One-way EPR steering beyond qubits

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Quantum steering has been exploited as an important resource in nowadays quantum information processing. It is inherently directional and doesn’t necessitate the state is of asymmetric form. However, some asymmetric states are revealed to be unidirectional which can manifest steering only in specific direction, thus are called one-way steering. Existing works focused on one-way steering in qubit system. Here we propose a family of two-partite states that is one-way steerable in arbitrary-dimensional system. In particular, we demonstrate how one-way steering manifests in two-qutrit system. Moreover, we develop a model for characterizing one-way steering with the experimental losses are counted, with which the tradeoff relation between losses and measurement settings in steering test in higher-dimensional system is investigated. Our loss-counted model works for any finite dimensional system with any finite measurement settings as well.

INTRODUCTION

Quantum entanglement, which has clear mathematical structure (that is, the state of the system is non-factorizable), is seen as a counter-intuitive effect in real physical process. This lead to the dispute on completeness of quantum mechanics that was raised from the famous Einstein–Podolsky–Rosen argument [1], and the dispute was later reformed into the discussion and certification of violation of Bell-types inequalities in the view of correlation [2]. Indeed, it is not revealed until recently that quantum entanglement has far richer structure between non-factorizable and Bell-nonlocal [3–5].

This mediate form of entanglement structure is named as steering [6], which describes the effect that one of the entangled parties, by performing local measurements, can affect the local state of another remote party. Different from the other two entanglement structures, quantum steering is inherently asymmetric, as the roles of steerer and steeree are distinct and cannot be interchanged. To capture the essence of quantum steering, one can interpret it in the view of theoretical information tasks, where steering is demonstrated if the steered ensembles at steeree’s side cannot be explained by local hidden state (LHS) models [7].

As steering has direct correspondence to information tasks, various application of steering has been developed in recent years, such as one-sided device-independent quantum key distribution [8], randomness certification [9, 10], subchannel discrimination [11], device-independent quantification of measurement incompatibility [12], secure quantum teleportation [13] and secret sharing [14].

In particular, the asymmetry of steering plays pivotal role in all those applications, which manifests in the different level of trust to the involved parties [15]. It is shown that the detection loophole can be effectively closed using loss-tolerant steering criteria, thus promising information-theoretical security [16]. However, the inherent asymmetry of steering does not only manifest assuming different level of trust, but rather with the state per se is of asymmetric form—known as one-way steering.

One-way steering, as identified in 2009 [3], was first studied in the Gaussian regime [17–20], and was subsequently studied in discrete variable regime. In specific, tailored two-qubit states are found exhibiting one-way steering, by using projective measurements [21] and positive-operator-valued measure (POVM) [5] respectively. Further, a family of two-parameter mixed states that exhibit one-way steering are characterized [22]. Recently, sufficient conditions for certifying one-way steering of two-qubit system has also been investigated [23, 24].

Following the theoretical works, experimental demonstrations of one-way steering have also been carried out. It was first observed using two-setting projective measurements [25], and later was extended to using multisetting projective measurements [26]. Using POVM, which is commonly realized by introducing lossy channel, one-way steering has also been demonstrated [27, 28]. Very recently, based on a newly-developed criterion [29], the authors show experimentally that projective measurement can fully characterize one-way steering in the two-qubit system [30].

However, the existing works on one-way steering focused on qubit system, the inherent asymmetry of high-dimensional steering has never been studied to date. Indeed, steering that is beyond qubit system is of special interests, since higher-dimensional steering possesses strong robustness against noise, which is theoretically predicted and experimentally demonstrated recently [31].

In this work, we for the first time extend the exploration of one-way steering to higher-dimensional system. We first introduce the \(d\)-dimensional partially entangled states, which is generalized from its two-qubit form. We then take three-dimensional partially entangled states for
example, to illustrate how one-way steering manifests in the higher-dimensional system. Next, we take the experimental loss into account, and build up a loss-counted model to characterize the tradeoff relation between the measurement settings and detection efficiencies in steering test. Finally we apply our model to characterize high-dimensional one-way steering in the loss-counted scenario.

**PARTIALLY ENTANGLED STATES**

Consider two-partite scenario, where Alice and Bob are sharing a quantum state $\rho_{AB}$. If Alice performs local measurements on her subsystem, which are denoted by a set of operators $\{M_{a|x}\}$, with $M_{a|x} \geq 0$ and $\sum_a M_{a|x} = 1$, where $x$ denotes her measurement setting and $a$ the corresponding outcome, then on Bob’s subsystem, conditioned on the measurement $x$ and outcome $a$, a collection of subnormalized density matrices $\{\sigma_{a|x}\}_{a,x}$ are generated, with

$$\sigma_{a|x} = \text{Tr}_A(M_{a|x} \otimes I \rho_{AB}).$$

This set of matrices is named as *assemblage* [32], and is proven to be useful in characterizing the steerability from Alice to Bob (or vice versa if measurements are performed on Bob’s side) for finite number of measurements. Specifically, the assemblage is called unsteerable if it can be reproduced by some LHS model, namely, it can be decomposed into

$$\sigma_{a|x}^{\text{LHS}} = \int \sigma_{\lambda p}(a|x, \lambda)d\lambda, \ \forall a, x,$$

where $\{\sigma_{\lambda}\}$ is a set of positive matrices with some probability distribution $p(a|x, \lambda)$. Any assemblage that refutes LHS model demonstrates EPR-steering. Interestingly, assemblages coming from entangled states do not guarantee steering. Indeed, given an assemblage, certifying whether it has LHS model is difficult, not mention to one has to consider the experimental imperfections when demonstrating steering in practice.

It can be seen that the role of steerer and steeree have already been designated for an assemblage, given a quantum state and the local measurement. Thus certifying steering is in itself directional, even if the given quantum state is of symmetric form. Traditionally, people use symmetric states to study steering, and the conclusion can be straightforward deduced when the role of two parties are interchanged. As illustrated in Fig. 1, for the symmetric states, if Alice is able to steer Bob, then Bob must be able to steering Alice. While if the states are in asymmetric form, then the conclusions may differ for different roles of the parties. It is pointed out that one-way steering can only occur for mixed states, as the pure entangled states can always be transformed to a symmetric form by local basis change [21].

![FIG. 1. The correspondence of symmetric states to two-way steering and asymmetric states to one-way steering. Note here the symmetric state doesn’t mean the state with symmetric form, but rather the ones can not demonstrate one-way steering. The black squares represent the uncharacterized devices, while the white squares denote the trusted devices.](image)

In two-qubit case, some mixed states in particular asymmetric form have been brought to sight, and were certified manifesting one-way steering. Recently, a family of states with two independent parameters is developed, and these states are well-characterized showing one-way steering for a range of combination of parameters, even for infinite many measurement settings [22]. The states are called partially entangled two-qubit state, with

$$\rho(p, \theta) = p|\psi_\theta\rangle\langle \psi_\theta| + (1 - p)\rho_0^A \otimes 1/2,$$

where $|\psi_\theta\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$, $\rho_0^A = \text{Tr}_B|\psi_0\rangle\langle \psi_0|$, $p \in [0, 1]$ and $\theta \in [0, \pi/4]$.

Here we generalize the above states to its higher-dimensional form, i.e.,

$$\rho(p, \mathbf{a}) = p|\psi_\mathbf{a}^+\rangle\langle \psi_\mathbf{a}^+| + (1 - p)\rho_0^A \otimes 1/d,$$

where $|\psi_\mathbf{a}^+\rangle = \sum_i a_i |ii\rangle$, $\rho_0^A = \text{Tr}_B|\psi_\mathbf{a}^+\rangle\langle \psi_\mathbf{a}^+|$, $p \in [0, 1]$, $d$ is the dimension of the system and $\mathbf{a}$ is a $d$-dimensional vector.

**ONE-WAY STEERING OF TWO-QUTRIT STATE**

With the asymmetric states being targeted, we start to consider the way characterizing the steerability of the state from both directions, respectively. Indeed, one finds that characterizing steerability for infinitely many measurement settings is tricky, since the variable $\mathbf{a}$ in Eq. (2) could take infinitely many values [33]. However, if we limit the number of measurements and outputs to finite values, the problem would become computationally feasible [34, 35]. In specific, if we further adopt the $d + 1$
tangled state, the lower bound of the cutoff value considered analytically proven that for $d$-dimensional isotropic state via some filtering operation [5], without affecting the steerability from Bob to Alice [22]. Thus for the general cases of partially entangled state defined in Eq. (4) with finite dimension $d$, one can always demonstrate one-way steering for certain combination of parameters.

The above results are based on four-setting mutually unbiased bases (see results and discussions for three and two-setting cases in supplemental materials). However, it is known that to faithfully characterize one-way steering, one needs to consider the general measurement. Specifically, increasing the number of measurement settings can effectively improve the steerability, which is shown both theoretically [4, 36, 37] and experimentally [16]. The intuitive reason for this is that the extra measurements brings more constraints to the LHS model in Eq. (5), thus leaves the LHS model of $\sigma_{a|x}^{\text{LHS}}$ with smaller proportion in Eq. (6), which directly gives the larger steering weight value.

**LOSS-COUNTED ONE-WAY STEERING**

To conclude the unsteerability, one in principle is required to test infinite-settings measurements, which is not feasible in the experiment. On the other hand, there is another prominent factor that affects one-way steering demonstration, which is the inevitable experimental losses. In photonic experiment for example, loss happens naturally, which manifests in no-click event is registered at the measurement devices. This should not be a
problem if the device is trusted. However, if the steerer, say Alice, is holding a cracked device, her measurement outcomes could be deliberately discarded, whenever the measurements she performs do not correspond to Bob’s announced measurements, thus faking the perfect correlations and pass the steering test. This opens the so-called detection loophole \[38\] which is also significant in Bell-nonlocality test \[39\].

From the view of LHS model, the discarded results grant more flexibility on the combination of local model in Eq. (5), which consequently reduces the steering weight value, thus indicating that steering is demolished—or equivalently, the criterion of demonstrating steering is pushed up. Indeed, the tradeoff relation between the losses (which is commonly characterized by the heralding efficiency \(\epsilon\), describes the probability that the steerer heralds steeree’s result by declaring a non-null prediction) and the number of measurement setting is investigated in two-qubit system \[16, 36\].

Here, we aim to give the general relation between the heralding efficiency and the number of measurement setting for any finite-dimensional system and any finite measurement settings. Then, we applied the relation to the one-way steering characterization. The influence of loss in the steering test can be interpreted as the extra outcome for each measurement setting \[40, 41\]. Taking the \(d+1\) MUB measurement for example, Alice performs \(d+1\) measurements, and in this loss-counted scenario, she obtains \(d+1\) outcomes. We denote the extra outcome which corresponds to no-event being registered by \(a = d\), as for now \(a = \{0,...,d\}\) and the assemblage in this scenario is called priori assemblage \[39\], which we denote as \(\{\sigma_{a|x}^{\text{pri}}\}\).

In this priori scenario, we naturally have

\[
\sigma_{a|x}^{\text{pri}} = \begin{cases} 
\epsilon \sigma_{a|x}, & \forall a, x \text{ with } a \neq d, \\
(1 - \epsilon) \rho^B, & \forall a, x \text{ with } a = d,
\end{cases}
\]

for the probability conservation, where \(\rho^B\) is the reduced state of Bob. With this form of \(\sigma_{a|x}^{\text{pri}}\), one straightforward way to understand the priori scenario is to consider Alice as the distributor, and she with some probability sends Bob a mixed state that won’t be discerned by Bob. In the meantime, she claims that those outcomes from mixed state are lost (discarded), then she can convince Bob that she can indeed steer his state—we assume that \(\{\sigma_{a|x}\}\) is a steerable assemblage—at a relatively low cost (with a discount of \(\epsilon\)), since preparing a steerable assemblage \(\{\sigma_{a|x}\}\) is obviously harder than preparing a local mixed state.

With this in mind, a decent way for Bob to deal with the losses is to assume that all the losses result from the trick of mixed state, but still believe that the rest of the experimental results are conclusive. Thus the LHS model in priori scenario can be reformed as

\[
\sigma_{a|x}^{\text{LHS}} = \frac{1}{1 - \epsilon} \sum_{\lambda=0}^{d^2-1} D(a|x, \lambda) \sigma_{a}^{\lambda}, \forall a, x \text{ with } a \neq d,
\]

\[
\rho^B = \frac{1}{1 - \epsilon} \sum_{\lambda=0}^{d^2-1} D(a|x, \lambda) \sigma_{a}^{\lambda}, \forall a, x \text{ with } a = d,
\]

where \(\Lambda = (d+1)^{(d+1)} - 1\).

It is worth noting that the normalization condition is automatically satisfied with \(\text{Tr} \sum_{\lambda} \sigma_{a}^{\lambda} = 1\), should the LHS model \(\{\sigma_{a|x}^{\text{LHS}}\}\) can fully reproduce the given assemblage \(\{\sigma_{a|x}\}\). Otherwise LHS model contributes partially to the reconstruction of \(\{\sigma_{a|x}\}\) as show in Eq. (6). Thus we can straightforwardly write down the SDP to solve the steering weight with loss-counted, namely, to maximize \(\text{Tr} \sum_{\lambda} \sigma_{a}^{\lambda}\) over \(\{\sigma_{a}^{\lambda}\}\), subject to the constraints in Eq. (8) \[42\].

Note here we assume that the probability of loss, or equivalently the heralding efficiency \(\epsilon\) independent of the measurement \(x\) that Alice performs, which means the action of discarding is completely random and not specified to particular measurement. However, if the efficiency \(\epsilon\) is dependent on the choice of measurement, one can simply replace \(\epsilon\) with \(\epsilon(x)\) in the above expressions.

Indeed, there are several analytical works talked about the steering test with loss-counted in two and higher-dimensional systems \[36, 40\]. The comparison between our results and those analytical bounds are provided in supplemental materials. It is important to note that the

\[
\epsilon\text{ setting}
\]

\[
\begin{array}{c|c|c|c|c}
\epsilon & 1 & 0.85 & 0.65 & 0.55 \\
\hline
\text{setting} & 1 & 4 & 3 & 2 \\
\end{array}
\]
analytical bounds are derived specifically for symmetric states, and our method is applicable for any input state, which is shown as follows.

In Fig. 3 we present the loss-counted one-way steering demonstration for two-qutrit partially entangled states, using two-, three- and four-setting MUB. For each setting of MUB, we test $\epsilon = 1, 0.85, 0.65$ respectively. Note here we only select the cross section with $\theta = \pi/4$ and we only consider the loss on Alice’s side, and Bob’s device is assumed to be well-characterized—still, Bob cannot steer Alice if the state is unsteerable from his side.

One can see clearly the tradeoff relation between the measurement settings and detection efficiencies on the steering threshold from Alice to Bob for the partially entangled states. Specifically, we see the state is two-way steerable at the point of $\phi = 0.9415$ (within numerical precision), and becomes completely one-way steerable when loss is counted. On the other hand, given certain amount of losses, increasing the number of measurement settings can effectively improve the steerability.

CONCLUSIONS

To conclude, we have proposed a generic form of high-dimensional two-partite state that can demonstrate one-way steering effect. We showed how this family of states is one-way steerable in the general cases, and taking the three-dimensional case for example, we have fully characterized the parameter space of one-way steering. Beyond merely the analysis on ideal case, we have also extended our model to the practical scenario with the experimental losses are counted. Specifically, we devised a concise interpretation accounting for the effect of experimental loss and accordingly developed the loss-counted version of steering weight measure. We tested our loss-counted model and compared it with known results for symmetric states. More importantly, the model is shown applicable for any finite dimensional system with any finite measurement setting in any conceivable form. Thus we have demonstrated how the tradeoff relation of measurement settings and losses manifests in two-qutrit one-way steering state. It is worth noting that though the examples are tested with $d+1$ MUB in our work, the model we developed is applicable to any finite measurement settings, such as Platonic solid measurements [37].

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[1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics Physique Fizika 1, 195 (1964).
[3] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. 98, 140402 (2007).
[4] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Experimental EPR-steering using Bell-local states, Nat. Phys. 6, 845 (2010).
[5] M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Inequivalence of entanglement, steering, and Bell nonlocality for general measurements, Phys. Rev. A 92, 032107 (2015).
[6] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, Rev. Mod. Phys. 92, 015001 (2020).
[7] S. Jones, H. Wiseman, and A. Doherty, Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering, Phys. Rev. A 76, 052116 (2007).
[8] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, Phys. Rev. A 85, 010301 (2012).
[9] P. Skrzypczyk and D. Cavalcanti, Maximal Randomness Generation from Steering Inequality Violations Using Qudits, Phys. Rev. Lett. 120, 260401 (2018).
[10] Y. Guo, S. Cheng, X. Hu, B.-H. Liu, E.-M. Huang, Y.-F. Huang, C.-F. Li, G.-C. Guo, and E. G. Cavalcanti, Experimental Measurement-Device-Independent Quantum Steering and Randomness Generation Beyond Qubits, Phys. Rev. Lett. 123, 170402 (2019).
[11] M. Piani and J. Watrous, All Entangled States are Useful for Channel Discrimination, Phys. Rev. Lett. 102, 250501 (2009).
[12] D. Cavalcanti and P. Skrzypczyk, Quantitative relations between measurement incompatibility, quantum steering, and nonlocality, Phys. Rev. A 93, 052112 (2016).
[13] M. Reid, Signifying quantum benchmarks for qubit teleportation and secure quantum communication using Einstein-Podolsky-Rosen steering inequalities, Phys. Rev. A 88, 062338 (2013).
[14] Y. Xiang, I. Kogias, G. Adesso, and Q. He, Multiparticle Gaussian steering: Monogamy constraints and quantum cryptography applications, Phys. Rev. A 95, 010101 (2017).
[15] E. G. Cavalcanti, M. J. W. Hall, and H. M. Wiseman, Entanglement verification and steering when Alice and Bob cannot be trusted, Phys. Rev. A 87, 032306 (2013).
[16] A. J. Bennet, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, Arbitrarily Loss-Tolerant Einstein-Podolsky-Rosen Steering Allowing a Demonstration over 1 km of Optical Fiber with No Detection Loophole, Phys. Rev. X 2, 031003 (2012).
[17] M. Reid, Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification, Phys. Rev. A 40, 913 (1989).
[18] S. L. W. Midgley, A. J. Ferris, and M. K. Olsen, Asym-
metric Gaussian steering: When Alice and Bob disagree, Phys. Rev. A 81, 022101 (2010).

[19] K. Wagner, J. Janousek, V. Delaubert, H. Zou, C. Harb, N. Treps, J. F. Morizur, P. K. Lam, and H. A. Bachor, Entangling the spatial properties of laser beams, Science 321, 541 (2008).

[20] V. Händchen, T. Eberle, S. Steinlechner, A. Sambloyski, T. Franz, R. F. Werner, and R. Schnabel, Observation of one-way Einstein–Podolsky–Rosen steering, Nat. Photonics 6, 596 (2012).

[21] J. Bowles, T. Vértesi, M. T. Quintino, and N. Brunner, One-way Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 112, 200402 (2014).

[22] J. Bowles, F. Hirsch, M. T. Quintino, and N. Brunner, Sufficient criterion for guaranteeing that a two-qubit state is unsteerable, Phys. Rev. A 93, 022121 (2016).

[23] T. J. Baker, S. Wollmann, G. J. Pryde, and H. M. Wiseman, Necessary condition for steerability of arbitrary two-qubit states with loss, J. Opt. 20, 034008 (2018).

[24] T. J. Baker and H. M. Wiseman, Necessary conditions for steerability of two qubits from consideration of local operations, Phys. Rev. A 101, 022326 (2020).

[25] K. Sun, X.-J. Ye, J.-S. Xu, X.-Y. Xu, J.-S. Tang, Y.-C. Wu, J.-L. Chen, C.-F. Li, and G.-C. Guo, Experimental Quantification of Asymmetric Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 116, 160404 (2016).

[26] Y. Xiao, X.-J. Ye, K. Sun, J.-S. Xu, C.-F. Li, and G.-C. Guo, Demonstration of Multisetting One-Way Einstein-Podolsky-Rosen Steering in Two-Qubit Systems, Phys. Rev. Lett. 118, 140404 (2017).

[27] S. Wollmann, N. Walk, A. J. Bennet, H. M. Wiseman, and G. J. Pryde, Observation of Genuine One-Way Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 116, 160403 (2016).

[28] N. Tischler, F. Ghafari, T. J. Baker, S. Slussarenko, R. B. Patel, M. M. Weston, S. Wollmann, L. K. Shalm, V. B. Verma, S. W. Nam, H. C. Nguyen, H. M. Wiseman, and G. J. Pryde, Conclusive Experimental Demonstration of One-Way Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 121, 100401 (2018).

[29] H. C. Nguyen, H.-V. Nguyen, and O. Gühne, Geometry of Einstein-Podolsky-Rosen Correlations, Phys. Rev. Lett. 122, 240401 (2019).

[30] Q. Zeng, J. Shang, H. C. Nguyen, and X. Zhang, Reliable experimental certification of one-way einstein-podolsky-rosen steering (2020), arXiv:2010.13014 [quant-ph].

[31] Q. Zeng, B. Wang, P. Li, and X. Zhang, Experimental High-Dimensional Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 120, 030401 (2018).

[32] M. F. Pusey, Negativity and steering: A stronger Peres conjecture, Phys. Rev. A 88, 032313 (2013).

[33] D. Cavalcanti and P. Skrzypczyk, Quantum steering: A review with focus on semidefinite programming, Rep. Prog. Phys. 80, 024001 (2017).

[34] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, Quantifying Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 112, 180404 (2014).

[35] M. Piani and J. Watrous, Necessary and Sufficient Quantum Information Characterization of Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. 114, 060404 (2015).

[36] D. A. Evans, E. G. Cavalcanti, and H. M. Wiseman, Loss-tolerant tests of Einstein-Podolsky-Rosen steering, Phys. Rev. A 88, 022106 (2013).

[37] D. A. Evans and H. M. Wiseman, Optimal measurements for tests of Einstein-Podolsky-Rosen steering with no detection loophole using two-qubit Werner states, Phys. Rev. A 90, 012114 (2014).

[38] P. M. Pearle, Hidden-Variable Example Based upon Data Rejection, Phys. Rev. D 2, 1418 (1970).

[39] C. Branciard, Detection loophole in Bell experiments: How postselection modifies the requirements to observe nonlocality, Phys. Rev. A 83, 032123 (2011).

[40] P. Skrzypczyk and D. Cavalcanti, Loss-tolerant Einstein-Podolsky-Rosen steering for arbitrary-dimensional states: Joint measurability and unbounded violations under losses, Phys. Rev. A 92, 022354 (2015).

[41] A. B. Sainz, Y. Gurynova, W. McCutcheon, and P. Skrzypczyk, Adjusting inequalities for detection-loophole-free steering experiments, Phys. Rev. A 94, 032122 (2016).

[42] Q. Zeng, Code for computing steering weight with loss counted, SteeringWeight_Loss (2022).
**I: One-way steering of two-qutrit partially entangled states with two- and three-setting MUB**

In Fig. 4 we present the one-way steering parameter space for two- and three-setting MUB. It is clear that the overall steering effect from both directions is underestimated comparing to four-setting case. Interestingly, the one-way steering is reversed for two-setting MUB given certain combination of parameter as show in Fig. 4A, which we see it resulting from the deficiency of two-setting measurement in characterizing three dimensional state.

**II: Comparison to existing analytical bounds**

In Fig. 5, we present the comparison between our loss-counted model and the known analytical bounds on the cutoff value of visibility of the $d$-dimensional isotropic state versus loss. The circles denote our numerical results for the specific dimension, measurement setting and heralding efficiency. The authors in Ref. [40] derived bounds for dimension up to five given $d+1$ MUB, and the bounds are denoted in blue, orange, red and magenta respectively. We can see clearly that our results indicate stronger steerability comparing to the known bounds when loss is low, and are approaching to the theoretical limit along with the increasing of loss. This is reasonable since the bounds in Ref. [40] consider the asymptotic behavior of the cutoff value given $d$ and $\epsilon$, which leads to the inexact $p^*$ when loss is low.

On the other hand, our results perfectly reproduced the optimal bounds (denoted in gray) derived in Ref. [37] for two-qubit case with two- and three-measurement settings, as indicated in the figure.