Large deviations for invariant measures of SPDEs with two reflecting walls

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Abstract
In this article, we established a large deviation principle for invariant measures of solutions of stochastic partial differential equations with two reflecting walls driven by space-time white noise.

Key Words: stochastic partial differential equations with two reflecting walls; deterministic obstacle problems; invariant measures; large deviations; skeleton equations.

MSC: Primary 60H15; 60F10; Secondary 60J35

1 Introduction

Consider reflected stochastic partial differential equations (SPDEs) of the following type:

\[
\begin{align*}
\frac{\partial u_\varepsilon(x,t)}{\partial t} & = \frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} - \alpha u_\varepsilon(x,t) + f(x, u_\varepsilon(x,t)) \\
& + \varepsilon \sigma(x, u_\varepsilon(x,t)) \dot{W}(x,t) + \eta_\varepsilon(x,t) - \xi_\varepsilon(x,t) \\
K_1(x) & \leq u_\varepsilon(x,t) \leq K_2(x)
\end{align*}
\]

in \((x,t) \in Q := [0,1] \times \mathbb{R}_+\) while \(K_1(x) \leq u_\varepsilon(x,t) \leq K_2(x)\). Here \(\dot{W}\) is a space-time white noise. When \(u_\varepsilon(x,t)\) hits \(K_1(x)\) or \(K_2(x)\), the additional forces are added to prevent \(u_\varepsilon\) from leaving \([K_1, K_2]\). These forces are expressed by random measures \(\xi_\varepsilon\) and \(\eta_\varepsilon\) in equation (1.1) which play a similar role as the local time in the usual Skorokhod equation constructing Brownian motions with reflecting barriers.
Parabolic SPDEs with reflection are natural extension of the widely studied deterministic parabolic obstacle problems. They also can be used to model fluctuations of an interface near a wall, see Funaki and Olla [6]. In recent years, there is a growing interest on the study of SPDEs with reflection. Several works are devoted to the existence and uniqueness of the solutions. In the case of a constant diffusion coefficient and a single reflecting barrier $K_1 = 0$, Nualart and Pardoux [8] proved the existence and uniqueness of the solutions. In the case of a non-constant diffusion coefficient and a single reflecting barrier $K_1 = 0$, the existence of a minimal solution was obtained by Donati-Martin and Pardoux [4]. The existence and particularly the uniqueness of the solutions for a fully non-linear SPDE with reflecting barrier $0$ were solved by Xu and Zhang [12]. In the case of double reflecting barriers, Otobe [9] obtained the existence and uniqueness of the solutions of a SPDE driven by an additive white noise.

In addition to the existence and uniqueness, various other properties of the solution have been studied by several authors, see Donati-Martin and Pardoux [5], Zambotti [15], Dalang et al. [3] and Zhang [16].

The purpose of this paper is to establish a large deviation principle for invariant measures of the solutions of fully non-linear SPDEs with two reflecting walls (1.1). Large deviations for invariant measures of the solutions of SPDEs were previously studied in [10] and [2]. Our approach will be along the same lines as that in [10] and [2]. However, the extension is non-trivial. The extra difficulty arises from the appearance of the random measures (local times) $\eta_\varepsilon$ and $\xi_\varepsilon$ in the equation (1.1). We need to carefully analyze the local time terms in the skeleton equations and provide some uniform estimates for the penalized approximating equations.

The rest of the paper is organized as follows. In Section 2 we introduce the SPDEs with reflecting walls and state the precise conditions on the coefficients. In Section 3 we recall some results on the deterministic obstacle problems which will be used later. In Section 4, we study the skeleton equations and the rate functional. We provide some estimates for the extra measures (local times) in the equation and prove equivalent characterizations of the rate functional. In Section 5, we prove the exponential tightness for the invariant measures. The main result is stated in Section 6. The lower bound of the large deviation is established in Section 7 and upper bound is obtained in Section 8.
2 Reflected SPDEs

In this section, we introduce reflected stochastic partial differential equations (SPDEs) and state the precise conditions on the coefficients.

Consider the following SPDE with two reflecting walls:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - \alpha u + f(x, u(x, t)) + \sigma(x, u(x, t)) W(x, t) + \eta - \xi; \\
\frac{\partial u}{\partial x}(0, t) &= 0, \quad \frac{\partial u}{\partial x}(1, t) = 0, \quad \text{for } t \geq 0; \\
u(x, 0) &= u_0(x) \in C([0, 1]); \quad K_1(x) \leq u_0(x) \leq K_2(x), \\
K_1(x) &\leq u(x, t) \leq K_2(x), \quad \text{for } (x, t) \in Q,
\end{aligned}
\]

(2.1)

here \( W(x, t) \) is a space-time Brownian sheet on a filtered probability space \( (\Omega, P, \mathcal{F}; \mathcal{F}_t, t \geq 0) \).

Throughout the paper, the reflecting walls \( K_1(x), K_2(x) \) are assumed to be continuous functions satisfying

(H1) \( K_1(x) < 0 < K_2(x) \) for \( x \in [0, 1] \);

(H2) \( \frac{\partial^2 K_i}{\partial x^2} \in L^2([0, 1]) \), where \( \frac{\partial^2}{\partial x^2} \) are interpreted in a distributional sense.

Introduce the following conditions:

(F1) the coefficients: \( f, \sigma : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) are bounded and there exists \( C > 0 \) such that

\[ |f(x, y) - f(x, \hat{y})| + |\sigma(x, y) - \sigma(x, \hat{y})| \leq C|y - \hat{y}|, \]

for \( x \in [0, 1] \) and \( y, \hat{y} \in \mathbb{R} \).

(F2) \( \sigma(x, y) \) is continuous in both variables and there exists \( m > 0 \) such that

\[ |\sigma(x, y)| \geq m. \]

**Remark 2.1.** Here in the equation (2.1) we choose the Neumann boundary condition for the Laplacian operator for convenience. The results in this paper are also valid for other boundary conditions, e.g., periodic boundary condition, Dirichlet boundary condition.

The following is the definition of the solution of a SPDE with two reflecting walls \( K_1, K_2 \).

**Definition 2.1.** A triplet \( (u, \eta, \xi) \) is a solution to the SPDE (2.1) if

(i) \( u = \{u(x, t); (x, t) \in Q\} \) is a continuous, adapted random field (i.e., \( u(x, t) \) is \( \mathcal{F}_t \)-measurable \( \forall t \geq 0, x \in [0, 1] \)) satisfying \( K_1(x) \leq u(x, t) \leq K_2(x), \ a.s; \)
We first present a precise definition of the solution for equation (3.1).

\[
(u(t), \phi) - \int_0^t (u(s), \phi') ds - \int_0^t (f(\cdot, u(s)), \phi) ds - \int_0^t \int_0^1 \phi(x) \sigma(x, u(x, s)) W(dx, ds)
= (u_0, \phi) - \alpha \int_0^t (u(s), \phi) ds + \int_0^t \int_0^1 \phi(x) dx, ds - \int_0^t \int_0^1 \phi(x) \xi(dx, ds), \text{ a.s. (2.2)}
\]

where \((\cdot, \cdot)\) denotes the inner product in \(L^2([0,1])\) and \(u(t)\) denotes \(u(\cdot, t)\);

(iv)

\[
\int_Q (u(x, t) - K_1(x)) \eta(dx, dt) = \int_Q (K_2(x) - u(x, t)) \xi(dx, dt) = 0.
\]

3 Deterministic obstacle problems

Let \(K_1, K_2\) be as in Section 2 and \(u_0 \in C([0,1])\) with \(K_1(x) \leq u_0(x) \leq K_2(x)\). Let \(v(x, t) \in C(Q)\) with \(v(x, 0) = u_0(x)\). Consider a deterministic PDE with two reflecting walls:

\[
\begin{cases}
\frac{\partial z(x,t)}{\partial t} - \frac{\partial^2 z(x,t)}{\partial x^2} + \alpha z(x,t) = \eta(x,t) - \xi(x,t); \\
\frac{\partial z}{\partial t}(0,t) = \frac{\partial z}{\partial x}(1,t) = 0, \text{ for } t \geq 0; \\
z(x,0) = 0, \text{ for } x \in [0,1]; \\
K_1(x) - v(x,t) \leq z(x,t) \leq K_2(x) - v(x,t), \text{ for } (x,t) \in Q.
\end{cases}
\]

We first present a precise definition of the solution for equation (3.1).

Definition 3.1. A triplet \((z, \eta, \xi)\) is called a solution to the PDE (3.1) if

(i) \(z = z(x,t); (x,t) \in Q\) is a continuous function satisfying \(K_1(x) \leq z(x,t) + v(x,t) \leq K_2(x), z(x,0) = 0;\)
(ii) \(\eta(dx, dt)\) and \(\xi(dx, dt)\) are measures on \([0,1] \times \mathbb{R}_+\) satisfying

\[
\eta([0,1] \times [0,T]) < \infty, \quad \xi([0,1] \times [0,T]) < \infty
\]
(iii) for all $t \geq 0$ and $\phi \in C^\infty[0,1]$ with $\frac{\partial \phi}{\partial x}(0) = \frac{\partial \phi}{\partial x}(1) = 0$ we have

$$
(z(t), \phi) = \int_0^t (z(s), \phi''(s)) \, ds + \alpha \int_0^t (z(s), \phi(s)) \, ds
$$

where $z(t)$ denotes $z(\cdot, t)$.

(iv)

$$
\int_Q (z(x, t) + v(x, t) - K_1(x)) \eta(dx, dt) = \int_Q (K_2(x) - z(x, t) - v(x, t)) \xi(dx, dt) = 0.
$$

The following result is the existence and uniqueness of the solutions. We refer the reader to [13] for the proof.

**Theorem 3.1.** Suppose (F1) holds. Equation (3.1) admits a unique solution $(z, \eta, \xi)$.

**Remark 3.1.** Let $\hat{z}$ be the solution to equation (3.1) replacing $v$ by $\hat{v}$, where $\hat{v}(x, t)$ is another continuous function on $Q$. It is proved in Otobe [9] that

$$
\|z - \hat{z}\|_\infty^T \leq \|v - \hat{v}\|_\infty^T,
$$

for any $T > 0$, where $\|\omega\|_\infty^T := \sup_{x \in [0,1], t \in [0,T]} |\omega(x, t)|$.

**4 Skeleton equations and rate functional**

The Cameron-Martin space associated with the Brownian sheet $\{W(x, t), x \in [0, 1], t \in \mathbb{R}_+\}$ is given by

$$
\mathcal{H} = \{h = \int_0^T \int_0^1 \hat{h}(x, s) \, dx \, ds; \int_0^T \int_0^1 \hat{h}^2(x, s) \, dx \, ds < \infty, T > 0\}.
$$
Proof. Consider the approximating equations:

\[
\begin{align*}
\frac{\partial u^h(x,t)}{\partial t} &- \frac{\partial^2 u^h(x,t)}{\partial x^2} + \alpha u^h(x,t) + f(u^h(x,t)) + \sigma(u^h(x,t))\hat{h}(x,t) + \eta^h; \\
K_1(x) \leq u^h(x,t) &\leq K_2(x); \\
\int_0^T \int_0^1 (u^h(x,t) - K_1(x))\eta^h(dx,dt) &+ \int_0^T \int_0^1 (K_2(x) - u^h(x,t))\xi^h(dx,dt) = 0, T > 0 \quad (4.1) \\
u^h(\cdot,0) &= u_0; \\
\frac{\partial u^h}{\partial x}(0,t) &= \frac{\partial u^h}{\partial x}(1,t) = 0.
\end{align*}
\]

Theorem 4.1. Assume (F1). Then, equation (4.1) admits a unique solution. Moreover, the measures \(\eta^h, \xi^h\) are absolutely continuous w.r.t. the Lebesgue measure \(dx \times dt\) and for \(T > 0\),

\[
\int_0^T \int_0^1 |\eta^h(x,t)|^2 dxdt \leq C_{1,T,h}, \quad \int_0^T \int_0^1 |\xi^h(x,t)|^2 dxdt \leq C_{2,T,h}, \quad (4.2)
\]

where \(C_{1,T,h}, C_{2,T,h}\) are constants only depending on the bounds of \(f, \sigma\), and the norm \(||h||\). Furthermore, if \(||h||_2^\infty = \int_0^\infty \int_0^1 h^2(x,s)dxds < \infty\), then we have

\[
\int_0^T e^{-\alpha(T-t)} \int_0^1 |\eta^h(x,t)|^2 dxdt \leq C(1 + ||h||_2^\infty), \quad (4.3)
\]

\[
\int_0^T e^{-\alpha(T-t)} \int_0^1 |\xi^h(x,t)|^2 dxdt \leq C(1 + ||h||_2^\infty), \quad (4.4)
\]

where \(C\) is a constant independent of \(T\).

Proof. Consider the approximating equations:

\[
\begin{align*}
\frac{\partial u^h_{\varepsilon,\delta}(x,t)}{\partial t} &= \frac{\partial^2 u^h_{\varepsilon,\delta}(x,t)}{\partial x^2} - \alpha u^h_{\varepsilon,\delta}(x,t) + f(u^h_{\varepsilon,\delta}(x,t)) + \sigma(u^h_{\varepsilon,\delta}(x,t))\hat{h}(x,t) \\
&\quad + \frac{1}{\varepsilon}(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- - \frac{1}{\delta}(u^h_{\varepsilon,\delta}(x,t) - K_2(x))^+; \quad (4.5) \\
u^h_{\varepsilon,\delta}(x,0) &= u_0(x); \\
u^h_{\varepsilon,\delta}(0,t) &= u^h_{\varepsilon,\delta}(1,t) = 0.
\end{align*}
\]

Here for simplicity, we write \(f(u^h_{\varepsilon,\delta}(x,t))\) for \(f(x, u^h_{\varepsilon,\delta}(x,t))\) and \(\sigma(u^h_{\varepsilon,\delta}(x,t))\) for \(\sigma(x, u^h_{\varepsilon,\delta}(x,t))\). As shown in the SPDE case in [9, 13], the solution \(u^h_{\varepsilon,\delta}\) of equation (4.5) converges to the unique solution \(u^h\) of (4.1) as \(\delta, \varepsilon \to 0\). Moreover

\[
\eta^h(dx,dt) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{1}{\delta} (u^h_{\varepsilon,\delta}(x,t) - K_1(x))^-. \quad (4.6)
\]
\[
\xi^h(dx, dt) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (u_{\varepsilon, \delta}^h(x, t) - K_2(x))^+ dx dt. \tag{4.7}
\]

Now we show that the measures \(\eta^h, \xi^h\) are absolutely continuous w.r.t. the Lebesgue measure \(dx \times dt\). Observe that the following equation holds:

\[
\frac{\partial}{\partial t} (u_{\varepsilon, \delta}^h(x, t) - K_1(x)) = \frac{\partial^2 (u_{\varepsilon, \delta}^h(x, t) - K_1(x))}{\partial x^2} + f(u_{\varepsilon, \delta}^h(x, t)) + \sigma(u_{\varepsilon, \delta}^h(x, t)) \dot{h}(x, t) - \alpha (u_{\varepsilon, \delta}^h(x, t) - K_1(x))
\]

\[+ \frac{1}{\delta} (u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+ - \frac{1}{\varepsilon} (u_{\varepsilon, \delta}^h(x, t) - K_2(x))^+ \tag{4.8}\]

\[+ \frac{\partial^2 K_1(x)}{\partial x^2} - \alpha K_1(x).\]

Multiplying both sides of the above equation by \((u_{\varepsilon, \delta}^h(x, t) - K_1(x))^-\) and integrating against \(dx\) we obtain

\[
- \frac{\partial}{\partial t} \int_0^1 [(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+]^2 dx
\]

\[= \int_0^1 \left| \frac{\partial}{\partial x} (u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+ \right|^2 dx + \alpha \int_0^1 [(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+]^2 dx
\]

\[+ \int_0^1 f(u_{\varepsilon, \delta}^h(x, t))(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^- dx
\]

\[+ \int_0^1 \sigma(u_{\varepsilon, \delta}^h(x, t))(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^- h(x, t) dx
\]

\[+ \frac{1}{\delta} \int_0^1 [(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+]^2 dx - \frac{1}{\varepsilon} \int_0^1 (u_{\varepsilon, \delta}^h(x, t) - K_2(x))^+(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+ dx
\]

\[+ \int_0^1 \frac{\partial^2 K_1(x)}{\partial x^2} (u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+ dx - \alpha \int_0^1 K_1(x)(u_{\varepsilon, \delta}^h(x, t) - K_1(x))^+ dx. \tag{4.9}\]
Applying the chain rule and integrating w.r.t. \( t \) from 0 to \( T \) yield

\[
\int_0^1 [(u^h_{\varepsilon,\delta}(x,T) - K_1(x))^-] dx + \frac{1}{\delta} \int_0^T e^{-\alpha(T-t)} \int_0^1 [(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^-] dx dt \\
+ \int_0^T e^{-\alpha(T-t)} \int_0^1 \left| \frac{\partial (u^h_{\varepsilon,\delta}(x,t) - K_1(x))^-}{\partial x} \right|^2 dx dt \\
= - \int_0^T e^{-\alpha(T-t)} \int_0^1 f(u^h_{\varepsilon,\delta}(x,t))(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- dx dt \\
- \int_0^T e^{-\alpha(T-t)} \int_0^1 \sigma(u^h_{\varepsilon,\delta}(x,t))h(x,t)(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- dx dt \\
+ \frac{1}{\varepsilon} \int_0^T e^{-\alpha(T-t)} \int_0^1 (u^h_{\varepsilon,\delta}(x,t) - K_2(x))^+(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- dx dt \\
- \int_0^T e^{-\alpha(T-t)} \int_0^1 \frac{\partial^2 K_1(x)}{\partial x^2}(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- dx dt \\
+ \alpha \int_0^T e^{-\alpha(T-t)} \int_0^1 K_1(x)(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- dx dt. \tag{4.10}
\]

Note that

\[
(u^h_{\varepsilon,\delta}(x,t) - K_2(x))^+(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^- \leq 0,
\]

and for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that

\[
ab \leq \varepsilon \frac{1}{\delta} a^2 + C_\varepsilon b^2, \quad a, b \in \mathbb{R} \tag{4.11}
\]

From (4.10) and (4.11) we deduce that

\[
\frac{1}{\delta} \int_0^T e^{-\alpha(T-t)} \int_0^1 [(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^-] dx dt \\
\leq \frac{1}{\delta} \int_0^T e^{-\alpha(T-t)} \int_0^1 [(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^-] dx dt \\
+ C\delta \int_0^T e^{-\alpha(T-t)} \int_0^1 \sigma(u^h_{\varepsilon,\delta}(x,t))^2 h(x,t)^2 dx dt \\
+ C\delta \int_0^T e^{-\alpha(T-t)} \int_0^1 f(u^h_{\varepsilon,\delta}(x,t))^2 dx dt + C\delta \int_0^T e^{-\alpha(T-t)} \int_0^1 \left( \frac{\partial^2 K_1(x)}{\partial x^2} \right)^2 dx dt \\
+ C\delta \int_0^T e^{-\alpha(T-t)} \int_0^1 (K_1(x))^2 dx dt. \tag{4.12}
\]

In particular, we obtain that

\[
\frac{1}{\delta^2} \int_0^T e^{-\alpha(T-t)} \int_0^1 [(u^h_{\varepsilon,\delta}(x,t) - K_1(x))^-] dx dt \leq C_{T,h},
\]

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Since \( f, \sigma \) are bounded, we see that if \( \|h\|_2^\infty = \int_0^\infty \int_0^1 \dot{h}^2(x,s)dxds < \infty \), then

\[
C_{T,h} \leq C(1 + \|h\|_2^\infty)
\]

for some constant \( C \) independent of \( T \). Subtracting a weak convergent subsequence if necessary, the above inequality together with (4.6) implies that \( \eta^h \) is absolutely continuous w.r.t. \( dxdt \) and

\[
\int_0^T e^{-\alpha(T-t)} \left( \int_0^1 \eta^h(x,t)^2 \right) dxdt \\
\leq \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{1}{\delta^2} \int_0^T e^{-\alpha(T-t)} \left( \int_0^1 |u^h_{\varepsilon,\delta}(x,t) - K_1(x)|^2 \right) dxdt \\
\leq C_{T,h}.
\]

(4.13)

The proof of the corresponding conclusion for \( \xi^h \) is similar. ■

Let \( C^\gamma([0,1]) \) denote the Banach space of \( \gamma \)-Hölder continuous functions on \([0,1]\) equipped with the Hölder norm \( \| \cdot \|_\gamma \).

**Proposition 4.1.** Assume (F1). Let \( u^h(x,t) \) be the solution of equation (4.1). For \( 0 < \gamma < \frac{1}{2} \), we have

\[
\|u^h(\cdot,t)\|_\gamma \leq C(1 + \frac{1}{\sqrt{t}})(1 + \|h\|_2^\infty).
\]

(4.14)

**Proof.** Since \( \eta^h, \xi^h \) are absolutely continuous w.r.t. \( dxdt \), it follows that \( u^h \) has the following mild form:

\[
u^h(x,t) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(y,u^h(y,s))dyds \\
+ \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,u^h(y,s))\dot{h}(y,s)dyds + \int_0^t \int_0^1 G_{t-s}(x,y)\eta^h(y,s)dyds \\
- \int_0^t \int_0^1 G_{t-s}(x,y)\xi^h(y,s)dyds.
\]

(4.15)

Where \( G_t(x,y) = e^{-\alpha t}P_t(x,y) \) and \( P_t(x,y) \) is the heat kernel of the Neumann Laplacian on \([0,1]\). The Proposition will follow if we prove that each of the five terms on the right has the bound (4.14). Recall the following inequality proved in [11]: for \( 0 < \gamma < \frac{1}{2} \),

\[
\left( \int_0^\infty \int_0^1 |P_u(x_1,y) - P_u(x_2,y)|^2 dydu \right)^{\frac{1}{2}} \leq C|x_1 - x_2|^{\gamma}.
\]

(4.16)
By the property of the heat kernel, it holds that
\[ |\int_0^1 G_t(x_1, y)u_0(y)dy - \int_0^1 G_t(x_2, y)u_0(y)dy| \leq Ce^{-\alpha t} \frac{1}{\sqrt{t}} |x_1 - x_2|. \]

The remaining terms can be treated in a similar way. Let us look at one of the terms, say, the fourth term \( F(x, h) := \int_0^t \int_0^1 G_{t-s}(x, y)\eta^h(y, s)dyds. \) By Theorem 4.1, we have
\[
|F(x, t)| \leq C(1 + ||h||^\infty),
\]
where (4.16) and (4.3) have been used. Combining (4.17) and (4.18) we conclude
\[ ||F(\cdot, t)||_\gamma \leq C(1 + ||h||^\infty). \]

The proof is complete.

Let \( v(\cdot, \cdot) \in C([0, 1] \times R). \) For \( t_1 < t_2, \) define
\[
I_{t_1}^{t_2}(v) = \inf\{\frac{1}{2}\hat{h}_{t_1}^2([0,1] \times [t_1, t_2]); v = u^h\},
\]
(4.19)
where \( u^h \) is the solution of the equation (4.1) on the time interval \([t_1, t_2].\)

Introduce
\[ E = \{z \in C([0, 1]); \quad K_1(x) \leq z(x) \leq K_2(x), x \in [0, 1]\}. \]
(4.20)

\( E \) is a complete metric space equipped with the metric deduced from the maximum norm \( \| \cdot \|_\infty \) on \( C([0, 1]). \) Let \( s > 0, \) \( t > 0. \) Set
\[ K_{0, t}(s) = \{\phi \in C([0, t]; E); I_0^t(\phi) \leq s\}, \]
and
\[ K_{z,0,t}(s) = \{\phi \in C([0, t]; E); \phi(0) = z, I_0^t(\phi) \leq s\}. \]

For \( z \in E, \) define
\[ J(z) = \inf\{I_0^t(v); v \in C([0, t]; E), v(\cdot, 0) = 0, v(\cdot, t) = z, t > 0\}. \]
(4.21)
Theorem 4.2. We have

\[ J(z) = \inf \{ J_{-\infty}(v); v(\cdot, 0) = z, \lim_{t \to -\infty} v(\cdot, t) = 0 \} \quad (4.22) \]

Proof. Let \( t > 0 \) and \( v \in C([0, t]; E) \) with \( v(0) = 0 \), \( v(t) = z \). Define

\[ \bar{v}(s) = \begin{cases} v(s + t) & \text{if } s \in [-t, 0], \\ 0 & \text{if } s \leq -t. \end{cases} \]

Then \( \bar{v}(0) = z \), \( \lim_{s \to -\infty} \bar{v}(\cdot, s) = 0 \) and \( J^0_{-\infty}(\bar{v}) = I^0_v(v) \). Consequently,

\[ \inf \{ J_{-\infty}(v); v(\cdot, 0) = z, \lim_{t \to -\infty} v(\cdot, t) = 0 \} \leq I^0_{-\infty}(\bar{v}) = I^0_v(v) \]

As \( t, v \) are arbitrary, we deduce that

\[ J(z) \geq \inf \{ J_{-\infty}(v); v(\cdot, 0) = z, \lim_{t \to -\infty} v(\cdot, t) = 0 \}. \]

To prove the opposite inequality, we may assume

\[ \inf \{ J_{-\infty}(v); v(\cdot, 0) = z, \lim_{t \to -\infty} v(\cdot, t) = 0 \} < \infty. \]

In this case, following the same method as in [10], [2] we can show that the inf can be attained, i.e., there exists \( v_0 \) with \( v_0(\cdot, 0) = z \), \( \lim_{t \to -\infty} v_0(\cdot, t) = 0 \) such that

\[ I^0_{-\infty}(v_0) = \inf \{ J^0_{-\infty}(v); v(\cdot, 0) = z, \lim_{t \to -\infty} v(\cdot, t) = 0 \}. \quad (4.23) \]

In view of the assumptions on \( K_1(x) \) and \( K_2(x) \), there exists \( \varepsilon_0 > 0 \) such that

\[ K_1(x) < -\varepsilon_0 < 0 < \varepsilon_0 < K_2(x). \quad (4.24) \]

For any \( \varepsilon > 0 \), there exists \( n_0 \) such that \( \|v_0(\cdot, t)\|_\infty \leq \varepsilon_0 \wedge \varepsilon \) for \( t \leq -n_0 + 2 \) and \( J_{-n_0 + 1}(v_0) \leq \varepsilon \). Write \( v_0(t) \) for \( v_0(\cdot, t) \) and define

\[ v_{n_0}(t) = \begin{cases} v_0(t) & \text{if } t \in [-n_0 + 1, 0], \\ (t + n_0)v_0(t) & \text{if } -n_0 \leq t \leq -n_0 + 1. \end{cases} \]

Set \( \bar{v}_{n_0}(t) = v_{n_0}(t - n_0) \), for \( 0 \leq t \leq n_0 \). Then \( \bar{v}_{n_0}(0) = 0 \), \( \bar{v}_{n_0}(n_0) = z \). For \( v(\cdot, \cdot) \in C([0, 1] \times \mathbb{R}) \) and \( t_1 < t_2 \), define

\[ S^{t_2}_{t_1}(v) = \inf \left\{ \frac{1}{2} \| \hat{h} \|_2^2(0, 1 \times [t_1, t_2]); v = \hat{v} \} \right\}, \quad (4.25) \]
where \( v^h \) is the solution of the following skeleton equation:

\[
\begin{cases}
\frac{\partial v^h(x,t)}{\partial t} - \frac{\partial^2 v^h(x,t)}{\partial x^2} + \alpha v^h(x,t) = f(x,v^h) + \sigma(x,v^h)\dot{h}(x,t); \\
u^h(\cdot,0) = u_0; \\
\frac{\partial u^h}{\partial x}(0,t) = \frac{\partial u^h}{\partial x}(1,t) = 0.
\end{cases}
\] (4.26)

If \( ||v(t)||_\infty \leq \varepsilon_0 \) for \( t_1 \leq t \leq t_2 \), it is clear that \( S^0_{v_0}(v) = I^0_{v_0}(v) \) because in this case, the extra forces \( \eta^h, \xi^h \) in (4.1) disappear. From the proof of Proposition 7 in [10], we know that there exists a constant \( C \) such that

\[
S_{-n_0+1}(v_{n_0}) \leq C[S_{-n_0+1}(v_0) + \sup_{-n_0 \leq t \leq n_0+1} (||v_0(t)||_\infty)^2].
\]

Taking into account of the choice of the choice of \( n_0 \) it follows that

\[
\begin{align*}
J(z) &\leq I^0_{-\infty}(v_0) = I^0_{-n_0}(v_{n_0}) \\
&\leq I^0_{-n_0+1}(v_{n_0}) + I^0_{-n_0+1}(v_0) \\
&\leq I^0_{-\infty}(v_0) + S^0_{-n_0+1}(v_0) \\
&\leq I^0_{-\infty}(v_0) + C[S^0_{-n_0+1}(v_0) + \sup_{-n_0 \leq t \leq n_0+1} (||v_0(t)||_\infty)^2] \\
&\leq I^0_{-\infty}(v_0) + C[I^0_{-n_0+1}(v_0) + \sup_{-n_0 \leq t \leq n_0+1} (||v_0(t)||_\infty)^2] \\
&\leq I^0_{-\infty}(v_0) + C(\varepsilon + \varepsilon^2),
\end{align*}
\] (4.27)

where \( C \) is an independent constant. Since \( \varepsilon \) is arbitrary, this proves

\[
J(z) \leq \inf \{I^0_{-\infty}(v); v(\cdot, 0) = z, \lim_{t \to -\infty} v(\cdot, t) = 0\}
\]

by the choice of \( v_0 \).  

\[\blacksquare\]

**Proposition 4.2.** The functional \( J(\cdot) : E \to [0, +\infty] \) is lower semi-continuous with compact level sets, i.e., for \( r \geq 0 \), \( K(r) = \{z \in E; J(z) \leq r\} \) is compact.

The proof of this proposition is very similar to that of Theorem 5.5 in [2] (see also Section 6 in [10]). We omit the details.

## 5 Exponential tightness of invariant measures

Let \( \mu_\varepsilon \) denote the unique invariant probability measure of the solution \( u_\varepsilon(t), t \geq 0 \) of equation (1.1). The following result is an exponential tightness for the invariant measures.
Theorem 5.1. Assume (F1). For any $L > 0$, there exists a compact subset $K_L \subset C([0, 1])$ such that

$$\exp\left(-\frac{L}{\varepsilon^2}\right)$$

(5.1) for $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0$ is a positive constant.

Proof. By the invariance of $\mu_{\varepsilon}$, we have

$$\mu_{\varepsilon}(K_L^c) = \int_E P(u_{\varepsilon}(\cdot, 1, g) \in K_L^c)\mu_{\varepsilon}(dg).$$

(5.2)

Thus it is sufficient to prove that there exists a compact subset $K_L \subset C([0, 1])$ such that

$$P(u_{\varepsilon}(\cdot, 1, g) \in K_L^c) \leq \exp\left(-\frac{L}{\varepsilon^2}\right)$$

(5.3) for $\varepsilon \leq \varepsilon_0$ and all $g \in C([0, 1])$ with $K_1 \leq g \leq K_2$. Put

$$v_{\varepsilon}(x, t, g) = \int_0^t \int_0^1 G_{t-s}(x, y)f(y, u_{\varepsilon}(y, s, g))dyds + \varepsilon \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(y, u_{\varepsilon}(y, s, g))W(dy, ds).$$

(5.4)

Then $u_{\varepsilon}$ can be written in the form

$$u_{\varepsilon}(x, t, g) - \int_0^1 G_t(x, y)g(y)dy = v_{\varepsilon}(x, t, g) + \int_0^t \int_0^1 G_{t-s}(x, y)\eta_{\varepsilon}(g)(dx, dt) - \int_0^t \int_0^1 G_{t-s}(x, y)\xi_{\varepsilon}(g)(dx, dt),$$

where $\eta_{\varepsilon}(g), \xi_{\varepsilon}(g)$ indicates the dependence of the random measures on the initial condition $g$. Put

$$\bar{u}_{\varepsilon}(x, t, g) = u_{\varepsilon}(x, t, g) - \int_0^1 G_t(x, y)g(y)dy.$$

Then $(\bar{u}_{\varepsilon}, \eta_{\varepsilon}, \xi_{\varepsilon})$ solves a random obstacle problem (5.1) with $v(x, t)$ replaced by $v_{\varepsilon}(x, t, g)$. As shown in [14], there exists a continuous functional $\Phi_1$ from $C([0, 1] \times [0, 1])$ to $C([0, 1])$ such that $\bar{u}_{\varepsilon}(\cdot, 1, g) = \Phi_1(v_{\varepsilon}(\cdot, g))$, where
\( v_\varepsilon(\cdot, g) = v_\varepsilon(\cdot, \cdot, g) \). Since \( f(u_\varepsilon(y, s, g)), \sigma(u_\varepsilon(y, s, g)) \) are bounded by a constant independent of \( g \), by Proposition 4 in [10], for \( L > 0 \) there exists a compact subset \( K'_L \subset C([0, 1] \times [0, 1]) \) such that

\[
P(v_\varepsilon(\cdot, \cdot, g) \in (K'_L)^c) \leq \exp(-\frac{L}{\varepsilon^2}) \tag{5.5}
\]

for \( \varepsilon \leq \varepsilon_0 \) and all \( g \in C([0, 1]) \) with \( K_1 \leq g \leq K_2 \). On the other hand, it is easy to see that there is a compact subset \( K_0 \subset C([0, 1]) \) such that

\[
\{ \int_0^1 G_1(x, y)g(y)dy; \ K_1 \leq g \leq K_2 \} \subset K_0.
\]

Let \( \Phi_1(K'_L) \) denote the image of \( K'_L \) under the map \( \Phi_1 \). Put \( K_L = \Phi_1(K'_L) + K_0 \). Then we have

\[
P(u_\varepsilon(\cdot, 1, g) \in K^c_L) \leq P(v_\varepsilon(\cdot, \cdot, g) \in (K'_L)^c) \leq \exp(-\frac{L}{\varepsilon^2}) \tag{5.6}
\]

for \( \varepsilon \leq \varepsilon_0 \) and all \( g \in C([0, 1]) \) with \( K_1 \leq g \leq K_2 \). This finishes the proof. \( \blacksquare \)

## 6 Statement of large deviations

The following result is a large deviation principle of \( u_\varepsilon \) (the solution of the equation (1.1)) on the path space \( C([0, 1] \times [0, T]) \). The proof of the theorem is very similar to that of Theorem 5.1 in [12] where a large deviation principle was proved for SPDEs with reflection at 0. We just need to use Theorem 7 in [1] to improve Theorem 5.1 in [12] to a uniform large deviation principle on compact sets.

**Theorem 6.1.** Assume (F1). Then, the laws \( \nu^0_\varepsilon \) of \( \{u_\varepsilon(\cdot, \cdot, g)\}_{\varepsilon > 0} \) satisfy a large deviation principle on \( C([0, 1] \times [0, T]) \) with the rate function \( I^T_0(\cdot) \) uniformly on compact sets, i.e., given any compact subset \( K \), we have

(i) for any closed subset \( C \subset C([0, 1] \times [0, T]) \),

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \sup_{g \in K} \nu^0_\varepsilon(C) \leq -\inf_{f \in C} I^T_0(f).
\]

(ii) for any open subset \( G \subset C([0, 1] \times [0, T]) \),

\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \inf_{g \in K} \nu^0_\varepsilon(G) \geq -\inf_{f \in G} I^T_0(f).
\]

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Let \( \mu_\varepsilon \) denote the unique invariant probability measure of the solution \( u_\varepsilon(t), t \geq 0 \). Introduce the following assumption:

(H). Assume \( f(x,0) = 0 \) and that there exists a constant \( c < \alpha \) such that
\[
| f(x, u_1) - f(x, u_2) | \leq c|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}.
\] (6.1)

Here is the main result of this paper:

**Theorem 6.2.** Suppose the conditions (F1), (F2) and (H) hold. Then \( \mu_\varepsilon, \varepsilon > 0 \) satisfies a large deviation principle on \( E \) with the rate function \( J(\cdot) \), i.e.,

(i) for any closed subset \( C \subset E \),
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_\varepsilon(C) \leq - \inf_{z \in C} J(z).
\]

(ii) for any open subset \( G \subset E \),
\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu_\varepsilon(G) \geq - \inf_{z \in G} J(z).
\]

To prove this theorem, it is well known (see e.g. [7], [10] and [2]) that it suffices to establish the following:

1. lower bounds: for any \( \delta > 0, \gamma > 0 \) and \( z^* \in E \) there exists \( \varepsilon_0 > 0 \) such that
\[
\mu_\varepsilon(\{z \in E : \|z - z^*\|_\infty < \delta \}) \geq \exp(-\frac{J(z^*) + \gamma}{\varepsilon^2}), \quad \varepsilon \leq \varepsilon_0,
\] (6.2)

2. upper bounds: for any \( s \geq 0, \delta > 0, \gamma > 0 \) there exists \( \varepsilon_0 > 0 \) such that
\[
\mu_\varepsilon(\{z \in E; \text{dis}_E(z, K(s)) \geq \delta \}) \leq \exp(-\frac{s - \gamma}{\varepsilon^2}), \quad \varepsilon \leq \varepsilon_0,
\] (6.3)

where \( K(s) = \{z \in E; J(z) \leq s\} \).

These will be proved in Section 7 and 8.

### 7 Lower bounds

Consider
\[
\frac{\partial u^0(x,t)}{\partial t} = \frac{\partial^2 u^0(x,t)}{\partial x^2} - \alpha u^0(x,t) + f(x, u^0) + \eta^z - \xi^z;
\]
\[
K_1(x) \leq u^0(x,t) \leq K_2(x);
\]
\[
\int_0^T \int_0^1 (u^0(x,t) - K_1(x))\eta^z(dx, dt) = \int_0^T \int_0^1 (K_2(x) - u^0(x,t))\xi^z(dx, dt) = 0
\] (7.1)
\[
u^0(0,0) = z; \quad \frac{\partial \nu^0}{\partial x}(0, t) = \frac{\partial \nu^0}{\partial x}(1, t) = 0.
\]
Write the solution of (7.1) as $u^0(z, x, t)$ to stress the dependence on the initial function $z$. Denote by $B$ the Banach space $C([0, 1])$ with the maximum norm $\| \cdot \|_\infty$.

**Lemma 7.1.** Assume the conditions (F1), (F2) and (H). Then it holds that
\[
\|u^0(z, \cdot, t)\|_\infty \leq e^{-\alpha_1 t}\|z\|_\infty, \tag{7.2}
\]
where $\alpha_1 = \alpha - c > 0$.

**Proof.** Set
\[ A = \frac{\partial^2}{\partial x^2} - \alpha I. \]

We write $f(g)(x)$ for $f(x, g(x))$ for brevity. A similar notation will be used for $\sigma(x, g(x))$. First we claim that for any $g \in E \cap D(A)$, there exists $l_g \in \partial\|g\|_B := \{l \in B^* ; \|l\|_{B^*} = 1, <l, g> = \|g\|_B\}$ such that
\[
<A g, l_g> + <f(g), l_g> \leq -\alpha_1\|g\|_B. \tag{7.3}
\]
In fact, choose $l_g = \delta_{x_0}$ if $g(x_0) = \max_{x \in [0, 1]}|g(x)|$ and $l_g = -\delta_{x_0}$ if $g(x_0) = -\max_{x \in [0, 1]}|g(x)|$. Then
\[
<A g, l_g> + <f(g), l_g> \leq -\alpha\|g\|_B + f(x_0, g(x_0)) \leq (c - \alpha)\|g\|_B. \tag{7.4}
\]
Furthermore, if $g(x_0) = \max_{x \in [0, 1]}|g(x)|$, then, as $g \in E$,
\[
<(g - K_1)^-, l_g> = (g(x_0) - K_1(x_0))^- = 0.
\]
If $g(x_0) = -\max_{x \in [0, 1]}|g(x)|$, then
\[
<(g - K_1)^-, l_g> = -(g(x_0) - K_1(x_0))^- \leq 0,
\]
\[
<(g - K_2)^+, l_g> = 0.
\]
So in all cases we have
\[
<A g, l_g> + <f(g), l_g> \\
+ \frac{1}{\delta} <(g - K_1)^-, l_g> - \frac{1}{\varepsilon} <(g - K_2)^+, l_g> \leq (c - \alpha)\|g\|_B, \tag{7.5}
\]
for all $\varepsilon > 0$, $\delta > 0$. Consider the approximating equations:

$$\begin{align*}
\frac{\partial u_{0,\varepsilon,\delta}(x,t)}{\partial t} &= \frac{\partial^2 u_{0,\varepsilon,\delta}(x,t)}{\partial x^2} + f(u_{0,\varepsilon,\delta}(x,t)) - \alpha u_{0,\varepsilon,\delta}(x,t) \\
&\quad + \frac{1}{\delta}(u_{0,\varepsilon,\delta}(x,t) - K_1(x))^- - \frac{1}{\varepsilon}(u_{0,\varepsilon,\delta}(x,t) - K_2(x))^+; \\
\frac{\partial u_{0,\varepsilon,\delta}(x,0)}{\partial x} &= u_0(x); \\
\frac{\partial u_{0,\varepsilon,\delta}(0,t)}{\partial x} &= \frac{\partial u_{0,\varepsilon,\delta}(1,t)}{\partial x} = 0.
\end{align*}$$

(7.6)

By the chain rule, we have

$$\frac{d}{dt} \|u_{0,\varepsilon,\delta}(t)\|_B \leq <Au_{0,\varepsilon,\delta}(t),l_{u_{0,\varepsilon,\delta}(t)}> + f(u_{0,\varepsilon,\delta}(t),l_{u_{0,\varepsilon,\delta}(t)}) > + \frac{1}{\delta} <u_{0,\varepsilon,\delta}(t) - K_1, l_{u_{0,\varepsilon,\delta}(t)}> - \frac{1}{\varepsilon} <u_{0,\varepsilon,\delta}(t) - K_2, l_{u_{0,\varepsilon,\delta}(t)}> \leq -\alpha_1 \|u_{0,\varepsilon,\delta}(t)\|_B.$$ 

(7.7)

This yields

$$\|u_{0,\varepsilon,\delta}(t)\|_B \leq e^{-\alpha_1 t} \|z\|_B,$$

where $z$ is the initial function. Because the constants on the right side are independent of $\delta, \varepsilon$, let $\delta \to 0$ and $\varepsilon \to 0$ to get (7.2). \hfill \blacksquare

Fix $z^* \in E$ with $J(z^*) < \infty$. For any $\gamma > 0$, by the definition of $J(z^*)$ there exists a function $\psi$ and $T_0 > 0$ such that $\psi(0) = 0$, $\psi(T_0) = z^*$ and $\psi = u^\bar{h}$ for some $\bar{h}$ with $\frac{1}{2} \int_{[0,1] \times [0,T_0]} |\hat{h}|^2 < J(z^*) + \frac{\gamma}{2}$. Define for $T > 0$,

$$\hat{h}^T(x,t) = \begin{cases} 0, & (x,t) \in [0,1] \times [0,T], \\
\hat{h}(x,t-T), & (x,t) \in [0,1] \times [T,T+T_0].
\end{cases}$$

(7.8)

Consider the following PDE with reflection:

$$\begin{align*}
\frac{\partial u^T(z,x,t)}{\partial t} &= \frac{\partial^2 u^T(z,x,t)}{\partial x^2} + f(x,u^T) + \sigma(x,u^T)\hat{h}^T(x,t) \\
&\quad + \eta^T - \xi^T; \\
u^T(\cdot,0) &= z, \quad u^h(0,t) = u^h(1,t) = 0.
\end{align*}$$

(7.9)

Clearly,

$$\begin{align*}
u^T(z,x,t) &= \begin{cases} u^0(z,x,t), & (x,t) \in [0,1] \times [0,T], \\
u^h(u^0(z,\cdot,T),x,t-T), & (x,t) \in [0,1] \times [T,T+T_0].
\end{cases}
\end{align*}$$

(7.10)
Set $\tilde{\psi}(x,t) = u^T(z,x,t+T)$ for $t \geq 0$. Then we have
\[
\frac{\partial \tilde{\psi}(z,x,t)}{\partial t} = \frac{\partial^2 \tilde{\psi}(z,x,t)}{\partial x^2} + f(x,\tilde{\psi}) + \sigma(x,\tilde{\psi})\dot{h}(x,t) + \bar{\eta} - \bar{\xi};
\]
(7.11)
Recall that $\psi = u^h$ satisfies the following reflected PDE:
\[
\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial^2 \psi(x,t)}{\partial x^2} + f(x,\psi) + \sigma(x,\psi)\dot{h}(x,t) + \eta - \xi;
\]
(7.12)
$\psi(x,0) = 0$.

Put
\[
F(t) := \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} |\psi(x,s) - \tilde{\psi}(x,s)| = \sup_{0 \leq s \leq t} ||\psi(s) - \tilde{\psi}(s)||_{\infty}.
\]
We have the following result:

**Proposition 7.1.** Assume (F1). We have
\[
F(T_0) \leq C(T_0, ||\tilde{h}||) \sup_{x \in [0,1]} |u^0(z,x,T)|.
\]
(7.13)

**Proof.** Set
\[
A(x,t) = \int_0^t \int_0^1 G_{t-s}(x,y)f(y,\psi(y,s))dyds
+ \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,\psi(y,s))\dot{h}(y,s)dyds.
\]
(7.14)
\[
\dot{A}(x,t) = P_t(u^0(z,\cdot,T))(x) + \int_0^t \int_0^1 G_{t-s}(x,y)f(y,\tilde{\psi}(y,s))dyds
+ \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,\tilde{\psi}(y,s))\dot{h}(y,s)dyds.
\]
(7.15)
Then it follows from Theorem 3.1 in [13] (also see Remark 3.1) that for
\[ t \leq T_0, \]
\[
F(t) \leq 2 \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} |A(x, s) - \bar{A}(x, s)| \\
\leq 2 \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} |P_s(u^0(z, \cdot), T))(x)| \\
+ 2 \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s \int_0^1 G_{s-u}(x, y)\left| f(y, \psi(y, u)) - f(y, \bar{\psi}(y, u)) \right| dy du \right| \\
+ 2 \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s \int_0^1 G_{s-u}(x, y)\left| \sigma(y, \psi(y, u)) - \sigma(y, \bar{\psi}(y, u)) \right| \hat{h}(y, u) dy du \right|.
\]
\[
\leq C_{T_0} \sup_{x \in [0,1]} |u^0(z, \cdot, T)(x)| \\
+ 2C \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s \sup_{y \in [0,1]} \left| \psi(y, u) - \bar{\psi}(y, u) \right| \int_0^1 G_{s-u}(x, y) dy du \right| \\
+ 2C \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s \int_0^1 G_{s-u}(x, y) \sup_{y \in [0,1]} \left| \psi(y, u) - \bar{\psi}(y, u) \right| |\hat{h}(y, u)| dy du \right|
\]
\[
\leq C_{T_0} \sup_{x \in [0,1]} |u^0(z, \cdot, T)(x)| + C \int_0^T F(u) du \\
+ 2C \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s F(u) \int_0^1 G_{s-u}(x, y) |\hat{h}(y, u)| dy du \right|. \tag{7.16}
\]

Now,
\[
\sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s F(u) \int_0^1 G_{s-u}(x, y) |\hat{h}(y, u)| dy du \right|
\]
\[
\leq \sup_{0 \leq s \leq t} \sup_{x \in [0,1]} \left| \int_0^s F(u) du \int_0^1 G_{s-u}(x, y)^2 dy \right|^{\frac{1}{2}} \left( \int_0^1 |\hat{h}(y, u)|^2 dy \right)^{\frac{1}{2}}
\]
\[
\leq C \sup_{0 \leq s \leq t} \left( \int_0^s F(u) \frac{1}{\sqrt{s-u}} du \right)^{\frac{1}{2}} \left( \int_0^1 |\hat{h}(y, u)|^2 dy \right)^{\frac{1}{2}}
\]
\[
\leq C \sup_{0 \leq s \leq t} \left( \int_0^s F(u)^{\frac{1}{2}} du \frac{1}{\sqrt{s-u}} \right)^{\frac{1}{2}} \left( \int_0^1 |\hat{h}(y, u)|^2 dy \right)^{\frac{1}{2}}
\]
\[
\leq C |\hat{h}|_{L^2([0,T_0] \times [0,1])} \sup_{0 \leq s \leq t} \left( \int_0^s F(u)^{\frac{1}{2}} du \right)^{\frac{1}{2}} \left( \int_0^1 \frac{1}{\sqrt{s-u}} \right)^{\frac{1}{2}}
\]
\[
\leq C |\hat{h}|_{L^2([0,T_0] \times [0,1])} C_{T_0, q} \left( \int_0^s F(u)^{\frac{1}{2}} du \right)^{\frac{1}{2}}, \tag{7.17}
\]

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here $p > 2$, $\frac{1}{p} + \frac{1}{q} = 1$. Combining (7.16) and (7.17) we obtain for $t \leq T_0$,

$$F(t)^{2p} \leq C_{T_0,p} \sup_{x \in [0,1]} |u^0(z, \cdot, T)(x)|^{2p} + C_{T_0,p} \int_0^t F(u)^{2p} du$$

$$+ C_{T_0,p} |\dot{h}|_{L^2([0,T_0] \times [0,1])}^{2p} \int_0^t F(u)^{2p} du$$

(7.18)

Applying Gronwall’s inequality yields

$$F(T_0)^{2p} \leq C(T_0, p, ||h||) \sup_{x \in [0,1]} |u^0(z, \cdot, T)(x)|^{2p},$$

giving (7.13). □

Let $\mu_\varepsilon$ be the invariant measure of the solution $u_\varepsilon$ of the reflected SPDE (1.1). We have

**Theorem 7.1.** Let the assumptions (F1), (F2) and (H) hold. For any $\delta > 0, \gamma > 0$ and $z^* \in E$ there exists $\varepsilon_0 > 0$ such that

$$\mu_\varepsilon(\{z \in E : ||z - z^*||_{\infty} < \delta\}) \geq \exp\left(-\frac{J(z^*) + \gamma}{\varepsilon^2}\right), \quad \varepsilon \leq \varepsilon_0.$$  

(7.19)

**Proof.** Without loss of generality, we assume $J(z^*) < \infty$. For $\gamma > 0$, there exists a function $\psi$ and $T_0 > 0$ such that $\psi(0) = 0$, $\psi(T_0) = z^*$ and $\psi = \mathbb{u}^{\bar{h}}$ for some $\bar{h}$ with $\frac{1}{2} |\bar{h}|_{L^2([0,1] \times [0,T_0])}^2 < J(z^*) + \frac{\gamma}{2}$. Combining Lemma 7.1 and Proposition 7.1, we see that there exists a sufficiently big constant $T > 0$ such that

$$||z^* - u^T(z, T_0 + T)||_{\infty} = ||\psi(T_0) - u^T(z, T_0 + T)||_{\infty} \leq \frac{\delta}{2}.$$  

Thus for any $z \in E$ we have

$$||u_\varepsilon(z, T_0 + T) - z^*||_{\infty} \leq ||u_\varepsilon(z, T_0 + T) - u^T(z, T_0 + T)||_{\infty} + ||z^* - u^T(z, T_0 + T)||_{\infty} \leq ||u_\varepsilon(z, T_0 + T) - u^T(z, T_0 + T)||_{\infty} + \frac{\delta}{2}.$$  

(7.20)

On the other hand, using Theorem 5.1, we can find a compact subset $K$ such that for $\varepsilon \leq \varepsilon_0$,

$$\mu_\varepsilon(K) \geq \frac{1}{2}.$$  

(7.21)
By the invariance of $\mu_\varepsilon$, we have, in view of (7.20),
\[
\mu_\varepsilon(\{z \in E : ||z - z^*||_\infty < \delta\}) \\
= \int_E P(||u_\varepsilon(z, T_0 + T) - z^*||_\infty < \delta) \mu_\varepsilon(dz) \\
\geq \int_E P(||u_\varepsilon(z, T_0 + T) - u^T(z, T_0 + T)||_\infty < \frac{\delta}{2}) \mu_\varepsilon(dz) \\
\geq \int_E P(||u_\varepsilon(z) - u^T(z)||_{C([0,1] \times [0, T_0 + T])} < \frac{\delta}{2}) \mu_\varepsilon(dz) \\
\geq \int_K P(||u_\varepsilon(z) - u^T(z)||_{C([0,1] \times [0, T_0 + T])} < \frac{\delta}{2}) \mu_\varepsilon(dz).
\tag{7.22}
\]
By the uniform large deviation principle satisfied by $u_\varepsilon$ (Theorem 6.1), there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, $z \in K$,
\[
P(||u_\varepsilon(z) - u^T(z)||_{C([0,1] \times [0, T_0 + T])} < \frac{\delta}{2}) \geq \exp\left(-\frac{\varepsilon^2}{2}\frac{\|J(z^*)\|_{L^2([0,1] \times [0, T_0])} + \gamma}{\varepsilon^2}\right) \\
\geq \exp\left(-\frac{J(z^*) + \gamma}{\varepsilon^2}\right), \quad \varepsilon \leq \varepsilon_0.
\]
Putting (7.22), (7.21) and (7.23) together, we obtain
\[
\mu_\varepsilon(\{z \in E : ||z - z^*||_\infty < \delta\}) \geq \frac{1}{2} \exp\left(-\frac{J(z^*) + \gamma}{\varepsilon^2}\right), \quad \varepsilon \leq \varepsilon_0.
\]
\[\blacksquare\]

8 Upper bounds

Lemma 8.1. Assume (F1), (F2) and (II). For any $\delta > 0, s > 0$, there exists $T_0 > 0$ such that
\[
\{g(t); g \in K_{0,t}(s)\} \subset \{z \in E; \text{dist}_E(z, K(s)) \leq \frac{\delta}{2}\}, \quad t \geq T_0. \tag{8.1}
\]

The same proof as that of Lemma 7.1 in [2] works here. We refer the reader to [2] for details.

After the necessary preparations, using arguments analogous to those employed by Sowers in [10] and by Cerrai, Röckner in [2], we can prove the following upper bounds. Put $K(s) = \{z \in E; J(z) \leq s\}$. 

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Theorem 8.1. Assume (F1), (F2) and (H). For any \( s \geq 0, \delta > 0, \gamma > 0 \) there exists \( \varepsilon_0 > 0 \) such that
\[
\mu_{\varepsilon}(\{z \in E; \text{dis}_E(z, K(s)) \geq \delta \}) \leq \exp\left(-\frac{s - \gamma}{\varepsilon^2}\right), \quad \varepsilon \leq \varepsilon_0.
\] (8.2)

Proof. Let \( L > s - \gamma \). By Theorem 5.1 there exists a compact subset \( K_L \subset E \) and \( \varepsilon_1 > 0 \) such that for \( \varepsilon \leq \varepsilon_1 \),
\[
\mu_{\varepsilon}(K_L^c) \leq \exp\left(-\frac{L}{\varepsilon^2}\right).
\] (8.3)

By the invariance of the measure \( \mu_{\varepsilon} \), for any \( t \geq 0 \), we have
\[
\mu_{\varepsilon}(\{z \in E; d(z, K(s)) \geq \delta \}) = \int_E P(d(\varepsilon(z, t), K(s)) \geq \delta) \mu_{\varepsilon}(dz). \quad (8.4)
\]

Thus,
\[
\mu_{\varepsilon}(\{z \in E; d(z, K(s)) \geq \delta \}) \\
\leq \mu_{\varepsilon}(K_L^c) + \int_{K_L} P(d(\varepsilon(z, t), K(s)) \geq \delta) \mu_{\varepsilon}(dz). \quad (8.5)
\]

By Lemma 8.1, there exists \( T_1 > 0 \) such that for \( t \geq T_1 \),
\[
P(d(\varepsilon(z, t), K(s)) \geq \delta) \\
\leq P(d(\varepsilon(z, t), K_0, t(s)) \geq \delta) \\
\leq P(d(\varepsilon(z, t), K_{z, 0, t}(s)) \geq \delta). \quad (8.6)
\]

The uniform large deviation principle of \( \varepsilon(t) \) on the path space implies that there exists \( \varepsilon(t) > 0 \) such that for \( z \in K_L \),
\[
P(d(\varepsilon(z, t), K_{z, 0, t}(s)) \geq \delta) \leq \exp\left(-\frac{s - \gamma/2}{\varepsilon^2}\right), \quad \varepsilon \leq \varepsilon(t). \quad (8.7)
\]

Choosing \( t = T_1 \) we obtain
\[
\int_{K_L} P(d(\varepsilon(z, t), K(s)) \geq \delta) \mu_{\varepsilon}(dz) \leq \exp\left(-\frac{s - \gamma/2}{\varepsilon^2}\right), \quad \varepsilon \leq \varepsilon(t). \quad (8.8)
\]

for \( \varepsilon \leq \varepsilon_0 \), where \( \varepsilon_0 > 0 \).

Combining (8.2), (8.5) with (8.8) it follows that
\[
\mu_{\varepsilon}(\{z \in E; d(x, K(s)) \geq \delta \}) \leq 2\exp\left(-\frac{s - \gamma/2}{\varepsilon^2}\right),
\]

which gives the upper bound. ■

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