Quantum “az+b” group at roots of unity: unitary representations

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Abstract

All unitary representations of the quantum “az+b” group are found. It turns out that this quantum group is self dual i.e. all unitary representations are ‘numbered’ by elements of the same group. Moreover, the formula for all unitary representations involving the quantum exponential function is proven.

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1 Introduction

Locally compact quantum groups are nowadays studied extensively by many scientists [2]. Although at the moment there is no commonly accepted definition of a locally compact quantum group, there are promising approaches and interesting examples have been worked out. One of the most remarkable ones is the quantum “az+b” group constructed by S.L. Woronowicz in [15]. According to the recent computation by A. van Daele [10], this group is an example of an interesting phenomenon foreseen by Vaes and Kustermans in [4]. In this paper we study the quantum ’az+b’ group from the point of view of unitary representations and duality theory. The aim of this paper is to derive a formula for all unitary representations of the quantum ’az+b’ group.

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In Section 3 we recall relevant information on the quantum “az+b” group. We discuss in details the structure of the $C^*$-algebra of all continuous functions vanishing at infinity on the quantum “az+b” group. We introduce also the corresponding $C^*$- and $W^*$-dynamical systems. We prove a couple of corollaries we use later on to find all unitary representations of the quantum “az+b” group. The formula for this representations will be found and proved in Section 4. The important theorem we use was proved in our previous paper [9], where we investigated unitary representations of some braided quantum group related to the quantum “az+b”.

We use methods similar to those introduced by S.L. Woronowicz in the case of the quantum $E(2)$ group [13] and then applied by us to giving the formula for all unitary representations of the quantum $\"ax + b\"$ group in [8] and in Chapter 3 of [6].

In the next Section we will fix the notation.

2 Notation

We consider only concrete $C^*$-algebras, i.e. embedded into $C^*$-algebra of all bounded operators acting on Hilbert space $\mathcal{H}$, denoted by $B(\mathcal{H})$. The $C^*$-algebra of all compact operators acting on $\mathcal{H}$ will be denoted by $CB(\mathcal{H})$. All algebras we consider are separable with the exception of multiplier algebras (see definition of multiplier algebra below).

Let $A$ be $C^*$-algebra. Then $M(A)$ will denote the multiplier algebra of $A$, i.e.

$$ M(A) = \{ m \in B(\mathcal{H}) : \text{ ma, am } \in A \text{ for any } a \in A \} . $$

Observe that $A$ is an ideal in $M(A)$. If $A$ is a unital $C^*$-algebra , then $A = M(A)$, in general case $A \subset M(A)$. For example the multiplier algebra of $CB(\mathcal{H})$ is the algebra $B(\mathcal{H})$ and the multiplier algebra of $C^*$-algebra $C_\infty(\mathbb{R})$ of all continuous vanishing at infinity functions on $\mathbb{R}$ is the algebra of all continuous bounded functions on $\mathbb{R}$ denoted by $C_{\text{bounded}}(\mathbb{R})$. The natural topology on $M(A)$ is the strict topology, i.e. we say that a sequence $(m_n)_{n \in \mathbb{N}}$ of $m_n \in M(A)$ converges strictly to 0 if for every $a \in A$, we have $\|m_n a\| \to 0$ and $\|am_n\| \to 0$, when $n \to +\infty$. Whenever we will consider continuous maps from or into $M(A)$, we will mean this topology.

For any $C^*$-algebras $A$ and $B$, we will say that $\phi$ is a morphism and write $\phi \in \text{Mor}(A, B)$ if $\phi$ is a * - algebra homomorphism acting from $A$ into $M(B)$ and such that $\phi(A)B$ is dense in $B$. Any $\phi \in \text{Mor}(A, B)$ admits unique extension to a * - algebra homomorphism acting from $M(A)$ into $M(B)$. For any $S \in M(A)$, operator $\phi(S)$ is given by

$$ \phi(S)(\phi(a)b) = \phi(Ta)b , $$

where $a \in A$ and $b \in B$. 

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For any closed operator $T$ acting on $\mathcal{H}$ we define its $z$-transform by

$$z_T = T(I + T^*T)^{-\frac{1}{2}}.$$  

Observe that $z_T \in \mathcal{B}(\mathcal{H})$ and $||z_T|| \leq 1$. Moreover, one can recover $T$ from $z_T$

$$T = z_T(I - z_T^*z_T)^{-\frac{1}{2}}.$$  

A closed operator $T$ acting on $A$ is affiliated with a $C^*$-algebra $A$ if and only if $z_T \in M(A)$ and $(I - z_T^*z_T)A$ is dense in $A$. A set of all elements affiliated with $A$ is denoted by $A^\eta$. If $A$ is a unital $C^*$-algebra, then $A^\eta = M(A) = A$, in general case

$$A \subset M(A) = \{ T \eta A : ||T|| \leq \infty \} \subset A^\eta.$$  

The set of all elements affiliated with $C_\infty(\mathbb{R})$ is the set of all continuous functions on real line $C(\mathbb{R})$, and a set of all elements affiliated with $C^*$-algebra $CB(\mathcal{H})$ is a set of all closed operators $C(H)$. This last example shows that a product and a sum of two elements affiliated with $A$ may not be affiliated with $A$, since it is well known that a sum and a product of two closed operators may not be closed. Affiliation relation in $C^*$-algebra theory was introduced by Baaj and Julg in [1].

Observe, that if $\phi \in \text{Mor}(A, B)$, then one can extend $\phi$ to elements affiliated with $A$. Let us start with the observation, that for any $T \in M(A)$ we have

$$\phi(z_T) = z_{\phi(T)}.$$  

Hence for any $T \eta A$ we have $z_T \in M(A)$. Moreover, there exists a unique closed operator $S$ such that $\phi(z_T) = z_S$. This operator is given by

$$S = \phi(z_T)\phi(I - z_T^*z_T)^{-\frac{1}{2}}.$$  

From now on we will write $S = \phi(T)$.

We recall now a nonstandard notion of generation we use in this paper. This notion was introduced in [4], where a generalization of the theory of unital $C^*$-algebras generated by a finite number of generators was presented. It was proved in [3] that such $C^*$-algebras are isomorphic to algebras of continuous operator functions on compact operator domains (see Section 1.3 of [7] and references therein). In this approach, the algebra of all continuous vanishing at infinity functions on a compact quantum group is generated by matrix elements of fundamental representation. To use this approach to non compact quantum groups, one has to extend the notion of a generation of a $C^*$-algebra to non unital $C^*$-algebras and unbounded generators. According to the definition we recall below, $C^*$-algebra of continuous vanishing at infinity functions on a locally compact quantum group is generated by its fundamental representation. However, in this case the fundamental representation is not unitary and the generators are unbounded operators, so they are not in the $C^*$-algebra $A$. 

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Assume for a while, that were are given a \( C^\ast \)-algebra \( A \) and operators \( T_1, T_2, \ldots, T_N \) affiliated with \( A \). We say that \( A \) is *generated* by \( T_1, T_2, \ldots, T_N \) if for any Hilbert space \( \mathcal{H} \), a non degenerate \( C^\ast \)-algebra \( B \subset B(\mathcal{H}) \) and any \( \pi \in \text{Mor}(A, CB(\mathcal{H})) \) we have

\[
\pi(T_i) \text{ is affiliated with } A \quad \text{for any } i = 1, \ldots, N \quad \Rightarrow \quad \pi \in \text{Mor}(A, B).
\]

We stress that described above ‘generation’ is a relation between \( A \) and some operators \( T_1, T_2, \ldots, T_N \) and both have to be known in advance. There is no procedure to obtain \( A \) knowing only \( T_1, T_2, \ldots, T_N \) and it is even possible that there is no \( A \) generated by such operators.

For unital \( C^\ast \)-algebras generation in the sense introduced above is exactly the same as the classical notion of generation. More precisely, let \( A \) be a unital \( C^\ast \)-algebra and let \( T_1, T_2, \ldots, T_N \in A \). If \( A \) is the norm closure of all linear combinations of \( I, T_1, \ldots, T_N \), then \( A \) is generated by \( T_1, T_2, \ldots, T_N \) in the sense of the above definition. On the other hand, let \( A \) be a \( C^\ast \)-algebra generated by \( T_1, T_2, \ldots, T_N \) \( \eta A \) such that \( ||T_i|| < \infty \) for \( i = 1, 2, \ldots, N \). Then \( A \) contains unity , \( T_1, T_2, \ldots, T_N \in A \) and \( A \) is the norm closure of the set of all linear combinations of \( I, T_1, T_2, \ldots, T_N \).

An easy example of this relation is that \( C^\ast \)-algebra \( C_\infty(\mathbb{R}) \) of all continuous vanishing at infinity functions on \( \mathbb{R} \) is generated by function \( f(x) = x \) for any \( x \in \mathbb{R} \). The other example is \( C^\ast \)-algebra \( CB(L^2(\mathbb{R})) \) which is generated by momentum operator \( \hat{p} \) and multiplication-by-coordinate operator \( \hat{q} \).

Let \( A \) and \( B \) be \( C^\ast \)-algebras and assume that we know generators of \( A \). In order to describe \( \phi \in \text{Mor}(A, B) \) uniquely it is enough to know how \( \phi \) acts on generators of \( A \).

We will use exclusively the minimal tensor product of \( C^\ast \)-algebras and it will be denoted by \( \otimes \). We will also use the leg numbering notation. For example, if \( \phi \in \text{Mor}(A \otimes A, A \otimes A) \) then \( \phi_{12}(a \otimes b) = a \otimes b \otimes I_A \) and \( \phi_{13}(a \otimes b) = a \otimes I_A \otimes b \) for \( a, b \in A \). Clearly, \( \phi_{12}, \phi_{13} \in \text{Mor}(A \otimes A, A \otimes A \otimes A) \).

Let \( f \) and \( \phi \) be strongly commuting selfadjoint operators. Then, by the spectral theorem

\[
f = \int_{\Lambda} \! r dE(r, \rho) \quad \text{and} \quad \phi = \int_{\Lambda} \! \rho dE(r, \rho),
\]

where \( dE(r, \rho) \) denotes the common spectral measure associated with \( f \) and \( \phi \) and \( \Lambda \) stands for a joint spectrum of \( f \) and \( \phi \). Then

\[
F(f, \phi) = \int_{\Lambda} \! F(r, \rho) dE(r, \rho).
\]

Let \( b \) be a selfadjoint operator and let the symbol \( \chi \) denote the characteristic function defined on \( \mathbb{R} \). By \( \chi(b \neq 0) \) we mean the projection operator on the subspace \( \ker b^\perp \), by \( \chi(b < 0) \) - the projection onto the subspace on which \( b \) is negative, and so on.
3 Relevant information on the quantum “az+b” group

Let

\[ q = e^{2\pi i \frac{N}{N}}, \]  

where \( N \) is an even number and \( N \geq 6 \), i.e. \( q \) is a primitive root of unity: \( q^N = 1 \).

Let us introduce notation

\[ \hbar = 2\pi \frac{N}{N}. \]  

Note that \( \hbar < \pi \) and \( q = e^{i\hbar} \).

Let

\[ \Gamma = \bigcup_{k=0}^{N-1} q^k \mathbb{R}_+ . \]  

Observe that \( \Gamma \) is a multiplicative subgroup of \( \mathbb{C} \setminus \{0\} \).

Let \( \overline{\Gamma} \) denote a closure of \( \Gamma \) in \( \mathbb{C} \), i.e.

\[ \overline{\Gamma} = \Gamma \cup \{0\}. \]  

We introduce now the definition of the operator domain \( G_H \) \[9, 15\], related to the quantum "az+b" group (for the notion of operator domain we refer the Reader to \[7, 6\] and references therein)

\[ G_H = \left\{ (a, b) \in \mathcal{C}(H)^2 \mid \begin{array}{l} aa^* = a^*a, \\ bb^* = b^*b, \\ \ker a = \{0\} \\ \text{Sp}_a, \text{Sp}_b \subset \overline{\Gamma} \\ (\text{Phase } a)b = q\text{b(Phase a)} \\ \text{for any } t \in \mathbb{R} \\ |a|^t b = e^{-\frac{2\pi i}{N}} b |a|^t \end{array} \right\} . \]

Observe that the relationship between \( G_H \) and braided quantum group \( D_H \) considered in \[7, 8\] is the following

\[ \left( (b, a) \in D_H^2 \mid \ker a = \{0\} \right) \iff (a, b) \in G_H \]

Let \( A \) denote the \( C^* \)-algebra of all continuous functions vanishing at infinity on the quantum “az+b" group. The algebra \( A \) is generated (in the sense explained in Section \[7\]) by unbounded operators \( a, \ a^{-1} i b \), where \( (a, b) \in G_H \). It was proved in \[15\] that
the multiplicative unitary operator \( W \in B(\mathcal{H} \otimes \mathcal{H}) \) for the quantum 'az+b' group is given by
\[
W = F_N (ab^{-1} \otimes b) \chi(b^{-1} \otimes I, I \otimes a),
\]
where \( \chi \) is a symmetric bicharacter on \( \Gamma \), such that for any \( \gamma \in \Gamma \) and \( r \in \mathbb{R}_+ \)
\[
\chi(\gamma, q) = \text{Phase } \gamma \quad \text{and} \quad \chi(\gamma, r) = |\gamma|^\frac{2\pi}{|\gamma|} \log r.
\]

From the theory of multiplicative unitaries we know that operator \( W \) encodes the group structure of the quantum group. Precisely, for any \( d \in A \) comultiplication \( \Delta \in \text{Mor}(A, A \otimes A) \) is given by
\[
\Delta(d) = W(d \otimes \text{id})W^*.
\]
Comultiplication \( \Delta \) may be extended to unbounded operators affiliated with \( A \) and is given on generators of \( A \) by the same formula
\[
\Delta(a) = W(a \otimes \text{id})W^*
\]
\[
\Delta(b) = W(b \otimes \text{id})W^*
\]
It was also computed in [15] that for any \( (a, b) \in G_H \) we have
\[
\Delta(a) = a \otimes a
\]
\[
\Delta(b) = a \otimes b + b \otimes I,
\]
where \( a \otimes b + b \otimes I \) denotes the closure of the sum \( a \otimes b + b \otimes I \).

Thus defined \( \Delta \) is associative. Moreover, \( G \) equipped with thus defined \( \Delta \) is the quantum 'az+b' group. However, what we are mostly interested in in this paper is the \( C^* \)-algebra \( A \) of all continuous functions vanishing at infinity on the quantum "az+b" group.

We remind now the construction of a certain \( C^* \)-dynamical system investigated in [15]. It turns out that the \( C^* \)-algebra corresponding to this system is exactly our algebra \( A \).

Let
\[
B = \{ f \in C_\infty(\Gamma) \}
\]
and let
\[
b(\gamma) = \gamma
\]
for any \( \gamma \in \Gamma \). Then \( b \eta B \) and \( b \) generates \( B \) in the sense of [14].

Let us define an action \( \sigma \in \text{Aut}(M(B)) \) of the group \( \Gamma \) on any function \( f \in B \) by
\[
(\sigma_{t \tau}) f(t) = f(t \tau) \quad \text{where} \quad \tau \in \Gamma \text{ and } t \in \Gamma.
\]
\(^1\)To prove manageability of \( W \) we need a slightly more complicated formula [13]. However, for the purpose of this paper, we may consider a simpler formula [13].
Then \((B, \Gamma, \sigma)\) is a \(C^*\)-dynamical system (see e.g. [5]). Denote the \(C^*\)-crossed product algebra by \(A_{cp}\)
\[
A_{cp} = B \times_{\sigma} \Gamma .
\]
By definition of the \(C^*\)-crossed product algebra, \(M(A_{cp})\) contains a strictly continuous one-parameter group of unitary operators, implementing an action \(\sigma\) of the group \(\Gamma\) on an algebra \(B\): 
\[
\sigma_t f = U_t f U_{-t}
\]
for any \(f \in B\) and \(\gamma \in \Gamma\). By SNAG Theorem and Theorem 5.2 [15], every strictly continuous one-parameter group of unitary operators \((U_t)_{t \in \Gamma}\) contained in \(M(A_{cp})\) has form
\[
U_t = \chi(a, t),
\]
where \(a\) is a normal operator affiliated with \(A_{cp}\) and \(\text{Sp} a \subset \Gamma\). Moreover, \(a\) is invertible and \(a^{-1} \in A_{cp}\). Moreover, it can be easily seen that
\[
\sigma_t b = t b .
\]
Hence
\[
U_t b = t b U_t,
\]
for any \(t \in \Gamma\). It follows that \((a, b) \in G_H\).

It is also well-known that the linear envelope of a set
\[
\{ fg(a) : f \in B, \ g \in C_{\infty}(\Gamma) \}
\]
is dense in \(A_{cp} = B \times_{\sigma} \Gamma\). Hence \(B \subset M(A)\). It turns out ([5, Proposition 4.1]) that \(a, a^{-1}\) and \(b\) generate \(A_{cp}\) in the sense of [14]. Moreover, Proposition 3.2 [15] says that for any pair \((\tilde{a}, \tilde{b}) \in G_H\) there is unique representation \(\pi \in \text{Rep}(A_{cp}, H)\) such that \(\tilde{a} = \pi(a)\) and \(\tilde{b} = \pi(b)\). It was proven in [15] that \(A_{cp} = A\). From now on we will not distinguish \(A_{cp}\) and \(A\).

We proceed to construct a dual action of the group \(\Gamma\) on \(A = B \times_{\sigma} \Gamma\). To this end we consider a map
\[
\theta_{\gamma} = (\text{id} \otimes \varphi_{\gamma}) \Delta ,
\]
where a map \(\varphi \in \text{Mor}(A, C_{\infty}(\Gamma))\) is such that for any \(t \in \Gamma\)
\[
\varphi_{\gamma}(a) = \gamma \quad \text{and} \quad \varphi_{\gamma}(b) = 0 .
\]
Proposition 4.2 [15] implies that there is only one such a map \(\phi\). Observe that \(\theta_{\gamma}\) is an automorphism of \(A\) and that
\[
\theta_0 = \text{id} \quad \text{and} \quad \theta_{\gamma_1} \theta_{\gamma_2} = \theta_{\gamma_1 \gamma_2}
\]
for any \(\gamma_1, \gamma_2 \in \Gamma\). The map \(\gamma \to \theta_{\gamma}(d)\) is continuous for any \(d \in A\).

Moreover, for any \(\gamma, t \in \Gamma\)
\[
\theta_{\gamma}(U_t) = \chi(\gamma, t) U_t .
\]
Thus we showed that \((A, \Gamma, \theta)\) is a \(C^*\)-dynamical system and is dual to the \(C^*\)-dynamical system \((B, \Gamma, \sigma)\).

Using (1) and (2) we compute

\[
\theta_\gamma(a) = (\text{id} \otimes \varphi_\gamma) \Delta(a) = (\text{id} \otimes \varphi_\gamma)(a \otimes a) = \gamma a
\]

and

\[
\theta_\gamma(b) = (\text{id} \otimes \varphi_\gamma) \Delta(b) = b.
\]

Hence for any \(\gamma \in \Gamma\) and \(g \in \text{M}(B)\) we have

\[
\theta_\gamma(g) = g.
\]

Since \((B, \Gamma, \sigma)\) and \((A, \Gamma, \theta)\) are dual \(C^*\)-dynamical systems, it follows that \((B'', \Gamma, \sigma)\) and \((A'', \Gamma, \theta)\) are dual \(W^*\)-dynamical systems, if only \(\sigma\) and \(\theta\) are extended in the obvious way.

Let \(\mathcal{K}\) be a Hilbert space (this time we consider also finite-dimensional ones). Moreover, observe that it follows from the above remark that also \((B(\mathcal{K}) \otimes B'', \Gamma, I \otimes \sigma)\) is a \(W^*\)-dynamical system and its von Neumann \(W^*\)-crossed product algebra is \(B(\mathcal{K}) \otimes A''\). Analogously, the \(W^*\)-dynamical system dual to \((B(\mathcal{K}) \otimes B'', \Gamma, I \otimes \sigma)\) is \((B(\mathcal{K}) \otimes A'', \Gamma, I \otimes \theta)\).

In what follows we need Proposition 3.1, which is an obvious consequence of Theorem 7.10.4 [5]:

**Proposition 3.1** Let \(m \in B(\mathcal{K}) \otimes A''\) and let for any \(\gamma \in \Gamma\)

\[
(id \otimes \theta_\gamma)(m) = m.
\]

Then

\[
m \in B(\mathcal{K}) \otimes B''.
\]

We also need

**Proposition 3.2** Let \(B\) be a \(C^*\)-algebra and let \(w \in \text{M}(B \otimes A)\). Then a map \(\Gamma \ni \gamma \to (id \otimes \varphi_\gamma) w \in \text{M}(B)\) is strictly continuous.

**Proof:** We know that \(\varphi \in \text{Mor}(A, C_\infty(\Gamma))\) and therefore \((id \otimes \varphi)w \in \text{M}(B \otimes C_\infty(\Gamma))\).
S.L. Woronowicz proved in [14] that elements of \(\text{M}(B \otimes C_\infty(\Gamma))\) are bounded, strictly continuous functions on \(\Gamma\) with values in \(\text{M}(B)\). \(\Box\)
4 How do unitary representations of the quantum “az+b” group look like?

Definition 4.1 A unitary operator $V \in M(CB(\mathcal{K}) \otimes A)$ is called a (strongly continuous) unitary representation of the quantum 'az+b' group if

$$W_{23}V_{12} = V_{12}V_{13}W_{23},$$

or equivalently

$$(\text{id} \otimes \Delta)V = V_{12}V_{13}. \tag{6}$$

Observe that in case of the classical group condition (6) is equivalent to the usual definition of a unitary representation, i.e. a representation $U$ is a map

$$U : G \ni g \rightarrow U_g \in B(\mathcal{K})$$

such that $U_g$ is unitary for any $g \in G$ and for any $g, h \in G$ we have $U_gU_h = U_{gh}$.

It was proven in [15] that

Proposition 4.2 Let $(a, b) \in G_\mathcal{K}$ and $(c, d) \in G_\mathcal{K}$ and let $\ker b = \{0\}$. Then the operator $V \in M(CB(\mathcal{H}) \otimes A)$ given by

$$V(a, b) = F_N(d \otimes b)\chi(c \otimes I, I \otimes a)$$

is a unitary representation of the quantum 'az+b' group.

We now prove that all unitary representations of the quantum 'az+b' group are of this form.

Theorem 4.3 Every unitary representation of the quantum 'az+b' group (i.e. of $G$), acting on some Hilbert space $\mathcal{K}$ has form

$$V(a, b) = F_N(d \otimes b)\chi(c \otimes I, I \otimes a)$$

where $(c, d) \in G_\mathcal{K}$.

Proof: Let $V$ be a unitary representation of the quantum 'az+b' group acting on a Hilbert space $\mathcal{K}$. Then for any $\gamma \in \Gamma$ an operator $(\text{id} \otimes \varphi_\gamma)V \in B(\mathcal{K})$ is unitary. Applying $(\text{id} \otimes \varphi_s \otimes \varphi_t)$ to both sides of (6) we get

$$V(h(s + t), 0, 0) = V(hs, 0, 0)V(ht, 0, 0)$$

Hence

$$(\text{id} \otimes \varphi_{s+t})V = (\text{id} \otimes \varphi_s)V(\text{id} \otimes \varphi_t)V$$
i.e. $(\text{id} \otimes \varphi)V$ is a representation of $\Gamma$.

The strict topology on $B(H) = M(CB(H))$ coincides with the $\ast$-strong operator topology. Since the $\ast$-strong operator topology is stronger than the strong operator topology, then by Proposition 3.2 we obtain that the map

$$\Gamma \ni t \rightarrow (\text{id} \otimes \varphi_t)V \in B(K)$$

is strongly continuous. Therefore by the SNAG Theorem there is a normal operator $c$ acting on $K$ with spectrum contained in $\Gamma$ and such that

$$(\text{id} \otimes \varphi_\gamma)V = \chi(c, \gamma)$$

for any $t \in \Gamma$.

Note that from (1), (6) and (4) follows that

$$(\text{id} \otimes \theta_\gamma)V = V \chi(c \otimes I, \gamma I \otimes I)$$

Moreover, by

$$(\text{id} \otimes \theta_\gamma)\chi(c \otimes I, I \otimes a)^* = \chi(c \otimes I, I \otimes \gamma a)^* = \chi(c \otimes I, \gamma I \otimes I)^* \chi(c \otimes I, I \otimes a)^*.$$  

Hence

$$(\text{id} \otimes \theta_\gamma)V \chi(c \otimes I, I \otimes a)^* = V \chi(c \otimes I, I \otimes a)^*.$$  

Observe that

$$V \chi(c \otimes I, I \otimes a)^* \in B(K) \otimes A''.$$  

Hence by Proposition 3.1

$$V \chi(c \otimes I, I \otimes a)^* = f(b),$$

where $f \in B(K) \otimes B''$, i.e. $f$ is a Borel operator function on $\Gamma$ with values in bounded operators acting on $K$ (for explanation on operator functions see [4] [Section 1.3] and references therein). Hence

$$V = f(b)\chi(c \otimes I, I \otimes a).$$

Since $V$ is unitary operator, it follows from the above formula that $f(b) \in \text{Unit}(K \otimes H)$.

Compute

$$(\text{id} \otimes \Delta)V = (\text{id} \otimes \Delta)(f(b)\chi(c \otimes I, I \otimes a)) = f(\Delta(b))\chi(c \otimes \Delta(I), I \otimes \Delta(a)) =$$

$$= f(a \otimes b + b \otimes I)\chi(c \otimes I \otimes I, I \otimes a \otimes a).$$

Applying $(\text{id} \otimes \varphi_\gamma \otimes \text{id})$ to both sides of (3) we obtain, respectively,

$$(\text{id} \otimes \varphi_\gamma \otimes \text{id})V_{12}V_{13} = (\chi(c, \gamma) \otimes \text{id})V$$

and

$$(\text{id} \otimes \varphi_\gamma \otimes \text{id})(\text{id} \otimes \Delta)V = f(\gamma b)\chi(c \otimes I, \gamma \otimes a).$$
Comparing these results we see that

\[(\chi(c, \gamma I) \otimes \text{id})V = f(\gamma b)\chi(c \otimes I, \gamma I \otimes a) \]  
\tag{7}

On the other hand, applying \((\text{id} \otimes \text{id} \otimes \varphi_\gamma)\) to both sides of (\ref{eq:6}) we conclude that

\[(\text{id} \otimes \text{id} \otimes \varphi_\gamma)V_{12}V_{13} = V(\chi(c, \gamma I) \otimes \text{id})\]

and

\[(\text{id} \otimes \text{id} \otimes \varphi_\gamma)(\text{id} \otimes \Delta)V = f(b)\chi(c \otimes I, \gamma I \otimes a) .\]

Comparing these results we see that

\[V(\chi(c, \gamma I) \otimes \text{id}) = f(b)\chi(c \otimes I, \gamma I \otimes a) .\]
\tag{8}

Inserting \(a\) in the place of \(\gamma\) in formulas (7) and (8) we derive

\[(\chi(c \otimes I, I \otimes a) \otimes \text{id})V_{13} = f(a \otimes b)\chi(c \otimes I \otimes I, I \otimes a \otimes a) \]

and

\[V_{12}(\chi(c \otimes I, I \otimes a))_{13} = f(b \otimes I)\chi(c \otimes I \otimes I, I \otimes a \otimes a) .\]

Hence

\[V_{12} = f(b \otimes I)\chi(c \otimes I \otimes I, I \otimes a \otimes I)\]

\[V_{13} = \chi(c \otimes I \otimes I, I \otimes a \otimes I)^* f(a \otimes b)\chi(c \otimes I \otimes I, I \otimes a \otimes a) .\]

Therefore, since \(V\) satisfies \((\text{id} \otimes \Delta)V = V_{12}V_{13}\), we have

\[f(a \otimes b \otimes b \otimes I)\chi(c \otimes I \otimes I, I \otimes a \otimes a) =\]

\[= f(b \otimes I)f(a \otimes b)\chi(c \otimes I \otimes I, I \otimes a \otimes a) .\]

Hence obviously

\[f(a \otimes b \otimes b \otimes I) = f(b \otimes I)f(a \otimes b) .\]
\tag{9}

Let us introduce notation

\[R = a \otimes b \quad \text{and} \quad S = b \otimes I .\]
\tag{10}

We conclude that \((S, R) \in D^2\). Inserting notation (\ref{eq:10}) into formula (\ref{eq:9}) we derive

\[f(S^+R) = f(S)f(R).\]

By Theorem 7.1 \cite{9} if a function \(f\) is a Borel operator function \(f : D \rightarrow B(K \otimes \mathcal{H})\) and satisfies the above condition for \((S, R) \in D^2\), then it is given by

\[f(b) = F_N(d \otimes b) ,\]

where \(d\) is a normal operator with spectrum contained in \(\Gamma\).

What is left is to prove that \((c, d) \in G_K\). To this end observe that by (\ref{eq:4}) and from

\[V = F_N(d \otimes b)\chi(c \otimes I, I \otimes a),\]
follows that
\[ F_N(d \otimes I \otimes b) \chi(c \otimes I \otimes I, I \otimes I \otimes a) = \]
\[ = \chi(c \otimes I \otimes I, I \otimes a \otimes I)^* F_N(d \otimes a \otimes b) \chi(c \otimes I \otimes I, I \otimes a \otimes a) . \]

Hence
\[ (d \otimes \text{id}) = \chi(c \otimes I, I \otimes a)^* (d \otimes a) \chi(c \otimes I, I \otimes a) . \]

Inserting \( a = \gamma I \) in the formula \([3]\) we obtain that \((c, d, \delta) \in G_K\), which completes
the proof. \(\square\)

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