to
\[
(2.6) \quad \nabla_a W_{bcde} + \nabla_b W_{cade} + \nabla_c W_{abde} = 0
\]
which we call as Bianchi-like identity for \( W \)-curvature tensor.

Conversely, if \( W \)-curvature tensor satisfies the Bianchi-like identity (2.6), then equation (2.4) reduces to
\[
g_{bd}(\nabla_a R_{ce} - \nabla_c R_{ae}) + g_{cd}(\nabla_b R_{ae} - \nabla_a R_{be}) + g_{ad}(\nabla_c R_{be} - \nabla_b R_{ce}) = 0
\]
Contraction of the equation with \( g^{be} \) yields
\[
\nabla_a R_{dc} = \nabla_c R_{da}
\]
which shows that the Ricci tensor is Codazzi. Thus we have

**Theorem 2.1**: For a \( V_4 \), the Ricci tensor is of Codazzi type if and only if \( W \)-curvature tensor satisfies the Bianchi-like identity (2.5).

### 3. Spacetimes with vanishing \( W \)-curvature Tensor

Let \( V_4 \) be the spacetime of general relativity, then from equation (1.1a) we have
\[
(3.1) \quad W_{abcd} = R_{abcd} + \frac{1}{3}[g_{ac}R_{bd} - g_{bc}R_{ad}]
\]
\[
(3.2) \quad W_{bcd}^h = R_{bcd}^h + \frac{1}{3}[\delta_c^h R_{bd} - g_{bc}R_{d}^h]
\]
Contraction of equation (3.2) over \( h \) and \( d \) leads to
\[
(3.3) \quad W_{bc} = \frac{4}{3}(R_{bc} - \frac{1}{4}R g_{bc})
\]
Definition 3.1: A spacetime is said to be $W$-flat if its $W$-curvature tensor vanishes.

For a $W$-flat spacetime, equation (3.2) leads to

\begin{equation}
R_{bcd}^h = -\frac{1}{3}(\delta_c^h R_{bd} - g_{bc} R_d^h)
\end{equation}

which on contraction over $h$ and $d$ yields

\begin{equation}
R_{bc} = \frac{R}{4} g_{bc}
\end{equation}

Equation (3.6) shows that a $W$-flat spacetime is an Einstein space. Thus, we have

Theorem 3.1: A $W$-flat spacetime is an Einstein space and consequently the scalar curvature $R$ is a covariantly constant, i.e. $\nabla_l R = 0$.

Remark 3.1: The spaces of constant curvature play a significant role in cosmology. The simplest cosmological model is obtained by assuming that the universe is isotropic and homogenous. This is known as cosmological principle. This principle, when translated into the language of Riemannian geometry, asserts that the three dimensional position space is a space of maximal symmetry \[2, 11\], that is space of constant curvature whose curvature depends upon time. The cosmological solutions of Einstein equations which contain a three dimensional space-like surface of a constant curvature are the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe (c.f., \[11\]).

In general theory of relativity, the curvature tensor describing the gravitational field consists of two parts viz., the matter part and the free gravitational part. The interaction between these parts is described through Bianchi identities. For a given distribution of matter, the construction of gravitational potential satisfying Einstein field equations is the principal aim of all investigations in gravitational physics; and this has often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of spacetime are expressed through the equation

\begin{equation}
\mathcal{L}_\xi A - 2\Omega A = 0
\end{equation}

where $A$ represents a geometrical/physical quantity, $\mathcal{L}_\xi$ denotes the Lie derivative with respect to a vector field $\xi$ and $\Omega$ is a scalar.
Let the Einstein field equations with a cosmological term be

\[
R_{bc} - \frac{1}{2} R g_{bc} + \Lambda g_{bc} = k T_{bc}
\]

where \( R_{bc} \) is the Ricci tensor, \( R \), the scalar curvature, \( \Lambda \), the cosmological constant, \( k \), the non-zero gravitational constant and \( T_{bc} \), the energy-momentum tensor. From equation (3.6), equation (3.8) leads to

\[
(\Lambda - \frac{R}{4}) g_{bc} = k T_{bc}
\]

Since for \( W \)-flat spacetime \( R \) is constant, taking the Lie derivative of both sides of equation (3.9) we get

\[
(\Lambda - \frac{R}{4}) \mathcal{L}_\xi g_{bc} = k \mathcal{L}_\xi T_{bc}
\]

We thus have the following

**Theorem 3.2 :** For a \( W \)-flat spacetime satisfying Einstein field equations with a cosmological term, there exists a Killing vector field \( \xi \) if and only if the Lie derivative of the energy momentum tensor vanishes with respect to \( \xi \).

**Definition 3.2 :** A vector field \( \xi \) satisfying the equation

\[
\mathcal{L}_\xi g_{bc} = 2\Omega g_{bc}
\]

is called a conformal Killing vector field, where \( \Omega \) is a scalar. A spacetime satisfying equation (3.11) is said to admit a conformal motion.

From equation (3.10) and (3.11) we have

\[
2\Omega (\Lambda - \frac{R}{4}) g_{bc} = k \mathcal{L}_\xi T_{bc}
\]

which on using (3.9) leads to

\[
\mathcal{L}_\xi T_{bc} = 2\Omega T_{bc}
\]

The energy-momentum tensor \( T_{bc} \) satisfying equation (3.13) is said to posses the symmetry inheritance property (cf; [1]). Thus, we can state the following

**Theorem 3.3 :** A \( W \)-flat spacetime satisfying the Einstein field equations with a cosmological term admits a conformal Killing vector field if and only if the energy-momentum tensor has the symmetry inheritance property.
Consider now a perfect fluid spacetime with vanishing $W$-curvature tensor. The energy-momentum tensor $T_{bc}$ for a perfect fluid is given by

\begin{equation}
T_{bc} = (\mu + p)u_b u_c + p g_{bc}
\end{equation}

where $\mu$ is the energy density, $p$ the isotropic pressure, $u_a$ the velocity of fluid such that $u_a u^a = -1$ and $g_{bc} u^b = u_c$.

From equations (3.9) and (3.14) we have

\begin{equation}
(A - \frac{R}{4} - kp)g_{bc} = k(\mu + p)u_b u_c
\end{equation}

which on multiplication with $g^{bc}$ yields

\begin{equation}
R = k(\mu - 3p) + 4\Lambda
\end{equation}

Also, the contraction of equation (3.15) with $u^b u^c$ leads to

\begin{equation}
R = 4(k\mu + \Lambda)
\end{equation}

A comparison of equations (3.16) and (3.17) now yields

\begin{equation}
\mu + p = 0
\end{equation}

which means that either $\mu = 0$, $p = 0$ (empty spacetime) or the perfect fluid spacetime satisfies the vacuum-like equation of state [6]. We thus have the following

**Theorem 3.4**: For a $W$-flat perfect fluid spacetime satisfying Einstein field equations with a cosmological term, the matter contents of the spacetime obey the vacuum-like equation of state.

The Einstein field equations in the presence of matter are given by

\begin{equation}
R_{ab} - \frac{1}{2} R g_{ab} = k T_{ab}
\end{equation}

which on multiplication with $g^{ab}$ leads to

\begin{equation}
R = -k T
\end{equation}

Equation (3.19) may be expressed as

\begin{equation}
R_{ab} = k(T_{ab} - \frac{1}{2} T g_{ab})
\end{equation}

so that in the absence of matter, the field equations are

\begin{equation}
R_{ab} = 0
\end{equation}
Equations (3.22) are the field equations for empty spacetime.

It is known that [2, 11] the energy-momentum tensor for the electromagnetic field is given by

\begin{equation} \label{eq:3.23}
T_{ab} = -F_{ac}F^c_b + \frac{1}{4}g_{ab}F_{pq}F^{pq}
\end{equation}

Where \( F_{ac} \) represents the skew-symmetric electromagnetic field tensor satisfying Maxwell’s equations. From equation (3.23) it is evident that \( T^a_a = T = 0 \) and the Einstein equations for a purely electromagnetic distribution take the form

\begin{equation} \label{eq:3.24}
R_{ab} = kT_{ab}
\end{equation}

Moreover, from equation (3.19) we have

\begin{equation} \label{eq:3.25}
\nabla_c R_{ab} - \nabla_b R_{ac} = k(\nabla_c T_{ab} - \nabla_b T_{ac}) + \frac{1}{2}(g_{ab}\nabla_c R - g_{ac}\nabla_b R)
\end{equation}

If \( T_{ab} \) is of Codazzi type, then equation (3.25) becomes

\begin{equation} \label{eq:3.26}
\nabla_c R_{ab} - \nabla_b R_{ac} = \frac{1}{2}(g_{ab}\nabla_c R - g_{ac}\nabla_b R)
\end{equation}

which on multiplication with \( g^{ab} \), after simplification, leads to

\begin{equation} \label{eq:3.27}
\nabla_b R^{ab} = 0
\end{equation}

Thus we have

**Theorem 3.5**: For a \( V_4 \) satisfying Einstein field equations, the Ricci tensor is conserved if the energy-momentum tensor is Codazzi.

4. Divergence of \( W \)-curvature tensor and perfect fluid spacetimes

The Bianchi identities are given by

\begin{equation} \label{eq:4.1}
\nabla_e R^h_{bcd} + \nabla_c R^h_{bde} + \nabla_d R^h_{bec} = 0
\end{equation}

Contracting equation (4.1) over \( h \) and \( e \), using the symmetry properties of Riemann curvature tensor, we get

\begin{equation} \label{eq:4.2}
\nabla_h R^h_{bcd} = \nabla_d R_{bce} - \nabla_c R_{bd}
\end{equation}

It is known that Riemannian manifolds for which the divergence of the curvature tensor vanish identically are known as manifolds with harmonic curvature. The curvature of such manifolds occur as a special case of Yang-Mills fields. These
manifolds also form a natural generalization of Einstein spaces and of conformally flat manifolds with constant scalar curvature [3]. From equation (4.2) we thus have

**Theorem 4.1**: If the Ricci tensor is of Codazzi type then the manifold $V_4$ is of harmonic curvature and conversely.

**Remark 4.1**: The Ricci tensor $\text{Ric}(X, Y)$ is said to be parallel if

$$\nabla Z \text{Ric}(X, Y) - \nabla Y \text{Ric}(X, Z) = 0$$

which means that the simplest Codazzi tensors are parallel ones.

Now from equation (3.2) we have

$$(4.3) \quad \nabla_e W^h_{bcd} = \nabla_e R^h_{bcd} + \frac{1}{3}(\delta^h_c \nabla_e R_{bd} - g_{bc} \nabla_e R^h_d)$$

so that the divergence of $W$-curvature tensor is given by

$$(4.4) \quad \nabla_h W^h_{bcd} = \nabla_h R^h_{bcd} + \frac{1}{3}(\nabla_c R_{bd} - g_{bc} \nabla_h R^h_d)$$

which leads to the following

**Theorem 4.2**: For a spacetime possessing harmonic curvature with divergence-free $W$-tensor, the Ricci tensor is covariantly constant.

While from equations (4.2) and (4.4) we have

$$(4.5) \quad \nabla_h W^h_{bcd} = \nabla_d R_{bc} - \nabla_e R_{bd} + \frac{1}{3}(\nabla_c R_{bd} - g_{bc} \nabla_h R^h_d)$$

Thus we can state the following

**Theorem 4.3**: If for a $V_4$ the divergence of $W$-tensor vanishes and the Ricci tensor is covariantly constant then the spacetime is of constant curvature.

Now using equation (3.21) in equation (4.5) we get

$$(4.6) \quad \nabla_h W^h_{bcd} = k(\nabla_d T_{bc} - \frac{2}{3} \nabla_e T_{bd}) + \frac{k}{3}(g_{bd} \nabla_c T - \frac{5}{2} g_{bc} \nabla_d T)$$

so that for a purely electromagnetic distribution, we have

$$(4.7) \quad \nabla_h W^h_{bcd} = k(\nabla_d T_{bc} - \frac{2}{3} \nabla_e T_{bd})$$
which leads to

**Theorem 4.4 :** For a spacetime satisfying the Einstein equations for a purely electromagnetic distribution, the $W$-curvature tensor is conserved if the energy-momentum tensor is covariantly constant and conversely.

Consider now the energy-momentum tensor for a perfect fluid \([\text{cf, equation (3.14)}]\)

\[
T_{ab} = (\mu + p)u_a u_b + pg_{ab}
\]

which leads to

\[
T = -\mu + 3p
\]

If $T_{ab}$ is Codazzi and $\nabla_h W_{bcd} = 0$ then from equations (4.8) and (4.9), equation (4.6) leads to $(k = 1)$

\[
\frac{1}{3}[\nabla_c(\mu + p)u_b u_d + (\mu + p)\nabla_c u_b u_d + (\mu + p)u_b \nabla_c u_d + \nabla_c pg_{bd}] + \frac{1}{3}g_{bd} \nabla_c (-\mu + 3p) - \frac{5}{6}g_{bc} \nabla_d (-\mu + 3p) = 0
\]

Since $\nabla_b u_a u^a = 0$, contracting equation (4.10) with $g^{bd}$ we get

\[
\nabla_c(\mu - 3p) = 0
\]

that is

\[
(\mu - 3p) = \text{constant}
\]

Thus, we have

**Theorem 4.5 :** If for a perfect fluid spacetime, the divergence of $W$-curvature tensor vanishes and the energy-momentum tensor is of Codazzi type $(\mu - 3p)$ is constant.

It is known that [5] for a radiative perfect fluid spacetime $(\mu = 3p)$ the resulting universe is isotropic and homogenous. Thus by choosing the constant in equation (4.12) as zero, we have the following

**Corollary 4.1 :** If the energy-momentum tensor for a divergence-free $W$-fluid spacetime is of Codazzi type then the resulting spacetime is radiative and consequently isotropic and homogenous.

Now consider the spacetime for which the divergence of $W$-curvature tensor vanishes, then from equation (4.6), we have $(k = 1)$

\[
\nabla_d T_{bc} - \frac{5}{6}g_{bc} \nabla_d T = \frac{2}{3}\nabla_c T_{bd} - \frac{1}{3}g_{bd} \nabla_c T
\]
which on using equation (4.8) and (4.9) leads to

\[ \nabla_d(\mu + p)u_bu_c + (\mu + p)\nabla_d u_b u_c + (\mu + p)u_b \nabla_d u_c + \nabla_d p g_{bc}^\mu - \frac{\dot{g}_{bc}}{\dot{\theta}} \nabla_d(-\mu + 3p) = \frac{2}{3}[\nabla_c(\mu + p) u_b u_d + (\mu + p)u_b \nabla_c u_d + \nabla_c p g_{bd}] - \frac{1}{3}g_{bd} \nabla_c(-\mu + 3p) \]

Contracting this equation with \( u^d \), we get

\[ (\mu + p)u_b u_c + (\mu + p)\dot{u}_b u_c + (\mu + p)u_b \dot{u}_c + \dot{p}g_{bc} - \frac{\dot{g}_{bc}}{\dot{\theta}}(-\mu + 3p) = \frac{2}{3} \nabla_c(\mu + p) u_b + \frac{2}{3}(\mu + p) \nabla_c u_b \]

\[ -\frac{2}{3} \nabla_c p u_b + \frac{1}{3} \nabla_c(-\mu + 3p) u_b = 0 \]

where an over head dot denotes the covariant derivative along the fluid flow vector \( u_a \)
(that is, \( (\mu + p) = \nabla_c(\mu + p) u^c \), \( \dot{u}_b = \nabla_c u_b u^c \), \( \dot{p} = \nabla_d p u^d \), \( \nabla_b u_a u^a = 0 \), etc.).

Also, the conservation of energy-momentum tensor \( (\nabla_b T^{ab} = 0) \) leads to

\[ (\mu + p)\dot{u}_a = -\nabla_a p + \dot{p}u_a \]  

(force equation)

\[ \dot{\mu} = -(\mu + p)\nabla_a u^a = -(\mu + p)\theta \]  

(energy equation)

Moreover, the covariant derivative of the velocity vector can be splitted into kinematical quantities as

\[ \nabla_b u_a = \frac{1}{3} \theta (g_{ab} + u_a u_b) - \dot{u}_a u_b + \sigma_{ab} + \omega_{ab} \]

where \( \theta = \nabla a u^a \), is the expansion scalar, \( \dot{u}_a = \nabla_b u_a u^b \), the acceleration vector \( \sigma_{ab} = h_{a}^{c}h_{b}^{d}u_{(c;d)} - \frac{1}{3} \theta h_{ab} \), the symmetric shear tensor \( (h_{ab} = g_{ab} - u_a u_b) \) and \( \omega_{ab} = h_{a}^{c}h_{b}^{d}u_{(c;d)} \) is the skew symmetric vorticity or rotation tensor.

Using force equation (4.15) in equation (4.14), we get

\[ (\mu - p)u_b u_c - \nabla_b p u_c + \dot{p}g_{bc} + \frac{2}{3}(\mu + p) \nabla_c u_b - \frac{\dot{g}_{bc}}{\dot{\theta}}(-\mu + 3p) g_{bc} + \frac{1}{3} \nabla_c \mu u_b = 0 \]

Contracting this equation with \( u^b \), we get

\[ \frac{1}{6}(\mu - 3p)u_c - \frac{1}{3} \nabla_c \mu = 0 \]

Thus we have the following

**Theorem 4.6:** For a perfect fluid spacetime with divergence-free \( W \)-curvature tensor, the pressure and density of the fluid are constant.
Now contracting the equation (4.18) with $u^c$, we get

$$(4.20) \quad \frac{3}{2}(\mu - 3p)u_b + \frac{2}{3}(\mu + p)\dot{u}_b + \nabla_b p = 0$$

which on using force equation (4.15) leads to

$$(4.22) \quad \frac{3}{2}\ddot{u}_b - \frac{31}{6}\nabla d p u^d u_b + \frac{1}{3}\nabla_b p = 0$$

From energy equation (4.16), this equation yields

$$(4.21) \quad \frac{3}{2}[(\mu + p)\theta]u_b + \frac{31}{6}\nabla d p u^d u_b - \frac{2}{3}\nabla_b p = 0$$

We thus have the following

**Theorem 4.7**: For a perfect fluid spacetime having conserved $W$-curvature tensor, either pressure and density of the fluid are constant over the space-like hypersurface orthogonal to the fluid four velocity or the fluid is expansion-free.

Consider now a perfect fluid spacetime for which $\nabla_b W^b_{\text{bcd}} = 0$, then from equations (4.14) and (4.15), we have

$$(4.22) \quad (\mu + 3p)\dot{u}_b + \dot{p} g_{bc} - \nabla_b p u_c - \frac{5}{6}(-\mu + 3p)\dot{g}_{bc} + \frac{1}{3}\nabla_c(2\mu - 3p)u_b + \frac{2}{3}(\mu + p)\nabla_c u_b + \frac{1}{3}\nabla_c (-\mu + 3p)u_b = 0$$

which on contraction with $u^b$ leads to

$$(4.23) \quad (\mu + 3p)\dot{u}_c + \frac{5}{6}(-\mu + 3p)u_c + \frac{1}{3}\nabla_c(2\mu - 3p) + \frac{1}{3}\nabla_c (-\mu + 3p) = 0$$

This equation is satisfied only when

$$(4.24) \quad \nabla_c(2\mu - 3p) = -3(\mu + 3p)u_c$$

and

$$(4.25) \quad \nabla_c (-\mu + 3p) = -\frac{5}{2}(-\mu + 3p)u_c$$

Using force equation (4.15) in equation (4.24), we get

$$(4.26) \quad (\mu + p)\dot{u}_c = -\frac{1}{3}\ddot{u}_c - \frac{2}{9}\nabla_c (\mu + 3p)$$

Also, from force equation (4.15) and equation (4.25), we have

$$(4.27) \quad \nabla_c \mu = -3(\mu + p)\dot{u}_c - \frac{1}{2}(5\mu - 21p)\dot{u}_c$$
From equations (4.22) and (4.24) we have

\[ (4.28) \quad \dot{g}_{bc} - \nabla_b p u_c - \frac{5}{6}(-\mu + 3p) g_{bc} + \frac{1}{3} (\mu + p) \nabla_c u_b + \frac{1}{3} \nabla_c (-\mu + 3p) u_b = 0 \]

which on multiplication with $g^{bc}$ leads to

\[ (4.29) \quad \dot{p} = \frac{10}{9} (-\mu + 3p) - \frac{1}{9} (\mu + p) \theta - \frac{1}{9} \nabla_c (-\mu + 3p) u^c \]

Using energy equation (4.16) this equation yields

\[ (4.30) \quad \dot{p} = \frac{3}{7} \mu + \frac{1}{21} \nabla_c (-\mu + 3p) u^c \]

Also, from the equations (4.15) and (4.28), we have

\[ (4.31) \quad (\mu + p) \dot{u}_b u_c + \dot{p} (g_{bc} + u_b u_c) + \frac{1}{3} (\mu + p) \nabla_c u_b - \frac{5}{6} (-\mu + 3p) g_{bc} + \frac{1}{3} \nabla_c (-\mu + 3p) u_b = 0 \]

which on using equations (4.25) and (4.29) yields

\[ (4.32) \quad (\mu + p) \left[ 3 \ddot{u}_b u_c - \frac{1}{3} \theta (g_{bc} + u_b u_c) + \nabla_c u_b \right] = 0 \]

This equation suggests that either

\[ (4.33) \quad \mu + p = 0 \]

or

\[ (4.34) \quad \nabla_c u_b + 3 \ddot{u}_b u_c - \frac{1}{3} \theta (g_{bc} + u_b u_c) = 0 \]

From equation (4.33) either $\mu = 0$, $p = 0$ (neither matter nor radiation) or the perfect fluid spacetime with $\nabla_h W^h_{bcd} = 0$ satisfies the vacuum-like equation of state [6].

Moreover, from equations (4.16), (4.25) and (4.29) we have

\[ (4.35) \quad -\mu + 3p = \text{constant} \]

while from equations (4.17) and (4.34) we have

\[ (4.36) \quad 2 \dot{u}_b u_c + \sigma_{bc} + \omega_{bc} = 0 \]

which is satisfied only when $\dot{u}_b = 0$, $\sigma_{bc} = 0$, $\omega_{bc} = 0$
Therefore, if \( \mu + p \neq 0 \) then the above discussions show that the fluid is shear-free, rotation-free, acceleration-free and the energy density and pressure are constants over the space-like hypersurface orthogonal to the fluid four velocity.

It may be noted that vanishing of acceleration, shear and rotation and the constantness of energy density and pressure over the space-like hypersurface orthogonal to the fluid flow vector are the conditions for a spacetime to represent Friedmann-Robertson-Walker cosmological model provided that \( \mu + p \neq 0 \).

Hence, summing up these discussions, we can state the following

**Theorem 4.8:** A perfect fluid spacetime with conserved \( W \)-curvature tensor is either an Einstein spacetime \( (\mu + p) = 0 \) or a Friedmann-Robertson-Walker cosmological model satisfying \( \mu - 3p = \text{constant} \).

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