On Non Asymptotic Expansion of the MME in the Case of Poisson Observations.

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Abstract

The problem of parameter estimation by observations of inhomogeneous Poisson processes is considered. The method of moments estimator is studied and its stochastic expansion is obtained. This stochastic expansion is then used to obtain the expansion of the moments of this estimator and the expansion of the distribution function. The stochastic expansion, expansion of the moments and the expansion of distribution function are non asymptotic in nature. Several examples are considered.

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1 Introduction

Consider the problem of parameter estimation by \( n \) independent observations \( X^{(n)} = (X_1, \ldots, X_n) \). If we suppose that the density function \( f(\vartheta, x) \) of the observation \( X_j \) is \textit{regular}, i.e., is sufficiently smooth with respect to parameter \( \vartheta \), then the well-know estimators (MLE, Bayesian estimator, method of
moments estimators) are consistent, asymptotically normal and we have the convergence of polynomial moments ($p > 0$):

$$\bar{\vartheta}_n \longrightarrow \vartheta_0, \quad \sqrt{n} (\bar{\vartheta}_n - \vartheta_0) \Rightarrow \xi \sim \mathcal{N} \left( 0, D (\vartheta_0)^2 \right),$$

$$\lim_{n \rightarrow \infty} n^{p/2} E_{\vartheta_0} |\bar{\vartheta}_n - \vartheta_0|^p \longrightarrow D (\vartheta_0)^2 E |\zeta|^p, \quad \zeta \sim \mathcal{N} (0, 1).$$

Here we denoted $\vartheta_0$ the true value and $D (\vartheta_0)^2$ is the limit variance of the estimator $\bar{\vartheta}_n$. These relations can be written as follows

$$\bar{\vartheta}_n - \vartheta_0 = o (1),$$

$$\bar{\vartheta}_n - \vartheta_0 = \varphi_n \xi (1 + o (1)), \quad (1)$$

$$E_{\vartheta_0} |\bar{\vartheta}_n - \vartheta_0|^2 = \varphi_n^2 D (\vartheta_0)^2 (1 + o (1)), \quad (2)$$

$$P_{\vartheta_0} \left( D (\vartheta_0)^{-1} \varphi_n^{-1} (\bar{\vartheta}_n - \vartheta_0) < x \right) = F (x) + o (1). \quad (3)$$

Here $\varphi_n = n^{-1/2}$, we take $p = 2$ and $F (x)$ is distribution function of Gaussian law $\mathcal{N} (0, 1)$. Of course, the relation in $[1]$ is just a symbolique writing because the limit Gaussian variable $\xi$ is not defined on the same probability space, we have convergence in distribution only.

If the volume $n$ of observations is large, then the relations $[1]$--$[3]$ describe well the distribution of the error of estimation. For the moderate values of $n$ the real distribution of $\bar{\vartheta}_n - \vartheta_0$ and of the moments $E_{\vartheta_0} |\bar{\vartheta}_n - \vartheta_0|^2$ can be quite far from the given here limit values. The better approximations for the distribution function and the moments can be obtained with the help of well-known asymptotic expansion theory. There are at least three types of expansions:

**Stochastic expansion**

$$\bar{\vartheta}_n - \vartheta_0 = \varphi_n \xi_{n,1} + \ldots + \varphi_n^k \xi_{n,k} + o \left( \varphi_n^k \right), \quad (4)$$

where $\xi_{n,i}$ are bounded in probability random variables,

**Expansion of the moments**

$$E_{\vartheta_0} |\bar{\vartheta}_n - \vartheta_0|^2 = \varphi_n^2 P_{n,1} + \ldots + \varphi_n^{2k} P_{n,k} + o \left( \varphi_n^{2k} \right), \quad (5)$$

where $P_{n,i}$ are some bounded real values,

**Expansion of the distribution function**

$$P_{\vartheta_0} \left( \frac{\bar{\vartheta}_n - \vartheta_0}{D (\vartheta_0) \sqrt{n}} < x \right) = F (x) + \varphi_n p_1 (x) + \ldots + \varphi_n^k p_k (x) + o \left( \varphi_n^k \right), \quad (6)$$
where \( p_i(x) \) are some products of polynomials and \( f(x) \) (density function of \( \mathcal{N}(0,1) \)). The value of the parameter \( k \) depends on the smoothness of the model with respect to unknown parameter.

We can mention here the works devoted to asymptotic expansions of estimators for independent identically distributed observations of the random variables \([2], [6], [7], [10], [11], [16], [17], [18]\). In the last book there is an extensive list of references. Such asymptotic expansions are widely used in bootstrap too [11]. The difference between these works is in the conditions of regularity and in the estimates of the residuals \( o(\cdot) \) in (4)-(6). Note that in the majority of these works the results are asymptotic in nature, i.e., the residuals in expansions (3)-(5) are of the type \( o(\varphi_n^k) \) or \( \varphi_n^{k+\delta}o(1) \), \( \delta \in (0,1) \), where \( o(1) \to 0 \) as \( n \to \infty \) and nothing can be said about the term \( o(1) \) for finite \( n \).

The expansions of the errors of estimation by the powers of small parameters in the case of observations of continuous time stochastic processes were obtained in the works \([4]\) (signal in white Gaussian noise), \([12], [15]\) (inhomogeneous Poisson processes), \([13], [19], [20], [14]\), (diffusion processes), \([21]\) (martingales with jumps).

One of the goals of this work is to obtain such expansions (stochastic, moments and distribution function) for the recently introduced class of estimators: method of moments estimators (MME) in the case of observations of inhomogeneous Poisson processes. The method of moments allowed to have explicit expressions for many models with intensity functions non linearly depending on the unknown parameters, where the traditional MLE have no explicit expressions and therefore MME provides essential gain in the calculation of the consistent and asymptotically normal estimators. Note that the MME are used in One-step MLE construction to obtain the asymptotically efficient estimators too [9].

It is known that the publications related with the asymptotic expansions (4)-(6) in statistics are technically quite cumbersome (see, e.g., [18] and references therein). In the considered in this work case the exposition is essentially simplified because the random variables \( \xi_{n,i} \) in (4) have the form \( \xi_{n,i} = \eta_n^i \) with the same \( \eta_n \). Another advantage of this work with respect to traditional asymptotic expansions is the using in obtaining all expansions of the method of good sets, which in reality allows to obtain non asymptotic expansions. This method was developed by Burnashev in [4] and [6]. His approach allows to have expansions (4) and (5) non asymptotic in nature. This means, that the residuals in (4) and (5) for finite \( n \) can be estimated from above and from below with known rates and constants. Remark that this method was
already applied to obtain non asymptotic expansions in the works [12], [13], [14], [15].

2 Method of Moments Estimator

Let us consider the following problem of parameter estimation. Suppose that we have \( n \) independent observations of inhomogeneous Poisson processes \( X^{(n)} = (X_1, \ldots, X_n) \), where \( X_j = \{ X_j(t), t \in \mathcal{T} \} \) with intensity function \( \lambda(\vartheta, t), t \in \mathcal{T} \). Here \( \mathcal{T} \subset \mathcal{R} \) is some interval. It can be \( \mathcal{T} = [0, \tau], \mathcal{T} = [0, +\infty), \mathcal{T} = \mathcal{R} \) or any other interval on the line. As usual, \( X_j(\cdot) \) are counting processes (see details in [15]). We have to estimate the parameter \( \vartheta \in \Theta = (\alpha, \beta) \) by the observations \( X^{(n)} \) and to describe the properties of estimators in the asymptotic of large samples (\( n \to \infty \)).

For the study we take the method of moments estimator (MME) defined as follows. Introduce the functions \( g(t), t \in \mathcal{T} \) and the functions \( m(\vartheta) = \int_{\mathcal{T}} g(t) \lambda(\vartheta, t)dt, \vartheta \in \Theta \) and \( \bar{m}_n = \frac{1}{n} \sum_{j=1}^{n} \int_{\mathcal{T}} g(t)dX_j(t) \).

We suppose that we have such functions \( \lambda(\vartheta, t), \vartheta \in \Theta, t \in \mathcal{T} \), and \( g(t) \) that the function \( m(\vartheta), \alpha \leq \vartheta \leq \beta \) is monotone. Without loss of generality we assume that it is monotone increasing.

Let us introduce the following notations:

\[ \mathcal{M} = \{ m(\vartheta) : \vartheta \in [\alpha, \beta] \} = [m(\alpha), m(\beta)] \]

For \( y \in \mathcal{M} \) we write the solution of the equation \( m(\vartheta) = y \) as \( \vartheta = m^{-1}(y) = G(y) \). Therefore the function \( G(y) \) is inverse for \( m(\vartheta) \) and \( G(m(\vartheta)) = \vartheta \).

We define the method of moments estimator (MME) \( \hat{\vartheta}_n \) by the following equation

\[ \hat{\vartheta}_n = \arg \inf_{\vartheta \in \Theta} (m(\vartheta) - \bar{m}_n)^2. \]

This estimator admits the representation

\[ \hat{\vartheta}_n = \alpha \mathbb{1}_{\{m_n \leq m(\alpha)\}} + \bar{\vartheta}_n \mathbb{1}_{\{m(\alpha) < m_n < m(\beta)\}} + \beta \mathbb{1}_{\{m_n \geq m(\beta)\}}, \tag{7} \]

where \( \bar{\vartheta}_n \) is a solution of the equation \( m(\bar{\vartheta}_n) = \bar{m}_n \) when \( \bar{m}_n \in (m(\alpha), m(\beta)) \). The value \( \hat{\vartheta}_n = \alpha \) (respectively \( \hat{\vartheta}_n = \beta \)) corresponds to \( m \leq m(\alpha) \) closest
to \( \bar{m}_n \) (respectively \( m \geq m(\beta) \) closest to \( \bar{m}_n \)). Recall that with probability \( p_n = \exp(-n\Lambda(T)) \) we have no events (jumps) in all observations. Here

\[
\Lambda(T) = \int_T \lambda(\vartheta, t) \, dt.
\]

This estimator was recently studied in [9]. It was shown that as \( n \to \infty \) and under mild regularity conditions this estimator is consistent

\[
\hat{\vartheta}_n \longrightarrow \vartheta_0
\]

(here and in the sequel \( \vartheta_0 \) denotes the true value), asymptotically normal

\[
\sqrt{n} (\hat{\vartheta}_n - \vartheta_0) \Longrightarrow \mathcal{N}(0, D(\vartheta_0)^2)
\]

or, for any \( x \in \mathbb{R} \)

\[
P_{\vartheta_0} \left( D(\vartheta_0)^{-1/2} \sqrt{n} (\hat{\vartheta}_n - \vartheta_0) < x \right) \longrightarrow F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy.
\]

We have as well the convergence of all polynomial moments: for any \( p > 0 \)

\[
n^{\frac{p}{2}} E_{\vartheta_0} |\hat{\vartheta}_n - \vartheta_0|^p \longrightarrow E |\zeta|^p D(\vartheta_0)^{\frac{p}{2}}, \quad \zeta \sim \mathcal{N}(0,1).
\]

The advantage of the MME can be illustrated with the help of the following examples.

**Example 1** Suppose that we have an inhomogeneous Poisson process \( X^T = (X_t, 0 \leq t \leq T) \) with \( \tau \)-periodic intensity function \( \lambda(\vartheta, t), 0 \leq t \leq T \), i.e., \( \lambda(\vartheta, t+k\tau) = \lambda(\vartheta, t) \) for any \( k = 1, 2, \ldots \). Suppose that \( T = n\tau \) and introduce the independent Poisson processes

\[
X_j(t) = X_{(j-1)\tau+t} - X_{(j-1)\tau}, \quad t \in \mathcal{T} = [0, \tau], \quad j = 1, \ldots, n.
\]

Therefore we obtain the observations \( X^{(n)} = (X_1, \ldots, X_n) \) with intensity function \( \lambda(\vartheta, t) \) and can construct the MME \( \hat{\vartheta}_n \).

Suppose that we observe a \( \tau \)-periodic Poisson signal of intensity function \( S(\vartheta, t) = \vartheta h(t) \) (amplitude modulation) in the presence of the Poisson noise of intensity \( \lambda_0 > 0 \), i.e., we have

\[
\lambda(\vartheta, t) = \vartheta h(t) + \lambda_0, \quad 0 \leq t \leq \tau.
\]
Here $h(\cdot)$ is $\tau$-periodic known positive function and $\lambda_0 > 0$. Remember that the MLE $\hat{\vartheta}_n$ for this model has no explicit representation and is given as solution of the following equation

$$
\sum_{j=1}^{n} \int_{0}^{\tau} \frac{h(t)}{\hat{\vartheta}_n h(t) + \lambda_0} dX_j(t) = n \int_{0}^{\tau} h(t) dt.
$$

Despite the numerical difficulties of its calculation there is also the problem of definition of this stochastic integral. The calculation of the MLE has no such difficulties and for the wide class of functions $g(\cdot)$ (say, positive) we have

$$m(\vartheta) = \partial H_g + \lambda_0 G, \quad H_g = \int_{0}^{\tau} g(t) h(t) dt, \quad G = \int_{0}^{\tau} g(t) dt$$

and $\vartheta = H_g^{-1} [m(\vartheta) - \lambda_0 G]$. Hence $G(y) = H_g^{-1} [y - \lambda_0 G]$. The MME

$$\tilde{\vartheta}_n = \left( \int_{0}^{\tau} g(t) h(t) dt \right)^{-1} \left[ \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{\tau} g(t) dX_j(t) - \lambda_0 \int_{0}^{\tau} g(t) dt \right]$$

(12)

This estimator has the mentioned above properties (8)-(10) (see [9]).

**Example 2.** Suppose that the intensity function of the observed Poisson processes $X^{(n)} = (X_1, \ldots, X_n)$, $X_j = (X_j(t), t \in \mathcal{T} = [0, +\infty))$ is

$$\lambda(\vartheta, t) = f(t) e^{-\vartheta h(t)} + q(t), \quad t \geq 0.$$  

(13)

Here $f(\cdot) > 0, h(\cdot) \geq 0$ and $q(\cdot) \geq 0$ are known functions, $h(0) = 0, h(\infty) = \infty$ and $h(\cdot)$ has continuous derivative $h'(\cdot)$. For example, $f(t) = 1 + t^4, h(t) = t^2, q(t) = 1$. We have to estimate $\vartheta \in (\alpha, \beta), 0 < \alpha < \beta < \infty$. Recall that the MLE has no explicit expression. Let us put $g(t) = f(t)^{-1} h'(t)$, then we have

$$m(\vartheta) = \int_{0}^{\tau} g(t) \lambda(\vartheta, t) dt = \frac{1}{\vartheta} + R, \quad R = \int_{0}^{\tau} \frac{h'(t) q(t)}{f(t)} dt,$$

hence $\vartheta = (m(\vartheta) - R)^{-1}$ and the MME

$$\tilde{\vartheta}_n = \left( \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{\tau} \frac{h'(t) q(t)}{f(t)} dX_j(t) - R \right)^{-1}.$$ 

If we suppose that the corresponding integrals are finite then once more this estimator has the mentioned above properties (8)-(10) (see [9]).
The case \( h(0) \neq 0 \) can be treated by a similar way, but to define the MME we have to solve the equation

\[
\varphi^{-1} e^{-\varphi h(0)} = y + R, \quad \varphi \in \Theta.
\]

Of course, all derivatives of the solution \( \varphi = G(y) \) can be calculated without problems.

**Example 3** Consider Poisson processes \( X^{(n)} = (X_1, \ldots, X_n) \) with “Gaussian” intensity function

\[
\lambda(\varphi, t) = a e^{-\frac{(t-b)^2}{2\varphi^2}}, \quad t \in \mathcal{R}.
\]  

Here \( a > 0, b \) are supposed to be known and we have to estimate \( \varphi \in (\alpha, \beta) \), \( \alpha > 0 \). Let us take \( g_1(t) = (t-b)^2 \). Then

\[
m_1(\varphi) = \int_{\mathcal{R}} (t-b)^2 \lambda(\varphi, t) \, dt = \varphi^3 a \sqrt{2\pi}, \quad \varphi = \left( \frac{m(\varphi)}{a \sqrt{2\pi}} \right)^{1/3}
\]

and

\[
\hat{\varphi}_n = \left( \frac{1}{an \sqrt{2\pi}} \sum_{j=1}^{n} \int_{\mathcal{R}} (t-b)^2 \, dX_j(t) \right)^{1/3}.
\]

Another possibility is to take \( g_2(t) = |t-b| \). Then we obtain \( m_2(\varphi) = 2a \varphi^2 \) and

\[
\hat{\varphi}_n = \left( \frac{1}{2an} \sum_{j=1}^{n} \int_{\mathcal{R}} |t-b| \, dX_j(t) \right)^{1/2}.
\]

The both estimators are consistent and asymptotically normal.

### 3 Stochastic expansion

The properties \([9], [10]\) of the MME \( \hat{\varphi}_n \) can be written as \([1], [2]\) and our goal is to obtain the expansions like \([4]\) for these estimators too.

Recall, we have \( n \) independent observations of inhomogeneous Poisson processes \( X^{(n)} \) of the same intensity function \( \lambda(\varphi, t), t \in \mathcal{T} \), where \( \varphi \in \Theta = (\alpha, \beta) \). To estimate \( \varphi \) we use the MME \( \hat{\varphi}_n \) defined in \([7]\) with some function \( g(\cdot) \).
Introduce the notation:

\[
\psi_l(\vartheta_0) = \frac{G^{(l)}(m(\vartheta_0))}{l!}, \quad l = 1, \ldots, k,
\]

\[
\pi_n(t) = \sum_{j=1}^{n} X_j(t) - n \int_{s<t} \lambda(\vartheta_0, s) \, ds,
\]

\[
\eta_n(\vartheta_0) = \sqrt{n} \left( \bar{m}_n - m(\vartheta_0) \right) = \frac{1}{\sqrt{n}} \int_{T} g(t) \, d\pi_n(t).
\]

Remind that sufficient condition for the asymptotic normality of the stochastic integrals \( \eta_n(\vartheta) \), \( \vartheta \in \Theta \) is

\[
\sup_{\vartheta \in \Theta} \int_{T} g(t)^2 \lambda(\vartheta, t) \, dt < \infty.
\]

Below we introduce more strong condition \( \mathcal{L}_3 \).

In this work we suppose that the functions \( \lambda(\vartheta, t) \) and \( g(t) \) are such that the following conditions \( \mathcal{L} \) are fulfilled.

\( \mathcal{L}_1 \). The function \( m(\vartheta) \), \( \vartheta \in \Theta \) has \( k + 2 \) continuous bounded derivatives.

\( \mathcal{L}_2 \). The function \( m(\vartheta) \), \( \vartheta \in \Theta \) is monotone and

\[
\inf_{\vartheta \in \Theta} |\dot{m}(\vartheta)| > 0.
\]

\( \mathcal{L}_3 \). The function \( g(\cdot) \) is such that for any \( m > 0 \)

\[
\sup_{\vartheta \in \Theta} \int_{T} |g(t)|^m \lambda(\vartheta, t) \, dt < \infty. \tag{15}
\]

Without loss of generality, we suppose that the function \( m(\vartheta) \), \( \vartheta \in \Theta \) is increasing. We have the following first result concerning the stochastic expansion of the MME.

**Theorem 1** Let the conditions \( \mathcal{L} \) be fulfilled, then for any \( k = 1, 2, \ldots \) there exist the random variables \( r_{n,k}, \phi_{n,k} \) and the set \( \mathcal{B} \) such that the MME \( \hat{\vartheta}_n \) admits the representation

\[
\hat{\vartheta}_n = \vartheta_0 + \left\{ \sum_{l=1}^{k} \psi_l(\vartheta_0) \eta_n^l n^{-\frac{1}{4}} + r_n n^{-\frac{k}{4} - \frac{1}{4}} \right\} \mathbb{I}_{\{\mathcal{B}\}} + \phi_n \mathbb{I}_{\{\mathcal{B}^c\}}, \tag{16}
\]

where \( \eta_n = \eta_n(\vartheta_0), |r_n| \leq 1, \phi_n \in (\alpha - \vartheta_0, \beta - \vartheta_0) \) and for any \( Q > 0 \) and any compact \( \mathcal{K} \subset \Theta \) there exists a constant \( C = C(Q, \mathcal{K}) > 0 \) such that

\[
\sup_{\vartheta \in \mathcal{K}} \mathbb{P}(\vartheta_0(\mathcal{B}^c) \leq \frac{C}{nQ}. \tag{17}
\]
Proof. The proof of this theorem is based on the approach of good sets. Introduce the first good set

$$\mathbb{B}_1 = \left\{ \inf_{|\vartheta - \vartheta_0| < \delta} |m(\vartheta) - \bar{m}_n| < \inf_{|\vartheta - \vartheta_0| \geq \delta} |m(\vartheta) - \bar{m}_n| \right\},$$

where $\delta > 0$ is some small number satisfying the condition $\alpha + \delta < \vartheta_0 < \beta - \delta$. Then the MME $\bar{m}_n$ on the set $\mathbb{B}_1$ satisfies the relations

$$m(\tilde{\vartheta}_n) = \bar{m}_n, \quad \tilde{\vartheta}_n = G(\bar{m}_n) = G(m(\vartheta_0) + \varepsilon \eta_n).$$

Here $\varepsilon = n^{-1/2}$ and $\bar{m}_n = m(\vartheta_0) + n^{-1/2} \eta_n$.

The Taylor expansion of the function $G(\cdot)$ on the set $\mathbb{B}_1$ yields

$$\tilde{\vartheta}_n = G(m(\vartheta_0)) + \sum_{l=1}^{k} \frac{G^{(l)}(m(\vartheta_0))}{l!} \eta_n^l \varepsilon^l + \frac{G^{(k+1)}(\bar{m}_n)}{(k + 1)!} \varepsilon^{k+1} \eta_n^k \varepsilon^{k+1}$$

$$= \vartheta_0 + \sum_{l=1}^{k} \frac{G^{(l)}(m(\vartheta_0))}{l!} \eta_n^l \left( \frac{1}{\sqrt{n}} \right)^l + \frac{G^{(k+1)}(\bar{m}_n)}{(k + 1)!} \frac{\eta_n^k}{(\sqrt{n})^k} \left( \frac{1}{\sqrt{n}} \right)^{k + \frac{1}{2}}$$

$$= \vartheta_0 + \sum_{l=1}^{k} \psi_l(\vartheta_0) \eta_n^l n^{-\frac{l}{2}} + r_{n,k} n^{-\frac{k}{2} - \frac{1}{4}},$$

where $\bar{m}_n \in (m(\vartheta_0 - \delta), m(\vartheta_0 + \delta))$ and we denoted

$$r_{n,k} = \frac{G^{(k+1)}(\bar{m}_n) \eta_n^k (k + 1)! n^{-\frac{k}{2}}}{\eta_n^k (k + 1)! n^{-\frac{k}{2}}}. $$

Recall that the derivatives $G'(y), G''(y), G'''(y)$ of the inverse function $G(y)$ can be calculated using the equality $G(m(\vartheta)) = \vartheta$ as follows

$$G'(m(\vartheta)) \bar{m}(\vartheta) = 1, \quad G'(m(\vartheta)) = \frac{1}{\bar{m}(\vartheta)}, \quad G'(y) = \frac{1}{\bar{m}(G(y))},$$

$$G''(y) = -\frac{\bar{m}(G(y))}{\bar{m}^2(G(y))}, \quad G'''(y) = \frac{3\bar{m}(G(y))^2 - \bar{m}(G(y)) \bar{m}(G(y))}{\bar{m}^3(G(y))}.$$ 

Here dot means derivation w.r.t. $\vartheta$. The other derivatives can be calculated by the same rules. As it follows from conditions $\mathcal{L}$ all derivatives up to $G^{(k+1)}(\cdot)$ are bounded and we can write on the set $\mathbb{B}_1$ (with probability 1)

$$\left| \frac{\partial^{k+1} G(y)}{\partial y^{k+1}} \right|_{y = \bar{m}_n} \leq \sup_{m(\vartheta_0 - \delta) \leq y \leq m(\vartheta_0 + \delta)} \left| \frac{\partial^{k+1} G(y)}{\partial y^{k+1}} \right| = C_{k+1}.$$
where the constant $C_{k+1} = C_{k+1}(\vartheta_0, \delta) > 0$.

Introduce the second good set
\[ \mathcal{B}_2 = \left\{ |\eta_n|^{k+1} < n^{1/4}(k + 1)! \right\}. \]

On the set $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$ we have the estimate $|r_{n,k}| \leq 1$.

Therefore we obtain the stochastic expansion (16) on the set $\mathcal{B}$. The probability of the complement is estimated as follows
\[ P_{\vartheta_0}(\mathcal{B}^c) \leq P_{\vartheta_0}(\mathcal{B}_1^c) + P_{\vartheta_0}(\mathcal{B}_2^c). \]

To estimate $P_{\vartheta_0}(\mathcal{B}_1^c)$ we remark that
\[
P_{\vartheta_0}(\mathcal{B}_1^c) = P_{\vartheta_0}(|\hat{\vartheta}_n - \vartheta_0| \geq \delta) = P_{\vartheta_0} \left( \inf_{|\vartheta - \vartheta_0| < \delta} |m(\vartheta) - \bar{m}_n| \geq \inf_{|\vartheta - \vartheta_0| \geq \delta} |m(\vartheta) - \bar{m}_n| \right) \]
\[
= P_{\vartheta_0} \left( \inf_{|\vartheta - \vartheta_0| < \delta} |m(\vartheta) - m(\vartheta_0) + m(\vartheta_0) - \bar{m}_n| \geq \inf_{|\vartheta - \vartheta_0| \geq \delta} |m(\vartheta) - m(\vartheta_0) + m(\vartheta_0) - \bar{m}_n| \right) \]
\[
\leq P_{\vartheta_0} \left( 2|m_n - m(\vartheta_0)| \geq \inf_{|\vartheta - \vartheta_0| \geq \delta} |m(\vartheta) - m(\vartheta_0)| \right) \]
\[
\leq P_{\vartheta_0} \left( 2|m_n - m(\vartheta_0)| \geq \rho(\delta) \right),
\]
where we denoted $\rho(\delta) = \inf_{|\vartheta - \vartheta_0| \geq \delta} |m(\vartheta) - m(\vartheta_0)|$. Note that for any $\delta > 0$ by condition $L_2$ we can write
\[
\rho(\delta) = \inf_{|\vartheta - \vartheta_0| \geq \delta} |\dot{m}(\vartheta)| |\vartheta - \vartheta_0| \geq \kappa \delta > 0. \tag{18}
\]

Here $\kappa = \inf_{\vartheta \in \Theta} |\dot{m}(\vartheta)|$. Therefore we have
\[
P_{\vartheta_0}(\mathcal{B}_1^c) \leq P_{\vartheta_0} \left( \frac{2}{n} \sum_{j=1}^{n} \int_T g(t) [dX_j(t) - \lambda(\vartheta_0, t) dt] \geq \kappa \delta \right) \]
\[
\leq P_{\vartheta_0} \left( \frac{2}{\sqrt{n}} \left| \int_T g(t) d\pi_n(t) \right| \geq \kappa \delta \sqrt{n} \right),
\]
where
\[
d\pi_n(t) = dY_n(t) - n\lambda(\vartheta_0, t) dt, \quad Y_n(t) = \sum_{j=1}^{n} X_j(t).
\]
Remark that \( Y_n(t), t \in T \) is inhomogeneous Poisson process with intensity function \( n \lambda (\vartheta_0, t), t \in T \).

Further, by Markov inequality we can write for any integer \( q \geq 1 \)
\[
\mathbb{P}_{\varphi_0} \left( \frac{2}{\sqrt{n}} \left| \int_T g(t) \, d\pi_n(t) \right| \geq \kappa \delta \sqrt{n} \right) \\
\leq \left( \kappa \delta \sqrt{n} \right)^{-2q} \mathbb{E}_{\varphi_0} \left[ \left| \int_T g(t) \, d\pi_n(t) \right|^{2q} \right] \\
\leq C_1 4^q \left( \kappa \delta \sqrt{n} \right)^{-2q} n^{1-q} \int_T |g(t)|^{2q} \lambda (\vartheta_0, t) \, dt \\
+ C_2 4^q \left( \kappa \delta \sqrt{n} \right)^{-2q} \left( \int_T |g(t)|^2 \lambda (\vartheta_0, t) \, dt \right)^q \leq \frac{C}{n^q}. \tag{19}
\]

Here we used the property of stochastic integral
\[
\mathbb{E} \left[ \left| \int f(t) \, d\pi(t) \right|^{2q} \right] \leq C_1 \int |f(t)|^{2q} \lambda (t) \, dt + C_2 \left( \int |f(t)|^2 \lambda (t) \, dt \right)^q \tag{20}
\]
with obvious notation. The proof can be found, for example, in [15], Lemma 1.2.

Further, for the second probability we have
\[
\mathbb{P}_{\varphi_0} (B^c_2) = \mathbb{P}_{\varphi_0} \left( |\eta| \geq cn^{\frac{1}{2k+1}} \right) \leq c^{-2q} n^{-\frac{2q}{2k+1}} \mathbb{E}_{\varphi_0} |\eta|^{2q} \\
\leq C_1 c^{-2q} n^{-\frac{2q}{2k+1}} n^{1-q} \int_T |g(t)|^{2q} \lambda (\vartheta_0, t) \, dt \\
+ C_2 c^{-2q} n^{-\frac{2q}{2k+1}} \left( \int_T |g(t)|^2 \lambda (\vartheta_0, t) \, dt \right)^q \leq \frac{C}{n^{\frac{q}{2k+1}}} \tag{21}
\]
with the corresponding constant \( C > 0 \). From the estimates (19) and (21) we obtain
\[
\mathbb{P}_{\varphi_0} (B^c) \leq \mathbb{P}_{\varphi_0} (B^c_1) + \mathbb{P}_{\varphi_0} (B^c_2) \leq \frac{C}{n^{2k+2}} + \frac{C}{n^q} \leq \frac{C_*}{n^Q}, \tag{22}
\]
where for a given \( Q \) we put \( q = 2Q (k + 1) \). Note that having functions \( g(t) \) and \( \lambda (\vartheta, t) \) all mentioned constants can be calculated or estimated from above.

Remark, that the random variable \( \phi_{n,k} \) can take any value on the intervals \([\alpha - \vartheta_0, -\delta]\) and \([\beta - \vartheta_0, \delta]\). In any case we have the estimate \( |\phi_n| \leq \beta - \alpha \). It is important to mention that the representation (16) is valid for all \( n \).
4 Expansion of the moments

The stochastic expansion (16) we write as

\[
\hat{\vartheta}_n = \vartheta_0 + \Psi_n I(B) + \phi_{n,k} I(B^c),
\]

where \(\Psi_n\) can be written as follows

\[
\Psi_n = \Psi_{0,n} + r_{n,k} n^{-\frac{2k+1}{4}}.
\]

Here of course

\[
\Psi_{0,n} = \sum_{l=1}^{k} \psi_l(\vartheta_0) \eta_n^l n^{-\frac{l}{2}}.
\]

This presentation allows to write the expansion of the mean of the MME

\[
E_{\vartheta_0} \hat{\vartheta}_n = \vartheta_0 + \sum_{l=1}^{k} \psi_l(\vartheta_0) E_{\vartheta_0} \eta_n^l n^{-\frac{l}{2}} + O\left(n^{-\frac{2k+1}{4}}\right).
\]

The first terms are

\[
E_{\vartheta_0} \hat{\vartheta}_n = \vartheta_0 + \psi_2(\vartheta_0) E_{\vartheta_0} \eta_n^2 n^{-1} + \psi_3(\vartheta_0) E_{\vartheta_0} \eta_n^3 n^{-3/2} + O\left(n^{-\frac{7}{4}}\right)
\]

\[
= \vartheta_0 - \frac{\dot{m}(\vartheta_0)}{2\dot{m}(\vartheta_0)^3} \int_0^T g(t)^2 \lambda(\vartheta_0, t) dt \frac{1}{n} + O\left(n^{-\frac{7}{4}}\right).
\]

We see that if \(k = 3\) then the terms with \(\psi_1(\vartheta_0)\) and \(\psi_3(\vartheta_0)\) are absent in this expansion. These relations can be proved, but we will speak about more interesting problem of the expansion of the moments of the error of estimation.

**Theorem 2** Let conditions \(\mathcal{L}\) be fulfilled. Then for any \(p > 1\) there exists a constant \(C^* > 0\) such that

\[
\left| E_{\vartheta_0} |\hat{\vartheta}_n - \vartheta_0|^p - E_{\vartheta_0} |\Psi_{0,n}|^p \right| \leq C^* n^{-\frac{2k+1}{4}}.
\]

**Proof.** Without loss of generality we put \(r_{n,k} = 0\) on the set \(B^c\). Then for any \(p \geq 1\) we have

\[
E_{\vartheta_0} |\hat{\vartheta}_n - \vartheta_0|^p = E_{\vartheta_0} (|\Psi_n|^p I(B)) + E_{\vartheta_0} (|\phi_{n,k}|^p I(B^c))
\]

\[
= E_{\vartheta_0} |\Psi_n|^p + E_{\vartheta_0} (|\phi_{n,k}|^p - |\Psi_n|^p) I(B^c).
\]
Hence
\[ E_{\vartheta_0} |\tilde{\vartheta}_n - \vartheta_0|^p \leq E_{\vartheta_0} |\Psi_n|^p + E_{\vartheta_0} (|\phi_{n,k}|^p I_{B^c}) \]
and
\[ E_{\vartheta_0} |\tilde{\vartheta}_n - \vartheta_0|^p \geq E_{\vartheta_0} |\Psi_n|^p - E_{\vartheta_0} (|\Psi_n|^p I_{B^c}) . \]

We have
\[ E_{\vartheta_0} (|\phi_{n,k}|^p I_{B^c}) \leq \frac{C_* (\beta - \alpha)^p}{n^Q}, \]
where we used the estimate (22).

Note that for any $p > 0$ there exist constants $A > 0$ and $B > 0$ that
\[ E_{\vartheta_0} |\eta_n|^{2p} < A, \quad E_{\vartheta_0} |\Psi_n|^{2p} < B. \]

By Cauchy-Schwarz inequality we have the estimate
\[ E_{\vartheta_0} (|\Psi_n|^p I_{B^c}) \leq \left[ E_{\vartheta_0} |\Psi_n|^{2p} P_{\vartheta_0} (B^c) \right]^{\frac{1}{2}} \leq \frac{\sqrt{C_* B}}{n^{Q/2}} . \]

These two estimates allow us to write
\[ -\frac{\sqrt{C_* B}}{n^{Q/2}} \leq E_{\vartheta_0} |\tilde{\vartheta}_n - \vartheta_0|^p - E_{\vartheta_0} |\Psi_n|^p \leq \frac{C_* (\beta - \alpha)^p}{n^Q}. \] (24)

For $p > 1$ and small $x$ we have
\[ |a + x|^p = |a|^p + p \operatorname{sgn} (a + \tilde{x}) |a + \tilde{x}|^{p-1} x, \quad |\tilde{x}| \leq |x| . \]

Using this relation we can write
\[ \left| E_{\vartheta_0} |\Psi_n|^p - E_{\vartheta_0} |\Psi_{0,n}|^p \right| \leq p E_{\vartheta_0} \left| \Psi_{0,n} + \tilde{r}_n n^{-\frac{2k+1}{4}} \right|^{p-1} n^{-\frac{2k+1}{4}} \leq C' n^{-\frac{2k+1}{4}} . \]

Hence
\[ E_{\vartheta_0} |\tilde{\vartheta}_n - \vartheta_0|^p - E_{\vartheta_0} |\Psi_{0,n}|^p \leq \frac{C_* (\beta - \alpha)^p}{n^Q} + C n^{-\frac{2k+1}{4}} \]
and
\[ E_{\vartheta_0} |\tilde{\vartheta}_n - \vartheta_0|^p - E_{\vartheta_0} |\Psi_{0,n}|^p \geq -C n^{-\frac{2k+1}{4}} - \frac{\sqrt{C_* B}}{n^{Q/2}} . \]
Let us put \( Q = k + \frac{1}{2} \), then we obtain (23) with some constant \( C^* \).

**Case** \( p = 2 \) and \( k = 3 \). To illustrate (23) we consider the expansion of the moments in the case \( p = 2 \) and \( k = 3 \). We can write \( (\varepsilon = n^{-1/2}) \)

\[
\begin{align*}
  nE_{\vartheta_0} \Psi_{0,n}^2 &= E_{\vartheta_0} \left( \psi_1 \eta_n + \psi_2 \eta_n^2 \varepsilon + \psi_3 \eta_n^3 \varepsilon^2 \right)^2 \\
  &= \psi_1^2 E_{\vartheta_0} \eta_n^2 + 2\psi_1 \psi_2 E_{\vartheta_0} \eta_n^3 \varepsilon + \left[ \psi_2^2 + 2\psi_1 \psi_3 \right] E_{\vartheta_0} \eta_n^4 \varepsilon^2 + q_n \varepsilon^3 + p_n \varepsilon^4,
\end{align*}
\]

where \( q_n \) and \( p_n \) are bounded sequences. Remind that

\[
\begin{align*}
  E_{\vartheta_0} \eta_n^2 &= \int_0^T g(t)^2 \lambda(\vartheta_0, t) \, dt \equiv a_2, \\
  E_{\vartheta_0} \eta_n^3 &= \frac{1}{\sqrt{n}} \int_0^T g(t)^3 \lambda(\vartheta_0, t) \, dt \equiv a_3 \varepsilon, \\
  E_{\vartheta_0} \eta_n^4 &= 3 \left( \int_0^T g(t)^2 \lambda(\vartheta_0, t) \, dt \right)^2 + \frac{1}{n} \int_0^T g(t)^4 \lambda(\vartheta_0, t) \, dt \equiv 3a_2^2 + a_4 \varepsilon^2,
\end{align*}
\]

where we denoted

\[
a_m = \int_0^T g(t)^m \lambda(\vartheta_0, t) \, dt, \quad m = 2, 3, 4.
\]

Therefore the first terms are

\[
nE_{\vartheta_0} \Psi_{0,n}^2 = a_2 \psi_1^2 + \left[ 2a_3 \psi_1 \psi_2 + 3a_2^2 \left( \psi_2^2 + 2\psi_1 \psi_3 \right) \right] \varepsilon^2 + q_n \varepsilon^3 + p_n \varepsilon^4,
\]

and we can write

\[
\left| nE_{\vartheta_0} (\tilde{\vartheta}_n - \vartheta_0)^2 - a_2 \psi_1^2 - \left[ 2a_3 \psi_1 \psi_2 + 3a_2^2 \left( \psi_2^2 + 2\psi_1 \psi_3 \right) \right] \varepsilon^2 \right| \leq C \varepsilon^{5/2}
\]

or

\[
\frac{n}{\psi_1^2 a_2} E_{\vartheta_0} (\tilde{\vartheta}_n - \vartheta_0)^2 = 1 + \left[ \frac{2\psi_2 a_3}{\psi_1 a_2} + \frac{3\psi_1^2 a_2}{\psi_1^2} + \frac{6\psi_3 a_2}{\psi_1} \right] \frac{1}{n} + \frac{R_n}{n^2},
\]

(25)

where \( R_n \) is bounded sequence. We see that the term of order \( n^{-3/2} \) is absent.

**Example 4.** Consider the model of 1-periodic Poisson process \( X_t, 0 \leq t \leq T \) with intensity function

\[
\lambda(\vartheta, t) = 2 \sin(2\pi t + \vartheta) + 3, \quad 0 \leq t \leq T = n,
\]

where \( \vartheta \in (\alpha, \beta), \ 0 < \alpha < \beta < \frac{\pi}{2} \). Hence we have \( n \) independent observations \( X_j = (X_j(t), 0 \leq t \leq 1) \), where \( X_j(t) = X_{j-1+t} - X_{j-1}, \ j = 1, \ldots, n. \)
Let us take $g(t) = \cos(2\pi t)$. Then

$m(\theta) = \sin \theta$, \quad \tilde{\vartheta}_n = \arcsin \left( \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{1} \cos(2\pi t) \, dX_j(t) \right)$,

$G(y) = \arcsin y$, \quad a_2 = \frac{3}{2}$, \quad a_3 = \frac{3}{4} \sin \vartheta$,

$\psi_1 = \frac{1}{\cos \vartheta}$, \quad $\psi_2 = \frac{\sin \vartheta}{2 \cos^3 \vartheta}$, \quad $\psi_3 = \frac{1 + 2 \sin^2 \vartheta}{6 \cos^5 \vartheta}$.

Suppose that the true value is $\vartheta_0 = \frac{\pi}{3}$. Then

$a_2 = \frac{3}{2}$, \quad $a_3 = \frac{3\sqrt{3}}{8}$, \quad $\psi_1 = 2$, \quad $\psi_2 = 2\sqrt{3}$, \quad $\psi_3 = \frac{40}{3}$.

The expansion of the second moment is

$$nE_{\frac{\pi}{3}} \left( \hat{\vartheta}_n - \frac{\pi}{3} \right)^2 = 6 + \frac{450}{n} + \frac{R_n}{n^4}$$

**Simulation.** Let us check the last expansion by numerical simulations. The limit value of the right-hand side is 6. Suppose that $n = 1000$, then

$$1000E_{\frac{\pi}{3}} \left( \hat{\vartheta}_{1000} - \frac{\pi}{3} \right)^2 = 6 + \frac{450}{1000} + \frac{R_{1000}}{10^4} \approx 6.5.$$  

The estimator $\hat{\vartheta}_{1000}$ was simulated 10 000 times, $\hat{\vartheta}_{m,1000}$, $m = 1, \ldots, 10000$. The empirical error obtained by this simulation

$$\frac{1}{N} \sum_{m=1}^{N} \left( \hat{\vartheta}_{m,1000} - \frac{\pi}{3} \right)^2 = 6.53$$

corresponds well to the value obtained by asymptotic expansion.

**5 Expansion of the distribution function**

We discuss below the non asymptotic expansions of the distribution function of the MME $\hat{\vartheta}_n$. We consider the cases $k = 1$ and $k = 2$ only, which correspond to the representations

$$\tilde{\vartheta}_n = \vartheta_0 + \left( \psi_1 \eta_n \varepsilon + \psi_2 \eta_n^3 \varepsilon^2 + \tilde{r}_n \varepsilon^{5/2} \right) I_{\{A\}} + \phi_n I_{\{A^c\}},$$

$$\tilde{\vartheta}_n = \vartheta_0 + \left( \psi_1 \eta_n \varepsilon + \psi_2 \eta_n^2 \varepsilon^2 + \psi_3 \eta_n^3 \varepsilon^3 + \tilde{r}_n \varepsilon^{7/2} \right) I_{\{A\}} + \phi_n I_{\{A^c\}},$$

$$15$$
where $\varepsilon = n^{-1/2}$, $|\bar{r}_n| < 1$, $|\phi_n| < \beta - \alpha$ and we have estimates (17). Of course, $\psi_l = \psi_l (\vartheta_0)$, $l = 1, 2, 3$, $\eta_n = \eta_n (\vartheta_0)$ and $\bar{r}_n = \bar{r}_n (\vartheta_0)$ but we omit this dependence to simplify the exposition. Moreover we keep the same notation for the different random variables $\bar{r}_n, \phi_n$ and the sets $A$ in (26) and (27). Our goal is to describe the first non Gaussian terms of these expansions. The proof in the case $k = 2$ we give in details. The proof in the case $k = 3$ is much more cumbersome that is why we obtain just formally the first two terms after Gaussian and for detailed proofs propose to read the section 3.4 in [15], where the similar expansions were studied.

Introduce the notation

$$\xi_n = \frac{1}{\sqrt{a_2 n}} \sum_{j=1}^{n} \int_{\mathbb{T}} g(t) d\pi_j(t) = \frac{\eta_n}{\sqrt{a_2}} \Longrightarrow N(0,1),$$

$$b_2 = \frac{\psi_2 \sqrt{a_2}}{\psi_1}, \quad b_3 = \frac{\psi_3 a_2}{\psi_1}, \quad \bar{r}_n = \frac{\bar{r}_n}{\psi_1 \sqrt{a_2}}, \quad \tilde{\phi}_n = \frac{\phi_n \sqrt{n}}{\psi_1 \sqrt{a_2}}.$$  

Then the stochastic expansion can be written as follows

$$\sqrt{n} \psi_1 a_2 (\tilde{\vartheta}_n - \vartheta_0) = \{ \xi_n + b_2 \xi_n^2 \varepsilon + b_3 \xi_n^3 \varepsilon^2 + \bar{r}_n \varepsilon^{5/2} \} \mathbb{1}_A + \tilde{\phi}_n \mathbb{1}_{\tilde{A}^c}.$$  

Our goal is to obtain the expansion of the probability

$$F_n(x) = P_{\vartheta_0} \left( \sqrt{n} \psi_1 a_2 (\tilde{\vartheta}_n - \vartheta_0) < x \right) = P_{\vartheta_0} (\xi_n + b_2 \xi_n^2 \varepsilon + b_3 \xi_n^3 \varepsilon^2 + \bar{r}_n \varepsilon^{5/2} < x, A) + P_{\vartheta_0} (\tilde{\phi}_n < x, \tilde{A}^c)$$

by the powers of $\varepsilon$. Let us denote

$$\Phi_{n,\varepsilon} = \xi_n + b_2 \xi_n^2 \varepsilon + b_3 \xi_n^3 \varepsilon^2 + \bar{r}_n \varepsilon^{5/2}, \quad \Phi_{0,\varepsilon} = \xi_n + b_2 \xi_n^2 \varepsilon + b_3 \xi_n^3 \varepsilon^2.$$  

Then we can write

$$P_{\vartheta_0} (\Phi_{n,\varepsilon} < x, A) \leq F_n(x) \leq P_{\vartheta_0} (\Phi_{n,\varepsilon} < x, A) + P_{\vartheta_0} (\tilde{A}^c).$$

Therefore, it is sufficient to study the probability $P_{\vartheta_0} (\Phi_{n,\varepsilon} < x, A)$. Recall that on the set $A$ we have $|\bar{r}_n| \leq K = (\psi_1 \sqrt{a_2})^{-1}$. Hence

$$P_{\vartheta_0} (\Phi_{n,\varepsilon} < x, A) \leq P_{\vartheta_0} (\xi_n + b_2 \xi_n^2 \varepsilon + b_3 \xi_n^3 \varepsilon^2 < x + K \varepsilon^{5/2}, A) = P_{\vartheta_0} (\Phi_{0,\varepsilon} < x + K \varepsilon^{5/2}, A)$$  

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and
\[ P_{\vartheta_0}(\Phi_{n,\varepsilon} < x, A) \geq P_{\vartheta_0}(\xi_n + b_2\xi_n^2\varepsilon + b_3\xi_n^3\varepsilon^2 < x - K\varepsilon^{5/2}, A) = P_{\vartheta_0}(\Phi_{0,\varepsilon} < x - K\varepsilon^{5/2}, A). \]

Further, we can write
\[ P_{\vartheta_0}(\Phi_{0,\varepsilon} < y, A) \leq P_{\vartheta_0}(\Phi_{0,\varepsilon} < y), \quad P_{\vartheta_0}(\Phi_{0,\varepsilon} < y, A) \geq P_{\vartheta_0}(\Phi_{0,\varepsilon} < y) - P_{\vartheta_0}(A^c). \]

Let us study the probabilities
\[ P_{\vartheta_0}^{(1)}(y) = P_{\vartheta_0}(\xi_n + b_2\xi_n^2\varepsilon < y), \quad P_{\vartheta_0}^{(2)}(y) = P_{\vartheta_0}(\xi_n + b_2\xi_n^2\varepsilon + b_3\xi_n^3\varepsilon^2 < y) \]
separately.

The expansion of the probability \( P_{\vartheta_0}^{(1)}(\cdot) \) we give with proof and for the probability \( P_{\vartheta_0}^{(2)}(\cdot) \) we present a formal expansion without detailed description of the reminder term.

### 5.1 Expansion of \( P_{\vartheta_0}^{(1)}(y) \)

**Proposition 1** Suppose that the conditions \( \mathcal{L}, \psi_1 > 0 \) and
\[ \inf_{a_2|v| > 1} \int_0^T \sin^2(vg(t))\lambda(\vartheta_0, t)\, dt > 0 \] hold. Then we have the expansion
\[ P_{\vartheta_0}\left(\frac{\sqrt{n}(\vartheta_n - \vartheta_0)}{\psi_1\sqrt{a_2}} < y\right) = F(y) + \left[ a_3 - \frac{a_3 + 8a_3^{3/2}b_2}{6a_2^{3/2}}\right] y^2 f(y)\varepsilon + o(\varepsilon). \]

**Proof.** First we recall the expansion of the distribution function of the stochastic integral (see [15], Theorem 3.4). We use the same notation as above
\[ \xi_n = \frac{1}{\sqrt{a_2n}} \sum_{j=1}^n \int_0^T g(t)\, d\pi_j(t), \quad b_2 = \frac{\psi_2\sqrt{a_2}}{\psi_1}, \quad \varepsilon = \frac{1}{\sqrt{n}}, \]
\[ a_m = \int_0^T g(t)^m\lambda(\vartheta_0, t)\, dt, \quad m = 2, 3, 4. \]
Suppose that the function $g(t)$ is non constant and satisfies (15). Then

$$
P_{\vartheta_0}(\xi_n < x) = F(x) + \frac{a_3}{6a_2^{3/2} \sqrt{2\pi}} (1 - x^2) e^{-x^2/2} + o(\varepsilon).
$$

We have

$$
P_n^{(1)}(y) = \int_{x+b_2 x^2 \varepsilon < y} dP_{\vartheta_0}(\xi_n < x).
$$

The condition $x+b_2 x^2 \varepsilon < y$ is equivalent to $b_2 \varepsilon (x - x_1)(x - x_2) < 0$, where the roots are

$$
x_1 = \frac{-1 - \sqrt{1 + 4b_2 \varepsilon y}}{2b_2 \varepsilon}, \quad x_2 = \frac{-1 + \sqrt{1 + 4b_2 \varepsilon y}}{2b_2 \varepsilon}.
$$

Suppose that $b_2 > 0$ and $y \in \mathbb{Y}$, where $\mathbb{Y}$ is some compact. Then for sufficiently small $\varepsilon$ we have

$$
x_1 = -\frac{1}{b_2 \varepsilon} \left(1 + O(\varepsilon^2)\right), \quad x_2 = y - b_2 y^2 \varepsilon + O(\varepsilon^2),
$$

and

$$
P_n^{(1)}(y) = P_{\vartheta_0}(\xi_n < y - b_2 y^2 \varepsilon + O(\varepsilon^2))
- P_{\vartheta_0}(\xi_n < -(b_2 \varepsilon)^{-1} (1 + O(\varepsilon^2)))
= P_{\vartheta_0}(\xi_n < y - b_2 y^2 \varepsilon) + O(\varepsilon^2)
$$

because $P_{\vartheta_0}(\xi_n < -(b_2 \varepsilon)^{-1} (1 + O(\varepsilon^2)))$ is exponentially small.

Further

$$
F(y - b_2 y^2 \varepsilon) = F(y) - b_2 y^2 f(y) \varepsilon + O(\varepsilon^2).
$$

Hence for $b_2 > 0$ we have

$$
P_n^{(1)}(y) = F(y) + \left[\frac{a_3}{6a_2^{3/2}} (1 - y^2) - b_2 y^2\right] f(y) \varepsilon + o(\varepsilon).
$$

(30)

If $b_2 < 0$, then the condition $x+b_2 x^2 \varepsilon < y$ is equivalent to two conditions

$x < x_2 = y - b_2 y^2 \varepsilon + O(\varepsilon^2)$ and $x > x_1 = -(b_2 \varepsilon)^{-1} (1 + O(\varepsilon^2)) \to +\infty$.

Therefore

$$
P_n^{(1)}(y) = P_{\vartheta_0}(\xi_n < y - b_2 y^2 \varepsilon + O(\varepsilon^2))
+ P_{\vartheta_0}(\xi_n > -(b_2 \varepsilon)^{-1} (1 + O(\varepsilon^2)))
= P_{\vartheta_0}(\xi_n < y - b_2 y^2 \varepsilon) + O(\varepsilon^2)
$$

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and we obtain the same expansion (30).

The expansion of the corresponding density function we obtain by formal derivation of the first two terms of the right hand side of (29)

\[ f^{(1)}_{0,\varepsilon} (y) = f(y) + \left[ b_2 y + B_3 (y^3 - 3y) \right] f(y) \varepsilon + o(\varepsilon), \]

where we denoted \( B_3 = a_3 a_2^{-3/2} 6^{-1} + b_2 \).

### 5.2 Expansion of \( P^{(2)}_n(y) \)

Now we consider the probability

\[ P_{\varnothing_0} \left( \sqrt{n} \left( \frac{\varnothing_n - \varnothing_0}{\psi_1 \sqrt{a_2}} \right) < y \right) = P_{\varnothing_0} (\Phi_{0,\varepsilon} < y) + o(\varepsilon^2), \]

where \( P_{\varnothing_0} (\Phi_{0,\varepsilon} < y) = P_{\varnothing_0} (\xi_n + b_2 \xi_n^2 \varepsilon + b_3 \xi_n^3 \varepsilon^2 < y) = P^{(2)}_n(y) \).

The characteristic function \( M_\varepsilon(v) = E_{\varnothing_0} \exp(i \varepsilon \Phi_{0,\varepsilon}) \) can be expanded by the powers of \( \varepsilon \) as follows

\[
M_\varepsilon(v) = E_{\varnothing_0} \exp \left\{ \varepsilon - \frac{1}{2} \int_T \left( e^{ivg t} - 1 - ivg t \right) \lambda_t dt \right\} = E_{\varnothing_0} \exp \left\{ \varepsilon - \frac{1}{2} \int_T g_t \lambda_t dt \right\}
\]

(32)

Here \( h(\cdot) \) and \( H_n(\cdot) \) are the corresponding residuais.

By direct calculation we obtain the values (below \( g_t = a_2^{-1/2} g(t) \) and \( \lambda_t = \lambda(\varnothing_t, t) \))

\[
E_{\varnothing_0} e^{iv \xi_n} = \exp \left\{ \varepsilon^{-2} \int_T \left( e^{ivg t} - 1 - ivg t \right) \lambda_t dt \right\} \equiv E(v, \varepsilon),
\]

\[
E_{\varnothing_0} \xi_n e^{iv \xi_n} = \left[ \varepsilon^{-2} \int_T \left( e^{ivg t} - 1 \right) g_t \lambda_t dt \right] E(v, \varepsilon),
\]

\[
E_{\varnothing_0} \xi_n^2 e^{iv \xi_n} = \left[ \varepsilon^{-1} \int_T \left( e^{ivg t} - 1 \right) g_t^2 \lambda_t dt \right] E(v, \varepsilon),
\]

where we denoted

\[
J_\varepsilon(v) = \varepsilon^{-1} \int_T \left( e^{ivg t} - 1 \right) g_t \lambda_t dt.
\]
The similar but cumbersome expressions we have for expectations $E_{\theta_0} e^{3iv\xi_n}$ and $E_{\theta_0} e^{iv\xi_n}$. As we need just the first terms corresponding $\varepsilon^0$ we do not give here these expressions.

According to the expression (32) we need expansion of $E_{\theta_0} e^{iv\xi_n}$ up to $\varepsilon^2$, the expansion of $E_{\theta_0} e^{2iv\xi_n}$ up to $\varepsilon$ and expansions of $E_{\theta_0} e^{3iv\xi_n}$ and $E_{\theta_0} e^{4iv\xi_n}$ up to the first term $\varepsilon^0$ only. Let us denote

$$\hat{a}_m = a_2^{-m/2} \int_T g(t)^m \lambda(\theta_0, t) \, dt, \quad m = 3, 4, \quad (\hat{a}_2 = 1)$$

and remark, that $J_\varepsilon(v) = iv + \frac{(iv)^2}{2} \hat{a}_3 \varepsilon + O(\varepsilon^2)$.

We have

$$E_{\theta_0} e^{iv\xi_n} = e^{-\frac{v^2}{2}} \left( 1 + \frac{(iv)^3}{6} \hat{a}_3 \varepsilon + \frac{(iv)^4}{24} \hat{a}_4 \varepsilon^2 + \frac{(iv)^5}{72} \hat{a}_3^2 \varepsilon^2 + O(\varepsilon^3) \right),$$

$$E_{\theta_0} e^{2iv\xi_n} = e^{-\frac{v^2}{2}} \left( 1 + (iv)^2 + \left[ (iv) + \frac{7}{6} (iv)^3 + \frac{5}{6} \hat{a}_3 \varepsilon \right] \hat{a}_3 \varepsilon \right) + O(\varepsilon^2),$$

$$E_{\theta_0} e^{3iv\xi_n} = e^{-\frac{v^2}{2}} \left[ (iv)^3 + 3 (iv) \right] + O(\varepsilon),$$

$$E_{\theta_0} e^{4iv\xi_n} = e^{-\frac{v^2}{2}} \left[ (iv)^4 + 6 (iv)^2 + 3 \right] + O(\varepsilon).$$

These expressions allow us to write the expansion of the characteristic function (32)

$$M_\varepsilon(v) = e^{-\frac{v^2}{2}} + (iv) e^{-\frac{v^2}{2}} B_1 \varepsilon + (iv)^2 e^{-\frac{v^2}{2}} B_2 \varepsilon^2 + (iv)^3 e^{-\frac{v^2}{2}} B_3 \varepsilon^3 + (iv)^4 e^{-\frac{v^2}{2}} B_4 \varepsilon^4 + (iv)^6 e^{-\frac{v^2}{2}} B_6 \varepsilon^6 + R_\varepsilon(v) \varepsilon^3,$$

where $R_\varepsilon(v) \varepsilon^3$ is the corresponding residual and we used the notation

$$B_1 = b_2, \quad B_2 = b_2 \hat{a}_3 + \frac{3}{2} b_2^2 + 3 b_3, \quad B_3 = \frac{1}{6} a_3 + b_2,$$

$$B_4 = \frac{1}{24} \hat{a}_4 + \frac{7}{6} b_2 \hat{a}_3 + b_3 + 3 b_2^2, \quad B_6 = \frac{1}{72} \hat{a}_4^2 + \frac{1}{6} b_2 \hat{a}_3 + \frac{1}{2} b_2^2.$$

Recall the relation

$$\frac{1}{2\pi} \int_R e^{-ivx} (iv)^m e^{-\frac{x^2}{2}} \, dv = H_m(x) f(x), \quad m = 1, 2, \ldots \quad (33)$$
where $H_m(\cdot)$ are Hermite polynomials. We have
\[
H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \\
H_4(x) = x^4 - 6x^2 + 3, \quad H_6(x) = x^6 - 15x^4 + 45x^2 - 15.
\] (34)

Let us denote
\[
M_{0,\varepsilon}(v) = e^{-v^2/2} + (iv)e^{-v^2/2}B_1\varepsilon + (iv)^2e^{-v^2/2}B_2\varepsilon^2 + (iv)^3e^{-v^2/2}B_3\varepsilon \\
+ (iv)^4e^{-v^2/2}B_4\varepsilon^2 + (iv)^6e^{-v^2/2}B_6\varepsilon^2.
\]

Using these relations we obtain the inverse Fourier transform of $M_{0,\varepsilon}(v)$
\[
f_{0,\varepsilon}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixv} M_{0,\varepsilon}(v) dv = f(x) + B_1H_1(x) f(x) \varepsilon \\
+ B_3H_3(x) f(x) \varepsilon + [B_2H_2(x) + B_4H_4(x) + B_6H_6(x)] f(x) \varepsilon^2.
\]

Introduce the function
\[
F_{0,\varepsilon}(x) = \int_{-\infty}^{x} f_{0,\varepsilon}(y) dy.
\]

It is possible to show that
\[
F_{n,\varepsilon}(x) = P_{\vartheta_0} \left( \sqrt{\frac{n}{\psi_1^{2}\psi_2}} (\hat{\vartheta}_n - \vartheta_0) < x \right) = F_{0,\varepsilon}(x) + O\left(\varepsilon^{\frac{5}{2}}\right).
\]

The proof of the convergence $O\left(\varepsilon^{\frac{5}{2}}\right)$ follows the same steps as the proof of the Theorem 3.4 in [15].

Therefore the functions
\[
F_{0,\varepsilon}^{(1)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy + \int_{-\infty}^{x} [B_1H_1(y) + B_3H_3(y)] f(y) dy \frac{1}{\sqrt{n}}
\]
and
\[
F_{0,\varepsilon}^{(2)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy + \int_{-\infty}^{x} [B_1H_1(y) + B_3H_3(y)] f(y) dy \frac{1}{\sqrt{n}} \\
+ \int_{-\infty}^{x} [B_2H_2(y) + B_4H_4(y) + B_6H_6(y)] f(y) dy \frac{1}{n}
\]
can be considered as approximations of the distribution function of the MME with the corresponding precisions respectively. Note that the expression (35) corresponds well to the expansion (31) obtained before.
It is interesting to obtain the approximation of the second moment of the MME

$$\frac{n}{\psi_1^2 a_2} E_{\vartheta_0} (\tilde{\vartheta}_n - \vartheta_0)^2 = \int_{\mathbb{R}} x^2 dF_{0, \varepsilon} (x) + O \left( \varepsilon^{\frac{5}{2}} \right)$$

with the help of the approximation of the distribution function. We have

\[
\int_{\mathbb{R}} x^2 f_{0, \varepsilon} (x) \, dx = 1 + B_2 \int_{\mathbb{R}} x^2 H_2 (x) f (x) \, dx \frac{1}{n} = 1 + \frac{2B_2}{n} \\
= 1 + \left( 2b_2 \tilde{a}_3 + 3b_3^2 + 6b_3 \right) \frac{1}{n} \\
= 1 + \left( \frac{2\psi_2 a_3}{\psi_1 a_2} + \frac{3\psi_2^2 a_2}{\psi_1^2} + \frac{6\psi_3 a_2}{\psi_1} \right) \frac{1}{n}.
\]

Here we used the properties of the Hermite polynomials

\[
\int_{\mathbb{R}} x^2 H_m (x) p_1 (x) \, dx = 0, \quad m = 1, 3, 4, 6.
\]

Hence

\[
\frac{n}{\psi_1^2 a_2} E_{\vartheta_0} (\tilde{\vartheta}_n - \vartheta_0)^2 = 1 + \left( \frac{2\psi_2 a_3}{\psi_1 a_2} + \frac{3\psi_2^2 a_2}{\psi_1^2} + \frac{6\psi_3 a_2}{\psi_1} \right) \frac{1}{n} + O \left( n^{-\frac{5}{2}} \right). \quad (36)
\]

We see that the expressions (25) and (36) coincide.

**Example 4** Suppose that the observed Poisson process is from the Example 4 and the true value is \( \vartheta_0 = \frac{\pi}{3} \). Then

\[
B_1 \left( \frac{\pi}{3} \right) = \frac{3\sqrt{2}}{2}, \quad B_3 \left( \frac{\pi}{3} \right) = \frac{11\sqrt{2}}{6}
\]

and we can write the approximation

\[
P_{\vartheta_0} \left( \sqrt{\frac{2n \cos \vartheta_0}{3}} \left( \tilde{\vartheta}_n - \vartheta_0 \right) < x \right) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} \, dy + \frac{B_1 (\vartheta_0)}{\sqrt{2\pi}} \int_{-\infty}^{x} ye^{-\frac{y^2}{2}} \, dy \frac{1}{\sqrt{n}} \\
+ \frac{B_3 (\vartheta_0)}{\sqrt{2\pi}} \int_{-\infty}^{x} (y^3 - 3y) e^{-\frac{y^2}{2}} \, dy \frac{1}{\sqrt{n}} \\
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} \, dy + \frac{3}{2\sqrt{\pi}} \int_{-\infty}^{x} ye^{-\frac{y^2}{2}} \, dy \frac{1}{\sqrt{n}} \\
+ \frac{11}{6\sqrt{\pi}} \int_{-\infty}^{x} (y^3 - 3y) e^{-\frac{y^2}{2}} \, dy \frac{1}{\sqrt{n}}.
\]
We have no approximation of the density function but can write formally the density of the approximating distribution function

\[ f_{0,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \left[ \frac{3x}{2\sqrt{\pi}} e^{-\frac{x^2}{2}} + \frac{11(x^3 - 3x)}{6\sqrt{\pi}} e^{-\frac{x^2}{2}} \right] \frac{1}{\sqrt{n}}. \]

Figure 1: Densities: standard Gaussian \( f(\cdot) \) and proposed approximation \( f_{0,500}(\cdot) \).

6 Discussion

The presented here results admit several generalizations. The case of vector parameter \( \vartheta \) can be treated by a similar way, but the exposition will be more complicate. The main idea - to have non asymptotic description of estimators can be applied to many different statistical models and many statistical estimators. It was already realized for some models of continuous time processes (Gaussian, diffusion, inhomogeneous Poisson) but we suppose that many other models, for example, time series can be studied using the same ideas.
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