HIGHER ORDER NLS WITH ANISOTROPIC DISPERSION
AND MODULATION SPACES: A GLOBAL EXISTENCE
AND SCATTERING RESULT

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Abstract. In this paper we transfer a small data global existence and
scattering result by Wang and Hudzik to the more general case of mod-
ulation spaces $M^s_{p,q}(\mathbb{R}^d)$ where $q = 1$ and $s \geq 0$ or $q \in (1, \infty]$ and $s > \frac{d}{q}$
and to the nonlinear Schrödinger equation with higher order anisotropic
dispersion.

1. Introduction and main results

We are interested in the following Cauchy problem for the higher-order
nonlinear Schrödinger equation (NLS) with anisotropic dispersion

\begin{equation}
\begin{cases}
i \partial_t u + \alpha \Delta u + i \beta \frac{\partial^2 u}{\partial x^2_1} + \gamma \frac{\partial^4 u}{\partial x^4_1} + f(u) = 0, \\
u(t=0, \cdot) = u_0,
\end{cases}
\end{equation}

where $u = u(t, x), (t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}, d \geq 2, \alpha \in \mathbb{R} \setminus \{0\}$ and
$(\beta, \gamma) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We consider power-like and exponential non-linearities $f$. The former non-linearity,

\begin{equation}
f(u) = \pi^{m+1}(u),
\end{equation}

is any product of in total $m + 1 \in \mathbb{N}$ copies of $u$ and $\pi$ of any sign, e.g. $f(u) = -|u|^2 u$. The latter nonlinearity has the form

\begin{equation}
f(u) = \lambda \left(e^{\rho |u|^2} - 1\right) u,
\end{equation}

where $\lambda \in \mathbb{C}$ and $\rho > 0$. Such PDEs arise in the context of high-speed
soliton transmission in long-haul optical communication systems and were
introduced by Karpman, see [Kar96], [DC08] and [KH94]. The case where
the coefficients $\alpha, \beta, \gamma$ are time-dependent has been studied in [CPS15] in
one dimension for the cubic nonlinearity, $f(u) = |u|^2 u$ with initial data in
$L^2(\mathbb{R})$-based Sobolev spaces. Before we state our results, we need to recall
some concepts and make some definitions.

The initial data $u_0$ in our case comes from a modulation space. Modulation
spaces were introduced by Feichtinger in [Fei83] (see also [Grö01] or [WH07]
for a gentle introduction). Let $Q_0 := \left(-\frac{1}{2}, \frac{1}{2}\right]^d$ and $Q_k := k + Q_0$ for $k \in \mathbb{Z}^d$.
Consider any smooth partition of unity $(\sigma_k) \in C^\infty(\mathbb{R}^d)^{2d}$ which is adapted

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to \((Q_k)\), i.e. \(\text{supp}(\sigma_k) \leq B_{\sqrt{2}}(k)\) and \(|\sigma_k(\xi)| \geq C\) for some \(C > 0\), all \(k \in \mathbb{Z}^d\) and all \(\xi \in Q_k\). Define the isometric decomposition operators

\[
\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F} \quad \forall k \in \mathbb{Z}^d,
\]

which we also call the box operators. For \(p, q \in [1, \infty)\) and \(s \in \mathbb{R}\) we define the modulation space

\[
M^s_{p,q}(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) \mid \|f\|_{M^s_{p,q}(\mathbb{R}^d)} < \infty \right\},
\]

where

\[
\|f\|_{M^s_{p,q}(\mathbb{R}^d)} := \left\| \left( |k|^{s} \|\Box_k f\|_p \right)_{k \in \mathbb{Z}^d} \right\|_{l^q(\mathbb{Z}^d)},
\]

\(S'(\mathbb{R}^d)\) is the space of tempered distributions and \(\langle k \rangle := \sqrt{1 + |k|^2}\) denotes the Japanese bracket. We note that a different partition of unity leads to an equivalent norm. We shall sometimes shorten \(M^s_{p,q}(\mathbb{R}^d)\) to \(M^s_{p,q}\) and \(M^r_{p,q}\) to \(M^r_{p,q}\).

The solution \(u\) is sought in a Planchon-type space. Spaces of this type go back to [Pla00]. The following variant adapted to modulation spaces has been introduced in [WH07]. For \(p, q, r \in [1, \infty)\) and \(s \in \mathbb{R}\) we define

\[
I^s_{\square}(L^r(\mathbb{R}, L^p(\mathbb{R}^d)))) := \left\{ u \in S'(\mathbb{R}^{d+1}) \mid \|u\|_{I^s_{\square}(L^r(\mathbb{R}, L^p(\mathbb{R}^d))))} < \infty \right\},
\]

where

\[
\|u\|_{I^s_{\square}(L^r(\mathbb{R}, L^p(\mathbb{R}^d))))} := \left\| \left( |k|^{s} \|\Box_k u(t, x)\|_{L^r_t L^p_x} \right)_{k \in \mathbb{Z}^d} \right\|_{l^q},
\]

(1) The box operator acts in the space variable \(x\) only. As already done above we shall indicate in which variable which norm is taken by writing e.g. \(L^r_t L^p_x\) for the mixed-norm space \(L^r(\mathbb{R}, L^p(\mathbb{R}^d)))\). Also, we shall sometimes shorten \(I^s_{\square}(L^r(\mathbb{R}, L^p(\mathbb{R}^d))))\) to \(I^s_{\square}(L^r \mathbb{R} L^p)\).

Finally, we denote by \(\lceil \cdot \rceil\) the ceiling and by \(\lfloor \cdot \rfloor\) the floor functions.

We are now in the position to state our results.

**Theorem 1.1.**

Let \(d \in \mathbb{N}\) with \(d \geq 2\), \(f\) be a power-like nonlinearity from Equation (2) and assume

\[
m \geq m_0 := \left\lfloor \frac{4}{d - \frac{1}{c_\gamma}} \right\rfloor, \quad \text{where} \quad c_\gamma = \begin{cases} 2, & \text{for } \gamma \neq 0, \\ 3, & \text{for } \gamma = 0. \end{cases}
\]

Moreover, let \(\frac{1}{r} \in I_{m,d}\) and \(\frac{1}{p} \in J_{r,d}\), where

\[
I_{m,d} := \left[ \frac{1}{2(m + 1)} \frac{1}{m_0 + 1} \right],
\]

\[
J_{r,d} := \left[ \frac{1}{2} - \frac{2}{r \left( d - \frac{1}{c_\gamma} \right)} - \frac{1}{2(l + 1)} \left( l - \frac{4}{d - \frac{1}{c_\gamma}} \right) \right],
\]

\[
\frac{1}{2} - \frac{2}{r \left( d - \frac{1}{c_\gamma} \right)} \quad \text{and}
\]

\[
l := \min \left\{ \lfloor r \rfloor - 1, m \right\}.
\]

If \(q = 1\) let \(s \geq 0\) and if \(q > 1\) let \(s > \frac{d}{q}\).
Then, there exists a \( \delta > 0 \) such that for any \( u_0 \in M_{2,q}^s \) with \( \|u_0\|_{M_{2,q}^s} \leq \delta \) the Cauchy problem (1) has a unique global (mild) solution in
\[
X := l_{□}^{s,q}(L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))) \cap l_{□}^{s,q}(L^r(\mathbb{R}, L^p(\mathbb{R}^d))) \subseteq C(\mathbb{R}, M_{2,q}^s(\mathbb{R}^d)).
\]

**Theorem 1.2.**
Let \( d \geq 2 \), \( f \) be an exponential non-linearity from Equation (3). Moreover, let \( \frac{1}{r} \in I_{3,d} \) and \( \frac{1}{p} \in J_{r,d} \) (see Equation (9), (10) and (11)). If \( q = 1 \) let \( s \geq p \) and if \( q > 1 \) let \( s > \frac{4}{q} \).

Then, there exists a \( \delta > 0 \) such that for any \( u_0 \in M_{2,q}^s \) with \( \|u_0\|_{M_{2,q}^s} \leq \delta \) the Cauchy problem (1) has a unique global (mild) solution in \( X \) (given by Equation (11)).

**Corollary 1.3.**
If, in Theorem 1.2, one additionally has \( q \leq m + 1 \) then the scattering operator \( S \) carries a whole neighbourhood of 0 in \( M_{2,q}^s(\mathbb{R}^d) \) into \( M_{2,q}^s(\mathbb{R}^d) \). The same is true for Theorem 1.2 if \( q \leq 3 \).

## 2. Preliminaries

We start by the following observation.

**Lemma 2.1.**
Let \( l, p, q \in [1, \infty] \) with \( q \leq l \) and \( s \in \mathbb{R} \). Then
\[
l^{s,q}_{□}(L^l(\mathbb{R}, L^p(\mathbb{R}^d))) \hookrightarrow L^l(\mathbb{R}, M_{p,q}^s(\mathbb{R}^d)).
\]

**Proof.** This is just an application of Minkowski’s integral inequality to \( L^l \) and \( l^{s,q} \).

Let us denote by \( W(t) \) the **free Schrödinger propagator with higher-order anisotropic dispersion** at time \( t \in \mathbb{R} \), i.e.
\[
W(t) = \mathcal{F}^{-1} e^{i(\alpha|\xi|^2 + \beta|\xi|^4 + \gamma|\xi|^6)t} \mathcal{F}.
\]

We cite the **Strichartz estimates** for this propagator from [Bon08, Theorem 1.2]. Let us remark that the dispersive estimate (from which the Strichartz estimates follow), was obtained in [BKS00] for the isotropic case. Strichartz estimates hold for the so-called (dually) **admissible pairs**. We call the pair \( (p, r) \in [2, \infty] \times [2, \infty] \) admissible, if
\[
\frac{2}{r} + \left( d - \frac{1}{c_\gamma} \right) \frac{1}{p} = \left( d - \frac{1}{c_\gamma} \right) \frac{1}{2}.
\]

**Theorem 2.2** (Strichartz estimates).
Let \( (p, r) \) and \( (\tilde{p}', \tilde{r}') \) be admissible. For any \( u_0 \in L^2(\mathbb{R}^d) \) one has the homogeneous Strichartz estimate
\[
\|W(t)u_0\|_{L^r_t L^p_x} \lesssim \|u_0\|_{L^2}
\]
and for any \( F \in L^p_{t} L^\tilde{p}_x \) one has the inhomogeneous Strichartz estimate
\[
\left\| \int_0^t W(t - \tau) F(\tau, \cdot) d\tau \right\|_{L^r_t L^p_x} \lesssim \|F\|_{L_{t}^p L_{x}^{\tilde{p}}}.
\]

with implicit constants independent of \( u_0 \) and \( F \).
Strichartz estimates on Lebesque spaces immediately translate to the setting of modulation and Planchon-type spaces.

**Lemma 2.3.** Let \((p, r)\) and \((\tilde{p}', \tilde{r}')\) be admissible. Moreover, let \(q \in [1, \infty]\) and \(s \in \mathbb{R}\). For any \(u_0 \in M_{p, q}^s\) one has

\[
\|W(t)u_0\|_{L_{\tilde{r}'}^q(L_{\tilde{p}}^p)} \lesssim \|u_0\|_{M_{p, q}^s}
\]

and for any \(F \in l_{q}^{\tilde{s}}(L_{\tilde{r}}^{\tilde{p}}(\mathbb{R}, L_{\tilde{r}}^{p}(\mathbb{R}^d)))\) one has

\[
\left\| \int_0^t W(t - \tau)F(\tau, \cdot)d\tau \right\|_{l_{q}^{\tilde{s}}(L_{\tilde{r}}^{p}(\mathbb{R}^d))} \lesssim \|F\|_{l_{q}^{\tilde{s}}(L_{\tilde{r}}^{p}L_{\tilde{r}}^{p})}
\]

with implicit constants independent of \(u_0\) and \(F\).

**Proof.** The proof follows from Theorem 2.2 and the fact that \(W(t)\) and \(\square_k\) commute. \(\Box\)

**Lemma 2.4 (Hölder-like inequalities).** Let \(d, n \in \mathbb{N}\) and \(\tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_n, q, \tilde{r}, \tilde{r}_1, \ldots, \tilde{r}_n \in [1, \infty]\) be such that

\[
\frac{1}{\tilde{p}} = \frac{1}{\tilde{p}_1} + \cdots + \frac{1}{\tilde{p}_n}, \quad \frac{1}{\tilde{r}} = \frac{1}{\tilde{r}_1} + \cdots + \frac{1}{\tilde{r}_n}.
\]

For \(q > 1\) let \(s > d \left(1 - \frac{1}{q}\right)\) and for \(q = 1\) let \(s \geq 0\). Then, for any \((f_1, \ldots, f_n) \in l_{q}^{\tilde{s}}(L_{\tilde{r}_1}^{\tilde{p}_1}(\mathbb{R}, L_{\tilde{p}_1}^{p}(\mathbb{R}^d))) \times \cdots \times l_{q}^{\tilde{s}}(L_{\tilde{r}_n}^{\tilde{p}_n}(\mathbb{R}, L_{\tilde{p}_n}^{p}(\mathbb{R}^d)))\)

one has

\[
\left\| \prod_{j=1}^n f_j \right\|_{l_{q}^{\tilde{s}}(L_{\tilde{r}}^{p}(\mathbb{R}^d))} \lesssim \prod_{j=1}^n \|f_j\|_{l_{q}^{\tilde{s}}(L_{\tilde{r}}^{p}(\mathbb{R}^d))}
\]

with an implicit constant independent of \((f_1, \ldots, f_n)\).

Similarly, for \((g_1, \ldots, g_n) \in M_{p_1, q}^s(\mathbb{R}^d) \times \cdots \times M_{p_n, q}^s(\mathbb{R}^d)\)

one has

\[
\left\| \prod_{j=1}^n g_j \right\|_{M_{p, q}^s} \lesssim \prod_{j=1}^n \|g_j\|_{M_{p, q}^s}
\]

with an implicit constant independent of \((g_1, \ldots, g_n)\).

**Proof.** The special case of (22) for \(n = 2\) is proven in [Cha18, Theorem 4.3]. That proof is easily transferred to the case of Planchon-type spaces applying Hölder’s inequality for the time variable, also. \(\Box\)

**Lemma 2.5.** Let \(p_1, p_2, q, r \in [1, \infty]\) with \(p_1 \leq p_2\) and \(s \in \mathbb{R}\). Then

\[
l_{q}^{\tilde{s}}(L_{\tilde{r}}^{p}(\mathbb{R}, L_{\tilde{r}}^{p_1}(\mathbb{R}^d))) \hookrightarrow l_{q}^{\tilde{s}}(L_{\tilde{r}}^{p}(\mathbb{R}, L_{\tilde{r}}^{p_2}(\mathbb{R}^d))).
\]

**Proof.** The conclusion follows immediately from the definition of the norm and Bernstein’s multiplier estimate (see e.g. [Cha18, Corollary A.53]). \(\Box\)
Corollary 2.6.
Let \( d \in \mathbb{N} \), \( m \in \mathbb{N}_0 \) and \( l \in \{0, \ldots, m\} \). Moreover, let \( \tilde{p}, q, \tilde{r} \in [1, \infty] \). For \( q > 1 \) let \( s > \frac{d}{4} \) and for \( q = 1 \) let \( s > 0 \). Then
\[
\|\pi^{m+1}(u) - \pi^{m+1}(v)\|_{L_t^{\tilde{p}}(L_x^q)} \lesssim \|u - v\|_{L_t^{\tilde{p}}(L_x^q)} \|u\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})}^{m-1} + \|v\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})}^{m-1} \|u\|_{L_t^{\tilde{p}}(L_x^{\infty})}^{m-1}
\]
for any \( u, v \in S'(\mathbb{R}^{d+1}) \), where the implicit constant is independent of \( u, v \).

Proof. The base case \( m = 0 \) is trivial. For the induction step from \( m - 1 \) to \( m \) observe, that
\[
\|\pi^{m+1}(u) - \pi^{m+1}(v)\|_{L_t^{\tilde{p}}(L_x^q)} \lesssim \|u - v\|_{L_t^{\tilde{p}}(L_x^q)} \|\pi^m(u)\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})} \|\pi^m(v)\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})}.
\]
For the first summand one has, by the Hölder-like inequality (21),
\[
\|\pi^m(u)(u - v)\|_{L_t^{\tilde{p}}(L_x^q)} \lesssim \|u - v\|_{L_t^{\tilde{p}}(L_x^q)} \|\pi^m(u)\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})} \|\pi^m(v)\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})}.
\]
For \( l = 0 \) the second factor above is estimated via (21) and (23) against
\[
\|\pi^m(u)\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})} \lesssim \|u\|_{L_t^{\tilde{p}}(L_x^{\infty})} \lesssim \|u\|_{L_t^{\tilde{p}}(L_x^{\infty})}^{m-1} \|u\|_{L_t^{\tilde{p}}(L_x^{\infty})}^{m-1}.
\]
For \( l \geq 1 \) one uses the induction hypothesis instead and arrives at
\[
\|\pi^m(u)\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})} \lesssim \|u\|_{L_t^{\tilde{p}}(L_x^{q(1+s)})} \|u\|_{L_t^{\tilde{p}}(L_x^{\infty})}^{m-1} \|u\|_{L_t^{\tilde{p}}(L_x^{\infty})}^{m-1}.
\]
The second summand in (25) is treated via the same methods. This concludes the proof. □

3. PROOFS OF THE RESULTS

In this section we present the proofs of Theorem 1.1 of Theorem 1.2 and of Corollary 1.3.

Proof of Theorem 1.1. For a \( \delta > 0 \), which will be fixed later, consider
\[
M(\delta) := \left\{ f \in X \left| \|f\|_X \leq \delta \right. \right\},
\]
where \( X \) is given by (14) (the embedding claimed there immediately follows from Lemma 2.1 with \( l = \infty \) and \( p = 2 \)). By Banach’s fixed-point theorem, it suffices to show that the operator
\[
u \mapsto \text{T} \nu := W(t)u_0 + \int_0^t W(t - \tau)\pi^{m+1}(u(\tau))d\tau
\]
is a contractive self-mapping of \( M(\delta) \) for some \( \delta > 0 \). We only show the contractiveness of \( T \), as the proof of the self-mapping property follows along the same lines. Fix any \( u, v \in M(\delta) \).
By the definition of the norm \(\|\cdot\|_X\) one has
\[
\|T(u) - T(v)\|_X = \|T(u) - T(v)\|_{L^q(L^r L^p)} + \|T(u) - T(v)\|_{L^q(L^r L^p)}.
\]

To apply Strichartz estimates we require spaces with admissible indices in both summands. This is already the case for the first summand. For the second summand, observe that Assumption (11) implies
\[
p \geq p_a := \left( \frac{1}{2} - \frac{2}{r (d - \frac{1}{c_\gamma})} \right)^{-1}
\]
and hence, by Lemma 2.5
\[
\|T(u) - T(v)\|_{L^r L^{p_a}} \lesssim \|T(u) - T(v)\|_{L^r L^{p_a}}
\]
follows. Due to \(\frac{1}{r} \in I_{m,d} \subseteq \left[0, \frac{1}{2}\right]\), the pair \((p_a, r)\) is indeed admissible.

By the inhomogeneous Strichartz estimate (20) we therefore have
\[
\|T u - T v\|_X = \left\| \int_0^t W(t - \tau) \left( \pi^{m+1}(u(\tau)) - \pi^{m+1}(v(\tau)) \right) d\tau \right\|_X
\]
\[
\lesssim \|\pi^{m+1}(u) - \pi^{m+1}(v)\|_{L^r L^{p_a}}
\]
for any pair \((\tilde{p}, \tilde{r})\) in \([1, 2]\) such that \((\tilde{p}', \tilde{r}')\) is admissible, which we will fix in the following. Observe, that
\[
I_{m,d} = \bigcup_{k=m_0}^m \left[ \frac{1}{2(k+1)}, \frac{1}{k+1} \right]
\]
and that the effective non-linearity \(l\) satisfies
\[
(28) \quad l = \max \left\{ k \in \{m_0, \ldots, m\} \mid \frac{1}{r} \in \left[ \frac{1}{2(k+1)}, \frac{1}{k+1} \right] \right\}.
\]
Hence, \(\tilde{r} := \frac{r}{l+1} \in [1, 2]\) and thus \((\tilde{p}', \tilde{r}')\), where
\[
(29) \quad \tilde{p} := \left( \frac{1}{2} + \frac{4 (1 - \frac{1}{l})}{d - \frac{1}{c_\gamma}} \right)^{-1},
\]
indeed form an admissible pair.

We now apply Estimate (24) to arrive at
\[
\|\pi^{m+1}(u) - \pi^{m+1}(v)\|_{L^r L^{\tilde{p}}}
\]
\[
\lesssim \|u - v\|_{L^{r L_l (l+1) \tilde{p}}}
\]
\[
\cdot \left( \|u\|_{L^r L_{(l+1) \tilde{p}}}^{m-l} \|u\|_{L^r L^2}^{m-l} + \|v\|_{L^r L_{(l+1) \tilde{p}}}^{m-l} \|v\|_{L^r L^2}^{m-l} \right).
\]
A short calculation yields
\[
(l + 1)\tilde{p} = \left( \frac{1}{2} - \frac{2}{r (d - \frac{1}{c_\gamma})} - \frac{1}{2(l+1)} \left( l - \frac{4}{d - \frac{1}{c_\gamma}} \right) \right)^{-1},
\]
and thus, due to Assumption (11), \((l + 1)\tilde{p} \geq p\). Therefore, invoking Embedding (23) once again, one finally obtains
\[
\|T(u) - T(v)\|_X \lesssim \|u - v\|_{\tilde{L}^{\infty,q}(L^p)} \cdot (\|u\|_{\tilde{L}^{\infty,q}(L^p)} + \|v\|_{\tilde{L}^{\infty,q}(L^p)}) \lesssim \|u - v\|_{\tilde{L}^{\infty,q}(L^p)} \delta^m.
\]
Choosing \(\delta\) small enough finishes the proof. \(\square\)

**Proof of Theorem 1.3.** By the definition of the exponential nonlinearity (3) we have
\[
f(u) = \lambda \sum_{m=1}^{\infty} \frac{\rho^m}{m!} |u|^{2m} u.
\]
We have to show that the operator
\[
u \mapsto T(u) := W(t)u_0 + \int_0^t W(t - \tau)f(u(\tau))d\tau
\]
is a contractive self-mapping of the complete metric space \(M(\delta)\) as defined in (26). We only briefly sketch the argument for the self-mapping. We have
\[
\|T u\|_X \lesssim \|u_0\|_{M_{2,q}^d} + \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \left\| \int_0^t W(t - \tau)\pi^{2m+1}(u)d\tau \right\|_X \\
\lesssim \|u_0\|_{M_{2,q}^d} + \sum_{m=1}^{\infty} \frac{\rho^m}{m!} \|u\|_{X}^{2m+1} \leq \|u_0\|_{M_{2,q}^d} + \left( \delta e^{\rho^2} - 1 \right) \leq \delta,
\]
where we set \(\pi^{2m+1}(u) := |u|^{2m} u\) and proceeded as in the proof of Theorem 1.1. This is justified because the assumptions on \(r\) and \(p\) are exactly such that Theorem 1.1 can be applied to \(\pi^{2m+1}(u)\) with \(m \geq 1\) \((I_{m,d} \subseteq I_{n,d})\) for \(m \leq n\). Thus, if \(\|u_0\|_{M_{2,q}^d} \lesssim \frac{\delta}{2}\) and \(\delta > 0\) is sufficiently small, the operator \(T\) is a self-mapping of \(M(\delta)\). \(\square\)

**Proof of Corollary 1.3.** Given initial data \(u_0\) at \(-\infty\) of sufficiently small \(M_{2,q}^d(\mathbb{R}^d)\)-norm we show that the operator \(S_-\) given by
\[
(S_- u)(t) := W(t)u_0 + \int_{-\infty}^t W(t - \tau)f(u(\tau))d\tau, \quad t \in \mathbb{R},
\]
is a contractive self-mapping of \(M\) given in (26). As the only difference between \(S_-\) and \(T\) is the lower limit of the integral being \(-\infty\) instead of 0, the argument is very similar to the proof of Theorem 1.1 and we omit the details. We denote the unique fixed-point of \(S_-\) by \(u\). Following the proof of Theorem 1.1 we notice, that \(u \in \tilde{L}^{\infty,q}(L^p)\) for any \(\rho, \sigma \in [2, \infty]\) satisfying
\[
\frac{2}{\rho} + \left( d - \frac{1}{c_\gamma} \right) \frac{1}{\sigma} \leq \left( d - \frac{1}{c_\gamma} \right) \frac{1}{2},
\]
This is because we can replace the $\sigma$ by $\sigma_a$ such that $(\sigma_a, \rho)$ is admissible, i.e.

$$\sigma \geq \sigma_a := \left( \frac{1}{2} - \frac{2}{\rho \left( d - \frac{1}{c_s} \right)} \right)^{-1},$$

and hence

$$\|u\|_{L_t^q(L_x^p)} \lesssim \|S_- u\|_{L_t^q(L_x^p)} \lesssim \|u_0^-\|_{M_{2,q}^q} + \|\pi^{m+1}(u)\|_{L_t^q(L_x^p)}$$

$$\lesssim \|u\|_{X_{\infty}^{\rho,\sigma}}^{m+1} < \infty,$$

where above we used Lemma 2.5 Strichartz estimates and the Hölder-like inequality for Planchon-type spaces. Notice that, as $t \to -\infty$, one has

$$\|u(t) - W(t)u_0\|_{M_{2,q}^q} \to 0.$$  \hspace{1cm} (34)

This is because by Lemma 2.1 and the assumption $q \leq m + 1$ one has

$$\int_{-\infty}^{t} \|\pi^{m+1}(u)(\cdot, \tau)\|_{M_{2,q}^q} \, d\tau \leq \int_{-\infty}^{\infty} \|u(\cdot, \tau)\|_{L_{\infty}^{2(m+1),q}} \, d\tau$$

$$= \|u\|_{L_{t}^{\infty}(M_{2(m+1),q})} \leq \|u\|_{L_{t}^{\infty}(L_{\infty}^{2(m+1),L_{x}^{2(m+1)}})}$$

and, as $\rho = m + 1$ and $\sigma = 2(m + 1)$ satisfy the condition (33) by the prerequisite $m \geq m_0$, the norm above is finite. Finally, we define

$$u_0^+ := u_0^- + \int_{-\infty}^{\infty} W(-\tau)\pi^{m+1}(u(\tau)) \, d\tau \in M_{2,q}^q(\mathbb{R}^d)$$

and notice that $\|u(t) - W(t)u_0^+\|_{M_{2,q}^q} \to 0$ as $t \to +\infty$, because

$$\|u(t) - W(t)u_0^+\|_{M_{2,q}^q} = \left\|W(-t)u(t) - u_0^+\right\|_{M_{2,q}^q}$$

$$= \left\|\int_{t}^{\infty} W(-\tau)\pi^{m+1}(u(\tau)) \, d\tau\right\|_{M_{2,q}^q}$$

and the same argument as for the convergence to $u_0^-$ applies. Hence, the scattering operator $u_0^- \overset{S}{\to} u_0^+$ indeed carries a whole neighborhood of 0 in $M_{2,q}^q$ into $M_{2,q}^q$.  \hspace{1cm} \Box

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