Computer proof of ergodicity breaking in expanding coupled maps

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Abstract. From a dynamical viewpoint, basic phase transitions of statistical mechanics can be regarded as some ergodicity breaking. While many random models exhibiting such transitions exist, deterministic examples, which would only use chaotic dynamics as the source of randomness, are rare, if at all existent. Here, the dynamics of a family of $N$ coupled expanding circle maps is investigated in a parameter regime where their attractor supports absolutely continuous measures. Empirical evidence is given for symmetry breaking of the ergodic components upon increase of the coupling strength. An automated, $N$-independent, procedure is proposed, based on exact computer arithmetic, which aims to rigorously demonstrate the existence of asymmetric ergodic components of positive Lebesgue measure. This algorithm successfully terminates for only small values of $N$, due to the explosive growth of the computational resources required. However, this approach suggests that phase transitions in systems with erratic dynamics should be provable in a purely deterministic setting, without any reference to random processes.

1 Introduction

In statistical mechanics, the term phase transition originally refers to the emergence of multiple Gibbs states for the Hamiltonian associated with a collective system [23], most notably accompanied with symmetry breaking as in the Ising model. While this phenomenon involves equilibrium measures, from the dynamical viewpoint, it materializes in the breaking of ergodicity in a corresponding dynamical process [27]. Using out-of-equilibrium procedures, such ergodicity failures have been rigorously obtained for suitably designed Markov processes and probabilistic cellular automata (PCA) [12, 17]. Metropolis rules, Glauber dynamics and the like have been designed to account for relaxation to pre-constructed equilibrium states, including the case when several such states coexist [21].

Notwithstanding the success of this dynamical approach to phase transitions, the fact that classical mechanics is ruled by deterministic laws of motion calls for evidence in deterministic systems, irrespective of any consideration of random processes. Barring the use of statistical mechanics, can chaotic attractors in deterministic analogues to random models of interacting particles systems exhibit ergodicity breaking (associated with symmetry breaking)? Despite having generated considerable attention, this question still eludes a satisfactory response that would surely excludes numerical flaws or unverifiable theoretical assumptions, even in basic examples such as networks of coupled expanding or hyperbolic maps.

Indeed, some examples of (infinite) lattices of interacting chaotic maps have been designed so that their dual dynamics acting on measures consist of PCA exhibiting phase transitions [11, 13]. Thus, out-of-equilibrium approaches can in principle be lifted to deterministic dynamical systems. However, this operation requires explicit knowledge of a coupling-intensity-independent Markov partition, in particular one that ensures a pre-selected Markov chain/PCA for the dual measure evolution. Yet, such a complete understanding of Markovian properties is rare for realistic deterministic systems, and especially for chaotic collective systems with homogenizing interactions [27], which typically fail to fit the standard assumptions of the theory of dynamical systems, such as being diffeomorphic. In general, very little is known about the symbolic dynamics, and, if at all, only for weak interaction intensity [14, 7]. This shortcoming calls for verifications of non-ergodicity in a purely deterministic setting, that are independent of any knowledge of the associated symbolic dynamics.

Various studies have reported changes in the global dynamics of coupled chaotic maps as their interaction strength is incremented. Phenomenological changes in finite systems have been always observed to be preceded by a reduction in the Lyapunov dimension; further, these changes have repeatedly resulted in stationary or periodic behaviors in spatial averages of symmetry-related observables, see e.g. [4, 6, 15, 19]. Such features are reminiscent of synchronization-like scenarios in which trajectories asymptotically shrink...
to lower dimensional subspaces. Hardly compatible with a coupling-independent symbolic grammar, they cast doubt on the nature of phase transitions in dynamical systems: do they necessarily require a lowering of the dimension of the attractor, or can they take place while hyperbolicity properties remain unchanged, and absent knowledge of a Markov partition?

To address these issues, we consider a family of piecewise affine mappings \( F_{\epsilon,N} \) of the \( N \)-dimensional torus, that mimic population dynamics driven by chaotic individual stirring and homogenizing interactions of adjustable strength \( \epsilon \in [0,1) \). They are defined as follows \cite{10}

\[
(F_{\epsilon,N}(u))_i = 2u_i + 2\epsilon \sum_{j=1}^N g(u_j - u_i) \mod 1, \quad 1 \leq i \leq N,
\]

where \( g(u) = u - h(u) \) for all \( u \in \mathbb{T}^1 \) with

\[
h(u) = \left\{ \begin{array}{ll}
[u + \frac{1}{2}] & \text{if } u \notin \frac{1}{2} + \mathbb{Z} \\
0 & \text{if } u \in \frac{1}{2} + \mathbb{Z}
\end{array} \right.
\]

mimics pairwise elastic interactions on the circle \cite{13}. Mean field coupling implies that \( F_{\epsilon,N} \) commutes with the map \((u_i)_{i=1}^N \mapsto (u_{\pi(i)})_{i=1}^N\); for every permutation \( \pi \) of \( \{1, \ldots, N\} \) while \( g(-u) = -g(u) \mod 1 \) implies commutativity with the map \((u_i)_{i=1}^N \mapsto (-u_i \mod 1)_{i=1}^N\); hence with all maps in the symmetry group \( \mathbb{Z}_2 \times S_N \).

The mappings \( F_{\epsilon,N} \) are non-singular; hence, for the analysis of ergodicity we can ignore sets of vanishing Lebesgue measure, and in particular, those points with coordinates lying on discontinuity planes and their pre-images. Away from these discontinuities, \( F_{\epsilon,N} \) is piecewise affine with constant derivative

\[
(DF_{\epsilon,N}v)_i = 2(1-\epsilon)v_i + 2\epsilon \sum_{j=1}^N v_j, \quad 1 \leq i \leq N,
\]

from which it follows that \( F_{\epsilon,N} \) is expanding for \( \epsilon < \frac{1}{4} \). In this regime, the fact that these maps are piecewise affine implies that their Milnor attractor \cite{2} \cite{20} must consist of a finite union of Lebesgue ergodic components \cite{28}, viz. the attractor of almost every trajectory must be a set of positive Lebesgue measure, thereby excluding any dimension reduction.

Focusing on this expanding regime, we aim to address ergodicity of the attractor, that is whether the Lebesgue ergodic components are unique or not. Up to semi-conjugacy, this question is equivalent to the same one for a more convenient family of piecewise affine mappings, of the \( D = (N-1) \)-dimensional torus. The reduced mappings \( G_{\epsilon,D} \) (defined below) have equivalent features to the originals. In particular, they commute with the inverting of all coordinates \((x_i)_{i=1}^D \mapsto (-x_i)_{i=1}^D\), their symmetry group consists of \( \mathbb{Z}_2 \times S_D \), and their asymptotic dynamics for \( \epsilon \in [0, \frac{1}{2}) \) must lie in finitely many ergodic components of positive Lebesgue measure.

More precisely, the transformation \( \pi_N \) of \( \mathbb{T}^N \) defined by \cite{25}

\[
(\pi_N u)_i = \left\{ \begin{array}{ll}
u_i - u_{i+1} \mod 1 & \text{if } 1 \leq i \leq N - 1 \\
\sum_{j=1}^N u_j \mod 1 & \text{if } i = N
\end{array} \right.
\]

(semi-)conjugates \( F_{\epsilon,N} \) to the direct product \( G_{\epsilon,N-1} \times F_{\epsilon,1} \) (viz. \( \pi_N \circ F_{\epsilon,N} = (G_{\epsilon,N-1} \times F_{\epsilon,1}) \circ \pi_N \)), where \( F_{\epsilon,1}(u) = 2u \mod 1 \) (acts on the coordinate sum \( (\pi_N u)_N \)) does not depend on \( \epsilon \) and is ergodic with respect to the Lebesgue measure on \( \mathbb{T}^1 \). Therefore, any failure of ergodicity for \( F_{\epsilon,N} \) has to be concomitant with the same phenomenon for \( G_{\epsilon,N-1} \).

The mapping \( G_{\epsilon,N-1} \) does not involve the coordinate sum, and its (constant) derivative conveniently turns out to be a multiple of the identity in \( \mathbb{T}^{N-1} \), i.e. \( DG_{\epsilon,N-1} = 2(1-\epsilon)I_{\mathbb{T}^{N-1}} \). Explicitly, we have \( G_{\epsilon,D} = 2(1-\epsilon)I_{\mathbb{T}^D} + \frac{2\epsilon}{D+1}B_D \mod 1 \), where we have for \( i \in \{1, \ldots, D\} \)

\[
(B_D(x))_i = 2h(x_i) + \sum_{j=1}^{i-1} h(\sum_{k=j}^i x_k) - h(\sum_{k=j}^{i-1} x_k) + \sum_{k=j+1}^{D} h(\sum_{k=i}^j x_k) - h(\sum_{k=i+1}^j x_k).
\]

For \( \epsilon = 0 \), individual units are decoupled and evolve independently of each other on the normalized circle, as given by iterations of the map \( u \mapsto 2u \mod 1 \). Perturbative arguments at the uncoupled limit, \( \epsilon = 0 \), applied
to the related transfer operator acting on measure densities, demonstrate ergodicity for $\epsilon > 0$ sufficiently small, for every integer $D$. Accordingly, we ask if that property can fail, especially due to symmetry breaking, at the other end of the expanding range, when $\epsilon$ is close enough to $\frac{1}{2}$. Continuation arguments appear inapplicable in this case because, unlike for the uncoupled limit, no basic situation is available to refer to. In particular, for any $D \in \mathbb{N}$, the mappings $G^D_1$ consist of piecewise isometries, and their asymptotic dynamical properties remain extremely elusive. These concerns lead us to begin by collecting numerical evidence.

Figure 1: Direct empirical evidence of ergodicity/symmetry breaking for the maps $G_{\epsilon, 2}$ (top row) and $G_{\epsilon, 3}$ (bottom row). Superimposed plots of $n_1$ consecutive points of orbits (one color for each orbit), after projection on $(0, 1)^D$ and discarding of the first $n_0$ iterates ($n_0 = 15 \times 10^3$ and $n_1 = 5 \times 10^3$ for $D = 2$, $n_0 = 40 \times 10^3$ and $n_1 = 10 \times 10^3$ for $D = 3$). In each image, one orbit is started from a representative initial condition and the other ones follow by symmetry. For $\epsilon < \epsilon_D$ (where $\epsilon_2 \simeq 0.417$ and $\epsilon_3 \simeq 0.393$), all orbits appear to cover the same set, suggesting that ergodicity holds. However, for $\epsilon > \epsilon_D$, no two points from distinct orbits overlap, suggesting the existence of multiple Lebesgue ergodic components. For $D = 2$, discontinuity lines $x_1, x_2 = \frac{1}{2}$ and $x_1 + x_2 = \frac{1}{2}$ are also shown. For $D = 3$, several color schemes are used to differentiate groups of distinct symmetry type: black for the symmetric trajectory and rainbow (resp. neon) colors for the 6 (resp. 8) element group, which exist for $\epsilon > \epsilon_3$ (resp. $\epsilon > 0.437$).

2 Empirical results from numerical trajectories

Direct visualization of asymptotic-orbit traces in phase space can offer a straightforward evaluation of ergodicity/symmetry and their failure in low dimension, $D = 2$ and 3 (NB: the Milnor attractor of $G_{\epsilon, 1}$ is transitive, and hence ergodic, for every $\epsilon \in [0, \frac{1}{2}]$ [10, 22]). Late iterates of orbits started from representative initial conditions, and their images under symmetries, are plotted for $\epsilon$ across the interval $[0, \frac{1}{2}]$ (Fig. 1). Initial conditions were selected using random sampling of phase space, in a way to render, if not all, then the most essential attractor features, with an emphasis on detecting asymmetry.

Although the bifurcation scenarios differ for the two cases, the pictures reveal that when increasing $\epsilon$ from 0, ergodicity persists for $\epsilon$ up to some $\epsilon_D$ and then fails beyond that threshold. In the $D = 2$ case, the transitive and fully symmetric attractor continuously splits at $\epsilon = \epsilon_2$ into 6 disjoint and asymmetric invariant pieces. Each emerging piece breaks all map symmetries except one (Appendix A). In the $D = 3$ case, a fully symmetric invariant set exists throughout the expanding domain. In addition, 6 asymmetric invariant components discontinuously appear at $\epsilon = \epsilon_3$, away from the symmetric set, and persist from thereon as $\epsilon$ continues to increase. Then, at $\epsilon \simeq 0.437$, this group of partly asymmetric orbits is augmented in a similarly discontinuous way by an additional analogously persisting group, composed of 8 asymmetric orbits (see the
involved symmetries in Appendix B). In all cases, the asymmetric components always appear to be disjoint from their image under all-coordinate inversion.

Following standard diagnostics in statistical physics, ergodicity/symmetry breaking for arbitrary $D$ can be assessed by means of order parameter (OP) empirical estimates, viz. unsigned averages over consecutive iterates of asymmetry quantifying observables, which suggest the existence of several ergodic components when positive. Here, our focus is on establishing failure of symmetry under coordinate inversion. To that end, we use the "central" coordinate $x_{\left\lceil D/2 \right\rceil}$ as an observable (Fig. 2 and 5); other estimates based on different quantifiers e.g. any single coordinate, two/all coordinate mean values, etc., all yield similar plots with identical properties, in particular as bifurcation values are concerned (data not shown). Finite time effects are accounted for by superimposing results from averages over increasing numbers of iterates. Similarly, dependence on the initial condition is evaluated using multiple runs based upon randomly drawn inputs.

In agreement with ergodicity, the OP vanishes at small coupling. However, as soon as $\epsilon$ exceeds $\epsilon_D$, this quantity takes on positive values for a positive fraction of initial conditions. This was observed for all investigated values of the dimension, from $D = 2$ up to $D = 200$. For $D = 2$ and 3, the data were consistent with the phase space plots of Fig. 1; the emergence of positive OP coincides with the appearance of asymmetric ergodic components (NB: Surprisingly, for $D = 3$, the additional asymmetric group emerging at $\epsilon \approx 0.437 > \epsilon_3$ in Fig. 1 - signs of the corresponding transition are barely visible on Fig. 2 - shows OP values that are very close, if not identical, to the primary asymmetric group). Furthermore, the bifurcation diagrams show distinctive characteristics according to the parity dimension (Fig. 2 and 5). For $D$ odd, the bifurcation values $\epsilon_D \sim 0.42$ appears to be almost insensitive to $D > 3$, and OP estimates are localized around a single $(\epsilon$-dependent) value. Moreover, the fraction of initial conditions that yield non-zero estimates decreases with $D$, making it more difficult to collect marked evidence of ergodicity breaking. For $D$ even, $\epsilon_D$ increases with $D$ and seems to approach a limit value $\epsilon_\ast < \frac{1}{2}$. OP estimates appear to be uniformly spread between 0 and the $(\epsilon$-dependent) maximal value, making it difficult to discriminate limit values from finite size fluctuations.

Independently of parity considerations, maximal OP values for $\epsilon > \epsilon_D$ - which show a linear coupling dependence of tiny slope, and eventually become independent of $\epsilon$ for large $D$ - decrease as $D$ increases, and asymptotically behave as $\frac{1}{2D}$ for $D$ even (resp. $\frac{1}{D}$ for $D$ odd), see Fig. 2 and Table D.2. Accordingly, to unambiguously distinguish asymmetry from short time fluctuations requires longer averages. In particular, $T = 10^8$ averages do not suffice to identify the emergence of positive OP for dimensions $D = 70$ and beyond (Fig. 5). This issue, when combined with linear increase of the dimension of the variables in the numerical iterations, substantially increases the computation time required for conclusive evidence. For instance, to obtain the $100 \times 100$ averages over $T = 10^8$ iterates in each of the Fig. 5 pictures for $D \geq 70$ required running times of several hundred of hours on a 2.4 GHz multi-processor computer (compared to a few
minutes for \( D = 2 \)). Therefore, to show ergodicity breaking for dimensions that are commensurate with realistic physical systems appears to be a considerable challenge. However, the numerical evidence of the emergence of asymmetry at strong coupling calls for rigorous confirmation, in particular to exclude transient effects and other computer round-off shortcomings.

![Figure 3: Maximum order parameter \( \times D \) vs. the dimension \( D \), respectively for even and odd \( D \). Data from Table D2.](image)

### 3 Algorithmic construction of asymmetric invariant sets

In order to prove the existence of asymmetric Lebesgue ergodic components in piecewise affine and expanding maps such as those presented here, it suffices to identify forward invariant, finite unions of polytopes that are disjoint from their symmetric image. For convenience, these sets are denoted InAsUP (an acronym for Invariant Asymmetric Union of Polytopes). Such proofs exist for \( D = 2 \) and \( \epsilon > \frac{\sqrt{10}}{2} \approx 0.419 \) \([10, 25]\), and for \( D = 3 \) and \( \epsilon > \frac{1}{6} \left( 7 - \sqrt[3]{\frac{43 + 3\sqrt{177}}{2}} - \sqrt[3]{\frac{43 - 3\sqrt{177}}{2}} \right) \approx 0.397 \) \([24]\) (NB: both these thresholds are remarkably close to the empirical ones). The arguments share similar features that suggest possible extensions to larger dimensions. However, the extensions would involve solving polynomial inequalities of large degree, and hence rely on numerical methods.

#### 3.1 Algorithmic procedure

Having to imply numerical computations, we opted to develop a fully computational rigorous approach to the existence of InAsUP. The approach consists of generating forward invariant sets by computing subsequent iterates of selected asymmetric initial polytopes, while testing that the sets under construction remain disjoint from their symmetric image for the coordinate inversion \(-\Id_T\). It is based on the following elementary operations:

- Choose an initial polytope \( P^0 \subset T^D \) such that \( P^0 \cap -P^0 = \emptyset \) (NB: \(-P^0\) denotes the image set \(-\Id_T(P^0))\).
- Compute consecutive iterates \( P^t := G_{t,D}(P^0) \) for \( t \in \mathbb{N} \) – which consist of unions of polytopes – until either every polytope of \( P^t \) is included in a polytope in \( \bigcup_{k=0}^{t-1} P^k \) or a polytope of \(-P^t\) intersects one in \( \bigcup_{k=0}^{t-1} P^k \).

The purpose of the second terminating condition is to prematurely interrupt the procedure when one cannot guarantee that the constructed sets will be asymmetric. Otherwise, if the procedure terminates in the first instance, then both \( \bigcup_{k=0}^{t} P^k \) and \( -\bigcup_{k=0}^{t} P^k \) must be disjoint InAsUP, as desired.

Without \textit{a priori} control of the dynamics, such as Markov-type symbolic dynamics, no formal argument can be given that such an algorithm will eventually terminate. However, analytic proofs for \( D = 2, 3 \) and the computer statistics in Table 3.4 suggest that it should conclude in finite time whenever a trajectory with asymmetric order parameter is observed.

The algorithm may be assumed to deal only with convex polytopes whose facets are aligned with discontinuity planes (see section 3.2 below), viz. polytope dynamics can be reduced to a \( D(D + 1) \)-dimensional setting. While this reduction has considerable virtue for computational purposes (as explained in section 3.3), it may...
happen that a polytope \( P \subset P' \) under consideration intersects a previously existing one, \( P' \subset \bigcup_{k=0}^{t} P_k \), in such a way that \( P \setminus P' \) is not of the specified type. This may be problematic if, for the sake of efficiency, one aims to exclude already existing parts from to-be-added polytopes. It suggests to introduce a procedure for decomposing \( P \setminus P' \) into a union (mod 0) of non-overlapping polytopes as desired. For simplicity, we instead opted to evaluate whether or not the complementary component \( P \setminus P' \) is of specified type, and to only retain that component in case of a positive return, or to retain both polytopes otherwise (NB: This includes discarding \( P' \) when \( P \supset P' \); we obviously discard \( P \) when \( P \subset P' \)). Therefore, polytopes in InAsUP can overlap with each other, and the same invariant set can result when starting from two distinct polytopes, even though the collections of constituting pieces themselves differ.

### 3.2 Cylinder sets as adapted polytopes

Initial polytopes \( P_0 \) in the algorithm consist of cylinder sets, viz. intersections of pre-images of domains on which \( G_{\epsilon,D} \) reduces to an affine map [16]. In technical terms, when for convenience, \( G_{\epsilon,D} \) is regarded as a mapping from \((0,1)^D\) into itself, and letting \((0,1)^D = \bigcup_{a=1}^{N_D} A_a \) mod 0 denote the partition of \((0,1)^D\) into polytopes \( A_a \) – called atoms – on which the mapping is affine, a cylinder set can be characterized as the intersection
\[
A_{a_0...a_t} := \bigcap_{k=0}^{t} G_{\epsilon,D}^{-k}(A_{a_k}),
\]
where \( \{a_k\}_{k=0}^{t} \) is a word with letters \( a_k \in \{1, \ldots, N_D\} \).

In practice, non-empty cylinders are obtained from empirical trajectories, by sequentially recording the labels of atoms as they are visited. In addition, the expression for \( G_{\epsilon,D} \) above implies that every atom \( A_a \) has to be disjoint from its symmetric image \(-A_a + 1 \subset (0,1)^D\). By inclusion, the same property also holds for every cylinder set.

### 3.3 Polytope related vector dynamics

Given that initial polytopes are chosen from among the cylinder sets, and that the derivative \( DG_{\epsilon,D} \) is a multiple of the identity, our implementation exclusively deals with convex polytopes for which each facet is of specified type. This substitution by vector dynamics extends to arbitrary polytopes \( P_m \subset (0,1)^D \), by means of atomic refinement. The decomposition \( P_m = \bigcup_{a \in A_m} P_{m\cap a} \mod 0 \) – where given any atom label \( a \), the polytope \( P_{m\cap a} \mod 0 \), and therefore the vector \( m \cap a \) (see Appendix \([\text{C}]\), are defined by the relation \( P_{m\cap a} := P_m \cap A_a \) – combined with the previous affine rule in each atom gives
\[
G_{\epsilon,D}(P_m) = \bigcup_{a \in A_m} G_{\epsilon,D}(P_{m\cap a}) \mod 0
\]
for the image of an arbitrary \( P_m \), and thus provides an implicit definition for the image of an arbitrary vector, \( \Gamma_{\epsilon} D(m) = \{ \Gamma_{\epsilon,B_a}(m \cap m_a) \}_{a} \) where \( a \) runs over all labels for which \( P_{m \cap m_a} \) is not empty.

Furthermore, for \( \epsilon \in \mathbb{Q} \) a rational number, every map \( \Gamma_{\epsilon,B_a} \) has the properties

\[
\Gamma_{\epsilon,B_a}(\mathbb{Q}^{D(D+1)}) \subset \mathbb{Q}^{D(D+1)} \quad \text{and} \quad \Gamma_{\epsilon,B_a}^{-1}(\mathbb{Q}^{D(D+1)}) \subset \mathbb{Q}^{D(D+1)}.
\]

In particular, for \( \epsilon \in \mathbb{Q} \), the vector dynamics associated with cylinder sets \( A_{a_0 \ldots a_t} \) remains limited to \( \mathbb{Q}^{D(D+1)} \). Exact computer arithmetic on rational numbers then implies that the numerical procedure perfectly mimics the original set dynamics (NB: in practice, we use the GNU arithmetic library GMP). For such \( \epsilon \), any positive outcome of the algorithm corresponding to \( P^{t} \subset \bigcup_{k=0}^{t-1} P^{k} \) therefore provides a full rigorous proof of InAsUP existence. (NB: Analytic proofs for \( D = 2 \) and \( D = 3 \) show that some cylinder set facets can be exactly mapped onto other ones, even when \( \epsilon \) is an irrational number. Therefore, analytic cancellations that yield such exact matchings must be taken into account when dealing instead with floating-point arithmetic.)

### 3.4 Computer results on InAsUP

Computer proofs of existence of InAsUP have been obtained for values of \( D \) up to 5 and in particular, in dimensions \( D = 4 \) and \( D = 5 \), for which no analytic proof is available (Table 1). For each \( D < 5 \) (resp. for \( D = 5 \)), InAsUP were successfully constructed for several (resp. one) value(s) of \( \epsilon \) in the interval where a positive order parameter was observed in Fig. 2. For completeness, the dependence on the initial cylinder has also been evaluated. For a given value of \( \epsilon \), we ran the algorithm starting from each cylinder in an empirical collection of a given length, obtained using a sample trajectory exhibiting positive order parameter (NB: for the sake of CPU time, for \( D = 4 \), the construction only considered cylinders inside a unique atom, and for \( D = 5 \), we proceeded with only 5 cylinders).

For short lengths, InAsUP construction was successful only for relatively few cylinders (Table 1). In fact, analytic proofs for \( D = 2 \) and \( D = 3 \) show that no InAsUP can be obtained starting from cylinders of length 1, i.e. from basic atoms (NB: for \( D = 2 \), this fact can be anticipated from Fig. 1, which indicates that the asymmetric ergodic components intersect discontinuity lines). For \( D = 4 \) (and \( \epsilon = 0.47 \)), no InAsUP results when the construction starts from any cylinder of length 5. The minimal cylinder length for InAsUP construction for \( D = 5 \) remains unknown. Besides, the success ratio increases with cylinder length and rapidly approaches – or attains – 1. Hence, a small enough adapted neighborhood of any point of an

### Table 1: Data summary of InAsUP’s exact numerical construction

| \( D \) | # atoms in \( (0,1)^D \) | \( \epsilon \) | Cyl. length | InAsUP success ratio | # InAsUP | Run. time for InAsUP |
|------|----------------|--------|-------------|-------------------|---------|---------------------|
| 2    | 6              | 0.44   | 2           | 2/6               | 10 - 70 | ~1s                 |
|      |                 |        | 3           | 8/10              |         |                     |
|      |                 |        | 5           | 24/24             |         |                     |
| 3    | 26             | 0.4    | 2           | 12/61             | 500 - 10^4 | 3 - 60s           |
|      |                 |        | 4           | 398/585           |         |                     |
|      |                 |        | 9           | 11584/11706       |         |                     |
| 4    | 75             | 0.47   | 5           | 0/29              | 1 - 7×10^5 | 10 - 50h          |
|      |                 |        | 6           | 7/43              |         |                     |
|      |                 |        | 11          | 355/373           |         |                     |
| 5    | 541            | 0.44   | 4           | N/A               | 4 - 6×10^6 | 2.5 - 4×10^3h |
|      |                 |        | 5           | 3/3               |         |                     |
|      |                 |        | 7           | 2/2               |         |                     |

1st column: Dimension \( D \). 2nd col.: Atomic partition cardinality. 3rd col.: Tested value for the coupling parameter \( \epsilon \) (other values have been tested, data not shown). 4th col.: Length, \( t + 1 \), of the initial cylinders \( A_{a_0 \ldots a_t} \). 5th col.: Numb. of cylinders of a given length for which computations resulted in InAsUP / total number of tested cylinders of the same length (For \( D = 5 \) and cyl. length = 4, runs had to be stopped before reaching any conclusion). 6th col.: Cardinality range for the resulting unions of polytopes \( P_m \) that defined InAsUP. 7th col.: Computation time range for InAsUP completion (CPU frequency = 2.2 GHz).
empirical asymmetric trajectory can be chosen to generate InAsUP. Not only is ergodicity breaking not a
spurious numerical illusion, but transient behaviors and round-off errors do not significantly impact this
phenomenon.

As mentioned in Section 3.1, the numerical procedure has not been optimized to reduce overlapping inter-
sections between polytopes. Accordingly, InAsUP cardinality – which depends on the initial cylinder and
shows considerable variation over a substantial range, independent of cylinder length (Table 1) – has limited
significance. However, this number shows formidable increase with the dimension $D$, as does the atomic
partition cardinality, and also in agreement with the corresponding augmentation in the observed number
of connected elements of each ergodic component in Fig. 1. More importantly, together with dealing with
longer vectors, this increase of cardinality substantially increases the computation time for InAsUP comple-
tion. The time becomes unreasonably excessive for $D = 5$ (viz. 100 days or more on a standard workstation).
As for empirical OP computations, this suggests that rigorous evidence of InAsUP existence for more re-
alistic number of individuals requires major algorithmic improvements and the dedication of considerable
computer resources.

4 Discussion

The literature on the deterministic dynamics of collective systems mostly describes phenomenological changes
that deal with pattern formation and synchrony phenomena and that usually correspond to bifurcations of
steady states or periodic solutions \cite{1, 9, 8, 22}. These transitions – which may also imply a reduction in the
phase space to lower dimensional subspaces \cite{2} – can be regarded as the breaking of ergodicity of an atomic
or singular measure. Instead, in order to preserve all the degrees of freedom of the dynamics in a chaotic
attractor, analogues of phase transitions with spontaneous symmetry breaking should involve absolutely
continuous invariant measures (ACIM). Rigorously articulated examples of this are few, especially when
appeal to a underlying ad hoc random process possessing the desired property is excluded.

We have provided numerical evidence for symmetry breaking of a chaotic attractor of full dimension in a
model of a collective system of interacting individuals (whose Markov partition remains elusive). Further,
we have developed and tested an algorithm for exact computer proof of the corresponding breaking of ACIM
ergodicity, which exploits the characteristics of the system. Even though, for the sake of computational
resources, the algorithm has only been benchmarked against a limited number of individuals, it indicates
that phase transitions can be rigorously proven in a purely deterministic setting, absent any reference to
statistical mechanics.

In order to improve their physical relevance, such features should be confirmed for systems with larger
numbers of individuals and for more realistic models. For our systems, such confirmation might require
algorithmic improvements, such as, for instance, parallel implementation – even though the iterative con-
struction of an invariant set is an intrinsically sequential process – or automated decomposition into unions
of non-overlapping polytopes, and the use of optimized libraries for rational arithmetic, as well as dedicated
computational resources.

Besides the existence of InAsUP, alternative proofs of the breaking of ACIM ergodicity could be obtained
using spectral properties of the transfer operator associated with the coupled map system. Since this operator
governs the dynamics of the densities associated with measures, it suffices to prove that it acquires multiple
fixed points as the coupling intensity increases. Likewise, one could aim at delineating (coupling dependent)
Markov partitions, which are compatible with ergodicity at small coupling and simultaneously imply splitting
into asymmetric ergodic components as interactions become strong. To our best knowledge, these approaches
have not been considered in the literature.

In any case, while transitions in statistical mechanics only occur at the thermodynamic limit, $N \to +\infty$
– especially because irreducible Markov chains, which govern the dynamics for finite $N$, must be uniquely
ergodic – together with \cite{10, 21, 23}, this paper shows that without the Markov assumption, deterministic
models do not require us to consider this limit, and can display similar non-ergodic behaviors and symmetry
breaking in finite dimension.
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References

[1] J.A. Acebron, L.L. Bonilla, C.J. Perez-Vicente, F. Ritort, and R. Spigler. The Kuramoto Model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77:137–185, 2005.

[2] J. Buescu. *Exotic Attractors*. Birkhäuser, 1997.

[3] L. Bunimovich. Coupled map lattices: at the age of maturity. In *Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems*, volume 671 of *Lect. Notes Phys.*, pages 9–32. Springer-Verlag, 2005.

[4] C. Boldrighini, L. Bunimovich, G. Cosimi, S. Frigio, and A. Pellegrinotti. Ising-type transition in coupled map lattices. *J. Stat. Phys.*, 80:1185–1205, 1995.

[5] S.P. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[6] H. Chaté and P. Manneville. Collective behavior in spatially extended systems with local interaction and synchronous updating. *Prog. Theo. Phys.*, 87(1):1–60, 1993.

[7] R. Coutinho and B. Fernandez. Extensive bounds on the entropy of repellers in expanding coupled map lattices. *Ergod. Th. & Dynam. Sys.*, 33:870–895, 2013.

[8] M. Cross and P. Hohenberg. Pattern formation outside equilibrium. *Rev. Mod. Phys.*, 65:85–1112, 1993.

[9] H. Dietert and B. Fernandez. The mathematics of asymptotic stability in the Kuramoto model. *Proc. Roy. Soc. A*, 474:0467, 2018.

[10] B. Fernandez. Breaking of ergodicity in expanding systems of globally coupled piecewise affine circle maps. *J. Stat. Phys.*, 154:999–1029, 2014.

[11] G. Gielis and R. MacKay. Coupled map lattices with phase transitions. *Nonlinearity*, 13:867–888, 2000.

[12] H. Hinrichsen. Non-equilibrium critical phenomena and phase transitions into absorbing states. *Adv Phys*, 49:815–958, 2000.

[13] J. Koiller and L-S. Young. Coupled map networks. *Nonlinearity*, 23:1121–1141, 2010.

[14] M. Jiang and Y. Pesin. Equilibrium Measures for Coupled Map Lattices: Existence, Uniqueness and Finite-Dimensional Approximations. *Commun. Math. Phys.*, 193:675–711, 1998.

[15] W. Just. Globally coupled maps: phase transitions and synchronization. *Physica D*, 81:317–340, 1995.

[16] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[17] T.M. Liggett. *Interacting Particle Systems*. Springer, 2005.

[18] R. MacKay. Indecomposable coupled map lattices with non-unique phases. In *Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems*, volume 671 of *Lect. Notes Phys.*, pages 65–94. Springer-Verlag, 2005.

[19] J. Miller and D.A. Huse. Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled-map lattice. *Phys. Rev. E*, 48:2528–2535, 1993.

[20] J. Milnor. On the concept of attractor. *Commun. Math. Phys.*, 99:177–195, 1985.
A Symmetry breaking for $G_{\epsilon,2}$

The map $G_{\epsilon,2}$'s explicitly expression is given by

\[
\begin{align*}
(G_{\epsilon,2}(x))_1 &= 2(1-\epsilon)x_1 + \frac{2\epsilon}{3} (2h(x_1) - h(x_2) + h(x_1 + x_2)) \\
(G_{\epsilon,2}(x))_2 &= 2(1-\epsilon)x_2 + \frac{2\epsilon}{3} (2h(x_2) - h(x_1) + h(x_1 + x_2)) \\
&\quad \pmod{1}
\end{align*}
\]

$G_{\epsilon,2}$ commutes with each transformation in the natural representation of $\mathbb{Z}_2 \times S_3$ on $T^2$. This group can be generated by the coordinate inversion $-\text{Id}$ and the transformations $\sigma_{213}$, $\sigma_{132}$ and $\sigma_{321}$, which are induced by the transpositions in $S_3$ (NB: subscripts here denote transposition images of the ordered list $\{1, 2, 3\}$ - abbreviated as 123). The coordinate inversion writes

\[
-\text{Id}x = (-x_1, -x_2) \pmod{1},
\]

and it corresponds to the central symmetry wrt $(\frac{1}{2}, \frac{1}{2})$ in the square $(0, 1)^2$. Similarly, we have

\[
\sigma_{321}x = (-x_2, -x_1) \pmod{1},
\]

which corresponds to the orthogonal reflection wrt to the anti-diagonal $x_1 + x_2 = 1$. Moreover, $\sigma_{213}$ (resp. $\sigma_{132}$) is defined by

\[
\sigma_{213}x = (-x_1, x_1 + x_2) \pmod{1} \quad \text{(resp. } \sigma_{132}x = (x_1 + x_2, -x_2) \pmod{1}),
\]

is the reflection wrt the axis $x_1 = 0$ along the direction $x_1 + 2x_2 = \text{cst}$ (resp. wrt the axis $x_2 = 0$ along the direction $2x_1 + x_2 = \text{cst}$).

The representation on $T^2$ of $\mathbb{Z}_2 \times S_3$ itself can be listed as follows

\[
\{\text{Id}, -\text{Id}, \sigma_{213}, \sigma_{132}, \sigma_{321}, \sigma_{213} \circ \sigma_{321}, \sigma_{321} \circ \sigma_{213}, -\sigma_{213}, -\sigma_{132}, -\sigma_{321}, -\sigma_{213} \circ \sigma_{321}, -\sigma_{321} \circ \sigma_{213}\}.
\]

Clearly, all ergodic components for $\epsilon > \epsilon_2$ break the coordinate inversion symmetry (Fig. 1). More precisely, each component remains invariant under the action of one transformation above and is mapped onto another component under any other transformation. In particular, the chartreuse and dark green components are invariant under $\sigma_{213}$. Blue and light green components are invariant under $\sigma_{132}$ and red and orange components are invariant under $\sigma_{321}$. 

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B  Symmetry breaking for $G_{\epsilon,3}$

The map $G_{\epsilon,3}$ writes

\[
\begin{align*}
(G_{\epsilon,3}(x))_1 &= 2(1-\epsilon)x_1 + \frac{\epsilon}{2}(2h(x_1) - h(x_2) + h(x_1 + x_2) - h(x_2 + x_3) + h(x_1 + x_2 + x_3)) \\
(G_{\epsilon,3}(x))_2 &= 2(1-\epsilon)x_2 + \frac{\epsilon}{2}(2h(x_2) - h(x_1) + h(x_1 + x_2) + h(x_2 + x_3)) \\
(G_{\epsilon,3}(x))_3 &= 2(1-\epsilon)x_3 + \frac{\epsilon}{2}(2h(x_3) - h(x_2) - h(x_1 + x_2) + h(x_2 + x_3) + h(x_1 + x_2 + x_3))
\end{align*}
\]

(mod 1)

As before, any transformation in the representation of $S_4$ on $\mathbb{T}^3$ can be obtained as a composition of 2-element-permutation representations, whose expressions are given by

\[
\begin{align*}
\sigma_{2134} &= (-x_1, x_1 + x_2, x_3) \pmod{1}, & \sigma_{3214} &= (-x_2, -x_1, x_1 + x_2 + x_3) \pmod{1} \\
\sigma_{4231} &= (-x_2 - x_3, x_2, -x_1 - x_2) \pmod{1}, & \sigma_{1324} &= (x_1 + x_2, -x_2, x_2 + x_3) \pmod{1} \\
\sigma_{1432} &= (x_1 + x_2 + x_3, -x_3, -x_2) \pmod{1}, & \sigma_{1243} &= (x_1, x_2 + x_3, -x_3) \pmod{1}
\end{align*}
\]

The ergodic components that emerge at $\epsilon = \epsilon_3$ all break the coordinate inversion symmetry $-\text{Id}$ in $\mathbb{T}^3$. Similarly to as before, these 6 components are only partly asymmetric; they remain invariant under the action of 7 transformations in $\mathbb{Z}_2 \times S_4$, and they are exchanged under other transformations. In particular, the blue component is invariant under

\[
\sigma_{3214}, \sigma_{1432}, -\sigma_{1243} \circ \sigma_{2134}, -\sigma_{1324} \circ \sigma_{4231}, -\sigma_{1324} \circ \sigma_{1432} \circ \sigma_{2134} \quad \text{and} \quad -\sigma_{1243} \circ \sigma_{4231} \circ \sigma_{2134},
\]

and, obviously, under the composition $\sigma_{3214} \circ \sigma_{1432} = \sigma_{1432} \circ \sigma_{3214}$.

As for the 8 components emerging at $\epsilon \approx 0.437$, their asymmetries are stronger than for the previous components, as they remain invariant under only 5 transformations. For instance, the fuchsia component is invariant under

\[
\sigma_{2134}, \sigma_{4231}, \sigma_{1432},
\]

and, obviously, under the compositions $\sigma_{4231} \circ \sigma_{2134} = \sigma_{2134} \circ \sigma_{4231} = \sigma_{1432} \circ \sigma_{1432}$. $\sigma_{1432} \circ \sigma_{2134} = \sigma_{4231} \circ \sigma_{1432}$.

C  Polytope induced vector operations and constraint optimization

Recall from the main text that the algorithm for InAsUP construction relies on the following convex polytopes $P_m$

\[
P_m = \left\{ x \in \mathbb{R}^D : \underline{m}_{i,j} < \sum_{k=i}^{j} x_k < \overline{m}_{i,j}, 1 \leq i \leq j \leq D \right\},
\]

and substitutes their dynamics by one acting on the $D(D+1)$-dimensional vectors $m$.

C.1  Polytope induced vector operations

Substitution by vector dynamics requires expressing algebraic operations on the polytopes $P_m$ in terms of operations on the vectors $m$. The following operations are of special interest:

- **Inclusion**: $P_m \subset P_{m'}$ if $\underline{m}_{i,j} \leq \underline{m}_{i,j}$ and $\overline{m}_{i,j} \leq \overline{m}_{i,j}$ for all $1 \leq i \leq j \leq D$.
- **Intersection**: $P_m \cap P_{m'} = P_{m''}$ where $\underline{m''}_{i,j} = \max(\underline{m}_{i,j}, \underline{m'}_{i,j})$ and $\overline{m''}_{i,j} = \min(\overline{m}_{i,j}, \overline{m'}_{i,j})$ for all $1 \leq i \leq j \leq D$.
- **Translation**: $P_m + x = P_{m'}$ where $\overline{m'}_{i,j} = \overline{m}_{i,j} + \sum_{k=i}^{j} x_k$ for all $1 \leq i \leq j \leq D$.
- **Symmetry**: $-P_m = P_{m'}$ where $\underline{m'}_{i,j} = -\underline{m}_{i,j}$ and $\overline{m'}_{i,j} = -\overline{m}_{i,j}$ for all $1 \leq i \leq j \leq D$.  

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C.2 Polytope constraint optimization

Overdetermination in expression (1) is a convenient, sometimes optimal (Fig. 4 left), way to encode in a single formal expression, any polytope under consideration. However, this definition does not systematically imply existence. Indeed, $P_m$ can perfectly be empty if, for instance, two coordinate sum constraints in (1) turn out incompatible with each other.

Likewise, overdetermination does not imply uniqueness of the constraining vector $m$, because $P_m$ can be unambiguously specified and yet, as many as $D(D - 1)$ constraints can remain inactive (Fig. 4 right).

Inactive constraints can be problematic when involved in the algorithmic implementation, because automated refinement can yield spurious polytopes via generation of incompatible constraints.

Figure 4: Illustrations of type $P_m$ polytopes for $D = 2$. Left. Hexagonal example for which all constraints in expression (1) must be active. Right. Rectangular example for which the constraints $\underline{m}_{1,2} \leq x_1 + x_2 \leq \overline{m}_{1,2}$ need not be active. Indeed, any vector $(\underline{m}_{i,j}, \overline{m}_{i,j})_{1 \leq i \leq j \leq D}$ for which $\underline{m}_{1,2} \leq \underline{m}_{1,1} + \underline{m}_{2,2}$ and $\overline{m}_{1,1} + \overline{m}_{2,2} \leq \overline{m}_{1,2}$ defines the same rectangle.

An algorithmically appropriate way to address this issue - and also to test existence - is to optimize constraints, viz. to compute, for every $m \in \mathbb{R}^{D(D+1)}$ for which $P_m$ is not empty, the sharpest vector $m'$ such that $P_{m'} = P_m$. The optimization procedure which, for the sake of notation, is presented for more general systems of inequality constraints, relies on standard Linear Programming arguments.

Suppose that a collection $(\alpha_{ij})_{1 \leq i \leq D', 1 \leq j \leq D}$ of non-negative elements ($D' \geq D$) together with a constraint vector $m = (\underline{m}_i, \overline{m}_i)_{i=1}^{D'}$ are given. We aim to compute the sharpest constraint vector that defines the set

$$\left\{ x \in \mathbb{R}^D : \underline{m}_i < \sum_{j=1}^{D} \alpha_{ij} x_j < \overline{m}_i, \quad 1 \leq i \leq D' \right\}.$$  

To that goal, we consider, for each $i \in \{1, \ldots, D'\}$, the set $\Lambda_i$ of vectors $(\lambda_k)_{k=1}^{D'}$ that solve the system of equations

$$\sum_{k=1}^{D'} \lambda_k \alpha_{kj} = \alpha_{ij}, \quad 1 \leq j \leq D.$$  

Importantly for our purpose, vectors in $\Lambda_i$ can be chosen with $D' = D$ or more vanishing coordinates and the corresponding subset $\Lambda_i^o$ can be generated via algorithmic procedure (Appendix D.2).

Besides, given $\lambda \in \mathbb{R}$ and a vector $m$ of component pairs, we introduce the vector $(\varepsilon_k(\lambda, m))_{k=1}^{D'}$ defined by

$$\varepsilon_k(\lambda, m) = \begin{cases} \frac{m_k}{\overline{m}_k} & \text{if } \lambda > 0 \\ \frac{m_k}{\underline{m}_k} & \text{if } \lambda < 0 \end{cases}$$

and $(\varepsilon_k(\lambda, m))_{k=1}^{D'}$ obtained by exchanging the lower and upper bars in the RHS. Using the Karush-Kuhn-Tucker (KKT) approach to linear programming, which is an extension of the method of Lagrange multipliers to the case of inequalities constraints [B], one can prove (Appendix D.1) that the desired sharpest vector of component pairs $O(m) = (\underline{O}_i(m), \overline{O}_i(m))_{i=1}^{D'}$ is given by

$$O(m) = \max_{(\lambda_k) \in \Lambda_i^o} \sum_{k=1}^{D'} \lambda_k \varepsilon_k(\lambda, m)$$  

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and again, $O(m)_i$ is defined by taking a minimum and replacing $\xi_k$ by $\bar{\xi}_k$ in the RHS. Together with (2), equation (3) indicates that, for each index $i$, one has to select the combination of constraints that maximises the functional $\sum_{j=1}^{D} \alpha_{ij} x_j$. While this definition is intuitively clear, the purpose of the KKT approach is to prove that it is actually the best choice. In fact, one could imagine, for instance, that replacing $m_j$ by $O(m)_i$ could affect the value of $O(m)_i$ or $\overline{O(m)}_i$ for some $i' \neq i$. The proof below implies that this is not the case. Moreover, we shall also prove that all constraints in the definition of $P_{O(m)} = P_m$ have to be active, hence the following necessary and sufficient condition result for existence of $P_m$

$$O(m)_i < \overline{O(m)}_i, \quad 1 \leq i \leq D'. \quad (4)$$

## D Optimization procedure: proof and algorithmic implementation

### D.1 Proof of the optimal expression (3)

Suppose first that, given $i \in \{1, \ldots, D'\}$, we aim to maximize the quantity $\sum_{j=1}^{D} \alpha_{ij} x_j$ given the $2D'$ linear inequality constraints

$$\sum_{j=1}^{D} \alpha_{kj} x_j \leq \overline{m}_k \quad \text{and} \quad -\sum_{j=1}^{D} \alpha_{kj} x_j \leq -\overline{m}_k \quad \text{and} \quad m_k < \overline{m}_k, \quad 1 \leq k \leq D'. \quad (5)$$

This problem perfectly fits the KKT approach to linear programming. The corresponding KKT conditions first state that every maximizer $(x^*_j)_{j=1}^{D}$ must cancel the gradient of the following Lagrangian

$$\sum_{j=1}^{D} \alpha_{ij} x_j - \sum_{k=1}^{D'} \gamma_k \left( \sum_{j=1}^{D} \alpha_{kj} x_j - \overline{m}_k \right) + \sum_{k=1}^{D'} \gamma'_k \left( \sum_{j=1}^{D} \alpha_{kj} x_j + m_k \right),$$

for a unique pair of vectors $(\gamma_k)_{k=1}^{D'}, (\gamma'_k)_{k=1}^{D'}$ with non-negative components. In other words, these vectors must satisfy the equation

$$\sum_{k=1}^{D'} (\gamma_k - \gamma'_k) \alpha_{kj} = \alpha_{ij}, \quad 1 \leq j \leq D.$$  

The KKT conditions also state that we must have

$$\gamma_k \left( \sum_{j=1}^{D} \alpha_{kj} x_j - \overline{m}_k \right) = 0 \quad \text{and} \quad \gamma'_k \left( \sum_{j=1}^{D} \alpha_{kj} x_j + m_k \right) = 0, \quad 1 \leq k \leq D. \quad (5)$$

Since we assume $m_k < \overline{m}_k$, the constraints $\sum_{j=1}^{D} \alpha_{kj} x_j \leq \overline{m}_k$ and $-\sum_{j=1}^{D} \alpha_{kj} x_j \leq -m_k$ cannot be simultaneously saturated, viz. we must have $\gamma_k \gamma'_k = 0$ for all $k$. A single multiplier $\lambda_k \geq 0$ effectively results for each $k$ and the vector $(\lambda_k)_{k=1}^{D'}$ must solve equation (2).

Together with (5), equation (2) implies that every maximum $\sum_{j=1}^{D} \alpha_{ij} x^*_j$ can be expressed as

$$\sum_{j=1}^{D} \alpha_{ij} x^*_j = \sum_{k=1}^{D'} \lambda_k \sum_{j=1}^{D} \alpha_{kj} x^*_j = \sum_{k=1}^{D'} \lambda_k \xi_k (\lambda_k, m).$$

It follows that the the global maximum must occur in $\Lambda_i$; hence the definition of $\overline{O(m)}_i$ when additionally shown that it must in fact occur in $\Lambda_i^0$. A similar reasoning on minimisers yields $O(m)_i$.

Moreover, the reasoning above implies that $\lambda_k = 0$ when the corresponding constraint is not saturated. This yields that $O$ must be a projection operator; hence $O(m)$ must be the sharpest vector such that $P_{O(m)} = P_m$. The same argument shows that every constraint given by $O(m)$ has to be saturated for a point in $P_{O(m)}$; hence condition (4) is necessary and sufficient for existence of $P_m$. 

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D.2 Algorithmic solving of equation (2)

Consider the partition of \( \mathbb{R}^{D'} \) into orthants where the sign of every coordinate \( \lambda_k \) is fixed. Take any such orthant that intersects \( \Lambda_i \) and take any \((\lambda_k)_{k=1}^{D'}\) in the corresponding intersection, for which at least 3 coordinates do not vanish. Then, at least two of these coordinates must have the same sign. Moreover, equation (2) implies that these coordinates must have opposite variations when \((\lambda_k)_{k=1}^{D'}\) varies in the intersection and all other coordinates remain fixed. Therefore, the function \( \sum_{k=1}^{D'} \lambda_k e_k(\lambda_k, m) \) must reach its maximum when one of these coordinates vanishes, because its gradient remains constant in this set.

This argument shows that (for \( D \geq 2 \)) the maximum must be reached on vectors \((\lambda_k)_{k=1}^{D'}\) with \( D' - D \) vanishing coordinates. We now describe an algorithmic procedure that exhausts all such solutions of equation (2).

- Take any \( D' - D \)-tuple of vanishing coordinates for \((\lambda_k)_{k=1}^{D'}\). Then (2) reduces to a system of \( D \) equations with \( D \) unknowns.
- If the corresponding determinant does not vanish, then, store the unique solution if it exists and, consider the previous instruction with another \( D' - D \)-nuple, until all such combinations have been exhausted.
- If the determinant vanishes (and the number of equations/unknown exceeds 2), then remove a degenerate equation, set an additional coordinate to 0, and repeat the instructions for every such subsystem of \( D - 1 \) equations/unknowns. Repeat these instructions for every subsequent lower dimensional subsystem (corresponding to vectors with larger numbers of vanishing coordinates) as long a vanishing determinant results.
- In each case, the procedure stops when no or unique solution results, or eventually yields a system of 2 equations/unknowns. As before, the only problematic case is when this system is degenerate. Investigating all situations, it could be only problematic when the two coordinates have opposite signs and when the function \( \sum_{k=1}^{D'} \lambda_k e_k(\lambda_k, m) \) increases when the positive one increases (otherwise the maximum is reached when one coordinate vanishes) : the maximum would be \(+\infty\) in this case, which is certainly impossible. Altogether, we conclude that there are finitely many relevant solutions to equation (2), which can be systematically exhausted.

Figure 5: Empirical estimates of all-coordinate inversion asymmetry observable, cont. from Fig 2. For \( D = 70 \) and beyond, changes in the estimates obtained from \( T = 10^5 \) averages (blue points) are not sufficiently sharp to support clear identification of symmetry breaking when \( \epsilon \) increases. For the sake of clarity, these estimates are not reported on the pictures for \( D = 86 \) and beyond.
Table 2: Maximal order parameter range in the domain $\epsilon > \epsilon_D$ for even and odd dimensions respectively

| Dim. | Mean $\times 10^4$ | Ampl. $\times 10^4$ | Dim. | Mean $\times 10^4$ | Ampl. $\times 10^4$ |
|------|-------------------|-------------------|------|-------------------|-------------------|
| 2    | 1560              | ± 110             | 5    | 1630              | ± 80              |
| 4    | 970               | ± 30              | 7    | 1230              | ± 30              |
| 6    | 700               | ± 20              | 9    | 990               | ± 30              |
| 8    | 550               | ± 10              | 11   | 820               | ± 20              |
| 10   | 450               | ± 5               | 15   | 620               | ± 10              |
| 14   | 333               | ± 3               | 19   | 490               | ± 10              |
| 18   | 263               | ± 3               | 23   | 413               | ± 8               |
| 22   | 217               | ± 3               | 27   | 352               | ± 3               |
| 26   | 185               | < ± 1             | 31   | 315               | ± 5               |
| 30   | 160               | < ± 1             | 39   | 247               | ± 3               |
| 38   | 130               | < ± 1             | 47   | 207               | ± 3               |
| 46   | 105               | < ± 1             | 55   | 183               | ± 3               |
| 54   | 94                | < ± 1             | 63   | 157               | < ± 1             |
| 62   | 82                | < ± 1             | 71   | 136               | ± 1               |
| 70   | 72                | < ± 1             | 86   | 59                | < ± 1             |
| 100  | 51                | < ± 1             | 128  | 40                | < ± 1             |
| 150  | 38                | < ± 1             | 150  | 38                | < ± 1             |
| 200  | 25                | < ± 1             |

Lists the mean values and amplitudes of black linear segments in Fig. 2 and 5 and from other data not shown.