Inhomogeneous and anisotropic cosmologies

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1 Introduction

Although the purpose of this workshop is to discuss the structures in the universe, which are inhomogeneous, homogeneous models have been used in considering many of the cosmological issues raised in that discussion, so I have also included in this survey the anisotropic homogeneous models and their implications. Only exact solutions will be covered: other speakers at Pont d’Oye (e.g. Bardeen, Brandenberger, Dunsby and Ellis) gave very full discussions of perturbation theory. Here I continue my previous practice (MacCallum 1979, 1984), by using the mathematical classification of the solutions as an overall scheme of organization; the earlier reviews give additional details and references. (A survey organized by the nature of the applications is to appear in the proceedings of Dennis Sciama’s 65th birthday meeting.) Other useful reviews are: Ryan and Shepley (1975) on homogeneous anisotropic models; Krasinski (1990) on inhomogeneous models; and Verdaguer (1985, 1992) on models of solitonic character.

In section 2 I will consider the spatially-homogeneous but anisotropic models. These are the Bianchi models, in general, the exceptions being the Kantowski-Sachs models with an $S^2 \times R^2$ topology. Such models could be significant in understanding the background in which structure is formed, but they do not themselves model that structure. However, I will include here some remarks about inhomogeneous models which are closely related to calculations done with Bianchi models. Then in section 3 I will consider the inhomogeneous models, which fall into several classes. They can be used both as local models of structure and as possible global models of the background in which structure forms (and are in some cases used for both purposes simultaneously). A final section attempts a synthesis and makes some summarizing remarks.

What is it that a cosmological model should explain? There are the following
main features:

1. Lumpiness, or the clumping of matter. The evidence for this is obvious.
2. Expansion, shown by the Hubble law.
3. Evolution, shown by the radio source counts and more recently by galaxy counts.
4. A hot dense phase, to account for the cosmic microwave background radiation (CMWBR) and the abundances of the chemical elements.
5. Isotropy, shown to a high degree of approximation in various cosmological observations, but especially in the CMWBR.
6. Possibly, homogeneity. (The doubt indicated here will be explained later.)
7. The numerical values of parameters of the universe and its laws, such as the baryon number density, the total density parameter $\Omega$, the entropy per baryon, and the coupling constants
8. (Perhaps) such features as the presence of life.

Originally, the standard big-bang models were the Friedman-Lemaître-Robertson-Walker (FLRW) models characterized as:

1. Isotropic at all points and thus necessarily...
2. Spatially-homogeneous, implying Robertson-Walker geometry.
3. Satisfying Einstein’s field equations
4. At recent times (for about the last $10^{10}$ years) pressureless and thus governed by the Friedman-Lemaître dynamics.
5. At early times, radiation-dominated, giving the Tolman dynamics and a thermal history including the usual account of nucleogenesis and the microwave background.

To this picture, which was the orthodox view from about 1965-80, the last decade has added the following extra orthodoxies:

6. $\Omega = 1$. Thus there is dark matter, for which the Cold Dark Matter model was preferred.
7. Inflation – a period in the early universe where some field effectively mimics a large cosmological constant and so causes a period of rapid expansion long enough to multiply the initial length scale many times.
8. Non-linear clustering on galaxy cluster scales, modelled by the $N$-body simulations which fit correlation functions based on observations.

and also added, as alternatives, such concepts as cosmic strings, GUTs or TOEs\footnote{Why so anatomical?} and so on.

The standard model has some clear successes: it certainly fits the Hubble law, the
source count evolutions (in principle if not in detail), the cosmic microwave spectrum, the chemical abundances, the measured isotropies, and the assumption of homogeneity. Perhaps its greatest success was the prediction that the number of neutrino species should be 3 and could not be more than 4, a prediction now fully borne out by the LEP data.

However, the model still has weaknesses [MacCallum, 1987]. For example, the true clumping of matter on large scales, as shown by the QDOT data [Saunders et al., 1991] and the angular correlation functions of galaxies [Maddox et al., 1990], is too strong for the standard cold dark matter account. The uniformity of the Hubble flow is under question from the work of the “Seven Samurai” [Lynden-Bell et al., 1988] and others. The question of the true value of $\Omega$ has been re-opened, partly because theory has shown that inflation does not uniquely predict $\Omega = 1$ (cf. Ellis’ talk at Pont d’Oye) and partly because observations give somewhat variant values. Some authors have pointed out that our knowledge of the physics valid at nucleogenesis and before is still somewhat uncertain, and that we should thus retain some agnosticism towards our account of those early times.

Finally, we should recognize that our belief in homogeneity on a large scale has very poor observational support. We have data from our past light cone (and those of earlier human astronomers) and from geological records [Hoyle, 1962]. Studying spatial homogeneity requires us to know about conditions at great distances at the present time, whereas what we can observe at great distances is what happened a long time ago, so to test homogeneity we have to understand the evolution both of the universe’s geometry and of its matter content. Thus we cannot test homogeneity, only check that it is consistent with the data and our understanding of the theory. The general belief in homogeneity is indeed like the zeal of the convert, since until the 1950s, when Baade revised the distance scale, the accepted distances and sizes of galaxies were not consistent with homogeneity.

These comments, however, are not enough to justify examination of other models. Why do we do that? The basic reason is to study situations where the FLRW models, even with linearized perturbations, may not be adequate. Three types of situation come to mind: the fully non-linear modelling of local processes; exploration of the uniqueness of features of the FLRW models; and tests of the viability of non-FLRW models. The uniqueness referred to here may lie in characteristics thought to be peculiar to the standard model; in attempted proofs that no model universe could

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2 These discoveries made it possible for disagreement with the 1980s dogmatism on such matters to at last be listened to.

3 Local measures of homogeneity merely tell us that the spatial gradients of cosmic quantities are not too strong near us.
be anisotropic or inhomogeneous, by proving that any strong departures from the standard model decay away during evolution; or in comparisons with observation, to show that only the standard models fit.

Some defects of the present survey should be noted. One is that the matter content is generally assumed to be a perfect fluid, although this is strictly incompatible with the other assumed physical properties. Attempting to remedy this with some other mathematically convenient equation of state is not an adequate response; one must try to base the description of matter on a realistic model of microscopic physics or thermodynamics, and few have considered such questions [Bradley and Sviestins, 1984, Salvati et al., 1987, Bona and Coll, 1988, Romano and Pavon, 1992].

A second limitation is that we can only explore the mathematically tractable subsets of models⁴ which may be far from representative of all models. To avoid this restriction, we will ultimately have to turn to numerical simulations, including fully three-dimensional variations in the initial data. Some excellent pioneering work has of course been done, e.g. Anninos et al. (1991b), but capabilities are still limited (for example Matzner (1991) could only use a space grid of $31^3$ points and 256 time steps). Moreover, before one can rely on numerical simulations one needs to prove some structural stability results to guarantee that the numerical and exact answers will correspond.

As a final limitation, in giving this review I only had time to mention and discuss some selected papers and issues, not survey the whole vast field. For his mammoth survey of all inhomogeneous cosmological models which contain, as a limiting case, the FLRW models, Krasinski now has read about 1900 papers (as reported at the GR13 conference in 1992)⁵. Thus the bibliography is at best a representative selection from many worthy and interesting papers, and authors whose work is unkindly omitted may quite reasonably feel it is unrepresentative. In particular, I have not attempted to cover the higher-dimensional models discussed by Demaret and others.

2 **Spatially-homogeneous anisotropic models**

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⁴Kramer et al. (1980) provides a detailed survey of those classes of relativistic spacetimes where the Einstein field equations are sufficiently tractable to be exactly solved.

⁵The survey is not yet complete and remains to be published, but interim reports have appeared in some places, e.g. Krasinski (1990).
2.1 Metrics and field equations

As already mentioned, this class consists of the Bianchi and Kantowski-Sachs models. They have the advantage that the Einstein equations reduce to a system of ordinary differential equations, enabling the use of techniques from dynamical systems theory, and there is thus again a vast literature, too big to fully survey here.

The Bianchi models can be defined as spacetimes with metrics

\[ ds^2 = -dt^2 + g_{\alpha\beta}(t) (e^\alpha_\mu dx^\mu)(e^\beta_\nu dx^\nu) \]

where the corresponding basis vectors \( \{ e_\alpha \} \) obey

\[ [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma \]

in which the C’s are the structure constants of the relevant symmetry group. The different Bianchi-Behr types I-IX are then defined (see e.g. Kramer et al. (1980)) by algebraic classification of these sets of structure constants.

The Kantowski-Sachs metric is

\[ ds^2 = -dt^2 + a^2(t) dx^2 + b^2(t)(d\theta^2 + \sin^2 \theta d\phi^2). \]

(The other metric given in the original paper of Kantowski and Sachs was in fact a Bianchi metric, as pointed out by Ellis.)

The adoption of methods from the theory of dynamical systems has considerably advanced the studies of the behaviour of Bianchi models, beginning in the early 70s with the discussion of phase portraits for special cases [Collins, 1971]. Subsequently, more general cases were discussed using a compactified phase space. In the last decade these methods have been coupled with the parametrization of the Bianchi models using automorphism group variables [Collins and Hawking, 1973, Harvey, 1979, Jantzen, 1979, Siklos, 1980, Roque and Ellis, 1983, Jaklitsch, 1987].

The automorphism group can be briefly described as follows. Take a transformation

\[ \hat{e}^\alpha = M^\alpha_\beta e^\beta, \]

This is an automorphism of the symmetry group if the \( \{ \hat{e}_\alpha \} \) obey the same commutation relations as the \( \{ e_\alpha \} \). The matrices \( M \) are time-dependent and are chosen so that the new metric coefficients \( \hat{g}_{\alpha\beta} \) take some convenient form, for example, become diagonal. The real dynamics is in these metric coefficients. This idea is
present in earlier treatments which grew from Misner’s methods for the Mixmaster case [Ryan and Shepley, 1975] but unfortunately the type IX case was highly misleading in that for Bianchi IX (and no others except Bianchi I) the rotation group is an automorphism group.

The compactification of phase space, introduced for general cases by S.P. Novikov and Bogoyavleneskii (see Bogoyavleneskii (1985)) entailed the normalization of configuration variables to lie within some bounded region, which was then exploited by (a) finding Lyapunov functions, driving the system near the boundaries of the phase space and (b) using analyticity, together with the behaviour of critical points and separatrices, to derive the asymptotic behaviour.

Three main groups have developed these treatments: Bogoyavleneskii and his colleagues (op. cit); Jantzen, Rosquist and collaborators (e.g. Jantzen (1984), Rosquist et al. (1990)) who have coupled the automorphism variables with Hamiltonian treatments in a powerful formalism; and Wainwright and colleagues (e.g. Wainwright and Hsu (1989)) who have used a different, and in some respects simpler, set of automorphism variables, which are well-suited for studying asymptotic behaviour because their limiting cases are physical evolutions of simpler models rather than singular behaviours. Similar ideas can be used for the Kantowski-Sachs models too. As well as qualitative results, some of them described below, these methods have enabled new exact solutions to be found, and some general statements about the occurrence of these solutions to be made.

Many of the geometrical properties of Bianchi cosmologies can be carried over to cases where the 3-dimensional symmetry group (which is still classifiable by Bianchi type) acts on timelike surfaces. A number of authors have considered such metrics, for example Harness (1982) and myself and Siklos (1992). Although of less interest, since they do not evolve in time, than the usual Bianchi models, some of these models reappear as (spatially) inhomogeneous static or stationary models below.

Since the present-day universe is not so anisotropic that we can readily detect its shear and vorticity, the Bianchi models can be relevant to cosmology only as models of asymptotic behaviour, in the early or late universe, or over long time scales, such as the time since the “last scattering”. They have also been used, in these contexts, as approximations in genuinely inhomogeneous universes, and one has to be careful to distinguish the approximate and exact uses.
2.2 Asymptotic behaviour: the far past and future

The earliest use of anisotropic cosmological models to study a real cosmological problem was the investigation by Lemaître (1933) of the occurrence of singularities in Bianchi type I models. The objective was to explore whether the big-bang which arose in FLRW models was simply a consequence of the assumed symmetry: it was of course found not to be.

One can argue that classical cosmologies are irrelevant before the Planck time, but until a theory of quantum gravity is established and experimentally verified (if indeed that will ever be possible) there will be room for discussions of the behaviour of classical models near their singularities.

In the late 1950s and early 60s Lifshitz and Khalatnikov and their collaborators showed (a) that singularities in synchronous coordinates in inhomogeneous cosmologies were in general ‘fictitious’ and (b) that a special subclass gave real curvature singularities, with an asymptotic behaviour like that of the Kasner (vacuum Bianchi I) cosmology [Lifshitz and Khalatnikov, 1963]. From these facts they (wrongly) inferred that general solutions did not have singularities. This contradicted the later singularity theorems (for which see Hawking and Ellis (1973)), a disagreement which led to the belief that there were errors in LK’s arguments. They themselves, in collaboration with Belinskii, and independently Misner, showed that Bianchi IX models gave a more complicated, oscillatory, behaviour than had been discussed in the earlier work, and Misner christened this the ‘Mixmaster’ universe after a brand of food mixer. The broad picture of the roles of the Kasner-like and oscillatory behaviours has been borne out by the more rigorous studies by the methods described in the previous section. There is also an interesting and as yet incompletely explored result that after the oscillatory phase many models approximate one of a few particular power-law (self-similar) solutions [Bogoyavlenskii, 1985].

The detailed behaviour of the Mixmaster model has been the subject of still-continuing investigations: some authors argue that the evolution shows ergodic and chaotic properties, while others have pointed out that the conclusions depend crucially on the choice of time variable [Barrow, 1982, Burd et al., 1990, Berger, 1991]. Numerical investigations are tricky because of the required dynamic range if one is to study an adequately large time-interval, and the difficulties of integrating chaotic systems.

The extension of these ideas to the inhomogeneous case, by Belinskii, Lifshitz and Khalatnikov, has been even more controversial, though prompting a smaller
literature. It was strongly attacked by Barrow and Tipler (1979) on a number of technical grounds, but one can take the view that these were not as damaging to the case as Barrow and Tipler suggested [Belinskii et al., 1980, MacCallum, 1982]. Indeed the ‘velocity-dominated’ class whose singularities are like the Kasner cosmology have been more rigorously characterized and the results justified [Eardley et al., 1971, Holmes et al., 1990]. Sadly this does not settle the more general question, and attempts to handle the whole argument on a completely rigorous footing have so far failed.

General results about singularity types have been proved. The ‘locally extendible’ singularities, in which the region around any geodesic encountering the singularity can be extended beyond the singular point, can only exist under strong restrictions [Clarke, 1976], while the ‘whimper’ singularities [King and Ellis, 1973], in which curvature invariants remain bounded while curvature components in some frames blow up, have been shown to be non-generic and unstable [Siklos, 1978]. Examples of these special cases were found among Bianchi models, and both homogeneous and inhomogeneous cosmologies have been used as examples or counter-examples in the debate.

A further stimulus to the study of singularities was provided by Penrose’s conjecture that gravitational entropy should be low at the start of the universe and this would correspond to a state of small or zero Weyl tensor [Penrose, 1979, Tod, 1992].

Many authors have also considered the far future evolution (or, in closed models, the question of recollapse, whose necessity in Bianchi IX models lacked a rigorous proof until recently [Lin and Wald, 1991]). From various works [MacCallum, 1973, Collins and Hawking, 1973, Barrow and Tipler, 1978] one finds that the homogeneous but anisotropic models do not in general settle down to an FLRW-like behaviour but typically generate shears of the order of 25% of their expansion rates; see also [Uggla et al., 1991]. From the dynamical systems treatments, it is found that certain exact solutions (which in general have self-similarity in time) act as attractors of the dynamical systems in the future [Wainwright and Hsu, 1989]. (All such exact solutions are known: see Hsu and Wainwright (1986) and Jantzen and Rosquist (1986).)

This last touches on an interesting question about our account of the evolution of the universe: is it structurally stable, or would small changes in the theory of the model parameters change the behaviour grossly? Several instances of the latter phenomenon, ‘fragility’, have recently been explored by Tavakol, in collaboration with

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6One of them made by Smallwood and myself.
2.3 Long time effects: the cosmic microwave background

To test the significance of the observed isotropy of the CMWBR, many people in the 1960s and 70s computed the angular distribution of the CMWBR temperature in Bianchi models (e.g. Thorne (1967), Novikov (1968), Collins and Hawking (1972), and Barrow et al. (1983)). These calculations allow limits to be put on small deviations from isotropy from observation, and also enabled, for example, the prediction of ‘hot spots’ in the CMWBR in certain Bianchi models, which could in principle be searched for, if there were a quadrupole component, as there is in the COBE data (though perhaps not for this reason), to see if the quadrupole verifies one of those models.

Similar calculations, by fewer people, considered the polarization [Rees, 1968, Anile, 1974, Tolman and Matzner, 1984] and spectrum [Rees, 1968, Rasband, 1971]. More recently still, work has been carried out on the microwave background in some inhomogeneous models [Saéz and Arnau, 1990]. It has been shown that pure rotation (without shear) is not ruled out by the CMWBR [Obukhov, 1992], but this result may be irrelevant to the real universe where shear is essential to non-trivial perturbations [Goode, 1983, Dunsby, 1992]; in any case shearfree models in general relativity are a very restricted class [Ellis, 1967].

An example of the problem with assuming a perfect fluid is that in Bianchi models, as soon as matter is in motion relative to the homogeneous surfaces (i.e. becomes ‘tilted’) it experiences density gradients which should lead to heat fluxes [Bradley and Sviestins, 1984]: similar remarks apply to other simple models. Such models have recently been used to fit the observed dipole anisotropy in the CMWBR [Turner, 1992], though other explanations seem to me more credible.

2.4 Early universe effects

Galaxy formation in anisotropic models has been studied to see if by this means one could overcome the well-known difficulties in FLRW models (without inflation), but with negative results [Perko et al., 1972].

A similar investigation was to see if the helium abundance, as known in the 1960s, could be fitted better by anisotropic cosmologies than by FLRW models, which at the
time appeared to give discrepancies. The reason this might happen is that anisotropy speeds up the evolution between the time when deuterium can first form, because it is no longer dissociated by the photons, and the time when neutrons and protons are sufficiently sparse that they no longer find each other to combine. Hawking and Tayler (1966) were pioneers in this effort, which continued into the 1980s but suffered some mutations in its intention.

First the argument was reversed, and the good agreement of FLRW predictions with data was used to limit the anisotropy during the nucleogenesis period (see e.g. Barrow (1976), Olson (1977)). Later still these limits were relaxed as a result of considering the effects of anisotropic neutrino distribution functions [Rothman and Matzner, 1982] and other effects on reaction rates [Juszkiewicz et al., 1983]. It has even been shown [Matravers et al., 1984, Barrow, 1984] that strongly anisotropic Bianchi models, not obeying the limits deduced from perturbed FLRW models, can produce correct element abundances, though they may violate other constraints [Matravers and Madsen, 1983, Matravers et al., 1985].

3 Inhomogeneous cosmologies

3.1 Self-similar models

Some of the self-similar models, especially those relevant to modelling structure formation, are reviewed in much greater detail in a complementary talk by Carr, so I will give here only a few details of other cases.

The geometry of the self-similar models first considered in cosmology is somewhat like that of the Bianchi models, except that one of the isometries is replaced by a homothety, that is to say by a vector field satisfying

$$\xi_{(a;b)} = 2kg_{ab}$$

where $k$ is a constant. This class, where the homothety and two independent symmetries act, was considered by a number of authors [Eardley, 1974, Luminet, 1978, Wu, 1981, Hanquin and Demaret, 1984], and many details, parallel in nature to those covered by the detailed studies of Bianchi models, can be found in those works.

Due to Western confusion over Chinese name order, Wu Zhong-Chao is sometimes incorrectly referred to as W.Z. Chao rather than Wu, Z-C.
More recently Wainwright, Hewitt and colleagues \cite{Hewitt et al., 1988, Hewitt and Wainwright, 1990, Hewitt et al., 1991} have considered cases where the homothety has a timelike rather than spacelike generator. Like the former class, these solutions are in fact special cases of “$G_2$ solutions” (discussed below) with perfect fluid matter content. It is found that the spatial variations can be periodic or monotone; the asymptotic behaviour may be a vacuum or spatially homogeneous model; the periodic cases are unstable to increases in the anisotropy; and the singularities can be acceleration-dominated.

### 3.2 Spherically symmetric models

These have a metric

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where $\nu$, $\lambda$ and $R$ are functions of $r$ and $t$. The precise functional forms in the metric depend on the choice of coordinates and the additional restrictions assumed. It should be noted that there are so few undetermined functions that a sufficiently-complicated energy-momentum will fit a totally arbitrary choice of the remaining functions: in my view this should not be regarded as a solution, since no equation is actually solved!

Some important subcases have been studied, notably:

1. The dust (pressureless perfect fluid) cases, originally studied by Lemaître, but usually named after Tolman and Bondi;
2. McVittie’s 1933 solution representing a black hole in an FLRW universe;
3. The “Swiss cheese” model constructed by matching a Schwarzschild vacuum solution inside some sphere to an exterior FLRW universe;
4. Shearfree fluid solutions \cite{Wyman, 1940, Kustaanheimo and Qvist, 1948, Stephani, 1983, McVittie, 1984};
5. Self-similar solutions, discussed in Carr’s contribution at Pont d’Oye.

Spherically symmetric models, especially Tolman-Bondi, have often been used to model galactic scale inhomogeneities, in various contexts. Galaxy formation has been studied (e.g. Tolman (1934), Carr and Yahil (1990)): Meszaros (1991) developed a variation on the usual approach by considering the shell-crossings, with the aim of producing “Great Wall” like structures, rather than the collapse to the centre producing a spherical cluster or galaxy. Some authors have used spherically symmetric lumps to estimate departures from the simple theory of the magnitude-redshift relations based on a smoothed out model \footnote{The point is that the beams of light we observe are focussed only by the matter actually inside the beam, not the matter that would be there in a completely uniform model.} (e.g. Dyer (1976), Kantowski (1969a) and
Newman (1979)): note that these works show that the corrections depend on the choice of modelling, since Newman’s results from a McVittie model differ from the ones based on Swiss cheese models. The metrics also give the simplest models of gravitational lenses and have also been used to model the formation of primordial black holes Carr and Hawking, 1974.

On a larger scale, inhomogeneous spherical spacetimes have been used to model clusters of galaxies Kantowski, 1969b, variations in the Hubble flow due to the super-cluster Mavrides, 1977, the evolution of cosmic voids Sato, 1984, Hausman et al., 1983, Bonnor and Chamorro, 1990 and 1991, the observed distribution of galaxies and simple hierarchical models of the universe Bonnor, 1972, Wesson, 1978, Wesson, 1979, Ribeiro, 1992a. Most of this work used Tolman-Bondi models, sometimes with discontinuous density distributions.

Recent work by Ribeiro (1992b), in the course of an attempt to make simple models of fractal cosmologies using Tolman-Bondi metrics, has reminded us of the need to compare data with relativistic models not Newtonian approximations. Taking the Einstein-de Sitter model, and integrating down the geodesics, he plotted the number counts against luminosity distances. At small distances, where a simple interpretation would say the result looks like a uniform density, the graph is irrelevant because the distances are inside the region where the QDOT survey shows things are lumpy Saunders et al., 1991, while at greater redshifts the universe ceases to have a simple power-law relation of density and distance. Thus even Einstein-de Sitter does not look homogeneous!

One must therefore ask in general “do homogeneous models look homogeneous?” Of course, they will if the data is handled with appropriate relativistic corrections, but to achieve such comparisons in general requires the integration of the null geodesic equations in each cosmological model considered, and, as those who have tried it know, even when solving the field equations is simple, solving the geodesic equations may not be.

Many other papers have considered spherically symmetric models, but there is not enough space here to review them all, so I will end by mentioning a jeu d’esprit in which it was shown that in a “Swiss cheese” model, made by joining two FLRW exteriors at the two sides of a Kruskal diagram for the Schwarzschild solution, one can have two universes each of which can receive (but not answer) a signal from the other Sussman, 1983.

9The very detailed modern work interpreting real lenses to study various properties of individual sources and the cosmos mostly uses linearized approximations.
3.3 Cylindrically symmetric and plane symmetric (static) models

These have been used to model cosmic strings and domain walls. One should note that locally the metrics may be the same for these two cases, the difference lying in whether there is or is not a Killing vector whose integral curves have spatial topology $S^1$. Plane symmetric metrics should have a rotational symmetry in the plane, but to add to the possible confusions some authors use the term “plane” for solutions without such a rotation: the term “planar” would be a useful alternative.

The usual (though not the only) form for the cylindrically symmetric metrics is

$$ds^2 = -T^2 dt^2 + R^2 dr^2 + Z^2 dz^2 + 2W dz d\phi + \Phi^2 d\phi^2$$

where $T$, $R$, $Z$, $W$ and $\Phi$ depend on $r$ (and, in the non-stationary case, $t$) and $\phi$ is periodic, and, for the plane symmetric case,

$$ds^2 = -T^2 dt^2 + R^2 dr^2 + X^2(dx^2 + x^2 d\phi^2)$$

where $T$, $R$ and $X$ are functions of $r$ (and perhaps $t$). The static cases all belong in Harness’s (1982) general class.

Plane symmetric models, usually static, solutions have been used to model domain walls [Vilenkin, 1983, Ipser and Sikivie, 1984, Goetz, 1990, Wang, 1991]. The cylindrically symmetric models have been used for cosmic strings, starting with the work of Gott, Hiscock and Linet in 1985. These studies have usually been done with static strings$^{11}$, and have considered such questions as the effects on classical and quantum fields in the neighbourhood of the string.

3.4 $G_2$ cosmologies

I use the above title as a general name for all cosmological metrics with two spacelike Killing vectors (and hence two essential variables). The cylindrical and plane metrics, and many of the Bianchi metrics, are special cases of $G_2$ cosmologies.

$G_2$ cosmologies admit a number of specializations, such as:

$^{10}$Note that since the sources usually have a boost symmetry in the timelike surface giving the wall, corresponding solutions have timelike surfaces admitting the $(2+1)$-dimensional de Sitter group.

$^{11}$There is some controversy about whether these can correctly represent strings embedded in an expanding universe [Clarke et al., 1990].
[1] the Killing vectors commute;
[2] the orbits of the $G_2$ are orthogonal to another set of 2-dimensional surfaces $V_2$;
[3] the Killing vectors individually are hypersurface-orthogonal;
[4] the matter content satisfies conditions allowing generating techniques.

Among the classes of metrics covered here are colliding wave models, cosmologies with superposed solitonic waves, and what I call “corrugated” cosmologies with spatial irregularities dependent on only one variable.

The metrics where the Killing vectors do not commute have been very little studied: it is known they cannot admit orthogonal $V_2$ if the fluid flows orthogonal to the group surfaces (unless they have an extra symmetry) and that if the fluid is thus orthogonal it is non-rotating [Bugalho, 1987; van den Bergh, 1988]. So we now take only cases where the Killing vectors commute.

The case without orthogonal $V_2$ has also been comparatively little studied, but recently some exact solutions which have one hypersurface-orthogonal Killing vector and in which the metric coefficients are separable, have been derived and studied [van den Bergh et al., 1991; van den Bergh, 1991]. One class consists of metrics of the form

$$ds^2 = e^{2(K+k)}(-dt^2 + dx^2) + e^{2(S+s)}[e^{F+f}dy]^2 + (e^{-(F+f)}\theta)^2$$

where: $K$, $S$ and $F$ depend on $t$; $k$, $s$ and $f$ depend on $x$; $\theta = dz + 2\omega dx$; and $\omega$ depends on $t$ and $x$. Some perfect fluid solutions are known explicitly but usually turned out to be self-similar, with big-bang singularities of the usual types. The “stiff fluid” ($\gamma = 2$) is a special case, discussed in detail by van den Bergh (1991). Most of the solutions have singularities at finite spatial distances or can be regarded as inhomogeneous perturbations of the Bianchi $VI_{-1}$ models.

The cases with orthogonal $V_2$ were classified by Wainwright (1979,1981), and a number of specific examples are known (e.g. Wainwright and Goode (1980); Kramer (1984)). A recent solution found by Senovilla (1990) attracted much attention, because it is non-singular [Chinea et al., 1991], evading the focussing conditions in the singularity theorems by containing matter that is too diffuse: it is closely related to an earlier solution of Feinstein and Senovilla (1989). The metrics investigated in this class generally have Kasner-like behaviour near the singularity (though some have a plane-wave asymptotic behaviour [Wainwright, 1983]) and become self-similar or spatially homogeneous in the far future.

Finally we come to the most-studied class, those where the generating techniques
are applicable. The matter content must have characteristic propagation speeds equal to the speed of light, so attention is restricted to vacuum, electromagnetic, neutrino and “stiff fluid” (or equivalently, massless scalar field with a timelike gradient) cases. However, FLRW fluid solutions can be obtained by using the same methods in higher-dimensions and using dimensional reduction. There are useful reviews covering the cosmological, cylindrical, and colliding wave sub-classes [Carmeli et al., 1981, Verdaguer, 1985, Verdaguer, 1992, Ferrari, 1990, Griffiths, 1991]. The metrics can be written in a form covering also the related stationary axisymmetric metrics as

\[ ds^2 = \epsilon f_{AB} dx^A dx^B + \delta \epsilon^{2\gamma} ((dx^1)^2 - \epsilon(dx^3)^2)/f \]

where \( A, B \) take values 1, 2 and the values of \( f_{AB} \) can be written as a matrix

\[
\begin{pmatrix}
 f & -f \omega \\
 -f \omega & f \omega^2 + \epsilon (x^3)^2/f
\end{pmatrix}
\]

The case \( \delta = -\epsilon = 1 \) gives the stationary axisymmetric metrics, the case \( \delta = \epsilon = 1 \) the cylindrical cases and \( \epsilon = -\delta = 1 \) the cosmological cases. Physically these classes differ in the timelike or spacelike nature of the surfaces of symmetry and the nature of the gradient of the determinant of the metric in those surfaces.

Some studies have focussed on the mathematics, showing how known vacuum solutions can be related by solution-generating techniques [Kitchingham, 1984], while others have concentrated on the physics of the evolution and interpretative issues. The generating techniques use one or more of a battery of related methods: Bäcklund transformation, inverse scattering, soliton solutions and so on. One interesting question that has arisen from recent work is whether solitons in relativity do or do not exhibit non-linear interactions: Boyd et al. (1991), in investigations of solitons in a Bianchi I background, found no non-linearity, while Belinskii (1991) has claimed there is a non-linear effect (see also Verdaguer (1992)).

The applications in cosmology, which have generated far too many papers to list them all here, have been pursued by a number of groups, notably by Carmeli, Charach and Feinstein, by Verdaguer and colleagues, by Gleiser, Pullin and colleagues, and Belinski, Curir and Francaviglia, with important contributions by Ibanez, Kitchingham, Yurtsever, Ferrari, Chandrasekhar and Xanthopoulos, Letelier, Tsoubelis and Wang and many others.

One use of these metrics is to provide models for universes with gravitational waves. It emerges that the models studied are typically Kasner-like near the singularity (agreeing with the LK arguments), and settle down to self-similar or spatially homogeneous models with superposed high-frequency gravitational waves at
late times [Adams et al., 1982, Carmeli and Feinstein, 1984, Feinstein, 1988]. Another use is to model straight cosmic strings in interaction with gravitational or other waves (e.g. Economou and Tsoubelis (1988), Verdaguer (1992)). One can also examine the gravitational analogue of Faraday rotation [Piran and Saifer, 1985, Tomimatsu, 1989, Wang, 1991a] and there are even solutions whose exact behaviour agrees precisely with the linearized perturbation calculations for FLRW universes [Carmeli et al., 1983].

3.5 Other models

Solutions with less symmetry than those above have been little explored. Following Krasinski one can divide the cases considered into a number of classes (in which I only mention a few important special subcases).

1. The Szekeres-Szafron family (also independently found by Tomimura). These have in general no symmetries, and contain an irrotational non-accelerating fluid. Tolman-Bondi universes are included in this class, as are the Kantowski-Sachs metrics; some generalizations are known, such as the rotating inhomogeneous model due to Stephani (1987). Like the $G_2$ solutions mentioned earlier, some Szekeres models obey exactly the linearized perturbation equations for the FLRW models [Goode and Wainwright, 1982].

2. Shearfree irrotational metrics [Barnes, 1973] which include the conformally flat fluids (Stephani 1967a, 1967b) and McVittie’s spherically symmetric metric. Bona and Coll (1988) have recently argued that the Stephani cases can only have acceptable thermodynamics if the metrics admit three Killing vectors.

3. The Vaidya-Patel-Koppar family, which represent an FLRW model containing a “Kerr” solution using null radiation and an electromagnetic field. The physical significance of these metrics is dubious.

4. Some other special cases such as Oleson’s Petrov type N fluid solutions.

4 Syntheses and conclusions: what have we learnt?

Here I collect up the outcome of the work surveyed above, without repeating all details, and review some relevant extra references, but many interesting aspects are
still omitted. For example, the literature covers such issues as models for interactions between different forms of matter, and generation of gravitational radiation.

### 4.1 The classical singularity

The occurrence of a “big-bang” in FLRW models is not just a consequence of the high symmetry. Its nature in general models is probably a curvature singularity, and the best guess so far is that the asymptotic behaviour would be oscillatory but other possibilities exist. The Penrose conjecture, which would be a selection principle on models, has been particularly developed, using exact solutions as examples, by Wainwright and Goode, who have given a precise definition of the notion of an ‘isotropic singularity’ [Goode et al., 1992, Tod, 1992].

### 4.2 Occurrence of inflation

In “old” inflation in Bianchi I models, inflation need not occur [Barrow and Turner, 1981], but in “new” inflation it was predicted [Steigman and Turner, 1983]. In a large class of chaotic inflation models it is also expected [Moss and Sahni, 1986]. Further papers by a number of authors have suggested that inflation need not always occur (see Rothman and Ellis (1986) for some criticisms of earlier papers).

### 4.3 Removal of anisotropy and inhomogeneity

Three means of smoothing the universe have been explored over the years: the use of viscosity in the early universe; the removal of horizons in the Mixmaster universes; and removal during inflation. The first two of these ingenious suggestions are due to Misner.

Attempts to smooth out anisotropies or inhomogeneities by any process obeying deterministic sets of differential equations satisfying Lipschitz-type conditions are doomed to fail, as was first pointed out by Collins and Stewart (1971) in the context of viscous mechanisms. The argument is simply that one can impose any desired amount of anisotropy or inhomogeneity now and evolve the system backwards in time to reach initial conditions at some earlier time whose evolution produces the chosen present-day values.
The same argument also holds for inflationary models. Inflation in itself, without the use of singular equations or otherwise indeterminate evolutions, cannot wholly explain present isotropy or homogeneity, although it may reduce deviations by large factors [Sirousse-Zia, 1982, Wald, 1983, Moss and Sahni, 1986, Futamase et al., 1989]. Although one can argue that anisotropy tends to prolong inflation, this does not remove the difficulty.

Since 1981 I have been arguing a heretical view about one of the grounds for inflation, namely the ‘flatness problem’, on the grounds that the formulation of this problem makes an implicit and unjustified assumption that the \emph{a priori} probabilities of values of Ω is spread over some range sufficient to make the observed closeness to 1 implausible. Unless one can justify the \emph{a priori} distribution, there is no implausibility[13] [Ellis, 1991].

However, if one accepts there is a flatness problem, then there is also an isotropy problem, since at least for some probability distributions on the inhomogeneity and anisotropy the models would not match observation. Protagonists of inflation cannot have it both ways. Perhaps, if one does not want to just say “well, that’s how the universe was born”, one has to explain the observed smoothness by appeal to the ‘speculative era’, as Salam (1990) called it, i.e. by appeal to one’s favourite theory of quantum gravity.

If inflation works well at early times, then inflation actually enhances the chance of an anisotropic model fitting the data, and since the property of anisotropy cannot be totally destroyed in general (because it can be coded into geometric invariants which cannot become zero by any classical evolution) the anisotropy could reassert itself in the future! (This of course will not happen if a non-zero Λ term persists, as the “cosmic no-hair” theorems show [Wald, 1983, Morrow-Jones and Witt, 1988].)

The Mixmaster horizon removal suggestion was shown to fail when more detailed computations than Misner’s were made [MacCallum, 1971, Doroshkevich et al., 1971, Chitre, 1972]. Incidentally, one may note that inflation does not solve the original form of the ‘horizon problem’, which was to account completely for the similarity of points on the last scattering surface governed by different subsets of the initial data surface. Inflation leads to a large overlap between these initial data subsets, but not to their exact coincidence. Thus one still has to assume that the non-overlap regions are not too different. While this may give a more plausible model, it does not remove

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[13] One can however argue that only Ω = 1 is plausible, on the grounds that otherwise the quantum theory before the Planck time would have to fix a length-scale parameter much larger than any quantum scale, only the Ω = 1 case being scale-free. I am indebted to Gary Gibbons for this remark.
the need for assumptions on the initial data.

4.4 The exit from inflation

A further interesting application of non-standard models has come in a recent attempt to answer the question posed by Ellis and Rothman (unpublished) of how the universe can choose a uniform reference frame at the exit from inflation when a truly de Sitter model has no preferred time axis. Anninos et al. (1991a) have shown by taking an inflating Bianchi V model that the answer is that the memory is retained and the universe is never really de Sitter.

4.5 The helium abundance

This is still used to set limits on anisotropy during the nucleosynthesis phase.

4.6 The cosmic microwave background

Observations limit the integrated effect since “last scattering”: note this can in principle permit large but compensating excursions from FLRW. One intriguing possibility raised by Ellis et al. (1978) is that the observed sphere on the last scattering surface could lie on a timelike (hyper)cylinder of homogeneity in a static spherically symmetric model. This makes the CMWBR isotropic at all points not only at the centre, and although it cannot fit all the other data, the model shows how careful one must be, in drawing conclusions about the geometry of the universe from observations, not to assume the result one wishes to prove.

There is a theorem by Ehlers, Geren and Sachs (1968) showing that if a congruence of geodesically-moving observers all observe an isotropic distribution of collisionless gas the metric must be Robertson-Walker. Treciokas and Ellis (1971) have investigated the related problem with collisions. Recently Ferrando et al. (1992) have investigated inhomogeneous models where an isotropic gas distribution is possible. These studies throw into focus a conjecture which is usually assumed, namely that an approximately isotropic gas distribution, at all points, would imply an approximately Robertson-Walker metric. (It is this assumption which underlies the arguments normally used in analysing data like that from COBE to get detailed information on
4.7 The far future

Anisotropy will in general become apparent, if it is present and if the cosmological constant $\Lambda$ is zero: isotropy is not stable. Inhomogeneities may become significant even faster.

4.8 The origin of structure

None of the work discussed above accounts for the origin of structure, although it offers suitable descriptions for the evolution, or the background spacetime in which the evolution takes place. I feel it does, however, indicate strongly that the true origin lies in the perhaps unknowable situation in the Speculative Era, and the resulting initial conditions for the later evolution.

4.9 A genuinely anisotropic and inhomogeneous universe?

While I do not think one can give a definitive answer to this question, I would personally be very surprised if anisotropic but homogeneous models turned out to be anything more than useful examples. However, the status of fully inhomogeneous models is less clear.

One argument is that while the standard models may be good approximations at present, they are unstable to perturbations both in the past and the future. The possible alternative pasts are quite varied, as shown above, even without considering quantum gravity. Similarly, the universe may not be isotropic in the far future. Moreover, we have no knowledge of conditions outside our past null cone, where some inflationary scenarios would predict bubbles of differing FLRW universes, and perhaps domain walls and so on.

If the universe were FLRW, or very close to that, this means it is in a region, in the space of all possible models, which almost any reasonable measure is likely to say has very low probability (though note the earlier remarks on assignments of probabilities). One can only evaluate, and perhaps explain, this feature by considering non-FLRW
models. It is noteworthy that many of the “problems” inflation claims to tackle are not problems if the universe simply is always FLRW. Hence, as already argued above, one has a deep problem in explaining why the universe is in the unlikely FLRW state if one accepts the arguments about probabilities current in work on inflation.

Suppose we speculated that the real universe is significantly inhomogeneous at the present epoch (at a level beyond that arising from perturbations in FLRW). What would the objections be? There are only two relevant pieces of data, as far as I can see. One is the deep galaxy counts made by the automatic plate measuring machines, which are claimed to restrict variations to a few percent, and the other is the isotropy of the CMWBR. Although the latter is a good test for large lumps in a basically FLRW universe, one has to question (recalling the results of Ellis et al. (1978)) whether it really implies homogeneity.

References

[Adams et al., 1982] Adams, P.J., Hellings, R.W., Zimmerman, R.L., Farshoosh, H., Levine, D.I., and Zeldich, S. (1982). Inhomogeneous cosmology: gravitational radiation in Bianchi backgrounds. Astrophys. J. 253, 1.

[Anile, 1974] Anile, A.M. (1974). Anisotropic expansion of the universe and the anisotropy and linear polarization of the cosmic microwave background. Astrophys. Sp. Sci. 29, 415.

[Anninos et al., 1991a] Anninos, P., Matzner, R.A., Rothman, T., and Ryan, M.P. (1991a). How does inflation isotropize the universe? Phys. Rev. D 43, 3821–3832.

[Anninos et al., 1991b] Anninos, P., Matzner, R.A., Tuluie, R., and Centrella, J. (1991b). Anisotropies of the cosmic background radiation in a “hot” dark matter universe. Astrophys. J. 382, 71–78.

[Barnes, 1973] Barnes, A. (1973). On shearfree normal flows of a perfect fluid. Gen. Rel. Grav. 4, 105.

[Barrow, 1976] Barrow, J.D. (1976). Light elements and the isotropy of the universe. Mon. Not. R.A.S. 175, 359.

[Barrow, 1982] Barrow, J.D. (1982). Chaotic behaviour in general relativity. Phys. Repts. 85, 1.

[Barrow, 1984] Barrow, J.D. (1984). Helium formation in cosmologies with anisotropic curvature. Mon. Not. R.A.S. 211, 221.
Barrow et al., 1983] Barrow, J.D., Juszkiewicz, R., and Sonoda, D. (1983). Reply to “The effect of “spottiness” in large-scale structure of the microwave background” by V.N. Lukash and I.D. Novikov. Nature 316, 48.

[Barrow and Tipler, 1978] Barrow, J.D. and Tipler, F.J. (1978). Eternity is unstable. Nature 278, 453.

[Barrow and Tipler, 1979] Barrow, J.D. and Tipler, F.J. (1979). An analysis of the generic singularity studies by Belinskii, Lifshitz and Khalatnikov. Phys. Repts. 56, 371.

[Barrow and Turner, 1981] Barrow, J.D. and Turner, M.S. (1981). Inflation in the Universe. Nature 292, 35.

[Belinskii et al., 1980] Belinskii, V.A., Lifshitz, E.M., and Khalatnikov, I.M. (1980). On the problem of the singularities in the general cosmological solution of the Einstein equations. Phys. Lett. A 77, 214.

[Belinsky, 1991] Belinsky, V. (1991). Gravitational breather and topological properties of gravisolitons. Phys. Rev D 44, 3109–3115.

[Berger, 1991] Berger, B.K. (1991). Comments on the calculation of Liapunov exponents for the Mixmaster universe. Gen. Rel. Grav. 23, 1385.

[Bogoyavlenskii, 1985] Bogoyavlenskii, O.I. (1985). Methods of the qualitative theory of dynamical systems in astrophysics and gas dynamics. Springer-Verlag, Berlin & Heidelberg. [Russian original published by Nauka, Moscow, 1980.]

[Bona and Coll, 1988] Bona, C. and Coll, B. (1988). On the Stephani universes. Gen. Rel. Grav. 20, 297–303.

[Bonnor, 1972] Bonnor, W.B. (1972). A non-uniform relativistic cosmological model. Mon. Not. R.A.S. 159, 261.

[Bonnor and Chamorro, 1990 and 1991] Bonnor, W.B. and Chamorro, A. (1990). Models of voids in the expanding universe. Astrophys. J. 361, 21–26.

[Bonnor and Chamorro, 1991] Bonnor, W.B. and Chamorro, A. (1991). Models of voids in the expanding universe II. Astrophys. J. 378, 461–465.

[Boyd et al., 1991] Boyd, P.T., Centrella, J.M., and Klasky, S.A. (1991). Properties of gravitational “solitons”. Phys. Rev. D 43, 379–390.

[Bradley and Sviestins, 1984] Bradley, M. and Sviestins, E. (1984). Some rotating, time-dependent Bianchi VIII cosmologies with heat flow. Gen. Rel. Grav. 16, 1119–1133.
[Bugalho, 1987] Bugalho, M.H. (1987). Orthogonality transitivity and cosmologies with a non-Abelian two-parameter isometry group. *Class. Quant. Grav.*** 4, 1043.

[Burd et al., 1990] Burd, A.B., Buric, N., and Ellis, G.F.R. (1990). A numerical analysis of chaotic behaviour in Bianchi IX models. *Gen. Rel. Grav.* **22**, 349–363.

[Carmeli et al., 1983] Carmeli, M., Charach, C., and Feinstein, A. (1983). Inhomogeneous mixmaster universes: some exact solutions. *Ann. Phys. (N.Y.)* **150**, 392.

[Carmeli et al., 1981] Carmeli, M., Charach, C., and Malin, S. (1981). Survey of cosmological models with gravitational, scalar and electromagnetic waves. *Phys. Repts.* **76**, 79.

[Carmeli and Feinstein, 1984] Carmeli, M. and Feinstein, A. (1984). The cosmological peeling-off property of gravity. *Phys. Lett. A* **103**, 318–320.

[Carr and Hawking, 1974] Carr, B.J. and Hawking, S.W. (1974). Black holes in the early universe. *Mon. Not. R.A.S.* **168**, 399.

[Carr and Yahil, 1990] Carr, B.J. and Yahil, A. (1990). Self-similar perturbations of a Friedmann universe. *Astrophys. J.* **360**, 330–342.

[Chinea et al., 1991] Chinea, F.J., Fernandez-Jambrina, F., and Senovilla, J.M.M. (1991). A singularity-free spacetime. Madrid/Barcelona preprint FT/UCM/16/91.

[Chitre, 1972] Chitre, D.M. (1972). High-frequency sound waves to eliminate a horizon in the Mixmaster universe. *Phys. Rev. D* **6**, 3390–3396.

[Clarke, 1976] Clarke, C.J.S. (1976). Space-time singularities. *Comm. math. phys.* **49**, 17.

[Clarke et al., 1990] Clarke, C.J.S., Ellis, G.F.R., and Vickers, J.A. (1990). The large-scale bending of cosmic strings. *Class. Quant. Grav.* **7**, 1–14.

[Coley and Tavakol, 1992] Coley, A.A. and Tavakol, R.K. (1992). Fragility in cosmology. Preprint, Queen Mary and Westfield College.

[Collins, 1971] Collins, C.B. (1971). More qualitative cosmology. *Comm. math. phys.* **23**, 137.

[Collins and Hawking, 1972] Collins, C.B. and Hawking, S.W. (1972). The rotation and distortion of the universe. *Mon. Not. R.A.S.* **162**, 307.

[Collins and Hawking, 1973] Collins, C.B. and Hawking, S.W. (1973). Why is the universe isotropic? *Astrophys. J.* **180**, 37.
[Collins and Stewart, 1971] Collins, C.B. and Stewart, J.M. (1971). Qualitative cosmology. *Mon. Not. R.A.S.* **153**, 419–434.

[Doroshkevich et al., 1971] Doroshkevich, A.G., Lukash, V.N., and Novikov, I.D. (1971). Impossibility of mixing in a cosmological model of the Bianchi IX type. *Zh. E. T. F.* **60**, 1201. (Translation in *Sov. Phys. – J.E.T.P.* **33**, 649 (1971)).

[Dunsby, 1992] Dunsby, P.K.S. (1992). Perturbations in general relativity and cosmology. Ph.D. thesis, Queen Mary and Westfield College, University of London.

[Dyer, 1976] Dyer, C.C. (1976). The gravitational perturbation of the cosmic background radiation by density concentrations. *Mon. Not. R.A.S.* **175**, 429.

[Eardley, 1974] Eardley, D.M. (1974). Self-similar spacetimes: geometry and dynamics. *Comm. math. phys.* **37**, 287.

[Eardley et al., 1971] Eardley, D.M., Liang, E., and Sachs, R.K. (1971). Velocity-dominated singularities in irrotational dust cosmologies. *J. Math. Phys.* **13**, 99.

[Economou and Tsoubelis, 1988] Economou, A. and Tsoubelis, D. (1988). Rotating cosmic strings and gravitational soliton waves. *Phys. Rev. D* **38**, 498.

[Ehlers et al., 1968] Ehlers, J., Geren, P., and Sachs, R.K. (1968). Isotropic solutions of the Einstein-Liouville equation. *J. Math. Phys.* **9**, 1344.

[Ellis, 1967] Ellis, G.F.R. (1967). Dynamics of pressure-free matter in general relativity. *J. Math. Phys.* **8**, 1171.

[Ellis, 1991] Ellis, G.F.R. (1991). Standard and inflationary cosmologies. In Mann, R. and Wesson, P., editors, *Gravitation: a Banff summer institute*. World Scientific, Singapore.

[Ellis et al., 1978] Ellis, G.F.R., Maartens, R., and Nel, S.D. (1978). The expansion of the universe. *Mon. Not. R.A.S.* **184**, 439.

[Feinstein, 1988] Feinstein, A. (1988). Late-time behaviour of primordial gravitational waves in expanding universe. *Gen. Rel. Grav.* **20**, 183–190.

[Feinstein and Senovilla, 1989] Feinstein, A. and Senovilla, J.M.M. (1989). A new inhomogeneous cosmological perfect fluid solution with $p = \rho/3$. *Class. Quant. Grav.* **6**, L89–91.

[Ferrando et al., 1992] Ferrando, J.J., Morales, J.A., and Portilla, M. (1992). Inhomogeneous space-times admitting isotropic radiation. Preprint, University of Valencia.
[Ferrari, 1990] Ferrari, V. (1990). Colliding waves in general relativity. In N. Ashby, D.F. Bartlett and Wyss, W., editors, *General Relativity and Gravitation, 1989*, pp. 3–20. Cambridge University Press, Cambridge, New York and Melbourne.

[Futamase et al., 1989] Futamase, T., Rothman, T., and Matzner, R. (1989). Behaviour of chaotic inflation in anisotropic cosmologies with nonminimal coupling. *Phys. Rev. D* **39**, 405–411.

[Goetz, 1990] Goetz, G. (1990). Gravitational field of plane symmetric thick domain walls. *J. Math. Phys.* **31**, 2683–2687.

[Goode, 1983] Goode, S.W. (1983). Spatially inhomogeneous cosmologies and their relation with the FRW models. Ph.D. thesis, University of Waterloo.

[Goode et al., 1992] Goode, S.W., Coley, A.A., and Wainwright, J. (1992). The isotropic singularity in cosmology. *Class. Quant. Grav.* **9**, 445–455.

[Goode and Wainwright, 1982] Goode, S.W. and Wainwright, J. (1982). Friedman-like singularities in Szekeres cosmological models. *Mon. Not. R.A.S.* **198**, 83.

[Gott, 1985] Gott, J.R. (1985). Gravitational lensing effects of vacuum strings: exact solutions. *Astrophys. J.* **288**, 422.

[Griffiths, 1991] Griffiths, J.B. (1991). *Colliding plane waves in general relativity*. Oxford mathematical monographs. Oxford University Press, Oxford.

[Hanquin and Demaret, 1984] Hanquin, J.-L. and Demaret, J. (1984). Exact solutions for inhomogeneous generalizations of some vacuum Bianchi models. *Class. Quant. Grav.* **1**, 291.

[Harness, 1982] Harness, R.S. (1982). Spacetimes homogeneous on a timelike hypersurface. *J. Phys. A* **15**, 135.

[Harvey, 1979] Harvey, A.L. (1979). Automorphisms of the Bianchi model Lie groups. *J. Math. Phys.* **20**, 251.

[Hausman et al., 1983] Hausman, M.A., Olson, D.W., and Roth, B.D. (1983). The evolution of voids in the expanding universe. *Astrophys. J.* **270**, 351.

[Hawking and Ellis, 1973] Hawking, S.W. and Ellis, G.F.R. (1973). *The large-scale structure of space-time*. Cambridge University Press, Cambridge.

[Hawking and Tayler, 1966] Hawking, S.W. and Tayler, R.J. (1966). Helium production in an anisotropic big-bang cosmology. *Nature* **209**, 1278.
[Hewitt and Wainwright, 1990] Hewitt, C.G. and Wainwright, J. (1990). Orthogonally transitive $G_2$ cosmologies. *Class. Quant. Grav.* 7, 2295–2316.

[Hewitt et al., 1991] Hewitt, C.G., Wainwright, J., and Glaum, M. (1991). Qualitative analysis of a class of inhomogeneous self-similar cosmological models II. *Class. Quant. Grav.* 8, 1505–1518.

[Hewitt et al., 1988] Hewitt, C.G., Wainwright, J., and Goode, S.W. (1988). Qualitative analysis of a class of inhomogeneous self-similar cosmological models. *Class. Quant. Grav.* 5, 1313–1328.

[Hiscock, 1985] Hiscock, W.A. (1985). Exact gravitational field of a string. *Phys. Rev. D* 31, 3288–90.

[Holmes et al., 1990] Holmes, G., Joly, G.C., and Smallwood, J. (1990). On the application of computer algebra to velocity dominated approximations. *Gen. Rel. Grav.* 22, 749–764.

[Hoyle, 1962] Hoyle, F. (1962). Cosmological tests of gravitational theories. In Moller, C., editor, *Evidence for gravitational theories*, Enrico Fermi Corso XX, Varenna, page 141. Academic Press, New York.

[Hsu and Wainwright, 1986] Hsu, L. and Wainwright, J. (1986). Self-similar spatially homogeneous cosmologies: orthogonal perfect fluid and vacuum solutions. *Class. Quant. Grav.* 3, 1105–1124.

[Ipser and Sikivie, 1984] Ipser, J. and Sikivie, P. (1984). Gravitationally repulsive domain walls. *Phys. Rev. D* 30, 712–9.

[Jaklitsch, 1987] Jaklitsch, M.J. (1987). First order field equations for Bianchi types $II - VI_h$. Preprint 87-6, University of Capetown.

[Jantzen, 1979] Jantzen, R.T. (1979). The dynamical degrees of freedom in spatially homogeneous cosmology. *Comm. math. phys.* 64, 211.

[Jantzen, 1984] Jantzen, R.T. (1984). Spatially homogeneous dynamics: a unified picture. In Ruffini, R. and L.-Z., Fang, editors, *Cosmology of the early universe*, pp. 233–305. World Scientific, Singapore. Also in “Gamow cosmology”, (Proceedings of the International School of Physics 'Enrico Fermi', Course LXXXVI) ed. R. Ruffini and F. Melchiorri, pp. 61-147, North Holland, Amsterdam, 1987.

[Jantzen and Rosquist, 1986] Jantzen, R.T. and Rosquist, K. (1986). Exact power law metrics in cosmology. *Class. Quant. Grav.* 3, 281.
[Juszkiewicz et al., 1983] Juszkiewicz, R., Bajtlik, S., and Gorski, K. (1983). The helium abundance and the isotropy of the universe. *Mon. Not. R.A.S.* **204**, 63P.

[Kantowski, 1969a] Kantowski, R. (1969a). Corrections in the luminosity-redshift relations of the homogeneous Friedman models. *Astrophys. J.* **155**, 59.

[Kantowski, 1969b] Kantowski, R. (1969b). The Coma cluster as a spherical inhomogeneity in relativistic dust. *Astrophys. J.* **155**, 1023.

[King and Ellis, 1973] King, A.R. and Ellis, G.F.R. (1973). Tilted homogeneous cosmological models. *Comm. math. phys.* **31**, 209.

[Kitchingham, 1984] Kitchingham, D.W. (1984). The use of generating techniques for space-times with two non-null commuting Killing vectors in vacuum and stiff perfect fluid cosmological models. *Class. Quant. Grav.* **1**, 677–694.

[Kramer, 1984] Kramer, D. (1984). A new inhomogeneous cosmological model in general relativity. *Class. Quant. Grav.* **1**, L3-7.

[Krasinski, 1990] Krasinski, A. (1990). Early inhomogeneous cosmological models in Einstein’s theory. In Bertotti, B., Bergia, S., Balbinot, R., and Messina, A., editors, *Modern cosmology in retrospect*. Cambridge University Press, Cambridge.

[Kustaanheimo and Qvist, 1948] Kustaanheimo, P. and Qvist, B. (1948). A note on some general solutions of the Einstein field equations in a spherically symmetric world. *Comment. Math. Phys. Helsingf.* **13**, 12.

[Lemaitre, 1933] Lemaitre, G. (1933). L’univers en expansion. *Ann. Soc. Sci. Bruxelles A* **53**, 51.

[Lifshitz and Khalatnikov, 1963] Lifshitz, E.M. and Khalatnikov, I.M. (1963). Investigations in relativistic cosmology. *Adv. Phys.* **12**, 185.

[Lin and Wald, 1991] Lin, X.-F. and Wald, R.M. (1991). Proof of the closed universe recollapse conjecture for general Bianchi IX cosmologies. *Phys. Rev D* **41**, 2444.

[Linet, 1985] Linet, B. (1985). The static metrics with cylindrical symmetry describing a model of cosmic strings. *Gen. Rel. Grav.* **17**, 1109.

[Luminet, 1978] Luminet, J. (1978). Spatially homothetic cosmological models. *Gen. Rel. Grav.* **9**, 673.

[Lynden-Bell et al., 1988] Lynden-Bell, D., Faber, S.M., Burstein, D., Davies, R.L., Dressler, A., Terlevich, R.J., and Wegener, G. (1988). Spectroscopy and photometry of elliptical galaxies V. Galaxy streaming toward the new supergalactic center. *Astrophys. J.* **326**, 19.
[MacCallum, 1971] MacCallum, M.A.H. (1971). On the mixmaster universe problem. Nature (Phys. Sci.) 230, 112–3.

[MacCallum, 1979] MacCallum, M.A.H. (1979). Anisotropic and inhomogeneous relativistic cosmologies. In Hawking, S.W. and Israel, W., editors, General relativity: an Einstein centenary survey, pp. 533–580. Cambridge University Press, Cambridge. (Russian translation: In ‘Obshchaya teoria otnositel’nosti’ edited by Ya. A. Smorodinskii and V.B. Braginskii, Mir, Moscow, 1983. Also reprinted on pp. 179-236 in “The early universe: reprints”, ed. E.W. Kolb and M.S. Turner, Addison-Wesley, Reading, Mass. 1988.).

[MacCallum, 1982] MacCallum, M.A.H. (1982). Relativistic cosmology for astrophysicists. In V. de Sabbata (ed.) Origin and evolution of the galaxies, World Scientific, Singapore, pp. 9–33. [Also, in revised form, in “Origin and evolution of the galaxies”, ed. B.J.T. and J.E. Jones, Nato Advanced Study Institute Series, B97, pp. 9-39, D.Reidel and Co., Dordrecht, 1983.

[MacCallum, 1984] MacCallum, M.A.H. (1984). Exact solutions in cosmology. In Hoenselaers, C. and Dietz, W., editors, Solutions of Einstein’s equations: techniques and results (Retzbach, Germany, 1983), Lecture Notes in Physics, 205, 334–366. Springer Verlag, Berlin and Heidelberg.

[MacCallum, 1987] MacCallum, M.A.H. (1987). Strengths and weaknesses of cosmological big-bang theory. In W.R. Stoeger, S.J., editor, Theory and observational limits in cosmology, pp. 121–142. Specola Vaticana, Vatican City.

[MacCallum and Siklos, 1992] MacCallum, M.A.H. and Siklos, S.T.C. (1992). Algebraically-special hypersurface-homogeneous Einstein spaces in general relativity. J. Geom. Phys. 8, 221–242.

[Maddox et al., 1990] Maddox, S.J., Efstathiou, G.P., Sutherland, W.J., and Loveday, J. (1990). Galaxy correlations on large scales. Mon. Not. R.A.S. 242, 43P.

[Matravers and Madsen, 1985] Matravers, D.R. and Madsen, M.S. (1985). Baryon number generation in a class of anisotropic cosmologies. Phys. Lett. B 155, 43–46.

[Matravers et al., 1985] Matravers, D.R., Madsen, M.S., and Vogel, D.L. (1985). The microwave background and (m, Z) relations in a tilted cosmological model. Astrophys. Sp. Sci. 112, 193–202.

[Matravers et al., 1984] Matravers, D.R., Vogel, D.L., and Madsen, M.S. (1984). Helium formation in a Bianchi V universe with tilt. Class. Quant. Grav. 1, 407.

[Matzner, 1991] Matzner, R.A. (1991). Three-dimensional numerical cosmology. Ann. N.Y. Acad. Sci. 631, 1–14.
[Mavrides, 1977] Mavrides, S. (1977). Anomalous Hubble expansion and inhomogeneous cosmological models. *Mon. Not. R.A.S.* **177**, 709.

[McVittie, 1984] McVittie, G.C. (1984). Elliptic functions in spherically symmetric solutions of Einstein’s equations. *Ann. Inst. Henri Poincaré* **40**, 235–271.

[Meszaros, 1991] Meszaros, A. (1991). On shell crossing in the Tolman metric. *Mon. Not. R. Astr. Soc.* **253**, 619–624.

[Morrow-Jones and Witt, 1988] Morrow-Jones, J. and Witt, D.M. (1988). Proof of the cosmic no-hair conjecture. Santa Barbara preprint UCSBTH-88-08.

[Moss and Sahni, 1986] Moss, I. and Sahni, V. (1986). Anisotropy in the chaotic inflationary universe. *Phys. Lett. B* **178**, 159.

[Newman, 1979] Newman, R.P.A.C. (1979). Singular perturbations of the empty Robertson-Walker cosmologies. Ph. D. thesis, University of Kent.

[Novikov, 1968] Novikov, I.D. (1968). An expected anisotropy of the cosmological radioradiation in homogeneous anisotropic models. *Astr. Zh.* **45**, 538. Translation in *Sov. Astr.-A.J.* **12**, 427.

[Obukhov, 1992] Obukhov, Yu.N. (1992). Rotation in cosmology. *Gen. Rel. Grav.* **24**, 121–128.

[Olson, 1977] Olson, D.W. (1977). Helium production and limits on the anisotropy of the universe. *Astrophys. J.* **219**, 777.

[Penrose, 1979] Penrose, R. (1979). Singularities and time-asymmetry. In Hawking, S.W. and Israel, W., editors, *General relativity: an Einstein centenary survey*, pp. 581–638. Cambridge University Press, Cambridge.

[Perko et al., 1972] Perko, T.E., Matzner, R.A., and Shepley, L.C. (1972). Galaxy formation in anisotropic cosmologies. *Phys. Rev. D* **6**, 969.

[Piran and Safier, 1985] Piran, T. and Safier, P.D. (1985). A gravitational analogue of Faraday rotation. *Nature* **318**, 271.

[Rasband, 1971] Rasband, S.N. (1971). Expansion anisotropy and the spectrum of the cosmic background radiation. *Astrophys. J.* **170**, 1.

[Rees, 1968] Rees, M.J. (1968). Polarization and spectrum of the primeval radiation in an anisotropic universe. *Astrophys. J.* **153**, 1.

[Ribeiro, 1992a] Ribeiro, M.B. (1992a). On modelling a relativistic hierarchical (fractal) cosmology by Tolman’s spacetime. I. Theory. *Astrophys. J.* **388**, 1.
[Ribeiro, 1992b] Ribeiro, M.B. (1992b). On modelling a relativistic hierarchical (fractal) cosmology by Tolman’s spacetime. II. Analysis of the Einstein-de Sitter model. *Astrophys. J.* **395**, 29-33.

[Romano and Pavon, 1992] Romano, V. and Pavon, D. (1992). Causal dissipative Bianchi cosmology. Catania/Barcelona preprint.

[Roque and Ellis, 1985] Roque, W.L. and Ellis, G.F.R. (1985). The automorphism group and field equations for Bianchi universes. In MacCallum, M.A.H., editor, *Galaxies, axisymmetric systems and relativity: essays presented to W.B. Bonnor on his 65th birthday*, pp. 54–73. Cambridge University Press, Cambridge.

[Rosquist et al., 1990] Rosquist, K., Uggla, C., and Jantzen, R.T. (1990). Extended dynamics and symmetries in perfect fluid Bianchi cosmologies. *Class. Quant. Grav.* **7**, 625–637.

[Rothman and Matzner, 1982] Rothman, T. and Matzner, R.A. (1982). Effects of anisotropy and dissipation on the primordial light isotope abundances. *Phys. Rev. Lett.* **48**, 1565.

[Ruiz and Senovilla, 1992] Ruiz, E. and Senovilla, J.M.M (1992). A general class of inhomogeneous perfect-fluid solutions. *Phys. Rev. D* **45**, 1995–2005.

[Ryan and Shepley, 1975] Ryan, M.P. and Shepley, L.C. (1975). *Homogeneous relativistic cosmologies*. Princeton University Press, Princeton.

[Saez and Arnau, 1990] Saez, D. and Arnau, J.V. (1990). On the Tolman Bondi solution of Einstein equations. Numerical applications. In E. Verdaguer, J. Garriga and Cespedes, J., editors, *Recent developments in gravitation (Proceedings of the “Relativity Meeting – 89”)*, pp. 415–422. World Scientific, Singapore.

[Salam, 1990] Salam, A. (1990). *Unification of fundamental forces*. Cambridge University Press, Cambridge.

[Salvati et al., 1987] Salvati, G.A.Q., Schelling, E.E., and van Leeuwen, W.A. (1987). Homogeneous viscous universes with magnetic field. II Bianchi type I spaces. *Ann. Phys. (N.Y.)* **179**, 52–75.

[Sato, 1984] Sato, H. (1984). Voids in the expanding universe. In Bertotti, B., de Felice, F., and Pascolini, A., editors, *General relativity and gravitation: Proceedings of the 10th international conference on general relativity and gravitation*, pp. 289–312. D. Reidel and Co., Dordrecht.
[Saunders et al., 1991] Saunders, W., Frenk, C., Rowan-Robinson, M., Efstathiou, G., Lawrence, A., Kaiser, N., Ellis, R.S., Crawford, J., and Parry, I. (1991). The density field of the local universe. *Nature* **349**, 32.

[Senovilla, 1990] Senovilla, J.M.M. (1990). New class of inhomogeneous cosmological perfect-fluid solutions without big-bang singularity. *Phys. Rev. Lett.* **64**, 2219–2221.

[Siklos, 1978] Siklos, S.T.C. (1978). Occurrence of whimper singularities. *Comm. math. phys.* **58**, 255.

[Siklos, 1980] Siklos, S.T.C. (1980). Field equations for spatially homogeneous space-times. *Phys. Lett. A* **76**, 19.

[Sirousse-Zia, 1982] Sirousse-Zia, H. (1982). Fluctuations produced by the cosmological constant in the empty Bianchi IX universe. *Gen. Rel. Grav.* **14**, 751.

[Steigman and Turner, 1983] Steigman, G. and Turner, M.S. (1983). Inflation in a shear- or curvature-dominated universe. *Phys. Lett. B* **128**, 295.

[Stephani, 1967a] Stephani, H. (1967a). Konform flache Gravitationsfelder. *Comm. math. phys.* **5**, 337.

[Stephani, 1967b] Stephani, H. (1967b). Über Lösungen der Einsteinschen Feldgleichungen, die sich in einen fünfdimensionalen flachen Raum einbetten lassen. *Comm. math. phys.* **4**, 137.

[Stephani, 1983] Stephani, H. (1983). A new interior solution of Einstein’s field equations for a spherically symmetric perfect fluid in shearfree motion. *J. Phys. A* **16**, 3529–3532.

[Stephani, 1987] Stephani, H. (1987). Some perfect fluid solutions of Einstein’s field equations without symmetries. *Class. Quant. Grav.* **4**, 125.

[Sussman, 1985] Sussman, R.A. (1985). Conformal structure of a Schwarzschild black hole immersed in a Friedman universe. *Gen. Rel. Grav.* **17**, 251–292.

[Thorne, 1967] Thorne, K.S. (1967). Primordial element formation, primordial magnetic fields and the isotropy of the universe. *Astrophys. J.* **48**, 51.

[Tod, 1992] Tod, K.P. (1992). Mach’s principle and isotropic singularities. Preprint, University of Oxford.

[Tolman and Matzner, 1984] Tolman, B.W. and Matzner, R.A. (1984). Large scale anisotropies and polarization of the microwave background in homogeneous cosmologies. *Proc. Roy. Soc. A* **392**, 391.
[Tolman, 1934] Tolman, R. (1934). Effect of inhomogeneity on cosmological models. *Proc. Nat. Acad. Sci. (Wash.)* **20**, 169.

[Tomimatsu, 1989] Tomimatsu, A. (1989). The gravitational Faraday rotation for cylindrical gravitational solitons. *Gen. Rel. Grav.* **21**, 613–622.

[Treciokas and Ellis, 1971] Treciokas, R. and Ellis, G.F.R. (1971). Isotropic solutions of the Einstein-Boltzmann equations. *Comm. math. phys.* **23**, 1.

[Turner, 1992] Turner, M.S. (1992). The tilted universe. *Gen. Rel. Grav.* **24**, 1–7.

[Uggl et al., 1991] Uggl, C., Jantzen, R.T., Rosquist, K., and von Zur-Mühlen, H. (1991). Remarks about late stage homogeneous cosmological dynamics. *Gen. Rel. Grav.* **23**, 947–966.

[van den Bergh, 1988] van den Bergh, N. (1988). Perfect-fluid models admitting a non-Abelian and maximal two-parameter group of isometries. *Class. Quant. Grav.* **5**, 861–870.

[van den Bergh, 1991] van den Bergh, N. (1991). A qualitative discussion of the Wils inhomogeneous stiff fluid cosmologies. Brussels preprint.

[van den Bergh and Skea, 1992] van den Bergh, N. and Skea, J.E.F. (1992). Inhomogeneous perfect fluid cosmologies. *Class. Quant. Grav.* **9**, 527.

[van den Bergh et al., 1991] van den Bergh, N., Wils, P., and Castagnino, M. (1991). Inhomogeneous cosmological models of Wainwright class A1. *Class. Quant. Grav.* **8**, 947–954.

[Verdaguer, 1985] Verdaguer, E. (1985). Solitons and the generation of new cosmological solutions. In Sanz, J.L. and Goicoechea, L.J., editors, *Observational and theoretical aspects of relativistic astrophysics and cosmology (Proceedings of the 1984 Santander School)*, pp. 311–350. World Scientific, Singapore.

[Verdaguer, 1992] Verdaguer, E. (1992). Soliton solutions in spacetimes with two spacelike Killing vectors. Barcelona preprint UAB-FT-258.

[Vilenkin, 1983] Vilenkin, A. (1983). Gravitational field of vacuum domain walls. *Phys. Lett. B* **133**, 177–179.

[Wainwright, 1979] Wainwright, J. (1979). A classification scheme for non-rotating inhomogeneous cosmologies. *J. Phys. A* **12**, 2015.

[Wainwright, 1981] Wainwright, J. (1981). Exact spatially inhomogeneous cosmologies. *J. Phys. A* **14**, 1131.
[Wainwright, 1983] Wainwright, J. (1983). A spatially homogeneous cosmological model with plane-wave singularity. *Phys. Lett. A* 99, 301.

[Wainwright and Goode, 1980] Wainwright, J. and Goode, S.W. (1980). Some exact inhomogeneous cosmologies with equation of state $p = \gamma \mu$. *Phys. Rev. D* 22, 1906.

[Wainwright and Hsu, 1989] Wainwright, J. and Hsu, L. (1989). A dynamical systems approach to Bianchi cosmologies: orthogonal models of Class A. *Class. Quant. Grav.* 6, 1409–1431.

[Wald, 1983] Wald, R.M. (1983). Asymptotic behaviour of homogeneous cosmological models in the presence of a positive cosmological constant. *Phys. Rev. D* 28, 211.

[Wang, 1991a] Wang, A. (1991a). A gravitational analogue of the Faraday rotation for interacting gravitational plane waves. Ioannina preprint.

[Wang, 1991b] Wang, A.-Z. (1991b). Planar domain walls emitting and absorbing electromagnetic radiation. Ioannina preprint IOA-258/91.

[Wesson, 1978] Wesson, P.S. (1978). General relativistic hierarchical cosmology: an exact model. *Astrophys. Sp. Sci.* 54, 489.

[Wesson, 1979] Wesson, P.S. (1979). Observable relations in an inhomogeneous self-similar cosmology. *Astrophys. J.* 228, 647.

[Wu, 1981] Wu Z.-C. (1981). Self-similar cosmological models. *Gen. Rel. Grav.* 13, 625. Also see J. China Univ. Sci. Tech. 11(2), 25 and 11 (3), 20.

[Wyman, 1946] Wyman, M. (1946). Equations of state for radially symmetric distributions of matter. *Phys. Rev.* 70, 396.