Determination for minimum symbol-pair and RT weights via torsional degrees of repeated-root cyclic codes

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Abstract
There are various metrics for researching error-correcting codes. Especially, high-density data storage system gives the existence of the inconsistency for the reading and writing process. The symbol-pair metric is motivated for outputs that have overlapping pairs of symbols in a certain channel. The Rosenbloom-Tsfasman (RT) metric is introduced since there exists a problem that is related to transmission over several parallel communication channels with some channels not available for the transmission. In this paper, we determine the minimum symbol-pair weight and RT weight of repeated-root cyclic codes over $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^4 \rangle$ of length $n = p^k$. For the determination, we explicitly present third torsional degree for all different types of cyclic codes over $\mathcal{R}$ of length $n$.

Keywords: cyclic code, torsional degree, symbol-pair weight, RT weight.
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1. Introduction
In coding theory, one of the interesting goals is for constructing codes that give easier encoding and decoding processes. By the perspective, cyclic codes have importance in this theory because they have efficient encoding and decoding algorithms. Moreover, the cyclic codes give many important applications in the other areas, such as cryptography and sequence design [3, 6, 7]. In particular, the repeated-root cyclic codes are first introduced in [4, 19]. They present that repeated-root cyclic codes have a concatenated construction and many optimal codes; from this, many researchers have been studying repeated-root cyclic codes over various rings.

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There are various metrics for researching error-correcting codes; such as Hamming, Lee, Euclidean, symbol-pair, and RT metrics. Especially, high-density data storage system gives the existence of the inconsistency for the reading and writing process. The symbol-pair metric is motivated for outputs that have overlapping pairs of symbols in a certain channel [1, 2]. The Rosenbloom-Tsfasman (RT) metric is introduced [18] since there exists a problem that is related to transmission over several parallel communication channels with some channels not available for the transmission.

For many years, linear codes have active developments over various rings [9, 10, 11, 12, 13, 14, 17, 21, 23]. Recently, finding symbol-pair weight and RT weight is a great challenge in coding theory [5, 16, 22]. In [8], they study MDS symbol-pair repeated-root constacyclic codes over \( \mathbb{F}_{p^m}[u]/\langle u^3 \rangle \). The author gives explicit classification for cyclic codes over \( \mathbb{F}_{p^m}[u]/\langle u^3 \rangle \) with torsional degrees [15]. Sharma and Sidana [20] consider symbol-pair weight and RT weight for repeated-root constacyclic codes over finite commutative chain rings. They give the relationship between a certain minimum weight and torsional degrees for constacyclic codes. In detail, if we know about the \( i \)-th torsional degree for constacyclic codes over \( \mathbb{F}_{p^m}[u]/\langle u^i \rangle \), then we can determine the symbol-pair weight and RT weight. Our results are motivated by this fact. The determination for \( i \)-th torsional degree is very difficult in general. So we focus on the specific ring \( \mathbb{F}_{p^m}[u]/\langle u^4 \rangle \), and we explicitly determine the symbol-pair weight and RT weight by using third torsional degree.

In this paper, we determine the third torsional degrees for all different types of repeated-root cyclic codes over \( \mathbb{R} = \mathbb{F}_{p^m}[u]/\langle u^4 \rangle \) of length \( n = p^k \) (Theorems 3.1, 3.3 and 3.4). Through this information, we give the symbol-pair weight and RT weight for the cyclic codes (Theorem 3.5). We present some examples for supporting our results (Examples 1, 2 and 3).

2. Preliminaries

Let \( \mathbb{R} \) be a finite commutative ring with unity. An \( \mathbb{R} \)-submodule of \( \mathbb{R}^n \) is called a code \( C \) of length \( n \) over a ring \( \mathbb{R} \). Any element \( c = (c_1, \ldots, c_n) \) in \( C \) is called a codeword. Henceforth, a code means a linear code, and \( \mathbb{R} \) is the ring \( \mathbb{F}_{p^m}[u]/\langle u^4 \rangle \), where \( p \) is a prime number and \( m \geq 1 \); the ring \( \mathbb{F}_{p^m}[u]/\langle u^4 \rangle \) is a finite commutative Frobenius ring.

**Definition 2.1.** Let \( \mathbb{R} \) be a finite commutative Frobenius ring, and \( g(x) \) be a polynomial in \( \mathbb{R}[x] \). A poly-cyclic code over \( \mathbb{R} \) is an ideal in \( \mathbb{R}[x]/(g(x)) \).

(i) If \( g(x) = x^n - 1 \), then the code is called a cyclic code of length \( n \).

(ii) If \( g(x) = x^n + 1 \), then the code is called a negacyclic code of length \( n \).

(iii) If \( g(x) = x^n + \lambda \), then the code is called a constacyclic code of length \( n \), where \( \lambda \) is a unit in \( \mathbb{R} \).

In this paper, we deal with a repeated-root cyclic code over \( \mathbb{R} = \mathbb{F}_{p^m}[u]/\langle u^4 \rangle \) of length \( n = p^k \). We present a cyclic code \( C \) over \( \mathbb{R} \) of length \( n \) as an ideal in
\(\mathfrak{R}[x]/(x^n - 1)\) by Definition 2.3. We consider a codeword in \(\mathfrak{C}\) as a polynomial in \(\mathfrak{R}[x]/(x^n - 1)\). Furthermore, the generator polynomials for a cyclic code over \(\mathfrak{R}\) are given in Lemma 2.4. Before we give the generator polynomials, we introduce a torsion code of a cyclic code over \(\mathfrak{R}\) of length \(n\).

**Definition 2.2.** Let \(C\) be a cyclic code over \(\mathfrak{R}\) of length \(n = p^k\). Then the \(i\)-th torsion code of \(C\) is defined as

\[
\text{Tor}_i(C) = \{\mu(g(x)) \in \mathbb{F}_{p^m}[x]/(x^n - 1) : u^i g(x) \in C\},
\]

where \(\mu\) is a natural projection map from \(\mathfrak{R}[x]/(x^n - 1)\) to \(\mathbb{F}_{p^m}[x]/(x^n - 1)\) (0 \(\leq i \leq 3\)). Especially, when \(i = 0\), the code \(\text{Tor}_0(C)\) is called the residue code of \(C\).

In [20], the authors give the following results, Lemmas 2.3 and 2.4, for the ring \(\mathfrak{R} = \mathbb{F}_{p^m}[u]/(u^t)\). In this current work, we deal with the ring \(\mathfrak{R} = \mathbb{F}_{p^m}[u]/(u^4)\), thus the lemmas are rewritten for this ring \(\mathfrak{R}\).

**Lemma 2.3.** [20] Let \(C\) be a cyclic code over \(\mathfrak{R}\) of length \(n = p^k\).

(i) The \(i\)-th torsion code \(\text{Tor}_i(C)\) of \(C\) is a cyclic code over \(\mathbb{F}_{p^m}\), and

\[
\text{Tor}_i(C) = \langle (x - 1)^{t_i} \rangle,
\]

where \(t_i\) is an integer satisfying 0 \(\leq t_i \leq p^k\). The integer \(t_i\) is called the \(i\)-th torsional degree of \(C\) (0 \(\leq i \leq 3\)).

(ii) We have 0 \(\leq t_3 \leq t_2 \leq t_1 \leq t_0 \leq p^k\).

**Lemma 2.4.** [20] Let \(C\) be a cyclic code over \(\mathfrak{R}\) of length \(n = p^k\). Let \(\text{Tor}_i(C)\) be the \(i\)-th torsional code of \(C\) with the \(i\)-th torsional degree \(t_i\) (0 \(\leq i \leq 3\)). Then the code \(C\) is uniquely generated by the following four polynomials in \(\mathfrak{R}[x]/(x^n - 1)\):

\[
\begin{align*}
g_0(x) &= (x - 1)^{t_0} + u(x - 1)^{k_1}p_1(x) + u^2(x - 1)^{k_2}p_2(x) + u^3(x - 1)^{k_3}p_3(x), \\
g_1(x) &= u(x - 1)^{t_1} + u^2(x - 1)^{k_4}p_4(x) + u^3(x - 1)^{k_5}p_5(x), \\
g_2(x) &= u^2(x - 1)^{t_2} + u^3(x - 1)^{k_6}p_6(x), \\
g_3(x) &= u^3(x - 1)^{t_3},
\end{align*}
\]

where \(k_1 < t_1, k_2 < t_2, k_3 < t_3\), and \(p_i(x)\) is a unit or zero in \(\mathbb{F}_{p^m}[x]/(x^n - 1)\) (\(i = 2, 4, j = 3, 5, 6\) and \(\ell = 1, \ldots, 6\)). Especially, if \(t_i = p^k\) (resp. \(t_i = 0\)), then \(g_i(x) = 0\) (resp. \(g_i(x) = u^i\)) for all 0 \(\leq i \leq 3\).

Let \(g(x) = (x - 1)^{s_0}h_0(x) + u(x - 1)^{s_1}h_1(x) + u^2(x - 1)^{s_2}h_2(x) + u^3(x - 1)^{s_3}h_3(x)\) be an arbitrary polynomial in \(\mathfrak{R}[x]/(x^n - 1)\), where \(h_0(x) = 0\) or 1, \(h_i(x)\) is a unit or zero in \(\mathbb{F}_{p^m}[x]/(x^n - 1)\) and \(s_i \geq 0\) (1 \(\leq i \leq 3\)). In the polynomial \(g(x)\), the term \(u(x - 1)^{s_1}h_1(x)\) (resp. \(u^2(x - 1)^{s_2}h_2(x), u^3(x - 1)^{s_3}h_3(x)\)) of \(g(x)\) is called a \(u\)-part (resp. \(u^2\)-part, \(u^3\)-part) of \(g(x)\).

We note that, for any codeword \(f(x) = \sum_{i=0}^{n-1} c_i(x - 1)^i\) in \(C\), we also consider the codeword \(f(x)\) as the vector \(c = (c_0, \ldots, c_{n-1})\).

The next definition gives various metrics for the weight of a codeword in a code.
Definition 2.5. Let $C$ be a code over $\mathcal{R}$ of length $n$, and $c = (c_0, \ldots, c_{n-1})$ be a codeword in $C$.

(i) The Hamming weight of $c$ is the number of $i$ such that $c_i \neq 0$ in $c$ for $0 \leq i \leq n-1$.

(ii) The symbol-pair weight of $c$ is the number of $i$ such that $(c_i, c_{i+1}) \neq (0, 0)$ in $\pi(c)$ for $0 \leq i \leq n-1$, where $\pi(c) := ((c_1, c_2), (c_2, c_3), \ldots, (c_{n-1}, c_0))$ is the symbol-pair vector of the codeword $c$.

(iii) The Rosenbloom-Tsfasman (RT) weight of $c$ is $1 + \max\{i : 0 \leq i \leq n-1, c_i \neq 0\}$ in $c$ when $c \neq 0$. If $c = 0$, then the RT weight is equal to 0.

The minimum weight is the minimum value among the weights for all nonzero codewords in a code $C$. The minimum Hamming weight, minimum symbol-pair weight and minimum RT weight of a code $C$ are denoted by $\text{wt}_H(C)$, $\text{wt}_{sp}(C)$ and $\text{wt}_{RT}(C)$, respectively.

3. Third torsional degree for a cyclic code over $\mathcal{R}$ of length $n = p^k$

In this section, we determine the third torsional degree for a cyclic code over $\mathcal{R}$ of length $n = p^k$. After that, we present the minimum symbol-pair weight and RT weight for the code. From this time on, we use the following notations.

Notation.

- $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^4 \rangle$
- $n = p^k$, where $p$ is prime and $k \geq 1$.
- $\binom{n}{k_1}$ the binomial coefficient $s_1$ choose $s_2$
- $g_0(x) = (x-1)^r + u(x-1)^{k_1}p_1(x) + u^2(x-1)^{k_2}p_2(x) + u^3(x-1)^{k_3}p_3(x)$
- $g_1(x) = u(x-1)^{r_1} + u^2(x-1)^{k_1}p_1(x) + u^3(x-1)^{k_2}p_2(x)$
- $g_2(x) = u^2(x-1)^{r_2} + u^3(x-1)^{k_2}p_2(x)$
- $g_3(x) = u^3(x-1)^{r_3}$
where $0 \leq r_3 \leq r_2 \leq r_1 \leq r < n, k_1 < r_1, k_2 < r_2, k_3 \leq r_3$, and $p_i(x)$ is a unit or zero in $\mathcal{R}[x]/\langle x^n - 1 \rangle$ ($i = 2, 4, j = 3, 5, 6$ and $\ell = 1, \ldots, 6$).

Remark 1. Over the ring $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^4 \rangle$, all different types of cyclic codes of length $n = p^k$ are given as follows:

- Principal ideals $(g_0(x), (g_1(x)), (g_2(x)), (g_3(x)))$.
- Non-principal ideals $(g_0(x), (g_1(x)), (g_2(x)), (g_3(x))), (g_1(x), g_2(x)), (g_1(x), g_3(x)), (g_2(x), g_3(x)), (g_0(x), (g_1(x), g_2(x)), (g_0(x), (g_1(x), g_3(x)), (g_0(x), (g_2(x), g_3(x)),$
- $(g_1(x), g_2(x), g_3(x)), (g_0(x), g_1(x), g_2(x)), (g_0(x), g_1(x), g_3(x)), (g_0(x), g_2(x), g_3(x)),$
- $(g_1(x), g_2(x), g_3(x)), (g_0(x), g_1(x), g_2(x), g_3(x)).$

The third torsional degree for a cyclic code $C$ over $\mathcal{R}$ is defined in Lemma 2.3.

As the other perspective, the third torsional degree for a cyclic code $C$ over $\mathcal{R}$ is the smallest non-negative integer $s$ such that $u^3(x-1)^s \in C$.
Theorem 3.1. \( (i) \) For a cyclic code \( C = \langle g_1(x) \rangle \) over \( \mathfrak{R} \) of length \( n \), the third torsional degree \( t_3 \) is given as follows:

(a) If \( n - r_1 + k_4 \geq r_1 \), then the third torsional degree \( t_3 \) is

\[
t_3 = \begin{cases} 
\min\{r_1, r_1\} & \text{if } \hat{h}_1(x) \text{ is a unit,} \\
r_1 & \text{if } \hat{h}_1(x) = 0,
\end{cases}
\]

where \( u^3(x-1)^r \hat{h}_1(x) = u^3(x-1)^{n-2r_1+2k_4}p_2^2(x) - u^3(x-1)^{n-r_1+k_4}p_5(x) \), with a unit or zero \( \hat{h}_1(x) \) in \( \mathfrak{R}[x]/\langle x^n - 1 \rangle \).

(b) If \( n - r_1 + k_4 < r_1 \), then the third torsional degree \( t_3 \) is

\[
t_3 = \begin{cases} 
\min\{n - r_1 + k_4, r_2\} & \text{if } \hat{h}_2(x) \text{ is a unit,} \\
n - r_1 + k_4 & \text{if } \hat{h}_2(x) = 0,
\end{cases}
\]

where \( u^3(x-1)^{\tau_2} \hat{h}_2(x) = u^3(x-1)^{k_4}p_2^2(x) - u^3(x-1)^{\tau_1-k_4+k_5}p_5(x) \), with a unit or zero \( \hat{h}_2(x) \) in \( \mathfrak{R}[x]/\langle x^n - 1 \rangle \).

(ii) For a cyclic code \( C = \langle g_1(x), g_2(x) \rangle \) over \( \mathfrak{R} \) of length \( n \), the third torsional degree \( t_3 \) is obtained as follows: We consider

\[
u^3(x-1)^{\tau_3} \hat{h}_3(x) \quad \text{as}
\]

\[
\begin{cases} 
u^3(x-1)^{n-r_1+k_5}p_5(x) - \nu^3(x-1)^{n-r_1-r_2+k_4+k_5}p_4(x)p_6(x) & \text{if } n - r_1 + k_4 > r_2, \\
u^3(x-1)^{r_2-k_4+k_5}p_5(x) - \nu^3(x-1)^{k_4}p_4(x)p_6(x) & \text{if } n - r_1 + k_4 \leq r_2,
\end{cases}
\]

where \( \hat{h}_3(x) \) is a unit or zero in \( \mathfrak{R}[x]/\langle x^n - 1 \rangle \), and \( \tau_3 \geq 0 \). Set

\[
u^3(x-1)^{\tau_4} \hat{h}_4(x) = \nu^3(x-1)^{k_4}p_4(x) - \nu^3(x-1)^{\tau_1-r_2+k_5}p_6(x),
\]

where \( \hat{h}_4(x) \) is a unit or zero in \( \mathfrak{R}[x]/\langle x^n - 1 \rangle \), and \( \tau_4 \geq 0 \). Then, for \( j = 3, 4 \),

\[
t_3 = \begin{cases} 
\min\{\kappa, t, r_2, n - r_1 + k_4, n - k_4 + k_5, n - r_2 + k_6\} & \text{if } \hat{h}_j(x) \text{ is a unit for some } j, \\
\min\{t, r_2, n - r_1 + k_4, n - k_4 + k_5, n - r_2 + k_6\} & \text{if } \hat{h}_j(x) = 0 \text{ for all } j,
\end{cases}
\]

where \( \kappa = \min\{\tau_j : \hat{h}_j(x) \text{ is a unit in } \mathfrak{R}[x]/\langle x^n - 1 \rangle \text{ for } j = 3, 4\} \), and \( t \) is the third torsional degree for the code \( \langle g_1(x) \rangle \) obtained in (i).

(iii) For a cyclic code \( C = \langle g_2(x) \rangle \) over \( \mathfrak{R} \) of length \( n \), the third torsional degree \( t_3 \) is

\[
t_3 = \min\{n - r_2 + k_6, r_2\}.
\]

(iv) For a cyclic code \( C = \langle g_3(x) \rangle \) over \( \mathfrak{R} \) of length \( n \), then the third torsional degree \( t_3 \) is \( r_3 \).
Proof. (i) For $C = \langle g_1(x) \rangle$, the following two polynomials are all polynomials which have $u^2$-part generated by $g_1(x)$:

$$ug_1(x) = u^2(x - 1)^{r_1} + u^3(x - 1)^{k_4}p_4(x), \quad (3)$$

and

$$(x - 1)^{n-r_1}g_1(x) = u^2(x - 1)^{n-r_1+k_4}p_4(x) + u^3(x - 1)^{n-r_1+k_5}p_5(x). \quad (4)$$

If $n - r_1 + k_4 \geq r_1$, then for deleting $u^2$-part in both polynomials $(3)$ and $(4)$, we compute

$$(x - 1)^{n-2r_1+k_4}p_4(x)(u^2(x - 1)^{r_1} + u^3(x - 1)^{k_4}p_4(x)) - (x - 1)^{n-r_1}g_1(x),$$

where $\tilde{h}_1(x)$ is a unit or zero in $\mathbb{R}[x]/\langle x^n - 1 \rangle$, and $\tau_1 \geq 0$.

So, if $\tilde{h}_1(x)$ is a unit, then $t_3 = \min\{\tau_1, r_1\}$: this is because $u^2g_1(x) = u^3(x - 1)^{r_1} \in C$, $u(x - 1)^{n-r_1}g_1(x) = u^3(x - 1)^{n-r_1+k_4} \in C$ and $r_1 \leq n - r_1 + k_4$ by the assumption. If $\tilde{h}_1(x) = 0$, then $t_3 = r_1$.

For $n - r_1 + k_4 < r_1$, we have the result by the same process as above.

(ii) For $C = \langle g_1(x), g_2(x) \rangle$, the all polynomials which have $u^2$-part are $(3)$, $(4)$ and

$$g_2(x) = u^2(x - 1)^{r_2} + u^3(x - 1)^{r_6}p_6(x). \quad (5)$$

By the same reasoning as (i), using equations $(3)$, $(4)$, we have the third torsional degree $t$ for $\langle g_1(x) \rangle$. From equations $(4)$ and $(5)$, we have the equation $(1)$. Moreover, through equations $(3)$ and $(5)$, we obtain the equation $(2)$.

We have polynomials $u^3(x - 1)^{r_1}$, $u^3(x - 1)^{n-r_1+k_4}$, $u^3(x - 1)^{r_2}$, $u^3(x - 1)^{n-k_4+k_5}$ and $u^3(x - 1)^{n-r_2+k_6}$; in detail, we multiply $u$ to $(3)$, $(4)$ and $(5)$. And we consider multiplying $(x - 1)^{n-r_1}$ to $(3)$, $(x - 1)^{r_1-k_4}$ to $(4)$ and $(x - 1)^{n-r_2}$ to $(5)$. Then we can get the above five polynomials. Moreover, the five polynomials are all polynomials such that $u$-part and $u^2$-part are zeros with nonzero $u^3$-part generated by $g_1(x)$ and $g_2(x)$. We note that $r_2 \leq r_1$.

As a result, we get the third torsional degree for $\langle g_1(x), g_2(x) \rangle$ as in the statement.

(iii), (iv) Clearly, the result follows. 

We give some examples for Theorem 3.4
Example 1.  
(i) Let $C = \langle u(x-1)^6 + u^2(x-1)(1+(x-1)) + u^3\omega(x-1)^2 \rangle$ be a cyclic code over $R = \mathbb{F}_4[u]/\langle u^4 \rangle$ of length $n = 8$; here, $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$. Then, by Theorem 3.1 (i), we calculate first
\[
u(x-1)^\tau h_1(x) = \nu(x-1)((1+(x-1))^2 - \omega(x-1)^6)
\]
since $n - r_1 + k_4 = 3 < r_1 = 6$; hence, $\tau_2 = 1$ and $\tilde{h}_2(x) = (1+(x-1))^2 - \omega(x-1)^6$ is a unit in $R[x]/(x^n - 1)$. The third torsional degree $t_3$ of $C$ is
\[
\min\{n - r_1 + k_4, \tau_2\} = \{3, 1\} = 1.
\]
(ii) Let $C = \langle u(x-1)^6 + u^2(x-1)(1+(x-1)) + u^3\omega(x-1)^2, u^2(x-1)^4 + u^3(1+\omega(x-1)+(x-1)^2) \rangle$ be a cyclic code over $R = \mathbb{F}_4[u]/\langle u^4 \rangle$ of length $n = 8$. We note that $(k_4, k_5, k_6) = (1, 2, 0)$ and $(p_4(x), p_5(x), p_6(x)) = (1+(x-1), \omega, 1+\omega(x-1)+(x-1)^2)$. By (1), we have
\[
u(x-1)^\tau h_3(x) = \nu(x-1)^3 - (1+\omega(x-1)+(x-1)^2)(1+(x-1)).
\]
By (2), we get
\[
u(x-1)^\tau h_4(x) = \nu(x-1)(1+(x-1)-(x-1))(1+\omega(x-1)+(x-1)^2)).
\]
Hence, the third torsional degree $t_3$ of $C$ is
\[
t_3 = \min\{0, 1, 3, 4, 9\} = 0
\]
since $\kappa = \min\{0, 3\} = 0$ and $t = 1$; $t$ is the third torsional degree of $\langle g_0(x) \rangle$ obtained in (i).

For determining the third torsional degrees for the other cyclic codes over $R$, we use the following lemma.

Lemma 3.2. Let $C$ be a cyclic code over $R$ of length $n$. An arbitrary polynomial $g(x)$ in $C$ which has $u^2$-part in $C$ is one of the following forms:

(i) If $C = \langle g_0(x) \rangle$, then the polynomial $g(x)$ is an element of the set $T_1 = \{9, 10, 12\}$.

(ii) If $C = \langle g_0(x), g_1(x) \rangle$, then the polynomial $g(x)$ is an element of the set $T_2 = \{6, 7, 8, 9, 10, 11, 12, 13\}$.

(iii) If $C = \langle g_0(x), g_1(x), g_2(x) \rangle$, then the polynomial $g(x)$ is an element of the set $T_3 = \{6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

(iv) If $C = \langle g_0(x), g_2(x) \rangle$, then the polynomial $g(x)$ is an element of the set $T_4 = \{6, 10, 12, 14\}$,
\[ u^2(x - 1)^{\epsilon_1} h_1(x) + u^3(x - 1)^{\epsilon_2} p_3(x) - u^3(x - 1)^{\epsilon_3} p_1(x)p_2(x), \quad (6) \]
\[ u^2(x - 1)^{\epsilon_4} h_2(x) + u^3(x - 1)^{\epsilon_5} p_3(x) - u^3(x - 1)^{\epsilon_6} p_1(x)p_5(x), \quad (7) \]
\[ u^2(x - 1)^{\min\{k_1, r - r_1 + k_4\} + \hat{\epsilon}_3} h_3(x) + u^3(x - 1)^{k_2} p_2(x) - u^3(x - 1)^{r - r_1 + k_5} p_5(x), \quad (8) \]
\[ u^2(x - 1)^{n - k_1 + k_2} p_2(x) + u^3(x - 1)^{n - r_1 + k_3} p_3(x), \quad (9) \]
\[ u^2(x - 1)^{n - r + k_1} p_1(x) + u^3(x - 1)^{n - r + k_2} p_2(x), \quad (10) \]
\[ u^2(x - 1)^{n - r + k_4} p_4(x) + u^3(x - 1)^{n - r_1 + k_5} p_5(x), \quad (11) \]
\[ u^2(x - 1)^{\epsilon_4} + u^3(x - 1)^{k_1} p_1(x), \quad (12) \]
\[ u^2(x - 1)^{\epsilon_5} + u^3(x - 1)^{k_4} p_4(x), \quad (13) \]
\[ u^2(x - 1)^{\epsilon_6} + u^3(x - 1)^{k_6} p_6(x), \quad (14) \]

with

\[
(\epsilon_1, \epsilon_2, \epsilon_3) = \begin{cases} 
\min\{n - r + k_2, n - 2r + 2k_1\} + \hat{\epsilon}_1, n - r + k_3, & \text{if } n - r + k_1 > r, \\
\min\{k_1, r - k_1 + k_2\} + \hat{\epsilon}_1, r - k_1 + k_3, k_2, & \text{if } n - r + k_1 \leq r,
\end{cases}
\]
\[
(\epsilon_4, \epsilon_5, \epsilon_6) = \begin{cases} 
\min\{n - r + k_2, n - r + k_1 - r_1 + k_4\} + \hat{\epsilon}_2, & \text{if } n - r + k_1 > r_1, \\
\min\{k_1, r_1 - k_1 + k_2\} + \hat{\epsilon}_2, r_1 - k_1 + k_3, k_5, & \text{if } n - r + k_1 \leq r_1,
\end{cases}
\]
\[
(x - 1)^{\epsilon_i} h_i(x) = \begin{cases} 
(x - 1)^{n - r + k_2} p_2(x) - (x - 1)^{n - 2r + 2k_1} p_1^2(x) & \text{if } n - r + k_1 > r, \\
(x - 1)^{r - k_1 + k_2} p_2(x) - (x - 1)^{k_1} p_1^2(x) & \text{if } n - r + k_1 \leq r,
\end{cases}
\]
\[
(x - 1)^{\epsilon_i} h_i(x) = \begin{cases} 
(x - 1)^{n - r + k_2} p_2(x) - (x - 1)^{n - r + k_1 - r_1 + k_4} p_1 p_4(x) & \text{if } n - r + k_1 > r_1, \\
(x - 1)^{r_1 - k_1 + k_2} p_2(x) - (x - 1)^{k_4} p_1^2(x) & \text{if } n - r + k_1 \leq r_1,
\end{cases}
\]
\[
(x - 1)^{\min\{k_1, r - r_1 + k_4\} + \hat{\epsilon}_3} h_3(x) = (x - 1)^{k_1} p_1(x) - (x - 1)^{r_1 + k_4} p_4(x),
\]
\[\hat{\epsilon}_i\text{ is a non-negative integer, and } h_i(x)\text{ is a unit or zero in } \mathbb{F}_p[x]/(x^n - 1) \text{ for } i = 1, 2, 3.\]

**Proof.** Suppose that \( C = \angle g_0(x), g_1(x) \). Then any element in \( C \) is written as

\[
g_0(x) F_1(x) + g_1(x) F_2(x)
= ((x - 1)^{\epsilon_i} + u(x - 1)^{k_1} p_1(x) + u^2(x - 1)^{k_2} p_2(x) + u^3(x - 1)^{k_3} p_3(x)) F_1(x)
+ (u(x - 1)^{\epsilon_i} + u^2(x - 1)^{k_4} p_4(x) + u^3(x - 1)^{k_5} p_5(x)) F_2(x),
\]

(15)

where \( F_1(x) = (f_{00}(x) + u f_{01}(x) + u^2 f_{02}(x) + u^3 f_{03}(x)), F_2(x) = (f_{10}(x) + u f_{11}(x) + u^2 f_{12}(x) + u^3 f_{13}(x)) \) and \( f_{ij}(x) \in \mathbb{F}_p[x]/(x^n - 1) \) for \( i = 0, 1 \) and \( j = 0, 1, 2, 3. \)

If an element \( g(x) \) in \( C \) is divisible by \( u \), then \( f_{00}(x) \) must be equal to \( (x - 1)^{n - r} f_{00}(x) \) in \([15]\) for some \( f_{00}(x) \in \mathbb{F}_p[x]/(x^n - 1).\)
Then the polynomial \( g(x) \) which has nonzero \( u \)-part in \( C \) is one of the following form:

\[
(u(x-1)^{n-r+k_1}p_1(x) + u^2(x-1)^{n-r+k_2}p_2(x) + u^3(x-1)^{n-r+k_3}p_3(x))f_{00}(x), \quad (16)
\]

\[
(u(x-1)^r + u^2(x-1)^{k_1}p_1(x) + u^3(x-1)^{k_2}p_2(x))f_{01}(x), \quad (17)
\]

\[
(u(x-1)^{r_1} + u^2(x-1)^{k_4}p_4(x) + u^3(x-1)^{k_5}p_5(x))f_{10}(x) \quad (18)
\]

from (14). We need to annihilate \( u \)-part in each polynomials (16), (17) and (18). First, we calculate the following using (17) and (18) since \( r > r_1 \):

\[
u(x-1)^r + u^2(x-1)^{k_1}p_1(x) + u^3(x-1)^{k_2}p_2(x)
\]

\[
- (x-1)^{r-r_1}(u(x-1)^{r_1} + u^2(x-1)^{k_1}p_4(x) + u^3(x-1)^{k_5}p_5(x))
\]

\[
= u^2(x-1)^{k_1}p_1(x) - u^2(x-1)^{r-r_1+k_4}p_4(x)
\]

\[
+ u^3(x-1)^{k_2}p_2(x) - u^3(x-1)^{r-r_1+k_5}p_5(x)
\]

\[
= u^2(x-1)^{\min\{k_1, r-r_1+k_4\}+\varepsilon_3}h_3(x) + u^3(x-1)^{k_2}p_2(x) - u^3(x-1)^{r-r_1+k_5}p_5(x),
\]

where \( (x-1)^{\min\{k_1, r-r_1+k_4\}+\varepsilon_3}h_3(x) = (x-1)^{k_1}p_1(x) - (x-1)^{r-r_1+k_4}p_4(x) \); this calculating result is equation (3).

(a) Assume \( n-r+k_1 > r_1 \). For the same reasoning as above, we have the followings using (10) and (18):

\[
u(x-1)^{n-r+k_1}p_1(x) + u^2(x-1)^{n-r+k_2}p_2(x) + u^3(x-1)^{n-r+k_3}p_3(x)
\]

\[
- (x-1)^{n-r+k_1-r_1}p_1(x)(u(x-1)^{r_1} + u^2(x-1)^{k_4}p_4(x) + u^3(x-1)^{k_5}p_5(x))
\]

\[
= u^2(x-1)^{n-r+k_1}p_1(x) - u^2(x-1)^{n-r+k_1-r_1+k_4}p_1(x)p_4(x)
\]

\[
+ u^3(x-1)^{n-r+k_2}p_2(x) - u^3(x-1)^{n-r+k_1-r_1+k_5}p_2(p_1(x)p_4(x),
\]

\[
= u^2(x-1)^{\min\{n-r+k_2, n-r+k_1-r_1+k_4\}+\varepsilon_3}h_3(x)
\]

\[
+ u^3(x-1)^{n-r+k_3}p_3(x) - u^3(x-1)^{n-r+k_1-r_1+k_5}p_3(x)p_5(x),
\]

where \( h_3(x) \) is a unit or zero in \( \mathbb{F}_p^{n_1}[x]/(x^{n_1}-1) \) such that \( (x-1)^{n-r+k_1}p_1(x) - (x-1)^{n-r+k_1-r_1+k_4}p_1(x)p_4(x) = (x-1)^{\min\{n-r+k_2, n-r+k_1-r_1+k_4\}+\varepsilon_3}h_3(x) \).

(b) Assume \( n-r+k_1 \leq r_1 \). We get the followings using (10) and (18):

\[
(x-1)^{r_1-n-r+k_1}(u(x-1)^{n-r+k_1}p_1(x) + u^2(x-1)^{n-r+k_2}p_2(x)
\]

\[
+ u^3(x-1)^{n-r+k_3}p_3(x) - p_1(x)(u(x-1)^{r_1} + u^2(x-1)^{k_4}p_4(x) + u^3(x-1)^{k_5}p_5(x))
\]

\[
= u^2(x-1)^{r_1-k_1+k_2}p_2(x) - u^2(x-1)^{k_4}p_1(x)p_4(x) + u^3(x-1)^{r_1-k_1+k_3}p_3(x)
\]

\[
- u^3(x-1)^{k_5}p_1(x)p_5(x),
\]

\[
= u^2(x-1)^{\min\{r_1-k_1+k_2, k_4\}+\varepsilon_3}h_2(x) + u^3(x-1)^{r_1-k_1+k_3}p_3(x)
\]

\[
- u^3(x-1)^{k_5}p_1(x)p_5(x),
\]

where \( h_2(x) \) is a unit or zero in \( \mathbb{F}_p^{n_1}[x]/(x^{n_1}-1) \) such that \( (x-1)^{r_1-k_1+k_2}p_2(x) - (x-1)^{k_4}p_1(x)p_4(x) = (x-1)^{\min\{r_1-k_1+k_2, k_4\}+\varepsilon_3}h_2(x) \).

From (a) and (b), we obtain the equation (5).

By the same process as (a) and (b), according to \( n-r+k_1 > r \) or \( n-r+k_1 \leq r \), we obtain the equation (17) using equations (10) and (17).
Moreover, from equation (13), we obtain all polynomials such that $u$-part is zero with nonzero $u^2$-part:

\[
\begin{align*}
equation 15 &= g_0(x)(x-1)^{n-k_1}\hat{f}_{00}(x), \\
equation 16 &= g_0(x)(x-1)^{n-r}(u\hat{f}_{01}(x)), \\
equation 17 &= g_1(x)(x-1)^{n-r_1}\hat{f}_{10}(x), \\
equation 18 &= g_0(x)(u^2\hat{f}_{02}(x)), \\
equation 19 &= g_1(x)(u\hat{f}_{11}(x)),
\end{align*}
\]

where $f_{00}(x) = (x-1)^{n-r}\hat{f}_{00}(x) = (x-1)^{n-r}(x-1)^{r-k_1}\hat{f}_{00}(x) = (x-1)^{n-k_1}\hat{f}_{00}(x)$, $f_{01}(x) = (x-1)^{n-r_1}\hat{f}_{01}(x)$ and $f_{10}(x) = (x-1)^{n-r_1}\hat{f}_{10}(x)$ for some polynomials $\hat{f}_{00}(x), \hat{f}_{01}(x), \hat{f}_{02}(x)$ and $\hat{f}_{10}(x)$ in $\mathfrak{R}[x]/(x^n-1)$. Hence, for $C = \langle g_0(x), g_1(x) \rangle$, any polynomial which has nonzero $u^2$-part in $C$ is an element of the set $T_2$.

The other parts can be proved similar way as above.

We note that every polynomial in Lemma 3.2 has the following form:

\[
u^2(x-1)^k h_1(x) + u^3(x-1)^\tilde{k} h_2(x), \tag{19}\]

where $k$ and $\tilde{k}$ are non-negative integers, and $h_i(x)$ is a unit or zero in $\mathbb{F}_p[x]/(x^n-1)$ for $i = 1, 2$. In (19), the degree $k$ (resp. $\tilde{k}$) is called a $u^2$-degree (resp. $u^3$-degree) when $h_1(x) \neq 0$ (resp. $h_2(x) \neq 0$).

In Theorem 3.3 we determine the third torsional degree for cyclic codes $\langle g_0(x) \rangle$, $\langle g_0(x), g_1(x) \rangle$, $\langle g_0(x), g_2(x) \rangle$ and $\langle g_0(x), g_1(x), g_2(x) \rangle$ in $\mathfrak{R}[x]/(x^n-1)$.

**Theorem 3.3.** Let $C$ be a cyclic code over $\mathfrak{R}$ of length $n$, and $T$ be the set defined as in Lemma 3.2:

\[
T = \begin{cases} 
T_1 & \text{if } C = \langle g_0(x) \rangle, \\
T_2 & \text{if } C = \langle g_0(x), g_1(x) \rangle, \\
T_3 & \text{if } C = \langle g_0(x), g_1(x), g_2(x) \rangle, \\
T_4 & \text{if } C = \langle g_0(x), g_2(x) \rangle.
\end{cases}
\]

Let $\nu$ be the number of polynomials in $T_i$ $(1 \leq i \leq 4)$ which has a non-negative $u^2$-degree for each cases. Let $f_i = u^2(x-1)^{\omega_i} h_{i1}(x) + u^3(x-1)^{\tilde{\omega}_i} h_{i2}(x)$ be an element of the set $T$, where $\omega_i$ (resp. $\tilde{\omega}_i$) is the $u^2$-degree (resp. $u^3$-degree) and $h_{ij}(x)$ is a unit or zero in $\mathbb{F}_{p^m}[x]/(x^n-1)$ $(1 \leq i \leq |T|, j = 1, 2)$. Then the third torsional degree $t_3$ of the code $C$ is obtained as follows:

(i) If $\nu = 0$, then the third torsional degree $t_3 = \min \{ \tilde{\omega}_i : 1 \leq i \leq |T| \}$.

(ii) If $\nu = 1$, we suppose that $f_1(x)$ has a non-negative $u^2$-degree. Then the third torsional degree $t_3$ is

\[
t_3 = \min \{ n - \omega_1 + \omega_1, \omega_2, \ldots, \omega_{|T|} \}.
\]
polynomials in \( \hat{g} \) as follows:

\[
f_j - (x - 1)^{\omega_j - \omega_i} f_i(x) h_{j_1}^{-1}(x) h_{j_1}(x) = u^3(x - 1)^{\tau_k} \tilde{h}_k(x),
\]

where \( \tilde{h}_k(x) \) is a unit or zero in \( \mathcal{R}[x]/(x^n - 1) \) (1 \( \leq \) \( i \) \( \leq \) \( \nu \), 1 \( \leq \) \( k \) \( \leq \) \( \left( \begin{array}{c} \nu \\ 2 \end{array} \right) \)).

Let \( \mathcal{M} = \{ \tau_k : \tilde{h}_k(x) \neq 0 \} \), and \( m \) be the minimum value in \( \mathcal{M} \). Then the third torsional degree \( t_3 \) is

\[
t_3 = \begin{cases} 
\min \{ m, \omega_1, \ldots, \omega_{\nu}, n - \omega_1 + \hat{\omega}_1, \ldots, n - \omega_{\nu} + \hat{\omega}_{\nu+1}, \ldots, \hat{\omega}_{|T|} \} & \text{if } \mathcal{M} \neq \emptyset, \\
\min \{ \omega_1, \ldots, \omega_{\nu}, n - \omega_1 + \hat{\omega}_1, \ldots, n - \omega_{\nu} + \hat{\omega}_{\nu+1}, \ldots, \hat{\omega}_{|T|} \} & \text{if } \mathcal{M} = \emptyset.
\end{cases}
\]

**Proof.**

(i) Suppose that \( \nu = 0 \). It means that every polynomial in the set \( T \) has zero \( u^2 \)-degree. Hence we only consider \( u^3 \)-degrees \( \hat{\omega}_i \) (1 \( \leq \) \( i \) \( \leq \) \( |T| \)) for all polynomials in \( T \). Easily, the third torsional degree for the code is

\[
\min \{ \hat{\omega}_i : 1 \leq i \leq |T| \}.
\]

(ii) Suppose that \( \nu = 1 \). Then for removing \( u^2(x - 1)^{\omega_1} \hat{h}_{11}(x) \) for \( f_1(x) \), we multiply \( (x - 1)^{n - \omega_1} \) to \( f_1(x) \). We obtain that \( (x - 1)^{n - \omega_1} f_1(x) = u^3(x - 1)^{n - \omega_1 + \hat{\omega}_1} h_{12}(x) \). Furthermore, \( u f_1(x) = u^3(x - 1)^{\omega_1} h_{11}(x) \) is also an element in \( C \). These imply the results.

(iii) We assume that \( \nu \geq 2 \). Then for the polynomials \( f_1, \ldots, f_{\nu} \), we obtain the same result as the previous case (ii). On the other way, for deleting \( u^2 \)-parts of \( f_i(x) \) and \( f_j(x) \) each other, we compute

\[
f_j(x) - (x - 1)^{\omega_j - \omega_i} f_i(x) h_{j_1}^{-1}(x) h_{j_1}(x) = u^3(x - 1)^{\tau_k} \tilde{h}_k(x),
\]

where 1 \( \leq \) \( i \) \( \leq \) \( j \) \( \leq \) \( \nu \) and 1 \( \leq \) \( k \) \( \leq \) \( \left( \begin{array}{c} \nu \\ 2 \end{array} \right) \). Through this process, we have that \( u^3 \)-degree \( \tau_k \) only when \( \tilde{h}_k(x) \neq 0 \). Hence we get the result. The items (i), (ii) and (iii) therefore implies the third torsional degree \( t_3 \) for the code \( C \) over \( \mathcal{R} \).

Finally, in Theorem 3.4 we get the third torsional degree of a cyclic code which is a non-principal ideal containing \( g_3(x) = u^3(x - 1)^{r_3} \) using Theorems 3.1 and 3.3.

**Theorem 3.4.** Let \( \tilde{C} \) be a cyclic code given in Theorems 3.1 and 3.3 except for the case \( \tilde{C} = \langle g_3(x) \rangle \). Let \( t \) be the third torsional degree for a cyclic code \( \tilde{C} \). Let \( C \) be a cyclic code over \( \mathcal{R} \) of length \( n \) generated by \( g_3(x) \) and all generator polynomials in \( \tilde{C} \). Then the third torsional degree \( t_3 \) for the code \( C \) is obtained as follows:

(i) If \( r_3 \geq t \), then the third torsional degree \( t_3 \) of \( C \) is equal to \( t \).

(ii) If \( r_3 < t \), then the third torsional degree \( t_3 \) of \( C \) is equal to \( r_3 \).
**Proof.** If \( r_3 \geq t \), then \( C = \hat{C} \), that is \( g_3(x) \in \hat{C} \) since \( u^3(x-1)^t \in \hat{C} \) and \( u^3(x-1)^t(x-1)^{r_3-t} = u^3(x-1)^{r_3} \). Hence, in this case, \( t_3 = t \).

If \( r_2 < t \), then \( r_3 \) is the smallest non-negative integer such that \( u^3(x-1)^{r_3} \in C \). Thus, it follows the result.

The next example gives the third torsional degree for certain cyclic codes by using Theorem 3.3.

**Example 2.** (i) Let \( C = \langle (x-1)^3 + u(x-1) + u^2 + u^3(1 + (x-1)), u(x-1)^2 + u^2 + u^3(x-1) \rangle \) be a cyclic code over \( \mathbb{F}_2[u]/(u^4) \) of length 4. We note that \( (k_1, k_2, k_3, k_4, k_5) = (1, 0, 0, 0, 1) \) and \( (p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)) = (1, 1, 1 + (x-1), 1, 1) \). We have the followings

\[
\begin{align*}
&\bullet \ (x-1)^r h_1(x) = (x-1)(1 + (x-1)), \\
&\bullet \ (x-1)^r h_2(x) = -1 + (x-1), \\
&\bullet \ (x-1)^{\min(k_1, r-r_1+k_4)} h_3(x) = 0, \\
&\bullet \ (\epsilon_2, \epsilon_3) = (r - k_1 + k_3, k_2) = (2, 0), \\
&\bullet \ (\epsilon_5, \epsilon_6) = (r_1 - k_1 + k_3, k_5) = (1, 1),
\end{align*}
\]

since \( n - r + k_1 = 2 < r \) and \( n - r + k_1 \leq r_1 \). Thus, by Lemma 3.2, the set \( T_3 \) consists of the following polynomials:

\[
\begin{align*}
&u^2(x-1)(1 + (x-1)) + u^3(x-1)^2(1 + (x-1)) - u^3, \\
u^2(-1 + (x-1)) + u^3(x-1)(1 + (x-1)) - u^3(x-1), \\
u^3 - u^3(x-1)^2, \\
u^2(x-1)^3 + u^3(x-1)^2(1 + (x-1)), \\
u^2(x-1)^2 + u^3(x-1), \\
u^2(x-1)^2 + u^3(x-1)^3, \\
u^2(x-1)^3 + u^3(x-1), \\
u^2(x-1)^2 + u^3.
\end{align*}
\]

In Theorem 3.3 we get \( t_3 = 0 \) since there is the polynomial \( u^3 - u^3(x-1)^2 = u^3(1 + (x-1)^2) \). It means that \( u^3 \in C \).

(ii) Let \( C = \langle (x-1)^3 + u(x-1)^2(1 + 2(x-1)) + u^3(x-1)(3 + (x-1)) \rangle \) be a cyclic code over \( \mathbb{F}_3[u]/(u^4) \) of length 9. We note that \( (k_1, k_2, k_3) = (2, 0, 1) \) and \( (p_1(x), p_2(x), p_3(x)) = (1 + (2(x-1)), 0, 3 + (x-1)) \). We have that

\[
\begin{align*}
&\bullet \ (x-1)^r h_1(x) = 0, \\
&\bullet \ (\epsilon_2, \epsilon_3) = (n - r + k_3 = 5, n - 2r + k_1 + k_2 = 1),
\end{align*}
\]

since \( n - r + k_1 = 6 > r \). Then the set \( T_3 \) consists of

\[
\begin{align*}
u^3(x-1)^5(1 + 2(x-1))(3 + (x-1)), \\
u^2(x-1)^6(1 + 2(x-1)), \\
u^2(x-1)^5 + u^3(x-1)^2(1 + 2(x-1)).
\end{align*}
\]

Hence \( \nu = 2 \) here. Let \( f_1(x) = u^2(x-1)^6(1 + 2(x-1)) \) and \( f_2(x) = u^2(x-1)^5 + u^3(x-1)^2(1 + 2(x-1)) \). Then \( f_1(x) - (x-1)(1 + 2(x-1))f_2(x) = -u^3(x-1)^5(1 + 2(x-1))^2 \), that is, \( \tau_1 = 3 \) in equation (20). Thus, the third torsional degree \( t_3 \) of \( C \) is \( \min\{3, 5, 6\} = 3 \) by Theorem 3.3.

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By Theorems 18 and 22, the minimum symbol-pair weight and RT-weight for a cyclic code $C$ over $R$ can be given by the third torsional degree $t_3$ of $C$.

**Theorem 3.5.** Let $C$ be a cyclic code over $R = \mathbb{F}_{2^m}[u]/\langle u^4 \rangle$ of length $n = p^k$. Let $\text{Tor}_3(C) = \langle (x-1)^{t_3} \rangle$ be the third torsion code of $C$ over $\mathbb{F}_{2^m}$ with the third torsional degree $t_3$ (the third torsional degree $t_3$ is obtained in Theorems 3.7, 3.9, and 3.10).

(i) The minimum symbol-pair weight $\text{wt}_{sp}(C)$ of $C$ is

$$
\begin{align*}
\text{wt}_{sp}(C) &= \begin{cases} 
2 & \text{if } t_3 = 0, \\
3p^\ell & \text{if } t_3 = p^k - p^{k-\ell} + 1, \text{ where } 0 \leq \ell \leq k - 2, \\
4p^\ell & \text{if } p^k - p^{k-\ell} + 2 \leq t_3 \leq p^k - p^{k-\ell} + p^{k-\ell-1}, \\
2(\mu + 2)p^\ell & \text{if } p^k - p^{k-\ell} + \mu p^{k-\ell-1} + 1 \leq t_3 \leq p^k - p^{k-\ell} + (\mu + 1)p^{k-\ell-1}, \\
(\mu + 2)p^{k-1} & \text{if } t_3 = p^k - p + \mu, \text{ where } 1 \leq \mu \leq p - 2, \\
p^k & \text{if } t_3 = p^k - 1, \\
0 & \text{if } t_3 = p^k.
\end{cases}
\end{align*}
$$

(ii) The RT weight $\text{wt}_{RT}(C)$ of $C$ is $t_3 + 1$.

We close this section with the next example; we obtain symbol-pair weight and RT weight for certain cyclic codes.

**Example 3.** (i) In Examples 1 and 2 we obtain symbol-pair weight and RT weight for certain cyclic codes using Theorem 3.5.

$$(\text{wt}_{sp}(C), \text{wt}_{RT}(C)) = \begin{cases} 
(3, 2) & \text{if } C \text{ is given in Example 1 (i)}, \\
(2, 1) & \text{if } C \text{ is given in Examples 1 (ii) and 2 (i)}, \\
(4, 4) & \text{if } C \text{ is given in Example 2 (ii)}.
\end{cases}$$

(ii) Let $C = \langle u^2(x-1)^{51} + u^3(x-1)^{67}h(x) \rangle$ be a cyclic code over $R = \mathbb{F}_{2^2}[u]/\langle u^4 \rangle$ of length $n = 5^3$, where $h(x)$ is an arbitrary unit in $R[x]/\langle x^n - 1 \rangle$. Then, by Theorem 3.1, the third torsional degree $t_3$ of $C$ is $t_3 = \min \{141, 51\} = 51$ since $n - r_2 + k_0 = 141$ and $r_2 = 51$. From Theorem 3.5 we have $\text{wt}_{sp}(C) = 8$ and $\text{wt}_{RT}(C) = 52$.

(iii) Generally, let $C = \langle u^2(x-1)^{r_2} + u^3(x-1)^{k_0}p_9(x) \rangle$ be a cyclic code over $\mathbb{F}_{2^2}[u]/\langle u^4 \rangle$ of length $n = 5^3$. For each case, we obtain the symbol-pair weight and RT weight for the code $C$ from Theorem 3.5.

(a) If $0 \leq r_2 \leq 62$, then the third torsional degree $t_3$ is equal to $r_2$ by
Theorem 3.1. Thus $wt_{RT}(C) = r_2 + 1$, and

\[ wt_{sp}(C) = \begin{cases} 
2 & \text{if } t_3 = 0, \\
3 & \text{if } t_3 = 1, \\
4 & \text{if } 2 \leq t_3 \leq 25, \\
6 & \text{if } 26 \leq t_3 \leq 50, \\
8 & \text{if } 51 \leq t_3 \leq 62. 
\end{cases} \]  \hfill (21)

(b) If $63 \leq r_2 \leq 124$ with $2r_2 - k_0 \leq 125$, so the third torsional degree $t_3$ is $r_2$. Then $wt_{RT}(C) = r_2 + 1$, and

\[ wt_{sp}(C) = \begin{cases} 
8 & \text{if } 63 \leq t_3 \leq 75, \\
10 & \text{if } 76 \leq t_3 \leq 100, \\
15 & \text{if } t_3 = 101, \\
20 & \text{if } 102 \leq t_3 \leq 105, \\
30 & \text{if } 106 \leq t_3 \leq 110, \\
40 & \text{if } 111 \leq t_3 \leq 115, \\
50 & \text{if } 116 \leq t_3 \leq 120, \\
75 & \text{if } t_3 = 121, \\
100 & \text{if } t_3 = 122, \\
125 & \text{if } t_3 = 123, \\
125 & \text{if } t_3 = 124, \\
0 & \text{if } t_3 = 125. 
\end{cases} \]  \hfill (22)

(c) If $63 \leq r_2 \leq 124$ with $2r_2 - k_0 < 125$, then $t_3 = n - r_2 + k_0$. Hence, $wt_{RT}(C) = n - r_2 + k_0 + 1$. By (21) and (22), according to the value $t_3 = n - r_2 + k_0$, we can obtain $wt_{sp}(C)$.

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