DIRECT SIMULATIONS OF HELICAL HALL-MHD TURBULENCE AND DYNAMO ACTION

PABLO D. MININNI
Advanced Study Program, National Center for Atmospheric Research,
P.O. Box 3000, Boulder, CO 80307; mininni@df.uba.ar

DANIEL O. GÓMEZ
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires,
Ciudad Universitaria, 1428 Buenos Aires, Argentina

AND

SWADESH M. MAHAJAN
Institute for Fusion Studies, University of Texas, Austin, TX 78712
Received 2004 June 10; accepted 2004 October 6

ABSTRACT

Direct numerical simulations of turbulent Hall dynamos are presented. The evolution of an initially weak and small-scale magnetic field in a system maintained in a stationary turbulent regime by a stirring force at a macroscopic scale is studied to explore the conditions for exponential growth of the magnetic energy. Scaling of the dynamo efficiency with Reynolds numbers is studied, and the resulting total energy spectra are found to be compatible with a Kolmogorov-type law. A faster growth of large-scale magnetic fields is observed at intermediate intensities of the Hall effect.

Subject headings: accretion, accretion disks — magnetic fields — MHD — stars: magnetic fields — stars: neutron

1. INTRODUCTION

In recent years, the relevance of two-fluid effects has been pointed out in several astrophysical (Balbus & Terquem 2001; Sano & Stone 2002; Mininni et al. 2002, 2003a), as well as laboratory, plasmas (Mirnov et al. 2003). The standard magnetohydrodynamic (MHD) framework for the study of astrophysical plasmas may not be adequate in the presence of strong magnetic fields and/or low ionization; the electric conductivity then is not isotropic, and nonlinear effects arise in Ohm’s law. In low-temperature accretion disks around young stellar objects or in dwarf nova systems in quiescence, for example, the plasma is only partially ionized, with a small abundance of charged particles. As a result, two new effects appear in Ohm’s law: ambipolar diffusion and Hall currents. The predominance of either of these effects is determined by the ionization fraction and the plasma density (Sano & Stone 2002; Braginskii 1965). The impact of ambipolar diffusion on dynamo action has already been studied by Zweibel (1988; see also Brandenburg & Subramanian 2000). The Hall effect affects the dynamics of protostellar disks (Balbus & Terquem 2001). In neutron stars magnetic fields are so strong that the Hall term can be even more important than the induction term for the magnetic field evolution (Muslimov 1994). The Hall effect is also known to be relevant in other astrophysical scenarios (Sano & Stone 2002; Mininni et al. 2003b and references therein).

The Hall effect, when strong, is expected to seriously affect the MHD results on the generation of magnetic fields in astrophysical and laboratory plasmas by inductive motions in a conducting fluid (dynamo effect). More specifically, it is expected to modify the growth and evolution of magnetic energy, since the addition of the Hall term to the MHD equations leads to the freezing of the magnetic field to the electron flow (in the nondissipative limit) rather than to the bulk velocity field. The first studies on the impact of Hall currents on dynamo action (Helmis 1968, 1971) were carried out using mean-field theory and the first-order smoothing approximation (Krause & Rädler 1980). Helmis obtained decreasing dynamo action as the strength of the Hall terms increased. Recently, the impact of the Hall effect has been studied using the kinematic approximation or focusing on particular geometries, both in numerical simulations (Galanti et al. 1994) and in experimental and theoretical studies (Heintzmann 1983; Ji 1999; Rheinhardt & Geppert 2002; Mirnov et al. 2003). A general closure scheme was also proposed to compute the contribution of the Hall term to the dynamo action \( \alpha \)–effect (Mininni et al. 2002) using mean-field theory and the reduced smoothing approximation (Blackman & Field 1999); it was found that the Hall effect could either suppress or enhance dynamo action. Expressions for the turbulent diffusivity using this closure were also derived for particular cases in Mininni et al. (2003a).

In the present work we report results from direct numerical simulations of the dynamo action in MHD and Hall-MHD at moderate Reynolds numbers with strong kinetic helical forcing. The study of the scaling of the dynamo efficiency with increasing Reynolds numbers is the main aim of this paper. In a previous paper (Mininni et al. 2003b), we showed that three distinct dynamo regimes can be clearly identified: (1) dynamo activity is enhanced (Hall-enhanced regime), (2) it is inhibited (Hall-suppressed regime), and (3) it asymptotically approaches the MHD value (MHD regime). These regimes arise as a result of the relative ordering between the relevant length scales of the problem, namely, the energy-containing scale of the flow, the Hall length, and the correlation length of the magnetic seed.
Simulations in Mininni et al. (2003b) were performed with \(64^3\) spatial grid points and fixed Reynolds numbers. This study left unanswered the question of whether the enhancement of dynamo action by the Hall effect would increase or decrease with increasing Reynolds numbers and scale separation.

Theoretical estimates using mean-field theory and a particular choice for the small-scale fields suggest that the efficiency of Hall-MHD dynamos compared with the MHD counterpart will increase as the scale separation is increased (Mininni et al. 2002). In this work we present direct simulations with higher spatial resolutions and Reynolds numbers that confirm this result. In addition, the magnetic, kinetic, and total energy spectra developed in Hall-MHD turbulence are calculated and studied. These spectra correspond to the dynamo regime, in which no imposed currents are present, i.e., the magnetic fields are purely self-generated. This regime is rather general and probably common to most astrophysical flows (Haugen et al. 2004).

The paper is organized as follows: In § 2 we present the general equations describing the evolution of the fields. In § 3 the code used to numerically integrate the Hall-MHD system is described, and all the simulations made with different resolutions and parameters are listed. Section 4 presents the results obtained in the MHD limit of our equations; the results obtained are similar to those of other authors (Meneguzzi et al. 1981; Brandenburg 2001) and provide a reference set with which to compare the Hall-MHD solutions. Section 5 is devoted to the results of Hall-MHD simulations. In § 6 we discuss a subset of MHD and Hall-MHD simulations made to study the evolution of large-scale magnetic fields. Finally, in § 7 we give a brief summary of the current effort.

2. THE HALL-MHD SYSTEM

Incompressible Hall-MHD is described by the modified induction and the dissipative Navier-Stokes equations,

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [(\mathbf{U} - \epsilon \nabla \times \mathbf{B}) \times \mathbf{B}] + \eta \nabla^2 \mathbf{B},
\]

\[
\frac{\partial \mathbf{U}}{\partial t} = - (\mathbf{U} \cdot \nabla) \mathbf{U} + (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left( P + B^2 \right) + F + \nu \nabla^2 \mathbf{U},
\]

\[
\nabla \cdot \mathbf{B} = 0 = \nabla \cdot \mathbf{U},
\]

where \(F\) denotes a solenoidal external force. The velocity \(\mathbf{U}\) and the magnetic field \(\mathbf{B}\) are expressed in units of a characteristic speed \(U_0\), \(\epsilon\) measures the relative strength of the Hall effect, and \(\eta\) and \(\nu\) are the (dimensionless) magnetic diffusivity and kinematic viscosity, respectively. Note that the measure of the Hall effect \(\epsilon\) can be written as

\[
\epsilon = \frac{L_{\text{Hall}}}{L_0},
\]

where \(L_0\) is a characteristic length scale (the size of the box in our simulations is equal to \(2\pi\)) and the Hall length,

\[
L_{\text{Hall}} = \frac{c U_A}{\omega_p U_0}.
\]

is given in terms of the characteristic speed \(U_0\) and the characteristic Alfvénic speed \(U_A\). In particular, we are free to choose \(U_0 = U_A\) as our characteristic velocity, reducing \(L_{\text{Hall}}\) to the ion skin depth.

Equation (5) is valid in a fully ionized plasma. We work under this assumption without any loss of generality. In the more general case of partially ionized plasmas, the values of \(L_{\text{Hall}}\) and \(\epsilon\) are those reported by Sano & Stone (2002) or by Mininni et al. (2003b). Typical values of \(\epsilon\) in astrophysics are also mentioned in these papers.

To keep in mind astrophysical scenarios, we recall just three examples. In a protostellar disk the Hall scale is larger than the dissipation scale, typically by 2 orders of magnitude, but smaller than the largest scales of the system (Balbus & Terquem 2001). Therefore, we expect \(\epsilon < 1\) but with \(L_{\text{Hall}}\) larger than the dissipation scale. In some dwarf nova disks and protoplanetary disks \(\epsilon \approx 1\) (Sano & Stone 2002). As previously mentioned, in neutron stars \(\epsilon > 1\) (Muslimov 1994).

The Hall-MHD system has three well-known ideal (\(\eta = \nu = 0\)) quadratic invariants,

\[
E = \frac{1}{2} \int \left( U^2 + B^2 \right) dV,
\]

\[
H_m = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{B} dV,
\]

\[
K = \frac{1}{2} \int (\mathbf{B} + \epsilon \mathbf{\omega}) \cdot (\mathbf{A} + \epsilon \mathbf{U}) dV.
\]

Here \(E\) is the energy, \(H_m\) is the magnetic helicity, and \(K\) is the hybrid helicity, which replaces the cross-helicity from MHD. The vector potential \(\mathbf{A}\) is defined by \(\mathbf{B} = \nabla \times \mathbf{A}\), and \(\mathbf{\omega}\) is the vorticity. Conservation of these ideal invariants during the evolution of the system provides a check on the simulation.

3. THE CODE

The pseudospectral code used in Mininni et al. (2003b) was modified to run in a Beowulf cluster using a message-passing interface. We integrated the Hall-MHD equations (1)–(3) in a cubic box with periodic boundary conditions. The equations were evolved in time using a second-order Runge-Kutta method. The total pressure \(P_T = P + B^2/2\) was computed in a self-consistent fashion at each time step to ensure the incompressibility condition \(\nabla \cdot \mathbf{U} = 0\) (Canuto et al. 1998). In Fourier space, taking the divergence of equation (2) we obtain

\[
(P_T)_k = \frac{i}{k^2} \mathbf{k} \cdot \left[ (\mathbf{U} \cdot \nabla) \mathbf{U} - (\mathbf{B} \cdot \nabla) \mathbf{B} \right]_k,
\]

where \((\ldots)_k\) denotes a spatial Fourier transform of the argument and \(\mathbf{k}\) is the wavenumber vector.

To satisfy the divergence-free condition for the magnetic field, the induction equation (1) was replaced by an equation for the vector potential:

\[
\frac{\partial \mathbf{A}}{\partial t} = (\mathbf{U} - \epsilon \nabla \times \mathbf{B}) \times \mathbf{B} + \epsilon \nabla p_e + \epsilon \nabla^2 \mathbf{A},
\]

where \(p_e\) (electron pressure) was computed at each time step to satisfy the Coulomb gauge \(\nabla \cdot \mathbf{A} = 0\), solving an equation similar to equation (9).

We present results from different runs with \(\eta = 0.05, 0.02,\) and 0.011. For the first value of \(\eta\), simulations with \(64^3\) and \(128^3\) spatial grid points were performed to check convergence. The rest of the simulations were made with \(128^3\) grid points \((\eta = 0.02)\) and \(256^3\) grid points \((\eta = 0.011)\). All the runs were made with magnetic Prandtl number \(\nu/\eta = 1\). Therefore,
hereafter we consider only a single Reynolds number (i.e., both kinetic and magnetic), defined as

$$\text{Re} = \frac{UL_0}{\eta}. \quad (11)$$

In the study of turbulent flows, the number

$$\text{Re}_\lambda = \frac{UL_0}{\eta}. \quad (12)$$

constructed from Taylor’s length scale \(\lambda = \left(\left\langle U^2 \right\rangle / \left\langle \omega^2 \right\rangle \right)^{1/2}\), is often considered. Note that this definition of Taylor’s microscale might differ from other definitions (for instance, in connection with experiments on fluid turbulence) by factors of order unity (Pope 2000). These two Reynolds numbers were respectively \(\text{Re} \approx 100\) and \(\text{Re}_\lambda \approx 20\) for the first value of \(\nu\) and \(\eta\), \(\text{Re} \approx 300\) and \(\text{Re}_\lambda \approx 40\) for the second, and \(\text{Re} \approx 560\) and \(\text{Re}_\lambda \approx 60\) in the last case. The energy injection rate was approximately the same for all these runs.

The simulation begins by subjecting the Navier-Stokes equation to a stationary helical force \(F\) (given by eigenfunctions of the curl operator) operating at a macroscopic scale \(k_{\text{force}} = 3\) (Mininni et al. 2003b) to reach a hydrodynamic turbulent steady state. The resulting statistically steady state is characterized by a positive kinetic helicity. The relative helicity in runs with \(\text{Re} = 300\) is \(2H_k / \left(\left\langle U^2 \right\rangle / \left\langle \omega^2 \right\rangle \right)^{1/2} \approx 0.4\), and this value decreases slightly for larger Reynolds numbers. The kinetic helicity is defined as

$$H_k = \frac{1}{2} \int U \cdot \omega \, dV. \quad (13)$$

Once the hydrodynamic stage of the simulation reached a steady state, a nonhelical but small magnetic seed was introduced. This initial magnetic seed was generated by a \(\delta\)-correlated vector potential centered at \(k_{\text{seed}} = 13\) for the \(\text{Re} = 100\) runs and \(k_{\text{seed}} = 35\) for the \(\text{Re} = 300\) and 560 simulations. The run was continued with the same external helical force in the Navier-Stokes equation, to study the growth of magnetic energy due to dynamo action.

Another set of simulations was made under the same conditions but with \(\nu = \eta = 0.02\) and \(k_{\text{force}} = 10\) (128³ grid points), to study the changes in the growth of the large-scale magnetic field in the presence of the Hall effect. In this case the Reynolds numbers are smaller (\(\text{Re} \approx 220\) and \(\text{Re}_\lambda \approx 20\)), and the turbulence is weaker, since there are not enough modes in Fourier space for a direct cascade to develop properly. On the other hand, there are more Fourier modes to study the inverse cascade and the growth of large-scale fields. The results of these simulations are discussed in § 6.

In all our simulations the Kolmogorov kinetic and magnetic dissipation length scales were properly resolved in the computational domain, i.e., we made sure that the dissipation wavenumbers remained smaller than the maximum wavenumber allowed by the de-aliasing step, namely, \(k_{\text{max}} = 128/3\).

4. MHD DYNAMOS

In this section we briefly present the results from MHD simulations. The results are in good agreement with previous simulations of dynamo action under periodic boundary conditions (Meneguzzi et al. 1981; Brandenburg 2001). These simulations are intended for comparison with the Hall-MHD runs, and therefore some specific results of the MHD simulations are discussed in more detail in the following subsections.

4.1. Magnetic Energy Evolution

In Figure 1 we show the magnetic and kinetic energy as a function of time in MHD runs \((\epsilon = 0)\) for different Reynolds numbers. The turnover time for \(\text{Re} = 300\) is \(\tau = 2\pi / \left(\left\langle k_{\text{force}} U^2 \right\rangle / \left\langle \omega^2 \right\rangle \right)^{1/2} \approx 0.3\).

Two phases can be clearly identified in the evolution of the magnetic energy. After a first stage with exponential growth (which can be considered the kinematic dynamo stage), the magnetic energy saturates and reaches equipartition with the kinetic energy (see Fig. 1). In the first stage, the magnetic energy is still weak, and the velocity field is not strongly affected by the Lorentz force. Note that during this exponential growth of magnetic energy, the kinetic energy remains approximately constant.

This kinematic dynamo stage can be understood at least qualitatively considering the mean-field induction equation (Krause & Rädler 1980)

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times \left(\bar{U} \times \bar{B} + \alpha \bar{B}\right) + \eta_{\text{eff}} \nabla^2 \bar{B}. \quad (14)$$

Here the overbar denotes mean-field quantities, and \(\eta_{\text{eff}}\) is the magnetic plus turbulent diffusivity. The MHD \(\alpha\)-effect (Pouquet et al. 1976),

$$\alpha = \frac{\tau}{3} \left(-\bar{u} \cdot \nabla \times \bar{u} + \bar{b} \cdot \nabla \times \bar{b}\right), \quad (15)$$

represents the back-reaction of the turbulent motions in the mean field and gives the exponential growth of magnetic energy in the kinematic regime for helical turbulence. Here \(\bar{u}\) and \(\bar{b}\) are respectively the fluctuating velocity and magnetic fields, and \(\tau\) is the typical correlation time for the turbulent motions. Attempts to measure this quantity in direct simulations were made by Cattaneo & Hughes (1996) and Brandenburg (2001).

As the Reynolds number increases, the saturation field strength increases, although it seems to reach an asymptotic value. After the saturation, the magnetic energy keeps growing slowly on a resistive timescale (Brandenburg 2001). This late growth takes place mainly at large scales, as is shown in the energy spectrum. As far as we know, there are no simulations of MHD dynamos with periodic boundary conditions showing generation of large-scale fields on shorter times.
Relations is of the form $Re = \frac{v}{c}$ and in good agreement with simulations of helical MHD scale structures, as is shown in the kinetic helicity. This helicity is located mostly in large scales. The mean-field helicity grows with the same sign as the $\alpha$-coefficient (opposite sign from the kinetic helicity). Our results are in good agreement with this relation, as well as with previous simulations of MHD dynamo action (Brandenburg 2001; Mininni et al. 2003b).

Although this generation of magnetic helicity by dynamo action is expected to decrease as the magnetic Reynolds number $Re_m$ increases, in simulations with higher Reynolds numbers the growth of magnetic helicity seems to reach an asymptotic value.

## 5. HALL DYNAMOS

To quantitatively assess the role of the Hall effect on dynamo action, we display results from runs with different values of $\epsilon$: the MHD run ($\epsilon = 0$) and the Hall-MHD runs with $\epsilon = 0.066, 0.1$, and $0.2$. We focus on the Hall-enhanced dynamo regime (Mininni et al. 2003b). Note that all these values of $\epsilon$ correspond to dynamos in which the Hall effect is only relevant in a fraction of the scales involved. The Hall inverse length scale for these runs is measured by $k_H = 15, 10, \text{and } 5$, respectively ($k_H = 1/\epsilon$). All length scales smaller than the Hall scale are expected to be strongly affected by the Hall effect. The Kolmogorov kinetic dissipation scale $k_v = (\nu^3/\nu^2)^{1/4}$ is $k_v \approx 20$ when $Re = 100$, $k_v \approx 40$ when $Re = 300$, and $k_v \approx 75$ when $Re = 560$.

### 5.1. Magnetic Energy Evolution

The magnetic helicity $H_m$ is displayed in Figure 3. The initial magnetic field is nonhelical, but during the dynamo process net magnetic helicity is generated with a sign opposite to that of the kinetic helicity. This helicity is located mostly in large-scale structures, as is shown in § 6.

From mean-field equations we obtain for the large-scale magnetic helicity (Mininni et al. 2003b)

$$\frac{dH_m}{dt} = 2 \int (\alpha \nabla \times B) \cdot \hat{B} \, dV,$$

where $\nabla \times B$ is the electric current density. This equation represents a transfer of magnetic helicity from small scales to large scales. The mean-field helicity grows with the same

sign as the $\alpha$-coefficient (opposite sign from the kinetic helicity). Our results are in good agreement with this relation, as well as with previous simulations of MHD dynamo action (Brandenburg 2001; Mininni et al. 2003b).

Although this generation of magnetic helicity by dynamo action is expected to decrease as the magnetic Reynolds number $Re_m$ increases, in simulations with higher Reynolds numbers the growth of magnetic helicity seems to reach an asymptotic value.

Figure 4 shows the kinetic and magnetic energy as a function of time for the MHD run and one Hall-MHD run with $Re = 300$ and $\epsilon = 0$. At early times, the evolution of magnetic energy in MHD and Hall-MHD is similar. Using mean-field
theory, the induction equation for the mean magnetic field reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( (\bar{U} - \epsilon \nabla \times \mathbf{B}) \times \mathbf{B} + \alpha \mathbf{B} \right) + \eta_{\text{eff}} \nabla^2 \mathbf{B},$$

with the Hall-MHD $\alpha$-effect now given by (Mininni et al. 2002)

$$\alpha = \frac{\tau}{3} \left( -\bar{u}^e \cdot \nabla \times \bar{u}^e + \mathbf{b} \cdot \nabla \times \mathbf{b} - \epsilon \mathbf{b} \cdot \nabla \times \nabla \times \bar{u}^e \right).$$

Here $\bar{u}^e = \mathbf{u} - \epsilon \nabla \times \mathbf{b}$ is the fluctuating electron flow velocity. When the fluctuating magnetic field is weak, this expression reduces to equation (15), and the Hall effect can be dropped. Therefore, the first stage corresponds to a kinematic dynamo during which the Hall effect is negligible.

After this stage, and when the dynamo-generated magnetic fields are strong enough for the Hall effect to become non-negligible, the evolution changes, and the magnetic energy keeps growing but at a different pace (see also Fig. 5). A third stage can be identified, when the velocity field is affected by the Lorentz force and the dynamo reaches saturation. The kinetic energy drop is not as intense as in the MHD case, and the increase of magnetic energy in this final stage is larger than in the MHD case for moderate values of $\epsilon$. Finally, a state with more magnetic energy than its MHD counterpart is reached (by a factor of 2.3 when $\text{Re} = 300$ and $\epsilon = 0.1$).

Figure 5 shows the evolution for a Hall-MHD run with $\text{Re} = 300$ and $\epsilon = 0.2$. Here, after the exponentially growing stage, the magnetic field saturates at an amplitude smaller than the previous run. Note that equipartition between the kinetic and magnetic energy is not reached.

When a turbulent stationary state is attained, the sum of magnetic and kinetic energy is not equal to the initial kinetic energy. This is related to the fact that, when the magnetic seed is introduced, a new channel for energy dissipation arises,

$$\frac{dE}{dt} = -\nu \int \omega^2 dV - \eta \int J^2 dV.$$  

As in previous simulations (Mininni et al. 2003b), it is found that the final energy reached for the Hall-MHD runs is larger than the value obtained for the MHD runs, revealing that Hall-MHD dynamos can be more efficient (in the sense that they generate more magnetic energy and dissipate less total energy).

Figure 6 shows the maximum value attained by the magnetic energy as a function of $\epsilon$ for several simulations with different Reynolds numbers and scale separations. When $\text{Re} = 560$ the final amplitude reached by the magnetic energy is unknown. Given the stringent quadratic Courant-Friedrich-Lewy condition imposed by dispersive waves in Hall-MHD, simulations were only carried up to saturation of the dynamo. The maximum value of the energy in Figure 6 is normalized with the value obtained in an MHD run with the same Reynolds numbers and initial kinetic energy and helicity. As previously mentioned, considering the three different regimes of the Hall dynamo discussed in Mininni et al. (2003b), we focus on the Hall-enhanced case. As the Reynolds numbers are increased, the efficiency of the Hall-MHD dynamo grows. In addition, the value of $\epsilon$ at which maximum efficiency is obtained decreases as the Reynolds numbers are increased. Note that the growth of the efficiency of the Hall-MHD dynamo with increasing scale separation was predicted analytically by Mininni et al. (2002). The shift of the most efficient $\epsilon$ to smaller values as $\text{Re}$ increases is also obtained from analytical estimates (Mininni et al. 2005).

5.2. Energy Spectrum

Figure 7 shows the kinetic, magnetic, and total energy spectra at different times for $\text{Re} = 300$ and $\epsilon = 0.1$. Barring $\epsilon$, all the parameters and initial conditions in this run are the same as those corresponding to Figure 2. Therefore, a direct comparison between the evolution of both spectra can be made. During the first few time steps, the evolution is similar to the MHD run, with the entire magnetic spectrum growing at almost the same rate. The difference observed in Mininni et al. (2003b), that the large-scale magnetic field is slightly larger than its MHD counterpart, is now increased as a result of the larger Reynolds number and larger scale separation.

The rate of increase of the large-scale magnetic field changes in the presence of the Hall effect. While in MHD the buildup of this field proceeds on a resistive timescale, in Hall-MHD it grows faster. Note that the magnetic energy in the shell $k = 1$ in a Hall-MHD ($\epsilon = 0.1$) simulation at $t = 14.1$ is a factor of 2 larger than the magnetic energy in the same shell in the MHD run at $t = 18.4$. In addition, the kinetic energy in the same shell is larger. This can also be observed in Figure 8, which shows...
the energy contained in the large-scale magnetic field (in the shell \( k = 1 \)) as a function of time for several values of \( \epsilon \). However, the filling factor \( \bar{h} = \frac{\langle B^2 \rangle}{\langle B^2 \rangle^{1/2}} \) becomes smaller as \( \epsilon \) is increased. At \( t = 14 \) the filling factor is \( \approx 0.23 \) for \( \epsilon = 0 \), \( \approx 0.17 \) for \( \epsilon = 0.1 \), and \( \approx 0.1 \) for \( \epsilon = 0.2 \).

Note also that, while the MHD spectrum shows super-equipartition at small scales (the magnetic energy is larger than the kinetic energy at large wavenumbers), the Hall-MHD leads to equipartition at these scales. The evolution of both the large-scale and small-scale magnetic fields are therefore clearly affected by the Hall effect, even though the Hall effect operates effectively only at small scales.

Figure 9 shows the spectrum for \( \Re = 300 \) and \( \epsilon = 0.2 \). In this case all wavenumbers larger than \( k_{\text{Hall}} = 5 \) are affected by the Hall effect. However, the total energy spectrum in the saturated state seems to obey a Kolmogorov-type law, although the Hall length scale is placed in the middle of the inertial range.

Figure 10 shows the compensated energy spectrum \( E(k) / (\epsilon^{2/3} k^{5/3}) \) for higher spatial resolution runs with \( \Re = 560 \) and \( \epsilon = 0 \) and 0.1, using 256\(^3\) grid points. If the spectrum obeys a Kolmogorov-type law, the compensated spectrum should be flat over a certain range, and the amplitude of the spectrum in this range should give the Kolmogorov constant \( C_K \). The mild hump observed for the total energy before entering the dissipative range might be indicative of the presence of a “bottle-neck effect” for the energy cascade, as was recently discussed by Haugen et al. (2003).

The spectra displayed in Figure 10 correspond to the era when the dynamo is saturated. The first example (Fig. 10a) represents the MHD limit (i.e., \( \epsilon = 0 \)), while the second one (Fig. 10b) has \( k_{\text{Hall}} = 10 \) (\( \epsilon = 0.1 \)). All the length scales smaller than \( 1/k_{\text{Hall}} \) are expected to be dominated by the Hall effect. For the MHD spectrum, the Kolmogorov constant \( C_K \approx 1.39 \), a value slightly larger than the one found by Haugen et al. (2003), \( C_K \approx 1.3 \) (see also Haugen et al. 2004), but smaller than the one obtained by Kida et al. (1991), \( C_K \approx 2.1 \). No clear change in the
slope can be identified in the Hall-MHD case, and the spectrum is compatible with a Kolmogorov-type law in its inertial range. However, the spectrum could really be a little shallower. Higher spatial resolution simulations are needed to settle this point. In this case Kolmogorov’s constant turns out to be $C_k \approx 1.66$, somewhat larger than its MHD counterpart. The Kolmogorov dissipation wavenumbers in the last stages of these runs are $k_\eta \approx C_k \approx 70$ [where $k_\eta = \langle (\mathbf{\omega}^2) / \eta^2 \rangle^{1/4}$].

5.3. Magnetic Helicity

In Mininni et al. (2003b), the Hall effect was observed to inhibit the creation of net magnetic helicity by the dynamo process. This effect is enhanced as we increase the Reynolds numbers. While the MHD dynamo is an efficient generator of magnetic helicity, with most of the helicity concentrated in the larger scales, the Hall dynamo is somewhat sluggish; the growth of net magnetic helicity is slower and in some cases oscillates around zero (see Fig. 11). This result is in good agreement with theoretical estimates suggesting that in the presence of the Hall effect reconnection events are faster (Priest & Forbes 1998; Ji 1999) and therefore dissipate less magnetic helicity (see § 6).

5.4. Kinetic Helicity

In MHD, the relative kinetic helicity is known to change only slightly during dynamo action (Brandenburg 2001). In all these simulations, kinetic energy and kinetic helicity are injected at the same length scale by the stirring force acting at $k_{\text{force}} = 3$. For homogeneous hydrodynamic turbulence, kinetic helicity directly cascades to smaller scales. On dimensional grounds (Moffat 1978), the spectrum of kinetic helicity in the inertial range is also expected to follow Kolmogorov’s law (Chen et al. 2003; Gómez & Mininni 2004),

$$H_k(k) = C_k k_{\text{hel}}^{\varepsilon_{2/3}} k^{-5/3}$$ \hspace{1cm} (21)

where $C_k$ is another Kolmogorov constant and $k_{\text{hel}}$ is the scale at which kinetic helicity is injected. Using the inertial range kinetic energy expression ($C_k$ is the standard Kolmogorov’s constant)

$$E_k(k) = C_k \varepsilon^{2/3} k^{-5/3}$$ \hspace{1cm} (22)

the ratio between the total kinetic helicity and kinetic energy in hydrodynamic turbulence comes out to be

$$\frac{H_k}{E_k} = \frac{\int \mathbf{U} \cdot \mathbf{\omega} dV}{\int \mathbf{U}^2 dV} \approx k_{\text{hel}}, \hspace{1cm} (23)$$

where $k_{\text{hel}} = k_{\text{force}}$ in our case. What happens after we introduce the magnetic seed?

Figure 12 shows the ratio $H_k/E_k$ for the MHD and the Hall-MHD runs with $Re = 300$. The evolution of the relative kinetic helicity is similar. In the MHD run $H_k/E_k \approx 2.5$, a value close to $k_{\text{force}} = 3$. Note that, although the total kinetic energy (and the kinetic helicity) decreases during the time evolution as a result of the increasing Lorentz force, the ratio $H_k/E_k$ remains nearly constant. On the other hand, this ratio grows with $\varepsilon$ in the Hall-MHD runs. The growth takes place just after the exponentially growing stage, when a large-scale magnetic field is developing in the box. This result suggests that a new source of kinetic helicity has appeared and is in good agreement with theoretical estimates suggesting that the Hall effect introduces handedness into the fluid motions (Mininni et al. 2005). However, we want to point out that this handedness does not, by itself, generate a net $\alpha$-effect. As previously mentioned, the relative kinetic helicity $H_k/(\langle U^2 \rangle \langle \omega^2 \rangle)$ in Hall-MHD also shows the same behavior, changing from a relative kinetic helicity of 0.4 when $\varepsilon = 0$ up to about 0.6 when $\varepsilon = 0.1$.

6. LARGE-SCALE MAGNETIC FIELD GENERATION

In this section we present MHD and Hall-MHD results for an external force located at $k_{\text{force}} = 10$. Simulations were carried out with $\nu = \eta = 0.02$ and $128^3$ grid points. In this case the Reynolds numbers are smaller (Re $\approx 220$ and Re$_e$ $\approx 20$), and the turbulence is weaker, since there are not enough modes in Fourier space for a direct cascade to develop properly. On the other hand, there is more room for the generation of a large-scale magnetic field through inverse cascade. Simulations were
performed for $\epsilon = 0$, 0.1, and 0.2, corresponding respectively to the MHD case and $k_{\text{Hall}} = 10$ and 5.

Figure 13 shows the evolution of the magnetic and kinetic energy in the MHD simulation. After saturation, a large-scale magnetic field grows on a resistive timescale, as is shown in the energy spectrum and was previously observed in large-scale dynamo simulations (Brandenburg 2001). Note that the system finally reaches a state of superequipartition, i.e., a level of magnetic energy that is larger than the kinetic energy. Figure 14 shows the counterpart of this evolution when $\epsilon = 0.1$ and 0.2. In the simulation with $\epsilon = 0.1$, the growth of magnetic energy after the saturation is clearly faster, and the system reaches a final state with more magnetic energy than in the MHD run.

Figures 15 and 16 show the energy spectrum at different times for runs with $\epsilon = 0$ and 0.1, respectively. Note that in both simulations, the magnetic energy at intermediate scales ($2 < k < 8$) starts to decay after saturation ($t > 3.5$), while magnetic energy at the largest scale ($k = 1$) keeps growing. As a result, after $t = 8$ the system reaches the state of superequipartition in the MHD case. This is even more clear in Fourier space (Fig. 15), where the magnetic energy in the shell $k = 1$ is 2 orders of magnitude larger than the kinetic energy at $t = 17.5$. An excess of magnetic energy can also be observed at small scales.

In the Hall-MHD case, we observe again a faster growth of the large-scale magnetic field (Fig. 16) but with a final state with superequipartition only at large scales. Moreover, the kinetic energy in the shell $k = 1$ is 1 order of magnitude larger than in the MHD case, and at small scales we obtain subequipartition between the magnetic and kinetic energy.

Figure 17 shows the magnetic helicity spectrum as a function of time for different values of $\epsilon$. As mentioned in § 4, the $\alpha$-effect creates magnetic helicity of a sign opposite to that of the kinetic helicity at large scales. This effect is balanced by the creation of an opposite amount of magnetic helicity at small scales. Therefore, the diffusion preferentially destroys the short-scale magnetic helicity in reconnection events, leaving a net helicity of opposite sign at large scales (Brandenburg 2001). Mininni et al. (2003b) suggested that while the Hall-MHD dynamo process also creates equal and opposite amounts of magnetic helicity at large and small scales, the dissipation of magnetic helicity at small scales is less efficient as $\epsilon$ is

![Figure 14](image1.png)

**Fig. 14.** Magnetic (lower curves) and kinetic energy (upper curves) as a function of time for two runs with $\epsilon = 0.2$, 0.1, and 0 ($k_{\text{force}} = 10$).

![Figure 15](image2.png)

**Fig. 15.** Mean kinetic energy spectrum (solid line), total energy spectrum (thick dashed line), and magnetic energy spectrum at different times ($\epsilon = 0$ and $k_{\text{force}} = 10$).

![Figure 16](image3.png)

**Fig. 16.** Mean kinetic energy spectrum (solid line), total energy spectrum (thick dashed line), and magnetic energy spectrum at different times ($\epsilon = 0.1$ and $k_{\text{force}} = 10$).

![Figure 17](image4.png)

**Fig. 17.** Magnetic helicity spectra for (a) $\epsilon = 0$ and (b) $\epsilon = 0.2$ (Re = 300).
increased. Figure 17 shows that when the Hall effect is present, even at late times, an excess of positive magnetic helicity at small scales \( k \gtrsim 10 \) can be readily identified in the spectra.

7. DISCUSSION

In this paper we have presented the results of direct numerical simulations of turbulent dynamo action in Hall-MHD. We find that with increasing Reynolds number and scale separation, the Hall MHD dynamo works more efficiently when the Hall length is close to but larger than the dissipation scale (Hall-enhanced regime). For larger values of \( \epsilon \) the Hall MHD dynamo is less efficient. In addition, the value of \( \epsilon \) (which measures the strength of the Hall term) at which the dynamo is most efficient decreases at higher Reynolds numbers.

An acceleration of the process responsible for the growth of a large-scale magnetic field is observed at moderate values of \( \epsilon \). Although these simulations are made at Reynolds numbers that are far away from realistic values for astrophysical plasmas, the results obtained are encouraging; the dynamos tend to work better at high Reynolds numbers. By calculating the magnitude and nature of the generated magnetic field as the amplitude of the Hall term is varied, we obtain new evidence showing that the Hall dynamo can be fundamentally different from its classical MHD counterpart.

Use of the Beowulf cluster BOCHA at Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires is acknowledged. The authors gratefully acknowledge A. Brandenburg and A. Pouquet for very fruitful and enlightening comments while reviewing the manuscript and the Abdus Salam International Centre for Theoretical Physics, where the initial stages of this study were performed. The research of S. M. M. was supported by US Department of Energy contract DE-FG03-96ER-54366. The research of D. O. G. and P. D. M. has been partially funded by grant X209/01 from the University of Buenos Aires and by grant PICT 03-9483 from ANPCyT. P. D. M. is a fellow of CONICET, and D. O. G. is a member of the Carrera del Investigador Científico of CONICET.

REFERENCES

Balbus, S. A., & Terquem, C. 2001, ApJ, 552, 235
Blackman, E. G., & Field, G. B. 1999, ApJ, 521, 597
Braginskii, S. I. 1965, Rev. Plasma Phys., 1, 205
Brandenburg, A. 2001, ApJ, 550, 824
Brandenburg, A., & Subramanian, K. 2000, A&A, 361, L33
Canuto, C., Hussaini, M. Y., Quarteroni, A., & Zang, T. A. 1988, Spectral Methods in Fluid Dynamics (Berlin: Springer)
Cattaneo, F., & Hughes, D. W. 1996, Phys. Rev. E, 54, 4532
Chen, Q., Chen, S., & Eyink, G. 2003, Phys. Fluids, 15, 361
Galanti, B., Klecorin, N., & Rogachevskii, I. 1994, Phys. Plasmas, 1, 3843
Gómez, D., & Mininni, P. 2004, Phys. A, 342, 69
Haugen, N. E. L., Brandenburg, A., & Dobler, W. 2003, ApJ, 597, L141
———. 2004, Phys. Rev. E, 70, 16308
Heintzmann, H. 1983, J. Exp. Theor. Phys., 57, 251
Helmis, G. 1968, Monatsh. Deutsch Akad. Wiss. Berlin, 10, 280
———. 1971, Beitr. Plasma Phys., 11, 417
Ji, H. 1999, Phys. Rev. Lett., 83, 3198
Kida, S., Yanase, S., & Mizushima J. 1991, Phys. Fluids A, 3, 457
Krause, F., & Rädler, K.-H. 1980, Mean-Field Magnetohydrodynamics and Dynamo Theory (Oxford: Pergamon)
Meneguzzi, M., Frisch, U., & Pouquet, A. 1981, Phys. Rev. Lett., 47, 1060
Mininni, P. D., Gómez, D. O., & Mahajan, S. M. 2002, ApJ, 567, L81
———. 2003a, ApJ, 584, 1120
———. 2003b, ApJ, 587, 472
———. 2005, ApJ, 619, 1014
Mirnov, V. V., Hegna, C. C., & Prager, S. C. 2003, Plasma Phys. Rep., 29, 566
Moffat, H. K. 1978, Magnetic Field Generation in Electrically Conducting Fluids (Cambridge: Cambridge Univ. Press)
Muslimov, A. G. 1994, MNRAS, 267, 523
Pope, S. B. 2000, Turbulent Flows (Cambridge: Cambridge Univ. Press)
Pouquet, A., Frisch, U., & Lecorat, J. 1976, J. Fluid Mech., 77, 321
Priest, E., & Forbes, T. 1998, Magnetic Reconnection (Cambridge: Cambridge Univ. Press)
Rheinhardt, M., & Geppert, U. 2002, Phys. Rev. Lett., 88, 101103
Sanin, T., & Stone, J. M. 2002, ApJ, 570, 314
Zweibel, E. G. 1988, ApJ, 329, 384