QUANTITATIVE INDUCTIVE ESTIMATES FOR GREEN’S FUNCTIONS OF NON-SELF-ADJOINT MATRICES

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ABSTRACT. We provide quantitative inductive estimates for Green’s functions of matrices with (sub)exponentially decaying off diagonal entries in higher dimensions. Together with Cartan’s estimates and discrepancy estimates, we establish explicit bounds for the large deviation theorem for non-self-adjoint Toeplitz operators. As applications, we obtain the modulus of continuity of the integrated density of states with explicit bounds and the pure point spectrum property for analytic quasi-periodic operators. Moreover, our inductions are self-improved and work for perturbations with low complexity interactions.

1. INTRODUCTION

The dynamics and spectral theory for quasi-periodic operators have been made significant progress in the last 40 years, through earlier perturbative methods [28, 31, 33, 34, 64, 69], and then non-perturbative methods by controlling Green’s functions/transfer matrices [15, 18, 19, 21, 54, 55] or by reducibility [4, 43]. The case of one dimensional lattice and one frequency potential, has been well understood for both small and large coupling constants, with the recent discovery of global theory [2] and universal structure [49, 50]. In particular, remarkable developments have been achieved for several models motivated by physics: the almost Mathieu operator (the Harper’s model), the extended Harper’s model and the Maryland model [1, 3, 5, 8, 9, 41, 44–50, 52, 53, 56–58, 68]. We refer readers to [62, 74] and references therein for more details.

Problems are known to be much more complicated if one increases the underlying dimension \( b \) of the torus or the dimension \( d \) of the lattice. The higher dimension picture is still far from clear. For the one dimensional lattice \( d = 1 \) and multi-frequencies \( b \geq 1 \), some special cases have been studied by transfer matrices or Schrödinger cocycles [15, 26, 30, 34, 38, 40]. The first multi-dimensional localization result was obtained by perturbative (KAM) methods by Chulaevsky-Dinaburg for operators on lattices \( \mathbb{Z}^d \) and torus \( \mathbb{T} \) for arbitrary \( d \) [27]. Bourgain-Goldstein-Schlag developed a celebrated method in the spirit

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of non-perturbative approaches from Bourgain-Goldstein \[18\] to handle the two-dimension and two-frequency case \(b = d = 2\) and established the Anderson localization for large coupling constants. This is the first higher dimension lattice and multi-frequency result. Moreover, the large deviation theorem in \[20\], which is a key ingredient to prove the Anderson localization, is purely arithmetic in the sense that removed sets of frequencies are independent of the potential. Roughly speaking, by imposing some purely arithmetic condition on \((\omega_1, \omega_2) \in \mathbb{R}^2\), for any algebraic curve \(\Gamma \subset [0, 1]^2\) with degree at most \(N_C\), the number of lattice points

\[
\{(n_1, n_2) \in \mathbb{Z}^2 : |n_1| \leq N, |n_2| \leq N, (n_1 \omega_1, n_2 \omega_2) \mod \mathbb{Z}^2 \in \Gamma \}
\]

is bounded by \(N^{1-\delta}\) for some \(\delta > 0\), where \(\Gamma_\tau\) is the \(e^{-N\tau}\) neighborhood of \(\Gamma\). The quantity \(N^{1-\delta}\) is referred to as the sublinear bound. It is still open whether the analogy for \(d \geq 3\) is true or not.

In \[20\], Bourgain developed a new scheme to prove the large deviation theorem for arbitrary \(b = d\) \[17\] by a delicate study of the semi-algebraic sets. Jitomirskaya-Liu-Shi extended Bourgain’s result to the case of arbitrary \(b\) and \(d\) \[51\]. However, the removed set of frequencies in \[17, 51\] depends on the potential.

Bourgain, Goldstein and Schlag \[20\] mentioned that the sub-linear bound \((1)\) is the only obstruction to establish an arithmetic version of the large deviation theorem in higher dimensions. However, there is no detailed proof available yet.

Our first goal of this paper is to provide such a proof. Moreover, we are going to establish the quantitative version of the main results in \[20\] with generalizations, in particular it can be applied to quasi-periodic operators on arbitrary lattices \(\mathbb{Z}^d\) driven by any dynamics on tori \(\mathbb{Z}^b\) under the assumption on sub-linear bounds.

Instead of Laplacians or long range operators, we will study Toeplitz matrices with (sub)exponentially decaying off diagonal entries. Among all the motivations of our generalizations, we want to highlight one. Anderson localization receives a lot of attentions from both mathematics and physics. The approach to establish Anderson localization for quasi-periodic operators with analytic potentials turns out to be a breakthrough component to construct quasi-periodic solutions for nonlinear Schrödinger equations and nonlinear wave equations \[13, 72\]. It is known that the quasi-periodic solutions in PDEs are only sub-exponentially, not exponentially decaying \[12, 13, 72\]. Therefore, the (sub)exponentially decaying matrices are more natural settings in PDEs.

In our arguments, the matrices are not necessarily self-adjoint and every entry of the matrices is allowed to be a function. For \(d \geq 2\), this is the first time to study operators that beyonds long range cases. For \(d = 1\), our assumptions are weaker than Bourgain’s \[14\]. See Remark \[3\] for details. Moreover, our arguments hold under perturbations with low complexity.

Our proof is definitely inspired by \[20\]. However, there are a lot of important ingredients being added into the arguments to make it quantitative in our more
general settings. Moreover, we significantly simplify the arguments even for the case appearing in [20]. The analysis of [20] required dealing with many different types of elementary regions, say rectangles and \(L\)-shapes in \(\mathbb{Z}^2\). We largely reduced the elementary regions to be square related. See Fig.1. Two novelties are added here. Firstly, we introduce the concept of width of subsets of lattices. In our augments, we always keep the involved regions \(\Lambda\) having large width so that every lattice point in \(\Lambda\) can be covered by a square related elementary region with presetting size contained in \(\Lambda\). For example, the region like Fig.2 was not allowed because the width determined by the distance between \(B\) and \(C\) is too small. Secondly, we reconstruct the exhaustion of \(x\) in every elementary region. In our new construction, the annuli with small width are absorbed into bigger ones. See Fig.3.

There are several other technical improvements in this paper, which we believe to be of independent interest. For example, we establish the Cartan’s estimates for non-self-adjoint matrices.

We will prove a quantitatively inductive theorem about the Green’s functions in higher dimensions as stated in Theorem 2.1. This is a deterministic statement, which can be applied to study operators even without dynamics. Based on matrix-valued Cartan-type theorem (estimates on subharmonic functions) in [20] with further developments in [15, 36, 51], we will establish the measure estimates in Theorem 2.2. Imposing proper dynamics on tori, the quantitative inductive estimate for Green’s functions is obtained (Theorem 2.3). Moreover, the relation among all constants and parameters is displayed clearly so that the whole picture becomes extremely transparent. We will see how arithmetic conditions on frequencies effect the discrepancy, how structures of semi-algebraic sets effect the number of bad Green’s functions, and how the dimensions of lattices and frequencies contribute to bounds.

Finally, we want to talk about the applications. As far as we know, there is no explicit bound yet for the large deviation theorem except for the case \(d = 1\) and \(b = 1, 2\). Our approaches (Theorems 2.1, 2.2 and 2.3) are the first time to establish the explicit bounds in higher dimensions and multi-frequencies. We show that in the arithmetic sense, for \(d = 1\) and any \(b\), the bound is arbitrarily close to \(\frac{1}{b}\) for shift dynamics and \(\frac{1}{4^{d-1}b}\) for skew-shift dynamics. For \(b = 1\) and arbitrary \(d\), we show that the bound is arbitrarily close to 1.

Another application we want to mention is the regularity of the integrated density of states (IDS) of quasi-periodic operators. The log-Hölder continuity of the integrated density of states is quite general [23, 26]. The Hölder continuity in one dimensional settings was well established [6, 7, 13, 15, 26, 36, 40, 42, 61, 75] for both large and small coupling constants. What we will investigate in this paper is the modulus of continuity \(f(x) = e^{-c[\log x]^\tau}\). Unfortunately, like the large deviation theorem, except for the case \(d = 1\) and \(b = 1, 2\), there
are no explicit bounds of $\tau$ in the region of large coupling constants. Based on the ingredients from $[13, 65]$ and the large deviation theorem, the modulus of continuity of the integrated density of states with explicit estimates will be obtained in Theorem 2.3.

2. Main results

Let $A$ be a (operator) matrix on $\ell^2(\mathbb{Z}^d)$ satisfying,

$$|A(n, n')| \leq Ke^{-c_1|n-n'|^{\theta}}, K > 0, c_1 > 0, 0 < \tilde{\sigma} \leq 1,$$

where $|n| := \max_{1 \leq i \leq d} |n_i|$ for $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$. We say that the off diagonal entries of $A$ are subexponentially decaying if $A$ satisfies (2). Sometimes, we just say $A$ is subexponentially decaying for simplicity.

For $d = 1$, the elementary region of size $N$ centered at 0 is given by

$$Q_N = [-N, N].$$

For $d \geq 2$, denote by $Q_N$ an elementary region of size $N$ centered at 0, which is one of the following regions,

$$Q_N = [-N, N]^d$$

or

$$Q_N = [-N, N]^d \setminus \{n \in \mathbb{Z}^d : n_i \leq 0, 1 \leq i \leq d\},$$

where for $i = 1, 2, \cdots, d$, $\varsigma_i \in \{<, >, \emptyset\}^d$ and at least two $\varsigma_i$ are not $\emptyset$.

Denote by $\mathcal{E}_N$ the set of all elementary regions of size $N$ centered at 0. Let $\mathcal{E}_N$ be the set of all translates of elementary regions with center at 0, namely,

$$\mathcal{E}_N := \{n + Q_N : n \in \mathbb{Z}^d, Q_N \in \mathcal{E}_N^0\}.$$

We call elements in $\mathcal{E}_N$ elementary regions.

**Example 1:** For $d = 2$, there are five types of elementary regions.

![Fig.1: elementary regions in $\mathbb{Z}^2$](image)

The width of a subset $\Lambda \subset \mathbb{Z}^d$, is defined by maximum $M \in \mathbb{N}$ such that for any $n \in \Lambda$, there exists $\tilde{M} \in \mathcal{E}_M$ such that

$$n \in \tilde{M} \subset \Lambda$$

and

$$\text{dist}(n, \Lambda \setminus \tilde{M}) \geq M/2.$$

**Example 2:** In Fig.2, the width of $\Lambda$ is determined by the distance between B and C.
A generalized elementary region is defined to be a subset $\Lambda \subset \mathbb{Z}^d$ of the form
$$\Lambda := R - (R + z),$$
where $z \in \mathbb{Z}^d$ is arbitrary and $R$ is a rectangle,
$$R = \{ n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d : |n_1 - n'_1| \leq M_1, \ldots, |n_d - n'_d| \leq M_d \}.$$ 

For $\Lambda \subset \mathbb{Z}^d$, we introduce its diameter,
$$\text{diam}(\Lambda) = \sup_{n, n' \in \Lambda} |n - n'|.$$ 

Denote by $\mathcal{R}_N$ all generalized elementary regions with diameter less than or equal to $N$. Denote by $\mathcal{R}_M^N$ all generalized elementary regions in $\mathcal{R}_N$ with width larger than or equal to $M$. For $\Lambda \subset \mathbb{Z}^d$, let $R_\Lambda$ be the restriction operator, i.e.,
$$(R_\Lambda u)(n) = u(n) \text{ for } n \in \Lambda, \text{ and } (R_\Lambda u)(n) = 0 \text{ for } n \notin \Lambda.$$ 

We say an elementary region $\Lambda \in \mathcal{E}_N$ is in class $G$ (Good) if
$$|(R_\Lambda A R_\Lambda)^{-1}(n, n')| \leq e^{-c_2|n-n'|^\sigma}, \text{ for } |n - n'| \geq \frac{N'}{10},$$
where $0 < c_2 \leq \frac{5^\sigma - 1}{5^{\sigma - 1}}c_1$ and $0 < \tilde{\sigma} \leq 1$. We mentioned that the upper bound $\frac{5^\sigma - 1}{5^{\sigma - 1}}c_1$ is chosen for technical convenience. See (72) for the explanation.

Denote by $\lfloor x \rfloor$ the largest integer smaller than or equal to $x$.

**Theorem 2.1.** Assume $A$ satisfies (2). Let $\varsigma, \sigma, \xi \in (0, 1)$ and $\sigma < \tilde{\sigma} \leq 1$. Let $\Lambda_0 \in \mathcal{E}_N$ be an elementary region with the property that for all $\Lambda \subset \Lambda_0$, $\Lambda \in \mathcal{R}_{N^\xi}^L$ with $N^\xi \leq L \leq N$, the Green’s function $(R_\Lambda A R_\Lambda)^{-1}$ satisfies
$$\| (R_\Lambda A R_\Lambda)^{-1} \| \leq e^{L^\varsigma}.$$ 

Assume that for any family $\mathcal{F}$ of pairwise disjoint elementary regions in $\Lambda_0$ with size $M = \lfloor N^\xi \rfloor$, 
$$\# \{ \Lambda \in \mathcal{F} : \Lambda \text{ is not in class G} \} \leq \frac{N^\xi}{N^\xi}.$$
Then for large $N$ (depending on $K, \varsigma, \sigma, \tilde{\sigma}, \xi, c_1$ and the lower bound of $c_2$),

$$|(R_{\tilde{\Lambda}} A R_{\tilde{\Lambda}})^{-1}(n, n')| \leq e^{-(c_2-N^{-\vartheta})|n-n'|^\varrho}, \text{ for } |n-n'| \geq \frac{N}{10},$$

where $\vartheta = \vartheta(\sigma, \tilde{\sigma}, \xi, \varsigma) > 0$.

Here are several comments about Theorem 2.1.

**Remark 1.**

(1) For $d = 1$ and $\tilde{\sigma} = 1$, a similar statement was proved by Bourgain [14]. For $d = 2$ and $\tilde{\sigma} = 1$, a similar statement was proved for the particular case where $A$ is given by the discrete Laplacian [20].

(2) The statement in Theorem 2.1 is a robust approach to deal with the spectral theory for quasi-periodic operators and also the construction of quasi-periodic solutions for nonlinear Schrödinger/wave equations. See [14, 19, 20, 24, 25] for applications. Some particular cases of Theorem 2.1 have been used as ingredients to construct quasi-periodic solutions for PDEs and have been stated in [24, 25, 72] without detailed proof. There are no explicit bound estimates in their arguments either.

(3) In applications, $\varsigma$ is chosen to be arbitrarily close to 1, namely $\varsigma = 1 - \varepsilon$ with arbitrarily small $\varepsilon > 0$. Then the upper bound in (5) equals $N^{1-\xi-\varepsilon}$. Theorem 2.1 says that the “goodness” of Green’s functions at small size $N^\xi$ will ensure the “goodness” of Green’s functions at larger size $N$ under the following two conditions:

- The number of bad Green’s functions of size $N^\xi$ in $[-N, N]^d$ is less than $N^{1-\xi-\varepsilon}$ (referred to as the sub-linear bound).
- The Green’s functions cannot be “super bad” in the sense that they are controlled by $e^{L^\sigma}$ with $\sigma < 1$ is referred to as the sub-exponential bound.

Let $b = \sum_{i=1}^k b_i$, where $b_i \in \mathbb{N}$. Let $x = (x_1, x_2, \cdots, x_k)$, where $x_i \in \mathbb{T}^{b_i} = (\mathbb{R}/\mathbb{Z})^{b_i}$, $i = 1, 2, \cdots, k$. For any $x \in \mathbb{T}^b$ and $1 \leq i \leq k$, let

$$x_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \in \mathbb{T}^{b-b_i}.$$

For any $y \in \mathbb{T}^{d_1}$ and $X \subset \mathbb{T}^{d_1+d_2}$, denote the $y$-section of $X$:

$$X(y) := \{z \in \mathbb{T}^{d_2} : (y, z) \in X\}.$$

Write $\text{Leb}(S)$ for the Lebesgue measure.

Assume each element of the operator $A$ is a function on $\mathbb{T}^b$. Sometimes, we indicate the dependence and denote by the element $A(x; n, n')$. Assume every element $A(z; n, n')$ is analytic in the strip $\{z \in \mathbb{C}^b : |\Im z| \leq \rho\}$, $\rho > 0$, and satisfies for any $n, n' \in \mathbb{Z}^d$ and $x \in \mathbb{T}^b$,

$$|A(x; n, n')| \leq Ke^{-c_1|n-n'|^\varrho}, \quad K > 0, c_1 > 0, 0 < \tilde{\sigma} \leq 1.$$
Assume that there exists $K_1 > 1$ such that for any $x \in \mathbb{T}^b$ and $z \in \{z \in \mathbb{C}^b : |\Im z| \leq \rho\}$ with $||x - z|| \leq e^{-(\log(|n|+|n'|+2)/K_1)}$,

\begin{equation}
|A(x; n, n') - A(z; n, n')| \leq K||x - z||^\gamma, \tag{8}
\end{equation}

where $||z|| = \text{dist}(z, \mathbb{Z}^b)$.

\textbf{Example 3.} If $A$ satisfies (7) and for any $n, n' \in \mathbb{Z}^d$, $A(x; n, n')$ is a trigonometric polynomial of degree at most $e^{(\log(|n|+|n'|+2))}$, then (8) holds.

We say an elementary region $\Lambda \in \mathcal{E}_N$ is in class $SG_N$ (strongly good with size $N$) if

\begin{equation}
||R_AR_A^{-1}|| \leq e^{N\sigma}, \tag{9}
\end{equation}

and

\begin{equation}
|(R_AR_A^{-1})^{-1}(n, n')| \leq e^{-c|n-n'|^\theta}, \text{ for } |n - n'| \geq \frac{N}{10}, \tag{10}
\end{equation}

where $0 < c_2 < \frac{5(1-\sigma)}{3\rho}c_1$ and $0 < \sigma < \bar{\sigma} \leq 1$. When there is no confusion, we drop the dependence of $N$ from the notation $SG_N$.

\textbf{Theorem 2.2.} Assume $A$ satisfies (7) and (8). Fix $\sigma, \delta, \bar{\sigma}, \zeta \in (0, 1)$ and $\mu \in (1 - \delta, 1)$, $\sigma < \bar{\sigma}$. Suppose $\mathcal{R} \subset [-N_3, N_3]^d$ has width at least $N_2$. For $x \in \mathbb{T}^b$, define $\mathcal{B}\_R(x)$ as

\[ \mathcal{B}\_R(x) = \{n \in \mathcal{R} : \text{there exists } Q_{N_1} \in \mathcal{E}_{N_1}^0 \text{ such that } n + Q_{N_1} \notin SG_{N_1}\} \]

Assume that for any $x \in \mathbb{T}^b$,

\begin{equation}
\#\mathcal{B}\_R(x) \leq L^{1-\delta}. \tag{11}
\end{equation}

Assume that there exists a subset $X_{N_2} \subset \mathbb{T}^b$, such that

\begin{equation}
\sup_{1 \leq i \leq k, x_i \in \mathbb{T}^{b-b_i}} \text{Leb}(X_{N_2}(x_i^\gamma)) \leq e^{-N_2\zeta}, \tag{12}
\end{equation}

and for any $Q_{N_2} \in \mathcal{E}_{N_2}^0$, $x \notin X_{N_2}$ and $n \in \mathcal{R}$, the region $n + Q_{N_2}$ is in class $SG_{N_2}$. Let

\[ \mathcal{X}\_R(x) = \{x \in \mathbb{T}^b : ||(R_AR_A^{-1})^{-1}|| \geq e^{L^\mu}\} \]

Suppose $N_3 \geq e^{N_{1,1}}$, $N_2 \geq N_{1,1}^2$ and $L \geq N_{2,1}^{2d+b_i+2}$. Then there exists $N_0 = N_0(K_1, K, c_1, c_2, \bar{\sigma}, \sigma, \delta, \gamma, \rho, \mu)$ such that for any $N_1 \geq N_0$ and $i = 1, 2, \cdots, k$,

\begin{equation}
\sup_{x_i \in \mathbb{T}^{b-b_i}} \text{Leb}(\mathcal{X}\_R(x_i^\gamma)) \leq e^{-(\frac{L^{1-\delta}}{N_2^{2d+b_i+2}})^{1/b_i}}. \tag{13}
\end{equation}

\footnote{It depends on the lower bound of $c_2$.}
Let $f$ be a function from $\mathbb{Z}^d \times \mathbb{T}^b$ to $\mathbb{T}^b$. Assume for any $m_1, m_2, \ldots, m_d \in \mathbb{Z}^d$ and $n_1, n_2, \ldots, n_d \in \mathbb{Z}^d$,
\[ f(m_1 + n_1, m_2 + n_2, \ldots, m_d + n_d, x) = f(m_1, m_2, \ldots, m_d, f(n_1, n_2, \ldots, n_d, x)) . \]
Sometimes, we write down $f^n(x)$ for $f(n, x)$ for convenience, where $n \in \mathbb{Z}^d$ and $x \in \mathbb{T}^b$. We say $A$ is a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to $f$, if
\[ A(x; n + k, n' + k) = A(f^k(x); n, n') , \]
for any $n \in \mathbb{Z}^d$, $n' \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$. We note that $A$ is not necessarily self-adjoint.

We say the Green's function of an operator $A(x)$ satisfies property $P$ with parameters $(\mu, \zeta, c_2)$ at size $N$ if the following statement is true: there exists a subset $X_N \subset \mathbb{T}^b$ such that
\[ \sup_{1 \leq i \leq k, x_i \in \mathbb{T}^{b_i}} \text{Leb}(X_N(x_i^{-})) \leq e^{-N^c} , \]
and for any $x \notin X_N \mod \mathbb{Z}^b$ and $Q_N \in \mathcal{E}_N^0$,
\[ \|(R_{Q_N} A(x) R_{Q_N})^{-1}\| \leq e^{N\mu} , \]
\[ |(R_{Q_N} AR_{Q_N})^{-1}(x; n, n')| \leq e^{-c_2|n-n'|^\theta}, \text{ for } |n-n'| \geq \frac{N}{10} . \]

**Theorem 2.3.** Assume $A(x)$ satisfies (7), (8) and (14), and
\[ 0 < c_2 < (1 - 5^{-\tilde{\delta}}) c_1, 1 - \delta < \sigma < \tilde{\sigma} \leq 1, \delta > \iota > 0, \text{ and } 0 < \mu < \tilde{\sigma}. \]

Let $c = \frac{1}{2} \min\{\frac{1}{K_1}, \tilde{\sigma}\}$. Fix any sufficiently small $\varepsilon > 0$. There exists a large constant $C$ depending on all parameters such that the following statements are true. Let $N_1$ be sufficiently large, $N_2 \in [N_1^{1/2}, e^{N_1^{1/2}}]$ and $N_3 \in [N_3^{1/2}, e^{N_3^{1/2}}]$. Assume that the Green's function satisfies the property $P$ with parameters $(\mu, \zeta, c_2)$ at sizes $N_1$ and $N_2$. Assume for any $L \in [N_3^{1/2}, N_3]$ and any $x \in \mathbb{T}^b$,
\[ \# \{ n \in \mathbb{Z}^d : |n| \leq L, f(n, x) \in X_{N_1} \mod \mathbb{Z}^b \} \leq L^{1-\delta} . \]
Then there exists $X_{N_3} \subset \mathbb{T}^b$ such that
\[ \sup_{1 \leq i \leq k, x_i^{-} \in \mathbb{T}^{b_i}} \text{Leb}(X_{N_3}(x_i^{-})) \leq e^{-\frac{\sigma (\delta_1 \delta_2 + 2) N_3}{4} - \varepsilon} , \]
and for any $x \notin X_{N_3}$ and $Q_{N_3} \in \mathcal{E}_{N_3}^0$,
\[ \|(R_{Q_{N_3}} A(x) R_{Q_{N_3}})^{-1}\| \leq e^{N_{N_3}^{1/2}} , \]
and for $|n-n'| \geq \frac{N_3}{10}$,
\[ |(R_{Q_{N_3}} AR_{Q_{N_3}})^{-1}(x; n, n')| \leq e^{-(c_2-2N_3^{-\delta_1}-N_3^{-\delta_2})|n-n'|^{\theta}}, \]
where $\vartheta_1 = \vartheta_1(\tilde{\sigma}, \mu, c)$ and $\vartheta_2 = \vartheta_2(\tilde{\sigma}, \sigma, \delta, \varepsilon)$. 
Our Theorems work for Toeplitz matrices with low complexity interactions. Let $U$ be an operator on $\ell^2(\mathbb{Z}^d)$ satisfying
\[ |U(n, n')| \leq Ke^{-c_1|n-n'|^\theta}. \]
Given $m \in \mathbb{Z}^d$, define the operator $U^m$ by
\[ U^m(n, n') = U(m + n, m + n'), n, n' \in \mathbb{Z}^d. \]
We say $U$ has low complexity if there exists $0 < a < 1$ such that for any $N > 1$,
\[ \#\{R_{Q_N}U^mR_{Q_N} : m \in \mathbb{Z}^d, Q_N \in \mathcal{E}_N^0\} \leq Ke^{Na}. \]
For any $m \in \mathbb{Z}^d$, denote by
\[ \tilde{A}^m(x; n, n') = A(x; n, n') + U^m(n, n'). \]
We say that the Green’s function of an operator $A(x)$ satisfies property $\tilde{P}$ with parameters $(\mu, \zeta, c_2)$ at size $N$ if the following statement is true: there exists a set $X_N \subset \mathbb{T}^b$ such that
\[ \sup_{1 \leq i \leq k, x_i \in \mathbb{T}^b} \text{Leb}(X_N(x_i^-)) \leq e^{-N\zeta}, \]
and for any $x \notin X_N \mod \mathbb{Z}^b$, $m \in \mathbb{Z}^d$, and $Q_N \in \mathcal{E}_N^0$
\[ ||(R_{Q_N}\tilde{A}^m(x)R_{Q_N})^{-1}|| \leq e^{N\mu}, \]
\[ |(R_{Q_N}\tilde{A}^mR_{Q_N})^{-1}(x; n, n')| \leq e^{-c_2|n-n'|^{\theta}}, \text{ for } |n-n'| \geq N/10. \]
We have
\[ \textbf{Theorem 2.4.} \text{ Assume } A(x) \text{ satisfies (7), (8) and (14), } U \text{ has low complexity,} \]
\[ 0 < c_2 < (1 - 5^{-\delta})c_1, 1 - \delta < \sigma < \tilde{\sigma} \leq 1, \delta > \iota > 0, 0 < \mu < \tilde{\sigma}, \]
and
\[ a \leq \frac{1}{2} \min_i \left\{ \frac{\sigma - 1}{b_i} \delta + \frac{\delta^2}{b^*_i} \right\}. \]
Let $\tilde{A}^m$ be given by (21) and $c = \frac{1}{2} \min\{\frac{1}{K_1}, \tilde{\sigma}\}$. Fix any sufficiently small $\varepsilon > 0$. Then there exists a large constant $C$ depending on all parameters such that the following statements are true. Let $N_1$ be sufficiently large, $N_2 \in [N_1^C, e^{N_1^{c_2/2}}]$ and $N_3 \in [N_2^C, e^{N_2^\delta}]$. Assume the Green’s function satisfies the property $\tilde{P}$ with parameters $(\mu, \zeta, c_2)$ at sizes $N_1$ and $N_2$. Assume for any $L \in [N_3^{d-\iota}, N_3]$ and any $x \in \mathbb{T}^b$,
\[ \#\{n \in \mathbb{Z}^d : |n| \leq L, f(n, x) \in X_{N_i} \mod \mathbb{Z}^b\} \leq L^{1-\delta}. \]
Then there exists a subset $X_{N_3} \subset \mathbb{T}^b$ such that

$$\sup_{1 \leq i \leq k, x_i \in \mathbb{T}^{b-b_i}} \text{Leb}(X_{N_3}(x_i^-)) \leq e^{-N_3 \frac{\sigma - 1 - \frac{\delta^2}{b_i} - \varepsilon}{b_i}}$$

and for any $x \notin X_{N_3}$, $m \in \mathbb{Z}^d$ and $Q_{N_3} \in \mathcal{E}_{N_3}$,

$$\| (R_{Q_{N_3}} \tilde{A}^m(x) R_{Q_{N_3}})^{-1} \| \leq e^{N_3^2}$$

and for $|n - n'| \geq N_3$,

$$\| (R_{Q_{N_3}} \tilde{A}^m R_{Q_{N_3}})^{-1}(x; n, n') \| \leq e^{-(c_2 - N_1^{-\theta_1} - N_3^{-\theta_2})|n - n'|^\phi},$$

where $\vartheta_1 = \vartheta_1(\sigma, \mu, c)$ and $\vartheta_2 = \vartheta_2(\sigma, \delta, \varepsilon)$.

**Remark 2.**

1. Theorem 2.3 improves the parameters from $(\mu, \zeta, c_2)$ to $(\sigma, \sigma - 1 - \frac{\delta^2}{b_i} - \varepsilon, c_2 - N_1^{-\theta_1} - N_3^{-\theta_2})$.

   Theorem 2.3 gives us opportunities to combine perturbative approaches with non-perturbative approaches. After establishing the property P for initial scales by non-perturbative methods, we can adapt the parameters to establish property P with explicit bounds for larger scales. See Theorems 3.1, 3.2 and 3.3 and Corollaries 3.4, 3.5 and 3.6 for examples.

2. Roughly speaking Theorem 2.3 says that under the assumption on the sublinear bound, the large deviation theorem at sizes $N = N_1$ and $N = N_2$ will ensure the large deviation theorem at size $N = N_3$.

We are going to discuss the modulus of continuity of the integrated density of states (IDS). In order to make it as general as possible, we do not require the existence of the integrated density of states first. Let $E_1 < E_2$ and define

$$k(x, E_1, E_2) = \limsup_{N \to \infty} \frac{1}{(2N + 1)^d} \# \{ \text{eigenvalues of } R_{[-N,N]^d} A(x) R_{[-N,N]^d} \text{ in } [E_1, E_2] \}.$$  

Fix $x \in \mathbb{T}^b$. Assume for any measurable set $\mathcal{S} \subset \mathbb{T}^b$, we have

$$\limsup_{N \to \infty} \frac{1}{(2N + 1)^d} \# \{ n \in \mathbb{Z}^d : |n| \leq N, f(n_1, n_2, \ldots, n_d, x) \in \mathcal{S} \} \leq \text{Leb}(\mathcal{S}).$$

For an operator $A(x)$ on $\ell^2(\mathbb{Z}^d)$, denote by the energy dependent Green’s functions

$$G_\Lambda(E, x) = (R_\Lambda(A(x) - E) R_\Lambda)^{-1}.$$ 

Instead of $G_\Lambda(E, x)$, we will write $G_\Lambda$, $G_\Lambda(E)$, or $G_\Lambda(x)$ when there is no ambiguity. We will write $G_\Lambda(n, n')$, $G_\Lambda(E; n, n')$, $G_\Lambda(x; n, n')$, or $G_\Lambda(E, x; n, n')$ for the element of matrices.
Theorem 2.5. Assume $A(x)$ is a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to $f$ in the sense of (14). Let $\zeta \in (0,1)$ and $0 < \sigma < \bar{\sigma} \leq 1$. Assume for any $E \in \mathbb{R}$, there exists a set $X_N \subset \mathbb{T}_b$ such that

$$\text{Leb}(X_N) \leq e^{-N^c}$$

and for any $x \not\in X_N$ and any $Q_N \in \mathbb{E}_N^0$,

$$\|G_{Q_N}(E, x)\| \leq e^{N^z}$$

$$|G_{Q_N}(E, x; n, n')| \leq e^{-c|n-n'|^{\sigma}} \text{ for } |n - n'| \geq \frac{N}{10},$$

where $c > 0$. Assume (23) holds for some $x_0 \in \mathbb{T}_b$. Then for any $\varepsilon > 0$, we have

$$|k(x_0, E_1, E_2)| \leq e^{-\|\log|E_1 - E_2||^{-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

The rest of this paper is organized as follows. Except for some statements in applications (Section 3), this paper is entirely self contained. We will introduce many applications to quasi-periodic operators in Section 3. Sections 4, 5, 6, 7 are devoted to prove Theorems 2.1, 2.2, 2.3, 2.4 and 2.5. We will introduce the discrepancy for semi-algebraic sets in Section 8. In Section 9, we will give the proof for all the results in Section 3.

3. Applications

Let $S$ be a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to $f$, namely,

$$S(x; n + k, n' + k) = S(f^k(x); n, n'),$$

for any $n \in \mathbb{Z}^d, n' \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$. Assume every element $S(z; n, n')$, $n, n' \in \mathbb{Z}^d$, is analytic in a strip $\{z : |\Im z| \leq \rho\}$ with $\rho > 0$ and satisfies for any $x \in \mathbb{R}$ and $n, n' \in \mathbb{Z}^d$,

$$|S(x; n, n')| \leq Ke^{-c_1|n-n'|}, K > 0, c_1 > 0.$$

Assume that there exists $K_1 > 1$ such that for any $x \in \mathbb{T}_b$ and $z \in \{z \in \mathbb{C}^b : |\Im z| \leq \rho\}$ with $||x - z|| \leq e^{-(\log(|n| + |n'| + 2))^K_1}$,

$$|A(x; n, n') - A(z; n, n')| \leq K||x - z||^\gamma.$$

Assume for any $N > 1, n, n' \in \mathbb{Z}^d$ with $|n| \leq N$ and $|n'| \leq N$, there exists a trigonometric polynomial $\tilde{S}(x; n, n')$ of degree less than $e^{(\log N)K_1}$ such that

$$\sup_{x \in \mathbb{T}_b} |S(x; n, n') - \tilde{S}(x; n, n')| \leq Ke^{-N^2}.$$

Define a family of operators $H(x)$ on $\ell^2(\mathbb{Z}^d)$:

$$H(x) = \lambda^{-1}S + v(f(n, x))\delta_{nn'},$$

where $v$ is an analytic function on $\mathbb{T}_b$. 
In this section, we always assume
- \( v \) is non-constant,
- \( f \) is a frequency shift or skew-shift,
- except for subsection 3.6, \( S \) is a Toeplitz (operator) matrix on \( \ell^2(\mathbb{Z}^d) \) with respect to \( f \) and satisfies (25)-(28).

**Example 4:**
- If \( S \) is a long range operator, namely, \( S \) does not depend on \( x \) and
  \[
  S(n, n') \leq Ke^{-c|n-n'|}, \quad n, n' \in \mathbb{Z}^d,
  \]
  then (26), (27) and (28) hold.
- Let \( \phi_k(x), k \in \mathbb{Z} \), be a trigonometric polynomial on \( \mathbb{T}^d \) of degree less than \( e^{(\log(1+|k|))K_1} \) satisfying
  \[
  \left| \phi_k(x) \right| \leq Ke^{-c|k|}.
  \]
  Let
  \[
  S(x; n, n') = \phi_{n-n'}(f(n, x)) + \phi_{n'-n}(f(n', x)).
  \]
  Then (26), (27) and (28) hold.

**Remark 3.** For \( d \geq 2 \), our settings (25)-(28) is the first time to allow every entry of \( S \) to depend on \( x \), which beyonds the long range operators. For \( d = 1 \), Bourgain [14] studied the case in Example 4 under the assumption that \( \phi_k(x) \) is a trigonometric polynomial of degree at most \( N^C \).

We will apply Theorems 2.1, 2.2, 2.3 and 2.5 to operators
\[
A(x) = H(x) = \lambda^{-1}S + v(f(n, x))\delta_{nn'}.
\]
In this section, the Green’s functions always depend on energy \( E \). See [24].

The IDS appearing in applications is always existed, namely, the following limit
\[
k(x, E) = \lim_{N \to \infty} \frac{1}{(2N + 1)^d} \# \{ \text{eigenvalues of } R_{[-N,N]^d}A(x)R_{[-N,N]^d} \text{ smaller than } E \},
\]
converges to \( k(E) \) for almost every \( x \). We write \( k(E) \) for the IDS when it exists.

For the large deviation theorem, \( S \) is not necessarily self-adjoint. However, in order to establish pure point spectrum property, self-adjointness is necessary because of the energy elimination.

3.1. **Shifts: \( d = 1 \), arbitrary \( b \).** Denote by \( \Delta \) the discrete Laplacian on \( \ell^2(\mathbb{Z}) \), that is, for \( \{u(n)\} \in \ell^2(\mathbb{Z}) \),
\[
(\Delta u)(n) = \sum_{|n-n'|=1} u(n').
\]
We say that \( \omega = (\omega_1, \omega_2, \cdots, \omega_b) \) satisfies Diophantine condition \( \text{DC}(\kappa, \tau) \), if
\[
||k\omega|| \geq \frac{\tau}{|k|^\kappa}, \quad k \in \mathbb{Z}^b \setminus \{(0, 0, \cdots, 0)\}.
\]

By the Dirichlet principle, one has \( \kappa \geq b \). When \( \kappa > b \), \( \cup_{\tau > 0} \text{DC}(\kappa, \tau) \) has full Lebesgue measure.

We say that \( \omega \in \mathbb{R} \) satisfies strong Diophantine conditions if there exist \( \kappa > 1 \) and \( \tau > 0 \) such that
\[
||k\omega|| \geq \frac{\tau}{k(1 + \log k)^\kappa} \quad \text{for all } k \in \mathbb{N}.
\]

It is easy to see that almost every \( \omega \) satisfies strong Diophantine conditions.

Let
\[
f^n(x) = x + n\omega = (x_1 + n\omega_1, x_2 + n\omega_2, \cdots, x_b + n\omega_b) \mod \mathbb{Z}^b,
\]
where \( x = (x_1, x_2, \cdots, x_b) \in \mathbb{T}^b \), \( n \in \mathbb{Z} \) and \( \omega = (\omega_1, \omega_2, \cdots, \omega_b) \in \mathbb{R}^b \).

Let \( H(x) \) on \( \ell^2(\mathbb{Z}) \) be given by
\[
H(x) = \Delta + v(f^n(x)) = \Delta + v(x_1 + n\omega_1, x_2 + n\omega_2, \cdots, x_b + n\omega_b)\delta_{nn'},
\]
where \( n, n' \in \mathbb{Z} \).

Let
\[
A_k^E(x) = \prod_{j=k-1}^{0} A^E(x + j\omega) = A^E(x + (k-1)\omega)A^E(x + (k-2)\omega) \cdots A^E(x)
\]
and
\[
A_{-k}^E(x) = (A_k^E(x - k\omega))^{-1}
\]
for \( k \geq 1 \), where \( A^E(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix} \). \( A_k^E \) is called the (k-step) transfer matrix. The Lyapunov exponent is given by
\[
L(E) = \lim_{k \to \infty} \frac{1}{k} \int_{\mathbb{T}^b} \ln \|A_k^E(x)\| dx.
\]

**Theorem 3.1.** Let \( \omega \in \text{DC}(\kappa, \tau) \) and \( 1 - \frac{1}{b\kappa} < \sigma < 1 \). Let \( H(x) \) be given by (32). Assume the Lyapunov exponent \( L(E) \) is positive. Then for any \( \varepsilon > 0 \) and large \( N \), there exists a subset \( X_N \subset \mathbb{T}^b \) such that
\[
\text{Leb}(X_N) \leq e^{-N^{\frac{\sigma-1}{\sigma} + \frac{1}{b\kappa} - \varepsilon}},
\]
and for any \( x \notin X_N \), we have
\[
||G_{[-N,N]}(E, x)|| \leq e^{N^\sigma},
\]
and
\[
|G_{[N,-N]}(E, x; n, n')| \leq e^{-(L(E) - \varepsilon)|n-n'|} \quad \text{for } |n - n'| \geq N/10.
\]
Theorem 3.2. Let \( \omega \in \text{DC}(\kappa, \tau) \) and \( H(x) \) be given by (32). Suppose the Lyapunov exponent \( L(E) > 0 \) for every \( E \) in an interval \( I \). Then for any \( \varepsilon > 0 \),
\[
|k(E_1) - k(E_2)| \leq e^{-\left( \log \frac{1}{|E_1 - E_2|} \right)^{1/3b^2\kappa^2} - \varepsilon}
\]
provided that \( |E_1 - E_2| \) is sufficiently small and \( E_1, E_2 \in I \).

Theorem 3.3. Let \( H(x) \) be given by (32). Then the following statement is true for almost every \( \omega \). Assume the Lyapunov exponent \( L(E) > 0 \) for every \( E \) in an interval \( I \). Then for any \( \varepsilon > 0 \),
\[
|k(E_1) - k(E_2)| \leq e^{-\left( \log \frac{1}{|E_1 - E_2|} \right)^{1/3b^2\kappa^2} - \varepsilon},
\]
provided that \( |E_1 - E_2| \) is sufficiently small and \( E_1, E_2 \in I \).

Remark 4. Under the same assumptions, the large deviation theorem and the modulus of continuity of the IDS were shown in [35] (also see [15]). When \( b = 2 \), a better bound \( b = 1/3 \) was obtained in [35]. However, there are no explicit bounds in [15, 35] when \( b \geq 3 \).

Putting a coupling constant \( \lambda^{-1} \) in front of the Laplacian \( \Delta \), the operator given by (32) becomes
\[
H(x) = \lambda^{-1} \Delta + v(x + n\omega)\delta_{nn'}.
\]
For large \( \lambda \) only depending on the potential \( v \), the Lyapunov exponent \( L(E) \) is positive for every \( E \) [16]. Therefore, we have the following three corollaries

Corollary 3.4. Assume \( \omega \in \text{DC}(\kappa, \tau) \) and \( 1 - \frac{1}{b\kappa} < \sigma < 1 \). Let \( H(x) \) be given by (36). Then there exists \( \lambda_0 = \lambda_0(v) \) such that for any \( \varepsilon > 0 \), \( \lambda > \lambda_0 \) and large \( N \), there exists \( X_N \subset \mathbb{T}^b \) such that
\[
\text{Leb}(X_N) \leq e^{-N \frac{\sigma - 1}{3b^2\kappa^2} - \frac{1}{3b^2\kappa^2} - \varepsilon},
\]
and for any \( x \not\in X_N \), we have
\[
||G_{[-N,N]}(E, x)|| \leq e^{N\sigma},
\]
and
\[
|G_{[N,-N]}(E, x; n, n')| \leq e^{-(L(E) - \varepsilon)|n - n'|} \text{ for } |n - n'| \geq N/10.
\]

Corollary 3.5. Let \( \omega \in \text{DC}(\kappa, \tau) \) and \( H(x) \) be given by (36). Then there exists \( \lambda_0 = \lambda_0(v) \) such that for any \( \varepsilon > 0 \) and \( \lambda > \lambda_0 \),
\[
|k(E_1) - k(E_2)| \leq e^{-\left( \log \frac{1}{|E_1 - E_2|} \right)^{1/3b^2\kappa^2} - \varepsilon}
\]
provided that \( |E_1 - E_2| \) is sufficiently small.
Corollary 3.6. Let $H(x)$ be given by (36). Then there exists $\lambda_0 = \lambda_0(v)$ such that the following statement is true for almost every $\omega$. For any $\varepsilon > 0$ and $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \leq e^{-\left(\frac{1}{\log |E_1 - E_2|}\right)^{\frac{1}{4}} - \varepsilon},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Let $H(x)$ on $\ell^2(\mathbb{Z})$ be given by

(38) \hspace{1cm} H(x) = \lambda^{-1}S + v(f^n(x)) = \lambda^{-1}S + v(x + n\omega)\delta_{nn'},

where $x, \omega \in \mathbb{R}^b$.

Theorem 3.7. Let $H(x)$ be given by (38). Assume $\omega \in \text{DC}(\kappa, \tau)$ and $1 - \frac{1}{b\kappa} < \sigma < 1$. Then for any $\varepsilon > 0$, there exists

$$\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \sigma, \gamma, K, K_1, c_1, v)$$

such that for any $\lambda > \lambda_0$ and any $N$, there exists $X_N \subset \mathbb{T}^b$ such that

$$\text{Leb}(X_N) \leq e^{-N^{\sigma^2} + \frac{1}{\sigma^2} - \varepsilon},$$

and for any $x \notin X_N$, we have

$$||G_{[-N,N]}(E, x)|| \leq e^{N^\sigma},$$

and

$$|G_{[N,-N]}(E, x; n, n')| \leq e^{-\frac{1}{2}c_1|n-n'|} \text{ for } |n - n'| \geq N/10.$$

Theorem 3.8. Assume $S$ is self-adjoint and $\omega \in \text{DC}(\kappa, \tau)$. Let $H(x)$ be given by (38). Then for any $\varepsilon > 0$, there exists

$$\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \gamma, K, K_1, c_1, v)$$

such that for any $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \leq e^{-\left(\frac{1}{\log |E_1 - E_2|}\right)^{\frac{1}{4}} - \varepsilon},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Theorem 3.9. Assume $S$ is self-adjoint. Let $H(x)$ be given by (38). Then for almost every $\omega \in \mathbb{R}^b$ the following is true. For any $\varepsilon > 0$, there exists

$$\lambda_0 = \lambda_0(\varepsilon, \omega, \rho, \gamma, K, K_1, c_1, v)$$

such that for any $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \leq e^{-\left(\frac{1}{\log |E_1 - E_2|}\right)^{\frac{1}{4}} - \varepsilon},$$

provided that $|E_1 - E_2|$ is sufficiently small.
Theorem 3.10. Let $H(x)$ be given by (38). Then for any $\varrho > 0$, there is $\lambda_0 = \lambda_0(\varrho, \rho, \gamma, K, K_1, c_1, v) > 0$ such that the following statement holds. For any $\lambda > \lambda_0$ and any $x \in \mathbb{T}$, there exists $\Omega = \Omega(x, \lambda, S, v, \varrho) \subset \mathbb{T}^b$ with $\text{Leb}(\mathbb{T}^b \setminus \Omega) \leq \varrho$ such that for any $\omega \in \Omega$, $H(x)$ satisfies Anderson localization.

3.2. Shifts: $b = 1, \text{ arbitrary } d$. Let $v$ be analytic on $\mathbb{T}$. Let

$$f^n(x) = x + n\omega = x + n_1\omega_1 + n_2\omega_2 + \cdots + n_d\omega_d \mod \mathbb{Z},$$

where $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$ and $x \in \mathbb{T}$. Let $H(x)$ on $\ell^2(\mathbb{Z}^d)$ be given by

$$H(x) = \lambda^{-1}S + v(f^n(x))\delta_{nn'} = \lambda^{-1}S + v(x + n_1\omega_1 + n_2\omega_2 + \cdots + n_d\omega_d)\delta_{nn'}.$$

Theorem 3.11. Let $\omega \in \text{DC}(\kappa, \tau)$ and $H(x)$ be given by (39). Then for any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \sigma, \gamma, K, K_1, c_1, v)$ such that for any $\lambda > \lambda_0$ and any $N$, there exists $X_N \subset \mathbb{T}$ such that

$$\text{Leb}(X_N) \leq e^{-N^\sigma - \varepsilon},$$

and for any $x \not\in X_N$ and any $Q_N \in \mathcal{E}_N^0$, we have

$$||G_{Q_N}(E, x)|| \leq e^{N^\sigma},$$

and

$$|G_{Q_N}(E, x; n, n')| \leq e^{-\frac{1}{2}c_1|n - n'|} \text{ for } |n - n'| \geq N/10.$$

Theorem 3.12. Assume $S$ is self-adjoint and $\omega \in \text{DC}(\kappa, \tau)$. Let $H(x)$ be given by (39). Then for any $\varepsilon > 0$, there exists

$$\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \gamma, K, K_1, c_1, v)$$

such that for any $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \leq e^{-\left(\log \frac{1}{|E_1 - E_2|}\right)^{1-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Theorem 3.13. Assume $S$ is self-adjoint. Let $H(x)$ be given by (39). Then for any $\varrho > 0$, there is $\lambda_0 = \lambda_0(\varrho, \rho, \gamma, K, K_1, c_1, v) > 0$ such that the following statement holds. For any $\lambda > \lambda_0$ and any $x \in \mathbb{T}$, there exists $\Omega = \Omega(x, \lambda, S, v, \varrho) \subset \mathbb{T}^d$ with $\text{Leb}(\mathbb{T}^d \setminus \Omega) \leq \varrho$ such that for any $\omega \in \Omega$, $H(x)$ satisfies Anderson localization.

Remark 5. Theorem 3.13 is a generalization of Theorem 2 in p.138 of [15] and main result in [27].
3.3. Skew-shifts: \( d = 1, \) arbitrary \( b \). Let \( f : \mathbb{T}^b \to \mathbb{T}^b \) be the skew-shift defined as follows

\[
f(x_1, x_2, \ldots, x_b) = (x_1 + \omega, x_2 + x_1, \ldots, x_b + x_{b-1})
\]

Let \( H(x) \) on \( \ell^2(\mathbb{Z}) \) be given by

\[
H(x) = \lambda^{-1} S(x) + v(f^n(x)) \delta_{nn'},
\]

where \( v \) is analytic on \( \mathbb{T}^b \).

**Theorem 3.14.** Let \( H(x) \) be given by (43). Assume \( \omega \in \text{DC}(\kappa, \tau) \) and \( \frac{1}{2^b-1}\beta - \sigma < 1 \). Then for any \( \varepsilon > 0 \), there exists \( \lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \gamma, K, K_1, c_1, v) \) such that for any \( \lambda > \lambda_0 \) and any \( N \), there exists \( X_N \subset \mathbb{T}^b \) such that

\[
\text{Leb}(X_N) \leq e^{-N \frac{\varepsilon}{2^b-1}\beta - \varepsilon},
\]

and for any \( x \notin X_N \), we have

\[
||G_{[-N,N]}(E, x)|| \leq e^{N\sigma}
\]

and

\[
|G_{[N-\varepsilon,N]}(E, x; n, n')| \leq e^{-\varepsilon|n-n'|} \text{ for } |n - n'| \geq N/10.
\]

**Remark 6.** Under stronger assumptions that \( \omega \in \text{DC}(2, \tau) \), \( v \) and each element of \( S \) are nonconstant trigonometric polynomials, the large deviation theorem appearing in Theorem 3.14 without explicit bounds was proved for \( d = 2 \) [15] and arbitrary \( d \) [67].

**Theorem 3.15.** Assume \( S \) is self-adjoint and \( \omega \in \text{DC}(\kappa, \tau) \). Let \( H(x) \) be given by (43). Then for any \( \varepsilon > 0 \), there exists

\[
\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \gamma, K, K_1, c_1, v)
\]

such that for any \( \lambda > \lambda_0 \), we have

\[
|k(E_1) - k(E_2)| \leq e^{-\left(\log \frac{1}{|E_1 - E_2|}\right) \frac{\varepsilon}{2^b-1}\beta - \varepsilon},
\]

provided that \( |E_1 - E_2| \) is sufficiently small.

**Corollary 3.16.** Assume \( S \) is self-adjoint. Let \( H(x) \) be given by (43). Then for almost every \( \omega \in \mathbb{R} \) the following is true. For any \( \varepsilon > 0 \), there exists \( \lambda_0 = \lambda_0(\varepsilon, \omega, \rho, \gamma, K, K_1, c_1, v) \) such that for any \( \lambda > \lambda_0 \),

\[
|k(E_1) - k(E_2)| \leq e^{-\left(\log \frac{1}{|E_1 - E_2|}\right) \frac{\varepsilon}{2^b-1}\beta - \varepsilon},
\]

provided that \( |E_1 - E_2| \) is sufficiently small.

Assume \( S \) is taken the particular case, i.e., \( S = \Delta \). Let \( b = 2 \). In this case, by Corollary 3.16, \( \frac{1}{2^b-1}\beta = \frac{1}{12} \). A bound \( \frac{1}{32} \) was shown by Bourgain, Goldstein and Schlag [19]. By combining the arguments in Bourgain, Goldstein and Schlag [19] with the proof of Corollary 3.16, we are able to improve the bound.
Corollary 3.17. Assume $S$ is self-adjoint. Let $b = 2$ and $H(x)$ be given by (43). Then for almost every $\omega \in \mathbb{R}$ the following is true. For any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon, \omega, \rho, \gamma, K, K_1, c_1, v)$ such that for any $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \leq e^{-\left(\frac{1}{|E_1 - E_2|^{1/2}}\right)^{1/4}}$$

provided that $|E_1 - E_2|$ is sufficiently small.

3.4. Skew-shifts: $d = b = 1$. Let $P_b$ be the projection on the $b$th coordinate of $\mathbb{T}^b$, namely, $P_b(x_1, x_2, \cdots, x_b) = x_b$, where $(x_1, x_2, \cdots, x_b) \in \mathbb{R}^b$. Define $H(x)$ on $\ell^2(\mathbb{Z})$,

$$H(x) = \lambda^{-1} \Delta + v(P_b(f^n(x)))\delta_{nn'},$$

where $v$ is analytic on $\mathbb{T}$ and $f$ is the skew-shift on $\mathbb{T}^b$.

Theorem 3.18. Let $H(x)$ be given by (44). Assume $\omega$ is strong Diophantine and $1 - \frac{1}{2b^3} < \sigma < 1$. Then there exists $\lambda_0 = \lambda_0(v)$ such that for any $\varepsilon > 0$, $\lambda > \lambda_0$ and large $N$, there exists $X_N \subset \mathbb{T}^b$ such that

$$\text{Leb}(X_N) \leq e^{-N^{\frac{\sigma - 1}{2b^3 - 1} - \frac{1}{4b^3 - 1} - \varepsilon}},$$

and for any $x \notin X_N$, we have

$$||G_{[-N,N]}(E, x)|| \leq e^{N\sigma}$$

and

$$|G_{[-N,N]}(E, x; n, n')| \leq e^{-\frac{1}{2}c_1|n - n'|} \text{ for } |n - n'| \geq N/10.$$

Theorem 3.19. Let $\omega$ be strong Diophantine and $H(x)$ be given by (44). Then there exists $\lambda_0 = \lambda_0(v)$ such that for any $\varepsilon > 0$ and $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \leq e^{-\left(\frac{1}{|E_1 - E_2|^{1/2}}\right)^{1/4}}$$

provided that $|E_1 - E_2|$ is sufficiently small.

Remark 7. • Comparing to Theorems 3.14 and 3.15 there is no dimension ($b^3$) loss in the bounds of Theorems 3.18 and 3.19. This is because the potential $v$ is defined on $\mathbb{T}$.

• The large deviation theorem and the modulus of continuity of Lyapunov exponents (the IDS) without explicit bounds was obtained in [70].

• Let $b = 2$. The constant in (45) becomes $\frac{1}{4} - \varepsilon = \frac{1}{4}$. It is possible to improve the bound from $1/4$ to $1/3$ by incorporating the arguments in [19]. A weaker result was proved by Tao [71], where a constant $\frac{1}{30}$ was obtained.
3.5. Shifts: \( d = b = 2 \). Assume \( v \) is analytic on \( \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2 \). Let
\[
 f^n(x) = (x_1 + n_1\omega_1, x_2 + n_2\omega_2) \mod \mathbb{Z}^2,
\]
where \( n = (n_1, n_2) \in \mathbb{Z}^2 \), \( \omega = (\omega_1, \omega_2) \in \mathbb{R}^2 \) and \( x = (x_1, x_2) \in \mathbb{T}^2 \). Let \( H(x) \) on \( \ell^2(\mathbb{Z}^2) \) be given by
\[
 H(x) = \lambda^{-1}S(x) + v(f^n(x))\delta_{n'} = \lambda^{-1}S(x_1, x_2) + v(x_1 + n_1\omega_1, x_2 + n_2\omega_2)\delta_{n'}.
\]

**Theorem 3.20.** Let \( H(x) \) be given by (46). Suppose \( v \) is nonconstant on any line segment contained in \([0, 1]^2\), \( \omega_1 \in \text{DC}(\kappa, \tau) \) and \( \omega_2 \in \text{DC}(\kappa, \tau) \) with \( 1 \leq \kappa < \frac{13}{12} \). Assume
\[
 3\kappa - \frac{9}{4} < \sigma < 1.
\]
Then there exists \( \lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \sigma, \gamma, K, K_1, c_1, v) \) such that for any \( \lambda > \lambda_0 \) and any \( N \), there exists \( X_N \subset \mathbb{T}^2 \) such that for any line segment \( L \subset [0, 1)^2 \),
\[
 \text{Leb}(X_N \cap L) \leq e^{-N(\sigma - 1)(\frac{14}{3} - 3\kappa) + (\frac{13}{4} - 3\kappa)^2 - \varepsilon},
\]
and for any \( x \notin X_N \) and \( Q_N \in \mathcal{E}_N^0 \), we have
\[
 ||G_{Q_N}(E, x)|| \leq e^{N\sigma}
\]
and
\[
 |G_{Q_N}(E, x; n, n')| \leq e^{-\frac{1}{2}c_1|n-n'|} \text{ for } |n - n'| \geq N/10.
\]

**Theorem 3.21.** Assume \( S \) is self-adjoint, \( v \) is nonconstant on any line segment contained in \([0, 1]^2\), \( \omega_1 \in \text{DC}(\kappa, \tau) \) and \( \omega_2 \in \text{DC}(\kappa, \tau) \) with \( 1 \leq \kappa < \frac{13}{12} \). Let \( H(x) \) be given by (46). Then for any \( \varepsilon \), there exists \( \lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \gamma, K, K_1, c_1, v) \) such that for any \( \lambda > \lambda_0 \),
\[
 |k(E_1) - k(E_2)| \leq e^{-\left(\log \left[\frac{1}{|E_1 - E_2|}\right]\right)\frac{(\frac{13}{4} - 3\kappa)^2 - \varepsilon}{2}},
\]
provided that \( |E_1 - E_2| \) is sufficiently small.

**Corollary 3.22.** Assume \( S \) is self-adjoint and \( v \) is nonconstant on any line segments contained in \([0, 1]^2\). Let \( H(x) \) be given by (46). Then for almost every \( \omega \in \mathbb{R}^2 \) the following is true. For any \( \varepsilon > 0 \), there exists \( \lambda_0 = \lambda_0(\varepsilon, \omega, \rho, \gamma, K, K_1, c_1, v) \) such that for any \( \lambda > \lambda_0 \),
\[
 |k(E_1) - k(E_2)| \leq e^{-\left(\log \left[\frac{1}{|E_1 - E_2|}\right]\right)\frac{1}{2} - \varepsilon},
\]
provided that \( |E_1 - E_2| \) is sufficiently small.

**Remark 8.** Theorems 3.20 and 3.21 follow the arguments in (22). Our quantitative approaches developed in the paper allow us to obtain the explicit bounds.
3.6. Sub-exponentially decaying matrices with interactions. Our applications can be wider. Here are several examples. Instead of (26), assume
\begin{equation}
|S(x; n, n')| \leq K e^{-c_1 |n-n'|^b}, 0 < \tilde{\sigma} \leq 1, c_1 > 0,
\end{equation}
for any $n, n' \in \mathbb{Z}^d$.

Assume for any $N > 1$, $n, n' \in \mathbb{Z}^d$ with $|n| \leq N$ and $|n'| \leq N$, there exists a trigonometric polynomial $\tilde{S}(x; n, n')$ of degree less than $e^{N^a}$ such that
\begin{equation}
\sup_{x \in \mathbb{T}^b} |S(x; n, n') - \tilde{S}(x; n, n')| \leq K e^{-N^2}.
\end{equation}

In this subsection, assume $S$ satisfies (27), (48) and (49).

Let $\tilde{U}$ be a diagonal matrix on $\ell^2(\mathbb{Z}^d)$ satisfying
\[||U|| \leq K.\]

Given $m \in \mathbb{Z}^d$, define the diagonal matrix $\tilde{U}^m$ on $\ell^2(\mathbb{Z}^d)$ by
\[\tilde{U}^m(n) = \tilde{U}(m + n), n \in \mathbb{Z}^d.\]

We say $\tilde{U}$ has low complexity if there exists $0 < a < 1$ such that for any $N > 1$,
\begin{equation}
\# \{R_{Q_N} U^m(n) \delta_{nn'} : m \in \mathbb{Z}^d, Q_N \in E_N \} \leq K e^{N^a}.
\end{equation}

Let
\begin{equation}
\tilde{H}(x) = H(x) + \lambda^{-1} U + \tilde{U} = \lambda^{-1} (S + U) + (\tilde{U}(n) + v(f(n, x))) \delta_{nn'}.
\end{equation}

For any $m \in \mathbb{Z}^d$, let
\begin{equation}
\tilde{H}^m(x) = H(x) + \lambda^{-1} U^m + \tilde{U}^m = \lambda^{-1} (S + U^m) + (\tilde{U}^m + v(f(n, x))) \delta_{nn'}.
\end{equation}

Denote by $\tilde{G}^m$ the Green’s function of $\tilde{H}^m$.

**Theorem 3.23.** Assume $\alpha$ is strong Diophantine, and $U$ and $\tilde{U}$ have low complexity in the sense of (19) and (50) respectively. Assume
\[1 - \frac{1}{b} < \sigma < \tilde{\sigma} \text{ and } a \leq \frac{1}{4} \left\{ \frac{1}{K_1}, \frac{\sigma - 1}{b^2} + \frac{1}{b^3} \right\}.\]

Let $H(x)$ and $\tilde{H}^m(x)$ be given by (38) and (52) respectively. Then for any $\varepsilon > 0$, there exists
\[\lambda_0 = \lambda_0(\varepsilon, \alpha, \rho, c_1, \sigma, \tilde{\sigma}, \gamma, K, K_1, c_1, v)\]
such that for any $\lambda > \lambda_0$ and any $N$, there exists $X_N \subset \mathbb{T}^b$ such that
\[\text{Leb}(X_N) \leq e^{-N^{\frac{1}{2b} + 1/b^2 - \varepsilon}},\]
and for any $x \notin X_N$ and $m \in \mathbb{Z}$, we have
\[||\tilde{G}^m_{[-N,N]}(E, x)|| \leq e^{N^a},\]
and
\[||\tilde{G}^m_{[-N,N]}(E, x; n, n')|| \leq e^{-\frac{1}{2}|n-n'|} \text{ for } |n - n'| \geq N/10,\]
for any $n, n' \in \mathbb{Z}^d$. 

where \( c = \frac{5^\delta - 1}{5^\sigma}. \)

**Theorem 3.24.** Assume \( \omega \in DC(\kappa, \tau), U \) and \( \tilde{U} \) have low complexity. Assume \( 0 < \sigma < \tilde{\sigma} \leq 1 \) and \( a \leq \frac{1}{4} \min\{\frac{1}{K_1}, \sigma\}. \) Let \( H(x) \) and \( \tilde{H}^m(x) \) be given by (39) and (52) respectively. Then for any \( \varepsilon > 0, \) there exists
\[
\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \sigma, \tilde{\sigma}, \rho, \gamma, K, K_1, c, v)
\]
such that for any \( \lambda > \lambda_0 \) and any \( N, \) there exists \( X_N \subset \mathbb{T} \) such that
\[
\text{Leb}(X_N) \leq e^{-N^{\sigma - \varepsilon}},
\]
and for any \( x \notin X_N, \) any \( m \in \mathbb{Z}^d \) and any \( Q_N \in \mathcal{E}_N^0, \) we have
\[
||\tilde{G}^m_{Q_N}(E, x)|| \leq e^{N^\sigma},
\]
and
\[
|\tilde{G}^m_{Q_N}(E, x; n, n')| \leq e^{-\frac{1}{4}|n - n'|^\rho} \text{ for } |n - n'| \geq N/10,
\]
where \( c = \frac{5^\delta - 1}{5^\sigma}. \)

**Theorem 3.25.** Assume \( S \) is self-adjoint, \( \omega \in DC(\kappa, \tau) \) and \( a \leq \frac{1}{4} \min\{\frac{1}{K_1}, \tilde{\sigma}\}. \) Let \( H(x) \) be given by (39). Then for any \( \varepsilon > 0, \) there exists
\[
\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \tilde{\sigma}, \rho, \gamma, K, K_1, c, v)
\]
such that for any \( \lambda > \lambda_0, \)
\[
|k(E_1) - k(E_2)| \leq e^{-\left(\log \frac{|E_1 - E_2|}{|E_1|}|^\rho\right)^{1-\varepsilon}},
\]
provided that \( |E_1 - E_2| \) is sufficiently small.

Using Theorem 2.4 instead of Theorem 2.3, the proof of Theorems 3.24, 3.25 and 4.1 follows from that of Theorems 3.7, 3.11 and 3.12 respectively. In order to avoid repetitions, we skip the details.

**4. Multi-scale analysis**

**4.1. Exhaustion construction for an elementary region.** For \( m \in \mathbb{Z}^d \) and \( \Lambda \subset \mathbb{Z}^d, \) define the distance by
\[
\text{dist}(m, \Lambda) = \inf_{n \in \Lambda} |m - n|.
\]

Fix an elementary region \( \Lambda \in \mathcal{E}_N. \) Let \( x \in \Lambda. \) Given \( M \leq N/10, \) we will construct exhaustion at \( x \) with width \( M. \) Set
\[
\tilde{S}_0(x) = (x + [-2M, 2M]^d) \cap \Lambda,
\]
\[
\tilde{S}_j(x) = \bigcup_{y \in \tilde{S}_{j-1}(x)} (y + [-4M, 4M]^d) \cap \Lambda, 1 \leq j \leq \tilde{l}
\]
where \( \tilde{l} \) is the minimum such that \( \tilde{S}_j(x) = \Lambda. \) We set \( S_{-1}(x) = \emptyset \) for convenience.
When $\tilde{S}_{j-1}(x)$ is very close to the boundary of $\Lambda$, $\tilde{A}_j(x) = \tilde{S}_j(x)\backslash \tilde{S}_{j-1}(x)$ and $\tilde{S}_j$ may have width less than $M$. However, there are at most finitely many $j$ with $0 \leq j \leq \tilde{l}$, saying $C(d)$, such that $A_j(x) = \tilde{S}_j(x)\backslash \tilde{S}_{j-1}(x)$ has width less than $M$, where $C(d)$ is a constant depending on $d$.

We will delete $j$ if $A_j(x) = \tilde{S}_j(x)\backslash \tilde{S}_{j-1}(x)$ has small width and then rearrange exhaustion. Here are the details. Let $j_0 \in \{0, 1, \ldots, \tilde{l} - 1\}$ be the possibly smallest number such that both $\tilde{S}_{j_0}(x)$ and $\tilde{S}_l(x)\backslash \tilde{S}_{j_0}(x)$ have width at least $M$. Otherwise, set $j_0 = \tilde{l}$. Let $S_0(x) = \tilde{S}_{j_0}(x)$. Let $j_1 \in \{j_0, j_0 + 1, \ldots, \tilde{l} - 1\}$ be the possibly smallest number such that both $\tilde{S}_{j_1}(x)\backslash \tilde{S}_{j_0}(x)$ and $\tilde{S}_{l}(x)\backslash \tilde{S}_{j_1}(x)$ have width at least $M$. Otherwise, set $j_1 = \tilde{l}$. Let $S_1(x) = \tilde{S}_{j_1}(x)$. Suppose we have defined $j_0, j_1, \ldots, j_k$ and corresponding $S_0(x), S_1(x), \ldots, S_k(x)$. Let $j_k+1 \in \{j_k, j_k + 1, \ldots, \tilde{l} - 1\}$ be the possibly smallest number such that $\tilde{S}_{j_{k+1}}(x)\backslash \tilde{S}_{j_k}(x)$ and $\tilde{S}_{l}(x)\backslash \tilde{S}_{j_{k+1}}(x)$ have width at least $M$. Otherwise, set $j_{k+1} = \tilde{l}$. Let $S_{k+1}(x) = \tilde{S}_{j_{k+1}}(x)$. Let $l$ be such that $S_l(x) = \Lambda$. By our constructions, $\tilde{l} - C(d) \leq l \leq \tilde{l}$.

Here is an example. Assume $x$ locates exactly at the left upmost corner. In Fig.3, $A_k(x) = \tilde{S}_k(x)\backslash \tilde{S}_{k-1}(x)$ and $\tilde{S}_l(x) = \tilde{S}_l(x)\backslash \tilde{S}_{l-1}(x)$ are the only two annuli which have width less than $M$. Therefore,

- $l = \tilde{l} - 2$
- For $j = 0, 1, 2, \ldots, k - 2$, $S_j(x) = \tilde{S}_j(x)$.
- For $j = k - 1, k - 2, \ldots, l - 3$, $S_j(x) = \tilde{S}_{j+1}(x)$. $S_{l-2}(x) = \tilde{S}_l(x)$.

For any elementary region $\Lambda$, $x \in \Lambda$ and $M$, we call $\{S_j(x)\}_{j=0}^l$ the exhaustion of $\Lambda$ at $x$ with width $M$. We call $A_j(x) = S_j(x)\backslash S_{j-1}(x)$ the $j$th annulus. For any $y \in S_j(x)\backslash S_{j-1}(x)$, $j = 1, 2, \ldots, l$, one has

\[4(j - 1)M \leq |y - x| \leq 4jM + C(d)M.\]

By our constructions, any $\{A_j(x)\}$ has width at least $M$. Namely, for any $n \in A_j(x)$ there exists $W(n) \in E_M$ such that

\[n \in W(n) \subset A_j(x)\]

and

\[\text{dist} \ (n, A_j(x)\backslash W(n)) \geq M/2.\]
4.2. Resolvent identities. For simplicity, assume $K = 1$, namely

$$|A(n, n')| \leq e^{-c_1|n-n'|^\bar{\sigma}}, \ 0 < \bar{\sigma} \leq 1, \ c_1 > 0,$$

for any $n, n' \in \mathbb{Z}^d$. For any $\Lambda \subset \mathbb{Z}^d$, denote by $A_\Lambda = R_\Lambda A R_\Lambda$, where $R_\Lambda$ is the restriction on $\Lambda$, and the Green’s function

$$G_\Lambda = (R_\Lambda A R_\Lambda)^{-1},$$

provided $R_\Lambda A R_\Lambda$ is invertible. Denote by $G_\Lambda(n, n')$ its elements, $n, n' \in \Lambda \subset \mathbb{Z}^d$.

Assume $\Lambda_1$ and $\Lambda_2$ are two disjoint subsets of $\mathbb{Z}^d$. Namely, $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. Suppose that $R_\Lambda A R_\Lambda$, $i = 1, 2$ are invertible. Then

$$G_\Lambda = G_{\Lambda_1} + G_{\Lambda_2} - (G_{\Lambda_1} + G_{\Lambda_2})(A_\Lambda - A_{\Lambda_1} - A_{\Lambda_2})G_\Lambda.$$

If $m \in \Lambda_1$ and $n \in \Lambda$, we have

$$|G_\Lambda(m, n)| \leq |G_{\Lambda_1}(m, n)|\chi_{\Lambda_1}(n) + \sum_{n' \in \Lambda_1, n'' \in \Lambda_2} e^{-c_1|n'-n''|^\bar{\sigma}}|G_{\Lambda_1}(m, n')||G_{\Lambda}(n'', n)|.$$
Lemma 4.1 (Schur test). Suppose $A = A_{ij}$ is a matrix. Then

$$
\|A\| \leq \sqrt{\left( \sup_i \sum_j |A_{ij}| \right) \left( \sup_j \sum_i |A_{ij}| \right)}.
$$

The following lemma is a generalization of Lemma 3.2 in [51].

**Lemma 4.2.** Let $c_2 \in [\tilde{c}_1, c_1]$, $\sigma < \tilde{\sigma}$ and $M_0 \leq M_1 \leq N$. Assume $\Lambda$ is a subset of $\mathbb{Z}^d$ with $\text{diam}(\Lambda) \leq 2N + 1$. Suppose that for any $n \in \Lambda$, there exists some $W = W(n) \in \mathcal{E}_M$ with $M_0 \leq M \leq M_1$ such that $n \in W \subset \Lambda$, $\text{dist}(n, \Lambda \setminus W) \geq \frac{M}{2}$ and

$$
\|G_{W(n)}\| \leq 2e^{M^\sigma},
$$
$$
|G_{W(n)}(n, n')| \leq 2e^{-c_2|n-n'|^{\sigma}} \text{ for } |n - n'| \geq \frac{M}{10}.
$$

We assume further that $M_0$ is large enough so that

$$
\sup_{M_0 \leq M \leq M_1} \sup_{c_2 \in [\tilde{c}_1, c_1]} 2e^{M^\sigma} (2M + 1)^d e^{\frac{c_2}{10^d} M^\sigma} \sum_{j=0}^{\infty} (M + 2j + 1)^d e^{-c_2(j+M/2)^\sigma} \leq \frac{1}{2}.
$$

Then

$$
\|G_\Lambda\| \leq 4(2M_1 + 1)^d e^{M_1^\sigma}.
$$

**Proof.** Under the assumption of (61), it is easy to check that for any $M$ with $M_0 \leq M \leq M_1$ and any $n \in \Lambda$,

$$
2(2M + 1)^d e^{M^\sigma + \frac{c_2}{10^d} M^\sigma} \sum_{n_2 \in \Lambda} e^{-c_2|n-n_2|^{\sigma}} \leq \frac{1}{2}.
$$

By (59) and (60), one has

$$
|G_{W(n)}(n, n')| \leq 2e^{M^\sigma + \frac{c_2}{10^d} M^\sigma} e^{-c_2|n-n'|^{\sigma}}.
$$

For each $n \in \Lambda$, applying (57) with $\Lambda_1 = W(n)$, one has

$$
|G_\Lambda(n, n')| \leq |G_{W(n)}(n, n')| \chi_{W(n)}(n') + \sum_{n_2 \in W(n)} e^{-c_1|n_1-n_2|^{\sigma}} |G_{W(n)}(n, n_1)| |G_\Lambda(n_2, n')|.
$$

It is easy to see for $0 < \tilde{\sigma} \leq 1$,

$$
|x + y|^{\tilde{\sigma}} \leq |x|^{\tilde{\sigma}} + |y|^{\tilde{\sigma}}.
$$
Proof. Choose a constant \( \rho \in (1, 1 + \tilde{\sigma} - \sigma) \). Calculation shows \( \rho \sigma < \tilde{\sigma} \).

Define inductively \( M_{j+1} = \lceil M_j^\rho \rceil \), \( M_0 = M \). Let \( \gamma_0 = c_2 \). Fix an elementary region \( \tilde{\Lambda}_1 \in \mathcal{E}_{M_1} \) and \( \tilde{\Lambda}_1 \subset \Lambda_0 \). For any \( x \in \tilde{\Lambda}_1 \), consider the exhaustion \( \{ S_j(x) \}_{j=0}^\infty \) of \( \tilde{\Lambda}_1 \) at \( x \) with width \( M_0 \). Denote by \( \{ A_k(x) \} \) the annuli.

We call the annulus \( A_k(x) \) good, if for any \( y \in A_k(x) \), there exists \( W(y) \in \mathcal{E}_{M_0} \) such that

\[
y \in W(y) \subset A_k(x), \quad \text{dist}(y, A_k(x) \setminus W(y)) \geq M_0/2,
\]

and for \( |n - n'| \geq \frac{M_0}{10} \),

\[
|(R_{W(y)} A_{W(y)})^{-1}(n, n')| \leq e^{-\gamma_0|n-n'|^\rho}.
\]

Otherwise, we call the annulus \( A_k \) bad.
Fix $\kappa > 0$, which will be determined later. An elementary region $\tilde{\Lambda}_1 \subset \tilde{\Lambda}_0$ is called bad if there exists $x \in \tilde{\Lambda}_1$ such that the number of bad annuli $\{A_k(x)\}$ exceeds

$$B_1 := \frac{M_1}{M_0}.$$ 

Otherwise, we call $\tilde{\Lambda}_1$ good. Let $\mathcal{F}_1$ be an arbitrary family of pairwise disjoint bad elementary regions in $\mathcal{E}_{M_1}$ contained in $\tilde{\Lambda}_0$. Since every annulus in $\{A_k\}$ has width at least $M_0$ by our construction, one has that every bad annulus contains at least one elementary region in $\mathcal{E}_{M_0}$ without satisfying (68) and hence

$$\# \mathcal{F}_1 \leq \frac{N^c}{\kappa M_1}.$$ 

Assume that $\tilde{\Lambda}_1 \subset \tilde{\Lambda}_0$ is a good elementary region in $\mathcal{E}_{M_1}$. We will first show that $\tilde{\Lambda}_1$ is in class $G$ with slightly smaller $\gamma_0$. Consider the exhaustion $\{S_j(x)\}$ of $\tilde{\Lambda}_1$ at $x$ with width $M_0$. By the assumption, there are no more than $B_1$ bad annuli in this exhaustion. Denote by $\{A_j(x)\}_{j=0}^l$ annuli. By putting adjacent good annuli or bad annuli together, we obtain a new exhaustion

$$\emptyset = J_{-1} \subset J_0 \subset J_1 \subset \cdots \subset J_g = \tilde{\Lambda}_1.$$ 

More precisely, $\{J_s(x)\}$, $s = 0, 1, 2, \cdots, g$, satisfies the following rules.

- $J_s(x) \setminus J_{s-1}(x) = \{A_j(x)\}_{j=t_s}^{t_s+1}$ for some $t_s < t'_s$.
- $x \in J_0(x)$.
- The annuli $A_j(x)$, $j = t_s, t_s + 1, \cdots, t'_s$ are either all good or all bad.
- Take $J_s(x)$ maximal with the above three properties.

We remind that $J_s(x) \setminus J_{s-1}(x)$ has width at least $M_0$ for any $s = 0, 1, 2, \cdots, g$. By our construction, if all annuli in $J_s(x) \setminus J_{s-1}(x)$ are good (bad), then all annuli in $J_{s+1}(x) \setminus J_s(x)$ are bad (good).

For any $n \in \tilde{\Lambda}_1$, let $k(n)$ be the number of good annuli between $x$ and $n$. Namely, for any $n \in A_j(x)$,

$$k(n) = \# \{A_t(x) : A_t(x) \text{ is a good annulus}, 0 \leq t \leq j\}.$$ 

Before we start the estimates, let us give several facts first, which will be used constantly in the later proof. By our constructions, $J_s$ is a generalized elementary region, $s = 0, 1, \cdots, g$. By the assumption (64), one has for all $s = 0, 1, \cdots, g$,

$$(R_{J_s}AR_{J_s})^{-1} \leq e^{M_i^r}.$$ 

Assume

$$0 < c \leq (1 - 5^{-\sigma})c_1.$$ 

If $|n - n_2| \geq \frac{M}{2}$ and $|n - n_1| \leq \frac{M}{4}$, one has

$$c_1|n_1 - n_2|^{\sigma} \geq c_1(|n - n_2|^{\sigma} - |n - n_1|^{\sigma})$$

$$\geq c|n - n_2|^{\sigma}.$$
It is clear that for any $n_1, n_2 \in \tilde{\Lambda}_1$,

\begin{equation}
4M_0 k(n_1) + |n_1 - n_2| \geq 4M_0 k(n_2) - 4M_0. \tag{73}
\end{equation}

Without loss of generality, assume all the annuli in $J_0$ are bad (the other case is similar).

For any $n \in \tilde{\Lambda}_1$, define

$$
\Gamma_s(n) = \max\{4M_0 k(n) - 10(s + 1)M_0, 0\}.
$$

By (73), we have for any $n_1 \in \tilde{\Lambda}_1$ and $n_2 \in \tilde{\Lambda}_1$,

\begin{equation}
\Gamma_s(n_1) + |n_2 - n_1| \geq \max\{\Gamma_s(n_2) - 4M_0, 0\}. \tag{74}
\end{equation}

We shall inductively obtain estimates of the form

\begin{equation}
|G_{J_s(x)}(x, z)| \leq T_s e^{-\gamma_0 \Gamma_s(z)}, \tag{75}
\end{equation}

where $z \in J_s$, $s = 0, 1, \cdots, g$.

**First step:** $s = 0$

Since all annuli in $J_0$ are bad, one has $k(z) = 0$ and hence

\begin{equation}
\Gamma_0(z) = 0. \tag{76}
\end{equation}

By (71) and (76), one has for $z \in J_0(x)$,

$$
|G_{J_s(x)}(x, z)| \leq e^{M_1} = e^{M_1} e^{-\gamma_0 \Gamma_0(z)}.
$$

It implies that (75) holds for

\begin{equation}
T_0 = e^{M_1}. \tag{77}
\end{equation}

Assume (75) holds at $s$th step for a proper $T_s$.

**Case 1:** All annuli in $J_{s+1}\backslash J_s$ are bad.

Pick any $z \in J_{s+1}$. Let $\tilde{n}_1 \in J_s$ and $\tilde{n}_2 \in J_{s+1}\backslash J_s$ be such that

$$
\Gamma_s(\tilde{n}_1) + |\tilde{n}_1 - \tilde{n}_2| = \inf_{n_1 \in J_s \atop n_2 \in J_{s+1}\backslash J_s} (\Gamma_s(n_1) + |n_1 - n_2|).
$$

**Case 1:** $z \in J_{s+1}\backslash J_s$. In this case, for any $n_2 \in J_{s+1}\backslash J_s$, one has

\begin{equation}
k(z) = k(n_2), \Gamma_s(z) = \Gamma_s(n_2), \tag{78}
\end{equation}

since all annuli in $J_{s+1}\backslash J_s$ are bad.
Applying (57) \((\Lambda_1 = J_s \text{ and } \Lambda_2 = J_{s+1} \setminus J_s)\), one has

\[
|G_{J_{s+1}}(x, z)| \leq \sum_{n_1 \in J_s} |G_{J_s}(x, n_1)| e^{-c_1|n_1-n_2|} |G_{J_{s+1}}(n_2, z)|
\]

\[
\leq \sum_{n_1 \in J_s} T_s e^{-\gamma_0 \Gamma_s^\theta(n_1)} e^{-\gamma_0|n_1-n_2|} |G_{J_{s+1}}(n_2, z)|
\]

\[
\leq e^{M_s^\theta} T_s \sum_{n_1 \in J_s} e^{-\gamma_0 \Gamma_s^\theta(n_1)} e^{-\gamma_0|n_1-n_2|}
\]

\[
\leq (2M_1 + 1)^{2d} e^{M_s^\theta} T_s \sup_{n_1 \in J_s} e^{-\gamma_0 \Gamma_s^\theta(n_1)} e^{-\gamma_0|n_1-n_2|}
\]

\[
\leq (2M_1 + 1)^{2d} e^{M_s^\theta} T_s e^{-\gamma_0(\Gamma_s(n) + |n_1-n_2|)}
\]

\[
\leq (2M_1 + 1)^{2d} e^{M_s^\theta} T_s e^{-\gamma_0(\max(\Gamma_s(n), -4M_0, 0))},
\]

\[
(79)
\]

where the second inequality holds by the induction (75) and \(\gamma_0 \leq c_1\), the third inequality holds by (71), the fifth inequality holds by (64), the sixth inequality holds (74), and the last inequality holds by (78).

**Case 12**: \(z \in J_s\).

In this case, we have for any \(n_2 \in J_{s+1} \setminus J_s\),

\[
(80)
\]

\[
k(n_2) \geq k(z).
\]

Applying (57) \((\Lambda_1 = J_s \text{ and } \Lambda_2 = J_{s+1} \setminus J_s)\), one has

\[
|G_{J_{s+1}}(x, z)| \leq |G_{J_s}(x, z)| + \sum_{n_1 \in J_s} |G_{J_s}(x, n_1)| e^{-c_1|n_1-n_2|} |G_{J_{s+1}}(n_2, z)|
\]

\[
\leq T_s e^{-\gamma_0 \Gamma_s^\theta(z)} + \sum_{n_1 \in J_s} T_s e^{-\gamma_0 \Gamma_s^\theta(n_1)} e^{-\gamma_0|n_1-n_2|} |G_{J_{s+1}}(n_2, z)|
\]

\[
\leq T_s e^{-\gamma_0 \Gamma_s^\theta(z)} + (2M_1 + 1)^{2d} e^{M_s^\theta} T_s \sup_{n_1 \in J_s} e^{-\gamma_0(\Gamma_s(n_1) + |n_1-n_2|)}
\]

\[
= T_s e^{-\gamma_0 \Gamma_s^\theta(z)} + (2M_1 + 1)^{2d} e^{M_s^\theta} T_s e^{-\gamma_0(\Gamma_s(n) + |n_1-n_2|)}
\]

\[
\leq T_s e^{-\gamma_0 \Gamma_s^\theta(z)} + (2M_1 + 1)^{2d} e^{M_s^\theta} T_s e^{-\gamma_0(\max(\Gamma_s(n), -4M_0, 0))},
\]

\[
\leq T_s e^{-\gamma_0 \Gamma_s^\theta(z)} + (2M_1 + 1)^{2d} e^{M_s^\theta} T_s e^{-\gamma_0(\max(\Gamma_s(z), -4M_0, 0))},
\]

\[
(81)
\]

\[
2(2M_1 + 1)^{2d} e^{M_s^\theta} T_s e^{-\gamma_0(\max(\Gamma_s(z), -4M_0, 0))}.
\]
where the second inequality holds by the induction (75), the third inequality holds by (71) and (64), the forth inequality holds by (74), and the fifth inequality holds by (80).

**Case 2:** All the annuli in $J_{s+1}\setminus J_s$ are good.

By our constructions, $J_{s+1}\setminus J_s$ has width at least $M_0$. Therefore, for any $k \in J_{s+1}\setminus J_s$, there exists some $W = W(k) \in \mathcal{E}_{M_0}$ such that $k \in W \subset \Lambda$,

\[(82) \quad \text{dist}(k, J_{s+1}\setminus J_s \setminus W) \geq \frac{M_0}{2},\]

and

\[(83) \quad \|G_{W(k)}\| \leq e^{M_0^\sigma},\]
\[(84) \quad |G_{W(k)}(n_1, n_2)| \leq e^{-\gamma_0|n_1 - n_2|^\sigma} \quad \text{for} \quad |n_1 - n_2| \geq \frac{M_0}{10},\]

where (83) holds by the assumption (4).

Since $M_0$ is large enough, one has (61) is satisfied. Applying Lemma 4.2, we have

\[(85) \quad \|G_{J_{s+1}\setminus J_s}\| \leq 4(2M_0 + 1)^d e^{M_0^\sigma}.\]

We remark that we can not use the assumption (4) to bound $G_{J_{s+1}\setminus J_s}$ since $J_{s+1}\setminus J_s$ is not necessary to be a generalized elementary region. It is worth to point out that $J_{s+1}\setminus J_s$ may not be connected.

We will first prove that for any $m, n \in J_{s+1}\setminus J_s$,

\[(86) \quad |G_{J_{s+1}\setminus J_s}(m, n)| \leq M_1^{10dM_0^\sigma M_0^{-\sigma} e^{-\gamma_0(max\{|m-n|-2M_0,0\})^\sigma}}.\]

Assume $|m - n| \leq 2M_0$. (86) holds by (85).

Assume $|m - n| > 2M_0$. Applying (57) with $\Lambda_1 = W(m)$ and using that $|m - n| > 2M_0$, one has

\[(87) \quad |G_{J_{s+1}\setminus J_s}(m, n)| \leq \sum_{\substack{n_1 \in W(m) \\ n_2 \in J_{s+1}\setminus J_s \setminus W(m)}} e^{-c_1|n_1 - n_2|^\sigma}|G_{W(m)}(m, n_1)||G_{\Lambda}(n_2, n)|.\]

Applying (82) with $k = m$ and by (72), one has for any $n_1$ with $|n_1 - m| \leq \frac{M_0}{10}$ and $n_2 \in J_{s+1}\setminus J_s \setminus W(m)$, one has

\[(88) \quad c_1|n_1 - n_2|^\tilde{\sigma} \geq c_2|m - n_2|^\tilde{\sigma}.\]
By (83), (84) and (87), we have

\[ |G_{J_{s+1}\setminus J_s}(m, n)| \]
\[ \leq \sum_{n_1 \in W(m), |n_1 - m| \leq M_0} e^{-c_1|n_1 - n_2|^\theta} |G_{W(m)}(m, n_1)||G_{J_{s+1}\setminus J_s}(n_2, n)| \]
\[ + \sum_{n_2 \in J_{s+1}\setminus J_s \setminus W(m)} e^{-c_1|n_1 - n_2|^\theta} |G_{W(m)}(m, n_1)||G_{J_{s+1}\setminus J_s}(n_2, n)| \]
\[ \leq \sum_{n_1 \in W(m), |n_1 - m| \leq M_0} e^{-c_1|n_1 - n_2|^\theta} |G_{J_{s+1}\setminus J_s}(n_2, n)| \]
\[ + \sum_{n_2 \in J_{s+1}\setminus J_s \setminus W(m), |n_2 - |n_1 - m|| \geq M_0} e^{-c_1|n_1 - n_2|^\theta} e^{-\gamma_0|m-n|^\theta} |G_{J_{s+1}\setminus J_s}(n_2, n)| \]
\[ \leq \sum_{n_1 \in W(m), |n_1 - m| \leq M_0} e^{-\gamma_0|m-n|^\theta} |G_{J_{s+1}\setminus J_s}(n_2, n)| \]
\[ + \sum_{n_2 \in J_{s+1}\setminus J_s \setminus W(m), |n_2 - |n_1 - m|| \geq M_0} e^{-\gamma_0|m-n|^\theta} |G_{J_{s+1}\setminus J_s}(n_2, n)| \]
\[ \leq (2M_1 + 1)^{2d} e^{M_0^\theta} \sup_{n_2 \in J_{s+1}\setminus J_s \setminus W(m)} e^{-\gamma_0|m-n|^\theta} |G_{J_{s+1}\setminus J_s}(n_2, n)|, \]  

(89)

where the third inequality holds because of (88).

Recall that \(|m - n_2| \geq M_0/2\). Iterating (89) until \(|n_2 - n| \leq 2M_0\) or at most \(\left\lfloor \frac{2^d|m-n|^\theta}{M_0^\theta} \right\rfloor + 1\) times, we have

\[ |G_{J_{s+1}\setminus J_s}(m, n)| \leq e^{M_0^\theta \left( \frac{2^d|m-n|^\theta}{M_0^\theta} + 1 \right) \left( 2M_1 + 1 \right)^{2d} \left( \frac{2^d|m-n|^\theta}{M_0^\theta} + 1 \right)} e^{-\gamma_0(|m-n|-2M_0)^\theta} \|G_{J_{s+1}\setminus J_s}\| \]
\[ \leq M_1^{9dM_1^\theta M_0^\theta} e^{-\gamma_0(|m-n|-2M_0)^\theta} \|G_{J_{s+1}\setminus J_s}\| \]
\[ \leq M_1^{9dM_1^\theta M_0^\theta} e^{-\gamma_0(|m-n|-2M_0)^\theta} 4(2M_0 + 1)^d e^{M_0^\theta} \]
\[ \leq M_1^{10dM_1^\theta M_0^\theta} e^{-\gamma_0(|m-n|-2M_0)^\theta}, \]  

(90)

where the first inequality holds by \(|m - n| \leq 2M_1\) and the third inequality holds by (85).
**Case 2₁: z ∈ Jₙ.** For this case, following the proof of Case 1₂ (see (81)), one has

\[(91) \quad |G_{J_{n+1}}(x, z)| \leq 2(2M_1 + 1)^{2d} e^{M_1^\theta} T_s e^{-\gamma_0(\max\{\Gamma_s(z) - 4M_0, 0\})^\theta}.
\]

**Case 2₂: z ∈ Jₙ₊₁ \setminus Jₙ.**

Applying (58) (Λ₁ = Jₙ and Λ₂ = Jₙ₊₁ \setminus Jₙ), one has

\[|G_{J_{n+1}}(x, z)| \leq 2(2M_1 + 1)^{2d} e^{M_1^\theta} T_s \]

\[\times \sum_{n_1 \in Jₙ, n_2 \in Jₙ₊₁ \setminus Jₙ} |G_{J_{n+1}\setminus Jₙ}(n_2, z)| e^{-\gamma_0(\max\{\Gamma_s(n_1) - 4M_0, 0\})^\theta}
\]

\[\leq 2(2M_1 + 1)^{4d} e^{M_1^\theta} M_1^{10dM_1^\theta} T_s \sup_{n_1 \in Jₙ} e^{-\gamma_0(\max\{|\Gamma_s(z) - 2M_0, 0\}|^\theta) e^{-\gamma_0(\max\{\Gamma_s(n_1) - 4M_0, 0\})^\theta}
\]

\[\leq M_1^{11dM_1^\theta} M_0^{\sigma-\theta} T_s e^{-\gamma_0(\max\{\Gamma_s(z) - 10M_0, 0\})^\theta},
\]

where the second inequality holds by (91), the third inequality holds by (86), the fourth inequality holds by (61) and the fifth inequality holds by (64) and (72).

Putting all cases together and by (79), (81), (91) and (92), one has if (75) holds at sth step, then

\[|G_{J_{n+1}}(x, z)| \leq M_1^{12dM_1^\theta} M_0^{\sigma-\theta} T_s e^{-\gamma_0(\max\{\Gamma_s(z) - 10M_0, 0\})^\theta}
\]

\[(93) \quad |G_{J_{n+1}}(x, z)| \leq M_1^{12dM_1^\theta} M_0^{\sigma-\theta} T_s e^{-\gamma_0(\max\{\Gamma_s(z) - 10M_0, 0\})^\theta}.
\]

By (77) and (93), we obtain that (75) is true for

\[(94) \quad T_0 = e^{M_1^\theta},
\]

and

\[(95) \quad T_{s+1} = M_1^{12dM_1^\theta} M_0^{\sigma-\theta} T_s.
\]

By (75), (94) and (95), one has

\[|G_{J_n}(x, z)| \leq M_1^{13dM_1^\theta} M_0^{\sigma-\theta} e^{-\gamma_0\Gamma_s^\theta(z)}.
\]
By the assumption that $\tilde{\Lambda}_1$ is good, one has

\[(97) \quad g \leq 2B_1 = 2\kappa \frac{M_1}{M_0},\]

and hence (by (55))

\[(98) \quad k(z) \geq \frac{|x - z|}{4M_0} - 2B_1 - C(d) \geq \frac{|x - z|}{4M_0} - 2\kappa \frac{M_1}{M_0} - C(d).\]

By (98) and the definition of $\Gamma_s$, we have for $|x - z| \geq \frac{M_1}{10}$,

\[(99) \quad \Gamma_g(z) \geq (|x - z| - 8\kappa M_1 - 10(g + 1)M_0)^\sigma \]

\[\geq (|x - z| - 30\kappa M_1)^\sigma \]

\[\geq |x - z|^\sigma (1 - 160\kappa)^\sigma \]

\[\geq |x - z|^\sigma (1 - 200\tilde{\sigma}\kappa),\]

where $\kappa$ will be chosen to be sufficiently small.

By (96), (97) and (99), we have for $|x - z| \geq \frac{M_1}{10}$,

\[(100) \quad |G_{\tilde{\Lambda}_1}(x, z)| = |G_{J_g(x, z)}| \leq M_1^{13gDM_0^{\sigma - 3}e^{-\gamma_0\Gamma_g(z)}} \leq e^{-\gamma_0(1 - 200\kappa\tilde{\sigma} - 300d\kappa(1 - \sigma)\rho_0^{-1}\log M_0)\frac{\log M_0}{M_1^{1 - \rho + \tilde{\sigma} - \sigma}}}|x - z|^\sigma.\]

**Inductions:** Define

\[(101) \quad \gamma_m = \prod_{i=0}^{m-1} \gamma_0(1 - C(d)\kappa\tilde{\sigma} - C(d)\kappa\rho_\gamma^{-1}\frac{\log M_i}{M_i^{1 - \rho + \tilde{\sigma} - \sigma}}).\]

We remind that $1 - \rho + \tilde{\sigma} - \sigma > 0$. Fix an elementary region $\tilde{\Lambda}_1 \in E_{M_m}$ and $\tilde{\Lambda}_1 \subset \tilde{\Lambda}_0$. For any $x \in \tilde{\Lambda}_1$, consider the exhaustion $\{S_j(x)\}_{j=0}^l$ of $\tilde{\Lambda}_1$ at $x$ with width $M_{m-1}$. We say the annulus $A_j(x)$ is good, if for any $y \in A_j(x)$, there exists $W(y) \in E_{M_{m-1}}$ such that

\[y \in W(y) \subset A_j(x), \text{ dist } (y, A_j(x) \setminus W(y)) \geq M_{m-1}/2,\]

and for $|n - n'| \geq \frac{M_{m-1}}{10}$,

\[|(R_{W(y)}A_{W(y)})^{-1}(n, n')| \leq e^{-\gamma_m|n - n'|^{\sigma}}.\]

Otherwise, we call the annulus bad. An elementary region $\tilde{\Lambda}_1 \subset \tilde{\Lambda}_0$ is called bad provided for some $x \in \tilde{\Lambda}_1$ the number of bad annuli $\{A_j(x)\}$ exceeds

\[B_m := \kappa \frac{M_m}{M_{m-1}}.\]
Otherwise, we call $\tilde{\Lambda}_1$ good. Let $\mathcal{F}_m$ be an arbitrary family of pairwise disjoint bad elementary regions in $E_{M_m}$ contained in $\tilde{\Lambda}_0$. By induction, it is easy to see that

$$\# \mathcal{F}_m \leq \frac{1}{\kappa^m} \frac{N^c}{M_m}.$$  

Replace $M_0, M_1, \gamma_0, B_1$ with $M_{m-1}, M_m, \gamma_{m-1}, B_m$. By induction and following the proof of (100), we have for good elementary regions $\tilde{\Lambda}_1 \subset \tilde{\Lambda}_0$ and $\tilde{\Lambda}_1 \in E_{M_m}$, we have for $|x - z| \geq \frac{M_m}{10}$,

$$|G_{\tilde{\Lambda}_1}(x, z)| \leq e^{-\gamma_m |x-z|^\theta}.$$  

In order to reach $\tilde{\Lambda}_0$ after $k$ step, we need

$$M_k = M^\rho_k = N,$$

hence

$$\rho^k \approx \frac{1}{\xi}.$$  

We may modify $\rho$ a little bit at the last few steps to ensure that $k$ is a positive integer. However, this issue is rather small. Choose $\kappa = N^{-\delta}$ and

$$\delta = -\frac{1}{2} (1 - \varsigma) \frac{\log \rho}{\log \xi^{-1}}.$$  

Direct computations show that

$$\# \mathcal{F}_k \leq \frac{1}{\kappa^k} \frac{N^c}{M_k} < 1.$$  

(104) implies that $M_k = N$ is good. Therefore, (6) holds for

$$c_3 = \gamma_k,$$

where $k$ solves (103). Computations show that

$$c_3 = c_2 - N^{-\vartheta},$$

where $\vartheta = \vartheta(\sigma, \bar{\sigma}, \xi, \varsigma) > 0$.

\[\square\]

5. **Proof of Theorem 2.2**

The proof of Theorem 2.2 is based on matrix-valued Cartan-type estimates [15, 20, 36, 51]. For our purpose, a new version of Cartan’s estimate, which works for non-self-adjoint matrices, is necessary. For convenience, we include a proof in the Appendix.

**Lemma 5.1.** Let $T(x)$ be a $N \times N$ matrix function of a parameter $x \in [-\delta, \delta]^J \quad (J \in \mathbb{N})$ satisfying the following conditions:
(i) $T(x)$ is real analytic in $x \in [-\delta, \delta]^J$ and has a holomorphic extension to

$$\mathcal{D}_{\delta, \delta_1} = \left\{ x = (x_i)_{1 \leq i \leq J} \in \mathbb{C}^J : \sup_{1 \leq i \leq J} |\Re x_i| \leq \delta, \sup_{1 \leq i \leq J} |\Im x_i| \leq \delta_1 \right\}$$

satisfying

(105) \[ \sup_{x \in \mathcal{D}_{\delta, \delta_1}} \|T(x)\| \leq B_1, B_1 \geq 1. \]

(ii) For all $x \in [-\delta, \delta]^J$, there is subset $V \subset [1, N]$ with

$$|V| \leq M,$$

and

(106) \[ \|(R_{[1,N]} \setminus V)T(x)R_{[1,N]} \setminus V)^{-1}\| \leq B_2, B_2 \geq 1. \]

(iii) \[ \mes\{x \in [-\delta, \delta]^J : \|T^{-1}(x)\| \geq B_3\} \leq 10^{-3J}J^{-J}B_1^{-J}(1+B_1)^{-J}(1+B_2)^{-J}. \]

Let

(107) \[ 0 < \epsilon \leq (1 + B_1 + B_2)^{-10M}. \]

Then

(108) \[ \mes\{x \in [-\delta, \delta]^J : \|T^{-1}(x)\| \geq \epsilon^{-1}\} \leq C\epsilon^J e^{-\epsilon \left( \frac{\log_B(1+1)^{-1}}{M \log_B(1+1+B_2+B_3)} \right)^{1/J}}, \]

where $C = C(J), c = c(J) > 0$.

**Proof of Theorem 2.2.** Without loss of generality, we assume $i = 1$. Fix $x_1 \in \mathbb{T}^d$ and $x_1^- \in \mathbb{T}^{d-b_1}$. Recall that $x = (x_1, x_1^-) \in \mathbb{T}^d$.

Let $\Lambda = \mathcal{R} \subset [-N_3, N_3]^d$. By making $B_{\mathcal{R}}(x)$ slightly larger, we have there exists $\bar{\Lambda} \subset \Lambda$ such that for any $j \in \Lambda \setminus \bar{\Lambda}$, there exists $W(j) \in \mathcal{E}_{N_1}$ such that $W(j) \subset \Lambda \setminus \bar{\Lambda}$,

$$\dist(j, \Lambda \setminus \bar{\Lambda} \setminus W(j)) \geq N_1/2$$

and

(109) \[ \|G_{W(j)}\| \leq e^{N_1 \sigma}, \]

(110) \[ |G_{W(j)}(n, n')| \leq e^{-c_2|n-n'|^\delta} \text{ for } |n-n'| \geq \frac{N_1}{10}, \]

and

(111) \[ |\bar{\Lambda}| \leq C(d) L^{1-\delta} N_1^{2d}. \]

Indeed, $\bar{\Lambda}$ can be chosen so that

(112) \[ \bar{\Lambda} \subset \{ n \in \mathbb{Z}^d : \dist(n, B_{\mathcal{R}}(x)) \leq C(d) N_1 \}. \]
Let \( \eta = \frac{c_1}{b_1} \). Let \( \mathcal{D} \) be the \( e^{-\eta N_1} \) neighborhood of \( x_1 \) in the complex plane, i.e.,
\[
\mathcal{D} = \{ z \in \mathbb{C}^{b_1} : |\Im z| \leq e^{-\eta N_1}, |\Re z - x_1| \leq e^{-\eta N_1} \}.
\]

By the assumption that \( N_3 \leq e^{N_1} \frac{K_1}{b_1} \), one has for any \( y \in \mathcal{D} \),
\[
||x - y|| \leq e^{-e(\log(2N_3+2))K_1}
\]
and hence (by (8))
\[
|A(x; n, n') - A(y; n, n')| \leq K||x - y||^\gamma
\]
for \( n, n' \in [-N_3, N_3]^d \) and large \( N_1 \). By (110), (111), (114), and standard perturbation arguments, we have for any \( y \in \mathcal{D} \), and \( j \in \Lambda \setminus \bar{\Lambda} \),
\[
\|G_{W(j)}(x_1 + y, x_1^-)\| \leq 2e^{N_1\sigma},
\]
(115)
\[
\|G_{W(j)}(x_1 + y, x_1^-; n, n')\| \leq 2e^{-c_2|n - n'|^\sigma} \text{ for } |n - n'| \geq \frac{N_1}{10}.
\]
(116)
Substituting \( \Lambda \) with \( \Lambda \setminus \bar{\Lambda} \) in Lemma 4.2, one has for any \( y \in \mathcal{D} \),
\[
\|G_{\Lambda \setminus \bar{\Lambda}}(x_1 + y, x_1^-)\| \leq e^{2N_1\sigma}.
\]
(117)
We want to use Lemma 5.1. For this purpose, let
\[
T(y) = R_{\Lambda} A R_{\Lambda}, J = b_1, \delta = \delta_1 = e^{-n N_1}.
\]
Now we are in the position to check the assumptions of Lemma 5.1. By (114) and (7), one has \( B_1 = O(1) \).

Let \( V = \bar{\Lambda} \). By (112) and (117), one has
\[
M = |\bar{\Lambda}| \leq C(d)L^{1-\delta} N_1^{2d}, B_2 = e^{2N_1\sigma}.
\]
(118)
Applying Lemma 4.2 with \( M_0 = M_1 = N_2 \) and (12), one has
\[
\|T^{-1}(y)\| \leq 4(2N_2 + 1)^d e^{N_2 \sigma} \leq e^{2N_2 \sigma} =: B_3,
\]
except on a set of \( y \in \mathbb{T}^{b_1} \) with measure less than \( e^{-N_2^2} \).

Since \( N_2 \geq N_1 \), direct computation shows that
\[
10^{-2b_1} b_1 A_{b_1}(1 + B_1)^{-b_1} (1 + B_2)^{-b_1} \geq e^{-N_2^2}.
\]
This verifies (iii) in Lemma 5.1.

Let \( \epsilon = e^{-L\mu} \). By (118) and the assumption that \( L \geq N_2 \frac{2d+b_1 + \delta}{\mu+1+b_1} \), one has
\[
\epsilon < (1 + B_1 + B_2)^{-10M}.
\]
Let
\[
Y = \{ y \in \mathcal{D} : \|T^{-1}(y)\| \geq e^{L\mu} \}.
\]
By (109) of Lemma 5.1

$$\text{mes}(Y) \leq C e^{-c \left( \frac{L^{\mu-1+\delta}}{N^2 N_1^{2d+\sigma}} \right)^{1/b_1}}.$$  
(119)

By covering $T_{b_1}$ with balls with radius $e^{-\eta N_1}$, we have

$$\text{Leb}(\tilde{X}_R(x_1^n)) \leq e^{CN_1} e^{-c \left( \frac{L^{\mu-1+\delta}}{N^2 N_1^{2d+\sigma}} \right)^{1/b_1}} \leq e^{-c \left( \frac{L^{\mu-1+\delta}}{N^2 N_1^{2d+\sigma}} \right)^{1/b_1}},$$

where the second inequality holds by the assumption $L \geq N^{2d+4+2\mu+1+c_1}$. It implies (13).

\[\square\]

6. Proof of Theorems 2.3 and 2.4

**Theorem 6.1.** Let $\sigma, \tilde{\sigma}, \kappa, s \in (0, 1)$ and $\tilde{\sigma} > \kappa$. Assume $\text{diam}(\Lambda) \leq 2N + 1$. Let $M_0 = (\log N)^{1/s}$. Assume

$$c_2 \in (0, (1-5^{-\tilde{\sigma}})c_1].$$
(120)

Suppose that for any $n \in \Lambda$, there exists some $W = W(n) \in E_M$ with $M_0 \leq M \leq N^s$ such that $n \in W$, $\text{dist}(n, \Lambda \setminus W) \geq M^{1/2}$, $W \subset \Lambda$ and

$$\|G_W\| \leq 2e^{M^\sigma},$$

$$|G_W(n, n')| \leq 2e^{-c_2|n-n'|^{\tilde{\sigma}}} \text{ for } |n-n'| \geq \frac{M}{10}.$$  
(121)

Then

$$||G_\Lambda|| \leq 4(1 + 2N^{\kappa})^d e^{N^{\kappa} \sigma},$$

and

$$|G_\Lambda(n, n')| \leq e^{-\tilde{c}|n-n'|^{\tilde{\sigma}}} \text{ for } |n-n'| \geq N/10,$$

where

$$\tilde{c} = c_2 - \frac{O(1)}{M_0^{\sigma-s}} - \frac{O(1)}{M_0^{\sigma-\sigma}} - \frac{O(1)}{N^{\tilde{\sigma}-\kappa}}.$$  
(122)

**Proof.** (121) follows from Lemma 4.2 immediately.

Assume $|n-n'| \geq \frac{N}{10}$. Applying (57) with $\Lambda_1 = W = W(n)$, one has $n' \notin W(n)$ and

$$|G_\Lambda(n, n')| \leq \sum_{n_1 \in W \atop n_2 \in \Lambda \setminus W} e^{-c_1|n_1-n_2|^{\tilde{\sigma}}} |G_W(n, n_1)||G_\Lambda(n_2, n')|. $$
It implies

\[
|G_{\Lambda}(n, n')| \leq \sum_{n_1 \in W, |n_1 - n| \leq \frac{M}{\alpha}} e^{-c_1|n_1 - n_2|^\beta} |G_W(n, n_1)||G_{\Lambda}(n_2, n')|
\]

\[
+ \sum_{n_1 \in W, |n_1 - n| \geq \frac{M}{\alpha}} e^{-c_1|n_1 - n_2|^\beta} |G_W(n, n_1)||G_{\Lambda}(n_2, n')|
\]

\[
\leq \sum_{n_1 \in W, |n_1 - n| \leq \frac{M}{\alpha}} e^{M^\sigma} e^{-c_1|n_1 - n_2|^\beta} |G_{\Lambda}(n_2, n')|
\]

\[
+ \sum_{n_1 \in W, |n_1 - n| \geq \frac{M}{\alpha}} e^{-c_2|n - n_2|^\beta} |G_{\Lambda}(n_2, n')|
\]

\[
\leq e^{M^\sigma} \sum_{n_1 \in W, |n_1 - n| \leq \frac{M}{\alpha}} e^{-c_2|n - n_2|^\beta} |G_{\Lambda}(n_2, n')|
\]

\[
+ \sum_{n_1 \in W, |n_1 - n| \geq \frac{M}{\alpha}} e^{-c_2|n - n_2|^\beta} |G_{\Lambda}(n_2, n')|
\]

\[
\leq e^{M^\sigma} (2N + 1)^{2d} \sup_{n_2 \in A \setminus W} e^{-c_2|n - n_2|^\beta} |G_{\Lambda}(n_2, n')|
\]

(123)

\[
\leq (2N + 1)^{2d} \sup_{n_2 \in A \setminus W} e^{-c_2|n - n_2|^\beta} |G_{\Lambda}(n_2, n')|
\]

where the third inequality holds by (64) and the fourth inequality holds by (72).

Iterating (123) until \(|n_2 - n'| \leq 4N^\kappa\) (but at most \(\left\lceil \frac{2^d|n-n'|\beta}{M_0}\right\rceil\) times) and applying (121), we have for \(|n - n'| \geq \frac{N}{10}\),

\[
|G_{\Lambda}(n, n')| \leq (2N + 1)^{2d} e^{-c_2 \frac{\log N}{M_0}} |n - n'|^{\beta} e^{-\frac{c_1 \log \frac{1}{\alpha}}{M_0}} (|n - n'| - 4N^\kappa)^\beta 4(1 + 2N^\kappa)^d e^{N^\kappa}
\]

\[
\leq e^{-c_1|n - n'|^\beta} 4(1 + 2N^\kappa)^d e^{N^\kappa}
\]

\[
\leq e^{-c_1|n - n'|}
\]
It is easy to see that the number of generalized elementary regions in \([-N, N]^d\) with width larger or equal to \(N^\xi\) is bounded by \(N^{C(d)}\), more precisely for any \(\xi > 0\),

\[
#\{\Lambda \subset \mathbb{R}^d : \Lambda \subset [-N, N]\} \leq N^{C(d)}.
\]

**Proof of Theorem 2.3.** Since the Green’s function satisfies properties \(P\) with parameters \((\mu, \zeta, c_2)\) at size \(N_2\), we have there exists \(X_{N_2} \subset \mathbb{T}^d\) with

\[
\sup_{1 \leq i \leq k, x_i \in \mathbb{T}^d} \text{Leb}(X_{N_2}(x_i^-)) \leq N_3^{C(d)} e^{-N_2^\xi},
\]

such that

\[
\|G_{m+Q_{N_2}}(x)\| \leq e^{N_2^\mu},
\]

and for \(|n - n'| \geq N_2/10\),

\[
|G_{m+Q_{N_2}}(x; n, n')| \leq e^{-c_2|n - n'|^\theta},
\]

for any \(Q_{N_2} \in \mathcal{E}_{N_2}^0\) and \(|m| \leq N_3\). Indeed, we only need to set

\[
\tilde{X}_{N_2} = \bigcup_{|m| \leq N_3} X_{N_2}(f^m(x)).
\]

By the assumption \(N_3 \geq N_2^C\) and \(N_2 \geq N_3^C\) with large \(C\) depending on \(\varepsilon\), one has

\[
N_2 \leq N_3^\varepsilon, N_1 \leq N_2^\varepsilon.
\]

(126) Let \(\xi = \delta - 5\varepsilon\). Applying (15) to Theorem 2.2 and by (124) and (126), there exists \(X_{N_2} \subset [0, 1)^d\) such that

\[
\sup_{x_i^- \in \mathbb{T}^d} \text{Leb}(X_{N_2}(x_i^-)) \leq N_3^{C(d)} e^{-N_3^\xi(\frac{\sigma_1 - 1}{\sigma_1} + \frac{\sigma_2}{\sigma_1}) - \varepsilon}
\]

(127) and for any \(x \notin X_{N_3}, \mathcal{R} \subset \mathcal{R}_{N_3}^\infty\) with \(N_3 \leq L \leq N_3\),

\[
||G_{\mathcal{R}}(x)|| \leq e^{L^\theta}.
\]

(128) Let \(\tilde{F}\) be any pairwise disjoint elementary regions in \([-N_3, N_3]^d\) with size \([N_3^\xi]\). By (15), it is easy to see that there are at most \(N_1^{C(d)} N_3^{1-\delta} = N_3^{1-\delta+\varepsilon}\) in \(\tilde{F}\) will intersect elementary regions not in \(SG_{N_1}\). By Theorem 6.1, any elementary region in \([-N_3, N_3]^d\) with size \([N_3^\xi]\), without intersecting any non-\(SG_{N_1}\) elementary regions, will satisfy (3). It implies (3) is true for \(\varsigma = 1 - \varepsilon\). Applying Theorem 2.1 and (128), we obtain Theorem 2.3. Let us explain where the bound \(c_2 - N_1^{-\theta_1} - N_3^{-\theta_2}\) in (18) is from. Since \(N_3^\xi \leq e^{\xi N_3^\xi}\), one has \(s = \frac{11}{10} c\) in Theorem
6.1 Applying $M_0 = N_1$, $N = N_1^N$, $\sigma = \mu$ to Theorem 6.1, we obtain the bound $c_2 - O(1)N_1^{-1} - O(1)N_1^{-1} - N_1^{-\vartheta_2}$. Theorem 2.4 will only contribute $N_3^{-\vartheta_2}$.

Proof of Theorem 2.4. Fix any $m \in \mathbb{Z}^d$. Applying Theorem 2.3 with $\tilde{A}_m$, one has there exists a subset $X_{N_3}^m \subset \mathbb{T}^b$ such that

$$\sup_{1 \leq i \leq k, x \in \mathbb{T}^b - b_i} \text{Leb}(X_{N_3}^m(x_i)) \leq e^{-\frac{N_3}{2} + \frac{\vartheta_3^2}{2} - \epsilon},$$

and for any $x \notin X_{N_3}^m$ and $Q_{N_3} \in \mathcal{E}_{N_3}^0$,

$$\|(R_{Q_{N_3}} \tilde{A}_m^m(x) R_{Q_{N_3}})^{-1}\| \leq e^{N_3^\sigma},$$

and for $|n - n'| \geq \frac{N_3}{10}$,

$$|(R_{Q_{N_3}} \tilde{A}_m^m R_{Q_{N_3}})^{-1}(x; n, n')| \leq e^{-(c_2 - (N_1^{\vartheta_2} - N_1^{-\vartheta_2}))(n - n')^\vartheta}.$$

Let

$$X_{N_3} = \bigcup_{m \in \mathbb{Z}^d} X_{N_3}^m.$$

By (19) and (21), we have

$$\sup_{1 \leq i \leq k, x \in \mathbb{T}^b - b_i} \text{Leb}(X_{N_3}^m(x_i)) \leq e^{N_3^\sigma} e^{-\frac{N_3}{2} + \frac{\vartheta_3^2}{2} - \epsilon} \leq e^{-N_3^{-\vartheta_2}}.$$

7. Proof of Theorem 2.5

Proof. Once we have the LDT at hand, the modulus of continuity of the IDS is standard. The proof here follows from the corresponding part in [13, 65]. Let $N = |\log |E_1 - E_2||^{-\vartheta - \epsilon}$. Without loss of generality, assume $E_1 < E_2$ and let $E$ be the center of $[E_1, E_2]$. Therefore,

(129) $$|E_1 - E_2| \leq e^{-N^\sigma + \epsilon}.$$ 

By the assumption, there exists a set $X_N \subset \mathbb{T}^b$ such that

$$\text{Leb}(X_N) \leq e^{-N^c},$$

and for any $x \notin X_N$ and any $Q_N \in \mathcal{E}_{N}^0$,

$$||G_{Q_N}(E, x)|| \leq e^{N^\sigma}$$

$$|G_{Q_N}(E, x; n, n')| \leq e^{-c|n - n'|^\vartheta} \text{ for } |n - n'| \geq \frac{N}{10}.$$

where $c > 0$. We should mention that $X_N$ depends on $E$. By the assumption (23), for large $N_1$, one has
\[
\#\{n \in \mathbb{Z}^d : |n| \leq N_1, f^n(x) \in X_N\} \leq 2(2N_1 + 1)^{d-e-N^c}.
\]
Let $\Lambda = [-N_1, N_1]^d$. By making $\#\{n \in \mathbb{Z}^d : |n| \leq N_1, f^n(x) \in X_N\}$ slightly larger, we have there exists $\tilde{\Lambda} \subset \Lambda$ such that for all $j \in \Lambda \backslash \tilde{\Lambda}$, there exists $W(j) \in \mathcal{E}_N$ such that $W(j) \subset \Lambda \backslash \tilde{\Lambda}$, $\text{dist}(j, \Lambda \backslash \tilde{\Lambda} \backslash W(j)) \geq N/2$ and
\[
\|G_{W(j)}\| \leq e^{N^c},
\]
\[
|G_{W(j)}(n, n')| \leq e^{-c|n-n'|^\theta} \text{ for } |n - n'| \geq \frac{N}{10}.
\]
and
\[
|\tilde{\Lambda}| \leq C(d)N^{2d}(2N_1 + 1)^{d-e-N^c}.
\]
Here, $\tilde{\Lambda}$ is obtained in a similar way as (113).

Substituting $\Lambda$ with $\Lambda \backslash \tilde{\Lambda}$ in Lemma 4.2, we have
\[
\|G_{\Lambda \backslash \tilde{\Lambda}}(E, x)\| \leq 4(2N_1 + 1)^{d-eN^c}.
\]
By standard perturbation arguments, we have for any $\tilde{E} \in [E_1, E_2]$,
\[
\|G_{\Lambda \backslash \tilde{\Lambda}}(\tilde{E}, x)\| \leq 8(2N_1 + 1)^{d-eN^c}.
\]
Denote by $\xi_j$, $j = 1, 2, \cdots, M$, the normalized eigenfunctions of $H_\Lambda$ with eigenvalues falling into the interval $[E_1, E_2]$. Let $\xi$ be one of them with eigenvalue $\tilde{E}$. By definition,
\[
R_{\Lambda \backslash \tilde{\Lambda}}(H_\Lambda - E)R_{\Lambda \backslash \tilde{\Lambda}}\xi = (\tilde{E} - E)R_{\Lambda \backslash \tilde{\Lambda}}\xi.
\]
Applying $G_{\Lambda \backslash \tilde{\Lambda}}(E, x)$ to (133), one has
\[
R_{\Lambda \backslash \tilde{\Lambda}}\xi + G_{\Lambda \backslash \tilde{\Lambda}}(E, x)R_{\Lambda \backslash \tilde{\Lambda}}(H_\Lambda - E)R_{\Lambda \backslash \tilde{\Lambda}}\xi = (\tilde{E} - E)G_{\Lambda \backslash \tilde{\Lambda}}(E, x)R_{\Lambda \backslash \tilde{\Lambda}}\xi.
\]
Denote by $P$ the projection onto the range of $G_{\Lambda \backslash \tilde{\Lambda}}(E, x)$. Clearly, the dimension of this range does not exceed $\tilde{\Lambda}$. Thus rank$(P) \leq \tilde{\Lambda}$. By (129) and (132), one has
\[
\|(\tilde{E} - E)G_{\Lambda \backslash \tilde{\Lambda}}(E, x)R_{\Lambda \backslash \tilde{\Lambda}}\xi\| \leq \frac{1}{100}\|\xi\|.
\]
Applying $I - P$ to (134) and by (133), we have
\[
\|R_{\Lambda \backslash \tilde{\Lambda}}\xi - PR_{\Lambda \backslash \tilde{\Lambda}}\xi\| \leq \frac{1}{100}\|\xi\|.
\]
Applying (136) to each $\xi_j$, we have

$$M = \sum_{j=1}^{N} ||\xi_j||^2$$

$$\leq \frac{M}{2} + 4 \sum_{j=1}^{M} ||PR_{\Lambda \Lambda} \xi_j||^2 + 2 \sum_{j=1}^{M} ||R_{\Lambda} \xi_j||^2$$

$$\leq \frac{M}{2} + 4 \text{Trace}(PR_{\Lambda \Lambda}) + 2 \text{Trace}(R_{\Lambda})$$

$$\leq \frac{M}{2} + 6|\Lambda|$$

$$\leq \frac{M}{2} + C(d)N^{2d}(2N_1 + 1)^d e^{-N^\zeta}.$$ 

Therefore,

$$M \leq C(d)N^{2d}(2N_1 + 1)^d e^{-N^\zeta}.$$ 

It implies

$$k(x, E_1, E_2) \leq C(d)N^{2d}e^{-N^\zeta} \leq e^{-(\log \frac{1}{2N_1})^{d/2} - \epsilon}.$$ 

\[ \square \]

8. The discrepancy and semi-algebraic sets

8.1. Discrepancy. Let $\bar{x}_1, ..., \bar{x}_N \in [0, 1)^b$ and $S \subset [0, 1)^b$. Let $A(S; \{\bar{x}_n\}_{n=1}^N)$ be the number of $\bar{x}_n$ ($1 \leq n \leq N$) such that $\bar{x}_n \in S$. We define the discrepancy of the sequence $\{\bar{x}_n\}_{n=1}^N$ by

$$D_N(\{\bar{x}_n\}_{n=1}^N) = \sup_{S \subset C} \left| \frac{A(S; \{\bar{x}_n\}_{n=1}^N)}{N} - \text{Leb}(S) \right|,$$

where $C$ is the family of all intervals in $[0, 1)^b$, namely $S$ has the form of

$$S = [\theta_1, \beta_1] \times [\theta_2, \beta_2] \times \cdots \times [\theta_b, \beta_b]$$

with $0 \leq \theta_n < \beta_n < 1$, $n = 1, 2, \cdots, b$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_b) \in [0, 1)^b$. The $b$-dimensional sequence $\bar{x}_n = (n\alpha_1, n\alpha_2, \cdots, n\alpha_b) \mod \mathbb{Z}^b$ ($n\alpha$ for short), $n = 1, 2, \cdots$, is called the Kronecker sequence. We denote by the discrepancy of $\{n\alpha\}_{n=1}^N$, $D_N(\alpha)$. The following Lemmas are well known.

**Lemma 8.1.** [32] Assume $\alpha \in \text{DC}(\kappa, \tau)$. Then

$$D_N(\alpha) \leq C(b, \kappa, \tau)N^{-\frac{3}{2} \log N^2}.$$ 

**Lemma 8.2.** [32] For almost every $\alpha$, we have

$$D_N(\alpha) \leq C(\alpha)N^{-1}(\log N)^{b+2}.$$ 

Let \( f: \mathbb{T}^b \to \mathbb{T}^b \) be defined as follows
\[
T(y_1, y_2, \ldots, y_b) = (y_1 + \alpha, y_2 + y_1, \ldots, y_b + y_{b-1}).
\]
Let \( T^n \) be the \( n \)th iteration of \( T \) and \( \bar{Y}_n = T^n(y_1, \ldots, y_b) \).

**Lemma 8.3.** Assume \( \alpha \in DC(\kappa, \tau) \). Then for any \( \varepsilon > 0 \),
\[
D_N(\{\bar{Y}_n\}_{n=1}^N) \leq C(b, \kappa, \tau, \varepsilon) N^{-\frac{1}{2^{b-1} \kappa} + \varepsilon}.
\]

**Remark 9.** Lemma 8.3 follows from the Erdős-Turán inequality (see Corollary 1.1 in p.8 of [63]) and the Weyl’s method (Theorem 2 in p.41 of [63]).

The Erdős-Turán inequality and Weyl’s method also imply

**Lemma 8.4.** Assume \( \alpha \in DC(\kappa, \tau) \). Let \( Y_n = P_b(T^n(y_1, \ldots, y_b)) \), where \( P_b \) is the \( b \)th coordinate projection. Then for any \( \varepsilon > 0 \),
\[
D_N(\{Y_n\}_{n=1}^N) \leq C(b, \kappa, \tau, \varepsilon) N^{-\frac{1}{2^{b-1} \kappa} + \varepsilon}.
\]

**8.2. Semi-algebraic sets.** A set \( S \subset \mathbb{R}^n \) is called a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let \( \{P_1, \cdots, P_s\} \subset \mathbb{R}[x_1, \cdots, x_n] \) be a family of real polynomials whose degrees are bounded by \( d \). A (closed) semi-algebraic set \( S \) is given by an expression
\[
S = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^n : P_\ell(x) \varsigma_{j\ell} 0\},
\]
where \( \mathcal{L}_j \subset \{1, \cdots, s\} \) and \( \varsigma_{j\ell} \in \{\geq, \leq, =\} \). Then we say that \( S \) has degree at most \( sd \). In fact, the degree of \( S \) which is denoted by \( \deg(S) \), means the smallest \( sd \) over all representations as in (138).

The following lemma is a special case appearing [10]. It is restated in [15].

**Lemma 8.5.** [15, Theorem 9.3] [10, Theorem 1] Let \( S \subset [0,1]^n \) be a semi-algebraic set of degree \( B \). Then the number of connected components of \( S \) does not exceed \( (1 + B)^C(n) \).

The following lemma follows from the Yomdin-Gromov triangulation theorem [39, 73], which has been stated in [15]. We refer readers to [11] and references therein for the complete proof of the Yomdin-Gromov triangulation theorem.

**Lemma 8.6.** [13, Corollary 9.6] Let \( S \subset [0,1]^n \) be a semi-algebraic set of degree \( B \). Let \( \varepsilon > 0 \) be a small number and \( \text{Leb}(S) \leq \varepsilon^n \). Then \( S \) can be covered by a family of \( \varepsilon \)-balls with total number less than \( \frac{(1 + B)^C(n)}{\varepsilon^{n-1}} \).

**Theorem 8.7.** Assume that the discrepancy of the sequence \( \{\vec{x}_j\}_{j=1}^N \) satisfies
\[
D_N(\{\vec{x}_j\}_{j=1}^N) \leq N^{-\varepsilon},
\]
for some $\varsigma > 0$. Let $S \subset [0, 1]^n$ be a semi-algebraic set with degree less than $B$. Suppose

$$\text{Leb}(S) \leq N^{-\varsigma}.$$  

Then

$$A(S; \{\bar{x}_j\}_{j=1}^N) \leq (1 + B)^{C(n)} N^{1-\frac{\varsigma}{n}}.$$  

Proof. Let $\epsilon = N^{-\frac{\varsigma}{n}}$. By Lemma 8.6, $S$ can be covered, at most $(1 + \frac{1+B)^{C(n)}}{\epsilon^{n-1}}$, $\epsilon$-balls. Pick one $\epsilon$-ball, say $J$. By the fact $D_N(\{\bar{x}_j\}_{j=1}^N) \leq N^{-\varsigma}$, one has

$$A(J; \{\bar{x}_j\}_{j=1}^N) \leq CN\epsilon^n + N^{1-\varsigma} \leq CN^{1-\varsigma},$$  

where $C$ depends on the dimension $n$. Since there are at most $(1 + \frac{1+B)^{C(n)}}{\epsilon^{n-1}}$ balls, we have

$$A(S; \{\bar{x}_j\}_{j=1}^N) \leq (1 + B)^{C(n)} \frac{1}{\epsilon^{n-1}} N^{1-\varsigma} = (1 + B)^{C(n)} N^{\frac{n-1}{n}} \varsigma N^{1-\varsigma} = (1 + B)^{C(n)} N^{1-\frac{\varsigma}{n}}.$$  

\[\square\]

Remark 10.  

- Theorem 8.7 says that there is a factor $b$ loss (referred to as dimension loss) when passing discrepancy from intervals to semi-algebraic sets. The dimension loss is not surprising. For example, there is also a dimension loss passing the discrepancy to the isotropic discrepancy [56, Theorem 1.6].

- The proof of Theorem 8.7 is taken from Bourgain [15], where no explicit bounds are given.

For a set $S \subset [0, 1)^2$, denote by $l(S)$ the length of the longest line segment contained in $S$.

Lemma 8.8. [22, Theorem 5.1] Assume $\alpha_1 \in \text{DC}(\kappa, \tau)$ and $\alpha_2 \in \text{DC}(\kappa, \tau)$. Let $S \subset [0, 1)^2$ be a semi-algebraic set with degree less than $B$ and

$$l(S) \leq \frac{1}{2} \min_{1 \leq |k| \leq 2N} ||k\alpha||.$$  

Then

$$\#\{k = (k_1, k_2) \in \mathbb{Z}^2 : |k| \leq N, (k_1\alpha_1, k_2\alpha_2) \in S \mod \mathbb{Z}^2\} \leq (1 + B)^{C(d)} C(\kappa, \tau) N^{3\kappa - \frac{3}{2}}.$$  

(139)
9. Proof of all the results in Section 3

Applying Theorem 2.5 with \( \sigma = 1 - \varepsilon \), Theorem 3.2 follows from Theorem 3.1, Theorem 3.8 follows from Theorem 3.7, Theorem 3.12 follows from Theorem 3.11, Theorem 3.15 follows from Theorem 3.14, Theorem 3.19 follows from Theorem 3.18 and Theorem 3.21 follows from Theorem 3.20.

Applying strong Diophantine frequencies to Theorems 3.15 and 3.21, we obtain Corollaries 3.16 and 3.22.

With large deviation theorems 3.7 and 3.11 at hand, the proof of Theorems 3.10 and 3.13 is rather standard. We refer the readers to [17, Section 3], [20, Section 6] and [15, Chapter XV] for details. We note that the only difference is that the degree of semi-algebraic sets is at most \( e^{(\log N)^C} \) in our cases, not \( N^C \).

By the discussion above, in order to prove all the results in Section 3, it suffices to prove Theorems 3.1, 3.3, 3.7, 3.9, 3.11, 3.14, 3.18, 3.20 and Corollary 3.17.

In this section, \( C(c) \) is always a large (small) constant. It may change even in the same formula.

Lemma 9.1. [15, Prop.7.19] Let \( H(x) \) be given by (32) and the Lyapunov exponent is given by (35). Suppose \( L(E) > 0 \). Then there exist \( 0 < \sigma < 1 \) and \( \zeta > 0 \) such that for large \( N \), there exists \( X_N \subset \mathbb{T}^b \) such that \( \text{Leb}(X_N) \leq e^{-N\zeta} \) and for \( x \notin X_N \), one of the intervals

\[
\Lambda = [1, N]; [1, N - 1]; [2, N]; [2, N - 1]
\]

will satisfy

\[
|G_{\Lambda}(n_1, n_2)| \leq e^{-L(E)|n_1-n_2|+N\sigma}.
\]

Proof of Theorem 3.1. By Lemma 9.1, there exist \( 0 < \sigma_1 < 1 \) and \( \zeta_1 > 0 \) such that for any large \( N_1 \), there exists \( X_{N_1} \subset \mathbb{T}^b \) such that \( \text{Leb}(X_{N_1}) \leq e^{-N_1^{\zeta_1}} \) and for \( x \notin X_{N_1} \), one of the intervals

\[
\Lambda(N_1) = [1, N_1]; [1, N_1 - 1]; [2, N_1]; [2, N_1 - 1]
\]

will satisfy

\[
|G_{\Lambda(N_1)}(n_1, n_2)| \leq e^{-L(E)|n_1-n_2|+N_1^{\sigma_1}}.
\]

By approximating the analytic function with trigonometric polynomials given by (28) and using Taylor expansions, we can further assume that \( X_{N_1} \) is a semi-algebraic set with degree less than \( e^{(\log N_1)^C} \). This argument is quite standard. We refer to [15] for details. By Lemma 8.4 and Theorem 8.7 for any \( e^{(\log N_1)^C} \leq N_3 \leq e^{N_1^c} \),

\[
A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq N_3^{1-\frac{1}{C}} + \varepsilon.
\]

Let \( N_2 = N_3^{\frac{1}{C}} \). Applying (141) to \( N_2 \), one has

\[
|G_{\Lambda(N_2)}(n_1, n_2)| \leq e^{-L(E)|n_1-n_2|+N_2^{\sigma_1}},
\]
except for a set of $x$ with measure less than $e^{-N_0^{2\gamma}}$. Now Theorem 3.1 follows from Theorem 2.3. We should mention that the elementary region is $[-N_1, N_1]$ in Theorem 2.3 which is slightly different from (140). However, the same statement is true.

**Proof of Theorem 3.18.** The proof of Theorem 3.18 is similar to that of Theorem 3.1. The difference is that instead of Lemma 9.1, we need to use the corresponding statements in p.3575 [70] for initial scales. We also need to use Lemma 8.4 instead of Lemma 8.3.

**Proof of Theorem 3.11.** Let $N_2 = e^{N_1^\epsilon}$. Assume the Green’s function in Theorem 3.11 satisfies properties $P$ with parameters $(\mu, \zeta, c_2)$ at sizes $N_1$ and $N_2$. Let $N_3 = N_2^{C}$. We can assume that $X_{N_1}$ is a semi-algebraic set with degree less than $e^{(\log N_1)^C}$. By Lemma 8.5, $X_{N_1}$ is consisted of at most $e^{(\log N_1)^C}$ intervals with measure less than $e^{-N_1^{\zeta}}$. Let $I$ be one of the intervals. Since $\omega$ satisfies Diophantine condition, for any $x \in T$, there is at most one $n \in \mathbb{Z}^d$ with $|n| \leq N_3$ such that $x + n\omega \mod\mathbb{Z} \in I$. Therefore,

\[(143)\quad A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq e^{(\log N_1)^C} \leq N_3^\epsilon.\]

By Theorem 2.3, we have the Green’s function satisfies properties $P$ with parameters $(\sigma, \sigma - \epsilon, c_2 - N_3^{\theta})$ at size $N_3$. Standard Neumann series expansion ensures that for any large $N_0$, there exists $\lambda_0$ such that for any $\lambda > \lambda_0$, the Green’s functions have properties $P$ with parameters $(\sigma, \sigma - \epsilon, \frac{4}{5}c_1)$ at all sizes smaller than $N_0$ [51, Theorem 4.3]. Now Theorem 3.11 follows by standard induction. See pages 15 and 16 in [51] for details.

**Proof of Theorem 3.7.** Fix $N_1$. Let $N_2 = e^{N_1^\epsilon}$ and $N_3 = N_2^{C}$. Assume the Green’s function in Theorem 3.7 satisfies properties $P$ with parameters $(\mu, \zeta, c_2)$ at sizes $N_1$ and $N_2$. We can again assume that $X_{N_1}$ is a semi-algebraic set with degree less than $e^{(\log N_1)^C}$. By Lemma 8.1 and Theorem 8.7

\[(144)\quad A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq N_3^{\frac{1-\frac{d}{2\kappa} + \epsilon}.\]

By Theorem 2.3, we have the Green’s function satisfies properties $P$ with parameters

\[
\left(\sigma, \frac{\sigma - 1}{b^2 \kappa} + \frac{1}{b^2 \kappa^2} - \epsilon, c_2 - N_3^{\theta}\right)
\]

at size $N_3$. As the arguments at the end of proof of Theorem 3.11 large $\lambda$ will ensure the initial scales and hence Theorem 3.7 follows by induction.

**Proof of Theorems 3.3 and 3.9.** The proof of Theorems 3.3 and 3.9 closely follow that of Theorems 3.2 and 3.8. The difference is that we need to use Lemma 8.2 instead of Lemma 8.1. □
Proof of Theorem 3.14. Replacing Lemma 8.1 with Lemma 8.3, Theorem 3.14 follows Theorem 3.7.

Proof of Corollary 3.17. By formula (3.53) in [19], one has for almost every $\alpha$,

$$A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq N_3^{1-\frac{1}{3}+\varepsilon}.$$  \hspace{1cm} (145)

Let $\delta = 1/3 - \varepsilon$. Applying $\tilde{\sigma} = 1$, $\sigma = 1 - \varepsilon$ and $b_i = 2$ in Theorem 2.3 and then Theorem 2.5 we obtain Corollary 3.17. Indeed, 1/18 comes from $(1/3)^2/b$.

Proof of Theorem 3.20. The proof of Theorem 3.20 is similar to that of Theorems 3.11 and 3.7. We only point out the modifications.

- The induction goes in the following way. The semi-algebraic set $X_N$ intersecting with any line segments contained in $[0,1)^2$ has Lebesgue measure at most $e^{-N^\varepsilon}$. The assumption that $v$ is not constant on any line segments ensure the initial scales.
- Replace (143) or (144) with (139).
- Since the induction is based on semi-algebraic sets only on line segments, the Cartan’s estimate will not lead to dimension loss. In other words, when (16) is used to do the induction, $b_i = 1$.

Remark 11. (1) The calculation of the bound in Theorem 3.20 goes in the following way. By (139), the sublinear bound is

$$3\kappa - \frac{9}{4} = 1 - \delta,$$  \hspace{1cm} where $\delta = \frac{13}{4} - 3\kappa$.

Therefore, the bound in (16) becomes $(b_i = 1)$

$$\frac{\sigma - 1}{b_i} \delta + \frac{\delta^2}{b_i} = (\sigma - 1)\delta + \delta^2$$

$$= (\sigma - 1) \left( \frac{13}{4} - 3\kappa \right) + \left( \frac{13}{4} - 3\kappa \right)^2.$$  

(2) The induction of Theorem 3.20 follows the corresponding parts in [22].

Our quantitative approaches developed in the paper allow us to obtain the explicit bound.

APPENDIX A. CARTAN’S ESTIMATES FOR NON-SELF-ADJOINT MATRICES

In the following, we will prove the several variables matrix-valued Cartan estimate (Lemma 5.1). The proof is similar to that in [14, 15, 17, 20, 51]. The improvement is that we do not assume the matrix is self-adjoint.
Lemma A.1. Let $T$ be the matrix

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $T_1$ is an invertible $n \times n$ matrix, $T_2$ is an $n \times k$ matrix, $T_3$ is a $k \times n$ matrix, and $T_4$ is a $k \times k$ matrix. Let

$$S = T_4 - T_3 T_1^{-1} T_2.$$

Then $T$ is invertible if and only if $S$ is invertible, and

\begin{equation}
\|S^{-1}\| \leq \|T^{-1}\| \leq C(1 + \|T_2\|)(1 + \|T_3\|)(1 + \|T_1^{-1}\|)^2(1 + \|S^{-1}\|),
\end{equation}

where $C$ is an absolute constant.

Proof. It is easy to check that

\begin{equation}
T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ T_3 T_1^{-1} & I \end{pmatrix} \begin{pmatrix} I & T_2 \\ 0 & S \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}.
\end{equation}

It implies $T$ is invertible if and only if $S$ is invertible. By (147), one has

\begin{equation}
T^{-1} = \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} I & T_2 \\ 0 & S \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ T_3 T_1^{-1} & I \end{pmatrix}^{-1}
\end{equation}

\begin{equation}
= \begin{pmatrix} T_1^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -T_2 S^{-1} \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_3 T_1^{-1} & I \end{pmatrix}
\end{equation}

\begin{equation}
= \begin{pmatrix} * & * \\ * & S^{-1} \end{pmatrix}.
\end{equation}

Now the second inequality of (146) follows from (148) and the first one follows from (149). □

Denote by $D(z, r)$ the standard disk on $\mathbb{C}$ of center $z$ and radius $r > 0$.

Lemma A.2. [37, Lemma 2.15] Let $f(z_1, \ldots, z_J)$ be an analytic function defined in a polydisk $P = \prod_{1 \leq i \leq J} D(z_{i,0}, 1/2)$ and $\phi = \log |f|$. Let $\sup_{\bar{z} \in P} \phi(\bar{z}) \leq M$, $m \leq \phi(z_0)$, $z_0 = (z_{1,0}, \ldots, z_{J,0})$. Given sufficiently large $F$, there exists a set $B \subset P$ such that

\begin{equation}
\phi(z) > M - C(J) F(M - m), \text{ for any } z \in \prod_{1 \leq i \leq J} D(z_{i,0}, 1/4) \setminus B,
\end{equation}

and

\begin{equation}
\text{mes}(B \cap \mathbb{R}^J) \leq C(J)e^{-F^{1/J}}.
\end{equation}

Proof of Lemma 5.1. The proof is similar to that of Lemma 3.4 in [15]. In the following proof, $C = C(J)$ and $c = c(J)$.

Let

$$\mu = 10^{-2} J^{-1} \delta_1 (1 + B_1)^{-1} (1 + B_2)^{-1}.$$
Fix 
\[ x_0 \in [-\delta/2, \delta/2]^J \]
and consider \( T(z) \) with \( |z - x_0| = \sup_{1 \leq i \leq J} |z_i - x_{0,i}| < \mu \). Thanks to Cauchy’s estimate and (105), one obtains for \( |z - x_0| < \mu \),
\[
\|\partial_{z_i} T(z)\| \leq \frac{4B_1}{\delta_1}, i = 1, 2, \ldots, J,
\]
which implies
\[
\|T(z) - T(x_0)\| \leq \frac{4J B_1 \mu}{\delta_1} \leq 25^{-1}(1 + B_2)^{-1}.
\]
From the assumption (ii) of Lemma 5.1, we can find \( V = V(x_0) \) so that \( |V| = M \leq M \) and (106) is satisfied. Denote by \( V^c = [1, N] \setminus V \). Thus using the standard Neumann series argument and (106), one has
\[
(R_V T(z) R_V^{-1})^{-1} \leq 2B_2 \text{ for } |z - x_0| < \mu.
\]
We define for \( |z - x_0| < \mu \) the analytic function
\[
S(z) = R_V T(z) R_V - R_V T(z) R_V (R_V T(z) R_V^{-1})^{-1} R_V T(z) R_V.
\]
Then by (152) and (153), we have
\[
\|S(z)\| \leq 3B_1^2 B_2.
\]
Recalling Lemma A.1 if \( S(z) \) is invertible, so is \( T(z) \) and by (146),
\[
\|S^{-1}(z)\| \leq C \|T^{-1}(z)\| \leq C B_1^2 B_2^2 (1 + \|S^{-1}(z)\|).
\]
For \( x \in \mathbb{R}^J \), one has
\[
\|S(x)\| \geq \det S(x).
\]
Let \( \lambda = \min\{|\lambda| : \lambda \in \sigma(S(x))\} \). We have
\[
|\det S(x)| \geq \lambda^M \geq \|S^{-1}(x)\|^{-\tilde{M}}.
\]
By Cramer’s rule, one has every entry of \( S^{-1}(x) \) is bounded by
\[
\frac{|\det S(x)|^{\tilde{M}-1}}{|\det S(x)|}
\]
and hence (by (154))
\[
\|S^{-1}(x)\| \leq \frac{\tilde{M}(3B_1^2 B_2)^{\tilde{M}}}{|\det S(x)|}.
\]
Let
\[
\phi(z) = \log |\det S(x_0 + \mu z)|, \ |z| < 1.
\]
Then by (156) and (154),
\begin{equation}
\sup_{|z|<1} \phi(z) \leq C \tilde{M} \log(B_1 + B_2).
\end{equation}

By (107) and the definition of \(\mu\), there is some \(x_1\) with \(|x_0 - x_1| < \mu/10\) such that
\begin{equation}
\|T^{-1}(x_1)\| \leq B_3.
\end{equation}

Hence by (155), \(\|S^{-1}(x_1)\| \leq CB_3\), and from (157),
\begin{equation}
\phi(a) \geq -C \tilde{M} \log B_3,
\end{equation}

where \(a = \frac{x_1 - x_0}{\mu}\), so \(|a| < 1/10\). Let
\[ \mathcal{P} = \prod_{1 \leq i \leq J} D(a_i, 1/2). \]

Therefore, one has
\[ \sup_{z \in \mathcal{P}} \phi(z) \leq C \tilde{M} \log(B_1 + B_2), \phi(a) \geq -C \tilde{M} \log B_3. \]

Applying Lemma A.2 and recalling (150), (151), for any \(F \gg 1\), there is some set \(B \subset \prod_{1 \leq i \leq J} D(a_i, 1/4)\) with
\begin{equation}
\phi(z) \geq -CF \tilde{M} \log(B_1 + B_2 + B_3) \text{ for } z \in \prod_{1 \leq i \leq J} D(a_i, 1/4) \setminus B,
\end{equation}

and
\begin{equation}
\mes(B \cap \mathbb{R}^J) \leq Ce^{-F^{1/J}}.
\end{equation}

For \(0 < \epsilon < 1\), let
\[ F = \frac{-c \log \epsilon}{\tilde{M} \log(B_1 + B_2 + B_3)}. \]

Then by (162) and (163),
\[ \mes \left\{ x \in \mathbb{R}^J : |x - x_1| < \mu/4 \text{ and } |\det(S(x))| \leq \epsilon \right\} = \mu^J \mes \left\{ x \in \mathbb{R}^J : |x - a| < 1/4 \text{ and } \phi(x) \leq \log \epsilon \right\} \leq C \mu^J e^{-F^{1/J}}. \]

Since \(|x_0 - x_1| < \mu/10\), we have
\begin{equation}
\mes \left\{ x \in \mathbb{R}^J : |x - x_0| < \mu/8 \text{ and } |\det(S(x))| \leq \epsilon \right\} \leq C \mu^J e^{-c \left( \frac{\log \epsilon^{-1}}{M \log(B_1 + B_2 + B_3)} \right)^{1/J}}.
\end{equation}

Recalling (155), (158) and (108), one has for \(|x - x_0| < \mu/8\) and \(|\det S(x)| \geq \epsilon\),
\begin{equation}
\|T^{-1}(x)\| \leq CB_1^2 B_2 \epsilon^{-1} \tilde{M}(3B_3^2 B_2)^M \leq \epsilon^{-2}.
\end{equation}
Covering $[-\frac{\delta}{2}, \frac{\delta}{2}]^J$ by cubes of side $\mu/4$, and combining (164) and (165), one has

$$\text{mes} \left\{ x \in [-\delta/2, \delta/2]^J : \|T^{-1}(x)\| \geq \epsilon^{-2} \right\} \leq C\delta^J e^{-\epsilon \left( \frac{\log \epsilon^{-1} \log (M_B \log (B_1 + B_2 + B_3))}{M_B \log (B_1 + B_2 + B_3)} \right)^{1/J}}$$

$$\leq C\delta^J e^{-\epsilon \left( \frac{\log \epsilon^{-1} \log (M_B \log (B_1 + B_2 + B_3))}{M_B \log (B_1 + B_2 + B_3)} \right)^{1/J}}.$$  

\[ \square \]

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