Analytic Perturbation Theory in analyzing some QCD observables

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Abstract

The paper is devoted to application of recently devised ghost–free Analytic Perturbation Theory (APT) for analysis of some QCD observables. We start with the discussion of the main problem of the perturbative QCD — ghost singularities and with the resume of its resolving within the APT.

By a few examples in the various energy and momentum transfer regions (with the flavor number \( f = 3, 4 \) and 5) we demonstrate the effect of improved convergence of the APT modified perturbative QCD expansion.

Our first observation is that in the APT analysis the three-loop contribution \( (\sim \alpha_s^3) \) is as a rule numerically inessential. This gives raise a hope for practical solution of the well–known problem of asymptotic nature of common QFT perturbation series.

The second result is that a usual perturbative analysis of time-like events with the big \( \pi^2 \) term in the \( \alpha_s^3 \) coefficient is not adequate at \( s \leq 2 \text{ GeV}^2 \). In particular, this relates to \( \tau \) decay.

Then, for the “high” \( (f = 5) \) region it is shown that the common two-loop (NLO, NLLA) perturbation approximation widely used there \((10 \text{ GeV} \lesssim \sqrt{s} \lesssim 170 \text{ GeV})\) for analysis of shape/events data contains a systematic negative error of a 1–2 per cent level for the extracted \( \bar{\alpha}_s^{(2)} \) values.

Our physical conclusion is that the \( \bar{\alpha}_s(M_Z^2) \) value averaged over the \( f = 5 \) data appreciably differs \( <\bar{\alpha}_s(M_Z^2) >_{f=5} \simeq 0.124 \) from the currently accepted “world average” \( (= 0.118) \).

1 Preamble

In QCD, a dominant means of theoretical analysis is based on perturbation power expansion supported by appropriate renormalization group (RG) summation. This perturbative QCD (pQCD) satisfactorily correlates the bulk of experimental data in spite of the fact that the RG invariant power expansion parameter \( \bar{\alpha}_s \) is not a “small enough” quantity. Nowadays, the physically accessible region corresponds to three, four and five \( (f = 3, 4, 5) \) flavor numbers (of active quarks). Just in the three–flavor region there lie unphysical singularities of the central theoretical object — invariant effective coupling \( \bar{\alpha}_s \). These singularities, associated with the QCD scale parameter \( \Lambda_{f=3} \simeq 400 \text{ MeV} \), complicate theoretical interpretation of data in the “small energy” and “small momentum transfer” regions \( (\sqrt{s}, q \equiv \sqrt{q^2} \lesssim 1 \div 1.5 \text{ GeV}) \). On the other hand, as it is well known, their existence contradicts some general statements of the local QFT.

In this paper, we first discuss this main problem of the pQCD, the singularities lying in the physically accessible domain, and then give a resume of its solution within the recently devised ghost–free Analytic Perturbation Theory (APT) that resolves the problem without using any additional adjustable parameters. Then, we give some impressive results of the APT application for analysis of QCD observables.
1.1 Invariant QCD coupling and observables

Usually, the perturbative QCD part of theoretical contribution to observables in both the space– and time–like channels is presented in the form of two– or three–term power expansion

\[
\frac{O(x)}{O_0} = 1 + r(x); \quad r(x) = c_1 \bar{\alpha}_s(x) + c_2 \bar{\alpha}_s^2 + c_3 \bar{\alpha}_s^3 + \ldots; \quad x = q^2 \text{ or } s = \frac{1}{q^2}
\]

(our coefficients are normalized \( c_k = C_k \pi^{-k} \) differently from the commonly adopted \( C_k \), like in Refs. [1, 2, 3]) over powers of the effective QCD coupling \( \bar{\alpha}_s \) which is supposed ad hoc to be of the same form as in both the channels, e.g., in the massless three–loop case

\[
\bar{\alpha}_s^{(3)}(x) = \frac{1}{\beta_0 L} - \frac{b_1 \ln L}{\beta_0 L^2} + \frac{1}{\beta_0^2 L^3} \left[ b_1^2 (\ln^2 L - \ln L - 1) + b_2 \right] + \frac{1}{\beta_0^3 L^4} \left[ b_3^3 \left( -\ln^3 L + \frac{5}{2} \ln^2 L + 2 \ln L - \frac{1}{2} \right) - 3 b_1 b_2 \ln L + \frac{b_3}{2} \right].
\]

(2)

Here, \( L = \ln(x/\Lambda^2) \), and for the beta–function coefficients we use the normalization

\[
\beta(\alpha) = -\beta_0 \alpha^2 - \frac{1}{\beta_0} \ln L - \frac{1}{\beta_0^2 L^3} \left[ b_1^2 (\ln^2 L - \ln L - 1) + b_2 \right] + \frac{1}{\beta_0^3 L^4} \left[ b_3^3 \left( -\ln^3 L + \frac{5}{2} \ln^2 L + 2 \ln L - \frac{1}{2} \right) - 3 b_1 b_2 \ln L + \frac{b_3}{2} \right].
\]

(2)

that is also free of the \( \pi \) powers. Numerically, they are of an order of unity

\[
\beta_0(f) = \frac{33 - 2f}{12\pi}; \quad b_1(f) = \frac{153 - 19f}{2\pi(33 - 2f)}; \quad \beta_0(4 \pm 1) = 0.875 \pm 0.005; \quad b_1(4 \pm 1) = 0.490^{-0.089}_{+0.076}.
\]

Meanwhile, the RG notion of invariant coupling was first introduced in QED [4] in the space-like region in terms of a real constant \( z_3 \) of the finite Dyson renormalization transformation. Just this QED Euclidean invariant charge \( \bar{e}(q) \) is the Fourier transform of the space distribution \( e(r) \) of electric charge (arising due to vacuum fluctuations around a point “bare” electron) discussed by Dirac [5] in 30s — see Appendix IX in the textbook [6].

Generally, in the RG formalism (for details, see, e.g., the chapter “Renormalization group” in the monograph [7] and/or Section 1 in Ref. [8]) the notion of invariant coupling \( \bar{g}(q) \) is defined only in the space–like domain.

In particular, this means that if some observable \( O(q^2) \) is physically a function of one kinematic Lorentz–invariant space–like argument \( q^2 \), then, due to its renormalization invariance, it should be a function of RG invariants only. E.g., in the one–coupling massless case

\[
O(q^2/\mu^2, g_\mu) = F \left( \bar{g}(q^2/\mu^2, g_\mu) \right) \quad \text{with} \quad F(g) = O(1, g).
\]

Due to this important property, in the weak coupling case we deal with the functional expansion of an observable \( O(q^2) \) in powers of invariant coupling \( \bar{g} \). This is a real foundation of QCD power expansion [1] in the Euclidean case with \( x = q^2 \). At the same time, inside the RG formalism, there is no natural means for defining invariant coupling \( \bar{g}(s) \); \( s = -q^2 \) and perturbative expansion for an observable \( \bar{O}(s) \) in the time–like region.
Nevertheless, in modern practice, people commonly use the same singular expression for the QCD effective coupling $\tilde{\alpha}_s$, like (2), in both the space- and time-like domains. The only price usually paid for this transferring from the Euclidean to Minkowskian region is the change of numerical expansion coefficients. The time-like ones $c_k \geq 3 = d_k - \delta_k$ include negative “$\pi^2$ terms” proportional to $\pi^2$ and lower expansion coefficients $c_k$

$$\delta_3 = \frac{(\pi^2 \beta_0(f))^2}{3} c_1, \quad \delta_4 = (\pi^2 \beta_0)^2 (c_2 + \frac{5}{6} b_1 c_1) \ldots .$$

These (rather essential, as far as $\pi^2 \beta_0(f = 4 \pm 1) = 4.340^{+0.666}_{-0.723}$) structures $\delta_k$ arise in the course of analytic continuation from the Euclidean to Minkowskian region. The coefficients $d_k$ should be treated as genuine $k$th-order ones. Just they are calculated via the relevant Feynman diagrams.

Table 1

| Process   | $f$ | $c_1 = d_1$ | $c_2 = d_2$ | $c_3$ | $d_3$ | $\delta_3$ | $\delta_4$ |
|-----------|-----|-------------|-------------|-------|-------|-------------|-------------|
| $\tau$ decay | 3   | $1/\pi$ .526 | .852 | 1.389 | .537 | 5.01 |
| $e^+e^-$     | 4   | .318 | .155 | - .351 | .111 | .462 | 2.451 |
| $e^+e^-$     | 5   | .318 | .143 | - .413 | - .023 | .390 | 1.752 |
| $Z_0$ decay  | 5   | .318 | .095 | - .483 | - .094 | .390 | 1.576 |

To demonstrate the importance of the “$\pi^2$ terms”, we took the $f = 3$ case for $\tau$-decay, the $f = 4, 5$ cases for $e^+e^- \rightarrow$ hadron annihilation and the $Z_0$ decay (with $f = 5$) — see Table 1 in which we also give values for the $\pi^2$-terms. In the normalization (1), all coefficients $c_k$, $d_k$ and $\delta_k$ are of an order of unity. In the $f = 4, 5$ region the contribution $\delta_3$ prevails in $c_3$ and $|d_3| \ll |c_3|$ (see also Table II in Bjorken’s review [12]).

### 1.2 Unphysical singularities

Let us remind to the reader that the ghost-trouble first discovered in QED in the mid-50s (and quite soon in the renormalizable version of pion-nucleon interaction) was considered there as a serious argument in favor of inner inconsistency of the whole local QFT. In the QED case, the ghost singularity lies far above the mass of the Universe and has no pragmatic meaning.

However, in QCD it lies in the quite physical infrared (IR) region and we are forced to face it without any excusing arguments. This means that, if one believes in QCD as in a consistent, physically important theory, one has no other possibility as to consider the QCD unphysical singularities as an artefact of some approximations used in pQCD. This point of view is supported by some lattice simulations and solution of Schwinger–Dyson equations — see, e.g., Section 5.3. in a recent review [13].

For illustration of the fundamental inconsistency of current pQCD practice connected with unphysical singularities, take the well-known relation between the so-called Adler func-

\[\text{Table 1}
\]

Minkowskian $c_k$ and Euclidean $d_i = c_i + \delta_i$ expansion coefficients and their differences.
tion $D$ and the total cross–section ratio $R$ of a related process

$$D(q^2) = q^2 \int_0^\infty \frac{R(s) \, ds}{(s + q^2)^2}. \quad (4)$$

In the case of inclusive $e^+e^-$ annihilation into hadrons, $R(s)$ is the ratio of cross–sections presented in the form $R(s) = 1 + r(s)$ with a function $r$ expandable in powers of $\bar{\alpha}_s(s)$ like in Eq.(1). At the same time, the Adler function is also used to be presented in the form $D = 1 + d$ with $d$ expanded in powers of $\bar{\alpha}_s(q^2)$.

Here, we face two paradoxes. First, $\bar{\alpha}_s(q^2)$ and, hence, the perturbative $D(q^2)$ obeys — see eq.(2) — non-physical singularity at $q^2 = \Lambda^2$ in evident contradiction with representation (4). Second, the integrand $R(s)$, being expressed via powers of $\bar{\alpha}_s(s)$, obeys non-integrable singularities at $s = \Lambda^2$, which makes the r.h.s. of eq.(4) senseless.

This second problem is typical of inclusive cross–sections, e.g., for the $\tau$ hadronic decay. Generally, in the current literature it is treated in a very strange way — by shifting the contour of integration from the real axis with strong singularities on it into a complex plane. However, such a “physical” trick cannot be justified within the theory of complex variable.

### 1.3 The “ghost” problem resolving

Meanwhile, as it is known from the early 80s, the perturbation representation (4) for the Minkowskian observable with the coefficients modified by the $\pi^2$–terms is valid only at a small parameter $\pi^2/\ln^2(s/\Lambda^2)$ values, that is in the region of sufficiently high energies $W \equiv \sqrt{s} \gg \Lambda\pi/2 \simeq 2\text{ GeV}$.

Here, it is appropriate to remind the construction devised by Radyushkin [9] and Krasnikov–Pivovarov [10] (RKP procedure) about 20 years ago. These authors used the integral transformation

$$R(s) = \frac{i}{2\pi} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z} D_{pt}(-z) \equiv \mathbf{R} [D(q^2)]$$

reverse to the Adler relation (3) (that is treated now as integral transformation)

$$R(s) \rightarrow D(q^2) = q^2 \int_0^\infty \frac{R(s) \, ds}{(s + q^2)^2} \equiv \mathbf{D} \{R(s)\} \quad (6)$$

for defining modified expansion functions

$$\mathfrak{A}_k(s) = \mathbf{R}[\alpha_k^s(q^2)] \quad (7)$$

for the perturbative QCD contribution

$$r(s) = d_1\mathfrak{A}_1(s) + d_2\mathfrak{A}_2(s) + d_3\mathfrak{A}_3(s) \quad (8)$$

to an observable in the time–like region.

At the one-loop level, with the effective coupling $\bar{\alpha}_s^{(1)} = [\beta_0 \ln(q^2/\Lambda^2)]^{-1}$ one has

$$\mathfrak{A}^{(1)}_1(s) = \mathbf{R} \left[\bar{\alpha}^{(1)}_s\right] = \frac{1}{\pi \beta_0} \arccos \frac{L}{\sqrt{L^2 + \pi^2}} = \frac{1}{\beta_0} \left[\frac{1}{2} - \frac{1}{\pi} \arctan \frac{L}{\pi}\right] \quad ; \quad L = \ln \frac{s}{\Lambda^2} \quad (9)$$
and for higher functions

\begin{align*}
\mathcal{A}_2^{(1)}(s) &= \frac{1}{\beta_0^2 [L^2 + \pi^2]} ; \\
\mathcal{A}_3^{(1)}(s) &= \frac{L}{\beta_0^3 [L^2 + \pi^2]^2} ; \\
\mathcal{A}_4^{(1)}(s) &= \frac{L^2 - \pi^2/3}{\beta_0^4 [L^2 + \pi^2]^3} ,
\end{align*}

(10)

which are not powers of \( \mathcal{A}_1^{(1)}(s) \).

The r.h.s of (9) at \( L \geq 0 \) can also be presented in the form

\[ \mathcal{A}_1^{(1)}(s) = \frac{1}{\pi \beta_0} \arctan \frac{\pi}{L} \]

(9a)

convenient for the UV analysis. Just this form (9a) was discovered in the early 80s in Refs. [14] and [9], while eqs. (10) in Refs. [9] and [10]. All these papers dealt with HE behavior and did not pay proper attention to the region \( L \leq 0 \).

On the other hand, expression (9) was first discussed only 15 years later by Milton and Solovtsov [15]. Just these authors first made an important observation that expression (9) represents a continuous monotone function without unphysical singularity at \( L = 0 \) and proposed to use it as an effective "Minkowskian QCD coupling" \( \tilde{\alpha}(s) \equiv \mathcal{A}_1(s) \) in the time–like region.

For the two–loop case, to the popular approximation

\[ \beta_0 \tilde{\alpha}_{s,pop}^{(2)}(q^2) = \frac{1}{l} - b_1(f) \frac{\ln l}{l^2} ; \quad l = \ln \frac{q^2}{\Lambda^2} \]

there corresponds [9, 16]

\[ \tilde{\alpha}_{pop}^{(2)}(s) \equiv \mathcal{A}_{2,pop}^{(1,2)}(s) = \left( 1 + \frac{b_1 L}{L^2 + \pi^2} \right) \tilde{\alpha}^{(1)}(s) - \frac{b_1}{\beta_0} \ln \left[ \sqrt{L^2 + \pi^2} + 1 \right] \frac{L^2 + \pi^2}{L^2 + \pi^2} . \]

(11)

At \( L \gg \pi \), by expanding this expression and \( \mathcal{A}_2 \) from (10) in powers of \( \pi^2/L^2 \) we arrive at the \( \pi^2 \)--terms (3).

Both the functions (9) and (11) are monotonically decreasing with a finite IR value \( \tilde{\alpha}(0) = 1/\beta_0(f = 3) \approx 1.4 \). They have no singularity at \( L = 0 \). Higher functions go to zero \( \mathcal{A}_k(0) = 0 \) in the IR limit.

As it has first been noticed in [17, 18], by applying the transformation \( \mathbf{D} \) (3) to functions \( \mathcal{A}_k(s) \), instead of \( \tilde{\alpha}_s(q^2) \) powers, we obtain expressions \( \mathbf{D}[\mathcal{A}_k(s)] = \mathcal{A}_k(q^2) \) that are also free of unphysical singularities. These functions have been discussed at 90s [19] — 24] in the context of the so–called “Analytic approach” to perturbative QCD.

Therefore, this Analytic approach in the Euclidean region and the RKP formulation for Minkowskian observables can be united in the single scheme, the “Analytic Perturbation Theory” — APT, that has been formulated quite recently in our papers [17] and [18]. In the next Section, we give a short resume of this APT construction and then, in Sections 3 and 4, present the results of its practical applications.
2 The APT — a closed theoretical scheme

The APT scheme closely relates two ghost–free formulations of modified perturbation expansion for observables.

2.1 Relation between Euclidean and Minkowskian

The first one, that was initiated in the early eighties [9, 10] and outlined above, changes the standard power expansion (1) in the time-like region into the non power one (8). It uses operation Eq.(5), that is reverse $R = [D]^{-1}$ to the one defined by the “Adler relation” (6) and transforms a real function $R(s)$ of a positive (time–like) argument into a real function $D(q^2)$ of a positive (space–like) argument.

By operation $R$, one can define [15] the RG–invariant Minkowskian coupling $\tilde{\alpha}(s) = R[\alpha_s]$, and its “effective powers” (7) that are free of ghost singularities. Some examples are given by expressions (9), (10) and (11). At the one–loop level, they are related by the differential recursion relation

$$k\beta_0 A^{(1)}_{k+1} = -(d/dL)A^{(1)}_k$$

and are not powers of $A^{(1)}_{1}$.

By applying $D$ to $A_k(q^2)$, one can “try to return” to the Euclidean domain. However, instead of $\alpha_s$ powers, we arrive at some other functions $A^{(1)}_{k}(q^2) = D[A_{k}^{(1)}]$, analytic in the cut $q^2$–plane and free of ghost singularities. At the one–loop case

$$\beta_0 A^{(1)}_{1}(q^2) = \frac{1}{\ln(q^2/\Lambda^2)} - \frac{\Lambda^2}{q^2 - \Lambda^2}, \quad \beta_0^2 A^{(1)}_{2}(q^2) = \frac{1}{\ln^2(q^2/\Lambda^2)} + \frac{q^2 \Lambda^2}{(q^2 - \Lambda^2)^2}, \ldots$$

These expressions have been originally obtained by other means [13, 20] in the mid–90s. The first function $A_1 = \alpha_{an}(q^2)$, an analytic invariant Euclidean coupling, should now be treated as a counterpart of the invariant Minkowskian coupling $\tilde{\alpha}(s) = A^{(1)}_{1}(s)$. Both $\alpha_{an}$ and $\tilde{\alpha}$ are real monotonically decreasing functions with the same maximum value

$$\alpha_{an}(0) = \tilde{\alpha}(0) = 1/\beta_0(f = 3) \simeq 1.4$$

in the IR limit[1].

1 Note that the transition from the usual invariant $\overline{MS}$ coupling $\alpha_s$ to the Minkowskian $\tilde{\alpha}$ and Euclidean $\alpha_{an}$ ones can be understood as a transformation to new renormalization schemes. At the one–loop case

$$\alpha_s \to \tilde{\alpha}^{(1)} = \frac{1}{\pi\beta_0} \arctan(\pi\beta_0\alpha_s) \quad \text{and} \quad \alpha_s \to \alpha_{an}^{(1)} = \alpha_s + \frac{1}{\beta_0} \left(1 - e^{1/\beta_0\alpha_s}\right)^{-1}.$$ 

Here, the first transition looks “quite usual” as $\tilde{\alpha}$ can be expanded in powers of $\alpha_s$, while the second one in the weak coupling case behaves like the identity transformation as far as the second nonperturbative term $e^{-1/\beta_0\alpha_s}$ leaves no “footsteps” in the power expansion.

For both $\tilde{\alpha}^{(1)}$ and $\alpha_{an}^{(1)}$ the corresponding $\beta$ functions have zero at $\alpha = 1/\beta_0$ and are symmetric under reflection $[\alpha - 1/2\beta_0] \to -[\alpha - 1/2\beta_0]$. Moreover, the $\beta$ function for $\tilde{\alpha}(s)$ turns out to be equal to the spectral function for $\alpha_{an}(q^2)$ – see below Eq.(18) at $k = 1$. 

6
All higher functions vanish \( \mathcal{A}_k(0) = \mathfrak{A}_k(0) = 0 \) in this limit. For \( k \geq 2 \), they oscillate in the IR region and form \( \mathfrak{A}_k(0) = 0 \) in this limit. For \( k \geq 2 \), they oscillate in the IR region and form \( \mathfrak{A}_k(0) = 0 \) in this limit. For \( k \geq 2 \), they oscillate in the IR region and form \( \mathfrak{A}_k(0) = 0 \) in this limit. For \( k \geq 2 \), they oscillate in the IR region and form \( \mathfrak{A}_k(0) = 0 \) in this limit.

The same properties remain valid for a higher–loop case. Explicit expressions for \( \mathcal{A}_k \) and \( \mathfrak{A}_k \) at the two–loop case can be written down (see, Refs. [27] and [28]) in terms of a special Lambert function. They are illustrated below in Figs 1a and 1b. Note here that to relate Euclidean and Minkowskian functions, instead of integral expressions (5) and (6) one can use simpler relations, in terms of spectral functions \( \rho(\sigma) = \Im \mathcal{A}(\sigma) \),

\[
\mathcal{A}_k(q^2; f) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + q^2} \rho_k(\sigma; f); \quad \mathfrak{A}_k(s; f) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma; f),
\]

(15) equivalent to expressions \( \mathcal{A}_k(q^2) = D[\mathfrak{A}_k] \), and \( \mathfrak{A}_k(s) = R[\mathcal{A}_k] \).

Remarkably enough, the mechanism of liberation of unphysical singularities is quite different. While in the space-like domain it involves nonperturbative, power in \( q^2 \), structures, in the time-like region it is based only upon resummation of the “\( \pi^2 \) terms”. Figuratively, (non-perturbative!) analyticization [19, 20, 26] in the \( q^2 \)–channel can be treated as a quantitatively distorted reflection (under \( q^2 \rightarrow s = -q^2 \)) of (perfectly perturbative) \( \pi^2 \)–resummation in the \( s \)–channel. This effect of “distorting mirror”, first discussed in [15] and [29], is clearly seen in the figures 1a,b mentioned above.

This means also that introduction of nonperturbative \( 1/q^2 \) structures now has got another motivation, Eq.(12), independent of the analyticization prescription.

2.2 Global APT

In reality, a physical domain includes regions with various “numbers of active quarks”, i.e., with diverse flavor numbers \( f = 3, 4, 5 \) and 6. In each of these regions, we deal with a different amount of quark quantum fields, that is with distinct QFT models with corresponding Lagrangians. To combine them into a joint picture, the procedure of the threshold matching is in use. It establishes relations between renormalization procedures for a model with different \( f \) values.

For example, in the \( \overline{\text{MS}} \) scheme the matching relation has a simple form

\[
\bar{\alpha}_s(q^2 = M^2_f; f - 1) = \bar{\alpha}_s(q^2 = M^2_f; f).
\]

(16)

It defines a “global effective coupling”

\[
\bar{\alpha}_s(q^2) = \bar{\alpha}_s(q^2; f) \quad \text{at} \quad M^2_{f-1} \leq q^2 \leq M^2_f,
\]

continuous in the space-like region of positive \( q^2 \) values with discontinuity of derivatives at matching points \( q^2 = M^2_f \). To this global \( \bar{\alpha}_s \), there corresponds a discontinuous spectral density

\[
\rho_k(\sigma) = \rho_k(\sigma; 3) + \sum_{f \geq 4} \theta(\sigma - M^2_f) \{ \rho_k(\sigma; f) - \rho_k(\sigma; f - 1) \}
\]

(17)
with $\rho_k(\sigma; f) = \Im \tilde{\alpha}_{s}^k(-\sigma, f)$ which yields \cite{17, 18} via relations analogous to \cite{13}

$$
A_k(q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + q^2} \rho_k(\sigma); \quad A_k(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma),
$$

the smooth global Euclidean and spline–continuous global Minkowskian expansion functions.

**Figure 1:** a – Space-like and time-like global analytic couplings in a few GeV domain with $f = 3$ and $\Lambda^{(3)} = 350$ MeV; b – “Distorted mirror symmetry” for global expansion functions. All the curves in 1b correspond to exact two–loop solutions expressed in terms of Lambert function.

Here, in Fig.1a, by the dotted line we give a usual two-loop effective QCD coupling $\tilde{\alpha}_{s}(q^2)$ with a singularity at $q^2 = \Lambda^2$. Meanwhile, the dash–dotted curves represent the one-loop APT expressions \cite{9} and \cite{13}. The solid APT curves are based on the exact two-loop solutions of RG equations and approximate three–loop solutions in the $\overline{\text{MS}}$ scheme. Their remarkable coincidence (within the 2–4 per cent) demonstrates reduced sensitivity of the APT approach (see, also Refs. \cite{20, 21, 22}) with respect to higher–loop effects in the whole Euclidean and Minkowskian regions from IR to UV limits. Fig.1b shows higher two–loop functions in comparison with $\alpha_{an}$ and $\tilde{\alpha}$ powers.

Generally, functions $A_{k}$ and $A_{k}$ differ from the local ones with a fixed $f$ value. Minkowskian global functions $A_{k}$ can be presented via $A_{k}(s, f)$ by relations

$$
\tilde{\alpha}(s) = \tilde{\alpha}(s; f) + c(f); \quad A_2(s) = A_2(s; f) + c_2(f) \quad \text{at} \quad M_f^2 \leq s \leq M_{f+1}^2
$$

with shift constants $c(f), c_2(f)$ representable via integrals over $\rho_k(\sigma; f + n), n \geq 1$ with additional reservations, like $c(6) = 0$, related to the asymptotic freedom condition.

Numerical estimate performed in Ref.\cite{13} (see also Table 6 in Ref.\cite{27}) for traditional values of the QCD scale parameter $\Lambda_3 \sim 300 – 400$ MeV

$$
c(3) \sim 0.02, \quad c(4) \simeq 3.10^{-3}, \quad c(5) \simeq 3.10^{-4}; \quad c_2(f) \simeq 3 \alpha(M_f^2) c(f)
$$

reveals that these constants are essential in the $f = 3, 4$ region at a few per cent level for $\tilde{\alpha}$ and at ca 10% level for $A_2$. 
Meanwhile, global Euclidean functions $A_k(q^2)$ cannot be related to the local ones $A_k(q^2, f)$ by simple relations. Nevertheless, numerical calculation shows [27, 28] that in the $f = 3$ region one has approximately

$$\alpha_{an}(q^2) = \alpha_{an}(q^2; 3) + c(3); \quad A_2(q^2) = A_2(q^2; 3) + c_2(3) \quad \text{at} \quad M_3^2 \leq s \leq M_4^2.$$  \hspace{1cm} (20)

### 3 The APT applications

#### 3.1 General comments

In what follows, we abstract ourselves of recent successive use of the Analytic approach to hadronic formfactors [30] and concentrate on the QCD applications of APT.

To illustrate a quantitative difference between global APT scheme and common practice of data analysis in perturbative QCD, consider a few examples.

In the usual treatment — see, e.g., Ref. [1] — the (QCD perturbative part of) Minkowski observable, like $e^+e^-$ annihilation or $Z_0$ decay cross-section ratio, is presented in the form

$$R(s) = R_0 (1 + r(s)); \quad r_{PT}(s) = c_1 \bar{\alpha}_s(s) + c_2 \bar{\alpha}_s^2(s) + c_3 \bar{\alpha}_s^3(s) + \ldots.$$  \hspace{1cm} (21)

Here, the coefficients $c_1$, $c_2$ and $c_3$ are not diminishing numerically — see Table 1. A rather big negative $c_3$ value comes mainly from the $-c_1 \pi^2 \beta_0^2/3$ term. In the APT, we have instead

$$r_{APT}(s) = d_1 \bar{\alpha}(s) + d_2 A_2(s) + d_3 A_3(s) + \ldots$$  \hspace{1cm} (22)

with reasonably decreasing Feynman coefficients $d_{1,2} = c_{1,2}$ and $d_3 = c_3 + c_1 \pi^2 \beta_0^2/3$, the mentioned $\pi^2$ term of $c_3$ being “swallowed” by $\bar{\alpha}(s)$.

In the Euclidean channel, instead of power expansion similar to (21), we typically have

$$d_{APT}(q^2) = d_1 \alpha_{an}(q^2) + d_2 A_2(q^2) + d_3 A_3(q^2) + \ldots$$  \hspace{1cm} (23)

with the same coefficients $d_k$ extracted from Feynman diagrams. Here, the modification is related to nonperturbative, power in $q^2$, structures like in (13).

Table 2: Relative contributions (in %) of 1–, 2– and 3–loop terms to observables

| Process          | $q$ or $\sqrt{s}$ | $f$ | PT     | APT     |
|------------------|-------------------|-----|--------|---------|
| GLS sum rule     | 1.73 GeV          | 4   | 65     | 24      | 11      | 75 | 21 | 4 |
| Bjorken. s.r.    | 1.73 GeV          | 3   | 55     | 26      | 19      | 80 | 19 | 1 |
| Incl. $\tau$-decay | 0 - 2 GeV        | 3   | 55     | 29      | 16      | 88 | 11 | 1 |
| $e^+e^- \rightarrow$ hadr. | 10 GeV          | 4   | 96     | 8       | -4      | 92 | 7  | .5 |
| $Z_0 \rightarrow$ hadr. | 89 GeV          | 5   | 98.6   | 3.7     | -2.3    | 96.9    | 3.5 | - .4 |

In Table 2, we give values of the relative contribution of the first, second, and third terms of the r.h.s. in (21),(22) and (23) for the Gross–Llywellin-Smith [31] and Bjorken [32] sum...
rules, $\tau$ – decay in the vector channel \[^{[33]}\), as well as for $e^+e^−$ and $Z_0$ inclusive cross-sections. As it follows from this Table, in the APT case, the three–loop (last) term is very small, and being compared with data errors, numerically inessential. This means that, in practice, one can use the APT expansions (22) and (23) without the last term.

3.2 Semi–quantitative estimate

This conclusion can be valuable for the case when the three–loop contribution, i.e., $d_3$ is unknown. Here, people use the so-called NLLA approximation, that is common practice in the $f = 5$ region. For the Minkowskian observable, e.g., in the event–shape (see, e.g., Ref. \[^{[34]}\) analysis there corresponds the two-term expression

$$r(s) = c_1 \tilde{\alpha}_s(s) + c_2 \tilde{\alpha}_s^2(s). \quad (24)$$

On the basis of the numerical estimates of Table 1, in such a case, we recommend instead to use the two-term APT representation

$$r^{(2)}_{\text{APT}}(s) = d_1 \tilde{\alpha}(s) + d_2 \mathcal{A}_2(s) \quad (25)$$

which, at $L^2 \gg \pi^2$, is equivalent to the three-term expression

$$r^3(s) = d_1 \left\{ \tilde{\alpha}_s - \frac{\pi^2 \beta_0^2}{3} \tilde{\alpha}_s^3 \right\} + d_2 \tilde{\alpha}_s^2 = c_1 \tilde{\alpha}_s + c_2 \tilde{\alpha}_s^2 - \delta_3 \tilde{\alpha}_s^3, \quad (26)$$

i.e., to take into account the known predominant $\pi^2$ part of the next coefficient $c_3$. As it follows from the comparison of the last expression with the previous, two–term one (24), the $\tilde{\alpha}_s$ numerical value extracted from eq.(26), for the same measured value $r_{\text{obs}}$, will differ mainly by a positive quantity (e.g., in the $f = 5$ region with $\tilde{\alpha}_s \simeq 0.12 \div 0.15$

$$(\Delta \tilde{\alpha}_s)_3 = \frac{\pi \delta_3 \tilde{\alpha}_s^3}{1 + 2\pi d_2 \tilde{\alpha}_s} \bigg|^{f=5}_{20\div100\text{GeV}} \approx \frac{1.225 \tilde{\alpha}_s^3}{1 + 0.90 \tilde{\alpha}_s} \simeq 0.002 \div 0.003 \quad (27)$$

that turns out to be numerically important.

Moreover, in the $f = 4$ region, where the three-loop (NNLLA) approximation is commonly used in the data analysis, the $\pi^2$ term $\delta_4$ of the next order turns out also to be essential. Hence, we propose there, instead of (24), to use the APT three–term expression

$$r^{(3)}_{\text{APT}}(s) = d_1 \tilde{\alpha}(s) + d_2 \mathcal{A}_2(s) + d_3 \mathcal{A}_3(s) \quad (28)$$

approximately equivalent to the four-term one

$$r^4(s) = d_1 \tilde{\alpha}_s + d_2 \tilde{\alpha}_s^2 + c_3 \tilde{\alpha}_s^3 - \delta_4 \tilde{\alpha}_s^4; \quad c_3 = d_3 - \delta_3 \quad (29)$$

or to

$$r^4(s) = d_1 \left\{ \tilde{\alpha}_s - \frac{\pi^2 \beta_0^2}{3} \tilde{\alpha}_s^3 - b_1 \frac{5}{6} \pi^2 \beta_0^2 \tilde{\alpha}_s^4 \right\} + d_2 \left\{ \tilde{\alpha}_s^2 - \pi^2 \beta_0^2 \tilde{\alpha}_s^4 \right\} + d_3 \tilde{\alpha}_s^3 \quad (29)$$
with $\delta_3$ and $\delta_4$ defined in eq. (3).

The three– and two–term structures in braces are related to specific expansion functions $\tilde{\alpha} = \mathfrak{A}_1$ and $\mathfrak{A}_2$ defined above and entering into the non-power expansion (28).

To estimate roughly the numerical effect of using this last modified expression (29), we take the $e^+e^-$ inclusive annihilation. For $\sqrt{s} \simeq 3 \div 5$ GeV with $\bar{\alpha}_s \simeq 0.28 \div 0.22$ one has

$$\left(\Delta \bar{\alpha}_s\right)_4 = \frac{\pi \delta_4 \bar{\alpha}_s^4}{1 + 2\pi d_2 \bar{\alpha}_s}
\bigg|^{f=4}_{3\div5\text{GeV}} = \frac{1.07 \bar{\alpha}_s^4}{1 + 0.974 \bar{\alpha}_s} \simeq 0.005 \div 0.002$$

— an important effect on the level of ca $1 \div 2\%$.

Moreover, the $\left(\Delta \bar{\alpha}_s\right)_4$ correction turns out to be noticeable even in the lower part of the $f = 5$ region! Indeed, to $\sqrt{s} \simeq 10 \div 40$ GeV with $\bar{\alpha}_s \simeq 0.20 \div 0.15$ there corresponds

$$\left(\Delta \bar{\alpha}_s\right)_4|^{f=5}_{10\div40\text{GeV}} \simeq 0.71 \bar{\alpha}_s^4 \simeq (1.1 \div 0.3) \cdot 10^{-3} \quad (\lesssim 0.5\%).$$

### 3.3 Important warning

It is essential to note that approximate expressions eqs. (26) and (29) are equivalent to the exact ones (25) and (28) only in the region $L = \ln \left(\frac{s}{\Lambda^2}\right) \gg \pi$ as it is shown on Fig. 2.
One can see that the curve for approximate Minkowskian coupling
\[ \tilde{\alpha}_{\text{appr}}(s) = \tilde{\alpha}_s(s) - \left(\pi^2/3\right)\tilde{\alpha}_s^3, \]  
that precisely corresponds to the popular approximation (21) (and gives rise to the \(\pi^2\) term in the \(\alpha_s^3\) coefficient) has a rather peculiar behavior. In the region \(L > \pi\) it goes rather close to the curve for \(\tilde{\alpha}\). For instance, at \(L \approx \pi\) the relative error of approximation is about 5 per cent. On the other hand, below \(L \approx 0.8\pi\) (i.e., \(W \approx 1.0 - 1.4\) GeV) the distance \(\tilde{\alpha} - \tilde{\alpha}_{\text{appr}}\) between curves (error of approximation) increases and at \(L \approx 0.7\pi\) it blows up (better to say “comes down”).

In particular, at \(s \leq 2\) GeV\(^2\) it is rather desorienting to refer to \(\tilde{\alpha}_s(s)\) and it is erroneous to use \(\tilde{\alpha}_{\text{appr}}(s)\) and common expansion (21).

This means that below \(s = 2\) GeV\(^2\) it is nonadequate to use common \(\tilde{\alpha}_s(s)\) and power expansion eq.(21).

In other words, we claim that below \(s = 2\) GeV\(^2\) it is an intricate business to analyze data in terms of the “old good” (but singular) \(\alpha_s\) Here, approximate relation (30) does not work as it is illustrated in Fig.2.

In this, low–energy Minkowskian/Euclidean region data have to be an alyzed in terms of nonpower expansion (22)/(23) and extracted parameter should be \(\alpha_{\text{an}}(s), \tilde{\alpha}(q^2)\) or \(\Lambda^{(3)}\). In Table 3 we give few numerical examples for the chain
\[ \alpha_{\text{an}}(M_\tau) \leftrightarrow \tilde{\alpha}(M_\tau) \leftrightarrow \Lambda^{(3)} \leftrightarrow \tilde{\alpha}_s(M_Z) \]
that allows to study the QCD theoretical compatibility of LE data with the HE ones in the APT analysis.

| \(\tilde{\alpha}(M_\tau)\) | \(\alpha_{\text{an}}(M_\tau)\) | \(\Lambda^{(3)}\) | \(\Lambda^{(5)}\) | \(\tilde{\alpha}_s(M_Z)\) |
|------------------------|------------------------|------------------|------------------|------------------------|
| 0.309                  | 0.332                  | 450 MeV          | 303 MeV          | 0.125                  |
| 0.292                  | 0.314                  | 400 MeV          | 260 MeV          | 0.121                  |
| 0.278                  | 0.299                  | 350 MeV          | 218 MeV          | 0.119                  |
| 0.266                  | 0.286                  | 300 MeV          | 180 MeV          | 0.116                  |

Here, the main element of correlation is the chain \(\Lambda^{(3)} \leftrightarrow \Lambda^{(3)} \leftrightarrow \Lambda^{(5)}\) that follows from the matching condition (16).

2In particular, this relates to analysis of \(\tau\) decay. In this connection we would like to attract attention to the important paper \(33\) that treats the \(\tau\) decay within the APT approach (with effective mass of light quarks and the threshold resummation factor) and results in \(\Lambda^{(3)} = 420\) MeV that corresponds to \(\alpha_{\text{an}}(M_\tau^2) = 0.32\) or \(\tilde{\alpha}(M_\tau^2) = 0.30\). At the same time, attempts to interpret results of APT for \(\tau\) decay in terms of \(\alpha_s\), like, e.g., in Ref.\(33\), needs some special precaution — see next footnote. A more detailed comment on the \(\tau\) decay theoretical analysis will be published elsewhere.

3Generally, it is possible to use correspondence between \(\alpha_{\text{an}}\), \(\tilde{\alpha}\) and \(\alpha_s\) as expressed by relations (14). However, the use of \(\alpha_{\text{an}}^{\text{MS}}(\mu^2)\) at \(\mu \lesssim 1\) GeV as a QCD parameter could be misleading due to vicinity to singularity. For example, at \(\Lambda^{(3)} = 400\) MeV one has \(\alpha_s(M_\tau^2) \approx 0.35\) and \(\alpha_s(1\) GeV\(^2\)) \approx 0.55\) to be compared with \(\alpha_{\text{an}}(M_\tau^2) \approx 0.31\) and \(\alpha_{\text{an}}(1\) GeV\(^2\)) \approx 0.40\).
4 Quantitative illustration

Consider now a few cases in the $f = 5$ region.

$\Upsilon$ decay. According to the Particle Data Group (PDG) overview (see their Fig.9.1 on page 88 of Ref.[1]), this is (with $\alpha_s(M_\Upsilon^2) \simeq 0.170$ and $\bar{\alpha}_s(M_\Upsilon^2) = 0.114$) one of the most “annoying” points of their summary of $\bar{\alpha}_s(M_\Upsilon^2)$ values. It is also singled out theoretically. The expression for the ratio of decay widths starts with the cubic term

$$R(\Upsilon) = R_0 \alpha_s^3(\xi M_\Upsilon^2)(1 - e_1 \alpha_s) \quad \text{with} \quad \xi \lesssim 0.5 \quad \text{and} \quad c_1(\xi) \simeq 1.$$  

Due to this, the $\pi^2$ corrections corresponding to the APT expression

$$R_{\text{APT}}(\Upsilon) = R_0 A_3(\xi M_\Upsilon^2)(1 - e_1 A_4)  \tag{31}$$

are rather big $A_3 \simeq \alpha_s^3(1 - 2(\pi \beta_0)^2 \alpha_s^2)$, $A_4 \simeq \alpha_s^4 [1 - (10/3)(\pi \beta_0)^2 \alpha_s^2]$ in the region with $\pi^2 \beta_0^2(5) = 3.57$ and $\alpha_s(\xi M_\Upsilon^2) \simeq 0.2$. As a crude estimate (taken from $\alpha_s^3 \to A_3$ only),

$$\Delta \alpha_s(M_\Upsilon^2) = \frac{2}{3} (\pi \beta_0)^2 \alpha_s^3(M_\Upsilon^2) \simeq 0.0123,$$

which corresponds to

$$\Delta \bar{\alpha}_s(M_\Upsilon^2) = 0.006 \quad \text{with resulting} \quad \bar{\alpha}_s(M_\Upsilon^2) = 0.120.  \tag{32}$$

One should note here, that this estimate is rather crude and gives only indication of the order of magnitude.

The NNLO case. Now, let us turn to a few cases analyzed by the three-term expansion formula [2]. For the first example, take $e^+ e^- \text{ hadron annihilation}$ at $\sqrt{s} = 42 \text{ GeV}$ and $11 \text{ GeV}$.  

A common form (see, e.g., Eq.(15) in Ref.[2]) of theoretical presentation of the QCD correction in our normalization looks like

$$r_{e^+ e^-}(\sqrt{s}) = 0.318 \bar{\alpha}_s(s) + 0.143 \alpha_s^2(s) - 0.413 \alpha_s^3(s).$$

In the standard PT analysis, one has (see, e.g., Table 3) $\bar{\alpha}_s(42^2) = 0.144$ that corresponds to $r_{e^+ e^-}(42) \simeq 0.0476$. Along with the APT prescription, one should use

$$r_{e^+ e^-}(\sqrt{s}) = 0.318 \bar{\alpha}(s) + 0.143 \alpha_2(s) - 0.023 \alpha_3(s),  \tag{33}$$

which yields $\bar{\alpha}(42^2) = 0.142 \Rightarrow \alpha_s(42^2) = 0.145$ and $\bar{\alpha}_s(M_\Upsilon^2) = 0.127$ to be compared with $\bar{\alpha}_s(M_\Upsilon^2) = 0.126$ under a usual analysis.

Quite analogously, with $\bar{\alpha}_s(11^2) = 0.200$ and $r_{e^+ e^-}(11) \simeq 0.0661$ we obtain via $\bar{\alpha}(11^2) = 0.190$ that corresponds to $\bar{\alpha}_s(M_\Upsilon^2) = 0.129$ instead of 0.130.

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4 See, e.g., eq.(9.16) in Ref.[1].

5 First proposal of taking into account this effect in the $\Upsilon$ decay was discussed [10] more than a quarter of century ago. Nevertheless, in current practice it is completely forgotten.
For the next example, take the $Z_0$ inclusive decay. The observed ratio

$$R_Z = \frac{\Gamma(Z_0 \rightarrow \text{hadrons})}{\Gamma(Z_0 \rightarrow \text{leptons})} = 20.783 \pm 0.029$$

can be written down as follows: $R_Z = R_0 (1 + r_Z(M_Z^2))$ with $R_0 = 19.93$. A common form (see, e.g., Eq.(15) in Ref.[3]) of presenting the QCD correction $r_Z$ looks like

$$r_Z(M_Z^2) = 0.3326\bar{\alpha}_s + 0.0952\bar{\alpha}_s^2 - 0.483\bar{\alpha}_s^3.$$  

To $[r_Z]_{obs} = 0.04184$ there corresponds $\bar{\alpha}_s(M_Z^2) = 0.124$ with $\Lambda^{(5)}_{\overline{\text{MS}}} = 292 \text{ MeV}$. In the APT case, from

$$r_Z^{\text{APT}}(M_Z^2) = 0.3326\bar{\alpha}(M_Z^2) + 0.0952\bar{\alpha}_2(M_Z^2) - 0.094\bar{\alpha}_3(M_Z^2)$$

we obtain $\bar{\alpha}(M_Z^2) = 0.122$ and $\bar{\alpha}_s(M_Z^2) = 0.124$. Note that here the three-term approximation (3) gives the same relation between the $\bar{\alpha}_s(M_Z^2)$ and $\bar{\alpha}(M_Z^2)$ values.

Nevertheless, in accordance with our preliminary estimate for the $(\Delta\bar{\alpha}_s)_4$ role, even the so-called NNLO theory needs some $\pi^2$ correction in the $W = \sqrt{s} \lesssim 50 \text{ GeV}$ region.

The NLO case. Now, turn to those experiments in the HE ($f = 5$) Minkowskian region (mainly with a shape analysis) that usually are confronted with the two-term expression (24). As it has been shown above (27), the main theoretical error here can be expressed in the form

$$|\Delta\bar{\alpha}_s|_4 \lesssim 1.225\bar{\alpha}_s(s) \simeq 0.002 \div 0.003.$$  

### Table 4: The APT revised part ($f = 5$) of Bethke’s [2] Table 6

| Process | $\sqrt{s}$ (GeV) | loops | $\bar{\alpha}_s$ (s) | $\bar{\alpha}_s (m_Z^2)$ | $\bar{\alpha}_s$ (s) | $\bar{\alpha}_s (m_Z^2)$ |
|---------|------------------|-------|----------------------|--------------------------|----------------------|--------------------------|
| $\Upsilon$-decay | 9.5 | 2 | .170 | .114 | .182 | .120 (+6) |
| $e^+e^-[\sigma_{\text{had}}]$ | 10.5 | 3 | .200 | .130 | .198 | .129(-1) |
| $e^+e^-[j \& sh]$ | 22.0 | 2 | .161 | .124 | .166 | .127(+3) |
| $e^+e^-[j \& sh]$ | 35.0 | 2 | .145 | .123 | .149 | .126(+3) |
| $e^+e^-[\sigma_{\text{had}}]$ | 42.4 | 3 | .144 | .126 | .145 | .127(+1) |
| $e^+e^-[j \& sh]$ | 44.0 | 2 | .139 | .123 | .142 | .126(+3) |
| $e^+e^-[j \& sh]$ | 58 | 2 | .132 | .123 | .135 | .125(+2) |
| $Z_0 \rightarrow \text{had.}$ | 91.2 | 3 | .124 | .124 | .124 | .124(0) |
| $e^+e^-[j \& sh]$ | 91.2 | 2 | .121 | .121 | .123 | .123(+2) |
| $Z_0 \rightarrow \text{had.}$ | 189 | 2 | .110 | .123 | .112 | .125(+2) |

Averaged $<\bar{\alpha}_s(M_Z^2)>_{f=5}$ values  

\[0.12; \quad 0.124\]

\[\text{a}^{\text{a}} \text{"j & sh"} = \text{jets and shapes; Figures in brackets in the last column give the difference } \Delta\bar{\alpha}_s(M_Z^2) \text{ between common and APT values.} \]

\[\text{b}^{\text{b}} \text{Taken from Ref.[1].} \]

the form

$$|\Delta\bar{\alpha}_s(s)|_{f=5}^{20 \div 100 \text{ GeV}} \simeq 1.225\bar{\alpha}_s(s) \simeq 0.002 \div 0.003.$$ (35)
An adequate expression for the equivalent shift of the $\bar{\alpha}_s(M_Z^2)$ value is
\[
[\Delta \bar{\alpha}_s(M_Z^2)]_3 = 1.225 \bar{\alpha}_s(s) \bar{\alpha}_s(M_Z^2)^2 .
\] (36)

We give the results of our approximate APT calculations, mainly by Eqs. (35) and (36), in the form of Table 4 and Figure 3. In the last column of Table 4 in brackets, we indicate the difference between the APT and usual analysis. The results of the three–loop analysis are marked by bold figures. Dots in the lower part of the Table correspond to shape–events data for energies $W = 133, 161, 172$ and $183$ GeV with the same positive shift 0.002 for the the extracted $\bar{\alpha}_s$ values.

![Figure 3: The APT analysis for $\bar{\alpha}_s$ in the $f = 5$ time–like region. Crosses (+) differ from circles by the $\pi^2$ correction (35). Solid APT curve relates to $\Lambda_{MS} = 270$ MeV and $\bar{\alpha}_s(M_Z^2) = 0.124$. By dot-and-dash curve, we give the standard $\bar{\alpha}_s$ (at $\Lambda^{(5)} = 213$ MeV and $\bar{\alpha}_s(M_Z^2) = 0.118$) taken from Fig.10 of paper [2].](image)

In Fig.3, by open and hatched circles we give two– and three–loops data from Fig.10 of paper [2]. The only exclusion is the $\Upsilon$ decay taken from Table X of the same paper. By crosses, we marked the “APT values” calculated approximately by Eq.(35).

For clearness of the $\pi^2$ effect, we skipped the error bars. They are the same as in the mentioned Bethke’s figure and we used them for calculating $\chi^2$.

Let us note that our average 0.121 over events from Table 6 of Bethke’s review [2] nicely correlates with recent data of the same author (see Summary of Ref.[36]). The best $\chi^2$ fit yields $\bar{\alpha}_s(M_Z^2)_{[2]} = 0.1214$ and [4]

\[
\bar{\alpha}_s(M_Z^2)_{APT} = 0.1235 .
\]

6This value, corresponding to $\Lambda^{(5)} = 290$ MeV, is supported by recent analysis [33] of $\tau$ decay that gives $\Lambda^{(3)} = 420$ MeV; compare with Table 3.
This new $\chi^2_{\text{APT}}$ is smaller $\chi^2_{\text{APT}}/\chi^2_{\text{PT}} \simeq 0.73$ than the usual one. This illustrates the effectiveness of the APT procedure in the region far enough from the ghost singularity.

5 Conclusion

It is a common standpoint that in QCD it is legitimate to use the power in $\alpha_s$ expansion for observables in the low energy (low momentum transfer) region. At the same time, there exist rather general (and old [37]) arguments in favor of nonanalyticity of the $S$ matrix elements at the origin [38] of the complex plane of the expansion parameter $\alpha$ variable. This, in turn, implies that common perturbation expansion has no domain of convergence. Technically, this corresponds to the factorial growth ($\sim n!$) of expansion coefficients (like $d_n$ or $r_n$) at large $n$ [39, 40]. In QCD, with its “not small enough” $\alpha_s$ values in the region below 10 GeV it is a popular belief that one does face an asymptotic nature of perturbation expansion by observing approximate equality of relative contributions of the second ($\alpha_s^2$) and the third ($\alpha_s^3$) terms into observable, like in all PT columns of Table 2.

Our first qualitative result consists in observation that convergence properties of the APT expansions drastically differ from the usual PT ones.

The evidently better practical convergence of the APT series for the Euclidean observable, as it has been demonstrated in the right part of Table 2, probably means that essential singularity at $\alpha_s = 0$ is adequately taken into account by new expansion functions $A_k(q^2)$.

On the other hand, in the time-like region the improved approximation property of the APT expansion over $A_k(s)$ has a bit different nature, being related, in our opinion, to the non-uniform convergence of the standard PT expansion for Minkowskian observables. In any case, from a practical point of view:

1. In the APT, one can use the nonpower expansions (22) and (23) without the last term.

The next point, discussed in Section 3.3, refers to a more specific issue connected with current practice of the Minkowskian observable analysis in the low–energy ($s \lesssim 3 \text{ GeV}^2$) region (like, e.g., inclusive $\tau$ decay). As it has been shown – see Fig. 2 —

2. Below 2 GeV$^2$ it is impossible to use the common power expansion (11) for a time–like observable.

Second group of our results is of a quantitative nature:

3. Effective positive shift $\Delta \alpha_s \simeq +0.002$ in the upper half ($\geq 50 \text{ GeV}$) of the $f = 5$ region for all time-like events that have been analyzed up to now in the NLO mode.

4. Effective shift $\Delta \alpha_s \gtrsim +0.003$ in the lower half ($10 \div 50 \text{ GeV}$) of the $f = 5$ region for all time-like events that have been analyzed in the NLO mode.

5. The new value

$$\bar{\alpha}_s(M_Z^2) = 0.124,$$

obtained by averaging new APT results over the $f = 5$ region.
The quantitative results are based on the new APT nonpower expansion (8) and plausible hypothesis on the \( \pi^2 \)–term prevalence in common expansion coefficients for observables in the Minkowskian domain. The hypothesis has some preliminary support — see Table 1 — but needs to be checked in more detail.

Nevertheless, our result (37) being taken as granted raises two physical questions:

– The issue of self-consistency of QCD invariant coupling behavior between the “medium \((f = 3, 4)\)” and “high \((f = 5, 6)\)” regions.

Here, the more detailed APT analysis of data on DIS, heavy quarkonium decays and some other processes are in order. As it has been mentioned above, fresh APT analysis of \( \tau \)–decay \[33\] seems to support such a correlation with \( \Lambda_3 \sim 400 \div 450 \text{ MeV} \) and \( \Lambda_5 \sim 290 \text{ MeV} \).

– The new “enlarged value” (37) can influence various physical speculations in the very HE region, in particular on superpartner masses in MSSM GUT constructions – compare, e.g., with recent attempts \[41\] in this direction.

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