Fractal Properties and Small-scale Structure of Cosmic String Networks

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We present results from a detailed numerical study of the small-scale and loop production properties of cosmic string networks, based on the largest and highest resolution string simulations to date. We investigate the non-trivial fractal properties of cosmic strings, in particular, the fractal dimension and renormalised string mass per unit length, and we also study velocity correlations. We demonstrate important differences between string networks in flat (Minkowski) spacetime and the two very similar expanding cases. For high resolution matter era network simulations, we provide strong evidence that small-scale structure has converged to ‘scaling’ on all dynamical length scales, without the need for other radiative damping mechanisms. We also discuss preliminary evidence that the dominant loop production size is also approaching scaling.

I. INTRODUCTION

Topological defects are an unavoidable consequence of phase transitions in the early universe (for a review see ref. \cite{1}). Cosmic strings, in particular, will form in a range of cosmological scenarios, including at the end of brane inflation \cite{2,3,4}. Since they are intrinsically non-linear objects, one is ultimately forced to resort to high resolution numerical simulations \cite{5,6} if one wants to study their evolution and cosmological consequences in detail.

In recent years, while some authors have exploited ever-increasing computing capabilities \cite{7,8,9}, many others have tried to simplify the problem. For example, a number of flat (Minkowski) spacetime string simulations have been performed \cite{10,11,12,13,14,15}, which it has been argued are a good approximation to the expanding case. (The idea is that the string correlation length is significantly smaller than the horizon size and so the network should not feel the expansion.) Another popular approach is to generate artificial networks \cite{16,17} made of a (variable) number of segments with a given size and velocity, and aiming to mimic the evolution of a real network by enforcing (by hand) suitable variations of these parameters (e.g., as predicted by quantitative analytic string evolution models \cite{18,19,20,21}). Finally, doubts have also been raised as to whether the Nambu action accurately reproduces the dynamics of vortex-strings in the underlying field theory \cite{22,23} (though we will not discuss this issue further here, see \cite{8}). The problem is that different approaches yield quantitatively different and even sometimes contradictory cosmological consequences. This raises the question of how reliable all these approximations really are. One of the goals of this letter is to clarify these issues. In particular, we argue that there are important differences between string networks evolved in flat spacetime and in an expanding universe.

There is also considerable confusion in the literature on the issue of scaling. This is partially due to the fact that people define scaling in different ways. The weakest definition – large-scale scaling – is simply to say that one wants the energy density to be a constant fraction of the total density in the universe. This is the attractor solution of the evolution of long-string networks, and is strongly confirmed by both analytical and numerical evidence. On the other hand, this by itself does not guarantee that small scale features of the network are also scaling, such as the typical scale of small scale wiggles, that of loop production, or even the correlation length itself. In this letter we aim to clarify these different concepts. We also present strong evidence for scaling of the small-scale features of the string network (as measured by its multifractal dimension \cite{24,25} and renormalized mass per unit length) and we discuss the scaling of the characteristic loop production size.

II. AVERAGED AND MICROSCOPIC PROPERTIES

We have performed many thousands of CPU hours of cosmic string simulations, using an upgraded version of the Allen-Shellard string code on the COSMOS supercomputer. Some preliminary, large-scale results have already been discussed in \cite{8,9}, and a thorough description of the simulations will be presented elsewhere \cite{21}. Here, however, we take advantage of the extremely high resolution of our runs to provide a detailed description of the averaged and small-scale properties of cosmic string networks, as well as of the processes that determine loop production. This goes substantially further than the previous analyses of refs. \cite{21,22} and it is interesting to compare with the more recent flat spacetime analysis of ref. \cite{15}. Ultra-high resolution simulations were performed in the matter and radiation epochs and in flat (Minkowski) spacetime. The initial Vachaspati-Vilenkin
networks [29] have resolutions between 75 and 256 points per correlation length (PPCL), and we subsequently enforce a constant resolution in physical coordinates. This resolution is a substantial improvement over currently published simulations. These networks are then evolved for a dynamic range (in conformal time) of order 3 (and up to 6), each taking several weeks of processor time.

We note that not everybody agrees on when a network reaches the scaling regime. For example, numerically we find that the number of strings per horizon $\rho t^2/\mu_0$ becomes a constant (a plausible definition of linear scaling) before the correlation length $\xi$ becomes a constant fraction of physical time $t$ (which is another possibility). This, in turn, happens before other properties of the network reach scaling—indeed, there are a number of different timescales involved in the approach to scaling. We have developed our own statistics, based entirely on small-scale properties, to quantify how far a network is from scaling. We will describe them in detail elsewhere [26], but for the purposes of this letter it is enough to think of the two definitions above as roughly measuring large-scale and small-scale scaling. Our high resolution and large dynamic range allow us to go well beyond the point where large-scale scaling is reached, and even to reach small-scale scaling (under certain circumstances).

We must emphasise that we have chosen initial conditions for the radiation and matter era boxes for which we knew (by performing other smaller simulations beforehand [8, 9] and by using analytic modelling [20, 21]) that the approach to scaling would be fastest. Also with this aim, we have started the runs giving the strings an initial RMS velocity $v_i \approx 0.6$. For other choices of initial conditions, the approach to scaling would be considerably slower. By the end of each run, the error made in energy conservation was less than half of a percent.

Consider first some averaged properties of the networks, displayed in Fig. 1. This shows the evolution of the ‘characteristic’ lengthscale

$$\rho_\infty \equiv \mu_0/L^2,$$

with $\mu_0$ being the ‘bare’ string mass per unit length, the correlation (or persistence) length $\xi$ (to be defined below), the RMS string velocity, the coherent velocity at the scale of the correlation length $v_c$, the dimensionless parameter $\rho_\infty t^2/\mu_0$ (essentially a measure of the number of long strings inside one horizon volume), and the ‘renormalised’ dimensionless string mass per unit length defined as

$$\mu_{\text{eff}} \equiv \mu/\xi.$$

These properties have strikingly different behaviours in the three different regimes. Interestingly, the only property which seems to be relatively independent of the cosmological scenario is the coherent velocity at the scale of the correlation length.

We have also performed a detailed analysis of the small-scale properties of these networks. This involves a significant amount of data analysis, the details of which will be provided elsewhere [26]. Here, we will analyse a typical ‘late-time’ box of each of the three simulations above. (The analysis of the time evolution of these properties for different epochs in the approach to scaling can also be done.) In order to have a more meaningful comparison, we have chosen boxes that have evolved for a dynamic range (in conformal time) just above 3 (the latest time-step we obtained for the radiation era run). We have exhaustively analysed boxes for other nearby time-
steps, and find that the results to be presented below remain unchanged, within errors, so we can confidently say that these properties are indeed representative of the epoch in question (with one exception that we will point out below). The results are presented in Fig. 2 as a number of quantities measured on a range of physical scales relative to the correlation length $\xi$. Each point on the graphs results from an average over (typically) a few thousand measurements performed on randomly selected points on the simulation box. The plots do not show the entire range of scales we probe—only those which we believe to be accurately sampled and free of systematic errors (e.g., box-size effects).

The top two panels show the correlation function for the tangent vectors,

$$corr_x(s) = \langle \hat{x}'(s_0) \cdot \hat{x}'(s_0 + s) \rangle$$

and its cumulative sum ($C_x(s) = \int_0^s corr_x(s')ds'$, plotted relative to physical time $t$). Notice the differences between the three cases for scales around and slightly below the correlation length. In particular, this confirms that the correlation length $\xi$ (which is defined as the smallest scale at which the correlator vanishes, and is therefore roughly the height of the peak in the $C_x/t$ curve) is quite different in the three cases, as we already saw in Fig. 1.

The middle panels show the analogous functions for the velocity vectors, $\hat{x}$. A first striking feature is that in the expanding universe cosmic string velocities are anti-correlated on scales between the correlation length and the horizon. However, such a feature is not present in flat spacetime. This anti-correlation is the result of a ‘memory’ of the network for recent intercommutings, and its absence in the flat spacetime case highlights the fact that the loop production mechanism is different in the expanding and non-expanding cases. Indeed, such an effect was discussed in [18] (see also [27]). If one defines a ‘velocity coherence length’, this will be significantly smaller than $\xi$ itself.

Finally, the bottom two panels show what we believe to be the most fundamental microscopic properties of the string networks. On the left we have plotted the generalized fractal (also referred to as ‘multi-fractal’ or ‘differential fractal’, depending on the context) dimension of the network [23, 24, 25], again as a function of scale relative to the correlation length. On the right we have the ‘renormalized’ (dimensionless) string mass per unit length, $\mu$. The fractal dimension is obviously unity on small scales (strings are roughly straight lines) and two on very large scales (strings are roughly random walks). The interesting question, however, is what happens at intermediate scales. In particular, one should expect a range of scales where strings should behave as self-avoiding random walks, and these have a fractal dimension $d_s = 5/3$ in three spatial dimensions.

We can immediately see that the radiation and matter era boxes have strikingly similar fractal profiles (recall that we have rescaled by the respective correlation lengths, so the physical scales involved will be different). The differences result in a slightly higher mass per unit length on a given scale (as one can confirm on the right). The flat spacetime profile, however, is qualitatively different, and shows a rather high ‘step’ around the scale of the correlation length. We have evidence that this step in the fractal dimension tends to diminish and move very slowly to smaller scales as the simulation progresses, indicating that flat space simulations have not really achieved small-scale structure scaling. Nevertheless, we have performed moderately high resolution simulations with a dynamic range of 10 and the ‘step’ is still clearly visible, so we believe it is to some extent a persistent feature. This difference in the fractal properties must reflect the differing efficiencies of loop production in flat space and the ex-

FIG. 2: Some characteristic small-scale properties of cosmic string networks in the linear scaling regime, for 75 PPCL matter (solid lines), radiation (dashed) and flat-spacetime (dotted) runs. In all cases, the properties are measured after evolution by a dynamic range of about 3. In all plots the horizontal axis represents the logarithm of the physical lengthscale relative to the correlation length of the network. All plotted quantities are defined in the text.
panding universe. While the integrated loop production efficiency is much greater in flat spacetime $c = 0.57$, as opposed to the expanding $c = 0.23$ [8, 9], it appears to be relatively less effective around the correlation scale with energy “stuck” on fairly large scales. This intuitive picture can be confirmed by noting that the renormalised mass per unit length $\mu$ is smaller than in the expanding case on small scales, but is larger on large scales.

Relevant to this issue, we note that a different (functional forms) approach to flat spacetime simulations was recently presented by [12]. Simple comparisons indicate that their results appear to be in good agreement with our flat spacetime ones, though note that an allowance has to be made for the fact that the two codes use different definitions when calculating the correlation length $\xi$. Another difference is that our characterization of the statistical properties of the network is based on fractal properties, whereas theirs is based on Fourier space techniques. In fact, there are very interesting relations between the two approaches, that will be explored elsewhere [20].

Finally, we point out that in all cases the fractal dimension of the network at the scale of the correlation length is well below two: typical values are 1.2 in the expanding case and 1.4 in the flat case. (Note that this is to be expected if one interprets $\xi$ as a persistence length.) So the intuitive picture that a string network looks Brownian at the scale of the correlation length is clearly incorrect. Indeed, it’s not even strictly true at the scale of the horizon—here the network looks more like a self-avoiding random walk (an obvious consequence of intercommutings). The Brownian picture is only valid on significantly larger scales. This is relevant in a number of contexts, in particular for the ‘toy model’ approach discussed in the introduction, which implicitly models the network under the former assumption.

III. WHEN ARE WE SCALING?

The key question is now whether one can see scaling of the small-scale features of the cosmic string network. Detailed analysis indicates that quantities like the multifractal dimension $d_m$, renormalized mass $\mu$ or coherent velocity as a function of scale evolve significantly at the beginning of the simulations, and even continue to do so after the large-scale properties of the network reach scaling (say, as measured by $L/t = \text{const}$, so that there in a constant number of strings—about 40—per Hubble volume). On the other hand, if one has high enough resolution (to probe lengthscales well below the correlation length) as well as dynamical range, one will see the evolution of $d_m(\ell)$ and $\mu(\ell)$ stop, hence achieving scaling. The first evidence for this is shown in the left panels of Fig. 4, where these quantities are plotted for the late-time evolution of five different matter era runs. Note that the initial conditions in each are very different. Not only do the resolutions vary between 75 and 256 PPCL, but the 128 and 256 PPCL boxes are two very different simulations, one with $v_i = 0$ and the other with $v_i = 0.6$ (and with different initial densities). Despite this, one sees that the $d_m(\ell)$ and $\mu(\ell)$ profiles are virtually identical, within errors. We have similar evidence of scaling for the radiation era, although the convergence is somewhat slower—this will be discussed in more detail elsewhere 20.

These matter era results, in particular, now offer unambiguous evidence that small-scale structure has achieved scaling on all lengthscales accessible to these high resolution numerical simulations. As indicated by and conjectured from an earlier study [27], it now seems clear that the dynamical processes of reconnection and loop production are sufficient to govern the build-up of small-scale structure on cosmic string networks, without requiring additional damping mechanisms such as gravitational radiation. Of course, simulations of even higher resolution and longer dynamical range will inevitably acquire small-scale structure on even smaller scales, but it seems reasonable to suppose, given the convergence of the renormalised tension, that the integrated effect of these additional modes will be relatively small. We note that similar evidence for the scaling of small-scale structure on particular lengthscales has also been presented from flat spacetime simulations [15], though this is somewhat weaker because, like our own flat space results, a wide range of other lengthscales continue to evolve slowly in this case.

A related but different issue is that of loop production. On the right-hand panels of Fig. 4 we quantify the net loop production function of each of these networks for the timesteps corresponding to the left-hand plots. We show both the overall net production function (top panel, summing all loop creating or destroying events) and the production from long strings only (bottom panel). The dominant overall loop production mechanism is that of the self-intersection of larger loops, so note that direct loop production by long strings typically happens on larger lengthscales (above the resolution of the simulations). It is important to note that these are the time-dependent loop production functions, that is, the loops produced or destroyed on each lengthscale in a very narrow time span centered around the corresponding timestep. This is a much finer diagnostic of loop generation mechanisms than the overall string loop density, and it also indicates the typical lengthscales of the long-lived loops which are relevant, for example, for gravitational wave background estimates. Unfortunately, Fig. 4 indicates that the dominant loop production scale remains at least weakly dependent on the simulation resolution – there is not the same strong agreement as between the fractal profiles on the left-hand side. This is even clearer from the equally-spaced time evolution of the loop production function shown in Fig. 4 for the 75 PPCL matter era. This scale starts out being about the size of the correlation length, but becomes progressively smaller as small-scale structure builds up on the strings. The evo-
lution of the peak of the loop distribution, however, is clearly beginning to slow down at late times indicating that it is rising above the minimum simulation resolution and will approach scaling.

It is clear from Fig. 3 that even though all the networks are statistically almost identical on a wide range of lengthscales, the dominant loop production scales are much more sensitive to variations on very small scales. Indeed we find that even when small-scale structure has reached scaling (as measured by $d_m(\ell)$, $\mu(\ell)$ or other diagnostics to be discussed in [28]) the typical loop production scale can still evolve significantly towards smaller scales. However, we do have some evidence that this migration is significantly slowing down, particularly in the matter era, so that the loop production scale will itself reach scaling.

FIG. 3: The multi-fractal dimension (top left), renormalized mass per unit length (bottom left), overall net loop production function (top right) and net loop production function from long strings only (bottom right) for a series of 5 different matter era runs. The resolutions (in PPCL) and dynamical ranges (in conformal time) are indicated in the legend.

IV. DISCUSSION AND CONCLUSIONS

We have presented results from a detailed analysis of the small-scale and loop production properties of cosmic string networks, in both flat spacetime and the expanding universe. We have shown that, while the differences between the radiation and matter epochs are largely 'quantitative', the evolution in flat spacetime is fundamentally different from the expanding case. This indicates that flat spacetime simulations, while qualitatively useful, will not provide an accurate approximation for cosmic strings in an expanding universe.

The two distinguishing characteristics of string evolution in Minkowski spacetime are the absence of velocity anti-correlations on scales around the correlation length, and the apparent existence of a ‘preferred’ scale (roughly around the correlation length $\xi$) from which energy does not move to smaller scales. We believe that these can have a substantial influence, for example when calculating unequal time correlators for the network, which we are currently investigating in more detail. Another result of our detailed analysis is that cosmic string networks are only random walks on length scales well outside the horizon, indicating that the semi-analytic models for strings need to be improved.

Finally, from high resolution matter era simulations we have presented comprehensive evidence over all length-scales that small-scale structure converges to a scaling solution, that is, through reconnections alone without the need for additional damping mechanisms such as gravitational radiation. Evidence for similar convergence has also been found from radiation era simulations. While loop production is clearly more sensitive to the presence of structure on the smallest scales probed by the sim-
ulations, there are good indications that this will also achieve scaling. We will present a more detailed and quantified analysis of these results elsewhere [26].

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Note added: While our paper was being completed, ref. [30] appeared on the archive. This paper presents some evidence for the scaling of the overall loop distribution on intermediate length scales below the correlation length but still near it, roughly loop lengths $\xi/10$ – $\xi$. A comparison with the net loop production function (Fig. 4) indicates that a population of short-lived relatively large loops is being characterised. These loops will soon cascade through self-intersection down to much smaller sizes, presumably determined at present by their simulation resolution.

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