Lubrication theory applied to the convergent flows of two stacked liquid layers

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Abstract. With the aim of describing the mountain building process, we have previously applied the lubrication approximation to obtain the evolution equations of the problem of two stacked layers of viscous fluids with different densities and different viscosities. The lubrication approximation is a perturbation method where the small parameter is the aspect ratio (thickness/lenght) of the current. This approximation is widely used to study the slow flow of one layer of a viscous fluid, but it is not well known under which conditions it can be applied in more general settings. Here we analyze in detail the assumptions needed to apply the lubrication theory to study the flow of two stacked viscous fluid layers. We employ the same perturbation method and we found that, besides the usual conditions (low Reynolds number and gentle slope), we must require that the viscosity and density ratios are of the order of unity. These requirements determine the range of validity of the equations of our model of the mountain building.

1. Introduction

Mountain ranges are one of the most striking features of the Earth and their origin and evolution have been investigated for a long time. It is known that the lithosphere (the outer solid layer of the Earth) is a two layer structure in which the crust rests on the denser lithospheric mantle (see Fig. 1), being separated by the Mohorovičić discontinuity (called Moho). The lithosphere is divided into several approximately rigid plates that rest on the hotter and more fluid asthenosphere. The relative motion of these plates is the cause of mountain building, because of the shortening and consequent thickening of the crust that occurs when two continental plates collide or when an oceanic plate is subducted beneath a continent. On the timescale of the orogenic processes the lithosphere is in local hydrostatic equilibrium (a condition called isostasy) that implies that the visible regional topography is accompanied by a corresponding anti-topography (called root) of the Moho (see Fig. 1).

Clearly mountain building is an important problem that involves many disciplines and interests a broad range of scientists. To attempt a realistic and detailed theoretical description of mountain building is an exceedingly complex task because across the lithosphere there are large variations of the temperature, density and rheological parameters as well as other properties (many of which, to compound the issue, are poorly known). To this should be added the complications due to the geometry and the time dependence of the motion of the plates.
Figure 1. Formation of a ridge due to the convergent motion of a two-layer system.

The basic phenomena that govern the large scale evolution of mountain belts are the spreading flow at the depth of the roots together with isostasy and crustal shortening. The profile of the ridge is determined by the dynamic balance between buoyancy and viscous forces. Based on these ideas we have recently developed a model to describe the process of mountain building which showed a very good agreement when it was compared with real topographical data from approximately straight segments of several mountain belts [1]. In this model we describe the lithospheric plates as two stacked layers of Newtonian liquids. The upper layer, which is the lightest, represents the crust, and the lower one represents the lithospheric mantle. The motion of the plates is driven by the basal traction that is assumed it is exerted by the astenosphere at the bottom of the lithospheric mantle. We introduced the basal traction in our model as a boundary condition which states that below the plates there is a prescribed horizontal velocity. When the motion due to the basal traction is convergent, the crust is shortened and thus its thickness increases in the region of convergence producing a ridge. We also assume in our model the isostasy condition, which is introduced assuming that at the bottom of the lithospheric mantle the pressure is uniform. From these hypothesis and using the lubrication approximation we derive a partial differential equation for the thickness of the crust.

The lubrication approximation is a well known and established treatment to study the viscous slow flow of a thin current whose typical length is much larger than its thickness. Depending on the problem under study, the driven forces can be the gravity, centrifugation, surface tension, Marangoni effect, Van der Waals forces, thermocapillarity, condensation and evaporation, etc. It was used to study various problems with very different spatial and temporal scales, from lava flows to tear films in the eye (see for example the reviews by Oron et al. [2] and Myers [3]). However, in most of the works where the lubrication approximation was applied, the problem involves only one layer of liquid over a rigid substrate. Much less is known about the lubrication approximation when it is applied to study the flow of more than one layer of fluid, as it happens in our model of mountain building. Here we discuss the conditions and hypotheses needed to apply the lubrication approximation to study the flow of two stacked viscous layers under the action of gravity.

This paper is organized as follow. In the next Section we state the problem and the full system of equations that describes it. In Section 3 we reduce the mathematical problem by means of a set of simplifications which are the bases of the lubrication approximation. In Section 4 we particularize the problem to derive the equations of our two-layer model of mountain building [1]. Finally in Section 5 we present the conclusions.

2. The complete system of equations
We consider a system which consists of two layers of Newtonian liquids with different viscosities and densities. The upper layer has viscosity $\mu_c$ and density $\rho_c$, and the lower one has viscosity $\mu_m$ and density $\rho_m$. The motion is driven by the basal traction $U_b$ and is constrained by the isostasy condition at the bottom of the lithospheric mantle. The lubrication approximation allows us to derive a partial differential equation for the thickness of the crust.
\( \mu_m \) and density \( \rho_m \). For simplicity we assume planar symmetry, and \( X \) and \( Z \) are the horizontal and vertical coordinates respectively (see Fig. 1). The domain of the two–layer system is \( Z \geq 0 \). The thickness of the upper and lower layers are \( H_c \) and \( H_m \) respectively, so that the system has two interfaces, one at \( Z = H_c + H_m \) and the other at \( Z = H_m \). For \( H_m \leq Z \leq H_m + H_c \) the Navier–Stokes equation takes the form

\[
\rho_c \left( \frac{\partial U_c}{\partial t} + U_c \frac{\partial U_c}{\partial X} + W_c \frac{\partial U_c}{\partial Z} \right) = -\frac{\partial P_c}{\partial X} + \mu_c \left( \frac{\partial^2 U_c}{\partial X^2} + \frac{\partial^2 U_c}{\partial Z^2} \right),
\]

(1)

and for \( 0 \leq Z \leq H_m \)

\[
\rho_m \left( \frac{\partial U_m}{\partial t} + U_m \frac{\partial U_m}{\partial X} + W_m \frac{\partial U_m}{\partial Z} \right) = -\frac{\partial P_m}{\partial X} + \mu_m \left( \frac{\partial^2 U_m}{\partial X^2} + \frac{\partial^2 U_m}{\partial Z^2} \right),
\]

(3)

where \( T \) is the time, \( \Phi \) is the potential of the gravitational force, \( P \) is the pressure, \( U \) and \( W \) are horizontal and vertical components of the velocity respectively, and the subscripts \( c \) and \( m \) indicate to which layer each magnitude belongs.

At the interface at \( Z = H_c + H_m \), the usual kinematic condition holds

\[
\frac{\partial}{\partial t} (H_c + H_m) + U_c \frac{\partial}{\partial X} (H_c + H_m) = W_c,
\]

(5)

and we assume that the normal and tangential components of stress vanish (we neglect surface tension effects)

\[
\hat{n}_c \cdot \Sigma_c \cdot \hat{n}_c = 0, \quad \hat{t}_c \cdot \Sigma_c \cdot \hat{n}_c = 0,
\]

(6)

where \( \Sigma_c \) is the stress tensor of the upper layer and \( \hat{n}_c \) and \( \hat{t}_c \) are the unitary vector normal and tangential to the upper surface.

At the interface between the two layers at \( Z = H_m \) we have the kinematic condition

\[
\frac{\partial H_m}{\partial t} + U_c \frac{\partial H_m}{\partial X} = W_c.
\]

(7)

and we assume continuity of the components of the velocity and the stress

\[
U_m = U_c,
\]

(8)

\[
W_m = W_c,
\]

(9)

\[
\hat{n}_m \cdot \Sigma_m \cdot \hat{n}_m = \hat{n}_m \cdot \Sigma_m \cdot \hat{n}_m,
\]

(10)

\[
\hat{t}_m \cdot \Sigma_m \cdot \hat{n}_m = \hat{t}_m \cdot \Sigma_m \cdot \hat{n}_m,
\]

(11)

where \( \Sigma_m \) is the stress tensor of the lower layer and \( \hat{n}_m \) and \( \hat{t}_m \) are the unitary vector normal and tangential to the interface. We impose at \( Z = 0 \) a prescribed horizontal velocity \( U_b \), so that

\[
U_m = U_b, \quad \text{for } Z = 0.
\]

(12)

Equations (1-12) together with the continuity equation and an additional condition at \( Z = 0 \) (usually this condition is \( W_m = 0 \), however we shall use a different one more appropriate for our problem) describe without any approximation the flow of the two layers of Newtonian liquids under the action of body forces with any initial conditions. However, this system of equations is difficult to solve, even numerically, so that in order to gain insight about a particular problem it is convenient to introduce some simplifying hypotheses and to reduce the system of equations.
3. The lubrication approximation

We are interested in flows which are slow and almost horizontal. Let $Z_0$ and $X_0$ be the characteristic vertical and horizontal scales, respectively. The main hypothesis of the lubrication approximation is that $X_0 \gg Z_0$, so that

$$
\varepsilon \equiv \frac{Z_0}{X_0} \ll 1.
$$

(13)

This means that both interfaces have gentle slopes. Calling $W_0$ and $U_0$ the characteristic vertical and horizontal velocities, form the continuity equation and (13) one has that

$$
W_0 = \varepsilon U_0 \ll U_0,
$$

(14)

so that the flow is predominately horizontal.

Now we introduce the following dimensionless quantities:

$$
x = \frac{X}{X_0} = \varepsilon \frac{X}{Z_0}, \quad z = \frac{Z}{Z_0}, \quad t = \frac{T}{T_0} = \frac{T U_0}{X_0} = \varepsilon T U_0, \quad h_c = \frac{H_c}{Z_0},
$$

$$
h_m = \frac{H_m}{Z_0}, \quad u_c = \frac{U_c}{U_0}, \quad u_m = \frac{U_m}{U_0}, \quad w_c = \frac{W_c}{W_0} = \frac{W_c}{\varepsilon U_0},
$$

$$
w_m = \frac{W_m}{W_0} = \frac{W_m}{\varepsilon U_0}, \quad p_c = \frac{P_c}{P_0}, \quad p_m = \frac{P_m}{P_0}, \quad \phi = \frac{\phi}{P_0}.
$$

(15)

The scale of the pressure $P_0$ is determined from the understanding that the horizontal gradient of pressure is equilibrated by the vertical gradient of the viscous stress so that $\partial P/\partial X \sim \mu \partial^2 U/\partial Z^2$. Then we choose

$$
P_0 = \frac{\mu_c U_0 X_0}{Z_0^2} = \frac{\mu_c U_0}{\varepsilon Z_0}.
$$

(16)

Finally, we define the Reynolds number as

$$
Re = \frac{\rho_c U_0 Z_0}{\mu_c}.
$$

(17)

In terms of the dimensionless quantities the governing equations take the form

$$
\varepsilon Re \left( \frac{\partial u_c}{\partial t} + u_c \frac{\partial u_c}{\partial x} + w_c \frac{\partial u_c}{\partial z} \right) = - \frac{\partial p_c}{\partial x} + \varepsilon^2 \frac{\partial^2 u_c}{\partial x^2} + \frac{\partial^2 u_c}{\partial z^2},
$$

(18)

$$
\varepsilon^3 Re \left( \frac{\partial w_c}{\partial t} + u_c \frac{\partial w_c}{\partial x} + w_c \frac{\partial w_c}{\partial z} \right) = - \frac{\partial p_c}{\partial z} + \varepsilon^4 \frac{\partial^2 w_c}{\partial x^2} + \varepsilon^2 \frac{\partial^2 w_c}{\partial z^2} - \frac{\partial \phi}{\partial z},
$$

(19)

for $h_m \leq z \leq h_m + h_c$, and

$$
\varepsilon Re \frac{\rho_m}{\rho_c} \left( \frac{\partial u_m}{\partial t} + u_m \frac{\partial u_m}{\partial x} + w_m \frac{\partial u_m}{\partial z} \right) = - \frac{\partial p_m}{\partial x} + \varepsilon \frac{\partial^2 w_m}{\partial x^2} + \frac{\mu_m}{\mu_c} \frac{\partial^2 u_m}{\partial z^2},
$$

(20)

$$
\varepsilon^3 Re \frac{\rho_m}{\rho_c} \left( \frac{\partial w_m}{\partial t} + u_m \frac{\partial w_m}{\partial x} + w_m \frac{\partial w_m}{\partial z} \right) = - \frac{\partial p_m}{\partial z} + \varepsilon^4 \frac{\partial^2 w_m}{\partial x^2} + \varepsilon^2 \frac{\mu_m}{\mu_c} \frac{\partial^2 w_m}{\partial z^2} - \frac{\partial \phi}{\partial z},
$$

(21)

for $0 \leq z \leq h_m$.

The boundary conditions at $z = h_m + h_c$ are given by

$$
\frac{\partial}{\partial t} (h_m + h_c) + u_c \frac{\partial}{\partial x} (h_m + h_c) = w_c,
$$

(22)
The matching conditions at $z = h_m$ are given by

\[ \frac{\partial h_m}{\partial t} + u_c \frac{\partial h_m}{\partial x} = w_c, \]

\[ u_c = u_m, \]

\[ w_c = w_m, \]

\[ p_c = p_m + \varepsilon^2 \left\{ (p_m - p_c) \left( \frac{\partial h_m}{\partial x} \right)^2 \right. \]
\[ + \frac{2}{\varepsilon} \frac{\partial}{\partial z} \left. \left( \frac{\partial h_m}{\partial x} \right) \frac{\partial}{\partial z} \left( u_c - \frac{\mu_m}{\mu_c} u_m \right) - \frac{\partial}{\partial z} \left( w_c - \frac{\mu_m}{\mu_c} w_m \right) \right\} \]
\[ + \varepsilon^4 \frac{\partial h_m}{\partial x} \left[ \frac{\partial}{\partial z} \left( u_c - \frac{\mu_m}{\mu_c} u_m \right) - \frac{\partial}{\partial z} \left( w_c - \frac{\mu_m}{\mu_c} w_m \right) \right] \]
\[ + 2 \frac{\partial h_m}{\partial x} \left( \frac{\partial}{\partial z} \left( u_c - \frac{\mu_m}{\mu_c} u_m \right) - \frac{\partial}{\partial z} \left( w_c - \frac{\mu_m}{\mu_c} w_m \right) \right) \]
\[ + \varepsilon^4 \left( \frac{\partial h_m}{\partial x} \right)^2 \frac{\partial}{\partial z} \left( w_c - \frac{\mu_m}{\mu_c} w_m \right). \]

At $z = 0$ we have $u_m = u_b$.

Now we seek the solution as a perturbation series in powers of the small parameter $\varepsilon$:

\[ u_c = u_{c0} + \varepsilon u_{c1} + \varepsilon^2 u_{c2} + \cdots, \]

\[ w_c = w_{c0} + \varepsilon w_{c1} + \varepsilon^2 w_{c2} + \cdots, \]

\[ p_c = p_{c0} + \varepsilon p_{c1} + \varepsilon^2 p_{c2} + \cdots, \]

\[ u_m = u_{m0} + \varepsilon u_{m1} + \varepsilon^2 u_{m2} + \cdots, \]

\[ u_m = u_{m0} + \varepsilon u_{m1} + \varepsilon^2 u_{m2} + \cdots, \]

\[ u_m = u_{m0} + \varepsilon u_{m1} + \varepsilon^2 u_{m2} + \cdots. \]

To the leading order in $\varepsilon$ the governing equations reduce to (after omitting the subscript 0)

\[ \frac{\partial p_c}{\partial x} = \frac{\partial^2 u_c}{\partial z^2}, \]

\[ \frac{\partial p_c}{\partial z} = - \frac{\partial \phi}{\partial z}, \]

\[ \frac{\partial p_m}{\partial x} = \frac{\mu_m}{\mu_c} \frac{\partial^2 u_m}{\partial z^2}, \]

\[ \frac{\partial p_m}{\partial z} = - \frac{\partial \phi}{\partial z}, \]

\[ \frac{\partial}{\partial t} (h_m + h_c) + u_c \frac{\partial}{\partial x} (h_m + h_c) = w_c, \quad \text{for } z = h_m + h_c, \]

\[ p_c = 0, \quad \text{for } z = h_m + h_c, \]
\[
\frac{\partial u_c}{\partial z} = 0, \quad \text{for } z = h_m + h_c, \quad (42)
\]

\[
\frac{\partial h_m}{\partial t} + u_c \frac{\partial h_m}{\partial x} = w_c, \quad \text{for } z = h_m, \quad (43)
\]

\[
u_c = u_m, \quad \text{for } z = h_m, \quad (44)
\]

\[
w_c = w_m, \quad \text{for } z = h_m, \quad (45)
\]

\[
p_c = p_m, \quad \text{for } z = h_m, \quad (46)
\]

\[
\frac{\partial u_c}{\partial z} = \frac{\mu_m}{\mu_c} \frac{\partial u_m}{\partial z}, \quad \text{for } z = h_m, \quad (47)
\]

\[
u_m = u_b, \quad \text{for } z = 0. \quad (48)
\]

It can be mentioned that to obtain this leading order approximation it is necessary to add the following condition

\[
Re \lesssim O(1), \quad \text{as } \varepsilon \to 0, \quad (49)
\]

as usual in the lubrication approximation for a single fluid layer. In addition to this we must require that

\[
\rho_m / \rho_c \sim O(1) \quad \text{and} \quad \mu_m / \mu_c \sim O(1), \quad \text{for } \varepsilon \to 0. \quad (50)
\]

These additional conditions are necessary due to the presence of the two layers.

Notice that the system (36–48) is much simpler than the original system of equations. In fact, some important results can be obtained immediately. First, from Eqs. (37) and (39) is clear that the pressure in both layers is hydrostatic, and with the conditions (41) and (46) the expressions for \(p_c\) and \(p_m\) can be completely determined. Second, once the pressure is found one can integrate (36) and (38) and use (42), (44), (47) and (48) to determine the horizontal velocities \(u_c\) and \(u_m\). The velocity profiles thus obtained are parabolic in both layers. However, notice that the pressure and the velocity field in both layers are expressed in terms of \(h_m\) and \(h_c\), which remain unknown.

Integrating in \(z\) the continuity equation from \(h_m\) to \(h_m + h_c\), and using the kinematic conditions (40) and (43) it is possible to obtain

\[
\frac{\partial h_c}{\partial t} = -\frac{\partial}{\partial x} \left( \int_{h_m}^{h_m+h_c} u_c \, dz \right), \quad (51)
\]

which expresses the conservation of the mass of the upper layer. However, an analog expression for the lower layer can not be obtained until an additional condition at \(z = 0\) be adopted. If the usual condition \(w_m = 0\) is chosen, an expression for \(h_m\) equivalent to (51) will be obtained.

**4. The two–layer model of mountain building**

In the context of our model of mountain building the upper layer represents the Earth’s crust and the lower one represents the upper mantle. We call \(C\) and \(M\) the initial thickness of the crust and lithospheric mantle respectively. We choose the characteristic scales as (see Ref. [1] for details)

\[
Z_0 = C, \quad X_0 = (1 - \frac{\rho_c}{\rho_m}) \frac{\rho_c g M C^2}{\mu_m U_0}, \quad U_0 = \max(|U_b|). \quad (52)
\]

As it was said before, to close the system and derive evolution equations for \(h_c\) and \(h_m\) it is needed to add a boundary condition at \(z = 0\). In our two–layer model of mountain building we introduce the isostasy by means of the condition

\[
\frac{\partial p_c}{\partial x} = 0, \quad \text{at } z = 0. \quad (53)
\]
From this is immediately obtained that

\[ h_m = \frac{M}{C} + \frac{\rho_c}{\rho_m}(1 - h_c). \]  

(54)

This simple linear relation between \( h_m \) and \( h_c \) allows us to reduce even more the problem because in the following the only unknown is \( h_c \). On the hand, notice that the boundary condition (53) does not guarantee the mass conservation of the lower layer. In other words, flow of lithospheric mass trough the boundary \( z = 0 \) is allowed.

5. Conclusions
Here we applied the lubrication approximation to the flow of two stacked layers of Newtonian liquids. We conclude that the needed hypothesis are the same as in the usual case of only one layer: the ratio between the characteristic horizontal and vertical scales must be much lesser than one, the horizontal gradient of the pressure and the vertical gradient of the viscous stress are balanced, and the Reynolds number must be of the order of the unity or less. But, to apply the lubrication approximation to both layers we must require in addition that the viscosity and density ratios are of the order of unity.

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