The hydrogen atom as relativistic bound system

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The hydrogen atom as relativistic bound-state system of a proton and an electron in the complex-mass scheme is investigated. Interaction of a proton and an electron in the atom is described by the Lorentz-scalar Coulomb potential; the proton structure is taken into account. The concept of position dependent particle mass is developed. Relativistic wave equation for two interacting spinless particles is derived; asymptotic method is used to solve the equation. Complex eigenmasses for the H atom are obtained. The spin center-of-gravity energy levels for the H atom are calculated and compared with ones obtained from solution of some known relativistic wave equations and tabulated NIST data.

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I. INTRODUCTION

Hydrogen is the most popular basic element in the Universe. It is the simplest atom, comprising only a proton and an electron which are stable particles. This simplicity means its properties can be calculated theoretically with impressive accuracy. The spherically symmetric Coulomb potential, with interaction strength parametrized by dimensionless coupling (“fine structure”) constant, $\alpha$, is of particular importance in many realms of physics.

In the latest measurements, the Rydberg constant $R$ (the ground-state energy), the scaling factor for all transition frequencies in $H$ atom, has been determined with a precision approaching $10^{-11}$ [1]. This means that all other transitions in the $H$ atom, ranging from the microwave to the ultraviolet, can be considered as metrological standards to this same level of precision [2, 3].

Rapid advances in the techniques of laser spectroscopy have shown the way for experimental measurements at a comparable level of precision. Many highly precise spectroscopic data had been obtained experimentally, and these data could now be fully understood in terms of atomic structure and quantum numbers. Thus, laboratory and astrophysical spectra could be interpreted as specific classified transitions between energy levels of atoms [2].

At the present time, there is no strict theory of the $H$ atom as relativistic two-body system. An outline of the problems encountered in the theoretical calculations and explanations where the current uncertainties lie have been reviewed in [4]. The description of bound states in a way fully consistent with all requirements imposed by special relativity and within the framework of relativistic Quantum Field Theory is one of the great challenges in theoretical elementary particle physics. Comprehensive description of the $H$ atom is reduced to relativistic bound state problem.

In this work we consider the $H$ atom as relativistic two-body system in the potential approach; here we have two fundamental problems: 1) two-particle relativistic equation of motion and 2) absence of a strict definition of the potential in relativistic theory. There are various relativistic effects such as fine and hyperfine splitting of the energy levels that should be taken into account in more accurate analysis. An objective of this work is to calculate the spin center-of-gravity energy levels for the $H$ atom in the framework of developed approach and compare them with ones obtained from solution of the Klein-Gordon (KG), spinless Salpeter (SS) equations and tabulated NIST data.
The Coulomb potential is treated as the Lorentz-scalar function of the spatial variable \( r \) additive to particle’s masses. We develop the concept of distance-dependent particle mass and derive a two-particle equation of motion. We use the fact that in relativistic kinematics the spatial two-particle relative momentum is relativistic invariant, spatial part of Minkowski force are relativistic invariants. We derive relativistic two-particle wave equation and give its analytic solution in the form of a standing wave. The free particle hypothesis for the bound state is developed: proton and electron move as free particles inside atom on the ends of a 3D string [5].

II. NONRELATIVISTIC APPROXIMATION AND RELATIVISTIC CORRECTIONS

Description of the \( H \) atom in nonrelativistic formulation is very accurate by engineering standards; however, it is not exact. The usual approach is to take the nonrelativistic approximation as the starting point. Then corrections are applied using perturbation theory.

The energy levels of \( H \)-like atoms are determined mainly by the Dirac eigenvalue, Quantum ElectroDynamic effects such as self energy and vacuum polarization, nuclear size and motion effects. The binding energy of an electron in a static Coulomb field (the external electric field of a point nucleus of charge \( Z \epsilon \) with infinite mass) is determined predominantly by the Dirac eigenvalue. A brief summary of the theory of the energy levels of the \( H \) atom relevant to the determination of the Rydberg constant \( R \) based on measurements of transition frequencies is given in [2, 3].

In nonrelativistic formulation, the \( H \) atom is described by the Schrödinger equation and is usually considered as an electron moving in the external field generated by the proton static electric field given by the Coulomb potential,

\[
V(r) = -\frac{\alpha}{r}, \quad \alpha = \frac{e^2}{4\pi \epsilon_0 \hbar c} \approx \frac{1}{137},
\]

(1)

The fine structure constant, \( \alpha \), combines the constants \( e^2/4\pi \epsilon_0 \) from electromagnetism, \( \hbar \) from quantum mechanics, and the speed of light \( c \) from relativity into the dimensional number, which one of the most important numbers in all of physics.

Solution of the Schrödinger’s equation for the potential (1) gives, for the energy eigen-
values,
\[ E_N = \frac{m_e}{2} v_N^2, \quad v_N = \frac{i\alpha c}{N}, \]  \hspace{1cm} (2)

where \( N = k + l + 1 \) is principal quantum number, \( k \) and \( l \) being the radial and orbit quantum numbers. The total energy eigenvalues \( E_N \) has the form of the kinetic energy for a free particle with the imaginary discrete velocity, \( v_N = i\alpha c/N \). This means, that the motion of the electron in a hydrogen atom is free, but restricted by the “walls” of the potential \([5, 6]\).

The Rydberg constant,
\[ R_\infty = \frac{m_e(\alpha c)^2}{2} \frac{1}{\hbar c} = 10.973\,731.568\,549(83) \text{m}^{-1}, \]  \hspace{1cm} (3)

is defined in terms of a free electron mass, \( m_e \). The dimension of the Rydberg is inverse distance, so it is directly related to wave number (inverse of wavelength) measurements. The ionization energy of the hydrogen ground state (the Rydberg energy),
\[ E_0 = \frac{m_e(\alpha c)^2}{2} \equiv R_\infty \hbar c = 13.598\,434\,485\,644(12) \text{eV}, \]  \hspace{1cm} (4)

is not exactly 1 Rydberg multiplied by \( \hbar c \). In nonrelativistic formulation, the \( H \) atom is modeled as a particle of reduced mass \( \mu = \frac{m_em_p}{m_e + m_p} \), which is used for calculations in (3) instead of \( m_e \) as a correction.

According to the expression (2), all energy eigenfunctions \( \psi_{nlm} \) with the same value of \( n \) have the same energy \( E_n \) and they should show up as a single line in an experimental line spectrum; this is the spin center-of-gravity energy level. When these spectra are examined very precisely, the energy levels for a given value of \( n \) are found to consist of several closely spaced lines, rather than a single one; that is the \( H \) atom fine structure.

There are three types of relativistic corrections in the \( H \) atom energy levels which can be listed in order of decreasing magnitude: \( \alpha \) fine structure (includes three effects: Einstein’s relativistic correction, spin-orbit interaction, the Darwin term for states of zero angular momentum); \( \beta \) Lamb shift (includes virtual photons and virtual \( e^+e^- \) pairs); \( \gamma \) hyperfine splitting (neglects the spins of proton and electron).

The spectroscopic data are usually analyzed with the use of the Sommerfeld’s fine-structure formula \([7]\),
\[ E_{nj} = m_e c^2 \left\{ 1 + \left[ \frac{Z\alpha}{n - (j + \frac{1}{2}) + \lambda(j)} \right]^2 \right\}^{-1/2}, \]  \hspace{1cm} (5)
where \( \lambda(j) = \sqrt{(j + \frac{1}{2})^2 - (Z \alpha)^2} \), \( n = 1, 2, \ldots \) is principal and \( j \) is total quantum numbers.

We are interested in the spectrum of photons emitted by a \( H \) atom when an electron transfers from one energy level to another. This means that the rest mass of the electron is not of concern. One defines the observed energy levels of the \( H \) atom to be:

\[
E_{nj} = E_{nj} - m_e c^2. 
\]

Note the following in (5). The term \( \lambda(j) \) becomes complex if \( Z \alpha > j + \frac{1}{2} \). This means that the \( S \) states start to be destroyed above \( Z = 137 \), and that the \( P \) states begin being destroyed above \( Z = 274 \). Note that this differs from the result of the KG equation, which predicts \( S \) states being destroyed above \( Z = 68 \) and \( P \) states destroyed above \( Z = 82 \). Besides, the radial \( S \)-wave function \( R(r) \) diverges as \( r \to 0 \), i.e., \( R(r) \sim r^\beta \), with \( \beta = -\frac{1}{2} + \lambda(0) \). What is the reason of the problems?

The formula (5), as well as the one predicted from solution of the KG wave equation, obtained for the Lorentz-vector Coulomb potential. In general there are two different relativistic versions: the potential is considered either as the zero component of a four-vector, or as a Lorentz-scalar [8, 9]. The relativistic correction for the case of the Lorentz-vector potential is different from that for the case of the Lorentz-scalar potential [10].

III. THE \( H \) ATOM POTENTIAL

The potential is correctly defined in nonrelativistic theory, that gives reasonable results for the \( H \) atom spectra. The use of the same nonrelativistic potential [11] in relativistic kinematics gives even more accurate results for the \( H \) atom spectra. The non-relativistic potential model has proven extremely successful for the description not only of nonrelativistic systems but also relativistic (bound) states. This success is somewhat puzzling in that it persists even when the model is applied to relativistic systems like hadrons. Potential models work much better than one would naively expect [11].

There is no a rigorous definition of the potential in relativistic theory. Relativistically, the electron and proton are affected by virtual photons and virtual electron-positron pairs. For example, derivation of the Darwin term does succeed in its explaining fully within the nonrelativistic picture alone. First assumption is that the electric potential of the proton does not really become infinite as \( 1/r \) at \( r \to 0 \), but is smoothed out over some finite proton
size. Secondly, the electron "does not see" this potential sharply, but at a smallest typical distance equal to the Compton wave length \( \hbar/m_c c \); its uncertainty in position is of the order of the Compton wave length. The electron just cannot figure out where the right potential is at \( r \to 0 \).

An electron in physics is considered to be structureless point-like particle, but a proton has a complicate structure characterized by its form-factor; this structure should be taken into account in accurate calculations. This means that in the fine structure constant, \( \alpha = e^2/(4\pi\epsilon_0\hbar c) \), where \( e^2 = q_e q_p \), the proton charge \( q_p \) should be replaced by the proton form-factor, \( eF_p^E(Q) \). We use the proton electric form-factor which is very well described [up to \( Q^2 \approx 30 (GeV/c)^2 \)] by the dipole formula: \( F_p^E(Q) = [\Lambda^2/(Q^2 + \Lambda^2)]^2 \) where \( \Lambda \approx 0.84 GeV/c \). This results in the modification of the fine structure constant for the \( H \) atom in momentum space (hereafter \( \hbar = c = 1 \)),

\[
\alpha_{H}(Q) = \alpha \left( \frac{\Lambda^2}{Q^2 + \Lambda^2} \right)^2. \tag{7}
\]

What is important in this modification? We are interesting in asymptotic properties of \( \alpha_{H}(Q) \), which are: \( \alpha_{H}(Q) \to \alpha \) at \( Q \to 0 \) (\( r \to \infty \)) and \( \alpha_{H}(Q) \to 0 \) at \( Q \to \infty \) (\( r \to 0 \)). Thus, using the mnemonic rule \( Q \to 1/r \), we come to the following ansatz, for \( \alpha_{H}(r) \) in configuration space:

\[
\alpha_{H}(r) = \alpha \left( \frac{\Lambda^2 r^2}{1 + \Lambda^2 r^2} \right)^2. \tag{8}
\]

Therefore, the Coulomb potential (for the \( H \) atom) is modified as follows:

\[
V_{H}(r) = -\frac{\alpha_{H}(r)}{r}. \tag{9}
\]

The form of the proton form-factor is not important in our asymptotic approach; important is its asymptotic behavior. The running coupling \( \alpha_{H}(r) \) has the properties: \( \alpha_{H}(r \to 0) \to 0 \) and \( \alpha_{H}(r \to \infty) \to \alpha \). We see similarity with the running coupling in QCD: \( \alpha_s(r) \) is frozen at \( r \to \infty \) and is in agreement with the asymptotic freedom properties \( [\alpha_s(r \to 0) \to 0] \).

In case of a bound state, we have a closed system, no external field and any particle of the system can be considered as moving source of the interaction field. In this case the interacting particles and the potential is a unified system.

A serious problem of relativistic potential models is definition and the nature of the potential: whether it is Lorentz-vector or Lorentz-scalar or their mixture? This problem
is very important, for example, in hadron physics where, for the vector-like confining potential, there are no normalizable solutions; there are normalizable solutions for scalarlike potentials [13, 14]. This issue was investigated in [15, 16]; it was shown that the effective interaction has to be Lorentz-scalar in order to confine quarks and gluons.

A Schrödinger-like relativistic wave equation of motion for the Lorentz-scalar potential was formulated in [15]. Though the physical meaning of the Lorentz-scalar potential is not well described, it is generally accepted that the Lorentz-scalar potential is coupled to the rest mass of the particle such that the relativistic energy-momentum relation is given as [8, 9]

$$E^2 - p^2 = [m + S(r)]^2.$$  (10)

This relation is a consequence of the Lagrange equations of relativistic motion, with a scaled time as the evolution parameter [15].

The Lorentz-scalar potentials result in concept of the position-dependent masses, as usually investigated; one assumes that the Lorentz-scalar potential $S(r)$ depends only on position $r$. Application of this Schrödinger-like formalism for the Lorentz-scalar square-step potential was shown to be free from the Klein’s paradox [15]. The predictions are free from not only the Klein’s paradox, but also the paradoxical results predicted for the Lorentz-vector potential [15].

For the Coulomb potential, this formalism yields the exact bound-state eigenfunctions and eigenenergies [15],

$$E_{nl} = mc^2 \left\{ 1 - \left[ \frac{Z\alpha}{n - (l + \frac{1}{2}) + \lambda'(l)} \right]^2 \right\}^{1/2},$$  (11)

where $\lambda'(l) = \sqrt{(l + \frac{1}{2})^2 + (Z\alpha)^2}$. These eigenenergies are the same as those predicted exactly from solution of the relativistic semi-classical wave equation [3, 10]. Comparisons between the cases of the Lorentz-scalar potential and the Lorentz-vector potential was given in [10, 15]. In contrast to the Lorentz-vector potential, the radial $S$-wave function $R(r)$ for the Lorentz-scalar one is regular as $r \to 0$, i.e., $R(r) \sim r^\beta$ with $\beta = -\frac{1}{2} + \lambda'(0)$ [15].

IV. RELATIVISTIC BOUND-STATE EQUATION

There are several forms of relativistic two-particle wave equations such as the KG equation, the Dirac equation, quasipotential equations. The appropriate tool to the description
of bound states is QFT. Within the framework of QFT the adequate Poincaré-covariant
description of bound states is the Bethe-Salpeter (BS) formalism [17–19]. Description of the
bound system is founded on the four-dimensional covariant BS equation [18]. The homoge-
neous BS equation governs all the bound states.

However, attempts to apply the BS formalism to relativistic bound-state problems give
series of difficulties [20]. Its inherent complexity usually prevents to find the exact solutions
or results in the appearance of excitations in the relative time variable of the bound-state
constituents (abnormal solutions), which are difficult to interpret in the framework of quan-
tum physics. The relative time \( t \) in BS formalism causes troubles because the BS wave
functions with time-like relative coordinate have no physical meaning. The BS equation in
the ladder approximation possesses negative norm or ghost states, due to its treatment of
the relative time degree of freedom — spoiling the naive interpretation of it as a quantum
wave equation.

For various practical reasons and applications to both QED and QCD simplified equa-
tions, situated along a path of nonrelativistic reduction, are used. More valuable are methods
which provide either exact or approximate analytic solutions for various forms of differential
equations. They may be remedied in three-dimensional reductions of the BS equation; the
most well-known of the resulting bound-state equations is the one proposed by Salpeter [21].

The most straightforward way out is the reduction of the BS equation to the Salpeter
equation by a series of approximations, which rely on the simplifying assumptions [20]. In
order to avoid the problem of relative time, the constant \( t \) formalism can be used. When
one of the constituents is infinitely heavy, the solution with the retarded propagator gives
the correct results. After applying all these assumptions and approximations to the BS
equation, one comes to the SS Schrödinger-like relativistic wave equation [20], \( \hat{H}\psi = M\psi \),
with the Hamiltonian in the c.m. rest frame,

\[
\hat{H} = \sqrt{\hat{p}^2 + m_1^2} + \sqrt{\hat{p}^2 + m_2^2} + W(r), \tag{12}
\]

where \( \mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2 \). The potential \( W(r) \) in (12) arises as the Fourier transform of the BS
kernel \( K(q) \).

The causal condition leads to the BS equation that reduces to the SS equation (12) for an
instantaneous interaction. The SS equation (12) is the conceptually simplest bound-state
wave equation incorporating to some extent relativistic effects. This equation has to be
regarded as a well-defined standard approximation to the BS formalism. When applied, to the $H$ atom, the SS equation contains a projection operator onto positive energy states and is different from the Dirac equation.

However, even this simplest SS equation (12) leads to difficulties. The square root of the operators cannot be used as it stands; it would have to be expanded in a power series before the momentum operator, raised to a power in each term, could act on $\psi$. There is another problem: the $S$-wave solution of the radial equation for the Coulomb potential diverges at the spatial origin and behaves as $\psi \sim r^{-\alpha/\pi}$ at $r \to 0$ [22]. This divergence at the spatial origin is actually a general problem affecting relativistic wave equations with the Lorentz-vector potential. For example, the solution of the Dirac equation with the Coulomb potential for the $S$-wave states behaves at $r \to 0$ as $\psi(r) \sim r^\beta$, $\beta = \sqrt{1 - \alpha^2} - 1$.

A. Transformation of the SS equation

The corresponding to (12) integral of motion (invariant bound-state mass) is

$$M = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + W(r).$$  \hspace{1cm} (13)

Relativistic total energy of a particle $i$, $\epsilon_i(p) = \sqrt{p^2 + m_i^2}$ ($i = 1, 2$), can be represented as sum of the kinetic energy, $\tau_i(p)$, and the rest mass, $m_i$, i.e., $\epsilon_i(p) = \tau_i(p) + m_i$. Kinetic and potential energies are different types of the total energy. The potential $W(r)$ is Lorentz-scalar as well as particle masses. This is why, it is natural to combine the scalar potential $W(r)$ with masses of particles.

So, (13) can be rewritten in the form, (see also [8, 9, 15]),

$$M = [\tau_1(p) + w_1(r)] + [\tau_2(p) + w_2(r)],$$  \hspace{1cm} (14)

where we have introduced the position-dependent particle masses $w_i(r) = m_i + \frac{1}{2}W(r)$ [23, 24]. The weight coefficient $\frac{1}{2}$ is a subject of discussion. We choose $\frac{1}{2}$ because forces acting on bound particles in the c.m. rest frame are equal and opposite each other. The Minkowsky force acting on proton and electron is expressed via the potential $W(r)$. Another ground to the weight coefficient $\frac{1}{2}$ gives the siring theory [25, 26].

Thus, the total energies of the interacting particles can be written as $\epsilon_i(p) = \sqrt{p^2 + w_i^2(r)}$ in agreement with (10) as shown in [8, 9], and the classic dynamical equation for the total
invariant mass (14) (the system’s total energy in the c.m. rest frame) takes the form:

\[ M = \sqrt{p^2 + w_1^2(r)} + \sqrt{p^2 + w_2^2(r)}. \]  

(15)

This expression can be transformed to the squared relative momentum,

\[ p^2 = \frac{1}{4M^2} \left[ M^2 - m_+^2 \right] \left[ M^2 - (m_+ + W)^2 \right], \]  

(16)

where \( m_+ = m_1 + m_2 \), \( m_- = m_1 - m_2 \).

The equation (16) with the help of the fundamental correspondence principle (according to which physical quantities are replaced by operators acting onto the wave function) [5, 10] results in the two-particle spinless wave equation,

\[ \left\{ \vec{\nabla}^2 + K(M^2) \left[ M^2 - \mathcal{M}^2(r) \right] \right\} \psi(r) = 0, \]  

(17)

where \( K(M^2) = (1 - m_-^2/M^2)/4 \), \( \mathcal{M}(r) = m_+ + W(r) \). This equation coincides with the one obtained from the two-body Dirac Hamiltonian in the rest frame [14],

\[ H = (\bar{\alpha}_1 - \bar{\alpha}_2)\vec{p} + \beta_1 m_1 + \beta_2 m_2 + \frac{1}{2}(\beta_1 + \beta_2)V(r), \]  

(18)

for the scalar potential (note here the weight multiplier 1/2, which has been used for derivation of (17)). Wave equation (17) describes the system of two relativistic spinless particles interacting by Lorentz-scalar central potential.

V. SOLUTION OF THE QC WAVE EQUATION

It is a problem to solve (17) by known methods for the potential (9). Instead, we use the asymptotic quasiclassical (QC) method which is the mathematical realization of the correspondence principle. The QC method developed in [5, 10, 27, 28] was tested as the exact for all one-particle solvable spherically symmetric potentials. The corresponding eigenfunctions have the same behavior as the asymptotes of the exact solutions [5].

Derivation of the QC equation in our case is reduced to replacement of the operator \( \vec{\nabla}^2 \) by the canonical operator \( \Delta^c \) [5] without the first derivatives, acting onto the state function

\[ \Psi(r) = \sqrt{\text{det} g_{ij}} \psi(r), \]  

(19)

where \( g_{ij} \) is the metric tensor.
In particular case of the spherical coordinates \( q = \{ r, \theta, \varphi \} \) (the determinant of metric tensor, \( \det g_{ij} = r^2 \sin \theta \)), the canonical operator is

\[
\Delta^c = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2},
\]

and the relation (19) follows from the identity for the normalization condition,

\[
\int |\Psi(r)|^2 \, dr \, d\theta \, d\varphi \equiv \int |\psi(r)|^2 \det g_{ij} \, dr \, d\theta \, d\varphi = 1.
\]

The generalized two-particle QC wave equation in arbitrary coordinates has the form

\[
\left\{ \sum_{i=1}^{3} \left( \frac{\partial}{\partial q_i} \right)^2 + K(M^2) \left[ M^2 - M^2_H(r) \right] \right\} \Psi(r) = 0.
\]

The QC wave equation (21) is the second-order differential equation of the Schrödinger’s type in canonical form.

Thus, instead of (17) we solve the QC equation, which is, for the spherically symmetric potentials,

\[
\left\{ \Delta^c + K(M^2) \left[ M^2 - M^2_H(r) \right] \right\} \Psi(r) = 0.
\]

One of important feature of this equation is that, for two and more turning-point problems, it can be solved exactly (for eigenvalues) by the conventional leading order in \( \hbar \) WKB method [5, 27].

The appropriate solution method of the QC wave equation, which is the same for non-relativistic and relativistic systems, was developed in [5, 27]. In this method, each of the one-dimensional equations obtained after separation of the QC wave equation is solved by the same QC method. Equation (22) for the spherical potential (9) is separated that gives the radial \([ s = M^2, \mathcal{M}_H(r) = m_p + m_e - \alpha_H(r)/r ]\),

\[
\left\{ \frac{d^2}{dr^2} + K(s) \left[ s - \mathcal{M}_H^2(r) \right] - \frac{\vec{M}^2}{r^2} \right\} R(r) = 0,
\]

and the angular,

\[
\left[ \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \vec{M}^2 \right] Y(\theta, \varphi) = 0,
\]

equations.

Solution of the angular equation (24) was obtained in [5] by the QC method in the complex plane, that gives \( |\vec{M}| = M_l = \left( l + \frac{1}{2} \right) \hbar \), for the angular momentum eigenvalues. These
angular eigenmomenta are universal for all spherically symmetric potentials in nonrelativistic and relativistic cases \[27\].

The radial equation (23) cannot be solved analytically by standard methods. It can be solved by the same QC method in the complex plane. The equation (23) has two turning points and QC quantization condition is

$$\oint_C \sqrt{K(s)\left[s - M_H^2(r)\right] - \frac{M_H^2}{r^2}} dr = 2\pi \left(k + \frac{1}{2}\right). \quad (25)$$

To calculate the phase-space integral (25) in the complex plane we choose a contour $C$ enclosing the cut between turning points $r_1, r_2$ at $r > 0$, and zeros of the radial state function $R(r)$. Outside the contour $C$, the problem has two singularities, i.e., at $r = 0$ and $\infty$. Using the standard method of stereographic projection and residue theory, we should exclude the singularities outside the contour $C$ \[5\]. Excluding these singularities we have, for the integral (25), $I = I_0 + I_\infty$, where $I_0 = -2\pi |l + \frac{1}{2}|$ is contribution of the centrifugal term. The phase-space integral $I_\infty$ is calculated with the help of the replacement of variable, $z = 1/r$. Here we have took into account the asymptotic properties of the fine coupling (8) and its derivative: $\alpha_H(r) = 0, \alpha_H'(r) = 0$ at $r \to 0$ \[6, 29\]. The integration result is

$$I_\infty = \pi \alpha m_+ \sqrt{\frac{s - m_-}{s - s + m_+^2}}, \quad (26)$$

where $m_+ = m_p + m_e, m_- = m_p - m_e$. Combining the integration results and condition (25) we obtain the square equation,

$$s^2 - 4e_N^2 s - (2m_am_{-}v_N)^2 = 0, \quad (27)$$

where $e_N^2 = m_a^2 (1 - v_N^2)$, $m_a = m_+/2, v_N = \frac{1}{2}\alpha/N$, and the principal quantum number in our asymptotic approach is $N = (k + \frac{1}{2}) + |l + \frac{1}{2}|$. Solution of (27) gives, for the squared eigenmasses of the $H$ atom,

$$s_N \equiv M_N^2 = 2e_N^2 \pm 2\sqrt{(e_N^2)^2 + (m_am_{-}v_N)^2}. \quad (28)$$

Using the identity in (28),

$$e_N^2 = e_N^2 + im_am_{-}v_N, \quad (e_N^2)^* = e_N^2 - im_am_{-}v_N.$$
we have

$$M^2_N = (\sqrt{\epsilon^2_N} \pm \sqrt{\epsilon^2_N^*})^2, \quad (29)$$

$$M_N = \pm (\epsilon_N \pm \epsilon_N^*), \quad (30)$$

$$\epsilon_N = \pm \sqrt{\epsilon^2_N} = \pm \left(\text{Re}\{\epsilon_N\} + i\xi \text{Im}\{\epsilon_N\}\right), \quad (31)$$

$$\text{Re}\{\epsilon_N\} = \sqrt{\left(|\epsilon^2_N| + \text{Re}\{\epsilon^2_N\}\right)/2}, \quad (32)$$

$$\text{Im}\{\epsilon_N\} = \sqrt{\left(|\epsilon^2_N| - \text{Re}\{\epsilon^2_N\}\right)/2}; \quad (33)$$

here in (31) $\xi = \text{sgn}(\text{Im}\{\epsilon_N^2\})$.

Thus, the $H$-atom’s eigenmasses are complex. Simple expression (30) defines the $H$ atom’s positions in the Riemann $M_N$-surface [29–31]; negative eigenmasses correspond to the $H$ anti atom. The centered eigenmasses, $M^\text{Re}_N$, and the total widths, $M^\text{Im}_N$, are

$$M^\text{Re}_N = \pm 2\text{Re}\{\epsilon_N\} \equiv \pm \sqrt{2 \left(|\epsilon_N^2| + \text{Re}\{\epsilon_N^2\}\right)}, \quad (34)$$

$$M^\text{Im}_N = \pm 2\text{Im}\{\epsilon_N\} \equiv \pm \sqrt{2 \left(|\epsilon_N^2| - \text{Re}\{\epsilon_N^2\}\right)}. \quad (35)$$

Note, in case of equal particle masses, $m_1 = m_2$ (positronium), the imaginary-part mass disappears.

It is important to note that (30) has the form of the expression for the total energy of two free particles: a proton and an electron move as free particles inside the $H$ atom. The result is in accordance with the free particle hypothesis for bound states. The total kinetic energy of the proton and the electron is

$$T^H_N = |M^\text{Re}_N| - m_p - m_e. \quad (36)$$

The spin center-of-gravity energy levels for the $H$ atom calculated with the use of (36) are shown in Table II. Here we compare our calculations $T^{\text{QC}}_N$ obtained from solution of the QC wave equation (22) with the ones given by the static KG ($T^{\text{KG}}_N$) equation, SS equation ($T^{\text{SS}}_N$) and the tabulated NIST data ($T^{\text{NIST}}_N$) [1].

Our calculations of the $H$ atom spectrum ($T^{\text{QC}}_N$) have precision 3–6 positions and give better description of the NIST data in comparison with those obtained from solution of the KG ($T^{\text{KG}}_N$) and SS ($T^{\text{SS}}_N$) equations, which give precision 1–3 positions.

In Table II we represent the relative calculation precisions corresponding to the states shown in Table II; they are given by $\epsilon_{\text{eq}} = \left|\left(T^{\text{eq}}_N - T^{\text{NIST}}_N\right)/T^{\text{NIST}}_N\right| \times 100\%$. Also, we bring the
Table I. The $H$ atom energy levels and known results.

| State | $\mathcal{T}^{\text{KG}}_N$ | $\mathcal{T}^{\text{SS}}_N$ | $\mathcal{T}^{\text{QC}}_N$ | $\mathcal{T}^{\text{NIST}}_N$ |
|-------|-----------------|-----------------|-----------------|-----------------|
| 1S    | -13.60659871    | -13.60442520    | -13.59810653    | -13.59843445    |
| 1P    | -3.4014965      | -3.40137418     | -3.39956046     | -3.39959812     |
| 1D    | -1.51174769     | -1.51173516     | -1.51091854     | -1.51092434     |
| 1F    | -0.85035692     | -0.85035328     | -0.84989222     | -0.84989357     |
| 1G    | -0.54422814     | -0.54393117     | -0.54393115     | -0.54393196     |
| 2S    | -3.40157042     | -3.40125344     | -3.39956046     | -3.39962387     |
| 2P    | -1.51175484     | -1.51172801     | -1.51091854     | -1.51093197     |
| 2D    | -0.85035822     | -0.85035199     | -0.84989222     | -0.84989548     |
| 3S    | -1.51179063     | -1.51169223     | -1.51091854     | -1.51093960     |
| 3P    | -0.85036123     | -0.85034897     | -0.84989222     | -0.84989834     |

imaginary-part masses of the $H$ atom, $M^\text{Im}_N$, given by (35). If sign is negative, then we come to the relation for the total widths, $\Gamma^\text{TOT}_N \equiv \Gamma^\text{QC}_N = -2M^\text{Im}_N$. As one can see, our calculation precisions are better of known results. More accurate calculations require accounting for the spin corrections, i.e., fine and hyperfine splittings of the energy levels. These corrections

Table II. The relative accuracies of the $H$ atom’s energy levels and the levels’ total width.

| State | $\varepsilon_\text{KG}^N(\%)$ | $\varepsilon_\text{SS}^N(\%)$ | $\varepsilon_\text{QC}^N(\%)$ | $M^\text{Im}_N$ |
|-------|-------------------|-------------------|-------------------|----------------|
| 1S    | 6.00$^{-2}$       | 4.41$^{-2}$       | 2.41$^{-3}$       | 3.421587       |
| 1P    | 5.45$^{-2}$       | 5.22$^{-2}$       | 1.11$^{-3}$       | 1.710793       |
| 1D    | 5.45$^{-2}$       | 5.37$^{-2}$       | 2.41$^{-3}$       | 1.140530       |
| 1F    | 5.45$^{-2}$       | 5.41$^{-2}$       | 3.84$^{-4}$       | 0.855397       |
| 1G    | 5.45$^{-2}$       | 5.42$^{-2}$       | 1.59$^{-4}$       | 0.684317       |
| 2S    | 5.73$^{-2}$       | 4.79$^{-2}$       | 1.87$^{-4}$       | 1.710793       |
| 2P    | 5.45$^{-2}$       | 5.27$^{-2}$       | 8.89$^{-3}$       | 1.140530       |
| 2D    | 5.54$^{-2}$       | 5.37$^{-2}$       | 3.84$^{-4}$       | 0.855397       |
| 3S    | 5.63$^{-2}$       | 4.98$^{-2}$       | 1.39$^{-3}$       | 1.140530       |
| 3P    | 5.45$^{-2}$       | 5.30$^{-2}$       | 7.20$^{-4}$       | 0.855397       |
were not an objective of our work; they will be considered somewhere else.

VI. DISCUSSION AND CONCLUSION

The potential approach, in spite of nonrelativistic phenomenological nature the potential, is used with success to describe the $H$ atom spectrum. We have considered the $H$ atom as a relativistic two-body problem for the static Lorentz-scalar Coulomb potential and took into account the proton structure, using the dipole form factor, and motion effects. We have derived relativistic two-body wave equation and found, from its solution, the complex-mass expression for the $H$ atom’s eigenstates. In the framework of developed approach we have calculated the spin center-of-gravity energy levels for the $H$ atom and compared them with the ones obtained from solution of the Klein-Gordon, spinless Salpeter equations and tabulated NIST data.

In our consideration, we have came to free particle hypothesis and those relevant to particles subject to non-trivial potentials like the modified Coulomb potential. The $H$ atom can be used as a tool for testing any relativistic two-body theory, because latest measurements for transition frequencies have been determined with a highest precision\cite{1-3}.

Description of the $H$ atom with the use of spinless equations is not exact. There are various relativistic effects such as fine and hyperfine splitting of the energy levels that should be taken into account in more accurate analysis. The corrections are applied using perturbation theory. Hyperfine splitting couples the spins of proton and electron, and in the ground state, they combine in the singlet state. A slightly higher energy level occurs when they are in a spin-one triplet state. Transitions between these states radiate very low energy photons with a wave length of 21 cm. This is the source of the 21 centimeter line or “hydrogen line radiation” that is of great importance in cosmology. It has been used to analyze the spiral arms of the galaxy, and can shed light on the so called dark ages that the universe went through. Errors in the ground state energy of hydrogen that are $\sim 10^{-6}$ the energy itself can be of critical importance. This effect is a factor $m_e/m_p \propto 10^{-3}$ smaller still than the fine structure corrections, making the associated energy changes about two orders of magnitude smaller.
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