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Almost Automorphic Solutions in the Sense of Besicovitch to Nonautonomous Semilinear Evolution Equations

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Abstract: In this paper, we study the existence of almost automorphic solutions in the sense of Besicovitch for a class of semilinear evolution equations. Firstly, we study some basic properties of almost automorphic functions in the sense of Besicovitch, including the composition theorem. Then, by using the Banach fixed point theorem, the existence of almost automorphic solutions in the sense of Besicovitch to the semilinear equation is obtained. Finally, we give an example of partial differential equations to illustrate the applicability of our results.

Keywords: nonautonomous semilinear evolution equation; almost automorphic solution in the sense of Besicovitch; fixed point theorem

MSC: 34C27, 43A60, 34G20

1. Introduction

Consider the nonautonomous semilinear evolution equation:

\[ x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{R}, \]  

(1)

where for each \( t \in \mathbb{R} \), \( A(t) : D(A(t)) \subset X \to X \) is a closed and densely defined linear operator on \( D = D(A(t)) \) satisfying the so-called Acquistapace and Terreni conditions:

\((S_1)\) There are constants \( \lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi) \) and \( K_1 \geq 0 \) such that \( \Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0) \) and for all \( x \in \Sigma_\theta \cup \{0\}, t \in \mathbb{R}, \)

\[ \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K_1}{1 + |\lambda|}. \]

\((S_2)\) There are constants \( K_2 \geq 0 \) and \( a, \beta \in (0, 1) \) with \( a + \beta > 1 \) such that for all \( \lambda \in \Sigma_\theta \) and \( t, s \in \mathbb{R}, \)

\[ \|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \frac{K_2|t - s|^\beta}{|\lambda|^\beta}, \]

where \( R(\lambda, L) = (\lambda I - L)^{-1} \) for all \( \lambda \in \rho(L), \Sigma_\theta = \{ \lambda \in \mathbb{C}/\{0\} : \arg \lambda \leq \theta \}. \)

According to [1], under assumptions \((S_1)\) and \((S_2)\), there is a unique evolution family \( \{U(t, s)\}_{t \geq s} \) generated by \( A(t) \) such that \( U(t, s)x \subset D(A(t)) \) for all \( t \geq s \), in the sense that:

\((i)\) \( U(t, \theta)U(\theta, s) = U(t, s) \) and \( U(s, s) = I \) for all \( t \geq \theta \geq s. \)
\((ii)\) The mapping \( (t, s) \mapsto U(t, s)x \) is continuous for all \( x \in X \) and \( t \geq s. \)
\((iii)\) \( U(\cdot, s) \in C^1((s, \infty), L(X)), \frac{\partial u}{\partial t}(t, s) = A(t)U(t, s) \) and

\[ \|A(t)U(t, s)\| \leq \lambda(t - s)^{-1}, \quad \text{for} \quad 0 < t - s < 1 \quad \text{and} \quad t = 0, 1. \]
We make the following assumptions:

(H1) Function $f \in AAB^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfies the Lipschitz condition with respect to its second argument and uniformly in its first argument, that is, there exists a positive constant $L$ such that for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}$,

$$\|f(t, x) - f(t, y)\|_{\mathbb{X}} \leq L\|x - y\|_{\mathbb{X}}$$

and $f(t, 0) = 0$.

(H2) The evolutionary family $\{U(t, s), t \geq s\}$ is exponentially stable, that is, there exist numbers $M, \lambda > 0$ such that $\|U(t, s)\| \leq Me^{-\lambda(t-s)}, t \geq s$;

(H3) $U(t, s)x \in BAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ uniformly for $x$ in any bounded subset of $\mathbb{X}$;

(H4) The constant $\kappa := \frac{M\lambda}{\lambda} < 1$, where $M$ is mentioned in (H3).

**Definition 1.** A mild solution of Equation (1) is a continuous function $x : \mathbb{R} \to \mathbb{X}$ satisfying:

$$x(t) = U(t, s)x(s) + \int_{s}^{t} U(t, \theta)f(\tau, x(\tau))d\theta, \quad t \geq s, \quad s \in \mathbb{R}.$$ 

If $U(t, s)$ is exponentially stable, then we have:

$$x(t) = \int_{-\infty}^{t} U(t, \theta)f(\theta, x(\theta))d\theta, \quad t \geq s, \quad s \in \mathbb{R}.$$ 

The concept of almost automorphic functions as a generalization of the concept of almost periodic functions was introduced into mathematics by Bochner [2], and once this concept was put forward, its various generalizations were constantly put forward, such as asymptotic almost automorphic functions, pseudo almost automorphic functions, Stepanov almost automorphic functions and so on [3–6]. At the same time, the study of almost periodic solutions and almost automorphic solutions of differential equations has become an important part of the qualitative theory of differential equations. Almost automorphy in the sense of Besicovitch is a generalization of those concepts mentioned above, but so far, the results of almost automorphic solutions in the sense of Besicovitch of differential equations are still very rare [7]. It is worth mentioning that Reference [7] studies the existence of 1-almost automorphic solutions in the sense of Besicovitch for a class of first-order nonautonomous linear differential equations. However, Reference [7] does not involve the concept of uniform almost automorphic functions in the sense of Besicovitch, nor does it give the composite theorem of almost automorphic functions in the sense of Besicovitch, which are necessary concepts and tools to study the existence of almost automorphic solutions in the sense of Besicovitch for nonlinear differential equations. Therefore, it is very meaningful to study the composition theorem of almost automorphic functions in the sense of Besicovitch and the existence of almost automorphic solutions in the sense of Besicovitch of nonlinear differential equations.

Motivated by the above discussion, in this paper, we first give the concepts of almost automorphic and uniform almost automorphic functions in the sense of Besicovitch defined by Bochner property, and study some of their basic properties, including composition theorems. Then, we study the existence of p-almost automorphic solutions in the sense of Besicovitch of system (1) by using the Banach fixed point theorem. The results of our paper are new.

The rest of this paper is arranged as follows: in Section 2, we study some basic properties of almost automorphic functions in the sense of Besicovitch. In Section 3, we use the results obtained in Section 2 and the Banach fixed point theorem to establish the existence of almost automorphic solutions in the sense of Besicovitch of system (1). In Section 4, we provide an example to illustrate the applicability of our results. Finally, in Section 5, we present a brief conclusion.
2. Besicovitch Almost Automorphic Functions and Their Some Properties

Let \( (X, \| \cdot \|_X) \) be a Banach space, \( C(\mathbb{R}, X) \) be the set of all continuous functions from \( \mathbb{R} \) to \( X \) and \( BC(\mathbb{R}, X) \) be the space of all bounded continuous functions from \( \mathbb{R} \) to \( X \). We denote by \( L^\infty(\mathbb{R}, X) \) the set of all functions \( f : \mathbb{R} \to X \) that are measurable and essentially bounded. The space \( L^\infty(\mathbb{R}, X) \) is a Banach space with the norm
\[
\| f \|_\infty := \inf \{ D \geq 0 : \| f(t) \|_X \leq D \text{ a.e. } t \in \mathbb{R} \}.
\]

Definition 5 \((8)\). Let \( f \in C(\mathbb{R}, X) \), then \( f \) is called (Bohr) almost periodic if for each \( \varepsilon > 0 \), there exists \( l = l(\varepsilon) > 0 \) such that in every interval of length \( l \) of \( \mathbb{R} \) one can find a number \( \tau \in (a, a + l) \) with the property (Bohr property):
\[
\sup_{t \in \mathbb{R}} \| f(t + \tau) - f(t) \|_X < \varepsilon.
\]
The collection of such Bohr almost periodic functions will be denoted by \( AP(\mathbb{R}, X) \).

Definition 3 \((5)\). A function \( f \in C(\mathbb{R}, X) \) is almost automorphic in Bochner’s sense if for every sequence of real numbers \( (S_n)_{n \in \mathbb{N}} \), there exists a subsequence \( (S_{n_k})_{k \in \mathbb{N}} \) such that:
\[
g(t) := \lim_{n \to \infty} f(t + S_n)
\]
is well defined for each \( t \in \mathbb{R} \), and
\[
f(t) = \lim_{n \to \infty} g(t - S_n)
\]
for each \( t \in \mathbb{R} \). The set of all such functions will be denoted by \( AA(\mathbb{R}, X) \).

If (2) and (3) are uniformly on compact subsets of \( \mathbb{R} \), then \( f \) is called compact almost automorphic in Bochner’s sense.

Lemma 1 \((5)\). The space \( AA(\mathbb{R}, X) \) is a Banach space when it is endowed with the norm
\[
\| f \|_0 = \sup_{t \in \mathbb{R}} \| f(t) \|_X
\]
for \( f \in AA(\mathbb{R}, X) \). And if \( f \in AA(\mathbb{R}, X) \), then the function \( f \) is bounded on \( \mathbb{R} \) with respect to the norm \( \| \cdot \|_0 \).

Definition 4 \((19)\). A continuous function \( f : \mathbb{R} \times \mathbb{R} \to X \) is said to be bi-almost automorphic if for every sequence of real numbers, \( (S_n)_{n \in \mathbb{N}} \), there exists a subsequence \( (S_{n_k})_{k \in \mathbb{N}} \) such that:
\[
g(t, s) := \lim_{n \to \infty} f(t + S_n, s + S_n)
\]
is well defined in \( t, s \in \mathbb{R} \), and
\[
\lim_{n \to \infty} g(t - S_n, s - S_n) = f(t, s)
\]
for each \( t, s \in \mathbb{R} \). The collection of all such functions will be denoted by \( BAA(\mathbb{R} \times \mathbb{R}, X) \).

For \( p \in [1, \infty) \), let \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) be the collection of all locally \( p \)-integrable functions from \( \mathbb{R} \) to \( X \). For \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \), we consider the following seminorm:
\[
||f||_{B^p} = \left\{ \limsup_{l \to \infty} (2l)^{-1} \int_{-l}^{l} \| f(t) \|^p dt \right\}^{\frac{1}{p}}.
\]

Definition 5. A function \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) is called \( B^p \)-bounded if \( ||f||_{B^p} < \infty \). We denote by \( B\!B^p(\mathbb{R}, X) \) the set of all such functions.
Similar to the definitions of the corresponding concepts in [5,7,8,10–12], we give the following definition:

**Definition 6.** A function \( f \in \mathcal{B}^p(\mathbb{R}, X) \) is said to be \( p \)-almost automorphic in the sense of Besicovitch, if for every sequence of real numbers \( (S_n')_{n \in \mathbb{N}} \), there exists a subsequence \( (S_n)_{n \in \mathbb{N}} \) such that

\[
\hat{f}(t) := \lim_{n \to \infty} f(t + S_n)
\]

is well defined for each \( t \in \mathbb{R} \), and

\[
\lim_{n \to \infty} \hat{f}(t - S_n) = f(t)
\]

for each \( t \in \mathbb{R} \). That is,

\[
\lim_{n \to \infty} \|f(t + S_n) - \hat{f}(t)\|_{\mathcal{B}^p} = 0
\]

and

\[
\lim_{n \to \infty} \|\hat{f}(t - S_n) - f(t)\|_{\mathcal{B}^p} = 0.
\]

We denote by \( AAB^p(\mathbb{R}, X) \) the collection of all such functions.

From Definitions 2, 3 and 6, it is easy to see that:

\[
AP(\mathbb{R}, X) \subset AA(\mathbb{R}, X) \subset AAB^p(\mathbb{R}, X).
\]

**Remark 1.** Obviously, the convergence in Definition 6 is uniformly in \( t \in \mathbb{R} \), so the almost automorphy defined in Definition 6 is corresponding to the compact almost automorphy of Definition 3.

**Remark 2.** Because the functions that are asymptotic to zero and the functions whose integral averages are zero belong to the zero space of semi norm \( \| \cdot \|_{\mathcal{B}^p} \). Therefore, for the almost automorphic functions in the sense of Besicovitch, there are no concepts of asymptotic almost automorphic functions and pseudo almost automorphic functions.

**Example 1.** According to Example 4.4 in [8], we see that:

\[
\cos \left( \frac{1}{2 + \sin t + \sin \sqrt{5}t} \right) \in AA(\mathbb{R}, \mathbb{R}).
\]

Since \( \|f\|_{\mathcal{B}^p} = 0 \) and \( \|g\|_{\mathcal{B}^p} = 0 \), where \( f(t) = e^{-|t|} \) and \( g(t) = \frac{1}{1 + t^2} \), we have:

\[
\cos \left( \frac{1}{2 + \sin t + \sin \sqrt{5}t} \right) + e^{-|t|} + \frac{1}{1 + t^2} \in AAB^p(\mathbb{R}, \mathbb{R}).
\]

**Lemma 2.** If a function \( f \in \mathcal{B}^p(\mathbb{R}, X) \) is almost automorphic in the sense of Besicovitch, then \( \hat{f} \) is \( \mathcal{B}^p \)-bounded, where \( \hat{f} \) is mentioned in Definition 6.

**Proof.** Since \( f \in \mathcal{B}^p(\mathbb{R}, X) \), for every sequence of real numbers \( (S_n')_{n \in \mathbb{N}} \), we can extract a subsequence \( (S_n)_{n \in \mathbb{N}} \) such that for each \( t \in \mathbb{R} \) and \( \varepsilon = 1 \), we have

\[
\|f(t + S_n) - \hat{f}(t)\|_{\mathcal{B}^p} < 1.
\]

Then,

\[
\|\hat{f}(t)\|_{\mathcal{B}^p} \leq \|f(t + S_n) - \hat{f}(t)\|_{\mathcal{B}^p} + \|f(t + S_n)\|_{\mathcal{B}^p} < \|f\|_{\mathcal{B}^p} + 1 < \infty.
\]

The proof is complete. \( \square \)

**Lemma 3.** If \( f, g \in AAB^p(\mathbb{R}, X) \) and \( \lambda \in \mathbb{R} \), then we have \( f + g, \lambda f \in AAB^p(\mathbb{R}, X) \).
Theorem 1. Since \( f, g \in AAB^p(\mathbb{R}, \mathcal{X}) \), for every sequence of real numbers \((S_n')_{n \in \mathbb{N}}\), we can extract a subsequence \((S_n')_{n \in \mathbb{N}}\) such that for each \( t \in \mathbb{R} \) and any \( \varepsilon > 0 \), there exists \( N(t, \varepsilon) \in \mathbb{N} \), when \( n > N(t, \varepsilon) \),
\[
\|f(t + S_n') - \tilde{f}(t)\|_{B^p} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\tilde{f}(t - S_n') - f(t)\|_{B^p} < \frac{\varepsilon}{2}.
\]

Meanwhile, for \((S_n')_{n \in \mathbb{N}}\), there exist a subsequence \((S_n')_{n \in \mathbb{N}}\) of \((S_n')_{n \in \mathbb{N}}\) and \( N_2(t, \varepsilon) \in \mathbb{N} \), when \( n > N_2(t, \varepsilon) \),
\[
\|g(t + S_n') - \tilde{g}(t)\|_{B^p} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\tilde{g}(t - S_n') - g(t)\|_{B^p} < \frac{\varepsilon}{2}.
\]

Denote \( N_0 = \max(N_1, N_2) \), so when \( n > N_0 \), we have
\[
\|f(t + S_n) - \tilde{f}(t)\|_{B^p} < \frac{\varepsilon}{2} \quad \text{and} \quad \|g(t + S_n) - \tilde{g}(t)\|_{B^p} < \frac{\varepsilon}{2}.
\]

Consequently, we arrive at, for \( n > N_0 \),
\[
\|(f(t + S_n) + g(t + S_n)) - \tilde{f}(t) - \tilde{g}(t)\|_{B^p}
\leq\|f(t + S_n) - \tilde{f}(t)\|_{B^p} + \|g(t + S_n) - \tilde{g}(t)\|_{B^p} < \varepsilon.
\]

Similarly, for \( n > N_0 \), one can get
\[
\|\tilde{f}(t - S_n) + \tilde{g}(t - S_n) - f(t) - g(t)\|_{B^p} < \varepsilon.
\]

So \( f + g \in AAB^p(\mathbb{R}, \mathcal{X}) \) is proved. The proof of \( \lambda f \in AAB^p(\mathbb{R}, \mathcal{X}) \) is trivial and we will omit it here. The proof of Lemma 3 is complete. \( \square \)

Lemma 4. If \( f \in AAB^p(\mathbb{R}, \mathcal{X}) \), then \( f(\cdot + \lambda) \in AAB^p(\mathbb{R}, \mathcal{X}) \) for each \( \lambda \in \mathbb{R} \).

Proof. Since \( f \in AAB^p(\mathbb{R}, \mathcal{X}) \), for every sequence of real numbers \((S_n')_{n \in \mathbb{N}}\), we can extract a subsequence \((S_n)_{n \in \mathbb{N}}\) such that
\[
\lim_{n \to \infty} \|f(s + S_n) - \tilde{f}(s)\|_{B^p} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\tilde{f}(s - S_n) - f(s)\|_{B^p} = 0
\]
for each \( s \in \mathbb{R} \). Letting \( s = t + \lambda \), then
\[
\lim_{n \to \infty} \|f(t + \lambda + S_n) - \tilde{f}(t + \lambda)\|_{B^p} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\tilde{f}(t + \lambda - S_n) - f(t + \lambda)\|_{B^p} = 0
\]
for each \( t \in \mathbb{R} \).

Hence, \( f(\cdot + \lambda) \in AAB^p(\mathbb{R}, \mathcal{X}) \). The proof is completed. \( \square \)

Theorem 1. If \( f \in C(\mathbb{R}, \mathcal{X}) \) satisfies the Lipschitz condition and \( x \in AAB^p(\mathbb{R}, \mathcal{X}) \), then \( f(x(\cdot)) \) belongs to \( AAB^p(\mathbb{R}, \mathcal{X}) \).

Proof. It is easy to obtain that \( f(x(\cdot)) \in BB^p(\mathbb{R}, \mathcal{X}) \). Since \( x \in AAB^p(\mathbb{R}, \mathcal{X}) \), for every sequence of real numbers \((S_n')_{n \in \mathbb{N}}\), there exists a subsequence \((S_n)_{n \in \mathbb{N}}\) such that for any \( \varepsilon > 0 \) and each \( t \in \mathbb{R} \), there exists a positive number \( N(\varepsilon, t) \), when \( n > N(\varepsilon, t) \),
\[
\|x(t + S_n) - \bar{x}(t)\|_{B^p} < \varepsilon \quad \text{and} \quad \|\bar{x}(t - S_n) - x(t)\|_{B^p} < \varepsilon.
\]
Consequently,
\[\|f(x(t+S_n))-f(\bar{x}(t))\|_{BF} = \left(\limsup_{l \to \infty} \frac{1}{2L} \int_{-L}^{L} |f(x(t+S_n))-f(\bar{x}(t))|^p dt\right)^{1/p}\]
\[\leq \left(\limsup_{l \to \infty} \frac{1}{2L} \int_{-L}^{L} L_f^p |x(t+S_n)-\bar{x}(t)|^p dt\right)^{1/p}\]
\[\leq L_f \|x(t+S_n)-\bar{x}(t)\|_{BF} < L_f \epsilon.\]

Similarly, one can prove that
\[\|f(\bar{x}(t-S_n))-f(x(t))\|_{BF} < \epsilon.\]

The proof is completed. □

Lemma 5. Let $\mathcal{X}$ be a Banach algebra. If $f \in AP(\mathbb{R}, \mathcal{X})$ and $g \in AAB^p(\mathbb{R}, \mathcal{X})$, then $f \cdot g \in AAB^p(\mathbb{R}, \mathcal{X})$.

Proof. Noting that:
\[\|f \cdot g\|_{BF} = \left(\limsup_{l \to \infty} \frac{1}{2L} \int_{-L}^{L} |f(t)g(t)|^p dt\right)^{1/p}\]
\[\leq \left(\limsup_{l \to \infty} \frac{1}{2L} \int_{-L}^{L} |f(t)|^p |g(t)|^p dt\right)^{1/p}\]
\[= \|f\|_0 \left(\limsup_{l \to \infty} \frac{1}{2L} \int_{-L}^{L} |g(t)|^p dt\right)^{1/p}\]
\[= \|f\|_0 \|g\|_{BF} < \infty,\]
we have $f \cdot g \in BB^p(\mathbb{R}, \mathcal{X})$.

On the other hand, since $f \in AP(\mathbb{R}, \mathcal{X})$, by Bochner property of Bohr almost periodic functions [8], for any sequence $(S_n)_{n \in \mathbb{N}}$ of real numbers, there exists a subsequence $(S'_n)_{n \in \mathbb{N}}$ of $(S''_n)_{n \in \mathbb{N}}$ such that for every $\epsilon > 0$, there is $N_1 > 0$ such that
\[\|f(t+S'_n)-\bar{f}(t)\|_{\mathcal{X}} < \epsilon \quad \text{and} \quad \|\bar{f}(t)-S'_n\| \leq \epsilon\] (4)

for $n > N_1$.

Moreover, since $g \in AAB^p(\mathbb{R}, \mathcal{X})$, there exists a subsequence $(S'_n)_{n \in \mathbb{N}}$ of $(S''_n)_{n \in \mathbb{N}}$ and $N_2 > 0$ such that
\[\|g(t+S_n)-\bar{g}(t)\|_{BF} < \epsilon \quad \text{and} \quad \|\bar{g}(t)-S_n\| \leq \epsilon.\] (5)

Noting that
\[\|f(t+S_n)g(t+S_n) - f(t)\bar{g}(t)\|_{\mathcal{X}}^p\]
\[= \|f(t+S_n)(g(t+S_n) - \bar{g}(t)) + f(t+S_n)\bar{g}(t) - f(t)\bar{g}(t)\|_{\mathcal{X}}^p\]
\[\leq 2^p \|f(t+S_n)(g(t+S_n) - \bar{g}(t))\|_{\mathcal{X}}^p + 2^p \|f(t+S_n)\bar{g}(t) - f(t)\bar{g}(t)\|_{\mathcal{X}}^p\]
\[\leq 2^p \|f\|_0 \|g(t+S_n) - \bar{g}(t)\|_{\mathcal{X}}^p + 2^p \|\bar{g}(t)\|_{\mathcal{X}}^p \|f(t+S_n) - f(t)\|_{\mathcal{X}}^p.\]
By (4), (5) and Lemma 2, we derive that:

\[
\| f(t + S_n)g(t + S_n) - \bar{f}(t)\bar{g}(t)\|_{\mathcal{B}^p}
\]

\[
= \limsup_{l \to \infty} \frac{1}{2l} \int_{l-1}^{l} \| f(t + S_n)g(t + S_n) - (f + S_n)\bar{g}(t) + f(t + S_n)\bar{g}(t) - \bar{f}(t)\bar{g}(t)\|_{\mathcal{B}^p} dt
\]

\[
\leq 2^p \limsup_{l \to \infty} \frac{1}{2l} \int_{l-1}^{l} \left[ \| f(t + S_n)(g(t + S_n) - \bar{g}(t))\|_{\mathcal{B}^p} + \| \bar{g}(t)(f(t + S_n) - \bar{f}(t))\|_{\mathcal{B}^p} \right] dt
\]

\[
\leq 2^p \limsup_{n \to \infty} \| f(t)\|_{p} \| g(t)\|_{\mathcal{B}^p} + \epsilon^p \| g(t)\|_{\mathcal{B}^p}
\]

\[
\leq 2^p \epsilon^p \| f(t)\|_{p} + \| g(t)\|_{\mathcal{B}^p},
\]

which implies that:

\[
\| f(t + S_n)g(t + S_n) - \bar{f}(t)\bar{g}(t)\|_{\mathcal{B}^p} \to 0 \quad \text{as} \quad n \to \infty.
\]

Similarly, we can get:

\[
\lim_{n \to \infty} \| \bar{f}(t)(t - S_n)\bar{g}(t - S_n) - f(t)\bar{g}(t)\|_{\mathcal{B}^p} = 0.
\]

Consequently, \( f \cdot g \in \text{AAB}^p(\mathbb{R}, \mathbb{X}) \). The proof is completed. \( \square \)

**Lemma 6** ([8]). The space \((\text{BB}^p(\mathbb{R}, \mathbb{X}), \| \cdot \|_{\mathcal{B}^p})\) is a linear space, which is complete with respect to the seminorm \( \| \cdot \|_{\mathcal{B}^p} \).

**Lemma 7.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of \(p\)-almost automorphic functions in the sense of Besicovitch such that \( \lim_{n \to \infty} f_n(t) = f(t) \) uniformly in \( t \in \mathbb{R} \) with respect to the seminorm \( \| \cdot \|_{\mathcal{B}^p} \). Then \( f \) is \( p \)-almost automorphic in the sense of Besicovitch.

**Proof.** Let \((S'_n)_{n \in \mathbb{N}}\) be an arbitrary sequence of real numbers. By the diagonal procedure we can extract a subsequence \((S_k)_{n \in \mathbb{N}}\) of \((S'_n)_{n \in \mathbb{N}}\) such that:

\[
\| f_n(t + S_k) - \bar{f}_n(t)\|_{\mathcal{B}^p} = 0 \quad \text{as} \quad k \to \infty
\]

for each \( n \in \mathbb{N} \) and each \( t \in \mathbb{R} \). Noting that

\[
\| \bar{f}_n(t) - \bar{f}_m(t)\|_{\mathcal{B}^p}
\]

\[
\leq \| \bar{f}_n(t) - f(t)\|_{\mathcal{B}^p} + \| f(t + S_k) - f_m(t)\|_{\mathcal{B}^p} + \| f_m(t + S_k) - \bar{f}_m(t)\|_{\mathcal{B}^p}. \quad (6)
\]

For any \( \epsilon > 0 \). By the uniform convergence of \((f_n)_{n \in \mathbb{N}}\), we can find a positive integer \( N \) such that when \( n, m > N \), for all \( t \in \mathbb{R} \) and all \( k \in \mathbb{N} \), we have

\[
\| f_n(t + S_k) - f_m(t + S_k)\|_{\mathcal{B}^p} < \epsilon.
\]

This, combined with (6), implies that \((f'_n)_{n \in \mathbb{N}}\) is a Cauchy sequence. Hence, by Lemma 6, we can deduce the pointwise convergence of the sequence \((f_n)_{n \in \mathbb{N}}\), say to a function \( \bar{f}(t) \).

Let us prove now that:

\[
\lim_{k \to \infty} \| f(t + S_k) - \bar{f}(t)\|_{\mathcal{B}^p} = 0
\]

and

\[
\lim_{k \to \infty} \| \bar{f}(t - S_k) - f(t)\|_{\mathcal{B}^p} = 0
\]

pointwise on \( \mathbb{R} \).
Indeed, for each \( n \in \mathbb{N} \), we get:

\[
\parallel f(t + S_k) - \hat{f}(t) \parallel_{BP} \\
\leq \parallel f(t + S_k) - f_n(t + S_k) \parallel_{BP} + \parallel f_n(t + S_k) - \hat{f}_n(t) \parallel_{BP} + \parallel \hat{f}_n(t) - \hat{f}(t) \parallel_{BP}.
\]

Again, by the uniform convergence of Theorem 2, for an arbitrary \( \epsilon > 0 \), we can find some positive integer \( N_0 \) such that for every \( t \in \mathbb{R} \) and \( k \in \mathbb{N} \),

\[
\parallel f(t + S_k) - f_{N_0}(t + S_k) \parallel_{BP} < \epsilon \quad \text{and} \quad \parallel \hat{f}_{N_0}(t) - \hat{f}(t) \parallel_{BP} < \epsilon.
\]

Consequently, for every \( t \in \mathbb{R} \) and \( k \in \mathbb{N} \), we have

\[
\parallel f(t + S_k) - \hat{f}(t) \parallel < 2\epsilon + \parallel f_{N_0}(t + S_k) - \hat{f}_{N_0}(t) \parallel_{BP}.
\]

Now for every \( t \in \mathbb{R} \), we can find some positive integer \( N_1 = N(t, N_0) \) such that

\[
\parallel f_{N_0}(t + S_k) - \hat{f}_{N_0}(t) \parallel_{BP} < \epsilon
\]

for every \( k > N_1 \).

Finally, we get:

\[
\parallel f(t + S_k) - \hat{f}(t) \parallel_{BP} < 3\epsilon,
\]

for \( n \geq N \), where \( N \) is some positive integer depending on \( t \) and \( \epsilon \). That is, we have proven that \( \parallel f(t + S_k) - \hat{f}(t) \parallel_{BP} = 0 \) as \( k \to \infty \) for each \( t \in \mathbb{R} \).

By the same method, one can prove that

\[
\parallel \hat{f}(t - S_k) - f(t) \parallel_{BP} = 0 \quad \text{as} \quad k \to \infty
\]

for each \( t \in \mathbb{R} \). The proof is complete. \( \square \)

**Theorem 2.** The space \( (AAB^p(\mathbb{R}, X), \parallel \cdot \parallel_{BP}) \) is a linear space, which is complete with respect to the seminorm \( \parallel \cdot \parallel_{BP} \).

**Proof.** According to Lemma 6, \( BB^p(\mathbb{R}, X) \) is complete with respect to the seminorm \( \parallel \cdot \parallel_{BP} \). Since \( AAB^p(\mathbb{R}, X) \subset BB^p(\mathbb{R}, X) \), by Lemma 7, \( AAB^p(\mathbb{R}, X) \) is a closed subset of \( BB^p(\mathbb{R}, X) \). Consequently, \( AAB^p(\mathbb{R}, X) \) is complete with respect to the seminorm \( \parallel \cdot \parallel_{BP} \). The proof is completed. \( \square \)

**Definition 7.** A function \( f : \mathbb{R} \times X \to X, (t, x) \mapsto f(t, x) \) with \( f(\cdot, x) \in BB^p(\mathbb{R}, X) \) for each \( x \in X \), is said to be Besicovitch almost automorphic in \( t \in \mathbb{R} \) uniformly in \( x \in X \) if for every sequence of real numbers \( (S_n)_{n \in \mathbb{N}} \), there exists a subsequence \( (S'_{n})_{n \in \mathbb{N}} \) such that

\[
\hat{f}(t, x) := \lim_{n \to \infty} f(t + S_n, x)
\]

is well defined for each \( t \in \mathbb{R} \), and

\[
\lim_{n \to \infty} \hat{f}(t - S_n, x) = f(t, x)
\]

for each \( t \in \mathbb{R} \), uniformly in \( x \in X \). That is,

\[
\lim_{n \to \infty} \parallel f(t + S_n, x) - \hat{f}(t, x) \parallel_{BP} = 0
\]

and

\[
\lim_{n \to \infty} \parallel \hat{f}(t - S_n, x) - f(t, x) \parallel_{BP} = 0
\]

for each \( t \in \mathbb{R} \), uniformly in \( x \in X \). The collection of these functions will be denoted by \( AAB^p(\mathbb{R} \times X, X) \).
Theorem 3. If \( f \in AAB^p(\mathbb{R} \times \mathcal{X}, \mathcal{X}) \) satisfies the Lipschitz condition respect to its second argument and uniformly in its first argument, and \( g \in AAB^p(\mathbb{R}, \mathcal{X}) \), then \( f(\cdot, g(\cdot)) \) belongs to \( AAB^p(\mathbb{R}, \mathcal{X}) \).

Proof. In view of the definition of the seminorm \( \| \cdot \|_{B^p} \) and by the Lipschitz condition, one can easily get \( f(\cdot, g(\cdot)) \in BB^p(\mathbb{R}, \mathcal{X}) \). Since \( g \in AAB^p(\mathbb{R}, \mathcal{X}) \) and \( f \in AAB^p(\mathbb{R} \times \mathcal{X}, \mathcal{X}) \), for every sequence of real numbers \((S_n)_{n \in \mathbb{N}}\), one can extract a subsequence \((S'_n)_{n \in \mathbb{N}} \subset (S_n)_{n \in \mathbb{N}}\) such that for every \( \varepsilon > 0 \), \( t \in \mathbb{R} \) and every bounded subset \( B \subset \mathcal{X} \), there exists a positive number \( N(\varepsilon, t, B) \) satisfying for \( n > N \),

\[
\|g(t + S_n) - g(t)\|_{B^p} < \varepsilon, \quad \|g(t - S_n) - g(t)\|_{B^p} < \varepsilon,
\]

\[
\|f(t + S_n, x) - f(t, x)\|_{B^p} < \varepsilon, \quad \|f(t - S_n, x) - f(t, x)\|_{B^p} < \varepsilon
\]

for each \( t \in \mathbb{R} \) and \( x \in B \). Therefore,

\[
\|f(t + S_n, g(t + S_n)) - \tilde{f}(t, \tilde{g}(t))\|_{B^p}
\]

\[
\leq \|f(t + S_n, g(t + S_n)) - f(t + S_n, \tilde{g}(t))\|_{B^p} + \|f(t + S_n, \tilde{g}(t)) - \tilde{f}(t, \tilde{g}(t))\|_{B^p}
\]

\[
= \left( \limsup_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \|f(t + S_n, g(t + S_n)) - f(t + S_n, \tilde{g}(t))\|_{\mathcal{W}}^p dt \right)^{1/p}
\]

\[
+ \|f(t + S_n, \tilde{g}(t)) - \tilde{f}(t, \tilde{g}(t))\|_{B^p}
\]

\[
\leq L_f \left( \limsup_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \|g(t + S_n) - \tilde{g}(t)\|_{\mathcal{W}}^p dt \right)^{1/p} + \|f(t + S_n, \tilde{g}(t)) - \tilde{f}(t, \tilde{g}(t))\|_{B^p}
\]

\[
< (L_f + 1)\varepsilon,
\]

that is, \( \lim_{n \to \infty} \|f(t + S_n, g(t + S_n)) - \tilde{f}(t, \tilde{g}(t))\|_{B^p} = 0 \).

Similarly, one can prove that

\[
\lim_{n \to \infty} \|\tilde{f}(t - S_n, g(t - S_n)) - f(t, g(t))\|_{B^p} = 0.
\]

The proof is completed. \( \square \)

Remark 3. Theorems 1 and 3 are called the composition theorems.

3. Besicovitch Almost Automorphic Solutions

Before stating and proving our existence theorem, we need to give two lemmas.

Let

\[
\mathcal{W} = \{ g \in L^\infty(\mathbb{R}, \mathcal{X}) \cap AAB^p(\mathbb{R}, \mathcal{X}) \}
\]

with the norm \( \| \cdot \|_{\mathcal{W}} := \| \cdot \|_{\infty} \).

Lemma 8. The space \( (\mathcal{W}, \| \cdot \|_{\mathcal{W}}) \) is a Banach space.

Proof. Let \( \{f_n; n \geq 1\} \) be an arbitrary Cauchy sequence in \( \mathcal{W} \). Since \( L^\infty(\mathbb{R}, \mathcal{X}) \), \( \| \cdot \|_{\infty} \) is a Banach space and \( \{f_n\} \subset L^\infty(\mathbb{R}, \mathcal{X}) \) it follows that there exists \( f \in L^\infty(\mathbb{R}, \mathcal{X}) \) such that

\[
\|f_n - f\|_{\infty} \to 0 \quad \text{as} \quad n \to \infty. \quad (7)
\]

Hence, to show \( (\mathcal{W}, \| \cdot \|_{\mathcal{W}}) \) is a Banach space, it suffices to show \( f \in AAB^p(\mathbb{R}, \mathcal{X}) \). Noting that \( \|f_n - f\|_{B^p} \leq \|f_n - f\|_{\infty} \), consequently,

\[
\|f_n - f\|_{B^p} \to 0 \quad \text{as} \quad n \to \infty. \quad (8)
\]

By Lemma 7, we conclude that \( f \in AAB^p(\mathbb{R}, \mathcal{X}) \). The proof is complete. \( \square \)
Lemma 9. If \((H_1)\) holds and \(x \in \mathbb{W}\), then \(f(\cdot, x(\cdot)) \in \mathbb{W}\).

**Proof.** By Theorem 3, we see that \(f(\cdot, x(\cdot)) \in AAB^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})\). Since \(f\) satisfies the Lipschitz condition, for all \(t \in \mathbb{R}\), we have
\[
\|f(t, x(t))\|_\mathbb{X} \leq \mathcal{L} \|x(t)\|_\mathbb{X}.
\]
The above inequality implies \(f(\cdot, x(\cdot)) \in L^\infty(\mathbb{R}, \mathbb{X})\). Consequently, we have \(f(\cdot, x(\cdot)) \in \mathbb{W}\). The proof is complete. □

Lemma 10. Assume that \((H_1)-(H_3)\) hold. Let \(x \in \mathbb{W}\), then function \(\Phi x : \mathbb{R} \to \mathbb{X}\) defined by
\[
(\Phi x)(t) = \int_{-\infty}^t U(t, \theta)f(\theta, x(\theta))d\theta, \quad t \geq s, \ s \in \mathbb{R}
\]
belongs to \(\mathbb{W}\).

**Proof.** In view of Lemma 9, we know that \(f(\cdot, x(\cdot)) \in \mathbb{W}\). Our first task is to show that the integral on the right hand of formula (9) exists. Since \((H_2)\) and \((H_3)\), we have:
\[
\|\Phi x\|_\mathbb{X} = \left\| \int_{-\infty}^t U(t, s)f(s, x(s))ds \right\|_\mathbb{X}
\leq \int_{-\infty}^t \|U(t, s)\|_\mathbb{X} \|f(s, x(s))\|_\mathbb{X} ds
\leq \int_{-\infty}^t \mathcal{M} e^{-\lambda(t-s)} \|x(s)\|_\mathbb{X} ds
\leq \frac{\mathcal{G}}{\lambda} \|x\|_\mathbb{W},
\]
which yields that
\[
\|\Phi x\|_\mathbb{W} \leq \frac{\mathcal{G}}{\lambda} \|x\|_\mathbb{W},
\]
that is to say, (9) is well defined and, as a byproduct, we have obtained that \(\Phi x \in L^\infty(\mathbb{R}, \mathbb{X})\).

Next, we will show that \(\Phi x \in AAB^p(\mathbb{R}, \mathbb{X})\). Because \(x \in \mathbb{W}(\mathbb{R}, \mathbb{X})\), then by Lemma 9, we have \(f(\cdot, x(\cdot)) \in \mathbb{W} \subset AAB^p(\mathbb{R}, \mathbb{X})\), which combines with \((H_3)\), for given any bounded subset \(\tilde{\Omega} \subset \mathbb{X}\) and every sequence \(\{S_n\}\), we can select a subsequence \(\{\tilde{S}_n\} \subset \{S_n\}\) such that:
\[
\lim_{n \to \infty} \|f(t + S_n, x(t + S_n)) - \tilde{f}(t, x(t))\|_{B^p} = 0, \tag{10}
\]
\[
\lim_{n \to \infty} \|\tilde{f}(t - S_n, x(t - S_n)) - f(t, x(t))\|_{B^p} = 0 \tag{11}
\]
and
\[
\lim_{n \to \infty} \|U(t + S_n, x + S_n)x - \tilde{U}(t, s)x\|_\mathbb{X} = 0, \quad \text{for} \quad x \in \tilde{\Omega}, \tag{12}
\]
\[
\lim_{n \to \infty} \|\tilde{U}(t - S_n, s - S_n)x - U(t, s)x\|_\mathbb{X} = 0, \quad \text{for} \quad x \in \tilde{\Omega}. \tag{13}
\]
Further, by the Hölder inequality, we have:

\[
\| (\Phi x)(t + S_n) - (\Phi x)(t) \|_X^p \\
\leq \left\| \int_{-\infty}^{t+S_n} U(t + S_n, s) f(s, x(s)) ds - \int_{-\infty}^{t} \bar{U}(t, s) f(s, x(s)) ds \right\|_X^p \\
= \left\| \int_{-\infty}^{t} U(t + S_n, s + S_n) f(s + S_n, x(s + S_n)) ds - \int_{-\infty}^{t} \bar{U}(t, s) f(s, x(s)) ds \right\|_X^p \\
\leq 2^{p-1} \left\| \int_{-\infty}^{t} U(t + S_n, s + S_n) \| f(s + S_n, x(s + S_n)) \|_X^p ds \right\|_X^p \\
\times \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right\|_X^p \\
\leq 2^{p-1} \left\| \int_{-\infty}^{t} U(t + S_n, s + S_n) \| f(s + S_n, x(s + S_n)) \|_X^p ds \right\|_X^p \\
\times \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right\|_X^p \\
\leq 2^{p-1} M^p \left[ \int_{-\infty}^{t} e^{-\frac{p}{\lambda} \lambda(t-s)} ds \right] \left[ \int_{-\infty}^{t} e^{-\frac{p}{\lambda} \lambda(t-s)} \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right] \\
\times \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right\|_X^p \\
\times \left[ \int_{-\infty}^{t} \| U(t + S_n, s + S_n) - \bar{U}(t, s) \|_X^p \right|_X^p ds \right\|_X^p \\
\times \left[ \int_{-\infty}^{t} \| U(t + S_n, s + S_n) - \bar{U}(t, s) \|_X^p \right|_X^p ds \right\|_X^p \\
\times \left[ \int_{-\infty}^{t} \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right] \\
\times \left[ \int_{-\infty}^{t} \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right] \\
\times \left[ \int_{-\infty}^{t} \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right] \\
\times \left[ \int_{-\infty}^{t} \| f(s + S_n, x(s + S_n)) - \bar{f}(s, x(s)) \|_X^p ds \right],
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).
By a change of variables and Fubini’s theorem, from the inequality above, we have:

\[
\limsup_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \| (\Phi x)(t + S_n) - (\Phi x)(t) \|_{\mathcal{X}}^p dt \\
\leq 2^{n-1} M_p \left( \frac{p}{\lambda q} \right) \limsup_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \left[ \int_{-\infty}^{t} e^{-\frac{t}{\lambda(l-s)}} \| f(s + S_n, x(s + S_n)) - \tilde{f}(s, x(s)) \|_{\mathcal{X}}^p ds \right] dt \\
+ 2^{n-1} \limsup_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \left\{ \int_{-\infty}^{t} \| U(t + S_n, s + S_n) - \tilde{U}(t, s) \|_{\mathcal{X}}^p ds \right\}^{\frac{p}{q}} dt \\
\leq 2^{n-1} M_p \left( \frac{p}{\lambda q} \right) \int_{0}^{\infty} e^{-\frac{t}{\lambda S}} \left[ \int_{-l}^{l} \| f(t - s + S_n, x(t - s + S_n)) - \tilde{f}(t - s, x(t - s)) \|_{\mathcal{X}}^p ds \right] dt \\
+ 2^{n-1} \limsup_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \left\{ \int_{-\infty}^{t} \| U(t + S_n, S_n) - \tilde{U}(t, s) \|_{\mathcal{X}}^p ds \right\}^{\frac{p}{q}} dt.
\]

thus by (H2), Lebesgue’s dominated convergence theorem, (10) and (12), we deduce that

\[
\lim_{n \to \infty} \| (\Phi x)(t + S_n) - (\Phi x)(t) \|_{\mathcal{Y}}^p = 0.
\]

Similarly, from (11) and (13), one can get

\[
\lim_{n \to \infty} \| (\Phi x)(t - S_n) - (\Phi x)(t) \|_{\mathcal{Y}}^p = 0,
\]

which means that \( \Phi x \in \mathcal{W} \). In conclusion, the proof is complete. \( \square \)

**Theorem 4.** If conditions (H1)–(H4) hold, then system (10) has a unique \( p \)-almost automorphic mild solution in the sense of Besicovitch in \( \mathcal{W} \).

**Proof.** Define an operator \( \Psi : \mathcal{W} \to \mathcal{W} \) by

\[
(\Psi x)(t) = \int_{-\infty}^{t} U(t, s) f(s, x(s)) ds, \ x \in \mathcal{W}, \ t \in \mathbb{R}.
\]
Obviously, $Ψ$ is well-defined and maps $\mathbb{W}$ into $\mathbb{W}$ according to Lemma 10. We just have to show that $Ψ : \mathbb{W} \rightarrow \mathbb{W}$ is a contraction mapping. In fact, for any $x, y \in \mathbb{W},$

$$\| (Ψx)(t) - (Ψy)(t) \|_X = \left\| \int_{-\infty}^{t} U(t, s)(f(s, x(s)) - f(s, y(s)))ds \right\|_X$$

$$\leq \int_{-\infty}^{t} Me^{-\lambda(t-s)}\|x(s) - y(s)\|_X ds$$

$$\leq \frac{M\lambda}{\lambda} \|x - y\|_{\mathbb{W}}, \quad t \in \mathbb{R},$$

which, combined with $(H_4),$ yields

$$\|Ψx - Ψy\|_{\mathbb{W}} \leq \kappa \|x - y\|_{\mathbb{W}},$$

Hence, $Ψ$ is a contraction mapping from $\mathbb{W}$ to $\mathbb{W}.$ Noting the fact that $(\mathbb{W}, \| \cdot \|_{\mathbb{W}})$ is a Banach space; therefore, according to the Banach fixed point theorem, $Ψ$ has a unique fixed point $z^* \in \mathbb{W}$ such that $Tz^* = z^*.$ Consequently, system (10) has a unique $p$-almost automorphic mild solution $z^*$ in the sense of Besicovitch. The proof is complete. \(\square\)

4. An Example

Consider the following partial differential equation with Dirichlet boundary conditions:

$$\frac{∂u(t, ξ)}{∂t} = \frac{∂^2}{∂ξ^2}u(t, ξ) - 2u(t, ξ) + \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{2}t}\right)u(t, ξ)$$

$$+ \frac{1}{2}\cos\left(\frac{1}{2 + \sin t + \sin \sqrt{2}t}\right) + e^{-|t|} + \frac{1}{1 + t^2} \quad \forall ξ \in [0, π],$$

$$u(t, 0) = u(t, π) = 0, \quad t \in \mathbb{R}. \quad (14)$$

Take $X = L^2[0, π]$ with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)_2.$ Define: $A : D(A) \subset X \rightarrow X$ given by

$$Ax = \frac{∂^2 x(ξ)}{∂ξ^2} - 2x,$$

with domain

$$D(A) = \{x(\cdot) \in X : x'' \in X, x' \in X \text{ is absolutely continuous on } [0, π], \ x(0) = x(π) = 0\}.$$ 

According to [13], we know that $A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t≥0}$ on $X$ satisfying

$$\|T(t)\| \leq e^{-3t} \quad \text{for } \quad t > 0.$$ 

In addition,

$$T(t)x = \sum_{n=1}^{+∞} e^{(-n^2+2)t} (x, y_n)_2 y_n, \quad t \geq 0, \ x \in X,$$

where $y_n(x) = \sqrt{\frac{2}{π}} \sin(nx).$ Define a family of linear operators $A(t)$ by $D(A(t)) = D(A),$

$$A(t)x(ξ) = \left( A + \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{2}t}\right) \right) x(ξ), \quad ∀ξ \in [0, π], \ x \in D(A).$$

Then, the system

$$\begin{cases}
    x'(t) = A(t)x(t), \ t > s,
    
    x(s) = x \in X
\end{cases}$$
has an associated evolution family \( \{U(t,s)\}_{t \geq s} \) on \( X \), which can be explicitly express by

\[
U(t,s)x = \left( T(t-s) e^{\int_t^s \sin \left( \frac{r}{2} + \cos \cos \sqrt{2r} \right) dr } \right) x.
\]

It is easy to see that for any sequence \( \{S_n\} \subset \mathbb{R} \), we have:

\[
\|U(t+S_n,s+S_n) - U(t,s)\| \leq e^{-(t-s)} \left| \sin \left( \frac{1}{2 + \cos(t + S_n) + \cos \sqrt{2(t + S_n)} } \right) - \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2t} } \right) \right|.
\]

Since, as mentioned before, the function \( \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2t} } \right) \) is almost automorphic, \( U(t,s) \) is bi-almost automorphic. Thus, \( (H_3) \) is verified. Moreover,

\[
\|U(t,s)\| \leq e^{-2(t-s)} \quad \text{for} \quad t \geq s.
\]

By [14], we see that \( A(t) \) satisfies conditions \( (S_1) \) and \( (S_2) \).

Let

\[
f(t,x(t)) = \frac{1}{3} \left( \cos \left( \frac{1}{2 + \sin t + \sin \sqrt{5t} } \right) + e^{-|t|} + \frac{1}{1 + t^2} \right) \sin x(t),
\]

then (14) can be transformed into the abstract Equation (1).

Noticing that Example 1, it is easy to verify that \((H_1)-(H_4)\) hold with \( M = 1, \lambda = 2 \) and \( \mathcal{L} = 1 \).

Consequently, by Theorem 4, system (14) has a unique almost automorphic mild solution in the sense of Besicovitch.

5. Conclusions

In this paper, some basic properties of almost automorphic functions in the sense of Besicovitch defined by Bochner properties are studied, and on this basis, the existence of \( p \)-almost automorphic solutions in the sense of Besicovitch for a class of semilinear evolution equations is established. The results of this paper are new. At the same time, the results and methods of this paper can be used to study the existence of \( p \)-almost automorphic solutions in the sense of Besicovitch for other types of semilinear evolution equations. For example, semilinear evolution equations with time delays and semilinear differential integral equations.

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