New Pentagon Identities Revisited

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Abstract. We present a new solution to the pentagon identity in terms of gamma function. We obtain this solution by taking the gamma function limit from the pentagon identity related to the three-dimensional index. This limit corresponds to the identification of the sphere partition function of dual theories and being equivalent to the star-triangle relation in statistical mechanics which corresponds to the “strongly coupled” regime of the Faddeev-Volkov model.

1. Introduction
Following identity is called the pentagon identity [1] for non-commutative operators \( xy = qyx \)

\[ l(y)l(x) = l(x)l(-xy)l(y), \] (1)

where \( l(x) = \prod_{i=1}^{\infty} (1 - qx^i) \) and \(|q| < 1\) for complex variable \( x \).

This is the quantum generalization of the classical pentagon identity

\[ L(x) + L(y) - L(xy) = L\left(\frac{x - xy}{1 - xy}\right) + L\left(\frac{y - xy}{1 - xy}\right), \] (2)

where Rogers’ dilogarithm \( L(x) \) is defined as

\[ L(x) := -\int_0^x \frac{\log(1 - z)}{z} dz + \log(1 - x) \log(x)/2. \] (3)

Notice that, pentagon identity for the quantum dilogarithm has been interpreted as the restricted star-triangle relation in statistical mechanics [1] and Pachner-moves [3].

Consider the following five-term relation \(^1\)

\[ \int d[\sigma] \mathcal{B} \mathcal{B} \mathcal{B} = \mathcal{B} \mathcal{B}. \] (4)

Recently, one solution of this identity is observed in terms of hyperbolic gamma function and plays very important role in shaped triangulation \(^2\). \(^2\)

Integral identity for this case reads

\[ \int_{-i\infty}^{i\infty} \frac{du}{i\sqrt{\omega_1 \omega_2}} \prod_{i=1}^{3} B(a_i + u, b_i - u) = B(a_1 + b_2, a_3 + b_1)B(a_2 + b_1, a_3 + b_2), \] (5)

\(^1\) Here we define pentagon identity in the integral form.

\(^2\) For the case of ideal triangulation, see [7, 17]
where $B(x, y)$ is defined as
\[
B(x, y) := \frac{\gamma(2)(x; \omega_1, \omega_2)\gamma(2)(y; \omega_1, \omega_2)}{\gamma(2)(x + y; \omega_1, \omega_2)}.
\]
(6)

The hyperbolic gamma function is defined as
\[
\gamma(2)(u; \omega_1, \omega_2) := e^{-\pi i B_{2, 2}(u; \omega)} \left( e^{2\pi i u/\omega_1} \tilde{q}, \tilde{q} \right)
\]
with $q := e^{2\pi i \omega_1/\omega_2}$, $\tilde{q} := e^{-2\pi i \omega_2/\omega_1}$,
(7)
where $B_{2, 2}(u; \omega)$ is the second order Bernoulli polynomial,
\[
B_{2, 2}(u; \omega) := \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6 \omega_2} + \frac{\omega_2}{6 \omega_1} + \frac{1}{2}.
\]
(8)

Identity (5) describes the matching of $S^3$ supersymmetric partition functions [11] of 3-d duality with $U(1)$ gauge theories and six chiral multiplet for electric part, while no gauge degree of freedom and 9 chiral multiplets for magnetic part. 3

Considering the same duality for three dimensional index leads to the following pentagon identity [14] 4
\[
\sum_{m=-\infty}^{\infty} \oint \frac{dz}{2\pi i z} \prod_{i=1}^{3} B(a_i z, n_i + m; b_i z^{-1}, m_i - m)
\]
\[
= B(a_1 b_2, n_1 + m_2; a_3 b_1, n_3 + m_1) B(a_2 b_1, n_2 + m_1; a_3 b_2, n_3 + m_2),
\]
(9)
where the corresponding $B(a, n; b, m)$ is defined as
\[
B(a, n; b, m) := \frac{(q^{1+\frac{3}{2}} a^{-1}; q)_\infty}{(q^3 a; q)_\infty} \frac{(q^{1+\frac{3}{2}} b^{-1}; q)_\infty}{(q^3 b; q)_\infty} \frac{(q^{1+n+m} ab; q)_\infty}{(q^{1+n+m} (ab)^{-1}; q)_\infty}.
\]
(10)

In this letter we study gamma function limit of identity (9) and present connection to statistical mechanics and superymmetric sphere partition function. The structure of the paper starts with gamma function limit of three dimensional index in section 2. Afterwards, we study different gamma function limits of $S^3$ partition function in section 3.

2. Gamma function limit

In this section we consider the gamma function limit of the pentagon identity related to the three dimensional index. This limit corresponds to the new pentagon identities in terms of gamma function. This pentagon identity have interpretation as the identification of the sphere partition function for the dual theories.

In order to consider the gamma function limit we take the following limit
\[
\lim_{q \to 1} \frac{1 - q}{(q^b; q)_\infty} (1 - q)^{\alpha - \beta} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}
\]
(11)
from the pentagon identity (9) which gets the following form
\[
\sum_{m=-\infty}^{\infty} \oint \prod_{i=1}^{3} \frac{(q^{1+(m+n_i)/2} a_i z, q^{1+(n_i-m)/2} z/b_i; q)_\infty}{(q^{m+n_i}/2 a_i z, q^{m-n_i}/2 b_i z; q)_\infty} (1 - q^m z^2)(1 - q^m z^{-2}) \frac{dz}{2\pi i z}
\]
\[
= \frac{2}{\prod_{i,j=1}^{3} a_i^{m_j} b_i^{n_j}} \prod_{i,j=1}^{3} \frac{(q^{1+(m+n_j)/2} a_i b_j; q)_\infty}{(q^{m+n_i}/2 a_i b_j; q)_\infty},
\]
(12)
\[3\text{See [7, 10, 16] for pentagon identities from gauge theories.}\]
\[4\text{Special case of this identity corresponds to the pentagon identity related to the tetrahedron index [9].}\]
with the conditions \( \prod_{i=1}^{3} a_i = \prod_{i=1}^{3} b_i = q^{\frac{1}{2}} \) and \( \sum_{i=1}^{3} m_i = \sum_{i=1}^{3} n_i = 0 \). After the limit we obtain the following identity

\[
\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{du}{2\pi} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{m+n_i}{2} + \alpha_i + iu\right)}{\Gamma(1 + \frac{m+n_i}{2} - \alpha_i - iu)} \frac{\Gamma\left(\frac{m+m_i}{2} + \beta_i + iu\right)}{\Gamma(1 + \frac{m+m_i}{2} - \beta_i + iu)} = \prod_{i,j=1}^{3} \frac{\Gamma(\alpha_i + \beta_j + \frac{n_i+m_j}{2})}{\Gamma(1 - \alpha_i - \beta_j - \frac{n_i+m_j}{2})},
\]

(13)

with the conditions \( \sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \beta_i = \frac{3}{2} \) and \( \sum_{i=1}^{3} m_i = \sum_{i=1}^{3} n_i = 0 \). It is interesting that this identity corresponds to the identification of the sphere partition functions [13] for the three-dimensional duality which we discussed below the formula (8). It would be interesting to study such duality in two dimension.

Identity (13) is equivalent to the following pentagon identity

\[
\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{du}{2\pi} \prod_{i=1}^{3} B(\alpha_i + iu, n_i + m; \beta_i - iu, m_i - m)
= B(\alpha_1 + \beta_2, n_1 + m_2; \alpha_3 + \beta_1, n_3 + m_1) B(\alpha_2 + \beta_1, n_2 + m_1; \alpha_3 + \beta_2, n_3 + m_2),
\]

(14)

where \( B(a, n; b, m) \) reads

\[
B(a, n; b, m) := \frac{\Gamma(a + \frac{n}{2})}{\Gamma(1 - a + \frac{n}{2})} \frac{\Gamma(b + \frac{m}{2})}{\Gamma(1 - b + \frac{m}{2})} \frac{\Gamma(1 - a - b + \frac{n+m}{2})}{\Gamma(a + b + \frac{n+m}{2})}.
\]

(15)

Statistical mechanics interpretation of this pentagon identity is equivalent to the star-triangle relation with continuous and discrete spin parameters.5

3. Other Gamma Function Limits

In this section we will consider other gamma function solutions of pentagon identity. There are two different gamma function limits of \( S_b^3 \) partition function for dual theories.

If we consider the following limit

\[
\lim_{\omega_2 \to -\infty} \gamma^{(2)}(z; \omega_1, \omega_2) = \left( \frac{\omega_2}{2\pi \omega_1} \right)^{\frac{1}{2}} \left( \frac{z}{\omega_1} \right)^{\frac{1}{2}} \frac{\Gamma(z/\omega_1)}{2\pi}.
\]

(16)

for the pentagon identity (5), we will get the pentagon identity [8] in terms of gamma function. Pentagon identity (4) gets the following form with \( B(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) [18]

\[
\int_{-\infty}^{\infty} \frac{du}{2\pi i} \prod_{i=1}^{3} B(a_i + u, b_i - u) = B(a_1 + b_2, a_3 + b_1) B(a_2 + b_1, a_3 + b_2).
\]

(17)

From statistical mechanics point of view identity (5) is equivalent to the star-triangle relation [6] for Faddeev-Volkov model [2] with continuous spin. In [4], authors also studied one of the main physical regime so-called “strongly coupled” regime of Faddeev-Volkov model corresponds to \( \omega_1 \to 1 \). We note that in that case star-triangle relation is equivalent to the identity (13) obtained in this paper.

5 See [6, 12, 15] for the relation between supersymmetric dualities and star-triangle relation.
4. Conclusion
We study different gamma function limits of the pentagon identities related to three-dimensional index and $S^3_b$ partition function. We obtain a new solution to the pentagon identity in terms of gamma function (15) by considering $q \to 1$ limit of the index. This solution is different from the one in literature (17). Pentagon identity (13) is equivalent to the “strongly coupled” regime of Faddeev-Volkov model. We conclude that $|b| \to 1$ limit of $S^3_b$ partition function for the duality is equivalent to $q \to 1$ of three dimensional index for the same duality.

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