Two-flavor QCD phases and condensates at finite isospin chemical potential

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Abstract

We study the phase structure and condensates of two-flavor QCD at finite isospin chemical potential in the framework of a confining, Dyson-Schwinger equation model. We find that the pion superfluidity phase is favored at high enough isospin chemical potential. A new gauge invariant mixed quark-gluon condensate induced by isospin chemical potential is proposed based on Operator Product Expansion. We investigate the sign and magnitude of this new condensate and show that it’s an important condensate in QCD sum rules at finite isospin density.

PACS numbers: 12.39.Fe; 11.30.RD; 12.38.Lg;

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1. Introduction

The phase structure of QCD at non-zero temperature and baryon chemical potential has been intensively investigated throughout the last decade. In reality, dense baryonic matter obeys an isospin asymmetry, i.e., in the case of two light flavors, the densities of u and d quark are different. In order that QCD adequately describe the isotopically asymmetric matter, such as compact star, isospin asymmetric nucleon matter and heavy ion collisions, usually the isospin chemical potential $\mu_I = (\mu_u - \mu_d)$ is introduced in the theory. Different approaches, such as Lattice QCD, chiral perturbation theory, Ladder QCD, Nambu-Jona-Lasinio type model, and random matrix model, has been used to explore the QCD phase structure at finite isospin density. It has been widely confirmed that there is a phase transition from the normal phase to the pion superfluidity phase which is characterized by a pion condensate $\langle d\gamma_5 u + H.c. \rangle$ at high enough isospin chemical potential. It is also found that the kaon superfluidity phase characterized by kaon condensate $\langle s\gamma_5 u + H.c. \rangle$ appears at high isospin and strangeness chemical potential in the three light flavors case.

The previous studies on the effects of finite isospin chemical potential and strangeness are mostly focused on two types of condensates $\langle u\gamma_5 d + H.c. \rangle$ and $\langle s\gamma_5 d + H.c. \rangle$, which are order parameters for the corresponding superfluidity phase transitions. It is generally believed that the vacuum of QCD has complicated structure and it is expected that all gauge invariant Lorentz singlet local operators built of the quarks and/or gluons have non-vanishing vacuum expectation values according to QCD sum rules. For example, the well-known low-dimensional condensates, such as quark condensate $\langle \bar{q}q \rangle$, gluon condensate $g^2\langle GG \rangle$, mixed quark gluon condensate $g\langle \bar{q}\sigma Guq \rangle$ and four quark condensate $\langle \bar{q}\Gamma_1 q\bar{q}\Gamma_2 q \rangle$, play significant roles in the hadronic studies based on QCD sum rules.

Due to the presence of the flavor mixed condensates $\langle u\gamma_5 d + H.c. \rangle$ and $\langle s\gamma_5 d + H.c. \rangle$ at finite isospin and strangeness chemical potential, it is natural to expect that there should exist other new types of flavor mixed condensates induced by isospin chemical potential and strangeness chemical potential according to Operator Product Expansion (OPE). Besides the pion condensate and kaon condensate, the possible low-dimensional flavor mixed condensates are mixed quark-gluon condensate $g\langle d\gamma_5 \sigma Gu + H.c. \rangle$ induced by isospin density, and $g\langle s\gamma_5 \sigma Gu + H.c. \rangle$ induced by strangeness density (for convenience, we call the former pion mixed quark-gluon condensate and the later kaon mixed quark-gluon condensate). In addition, new forms of four quark condensates, such as $\langle \bar{q}\gamma_5 \Gamma_1 q\bar{q}\Gamma_2 q \rangle$, may also appear in OPE. We expect these induced low-dimensional condensates also play important roles on the hadronic physical observables in the framework of QCD sum rules.

It is well-known that, in the chiral limit, both chiral condensate and mixed quark-gluon condensate are ideal order parameters for the chiral phase transition of QCD. Similarly, pion mixed quark-gluon condensate and kaon mixed quark-gluon condensate can play the roles of order parameters for the pion superfluidity phase transition and the kaon superfluidity phase transition at finite isospin chemical potential and strangeness chemical potential, respectively. Though both pion condensate and pion mixed quark-gluon condensate can be used as order
parameters to describe the pion superfluidity phase transition, they reflect different aspects of the non-perturbative structure of the ground state: the former reflects the correlation between different flavors with color-singlet component, while the later reflects the correlation between different flavors with color-octet components. Therefore, pion mixed quark-gluon condensate will give new and important information on the pion superfluidity phase transition. The same thing is true for kaon condensate and kaon mixed quark-gluon condensate.

Therefore, it is interesting to investigate the thermal and dense properties of these new types of low-dimensional condensates and their effects on the physical hadronic observables. Since the Global Color Model (GCM)\[16, 17, 18, 19\] is an effective quark and gluon fields theory and has been successfully used to investigate the property of the traditional mixed quark-gluon condensate\[20, 21\] and other QCD condensates\[22\], we will adopt this model to explore the thermal and dense properties of above induced mixed condensates. In this paper, we only consider the possible pion superfluidity phase transition and pion mixed quark-gluon condensate.

2. Mean field theory of GCM at finite isospin chemical potential

In Euclidean metric, with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $\gamma^+_\mu = \gamma_\mu$, the generating functional of GCM with quark and gluon degrees is

$$Z[J, \overline{\eta}, \eta] = \int D\overline{q}DqDA \exp(-S_{GCM}[\overline{q}, q, A^a_\mu] + \overline{q}\eta + \overline{\eta}q + J^a_\mu A^a_\mu)$$

with the action

$$S_{GCM}[\overline{q}, q, A^a_\mu] = \int (\overline{q}(\gamma \cdot \partial + M - igA^a_\mu \lambda^a_\mu/2)q + \frac{1}{2}A^a_\mu D_{\mu\nu}(i\partial)A^a_\nu).$$

The essence of GCM is that it models the QCD local gluonic action $\int F^a_{\mu\nu}F^a_{\mu\nu}$ which has local color symmetry, by a highly nonlocal action which has a global color symmetry. The main aspects of GCM have been reviewed in \[17, 18, 19\].

Integrating over the gluon degrees, the partition function of the GCM with two quark flavors at finite baryon and isospin chemical potential (we only consider the case with temperature $T = 0$ in this paper) has the form

$$Z(\mu, \mu_I) = \int D\overline{q}(x)Dq(x)\exp[-\int x(\overline{q}(x)[\gamma_\mu \partial_\mu + M - \mu\gamma_4 - \delta_\mu \tau_3 \gamma_4]q(x)] - \frac{1}{2} \int x \int y j^a_\mu(x)g^2 D_{\mu\nu}(x - y; \mu, \mu_I)j^a_\nu(y)],$$

where $M = \text{diag}(m_u, m_d)$, $\int x = \int d^4x$ and $j^a_\mu(x) = \overline{q}(x)\gamma_\mu \lambda^a_\mu q(x)$. In Eq.(3), $\tau_i(i = 1, 2, 3)$ are the Pauli matrices in flavor place, $\mu \equiv \mu_B/3$ is the chemical potential associated with baryon number and the quantity $\delta_\mu$ is a half of the isospin chemical potential, i.e. $\delta_\mu = \mu_I/2$ (In this paper, we only consider $\mu_I > 0$). The effective gluon propagator $g^2 D_{\mu\nu}(x - y; \mu, \mu_I)$
is generally a \((\mu, \mu_I)\)-dependent function, which is parameterized to model the low energy dynamics of QCD.

In this study, we will take \(m_u = m_d = m\). Evidently, the above Lagrangian is invariant under the baryon \(U_B(1)\) symmetry and the parity symmetry transformation \(P\). In the case with \(\mu_I \neq 0\), the traditional isospin \(SU_I(2)\) symmetry is reduced to \(U_{I_S}(1)\) symmetry. Usually, the quark condensate \(\langle \bar{q}q \rangle\) is responsible for the chiral symmetry breaking of the ground state and does not spoil the parity and isotopical symmetry, while the nonzero pion condensate \(\langle \bar{q}\gamma_5\tau_3q \rangle\) breaks both the parity and isotopical symmetry of the ground state.

Within the GCM formalism, the ground state of QCD is defined by the saddle point of the action and the quark gap equation at the mean field level is determined by the rainbow truncated quark Dyson-Schwinger equation (DSE) (The application of the DSE model to finite temperature and density is reviewed in \([23]\))

\[
\Sigma(p) = \frac{4}{3} \int \frac{d^3q}{(2\pi)^3} g^2 D_{\mu\nu}(p - q) \gamma_\mu S(q) \gamma_\nu. \tag{4}
\]

At finite \((\mu, \mu_I)\) with \(u, d\) quarks, the inverse of quark propagator can be written in the form

\[
S^{-1}(p, \mu, \mu_I) = S_0^{-1}(p, \mu, \mu_I) + \left( \frac{\Sigma_{uu}(p, \mu, \mu_I)}{\Sigma_{du}(p, \mu, \mu_I)} \frac{\Sigma_{ud}(p, \mu, \mu_I)}{\Sigma_{dd}(p, \mu, \mu_I)} \right), \tag{5}
\]

where

\[
S_0^{-1}(p, \mu, \mu_I) = \begin{pmatrix}
  i\vec{\gamma} \cdot \vec{p} + i\gamma_4 w_u + m \\
  i\vec{\gamma} \cdot \vec{p} + i\gamma_4 w_d + m
\end{pmatrix}, \tag{6}
\]

with \(w_u = (p_4 + \mu + \delta\mu), w_d = (p_4 + \mu - \delta\mu), \)

\[\Sigma_{aa}(p, \mu, \mu_I) = i\vec{\gamma} \cdot \vec{p} A_a(\vec{p}, w_u, w_d) + i\gamma_4 w_u B_a(\vec{p}, w_u, w_d) + C_a(\vec{p}, w_u, w_d), \tag{7}\]

\[\Sigma_{ad}(p, \mu, \mu_I) = \Sigma_{da}(p, \mu, \mu_I) = i\gamma_5 D(\vec{p}, w_u, w_d), \tag{8}\]

and \(A_a, B_a, C_a, D\) are momentum-dependent scalar functions. Nonzero \(C_a\) and \(D\) are responsible for the dynamical chiral symmetry breaking and isotopical symmetry breaking, respectively.

Note that the possible diquark condensation is not considered here and only single Lorentz structure is concerned in \(\Sigma_{ud}\) and \(\Sigma_{du}\). The four matrix elements of the momentum dependent quark propagator

\[
S(p, \mu, \mu_I) = \begin{pmatrix}
  S_{uu}(p, \mu, \mu_I) & S_{ud}(p, \mu, \mu_I) \\
  S_{du}(p, \mu, \mu_I) & S_{dd}(p, \mu, \mu_I)
\end{pmatrix}, \tag{9}
\]

take the form

\[
S_{uu} = [(X_u C_u + D^2 C_d) - i\vec{\gamma} \cdot \vec{p}(X_d A_u + D^2 A_d) - i\gamma_4 (X_d w_u B_u + D^2 w_d B_d)]/H, \tag{10}
\]

\[
S_{dd} = [(X_u C_d + D^2 C_u) - i\vec{\gamma} \cdot \vec{p}(X_d A_d + D^2 A_u) - i\gamma_4 (X_u w_d B_d + D^2 w_u B_u)]/H, \tag{10}
\]

\[
S_{ud} = -iD[\gamma_5 Y - i\gamma_5 \vec{p} S_{ud}^{\gamma_5}] - i\gamma_5 \gamma_4 S_{ud}^{\gamma_5 \gamma_4} + \gamma_5 \vec{p} \gamma_4 S_{ud}^{\gamma_5 \gamma_4}]/H, \tag{10}
\]

\[
S_{du} = -iD[\gamma_5 Y - i\gamma_5 \vec{p} S_{du}^{\gamma_5}] - i\gamma_5 \gamma_4 S_{du}^{\gamma_5 \gamma_4} - \gamma_5 \vec{p} \gamma_4 S_{du}^{\gamma_5 \gamma_4}]/H, \tag{10}
\]
With

\[ S_{\gamma\gamma}^{\gamma} \vec{\gamma} = -S_{\gamma\gamma}^{\gamma} \vec{\gamma} = A_u C_d - C_u A_d, \]
\[ S_{\gamma\gamma}^{\gamma} = -S_{\gamma\gamma}^{\gamma} = w_u B_u C_d - w_d B_d C_u, \]
\[ S_{\gamma\gamma}^{\gamma} = -S_{\gamma\gamma}^{\gamma} = A_u w_d B_d - A_d w_u B_u, \]

and

\[ X_a = A_a p^2 + B_a w_a^2 + C_a^2, \quad Y = C_u C_d + A_u A_d p^2 + w_u w_d B_u B_d + D^2, \]
\[ H = X_u X_d + D^4 + 2D^2(C_u C_d + A_u A_d p^2 + w_u w_d B_u B_d). \]

With above decomposition, the gap equation can be expressed as

\[ \Sigma_{ij}(p) = \frac{4}{3} \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p - q) \gamma^\mu S_{ij}(q) \gamma^\nu. \]

Since each term of Eq. (11) is nonzero at finite \( \mu_I \), more Lorentz structures with new scalar functions should also be considered in \( \Sigma_{ud} \) and \( \Sigma_{du} \) to guarantee the self-consistent treatment of the gap equation. However, introducing more Lorentz structures will complicate the resolving of the gap equations. Just as the Lorentz tensor structure is not concerned in \( \Sigma_{aa} \) at the traditional treatment of DSE at finite \((T, \mu)\) [23], we suppose that \( i\gamma_5 D \) is the leading order term of \( \Sigma_{ud} \) and \( \Sigma_{du} \) and other Lorentz structures has small impact on the determination of quark self energy. Because there are no structures \( \gamma_5 \vec{\gamma}, \gamma_5 \gamma_4 \) and \( \gamma_5 \vec{\gamma} \gamma_4 \) in \( \Sigma_{ud} \) and \( \Sigma_{du} \), the corresponding structures associated with Eq. (11) in \( S_{ud} \) and \( S_{du} \) are ignored in the following. At least, this is a good approximation in the case with small \( \mu_I \).

Due to the phenomenological nature of this effective theory, for simplicity, the Feynman-like gauge \( g^2 D_{\mu\nu}(p-q) = \delta_{\mu\nu} g^2 D(p-q) \) was adopted in our calculation. With above approximation, the gap equation (11) is reduced to seven coupled integral equations, which is still complicated to solve. To get a qualitative understanding of the phase diagram and the structure of the ground state at finite \( \mu_I \), a pedagogical model first introduced by Munczek and Nemirovsky [24] for the modelling of confinement in QCD is favored in this study. Munczek-Nemirovsky (MN) model has been extensively used to explore the properties of strong QCD both at zero \((T, \mu)\) and nonzero \((T, \mu)\) [25, 21], which can always give qualitatively consistent results with the more sophisticated models. The effective gluon propagator of MN model takes the form

\[ g^2 D_{\mu\nu}(p-q) = \delta_{\mu\nu} \frac{3}{16} (2\pi)^4 \eta^2 \delta^4(p-q), \]

with the single parameter \( \eta \) determined by \( \pi \) and \( \rho \) masses in vacuum. The scale parameter \( \eta \) has relation with the string tension of QCD, and in the more real world, it should been the function of \( T \) and \( \mu \). Using Eq. (15), the complete expressions of Eq. (13) are simplified as
\[
(A_u - 1) = \frac{1}{2} \eta^2 [X_u A_u + D^2 A_u]/H, \quad (A_d - 1) = \frac{1}{2} \eta^2 [X_d A_d + D^2 A_d]/H,
\]
\[
(B_u - 1) = \frac{1}{2} \eta^2 [X_u B_u + \frac{w_u}{w_d} D^2 B_u]/H, \quad (B_d - 1) = \frac{1}{2} \eta^2 [X_d B_d + \frac{w_u}{w_d} D^2 B_u]/H,
\]
\[
C_u - m = \eta^2 [X_u C_u + D^2 C_u]/H, \quad C_d - m = \eta^2 [X_d C_d + D^2 C_u]/H,
\]
\[
D = \eta^2 D[C_u C_d + A_u A_d p^2 + w_u w_d B_u B_d + D^2]/H.
\]

Eq. (19) illustrates that there are two distinctive solutions to \(D\): one characterized by \(D \equiv 0\), which describes the normal phase; the alternative, characterized by \(D \neq 0\), which describes the pion superfluidity phase. The phase with small free energy is favored in nature.

### 3. Thermal Potential and Condensates

In GCM/DSE formalism, whether the normal phase or the pion superfluidity phase is stable is determined by evaluating the \((\mu, \mu)\)-dependent pressure difference

\[
\delta P(\mu, \mu) = P[\mu, \mu, S[D \neq 0]] - P[\mu, \mu, S[D = 0]],
\]

where the pressure is calculated by using a steepest-descent approximation

\[
P[T, \mu, \mu, S] = -\Omega[S] = \frac{1}{\beta V} Tr \text{Ln}[\beta S^{-1}] - \frac{1}{\beta V} Tr \text{Ln}[\Sigma S].
\]

Using the technique

\[
\text{Det}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= \text{Det}(A)\text{Det}(B)\text{Det}(C)\text{Det}(C^{-1}DB^{-1} - A^{-1}),
\]

the pressure at finite \((\mu, \mu)\) can be expressed as

\[
P[S] = \frac{1}{2} \int 2 \text{Ln}[H] + 2N_c \int \left\{ D^2 p^2 (A_u + A_d) + w_u w_d (B_u + B_d) \right\} + X_u [p^2 A_u + w_d^2 B_u] + X_d [p^2 A_d + w_u^2 B_d] + m [X_u C_u + X_d C_d + D^2 (C_u + C_d)] \right\}/H,
\]

where \(\int p = \int \frac{d^3 p}{(2\pi)^3}\). Note that a constant term has been ignored in Eq. (23). Though the pressure (or thermal potential) calculated through Eq. (23) is ultraviolet divergent, the pressure difference or the “bag constant” \(\delta P\) is finite.

From the GCM generating functional, it is straightforward to calculate the vacuum expectation value (VEV) of any quark operator with the forms

\[
\mathcal{O}_n \equiv \langle \bar{q}_{j_1, \Lambda^{(1)}_{j_1, i_1}} q_{i_1} \rangle \langle \bar{q}_{j_2, \Lambda^{(2)}_{j_2, i_2}} q_{i_2} \rangle \cdots \langle \bar{q}_{j_n, \Lambda^{(n)}_{j_n, i_n}} q_{i_n} \rangle,
\]

in the mean field vacuum. Here \(\Lambda^{(i)}\) stands for an operator in Dirac, flavor, and color space. The VEV of the operator \(\mathcal{O}_n\) has the form

\[
\langle \mathcal{O}_n \rangle = (-1)^n \sum_p (-)^p \Lambda^{(1)}_{j_1 i_1} \cdots \Lambda^{(n)}_{j_n i_n} S_{ij_1} S_{i_1 j_2} \cdots S_{i_n j_p}. \]

where \( p \) stands for a permutation of the \( n \) indices. Based on formula (25), the low-dimensional condensates, such as chiral condensate and pion condensate can be expressed as

\[
\langle \bar{u}u \rangle = -N_c \int_p \text{Tr} D_{C}[S_{uu}(x, x) - \sigma_{UV}(p)],
\]

(26)

\[
\langle \bar{d}d \rangle = -N_c \int_p \text{Tr} D_{C}[S_{dd}(x, x) - \sigma_{UV}(p)],
\]

(27)

\[
\langle \bar{q}i\gamma_5\tau_1q \rangle = -N_c \int_p \text{Tr} D_{C,F}[i\gamma_5\tau_1S(x, x) - \sigma_{UV}(p)].
\]

(28)

To get a convergent condensate integral, a subtracting term \( \sigma_{UV}(p) \) which simulates the ultraviolet behavior of the quark propagator is introduced in the definition of the quark condensate. In the case with nonzero current quark mass and zero chemical potential, for MN model one has, with \( s = p^2 \),

\[
A(s) = B(s) = 1 + \frac{1}{2s}, \quad C(s) = m(1 + \frac{2}{s}),
\]

(29)

with corrections of high order in \((1/s)\). From these approximate expressions one can construct

\[
\sigma_{UV}(s) = \frac{C(s)}{sA(s)^2 + C(s)^2}.
\]

(30)

Since \( \sigma_{UV}(s) \to m/s \) as \( s \to \infty \) and \( \sigma_{UV}(s) \to 0 \) as \( s \to 0 \), this prescription will provide an absolutely convergent result with no need for a cutoff. The above definition can be generalized to the case with nonzero quark chemical potential. There is no need to introduce a subtracting term in the definition of the pion condensate since \( D(p) \) keeps zero in the large momentum region in MN model. The formula for evaluating the 6-dimensional four-quark condensates can also be directly derived from Eq.(25), which is consistent with the vacuum saturation approximation at zero quark chemical potential. At finite isospin density, the new type four-quark condensates, such as \( \langle \bar{q}i\gamma_5\tau_1qq \rangle \), will appear in OPE. It is easily to prove that \( \langle \bar{q}i\gamma_5\tau_1qq \rangle \sim \langle \bar{q}i\gamma_5\tau_1q \rangle \langle \bar{q}q \rangle \) with the approximation that only \( \gamma_5 \) structure is hold in \( \Sigma_{ud(du)} \) and \( S_{ud(du)} \), with \( \langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d \rangle \).

Since the functional integration over the gluon field \( A_\mu^a \) is quadratic in the framework of GCM, one can perform the integration over gluon field analytically. Using the technique introduced by Meissner[20], through the following integral formulae

\[
\int D A e^{-\frac{1}{4} AD^{-1}A + jA} = e^{\frac{1}{4} jDj},
\]

\[
\int DAA e^{-\frac{1}{4} AD^{-1}A + jA} = (jD) e^{\frac{1}{4} jDj},
\]

\[
\int DAA^2 e^{-\frac{1}{4} AD^{-1}A + jA} = [D + (jD)^2] e^{\frac{1}{4} jDj},
\]

\[
\ldots .
\]

(31)
the gluon fields vacuum average can be replaced by the quark current \( j^{\mu}_q \) with the effective gluon propagator \( D(x - y) \). At the mean field level, according to Eq. (25), one can in principle obtain the VEVs for any gluon fields. This technique provides a feasible way to calculate the VEVs of operators with low-dimensional gluon fields such as the traditional mixed quark-gluon condensate and the isospin density induced pion mixed quark-gluon condensate. Since the number of terms produced by Eq. (25) will increased rapidly with the number of gluonic fields, this technique is not suitable for the evaluation of the VEV of the operator involving high powers of gluonic field \( A \). For instance for the gluon condensate \( \langle GG \rangle \), which contains a \( A^4 \) term, the calculation gets already rather involved.

Applying the method described above, we obtain the expression

\[
g\langle \tilde{q}i\gamma_5\tau_1\sigma_{\mu\nu}G^{\mu\nu}_{\lambda^a}\frac{\lambda^a}{2}q \rangle = -2iN_c \int \frac{4}{3}[\partial_\mu^2g^2D(y - x)]Tr_{D,F}[S(y - x)i\gamma_5\tau_1\sigma_{\mu\nu}S(x - y)\gamma_\nu]
+ 4iN_c \int \int g^2D(y - x)g^2D(z - x)Tr_{D,F}[S(z - x)i\gamma_5\tau_1\sigma_{\mu\nu}S(x - y)\gamma_\nu].
\]

(32)

The similar expression for evaluating the traditional quark-gluon condensate \( g\langle \tilde{q}\sigma_{\mu\nu}G^{\mu\nu}_{\lambda^a}\frac{\lambda^a}{2}q \rangle \) can be got by replacing the structure \( i\gamma_5\tau_1\sigma_{\mu\nu} \) in (32) with \( \sigma_{\mu\nu} \).

Using the Eq. (14) and the formulae

\[
Tr_c \left[ \frac{\lambda^a}{2} \frac{\lambda^b}{2} \frac{\lambda^c}{2} - \frac{\lambda^a}{2} \frac{\lambda^c}{2} \frac{\lambda^b}{2} \right] = i \frac{f_{abc}}{2}, \quad f_{abc}f^{abc} = N_c\delta^{aa},
\]

the expression for the pion mixed quark-gluon condensate can be simplified as

\[
g\langle \tilde{q}i\gamma_5\tau_1\sigma_{\mu\nu}G^{\mu\nu}_{\lambda^a}\frac{\lambda^a}{2}q \rangle = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\]

(34)

where

\[
I_1 = -72 \int \frac{D}{H} Y \left[ (A_u + A_d - 2)\bar{p}^2 + (B_u - 1)w_u^2 + (B_d - 1)w_d^2 \right],
\]

(35)

\[
I_2 = 36 \int \frac{D}{H} X_d A_u + D^2 A_d + X_u A_d + D^2 A_u \bar{p}^2,
\]

(36)

\[
I_3 = 36 \int \frac{D}{H} X_d B_u w_u^2 + D^2 w_u w_d B_d + X_u B_d w_d^2 + D^2 w_u w_d B_u,
\]

(37)

\[
I_4 = \frac{81}{2} \int \frac{D}{H} Y \left[ D^2 - (C_u - m)(C_d - m) \right],
\]

(38)

\[
I_5 = \frac{81}{2} \int \frac{D}{H} \left[ (C_u - m)(X_d C_u + D^2 C_d) + (C_d - m)(X_u C_d + D^2 C_u) \right],
\]

(39)

\[
I_6 = \frac{81}{2} \int \frac{D}{H} \left[ (A_u - 1)(X_d A_u + D^2 A_d) + (A_d - 1)(X_u A_d + D^2 A_u) \right] \bar{p}^2,
\]

(40)

\[
I_7 = \frac{81}{2} \int \frac{D}{H} \left[ (B_u - 1)(X_d B_u w_u^2 + D^2 B_d w_d w_d) + (B_d - 1)(X_u B_d w_d^2 + D^2 B_u w_u w_d) \right].
\]

(41)
Note that the scalar functions $A_a, B_a, C_a, D$ are all momentum dependent in the DSEs formalism. The above integral expressions explicitly show that the pion mixed quark-gluon condensate is an order parameter for the pion superfluidity phase transition. By the way, it should be mentioned that the expression in Eq. (34) for pion mixed quark-gluon condensate is only valid in Feynman-like gauge.

4. Numerical Results and Discussions

To get a qualitative understanding of the finite $\mu_I$ effects on the ground state, the numerical study below are all based on the MN model. Due to the defect of the supposed Lorentz structure of quark self-energy, the isospin chemical potential concerned below is limited within the range $|\mu_I/2| < 0.2$ (GeV). In this paper, for simplicity, we only concern the case with zero temperature and baryon chemical potential.

A. The critical isospin chemical potential for pion condensate

The effective field theory arguments [1] indicate that critical isospin chemical potential $\mu_I^c$ for the pion superfluidity phase is exactly the vacuum pion mass $m_\pi$ at $T = \mu_B = 0$. Whether the MN model can algebraically reproduce this result within above formalism is discussed below.

For the solution with $D \neq 0$, the gap equation (19) is reduced to

$$H = \eta^2 [C_u C_d + A_u A_d p^2 + B_u B_d w_u w_d + D^2].$$

(42)

In the neighborhood of $\mu_I^c$, one can probably neglect the $D^2$ and $D^4$ in above expression. According to gap equations (16) and (17), $A_{u(d)}$ and $B_{u(d)}$ are identical for $D = 0$. Therefore, the gap equation (42) can be further reduced to

$$X(p + P'^2)X(p - P'^2) = \eta^2 \left[ C(p + P'^2)C(p - P'^2) + A(p + P'^2)A(p - P'^2)(p + P'^2)\cdot (p - P'^2) \right].$$

(43)

where $P' = (\vec{0}, i\mu_I^c)$ and the flavor subscript has been ignored. Within the feynman-like gauge and only considering the $\gamma_5$ structure, the Bethe-Salpeter equation (BSE) for the vacuum pseudoscalar amplitude $\Gamma_j^\pi(p, P)$ in GCM takes the form [17]

$$\Gamma_j^\pi(p, P) = -\frac{2}{9} \int \frac{d^4q}{(2\pi)^4} D(p - q) Tr_{D,C,F} [i\gamma_5 \tau_3^+ S(q + \frac{P}{2}) i\gamma_5 \tau_3 S(q - \frac{P}{2})] \Gamma_j^\pi(q, P),$$

(44)

where $P^2 = -m_\pi^2$. Using $Tr_F[\tau_3^+ \tau_3] = 2$, the BSE (44) is greatly simplified in the MN model

$$X(p + \frac{P}{2})X(p - \frac{P}{2}) = \eta^2 \left[ A(p + \frac{P}{2})A(p - \frac{P}{2})(p + \frac{P}{2})\cdot (p - \frac{P}{2}) + C(p + \frac{P}{2})C(p - \frac{P}{2}) \right].$$

(45)

which has the same form as the gap equation (43). Therefore, if the BSE (44) can produce the vacuum pion mass, we algebraically prove $-P'^2 = m_\pi^2$ and get the conclusion $\mu_I^c = m_\pi$ at $T = \mu = 0$. 


However, the above proof is only true for the chiral limit case since the MN model doesn’t support the pion bound state beyond the chiral limit if only the $\gamma_5$ structure is considered in the pseudoscalar meson Bethe-Salpeter amplitude [24]. Therefore, to produce the result $\mu_c^\chi = m_\pi$ beyond chiral limit in the MN model, other Dirac amplitudes beyond $\gamma_5D$ should also be included in the off-diagonal term of the inverse quark propagator, which will make solving the gap equation more involved.

Note that for other improved effective gluon propagators such as those used in [20] [28] [29], the pion Bethe-Salpeter amplitude with only $\gamma_5$ structure is a good approximation to obtain the vacuum pion mass. In principle, using these improved effective gluon propagator to explore the pion superfluidity within the DSE formalism should give more quantitatively reasonable results in contrast with the simple MN model. However, the set of seven coupled algebraic gap equations (16)-(19) will be replaced by a set of seven coupled integral equations which leads to the numerical calculation more difficult.

For simplicity and to get a qualitatively understanding of the pion superfluidity within the DSE formalism, we still use the MN model in the following and only contains the $\gamma_5$ structure in the off-diagonal term of the inverse quark propagator.

**B. The Chiral limit**

Figure 1: The pressure difference $\delta P$, pion condensate $\pi$ and pion mixed quark-gluon condensate Mix in the chiral limit. $B_0 = (0.1\eta)^4$, $\langle \bar{q}q \rangle_0 = \langle \bar{u}u + \bar{d}d \rangle_0$ and $\text{Mix}_0 = (310\text{MeV})^5$ are the bag constant, chiral condensate and mixed quark-gluon condensate of the vacuum obtained from the MN model with $m = 0$ and $\eta = 1.06\text{GeV}$, respectively.

In the chiral limit, there are four possible solutions according to the gap equations (16)-(19):

\[
C_u = C_d = 0, D = 0; \quad C_u \neq 0, C_d \neq 0, D = 0;
\]
\[
C_u = C_d = 0, D \neq 0; \quad C_u \neq 0, C_d \neq 0, D \neq 0,
\]

which characterize four possible phases of QCD at finite isospin density, respectively. However,
the solution with both nonzero quark condensate and nonzero pion condensate is not found in our numerical study. By comparing the corresponding free energies of the former three possible phases, it is found that the pion superfluidity phase is favored in the chiral limit at nonzero isospin chemical potential. It seems that this is a universal result which has been confirmed by many former studies\[1, 2\]. In this case, chiral symmetry is not broken and both quark condensate and mixed quark-gluon condensate disappear in OPE. In contrast with the vanishing of above chiral condensates, new condensates induced by isospin density, such as pion condensate and pion mixed quark-gluon condensate appear. Note that the new four-quark condensates, such as $\langle \bar{q}i\gamma_5\tau_1qq \rangle$, also vanish in this case because these condensates factorize at mean field level within our formalism with the approximation used in Sec. 2.

![Figure 2: The isospin chemical potential dependence of the ratio between pion mixed quark-glue condensate and pion condensate from MN model with $m=0$ and $\eta=1.06\text{GeV}$.](image)

The $\mu_I$ dependence of the pressure difference between the pion superfluid phase and the normal phase, pion condensate and pion mixed quark-gluon condensate is shown in Fig.1. For the case with $\mu_I = 0$, the zero pressure difference of $\delta P$ suggests that the nonzero pion condensate and nonzero quark condensate correspond to equivalent but distinct vacua, which is guaranteed by the chiral symmetry (A small quark mass will destabilize the superfluidity phase). In contrast with this event, for the whole domain of nonzero $\mu_I$ concerned, the positive pressure difference $\delta P$ suggests the pion superfluidity phase is the stable ground state in the chiral limit for two-flavor QCD. Fig.1 shows that the magnitudes of both induced condensates and the pressure difference are monotonically increasing functions of $\mu_I$.

Fig.2 shows the $\mu_I$-dependent behavior of the ratio of pion mixed quark-gluon condensate to pion condensate from MN model. In Fig.2, The ratio ranges from $1.56\text{(GeV)}^2$ to $1.66\text{(GeV)}^2$, which suggests that the induced mixed quark-gluon condensate has the same magnitude of the traditional mixed condensate in the concerning domain of $\mu_I$ (in MN model, the ratio of the mixed quark-gluon condensate and the chiral condensate is 1.92 in the vacuum[21]). Note that
numerical study suggests that the pion mixed quark-gluon condensate defined in Eq. (32) is positive, which is consistent with the sign of the traditional mixed quark-gluon condensate.

C. Finite current mass

![Graph showing the relationship between delta P and mu_I](image)

Figure 3: The pressure difference $\delta P$ (plotted only in the region $\mu_I \geq \mu_I^C$), quark condensate $\langle \bar{u}u \rangle$, pion condensate $\pi$ and pion mixed quark-gluon condensate Mix obtained from MN model with $m = 12$ MeV and $\eta = 1.06$ GeV.

With finite current quark mass $m = 12(\text{MeV})$, apparently, only two types of solutions exist: the normal phase with nonzero quark condensate and zero pion condensate and the pion superfluidity phase with nonzero pion condensate and nonzero quark condensate. In contrast to the chiral limit, for finite current mass, there is no solution with zero quark condensate to the gap equations due to the explicit chiral symmetry broken term in the lagrangian.

According to the numerical study, in the small isospin chemical potential region $\mu_I < 64$ MeV, only the normal phase solution exists. For the region $\mu_I > 64$ MeV, there also exist the solution corresponding the pion superfluidity phase beside the normal phase solution. The stable ground state in the region $\mu_I > 64$ MeV is again determined by the difference of the pressure $\delta P$ in Eq. (23).

It is shown in Fig. 3 that at the point $\mu_I = 64$ MeV, the scaled pressure difference of two solutions is close to zero and for the region $\mu_I > 64$ MeV, the pressure difference is positive and monotonically increase with the increase isospin chemical potential, which suggests that the pion superfluidity phase is favored in $\mu_I > 64$ MeV region. From the $\mu_I$-dependent behavior of both pressure difference $\delta P$ and the two order parameters, pion condensate and pion mixed quark-gluon condensate, one can judge the phase transition from normal phase to pion superfluidity phase is second order. This conclusion is consistent with the result obtained from lattice simulation and other model studies.

Note that at zero $\mu$, the scalar functions $A(B, C)_u$ in $S_{uu}$ and $A(B, C)_d$ in $S_{dd}$ at the same
Figure 4: The $\mu_I$-dependence of the ratio between pion mixed quark-gluon condensate and pion condensate obtained from MN model with $m = 12\text{MeV}$ and $\eta = 1.06\text{GeV}$.

point $(\vec{p}, w_u, w_d)$ are complex conjugates, therefore the relation $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ always holds. Fig. 3 shows that the magnitude of quark condensate monotonically increases with the increase of $\mu_I$ in the normal phase region, which is consistent with the dependence of the chiral condensate on the baryon chemical potential obtained within DSE formalism [23]. In the pion superfluidity phase, the magnitude of quark condensate monotonically decreases with the increase of $\mu_I$, which is an anticipated result due to the monotonically increasing behavior of pion condensate with respect to $\mu_I$. This behavior is similar to the dependence of the chiral condensate on the baryon chemical potential for the appearance of diquark condensate [23]. It is expected that the traditional mixed quark-gluon condensates, $g\langle \bar{u}\sigma Gu \rangle$ and $g\langle \bar{d}\sigma Gd \rangle$, have the similar $\mu_I$-dependent behavior. Fig.3 manifests a competitive relationship between the induced condensates and their corresponding traditional partners.

In Fig. 4, we display the numerical result of the isospin chemical potential dependence of the ratio of the pion mixed quark-gluon condensate to pion condensate. Such a value ranges from $2.2(\text{GeV})^2$ to $1.7(\text{GeV})^2$, which is also close to the ratio of the traditional mixed quark-gluon condensate to quark condensate obtained in the vacuum[21]. The large magnitude of the ratio suggests that the induced mixed quark-gluon condensate is an important parameter within the QCD sum rules at finite $\mu_I$. In addition, in contrast with the chiral limit case, the nonzero pion condensate and nonzero quark condensate suggests that the new four-quark condensates also have nonzero value in the superfluidity phase, even at the mean field level.

The critical chemical potential 64MeV is relatively small in contrast with the vacuum pion mass. As mentioned above, the main reason for this discrepancy arises from the fact that only $\gamma_5$ structure is considered in the off-diagram part of the inverse quark propagator, while the MN model doesn’t support the pseudoscalar bound state beyond the chiral limit when only $\gamma_5$ structure is contained in the pseudoscalar meson Bethe-Salpeter amplitude. One can expects
that this discrepancy will become small when we either adopt the other improved effective
gluon propagators in the calculation or include other allowed Dirac structures such as $ \gamma_5 \vec{\gamma} \cdot \vec{p} $ in
the off-diagonal part of the inverse quark propagator within the MN model.

The dependence of the critical isospin potential $ \mu_I^c $ on the current quark mass is plotted in
Fig. 5. It is shown that $ \mu_I^c $ monotonically increases with the value of current quark mass.
Though $ \mu_I^c $ obtained in MN model with only $ \gamma_5 $ structure considered is markedly smaller than $ m_\pi $, the critical point is still roughly proportional to the square root of current quark mass in the range (0-20) MeV.

5. Summary and Remarks

Using a pedagogical confining model within the framework of GCM, we have qualitatively
investigated the phase structure and condensates of two-flavor QCD at finite isospin density
with zero temperature and baryon chemical potential. By solving the quark gap equation
through the DSE formalism, we obtained that the truncated DSE type model supports the
pion superfluidity phase transition at high enough isospin chemical potential. In contrast with
the previous model studies, the obtained gaps responsible for both the chiral condensate and
pion condensate are all momentum dependent within the DSE formalism, which are more close
to the real world. In addition, some new types of low-dimensional condensates of QCD induced
by finite isospin chemical potential, such as pion mixed quark-gluon condensate and mixed four
quark condensate, are proposed and investigated in this paper.

In the chiral limit with finite isospin chemical potential, the normal phase is unfavored and
the pion superfluidity phase is the stable ground state. For the case with finite current quark
mass, only the solution corresponding to the normal phase is found in the gap equations in
the region $ \mu_I < \mu_I^c $; while for the region $ \mu_I > \mu_I^c $, the normal phase is unfavored and the pion
superfluidity phase is the stable ground state. The distinctly different phase structure between

Figure 5: The dependence of the critical isospin potential $ \mu_I^c $ on the current quark mass from
MN model with $ \eta = 1.06 $GeV. The current quark mass ranges from 0 to 20MeV.
the chiral limit and the finite current quark mass suggests that the value of the critical isospin chemical has close relation with the pion mass. Even though for the simplicity of numerical study, the obtained critical point $\mu_I^c$ in this paper is not exactly the pion mass for the case beyond chiral limit, we point out that the improved calculation (more involved) within the DSE formalism should confirm $\mu_I^c = m_\pi$.

Furthermore, our calculation shows that in the chiral limit with finite isospin chemical potential, both quark condensate and traditional mixed quark-gluon condensate vanish in OPE with the appearance of isospin density induced pion condensate and pion mixed quark-gluon condensate. In the real world, for $\mu_I < \mu_I^c$, quark condensate and mixed quark-gluon condensate exist in OPE with the vanishing of pion and pion mixed quark-gluon condensate; for $\mu_I > \mu_I^c$, the magnitudes of both isospin density induced condensates increase with the increasing of isospin chemical potential, while the magnitude of quark condensate decreases with the increasing of isospin chemical potential (It is expected that the mixed quark-gluon condensate has the similar behavior). Meanwhile, numerical calculations suggest that the induced pion condensate and pion mixed quark-gluon condensate have the same signs with their corresponding traditional chiral condensates. We also obtained that the magnitude of the ratio of pion mixed quark-gluon condensate to pion condensate is close to the one of traditional quark-gluon condensate to quark condensate in the vacuum, both for the chiral limit and for the real world in the pion superfluidity phase.

Since there is no Fermion sign problem at finite isospin chemical potential with zero baryon chemical potential, in principle, the evaluation of the induced mixed quark-gluon condensate and four-quark condensate can be investigated through the lattice Monte Carlo method. The effect of these isospin chemical potential induced condensates on the hadron properties can be investigated in the framework of QCD sum rules. However, such effects on the hadron properties can more directly be explored using suitable Bethe-Salpeter equations in conjunction with the solutions of the quark gap equation. A natural extension of the present work is to investigate two flavor QCD phase diagram, condensates and hadron properties at finite isospin chemical potential for both nonzero temperature and baryon chemical potential.

Acknowledgements

This work was supported by National Natural Science Foundation of China under contract 10425521, 10305030 and 10575004, the key Grant Project of Chinese Ministry of Education (CMOE) under contract No.305001 and the Research Fund for the Doctoral Program of Higher Education of China under Grant No.20040001010. One of the author (LYX) thanks also the support of the Foundation for University Key Teacher by the CMOE.

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