ASYMPTOTIC BEHAVIOR OF A SCHRÖDINGER EQUATION UNDER A CONSTRAINED BOUNDARY FEEDBACK

HAOYUE CUI
Department of Mathematics, Tianjin University of Commerce
Tianjin 300134, China

DONGYI LIU* AND AND GENQI XU
School of Mathematics, Tianjin University
Tianjin 300354, China

(Communicated by Xu Liu)

Abstract. Design of controller subject to a constraint for a Schrödinger equation is considered based on the energy functional of the system. Thus, the resulting closed-loop system is nonlinear and its well-posedness is proven by the nonlinear monotone operator theory and a complex form of the nonlinear Lax-Milgram theorem. The asymptotic stability and exponential stability of the system are discussed with the LaSalle invariance principle and Riesz basis method, respectively. In the end, a numerical simulation illustrates the feasibility of the suggested feedback control law.

1. Introduction. The Schrödinger equation is an efficient model to describe optical propagation and dynamics of quantum system, and especially, it can simulate the behavior of micro mechanics precisely ([9, 13, 19, 32]), so, it has drawn a great deal of attention. There are extensive literatures on Schrödinger equations, involving solvability, controllability, stabilization and so on ([1, 6, 7, 10, 11, 14, 20, 22, 23, 24, 25, 28, 29, 30]). In this paper, we address the following Schrödinger equation

\[
\begin{aligned}
    w_t(x,t) &= -iw_{xx}(x,t), \quad x \in (0,1), \quad t > 0, \\
    w(0,t) &= 0, \quad w_x(1,t) = u(t), \\
    w(x,0) &= w_0(x),
\end{aligned}
\]

where \(i\) is the pure imaginary unit, \(u(t)\) is the control (input) and \(w_0(x)\) is the initial state. In some biological and chemical experiments, the control signals are often limited in an interval due to some constraint conditions, e.g., the restriction of concentration in some types of chemical reactions. Therefore, it is very interesting to study the system (1) with a bounded control, i.e., \(|u(t)| \leq \beta\), where \(\beta\) is a positive constant.

Based on the \(L^2\)-energy of the system (1)

\[
E(t) = \frac{1}{2} \int_0^1 |w(x,t)|^2 dx,
\]

2010 Mathematics Subject Classification. Primary: 93D15, 93D20; Secondary: 35B40.
Key words and phrases. Schrödinger equation, constrained boundary control, collocated observation, maximal monotone operator, invariant principle, stability.

This research is supported by the Natural Science Foundation of China grant NSFC-61573252.

* Corresponding author: Dongyi Liu.
its derivative with respect to $t$ along the locus of the system (1) can be formulated by
\[
\frac{dE(t)}{dt} = -\Im \left[ u(t)w(1, t) \right],
\]
where the notation "\(\Im\)" is the imaginary part of a complex number. Thus, the feedback control is taken as follows:
\[
u(t) = -i \max\{\frac{|w(1, t)|}{\beta}, 1\} w(1, t),
\]
when \(|w(1, t)| \leq \beta, \]
\[
-\beta \frac{w(1, t)}{|w(1, t)|}, \text{ when } |w(1, t)| > \beta.
\]
(3)

Thus, \(|u(t)| \leq \beta\) and
\[
\frac{dE(t)}{dt} = \begin{cases} 
-|w(1, t)|^2, \text{ when } |w(1, t)| \leq \beta, \\
-\beta|w(1, t)|, \text{ when } |w(1, t)| > \beta.
\end{cases}
\]
(4)

Hence, the closed-loop system (1) with (3) can be formulated by
\[
\begin{aligned}
&w_t(x, t) = -iw_{xx}(x, t), \quad x \in (0, 1), \quad t > 0, \\
&w(0, t) = 0, \quad w_x(1, t) = -iw_\beta(1, t), \quad t > 0, \\
&w(x, 0) = w_0(x), \quad x \in (0, 1),
\end{aligned}
\]
(5)

where \(w_\beta(1, t) = \max\{\frac{|w(1, t)|}{\beta}, 1\}\) and \(|w_\beta(1, t)| \leq \beta\).

Since the dynamic system (5) is nonlinear, the premier difficulty is its solvability. A similar problem was discussed in [23]. In this paper, the well-posedness is proven via the maximal monotone operator theory and a complex form of the nonlinear Lax-Milgram theorem. Let \(y(t) = w(1, t)\) be the collocated observation (output), \(y(t)\) is obviously not continuous with respect to \(w\) (refer to [21, Theorem 1]). Thus, the second key point is the asymptotical stability of the output, \(\lim_{t \to \infty} y(t) = 0\), which is an important step for the exponentially stable analysis. Hence, the novelties in this paper are to provide a complex form of the Lax-Milgram theorem and to prove the exponential stability of the system (5) without the assumption of Theorem 1 in [21] that the collocated output of the system is uniformly continuous on its state.

In this paper, the notation "\(\Re\)" and "\(\Im\)" stand for the real and imaginary part of a complex number, respectively. \(\dot{w}(t)\) is the time derivative of \(w(t)\). \(f'(x)\) and \(f''(x)\) are the first and second derivative of \(f(x)\) with respect to the spatial variable \(x\), respectively.

The remainder of the paper is proceeded as follows. In section 2, the well-posedness of the system (5) is discussed with the maximal monotone operator theory and a complex form of the nonlinear Lax-Milgram theorem, which is proven in Appendix for completeness. In section 3, the asymptotic stability, the asymptotical stability of the output and the exponential stability are proven with the invariant principle in infinite dimensional space [8], an estimation of the upper bound for \(w(1, t)\) and Riesz basis method [20, 35, 36], respectively. In section 4, a numerical simulation is given to verify the theoretical results and the feasibility of the presented feedback law (3). Finally, some concluding remarks are presented in section 5.

2. Well-posedness of the system. This section discusses the well-posedness of the system (5). For this, the state space \(\mathcal{H}\) is chosen as \(L^2(0, 1)\) with the norm
\[
\|f\|^2 = \int_0^1 f(x)\overline{f(x)}dx.
\]
The system operator $\mathcal{A}$ in $\mathcal{H}$ is defined by
\[ \mathcal{A}f(x) = -if''(x), \] (6)
with its domain
\[ \text{dom}(\mathcal{A}) = \{ f \in H^2(0,1) \mid f(0) = 0, f'(1) = -if_\beta(1) \}, \] (7)
where $f_\beta(1) = \max \{ |f(1)|/\beta, 1 \}$ and $H^2(0,1)$ is the usual Sobolev space [2, Chapter 3]. Then the system (5) can be rewritten as a nonlinear evolutionary equation in $\mathcal{H}$
\[ \begin{cases} \dot{w}(t) = \mathcal{A}w(t), & t > 0, \\ w(0) = w_0, \end{cases} \] (8)
where $w(t) = w(\cdot, t)$ and $w_0$ is an initial state.

In addition, if a new operator $\mathcal{A}_0$ is defined by
\[ \mathcal{A}_0f(x) = -if''(x) \]
with the domain
\[ \text{dom}(\mathcal{A}_0) = \{ f \in H^2(0,1) \mid f(0) = 0, f'(1) = 0 \}, \]
the system (1) can be formulated as
\[ \begin{cases} \dot{w}(t) = \mathcal{A}_0w(t) + \mathcal{B}_0u(t), & t > 0, \\ w(0) = w_0, \end{cases} \] (9)
where $\mathcal{B}_0 = \delta(\cdot - 1)$ is the Dirac Delta distribution. Obviously, input operator $\mathcal{B}_0$ is unbounded and the nonhomogeneous term $\mathcal{B}_0u(t)$ with $u(t)$ defined by (3) does not satisfy the Lipschitz continuous condition on $w(t)$. Therefore, the linear nonhomogeneous theory of the Cauchy problem is not suitable for (9). Thus, it is necessary to apply the theory of nonlinear monotone operators to prove the existence and uniqueness of the solution of the system (8) (i.e., (5)).

According to the definition of $f_\beta(1)$ in (7), it can be guessed that the operator $-\mathcal{A}$ defined by (6) and (7) is monotone. Furthermore, $\mathcal{A}$ satisfies the following proposition.

**Proposition 1.** $-\mathcal{A}$ is a maximal monotone operator.

Thus, according to Proposition 1, and Kōmura-Kato Theorem ([5, Chapter 4], [17], [18], [31, Chapter 2], [33, Chapter 4], [37, Chapter 31]) we can obtain the following theorem, which shows the well-posedness of the closed-loop system (8) (i.e., (5)).

**Theorem 2.1.** Let $\mathcal{A}$ be defined by (6) and (7). Then
(1): $\mathcal{A}$ is a generator of $S(t)$, a nonexpansive semigroup on $\text{dom}(\mathcal{A})$, which can be uniquely extended on $\text{dom}[\mathcal{A}]$, the closure of $\text{dom}(\mathcal{A})$.
(2): For each given $w_0 \in \text{dom}(\mathcal{A})$, $w(t) = S(t)w_0 \in \text{dom}(\mathcal{A})$ is exactly one solution of (8), which is Lipschitz continuous function on $[0, \infty)$, where the derivative $\dot{w}(t)$ is to be understood in the sense of weak convergence.
(3): For each given $w_0 \in \text{dom}(\mathcal{A})$, $w(t) = S(t)w_0 \in \text{dom}(\mathcal{A})$ also is the unique strong solution such that $w \in W^{1,\infty}(0, +\infty, \mathcal{H})$.
(4): $\|\mathcal{A}w(t)\|$ and $\|w(t)\|$ are non-increasing with respect to time $t$.

To prove the maximality of $-\mathcal{A}$ in Proposition 1, a generalization form of the nonlinear Lax-Milgram theorem needs to be introduced, whose proof will be given in Appendix.
Lemma 2.2. Let \( Z \) be a complex Hilbert space and \( L \) a bounded linear functional on it. Assume that \( B(x, z) = B_1(x, z) + iB_2(x, z) \) is a complex-valued functional defined on the product space \( Z \times Z \), and satisfies the following inequality: \( \forall z \in Z, \)
\[
\Re(B(z, z)) \geq c_B \|z\|_2^2, \quad \text{with } c_B > 0,
\]
where \( B_1(x, z) \) and \( B_2(x, z) \) are bounded sesqui-linear forms. And assume that \( N(x, z) \) is a complex-valued nonlinear functional defined on \( Z \times Z \), and satisfies the following conditions:

(1): For every \( x, N(x, z) \) is a bounded linear functional with respect to \( z \), and \( N(x, z) \) is sequentially continuous with respect to \( x \), that is, \( x_n \to x \) implies that \( N(x_n, z) \to N(x, z) \) for each \( z \in Z \).

(2): \( \Re(N(z, z) + N(x, x) - N(x, z) - N(z, x)) \geq 0 \).

Then, there exists a unique \( x \in Z \) such that
\[
B(x, z) + N(x, z) = L(z), \forall z \in Z.
\]

In what follows, the proof of Proposition 1 is divided into two steps.

Step 1. monotonicity

Proof. For any \( w, \phi \in \text{dom}(A) \), it follows from (6), (7) and the integration by parts that
\[
(Aw, \phi) = \int_0^1 -iw''(x)\bar{\phi}(x)dx
\]
\[
= -iw'(x)\bar{\phi}(x)|_0^1 + iw(x)\bar{\phi}'(x)|_0^1 - i\int_0^1 w(x)\bar{\phi}''(x)dx
\]
\[
= -w_\beta(1)\bar{\phi}(1) - w(1)\bar{\phi}_\beta(1) - (w, A\phi),
\]
where
\[
w_\beta(1) = \frac{w(1)}{\max\{|w(1)|/\beta, 1\}} \quad \text{and} \quad \phi_\beta(1) = \frac{\phi(1)}{\max\{|\phi(1)|/\beta, 1\}}.
\]

Hence,
\[
(Aw, w) + (w, Aw) = -w_\beta(1)\bar{w}(1) - w(1)\bar{w}_\beta(1),
\]
and
\[
\Re(Aw - A\phi, w - \phi) = \Re(Aw, w) + \Re(A\phi, \phi) - \Re(Aw, \phi) - \Re(A\phi, w)
\]
\[
= -\Re\left\{[w(1) - \phi(1)] [w_\beta(1) - \phi_\beta(1)]\right\} \triangleq -\text{RH}.
\]

When \( |w(1)| > \beta \) and \( |\phi(1)| > \beta \),
\[
\text{RH} = \Re\left\{[w(1) - \phi(1)] [w_\beta(1) - \phi_\beta(1)]\right\}
\]
\[
= \beta\Re\left\{[w(1)| + |\phi(1)|]\phi(1) - \frac{\phi(1)}{|\phi(1)|} [w(1)|/|\phi(1)|]\right\}
\]
\[
= \beta(|w(1)| + |\phi(1)|) \left(1 - \frac{\Re(w(1)|\phi(1)|}{|w(1)||\phi(1)|}\right) \geq 0.
\]

When \( |w(1)| \leq \beta \) and \( |\phi(1)| \leq \beta \),
\[
\text{RH} = |w(1) - \phi(1)|^2 \geq 0.
\]
From the following inequalities:  

\[
\Re \left( \frac{\beta}{|\phi(1)|} |w(1)|^2 \right) \geq 0 \quad \text{or} \quad \Re \left( \frac{\beta}{|\phi(1)|} |\phi(1) - w(1)|^2 \right) \geq 0
\]

Thus,  

\[
\Re (Aw - A \phi, w - \phi) \leq 0,
\]

which implies that \(-A\) is monotone. \(\square\)

**Step 2. maximality**

*Proof.* To prove the maximality, we only need to verify that \(R(I - A) = \mathcal{H}\), that is, the range of \(I - A\) is the total space \(\mathcal{H}\).

\(\forall f \in \mathcal{H}\), we want to establish the solvability of the resolvent equation \((I - A)w = f\), that is, there exists \(w \in \text{dom}(A)\), such that,

\[
w''(x) = iw(x) - if(x).
\]

We introduce a Hilbert space \(\mathcal{Z} = \{ \phi \in H^1(0,1) | \phi(0) = 0 \}\), whose inner product and norm are defined by

\[
(\varphi, \psi) = \int_0^1 \varphi'(x)\overline{\psi'(x)}dx \quad \text{and} \quad ||\varphi||_{\mathcal{Z}} = \sqrt{(\varphi, \varphi)},
\]

respectively, where \(H^1(0,1)\) is the usual Sobolev space. Thus for \(\phi \in \mathcal{Z}\), multiplying both sides of the resolvent equation (12) by \(\bar{\phi}(x)\) and integrating from 0 to 1, we obtain that

\[
i \int_0^1 w(x)\overline{\phi(x)}dx + iw_\beta(1)\overline{\phi(1)} + \int_0^1 \phi(x)\overline{w'(x)}dx = i \int_0^1 f(x)\overline{\phi(x)}dx.
\]

Denote by \(N(w, \phi) = iw_\beta(1)\overline{\phi(1)}\), \(L(\phi) = i \int_0^1 f(x)\overline{\phi(x)}dx\) and

\[
B(w, \phi) = \int_0^1 w'(x)\overline{\phi'(x)}dx + i \int_0^1 w(x)\overline{\phi(x)}dx \triangleq B_1(w, \phi) + iB_2(w, \phi),
\]

then, from (13), we can obtain the following variational equation

\[
B(w, \phi) + N(w, \phi) = L(\phi)
\]

and \(\Re(B(w, w)) = ||w||^2_{\mathcal{Z}}\).

Similar to the proof of the monotonicity of \(-A\), we can deduce that

\[
\Re[N(w, w) + N(\phi, \phi) - N(\phi, w) - N(w, \phi)]
\]

\[
= \Re[i(w_\beta(1)\overline{\phi(1)} - w_\beta(1)\overline{\phi(1)} - \phi_\beta(1)\overline{w(1)})]
\]

\[
= \Re[i(w_\beta(1) - \phi_\beta(1))(\overline{w(1)} - \overline{\phi(1)})] = 0.
\]

From the following inequalities:

\[
\int_0^1 |\phi(x)|^2dx \leq \int_0^1 |\phi'(x)|^2dx \quad \text{and} \quad |\phi(1)|^2 \leq \int_0^1 |\phi'(x)|^2dx, \forall \phi \in \mathcal{Z},
\]

it can be deduced that \(B_1(w, \phi)\) and \(B_2(w, \phi)\) are bounded sesqui-linear forms on \(\mathcal{Z}\), and that \(N(w, \phi)\) is bounded linear functional with respect to \(\phi\) and is sequentially continuous with respect to \(w\). Thus, according to Lemma 2.2, there exists \(w \in \mathcal{Z}\) such that (13) holds. Further restricting \(\phi\) so that \(\phi \in C_0^\infty(0,1)\), we can derive from (13) that

\[
i \int_0^1 w(x)\overline{\phi(x)}dx - \int_0^1 \phi(x)\overline{w'(x)}dx = i \int_0^1 f(x)\overline{\phi(x)}dx.
\]
which implies that (12) holds in the sense of distributions. Thus, \( w \in Z \subset H \) and \( f \in H \) leads to \( w' \in H \). So, \( w \in Z \cap H^2(0,1) \), according to the Sobolev spaces theory [2].

To argue \( w'(1) = -iw_\beta(1) \), we return to (13) which is true for \( \phi \in Z \). Integrating by parts in the third term and using the above equation, we get

\[
\frac{w_\beta(1)}{\phi(1)} - iw'(1)\phi(1) = 0.
\]

Thus, \( w'(1) = -iw_\beta(1) \), which shows that there exists a \( w \in \text{dom}(A) \) such that (12) holds. So \( R(I - A) = H \).

3. **Stability of the system.** In this section, we first study the asymptotical stability, using the LaSalle invariance principle in infinite dimensional space, which is a very useful tool to have been applying to the asymptotical stability of nonlinear systems extensively (\cite{3, 4, 34}, etc.). Then, we discuss the asymptotical stability of the output and the exponential stability. For this, we need the following proposition.

**Proposition 2.** \( (I - A)^{-1} : H \mapsto \text{dom}(A) \subset H \) is compact.

**Proof.** Since \( -A \) is monotone, it can be easily obtained that

\[
(w - \phi - Aw + A\phi, w - \phi - Aw + A\phi) \geq (w - \phi, w - \phi),
\]

which implies that \( I - A \) is an injective (one-to-one) map. Thus, it can be concluded from \( R(I - A) = H \) that \( (I - A)^{-1} \) exists. It follows from \( H^2(0,1) \subset H = L^2(0,1) \) with compact embedding \cite[Chapter 6]{2} that \( (I - A)^{-1} \) is compact.

Next, we study the asymptotical stability of the system (5) (i.e., (8)).

3.1. **Asymptotical stability.** It follows from Proposition 2 that the positive trajectory through \( w \) of the system (5) (i.e., (8)) defined by

\[
\mathcal{O}^+(w) = \cup_{t \geq 0} \{ S(t)w \}
\]

is relative compact for any \( w \in \text{dom}(A) \), according to Theorem 3 in \cite{12}. So, we can apply LaSalle invariance principle in the infinite dimensional space to the system (5) (see Theorem 9.2.7 and Corollary 9.2.9 in \cite{8}, also \cite{12, 15, 34}). We denote the set of equilibrium points of \( \{ S(t) \}_{t \geq 0} \) by \( E \), then

\[
E = \{ w \in H | \dot{E}(S(t)w) = 0 \} = \{ w \in H | w(1, t) = 0 \}
\]

where \( w(x, t) \) satisfies the following system

\[
\begin{cases}
w_t(x, t) = -iw_{xx}(x, t), & x \in (0, 1), \ t > 0, \\
w(0, t) = 0, w(1, t) = 0, & w_x(1, t) = 0, \ t > 0, \\
w(x, 0) = w_0, & x \in (0, 1).
\end{cases}
\]  

(14)

Obviously, (14) only has the trivial solution, which, together with the LaSalle invariance principle, shows that the system (5) is asymptotically stable. Thus, we can draw the following conclusion:

**Theorem 3.1.** Let \( A \) be defined by (6) and (7), and the initial state \( w_0 \in \text{dom}(A) \). Then, the \( L^2 \)-energy function (2) decays to zero as \( t \to \infty \), i.e., \( \lim_{t \to \infty} E(w(\cdot, t)) = 0 \), where \( w(x, t) \) is a solution of (5).
3.2. Asymptotical stability of the output. From (2), (4) and Theorem 2.1, it can be deduced that
\[ \min\{|w(1,t)|^2, \beta |w(1,t)|\} = -\frac{dE(t)}{dt} = \langle Aw(\cdot,t), w(\cdot,t) \rangle \leq c_1 \|w_0\|, \quad \forall t > 0, \] (15)
where \(w_0 \in \text{dom}(A)\) and \(c_1 = \|Aw_0\|\). Thus, it can be derived from (15) that
\[ |w(1,t)| \leq \frac{\beta - 1}{c_1} \|w_0\| \] for \(|w(1,t)| > \beta\), so,
\[ c_0 |w(1,t)|^2 \leq c_1 \|w(\cdot,t)\|, \quad \forall t > 0. \] (16)

According to Theorem 3.1, we have

**Theorem 3.2.** Let \(A\) be defined by (6) and (7), and the initial state \(w_0 \in \text{dom}(A)\). Then, for \(y(t) = w(1,t)\), the collocated output of system (5), \(\lim_{t \to \infty} y(t) = 0\).

3.3. Exponential stability. It follows from Theorem 3.2 that \(|w(1,t)| \leq \beta\) always holds for \(t > 0\) large enough. So, the nonlinear system (5) reduces to a linear one for \(t > t_0\) with \(t_0 > 0\) large enough, that is,
\[ \begin{cases} w_0(x,t) = -iw_{xx}(x,t), \quad x \in (0,1), \quad t > t_0, \\ w(0,t) = 0, \quad w_x(1,t) = -iw(1,t), \quad t > t_0. \end{cases} \] (17)

For the system (17), we can define its system operator \(L\) by
\[ Lf = -if''(x) \] (18)
with domain
\[ \text{dom}(L) = \{ f \in H^2(0,1) | f(0) = 0, \quad f'(1) = -if(1) \}. \] (19)

In what follows, we investigate the asymptotic spectra of \(L\). Assume that \(\lambda\) is an eigenvalue for \(L\), and \(f \in \text{dom}(L)\) is the corresponding eigenfunction, then
\[ \begin{cases} f''(x) = i\lambda f(x), \\ f(0) = 0, \quad f'(1) = -if(1). \end{cases} \] (20)

By a simple calculation, we can obtain from (20) that \(f(x) = \sinh(\sqrt{i\lambda}x)\) and
\[ e^{2\sqrt{i\lambda}} = -1 - \frac{2}{i\sqrt{i\lambda} - 1}. \] (21)

Denote by
\[ 2\sqrt{i\lambda} = i(2n + 1)\pi + \varepsilon_n \] (22)
where \(\varepsilon_n \to 0\) as \(n \to \infty\). Substituting (22) into (21), we get
\[ \varepsilon_n = -\frac{4}{2 + \pi(2n + 1)} + O(\varepsilon_n^2). \]

Then, the eigenvalues of operator \(L\) are
\[ \lambda_n = -2 + i\frac{(2n + 1)^2\pi^2}{4} + O\left(\frac{1}{n}\right), \]
where \(n\) is a sufficiently large integer. Correspondingly, the asymptotic expression of eigenfunction is
\[ f_n(x) = \sin((n + 1/2)\pi x). \]

Similar to Theorem 2.2 in [20], we can deduce the following theorem.
Theorem 3.3. For the operator $\mathcal{L}$ defined by (18) and (19), there exists a sequence of (generalized) eigenfunctions of $\mathcal{L}$, which forms a Riesz basis for $\mathcal{H}$. Additionally, the following assertions are true.

1. The eigenvalues with sufficiently large module are algebraically simple.
2. The spectrum determined growth condition holds for the $C_0$ semigroup $e^{Lt}$: $s(\mathcal{L}) = \omega(\mathcal{L})$, where $s(\mathcal{L})$ is the spectral bound of $\mathcal{L}$ and $\omega(\mathcal{L})$ is the growth order of $e^{Lt}$.
3. System (17) is exponentially stable for $w_0 \in \text{dom}(\mathcal{L})$, i.e., there exist constants $\rho > 0$ and $M > 0$, such that
   $$\|e^{Lt}\| \leq Me^{-\rho t}, \text{ for } t > t_0,$$
   and the $L^2$-energy of the system decays exponentially, i.e., $E(t) \leq ME(0)e^{-\rho t}$, for $t > t_0$, where $t_0 > 0$ is large enough.

So, using Theorem 3.2 and Theorem 3.3, we can obtain the following corollary.

Corollary 1. Let $\mathcal{A}$ be defined by (6) and (7), and the initial state $w_0 \in \text{dom}(\mathcal{A})$. Then, the system (5) is exponentially stable after sufficiently long time.

Remark 1. In fact, the system (5) can be regarded as a combination of two systems: one is a linear system governed by (17) with $|w(1,t)| \leq \beta$, the other is a nonlinear system as follows
   $$\begin{cases}
   w_t(x,t) = -iw_{xx}(x,t), \ x \in (0,1), \ t > 0, \\
   w(0,t) = 0, \ w_x(1,t) + i\beta \frac{w(1,t)}{|w(1,t)|} = 0, \ t > 0, \\
   w(x,0) = w_0(x), \ x \in (0,1),
   \end{cases}$$
   with $|w(1,t)| > \beta > 0$. It follows from (4) that the system (23) is dissipative, but it has not steady state when $|w(1,t)| > \beta > 0$. If we divide the state space into two regions, $\Omega_L = \{w(x,t)||w(1,t)| \leq \beta\}$ and $\Omega_N = \{w(x,t)||w(1,t)| > \beta\}$, then the trajectory of the system (5) will enter into $\Omega_L$ eventually.

4. Numerical simulation. In this section, we simulate the dynamical behavior of the system (5) using the finite-difference method. In the numerical experiment, the parameter of boundary restriction on the control $u(t)$, $\beta = 1$, and the initial function in the system (5) is taken as $w_0(x) = 8 \sin(5\pi x) + 2ix^3$. The stepsizes of time and space are taken as $\tau = 0.01$ and $h = 0.01$, respectively.

Figure 1 and 2 show the dynamic behaviors of real and imaginary parts of $w(x,t)$, respectively. Figure 3 displays the asymptotic behaviors of $w(x,t)$ at the right end $x = 1$. For comparison, we also draw the asymptotic curve of $w(1,t)$, the system at the right end $x = 1$ without restriction on the boundary control $u(t)$, i.e., $u(t) = -iw(1,t)$. The profiles in Figure 3 indicate that the simple modified feedback law (3) is feasible and practical, although the restriction on $u(t)$ reduces the performance of the feedback law, $u(t) = -iw(1,t)$.

5. Conclusions. The nonlinearity of dynamic system (5) only results from the external restriction, and after a period of time, the motion of (5) restores the linear one, again. This conclusion seems to be commonplace but the constraint causes a certain trouble to analyze the system. In engineering, the control strategy (3) is called the so-called “saturating control law” [21, 34]. The method of stability analysis is slightly different with the ones used in [21, 34]. In fact, the assumption in Theorem 1 is simplified and weakened, that is to say, we generalize Theorem 1 in [21].
When the boundary conditions in the system (1) are replaced by $w_x(0, t) = 0$, $w_x(1, t) = u(t)$, it can easily be proven that the control law (3) make the energy $E(t)$ of the corresponding system decay. If the boundary conditions are $w(0, t) = 0$, $w(1, t) = u(t)$ and the control law $u(t)$ is chosen as follows:

$$u(t) = \frac{iw_x(1, t)}{\max\{|w_x(1, t)|/\beta, 1\}} = \begin{cases} \frac{iw_x(1, t)}{|w_x(1, t)|}, & \text{when } |w_x(1, t)| \leq \beta, \\ i\beta \frac{w_x(1, t)}{|w_x(1, t)|}, & \text{when } |w_x(1, t)| > \beta, \end{cases}$$
then, it can be deduced that $|u(t)| \leq \beta$ and
\[
\frac{dE(t)}{dt} = \begin{cases} 
-|w_x(1, t)|^2, & \text{when } |w_x(1, t)| \leq \beta, \\
-\beta|w_x(1, t)|, & \text{when } |w_x(1, t)| > \beta.
\end{cases}
\]

Thus, similar to (16), there exists $c > 0$ such that
\[
|w_x(1, t)|^2 \leq c\|w(\cdot, t)\|, \forall t > 0.
\]

Therefore, for the boundary conditions $w_x(0, t) = 0$, $w_x(1, t) = u(t)$ or $w(0, t) = 0$, $w(1, t) = u(t)$, the similar results may be obtained. Contrasting with the Timoshenko beam with the saturating control [27], we discussed a first order complex system here. For this, the complex form of the Lax-Milgram theorem was introduced, titled Lemma 2.2, to prove the well-posedness of the closed-loop system. It, a generalization form of the nonlinear Lax-Milgram theorem, is also true for the real Hilbert space only if the imaginary part $B_2(x, z)$ is deleted, and it can be applied to other nonlinear systems. Moreover, we further improved and perfected the theoretical analysis of [27] here. Recently, the constrained systems begun to absorb attentions in engineering [16, 26]. So, the general theory for constrained systems is very interesting and meaningful, which will be discussed in our future work.

Appendix. In this appendix, we give a proof of Lemma 2.2, which is divided into four steps.

Proof. Step 1. The equivalent form of (11).

According to the assumption conditions of $B(x, z)$, there exists a bounded linear operator $B$ defined on $\mathcal{Z}$ such that $\langle Bx, z \rangle = B(x, z)$. Since $N(x, z)$ is a bounded linear functional with respective to $z$, there exists an operator $\mathcal{N} : \mathcal{Z} \mapsto \mathcal{Z}$ such that $N(x, z) = \langle \mathcal{N}(x), z \rangle$ for each $z \in \mathcal{Z}$. In addition, the continuity of $N(x, z)$ with
respective to $x$ implies that $\mathcal{N}$ is weak* sequentially continuous. Moreover,
\[\Re(\mathcal{N}(x) - \mathcal{N}(z), x - z) = \Re(N(z, z) + N(x, x) - N(x, z) - N(z, x)) \geq 0, \forall x, z \in \mathcal{Z},\]
which shows that $\mathcal{N}(x)$ is monotone. The Riesz’ representation theorem means that there is a $f \in \mathcal{Z}$ such that $L(z) = \langle f, z \rangle, \forall z \in \mathcal{Z}$. Thus (11) can be rewritten as follows
\[\langle Bx, z \rangle + \langle \mathcal{N}(x), z \rangle = \langle f, z \rangle, \forall z \in \mathcal{Z}.\]

**Step 2.** The range of $\mathcal{B} + \mathcal{N}$, $\text{Range}(\mathcal{B} + \mathcal{N})$, is close in $\mathcal{Z}$.

Let $y \in \text{Range}(\mathcal{B} + \mathcal{N})$, then there are sequences $\{x_n\}, \{y_n\} \subset \mathcal{Z}$ such that $y_n = B(x_n) + \mathcal{N}(x_n)$ and $\lim_{n \to \infty} y_n = y$. So, the monotonicity of $\mathcal{N}(z)$ and (10) lead to
\[
\Re(y_n - y_m, x_n - x_m) = \Re(B(x_n) + \mathcal{N}(x_n) - (B(x_m) + \mathcal{N}(x_m)), x_n - x_m)
\]
\[
= \Re(B(x_n - x_m), x_n - x_m) + \Re(N(x_n) - \mathcal{N}(x_m), x_n - x_m)
\]
\[
\geq c_B \|x_n - x_m\|^2,
\]
which implies that $\lim_{n \to \infty} x_n = x \in \mathcal{Z}$. Thus
\[\langle y, z \rangle = \lim_{n \to \infty} \langle y_n, z \rangle = \lim_{n \to \infty} \langle Bx_n, z \rangle + \langle \mathcal{N}(x_n), z \rangle = \langle Bx, z \rangle + \langle \mathcal{N}(x), z \rangle, \forall z \in \mathcal{Z}.
\]
Hence, $y \in \text{Range}(\mathcal{B} + \mathcal{N})$.

**Step 3.** $\text{Range}(\mathcal{B} + \mathcal{N}) = \mathcal{Z}$. Assume that $\text{Range}(\mathcal{B} + \mathcal{N})$ is a proper subspace of $\mathcal{Z}$, then there is a nonzero element $f \in \text{Range}(\mathcal{B} + \mathcal{N})^\perp$ (the orthogonal complement subspace of $\text{Range}(\mathcal{B} + \mathcal{N})$ in $\mathcal{Z}$) such that
\[0 = \langle B(z) + \mathcal{N}(z), f \rangle = B(z, f) + \mathcal{N}(z, f), \forall z \in \mathcal{Z}.
\]
But, the monotonicity of $\mathcal{N}(z)$ and the coercivity of $B(z, z)$ imply that
\[0 = \Re(B(f, f) + \mathcal{N}(f, f)) = \Re(B(f, f)) + \Re(\mathcal{N}(f), f) \geq c_B \|f\|^2.
\]
So, $f = 0$, which is a contradiction. Thus, $\text{Range}(\mathcal{B} + \mathcal{N}) = \mathcal{Z}$, that is to say, there exists $x \in \mathcal{Z}$ such that (11) holds.

**Step 4.** Uniqueness.

Assume that there are two elements $x$ and $y$ such that (11) holds, then
\[0 = B(x, z) + \mathcal{N}(x, z) - B(y, z) - \mathcal{N}(y, z)
\]
\[= \langle B(x - y), z \rangle + \langle \mathcal{N}(x) - \mathcal{N}(y), z \rangle, \forall z \in \mathcal{Z}.
\]
Substituting $x - y$ for $z$ in above expressions yields
\[0 = \Re(B(x - y, x - y) + \Re(\mathcal{N}(x) - \mathcal{N}(y), x - y) \geq c_B \|x - y\|^2,
\]
which implies that $x = y$. That is to say, the element of $\mathcal{Z}$ that satisfies (11) is unique.

**Acknowledgments.** The authors are grateful to the anonymous referees and the editor for their valuable comments and suggestions on the original version of this paper.
REFERENCES

[1] M. Aassila, Exact controllability of the Schrödinger equation, Applied Mathematics and Computation, 144 (2003), 89–106.
[2] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, 2nd edition, Elsevier/Academic Press, Amsterdam, 2003.
[3] I. Aksikas, J. J. Winkin and D. Dochain, Asymptotic stability of infinite-dimensional semilinear systems: Application to a nonisothermal reactor, Systems & Control Letters, 56 (2007), 122–132.
[4] B. d’Andréa-Novel and J. M. Coron, Exponential stabilization of an overhead crane with flexible cable via a back-stepping approach, Automatica, 36 (2000), 587–593.
[5] V. Barbu, Nonlinear Differential Equations of Monotone types in Banach Spaces, Springer New York Dordrecht Heidelberg London, 2010.
[6] P. Bégout, Necessary conditions and sufficient conditions for global existence in the nonlinear Schrödinger equation, Advances in Mathematical Sciences and Applications, 12 (2002), 817–827.
[7] P. Bégout, Maximum decay rate for the nonlinear Schrödinger equation, Nonlinear Differential Equations and Applications NoDEA, 11 (2004), 451–467.
[8] T. Cazenave and A. Haraux, An Introduction to Semilinear Evolution Equations, Oxford University Press, New York, 1998.
[9] C. Chen and D. S. Elliott, Measurements of optical phase variations using interfering multiphoton ionization processes, Physical Review Letters, 65 (1990), 1737–1740.
[10] R. Cipolatti, E. Machtyngier and E. San Pedro Siqueira, Nonlinear boundary feedback stabilization for Schrödinger equations, Differential and Integral Equations, 9 (1996), 137–148.
[11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, A refined global well-posedness result for Schrödinger equations with derivative, SIAM Journal on Mathematical Analysis, 34 (2002), 64–86.
[12] C. M. Dafermos and M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups, Journal of Functional Analysis, 13 (1973), 97–106.
[13] P. Gross, D. Neuhauser and H. Rabitz, Teaching lasers to control molecules in the presence of laboratory field uncertainty and measurement imprecision, The Journal of Chemical Physics, 98 (1993), 4557–4566.
[14] B. Guo and J. Liu, Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrödinger equation subject to boundary control matched disturbance, International Journal of Robust and Nonlinear Control, 24 (2014), 2194–2212.
[15] A. Haraux, Nonlinear Evolution Equations: Global Behavior of Solutions, Lecture Notes in Mathematics, Vol. 841, Springer-Verlag, New York, 1981.
[16] W. He, X. He and S. S. Ge, Vibration Control of Flexible Marine Riser Systems with Input Saturation, IEEE/ASME Transactions on Mechatronics, 21 (2016), 254–265.
[17] T. Kato, Nonlinear semigroups and evolution equations, Journal of the Mathematical Society of Japan, 19 (1967), 493–507.
[18] Y. Kômura, Nonlinear semi-groups in Hilbert space, Journal of the Mathematical Society of Japan, 19 (1967), 493–507.
[19] R. Kosloff, S. A. Rice, P. Gaspard, S. Tersigni and D. J. Tannor, Wavepacket dancing: Achieving chemical selectivity by shaping light pulses, Chemical Physics, 139 (1989), 201–220.
[20] M. Krstic and B. Guo, Boundary controllers and observers for the linearized schrödinger equation, SIAM Journal on Control and Optimization, 49 (2011), 1479–1497.
[21] I. Lasiecka and T. Seidman, Strong stability of elastic control systems with dissipative saturating feedback, System and Control Letters, 48 (2003), 243–252.
[22] I. Lasiecka and R. Triggiani, Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control, Differential and Integral Equations, 5 (1992), 521–535.
[23] I. Lasiecka and R. Triggiani, Well-posedness and sharp uniform decay rates at the $L_2(\Omega)$-level of the Schrödinger equation with nonlinear boundary dissipation, Journal of Evolution Equations, 6 (2006), 485–537.
[24] I. Lasiecka, R. Triggiani and X. Zhang, Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. Part I: $H_1(\Omega)$-estimates, Journal of Inverse Ill-posed Problems, 12 (2004), 43–123.
I. Lasiecka, R. Triggiani and X. Zhang, Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. Part II: $L_2(\Omega)$-estimates, Journal of Inverse and Ill-posed Problems, 12 (2004), 183–231.

Z. Liu, J. Liu and W. He, Partial differential equation boundary control of a flexible manipulator with input saturation, International Journal of Systems Science, 48 (2017), 53–62.

D. Liu, L. Zhang, Z. Han and G. Xu, Stabilization of the Timoshenko beam system with restricted boundary feedback controls, Acta Applicandae Mathematicae, 141 (2016), 149–164.

E. Machtyngier and E. Zuazua, Stabilization of the Schrödinger equation, Portugaliae Mathematica, 51 (1994), 243–256.

E. Machtyngier, Exact controllability for the Schrödinger equation, SIAM Journal Control and Optimization, 32 (1994), 24–34.

S. Nicaise and S. Rebiai, Stabilization of the Schrödinger equation with a delay term in boundary feedback or internal feedback, Portugaliae Mathematica, 68 (2011), 19–39.

N. H. Pavel, Nonlinear Evolution Operators and Semigroups, Springer-Verlag, Berlin, Heidelberg, 1987.

S. Shi, A. Woody and H. Rabitz, Optimal control of selective vibrational excitation in harmonic linear chain molecules, The Journal of Chemical Physics, 88 (1988), 6870–6883.

R. Showalter, Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations, American Mathematical Society, Providence, RI, 1997.

M. Slemrod, Feedback Stabilization of a Linear Control System in Hilbert Space with an a priori Bounded Control, Mathematics of Control, Signals and Systems, 2 (1989), 265–285.

G. Xu and B. Guo, Riesz basis property of evolution equations in Hilbert spaces and application to a coupled string equation, SIAM Journal on Control and Optimization, 42 (2003), 966–984.

G. Xu and D. Feng, The Riesz basis property of a Timoshenko beam with boundary feedback and application, IMA Journal of Applied Mathematics, 67 (2002), 357–370.

E. Zeidler, Nonlinear Functional Analysis and Its Applications, II/ B: Nonlinear Monotone Operators, Springer-Verlag, New York, 1990.