A general surplus decomposition principle in life insurance

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ABSTRACT
In with-profit life insurance, the prudent valuation of future insurance liabilities leads to systematic surplus that mainly belongs to the policyholders and is redistributed as bonus. For a fair and lawful redistribution of surplus, the insurer needs to decompose the total portfolio surplus with respect to the contributions of individual policies and with respect to different risk sources. For this task, actuaries have a number of heuristic decomposition formulas, but an overarching decomposition principle is still missing. This paper fills that gap by introducing a so-called ISU decomposition principle that bases on infinitesimal sequential updates of the insurer’s valuation basis. It is shown that the existing heuristic decomposition formulas can be replicated as ISU decompositions. Furthermore, alternative decomposition principles and their relation to the ISU decomposition principle are discussed. The generality of the ISU concept makes it a useful tool also beyond classical surplus decompositions in life insurance.

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1. Introduction
The long-term nature of life insurance contracts causes a significant trend risk that is non-diversifiable in the insurance portfolio. In with-profit life insurance, the insurer uses a conservative trend scenario upfront and successively replaces it with the empirically observed trend. This updating of the valuation basis produces surplus. Insurance regulation requires that this surplus is for the most part redistributed to the policyholders. In each reporting period, the insurer determines at first the total profits and losses in the life insurance portfolio, which includes surplus from non-diversifiable as well as diversifiable risks. In a second step, the profits and losses are shared between the insurer and each of the policyholders subject to regulation. Surplus from diversifiable risk is commonly credited or debited to the insurer. Surplus that arises from conservative trend scenario calculations mainly belongs to the policyholders. In Germany, the regulator moreover requires to distinguish between surplus due to demographic trends, investment success and changes of administration costs. For such splitting of the total surplus, the insurer needs an additive and risk-based decomposition method.

The life insurance literature knows various surplus decomposition formulas, cf. Ramlau-Hansen (1988), Ramlau-Hansen (1991) and Norberg (1999) for continuous time modelling and Milbrodt & Helbig (1999) for discrete time modelling. All these formulas are derived in a heuristic manner. An overarching general decomposition principle is still missing, and this paper closes that gap. We introduce a so-called infinitesimal sequential updating (ISU) decomposition principle, which can reproduce the existing decomposition formulas and put them into a general and consistent framework. The ISU concept is an advancement of sequential updating (SU) decomposition principles, which are used in various fields of economics. For example in labor economics, Blinder (1973)
and Oaxaca (1973) (‘Blinder-Oaxaca-Decomposition’) decompose mean differentials of U.S. wages in linear regression models via SU schemes. Similarly, DiNardo et al. (1996) apply the SU technique for decomposing distributional changes (for the U.S. distribution of wage) into its different components. In actuarial economics, Candland & Lotz (2014) provide a profit and loss attribution by sequentially updating the risk factors. However, as elaborated by Fortin et al. (2011) and Biewen (2014), the main disadvantage of SU decompositions is that they depend on the formal ordering of the different risk factors. The ISU concept overcomes this drawback of the SU concept by pushing the lengths of the reporting periods down to zero so that the impact of the ordering vanishes. To our knowledge, this asymptotic approach is completely new in the literature. An alternative to sequential decompositions are one-at-a-time (OAT) principles (cf. Biewen (2014)), which avoid the ordering problem but suffer from interaction effects between the parameters, undermining the desired additivity of surplus decompositions. We show that the asymptotic approach can help also here and find that the resulting infinitesimal OAT decompositions are largely equivalent to ISU decompositions.

A recent trend in insurance is to reward risk averse behaviour of the insured by means of individual activity tracking. The difference between the expected activity and the real activity of an insured leads to surplus. Therefore, advanced decomposition formulas that separate this activity surplus from the classical surplus sources are needed. The ISU decomposition principle offers the necessary basic tools for solving that problem, but a detailed study of these new insurance forms is beyond the scope of this paper and is left for future research.

The paper is structured as follows. In Section 2, we formally define the surplus decomposition problem. Section 3 describes the life insurance modelling framework and recalls the definition of the total surplus. In Section 4, we introduce the ISU decomposition principle. Section 5 is rather technical and develops integral representation results for ISU decompositions, which are needed and applied in Section 6, where we illustrate the ISU decomposition concept for typical life insurance applications. Section 7 discusses alternatives to the ISU decomposition concept and explains their relations. Section 8 briefly summarizes our findings.

2. The surplus process of an individual insurance contract

We generally assume that we have a complete probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with a right-continuous and complete filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\). We consider an individual insurance policy on a finite contract period \([0, T]\). For each \(t \geq 0\) let \(B(t)\) be the aggregated insurance cash flow on \([0, t]\) between insurer and insured. We use the convention that premiums have a negative sign and benefits have a positive sign. Let \(\kappa\) be a semimartingale with \(\kappa(0) = 1\) that describes the value process of the insurer’s self-financing investment portfolio. Then the value \(A(t)\) of the assets accrued at time \(t\) is given by

\[
A(t) = -\int_{[0,t]} \frac{\kappa(t)}{\kappa(s)} dB(s),
\]

assuming that \(B\) is a finite variation semimartingale and that \(\kappa\) is strictly positive. In the hypothetical case that the insurer knew the future, the liabilities at time \(t\) would be likewise calculated as

\[
L^h(t) = \int_{(t,T]} \frac{\kappa(t)}{\kappa(s)} dB(s).
\]

The difference between assets and liabilities is the surplus

\[
S^h(t) = A(t) - L^h(t) = \kappa(t)(A(0) - L^h(0)).
\]

In this hypothetical setting, the actual surplus emerges at time zero and any dynamics after zero just comes from the compounding factor \(\kappa(t)\). By defining

\[
d\Phi(t) = \frac{d\kappa(t)}{\kappa(t)}
\]
as the return on investment of the insurer’s investment portfolio, the process $S^h$ satisfies

$$dS^h(t) = S^h(t-)d\Theta(t)$$

for $t > 0$, which shows again that the dynamics of $S^h$ on $(0, \infty)$ stems solely from investment gains earned on the existing surplus. Since $A(0) - L^h(0)$ depends on the future and is nowhere adapted to the available information, in real life the insurer has to replace $A(0) - L^h(0)$ at each time $t$ by an $\mathcal{F}_t$-measurable proxy $R(t)$. We denote

$$R = (R(t))_{t \geq 0}$$

as the revaluation surplus process since it describes profits and losses that result from the continuous revaluation of $A(0) - L^h(0)$ as the information $\mathcal{F}_t$ increases with time $t$. Now the total surplus process is given by

$$S(t) = \kappa(t)R(t), \quad t \geq 0,$$

and its dynamics is driven by both the compounding factor $\kappa$ and the revaluation surplus process $R$. The aim of this paper is to decompose $R$ with respect to a given set of risk sources. We assume that the life insurance model rests on a so-called risk basis

$$X = (X_1, \ldots, X_m),$$

which is a multivariate adapted process composed of so-called risk factors $X_1, \ldots, X_m$ such that $R$ is adapted to the right-continuous and complete filtration generated by $X$. The risk basis is assumed to be fixed, but depending on $R$, different choices of $X$ are conceivable. The right-continuous and complete filtration generated by $X$ does not necessarily coincide with $\mathcal{F}$ but may be a strict sub-filtration of $\mathcal{F}$ in such a way that at least $R$ is adapted. The information provided by $X$ at time $t$ can be represented by the stopped process $X^t$, formally defined by

$$X^t(s) = 1_{s \leq t}X(s) + 1_{s > t}X(t).$$

So at each time $t$ the proxy $R(t)$ of $A(0) - L^h(0)$ can be interpreted as the value of a mapping

$$(t, X^t) \mapsto R(t)$$

that assigns at each time $t$ to the current information $X^t$ the random variable $R(t)$. In this paper, we assume that there even exists a mapping $\varrho$ such that

$$\varrho(X^t) = R(t), \quad t \geq 0.$$

In the latter equation, the time parameter $t$ itself is not an argument of $\varrho$ and only appears as stopping parameter in $X^t$. That means that the dynamics of $R$ are solely driven by the increase of information through $X^t$.

The central aim of this paper is to decompose $R$ as

$$R(t) = R(0) + D_1(t) + \cdots + D_m(t), \quad t \geq 0,$$

where $D_1, \ldots, D_m$ are adapted processes that start at zero and describe the contributions of each risk factor $X_1, \ldots, X_m$ to the dynamics of $R$. The first addend $R(0)$ represents initial surplus, which is not decomposed here. Equation (5) is equivalent to the additive decomposition

$$S(t) = \kappa(t)S(0) + \kappa(t)D_1(t) + \cdots + \kappa(t)D_m(t), \quad t \geq 0,$$

for the total surplus process. The first addend $\kappa(t)S(0)$ represents the time-$t$ value of the initial surplus $S(0) = R(0)$, and the addends $\kappa(t)D_1(t), \ldots, \kappa(t)D_m(t)$ describe the time-$t$ values of the
contributions that the risk factors $X_1, \ldots, X_m$ make to the dynamics of $S$. The additivity of the decompositions (5) and (6) allows us to distribute the surplus among different parties.

The dynamics of the total surplus in (3) is driven by investment gains on the surplus itself and by revaluation gains. In (6) the investment gains are subdivided among the different surplus contribution addends according to their shares in the total investment earnings. It is not uncommon in the actuarial literature to collect all the investment gains in a separate term, see for example Norberg (1999, formula (5.3)). The idea is to apply Itô’s product rule on $S(t) + \kappa(t)R(t)$ and then to identify each of the resulting addends either as investment gains or as revaluation gains. However, this approach mixes up the investment earnings of the carefully separated surplus contribution addends, so it is not helpful in our opinion and therefore it is not further considered in this paper.

3. The revaluation surplus in multi-state models

Let the random pattern $Z$ of the insured be a right-continuous and adapted jump process on a finite state space $\mathcal{Z}$ with starting value $Z(0) = a \in \mathcal{Z}$. We define corresponding state processes $(I_j)_j$ and counting processes $(N_{jk})_{jk \neq k}$ by $I_j(t) := 1_{\{Z(t) = j\}}$ and

$$N_{jk}(t) = \sharp\{s \in (0, t) : Z(s) = j, Z(s) = k\}, \quad j, k \in \mathcal{Z}, \quad j \neq k, \quad t \geq 0.$$ 

Additionally, we define $N_{jj} := -\sum_{k \neq j} N_{kj}, j \in \mathcal{Z}$, and the vector-valued process $N = (N_{jk})_{jk \neq k}$.

We call a pair $(\Phi, \Lambda)$ a valuation basis if the following properties hold:

- $\Phi$ is semimartingale with $\Phi(0) = 0$ and $\Delta \Phi(t) > -1$ for all $t > 0$,
- $\Lambda = (\Lambda_{jk})_{jk \neq k}$ is a vector-valued, right-continuous finite variation process with $\Lambda(0) = 0$,
- the processes $\Lambda_{jk}, j \neq k$ are non-decreasing and $\sum_{k \neq j} \Delta \Lambda_{jk}(t) \leq 1$ for every $t > 0$ and every $j$.

The process $\Phi$ represents cumulative returns on investment, and the solution $\overline{\kappa} = (\overline{\kappa}(t))_{t \geq 0}$ of the stochastic differential equation

$$d\overline{\kappa}(t) = \overline{\kappa}(t-) \, d\Phi(t), \quad \overline{\kappa}(0) = 1, \quad (7)$$

is the value process of a self-financing investment portfolio with respect to $\Phi$.

Furthermore, given the valuation basis $(\Phi, \Lambda)$, let $\overline{p} = (\overline{p}(t, s))_{0 \leq t \leq s}$ with $\overline{p}(t, s) = (\overline{p}_{jk}(t, s))_{jk}$ denote the solution of the stochastic differential equation system

$$\overline{p}_{jk}(t, ds) = \sum_i \overline{p}_{ij}(t, s-) \, d\Lambda_{ik}(s), \quad \overline{p}_{jk}(t, t) = \delta_{jk}, \quad s > t. \quad (8)$$

Observe that we may pick $N$ itself for $\Lambda$. In this case, the solution of (8) satisfies $p_{aj}(0, s) = I_j(s)$, since $I_j(0) = \delta_{aj}$ and

$$dI_j(s) = \sum_{k \neq j} (dN_{kj}(s) - dN_{jk}(s)) = \sum_k I_k(s-) \, dN_{kj}(s). \quad (9)$$

Throughout this paper, let the valuation basis $(\Phi, \Lambda)$ represent the so-called second order valuation basis. The process $\Phi$ describes the real return in investment in the insurer’s investment portfolio. Let $\kappa$ denote the solution of (7) with respect to $\Phi$. For the second-order basis we additionally assume that

(S.1) $\Lambda$ is a predictable process,
(S.2) conditional on $(\Phi, \Lambda) = (E, F)$ the process $Z$ is a Markov process under $\mathbb{P}$ with cumulative transition intensity matrix $F$. 
So the process \( I_j(t) \) is a \( \mathbb{P} \)-compensator of \( dN_{jk} \) with respect to the natural completed filtration of the random vector \((Z^* \!, \Phi, \Lambda)_{t \geq 0}\). Due to the conditional Markov property, the stochastic difference equation (8) with respect to \( \Lambda \) corresponds to the Kolmogorov forward equation of \( Z \) conditional on \((\Phi, \Lambda)\), and its solution \( p(t, s) = (p_{jk}(t, s))_{jk} \) is the transition probability matrix of \( Z \) conditional on \((\Phi, \Lambda)\). 

Furthermore, let the valuation basis \((\Phi^*, \Lambda^*)\) represent the so-called first order valuation basis. For this specific valuation basis, we additionally assume that 

\[
\begin{align*}
\text{F.1) } & \Phi^* \text{ and } \Lambda^* \text{ are deterministic,} \\
\text{F.2) } & Z \text{ is a Markov process under a prudent probability measure } \mathbb{P}^* \text{ with cumulative transition} \\
& \text{intensities } \Lambda_{jk}^*, j \neq k, \\
\text{F.3) } & (\mathbb{I} + \Delta \Lambda_M^*(s))^{-1} \text{ exists for every } s > 0,
\end{align*}
\]

where \( \Lambda_M^* \) denotes the matrix-valued process \( \Lambda^*_M = (\Lambda^*_M)_{jk} \) with \( \Lambda_{jj}^* := -\sum_{k \neq j} \Lambda_{jk}^* \). Let \( \kappa^* \) and \( p^* \) be the solutions of (7) and (8) with respect to \( \Phi^* \) and \( \Lambda^* \), respectively. Under the first order valuation basis, (8) is the classical Kolmogorov forward equation and \( p^* \) is the classical transition probability matrix of \( Z \) under \( \mathbb{P}^* \). The existence of \( (\mathbb{I} + \Delta \Lambda_M^*(s))^{-1} \) for every \( s > 0 \) ensures that the matrix \( p^*(t, s) \) has an inverse for each \( s > t \), denoted as \( q^*(t, s) \), cf. Lemma A.1 in the Appendix. In particular, \( q^* \) satisfies the stochastic differential equation

\[
q^*(t, ds) = -(dG(s))q^*(t, s-), \quad q^*(t, t) = \mathbb{I}, \quad s > t,
\]

where \( G(s) = \Lambda^*_M(s) - \sum_{0 < u \leq s} (\Delta \Lambda^*_M(u))^2 (\mathbb{I} + \Delta \Lambda^*_M(u))^{-1} \) (cf. Lemma A.1).

Recall that the insurance policy shall have a finite contract horizon in \([0, T]\). We assume that the insurance cash flow \( B \) has the form

\[
dB(t) = \sum_j I_j(t) \, dB_j(t) + \sum_{j \neq k} b_{jk}(t) \, dN_{jk}(t),
\]

where \((B_j)\) are right-continuous finite variation functions that satisfy \( dB_j(t) = 0 \) for \( t > T \), and \((b_{jk})_{j \neq k} \) are bounded and measurable functions with \( b_{jk}(t) = 0 \) for \( t > T \).

We generally assume that

\[
\text{J) } \text{the processes } \Phi^*, \Phi \text{ and } (N, \Lambda^*, \Lambda, (B_j)) \text{ have no simultaneous jumps.}
\]

The latter condition implies that the covariation between the investment risk and all other risk drivers is zero. This fact will help us to build additive decompositions by applying Itô’s formula, cf. Lemma 5.1.

**Individual revaluation surplus**

In with-profit life insurance, the remaining future liabilities of the individual insurance contract at time \( t \) are commonly evaluated as

\[
\sum_j I_j(t) V_j^*(t),
\]
where \( V_j^*(t) \) shall be the prospective reserve at time \( t \) in state \( j \) with respect to the first-order valuation basis, cf. Norberg (1999). According to Milbrodt and Helbig (1999, Chapter 10.A) it holds that

\[
V_j^*(t) = \mathbb{E}^* \left[ \int_t^T \frac{\kappa^*(t)}{\kappa^*(s)} dB(s) \left| Z(t) = j \right. \right] \\
= \sum_k \int_{(t,T]} \frac{\kappa^*(t)}{\kappa^*(s)} \rho^*_j(t, s-) \, dB_k(s) \\
+ \sum_{k,l,k \neq l} \int_{(t,T]} \frac{\kappa^*(t)}{\kappa^*(s)} \rho^*_j(t, s-) b_{kl}(s) \, d\Lambda^*_k(s),
\]

where \( \mathbb{E}^* \) denotes the expectation with respect to \( \mathbb{P}^* \) (cf. (F.2)). The accrued assets of the individual insurance contract at time \( t \) equal (1), so the total surplus of the individual policy at time \( t \) is

\[
S(t) = -\int_{[0,t]} \frac{\kappa(t)}{\kappa(s)} dB(s) - \sum_j I_j(t) V_j^*(t), \quad (11)
\]

cf. Norberg (1999). The corresponding revaluation process \( R \) equals

\[
R(t) = \frac{S(t)}{\kappa(t)} = -\int_{[0,t]} \frac{1}{\kappa(s)} dB(s) - \sum_j \frac{1}{\kappa(t)} I_j(t) V_j^*(t). \quad (12)
\]

**Proposition 3.1:** For \( R \) defined by (12) and \( t \in [0,T] \) it holds that

\[
R(t) = -H((\Phi^*, \Lambda^*) + (\Phi - \Phi^*, N - \Lambda^*))^t), \quad (13)
\]

where \( (\cdot)^t \) denotes the corresponding stopped process (cf. (4)) and where for any valuation basis \( (\Phi, \Lambda) \) the mapping \( H \) is defined by

\[
H((\Phi, \Lambda)) := \sum_j \int_{[0,T]} \frac{1}{\kappa(s)} p^*_j(0, s-) \, dB_j(s) + \sum_{j,k: j \neq k} \int_{(0,T]} \frac{1}{\kappa(s)} p^*_j(0, s-) b_{jk}(s) \, d\Lambda_{jk}(s) \quad (14)
\]

with \( p^*_j(0, 0-) := \delta_{aj} \).

**Proof:** The solution of (8) with respect to the cumulative transition intensity vector \( \Lambda^* + (N - \Lambda^*)^t \) is

\[
\begin{cases}
I_j(s), & s \leq t, \\
\sum_l I_l(t) p^*_j(t, s), & s > t,
\end{cases}
\]

where \( p^*_j(t, s) \) is the solution of (8) with respect to the first-order valuation basis. The solution of (7) with respect to \( \Phi^* + (\Phi - \Phi^*)^t \) is

\[
\begin{cases}
\kappa(s), & s \leq t, \\
\frac{\kappa^*(s)}{\kappa^*(t)}, & s > t,
\end{cases}
\]

where \( \kappa^* \) is the solution of (7) with respect to the first-order valuation basis. By plugging these solutions into (14), we obtain the desired result.
By setting \( R(t) = \varrho(X^t), \quad t \geq 0, \)
for various choices of \( X \) and \( \varrho \). For example, we may define the risk basis \( X \) and the mapping \( \varrho \) by means of the mapping \( H \) (cf. (14)) as follows:

**Example 3.2:** By setting

\[
X = (X_\Phi, X_u, X_\iota) = (\Phi - \Phi^*, N - \Lambda, \Lambda - \Lambda^*),
\]
we distinguish between financial risk, unsystematic biometric risk and systematic biometric risk, and we may define \( \varrho \) by

\[
\varrho(X^t) = -H\left((\Phi^*, \Lambda^*) + (X_\Phi^t, X_u^t + X_\iota^t)\right).
\]

**Example 3.3:** By setting

\[
X = (X_\Phi, (X_{jk})_{jk \neq k}) = (\Phi - \Phi^*, (N_{jk} - \Lambda^*_{jk})_{jk \neq k}),
\]
we distinguish between financial risk and transition-wise biometric risks, and we may define \( \varrho \) by

\[
\varrho(X^t) = -H\left((\Phi^*, \Lambda^*) + (X_\Phi^t, (X_{jk}^t)_{jk \neq k})\right).
\]

**Example 3.4:** Let the processes \( (\Phi_j)_j \) and \( (\Phi^*_j)_j \) be defined by \( d\Phi_j(t) = I_j(t-)d\Phi(t), \Phi_j(0) = 0 \), and \( d\Phi^*_j(t) = I_j(t-)d\Phi^*(t), \Phi^*_j(0) = 0 \), respectively. Further, we denote \( \Lambda_j = (\Lambda_{jk})_{k \neq j} \) and \( \Lambda^*_j = (\Lambda^*_{jk})_{k \neq j} \). By setting

\[
X = (X_\Phi, (X_j)_j) = (X_\Phi, (X_{j,1}, X_{j,2})_j) = (N - \Lambda, (\Phi_j - \Phi^*_j, \Lambda_j - \Lambda^*_j)_j)
\]
we distinguish between unsystematic biometric risk and state-wise remaining risks, and we may define \( \varrho \) by

\[
\varrho(X^t) = -H\left((\Phi^*, \Lambda^*_{jk}) + (0, X^t_\Phi) + \left(\sum_j X^t_{j,1}, (X^t_{j,2})_j\right)\right).
\]

**Mean portfolio revaluation surplus**

In actuarial practice, it is not uncommon to focus on mean portfolio values only. We can replicate this perspective by applying the expectation \( \mathbb{E}[\cdot | \Phi, \Lambda] \) on the individual values (11) and (12). The revaluation surplus takes then the form

\[
R(t) = \mathbb{E} \left[ -\int_{[0,t]} \frac{1}{\kappa(s)} dB(s) - \sum_j \frac{1}{\kappa(t)} I_j(t) V^*_j(t) \right| \Phi, \Lambda, \right], \tag{15}
\]
and the corresponding total surplus still satisfies the equation

\[
S(t) = \kappa(t) R(t). \tag{16}
\]
Note that Norberg (1999) applies the expectation \( \mathbb{E}[\cdot | \Phi^t, \Lambda^t] \) instead, but his definition is equivalent since

\[
S(t) = -\int_{[0,t]} \frac{\kappa(t)}{\kappa(s)} \sum_j \left( p_{aj}(0, s-) dB_j(s) + \sum_{k \neq j} b_{jk}(s)p_{aj}(0, s-) d\Lambda_{jk}(s) \right) - \sum_j p_{aj}(0, t) V^*_j(t)
\]
is \( \sigma(\Phi^t, \Lambda^t) \)-measurable. The following corollary is a direct consequence of Proposition 3.1.
Corollary 3.5: For $R$ defined by (15) and $t \in [0, T]$ it holds that

$$R(t) = \mathbb{E}\left[ -H((\Phi^*, \Lambda^*) + (\Phi - \Phi^*, N - \Lambda^*))^t | \Phi, \Lambda \right],$$

(17)

where $H$ is given by (14).

Because of the latter corollary, in the Examples 3.2 to 3.4 we just need to add the conditional expectation $\mathbb{E}[ \cdot | \Phi, \Lambda]$ to the definition of $\varrho$ to get to the mean portfolio perspective.

The next example is in particular relevant in German life insurance.

Example 3.6: Consider a life insurance contract with the states active, surrendered and dead,

$$Z = \{a, s, d\},$$

of an $x$-year old insured. We assume that $\Lambda^*$ and $\Lambda$ are absolutely continuous with densities $\lambda^*$ and $\lambda$, respectively. Let

$$k-lp_{x+l} = p_{ad}^*(x + l, x + k), \quad q_{x+k-1}^* = p_{sd}^*(x + k - 1, x + k), \quad r_{x+k-1}^* = p_{as}^*(x + k - 1, x + k),$$

and define $k-lp_{x+l}$, $q_{x+k-1}$ and $r_{x+k-1}$ likewise for the second-order valuation basis. We assume that sojourn payments occur only in state active and only as lump sum payments $b_k$ at integer times $k$. Furthermore, we assume that the death benefit function and the surrender benefit function have the form

$$b_{ad}(t) = \frac{\kappa([t])}{\kappa(t)} d_{[t]}, \quad b_{as}(t) = \frac{\kappa([t])}{\kappa(t)} s_{[t]},$$

where $d_{[t]}$ and $s_{[t]}$ represent the death benefit and surrender benefit in year $[t]$. This definition of $b_{ad}$ and $b_{as}$ discounts death benefits and surrender benefits as if they are paid out at the end of the year, so that $V_a^*$ has at integer times $l$ the representation

$$V_a^*(l) = \sum_{k=l+1}^T \frac{\kappa^*(l)}{\kappa^*(k)} k-lp_{x+l}^* b_k + \sum_{k=l+1}^T \frac{\kappa^*(l)}{\kappa^*(k)} k-l p_{x+l}^* (d_k q_{x+k-1}^* + s_k r_{x+k-1}^*).$$

We define yearly interest rates of first order and second order by

$$i_k^* = e^{(k+1)(\varphi(u)) du} - 1, \quad i_k = e^{(k+1)(\varphi(u)) du} - 1, \quad k \in \mathbb{N}_0.$$

One can show that the yearly increments of the mean portfolio revaluation surplus process equal

$$R(k+1) - R(k) = e^{-\int_0^{k+1} \varphi(u) du} \sum_{k=1}^l \left( V_a^* (k+1) - V_a^*(k) \right) - q_{s+k} d_{k+1} - r_{s+k} s_{k+1} - p_{s+k} (b_{k+1} + V_a^*(k+1)).$$

This formula is commonly used in German life insurance, cf. Milbrodt & Helbig (1999, Section 11.B). It is common in Germany to decompose the increments $R(k+1) - R(k)$ into investment surplus, mortality surplus and lapse surplus. For that purpose, analogously to Example 3.3 we choose

$$X = (X_{\Phi}, X_{ad}, X_{as}) = (\Phi - \Phi^*, N_{ad} - \Lambda_{ad}^*, N_{as} - \Lambda_{as}^*)$$

as risk basis.
4. The ISU decomposition principle

Recall that the $t$-stopped process $X^t = (X^t_1, \ldots, X^t_m)$ represents the currently available information on the risk factors $X_1, \ldots, X_m$ at time $t$. Suppose that the information updates of the risk factors $X_1, \ldots, X_m$ are asynchronously delayed with $t_1, \ldots, t_m \leq t$ being the current update statuses of each risk factor. Then

\[ U(t_1, \ldots, t_m) := \varrho((X^t_1, \ldots, X^t_m)) \]  

(18)

is the value of the delayed revaluation process at time points $t_1, \ldots, t_m$. Furthermore, we denote $U = (U(t_1, \ldots, t_m))_{t_1,\ldots, t_m \geq 0}$ as the revaluation surplus surface with respect to $X$. We can recover the revaluation surplus process $R$ from the revaluation surplus surface $U$ as

\[ R(t) = U(t, \ldots, t), \quad t \geq 0. \]

For any partition $T(t) = \{0 = t_0 < t_1 < \cdots < t_k = t\}$ of the interval $[0, t]$ we can build the telescoping series

\[ R(t) - R(0) = U(t, \ldots, t) - U(0, \ldots, 0) \]

\[ = \sum_{l=0}^{k-1} \left( U(t_{l+1}, t_l, \ldots, t_l) - U(t_l, \ldots, t_l) \right) \]

\[ + \sum_{l=0}^{k-1} \left( U(t_{l+1}, t_{l+1}, t_l, \ldots, t_l) - U(t_{l+1}, t_l, \ldots, t_l) \right) \]

\[ + \cdots \]

\[ + \sum_{l=0}^{k-1} \left( U(t_{l+1}, \ldots, t_{l+1}) - U(t_{l+1}, \ldots, t_{l+1}, t_l) \right). \]

It is natural here to interpret the $m$ different sums on the right-hand side as an additive decomposition $R(t) - R(0) = D_1(t) + \cdots + D_m(t)$, since the $i$th sum collects exactly the information updates for the $i$th risk factor.

**Definition 4.1:** The random vector $D(t) = (D_1(t), \ldots, D_m(t))$ defined by

\[ D_1(t) = \sum_{l=0}^{k-1} \left( U(t_{l+1}, t_l, \ldots, t_l) - U(t_l, \ldots, t_l) \right), \]

\[ \ldots \]

\[ D_m(t) = \sum_{l=0}^{k-1} \left( U(t_{l+1}, \ldots, t_{l+1}) - U(t_{l+1}, \ldots, t_{l+1}, t_l) \right), \]  

(19)

is called the SU (sequential updating) decomposition of $R(t) = \varrho(X^t)$ with respect to $T(t)$.  

The SU decomposition principle is used in various fields of economics, see, e.g. Fortin et al. (2011) and Biewen (2014). In the definition formula (19), we update the information on $X$ in a specific order, starting with risk factor $X_1$, then updating $X_2$, and so on. Unfortunately, the decomposition is not invariant with respect to this update order, which is a major drawback of the SU concept. We can reduce the impact of the update order by increasing the number of updating steps, i.e. refining the partition $T_n(t)$. In a next step, we push such refinements to the limit.
Let \( \mathcal{T}_n(t) = \{0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\}, \ n \in \mathbb{N}, \) be a sequence of partitions of \([0, t]\) with vanishing step lengths (i.e. \(\lim_{n \to \infty} \max_{1 \leq i \leq k_n} |t_i^n - t_{i-1}^n| = 0\)). For each \( n \in \mathbb{N} \) let \( D_n(t) = (D_1^n(t), \ldots, D_{m_n}^n(t)) \) be the SU decomposition of \( R(t) = \varrho(X_t) \) with respect to \( \mathcal{T}_n(t) \). We are looking for a random vector \( D(t) \) that satisfies

\[
D_i(t) = \plim_{n \to \infty} D_i^n(t), \quad i \in \{0, \ldots, m\}.
\]

**Definition 4.2:** Let \( (\mathcal{T}_n(t))_{n \in \mathbb{N}} \) be a sequence of partitions of \([0, t]\) with vanishing step lengths. If \( D(t) \) satisfies (20), then we call \( D(t) \) the ISU (infinitesimal sequential updating) decomposition of \( R(t) = \varrho(X_t) \) with respect to \( (\mathcal{T}_n(t))_{n \in \mathbb{N}} \).

### 5. ISU decompositions in multi-state life insurance

This section contains general technical results that will be needed for the examples in the next section. The proofs can be found in the appendix. For any valuation basis \((\Phi, \Lambda)\), we write

\[
\tilde{\Phi}(t) = \Phi(t) - [\Phi, \Phi]^c(t) \quad - \sum_{0 < s \leq t} (1 + \Delta \Phi(s))^{-1}(\Delta \Phi(s))^2,
\]

where \([\Phi, \Phi]^c(t)\) signifies the continuous part of \([\Phi, \Phi]\).

Moreover, let \( R_{jk}^n(t, s) \) denote the first-order sum at risk, i.e.

\[
R_{jk}^n(t) = b_{jk}(t) + V_{jk}^n(t) - V_j^*(t).
\]

Recalling that

\[
H(\Phi, \Lambda) := \sum_j \int_{[0, t]} \frac{1}{\kappa(s)} \bar{p}_{aj}(0, s) \, d\Lambda_j(s) + \sum_{j, k : j \neq k} \int_{[0, t]} \frac{1}{\kappa(s)} \bar{p}_{aj}(0, s) b_{jk}(s) \, d\Lambda_jk(s),
\]

for any valuation basis \((\Phi, \Lambda)\) (cf. (14)), we can pose the following results.

**Lemma 5.1:** Let \((\Phi, \Lambda)\) be a valuation basis such that \((\Phi^*, \bar{\Phi})\) and \((\Lambda^*, \bar{\Lambda}, (B_{jk}))\) have no simultaneous jumps. Then it holds that

\[
H((\Phi^*, \Lambda^*) + (\bar{\Phi} - \Phi^*, \bar{\Lambda} - \Lambda^*)) = \int_{[0, t]} \frac{1}{\kappa(s)} \sum_j \bar{p}_{aj}(0, s) V_j^*(s) - d(\bar{\Phi} - \Phi^* + [\bar{\Phi}, \Phi^*])(s) - \sum_{j, k : j \neq k} \int_{[0, t]} \frac{1}{\kappa(s)} \bar{p}_{aj}(0, s) R_{jk}^n(s) - d(\bar{\Lambda}_{jk} - \Lambda_{jk}^*)(s).
\]

**Theorem 5.2:** Let the processes \((\Phi_j)_j\) and \((\Phi_{jk}^*)_j\) be defined by \(d\Phi_j(t) = I_j(-)d\Phi(t)\), \(\Phi_j(0) = 0\), and \(d\Phi_{jk}^*(t) = I_j(-)d\Phi^*(t)\), \(\Phi_{jk}^*(0) = 0\), respectively. For \(j, k \in \mathcal{Z} \) let

\[
X_{\Phi, j} = \Phi_j - \Phi_j^*,
\]

\[
X_{\Lambda, jk} = \Lambda_{jk} - \Lambda_{jk}^*,
\]

and set \(X = ((X_{\Phi, j})_j, (X_{\Lambda, jk})_{j, k \neq k}, (X_{\Lambda, jk})_{j, k \neq k})\). Then

\[
\varrho(X) = -H \left( \left(\Phi^*, \Lambda^*\right) + \left( \sum_j X_{\Phi, j}, X_{\Lambda, jk} + X_{\Lambda, jk})_{j, k \neq k} \right) \right).
\]
has the ISU decomposition

\[ D_{\Phi,j}(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} I_j(s-) V^*_j(s-\) d(\Phi - \Phi^*)(s), \]

\[ D_{u,ijk}(t) = -\int_{(0,t]} \frac{1}{\kappa(s)} I_j(s-) R^*_jk(s) d(N_{jk} - \Lambda_{jk})(s), \]

\[ D_{s,ijk}(t) = -\int_{(0,t]} \frac{1}{\kappa(s)} I_j(s-) R^*_jk(s) d(\Lambda_{jk} - \Lambda^*_{jk})(s). \]

In particular, the ISU decomposition does not depend on the update order or the choice of partitions.

**Lemma 5.3:** Let \( X = (X_1, \ldots, X_m) \) be a given risk basis with

\[ R(t) = \varrho((X_1, \ldots, X_m)^t) \]

for a suitable mapping \( \varrho \), generating the ISU decomposition \( D(t) = (D_1(t), \ldots, D_m(t)) \) with respect to \((T_n(t))_n\), and let \( \mathcal{G} \) be a sub-\( \sigma \)-algebra of \( \mathcal{A} \). Suppose that the SU decomposition \( D^\mu(t) = (D^\mu_1(t), \ldots, D^\mu_m(t)) \) of \( R(t) - R(0) \) with respect to \( T_n(t) \) satisfies \( |D^\mu_i(t)| \leq Y, i = 1, \ldots, m, n \in \mathbb{N} \), for some integrable random variable \( Y \). Then the ISU decomposition of

\[ \tilde{R}(t) = \tilde{\varrho}((X_1, \ldots, X_m)^t) := \mathbb{E}[\varrho((X_1, \ldots, X_m)^t) | \mathcal{G}] \]

is given by

\[ \tilde{D}(t) = (\mathbb{E}[D_1(t) | \mathcal{G}], \ldots, \mathbb{E}[D_m(t) | \mathcal{G})]. \]

**Proof:** Since the revaluation surplus surfaces \( U \) and \( \tilde{U} \) are linked via the equation

\[ \tilde{U}(t_1, \ldots, t_m) = \mathbb{E}[U(t_1, \ldots, t_m) | \mathcal{G}], \]

the SU decomposition of \( \tilde{R}(t) - \tilde{R}(0) \) is given by \( \tilde{D}^\mu(t) = (\mathbb{E}[D^\mu_1(t) | \mathcal{G}], \ldots, \mathbb{E}[D^\mu_m(t) | \mathcal{G})] \). Using that \( |D^\mu_i(t)| \leq Y, i = 1, \ldots, m \), for some integrable random variable \( Y \) and the fact that stochastically converging sequences have almost surely converging subsequences, the dominated convergence theorem for conditional expectations almost surely yields

\[ \tilde{D}_i(t) = \lim_{n \to \infty} \mathbb{E}[D^\mu_i(t) | \mathcal{G}] = \mathbb{E}[D_i(t) | \mathcal{G}], \quad i = 1, \ldots, m. \]

**Theorem 5.4:** Let \( X \) be defined as in Theorem 5.2. Then

\[ \varrho(X^t) = \mathbb{E} \left[ -H \left( (\Phi^*, \Lambda^*) + \left( \sum_j X_{\Phi,j} + X_{\Lambda,ijk} \right) \right) \mid \Phi, \Lambda \right] \]

has the ISU decomposition

\[ D_{\Phi,j}(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} p_{aj}(0,s-) V^*_j(s-) d(\Phi - \Phi^*)(s), \]

\[ D_{u,ijk}(t) = 0, \]

\[ D_{s,ijk}(t) = -\int_{(0,t]} \frac{1}{\kappa(s)} p_{aj}(0,s-) R^*_jk(s) d(\Lambda_{jk} - \Lambda^*_{jk})(s). \]

In particular, the ISU decomposition does not depend on the update order or the choice of partitions.
Proposition 5.5: Let $X = (X_1, \ldots, X_m)$ be a given risk basis with

$$R(t) = \varrho((X_1 + X_2, X_3, \ldots, X_m)^t)$$

for a suitable mapping $\varrho$, generating the ISU decomposition $D(t) = (D_1(t), \ldots, D_m(t))$. Then the partially aggregated risk basis

$$\tilde{X} = (X_1 + X_2, (X_3, X_4), X_5 \ldots, X_m)$$

generates the ISU decomposition

$$\tilde{D}(t) = (D_1(t) + D_2(t), D_3(t) + D_4(t), D_5(t) \ldots, D_m(t)).$$

Proof: Since the revaluation surplus surfaces $\tilde{U}$ and $\tilde{U}$ are linked via the equation

$$\tilde{U}(t_1, t_2, t_3 \ldots, t_m) = U(t_1, t_1, t_3, t_4, t_5 \ldots, t_m),$$

the SU decompositions $D^n$ and $\tilde{D}^n$ with respect to $T_n(t)$ satisfy

$$\tilde{D}^n(t) = (D^n_1(t) + D^n_2(t), D^n_3(t) + D^n_4(t), D^n_5(t) \ldots, D^n_m(t)).$$

The latter equation carries through the limit (20) to the ISU decompositions.

6. Examples

We continue with the examples for the risk basis $X$ and the mapping $\varrho$ from Section 3 and present the corresponding ISU decompositions.

Decomposition of the individual revaluation surplus

Let $R$ be the individual revaluation surplus according to (12).

Example 6.1: Suppose that we are in the setting of Example 3.2, where we distinguish between financial risk, unsystematic biometric risk and systematic biometric risk. By applying Theorem 5.2 and Proposition 5.5, we obtain the ISU decomposition

$$D_\Phi(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} \sum_j I_j(s-) V_j^*(s-) \, d(\tilde{\Phi} - \Phi^*)(s),$$

$$D_u(t) = - \sum_{jk,j\neq k} \int_{(0,t]} \frac{1}{\kappa(s-)} I_j(s-) R_{jk}^s(s) \, d(N_{jk} - \Lambda_{jk})(s),$$

$$D_s(t) = - \sum_{jk,j\neq k} \int_{(0,t]} \frac{1}{\kappa(s-)} I_j(s-) R_{jk}^s(s) \, d(\Lambda_{jk} - \Lambda_{jk}^*)(s).$$

Example 6.2: Suppose that we are in the setting of Example 3.3, where we distinguish between financial risk and transition-wise biometric risks. By applying Theorem 5.2 and Proposition 5.5, we obtain the ISU decomposition

$$D_\Phi(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} \sum_j I_j(s-) V_j^*(s-) \, d(\tilde{\Phi} - \Phi^*)(s),$$

$$D_{jk}(t) = - \int_{(0,t]} \frac{1}{\kappa(s-)} I_j(s-) R_{jk}^s(s) \, d(N_{jk} - \Lambda_{jk}^*)(s), \quad j, k \in Z, \quad j \neq k.$$

As a special case, this ISU decomposition includes the heuristic approach of Ramlau-Hansen (1988, formula (4.7)) for subdividing biometric surplus in a transition-wise way.
Example 6.3: Suppose that we are in the setting of Example 3.4, where we distinguish unsystematic biometric risk and state-wise remaining risks. By applying Theorem 5.2 and Proposition 5.5, we obtain the ISU decomposition

\[ D_u(t) = - \sum_{j:k \neq k} \int_{(0,t]} \frac{1}{\kappa(s)} I_j(s-) R^*_{jk}(s) d(N_{jk} - \Lambda_{jk})(s), \]

\[ D_j(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} I_j(s-) \left( V_j^*(s-) d(\Phi^* - \Phi)(s) - \sum_{k:k \neq j} R^*_{jk}(s) d(\Lambda_{jk} - \Lambda_{jk}^*)(s) \right), \quad j \in \mathbb{Z}. \]

As a special case, this ISU decomposition includes heuristic approaches of Ramlau-Hansen (1988, formula before (4.10)) and Norberg (1999, formula (5.4)) for splitting off unsystematic biometric surplus and then subdividing the remaining surplus in a state-wise way.

In Examples 6.1 and 6.3, we split off the surplus contribution of the unsystematic biometric risk. Since this unsystematic biometric risk is diversifiable in the insurance portfolio, its contribution \( \kappa(t) D_u(t) \) to the total surplus \( S(t) \), cf. (6), is typically credited or debited to the insurer. Møller and Steffensen (2007, Chapter 6.3) denote the remaining surplus \( S(t) - \kappa(t) D_u(t) \) as the ‘systematic surplus’. This systematic surplus mainly belongs to the policyholder.

Asmussen and Steffensen (2020, Chapter VI.4) split also the financial risk into an unsystematic part and a systematic part and argue that the unsystematic financial risk surplus contribution should be fully credited or debited to the insurer, similarly to the unsystematic biometric risk surplus contribution. They distinguish unsystematic and systematic financial risks by splitting \( \Phi \) into a martingale part and a remaining systematic part. If we likewise split \( \Phi - \Phi^* \) in the risk basis \( X \) into a martingale part and a remaining systematic part, then the resulting ISU decomposition allows us to distinguish between systematic and unsystematic surplus contributions. If we then collect the systematic biometrical and systematic financial surplus contributions, then we just end up with the systematic surplus formula of Asmussen & Steffensen (2020, Chapter VI.4). We do not show the detailed calculations here but leave them to the reader.

**Decomposition of the mean portfolio revaluation surplus**

Let \( R \) be the mean portfolio revaluation surplus according to (15).

Example 6.4: We choose the setting from Example 3.2 but adopt the mean portfolio perspective. By applying Theorem 5.4 and Proposition 5.5, we obtain the ISU decomposition

\[ D_\Phi(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} \sum_j p_{aj}(0, s-) V_j^*(s-) d(\Phi - \Phi^*)(s), \]

\[ D_u(t) = 0, \]

\[ D_s(t) = - \sum_{j:k \neq k} \int_{(0,t]} \frac{1}{\kappa(s)} p_{aj}(0, s-) R^*_{jk}(s) d(\Lambda_{jk} - \Lambda_{jk}^*)(s). \]

The conditional expectation in (15) and (16) completely eliminates the unsystematic biometric risk, which explains why we have \( D_u(t) = 0 \) here.
Example 6.5: Here we choose the setting from Example 3.3 but adopt the mean portfolio perspective. By applying Theorem 5.4 and Proposition 5.5, we obtain the ISU decomposition

\[
D_\Phi(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} \sum_j p_{aj}(0,s-) V_j^s(s-) \, d(\Phi - \Phi^*)(s),
\]

\[
D_jk(t) = -\int_{(0,t]} \frac{1}{\kappa(s-)} p_{aj}(0,s-) R_{jk}^s(s) \, d(\Lambda_{jk} - \Lambda_{jk}^*)(s), \quad j, k \in \mathbb{Z}, \quad j \neq k.
\]

The next example shows an application of this formula.

Example 6.6: We continue with the previous example but focus here on the specific setting of Example 3.6. One can show that the SU decomposition of \(R(k+1) - R(k)\) with respect to an integer partition equals

\[
U(k+1, k, k) - U(k, k, k) = e^{-\int_0^{k+1} \phi(u) \, du} k\rho_x V_a^s(k) \left( i_k - i_k^* \right),
\]

\[
U(k+1, k+1, k) - U(k+1, k, k)
\]

\[
= e^{-\int_0^{k+1} \phi(u) \, du} k\rho_x \left( V_a^s(k+1) - d_{k+1} \right) \left( q_{x+k} - q_{x+k}^* \right),
\]

\[
U(k+1, k+1, k+1) - U(k+1, k+1, k)
\]

\[
= e^{-\int_0^{k+1} \phi(u) \, du} k\rho_x \left( V_a^s(k+1) - s_{k+1} \right) \left( r_{x+k} - r_{x+k}^* \right),
\]

see the appendix. This decomposition is the standard surplus decomposition formula used in German life insurance, cf. Milbrodt & Helbig (1999, Section 11.B). We can interpret the latter SU decomposition as an approximation of the ISU decomposition of \(R'(k+1) - R'(k)\), which equals here

\[
D_\Phi(k+1) - D_\Phi(k) = \int_{(k,k+1]} e^{-\int_0^u \phi(u) \, du} s\rho_x V_a^s(s) \, d(\Phi - \Phi^*)(s),
\]

\[
D_{ad}(k+1) - D_{ad}(k) = \int_{(k,k+1]} e^{-\int_0^u \phi(u) \, du} s\rho_x \left( V_a^s(s) - b_{ad}(s) \right) \, d(\Lambda_{ad} - \Lambda_{ad}^*)(s),
\]

\[
D_{as}(k+1) - D_{as}(k) = \int_{(k,k+1]} e^{-\int_0^u \phi(u) \, du} s\rho_x \left( V_a^s(s) - b_{as}(s) \right) \, d(\Lambda_{as} - \Lambda_{as}^*)(s).
\]

The latter decomposition is invariant with respect to a reordering of the components of \(X\), whereas the SU decomposition changes. Therefore, we recommend to replace the traditional SU decomposition (21) by the ISU decomposition (22).

Example 6.7: We choose the setting from Example 3.4 but adopt the mean portfolio perspective. By applying Theorem 5.4 and Proposition 5.5, we obtain the ISU decomposition

\[
D_u(t) = 0,
\]

\[
D_j(t) = \int_{(0,t]} \frac{1}{\kappa(s-)} p_{aj}(0,s-) \left( V_j^s(s-) \, d(\Phi - \Phi^*)(s) - \sum_{k: k \neq j} R_{jk}^s(s) \, d(\Lambda_{jk} - \Lambda_{jk}^*)(s) \right), \quad j \in \mathbb{Z}.
\]

As a special case, this ISU decomposition includes heuristic approaches of Ramlau-Hansen (1991, formula (3.2)) and Norberg (1999, formula (5.7)) for subdividing mean portfolio surplus in a state-wise manner.
7. Alternative decomposition principles

In Section 4, we already mentioned that the ISU decomposition may depend on the update order. In this section, we want to elaborate on that point by discussing two alternative decomposition principles in the setup of Section 4. Instead of updating the sources of risk sequentially, we could also update only one source of risk at a time and quantify its impact on total revaluation surplus \( R(t) - R(0) \), which will lead us to the OAT (one-at-a-time) decomposition.

Recall that \( U(t_1, \ldots, t_m) = \varrho((X^{t_1}_1, \ldots, X^{t_m}_m)) \) is the value of the delayed revaluation process at time points \( t_1, \ldots, t_m \) (cf. (18)). For any partition \( T(t) = \{0 = t_0 < t_1 < \cdots < t_k = t\} \) of the interval \([0, t]\), we can decompose

\[
R(t) - R(0) = U(t, \ldots, t) - U(0, \ldots, 0)
= \sum_{l=0}^{k-1} \left( U(t_{l+1}, t_l, \ldots, t_l) - U(t_l, \ldots, t_l) \right)
+ \sum_{l=0}^{k-1} \left( U(t_l, t_{l+1}, t_l, \ldots, t_l) - U(t_l, \ldots, t_l) \right)
+ \ldots
+ \sum_{l=0}^{k-1} \left( U(t_l, \ldots, t_l, t_{l+1}) - U(t_l, \ldots, t_l) \right)
+ \sum_{l=0}^{k-1} \left( U(t_{l+1}, \ldots, t_{l+1}) - U(t_l, \ldots, t_l) \right)
- \sum_{l=0}^{k-1} \left( U(t_{l+1}, t_l, \ldots, t_l) - U(t_l, \ldots, t_l) \right)
+ \ldots
+ U(t_l, \ldots, t_l, t_{l+1}) - U(t_l, \ldots, t_l)
\]

Here, the first \( m \) sums quantify the single effect of the corresponding source of risk. Following Biewen (2014), we call them the ceteris paribus effects. Since the ceteris paribus effects do not necessarily add up to the total revaluation surplus \( R(t) - R(0) \), we get an extra term in the last two lines, which is called the interaction effect (cf. Biewen (2014)). Based on this construction, we get a decomposition principle with a joint risk factor.

**Definition 7.1:** The random vector \( D(t) = (D_1(t), \ldots, D_m(t), \overline{D}(t)) \) defined by

\[
D_1(t) = \sum_{l=0}^{k-1} \left( U(t_{l+1}, t_l, \ldots, t_l) - U(t_l, t_l, \ldots, t_l) \right),
\]

\[
D_m(t) = \sum_{l=0}^{k-1} \left( U(t_l, \ldots, t_l, t_{l+1}) - U(t_l, \ldots, t_l) \right),
\]

\[
\overline{D}(t) = R(t) - R(0) - \sum_{j=1}^{m} D_j(t)
\]

is called the OAT (one-at-a-time) decomposition of \( R(t) = \varrho(X^t) \) with respect to \( T(t) \).
The OAT decomposition principle is also known in economics, see, for example Biewen (2014). In contrast to the SU decomposition, the OAT decomposition is order invariant, i.e. it does not depend on the order of the risk basis (cf. Schilling et al. (2020), Section 2.2). Nevertheless, we get a joint risk factor that cannot be assigned to any source of risk. In Section 4, we faced the order dependence of the SU decomposition by considering increasing sequences of partitions of \([0, t]\). Similarly, we face the unassignable interaction effect in the OAT decomposition.

Let \(T_n(t) = \{0 = t_0^n < t_1^n < \cdots < t_n^n = t\}, n \in \mathbb{N}\), be a sequence of partitions of \([0, t]\) with vanishing step lengths (i.e. \(\lim_{n \to \infty} \max_{1 \leq i \leq n} |t_i^n - t_{i-1}^n| = 0\)). For each \(n \in \mathbb{N}\) let \(D^n(t) = (D_1^n(t), \ldots, D_m^n(t), D^n(t))\) be the OAT decomposition of \(R(t) = \varrho(X)\) with respect to \(T_n(t)\). We are looking for a random vector \(D(t) = (D_1(t), \ldots, D_m(t), D(t))\) that satisfies

\[
D_i(t) = \lim_{n \to \infty} D_i^n(t), \quad i \in \{1, \ldots, m\},
\]

\[
D(t) = \lim_{n \to \infty} D^n(t).
\]

Definition 7.2: Let \((T_n(t))_{n \in \mathbb{N}}\) be a sequence of partitions of \([0, t]\) with vanishing step lengths. If \(D(t) = (D_1(t), \ldots, D_m(t), D(t))\) satisfies (24), then we call \(D(t)\) the IOAT (infinitesimal one-at-a-time) decomposition of \(R(t) = \varrho(X)\) with respect to \((T_n(t))_{n \in \mathbb{N}}\).

The next theorem characterizes the relation between the ISU decomposition and the IOAT decomposition.

Theorem 7.3: The following statements are equivalent:

(a) The ISU decomposition is independent of update order.

(b) For each update order, the ISU decomposition is equal to the ceteris paribus effects of the IOAT decomposition.

In both cases, the interaction effect is zero.

Proof: The proof follows Biewen (2014). Let us fix a source of risk \((i = 1, \ldots, m)\). Choosing an update order, such that this source of risk is updated first, the corresponding risk factor of the ISU decomposition coincides per definition with the ceteris paribus effect of the IOAT decomposition. If the ISU decomposition is independent of update order, the risk factor, corresponding to the fixed source of risk, equals the ceteris paribus effect of the IOAT decomposition for each update order.

On the other hand, the statement in (b) directly implies that the ISU decomposition is independent of update order. Furthermore, if the ISU decomposition equals the IOAT decomposition, then the ceteris paribus effects sum up to total risk \(R(t) - R(0)\), therefore the interaction effect is zero.

By subdividing the interaction effect into different groups of interaction effects (depending on the number of involved risk factors), Biewen (2014) even shows that the particular interaction effects are zero if and only if the ISU decomposition is independent of update order.

If the interaction effect is non-zero, neither the ISU decomposition nor the IOAT decomposition yields an order-invariant decomposition satisfying (5). One possible solution for this problem is to build a decomposition principle based on the ISU decomposition principle that is symmetric with respect to the sources of risk. For that, let \(\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}\) be a permutation that represents an update order for the ISU decomposition. The set of all possible permutations on \(\{1, \ldots, m\}\) is denoted by \(\sigma_m\).

Definition 7.4: Let \((T_n(t))_{n \in \mathbb{N}}\) be an increasing sequence of partitions of \([0, t]\) with vanishing step lengths and let \(\pi \in \sigma_m\). Further, let \(D^n(\pi)(t) = (D^n_1(\pi)(t), \ldots, D^n_m(\pi)(t))\) denote the ISU decomposition of \(R(t) = \varrho(X)\) with respect to \(\pi\) and with respect to \((T_n(t))_{n \in \mathbb{N}}\). The random vector \(D(t) =\)
(\(D_1(t), \ldots, D_m(t)\)) defined by

\[
D_1(t) = \frac{1}{m!} \sum_{\pi \in \sigma_m} D_\pi^{(1)}(t), \\
\ldots \\
D_m(t) = \frac{1}{m!} \sum_{\pi \in \sigma_m} D_\pi^{(m)}(t),
\]

is called the \textit{averaged ISU decomposition} of \(R(t) = \varrho(X_t)\) with respect to \((T_n(t))_{n \in \mathbb{N}}\).

In a similar manner, Shorrock\(s (2013)\) proposes the averaged SU decomposition (without taking limits) in economics literature. By construction, the averaged ISU decomposition principle is symmetric with respect to the risk basis and therefore gives an order-invariant surplus decomposition satisfying (5) even if the interaction effect is non-zero. Furthermore, the averaged ISU decomposition is in line with the previously proposed decomposition principles as the next theorem shows.

**Theorem 7.5:** If the ISU decomposition is independent of update order, then ISU (for each update order), IOAT and averaged ISU yield the same decomposition.

**Proof:** Assume that the ISU decomposition principle yields a decomposition \((D_1(t), \ldots, D_m(t))\) for each update order. Then, by Theorem 7.3, the ISU decomposition is equal to the IOAT decomposition for each update order. Furthermore, it holds \(D_\pi^{(i)}(t) = D_i(t), i = 1, \ldots, m\) for every permutation \(\pi\). Since \#\(\sigma_m = m!\), the averaged ISU decomposition is also given by \((D_1(t), \ldots, D_m(t))\). \(\blacksquare\)

As shown in Section 5, the ISU decompositions in our life insurance model do not depend on the update order. Thus we directly get the following result.

**Corollary 7.6:** For all examples in Section 6 the IOAT decomposition and the averaged ISU decomposition are both equal to the ISU decomposition.

### 8. Conclusion and outlook

The ISU decomposition principle allows us to unite the various surplus decomposition formulas from the life insurance literature under one banner. By doing so, we replace the common heuristic constructions by a general and consistent decomposition principle.

We focussed here on recreating surplus decompositions that are already known in the actuarial literature, but the generality of the ISU concept makes it a promising tool for creating new results for various open decomposition problems that need yet to be solved. For example, the classical surplus decomposition formulas in with-profit life insurance all based on expectation-based evaluation in combination with implicit risk margins, and it is an open question if and how we can switch to other risk measures while maintaining the with-profit concept. Another open problem is the actuarial calculation of behaviour-based insurance. New insurance forms are on the rise that digitally track certain activities of the insured and reward risk averse behaviour. The individual activity of the insured influences the total surplus, and the open question is how to determine the fair share of the total surplus that may be attributed to the observed behaviour of the insured. The ISU concept is also useful beyond life insurance. Whenever the profits and losses of a financial entity shall be decomposed with respect to their sources, the ISU decomposition is a helpful tool for that. One of the key ideas in this paper is to overcome the well-known limitations of SU and OAT decompositions by an infinitesimal approach. So our ISU, average ISU and IOAT concepts may be of help in all that applications where SU and OAT decompositions are currently in use, see our references to the economics in the introduction. A strength of the ISU concept is that it guarantees additivity under all circumstances.
As the general definition of ISU decompositions involves a stochastic limit, one might think that they are rather difficult to calculate in practice. However, the asymptotic definition allows us to approximate ISU decompositions by SU decompositions, and these approximations are in fact rather easy to implement in practice.

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**Appendix**

**A.1 Proofs**

**Proof of Lemma 5.1:** As a shorthand notation, we define multivariate processes \( C^* = (C^*_1, \ldots, C^*_n)^T \) and \( C = (C_1, \ldots, C_n)^T \) by

\[
dC^*_j(s) = dB_j(s) + \sum_{kk' \neq j} b_{jk}(s) d\Lambda_{jk}^*(s), \quad C^*_j(0) = 0,
\]

\[
dC_j(s) = dB_j(s) + \sum_{kk' \neq j} b_{jk}(s) d\Lambda_{jk}(s), \quad C_j(0) = 0.
\]

Note that \( C^* \) and \( C \) are column vectors. The vectorial process \( I = (I_1) \) shall combine all state processes as a row vector. We further define

\[
W(s) := -H((\Phi^*, \Lambda^*) + (\Phi - \Phi^*, \Lambda - \Lambda^*))^T,
\]

where \( H \) is given by (14)). Due to the assumptions made on the first-order valuation basis in Section 3 (cf. (F.3) and the follow-up remarks), \( \rho^*(0, s) \) is invertible with inverse \( q^*(0, s) \).
Thus, for $s \in (0, t]$, we get
\[
W(s) = - \int_{[0,s]} \frac{1}{k(s)} I(0) \bar{p}(0, u-) \, d\bar{C}(u) - \int_{(s,T]} k^*(u) I(0) \bar{p}(0, s) p^*(s, u-) \, dC^*(u)
\]
\[
= - \int_{[0,s]} \frac{1}{k(s)} I(0) \bar{p}(0, u-) \, d\bar{C}(u) - \int_{(s,T]} \frac{k^*(s)}{k(s)} I(0) \bar{p}(0, s) q^*(s, 0) \, Y(s),
\]
for $Y(s) = \int_{[s,T]} \frac{1}{k(s)} p^*(0, u-) \, dC^*(u)$. Analogously to $\Lambda^*_M$, let $\Lambda_M$ denote the matrix-valued process $\Lambda_M = (\Lambda_{jk})_k$ with $\Lambda_{jk} = -\sum_{j' \neq j} \Lambda_{jk}$. By applying Itô’s formula and using the assumption that $(\Phi^*, \bar{\Phi})$ and $(\Lambda^*, \bar{\Lambda}, (B_j)_j)$ have no common jumps, we can show that
\[
dW(s) = - \frac{1}{k(s)} I(0) \bar{p}(0, s-) \, d(\bar{C} - C^*)(s)
\]
\[
- I(0) \frac{k^*(s)}{k(s)} \bar{p}(0, s-) q^*(s, 0-) \, d(\Phi^* + \bar{\Phi} - [\Phi^*, \bar{\Phi}]) (s)
\]
\[
- I(0) \frac{k^*(s)}{k(s)} \bar{p}(0, s-) \, d(\Lambda_M - \Lambda^*_M) (s) q^*(s, 0) \, Y(s)
\]
\[
= - \frac{1}{k(s)} I(0) \bar{p}(0, s-) \, d(\bar{C} - C^*)(s)
\]
\[
- \frac{1}{k(s)} I(0) \bar{p}(0, s-) \left( \int_{[s,T]} \frac{k^*(s)}{k(u)} p^*(s, u-) \, dC^*(u) \right) \, d(\Phi^* + \bar{\Phi} - [\Phi^*, \bar{\Phi}]) (s)
\]
\[
- \frac{1}{k(s)} I(0) \bar{p}(0, s-) \, d(\Lambda_M - \Lambda^*_M)(s) \left( \int_{(s,T]} \frac{k^*(s)}{k(u)} p^*(s, u-) \, dC^*(u) \right)
\]
where we used Lemma A.2 to get
\[
d \left( \frac{k^*(s)}{k(s)} \bar{p}(0, s) q^*(0, s) \right)
\]
\[
= \frac{k^*(s)}{k(s)} \, d \left( \bar{p}(0, s) q^*(0, s) \right) + \bar{p}(0, s-) q^*(0, s-) \, d \left( \frac{k^*(s)}{k(s)} \right) + \frac{d}{d s} \left( \frac{k^*(s)}{k(s)} \bar{p}(0, s-) q^*(0, s-) \right)
\]
\[
= \frac{k^*(s)}{k(s)} \, d \left( \Lambda_M - \Lambda^*_M \right)(s) q^*(0, s) + \frac{k^*(s)}{k(s)} \bar{p}(0, s-) q^*(0, s-) \, d(\Phi^* + \bar{\Phi} - [\Phi^*, \bar{\Phi}]) (s)
\]
with $\tilde{\Phi}(s) = [\Phi, \bar{\Phi}]_s - \sum_{s_{i-1} < s \leq r} (\alpha + \Delta \bar{\Phi}(u))^{-1}(\Delta \bar{\Phi}(u))^2$. Component-wise evaluation and integration on $(0, t]$ gives us the assertion.

**Proof of Theorem 5.2:** Let $J \subseteq \mathcal{Z}$ and $J_u, J_s \subseteq \mathcal{J} := \{(j, k) \in \mathbb{Z}^2 : j \neq k \}$. For $r \leq s$, we define
\[
X^{r,s}_{J_u, j} := \begin{cases} X_{J_u,j} & j \notin J_s, \\ X_s^{r,j} & j \in J_s, \end{cases} \quad X^{r,s}_{J_s, j} := \begin{cases} X^{r,j} & j \notin J_u, \\ X_{J_s, j} & j \in J_u, \end{cases}
\]
as well as
\[
X^{r,s}_{J_u, j} := \begin{cases} X^{r,j} & (j, k) \notin J_u, \\ X_{J_u, j} & (j, k) \in J_u, \end{cases} \quad X^{r,s}_{J_s, j} := \begin{cases} X_{J_s, j} & (j, k) \notin J_s, \\ X^{r,j} & (j, k) \in J_s. \end{cases}
\]
We further set
\[
X_{J_u} := \sum_{J_u, j} X_{J_u,j}^{0,T}, \quad X_{J_s} := \sum_{J_s, j} X_{J_s,j}^{0,T}, \quad X_{J_s, j}^{0,T} := (X^{0,T}_{J_s, j})_j, \quad X_{J_u, j}^{0,T} := (X^{0,T}_{J_u, j})_j,
\]
where $X_{J_u, j}^{0,T} = -\sum_{j \neq j'} X_{J_u, j'}^{0,T}$ and $X_{J_s, j}^{0,T} = -\sum_{j \neq j'} X_{J_s, j'}^{0,T}$. Let $\Phi^{\lambda o} := X_{J_u} + \Phi^*$ and let $\lambda_{\text{io}}$ denote the solution of $d\lambda = \kappa_{\text{io}}(s-) = \Phi^{\lambda o}(s)$ with $\kappa_{\text{io}}(0) = 1$. Similarly, for $J = (J_u, J_s)$ let $\Lambda^{\lambda} := X_{J_u, j} + X_{J_s, j} + \Lambda^*_M$ and let $p^t = (p_{jk})_j$ denote the solution of $p^t(r, dz) = r + s \, d\Lambda^*(s)$ with $p^t(r, r)$ being the identity matrix.

Let $t \in [0, T]$ and let $(T_n(t))_n$ be a sequence of partitions of $[0, t]$. For a simpler notation, we only write $t_n$ instead of $t^n_k$ for the grid points in $T_n$. Throughout the proof, let $s_{\alpha_n}(s)$ be the left point of $s$ in $T_n(t)$, i.e. $s_{\alpha_n} := \min\{s \in (t_k, t_{k+1}] \mid t_k < s < t_{k+1} \}$.

For notational convenience, we write
\[
\bar{e}_{J_u, J_s, j} := e((X_{J_u, J_s, j})_j (X_{J_u, J_s, j})_j)_{j \neq k}, (X^{s_{\alpha_n}, t_{n+1}}_{J_u, J_s, j})_{j \neq k}.
\]

It is sufficient to show that
(i) \( \lim_{n \to \infty} \sum_{t_k, j_k+1 \in \mathcal{T}_n(t)} (\varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n) = D_{\Phi_{j_k}}(t), j_0 \in \mathbb{Z} \setminus J_B, \)
(ii) \( \lim_{n \to \infty} \sum_{t_k, j_k+1 \in \mathcal{T}_n(t)} (\varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n) = D_{\Phi_{j_k}}(t), (j_0, k_0) \in J \setminus J_B, \)
(iii) \( \lim_{n \to \infty} \sum_{t_k, j_k+1 \in \mathcal{T}_n(t)} (\varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n) = D_{\Phi_{j_k}}(t), (j_0, k_0) \in J \setminus J_B. \)

We prove the convergence consecutively.

(i) Let \( \bar{T}_B = J_B \cup \{j_0\}, j_0 \in \mathbb{Z} \setminus J_B \) and let

\[
\Delta(r, s) = \frac{\kappa^B_t(r)}{\kappa^B_t(s)} - \frac{\kappa^B_t(s)}{\kappa^B_t(s)}, \quad r < s.
\]

We define stochastic processes

\[
\xi_{\Phi, j_0, n}(s) = \frac{1}{\kappa(\alpha_n(s))} \kappa^B_t(\alpha_n(s)) \sum_j I^j(s) \sum_j \varphi^j(\alpha_n(s), s) V_j(s) - I_{j_0}(s),
\]

\[
\xi_{\Phi, j, n}(s) = \frac{\Delta(\alpha_n(s), s)}{\kappa(\alpha_n(s))} \sum_j I^j(s) \sum_j \varphi^j(\alpha_n(s), s) V_j(s) - I_{j_0}(s), \quad j \in J_B,
\]

\[
\xi_{\Phi, j, n}(s) = - \sum_j I^j(s) \sum_j \varphi^j(\alpha_n(s), s) \frac{\Delta(\alpha_n(s), s)}{\kappa(\alpha_n(s))} \varphi^j(\alpha_n(s), s) - I_{j_0}(s), \quad j \in J_B.
\]

where \( s \in [0, t] \). Due to Assumption 3, we can apply Lemma 5.1, which gives us

\[
\sum_{t_k, j_k+1 \in \mathcal{T}_n(t)} (\varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n)
\]

\[
= \sum_{t_k, j_k+1 \in \mathcal{T}_n(t)} (\varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n)
\]

\[
= \sum_{j \in J_B} \int_{[0, t]} \xi_{\Phi, j, n}(s) d(\bar{\Phi} - \Phi^*) + \sum_{(j, k) \in J_B} \int_{[0, t]} \xi_{\Phi, j, n}(s) d(N_{j k} - \Lambda_{j k})(s)
\]

\[
+ \sum_{(j, k) \in J_B} \int_{[0, t]} \xi_{\Phi, j, n}(s) d(\Lambda_{j k} - \Lambda_{j k}^*)
\]

where \( \bar{\Phi}(s) = \Phi(s) - \left[\Phi(s) \Phi(s) - \sum_{0 < u \leq s} (1 + \Delta(\Phi(u))^{-1} - \Delta(\Phi(u)))^2 \right]. \) Here, we used that

\[
d(\bar{\Phi} - \Phi^* + [\bar{\Phi} - \Phi^*]) = \sum_{j \in J_B} I_j(s) d(\bar{\Phi} - \Phi^*)(s),
\]

exploiting that \( \Phi \) and \( \Phi^* \) have no common jumps (cf. Assumption 3). Since for every \( s \in [0, t] \) we almost surely have

\[
\lim_{n \to \infty} \xi_{\Phi, j_0, n}(s) = \frac{1}{\kappa(s)} I_{j_0}(s) V_{j_0}(s),
\]

\[
\lim_{n \to \infty} \xi_{\Phi, j, n}(s) = 0, \quad j \in J_B,
\]

\[
\lim_{n \to \infty} \xi_{\Phi, j_0, n}(s) = I_{j}(s) \Delta(s, s) R_{j k}(s), \quad (j_0, k_0) \in J_B \cup J_B,
\]

and since \( \Delta(s, s) d(N_{j k} - \Lambda_{j k})(s) - \Delta(s, s) d(\Lambda_{j k} - \Lambda_{j k}^*) = 0 \) almost surely, the dominated convergence theorem for stochastic integrals (cf. Protter, 2005, Chapter IV, Theorem 32) yields

\[
\lim_{n \to \infty} \sum_{t_k, j_k+1 \in \mathcal{T}_n(t)} (\varphi_{j_k, j_{k+1}}^n - \varphi_{j_k, j_{k+1}}^n) = D_{\Phi_{j_k}}(t).
\]
(ii) Let $\overline{J} = (J_u \cup \{j_0, k_0\}, J_u )$, $(j_0, k_0) \in \mathcal{J} \setminus J_u$ and let

$$\Delta_{jk}(r, s) := p^\overline{J}_{jk}(r, s) - p^J_{jk}(r, s), \quad r \leq s.$$ 

We define stochastic processes

$$\xi_{\Phi,j,n}(s) = \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) \sum_g I_g(\alpha_n(s)) \sum_j J_j_0(\alpha_n(s), s-) V_j^s(s-) I_j(s-), \quad j \in J_0,$$

$$\xi_{\Phi,j,k,n}(s) = -\sum_g I_g(\alpha_n(s)) \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) \Delta_{\Phi,s}^j(\alpha_n(s), s-) R^s_{jk}(s), \quad (j, k) \in J_u \cup J_s,$$

$$\xi_{\Phi,j,k_0,n}(s) = -\sum_g I_g(\alpha_n(s)) \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) p^\overline{J}_{j_0}(\alpha_n(s), s-) R^s_{jk_0}(s),$$

where $s \in [0, t]$. Again with Lemma 5.1, we have

$$\lim_{n \to \infty} \xi_{\Phi,j,k_0,n}(s) = \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) \sum_g D^g_{j_0,k_0}(s)$$

$$= -\frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) \Delta_{\Phi,s}^j(\alpha_n(s), s-) R^s_{jk_0}(s),$$

$$\lim_{n \to \infty} \xi_{\Phi,j,n}(s) = \lim_{n \to \infty} \xi_{\Phi,j,k,n}(s) = 0, \quad j \in J_0, \quad (j, k) \in J_u \cup J_s,$$

and since $\frac{1 + \Delta_{\Phi,s}^j(\alpha_n(s), s-)}{1 + \Delta_{\Phi,s}^j(\alpha_n(s), s-)} d(N_{j_0k_0} - \Lambda_{j_0k_0})(s) = d(N_{j_0k_0} - \Lambda_{j_0k_0})(s)$ almost surely, the dominated convergence theorem for stochastic integrals (cf. Protter, 2005, Chapter IV, Theorem 32) yields

$$\lim_{n \to \infty} \sum_{t_k \in I} \left( \xi^t_{\Phi,j,k_0,n}(t) - \xi^t_{\Phi,j,k_0,n}(t) \right) = D^t_{\Phi,j,k_0}(t),$$

(iii) Let $\overline{J} = (J_u \cup \{j_0, k_0\}, J_u )$, $(j_0, k_0) \in \mathcal{J} \setminus J_s$ and let

$$\Delta_{jk}(r, s) := p^\overline{J}_{jk}(r, s) - p^J_{jk}(r, s), \quad r \leq s.$$ 

We define stochastic processes

$$\xi_{\Phi,j,n}(s) = \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) \sum_g I_g(\alpha_n(s)) \sum_j J_j_0(\alpha_n(s), s-) V_j^s(s-) I_j(s-), \quad j \in J_0,$$

$$\xi_{\Phi,j,k,n}(s) = -\sum_g I_g(\alpha_n(s)) \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) \Delta_{\Phi,s}^j(\alpha_n(s), s-) R^s_{jk}(s), \quad (j, k) \in J_u \cup J_s,$$

$$\xi_{\Phi,j,k_0,n}(s) = -\sum_g I_g(\alpha_n(s)) \frac{1}{\kappa(\alpha_n(s))} \kappa^{J_0}(\alpha_n(s)) p^\overline{J}_{j_0}(\alpha_n(s), s-) R^s_{jk}(s),$$
Lemma 5.3 for the SU decomposition (cf. proof of Theorem 5.2) is satisfied. Thus applying Lemma 5.3 with

**Proof of Theorem 5.4:**

Furthermore, for the risk basis

\[ \Phi = \sigma(\Phi, \Lambda) \]

we get the first equation.

Since

\[ \lim_{t \to \infty} \xi_{\Phi, \infty}(s) = \lim_{t \to \infty} \xi_{\Phi, \infty}(s) = 0, \quad j \in J_{\Phi}, \quad (j, k) \in J_{\Phi} \cup J_{I}, \]

and since \( \frac{1+\Delta \Phi(s)}{1+\Delta \Phi^\infty(s)} \) almost surely, the dominated convergence theorem for stochastic integrals (cf. Protter, 2005, Chapter IV, Theorem 32) yields

\[ \text{plim}_{n \to \infty} \sum_{t_k, t_{k+1} \in T_n(t)} (\xi_{\Phi, t_{k+1}}(\eta_{\Phi, t_{k}, t_{k+1}}) - \xi_{\Phi, t_{k}}(\eta_{\Phi, t_{k}, t_{k+1}})) = D_{\Phi, t_{k}}(t). \]

**Proof of Theorem 5.4:** The model framework, introduced in Section 3, entails that the integrability assumption in Lemma 5.3 for the SU decomposition (cf. proof of Theorem 5.2) is satisfied. Thus applying Lemma 5.3 with \( G = \sigma(\Phi, \Lambda) \) to the ISU decomposition in Theorem 5.2 and using the martingale property of \( dN_{jk}(t) - I_{j}(t-)d\Lambda_{jk}(t) \) with respect to the natural completed filtration of the random vector \( (Z', \Phi, \Lambda)_{t \geq 0} \) (cf. Assumption 3) give the desired result.

**Proof of the SU decomposition in the time-discrete case (Example 3.6):** In the setting of Example 3.6, the functional \( H \) in (14) takes the form

\[ H(\Phi, \Lambda_{ad}, \Lambda_{as}) = \sum_{l=0}^{T} e^{-f_{l}^{\Phi} \varphi(u)du} \Pi_{l} b_{l} + \sum_{l=1}^{T} e^{-f_{l}^{\Phi} \varphi(u)du} I_{l} \Pi_{l} (\eta_{l+1} + \Pi_{x+l} \eta_{l}) \]

Furthermore, for the risk basis \( X = (\Phi - \Phi^{*}, \Lambda_{ad} - \Lambda_{ad}^{*}, \Lambda_{as} - \Lambda_{as}^{*}) \), the mapping \( \varphi \) is given by \( \varphi(X) = -H((\Phi^{*}, \Lambda_{ad}^{*}, \Lambda_{as}^{*}), X') \). We prove the three equations consecutively.

(i) We have that

\[ U(k+1, k, k) - U(k, k, k) \]

\[ = \varphi(\Phi^{k+1}, \Lambda_{ad}^{k+1}, \Lambda_{as}^{k+1}) - \varphi(\Phi^{k}, \Lambda_{ad}^{k}, \Lambda_{as}^{k}) \]

\[ = e^{-f_{k}^{\Phi} \varphi(u)du} k_{p_{x}} ((1 + ik) V_{a}^{k}(k) - p_{x+k}^{*} b_{k+1} - q_{x+k}^{*} d_{k+1} - r_{x+k}^{*} s_{k+1} - p_{x+k}^{*} V_{a}^{k}(k+1)) \]

Since

\[ -p_{x+k}^{*} b_{k+1} - q_{x+k}^{*} d_{k+1} - r_{x+k}^{*} s_{k+1} - p_{x+k}^{*} V_{a}^{k}(k+1) = -(1 + ik) V_{a}^{k}(k), \]

we get the first equation.
(ii) With similar calculations as in (i), we get
\[ U(k + 1, k, k) - U(k, k, k) \]
\[ = \varphi(\Phi_{k+1}, \Lambda_{(k+1)}^{n}, \Lambda_{a}) - \varphi(\Phi_{k}, \Lambda_{k}^{n}, \Lambda_{a}) \]
\[ = e^{-\int_{0}^{k+1} \phi(u) du} k_{0} ((1 - q_{x+k} - r_{x+k}^{a}) b_{k+1} + q_{x+k} d_{k+1} + r_{x+k}^{a} s_{k+1}) \]
\[ \quad - e^{-\int_{0}^{k+1} \phi(u) du} k_{0} (1 - q_{x+k} - r_{x+k}^{a}) V_{a}(k + 1) + e^{-\int_{0}^{k} \phi(u) du} k_{0} V_{a}(k) \]
\[ = e^{-\int_{0}^{k+1} \phi(u) du} k_{0} (V_{a}(k + 1 -) - d_{k+1} (q_{x+k} - d_{x+k}) + e^{-\int_{0}^{k} \phi(u) du} k_{0} V_{a}(k) (i_{k} - i_{k}^{d}) \]

The second equality follows then by subtracting \( U(k + 1, k, k) - U(k, k, k) \) (cf. (i)) from \( U(k + 1, k, k) - U(k, k, k) \).

(iii) For the third equality, we can use the results from (i) and (ii) to obtain
\[ U(k + 1, k, k + 1) - U(k + 1, k, k) \]
\[ = R(k + 1) - R(k) - (U(k + 1, k, k + 1) - U(k + 1, k, k)) - (U(k + 1, k, k) - U(k, k, k)) \]
\[ = e^{-\int_{0}^{k+1} \phi(u) du} k_{0} (V_{a}(k + 1 -) - s_{k+1}) (r_{x+k} - r_{x+k}^{a}) \]

\[ \blacksquare \]

### A.2 Technical results

Analogously to \( \Lambda_{M}^{*} \), let \( \Lambda_{M} \) denote the matrix-valued process \( \Lambda_{M} = (\Lambda_{jk})_{jk} \) with \( \Lambda_{ij} := -\sum_{k \neq j} \Lambda_{jk} \), and define \( \Lambda'_{M} \) likewise.

**Lemma A.1:** Let \( (\Phi, \Lambda) \) be a valuation basis.

1. Let \( \Phi(s) \) be the solution of the stochastic differential \( d\Phi(s) = \Phi(s-)d\Phi(s) \) with \( \Phi(0) = 1 \). Then it holds that
\[
\frac{d}{ds} \left( \frac{1}{\Phi(s)} \right) = -\frac{1}{\Phi(s-)} d\Phi(s),
\]
where \( \Phi(s) = \Phi(s) - [\Phi, \Phi]^{c}(s) = \sum_{0 \leq u \leq s} (1 + \Delta \Phi(u))^{-1} (\Delta \Phi(u))^{2} \).

2. Let \( \tilde{\Phi}(t, s) \) be the solution of the matrix-valued stochastic differential equation \( \tilde{\Phi}(t, s) = \tilde{\Phi}(t, s-)d\Lambda_{M}(s) \) with \( \tilde{\Phi}(t, t) = I \). Assume that \( (I + \Delta \Lambda_{M}(s))^{-1} \) exists for all \( s > 0 \). Then \( \tilde{\Phi}(t, s) \) is invertible, and the inverse \( \tilde{\Phi}(t, s) \) solves the SDE
\[
\tilde{\Phi}(t, s) = -(dG(s))\tilde{\Phi}(t, s-),
\]
where \( G(s) = \Lambda_{M}(s) - \sum_{0 \leq u \leq s} (\Delta \Lambda_{M}(u))^{2} (I + \Delta \Lambda_{M}(u))^{-1} \).

**Proof:**

(a) Due to the properties of a valuation basis, \( \Phi \) is a well-defined semimartingale. Thus with Theorem V.10.63 of Protter (2005), the assertion follows.

(b) Without loss of generality, we prove the case \( t = 0 \). For applying Theorem V.10.63 of Protter (2005) later again, we first have to show that \( G \) is a well-defined semimartingale. Since \( \Lambda \) is a càdlàg finite variation process, it suffices to show \( (\sum_{0 \leq u \leq s} (\Delta \Lambda_{M}(u))^{2} (I + \Delta \Lambda_{M}(u))^{-1})_{jk} < \infty \) for all \( s > 0 \) and \( j, k \). Let \( \| \cdot \| \), defined by \( \| A \| = n \cdot \max_{jk} \| a_{jk} \|$ for a matrix \( A = (a_{jk})_{jk} \in \mathbb{R}^{n \times n} \), denote the maximum norm on \( \mathbb{R}^{n \times n} \). If \( \| \Delta \Lambda_{M}(u) \| \leq 1/2 \), then it holds
\[
\| (I + \Delta \Lambda_{M}(u))^{-1} \| \leq \frac{1}{1 - \| \Delta \Lambda_{M}(u) \|} \leq 2,
\]
see, e.g., Werner (2018, Theorem II.1.12). Using this upper bound, the subadditivity and the submultiplicity of the norm, we get

\[
\left\| \sum_{0 < u < s} (\Delta \overline{\Lambda}_M(u))^2 (I + \Delta \overline{\Lambda}_M(u))^{-1} \right\|
\]

\[
\leq \sum_{0 < u < s, \|\Lambda_M(u)\| \leq 1/2} (\Delta \overline{\Lambda}_M(u))^2 (I + \Delta \overline{\Lambda}_M(u))^{-1} + \sum_{0 < u < s, \|\Lambda_M(u)\| > 1/2} (\Delta \overline{\Lambda}_M(u))^2 (I + \Delta \overline{\Lambda}_M(u))^{-1}
\]

\[
\leq \sum_{0 < u < s, \|\Lambda_M(u)\| > 1/2} (\Delta \overline{\Lambda}_M(u))^2 (I + \Delta \overline{\Lambda}_M(u))^{-1} + \sum_{0 < u < s, \|\Lambda_M(u)\| \leq 1/2} \|\Delta \overline{\Lambda}_M(u)\|
\]

The first sum in the latter expression is finite, since \(\|\Delta \overline{\Lambda}_M(u)\| > 1/2\) occurs only for finitely many \(u \in [0, s]\). For the second term, observe that

\[
\sum_{0 < u < s} \|\Lambda_M(u)\| \leq \sum_{j, k} \sum_{0 < u < s} |\Delta \overline{\Lambda}_{jk}(u)| < \infty,
\]

on account of the fact that \(\overline{\Lambda}\) is a finite variation process. Thus \(G\) is a well-defined semimartingale.

For a matrix-valued semimartingale \(Z\), let \(\mathcal{E}(Z)\) denote the (matrix-valued) exponential of \(Z\) and let \(\mathcal{E}^R(Z)\) denote the (matrix-valued) right-stochastic exponential of \(Z\) (cf. Chapter V in Protter (2005)). By applying Theorem V.10.63 of Protter (2005), we get

\[
\mathcal{E}(F)(s)\mathcal{E}^R(\overline{\Lambda}_M^T)(s) = I
\]

for \(F(s) = -\overline{\Lambda}_M(s) + \sum_{0 < u < s} (I + \Delta \overline{\Lambda}_M(u))^{-1}(\Delta \overline{\Lambda}_M(u))^2\). Because of \(\mathcal{E}^R(Z) = \mathcal{E}(Z)^T\) and \(F^T = -G\), the latter equation is equivalent to

\[
\mathcal{E}(\overline{\Lambda}_M)(s)\mathcal{E}^R(-G)(s) = I,
\]

which proves the first equation of the assertion. In particular, we verified that \(\overline{\eta}(0, s) - \overline{\eta}(0, s-) = -\Delta G(s)\overline{\eta}(0, s-)\), which implies that

\[
-\Delta \overline{\Lambda}_M(s)\overline{\eta}(0, s) = -\Delta \overline{\Lambda}_M(s)\overline{\eta}(0, s-) + (\Delta \overline{\Lambda}_M(s))(\Delta G(s))\overline{\eta}(0, s-)
\]

\[
= -\Delta \overline{\Lambda}_M(s)(I - \Delta G(s))\overline{\eta}(0, s-) = -\Delta G(s)\overline{\eta}(0, s-).
\]

Thus the second equation of the assertion is also true.

\[\Box\]

**Lemma A.2:** Let \((\Phi', (\Lambda'_{jk})_{jk \neq k}), (\overline{\Phi}, (\overline{\Lambda}_{jk})_{jk \neq k})\) be valuation bases.

(a) Let \(d\kappa'(s) = \kappa'(s-)d\Phi'(s)\) with \(\kappa'(0) = 1\) and \(d\overline{\kappa}(s) = \overline{\kappa}(s-)d\overline{\Phi}(s)\) with \(\kappa'(0) = 1\). Then it holds that

\[
d\left(\frac{\kappa'(s)}{\overline{\kappa}(s)}\right) = \frac{\kappa'(s-)}{\overline{\kappa}(s-)} \left( d\Phi'(s) - d\overline{\Phi}(s) - d[\Phi', \overline{\Phi}](s) \right),
\]

where \(\overline{\Phi}(s) = \overline{\Phi}(s) - [\overline{\Phi}, \overline{\Phi}]^c(s) - \sum_{0 < u < s} (1 + \Delta \overline{\Phi}(u))^{-1}(\Delta \overline{\Phi}(u))^2\).

(b) Let \(p'(t, s) = p'(t, s-)d\Lambda'_M(s)\) with \(p'(t, s) = \overline{p}(t, s-)d\overline{\Lambda}_M(s)\) with \(\overline{p}(t, s) = \overline{p}(t, s-)d\overline{\Lambda}_M(s)\) with \(\overline{p}(t, s) = \overline{p}(t, s-)\). Suppose that \(\overline{p}(t, s)\) is invertible with inverse \(\overline{q}(t, s)\). Then it holds that

\[
d_s p'(t, s)\overline{q}(t, s) = p'(t, s-)d(\Lambda'_M - \overline{\Lambda}_M)(s)\overline{q}(t, s).
\]

**Proof:** (a) Integration by parts (Protter, 2005, Corollary II.6.2) and Lemma A.1(a) yields

\[
d\left(\frac{\kappa'(s)}{\overline{\kappa}(s)}\right) = \frac{\kappa'(s-)}{\overline{\kappa}(s-)} d\overline{\Phi}(s) + \frac{\kappa'(s-)}{\overline{\kappa}(s-)} d\Phi'(s) - \frac{\kappa'(s-)}{\overline{\kappa}(s-)} d[\overline{\Phi}, \Phi'](s).
\]
(b) Integration by parts (Protter, 2005, Corollary II.6.2) and Lemma A.1(b) yields
\[
\frac{d}{ds} \left( (p'(t,s)\overline{q}(t,s)) \right) = p'(t,s-)(\overline{q}(t,ds) + p'(t,ds)\overline{q}(t,s-)) + d \left[ p'(t,\cdot),\overline{q}(t,\cdot) \right] (s)
\]
\[
= -p'(t,s-)(d\overline{\Lambda}_M(s))\overline{q}(t,s) + p'(t,s-)(d\Lambda'_M(s))\overline{q}(t,s)
\]
\[
= p'(t,s-)d(\Lambda'_M - \overline{\Lambda}_M)(s)\overline{q}(t,s).
\]