DISCRETE SCALE INVARIANCE IN SCALE FREE GRAPHS

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Abstract. In this work we introduce an energy function in order to study finite scale free graphs generated with different models. The energy distribution has a fractal pattern and presents log periodic oscillations for high energies. This oscillations are related to a discrete scale invariance of certain graphs, that is, there are preferred scaling ratios suggesting a hierarchical distribution of node degrees. On the other hand, small energies correspond to graphs with evenly distributed degrees.

1. Introduction

In the last few years a huge amount of research on graphs has been achieved in physics and mathematics, mainly due to the increasing importance of complex networks like the Internet and the World Wide Web, and the role of social networks in the propagation of diseases, infections, rumors and news, in both real and virtual environments such as populations or web-based communities.

Despite the different origins, methods and perspectives in those works, certain concepts appeared and pervaded the network literature, like small worlds, clustering, centrality, scale free degree distribution, and preferential attachment, among others. They suggest that simple underlying mechanisms could describe and explain the network growth and their emerging properties. To this end, several networks based on real data were analyzed (see [4], [5] and the references therein), and many theoretical models of network growth were proposed (see Table 3 in [2]).

Also, there are still many questions and problems which remain unanswered. Perhaps the main one is the origin of the power law degree distribution of nodes in real networks. This distribution seems to be observed in several real networks (although some criticism appeared in the last year, [6], [7], [8]). Roughly speaking, the power law distribution says that the number of nodes in a graph with $k$ links for $k > k_0$ has a decay proportional to $k^{-\alpha}$, for certain $\alpha > 1$. Several theoretical models were proposed to explain this phenomenon. Among them, let us mention preferential attachment of Barabasi and Albert [9], edge redirection of Krapivsky and Redner [10], and attachment to edges of Dorogovtsev, Mendes and Samukhin [11].

Although much attention was paid in the theoretical literature in the thermodynamical limit (when time and number of nodes tend to $\infty$) in these models, not too much is known for graphs with a bound in their size ([11] being an interesting exception). Indeed, many of the computer-based simulations tend to concentrate on large graphs, while small ones usually get ignored. Yet, real networks are finite, and in many cases they are very small (like networks of metabolic reactions,

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proteins interactions, digital electronic circuits, sexual interactions or food webs). Our intention in this paper is to concentrate on these small graphs. Specifically, we concentrated on graphs with a number of nodes ranging from 50 to 100,000. On the other hand, we were able to generate large amounts of graphs for each model, which reduced the noise and statistical fluctuations, and helped see some otherwise hidden properties.

Another question, which motivates our research, was the discrimination among the different graphs generated from these models and the analysis of their properties. We may consider this question as a basic one concerning the internal topology of the network, and the quality of their connections. Let us note that different networks with the same node distribution respond in distinct ways to edges or nodes removal depending on the connectivity between hubs, that is, if the nodes with high number of links are connected or not between them. This question was considered by Newman in [12], who coined the term assortativity in order to describe this phenomenon.

In order to study this question, we introduce the linking energy of a graph $G$, defined as

$$E_2(G) = \sum_{x \sim y} |d(x) - d(y)|^2,$$

where $d(x)$ is the degree of the node $x$, i.e., its number of links, and the sum runs over the pairs of connected nodes.

Our original intention was to compute the expected linking energy for certain scale-free network generating models, which enabled us to classify the graphs generated for each models as either assortative or not. One of the first observations, was that the underlying mechanism responsible of networks generation in each model could generate both assortative and non-assortative networks. Another interesting fact is the correlation of the assortativity and the linking energy $E_2$, and the fractal structure of this correlation. These results will be published in a separated work.

However, we also obtained an unexpected result when we computed $E_2$ for the models mentioned above, since the distribution of energies presents log-periodic oscillations in the tail. This oscillatory behavior is more apparent and becomes amplified within models which tend to produce a low number of highly connected hubs. The oscillatory behavior is not present for random or regular graphs, even when they are small.

Surprisingly, this shed some light on the first problem, the node distribution. The presence of log-periodic oscillations is associated to fractal complex dimensions, which are due to discrete scale invariance instead of a scale free similarity ([13], [14]). Indeed, this suggests a stronger order in the networks other than that predicted by the scale free models, since fractal complex dimensions are related to (mathematical) self similar fractals such as the triadic Cantor set. Hence, we may classify the graphs generated with any of those models as continuous or lacunary scale free. A similar dichotomy is present for fractals generated with iterated functions systems [15].

We conjecture that the nodes distribution itself must also exhibit the oscillations. Further work is required to settle this question. However, several real networks that appeared in the literature seem to show such oscillations, like the World Trade Web [16], or web-based communities (see Fig. 1, 5 and 8 in [17]). Also, two theoretical models of networks without growth were recently proposed in [18] and [19], where
oscillations are apparent. The existence of oscillations on the degree distribution was related to the existence of different hierarchies among the nodes of the graph, see [20], [21] where theoretical models were proposed to study this kind of networks. Hence, our results show that classical models based on preferential attachment or edge redirection can be used to generate non homogeneous networks.

2. Main Results

2.1. The linking energy. Let $G$ be any graph, and let us denote $x \sim y$ whenever there exists a link between the nodes $x$ and $y$. Let $d(x)$ be the degree of $x$, that is, the number of links attached to $x$.

We define the linking energy of $G$ as

$$E_2(G) = \sum_{x \sim y} |d(x) - d(y)|^2.$$  

This is a discrete version of the energy which arises for the tension of a system of strings attached at the nodes, with heights proportional to the number of connections. For regular networks, it coincides with the variational form of the Dirichlet energy of a discrete Laplacian.

Let us note that the energy spectra for graphs with $N$ nodes is finite. For a tree on $N$ nodes, the maximum value of the energy is attained for the central tree where all the nodes are connected to one of them, with an energy of

$$(N - 1)(N - 2)^2 \sim N^3.$$  

Let us note as well that the energy is an even number:

$$E_2(G) \equiv \sum_{x \sim y} d_x - d_y \equiv \sum_{x \sim y} d_x + d_y = \sum_x d_x^2 \equiv \sum_x d_x = \sum_{x \sim y} 2 \equiv 0 \pmod{2}$$

For regular lattices, such as the triangular, square or hexagonal lattice, this energy is proportional to the number of boundary nodes, since the degree is the same for all the internal nodes. Also, a random (Erdős-Renyi) graph has lower energy than a scale free one due to the small number of hubs in it.

For networks with power law degree distribution, this degree sequence is not enough to characterize several topological and dynamical properties. Graphs with the same degree sequence behave differently, depending on whether or not the hubs are connected among them, a problem considered by Newman who introduced the assortativity to measure the connectivity between hubs. The linking energy could be used in order to study this problem.

2.2. $E_2$ in a random (Erdős-Renyi) graph. Let $G$ be a graph in the Erdős-Renyi model, with $d$ nodes. Here, two nodes connect each other with probability $p$. Let us fix an order for the nodes, say $x_1, \ldots, x_d$, and for $1 \leq i < j \leq d$ let $X_{ij} = \# \{ k \sim i, k \neq j \}$ and let $Y_{ij} = \# \{ k \sim j, k \neq i \}$. In other words, $X_{ij}$ is the degree of the node $i$ disregarding a possible connection to node $j$, and similarly for $Y_{ij}$. Then, $X_{ij}$ and $Y_{ij}$ are binomial, with probability

$$p(X_{ij} = k) = p(X_{jj} = k) = \binom{d-2}{k} p^k (1-p)^{d-2-k}.$$
Thus, the mean and the variance are given by

\[ < X_{ij} > = p(d - 2), \]

\[ \sigma^2_{X_{ij}} = < X_{ij}^2 > - < X_{ij} >^2 = (d - 2)p(1 - p). \]

Furthermore, the variables \( X_{ij} \) and \( Y_{ij} \) are independent. Hence,

\[ < (X_{ij} - Y_{ij})^2 > = 2 < X_{ij}^2 > - 2 < X_{ij} >^2 = 2(d - 2)p(1 - p). \]

Now, the mean value of \( E_2 \) is given by

\[ < E_2 > = \sum_{i<j} p < (X_{ij} - Y_{ij})^2 > = \binom{d}{2} p \cdot 2(d - 2)p(1 - p) = d(d - 1)(d - 2)p^2(1 - p). \]

For other models of graphs, the distribution of energies gives better information than the mean value of \( E_2 \).

### 2.3. Models of graph generation.

We explain now the studied models. All of them consist of the following steps:

- One begins with a fixed seed of nodes and links.
- At each step, a new node is added to the graph, and
- a fixed number of links emanate from the new node to the existing ones.

These models differ in the way the targets for the new nodes are chosen. We considered the following models:

**BA Preferential attachment** (or linear P.A.): one target node is selected at random, with a probability proportional to the degree of the nodes.

**KR Edge redirection**: one target node is selected at random with uniform probability. However, with probability \( 1 - r \), this target is changed by the node it points to.

**DMS Attach to edges**: one link is selected at random with uniform probability (among the links). The new node connects to both ends of the chosen link.

The labels are put after the authors of these models, see [9], [10], [11].

### 2.4. Numerical Experiments.

We computed the energy \( E_2 \) for several graphs constructed according to these models. We present in this subsection some of these computations. For the KR model, we used \( r = 0.5 \), which gives a power law with exponent 3 for the degree distribution, the same exponent of the other two models (see [2]).

We present the results in Figures [I] to [IV]. In Figure [I] we show \( E_2 \) for \( 10^7 \) graphs on 100 nodes produced with BA model. In Figure [II] we show \( E_2 \) for \( 10^7 \) graphs on 100 nodes, produced with KR model, with \( r = 0.5 \). In Figure [III] we show \( E_2 \) for \( 10^8 \) graphs on 100 nodes, produced with DMS model. Oscillations in this case become more apparent using logarithmic scale, shown in Figure [IV].

Here, the energy spectra has a maximum close to \( 100^3 \), although the observed energy levels are far from this value in the examples. Moreover, there are no forbidden levels at least for \( E_2 \leq 40,000 \). This fact also shows that the oscillations are not caused by the discreteness nor the sparsity at high values of the energy.
The oscillations persist even varying the width of the bins in the histogram. This could be caused by the fractal pattern in the energy distribution, since each
Figure 3. Attach to edges, 100 nodes, histogram over $10^8$ graphs

oscillation exhibits a strong self-similarity with a sequence of sub-oscillations, see Figure 3.

Figure 4. Attach to edges, 100 nodes, histogram over $10^8$ graphs, logarithmic scale in $E_2$
It is interesting to note that the three models shown above share the same node-degree distribution. In all of the cases, degrees decay as a power-law of exponent 3. However, the energy distribution presents some differences. For instance, it is clear that their modes differ: they are close to 4400 (BA), 5700 (KR), and 20000 (DMS). Their oscillations also differ both in the place where they begin and in their amplitudes.

Also, for \( r \) close to zero, the KR model gives high energy graphs, together with interesting behavior of the oscillations, which become highly amplified. We conjecture that a phase transition occurs for some specific value \( r = r_c \), a phenomenon similar to that in [22], which deserves further attention. We show in Figure 6 the energy distribution for \( r = 0.25 \).

The fractal pattern is clearly shown in the integrated density of energies,

\[
N_2(E) = \int_0^E d\delta(E - E).
\]

As was pointed out in [5], the analysis of the cumulative distribution is better than binning in order to avoid statistical fluctuations. In Figure 7, Figure 8, and Figure 9 we show different amplifications of \( N_2(E) \) for KR with \( r = 0.25 \). Clearly, the structure resembles a Cantor devil staircase.

2.5. **Discrete Scale Invariance and log-periodic oscillations.** Let us consider a model in which the first node has no outgoing edge, while the others have exactly one, as it happens in KR or BA models. Then, a graph with \( d \) nodes has \( d - 1 \) edges, and for each edge \( x \sim y \), \( |d(x) - d(y)| \leq d - 2 \). Thus, \( E_2(G) \leq E_2^{\text{max}} := (d-1)(d-2)^2 \). This level of energy is reached with probability 1 in KR when \( r = 0 \).
Now, consider the graph $G_k$ depicted in figure 10. It has 1 node at the center, $k$ nodes with degree 2, $k$ nodes “in a second row”, attached to the previous $k$ nodes, and $d - 2k - 1$ nodes with degree 1 and at distance 1 from the origin.
The energy of $G_k$ is given by

$$e_k := E_2(G_k) = (d - 2k - 1)(d - k - 2)^2 + k(d - k - 3)^2 + k.$$
If $k \ll d$,

$$e_k \sim (d - k - 1)^3.$$  

Within the KR model, if $r$ is close to 0, this sort of graphs have non-negligible probability, and they explain the peaks at energies close to $E_{2_{\text{max}}}^2$, which can be seen in Figure 11. If one pays a closer attention to one of these peaks, one will see that each of them has sub-peaks with smaller probability (see Figure 12).

![Figure 10. Graph with $k$ edges in a second row](image)

**Figure 10.** Graph with $k$ edges in a second row

![Figure 11. Sub-peaks at high energies. KR, 200 nodes, $r = 0.25$, $E_2$.](image)

**Figure 11.** Sub-peaks at high energies. KR, 200 nodes, $r = 0.25$, $E_2$.

These sub-peaks are due to graphs similar to $G_k$, but for which the $k$ nodes at second row do not attach to those in first row so regularly. The number of energy levels around the peak associated to $G_k$ is related to the number of graphs on $k$ nodes (more precisely, to the number of forests, i.e. disjoint union of trees, on $k$ leaves). This means that when $k$ grows, these peaks get wider, and eventually
they become close enough to seem oscillations. Such a phenomenon can be seen in Figure 12.

![Graph](image)

**Figure 12.** Sub-peaks become close. KR, 2000 nodes, \( r = 0.20, E_2 \).

Since, for \( k \ll d \),

\[
\ln(e_k) - \ln(e_{k+1}) \sim \frac{3}{d - k - 2} \sim \frac{3}{d - 2},
\]

the peaks can be considered a log periodic structure. The log periodicity of oscillations was related to the existence of a discrete scale invariance (as opposed to a continuous one), which is due to the existence of preferred scaling ratios. For self similar fractals like the triadic Cantor set or the Sierpinski gasket, the existence of complex fractal dimensions were proposed, since the imaginary part leads to log periodic corrections of the scaling.

In our setting, the existence of preferred scaling ratios correspond to the existence of different hierarchies in the distribution of node degrees in a given graph. That is, the degrees are not evenly distributed, but lacunary.

We wish to stress that this fact was obtained independently for the degree distribution of hierarchical scale free graphs in [21], and some models of graph generation were proposed. Here, the energy shows that those kind of networks are obtained also from classical models of graphs generation like the BA, KR, or DMS.

3. Conclusions

In this work we introduced the linking energy \( E_2 \), and computed it for several graphs from 50 to 100,000 nodes. We generated the graphs with preferential attachment, edge redirection and attachment to edges models. These models are known as responsible of power law distribution of the nodes degree, and the graphs
generated are considered scale free. We also consider random (Erdős-Rényi) and regular lattice graphs.

The frequency distribution of energies shows log periodic oscillations in the tail for the graphs generated with scale free models, which are associated to complex fractal dimensions and discrete scale invariance. They are not present for random or regular graphs.

There are two interpretations of the terms scale free. In the first one, the range of the nodes degree distribution crosses several scales, instead of being concentrated around the mean number of connections. The second one suggest that the nodes degree distribution presents details at every interval of the range following a power law.

Our results show the presence of discrete scale invariance in high energies graphs: they are scale free in the first sense, although the nodes degree are not evenly distributed between the minimum and maximum values, but clustered in hierarchies. This is a fact observed in small networks gathered from real data. Also, for small energies, the distribution does not show the oscillations and the graphs seems to be scale free in the second sense. Hence, the models considered are able to generate both kind of graphs, and the energy could be used to discriminate between them.

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