ON THE QUANTUM 'AX+B' GROUP

PIOTR STACHURA

Abstract. The more detailed description of the 'ax+b' group of Baaj and Skandalis is presented. In particular we give generators and present formulae for the action of the comultiplication on them. We prove that this quantum group is a defined by the twist.

1. Introduction

The purpose of this work is to give a more detailed description of the quantum 'ax+b' group of Baaj and Skandalis \[1, 2, 3\]. In particular we describe the C*-algebra by generators and relations, show that the comultiplication is given by a twist and compute an action of the comultiplication on generators. We also show that this quantum group is a quantization of a Poisson-Lie structure on the classical 'ax+b' group.

On the Hopf C*-algebra level this group is given by generators $A$, $A^{-1}$, $Z$ and relations [2]:

\[ A = A^* , \quad Z = -Z^* , \quad [A, Z] = A(1 - A) \]

Together with comultiplication $\Delta$, counit $\epsilon$ and antipode $S$:

\[ \Delta(A) = A \otimes A , \quad \Delta(Z) = Z \otimes A + 1 \otimes Z , \quad \epsilon(A) = 1 , \quad \epsilon(Z) = 0 , \quad S(A) = A^{-1} , \quad S(Z) = -ZA^{-1} \]

If one wants to find a C*-algebra with affiliated elements $A, Z$ that satisfy these commutation relations, first step is to represent them on a Hilbert space. One hopes to find $A, A^{-1}$–unbounded, invertible, selfadjoint operators and $iZ$–selfadjoint, satisfying in some reasonable sense the relation $[A, Z] = A(1 - A)$. This is possible but the most natural choice leads to the situation where $\Delta(iZ)$ is symmetric but not selfadjoint.

One can rewrite these relations using different generators and this is the way we choose. Let us define $Y := A^{-1} - I$ and $X := iZ$. These elements satisfy:

\[ Y = Y^* , \quad X = X^* , \quad [X, Y] = iY , \]

\[ \Delta(Y) = Y \otimes Y + I \otimes Y + Y \otimes I , \quad \Delta(X) = X \otimes (Y + I)^{-1} + I \otimes X \] (1)

These are relations we are going to give a meaning to.

In the following we will use groupoid algebras, so now we recall basic facts and establish the relevant notation. The category of groupoids used here is described in [4] and in a differential setting in [5]. A groupoid is a set $\Gamma$ together with a subset $E \subset \Gamma$ (the set of identities), an associative relation (multiplication) $m : \Gamma \times \Gamma \rightarrow \Gamma$ and an involutive mapping (inverse) $s : \Gamma \rightarrow \Gamma$. They satisfy certain relations that entail the existence of two projections $e_L, e_R : \Gamma \rightarrow E$ (target and source projections) and the fact that $m$ is a mapping defined on the set $\{(x, y) \in \Gamma \times \Gamma : e_R(x) = e_L(y)\}$ of composable pairs; this is the standard notion of groupoid. The definition of a morphism of groupoids used here is different, however. By a morphism of groupoids $\Gamma, \Gamma'$ we mean a relation $h : \Gamma \rightarrow \Gamma'$ which satisfies: $hm = m'(h \times h)$, $hs = s'h$ and $hE = E'$ (see [3, 4]). In a differential setting $\Gamma$ is a smooth(Hausdorff) manifold, the set of units and the set of composable pairs are closed submanifolds, $s$ is a diffeomorphism, $m$ is a differential reduction and $e_L, e_R$ are surjective submersions.
A bisection of a differential groupoid $\Gamma$ is a submanifold $B \subset \Gamma$ such that $e_L|_B$ and $e_R|_B$ are diffeomorphisms $B \to E$. If $B \subset \Gamma$ is a bisection and $h : \Gamma \to \Gamma'$ is a morphism then $h(B)$ is a bisection of $\Gamma'$.

Let $\Gamma$ be a differential groupoid and let $\Omega^{1/2}_L, \Omega^{1/2}_R$ denote the bundles of complex half densities along left and right fibers. A groupoid $*$-algebra $A(\Gamma)$ is a vector space of compactly supported, smooth sections of $\Omega^{1/2}_L \otimes \Omega^{1/2}_R$ together with a convolution and $*$-operation. To write explicit formulae let us choose $\lambda_0$ - a real, nonvanishing, left invariant half density along left fibers (in that fact that means we choose a Haar system on $\Gamma$, the “choice-free” definition and other details are given in [7]). Let $\rho_0 := s(\lambda_0)$ be the corresponding right invariant half density, and $\omega_0 := \lambda_0 \otimes \rho_0$. Then any $\omega \in A(\Gamma)$ can be written as $\omega = f\omega_0$ for a function $f \in D(\Gamma)$ (smooth, compactly supported). With such a choice we write $(f_1\omega_0)(f_2\omega_0) := (f_1 * f_2)\omega_0$, $(f\omega_0)^* := (f^*)\omega_0$ and:

\[
(f_1 * f_2)(\gamma) := \int \lambda_0^2(\gamma') f_1(\gamma') f_2(s(\gamma')\gamma) = \int \rho_0^2(\gamma') f_1(\gamma') f_2(2) \ , \ f^*(\gamma) := \overline{f(s(\gamma))}
\]

The first integral is over the left fiber passing through $\gamma$, the second over the right one. The choice of $\omega_0$ defines a norm on $A(\Gamma)$:

\[
\|f\omega_0\|_0 := \|f\|_0 = \max \left\{ \sup_{e \in E} \int_{e^{-1}_{s}(e)} \lambda_0^2(\gamma)|f(\gamma)|, \sup_{e \in E} \int_{e^{-1}_{r}(e)} \rho_0^2(\gamma)|f(\gamma)| \right\}
\]

With this norm $A(\Gamma)$ is a normed $*$-algebra.

There is a faithful representation $\pi_{id}$ of $A(\Gamma)$ on $L^2(\Gamma)$ described as follows: choose $\nu_0$ - a real, nonvanishing half density on $E$; since $e_{q_2}$ is a submersion one can define $\psi_0 := \rho_0 \otimes \nu_0$ - this is a real, nonvanishing, half density on $\Gamma$. For $\psi = f_2\psi_0, f_2 \in D(\Gamma)$ the representation is given by $\pi_{id}(f_1\omega_0)(f_2\psi_0) := (\pi_{id}(f_1)f_2)\psi_0$ and $\pi_{id}(f_1)f_2 = f_1 * f_2$ is as in [2]. The estimate $\|\pi_{id}(\omega)\| \leq \|\omega\|_0$ makes possible the definition: The reduced $C^*$-algebra of a groupoid is the completion of $A(\Gamma)$ in the norm $\|\omega\| := \|\pi_{id}(\omega)\|$. We will also use the following fact which is a direct consequence of the definition of the norm $\|f\|_0$:

Lemma 1.1. Let $U \subset \Gamma$ be an open set with compact closure. There exists $M$ such that $\|f\|_0 \leq M \sup|f(\gamma)|$ for any $f \in D(\Gamma)$ with support in $U$. If $f_n \in D(\Gamma)$ have supports in a fixed compact set and $f_n$ converges to $f \in D(\Gamma)$ uniformly then $f_n\omega_0$ converges to $f\omega_0$ in $C^*_r(\Gamma)$.

With a morphism $h : \Gamma \to \Gamma'$ of differential groupoids there is associated a mapping $A(\Gamma) \otimes A(\Gamma') \to A(\Gamma')$, this mapping commutes with a (right) multiplication in $A(\Gamma')$ and we use notation $h(\omega)\omega'$; there is also a representation $\pi_h$ of $A(\Gamma)$ on $L^2(\Gamma')$: these objects satisfy some obvious compatibility conditions with respect to multiplication and $*$-operation (see [7] for a detailed exposition).

Now we recall some facts about double groups. Let $G$ be a group and $A, B \subset G$ subgroups such that $A \cap B = \{e\}$. Every element $g$ in the set $\Gamma := AB \cap BA$ can be written uniquely as

\[
g = a_L(g)b_R(g) = b_L(g)a_R(g) \ , \ a_L(g), a_R(g) \in A, b_L(g), b_R(g) \in B.
\]

These decompositions define surjections: $a_L, a_R : \Gamma \to A$ and $b_L, b_R : \Gamma \to B$ (in fact $a_L, b_R$ are defined on $AB$ and $b_L, a_R$ on $BA$, we will denote these extensions by the same symbols). The formulae:

\[
(3) \quad E := A, \quad s(g) := b_L(g)^{-1}a_L(g) = a_R(g)b_R(g)^{-1}, \quad Gr(m) := \{(b_1a_2; b_1a, b_2) : b_1a, b_2 \in \Gamma\}
\]

define the structure of the groupoid $\Gamma_A$ over $A$ on $\Gamma$. The analogous formulae define the groupoid $\Gamma_B$ over $B$. On the other hand for a subgroup $B \subset G$ there is a (right) transformation groupoid $(B \setminus G) \times B$.

The following lemma explains relation between these groupoids.
Lemma 1.2. The map:

\[ \Phi : \Gamma_A \ni g \mapsto ([a_L(g)], b_R(g)) \in (B \setminus G) \times B \]

is an isomorphism of the groupoid \( \Gamma_A \) over \( A \) with the restriction of \( (\text{a right}) \) transformation groupoid \( (B \setminus G) \times B \) to the set \( \{ [a] : a \in A \} \subset B \setminus G \).

Proof: We give only the sketch of the proof. Let \( \Gamma_A \ni g = ab = b'a' \). Then \( \Phi(g) = ([a], b) \) and \([a] \cdot b = [ab] = [a'] \), so \( \Phi(\Gamma_A) \) is really contained in the restriction. On the other hand, if \((g], b)\) is an element of the restriction i.e \([g] = [a]\) and \([gb] = [a']\) then the mapping \((g], b) \mapsto a_R(g)b \in G\) is well defined, has image in \( \Gamma_A \) and is the inverse of \( \Phi \). Direct computations show that \( \Phi \) is an isomorphism of groupoids.

If \( AB = G \) (i.e. \( \Gamma = G \)) the triple \((G; A, B)\) is called a double group and in this situation we will denote groupoids \( \Gamma_A, \Gamma_B \) by \( G_A, G_B \). It turns out that \( \delta := \rho^T_B : G_A \rightrightarrows G_A \times G_A \) is a coassociative morphism of groupoids. Applying the lemma 1.2 to the groupoid \( G_A \) we can identify it with the transformation groupoid \( (B \setminus G) \times B \). So \( G_A = A \times B \) is a right transformation groupoid for the action \((a, b) \mapsto a_R(ab)\) i.e the structure is given by:

\[ E := \{(a, e) : a \in A\}, \quad s(a, b) := (aR(ab), b^{-1}), \]
\[ m := \{(a_1, b_1 b_2 ; a_1, b_1, a_R(a_1 b_1), b_2) : a_1 \in A, b_1, b_2 \in B\} \]

In the formula above, we identify a relation \( m : \Gamma \times \Gamma \rightrightarrows \Gamma \) with its graph, i.e. subset of \( \Gamma \times \Gamma \). We will use such notation throughout the paper. If \( G \) is a Lie group, \( A, B \) are closed subgroups, \( A \cap B = \{e\} \), \( AB = G \) then \((G; A, B)\) is called a double Lie group. It turns out that the mapping \( \delta \) extends to the coassociative morphism \( \Delta \) of \( C^\ast_r(G_A) \) and \( C^\ast_r(G_A \times G_A) = C^\ast_r(G_A) \otimes C^\ast_r(G_A) \) which satisfies density conditions (cls denotes the closed linear span):

\[ \text{cls} \{ \Delta(a) (I \otimes b) : a, b \in C^\ast_r(G_A) \} = \text{cls} \{ \Delta(a) (b \otimes I) : a, b \in C^\ast_r(G_A) \} = C^\ast_r(G_A) \otimes C^\ast_r(G_A) \]

There are other objects that make the pair \((C^\ast_r(G_A), \Delta)\) a locally compact quantum group, we refer to \( \text{[1]} \) for details.

The quantum 'ax+b' does not completely fit into this framework, but as we will see, it is possible to describe it using this approach as a guiding line. The main technical problem is that vector fields that "should" define operators affiliated to a groupoid \( C^\ast \)-algebra are not complete, so the operators are not essentially selfadjoint on their "natural" domains. So one has to choose right domains or overcome this problem in a different way.

The next section "sets the stage": in the third one we consider a general situation in which the twist can be defined and apply results to the 'ax+b' group in the fourth section. (The situation from the third section also appears in the \( \kappa \)-Poincare Group, this will be described in a forthcoming paper). In the fifth section we give generators and relations and compute the action of comultiplication on them; we also express the twist as a function of generators. In the last section we consider our group as a deformation of a Poisson-Lie group. In the appendix we collect some formulae used in the paper.

2. Setup

Let \( G \) be the 'ax+b' group i.e. \( G := \mathbb{R} \times \mathbb{R}_s \) with the multiplication \((b_1, a_1)(b_2, a_2) := (b_1 + a_1 b_2, a_1 a_2)\). For \( s \in \mathbb{R} \cup \{\infty\} \) let us define the closed subgroup of \( G \):

\[ C_s := \{(b, 1 + sb) : 1 + sb \neq 0\}, \quad s \in \mathbb{R}; \quad C_\infty := \{(0, a) : a \in \mathbb{R}_s\}. \]

(If we treat \( \mathbb{R} \cup \{\infty\} \) as one point compactification of \( \mathbb{R} \) the mapping \( s \mapsto C_s \) is continuous as a mapping into closed subgroups of \( G \) with Fell topology). Let \( B := C_0 \) and \( A := C_\infty \). Note that for \( s \neq t \) we have \( C_s \cap C_t = \{e\} \) and for \( s \in \mathbb{R} \setminus \{0\} \): \( C_s = \{ (\frac{c-1}{s}, c) : c \in \mathbb{R}_s \} \).
For $s \neq t$ let:
\[ \Gamma_{st} := C_s C_t \cap C_t C_s = \{(b, a) : stb - sa + t \neq 0, \ stb - ta + s \neq 0\} = \Gamma_{ts} \]

It is straightforward to check that each $\Gamma_{st}$ is open and dense in $G$, and for $t \neq 0$ $\Gamma_{0t} = G$. $\Gamma_{st}$ is a differential groupoid over $C_s$ and $C_t$.

$G$ acts on $\{\Gamma_s : s \in \mathbb{R} \cup \{\infty\}\}$ by adjoint action and this action induces isomorphisms of corresponding differential groupoids. So it is sufficient to consider the family
\[ \Gamma_s := \Gamma_{s\infty} = C_s A \cap AC_s = \{(b, a) : (1 + sb)(a - sb) \neq 0\} \]

and groupoid structures over $C_s$ and $A$. Projections in $\Gamma_s$ on $C_s$ and $A$ will be denoted by $\tilde{a}_L, \tilde{a}_R : \Gamma_s \to A$ and $\tilde{c}_L, \tilde{c}_R : \Gamma_s \to C_s$, they are given by:
\[ \tilde{a}_L(b, a) := (0, a - sb), \quad \tilde{a}_R(b, a) := \left(0, \frac{a}{1 + sb}\right), \]
\[ \tilde{c}_L(b, a) := (b, 1 + sb), \quad \tilde{c}_R(b, a) := \left(\frac{b}{a - sb}, \frac{a}{a - sb}\right) \]

Remaining parts of structures of groupoids are as follows.
For $\Gamma_s \ni C_s$: the inverse $\tilde{s}_C(b, a) := \left(\frac{b}{a - sb}, \frac{1 + sb}{a - sb}\right)$ and the multiplication relation:
\[ \tilde{m}_C := \{(b_1, \frac{a_1 a_2}{1 + sb_2}; b_1, a_1, b_2, a_2) : b_1 = b_2(a_1 - sb_1)\} \subset \Gamma_s \times \Gamma_s \times \Gamma_s \]

For $\Gamma_s \ni A$: the inverse $\tilde{s}_A(b, a) := \left(-\frac{b}{1 + sb}, \frac{a - sb}{1 + sb}\right)$ and the multiplication relation:
\[ \tilde{m}_A := \{(b_1 + (1 + sb_1)b_2, (1 + sb_1)a_2; b_1, a_1, b_2, a_2) : a_1 = (1 + sb_1)(a_2 - sb_2)\} \subset \Gamma_s \times \Gamma_s \times \Gamma_s \]

Straightforward computations show that, for $s \neq 0$ the map:
\[ \Gamma_s \ni (b, a) \mapsto (-\frac{1}{s}, -1)(b, a)(-\frac{1}{s}, -1)^{-1} = (-\frac{1}{s} - b + \frac{a}{s}, a) \in \Gamma_s \]

is an isomorphism of $\Gamma_s \ni C_s$ and $\Gamma_s \ni A$.

Groupoid structures on $\Gamma_0 = G$ are given by the double Lie group $(G; A, B)$; for $s \neq 0$ the map:
\[ \Gamma_s \ni (b, a) \mapsto (0, s)(b, a)(0, s)^{-1} = (sb, a) \in \Gamma_1 \]

gives isomorphisms of both groupoid structures, so it is enough to consider $s = 1$.

Let us now denote $C := C_1$ and $\Gamma := \Gamma_1$. On $\Gamma$ there are two (isomorphic) groupoid structures:
$\Gamma_C : \Gamma \rightrightarrows C$ with structure
\[ \tilde{c}_L(b, a) = (b, 1 + b), \quad \tilde{c}_R(b, a) = \left(\frac{b}{a - b}, \frac{a}{a - b}\right), \quad \tilde{s}_C(b, a) := \left(\frac{b}{a - b}, \frac{1 + b}{a - b}\right) \]
\[ \tilde{m}_C := \{(b_1, \frac{a_1 a_2}{1 + b_2}; b_1, a_1, b_2, a_2) : b_1 = b_2(a_1 - b_1)\} \]

$\Gamma_A : \Gamma \rightrightarrows A$ with structure
\[ a_L(b, a) = (0, a - b), \quad a_R(b, a) = \left(0, \frac{a}{1 + b}\right), \quad s_A(b, a) := \left(-\frac{b}{1 + b}, \frac{a - b}{1 + b}\right), \]
\[ \tilde{m}_A := \{(b_1 + (1 + b_1)b_2, (1 + b_1)a_2; b_1, a_1, b_2, a_2) : a_1 = (1 + b_1)(a_2 - b_2)\} \]

Since $(G; B, C)$ is a double Lie group there are two groupoid structures on $G$:
$G_B : G \rightrightarrows B$:
\[ b_L(b, a) := (b - a + 1, 1), \quad b_R(b, a) := \left(\frac{b - a + 1}{a}, 1\right), \quad s_B(b, a) := \left(\frac{2 - 2a + b}{a}, \frac{1}{a}\right) \]
\[ \tilde{m}_B := \{(b_1 + a_1(a_2 - 1), a_1a_2; b_1, a_1, b_2, a_2) : 1 + b_1 - a_1 = a_1(1 + b_2 - a_2)\} \]
As described above, in this situation, there is the groupoid $G$. This is investigated in the next section.

Let us denote $\delta_0 := m^T_C$; this is a coassociative morphism: $G_B \to G_B \times G_B$.

To summarize, we arrive at the following situation: there is a Lie group $G$ and three closed subgroups $A, B, C$ satisfying conditions

$$G = BC, \ A \cap C = B \cap C = \{e\}$$

This is investigated in the next section.

### 3. The twist

Let $G$ be a group and $A, B, C \subset G$ subgroups satisfying conditions:

$$A \cap B = \{e\} = C \cap B, \ AB = G.$$  

As described above, in this situation, there is the groupoid $G_A$, and the (coassociative) morphism $\delta_0 : G_A \to G_A \times G_A$, explicitly the graph of $\delta_0$ is equal to:

$$\delta_0 = \{(a_1b, ba_2; a_1a_2) : a_1, a_2 \in A, \ b \in B\}$$

Let us note that $b_L(g), a_L(g)$ and $b_R(g), a_R(g)$ determine $g$ uniquely:

$$g = a_R(a_R(g)b_R(g)^{-1})b_R(g) = b_L(g)a_L(b_L(g)^{-1})a_L(g).$$

Using the lemma [1.2] we see that this is a transformation groupoid $(B \setminus G) \times B$ and the isomorphism is

$$(B \setminus G) \times B \ni ([g], b) \mapsto a_R(g)b \in G$$

Let $\Gamma := CB \cap BC$ and consider on $\Gamma$ the groupoid structure $\Gamma_C$ over $C$ described above together with a relation

$$\tilde{m}^T_B := \{(c_1b_1, b_1c_2; c_1b_1c_2) : c_1b_1, b_1c_2 \in \Gamma\} \subset \Gamma \times \Gamma \times \Gamma.$$ 

The corresponding projections will be denoted by $\tilde{b}_L, \tilde{b}_R$ and $c_R, c_L$. Again, by the lemma [1.2] we identify the groupoid $\Gamma \to C$ with the restriction of $(B \setminus G) \times B$ and then with the restriction of $G_A$ to the set $A' := A \cap BC$, i.e. with $a_L^{-1}(A') \cap a_R^{-1}(A')$. This isomorphism is given by:

$$\Gamma_C \ni cb \mapsto a_R(c)b \in G_A$$

The image of $\tilde{m}^T_B$ is equal to:

$$\{(a_R(c_1)b_1, b_1c_2; b_R(c_1c_2)b_2) : c_1b_1 = \tilde{b}_1c_1, b_1c_2 = c_2b_2\}$$

Let us now define the basic object of this section

$$T := \{(g, a) : b_R(g)a \in C\} = \{(a_1b_L(a_2)^{-1}, a_2) : a_1 \in A, a_2 \in A'\} \subset G_A \times G_A.$$ 

Using the definition (9) of $\delta_0$ one easily computes images of $T$ by relations $id \times \delta_0$ and $\delta_0 \times id$:

$$\text{id} \times \delta_0 : \{(g_1, a_2, a_3) : a_2a_3 \in A', \ b_R(g_1) = \tilde{b}_L(a_2a_3)^{-1}\}$$

$$\delta_0 \times id : \{(g_1, g_2, a_3) : b_R(g_2) = b_L(g_2), \ a_3 \in A', \ b_R(g_2) = \tilde{b}_L(a_3)^{-1}\}$$

Let us also denote $T_{12} := T \times A \subset G_A \times G_A \times G_A$ and $T_{23} := A \times T \subset G_A \times G_A \times G_A$.

The main properties of $T$ are listed in the following

**Proposition 3.1.**

1. $T$ is a section of left and right projections (in $G_A \times G_A$) over the set $A \times A'$ and a bisection of $G_A \times \Gamma_C$;
(2) \((\text{id} \times \delta_0) T\) is a section of left and right projections \((\text{in } G_A \times G_A \times G_A)\) over the set \(A \times \delta_0(A') = \{(a_1, a_2, a_3) : a_2 a_3 \in A'\};\)

(3) \((\delta_0 \times \text{id}) T\) is a section of left and right projections over the set \(A \times A \times A';\)

(4) \(T_{23}(\text{id} \times \delta_0) T = T_{12}(\delta_0 \times \text{id}) T\) (equality of sets in \(G_A \times G_A \times G_A\)), moreover this set is a section of the right projection over \(A \times (\delta_0(A') \cap (A \times A'))\) and the left projection over \(A \times A' \times A'.\)

Proof: 1) Since the “right leg” of \(T\) is \(A'\), it is clear that if \(T\) is a section of left and right projection over \(A \times A'\) it is a bisection of \(G_A \times \Gamma_C\). The formula (10) implies that \(T\) is a section of \(a_L \times a_L\) over \(A \times A'\). But since \(g \in G\) is determined by \(a_R(g)\) and \(b_R(g)\), it is enough to show that for \((a_1, a_2) \in A \times A'\) there exists \(g\) with \(a_R(g_1) = a_1\) and \((g_1, a_2) \in T\). So let \(a_2 = b_0 c_0\) then for \(g_1 := s_A(a_1 b_0)\) we have \(a_R(g_1) = a_1\) and \(b_R(g_1) = b_0^{-1}\) so \((g_1, a_2)\) belongs to \(T\).

2) From formula (11) it follows that \((\text{id} \times \delta_0) T\) is a section of the left projection over \(A \times \delta_0(A')\). As in point 1), for \((a_1, a_2, a_3) \in A \times \delta_0(A')\) with \(a_2 a_3 =: b_0 c_0\) one has \(a_R(s_A(a_1 b_0)) = a_1\) and \((s_A(a_1 b_0), a_2, a_3) \in (\text{id} \times \delta_0) T\).

3) Follows directly from formula (12).

4) By direct computation one gets:

\[ T_{23}(\text{id} \times \delta_0) T = \{(g_1, g_2, a_3) : a_3, a_R(g_2)a_3 \in A', b_R(g_2) = \tilde{b}_L(a_3)^{-1}, b_R(g_1) = \tilde{b}_L(a_R(g_2)a_3)^{-1}\} \]

and the same result for \(T_{12}(\delta_0 \times \text{id}) T\). This formula implies that for any \((a_1, a_2, a_3) \in A \times (\delta_0(A') \cap (A \times A'))\) there exists exactly one pair \((g_1, g_2)\) such that \(a_1 = a_R(g_1), a_2 = a_R(g_2)\) and \((g_1, g_2, a_3) \in T_{23}(\text{id} \times \delta_0) T\).

It remains to prove that it is a section of the left projection over \(A \times A' \times A'\). It is clear that for a given \((a_1, a_2, a_3)\) there exists at most one pair \((g_1, g_2)\) such that \(a_1 = a_L(g_1), a_2 = a_L(g_2)\) and \((g_1, g_2, a_3) \in T_{23}(\text{id} \times \delta_0) T\). The rest follows from the following observation:

Let \(a_3 \in A'\), \(a_3 =: b_0 c_0\) then

\[ a \in A' \iff a_R(ab_0^{-1})a_3 \in A' \]

Indeed if \(a =: \tilde{b}_0 \tilde{c}_0\) then

\[ a_R(ab_0^{-1})a_3 = b_L(ab_0^{-1})^{-1}ab_0^{-1}a_3 = b_L(ab_0^{-1})^{-1}ac_0 = b_L(ab_0^{-1})^{-1}b_0 \tilde{c}_0 c_0 \in BC \]

On the other direction,

\[ a_R(ab_0^{-1})a_3 \in BC \Rightarrow ab_0^{-1}a_3 \in BC \Rightarrow ac_0 \in BC \Rightarrow a \in BC \]

Because of the prop \([3.1]\) the left multiplication by \(T\), which we denote by the same symbol, is a bijection of \(G_A \times a_L^{-1}(A')\); also the left multiplication by \((\delta_0 \times \text{id}) T\), which will be denoted by \(T_1\), is a bijection of \(G_A \times G_A \times a_L^{-1}(A')\), and the left multiplication by \((\text{id} \times \delta_0) T\), denoted by \(T_2\), is a bijection of \(G_A \times (a_L \times a_L)^{-1}(\delta_0(A'))\). These mappings are given by:

\[ T : (a_1 b_1, a_2 b_2) \mapsto (s_A(a_1 \tilde{b}_L(a_2))b_1, a_2 b_2) = (b_L(a_1 \tilde{b}_L(a_2))^{-1} a_1 b_1, a_2 b_2) = (a_R(a_1 \tilde{b}_L(a_2))\tilde{b}_L(a_2)^{-1} a_1 b_1, a_2 b_2) \]

\[ T^{-1} : (a_1 b_1, a_2 b_2) \mapsto (s_A(a_1 \tilde{b}_L(a_2)^{-1})b_1, a_2 b_2) \]

\[ T_1 : (a_1 b_1, a_2 b_2, a_3 b_3) \mapsto (s_A(a_1 b_L(a_2 \tilde{b}_L(a_3)))b_1, s_A(a_2 \tilde{b}_L(a_3))b_2, a_3 b_3) = (b_L(a_1 a_2 \tilde{b}_L(a_3))^{-1} a_1 b_1, b_L(a_2 \tilde{b}_L(a_3))^{-1} a_2 b_2, a_3 b_3) \]

\[ T_1^{-1} : (a_1 b_1, a_2 b_2, a_3 b_3) \mapsto (s_A(a_1 b_L(a_2 \tilde{b}_L(a_3)^{-1}))b_1, s_A(a_2 \tilde{b}_L(a_3)^{-1})b_2, a_3 b_3) \]

\[ T_2 : (a_1 b_1, a_2 b_2, a_3 b_3) \mapsto (s_A(a_1 \tilde{b}_L(a_2 a_3))b_1, a_2 b_2, a_3 b_3) = (b_L(a_1 \tilde{b}_L(a_2 a_3))^{-1} a_1 b_1, a_2 b_2, a_3 b_3) \]
The composition $T_2T_1 = T_2T_2$ is a bijection from $G_A \times (A_L \times A_L)^{-1} (\delta_0(A') \cap (A \times A'))$ to $G_A \times (A_L \times A_L)^{-1} (A' \times A')$ and is given by:

$$T_2T_1 : (a_1 b_1, a_2 b_2, a_3 b_3) \mapsto (s_A (a_1 b_L (a_2 a_3))^{-1} b_1, a_2 b_2, a_3 b_3)$$

Let $Ad_T : G_A \times G_A \to G_A \times G_A$ be a relation defined by:

$$(g_1, g_2, g_3, g_4) \in Ad_T \iff \exists t_1, t_2 \in T : (g_1, g_2) = t_1 (g_3, g_4) (s_A \times s_A) (t_2).$$

Using the definition of $T$ \[10\] one gets:

$$Ad_T = \{(aR (a_3 b_L (a_4)) b_L (a_4)^{-1} b_3 b_L (aR (a_4 b_4)), aR b_4; aR b_4, aR b_4) : aR, aR b_4 \in A'\}.$$ 

Finally, let us define the relation $\delta := Ad_T \cdot \delta_0 : G_A \to G_A \times G_A.$

$$(16)\quad \delta = \{(aR (a_3 a_2 b_L (a_2)) b_L (a_2)^{-1} b_2 b_L (aR (a_2 b_2)), aR b_2, aR b_2) : aR, aR b_2 \in A'\}.$$ 

**Lemma 3.2.** $\delta$ is an extension of $\bar{m}_B^T$ i.e. $\bar{m}_B^T \subset \delta$.

**Proof:** Recall that

$$\bar{m}_B^T = \{(aR (c_1) b_1, aR (c_2) b_2; aR (c_1 c_2) b_2) : c_1 b_1 = b_1 \tilde{c}_1, b_1 \tilde{c}_2 = c_2 b_2\}.$$ 

By lemma \[12\] $aR (c_2)$ and $aR (aR (c_2) b_2)$ are in $A'$. So it remains to prove that:

$$aR (aR (c_1 c_2) aR (c_2) b_2) b_L (aR (c_2) b_2) b_L (aR (aR (c_2) b_2)) = aR (c_1) b_1$$

Let $c_2 = b_0 a_0$ then $aR (c_2) = a_0$ and $b_L (aR (c_2)) = b_0^{-1}$. We have:

$$aR (aR (c_1 c_2) aR (c_2) b_2) b_L (aR (c_2) b_2) = aR (c_1 c_2 a_0^{-1} b_0^{-1}) = aR (c_1)$$

and

$$aR (c_1) b_L (aR (c_2) b_2) b_L (aR (aR (c_2) b_2)) = aR (c_1) b_0 b_L (a_0 b_2) b_L (aR (aR (c_2) b_2)) =$$

$$= aR (c_1) b_L (b_0 a_0 b_2) b_L (aR (aR (c_2) b_2)) = aR (c_1) b_L (b_0 a_0 b_2) aR (b_0 a_0 b_2) =$$

$$= aR (c_1) b_L (c_2 b_2)) = aR (c_1) b_1$$

Everything above was purely algebraic. Now we add some differential conditions, that enable us to use $T$ to twist the comultiplication on $C^*_v(G_A)$.

The following are standing assumptions for the rest of this section.

**Assumptions 3.3.**

1. $G$ is a Lie group and $A, B, C$ are closed Lie subgroups such that $A \cap B = \{e\} = C \cap B, AB = G$.

2. The set $\Gamma := BC \cap CB$ is open and dense in $G$.

3. Let $U := a_2^{-1} (A')$ and $A(U)$ be the linear space of elements from $A(G_A)$ supported in $U$. We assume that $A(U)$ is dense in $C^*_v(G_A)$.

4. For a compact set $K_B \subset B$, open $V \subset A$ and $(a_1, a_2) \in A \times A'$ let us define a set $Z(a_1, a_2, K_B; V) := K_B \cap \{b \in B : aR (a_1 b) a_2 \in V\}$ and a function:

$$A \times A' \ni (a_1, a_2) \mapsto \mu(a_1, a_2, K_B; V) := \int_{Z(a_1, a_2, K_B; V)} d_1 b.$$ 

For compact sets $K_1 \subset A$ and $K_2 \subset A'$ let $\mu(K_1, K_2, K_B; V) := \sup \{\mu(a_1, a_2, K_B; V) : a_1 \in K_1, a_2 \in K_2\}$ We assume that

$$(17)\quad \forall \epsilon > 0 \exists V - \text{ a neighbourhood of } A \setminus A' \text{ in } A : \mu(K_1, K_2, K_B; V) \leq \epsilon.$$
These technical assumptions are sufficient to prove all we need and are satisfied in examples we are interested in (probably they can be weakened and some of them imply others). The strategy is to interpret all in \( C_\ast^\prime(G_A) \) where there is a well defined comultiplication. The first simple result is:

**Lemma 3.4.** \( C_\ast^\prime(\Gamma_C) = C_\ast^\prime(G_A) \)

Proof: Let \( U \) and \( A(U) \) be as in assumptions 3.3. Then \( J := A(U) \) is a right ideal in \( A(G_A) \). Then \( J^\ast \) is a left ideal which is also dense in \( C_\ast^\prime(G_A) \). The subalgebra \( J J^\ast \) is contained in \( A(\Gamma) \). Every positive element in \( C_\ast^\prime(G_A) \) can be approximated by elements from \( J J^\ast \), so the same is true for every element in \( C_\ast^\prime(G_A) \).

It is straightforward to check that assumptions 3.3 guarantee that all sets appearing in prop 3.1 are submanifolds and the corresponding mappings are diffeomorphisms. Since \( T \) is a bisection over the open set \( A \times A' \) it defines (by a push-forward) mapping of \( A(G_A \times U) \) which will be denoted again by \( T \). The prop 3.1 suggests that it can be used to twist the comultiplication \( \Delta_0 \) on \( C_\ast^\prime(G_A) \). This is the content of the next proposition.

**Proposition 3.5.** a) The mapping \( T : A(G_A \times U) \to A(G_A \times U) \) extends to a unitary \( \hat{T} \in M(C_\ast^\prime(G_A) \otimes C_\ast^\prime(G_A)) \) which satisfies:
\[(\hat{T} \otimes I)(\Delta_0 \otimes id)\hat{T} = (I \otimes \hat{T})(id \otimes \Delta_0)\hat{T}\]

b) Because of a), the formula \( \Delta(a) := \hat{T}\Delta_0(a)\hat{T}^{-1} \) defines a coassociative morphism. For this morphism:
\[cls\{\Delta(a)(I \otimes b) : a, b \in C_\ast^\prime(G_A)\} = cls\{\Delta(a)(b \otimes I) : a, b \in C_\ast^\prime(G_A)\} = C_\ast^\prime(G_A) \otimes C_\ast^\prime(G_A)\]

("cls" stands for "closed linear span").

The rest of this section is dedicated to the proof of this proposition.

**Proof of the statement a):**
Mappings \( T, T_1, T_2, T_{12}, T_{23}, T_{12}T_1 = T_{23}T_2 \) are originally defined on linear subspaces of \( A(G_A \times G_A) \) and \( A(G_A \times G_A \times G_A) \). So the first task is to extend them to multipliers. To this end we will use the following simple lemma:

**Lemma 3.6.** Let \( A \) be a \( C^\ast \)-algebra and \( V_I, V_2 \) be dense linear subspaces of \( A \). Suppose that a linear bijection \( T : V_I \to V_2 \) satisfies \( \omega_2^2(T\omega_1) = (T^{-1}\omega_2)^*\omega_1 \) for any \( \omega_2 \in V_2, \omega_1 \in V_1 \). Then \( T \) extends to a unitary multiplier of \( A \).

We need also the lemma which is a straightforward generalization of the similar fact for a bisection which was proven in [7].

**Lemma 3.7.** Let \( \Gamma \) be a differential groupoid and \( C \subset \Gamma \) be a submanifold such that \( \epsilon_L|_C : C \to C_l \) and \( \epsilon_R|_C : C \to C_r \) are diffeomorphisms onto open sets \( C_l, C_l \subset E \). Then for \( \omega_1 \in A(\epsilon_L^{-1}(C_l)) \), \( \omega_2 \in A(\epsilon_R^{-1}(C_l)) \) we have:
\[\omega_2^2(C\omega_1) = (s(C)\omega_2)^*\omega_1 \text{ and } (C\omega_1)^*(C\omega_1) = \omega_1^2\omega_1.\]

Using the lemma we obtain equalities:
\[
\begin{align*}
\omega_2^2(T\omega_1) &= (T^{-1}\omega_2)^*\omega_1, \quad \omega_1, \omega_2 \in A(G_A \times U), \\
\omega_2^2(T_1\omega_1) &= (T_1^{-1}\omega_2)^*\omega_1, \quad \omega_1, \omega_2 \in A(G_A \times G_A \times U), \\
\omega_2^2(T_2\omega_1) &= (T_2^{-1}\omega_2)^*\omega_1, \quad \omega_1, \omega_2 \in A(U_2), \\
\omega_2^2(T_{23}\omega_1) &= (T_{23}^{-1}\omega_2)^*\omega_1, \quad \omega_1, \omega_2 \in A(G_A \times G_A \times U), \\
\omega_2^2(T_{12}\omega_1) &= (T_{12}^{-1}\omega_2)^*\omega_1, \quad \omega_1, \omega_2 \in A(G_A \times U \times G_A), \\
\end{align*}
\]
where \( U_2 := G_A \times (a_L \times a_L)^{-1}(\delta_0(A')) \)
Now, by assumptions (3) it is clear that $A(G \times U), A(G \times A \times U)$ and $A(G \times U \times G)$ are dense in $C_\mu^r(G \times A) \otimes C^*_\nu(G_A)$ and $C_\mu^r(G_A) \otimes C^*_\nu(G_A)$ respectively, so we have multipliers $\hat{T}, \hat{T}_1, \hat{T}_{12}$ and $\hat{T}_{23}$.

To extend $T_2$ we need density of $A(U_2)$ i.e density of $A((a_L \times a_L)^{-1}(\delta(A'))$. By the results of [6], the set $\delta_0(A(G_A))A(G_A \times A)$ is linearly dense in $C^*_\nu(G_A)$ since $A(U)$ is dense, the same is true for the set $\delta_0(A(U))A(G_A \times A)$, using formula [13] one checks that this set is contained in $A((a_L \times a_L)^{-1}(\delta(A'))$. Therefore $T_2$ extends to unitary multiplier.

Now, we know that multipliers $\hat{T}_{12} \hat{T}_1$ and $\hat{T}_{23} \hat{T}_2$ restricted to the linear space $A(G_A \times (a_L \times a_L)^{-1}(\delta(A') \setminus (A \times A'))$ are equal and defines bijection to $A(G_A \times U \times U)$ Since the last set is dense in $C^*_\nu(G_A) \otimes C^*_\nu(G_A)$ they must be equal.

Now we have multipliers $\hat{T}, \hat{T}_1, \hat{T}_2, \hat{T}_{12}$ and $\hat{T}_{23}$ together with the equation $\hat{T}_{12} \hat{T}_1 = \hat{T}_{23} \hat{T}_2$. So to prove the statement a) of the proposition, it remains to show equalities:

$$\hat{T}_{12} = (\hat{T} \otimes I), \hat{T}_1 = (\delta_0 \otimes I) \hat{T}, \hat{T}_{23} = (I \otimes \hat{T}), \hat{T}_2 = (i \otimes \delta_0) \hat{T}.$$  

For $\hat{T}_{23}$ and $(I \otimes \hat{T})$ this is straightforward, since both multipliers agree on a dense set $A(G_A) \otimes A(G_A \times U)$. The same arguments work for $\hat{T}_{12}$ and $(\hat{T} \otimes I)$ because they agree on a dense set $A(G_A \times U) \otimes A(G_A)$. For two other equalities we will use two lemmas that will be proven in the end of the proof of the statement a) of the proposition.

**Lemma 3.8.** Let $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ be differential groupoids, and $h_1 : \Gamma_1 \rightarrow \Delta_1, h_2 : \Gamma_2 \rightarrow \Delta_2$ be morphisms. Assume that the mappings $\hat{h}_1, \hat{h}_2$ extends to morphisms $\phi_1, \phi_2$ of corresponding reduced $C^*$-algebras. Then $(\phi_1 \otimes \phi_2)(\omega) = (h_1 \otimes h_2)(\omega)$ for $\omega \in A(\Gamma_1 \times \Gamma_2)$

**Lemma 3.9.** For $\omega_1 \in A(G_A \times U)$ and $\omega_2 \in A(G_A \times G_A \times G_A)$:

$$T_1[(\delta_0 \otimes i \delta_0)(\omega_1)\omega_2] = (\delta_0 \otimes i \delta_0)(T\omega_1)\omega_2,$$

$$T_2[(i \delta_0 \otimes i \delta_0)(\omega_1)\omega_2] = (i \delta_0 \otimes i \delta_0)(T\omega_1)\omega_2.$$ 

Now, the multiplier $(i \otimes \delta_0) \hat{T}$ is defined by

$$[(i \otimes \delta_0) \hat{T}][(i \otimes \delta_0)(a)\omega] := (i \otimes \delta_0)(\hat{T} \omega)(a), a \in C^*_\nu(G_A \times G_A), b \in C^*_\nu(G_A \times G_A \times G_A).$$

In fact, it is enough to take a $a = \omega_1 \in A(G_A \times U) and b = \omega_2 \in A(G_A \times G_A \times G_A)$. Let us compute:

$$[(i \otimes \delta_0) \hat{T}][(i \otimes \delta_0)(\omega_1)\omega_2] = (i \otimes \delta_0)(\hat{T} \omega_1)\omega_2 =$$

$$= (i \otimes \delta_0)(T\omega_1)\omega_2 = T_2[(i \otimes \delta_0)(\omega_1)\omega_2] = T_2[(i \otimes \delta_0)(\omega_1)\omega_2]$$

Therefore $(i \otimes \delta_0) \hat{T} = \hat{T}_2$. In the same way the second equality can be proven. So to complete the proof of the statement a) it remains to prove lemmas 3.8, 3.9.

**Proof of lemmas:**

**Lemma 3.8** It is known [6], that for $\omega \in A(\Gamma_1 \times \Gamma_2)$ the mapping $(h_1 \otimes h_2)(\omega)$ defines a multiplier of $C^*_\mu(\Delta_1 \times \Delta_2) = C^*_\mu(\Delta_1) \otimes C^*_\mu(\Delta_2)$. Let $\pi_{12}$ denotes the representation defined by $h_1 \otimes h_2$ on $H := L^2(\Delta_1 \times \Delta_2)$. For $\omega' \in A(\Delta_1 \times \Delta_2)$ we have:

$$||\pi_{12}(\omega)\pi_{12}(\omega')||_{B(H)} \leq ||\pi_{12}(\omega)||_{B(H)} ||\pi_{12}(\omega')||_{B(H)} = ||\pi_{12}(\omega)||_{B(H)} ||\omega'||_{C^*_\mu}$$

Therefore we have the estimate:

$$||\pi_{12}(\omega)||_{M(C^*_\mu)} \leq ||\pi_{12}(\omega)||_{B(H)} \leq ||\omega||_0$$

Now take a sequence $\omega_n \in A(\Gamma_1) \otimes A(\Gamma_2)$ that converges to $\omega$ in a norm $||.|||_0$, so also in a $C^*_\mu(\Gamma_1) \otimes C^*_\mu(\Gamma_2)$. So $(h_1 \otimes h_2)(\omega_n)$ converges to $(h_1 \otimes h_2)(\omega)$ in $M(C^*_\mu(\Delta_1) \otimes C^*_\mu(\Delta_2))$. On the other hand $(h_1 \otimes h_2)(\omega_n) = (\phi_1 \otimes \phi_2)(\omega_n)$ so it converges to $(\phi_1 \otimes \phi_2)(\omega)$.
Lemma To prove those formulae we need the explicit form of the action of $T$, $T_1$ and $T_2$. Let us start with $T$. Writing $\omega := F\omega_0$ for $F \in D(G_A \times U)$ we define: $T(F\omega_0) := (TF)\omega_0$. As for any bisection $TF$ is given by \[ (TF)(T(g_1,g_2)) = F(g_1,g_2) \frac{(\rho_0 \otimes \rho_0)(vg_1 \otimes wg_2)}{(\rho_0 \otimes \rho_0)(T(vg_1 \otimes wg_2))}, \quad v, w \in \Lambda^{\text{max}}(T_e B) \]

Let $(g_1, g_2) = (a_1 b_1, a_2 b_2)$ and $T(g_1, g_2) =: (g_1, g_2)$. Using the definition of $T$ we obtain, for a curve $b_0(t) \subset B$ with $b_0(0) = e$, $T(b_0(t)g_1, g_2) = (b_3(t)\tilde{g}_1, g_2)$, where \[
 b_3(t) = b_L \left[ s_A(a_1 \tilde{b}_L(a_2))a_L(a_1^{-1}b_0(t)a_1)s_A(a_1 \tilde{b}_L(a_2)^{-1}) \right]^{-1} b_L(a_1 \tilde{b}_L(a_2))^{-1} b_0(t)b_L(a_1 \tilde{b}_L(a_2)), \]

and $T(g_1, b_0(t)g_2) = (b_4(t)\tilde{g}_1, b_0(t)g_2)$.

Identifying tangent spaces to right fibers with $T_e B$ in corresponding points, one sees that the map we have to consider has the form \[
 \begin{pmatrix} M_1 & M_2 \\ 0 & I \end{pmatrix}, \]

so its action on densities is determined by $M_1$ i.e. (derivative of) $b_0(t) \mapsto b_3(t)$ and this is given by: \[
 M_1 : T_e B \ni \hat{b} \mapsto -P_B \cdot \text{Ad}(s_A(a_1 \tilde{b}_L(a_2))) \cdot P_A \cdot \text{Ad}(a_1)^{-1} \hat{b} + \text{Ad}(b_L(a_1 \tilde{b}_L(a_2)))^{-1} \hat{b} \]

Straightforward computation gives: $M_1 = \text{Ad}(b_L(a_1 \tilde{b}_L(a_2)))^{-1}$. Finally using the definition of $\rho_0$ and modular functions \[
(18) \quad (TF)(g_1, g_2) = F(T^{-1}(g_1, g_2))j_B(\tilde{b}_L(a_L(g_2)))^{1/2}, \\
(T^-1F)(g_1, g_2) = F(T(g_1, g_2))j_B(\tilde{b}_L(a_L(g_2)))^{-1/2} \]

$T_2$ defines a map, again denoted by the same letter, of $A(G_A \times (a_L \times a_L)^{-1}(\hat{d}_0(A')))$. In the same way as for $T$ we have to compute the action of $T_2$ on $\rho_0 \otimes \rho_0 \otimes \rho_0$. So let $(g_1, g_2, g_3) =: (a_1 b_1, a_2 b_2, a_3 b_3)$ be in the domain of $T_2$ and $T_2(g_1, g_2, g_3) = (s_A(a_1 \tilde{b}_L(a_2 a_3) b_1, g_2, g_3) =: (g_1, g_2, g_3)$. For a curve $b_0(t) \subset B$ with $b_0(0) = e$, we obtain: $T_2(b_0(t)g_1, g_2, g_3) = (b_1(t)\tilde{g}_1, g_2, g_3)$ and \[
 b_1(t) = b_R \left[ b_L(a_1 \tilde{b}_L(a_2 a_3))^{-1} a_1 b_R(a_1^{-1}b_0(t)a_1) a_1^{-1} b_L(a_1 \tilde{b}_L(a_2 a_3)) \right] \]

Reasoning in the same way as for $T$, the action on densities is determined by the map: \[
 T_e B \ni \hat{b} \mapsto -P_B \cdot \text{Ad}(s_A(a_1 \tilde{b}_L(a_2 a_3))) \cdot P_A \cdot \text{Ad}(a_1)^{-1} \hat{b} \]

Using the definition of modular functions \[
(19) \quad (T_2 F)(g_1, g_2, g_3) = F(T_2^{-1}(g_1, g_2, g_3))j_B(\tilde{b}_L(a_2 a_3))^{1/2} \\
(T_2^{-1}F)(g_1, g_2, g_3) = F(T_2(g_1, g_2, g_3))j_B(\tilde{b}_L(a_2 a_3))^{-1/2} \]

Finally, similar computations as for $T_2$ give for $T_1$: \[
(20) \quad (T_1 F)(a_1 b_1, a_2 b_2 a_3 b_3) = F(T_1^{-1}(a_1 b_1, a_2 b_2 a_3 b_3))j_B(\tilde{b}_L(a_3))^{1/2}j_B(b_L(a_2 b_L(a_3)^{-1}))^{-1/2} \\
(T_1^{-1}F)(a_1 b_1, a_2 b_2 a_3 b_3) = F(T_1(a_1 b_1, a_2 b_2 a_3 b_3))j_B(\tilde{b}_L(a_3))^{-1/2}j_B(b_L(a_2 b_L(a_3)^{-1}))^{1/2} \]

Now we can prove formulae in the lemma. Let us compute the left hand side of the first equality:

\[
 T_1[(\hat{d}_0 \times id)(F_1)F_2](a_1 b_1, a_2 b_2, a_3 b_3) = \\
 = [(\hat{d}_0 \times id)(F_1)F_2] \left( s_A(a_1 b_1 (a_2 \tilde{b}_L(a_3)^{-1})) b_1, s_A(a_2 \tilde{b}_L(a_3)^{-1}) b_2, a_3 b_3 \right) \frac{j_B(\tilde{b}_L(a_3))^{1/2}}{j_B(b_L(a_2 b_L(a_3)^{-1}))^{1/2}} = \\
 = \frac{j_B(\tilde{b}_L(a_3))^{1/2}}{j_B(b_L(a_2 b_L(a_3)^{-1}))^{1/2}} \int_{B \times B} d'b' b'' F_1(a_R(a_1 a_2 \tilde{b}_L(a_3)^{-1}) b', a_3 b'') \times 
\]
In this way the proof of the statement a) of the prop. 3.5 is completed.

Finally we prove density conditions.

To prove the second formula we compute using formulae (19) for
$$T_2[(\hat{id} \times \delta_0)(TF_1)F_2](a_1 b_1, a_2 b_2, a_3 b_3) =$$
$$= j_B(b_L(a_2 b_3))^{\frac{1}{2}} [(\hat{id} \times \delta_0)(F_1)F_2](s_A(a_1 b_L(a_2 b_3))^{-1}) b_1, a_2 b_2, a_3 b_3) =$$
$$= j_B(b_L(a_2 b_3))^{\frac{1}{2}} \int_{B \times B} d_1 b_1 d_2 b_2 \hat{j}_B(b_L(a_3 b''))^{-\frac{1}{2}} F_1(a_1 a_2 b_L(a_2 b_3))^{-1} b_3, a_3 b'') \times$$
$$\times F_2(b_L(a_1 a_2 b_L(a_2 b_3))^{-1} a_1 b_1, a_2 a_3 b''^{-1} b_3, b_L(a_3 b''))^{-1} a_3 b_3)$$

The first argument of $F_2$ reads:
$$b_L(a_1 a_2 b_L(a_2 b_3))^{-1} a_1 b_L(a_2 b_3) b_1 =$$
$$= b_L(a_1 a_2 b_L(a_2 b_3))^{-1} a_1 b_L(a_2 b_3) b_1 =$$
$$= b_L(a_1 a_2 b_L(a_2 b_3))^{-1} a_1 b_L(a_2 b_3) b_1 =$$

Now, the right hand side:
$$\left[(\hat{id} \times \delta_0)(TF_1)F_2\right](a_1 b_1, a_2 b_2, a_3 b_3) = \int_{B \times B} d_1 b_1 d_2 b_2 \hat{j}_B(b_L(a_3 b''))^{-\frac{1}{2}} \times$$
$$\times (TF_1)(a_1 b'_1, a_2 a_3 b''_2) F_2(b_L(a_3 b''))^{-1} a_1 b_1, b_L(a_3 b'')^{-1} a_2 b_2, a_R(a_3 b'')^{-1} b''^{-1} b_3) =$$
$$= \int_{B \times B} d_1 b_1 d_2 b_2 \hat{j}_B(b_L(a_3 b''))^{-\frac{1}{2}} \times$$
$$\times F_1(b_L(a_1 a_2 b_L(a_2 b_3))^{-1} a_1 b_1, a_2 a_3 b''^{-1} a_1 b_1, b_L(a_3 b''))^{-1} a_2 b_2, a_R(a_3 b'') b''^{-1} b_3)$$

Now, after the change the variables in this integral $(b', b'') \mapsto (\hat{b}_L(a_2 b_3) b', b'')$ we get the equality.

In this way the proof of the statement a) of the prop. 3.5 is completed.

Now we pass to statement b) i.e. density conditions for $\Delta$. Since $\Delta(a)(I \otimes b) = \hat{T} \Delta_0(a) \hat{T}^{-1}(I \otimes b)$ and $\hat{T}$ is a unitary multiplier, it is enough to prove that $\Delta_0(a) \hat{T}^{-1}(I \otimes b)$ is linearly dense in $C^*_\tau(G_A) \otimes C^*_\tau(G_A)$; the same is true for $\Delta_0(a) \hat{T}^{-1}(b \otimes I)$. As in (3) we give explicit formulae for $\Delta_0(a) \hat{T}^{-1}(I \otimes b)$ and $\Delta_0(a) \hat{T}^{-1}(b \otimes I)$ for $a, b \in A(G_A)$ supported in some subsets. Then, by continuity we obtain inclusions $\Delta_0(a) \hat{T}^{-1}(I \otimes b) \subset C^*_\tau(G_A) \otimes C^*_\tau(G_A)$ and $\Delta_0(a) \hat{T}^{-1}(b \otimes I) \subset C^*_\tau(G_A) \otimes C^*_\tau(G_A)$ for all $a, b \in C^*_\tau(G_A)$. Finally we prove density conditions.
Lemma 3.10. Sets $D(\Phi_1), D(\Psi_1), D(\Phi_2), D(\Psi_2)$ are open. $\Phi_1$ is a diffeomorphism of $D(\Phi_1)$ and $D(\Psi_1)$ and $\Psi_1 = \Phi_1^{-1}$. $\Phi_2$ is a diffeomorphism of $D(\Phi_2)$ and $D(\Psi_2)$ and $\Psi_2 = \Phi_2^{-1}$.

Proof: Short inspection of definitions shows that really these sets are open and mappings are smooth. So it is sufficient to prove that $\Phi_1, \Psi_1$ and $\Phi_2, \Psi_2$ are pairs of mutually inverse mappings. The proof will be given for the first pair, for the second one it is similar.

1) The inclusion $\Phi_1(D(\Phi_1)) \subset D(\Psi_1)$. Let $(a_1 b_1, a_2 b_2) \in D(\Phi_1), a_2^{-1} b_1 =: b_0 c_0$.

$$\Phi_1(a_1 b_1, a_2 b_2) = (a_1 a_2 b_0, a_R(a_2 b_0) b_0^{-1} b_2) =: (\tilde{a}_1 b_1, \tilde{a}_2 b_2).$$

And we have: $s_A(\tilde{a}_2 b_1^{-1}) = s_A(a_R(a_2 b_0) b_0^{-1}) = a_2 b_0 = b_1 c_0^{-1} \in BC$.

2) The inclusion $\Psi_1(D(\Psi_1)) \subset D(\Phi_1)$. Let $(a_1 b_1, a_2 b_2) \in D(\Psi_1), s_A(a_2 b_1^{-1}) = a_R(a_2 b_1^{-1}) b_1 =: b_0 c_0$.

$$\Psi_1(a_1 b_1, a_2 b_2) = (a_1 a_2 b_1^{-1} b_0, a_R(a_2 b_1^{-1}) b_1 b_2) =: (\tilde{a}_1 b_1, \tilde{a}_2 b_2).$$

So $\tilde{a}_2 b_1 = a_R(a_2 b_1^{-1}) b_0 = b_1 c_0^{-1} \in BC$.

3) Composition $\Psi_1 \Phi_1$:

$$\Psi_1 \Phi_1(a_1 b_1, a_2 b_2) = \Psi_1(a_1 a_2 b_0, a_R(a_2 b_0) b_0^{-1} b_2) =$$

$$= \Psi_1(\tilde{a}_1 b_1, \tilde{a}_2 b_2) = (\tilde{a}_1 a_R(\tilde{a}_2 b_1^{-1}) b_0, a_R(\tilde{a}_2 b_1^{-1}) b_1 b_2), \text{ where } s_A(\tilde{a}_2 b_1^{-1}) =: \tilde{b}_0 \tilde{c}_0.$$

But $s_A(\tilde{a}_2 b_1^{-1}) = s_A(a_R(a_2 b_0) b_0^{-1}) = a_2 b_0$ i.e. $\tilde{b}_0 = b_1$, so we have:

$$\tilde{a}_1 a_R(\tilde{a}_2 b_1^{-1}) b_0 = a_1 a_2 a_R(a_R(a_2 b_0) b_0^{-1}) b_1 = a_1 b_1$$

$$a_R(\tilde{a}_2 b_1^{-1}) b_1 b_2 = a_2 b_0 b_0^{-1} b_2 = a_2 b_2$$

4) Composition $\Phi_1 \Psi_1$:

$$\Phi_1 \Psi_1(a_1 b_1, a_2 b_2) = \Phi_1(a_1 a_R(a_2 b_1^{-1}) b_0, a_R(a_2 b_1^{-1}) b_1 b_2) =$$

$$= \Phi_1(\tilde{a}_1 b_1, \tilde{a}_2 b_2) = (\tilde{a}_1 \tilde{a}_2 \tilde{b}_0, a_R(\tilde{a}_2 b_0) b_0^{-1} b_2),$$

where $s_A(a_2 b_1^{-1}) =: b_0 c_0$ and $\tilde{a}_2^{-1} b_1 =: \tilde{b}_0 \tilde{c}_0$. But as shown in point 2) $\tilde{b}_0 = b_1$, so

$$\tilde{a}_1 \tilde{a}_2 \tilde{b}_0 = a_1 a_R(a_2 b_1^{-1}) a_R(a_2 b_1^{-1}) b_1 = a_1 b_1$$

$$a_R(\tilde{a}_2 b_0) b_0^{-1} b_2 = a_R(a_R(a_2 b_1^{-1}) b_1 b_2 = a_2 b_2$$
By the lemma $\Phi_1^*: \mathcal{D}(D(\Psi_1)) \to \mathcal{D}(D(\Phi_1))$ and $\Phi_2^*: \mathcal{D}(D(\Psi_2)) \to \mathcal{D}(D(\Phi_2))$ are isomorphisms of vector spaces.

**Lemma 3.11.** There exists smooth, positive function $k_1 : D(\Phi_1) \to \mathbb{R}$ such that

$$(k_1 \Phi_1^*(f_1 \otimes f_2)) \ast (f_3 \otimes f_4) = \hat{\delta}_0(f_1)T^{-1}(f_3 \otimes f_2 \ast f_4)$$

for $f_1, f_3, f_4 \in D(G_A), f_2 \in D(a_L^{-1}(A'))$.

There exists smooth, positive function $k_2 : D(\Phi_2) \to \mathbb{R}$ such that

$$(k_2 \Phi_2^*(f_1 \otimes f_2)) \ast (f_3 \otimes f_4) = \hat{\delta}_0(f_1)T^{-1}(f_2 \ast f_3 \otimes f_4)$$

for $f_3 \in D(G_A), f_2, f_4 \in D(a_L^{-1}(A'))$, $f_1 \in D(a_R^{-1}(A'))$.

Since in both cases the set of possible $f_3 \otimes f_4$ is linearly dense in $C^*_r(G_A) \otimes C^*_r(G_A)$, the lemma means that

$$k_1 \Phi_1^*(f_1 \otimes f_2) = \hat{\delta}_0(f_1)T^{-1}(I \otimes f_2), f_1, \in D(G_A), f_2 \in D(a_L^{-1}(A'))$$

$$k_2 \Phi_2^*(f_1 \otimes f_2) = \hat{\delta}_0(f_1)T^{-1}(f_2 \otimes I), f_1 \in D(a_R^{-1}(A')), f_2 \in D(a_L^{-1}(A'))$$

and by density of subspaces $D(G_A), D(a_L^{-1}(A')), D(a_R^{-1}(A'))$ in $C^*_r(G_A)$ we obtain

$$\Delta(C^*_r(G_A))(I \otimes C^*_r(G_A)), \Delta(C^*_r(G_A))(C^*_r(G_A) \otimes I) \subset C^*_r(G_A) \otimes C^*_r(G_A).$$

**Proof of the lemma:** Let $f_1, f_2, f_3, f_4$ be as stated. Let us compute:

$$[k_1 \Phi_1^*(f_1 \otimes f_2)] \ast (f_3 \otimes f_4)(a_1b_1, a_2b_2) = \int_{B \times B} db db' k_1(a_1b, a_2b) \times$$

$$\times [\Phi_1^*(f_1 \otimes f_2)](a_1b, a_2b) f_3(b_L^{-1}(a_1b)a_1b) f_4(b_L^{-1}(a_2b) a_2b2)$$

Since $\Phi_1^*(f_1 \otimes f_2) \in \mathcal{D}(D(\Phi_1))$ the integral over $b$ can be restricted to the set $B_{a_2^{-1}} := \{b : a_R(a_2^{-1}b) \in A'\}$.

Now, the map $B_{a_2^{-1}} \times B \ni (b, \tilde{b}) \mapsto (b' := \tilde{b}_L(a_2^{-1}b), \tilde{b}) \in B_{a_2} \times B$ is a diffeomorphism, as well as the map $B_{a_2} \times B \ni (b', \tilde{b}) \mapsto (b, b' := b^{-1}\tilde{b}) \in B_{a_2} \times B$. Their composition $B_{a_2^{-1}} \times B \ni (b, \tilde{b}) \mapsto (b' := \tilde{b}_L(a_2^{-1}b), \tilde{b} = b'b'' \in B_{a_2} \times B$ has the inverse:

$$\Lambda_{a_2} : B_{a_2} \times B \ni (b', b'') \mapsto (b = \tilde{b}_L(a_2b'), \tilde{b} = b'' \in B_{a_2} \times B$$

Using this change of variables we write the integral as:

$$\int_{B_{a_2} \times B} db db' |\det \Lambda_{a_2}^t| k_1(a_1 b_L(a_2b'), a_2 b'b'') \times$$

$$\times f_1(a_1a_2b') f_2(a_R(a_2b')b'') f_3(b_L^{-1}(a_1b_L(a_2b')a_1b_1) f_4(b_L^{-1}(a_2b'b'')a_2b_2),$$

moreover $|\det \Lambda_{a_2}^t|$ is a smooth function of $a_2$.

On the other hand, using (13) and (43) we compute:

$$(\hat{\delta}_0(f_1)T^{-1}(f_3 \otimes f_2 \ast f_4))(a_1b_1, a_2b_2) =$$

$$= \int_B db' j_B(b_L(a_2b'))^{-1/2} f_1(a_1a_2b') |T^{-1}(f_3 \otimes f_2 f_4)(b_L(a_1a_2b')^{-1}a_1b_1, a_R(a_2b')b^{-1}b_2) =$$

$$\int_B db' j_B(b_L(a_2b'))^{-1/2} j_B(b_L(a_R(a_2b')))^{-1/2} f_1(a_1a_2b') \times$$

$$\times f_3(b_L(a_1b_L(a_2b')b_L(a_R(a_2b')))^{-1}a_1b_1) (f_2 \ast f_4)(a_R(a_2b')b^{-1}b_2)$$

Using assumption about $f_2$ we can restrict domain of integration to the set $B_{a_2}$, and using formula for multiplication (17) we expand the integral as:

$$= \int_{B_{a_2} \times B} db' db'' j_B(b_L(a_2b'))^{-1/2} j_B(b_L(a_R(a_2b')))^{-1/2} f_1(a_1a_2b') f_2(a_R(a_2b')b'') \times$$
Lemma 3.12. Following:

\[ \times f_3(b_L(a_1 b_L(a_2 b'))^{-1} a_1 b_1) f_4(b_L(a_2 b'')^{-1} a_2 b_2) \]

but \( b_L(a_2 b') \tilde b_L(a_R(a_2 b')) = \tilde b_L(a_2 b') \) so we get:

\[
= \int_{B_{a_2} \times B} d b' d b'' j_B(\tilde b_L(a_2 b'))^{-1/2} f_1(a_1 a_2 b') f_2(a_R(a_2 b') b'') \times \\
\times f_3(b_L(a_1 \tilde b_L(a_2 b'))^{-1} a_1 b_1) f_4(b_L(a_2 b'')^{-1} a_2 b_2)
\]

Comparison of this expression with \( (23) \) gives the function \( k_1 \) that appears in the lemma.

Now we pass to the second statement of the lemma \([3.11]\). For \( f_1, f_2, f_3, f_4 \) as in the lemma let’s compute:

\[
[k_2 \Phi_2^*(f_1 \otimes f_2)] * (f_3 \otimes f_4)(a_1 b_1, a_2 b_2) = \int_{B \times B} d b' d b'' k_2(a_1 b, a_2 \tilde b) \times \\
\times [\Phi_2^*(f_1 \otimes f_2)](a_1 b, a_2 \tilde b) f_3(b_L^{-1}(a_1 b)a_1 b_1) f_4(b_L^{-1}(a_2 \tilde b)a_2 b_2)
\]

Since \( \Phi_2^*(f_1 \otimes f_2) \in \mathcal{D}(D(\Phi_2)) \) the integral over \( \tilde b \) can be restricted to the set \( B_{a_2} \), and using formula \((27)\) we obtain:

\[
\int_{B \times B_{a_2}} d b' d b'' k_2(a_1 b, a_2 \tilde b) f_1(a_1 a_2 \tilde b) \times \\
\times f_2(a_R(a_1 \tilde b_L(a_2 \tilde b)) b_L^{-1}(a_2 \tilde b) b) f_3(b_L^{-1}(a_1 b)a_1 b_1) f_4(b_L^{-1}(a_2 \tilde b)a_2 b_2)
\]

The mapping

\[
B \times B_{a_2} \ni (b, \tilde b) \mapsto (b' := \tilde b_L^{-1}(a_2 \tilde b) b, b' := \tilde b) \in B \times B_{a_2}
\]

is a diffeomorphism with the inverse:

\[
\tilde \Lambda_{a_2} : B \times B_{a_2} \ni (b', b'') \mapsto (b := \tilde b_L(a_2 b'') b', \tilde b := b') \in B \times B_{a_2}
\]

Therefore our integral is equal to:

\[
\int_{B \times B_{a_2}} d b' d b'' |\det \tilde \Lambda_{a_2}| k_2(a_1 b_L(a_2 b') b', a_2 b'') f_1(a_1 a_2 b'') \times \\
\times f_2(a_R(a_1 \tilde b_L(a_2 b'')) b') f_3(b_L^{-1}(a_1 b_L(a_2 b'')) b') f_4(b_L^{-1}(a_2 b') a_2 b_2),
\]

and again \( |\det \tilde \Lambda_{a_2}| \) is a smooth function of \( a_2 \).

On the other hand using \([13] \) and \([13] \) we compute:

\[
(\hat \delta_0(f_1) T^{-1}(f_2 * f_3 \otimes f_4))(a_1 b_1, a_2 b_2) = \\
= \int_{B} d b'' j_B(b_L(a_2 b''))^{-1/2} j_B(\tilde b_L(a_R(a_2 b'')))^{-1/2} f_1(a_1 a_2 b'') \times \\
\times f_2(a_R(a_1 b')(a_2 \tilde b_L(a_R(a_2 b'))^{-1} a_1 b_1) f_4(a_R(a_2 b') b_L^{-1} a_2 b_2).
\]

Again using the assumption about the support of \( f_4 \) and expanding the product \( f_2 * f_3 \) we can write this integral as:

\[
\int_{B_{a_2} \times B} d b' d b'' j_B(b_L(a_2 b''))^{-1/2} f_1(a_1 a_2 b'') f_2(a_R(a_1 \tilde b_L(a_2 b'')) b') \times \\
\times f_3(b_L(a_1 \tilde b_L(a_2 b'')) b')^{-1} a_1 b_1) f_4(b_L(a_2 b'')^{-1} a_2 b_2)
\]

Again comparison of this expression with \([3] \) gives the function \( k_2 \) that appears in the lemma.

Finally, from the previous lemma and lemma \([3.11]\) to prove density conditions it is sufficient to prove the following:

**Lemma 3.12.** The linear spaces \( \mathcal{A}(D(\Phi_1)) \) and \( \mathcal{A}(D(\Phi_2)) \) are dense in \( C^*_r(G_A) \otimes C^*_r(G_A) \).
Proof: It is sufficient to show that a function \( F \in \mathcal{D}(G_A \times G_A) \) can be approximated in norm \( \| \cdot \|_0 \) by functions supported in \( \mathcal{D}(\Phi_1) \) and a function \( F \in \mathcal{D}(G_A \times a_R^{-1}(A')) \) (by \( \| \cdot \| \)) by functions supported in \( \mathcal{D}(\Phi_2) \).

Let us begin with \( \mathcal{D}(\Phi_1) \). For an open \( V \subset A \) such that \( A \setminus A' \subset V \) let \( \tilde{\chi} \) be a smooth function on \( A \) satisfying conditions: \( \tilde{\chi} = 1 \) on some neighbourhood of \( A \setminus A' \); \( 0 \leq \tilde{\chi} \leq 1 \) and \( \text{supp}(\tilde{\chi}) \subset V \). Define:

\[
\chi(a_1b_1, a_2b_2) := \tilde{\chi}(aR(a_2^{-1}b_1))
\]

Let \( F \in \mathcal{D}(G_A \times G_A) \), \( \text{supp}(F) =: K_F \), then \( F = (F - F\chi) + F\chi \) and \( (F - F\chi) \in \mathcal{D}(\mathcal{D}(\Phi_1)) \) so we have to prove that, by choosing \( V \), the \( C^* \)-norm of \( F\chi \) can be made as small as we wish. It is sufficient to prove that for norms \( \| F\chi \|_l \) and \( \| F\chi \|_r = \| (F\chi)^* \|_l \). For the left norm we need to estimate (compare \ref{eq:42}) the integral

\[
I_l := \int d_1b_1d_2b_2|F\chi(a_1b_1, a_2b_2)| \quad \text{for} \quad (a_1, a_2) \in (a_L \times a_L)(K_F).
\]

First we estimate this integral from above by \( \sup |F| \int_{C(a_1,a_2)} d_1b_1d_2b_2 \), where

\[
C(a_1,a_2) := \{(b_1, b_2) : (a_1b_1, a_2b_2) \in K_F, aR(a_2^{-1}b_1) \in \text{supp}(\tilde{\chi})\};
\]

Now we have the chain of inclusions:

\[
C(a_1,a_2) \subset \{(b_1, b_2) : (a_1b_1, a_2b_2) \in K_F \} \cap \{(b_1, b_2) : aR(a_2^{-1}b_1) \in \text{supp}(\tilde{\chi})\} \subset \{(b_1, b_2) : b \in B : aR(a_2^{-1}b) \in \text{supp}(\tilde{\chi})\} \times B \subset \{(b_1, b_2) : aR(a_2^{-1}b) \in \text{supp}(\tilde{\chi})\} \times B = (K_B \cap \{ b : aR(a_2^{-1}b) \in \text{supp}(\tilde{\chi}) \}) \times K_B \subset Z(a_2^{-1}, e, K_B; V) \times K_B
\]

where \( K_B \subset B \) is a compact set such that \( (b_1 \times b_2)(K_F) \subset K_B \times K_B \); note that \( K_B \) depends only on \( F \).

In this way we get the estimate for \( (a_1, a_2) \in (a_L \times a_L)(K_F) \):

\[
I_l \leq \sup |F| \left( \int_{K_B} d_1b \mu(a_2^{-1}, e, K_B; V) \right)
\]

and, if \( K_1 \) is a compact subset of \( A \) such that \( (a_L \times a_L)(K_1) \subset K_1 \times K_1 \), we obtain

\[
\| F\chi \|_l \leq \sup |F| \left( \int_{K_B} d_1b \mu(K_1^{-1}, \{e\}, K_B; V) \right)
\]

Now the right norm of \( F\chi \) i.e. the left norm of \( (F\chi)^* \). The integral to estimate is:

\[
I_r := \int d_1b_1d_2b_2|(F\chi)^*(a_1b_1, a_2b_2)|
\]

As above, first we estimate this integral by \( \sup |F| \int_{\tilde{C}(a_1,a_2)} d_1b_1d_2b_2 \), where

\[
\tilde{C}(a_1,a_2) := \{(b_1, b_2) : (a_1b_1, a_2b_2) \in \text{supp}(F\chi)^*\}.
\]

Now,

\[
(a_1b_1, a_2b_2) \in \text{supp}(F\chi)^* \iff (sA(a_1b_1), sA(a_2b_2)) \in \text{supp}(F\chi) \iff \quad [sA(a_1b_1), sA(a_2b_2)) \in K_F \cap aR(a_2^{-1}b_1) \in \text{supp}(\tilde{\chi})]
\]

Let us denote \( \tilde{K}_F := (sA \times sA)(K_F) \), so we have to estimate the integral \( \int_{\tilde{C}(a_1,a_2)} d_1b_1d_2b_2 \) for \( (a_1, a_2) \in (a_L \times a_L)(\tilde{K}_F) \). Again there is an inclusion:

\[
\tilde{C}(a_1,a_2) \subset (\tilde{K}_B \times \tilde{K}_B) \cap \{(b_1, b_2) : aR(a_2^{-1}b_1) \in \text{supp}(\tilde{\chi})\}.
\]
where $\tilde{K}_B$ is a compact set such that $(b_R \times b_R)(\tilde{K}_F) \subset \tilde{K}_B \times \tilde{K}_B$.

For fixed $b_2 \in \tilde{K}_B$ define $\tilde{a} := a_R(a_2 b_2)$ and consider the set

$$\{b \in \tilde{K}_B : a_R(\tilde{a}^{-1}b^{-1}) \in \text{supp}(\tilde{\chi})\} = \{b \in \tilde{K}_B^{-1} : a_R(\tilde{a}^{-1}b) \in \text{supp}(\tilde{\chi})\} \subset Z(\tilde{a}, e, \tilde{K}_B^{-1}, V)^{-1}$$

Since all the sets $Z(\tilde{a}, e, \tilde{K}_B^{-1}, V)$ are subsets of the fixed compact set $\tilde{K}_B^{-1}$ there exists a constant $M(\tilde{K}_B)$ depending only on $\tilde{K}_B$, such that

$$\int_{Z(\tilde{a}, e, \tilde{K}_B^{-1}, V)^{-1}} d\tilde{b} \leq M(\tilde{K}_B) \int_{Z(\tilde{a}, e, \tilde{K}_B^{-1}, V)} d\tilde{b} = M(\tilde{K}_B) \mu(\tilde{a}, e, \tilde{K}_B^{-1}, V)$$

But $\tilde{a} = a_R(a_2 b_2)$ belongs to a fixed compact set $K_2 = a_R(\tilde{K}_A \tilde{K}_B)$, where $\tilde{K}_A$ is a compact set such that $(a_L \times a_L)\tilde{K}_F \subset \tilde{K}_A \times \tilde{K}_A$ Putting all together we get the estimate:

$$(31) \quad ||F\chi||_r \leq \sup F M(\tilde{K}_B) \mu(K_2, \{e\}, \tilde{K}_B^{-1}, V) \left(\int_{\tilde{K}_B} d\tilde{b}\right)$$

This inequality together with (30) and the condition (4) of assumptions (3.3) gives the density of $A(D(\Phi_1))$.

Now we pass to $D(\Phi_2)$. Let $\chi$ be as above and define $\chi(a_1 b_1, a_2 b_2) := \tilde{\chi}(a_R(a_1 a_2 b_2))$. It is sufficient to approximate functions $F \in D(G_K \times a_R^{-1}(A'))$ and to this end we need an estimate for $||F\chi||_l$ and $||(F\chi)^*||_l$. Let $K_F := \text{supp}(F)$ and $(a_1, a_2) \in (a_L \times a_L)(K_F)$. As above we have inequality

$$\int \left| d_1 b_1 d_2 b_2 |F\chi| \right| \leq \sup |F| \int_{(a_1, a_2)} d_1 b_1 d_2 b_2,$$

where

$$C(a_1, a_2) := \{(b_1, b_2) : (a_1 b_1, a_2 b_2) \in K_F, a_R(a_1 a_2 b_2) \in \text{supp}(\tilde{\chi})\} \subset (K_B \times K_B) \cap (B \times \{b \in B : a_R(a_1 a_2 b) \in \text{supp}(\tilde{\chi})\}) = \tilde{K}_B \times \tilde{K}_B \cap \{b \in B : a_R(a_1 a_2 b) \in \text{supp}(\tilde{\chi})\} = \tilde{K}_B \times Z(a_1 a_2, e, K_B, V),$$

where $\tilde{K}_B \subset B$ is a compact such that $(b_R \times b_R)(K_F) \subset \tilde{K}_B \times \tilde{K}_B$; In this way we obtain the estimate:

$$(32) \quad ||F\chi||_l \leq \sup |F| \int_{\tilde{K}_B} d\tilde{b} \mu(K_1, \{e\}, K_B; V), \text{ where }$$

$K_1 = K_A K_A$ for a compact $K_A \subset A$ with $(a_L \times a_L)(K_F) \subset K_A \times K_A$.

Finally we pass to $(F\chi)^*$. As above, we consider the set

$$\tilde{C}(a_1, a_2) := \{(b_1, b_2) : (s_A(a_1 b_1), s_A(a_2 b_2)) \in \text{supp}(F\chi)\}$$

Let $\tilde{K}_F := (s_A \times s_A)(K_F)$ and $\tilde{K}_B \subset B$ be a compact set such that $(b_R \times b_R)(\tilde{K}_F) \subset \tilde{K}_B \times \tilde{K}_B$. It is sufficient to consider $(a_1, a_2) \in (a_L \times a_L)(\tilde{K}_F)$.

$$\tilde{C}(a_1, a_2) \subset (\tilde{K}_B \times \tilde{K}_B) \cap \{(b_1, b_2) : a_R(a_1 b_1) b_L(a_2 b_2)^{-1} a_2 \in V\}$$

Let us fix $b_1 \in \tilde{K}_B$, denote $a_R(a_1 b_1) := \tilde{a}$ and consider the set

$$S(a_1, a_2, b_1) := \tilde{K}_B \cap \{b \in B : a_R(\tilde{a} b_L(a_2 b_2)^{-1}) a_2 \in V\}.$$ 

The map $\varphi_{a_2} : b \mapsto b_L(a_2 b)$ is a diffeomorphism so

$$\{b \in B : a_R(\tilde{a} b_L(a_2 b_2)^{-1}) a_2 \in V\} = \varphi_{a_2}^{-1}\{b \in B : a_R(\tilde{a} b^{-1}) a_2 \in V\}$$

and

$$\tilde{K}_B \cap \{b \in B : a_R(\tilde{a} b_L(a_2 b_2)^{-1}) a_2 \in V\} = \varphi_{a_2}^{-1}\{\varphi_{a_2}(\tilde{K}_B) \cap \{b \in B : a_R(\tilde{a} b_L(a_2 b_2)^{-1}) a_2 \in V\}\} \subset \varphi_{a_2}^{-1}\{\{b \in H_B : a_R(\tilde{a} b^{-1}) a_2 \in V\}\} = \varphi_{a_2}^{-1}\{Z(\tilde{a}, a_2, H_B^{-1}, V)^{-1}\}$$
Since $\varphi_{a_2}^{-1} = \varphi_{a_2}^{-1}$ and $a_2$ is contained in a fixed compact set depending only on $\tilde{K}_F$ there exists a constant $W(\tilde{K}_F)$ such that
\[
\int_{\varphi_{a_2}^{-1}(K)} d\tilde{b} \leq W(\tilde{K}_F) \int_K \tilde{d}b
\]
in this way, for fixed $b_1$ we obtain that
\[
\int_{S(a_1,a_2,b_1)} d\tilde{b} \leq W(\tilde{K}_F) M(H_B) \mu(\tilde{a}, a_2, H_B^{-1}; V),
\]
where $M(H_B)$ is as before (31). Finally, since $\tilde{a} = a_2(a_1b_1)$ is contained in a compact set $\tilde{K}_A$ depending only on $\tilde{K}_F$ and $a_2$ is contained in a fixed compact $\tilde{K}_1 \subset A'$ we obtain the estimate
\[
\|\{F\chi\}|_{l} \leq \sup |F| W(\tilde{K}_F) M(H_B) \mu(\tilde{K}_A, \tilde{K}_1, H_B^{-1}; V) \left( \int_{\tilde{K}_B} \tilde{d}b \right)
\]
This estimate together with (32) and condition (4) of assumptions (3.3) proves the density of $\mathcal{A}(D(\Phi_2))$.

This completes the proof of the statement b) of prop 3.5.

4. ’AX+B’ AGAIN

Now we apply results of the previous section to the ‘ax+b’ group. In this section we keep notation from section 2. $G$ is the ‘ax+b’ group:
\[
B := \{(b, 1), b \in \mathbb{R}\}, \quad C := \{(c - 1, c), c \in \mathbb{R}_+\}, \quad A := \{(0, a), a \in \mathbb{R}_+\}
\]
We have $A \cap C = B \cap C = \{e\}$ and $G = BC$. We apologize the reader for changing roles of subgroup and present the short ‘dictionary’ from section 3 to ‘ax+b’ notation.

| section 3 | $A$ | $B$ | $C$ | $A' = A \cap BC$ | $a_L, a_R$ | $b_L, b_R$ | $\tilde{b}_L, \tilde{b}_R$ | $\delta_0 = m_B^T \cdot \hat{m}_B$ |
|-----------|----|----|----|-----------------|-------------|-------------|------------------|------------------|
| ax+b      | $B$ | $C$ | $A$ | $B' = B \cap CA$ | $b_L, b_R$ | $c_L, c_R$ | $\tilde{c}_L, \tilde{c}_R$ | $\delta_0 = m_C^T \cdot \hat{m}_C$ |

So our main object here is a groupoid $G_B$ related to the double Lie group $(G; B, C)$ together with coassociative morphism $\delta_0 = m_B^T : G_B \longrightarrow G_B \times G_B$. We identify $G$ with $B \times C = \mathbb{R} \times \mathbb{R}_+$ by:
\[
\mathbb{R} \times \mathbb{R}_+ \ni (z, c) \mapsto (z, 1)(c - 1, c) = (z + c - 1, c) \in G
\]

In this presentation the relevant objects are given by:
\[
\begin{align*}
 b_L(z, c) &= (z, 1), \quad b_R(z, c) = \left( \frac{z}{c}, 1 \right), \quad c_L(z, c) = c_R(z, c) = (0, c), \quad \tilde{c}_L(z, c) = (0, z + c) \\
 m_B &= \{(z, c_1c_2; z, c_1, \frac{z}{c_1})\}, \quad \delta_0 = \{(z_1, c, z_2; c_1 + z_1, z_2, c)\}, \quad B' = \{(z, 1) : z + 1 \neq 0\} \\
 T &= \{(z_1, 1 + z_2, z_2, 1) : z_2 + 1 \neq 0\} \\
 \hat{m}_C &= \{(z_1 + z_2 + z_1z_2, c, z_1, \frac{c + z_2}{1 + z_2}, z_2, c) : (1 + z_1)(1 + z_2)(z_2 + c)(z_1 + z_2 + z_1z_2 + c) \neq 0\}
\end{align*}
\]

We will also need explicit formulæ for operations in $\mathcal{A}(G_B)$.

The choice of $\omega_0$. Let $\lambda_0(z, 1)(\partial_c) := 1$. This choice leads to the left invariant half density $\lambda_0$, the corresponding right invariant half density $\rho_0$ and $\omega_0 := \lambda_0 \otimes \rho_0$. The formulæ are:
\[
\lambda_0(z, c)(c\partial_c) = \rho_0(z, c)(z \partial_z + c \partial_c) = 1
\]

The multiplication and $*$-operation in $\mathcal{A}(G_B)$ are given by:
\[
(f_1 * f_2)(z, c) := \int_{\mathbb{R}_+} \frac{dc_1}{|c_1|} f_1(z, c_1) f_2(zc_1^{-1}, cc_1^{-1}) , \quad f^*(z, c) := \overline{f(zc^{-1}, c^{-1})}.
\]
The choice of \( \psi_0 \). We choose \( \nu_0(z) := |dz|^{1/2} \) and obtain \( \psi_0(z,c)(\partial_z, \partial_c) = \frac{1}{|c|} \), and the formula for the scalar product in \( D(G_B) \) reads:

\[
(f_1, f_2) = \int f_1 dz dc \frac{f_1(z,c)}{c^2} f_2(z,c)
\]

and the formula for \( \pi_{id} \):

\[
(\pi_{id}(f)\psi)(z,c) = \int \frac{dc}{|c|} f(z,c) \psi(zc^{-1}, c^{-1}) \quad f \in A(G_B), \quad \psi \in D(G_B)
\]

Now we are going to verify assumptions (3.3). The first and the second one are obvious, so let’s pass to the third one. We have to verify that \( A(b^*_L(B')) \) is dense in \( C^*_r(G_B) \). Take \( f, \psi \in D(G_B) \) and compute:

\[
||\pi_{id}(f)\psi||^2 = \int \frac{dy dc}{c^2} |\pi_{id}(f)\psi(y,c)|^2 = \int \frac{dy dc}{c^2} \left| \int \frac{db}{|b|} f(y, b) \psi(yb^{-1}, cb^{-1}) \right|^2
\]

Now we use Schwartz inequality for functions: \( b \mapsto f(y, b) \) and \( b \mapsto \psi(yb^{-1}, cb^{-1}) \):

\[
\left| \int \frac{db}{|b|} f(y, b) \psi(yb^{-1}, cb^{-1}) \right|^2 \leq \left( \int \frac{db}{|b|} |f(y, b)|^2 \right) \left( \int \frac{db}{|b|} |\psi(yb^{-1}, cb^{-1})|^2 \right)
\]

and get the estimate:

\[
||\pi_{id}(f)\psi||^2 \leq \int \frac{dy}{|y|} \left( \int \frac{db}{|b|} |f(y, b)|^2 \right) \int \frac{dc db}{|bc|^2} |\psi(yb^{-1}, cb^{-1})|^2
\]

Note that for \( y \neq 0 \) applying the change of variables \( (b, c) \mapsto (yb^{-1}, cb^{-1}) \) to the integral on the right we obtain:

\[
\int \frac{dc db}{|bc|^2} |\psi(yb^{-1}, cb^{-1})|^2 = \frac{1}{|y|} \int \frac{dc db}{|bc|^2} |\psi(b, c)|^2
\]

Therefore if \( supp \psi \subset \{(z, c) : z \neq 0 \} \) we get:

\[
(35) \quad ||\pi_{id}(f)\psi||^2 \leq ||\psi||^2 \left( \int \frac{db}{|yb|} |f(y, b)|^2 \right) =: ||\psi||^2 ||f||^2
\]

Since the set of such \( \psi \)'s is dense the estimate is valid for any \( \psi \) and we get \( ||f|| \leq ||f||_2 \).

For \( \epsilon > 0 \), let \( \chi \epsilon : \mathbb{R} \to [0, 1] \) be a smooth function that is 1 on some neighbourhood of 0 and 0 on the set \( \{x \in \mathbb{R} : |x| \geq \epsilon \} \). Let \( f \in D(G_B) \) and for \( z_0 \neq 0 \) let \( f_\epsilon (z,c) := \chi \epsilon (z - z_0) f(z,c) \). Now the estimate given above implies \( f_\epsilon \to 0 \) in \( C^*_r(G_B) \) as \( \epsilon \to 0 \). But \( f = (f - f_\epsilon) + f_\epsilon \) and \( f - f_\epsilon \in D(b^*_L(\mathbb{R} \setminus \{z_0\})) \).

This proves that statement (3) of (3.3) is true in our situation.

Now we are going to verify the fourth condition in assumptions (3.3) which in our situation takes form: Let \( K_C \subset C \) be compact, \( V \subset B \) open and \( (z_1, z_2) \in B \times B' \).

Let \( Z(z_1, z_2, K_C; V) := K_C \cap \{ c \in C : b_R(z_1,c)(z_2,1) \in V \} \),

and the function \( \mu(z_1, z_2, K_C; V) \) be given by:

\[
B \times B' \ni (z_1, z_2) \mapsto \mu(z_1, z_2, K_C; V) := \int_{Z(z_1,z_2,K_C;V)} dL_c.
\]

Let \( K_1 \subset B \) and \( K_2 \subset B' \) be compact and \( \mu(K_1, K_2, K_C; V) := \sup \{ \mu(z_1, z_2, K_C; V) : z_1 \in K_1, z_2 \in K_2 \} \). Then

\[
(36) \quad \forall \epsilon > 0 \exists V - a \text{ neighbourhood of } B \setminus B' \text{ in } B : \mu(K_1, K_2, K_C; V) \leq \epsilon
\]

It is sufficient to check this condition for \( K_C = K_m := \{ c : m \leq |c| \leq \frac{1}{m} \} \), \( m < 1 \). Using formula for \( b_R \) we get \( b_R(z_1,c)(z_2,1) = (\frac{z_1}{c} + z_2, 1) \) and

\[
Z(z_1, z_2, K_m; V) = \{ c \in \mathbb{R} : m \leq |c| \leq \frac{1}{m}, \frac{z_1}{c} + z_2 \in V \}, \quad 1 + z_2 \neq 0.
\]
The left invariant measure on $C$ is $\frac{dc}{|c|}$ and we obtain:

$$\mu(z_1, z_2, K_m; V) := \int_{Z(z_1, z_2, K_m; V)} \frac{dc}{|c|} \leq \frac{1}{m} \int_{Z(z_1, z_2, K_m; V)} dc$$

We will look for $V = V_\delta := \{ z \in \mathbb{R} : |z + 1| < \delta \}$. In this situation $Z(z_1, z_2, K_m; V_\delta)$ is given by $c$ satisfying inequalities:

$$m \leq |c| \leq \frac{1}{m}, \left| \frac{z_1}{c} + z_2 + 1 \right| < \delta, (1 + z_2 \neq 0)$$

The second inequality is for $\delta < |1 + z_2|$ equivalent to:

$$\frac{-\text{sgn}(c)\text{sgn}(1 + z_2)z_1}{\delta + |1 + z_2|} < |c| < \frac{-\text{sgn}(c)\text{sgn}(1 + z_2)z_1}{|1 + z_2| - \delta}$$

and we get for $\text{sgn}(1 + z_2)z_1 \geq 0$:

$$c < 0, m \leq |c| \leq \frac{1}{m}, \frac{|z_1|}{\delta + |1 + z_2|} < |c| < \frac{|z_1|}{|1 + z_2| - \delta}$$

and for $\text{sgn}(1 + z_2)z_1 < 0$:

$$c > 0, m \leq |c| \leq \frac{1}{m}, \frac{|z_1|}{\delta + |1 + z_2|} < |c| < \frac{|z_1|}{|1 + z_2| - \delta}$$

Therefore the integral $\int_{Z(z_1, z_2, K_m; V)} dc$ is majorized by:

$$\frac{|z_1|}{|1 + z_2| - \delta} - \frac{|z_1|}{\delta + |1 + z_2|} = \frac{2|z_1|}{|1 + z_2|^2 - \delta^2}$$

and there is an estimate (for $\delta < |1 + z_2|$):

$$\mu(z_1, z_2, K_m; V_\delta) \leq \delta \frac{2|z_1|}{m(|1 + z_2|^2 - \delta^2)}$$

It is sufficient to consider $K_1 = K_M := \{ z \in \mathbb{R} : |z| \leq M \}$ and $K_2 = \hat{K}_M := \{ z \in \mathbb{R} : \frac{1}{M} \leq |z + 1| \leq M \}$ for $M > 1$. For $(z_1, z_2) \in K_M \times \hat{K}_M$ and $\delta \leq \frac{1}{2M}$ we have:

$$\mu(z_1, z_2, K_M; V_\delta) \leq \frac{2M}{m(|1 + z_2|^2 - \delta^2)} \leq \frac{2M}{m(1/M^2 - 1/(4M^2))} = \delta \frac{8M^3}{3m}$$

So there is an estimate:

$$\mu(K_M, \hat{K}_M, K_m; V_\delta) \leq \delta \frac{8M^3}{3m}$$

and the fourth statement of assumptions (3.3) is fulfilled. Therefore we have

**Proposition 4.1.** Let $G$ be the 'ax+b' group and $A := \{(0,a) : a \in \mathbb{R}\}$, $B := \{(b,1) : b \in \mathbb{R}\}$, $C := \{(c-1,c) : c \in \mathbb{R}_+\}$. The reduced $C^*$ algebra of a differential groupoid $\Gamma = AC \cap CA$ over $A$ is isomorphic to $C^*_r(G_B)$ – the reduced $C^*$ algebra of a differential groupoid related to a double Lie group $(G;B,C)$. It is equipped with a comultiplication $\Delta = \hat{T} \Delta_0 \hat{T}^{-1}$ satisfying the density conditions, where $\hat{T}$ is a unitary multiplier and satisfies:

$$(\hat{T} \otimes I)(\Delta_0 \otimes \text{id}) \hat{T} = (I \otimes \hat{T})(\text{id} \otimes \Delta_0) \hat{T}$$
5. Generators and relations

In this section we denote the groupoid $G_R$ by $\Gamma$ and identify it with the right transformation groupoid $\mathbb{R} \times \mathbb{R}$, with the action $(z,c) \mapsto z/c$. The set of units will be denoted by $E$ and the set \{z : z + 1 \neq 0\} by $E'$. For $t \in \mathbb{R}$ let $B_t := \{(z,e^t) : z \in \mathbb{R} \}$ $\subset \Gamma$. Then $B_t$ is a one parameter group of bisections of $\Gamma$, therefore it defines one parameter group $\hat{B}_t$ of unitaries on $L^2(\Gamma)$ which are multipliers of $C^*_r(\Gamma)$. Easy computations show that $B_t \psi_0 = \psi_0$ (where $\psi_0$ was defined in section 2) so $\hat{B}_t(\mathcal{F}) = (\mathcal{F})$ and

$$(\hat{B}_t f)(z,c) = f(e^{-t},z,e^{-t}c), \quad f \in \mathcal{D}(\Gamma).$$

The set $J := \{(z,-1) : z \in \mathbb{R}\}$ is a bisection, $J^2 = E$, so it defines unitary, selfadjoint operator $\hat{J}$ which is a multiplier of $C^*_r(\Gamma)$. Again $J \psi_0 = \psi_0$ so

$$(\hat{J} f)(z,c) := f(-z,-c), \quad f \in \mathcal{D}(\Gamma)$$

Let $Y : \mathbb{R} \ni z \mapsto z \in \mathbb{R}$ and let the operator $\hat{Y}$ be defined by:

$$(\hat{Y} f)(z,c) := Y(z) f(z,c) = z f(z,c), \quad f \in \mathcal{D}(\Gamma)$$

Then the closure of $\hat{Y}$, which will be denoted by the same letter, is an unbounded, selfadjoint operator affiliated with $C^*_r(\Gamma)$. On $\mathcal{D}(\Gamma)$ these operators satisfy commutation relations:

$$\hat{J} \hat{B}_t = \hat{B}_t \hat{J}, \quad \hat{J} \hat{Y} + \hat{Y} \hat{J} = 0, \quad e^t \hat{B}_t \hat{Y} = \hat{Y} \hat{B}_t$$

Note that the last relation gives precise meaning to the relation $[\hat{X},\hat{Y}] = i\hat{Y}$, for $\hat{X}$- generator of $\hat{B}_t$. Additionaly if $\hat{Y} = \text{sgn}(\hat{Y})|\hat{Y}|$ is a polar decomposition then :

$$|\hat{Y}| \hat{J} = \hat{J} |\hat{Y}|, \quad e^t |\hat{B}_t| \hat{Y} = |\hat{Y}| \hat{B}_t$$

**Lemma 5.1.** The group $\hat{B}_t$ is strictly continuous. Let $\hat{X}$ denote the generator of $\hat{B}_t$, then $\hat{X}$ is affiliated with $C^*_r(\Gamma)$. On $\mathcal{D}(\Gamma)$ $\hat{X} f = i(z\partial_z + c\partial_c) f$ and $\mathcal{D}(\Gamma)$ is a core for $\hat{X}$.

**Proof:** Since $\hat{B}_t$ is a group of unitary multipliers, strict continuity is equivalent to continuity of the mapping $\mathbb{R} \ni t \mapsto B_t f \in \mathcal{A}(\Gamma)$ in $t = 0$ for any $f \in \mathcal{A}(\Gamma)$. The mapping $\Phi : \mathbb{R} \times \Gamma \ni (t,\gamma) \mapsto B_t(\gamma) \in \Gamma$ is continuous. Therefore if $|t| \leq \delta$ support of $B_t f$ is contained in a compact set $\Phi([-\delta,\delta] \times \text{supp}(f)$ and $B_t f$ converges to $f$ uniformly as $t$ goes to 0. So, by the lemma 1.1, also $\hat{X}$ is affiliated with $C^*_r(\Gamma)$. By the general result $\hat{X}$ is affiliated with $C^*_r(\Gamma)$. To prove the second claim it is enough to show that $\frac{1}{t}(B_t f - f)$ converges to $s(z\partial_z + c\partial_c) f$ uniformly, but since $s$ is smooth this is clear. Finally, since $\mathcal{D}(\Gamma)$ is $B_t$-invariant, it is a core for $\hat{X}$.

**Proposition 5.2.** $C^*_r(\Gamma)$ is generated by $\hat{X}, \hat{Y}, \hat{J}$.

**Proof:** The proof is based on identification of $C^*_r(\Gamma)$ with crossed product $C^0(\mathbb{R}) \times_{\alpha} \mathbb{R}$. This is known, but we need explicit form of the isomorphism so we present relevant formulae. Let a Lie group $H$ acts from the right on a manifold $X$. Let $\alpha$ be the corresponding (left) action on $C^0(X)$ ( i.e. $(\alpha_h f)(x) := f(xh)$). Since on a Lie group the Haar measure is given by a left invariant density, we choose such a real half density $\lambda$. Then we have $\int_H f(h)d\lambda(h) = \int_H f \lambda^2$, $f \in \mathcal{D}(H)$. The modular function is given by $\delta(h) = |\text{det}A_d h|^{-1}$. These data define the $*$-algebra structure on $\mathcal{D}(X \times H)$ by:

$$(F * G)(x,h) := \int_H \lambda^2(h') F(x,h') G(xh', h'\lambda h) \quad (F^*)(x,h) := \delta(h)^{-1} F(xh, h^-1)$$

The algebra $C^0(X)$ is represented faithfully on $L^2(X)$ by multiplication. The induced representation $\Pi$ of the $*$-algebra $\mathcal{D}(X \times H)$ in $L^2(X \times H)$ is given by: $\Pi(F)(G \Psi_0) =: (\pi(F)G)\Psi_0$, $G \in \mathcal{D}(X \times H)$, and

$$\Pi(F)(x,h) := \int \lambda^2(k) F(xh^{-1}, k) G(x, k^{-1}h),$$
where \( \Psi_{0} = \nu_{0} \otimes \lambda \) for some real, nonvanishing half density \( \nu_{0} \) on \( X \). The reduced crossed product \( C_{0}(X) \times_{\alpha} H \) is the completion of the \( * \)-algebra \( D(X \times H) \) in the the norm coming from this representation. The canonical morphisms \( i_{H} \in \text{Mor}(C^{*}(H), C_{0}(X) \times_{\alpha} H) \) and \( i_{A} \in \text{Mor}(C_{0}(X), C_{0}(X) \times_{\alpha} H) \) are given by (the extension of):

\[
(i_{H}(g)F)(x, h) := F(xg, g^{-1}h), \quad (i_{A}(f)F)(x, h) := f(x)F(x, h), \quad f \in D(X)
\]

On the other hand we have the transformation groupoid \( \Delta := X \times H \), the \( * \)-algebra \( A(\Delta) \) and the \( C^{*} \)-algebra \( C^{*}_{\omega}(\Delta) \). Let us choose \( \omega_{0} := \lambda_{0} \otimes \rho_{0} \), where \( \lambda_{0} \) is defined by \( \lambda_{0}(x, f)(v) := \lambda(e)(v) \) and let us define \( \Psi := \nu \otimes \rho_{0} \). With such a choice the formulae for operations in \( A(\Delta) \) and the representation \( \pi_{id} \) are as follows:

\[
(f \ast g)(x, h) := \int \lambda^{2}(k)f(x, k)g(xk, k^{-1}h), \quad f^{*}(x, h) := \overline{f(x, h^{-1})}, \quad \pi_{id}(f)(g\Psi) = (f \ast g)\Psi
\]

For \( h \in H \) let \( B_{h} \) denotes the operator acting on \( A(\Delta) \) defined by a bisection \( \{(x, h) : x \in X\} \). Now define \( \varphi : A(\Gamma) \to D(X \times H) \) by \( \varphi(f\omega_{0}))(x, h) := \delta(h)^{-1/2}f(x, h) \) and let \( V : L^{2}(X \times H) \to L^{2}(X \times H) \) be a unitary defined as a push-forward by a diffeomorphism \( (x, h) \mapsto (xh^{-1}, h) \). Using the definitions above one proves:

**Lemma 5.3.** The mapping \( \varphi \) is an isomorphism of \( * \)-algebras. For \( h \in H, k \in D(X) \) and \( \omega \in A(\Delta) \):

\[
i_{H}(h)\varphi(\omega) = \varphi(B_{h}\omega) \quad \text{and} \quad i_{A}(k)\varphi(\omega) = \varphi(k\omega) \quad \text{and} \quad \text{VII}(\varphi(\omega))V^{*} = \pi_{id}(\omega).
\]

Therefore \( \varphi \) extends to an isomorphism of \( C^{*}_{\omega}(\Delta) \) and \( C_{0}(X) \times_{\alpha} H \). Since \( \mathbb{R}_{\omega} \) is abelian the reduced and universal crossed product coincide and we have \( C^{*}_{\omega}(\Gamma) = C_{0}(\mathbb{R}_{\omega}) \times_{\alpha} \mathbb{R}_{\omega} \), where \( (\alpha_{c}f)(x) := f(x/c) \). By the universality of crossed product we have the following:

**Lemma 5.4.** Let \( A \) be a \( C^{*} \)-algebra, \( G \)-locally compact group and \( (A, G, \alpha) \) be a dynamical system. Let \( B := A \times_{\alpha} G \) be the corresponding crossed product with canonical morphisms \( i_{A} \in \text{Mor}(A, B) \) and \( i_{G} \in \text{Mor}(C^{*}(G), B) \). If \( A \) is generated by \( X_{1}, X_{2}, \ldots, X_{k} \) and \( C^{*}(G) \) by \( Y_{1}, \ldots, Y_{n} \), then \( B \) is generated by \( i_{A}(X_{1}), \ldots, i_{A}(X_{k}), \) and \( i_{G}(Y_{1}), \ldots, i_{G}(Y_{n}) \).

Now it is clear that \( C^{*}_{\omega}(\mathbb{R}_{\omega}) \) is generated by the the generator of one parameter group \( \mathbb{R} \ni t \mapsto e^{t} \in \mathbb{R}_{\omega} \) acting on \( L^{2}(\mathbb{R}_{\omega}) \) and the unitary corresponding to element \(-1 \in \mathbb{R}_{\omega} \). Also \( C_{0}(\mathbb{R}) \) is generated by the function \( \mathbb{R} \ni x \mapsto x \in \mathbb{R} \). Using two previous lemmas one gets the proof of the proposition.

In the remaining part of this section we express the twist \( \hat{T} \) as a function of generators and compute the action of the comultiplication on generators. It turns out that it is given by formulæ (33).

For the morphism \( \delta_{0} \) defined in (33), we have:

\[
\delta_{0}(B_{t}) = B_{t} \times B_{t} \quad \delta_{0}(J) = J \times J \quad \delta_{0}(Y) = I \otimes Y + Y \otimes I
\]

so

\[
\Delta_{0}(\hat{X}) = \hat{X} \otimes I + I \otimes \hat{X} \quad \Delta_{0}(\hat{J}) = \hat{J} \otimes \hat{J} \quad \Delta_{0}(\hat{Y}) = I \otimes \hat{Y} + \hat{Y} \otimes I
\]

The expressions for \( \Delta_{0}(\hat{X}) \) and \( \Delta_{0}(\hat{Y}) \) should be read in the following sense. Since \( \hat{X} \) and \( \hat{Y} \) are essentially selfadjoint on \( D(\Gamma) \) the sums are essentially selfadjoint on \( D(\Gamma) \otimes D(\Gamma) \) and their closures are to equal the left hand sides.

For \( t \in \mathbb{R} \) let us define sets:

\[
\mathcal{T}_{t} := \{(z_{1}, |1 + z_{2}|^{-t}, z_{2}, 1) : 1 + z_{2} \neq 0\} \quad \text{and} \quad K := \{(z_{1}, \text{sgn}(1 + z_{2}), z_{2}, 1) : 1 + z_{2} \neq 0\}
\]

Both of these sets are sections of the right and left projections over \( E \times E' \), therefore they define (by the left multiplication) diffeomorphisms of \( \Gamma \times U \), where \( U := \{(z, c) : z \in E'\} \), and corresponding mappings of \( A(\Gamma \times U) \), which will be denoted by \( T_{t} \) and \( K \). Explicitly:

\[
T_{t}(z_{1}, c_{1}, z_{2}, c_{2}) = (z_{1}|1 + z_{2}|^{t}, c_{1}|1 + z_{2}|^{t}, z_{2}, c_{2}),
\]
Lemma 5.5. We have the equality of sets:

\[ K(z_1, c_1, z_2, c_2) = (z_1 \text{sgn}(1 + z_2), c_1 \text{sgn}(1 + z_2), z_2, c_2) \]

and for \( f \in \mathcal{A}(\Gamma \times U) \):

\[ (T_t f)(z_1, c_1, z_2, c_2) = f(T_{-t}(z_1, c_1, z_2, c_2)), \ (K f)(z_1, c_1, z_2, c_2) = f(K(z_1, c_1, z_2, c_2)) \]

The formulae above define also unitary operators \( \hat{T}_t \) and \( \hat{K} \) on \( L^2(\Gamma \times \Gamma) \).

Lemma 5.6. We have the equality of sets: \( T_t T_r = T_{t+r} \), \( T_1 K = K T_1 = T \). The family \( \hat{T}_t \) is a strictly continuous group of unitary multipliers of \( C^*_r(\Gamma) \otimes C^*_r(\Gamma) \). Let \( Z \) be its generator, then on \( \mathcal{A}(\Gamma \times U) \) \( Z = \hat{X} \otimes \log |\hat{Y} + I| \) and \( D(\Gamma \times U) \) is a core for \( Z \).

Proof: First statement is straightforward. Also the fact that \( \hat{T}_t \) is a group of unitary multipliers is clear. So only strict continuity and a statement about \( Z \) require a proof. But this can be done exactly as in lemma 5.1 \[ \square \]

Recall that for a group \( G \) a bicharacter \( c \) is a map \( c : G \times G \to S^1 \) with the properties \( c(g_1, g_2 g_3) = c(g_1, g_2)c(g_1, g_3) \) and \( c(g_1 g_2, g_3) = c(g_1, g_3)c(g_1 g_2, g_3) \). The function \( Ch : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R} \) defined by

\[ Ch(\epsilon_1, \epsilon_2) := \begin{cases} \ -1 & \text{if} \quad \epsilon_1 = \epsilon_2 = -1 \\ \ 1 & \text{otherwise} \end{cases} \]

is a bicharacter and the function \( \chi : (\mathbb{R} \times \mathbb{Z}_2) \times (\mathbb{R} \times \mathbb{Z}_2) \to \mathbb{C} \)

\[ \chi(x, \epsilon_1, y, \epsilon_2) := \exp(ixy) \, Ch(\epsilon_1, \epsilon_2) \]

is a bicharacter.

Let us define the unitary

\[ V := \chi(I \otimes \log |\hat{Y} + I|, I \otimes \text{sgn}(\hat{Y} + I), \hat{X} \otimes I, \hat{J} \otimes I), \]

i.e. \( V = \exp(i\hat{X} \otimes \log |\hat{Y} + I|) \, Ch(I \otimes \text{sgn}(\hat{Y} + I), \hat{J} \otimes I) \).

Lemma 5.6. \( \hat{T} = V \)

Proof: By the previous lemma \( \hat{T} = \hat{T}_1 \hat{K} \) so it remains to prove that \( \hat{K} = Ch(I \otimes \text{sgn}(\hat{Y} + I), \hat{J} \otimes I) \). If \( A, B \) are commuting operators with spectrum contained in \( \{-1, 1\} \) then easy computations show that \( Ch(A, B) = \frac{1}{2}(I + A + B - AB) \). Applying this formula to operators \( A := I \otimes \text{sgn}(\hat{Y} + 1) \) and \( B := \hat{J} \otimes I \) one obtains:

\[ Ch(I \otimes \text{sgn}(\hat{Y} + 1), \hat{J} \otimes I) = \frac{1}{2}(I \otimes (I + \text{sgn}(I + \hat{Y})) + \hat{J} \otimes (I - \text{sgn}(I + \hat{Y}))) \]

and checks that this is exactly \( \hat{K} \). \[ \square \]

In this way we obtain formulae for comultiplication on generators:

\[ \Delta(\hat{Y}) = V(\hat{Y} \otimes I + I \otimes \hat{Y})V^*, \]
\[ \Delta(\hat{X}) = V(\hat{X} \otimes I + I \otimes \hat{X})V^*, \]
\[ \Delta(\hat{J}) = V(\hat{J} \otimes \hat{J})V^* \]

On \( \mathcal{A}(\Gamma \times U) \) we have:

\[ \Delta(\hat{X}) = \hat{X} \otimes (\hat{Y} + I)^{-1} + I \otimes \hat{X}, \ \Delta(\hat{Y}) = I \otimes \hat{Y} + \hat{Y} \otimes I + \hat{Y} \otimes \hat{Y} \]

and these are relations \[ \square \].
In this section we consider the family of groupoids $\Gamma_s$, $s \in \mathbb{R}$ over $A$ defined in the section (2). We define $\lambda_s(0, a)(\partial_b + s\partial_a) := 1$. The corresponding left and right invariant half densities on $\Gamma_s$ are given by:

$$\lambda_s(b, a)(\partial_b + s\partial_a) = \rho_s(b, a)(\partial_b + s\frac{a}{1 + sb}\partial_a) := |1 + sb|^{-1/2}$$

We put $\omega_s := \lambda_s \otimes \rho_s$ and identify $\mathcal{A}(\Gamma_s)$ with $\mathcal{D}(\Gamma_s)$ with multiplication and involution defined by $(f\omega_s)(g\omega_s) := (f \ast g)\omega_s$ and $(f\omega_s)^* := (f^*\omega_s)$:

$$\begin{align*}
(f \ast g)(b, a) &:= \int \frac{dc}{|1 + sc|}f(c, a + s(c - b))g\left(\frac{b - c}{1 + sc}, \frac{a}{1 + sc}\right) \\
&= \int \frac{dc}{|1 + sc|}f\left(\frac{b - c}{1 + sc}, \frac{a - sc}{1 + sc}\right)g(c, \frac{a + sc}{1 + sb})
\end{align*}$$

and $f^*(b, a) := f\left(\frac{-b}{1 + sb}, \frac{a}{1 + sb}\right)$. We will write $f \ast g$ and $f^*$ instead of $f \ast_0 g$ and $f^0$. For $M > 1$ let $K_M := \{(b, a) \in \Gamma_0 : |b| \leq M, \frac{1}{M} \leq |a| \leq M\}$, it is clear that any $f \in \mathcal{D}(\Gamma_0)$ is supported in $K_M$ for sufficiently large $M$. The product $\ast_s$ is in fact defined for all $f, g \in \mathcal{D}(\Gamma_0)$:

**Lemma 6.1.** Let $f, g \in \mathcal{D}(\Gamma_0)$ have supports in $K_M$. Then for any $s \in \mathbb{R}$ the function $f \ast_s g$ defined by (38) is smooth and has the support in the set $(b, a) \in \Gamma_0 : |b| \leq M(2 + |s|M), \frac{1}{M(1 + M^2|s|)} \leq |a| \leq M(1 + |s|M))$. In particular, for $|s| \leq \delta$ functions $f \ast_s g$ have supports contained in a fixed compact set.

**Proof:** Let $(b, a, s) \in \Gamma_0 \times \mathbb{R}$. It is straightforward that the function

$$\mathbb{R} \ni b' \mapsto f(b', a + s(b' - b))g\left(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}\right)$$

is smooth and has compact support contained in the set $\{b' \in \mathbb{R} : 1 + sb' \neq 0\}$, so smoothness follows. For $(f \ast_s g)(b, a) \neq 0$ it is necessary that there exists $b'$ such that

$$|b'| \leq M, \frac{1}{M} \leq |a + s(b' - b)| \leq M, \left|\frac{b - b'}{1 + sb'}\right| \leq M, \frac{1}{M} \leq \left|\frac{a}{1 + sb'}\right| \leq M.$$ If $|a| < \frac{1}{M(1 + M^2|s|)}$ then

$$\frac{1}{M} \leq |a + s(b' - b)| \leq |a| + |s||b' - b| < \frac{1}{M(1 + M^2|s|)} + |s|M|1 + sb'| \leq \frac{1}{M(1 + M^2|s|)} + |s|M^2|a| < \frac{1}{M(1 + M^2|s|)} + \frac{1}{M(1 + M^2|s|)} = \frac{1}{M}$$

In a similar way, if $|a| > M(1 + |s|M)$ then:

$$1 + |s|M \geq 1 + |s||b'| \geq |1 + sb'| \geq \frac{|a|}{M} > 1 + |s|M$$

The estimates for $|a|$ are proven. If $|b| > M(2 + |s|M)$ then

$$M(2 + |s|M) < |b| \leq |b - b'| + |b'| \leq M|1 + sb'| + M \leq M(1 + |s|M) + M = M(2 + |s|M).$$

The choice $\nu(a)(\partial_a) := \frac{1}{|a|}$ defines real, non vanishing half density $\Psi_s := \rho_s \otimes \nu$ on $\Gamma_s$ and short calculation gives: $\Psi_s(b, a)(\partial_b, \partial_a) := \Psi_0(b, a)(\partial_b, \partial_a) = \frac{1}{|a|}$. This makes possible identification of all spaces $L^2(\Gamma_s)$ with $L^2(\Gamma_0)$. The identity representation of $\mathcal{D}(\Gamma_s)$ is then given by: $\pi_s(f)(g\Psi_0) = (f \ast_s g)\Psi_0$ for $f, g \in \mathcal{D}(\Gamma_s)$. The norms on $\mathcal{D}(\Gamma_s)$ defined by $\omega_s$ will be denoted by $\|f\|_{l,s}, \|f\|_{r,s}, \|f\|_{0,s}$ and $\|f\|_s$ is the norm of $\pi_s(f)$. Finally let us define for $f, g \in \mathcal{D}(\Gamma_0)$:

$$\{f, g\} := (a - 1)([(\partial_a f) \ast (bg)] - (\partial_a g) \ast (bf)).$$

In this formula $(a - 1)f$ and $bf$ denote functions $((a - 1)f)(b, a) := (a - 1)f(b, a)$ and $(bf)(b, a) := bf(b, a)$. 

\[\square\]
Lemma 6.2. \{\cdot, \cdot\} is a Poisson bracket on \(\mathcal{D}(\Gamma_0)\) and \(\{f_1^*, f_2^*\} = \{f_2, f_1\}^*\).

Proof: The mappings \(f \mapsto (a-1)\partial_a f\) and \(f \mapsto bf\) are commuting derivations of a commutative algebra \((\mathcal{D}(\Gamma_0),\ast)\), moreover \((a-1)\partial_a (f^*) = ((a-1)\partial_a f)^*\) and \(bf^* = -(bf)^*\).

It is clear that if \(f \in \mathcal{D}(\Gamma_0)\), then there exists \(\delta > 0\) such that \(f \in \mathcal{D}(\Gamma_\delta)\) for all \(|s| \leq \delta\), e.g. if support of \(f\) is contained in a set \(K_M\), take any \(\delta < \frac{1}{M^2}\). Let us define linear spaces

\[D(Q_s) := \{f \in \mathcal{D}(\Gamma_0) : |\delta| \leq |s| \Rightarrow f \in \mathcal{D}(\Gamma_\delta)\}.
\]

This family of subspaces has properties:

\[|r| \leq |s| \Rightarrow D(Q_s) \subset D(Q_r), D(Q_s)^* = D(Q_s), \bigcup_{s \neq 0} D(Q_s) = \mathcal{D}(\Gamma_0).
\]

Let us define linear map \(Q_s : \mathcal{D}(\Gamma_0) \supset D(Q_s) \ni f \mapsto f \in \mathcal{D}(\Gamma_s)\). With this definition we have

**Proposition 6.3.**

1. \(Q_0 = id\);
2. \(\lim_{s \to 0} ||Q_s(f^*) - Q_s(f)^*||_s = 0\);
3. \(\lim_{s \to 0} ||Q_s(f) \ast_Q Q_s(g) - Q_s(f \ast g)||_s = 0\);
4. \(\lim_{s \to 0} ||\frac{1}{2}[Q_s(f), Q_s(g)] - Q_s(\{f,g\})||_s = 0\);
5. The function \(s \mapsto ||Q_s(f)||_s\) is continuous for \(s \neq 0\) and lower semicontinuous at \(s = 0\).

Proof: First statement is evident. To prove statements (2)-(4) it is enough to prove that norms \(||\cdot||_{0,s}\) converge to 0. In fact we claim that convergence of norms \(||\cdot||_{0,s}\) is sufficient. In the following to simplify notation we write \(Q_s(f)Q_s(g)\) instead of \(Q_s(f) \ast_Q Q_s(g)\) and \(Q_s(f)^*\) instead of \(Q_s(f)^*\). Let us compute:

\[||Q_s(f^*)^* - Q_s(f)^*||_{r,s} = ||Q_s(f^*) - Q_s(f)^*||_{l,s} = ||Q_s((f^*)^*) - Q_s(f^*)||_{l,s} = ||Q_s(f_1^*) - Q_s(f_1)||_{l,s},\]

where \(f_1 := f^*\). So once we know that \(\lim_{s \to 0} ||Q_s(f^*) - Q_s(f)||_{l,s} = 0\) for any \(f \in \mathcal{D}(\Gamma_0)\), the second statement is proven.

In a similar way:

\[||Q_s(f)Q_s(g) - Q_s(f \ast g)||_{r,s} = ||Q_s(g)^*Q_s(f)^* - Q_s(f \ast g)^*||_{l,s}\]

Let \(f_1 := f^*, g_1 := g^*,\) then

\[Q_s(g_1)^*Q_s(f_1)^* - Q_s(f \ast g)^* = Q_s(g_1)^*Q_s(f_1)^* - Q_s(f_1^* \ast g_1^*) =
\]

\[(Q_s(g_1)^* - Q_s(g_1)) (Q_s(f_1)^* - Q_s(f_1)) + (Q_s(g_1)^* - Q_s(g_1)) Q_s(f_1) + Q_s(g_1) (Q_s(f_1)^* - Q_s(f_1)) +
\]

\[+ (Q_s(g_1 \ast f_1) - Q_s((g_1 \ast f_1)^*)) + Q_s(g_1)Q_s(f_1) - Q_s(g_1 \ast f_1)\]

And once more:

\[||[Q_s(f), Q_s(g)] - sQ_s(\{f,g\})||_{r,s} = ||[Q_s(f), Q_s(g)]^* - sQ_s(\{f,g\})^*||_{l,s}\]

\[|Q_s(f), Q_s(g)]^* - sQ_s(\{f,g\})^* = Q_s(g)^*Q_s(f)^* - Q_s(f)^*Q_s(g)^* - sQ_s(\{f,g\})^* =
\]

\[= Q_s(g_1)^*Q_s(f_1)^* - Q_s(f_1^* \ast g_1^*) - sQ_s(\{f_1, g_1\})^* = [Q_s(g_1)^*, Q_s(f_1)^*] - sQ_s(\{g_1, f_1\})^* =
\]

\[= [Q_s(g_1)^* - Q_s(g_1), Q_s(f_1)^* - Q_s(f_1)] + (Q_s(g_1)^* - Q_s(g_1))Q_s(f_1) + [Q_s(g_1), Q_s(f_1)^* - Q_s(f_1)] +
\]

\[- s[Q_s(\{g_1, f_1\})^* - Q_s(\{g_1, f_1\})] + ([Q_s(g_1), Q_s(f_1)] - sQ_s(\{g_1, f_1\}))\]

Using the equalities above one can see that to prove statements 2-4 of the proposition it is enough to prove
Lemma 6.4. For any $f \in \mathcal{D}(\Gamma_0)$ there is $\delta > 0$, $C > 0$ such that $\|Q_s(f)\|_{l,s} \leq C$ for $|s| \leq \delta$; Moreover for any $f, g \in \mathcal{D}(\Gamma_0)$:

$$\lim_{s \to 0} \|Q_s(f^*)_s - Q_s(f)\|_{l,s} = \lim_{s \to 0} \|Q_s(f)_s Q_s(g) - Q_s(f \ast g)\|_{l,s} = 0.$$  

Proof: Let $\text{supp} \{f\} \subset K_M \subset \Gamma_s$ for $|s| \leq \delta$. Then $\|Q_s(f)\|_{l,s} = \sup_{a \in A} \int_{\mathcal{B}_s} \frac{db'}{1 + |s|b'} |f(b', a + sb')|$. For $\delta < \frac{1}{M}$: $\frac{1}{1 + |s|b'} \leq \frac{1}{1 + |s|} |b'|$ on the support of $f$, therefore $\|Q_s(f)\|_{l,s} \leq \frac{2M}{1 - |s|} \sup |f|$ and the estimate proves the first statement.

Let $\text{supp} f \cup \text{supp} f^* \subset K_M$, for any $|s| < \frac{1}{2M} \sup |f|$, $\text{supp} f^* \subset \Gamma_s$. Define $F(b, a) := (Q_s(f^*)_s - Q_s(f))(b, a) = f(\frac{b}{1 + sb'}, \frac{a}{1 + sb'}) - f(b, a)$. Fix $a \in A$ and consider the function $b' \mapsto |F(b', a + sb')|$. Its support is contained in $S_1 = \{ b' : |b'| \leq M, \frac{1}{M} \leq |a + sb'| \leq M \}$. \begin{equation*} S_2 := \{ b' : |b'| \leq M, \frac{1}{M} \leq |a + sb'| \leq M \} \end{equation*}

Now, for $|s| < \frac{1}{2M}$ we get that union of these sets is contained in the set $S_3 := \{ b' : |b'| \leq 2M \}$ and on this set we have estimates:

$$\frac{1}{1 + |s|b'} \leq 2 \text{ and } \frac{1}{M} \leq \frac{a + sb'}{1 + sb'} \leq 2M.$$  

Using these estimates and equalities:

$$\frac{b}{1 + sb'} = b' + \frac{-sb^2}{1 + sb'} \frac{a + sb}{1 + sb} = b' + \frac{-sb(a + sb)}{1 + sb}$$

we get:

$$|F(b', a + sb')| \leq \sup |f'(a)| \left( \frac{-sb^2}{1 + sb'} \frac{a + sb}{1 + sb} \right) \leq |s| \sup |f'(a)| |(2|b'|^2 + 2M|b'|)|$$

It follows that $\lim_{s \to 0} |F(b', a + sb')|_{l,s} = 0$, so $\lim_{s \to 0} \|Q_s(f^*)_s - Q_s(f)\|_{l,s} = 0$.

For sufficiently small $|s|$ the functions $(Q_s(f)Q_s(g) - Q_s(f \ast g)$ have support in a fixed compact subset of $\Gamma_s$, so we have to show that $Q_s(f)Q_s(g) - Q_s(f \ast g)$ converges to 0 uniformly. So choose $M$ such that $\text{supp} f \cup \text{supp} g \subset K_M$, $|s|M^2 < 1$. Then

$$(Q_s(f)Q_s(g) - Q_s(f \ast g))(b, a) = \int_{-M}^{M} db' \left( \frac{1}{1 + sb'} f(b', a + s(b - b')) g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - f(b', a) g(b - b', a) \right).$$

For $|b'| \leq M$ we have

$$\left| \frac{1}{1 + sb'} f(b', a + s(b - b')) g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - f(b', a) g(b - b', a) \right| \leq (1 + sb')^{-1} \left| f(b', a + s(b - b')) g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - f(b', a) g(b - b', a) - sb' f(b', a) g(b - b', a) \right| \leq (1 - |s| M)^{-1} \left| f(b', a + s(b - b')) g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - f(b', a) g(b - b', a) \right| + |s| M \sup |f g|$$

For the first term there is an estimate:

$$\left| f(b', a + s(b - b')) g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - f(b', a) g(b - b', a) \right| \leq |s| \sup |g| \sup |\partial_a f| |b - b'| + \sup |f| \left| g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - g(b - b', a) \right|$$

and finally:

$$\left| g(\frac{b - b'}{1 + sb'}, \frac{a}{1 + sb'}) - g(b - b', a) \right| \leq |s| M \sup |g'| \left( \left| \frac{b - b'}{1 + sb'} \right| + \left| \frac{a}{1 + sb'} \right| \right) \leq |s| M \sup |g'| (1 - |s| M)^{-1} (|b| + |a| + M) ,$$

since $b, a$ are in a fixed compact set, convergence is uniform and $\lim_{s \to 0} \|Q_s(f^*)_s Q_s(g) - Q_s(f \ast g)\|_{l,s} = 0.$
In a similar way, to prove the third equality we have to show that the functions \( \frac{1}{s}[Q_s(f), Q_s(g)] - Q_s(\{f, g\}) \) converge to 0 uniformly. Again, let \( f, g, \{f, g\} \) be supported in \( K_M \subset \Gamma_s^0 \):

\[
(Q_s(f)Q_s(g))(b, a) = \int_{-M}^{M} \frac{db'}{1 + sb'} f(b', a + s(b' - b)) g(b + b', a + s(b' - b)) - a \frac{b - b'}{1 + sb'}
\]

\[
(Q_s(g)Q_s(f))(b, a) = \int_{-M}^{M} \frac{db'}{1 + sb'} g(b - b', a - sb) f(b', a + s(b' - b)) - a \frac{b - b'}{1 + sb'}
\]

\[
(Q_s(\{f, g\}))(b, a) = (a - 1) \int_{-M}^{M} db' \left[ (\partial_a f)(b', a) (b - b') g(b', a; \partial_a g)(b - b', a) \right]
\]

(in the formula for \( Q_s(g)Q_s(f) \) we use second expression of \( \{SS\} \)). This computation is straightforward and can be done as in the previous point. This ends the prove of the lemma and statements (2)-(4) of the proposition \([6,3]\).

Now we come to the last point. Let’s start with lower semicontuity at \( s = 0 \). So we have to prove that:

\[
\forall f \in D(\Gamma_0) \exists \epsilon > 0 \exists \delta > 0 \ (|s| < \delta \Rightarrow ||Q_s(f)||_s > ||f|| - \epsilon) \quad (||f||-denotes \ the \ operator \ norm \ of \ \pi_0(f) \ on \ L^2(\Gamma_0)).
\]

The following simple lemma reduces the problem to strong convergence of operators \( Q_s(f) \):

Lemma 6.5. Let \( H \) be a Hilbert space, \( A \in B(H), A_s - family \ of \ bounded \ operators \ defined \ for \ some \ neighbourhood \ of \ 0 \in \mathbb{R} \ and \ V \subset H \ a \ dense \ subspace. Assume \ that \ for \ every \ v \in V \ \lim_{s \to 0} A_s v = Av. \ Then \ \forall \epsilon > 0 \exists \delta > 0 \ (|s| < \delta \Rightarrow ||A_s|| > ||A|| - \epsilon)\]

So we have to prove that for \( f, g \in D(\Gamma_0) \) \( f \ast_s g - f \ast g \) converges to 0 in \( L^2(\Gamma_0) \) as \( s \) goes to 0. But for sufficiently small \( |s| \) these functions have supports in a fixed compact set, so it is enough to prove uniform convergence and this can be done as in the previous points.

Now, for \( s, r \neq 0 \), consider the map \( \Phi_{sr} : \Gamma_r \ni (b, a) \mapsto (sb, a) \in \Gamma_s \). This is a diffeomorphism and an isomorphism of groupoids. It defines a \( * \)-isomorphism \( \Phi_{sr} : D(\Gamma_r) \to D(\Gamma_s) \) and unitary operator \( V_{sr} \) on \( L^2(\Gamma_0) \). The formulae are:

\[
(\Phi_{sr}f)(b, a) := \frac{s}{r} f(srb, a), \quad V_{sr}(f \psi_0) = \sqrt{\frac{s}{r}} (\Phi_{sr} f) \psi_0, \quad f \in D(\Gamma_r)
\]

Moreover, we have:

\[
V_{rs} V_{sr} = I, \quad V_{rs} \pi_s(f) V_{sr} = \pi_r(\Phi_{rs}(f)) \quad f \in \mathcal{D}(\Gamma_s)
\]

Let \( f \in \mathcal{D}(\Gamma_s) \) then there exists \( |s| > \delta > 0 \) such that \( f \in \mathcal{D}(\Gamma_{s+\epsilon}) \) for \( |\epsilon| < \delta \).

\[
||Q_s+\epsilon(f)||_{s+\epsilon} = ||\pi_{s+\epsilon}(f)|| = ||V_{s+\epsilon} \pi_{s+\epsilon}(f) V_{s+\epsilon||s}|| = ||\pi_s(\Phi_{ss+\epsilon}(f))|| = ||Q_s(\Phi_{ss+\epsilon}(f))||_s
\]

Now, it is straightforward to check that the functions \( \Phi_{ss+\epsilon}(f) - f \) have support in a fixed compact set and tend to 0 uniformly as \( \epsilon \) goes to 0. This ends the proof of the proposition \([6,3]\).

The groupoid \( \Gamma_0 \) is a (trivial) bundle of groups \( \mathbb{R} \), as such it can be identified with its Lie algebroid \( \mathbb{R} \times A \). For \( f \in \mathcal{D}(\Gamma_0) \) let \( (F(f))(\beta, a) := \int db e^{-2\pi i \beta b} f(b, a) \) be the (partial) Fourier transform. Let \( \mathcal{B} := \mathcal{F}(D(\Gamma_0)) \), then it is a \( * \)-subalgebra of smooth functions (with respect to pointwise multiplication and complex conjugation as an involution) on a dual bundle. For \( F, G \in \mathcal{B} \) let us define:

\[
\{F, G\} := -i \mathcal{F}(\{F^{-1}F, F^{-1}G\})
\]

Then straightforward calculation gives:

\[
(39) \quad \{F, G\}(\beta, a) = \frac{a - 1}{2\pi} ((\partial_a F)(\partial_\beta G)(\beta, a) - (\partial_a G)(\partial_\beta F)(\beta, a))
\]

On the other hand recall the morphism \( m^F_B : \Gamma_0 \to \Gamma_0 \times \Gamma_0 \) defined in section 2

\[
Gr(m^F_B) := \{(b_2a_1, a_1, b_2, a_2; b_2a_1, a_1, a_2)\}
\]
Applying the cotangent lift (see [5]) we obtain the morphism (of symplectic groupoids) $T^*(\Gamma_0)$ and $T^*(\Gamma_0) \times T^*(\Gamma_0)$, its base map is nothing but the multiplication in the subgroup $(TA)^0 \subset T^*(\Gamma_0)$. The map $\mathbb{R} \times A \ni (\beta, a) \mapsto \beta db(0,a) \in (TA)^0$ identifies the group $(TA)^0$ with $\mathbb{R} \times A$ with the multiplication:

$$(\beta_1, a_1)(\beta_2, a_2) := (\beta_1 + a_1^{-1}\beta_2, a_1a_2)$$

(which is again 'ax+b' group in different presentation). One easily checks that the bracket (39) is a Poisson-Lie bracket on this group.

7. Appendix

Here we collect some formulae proven in [6] and used in this paper. $(G; A, B)$ is a double Lie group, $g, a, b$ are corresponding Lie algebras and $g = a \oplus b$ (direct sum of vector spaces).

Modular functions. Let $P_A, P_B$ be projections in $g$ corresponding to the decomposition $g = a \oplus b$. Let us define:

$$j_A(g) := |\det(P_A Ad(g)|_a|), \quad j_B(g) := |\det(P_B Ad(g)|_b|)$$

The choice of $\omega_0$. Choose a real half-density $\mu_0 \neq 0$ on $T_G B$ and define left-invariant half-density on $G_A$

$$\lambda_0(g)(v) := \mu_0(g^{-1}v), \quad v \in \Lambda^{max}T^*_g G_A$$

Then the corresponding right-invariant half-density is given by:

$$\rho_0(g)(w) := j_B(a_L(g))^{-1/2}\mu_0(wg^{-1}), \quad w \in \Lambda^{max}T^*_g G_A.$$

Multiplication and comultiplication in $\mathcal{A}(G_A)$. After a choice of $\omega_0$ as above, the multiplication in $\mathcal{A}(G_A)$ reads: $(f_1 \omega_0)(f_2 \omega_0) =: (f_1 * f_2)\omega_0$ and

$$(f_1 * f_2)(g) = \int_B db f_1(a_L(g)b)f_2(bL(a_L(g))^{-1}g) =$$

$$\int_B db_b j_B(a_L(ba_R(g))^{-1}f_1(gb_R(ba_R(g))^{-1})f_2(ba_R(g)),$$

where $db$ and $db_b$ are left and right Haar measures on $B$ defined by $\mu_0$.

The $|| \cdot ||_l$ defined by this $\omega_0$ is given by:

$$||f||_l = \sup_{a \in A} \int_B db |f(ab)|$$

The formula for $\hat{\delta}_0$ reads

$$(\hat{\delta}_0(f)F)(a_1b_1, a_2b_2) = \int_B db j_B(b_L(a_2b))^{-1/2}f(a_1a_2b)F(b_L(a_1a_2b)^{-1}a_1b_1, a_R(a_2b)^{-1}b^{-1}a_2b_2)$$

The mappings $(\bar{id} \times \bar{\delta}_0)$ and $(\bar{\delta}_0 \times id)$ are given by:

$$\begin{align*}
\left[(\bar{id} \times \bar{\delta}_0)(F_1)F_2\right](a_1, a_2, b_1, b_2, b_3) &= \int_{B \times B} db dB F_1(a_1b', a_2a_3b''b) \\
\times F_2(b_L(a_1b')^{-1}a_1b_1, b_L(a_2a_3b'')^{-1}a_2b_2, a_R(a_3b'')^{-1}b''^{-1}b_3)j_B(b_L(a_3b''))^{-\frac{1}{2}}
\end{align*}$$

$$\begin{align*}
\left[\bar{\delta}_0 \times id\right](F_1)F_2(a_1, a_2, b_1, b_2, b_3) &= \int_{B \times B} db dB F_1(a_1a_2b', a_3b) \\
\times F_2(b_L(a_1a_2b')^{-1}a_1b_1, a_R(a_2b')b^{-1}b_2, b_L(a_3b'')^{-1}a_3b_3)j_B(b_L(a_2b'))^{-\frac{1}{2}}
\end{align*}$$
References

[1] G. Skandalis, Duality for locally compact ’quantum groups’ (joint work with S. Baaj), Mathematisches Forschungsinstitut Oberwolfach, Taungsbericht 46/1991, C*-algebren, 20,10-26.10.1991, p. 20;
[2] S. Vaes, L. Vainerman, Extensions of locally compact quantum groups and the bicrossed product construction, Adv. in Math 175 (1) (2003), 1-101;
[3] S. Baaj, G. Skandalis, S. Vaes Non-semi-regular quantum groups coming from number theory, Comm. Math. Phys. 235 (1) (2003), 139-167;
[4] S. Zakrzewski Quantum and Classical pseudogroups I Comm. Math. Phys. 134 (1990), 347-370;
[5] S. Zakrzewski Quantum and classical pseudogroups. II. Differential and symplectic pseudogroups, Comm. Math. Phys., 134 (2) (1990), 371-395;
[6] P. Stachura, From double Lie groups to quantum groups, Fund. Math. 188 (2005), 195-240.
[7] P. Stachura, Differential groupoids and C*-algebras, [math.QA/9905097] for a short exposition see: C*-algebra of a differential groupoid, Banach Center Publ 51, Inst. Math. Polish Acad. Sci., 2000, 263-281.
[8] S. L. Woronowicz, K. Napiórkowski, Operator theory in the C*-algebra framework, Reports on Math. Phys. 31 (3) (1992), 353-371.

Faculty of Applied Informatics and Mathematics, Warsaw University of Life Sciences-SGGW, ul Nowoursynowska 166, 02-787 Warszawa, Poland, e-mail: stachura@fuw.edu.pl