EXISTENCE OF SOLUTIONS FOR DUAL SINGULAR INTEGRAL EQUATIONS WITH CONVOLUTION KERNELS IN CASE OF NON-NORMAL TYPE

Pingrun Li¹,†

Abstract This paper is devoted to the study of dual singular integral equations with convolution kernels in the case of non-normal type. Via using the Fourier transforms, we transform such equations into Riemann boundary value problems. To solve the equation, we establish the regularity theory of solvability. The general solutions and the solvable conditions of the equation are obtained. Especially, we investigate the asymptotic property of solutions at nodes. This paper will have a significant meaning for the study of improving and developing complex analysis, integral equations and Riemann boundary value problems.

Keywords Singular integral equations, Riemann boundary value problems, convolution kernel, regularity theory, dual equations.

MSC(2010) 45E10, 45E05, 30E25.

1. Introduction

It is well known that there are rather complete investigations on the method of solution for equations of Cauchy type as well as integral equations of convolution type. Singular integral equations and Riemann boundary value problems have a lot of applications, e.g. in elasticity theory, fluid dynamics, quantum mechanics. In recent years, many mathematicians have studied singular integral equations and formed a relatively systematic theoretical system (see [1, 4, 6, 7, 10, 29, 30] and references therein). [5] first began to study singular integral equation of Wiener-Hopf type with continuous coefficients. [11] discussed the Noether theory of singular integral equations of convolution type. [14, 16, 18, 19, 25–27] dealt with the invertibility of singular integral operators with discontinuous coefficients, and then considered the solvability theory and the general solutions for some classes of singular integral equations with convolution kernels on the whole real axis (or, on the unit circle) in the case of normal-type. For operators containing both Cauchy principal value integral and convolution, the conditions of their Noethericity were discussed in [8, 23, 28, 33] in more general cases. For applications, the problems to find their solutions is very important. Therefore, singular integral equations of convolution type, mathematically, belong to an interesting subject in the theory of integral equations.

¹The corresponding author. Email address: lipingrun@163.com(P. Li)
²School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
Motivated by the above works, we investigate the existence of solutions for one class of dual singular integral equations with convolution kernels in the cases of non-normal type. In the process of studying equations, we find that the methods used in [5, 10, 29] are no longer suitable for the case of non-normal type, that is, it is difficult to use only the Fourier transform technique to study the case of non-normal type. Hence, we shall introduce a new method to complete our research. In this paper we apply Fourier analysis theory and boundary value method in the theory of analytic functions to deal with the solvability of the equations. Our approach is novel and effective, different from the ones in classical cases. Therefore, this paper generalizes and improves the theories of integral equations and the classical Riemann boundary value problems.

2. Some classes of functions and Fourier transforms

In this section, we present some definitions and lemmas, and we mainly introduce the concepts of classes \{\{0\}\} (\{0\}, <0 \gg) and \{0\} (\{0\}, <0 >).

**Definition 2.1.** The Fourier transform \( \mathcal{F} \) and the inverse transform \( \mathcal{F}^{-1} \) are defined as follows

\[
(\mathcal{F} f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{ist} dt; \quad (\mathcal{F}^{-1} f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s) e^{-ist} ds. \tag{2.1}
\]

For simplification, in (2.1), we denote them as \( F(s) = (\mathcal{F} f)(s), \ f(t) = (\mathcal{F}^{-1} f)(t) \), respectively.

**Definition 2.2.** We say that \( F(s) \in \{\{0\}\} \), if (1) \( F(s) \in \mathcal{H} \), i.e., it satisfies the Hölder condition on the whole real domain \( \mathbb{R} = \mathbb{R} \cup \{\infty\} \); (2) \( F(s) \in L^2(\mathbb{R}) \).

**Definition 2.3.** A function \( f(t) \in \{0\} \), if its Fourier transform \( F(s) \) belongs to \{\{0\}\}.

**Definition 2.4.** Let \( F(s) \) be continuous on \( \mathbb{R} \), if the following conditions are fulfilled: (1) \( F(s) \in \mathcal{H} \); (2) \( F(s) = O(|s|^{-\sigma}), \sigma > \frac{1}{2} \), where \( |s| \) is sufficiently large, then we call \( F(s) \in ((0))^{\sigma} \) or \((0))\).

If \( F(s) \in ((0))^{\sigma} \) or \((0))\), we call that \( f(t) \in (0)^{\sigma} \) or \((0))\).

**Definition 2.5.** If (1) \( F(s) \in \mathcal{H} \); (2) \( F(s) \in H^1(\mathbb{R}) \), \( \sigma > \frac{1}{2} \), i.e., it belongs to \( \mathcal{H} \) in the neighborhood \( N_{\infty} \) of \( \infty \), and \( F(\infty) = 0 \), then we call \( F(s) \in \ll 0 \gg^{\sigma} \) or \( \ll 0 \gg \), and \( f(t) \in <0 >^{\sigma} \) or \( <0 > \).

**Definition 2.6.** For two functions \( k(t) \) and \( f(t) \), their convolution is defined by

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} k(t-\tau) f(\tau) d\tau, \quad -\infty < t < +\infty, \tag{2.2}
\]

we denote it as \( k \ast f \). It is well known that [5, 29]

\[
\mathcal{F}(k \ast f(t)) = \mathcal{F}k(t) \cdot \mathcal{F}f(t) = K(s)F(s). \tag{2.3}
\]

**Definition 2.7.** We also introduce the operator \( \epsilon \) of Cauchy principal value integral

\[
\epsilon f(t) = \text{P.V.} \frac{1}{\pi t} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau-t} d\tau = \lim_{\epsilon \to 0} \frac{1}{\pi t} \int_{|\tau-t|>\epsilon} \frac{f(\tau)}{\tau-t} d\tau, \quad -\infty < t < +\infty. \tag{2.4}
\]
It follows from [8,16,33] that \( \epsilon \) maps \( \{0\} \) and \( < 0 > \) into themselves respectively and \( \epsilon^2 = I \) (identity).

**Definition 2.8.** We define operators \( N \) and \( S \) as follows

\[
Nf(t) = f(-t), \quad Sf(t) = f(t)sgn t, \quad -\infty < t < +\infty.
\] (2.5)

Lemmas 2.1 and 2.2 are obvious facts and we omit their proof here.

**Lemma 2.1.** (1) If \( k, f \in \{0\} \ ( < 0 >) \), then \( k * f \in \{0\} \ ( < 0 >) \); (2) If \( f \in \{0\} \) and \( k \in (0) \ ( < 0 >) \), then \( k * f \in (0) \ ( < 0 >) \).

**Lemma 2.2** (see [8,33]). The operators \( \mathcal{F}, \mathcal{F}^{-1}, \epsilon, N, S \) are as the before, then we have

(1) \( N^2 = S^2 = I \); (2) \( \mathcal{F}^2 = N \); (3) \( \mathcal{F}S = \epsilon \mathcal{F} \); (4) \( SN = -NS \); (5) \( \mathcal{F}^{-1} = NF = FN \).

The following lemma 2.3 plays an important role and it is used to get our some results in this paper.

**Lemma 2.3.** Let \( f(t) \in \{0\} \), \( F(s) = \mathcal{F}f(t) \), then we have

\[
\mathcal{F}[\epsilon f(t)] = -SF(s), \text{ i.e., } \mathcal{F}\epsilon = -SF.
\] (2.7)

**Proof.** By Lemma 2.3, we have \( \epsilon = \mathcal{F}SF^{-1} \), but \( \mathcal{F}^{-1} = NF = FN \), \( \mathcal{F}^2 = N \), thus we obtain \( \mathcal{F}\epsilon = -SF \).

**Lemma 2.4.** If \( f \in \{0\} \), \( (0) \ or \ < 0 > \ and \ \mathcal{F}f(0) = 0 \), then \( \epsilon f \) belongs to the same class.

**Proof.** By Lemma 2.3 and assumptions, and note that

\[
\mathcal{F}f(\infty) = \mathcal{F}f(0) = 0,
\] (2.8)

thus Lemma 2.4 can be proved.

In Lemma 2.4, note that \( F(0) = 0 \) is a necessary condition, otherwise the lemma is invalid.

In the following section, we shall focus on the theory of Noether solvability and the methods of solution for dual singular integral equations with convolution kernels in the non-normal type case.

### 3. Singular integral equations of dual type

Consider the equation

\[
\begin{aligned}
& \begin{cases}
  a_1\omega(t) + \frac{b_1}{\pi} \int_{-\infty}^{+\infty} \frac{\omega(\tau)}{\tau-t}d\tau + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} k_1(t-\tau)\omega(\tau)d\tau = g(t), & 0 < t < +\infty; \\
  a_2\omega(t) + \frac{b_2}{\pi} \int_{-\infty}^{+\infty} \frac{\omega(\tau)}{\tau-t}d\tau + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} k_2(t-\tau)\omega(\tau)d\tau = g(t), & -\infty < t < 0.
\end{cases}
\end{aligned}
\] (3.1)

where \( a_j, b_j (j = 1, 2) \) are constants and \( b_1, b_2 \) are not equal to zero simultaneously. \( k_1, k_2, g \leq 0 > \beta \) ( or \( 0^\beta > \beta < 1 \) and the unknown function \( \omega(t) \) is required to be in \( \{0\} \). After simplification, (3.1) may be written as

\[
\begin{aligned}
& \begin{cases}
  a_1\omega(t) + b_1\epsilon\omega(t) + k_1 * \omega(t) = g(t), & 0 < t < +\infty; \\
  a_2\omega(t) + b_2\epsilon\omega(t) + k_2 * \omega(t) = g(t), & -\infty < t < 0.
\end{cases}
\end{aligned}
\] (3.2)
Extending $t$ in the first equation of (3.2) to $-\infty < t < 0$, and in the second one of (3.2) to $0 < t < +\infty$, i.e., we add $-\phi_-(t)$ and $+\phi_+(t)$ to (3.2), then (3.2) can be rewritten as

$$
\begin{align*}
\begin{cases}
    a_1\omega(t) + b_1\epsilon\omega(t) + k_1 \ast \omega(t) = g(t) - \phi_-(t); \\
    a_2\omega(t) + b_2\epsilon\omega(t) + k_2 \ast \omega(t) = g(t) + \phi_+(t),
\end{cases}
\end{align*}
$$

(3.3)

where

$$
\phi_+(t) = \begin{cases}
    \phi(t), & t \geq 0, \\
    0, & t < 0,
\end{cases}
\quad \phi_-(t) = \begin{cases}
    0, & t \geq 0, \\
    -\phi(t), & t < 0,
\end{cases}
$$

and $\phi \in \{0\}$ is an undetermined function, obviously $\phi(t) = \phi_+(t) - \phi_-(t)$.

We firstly use the Fourier transform to convert Eq.(3.3) into a Riemann boundary value problem. By Lemmas 2.2 and 2.3, we get

$$
\begin{align*}
\begin{cases}
    \Psi^+(s) + G(s) = E_2(s)\Omega(s); \\
    \Psi^-(s) + G(s) = E_1(s)\Omega(s),
\end{cases}
\end{align*}
$$

(3.4)

where

$$
\Omega = \mathcal{F}\omega, \quad G = \mathcal{F}g, \quad K_j = \mathcal{F}k_j, \quad \Psi^\pm = \mathcal{F}\phi^\pm, \quad E_j(s) = a_j - b_j\text{sgn}s + K_j(s), \quad j = 1, 2.
$$

Note that, from equation (3.3) to equation (3.4), by taking the Fourier transform to $-\phi_-(t)$ we have

$$
\begin{align*}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-\phi_-(t)) e^{ist} dt &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (-\phi_-(t)) e^{ist} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} (-\phi_-(t)) e^{ist} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \phi(t) e^{ist} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi_-(t) e^{ist} dt = \Psi^-(s).
\end{align*}
$$

From (3.4) we have

$$
\frac{\Psi^+(s) + G(s)}{E_2(s)} = \frac{\Psi^-(s) + G(s)}{E_1(s)} = \Omega(s).
$$

(3.5)

Thus, we should only solve the following Riemann boundary value problem (3.6) in place of (3.1).

$$
\Psi^+(s) = E(s)\Psi^-(s) + W(s), \quad -\infty < s < +\infty,
$$

(3.6)

in which

$$
E(s) = \frac{E_2(s)}{E_1(s)}, \quad W(s) = (E(s) - 1)G(s).
$$

(3.7)

Now we assume that $E_1(s)$ has some zero-points $c_1, c_2, \cdots, c_n$ with the orders $\xi_1, \xi_2, \cdots, \xi_n$ respectively; $E_2(s)$ has some zero-points $c_1, c_2, \cdots, c_q$ with the orders $\eta_1, \eta_2, \cdots, \eta_q$ respectively, where $\xi_j, \eta_j$ are the non-negative integers. In this case, we say that (3.6) is a Riemann boundary value problem of non-normal type.

Put

$$
\sum_{j=1}^{n} \xi_j = n_1, \quad \sum_{j=1}^{q} \eta_j = n_2, \quad V_1(s) = \Pi_{j=1}^{n}(s - c_j)^{\xi_j}, \quad V_2(s) = \Pi_{j=1}^{q}(s - c_j)^{\eta_j},
$$
then we can rewrite (3.6) in the form
\[ \Psi^+(s) = \frac{V_2(s)}{V_1(s)} D(s) \Psi^-(s) + W(s), \quad -\infty < s < +\infty, \] (3.8)
where \( E(s) = \frac{V_2(s)}{V_1(s)} D(s) \) and \( D(s) \neq 0 \). In view of the values of \( a_j \pm b_j \), we have the following several cases.

1. If \( a_1 \pm b_1 \neq 0 \), and \( a_2 \pm b_2 \) are not equal to zero simultaneously, then (3.6) is a Riemann boundary value problem with nodes \( s = 0, \infty \).
2. If \( a_j - b_j = 0, a_j + b_j \neq 0 (j = 1, 2) \), then (3.6) is a Riemann boundary value problem with node \( s = 0 \).
3. If \( a_1 \pm b_1 \neq 0, a_2 \pm b_2 = 0 \), then (3.6) is a Riemann boundary value problem with node \( s = \infty \).

Without loss of generality, we only consider the case (1). Other cases can be discussed similarly. Since \( \omega(t) \in \{0\} \), thus \( \Omega(s) = \mathcal{F}\omega(t) \in \{0\} \), and by [10] we must have \( \mathcal{F}\omega(0) = 0 \).

Thus the solution \( \Psi(s) \) of (3.6) should be at least continuous along the whole real axis and
\[ \Psi^\pm(0) = -G(0). \] (3.9)
We denote
\[ \gamma_0 = \alpha_0 + i\beta_0 = \frac{1}{2\pi i} \ln D(s)|^{s=0}_{-0}. \] (3.10)
Define by \( \kappa = [\alpha_0] \) the index of the problem (3.6), then we have \( 0 \leq \alpha = \alpha_0 - \kappa < 1 \).

Set
\[ \gamma = \gamma_0 - \kappa = \alpha + i\beta_0. \] (3.11)

Next we discuss the solvability of (3.8). We first define a sectionally holomorphic function \( X(z) \):
\[ X(z) = \begin{cases} e^{\Gamma(z)}, & z \in \mathbb{C}^+; \\ \frac{(z+i)^{n_1}}{(z-i)^{n_2}} e^{\Gamma(z)}, & z \in \mathbb{C}^- \end{cases} \] (3.12)
in which we have put
\[ \Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln D_0(t) \frac{dt}{t-z}, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^- \] (3.13)
and
\[ D_0(t) = \left( \frac{t+i}{t-i} \right)^\kappa D(t), \]
here we have taken the definite branch of
\[ \ln D_0(t) = \kappa \ln \frac{t+i}{t-i} + \ln D(t), \]
provided we have chosen \( \ln \frac{t+i}{t-i} \big|_{t=\pm i} = \pm i\pi \). It is easy to verify that \( X(z) \) is a canonical function and its boundary values satisfy
\[ \frac{X^+(s)}{X^-(s)} = \frac{(s-i)^{n_2}}{(s+i)^{n_1}} D_0(s). \] (3.14)
Thus, (3.8) could also be rewrite as

$$\Psi^+(s) = \frac{V_2(s)(s + i)^{n_1}X^+(s)}{V_1(s)(s-i)^{n_2}X^-(s)}\Psi^-(s) + W(s), \quad -\infty < s < +\infty. \quad (3.15)$$

We again put

$$\gamma_\infty = \alpha_\infty + i\beta_\infty = \frac{1}{2\pi i}\ln D(s)\bigg|_{-\infty}^{+\infty}, \quad (3.16)$$

where \(\ln D(s)\) is taken to be continuous branch for \(s > 0\) and \(s < 0\) respectively such that it is continuous at \(s = \infty\), and \(0 \leq \alpha_\infty < 1\). Without loss of generality, we assume \(a_1b_2 \neq a_2b_1\), then \(\gamma_\infty \neq 0\). If \(a_1b_2 = a_2b_1\), the only difference lies in that \(\gamma_\infty\) and \(\gamma\) may be zero, then in which cases the analysis will be even simpler, here we do not discuss it. We first consider the homogeneous problem of (3.15) given by

$$\Psi^+(s) = \frac{V_2(s)(s + i)^{n_1}X^+(s)}{V_1(s)(s-i)^{n_2}X^-(s)}\Psi^-(s). \quad (3.17)$$

Via using the principle of analytic continuation \([16, 27]\), we obtain an analytic solution of (3.17):

$$Y_1(z) = \begin{cases} X(z)V_2(z)(z+i)^{n_1}P_\vartheta(z), & z \in \mathbb{C}^+, \\ X(z)V_1(z)(z-i)^{n_2}P_\vartheta(z), & z \in \mathbb{C}^- \end{cases}, \quad (3.18)$$

in (3.18), when \(\vartheta \geq 0\), \(P_\vartheta(z)\) is a polynomial of degree \(\vartheta\) with arbitrary complex coefficients; when \(\vartheta < 0\), \(P_\vartheta(z) \equiv 0\), where \(\vartheta = \kappa - n_1 - n_2\).

Now we solve the non-homogeneous problem (3.15) in class \(\{0\}\). To do this, we consider the following function

$$\eta(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{V_1(s)W(s)}{(s+i)^{n_1}X^+(s)(s-z)}ds, \quad \forall z \in \mathbb{C}^+ \cup \mathbb{C}^- \quad (3.19)$$

We will apply Sokhotski-Plemelj formula and generalized Liouville theorem \([16, 33]\) to the boundary value problem (3.15), which has a singularity at \(e_j, c_k\). Therefore, we need to construct a Hermite interpolation polynomial \(H_\rho(z)\) with the degree \(\rho\), and we can assume that

$$H_\rho(z) = \sum_{j=0}^{\rho} A_j z^{\rho-j}, \quad \rho = n_1 + n_2 - 1$$

which has some zero-points of the orders \(\xi_j, \eta_k\) (1 \(\leq j \leq n, 1 \leq k \leq q\) at \(e_j, c_k\), respectively, where \(A_j (0 \leq l \leq \rho)\) are constants.

Making use of (3.19) and \(H_\rho(z)\), we can define the following function:

$$Y_2(z) = \begin{cases} \frac{X(z)(z+i)^{n_1}[\eta(z)(z+i)^{n_1}H_\rho(z)]}{V_1(z)}, & z \in \mathbb{C}^+, \\ \frac{X(z)(z-i)^{n_2}[\eta(z)(z+i)^{n_1}H_\rho(z)]}{V_2(z)}, & z \in \mathbb{C}^- \end{cases}, \quad (3.20)$$

By means of the classical Riemann boundary value problem, we can verify that (3.20) is the particular solution of (3.15). In view of the solvability of linear equations, we obtain a general solution of (3.15):

$$\Psi(z) = \sum_{j=1}^{2} Y_j(z). \quad (3.21)$$
From (3.18) and (3.20), \( \Psi(z) \) can also be written as the explicit solution:

\[
\Psi(z) = \begin{cases} 
\frac{X(z)(z+i)^{\eta}}{(z+i)-V_1(z)}[\eta(z)(z+i)^{\eta} - H_{\rho}(z) + V_1(z)V_2(z)P_0(z)], & z \in \mathbb{C}^+; \\
\frac{X(z)(z-i)^{\eta}}{(z-i)+V_2(z)}[\eta(z)(z+i)^{\eta} - H_{\rho}(z) + V_1(z)V_2(z)P_0(z)], & z \in \mathbb{C}^-.
\end{cases}
\]

(3.22)

In the following, we discuss the conditions of solvability and the properties of solution for Eq. (3.15).

First, we consider the behaviors of solution near \( s = 0 \). Similar to the discussion in [2, 14], by using Sokhotski-Plemelj formula to \( X(z) \) in (3.12), we can obtain

\[
X^+(s) = \sqrt{D_0(s)}e^{\Gamma(s)}, \quad X^-(s) = \frac{e^{\Gamma(s)}}{\sqrt{D_0(s)}},
\]

(3.23)

where \( \sqrt{D_0(s)} = \exp\{\frac{1}{2}\log D_0(s)\} \) has a definite value.

If \( s = 0 \) is an ordinary node, then \( 0 < \alpha < 1 \) and \( \gamma \neq 0 \). It is easy to verify that, in the neighborhood of \( s = 0 \),

\[
\Psi^+(+0) = \frac{e^{3\gamma i}W(+0) - W(-0)}{2i\sin\gamma\pi}e^{-2\gamma i},
\]

\[
\Psi^+(-0) = \frac{e^{3\gamma i}W(+0) - W(-0)}{2i\sin\gamma\pi}e^{-\gamma i}.
\]

(3.24)

By using (3.9) and \( e^{\gamma i} \neq 1 \), from (3.24) we can obtain

\[
\frac{W(+0)}{W(-0)} = e^{-3\gamma i}.
\]

(3.25)

If \( s = 0 \) is a special node, since \( \Psi(s) \) is continuous at \( s = 0 \), we should have the following conditions of solvability

\[
u_0 = \frac{1}{V_1(0)V_2(0)}[v_0 - \frac{i^{\varepsilon-1}}{2\pi} \int_{-\infty}^{+\infty} \frac{V_1(s)W(s)}{X^+(s)(s+i)^{\eta_1}s}ds]
\]

(3.26)

as well as

\[
\mathcal{F}g(0) = 0, \text{ i.e., } G(0) = 0,
\]

(3.27)

where \( u_0, v_0 \) are the constant terms of \( P_0(z), H_{\rho}(z) \), respectively.

Second, we consider the property of solution at \( s = \infty \). Note that, it follows from [10, 26, 28] that, near \( s = \infty \),

\[
X(s) = \frac{\chi(s)}{|s|^{\alpha_{\infty}}}
\]

(3.28)

and \( \chi(s) \in H(N_{\infty}) \), i.e., \( \chi(s) \) satisfies the Hölder condition in the neighbourhood \( N_{\infty} \) of \( \infty \).

If \( s = \infty \) is an ordinary node, i.e., \( 0 \leq \alpha_{\infty} < 1 \), \( \gamma_{\infty} \neq 0 \). Due to (3.27) and \( W(s) \in \mathcal{H} \), we have \( \eta(s) \in \mathcal{H} \), so, when \( \frac{1}{2} < \alpha_{\infty} < \beta < 1 \), we have \( \lim_{s \to \infty} X(s)\eta(s)s^{\alpha_{\infty}} = 0 \). This implies

\[
X(s)\eta(s) = o\left(\frac{1}{|s|^{\alpha_{\infty}}} \right) \quad (s \to \infty).
\]

(3.29)
When \( \frac{1}{2} < \beta \leq \alpha_{\infty} < 1 \), by [8, 19] we know that \( X(s)\eta(s)s^{-\varepsilon+\alpha_{\infty}} \) is bounded at \( s = \infty \), thus we have
\[
X(s)\eta(s) = O\left( \frac{1}{|s|^{-\varepsilon+\alpha_{\infty}}} \right) \quad (s \to \infty),
\]
(3.30)

where \( \varepsilon > 0 \) is arbitrarily small such that \( -\varepsilon + \alpha_{\infty} > \frac{1}{2} \). Again set
\[
B(s) = \frac{X^+(s)(s+i)^{n_1-\kappa}}{V_1(s)}[V_1(s)V_2(s)P\rho(s) - H\rho(s)].
\]
(3.31)

We now discuss the asymptotic property of \( B(s) \) at \( s = \infty \), and when \( \vartheta \geq 0 \), we know that \( \kappa \geq n_1 + n_2 \), and \( \kappa > \rho = n_1 + n_2 - 1 \), so we obtain that the following formula
\[
\frac{(s+i)^{n_1-\kappa}}{V_1(s)}[V_1(s)V_2(s)P\rho(s) - H\rho(s)]
\]
is bounded at \( s = \infty \). From (3.12), (3.23), (3.28), and again by [11, 29], we have
\[
|X^+(s)| \leq A \left| \frac{1}{s^{\alpha_{\infty}}} \right|, \quad (s \to \infty),
\]
where \( A \in \mathbb{R}^+ \). Therefore, we get
\[
|B(s)X^+(s)| \leq A,
\]
that is,
\[
B(s) = O\left( \frac{1}{|s|^{\alpha_{\infty}}} \right) \quad (s \to \infty).
\]
(3.32)

Similar to the previous discussion, we have the following results:
when \( \vartheta < 0 \), since \( \Psi(z) \) is bounded at \( z = \infty \), one must have
\[
A_j = 0, \quad \forall j \in \{0, 1, \ldots, -\vartheta - 1\},
\]
(3.33)

moreover, when \( \kappa > 0 \), we get
\[
B(s) = o\left( \frac{1}{|s|^{\alpha_{\infty}}} \right) \quad (s \to \infty);
\]
(3.34)

when \( \kappa < 0 \), we require that (3.26) holds, and to eliminate the singularity of \( \Psi(z) \) at \( c_k, d_j \), we also have
\[
\int_{-\infty}^{+\infty} \frac{V_1(s)W(s)ds}{X^+(s)(s+i)^{n_1}(s-c_k)^r} = 0, \quad r = 1, 2, \ldots, \eta_k, \quad k = 1, 2, \ldots, q,
\]
\[
\int_{-\infty}^{+\infty} \frac{V_1(s)W(s)ds}{X^+(s)(s+i)^{n_1}(s-e_j)^p} = 0, \quad p = 1, 2, \ldots, \xi_j, \quad j = 1, 2, \ldots, n;
\]
(3.35)

when \( \kappa = 0 \), we require that (3.35) and the following (3.36) are fulfilled
\[
\eta(c_k) = \frac{u_0}{c_k + i}, \quad \forall k = 1, 2, \ldots, q;
\]
\[
\eta(e_j) = \frac{v_0}{d_j + i}, \quad \forall j = 1, 2, \ldots, n.
\]
(3.36)

Thus, when \( \alpha_{\infty} > \frac{1}{2} \), we have
\[
\Psi(s) = o\left( \frac{1}{|s|^\vartheta} \right) \quad (s \to \infty),
\]
(3.37)
where \( v > \min\{\beta, -\varepsilon + \alpha_\infty\} \); when \( \alpha_\infty \leq \frac{1}{2} \), discussions may be made fully analogous to those in [2, 9, 19, 33].

If \( s = \infty \) is a special node, then \( \alpha_\infty = 0, \gamma_\infty \neq 0 \), one can translate it into the case that \( \alpha_\infty < \frac{1}{2} \). Similar arguments can be used [14, 31, 33, 34]. Note that when \( \kappa < 0 \), in order to eliminate a singularity of \( \Psi(z) \) at \( z = -i \), one must have

\[
\int_{-\infty}^{+\infty} \left( \frac{E_2(s)}{E_1(s)} - 1 \right) \frac{V_1(s)}{X'(s)} \frac{G(s)}{(s+i)^{n+\tau}} ds = 0, \quad r = 1, 2, \ldots, -\kappa. \tag{3.38}
\]

In conclusion, we can formulate the main results about solutions of Eq. (3.1) in the following form.

**Theorem 3.1.** Under suppositions \( a_1 \pm b_1 \neq 0 \), in the case of non-normal type, the necessary condition of solvability to Eq. (3.1) is (3.27) in class \( \{\varnothing\} \). Assume that this is fulfilled.

1. If \( s = 0 \) is an ordinary node, then (3.25) holds; if \( s = 0 \) is a special node, then (3.26) and (3.27) hold.

2. Let \( s = \infty \) be an ordinary node, if \( \alpha_\infty > \frac{1}{2} \), one require that (3.29), (3.30), and (3.37) hold, then Eq. (3.1) has a solution; if \( \alpha_\infty \leq \frac{1}{2} \), when \( \vartheta > 0 \), we rewrite \( P_{\vartheta-1}(s) \) instead of \( P_0(s) \) in (3.22), then Eq. (3.1) has \( \vartheta - 1 \) linearly independent solutions; when \( \vartheta \leq 0 \), (3.33) holds. Moreover, when \( \kappa > 0 \), (3.34) holds; when \( \kappa < 0 \), (3.26) and (3.35) hold; when \( \kappa = 0 \), (3.35) holds, then Eq. (3.1) has the unique solution.

Let \( s = \infty \) be a special node, if \( \kappa > 0 \), then \( A_{\vartheta-1} = 0 \); if \( \kappa < 0 \), then (3.35) and (3.38) hold; if \( \kappa = 0 \), the discuss is similar to the case that \( \alpha_\infty > \frac{1}{2} \), then Eq. (3.1) has a unique solution.

3. If \( \vartheta > 0 \), Eq. (3.1) has \( \vartheta \) linearly independent solutions; if \( \vartheta \leq 0 \), Eq. (3.1) has a unique solution.

Thus (3.1) has the general solution

\[
\omega(t) = F^{-1}\Omega(s), \tag{3.39}
\]

where \( \Omega(s) \) is given by (3.5).

Finally, we give the following two remarks.

**Remark 3.1.** In Eq.(3.1), if \( k_1, k_2, g \in < 0 > \), then \( \omega \in < 0 > \). Similarly, if \( k_1, k_2, g \in < 0 >^\sigma \), then \( \omega \in < 0 >^\sigma \), where \( 0 < \sigma < 1 \).

**Remark 3.2.** Indeed, we can also investigate the solvability of Eq. (3.1) in Clifford analysis, and the stability of solution for Eq. (3.1) (see [3, 12, 13, 15, 17, 20–22, 24, 32]). Further discussion is omitted here.

**Acknowledgment**

The authors are very grateful to the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper.

**References**

[1] H. Begehr and T. Vaitekhovich, *Harmonic boundary value problems in half disc and half ring*, Functions et Approximation, 2009, 40(2), 251–282.
[2] I. Belmoulouda and A. Memoub, *On the solvability of a class of nonlinear singular parabolic equation with integral boundary condition*, Appl. Math. Comput., 2020, 373, 124999.

[3] Z. Blocki, *Suita conjecture and Ohsawa-Takegoshi extension theorem*, Invent. Math., 2013, 193, 149–158.

[4] L. Chuan, N. V. Mau and N. Tuan, *On a class of singular integral equations with the linear fractional Carleman shift and the degenerate kernel*, Complex Var. Elliptic Equ., 2008, 53(2), 117–137.

[5] R. V. Duduchava, *Integral equations of convolution type with discontinuous coefficients*, Math. Nachr., 1977, 79, 75–78.

[6] M. C. De-Bonis and C. Laurita, *Numerical solution of systems of Cauchy singular integral equations with constant coefficients*, Appl. Math. Comput., 2012, 219, 1391–1410.

[7] H. Du and J. Shen, *Reproducing kernel method of solving singular integral equation with cosecant kernel*, J. Math. Anal. Appl., 2008, 348(1), 308–314.

[8] C. Gomez, H. Prado and S. Trofimchuk, *Separation dichotomy and wavefronts for a non-linear convolution equation*, J. Math. Anal. Appl., 2014, 420, 1–19.

[9] K. Kant and G. Nelakanti, *Approximation methods for second kind weakly singular Volterra integral equations*, J. Comput. Appl. Math., 2020, 368, 112531.

[10] G. S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift*, London: Kluwer Academic Publishers, 2004.

[11] J. Lu, *Boundary Value Problems for Analytic Functions*, Singapore: World Scientific, 2004.

[12] P. Li, *One class of generalized boundary value problem for analytic functions*, Bound. Value Probl., 2015, 2015, 40.

[13] P. Li, *Generalized boundary value problems for analytic functions with convolutions and its applications*, Math. Meth. Appl. Sci., 2019, 42, 2631–2645.

[14] P. Li and G. Ren, *Some classes of equations of discrete type with harmonic singular operator and convolution*, Appl. Math. Comput., 2016, 284, 185–194.

[15] P. Li, *Some classes of singular integral equations of convolution type in the class of exponentially increasing functions*, J. Inequal. Appl., 2017, 2017, 307.

[16] P. Li, *Generalized convolution-type singular integral equations*, Appl. Math. Comput., 2017, 311, 314–323.

[17] P. Li, *Two classes of linear equations of discrete convolution type with harmonic singular operators*, Complex Var. Elliptic Equ., 2016, 61(1), 67–75.

[18] P. Li and G. Ren, *Solvability of singular integro-differential equations via Riemann-Hilbert problem*, J. Differential Equations, 2018, 265, 5455–5471.

[19] P. Li, *Singular integral equations of convolution type with Cauchy kernel in the class of exponentially increasing functions*, Appl. Math. Comput., 2019, 344–345, 116–127.

[20] P. Li, *Singular integral equations of convolution type with Hilbert kernel and a discrete jump problem*, Adv. Difference Equ., 2017, 2017, 360.

[21] P. Li, *Solvability of some classes of singular integral equations of convolution type via Riemann-Hilbert problem*, J. Inequal. Appl., 2019, 2019, 22.
[22] P. Li, Singular integral equations of convolution type with cosecant kernels and periodic coefficients, Math. Probl. Eng., 2017, https://doi.org/10.1155/2017/6148393.

[23] P. Li, Singular integral equations of convolution type with reflection and translation shifts, Numer. Func. Anal. Opt., 2019, 40(9), 1023–1038.

[24] P. Li, Linear BVPs and SIEs for generalized regular functions in Clifford analysis, J. Funct. Spaces, 2018, https://doi.org/10.1155/2018/6967149.

[25] P. Li, Linear BVPs and SIEs for generalized regular functions in Clifford analysis, J. Funct. Spaces, 2018, https://doi.org/10.1155/2018/6967149.

[26] P. Li, Solvability theory of convolution singular integral equations via Riemann-Hilbert approach, J. Comput. Appl. Math., 2020, 370(2), 112601.

[27] P. Li, On solvability of singular integral-differential equations with convolution, J. Appl. Anal. Comput., 2019, 9(3), 1071–1082.

[28] P. Li, The solvability and explicit solutions of singular integral-differential equations of non-normal type via Riemann-Hilbert problem, J. Comput. Appl. Math., 2020, 374(2), 112759.

[29] N. I. Muskhelishvilli, Singular Integral Equations, NauKa, Moscow, 2002.

[30] T. Nakazi and T. Yamamoto, Normal singular integral operators with Cauchy kernel, Integral Equations Operator Theory, 2014, 78, 233–248.

[31] E. Najafi, Nyström-quasilinearization method and smoothing transformation for the numerical solution of nonlinear weakly singular Fredholm integral equations, J. Comput. Appl. Math., 2020, 368, 112538.

[32] G. Ren, U. Kaehler, J. Shi and C. Liu, Hardy-Littlewood inequalities for fractional derivatives of invariant harmonic functions, Complex Anal. Oper. Theory., 2012, 6(2), 373–396.

[33] N. Tuan and N. T. Thu-Huyen, The solvability and explicit solutions of two integral equations via generalized convolutions, J. Math. Anal. Appl., 2010, 369, 712–718.

[34] Q. Wen and Q. Du, An approximate numerical method for solving Cauchy singular integral equations composed of multiple implicit parameter functions with unknown integral limits in contact mechanics, J. Math. Anal. Appl., 2020, 482, 123530.