Coarse Homotopy on metric Spaces and their Corona

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July 9, 2019

Abstract

This paper discusses properties of the Higson corona by means of a quotient on coarse ultrafilters on a proper metric space. We use this description to show that the corona functor is faithful. This study provides a Künneth formula for twisted coarse cohomology. We obtain the Gromov boundary of a hyperbolic proper geodesic metric space as a quotient of its Higson corona.

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0 Introduction

The corona $\nu'(X)$ of a metric space $X$ has been introduced in [1] and studied in [2], [3], [4], [5], [6], [7].

The Stone-Čech compactification is a functor $\beta$ from the category of completely regular spaces to the category of compact Hausdorff spaces. Note that by [8, Theorem 2.1] if $X$ is a completely regular space and $G$ a group then

$$\tilde{H}_n^p(X; G) = \hat{H}_n(\beta X, G)$$

The left side denotes $n$-dimensional Čech type functional cohomology based on finite open covers and the right side denote $n$-dimensional Čech cohomology.

This resembles [7, Corollary 35] where sheaf cohomology based on finite coarse covers of a metric space $X$ is related to sheaf cohomology on the corona $\nu'(X)$. This property and other properties which we are going to discuss in this paper suggest that the corona functor is the Stone-Čech boundary version of a space in the coarse category.

We start with the first quite elementary property:
Theorem A. If $m\text{Coarse}$ denotes the category of metric spaces and coarse maps modulo close and $\text{Top}$ the category of topological spaces and continuous maps then the functor 

$$\nu' : m\text{Coarse} \to \text{Top}$$

is faithful.

A direct consequence of this result is that $\nu'$ reflects isomorphisms.

We examine in which way the corona functor $\nu'$ is related to the Higson corona $\nu$ of $[9]$. Originally the Higson corona has been defined on a proper metric space $X$ as the boundary of the compactification determined by an algebra of bounded functions called the Higson functions. Already $[2]$ showed that there exists a homeomorphism $\nu(X) = \nu'(X)$. We provide an explicit homeomorphism and show $\nu, \nu'$ agree on morphisms too.

Theorem B. If $X$ is a proper metric space then there is a homeomorphism 

$$\nu'(X) \to \nu(X).$$

Here the right side denote the Higson corona of $[9]$. If $f : X \to Y$ is a coarse map between proper metric spaces then $\nu'(f), \nu(f)$ are homeomorphic (the same map pre-and postcomposed by a homeomorphism).

The asymptotic product of two metric spaces has been introduced in $[10]$ as the limit of a pullback diagram in the coarse category. Note $[11]$ Theorem 1] shows the following: If $X, Y$ are hyperbolic coarsely proper coarsely geodesic metric spaces then $X * Y$ is hyperbolic coarsely proper coarsely geodesic and therefore its Gromov boundary $\partial(X * Y)$ is defined. There is a homeomorphism $\partial(X * Y) = \partial(X) \times \partial(Y)$ which is the main result of $[11]$.

This paper shows if $X, Y$ are metric spaces then there is a homeomorphism $\nu'(X) \times \nu'(Y) = \nu'(X \times Y)$. If $Y$ is coarsely geodesic coarsely proper then $\nu'(X * Y)$ is the pullback of 

$$\nu'(Y) \quad \nu'(X) \quad \nu'(\mathbb{Z}^+)$$

Here $p \in X, q \in Y$ denote fixed points. Thus $\nu'$ preserves limits of this type. We obtain a coarse version of a Künneth formula for coarse cohomology with twisted coefficients:

Theorem C. (Künneth formula) Let $X, Y$ be metric spaces, $\mathcal{F}$ a sheaf on $X$ and $\mathcal{G}$ a sheaf on $Y$. Define a presheaf $\mathcal{F}'$ on $X \times Y$ by

$$U \mapsto \mathcal{F}(p_1(U)).$$

Then $\mathcal{F}'$ is a sheaf on $X \times Y$. Similarly we can define a sheaf $\mathcal{G}'$ on $X \times Y$. There is a homomorphism

$$\bigoplus_{p+q=n} \hat{H}^p(X, \mathcal{F}) \otimes \hat{H}^q(Y, \mathcal{G}) \to \hat{H}^n(X \times Y, \mathcal{F}' \otimes \mathcal{G}')$$

Here $\mathcal{F}' \otimes \mathcal{G}'$ denotes the sheaf associated to the presheaf $U \mapsto \mathcal{F}'(U) \otimes \mathcal{G}'(U)$ for $U \subseteq X \times Y$. If there is a $\mathcal{F}$-acyclic coarse cover $\mathcal{U}$ of $X$ and a $\mathcal{G}$-acyclic coarse cover $\mathcal{V}$ of $Y$ such that $\hat{C}^q(\mathcal{V}, \mathcal{G})$ is torsion free for every $q$ and $\hat{H}^p(\mathcal{U}, \mathcal{F})$ is torsion free for every $p$ then the homomorphism is an isomorphism.
If \( X \) is a hyperbolic proper geodesic metric space its Gromov boundary \( \partial(X) \) is defined \([12]\). Since every Gromov function is a Higson function the Gromov boundary arises as a quotient of the Higson corona \([9]\). We provide an explicit description of the quotient map and the induced topology on \( \partial(X) \).

**Theorem D.** Let \( X \) be a proper geodesic hyperbolic metric space. The relation \( \mathcal{F} \sim \mathcal{G} \) if \( \mathcal{F}, \mathcal{G} \in \text{cl}(\rho(\mathbb{Z}_+)) \) for some coarsely injective coarse map \( \rho : \mathbb{Z}_+ \to X \) is an equivalence relation on coarse ultrafilters and the mapping

\[
q_X : \nu'(X) \to \partial(X)
\]

\[
\mathcal{F} \mapsto [\rho] \quad \mathcal{F} \in \text{cl}(\rho(\mathbb{Z}_+))
\]

to the Gromov boundary \( \partial(X) \) of \( X \) is continuous and respects \( \sim \). The induced map on the quotient associated to \( \sim \) is a homeomorphism.

If \( A \subseteq X \) is a subset then \( \partial_X A := \{[\rho] : \rho(\mathbb{Z}_+) \wedge A\} \) is closed in \( \partial(X) \). The \( ((\partial_X A) \cap \mathbb{X}) \) constitute a basis for the topology on \( \partial(X) \).

This result implies there is a larger class of morphisms in the coarse category for which the Gromov boundary is a functor. Originally coarse equivalences were shown to induce continuous maps between Gromov boundaries. If \( f : X \to Y \) is a coarse map between hyperbolic proper geodesic metric spaces with the property that for every coarsely injective coarse map \( \rho : \mathbb{Z}_+ \to X \) the map \( f \circ \rho \) is coarsely injective coarse then \( f \) induces a map between Gromov boundaries.

1 Metric Spaces

**Definition 1.** Let \( (X,d) \) be a metric space. Then the **coarse structure associated to** \( d \) on \( X \) consists of those subsets \( E \subseteq X^2 \) for which

\[
\sup_{(x,y) \in E} d(x,y) < \infty.
\]

We call an element of the coarse structure **entourage**. In what follows we assume the metric \( d \) to be finite for every \( (x,y) \in X^2 \).

**Definition 2.** A map \( f : X \to Y \) between metric spaces is called **coarse** if

- \( E \subseteq X^2 \) being an entourage implies that \( f^{\times 2}(E) \) is an entourage (coarsely uniform);
- and if \( A \subseteq Y \) is bounded then \( f^{-1}(A) \) is bounded (coarsely proper).

Two maps \( f, g : X \to Y \) between metric spaces are called **close** if

\[
f \times g(\Delta_X)
\]

is an entourage in \( Y \). Here \( \Delta_X \) denotes the diagonal in \( X^2 \).

**Notation 3.** A map \( f : X \to Y \) between metric spaces is called

- **coarsely surjective** if there is an entourage \( E \subseteq Y^2 \) such that
  \[
  E[\text{im } f] = Y;
  \]
coarsely injective if for every entourage \( F \subseteq Y^2 \) the set \((f^{x,2})^{-1}(F)\) is an entourage in \( X \).

Two subsets \( A, B \subseteq X \) are called not coarsely disjoint if there is an entourage \( E \subseteq X^2 \) such that the set

\[ E[A] \cap E[B] \]

is not bounded. We write \( A \not\lambda B \) in this case.

Two subsets \( A, B \subseteq X \) are called asymptotically alike if there is an entourage \( E \subseteq X^2 \) such that

\[ E[A] = B. \]

We write \( A \lambda B \) in this case.

**Remark 4.** We study metric spaces up to coarse equivalence. A coarse map \( f : X \to Y \) between metric spaces is a coarse equivalence if

- There is a coarse map \( g : Y \to X \) such that \( f \circ g \) is close to \( id_Y \) and \( g \circ f \) is close to \( id_X \).
- or equivalently if \( f \) is both coarsely injective and coarsely surjective.

**Definition 5.** A metric space is called coarsely proper if it is coarsely equivalent to a proper metric space. It is called coarsely geodesic if it is coarsely equivalent to a geodesic metric space.

**Notation 6.** If \( X \) is a metric space and \( U_1, \ldots, U_n \subseteq X \) are subsets then \((U_i)_i\) are said to coarsely cover \( X \) if for every entourage \( E \subseteq X^2 \) the set

\[ E[U_1] \cap \cdots \cap E[U_n] \]

is bounded.

## 2 The Corona Functor

**Definition 7.** If \( X \) is a metric space a system \( F \) of subsets of \( X \) is called a coarse ultrafilter if

1. \( A, B \in F \) then \( A \not\lambda B \).
2. \( A, B \subseteq X \) are subsets with \( A \cup B \in F \) then \( A \in F \) or \( B \in F \).
3. \( X \in F \).

**Lemma 8.** If \( f : X \to Y \) is a coarse map between metric spaces and \( F \) is a coarse ultrafilter on \( X \) then

\[ f_* F := \{ A \subseteq Y : f^{-1}(A) \in F \} \]

is a coarse ultrafilter on \( Y \).

**Proof.** see [7]. \[ \square \]

**Definition 9.** We define a relation on coarse ultrafilters on \( X \): two coarse ultrafilters \( F, G \in \mathcal{G} \) are asymptotically alike, written \( A \not\lambda B \) if for every \( A \in F, B \in G \):

\[ A \not\lambda B. \]

**Remark 10.** By [7] the relation \( \not\lambda \) is an equivalence relation on coarse ultrafilters on \( X \). If two coarse ultrafilters \( F, G \) on \( X \) are asymptotically alike and \( f : X \to Y \) is a coarse map to a metric space \( Y \) then \( f_* F \not\lambda f_* G \) on \( Y \).
**Definition 11.** Let $X$ be a metric space. Denote by $\nu'(X)$ the set of coarse ultrafilters modulo asymptotically alike on $X$. The relation $\lambda$ on subsets of $\nu'(X)$ is defined as follows: Define for a subset $A \subseteq X$:

$$cl(A) = \{[\mathcal{F}] \in \nu'(X) : A \in \mathcal{F}\}$$

Then $\pi_1 \neq \pi_2$ if and only if there exist subsets $A, B \subseteq X$ such that $A \neq B$ and $\pi_1 \subseteq cl(A), \pi_2 \subseteq cl(B)$.

**Remark 12.** The relation $\lambda$ on subsets of $\nu'(X)$ defines a proximity relation on $\nu'(X)$ which induces a compact topology. By [7] the mapping $\nu_*$ between coarse ultrafilters induces a continuous map $\nu'(f)$ between the quotients. Thus $\nu'$ is a functor mapping coarse metric spaces to compact topological spaces.

The topology on $\nu'(X)$ is generated by $(cl(A))_{A \subseteq X}$. Coarse covers determine the finite open covers [2].

### 3 On Morphisms

**Lemma 13.** Let $f : X \to Y$ be a map between metric spaces. Then

1. $f$ is a coarse map if
   - $B \subseteq X$ is bounded then $f(B)$ is bounded.
   - if for every subsets $A, B \subseteq X$ the relation $A \lambda B$ implies $f(A) \lambda f(B)$.
2. if $f$ is coarse then $f$ is coarsely injective if $A \not\lambda B$ implies $f(A) \not\lambda f(B)$.
3. $f$ is coarsely surjective if the relation $f(X) \not\lambda B$ in $Y$ implies $B$ is bounded.

**Proof.**

1. First we show $f$ is coarsely proper. If $B \subseteq Y$ is bounded then $B \not\lambda Y$. This implies $f^{-1}(B) \not\lambda X$. Thus $f^{-1}(B)$ is bounded.

   Now we show $f$ is coarsely uniform: Suppose $A, B \subseteq X$ are two subsets with $f(A) \lambda f(B)$. Then there is an unbounded subset $S \subseteq f(A)$ with $S \not\lambda f(B)$ or there is an unbounded subset $T \subseteq f(B)$ with $T \not\lambda f(A)$. Assume the former. Then $f^{-1}(S) \not\lambda B$. Since $f$ maps bounded sets to bounded sets the set $f^{-1}(S) \cap A$ is unbounded. Thus $\lambda \lambda B$. Thus we have shown $\lambda \lambda B$ implies $f(A) \lambda f(B)$. By [13] Theorem 2.3 we can conclude that $f$ is coarsely uniform.

2. This is [7] Lemma 41.

3. easy. \qedsymbol

**Theorem 14.** If $f, g : X \to Y$ are two coarse maps between metric spaces and $\nu'(f) = \nu'(g)$ then $f, g$ are close.

**Proof.** Suppose $f, g$ are not close. By [13] Proposition 2.15 there is a subset $A \subseteq X$ with $f(A) \lambda g(A)$. This implies there is a subset $S \subseteq A$ with $f(S) \not\lambda g(S)$. Now by [6] Proposition 4.7 there is a coarse ultrafilter $\mathcal{F}$ on $X$ with $S \in \mathcal{F}$. Then $f(S) \in f_* \mathcal{F}$ and $g(S) \in g_* \mathcal{F}$. Since $f(S) \not\lambda g(S)$ this implies $f_* \mathcal{F} \neq g_* \mathcal{F}$. Thus $\nu'(f), \nu'(g)$ are not the same map. \qedsymbol
Corollary 15. If $\text{mCoarse}$ denotes the category of metric spaces and coarse maps modulo close and $\text{Top}$ the category of topological spaces and continuous maps then the functor

$$\nu': \text{mCoarse} \to \text{Top}$$

is faithful.

Corollary 16. The functor $\nu': \text{mCoarse} \to \text{Top}$ reflects epimorphisms and monomorphisms.

Proof. It is general theory that a faithful functor reflects epimorphisms and monomorphisms. This fact can also be found in [13, Exercise 1.6.vii]. Since by Corollary 15 the functor $\nu'$ is faithful the result follows.

Corollary 17. The functor $\nu': \text{mCoarse} \to \text{Top}$ reflects isomorphisms.

Proof. Suppose $f: X \to Y$ is a coarse map between metric spaces such that $\nu'(f)$ is an isomorphism in $\text{Top}$. Then $\nu'(f)$ is both a monomorphism and an epimorphism. The proof of [7, Theorem 40] can be generalized to hold for metric spaces. By Corollary 16 the map $f$ is a monomorphism in $\text{mCoarse}$. By a proof similar to the one of [15, Proposition 3.A.16] every monomorphism is coarsely injective. Since $f$ is coarsely injective and coarsely surjective it is a coarse equivalence.

Theorem 18. If $X$ is a proper metric space then there is a homeomorphism

$$\nu'(X) \to \nu(X).$$

Here the right side denote the Higson corona of [9]. If $f: X \to Y$ is a coarse map between proper metric spaces then $\nu'(f), \nu(f)$ are homeomorphic (the same map pre-and postcomposed by a homeomorphism).

Proof. Let $X$ be a proper metric space. First we show that $h'(X) := X \cup \nu'(X)$ is a compactification of $X$: Closed sets on $h'(X)$ are generated by $(\overline{A} \cup \cl(A))_{A \subseteq X}$. We show this topology is compact: If $(\overline{A}_i \cup \cl(A_i))_{i=1}^\infty$ is an open cover of $h'(X)$ then there is a subcover

$$(\overline{A}_1 \cup \cl(A_1))^c, \ldots, (\overline{A}_n \cup \cl(A_n))^c$$

such that $\cl(A_{1})^c, \ldots, \cl(A_{n})^c$ is a cover of $\nu'(X)$. Now this implies $A_{1}^c, \ldots, A_{n}^c$ are a coarse cover of $X$. Thus $\overline{A}_1 \cap \cdots \cap \overline{A}_n$ is both bounded and closed. Then there is a subcover

$$(\overline{A}_{n+1} \cup \cl(A_{n+1}))^c, \ldots, (\overline{A}_{n+m} \cup \cl(A_{n+m}))^c$$

of $(\overline{A}_i \cup \cl(A_i))^c_i$ such that $\overline{A}_{n+1}^c, \ldots, \overline{A}_{n+m}^c$ covers $\overline{A}_1 \cap \cdots \cap \overline{A}_n$. Then

$$(\overline{A}_1 \cup \cl(A_1))^c, \ldots, (\overline{A}_{n+m} \cup \cl(A_{n+m}))^c$$

are a subcover of $(\overline{A}_i \cup \cl(A_i))^c_i$ that cover $h'(X)$.

Now $X, \nu'(X)$ both appear as subspaces of $h'(X)$. We show the inclusion $X \to h'(X)$ is dense:

$$\overline{X}^h' = \bigcap_{\overline{A} \cup \cl(A) \supseteq X} (\overline{A} \cup \cl(A)) = X \cup \cl(X) = h'(X).$$
The Higson compactification $h(X)$ is determined by the $C^*$-algebra of Higson functions whose definition we now recall from [9]: A bounded continuous function $\varphi : X \to \mathbb{R}$ is called Higson if the function

$$d\varphi : X^2 \to \mathbb{R}$$

$$(x, y) \mapsto \varphi(y) - \varphi(x)$$

when restricted to $E$ vanishes to infinity for every entourage $E \subseteq X^2$.

Note [2, Proposition 1] shows Higson functions on $X$ can be extended to $h'(X)$. For the convenience of the reader we recall it.

Without loss of generality assume that $X$ is $R$-discrete for some $R > 0$. Then every coarse ultrafilter $\mathcal{F}$ on $X$ is determined by an ultrafilter $\sigma$ on $X$ by the proof of [7, Theorem 17]. If $\sigma$ is an ultrafilter on $X$ then a bounded continuous function $\varphi : X \to \mathbb{R}$ determines an ultrafilter $\varphi_\sigma := \{A : \varphi^{-1}(A) \in \sigma\}$ on $\mathbb{R}$. Since the image of $\varphi$ is bounded and therefore relatively compact the ultrafilter $\varphi_\sigma$ converges to a point $\sigma = \lim \varphi \in \mathbb{R}$.

If two ultrafilters $\sigma, \tau$ induce asymptotically alike coarse ultrafilters and $\varphi$ is a Higson function then $\sigma = \lim \varphi = \tau = \lim \varphi$. Suppose $\sigma = \lim \varphi \neq \tau = \lim \varphi$. Then there exist neighborhoods $U \ni \sigma - \lim \varphi$ and $V \ni \tau - \lim \varphi$ such that $d(U, V) > 0$. Let $E \subseteq X^2$ be an entourage. Then

$$d\varphi : \varphi^{-1}(U) \times \varphi^{-1}(V) \cap E \to \mathbb{R}$$

$$(x, y) \mapsto \varphi(y) - \varphi(x)$$

vanishes at infinity. Since $d(U, V) > 0$ this implies that $\varphi^{-1}(U) \times \varphi^{-1}(V) \cap E$ is bounded. Now $E$ was an arbitrary entourage thus $\varphi^{-1}(U), \varphi^{-1}(V)$ are coarsely disjoint. Since $\varphi^{-1}(U) \in \sigma, \varphi^{-1}(V) \in \tau$ the ultrafilters $\sigma, \tau$ induce coarse ultrafilters which are not asymptotically alike.

If $\mathcal{F}$ is a coarse ultrafilter on $X$ induced by an ultrafilter $\sigma$ and $\varphi$ a Higson function then denote by $\mathcal{F} - \lim \varphi$ the point $\sigma - \lim \varphi$ in $\mathbb{R}$. By the above $\mathcal{F} - \lim \varphi$ is well defined modulo asymptotically alike of $\mathcal{F}$.

If $\varphi : X \to \mathbb{R}$ is a Higson function then there is an extension

$$\hat{\varphi} : h'(X) \to \mathbb{R}$$

$$x \mapsto \begin{cases} \varphi(x) & x \in X \\ \mathcal{F} - \lim \varphi & x = \mathcal{F} \in \nu'(X) \end{cases}$$

we have shown $\hat{\varphi}$ is well defined. Now we show $\hat{\varphi}$ is continuous: Let $A \subseteq \mathbb{R}$ be a closed set. If $\mathcal{F} - \lim \varphi \in A$ fix an ultrafilter $\sigma$ on $X$ that induces $\mathcal{F}$. Then $\varphi^{-1}(A) \in \sigma$. This implies $\mathcal{F} \in \text{cl}(\varphi^{-1}(A))$. On the other hand if $\mathcal{F} \in \text{cl}(\varphi^{-1}(A))$ then there is an ultrafilter $\sigma$ on $X$ with $\varphi^{-1}(A) \in \sigma$ that induces $\mathcal{F}$. This implies $\sigma - \lim \varphi \in A$, thus $\mathcal{F} - \lim \varphi \in A$. Now

$$\hat{\varphi}^{-1}(A) = \varphi^{-1}(A) \cup \{\mathcal{F} : \mathcal{F} - \lim \varphi \in A\}$$

$$= \varphi^{-1}(A) \cup \text{cl}(\varphi^{-1}(A))$$

is closed.

Denote by $(C_h(X))^{h'}$ the set of extensions of Higson functions on $X$ to $h'(X)$. By [16] the $C^*$-algebra of Higson functions $C_h(X)$ determines the compactification $h'(X)$ if and only if $(C_h(X))^{h'}$ separates points of $\nu'(X)$.

We show $(C_h(X))^{h'}$ separates points of $\nu'(X)$: Let $\mathcal{F}, \mathcal{G} \in \nu'(X)$ be two coarse ultrafilters with $\mathcal{F} \mathcal{G}$. Then there exist elements $U \in \mathcal{F}, V \in \mathcal{G}$ with $U \not\parallel V$. Without loss of generality
assume that \( U, V \) are disjoint such that \( d(x, U) + d(x, V) \neq 0 \) for every \( x \in X \). Then define a function

\[
\varphi : X \to \mathbb{R} \\
\quad x \mapsto \frac{d(x, U)}{d(x, U) + d(x, V)}
\]

By [17, Lemma 2.2] the function \( d\varphi|_E \) vanishes to infinity for every entourage \( E \subseteq X^2 \). Now \( \varphi|_U \equiv 0 \) and \( \varphi|_V \equiv 1 \). This implies \( F - \lim \varphi = 0 \) and \( G - \lim \varphi = 1 \).

If \( f : X \to Y \) is a coarse map between \( R \)-discrete for some \( R > 0 \) proper metric spaces and \( \varphi : Y \to \mathbb{R} \) a Higson function then \( \varphi \circ f : X \to \mathbb{R} \) is a Higson function: Since \( X \) is \( R \)-discrete the map \( f \) is continuous, therefore \( \varphi \circ f \) is continuous. The map \( \varphi \circ f \) is bounded since \( \varphi \) is bounded. Let \( E \subseteq X^2 \) be an entourage and \( \varepsilon > 0 \) a number. Then \( f^{-1}(E) \subseteq Y^2 \) is an entourage. This implies \( (d\varphi)|_{f^{-1}(E)} \) vanishes to infinity. Thus there is a compact set \( K \subseteq Y \) such that

\[
|d(\varphi(x, y))| < \varepsilon
\]

whenever \( (x, y) \in f^{-1}(E) \cap (K^1)^\varepsilon \). Since \( K \) is bounded the set \( f^{-1}(K) \subseteq X \) is bounded. The set \( f^{-1}(K) \) is finite since \( X \) is \( R \)-discrete and therefore \( f^{-1}(K) \) is compact. Then

\[
|d(\varphi \circ f)(x, y)| < \varepsilon
\]

whenever \( (x, y) \in E \cap (f^{-1}(K))^2 \).

Now we provide an explicit homeomorphism \( \nu(X) \to \nu'(X) \): Denote by

\[
e_{C_\nu(X)} : Z \to \mathbb{R}^{C_\nu(X)} \\\ne(\varphi(x))\varphi
\]

the evaluation map for \( X \).

Note \( e_{C_\nu(X)} \) is a topological embedding and \( \nu(X) : = \overline{e_{C_\nu(X)}}(X) \setminus e_{C_\nu(X)}(X) \) by [16]. A point \( p \in \nu(X) \) is represented by a net \( (x_i)_i \) such that for every Higson function \( \varphi \in C_\nu(X) \) the net \( \varphi(x_i)_i \) converges in \( \mathbb{R} \). Define \( F_i : = \{ x_j : j \geq i \} \) for every \( i \). Then \( \sigma : = \{ F_i \} \) is a filter on \( X \) such that \( \varphi \circ \sigma \) converges to \( \lim \varphi(x_i) \) for every Higson function \( \varphi \) on \( X \). An ultrafilter \( \sigma' \) which is finer that \( \sigma \) determines a coarse ultrafilter \( F \). We have shown above that the association \( \Phi_X : p \mapsto F \) is well defined modulo asymptotically alike.

Now we show the map \( \Phi_X \) is injective: Let \( p, q \in \nu(X) \) be two points. If \( \Phi_X(p) = \Phi_X(q) \) then \( \Phi_X(p) - \lim \varphi = \Phi_X(q) - \lim \varphi \) for every Higson function \( \varphi \). This implies \( p = q \) in \( \mathbb{R}^{C_\nu(X)} \).

We show \( \Phi_X \) is surjective: If \( \sigma \) is an ultrafilter on \( X \) that determines a coarse ultrafilter on \( X \) then there is a net \( (x_i)_i \) on \( X \) which constitutes a section of \( \sigma \). Since \( \varphi(x_i)_i \) is a section of \( \varphi \circ \sigma \) for every Higson function \( \varphi \) the net \( \varphi(x_i)_i \) converges to \( \sigma \) - \( \lim \varphi \) in \( \mathbb{R} \). Thus \( (x_i)_i \) converges to a point in \( \nu(X) \).

Now we show \( \Phi_X \) is continuous: If \( A \subseteq X \) is a subset then \( \Phi_X^{-1}(\text{cl}(A)) \) is a subset of \( \nu(X) \). We show it is closed. If \( p \in \Phi_X^{-1}(\text{cl}(A)) \) then there is a net \( (x_i)_i \subseteq X \) that converges to \( p \). The net \( (x_i)_i \) is a section of an ultrafilter \( \sigma \) with \( A \in \sigma \). Thus there exists \( i \) with \( x_j \in A \) for every \( j \geq i \). If on the other hand \( (x_i)_i \) is a net in \( X \) and there exists \( i \) with \( x_j \in A \) for every \( j \geq i \) then \( (x_i)_i \) is a section of an ultrafilter \( \sigma \) on \( X \) with \( A \in \sigma \). This implies if \( (x_i)_i \) converges to \( p \in \nu(X) \) then \( p \in \Phi_X^{-1}(\text{cl}(A)) \). Thus we have shown

\[
\Phi_X^{-1}(\text{cl}(A)) = \overline{e_{C_\nu(X)}}(A) \setminus e_{C_\nu(X)}(A)
\]

is closed. This way we have obtained that \( \Phi_X \) is a homeomorphism.
Now we define a map $f^\ast : \mathbb{R}^C(X) \to \mathbb{R}^C(Y)$

$(x\varphi)_{\varphi \in C_h(X)} \mapsto (x\varphi \circ f)_{\varphi \in C_h(Y)}$

We show $f^\ast (\mathbb{R}^C_h(X)) \subseteq \mathbb{R}^C_h(Y)$: If $(x\varphi)_{\varphi \in C_h(X)}$ then there is a net $(x_i) \subseteq X$ such that $\lim_i \varphi(x_i) = x\varphi$ for every $\varphi \in C_h(X)$. Then $f(x_i) \subseteq Y$ is a net such that $\lim_i \varphi(f(x_i)) = x\varphi \circ f$ for every $\varphi \in C_h(Y)$.

Now $\nu(f) := f^\ast (\mathbb{R}^C_h(X)) \subseteq \mathbb{R}^C_h(Y)$.

4 A Künneth Formula

This is [10, Definition 25]:

Definition 19. (asymptotic product) If $X$ is a metric space and $Y$ a coarsely geodesic coarsely proper metric space pick points $p \in X$ and $q \in Y$ and a constant $R \geq 0$ large enough. Then the asymptotic product $X \ast Y$ of $X$ and $Y$ is defined by

$$X \ast Y := \{(x, y) \in X \times Y : |d(p, x) - d(q, y)| \leq R\}$$

as a subspace of $X \times Y$. We define the projection $p_1 : X \ast Y \to X$ by $(x, y) \mapsto x$ and the projection $p_2 : X \ast Y \to Y$ by $(x, y) \mapsto y$. Note that the projections are coarse maps. In what follows we denote by $d(p, \cdot), d(q, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ coarse maps $X \to \mathbb{R}^+, Y \to \mathbb{R}^+$ defined by $x \in X \mapsto d(p, x), y \in Y \mapsto d(q, y)$.

Remark 20. Let $X, Y$ be metric spaces of which $Y$ is coarsely geodesic coarsely proper. Now $X \ast Y$ of Definition [10] is determined by points $p \in X, q \in Y$ and constant $R \geq 0$. By [10, Lemma 26] the space $X \ast Y$ does not depend on the choice of $p, q, R$ up to coarse equivalence. By [10, Lemma 27] the diagram

$$
\begin{array}{ccc}
X \ast Y & \xrightarrow{p_1} & X \\
\downarrow{p_2} & & \downarrow{d(p, \cdot)} \\
X & \xrightarrow{d(q, \cdot)} & \mathbb{R}^+
\end{array}
$$

is a pullback diagram in Coarse.

Lemma 21. Let $X, Y$ be metric spaces with $Y$ coarsely geodesic coarsely proper. The following statements hold:

1. If $A \subseteq X, B \subseteq Y$ are subsets then $(A \times B) \cap (X \ast Y)$ is bounded if $A$ is bounded or $B$ is bounded.

2. If $(U_i)_i$ is a coarse cover of $X$ and $(V_j)_j$ a coarse cover of $Y$ then $((U_i \times V_j) \cap (X \ast Y))_{ij}$ is a coarse cover of $X \ast Y$.
3. Let $\mathcal{F}, \mathcal{G}$ be coarse ultrafilters on $X, Y$ respectively with $d(p, \cdot), \mathcal{F}, d(q, \cdot), \mathcal{G}$. Choose the constant of $X \times Y$ large enough. Then

$$\mathcal{F} \ast \mathcal{G} := \{(A \times B) : A \in \mathcal{F}, B \in \mathcal{G}\}$$

is a coarse ultrafilter on $X \times Y$.

**Proof.** 1. Suppose $A$ is bounded. Then $(x, y) \in A \times Y$ implies $x \in A$ and $|d(x, p) - d(y, q)| \leq R$. Let $S \geq 0$ be such that $A \subseteq B(p, S)$. Then $y \in B(q, R + S)$. Thus $A \times Y$ is bounded. Similarly if $B$ is bounded then $X \times B$ is bounded.

2. Let $E \subseteq (X \times Y)^2$ be an entourage. Then

$$\bigcap_{ij} E[(U_i \times V_j)^c] \cap (X \times Y) \subseteq \bigcap_{ij} E[(U_i \times V_j)^c] \cap (X \times Y)$$

is bounded. Thus $((U_i \times V_j) \cap (X \times Y))$ is a coarse cover of $X \times Y$.

Alternative proof: $(p^{-1}(U_i) \cap p^{-1}(V_j))_{ij}$.

3. Let $i : X \times Y \to X \times Y$ be the inclusion. At first we prove

$$i_* (\mathcal{F} \ast \mathcal{G}) = \{A \times B : A \in \mathcal{F}, B \in \mathcal{G}\}$$

is a coarse ultrafilter on $X \times Y$. We check the axioms of a coarse ultrafilter on $i_* (\mathcal{F} \ast \mathcal{G})$:

(a) If $A_1 \times B_1, A_2 \times B_2 \in i_* (\mathcal{F} \ast \mathcal{G})$ then $A_1, A_2 \in \mathcal{F}, B_1, B_2 \in \mathcal{G}$. This implies $A_1 \times A_2$ in $X$ and $B_1 \times B_2$ in $Y$. Then $A_1 \times B_1 \times A_2 \times B_2$ in $X \times Y$.

(b) Let $A_1 \times B_1, A_2 \times B_2 \subseteq X \times Y$ be two subsets with $(A_1 \times B_1) \cup (A_2 \times B_2) \in i_* (\mathcal{F} \ast \mathcal{G})$. Since $(A_1 \cup A_2) \times (B_1 \cup B_2) \supseteq (A_1 \times B_1) \cup (A_2 \times B_2)$ this implies $(A_1 \cup A_2) \times (B_1 \cup B_2) \in i_* (\mathcal{F} \ast \mathcal{G})$. Thus $(A_1 \cup A_2) \in \mathcal{F}, (B_1 \cup B_2) \in \mathcal{G}$. This implies $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$. Then $A_1 \times (B_1 \cup B_2) \in i_* (\mathcal{F} \ast \mathcal{G})$ or $A_2 \times (B_1 \cup B_2) \in i_* (\mathcal{F} \ast \mathcal{G})$. Suppose $A_1 \times (B_1 \cup B_2) \in i_* (\mathcal{F} \ast \mathcal{G})$. Since $A_1 \times B_1$ is maximal among factors of two subsets of $X, Y$ contained in $A_1 \times (B_1 \cup B_2), (A_1 \times B_1) \cup (A_2 \times B_2) \in i_* (\mathcal{F} \ast \mathcal{G})$ we obtain $A_1 \times B_1 \in i_* (\mathcal{F} \ast \mathcal{G})$.

(c) $X \times Y \in i_* (\mathcal{F} \ast \mathcal{G})$ since $X \in \mathcal{F}, Y \in \mathcal{G}$.

Let $A \times B \in i_* (\mathcal{F} \ast \mathcal{G})$ be an element. Since $d(p, \cdot), \mathcal{F}, d(q, \cdot), \mathcal{G}$ the sets $d(p, \cdot)(A), d(q, \cdot)(B)$ are close in $\mathbb{R}_+$. Thus there exists an $R \geq 0$ and unbounded subsets $A' \subseteq A, B' \subseteq B$ with

$$|d(p, a) - d(q, b)| \leq R$$

for $a \in A', b \in B'$. Thus we have shown $A \times B \times X \times Y$. Choose the constant of $X \times Y$ large enough then $X \times Y \in i_* (\mathcal{F} \ast \mathcal{G})$. We can thus restrict $i_* (\mathcal{F} \ast \mathcal{G})$ to $X \times Y$ and obtain $\mathcal{F} \ast \mathcal{G} = (i_* (\mathcal{F} \ast \mathcal{G}))_{|X \times Y}$. This way we have shown $\mathcal{F} \ast \mathcal{G}$ is a coarse ultrafilter.

$\square$
Theorem 22. Let $X, Y$ be metric spaces with $Y$ coarsely geodesic coarsely proper. Define
\[
\nu'(X) \ast \nu'(Y) := \{ (\mathcal{F}, \mathcal{G}) \in \nu'(X) \times \nu'(Y) : \nu'(d(p, \cdot))(\mathcal{F}) = \nu'(d(q, \cdot))(\mathcal{G}) \}
\]
Then the map
\[
(\nu'(p_1), \nu'(p_2)) : \nu'(X \ast Y) \to \nu'(X) \ast \nu'(Y)
\]
is a homeomorphism.

Proof. We prove $(\nu'(p_1), \nu'(p_2))$ is well defined: Let $\mathcal{F}$ be a coarse ultrafilter on $X \ast Y$ then $p_1 \mathcal{F}, p_2 \mathcal{F}$ are coarse ultrafilters on $X, Y$, respectively. Since $d(p, \cdot) \circ p_1, d(q, \cdot) \circ p_2$ are close the coarse ultrafilters $d(p, \cdot), p_1 \mathcal{F}, d(q, \cdot), p_2 \mathcal{F}$ are asymptotically alike. Thus we have shown $(p_1 \mathcal{F}, p_2 \mathcal{F}) \in \nu'(X) \ast \nu'(Y)$.

Now we prove $(\nu'(p_1), \nu'(p_2))$ is surjective: Let $(\mathcal{F}, \mathcal{G}) \in \nu'(X) \ast \nu'(Y)$ be a point. By Lemma 21 the system of subsets $\mathcal{F} \ast \mathcal{G}$ is a coarse ultrafilter on $X \ast Y$. Denote by $p_1' : X \times Y \to X, p_2' : X \times Y \to Y$ the projection to the first, second factor, respectively and by $i : X \ast Y \to X \times Y$ the inclusion. Then $p_1 = p_1' \circ i, p_2 = p_2' \circ i$. Since $i_* (\mathcal{F} \ast \mathcal{G}) = \{ A \times B : A \in \mathcal{F}, B \in \mathcal{G} \}$ we obtain the relations $p_1' _* i_* (\mathcal{F} \ast \mathcal{G}) \mathcal{F}, p_2' _* i_* (\mathcal{F} \ast \mathcal{G}) \mathcal{G}$. Thus we have proved $(\nu'(p_1), \nu'(p_2))(\mathcal{F} \ast \mathcal{G}) = (\mathcal{F}, \mathcal{G})$.

Now we prove $(\nu'(p_1(\mathcal{F})), (\nu'(p_2(\mathcal{F})) = \mathcal{F}$ for every point $\mathcal{F} \in \nu'(X \ast Y)$: Let $A \in \mathcal{F}$ be an element. Then $(p_1(\mathcal{A}) \times p_2(\mathcal{A}))(\mathcal{F} \ast \mathcal{G}) \subseteq (p_2 \mathcal{F}) \ast (p_2 \mathcal{G})$. Since $A \subseteq (p_1(\mathcal{A}) \times p_2(\mathcal{A})) \cap (X \ast Y)$ we obtain $(p_1 \mathcal{F}) \ast (p_2 \mathcal{F}) \subseteq \mathcal{F}$. Thus $(p_1 \mathcal{F}) \ast (p_2 \mathcal{F}) \supseteq \mathcal{F}$. This way we have shown $(\nu'(p_1), \nu'(p_2))$ is bijective.

Since $\nu'(X \ast Y)$ is compact and $\nu'(X) \ast \nu'(Y)$ is Hausdorff we obtain that $(\nu'(p_1), \nu'(p_2))$ is a homeomorphism.

Lemma 23. Let $X, Y$ be metric spaces. There is a homeomorphism
\[

\nu'(X) \times \nu'(Y) \to \nu'(X \times Y)
\]
\[
(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \ast \mathcal{G}
\]
where $\mathcal{F} \ast \mathcal{G} := \{ A \times B : A \in \mathcal{F}, B \in \mathcal{G} \}$.

Proof. We already showed in the proof of Lemma 21 that $\mathcal{F} \ast \mathcal{G}$ is a coarse ultrafilter on $X \times Y$. It remains to show that the map is bijective and continuous.

Let $\mathcal{F}_1, \mathcal{F}_2 \in \nu'(X)$, $\mathcal{G}_1, \mathcal{G}_2 \in \nu'(Y)$ be coarse ultrafilters. Suppose $(\mathcal{F}_1 \times \mathcal{G}_1) \lambda (\mathcal{F}_2 \times \mathcal{G}_2)$ Let $A \in \mathcal{F}_1, B \in \mathcal{F}_2$ be elements. Then $A \times Y \in \mathcal{F}_1 \times \mathcal{G}_1, B \times Y \in \mathcal{F}_2 \times \mathcal{G}_2$. Thus $A \times Y \lhd B \times Y$. This implies $A \lhd B$ in $X$, thus $\mathcal{F}_1 \lhd \mathcal{F}_2$.

Let $\mathcal{F} \in \nu'(X \times Y)$ be a coarse ultrafilter. Define
\[
\mathcal{F}_i := \{ p_i(A) : A \in \mathcal{F} \}
\]
for $i = 1, 2$. Here $p_i$ denotes the projection to the $i$th factor. Then $\mathcal{F}_1$ is a coarse ultrafilter on $X$:

1. If $A, B \in \mathcal{F}_1$ then $A \times Y, B \times Y \in \mathcal{F}$. This implies $A \lhd B$.

2. If $A, B \subseteq X$ with $A \cup B \in \mathcal{F}_1$ then $(A \cup B) \times Y \in \mathcal{F}$. Thus $A \times Y \in \mathcal{F}$ or $B \times Y \in \mathcal{F}$.

Then $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_1$.

3. Since $X \times Y \in \mathcal{F}$ the set $X \in \mathcal{F}_1$ is contained.
Since \( A \subseteq p_1(A) \times p_2(A) \) we have an inclusion \( \mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{F} \). Thus \( (\mathcal{F}_1 \times \mathcal{F}_2)\lambda \mathcal{F} \).

Fix a coarse ultrafilter \( \mathcal{G} \in \mathcal{V}(Y) \). We show the map

\[
\mathcal{V}(X) \to \mathcal{V}(X \times Y)
\]

\[
\mathcal{F} \mapsto \mathcal{F} \times \mathcal{G}
\]

is continuous: Let \( \pi_1, \pi_2 \subseteq \mathcal{V}(X) \) be subsets with \( \pi_1 \times \mathcal{G} \not\subseteq (\pi_2 \times \mathcal{G}) \). Then there exist subsets \( A, B \subseteq X \times Y \) with \( \pi_1 \times \mathcal{G} \subseteq \text{cl}(A), \pi_2 \times \mathcal{G} \subseteq \text{cl}(B) \) and \( A \not\subseteq B \). Since the left side is a product we can assume \( A = A_1 \times A_2, B = B_1 \times B_2 \) also. Then \( \pi_1 \subseteq \text{cl}(A_1), \pi_2 \subseteq \text{cl}(B_1) \) with \( A_1 \not\subseteq B_1 \).

If \( X \) is a metric space we associate to \( X \) a Grothendieck topology determined by coarse covers. Sheaf cohomology on coarse covers is coined coarse cohomology with twisted coefficients in [18]. Now coarse covers on \( X \) determine the finite open covers on \( \mathcal{V}(X) \). Thus sheaf cohomology on \( \mathcal{V}(X) \) equals twisted cohomology on \( X \) as a coarse space. We compose a Künneth formula for coarse cohomology with twisted coefficients.

**Theorem 24. (Künneth formula)** Let \( X, Y \) be metric spaces, \( \mathcal{F} \) a sheaf on \( X \) and \( \mathcal{G} \) a sheaf on \( Y \). Define a presheaf \( \mathcal{F}' \) on \( X \times Y \) by

\[
U \mapsto \mathcal{F}(p_1(U)).
\]

Then \( \mathcal{F}' \) is a sheaf on \( X \times Y \). Similarly we can define a sheaf \( \mathcal{G}' \) on \( X \times Y \). There is a homomorphism

\[
\bigoplus_{p+q=n} \check{H}^p(X, \mathcal{F}) \otimes \check{H}^q(Y, \mathcal{G}) \to \check{H}^n(X \times Y, \mathcal{F}' \otimes \mathcal{G}')
\]

Here \( \mathcal{F}' \otimes \mathcal{G}' \) denotes the sheaf associated to the presheaf \( U \mapsto \mathcal{F}'(U) \otimes \mathcal{G}'(U) \) for \( U \subseteq X \times Y \). If there is a \( \mathcal{F} \)-acyclic coarse cover \( \mathcal{U} \) of \( X \) and a \( \mathcal{G} \)-acyclic coarse cover \( \mathcal{V} \) of \( Y \) such that \( \check{C}^q(\mathcal{V}, \mathcal{G}) \) is torsion free for every \( q \) and \( \check{H}^p(\mathcal{U}, \mathcal{F}) \) is torsion free for every \( p \) then the homomorphism is an isomorphism.

**Proof.** There is a Čech cohomology version of the Eilenberg-Zilber theorem. If \( \mathcal{U}, \mathcal{V} \) are coarse covers of \( X, Y \), respectively then

\[
\mathcal{U} \times \mathcal{V} := \{ U_i \times V_j : U_i \in \mathcal{U}, V_j \in \mathcal{V} \}
\]

is a coarse cover of \( X \times Y \). Then there is a homomorphism

\[
\bigoplus_{p+q=n} \check{C}^p(\mathcal{U}, \mathcal{F}) \otimes \check{C}^q(\mathcal{V}, \mathcal{G}) \to \check{C}^n(\mathcal{U} \times \mathcal{V}, \mathcal{F}' \otimes \mathcal{G}')
\]

for every \( n \geq 0 \) which maps \( (s_{i_0 \cdots i_p}) \in \prod (\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})), (t_{j_0 \cdots j_q}) \in \prod (\mathcal{G}(V_{j_0} \cap \cdots \cap V_{j_q})) \) to \( (s_{i_0 \cdots i_p} \otimes t_{j_0 \cdots j_q}) \in \prod [(\mathcal{F} \otimes \mathcal{G})((U_{i_0} \cap \cdots \cap U_{i_p}) \times (V_{j_0} \cap \cdots \cap V_{j_q}))]. \) This induces an isomorphism of cochain complexes. We can now apply [10] Section 2.8, Chapter 1 which gives the desired result in case of acyclic coarse covers. In the other case taking the direct limit over coarse covers gives the desired homomorphism.

\[ \square \]

## 5 Space of Rays

**Definition 25.** (space of rays) Let \( Y \) be a compact topological space. As a set the space of rays \( \mathcal{F}(Y) \) of \( Y \) is \( Y \times \mathbb{Z}_+ \). A subset \( E \subseteq Y^2 \) is an entourage if for every countable subset \( ((x_k, i_k), (y_k, j_k)) \) \( k \subseteq E \) the following properties hold:
1. The set \((i_k, j_k)_k\) is an entourage in \(\mathbb{Z}^+\).

2. If \((i_k)_k \to \infty\) then \((x_k)_k\) and \((y_k)_k\) have the same limit points.

This makes \(F(Y)\) a coarse space.

**Theorem 26.** If \(f : X \to Y\) is a continuous map between compact topological spaces

- then it induces a coarse map by

\[
F(f) : F(X) \to F(Y) \\
(x, i) \mapsto (f(x), i)
\]

- If \(f\) is a homeomorphism then \(F(f)\) is a coarse equivalence.

**Proof.**

- We show \(F(f)\) is coarsely uniform and coarsely proper. First we show \(F(f)\) is coarsely uniform: Suppose \(((x_i, n_i), (y_i, m_i))\) is a countable entourage in \(F(X)\) such that \((n_i)_i\) is a strictly monotone sequence in \(\mathbb{Z}^+\) and \((x_i)_i\) converges to \(x\). Then \((n_i, m_i)_i\) is an entourage in \(\mathbb{Z}^+\) and \((y_i)_i\) converges to \(x\). Since \(f\) is a continuous map \(f(x_i)_i\) and \(f(y_i)_i\) both converge to \(f(x)\). Thus we can conclude that

\[
((f(x_i), n_i), (f(y_i), m_i))_i
\]

is an entourage in \(F(Y)\).

Now we show \(F(f)\) is coarsely proper: If \(B \subseteq F(Y)\) is bounded we can write \(B = \bigcup_i B_i \times i\) with \(B_i \subseteq Y, i \in \mathbb{Z}^+\) where the number of \(i\) that appear is finite. Then

\[
f^{-1}(B) = \bigcup_i f^{-1}(B_i) \times i
\]

is bounded.

- if \(f\) is a homeomorphism then there is a topological inverse \(g : Y \to X\) of \(f\). Now \(f \circ g = id_Y\) and \(g \circ f = id_X\). Then

\[
F(f) \circ F(g) = F(f \circ g)
\]

\[
= F(id_Y)
\]

\[
= id_{F(Y)}
\]

and

\[
F(g) \circ F(f) = F(g \circ f)
\]

\[
= F(id_X)
\]

\[
= id_{F(X)}
\]

\(\square\)

**Corollary 27.** Denote by \(k\text{Top}\) the category of compact topological spaces and continuous maps and by \(\text{Coarse}\) denote the category of coarse spaces and coarse maps modulo close. Then \(F\) is a functor

\[
F : k\text{Top} \to \text{Coarse}
\]
Proposition 28. Denote by $\mathcal{F}_0$ a coarse ultrafilter on $\mathbb{Z}_+$, the choice is not important. For every $y \in Y$ denote by $i_y$ the inclusion $y \times \mathbb{Z}_+ \to F(Y)$. The map

$$
\eta_Y : Y \to \nu' \circ F(Y)
y \mapsto \nu'(i_y)(\mathcal{F}_0)
$$

for every metric space $Y$ defines a natural transformation $\eta : \mathbb{I}_{\text{top}} \to \nu' \circ F$.

Proof. If $f : Y \to Z$ is a continuous map between compact spaces we show the diagram

\begin{align*}
Y & \xrightarrow{f} Z \\
\eta_Y & \downarrow \quad \downarrow \eta_Z \\
\nu' \circ f(Y) & \xrightarrow{\nu' \circ f} \nu' \circ f(Z)
\end{align*}

commutes. down and then right: a point $y \in Y$ is mapped by $\eta_Y$ to $\nu'(i_y)(\mathcal{F}_0)$. Then

$$
\nu'(f(y))(\nu'(i_y)(\mathcal{F}_0)) = (f \circ i_y)(\mathcal{F}_0) = i_{f(y)}(\mathcal{F}_0)
$$

right and then down: a point $y \in Y$ is mapped by $f$ to $f(y)$. Then

$$
\eta_Z(f(y)) = \nu'(i_{f(y)})(\mathcal{F}_0)
$$

The map $\eta_Y$ is continuous for every compact space $Y$: Let $(y_i)_i$ be a net in $Y$ that converges to $y$. Then $(\nu'(i_{y_i})(\mathcal{F}_0))_i$ converges in $\eta_Y(Y)$ to $\nu'(i_y)(\mathcal{F}_0)$: Let $A \subseteq \nu' \circ f(Y)$ be a set such that $\nu'(i_y)(\mathcal{F}_0) \in \text{cl}(A)^\circ$. Thus there is some $B \in \mathcal{F}_0$ such that $y \times B \notin A$. Now for almost all $i$ the relation $(y_i \times B) \notin A$ holds, thus $\nu'(i_{y_i})(\mathcal{F}_0) \in \text{cl}(A)^\circ$ for almost all $i$. \qed

Lemma 29. Let $X$ be a coarsely geodesic coarsely proper metric space. If $\mathcal{F}$ is a coarse ultrafilter on $X$ there is a coarsely injective coarse map $\rho : \mathbb{Z}_+ \to X$ such that $\mathcal{F} \in \text{cl}(\rho(\mathbb{Z}_+))$.

Proof. Fix an ultrafilter $\sigma$ on $X$ that induces the coarse ultrafilter $\mathcal{F}$. Suppose $X$ is $R$-discrete and $c$-coarsely geodesic for $R, c > 0$. We will determine a sequence $(r_i)_i$ of points in $X$ and a sequence $(V_i)_i$ of subsets of $X$.

For a point $x_0 \in X$ and define $r_0 := x_0$ and $V_0 := X$. Then define for every $y \in X$ the number $d_0(y)$ to be the minimal length of a $c$-path joining $x_0$ to $y$. We define a relation on points of $X$: $y \leq z$ if $d_0(y) \leq d_0(z)$ and $y$ lies on a $c$-path of minimal length joining $x_0$ to $z$.

For every $i \in \mathbb{N}$ do: Denote by $C_i := \{ y \in X : d_0(y) = i \}$ and define $W_y := \{ z : y \leq z \} \cap V_{i-1}$ for every $y \in C_i \cap V_{i-1}$. Now $V_{i-1} \in \sigma$ and the $W_y$ cover $V_{i-1}$ except for a bounded set. Then there is one $y$ such that $W_y \in \sigma$. Define $V_i := W_y$ and $r_i := y$.

Define a map

$$
\rho : \mathbb{Z}_+ \to X
\quad i \mapsto r_i.
$$

Then $\rho$ is a coarsely injective coarse map with $(\rho(\mathbb{Z}_+)) \in \sigma$. \qed
6 An alternative Description of the Gromov Boundary

Theorem 30. Let $X$ be a proper geodesic hyperbolic metric space. The relation $\mathcal{F} \sim \mathcal{G}$ if $\mathcal{F}, \mathcal{G} \in \text{cl}(\rho(Z_+))$ for some coarsely injective coarse map $\rho : Z_+ \to X$ is an equivalence relation on coarse ultrafilters and the mapping

$$q_X : \nu'(X) \to \partial(X)$$
$$\mathcal{F} \mapsto [\rho] \quad \mathcal{F} \in \text{cl}(\rho(Z_+))$$

to the Gromov boundary $\partial(X)$ of $X$ is continuous and respects $\sim$. The induced map on the quotient associated to $\sim$ is a homeomorphism.

If $A \subseteq X$ is a subset then

$$\partial_X A := \{[\rho] : \rho(Z_+) \cap A\}$$

is closed in $\partial(X)$. The $((\partial_X A)^c)_{A \subseteq X}$ constitute a basis for the topology on $\partial(X)$.

Proof. Note the first part is already [9] Lemma 6.23] which shows the Gromov boundary appears as a quotient of the Higson corona by using the property that every Gromov function is a Higson function. The second part is already [6] Theorem 9.10] which defines a coarse proximity structure on $X$ that induces the Gromov compactification.

Every point $p$ in the Gromov boundary $\partial(X)$ is represented by a coarsely injective coarse map $\rho : Z_+ \to X$: A point in $\partial(X)$ is represented by a geodesic $r : \mathbb{R}_+ \to X$ as defined in [20] page 427]. By [20] Lemma 3.1] the point $p$ can be represented by a large-scale embedding $Z_+ \to X$. Since $Z_+, X$ are large-scale geodesic this is the same as a coarsely injective coarse map.

If $\rho, \sigma : Z_+ \to X$ are two coarsely injective coarse maps then either $\rho(Z_+), \sigma(Z_+)$ are finite Hausdorff distance apart or $\rho(Z_+), \sigma(Z_+)$: Suppose $\rho(Z_+) \cap \sigma(Z_+)$. Then there are subsequences $(j_i), (k_i) \subseteq Z_+$ and a constant $R > 0$ such that $d(\rho(j_i), \sigma(k_i)) \leq R$ for every $i$. By [9] Theorem 6.17] there exists $S > 0$ such that the geodesic joining $\rho(j_i)$ to $\rho(j_{i+1})$ has Hausdorff distance at most $S$ from $\rho(j_i), \rho(j_{i+1}), \ldots, \rho(j_{i+1})$ and from $\sigma(k_i), \sigma(k_{i+1}), \ldots, \sigma(k_{i+1})$ for every $i$. Thus we obtain $d(\rho(Z_+), \sigma(Z_+)) \leq 2S$.

By Lemma [20] for every coarse ultrafilter $\mathcal{F}$ there exists a coarsely injective coarse map $\rho : Z_+ \to X$ such that $\mathcal{F} \in \text{cl}(\rho(Z_+))$. This implies $\sim$ is an equivalence relation on coarse ultrafilters. Since the equivalence classes are closed the quotient is T1.

We recall [2] Definition 6.21]: If $\varphi : X \to \mathbb{R}$ is a continuous function then it is called Gromov if for every $\varepsilon > 0$ there exists $K > 0$ such that $(x|y) > K$ implies $|f(x) - f(y)| < \varepsilon$. We denote by $C_0(X)$ the algebra of Higson functions on $X$.

Now we provide the mapping $q_X$. Note that by [9] Lemma 6.23] every Gromov function is a Higson function. Thus there is a mapping

$$\Phi_X : R_{C_0(X)}(X) \to \mathbb{R}_{C_0(X)}(X)$$
$$(x_\varphi)_{\varphi \in C_0(X)} \mapsto (x_\varphi)_{\varphi \in C_0(X)}.$$

Now $\Phi_X(e_{C_0(X)}(X) \setminus e_{C_0(X)}(X)) \subseteq e_{C_0(X)}(X) \setminus e_{C_0(X)}(X)$. In fact this map is surjective. This map associates a net $(x_i)_i$ that is section of a coarse ultrafilter to a net $(x_i)_i$ such that $\lim \varphi(x_i) \in \mathbb{R}$ exists for every Gromov function $\varphi$. By [9] Lemma 6.24] every such net arises as $\rho(i)_i$ for some coarsely injective coarse map $\rho : Z_+ \to X$. Thus $\rho(i)_i$ is a section of some ultrafilter inducing $\mathcal{F}$ which translates to $\mathcal{F} \in \text{cl}(\rho(Z_+))$. Note the map $q_X$ maps $\mathcal{F}$ to $[\rho] \in \partial(X)$.

Now $q_X$ respects $\sim$ and by the above it induces a continuous bijection $\nu'(X)/\sim \to \partial(X)$.
We show the second part of the theorem: Denote by \( q : \nu'(X) \to \nu'(X)/\sim \) the quotient map associated to \( \sim \). Then

\[
q^{-1}(\text{cl}(A)) = \{ [\mathcal{F}] : \mathcal{F} \in \text{cl}(A) \}
= \{ \text{cl}(\rho(Z_+)) : \mathcal{F} \in \text{cl}(\rho(Z_+)), \mathcal{F} \in \text{cl}(A) \}
= \{ \text{cl}(\rho(Z_+)) : \rho(Z_+) \wedge A \}.
\]

Then \( \{ [\rho] : \rho(Z_+) \wedge A \} \) is closed in \( \partial(X) \). The \( \partial_X A = q_X^{-1}(\text{cl}(A)) \) generate the closed sets of \( \partial(X) \).

We define a topology on \( gX := X \cup \partial(X) \) by declaring

\[ \bar{A} \cup (\partial_X A) \]
as a base. Then \( gX \) is compact: Let \((x_i)_i\) be a net in \( gX \). If \((x_i)_i \cap X \) contains a bounded and infinite subsequence then there is a limit point \( x \in X \) to which a subsequence converges. If this is not the case and \((x_i)_i \cap X \) is infinite then by [10, Proposition 22] there exists a coarsely injective coarse map \( \rho : Z_+ \to X \) with \( \rho(Z_+) \wedge ((x_i)_i \cap X) \). Then a subsequence converges to \( [\rho] \).

Now \( X, \partial(X) \) appear as subspaces of \( gX \). Since \( \bar{X} = gX \) the space \( gX \) is a compactification of \( X \).

**Corollary 31.** If \( f : X \to Y \) is a coarse map between hyperbolic proper geodesic metric spaces and if for every coarsely injective coarse map \( \rho : Z_+ \to X \) the map

\[
f \circ \rho : Z_+ \to Y
\]
is coarsely injective then \( f \) induces a continuous map \( \partial(f) : \partial(X) \to \partial(Y) \).

**Proof.** Compare this result with [21, Theorem 2.8] where a visual large-scale uniform map induces a continuous map between Gromov boundaries.

Note that \( \nu'(f) \) maps equivalence classes of \( \sim \) in \( \nu'(X) \) to equivalence classes of \( \sim \) in \( \nu'(Y) \). Thus if \( \mathcal{F} \sim \mathcal{G} \) in \( \nu'(X) \) then \( q_Y \circ \nu'(f)(\mathcal{F}) = q_Y \circ \nu'(f)(\mathcal{G}) \). This implies there is a unique continuous map \( \tilde{f} : \partial(X) \to \partial(Y) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\nu'(X) & \xrightarrow{\nu'(f)} & \nu'(Y) \\
\downarrow q_X & & \downarrow q_Y \\
\partial(X) & \xrightarrow{f} & \partial(Y)
\end{array}
\]

Now the map

\[
\partial(f) : \partial X \to \partial Y \\
[\rho] \mapsto [f \circ \rho]
\]
also makes this diagram commute, thus \( \partial(f) = \tilde{f} \) is continuous by uniqueness. \( \square \)

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