Generalization of Hamilton-Jacobi method and its consequences in classical, relativistic, and quantum mechanics

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May 18, 2018

Abstract

The Hamilton-Jacobi method is generalized in classical and relativistic mechanics. The implications in quantum mechanics are discussed in the case of Klein-Gordon equation. We find that the wave functions of Klein-Gordon theory can be soundly interpreted as describing the motion of an ensemble of particles that move under the action of the electromagnetic field alone, without quantum potentials, hidden uninterpreted variables, or zero point fields.

pacs 03.20.+i, 03.20.+p, 03.65.-w

1 Introduction

In previous papers ([1], [2]) we have considered the possibility of reinterpreting Klein-Gordon and Dirac’s equations, as describing ensembles of particles moving under the action of the electromagnetic field, where the number of particles is not locally conserved. In order to do that for the Klein-Gordon equation, on the basis of some general electrodynamic considerations, we proved that it was sound to assume that the field of kinetic four-momentum

\[ p_\mu = \left( \frac{mc}{\sqrt{1 - v_\mu^2}}, \frac{m\vec{v}}{\sqrt{1 - v_\mu^2}} \right) \]

is given by

\[ p_\mu = -\partial_\mu S - \partial_\mu \Phi - \frac{q}{c}A_\mu; \quad (1) \]

where \( S \) is the phase of the wave function; \( A_\mu \) is the electrodynamic four-potential; and \( \Phi \) is a function of space-time coordinates, whose nature we left
completely undetermined. To deal with Dirac’s equation we used a similar approach.

In this paper we determine the nature of the function $\Phi$. Actually, we prove that the principle of relativity requires that we consider a general four-vector $\omega_{\mu}$, instead of the gradient of a potential. For this we start with an analysis of Helmholtz’s theorem on the representation of vectors fields and its extensions to $n$ dimensional spaces, in particular to space time.$^3$ In the first section we derive Helmholtz theorem:

$$\vec{f} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{x},$$

(2)

following a variational approach. This permits us to prove that if the potentials are chosen in a manner that $(\vec{\nabla}\phi - \vec{f}) \cdot \hat{n} = 0$ at the boundary of the region under consideration, then $\vec{\nabla}\phi$ is the best approximation to $\vec{f}$, in the sense of the quadratic norm—which we think is an original result.

Further we apply those conclusions to perform an analysis of the physical meaning of Hamilton-Jacobi theory. Taking it out of the mathematical realm of canonical transformations, allows us to find its necessary generalizations in classical and relativistic mechanics, laying thus a foundation for a complete explanation of the field $\omega_{\mu}$. In the second section we work out a complete example of this extension of Hamilton-Jacobi theory in a very simple classical situation, just to show how it works, as well as to prove that there are very simple and common dynamical scenarios that are being disregarded by the present theory and, therefore, that our extension is not only a theoretical speculation, but a necessary step to extend the scope and strength of a fundamental part of classical mechanics.

We consider a flux that corresponds to an infinite ensemble of particles with the initial velocities:

$$\vec{v}(\vec{r}, 0) = \vec{\omega} \times \vec{r},$$

(3)

where $\vec{\omega}$ is a constant vector. We prove that this flux is invertible, in the sense that we can obtain the initial position of the particle that passes through the point $\vec{x}$ at time $t$. However this flux cannot be derived from a potential in the form

$$\vec{v} = \frac{\nabla S}{m},$$

(4)

because

$$\vec{\nabla} \times \vec{v} \neq 0.$$

The generalization of Hamilton-Jacobi theory is even more necessary in the case of special relativity because—as we will prove—any field of four-velocities that describes the motion of an ensemble of particles has a vorticity, at least it describes a flux of free particles. Furthermore: all we have is to consider a relativistic ensemble of free particles, where the initial momentum is defined as

$$\vec{p}(\vec{r}, 0) = \vec{\omega} \times \vec{r},$$

(5)
for a particular observer, to have a system whose evolution cannot be described
by the Hamilton-Jacobi theory as it is presented in [4, pp. 24-29]. In some way
we follow Bohr when we state that if there is a reformulation of Hamilton-Jacobi
theory that widens its range of applicability it should be incorporated into the
body of mechanical knowledge, no matter if it is or not compatible with the
supremacy of Hamilton’s principal function.

In my opinion, there could be no other way to deem a logically consist-
ent mathematical formalism as inadequate than by demonstrating
the departure of its consequences from experience or by proving that
its predictions did not exhaust the possibilities of observation [5, pp.
200-241].

We consider this kind of fluxes, because they appear in the classical limit of
quantum mechanics:

In the classical approximation, $\psi$ describes a fluid of non-interacting
classical particles of mass $m$, and subject to the potential $V(\vec{r})$: the
density and current density of this fluid at each point of space are
respectively equal to the probability density $P$ and the probability
current $j$ of the quantum particle at that point.

Indeed, since the continuity equation of this fluid is satisfied [eq.
(VI.19)], it suffices to show that the velocity field

$$\vec{v} = \frac{j}{P} = \frac{\nabla S}{m}$$

of this fluid, actually follows the law of motion of the classical fluid
in question. [6, p. 223].

In this way, this paper demonstrates that there are classical scenarios, as
the one described by (5) that cannot be reproduced from quantum mechanics
in the classical limit.

In the last section we use our generalization of the Hamilton-Jacobi theory
to prove—as a mathematical fact, because our proof is logical, not ontological—
that the wave functions of Klein-Gordon theory can be soundly interpreted as
describing the motion of ensembles of particles under the action of the electro-
magnetic field, alone, without any quantum potentials, uninterpreted hidden
variables, or zero point fields. What Bohm called the quantum potential corre-
sponds in part to kinetic energy and in part to reaction forces associated to
local creation/annihilation processes. In this form, we address the only objec-
tion that Einstein raised against the de Broglie-Bohm interpretation of quantum
mechanics [7, 8, pp. 33-40]: that, according to Bohm’s theory, in the stationary
states of particles inside a box with rigid walls, the particles had to be at rest.

The reader should realize that to say that “the wave functions of Klein-
Gordon theory can be soundly interpreted as describing ensembles of particles
under the action of the electromagnetic field...” is not the same that to say “the
wave functions of Klein-Gordon theory describe ensembles of particles under the action of the electromagnetic field...” Therefore, we are not making any ontological assertions in this paper, which keeps us out of the, otherwise fascinating, realm of philosophical speculation.

We do not say, like Einstein-Podolsky-Rosen, that quantum mechanics is not a complete theory of motion[9]. Instead, we say that the set of terms that specify the wave function in Madelung’s representation, \( \{ \rho, S \} \), can be complemented introducing a four vector \( \omega_i \), in such manner that the same phenomena can be explained using the classical concept of particles that move along well defined trajectories. In addition, the logical necessity of introducing this four-vector will be mathematically proved beyond any doubt on the basis of Helmholtz theorem.

This four-vector appears in the equations as the potential of a kind of electromagnetic field, that does not produce any Lorentz force on the particles of the ensemble under examination. The electric force exactly cancels the magnetic force. However there is not any a priori condition on the divergence of the corresponding Faraday tensor. Therefore this field can have an associated virtual electric current.

2 The Helmholtz’s Theorem in Three Dimensions

To find the best approximation of a vector field \( \vec{f} \) by means of the gradient of a potential in a region of space we can follow a variational approach:

**Determine the extreme of the functional:**

\[
F = \frac{1}{2} \int_{\Omega} (\vec{\nabla} \phi - \vec{f})^2 dV. \tag{6}
\]

The first variation is:

\[
\delta F = \int_{\Omega} \delta \vec{\nabla} \phi \cdot (\vec{\nabla} \phi - \vec{f}) dV = \]

\[
\int_{\Omega} \vec{\nabla} \cdot [\delta \phi (\vec{\nabla} \phi - \vec{f})] dV - \int_{\Omega} \delta \phi \vec{\nabla} \cdot (\vec{\nabla} \phi - \vec{f}) dV. \tag{7}
\]

Making use of Gauss theorem, the first of the last integrals is replaced by an integral on \( \Omega \)’s boundary(\( \Gamma(\Omega) \)). Therefore,

\[
\delta F = \int_{\Gamma(\Omega)} \delta \phi (\vec{\nabla} \phi - \vec{f}) \cdot d\vec{S} - \int_{\Omega} \delta \phi \vec{\nabla} \cdot (\vec{\nabla} \phi - \vec{f}) dV. \tag{8}
\]

We do not fix the values of \( \phi \) on \( \Gamma(\Omega) \), so that, from the condition \( \delta F = 0 \), we can get a differential equation and a set of boundary conditions:

\[
\vec{\nabla}^2 \phi = \vec{\nabla} \cdot \vec{f}, \tag{9}
\]
\[ \forall \vec{x} \in \Gamma(\Omega) : (\nabla \phi - \vec{f}) \cdot \hat{n}(x) = 0. \quad (10) \]

(Where \( \hat{n}(\vec{x}) \) is the field of unitary normals on \( \Gamma(\Omega) \).)

From the theory of harmonic functions we know the last problem has a unique solution which we will not discuss further. The field \( \vec{f} \) is thus expressed as the sum of two fields:

\[ \vec{f} = \nabla \phi + \vec{t}, \quad (11) \]

where \( \vec{t} \) is a solenoidal field, such that

\[ \nabla \times \vec{f} = \nabla \times \vec{t}, \quad (12) \]

and

\[ \frac{1}{2} \int_{\Omega} \vec{t}^2 dV \]

has the minimum value compatible with equation (12).

Therefore, the field \( \vec{t} \) is a solution for another variational problem:

\[ \delta \int_{\Omega} \left[ \frac{1}{2} \vec{t}^2 + \vec{\lambda} \cdot \nabla \times (\vec{f} - \vec{t}) \right] dV, \quad (13) \]

where the components of \( \vec{\lambda} \) are Lagrange’s multipliers.

The variation with respect to \( \vec{\lambda} \) leads to the condition \( \nabla \times \vec{t} = \nabla \times \vec{f} \), which we already know.

The variation with respect to \( \vec{t} \) leads to:

\[ \int_{\Omega} \left[ \vec{t} \cdot \delta \vec{t} - \vec{\lambda} \cdot \nabla \times \delta \vec{t} \right] dV = 0. \quad (14) \]

Using the identity

\[ \nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot \nabla \times \vec{a} - \vec{a} \cdot \nabla \times \vec{b} \]

to obtain the substitution

\[ -\vec{\lambda} \cdot \nabla \times \delta \vec{t} = \nabla \cdot (\vec{\lambda} \times \delta \vec{t}) - \delta \vec{t} \cdot \nabla \times \vec{\lambda}, \]

equation (14) can be written as:

\[ \int_{\Omega} \left[ (\vec{t} - \nabla \times \vec{\lambda}) \cdot \delta \vec{t} + \nabla \cdot (\vec{\lambda} \times \delta \vec{t}) \right] dV = 0. \quad (15) \]

From this we get the differential equation:

\[ \vec{t} = \nabla \times \vec{\lambda} \quad (16) \]

From equation (12) we get:

\[ \nabla \times (\nabla \times \vec{\lambda}) = \nabla \times \vec{f}. \quad (17) \]
We can choose $\vec{\lambda}$ so that $\vec{\nabla} \cdot \vec{\lambda} = 0$. For this all we have to do is the substitution:

$$\vec{\lambda} \rightarrow \vec{\lambda} + \vec{\nabla} \psi,$$

where $\psi$ is a particular solution of a Poisson equation

$$\vec{\nabla}^2 \psi = -\vec{\nabla} \cdot \vec{\lambda}.$$

This transformation does not change the fundamental relation (16).

Equation (17) is transformed into:

$$\vec{\nabla}^2 \vec{\lambda} = -\vec{\nabla} \times \vec{f}.$$  \hfill (18)

with the extra and boundary conditions:

$$\vec{\nabla} \cdot \vec{\lambda} = 0 \quad (19)$$

$$\forall \vec{x} \in \Gamma(\Omega)(\vec{\nabla} \times \vec{\lambda} - \vec{f}) \times \hat{n} = 0 \quad (20)$$

Those relations univocally determine the field $\vec{\lambda}$.

We have thus an addition to Helmholtz’ theorem in the form:

Every vector field $\vec{f}$ with continuous partial derivatives inside a region $\Omega$ can be written in the form:

$$\vec{f} = \vec{\nabla} \phi + \vec{\nabla} \times \vec{\lambda},$$

where $\vec{\lambda}$ is solenoidal and $\phi$ and $\vec{\lambda}$ are solutions of the Poisson equations:

$$\vec{\nabla}^2 \phi = \vec{\nabla} \cdot \vec{f},$$

and

$$\vec{\nabla}^2 \vec{\lambda} = -\vec{\nabla} \times \vec{f}.$$  \hfill (18)

If $\phi$ and $\vec{\lambda}$ are chosen to meet the boundary condition

$$\forall \vec{x} \in \Gamma(\Omega) : (\vec{\nabla} \phi - \vec{f}) \cdot \hat{n}(\vec{x}) = 0,$$

then $\vec{\nabla} \phi$ is the best approximation of the field $\vec{f}$ as the gradient of a scalar function.

Consider now a vector field $(f_1, \cdots, f_n)$ in a space of $n$ dimensions. As before, we find the solutions of a variational problem $\delta F = 0$, where

$$F = \frac{1}{2} \int_{\Omega} (f_i - \frac{\partial \phi}{\partial x_i})(f_i - \frac{\partial \phi}{\partial x_i})dV,$$  \hfill (21)

where $\Omega$ is a region of a $n$-dimensional space limited by a hyper-surface $\Gamma(\Omega)$. (We are using the summation convention.)
The condition $\delta F = 0$ leads to a set of corresponding equations, analogous to (9, 10). As to the vector $t_i = f_i - \frac{\partial \phi}{\partial x_i}$, we know it is solenoidal, and that it is a solution of the variational problem

$$\delta \int_\Omega \frac{1}{2} t_i t_i + \Lambda_{ij} \left( \frac{\partial}{\partial x_j} (t_i - f_i) - \frac{\partial}{\partial x_i} (t_j - f_j) \right) dV = 0,$$

(22)

where $\Lambda_{ij}$ is an antisymmetric set of Lagrange’s multipliers.

The extreme condition is easily found to be:

$$t_i = \frac{\partial \Lambda_{ij}}{\partial x_j},$$

(23)

Therefore, $t_i$ is the divergence of an antisymmetric second order (cartesian) tensor.

Writing

$$\Phi_{ij} = \Lambda_{ij} + \delta_{ij} \phi,$$

(24)

we can see that

$$f_i = \frac{\partial \Phi_{ij}}{\partial x_j},$$

(25)

meaning that any vector field can be written as the divergence of a second order (cartesian) tensor.

### 3 Classical Mechanics

Let $p_i(q_1, \cdots, q_n, t)$ be a vector field, defined in the space-time of a mechanical system whose evolution is determined by a Hamilton function

$$H(q_1, \cdots, q_n, p_1, \cdots, p_n, t),$$

so that:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$  

(26)

(Given that functions defined in Cartan’s *espace des états* [12, 174] will be projected into functions defined in space-time, to avoid confusions, we will use symbols of the kind

$$G_{\alpha\beta\cdots}$$

(27)

to represent their derivatives. For example:

$$\frac{\partial G}{\partial q_i} = G_{q_i}.$$  

To represent the derivatives with respect to the space-time coordinates we will use the usual notation $\frac{\partial}{\partial q_i}, \frac{\partial}{\partial t}$.)
We suppose the $p_i$ are the momenta of an infinite ensemble of particles, of the kind used in Euler’s hydro-kinematics. The particle that occupies the position $q_i$ at time $t$, gets an increment of action in the interval $dt$, which is given by:

$$dS = \sum p_i dq_i - H dt.$$  

(28)

The conditions under which this is relation is integrable are

$$\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} = 0,$$  

(29)

and

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i} = 0.$$  

(30)

The derivative of $p_i$ along the corresponding trajectory in phase space is:

$$\dot{p}_i = -H_{q_i} + \frac{\partial p_i}{\partial t} + \sum \dot{q}_j \frac{\partial p_i}{\partial q_j}.$$  

(31)

From this we get the equation:

$$\frac{\partial p_i}{\partial t} = -H_{q_i} - \sum \dot{q}_j \frac{\partial p_i}{\partial q_j}.$$  

(32)

Also,

$$\frac{\partial H}{\partial q_i} = H_{q_i} + \sum p_j \frac{\partial p_i}{\partial q_i} = H_{q_i} + \sum \dot{q}_j \frac{\partial p_i}{\partial q_i}.$$  

(33)

Therefore:

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i} = \sum \dot{q}_j \left( \frac{\partial p_j}{\partial q_i} - \frac{\partial p_i}{\partial q_j} \right).$$

Thus, we get the important conclusion that the vorticity of $(p_1, \ldots, p_n)$ is zero in the configuration space if and only if the vorticity of $(-p_1, \ldots, -p_n, H)$ is zero in space-time.

According to equation (32)

$$\frac{\partial^2 p_i}{\partial t \partial q_j} = \frac{\partial H_{q_j}}{\partial q_i} - \sum \frac{\partial H_{p_k}}{\partial q_j} \frac{\partial p_i}{\partial q_k} = \sum \frac{\partial H_{p_k}}{\partial q_j} \frac{\partial p_i}{\partial q_k} - \sum \frac{\partial^2 p_i}{\partial q_k \partial q_j}.$$  

(34)

From this we can see that:

$$\frac{\partial}{\partial t} \left( \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} \right) + \sum \dot{q}_k \left( \frac{\partial p_i}{\partial q_k} - \frac{\partial p_j}{\partial q_k} \right) =$$  

$$\frac{\partial H_{q_j}}{\partial q_i} - \frac{\partial H_{q_i}}{\partial q_j} + \sum \frac{\partial H_{p_k}}{\partial q_i} \frac{\partial p_j}{\partial q_k} - \frac{\partial H_{p_k}}{\partial q_j} \frac{\partial p_i}{\partial q_k}.$$  

(35)

Considering the relations

$$\frac{\partial H_{q_i}}{\partial q_j} - \frac{\partial H_{q_j}}{\partial q_i} = \sum H_{q_i p_k} \frac{\partial p_k}{\partial q_i} - H_{q_j p_k} \frac{\partial p_k}{\partial q_j},$$  

(36)
and
\[ \sum \frac{\partial H_{pk}}{\partial q_i} \frac{\partial p_j}{\partial q_k} - \frac{\partial H_{pk}}{\partial q_j} \frac{\partial p_i}{\partial q_k} = (37) \]
\[ \sum \left( H_{pq_k} \frac{\partial p_j}{\partial q_k} - H_{pq_k} \frac{\partial p_i}{\partial q_k} \right) + \sum \sum H_{pq_{km}} \left( \frac{\partial p_m}{\partial q_i} \frac{\partial p_j}{\partial q_k} - \frac{\partial p_m}{\partial q_j} \frac{\partial p_i}{\partial q_k} \right), \]
we can write equation (35) in the form
\[ \frac{\partial}{\partial t} \left( \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} \right) + \sum \frac{\dot{q}_k}{\partial q_k} \left( \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} \right) = (38) \]
\[ \sum \sum H_{pq_{km}} \left( \frac{\partial p_m}{\partial q_i} \frac{\partial p_j}{\partial q_k} - \frac{\partial p_m}{\partial q_j} \frac{\partial p_i}{\partial q_k} \right). \]

Now we assume that the mass matrix is diagonal, so that
\[ \sum \sum H_{pq_{km}} \left( \frac{\partial p_m}{\partial q_i} \frac{\partial p_j}{\partial q_k} - \frac{\partial p_m}{\partial q_j} \frac{\partial p_i}{\partial q_k} \right) = \sum m_k \left( \frac{\partial p_k}{\partial q_k} \frac{\partial p_j}{\partial q_k} - \frac{\partial p_k}{\partial q_j} \frac{\partial p_j}{\partial q_k} \right) = (39) \]
\[ \sum m_k \left( \frac{\partial p_k}{\partial q_i} \frac{\partial p_j}{\partial q_k} - \frac{\partial p_k}{\partial q_j} \frac{\partial p_j}{\partial q_k} \right) = \sum m_k \left( \frac{\partial p_k}{\partial q_i} \frac{\partial p_j}{\partial q_k} + \frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial q_k} - \frac{\partial p_k}{\partial q_j} \frac{\partial p_j}{\partial q_k} \right) = \sum m_k \left( \frac{\partial p_k}{\partial q_i} - \frac{\partial p_i}{\partial q_k} \right), \]
\[ + \left( \frac{\partial p_j}{\partial q_k} - \frac{\partial p_k}{\partial q_j} \right) \frac{\partial p_i}{\partial q_k} \right). \]

Equation (38) gives us the material derivative of the vorticity of the field \( p_i \). From equation (39) we can see that, under this assumption—that the matrix of masses is diagonal—if the vorticity of \( (p_1, \ldots, p_n) \) is zero at \( t = t_0 \), it will be zero at any other time. Therefore, in those cases, there is a function \( \Phi_i \), defined in space-time such that:
\[ p_i = \frac{\partial \Phi}{\partial q_i}, \quad H = -\frac{\partial \Phi}{\partial t}. \]

Obviously, \( \Phi \) is a solution of the Hamilton Jacobi equation:
\[ H(q_1, \ldots, q_n, \frac{\partial \Phi}{\partial q_1}, \ldots, \frac{\partial \Phi}{\partial q_n}, t) + \frac{\partial \Phi}{\partial t} = 0. \] (41)

Here we are assuming that the trajectories of the particles in the ensemble do not cross each other, which cannot be granted from the sole condition on the vorticity. Therefore we can expect that the solution of (41) will have singularities, depending on the initial conditions.
But suppose that \((p_1, \cdots, p_n)\) is not a potential field. The vorticity of \((-p_1, \cdots, -p_n, H)\) will not be zero and, according to our previous conclusions, there is at least one field \((-A_1, \cdots, -A_n, \Theta)\) such that:

\[
\frac{\partial \Theta}{\partial t} - \sum \frac{\partial A_i}{\partial q_i} = 0, \tag{42}
\]

and \((-p_1-A_1, \cdots, -p_n-A_n, H+\Theta)\) is the best approximation of \((-p_1, \cdots, -p_n, H)\) as a potential field. Consequently, there is at least one function \(\Phi\) such that

\[
p_i = \frac{\partial \Phi}{\partial q_i} - A_i, H = -\frac{\partial \Phi}{\partial t} - \Theta. \tag{43}
\]

The function \(\Phi\) is a solution of the differential equation

\[
H(q_1, \cdots, q_n, \frac{\partial \Phi}{\partial q_1} - A_1, \cdots, \frac{\partial \Phi}{\partial q_n} - A_n, t) + \Theta + \frac{\partial \Phi}{\partial t} = 0, \tag{44}
\]

which is the equation we had obtained, had we started with a potential field \((p_1, \cdots, p_n)\) and another Hamilton’s function:

\[
H'(q_1, \cdots, q_n, p_1, \cdots, p_n, t) = H(q_1, \cdots, q_n, p_1 - A_1, \cdots, p_n - A_n, t) + \Theta \tag{45}
\]

It is clear that equation (44) is not enough to determine the fields \(\Phi\) and \(\vec{A}\), but this difficulty is easily surmounted. To show this in a way that is free of mathematical complexities, we will consider the case of a single particle, where the Hamilton’s function of the original system is:

\[
H(\vec{p}, \vec{q}) = \frac{\vec{p}^2}{2m} + V. \tag{46}
\]

The new Hamilton’s function is

\[
H'(\vec{q}, \vec{p}) = \frac{(\vec{p} - \vec{A})^2}{2m} + V + \Theta, \tag{47}
\]

which is formally analogous to the Hamilton’s function of a particle in an electromagnetic field. The corresponding Hamilton-Jacobi equation is:

\[
H'(\vec{q}, \vec{\nabla} \Phi) + \frac{\partial \Phi}{\partial t} = 0. \tag{48}
\]

From (47) we see that the function \(\Theta\) and the vector \(\vec{A}\) have to be chosen so that the corresponding Lorentz force is equal to zero for the given field:

\[
-\vec{\nabla} \Theta - \frac{\partial \vec{A}}{\partial t} + \frac{\partial H'}{\partial \vec{p}}(\vec{q}, \vec{\nabla} \Phi) \times (\vec{\nabla} \times \vec{A}) = 0. \tag{49}
\]

Equations (45) and (49) constitute the generalization of Hamilton-Jacobi theory to the consideration of fields with vorticity.
To make an example, we have considered an ensemble of free particles that satisfy the initial conditions:

$$\vec{v}(\vec{r}_0, t_0) = \vec{\omega} \times \vec{r}_0,$$  \hspace{1cm} (50)

where $\vec{\omega} = \hat{k}$ is a constant vector in the direction of the $z$ axis. The position at time $t$ of the particle that occupies position $\vec{r}_0$ at time $t_0$ is

$$\vec{r}(t) = \vec{r}_0 + (\vec{\omega} \times r_0)(t - t_0).$$  \hspace{1cm} (51)

Or, in open form:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 & -\omega(t - t_0) & 0 \\ \omega(t - t_0) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$  \hspace{1cm} (52)

The determinant of this system is greater than zero. Therefore, the transformation is invertible. Actually, we have this:

$$x_0(x, y, z, t) = \frac{x + \omega(t - t_0)y}{1 + \omega^2(t - t_0)^2},$$

$$y_0(x, y, z, t) = \frac{y - \omega(t - t_0)x}{1 + \omega^2(t - t_0)^2},$$

$$z_0(x, y, z, t) = z.$$  \hspace{1cm} (53)

From this and (51) we can get the field of momenta at time $t$:

$$p_x = -m\omega y_0 = \frac{-m\omega y + m\omega^2(t - t_0)x}{1 + \omega^2(t - t_0)^2},$$ \hspace{1cm} (54)

$$p_y = m\omega x_0 = \frac{m\omega x + m\omega^2(t - t_0)y}{1 + \omega^2(t - t_0)^2},$$

$$p_z = 0,$$

and the field of kinetic energy

$$K = \frac{1}{2} \frac{m\omega^2(x^2 + y^2)}{1 + \omega^2(t - t_0)^2}. \hspace{1cm} (55)$$

The potential and solenoidal parts of the field of momenta are clearly separated.

$$\vec{p} = \vec{\nabla} \Phi - \vec{A},$$  \hspace{1cm} (56)

where

$$\Phi = \frac{1}{2} \frac{m\omega^2(t - t_0)(x^2 + y^2)}{1 + \omega^2(t - t_0)^2},$$

$$- A_x = \frac{-m\omega y}{1 + \omega^2(t - t_0)^2}.$$  \hspace{1cm} (57)
\[-A_y = \frac{m\omega x}{1 + \omega^2(t - t_0)^2},\]

and

\[-A_z = 0.\]

Also:

\[\Theta = -K - \frac{\partial \Phi}{\partial t} = -\frac{m\omega^2(x^2 + y^2)}{1 + \omega^2(t - t_0)^2}.\]

At this moment we can see that the alternative Hamilton's function for this problem is:

\[H(\vec{r}, \vec{p}, t) = \frac{1}{2m} \left[ \left( p_x - \frac{m\omega y}{1 + \omega^2(t - t_0)^2} \right)^2 + \left( p_y + \frac{m\omega x}{1 + \omega^2(t - t_0)^2} \right)^2 + p_z^2 \right] \]

\[-\frac{m\omega^2(x^2 + y^2)}{1 + \omega^2(t - t_0)^2}.\]

Because of the way we have constructed this function, it is not difficult to see that (57) is a particular solution of the Hamilton-Jacobi Equation

\[\frac{1}{2m} \left[ \left( \frac{\partial \Phi}{\partial x} - \frac{m\omega y}{1 + \omega^2(t - t_0)^2} \right)^2 + \left( \frac{\partial \Phi}{\partial y} + \frac{m\omega x}{1 + \omega^2(t - t_0)^2} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] \]

\[-\frac{m\omega^2(x^2 + y^2)}{1 + \omega^2(t - t_0)^2} + \frac{\partial \Phi}{\partial t} = 0.\]

That (49) is also satisfied can be easily proved from the equalities:

\[-\vec{\nabla} \Theta - \frac{\partial A}{\partial t} = \frac{2m\omega^2(x\hat{i} + y\hat{j})}{1 + \omega^2(t - t_0)^2} - \frac{2m\omega^3(t - t_0)(y\hat{i} - x\hat{j})}{(1 + \omega^2(t - t_0)^2)^2},\]

\[\vec{v} = \frac{m\omega^2(t - t_0)x - \omega y\hat{i}}{1 + \omega^2(t - t_0)^2} + \frac{\omega x + m\omega^2(t - t_0)^2 y\hat{j}}{1 + \omega^2(t - t_0)^2},\]

and

\[\vec{\nabla} \times \vec{A} = \frac{-2m\omega\hat{k}}{1 + \omega^2(t - t_0)^2}.\]

Passing to special relativity, in previous papers [1] and [2], we have observed that in the case of a field of four-velocities that represents the motion of an infinite ensemble of particles, the derivative of the four-velocity along the corresponding world-lines

\[\frac{du_i}{ds} = u^j \frac{\partial u_i}{\partial x^j} \]

is determined by its vorticity. This is so because, from the condition \(u^iu_j = 1\), it follows that

\[u^j \frac{\partial u_i}{\partial x^j} = 0.\]
Combining this and (65) we get
\[ \frac{du_i}{ds} = u^j \left( \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x_i} \right). \] (66)

Then—we see now—the vorticity of a field of four-velocities cannot be zero at least it describes an ensemble of free particles. If the particles move under the action of an electromagnetic field we have:
\[ u^j \left( \frac{\partial P_j}{\partial x^i} - \frac{\partial P_i}{\partial x^j} \right) = 0, \] (67)
where
\[ P_i = mcu_i - \frac{q}{c} A_i, \] (68)
m is the rest mass of the particles, and \( q \) is the corresponding electric charge.

We can write the vector \( P_i \) in the form
\[ P_i = -\frac{\partial S}{\partial x^i} + \omega_i, \] (69)
where, because of (67):
\[ u^j \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) = 0, \] (70)
and
\[ \frac{\partial \omega_i}{\partial x_i} = 0. \] (71)

From (68), (69), (70), (71), we get a non-linear system of differential equations:
\[ \left( -\frac{\partial S}{\partial x^k} + \frac{q}{c} A^k + \omega^k \right) \left( -\frac{\partial S}{\partial x^k} + \frac{q}{c} A_k + \omega_k \right) = m^2 c^2, \] (72)
\[ \left( -\frac{\partial S}{\partial x^k} + \frac{q}{c} A^k + \omega^k \right) \left( \frac{\partial \omega^i}{\partial x^k} - \frac{\partial \omega_k}{\partial x^i} \right) = 0. \] (73)
\[ \frac{\partial \omega_i}{\partial x_i} = 0. \] (74)

These equations encompass the most reasonable generalization of Hamilton-Jacobi theory to special relativity since, as we said before, a potential field of four-velocities represents necessarily a field of free particles. Vorticity plays a special role in relativistic Hamilton-Jacobi theory, and there is not a physical reason to believe that the only real fields of four-velocities are those for which there is a scalar function \( \phi \) such that [11, p. 488-509]
\[ mcu^u - \frac{q}{c} A^u = \frac{\partial \phi}{\partial x^u}. \] (75)

Quite the contrary, this reformulation of Hamilton-Jacobi theory allows us to prove that it is possible to interpret the wave functions of Klein-Gordon theory as describing the motion of an ensemble of particles under the action of the electromagnetic field, alone, without quantum potentials or uninterpreted hidden variables, where the number of particles is not locally conserved.
4 The Klein-Gordon Field

From the Klein-Gordon equation

\[-\hbar^2 \frac{\partial^2}{\partial x_\mu \partial x^\mu} \Psi - \frac{2i\hbar}{c} A^\mu \frac{\partial \Psi}{\partial x^\mu} + q^2 A^\mu A_\mu \Psi = m^2 c^2 \Psi,\]  
(76)

we can easily show that

\[\frac{\partial}{\partial x_\mu} \left( i\hbar \left( \Psi^* \frac{\partial \Psi}{\partial x^\mu} - \Psi \frac{\partial \Psi^*}{\partial x^\mu} \right) - \frac{q}{c} A_\mu \Psi^* \Psi \right) = 0.\]  
(77)

From this, using the Madelung substitution

\[\Psi = \sqrt{\rho e^{(i/\hbar) S}},\]  
(78)

we prove that

\[\frac{\partial}{\partial x_\mu} \left( \rho \left( - \frac{\partial S}{\partial x^\mu} - \frac{q}{c} A_\mu \right) \right) = 0.\]  
(79)

This suggest that \( \rho \) could be interpreted as a density of particles, in the system of reference where they are at rest and that

\[v_\mu = -\frac{1}{mc} \frac{\partial S}{\partial x^\mu} - \frac{q}{mc^2} A_\mu,\]  
(80)

which is usually rejected on the grounds that \( v_\mu \) is not unitary time-like by definition. From the previous section of this paper we see now that we can complete the representation if we suppose that

\[mcv_\mu = -\frac{\partial S}{\partial x^\mu} - \frac{q}{c} A_\mu + \omega_\mu,\]  
(81)

where

\[\left( - \frac{\partial S}{\partial x_\mu} - \frac{q}{c} A^\mu + \omega_\mu \right) \left( \frac{\partial \omega_\mu}{\partial x^\mu} - \frac{\partial \omega_\mu}{\partial x'^\mu} \right) = 0,\]  
(82)

and

\[\left( - \frac{\partial S}{\partial x^\mu} - \frac{q}{c} A_\mu + \omega_\mu \right) \left( - \frac{\partial S}{\partial x_\mu} - \frac{q}{c} A^\mu + \omega_\mu \right) = m^2 c^2.\]  
(83)

The continuity equation (79) is transformed into:

\[\frac{\partial j_\mu}{\partial x^\mu} = \frac{\partial \rho \omega_\mu}{\partial x^\mu},\]  
(84)

that shows that the vector \( \omega_\mu \) is linked to local production/annihilation processes. This is not too strange if we consider that, after all, this four-vector represents a kind of electromagnetic field. Equation (82) is the condition it does not produce a Lorentz’ force. However, its very existence, implies the appearance of the corresponding conserved current.
From equations (76) and (81) we can show that

$$-2mc\nu\omega_{\mu} + \omega^\mu\omega_{\mu} = \hbar^2 \frac{\partial^2}{\sqrt{\rho} \partial x^\mu x_\mu} \sqrt{\rho}$$  \hspace{1cm} (85)$$

Equations (81) to (85) make up a complete description of an ensemble of particles that move under the action of the electromagnetic field, in such way that the number of particles is not locally conserved. Equations (84) and (85) are the only ones that include the density of particles and Planck’s constant $\hbar$, which is then interpreted as an empirical parameter that determines the local rate of creation/annihilation of matter. The reinterpretation of Dirac’s field is similar to the one we have exposed in [2].

In the low speed limit, we replace the four-vector $\omega_i$ by its best approximation in the sense we studied in the second section. This approximation does not meet the requirements of Lorentz invariance, but Lorentz invariance is not required in the low speed limit. Then we can follow the same line of thought we followed in [1] and [2] to recover Schrödinger’s and Pauli’s equations.

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