Two layer asymptotic model for the wave propagation in the presence of vorticity

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Abstract. In the present study, we consider the system of two layers of the immiscible constant density fluids which are modeled by the full Euler equations. The domain of the flow is infinite in the horizontal directions and delimited above by a free surface. Bottom topography is taken into account. This is a simple model of the wave propagation in the ocean where the upper layer corresponds to the (thin) layer of fluid above the thermocline whereas the lower layer is under the thermocline. Though even this simple framework is computationally too expensive and mathematically too complicated to describe efficiently propagation of waves in the ocean. Modeling assumption such as shallowness, vanishing vorticity and hydrostatic pressure are usually made to get the bi-layer shallow water models that are mathematically more manageable. Though, they cannot describe correctly the propagation of both internal and free surface waves and dispersive/non hydrostatic must be added. Our goal is to consider the regime of medium to large vorticities in shallow water flow. We present the derivation of the model for internal and surface wave propagation in the case of constant and general vorticities in each layer. The model reduces to the classical Green-Naghdi equations in the case of vanishing vorticities.

1. Introduction

We consider two-layer flow of inviscid incompressible and immiscible fluids of constant densities $\rho_1, \rho_2$, for the stable stratification $\rho_1 > \rho_2$. The flow area is bounded above by a free surface $z = \zeta_1(\vec{x}, t)$ and below by nonuniform bottom $z = -h_20 + \zeta_2(\vec{x}, t)$. Hereafter quantities with indexes ‘1’ are addressed to the upper layer, and indexes ‘2’ to the lower.

The governing equations is given by Euler equations for each layer (here $V_i = (\vec{u}_i(t, \vec{x}, z), w_i(t, \vec{x}, z))$, $\vec{u}_i(t, \vec{x}, z) = (u_i(t, \vec{x}, z), v_i(t, \vec{x}, z))$ and $p_1, p_2$ denote the velocities and pressure fields, respectively):

$$
\begin{cases}
(V_i)_t + V_i \cdot \nabla_{\vec{x}, z} V_i = -\frac{1}{\rho_i} \nabla_{\vec{x}, z} p_i - g\epsilon z, \\
\nabla_{\vec{x}, z} \cdot V_i = 0,
\end{cases} \quad (i = 1, 2)
$$
which are complemented with kinematic boundary conditions:

\[
\begin{align*}
\zeta_1 + \vec{u}_1 |_{z=h_0^1+\zeta_1} \cdot \nabla \zeta_1 &= w_1 |_{z=h_0^1+\zeta_1}, \\
\zeta_2 + \vec{u}_1 |_{z=\zeta_2} \cdot \nabla \zeta_2 &= w_1 |_{z=\zeta_2}, \\
\zeta_2 + \vec{u}_2 |_{z=\zeta_2} \cdot \nabla \zeta_2 &= w_2 |_{z=\zeta_2}, \\
\vec{u}_2 |_{z=-h_0^2+b(x)} \cdot \nabla (b) &= w_2 |_{z=-h_0^2+b(x)}
\end{align*}
\]

and dynamic boundary conditions:

\[
p_1 |_{z=h_0^1+\zeta_1} = 0, \quad p_1 |_{z=\zeta_2} = p_2 |_{\zeta_2}.
\]

It is well known that under assumption of shallowness, vanishing vorticity and hydrostatic pressure the model can be reduced to the bi-layer shallow water models which are mathematically more manageable. Though, they cannot describe correctly the propagation of both internal and free surface waves since they do not take into account dispersive/non-hydrostatic effects. In [1, 2] these effects are added but still under the hypothesis of vanishing or small vorticity.

The goal of the present study is deriving a bi-layer model in the regime of medium to large vorticities with taking into account dispersive terms and non-hydrostatic pressure components. For that purpose we follow the strategy found in [3, 4]. We reduce the full two-layer Euler system describing the evolution of the flow by a non-dimensional system of averaged equations in each fluid layer that describes the evolution of the fluids height, the horizontal momentum in each layer and other averaged quantities (vorticities, internal energy). This system is exact but too complex to work with, because it is required a closure relations for the ‘Reynolds tensors’ and pressure gradients for each layer (we are following here the terminology using in [3]).

As no irrotationality assumptions are made, to find a closure we follow the strategy for the one layer found in [3] and introduce shear velocities representing the horizontal vorticities contribution to the horizontal velocities (which is zero for rotational case). The ‘Reynolds tensors’ and pressure gradients can be represented in terms of the shear velocities. By using vorticity and incompressible equations, we found an equation for shear velocities, and therefore our system become closed. The case of constant vorticities is considered as the first step of demonstration of the model applicability. The next step is to consider the case of general vorticities.

2. Asymptotic analysis

2.1. Dimensionless form of the averaged equations

For non-dimensionalization of the equations we use the following dimensionless parameters:

\[
\rho = \frac{\rho_1}{\rho_2} > 1, \quad \varepsilon_1 = \frac{a_1}{h_{10}}, \quad \varepsilon_2 = \frac{a_2}{h_{10}}, \quad \mu = \frac{h_{10}^2}{L^2}, \quad \delta = \frac{h_{10}}{h_{20}}, \quad \beta = \frac{\beta}{h_{10}}.
\]

To form these parameters several characteristic quantities should be specified: the typical amplitudes \(a_1\) of the surface waves and \(a_2\) of the interface waves, the typical amplitude \(\beta\) of the bottom variations and the typical depths \(h_{10}\) and \(h_{20}\) of two layers together with the typical wave length \(L\).

Parameters \(\varepsilon_1\) and \(\varepsilon_2\) are so-called nonlinearity parameters. The parameter \(\mu\) is the shallowness, or dispersion, parameter, that defines which wave scale is into account.

We are interested in shallow water flows and therefore we assume that shallowness parameter is small, \(\mu << 1\), but no smallness assumption is made for nonlinearity parameters \(\varepsilon_1, \varepsilon_2\), hence we allow for the large-amplitude waves.
In terms of dimensionless variables Euler equations take form:

\[
\begin{cases}
(V_i)_t + \varepsilon_i V_i \cdot \nabla V_i = -\frac{R_i}{\varepsilon_i M} (\nabla \cdot \varepsilon_i P_i + e_z), \\
\nabla \cdot V_i = 0,
\end{cases}
\]

where \( M = (0, 0, \mu)^T \), \( e_z = (0, 0, 1)^T \), \( R_1 = 1 \), \( R_2 = \rho \), coupled with dimensionless boundary conditions. Hereinafter unless indicated otherwise \( i = 1, 2 \), that is corresponding to the each layer.

Let us introduce the notations of the vertical averaging of the horizontal velocity components, here \( h_1 = 1 + \varepsilon_1 \zeta_1 - \varepsilon_2 \zeta_2 \) and \( h_2 = \varepsilon_2 \zeta_2 + \frac{1}{3} - \beta b(x) \) are changing layer depth:

\[
\bar{u}_1 = \frac{1}{h_1} \int_{\varepsilon_2 \zeta_2}^{1+\varepsilon_1 \zeta_1} \tilde{u}_1 \, dz, \quad \bar{u}_2 = \frac{1}{h_2} \int_{\frac{1}{3} + \beta b(x)}^{\varepsilon_2 \zeta_2} \tilde{u}_2 \, dz.
\]

Now we can construct decomposition of horizontal velocity field given by:

\[
\tilde{u}_1 = \bar{u}_1 + \sqrt{\mu} \tilde{u}_1^*, \quad \tilde{u}_2 = \bar{u}_2 + \sqrt{\mu} \tilde{u}_2^*.
\]

Integration of system (1) using boundary conditions leads to the system of averaging equations on changing depths \( h_1 \) and \( h_2 \) and average velocities (implying decomposition (2)):

\[
\begin{cases}
(h_i)_t + \varepsilon_i \nabla \cdot (h_i \bar{u}_i) = 0, \\
(h_i \bar{u}_i)_t + \varepsilon_i \nabla \cdot (h_i \bar{u}_i \otimes \bar{u}_i) + \varepsilon_i \mu \nabla \cdot \left( \int_{A_i}^{B_i} \tilde{u}_i^* \otimes \tilde{u}_i^* \, dz \right) + \frac{R_i}{\varepsilon_i} \int_{A_i}^{B_i} \nabla p_i \, dz = 0, \\
(w_i)_t + \varepsilon_i (u_i \nabla \cdot \bar{u}_i) + \varepsilon_i w_i(w_i)_z = \frac{1}{\mu \varepsilon_i} (R_i(p_i)_z + 1),
\end{cases}
\]

it should be noted that \( \nabla = (\partial_x, \partial_y) \) here is two-dimensional operator, and \( A_1 = \varepsilon_2 \zeta_2; B_1 = 1 + \varepsilon_1 \zeta_1, A_2 = -\frac{1}{3} + \beta b(x), B_2 = \varepsilon_2 \zeta_2 \) denote the layer boundaries.

The system (3) is exact, but it requires the closure for ‘Reynolds tensors’ \( \int_{A_i}^{B_i} \tilde{u}_i^* \otimes \tilde{u}_i^* \, dz \) and pressure terms \( \frac{R_i}{\varepsilon_i} \int_{A_i}^{B_i} \nabla p_i \, dz \) in terms of \( \bar{u}_i, h_i \). The closure is obtained within the framework of procedure introduced in [3]. For this purpose we now introduce vorticities \( \Omega_i = \nabla \times u_i \) that can be represented in accordance with velocity fields decomposition as:

\[
\Omega_i = \begin{pmatrix}
w_{iy} - v_{iz} \\
w_{iz} - w_{ix} \\
v_{ix} - w_{iy}
\end{pmatrix} = \frac{\varepsilon_i \sqrt{gh_{10}}}{h_{10}} \sqrt{\mu} \begin{pmatrix}
\tilde{v}_{ix}^* + \sqrt{\mu} \tilde{w}_{iy} \\
\tilde{v}_{iz}^* - \sqrt{\mu} \tilde{w}_{ix} \\
\tilde{v}_{ix}^* \cdot \tilde{u}_i
\end{pmatrix} = \frac{\varepsilon_i \sqrt{gh_{10}}}{h_{10}} \sqrt{\mu} \begin{pmatrix}
\tilde{\omega}_b^x \\
\tilde{\omega}_b^y \\
\tilde{\omega}_b^z
\end{pmatrix},
\]

the tildes here correspond to the dimensionless functions, orthogonal operator \( \nabla^\perp = (-\partial_y, \partial_z) \).

2.2. Velocity fields decompositions

Now we can find a representation for the velocity fields in terms of vorticities. Indeed, integration of the discontinuity equations yields to the expressions for the vertical velocities components in terms of horizontal velocities:

\[
\begin{align*}
w_1 &= -\nabla \cdot \left[ \bar{u}_1 (z - \varepsilon_2 \zeta_2) \right] - \sqrt{\mu} \nabla \cdot \int_{\varepsilon_2 \zeta_2}^{z} \tilde{u}_1^* \, dz + \frac{\varepsilon_2}{\varepsilon_1} \zeta_1, \\
w_2 &= -\nabla \cdot \left[ \bar{u}_2 \left( z - \beta b(x) + \frac{1}{3} \right) \right] - \sqrt{\mu} \nabla \cdot \int_{\beta b(x)-\frac{1}{3}}^{z} \tilde{u}_2^* \, dz,
\end{align*}
\]

3
boundary conditions is implied.

Definition of the vorticities (4) give us an equations for the components of horizontal velocity fields:

\[ \partial_z \tilde{u}_i^\perp = \sqrt{\mu} \nabla w_i - (\tilde{\omega}_1^\perp)^\perp. \]

Integrating it with taking into account expressions for vertical velocities (5), and introduce the operators for the sake of simplicity:

\[ T_1 V = \int_1^{1+\varepsilon_1 \zeta_1} \nabla \nabla \cdot V, \quad T_2 V = \int_1^{\varepsilon_2 \zeta_2} \nabla \nabla \cdot V, \]

\[ T_1^* V = (T_1 V)^*, \quad T_2^* V = (T_2 V)^*, \]

we found:

\[ (1 - \mu T_1^*) \tilde{u}_1^\perp = \tilde{u}_{1sh}^\perp + \sqrt{\mu} \left( T_1^* \Pi_1 - \left( \frac{\varepsilon_2}{\varepsilon_1} \nabla \zeta_2, (1 + \varepsilon_1 \zeta_1 - z) \right)^* \right), \]

\[ (1 - \mu T_2^*) \tilde{u}_2^\perp = \tilde{u}_{2sh}^\perp + \sqrt{\mu} \left( T_2^* \Pi_2 \right). \]

Here we introduce shear velocities representing the horizontal vorticities contribution to the horizontal velocities:

\[ \tilde{u}_{1sh}^\perp = \int_1^{1+\varepsilon_1 \zeta_1} (\tilde{\omega}_1^\perp)^\perp, \quad \tilde{u}_{2sh}^\perp = \int_1^{\varepsilon_2 \zeta_2} (\tilde{\omega}_2^\perp)^\perp. \tag{6} \]

Then together with the velocity field decompositions (2) we found the following first order approximations:

\[ u_1 = \Pi_1 + \sqrt{\mu} u_{1sh}^\perp + O(\mu), \quad u_2 = \Pi_2 + \sqrt{\mu} u_{2sh}^\perp + O(\mu), \]

and the second order approximations:

\[ u_1 = \Pi_1 + \sqrt{\mu} u_{1sh}^\perp + \mu \left( T_1^* \Pi_1 - \left( \frac{\varepsilon_2}{\varepsilon_1} \nabla \zeta_2, (1 + \varepsilon_1 \zeta_1 - z) \right)^* \right) + \frac{\varepsilon_2}{\varepsilon_1} u_{1sh}^\perp + O(\mu^2), \]

\[ u_2 = \Pi_2 + \sqrt{\mu} u_{2sh}^\perp + \mu \left( T_2^* \Pi_2 \right) + \frac{\varepsilon_2}{\varepsilon_1} u_{2sh}^\perp + O(\mu^2). \]

2.3. Equations for the shear velocities

It is required to find an equations for shear velocities to close the system. For this purpose we use the dimensionless vorticity equations:

\[ \omega_i^\perp + \varepsilon_i (\tilde{u}_i^\perp \nabla x, z) \omega_i^\perp = \varepsilon_i \omega_h^\perp \nabla u_i + \frac{\varepsilon_i}{\sqrt{\mu}} \omega_h^\perp (u_i)_z. \]

Recalling the definition (6) of \( u_{1sh} \), we integrate this equation for the upper layer with respect to \( z \) (we use here vectorial identity):

\[ \partial_t u_{1sh} + \varepsilon_1 \Pi_1 \cdot \nabla u_{1sh} + \varepsilon_1 u_{1sh} \cdot \nabla \Pi_1 - (\varepsilon_1 \nabla \cdot (\Pi_1 (z - \varepsilon_2 \zeta_2)) - \varepsilon_2 \zeta_2) \partial_z u_{1sh} = O(\varepsilon_1 \sqrt{\mu}). \]

We found the equation for the average component \( \Pi_{1sh} \) by integration the equation obtained above. Substraction one from another gives us the following equation for \( u_{1sh}^\perp \):

\[ \partial_t u_{1sh}^\perp + \varepsilon_1 \Pi_1 \cdot \nabla u_{1sh}^\perp + \varepsilon_1 u_{1sh}^\perp \cdot \nabla \Pi_1 - (\varepsilon_1 \nabla \cdot (\Pi_1 (z - \varepsilon_2 \zeta_2)) - \varepsilon_2 \zeta_2) \partial_z u_{1sh}^\perp = O(\varepsilon_1 \sqrt{\mu}) \tag{7} \]

Analogously, we obtain the equation for the shear velocity of lower layer:

\[ \partial_t u_{2sh}^\perp + \varepsilon_2 \Pi_2 \cdot \nabla u_{2sh}^\perp + \varepsilon_2 u_{2sh}^\perp \cdot \nabla \Pi_2 - (\varepsilon_2 \nabla \cdot (\Pi_2 (z - \beta h(x) + \frac{1}{\delta})) \partial_z u_{2sh}^\perp = O(\varepsilon_2 \sqrt{\mu}). \tag{8} \]

It is should be noted that the order \( O(\varepsilon_1 \sqrt{\mu}) \) of equations obtained in this section is enough to derive the models of the order \( O(\mu) \) in the case of general vorticities. But to obtain models of higher order the more precise equations should be found.
2.4. Pressure fields decompositions

The pressure fields representation in terms of the vorticities is obtained by integrating vertical component of Euler equations (3). Imposing dynamic boundary condition we have:

$$\frac{1}{\varepsilon_1} \nabla p_1 = \nabla \int_z^{1+\varepsilon_1 \zeta_1} \left( -\frac{1}{\varepsilon_1} \partial_z p_1 \right) dz = \nabla \zeta_1 + \mu \nabla \int_z^{1+\varepsilon_1 \zeta_1} \left( w_{1t} + \varepsilon_1 (u_1 \nabla \cdot) w_1 + \varepsilon_1 w_1 w_{1z} \right) dz,$$

$$\frac{\rho}{\varepsilon_2} \nabla p_2 = \nabla \left( \int_z^{\varepsilon_2 \zeta_2} \left( -\frac{\rho}{\varepsilon_2} \partial_z p_2 \right) dz + \frac{\rho}{\varepsilon_2} p_1 \right) = \nabla \zeta_2 + \mu \nabla \int_z^{\varepsilon_2 \zeta_2} \left( w_{2t} + \varepsilon_2 (u_2 \nabla \cdot) w_2 + \varepsilon_2 w_2 w_{2z} \right) dz + \nabla \left( \frac{\rho}{\varepsilon_2} p_1 \right).$$

The first terms on the right-hand side in both representations related to the hydrostatic pressure. An expansion for non-hydrostatic terms we will find in the next section in the case of constant vorticities, and further for general vorticity case.

3. Constant vorticities 1d case

As the first step of demonstration of the model applicability let us consider the case of constant vorticities. For the sake of simplicity the calculations are performed in the one dimensional case $x \in \mathbb{R}^1$.

In this case the only one component of each vorticities is non-zero, we assume that these components remain constant for all time:

$$\Omega_i = \begin{pmatrix} 0 \\ \omega_i \\ 0 \end{pmatrix} , \quad \omega_i = \partial_x \vec{u}_i \equiv const.$$

To define the shear velocities we use (6) with $(\omega_h) = - (\omega_i, 0)^T$:

$$u_{ish} = \int_{A_i}^{B_i} (\omega_h) dz = -\omega_i (B_i - x - \frac{h_i}{2}), \quad (9)$$

$A_i, B_i$ are boundaries of the layers defined above. The vertical velocities, therefore, is defined as:

$$w_1 = \partial_x (\bar{u}_1 (z - \varepsilon_2 \zeta_2)) + \frac{\sqrt{\mu}}{2} \partial_x (\omega_1 (z - \varepsilon_2 \zeta_2) (1 + \varepsilon_1 \zeta_1 - z)) + \frac{\varepsilon_2}{\varepsilon_1} \zeta_4,$$

$$w_2 = \partial_x (\bar{u}_2 (z - \beta b(x) + \frac{1}{\delta})) + \frac{\sqrt{\mu}}{2} \partial_x (\omega_2 (z - \varepsilon_2 \zeta_2) (\beta b(x) - \frac{1}{\delta} - z)). \quad (10)$$

3.1. ‘Reynolds tensor’ contributions for the constant vorticity case

Using velocity fields decomposition (2) and (9) we define ‘Reynolds tensor’ contribution with the second order for the upper layer:

$$\varepsilon_1 \mu \frac{1}{\varepsilon_1} \int_{\varepsilon_2 \zeta_2}^{1+\varepsilon_1 \zeta_1} |u_1|^2 dz = \varepsilon_1 \mu \int_{\varepsilon_2 \zeta_2}^{1+\varepsilon_1 \zeta_1} |u_{ish}^*|^2 + 2 \varepsilon_1 \mu \frac{3}{2} \partial_x \int_{\varepsilon_2 \zeta_2}^{1+\varepsilon_1 \zeta_1} u_{ish}^* T_1 \bar{u}_1 - 2 \varepsilon_2 \mu \frac{3}{2} \partial_x \left( \zeta_4 \int_{\varepsilon_2 \zeta_2}^{1+\varepsilon_1 \zeta_1} u_{ish}^* (1 + \varepsilon_1 \zeta_1 - z) dz \right) =$$

$$= \frac{\varepsilon_1 \mu}{12} \omega_1^2 (\bar{h}_1^3)_x - \frac{\varepsilon_1 \mu \frac{3}{2}}{12} \partial_x (\omega_1^3 (h_1^3 \partial_x^2 \bar{u}_1 - 4 \varepsilon_2 \partial_x \zeta_2 \partial_x \bar{u}_1 - 2 \varepsilon_2 \partial_x^2 \zeta_2 \bar{u}_1)) + \frac{\varepsilon_1 \mu \frac{3}{2}}{6} \omega_1 \partial_x (h_1^3 \zeta_4). \quad (11)$$
and for the lower layer:

\[\varepsilon_2\mu\partial_x\int_{-\frac{1}{4}+\beta b(x)}^{\varepsilon_2\zeta_2} |u_x^2|^2 dz = \]

\[\frac{\varepsilon_2\mu}{12} \omega_2^2 (h_2^3)_{xx} - \frac{\varepsilon_2\mu}{12} \omega_2 \partial_x (h_2^3 (h_2^2 \partial_x^2 \pi_2 - 4\beta b'(x) \partial_x \pi_2 - 2\beta b''(x) \pi_2)).\] (12)

These expressions are required closures for ‘Reynold tensor’ as long as they depends only on \(h_i\), \(\pi_i\) and on the other known task parameters such as a bottom parametrization \(b(x)\) and constant vorticities \(\omega_i\).

### 3.2. Pressure contribution

To compute the pressure contributions we represent the vertical velocity (10) in the following form:

\[w_1 = f_1(x,t) + zg_1(x,t), \quad w_2 = f_2(x,t) + zg_2(x,t),\]

here \(f_i(x,t), g_i(t,x)\) are defined as:

\[f_1(x,t) = \frac{\varepsilon_2}{\varepsilon_1} \zeta_1 + \varepsilon_2 \partial_x (\pi_1 \zeta_2) - \frac{\sqrt{\mu}}{2} \omega_1 (\varepsilon_1 \varepsilon_2 \partial_x (\zeta_1 \zeta_2) + \varepsilon_2 \zeta_2),\]

\[g_1(x,t) = \frac{\sqrt{\mu}}{2} \omega_1 \partial_x (\zeta_1 \zeta_1 + \varepsilon_2 \zeta_2) - \partial_x \pi_1,\]

\[f_2(x,t) = \beta \partial_x (\pi_1 b(x)) - \frac{1}{\delta} \partial_x \pi_2 - \frac{\sqrt{\mu}}{2} \omega_2 (\varepsilon_2 \beta \partial_x (z_2 b(x)) - \frac{1}{\delta} \varepsilon_2 \zeta_2),\]

\[g_2(x,t) = \frac{\sqrt{\mu}}{2} \omega_2 \partial_x (\varepsilon_2 \zeta_2 + \beta b(x)) - \partial_x \pi_2.\]

With such notations integrals take form:

\[\frac{1}{\varepsilon_1} \int_{\varepsilon_2 \zeta_2}^{1+\varepsilon_1 \zeta_1} \nabla p_1 = h_1 \zeta_1 + \mu \left( \varepsilon_2 \zeta_2, h_1 \left( D_1 f_1 + \varepsilon_1 f_1 g_1 + \frac{1}{2} (D_1 g_1 + \varepsilon_1 g_1^2) (1 + \varepsilon_1 \zeta_1 + \varepsilon_2 \zeta_2) \right) + \partial_x \left( \frac{h_1^2}{2} (D_1 f_1 + \varepsilon_2 \zeta_2 D_1 g_1 + \varepsilon_1 f_1 g_1 + \varepsilon_2 \zeta_2 g_1^2) + \frac{h_1^3}{3} (D_1 g_1 + \varepsilon_1 g_1^2) \right) \right) - \mu^2 \omega_1 \varepsilon_1 \left( \frac{h_1^3}{12} \varepsilon_2 \zeta_2, \partial_x^2 \pi_1 + \partial_x \left( \frac{h_1^3}{12} (\partial_x f_1 + (1 + \varepsilon_1 \zeta_1 - \frac{h_1}{2}) \partial_x g_1) + \frac{h_1^4}{24} \partial_x g_1 \right) \right),\]

\[\frac{1}{\varepsilon_2} \int_{-\frac{1}{4}+\beta b(x)}^{\varepsilon_2 \zeta_2} \nabla p_2 = h_2 \zeta_2 + \mu \left( \beta b'(x) h_2 \left( D_2 f_2 + \varepsilon_2 f_2 g_2 + \frac{1}{2} (D_2 g_2 + \varepsilon_2 g_2^2) \zeta_2 - \frac{1}{\delta} + \beta b(x) \right) \right) \partial_x \left( \frac{h_1^2}{2} \left( D_2 f_2 - \frac{1}{\delta} - \beta b(x) \right) D_2 g_2 + \varepsilon_2 f_2 g_2 - \varepsilon_2 \left( \frac{1}{\delta} - \beta b(x) \right) g_2^2 \right) + \frac{h_1^3}{3} (D_2 g_2 + \varepsilon_2 g_2^2) \right) - \mu^2 \omega_2 \varepsilon_2 \partial_x \left( \frac{h_1^3}{12} \beta b'(x) \partial_x^2 \pi_2 + \partial_x \left( \frac{h_1^3}{12} (\partial_x f_2 + (\varepsilon_2 \zeta_2 - \frac{h_2}{2}) \partial_x g_2) + \frac{h_1^4}{24} \partial_x g_2 \right) \right),\]

with operators are given by:

\[D_1 f = \partial_t f + \varepsilon_1 \pi_1 \partial_x f, \quad D_2 f = \partial_t f + \varepsilon_2 \pi_2 \partial_x f.\]
Thus, summing ‘Reynolds tensor’ and pressure contributions, we finally obtained a model
with first order of approximation:
\[
\begin{aligned}
&h_{1t} + \varepsilon_1 (h_{1\tau})_t = 0,
&(h_1\tau)_t + \varepsilon_1 (h_1\tau^2)_x + \frac{\mu_1}{12} \omega_1 (h_1^3)_x + h_1\zeta_1 + \frac{1}{6}\mu h_1
&(3\varepsilon_1 \epsilon_1^2 \omega_1^2 \mathcal{G}(h_1, \zeta_1, \zeta_2) + 2h_1^2 (\varepsilon \tau_{1x} - \tau_{1xx}) + \tau_{1} T_1(\tau_1) + 6\varepsilon_2 \zeta_1 (\zeta_{itt} + \varepsilon_1 \zeta_2 \tau_{1x}) + 3h_1 T_2(\tau_1)) = 0
&h_{2t} + \varepsilon_2 (h_2\tau)_x = 0,
&(h_2\tau)_t + \varepsilon_2 (h_2\tau^2)_x + \frac{\mu_2}{12} \omega_2 (h_2^3)_x + h_2\zeta_2 + \int_{-\frac{1}{2} + \beta b(x)}^{\varepsilon_2 \omega_2} \nabla \left( \frac{\rho}{\varepsilon_2} p_1 |_{\varepsilon_2 \omega_2} \right) + \frac{\varepsilon_2}{6}\mu h_2
&(3\beta \varepsilon_2 \tau_2^2 \mathcal{B}(h_2, b) + \varepsilon_2 \tau_2 R_1(\tau_2) + 6\beta \beta'(\beta'b + h_2) \tau_{2x} + h_2 R_2(\tau_2)) = 0.
\end{aligned}
\]
In italic here we denote following operators:
\[
\begin{aligned}
\mathcal{T}_1(V) &= -6\varepsilon_1 h_{1t} (\varepsilon \zeta_1, \partial_x V + h_1 \partial_{xx} V) + \partial_x V (6\varepsilon_1^3 (\zeta_1)_x^2) + 9\varepsilon_1 \varepsilon_2 h_1 \zeta_{xxxx} - 2\varepsilon_1 h_1^2 \partial_{xx} V + 12\varepsilon_1 \varepsilon_2 \zeta_1 \zeta_{xx} + 6 \varepsilon_2 h_1 \zeta_{xxxx},
\mathcal{T}_2(V) &= 2\varepsilon_1 (\partial_x V)_x^2 \zeta_1 + \varepsilon_2 \zeta_{xx} (\partial_t V - 2h_1 \partial_{xx} V + 2\varepsilon_2 \partial_x V \zeta_{xx} + \varepsilon_2 \zeta_{xxxx},
\mathcal{R}_1(V) &= (6\beta^2 \beta' + 9h_2 \beta') \partial_x V - 6h_2 \partial_{xx} V + h_2 (2\varepsilon_2 \partial_x V \partial_{xx} V - 2\partial_{xxxx} V),
\mathcal{R}_2(V) &= 6 \varepsilon_2 (\beta'b + h_2) (\partial_x V)^2 + 3 \beta b' \partial_t V - 6h_2 \partial_{xx} V + h_2 (2\varepsilon_2 \partial_x V \partial_{xx} V - 2\partial_{xxxx} V),
\mathcal{G}(h_1, \zeta_1, \zeta_2) &= 2\varepsilon_1 \zeta_1 \zeta_{xx} + h_1 \zeta_{xxxx},
\mathcal{B}(h_2, b) &= (2\beta'b + h_2 \beta') \beta' + h_2 \beta').
\end{aligned}
\]
The boundary term is defined as:
\[
\int_{-\frac{1}{2} + \beta b(x)}^{\varepsilon_2 \omega_2} \nabla \left( \frac{\rho}{\varepsilon_2} p_1 \right) = \frac{\varepsilon_1}{\varepsilon_2} \partial_x \mu \left( (\mathcal{D}(A_1) + \varepsilon_1 A_1 B_1) h_1 + (\mathcal{D}(B_1) + \varepsilon_1 B_1^3 h_1 (1 + \varepsilon_1 \zeta_1 + \varepsilon_2 \zeta_2) \right)
+ \frac{\rho}{\varepsilon_2} h_{1x} + \frac{\varepsilon_1}{\varepsilon_2} \mu \partial_x (\frac{h_1^3}{12} \omega_1 B_{1x})
\]

The model of the second order can be written directly from the contribution calculated above
with additional terms in the boundary term.

4. 1d Equations with general vorticity
In this section we derive the generalized Green-Naghdi equations in the presence of general
vorticities in each layer. Pressure fields contributions can be calculated explicitly in the first
order but it requires the closure in the second order. In contrast, ‘Reynolds tensor’ contribution
can not be computed even in the first order and we should find an equation for component
of the tensor. This implies that we limit ourselves to first order model to simplify computations.
Now vorticities \( \omega_i = \omega_i(t, x, z) \) depend on space and time.

4.1. Computation of the ‘Reynolds tensor’ contribution
In the first order the ‘Reynolds tensor’ contribution can be written as:
\[
\varepsilon_1 \mu \partial_x \int_{\varepsilon_2 \omega_2}^{1+\varepsilon_1 \zeta_1} |u_1|^2 dz = \varepsilon_1 \mu \partial_x \int_{\varepsilon_2 \omega_2}^{1+\varepsilon_1 \zeta_1} |u_{1sh}|^2 dz + O(\mu),
\]
The presented models are derived in the one dimensional case for constant vorticities in each layer with the first and second order of approximation and for general vorticities case with first order.

The natural perspective of this work is to derive the second order model for the general vorticities case in order to correct non-hydrostatic part with additional terms. The other direction of the research is to perform the numerical simulation for obtained models.
References
[1] Barros R, Gavrilyuk S L and Teshukov V M 2007 *Stud. Appl. Math.* 119(3) 191-211
[2] Duchene V 2010 *SIAM J. Math. Anal.* 42, 5 2229-60
[3] Castro A and Lannes D 2014 *J. Fluid Mech.* 759 642–75
[4] Richard G L and Gavrilyuk S L 2015 *J. Fluid Mech.* 773 49-74