HARMONIC MORPHISMS AND EIGENFAMILIES ON THE EXCEPTIONAL LIE GROUP $G_2$

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Abstract. We construct harmonic morphisms on the compact simple Lie group $G_2$. The construction uses eigenfamilies in a representation theoretic scheme.

1. Introduction

A harmonic morphism is a map between two Riemannian manifolds that pulls back local real-valued harmonic functions to local real-valued harmonic functions. The simplest examples of harmonic morphisms are constant maps, real-valued harmonic functions and isometries. A characterization of harmonic morphisms are constant maps, real-valued harmonic functions and isometries. A characterization of harmonic morphisms was given by Fuglede and Ishihara, they show in [3] and [9], respectively, that the harmonic morphisms are exactly the harmonic horizontally weakly conformal maps. If we restrict our attention to the maps where the codomain is a surface then the harmonic morphisms are horizontally weakly conformal maps with minimal fibers. Good references for harmonic morphisms and there properties are [1] and [7].

In [5], Gudmundsson and Sakovich develop a method for constructing complex-valued harmonic morphisms on the classical compact Lie groups by looking at eigenfamilies, simultaneous eigenfunction of the Laplacian and the conformality operator, see Theorem 2.5. We give some further properties of such eigenfamilies and extend the concept to maximal eigenfamilies.

In [6], the authors show how to construct these eigenfamilies using representation theory. The Peter-Weyl theorem gives the eigenfunctions of the Laplacian and decompositions into irreducible representation and Schur’s lemma gives eigenfunctions of the conformality operator. We show in this paper that the method works for the exceptional Lie group $G_2$ as well. We make use of the 7-dimensional cross product to control the behavior of the conformality operator. Our main result is the following.

Theorem 1.1. Let $\rho : G_2 \to \text{Aut}(\mathbb{C}^7)$ be the standard representation of the exceptional compact Lie group $G_2$. For any $a, b \in \mathbb{C}^7$ define

$$\phi_{a\mathbb{C}^7}(g) = \langle \rho(g)a, b \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the complex-bilinear extension of the standard scalar product on $\mathbb{R}^7$. Suppose $p \in \mathbb{C}^7$ satisfies $\langle p, p \rangle = 0$. Then

$$\mathcal{E}_p = \{\phi_{a\mathbb{C}^7} | a \in \mathbb{C}^7\}$$

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is an eigenfamily on $G_2$, i.e. there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\Delta(\phi) = \lambda \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \phi \psi$$

for all $\phi, \psi \in E_p$.

2. Maximal eigenfamilies

In this section we present the notion of an eigenfamily introduced in [5]. We give some further properties of eigenfamilies and extend the notion to maximal eigenfamilies, with the hope that this may help with future classifications.

Let $(M, g)$ be a Riemannian manifold. For functions $\phi, \psi : (M, g) \to \mathbb{C}$ we define the Laplacian $\Delta$ and the conformality operator $\kappa$ by

$$\Delta(\phi) = \operatorname{div}(\operatorname{grad}(\phi))$$

and

$$\kappa(\phi, \psi) = g(\operatorname{grad}(\phi), \operatorname{grad}(\psi)),$$

here $g$ is the complex-bilinear extension of the metric $g$ and $\operatorname{grad}(\phi), \operatorname{grad}(\psi)$ are sections of the complexified tangent bundle $T^C M$. It is well known that the Laplacian acts as

$$\Delta(\phi \psi) = \psi \Delta(\phi) + 2 \kappa(\phi, \psi) + \phi \Delta(\psi)$$
on a product of two functions. If $\kappa(\phi, \phi) = 0$ then $\phi$ is horizontally conformal and if $\Delta(\phi) = 0$ then $\phi$ is a harmonic map and if $\kappa(\phi, \phi) = \Delta(\phi) = 0$ then $\phi$ is a harmonic morphism.

Given $\lambda \in \mathbb{C}$ let $F_\lambda$ be the eigenfunctions of the Laplacian with eigenvalue $\lambda$

$$F_\lambda = \{ \phi : (M, g) \to \mathbb{C} | \Delta(\phi) = \lambda \phi \}$$

and given $\lambda, \mu \in \mathbb{C}$ let $F_{\lambda, \mu}$ be the simultaneous eigenfunctions of the Laplacian and the conformality operator with eigenvalues $\lambda$ and $\mu$ respectively,

$$F_{\lambda, \mu} = \{ \phi \in F_\lambda | \kappa(\phi, \phi) = \mu \phi \phi \}.$$

Even if $\phi, \psi \in F_{\lambda, \mu}$ it may still happen that $\kappa(\phi, \psi) \neq \mu \phi \psi$.

**Definition 2.1.** [5] Let $(M, g)$ be a Riemannian manifold and $\lambda, \mu \in \mathbb{C}$. A subset $E_{\lambda, \mu}$ of $F_{\lambda, \mu}$ such that $\kappa(\phi, \psi) = \mu \phi \psi$ for all $\phi, \psi \in E_{\lambda, \mu}$ is said to be an eigenfamily.

In fact, a subset $E \subseteq F_{\lambda, \mu}$ is an eigenfamily if and only if, $\phi, \psi \in E$ imply $\phi + \psi \in F_{\lambda, \mu}$, since

$$\kappa(\phi, \psi) = \frac{1}{2}(\kappa(\phi + \psi, \phi + \psi) - \kappa(\phi, \phi) - \kappa(\psi, \psi)) = \mu \phi \psi.$$

$F_{\lambda, \mu}$ is closed under composition by isometries, i.e. if $\phi \in F_{\lambda, \mu}$ and $F$ is an isometry then $F^*(\phi) = \phi \circ F \in F_{\lambda, \mu}$.

**Proposition 2.2.** Let $(M, g)$ be a Riemannian manifold and $E_{\lambda, \mu}$ be an eigenfamily on $M$. Then $\operatorname{span}_C E_{\lambda, \mu}$ and $\overline{E_{\lambda, \mu}} = \{ \overline{\phi} | \phi \in E_{\lambda, \mu} \}$ are also eigenfamilies.

**Proof.** This follows from the fact that $\Delta$ is linear and $\kappa$ is bilinear. □

Given an eigenfamily it is easy to produce more eigenfamilies by taking the symmetric product, as in the next definition and lemma.
Definition 2.3. Let \((M,g)\) be a Riemannian manifold and \(\mathcal{E}_{\lambda,\mu}\) be an eigenfamily on \(M\). Define \(\otimes^n\mathcal{E}_{\lambda,\mu}\) by
\[
\otimes^n\mathcal{E}_{\lambda,\mu} = \text{span}_C\{\phi_1 \cdots \phi_n | \phi_i \in \mathcal{E}_{\lambda,\mu} \text{ for } i = 1 \ldots n\}.
\]

Lemma 2.4. \([5]\) Let \((M,g)\) be a Riemannian manifold and \(\mathcal{E}_{\lambda,\mu}\) be an eigenfamily on \(M\). If \(\phi \in \otimes^k\mathcal{E}_{\lambda,\mu}\) and \(\psi \in \otimes^l\mathcal{E}_{\lambda,\mu}\) then
\[
\kappa(\phi,\psi) = kl\mu\phi\psi \quad \text{and} \quad \Delta(\phi) = k(\lambda + (k-1)\mu)\phi.
\]
In particular, \(\otimes^n\mathcal{E}_{\lambda,\mu}\) is an eigenfamily.

Proof. The proof is by induction. The statement is obvious for \(k = l = 1\). Assume \(\kappa(\phi_i,\psi_j) = ij\mu\phi_i\psi_j\), for all \(\phi_i \in \otimes^i\mathcal{E}\) and all \(\psi_j \in \otimes^j\mathcal{E}\) for \(i \leq k\) and \(j \leq l\). We will show that the statement is true for all \(i \leq k+1\) and all \(j \leq l+1\). Let \(\theta, \Theta \in \mathcal{E}_{\lambda,\mu}\) then
\[
\kappa(\phi_k\theta,\psi_l) = g(\theta\text{grad}(\phi_k) + \phi_k\text{grad}(\theta), \text{grad}(\psi_l))
\]
\[
= \theta g(\text{grad}(\phi_k), \text{grad}(\psi_l)) + \phi_k g(\text{grad}(\theta), \text{grad}(\psi_l))
\]
\[
= (kl\mu + l\mu)\theta\phi_k\psi_l
\]
\[
= \mu(k+1)\theta\phi_k\psi_l.
\]
Similarly \(\kappa(\phi_k,\theta\psi_l) = k(l+1)\mu\phi_k\psi_l\), \(\kappa(\phi_k\Theta,\psi_l) = (k+1)(l+1)\mu\Theta\phi_k\psi_l\). For the Laplacian we have
\[
\Delta(\phi_k\theta) = \phi_k\Delta(\theta) + 2\kappa(\phi_k,\theta) + \theta\Delta(\phi_k)
\]
\[
= \lambda\phi_k\theta + 2\mu k\phi_k\theta + k(\lambda + (k-1)\mu)\phi_k\theta
\]
\[
= (k+1)(\lambda + k\mu)\phi_k\theta. \quad \square
\]

The following result from \([5]\) details how eigenfamilies are used to produce harmonic morphisms.

Theorem 2.5. Let \((M,g)\) be a Riemannian manifold, \(\mathcal{E}_{\lambda,\mu}\) an eigenfamily on \(M\) and \(\{\phi_1, \ldots, \phi_n\}\) a basis for \(\text{span}_C\mathcal{E}_{\lambda,\mu}\). Then
\[
\frac{P(\phi_1, \ldots, \phi_n)}{Q(\phi_1, \ldots, \phi_n)}
\]
is a harmonic morphism on \(M\backslash\{p \in M | Q(\phi_1, \ldots, \phi_n)(p) = 0\}\) for any linearly independent homogeneous polynomials \(P, Q : \mathbb{C}^n \to \mathbb{C}\) of the same degree \(d > 0\).

In order to obtain non-constant harmonic morphisms we need eigenfamilies with at least two dimensional spans.

Now we will state some further properties of eigenfamilies on compact manifolds, we will start with the possible eigenvalues.

Proposition 2.6. Let \((M,g)\) be a compact Riemannian manifold. If \(\mathcal{F}_{\lambda,\mu} \neq \{0\}\) then \(\lambda, \mu \in \mathbb{R}\) and \(\lambda, \mu \leq 0\).

Proof. With the sign convention, for the Laplacian, that we have chosen it is well known that for compact manifolds any eigenvalue \(\lambda\) has to be real and \(\lambda > 0\) imply \(\mathcal{F}_{\lambda} = \{0\}\).
Let $\phi \in \mathcal{F}_{\lambda,\mu}$ be a non-zero function. Then $\phi^2 \in \mathcal{F}_{2(\lambda+\mu),4\mu}$ is non-zero. If $\mu$ is not real then $\phi^2$ will have a complex eigenvalue of the Laplacian, which is a contradiction. If $\mu > 0$ then there exist a $k_0 \in \mathbb{Z}^+$ such that $\lambda + (2k_0 - 1)\mu > 0$. Then
\[
\Delta(\phi^{2k_0}) = 2k_0(\lambda + (2k_0 - 1)\mu)\phi^{2k_0}
\]
so $\phi^{2k_0} \in \mathcal{F}_{2k_0(\lambda+(2k_0-1)\mu)}$ but $\mathcal{F}_{2k_0(\lambda+(2k_0-1)\mu)} = \{0\}$ since $2k_0(\lambda + (2k_0 - 1)\mu) > 0$. This is a contradiction. 

The following result shows the importance of letting the functions be complex valued.

**Proposition 2.7.** Let $(M,g)$ be a compact Riemannian manifold. If a function $\phi \in \mathcal{F}_{\lambda,\mu}$ is real-valued then it is constant.

**Proof.** Since $\phi$ is real-valued $\kappa(\phi,\phi) = |\nabla(\phi)|^2 = \mu|\phi|^2$, so $\mu \geq 0$. But we know that $\mu = 0$. So $\mu = 0$ which imply $|\nabla(\phi)|^2 = 0$ and $\phi$ is constant. 

Any subset of an eigenfamily is again an eigenfamily, so to classify them we want to find maximal subspaces of $\mathcal{F}_\lambda$ that are eigenfamilies.

**Definition 2.8.** Suppose $\mathcal{E}_{\lambda,\mu}$ is an eigenfamily and that for all $\theta \in \mathcal{F}_{\lambda,\mu}|\mathcal{E}_{\lambda,\mu}$ there exists a $\phi \in \mathcal{E}_{\lambda,\mu}$ such that $\kappa(\theta,\phi) \neq \mu \theta \phi$. Then we call $\mathcal{E}_{\lambda,\mu}$ a maximal eigenfamily.

Since the span of an eigenfamily is also an eigenfamily, maximal eigenfamilies are vector spaces. In fact they are the largest possible linear subsets of $\mathcal{F}_{\mu,\lambda}$.

**Proposition 2.9.** Let $(M,g)$ be a compact Riemannian manifold and $\lambda < 0$ and $\mathcal{E}_{\lambda,\mu}$ an eigenfamily. Then $\mathcal{E}_{\lambda,\mu} \cap \mathcal{E}_{\lambda,\mu} = \{0\}$.

**Proof.** Let $\phi \in \mathcal{E}_{\lambda,\mu} \cap \mathcal{E}_{\lambda,\mu}$. Then $\phi, \bar{\phi} \in \mathcal{E}_{\lambda,\mu}$ so $|\phi|^2 = \bar{\phi}\phi \in \mathcal{F}_{2(\lambda+\mu),4\mu}$ but since it is real-valued it must be constant. Then $\lambda < 0$ imply $\phi = 0$. 

We see that the dimension of a maximal eigenfamily $\mathcal{E}_{\lambda,\mu}$ must be less than or equal to half the dimension of $\mathcal{F}_{\lambda}$.

### 3. Constructing Eigenfamilies on Compact Lie Groups

In this section we describe the method to obtain eigenfamilies using unitary representations of Lie groups given in [6]. Let $G$ be a compact Lie group with a bi-invariant metric. By the Theorem of Highest Weight, see Theorem 5.110 in [10], the irreducible representations of $G$ are in one-to-one with the dominant algebraically integral roots, we denote this set of roots by $\Gamma$. The correspondence is such that each $\gamma \in \Gamma$ is the highest weight for the representation $\rho_\gamma : G \to \text{Aut}(V_\gamma)$. By the Peter-Weyl theorem, see Theorem 4.20 in [10], we have
\[
L^2(G) = \bigoplus_{\gamma \in \Gamma} M(V_\gamma),
\]
where $M(V_\gamma)$ is the set of functions spanned by the elements of the matrix $\rho_\gamma(g) : V_\gamma \to V_\gamma$. From now on we suppress $\gamma$ and $\rho$ denotes any irreducible representation of $G$.

Given $a, b \in V^C$ define $\phi_{ab} : G \to \mathbb{C}$ by $\phi_{ab}(g) = \langle \rho(g)a, b \rangle$. Then $\phi_{ab} \in M(V)$, in fact these function span all of $M(V)$. Let $\sigma : \mathfrak{g} \to \text{End}(V)$ denote the Lie algebra representation related to $\rho$. Then we have $X_\sigma(\phi_{ab}) = \langle \rho(g)\sigma(X)a, b \rangle$ for any left-invariant vector field $X \in \mathfrak{g}$. 



Let \( \{X_i\}_{i=1}^n \) be an orthonormal basis for the Lie algebra \( \mathfrak{g} \) of \( G \). Then \( \{-X_i\}_{i=1}^n \) is its dual basis with respect to the Killing form. The Laplacian acts on \( \phi_{ab} \) as
\[
\Delta(\phi_{ab})(g) = \sum_i X_i(X_i(\phi_{ab}))(g)
= \sum_i \langle \rho(g)\sigma(X_i)^2a,b \rangle
= \langle \rho(g)\sum_i \sigma(X_i)^2a,b \rangle
= -\langle \rho(g)\sum_i \sigma(X_i)\sigma(-X_i)a,b \rangle.
\]
Now \( \sum_i \sigma(X_i)\sigma(-X_i) \) is the Casimir operator of the representation \( \sigma \). Since the representation is irreducible it is a scalar times the identity. The scalar is in fact, see Theorem 5.28 in [10], \( \langle \gamma,\gamma + 2\delta \rangle \), where \( \gamma \) is the highest weight and \( \delta \) is half the sum of the positive roots. Thus
\[
\Delta(\phi_{ab}) = \lambda \phi_{ab},
\]
where \( \lambda = -\langle \gamma,\gamma + 2\delta \rangle \). In the notation of the previous section \( F_\lambda = M(V_\gamma) \).

Similarly \( \kappa \) is given by
\[
\kappa(\phi_{ab},\phi_{cd})(g) = \sum_{i=1}^n (X_i)_g(\phi_{ab})(X_i)_g(\phi_{cd})
= \sum_{i=1}^n \langle \rho(g)\sigma(X_i)a,b \rangle \langle \rho(g)\sigma(X_i)c,d \rangle.
\]
Define \( Q \) to be the value of \( \kappa(\phi_{ab},\phi_{cd}) \) in the identity element \( e \in G \), i.e.
\[
Q(a,b,c,d) = \kappa(\phi_{ab},\phi_{cd})(e) = \sum_i \langle \sigma(X_i)a,b \rangle \langle \sigma(X_i)c,d \rangle.
\]
If we assume that the representation \( \rho \) is of real type then
\[
\kappa(\phi_{ab},\phi_{cd})(g) = \sum_i \langle \rho(g)\sigma(X_i)a,b \rangle \langle \rho(g)\sigma(X_i)c,d \rangle
= \sum_i \langle \rho(g)\sigma(X_i)\rho(g)^{-1}\rho(g)a,b \rangle \langle \rho(g)\sigma(X_i)\rho(g)^{-1}\rho(g)c,d \rangle
= \sum_i \langle \sigma(Y_i)\rho(g)a,b \rangle \langle \sigma(Y_i)\rho(g)c,d \rangle
= Q(\rho(g)a,b,\rho(g)c,d),
\]
since \( \{Y_i = gX_ig^{-1} \} \) is another orthonormal basis for \( \mathfrak{g} \).

\( Q \) is skew-symmetric in \( a,b \) and in \( c,d \) and symmetric in the pairs \( (a,b), (c,d) \), thus we can consider \( Q \) as a symmetric map \( Q : \Lambda^2 V_\lambda \times \Lambda^2 V_\lambda \rightarrow \mathbb{R} \). By extending the scalar product to be complex bilinear we can see \( Q \) as a symmetric map \( Q : \Lambda^2 V_\lambda^c \times \Lambda^2 V_\lambda^c \rightarrow \mathbb{C} \).

**Theorem 3.1.** [6] Let \( G \) be a compact Lie group and \( \rho : G \to \text{Aut}(V) \) be a unitary representation of real type. If there exists a \( \mu \in \mathbb{R} \) such that \( Q(a \wedge p,b \wedge p) = \mu \langle a \wedge p,b \wedge p \rangle \) for all \( a,b,p \in V^c \), then \( \mathcal{E}(q) = \{ \phi_{aq}(g) = \langle \rho(g)a,q \rangle | a \in V^c \} \) is an eigenfamily for any \( q \in V^c \) such that \( \langle q,q \rangle = 0 \).
Proof. We already know that $\Delta(\phi_{ab}) = \lambda \phi_{ab}$ by the Peter-Weyl theorem. For any $g \in G$

$$\kappa(\phi_{aq}, \phi_{ bq})(g) = Q(\rho(g) a \wedge q, \rho(g) b \wedge q)
= \mu \langle \rho(g) a \wedge q, \rho(g) b \wedge q \rangle
= \mu (\langle \rho(g) a, \rho(g) b \rangle q, q - \langle \rho(g) a, q \rangle \langle \rho(g) b, q \rangle)
= - \mu \phi_{aq}(g) \phi_{bq}(g).$$

For a Euclidean vector space $V$ define the isomorphism $R : \wedge^2 V \to \mathfrak{so}(V)$ by

$$R(a \wedge b)(v) = \langle a, v \rangle b - \langle b, v \rangle a.$$  

Define the exterior square representation $\tilde{\rho} : G \to \text{Aut}(\wedge^2 V)$ by

$$\tilde{\rho}(g)(a \wedge b) = (\rho(g) a) \wedge (\rho(g) b)$$

and the representation $\hat{\rho} : G \to \text{Aut}(\mathfrak{so}(V))$ by

$$\hat{\rho}(g)(A) = \rho(g) A \rho(g)^{-1}.$$  

These representations are equivalent since

$$(\hat{\rho}(g) R(a \wedge b))(v) = \rho(g) R(a \wedge b)(\rho(g)^{-1} v)
= \rho(g)(\langle a, \rho(g)^{-1} v \rangle b - \langle b, \rho(g)^{-1} v \rangle a)
= \langle \rho(g) a, v \rangle \rho(g) b - \langle \rho(g) b, v \rangle \rho(g) a
= R(\tilde{\rho}(g)(a \wedge b))(v).$$

The scalar product on $\wedge^2 V$ is given by

$$\langle a \wedge b, c \wedge d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle$$

and the scalar product on $\mathfrak{so}(V)$ by

$$\langle A, B \rangle = \sum_i \langle A(e_i), B(e_i) \rangle = \text{trace}(A^t B).$$

Lemma 3.2. Let $V$ be a Euclidean space, $A \in \mathfrak{so}(V)$ and $a, b, c, d \in V$. Then

(i) $\{ A, R(a \wedge b) \} = 2 \{ A(a), b \}$, and

(ii) $\{ R(a \wedge b), R(c \wedge d) \} = 2 \langle a \wedge b, c \wedge d \rangle$.

Proof. By the definition of the scalar product

$$\{ A, R(a \wedge b) \} = \sum_i \langle A(e_i), \langle a, e_i \rangle b - \langle b, e_i \rangle a \rangle
= \sum_i \langle \langle A(\langle a, e_i \rangle e_i), b \rangle - \langle A(\langle b, e_i \rangle e_i), a \rangle \rangle
= 2 \langle A(a), b \rangle.$$  

Set $A = R(c \wedge d)$, then

$$\{ R(a \wedge b), R(c \wedge d) \} = 2 \langle R(a \wedge b)(c), d \rangle
= 2 \langle \{ a, c \} b - \{ b, c \} a, d \rangle
= 2 \langle \{ a, c \} \{ b, d \} - \{ b, c \} \{ a, d \} \rangle.$$
\[=2\langle a \wedge b, c \wedge d \rangle.\]

The next result tells us that \(Q\) acts as a projection onto the Lie algebra of \(G\).

**Theorem 3.3.** [6] Let \(G\) be a Lie group and \(\rho : G \to \text{Aut}(V)\) be a representation of real type. Then for all \(a, b, c, d \in V\)

\[Q(a \wedge b, c \wedge d) = \langle P_\rho(a \wedge b), c \wedge d \rangle\]

where \(P_\rho\) is the projection onto the Lie algebra \(g\) of \(G\).

**Proof.** Let \(a, b, c, d \in V\) and \(\{X_i\}_i^n\) be an orthonormal basis for \(g\). The scalar product on \(g\) is such that the Lie algebra representation \(\sigma : g \to \text{so}(V)\) is an isometry onto its image, meaning \((\sigma(X_i), \sigma(X_j)) = \delta_{ij}\).

\[Q(a \wedge b, c \wedge d) = \sum_{i=1}^n \langle \sigma(X_i)a, b \rangle \langle \sigma(X_i)c, d \rangle\]

\[= \frac{1}{4} \sum_{i=1}^n \langle \sigma(X_i), R(a \wedge b) \rangle \langle \sigma(X_i), R(c \wedge d) \rangle\]

\[= \frac{1}{4} \sum_{i=1}^n \langle \sigma(X_i), R(a \wedge b) \rangle \langle \sigma(X_i), R(c \wedge d) \rangle\]

\[= \frac{1}{4} \langle P_\rho(R(a \wedge b)), R(c \wedge d) \rangle\]

\[= \frac{1}{2} \langle P_\rho(a \wedge b), c \wedge d \rangle.\]

Everything will still hold if we take \(a, b, c, d \in V^C\) and use the complex bilinear extension of the scalar product.

**4. Eigenfamilies on \(G_2\)**

In this section we show that for \(G = G_2\) the conditions of Theorem 3.1 are satisfied, hence there exist eigenfamilies on \(G_2\). Let \(\rho : G_2 \to \text{Aut}(V)\) denote the standard representation of \(G_2\). The exterior square representation \(\tilde{\rho} : G_2 \to \text{Aut}(\wedge^2 V)\) has two irreducible subspaces \(\wedge^2 V = g_2 \oplus W\), see page 353 in [4]. As \(\dim(g_2) = 14\) we have \(\dim(W) = 7\).

The following result is standard, see page 408 in [2].

**Lemma 4.1.** Let \(V\) be a seven dimensional Euclidean space and \(\times : V \times V \to V\) be the seven dimensional cross-product. Then

\[\langle v \times u, w \rangle = -\langle u, v \times w \rangle\]

and

\[u \times (v \times w) + v \times (u \times w) = \langle u, w \rangle v + \langle v, w \rangle u - 2\langle u, v \rangle w\]

and for all \(u, v, w \in V\).

In fact \(G_2\) is exactly the set \(\{g \in SO(7) \mid g(v \times w) = (gv) \times (gw)\}\). Next we present 3 lemmas that describe the interaction between the seven dimensional cross product, the wedge product and the projection onto \(W\).
Lemma 4.2. Let $V$ be a seven dimensional Euclidean space and $\times : V \times V \rightarrow V$ be the seven dimensional cross product. Then

\[
(a \times u, b \times v) + (a \times v, b \times u) = \langle a \wedge u, b \wedge v \rangle + \langle a \wedge v, b \wedge u \rangle
\]

for all $a, b, u, v \in V$.

Proof. Using Lemma 4.1 we find that

\[
\langle a \times u, b \times v \rangle + \langle a \times v, b \times u \rangle = \langle a, u \times (b \times v) \rangle + \langle a, v \times (b \times u) \rangle
\]

\[
= \langle a, u \times (b \times v) + v \times (b \times u) \rangle
\]

\[
= -\langle a, u \times (v \times b) + v \times (u \times b) \rangle
\]

\[
= -\langle a, (u, b)v + (b, v)u - 2(u, v)b \rangle
\]

\[
= \langle a \wedge u, b \wedge v \rangle + \langle a \wedge v, b \wedge u \rangle.
\]

\[\square\]

Lemma 4.3. Let $V$ be a seven dimensional Euclidean space and $\times : V \times V \rightarrow V$ be the seven dimensional cross product. Define the operator $L : V \rightarrow \mathfrak{so}(V)$, $v \mapsto L_v$, where

\[
L_v(w) = v \times w
\]

for $v, w \in V$. Then $L(V) = R(W)$ and $\{L_v, L_w\} = \langle 6(v, w) \rangle$.

Proof. For the first part, $L_v = 0$ means that $v \times w = 0$ for all $w$ which implies $v = 0$, so $L$ is injective and therefore $L(V)$ is seven dimensional. Now

\[
(\hat{\rho}(g)L_v)(w) = (\rho(g)L_v\rho(g)^{-1})(w)
\]

\[
= \rho(g)(v \times (\rho(g)^{-1}w))
\]

\[
= \langle \rho(g)v \rangle \times w
\]

\[
= L_{\rho(g)v}(w).
\]

This means that $\hat{\rho}(g)L_v \in L(V)$ for all $g \in G_2$ and $v \in V$ thus $L(V)$ is an invariant 7-dimensional subspace of $\mathfrak{so}(V)$. Since $\mathfrak{g}_2$ is an irreducible 14-dimensional invariant subspace we must have $L(V) = R(W)$.

For the second part, let $\{e_i\}_{i=1}^7$ be an orthonormal basis for $V$. Then

\[
\{L_v, L_w\} = \sum_i \{L_v(e_i), L_w(e_i)\}
\]

\[
= \sum_i \{v \times e_i, w \times e_i\}
\]

\[
= \sum_i \{e_i \times v, e_i \times w\}
\]

\[
= -\sum_i \{v, e_i \times (e_i \times w)\}
\]

\[
= -\sum_i \{v, -(e_i, e_i)w + \{e_i, w\}e_i\}
\]

\[
= \sum_i \{(v, w)\{e_i, e_i\} - \{v, e_i\}\{w, e_i\}\}
\]

\[
= 7(v, w) - \{v, w\}
\]

\[
= 6(v, w).
\]

\[\square\]
Lemma 4.4. Let \( V \) be a seven dimensional Euclidean space and \( \times : V \times V \to V \) be the seven dimensional cross-product. Let \( a, b \in V \) then

\[
P_{R(W)}(R(a \wedge b)) = \frac{1}{3}L_{axb}.
\]

Proof. Let \( \{e_k\} \) be an orthonormal basis for \( V \). Then \( \{\frac{1}{\sqrt{6}}L_{e_k}\} \) is an orthonormal basis for \( R(W) \) so

\[
P_{R(W)}(R(a \wedge b)) = \frac{1}{6}\sum_{i,k}\{R(a \wedge b)(e_i), e_k \times e_i\}L_{e_k}
\]

\[
= \frac{1}{6}\sum_{i,k}\{R(a \wedge b)(e_i), e_i \times e_k\}L_{e_k}
\]

\[
= \frac{1}{6}\sum_{i,k}\{e_i \times R(a \wedge b)(e_i), e_k\}L_{e_k}
\]

\[
= \frac{1}{6}\sum_{i,k}\{(a, e_i)b - (b, e_i)a, e_k\}L_{e_k}
\]

\[
= \frac{1}{6}\sum_{i,k}\{(a, e_i)e_i \times b - (b, e_i)e_i \times a, e_k\}L_{e_k}
\]

\[
= \frac{1}{6}\sum_{i,k}\{2a \times b, e_k\}L_{e_k}
\]

\[
= \frac{1}{3}L_{axb}.
\]

Thus for all \( a, b, c, d \in V \)

\[
Q(a \wedge b, c \wedge d) = \frac{1}{4}\{P_{g_2}(R(a \wedge b)), R(c \wedge d)\}
\]

\[
= \frac{1}{4}\{R(a \wedge b) - P_{R(W)}(R(a \wedge b)), R(c \wedge d)\}
\]

\[
= \frac{1}{4}\{(R(a \wedge b), R(c \wedge d)) - \{P_{R(W)}(R(a \wedge b)), P_{R(W)}(R(c \wedge d))\}\}
\]

\[
= \frac{1}{2}\langle a \wedge b, c \wedge b \rangle - \frac{1}{4}\langle \frac{1}{3}L_{axb}, \frac{1}{3}L_{cxd} \rangle
\]

\[
= \frac{1}{2}\langle a \wedge b, c \wedge b \rangle - \frac{1}{6}\langle a \times b, c \times d \rangle.
\]

The next result details how \( Q \) works on the set \( \{a \wedge p | a \in V^C\} \) where \( p \in V^C \) is given.

Theorem 4.5. Let \( \rho : g_2 \to \text{Aut}(V) \) be the standard representation of \( g_2 \) and \( \sigma : g_2 \to \text{End}(V) \) be the corresponding representation of its Lie algebra \( g_2 \). Define \( Q : \Lambda^2 V^C \to \Lambda^2 V^C \) by

\[
Q(a \wedge b, c \wedge d) = \sum_{i=1}^{14}\langle \sigma(X_i)a, b \rangle \langle \sigma(X_i)c, d \rangle
\]
for \(a, b, c, d \in V^c\), where \(\{X_i\}_{i=1}^{14}\) is an orthonormal basis for \(\mathfrak{g}_2\). Then

\[
Q(a \wedge p, b \wedge p) = \frac{1}{3} \langle a \wedge p, b \wedge p \rangle
\]

for all \(a, b, p \in V^c\).

**Proof.** Let \(p = u + iv \in V^c\) and \(a, b \in V\). Then

\[
Q(a \wedge (u + iv), b \wedge (u + iv)) = Q(a \wedge u, b \wedge u) - Q(a \wedge v, b \wedge v) + i(Q(a \wedge u, b \wedge v) + Q(a \wedge v, b \wedge u))
\]

\[
= \left(\frac{1}{2} \langle a \wedge u, b \wedge u \rangle - \frac{1}{6} \langle a \wedge u, b \wedge u \rangle \right)
\]

\[
- \left(\frac{1}{2} \langle a \wedge v, b \wedge v \rangle - \frac{1}{6} \langle a \wedge v, b \wedge v \rangle \right)
\]

\[
+ i\left(\frac{1}{2} \langle a \wedge u, b \wedge v \rangle - \frac{1}{6} \langle a \wedge u, b \wedge v \rangle \right)
\]

\[
+ i\left(\frac{1}{2} \langle a \wedge v, b \wedge u \rangle - \frac{1}{6} \langle a \wedge v, b \wedge u \rangle \right)
\]

\[
= \frac{1}{3} \langle a \wedge u, b \wedge u \rangle - \frac{1}{3} \langle a \wedge v, b \wedge v \rangle
\]

\[
+ i\left(\frac{1}{3} \langle a \wedge u, b \wedge v \rangle + \frac{1}{3} \langle a \wedge v, b \wedge u \rangle \right)
\]

\[
= \frac{1}{3} \langle a \wedge (u + iv), b \wedge (u + iv) \rangle.
\]

Since the expression is complex linear in \(a, b\) the formula holds for \(a, b \in V^c\) as well. \(\square\)

We get our main result Theorem 1.1 by combining Theorem 3.1 and Theorem 4.5.

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